MAT258S25 Proof 3

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Lemma 1 ($\mathcal{L}_1$)**.** The nth triangular number can be calculated by the explicit formula:

$$T_n = \sum_{i=1}^{n} i$$

Lemma 2 ($\mathcal{L}_2$)**.** If some number m^2 is both odd and a perfect square, m must be odd.

Lemma 3 ($\mathcal{L}_3$)**.** For any odd positive integer m, $\exists n \in \mathbb{Z}, n \geq 0 : m = 2n + 1$.

Lemma 4 ($\mathcal{L}_4$)**.** $\mathbb{Z}^+$ is closed under addition, subtraction, and multiplication.

Let $T_m = 1 + 2 + 3 + \cdots + m$ where $m \in \mathbb{Z}$ with $m \geq 1$ (T_m is the mth triangular number).

1. Use induction to prove $\forall m \in (\mathbb{Z}^+ \setminus \{0\}) : T_m = \frac{m(m+1)}{2}$.

   Proof. Let $P_1(m)$ be the proposition that $T_m = \frac{m(m+1)}{2}$ is true.

$$\begin{aligned}
\mathcal{L}_1 &\implies T_1 = \sum_{i=1}^{1} i = 1 \\
P_1(1) &\implies T_1 = \frac{m(m+1)}{2} \\
&\qquad = \frac{1(1+1)}{2} \\
&\qquad = 1 \\
&\qquad \therefore\ P_1(1) \text{is true.}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_1 \implies T_{k+1} &= \sum_{i=1}^{k+1} i \\
&= \sum_{i=1}^{k} i + (k+1) \\
\therefore T_{k+1} &= T_k + (k+1)
\end{aligned}$$

$$\begin{aligned}
P_1(k) &\implies T_k = \frac{k(k+1)}{2} \\
P_1(k+1) &\implies T_{k+1} = \frac{(k+1)(k+2)}{2}
\end{aligned}$$

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\begin{aligned}
P_1(k) \wedge \mathcal{L}_1 \implies T_{k+1} &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{k^2 + k + 2k + 2}{2} \\
&= \frac{k^2 + 3k + 2}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
\therefore P_1(k) \wedge \mathcal{L}_1 &\implies P_1(k+1)
\end{aligned}
$$

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\begin{aligned}
\mathcal{L}_1 &\implies P_1(1) \\
\mathcal{L}_1 \wedge P_1(k) &\implies P_1(k+1) \\
\therefore \forall m \in \operatorname{dom}(\mathcal{L}_1) &: P_1(m) \text{ is true.}
\end{aligned}
$$

QED

2. Prove that the positive integer n is a triangular number (P_2) if and only if $8n+1$ is a perfect square (P_3).

Proof.

$$P_3 \implies m \in \mathbb{Z}^+ : m^2 = 8n+1$$

$$\begin{aligned} \text{let } a &= 4n \\ 8n+1 &= 2a+1 \\ \mathcal{L}_3 &\implies 8n+1 \text{ is odd.} \\ \mathcal{L}_2 &\implies m \text{ is odd.} \end{aligned}$$

$$\begin{aligned} m^2 &= 8n+1 \\ \text{let } b \in \mathbb{Z}, b \geq 0 : m &= 2b+1; \\ (2b+1)^2 &= 8n+1 \\ 4b^2+4b+1 &= 8n+1 \\ n &= \frac{4b^2+4b}{8} \\ n &= \frac{b^2+b}{2} \\ n &= \frac{b(b+1)}{2} \\ n = T_b &\implies P_2 \\ \therefore P_3 &\implies P_2 \end{aligned}$$

$$\begin{aligned} P_2 \implies \exists k \in \mathbb{Z}^+ : n &= \frac{k(k+1)}{2} \\ 8n+1 &= 8\frac{k(k+1)}{2}+1 \\ &= 4k(k+1)+1 \\ &= 4k^2+4k+1 \\ &= 4\left(k+\frac{1}{2}\right)^2 \\ &= (16k+8)^2 \\ \text{let } m &= 16k+8 \\ \mathcal{L}_4 \wedge k \in \mathbb{Z}^+ &\implies m \in Z^+ \\ 8n+1 &= m^2 \\ \therefore P_2 &\implies P_3 \end{aligned}$$

$$[(P_2 \implies P_3) \wedge (P2 \impliedby P_3)] \implies P_2 \iff P_3$$

QED