ON THE STEADY ROTATION OF A SOLID OF REVOLUTION IN A VISCOUS FLUID

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THE only possible motion of a viscous fluid which is symmetrical about an axis, and for which the stream lines are circles having their centres on the axis and their planes perpendicular to the axis, is the motion generated by the rotation of two infinite concentric circular cylinders about their common axis.* If, however, the motion is slow so that the squares and products of the velocity components may be neglected, there are other possible solutions. Let ϖ, ϕ, z be cylindrical coordinates, and u, v, w the corresponding components of velocity, then neglecting the squares and products of u, v, w, the equations of steady motion in the symmetrical case are

$$\frac{1}{\nu}\frac{\partial\chi}{\partial\varpi}+\nabla^2 u-\frac{u}{\varpi^2}=0,$$

$$\nabla^2 v-\frac{v}{\varpi^2}=0,$$

$$\frac{1}{\nu}\frac{\partial\chi}{\partial z}+\nabla^2 w=0,$$

where ν is the kinematic viscosity, and $\chi=-V-p/\rho$, where V is the potential of the external forces, p the mean pressure, and ρ the density. These equations are satisfied by $\chi=\text{const.}$, $u=w=0$, while v is a function of ϖ, z only and

$$\nabla^2 v-\frac{v}{\varpi^2}=0. \tag{1}$$

Writing $$v=\psi/\sin\phi,$$

this equation becomes $$\nabla^2\psi=0.$$

* "The Equations of Motion of a Viscous Fluid," *Phil. Mag.*, April 1915, p. 445.

Hence if we have any solution of Laplace's equation of the form $f(\varpi, z)\sin\phi$, a solution of the present problem is given by

$$v = f(\varpi, z).$$

If the motion is generated by the rotation of a solid of revolution about the axis of z with angular velocity ω, we have to make v vanish for large values of ϖ while, on the hypothesis of no slipping at the solid-fluid surface, we have $v = \varpi\omega$ over the surface of the solid.

If α, β, γ are a system of orthogonal curvilinear coordinates, and u', v', w' the corresponding components of velocity, and if elements of arc measured along the normals to the surfaces $\alpha, \beta, \gamma =$ const., are $\delta\alpha/h_1$, $\delta\beta/h_2$, $\delta\gamma/h_3$ respectively, we have with the usual notation for stress components*

$$\widehat{\alpha\gamma} = \mu\left\{\frac{h_3}{h_1}\frac{\partial}{\partial\gamma}(h_1 u') + \frac{h_1}{h_3}\frac{\partial}{\partial\alpha}(h_3 w')\right\},$$

where μ is the coefficient of viscosity.

Let α, β be conjugate functions of ϖ, z, while $\gamma = \phi$, then

$$h_1 = h_2 = \left\{\left(\frac{\partial\alpha}{\partial\varpi}\right)^2 + \left(\frac{\partial\beta}{\partial\varpi}\right)^2\right\}^{\frac{1}{2}}, \qquad h_3 = \frac{1}{\varpi}.$$

If the solid is defined by one of the surfaces $\alpha =$ const., we have for the tangential stress on its surface in the direction perpendicular to the axis

$$\mu h_1 \varpi \frac{\partial}{\partial\alpha}\left(\frac{v}{\varpi}\right),$$

or the total couple exerted by the fluid on the solid is

$$G = 2\pi\mu\int \varpi^3 \frac{\partial}{\partial\alpha}\left(\frac{v}{\varpi}\right) d\beta, \tag{2}$$

the value of the integrand being taken on the surface of the solid and the integration extending round the contour of the solid.

If n, s are measured along the normal and the arc of the contour respectively

$$G = 2\pi\mu\int \varpi^3 \frac{\partial}{\partial n}\left(\frac{v}{\varpi}\right) ds. \tag{3}$$

We have a solution of Laplace's equation in spherical polar coordinates

$$\psi = \sin\phi \sum_{n=1}^{n=\infty} \{A_n r^n + B_n r^{-n-1}\} P_n^1(\cos\theta),$$

where $P_n^1(\cos\theta)$ is the associated Legendre function of the first kind, of degree n and of the first order. Hence we have a solution of the present

* Lamé, *Coordonnées Curvilignes*, p. 284.

problem

$$v = \sum_{n=1}^{\infty} \{A_n r^n + B_n r^{-n-1}\} P_n^1(\cos\theta).$$

As a particular case take $n = 1$. We have

$$v = \{Ar + B/r^2\} \sin\theta,$$

which is the well known solution for the motion of a fluid contained between two concentric spheres which are constrained to rotate about the same diameter. In the case of a single sphere of radius a rotating with angular velocity ω in an infinite fluid, we have

$$v = \frac{a^3\omega}{r^2}\sin\theta, \qquad G = 8\pi\mu\omega a^3. \tag{4}$$

The general case of the rotation of an ellipsoid of any shape about a principal axis has been solved in terms of ellipsoidal harmonics by Mr. Edwardes (*Quarterly Journal of Mathematics*, Vol. XXVI, p. 70). The simpler case of an ellipsoid of revolution can, however, be very readily solved by the method of this paper.

Rotation of an Ovary Ellipsoid.

Take coordinates defined by

$$z + i\varpi = c\cosh(\xi + i\eta),$$

so that $$z = c\cosh\xi\cos\eta, \qquad \varpi = c\sinh\xi\sin\eta.$$

The surfaces $\xi =$ const. are a series of confocal ovary ellipsoids, while the surfaces $\eta =$ const. are a set of confocal hyperboloids of revolution, the foci in each case being $z = \pm c$, $\varpi = 0$. Well known solutions of Laplace's equation are

$$P_n^m(\cos\eta)\, P_n^m(\cosh\xi)\sin m\phi,$$

$$P_n^m(\cos\eta)\, Q_n^m(\cosh\xi)\sin m\phi,$$

and hence we have solutions of the problem in hand

$$v = P_n^1(\cos\eta)\, P_n^1(\cosh\xi),$$

and $$v = P_n^1(\cos\eta)\, Q_n^1(\cosh\xi),$$

the former is finite when $\cosh\xi = 1$, *i.e.*, on the line joining the foci, but it becomes infinite with ξ. The second solution, on the other hand, becomes infinite on the line joining the foci, but vanishes at infinity. It is therefore applicable to the case of an ellipsoid in an infinite extent of fluid. We will, however, take the more general case of the motion of a fluid between two confocal ovary ellipsoids rotating with different angular velocities. For this purpose both solutions are required. Put $n = 1$ in the above solutions and assume

$$v = \sin\eta\left\{AP_1^1(\cosh\xi)+BQ_1^1(\cosh\xi)\right\}.$$

If the two surfaces are defined by $\xi = \xi_0, \xi_1$, and if their angular velocities are ω_0, ω_1 respectively, the boundary conditions give

$$AP_1^1(\cosh\xi_0)+BQ_1^1(\cosh\xi_0) = c\omega_0\sinh\xi_0,$$

$$AP_1^1(\cosh\xi_1)+BQ_1^1(\cosh\xi_1) = c\omega_1\sinh\xi_1.$$

We may conveniently write

$$f(\xi) = \log\coth\frac{\xi}{2}-\frac{\cosh\xi}{\sinh^2\xi}.$$

Then $\qquad P_1^1(\cosh\xi) = \sinh\xi \quad$ and $\quad Q_1^1(\cosh\xi) = \sinh\xi\, f(\xi).$

Solving for A, B, we have

$$v = c\sinh\xi\sin\eta\left\{\omega_0\frac{f(\xi)-f(\xi_1)}{f(\xi_0)-f(\xi_1)}+\omega_1\frac{f(\xi_0)-f(\xi)}{f(\xi_0)-f(\xi_1)}\right\}.$$

To calculate the couple on either ellipsoid necessary to maintain the rotations we may use the formula (2), and we have at once

$$G = \tfrac{16}{3}\pi\mu c^3\frac{\omega_0-\omega_1}{f(\xi_0)-f(\xi_1)}.$$

In the case of a single ellipsoid ($\xi = \xi_0$) in an infinite extent of fluid, we have, putting $\xi_1 = \infty$, $\omega_1 = 0$,

$$v = \omega_0 c\sinh\xi\sin\eta\frac{f(\xi)}{f(\xi_0)},$$

and
$$G = \tfrac{16}{3}\pi\mu\omega_0 c^3/f(\xi_0).$$

If a, b are the polar and equatorial radii respectively,

$$a = c\cosh\xi_0, \qquad b = c\sinh\xi_0,$$

and we have

$$G = \tfrac{16}{3}\pi\mu\omega_0 c^3\Big/\left\{\tfrac{1}{2}\log\left(\frac{a+c}{a-c}\right)-\frac{ac}{b^2}\right\}, \tag{5}$$

where
$$c^2 = a^2-b^2.$$

When $b \to a$, $c \to 0$, and

$$G = 8\pi\mu\omega_0 a^3,$$

which is the value already given for a sphere.

Rotation of a Planetary Ellipsoid.

In the case of an oblate or planetary ellipsoid we take coordinates defined by

$$z + i\varpi = c \sinh(\xi + i\eta),$$

so that $$z = c \sinh \xi \cos \eta, \qquad \varpi = c \cosh \xi \sin \eta.$$

The surfaces $\xi =$ const. and $\eta =$ const. are ellipsoids and hyperboloids of revolution having a common focal circle, $z = 0$, $\varpi = c$. We have as solutions of Laplace's equation*

$$P_n^m(\cos \eta)\, p_n^m(\sinh \xi) \sin m\phi, \qquad P_n^m(\cos \eta)\, q_n^m(\sinh \xi) \sin m\phi,$$

where $$p_n^m(\zeta) = (\zeta^2+1)^{\frac{1}{2}m} \frac{d^m}{d\zeta^m} p_n(\zeta), \qquad q_n^m(\zeta) = (\zeta^2+1)^{\frac{1}{2}m} \frac{d^m}{d\zeta^m} q_n(\zeta),$$

the functions p_n, q_n being connected with the ordinary Legendre functions by the relations

$$p_n(\zeta) = i^{-n} P_n(i\zeta), \qquad q_n(\zeta) = i^{n+1} Q_n(i\zeta).$$

Hence we have solutions

$$v = P_n^1(\cos \eta)\, p_n^1(\sinh \xi),$$

$$v = P_n^1(\cos \eta)\, q_n^1(\sinh \xi).$$

As in the previous case the former becomes infinite, while the latter vanishes at infinity.

Let the two ellipsoids ξ_0, ξ_1 have angular velocities ω_0, ω_1. Assume

$$v = \{A p_1^1(\sinh \xi) + B q_1^1(\sinh \xi)\} \sin \eta.$$

The boundary conditions give

$$A p_1^1(\sinh \xi_0) + B q_1^1(\sinh \xi_0) = c\omega_0 \cosh \xi_0,$$

$$A p_1^1(\sinh \xi_1) + B q_1^1(\sinh \xi_1) = c\omega_1 \cosh \xi_1.$$

Now $$p_1^1(\sinh \xi) = \cosh \xi,$$

* Lamb, *Hydrodynamics*, p. 136.

and writing $$F(\xi) = \frac{\sinh \xi}{\cosh^2 \xi} - \cot^{-1} \sinh \xi,$$

we have $$q_1^1 (\sinh \xi) = \cosh \xi \, F(\xi).$$

Solving for A, B,

$$v = c \cosh \xi \sin \eta \left\{ \omega_0 \frac{F(\xi_1) - F(\xi)}{F(\xi_1) - F(\xi_0)} + \omega_1 \frac{F(\xi) - F(\xi_0)}{F(\xi_1) - F(\xi_0)} \right\}.$$

Using equation (2) to evaluate the couple which must be applied to either ellipsoid to maintain the rotation, we have without difficulty

$$G = \tfrac{16}{3} \pi \mu c^3 \frac{\omega_1 - \omega_0}{F(\xi_1) - F(\xi_0)}.$$

If the fluid is not bounded externally, so that we have a single ellipsoid in an infinite extent of fluid, we may put $\omega_1 = 0$ and $\xi_1 = \infty$. In this case

$$v = c \cosh \xi \sin \eta \, \omega_0 \, F(\xi)/F(\xi_0),$$

and $$G = \tfrac{16}{3} \pi \mu c^3 \omega_0 \, F(\xi_0).$$

If a, b are the polar and equatorial radii respectively

$$a = c \sinh \xi_0, \qquad b = c \cosh \xi_0,$$

and we have

$$G = \tfrac{16}{3} \pi \mu \omega_0 c^3 \Big/ \left\{ \frac{ac}{b^2} - \cot^{-1} \frac{a}{c} \right\},$$

where $$b^2 - a^2 = c^2.$$

We may note that this is the form taken by the result (5) for an ovary ellipsoid when b is greater than a for

$$\tfrac{1}{2} \log \frac{a + i\sqrt{(b^2 - a^2)}}{a - i\sqrt{(b^2 - a^2)}} = i \cot^{-1} \frac{a}{\sqrt{(b^2 - a^2)}}.$$

The Rotation of a Circular Disc in an Infinite Fluid.

The solution of this problem may be obtained by putting $\xi_0 = 0$ in the solution for a planetary ellipsoid. Hence

$$v = -\frac{2\omega_0}{\pi} c \cosh \xi \sin \eta \, F(\xi), \tag{6}$$

and the couple necessary to maintain the rotation is

$$G = \tfrac{32}{3} \mu \omega_0 c^3,$$

where c is the radius of the disc.

From (6) the normal gradient of the velocity over the surface of the disc can be calculated,

$$\left.\frac{\partial v}{\partial z}\right)_{\xi=0} = -\frac{4}{\pi}\,\omega_0 \varpi \left.\frac{\partial \xi}{\partial z}\right)_{\xi=0}$$

$$= -\frac{4}{\pi}\,\omega_0 \frac{\varpi}{\sqrt{(c^2-\varpi^2)}}. \tag{7}$$

It appears that, while the velocity is everywhere finite, the velocity gradient, and therefore the shearing stress, become infinite at the edge of the disc. It is probable that in the neighbourhood of the sharp edge the condition of no relative motion between the solid and the fluid breaks down. It would be interesting to see whether this infinity disappeared in a more accurate solution which did not neglect the terms involving the squares of the velocity components.

Another solution of this problem may be obtained in terms of Bessel functions, for we have a solution of Laplace's equation

$$e^{-kz} J_1(k\varpi) \sin\phi,$$

and hence a solution of equation (1)

$$v = \int_0^\infty f(k) e^{-kz} J_1(k\varpi)\, dk.$$

In the case of the disc we have

$$v = \omega_0 \varpi \quad for \quad z = 0,\ \varpi < c, \tag{8}$$

$$\frac{\partial v}{\partial z} = 0 \quad for \quad z = 0,\ \varpi > c, \tag{9}$$

and hence $f(k)$ is determined by

$$\int_0^\infty f(k)\, J_1(k\varpi)\, dk = \omega_0 \varpi \quad (\varpi < c),$$

$$\int_0^\infty k f(k) J_1(k\varpi)\, dk = 0 \quad (\varpi > c).$$

It has not been found possible to solve these integral equations directly, but if we make use of (7), together with (9), and the theorem* that if

$$-\frac{\partial v}{\partial z} = F(\varpi) \quad for \quad z = 0,$$

then $$v = \int_0^\infty e^{-kz} J_1(k\varpi) \int_0^\infty F(\lambda)\, J_1(k\lambda)\, \lambda\, d\lambda\, dk,$$

* Gray and Mathews, *Bessel Functions*, p. 80. The infinity of F at $\varpi = c$ does not invalidate this theorem.

we obtain without difficulty

$$v = \frac{4}{\pi}\omega_0 \int_0^\infty e^{-kz}(\sin ck - ck\cos ck)\, J_1(k\varpi)\frac{dk}{k^2},$$

and this will be found to satisfy both conditions (8) and (9).

The Rotation of Two Non-Concentric Spheres.

This method may be applied to the problem of the motion of a viscous fluid generated by the rotation of any two spheres with different angular velocities about their common diameter. If we take coordinates defined by

$$\xi + i\eta = \log\frac{\varpi + i(z+a)}{\varpi + i(z-a)}, \tag{10}$$

in any meridian plane the curves $\xi = \text{const.}$, $\eta = \text{const.}$ are the systems of coaxial circles *about* the points $z = \pm a$ and *through* these points respectively. The surfaces $\xi = \text{const.}$ will be a family of coaxial spheres having the common radical plane z, $\xi = 0$, and we can choose the axes of reference and the constant a so that any two given spheres are members of this family. We have a solution of Laplace's equation in these coordinates of which the following is a particular case*

$$\sqrt{(\cosh\xi - \cos\eta)}\,\Sigma\left\{A_n\cosh(n+\tfrac{1}{2})\xi + B_n\sinh(n+\tfrac{1}{2})\xi\right\}P_n^1(\cos\eta)\sin\phi.$$

It follows that a solution of the present problem is given by

$$v = \sqrt{(\cosh\xi - \cos\eta)}\,\Sigma\left\{A_n\cosh(n+\tfrac{1}{2})\xi + B_n\sinh(n+\tfrac{1}{2})\xi\right\}P_n^1(\cos\eta).$$

Let the two spheres be defined by $\xi = \xi_1$, ξ_2, and let them have angular velocities ω_1, ω_2. Without loss of generality we may take ξ_1 positive, and ξ_2 positive or negative according as the two spheres do or do not lie one within the other. We have to determine A_n, B_n so that v is equal to $\varpi\omega_1$, when $\xi = \xi_1$, and equal to $\varpi\omega_2$, when $\xi = \xi_2$. From (10),

$$\varpi = \frac{a\sin\eta}{\cosh\xi - \cos\eta},$$

and we obtain

$$\begin{aligned} A_n\sinh(n+\tfrac{1}{2})(\xi_1-\xi_2) &\\ = -2\sqrt{2}\,a&\left[\omega_1 e^{-(n+\frac{1}{2})\xi_1}\sinh(n+\tfrac{1}{2})\xi_2 - \omega_2 e^{\pm(n+\frac{1}{2})\xi_2}\sinh(n+\tfrac{1}{2})\xi_1\right], \\ B_n\sinh(n+\tfrac{1}{2})(\xi_1-\xi_2) &\\ = 2\sqrt{2}\,a&\left[\omega_1 e^{-(n+\frac{1}{2})\xi_1}\cosh(n+\tfrac{1}{2})\xi_2 - \omega_2 e^{\pm(n+\frac{1}{2})\xi_2}\cosh(n+\tfrac{1}{2})\xi_1\right], \end{aligned}$$

* Heine, *Kugelfunctionen*, II, p. 268; see also a paper by the author "On a Form of the Solution of Laplace's Equation," *Proc. Roy. Soc.*, A, 87, 1912, p. 109.

where the upper or lower sign is taken according as ξ_2 is negative or positive.

We will investigate the resultant couple on either sphere

$$\frac{\partial}{\partial \xi}\left(\frac{v}{\varpi}\right) = \frac{3 \sinh \xi}{2a \sin \eta} v$$

$$+\frac{1}{a}(\cosh \xi - \zeta) \sum_{n=1}^{\infty} (n+\tfrac{1}{2}) \{A_n \sinh (n+\tfrac{1}{2})\xi + B_n \cosh (n+\tfrac{1}{2})\xi\} \frac{d}{d\zeta} P_n(\zeta),$$

where $$\zeta = \cos \eta.$$

The couple on the sphere ξ_1 is given by

$$G_1 = 2\pi\mu \int_0^{\pi} \varpi^3 \frac{\partial}{\partial \xi}\left(\frac{v}{\varpi}\right) d\eta \quad (\xi = \xi_1)$$

$$= 3\pi\mu a^3 \omega_1 \sinh \xi_1 \int_{-1}^{+1} \frac{(1-\zeta^2)\, d\zeta}{(\cosh \xi_1 - \zeta)^4}$$

$$+2\pi\mu a^2 \sum_{n=1}^{\infty} (n+\tfrac{1}{2}) \{A_n \sinh (n+\tfrac{1}{2})\xi_1 + B_n \cosh (n+\tfrac{1}{2})\xi_1\} I_n,$$

where $$I_n = \int_{-1}^{+1} \frac{1-\zeta^2}{(\cosh \xi_1 - \zeta)^{\frac{3}{2}}} \frac{dP_n(\zeta)}{d\zeta} d\zeta.$$

Now $$\int_{-1}^{+1} \frac{1-\zeta^2}{(\cosh \xi_1 - \zeta)^4} d\zeta = \tfrac{4}{3} \operatorname{cosech}^4 \xi_1,$$

and it remains to evaluate I_n. Now

$$\frac{1}{(\cosh \xi_1 - \zeta)^{\frac{1}{2}}} = \sqrt{2} \sum_{m=0}^{m=\infty} e^{-(m+\frac{1}{2})\xi_1} P_m(\zeta).$$

Differentiating with respect to ζ,

$$\frac{1}{(\cosh \xi_1 - \zeta)^{\frac{3}{2}}} = 2\sqrt{2} \sum_{m=1}^{m=\infty} e^{-(m+\frac{1}{2})\xi_1} \frac{dP_m(\zeta)}{d\zeta},$$

and so $$I_n = 2\sqrt{2} \sum_{m=1}^{\infty} e^{-(m+\frac{1}{2})\xi_1} \int_{-1}^{+1} P_m^1(\zeta)\, P_n^1(\zeta)\, d\zeta$$

$$= 2\sqrt{2}\, e^{-(n+\frac{1}{2})\xi_1} \frac{2}{2n+1} \frac{(n+1)!}{(n-1)!}$$

$$= 4\sqrt{2} \frac{n(n+1)}{2n+1} e^{-(n+\frac{1}{2})\xi_1}.$$

Hence

$$G_1 = 4\pi\mu a^3\omega_1 \operatorname{cosech}^3 \xi_1$$
$$+4\sqrt{2}\,\pi\mu a^2 \sum_{n=1}^{\infty} n(n+1)\left\{A_n \sinh(n+\tfrac{1}{2})\,\xi_1 + B_n \cosh(n+\tfrac{1}{2})\,\xi_1\right\} e^{-(n+\frac{1}{2})\xi_1},$$

or, inserting the values of the constants,

$$G_1 = 4\pi\mu a^3\omega_1 \operatorname{cosech}^3 \xi_1$$
$$+16\pi\mu a^3 \sum_{n=1}^{\infty} n(n+1)\left\{\omega_1 e^{-(2n+1)\xi_1} \coth(n+\tfrac{1}{2})(\xi_1-\xi_2)\right.$$
$$\left. -\omega_2 e^{-(n+\frac{1}{2})(\xi_1\mp\xi_2)} \operatorname{cosech}(n+\tfrac{1}{2})(\xi_1-\xi_2)\right\}.$$

The two series multiplying ω_1 and ω_2 are convergent, whatever ξ_2 may be, provided that ξ_1 is positive; they may therefore be summed separately.

Now, when $\xi > 0$, we have

$$\operatorname{cosech}^3 \xi = 8e^{-3\xi}/(1-e^{-2\xi})^3 = 4 \sum_{n=1}^{\infty} n(n+1)\,e^{-(2n+1)\xi},$$

and so the terms containing ω_1 in G_1 can be written

$$8\pi\mu a^3\omega_1 \operatorname{cosech}^3 \xi_1 + 16\pi\mu a^3\omega_1 \sum_{n=1}^{\infty} n(n+1)\,e^{-(2n+1)\xi_1}\left\{\coth(n+\tfrac{1}{2})(\xi_1-\xi_2)-1\right\}.$$

Also
$$\coth(n+\tfrac{1}{2})(\xi_1-\xi_2)-1 = 2\sum_{n=1}^{\infty} e^{-(2n+1)m(\xi_1-\xi_2)},$$

and so $\sum_{n=1}^{\infty} n(n+1)\,e^{-(2n+1)\xi_1}\left\{\coth(n+\tfrac{1}{2})(\xi_1-\xi_2)-1\right\}$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 2n(n+1)\,e^{-(2n+1)\{(m+1)\xi_1 - m\xi_2\}}.$$

The terms in this double series are all positive, and the series is convergent when summed in the present way; we may therefore interchange the order of summation, and we have

$$\tfrac{1}{2}\sum_{m=1}^{\infty} \operatorname{cosech}^3 \{(m+1)\,\xi_1 - m\xi_2\}.$$

We now see that the terms containing ω_1 in G_1 can be brought into the single sum

$$8\pi\mu a^3\omega_1 \sum_{m=0}^{\infty} \operatorname{cosech}^3 \{(m+1)\,\xi_1 - m\xi_2\}.$$

The terms containing ω_2 can be transformed similarly, using the identities

$$\sum_{n=1}^{\infty} n(n+1)\,e^{-(n+\frac{1}{2})(\xi_1\mp\xi_2)} \operatorname{cosech}(n+\tfrac{1}{2})(\xi_1-\xi_2)$$
$$= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} 2n(n+1)\,e^{-(2n+1)[(m+1)\,\xi_1-(m\pm\frac{1}{2}+\frac{1}{2})\,\xi_2]}$$
$$= \tfrac{1}{2}\sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\,\xi_1 - (m+\tfrac{1}{2}\pm\tfrac{1}{2})\,\xi_2].$$

On combining the results we readily obtain

$$G_1 = 8\pi\mu a^3\Big[\omega_1 \sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - m\xi_2] \\ - \omega_2 \sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - (m+\tfrac{1}{2}\pm\tfrac{1}{2})\xi_2]\Big].$$

If one sphere encloses the other, ξ_2 is positive, and the lower sign must be taken, and

$$G_1 = 8\pi\mu a^3(\omega_1 - \omega_2) \sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - m\xi_2]. \tag{11}$$

If, on the other hand, ξ_2 is negative, the spheres are separate and the upper sign must be taken. In this case

$$G_1 = 8\pi\mu a^3\Big[\omega_1 \sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - m\xi_2] \\ - \omega_2 \sum_{m=0}^{\infty} \operatorname{cosech}^3 (m+1)(\xi_1 - \xi_2)\Big]. \tag{12}$$

Suppose that the sphere ξ_2 is constrained to rotate with a given spin ω_2. There will be a certain value of ω_1 for which G_1 vanishes. This is the steady angular velocity with which the sphere ξ_1 would rotate if allowed to move freely. From (11) we see that in the case when one sphere encloses the other this gives $\omega_1 = \omega_2$. In this case, then, if one sphere be allowed to move freely it will acquire the same angular velocity as the other sphere, and the fluid will move as a rigid body. In the case of two non-enclosing spheres, if ξ_2 is constrained to rotate with angular velocity ω_2, while ξ_1 is left free to move under the fluid stresses, it will rotate with angular velocity

$$\omega_1 = \omega_2 \frac{\sum_{m=0}^{\infty} \operatorname{cosech}^3 (m+1)(\xi_1 - \xi_2)}{\sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - m\xi_2]}.$$

In the case of two equal spheres $\xi_2 = -\xi_1$, and from (10) we obtain

$$\frac{\text{diameter of either sphere}}{\text{distance apart of centres}} = \operatorname{sech} \xi_1.$$

The character of this influence of one sphere upon the other is exhibited in the following table:—

Two Equal Spheres—One free.

ξ_1	$\frac{\text{Diameter}}{\text{Centre Distance}}$	$\frac{\omega_1}{\omega_2}$
·2	·9803	·1278
·4	·9250	·1023
·6	·8435	·0759
·8	·7477	·0524
1·0	·6481	·0340
1·2	·5523	·0211
1·4	·4649	·0126
1·6	·3880	·0073
1·8	·3218	·0042
2·0	·2658	·0023
2·5	·1631	·0005
3·0	·0993	·0001

It appears that even when the distance between the spheres is as small as one fiftieth part of the distance between their centres, one sphere communicates only one eighth of its spin to the other sphere.

If we put $\xi_2 = 0$ and $\omega_2 = 0$, we obtain the solution for the rotation of a sphere in a viscous fluid in the presence of a fixed infinite plane perpendicular to the axis of rotation. The plane will cause the sphere to experience an increased resistance to its rotation, and we may compare the resisting couple with its value in the absence of the plane.

The Increase to the Resistance to the Rotation of a Sphere owing to the Presence of an Infinite Plane.

ξ_1	$\frac{\text{Radius of Sphere}}{\text{Distance of Centre from Plane}}$	Ratio of Increase of Couple due to Plane.
·2	·9803	1·171
·4	·9250	1·126
·6	·8435	1·087
·8	·7477	1·057
1·0	·6481	1·036
1·2	·5523	1·022
1·4	·4649	1·013
1·6	·3880	1·007
1·8	·3218	1·004
2·0	·2658	1·002
2·5	·1631	1·0005
3·0	·0993	1·0001

Here again the effect is surprisingly small. If the fixed plane is brought so close to the rotating sphere that their distance apart is but one-fiftieth of the radius of the latter, the couple required to maintain the rotation is only increased by 17 per cent. These results, it will be noted, are independent of the degree of viscosity of the fluid.