# Global Stability for Monotone Markov Chains

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- Consider $P$-Markov in discrete time
- Q: Under which condition, $P$ has a unique stationary distribution and the distributions can converge to it
- Finite state space: irreducible and aperiodic
- When $P$ is monotone
  - Compact state space: Hopenhayn and Prescott (1992)
  - General state space: Kamihigashi and Stachurski (2012; 2014)

- Let X be a normally ordered Polish space
- The left Markov operator maps $\mu \in \mathscr{D}(\mathsf{X})$ into $\mu P \in \mathscr{D}(\mathsf{X})$, where

$$(\mu P)(A) = \int P(x, A)\mu(\mathrm{d}x)$$

- Suppose that $P$ is **increasing**, in the sense that

$$\mu \preceq_F \mu' \quad \text{implies} \quad \mu P \preceq_F \mu' P$$

- The stochastic kernel $P$ is called **globally stable** if $P$ has a unique stationary distribution $\mu^*$ and $\mu P^t \to \mu^*$ for all $\mu \in \mathscr{D}(\mathsf{X})$, in the sense that

$$\langle \mu P^t, f \rangle \to \langle \mu^*, f \rangle \quad (f \in bc\mathsf{X})$$

# Stochastic Monotonicity and Stationary Distributions for Dynamic Economies

Hugo A. Hopenhayn & Edward C. Prescott

Econometrica, 1992

# THM 2 of Hopenhayn and Prescott (1992)

- Suppose X is compact with least element $a$ and greatest element $b$
- **Monotone Mixing Condition** (MMC): There exist some $\bar{x} \in$ X and $t \in \mathbb{N}$ such that $P^t(b, [a, \bar{x}]) > 0$ and $P^t(a, [\bar{x}, b]) > 0$
  - If you start from the top state $b$, then after $t$ steps, there is positive probability mass in the lower part of the state space $[a, \bar{x}]$
  - Similarly, starting from the bottom $a$, mass will move upward to $[\bar{x}, b]$

**Theorem 1**

If MMC holds, then $P$ is globally stable.

# Sketch of Proof

1. X is compact
   $\Rightarrow \mathscr{D}(\mathsf{X})$ is compact, i.e., convergent subsequence!!!

2. Sequence $\{\delta_a P^t\} \subset \mathscr{D}(\mathsf{X})$ is increasing
   $\Rightarrow \delta_a P^t \to \mu_a \in \mathscr{D}(\mathsf{X})$
   (Similarly, $\delta_b P^t \to \mu_b \in \mathscr{D}(\mathsf{X})$)

3. MMC helps to show $\mu_a = \mu_b$
   - $\mu P^t \to \mu_a$ since $\delta_a \preceq_F \mu \preceq_F \delta_b$
   - $\mu_a P = \mu_a$ since $\mu_a P^t \uparrow \mu_a$

# An order-theoretic mixing condition for monotone Markov chains

Takashi Kamihigashi & John Stachurski

Statistics & Probability Letters, 2012

# THM 3.1 of Kamihigashi and Stachurski (2012)

**Theorem 2**

If $P$ is order mixing, then for any $\mu, \nu \in \mathscr{D}(\mathsf{X})$, we have

$$\lim_{t \to \infty} |\langle \mu P^t, h \rangle - \langle \nu P^t, h \rangle| = 0 \quad (h \in ib\mathsf{X})$$

Existence implies Stability!

# Order Mixing

**Definition 1**

A stochastic kernel $P$ is called **order mixing** if for any $x, x' \in \mathsf{X}$ and any independent $P$-Markov process $\{X_t\}$ and $\{X'_t\}$ starting at $x$ and $x'$, respectively, we have

$$\mathbb{P}^{P\times P}_{x,x'} \bigcup_{t\geqslant 0} \{X_t \leqslant X'_t\} = 1$$

Independent $P$-Markov process $\{X_t\}$ and $\{X'_t\}$ starting at $x$ and $x'$ attain $X_t \leqslant X'_t$ eventually with probability 1

# Sketch of Proof

$\forall h \in (ib\mathsf{X})_+$, let $\tau = \inf\{t \in \mathbb{N} : X_t \leqslant X'_t\}$

$$\begin{aligned}\mathbb{E}h(X'_t) &\geqslant \mathbb{E}[\mathbb{1}_{[\tau \leqslant t]} h(X'_t)] \\ &= \mathbb{E}[\mathbb{1}_{[\tau \leqslant t]} (P^{t-\tau} h)(X'_\tau)] \\ &\geqslant \mathbb{E}[\mathbb{1}_{[\tau \leqslant t]} (P^{t-\tau} h)(X_\tau)] \\ &= \mathbb{E}[\mathbb{1}_{[\tau \leqslant t]} h(X_t)] \\ &= \mathbb{E}h(X_t) - \mathbb{E}[\mathbb{1}_{[\tau \geqslant t+1]} h(X_t)]\end{aligned}$$

Hence

$$\mathbb{E}h(X_t) - \mathbb{E}h(X'_t) \leqslant \mathbb{E}[\mathbb{1}_{[\tau \geqslant t+1]} h(X_t)] \leqslant \mathbb{P}(\tau \geqslant t+1) \sup_{x \in \mathsf{X}} h(x) \to 0$$

($\to 0$ since order mixing implies $\mathbb{P}(\tau < \infty) = 1$)

# Stochastic Stability in Monotone Economies

Takashi Kamihigashi & John Stachurski

TE, 2014

# THM 1 of Kamihigashi and Stachurski (2014)

**Theorem 3**

Suppose $P$ is order reversing. $P$ is globally stable iff

- C1 $P$ is bounded in probability, and
- C2 $P$ has either a deficient or an excessive distribution

Generalization

- MMC $\Rightarrow$ order reversing
- Compact $\Rightarrow$ bounded in probability, deficient, excessive

# Order Reversing

## Definition 2

A stochastic kernel $P$ is called **order reversing** if for any $x' \leqslant x \in \mathsf{X}$ and any independent $P$-Markov processes $\{X_t\}$ and $\{X'_t\}$ starting at $x$ and $x'$, respectively, there exists a $t \in \mathbb{N}$ with

$$\mathbb{P}\{X_t \leqslant X'_t\} > 0$$

MMC $\Rightarrow$ order reversing

$$\begin{aligned}
\mathbb{P}(X_t \leqslant X'_t) &\geqslant \mathbb{P}(X_t \leqslant \bar{x} \leqslant X'_t) \\
&= \mathbb{P}(X_t \leqslant \bar{x})\mathbb{P}(\bar{x} \leqslant X'_t) \\
&= P^t(x, [a, \bar{x}])P^t(x', [\bar{x}, b]) \\
&= \left\langle \delta_x P, \mathbb{1}_{[a,\bar{x}]} \right\rangle \cdot \left\langle \delta_{x'} P, \mathbb{1}_{[\bar{x},b]} \right\rangle \\
&\geqslant \left\langle \delta_b P, \mathbb{1}_{[a,\bar{x}]} \right\rangle \cdot \left\langle \delta_a P, \mathbb{1}_{[\bar{x},b]} \right\rangle \\
&= P^t(b, [a, \bar{x}])P^t(a, [\bar{x}, b])
\end{aligned}$$

# Bounded in Probability

**Definition 3**

A stochastic kernel $P$ is called **bounded in probability** if for any $x \in \mathsf{X}$, $\{P^t(x, \cdot)\}$ is **tight**, i.e., for any $\varepsilon > 0$, there exists a compact $K \subseteq \mathsf{X}$ such that

$$P^t(x, \mathsf{X} \backslash K) \leqslant \varepsilon \quad (t \in \mathbb{N})$$

$\mathsf{X}$ is compact $\Rightarrow$ $P$ is bounded in probability
(letting $K = \mathsf{X}$)

# Deficient & Excessive

**Definition 4**

- $\mu$ is called **deficient** if $\mu \preceq_F \mu P$
- $\mu$ is called **excessive** if $\mu P \preceq_F \mu$

X has a least element $a \quad \Rightarrow \quad \delta_a$ is deficient for $P$
($\delta_a \preceq_F \mu$ for any $\mu \in \mathscr{D}(\mathsf{X})$)

X has a greatest element $b \quad \Rightarrow \quad \delta_b$ is excessive for $P$

# Proof: GS $\Rightarrow$ C1 & C2

- GS $\Rightarrow$ C2: Trivial because $\mu^* P = \mu^*$
- GS $\Rightarrow$ C1: Trivial because $P^t(x, \cdot) = \delta_x P^t$ is convergent (Prohorov's theorem: tight $=$ closure is sequentially compact)

# Proof: C1 & C2 $\Rightarrow$ GS

1. Order reversing + bounded in probability $\Rightarrow$ Order mixing

$$\lim_{t\to\infty} |\langle \mu P^t, h\rangle - \langle \nu P^t, h\rangle| = 0 \quad (h \in ib\mathsf{X})$$

2. $\mu \preceq_F \mu P \Rightarrow$ Existence
3. **Then $\nu P^t \to \mu^*$ and hence uniqueness**
   - $\langle \nu P^t, h\rangle \to \langle \mu^*, h\rangle$, $\forall h \in ibc\mathsf{X}$
   - bounded in probability
     $\Rightarrow \{\nu P^t\}$ is tight
     $\Rightarrow$ subsequence $\langle \nu P^t, h\rangle \to \langle \nu^*, h\rangle$, $\forall h \in bc\mathsf{X}$
     $\Rightarrow \nu^* = \mu^*$ (normally partial ordered Polish space!)
   - every subsequence has a sub-subsequence converging to $\mu^*$
     $\Rightarrow \nu P^t \to \mu^*$

# Proof: C1 & C2 $\Rightarrow$ GS

1. Order reversing + bounded in probability $\Rightarrow$ Order mixing

$$\lim_{t\to\infty} |\left\langle \mu P^t, h \right\rangle - \left\langle \nu P^t, h \right\rangle| = 0 \quad (h \in ib\mathsf{X})$$

2. $\mu \preceq_F \mu P \Rightarrow$ **Existence**
    - tight increasing sequence $\{\mu P^t\}$ converges to its supremum $\mu^*$
    - $\left\langle \mu^* P^t, h \right\rangle \to \langle \mu^*, h \rangle$, $\forall h \in ib\mathsf{X} \Rightarrow \mu^* P^t \to \mu^*$
    - So $\mu^* \preceq_F \mu^* P \preceq_F \mu^*$
3. Then $\nu P^t \to \mu^*$ and hence uniqueness

# Proof: C1 & C2 $\Rightarrow$ GS

1. **Order reversing + bounded in probability $\Rightarrow$ Order mixing**
    - $P$ bounded in probability $\Rightarrow P \times P$ bounded in probability
    - $P$ order reversing $\Rightarrow$

    $$\forall \text{ compact } K \subseteq \mathsf{X} \times \mathsf{X}, \ \exists t \in \mathbb{N} \text{ s.t. } \inf_{(x,x') \in K} \mathbb{P}^{P \times P}_{x,x'} \{X_t \leqslant X'_t\} > 0$$

    - So order mixing
2. $\mu \preceq_F \mu P \Rightarrow$ Existence
3. Then $\nu P^t \to \mu^*$ and hence uniqueness

# THM 2 of Kamihigashi and Stachurski (2014)

**Theorem 4**

Suppose $P$ is order reversing and Feller. $P$ is globally stable iff $P$ is bounded in probability.

$\mu \preceq_F \mu P$ is only applied to show existence

Bounded in probability + Feller $\Rightarrow$ existence

HOPENHAYN, H. A. AND E. C. PRESCOTT (1992): "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies," *Econometrica*, 60, 1387–1406.

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——— (2014): "Stochastic stability in monotone economies," *Theoretical Economics*, 9, 383–407.