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The slow motion of a sphere through a viscous fluid towards a plane surface—II
Small gap widths, including inertial effects

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Abstract—Singular perturbation techniques are employed to calculate the hydrodynamic force experienced by a sphere moving, at small Reynolds numbers, perpendicular to a solid plane wall bounding a semi-infinite viscous fluid, for the limiting case where the gap width between the sphere and plane tends to zero. Two distinct, but related, analyses of the problem are presented. In the first analysis, the exact bipolar-coordinate expression for the force given independently by MAUDE [1] and BRENNER [2] for the quasistatic Stokes flow case is expanded via a novel asymptotic procedure. The second analysis, which is somewhat more general in scope, provides a perturbation solution of the *unsteady* Navier–Stokes equations for a more general axisymmetric particle than a sphere, taking account of the finiteness of the Reynolds number. When the Reynolds number is taken into account, the force on the particle differs, according as it moves towards or away from the wall.

Using the techniques developed in the first portion of this paper, formulas are derived for the Stokes couple required to maintain the symmetrical rotation of a dumbbell in an unbounded fluid, as well as the couples required to maintain the symmetrical rotation of a sphere touching a rigid plane wall and touching a planar free surface.

1. INTRODUCTION

A PREVIOUS paper [2] in this series presented an exact solution of the quasistatic Stokes equations for the axisymmetric translational motion of a sphere (with steady velocity U) perpendicular to a solid plane wall bounding a semi-infinite viscous fluid. An independent solution of this same problem is given by MAUDE [1]. Though the formal, bipolar-coordinate series solution given by these authors for the force, F', on the sphere is mathematically convergent for all ratios, $1 < b/h < \infty$, of sphere radius, b, to the distance, h, of the sphere center from the wall, numerical convergence is poor in the asymptotic limit as $b/h \to 1$. In the first portion of the present paper (Section 2), this difficulty is overcome by expanding the exact solution for the force on the sphere so as to obtain an alternate series for F' which is asymptotically valid as $b/h \to 1$. The leading term of this asymptotic series is identical with the "lubrication-theory" formula of G. I. Taylor,§

$$F'/6\pi\mu bU \sim b/(h-b) \quad \text{as} \quad b/h \to 1 \tag{1.1}$$

where, in accordance with Stokes law, $6\pi\mu bU$ is the Stokes force on the sphere in the absence of the wall. Here, μ is the fluid viscosity.

Because of the singular nature of the formal solution at the limit $b/h = 1$, the asymptotic development is performed via a novel method which involves decomposing the original infinite series into "inner" and "outer" portions and then "matching" them in their common domain of validity in a manner reminiscent of the techniques employed in singular perturbation theory [5]. By numerical comparison, the asymptotic development for the force is shown to be in good agreement with the formal series result

§ HARDY and BIRCUMSHAW [3] attribute this formula to Taylor. An elementary derivation of the formula is given by WALTON [4].

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near $b/h=1$. Since both the first-order lubrication-theory formula of Taylor [cf. 3] and the exact bipolar-coordinate solutions of MAUDE [1] and BRENNER [2] are known to accord well with experimental data [6, 7], this supplementary asymptotic analysis completes the theory of the motion of a solid sphere towards (or away from) a plane wall at Reynolds numbers which are sufficiently small to permit the neglect of inertial effects compared with viscous effects.

Section 3 of this paper deals with certain closely related asymptotic problems pertaining to the symmetrical *rotation* of a sphere near a solid plane wall and near a planar free surface, in the Stokes limit of zero Reynolds number, for the special case where $b/h=1$. These asymptotic formulas supplement the exact, bipolar-coordinate series expressions of JEFFERY [8], valid for all b/h, for the Stokes couple on a sphere rotating about an axis perpendicular to a solid plane wall or to a planar free surface bounding a semi-infinite viscous fluid. Jeffery's original solutions suffer from the same type of numerical convergence difficulties near $b/h=1$ as were outlined in the opening paragraph of this section. As a by-product of this investigation we are able to deduce an explicit formula for the Stokes couple acting upon a dumbbell rotating about an axis through the line-of-centers of the two equi-sized touching spheres in an *unbounded* fluid.

Small inertial effects are considered in Section 4 of this paper. In particular, singular perturbation methods are employed to calculate the first-order effects of fluid inertia upon the force experienced by the sphere in the small gap width limit, $(h-b)/b\to 0$. The small Reynolds number $[(h-b)U/\nu]$ effects incorporated into this analysis take account of both the convective and local† acceleration inertial terms in the Navier–Stokes equations, at least to the first-order in the Reynolds number. Here, ν is the kinematic viscosity. Both these inertial effects were neglected in the quasistatic Stokes analysis of MAUDE [1] and BRENNER [2], and hence, in the expression for the force in Section 2 of the present paper.

Since the length term appearing in the Reynolds number, $(h-b)U/\nu$, is based upon gap width rather than sphere radius, the sphere Reynolds number, $|Re|=b|U|/\nu$, may actually be quite large without violating the restrictions implicit in the final solution, provided that the nondimensional gap width, $(h-b)/b$, is sufficiently small.

† Local acceleration terms arise from the fact that the fluid motion is inherently unsteady, owing to the continual change in gap width as the sphere moves normal to the plane.

2. STOKES MOTION OF A SPHERE PERPENDICULAR TO A SOLID PLANE WALL

The exact expression for the Stokes force F' on a sphere moving perpendicular to a solid plane wall bounding a semi-infinite viscous fluid is given by MAUDE [1] and BRENNER [2] as†

$$F'/6\pi\mu bU=F \tag{2.1}$$

where F=function (b/h) is the correction to Stokes law arising from the presence of the plane wall:

$$F=\tfrac{4}{3}\sinh\alpha\sum_{n=1}^{\infty}\frac{n(n+1)}{(2n-1)(2n+3)}\left[\frac{2\sinh(2n+1)\alpha+(2n+1)\sinh 2\alpha}{4\sinh^2(n+\frac{1}{2})\alpha-(2n+1)^2\sinh^2\alpha}-1\right] \tag{2.2}$$

in which the bipolar-coordinate parameter α is related to b/h via the expression

$$\alpha=\cosh^{-1}\frac{h}{b}=\ln\left[\frac{h}{b}+\left\{\left(\frac{h}{b}\right)^2-1\right\}^{\frac{1}{2}}\right]. \tag{2.3}$$

Upon expanding Eq. (2.3) for small α, we find for small nondimensional gap widths $\varepsilon\ll 1$, that

$$\alpha\sim(2\varepsilon)^{\frac{1}{2}}\quad\text{as}\quad\varepsilon\to 0 \tag{2.4a}$$

or, more precisely,

$$\alpha\sim(2\varepsilon)^{\frac{1}{2}}[1-\tfrac{1}{12}\varepsilon+O(\varepsilon^2)]\quad\text{as}\quad\varepsilon\to 0 \tag{2.4b}$$

where the dimensionless gap width ε is defined as

$$\varepsilon=\delta/b \tag{2.5}$$

† In general, we shall employ primed symbols to represent dimensional quantities, the comparable nondimensional quantities being represented by the same symbol without the primed superscript.

in which

$$\delta = h - b \tag{2.6}$$

is the minimum gap width (see Fig. 1) between the sphere and plane. Thus, we shall be interested in the asymptotic expansion of Eq. (2.2) as $\alpha \to 0$.

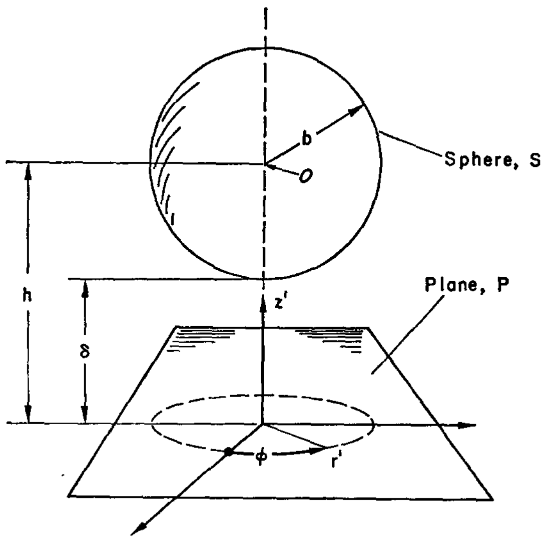

FIG. 1. Sphere near a plane boundary.

The expansion of Eq. (2.2) for small α is highly singular; for no matter how small α may be, the product $n\alpha$ appearing in the various terms in (2.2) will not be small for sufficiently large values of the summation index n. Hence, a conventional approach to the expansion of (2.2) for small α, involving the truncation of the infinite series (2.2) at a sufficiently large value of n, will be invalid when $n\alpha = O(1)$, i.e. when $n = O(\alpha^{-1})$. Accordingly, it becomes necessary to decompose the sum (2.2) into an "inner" sum,

$$F_i(\alpha;\, N) = \tfrac{4}{3}\sinh\alpha \sum_{n=1}^{N} \frac{n(n+1)}{(2n-1)(2n+3)} \left[\frac{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha}{4\sinh^2(n+\frac{1}{2})\alpha - (2n+1)^2\sinh^2\alpha} - 1\right] \tag{2.7}$$

and an "outer" sum,

$$F_o(\alpha;\, N) = \tfrac{4}{3}\sinh\alpha \sum_{n=N+1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \left[\frac{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha}{4\sinh^2(n+\frac{1}{2})\alpha - (2n+1)^2\sinh^2\alpha} - 1\right] \tag{2.8}$$

so that

$$F = F(\alpha) = F_i(\alpha;\, N) + F_o(\alpha;\, N). \tag{2.9}$$

Here, N is an arbitrary integer lying in the region of validity of both the inner expansion, where

$$n = O(\alpha^{-1})$$

and the outer expansion, where $n = O(\alpha^0)$. Thus, N must tend to ∞ as $\alpha \to 0$. Though it is immaterial what function N is of α, one could, for example, choose

$$N = c\alpha^{-(1-q)} \tag{2.10a}$$

where c and q are positive constants—independent of α—the constant q being chosen so that the inequality

$$0 < q < 1 \tag{2.10b}$$

is satisfied. Then N certainly lies in the region of validity of both the inner and outer expansions.

It will prove convenient in the subsequent analysis to introduce a slightly different parameter, X, in place of N. Let $X \ll 1$ be a small, positive parameter defined such that N is the integer immediately less than X/α, i.e.

$$(N+1)\alpha > X \geqslant N\alpha\,.$$

Note that $X \to 0$ as $\alpha \to 0$. In place of Eq. (2.7) one may then write

$$F(\alpha) = F_i(\alpha;\, X) + F_o(\alpha;\, X) \tag{2.11}$$

where F_i and F_o are defined by Eqs. (2.7) and (2.8), respectively. If, for example, N is defined as in Eq. (2.10a), then

$$X \approx c\alpha^q\,. \tag{2.12}$$

Because of the somewhat arbitrary manner in which it is defined, X cannot appear in the final expression for the nondimensional force F, though it can (and does) appear in the individual inner and outer contributions, F_i and F_o, respectively, to F. It is only the *sum* of these terms which must be independent of X, not the separate inner and outer terms themselves.

2.1 *Inner expansion*

The inner expansion corresponds to the case where $\alpha \to 0$ for n a fixed integer. Expansion of each of the various hyperbolic functions in Eq. (2.7) for small values of their arguments via the relation

$$\sinh\theta = \theta + \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots \tag{2.13}$$

ultimately yields

$$F_i = \tfrac{4}{3}[(\alpha^{-2} + \tfrac{1}{6})I_1 + I_2 - \alpha I_3] + O(\alpha^2) \tag{2.14}$$

where

$$I_1 = 48 \sum_{n=1}^{N} \frac{n(n+1)}{(2n-1)^2(2n+1)(2n+3)^2} \tag{2.15}$$

$$I_2 = \tfrac{12}{5} \sum_{n=1}^{N} \frac{n(n+1)[(2n+1)^2+2^2]}{(2n-1)^2(2n+1)(2n+3)^2} \tag{2.16}$$

$$I_3 = \sum_{n=1}^{N} \frac{n(n+1)}{(2n-1)(2n+3)}. \tag{2.17}$$

Each of these three summations will now be asymptotically summed.

Consider first I_1. As is readily verified, by identity

$$\frac{n(n+1)}{(2n-1)^2(2n+1)(2n+3)^2}$$

$$= \tfrac{1}{64}\left[\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right]$$

$$+ \tfrac{3}{128}\left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+3)^2}\right]. \tag{2.18}$$

Now,

$$\sum_1^N \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$$

$$+ \frac{1}{(2N-1)(2N+1)}$$

and

$$\sum_1^N \frac{1}{(2n+1)(2n+3)} = \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$$

$$+ \frac{1}{(2N-1)(2N+1)} + \frac{1}{(2N+1)(2N+3)}.$$

Subtracting these two intermediate sums yields

$$\sum_1^N \left[\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right]$$

$$= \tfrac{1}{3} - \frac{1}{(2N+1)(2N+3)}. \tag{2.19}$$

Similarly, since

$$\sum_1^N \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots + \frac{1}{(2N-1)^2}$$

and

$$\sum_1^N \frac{1}{(2n+3)^2} = \frac{1}{5^2} + \frac{1}{7^2} + \dots + \frac{1}{(2N-1)^2}$$

$$+ \frac{1}{(2N+1)^2} + \frac{1}{(2N+3)^2}$$

subtraction yields

$$\sum_1^N \left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+3)^2}\right]$$

$$= \tfrac{10}{9} - \frac{1}{(2N+1)^2} - \frac{1}{(2N+3)^2}. \tag{2.20}$$

Substitution of Eqs. (2.18), (2.19) and (2.20) into Eq. (2.15) therefore gives

$$I_1 = \tfrac{3}{2} - \tfrac{3}{8}\left[\frac{2}{(2N+1)(2N+3)}\right.$$

$$\left. + 3\left\{\frac{1}{(2N+1)^2} + \frac{1}{(2N+3)^2}\right\}\right]. \tag{2.21}$$

Hence, as $N\to\infty$,

$$I_1 \sim \tfrac{3}{2} - \tfrac{3}{4}N^{-2} + O(N^{-3})\,. \qquad (2.22)$$

Next, consider I_2. Since

$$(2n+1)^2+2^2=(2n+3)(2n-1)+8,$$

it is readily confirmed that

$$\frac{n(n+1)\,[(2n+1)^2+2^2]}{(2n-1)^2(2n+1)(2n+3)^2}=\frac{3}{32(2n-1)}+\frac{1}{16(2n+1)}+\frac{3}{32(2n+3)}+\frac{8n(n+1)}{(2n-1)^2(2n+1)(2n+3)^2}\,.$$

Upon summing both sides from $n=1$ to N and introducing the identities

$$\sum_1^N\frac{1}{2n+1}=\sum_1^N\frac{1}{2n-1}-1+\frac{1}{2N+1} \qquad (2.23)$$

and

$$\sum_1^N\frac{1}{2n+3}=\sum_1^N\frac{1}{2n-1}-\tfrac{4}{3}+\frac{1}{2N+1}+\frac{1}{2N+3} \qquad (2.24)$$

into the resulting sum, we obtain

$$\tfrac{5}{12}I_2=\tfrac{1}{4}\sum_1^N\frac{1}{2n-1}-\tfrac{3}{16}+\frac{5}{32(2N+1)}+\frac{3}{32(2N+3)}+\tfrac{1}{6}I_1$$

where I_1 is given by Eq. (2.21) or, asymptotically, by Eq. (2.22). Now [9, p. 16] as $N\to\infty$,

$$\sum_1^N\frac{1}{2n-1}\sim\tfrac{1}{2}(\gamma+\ln N)+\ln 2+\tfrac{1}{48}N^{-2}+O(N^{-4})$$

where $\gamma=0{\cdot}577216\ldots$ is Euler's constant. Consequently, as $N\to\infty$,

$$I_2\sim\tfrac{3}{10}\ln N+\tfrac{3}{10}\gamma+\tfrac{3}{20}+\tfrac{3}{5}\ln 2+O(N^{-1})\,. \qquad (2.25)$$

To evaluate I_3 in Eq. (2.17) we note that

$$\frac{n(n+1)}{(2n-1)(2n+3)}=\tfrac{1}{4}+\tfrac{3}{16}\left(\frac{1}{2n-1}-\frac{1}{2n+3}\right).$$

Summing this from $n=1$ to N and using Eq. (2.24) therefore yields

$$I_3=\tfrac{1}{4}N+\tfrac{1}{4}-\tfrac{3}{16}\left(\frac{1}{2N+1}+\frac{1}{2N+3}\right) \qquad (2.26)$$

whereupon, as $N\to\infty$,

$$I_3\sim\tfrac{1}{4}N+O(1)\,. \qquad (2.27)$$

Introduction of Eqs. (2.22), (2.25) and (2.27) into (2.14) thereby yields

$$F_i\sim 2\alpha^{-2}+\tfrac{2}{5}\ln N+(\tfrac{8}{15}+\tfrac{2}{5}\gamma+\tfrac{4}{5}\ln 2)-(\alpha N)^{-2}+O(\alpha N)+O(N^{-1})+O(\alpha^{-2}N^{-3})+O(\alpha) \quad (2.28)$$

as $N\to\infty$. To determine the ordering of the various error terms, we note from Eq. (2.10a) that

$$N^{-1}=O(\alpha^{1-q}).$$

Since, from Eq. (2.10b), $1>1-q>0$, it follows that the error term of $O(N^{-1})$ appearing in Eq. (2.28) is of $o(1)$ with respect to α. Similarly, since

$$\alpha N=O(\alpha^q)=o(1),$$

the error term of $O(\alpha N)$ in Eq. (2.28) is also of $o(1)$. Since the error term of $O(\alpha)$ is of higher order than $o(1)$, it may be neglected by comparison. Finally, we note that $\alpha^{-2}N^{-3}=O\{\alpha^{1-q}(\alpha N)^{-2}\}$, which is of higher order in α than is the term of $O(\alpha N)^{-2}$ which we have explicitly retained in Eq. (2.28). Accordingly, whatever may be the exact order of the $O(\alpha N)^{-2}$ term, the term of $O(\alpha^{-2}N^{-3})$ is negligible in comparison to it and may therefore be neglected. Hence, Eq. (2.28) may be written as

$$F_i\sim 2\alpha^{-2}+\tfrac{2}{5}\ln N+(\tfrac{8}{15}+\tfrac{2}{5}\gamma+\tfrac{4}{5}\ln 2)-(\alpha N)^{-2}+o(1)\,. \qquad (2.29)$$

Alternatively, by setting $N=X/\alpha$, this may be written in the form

$$F_i\sim 2\alpha^{-2}-\tfrac{2}{5}\ln\alpha+(\tfrac{8}{15}+\tfrac{2}{5}\gamma+\tfrac{4}{5}\ln 2)+\tfrac{2}{5}\ln X-X^{-2}+o(1)\,. \qquad (2.30)$$

2.2 *Outer expansion*

To obtain the outer expansion, set

$$m = n\alpha \tag{2.31}$$

in Eq. (2.8) and consider the case where $\alpha \to 0$ for a fixed value of m. Expressed in terms of m and α, Eq. (2.8) adopts the form

$$F_o = \tfrac{4}{3}\sinh\alpha \sum_{\substack{m=n\alpha \\ n=N+1}}^{\infty} \frac{m(m+\alpha)}{(2m-\alpha)(2m+3\alpha)} \left[\frac{2\alpha^2\sinh(2m+\alpha)+\alpha(2m+\alpha)\sinh 2\alpha}{4\alpha^2\sinh^2(m+\frac{1}{2}\alpha)-(2m+\alpha)^2\sinh^2\alpha} - 1\right]. \tag{2.32}$$

Expansion of the various functions for the case where $\alpha \to 0$ with m fixed yields

$$F_o = \tfrac{1}{3}\{1+O(\alpha^2)\} \sum_{\substack{m=n\alpha \\ n=N+1}}^{\infty} \alpha\{f(m)+O(\alpha)\}$$

where, for brevity, we have written

$$f(m) = \frac{\sinh 2m + 2m}{\cosh 2m - 1 - \frac{1}{2}(2m)^2} - 1 . \tag{2.33}$$

Define $\Delta m = m_{n+1} - m_n$. This makes

$$\Delta m = (n+1)\alpha - n\alpha = \alpha .$$

Hence, the outer force may be written in the form

$$F_o = \tfrac{1}{3}\{1+O(\alpha^2)\} \sum_{X}^{\infty} f(m)\Delta m + O(\alpha^2) \tag{2.34}$$

where we have noted from the definition of X that $m = X$ at the lower summation limit. Now, according to the Euler–Maclaurin summation formula [10, 11],

$$\sum_{X}^{\infty} f(m)\Delta m = \int_X^{\infty} f(m)\mathrm{d}m - \tfrac{1}{2}\Delta m\{f(\infty)-f(X)\} + \tfrac{1}{12}(\Delta m)^2\{f'(\infty)-f'(X)\} + \dots \tag{2.35}$$

where the primes here denote derivatives. This expression is asymptotically valid as $\Delta m \to 0$. Now, from Eq. (2.33), $f(\infty) = 0$. Furthermore, as $X \to 0$,

$$\frac{\sinh 2X + 2X}{\cosh 2X - 1 - \frac{1}{2}(2X)^2} - 1 \sim 6X^{-3} + \tfrac{6}{5}X^{-1} + O(X^0).$$

Thus, since $\Delta m = \alpha$, we have from Eq. (2.35) that as $\alpha \to 0$,

$$\sum_{X}^{\infty} f(m)\Delta m \sim \int_X^{\infty} f(m)\mathrm{d}m + O(\alpha X^{-3}) + O(\alpha X^{-1}) + O(\alpha) .$$

Consequently, from Eq. (2.34), as $\alpha \to 0$,

$$F_o \sim \tfrac{1}{6}\{1+O(\alpha^2)\}\left\{\int_{2X}^{\infty}\left[\frac{\sinh\lambda+\lambda}{\cosh\lambda - 1 - \frac{1}{2}\lambda^2} - 1\right]\mathrm{d}\lambda + O(\alpha X^{-3}) + O(\alpha X^{-1}) + O(\alpha)\right\} \tag{2.36}$$

where we have set $\lambda = 2m$.

The integral appearing above is convergent for large λ. Now, as $\lambda \to 0$,

$$\frac{\sinh\lambda+\lambda}{\cosh\lambda - 1 - \frac{1}{2}\lambda^2} - 1 \sim \frac{48}{\lambda^3} + \frac{12}{5\lambda} + O(\lambda^0) \tag{2.37}$$

Hence, it is convenient to write

$$\int_{2X}^{\infty}\left[\frac{\sinh\lambda+\lambda}{\cosh\lambda - 1 - \frac{1}{2}\lambda^2} - 1\right]\mathrm{d}\lambda = 6C_1 + \int_{2X}^{1} g(\lambda)\mathrm{d}\lambda + \int_{2X}^{1}\frac{12}{5\lambda}\mathrm{d}\lambda + \int_{2X}^{\infty}\frac{48}{\lambda^3}\mathrm{d}\lambda \tag{2.38}$$

where C_1 is the numerical constant

$$C_1 = \tfrac{1}{6}\int_1^{\infty}\left[\frac{\sinh\lambda+\lambda}{\cosh\lambda - 1 - \frac{1}{2}\lambda^2} - 1 - \frac{48}{\lambda^3}\right]\mathrm{d}\lambda = 0{\cdot}060797 \tag{2.39}$$

and

$$g(\lambda) = \frac{\sinh\lambda+\lambda}{\cosh\lambda - 1 - \frac{1}{2}\lambda^2} - 1 - \frac{48}{\lambda^3} - \frac{12}{5\lambda} . \tag{2.40}$$

The last two integrals on the right-hand side of Eq. (2.38) have the values $-(12/5)\ln 2X$ and $6X^{-2}$, respectively. Introducing these values into Eq.

(2.38) and substituting the resultant expression into Eq. (2.36) thereby yields

$$F_o=\{1+O(\alpha^2)\}\left\{C_1+\tfrac{1}{6}\int_{2X}^{1}g(\lambda)\mathrm{d}\lambda+X^{-2}-\tfrac{2}{5}\ln X\right.$$
$$\left.-\tfrac{2}{5}\ln 2+O(\alpha X^{-3})+O(\alpha X^{-1})+O(\alpha)\right\}.$$

However,

$$\tfrac{1}{6}\int_{2X}^{1}g(\lambda)\mathrm{d}\lambda=C_2-\tfrac{1}{6}\int_{0}^{2X}g(\lambda)\mathrm{d}\lambda$$

where C_2 is the numerical constant

$$C_2=\tfrac{1}{6}\int_{0}^{1}g(\lambda)\mathrm{d}\lambda=-0{\cdot}1590491\,. \qquad (2.41)$$

In view of Eq. (2.37), $g(\lambda)\sim O(\lambda^0)$ as $\lambda\to 0$. Consequently, as $X\to 0$,

$$\int_{0}^{2X}g(\lambda)\mathrm{d}\lambda\sim O(X).$$

Therefore,

$$F_o\sim(C-\tfrac{2}{5}\ln 2)+X^{-2}-\tfrac{2}{5}\ln X+O(X)+O(\alpha X^{-1})$$
$$+O(\alpha X^{-3})+O(\alpha) \qquad (2.42)$$

where

$$C=C_1+C_2=-0{\cdot}098252\,. \qquad (2.43)$$

By introducing the variable $X=\alpha N$ into the various error terms in Eq. (2.42), we see that the terms of $O(X)$, $O(\alpha X^{-1})$ and $O(\alpha X^{-3})$ become terms of $O(\alpha N)$, $O(N^{-1})$ and $O(\alpha^{-2}N^{-3})$, respectively. These are completely analogous to the comparable error terms in Eq. (2.28)†. Thus, we may utilise the arguments given in connection with that equation to conclude that as $\alpha\to 0$,

$$F_o\sim(C-\tfrac{2}{5}\ln 2)+X^{-2}-\tfrac{2}{5}\ln X+o(1)\,. \qquad (2.44)$$

2.3 *Combined expansions*

Upon adding together Eqs. (2.30) and (2.44) in accordance with Eq. (2.11) we obtain

$$F\sim 2\alpha^{-2}-\tfrac{2}{5}\ln\alpha+K+o(1) \qquad (2.45)$$

where

$$K=\tfrac{8}{15}+\tfrac{2}{5}\gamma+\tfrac{2}{5}\ln 2+C=0{\cdot}943226\,. \qquad (2.46)$$

Thereby, we obtain the final result for the Stokes law correction in the limit of small gap widths:

$$F\sim\frac{2}{\alpha^2}+\tfrac{2}{5}\ln\frac{1}{\alpha}+K+o(1) \qquad (2.47)$$

as $\alpha\to 0$.

The parameter α is explicitly expressed in terms of the dimensionless gap width ε by Eq. (2.4b). Hence, we have explicitly that

$$\frac{F'}{6\pi\mu bU}\sim\frac{1}{\varepsilon}\left(1+\tfrac{1}{5}\varepsilon\ln\frac{1}{\varepsilon}+k\varepsilon\right) \quad \text{as} \quad \varepsilon\to 0 \qquad (2.48)$$

where

$$k=K-\tfrac{1}{5}\ln 2+\tfrac{1}{6}=0{\cdot}971264\,. \qquad (2.49)$$

By comparison with Taylor's result, Eq. (1.1), it is clear that the leading term of Eq. (2.48) is identical with the result furnished by classical lubrication-theory techniques.

Insight into the range of validity of the asymptotic formula (2.47) may be gained by comparison with numerical results furnished by the exact formula, Eq. (2.2), for values of α near zero. Such a comparison is given in Table 1. It is clear that the asymptotic formula accords well with the exact result even for α-values as large as unity, the error in this case being only about 3 per cent.

† Actually, because of the somewhat arbitrary manner in which N (or X) is defined, these pairs of analogous terms occurring in both the inner and outer expansions must, in fact, be equal in magnitude and opposite in algebraic sign, so that their sum is zero. If this were not the case, the total force F would depend explicitly upon N (or X). And this cannot be, because of the arbitrary nature of these functions.

TABLE 1. COMPARISON OF EXACT AND ASYMPTOTIC FORMULAS FOR THE STOKES FORCE ON A SPHERE MOVING PERPENDICULAR TO A PLANE WALL

		Stokes law correction, F	
a	h/b	Exact Eq. (2.2)	Asymptotic Eq. (2.47)
0·050000	1·0012503	802·15	802·15
0·100000	1·0050042	201·863	201·864
0·200000	1·0200668	51·594	51·587
0·500000	1·1276260	9·2517663	9·220
1·000000	1·5430806	3·0360641	2·943
.	.	.	.
.	.	.	.
.	.	.	.
∞	∞	1·000	—

3. TORQUE ON A SPHERE ROTATING NEAR A PLANE BOUNDARY

The techniques developed in the preceding section for extracting a dominant, asymptotic series from a formally convergent series may be applied to related Stokes flow problems. In this section we apply these methods to the symmetrical *rotation* of a sphere about an axis perpendicular to a plane wall bounding a semi-infinite viscous fluid (Fig. 1). The two cases to be considered correspond to the situations where: (i) the plane is solid; (ii) the plane constitutes a planar free surface. Rather than carrying the asymptotic development as far as was done in Section 2, we content ourselves with calculating only the leading term in each of the two asymptotic series. Even this limited information is useful, however, since lubrication-theory arguments are inapplicable to these rotational problems, owing to the fact that the couple on the sphere is nonsingular —even in the limit where the sphere touches the plane. Hence, in contrast to the analogous translational problems, classical lubrication theory [12] is unable to furnish an independent calculation of the leading term of the asymptotic expansion as $\alpha \to 0$.

3.1 *Solid plane wall*

By employing bipolar coordinates, JEFFERY [8] succeeded in calculating the Stokes couple on a sphere rotating with angular velocity ω about an axis through its center lying perpendicular to a rigid plane wall. His formula is

$$T'/8\pi\mu b^3\omega = T \tag{3.1}$$

where T' is the torque about the sphere center in the presence of the plane; $8\pi\mu b^3\omega$ is the torque about the sphere center in an *unbounded* fluid; and $T=$ function (b/h) is the wall correction due to the presence of the rigid plane. This correction is given by JEFFERY [8] as

$$T = \sinh^3\alpha \sum_{n=1}^{\infty} \frac{1}{\sinh^3 n\alpha} \tag{3.2}$$

where α is defined by Eq. (2.3).

It can be shown *a priori* [13] that the torque on the sphere is finite even in the limit where it touches the wall. Hence, the inner expansion of Eq. (3.2) will, by itself, contain all the relevant information as to the limiting value of the couple at $\alpha=0$; therefore, no cognizance need be given the outer expansion in this limiting calculation.

The inner expansion of Eq. (3.2) may be obtained by expanding the hyperbolic functions as $\alpha \to 0$ for n fixed. Hence, for sufficiently small α we obtain

$$T \sim \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{as} \quad \alpha \to 0. \tag{3.3}$$

But [14, p. 212]

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3) = 1{\cdot}2020569 \tag{3.4}$$

where $\zeta(3)$ is the Riemann Zeta Function of argument 3. Consequently, when the sphere touches the wall

$$T'/8\pi\mu b^3\omega = 1{\cdot}2020569. \tag{3.5}$$

The couple required to maintain the steady rotation of the sphere in the presence of the plane is, therefore, only slightly more than 20 per cent greater than is required in the absence of the plane. This limiting result is compared in Table 2 with the exact values of T for various values of α near zero, as tabulated by JEFFERY [8] on the basis of Eq. (3.2). Jeffery's results appear to satisfactorily approach the limiting value of $T=1{\cdot}202$ at $\alpha=0$.

TABLE 2. WALL CORRECTION TO THE STOKES TORQUE ARISING FROM THE ROTATION OF A SPHERE NEAR A SOLID PLANE WALL

α	b/h	T Eq. (3.2)
0·0	1·0000	(1·202)
0·2	0·9803	1·171
0·4	0·9250	1·126
0·6	0·8435	1·087
1·0	0·6481	1·036
·	·	·
·	·	·
·	·	·
∞	0	1·000

3.2 *Planar free surface*

Similar arguments may be applied to the computation of the Stokes couple on a symmetrically rotating sphere just touching a planar free surface. The exact expression for the couple for an arbitrary separation distance may be obtained from JEFFERY'S [8] bipolar-coordinate calculations. It is given explicitly by BRENNER [15] as

$$T'/8\pi\mu b^3\omega=T \tag{3.6}$$

where the free-surface correction, $T=$function (b/h), is

$$T=\sinh^3\alpha\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sinh^3 n\alpha}. \tag{3.7}$$

The parameter α is defined as before. Expanding the the hyperbolic functions for small α with n fixed yields†

† We have here used the identity [16, p. 269] that if

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

is the Riemann Zeta Function of argument s, then

$$(1-2^{1-s})\zeta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$$

$$T=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}=\tfrac{3}{4}\zeta(3)\quad\text{as}\quad\alpha\to 0. \tag{3.8}$$

Therefore,

$$T=0{\cdot}90154268 \tag{3.9}$$

when the sphere is tangent to the planar free surface. The limiting couple is thus about 10 per cent *less* than obtains when the fluid is unbounded.

3.3 *Dumbbell*

When two equal spheres (radii$=b$) rotate as a unit about their common line-of-centers, the tangent plane midway between them is equivalent to a planar free surface. Accordingly, it follows from the results of the preceding paragraph that the couple on a dumbbell consisting of two equi-sized tangent spheres rotating as a single rigid body about the axis of revolution in an unbounded fluid is

$$T'=\tfrac{3}{2}\zeta(3)8\pi\mu b^3\omega \tag{3.10a}$$

$$=45{\cdot}316478\mu b^3\omega. \tag{3.10b}$$

4. INERTIAL EFFECTS

The results of Section 2 apply only to the case where the fluid motion satisfies the quasistatic Stokes equations. In this section we shall extend these results by taking account of first-order inertial effects for the special case where the gap width is small compared with the sphere radius. The inertial effects are of two types, resulting from the presence of both convective and local acceleration terms in the complete Navier–Stokes equations. The appearance of these local acceleration terms is a manifestation of the fact that the fluid motion is intrinsically *unsteady* due to the continuous change in gap width as the sphere moves perpendicular to the plane.

Singular perturbation techniques are employed in the analysis. However, by confining attention exclusively to the computation of only the *singular* terms in the expression for the force on the sphere (i.e. those terms which become infinite as the gap width tends to zero) we are able to avoid a detailed calculation of the outer expansion†. For when the gap width is small the dominant (i.e. singular) contribution to the force comes from the fluid motion in the immediate proximity of the pole of the sphere nearest the plane. And this region lies within the domain of validity of the inner expansion. Because of these considerations, our techniques have much in common with those utilised in classical lubrication theory [12]—specifically with that branch of the subject that gives recognition to nonzero inertial effects. It is assumed that the inertia effect resulting from any acceleration of the sphere may be neglected in comparison with that due to its velocity, since it may be seen that an investigation of the former effect would require a detailed analysis of the outer expansions.

The statement of the general problem is as follows (see Fig. 1): A solid plane wall (P), $z=0$, bounds a semi-infinite, incompressible, viscous fluid confined within the region $z>0$. A solid sphere (S) of radius b approaches the plane with steady, uniform speed $U>0$.‡ The instantaneous gap width (minimum separation distance) is δ, the nondimensional gap width δ/b being denoted by ε. The sphere Reynolds number

$$Re=bU/\nu$$

is taken to be of order unity. We seek to compute the instantaneous force on the sphere for the small gap width case, $\varepsilon \ll 1$, by obtaining an appropriate asymptotic solution of the Navier–Stokes and continuity equations in the neighborhood of the gap.

As before, dimensional quantities are represented by primed symbols, their nondimensional counterparts being represented by the same symbol without the primed superscript. Thus, we introduce a system of circular cylindrical coordinates

$$(r,\ \phi,\ z)=(r'/b,\ \phi',\ z'/b)$$

made dimensionless with the sphere radius. Because the fluid motion is axially symmetric, the azimuthal angle is irrelevant. The symbols u' and v' will be used to denote the z' and r' components of velocity, respectively. The corresponding nondimensional components are defined as

$$u=u'/U\,, \qquad v=v'/U\,.$$

Furthermore,

$$p=p'b/\mu U\,, \qquad t=t'U/b\,,$$

$$\nabla=b\nabla'\,, \qquad \nabla^2=b^2\nabla'^2$$

where p' and t' are the fluid dynamic pressure and the time, respectively.

In terms of these quantities the Navier–Stokes and continuity equations,

$$-\nabla\times(\nabla\times\mathbf{u})-\nabla p=Re\left(\mathbf{u}\,.\nabla\mathbf{u}+\frac{\partial\mathbf{u}}{\partial t}\right)$$

$$\nabla\,.\,\mathbf{u}=0$$

become

$$\frac{\partial^2 v}{\partial z^2}-\frac{\partial^2 u}{\partial r\partial z}-\frac{\partial p}{\partial r}=Re\left(v\frac{\partial v}{\partial r}+u\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}\right) \tag{4.1}$$

and

$$-\frac{\partial^2 v}{\partial r\partial z}+\frac{\partial^2 u}{\partial r^2}-\frac{1}{r}\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{\partial p}{\partial z}$$
$$=Re\left(v\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right) \tag{4.2}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial u}{\partial z}=0\,. \tag{4.3}$$

The boundary conditions are

$$u=-1\,, \qquad v=0\,, \qquad \text{on } S \tag{4.4a, b}$$

$$u=0\,, \qquad v=0\,, \qquad \text{on } P \tag{4.5a, b}$$

$$u\to 0\,, \qquad v\to 0\,, \qquad \text{at infinity.} \tag{4.6a, b}$$

† As will be seen, however, we do require knowledge of the asymptotic behavior of the outer expansion near the contact point between the sphere and plane. But this information may be gleaned from the inner expansion alone by utilising the "matching principle".

‡ Our analysis applies equally well to the case where the sphere moves *away* from the wall. In that case the scalar U will be negative [see boundary condition (4.4a), which is the only inhomogeneous boundary condition in the problem]. The Reynolds number $Re=bU/\nu$ will then be a negative scalar too.

4.1 *Inner expansion*

The equation of the sphere surface S may be written exactly as

$$z=1+\varepsilon-(1-r^2)^{\frac{1}{2}} \quad \text{on } S. \tag{4.7}$$

Since $r \ll 1$ in the neighborhood of the gap, we find upon expanding the last term in Eq. (4.7) by the binomial theorem that the equation of the sphere surface in the proximity of the gap is

$$z=\varepsilon+\tfrac{1}{2}r^2+\tfrac{1}{8}r^4+O(r^6) \quad \text{on } S. \tag{4.8}$$

This expansion suggests that to make the two independent variables of equal order in the neighborhood of the pole, one should introduce the following "stretched", inner, independent variables:

$$\tilde{z}=z/\varepsilon, \qquad \tilde{r}=r/\sqrt{\varepsilon}. \tag{4.9a, b}$$

In terms of these variables the sphere surface is described by the equation

$$\tilde{z}=1+\tfrac{1}{2}\tilde{r}^2+\tfrac{1}{8}\varepsilon\tilde{r}^4+O(\varepsilon^2) \quad \text{on } S. \tag{4.10}$$

Because of the boundary condition (4.4a), u will be of $O(1)$—at least near to the sphere. Thus, upon introducing the stretched variables (4.9) into the continuity equation (4.3), it follows that v will be of $O(1/\sqrt{\varepsilon})$, at least in the vicinity of the pole—where Eqs. (4.9) are pertinent. These considerations suggest introducing the "stretched", inner, dependent variables

$$\tilde{u}=u, \qquad \tilde{v}=\sqrt{\varepsilon}v. \tag{4.11a, b}$$

Upon substituting Eqs. (4.9) and (4.11) into Eqs. (4.1) and (4.2), the condition that, as $\varepsilon \to 0$, the pressure terms be of comparable order with the viscous terms (appearing on the left-hand sides of the latter equations) requires that the pressure be scaled in accordance with the relation

$$\tilde{p}=\varepsilon^2 p. \tag{4.12}$$

Upon introducing Eqs. (4.9), (4.11) and (4.12) into Eqs. (4.1)–(4.2), we obtain the complete Navier–Stokes equations in the stretched forms

$$\frac{\partial^2\tilde{v}}{\partial\tilde{z}^2}-\varepsilon\frac{\partial^2\tilde{u}}{\partial\tilde{r}\partial\tilde{z}}-\frac{\partial\tilde{p}}{\partial\tilde{r}}$$
$$=\varepsilon Re\left[\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{r}}+\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{z}}+\varepsilon^{\frac{3}{2}}\left(\frac{\partial}{\partial t}\right)_{r,z}(\varepsilon^{-\frac{1}{2}}\tilde{v})\right] \tag{4.13}$$

and

$$-\varepsilon\frac{\partial^2\tilde{v}}{\partial\tilde{r}\partial\tilde{z}}+\varepsilon^2\frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}-\varepsilon\frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\tilde{z}}+\varepsilon^2\frac{1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\tilde{r}}-\frac{\partial\tilde{p}}{\partial\tilde{z}}$$
$$=\varepsilon^2 Re\left[\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{r}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{z}}+\varepsilon\left(\frac{\partial}{\partial t}\right)_{r,z}\tilde{u}\right]. \tag{4.14}$$

As explicitly indicated by the subscript notation in the preceding relations, the time derivative appearing above refers to the rate of change when r and z are held constant. It will be more convenient, however, to utilise a time derivative with $\tilde{r}$ and $\tilde{z}$ held constant. To effect this transformation we apply the chain rule:

$$\left(\frac{\partial}{\partial t}\right)_{r,z}=\left(\frac{\partial\tilde{r}}{\partial t}\right)_{r,z}\frac{\partial}{\partial\tilde{r}}+\left(\frac{\partial\tilde{z}}{\partial t}\right)_{r,z}\frac{\partial}{\partial\tilde{z}}+\left(\frac{\partial}{\partial t}\right)_{\tilde{r},\tilde{z}} \tag{4.15}$$

However, from Eq. (4.9b),

$$\left(\frac{\partial\tilde{r}}{\partial t}\right)_{r,z}=\left(\frac{\partial}{\partial t}\right)_{r,z}(\varepsilon^{-\frac{1}{2}}r)=r\frac{d\varepsilon^{-\frac{1}{2}}}{dt}=-\tfrac{1}{2}r\varepsilon^{-\frac{3}{2}}\frac{d\varepsilon}{dt}$$
$$=-\tfrac{1}{2}\varepsilon^{-1}\tilde{r}\frac{d\varepsilon}{dt}. \tag{4.16}$$

But, since $U=-d\delta/dt'$ and $t'=tb/U$ and $\delta=b\varepsilon$, it follows that

$$\frac{d\varepsilon}{dt}=-1. \tag{4.17}$$

Substitution into Eq. (4.16) then yields

$$\left(\frac{\partial\tilde{r}}{\partial t}\right)_{r,z}=\tfrac{1}{2}\varepsilon^{-1}\tilde{r}. \tag{4.18}$$

An exactly similar procedure gives

$$\left(\frac{\partial\tilde{z}}{\partial t}\right)_{r,z}=\varepsilon^{-1}\tilde{z}. \tag{4.19}$$

Equation (4.15) thus furnishes the desired transformation:

$$\left(\frac{\partial}{\partial t}\right)_{r,z}=\tfrac{1}{2}\varepsilon^{-1}\tilde{r}\frac{\partial}{\partial\tilde{r}}+\varepsilon^{-1}\tilde{z}\frac{\partial}{\partial\tilde{z}}+\left(\frac{\partial}{\partial t}\right)_{\tilde{r},\tilde{z}}. \tag{4.20}$$

Introduction of the above relationship into Eqs. (4.13) and (4.14) now yields the following expressions for the stretched form of the complete Navier–Stokes equations:

$$\frac{\partial^2 \tilde{v}}{\partial \tilde{z}^2}-\varepsilon\frac{\partial^2 \tilde{u}}{\partial \tilde{r}\partial \tilde{z}}-\frac{\partial \tilde{p}}{\partial \tilde{r}}$$
$$=\varepsilon Re\left[\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{r}}+\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{z}}+\tfrac{1}{2}\tilde{r}\frac{\partial \tilde{v}}{\partial \tilde{r}}+\tilde{z}\frac{\partial \tilde{v}}{\partial \tilde{z}}+\varepsilon^{\frac{1}{2}}\left(\frac{\partial}{\partial t}\right)_{\tilde{r},\tilde{z}}(\varepsilon^{-\frac{1}{2}}\tilde{v})\right] \tag{4.21}$$

and

$$-\varepsilon\frac{\partial^2 \tilde{v}}{\partial \tilde{r}\partial \tilde{z}}+\varepsilon^2\frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2}-\varepsilon\frac{1}{\tilde{r}}\frac{\partial \tilde{v}}{\partial \tilde{z}}+\varepsilon^2\frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{r}}-\frac{\partial \tilde{p}}{\partial \tilde{z}}$$
$$=\varepsilon^2 Re\left[\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{r}}+\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{z}}+\tfrac{1}{2}\tilde{r}\frac{\partial \tilde{u}}{\partial \tilde{r}}+\tilde{z}\frac{\partial \tilde{u}}{\partial \tilde{z}}+\varepsilon\left(\frac{\partial \tilde{u}}{\partial t}\right)_{\tilde{r},\tilde{z}}\right]. \tag{4.22}$$

In addition to these, the continuity equation (in stretched form) must also be considered:

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{v})+\frac{\partial \tilde{u}}{\partial \tilde{z}}=0. \tag{4.23}$$

In stretched form, the exact boundary conditions are

$$\tilde{u}=-1, \quad \tilde{v}=0, \quad \text{on } S \tag{4.24a, b}$$

$$\tilde{u}=0, \quad \tilde{v}=0, \quad \text{on } P \tag{4.25a, b}$$

$$\tilde{u}\to 0, \quad \tilde{v}\to 0, \quad \text{at infinity}. \tag{4.26a, b}$$

Equations (4.21)–(4.26) are, of course, the *exact* governing equations and boundary conditions expressed in terms of stretched variables. In order to solve them in the neighborhood of the gap, we assume the existence of inner expansions of the forms

$$\tilde{u}(\tilde{r},\tilde{z};t)=\tilde{u}_1(\tilde{r},\tilde{z})+\varepsilon(t)\tilde{u}_2(\tilde{r},\tilde{z})+o(\varepsilon) \tag{4.27}$$

$$\tilde{v}(\tilde{r},\tilde{z};t)=\tilde{v}_1(\tilde{r},\tilde{z})+\varepsilon(t)\tilde{v}_2(\tilde{r},\tilde{z})+o(\varepsilon) \tag{4.28}$$

$$\tilde{p}(\tilde{r},\tilde{z};t)=\tilde{p}_1(\tilde{r},\tilde{z})+\varepsilon(t)\tilde{p}_2(\tilde{r},\tilde{z})+o(\varepsilon) \tag{4.29}$$

for a *fixed* value of Re, not necessarily small.

Upon substituting these expansions into Eqs. (4.21)–(4.25) and equating terms of $O(\varepsilon^0)$, one obtains the following set of differential equations governing the first-order inner fields:

$$\frac{\partial^2 \tilde{v}_1}{\partial \tilde{z}^2}-\frac{\partial \tilde{p}_1}{\partial \tilde{r}}=0. \tag{4.30}$$

$$\frac{\partial \tilde{p}_1}{\partial \tilde{z}}=0 \tag{4.31}$$

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{v}_1)+\frac{\partial \tilde{u}_1}{\partial \tilde{z}}=0 \tag{4.32}$$

These are to be solved subject to the following boundary conditions:

$$\tilde{u}_1=-1, \quad \tilde{v}_1=0, \quad \text{when } \tilde{z}=\tilde{z}_S \tag{4.33a, b}$$

$$\tilde{u}_1=0, \quad \tilde{v}_1=0, \quad \text{at } \tilde{z}=0 \tag{4.34a, b}$$

where

$$\tilde{z}_S=\tilde{z}_S(\tilde{r})=1+\tfrac{1}{2}\tilde{r}^2+\tfrac{1}{8}\Delta\tilde{r}^4. \tag{4.35}$$

Since the inner expansions are not expected to be valid outside the gap, we have not substituted them into the boundary conditions (4.26) at infinity. In the interests of generality, we have replaced the parameter ε in that part of Eq. (4.35) defining the sphere surface by the new, small parameter Δ [compare with Eq. (4.10)]. This new parameter is assumed to be of $O(\varepsilon)$ or of higher order in ε. This slight generalization permits us to simultaneously treat the problem of a *nonspherical* (but axisymmetric) body moving normal to the wall. The special case where the body is a sphere may be obtained by replacing Δ by ε in the final formulas. Equations (4.30)–(4.34) constitute a system of linear, homogeneous equations to be solved for $\tilde{u}_1$, $\tilde{v}_1$, $\tilde{p}_1$ in terms of the independent variables $\tilde{r}$, $\tilde{z}$ and the parameter Δ.

Determination of the forms of the second-order perturbation equations requires a somewhat more detailed analysis because of the added degree of generality entailed by having introduced the parameter Δ in place of ε. The dependent variables $\tilde{u}_1$, $\tilde{v}_1$, $\tilde{p}_1$ defined by the first-order perturbation Eqs. (4.30)–(4.34) will be functions of $\tilde{r}$, $\tilde{z}$ and of Δ, the latter quantity entering because of its appearance in the boundary conditions (4.33). Since Δ is a small parameter, the first-order solutions, when ultimately obtained, may be expanded into the Taylor series

$$\tilde{u}_1(\tilde{r}, \tilde{z}; \Delta) = \tilde{u}_1^*(\tilde{r}, \tilde{z}) + \Delta\tilde{u}_1^{**}(\tilde{r}, \tilde{z}) + o(\Delta) \tag{4.36}$$

$$\tilde{v}_1(\tilde{r}, \tilde{z}; \Delta) = \tilde{v}_1^*(\tilde{r}, \tilde{z}) + \Delta\tilde{v}_1^{**}(\tilde{r}, \tilde{z}) + o(\Delta) \tag{4.37}$$

$$\tilde{p}_1(\tilde{r}, \tilde{z}; \Delta) = \tilde{p}_1^*(\tilde{r}, \tilde{z}) + \Delta\tilde{p}_1^{**}(\tilde{r}, \tilde{z}) + o(\Delta) \tag{4.38}$$

where both the starred and double-starred fields are explicitly independent of the parameter Δ (and ε).

Combining the preceding relations with Eqs. (4.27)–(4.29) yields

$$\tilde{u} = \tilde{u}_1^*(\tilde{r}, \tilde{z}) + \Delta\tilde{u}_1^{**}(\tilde{r}, \tilde{z}) + \varepsilon\tilde{u}_2(\tilde{r}, \tilde{z}) + \dots \tag{4.39}$$

with similar expansions for $\tilde{v}$ and $\tilde{p}$. The quantities $\tilde{u}_1^*$, $\tilde{u}_1^{**}$, $\tilde{u}_2$, etc. depend only upon $\tilde{r}$ and $\tilde{z}$, being independent of ε and Δ (and of course, t). Upon substituting the expansions (4.39), etc. into Eqs. (4.21)–(4.23) and taking account of the Eqs. (4.30)–(4.32), satisfied by the first-order fields, we find upon equating terms of order ε (bearing in mind that Δ is of order ε or greater and that Re is independent of ε) that the second-order fields satisfy the following system of differential equations:†

$$\frac{\partial^2\tilde{v}_2}{\partial\tilde{z}^2} - \frac{\partial\tilde{p}_2}{\partial\tilde{r}} = \frac{\partial^2\tilde{u}_1^*}{\partial\tilde{r}\partial\tilde{z}} + Re\left(\tilde{v}_1^*\frac{\partial\tilde{v}_1^*}{\partial\tilde{r}} + \tilde{u}_1^*\frac{\partial\tilde{v}_1^*}{\partial\tilde{z}} + \tfrac{1}{2}\tilde{r}\frac{\partial\tilde{v}_1^*}{\partial\tilde{r}} + \tilde{z}\frac{\partial\tilde{v}_1^*}{\partial\tilde{z}} + \tfrac{1}{2}\tilde{v}_1^*\right) \tag{4.40}$$

$$\frac{\partial\tilde{p}_2}{\partial\tilde{z}} = -\frac{\partial^2\tilde{v}_1^*}{\partial\tilde{r}\partial\tilde{z}} - \frac{1}{\tilde{r}}\frac{\partial\tilde{v}_1^*}{\partial\tilde{z}} \tag{4.41}$$

† In arriving at Eq. (4.40) from Eq. (4.21) we have noted that

$$\varepsilon^{\frac{1}{2}}\left(\frac{\partial}{\partial t}\right)_{\tilde{r},\tilde{z}}(\varepsilon^{-\frac{1}{2}}\tilde{v})$$

$$= \varepsilon^{\frac{1}{2}}\left(\frac{\partial}{\partial t}\right)_{\tilde{r},\tilde{z}}\{\varepsilon^{-\frac{1}{2}}\tilde{v}_1^* + O(\varepsilon^{\frac{1}{2}})\} = \zeta, \text{ say}$$

However, since $\tilde{v}_1{}^*$ depends only upon $\tilde{r}$ and $\tilde{z}$, but not upon t, it remains constant in the differentiation. Hence

$$\zeta = \varepsilon^{\frac{1}{2}}\left[\tilde{v}_1^*\frac{d\varepsilon^{-\frac{1}{2}}}{dt} + \frac{d}{dt}\{O(\varepsilon^{\frac{1}{2}})\}\right].$$

In view of Eq. (4.17) this becomes

$$\zeta = \tfrac{1}{2}\tilde{v}_1^* + O(\varepsilon)$$

which explains the appearance of the last term on the right-hand side of Eq. (4.40).

and

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{v}_2) + \frac{\partial\tilde{u}_2}{\partial\tilde{z}} = 0. \tag{4.42}$$

By similar arguments, the boundary conditions satisfied by these second-order fields are found to be

$$\tilde{u}_2 = 0, \quad \tilde{v}_2 = 0, \quad \text{when } \tilde{z} = \tilde{z}_S^* \tag{4.43a, b}$$

$$\tilde{u}_2 = 0, \quad \tilde{v}_2 = 0, \quad \text{at } \tilde{z} = 0 \tag{4.44a, b}$$

in which

$$\tilde{z}_S^* = 1 + \tfrac{1}{2}\tilde{r}^2. \tag{4.45}$$

The differential Eqs. (4.40)–(4.42) and boundary conditions (4.43)–(4.44) constitute a set of linear, inhomogeneous equations to be solved for $\tilde{u}_2$, $\tilde{v}_2$, $\tilde{p}_2$ in terms of $\tilde{r}$, $\tilde{z}$ and Re. Observe that the derivation of these equations does not require that $|Re|$ be a small parameter, but only that $\varepsilon|Re|$ be small compared with unity [see the right-hand sides of Eqs. (4.21)–(4.22)]. Hence, the results will ultimately apply when the restrictions $\varepsilon \ll 1$ and $\varepsilon|Re| \ll 1$ are met. Since $\varepsilon Re = \delta U/\nu$ is the Reynolds number based on gap width, the latter condition will always be met as the gap width tends to zero, irrespective of how large $|Re|$ may be.

4.2 *First-order inner fields*

Equation (4.31) requires that $\tilde{p}_1$ be independent of $\tilde{z}$. Integration of Eq. (4.30) therefore yields

$$\tilde{v}_1 = \tfrac{1}{2}\frac{d\tilde{p}_1}{d\tilde{r}}\tilde{z}^2 + A(\tilde{r})\tilde{z} + B(\tilde{r})$$

where $A(\tilde{r})$ and $B(\tilde{r})$ are functions of $\tilde{r}$. The boundary condition (4.34b) requires that $B=0$. The boundary condition (4.33b) then furnishes the function A in terms of $d\tilde{p}_1/d\tilde{r}$. Thus,

$$\tilde{v}_1 = \tfrac{1}{2}\frac{d\tilde{p}_1}{d\tilde{r}}\tilde{z}(\tilde{z} - \tilde{z}_S) \tag{4.46}$$

where $\tilde{z}_S$ is given as a function of $\tilde{r}$ by Eq. (4.35). Introduction of Eq. (4.46) into (4.32) yields an explicit expression for $\partial\tilde{u}_1/\partial\tilde{z}$ which, when integrated with respect to $\tilde{z}$, gives

$$\tilde{u}_1 = \tfrac{1}{12}\left[\tilde{z}^2(3\tilde{z}_S - 2\tilde{z})\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}\left(\tilde{r}\frac{d\tilde{p}_1}{d\tilde{r}}\right) + 3\tilde{z}^2\frac{d\tilde{z}_S}{d\tilde{r}}\frac{d\tilde{p}_1}{d\tilde{r}}\right] + C(\tilde{r}). \tag{4.47}$$

Satisfaction of the boundary condition (4.34a) requires that the arbitrary function of integration, $C(\tilde{r})$, be zero. Boundary condition (4.33a) applied to Eq. (4.47) then furnishes the differential equation satisfied by $d\tilde{p}_1/d\tilde{r}$. Upon multiplying Eq. (4.47) by $12\tilde{r}$, the differential equation resulting from satisfaction of Eq. (4.33a) may be written as

$$-12\tilde{r}=\frac{d}{d\tilde{r}}\left(\tilde{r}\tilde{z}_S{}^3\frac{d\tilde{p}_1}{d\tilde{r}}\right)$$

the first integral of which is

$$\frac{d\tilde{p}_1}{d\tilde{r}}=-\frac{6\tilde{r}}{\tilde{z}_S{}^3}+\frac{\text{const.}}{\tilde{r}\tilde{z}_S{}^3}.$$

If the constant of integration in this expression were to be nonzero, there would exist a singularity in the pressure at $\tilde{r}=0$. Since this is impossible it follows that

$$\frac{d\tilde{p}_1}{d\tilde{r}}=-\frac{6\tilde{r}}{\tilde{z}_S{}^3}. \tag{4.48}$$

Though this expression is readily integrated in closed form to obtain $\tilde{p}_1$, it is unnecessary in subsequent calculations that we do so. Substitution of Eq. (4.48) into (4.47) now yields

$$\tilde{u}_1=\tilde{z}^2\tilde{z}_S{}^{-3}(2\tilde{z}-3\tilde{z}_S)-3\tilde{r}^2\tilde{z}^2\tilde{z}_S{}^{-4}(\tilde{z}-\tilde{z}_S)(1+\tfrac{1}{2}\Delta\tilde{r}^2). \tag{4.49}$$

Equations (4.46), (4.48) and (4.49) furnish the complete first-order inner fields. These fields may be expanded in powers of Δ as indicated in Eqs. (4.36)–(4.38) by utilising the fact that

$$\tilde{z}_S=\tilde{z}_S^*+\tfrac{1}{8}\Delta\tilde{r}^4.$$

Consequently,

$$\tilde{p}_1^*=3(\tilde{z}_S^*)^{-2} \tag{4.50†}$$

$$\tilde{u}_1^*=\tilde{z}^2(\tilde{z}_S^*)^{-3}(2\tilde{z}-3\tilde{z}_S^*)-3\tilde{r}^2\tilde{z}^2(\tilde{z}_S^*)^{-4}(\tilde{z}-\tilde{z}_S^*) \tag{4.51}$$

$$\tilde{v}_1^*=-3\tilde{r}\tilde{z}(\tilde{z}_S^*)^{-3}(\tilde{z}-\tilde{z}_S^*) \tag{4.52}$$

and

$$\tilde{p}_1^{**}=-9[(\tilde{z}_S^*)^{-1}-(\tilde{z}_S^*)^{-2}+\tfrac{1}{3}(\tilde{z}_S^*)^{-3}] \tag{4.53†}$$

$$\tilde{u}_1^{**}=-\tfrac{9}{4}\tilde{r}^4\tilde{z}^2(\tilde{z}_S^*)^{-4}(\tilde{z}-\tilde{z}_S)+\tfrac{3}{8}\tilde{r}^6\tilde{z}^2(\tilde{z}_S^*)^{-5}(4\tilde{z}-3\tilde{z}_S^*) \tag{4.54}$$

$$\tilde{v}_1^{**}=\tfrac{3}{8}\tilde{r}^5\tilde{z}(\tilde{z}_S^*)^{-4}(3\tilde{z}-2\tilde{z}_S^*). \tag{4.55}$$

4.3 *Second-order inner fields*

To solve the system of Eqs. (4.40)–(4.44), we note that Eq. (4.41) may be written in the form

$$\frac{\partial\tilde{p}_2}{\partial\tilde{z}}=-\frac{\partial}{\partial\tilde{z}}\left\{\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{v}_1^*)\right\}.$$

Partial integration with respect to $\tilde{z}$ yields

$$\tilde{p}_2=-\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{v}_1^*)+f(\tilde{r}) \tag{4.56}$$

where $f(\tilde{r})$ is a function of $\tilde{r}$ to be determined. From the continuity Eq. (4.32), the first term on the right-hand side is $-\partial\tilde{u}_1^*/\partial\tilde{z}$, whence

$$\tilde{p}_2=\frac{\partial\tilde{u}_1^*}{\partial\tilde{z}}+f(\tilde{r}). \tag{4.57}$$

Substituting this relation into Eq. (4.40) yields

$$\frac{\partial^2\tilde{v}_2}{\partial\tilde{z}^2}-\frac{df}{d\tilde{r}}=2\frac{\partial^2\tilde{u}_1^*}{\partial\tilde{r}\partial\tilde{z}}+Re\left(\tilde{v}_1^*\frac{\partial\tilde{v}_1^*}{\partial\tilde{r}}+\tilde{u}_1^*\frac{\partial\tilde{v}_1^*}{\partial\tilde{z}}+\tfrac{1}{2}\tilde{r}\frac{\partial\tilde{v}_1^*}{\partial\tilde{r}}+\tilde{z}\frac{\partial\tilde{v}_1^*}{\partial\tilde{z}}+\tfrac{1}{2}\tilde{v}_1^*\right). \tag{4.58}$$

Partial integration with respect to $\tilde{z}$ gives

$$\frac{\partial\tilde{v}_2}{\partial\tilde{z}}-\tilde{z}\frac{df}{d\tilde{r}}-g(\tilde{r})=2\frac{\partial\tilde{u}_1^*}{\partial\tilde{r}}+Re\int^{\tilde{z}}(\cdots)\partial\tilde{z}$$

where $g(\tilde{r})$ is a function of $\tilde{r}$ to be determined and $(\cdots)$ denotes the inertial function in parentheses

† In arriving at these explicit expressions for the pressure field, we have expanded Eq. (4.48) into the form

$$\frac{d\tilde{p}_1}{d\tilde{r}}=-6\tilde{r}(\tilde{z}_S^*)^{-3}+\Delta\tfrac{9}{4}\tilde{r}^5(\tilde{z}_S^*)^{-4}+O(\Delta^2)$$

and integrated each term on the right separately. The arbitrary constant arising in the integration of the pressure is irrelevant since we are dealing with incompressible fluids.

on the extreme right side of Eq. (4.58). One further partial integration with respect to $\tilde{z}$ yields

$$\tilde{v}_2-\tfrac{1}{2}\tilde{z}^2\frac{df}{d\tilde{r}}-\tilde{z}g-h(\tilde{r})$$
$$=2\frac{\partial}{\partial\tilde{r}}\int^{\tilde{z}}\tilde{u}_1^*\partial\tilde{z}+Re\int^{\tilde{z}}\partial\tilde{z}\int^{\tilde{z}}(\cdots)\partial\tilde{z}$$

where $h(\tilde{r})$ is a function of $\tilde{r}$ to be determined. Since $\tilde{u}_1^*$ and $\tilde{v}_1^*$ appearing in the integrands on the right-hand side of the above expression are known functions of $\tilde{r}$ and $\tilde{z}$ from Eqs. (4.51) and (4.52), one can perform the indicated integrations. A tedious, but straightforward, calculation eventually yields

$$\begin{aligned}\tilde{v}_2=&\tfrac{1}{2}\tilde{z}^2\frac{df}{d\tilde{r}}+\tilde{z}g+h-2\tilde{r}\tilde{z}^3(1+\tfrac{1}{2}\tilde{r}^2)^{-4}(3\tilde{z}-4-3\tilde{r}^2)\\&+2\tilde{r}^3\tilde{z}^3(1+\tfrac{1}{2}\tilde{r}^2)^{-5}[3\tilde{z}-4(1+\tfrac{1}{2}\tilde{r}^2)]\\&-\tfrac{1}{40}Re\,\tilde{r}\tilde{z}^4(1+\tfrac{1}{2}\tilde{r}^2)^{-7}[2\tilde{z}^2(2+7\tilde{r}^2)\\&-6\tilde{z}(2+7\tilde{r}^2)(1+\tfrac{1}{2}\tilde{r}^2)+30\tilde{r}^2(1+\tfrac{1}{2}\tilde{r}^2)^2]\\&-\tfrac{1}{4}Re\,\tilde{r}\tilde{z}^3(1+\tfrac{1}{2}\tilde{r}^2)^{-4}[3\tilde{z}-4(1+\tfrac{1}{2}\tilde{r}^2)].\end{aligned}\quad(4.59)$$

In order to satisfy the boundary condition (4.44b), we require that the arbitrary function of integration, $h(\tilde{r})$, satisfy the relation

$$h=0. \quad (4.60)$$

To find $\tilde{u}_2$, multiply Eq. (4.59) by $\tilde{r}$ and partially differentiate with respect to $\tilde{r}^{-1}\partial/\partial\tilde{r}$. From the continuity relation (4.42) this eventually yields an explicit expression for $\partial\tilde{u}_2/\partial\tilde{z}$ in terms of f, g, $\tilde{r}$, $\tilde{z}$ and Re. Partial integration of the resulting expression with respect to $\tilde{z}$ yields

$$\begin{aligned}\tilde{u}_2=&-\tfrac{1}{6}\tilde{z}^3\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}\left(\tilde{r}\frac{df}{d\tilde{r}}\right)-\tfrac{1}{2}\tilde{z}^2\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}(\tilde{r}g)+k(\tilde{r})\\&+\tfrac{2}{5}\tilde{z}^4(1+\tfrac{1}{2}\tilde{r}^2)^{-4}(6\tilde{z}-10+5\tilde{r}^2)\\&-\tfrac{2}{5}\tilde{r}^2\tilde{z}^4(1+\tfrac{1}{2}\tilde{r}^2)^{-5}(24\tilde{z}-20+5\tilde{r}^2)\\&+6\tilde{r}^4\tilde{z}^5(1+\tfrac{1}{2}\tilde{r}^2)^{-6}\\&-\tfrac{1}{20}Re\,\tilde{r}^2\tilde{z}^7(1+\tfrac{1}{2}\tilde{r}^2)^{-8}(2+7\tilde{r}^2)\\&+\tfrac{1}{140}Re\,\tilde{z}^6(1+\tfrac{1}{2}\tilde{r}^2)^{-7}[4\tilde{z}(1+7\tilde{r}^2)\\&+21(2\tilde{r}^2+7\tilde{r}^4)]-\tfrac{1}{20}Re\,\tilde{z}^5(1+\tfrac{1}{2}\tilde{r}^2)^{-6}\\&[2\tilde{z}(1+7\tilde{r}^2)+15\tilde{r}^4]+\tfrac{1}{20}Re\,\tilde{z}^4(1+\tfrac{1}{2}\tilde{r}^2)^{-4}\\&(6\tilde{z}+15\tilde{r}^2)-\tfrac{1}{2}Re\tilde{z}^4(1+\tfrac{1}{2}\tilde{r}^2)^{-3}.\end{aligned}\quad(4.61)$$

In order to satisfy the boundary condition (4.44a), we require that the arbitrary function of integration, $k(\tilde{r})$, satisfy the relation

$$k=0. \quad (4.62)$$

It remains to satisfy the two boundary conditions (4.43a, b). These boundary conditions serve to determine the unknown functions f and g. Boundary condition (4.43b) in conjunction with Eqs. (4.59)–(4.60) leads to the following relationship between f and g:

$$\tfrac{1}{4}(1+\tfrac{1}{2}\tilde{r}^2)^2\frac{df}{d\tilde{r}}+(1+\tfrac{1}{2}\tilde{r}^2)g$$
$$=-2\tilde{r}-\tfrac{3}{40}Re\,\tilde{r}(1+\tfrac{1}{2}\tilde{r}^2)^{-1}(6+\tilde{r}^2). \quad (4.63)$$

Similarly, boundary condition (4.43a) in conjunction with Eqs. (4.61)–(4.62) leads to a second relation between f and g; namely,

$$\tfrac{1}{6}(1+\tfrac{1}{2}\tilde{r}^2)^3\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}\left(\tilde{r}\frac{df}{d\tilde{r}}\right)+\tfrac{1}{2}(1+\tfrac{1}{2}\tilde{r}^2)^2\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}(\tilde{r}g)$$
$$=\tfrac{8}{5}(\tilde{r}^2-1)-\tfrac{1}{280}Re(1+\tfrac{1}{2}\tilde{r}^2)^{-1}(76-60\tilde{r}^2-7\tilde{r}^4). \quad (4.64)$$

These two equations are to be solved simultaneously for f and g. To do this we first solve Eq. (4.63) for g in terms of f and substitute the result into Eq. (4.64). In this way, f is found to satisfy the following differential equation:

$$\frac{d}{d\tilde{r}}\left[(1+\tfrac{1}{2}\tilde{r}^2)^3\tilde{r}\frac{df}{d\tilde{r}}\right]=-\tfrac{24}{5}\tilde{r}(1+4\tilde{r}^2)$$
$$-\tfrac{3}{70}Re\tilde{r}(1+\tfrac{1}{2}\tilde{r}^2)^{-1}(50+39\tilde{r}^2+7\tilde{r}^4). \quad (4.65)$$

Once this equation is solved, g may be obtained directly from Eq. (4.63). Knowledge of f and g suffices to determine $\tilde{p}_2$, $\tilde{v}_2$, $\tilde{u}_2$ in accordance with Eqs. (4.57) [in conjunction with Eq. (4.51)], (4.59)–(4.60) and (4.61)–(4.62), respectively.

Though Eq. (4.65) may be integrated exactly, in closed form, it proves sufficient for the purpose of computing the force on the body to only integrate it asymptotically as $\tilde{r}\to\infty$. In this limit, Eq. (4.65) becomes

$$\frac{d}{d\tilde{r}}\left[(1+\tfrac{1}{2}\tilde{r}^2)^3\tilde{r}\frac{df}{d\tilde{r}}\right]\sim-\tfrac{6}{5}(16+\tfrac{1}{2}Re)\tilde{r}^3.$$

Upon integrating and noting that $(1+\frac{1}{2}\tilde{r}^2)^3\sim\frac{1}{8}\tilde{r}^6$, we obtain the first integral

$$\frac{df}{d\tilde{r}}\sim-\tfrac{12}{5}(16+\tfrac{1}{2}Re)\tilde{r}^{-3}.$$

One further integration yields

$$f\sim\tfrac{6}{5}(16+\tfrac{1}{2}Re)\tilde{r}^{-2}. \tag{4.66}$$

For our ultimate purposes we do not require explicit knowledge of g.

4.4 *Outer expansion*

The solution of the unstretched system of differential equations (4.1)–(4.3) satisfying the boundary conditions (4.4)–(4.6) will explility contain the variable ε. For small values of this parameter, we assume that the dependent variables may be expanded as follows:

$$u(r,z;t)=u_1(r,z)+\varepsilon(t)u_2(r,z)+\dots \tag{4.67}$$

$$v(r,z;t)=v_1(r,z)+\varepsilon(t)v_2(r,z)+\dots \tag{4.68}$$

$$p(r,z;t)=p_1(r,z)+\varepsilon(t)p_2(r,z)+\dots \tag{4.69}$$

Such an expansion, which is valid only outside the gap, will be referred to as the "outer" expansion.

Substituting these expansions into Eqs. (4.1)–(4.6) and equating terms of $O(\varepsilon^0)$, we see that u_1, v_1, p_1 satisfy Eqs. (4.1)–(4.6) with u, v, p replaced by u_1, v_1, p_1, respectively, for the problem of a sphere in contact with a plane (i.e. for $\varepsilon=0$). Since the outer expansion is not expected to be valid in the inner region, i.e. within the gap, we need not be alarmed by the loss of physical meaning of this statement at the point of contact, arising from the inability of the sphere to penetrate the plane and thus satisfy Eq. (4.4a) at the contact point.

In terms of the original, unstretched, independent and dependent variables, we find from previous results, especially from Eqs. (4.50)–(4.52), (4.53), (4.57) and (4.66), that as $\tilde{r}\to\infty$,

$$u\sim\frac{4z^2}{r^6}(3r^2-8z)+l(r,z)+O(\varepsilon) \tag{4.70}$$

$$v\sim\frac{12z}{r^5}(r^2-2z)+m(r,z)+O(\varepsilon) \tag{4.71}$$

$$p\sim\frac{12}{r^4}-18\frac{\Delta}{\varepsilon}\frac{1}{r^2}-\frac{24z}{r^6}(4z-r^2)$$
$$+\tfrac{6}{5}(16+\tfrac{1}{2}Re)\frac{1}{r^2}+O(\varepsilon) \tag{4.72}$$

where $l(r,z)$ is a finite sum of terms of the form z^n/r^{2n-2} for $n>1$ and $m(r,z)$ is a sum of terms of the form z^n/r^{2n-1} for $n>1$. Hence, by the "matching principle", which requires that the inner and outer expansions be asymptotically equal in their common domain of validity, we require that the leading term of the outer expansion possess the following asymptotic form as $r\to0$ and $z\to0$:

$$u_1\sim\frac{4z^2}{r^6}(3r^2-8z)+l(r,z) \tag{4.73}$$

$$v_1\sim\frac{12z}{r^5}(r^2-2z)+m(r,z) \tag{4.74}$$

$$p_1\sim\frac{12}{r^4}-18\frac{\Delta}{\varepsilon}\frac{1}{r^2}-\frac{24z}{r^6}(4z-r^2)$$
$$+\tfrac{6}{5}(16+\tfrac{1}{2}Re)\frac{1}{r^2}. \tag{4.75}$$

Determination of the force on the body to the order in ε for which our analysis is valid does not require a detailed calculation of u_1, v_1, p_1. Rather, the asymptotic forms of these fields given by Eqs. (4.73)–(4.75) already furnish sufficient information about the singular "outer" force contribution to eliminate the need for an explicit computation of these fields.

4.5 *Force on the body*

The vector force $\mathbf{F}$ exerted by the fluid on the body is given by the general expression

$$\mathbf{F}=\frac{1}{6\pi}\int \mathrm{d}\mathbf{S}\,.\,\mathbf{\Pi} \tag{4.76}$$

where $\mathrm{d}\mathbf{S}$ is a directed element of surface area pointing into the fluid, and $\mathbf{\Pi}$ is the pressure tensor, all appropriately nondimensionalised via the substitutions $\mathbf{F}=\mathbf{F}'/6\pi\mu bU$, $\mathrm{d}\mathbf{S}=\mathrm{d}\mathbf{S}'/b^2$ and $\mathbf{\Pi}=\mathbf{\Pi}'b/\mu U$. The extraneous factor of 6π is introduced so as to make the notation of this section identical with that of Section 2. The integration in Eq. (4.76) is over the surface of the body.

If $\mathrm{d}\mathbf{R}$ is a directed element of arc length measured along the surface of the axisymmetric body in a meridian plane, then it is easily shown from the definition of a directed element of surface area in terms of a cross product that, for a body of revolution,

$$\mathrm{d}\mathbf{S}=(\mathbf{i}_\phi r\mathrm{d}\phi)\times \mathrm{d}\mathbf{R} \tag{4.77}$$

where $\mathbf{i}_\phi$ is a unit vector perpendicular to the meridian plane in which $\mathrm{d}\mathbf{R}$ lies. The sense of $\mathrm{d}\mathbf{R}$ is to be such that the cross product generates the outer, rather than inner, normal to the body (see Fig. 2). Now, in circular cylindrical coordinates [17, p. 493],

$$\mathrm{d}\mathbf{R}=\mathbf{i}_r\mathrm{d}r+\mathbf{i}_z\mathrm{d}z\,. \tag{4.78}$$

Substitution of this into Eq. (4.77) yields an explicit representation for $\mathrm{d}\mathbf{S}$ in circular cylindrical coordinates, applicable to any body of revolution. Since we deal only with axisymmetric motions, for which all of the dynamical and kinematical variables are independent of the azimuthal angle ϕ, we may integrate the resulting expression for $\mathrm{d}\mathbf{S}$ over ϕ to obtain

$$\mathrm{d}\mathbf{S}=\mathbf{i}_z\mathrm{d}S_z+\mathbf{i}_r\mathrm{d}S_r \tag{4.79}$$

where

$$\mathrm{d}S_z=-2\pi r\mathrm{d}r\,,\qquad \mathrm{d}S_r=2\pi r\mathrm{d}z\,. \tag{4.80a, b}$$

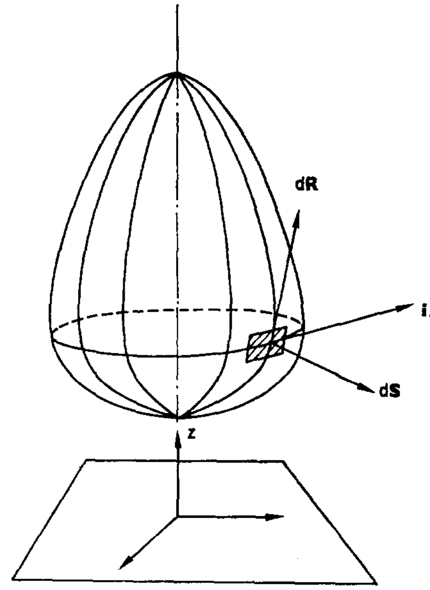

FIG. 2. Directed element of surface area on a body of revolution.

For axisymmetric motions, the only nonzero component of the vector force lies in the z-direction. Thus, from Eqs. (4.76) and (4.79), we obtain for the force $F=\mathbf{i}_z\,.\,\mathbf{F}$ on the body in the positive z-direction,

$$F=-\frac{1}{6\pi}\int p\mathrm{d}S_z+\frac{1}{6\pi}\int \tau_{zz}\mathrm{d}S_z+\frac{1}{6\pi}\int \tau_{zr}\mathrm{d}S_r\,.$$

Upon introducing the expressions relating the viscous stresses, τ_{ij}, to the velocity gradients (in cylindrical coordinates) for an incompressible Newtonian fluid and the formulas (4.80) for the elements of surface area, this becomes

$$F=\tfrac{1}{3}\int_S pr\mathrm{d}r-\tfrac{2}{3}\int_S \frac{\partial u}{\partial z}r\mathrm{d}r$$
$$+\tfrac{1}{3}\int_S\left(\frac{\partial v}{\partial z}+\frac{\partial u}{\partial r}\right)r\mathrm{d}z\,. \tag{4.81}$$

The integrations are to be performed over the surface, $z=z(r)$, of the body

Each integral in this equation may be decomposed into the sum of two integrals. The first represents integration over the "inner" portion of the body surface, in the immediate vicinity of the contact point—more precisely, for values of $0 \leqslant r \leqslant Y(z \ll 1)$, where Y is small compared with unity and arbitrarily chosen in any manner such that $r = Y$ may be considered as lying in *both* the inner and outer expansions. The second part of the integral represents integration over the remainder of the particle, $\bar{S}$, say, away from the point of contact.

According to the equation

$$z = \varepsilon + \tfrac{1}{2}r^2 + \tfrac{1}{8}\frac{\Delta}{\varepsilon}r^4 + \dots$$

defining the surface of the body, $r = O(\sqrt{\varepsilon})$ in the region of the inner expansion, whereas $r = O(\varepsilon^0)$ within the region of validity of the outer expansion. Accordingly, Y must tend to zero as $\varepsilon \to 0$. Though it is immaterial what explicit function Y is of ε, one could, for example, choose

$$Y = c\varepsilon^{q/2} \tag{4.82}$$

where c and q are any positive constants—independent of ε—the constant q being chosen so that the inequality

$$0 < q < 1 \tag{4.83}$$

is satisfied. Then $r = Y$ certainly lies within the regions of validity of both the inner and outer expansions. Because of the somewhat arbitrary manner in which Y is defined, Y cannot appear in the final expression for the force F, though it can (and does) appear in the individual inner and outer contributions, F_i and F_o, respectively, to F. It is only the *sum* of these contributions which must be independent of Y, not the inner and outer terms themselves.

In accordance with our decomposition of Eq. (4.81) into inner and outer parts, we may write

$$F = F(\varepsilon) = F_i(\varepsilon;\, Y) + F_o(\varepsilon;\, Y) \tag{4.84}$$

in which the inner contribution to the force is

$$F_i = \tfrac{1}{3}\int_{r=0}^{Y} pr\mathrm{d}r - \tfrac{2}{3}\int_{r=0}^{Y} \frac{\partial u}{\partial z} r\mathrm{d}r + \tfrac{1}{3}\int_{r=0}^{Y} \left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r}\right) r\mathrm{d}z \tag{4.85}$$

whereas the outer contribution is

$$F_o = \tfrac{1}{3}\int_{\bar{S}} pr\mathrm{d}r - \tfrac{2}{3}\int_{\bar{S}} \frac{\partial u}{\partial z} r\mathrm{d}r + \tfrac{1}{3}\int_{\bar{S}} \left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r}\right) r\,\mathrm{d}z\,. \tag{4.86}$$

4.6 *Inner expansion of force*

Equation (4.85) may be written in terms of stretched variables via the substitutions (4.9), (4.11) and (4.12). Since, on the surface of the body,

$$\tilde{z} = 1 + \tfrac{1}{2}\tilde{r}^2 + \tfrac{1}{8}\Delta\tilde{r}^4 + \dots \tag{4.87}$$

we have

$$\mathrm{d}\tilde{z} = (1 + \tfrac{1}{2}\Delta\tilde{r}^2 + \dots)\tilde{r}\,\mathrm{d}\tilde{r} \tag{4.88}$$

on the body. Substitution into Eq. (4.85) thus yields

$$F_i = \frac{1}{3\varepsilon}\int_{\tilde{r}=0}^{Y/\sqrt{\varepsilon}} \tilde{p}\tilde{r}\mathrm{d}\tilde{r} - \tfrac{2}{3}\int_{\tilde{r}=0}^{Y/\sqrt{\varepsilon}} \frac{\partial\tilde{u}}{\partial\tilde{z}}\tilde{r}\,\mathrm{d}\tilde{r} + \tfrac{1}{3}\int_{\tilde{r}=0}^{Y/\sqrt{\varepsilon}} \left(\frac{\partial\tilde{v}}{\partial\tilde{z}} + \varepsilon\frac{\partial\tilde{u}}{\partial\tilde{r}}\right)(1 + \tfrac{1}{2}\Delta\tilde{r}^2 + \dots)\tilde{r}^2\mathrm{d}\tilde{r} \tag{4.89}$$

where the integrands in these expressions are to be evaluated on the surface (4.87). Upon introducing the expansions (4.39), etc. into these expressions and retaining terms of only the lowest orders in ε and Δ, we obtain

$$F_i = \frac{1}{3\varepsilon}\int_0^{Y/\sqrt{\varepsilon}} \tilde{p}_1^*\tilde{r}\mathrm{d}\tilde{r} + \tfrac{1}{3}\frac{\Delta}{\varepsilon}\int_0^{Y/\sqrt{\varepsilon}} \tilde{p}_1^{**}\tilde{r}\mathrm{d}\tilde{r} + \tfrac{1}{3}\int_0^{Y/\sqrt{\varepsilon}} \tilde{p}_2\tilde{r}\mathrm{d}\tilde{r} - \tfrac{2}{3}\int_0^{Y/\sqrt{\varepsilon}} \frac{\partial\tilde{u}_1^*}{\partial\tilde{z}}\tilde{r}\mathrm{d}\tilde{r} + \tfrac{1}{3}\int_0^{Y/\sqrt{\varepsilon}} \frac{\partial\tilde{v}_1^*}{\partial\tilde{z}}\tilde{r}^2\mathrm{d}\tilde{r} + \dots \tag{4.90}$$

where the integrations are to be effected by setting

$$\tilde{z} = 1 + \tfrac{1}{2}\tilde{r}^2 \tag{4.91}$$

in the integrands and integrating over $\tilde{r}$. Each of the integrals in Eq. (4.90) will now be evaluated in turn.

From Eq. (4.50) we have that

$$\tilde{p}_1^* = 3(1+\tfrac{1}{2}\tilde{r}^2)^{-2}$$

whence

$$\int_0^{Y/\sqrt{\varepsilon}} \tilde{p}_1^* \tilde{r}\, d\tilde{r} = 3 - \frac{3}{1+(Y^2/2\varepsilon)}$$

$$\sim 3 - \frac{6\varepsilon}{Y^2} \quad \text{as } \varepsilon \to 0. \qquad (4.92)\dagger$$

To evaluate the second integral in Eq. (4.90), we note from Eq. (4.53) that

$$\tilde{p}_1^{**} = -9\left[\frac{1}{1+\frac{1}{2}\tilde{r}^2} - \frac{1}{(1+\frac{1}{2}\tilde{r}^2)^2} + \frac{1}{3(1+\frac{1}{2}\tilde{r}^2)^3}\right]. \qquad (4.93)$$

Though the appropriate integral of this function is easily evaluated in closed form, an exact integration is unnecessary for our purposes. We note that the upper limit of integration becomes infinite as $\varepsilon \to 0$. Accordingly, since $\tilde{p}_1^{**} = O(\tilde{r}^{-2})$ as $\tilde{r} \to \infty$, the dominant contribution to the integral occurs at the larger values of $\tilde{r}$. Hence, for the purpose of evaluating this dominant contribution, it suffices to employ the asymptotic expansion of $\tilde{p}_1^{**}$ for large $\tilde{r}$ in place of Eq. (4.93),‡ namely,

$$\tilde{p}_1^{**} \sim -18\tilde{r}^{-2} \quad \text{as } \tilde{r} \to \infty.$$

Integration of this yields

$$\int^{Y/\sqrt{\varepsilon}} \tilde{p}_1^{**} \tilde{r}\, d\tilde{r} \sim -18 \ln(Y/\sqrt{\varepsilon})$$

$$= 9 \ln \varepsilon - 18 \ln Y. \qquad (4.94)$$

† It follows from the definition of Y that $\varepsilon/Y^2 \to 0$ as $\varepsilon \to 0$. This is clear if, for example, Y is chosen as in Eq. (4.82).

‡ This can be explicitly demonstrated by noting that we have, exactly.

$$\int_0^{Y/\sqrt{\varepsilon}} \tilde{p}_1^{**} \tilde{r}\, d\tilde{r} = -9\left[\ln\left(1+\frac{Y^2}{2\varepsilon}\right) + \left\{\frac{1}{1+(Y^2/2\varepsilon)} - 1\right\} - \frac{1}{6}\left\{\frac{1}{[1+(Y^2/2\varepsilon)]^2} - 1\right\}\right].$$

Expanding this expression as $\varepsilon \to 0$ yields the same asymptotic result as is cited in Eq. (4.94).

To evaluate the third integral in Eq. (4.90) we note that $\tilde{p}_2$ is given by Eq. (4.57). However, from Eq. (4.51), on the surface of the body,

$$\left.\frac{\partial \tilde{u}_1^*}{\partial \tilde{z}}\right|_{\tilde{z}=1+\frac{1}{2}\tilde{r}^2} = -\frac{3\tilde{r}^2}{(1+\frac{1}{2}\tilde{r}^2)^2}$$

$$\sim -12\tilde{r}^{-2} \quad \text{as } \tilde{r} \to \infty. \qquad (4.95)$$

Using the asymptotic value of f in Eq. (4.66), we then obtain

$$\tilde{p}_2|_{\tilde{z}=1+\frac{1}{2}\tilde{r}^2} \sim \tfrac{6}{5}(6+\tfrac{1}{2}Re)\tilde{r}^{-2} \qquad (4.96)$$

Consequently,

$$\int^{Y/\sqrt{\varepsilon}} \tilde{p}_2 \tilde{r}\, d\tilde{r} \sim -\tfrac{3}{5}(6+\tfrac{1}{2}Re)\ln \varepsilon$$

$$+\tfrac{6}{5}(6+\tfrac{1}{2}Re)\ln Y. \qquad (4.97)$$

Using Eq. (4.95), the fourth integral in Eq. (4.90) is found to be

$$\int^{Y/\sqrt{\varepsilon}} \frac{\partial \tilde{u}_1^*}{\partial \tilde{z}} \tilde{r}\, d\tilde{r} \sim 6 \ln \varepsilon - 12 \ln Y. \qquad (4.97)$$

Finally, to evaluate the last integral in Eq. (4.90), we note from Eq. (4.52) that on the surface of the body:

$$\left.\frac{\partial \tilde{v}_1^*}{\partial \tilde{z}}\right|_{\tilde{z}=1+\frac{1}{2}\tilde{r}^2} = -\frac{3\tilde{r}}{(1+\frac{1}{2}\tilde{r}^2)^2}$$

$$\sim -12\tilde{r}^{-3} \quad \text{as } \tilde{r} \to \infty.$$

Therefore,

$$\int^{Y/\sqrt{\varepsilon}} \frac{\partial \tilde{v}_1^*}{\partial \tilde{z}} \tilde{r}^2\, d\tilde{r} \sim 6 \ln \varepsilon - 12 \ln Y. \qquad (4.99)$$

Collecting the various results together gives

$$F_i \sim \frac{1}{\varepsilon}\left[1 - \tfrac{1}{5}\left(16 - 15\frac{\Delta}{\varepsilon} + \tfrac{1}{2}Re\right)\varepsilon \ln \varepsilon\right]$$

$$-2Y^{-2} + \tfrac{2}{5}\left(16 - 15\frac{\Delta}{\varepsilon} + \tfrac{1}{2}Re\right)\ln Y. \qquad (4.100)$$

4.7 *Outer expansion of force*

Upon introducing the outer expansions (4.67)–(4.68) into Eq. (4.86) and retaining only those terms of $O(\varepsilon^0)$, we obtain

$$F_o=\tfrac{1}{3}\int_S p_1 r\,\mathrm{d}r-\tfrac{2}{3}\int_S \frac{\partial u_1}{\partial z} r\,\mathrm{d}r+\tfrac{1}{3}\int_S\left(\frac{\partial v_1}{\partial z}+\frac{\partial u_1}{\partial r}\right) r\,\mathrm{d}z+O(\varepsilon). \qquad (4.101)$$

In terms of unstretched variables, the surface of the body is defined by

$$z=\varepsilon+\tfrac{1}{2}r^2+\ \ldots \qquad (4.102)$$

whereupon

$$\mathrm{d}z=r\,\mathrm{d}r+\ \ldots \qquad (4.103)$$

on the body surface.

By definition, $Y\ll 1$. Furthermore, since the asymptotic behavior of u_1, v_1 and p_1 as $r\to 0$ (and $z\to 0$) is as indicated in Eqs. (4.73)–(4.75), it follows that the dominant contribution of the integrands to each of the three integrals in Eq. (4.101) occurs at the smaller values of r. Accordingly, even though u_1, v_1 and p_1 are explicitly known only in the neighborhood of $r=0$ and $z=0$, rather than throughout the entire outer region ($0<r<\infty$, $0<z<\infty$), this limited knowledge nevertheless suffices to obtain the dominant terms in the expression (4.101) for the outer force. Hence,

$$F_o\sim\tfrac{1}{3}\int_Y p_1 r\,\mathrm{d}r-\tfrac{2}{3}\int_Y \frac{\partial u_1}{\partial z} r\,\mathrm{d}r+\tfrac{1}{3}\int_Y\left(\frac{\partial v_1}{\partial z}+\frac{\partial u_1}{\partial r}\right) r^2\mathrm{d}r \qquad (4.104)$$

where u_1, v_1, p_1 are given to a sufficient degree of approximation by Eqs. (4.73)–(4.75).

To evaluate the first integral in the above expression we observe from Eqs. (4.75) and (4.102), on the surface of the body,

$$p_1\sim\frac{12}{r^4}-6\left[3\frac{\Delta}{\varepsilon}+2-\tfrac{1}{5}(16+\tfrac{1}{2}Re)\right]\frac{1}{r^2}.$$

Consequently,

$$\int_Y p_1 r\,\mathrm{d}r\sim 6Y^{-2}+6\left[3\frac{\Delta}{\varepsilon}+2-\tfrac{1}{5}(16+\tfrac{1}{2}Re)\right]\ln Y. \qquad (4.105)$$

Similarly, since from Eq. (4.73),

$$\partial u_1/\partial z\sim -12/r^2+\partial l/\partial z$$

on the body as $r\to 0$, and since $\partial l/\partial z=O(1)$, it follows that

$$\int_Y \frac{\partial u_1}{\partial z} r\,\mathrm{d}r\sim 12\ln Y. \qquad (4.106)$$

In addition, from Eqs. (4.73) and (4.74),

$$\partial v_1/\partial z\sim -12/r^3+\partial m/\partial z$$

and $\partial u_1/\partial r\sim 12/r+\partial l/\partial r$ on the surface of the body, where $\partial m/\partial z=O(r^{-1})$ and $\partial l/\partial r=O(r)$ as $r\to 0$. Hence, so far as the dominant terms are concerned, $\partial v_1/\partial z+\partial u_1/\partial r\sim -12/r^3$. This makes

$$\int_Y\left(\frac{\partial v_1}{\partial z}+\frac{\partial u_1}{\partial r}\right) r^2\,\mathrm{d}r\sim 12\ln Y. \qquad (4.107)$$

Combining these results yields

$$F_o\sim 2Y^{-2}-\tfrac{2}{5}\left(16-15\frac{\Delta}{\varepsilon}+\tfrac{1}{2}Re\right)\ln Y. \qquad (4.108)$$

4.8 *Total force*

In accordance with Eq. (4.84), Eqs. (4.100) and (4.108) furnish the following expression for the force on the body:

$$F\sim\frac{1}{\varepsilon}\left[1-\tfrac{1}{5}\left(16-15\frac{\Delta}{\varepsilon}+\tfrac{1}{2}Re\right)\varepsilon\ln\varepsilon\right]. \qquad (4.109)$$

This constitutes the principal result of our analysis. The variable Y has disappeared from the final expression for the force, as it must because of the somewhat arbitrary manner in which it was defined. This type of behavior is, of course, similar to that observed in Section 2 in connection with the comparable variable X.

5. Discussion

In dimensional form, Eq. (4.109) is of the form

$$\frac{F'}{6\pi\mu bU}\sim\frac{1}{\varepsilon}\left[1+\tfrac{1}{5}\left(16-15\frac{\Delta}{\varepsilon}+\tfrac{1}{2}Re\right)\varepsilon\ln\frac{1}{\varepsilon}\right] \quad (5.1)$$

this formula being valid for the case where $\varepsilon \ll 1$ and $\varepsilon|Re| \ll 1$. For a sphere ($\Delta/\varepsilon=1$) it reduces to

$$\frac{F'}{6\pi\mu bU}\sim\frac{1}{\varepsilon}\left[1+\tfrac{1}{5}(1+\tfrac{1}{2}Re)\varepsilon\ln\frac{1}{\varepsilon}\right]. \quad (5.2)$$

In the limit of a Stokes flow ($Re=0$) the latter becomes

$$\frac{F'}{6\pi\mu bU}\sim\frac{1}{\varepsilon}\left(1+\tfrac{1}{5}\varepsilon\ln\frac{1}{\varepsilon}\right). \quad (5.3)$$

To the order in ε indicated, Eq. (5.3) is identical with Eq. (2.48). In view of the very different techniques employed in the two independent methods of deriving the result, this agreement of the singular terms in the asymptotic expansion of the force provides confidence in the accuracy of the various computations. In fact, for the case of a sphere in a Stokes flow, we observe that not only are the overall results of the two techniques identical, but also that the individual inner and outer contributions to the force are identical, at least insofar as the singular terms are concerned. This may be seen by comparing Eq. (4.100) with Eq. (2.30) and Eq. (4.108) with Eq. (2.44), after setting $Y=X\sqrt{2}$ and introducing the relation (2.4a) connecting the parameters α and ε [see also Eqs. (2.12) and (4.82)].

Experimental data on the low Reynolds number fall of a sphere towards a plane wall are presented by MacKay, Suzuki and Mason [6, 7]. These data have been shown [6, 7] to be in excellent agreement with the Maude [1]–Brenner [2] quasistatic Stokes flow analysis at the "larger" gap widths, as well as with Taylor's [cf. 3] "lubrication-theory" analysis at the smaller gap widths. Data at substantially higher Reynolds numbers would be required to substantiate Eq. (5.2).

According to Eq. (5.2) the drag force on the sphere may be written as

$$\frac{|F'|}{6\pi\mu b|U|}\sim\frac{1}{\varepsilon}\left[1+\tfrac{1}{5}\left(1+\tfrac{1}{2}\frac{bU}{\nu}\right)\varepsilon\ln\frac{1}{\varepsilon}\right] \quad (5.4)$$

where $U>0$ when the sphere moves towards the wall and $U<0$ when it moves away from the wall. The coefficient $\varepsilon\ln(1/\varepsilon)$ is positive for the small values of ε to which the present theory is applicable. Thus, all other things being equal, the resistance of the sphere is greater when it approaches the wall than when it recedes from it, at least in so far as first-order inertial effects are concerned. This conclusion should be compared with results gleaned from ideal fluid theory [18], which state that a sphere moving perpendicular to a wall is repelled by the wall, whether the particle motion is directed towards or away from it. Thus, as in our present problem, inertial forces hinder the particle in the former case and assist it in the latter.

In the theory leading to the result given by Eq. (5.1) for the force on the particle, it was assumed that the particle moved towards or away from the plane with a *constant* velocity. For the more general problem in which the particle possesses a (dimensional) acceleration a' towards the plane in addition to its velocity U, one may define an "acceleration Reynolds number" as

$$Re_a=a'b^3/\nu^2 .$$

Inclusion of this particle acceleration would therefore require that in the inner expansion, as defined in Section 4, there now be an additional term of order $\varepsilon Re_a/Re$, whereas in the outer expansion terms of order Re_a/Re would appear. Hence, such an effect would be dominant in the outer expansion and its investigation would therefore necessitate a detailed analysis of the outer region of expansion. One may expect that particle accleration would produce a term of order Re_a/Re on the right-hand side of Eq. (5.1). Thus, in order to neglect such a term in comparison with terms of orders $\ln(1/\varepsilon)$ and $Re\ln(1/\varepsilon)$, one requires that

$$\left|\frac{Re_a}{Re}\right| \ll [\max.\ (1,\ |Re|)]\ln\left(\frac{1}{\varepsilon}\right)$$

i.e.

$$\left|\frac{a'b^2}{U\nu}\right| \ll \left[\max.\left(1, \frac{|U|b}{\nu}\right)\right]\ln\left(\frac{1}{\varepsilon}\right).$$

The present analysis, in conjunction with prior analyses of other, related, translational and rotational, sphere-plane wall Stokes motions (see Table 3), furnish all the intermediate results necessary for the completely general quasistatic treatment of a translating-rotating sphere in motion near a plane wall at an arbitrary angle of attack. That one may treat the general problem on the basis of the special configurations tabulated in Table 3 is a consequence of the linearity of Stokes equations. The intrinsic dyadic resistance scheme of Brenner [19] provides a general framework into which these special cases may be embedded. In particular, the force and torque about the sphere center (O) may be written quite generally as

$$\mathbf{F}' = -\mu(b\mathbf{K}\cdot\mathbf{U}_o + b^2\mathbf{C}_o^\dagger\cdot\boldsymbol{\omega}) \tag{5.5}$$

$$\mathbf{T}'_o = -\mu(b^2\mathbf{C}_o\cdot\mathbf{U}_o + b^3\boldsymbol{\Omega}_o\cdot\boldsymbol{\omega}) \tag{5.6}$$

where $\mathbf{U}_O$ and $\boldsymbol{\omega}$ are, respectively, the translational velocity of the sphere center and the angular velocity of the sphere; $\mathbf{K}$, $\boldsymbol{\Omega}_O$ and $\mathbf{C}_O$ are, respectively, the dimensionless translational, rotational and coupling resistance dyadics [19] for the sphere in the presence of the wall. By symmetry arguments these resistance dyadics have the following forms:

$$\mathbf{K} = \mathbf{ee}K_\perp + (\mathbf{I}-\mathbf{ee})K_\parallel \tag{5.7}$$

$$\boldsymbol{\Omega}_o = \mathbf{ee}\Omega_\perp + (\mathbf{I}-\mathbf{ee})\Omega_\parallel \tag{5.8}$$

$$\mathbf{C}_o = \boldsymbol{\varepsilon}\cdot\mathbf{e}C_\parallel \tag{5.9}$$

where $\mathbf{e}$ is a unit vector perpendicular to the wall, pointing *into* the fluid; $\mathbf{I}$ is the dyadic idemfactor; $\boldsymbol{\varepsilon}$ is the unit isotropic triadic. The five nondimensional scalar resistance coefficients each depend only upon b/h (or, equivalently, ε). They are each positive numbers and possess the following properties: $K_\perp/6\pi$, $K_\parallel/6\pi$, $\Omega_\perp/8\pi$ and $\Omega_\parallel/8\pi$ tend asymptotically to unity as $b/h \to 0$; $C_\parallel$ tends to zero in this limit. The values of these scalar resistance coefficients as a function of gap width are easily extracted from the particular solutions cited in Table 3. The particular resistance coefficient furnished by each of these solutions is tabulated in the last column of Table 3.

TABLE 3. LITERATURE REFERENCES TO THE QUASISTATIC STOKES MOTION OF A SPHERE IN PROXIMITY TO A PLANE WALL

	Reference			
Problem solved in the literature	Exact bipolar coordinate solution	Asymptotic "lubrication-theory" solution for small gap widths	Asymptotic "method of reflections" solution for large gap widths	Scalar resistance coefficients obtainable from solution
Sphere translating perpendicular to a plane wall	[1, 2]	[This paper]	[1, 2]	$K_\perp$
Sphere translating parallel to a plane wall	[20, 21, 22]	[21, 22]	[23]†	$K_\parallel$ and $C_\parallel$
Sphere rotating about an axis perpendicular to a plane wall	[8]	[This paper]	[15]	$\Omega_\perp$
Sphere rotating about an axis parallel to a plane wall	[13]‡	[22]	[24]	$\Omega_\parallel$ and $C_\parallel$

‡ Corrections to the numerical results cited in [13] are given in [22].
† MAUDE [24] also gives a result for this case; however, as pointed out in [22], his analysis contains a numerical error for the translational case.

As follows from Eq. (1.1) for sufficiently small gap widths, the hydrodynamic force on a sphere moving perpendicular to a plane wall is given by the expression

$$F' = 6\pi\mu b^2 U/\delta .$$

Setting $U = -\mathrm{d}\delta/\mathrm{d}t'$ and integrating for a constant force (equal to the net weight of the particle corrected for fluid buoyancy) yields

$$\ln\delta = -\frac{F'}{6\pi\mu b^2}t' + \text{const.}$$

Because of the logarithmic decrease of gap width with time, an effectively infinite time would be required for a settling sphere to actually attain contact with a wall towards which it was settling. Conversely, a particle literally in contact with a wall and subject to a force acting in a direction tending to move it away from the wall could never be separated from the wall by any finite force; rather, it would appear to a naive observer that the particle was held fast to the wall by some unknown "attractive" force. In practice, of course, neither of these conclusions would be literally true, since other factors would certainly intervene, e.g. surface roughness, cavitation [cf. 22], etc. Nevertheless, an experimental inquiry aimed at assessing the relative importance of these "nonhydrodynamic" factors would be welcome.

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Notation

a	acceleration of particle
$A(\tilde{r})$	function of $\tilde{r}$
b	sphere radius
$B(\tilde{r})$	function of $\tilde{r}$
c	positive constant
C	numerical constant defined in Eq. (2.43)
$C_{\parallel}$	scalar coupling coefficient defined in Eq. (5.9)
$C(\tilde{r})$	function of $\tilde{r}$
C_1	numerical constant defined in Eq. (2.39)
C_2	numerical constant defined in Eq. (2.41)
$\mathbf{C}$	coupling dyadic
$\mathrm{d}\mathbf{R}$	directed element of arc length
$\mathrm{d}S_j$	scalar element of surface area on coordinate surface $x_j = \text{const.}$
$\mathrm{d}\mathbf{S}$	directed element of surface area
$\mathbf{e}$	unit vector
$f(m)$	function of m defined in Eq. (2.33)
$f(\tilde{r})$	function of $\tilde{r}$
F	scalar force on body
$\mathbf{F}$	vector force on body
$g(\lambda)$	function of λ defined in Eq. (2.40)
$g(\tilde{r})$	function of $\tilde{r}$
h	distance from plane to sphere center
$h(\tilde{r})$	function of $\tilde{r}$
$(\mathbf{i}_r, \mathbf{i}_\phi, \mathbf{i}_z)$	unit vectors in circular cylindrical coordinates
I_1, I_2, I_3	summations defined in Eqs. (2.15)–(2.17)
$\mathbf{I}$	dyadic idemfactor
k	numerical constant defined in Eq. (2.49)
$k(\tilde{r})$	function of $\tilde{r}$
K	numerical constant defined in Eq. (2.46)
K	scalar translational resistance coefficient in Eq. (5.7)
$\mathbf{K}$	translational resistance dyadic
$l(r, z)$	function of r and z
m	summation index
$m(r, z)$	function of r and z
n	integral summation index
N	large positive integer lying in region of applicability of both inner and outer expansions
p	fluid dynamic pressure
P	refers to surface of plane wall
q	positive constant lying between zero and unity
r	cylindrical coordinate
Re	Reynolds number, bU/ν
Re_a	acceleration Reynolds number
S	refers to surface of body
t	time
T	torque on body
u	z-component of fluid velocity
$\mathbf{u}$	vector fluid velocity
U	speed of particle
$\mathbf{U}$	vector translational velocity of particle
v	r-component of fluid velocity
X	small, positive variable
Y	small radial distance in cylindrical coordinates lying in region of applicability of both inner and outer expansions
z	Cartesian coordinate measured normal to plane wall
α	bipolar coordinate defined in Eq. (2.3)

γ Euler's constant
δ minimum gap width between pole and plane, defined in Eq. (2.6)
Δ small parameter
Δm difference between two successive values of m
ε dimensionless gap width, defined in Eq. (2.5)
$\boldsymbol{\varepsilon}$ unit isotropic triadic
$\zeta(s)$ Riemann Zeta Function of argument s
θ variable in Eq. (2.13)
λ integration variable
μ viscosity
ν kinematic viscosity
ξ function
$\boldsymbol{\Pi}$ pressure dyadic
τ_{ij} viscous stress component
ϕ azimuthal angle
ω angular velocity of rotation
$\boldsymbol{\omega}$ angular velocity vector
Ω scalar rotational resistance coefficient
$\boldsymbol{\Omega}$ rotational resistance dyadic

Subscripts

i inner
o outer
O sphere center
S surface of body
1, 2 first- and second-order perturbation fields
$\parallel$ parallel motion to wall
$\perp$ perpendicular motion to wall

Superscripts

$'$ dimensional quantity
*, ** terms arising from the decomposition noted in Eqs. (4.36)–(4.38)
† transposition operator

Symbols over characters

$\sim$ stretched, inner variable

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Résumé—On utilise des techniques de perturbation simple pour calculer la force hydrodynamique d'une sphère en mouvement, à des nombres de Reynolds faibles, perpendiculaire à une paroi plane solide limitant un fluide visqueux semi-infini, dans le cas limite où la largeur de l'écartement entre la sphère et le plan de la paroi tend vers zéro. Deux analyses distinctes mais voisines du problème sont présentées. Dans la première analyse, l'expression exacte des coordonnées bipolaires pour la force exprimée indépendamment par MAUDE [1] et BRENNER [2] dans le cas du courant quasi statique de Stokes, est étendue par l'intermédiare d'un procédé asymptotique nouveau. La seconde analyse, dont le cadre est plus général, donne une solution de la perturbation des équations *instables* de Navier–Stokes pour une particle plus généralement symétrique par rapport à l'axe qu'une sphère, en tenant compte de la nature limitée du nombre de Reynolds. Quand on tient compte du nombre de Reynolds, la force agissant sur la particule change suivant que cette dernière se dirige vers la paroi ou s'en écarte.

En adoptant les techniques présentées dans la première partie de cet exposé, on tire des formules pour le couple de Stokes nécessaire pour maintenir la rotation symétrique d'une haltère dans un fluide non limité, ainsi que pour les couples nécessaires afin de maintenir la rotation symétrique d'une sphère touchant une paroi plane rigide et touchant une surface planar libre.

Zusammenfassung—Die Methode der einmaligen Störung wird benutzt, um die hydrodynamische Kraft au berechnen, die durch eine sich bei niedrigen Reynolds-Zahlen senkrecht auf eine feste, ebene Wand zu bewegende Kugel ausgeübt wird, wenn diese Wand eine einseitig-unendliche viskose Flüssigkeit abgrenzt und der Grenzfall eintritt, dass der Abstand zwischen Kugel und Ebene gegen Null geht. Zwei verschiedene, aber miteinander in Zusammenhang stehende Betrachtungen des Problems werden dargelegt. Bei der ersten wird der von MAUDE [1] und BRENNER [2] unabhängig angegebene exakte Bipolar-Koordinatenausdruck für die Kraft auf die quasi statische Stokes-Strömung erweitert mit Hilfe eines neuen asymptotischen Verfahrens. Die zweite Betrachtung mit allgemeiner gehaltener Zielsetzung liefert eine Störungslösung der nicht stationären Navier–Stokes–Gleichung für ein allgemeineres achsensymmetrisches Teilchen, als es die Kugel ist und berücksichtigt die Endlichkeit der Reynolds-Zahl. Wenn die Reynolds-Zahl mit einbezogen wird, ist die auf das Teilchen wirkende Kraft verschieden, je nach dem, ob es sich auf die Wand zu oder von ihr weg bewegt.

Unter Verwendung der in dem 1. Abschnitt des Artikels enthaltenen Technik werden Formeln sowohl für das Stokes-Kräftepaar abgeleitet, das erforderlich ist um die symmetrische Rotation einer Hantelform in einer unbegrenzten Flüssigkeit aufrecht zu erhalten, als auch für Kräftepaare, die zur Aufrechterhaltung der symmetrischen Rotation einer Kugel nötig sind, die eine starre, ebene Wand und eine planare Oberfläche berührt.