Converting Higher Order Tensors into Vectors and Matrices

In MATLAB, we can convert a second order tensor into a (9×1) vector using the reshape( ) function where elements are ordered column-wise in each representation.

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \leftrightarrow \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{12} & S_{22} & S_{32} & S_{13} & S_{23} & S_{33} \end{bmatrix}^T \tag{1}$$

This approach allows for the computation of higher order tensor products by matrix multiplication. For example, the product $E_{ij} = M_{ijk}^{EF} F_k$ becomes

$$\begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \\ E_{12} \\ E_{22} \\ E_{32} \\ E_{13} \\ E_{23} \\ E_{33} \end{bmatrix} = \begin{bmatrix} M_{111}^{EF} & M_{112}^{EF} & M_{113}^{EF} \\ M_{211}^{EF} & M_{212}^{EF} & M_{213}^{EF} \\ M_{311}^{EF} & M_{312}^{EF} & M_{211}^{EF} \\ \vdots & \vdots & \vdots \\ & & \\ & & \\ & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{2}$$

Similarly, the product $E_{ij} = M_{ijkl}^{ES} S_{kl}$ becomes

$$\begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \\ E_{12} \\ E_{22} \\ E_{32} \\ E_{13} \\ E_{23} \\ E_{33} \end{bmatrix} = \begin{bmatrix} M_{1111}^{ES} & M_{1121}^{ES} & M_{1131}^{ES} & \cdots \\ M_{2111}^{ES} & M_{2121}^{ES} & M_{2131}^{ES} & \\ M_{3111}^{ES} & M_{3121}^{ES} & M_{3131}^{ES} & \\ \vdots & & & \ddots \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} S_{11} \\ S_{21} \\ S_{31} \\ S_{12} \\ S_{22} \\ S_{32} \\ S_{13} \\ S_{23} \\ S_{33} \end{bmatrix} \tag{3}$$

In this way, the grand mobility matrix $\mathcal{M}$ can be written as a (15×15) matrix

$$\begin{bmatrix} \mathbf{U} \\ \mathbf{\Omega} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{UF} & \mathbf{M}_{UL} & \mathbf{M}_{US} \\ \mathbf{M}_{\Omega F} & \mathbf{M}_{\Omega L} & \mathbf{M}_{\Omega S} \\ \mathbf{M}_{EF} & \mathbf{M}_{EL} & \mathbf{M}_{ES} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F} \\ \mathbf{L} \\ \mathbf{S} \end{bmatrix} . \tag{4}$$

Inverting the Mobility Matrix

Importantly, the grand mobility matrix $\mathcal{M}$ as written is singular and cannot be inverted to obtain the corresponding resistance matrix $\mathcal{R}$. This is due to the fact that the stresslet **S** and the strain rate **E** are symmetric and traceless; therefore, they require only 5 independent components to specify them

$$\sum_i S_{ii} = 0 \text{ and } S_{ij} = S_{ji} . \tag{5}$$

Therefore, we introduce the vector $\mathbf{S}'$, which contains 5 linearly independent components derived by from the stresslet **S**

$$[\mathbf{S}'] = [\mathbf{a}][\mathbf{S}], \tag{6}$$

where **a** is a 5×9 matrix (to be determined). Additionally, the conditions (5) imply that

$$[\mathbf{0}] = [\mathbf{b}][\mathbf{S}], \tag{7}$$

where **b** is the following 4×9 matrix

$$\mathbf{b} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} . \tag{8}$$

Here, the coefficients are chosen such that each row of **b** is a unit vector. We can combine (6) and (7) to obtain the square matrix [**ab**]

$$\begin{bmatrix} \mathbf{S}' \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} [\mathbf{S}] \tag{9}$$

Substituting this relation into equation (4), we obtain

$$\begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{ab}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{\Omega} \\ \mathbf{E}' \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{UF} & \mathbf{M}_{UL} & \mathbf{M}_{US} \\ \mathbf{M}_{\Omega F} & \mathbf{M}_{\Omega L} & \mathbf{M}_{\Omega S} \\ \mathbf{M}_{EF} & \mathbf{M}_{EL} & \mathbf{M}_{ES} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{ab}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \mathbf{L} \\ \mathbf{S}' \\ \mathbf{0} \end{bmatrix}, \tag{10}$$

which can be rearranged to give

$$\begin{bmatrix} \mathbf{U} \\ \mathbf{\Omega} \\ \mathbf{E}' \\ \mathbf{0} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{ab} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{UF} & \mathbf{M}_{UL} & \mathbf{M}_{US} \\ \mathbf{M}_{\Omega F} & \mathbf{M}_{\Omega L} & \mathbf{M}_{\Omega S} \\ \mathbf{M}_{EF} & \mathbf{M}_{EL} & \mathbf{M}_{ES} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{ab}^{-1} \end{bmatrix}}_{\mathcal{M}'} \begin{bmatrix} \mathbf{F} \\ \mathbf{L} \\ \mathbf{S}' \\ \mathbf{0} \end{bmatrix}. \tag{11}$$

We can maintain the symmetry of $\mathcal{M}$ by choosing the matrix **a** such that [**ab**] is an orthogonal matrix satisfying

$$[\mathbf{ab}]^{-1} = [\mathbf{ab}]^{T}. \tag{12}$$

In particular, we find that the following matrix satisfies this condition

$$\mathbf{a} = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}. \tag{13}$$

Using this matrix, we compute the new symmetric mobility matrix as

$$\mathcal{M}' = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{ab} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{UF} & \mathbf{M}_{UL} & \mathbf{M}_{US} \\ \mathbf{M}_{\Omega F} & \mathbf{M}_{\Omega L} & \mathbf{M}_{\Omega S} \\ \mathbf{M}_{EF} & \mathbf{M}_{EL} & \mathbf{M}_{ES} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{ab}^{T} \end{bmatrix}, \tag{14}$$

and retain only the first (11×11) components, which are now nonsingular.