A SLOW MOTION OF VISCOUS LIQUID CAUSED BY THE ROTATION OF A SOLID SPHERE

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1. A steady motion of incompressible viscous liquid caused by the slow rotation of a rigid sphere of radius a is considered. The medium is bounded by an infinite rigid plane and the axis of rotation is parallel to, and at a distance d from, this plane. To complete the analysis the solution by successive approximation of an infinite set of linear equations is required. Satisfactory solutions have been found numerically for four values of d/a, of which $1\cdot 13$ is the smallest; we gratefully acknowledge valuable help from Miss S. M. Burrough in this part of the work.

The axi-symmetrical problem in which the axis of rotation is perpendicular to the plane was solved some years ago by G. B. Jeffery [1]; his solution can be combined with that given here to find the motion if the sphere rotates about an arbitrary axis.

2. The equations for a slow steady motion of viscous liquid of constant density ρ referred to cylindrical coordinates (r, θ, z) are

$$\frac{\partial}{\partial r}\left(\frac{p}{\mu_1}\right) = \left(\nabla^2 - \frac{1}{r^2}\right) u - \frac{2}{r^2}\frac{\partial v}{\partial \theta}, \tag{1}$$

$$\frac{\partial}{r\,\partial \theta}\left(\frac{p}{\mu_1}\right) = \left(\nabla^2 - \frac{1}{r^2}\right) v + \frac{2}{r^2}\frac{\partial u}{\partial \theta}, \tag{2}$$

$$\frac{\partial}{\partial z}\left(\frac{p}{\mu_1}\right) = \nabla^2 w, \tag{3}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2};$$

(u, v, w) are the components of the velocity of the liquid, p is the pressure, and μ_1 is the coefficient of viscosity. From the equation of continuity,

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \tag{4}$$

and the equations of motion it can be shown that

$$\nabla^2 p = 0. \tag{5}$$

A particular solution of the equations is found by writing

$$p = \mu_1 \Omega Q_1 \cos\theta, \quad u = \Omega(\tfrac{1}{2} r Q_1 + cu') \cos\theta, \tag{6}$$

$$v = \Omega c v' \sin\theta, \quad w = \Omega(\tfrac{1}{2} z Q_1 + c w_1) \cos\theta; \tag{7}$$

u', v', w_1 and Q_1 are functions of r, z only, Ω is the angular velocity of the sphere, and c is a length which is defined later. Since the first two equations of motion are satisfied if

$$0 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{2}{r^2} + \frac{\partial^2}{\partial z^2}\right) u' - \frac{2v'}{r^2},$$

$$0 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{2}{r^2} + \frac{\partial^2}{\partial z^2}\right) v' - \frac{2u'}{r^2},$$

it is convenient to write

$$u' - v' = U_0, \quad u' + v' = U_2.$$

Then the equations that must be satisfied are

$$L_0^2 U_0 = L_2^2 U_2 = 0,$$

where the operator is defined by

$$L_m^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2} \quad (m = 0, 1, 2).$$

The equations for w_1 and Q_1 are

$$L_1^2 w_1 = L_1^2 Q_1 = 0.$$

The equation of continuity is satisfied if

$$\left(3 + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}\right) Q_1 + c\left[\frac{\partial U_0}{\partial r} + \left(\frac{\partial}{\partial r} + \frac{2}{r}\right) U_2 + 2\frac{\partial w_1}{\partial z}\right] = 0. \tag{8}$$

3. The fluid motion is supposed due to the slow steady rotation of a rigid sphere about a diameter parallel to a fixed infinite plane. Referred to Cartesian coordinates (x, y, z), take $(0, 0, d)$ as the centre of the sphere, $(0, \Omega, 0)$ as the angular velocity, and $z = 0$ as the equation of the plane. The velocity $\mathbf{V}$ must vanish when $z = 0$ and must satisfy the condition

$$\mathbf{V} = \Omega(z - d), \quad 0, \quad -\Omega x$$

at any point (x, y, z) on the sphere. At the point (r, θ, z) on the sphere the cylindrical components of $\mathbf{V}$ must accordingly be

$$\Omega(z-d)\cos\theta, \quad -\Omega(z-d)\sin\theta, \quad -\Omega r\cos\theta.$$

It is now convenient to use coordinates (ξ, η) defined by

$$r = \frac{c\sin\eta}{\cosh\xi - \cos\eta}, \quad z = \frac{c\sinh\xi}{\cosh\xi - \cos\eta} \quad (0 \leqslant \eta \leqslant \pi);$$

this definition determines the constant c of §2. On the plane, $\xi = 0$; if $\xi = \alpha > 0$ defines the sphere, its radius is $c \operatorname{cosech}\alpha$ and the distance d of its centre from the plane is $c\coth\alpha$. The solution given in equations (6), (7) can clearly be used.

The boundary conditions are

$$u' = -\frac{Q_1 \sin\eta}{2(1-\cos\eta)}, \quad v' = w_1 = 0 \quad (\xi = 0), \tag{9}$$

$$u' + \frac{Q_1 \sin\eta}{2(\cosh\alpha - \cos\eta)} = -v' = \frac{\sinh\alpha}{\cosh\alpha - \cos\eta} - \coth\alpha \quad (\xi = \alpha),$$

and

$$w_1 = -\frac{Q_1 \sinh\alpha + 2\sin\eta}{2(\cosh\alpha - \cos\eta)} \quad (\xi = \alpha). \tag{10}$$

In terms of U_0, U_2 the conditions are

$$U_0 = U_2 = -\frac{Q_1 \sin\eta}{2(1-\cos\eta)} \quad (\xi = 0), \tag{11}$$

and

$$U_0 - \frac{2\sinh\alpha}{\cosh\alpha - \cos\eta} + 2\coth\alpha = U_2 = -\frac{Q_1 \sin\eta}{2(\cosh\alpha - \cos\eta)} \quad (\xi = \alpha). \tag{12}$$

Since $u = v = 0$ on the plane $\xi = 0$, it is clear from (4) that $\partial w/\partial z = 0$; hence we must also have

$$Q_1 = -2c\frac{\partial w_1}{\partial z} \quad (\xi = 0). \tag{13}$$

The equations for U_0, U_2, w_1 and Q_1 are satisfied [2] if

$$w_1 = (\cosh\xi - \mu)^{\frac{1}{2}} \sin\eta \sum_{n=1}^{\infty} [A_n \sinh(n+\tfrac{1}{2})\xi] P_n'(\mu), \tag{14}$$

$$Q_1 = (\cosh\xi - \mu)^{\frac{1}{2}} \sin\eta \sum_{n=1}^{\infty} [B_n \cosh(n+\tfrac{1}{2})\xi + C_n \sinh(n+\tfrac{1}{2})\xi] P_n'(\mu), \tag{15}$$

$$U_0 = (\cosh\xi - \mu)^{\frac{1}{2}} \sum_{n=0}^{\infty} [D_n \cosh(n+\tfrac{1}{2})\xi + E_n \sinh(n+\tfrac{1}{2})\xi] P_n(\mu), \tag{16}$$

$$U_2 = (\cosh\xi - \mu)^{\frac{1}{2}} \sin^2\eta \sum_{n=2}^{\infty} [F_n \cosh(n+\tfrac{1}{2})\xi + G_n \sinh(n+\tfrac{1}{2})\xi] P_n''(\mu); \tag{17}$$

μ denotes $\cos\eta$, $P_n(\mu)$ the Legendre polynomial of order n, and the accents differentiations with regard to μ. (The coefficient of viscosity has been denoted by μ_1.)

If Q_1 is known, the velocity functions U_0, U_2, w_1, are determined by the differential equations and the boundary conditions that they satisfy. In a first attempt at this problem the form (15) was assumed for Q_1; U_0, U_2, w_1 were then expressed in forms involving the constants B_n, C_n, and lastly these constants were found from the equation of continuity. It was later found to be much simpler to start from the form (14) for w_1; since then only a single series of constants has to be determined and some difficult expansions are avoided.

4. The condition that w_1 should vanish on the plane is clearly satisfied by (14). There are now conditions sufficient to determine Q_1, since

$$Q_1 = -2c\frac{\partial w_1}{\partial z} \quad (\xi = 0),$$

$$\tfrac{1}{2}zQ_1 + cw_1 = -r \quad (\xi = \alpha).$$

The equivalent conditions,

$$Q_1 = -\frac{2cw_1}{z} \qquad (\xi = 0),$$

$$= -\frac{2cw_1}{z} - \frac{2r}{z} \quad (\xi = \alpha),$$

where w_1/z represents the limit in the first equation, suggest using the function w_1/z. From (14),

$$-\frac{2cw_1}{z} = -\frac{2(\cosh\xi - \mu)}{\sinh\xi}(\cosh\xi - \mu)^{\frac{1}{2}} \sin\eta \sum_1^\infty A_n P_n' \sinh(n+\tfrac{1}{2})\xi.$$

The relation [3]

$$(2n+1)\mu P_n' = (n+1)P_{n-1}' + nP_{n+1}' \quad (n \geqslant 1)$$

can now be used to show that

$$-\frac{2cw_1}{z} = \frac{2(\cosh\xi-\mu)^{\frac{1}{2}}\sin\eta}{\sinh\xi}\sum_1^\infty\left[\frac{(n+1)P_{n-1}'+nP_{n+1}'}{2n+1} - P_n'\cosh\xi\right] \times A_n \sinh(n+\tfrac{1}{2})\xi,$$

$$= \frac{2(\cosh\xi-\mu)^{\frac{1}{2}}\sin\eta}{\sinh\xi}\sum_1^\infty\left[\frac{(n-1)A_{n-1}\sinh(n-\frac{1}{2})\xi}{2n-1} - A_n\cosh\xi\sinh(n+\tfrac{1}{2})\xi + \frac{(n+2)A_{n+1}\sinh(n+\frac{3}{2})\xi}{2n+3}\right]P_n'.$$

If $\alpha > 0$,

$$(\cosh\alpha - \mu)^{\frac{1}{2}} = \sum_0^\infty \lambda_n P_n,$$

$$(\cosh\alpha - \mu)^{-\frac{1}{2}} = -2\sum_0^\infty \lambda_n P_n',$$

where

$$\lambda_n = -\frac{1}{\sqrt{2}}\left[\frac{e^{-(n-\frac{1}{2})\alpha}}{2n-1} - \frac{e^{-(n+\frac{3}{2})\alpha}}{2n+3}\right].$$

The coefficients of P_n' can now be equated in the two conditions to show that

$$B_n = (n-1)A_{n-1} - (2n+1)A_n + (n+2)A_{n+1} \quad (n \geqslant 1), \tag{18}$$

$$B_n \cosh(n+\tfrac{1}{2})\alpha + C_n \sinh(n+\tfrac{1}{2})\alpha$$

$$= 2\operatorname{cosech}\alpha\left[2\lambda_n + \frac{(n-1)A_{n-1}\sinh(n-\frac{1}{2})\alpha}{2n-1} - A_n\cosh\alpha\sinh(n+\tfrac{1}{2})\alpha + \frac{(n+2)A_{n+1}\sinh(n+\frac{3}{2})\alpha}{2n+3}\right] \quad (n \geqslant 1).$$

The right-hand side of the last equation can be written as

$$4\lambda_n \operatorname{cosech}\alpha + 2\left[\frac{(n-1)A_{n-1}}{2n-1} - A_n + \frac{(n+2)A_{n+1}}{2n+3}\right]\coth\alpha \sinh(n+\tfrac{1}{2})\alpha$$
$$-2\left[\frac{(n-1)A_{n-1}}{2n-1} - \frac{(n+2)A_{n+1}}{2n+3}\right]\cosh(n+\tfrac{1}{2})\alpha;$$

from this and (18),

$$C_n = 4\lambda_n \operatorname{cosech}\alpha \operatorname{cosech}(n+\tfrac{1}{2})\alpha$$
$$-2k_n\left[\frac{(n-1)A_{n-1}}{2n-1} - A_n + \frac{(n+2)A_{n+1}}{2n+3}\right] \quad (n \geqslant 1), \tag{19}$$

where

$$k_n = (n+\tfrac{1}{2})\coth(n+\tfrac{1}{2})\alpha - \coth\alpha \quad (n \geqslant 0). \tag{20}$$

5. To determine U_0 there are the conditions

$$U_0 = \frac{c \sin\eta}{1-\mu}\frac{\partial w_1}{\partial z} \qquad (\xi = 0),$$

$$= \frac{\sinh\alpha}{\cosh\alpha-\mu} - \frac{\cosh\alpha-\mu}{\sinh\alpha} + \frac{w_1 \sin\eta}{\sinh\alpha} \qquad (\xi = \alpha),$$

which can be written as

$$U_0 = \frac{rw_1}{z} \qquad (\xi = 0),$$

$$= \frac{rw_1}{z} + \frac{\sinh\alpha}{\cosh\alpha-\mu} - \frac{\cosh\alpha-\mu}{\sinh\alpha} \qquad (\xi = \alpha),$$

w_1/z being as before interpreted as a limit when $\xi = 0$. Since

$$\frac{rw_1}{z} = \frac{(\cosh\xi-\mu)^{\frac{1}{2}}\sin^2\eta}{\sinh\xi}\sum_1^\infty A_n P_n' \sinh(n+\tfrac{1}{2})\xi$$

and [3]

$$P_n' \sin^2\eta = \frac{n(n+1)}{2n+1}(P_{n-1} - P_{n+1}) \quad (n \geqslant 1),$$

$$\frac{rw_1}{z} = -\frac{(\cosh\xi-\mu)^{\frac{1}{2}}}{\sinh\xi}\sum_0^\infty \left[\frac{(n-1)nA_{n-1}}{2n-1}\sinh(n-\tfrac{1}{2})\xi\right.$$
$$\left. - \frac{(n+1)(n+2)A_{n+1}}{2n+3}\sinh(n+\tfrac{3}{2})\xi\right] P_n.$$

Since also

$$\frac{\sinh\alpha}{(\cosh\alpha-\mu)^{3/2}} - \frac{(\cosh\alpha-\mu)^{\frac{1}{2}}}{\sinh\alpha} = \sum_0^\infty [(\sqrt{2})(2n+1)e^{-(n+\frac{1}{2})\alpha} - \lambda_n \operatorname{cosech}\alpha] P_n,$$

the coefficients of P_n can be equated to show that

$$D_n = -\tfrac{1}{2}(n-1)\,n\,A_{n-1} + \tfrac{1}{2}(n+1)(n+2)\,A_{n+1} \quad (n \geqslant 0), \tag{21}$$

$$\begin{aligned} &D_n \cosh (n+\tfrac{1}{2})\,\alpha + E_n \sinh (n+\tfrac{1}{2})\,\alpha \\ &= (\surd 2)(2n+1)\,e^{-(n+\frac{1}{2})\alpha} - \lambda_n \operatorname{cosech}\alpha - \operatorname{cosech}\alpha \left[\frac{(n-1)nA_{n-1}}{2n-1}\sinh (n-\tfrac{1}{2})\,\alpha \right. \\ &\qquad \left. - \frac{(n+1)(n+2)\,A_{n+1}}{2n+3}\sinh (n+\tfrac{3}{2})\,\alpha\right] \quad (n \geqslant 0). \end{aligned}$$

These equations are very similar in form to those for B_n, C_n, and it easily follows that

$$\begin{aligned} E_n = {} & [\surd 2(2n+1)\,e^{-(n+\frac{1}{2})\alpha} - \lambda_n \operatorname{cosech}\alpha] \operatorname{cosech}(n+\tfrac{1}{2})\,\alpha \\ & + k_n\left[\frac{(n-1)nA_{n-1}}{2n-1} - \frac{(n+1)(n+2)\,A_{n+1}}{2n+3}\right] \quad (n \geqslant 0). \end{aligned} \tag{22}$$

6. The conditions for U_2 can be written as

$$\begin{aligned} U_2 &= \frac{rw_1}{z} && (\xi = 0), \\ &= \frac{rw_1}{z} + \frac{\sin^2\eta}{\sinh\alpha(\cosh\alpha - \mu)} && (\xi = \alpha), \end{aligned}$$

and again rw_1/z can conveniently be used. Since [**3**]

$$(2n+1)\,P_n' = -P_{n-1}'' + P_{n+1}'' \quad (n \geqslant 1),$$

$$\frac{rw_1}{z} = \frac{(\cosh\xi - \mu)^{\frac{1}{2}}\sin^2\eta}{\sinh\xi}\sum_2^\infty \left[\frac{A_{n-1}}{2n-1}\sinh (n-\tfrac{1}{2})\,\xi - \frac{A_{n+1}}{2n+3}\sinh (n+\tfrac{3}{2})\,\xi\right] P_n''$$

Also

$$(\cosh\alpha - \mu)^{-3/2} = -4\sum_2^\infty \lambda_n P_n''.$$

It follows that

$$F_n = \tfrac{1}{2}(A_{n-1} - A_{n+1}) \quad (n \geqslant 2), \tag{23}$$

$$\begin{aligned} &F_n \cosh (n+\tfrac{1}{2})\,\alpha + G_n \sinh (n+\tfrac{1}{2})\,\alpha \\ &= \operatorname{cosech}\alpha\left[-4\lambda_n + \frac{A_{n-1}}{2n-1}\sinh (n-\tfrac{1}{2})\,\alpha - \frac{A_{n+1}}{2n+3}\sinh (n+\tfrac{3}{2})\,\alpha\right] \quad (n \geqslant 2), \end{aligned}$$

and hence that

$$G_n = -4\lambda_n \operatorname{cosech}\alpha \operatorname{cosech}(n+\tfrac{1}{2})\,\alpha - k_n\left(\frac{A_{n-1}}{2n-1} - \frac{A_{n+1}}{2n+3}\right) \quad (n \geqslant 2). \tag{24}$$

The constants B_n, C_n, ... have now been expressed in terms of A_n, and the equations of motion are satisfied. It remains to consider the equation of continuity.

7. It can be shown from the differential equations that each of the three functions

$$\left(r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}\right)Q_1,\quad c\frac{\partial U_0}{\partial r},\quad c\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)U_2,$$

is a solution of the equation $L_1{}^2=0$, as also are Q_1 and $c\,\partial w_1/\partial z$. Hence only coefficients need be equated to ensure that (8) is satisfied at all points of the field. The coefficient of

$$(\cosh\xi-\cos\eta)^{\frac{1}{2}}\sin\eta\, P_n' \quad (n\geqslant 1)$$

in $\left(r\dfrac{\partial}{\partial r}+z\dfrac{\partial}{\partial z}\right)Q_1$ is

$$-\tfrac{1}{2}[(n-1)B_{n-1}+B_n-(n+2)B_{n+1}]\cosh(n+\tfrac{1}{2})\xi$$
$$-\tfrac{1}{2}[(n-1)C_{n-1}+C_n-(n+2)C_{n+1}]\sinh(n+\tfrac{1}{2})\xi.$$

The coefficients in the other two functions are

$$-\tfrac{1}{2}(D_{n-1}-2D_n+D_{n+1})\cosh(n+\tfrac{1}{2})\xi$$
$$-\tfrac{1}{2}(E_{n-1}-2E_n+E_{n+1})\sinh(n+\tfrac{1}{2})\xi,$$

and

$$\tfrac{1}{2}[(n-2)(n-1)F_{n-1}-2(n-1)(n+2)F_n+(n+2)(n+3)F_{n+1}]\cosh(n+\tfrac{1}{2})\xi$$
$$+\tfrac{1}{2}[(n-2)(n-1)G_{n-1}-2(n-1)(n+2)G_n$$
$$+(n+2)(n+3)G_{n+1}]\sinh(n+\tfrac{1}{2})\xi.$$

The terms in $\cosh(n+\frac{1}{2})\xi$ give the first set of conditions that (8) should be satisfied. This is

$$\tfrac{5}{2}B_n-\tfrac{1}{2}(n-1)B_{n-1}+\tfrac{1}{2}(n+2)B_{n+1}-\tfrac{1}{2}D_{n-1}+D_n-\tfrac{1}{2}D_{n+1}$$
$$+\tfrac{1}{2}(n-2)(n-1)F_{n-1}-(n-1)(n+2)F_n+\tfrac{1}{2}(n+2)(n+3)F_{n+1}$$
$$-(n-1)A_{n-1}+(2n+1)A_n-(n+2)A_{n+1}=0\quad(n\geqslant 1).\qquad(25)$$

When $\xi=0$ the equation of continuity has already been used and it can be verified that if equations (18), (21) and (23) are used to express B_n, D_n, F_n in terms of A_n, this set of equations is identically satisfied. The other set of conditions is

$$\tfrac{5}{2}C_n-\tfrac{1}{2}(n-1)C_{n-1}+\tfrac{1}{2}(n+2)C_{n+1}-\tfrac{1}{2}E_{n-1}+E_n-\tfrac{1}{2}E_{n+1}$$
$$+\tfrac{1}{2}(n-2)(n-1)G_{n-1}-(n-1)(n+2)G_n+\tfrac{1}{2}(n+2)(n+3)G_{n+1}=0$$
$$(n\geqslant 1).\qquad(26)$$

These equations can be derived from (25) by replacing B_n, D_n, F_n by C_n, E_n, G_n, respectively, and leaving out the constants A_n. From (26) equations for the constants A_n can be found; these are

$$[(2n-1)k_{n-1}-(2n-3)k_n]\left[\frac{(n-1)A_{n-1}}{2n-1}-\frac{nA_n}{2n+1}\right]$$
$$-[(2n+5)k_n-(2n+3)k_{n+1}]\left[\frac{(n+1)A_n}{2n+1}-\frac{(n+2)A_{n+1}}{2n+3}\right]$$
$$=-\frac{\sqrt{2}\,e^{-(n+\frac{1}{2})\alpha}}{(2n+1)\sinh\alpha}\left[(2n+1)^2\left(\frac{e^{\alpha}}{2n-1}+\frac{e^{-\alpha}}{2n+3}\right)\operatorname{cosech}(n+\tfrac{1}{2})\alpha\right.$$
$$\left.-(2n-1)\operatorname{cosech}(n-\tfrac{1}{2})\alpha-(2n+3)\operatorname{cosech}(n+\tfrac{3}{2})\alpha\right]\quad(n\geqslant 1). \quad (27)$$

As defined earlier in (20),

$$k_n=(n+\tfrac{1}{2})\coth(n+\tfrac{1}{2})\alpha-\coth\alpha\quad(n\geqslant 0).$$

As $n\to\infty$,

$$(2n-1)k_{n-1}-(2n-3)k_n\to-2(\coth\alpha-1),$$
$$(2n+5)k_n-(2n+3)k_{n+1}\to-2(\coth\alpha+1),$$

and the term on the right of (27) tends to zero. If α is large the approach to these limits is rapid; in this case d/a, or $\cosh\alpha$, the distance of the centre of the sphere from the plane expressed as a multiple of the radius, is large, and the motion is not much affected by the presence of the plane.

The first r equations in the set (27) contain the $(r+1)$ constants $A_1, A_2, \ldots, A_{r+1}$, and thus determine $A_1, \ldots, A_r$ only if A_{r+1} is assumed to be zero. The equations have been solved on this assumption, the number r of equations required being, of course, larger the smaller the value of α. It is clear that, so far as the numerical work has gone, the significant solution has been found.

8. The components F_x, F_y, F_z of the total force exerted by the fluid on the fixed plane $\xi=0$ are given by

$$F_x=\mu_1\iint\left(\frac{\partial u}{\partial z}\cos\theta-\frac{\partial v}{\partial z}\sin\theta\right)r\,dr\,d\theta,\quad F_y=F_z=0,$$

and, in terms of Q_1, U_0,

$$F_x=\pi\mu_1\Omega\int_0^{\infty}\left(\tfrac{1}{2}r\frac{\partial Q_1}{\partial z}+c\frac{\partial U_0}{\partial z}\right)r\,dr.$$

If $\xi=0$,

$$c\frac{\partial}{\partial z}[(\cosh\xi-\mu)^{\frac{1}{2}}f(\xi)g(\eta)]=(1-\mu)^{3/2}f'(0)g(\eta),$$

hence

$$c\frac{\partial Q_1}{dz}=\tfrac{1}{2}(1-\mu)^{3/2}\sin\eta\sum_1^{\infty}(2n+1)C_nP_n',$$

$$c\frac{\partial U_0}{\partial z}=\tfrac{1}{2}(1-\mu)^{3/2}\sum_0^{\infty}(2n+1)E_nP_n$$

These results and the formulae

$$\int_{-1}^{1} \frac{P_n \, d\mu}{(1-\mu)^{\frac{1}{2}}} = \frac{2\sqrt{2}}{2n+1} \quad (n \geqslant 0),$$

$$\int_{-1}^{1} \frac{(P_{n-1}-P_{n+1}) \, d\mu}{(1-\mu)^{3/2}} = 4\sqrt{2} \quad (n \geqslant 1),$$

show that

$$\int_0^\infty \frac{\partial Q_1}{\partial z} r^2 \, dr = (2\sqrt{2}) \, c^2 \sum_1^\infty n(n+1) \, C_n, \quad \int_0^\infty \frac{\partial U_0}{\partial z} r \, dr = (\sqrt{2}) \, c \sum_0^\infty E_n.$$

The formula for the force component is therefore

$$F_x = (\sqrt{2}) \, \pi\mu_1 \Omega c^2 \sum_0^\infty [E_n + n(n+1) \, C_n]. \tag{28}$$

The forces acting on the fluid medium as a whole are exerted by the plane and sphere, and by the stresses over (say) a hemisphere of large radius. This must be a system of forces in statical equilibrium, since the equations in terms of the stress components are those for an elastic body in equilibrium under no body forces. If the sphere were rotating in an unbounded medium the velocity at a great distance R from its centre would be of order $\Omega a^3/R^2$ and the stress of order $\mu_1 \Omega a^3/R^3$. It seems clear that these orders of magnitude will not be altered by the presence of the plane, in which case the effect on the components of total force of the stresses over the large hemisphere can be ignored. The force exerted by the liquid on the sphere is then equal and opposite to the force it exerts on the plane, and is $-F_x$ in the positive x direction. As a check this result has been confirmed by integration over the surface of the sphere.

The force can be compared with

$$6\pi\mu_1 Ua,$$

Stokes's formula for the force on the sphere in a slow stream of velocity U. The radius a of the sphere is here c cosech α, and U can be represented by Ωc cosech α, the highest velocity of any point on the sphere. A non-dimensional force coefficient F can be defined by

$$\boxed{F = -\frac{F_x}{6\pi\mu_1 \Omega a^2} = -\frac{\sqrt{2}}{6} \sinh^2 \alpha \sum_0^\infty [E_n + n(n+1) \, C_n].} \tag{29}$$

It is clear that $F \to 0$ as $\alpha \to \infty$.

9. It follows from the last paragraph that the moment, say (G_x, G_y, G_z), about the centre of the sphere of the forces acting on its surface must be evaluated by direct surface integration, and a more complicated expression is therefore found.

It is easy to show that $G_x = G_z = 0$.

First suppose that (r_1, θ_1, θ) are spherical polar coordinates with the centre of the sphere as origin and the line $\theta_1 = \theta = \frac{1}{2}\pi$ as the axis of rotation; the coordinates r_1, θ_1 are new, but the angle of azimuth θ is unchanged. If the polar components of $\mathbf{V}$ are $\Omega U_1 \cos\theta$, $\Omega V_1 \cos\theta$, $\Omega W_1 \sin\theta$,

$$G_y = \pi\mu_1 \Omega a^3 \int_0^\pi \left(\frac{\partial V_1}{\partial r_1} - \cos\theta_1 \frac{\partial W_1}{\partial r_1} - \frac{V_1 - W_1\cos\theta_1}{a}\right) \sin\theta_1\, d\theta_1.$$

If the sphere is rotating in unbounded liquid it is known [1] that

$$G_y = -8\pi\mu_1 \Omega a^3,$$

and it is easy to show that one-third of this amount comes from the terms in V_1, W_1, the other two-thirds from the terms in $\partial V_1/\partial r_1$, $\partial W_1/\partial r_1$. The presence of the plane does not alter the surface values of V_1, W_1, but alters those of $\partial V_1/\partial r_1$, $\partial W_1/\partial r_1$.

A couple coefficient G can be defined by

$$\boxed{G = -\frac{G_y}{8\pi\mu_1 \Omega a^3};}$$

then

$$G = \tfrac{1}{3} - \tfrac{1}{8}\int_0^\pi \left(\frac{\partial V_1}{\partial r_1} - \frac{\partial W_1}{\partial r_1}\cos\theta_1\right)\sin\theta_1\, d\theta_1,$$

$$\equiv \tfrac{1}{3} - \frac{I}{8}.$$

In terms of the cylindrical components

$$I \equiv \int_0^\pi \left(\frac{\partial V_1}{\partial r_1} - \frac{\partial W_1}{\partial r_1}\cos\theta_1\right)\sin\theta_1\, d\theta_1,$$

$$= \int_0^\pi \left(\cos\theta_1 \frac{\partial}{\partial r_1}(\tfrac{1}{2}rQ_1 + cU_0) - \sin\theta_1 \frac{\partial}{\partial r_1}(\tfrac{1}{2}zQ_1 + cw_1)\right)\sin\theta_1\, d\theta_1.$$

Since in the integration $\xi = \alpha$,

$$\frac{a\,d\theta_1}{c} = \frac{d\eta}{\cosh\alpha - \mu}, \quad c\frac{\partial}{\partial r_1} = -(\cosh\alpha - \mu)\frac{\partial}{\partial\xi},$$

$$\frac{a\sin\theta_1}{c} = \frac{\sin\eta}{\cosh\alpha - \mu}, \quad \frac{a\cos\theta_1}{c} = \frac{\sinh\alpha}{\cosh\alpha - \mu} - \coth\alpha,$$

and it can be shown that

$$I\operatorname{cosech}^2\alpha = \int_{-1}^1 \left(\tfrac{1}{2}\frac{\partial Q_1}{\partial\xi}\cosh\alpha + \frac{\partial w_1}{\partial\xi}\sinh\alpha\right)\frac{\sin\eta\, d\mu}{(\cosh\alpha - \mu)^2}$$

$$- \int_{-1}^1 \frac{\partial U_0}{\partial\xi}\frac{(\mu\cosh\alpha - 1)\, d\mu}{(\cosh\alpha - \mu)^2},$$

To express these integrals in terms of the constants A_n, B_n, ... there are the relations

$$(2n+1)\int_{-1}^{1}\frac{\sin^2\eta\,P_n'\,d\mu}{(\cosh\alpha-\mu)^{3/2}}=3\sinh\alpha\int_{-1}^{1}\frac{\sin^2\eta\,P_n'\,d\mu}{(\cosh\alpha-\mu)^{5/2}}$$
$$=4\sqrt{2}\,n(n+1)\,e^{-(n+\frac{1}{2})\alpha},$$

$$(2n+1)\int_{-1}^{1}\frac{P_n\,d\mu}{(\cosh\alpha-\mu)^{\frac{1}{2}}}=\sinh\alpha\int_{-1}^{1}\frac{P_n\,d\mu}{(\cosh\alpha-\mu)^{3/2}}$$
$$=\frac{3\sinh^2\alpha}{2n+1+2\coth\alpha}\int_{-1}^{1}\frac{P_n\,d\mu}{(\cosh\alpha-\mu)^{5/2}}$$
$$=2\sqrt{2}\,e^{-(n+\frac{1}{2})\alpha}.$$

The formula found for the couple coefficient G is

$$4\sqrt{2}(1-3G)\operatorname{cosech}^3\alpha$$
$$=\sum_0^\infty[2+e^{-(2n+1)\alpha}][n(n+1)(2A_n+C_n\coth\alpha)-(2n+1-\coth\alpha)E_n]$$
$$+\sum_0^\infty[2-e^{-(2n+1)\alpha}][n(n+1)B_n\coth\alpha-(2n+1-\coth\alpha)D_n]. \quad (30)$$

The constants F_n, G_n in the form (17) for U_2 appear neither here nor in equation (29) for F.

10. The following tables show, in the cases $\alpha=2{\cdot}0$, $1{\cdot}5$, $1{\cdot}0$, $0{\cdot}5$, some values of the constants A_n and of the force- and couple-coefficients F and G.

α	A_1	A_2	A_3	A_4	A_5	A_6
2·0	—·02842	—·00044	—·00001	—	—	—
1·5	—·13316	—·00481	—·00025	—·00001	—	—
1·0	—·71814	—·05466	—·00723	—·00104	—·00016	—·00004
0·5	—7·14792	—·90880	—·24271	—·08073	—·02935	—·01107

α	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
0·5	—·00423	—·00162	—·00062	—·00023	—·00009	—·00003	—·00001

α	d/a	F	G
2·0	3·76	0·124	1·01
1·5	2·35	0·204	1·04
1·0	1·54	0·332	1·16
0·5	1·13	0·524	1·56

This table is wrong. See Brenner 1967.

If $d = a$, the sphere touches one point of the plane and the part of its surface near the point of contact is approximately in a motion of translation with velocity Ωa. Now consider the element of area $2\pi a^2 \sin\theta_1 \, d\theta_1$, where θ_1 is the polar coordinate of §9. The shearing force on the element is approximately

$$\frac{2\pi\mu_1 \Omega a^2 \sin\theta_1 \, d\theta_1}{1+\cos\theta_1}.$$

This shows that the coefficients F and G should tend to infinity as α tends to zero, and this conclusion is consistent with the numerical values above.

The slow rotation of the sphere about an axis perpendicular to the plane has been considered by Jeffery [1]. Suppose that in this case also the sphere touches the plane; then all points of the elementary area considered above move with velocity $\Omega a \sin\theta_1$. The moment about the axis of symmetry of the shearing force on the element is

$$2\pi\mu_1 \Omega a^3 (1-\cos\theta_1) \sin\theta_1 \, d\theta_1,$$

and hence the limiting value of G is finite. By symmetry $F = 0$ for all values of d/a.

Jeffery's solution can be combined with that given here to find the motion caused by the rotation of the sphere about an arbitrary axis. If the axis of rotation is inclined to the normal to the plane, the vectors representing the angular velocity and the moment will be in different directions; the direction of the latter will be nearly parallel to the plane if d is only just greater than a.

References.

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2. ———, *Proc. Roy. Soc.* (*A*), 87 (1912), 109–120.
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(*Received on the 7th of April*, 1963.)