4.2 LIMITS

Mlathematical quantities can be divided into two (i) constants and (ii) Variables

The quantitity which does not change in its value is called constant.

Constants can be divided (i) Absolute constants and (ii) Arbitrary constants

Absolute constants are those which retain in values at any time and at any place.

Examples:

Numbers 5, – 7, 3, $\frac{-11}{27}$, π, e, $\sqrt{2}$, $\sqrt{3}$, ... etc. are absolute constants.

Arbitrary constants are those which are constants in a particular problem but changes from problem to problem. They are represented as a, b, c, f, g, h, ... etc.

Examples:

(i) $ax + by + c = 0$ — a, b, c

(ii) $x^2 + y^2 + 2gx + 2fy + c = 0$ — f, g, h.........ex.

Variables are those which vary in their values. They are represented as u, v, w, x, y, z, θ, ...

Variables can be divided into two (i) Independent variable and (ii) Dependent varaible.

In the equation $y = x^2$,x is the independent variable and y is the dependent variable depending on x

For example, if x takes the value 2, y takes the value 4. If x takes –3, y takes 9 .... etc.

The relation or equation or connection between two variables is called a function and is denoted as f(x).

Example: $y = f(x) = x^2$ or sin x.

The value of f(x) at x = a is f(a) and is called the functional value of f(x) at x = a.

If for every function, if the functional value at every point of its domain exists, there is no need to study Limit chapter and from that the famous CALCULS.

For example, $f(x) = \frac{x^2-4}{x-2}$ at x = 2 i.e, f(2) does not exist, since $f(2) = \frac{0}{0}$, the indeterminant quantity. Similarly there are two more indeterminant quantities $\frac{\infty}{\infty}$ and $\infty - \infty$.

But, as x approaches 2, $f(x) = \frac{x^2-4}{x-2}$ will approach 4.

x	1	1.5	1.9	1.99	1.999	1.9999	......	→ 2
$y = \frac{x^2-4}{x-2}$	3	3.5	3.9	3.99	3.999	3.9999	......	→ 4

Hence the approachment value 4 of the function $\frac{x^2-4}{x-2}$ as x approaches 2 is called the limit value of the function.

Definition of Limit:

When the variable x approaches a constant a and if the function f(x) approaches a constant *l*, then *l* is called the limit value of f(x) as x approaches a and is denotd as

$$\boxed{\underset{x\to a}{\text{Lt}}\, f(x) = l}$$

Results :

1) $\underset{x\to a}{\text{Lt}}[f(x)+g(x)] = \underset{x\to a}{\text{Lt}} f(x) \pm \underset{x\to a}{\text{Lt}} g(x)$

2) $\underset{x\to a}{\text{Lt}}[k\ f(x)] = k \underset{x\to a}{\text{Lt}} f(x)$, where k is a constant

3) $\underset{x\to a}{\text{Lt}}[f(x).g(x)] = \underset{x\to a}{\text{Lt}} f(x). \underset{x\to a}{\text{Lt}} g(x)$

4) $\underset{x\to a}{\text{Lt}}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\underset{x\to a}{\text{Lt}} f(x)}{\underset{x\to a}{\text{Lt}} g(x)}$, provided $\underset{x\to a}{\text{Lt}} g(x) \neq 0$

Formulae:

1) $\underset{x\to a}{\text{Lt}} \dfrac{x^n - a^n}{x-a} = na^{n-1}$, for all values of x.

2) $\underset{x\to a}{\text{Lt}} \dfrac{\sin\theta}{\theta} = 1$

3) $\underset{x\to a}{\text{Lt}} \dfrac{\tan\theta}{\theta} = 1$, θ in radians

Now,

$$\underset{\theta\to 0}{\text{Lt}} \frac{\sin n\theta}{\theta} = \underset{\theta\to 0}{\text{Lt}}\left[\frac{\sin n\theta}{n\theta}.\ n\right]$$

$$= n \underset{\theta\to 0}{\text{Lt}}\left(\frac{\sin n\theta}{n\theta}\right) \qquad [\text{By } R_2$$

$$= n\ .\ 1 \qquad [\text{By } F_2$$

$$= n$$

Similarly, $\underset{\theta\to 0}{\text{Lt}} \dfrac{\tan n\theta}{\theta} = n$

WORKED EXAMPLES

PART – A

1. Evaluate $\underset{\theta\to 0}{\text{Lt}} \dfrac{3x^2+2x+1}{5x^2+6x+7}$.

Solution:

$$\underset{\theta\to 0}{\text{Lt}} \frac{3x^2+2x+1}{5x^2+6x+7} = \frac{3(0)^2+2(0)+1}{5(0)^2+6(0)+7}$$

$$= \frac{1}{7}$$

2. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{3x^2+4x}{5x-7x^2}$

Solution:

$$\underset{x\to 0}{\text{Lt}} \frac{3x^2+4x}{5x-7x^2} = \underset{x\to 0}{\text{Lt}} \frac{x(3x-4)}{x(5-7x)}$$

$$= \frac{3.0-4}{5-7.0}$$

$$= \frac{-4}{5} = -\frac{4}{5}$$

3. Evaluate: $\underset{\theta\to 0}{Lt}\dfrac{\sin 7\theta}{0}$

Solution:

$$\underset{\theta\to 0}{Lt}\frac{\sin 7\theta}{\theta}=\underset{\theta\to 0}{Lt}\left[\frac{\sin 7\theta}{7\theta}.7\right]$$

$$=7\underset{\theta\to 0}{Lt}\frac{\sin 7\theta}{7\theta}$$

$$= 7.1$$

$$= 7$$

PART – B

1. Evaluate: $\underset{x\to 1}{Lt}\dfrac{x^2+x-2}{x^2-4x+3}$

Solution:

$$\underset{x\to 1}{Lt}\frac{x^2+x-2}{x^2-4x+3} = \frac{1+1-2}{1-4(1)+3} = \frac{2-2}{4-4} = \frac{0}{0}\ \text{Indeterminent}$$

$$\underset{x\to 1}{Lt}\frac{x^2+x-2}{x^2-4x+3}=\underset{x\to 1}{Lt}\frac{(x+2)(x-1)}{(x-1)(x-3)}$$

$$=\underset{x\to 1}{Lt}\frac{x+2}{x-3}$$

$$=\frac{1+2}{1-3}=\frac{3}{-2}=-\frac{3}{2}$$

2. Evaluate $\underset{x\to 2}{Lt}\dfrac{x^{10}-2^{10}}{x-2}$.

Solution:

$$\underset{x\to 2}{Lt}\frac{x^{10}-2^{10}}{x-2}=10\,.\,2^{10-1}\qquad\left[\text{By F1 }\underset{x\to a}{Lt}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=10.2^9$$

$$=10\times 512$$

$$=5120$$

3. Evaluate $\underset{x\to a}{Lt}\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$.

Solution:

$$\underset{x\to a}{Lt}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\underset{x\to a}{Lt}\frac{x^{1/2}-a^{1/2}}{x-a}$$

$$=\frac{1}{2}\,.\,a^{1/2-1}$$

$$=\frac{1}{2}a^{-1/2}$$

$$=\frac{1}{2}\,.\,\frac{1}{a^{1/2}}=\frac{1}{2\sqrt{a}}$$

4. Evaluate $\underset{x\to 0}{\text{Lt}}\dfrac{\sin 5x}{4x}$

Solution:

$$\underset{x\to 0}{\text{Lt}}\frac{\sin 5x}{4x}=\frac{1}{4}\ \underset{x\to 0}{\text{Lt}}\frac{\sin 5x}{x}=\frac{1}{4}\ \underset{x\to 0}{\text{Lt}}\left[\frac{\sin 5x}{5x}\times 5\right]$$

$$=\frac{1}{4}\times 5\ \underset{x\to 0}{\text{Lt}}\frac{\sin 5x}{5x} \qquad \left[\begin{array}{l}\text{By } F_2\\ \underset{x\to 0}{\text{Lt}}\dfrac{\sin\theta}{\theta}=1\end{array}\right.$$

$$=\frac{5}{4}\ .\ 1=\frac{5}{4}$$

5. Evaluate $\underset{x\to 0}{\text{Lt}}\dfrac{\tan^2 5x}{x^2}$.

Solution:

$$\underset{x\to 0}{\text{Lt}}\frac{\tan^2 5x}{x^2}=\underset{x\to a}{\text{Lt}}\left(\frac{\tan 5x}{x}\right)^2$$

$$=\underset{x\to 0}{\text{Lt}}\left[\frac{\tan 5x}{5x}.5\right]^2$$

$$=\underset{x\to 0}{\text{Lt}}\left(\frac{\tan 5x}{5x}\right)^2\times 25 \qquad \left[\begin{array}{l}\text{By } F_3\\ \underset{\theta\to 0}{\text{Lt}}\dfrac{\tan\theta}{\theta}=1\end{array}\right.$$

$$=25\ \underset{x\to a}{\text{Lt}}\left(\frac{\tan 5x}{5x}\right)^2$$

$$=25.1^2=25$$

PART – C

1. Evaluate $\underset{x\to 3}{\text{Lt}}\dfrac{x^5-243}{x^2-9}$.

Solution:

$$\underset{x\to 3}{\text{Lt}}\frac{x^5-243}{x^2-9}=\underset{x\to 3}{\text{Lt}}\frac{x^5-3^5}{x^2-9}$$

$$=\underset{x\to 3}{\text{Lt}}\left[\frac{x^5-3^5}{x-3}\times\frac{x-3}{x^2-3^2}\right]$$

$$=\underset{x\to 3}{\text{Lt}}\left[\frac{x^5-3^5}{x-3}\Big/\frac{x^2-3^2}{x-3}\right]$$

$$=\frac{\underset{x\to 3}{\text{Lt}}\dfrac{x^5-3^5}{x-3}}{\underset{x\to 3}{\text{Lt}}\dfrac{x^2-3^2}{x-3}} \qquad [\text{By } R4$$

$$=\frac{5.3^{5-1}}{2.3^{2-1}}=\frac{5.3^4}{2.3^1}$$

$$=\frac{5}{2}\times 3^3=\frac{5}{2}\times 27=\frac{135}{2}$$

2. Evaluate : $\underset{x \to 4}{\text{Lt}} \dfrac{x^{7/3} - 4^{7/3}}{x^{2/3} - 4^{2/3}}$

Solution:

$$\underset{x \to 4}{\text{Lt}} \frac{x^{7/3} - 4^{7/3}}{x^{2/3} - 4^{2/3}} = \underset{x \to 4}{\text{Lt}} \left[\frac{x^{7/3} - 4^{7/3}}{x - 4} \times \frac{x - 4}{x^{2/3} - 4^{2/3}} \right]$$

$$= \underset{x \to 4}{\text{Lt}} \left[\frac{x^{7/3} - 4^{7/3}}{x - 4} \Big/ \frac{x^{2/3} - 4^{2/3}}{x - 4} \right]$$

$$= \frac{\underset{x \to 4}{\text{Lt}} \dfrac{x^{7/3} - 4^{7/3}}{x - 4}}{\underset{x \to 4}{\text{Lt}} \dfrac{x^{2/3} - 4^{2/3}}{x - 4}}$$

$$= \frac{\frac{7}{3} . 4^{\frac{7}{3} - 1}}{\frac{2}{3} . 4^{\frac{2}{3} - 1}} = \frac{\frac{7}{3} . 4^{4/3}}{\frac{2}{3} . 4^{-1/3}}$$

$$= \frac{7}{3} \times \frac{3}{2} \times 4^{4/3 + 1/3} = \frac{7}{2} 4^{5/3}$$

3. Evaluate $\underset{\theta \to 0}{\text{Lt}} \dfrac{\sin 8\theta}{\tan 3\theta}$.

Solution:

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin 8\theta}{\tan 3\theta} = \underset{\theta \to 0}{\text{Lt}} \left[\frac{\sin 8\theta}{8\theta} \times \frac{3\theta}{\tan 3\theta} \right] \times \frac{8}{3}$$

$$= \underset{\theta \to 0}{\text{Lt}} \left[\frac{\sin 8\theta}{8\theta} \Big/ \frac{\tan 3\theta}{3\theta} \right] \times \frac{8}{3}$$

$$= \frac{8}{3} \frac{\underset{\theta \to 0}{\text{Lt}} \dfrac{\sin 8\theta}{8\theta}}{\underset{\theta \to 0}{\text{Lt}} \dfrac{\tan 3\theta}{3\theta}}$$

$$= \frac{8}{3} . \frac{1}{1} = \frac{8}{3}$$

4.3 DIFFERENTIATION

Let y = f(x) —— (1) be a function of x.

Let Δx be a small increment in x and let Δy be the corresponding increment in y.

$\therefore$ y + Δy = f(x + Δx) ——— (2)

(2) – (1) Δy = f(x + Δx) – f(x)

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Taking the limit as Δx → 0,

$$\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x}$ is called the differential coefficient of y with respect to x and is denoted as $\frac{dy}{dx}$.

$$\therefore \boxed{\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

List of formulae:

(1) $\frac{d}{dx}(x^n) = n x^{n-1}$, n a real number

(2) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$ (3) $\frac{d}{dx}(e^x) = e^x$ (4) $\frac{d}{dx}(\log x) = \frac{1}{x}$

(5) $\frac{d}{dx}(\sin x) = \cos x$ (6) $\frac{d}{dx}(\cos x) = -\sin x$

(7) $\frac{d}{dx}(\tan x) = \sec^2 x$ (8) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(9) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (10) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

Results:

1) If u and v are functions of x,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

2) $\frac{d}{dx}(ku) = k\frac{du}{dx}$, where k is a constant.

3) $\frac{d}{dx}$(any constant) = 0

4) Product Rule of Differentiation

If u & v are two functions of x,

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

5) $\boxed{\frac{d}{dx}(u\ v\ w) = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}}$, where u, v and w are function of x.

6) Quotient Rule of Differentiation

If u & v are functions of x,

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

WORKED EXAMPLES

PART – A

1. Find $\frac{dy}{dx}$ if (i) $y = \frac{1}{x^2}$ (ii) $y = x^3 + 2$ (iii) $y = \frac{1}{\sqrt{x}}$

(iv) $y = \sqrt{x}$ (v) $y = \frac{1}{\sin x}$ (vi) $y = 9\sqrt{x} + x^2$

Solution:

(i) $y = \frac{1}{x^2} = x^{-2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^{-2}) = -2x^{-3} = -\frac{2}{x^3}$$

(ii) $y = x^3 + 2$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3 + 2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(2) = 3x^2 + 0 = 3x^2$$

(iii) $y = \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}$$

(iv) $y = \sqrt{x} = x^{1/2}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2} \cdot \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$$

(v) $y = \frac{1}{\sin x} = \text{cosec } x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$$

(vi) $y = 9\sqrt{x} + x^2$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(9\sqrt{x} + x^2) = 9\frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(x^2)$$

$$= 9 \cdot \frac{1}{2\sqrt{x}} + 2x$$

$$= \frac{9}{2\sqrt{x}} + 2x$$

PART –B

1. Find $\frac{dy}{dx}$ if $y = \frac{3}{x^2} + \frac{2}{x} + \frac{1}{4}$.

Solution:

$$y = \frac{3}{x^2} + \frac{2}{x} + \frac{1}{4} = 3x^{-2} + 2x^{-1} + \frac{1}{4}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^{-2} + 2x^{-1} + \frac{1}{4}\right) = 3(-2)x^{-3} + 2(-1)x^{-2} + 0$$

$$= \frac{-6}{x^3} - \frac{2}{x^2}$$

2. Find $\frac{dy}{dx}$ if $y = e^x \sin x$.

Solution:

$$y = e^x \sin x$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin x) = "u\frac{dv}{dx} + v\frac{du}{dx}"$$

$$= e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= e^x \cos x + \sin x . e^x$$

3. Find $\frac{dy}{dx}$ if $y = x^2 e^x \sin x$.

Solution:

$$y = x^2 e^x \sin x$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 e^x \sin x) = "uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}"$$

$$= x^2 e^x \frac{d}{dx}(\sin x) + e^x \sin x \frac{d}{dx}(x^2) + \sin x . x^2 \frac{d}{dx}(e^x)$$

$$= x^2 e^x \cos x + e^x \sin x . 2x + \sin x \,.\, x^2 e^x$$

4. Find $\frac{dy}{dx}$ if $y = \frac{\cos x}{\sqrt{x}}$.

Solution:

$$y = \frac{\cos x}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\sqrt{x}\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}(-\sin x) - \cos x . \frac{1}{2\sqrt{x}}}{x}$$

PART – C

1. Find $\frac{dy}{dx}$ if $y = (x^2 + 3) \cos x \log x$.

Solution:

$$y = (x^2 + 3) \cos x \log x$$

$$\frac{dy}{dx} = \frac{d}{dx}[(x^2+3)\cos x \log x] = \text{"}uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}\text{"}$$

$$= (x^2+3)\cos x\frac{d}{dx}(\log x) + \cos x \log x\frac{d}{dx}(x^2+3) + \log x(x^2+3)\frac{d}{dx}(\cos x)$$

$$= (x^2+3)\cos x.\frac{1}{x} + \cos x \log x \,.\, 2x + \log x \,.\, (x^2+3)(-\sin x)$$

$$= \frac{(x^2+3)\cos x}{x} + 2x\cos x\log x - (x^2+3)\sin x \log x$$

2. If $y = \frac{x^3 \tan x}{e^x + 1}$, find $\frac{dy}{dx}$.

Solution:

$$y = \frac{x^3 \tan x}{e^x + 1}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(e^x+1)\frac{d}{dx}(x^3\tan x) - (x^3\tan x)\frac{d}{dx}(e^x+1)}{(e^x+1)^2}$$

$$= \frac{(e^x+1)\left[x^3\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(x^3)\right] - x^3\tan x.e^x}{(e^x+1)^2}$$

$$= \frac{(e^x+1)[x^3\sec^2 x + \tan x.3x^2] - x^3\tan x \,.\, e^x}{(e^x+1)^2}$$

3. Find $\frac{dy}{dx}$ if $y = \frac{1+x+x^2}{1-x+x^2}$.

Solution:

$$y = \frac{1+x+x^2}{1-x+x^2}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1-x+x^2)\frac{d}{dx}(1+x+x^2) - (1+x+x^2)\frac{d}{dx}(1-x+x^2)}{(1-x+x^2)^2}$$

$$= \frac{(1-x+x^2)(1+2x) - (1+x+x^2)(-1+2x)}{(1-x+x^2)^2}$$

$$= \frac{1+2x-x-2x^2+x^2+2x^3+1-2x+x-2x^2+x^2-2x^3}{(1-x+x^2)^2}$$

$$= \frac{2-2x^2}{(1-x+x^2)^2} = \frac{2(1-x^2)}{(1-x+x^2)^2}$$

EXERCISE

PART –A

1. Find the principal value of

   (i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\cos^{-1}(0)$ (iii) $\tan^{-1}(\sqrt{3})$ (iv) $\cot^{-1}(-\sqrt{3})$ (v) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

   (vi) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ (vii) $\sec^{-1}(-1)$ (viii) $\cos^{-1}\left(-\frac{1}{2}\right)$ (ix) $\sin^{-1}(-1)$ (x) $\tan^{-1}(-1)$

2. Prove that $\sin^{-1}x+\sec^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{2}$.
3. Prove that $\sec^{-1}x+\sin^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{2}$.
4. Evaluate : $\underset{x\to 1}{\text{Lt}}\frac{px^2+qx+r}{ax^2+bx+c}$
5. Evaluate: $\underset{x\to 0}{\text{Lt}}\frac{5x-8x^2}{2x^2-3x}$
6. Evaluate : $\underset{\theta\to 0}{\text{Lt}}\frac{\sin 9\theta}{\theta}$
7. Evaluate : $\underset{\theta\to 0}{\text{Lt}}\frac{\tan 5\theta}{\theta}$
8. Evaluate : $\underset{x\to 2}{\text{Lt}}\frac{x^2+2x}{4x-3x^2}$
9. Find $\frac{dy}{dx}$ if $y = 3x^4 - 7$.
10. Find $\frac{dy}{dx}$ if $y=\frac{1}{x^3}$.
11. Find $\frac{dy}{dx}$ if $y=\frac{1}{\cot x}$.
12. Find $\frac{dy}{dx}$ if $y = 8e^x - 4\operatorname{cosec} x$.
13. Find $\frac{dy}{dx}$ if $y = 3\log x - 4\sqrt{x}$.
14. Find $\frac{dy}{dx}$ if $y=\frac{1}{x^{5/7}}$.
15. Find $\frac{dy}{dx}$ if $y=\frac{8}{x^{57}}$.

PART – B

1. Prove that $\cos^{-1}\left(\sqrt{1-x^2}\right)=\sin^{-1}x$.
2. Prove that $\operatorname{cosec}^{-1}\left(\sqrt{1+x^2}\right)=\cot^{-1}x$.
3. Prove that $\sec^{-1}\left(\sqrt{1+x^2}\right)=\tan^{-1}x$.
4. Prove that $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}$.

5. Evaluate : $\underset{x\to 1}{\text{Lt}}\dfrac{x^2-3x+2}{x^2-5x+4}$

6. Evaluate: $\underset{x\to 1}{\text{Lt}}\dfrac{x^2+4x-5}{x^2+x+4}$

7. Evaluate : $\underset{x\to 3}{\text{Lt}}\dfrac{x^6-3^6}{x-3}$

8. Evaluate: $\underset{x\to 0}{\text{Lt}}\dfrac{\tan 7x}{13x}$

9. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{\sin^2 13x}{x^2}$

10. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{4\sin 3x}{5x}$

11. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{\sin\left(\frac{x}{2}\right)}{x}$

12. Evaluate : $\underset{x\to 0}{\text{Lt}}\dfrac{\tan\left(\frac{x}{3}\right)}{x}$

13. Find $\dfrac{dy}{dx}$ if $y = x^3 - 4x^2 + 7x - 11$.

14. Find $\dfrac{dy}{dx}$ if $y = x^3 + \dfrac{2}{x^2} - \dfrac{1}{x} + \dfrac{3}{2}$.

15. Find $\dfrac{dy}{dx}$ if $y = 2\sqrt{x} - 3e^x + 7\sin x - 8\cos x + 11$

16. Find $\dfrac{dy}{dx}$ if $y = \sqrt{x}\,\log x$.

17. Find $\dfrac{dy}{dx}$ if $y = x^2\,\text{cosec}\, x$.

18. Find $\dfrac{dy}{dx}$ if $y = (x^3 - 4)\tan x$.

19. Find $\dfrac{dy}{dx}$ if $y = \dfrac{x-7}{x+3}$.

20. Find $\dfrac{dy}{dx}$ if $y = \sqrt{x}\,\log x\,\text{cosec}\, x$.

21. Find $\sqrt{x}$ if $y = x^3\cot x\log x$.

22. Find $\dfrac{dy}{dx}$ if $y = \dfrac{e^x}{\sin x}$.

23. Find $\dfrac{dy}{dx}$ if $y = \dfrac{\tan x}{x^4}$.

24. Find $\dfrac{dy}{dx}$ if $y = \dfrac{1-\cos x}{1+\sin x}$.

25. Find $\dfrac{dy}{dx}$ if $y = \dfrac{4}{x} + 7\cos x - 9\log x + \dfrac{8}{\text{cosec}\, x} - 3$.

PART – C

1. Show that $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$.
2. Show that $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1}x$.
3. Prove that $\sin^{-1}(3x - 4x^3) = 3\sin^{-1}x$.
4. Prove that $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x$.
5. Prove that $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$.
6. Prove that $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$.
7. Prove that $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$.
8. Show that $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$.
9. Show that $\tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right] = \frac{x}{2}$.
10. Show that $2\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{24}{7}\right)$.
11. Evaluate : $\cos\left[\sin^{-1}\left(\frac{5}{13}\right)\right]$
12. Evaluate : $\cos\left[\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right]$
13. Evaluate the following:

(i) $\underset{x\to 6}{\text{Lt}} \frac{x^3 - 6^3}{x^5 - 6^5}$ (ii) $\underset{x\to 4}{\text{Lt}} \frac{x^8 - 4^8}{x^5 - 4^5}$ (iii) $\underset{x\to 3}{\text{Lt}} \frac{x^{-\frac{7}{11}} - 3^{-\frac{7}{11}}}{x^{\frac{4}{11}} - 3^{\frac{4}{11}}}$ (iv) $\underset{x\to 2}{\text{Lt}} \frac{x^{32} - 2^{32}}{x^{17} - 2^{17}}$

(v) $\underset{\theta\to 0}{\text{Lt}} \frac{\sin 7\theta}{\sin 2\theta}$ (vi) $\underset{\theta\to 0}{\text{Lt}} \frac{\tan 6\theta}{\tan 11\theta}$ (vii) $\underset{\theta\to 0}{\text{Lt}} \frac{\sin 9\theta}{\tan 7\theta}$ (viii) $\underset{x\to 0}{\text{Lt}} \frac{1-\cos x}{x\sin x}$

14. Differentiate the following w.r.t. x.

(i) $y = (3x^2 + 2x + 1)\, e^x \tan x$.

(ii) $y = (2x + 1)(3x - 7)(4 - 9x)$.

(iii) $y = \frac{2\sin x + 3\cos x}{2\cos x + 3\sin x}$

(iv) $y = \frac{x^2 + 3}{x\cos x}$

(v) $y = \cos x - \frac{\sin x}{x + 7}$

(vi) $y = \frac{\sqrt{x} + \log x}{e^x + x^3}$

(vii) $y = \frac{e^x \sin x}{x^2 + 1}$

(viii) $y = \frac{\sqrt{x} + e^x}{x^3 \sin x}$

(ix) $y = \frac{x^2 e^x}{(x + 2)\tan x}$

(x) $y = \frac{x + \tan x}{e^x - \sin x}$

14. (i) $(3x^2 + 2x + 1)\, e^x \sec^2 x + e^x \tan x\,(6x + 2) + \tan x\,(3x^2 + 2x + 1)\, e^x$

(ii) $(2x + 1)(3x - 7)(-9) + (3x - 7)(4 - 9x) \,.\, 2 + (49x)(2x + 1)\, 3$

(iii) $\dfrac{(2\cos x + 3\sin x)(2\cos x - 3\sin x) - (2\sin x + 3\cos x)(-2\sin x + 3\cos x)}{(2\cos x + 3\sin x)^2}$

(iv) $\dfrac{x\cos x(2x) - (x^2+3)(-x\sin x + \cos x)}{x^2\cos^2 x}$ (v) $-\sin x - \left[\dfrac{(x+7)\cos x - \sin x.1}{(x+7)^2}\right]$

(vi) $\dfrac{(e^x + x^3)\left(\dfrac{1}{2\sqrt{x}} + \dfrac{1}{x}\right) - (\sqrt{x} + \log x)\,(e^x + 3x^2)}{(e^x + x^3)^2}$ (vii) $\dfrac{(x^2+1)(e^x\cos x + \sec x.e^x) - e^x \sin x.2x}{(x^2+1)^2}$

(viii) $\dfrac{x^3\sin x\left(\dfrac{1}{2\sqrt{x}} + e^x\right) - (\sqrt{x} + e^x)(x^3\cos x + 3x^2\sin x)}{x^6\sin^2 x}$

(ix) $\dfrac{(x+2)\,\tan x\,(x^2e^x + 2xe^x) - (x^2e^x)[(x+2)\sec^2 x + \tan x \,.\, 1]}{(x+2)^2\tan^2 x}$

(x) $\dfrac{(e^x - \sin x)(1 + \sec^2 x) - (x + \tan x)(e^x - \cos x)}{(e^x - \sin x)^2}$

ANSWERS

PART – A

1) (i) 60^o (ii) 90^o (iii) 60^o (iv) -30^o (v) 30^o (vi) -45^o

(vii) -180^o (viii) 120^o (ix) -90^o (x) -45^o

4) $\dfrac{p+q+r}{a+b+c}$ 5) $\dfrac{-5}{3}$ 6) 9 7) 5 8) –2 9) $12x^3$ 10) $-\dfrac{3}{x^4}$

11) $\sec^2 x$ 12) 8ex + 4 cosec x cot x 13) $\dfrac{3}{x} - \dfrac{2}{\sqrt{x}}$ 14) $\dfrac{-5}{7}x - \dfrac{12}{7}$ 15) $-\dfrac{456}{x^{58}}$

PART – B

5) $\dfrac{1}{3}$ 6) 2 7) 6.3^5 8) $\dfrac{7}{13}$ 9) 169 10) $\dfrac{12}{5}$ 11) $\dfrac{1}{2}$ 12) $\dfrac{1}{3}$ 13) $3x^2 - 8x + 7$

14) $3x^2 - \dfrac{4}{x^3} + \dfrac{1}{x^2}$ 15) $\dfrac{1}{\sqrt{x}} - 3e^x + 7\cos x + 8\sin x$ 16) $\dfrac{\sqrt{x}}{x} + \log x \,.\, \dfrac{1}{2\sqrt{x}}$

18) $(x^3 - 4)\sec^2 x + 3x^2\tan x$ 19) $\dfrac{10}{(x+3)^2}$ 20) $-\sqrt{x}\log x\ \operatorname{cosec} x \cot x + \log x\ \operatorname{cosec} x \,.\, \dfrac{1}{2\sqrt{x}} + \operatorname{cosec} x \,.\, \dfrac{\sqrt{x}}{x}$

21) $\dfrac{x^3\cot x}{x} + \cos x \log x \,.\, 3x^2 + \log x\ x^3(-\operatorname{cosec}^2 x)$ 22) $\dfrac{\sin x e^x - e^x\cos x}{\sin^2 x}$ 23) $\dfrac{x^4\sec^2 x - \tan x \,.\, 4x^3}{x^8}$

24) $\dfrac{(1+\sin x)\sin x - (1 - \cos x)\cos x}{(1+\sin x)^2}$ 25) $-\dfrac{4}{x^2} - 7\sin x - \dfrac{9}{x} + \sin x$

PART – C

11) $\dfrac{12}{13}$ 12) $\dfrac{16}{65}$ 13) (i) $\dfrac{1}{60}$ (ii) $\dfrac{512}{5}$ (iii) $-\dfrac{7}{12}$ (iv) $\dfrac{3.2^{15}}{17}$ (v) $\dfrac{7}{2}$

(vi) $\dfrac{6}{11}$ (vii) $\dfrac{9}{7}$ (vii) $\dfrac{1}{2}$

UNIT – V

DIFFERENTIAL CALCULUS-II

5.1 DIFFERENTIATION METHODS

Differentiation of function functions (chain rule), Inverse Trigonometric functions and Implict functions. Simple Problems.

5.2 SUCCESSIVE DIFFERENTIATION

Successive differentiation up to second order (parametric form not included). Definition of differential equation, order and degree, formation of differential equation. Simple problems.

5.3 PARTIAL DIFFERENTIATION

Definition – Partial differentiation of two variables up to second order only. Simple Problems.

5.1 DIFFERENTIATION METHODS

Function of Funtions Rule:

If 'y' is a function of 'u' and 'u' is a function of 'x' then the derivative of 'y' w.r.t 'x' is equal to the product of the derivative of 'y' w.r.t 'u' and the derivative of 'u' w.r.t 'x'.

i.e. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

It is called function of function rule. This rule can be extended which is known as 'Chain rule'.

Chain rule:

If 'y' is a function of 'u' and 'u' is a function of 'v' and 'v' is a function of 'x' then.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \quad \frac{dv}{dx}$$

WORKED EXAMPLES

PART –A

Find $\frac{dy}{dx}$ if

1) $y = (2x + 5)^3$ 2) $y = \sqrt{\sin x}$ 3) $y = \cos^4 x$ 4) $y = e^{\tan x}$ 5) $y = \log(\sec x)$

6) $y = \sin mx$ 7) $y = \sec\sqrt{x}$ 8) $y = \cos(2 - 3x)$

Solution:

1) $y = (2x + 5)^3$

$y = u^3$ where $u = 2x + 5$

$$\frac{dy}{du} = 3u^2 \quad \Big| \quad \frac{du}{dx} = 2$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (3u^2)(2)$$

$$= 6(2x + 5)^2$$

2) $y = \sqrt{\sin x}$

$y = \sqrt{u}$ where $u = \sin x$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} \quad \Big| \quad \frac{du}{dx} = \cos x$$

$$\frac{dy}{dx} = \left(\frac{1}{2\sqrt{u}}\right)(\cos x)$$

$$= \frac{\cos x}{2\sqrt{\sin x}}$$

3) $y = \cos^4 x$

$y = u^4$ where $u = \cos x$

$$\frac{dy}{du} = 4u^3 \quad \Big| \quad \frac{du}{dx} = -\sin x$$

$$\frac{dy}{dx} = 4u^3(-\sin x)$$

$$= -4\sin x \cos^3 x$$

4) $y = e^{\tan x}$

$y = e^u$ where $u = \tan x$

$$\frac{dy}{du} = e^u \quad \Big| \quad \frac{du}{dx} = \sec^2 x$$

$$\frac{dy}{dx} = (e^u)(\sec^2 x)$$

$$= e^{\tan x} \sec^2 x$$

5) y = log (sec x)

y = log u where u = sec x

$$\frac{dy}{du} = \frac{1}{u} \quad \Big| \quad \frac{du}{dx} = \sec x \ \tan x$$

$$\frac{dy}{dx} = \left(\frac{1}{u}\right)(\sec x \tan x)$$

$$= \frac{1}{\sec x}(\sec x \tan x)$$

$$= \tan x$$

6) y = sin mx

y = sin u where u = mx

$$\frac{dy}{du} = \cos u \quad \Big| \quad \frac{du}{dx} = m$$

$$\frac{dy}{dx} = (\cos u)\ (m)$$

$$= m \ \cos \ mx$$

7) y = cos (2 – 3x)

y = cos u where u = 2 – 3x

$$\frac{dy}{du} = -\sin u \quad \Big| \quad \begin{aligned} \frac{du}{dx} &= 0 - 3(1) \\ &= -3 \end{aligned}$$

$$\frac{dy}{dx} = (-\sin u)\ (-3)$$

$$= 3\sin(2 - 3x)$$

8) $y = \sec\sqrt{x}$

y = sec u where u = $\sqrt{x}$

$$\frac{dy}{du} = \sec u \ .\tan u \quad \Big| \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = (\sec u \tan u)\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x}} \ . \sec\sqrt{x} \ . \ \tan\sqrt{x}$$

Note:

1. $\frac{d}{dx}(e^{mx}) = me^{mx}$
2. $\frac{d}{dx}$ (sin mx) = m cos mx
3. $\frac{d}{dx}$ (cos mx) = – m sin mx
4. $\frac{d}{dx}$ (tan mx) = m $\sec^2 mx$

5. $\frac{d}{dx}$ (cot mx) = – m cosec^2 mx

6. $\frac{d}{dx}$ (sec mx) = m sec mx tan mx

7. $\frac{d}{dx}$ (cosec mx) = – m cosec mx cot mx

PART – B

Differentiate the following w.r.t 'x'.

1) $(2x^2 - 3x + 1)^3$ 2) $\cos(e^{5x})$ 3) $e^{\sin^2 x}$ 4) log (sec x + tan x) 5) $y = \text{cosec}^3(5x+1)$

Solution:

1) $y = (2x^2 - 3x + 1)^3$

$y = u^3$ where $u = 2x^2 - 3x + 1$

$$\frac{dy}{du} = 3u^2 \quad \Big| \quad \frac{du}{dx} = 4x - 3$$

$$\frac{dy}{dx} = (3u^2)(4x-3)$$

$$= 3(2x^2 - 3x + 1)^2(4x-3)$$

2) Let $y = \cos(e^{5x})$

$y = \cos u$ where $u = e^{5x}$

$$\frac{dy}{du} = -\sin u \quad \Big| \quad \frac{du}{dx} = 5e^{5x}$$

$$\frac{dy}{dx} = (-\sin u)(5e^{5x})$$

$$= -5e^{5x}\sin(e^{5x})$$

3) Let $y = e^{\sin^2 x}$

$y = e^u$ where $u = \sin^2 x$

$$\frac{dy}{du} = e^u \quad \Big| \quad u = v^2 \text{ where } v = \sin x$$

$$\frac{du}{dv} = 2v \quad \Big| \quad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= (e^u)(2v)(\cos x)$$

$$= 2e^{\sin^2 x}\sin x \cos x$$

$$= \sin 2x \cdot e^{\sin^2 x}$$

4) Let y = log (sec x + tan x)

y = log u where u = sec x + tan x

$$\frac{dy}{du} = \frac{1}{u} \quad \Bigg| \quad \frac{du}{dx} = \sec x \, . \, \tan x + \sec^2 x$$
$$= \sec x(\tan x + \sec x)$$

$$\frac{dy}{dx} = \left(\frac{1}{u}\right)[\sec x(\tan x + \sec x)]$$
$$= \frac{1}{(\sec x + \tan x)} \, . \, \sec x(\tan x + \sec x)$$
$$= \sec x$$

5) Let $y = \text{cosec}^3(5x+1)$

$y = u^3$ where $u = \text{cosec}\ (5x + 1)$

$$\frac{dy}{du} = 3u^2 \quad \Big| \quad u = \text{cosec } v \text{ where } v = 5x + 1$$
$$\frac{du}{dv} = -\text{cosec } v \cot v \quad \Big| \quad \frac{dv}{dx} = 5$$

$$\frac{dy}{dx} = (3u^2)(-\text{cosec } v \cot v)\ (5)$$
$$= -15 \ \text{cosec}^2(5x+1) \, . \, \text{cosec}\ (5x + 1) \cot\ (5x + 1)$$
$$= -15 \ \text{cosec}^3(5x+1)\cot(5x+1)$$

PART – C

Differentiate the following w.r.t x.

1) $y = \sin\ (e^x \log x)$ 2) $(x^2 + 5)\ e^{-2x} \sin x$ 3) $y = (x \cos x)^3$

4) $y = \log\left(\dfrac{1-\cos x}{1+\cos x}\right)$ 5) $y = \sqrt{\dfrac{1+x^3}{1+x^3}}$ 6) $y = e^{4x} + \sin\ (x^2 + 5)$

7) $y = x^3 e^{-5x} \log(\sec x)$

Solution:

1) $y = \sin\ (e^x \log x)$

$y = \sin u$ where $u = e^x \log x$

$$\frac{dy}{du} = \cos u \quad \Bigg| \quad \frac{du}{dx} = e^x \frac{1}{x} + \log x \times e^x$$
(using product rule)
$$= e^x\left(\frac{1}{x} + \log x\right)$$

$$\frac{dy}{dx} = (\cos u)\left[e^x\left(\frac{1}{x} + \log x\right)\right]$$
$$= \cos(e^x \log x)\left[e^x\left(\frac{1}{x} + \log x\right)\right]$$

2) $y = (x^2 + 5)\ e^{-2x} \sin x$

$$u = x^2 + 5 \quad \Big| \quad v = e^{-2x} \quad \Big| \quad w = \sin x$$
$$\frac{du}{dx} = 2x \quad \Big| \quad \frac{dv}{dx} = e^{-2x}(-2) = -2e^{-2x} \quad \Big| \quad \frac{dw}{dx} = \cos x$$

$$\frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}$$

$$= (x^2+5)e^{-2x}\cos x + e^{-2x}\sin x(2x) + (x^2-5)\sin x(-2e^{-2x})$$

3) $y = (x \cos x)^3$

$y = u^3$ where $u = x \cos x$

$\frac{dy}{du} = 3u^2$ | $\frac{du}{dx} = x(-\sin x) + \cos x(1)$

(using product rule)

$= -x\sin x + \cos x$

$$\frac{dy}{dx} = (3u^2)(-x\sin x + \cos x)$$

$$= 3(x\cos x)^2(-x\sin x + \cos x)$$

4) $y = \log\left(\frac{1-\cos x}{1+\cos x}\right)$

$y = \log u$ where $u = \frac{1-\cos x}{1+\cos x}$

$\frac{dy}{du} = \frac{1}{u}$ | $\frac{du}{dx} = \frac{(1+\cos x)(+\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$

(using Quotient rule)

$$\frac{dy}{dx} = \left(\frac{1}{u}\right)\left[\frac{\sin x(1+\cos x) + \sin x(1-\cos x)}{(1+\cos x)^2}\right]$$

$$= \left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{2\sin x}{(1+\cos x)^2}\right]$$

$$= \frac{2\sin x}{(1+\cos x)(1-\cos x)} = \frac{2\sin x}{1-\cos^2 x}$$

$$= \frac{2\sin x}{\sin^2 x}$$

$$= \frac{2}{\sin x}$$

5) $y = \sqrt{\frac{1+x^3}{1+x^3}}$

$y = \sqrt{u}$ | where $u = \frac{1-x^3}{1+x^3}$

$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$ | $\frac{du}{dx} = \frac{(1+x^3)(-3x^2) - (1-x^3)(3x^2)}{(1+x^3)^2}$

$= 3x^2\left[\frac{-(1+x^3) - (1-x^3)}{(1+x^3)^2}\right]$

$= \frac{-6x^2}{(1+x^3)^2}$

$$\frac{dy}{dx} = \left(\frac{1}{2\sqrt{u}}\right)\frac{-6x^2}{(1+x^3)^2}$$

$$= \frac{1}{2\sqrt{\frac{1-x^3}{1+x^3}}}\left[\frac{-6x^2}{(1+x^3)^2}\right]$$

$$= \frac{-3x^2}{(1+x^3)^2}\sqrt{\frac{1+x^3}{1-x^3}}$$

6) $y = e^{4x} + \sin(x^2 + 5)$

$$\frac{dy}{dx} = e^{4x} \times 4 + \cos(x^2 + 5) \times (2x + 0)$$

$$= 4e^{4x} + 2x\cos(x^2 + 5)$$

7) $y = x^3 e^{-5x} \log(\sec x)$

$$u = x^3 \qquad v = e^{-5x} \qquad w = \log(\sec x)$$

$$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = e^{-5x}(-5) \qquad \frac{dw}{dx} = \frac{1}{\sec x}\sec x \,.\, \tan x$$

$$= -5e^{-5x} \qquad = \tan x$$

$$\frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}$$

$$= x^3 e^{-5x}\tan x + e^{-5x}\log(\sec x)(3x^2) + x^3\log(\sec x)(-5e^{-5x})$$

5.1.2 DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

If $x = \sin y$ then we get a unique real value of x to the given angle y. But if x is given y has many values.

We can express y interms of x as $y = \sin^{-1}x$ "Inverse sine of x". The Quantities $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$ are called Inverse Trigonometric Functions.

Differential co-efficient of $\sin^{-1}x$:

Let $y = \sin^{-1}x$

$\Rightarrow \sin y = x$

Differentiating w.r.t x

$$\cos y.\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\text{i.e.,}\ \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

Similarly we get

(i) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(iii) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

(iv) $\frac{d}{dx}(\sec^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

(v) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

WORKED EXAMPLES

PART – A

Find $\frac{dy}{dx}$ if

(i) $y = \sin^{-1}\sqrt{x}$

(ii) $y = \cos^{-1}(2x)$

(iii) $y = \tan^{-1}(\log x)$

Solution:

(i) $y = \sin^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \sin^{-1}u \text{ where } u = \sqrt{x}$$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \quad \Bigg| \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-u^2}}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x(1-x)}}$$

(ii) $y = \cos^{-1}(2x)$

$y = \cos^{-1}u$ where $u = 2x$

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}} \quad \Bigg| \quad \frac{du}{dx} = 2$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}.2$$

$$= \frac{-1}{\sqrt{1-(2x)^2}} \,.\, 2$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-4x^2}}$$

(iii) $y = \tan^{-1}(\log x)$

$y = \tan^{-1}u$ where $u = \log x$

$$\frac{dy}{du} = \frac{1}{1+u^2} \quad \Bigg| \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \left(\frac{1}{1+u^2}\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{x[1+(\log x)^2]}$$

PART – B

Find $\frac{dy}{dx}$ if

(i) $y = \sin^{-1}(\cos x)$ (ii) $y = \cos^{-1}(2x + 3)$ (iii) $y = \cos^{-1}(2x^2 - 1)$ (iv) $y = \sin^{-1}(3x - 4x^3)$

Solution:

(i) $y = \sin^{-1}(\cos x)$

$y = \sin^{-1}u$ where $u = \cos x$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \quad \Bigg| \quad \frac{du}{dx} = -\sin x$$

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-u^2}}\right)(-\sin x)$$

$$= \frac{1}{\sqrt{1-\cos^2 x}}(-\sin x)$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}} = -1$$

(ii) $y = \cos^{-1}(2x + 3)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x+3)^2}} \cdot \frac{d}{dx}(2x+3)$$

$$= \frac{-1}{\sqrt{1-(2x+3)^2}} \times 2$$

$$= \frac{-2}{\sqrt{1-(2x+3)^2}}$$

(iii) $y = \cos^{-1}(2x^2 - 1)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x^2-1)^2}} \cdot \frac{d}{dx}(2x^2)$$

$$= \frac{-1}{\sqrt{1-(2x^2-1)^2}} \cdot 4x$$

$$= \frac{-4x}{\sqrt{1-(2x^2-1)^2}} = \frac{-4x}{\sqrt{1-4x^4+4x^2-1}} = \frac{-4x}{\sqrt{4x^2-4x^4}} = \frac{-4x}{2x\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

ALITER

$y = \cos^{-1}(2x^2 - 1)$

Put $x = \cos\theta$

$y = \cos^{-1}(2\cos^2\theta - 1)$

$= \cos{-1}(\cos 2\theta)$

$y = 2\theta$

$y = 2\cos^{-1}x \qquad x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$$\frac{dy}{dx} = 2 \times \frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

(iv) $y = \sin^{-1}(3x - 4x^3)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x-4x^3)^2}}.(3-12x^2)$$

ALITER

Put $x = \sin\theta$

$y = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta)$

$y = 3\theta$

$y = 3\sin^{-1}x \qquad x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$$

PART – C

(i) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(ii) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(iii) $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(iv) $y = (1+x^2)\tan^{-1}x$

Solution:

(i) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta$

$y = 2\tan^{-1}x$

$$\frac{dy}{dx} = 2\frac{1}{1+x^2}$$

(ii) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$y = 2\tan^{-1}x$$

$$\frac{dy}{dx} = 2\cdot\frac{1}{1+x^2}$$

(iii) $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$y = 2\theta$$

$$y = 2\tan^{-1}x$$

$$\frac{dy}{dx} = 2\cdot\frac{1}{1+x^2}$$

(iv) $y = (1+x^2)\tan^{-1}x$

$$\frac{dy}{dx} = (1+x^2)\frac{1}{1+x^2} + \tan^{-1}x(0+2x) \quad \text{(using product rule)}$$

$$= 1 + 2x\tan^{-1}x$$

5.1.3 IMPLICIT FUNCTION

If the variables x and y are connected by a relation $f(x, y) = 0$ then it is called an implicit function.

i.e If the dependent variable (y) cannot be expressed explicitly in terms of the independent variable x.

Example: (i) $x^2 + y^2 - 4x + y + 12 = 0$

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are implicit function

PART –B

1. Find $\frac{dy}{dx}$ if

(i) $x^2 + y^2 + 2y = 0$ (ii) $x^2 + y^2 = a^2$ (iii) $xy = c^2$ (iv) $\sqrt{y} + \sqrt{x} = \sqrt{a}$

Solution:

(i) $x^2 + y^2 + 2y = 0$

Differentiate w.r.t x

$$2x + 2y\frac{dy}{dx} + 2\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$(y+1)\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{1+y}$$

(ii) $x^2 + y^2 = a^2$

Differentiate w.r.t x

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

(iii) $xy = c^2$

Differentiate w.r.t x

$$x \cdot \frac{dy}{dx} + y(1) = 0 \text{ (using product rule)}$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

(iv) $\sqrt{y} + \sqrt{x} = \sqrt{a}$

Differentiate w.r.t x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}.\frac{dy}{dx} = 0$$

Multiply by 2

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}.\frac{dy}{dx} = 0$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = \frac{-1}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}$$

PART – C

1. Find $\frac{dy}{dx}$ if

(i) $\cos x + \sin y = c$ (ii) $x^2 \sin y = c$ (iii) $y = a + xe^y$

Solution:

(i) $\cos x + \sin y = c$

Differentiate w.r.t x

$$-\sin x + \cos y \cdot \frac{dy}{dx} = 0$$

$$\cos y \cdot \frac{dy}{dx} = \sin x$$

$$\frac{dy}{dx} = \frac{\sin x}{\cos y}$$

(ii) $x^2 \sin y = c$

Differentiate w.r.t x

$$x^2 \cos y . \frac{dy}{dx} + \sin y(2x) = 0 \quad \text{(using product rule)}$$

$$x^2 \cos y . \frac{dy}{dx} = -2x \sin y$$

$$\frac{dy}{dx} = \frac{-2x \sin y}{x^2 \cos y}$$

$$\frac{dy}{dx} = \frac{-2}{x} \tan y$$

(iii) $y = a + xe^y$

Differentiate w.r.t x

$$\frac{dy}{dx} = 0 + x\ e^y . \frac{dy}{dx} + e^y$$

$$\frac{dy}{dx} - xe^y \frac{dy}{dx} = e^y$$

$$\frac{dy}{dx}(1 - xe^y) = e^y$$

$$\frac{dy}{dx} = \frac{e^y}{1 - xe^y}$$

2. Find $\frac{dy}{dx}$ if

(i) $x^3 + y^3 = 3axy$ (ii) $x^2 + y^2 - 4x + 6y - 5 = 0$ (iii) $x^2 + y^2 = xy$

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(i) $x^3 + y^3 = 3axy$

Differentiate w.r.t x

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y(1)\right]$$

Divide by 3 (using product rule)

$$x^2 + y^2\frac{dy}{dx} = a\left[x\frac{dy}{dx} + y\right]$$

$$y^2\frac{dy}{dx} - ax\frac{dy}{dx} = ay - x^2$$

$$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

(ii) $x^2 + y^2 - 4x + 6y - 5 = 0$

Differentiating w.r.t x

$$2x + 2y\frac{dy}{dx} - 4 + 6\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} + 6\frac{dy}{dx} = -2x + 4$$

$$\frac{dy}{dx}(2y + 6) = -2x + 4$$

$$\frac{dy}{dx} = \frac{-2x + 4}{2y + 6} = \frac{2(-x + 2)}{2(y + 3)}$$

$$= \frac{-x + 2}{y + 3}$$

(iii) $x^2 + y^2 = xy$

uv rule

Differentiating w.r.t x,

$$2x + 2y\frac{dy}{dx} = x\frac{dy}{dx} + y(1)$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx}(2y - x) = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Differentiate w.r.t x

$$2ax + 2h\left[x.\frac{dy}{dx} + y(1)\right] + b(2y).\frac{dy}{dx} + 2g(1) + 2f.\frac{dy}{dx} + 0 = 0$$

Divide by 2

$$ax + hx\frac{dy}{dx} + hy + by\frac{dy}{dx} + g + f\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(hx + by + f) = -ax - hy - g$$

$$\frac{dy}{dx} = \frac{-ax - hy - g}{hx + by + f}$$

5.2.1 SUCCESSIVE DIFFERENTIATION

If $y = f(x)$ then $\frac{dy}{dx} = f'(x)$ is a function of x or constant which can be differentiated once again.

$\frac{dy}{dx} = f'(x)$ is the first order derivative of y w.r.t x.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the second order derivative of y w.r.t x

i.e the derivate of $\frac{dy}{dx}$ w.r.t x.

$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is the third order derivative of y w.r.t x and so on.

Notations of successive derivatives

$$\frac{dy}{dx} = y_1 = y' = f'(x) = D(y)$$

$$\frac{d^2y}{dx^2} = y_2 = y'' = f''(x) = D^2(y)$$

$$\frac{d^ny}{dx^n} = y_n = y^{(n)} = f^{(n)}(x) = D^n(y)$$

WORKED EXAMPLES

PART – A

1. If $y = e^{3x}$ find $\frac{d^2y}{dx^2}$.

Solution:

$$\frac{dy}{dx} = 3e^{3x}$$

$$\frac{d^2y}{dx^2} = 3(3e^{3x})$$

$$= 9e^{3x}$$

2. If $y = \sin 3x$ find $\frac{d^2y}{dx^2}$.

Solution:

$$\frac{dy}{dx} = 3\cos 3x$$

$$\frac{d^2y}{dx^2} = 3(-3\sin 3x)$$

$$= -9\sin 3x$$

PART –B

1. If $y = a \cos nx + b \sin nx$ find y_2.

Solution:

$$\begin{aligned} y_1 &= a(-n \sin nx) + b(n \cos nx) \\ &= n(-a \sin nx + b \cos nx) \\ y_2 &= n[-a(+n \cos nx) + b(-n \sin nx)] \\ &= n^2(a \cos nx + b \sin nx) \\ &= -n^2 y \end{aligned}$$

2. If $y = Ae^{3x} + Be^{-3x}$ find $\dfrac{d^2y}{dx^2}$.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= A(3e^{3x}) + B(-3e^{-3x}) \\ &= 3(A.e^{3x} - Be^{-3x}) \\ \frac{d^2y}{dx^2} &= 3[A(3e^{3x}) - B(-3e^{-3x})] \\ &= 3[3A\, e^{3x} + 3B\, e^{-3x}] \\ &= 9[Ae^{3x} + Be^{-3x}] \\ &= 9y \end{aligned}$$

3. If $y = \tan x$ find y_2.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \sec^2 x \\ \frac{d^2y}{dx^2} &= 2(\sec x) \times \sec x \tan x \\ &= 2 \sec^2 x \tan x \end{aligned}$$

PART – C

1. If $y = x^2 \cos x$ prove that $x^2y_2 - 4xy_1 + (x^2 + 6)\, y = 0$.

Solution:

$$\begin{aligned} y &= x^2 \cos x \\ y_1 &= x^2(-\sin x) + \cos x\,(2x) \qquad \text{(using product rule)} \\ &= -x^2 \sin x + 2x \cos x \\ y_2 &= -x^2(\cos x) + \sin x\,(-2x) + 2x(-\sin x) + \cos x\,(2) \\ &= -x^2 \cos x - 2x \sin x - 2x \sin x + 2 \cos x \\ &= -x^2 \cos x - 4x \sin x + 2 \cos x \end{aligned}$$

$$\begin{aligned} x^2y_2 - 4xy_1 + (x^2 + 6)\, y &= x^2[-x^2 \cos x - 4x \sin x + 2 \cos x] \\ &\quad - 4x[-x^2 \sin x + 2x \cos x] + (x^2 + 6)(x^2 \cos x) \\ &= -x^4 \cos x - 4x^3 \sin x + 2x^2 \cos x + 4x^3 \sin x - 8x^2 \cos x + x^4 \cos x + 6x^2 \cos x \\ &= 0 \end{aligned}$$

2. If $y = e^x \sin x$, prove that $y_2 - 2y_1 + 2y = 0$.

Solution:

$$y = e^x \sin x$$

$$y_1 = e^x (\cos x) + \sin x\,(e^x) \qquad \text{(using product rule)}$$

$$y_1 = e^x \cos x + y \qquad \text{———— (1)}$$

Differentiating w.r.t x

$$y_2 = e^x(-\sin x) + \cos x\, e^x + y_1$$

$$y_2 = -y + (y_1 - y) + y_1 \text{ from (1)}$$

$$y_2 = -y + y_1 - y + y_1$$

$$= -2y + 2y_1$$

$$y_2 - 2y_1 + 2y = 0$$

3. If $y = a \cos(\log x) + b \sin(\log x)$ prove that $x^2y_2 + xy_1 + y = 0$.

Solution:

$$y = a \cos(\log x) + b \sin(\log x)$$

Differentiating w.r.t x

$$y_1 = a\left[-\sin(\log x).\frac{1}{x}\right] + b\left[\cos(\log x).\,\frac{1}{x}\right]$$

$$y_1 = \frac{1}{x}[-a\sin(\log x) + b\cos(\log x)]$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating w.r.t x

$$xy_2 + y_1(1) = -a\cos(\log x)\,.\,\frac{1}{x} + b\left(-\sin(\log x)\,.\,\frac{1}{x}\right)$$

$$xy_2 + y_1 = \frac{-1}{x}[a\cos(\log x) + b\sin(\log x)]$$

$$x[xy_2 + y_1] = -y$$

$$x^2y_2 + xy_1 + y = 0$$

5.2.2 DIFFERENTIAL EQUATION

Definition: An equation containg differential co-efficients is called differential equation.

Examples:

1. $\dfrac{dy}{dx} + y = x$
2. $x\dfrac{d^2y}{dx^2} + 1 = 0$
3. $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = d$

Order of the differential equation:

The order of the highest derivative in the differential equation is called order of the differential equation.

Degree of the differential equation:

The power of the highest derivative in the differential equation is called degree of the differential equation.

Example:

(i) $7\left(\frac{d^2y}{dx^2}\right)^2 + 5\left(\frac{dy}{dx}\right)^5 + 7y = \sin x$

Order – 2, degree –2

Formation of Differential Equation:

A differential equation is obtained by differentiating the function f (x, y) = 0 as many times as the number of arbitrary constants, followed by elimination of arbitrary constants.

PART –A

1. Find the order and degree of the following differential equations.

(i) $\left(\frac{d^3y}{dx^3}\right)^4 + \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 7y = 0$

(ii) $\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} = k$

(iii) $\sqrt{1+\frac{dy}{dx}} = \frac{d^2y}{dx^2}$

Solution

(i) order = 3, degree = 4

(ii) order =1, degree = 2

(iii) $\sqrt{1+\frac{dy}{dx}} = \frac{d^2y}{dx^2}$

Squaring on both sides

$$\left(1+\frac{dy}{dx}\right) = \left(\frac{d^2y}{dx^2}\right)^2$$

Order –2, degree – 2.

2. From the differential equation by eliminating the constant.

(i) $xy = c$ (ii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Solution:

(i) $xy = c$

$x.\frac{dy}{dx} + y.1 = 0$ (using product rule)

$$x\frac{dy}{dx} + y = 0$$

(ii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}.\frac{dy}{dx} = 0$$

Multiply by 2

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

PART –B

From the different equation by eliminating the constants.

(i) $y = ax + b^2$ (ii) $y = m\,e^{5x}$ (iii) $\frac{x}{a}+\frac{y}{b}=1$ (iv) $y^2 = 4ax$

Solution:

(i) $y = ax + b^2$

$$\frac{dy}{dx}=a(1)+0$$

$$\frac{d^2y}{dx^2}=0$$

(ii) $y = m\,e^{5x}$ —— (1)

$$\frac{dy}{dx}=m.(5e^{5x})$$

$$\frac{dy}{dx}=5m(e^{5x}) \qquad \text{from (1) } me^{mx}=y$$

$$\frac{dy}{dx}=5y$$

(iii) $\frac{x}{a}+\frac{y}{b}=1$

Differentiating w.r.t x

$$\frac{1}{a}+\frac{1}{b} \qquad \frac{dy}{dx}=0$$

$$0+\frac{1}{b}\frac{d^2y}{dx^2}=0$$

$$\therefore \frac{d^2y}{dx^2}=0$$

(iv) $y^2 = 4ax$

$$2y.\frac{dy}{dx}=4a(0)$$

$$2y.\frac{dy}{dx}=4\left(\frac{y^2}{4x}\right) \qquad \left(\because y^2=4ax,\ \frac{y^2}{4x}=a\right)$$

$$2y\frac{dy}{dx}=\frac{y^2}{x}$$

5.3 PARTIAL DIFFERENTIATION

Functions of two or more variables:

In many applications, we come across function involving more than one independent variable. For example, Area of the rectangle depends on its length and breadth, volume of cuboid depends on its length, breadth and height.

Hence one variable u depends on more than one variable.

i.e. $u = f(x, y)$ $\quad$ $v = \varphi(x, y, z)$

Definition of partial differentiation:

Let $u = f(x, y)$ then partial differentiation of u w.r.t x is defined as differentiation of u w.r.t x treating y as constant and is denoted by $\frac{\partial u}{\partial x}$.

$$\therefore \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

Similarly partial differentiation of u w.r.t y is defined as differentiation of u w.r.t y treating x as constant and is denoted by $\frac{\partial u}{\partial y}$.

$$\therefore \frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Second order partial derivatives:

In General, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are function of x and y. They can be further dfferentiated partially w.r.t x and y as follows.

(i) $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial x^2}\right)$

(ii) $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)$

(iii) $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial y\, \partial x}\right)$

(iv) $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial^2 u}{\partial y^2}\right)$

For all ordinary functions $\left(\frac{\partial^2 u}{\partial x\, \partial y}\right) = \left(\frac{\partial^2 u}{\partial y\, \partial x}\right)$.

WORKED EXAMPLES

PART – A

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following:
   (i) $u = x + y$ (ii) $u = x^3 + y^3$ (iii) $u = e^{x+y}$ (iv) $u = e^{2x+3y}$

Solution:

i) $u = x + y$

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 1$$

ii) $u = x^3 + y^3$

$$\frac{\partial u}{\partial x} = 3x^2$$

$$\frac{\partial u}{\partial y} = 3y^2$$

(iii) $u = e^{x+y}$

$$\frac{\partial u}{\partial x} = e^{x+y}(1) = e^{x+y}$$

$$\frac{\partial u}{\partial y} = e^{x+y}(1) = e^{x+y}$$

(iv) $u = e^{2x+3y}$

$$\frac{\partial u}{\partial x} = e^{2x+3y}(2) = 2e^{2x+3y}$$

$$\frac{\partial u}{\partial y} = e^{2x+3y}(3) = 3e^{2x+3y}$$

PART – B

(i) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following:

(i) $u = x^3 + 4x^2y + 5xy^2 + y^3$

(ii) $u = 5 \sin x + 4 \tan y$

(iii) $u = y^2 \sec x$

(iv) $u = \sin 4x \cos 2y$

Solution:

(i) $u = x^3 + 4x^2y + 5xy^2 + y^3$

$$\frac{\partial u}{\partial x} = 3x^2 + 8xy + 5y^2 + 0$$

$$\frac{\partial u}{\partial y} = 0 + 4x^2 + 10xy + 3y^2$$

(ii) $u = 5 \sin x + 4 \tan y$

$$\frac{\partial u}{\partial x} = 5\cos x + 0$$

$$\frac{\partial u}{\partial y} = 0 + 4\sec^2 y$$

(iii) $u = y^2 \sec x$

$$\frac{\partial u}{\partial x} = y^2 \sec x \tan x$$

$$\frac{\partial u}{\partial y} = \sec x.2y$$

(iv) $u = \sin 4x \cos 2y$

$$\frac{\partial u}{\partial x} = 4\cos 4x.\cos 2y$$

$$\frac{\partial u}{\partial y} = 2\sin 4x \sin 2y$$

2. Find $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$ for the following:

(i) $u = x^3 + y^3$ (ii) $u = x^3 \tan y$

Solution:

(i) $u = x^3 + y^3$

$$\frac{\partial u}{\partial x} = 3x^2 + 0$$

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = 0 + 3y^2$$

$$\frac{\partial^2 u}{\partial y^2} = 6y$$

(ii) $u = x^3 \tan y$

$$\frac{\partial u}{\partial x} = 3x^2 \tan y$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \tan y$$

$$\frac{\partial u}{\partial y} = x^3 \sec^2 y$$

$$\frac{\partial^2 u}{\partial y^2} = x^3 (2\sec y \,.\, \sec y \tan y)$$

$$= 2x^3 \sec^2 y \,.\, \tan y$$

3. Find $\dfrac{\partial^2 u}{\partial x \partial y}$ and $\dfrac{\partial^2 u}{\partial y \partial x}$ for $u = x^2 e^{5y}$.

Solution:

$$u = x^2 e^{5y}$$

$$\frac{\partial u}{\partial x} = 2xe^{5y}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 2x \,.\, (5e^{5y}) = 10xe^{5y}$$

$$\frac{\partial u}{\partial y} = 5x^2 e^{5y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 10xe^{5y}$$

PART – C

1. If $u = \log (x^3 + y^3)$ find $\dfrac{\partial^2 u}{\partial x^2}$.

Solution:

$$u = \log (x^3 + y^3)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y3}(3x^2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^3 + y^3)(6x) - (3x^2)(3x^2 + 0)}{(x^3 + y^3)^2} \qquad \text{(using quotient rule)}$$

$$= \frac{6xy^3 - 3x^4}{(x^3 + y^3)^2}$$

2. If $u = \dfrac{x}{y^2} - \dfrac{y}{x^2}$ find $\dfrac{\partial^2 u}{\partial x \partial y}$.

Solution:

$$u = \frac{x}{y^2} - \frac{y}{x^2}$$

$$\frac{\partial u}{\partial y} = x.\left(\frac{-2}{y^3}\right) - \frac{1}{x^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{-2}{y^3}\right) - \left(\frac{-2}{x^3}\right)$$

$$= \frac{2}{x^3} - \frac{2}{y^3}$$

3. If $u = x^3 + 3x^2y + 3xy^2 + y^3$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$.

Solution:

$$u = x^3 + 3x^2y + 3xy^2 + y^3 \qquad \text{..............(A)}$$

$$\frac{\partial u}{\partial x} = 3x^2 + 6xy + 3y^2 + 0$$

$$x\frac{\partial u}{\partial x} = x(3x^2 + 6xy + 3y^2)$$

$$x\frac{\partial u}{\partial x} = 3x^3 + 6x^2y + 3xy^2 \qquad \text{.............(1)}$$

$$u = x^3 + 3x^2y + 3xy^2 + y^3$$

$$\frac{\partial u}{\partial y} = 0 + 3x^2 + 3x(2y) + 3y^2$$

$$= 3x^2 + 6xy + 3y^2$$

$$y\frac{\partial u}{\partial y} = y(3x^2 + 6xy + 3y^2)$$

$$= 3x^2y + 6xy^2 + 3y^3 \qquad \text{..............(2)}$$

$$(1) + (2) \Rightarrow$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3 + 6x^2y + 3xy^2 + 3x^2y + 6xy^2 + 3y^3$$

$$= 3x^3 + 9x^2y + 9xy^2 + 3y^3$$

$$= 3(x^3 + 3x^2y + 3xy^2 + y^3)$$

$$= 3u \text{ using (A)}$$

EXERCISE

PART – A

1. Find $\dfrac{dy}{dx}$ if

   (i) $y = (2x + 5)^3$ (ii) $y = \sqrt{ax + b}$

   (iii) $y = \sin 2x$ (iv) $y = \cos^3 x$

   (v) $y = \sin^2 5x$ (vi) $y = \tan\sqrt{x}$

   (vii) $y = \sin(\cos x)$ (viii) $y = \cos(\log x)$

   (ix) $y = \log(2x + 3)$ (x) $y = e^{x^2}$

2. Find $\dfrac{dy}{dx}$ if

   (i) $y = \sin^{-1}(2x)$ (ii) $y = \cos^{-1}(x^2)$ (iii) $y = \tan^{-1}(\sqrt{x})$ (iv) $y = \cos^{-1}(e^x)$

3. Find $\dfrac{dy}{dx}$ from the following equations

   (i) $y^2 = 4ax$ (ii) $2x^2 - y^2 - 9 = 0$ (iii) $x^3 + xy = 0$ (iv) $x \sin y = c$

4. If (i) $y = 2x^5 - 4x^2 + 3$

   (ii) $y = \sin^3 x$ (iii) $y = x + \cos x$ find $\dfrac{d^2y}{dx^2}$.

5. From the differential equation by eliminating the arbitrary constants from the following equations.

   (i) $y = ax + b$ (ii) $y^2 = 4ax$ (iii) $y = mx$

   (iv) $y = mx + \frac{1}{m}$ (v) $x^2 + y^2 = a^2$ (vi) $y = cx + c^3$

6. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ if $u = x^3 + 5x^2y + y^3$.

7. If $u = e^{x^2+y^2}$ find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

8. If $u = \tan(ax + by)$ find $\frac{\partial u}{\partial y}$.

9. If $u = \tan(ax + by)$ find $\frac{\partial u}{\partial x}$.

PART – B

1. Find $\frac{dy}{dx}$ if

   (i) $y = \operatorname{cosec}^3(5x + 1)$ (ii) $y = \tan(x^2 \log x)$

   (iii) $y = \cos\left[\frac{\log x}{x}\right]$ (iv) $y = e^{\cos^3 x}$

2. Find $\frac{dy}{dx}$ if

   (i) $y = \sqrt{x}\,\tan^{-1}x$ (ii) $y = x^3 + \tan^{-1}(x^2)$ (iii) $y = (\sin^{-1}x)^2$ (iv) $y = \cos^{-1}(4x^3 - 3x)$

3. If $x^2 - y^2 = 5y - 3x$ find $\frac{dy}{dx}$.

4. If $y^2 = x \sin y$ find $\frac{dy}{dx}$.

5. If $\sin y = x \sin(a + y)$ find $\frac{dy}{dx}$.

6. If $y = ae^x + be^{-x}$ that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$.

7. If $y = \frac{\cos x}{x}$ prove that $xy_2 + 2y_1 + xy = 0$.

8. If $u = x^3 - 2x^2y + 3xy^2 + y^3$ find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$.

9. If $u = x^y$ find $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$.

10. If $u = \sqrt{x^2 + y^2}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

PART – C

1. Find $\frac{dy}{dx}$ if

   (i) $y = e^{3x} \log x \sin 2x$ (ii) $y = \frac{1+\sin 2x}{1-\sin 2x}$

   (iii) $y = \frac{\sin(\log x)}{x}$ (iv) $(3x^2 - 2)^2 \cot x \,.\, \log \sin x.$

2. Find $\frac{dy}{dx}$ if

   (i) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ (ii) $y = x \sin^{-1}(\tan x)$ (iii) $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$

3. If $y = x^2 \sin x$ prove that $x^2y_2 - 4xy_1 + (x^2 + 6)\, y = 0$.

4. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the following functions:

   (i) $u = \frac{x}{y^2} - \frac{y}{x^2}$ (ii) $u = x \sin y + y \sin x$ (iii) $u = \log\left(\frac{x^2+y^2}{xy}\right)$

5. If $u = x^4 + 4x^3y + 3x^2y^2 + y^4$ find $x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2}$.

6. If $u = \log (x^2 + y^2)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$.

ANSWERS

PART – A

1. (i) $6\,(2x + 5)^2$ (ii) $\frac{a}{2\sqrt{ax+b}}$ (iii) $2 \cos 2x$ (iv) $-3 \cos^2 x \sin x$

   (v) $10 \sin 5x \cos 5x$ (vi) $\frac{1}{2\sqrt{x}}\sec^2 \sqrt{x}$ (vii) $-$ [cos (cois x)] sin x

   (viii) $-\frac{1}{x}\sin(\log x)$ (ix) $\frac{2}{2x+3}$ (x) $2x\, e^{x^2}$

2. (i) $\frac{2}{\sqrt{1-4x^2}}$ (ii) $-\frac{2x}{\sqrt{1-4^2}}$ (iii) $\frac{1}{2\sqrt{x}(1+x)}$ (iv) $\frac{-e^x}{\sqrt{1-e^{2x}}}$

3. (i) $\frac{dy}{dx} = \frac{2a}{y}$ (ii) $\frac{dy}{dx} = \frac{2x}{y}$ (iii) $\frac{-(3x^2+y)}{x}$ (iv) $\frac{-\tan y}{x}$

4. (i) $40x^3 - 8$ (ii) $3 \sin x\,[2 \cos^2 x - \sin^2 x]$ (iii) $-\cos x$

5. (i) $\frac{d^2y}{dx^2} = 0$ (ii) $y^2 = 2xy\frac{dy}{dx}$ (iii) $y = x\frac{dy}{dx}$ (iv) $y = x\frac{dy}{dx} + \frac{dx}{dy}$

   (v) $x + y\frac{dy}{dx} = 0$ (vi) $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3$

6. $3x^2 + 10xy,\ 5x^2 + 3y^2$

8. $2x\, e^{x^2+y^2},\ 2y\, e^{x^2+y^2}$

9. $a \sec^2(ax + by)$

PART – B

1. (i) $15 \operatorname{cosec}^3 (5x + 1) \cot (5x + 1)$ (ii) $x (1 + 2 \log x) \sec^2 (x^2 \log x)$

(iii) $\dfrac{-(1+\log x)}{x^2} \sin\left(\dfrac{\log x}{x}\right)$ (iv) $-3 \cos^2 x \sin x\, e^{\cos^3 x}$

2. (i) $\dfrac{1}{2\sqrt{x}} \tan^{-1} x + \dfrac{\sqrt{x}}{1+x^2}$ (ii) $3x^2 + \dfrac{2x}{1+x^4}$ (iii) $\dfrac{2\sin^{-1} x}{\sqrt{1-x^2}}$ (iv) $\dfrac{-3(4x^2-1)}{\sqrt{1-(4x^3-3x)^2}}$

3. $\dfrac{2x+3}{2y+5}$ 4. $\dfrac{\sin y}{2y - x\cos y}$ 5. $\dfrac{\sin(a+y)}{\cos y - x\cos(a+y)}$ 8. $6x - 4y,\ 6x + 6y$

9. $yx^{y-1},\ y(y-1)\, x^{y-2}$

PART – C

1. (i) $e^{3x}\left[2\log x \cos 2x + 3\log x \sin 2x + \dfrac{\sin 2x}{x}\right]$ (ii) $\dfrac{4\cos 2x}{(1-\sin 2x)^2}$

(iii) $\dfrac{\cos(\log x) - \sin(\log x)}{x^2}$

(iv) $12x (3x^2 - 2) \cot x \log (\sin x) + (3x^2 - 2)^2 \cot^2 x - (3x^2 - 2) \log (\sin x) \operatorname{cosec}^2 x$

2. (i) $\dfrac{2}{1+x^2}$ (ii) $\dfrac{x\sec^2 x}{\sqrt{1-\tan^2 x}}$ (iii) $\dfrac{\sqrt{1-x^2} - x\sin^{-1} x}{(1-x^2)^{3/2}}$

10. $6 (2x^4 + 4x^3 y + 2x^2 y^2 - xy^3 + 2y^4)$

UNIT – IV

INTEGRAL CALCULUS-I

4.1 Introduction:

Definition of integration – Integral values using reverse process of differentiation – Integration using decomposition method. Simple problems.

4.2 Integration by Substitution:

Integrals of the form $\int [f(x)]^n f'(x)dx,\ n \neq 1,\ \int \frac{f'(x)dx}{f(x)}$, and $\int [F(f(x)]f'(x)\ dx$, Simple problems.

4.3 Standard Integrals

Integrals of the form $\int \frac{dx}{a^2 \pm x^2}, \frac{dx}{x^2 - a^2}$ and $\int \frac{dx}{\sqrt{a^2 - x^2}}$, Simple problems.

4.1 INTRODUCTION

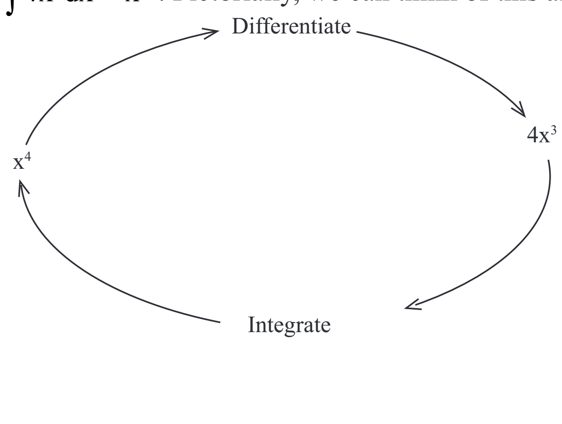
Sir Sardar Vallabhai Patel, called the Iron Man of India integrated several princely states together while forming our country Indian Nation after independence. Like that in Maths while finding area under a curve through integration, the area under the curve is divided into smaller rectangles and then integrating (i.e) summing of all the area of rectangles together. So, integraiton means of summatiion of very minute things of the same kind.

Integration as the reverse of differentiation:

Integration can also be introduced in another way, called integration as the reverse of differentiation.

Differentiation in reverse:

Suppose we differentiate the function y = x^4, then we have $\frac{5}{4}$. Now, we say that integral of $4x^3$ is x^4 and we write this as $\int 4x^3 dx = x^4$. Pictorially, we can think of this as follows:



Suppose we differentiate the functions, $y = x^4 + 5$, $y = x^4 - 23$, $y = x^4 + 100$, then also we get $\frac{dy}{dx} = 4x^3$. Now, what could we say about the integral of $4x^3$. Can we say that it is $x^4 + 5$ (or) $x^4 - 23$ (or) $x^4 + 100$? So, in general we say that $\int 4x^3 dx = x^4 + c$ where c is called constant of integration.

The symbol for integration is ∫, known as integral sign. Along with the integral sign there is a term dx which must always be written and which indicates the name of the variable involved, in this case 'x'. Technically integrals of this sort are called indefinite Integrals.

List of Formulae:

Sl.No.	Integration	Reverse Process of Differentiation
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$ $(n \neq -1)$	$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + c\right]$ $= \cancel{(n+1)} \frac{x^{n+1-1}}{\cancel{(n+1)}} + 0$ $= x^n$
2.	$\int \frac{1}{x} dx = \log x + c$	$\frac{d}{dx}(\log x + c) = \frac{1}{x} + 0 = \frac{1}{x}$
3.	$\int e^x dx = e^x + c$	$\frac{d}{dx}(e^x) = e^x$
4.	$\int \sin x dx = -\cos x + c$	$\frac{d}{dx}(-\cos x + c) = \sin x$
5.	$\int \cos x dx = \sin x + c$	$\frac{d}{dx}(\sin x + c) = \cos x$
6.	$\int \sec^2 x dx = \tan x + c$	$\frac{d}{dx}(\tan x + c) = \sec^2 x$
7.	$\int \text{cosec}^2 x dx = -\cot x + c$	$\frac{d}{dx}(-\cot x + c) = \text{cosec}^2 x$
8.	$\int \sec x \tan x \ dx = \sec x + c$	$\frac{d}{dx}(\sec x + c) = \sec x \tan x$
9.	$\int \text{cosec} x \cot x \ dx = -\text{cosec} \ x + c$	$\frac{d}{dx}(-\text{cosec} \ x + c) = \text{cosec} \ x \cot x$

Particular forms of $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ where $n \neq -1$.

1. $\int x \ dx = \frac{x^{1+1}}{1+1} = \frac{x^2}{2} + c$

2. $\int x^3 \ dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4} + c$

3. $\int dx = x + c \quad \because \frac{d}{dx} x = 1$

4. $\int \sqrt{x}\ dx = \int x^{1/2} dx = \dfrac{x^{1/2+1}}{\frac{1}{2}+1} = \dfrac{x^{3/2}}{\frac{3}{2}} = \dfrac{2}{3} x^{3/2} + c$

5. $\int \dfrac{1}{x^4}\ dx = \int x^{-4} dx = \dfrac{x^{-4+1}}{-4+1} = \dfrac{x^{-3}}{-3} + c$

6. $\int \dfrac{1}{x}\ dx = \log x + c \quad \because \dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

Note:

$$\boxed{\int \frac{1}{x} dx = \int x^{-1} dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \infty}$$

So we should not use the formula $\dfrac{x^{n+1}}{n+1}$

Two Basic Theorems on Integration (Without Proof):

1. If u, v, w etc. are functions of x, then $\int (u \pm v \pm w ....) dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx \pm ....$
2. If f(x) is any function of x and K any constant then $\int K\ f(x) dx = K \int f(x)\ dx$

Example:

1. Evaluate: $\int (4x - 3x^2) dx$

Solution:

$\int (4x - 3x^2) dx$

$= \int 4x\ dx - \int 3x^2 dx$

$= 4\int x\ dx - 3\int x^2 dx$

$= 4\dfrac{x^{1+1}}{1+1} - 3\dfrac{x^{2+1}}{2+1} + c$

$= \dfrac{4x^2}{2} - \dfrac{3x^3}{3} + c$

$= 2x^2 - x^3 + c$

Integration using decomposition method:

In integration, there is no rule for multiplication (or) division of algebraic or trigonometric function as we have in differentiation. Such functions are to be decomposed into addition and subtraction before applying integration.

For example $\dfrac{\sin^2 x}{1+\cos x}$ can be decomposed as follows

$$\frac{\sin^2 x}{1+\cos x} = \frac{1-\cos^2 x}{1+\cos x} = \frac{(1+\cancel{\cos x})\,(1-\cos x)}{(1+\cancel{\cos x})} = 1 - \cos x,$$

which can be integrated using above theorems.

Examples:

1. Evaluate: $\int (x+1)(x+2)\,dx$

Solution:

There is no *uv* rule in integration. So, we first multiply (x + 1) and (x + 2) and then integrate

$$\int (x+1)(x+2)\,dx$$
$$= \int (x^2+2x+x+2)\,dx$$
$$= \int (x^2+3x+2)\,dx$$
$$= \frac{x^3}{3}+\frac{3x^2}{2}+2x+c$$

2. Evaluate: $\int (1+x^2)^3\,dx$

Solution:

$$\boxed{\begin{aligned} \text{Formula: } (a+b)^3 &= a^3+b^3+3a^2b+3ab^2 \\ (1+x^2)^3 &= 1^3+(x^2)^3+3(1)\,x^2+3(1)\,(x^2)^2 \\ &= 1+x^6+3x^2+3x^4 \end{aligned}}$$

$$\int (1+x^2)^3\,dx = \int (1+x^6+3x^2+3x^4)\,dx$$
$$= x+\frac{x^7}{7}+\frac{3x^3}{3}+\frac{3x^5}{5}+c$$

3. Evaluate: $\int \frac{\sin x}{1+\sin x}\,dx$

Solution:

$$\int \frac{\sin x}{1+\sin x}\,dx$$
$$= \int \frac{\sin x}{1+\sin x}\times\frac{(1-\sin x)}{(1-\sin x)}\,dx \qquad \boxed{\begin{array}{c}\text{Multiply and divide by conjugate of}\\ 1+\sin x = 1-\sin x\end{array}}$$

$$(a+b)\,(a-b) = a^2-b^2$$
$$= \int \frac{\sin x(1-\sin x)}{1^2-\sin^2 x}\,dx$$
$$= \int \frac{\sin x-\sin^2 x}{\cos^2 x}\,dx \qquad \boxed{\begin{array}{l}\sin^2 x+\cos^2 x = 1\\ \therefore 1-\sin^2 x = \cos^2 x\end{array}}$$

Dividing separately

$$= \int \left(\frac{\sin x}{\cos^2 x}-\frac{\sin^2 x}{\cos^2 x}\right)dx$$
$$= \int \frac{1}{\cos x}\frac{\sin x}{\cos x}\,dx - \int \tan^2 x\,dx \qquad \boxed{\begin{array}{l}1+\tan^2 x = \sec^2 x\\ \Rightarrow \tan^2 x = \sec^2 x-1\end{array}}$$
$$= \int \sec x\tan x\,dx - \int (\sec^2 x-1)\,dx$$
$$= \sec x-(\tan x-x) \quad \text{Note: } \frac{d}{dx}(x) = 1 \Rightarrow \int 1dx = x$$
$$= \sec x-\tan x+x+c$$

3.1 WORKED EXAMPLES

PART – A

1. Evaluate : $\int \left(x^3 + e^x + \frac{1}{x} \right) dx$

Solution:

$$\int \left(x^3 + e^x + \frac{1}{x} \right) dx$$
$$= \frac{x^4}{4} + e^x + \log x + c$$

2. Evaluate: $\int (\tan^2 x + \cot^2 x) dx$

Solution:

$1 + \tan^2 x = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$1 + \cot^2 x = \operatorname{cosec}^2 x \Rightarrow \cot^2 x = \operatorname{cosec}^2 x - 1$

$$\int (\tan^2 x + \cot^2 x) dx$$
$$= \int (\sec^2 x - 1 + \operatorname{cosec}^2 x - 1)\, dx$$
$$= \int (\sec^2 x + \operatorname{cosec}^2 x - 2)\, dx$$
$$= \tan x - \cot x - 2x + c$$

3. Evaluate: $\int (e^{3x} + e^{-5x})\, dx$

Solution:

$$\int (e^{3x} + e^{-5x})\, dx$$
$$= \frac{e^{3x}}{3} + \frac{e^{-5x}}{-5}$$
$$= \frac{e^{3x}}{3} - \frac{e^{-5x}}{5} + c$$

> Hint : Divide by coefficient of x
>
> In general $\int e^{mx} dx = \frac{e^{mx}}{m}$ because
>
> $\frac{d}{dx}\left(\frac{e^{mx}}{m} \right) = \frac{1}{m} \frac{d}{dx}(e^{mx}) = \frac{1}{m} \cdot e^{mx} \cdot m = e^{mx}$

4. Evaluate: $\int \sec^2 (3 + 4x) dx$

Solution:

$$\int \sec^2 (3 + 4x) dx$$
$$= \frac{1}{4} \tan(3 + 4x) + c$$

> Hint : Divide by coefficient of x
>
> Shown as $\frac{1}{4}$
>
> $\frac{d}{dx}\left[\frac{1}{4} \tan(3 + 4x) \right]$
>
> $= \frac{1}{4} \sec^2 x(3 + 4x)\ 4 = \sec^2 (3 + 4x)$

5. Evaluate: $\int (3 - 2x)^4 dx$

Solution:

$$\int (3 - 2x)^4 dx$$
$$= \frac{1}{-2}\left[\frac{(3 - 2x)^5}{5} \right] + c$$

$\boxed{\text{Divide by coefficient of x}}$

shown as $\frac{1}{-2}$

6. Evaluate: $\int \sqrt{1-\sin 2x}\ dx$

Solution:

$$\int \sqrt{1-\sin 2x}\ dx \qquad \boxed{\begin{array}{l}\sin^2 x+\cos^2 x=1\\ \sin 2x=2\sin x\cos x\end{array}}$$

$$=\sqrt{\sin^2 x+\cos^2 x-2\sin x\cos x}\ \ dx$$

$$\boxed{a^2+b^2-2ab=(a-b)^2}$$

$$=\int \sqrt{(\sin x-\cos x)^2}\ dx$$

$$=\int (\sin x-\cos x)\ dx$$

$$=-\cos x-\sin x+c$$

7. Evaluate: $\int \sqrt{2x-3}\ dx$

Solution:

$$\int \sqrt{2x-3}\ dx$$

$$\int (2x-3)^{1/2} dx \qquad \boxed{\text{Divide by coefficient of x}}$$

$$=\frac{1}{2}\left[\frac{(2x-3)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]+c$$

$$=\frac{1}{2}\left[\frac{(2x-3)^{3/2}}{3/2}\right]+c=\frac{1}{2}\times\frac{2}{3}(2x-3)^{3/2}+c$$

$$=\frac{1}{3}(2x-7)^{3/2}+c$$

8. Evaluate: $\int \frac{dx}{\sqrt{2+x}}$

Solution:

$$\int \frac{dx}{\sqrt{2+x}}=\int \frac{dx}{(2+x)^{1/2}}=\int (2+x)^{-\frac{1}{2}}dx$$

$$=\frac{1}{1}\left[\frac{(2+x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]+c \qquad \boxed{\text{Divide by coefficient of x = 1}} \qquad \text{shown as } \frac{1}{1}$$

$$=\frac{(2+x)^{1/2}}{\frac{1}{2}}+c=2(2+x)^{1/2}+c$$

PART – B & C

1. Evaluate: $\int (x+1)\,(2x-3)\,dx$

Solution:

There is no uv rule in integraiton.

$$\int (x+1)\,(2x-3)\,dx$$
$$= \int (2x^2 - 3x + 2x - 3)\,dx$$
$$= \int (2x^2 - x - 3)\,dx$$
$$= \frac{2x^3}{3} - \frac{x^2}{2} - 3x + C$$

2. Evaluate: $\int \left(x + \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right) dx$

Solution:

$$\int \left(x + \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right) dx$$
$$= \int \left(x^3 - \frac{1}{x} + x - \frac{1}{x^3}\right) dx$$

$\boxed{\text{Do not write } \frac{1}{x} \text{ as } x^{-1}}$

$$= \int \left(x^3 - \frac{1}{x} + x - x^{-3}\right) dx$$
$$= \frac{x^4}{4} - \log x + \frac{x^2}{2} - \frac{x^{-3+1}}{-3+1} + c$$
$$= \frac{x^4}{4} - \log x + \frac{x^2}{2} - \frac{x^{-2}}{-2} + c$$
$$= \frac{x^4}{4} - \log x + \frac{x^2}{2} + \frac{1}{2x^2} + c$$

3. Evaluate: $\int \left(\frac{2}{x^2} - \frac{7}{x} + \frac{3}{\sin^2 x}\right) dx$

Solution:

$$\int \left(\frac{2}{x^2} - \frac{7}{x} + \frac{3}{\sin^2 x}\right) dx$$
$$= \int \left(2x^{-2} - \frac{7}{x} + 3\operatorname{cosec}^2 x\right) dx$$
$$= \frac{2x^{-2+1}}{-2+1} - 7\log x - 3\cot x + c$$
$$= \frac{2x^{-1}}{-1} - 7\log x - 3\cot x + c$$
$$= -\frac{1}{2x} - 7\log x - 3\cot x + c$$

Conjugate Model:

4. Evaluate: $\int \frac{\cos^2 x}{1+\sin x} dx$

Solution:

Multiply and Divide by the conjugate of $(1 + \sin x) = (1 - \sin x)$

$$\int \frac{\cos^2 x}{1+\sin x} \times \frac{(1-\sin x)}{(1-\sin x)} dx$$

$$(a+b)(a-b) = a^2 - b^2$$

$$= \int \frac{\cos^2 x(1-\sin x)}{1-\sin^2 x} dx$$

$$= \int \frac{\cos^2 x(1-\sin x)}{\cos^2 x} dx$$

$$\boxed{\begin{array}{l} \cos^2 x + \sin^2 x = 1 \\ \Rightarrow \cos^2 x = 1 - \sin^2 x \end{array}}$$

$$= \int (1-\sin x)\, dx$$

$$= x + \cos x + c$$

5. Evaluate: $\int \sin^2 x\, dx$

Solution:

$$\int \sin^2 x\, dx$$

$$= \frac{1}{2}\int (1-\cos 2x) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$$

$$\boxed{\begin{array}{l} \cos 2x = 1 - 2\sin^2 x \\ \Rightarrow 2\sin^2 x = 1 - \cos 2x. \\ \sin^2 x = \frac{1}{2}(1-\cos 2x) \end{array}}$$

6. Evaluate: $\int \cos^2 x\, dx$

Solution:

$$\int \cos^2 x\, dx$$

$$= \frac{1}{2}\int (1+\cos 2x) dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$$

$$\boxed{\begin{array}{l} \cos 2x = 2\cos^2 x - 1 \\ \Rightarrow 1 + \cos 2x = 2\cos^2 x \\ \Rightarrow \frac{1}{2}(1+\cos 2x) = \cos^2 x \end{array}}$$

7. Evaluate: $\int \sin^3 x dx$

Solution:

$$\int \sin^3 x\, dx$$

$$= \frac{1}{4}\int (3\sin x - \sin 3x)\, dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + c$$

$$\boxed{\begin{array}{l} \sin 3x = 3\sin x - 4\sin^3 x \\ 4\sin^3 x = 3\sin x - \sin 3x \\ \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x) \end{array}}$$

8. Evaluate: $\int \cos^3 x \, dx$

Solution:

$$\int \cos^3 x \, dx$$

$$= \frac{1}{4}\int (3\cos x + \cos 3x) \, dx$$

$$= \frac{1}{4}\left[3\sin x + \frac{\sin 3x}{3}\right] + c$$

$$\boxed{\begin{aligned} &\cos 3x = 4\cos^3 x - 3\cos x \\ &3\cos x + \cos 3x = 4\cos^3 x \\ &\frac{1}{4}(3\cos x + \cos 3x) = \cos^3 x \end{aligned}}$$

9. Evaluate: $\int \sin 5x \cos 3x \, dx$

Solution:

$$\therefore \int \sin 5x \cos 3x \, dx$$

$$= \frac{1}{2}\int [\sin(5x+3x) + \sin(5x-3x)] \, dx$$

$$= \frac{1}{2}\int (\sin 8x + \sin 2x) \, dx$$

$$= \frac{1}{2}\left[\frac{-\cos 8x}{8} - \frac{\cos 2x}{2}\right] + c$$

$$\boxed{\begin{aligned} &2SC = S + S \\ &SC = \frac{1}{2}[S+S] \end{aligned}}$$

Recall from Maths-I

$$\boxed{\begin{aligned} &S + S = 2SC \\ &S - S = 2CS \\ &C + C = 2CC \\ &C - C = -2SS \end{aligned}}$$

10. Evaluate: $\int \sin 7x \sin 4x \, dx$

Solution:

$$\int \sin 7x \sin 4x \, dx$$

$$= \frac{-1}{2}\int [\cos(7x+4x) - \cos(7x-4x)] \, dx$$

$$= -\frac{1}{2}\int (\cos 11x - \cos 3x) \, dx$$

$$= -\frac{1}{2}\left[\frac{\sin 11x}{11} - \frac{\sin 3x}{3}\right] + c$$

$$\boxed{\begin{aligned} &-2SS = C - C \\ &SS = -\frac{1}{2}[C-C] \end{aligned}}$$

11. Evaluate: $\int \sin^2 3x \, dx$

Solution:

$$\int \sin^2 3x \, dx$$

$$= \frac{1}{2}\int (1 - \cos 6x) \, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right] + c$$

Refer Problem 5

$$\boxed{\begin{aligned} \sin^2 x &= \frac{1}{2}[1 - \cos 2x] \\ \sin^2 3x &= \frac{1}{2}[1 - \cos 2(3x)] \\ &= \frac{1}{2}[1 - \cos 6x] \end{aligned}}$$

12. Evaluate: $\int \cos^3 5x \, dx$

Solution:

$$\int \cos^3 5x \, dx$$

$$= \frac{1}{4}\int (3\cos 5x + \cos 15x) \, dx$$

$$= \frac{1}{4}\left[\frac{3\sin 5x}{5} + \frac{\sin 15x}{15}\right] + c$$

Refer Problem 8

$$\boxed{\begin{aligned} \cos^3 x &= \frac{1}{4}(3\cos x + \cos 3(5x)) \\ \cos^3 5x &= \frac{1}{4}(3\cos 5x + \cos 3x) \\ &= \frac{1}{4}(3\cos 5x + \cos 15x) \end{aligned}}$$

4.2 INTEGRATION BY SUBSTITUTION

So far we have dealt with functions, either directly integrable using integration formula (or) integrable after decomposing the given functions into sums & differences.

But there are functions like $\frac{\sin(\log x)}{x}$, $\frac{2x+3}{x^2+3x+5}$ which cannot be decomposed into sums (or) differences of simple functions.

In these cases, using proper substitution, we shall reduce the given form into standard form, which can be integrated using basic integration formula.

When the integrand (the function to be integrated) is either in multiplication or in division form and if the derivative of one full or meaningful part of the function is equal to the other function then the integration can be evaluated using substitution method as given in the following examples.

1. $\int \frac{2x+3}{x^2+3x+5}$ since $\frac{d}{dx}$ $(x^2 + 3x + 5)$ is $2x + 3$ it can be integrated by taking $y = x^2 + 3x + 5$.

2. $\int \frac{\sin(\log x)}{x} dx = \int \sin(\log x) \frac{1}{x} dx$

Here $\frac{d}{dx}(\log x) = \frac{1}{x}$

The above integration can be evaluated by taking $y = \log x$.

Integrals of some standard forms:

Integrals of the form $\int [f(x)]^n f'(x)dx$, $\int \frac{f'(x)}{f(x)} dx$, $\int F\big(f(x)\big) f'(x)\, dx$ are all, more or less of the same type and the use of substitution $y = f(x)$ will reduce the given function to simple standard form which can be integrated using integration formulae.

4.2 WORKED EXAMPLES

PART – A

1. Evaluate: $I = \int \sin^3 x \boxed{\cos x\, dx}$.

Solution:

Put $y = \sin x$ ....(1)

$$\frac{dy}{dx} = \cos x$$

$$dy = \boxed{\cos x dx}$$

$$I = \int y^3 dy \quad \text{using (1)}$$

$$= \frac{y^4}{4} + c$$

$$= \frac{\sin^4 x}{x} + c$$

2. Evaluate: $I = \int \frac{\boxed{e^x - e^{-x}}}{e^x + e^{-x}} dx$

Solution:

$$y = e^x + e^{-x} \quad \text{.............(1)}$$

$$\frac{dy}{dx} = e^x + e^{-x}(-1)$$

$$= e^x - e^{-x}$$

$$dy = \boxed{(e^x - e^{-x})\, dx}$$

$$\therefore I = \int \frac{dy}{y} \text{ using (1)}$$

$$= \log y + c$$

$$= \log\ (e^x + e^{-x}) + c$$

3. Evaluate: $\int \tan x\ dx$

Solution:

$$I = \int \frac{\boxed{\sin x}}{\cos x} dx$$

$$\text{Put } y = \cos x \quad \text{............(1)}$$

$$\frac{dy}{dx} = -\sin x$$

$$dy = -\sin x\ dx$$

$$-dy = \boxed{\sin x\ dx}$$

$$I = \int \frac{-dy}{y} = -\log y = -\log\ (\cos x) + c$$

$$= \log(\cos x)^{-1} = \log\left(\frac{1}{\cos x}\right) = \log \sec x + c$$

Note:

$$\frac{d}{dx}\log(\sec x)$$

$$= \frac{1}{\sec x} . \sec x\ \tan x$$

$$= \tan x$$

$$\therefore \int \tan x\, dx = \log(\sec x) + c$$

4. Evaluate: $\int \cot x\ dx$

Solution:

$$I = \int \cot x\ dx$$

$$I = \int \frac{\boxed{\cos x}}{\sin x} dx$$

$$\text{Put } y = \sin x \quad \text{........(1)}$$

$$\frac{dy}{dx} = \cos x$$

$$dy = \boxed{\cos x\ dx}$$

$$\therefore I = \int \frac{dy}{y} = \log y = \log(\sin x) + c$$

Note:

$$\frac{d}{dx}\log(\sin x)$$

$$= \frac{1}{\sin x} . \cos x$$

$$= \cot x$$

$$\therefore \int \cot x\ dx = \log(\sin x) + c$$

5. Evaluate: $\int e^{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}}dx$

Solution:

$$I=\int e^{\sin^{-1}x}\boxed{\frac{1}{\sqrt{1-x^2}}dx}$$

Put $y=\sin^{-1}x$ ....(1)

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

$$dy=\boxed{\frac{1}{\sqrt{1-x^2}}dx}$$

$$\therefore I=\int e^y dy=e^y+c$$

$$=e^{\sin^{-1}x}+c$$

6. Evaluate: $\int \frac{(\log x)^5}{x}dx$

Solution:

$$I=\int (\log x)^5\boxed{\frac{1}{x}dx}$$

Put $y=\log x$ ....(1)

$$\frac{dy}{dx}=\frac{1}{x}$$

$$dy=\boxed{\frac{1}{x}dx}$$

$$I=\int y^5 dy$$

$$=\frac{y^6}{6}+c$$

$$=\frac{(\log x)^6}{6}+c$$

7. Evaluate: $\int \frac{\cos x}{2+3\sin x}dx$

Solution:

$$I=\int \frac{\boxed{\cos x}}{2+3\sin x}dx$$

$$y=2+3\sin x$$

$$\frac{dy}{dx}=3\cos x$$

$$dy=3\cos x dx$$

$$\frac{1}{3}dy=\boxed{\cos x dx}$$

$$\therefore I = \int \frac{1}{3}\frac{dy}{y}$$

$$= \frac{1}{3}\log y + c$$

$$= \frac{1}{y}\log(2 + 3\sin x) + c$$

8. Evaluate: $\int \sec x \, dx$

Solution:

$$I = \int \frac{\sec x(\sec x + \tan x)\, dx}{(\sec x + \tan x)}$$

Multiply and Divide by $\sec x + \tan x$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{(\sec x + \tan x)} dx$$

$$y = \sec x + \tan x$$

$$\frac{dy}{dx} = \sec x \tan x + \sec^2 x$$

$$dy = (\sec x \tan x + \sec^2 x)\, dx$$

$$\therefore I = \int \frac{dy}{y}$$

$$= \log y + c$$

$$= \log(\sec x + \tan x) + c$$

9. Evaluate: $\int \operatorname{cosec} x \, dx$

Solution:

$$I = \int \operatorname{cosec} x \, dx$$

$$= \int \frac{\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)} dx$$

$$= \int \frac{\boxed{(\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x)}}{(\operatorname{cosec} x + \cot x)} dx$$

$$y = \operatorname{cosec} x + \cot x$$

$$\frac{dy}{dx} = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$$

$$dy = -(\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x)\, dx$$

$$-dy = \boxed{(\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x)\, dx}$$

$$\therefore I = \int -\frac{dy}{y}$$

$$= -\log(\operatorname{cosec} x + \cot x) + c$$

10. Evaluate: $\int (x^2 - 5)^4 x\ dx$

Solution:

$$I = \int (x^2 - 5)^4 \boxed{x\ dx}$$

$$y = x^2 - 5$$

$$\frac{dy}{dx} = 2x$$

$$dy = 2x\ dx$$

$$\frac{1}{2} dy = \boxed{x\ dx}$$

$$\therefore I = \int y^4 \frac{1}{2} dy$$

$$= \frac{1}{2}\frac{y^5}{5} + c$$

$$= \frac{1}{10} y^5 + c$$

$$= \frac{1}{10}(x^2 - 5)^5 + c$$

PART B & C

1. Evaluate: $\int (2 + \sin x)^3 \boxed{\cos x\ dx}$

Solution:

$$y = 2 + \sin x$$

$$\frac{dy}{dx} = \cos x$$

$$dy = \boxed{\cos x\ dx}$$

$$\therefore I = \int y^3 dy$$

$$= \frac{y^4}{4} + c = \frac{(2 + \sin x)^4}{4} + c$$

2. Evaluate: $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

Solution:

$$I = \int \cos\sqrt{x} \boxed{\frac{1}{\sqrt{x}} dx}$$

$$y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$2dy = \boxed{\frac{1}{\sqrt{x}} dx}$$

$$\therefore I = \int \cos y 2dy$$
$$= 2\sin y + c$$
$$= 2\sin(\sqrt{x}) + c$$

3. Evaluate: $\int \frac{\sec^2 x}{(2+3\tan x)^3} dx$

Solution:

$$I = \int \frac{\boxed{\sec^2 x\ dx}}{(2+3\tan x)^3}$$
$$y = 2 + 3\tan x$$
$$\frac{dy}{dx} = 3\sec^2 x$$
$$dy = 3\sec^2 x\, dx$$
$$\frac{1}{3}dy = \boxed{\sec^2 x\ dx}$$

$$\therefore I = \frac{1}{3}\int \frac{dy}{y^3}$$
$$= \frac{1}{3}\int y^{-3} dy$$
$$= \frac{1}{3}\frac{y^{-3+1}}{-3+1} + c$$
$$= \frac{1}{3} \cdot \frac{y^{-2}}{-2} = \frac{1}{-6} y^{-2} + c$$
$$= -\frac{1}{6}(2+3\tan x)^{-2} + c$$

4. Evaluate: $\int x^2 \cos(x^3)\ dx$

Solution:

$$I = \int x^2 \cos(x^3)\ dx = \int \cos(x^3)\ \boxed{x^2 dx}$$
$$y = x^3$$
$$\frac{dy}{dx} = 3x^2$$
$$dy = 3x^2 dx$$
$$\frac{1}{3}dy = \boxed{x^2 dx}$$

$$\therefore I = \int \cos y \frac{1}{3} dy$$
$$= \frac{1}{3}\sin y$$
$$= \frac{1}{3}\sin(x^3) + c$$

5. Evaluate: $\int e^{\sin^2 x} \sin 2x \; dx$

Solution:

$$I = \int e^{\sin^2 x} \sin 2x \; dx \qquad \boxed{\therefore \sin 2A = 2\sin A \cos A}$$

$$= \int e^{\sin^2 x} \boxed{2\sin x \cos x \; dx}$$

$$y = \sin^2 x$$

$$\frac{dx}{dy} = 2\sin x \cos x$$

$$dy = \boxed{2\sin x \cos x \; dx}$$

$$\therefore I = \int e^y dy$$

$$= e^y + c$$

$$= e^{\sin^2 x} + c$$

6. Evaluate: $\int (2x^2 - 8x + 5)^{11} (x - 2) \; dx$

Solution:

$$I = \int (2x^2 - 8x + 5)^{11} \; \boxed{(x - 2) \; dx}$$

$$y = 2x^2 - 8x + 5$$

$$\frac{dy}{dx} = 4x - 8$$

$$\frac{dy}{dx} = 4(x - 2)$$

$$dy = 4(x - 2) dx$$

$$\frac{1}{4} dy = \boxed{(x - 2) \; dx}$$

$$\therefore I = \int y^{11} \frac{1}{4} dy$$

$$= \frac{1}{4} \frac{y^{12}}{12} + C$$

$$= \frac{1}{48} y^{12} + C$$

$$= \frac{1}{48} (2x^2 - 8x + 5)^{12} + C$$

7. Evaluate: $\int \frac{(\sin^{-1} x)^4 dx}{\sqrt{1 - x^2}}$

Solution:

$$I = \int (\sin^{-1} x)^4 \boxed{\frac{1}{\sqrt{1 - x^2}} dx}$$

$$y = \sin^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$$dy = \boxed{\frac{1}{\sqrt{1 - x^2}} dx}$$

$$I = \int y^4 dy$$

$$= \frac{y^5}{5} + C$$

$$= \frac{(\sin^{-1} x)^5}{5} + C$$

8. Evaluate: $\int \sqrt{\tan x} \sec^2 x \, dx$

Solution:

$$\int \sqrt{\tan x} \; \boxed{\sec^2 x \, dx}$$

$$y = \tan x$$

$$\frac{dy}{dx} = \sec^2 x$$

$$dy = \boxed{\sec^2 x \, dx}$$

$$\therefore I = \int \sqrt{y} dy$$

$$= \int y^{1/2} dy$$

$$= \frac{y^{1/2 + 1}}{\frac{1}{2} + 1} = \frac{y^{3/2}}{\frac{3}{2}} = \frac{2}{3} y^{3/2}$$

$$= \frac{2}{3} (\tan x)^{3/2} + C$$

9. Evaluate: $\int \frac{1}{x \log x} dx$

Solution:

$$I = \int \frac{1}{\log x} \boxed{\frac{1}{x} dx}$$

$$y = \log x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$dy = \boxed{\frac{1}{x} dx}$$

$$I = \int \frac{1}{y} dy$$

$$= \log y + C$$

$$= \log (\log x) + C$$

10. Evaluate: $\int \frac{\cot x}{\log(\sin x)} dx$

Solution:

$$I = \int \frac{1}{\log(\sin x)} \ . \ \boxed{\cot x \ dx}$$

$$y = \log(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{\sin x}\cos x$$

$$\frac{dy}{dx} = \cot x$$

$$dy = \boxed{\cot x \ dx}$$

$$\therefore I = \int \frac{1}{y} dy$$

$$= \log y + C$$

$$= \log(\log(\sin x)) + C$$

4.3 STANDARD INTEGRALS

Integrals of the form $\int \frac{dx}{a^2 \pm x^2}$, $\int \frac{dx}{x^2 - a^2}$ and $\int \frac{dx}{\sqrt{a^2 - x^2}}$.

1. Evaluate: $\int \frac{1}{a^2 + x^2} dx$

$$I = \int \frac{1}{a^2 + x^2} \boxed{dx}$$

Put $x = \tan\theta$

$$\frac{dx}{d\theta} = a\sec^2\theta$$

$$dx = \boxed{a \sec^2\theta d\theta}$$

$$a^2 + x^2$$
$$= a^2 + (a\tan\theta)^2$$
$$= a^2 + a^2\tan^2\theta$$
$$= a^2(1 + \tan^2\theta)$$
$$= a^2\sec^2\theta$$

$$\therefore I = \int \frac{1}{a^2\sec^2\theta} \ . \ a\sec^2\theta \ d\theta$$

$$= \int \frac{1}{a} d\theta$$

$$= \frac{1}{a}\theta$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\boxed{\begin{array}{l} x = a\tan\theta \\ \Rightarrow \tan\theta = \dfrac{x}{a} \\ \Rightarrow \theta = \tan^{-1}\left(\dfrac{x}{a}\right) \end{array}}$$

2. Evaluate: $\int \frac{dx}{a^2 - x^2}$

$$I = \int \frac{1 \, . \, dx}{(a+x)\,(a-x)}$$

$$\boxed{\begin{aligned}&(a+x)+(a-x)=2a\\ &\therefore \frac{1}{2a}[(a+x)+(a-x)]=1\end{aligned}}$$

$$= \int \frac{\frac{1}{2a}[(a+x)+(a-x)]}{(a+x)\,(a-x)} dx$$

$$= \frac{1}{2a}\int \frac{a+x+a-x}{(a+x)\,(a-x)}\, dx$$

Dividing Separately

$$= \frac{1}{2a}\int \frac{\cancel{(a+x)}}{\cancel{(a+x)}\,(a-x)} dx + \frac{1}{2a}\int \frac{\cancel{(a-x)}}{(a+x)\,\cancel{(a-x)}} dx$$

$$= \frac{1}{2a}\int \frac{dx}{a-x} + \frac{1}{2a}\int \frac{dx}{a+x}$$

$$\boxed{\text{Note}: \frac{d}{dx}(a-x) = -1 \;\; \text{and} \;\; \frac{d}{dx}(x+a) = 1}$$

$$I = \frac{-1}{2a}\int \frac{(-1)\, dx}{(a-x)} + \frac{1}{2a}\int \frac{1\, dx}{a+x}$$

$$= \frac{-1}{2a}\log(a-x) + \frac{1}{2a}\log(a+x)$$

$$= \frac{1}{2a}\log(a+x) - \frac{1}{2a}\log(a-x)$$

$$= \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$$

3. Evaluate: $\int \frac{dx}{x^2 - a^2}$

$$I = \int \frac{dx}{x^2 - a^2}$$

$$= \int \frac{dx}{(x+a)\,(x-a)}$$

$$\begin{aligned}&(x+a) - (x-a)\\ &= x+a-x+a\\ &= 2a\end{aligned}$$

$$= \int \frac{1}{2a}\frac{(x+a)-(x-a)}{(x+a)\,(x-a)} dx$$

$$= \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x+a)\,(x-a)} dx$$

Dividing Separately,

$$= \frac{1}{2a}\int \frac{\cancel{(x+a)}\; dx}{\cancel{(x+a)}\,(x-a)} - \frac{1}{2a}\int \frac{\cancel{(x-a)} dx}{(x+a)\,\cancel{(x-a)}}$$

$$= \frac{1}{2a}\int \frac{1}{x-a} dx - \frac{1}{2a}\int \frac{1}{x+a} dx$$

$$= \frac{1}{2a}\log(x-a) - \frac{1}{2a}\log(x+a)$$

$$= \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$$

4. Evaluate: $\int \frac{1}{\sqrt{a^2-x^2}}dx$

$$I = \int \frac{1}{\sqrt{a^2-x^2}}\boxed{dx}$$

Put $x = a\sin\theta$	$x = a\sin\theta$
$a^2-x^2 = a^2-(a\sin\theta)^2$	$\frac{dx}{d\theta} = a\cos\theta$
$= a^2 - a^2\sin^2\theta$	$dx = \boxed{a\cos\theta d\theta}$
$= a^2(1-\sin^2\theta)$	
$= a^2\cos^2\theta$	

$$\sqrt{a^2-x^2} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$

$$\therefore I = \int \frac{1}{a\cos\theta}\,.\,a\cos\theta\; d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\frac{x}{a} + c$$

$$\boxed{\begin{array}{l} x = a\sin\theta \\ \Rightarrow \sin\theta = \frac{x}{a} \\ \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right) \end{array}}$$

List of Standard Integrals Formulae:

Sl.No.	Integration	Result
1.	$\int \frac{dx}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}+c$
2.	$\int \frac{dx}{a^2-x^2}$	$\frac{1}{2a}\log\frac{a+x}{a-x}+c$
3.	$\int \frac{dx}{x^2-a^2}$	$\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c$
4.	$\int \frac{dx}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+c$

4.3 WORKED EXAMPLES

PART – A

1. Evaluate: $\int \frac{dx}{9+x^2}$

Solution:

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$\boxed{a^2 = 9 \Rightarrow a = 3}$$

$$\therefore \int \frac{dx}{9+x^2} = \int \frac{dx}{3^2+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$$

2. Evaluate: $\int \frac{dx}{25-x^2}$

Solution:

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$$

$$\boxed{\text{If } a=5, \text{ then } a^2=25}$$

$$\therefore \int \frac{dx}{25-x^2} = \int \frac{dx}{5^2-x^2}$$

$$= \frac{1}{2\times 5}\log\left(\frac{5+x}{5-x}\right)+c$$

$$= \frac{1}{10}\log\left(\frac{5+x}{5-x}\right)+c$$

3. Evaluate: $\int \frac{dx}{x^2-16}$

Solution:

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

$$\boxed{a^2=16 \Rightarrow a=4}$$

$$\therefore \int \frac{dx}{x^2-4^2} = \frac{1}{2\times 4}\log\left(\frac{x-4}{x+4}\right)+c$$

$$= \frac{1}{8}\log\left(\frac{x-4}{x+4}\right)+c$$

4. Evaluate: $\int \frac{dx}{\sqrt{36-x^2}}$

Solution:

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c$$

$$\boxed{a^2=36 \Rightarrow a=6}$$

$$\therefore \int \frac{dx}{\sqrt{6^2-x^2}} = \sin^{-1}\left(\frac{x}{6}\right)+c$$

PART – B & C

1. Evaluate: $\int \frac{dx}{(2x+3)^2+9}$

Solution:

$$I = \int \frac{\boxed{dx}}{(2x+3)^2+9}$$

$$y = 2x+3$$

$$\frac{dy}{dx} = 2$$

$$dy = 2dx$$

$$\frac{1}{2}dy = \boxed{dx}$$

$$\therefore I = \frac{1}{2}\int \frac{dy}{y^2+9} \qquad \boxed{a^2 = 9 \Rightarrow a = 3}$$

$$= \frac{1}{2} \times \frac{1}{a}\tan^{-1}\left(\frac{y}{a}\right) + c$$

$$= \frac{1}{2} \times \frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x+3}{3}\right) + c$$

2. Evaluate: $\int \frac{\boxed{dx}}{(3x-4)^2 - 25}$

Solution:

Put y = 3x – 4

$$\frac{dy}{dx} = 3$$

$$dy = 3dx$$

$$\frac{1}{3}dy = \boxed{dx}$$

$$= \frac{1}{3}\int \frac{dy}{y^2 - 5^2} \qquad a^2 = 25 \Rightarrow a = 5$$

$$= \frac{1}{3} \times \frac{1}{2a}\log\left[\frac{y-5}{y+5}\right]$$

$$= \frac{1}{3} \times \frac{1}{2 \times 5}\log\left[\frac{(3x-4)-5}{3x-4+5}\right] + c \qquad =$$

$$= \frac{1}{30}\log\left[\frac{3x-9}{3x+1}\right] + c$$

3. Evaluate: $\int \frac{dx}{49 - 4x^2}$

Solution:

$$I = \int \frac{\boxed{dx}}{49 - (2x^2)} \qquad \boxed{(2x)^2 = 4x^2}$$

$$y = 2x$$

$$\frac{dy}{dx} = 2$$

$$dy = 2dx$$

$$\frac{1}{2}dy = \boxed{dx}$$

$$\therefore I = \frac{1}{2}\int \frac{dy}{7^2 - y^2} \qquad \boxed{a^2 = 49 \Rightarrow a = 7}$$

$$= \frac{1}{2} \times \frac{1}{2a} \log\left[\frac{7+y}{7-y}\right]$$

$$= \frac{1}{2} \times \frac{1}{2\times 7} \log\left[\frac{7+2x}{7-2x}\right] + c$$

$$= \frac{1}{28} \log\left[\frac{7+2x}{7-2x}\right] + c$$

4. Evaluate: $\int \frac{dx}{\sqrt{121-(3x+5)^2}}$

Solution:

$$I = \int \frac{\boxed{dx}}{\sqrt{121-(3x+5)^2}}$$

$$y = 3x+5$$

$$\frac{dy}{dx} = 3$$

$$dy = 3dx$$

$$\frac{1}{3}dy = \boxed{dx}$$

$$\therefore I = \frac{1}{3}\int \frac{dy}{\sqrt{11^2 - y^2}} \qquad \boxed{a^2 = 121 \Rightarrow a = 11}$$

$$= \frac{1}{3} \times \sin^{-1}\left(\frac{y}{a}\right)$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{y}{11}\right) = \frac{1}{3}\sin^{-1}\left(\frac{3x+5}{11}\right) + c$$

5. Evaluate: $\int \frac{dx}{\sqrt{144-5x^2}}$

Solution:

$$I = \int \frac{dx}{\sqrt{144-5x^2}}$$

$$= \int \frac{\boxed{dx}}{\sqrt{12^2 - \left(\sqrt{5}x\right)^2}} \qquad \boxed{\left(\sqrt{5}x\right)^2 = 5x^2}$$

Put $y = \sqrt{5}x$

$$\frac{dy}{dx} = \sqrt{5}$$

$$dy = \sqrt{5}dx$$

$$\frac{1}{\sqrt{5}}dy = \boxed{dx}$$

$$I = \frac{1}{\sqrt{5}} \int \frac{dy}{\sqrt{12^2 - y^2}}$$

$$= \frac{1}{\sqrt{5}} \sin^{-1}\left(\frac{y}{a}\right) + c$$

$$= \frac{1}{\sqrt{5}} \sin^{-1}\left(\frac{\sqrt{5}x}{12}\right) + c$$

EXERCISES

PART – A

1. Evaluate the following:

(i) $\int \left(2x^3 - 4\sqrt{x}\right) dx$ t(ii) $\int \left(x^3 - \frac{7}{x^2} + \frac{1}{x}\right) dx$ (iii) $\int \left(\frac{7}{\cos^2 x} - \frac{3}{\sin^2 x}\right) dx$

(iv) $\int \frac{(e^{4x} + e^{-4x})}{2} dx$ (v) $\int \sec 5x \tan 5x \, dx$ (vi) $\int e^{8x-5} dx$

2. Evaluate the following:

(i) $\int (2 - 5x)^4 dx$ (ii) $\int \frac{dx}{\sqrt{4 - x^2}}$ (iii) $\int \frac{dx}{4 + x^2}$

(iv) $\int \sqrt{\frac{1 + \cos 2x}{2}} dx$ (v) $\int \sqrt{3 + 4x} \, dx$ (vi) $\int \frac{dx}{\sqrt{1 - x}}$

3. Evaluate the following:

(i) $\int \cos^5 x \sin x \, dx$ (ii) $\int \frac{e^{5x}}{1 + e^{5x}}$ (iii) $\int e^{\tan^{-1} x} \frac{1}{1 + x^2} dx$

(iv) $\int \frac{\log x}{x} dx$ (v) $\int \frac{\sin x}{3\cos x + 4} dx$ (vi) $\int (x^2 + 3)^4 x \, dx$

4. Evaluate the following:

(i) $\int \frac{dx}{49 + x^2}$ (ii) $\int \frac{dx}{4 - x^2}$ (iii) $\int \frac{dx}{x^2 - 36}$ (iv) $\int \frac{dx}{\sqrt{64 - x^2}}$

PART B and C

1. Evaluate the following:

(i) $\int (3x - 4)(2x + 5) \, dx$ (ii) $\int \left(x - \frac{1}{x}\right)\left(3 + \frac{5}{x^2}\right) dx$ (iii) $\int \frac{(x^5 - x^3 + x^2)}{x^3} dx$

(iv) $\int \frac{\sin^2 x \, dx}{1 - \cos x}$ (v) $\int \frac{1}{1 + \cos x} dx$ (vi) $\frac{1}{1 - \sin x} dx$

2. Evaluate the following:

(i) $\int \sin^2 4x \, dx$ (ii) $\int \cos^2 3x \, dx$ (iii) $\int \sin^3 6x \, dx$

(iv) $\int \cos^3 2x \, dx$ (v) $\int \cos 11x \sin 7x \, dx$ (vi) $\cos 6x \cos 2x \, dx$

3. Evaluate the following:

(i) $\int (3-\cos x)^5 \sin x \, dx$ (ii) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$ (iii) $\int \frac{\operatorname{cosec}^2 x \, dx}{4+5\cot x}$

(iv) $\int x^2 \sec^2(x^3)\, dx$ (v) $\int e^{\cos^2 x} \sin 2x \, dx$ (vi) $(x^2-6x+5)^7\,(x-3)\, dx$

4. Evaluate the following:

(i) $\int \frac{(\tan^{-1} x)^3}{1+x^2} dx$ (ii) $\int \sqrt{\sin x}\, \cos x \, dx$ (iii) $\int \frac{1}{x(\log x)^2} dx$

(iv) $\int \frac{\tan x \, dx}{\log(\sec x)}$ (v) $\int e^{2\sin^{-1} x} \frac{1}{\sqrt{1-x^2}} dx$ (vi) $\int \frac{(2x+3)}{\sqrt{x^2+3x-4}} dx$

5. Evaluate the following:

(i) $\int \frac{dx}{4+(3x+1)^2}$ (ii) $\int \frac{dx}{25+4x^2}$ (iii) $\int \frac{dx}{4-(7x-3)^2}$

(iv) $\int \frac{dx}{(5x+2)^2-4}$ (v) $\int \frac{dx}{\sqrt{25-(x+1)^2}}$ (vi) $\frac{dx}{\sqrt{169-4x^2}}$

ANSWERS

PART – A

1. (i) $\frac{2x^4}{4} - \frac{8}{3} x^{3/2} + c$ (ii) $\frac{x^4}{4} + \frac{7}{x} + \log x + c$ (iii) $7\tan x + 3\cot x + c$

(iv) $\frac{1}{2}\left[\frac{e^{4x}}{4} + \frac{e^{-4x}}{-4}\right] + c$ (v) $\frac{\sec 5x}{5} + c$ (vi) $\frac{1}{8} e^{8x-5} + c$

2. (i) $\frac{-1}{25}(2-5x)^5 + c$ (ii) $\sin^{-1}\left(\frac{x}{2}\right) + c$ (iii) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

(iv) $\sin x + c$ (v) $\frac{1}{6}(3+4x)^{3/2} + c$ (vi) $-2\sqrt{1-x} + c$

3. (i) $-\frac{\cos^6 x}{6} + c$ (ii) $\frac{1}{5}\log(1+e^{5x}) + c$ (iii) $e^{\tan^{-1} x} + c$

(iv) $\frac{1}{2}(\log x)^2 + c$ (v) $-\frac{1}{3}\log(3\cos x + 4) + c$ (vi) $\frac{1}{10}(x^2+3)^5 + c$

4. (i) $\frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right) + c$ (ii) $\frac{1}{4}\log\left(\frac{2+x}{2-x}\right) + c$

(iii) $\frac{1}{12}\log\left(\frac{x-6}{x+6}\right) + c$ (iv) $\sin^{-1}\left(\frac{x}{8}\right) + c$

PART – B & C

1. (i) $2x^3 + \frac{7x^2}{2} - 20x + c$ (ii) $\frac{3x^2}{2} + 2\log x + \frac{5}{2}x^{-2} + c$ (iii) $\frac{x^3}{3} - x + \log x + c$

(iv) $x + \sin x + c$ (v) $-\cot x + \operatorname{cosec} x + c$ (vi) $\tan x + \sec x + c$

2. (i) $\frac{1}{2}\left[x-\frac{\sin 8x}{8}\right]+c$ (ii) $\frac{1}{2}\left[x+\frac{\sin 6x}{6}\right]+c$ (iii) $\frac{1}{4}\left[\frac{-3\cos 6x}{6}+\frac{\cos 18x}{18}\right]+c$

(iv) $\frac{1}{4}\left[\frac{3\sin 2x}{2}+\frac{\sin 6x}{6}\right]+c$ (v) $\frac{1}{2}\left[\frac{-\cos 18x}{18}+\frac{\cos 4x}{4}\right]+c$ (vi) $\frac{1}{2}\left[\frac{\sin 8x}{8}+\frac{\sin 4x}{4}\right]+c$

3. (i) $-\frac{1}{6}(3-\cos x)^6+c$ (ii) $-2\cos\sqrt{x}+c$ (iii) $-\frac{1}{5}\log(4+5\cot x)+c$

(iv) $\frac{1}{3}\tan(x^3)+c$ (v) $-e^{\cos^2 x}+c$ (vi) $\frac{1}{16}(x^2-6x+5)^8+c$

4. (i) $\frac{1}{4}(\tan^{-1} x)^4+c$ (ii) $\frac{2}{3}(\sin x)^{3/2}+c$ (iii) $\frac{(\log x)^{-1}}{-1}$ (or) $\frac{-1}{\log x}+c$

(iv) Hint = y = log (sec x) $\log[\log(\sec x)]+c$ (v) $\frac{1}{2}e^{2\sin^{-1} x}+c$ (vi) $2(x^2+3x-4)^{3/2}+c$

5. (i) $\frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+c$ (ii) $\frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right)+c$ (iii) $\frac{1}{28}\log\left[\frac{2+(7x-3)}{2-(7x+3)}\right]+c$

(iv) $\frac{1}{20}\log\left[\frac{(5x+2)-2}{(5x+2)+2}\right]+c$ (v) $\sin^{-1}\left(\frac{x+1}{5}\right)+c$ (vi) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{13}\right)+c$

UNIT – V

INTEGRAL CALCULUS-II

5.1 INTEGRATION BY PARTS

Integrals of the form ∫x sin nx dx, ∫ x cos nx dx, ∫ x e^{nx} dx, ∫x^n log x dx and ∫ log x dx. Simple Problems.

5.2 BERNOULLI'S FORMULA

Evaluation of the integrals ∫x^m sin nx dx, ∫x^m, cos nx dx and ∫x^m enx dx where m ≤ 2 using Bernoulli's formula. Simple Problems.

5.3 DEFINITE INTEGRALS

Definition of definite Integral, Properties of definite Integrals – Simple Problems.

5.1 INTEGRATION BY PARTS

Introduction:

When the integrand is a product of two functions and the method of decomposition or substitution cannot be applied, then the method of by parts is used. In differentiation we have seen.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

i.e d(uv) = udv + vdu

Integrating both sides:

$$\int d(uv) = \int udv + \int vdu$$

$$uv = \int udv + \int vdu$$

$$\text{i.e} \int udv = uv - \int vdu$$

$\therefore \int udv = uv - \int vdu$ is called Integraiton by parts formula.

The above formula is used by taking proper choice of 'u' and 'dv'. 'u' should be chosen based on the following order of Preference.

1. Inverse trigonometric functions
2. Logarithmic functions
3. Algebraic functions
4. Trigonometric functions
5. Exponential functions

Simply remember ILATE.

WORKED EXAMPLES

PART – A

1. Evaluate: $\int x \cos x \, dx$

Solution:

$$\int u dv = uv - \int v du$$

choosing u = x and dv = cosx dx

$$du = dx \qquad \int dv = \int \cos x dx$$

$$v = \sin x$$

$$\therefore \int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + c$$

2. Evaluate: $\int \log x dx$

Solution:

Choosing u = log x and dv = dx

$$\int u dv = uv - \int v du$$

$$du = \frac{1}{x} dx \qquad \int dv = \int dx$$

$$v = x$$

$$\therefore \int \log x \, dx = \log x . \, x - \int x . \frac{1}{x} dx$$

$$= x \log x - \int dx$$

$$= x \log x - x + c$$

PART – B & C

1. Evaluate: $\int xe^{-x} dx$

Solution:

$$u = x \qquad dv = e^{-x} dx$$

$$\frac{du}{dx} = 1 \qquad v = -e^{-x}$$

$$du = dx$$

$$\int u dv = uv - \int v du$$

$$\therefore \int xe^{-x} dx = (x)\,(-e^{-x}) - \int -e^{-x} dx$$

$$= -xe^{-x} + \int e^{-x} dx$$

$$= -xe^{-x} - e^{-x} + c$$

$$= -e^{-x}(x+1) + c$$

2. Evaluate: $\int x \sin 2x \; dx$

Solution:

$$u = x \qquad dv = \sin 2x \; dx$$

$$\frac{du}{dx} = 1 \qquad v = \frac{-\cos 2x}{2}$$

$$du = dx$$

$$\int udv = uv - \int vdu$$

$$\int x \sin 2x = (x)\left(\frac{-\cos 2x}{2}\right) - \int \frac{-\cos 2x}{2} dx$$

$$= \frac{-x \cos 2x}{2} + \frac{1}{2}\int \cos 2x \; dx$$

$$= \frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} + c$$

3. Evaluate: $\int x \; \log x \; dx$

Solution:

Choosing $u = \log x$ and $dv = xdx$

$$du = \frac{1}{x}dx \qquad v = \frac{x^2}{2}$$

$$\therefore \int x \; \log x \; dx = \log x\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \times \frac{1}{x} dx$$

$$= \frac{x^2 \log x}{2} - \frac{1}{2}\int x \; dx$$

$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + c$$

5.2 BERNOULLI'S FORM OF INTEGRATION BY PARTS

If u and v are functions x, then Bernoulli's form of integration by parts formula is

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \text{...........}$$

Where u´, u´´,u´´´..... are successive differentiation of the function u and v, v_1, v_2, v_3, ... the successive integration of the function dv.

Note:

The function 'u' is differentiated upto constant.

PART B & C

Example:

1. Evaluate: $\int x^2 e^{2x} dx$

Solution:

Choosing $u = x^2$ and $dv = e^{2x} dx$

$$u' = 2x \qquad v = e^{2x}/2$$

$$u'' = 2$$

$$v_1 = \frac{e^{2x}}{4}$$

$$v_2 = \frac{e^{2x}}{8}$$

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \text{.........}$$

$$\int x^2 e^{2x} dx = x^2 \frac{e^{2x}}{2} - \frac{2xe^{2x}}{4} + \frac{2e^{2x}}{8} + c$$

2. Evaluate: $\int x^2 \sin 2x \; dx$

Solution:

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \text{.........}$$

$$u = x^2 \quad \text{and} \quad dv = \sin 2x \; dx$$

$$u' = 2x \qquad v = \frac{-\cos 2x}{2}$$

$$u'' = 2 \qquad v_1 = \frac{-\sin 2x}{4}$$

$$v_2 = \frac{\cos 2x}{8}$$

$$\therefore \int x^2 \sin 2x \; dx = (x^2)\left(\frac{-\cos 2xs}{2}\right) - (2x)\left(\frac{-\sin 2x}{4}\right) + (2)\left(\frac{\cos 2x}{8}\right) + c$$

$$= \frac{-x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$$

3. Evaluate: $\int x^2 \cos 3x \, dx$

Solution:

$$\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \text{.........}$$

$$u = x^2 \quad \text{and} \quad dv = \cos 3x \, dx$$

$$u' = 2x \qquad v = \frac{\sin 3x}{3}$$

$$u'' = 2 \qquad v_1 = \frac{-\cos 3x}{9}$$

$$v_2 = \frac{-\sin 2x}{27}$$

$$\therefore \int x^2 \cos 3x \, dx = (x^2)\left(\frac{\sin 3x}{3}\right) - (2x)\left(\frac{-\cos 3x}{9}\right) + (2)\left(\frac{-\sin 3x}{27}\right) + c$$

$$= \frac{x^2 \sin 3x}{3} + \frac{2x \cos 3x}{9} - \frac{2 \sin 3x}{27} + c$$

5.3 DEFINITE INTEGRALS

Definition of Definite Integrals:

Let $\int f(x)dx = F(x) + c$, where c is the arbitrary constant of integration. The value of the integral.

when x = b, is F(b) + c .... (1)

and when x = a, is F(a) + c .... (2)

Subtracting (2) from (1) we have

F(b) – F(a) = (the value of the integral when x = b) – (The value of the integral when x = a).

$$\text{i.e} \int_a^b f(x)dx = \left[F(x) + c\right]_a^b$$

$$= [F(b) + c] - F(a) + c$$

$$= F(b) + c - F(a) - c$$

$$= F(b) - F(a)$$

Thus

$\int_a^b f(x)dx$ is called the definite integral, here a and b are called the lower limit and upper limit of integral respectively.

Properties of Definite Integrals:

Certain properties of definite integral are useful in solving problems. Some of the often used properties are given below.

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

3. If $a < c < b$ in [a, b]

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

6. $\int_{-a}^{+a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if f(x) is even i.e f(– x) = f(x).

   = 0 u f(x) is odd   if f(x) is odd i.e f(–x) = – f(x)

7. $\int_0^{2a} f(x)\,dx = \int_0^{2a} f(2a-x)\,dx$

WORKED EXAMPLES

PART – A

1. Evaluate: $\int_1^2 \frac{1}{x}\,dx$

   Solution:

$$\text{Let } I = \int_1^2 \frac{1}{x}\,dx$$
$$= \left[\log x\right]_1^2$$
$$= \log 2 - \log 1 \qquad (\because \log 1 = 0)$$
$$I = \log 2$$

2. Evaluate: $\int_0^{\pi/2} \sin x\ dx$

   Solution:

$$\text{Let } I = \int_0^{\pi/2} \sin x\ dx$$
$$= \left[-\cos x\right]_0^{\pi/2}$$
$$= -\cos\frac{\pi}{2} + \cos 0$$
$$= 0 + 1$$
$$I = 1$$

3. Evaluate: $\int_0^{\pi/4} \sec^2 x \; dx$

Solution:

$$\text{Let } I = \int_0^{\pi/4} \sec^2 x \; dx$$

$$= [\tan x]_0^{\pi/4}$$

$$= \tan\frac{\pi}{4} - \tan 0$$

$$I = 1$$

4. Evaluate: $I = \int_1^2 (x + x^2)\, dx$

Solution:

$$\text{Let } I = \int_1^2 (x + x^2)\, dx$$

$$I = \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^2$$

$$= \left[\left(\frac{2^2}{2} + \frac{2^3}{3}\right) - \left(\frac{1^2}{2} + \frac{1^3}{3}\right)\right]$$

$$= \left(\frac{4}{2} + \frac{8}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= \left(\frac{12+16}{6}\right) - \left(\frac{3+2}{6}\right)$$

$$= \frac{28}{6} - \frac{5}{6}$$

$$= \frac{28-5}{6}$$

$$I = \frac{23}{6}$$

5. Evaluate: $\int_0^1 x^2 (1-x)^{10}\, dx$

Solution:

$$\text{By property : } \int_0^a f(x)\; dx = \int_0^a f(a-x)\; dx$$

$$\int_0^1 x^2 (1-x)^{10}\, dx = \int_0^1 (1-x)^2 [1-(1-x)]^{10}\, dx$$

$$= \int_0^1 (1 - 2x + x^2)\; x \,.\, dx$$

$$= \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) - 0$$

$$= \frac{6-8+3}{12} = \frac{1}{12}$$

PART - B & C

1. Evaluate: $\int_0^{\pi/2} \frac{\cos^2 x}{1+\sin x} dx$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \frac{\cos^2 x}{1+\sin x} dx$$

$$= \int_0^{\pi/2} \frac{1-\sin^2 x}{1+\sin x} dx$$

$$= \int_0^{\pi/2} \frac{(1-\sin x)(1+\sin x)}{1+\sin x} dx$$

$$= \int_0^{\pi/2} (1-\sin x)\ dx$$

$$= \left[x+\cos x\right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{2}+\cos\frac{\pi}{2}\right) - (0+\cos 0)$$

$$= \left(\frac{\pi}{2}+0\right) - (0+1)$$

$$I = \frac{\pi}{2} -$$

2. Evaluate: $\int_0^{\pi/2} \cos^2 x\ dx$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \cos^2 x\ dx$$

$$I = \int_0^{\pi/2} \cos^2 x\ dx \qquad \because \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\int_0^{\pi/2} (1+\cos 2x)\ dx$$

$$= \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right) - \left(0+\frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+0-0+0\right]$$

$$I = \frac{1}{2}\left[\frac{\pi}{2}\right]$$

$$I = \frac{\pi}{4}$$

3. Evaluate: $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

Solution:

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \text{ by property}$$

$$\therefore \text{Let } I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \qquad ......(1)$$

$$= \int_0^{\pi/2} \frac{\sin\left(\pi/2 - x\right)}{\sin\left(\pi/2 - x\right) + \cos\left(\pi/2 - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \qquad ......(2)$$

Adding (1) & (2)

$$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} dx$$

$$2I = \left[x\right]_0^{\pi/2}$$

$$2I = \frac{\pi}{2} - 0$$

$$I = \frac{\pi}{4}$$

4. Evaluate: $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx$

Solution:

The given Integrand $\sin^2 x \cos x$ is even.

$$\therefore \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx = 2\int_0^{\pi/2} \sin^2 x \cos x\, dx$$

$$= 2\left[\frac{\sin^3 x}{3}\right]_0^{\pi/2}$$

$$= \frac{2}{3}\left[\sin^3 \frac{\pi}{2} - \sin^3 0\right]$$

$$= \frac{2}{3}(1)$$

$$= \frac{2}{3}$$

$f(x) = \sin x$

$f'x = \cos x$

$n = 2$

using $\int [f(x)]^n f'(x)\,dx$

$$= \frac{[f(x)]^{n+1}}{n+1} + c$$

EXERCISE

PART – A

1. Evaluate: $\int xe^{x} dx$
2. Evaluate: $\int x\sin x \ dx$
3. Evaluate: $\int xe^{2x} dx$
4. Evaluate: $\int x\cos x \ dx$

PART – B

1. Evaluate: $\int x^{2} \sin x \ dx$
2. Evaluate: $\int x^{2} \cos x \ dx$
3. Evaluate: $\int x^{2} e^{x} \ dx$
4. Evaluate: $\int \log x \ dx$
5. Evaluate: $\int_{1}^{2} (x + x^{2}) \ dx$
6. Evaluate: $\int_{0}^{\pi/2} \cos x \ dx$
7. Evaluate: $\int_{0}^{\pi/2} \sin x \ dx$
8. $\int_{0}^{\pi} \cos x \ dx$

PART – C

Evaluate the following:

1) $\int x^{3} e^{x} dx$ 2) $\int x^{3} \sin x dx$

3) $\int x^{3} \cos dx$ 4) $\int \log x \ . \ x^{n} dx$

5) $\int_{0}^{\pi/2} \sin^{2} x \ dx$ 6) $\int_{0}^{\pi/2} \cos^{2} x \ dx$

7) $\int_{0}^{1} (2x+3)^{4} \ dx$ 8) $\int_{0}^{\pi/6} 2\sin 3x \ \cos 2x \ dx$

9) $\int_{0}^{\pi/2} \frac{\sin^{2} x}{1-\cos x} \ dx$ 10) $\int_{0}^{\pi/2} \sin^{7} x\cos x \ dx$

ANSWERS

PART – A

1) $xe^{x} - e^{x} + c$ 2) $- x \cos x + \sin x + c$

3) $x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + c$ 4) $x \sin x + \sin x + c$

PART – B

1) $-x^2 \cos x + 2x \sin x + 2 \cos x + c$
2) $x^2 \sin x + 2x \cos x - 2 \sin x + c$
3) $x^2 e^x - 2x e^x + 2e^x + c$
4) $x \log x - x + c$
5) $\frac{23}{6}$ 6) 1 7) 1 8) 0

PART – C

1) $x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x + c$
2) $-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$
3) $x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$
4) $\frac{x^{n+1} \log x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c$
5) $\frac{\pi}{4}$ 6) $\frac{\pi}{4}$ 7) 288.2 8) $\frac{6}{5}$ 9) $\frac{\pi}{2} + 1$ 10) $\frac{1}{8}$

UNIT – IV

APPLICATION OF INTEGRATION-I

4.1 Area and Volume

Area and Volume – Area of circle, Volume of Sphere and cone – Simple Problems.

4.2 First Order Differential Equation

Solution of first order variable separable type differential equation – Simple problems.

4.2 Linear Type Differential Equation

Solution of linear differential equation – Simple problems.

Introduction

In Engineering Mathematics-II, we discussed the basic concepts of integration. In Engineering Mathematics-I, we studied the formation of differential equation. In this unit, we shall study the application of integration and first order differential equation.

4.1 AREA AND VOLUME

Area and Volume:

We apply the concept of definite integral to find the area and volume.

Area:

The area under the curve y = f(x) between the x-axis and the ordinates x = a and x = b is given by the definite integral $\int_a^b f(x)\,dx$ (or) $\int_a^b y\,dx$.

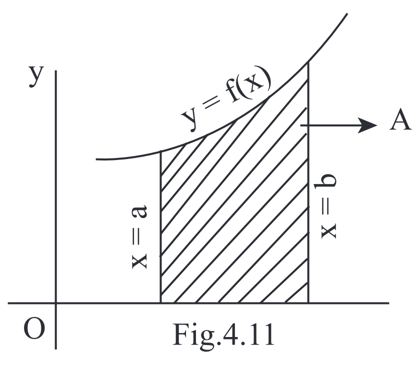


Fig.4.11

The area is shown as shaded region (A) in Fig.4.11

$$\text{Area} = A = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

Similarly, the area between the curve x = g(y), y-axis and the lines y = c and y = d shown as shaded region (A), in Fig. 4.12

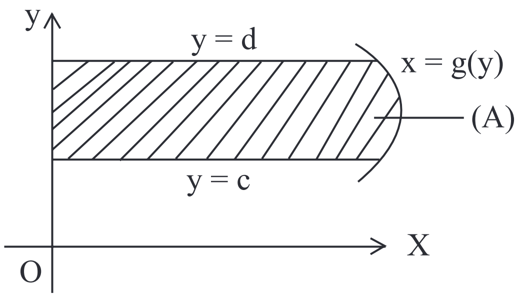


Fig 4.12

is given by

$$\text{Area} = A = \int_c^d g(y)\,dy \text{ (or)} \int_c^d x\,dy$$

Volume:

The volume of the solid obtained by rotating the area (shown in Fig.4.11) bounded by the curve y = f(x), x-axis and the lines x = a and x = b is given by

$$V = \pi\int_a^b [f(x)]^2 dx \quad \text{(or)}\ \pi \int_a^b y^2 dx$$

Similarly, the volume of solid obtained by rotating the area (shown in Fig.4.12) bounded by the curve x = g(y), y-axis and the lines y = c and y = d about the y-axis is given by

$$V = \pi\int_c^d g(y)^2 dy = \pi \int_c^d x^2 dy$$

4.1 WORKED EXAMPLES

PART –A

1. Find the area bounded by the curve $y = 4x^3$, the x-axis and the ordinates x = 0 and x = 1.

Solution:

$$\text{Area} = \int_a^b y dx = \int_0^1 4x^3 dx$$

$$= \left[\frac{4x^4}{4}\right]^1 = \left[x^4\right]_0^1$$

$$= (1)^4 - (0)^4 = 1 - 0 = 1 \text{ square units}$$

2. Find the area bounded by the curve $y = e^x$, the x-axis and the ordinates x = 0 and x = 6.

Solution:

$$\text{Area} = \int_0^6 e^x dx = \int_0^6 e^x dx = \left[e^x\right]_0^6$$

$$= e^6 - e^0 = e^6 - 1 \text{ square units}$$

3. Find the area bounded by the curve $x = 2y^2$, the y-axis and the lines $y = 0$ and $y = 3$.

Solution:

$$\text{Area} = \int_0^3 2y^2 dy = \left[\frac{2y^3}{3}\right]_0^3$$

$$= \left[\frac{2(3)^3}{3}\right] - 0 = \frac{54}{3} = 18 \text{ square units}$$

4. Find the volume of the solid formed when the area bounded by the area $y^2 = 4x$, the x-axis and the lines $x = 0$ and $x = 1$ is rotated about the x-axis.

Solution:

$$\text{Volume} = V = \pi\int_a^b y^2 dx$$

$$= \pi\int_0^1 4x \ dx = \pi\left[\frac{4x^2}{2}\right]_0^1$$

$$= \pi\left[2x^2\right]_0^1 = \pi\left[2(1)^2\right] - 0$$

$$= 2\pi \text{ cubic units}$$

PART – B

1. Find the area bounded by the curve $y = x^2 + x + 2$, x-axis and the lines $x = 1$ and $x = 2$.

Solution:

$$\text{Area} = \int_a^b y \ dx = \int_1^2 (x^2 + x + 2) \ dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_1^2$$

$$= \left[\frac{2^3}{3} + \frac{2^2}{2} + 2(2)\right] - \left[\frac{1^3}{3} + \frac{1^2}{2} + 2(1)\right]$$

$$= \left[\frac{8}{3} + \frac{4}{2} + 4\right] - \left[\frac{1}{3} + \frac{1}{2} + 2\right]$$

$$= \left[\frac{8}{3} + 6\right] - \left[\frac{2+3}{6} + 2\right]$$

$$= \left[\frac{8+18}{3}\right] - \left[\frac{5}{6} + 2\right]$$

$$= \left[\frac{26}{3}\right] - \left[\frac{5+12}{6}\right] = \frac{26}{3} - \frac{17}{6}$$

$$= \frac{52-17}{6} = \frac{35}{6} \text{ sq.units}$$

2. Find the area enclosed by one arch of the curve y = sin x, x-axis between x = 0 and x = π.

Solution:

$$\text{Area} = \int_a^b y\ dx = \int_0^\pi \sin x\ dx$$

$$= \left[-\cos x\right] = [-\cos \pi] - [-\cos 0]$$

$$= -(-1) + 1 = 1 + 1 = 2 \text{ square units}$$

3. Find the volume of the solid generated when the region enclosed by $y^2 = 4x^3 + 3x^2 + 2x$ between x = 1 and x = 2 is revolved about the x-axis.

Solution:

$$\text{Volume} = V = \pi\int_a^b y^2 dx$$

$$= \pi\int_1^2 (4x^3 + 3x^2 + 2x)\ dx$$

$$= \pi\left[\frac{4x^4}{4} + \frac{3x^3}{3} + \frac{2x^2}{2}\right]_1^2$$

$$= \pi\left[x^4 + x^3 + x^2\right]_1^2$$

$$= \pi\left[2^4 + 2^3 + 2^2\right] - \pi\left[1^4 + 1^3 + 1^2\right]$$

$$= \pi(16 + 8 + 4) - \pi(1 + 1 + 1)$$

$$= 28\pi - 3\pi = 25\pi \text{ cubic units}$$

4. Find the volume of the solid formed when the area bounded by the curve y = $\sqrt{10 + x}$ between x = 0 and x = 5 is rotated about x-axis.

Solution:

$$\text{Volume} = V = \pi\int_a^b y^2 dx = \pi\int_0^5 \left[\sqrt{10 + x}\right]^2 dx$$

$$= \pi\int_0^5 (10 + x)\ dx$$

$$= \pi\left[10x + \frac{x^2}{2}\right]_0^5$$

$$= \pi\left[10(5) + \frac{5^2}{2}\right] - \pi\left[10(0) + \frac{0^2}{2}\right]$$

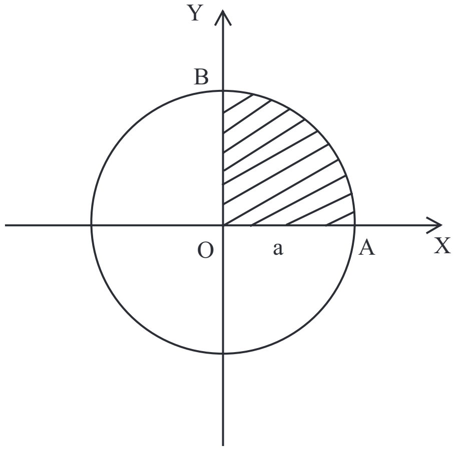
$$= \pi\left[50 + \frac{25}{2}\right] - 0$$

$$= \pi\left[\frac{100 + 25}{2}\right] = \frac{125\pi}{2} \text{ cubic units}$$

PART – C

1. Find the area of a circle of radius a, using integration.

Solution:



Area bounded by the circle $x^2 + y^2 = a^2$, the x-axis, and the lines x = 0 and x = a is given by

$$\text{Area OAB} = \int_0^a y\, dx$$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a \quad \text{[Formula]}$$

$$= \left[\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right] - [0 + 0]$$

$$= \frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1)$$

$$\boxed{\begin{array}{l} \sin 90 = 1 \\ 90^\circ = \sin^{-1}(1) \\ \dfrac{\pi}{2} = \dfrac{180}{2} = \sin^{-1}(1) \end{array}}$$

$$= 0 + \frac{a^2}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi a^2}{4}$$

Area of the circle = 4 × Area OAB

$$= 4 \times \frac{\pi a^2}{4}$$

$$= \pi a^2 \text{ sq.units}$$

Aliter:

$$I = \int_0^a \sqrt{a^2 - x^2}\,dx$$

Put $x = a \sin\theta$

when $x = 0$	when $x = a$
$a\sin\theta = 0$	$a\sin\theta = a$
$\sin\theta = 0 = \sin\theta$	$\sin\theta = 1 = \sin 90 = \sin\left(\frac{\pi}{2}\right)$
$\theta = 0$	$\theta = \frac{\pi}{2}$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta}$$

$$= \sqrt{a^2(1-\sin^2\theta)} \qquad x = a\sin\theta$$

$$= \sqrt{a^2\cos^2\theta} \qquad \frac{dx}{d\theta} = a\cos\theta$$

$$= a\cos\theta \qquad dx = a\cos\theta d\theta$$

$$\therefore I = \int_0^{\pi/2} (a\cos\theta)(a\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2} a^2\cos^2\theta\ d\theta$$

$$= \int_0^{\pi/2} a^2\frac{(1+\cos 2\theta)}{2}d\theta \qquad \boxed{\begin{aligned}&\frac{1+\cos 2\theta}{2}\\ &= \frac{\cancel{1} + 2\cos^2\theta - \cancel{1}}{2}\\ &= \cos^2\theta\end{aligned}}$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(1+\cos 2\theta)\ d\theta$$

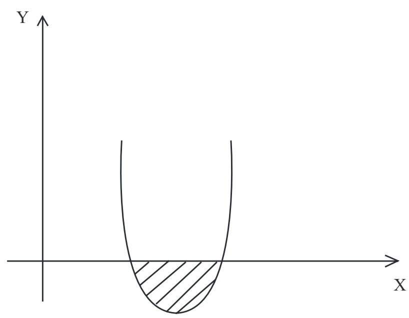
$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} + \frac{\sin(2\times\pi/2)}{2}\right] - \frac{a^2}{2}(0+0)$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} + \frac{\sin 180}{2}\right] = \frac{a^2}{2}\left[\frac{\pi}{2} + 0\right] = \frac{\pi a^2}{4}$$

2. Find the area bounded by the curve $y = x^2 - 6x + 8$ and the x-axis.

Solution:



To find limits

Put $y = 0$ as the curve meets the x-axis

$x^2 - 6x + 8 = 0$

$(x - 2)(x - 4) = 0$

$x = 2, 4$

$$\text{Area} = \int_{2}^{4} (x^2 - 6x + 8)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{6x^2}{2} + 8x\right]_2^4$$

$$= \left[\frac{4^3}{3} - \frac{6(4)^2}{2} + 8(4)\right] - \left[\frac{2^3}{3} - \frac{6(2)^2}{2} + 8(2)\right]$$

$$= \left[\frac{64}{3} - 48 + 32\right] - \left[\frac{8}{3} - 12 + 16\right]$$

$$= \left[\frac{64}{3} - 16\right] - \left[\frac{8}{3} + 4\right]$$

$$= \left[\frac{64 - 48}{3}\right] - \left[\frac{8 + 12}{3}\right]$$

$$= \frac{16}{3} - \frac{20}{3} = \frac{-4}{3} = \frac{4}{3}\text{sq.units (as Area is positive)}$$

3. Find the area bounded by the curve $y = 10 - 3x - x^2$ and the x-axis.

Solution:

To find limits

Put $y = 0$ as the curve cuts the x-axis

$10 - 3x - x^2 = 0$

$x^2 + 3x - 10 = 0$

$(x + 5)(x - 2) = 0$

$x = -5, x = 2$

$$\text{Area} = \int_{-5}^{2} (10 - 3x - x^2)\, dx$$

$$= \left[10x - \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-5}^{2}$$

$$= \left[10(2) - \frac{3(2)^2}{2} - \frac{2^3}{3}\right] - \left[10(-5) - \frac{3(-5)^2}{2} - \frac{(-5)^3}{3}\right]$$

$$= \left[20 - 6 - \frac{8}{3}\right] - \left[-50 - \frac{75}{2} + \frac{125}{3}\right]$$

$$= \left[14 - \frac{8}{3}\right] - \left[\frac{-100 - 75}{2} + \frac{125}{3}\right]$$

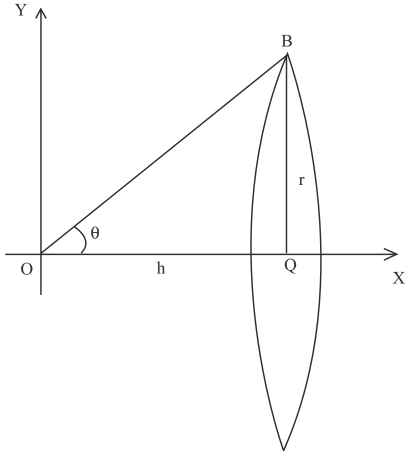
$$= \frac{34}{3} - \left[\frac{-525 + 250}{6}\right]$$

$$= \frac{68}{6} - \left[\frac{-275}{6}\right]$$

$$= \frac{68}{6} + \frac{275}{6} = \frac{343}{6}\ \text{sq.units}$$

4. Find the volume of a right circular cone of height h and base radius r by integration.

Solution:



Rotate a right angled triangle OAB with sides OA = h, AB = r about the x-axis. Then we get a right circular cone.

Volume of cone

$$V = \pi\int_0^h y^2 dx$$

$$= \pi\int_0^h \frac{r^2}{h^2} x^2 dx$$

$$= \left[\frac{\pi r^2}{h^2}\frac{x^3}{3}\right]_0^h$$

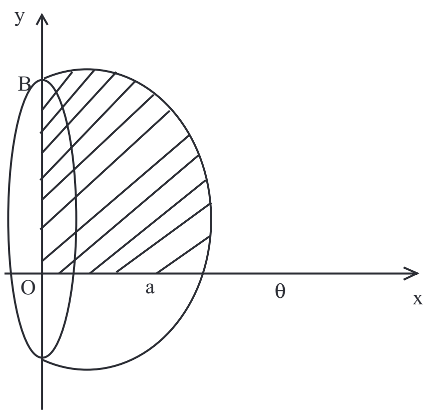
$$= \left[\frac{\pi r^2}{h^2}\frac{h^3}{3}\right] - [0]$$

$$= \frac{1}{3}\pi r^2 h$$

Equation of OB is
$y = mx$
$= \frac{r}{h}x$
$y^2 = \frac{r^2}{h^2}x^2$

5. Find the volume of a sphere of radius 'a' by integration.

Solution:



Rotate the area OAB (Quadrant of a circle) about OA, the x-axis. Then we get a hemi-sphere.

Volume of hemisphere

$$V = \pi\int_0^a y^2 dx$$

$$= \pi\int_0^a (a^2 - x^2)\,dx$$

$$= \pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \pi\left[a^2a - \frac{a^3}{3}\right] = \pi\left[a^3 - \frac{a^3}{3}\right]$$

$$= \pi\left[\frac{3a^3 - a^3}{3}\right] = \pi\left[\frac{2a^3}{3}\right] = \frac{2\pi a^3}{3}$$

Volume of sphere $= 2 \times \dfrac{2\pi a^3}{3} = \dfrac{4}{3}\pi a^3$

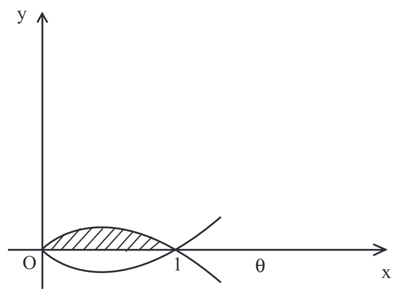
6. Find the volume generated by the area enclosed by the curve $y^2 = x\,(x-1)^2$ and the x-axis, when rotated about x-axis.

Solution:

To find limits

Put $y = 0$ as the curve cuts the x-axis.

$x\,(x-1)^2 = 0 \Rightarrow x = 0$ (or) $x - 1 = 0 \Rightarrow x = 1$.



$$V = \pi\int_0^1 y^2 dx$$

$$= \pi\int_0^1 x(x-1)^2\,dx$$

$$= \pi\int_0^1 x(x^2 - 2x + 1)\,dx$$

$$= \pi\int_0^1 (x^3 - 2x^2 + x)\,dx$$

$$= \pi\left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1$$

$$= \pi\left[\frac{1^4}{4} - \frac{2(1)^3}{3} + \frac{1^2}{2}\right] - [0 - 0 + 0]$$

$$= \pi\left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right]$$

$$= \pi\left[\frac{3 - 8 + 6}{12}\right] = \pi\left[\frac{1}{12}\right] = \frac{\pi}{12} \text{ cubic units}$$

4.2 FIRST ORDER DIFFERENTIAL EQUATION

Introduction:

Since the time of Newton, physical problems have been invetigated by formulating them mathematically as differential equations. Many mathematical models in engineering employ differential equations extensively.

In the first order differential equation, say $\frac{dy}{dx} = f(x,y)$, it is sometimes possible to group function of x with dx on one side and function of y with dy on the other side. This type of equation is called variables separable differential equations. The solution can be obtained by integrating both sides after separating the variables.

4.2 WORKED EXAMPLES

PART – A

1. Solve: $x\,dx + y\,dy = 0$

Solution:

$x\,dx = -y\,dy$

Integrating, $\int x\,dx = -\int y\,dy + c$

$$\frac{x^2}{2} = -\frac{y^2}{2} + c$$

2. Solve : $x\,dy + y\,dx = 0$

Solution:

$$x\,dy = -y\,dx$$

$$\frac{dy}{y} = \frac{-dx}{x}$$

Integrating, $\int \frac{dy}{y} = -\int \frac{dx}{x} + c$

$$\log y = -\log x + c$$

3. Solve: $x\frac{dy}{dx} = y$

Solution:

$$x\,dy = y\,dx$$

$$\frac{dy}{y} = \frac{dx}{x}$$

Integrating, $\int \frac{dy}{y} = \int \frac{dx}{x}$

$$\log y = \log x + c$$

4. Solve: $\frac{dy}{dx} - y\cos x = 0$

Solution:

$$\frac{dy}{dx} = y\cos x$$
$$dy = y\cos x \ dx$$
$$\frac{dy}{y} = \cos x\, dx$$

Integrating, $\int \frac{dy}{y} = \int \cos x \ dx$

$$\log y = \sin x + c$$

5. Solve: $\frac{dy}{dx} = e^x$

Solution:

$$dy = e^x dx$$

Integrating, $\int dy = \int e^x dx$

$$y = e^x + c$$

6. Solve: $\frac{dy}{dx} = \frac{1}{1+x^2}$

Solution:

$$dy = \frac{dx}{1+x^2}$$

Integrating, $\int dy = \int \frac{dx}{1+x^2}$

$$y = \tan^{-1} x + c$$

PART - B

1. Solve: $\frac{dy}{dx} = \frac{x}{1+x^2}$

Solution:

$$dy = \frac{x\ dx}{1+x^2}$$

Multiply both sides by 2, we get

$$2dy = \frac{2x\,dx}{1+x^2}$$

Integrating, $\int 2dy = \int \frac{2x\,dx}{1+x^2}$

Differentiation of Dr = $1 + x^2 = 2x$ = Nr Then result = log Dr = $\log(1 + x^2)$

$2y = \log(1 + x^2) + c$

2. Solve: $\dfrac{dy}{dx} = e^{x-5y}$

Solution:

$$dy = e^{x}e^{-5y}dx$$

$$\frac{dy}{e^{-5y}} = e^{x}dx$$

$$e^{5y}dy = e^{x}dx$$

Integrating, $\int e^{5y}dy = \int e^{x}dx$

$$\frac{e^{5y}}{5} = e^{x} + c$$

3. Solve: $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$

Solution:

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$$

Integrating, $\int \dfrac{dy}{\sqrt{1-y^2}} = \int \dfrac{dx}{\sqrt{1-x^2}}$

$$\sin^{-1} y = \sin^{-1} x + c$$

4. Solve: $\dfrac{dy}{dx} = \dfrac{3+x}{3+y}$

Solution:

$$(3 + y)\, dy = (3 + x)\, dx$$

Integrating, $\int (3+y)\, dy = \int (3+x)\, dx$

$$3y + \frac{y^2}{2} = 3x + \frac{x^2}{2} + c$$

PART – C

1. Solve: $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0.$

Solution:

$$\sec^2 x \tan y\, dx = -\sec^2 y \tan x\, dy$$

$$\frac{\sec^2 x\, dx}{\tan x} = -\frac{\sec^2 y\, dy}{\tan y}$$

Integrating, $\int \dfrac{\sec^2 x\, dx}{\tan x} = -\int \dfrac{\sec^2 y\, dy}{\tan y}$

> Note:
>
> If $\dfrac{d}{dx}(Dr) = Nr$, then the answer is log Dr.
>
> Here,
>
> $\dfrac{d}{dx}(\tan x) = \sec^2 x$ and $\dfrac{d}{dy}(\tan y) = \sec^2 y$

$\log \tan x = -\log \tan y + \log c$

$\log \tan x + \log \tan y = \log c$

$\log \tan x \tan y = \log c$

$\tan x \tan y = c$

2. Solve: $\dfrac{dy}{dx} = e^{x-y} + 3x^2e^{-y}$

Solution:

$$\frac{dy}{dx} = e^x e^{-y} + 3x^2 e^{-y}$$

$$\frac{dy}{dx} = e^{-y}(e^x + 3x^2)$$

$$dy = e^{-y}(e^x + 3x^2)\,dx$$

$$\frac{dy}{e^{-y}} = (e^x + 3x^2)\,dx$$

$$e^y dy = (e^x + 3x^2)\,dx$$

Integrating, $\int e^y dy = \int (e^x + 3x^2)\,dx$

$$e^y = e^x + \frac{3x^3}{3} + c$$

$$e^y = e^x + x^3 + c$$

3. Solve : $(1 - e^x)\sec^2 y\,dy + e^x \tan y\,dx = 0$

Solution:

$(1 - e^x)\sec^2 y\,dy = -e^x \tan y\,dx$

$$\frac{\sec^2 y\,dy}{\tan y} = \frac{-e^x dx}{1 - e^x}$$

Integrating, $\int \dfrac{\sec^2 y\,dy}{\tan y} = \int \dfrac{-e^x dx}{1 - e^x}$

$$\boxed{\text{As } \frac{d}{dy}(\tan y) = \sec^2 y \text{ and } \frac{d}{dx}(1 - e^x) = -e^x}$$

$\log \tan y = \log (1 - e^x) + \log c$

$\log (\tan y) - \log (1 - e^x) = \log c$

$$\log \frac{\tan y}{1 - e^x} = \log c$$

$$\frac{\tan y}{1 - e^x} = c$$

$$\tan y = c(1 - e^x)$$

4. Solve: $(x^2 - y)\,dx + (y^2 - x)\,dy = 0$

Solution:

$x^2dx - y\,dx + y^2dy - x\,dy = 0$

$x^2dx + y^2dy = x\,dy + y\,dx$

Note:
By uv rule, $d(xy) = x\,dy + y\,dx$

$x^2dx + y^2dy = d(xy)$

Integrating, $\int x^2dx + \int y^2dy = \int d(xy)$

$$\frac{x^3}{3} + \frac{y^3}{3} = xy + c$$

4.3 LINEAR DIFFERENTIAL EQUATION

A first order differential equation is said to be linear in y if the power of terms $\frac{dy}{dx}$ and y are unity. Differential equations of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x are called linear Differential Equations (LDE). The solution of linear differential equation is given by

$$ye^{\int pdx} = \int Q e^{\int pdx} dx + c$$

(or) shortly y (I.F) = ∫Q (IF) dx + c where IF = $e^{\int Pdx}$. IF is called Integrating Factor.

Note:
$e^{\log(f(x))} = f(x)$

Examples:

$e^{\log x} = x$; $e^{\log x^3} = x^3$; $e^{\log(\sin x)} = \sin x$

$e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$; $e^{-\log(\sin x)} = e^{\log(\sin x)^{-1}} = \operatorname{cosec} x$

4.3 WORKED EXAMPLES

PART – A

1. Find the integrating factor of $\frac{dy}{dx} + \frac{5}{x}y = x$.

Solution:

Compare with $\frac{dy}{dx} + Py = Q$

Here $P = \frac{5}{x}$

$$IF = e^{\int Pdx} = e^{\int \frac{5}{x}dx} = e^{5\log x}$$

$$= e^{\log x^5} = x^5$$

2. Find the integrating factor of $\frac{dy}{dx}+\frac{2x}{1+x^2}y=x^3$.

Solution:

Compare with $\frac{dy}{dx}+Py=Q$

$$P=\frac{2x}{1+x^2}$$

$$IF=e^{\int Pdx}=e^{\int\frac{2x\,dx}{1+x^2}}$$

$$=e^{\log(1+x^2)}=1+x^2$$

3. Find the integrating factor of $\frac{dy}{dx}-\frac{3}{x}y=x^3\cos x$.

Solution:

Compare with

$$.\frac{dy}{dx}+Py=Q$$

$$P=-\frac{3}{x}$$

$$\therefore IF=e^{\int Pdx}=e^{\int\frac{-3}{x}dx}=e^{-3\log x}=e^{\log x^{-3}}=x^{-3}=x^{-3}=\frac{1}{x^3}$$

4. Find the integrating factor of $\frac{dy}{dx}=-\frac{y}{x}$.

Solution:

The given equation is

$$\frac{dy}{dx}+\frac{y}{x}=0$$

Here $P=\frac{1}{x}$

$$\therefore IF=e^{\int Pdx}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$$

5. Find the integrating factor of $\frac{dy}{dx}+\frac{1}{1+x^2}y=1$.

Solution:

Here $\frac{1}{1+x^2}$

$$\therefore IF=e^{\int Pdx}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$$

PART – B

1. Find the integrating factor of $\frac{dy}{dx} + y \tan x = \sec^2 x$.

Solution:

Compare with $\frac{dy}{dx} + Py = Q$

Here P = tan x

$IF = e^{\int Pdx} = e^{\int \tan x\, dx}$

$$\boxed{\begin{array}{l} \text{Note:} \\ \frac{d}{dx}\log(\sec x) = \frac{1}{\sec x}\,.\sec x \tan x = \tan x \\ \therefore \int \tan x dx = \log \sec x \end{array}}$$

$IF = e^{\log \sec x} = \sec x$

2. Find the integrating factor of $\frac{dy}{dx} - y \tan x = \cot x$.

Solution:

Here P = – tan x

$$IF = e^{\int Pdx} = e^{\int \tan x\, dx}$$

$$= e^{-\log(\sec x)} = e^{\log(\sec x)^{-1}}$$

$$= (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$$

3. Find the integrating factor of $\frac{dy}{dx} + y \cot x = \sin x$.

Solution:

Here P = cot x

$$IF = e^{\int Pdx} = e^{\int \cot x\, dx}$$

$$\boxed{\begin{array}{l} \text{Note:} \\ \frac{d}{dx}\log(\sin x) = \frac{1}{\sin x}\cos x = \cot x \\ \therefore \int \cot x\ dx = \log(\sin x) \end{array}}$$

$$IF = e^{\log(\sin x)} = \sin x$$

4. Find the integration factor of $\frac{dy}{dx} - y \cot x = 4x^3$.

Solution:

Here P = – cot x

$$IF = e^{\int Pdx} = e^{\int \cot x\, dx}$$

$$= e^{-\log(\sin x)} = e^{\log(\sin x)^{-1}}$$

$$= (\sin x)^{-1} = \frac{1}{\sin x} = \text{cosec } x$$

PART – C

1. Solve: $\frac{dy}{dx} - \frac{3}{x}y = x^3 \cos x$

Solution:

Compare with $\frac{dy}{dx} + Py = Q$

Here $\boxed{P = -\frac{3}{x}}$ and $\boxed{Q = x^3 \cos x}$

$$IF = e^{\int Pdx} = e^{\int -\frac{3}{x}dx} = e^{-3\log x}$$

$$= e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3}$$

The required solution is

$$y(IF) = \int Q(IF)dx + c$$

$$y\left(\frac{1}{x^3}\right) = \int x^3 \cos x \frac{1}{x^3} dx + c$$

$$= \int \cos x dx + c$$

$$\frac{y}{x^3} = \sin x + c$$

2. Solve: $\frac{dy}{dx} + \frac{3x^2 y}{1+x^3} = \frac{2}{1+x^3}$

Solution:

Here $\boxed{P = \frac{3x^2}{1+x^3}}$ and $\boxed{Q = \frac{2}{1+x^3}}$

$$IF = e^{\int Pdx} = e^{\int \frac{3x^2 dx}{1+x^3}}$$

$$\boxed{\text{Note}: \frac{d}{dx}(1+x^3) = 3x^2 \quad \therefore \text{Ans} = \log(1+x^3)}$$

$$IF = e^{\log(1+x^3)} = 1+x^3$$

The required solution is

$$y(IF) = \int Q(IF)dx + c$$

$$y(1+x^3) = \int \frac{2}{(1+x^3)}(1+x^3)\, dx + c$$

$$y(1+x^3) = \int 2\, dx + c$$

$$y(1+x^3) = 2x + c$$

3. Solve: $(1+x^2)\frac{dy}{dx}+2xy=1$

Solution:

Divide both sides by $(1 + x^2)$

$$\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{1}{1+x^2}$$

Here $\boxed{P=\frac{2x}{1+x^2}}$ and $\boxed{Q=\frac{1}{1+x^2}}$

$$IF=e^{\int Pdx}=e^{\int\frac{2x}{1+x^2}dx}$$

$$=e^{\log(1+x^2)} \qquad \boxed{\therefore \frac{d}{dx}(1+x^2)=2x)}$$

$$=1+x^2$$

The required solution is

$$y(IF)=\int Q(IF)dx+c$$

$$y(1+x^2)=\int\frac{1}{(1+x^2)}(1+x^2)dx+c$$

$$=\int dx+c$$

$$y(1+x^2)=x+c$$

4. Solve: $\frac{dy}{dx}+y\cot x=2\cos x$

Solution:

Here $\boxed{P=\cot x}$ and $\boxed{Q=2\cos x}$

$$IF=e^{\int Pdx}=e^{\int \cot x dx} \qquad \boxed{\because \frac{d}{dx}\log(\sin x)=\frac{1}{\sin x}\cos x=\cot x}$$

$$=e^{\log(\sin x)}$$

$$=\sin x$$

The requried solution is

$$y(IF)=\int Q(IF)dx+c$$

$$y(\sin x)=\int 2\cos x\sin x\,dx+c$$

$$=\int \sin 2x\,dx \qquad \boxed{\therefore \sin 2A=2\sin A\cos A}$$

$$y\sin x=\frac{-\cos 2x}{2}+c$$

5. Solve: $\dfrac{dy}{dx} - y\tan x = e^x \sec x$

Solution:

Here $\boxed{P = -\tan x}$ and $\boxed{Q = e^x \sec x}$

$$IF = e^{\int P dx} = e^{\int -\tan x\, dx}$$

$$= e^{-\log(\sec x)}$$

$$= e^{\log(\sec x)^{-1}}$$

$$= (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$$

$$\boxed{\begin{aligned} \because \frac{d}{dx}\log(\sec x) &= \frac{1}{\sec x}\sec x\ \tan x \\ &= \tan x \end{aligned}}$$

The required solution is

$$y(IF) = \int Q(IF)dx + c$$

$$y(\cos x) = \int e^x \sec x \cos x\, dx + c$$

$$= \int e^x \frac{1}{\cos x}\cos x\ dx + c$$

$$y(\cos x) = \int e^x dx + c$$

$y \cos x = e^x + c$

EXERCISE

PART – A

1. Find the area bounded by the curve $y = 2x$, the x-axis and the lines $x = 0$ and $x = 1$.
2. Find the area bounded by the curve $y = x^2$, x-axis between $x = 0$ and $x = 2$.
3. Find the area bounded by the curve $y = \dfrac{x^2}{2}$, x-axis between $x = 1$ and $x = 3$.
4. Find the area bounded by the curve $xy = 1$, the y-axis and the lines $y = 1$ and $y = 5$.
5. Find the area bounded by the curve $x = 2y$, the y-axis and the lines $y = 1$ and $y = 2$.
6. Solve: $xy\dfrac{dy}{dx} = 1$
7. Solve: $e^x\, dx + e^y\, dy = 0$
8. Solve: $\dfrac{dy}{dx} = e^{3x}$
9. Solve: $\dfrac{dy}{dx} = \dfrac{\cos x}{y^2}$
10. Find the integrating factor of $\dfrac{dy}{dx} + 3y = 6$.
11. Find the integrating factor of $\dfrac{dy}{dx} + y\sin x = 0$.
12. Find the integrating factor of $\dfrac{dy}{dx} + \dfrac{y}{x} = x$.

PART – B

1. Find the area bounded by the curve $y = x^2 + x$, x-axis and the lines $x = 0$ and $x = 4$.
2. Find the area bounded by the curve $y = 3x^2 - x$, x-axis and the ordinates $x = 0$ and $x = 6$.
3. Find the volume of the solid generated when the area bounded by the curve $y^2 = 25x^3$ between $x=1$ and $x = 3$ is rotated about x-axis.
4. Find the volume of the solid formed when the area bounded by the curve $y^2 = 8x$ between $x = 0$ and $x = 2$ is rotated about x-axis.
5. Find the volume of the solid formed when the area bounded by the curve $x^2 = 3y^2$ between $y = 0$ and $y = 1$ is rotated about the y-axis.
6. Solve: $\tan x \sec^2 y\, dy + \tan y \sec^2 x\, dx = 0$
7. Solve: $\frac{dy}{dx} + \frac{1+x^2}{1+y^2} = 0$
8. Solve: $(1 + x^2) \sec^2 y\, dy = 2x \tan y\, dx$
9. Solve: $\frac{dy}{dx} = e^{2x+3y}$
10. Solve: $\sec x\, dy + \sec y\, dx = 0$

Find the integrating for the following linear differential equations:

11. $\frac{dy}{dx} + y \cot x = x \operatorname{cosec} x.$
12. $\frac{dy}{dx} - y \cot x = \sin x.$
13. $\frac{dy}{dx} + y \tan x = e^x \cos x.$
14. $\frac{dy}{dx} - y \tan x = x^3.$
15. $\frac{dy}{dx} + \frac{xy}{1+x^2} = \frac{1}{1+x^2}$

PART – C

1. Find the area enclosed by $y = 6 + x - x^2$ and the x-axis.
2. Find the area bounded by the curve $y = x + \sin x$, the x-axis and the ordinates $x = 0$ and $x = \frac{\pi}{2}$.
3. Find the area bounded by the curve $y = x^2 + x + 1$, x-axis and the ordinates $x = 1$ and $x = 3$.
4. Find the volume of the solid formed when the area of the loop of the curve $y^2 = 4x (x - 1)^2$ rotates about the x-axis.
5. Find the volume of the solid bound when the area bounded by the curve $y^2 = 2 + x - x^2$, the x-axis and the lines $x = -1$ and $x = 2$ is rotated about x axis.
6. Find the volume of the solid obtained by revolving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.
7. Solve: $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$
8. Solve: $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$
9. Solve : $\frac{dy}{dx} = \frac{1+\cos y}{1+\cos x}$

10. Solve: $\frac{dy}{dx} = \frac{4+y^2}{\sqrt{4-x^2}}$

Solve the following Linear Differential Equations:

11. $\frac{dy}{dx} + y \cot x = e^x \operatorname{cosec} x$.

12. $\frac{dy}{dx} - \frac{2y}{x} = x^2 \sin x$

13. $(1+x^2)\frac{dy}{dx} - 2xy = (1+x^2)^2$

14. $\frac{dy}{dx} - \frac{3y}{x} = x^3 e^{2x}$.

15. $2 \cos x \frac{dy}{dx} + 4y \sin x = \sin 2x$.

16. $\frac{dy}{dx} + y \tan x = \cos^3 x$.

ANSWERS

PART – A

1) 1 2) $\frac{8}{3}$ 3) $\frac{13}{3}$ 4) log 5 5) 3 6) $\frac{y^2}{2} = \log x + c$ 7) $e^x + e^y = c$

8) $y = \frac{e^{3x}}{3}$ 9) $\frac{y^3}{3} = \sin x + c$ 10) e^{3x} 11) $e^{-\cos x}$ 12) x

PART – B

1) $\frac{88}{3}$ 2) 198 3) 500 π 4) 16π 5) π

6) $\log \tan y = -\log \tan x + c$ (or) $\tan x \tan y = c$ 7) $y + \frac{y^3}{3} + x + \frac{x^3}{3} = c$

8) $\log \tan y = \log(1+x^2) + c$ 9) $\tan y = c(1+x^2)$ 10) $\sin y + \sin x = c$

11) sin x 12) cosec x 13) sec x 14) cos x 15) $\sqrt{1+x^2}$

PART – C

1) $\frac{125}{6}$ 2) $\frac{\pi^2}{8}$ 3) $\frac{44}{3}$ 4) 2π 5) $\frac{9}{2}$ 6) $\frac{4\pi}{3}ab^2$

7) $e^y = e^x + \frac{x^3}{y}$ 8) $\log \tan y - \log(1-e^x) = c$ (or) $\tan y = c(1-e^x)$

9) $2\tan\left(\frac{y}{2}\right) = -2\operatorname{cosec}\left(\frac{x}{2}\right) + c$ (or) $\tan\frac{y}{2} + \operatorname{cosec}\frac{x}{2} = c$ 10) $\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = \sin^{-1}\left(\frac{x}{2}\right) + c$

11) $y \sin x = e^x + c$ 12) $\frac{y}{x^2} = -\cos x + c$ 13) $\frac{y}{1+x^2} = x + \frac{x^3}{3} + c$ 14) $\frac{y}{x^3} = \frac{e^{2x}}{2} + c$

15) $y \sec^2 x = \sec x + c$ 16) $y \sec x = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$

UNIT – II

VECTOR ALGEBRA-I

2.1 Introduction:

Definition of vectors – types, addition and subtraction of vectors, Properties of addition and subtraction, Position Vector, Resolution of vector in two and three dimensions, Direction cosines, direction ratios – Simple problems.

2.2 Scalar Product of Vectors:

Definition of scalar product of two vectors – Properties – Angle between two vectors – Simple Problems.

2.3 Application of Scalar Product:

Geometrical meaning of scalar product. Workdone by Force – Simple Problems.

2.1 INTRODUCTION

A scalar quantity or briefly a scalar, has magnitude, but is not related to any direction in space. Examples of such are mass, volume, density, temperature, work, Real numbers.

A vector quantity, or briefly a vector, has magnitude and is related to a definite direction in space. Examples of such are Displacement, velocity, acceleration, momentum, force.

A vector is a directed line segment. The length of the segment is called magnitude of the vector. The direction is indicated by an arrow joining the initial and final points of the line segment. The vector AB, i.e, joining the initial point A and the final point B in the direction of AB is denoted as $\overrightarrow{AB}$. The magnitude of the vector $\overrightarrow{AB}$ is AB = $|\overrightarrow{AB}|$.

Zero vector or Null vector:

A zero vector is one whose magnitude is zero, but no definite direction associated with it. For example if A is a point, $\overrightarrow{AA}$ is a zero vector.

Unit Vector:

A vector of magnitude one unit is called an unit vector. If $\vec{a}$ is an unit vector, it is also denoted as $\hat{a}$. i.e, $|\hat{a}| = |\vec{a}| = 1$.

Negative Vector:

If $\overrightarrow{AB}$ is a vector, then the negative vector of $\overrightarrow{AB}$ is $\overrightarrow{BA}$. If the direction of a vector is changed, we can get the negative vector.

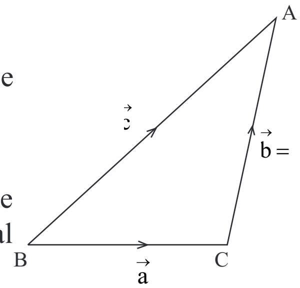
i.e, $\overrightarrow{BA} = -\overrightarrow{AB}$

Equal vectors:

Two vectors are said to be equal, if they have the same magnitude and the same direction, but it is not required to have the same segment for the two vectors.

For example, in a parallelogram ABCD, $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

Addition of two vectors:



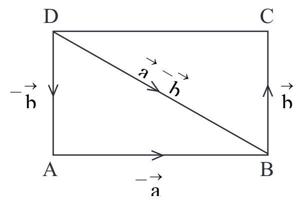
If $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{BA} = \vec{c}$, then $\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$ i.e, $\vec{a} + \vec{b} = \vec{c}$ [see figure].

If the end point of first vector and the initial point of the second vector are same, the addition of two vectors can be formed as the vector joining the initial point of the first vector and the end point of the second vector.

Properties of vector addition:

1) Vector addition is commutative i.e, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2) Vector addition is associative i.e, $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

Subtraction of two vectors:



If $\overline{AB} = \overline{a}$ and $\overline{BC} = \overline{b}$,

$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

$= \overrightarrow{AB} + \overrightarrow{CB}$

$= \overrightarrow{AB} + \overrightarrow{DA}$ [$\because \overrightarrow{CB}$ and $\overrightarrow{DA}$ are equal]

$= \overrightarrow{DA} + \overrightarrow{AB}$ [$\because$ addition is commutative]

$= \overrightarrow{DB}$

Multiplication by a scalar:

If $\vec{a}$ is a given vector and λ is a scalar, then $\lambda \vec{a}$ is a vector whose magnitude is $\lambda |\vec{a}|$ and whose direction is the same to that of $\vec{a}$, provided λ is a positive quantity. If λ is negative, $\lambda \vec{a}$ is a vector whose magnitude is $|\lambda||\vec{a}|$ and whose direction is opposite to that of $\vec{a}$.

Properties:

1) $(m + n)\vec{a} = m\vec{a} + n\vec{a}$
2) $m(n\vec{a}) = n(m\vec{a}) = mn\vec{a}$
3) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Collinear Vectors:

If $\vec{a}$ and $\vec{b}$ are such that they have the same or opposite directions, they are said to be collinear vectors and one is a numerical multiple of the other, i.e, $\vec{b} = k\vec{a}$ or $\vec{a} = k\vec{b}$.

Resolution of Vectors:

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be coplanar vectors such that no two vectors are parallel. Then there exists scalars α and β such that

$\vec{c} = \alpha\vec{a} + \beta\vec{b}$

Similarly, we can get cosntants (scalars) such that $\vec{a} = \alpha'\vec{b} + \beta\vec{c}$ and $\vec{b} = \alpha''\vec{c} + \beta''\vec{a}$.

If $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are four vectors, no three of which are coplanar, then there exist scalars l, m, n such that

$\vec{d} = l\vec{a} + m\vec{b} + n\vec{c}$,

Position Vector:

If P is any point in the space and 0 is the origin, then $\overrightarrow{OP}$ is called the position vector of the point P.

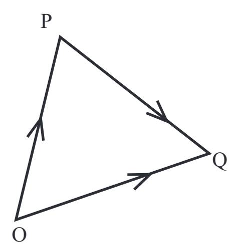
Let P be a point in a plane. Let 0 be the origin and $\vec{i}$ and $\vec{j}$ the unit vectors along the x and y axes in that plane. Then if P is (α, β), the position vector of the point P is $\overrightarrow{OP} = \alpha\vec{i} + \beta\vec{j}$.

Similarly if P is any point (x, y, z) in the space, $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors along the x, y, z axes in the space, the position vector of the point P is $\overrightarrow{OP} = x\vec{i}, y\vec{j}, z\vec{k}$. The magnitude of $\overrightarrow{OP}$ is $OP = |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.

Distance between two points:

If P and Q are two points in the space with co-ordinates P (x_1, y_1, z_1) and Q (x_2, y_2, z_2), then the posiiton vectors are $\overrightarrow{OP} = x_1\vec{i}, y_1\vec{j}, z_1\vec{k}$

and $\overrightarrow{OQ} = x_2\vec{i}, y_2\vec{j}, z_2\vec{k}$



Now, Distance between the points P & Q is

$$PQ = |\overrightarrow{PQ}| = |\overrightarrow{OQ} - \overrightarrow{OP}|$$

$$= |(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}|$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Direction Cosines and Direction Ratios:

Let AB be a straight line making angles α, β, γ with the co-ordinates axes x'ox, y'oy, z'oz respectively. Then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the line AB and denoted by l, m, n. Let OP be parallel to AB and P be (x, y, z). Then OP also makes angles α, β, γ with x, y and z axes. Now, $OP = r = \sqrt{x^2 + y^2 + z^2}$.

Then, $\cos\alpha = \frac{x}{r}$, $\cos\beta = \frac{y}{r}$, $\cos\gamma = \frac{z}{r}$.

Now, sum of squares of the direction cosines of any straight line is

$$l^2 + m^2 + n^2 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2$$

$$= \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1$$

Note:

Let $\hat{n}$ be the unit vector along OP.

$$\text{Then },\ \hat{n} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$$

$$= \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}$$

$$= l\vec{i} + m\vec{j} + n\vec{k}$$

Any three numbers p, q, r proportional to the direction cosines of the straight line AB are called the direction ratios of the straight line AB.

WORKED EXAMPLES

PART – A

1. If position vectors of the points A and B are $2\vec{i} + \vec{j} - \vec{k}$ and $5\vec{i} + 4\vec{j} + 3\vec{k}$, find $|\overrightarrow{AB}|$.

Solution:

Position vector of the point A,

i.e, $\overrightarrow{OA} = 2\vec{i} + \vec{j} - \vec{k}$

Position vector of the point B,

i.e, $\overrightarrow{OB} = 5\vec{i} + 4\vec{j} - 3\vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (5\vec{i} + 4\vec{j} - 3\vec{k}) - (2\vec{i} + \vec{j} - \vec{k})$$

$$= 3\vec{i} + 3\vec{j} - 2\vec{k}$$

$$\therefore AB = |\overrightarrow{AB}| = \sqrt{3^2 + 3^2 + (-2)^2}$$

$$= \sqrt{9 + 9 + 4} = \sqrt{22} \text{ units}$$

2. Find the unit vector along $4\vec{i} - 5\vec{j} + 7\vec{k}$.

Solution:

Let $\vec{a} = 4\vec{i} - 5\vec{j} + 7\vec{k}$

$$|\vec{a}| = \sqrt{4^2 + (-5)^2 + 7^2}$$

$$= \sqrt{16 + 25 + 49}$$

$$= \sqrt{90}$$

$$\therefore \text{ Unit vector along } \vec{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{4\vec{i} - 5\vec{j} + 7\vec{k}}{\sqrt{90}}$$

3. Find the direction cosines of the vector $2\vec{i}+3\vec{j}-4\vec{k}$.

Solution:

$$\text{Let } \vec{a} = 2\vec{i}+3\vec{j}-4\vec{k}$$

$$r = |\vec{a}| = \sqrt{2^2+3^2+(-4)^2}$$
$$= \sqrt{4+9+16}$$
$$= \sqrt{29}$$

Direction cosines of $\vec{a}$ are

$$\cos\alpha = \frac{x}{r} = \frac{2}{\sqrt{29}}$$

$$\cos\beta = \frac{y}{r} = \frac{3}{\sqrt{29}}$$

$$\cos\gamma = \frac{z}{r} = \frac{-4}{\sqrt{29}}$$

4. Find the direction ratios of the vector $\vec{i}+2\vec{j}-\vec{k}$.

Solution:

The direction ratios of $\vec{i}+2\vec{j}-\vec{k}$ are 1, 2, – 1.

PART – B

1. If the vectors $\vec{a} = 2\vec{i}-3\vec{j}$ and $\vec{b} = -6\vec{i}+m\vec{j}$ are collinear, find the value of them.

Solution:

$\vec{a} = 2\vec{i}-3\vec{j}$ and $\vec{b} = -6\vec{i}+m\vec{j}$. By given, $\vec{a}$ and $\vec{b}$ are collinear.

$$\therefore \quad \vec{a} = t\vec{b}$$

$$\vec{a} = 2\vec{i}-3\vec{j}$$
$$= t(-6\vec{i}+m\vec{j})$$

$$2\vec{i}-3\vec{j} = -6t\vec{i}+mt\vec{j}$$

Comparing coefficient of $\vec{i}$

$$2 = -6t \Rightarrow t = \frac{-1}{3}$$

Comparing coefficient of $\vec{j}$

$$-3 = mt$$

$$-3 = m\left(-\frac{1}{3}\right)$$

$$\therefore m = 9$$

2. If A (2, 3, – 4) and B (1, 0, 5) are two points, find the direction cosines of $\overrightarrow{AB}$.

Solution:

By given, the points are

A (2, 3, –4) and B (1, 0, 5)

$\therefore$ Position vectors are $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - 4\vec{k}, \quad \overrightarrow{OA} = \vec{i} + 5\vec{k}$,

$$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (\vec{i} + 5\vec{k}) - (2\vec{i} + 3\vec{j} - 4\vec{k})$$

$$= -\vec{i} - 3\vec{j} + 9\vec{k}$$

$$r = |\overrightarrow{AB}| = \sqrt{(-1)^2 + (-3)^2 + 9^2}$$

$$= \sqrt{1 + 9 + 81} = \sqrt{91}$$

$\therefore$ Direction cosines of $\overrightarrow{AB}$ are

$$\cos\alpha = \frac{-1}{\sqrt{91}} \qquad \cos\beta = \frac{-3}{\sqrt{91}} \qquad \cos\gamma = \frac{9}{\sqrt{91}}$$

PART – C

1. Show that the points whose position vectors $2\vec{i} + 3\vec{j} - 5\vec{k}$, $3\vec{i} + \vec{j} - 2\vec{k}$ and $6\vec{i} - 5\vec{j} + 7\vec{k}$ are collinear.

Solution:

$$\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - 5\vec{k}$$

$$\overrightarrow{OB} = 3\vec{i} + \vec{j} - 2\vec{k}$$

$$\overrightarrow{OC} = 6\vec{i} - 5\vec{j} + 7\vec{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (3\vec{i} + \vec{j} - 2\vec{k}) - (2\vec{i} + 3\vec{j} - 5\vec{k})$$

$$= \vec{i} - 2\vec{j} + 3\vec{k} \qquad \text{......... (1)}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= (6\vec{i} - 5\vec{j} + 7\vec{k}) - (3\vec{i} + \vec{j} - 2\vec{k})$$

$$= 3\vec{i} - 6\vec{j} + 9\vec{k}$$

$$= 3(\vec{i} - 2\vec{j} + 3\vec{k})$$

$$= 3\overrightarrow{AB} \qquad \text{[From (1)}$$

i.e, $\overrightarrow{BC} = 3\overrightarrow{AB}$

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors and B is the common point of these two vectors.

$\therefore$ The given points A, B and C are collinear.

2. Prove that the points A (2, 4, –1), B (4, 5, 1) and C (3, 6, – 3) form the vertices of a right angled isoceles triangle.

Solution:

$\overrightarrow{OA} = 2\vec{i} + 4\vec{j} - \vec{k}$ $\qquad \overrightarrow{OB} = 4\vec{i} + 5\vec{j} + \vec{k}$ $\qquad \overrightarrow{OC} = 3\vec{i} + 6\vec{j} - 3\vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (4\vec{i} + 5\vec{j} + \vec{k}) - (2\vec{i} + 4\vec{j} - \vec{k})$$
$$= 2\vec{i} + \vec{j} + 2\vec{k}$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= (3\vec{i} + 6\vec{j} - 3\vec{k}) - (4\vec{i} + 5\vec{j} + \vec{k})$$
$$= -\vec{i} + \vec{j} - 4\vec{k}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= (3\vec{i} + 6\vec{j} - 3\vec{k}) - (2\vec{i} + 4\vec{j} - \vec{k})$$
$$= \vec{i} + 2\vec{j} - 2\vec{k}$$

Now, $AB = |\overrightarrow{AB}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4+1+4} = \sqrt{9}$

$BC = |\overrightarrow{BC}| = \sqrt{(-1)^2 + 1^2 + (-4)^2} = \sqrt{1+1+16} = \sqrt{18}$

$AC = |\overrightarrow{AC}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9}$

$AB = AC = \sqrt{9} = 3$

& $AB^2 + AC^2 = 9 + 9 = 18 = BC^2$

$\therefore$ Triangle ABC is an isoceles triangle as well as a right angled triangle with $\hat{A} = 90^o$.

3. Prove that the position vectors $4\vec{i} + 5\vec{j} + 6\vec{k}$, $5\vec{i} + 6\vec{j} + 4\vec{k}$ and $6\vec{i} + 4\vec{j} + 5\vec{k}$ form the vertices of an equilateral triangle.

Solution:

Let $\overrightarrow{OA} = 4\vec{i} + 5\vec{j} + 6\vec{k}$ $\qquad \overrightarrow{OB} = 5\vec{i} + 6\vec{j} + 4\vec{k}$ $\qquad \overrightarrow{OC} = 6\vec{i} + 4\vec{j} + 5\vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5\vec{i} + 6\vec{j} + 4\vec{k}) - (4\vec{i} + 5\vec{j} + 6\vec{k})$$
$$= \vec{i} + \vec{j} - 2\vec{k}$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (6\vec{i} + 4\vec{j} + 5\vec{k}) - (5\vec{i} + 6\vec{j} + 4\vec{k})$$
$$= \vec{i} - 2\vec{j} + \vec{k}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (6\vec{i} + 4\vec{j} + 5\vec{k}) - (4\vec{i} + 5\vec{j} + 6\vec{k})$$
$$= 2\vec{i} - \vec{j} - \vec{k}$$

Now, $AB = |\overrightarrow{AB}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6}$

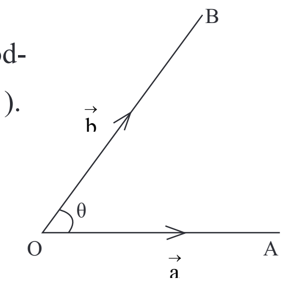
$BC = |\overrightarrow{BC}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6}$

$AC = |\overrightarrow{AC}| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{4+1+1} = \sqrt{6}$

Here, $AB = BC = CA = \sqrt{6}$

$\therefore$ The given triangle is an equilateral triangle, [since the sides are equal].

2.2 SCALAR PRODUCT OF VECTORS OR DOT PRODUCT



If the product of two vectors $\vec{a}$ and $\vec{b}$ gives a scalar, it is called scalar product of the vectors $\vec{a}$ and $\vec{b}$ and is denoted as $\vec{a} \cdot \vec{b}$ (pronounce as $\vec{a}$ dot $\vec{b}$).

If the angle between two vectors $\vec{a}$ and $\vec{b}$ is θ, then

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

Properties of scalar product:

1. If θ is an acute angle $\vec{a} \cdot \vec{b}$ is positive and if θ is an abtuse angle $\vec{a} \cdot \vec{b}$ is negative.
2. Scalar product is commutative.

   i.e, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = |\vec{b}||\vec{a}| \cos\theta = \vec{b} \cdot \vec{a}$
3. If $\vec{a}$ and $\vec{b}$ are (non zero) perpendicular vectors, then the angle θ between them is 90^o.

   $\therefore \; \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 90^o = 0$ [$\because \cos 90^o = 0$]

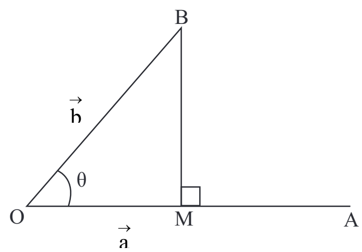
   If $\vec{a} \cdot \vec{b} = 0$, either $\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a}$ and $\vec{b}$ are perpendicular vector.

   ∴ The condition for two pwerpendicular vectors $\vec{a}$ and $\vec{b}$ is $\vec{a} \cdot \vec{b} = 0$.
4. If $\vec{a}$ and $\vec{b}$ are paraller vectors, θ = 0 or 180^o.

   $\therefore \; \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 0$

   $= |\vec{a}||\vec{b}|$ [$\because \cos 0 = 1$

   As a special case, $\therefore \; \vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}| = |\vec{a}|^2 = a^2$
5. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Draw BM perpendicular to OA.
   (BM perpendicular to OA)



   Let θ be the angle between $\vec{a}$ and $\vec{b}$.
   i.e, BOA = θ.

   Now, OM is the projection of $\vec{b}$ on $\vec{a}$.
   From the right angled triangle BOM,

$$\cos\theta = \frac{OM}{OB} = \frac{OM}{|\vec{b}|}$$

$$\therefore OM = |\vec{b}|\cos\theta$$

$$= \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{a}|} \qquad \text{[Multiplying Nr and Dr by } |\vec{a}|$$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \qquad \text{[By definition of scalar product}$$

∴ The projection of $\vec{b}$ on $\vec{a}$ = $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

Similarly, the projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

6. $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along the x, y and z axes respectively.

$\therefore \vec{i}\cdot\vec{i},\ \vec{j}\cdot\vec{j},\ \vec{k}\cdot\vec{k} = 1$ [using property 4]

$$\left.\begin{aligned} \text{Also } \vec{i}\cdot\vec{j} &= \vec{j}\cdot\vec{i} = 0 \\ \vec{j}\cdot\vec{k} &= \vec{k}\cdot\vec{j} = 0 \\ \vec{k}\cdot\vec{i} &= \vec{i}\cdot\vec{k} = 0 \end{aligned}\right\} \quad \text{[using property 3]}$$

Hence,

•	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0
$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

7. If, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors,

$$\vec{a}\bullet(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

8. If $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}\ \&\ \vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$

$\vec{a}\cdot\vec{b} = (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\cdot(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})$

$= a_1b_1\,\vec{i}\cdot\vec{i}+a_1b_2\,\vec{i}\cdot\vec{j}+a_1b_3\,\vec{i}\cdot\vec{k}+a_2b_1\,\vec{j}\cdot\vec{i}+$

$a_2b_2\,\vec{j}\cdot\vec{j}+a_2b_3\,\vec{j}\cdot\vec{k}+a_3b_1\,\vec{k}\cdot\vec{i}+a_3b_2\,\vec{k}\cdot\vec{j}+a_3b_3\,\vec{k}\cdot\vec{k}$

$= a_1b_1+0+0+0+a_2b_2+0+0+0+a_3b_3$ [By property 6]

$= a_1b_1+a_2b_2+a_3b_3$

9. Angle between two vectors

We know $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

If $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}\ \&\ \vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$

$$\text{then } \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}$$

10. $(\vec{a}+\vec{b}) = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = a^2+b^2+2\vec{a}\cdot\vec{b}$

11. $(\vec{a}-\vec{b}) = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = a^2+b^2-2\vec{a}\cdot\vec{b}$

12. $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = a^2-b^2$

WORKED EXAMPLES

PART – A

1. Find the scalar product of the two vectors $3\vec{i}+4\vec{j}+5\vec{k}$ and $2\vec{i}+3\vec{j}+\vec{k}$.

Solution:

$$\text{Let } \vec{a} = 3\vec{i}+4\vec{j}+5\vec{k}$$
$$\vec{b} = 2\vec{i}+3\vec{j}+\vec{k}$$
$$\vec{a}\cdot\vec{b} = (3\vec{i}+4\vec{j}+5\vec{k})\cdot(2\vec{i}+3\vec{j}+\vec{k})$$
$$= 3(2)+4(3)+5(1)$$
$$= 6+12+5$$
$$= 23$$

2. Prove that the vectors $3\vec{i}-\vec{j}+5\vec{k}$ and $-6\vec{i}+2\vec{j}+4\vec{k}$ are perpendicular.

Solution:

$$\text{Let } \vec{a} = 3\vec{i}-\vec{j}+5\vec{k}$$
$$\vec{b} = -6\vec{i}+2\vec{j}+4\vec{k}$$
$$\text{Now, } \vec{a}\cdot\vec{b} = (3\vec{i}-\vec{j}+5\vec{k})\cdot(-6\vec{i}+2\vec{j}+4\vec{k})$$
$$= 3(-6)+(-1)2+5(4)$$
$$= -18-2+20$$
$$= 0$$

$\therefore$ The vectors $\vec{a}$ and $\vec{b}$ are perpendicular vectors.

3. Find the value of 'p' if the vectors $2\overrightarrow{i}+p\overrightarrow{j}-\overrightarrow{k}$ and $3\overrightarrow{i}+4\overrightarrow{j}+2\overrightarrow{k}$ are perpendicular.

Solution:

$$\text{Let } \overrightarrow{a} = 2\overrightarrow{i}+p\overrightarrow{j}-\overrightarrow{k}$$
$$\overrightarrow{b} = 3\overrightarrow{i}+4\overrightarrow{j}+2\overrightarrow{k}$$

$\vec{a}$ and $\vec{b}$ are perpendicular

$\therefore\ \vec{a}\cdot\vec{b} = 0$

i.e, $(2\overrightarrow{i}+p\overrightarrow{j}-\overrightarrow{k})\cdot(3\overrightarrow{i}+4\overrightarrow{j}+2\overrightarrow{k}) = 0$

i.e, $2(3)+p(4)+(-1)2 = 0$

i.e, $6+4p-2 = 0$

i.e, $4p = 2-6 = -4$

$$\therefore\quad p = -\frac{4}{4} = -1$$

PART – B

1. Find the projection of $2\overrightarrow{i} + \overrightarrow{j} + 2\overrightarrow{k}$ on $\overrightarrow{i} + 2\overrightarrow{j} + 2\overrightarrow{k}$.

Solution:

Let $\overrightarrow{a} = 2\overrightarrow{i} + \overrightarrow{j} + 2\overrightarrow{k}$

$\overrightarrow{b} = \overrightarrow{i} + 2\overrightarrow{j} + 2\overrightarrow{k}$

$$\text{Projection of } \vec{a} \text{ on } \overrightarrow{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$= \frac{(2\vec{i}+\vec{j}+2\vec{k})\cdot(\vec{i}+2\vec{j}+2\vec{k})}{|\vec{i}+2\vec{j}+2\vec{k}|}$$

$$= \frac{2(1)+1(2)+2(2)}{\sqrt{1^2+2^2+2^2}} = \frac{2+2+4}{\sqrt{1+4+4}}$$

$$= \frac{8}{\sqrt{9}} = \frac{8}{3}$$

PART – C

1. Find the angle between the two vectors $\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$ and $3\overrightarrow{i} - \overrightarrow{j} + 2\overrightarrow{k}$.

Solution:

Let $\overrightarrow{a} = \overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$

$\overrightarrow{b} = 3\overrightarrow{i} - \overrightarrow{j} + 2\overrightarrow{k}$

$$\vec{a}\cdot\vec{b} = (\vec{i}+\vec{j}+\vec{k})\cdot(3\vec{i}-\vec{j}+2\vec{k})$$

$$= 1(3)+1(-1)+1(2)$$

$$= 3-1+2$$

$$= 4$$

$$|\overrightarrow{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$|\overrightarrow{b}| = \sqrt{3^2+(-1)^2+2^2} = \sqrt{9+1+4} = \sqrt{14}$$

Let θ be the angle between $\vec{a}$ & $\vec{b}$

$$\therefore \cos\theta = \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|}$$

$$= \frac{4}{\sqrt{3}.\sqrt{14}} = \frac{4}{\sqrt{42}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{4}{\sqrt{42}}\right)$$

2. Show that the vectors $-3\vec{i}+2\vec{j}-\vec{k}$, $\vec{i}-3\vec{j}+5\vec{k}$ and $2\vec{i}+\vec{j}-4\vec{k}$ form a right angled triangle, using scalar product.

Solution:

Let the sides of the triangle be

$\vec{a} = -3\vec{i}+2\vec{j}-\vec{k}$

$\vec{b} = \vec{i}-3\vec{j}+5\vec{k}$

$\vec{c} = 2\vec{i}+\vec{j}-4\vec{k}$

Now, $\vec{a}\cdot\vec{b} = (-3\vec{i}+2\vec{j}-\vec{k})\cdot(\vec{i}-3\vec{j}+5\vec{k})$

$= -3(1)+(+2)(-3)+(-1)(5)$

$= -3-6-5 = -14$

$\vec{b}\cdot\vec{c} = (\vec{i}-3\vec{j}+5\vec{k})\cdot(2\vec{i}+\vec{j}-4\vec{k})$

$= 1(2)+(-3)1+5(-4)$

$= 2-3-20 = -21$

$\vec{c}\cdot\vec{a} = (2\vec{i}+\vec{j}-4\vec{k})\cdot(-3\vec{i}+2\vec{j}-\vec{k})$

$= 2(-3)+1(+2)+(-4)(1)$

$= -6+2+4 = 0$

Now, $\vec{c}\cdot\vec{a} = 0$ and

$\vec{a}+\vec{c} = (-3\vec{i}+2\vec{j}-\vec{k})+(2\vec{i}+\vec{j}-4\vec{k})$

$= -\vec{i}+3\vec{j}-5\vec{k}$

$= -\vec{b}$

$\therefore$ The sides $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a triangle and the angle between $\vec{a}$ and $\vec{c}$ is 90^o hence right angled triangle.

3. Prove that the vectors $2\vec{i}-2\vec{j}+\vec{k}$, $\vec{i}+2\vec{j}+2\vec{k}$, $2\vec{i}+\vec{j}-2\vec{k}$ are perpendicular to each other. (one another).

Solution:

$\vec{a} = 2\vec{i}-2\vec{j}+\vec{k}$

$\vec{b} = \vec{i}+2\vec{j}+2\vec{k}$

$\vec{c} = 2\vec{i}+\vec{j}-2\vec{k}$

Now, $\vec{a}\cdot\vec{b} = (2\vec{i}-2\vec{j}+\vec{k})\cdot(\vec{i}+2\vec{j}+2\vec{k})$

$= 2(1)+(-2)2+1(2) = 2-4+2 = 0$

$\therefore \vec{a}$ is perpendicular to $\vec{b}$

$\vec{b}\cdot\vec{c} = (\vec{i}+2\vec{j}+2\vec{k})\cdot(2\vec{i}+\vec{j}-2\vec{k})$

$= 1(2)+2(1)+2(-2) = 2+2-4 = 0$

$\therefore \vec{b}$ is perpendicular to $\vec{c}$

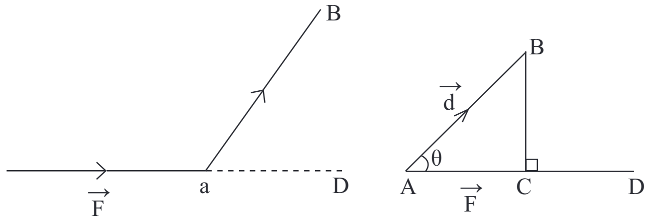
$\vec{c}\cdot\vec{a} = (2\vec{i}+\vec{j}-2\vec{k})\cdot(2\vec{i}-2\vec{j}+\vec{k})$

$= 2(2)+1(-2)+(-2).1 = 4-2-2 = 0$

$\therefore \vec{c}$ is perpendicular to $\vec{a}$

$\therefore$ The three vectors are perpendicular to one another.

2.3 APPLICATION OF SCALAR PRODUCT



A Force $\vec{F}$ acting on a particle, displaces that particle from the point A to the point B. Hence the vector $\overrightarrow{AB}$ is called the displacement vector $\vec{d}$ of the particle due to the force $\vec{F}$.

The force $\vec{F}$ acting on the particle does work when the particle is displaced in the direction which is not perpendiocular to the force $\vec{F}$. The workdone is a scalar quantity proportional to the force and the resolved part of the displacement in the direction of the force. We choose the unit quantity of the work as the work done when a particle, acted on by unit force, is displaced unit distance in the direction of the force.

Hence, if $\vec{F}$, $\vec{d}$ are the vectors representing the force and the displacement respectively, inclined at an angle θ, the measure of workdone is

$|\vec{F}||\vec{d}|\cos\theta = \vec{F}\cdot\vec{d}$

i.e, workdone, $W = \vec{F}\cdot\vec{d}$

WORKED EXAMPLES

PART – A

1. $3\vec{i} + 5\vec{j} + 7\vec{k}$ is the force acting on a particle giving the displacement $2\vec{i} - \vec{j} + \vec{k}$. Find the Workdone.

Solution:

The force $\vec{F} = 3\vec{i} + 5\vec{j} + 7\vec{k}$

Displacement $\vec{d} = 2\vec{i} - \vec{j} + \vec{k}$

$\therefore$ Workdone, $W = \vec{F}\cdot\vec{d} = (3\vec{i} + 5\vec{j} + 7\vec{k})\cdot(2\vec{i} - \vec{j} + \vec{k})$

$= 3(2) + 5(-1) + 7(1) = 6 - 5 + 7 = 8$

PART – B

1. A particle moves from the point (1, – 2, 5) to the point (3, 4, 6) due to the force $4\vec{i} + \vec{j} - 3\vec{k}$ acting on it. Find the workdone.

Solution:

The force $\vec{F} = 4\vec{i} + \vec{j} - 3\vec{k}$

The particle moves from A (1, – 2, 5) to B (3, 4, 6).

$\therefore$ Displacement vector, $\vec{d} = \overrightarrow{AB}$

$= \overrightarrow{OB} - \overrightarrow{OA}$

$= (3\vec{i} + 4\vec{j} + 6\vec{k}) - (\vec{i} - 2\vec{j} + 5\vec{k})$

$= 2\vec{i} + 6\vec{j} + \vec{k}$

$\therefore$ Workdone, $W = \vec{F}\cdot\vec{d}$

$= (4\vec{i} + \vec{j} - 3\vec{k})\cdot(2\vec{i} + 6\vec{j} + \vec{k})$

$= 4(2) + 1(6) + (-3)1 = 8 + 6 - 3 = 11$

PART – C

1. If a particle moves from $3\vec{i} - \vec{j} + \vec{k}$ to $2\vec{i} - 3\vec{j} + \vec{k}$ due to the forces $2\vec{i} + 5\vec{j} - 3\vec{k}$ and $4\vec{i} + 3\vec{j} + 2\vec{k}$. Find the workdone of the forces.

Solution:

The forces are $\vec{F_1} = 2\vec{i} + 5\vec{j} - 3\vec{k}$

$\vec{F_2} = 4\vec{i} + 3\vec{j} + 2\vec{k}$

$\therefore$ Total force, $\vec{F} = \vec{F_1} + \vec{F_2}$

$= (2\vec{i} + 5\vec{j} - 3\vec{k}) + (4\vec{i} + 3\vec{j} + 2\vec{k}) = 6\vec{i} + 8\vec{j} - \vec{k}$

The particle moves from $\vec{OA} = 3\vec{i} - \vec{j} + \vec{k}$ to $\vec{OB} = 2\vec{i} - 3\vec{j} + \vec{k}$

$\therefore$ The displacement vector,

$\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = (2\vec{i} - 3\vec{j} + \vec{k}) - (3\vec{i} - \vec{j} + \vec{k}) = -\vec{i} - 2\vec{j}$

$\therefore$ Workdone, $W = \vec{F} \cdot \vec{d}$

$= (6\vec{i} + 8\vec{j} - \vec{k})\ (-\vec{i} - 2\vec{j})$

$6(-1) + 8(-2) - 1(0) = -6 - 16 + 0 = -22$

EXERCISE

PART – A

1. If A and B are two points whose position vectors are $\vec{i} - 2\vec{j} + 2\vec{k}$ and $3\vec{i} + 5\vec{j} - 7\vec{k}$ respectively find $\vec{AB}$.
2. If $\vec{OA} = \vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{OB} = 2\vec{i} - 3\vec{j} + \vec{k}$, find $|\vec{AB}|$.
3. A and B are (3, 2, –1) and (7, 5, 2). Find $|\vec{AB}|$.
4. Find the unit vector along $2\vec{i} - \vec{j} + 4\vec{k}$.
5. Find the unit vector along $\vec{i} + 2\vec{j} - 3\vec{k}$.
6. Find the direction cosines of the vector $2\vec{i} - 3\vec{j} + 4\vec{k}$.
7. Find the modulus and direction cosines of the vector $4\vec{i} - 3\vec{j} + \vec{k}$.
8. Find the direction cosines and direction ratios of the vector $\vec{i} - 2\vec{j} + 3\vec{k}$.
9. Find the scalar product of the vectors.

   (i) $3\vec{i} + 4\vec{j} - 5\vec{k}$ and $2\vec{i} + \vec{j} + \vec{k}$

   (ii) $\vec{i} - \vec{j} + \vec{k}$ and $-2\vec{i} + 3\vec{j} - 5\vec{k}$

   (iii) $\vec{i} + \vec{j}$ and $\vec{k} + \vec{i}$

   (iv) $\vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{i} - 2\vec{j} + \vec{k}$

10. Prove that the two vectors are perpendicular to each other.

   (i) $3\vec{i} - \vec{j} + 5\vec{k}$ and $-\vec{i} + 2\vec{j} + \vec{k}$

   (ii) $8\vec{i} + 7\vec{j} - \vec{k}$ and $3\vec{i} - 3\vec{j} + 3\vec{k}$

   (iii) $\vec{i} - 3\vec{j} + 5\vec{k}$ and $-2\vec{i} + 6\vec{j} + 4\vec{k}$

   (iv) $2\vec{i} + 3\vec{j} + \vec{k}$ and $4\vec{i} - 2\vec{j} - 2\vec{k}$

11. If the two vectors are perpendicular, find the value of p.

    (i) $p\vec{i} + 3\vec{j} + 4\vec{k}$ and $2\vec{i} + 2\vec{j} - 5\vec{k}$

    (ii) $p\vec{i} + 2\vec{j} + 3\vec{k}$ and $-\vec{i} + 3\vec{j} - 4\vec{k}$

    (iii) $2\vec{i} + p\vec{j} - \vec{k}$ and $3\vec{i} - 4\vec{j} + \vec{k}$

    (iv) $\vec{i} + 2\vec{j} - \vec{k}$ and $p\vec{i} + \vec{j}$

    (v) $\vec{i} - 2\vec{j} - 4\vec{k}$ and $2\vec{i} - p\vec{j} + 3\vec{k}$

12. Define the scalar product of two vectors $\vec{a}$ and $\vec{b}$.
13. Write down the condition for two vectors to be perpendicular.
14. Write down the formula for the projection of $\vec{a}$ on $\vec{b}$.
15. If a force $\vec{F}$ acts on a particle giving the displacement $\vec{d}$, write down the formula for the workdone by the force.

PART – B

1. The position vectors of A and B are $\vec{i} + 3\vec{j} - 4\vec{k}$ and $2\vec{i} + \vec{j} - 5\vec{k}$ find unit vector along $\overrightarrow{AB}$.
2. If $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - 4\vec{k}$ and $\overrightarrow{OB} = \vec{i} + \vec{j} - 2\vec{k}$ find the direction cosines of the vector $\overrightarrow{AB}$.
3. If A is (2, 3, –1) and B is (4, 0, 7) find the direction ratios of $\overrightarrow{AB}$.
4. If the vectors $\vec{i} + 2\vec{j} + \vec{k}$ and $-2\vec{i} + k\vec{j} - 2\vec{k}$ are collinear, find the value of k.
5. Find the projection of

    (i) $2\vec{i} + \vec{j} - 2\vec{k}$ on $\vec{i} - 2\vec{j} - 2\vec{k}$

    (ii) $3\vec{i} + 4\vec{j} + 12\vec{k}$ on $\vec{i} + 2\vec{j} + 2\vec{k}$

    (iii) $\vec{j} + \vec{k}$ on $\vec{i} + \vec{j}$

    (iv) $8\vec{i} + 3\vec{j} + 2\vec{k}$ on $\vec{i} + \vec{j} + \vec{k}$

PART – C

1. Prove that the triangle having the following position vectors of the vertices form an equilateral triangle:

    (i) $4\vec{i} + 2\vec{j} + 3\vec{k},\ 2\vec{i} + 3\vec{j} + 4\vec{k},\ 3\vec{i} + 4\vec{j} + 2\vec{k}$

    (ii) $3\vec{i} + \vec{j} + 2\vec{k},\ \vec{i} + 2\vec{j} + 3\vec{k},\ 2\vec{i} + 3\vec{j} + \vec{k}$

    (iii) $2\vec{i} + 3\vec{j} + 4\vec{k},\ 5\vec{i} + 2\vec{j} + 3\vec{k},\ 3\vec{i} + 5\vec{j} + 2\vec{k}$

2. Prove that the triangles whose vertices have following position vectors form an isoceles triangles.

    (i) $3\vec{i} - \vec{j} - 2\vec{k},\ 5\vec{i} + \vec{j} - 3\vec{k},\ 6\vec{i} - \vec{j} - \vec{k}$

    (ii) $-7\vec{i} - 10\vec{k},\ 4\vec{i} - 9\vec{j} - 6\vec{k},\ \vec{i} - 6\vec{j} - 6\vec{k}$

    (iii) $7\vec{i} + 10\vec{k},\ 3\vec{i} - 4\vec{j} + 6\vec{k},\ 9\vec{i} - 4\vec{j} + 6\vec{k}$

3. Prove that the following position vectors of the vertices of a triangle form a right angled triangle.

   (i) $3\vec{i}+\vec{j}-5\vec{k},\ 4\vec{i}+3\vec{j}-7\vec{k},\ 5\vec{i}+2\vec{j}-3\vec{k}$

   (ii) $2\vec{i}-\vec{j}+\vec{k},\ 3\vec{i}-4\vec{j}-4\vec{k},\ \vec{i}-3\vec{j}-5\vec{k}$

   (iii) $2\vec{i}+4\vec{j}+3\vec{k},\ 4\vec{i}+\vec{j}-4\vec{k},\ 6\vec{i}+5\vec{j}-\vec{k}$

4. Prove that the following vectors are collinear.

   (i) $2\vec{i}+\vec{j}-\vec{k},\ 4\vec{i}+3\vec{j}-5\vec{k},\ \vec{i}+\vec{k}$

   (ii) $\vec{i}+2\vec{j}+4\vec{k},\ 4\vec{i}+8\vec{j}+\vec{k},\ 3\vec{i}+6\vec{j}+2\vec{k}$

   (iii) $2\vec{i}-\vec{j}+3\vec{k},\ 3\vec{i}-5\vec{j}+\vec{k},\ -\vec{i}+11\vec{j}+9\vec{k}$

5. Find the angle between the following the vectors.

   (i) $2\vec{i}-3\vec{j}+2\vec{k}$ and $-\vec{i}+\vec{j}-\vec{k}$

   (ii) $4\vec{i}+3\vec{j}+\vec{k}$ and $2\vec{i}-\vec{j}+2\vec{k}$

   (iii) $3\vec{i}+\vec{j}-\vec{k}$ and $\vec{i}-\vec{j}-2\vec{k}$

6. If the position vectors of A, B and C are $\vec{i}+2\vec{j}-\vec{k}, 2\vec{i}+3\vec{k}, 3\vec{i}-\vec{j}+2\vec{k}$, find the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

7. Show that the following position vectors of the points form a right angled triangle.

   (i) $3\vec{i}-2\vec{j}+\vec{k},\ \vec{i}-3\vec{j}+4\vec{k},\ 2\vec{i}+\vec{j}-4\vec{k}$

   (ii) $2\vec{i}+4\vec{j}-\vec{k},\ 4\vec{i}+5\vec{j}-\vec{k},\ 3\vec{i}+6\vec{j}-3\vec{k}$

   (iii) $3\vec{i}-2\vec{j}+\vec{k},\ \vec{i}-3\vec{j}+5\vec{k},\ 2\vec{i}+\vec{j}-4\vec{k}$

8. Due to the force $2\vec{i}-3\vec{j}+\vec{k}$, a particle is displaced from the point $\vec{i}+2\vec{j}+3\vec{k}$ to $-2\vec{i}+4\vec{j}+\vec{k}$. Find the work done.

9. A particle is displaced from A (3, 0, 2) to B (– 6, – 1, 3) due to the force $\vec{F}=15\vec{i}+10\vec{j}+15\vec{k}$. Find the work done.

10. $\vec{F}=2\vec{i}-3\vec{j}+4\vec{k}$ displaces a particle from origin to (1, 2, –1). Find the workdone of the force.

11. Two forces $4\vec{i}+\vec{j}-3\vec{k}$ and $3\vec{i}+\vec{j}-\vec{k}$ displaces a particle from the point (1, 2, 3) to (5, 4, 1). Find the work done.

12. A particle is moved from $5\vec{i}-5\vec{j}-7\vec{k}$ to $6\vec{i}+2\vec{j}-2\vec{k}$ due to the three forces $10\vec{i}-\vec{j}+11\vec{k},\ 4\vec{i}+5\vec{j}+6\vec{k}$ and $-2\vec{i}+\vec{j}-9\vec{k}$. Find the workdone.

13. When a particle is moved from the point (1, 1, 1) to (2, 1, 3) by a force $\vec{i}+\vec{j}+\vec{k}$, the work done is 4. FInd the value of A.

14. A force $2\vec{i}+\vec{j}+\lambda\vec{k}$ displaces a particle from the point (1, 1, 1) to (2, 2, 2) giving the workdone 5. Find the value of λ.

15. Find the value of p, if a force $2\vec{i}-3\vec{j}+4\vec{k}$ displaces a particle from (1, p, 3) to (2, 0, 5) giving the work done 17.

ANSWERS

PART - A

1) $2\vec{i}+7\vec{j}-9\vec{k}$ 2) $\sqrt{42}$ 3) $\sqrt{34}$ 4) $\dfrac{2\vec{i}-\vec{j}+4\vec{k}}{\sqrt{21}}$ 5) $\dfrac{\vec{i}+2\vec{j}-3\vec{k}}{\sqrt{14}}$

6) $\dfrac{2}{\sqrt{29}},\dfrac{-3}{\sqrt{29}},\dfrac{4}{\sqrt{29}}$ 7) $\sqrt{26},\dfrac{4}{\sqrt{26}},\dfrac{-3}{\sqrt{26}},\dfrac{1}{\sqrt{26}}$ 8) $\dfrac{1}{\sqrt{14}},\dfrac{-2}{\sqrt{14}},\dfrac{3}{\sqrt{14}}$, 1, –2, 3

9) i) 5 ii) –10 iii) 1 iv) –6 11) i) p = 7 ii) p = –6 iii) p = $5/4$ iv) p = –2, v) p = 5

12) $|\vec{a}||\vec{b}|\cos\theta$ 13) $\vec{a}\cdot\vec{b}=0$ 14) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ 15) $\vec{f}\cdot\vec{d}$

PART - B

1) $\dfrac{\vec{i}-2\vec{j}-\vec{k}}{\sqrt{6}}$ 2) $-\dfrac{1}{3},\dfrac{-2}{3},\dfrac{2}{3}$ 3) 2, –3, 8 4) K = –4,

5) i) $\dfrac{4}{3}$ ii) $\dfrac{35}{3}$ iii) $\dfrac{1}{\sqrt{2}}$ iv) $\dfrac{13}{\sqrt{3}}$

PART - C

5) i) $\mathrm{Cos}^{-1}\left(\dfrac{-7}{\sqrt{51}}\right)$ ii) $\mathrm{Cos}^{-1}\left(\dfrac{-7}{\sqrt{234}}\right)$ iii) $\mathrm{Cos}^{-1}\left(\dfrac{4}{\sqrt{16}}\right)$ 6) $\mathrm{Cos}^{-1}\left(\dfrac{20}{\sqrt{462}}\right)$

9) 14 10) 130 11) 8 12) 40 13) 37 14) $\lambda = 2$ 15) $\lambda = 2$ 16) p = $\dfrac{7}{3}$

UNIT – III

VECTOR ALGEBRA - III

3.1 VECTOR PRODUCT OF TWO VECTORS

Definition of vector product of two vectors. Geometrical meaning. Properties – Angle between two vectors – unit vector perpendicular to two vectors. Simple problems.

3.2 APPLICATION OF VECTOR PRODUCT OF TWO VECTORS & SCALAR TRIPLE PRODUCT

Definition of moment of a force. Definition of scalar product of three vectors – Geometrical meaning – Coplanar vectors. Simple problems.

3.3 VECTOR TRIPLE PRODUCT & PRODUCT OF MORE VECTORS

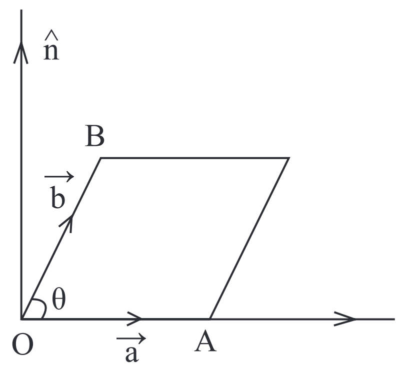
Definition of Vector Triple product, Scalar and Vector product of four vectors Simple Problems.

3.1 VECTOR PRODUCT OF TWO VECTORS

Definition:

Let $\vec{a}$ and $\vec{b}$ be two vectors and 'θ' be the angle between them. The vector product of $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \times \vec{b}$ and it is defined as a vector $|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{n}$ are in right handed system. Thus

$$\boxed{\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}}$$



The vector product is called as cross product.

Geometrical meaning of vector product

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and 'θ' be the angle between $\vec{a}$ and $\vec{b}$. Complete the parallelogram OACB with $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as its adjacent sides.

Draw BN ⊥OA.

From the right angled Δ ONB.

$$\frac{BN}{OB} = \sin\theta \Rightarrow BN = OB.\sin\theta$$

$$\Rightarrow BN = |\vec{b}|\sin\theta$$

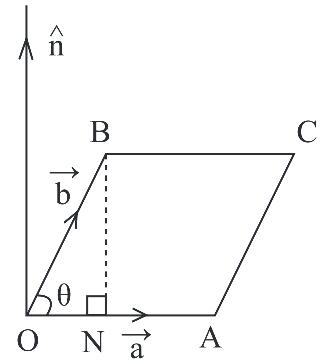
By definition, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$$

$= OA \times BN$

$=$ base $\times$ height

$=$ Area of parallelogram OACB.



$\therefore\ |\vec{a} \times \vec{b}| =$ Area of parallelogram with $|\vec{a}|$ and $|\vec{b}|$ as adjacent sides.

Also, area of Δ OAB $= \dfrac{1}{2}$ (area of parallelogram OACB)

$$= \frac{1}{2}\,|\overrightarrow{OA} \times \overrightarrow{OB}| = \frac{1}{2}\,|\vec{a} \times \vec{b}|$$

Results :

1. The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $A = |\vec{a} \times \vec{b}|$.
2. The vector area of parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$
3. The area of a triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is $A = \dfrac{1}{2}\,|\vec{a} \times \vec{b}|$
4. The area of triangle ABC is given by either $\dfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$ or $\dfrac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}|$ or $\dfrac{1}{2}|\overrightarrow{CA} \times \overrightarrow{CB}|$.

Properties of vector product

1. Vector product is not commutative.

If $\vec{a}$ and $\vec{b}$ are any two vectors then $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ however $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

2. Vector product of collinear or parallel vectors:

If $\vec{a}$ and $\vec{b}$ are collinear or parallel then $\theta = 0, \pi$

For $\theta = 0, \pi$ then $\sin\theta = 0$

$$\therefore\ \vec{a} \times \vec{b} = |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n} = \vec{0}$$

$$\Rightarrow\ \vec{a} \times \vec{b} = \vec{0}$$

Thus, $\vec{a}$ and $\vec{b}$ are collinear or parallel $\Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$

Note : If $\vec{a} \times \vec{b} = \vec{0}$ then

(i) $\vec{a}$ is a zero vector and $\vec{b}$ is any vector.

(ii) $\vec{b}$ is a zero vector and $\vec{a}$ is any vector.

(iii) $\vec{a}$ and $\vec{b}$ are parallel (collinear)

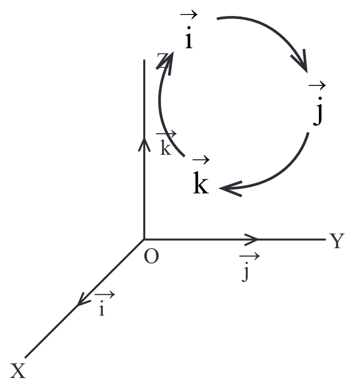
3. Cross product of equal vectors

$\vec{a} \times \vec{a} \;= |\vec{a}|\,|\vec{a}|\sin o\ \hat{n}$

$= |\vec{a}|\,|\vec{a}|\,(0)\ \hat{n}$

$= \vec{0}$

$\therefore\ \vec{a} \times \vec{a} = \vec{0}$ for every non zero vector $\vec{a}$.



4) Cross product of unit vectors $\vec{i}, \vec{j}, \vec{k}$

Let $\vec{i}, \vec{j}, \vec{k}$ be the three mutually perpendicular unit vectors. The involvement of these three unit vectors in vector product as follows : By property(3), $\vec{i} \times \vec{i} = \vec{0}$, $\vec{j} \times \vec{j} = \vec{0}$, $\vec{k} \times \vec{k} = \vec{0}$

also $\vec{i} \times \vec{j} = |\vec{i}\,\|\,\vec{j}|\ \sin 90^0\ \vec{k} = 1.1.1.\vec{k} = \vec{k}$ similarly, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and $\vec{j} \times \vec{i} = -\vec{k},\ \vec{k} \times \vec{j} = -\vec{i},\ \vec{i} \times \vec{k} = -\vec{j}$ etc. This can be shoron as follows.

x	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{i}$
$\vec{i}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{i}$	$-\vec{i}$	$\vec{0}$

5. If m is any scalar and $\vec{a}, \vec{b}$ are two vectors inclimed at an angle ‘θ’ then m $\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$

6. Distributing of vector product nor vector addition :

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors then

(i) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ [Left distributivity]

(ii) $(\vec{b} + \vec{c}) \times \vec{a} = (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{a})$ [Right distributivity]

7. Vector product in determinant from

Let $\vec{a} = a_1\vec{i} + a_2\vec{i} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{i} + b_3\vec{k}$ then $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{i}\ (a_2b_3 - b_2a_3) - \vec{i}\ (a_1b_3 - b_1a_3) + \vec{k}\ (a_1b_2 - b_1a_2)$

8. Angle between two vectors

Let $\vec{a}$, $\vec{b}$ be two vectors inclined at an angle 'θ' then,

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

$$\Rightarrow \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \theta \sin^{-1}\left[\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right]$$

9. Unit vector perpendicular to two vectors

Let $\vec{a}$, $\vec{b}$ be two non zero, non parallel vectors and 'θ' be the angle between them

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n} \text{ ———— } (1)$$

also, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ ———— (2)

$$\therefore (1) \div (2) \Rightarrow \frac{|\vec{a}||\vec{b}|\sin\theta\,\hat{n}}{|\vec{a}||\vec{b}|\sin\theta} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$\Rightarrow \hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Note that, $-\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is also a limit vector

perpendicular to $\vec{a}$ and $\vec{b}$.

$\therefore$Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ are $\pm\,\hat{n} = \pm\,\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

WORKED EXAMPLES

PART - A

1. Prove that $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2(\vec{b} \times \vec{a})$

Solution :

L.H.S. : $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$

$= \vec{a} \times \vec{a} \times - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$= \vec{0} + \vec{b} \times \vec{a} + \vec{b} \times \vec{a} - \vec{0}$

$= 2(\vec{b} \times \vec{a}) =$ R.H.S.

2. Prove that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.

Solution : L.H.S: $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$

$= \vec{a} \times \vec{b} - \vec{c} \times \vec{a} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} + \vec{c} \times \vec{a} - \vec{b} \times \vec{c}$

$= \vec{0}$ R.H.S.

3. If $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$ find $\vec{a} \times \vec{b}$.

Solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= \vec{i}(-3-2) - \vec{j}(6-1) + \vec{k}(4+1)$$

$$= \vec{i}(-5) - \vec{j}(5) + \vec{k}(5)$$

$$\Rightarrow \vec{a} \times \vec{b} = -5\vec{i} - 5\vec{j} + 5\vec{k}$$

4. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $|\vec{a} \times \vec{b}| = 7$ find the angle between $\vec{a}$ and $\vec{b}$.

Solution :

We have, $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \dfrac{7}{2 \times 7}$

$\Rightarrow \sin\theta = \dfrac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\dfrac{1}{2}\right) = 30^0$

PART - B

1. Find the area of the parallelogram whose adjacent sides are $\vec{i} + \vec{j} + \vec{k}$ and $3\vec{i} - \vec{k}$.

Solution :

Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ & $\vec{b} = 3\vec{i} - \vec{k}$

Formula :

The area of parallelogram from whose adjacent sides are

$\vec{a}$ & $\vec{b}$ is $A = |\vec{a} \times \vec{b}|$ square units. ———— (1)

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 3 & 0 & -1 \end{vmatrix}$

$$= \vec{i}\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= \vec{i}\,(-1-0) - \vec{j}\,(-1-3) + \vec{k}\,(0-3)$$

$$\Rightarrow \vec{a} \times \vec{b} = -\vec{i} + 4\vec{j} - 3\vec{k}$$

also, $\quad |\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + (4)^2 + (-3)^2} = \sqrt{1+16+9}$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{26}$$

area of parallelogram is $|\vec{a} \times \vec{b}| = \sqrt{26}$ square units.

2. Find the area of the triangle whose adjacent sides are $2\vec{i} - \vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} - \vec{k}$

Solution :

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$ & $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$

Formula : The area of the triangle whose adjacent sides are $\vec{a}$ & $\vec{b}$ is

$$A = \frac{1}{2}|\vec{a} \times \vec{b}| \quad \text{square units} \text{ ———— } (1)$$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 1 \\ 4 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= \vec{i}\,(1-4) - \vec{j}\,(-2-3) + \vec{k}\,(8+3)$$

$$= \vec{i}\,(-3) - \vec{j}\,(-5) + \vec{k}\,(11)$$

$$\Rightarrow \vec{a} \times \vec{b} = -3\vec{i} + 5\vec{j} + 11\vec{k}$$

also, $\left|\vec{a} \times \vec{b}\right| = \sqrt{(-3^2) + (5)^2 + (11)^2} = \sqrt{9+25+121}$

$$\Rightarrow \left|\vec{a} \times \vec{b}\right| = \sqrt{155}$$

$\therefore$ (1) becomes, area of triangle is $A = \frac{1}{2}\sqrt{155}$ square units.

3. Prove that $\left(\vec{a} \times \vec{b}\right)^2 + \left(\vec{a} \cdot \vec{b}\right)^2 = |\vec{a}|^2\,|\vec{b}|^2$

Solution :

L.H.S. $\left(\vec{a} \times \vec{b}\right)^2 + \left(\vec{a} \cdot \vec{b}\right)^2$

$$= \left[|\vec{a}||\vec{b}|\sin\theta\,\hat{n}\right]^2 + \left[|\vec{a}||\vec{b}|\cos\theta\right]^2$$

$= |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta (\hat{n})^2 + |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$

$= |\vec{a}|^2 |\vec{b}|^2 \left[\sin^2\theta + \cos^2\theta\right]$ $\quad \left[\because (\hat{n})^2 = 1\right.$

$= |\vec{a}|^2 |\vec{b}|^2 [1]$

$= |\vec{a}|^2 |\vec{b}|^2 =$ R.H.S.

4. If $|\vec{a}| = 13$, $|\vec{b}| = 5$ and $\vec{a}\,.\,\vec{b} = 60$ then find $|\vec{a} \times \vec{b}|$.

Solution :

We have,

$|\vec{a} \times \vec{b}|^2 + |\vec{a}\,.\,\vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$

$\Rightarrow |\vec{a} \times \vec{b}|^2 + (60)^2 = (13)^2\,.\,(5)^2$

$\Rightarrow |\vec{a} \times \vec{b}|^2 + 3600 = 169 \times 25$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = 4225 - 3600$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = 625$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{625} = 25.$

PART - C

1. Find the unit vector perpendicular to each of the vectors $\vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$. Also find the sine of the angle between these two vectors.

Solution :

Given : $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ & $\vec{b} = \vec{i} - \vec{j} - \vec{k}$

Formula : (i) Unit vector : $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

(ii) Sine of the angle : $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-2+3) - \vec{j}\,(-1-3) + \vec{k}\,(-1-2)$$

$$= \vec{i}\,(1) - \vec{j}\,(-4) + \vec{k}\,(-3)$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{i} + 4\vec{j} - 3\vec{k}$$

also, $|\vec{a} \times \vec{b}| = \sqrt{(1)^2 + (4)^2 + (-3)^2} = \sqrt{1+16+9}$

$\Rightarrow |\overrightarrow{a} \times \overrightarrow{b}| = \sqrt{26}$

Again, $|\overrightarrow{a}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}$

& $|\overrightarrow{b}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}$

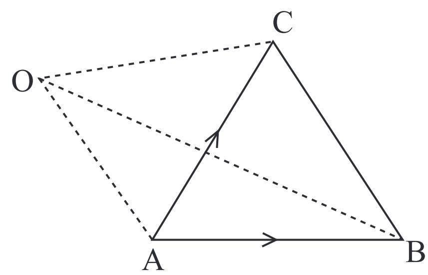
Hence,

(i) Unit vector : $\hat{n} = \dfrac{\vec{i} + 4\vec{j} - 3\vec{k}}{\sqrt{26}}$

(ii) $\sin\theta = \dfrac{\sqrt{26}}{\sqrt{14}\sqrt{3}} \Rightarrow \theta = \sin^{-1}\left[\dfrac{\sqrt{26}}{\sqrt{14}\sqrt{3}}\right]$

2. Find the area of the triangle formed by the points whose position vectors are $2\vec{i} + 3\vec{j} + 4\vec{k}$, $3\vec{i} + 4\vec{j} + 2\vec{k}$, $4\vec{i} + 2\vec{j} + 3\vec{k}$.

Solution :



Let $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

$\overrightarrow{OB} = 3\vec{i} + 4\vec{j} + 2\vec{k}$

& $\overrightarrow{OC} = 4\vec{i} + 2\vec{j} + 3\vec{k}$

be the given position vectors of the vertices of ΔABC.

Formula : Area of triangle ABC = $\dfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ ———— (1)

Now,

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (3\vec{i} + 4\vec{j} + 2\vec{k}) - (2\vec{i} + 3\vec{j} + 4\vec{k})$

$= 3\vec{i} + 4\vec{j} + 2\vec{k} - 2\vec{i} - 3\vec{j} - 4\vec{k}$

$\Rightarrow \overrightarrow{AB} = \overrightarrow{i} + \overrightarrow{j} - 2\overrightarrow{k}$

also, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (4\vec{i} + 2\vec{j} + 3\vec{k}) - (2\vec{i} + 3\vec{j} + 4\vec{k})$

$= 4\vec{i} + 2\vec{j} + 3\vec{k} - 2\vec{i} - 3\vec{j} - 4\vec{k}$

$\Rightarrow \overrightarrow{AC} = 2\overrightarrow{i} - \overrightarrow{j} - \overrightarrow{k}$

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= \overrightarrow{i}\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} \overrightarrow{j}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + \overrightarrow{k}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= \overrightarrow{i}\,(-1-2)\ \overrightarrow{j}\,(-1+4) + \overrightarrow{k}\,(-1-2)$$

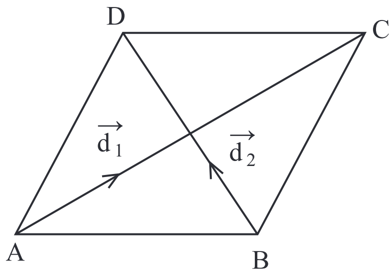
$$= \overrightarrow{i}\,(-3)\ \overrightarrow{j}\,(+3) + \overrightarrow{k}\,(-3)$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = -3\,\overrightarrow{i} - 3\,\overrightarrow{j} - 3\,\overrightarrow{k}$$

$$\text{and } |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-3)^2 + (-3)^2 + (-3)^2}$$

$$= \sqrt{9+9+9} = \sqrt{27}$$

$$\Rightarrow \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 3\sqrt{3}$$



Area of Δ ABC $= \dfrac{1}{2}(3\sqrt{3}) = \dfrac{3\sqrt{3}}{2}$ square units.

3. Find the area of the parallelogram whose diagonals are represented by $3\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{i} - 3\vec{j} + 4\vec{k}$.

Solution :

Let $\overrightarrow{d_1} = 3\vec{i} + \vec{j} - 2\vec{k}$ & $\overrightarrow{d_2} = \vec{i} - 3\vec{j} + 4\vec{k}$

be the given diagonals of parallelogram ABCD.

Formula : Area of parallelogram $= \dfrac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right|$ square units. ———— (1)

Now,

$$\overrightarrow{d_1} \times \overrightarrow{d_2} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= \overrightarrow{i}\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} - \overrightarrow{j}\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} + \overrightarrow{k}\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}$$

$$= \overrightarrow{i}\,(4-6) - \overrightarrow{j}\,(12+2) + \overrightarrow{k}\,(-9-1)$$

$$= \overrightarrow{i}\,(-2) - \overrightarrow{j}\,(14) + \overrightarrow{k}\,(-10)$$

$$\Rightarrow \overrightarrow{d_1} \times \overrightarrow{d_2} = -2\,\overrightarrow{i} - 14\,\overrightarrow{j} - 10\,\overrightarrow{k}$$

also, $\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2} = \sqrt{4+196+100} = \sqrt{300}$

$\Rightarrow \left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right| = 10\sqrt{3}$

$\therefore$ (1) becomes,

Area of parallelogram = $\frac{1}{2}\left[10\sqrt{3}\right] = 5\sqrt{3}$ square units

4. If $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = -\vec{i} + 3\vec{k}$ and $\vec{a} \times \vec{b}$. Verify that $\vec{a}$ is perpendicular to $\vec{a} \times \vec{b}$ and $\vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$.

Solution :

Given : $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$ & $\vec{b} = -\vec{i} + 3\vec{k}$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 3 & -2 \\ 0 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}$$

$$= \vec{i}\,(9+0) - \vec{j}\,(3-2) + \vec{k}\,(0+3)$$

$$= \vec{i}\,(9) - \vec{j}\,(1) + \vec{k}\,(3)$$

$$\Rightarrow \vec{a} \times \vec{b} = 9\vec{i} - \vec{j} + 3\vec{k}$$

(i) <u>To show $\vec{a} \perp (\vec{a} \times \vec{b})$:</u>

$$\vec{a}\,.(\vec{a} \times \vec{b}) = (\vec{i} + 3\vec{j} - 2\vec{k})\,.\,(9\vec{i} - \vec{j} + 3\vec{k})$$

$$= (1 \times 9) + (3 \times -1) + (-2 \times 3)$$

$$= 9 - 3 - 6$$

$$\therefore \vec{a}\,.(\vec{a} \times \vec{b}) = 0 \Rightarrow \vec{a} \perp (\vec{a} \times \vec{b})$$

(ii) <u>To show $\vec{b} \perp (\vec{a} \times \vec{b})$:</u>

$$\vec{b}\,.(\vec{a} \times \vec{b}) = (\vec{i} + 3\vec{k})\,.\,(9\vec{i} - \vec{j} + 3\vec{k})$$

$$= (-1 \times 9) + (3 \times 3)$$

$$= -9 + 9$$

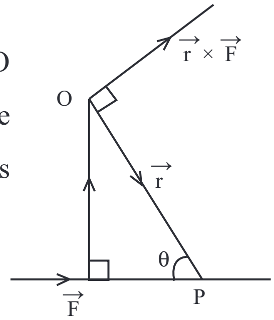
$$\therefore \vec{b}\,.(\vec{a} \times \vec{b}) = 0 \Rightarrow \vec{b} \perp (\vec{a} \times \vec{b})$$

3.2 APPLICATION OF VECTOR PRODUCT OF TWO VECTORS AND SCALAR TRIPLE PRODUCT

Definition :

Moment (or) Torque of a force about a point

Let O bc any point and $\vec{r}$ be the position vector relative to the point O of any point P on the line of action of the force $\vec{F}$. The moment of the force about the point O is defined as $\vec{M} = \vec{r} \times \vec{F}$. The magnitude of the moment is $\left|\vec{M}\right| = \left|\vec{r} \times \vec{F}\right| = \left|\vec{r}\right|\left|\vec{F}\right| \sin\theta$.



The moment of the force is also called as Torque of the force.

Definition : Scalar triple Product

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors. The scalar product of the two vectors $\vec{a} \times \vec{b}$ and $\vec{c}$. ie., $(\vec{a} \times \vec{b}) . \vec{c}$ is called the **scalar triple product** and it is denoted by $[\vec{a}, \vec{b}, \vec{c}]$. The scalar triple product is also called as box product (or) mixed product.

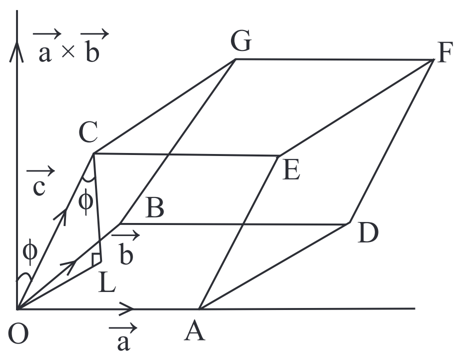
Geometrical meaning of Scalar triple product :

Let $\vec{a}, \vec{b}, \vec{c}$ be three non collinear vectors. Consider a parallelopiped having co-terminus edges OA, OB, OC so that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ & $\overrightarrow{OC} = \vec{c}$ Then $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

Let ϕ be the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$.

Draw CL perpendicular to the base OADB.

Let CL = the height of the parallelopiped. Here CL and $\vec{a} \times \vec{b}$ are perpendicular to the same plane.



$\Rightarrow CL \parallel \vec{a} \times \vec{b} \Rightarrow \angle OCL = \phi$

In right angled Δ OCL,

$$\cos\phi = \frac{CL}{OC} = \frac{CL}{\left|\vec{c}\right|} \Rightarrow CL = \left|\vec{c}\right| \cos\phi$$

Also, base area of the parallelopiped

= the area of the parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides $= |\vec{a} \times \vec{b}|$.

$\therefore$ By definition of scalar product,

$$(\vec{a} \times \vec{b}). \vec{c} = |\vec{a} \times \vec{b}|\,|\vec{c}| \cos\phi$$

= (Base area) × (height)

= V, the volume of the parallelopiped with co-terminous edges $\vec{a}, \vec{b}\ \vec{c}$.

Properties of Scalar triple product :

1. Let $\vec{a}, \vec{b}, \vec{c}$ represents co-terminous edges of a parallelopiped in right handed system then its volume $V = (\vec{a} \times \vec{b}) . \vec{c}$

   Similarly $\vec{b}, \vec{c}, \vec{a}$ and $\vec{c}, \vec{a}, \vec{b}$ are co-terminous edges of the same parallelopiped in right handed system then

   $V = (\vec{b} \times \vec{c}) . \vec{a}$ & $V = (\vec{c} \times \vec{a}) . \vec{b}$

Hence, $V = (\vec{a} \times \vec{c}) . \vec{c} = (\vec{b} \times \vec{c}) . \vec{a} = (\vec{c} \times \vec{a}) . \vec{b}$ ——— (1)

Since the scalar product is commutative then (1) becomes,

$V = \vec{c} . (\vec{a} \times \vec{b}) = \vec{a} .(\vec{b} \times \vec{c}) = \vec{b} .(\vec{c} \times \vec{a})$ ———— (2)

From (1) & (2),

$$(\vec{a} \times \vec{b}) . \vec{c} = \vec{a} . (\vec{b} \times \vec{c})$$
$$(\vec{b} \times \vec{c}) . \vec{a} = \vec{b} . (\vec{c} \times \vec{a})$$
$$(\vec{c} \times \vec{a}) . \vec{b} = \vec{c} . (\vec{a} \times \overline{b})$$

In scalar triple product the dot and cross are inter changeable. Due to this property,

$$(\vec{a} \times \vec{b}) . \vec{c} = \vec{a} . (\vec{b} \times \vec{c}) = [\vec{a} \ \vec{b} \ \vec{c}]$$
$$\therefore [\vec{a} \ \vec{b} \ \vec{c}] = [\vec{b} \ \vec{c} \ \vec{a}] = [\vec{c} \ \vec{a} \ \vec{b}]$$

2) The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude.

   $[\vec{a} \ \vec{b} \ \vec{c}] = -[\vec{b} \ \vec{a} \ \vec{c}] = [\vec{c} \ \vec{b} \ \vec{a}]$

3) The scalar triple product is zero if any two of the vectors are equal.

   i.e. $[\vec{a} \ \vec{a} \ \vec{b}] = 0$; $[\vec{a} \ \vec{b} \ \vec{a}] = 0$; $[\vec{b} \ \vec{b} \ \vec{c}] = 0$ etc.

4) For any three vectors $\vec{a}, \vec{b}, \vec{c}$ and the scalar λ then $[\lambda\vec{a} \ \vec{b} \ \vec{c}] = [\vec{a} \ \vec{b} \ \vec{c}]$

5) The scalar product of three vectors is zero if, any two of them are parallel (or) collinear.

6) Coplaner Vectors : The necessary and sufficient condition for three non-zero non-collinear vectors $\vec{a} \ \vec{b} \ \vec{c}$ to be coplanar is $[\vec{a} \ \vec{b} \ \vec{c}] = 0$

   i.e., $\vec{a} \ \vec{b} \ \vec{c}$ are coplanar $= [\vec{a} \ \vec{b} \ \vec{c}] = 0$.

Note : If $[\vec{a} \ \vec{b} \ \vec{c}] = 0$ then

(i) Atleast one of the vectors $\vec{a}, \vec{b}, \vec{c}$ is a zero vector.

(ii) Any two of the vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel.

(iii) The three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

7) For any three vectors $\vec{a}, \vec{b}, \vec{c}$ then $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{b} \times \vec{b}) + (\vec{a} \times \vec{c})$.

Scalar triple product in terms of Componants :

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{i} + b_3\vec{k}$ & $\vec{c} = c_1\vec{i} + c_2\vec{i} + c_3\vec{k}$ then

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof :

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Rightarrow \overline{b}\times\overline{c} = \overline{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \overline{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \overline{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot \left[\overline{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \overline{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \overline{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right]$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\Rightarrow \overline{a}\cdot\left(\overline{b}\times\overline{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \left[\overline{a}\ \overline{b}\ \overline{c}\right]$$

WORKED EXAMPLES

PART - A

1. If $\overline{F} = 2\overline{i} - 3\overline{j} + \overline{k}$ and $\overline{r} = \overline{i} + 2\overline{j} + 4\overline{k}$ find torque.

Solution :

Torque $= \overline{r}\times\overline{F}$

$$= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 1 & 2 & 4 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= \overline{i}\begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} - \overline{j}\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} + \overline{k}\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$$

$$= \vec{i}\ (2 + 12) - \vec{j}\ (1 - 8) + \vec{k}\ (-3 - 4)$$

$$= \vec{i}\ (14) - \vec{j}\ (-7) + \vec{k}\ (-7)$$

$\Rightarrow$ Torque $= 14\vec{i} + 7\vec{j} - 7\vec{k}$

2. Find the value of $[\vec{i}\ \ \vec{j}\ \ \vec{k}]$

Solution :

$$[\vec{i}\ \ \vec{j}\ \ \vec{k}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\,(1-0) = 1$$

3. Find the value of $[\vec{i}+\vec{j}\,,\ \vec{j}+\vec{k}\,,\ \vec{k}+\vec{i}\,]$

Solution :

$$[\vec{i}+\vec{j}\,,\ \vec{j}+\vec{k}\,,\ \vec{k}+\vec{i}\,] = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1\,(1-0) - 1\,(0-1)$$

$$= 1 + 1 \quad = 2$$

PART - B

1. Find the scalar triple product of the vectors $\vec{i} - 3\vec{j} + 3\vec{k}$, $2\vec{i} + \vec{j} - \vec{k}$ and $\vec{j} + \vec{k}$.

Solution :

Let $\vec{a} = \vec{i} - 3\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$ & $\vec{c} = \vec{j} + \vec{k}$

$$\therefore \qquad [\vec{a}\ \ \vec{b}\ \ \vec{c}\,] = \begin{vmatrix} 1 & -3 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1\,(1+1) + 3\,(2+0) + 3\,(2-0)$$

$$= 1\,(2) + 3\,(2) + 3\,(2)$$

$$= 2 + 6 + 6 = 14$$

2. If the edges $\vec{a} = -3\,\vec{i} + 7\,\vec{j} + 5\,\vec{k}$, $\vec{b} = -5\,\vec{i} + 7\,\vec{j} - 3\,\vec{k}$ and $\vec{c} = 7\,\vec{i} - 5\,\vec{j} - 3\,\vec{k}$ meet at a vertex, find the volume of the parallelopiped.

Solution :

Volume of parallelopiped : V = $[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]$

$$= \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$$= -3\begin{vmatrix} 7 & -3 \\ -5 & -3 \end{vmatrix} - 7\begin{vmatrix} -5 & -3 \\ 7 & -3 \end{vmatrix} + 5\begin{vmatrix} -5 & 7 \\ 7 & -5 \end{vmatrix}$$

$= -3\,(-21 - 15) - 7\,(15 + 21) + 5\,(25 - 49)$

$= -3\,(36) - 7\,(36) + 5\,(-24)$

$= 108 - 252 - 120$

$= -264$

$\therefore$ Volume, V = 264 cubic units.

3. Prove that the vectors $3\,\vec{i} + 2\,\vec{j} - 2\,\vec{k}$, $5\,\vec{i} - 3\,\vec{j} + 3\,\vec{k}$ and $5\,\vec{i} - \vec{j} + \vec{k}$ are coplanar.

Solution :

Let $\vec{a} = 3\,\vec{i} + 2\,\vec{j} - 2\,\vec{k}$

$\vec{b} = 5\,\vec{i} - 3\,\vec{j} + 3\,\vec{k}$

& $\vec{c} = 5\,\vec{i} - \vec{j} + \vec{k}$

$$\therefore [\vec{a}\ \ \vec{b}\ \ \vec{c}\,] = \begin{vmatrix} 3 & 2 & -2 \\ 5 & -3 & 3 \\ 5 & -1 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} -3 & 3 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 5 & 3 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 5 & -3 \\ 5 & -1 \end{vmatrix}$$

$= 3\,(-3 + 3) - 2\,(5 - 15) - 2\,(-5 + 15)$

$= 3\,(0) - 2\,(-10) - 2\,(10)$

$= 20 - 20 = 0$

$\therefore \vec{a}, \vec{b}, \vec{c}$ are coplanar.

4. If the three vectors $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} - 3\vec{k}$ and $3\vec{i} + m\vec{j} + 5\vec{k}$ are coplanar, find the value of m.

Solution :

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$

$\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$

& $\vec{c} = 3\vec{i} + m\vec{j} + 5\vec{k}$

Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\therefore \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & m & 5 \end{vmatrix} = 0$$

$$2\begin{vmatrix} 2 & -3 \\ m & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & m \end{vmatrix} = 0$$

$2(10 + 3m) + 1(5 + 9) + 1(m - 6) = 0$

$20 + 6m + 14 + m - 6 = 0$

$7m + 28 = 0$

$7m = -28$

$\boxed{m = -4}$

PART - C

1. Find the magnitiude of the moment about the point (1, –2, 3) of the force $2\vec{i} + 3\vec{i} + 6\vec{k}$ whose line of action passes through the origin.

Solution :

Given : $\vec{F} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

By data,

$\vec{r}$ = (Position Vector of 0) – (Position Vector of A)

= (0, 0, 0) – (1, –2, 3)

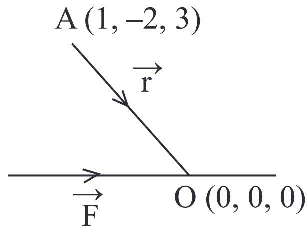
= (– 1, 2, – 3)

$\Rightarrow \quad \vec{r} = -\vec{i} + 2\vec{j} - 3\vec{k}$

$\therefore$ Moment of force,

$$\vec{M} = \vec{r} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -3 \\ 2 & 3 & 6 \end{vmatrix}$$



$$= \vec{i}\begin{vmatrix} 2 & -3 \\ 3 & 6 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & -3 \\ 2 & 6 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= \vec{i}\,(12+9) - \vec{j}\,(-6+6) + \vec{k}\,(-3-4)$$

$$= \vec{i}\,(21) - \vec{j}\,(0) + \vec{k}\,(-7)$$

$$= 21\vec{i} - 7\vec{k}$$

$$\Rightarrow \boxed{\vec{M} = 7(3\vec{i} - \vec{k})}$$

$\therefore$ Magnitude of moment,

$$\left|\vec{M}\right| = 7\sqrt{(3)^2 + (-1)^2}$$

$$= 7\sqrt{9+1}$$

$$\Rightarrow \vec{M} = 7\sqrt{10} \text{ units}$$

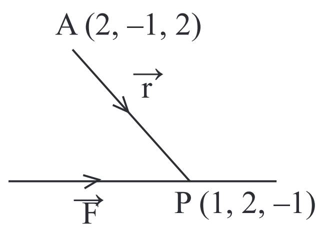
2. Find the moment of the force $3\vec{i} + \vec{k}$ acting along the point $\vec{i} + 2\vec{j} - \vec{k}$ about the point $2\vec{i} + \vec{j} - 2\vec{k}$

Solution :

Given : $\vec{F} = 3\vec{i} + \vec{k}$

$\overrightarrow{OP} = \vec{i} + 2\vec{j} - \vec{k}$

& $\overrightarrow{OA} = 2\vec{i} + \vec{j} - 2\vec{k}$



Now, $\vec{r} = \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$$= (\vec{i} + 2\vec{j} - \vec{k}) - (2\vec{i} + \vec{j} - 2\vec{k})$$

$$= \vec{i} + 2\vec{j} - \vec{k} - 2\vec{i} + \vec{j} - 2\vec{k}$$

$$\Rightarrow \boxed{\vec{r} = -\vec{i} + 3\vec{j} - 3\vec{k}}$$

Moment, $\vec{M} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & -3 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 3 & -3 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & -3 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 3 \\ 3 & 0 \end{vmatrix}$$

$$= \vec{i}\,(3+0) - \vec{j}\,(-1+9) + \vec{k}\,(0-9)$$

$$= \vec{i}\,(3) - \vec{j}\,(8) + \vec{k}\,(-9)$$

$$\Rightarrow \boxed{\vec{M} = 3\vec{i} - 8\vec{j} - 9\vec{k}}$$

$$\therefore |\vec{M}| = \sqrt{(3)^2 + (-8)^2 + (-9)^2} = \sqrt{9+64+81} \Rightarrow \boxed{\vec{M} = \sqrt{154}\text{ units}}$$

3. Prove that $[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

Solution :

L.H.S: $[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}]$

$$= (\vec{a}+\vec{b}).\left\{(\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})\right\}$$

$$= (\vec{a}+\vec{b}).\left\{\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}\right\}$$

$$= (\vec{a}+\vec{b}).\left\{\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{0} + \vec{c}\times\vec{a}\right\}$$

$$= \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a})$$

$$+ \vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{a})$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \vec{a}\right] + \left[\vec{a}\ \vec{c}\ \vec{a}\right]$$

$$+ \left[\vec{b}\ \vec{b}\ \vec{c}\right] + \left[\vec{b}\ \vec{b}\ \vec{a}\right] + \left[\vec{b}\ \vec{c}\ \vec{a}\right]$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] + 0 + 0 + 0 + 0 + \left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

$$= 2\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \text{R.H.S.}$$

4. Prove that the points given by the vectors $4\vec{i} + 5\vec{j} + \vec{k}$, $-\vec{j} - \vec{k}$, $3\vec{i} + 9\vec{j} + 4\vec{k}$ and $-4\vec{i} + 4\vec{j} + 4\vec{k}$ are coplanar.

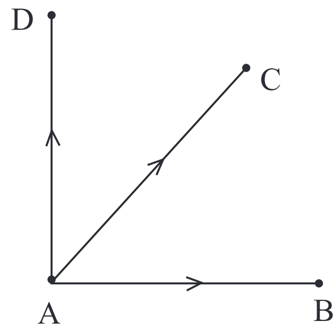
Solution :

Let $\overrightarrow{OA} = 4\vec{i} + 5\vec{j} + \vec{k}$

$\overrightarrow{OB} = -\vec{j} - \vec{k}$

$\overrightarrow{OC} = 3\vec{i} + 9\vec{j} + 4\vec{k}$

& $\overrightarrow{OD} = -4\vec{i} + 4\vec{j} + 4\vec{k}$



Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (-\vec{j} - \vec{k}) - (4\vec{i} + 5\vec{j} + \vec{k})$

$= -\vec{j} - \vec{k} - 4\vec{i} - 5\vec{j} - \vec{k}$

$\Rightarrow \boxed{\overrightarrow{AB} = -4\vec{i} - 6\vec{j} - 2\vec{k}}$

also, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (3\vec{i} + 9\vec{i} + 4\vec{k}) - (4\vec{i} + 5\vec{i} + \vec{k})$

$= 3\vec{i} + 9\vec{i} + 4\vec{k} - 4\vec{i} - 5\vec{i} - \vec{k}$

$\Rightarrow \boxed{\overrightarrow{AC} = -\vec{i} + 4\vec{j} + 3\vec{k}}$

and $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= (-4\vec{i} + 4\vec{i} + 4\vec{k}) - (4\vec{i} + 5\vec{i} + \vec{k})$

$= -4\vec{i} + 4\vec{i} + 4\vec{k} - 4\vec{i} - 5\vec{i} - \vec{k}$

$\Rightarrow \boxed{\overrightarrow{AD} = -8\vec{i} - \vec{j} + 3\vec{k}}$

$$\therefore \left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= -4\begin{vmatrix} 4 & 3 \\ -1 & 3 \end{vmatrix} + 6\begin{vmatrix} -1 & 3 \\ -8 & 3 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ -8 & -1 \end{vmatrix}$$

$= -4(12 + 3) + 6(-3 + 24) - 2(1 + 32)$

$= 4(15) + 6(21) - 2(33)$

$= -60 + 126 - 66$

$= 0$

$\therefore$ The points given by the vectors are coplanar.

3.3 VECTOR TRIPLE PRODUCT AND PRODUCT OF MORE VECTORS

Definition : Vector Triple Product

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors then the product $\vec{a}\times(\vec{b}\times\vec{c})$ & $(\vec{a}\times\vec{b})\times\vec{c}$ are called vector triple product of $\vec{a}, \vec{b}, \vec{c}$

Result :

(i) $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

(ii) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{b}\cdot\vec{c})\,\vec{a}$

(iii) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

Note : The vector triple product $(\vec{a}\times\vec{b})\times\vec{c}$ is perpendicular to $\vec{c}$ and lies in the plane which contain $\vec{a}$ and $\vec{b}$.

PRODUCT OF FOUR VECTORS

Definition : Scalar product of four Vectors

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors then the scalar product of these four factors is defined as $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$

Result : Determinant form of $(\overline{a}\times\overline{b})\cdot(\vec{c}\times\vec{d})$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

Proof :

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\cdot\vec{x}, \text{ where } \vec{x} = \vec{c}\times\vec{d}$$

$$= \vec{a}\cdot(\vec{b}\times\vec{x})$$

$$= \vec{a}\cdot[\vec{b}\times(\vec{c}\times\vec{d})]$$

$$= \vec{a}\cdot[(\vec{b}\cdot\vec{d})\,\vec{c} - (\vec{b}\cdot\vec{c})\,\vec{d}]$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\overline{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$\Rightarrow (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

Definition : Vector product of four vectors

If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are four vectors, then the vector product of the vectors $(\vec{a}\times\vec{b})$ and $(\vec{c}\times\vec{d})$ is defined as vector product of four vectors and is denoted by $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$.

Results :

1. If $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are four vectors then

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}$$

Proof :

$$\begin{aligned}
\text{L.H.S.} &= (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\\
&= \vec{x}\times(\vec{c}\times\vec{d}) \text{ where } \vec{x}=\vec{a}\times\vec{b}\\
&= (\vec{x}\,.\,\vec{d})\,\vec{c} - (\vec{x}\,.\,\vec{c})\,\vec{d}\\
&= [(\vec{a}\times\vec{b}).\vec{d}\,]\,\vec{c} - [(\vec{a}\times\vec{b})\,.\,\vec{c}\,]\,\vec{d}\\
&= [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} = \text{R.H.S.}
\end{aligned}$$

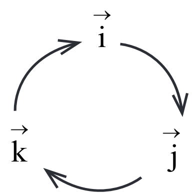
2. If $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are coplanar then $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{0}$.

3. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$

WORKED EXAMPLES

PART - A

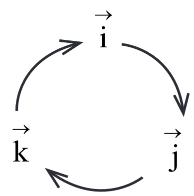
1. Find the value of $\vec{i}\times(\vec{j}\times\vec{k})$



Solution :

$$\begin{aligned}
\vec{i}\times(\vec{j}\times\vec{k}) &= \vec{i}\times\vec{i}\\
&= \vec{0}
\end{aligned}$$

2. Find the value of $\vec{k}\times(\vec{j}\times\vec{k})$



Solution :

$$\begin{aligned}
\vec{k}\times(\vec{j}\times\vec{k}) &= \vec{k}\times\vec{i}\\
&= \vec{j}
\end{aligned}$$

3. Find the value of $(\vec{a}\times\vec{m})\,.\,(\vec{b}\times\vec{n})$

Solution :

$$(\vec{a}\times\vec{m})\,.\,(\vec{b}\times\vec{n}) = \begin{vmatrix} \vec{a}.\vec{b} & \vec{a}.\vec{n} \\ \vec{m}.\vec{b} & \vec{m}.\vec{n} \end{vmatrix}$$

$$= (\vec{a}\,.\vec{b})(\overrightarrow{m}\,.\,\overrightarrow{n}) - (\overrightarrow{m}\,.\vec{b})(\vec{a}\,.\,\overrightarrow{n})$$

4. If $[\vec{a}\ \vec{b}\ \vec{d}] = 5$, $[\vec{a}\ \vec{b}\ \vec{c}] = 2$, $\vec{c} = \vec{i} - \vec{j} - \vec{k}$ and $\vec{d} = 2\vec{i} + 3\vec{j} - 4\vec{k}$ and $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

Solution :

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\ \vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{d}$$

$$= 5(\vec{i} - \vec{j} - \vec{k}) - 2(2\vec{i} + 3\vec{j} - 4\vec{k})$$

$$= 5\vec{i} - 5\vec{j} - 5\vec{k} - 4\vec{i} - 6\vec{j} + 8\vec{k}$$

$$\Rightarrow (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{i} - 11\vec{j} + 3\vec{k}$$

5. If $\vec{a} \times \vec{b} = 7\vec{i} - 3\vec{j} + 4\vec{k}$ & $\vec{c} \times \vec{d} = \vec{i} + 3\vec{j} - 2\vec{k}$ find $(\vec{a} \times \vec{b}) . (\vec{c} \times \vec{d})$

Solution :

$$(\vec{a} \times \vec{b}) . (\vec{c} \times \vec{d}) = 7\vec{i} - 3\vec{j} + 4\vec{k}\ .(\vec{i} + 3\vec{j} - 2\vec{k})$$

$$= (7 \times 1) + (-3 \times 3) + (4 \times -2)$$

$$= 7 - 9 - 8$$

$$= -10$$

PART - B

1. If $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = \vec{j} + \vec{k}$ & $\vec{c} = \vec{k} + \vec{i}$ find $\vec{a} \times (\vec{b} \times \vec{c})$

Solution :

$$\vec{b} \times \vec{c} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\ (1 - 0) - \vec{j}\ (0 - 1) + \vec{k}\ (0 - 1)$$

$$= \vec{i}\ (1) - \vec{i}\ (-1) + \vec{k}\ (-1)$$

$$\Rightarrow \boxed{\overrightarrow{b} \times \overrightarrow{c} = \overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}}$$

$$\therefore \overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \vec{i}\ (-1 - 0) - \vec{i}\ (-1 - 0) + \vec{k}\ (1 - 1)$$

$$= \vec{i}\ (-1) - \vec{i}\ (-1) + \vec{k}\ (0)$$

$$\Rightarrow \boxed{\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = -\overrightarrow{i} + \overrightarrow{j}}$$

2. For any vector $\vec{a}$ prove that $\vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k}) = 2\vec{a}$.

Solution :

We have for any vectors $\vec{a}$ then $(\vec{a} . \vec{i})\,\vec{i} + (\vec{a} . \vec{j})\,\vec{j} + (\vec{a} . \vec{k})\,\vec{k} = \vec{a}$ ——— (1)

Now,

$$\vec{i} \times (\overrightarrow{a} \times \vec{i}) = (\vec{i} . \vec{i})\vec{a} - (\vec{i} . \vec{a})\vec{i}$$

$$= \vec{a} - (\vec{a} . \vec{i})\,\vec{i}$$

Similarly, $\vec{j} \times (\vec{a} \times \vec{j}) = \vec{a} - (\vec{a} . \vec{j})\,\vec{j}$

& $\vec{k} \times (\vec{a} \times \vec{k}) = \overrightarrow{a} - (\overrightarrow{a} . \vec{k})\,\vec{k}$

L.H.S. $= \vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{i}) + \vec{k} \times (\vec{a} \times \vec{k})$

$= \vec{a} - (\vec{a} . \vec{i})\,\vec{i} + \vec{a} - (\vec{a} . \vec{j})\,\vec{j} + \vec{a} - (\vec{a} . \vec{k})\,\vec{k}$

$= 3\vec{a} - [(\vec{a} . \vec{i})\,\vec{i} + (\vec{a} . \vec{j})\,\vec{j} + (\vec{a} . \vec{k})\,\vec{k}]$

$= 3\vec{a} - \vec{a} = 2\vec{a}$ (R.H.S.)

PART - C

1. If $\vec{a} = \vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 2\vec{j}$, $\vec{c} = 2\vec{i} - \vec{j} + \vec{k}$ prove that $\overrightarrow{a} \times (\overrightarrow{b} \times \vec{c}) = (\vec{a} . \vec{c})\vec{b} - (\vec{a} . \vec{b})\vec{c}$.

Solution :

L.H.S.

Now, $\vec{b} \times \vec{c} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -2 & 0 \\ 2 & -1 & 1 \end{vmatrix}$

$$= \overrightarrow{i}\begin{vmatrix} -2 & 0 \\ -1 & 1 \end{vmatrix} - \overrightarrow{j}\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \overrightarrow{k}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-2+0) - \vec{j}\,(1-0) + \vec{k}\,(-1+4)$$

$$= \vec{i}\,(-2) - \vec{j}\,(1) + \vec{k}\,(3)$$

$$\Rightarrow \boxed{\overrightarrow{b} \times \overrightarrow{c} = -2\overrightarrow{i} - \overrightarrow{j} + 3\overrightarrow{k}}$$

$$\therefore \overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -1 & 1 \\ -2 & -1 & 3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -1 \\ -2 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-3+1) - \vec{j}\,(3+2) + \vec{k}\,(-1-2)$$

$$= \vec{i}\,(-2) - \vec{j}\,(5) + \vec{k}\,(-3)$$

$$\Rightarrow \boxed{\vec{a} \times (\vec{b} \times \vec{c}) = -2\vec{i} - 5\vec{j} - 3\vec{k}} \text{ ——— (1)}$$

R.H.S.

Now, $\vec{a}\cdot\vec{c} = (\vec{i} - \vec{j} + \vec{k})\cdot(2\vec{i} - \vec{j} + \vec{k})$
$= (1 \times 2) + (-1 \times -1) + (1 \times 1)$
$= 2 + 1 + 1 = 4$

also, $\vec{a}\cdot\vec{b} = (\vec{i} - \vec{j} + \vec{k})\cdot(\vec{i} - 2\vec{j})$
$= (1 \times 1) + (-1 \times -2)$
$= 1 + 2 = 3$

$\therefore (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = 4[\vec{i} - 2\vec{j}] - 3[2\vec{i} - \vec{j} + \vec{k}]$
$= 4\vec{i} - 8\vec{j} - 6\vec{i} + 3\vec{j} - 3\vec{k}$

$$\Rightarrow \boxed{(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = -2\vec{i} - 5\vec{j} - 3\vec{k}} \text{ ——— (2)}$$

From (1) & (2) we conclude that, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

2. If $\vec{a} = \vec{i} - \vec{j} + \vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} - \vec{k}$, $\vec{c} = \vec{i} + 2\vec{j}$ and $\vec{d} = \vec{i} - \vec{j} - 3\vec{k}$ find $(\vec{a} \times \vec{b})\cdot(\vec{c} \times \vec{d})$

Solution :

Given : $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ $\quad \vec{b} = -\vec{i} + 2\vec{j} - \vec{k}$ $\quad \vec{c} = \vec{i} + 2\vec{j}$ $\quad \vec{d} = \vec{i} - \vec{j} - 3\vec{k}$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= \vec{i}\,(1-2) - \vec{j}\,(-1+1) + \vec{k}\,(2-1)$$

$$= \vec{i}\,(-1) - \vec{j}\,(0) + \vec{k}\,(1)$$

$$\Rightarrow \boxed{\vec{a} \times \vec{b} = -\vec{i} + \vec{k}}$$

also, $\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 1 & -1 & -3 \end{vmatrix}$

$$= \vec{i}\begin{vmatrix} 2 & 0 \\ -1 & -3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-6+0) - \vec{j}\,(-3-0) + \vec{k}\,(-1-2)$$

$$= \vec{i}\,(-6) - \vec{j}\,(-3) + \vec{k}\,(-3)$$

$\Rightarrow \boxed{\vec{c} \times \vec{d} = -6\vec{i} + 3\vec{j} - 3\vec{k}}$

$\therefore \quad (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (-\vec{i} + \vec{k}) \cdot (6\vec{i} + 3\vec{j} - 3\vec{k})$

$$= (-1 \times -6) + (1 \times -3)$$

$$= 6 - 3$$

$\Rightarrow \boxed{(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 3}$

Alternate method :

$$\vec{a} \cdot \vec{c} = (\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{i} + 2\vec{j})$$

$$= (1 \times 1) + (-1 \times 2) + (1 \times 0)$$

$$= 1 - 2 + 0$$

$\Rightarrow \boxed{\vec{a} \cdot \vec{c} = -1}$

$$\vec{a} \cdot \vec{d} = (\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{i} - \vec{j} - 3\vec{k})$$

$$= (1 \times 1) + (-1 \times -1) + (1 \times -3)$$

$$= 1 + 1 - 3$$

$\Rightarrow \boxed{\vec{a} \cdot \vec{d} = -1}$

also, $\vec{b} \cdot \vec{c} = (-\vec{i} + 2\vec{j} - \vec{k}) \cdot (\vec{i} + 2\vec{j})$

$$= (-1 \times 1) + (2 \times 2) + (-1 \times 0)$$

$$= -1 + 4 + 0$$

$\Rightarrow \boxed{\vec{b} \cdot \vec{c} = 3}$

and $\vec{b} \cdot \vec{d} = (-\vec{i} + 2\vec{j} - \vec{k}) \cdot (\vec{i} - \vec{j} - 3\vec{k})$

$$= (-1 \times 1) + (2 \times -1) + (-1 \times -3)$$

$$= -1 - 2 + 3$$

$\Rightarrow \boxed{\vec{b} \cdot \vec{d} = 0}$

$$\therefore (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 \\ 3 & 0 \end{vmatrix}$$

$$= 0 + 3$$

$$\Rightarrow (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 3$$

3. If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - \vec{j} - \vec{k}$, $\vec{c} = -\vec{i} + \vec{j} + 2\vec{k}$, $\vec{d} = 2\vec{i} + \vec{j}$ find $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

Solution :

Given : $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ $\quad\quad$ $\vec{b} = \vec{i} - \vec{j} - \vec{k}$

$\vec{c} = -\vec{i} + \vec{j} + 2\vec{k}$ $\quad$ & $\quad$ $\vec{d} = 2\vec{i} + \vec{j}$

Now, $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-1+1) - \vec{j}\,(-1-1) + \vec{k}\,(-1-1)$$

$$= \vec{i}\,(0) - \vec{j}\,(-2) + \vec{k}\,(-2)$$

$$\Rightarrow \boxed{\vec{a} \times \vec{b} = 2\vec{j} - 2\vec{k}}$$

also, $$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= \vec{i}\,(0-2) - \vec{j}\,(0-4) + \vec{k}\,(-1-2)$$

$$= \vec{i}\,(-2) - \vec{j}\,(-4) + \vec{k}\,(-3)$$

$$\Rightarrow \boxed{\vec{c} \times \vec{d} = -2\vec{i} + 4\vec{j} - 3\vec{k}}$$

$$\therefore (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -2 \\ -2 & 4 & -3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & -2 \\ 4 & -3 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & -2 \\ -2 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 2 \\ -2 & 4 \end{vmatrix}$$

$$= \vec{i}\ (-6+8) - \vec{j}\ (0-4) + \vec{k}\ (0+4)$$

$$= \vec{i}\ (2) - \vec{j}\ (-4) + \vec{k}\ (4)$$

$$\Rightarrow \boxed{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 2\vec{i} + 4\vec{j} + 4\vec{k}}$$

Alternate method :

We have $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \ \vec{b}\ \ \vec{d}]\,\vec{c} - [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{d}$

$$[\vec{a}\ \ \vec{b}\ \ \vec{d}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1\ (0+1) - 1\ (0+2) + 1\ (1+2)$$

$$= 1\ (1) - 1\ (2) + 1\ (3) = 1 - 2 + 3$$

$$\Rightarrow \boxed{[\vec{a}\ \ \vec{b}\ \ \vec{d}] = 2}$$

$$\text{also, } [\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= 1\ (-2+1) - 1\ (2-1) + 1\ (1-1)$$

$$= 1\ (-1) - 1\ (1) + 1\ (0)$$

$$= -1 - 1$$

$$\Rightarrow \boxed{[\vec{a}\ \ \vec{b}\ \ \vec{c}] = -2}$$

$$\therefore\ (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})\ [\vec{a}\ \ \vec{b}\ \ \vec{d}]\,\vec{c} - [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{d} = 2\,(-\vec{i} + \vec{j} + 2\vec{k}) + 2\,(2\vec{i} + \vec{j})$$

$$= -2\vec{i} + 2\vec{j} + 4\vec{k} + 4\vec{i} + 2\vec{j}$$

$$\Rightarrow \boxed{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 2\vec{i} + 4\vec{j} + 4\vec{k}}$$

4. Prove that $[\vec{a}\times\vec{b}\,,\ \vec{b}\times\vec{c}\,,\ \vec{c}\times\vec{a}\,] = [\vec{a}\ \vec{b}\ \vec{c}\,]^2$

Solution :

L.H.S. : $[\vec{a}\times\vec{b}\,,\ \vec{b}\times\vec{c}\,,\ \vec{c}\times\vec{a}\,]$

$= \{(\vec{a}\times\vec{b}).\,[(\vec{b}\times\vec{c}) \times (\vec{c}\times\vec{a})]\}$

$= \{(\vec{a}\times\vec{b}).\ \{[\vec{b}\ \vec{c}\ \vec{a}\,]\ \vec{c} - [\vec{b}\ \vec{c}\ \vec{c}\,]\vec{a}\,\}\}$

$= \{(\vec{a}\times\vec{b}).\ \{[\vec{b}\ \vec{c}\ \vec{a}\,)\ \vec{c} - 0.\vec{a}\,\}\}$

$= \{[(\vec{a}\times\vec{b}).\vec{c}\,]\,[\vec{b}\ \vec{c}\ \vec{a}\,]\}$

$= [\vec{a}\ \vec{b}\ \vec{c}\,]\,[\vec{a}\ \vec{b}\ \vec{c}\,]$ $\qquad \left[\because \left[\overrightarrow{b}\ \overrightarrow{c}\ \overrightarrow{a}\right] = \left[\overrightarrow{a}\ \overrightarrow{b}\ \overrightarrow{c}\right]\right]$

$= [\vec{a}\ \vec{b}\ \vec{c}\,]^2 =$ R.H.S.

$\Rightarrow$ L.H.S. = R.H.S.

EXERCISE

PART - A

1. Prove that $(\vec{a}-\vec{b}) \times (\vec{a}+\vec{b}) = 2\,(\vec{a}\times\vec{b})$.

2. Find $\vec{a}\times\vec{b}$ if $\vec{a} = 2\vec{i} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$.

3. Find $\vec{a}\times\vec{b}$ if $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} - \vec{j} - \vec{k}$.

4. If $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\overline{b} = 3\vec{i} + 4\vec{j} - \vec{k}$ find $\vec{a}\times\vec{b}$.

5. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $|\vec{a}\times\vec{b}| = 7\sqrt{3}$, find the angle between $\vec{a}$ and $\vec{b}$.

6. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{a}\times\vec{b}| = 6$ find angle between $\vec{a}$ and $\vec{b}$.

7. Find the angle between the vectors $\vec{a}$ and $\vec{b}$ if $|\vec{a}\times\vec{b}| = \vec{a}\cdot\vec{b}$.

8. Find the value of $[\vec{j}\ \vec{k}\ \vec{i}\,]$.

9. Find the value of $[\vec{i} - \vec{j}\,,\ \vec{j} - \vec{k}\,, \vec{k} - \vec{i}\,]$

10. If $\vec{F} = 2\vec{i} + 3\vec{j} + 6\vec{k}$ and $\overline{r} = -\vec{i} + 2\vec{j} - 3\vec{k}$ find the moment of the force.

11. Find the value of

    (i) $\vec{i}\times(\vec{j}\times\vec{k})$ (ii) $\vec{k}\times(\vec{k}\times\vec{i})$ (iii) $\vec{i}\times(\vec{j}\times\vec{i})$ (iv) $\vec{k}\times(\vec{j}\times\vec{k})$

12. If $[\vec{a}\ \vec{b}\ \vec{d}\,] = 4$, $[\vec{a}\ \vec{b}\ \vec{c}\,] = -2$, $\vec{c} = \vec{i} - 2\vec{j}$ and $\vec{d} = 3\vec{j} + \vec{k}$ find $(\vec{a}\times\vec{b}) \times (\vec{c}\times\vec{d})$.

13. If $[\vec{a}\ \vec{c}\ \vec{d}\,] = 1$, $[\vec{b}\ \vec{c}\ \vec{d}\,] = 3$, $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ & $\vec{b} = \vec{i} - \vec{j} - \vec{k}$ find $(\vec{a}\times\vec{b}) \times (\vec{c}\times\vec{d})$.

PART - B

1. Find the area of the parallelogram whose adjacent sides are $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} - 3\vec{k}$

2. Find the area of the parallelogram whose adjacent sides are $2\vec{i} + 3\vec{j} + 6\vec{k}$ and $3\vec{i} - 6\vec{j} + 2\vec{k}$.

3. Find the area of the triangle whose adjacent sides are $\vec{i} + \vec{j} - 2\vec{k}$ and $2\vec{i} - \vec{j} - \vec{k}$.

4. Find area of the triangle whose adjacent sides are $2\vec{i} - \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} + 3\vec{k}$.

5. Find the volume of the parallelopiped whose edges are

   (i) $4\vec{i} - 8\vec{j} + \vec{k}$, $2\vec{i} - \vec{j} - 2\vec{k}$, $3\vec{i} - 4\vec{j} + 12\vec{k}$

   (ii) $2\vec{i} - 3\vec{j} + 4\vec{k}$, $\vec{i} + 2\vec{j} - \vec{k}$, $3\vec{i} - \vec{j} + 2\vec{k}$

6. Show that the following vectors are coplanar.

   (i) $2\vec{i} + \vec{j} + \vec{k}$, $3\vec{i} + 4\vec{j} + \vec{k}$, $\vec{i} - 2\vec{j} + \vec{k}$

   (ii) $2\vec{i} - 3\vec{j} + 5\vec{k}$, $\vec{i} + 2\vec{j} - \vec{k}$, $3\vec{i} - \vec{j} + 4\vec{k}$

   (iii) $3\vec{i} + 2\vec{j} - 5\vec{k}$, $\vec{i} - 2\vec{j} - 3\vec{k}$, $3\vec{i} + 10\vec{j} + 19\vec{k}$

7. Find the value of 'm' so that the vectors $2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{i} + \vec{j} + 3\vec{k}$ and $m\vec{i} + \vec{j}$ are coplanar.

8. If $\vec{a} = -4\vec{i} - 6\vec{j} - 2\vec{k}$, $\vec{b} = -\vec{i} + 4\vec{j} + 3\vec{k}$ & $\vec{c} = -8\vec{i} - \vec{j} + 3\vec{k}$ find $[\vec{a}\ \vec{b}\ \vec{c}]$.

9. If $\overline{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -2\vec{i} + 5\vec{k}$ & $\vec{c} = \vec{j} - 3\vec{k}$ find $\vec{a} \times (\vec{b} \times \vec{c})$.

10. If $\vec{a} = 3\vec{i} + 2\vec{j} - 4\vec{k}$, $\vec{b} = 5\vec{i} - 3\vec{j} + 6\vec{k}$ & $\vec{c} = 5\vec{i} - \vec{j} + 2\vec{k}$ find $(\vec{a} \times \vec{b}) \times \vec{c}$.

PART - C

1. Find the unit vector perpendicular to each of the vectors $2\vec{i} - \vec{j} + 2\vec{k}$ and $10\vec{i} - 2\vec{j} + 7\vec{k}$. Find also the sine of the angle between them.

2. Find the unit vector perpendicular to the vectors $-\vec{i} + \vec{j} + 2\vec{k}$ and $-4\vec{i} + 3\vec{j} + 2\vec{k}$. Also find the sine of the angle between them.

3. Find the unit vector perpendicular to each of the vectors $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} - 2\vec{j} + 4\vec{k}$. Also find the sine of the angle between them.

4. Find the area of the triangle formed by the points whose position vectors are

   (i) $\vec{i} + 3\vec{j} + 2\vec{k}$, $2\vec{i} - \vec{j} + \vec{k}$, $-\vec{i} + 2\vec{j} + 3\vec{k}$ (ii) (3, –1, 2) (1, –1, –3), (4, –3, 1)

5. Find the area of the parallelogram whose diagonals are represented by

   (i) $3\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{i} - 3\vec{j} + 4\vec{k}$ (ii) $2\vec{i} + \vec{j} - \vec{k}$ and $3\vec{i} - 4\vec{j} + \vec{k}$

6. Find the moment about the point $\vec{i} + 2\vec{j} - \vec{k}$ of the force represented by $3\vec{i} + \vec{k}$ acting through the point $2\vec{i} - \vec{j} - 3\vec{k}$

7. Show that torque about the point A (3, –1, 3) of the force $4\vec{i} + 2\vec{j} + \vec{k}$ through the point B (5, 2, 4) is $\vec{i} + 2\vec{j} + 8\vec{k}$

8. Find the moment of the force $3\vec{i} + \vec{j} + 2\vec{k}$ acting through the point $\vec{i} - \vec{j} + 2\vec{k}$ about the point $2\vec{i} - \vec{j} + 3\vec{k}$.

9. Show that the points given by the position vectors are coplanar.

(i) (1, 3, 1), (1, 1, –1), (–1, 1, 1), (2, 2, –1)

(ii) (1, 2, 2), (3, –1, 2) (–2, 3, 2), (6, –4, 2)

10. If $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -2\vec{i} + 5\vec{k}$ & $\vec{c} = \vec{j} - 3\vec{k}$ verify that $\vec{a} \times (\overline{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

11. If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$ & $\vec{c} = 2\vec{i} - \vec{j} - 4\vec{k}$ verify that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

12. If $\vec{a} = 3\vec{i} - 4\vec{j} + 5\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$ & $\vec{c} = 2\vec{i} - \vec{j} + \vec{k}$ show that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$.

13. If $\vec{a} = 2\vec{i} + 3\vec{j} - 5\vec{k}$, $\vec{b} = -\vec{i} + \vec{j} + \vec{k}$ & $\vec{c} = 4\vec{i} - 2\vec{j} + 3\vec{k}$ show that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$.

14. If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{k}$, $\vec{c} = 2\vec{i} + \vec{j} + \vec{k}$ & $\vec{d} = \vec{i} + \vec{j} + 2\vec{k}$ find $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

15. If $\vec{a} = \vec{i} - \vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} - 5\vec{k}$, $\vec{c} = 2\vec{i} + 3\vec{j} - \vec{k}$ & $\vec{d} = \vec{i} + \vec{j} - \vec{k}$ find $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

16. If $\vec{a} = \vec{i} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j}$, $\vec{c} = \vec{i} - \vec{k}$ & $\vec{d} = 2\vec{i} - \vec{j} + 3\vec{k}$ find $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

17. If $\vec{a} = 3\vec{i} + 4\vec{j} + 2\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{c} = 4\vec{i} + 2\vec{j} + 5\vec{k}$ & $\vec{d} = 4\vec{i} + 3\vec{j} + 7\vec{k}$ find $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.

18. If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{k}$, $\vec{c} = 2\vec{i} + \vec{j} + \vec{k}$ & $\vec{d} = \vec{i} + \vec{j} + 2\vec{k}$ verify that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}$

19. If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - \vec{j} - \vec{k}$ $\vec{c} = -\vec{i} - \vec{j} + 2\vec{k}$ & $\vec{d} = 2\vec{i} + \vec{j}$ verify that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{d})\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}$.

ANSWERS

PART – A

2) $-\vec{i}-\vec{j}+2\vec{k}$ 3) $\vec{i}+4\vec{j}-3\vec{k}$ 4) $-3\vec{i}+5\vec{j}+11\vec{k}$ 5) 60^o

6) 30^o 7) 45^o 8) 1 9) 0 10) $21\vec{i}-7\vec{k}$

11) (i) $\vec{0}$ (ii) $-\vec{i}$ (iii) $-\vec{j}$ (iv) $-\vec{j}$

12) $4\vec{i}-2\vec{j}+2\vec{k}$ 13) $-2\vec{i}-4\vec{j}-4\vec{k}$

PART – B

1) $A=\sqrt{42}$ sq. units 2) $A=\frac{49}{2}$ sq. units 3) $A=3\sqrt{3}$ sq. units

4) $A=5\sqrt{3}$ sq. units 5) (i) 155 cubic units (ii) 7 cubic units

7) $m=\frac{8}{5}$ 8) 0 9) $-12\vec{i}+9\vec{j}+3\vec{k}$ 10) $-95\vec{i}-95\vec{j}+190\vec{k}$

PART – C

1) $\hat{n}=\frac{-\vec{i}+2\vec{j}+2\vec{k}}{3}$; $\sin\theta=\frac{3}{\sqrt{153}}$ 2) $\hat{n}=\frac{-4\vec{i}-6\vec{j}+\vec{k}}{\sqrt{53}}$; $\sin\theta=\frac{\sqrt{53}}{\sqrt{6}\sqrt{29}}$

3) $\hat{n}=\frac{\vec{i}-\vec{j}-\vec{k}}{\sqrt{3}}$; $\sin\theta=\frac{4\sqrt{3}}{\sqrt{14}\sqrt{3}}$ 4) (i) $A=\frac{3\sqrt{14}}{2}$ sq. units (ii) $A=\frac{\sqrt{165}}{2}$ sq. units

5) (i) $A=5\sqrt{3}$ sq. units (ii) $A=\frac{\sqrt{155}}{2}$ 6) $\vec{m}=-3\vec{i}-7\vec{j}+9\vec{k}$; $|\vec{m}|=\sqrt{139}$ units

7) $\vec{m}=-\vec{i}-\vec{j}-\vec{k}$; $|\vec{m}|=\sqrt{3}$ units 14) – 4 15) – 2

16) $4\vec{i}-2\vec{j}+6\vec{k}$ 17) $-12\vec{i}-34\vec{j}-71\vec{k}$