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Slow viscous motion of a sphere parallel to a plane wall—I
Motion through a quiescent fluid†

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Abstract—Asymptotic solutions of the Stokes equations are derived for both the translational and rotational motions of a sphere parallel to a plane wall bounding a semi-infinite, quiescent, viscous fluid in the limit where the gap width tends to zero. In a numerical sense these solutions are shown to agree asymptotically with the 'exact' bipolar co-ordinate solutions of O'NEILL [1] and DEAN and O'NEILL [2] after correcting the numerical computations of the latter. The results are applied to the motion of a sphere 'rolling' down an inclined plane under the influence of gravity. It is demonstrated that the sphere cannot be in physical contact with the wall, and that it 'slips' as it rolls down the wall. Failure of the theory to agree with the experimental data of CARTY [3] is tentatively attributed to cavitation.

1. INTRODUCTION

USING spherical bipolar co-ordinates, DEAN and O'NEILL [2] succeeded in solving the Stokes equations for the rotation of a sphere about an axis parallel to a nearby plane wall bounding a semi-infinite fluid at rest at infinity. O'NEILL [1] solved the comparable problem for translatory motion of the sphere parallel to the wall. In the course of extending the analyses to the motion of a neutrally buoyant sphere near a wall in a Couette flow (GOLDMAN, COX and BRENNER [4]), it was found that their combined results failed to satisfy an Onsager-like reciprocity condition (BRENNER [5]). This symmetry condition manifests itself by the existence of a relationship between the force experienced by a rotating sphere, and the torque experienced by a translating sphere. Independent reworking of these problems showed the theoretical analyses to be correct, O'Neill's computations to be correct, and Dean and O'Neill's computations incorrect. Hence, one purpose of this paper is to present consistent and correct numerical results for these problems.

Apart from this, however, and more importantly, is the fact that these 'exact' solutions converge poorly (in a numerical sense) as the ratio of gap width to sphere radius, δ/a, tends to zero. They are, in fact, computationally useless for δ/a less than about 0·001. As such they are unsuited to establishing the limiting behavior when the sphere touches the wall. The major contribution of the present paper resides in providing asymptotic, lubrication-theory-like solutions for this limiting case, and in pointing out that such solutions are at odds with existing experimental data (CARTY [3]).

Section 2 outlines the translational problem. An asymptotic solution of the latter, valid as $\delta/a \rightarrow 0$, is presented in Section 2.1. The force and torque on the sphere thus obtained are compared in Section 2.2 with the 'exact' bipolar co-ordinate results of O'NEILL [1]. Section 3 outlines the comparable rotational problem. Asymptotic, lubrication-theory-type results for the latter are

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presented in Section 3.1, and compared in Section 3.2 with the corrected, 'exact' bipolar co-ordinate results of DEAN and O'NEILL [2]. The rotational and translational results are then shown to be consistent in regard to the 'cross effect.' Finally, in Section 4, we discuss the applicability of these results to the problem of a sphere rolling down an inclined plane wall bounding a semi-infinite fluid.

More complete details will be found in GOLDMAN's dissertation [6].

2. FORCE AND TORQUE ON A TRANSLATING SPHERE NEAR A PLANE

Following the notation of O'NEILL [1], consider the problem of a sphere translating *without rotation* ($\Omega=0$) parallel to a plane wall, as shown in Fig. 1; (x, y, z) constitute a right-handed system of rectangular Cartesian co-ordinates having their origin on the plane wall (P), $z=0$. The z axis passes through the sphere center, O, whose co-ordinates are $(x=0, y=0, z=h)$. The sphere radius is a, its surface being denoted by S. The minimum gap width is $\delta=h-a$.

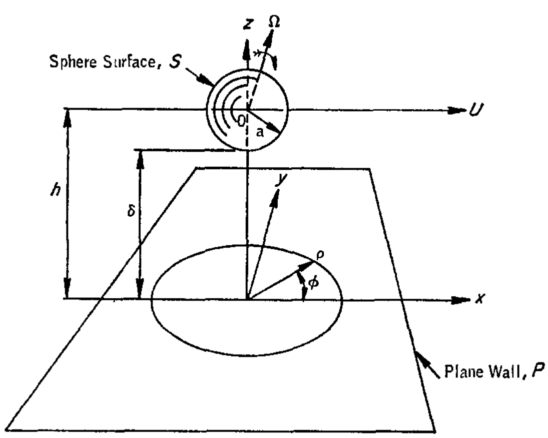

FIG. 1. Definition sketch.

The creeping motion and continuity equations are

$$\frac{1}{\mu}\nabla p^t=\nabla^2\mathbf{v}^t, \qquad \nabla\cdot\mathbf{v}^t=0 \qquad (2.1a, b)$$

whereas the boundary conditions require that

$$\mathbf{v}^t=\mathbf{u}_O=\mathbf{i}_x U \text{ on } S \qquad (2.2)$$

$$\mathbf{v}^t=\mathbf{0} \text{ on } P \qquad (2.3)$$

$$\mathbf{v}^t\to\mathbf{0} \text{ as } z\to\infty \qquad (2.4)$$

The superscript t pertains to the translational motion. Here, $\mathbf{i}_k$ denotes a unit vector in the direction of the kth co-ordinate axis.

In creeping flow the vector force $\mathbf{F}^t$ and torque $\mathbf{T}_O^t$ (about the sphere center) exerted by the fluid on the translating sphere have components only in the x and y directions, respectively. Accordingly, we may write

$$\mathbf{F}^t=\mathbf{i}_x F_x^t=\mathbf{i}_x 6\pi\mu a U F_x^{t*} \qquad (2.5)$$

$$\mathbf{T}_O^t=\mathbf{i}_y T_y^t=\mathbf{i}_y 8\pi\mu a^2 U T_y^{t*} \qquad (2.6)$$

in which the normalized scalar force and torque components, F_x^{t*} and T_y^{t*}, may be positive or negative scalars. These nondimensional quantities are functions only of a/h or, equivalently, δ/a.

2.1 *Translational lubrication-theory asymptote*

In this section we take up the problem of obtaining an asymptotic solution of the preceding equations for the limiting case $\delta/a\to 0$. Singular perturbation methods are employed, involving an 'inner' expansion in the neighborhood of the gap, $\rho/(a\delta)^{\frac{1}{2}}=0(1)$ and $z/\delta=0(1)$, and an 'outer' expansion outside of this region.

It is convenient to employ a system of cylindrical polar co-ordinates (ρ, ϕ, z) (see Fig. 1). Following O'NEILL [1], the boundary conditions suggest a trial solution of the form

$$p^t=\mu\lambda^t\cos\phi \qquad (2.7)$$

$$(v_\rho^t, v_\phi^t, v_z^t)=(\beta^t\cos\phi, \gamma^t\sin\phi, \varepsilon^t\cos\phi) \qquad (2.8a, b, c)$$

where $(v_\rho^t, v_\phi^t, v_z^t)$ are the appropriate velocity components. The auxiliary functions λ^t, β^t, γ^t, and ε^t are functions only of ρ and z.

Substitution into Eq. (2.1) furnishes the following set of differential equations for the auxiliary functions:

$$\frac{\partial \lambda^t}{\partial \rho}=\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \beta^t}{\partial \rho}\right)-\frac{2\beta^t}{\rho^2}+\frac{\partial^2 \beta^t}{\partial z^2}-\frac{2\gamma^t}{\rho^2} \quad (2.9)$$

$$-\frac{\lambda^t}{\rho}=\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \gamma^t}{\partial \rho}\right)-\frac{2\gamma^t}{\rho^2}+\frac{\partial^2 \gamma^t}{\partial z^2}-\frac{2\beta^t}{\rho^2} \quad (2.10)$$

$$\frac{\partial \lambda^t}{\partial z}=\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \varepsilon^t}{\partial \rho}\right)-\frac{\varepsilon^t}{\rho^2}+\frac{\partial^2 \varepsilon^t}{\partial z^2} \quad (2.11)$$

and

$$\frac{\partial \beta^t}{\partial \rho}+\frac{\beta^t}{\rho}+\frac{\gamma^t}{\rho}+\frac{\partial \varepsilon^t}{\partial z}=0 \quad (2.12)$$

Similarly, Eqs. (2.2)–(2.3) furnish the following set of boundary conditions on the sphere and plane:

$$(\beta^t, \gamma^t, \varepsilon^t)=(U, -U, 0) \text{ on } S \quad (2.13\text{a, b, c})$$

$$(\beta^t, \gamma^t, \varepsilon^t)=(0, 0, 0) \text{ on } P \quad (2.14\text{a, b, c})$$

These functions must also vanish at infinity.

The solution of this system of equations and boundary conditions must be a function of δ/a. Thus, for small values of this parameter, we assume that the dependent variables may be expanded as follows:

$$\lambda^t=\bar{\lambda}_0^t+(\delta/a)\bar{\lambda}_1^t+\cdots \qquad \beta^t=\bar{\beta}_0^t+(\delta/a)\bar{\beta}_1^t+\cdots \quad (2.15\text{a, b})$$

$$\gamma^t=\bar{\gamma}_0^t+(\delta/a)\bar{\gamma}_1^t+\cdots \qquad \varepsilon^t=\bar{\varepsilon}_0^t+(\delta/a)\bar{\varepsilon}_1^t+\cdots \quad (2.16\text{a, b})$$

Such an expansion, which is seen to be valid only outside the gap, shall be referred to as the 'outer' expansion.

Substituting the expansions (2.15) and (2.16) into Eqs. (2.9) to (2.14) and letting $\delta/a\to 0$, we see that $\bar{\lambda}_0^t$, $\bar{\beta}_0^t$, $\bar{\gamma}_0^t$ and $\bar{\varepsilon}_0^t$ satisfy Eqs. (2.9) to (2.14) with λ^t, β^t, γ^t and ε^t replaced by $\bar{\lambda}_0^t$, $\bar{\beta}_0^t$, $\bar{\gamma}_0^t$ and $\bar{\varepsilon}_0^t$, for the problem of a sphere in *contact* with a plane (i.e. for $\delta/a=0$).

In order to investigate the situation within the gap, where the expansions (2.15)–(2.16) are invalid, we note that near the point of contact the equation of the sphere surface, $z=z(\rho)$, is

$$z=\delta+(2a)^{-1}\rho^2+O(\rho^4) \quad (2.17)$$

Since $z=O(\delta)$ within the gap, this suggests the introduction of a stretched system of 'inner', independent variables (σ, χ) defined by the relations

$$\sigma=z/\delta, \qquad \chi=\rho/(a\delta)^{\frac{1}{2}} \quad (2.18\text{a, b})$$

Furthermore, in the neighborhood of the gap the dependent, auxiliary variables are assumed to possess expansions of the forms

$$\lambda^t=(\delta/a)^{-\frac{3}{2}}\{\lambda_0^t+(\delta/a)\lambda_1^t+(\delta/a)^2\lambda_2^t+\cdots\} \quad (2.19)$$

$$\beta^t=\{\beta_0^t+(\delta/a)\beta_1^t+(\delta/a)^2\beta_2^t+\cdots\} \quad (2.20)$$

$$\gamma^t=\{\gamma_0^t+(\delta/a)\gamma_1^t+(\delta/a)^2\gamma_2^t+\cdots\} \quad (2.21)$$

$$\varepsilon^t=(\delta/a)^{\frac{1}{2}}\{\varepsilon_0^t+(\delta/a)\varepsilon_1^t+(\delta/a)^2\varepsilon_2^t+\cdots\} \quad (2.22)$$

Such an expansion will be referred to as the 'inner' expansion.

Upon substituting Eqs. (2.18)–(2.22) into Eqs. (2.9)–(2.14), and equating terms of comparable order in δ/a, one obtains *inter alia* the following differential equations for the zero-order auxiliary functions:

$$\frac{\partial^2 \beta_0^t}{\partial \sigma^2}=a\frac{d\lambda_0^t}{d\chi}, \quad \frac{\partial^2 \gamma_0^t}{\partial \sigma^2}=-a\frac{\lambda_0^t}{\chi}, \quad \frac{\partial \lambda_0^t}{\partial \sigma}=0 \quad (2.23\text{a, b, c})$$

and

$$\frac{\partial \beta_0^t}{\partial \chi}+\frac{\beta_0^t}{\chi}+\frac{\gamma_0^t}{\chi}+\frac{\partial \varepsilon_0^t}{\partial \sigma}=0 \quad (2.24)$$

as well as the following zero-order boundary conditions:

$$(\beta_0^t, \gamma_0^t, \varepsilon_0^t)=(U, -U, 0) \text{ at } \sigma=\sigma_S \quad (2.25\text{a, b, c})$$

$$(\beta_0^t, \gamma_0^t, \varepsilon_0^t)=(0, 0, 0) \text{ at } \sigma=0 \quad (2.26\text{a, b, c})$$

where, from Eq. (2.17),

$$\sigma_S = 1 + \tfrac{1}{2}\chi^2 \tag{2.27}$$

corresponds to the sphere surface (to the appropriate order in δ/a).

The procedure which will be followed consists of finding an outer expansion of the form (2.15) and (2.16), and an inner expansion of the form (2.19)–(2.22) such that the form of the outer expansion for small z and ρ becomes identical with that of the inner expansion for large σ and χ. Should there exist terms in the inner and outer expansions other than those given by the relations (2.19)–(2.22) and (2.15) and (2.16), this will become evident in performing the above matching procedure.

In accordance with Eq. (2.23c), λ_0^t is a function only of χ. Hence, integration of Eqs. (2.23a, b) subject to the boundary conditions (2.25a, b) and (2.26a, b) leads to the following expressions:

$$\beta_0^t = \sigma\left[-\frac{a}{2}(\sigma_S - \sigma)\frac{d\lambda_0^t}{d\chi} + \frac{U}{\sigma_S}\right] \tag{2.28}$$

$$\gamma_0^t = \sigma\left[\frac{a}{2}(\sigma_S - \sigma)\frac{\lambda_0^t}{\chi} - \frac{U}{\sigma_S}\right] \tag{2.29}$$

Using these results, Eq. (2.24) may be integrated with respect to σ, and the boundary condition (2.26c) applied. This yields

$$\varepsilon_0^t = \frac{a\sigma^2}{12}\left[(3\sigma_S - 2\sigma)\frac{d^2\lambda_0^t}{d\chi^2} + \frac{1}{\chi}(3\chi^2 + 3\sigma_S - 2\sigma)\frac{d\lambda_0^t}{d\chi} - \frac{1}{\chi^2}(3\sigma_S - 2\sigma)\lambda_0^t + \frac{6U\chi}{a\sigma_S^2}\right] \tag{2.30}$$

where we have noted from Eq. (2.27) that $d\sigma_s/d\chi = \chi$. As now follows directly from Eq. (2.30), satisfaction of the remaining boundary condition (2.25c) requires that λ_0^t satisfy the following total differential equation:

$$\frac{d^2\lambda_0^t}{d\chi^2} + \left(\frac{3\chi}{\sigma_S} + \frac{1}{\chi}\right)\frac{d\lambda_0^t}{d\chi} - \frac{\lambda_0^t}{\chi^2} = -\frac{6U\chi}{a\sigma_S^3} \tag{2.31}$$

This relation is equivalent to the Reynolds equation occurring in classical lubrication theory.

Equation (2.31) cannot be solved in closed form over the entire range of χ. However, as shown in the following discussion, the solution need not be known over the entire range in order to predict the asymptotic forms of the force and torque as $\delta/a \to 0$. Rather, explicit knowledge of the limiting behavior of λ_0^t for large χ suffices to establish these asymptotic forms.

The vector force and torque (about the sphere center O) exerted on the sphere S are given by the expressions

$$\begin{aligned} \mathbf{F}^t &= \int_S\!\!\int d\mathbf{S}\cdot\mathbf{P}^t, \\ \mathbf{T}_O^t &= \int_S\!\!\int \mathbf{r}_{OS}\times(d\mathbf{S}\cdot\mathbf{P}^t) \end{aligned} \tag{2.32a, b}$$

where $d\mathbf{S}$ is a directed element of surface area pointing into the fluid, $\mathbf{r}_{OS}$ is a vector drawn from the sphere center to a point on the sphere surface, and

$$\mathbf{P}^t = -\mathbf{I}p^t + \mu[\nabla\mathbf{v}^t + (\nabla\mathbf{v}^t)^\dagger] \tag{2.33}$$

is the Newtonian pressure tensor arising from the translational motion ($\mathbf{v}^t, p^t$), in which $\mathbf{I}$ is the dyadic idemfactor, and † denotes the transposition operator. Each of the two surface integrals in Eq. (2.32) may be decomposed into the sum of two integrals. The first represents integration over the 'inner' sphere surface in the immediate vicinity of the contact point—more precisely for values of $\rho \le R$ $(z \ll a)$, where R is small compared with a and arbitrarily chosen in any manner such that $\rho = R$ may be considered as lying in both the inner and outer expansions.† The second part of the integrals (2.32)

† That is, R must tend to zero as $\delta/a \to 0$. Though it is immaterial what function R is of δ/a, one could, for example, take R to be equal to $K(\delta/a)^k$, where K and k are any positive constants–independent of δ/a—the constant k being chosen so that the inequality $0 < k < \frac{1}{2}$ is satisfied. Then $\rho = R$ certainly lies in the regions of validity of both the inner and outer expansions. Because of the arbitrary nature in which R is defined, the quantity R cannot appear in the final results for the force and couple; that is, it cannot appear in the final expressions for $\mathbf{F}^t$, and $\mathbf{T}_O^t$, though it can (and does) appear in the individual contributions to the force, $(\mathbf{F}^t)_{\text{inner}}$ and $(\mathbf{F}^t)_{\text{outer}}$, respectively, as well as in the individual contributions to the torque, $(\mathbf{T}_o^t)_{\text{inner}}$ and $(\mathbf{T}_o^t)_{\text{outer}}$, respectively. It is only the *sum* of the inner and outer terms which must be independent of R, not the separate inner and outer terms themselves.

represents integration over the remainder of the sphere, away from the point of contact. From the expansions (2.15) and (2.16) it is clear that the contribution from the outer region tends to a constant value (which may be a function of R) as $\delta/a\to 0$, since λ^t, β^t, γ^t and ε^t each tend, respectively, to $\bar{\lambda}^t_0$, $\bar{\beta}^t_0$, $\bar{\gamma}^t_0$ and $\bar{\varepsilon}^t_0$.

The element of surface area on the sphere surface is

$$d\mathbf{S}=(\mathbf{i}_\rho\sin\theta-\mathbf{i}_z\cos\theta)\{-a^2d(\cos\theta)d\phi\}$$

where $\sin\theta=\rho/a$ and $\cos\theta=[1-(\rho/a)^2]^{\frac{1}{2}}$. Since $\rho\ll a$ in the domain of integration, this becomes

$$d\mathbf{S}\sim\left(\mathbf{i}_\rho\frac{\rho}{a}-\mathbf{i}_z\right)\rho d\rho\, d\phi \tag{2.34}$$

The pressure tensor (2.33) is readily expressed in terms of cylindrical polar co-ordinates and then, via Eqs. (2.7) and (2.8), in terms of the auxiliary pressure and velocity variables. Upon taking account of the boundary conditions (2.13), we thus obtain for the element of force on any element of sphere surface area,

$$\begin{aligned} d\mathbf{S}\cdot\mathbf{P}^t\sim\mu d\phi d\rho \\ \Bigg\{\mathbf{i}_\rho\left[\frac{2}{a}\rho^2\frac{\partial\beta^t}{\partial\rho}-\rho\frac{\partial\beta^t}{\partial z}-\frac{\rho^2}{a}\lambda^t-\rho\frac{\partial\varepsilon^t}{\partial\rho}\right]_S\cos\phi \\ +\mathbf{i}_\phi\left[\frac{\rho^2}{a}\frac{\partial\gamma^t}{\partial\rho}-\rho\frac{\partial\gamma^t}{\partial z}\right]_S\sin\phi \\ +\mathbf{i}_z\left[\frac{\rho^2}{a}\frac{\partial\varepsilon^t}{\partial\rho}-2\rho\frac{\partial\varepsilon^t}{\partial z}+\frac{\rho^2}{a}\frac{\partial\beta^t}{\partial z}+\rho\lambda^t\right]_S\cos\phi\Bigg\}\end{aligned} \tag{2.35}$$

where S denotes the sphere surface. The x and y components of this vector force may be obtained from the relations

$$\begin{aligned}\mathbf{i}_\rho&=\mathbf{i}_x\cos\phi+\mathbf{i}_y\sin\phi\\ \mathbf{i}_\phi&=-\mathbf{i}_x\sin\phi+\mathbf{i}_y\cos\phi\end{aligned} \tag{2.36a, b}$$

In order to evaluate the 'inner' contribution to the force, we expand the dependent variables in Eq. (2.35) via Eqs. (2.19)–(2.22), introduce the inner independent variables (2.18), perform the ϕ-integration, and take account of Eq. (2.5). To terms of the lowest order in δ/a, this yields, for the inner portion of Eq. (2.32a),

$$(F^t_x)_{\text{inner}}\sim\pi\mu a\int_0^{R/(a\delta)^{\frac{1}{2}}}\left[-\chi\frac{\partial\beta^t_0}{\partial\sigma}-a\chi^2\lambda^t_0+\chi\frac{\partial\gamma^t_0}{\partial\sigma}\right]_{\sigma=\sigma_S}d\chi \tag{2.37}$$

Consider the above integrand as $\chi\to 0$. Its behavior can be predicted by determining the asymptotic behavior of λ^t_0 from the differential equation (2.31) for small χ. The resulting differential equation,

$$\frac{d^2\lambda^t_0}{d\chi^2}+\frac{1}{\chi}\frac{d\lambda^t_0}{d\chi}-\frac{\lambda^t_0}{\chi^2}=-\frac{6U\chi}{a} \tag{2.38}$$

has the general solution

$$\lambda^t_0=c_1\chi+c_2\chi^{-1}-\frac{3U\chi^3}{4a} \tag{2.39}$$

where c_1 and c_2 are constants. The term of order χ^{-1} is rejected because it yields an infinite pressure and velocity at $\chi=0$. Hence, $\lambda^t_0=O(\chi)$ as $\chi\to 0$; thus, from Eqs. (2·28) and (2·29), $\beta^t_0=O(1)$ and $\gamma^t_0=O(1)$ as $\chi\to 0$. Consequently, the integral (2.37) is convergent at the lower limit. Near the upper limit of this integral, $\chi\gg 1$, the differential equation (2.31) takes on the form

$$\frac{d^2\lambda^t_0}{d\chi^2}+\frac{7}{\chi}\frac{d\lambda^t_0}{d\chi}-\frac{\lambda^t_0}{\chi^2}=-\frac{48U}{a\chi^5} \tag{2.40}$$

The solution of this equation is

$$\lambda^t_0=c_3\chi^{(-3+\sqrt{10})}+c_4\chi^{(-3-\sqrt{10})}+\frac{24U}{5a\chi^3} \tag{2.41}$$

The term in $\chi^{(-3+\sqrt{10})}$ must be rejected since this term when written in outer variables is $c_3(\rho a^{-\frac{1}{2}})^{(-3+\sqrt{10})}\delta^{-(-3+\sqrt{10})/2}$, which is singular in δ and cannot therefore be matched onto the outer expansion (2.15) and (2.16). Thus,

$$\lambda_0^t \sim 24U/5a\chi^3 \text{ as } \chi \to \infty \tag{2·42}$$

Substitution into (2.28) and (2.29) therefore yields

$$\left(\frac{\partial \beta_0^t}{\partial \sigma}\right)_{\sigma s} \sim -\frac{8U}{5\chi^2} \text{ as } \chi \to \infty \tag{2.43a}$$

and

$$\left(\frac{\partial \gamma_0^t}{\partial \sigma}\right)_{\sigma s} \sim -\frac{16U}{5\chi^2} \text{ as } \chi \to \infty \tag{2.43b}$$

Substituting Eqs. (2.42) and (2.43) into (2.37) yields

$$(F_x^t)_{\text{inner}} \sim \pi\mu a \int^{R/(a\delta)^{\frac{1}{2}}} (-\tfrac{32}{5}U)\frac{d\chi}{\chi} \text{ as } \delta/a \to 0 \tag{2.44}$$

Hence, the asymptotic expansion of the inner contribution to the force for small δ/a is

$$(F_x^t)_{\text{inner}} \sim \tfrac{16}{5}\pi\mu U a \ln(\delta/a) + f(R)(\delta/a)^0 + \cdots \tag{2.45}$$

where

$$f(R) \sim -\tfrac{32}{5}\pi\mu U a \ln R \text{ as } R \to 0 \tag{2.46}$$

Now, the asymptotic forms of β_0^t, γ_0^t and ε_0^t for large χ may be deduced from Eqs. (2.28)–(2.30) and (2.42). Thus, as $\chi \to \infty$,

$$\beta_0^t \sim 2U\sigma\chi^{-2}[\tfrac{9}{5}(1-2\sigma\chi^{-2})+1] \tag{2.47}$$

$$\gamma_0^t \sim 2U\sigma\chi^{-2}[\tfrac{3}{5}(1-2\sigma\chi^{-2})-1] \tag{2.48}$$

$$\varepsilon_0^t \sim \tfrac{16}{5}U\sigma^2\chi^{-3}(1-2\sigma\chi^{-2}) \tag{2.49}$$

Hence, by the required matching conditions, it follows that as $z \to 0$ and $\rho \to 0$

$$\bar{\lambda}_0^t \sim \tfrac{24}{5}Ua^2\rho^{-3} \tag{2.50}$$

$$\bar{\beta}_0^t \sim 2Uaz\rho^{-2}[\tfrac{9}{5}(1-2az\rho^{-2})+1] \tag{2.51}$$

$$\bar{\gamma}_0^t \sim 2Uaz\rho^{-2}[\tfrac{3}{5}(1-2az\rho^{-2})-1] \tag{2.52}$$

$$\bar{\varepsilon}_0^t \sim \tfrac{16}{5}Uaz^2\rho^{-3}(1-2az\rho^{-2}) \tag{2.53}$$

From these asymptotic expansions and from the definition of the second part of the integral (2.32a), the 'outer' contribution to the force is

$$(F_x^t)_{\text{outer}} \sim g(R)(\delta/a)^0 + \cdots \text{ as } \delta/a \to 0 \tag{2.54}$$

where

$$g(R) \sim -\tfrac{32}{5}\pi\mu U a \int_R \frac{d\rho}{\rho} + O(R^0) \text{ as } R \to 0 \tag{2.55}$$

Therefore,

$$g(R) \sim \tfrac{32}{5}\pi\mu U a \ln R \text{ as } R \to 0 \tag{2.56}$$

From Eqs. (2.45), (2.46), (2.54) and (2.56), we find that the complete force on the body is

$$F_x^t \sim \tfrac{16}{5}\pi\mu U a \ln(\delta/a) + h(R)(\delta/a)^0 + \cdots \text{ as } \delta/a \to 0 \tag{2.57}$$

where $h(R)$ is bounded as $R \to 0$. Actually, $h(R)$ must be independent of R because of the arbitrary manner in which R was chosen. Accordingly, we may write

$$F_x^t = \tfrac{16}{5}\pi\mu U a \ln(\delta/a) + O(\delta/a)^0 \tag{2.58}$$

Alternatively, in terms of dimensionless variables

$$F_x^{t*} = \tfrac{8}{15}\ln(\delta/a) + O(\delta/a)^0 \tag{2.59}$$

Note, correctly, that $F_x^{t*} < 0$ since $\delta/a < 1$ in the region of applicability of Eq. (2.59).

The torque exerted on the sphere may be computed from Eq. (2.32b) by an analogous line of reasoning. Since, in general, $\mathbf{r}_{OS} = a(\mathbf{i}_\rho \sin\theta - \mathbf{i}_z \cos\theta)$, we have for $\rho \ll a$ that

$$\mathbf{r}_{OS} \sim \mathbf{i}_\rho \rho - \mathbf{i}_z a \tag{2.60}$$

The counterpart of Eq. (2.37) ultimately obtained in this way is

$$(T_y^t)_{\text{inner}} \sim \pi\mu a^2 \int_0^{R/(a\delta)^{\frac{1}{2}}} \left[-\chi\frac{\partial \gamma_0^t}{\partial \sigma} + \chi\frac{\partial \beta_0^t}{\partial \sigma}\right]_{\sigma=\sigma s} d\chi \tag{2.61}$$

The expression for the nondimensional torque which eventually results is

$$T_y^{t*} = -\tfrac{1}{10}\ln(\delta/a) + O(\delta/a)^0 \qquad (2.62)$$

Note that $T_y^{t*} > 0$ since $\delta/a < 1$.

2.2 *Discussion of translational results*

In order to compare the small gap approximation with O'NEILL's [1] bipolar co-ordinate solution, we have accurately recomputed O'Neill's numerical values. Results for the nondimensional force and torque are tabulated in Table 1 for normalized separation distances, h/a, ranging from 10·07 down to 1·0032.† The last-mentioned value required solution of 130 simultaneous linear equations to achieve the five digit accuracy noted. This represented the limiting capacity of the electronic computer.

† The bipolar coordinate parameter α appearing in Table 1 is related to h/a via the formula

$$\alpha = \ln\left\{\frac{h}{a} + \left[\left(\frac{h}{a}\right)^2 - 1\right]^{\frac{1}{2}}\right\}$$

TABLE 1. FORCES AND TORQUES ON A TRANSLATING SPHERE

α	h/a	F_x^{t*}	T_y^{t*}
3·0	10·0677	−1·0591	$8{\cdot}7744 \times 10^{-6}$
2·0	3·7622	−1·1738	$4{\cdot}2160 \times 10^{-4}$
1·5	2·3524	−1·3079	$2{\cdot}6423 \times 10^{-3}$
1·0	1·5431	−1·5675	$1{\cdot}4649 \times 10^{-2}$
0·5	1·1276	−2·1514	$7{\cdot}3718 \times 10^{-2}$
0·3	1·0453	−2·6475	$1{\cdot}4552 \times 10^{-1}$
0·1	1·005004	−3·7863	$3{\cdot}4187 \times 10^{-1}$
0·08	1·003202	−4·0223	$3{\cdot}8494 \times 10^{-1}$

The exact values tabulated in Table 1 are compared graphically in Figs. 2 and 3 with the lubrication-theory approximations, Eqs. (2.59) and (2.62). Also shown for comparison purposes are the approximate solutions of FAXÉN [7] (see also OSEEN [8]), obtained by the 'method of reflections,' valid for small a/h (i.e. large δ/a):

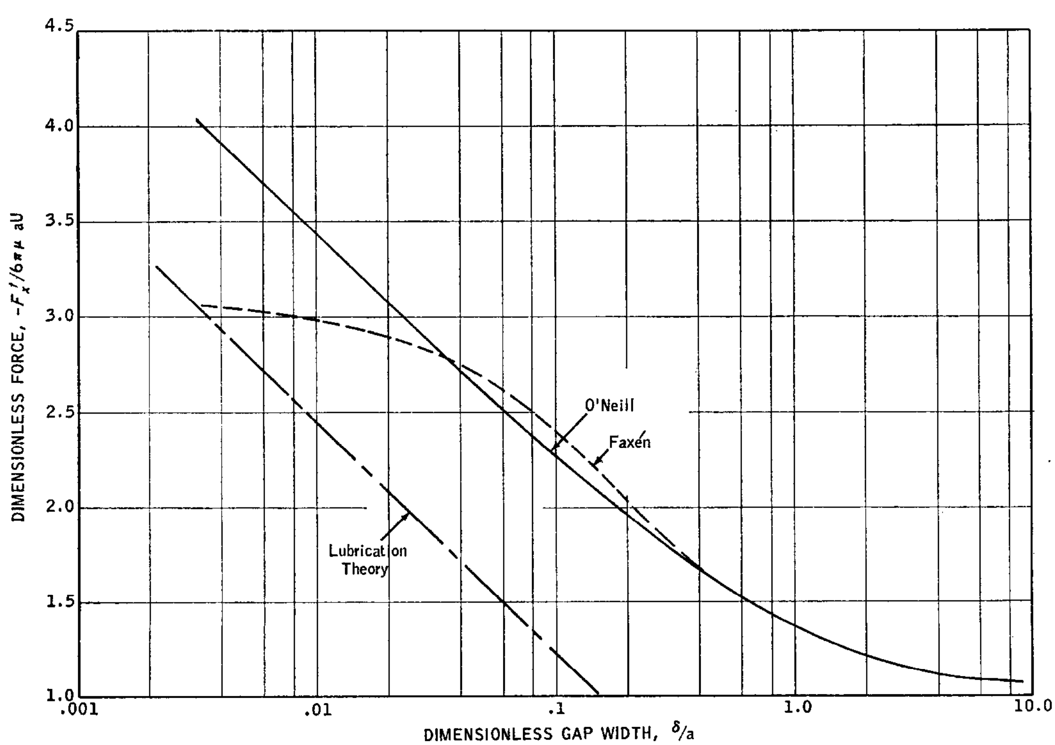

FIG. 2. Force on a translating sphere.

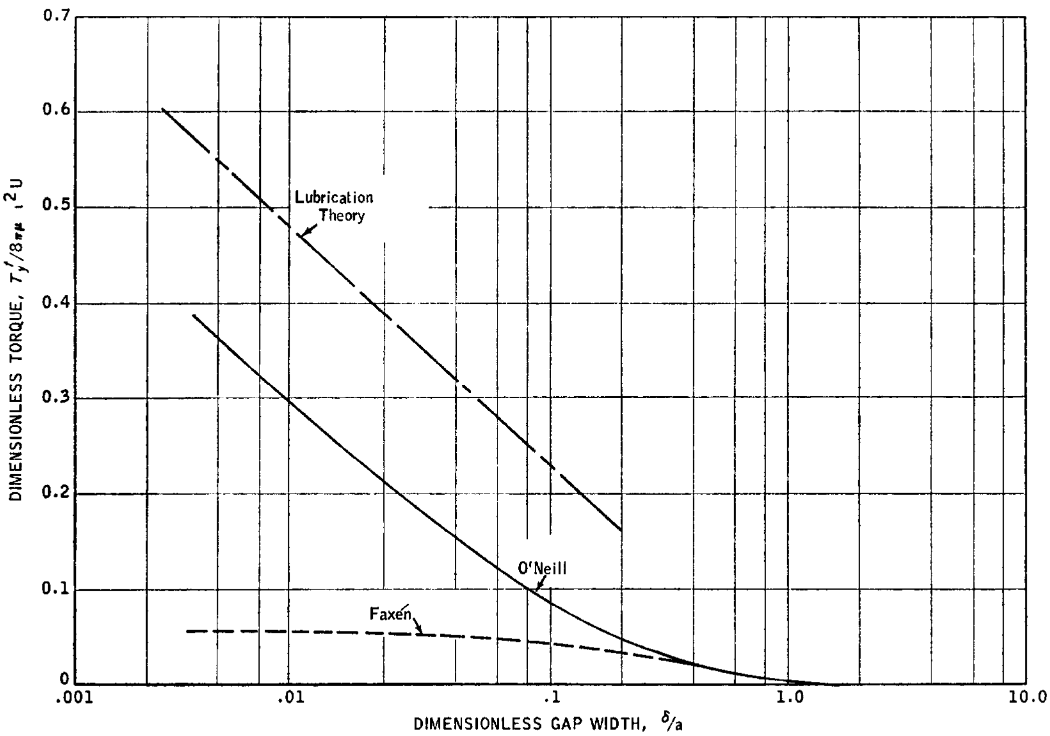

FIG. 3. Torque on a translating sphere.

$$F_x^{t*} = -\left[1-\frac{9}{16}\left(\frac{a}{h}\right)+\frac{1}{8}\left(\frac{a}{h}\right)^3-\frac{45}{256}\left(\frac{a}{h}\right)^4-\frac{1}{16}\left(\frac{a}{h}\right)^5\right]^{-1} \tag{2.63}$$

$$T_y^{t*} = \frac{3}{32}\left(\frac{a}{h}\right)^4\left(1-\frac{3}{8}\frac{a}{h}\right) \tag{2.64}$$

MAUDE [9] also gives an approximate, first-order formula for the force on a translating sphere, but it is incorrect, being inconsistent with Faxén's result, Eq. (2.63), and with O'Neill's numerical results (Table 1) for small a/h.

The lubrication-theory asymptotes are obviously quite poor for the range of δ/a values displayed in Figs. 2 and 3. This is, perhaps, not surprising in view of the very slow, logarithmic approach to the limit. The nonsingular, higher-order terms in the small gap width approximation would obviously go far towards improving the agreement, but their computation requires a more complete determination of the outer expansion. To form at least some quantitative criterion for judging the correctness of Eqs. (2.59) and (2.62), we have employed the last two data points in Table 1 to estimate the limiting slopes of F_x^{t*} and T_y^{t*} vs. $\ln(\delta/a)$. The slopes obtained in this way are 0·52859 and −0·09647, respectively, in good agreement with the theoretical slopes of 8/15 (0·5333 . . .) and −1/10 (−0·1000 . . .), respectively.

In subsequent calculations we require knowledge of the magnitudes of the nonsingular $(\delta/a)^0$ terms in Eqs. (2.59) and (2.62). In lieu of calculating them rigorously by continuing the singular perturbation analysis, they can be estimated by employing the theoretical asymptotic coefficients in Eqs. (2.59) and (2.62) in conjunction with the computed values of F_x^{t*} and T_y^{t*} for the final data point (i.e. at $\delta/a=0{\cdot}003202$) in Table 1. This leads to the asymptotic formulas

$$\boxed{\begin{aligned} F_x^{t*} &\sim \tfrac{8}{15}\ln(\delta/a)-0{\cdot}9588\,, \\ T_y^{t*} &\sim -\tfrac{1}{10}\ln(\delta/a)-0{\cdot}1895 \end{aligned}} \qquad (2.65\text{a, b})$$

3. Force and Torque on a Rotating Sphere Near a Plane

Following Dean and O'Neill [2], consider the problem of a sphere rotating, *without translating* ($U=0$), about an axis parallel to a plane wall, as depicted in Fig. 1. Let the superscript r represent rotation. The differential equations and boundary conditions are then identical to those set forth at the beginning of Section 2 (with r written in place of t), the only exception being that the boundary condition on the sphere is now

$$\mathbf{v}^r=\boldsymbol{\omega}\times\mathbf{r}_{OS}=(\mathbf{i}_y\times\mathbf{r}_{OS})\Omega \text{ on } S \qquad (3.1)$$

In creeping flow the vector force $\mathbf{F}^r$ and torque $\mathbf{T}_O^r$ (about the sphere center) exerted by the fluid on the rotating sphere are necessarily of the forms

$$\mathbf{F}^r=\mathbf{i}_xF_x^r=\mathbf{i}_x6\pi\mu a^2\Omega F_x^{r*} \qquad (3.2)$$

and

$$\mathbf{T}_O^r=\mathbf{i}_yT_y^r=\mathbf{i}_y8\pi\mu a^3\Omega T_y^{r*} \qquad (3.3)$$

in which the normalized scalar force and torque components, F_x^{r*} and T_y^{r*}, may be positive or negative scalars. These nondimensional parameters depend only upon a/h or, equivalently, δ/a.

3.1 *Rotational lubrication-theory asymptote*

Upon replacing t by r, the auxiliary functions which govern the rotational velocity and pressure fields are defined by Eqs. (2.7) and (2.8). These new auxiliary functions satisfy the same differential equations and boundary conditions as noted in Eqs. (2.9)–(2.14), except that in place of Eq. (2.13) the appropriate boundary condition on the sphere is now

$$(\beta^r,\gamma^r,\varepsilon^r)=(-\Omega a,\ \Omega a,\ -\Omega\rho) \text{ on } S \qquad (3.4\text{a, b, c})$$

Furthermore, Eqs. (2.15)–(2.27) are also applicable, the sole exception being that the zero-order boundary condition on the sphere [i.e. Eq. (2.25)] is now

$$(\beta_0^r,\gamma_0^r,\varepsilon_0^r) =(-\Omega a,\ \Omega a,\ -\Omega a\chi) \text{ at } \sigma=\sigma_S \qquad (3.5\text{a, b, c})$$

The expressions obtained for β_0^r, γ_0^r and ε_0^r are identical to those given by Eqs. (2.28)–(2.30) except that $-\Omega a$ now appears in place of U. Application of the boundary condition (3.5c) to the resulting expression for ε_0^r leads to a total differential equation for λ_0^r identical to Eq. (2.31), except that Ωa now appears in place of U. With this sole exception, the limiting solutions for λ_0^r as $\chi\to0$ and $\chi\to\infty$ are the same as before. Since Eqs. (2.37) and (2.61) continue to be applicable† to the rotational problem, the arguments employed in Section 2 eventually lead to the following asymptotic expressions for the nondimensional force and torque on the rotating sphere as $\delta/a\to0$:

$$F_x^{r*}=-\tfrac{2}{15}\ln(\delta/a)+O(\delta/a)^0 \qquad (3.6)$$

$$T_y^{r*}=\tfrac{2}{5}\ln(\delta/a)+O(\delta/a)^0 \qquad (3.7)$$

3.2 *Discussion of rotational results*

Fundamental considerations of 'cross effects' (Brenner [5]) require that the force (per unit angular speed) exerted on the rotating sphere, as considered herein, be equal to the torque (per unit translational speed) exerted on the translating sphere (Section 2). A comparison of Dean and O'Neill's [2] and O'Neill's [1] numerical results for these two, sphere motions uncovered an inconsistency in regard to this cross effect. Independent checks of both cases proved O'Neill's translating sphere calculations correct and Dean and O'Neill's rotating sphere calculations in error. Corrected numerical results for the latter are presented in Table 2. The integration procedure used to obtain these data are presented in detail by Goldman [6].

The results tabulated in Table 2 are plotted graphically in Figs. 4 and 5. Also shown is the

† Note that in the evaluation of these integrals the coefficients of Eqs. (2.43a, b) are now –28/5 and 4/5, respectively.

approximate, first-order formula given by MAUDE [9] for the torque,† valid for small a/h,

$$T_y^{r*} = -\left[1 + \frac{5}{16}\left(\frac{a}{h}\right)^3\right] \tag{3.8}$$

TABLE 2. FORCES AND TORQUES ON A ROTATING SPHERE

a	$\frac{h}{a}$	$F_x{}^{r*}$	$T_y{}^{r*}$
3·0	10·0677	$1{\cdot}1699 \times 10^{-5}$	$-1{\cdot}0003$
2·0	3·7622	$5{\cdot}6214 \times 10^{-4}$	$-1{\cdot}0059$
1·5	2·3524	$3{\cdot}5231 \times 10^{-3}$	$-1{\cdot}0250$
1·0	1·5431	$1{\cdot}9532 \times 10^{-2}$	$-1{\cdot}0998$
0·5	1·1276	$9{\cdot}8291 \times 10^{-2}$	$-1{\cdot}3877$
0·3	1·0453	$1{\cdot}9403 \times 10^{-1}$	$-1{\cdot}6996$
0·1	1·005004	$4{\cdot}5582 \times 10^{-1}$	$-2{\cdot}5056$
0·08	1·003202	$5{\cdot}1326 \times 10^{-1}$	$-2{\cdot}6793$

† WAKIYA [11] gives the same formula as Eq. (3.8), but with the coefficient 5/16 replaced by 3/16. This is almost certainly in error since it fails to agree with the corrected Dean–O'Neill results (Table 2) at the smaller values of a/h.

As shown in Fig. 4, this equation agrees well with the corrected Dean–O'Neill results (Table 2) for small a/h. The comparable approximation for the rotational force can be deduced by combining Eq. (2.64) with the reciprocity relation (3.12). The result is

$$F_x^{r*} = \frac{1}{8}\left(\frac{a}{h}\right)^4\left(1 - \frac{3}{8}\frac{a}{h}\right)$$

As shown in Fig. 5, this equation agrees well with the corrected Dean–O'Neill results for small a/h.

Discussion of the cross effect begins with the general formula of BRENNER [5], for the force and torque on a rigid particle of arbitrary shape in quasi-static Stokes motion near a rigid boundary. For the cross effect we have

$$\mathbf{F}^r = -\mu \mathbf{C}_O^\dagger \cdot \boldsymbol{\omega}, \qquad \mathbf{T}_O^t = -\mu \mathbf{C}_O \cdot \mathbf{u}_O \tag{3.9a, b}$$

where $\mathbf{C}_O$ is the coupling dyadic for the sphere-plane wall system (at the sphere center, O), $\mathbf{C}_O^\dagger$ is its transpose, and $\mathbf{u}_O$ and $\boldsymbol{\omega}$ are, respectively, the

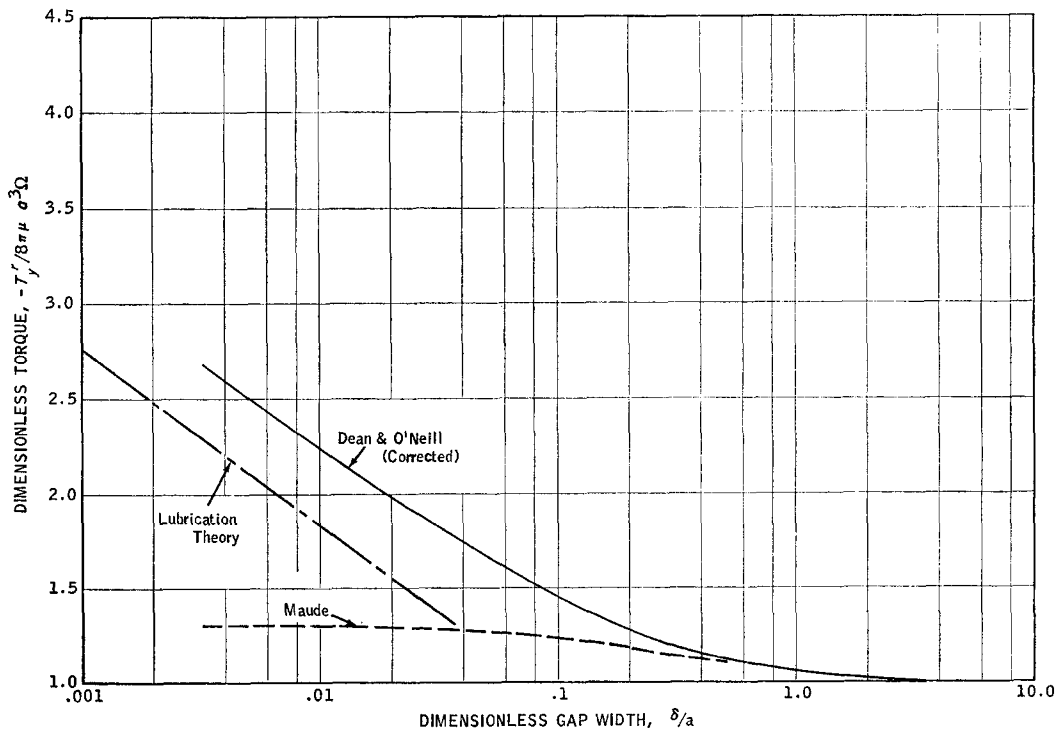

FIG. 4. Torque on a rotating sphere.

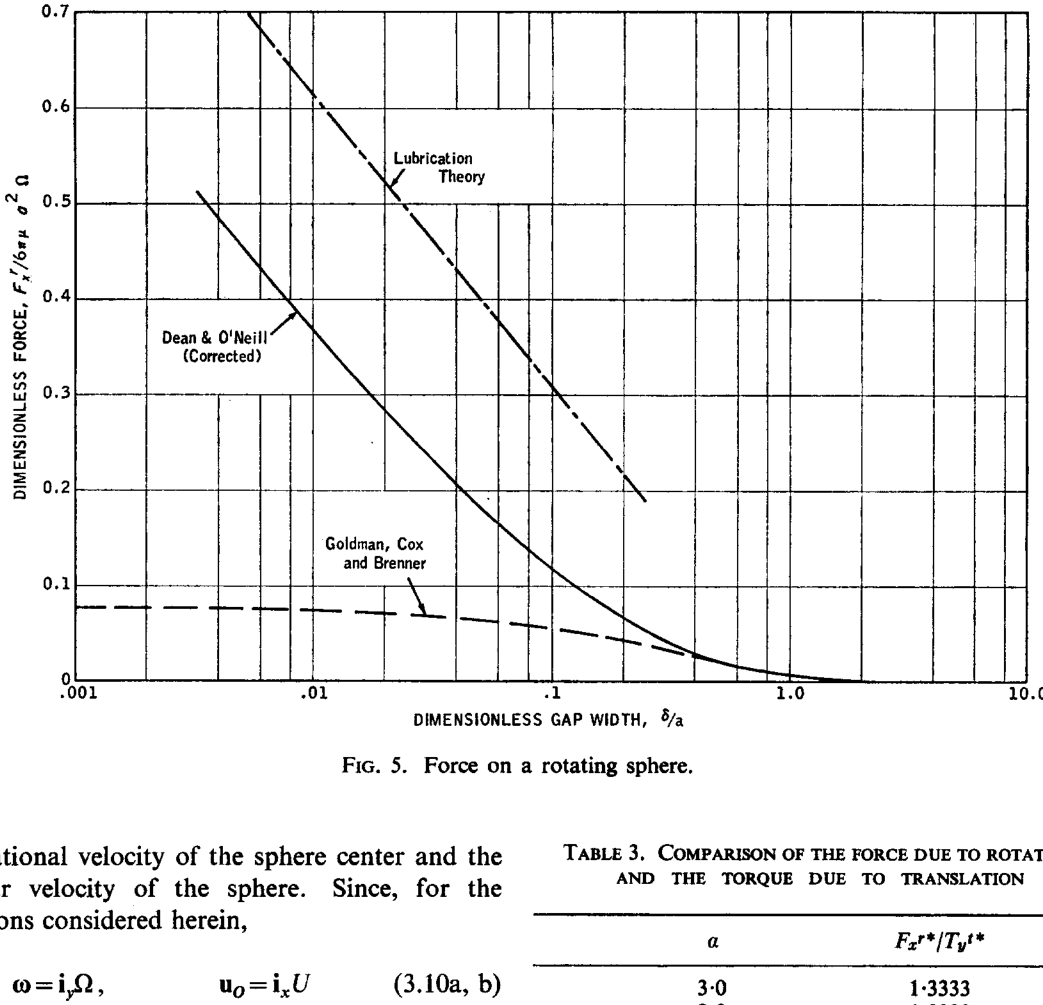

FIG. 5. Force on a rotating sphere.

translational velocity of the sphere center and the angular velocity of the sphere. Since, for the situations considered herein,

$$\boldsymbol{\omega} = \mathbf{i}_y \Omega, \qquad \mathbf{u}_O = \mathbf{i}_x U \qquad (3.10\text{a, b})$$

it follows that

$$F_x^r = -\mu \Omega C_{yx}, \qquad T_y^t = -\mu U C_{yx} \quad (3.11\text{a, b})$$

where C_{yx} is the appropriate Cartesian tensor component of $\mathbf{C}_O$. According to Eqs. (3.11a) and (3.11b) the cross-effect theory requires that

$$F_x^{r*}/T_y^{t*} = \tfrac{4}{3} \qquad (3.12)$$

A comparison of the appropriate forces and torques tabulated in Tables 1 and 2 is made in Table 3. The consistency requirement demanded by Eq. (3.12) is clearly met. This same test of internal consistency is satisfied by Eqs. (2.62) and (3.6).

TABLE 3. COMPARISON OF THE FORCE DUE TO ROTATION AND THE TORQUE DUE TO TRANSLATION

α	F_x^{r*}/T_y^{t*}
3·0	1·3333
2·0	1·3333
1·5	1·3333
1·0	1·3333
0·5	1·3333
0·3	1·3333
0·1	1·3333
0·08	1·3333

As before, the last two data points in Table 2 permit estimates of the limiting slopes of F_x^{r*} and T_y^{r*} vs. $\ln(\delta/a)$ as $\delta/a \to 0$. The limiting slopes thereby estimated are $-0{\cdot}12865$ and $0{\cdot}38905$, respectively, in satisfactory agreement with the theoretical slopes of $-\frac{2}{15}$ $(-0{\cdot}1333\ldots)$ and $\frac{2}{5}$ $(0{\cdot}4000\ldots)$, respectively. The nonsingular $(\delta/a)^0$ terms in Eqs. (3.6) and (3.7) may be estimated as previously described at the end of Section 2.2. This leads to the following approximate relations for small δ/a:

$$F_x^{r*} \sim -\tfrac{2}{15}\ln(\delta/a) - 0{\cdot}2526, \qquad T_y^{r*} \sim \tfrac{2}{5}\ln(\delta/a) - 0{\cdot}3817 \tag{3.13a, b}$$

4. COMBINED TRANSLATION AND ROTATION

Because of the linearity of the equations of motion, the stress equations, and the boundary conditions, the combined effect of a simultaneous translational and rotational motion is merely the vector sum of the separate effects. Hence,

$$F_x = 6\pi\mu a(UF_x^{t*} + a\Omega F_x^{r*}), \qquad T_y = 8\pi\mu a^2(UT_y^{t*} + a\Omega T_y^{r*}) \tag{4.1a, b}$$

In the absence of external torques on the sphere, Eq. (4.1b) requires that

$$a\Omega/U = -T_y^{t*}/T_y^{r*} \tag{4.2}$$

whence Eq. (4.1a) yields

$$F_x^* = \frac{F_x}{6\pi\mu Ua} = \frac{F_x^{t*}T_y^{r*} - F_x^{r*}T_y^{t*}}{T_y^{r*}} \tag{4.3}$$

Values of $a\Omega/U$ and F_x^* vs. h/a may be computed from Eqs. (4.2) and (4.3) by utilizing the data in Tables 1 and 2. In the case of a sphere moving under the influence of gravity, F_x is equal and opposite to the gravitational force on the sphere corrected for the buoyancy of the fluid.

Of special interest is the limiting case where the sphere approaches the plane. The asymptotic forms furnished by Eqs. (2.59), (2.62), (3.6), and (3.7) yield

$$\frac{a\Omega}{U} \sim \frac{1}{4}, \qquad F_x^* \sim \frac{1}{2}\ln\left(\frac{\delta}{a}\right) \quad \text{as} \quad \frac{\delta}{a} \to 0 \tag{4.4a, b}$$

Since the case of no slip at the wall requires that $a\Omega/U = 1$, Eq. (4.4a) shows that a sphere 'rolling' down an inclined plane wall must slip. Furthermore, the finiteness of the component of the net gravity force on the particle parallel to the wall, coupled with Eq. (4.4b), shows that the sphere cannot actually be in physical contact with the wall; that is, δ cannot be zero.

CARTY [3] made measurements of the terminal velocities of spherical particles rolling down smooth plane boundaries in the Reynolds number range $10^{-2} < \mathrm{Re} < 10^5 (\mathrm{Re} = 2aU\rho_f/\mu)$. Experiments were performed with Lucite, glass, acetate, and steel spheres, with water and oil on glass boundaries, and with water on a wooden boundary. A plot of the drag coefficient

$$\mathrm{C_D} = -2F_x/\pi a^2 \rho_f U^2 \tag{4.5}$$

vs. Re yielded a single, smooth curve for all particles, fluids, and boundaries. For $\mathrm{Re} < 60$ a log–log plot of $\mathrm{C_D}$ vs. Re was linear, corresponding to the empirical formula $\mathrm{C_D} = 215\mathrm{Re}^{-1}$. In our notation this is equivalent to $F_x^* = -8{\cdot}96$, and hence from Eq. (4.4b) to an *apparent* gap width† of $\delta/a \approx 10^{-8}$. For values of a of about one centimeter this gap width is of the order of atomic dimensions—well beyond the possible limits of applicability of the theory set forth in this paper. Accordingly, the idealized mathematical model of a smooth, rigid sphere moving through a fluid continuum of constant physical properties near a smooth, rigid, plane wall is incapable of providing a rational explanation of Carty's data. We discuss below possible reasons for this occurrence, and ways in which the theory might be amended to take account of these factors.

1. Surface roughness is a factor to be considered. Irregularities and roughness of the sphere and plane surfaces may cause mechanical interference and solid–solid friction. Standard mechanical

†A somewhat more accurate estimate of the apparent gap width may be obtained from Eq. (4.3) by employing the higher-order relations (2.65) and (3.13). Equation (4.4b) is then replaced by

$$F_x^* = \frac{1}{2}\left[\frac{\{\ln(\delta/a)\}^2 - 4{\cdot}325\ln(\delta/a) + 1{\cdot}591}{\ln(\delta/a) - 0{\cdot}9543}\right]$$

But for $F^* = -8{\cdot}96$ this yields virtually the same estimate for δ/a as does Eq. (4·4b).

finishing techniques produce surfaces with 'root-mean-square' irregularities, Δ, of the order of 2×10^{-6} to 10^{-4} in. (FRENCH and VIERCK [10]). Typically, for a $\frac{1}{4}$ in. diameter sphere this yields $\Delta/a=8\times10^{-6}$ to 4×10^{-4}. Nondimensional gap widths smaller than about 10^{-4} do not, therefore, lie within the purview of the analysis. In view of the different sphere and wall roughnesses encompassed by Carty's data it does not appear, however, that roughness can be invoked to explain the discrepancy between theory and experiment. For all of Carty's data lie on a *single* curve, irrespective of the different degrees of roughness likely to have been present in the various experiments.

2. In the region around the contact point, velocity gradients tend to become infinite since the velocity difference between the sphere and plane surfaces remains finite as the separation distance tends to zero. In this extreme situation it is questionable whether the fluid remains perfectly Newtonian. Non-Newtonian effects can drastically alter the theoretical behavior of the force and torque. However, non-Newtonian effects lead to nonlinear behavior. Such nonlinearities are clearly lacking in Carty's data in the 'viscous regime,' $\mathrm{Re}<60$. Accordingly, non-Newtonian behavior cannot be invoked to rationalize the observed discrepancy between theory and experiment.

3. Though inertial effects were not considered in the theoretical development, they too would lead to nonlinearities. The existence of a 'viscous regime' in the correlation of experimental data therefore suffices to rule out inertial effects as the source of the discrepancy in this region.

4. When the gap width approaches the mean free path or average spacing between molecules, the equations of continuum fluid mechanics cease to be applicable, as does the macroscopic no-slip boundary condition too. However, as the gap width tends towards zero, surface roughness will almost certainly become dominant long before the Knudsen or free-molecule flow regimes are approached. Thus, noncontinuum flow is unlikely to be relevant to any interpretation of experimental data.

5. It has been assumed in the analysis that the density and viscosity of the fluid are constant throughout the fluid domain. However, it follows from our calculations that as the gap width approaches zero, the pressure varies as

$$p=\mu\frac{U}{a}\left(\frac{a}{\delta}\right)^{\frac{1}{2}}\cos\phi[f(\chi)+O(\delta/a)] \tag{4.6}$$

where $p=p^t+p^r$. The positive function $f(\chi)$ is independent of δ, and approaches zero in the extreme ranges $\chi\to0$ and $\chi\to\infty$. Hence, the point of maximum (or minimum) pressure will occur at an intermediate value of χ, of order unity. The function $f(\chi)$ can be taken of order unity, in which case the pressure approaches plus or minus infinity as $\delta\to0$, depending upon the algebraic sign of $\cos\phi$. In particular, the pressure approaches $+\infty$ in the region immediately ahead of the sphere ($\pi/2>\phi>-\pi/2$) and $-\infty$ immediately behind the sphere ($\pi/2<\phi<3\pi/2$). The variation in fluid properties with these large local pressure variations has not been accounted for in the theoretical analysis. However, such effects result in nonlinear behavior, and are thus inconsistent with the observed linearity of Carty's data in the 'viscous regime.'

Another possible effect of pressure is to cause cavitation of the fluid. According to Eq. (4.6), the pressure at points immediately behind the sphere is less than the 'ambient' fluid pressure. Typically, if one considers a $\frac{1}{4}$ in. diameter sphere translating at 1 in./sec through a 100 cP viscosity fluid possessing a vapor pressure 1 atm. below 'ambient', the assumption $f(\chi)\approx1$ then leads to the conclusion that the fluid pressure is below the vapor pressure when $\delta/a\approx4\times10^{-4}$. Hence, cavitation is possible. Without further theoretical analysis, however, it is not clear whether or not cavitation is consistent with the linearity of Carty's data in the 'viscous regime.'

6. The large pressure gradients predicted near the point of contact may result in deformation of the sphere. However, in view of the different modulii of elasticity of the various spheres investigated by Carty, it hardly seems likely that this effect could be significant in rationalizing the experimental data.

On the basis of the above discussion, cavitation provides the most likely explanation of the failure of the theory. A careful experimental investigation of the rolling of spheres down inclined planes is clearly in order†

In the event that the rolling sphere does not touch the wall, the component of the gravitational force *normal* to the wall must be balanced by an equal and opposite hydrodynamic lift force. Such forces are necessarily inertial in origin; lift forces cannot arise in Stokes flow due to the symmetry of the sphere–plane wall configuration. At small non-zero Reynolds numbers, RUBINOW and KELLER [11] have already proved the existence of a lift force on a rotating–translating sphere acting at right angles to both $\mathbf{u}_O$ and $\boldsymbol{\omega}$, at least when the sphere moves through an *unbounded* fluid.‡ Even in the absence of rotation, a repulsive lift force is predicted on the basis of the Oseen equations for the case of a sphere translating parallel to a wall (OSEEN [8]).

In a subsequent paper (GOLDMAN, COX and BRENNER [4]), we shall discuss the case of a neutrally buoyant sphere moving parallel to a single, plane wall in a Couette flow.

† Such experiments are currently being pursued in the laboratories of the Pulp and Paper Research Institute of Canada, under the direction of Dr. S. G. Mason.

‡ Note, however, that the direction of their force is opposite to what would be required to hydrodynamically support the sphere. Wall effects could conceivably modify this conclusion.

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Résumé—Les solutions asymptotiques des équations de Stokes sont dérivées à la fois pour les mouvements de translation et de rotation d'une sphère parallèle à la paroi plane délimitant un fluide visqueux, semi-infini, au repos, dans la limite où la largeur de l'interstice tend vers zéro. On démontre que, d'un point de vue numérique, des solutions s'accordent asymptotiquement avec les solutions 'exactes' de co-ordonnées bipolaires de O'NEILL [1] et de DEAN et O'NEILL [2], rectification faite des calculs numériques de ces dernières. Les résultats sont appliqués au mouvement d'une sphère 'roulant' vers le bas d'une inclinaison plane sous l'influence de la gravité. Il est démontré que la sphère ne peut pas être physiquement en contact avec la paroi, et qu'elle 'dérape' à mesure qu'elle roule vers le bas de la paroi. On est tenté d'attribuer à la cavitation l'échec de l'accord de cette théorie avec les données expérimentales de CARTY [3].

Zusammenfassung—Asymptotische Lösungen der Stokes Gleichungen werden sowohl für die Translations als auch für die Rotationsbewegungen einer Kugel parallel zu einer ebenen Wand, die an einer halb unbegrenzten, ruhenden zähen Flüssigkeit angrenzt, in dem Bereiche, in dem sich die Abstandsweite der Grenze Null nähert, abgeleitet. Im numerischen Sinn erweist es sich, dass diese Lösungen asymptotisch mit den "präzisen" bipolaren, koordinierten Lösungen von O'Neill [1] und Dean und O'Neill [2] übereinstimmen, nachdem die zahlenmässigen Berechnungen der letzteren berichtigt wurden. Die Resultate werden auf die Bewegung einer Kugel, die infolge ihrer Schwerkraft an einer schrägen Ebene herunterrollt, bezogen. Es wird bewiesen, dass die Kugel mit der Wand nicht in physischem Kontakt sein kann, und dass sie, indem sie von der Wand herabrollt, "rutscht". Die Tatsache, dass diese Theorie mit den Versuchsergebnissen von Carty [3] nicht übereinstimmt, wird vorläufig der Kavitation zugeschrieben.