Sphere-Plane Resistance Tensor

The resistance tensor for a sphere moving and rotating near a plane surface relates the translational and rotational velocities to the force and torque on the sphere as

$$\begin{bmatrix} \mathbf{A} & \tilde{\mathbf{B}} \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \Omega \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{L} \end{bmatrix}. \tag{1}$$

To determine the various pieces of the resistance tensor, we can break the problem up into the following sub-problems:

1) Translation normal to the plane surface.
2) Translation parallel to the plane surface.
3) Rotation normal to the plane surface.
4) Rotation parallel to the plane surface.

Let the plane be oriented in the $\mathbf{e}_3$ direction. The resistance tensors take the following forms

$$\mathbf{A} = X^A \mathbf{e}_3\mathbf{e}_3 + Y^A(\boldsymbol{\delta} - \mathbf{e}_3\mathbf{e}_3)\,, \tag{2}$$

$$B_{ij} = \tilde{B}_{ji} = Y^B \varepsilon_{ij3} e_3\,, \tag{3}$$

$$\mathbf{C} = X^C \mathbf{e}_3\mathbf{e}_3 + Y^C(\boldsymbol{\delta} - \mathbf{e}_3\mathbf{e}_3)\,, \tag{4}$$

where the resistance functions X^A, Y^A, Y^B, X^C, and Y^C depend on the distance h between the plane and the sphere center. Below lengths are scaled by the sphere radius a and resistance functions by $6\pi\eta a^n$ where n = 1, 2, or 3.

Problem 1. Translation normal to the plane surface gives the function X^A. This problem has been worked out exactly by Brenner[1]

$$X^A = \frac{4}{3}\sinh\alpha \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \left[\frac{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha}{4\sinh^2(n+\frac{1}{2})\alpha - (2n+1)^2\sinh^2\alpha} - 1 \right], \tag{5}$$

where $\alpha = \cosh^{-1}(h)$. At small surface separations, this exact result can be approximated as

$$X^A = \xi^{-1} - \tfrac{1}{5}\ln\xi + 0.97128\,, \tag{6}$$

where $\xi = h - 1$ is the (dimensionless) surface separation.[2,3]

Problem 2. Translation parallel to the plane surface gives the functions Y^A and Y^B. This problem has been worked out exactly by O'Neill[4] (see MATLAB function *CalcYAYB*). At small surface separations, this exact result can be approximated[2,3] as

$$Y^A = -\tfrac{8}{15}\ln\xi + 0.9588\,, \tag{7}$$

$$Y^B = -\tfrac{2}{15}\ln\xi - 0.2526\,. \tag{8}$$

Problem 3. Rotation parallel to the plane surface gives the functions Y^B and Y^C. This problem has been worked out exactly by Dean & O'Neill[5] (see MATLAB function *CalcYBYC*). At small surface separations, this exact result can be approximated[3] as

$$Y^B = -\tfrac{2}{15}\ln\xi - 0.2526 \tag{9}$$

$$Y^C = -\tfrac{8}{15}\ln\xi + 0.5089 \tag{10}$$

Problem 4. Rotation normal to a place surface gives the function X^C. This problem has been worked out exactly by Jeffrey[6]

$$X^C = \frac{3}{4}\sinh^3\alpha\sum_{m=0}^{\infty}\operatorname{csch}^3\big((m+1)\alpha\big)\,. \tag{11}$$

At small surface separations, this exact result can be approximated[3] as

$$X^C = \tfrac{4}{3}\left[\zeta(3) - \tfrac{1}{2}\xi\ln\xi\right]. \tag{12}$$

References

[1] H. Brenner, Chem. Eng. Sci. **16**, 242 (1961).

[2] A.J. Goldman, R.G. Cox, and H. Brenner, Chem. Eng. Sci. **22**, 637 (1967).

[3] G. Bossis, A. Meunier, and J.D. Sherwood, Phys. Fluids A Fluid Dyn. **3**, 1853 (1991).

[4] M.E. O'Neill, Mathematika **11**, 67 (1964).

[5] W.R. Dean and M.E. O'Neill, Mathematika **10**, 13 (1963).

[6] G.B. Jeffrey, Proc. London Math. Soc. **14**, 327 (1915).