Question 1: (36 Marks)

a) [14 marks] Consider two vectors **u** = $[1, 2, 5]^T$ and **v** = $[7, 8, 6]^T$.
   - Normalize them.
   - Calculate the angle between the vectors.
   - Calculate the projection of **u** onto **v** after normalization.

Solution

The norms are

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 5^2} = \sqrt{30} = 5.4772$$ [2 marks]

$$\|\mathbf{v}\| = \sqrt{7^2 + 8^2 + 6^2} = \sqrt{149} = 12.2066$$ [2 marks]

The normalized vectors are

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \begin{bmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{bmatrix} = \begin{bmatrix} 0.1826 \\ 0.3651 \\ 0.9129 \end{bmatrix}$$ [2 marks]

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} \frac{7}{\sqrt{149}} \\ \frac{8}{\sqrt{149}} \\ \frac{6}{\sqrt{149}} \end{bmatrix} = \begin{bmatrix} 0.5735 \\ 0.6554 \\ 0.4915 \end{bmatrix}$$ [2 marks]

To calculate the angle

$$\hat{\mathbf{u}} \bullet \hat{\mathbf{v}} = \frac{1}{\sqrt{30}} \times \frac{7}{\sqrt{149}} + \frac{2}{\sqrt{30}} \times \frac{8}{\sqrt{149}} + \frac{5}{\sqrt{30}} \times \frac{6}{\sqrt{149}} = \frac{53}{\sqrt{4470}} = 0.7927$$ [2 marks]

$$\theta = \cos^{-1}(\hat{\mathbf{u}} \bullet \hat{\mathbf{v}}) = \cos^{-1}\left(\frac{53}{\sqrt{4470}}\right) = 37.5592^\circ$$ [2 marks]

Calculating the angle as follows is also correct.

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

The projection of u onto v after normalization:

$$|\mathrm{proj}_{\hat{\mathbf{v}}}\hat{\mathbf{u}}| = \hat{\mathbf{u}} \bullet \hat{\mathbf{v}} = 0.7927$$ [2 marks]

Question 1 [Math Basics]:

a) [4 marks] Suppose that two vectors **u** and **v** are emitted from the origin $[0, 0, 0]^T$ to the points $[2,-4, 0]^T$ and $[5,-3, 0]^T$ respectively. Using the dot product, determine the length of the projection of **u** onto **v**.

Answers to Question 1 a):

$\mathbf{u} = [2,-4, 0]^T$
$\mathbf{v} = [5,-3, 0]^T$

$\|\mathbf{v}\|$ = sqrt(34) [2 marks]

Proj = **u . v** /$\|\mathbf{v}\|$ = (2*5 + (-4)*(-3) + 0*0) / sqrt(34) = 3.773 [2 marks]

b) [8 marks] Consider a triangle $P_1P_2P_3$ where $P_1 = [3, 0, 0]^T$, $P_2 = [0, 1, 2]^T$ and $P_3 = [1.5, 0, 1.5]^T$. Estimate the normal to this triangle.

Answers to Question 1 b):

$P_2P_1 = [0, 1, 2]^T - [3, 0, 0]^T = [-3, 1, 2]^T$ [2 marks]

$P_3P_1 = [1.5, 0, 1.5]^T - [3, 0, 0]^T = [-1.5, 0, 1.5]^T$ [2 marks]

$\mathbf{n} = P_3P_1 \times P_2P_1 = [-1.5, 0, 1.5]^T \times [-3, 1, 2]^T$ [4 marks]
$=[-1.5, -1.5, -1.5]^T$

Question 1 [Math Basics]:

[10 points] Consider a pyramid whose vertices are **a** = $[2,2,1]^T$, **b** = $[5,4,1]^T$, **c** = $[3,7,1]^T$, and **d** = $[3,3,5]^T$. What is the normal to the face **abd**?

Answers to Question 1:

Get the direction vectors emitting from **a**:

u=b-a= $[5,4,1]^T$- $[2,2,1]^T$= $[3,2,0]^T$
v=d-a= $[3,3,5]^T$- $[2,2,1]^T$= $[1,1,4]^T$

$$\mathbf{u} \times \mathbf{v} = [\mathbf{u}]_\times \mathbf{v}$$

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \\ 1 \end{bmatrix}$$

Getting **u** and **v** ➔ 4 points
Cross product ➔ 4 points
Final result ➔ 2 points

It is OK if they get the inward vector $[-8,12,-1]^T$.

Question 1 (Math Basics):

[10 points] Using the parametric equation of a line, determine the intersection point between the two line segments <p_1,p_2> and <p_3,p_4> where $p_1=[1,1]^T$, $p_2=[9,9]^T$, $p_3=[3,0]^T$ and $p_4=[3,9]^T$.

Answers to Question 1:

The parametric equations of the two lines:

$$\mathbf{p_1} + t_a(\mathbf{p}_2 - \mathbf{p_1}) \text{ and } \mathbf{p}_3 + t_b(\mathbf{p}_4 - \mathbf{p}_3)$$

At the point of intersection, the two equations are equal; thus:

$$\mathbf{p_1} + t_a(\mathbf{p}_2 - \mathbf{p_1}) = \mathbf{p}_3 + t_b(\mathbf{p}_4 - \mathbf{p}_3)$$

$$\begin{bmatrix}1\\1\end{bmatrix} + t_a\left(\begin{bmatrix}9\\9\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\0\end{bmatrix} + t_b\left(\begin{bmatrix}3\\9\end{bmatrix} - \begin{bmatrix}3\\0\end{bmatrix}\right)$$

$$\begin{bmatrix}1\\1\end{bmatrix} + t_a\begin{bmatrix}8\\8\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix} + t_b\begin{bmatrix}0\\9\end{bmatrix}$$

This can be written as :

$$1 + 8t_a = 3 \Rightarrow t_a = 0.25 \Rightarrow \begin{bmatrix}1\\1\end{bmatrix} + t_a\begin{bmatrix}8\\8\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix}$$

This can be added (but not required)

$$1 + 8t_a = 0 + 9t_b \Rightarrow t_b = \frac{1}{3} \Rightarrow \begin{bmatrix}3\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}0\\9\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix}$$

Question 1: (6 Marks)

Consider a pyramid whose vertices are $\dot{\mathbf{A}} = [2,2,1]^T$, $\dot{\mathbf{B}} = [5,4,1]^T$, $\dot{\mathbf{C}} = [3,7,1]^T$ and $\dot{\mathbf{D}} = [3,3,5]^T$. What is the normal to the face $\dot{\mathbf{A}}\dot{\mathbf{B}}\dot{\mathbf{D}}$?

Solution

Direction vectors [2 marks each]

$$
\begin{aligned}
\mathbf{u} &= \dot{\mathbf{B}} - \dot{\mathbf{A}} \\
&= \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v} &= \dot{\mathbf{D}} - \dot{\mathbf{A}} \\
&= \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.
\end{aligned}
$$

The normal to the face $\dot{\mathbf{A}}\dot{\mathbf{B}}\dot{\mathbf{D}}$ is the cross product $\mathbf{u}\times\mathbf{v}$: [2 marks]

$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \\
&= \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
&= \begin{bmatrix} u_2 v_3 - v_2 u_3 \\ v_1 u_3 - u_1 v_3 \\ u_1 v_2 - v_1 u_2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \times 4 - 1 \times 0 \\ 1 \times 0 - 3 \times 4 \\ 3 \times 1 - 1 \times 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \\ 1 \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \underbrace{\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}}_{[\mathbf{u}]_\times} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\mathbf{v}} \\
&= \begin{bmatrix} u_2 v_3 - v_2 u_3 \\ v_1 u_3 - u_1 v_3 \\ u_1 v_2 - v_1 u_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \\ 1 \end{bmatrix}
\end{aligned}
$$

Any of the above methods is acceptable.

Question 1: [Math Basics]

a) [4 marks] Suppose that two vectors **u** and **v** are emitted from the origin $[0, 0, 0]^T$ to the points $[2,-4, 0]^T$ and $[5,-3, 0]^T$ respectively. Using the dot product, determine the length of the projection of **u** onto **v**.

Answers to Question 1a):

Pr A.12
$\mathbf{u} = [2,-4, 0]^T$
$\mathbf{v} = [5,-3, 0]^T$

$\|\mathbf{v}\|$ = sqrt(34) [2 marks]

Proj $= \mathbf{u} \cdot \mathbf{v} / \|\mathbf{v}\|$ = (2*5 + (-4)*(-3) + 0*0) / sqrt(34) = 3.773 [2 marks]

b) [8 marks] Consider a triangle $P_1P_2P_3$ where $P_1 = [3, 0, 0]^T$, $P_2 = [0, 1, 2]^T$ and $P_3 = [1.5, 0, 1.5]^T$. Estimate the normal to this triangle.

Answers to Question 1b):

$P_2P_1 = [0, 1, 2]^T - [3, 0, 0]^T = [-3, 1, 2]^T$ [2 marks]

$P_3P_1 = [1.5, 0, 1.5]^T - [3, 0, 0]^T = [-1.5, 0, 1.5]^T$ [2 marks]

$\mathbf{n} = P_3P_1 \times P_2P_1 = [-1.5, 0, 1.5]^T \times [-3, 1, 2]^T = [-1.5, -1.5, -1.5]^T$ [4 marks]

German University in Cairo
Department of Digital Media Engineering and Technology
DMET 502 - Computer Graphics Quiz 1.1
Quiz Duration: 45 minutes

Name :	ID :	Tut No.:

Question:

1. Use Bresenham's line drawing algorithm (found in the formula sheet) to find out the location of the pixels that constitute the line segment from $[6,5]^T$ to $[3,1]^T$ (Total 7 marks)

Steep = | 1-5| > |3-6| = true
If(steep=true)
Swap(X_0, Y_0) → $X_0 = 5$, $Y_0 = 6$ (0.25 marks)
Swap(X_1, Y_1) → $X_1 = 1$, $Y_1 = 3$ (0.25 marks)

If($X_0 > X_1$)
Swap(X_0, X_1) → $X_0 = 1$, $X_1 = 5$ (0.25 marks)
Swap(Y_0, Y_1) → $Y_0 = 3$, $Y_1 = 6$ (0.25 marks)
End if

$\Delta X = X_1 - X_0 = 4$ (0.5 marks)
$\Delta Y = |Y_1 - Y_0| = 3$ (0.5 marks)
Carry forward if above operations are incorrect
Y= 3, error =0 (1 mark, 0.5 mark each)
X coordinates go from 3 to 6 (1 mark)
Error column (1 mark), new error column (1 mark)
Y coordinates go from 1 to 5 (1 mark)

X	Error	Y	New error
1	0	3	0
2	3	4	-1
3	2	5	-2
4	1	5	1
5	1	6	0

German University in Cairo
Department of Digital Media Engineering and Technology
DMET 502 - Computer Graphics Quiz 1.2
Quiz Duration: 45 minutes

Name :	ID :	Tut No.:

Question:

1. Consider the line from [0,0] to [-8,-4], use general Bresenham's line drawing algorithm to rasterize this line. Evaluate and tabulate all the steps involved. (Total 7 marks)

Steep = | -4-0| > |-8-0| = false

If($X_0 > X_1$)
Swap(X_0, X_1) → $X_0 = -8$, $X_1 = 0$ (0.5 marks)
Swap(Y_0 ,Y_1) → $Y_0 = -4$,$Y_1 = 0$ (0.5 marks)
End if

$\Delta X = X_1 - X_0 = 8$ (0.5 marks)
$\Delta Y = |Y_1 - Y_0| = 4$ (0.5 marks)
Carry forward if above operations are incorrect
Y= -4, error =0 (1 mark, 0.5 each)
X coordinates go from -8 to 0 (1 mark)
Error column (1 mark), new error column (1 mark)
Y coordinates go from -4 to 0 (1 mark)

X	Error	Y	New error
-8	0	-4	0
-7	4	-3	-4
-6	0	-3	0
-5	4	-2	-4
-4	0	-2	0
-3	4	-1	-4
-2	0	-1	0
-1	4	0	-4
0	0	0	0

b) [10 marks] Consider a clip rectangle spanning from $[5, 3]^T$ to $[15, 15]^T$. This clip rectangle is used to clip a group of lines; among them is a line that extends from $[7, 18]^T$ to $[10, 10]^T$. Determine the intersection point between this line and the top border of the clip rectangle.

Solution

This problem has different solutions; any of them is acceptable. The following is from Solution 2.13.

The top border = $[x_{min}, y_{max}]^T \leftrightarrow [x_{max}, y_{max}]^T == [5,15]^T \leftrightarrow [15,15]^T$

The line = $[x_0, y_0]^T \leftrightarrow [x_1, y_1]^T == [7,18]^T \leftrightarrow [10,10]^T$

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{10 - 18}{10 - 7} = -\frac{8}{3} = -2.6667$$ [4 marks]

$$x = x_0 + \frac{y_{max} - y_0}{m} = 7 + \frac{15 - 18}{-\frac{8}{3}} = 8.125$$ [4 marks]

$y = y_{max} = 15$ [2 marks]

Thus, the intersection point is $[8.125,15]^T$

c) [12 marks] The listed is the digital differential analyzer (DDA) algorithm used to draw lines when the absolute magnitude of slope is less than or equal to 1. Modify it to accommodate |*m*|>1.

Algorithm
Input: x_0, y_0, x_1, y_1
1: $m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$
2: $y = y_0$
3: **for** $(x = x_0 \text{ to } x_1)$ **do**
4: Plot $[x, \lfloor y + 0.5 \rfloor]^T$
5: $y = y + m$
6: **end for**
end

__Solution__

It can be written that

$$x_i = \frac{1}{m}(y_i - B)$$

Consequently, x_{i+1} is calculated as by incrementing the value of *y* by 1; so, we can write

$$x_{i+1} = \frac{1}{m}(y_{i+1} - B) = \frac{1}{m}(y_i + 1 - B) = \underbrace{\frac{1}{m}(y_i - B)}_{x_i} + \frac{1}{m} = x_i + \frac{1}{m}$$

Thus, the roles of *x* and *y* are reversed by assigning an increment of 1 to *y* and an increment of 1/*m* to *x*.

Algorithm
Input: x_0, y_0, x_1, y_1
1: $m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$
2: $x = x_0$
3: **for** $(y = y_0 \text{ to } y_1)$ **do**
4: Plot $[\lfloor x + 0.5 \rfloor, y]^T$
5: $x = x + \frac{1}{m}$
6: **end for**
end

[The idea: 4 marks; lines from 2 thru 5: 8 marks]

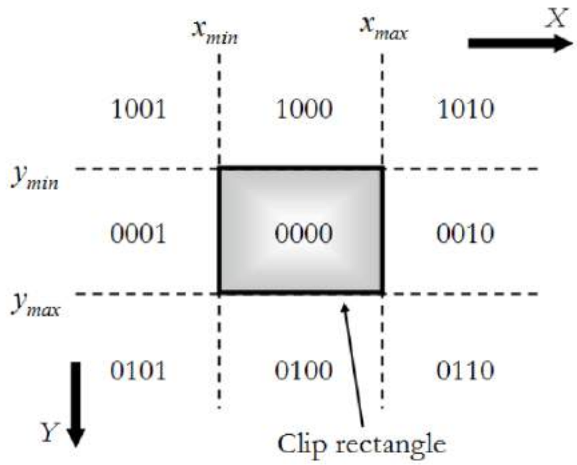
Question 1: [2D Graphics]

a) [16 marks] Assuming a right-handed coordinate system with Cohen-Sutherland clipping algorithm, the following table may be used to assign the outcode values for the 9 regions where bit 3 represents the most-significant bit while bit 0 represents the least-significant bit.

Bit	Value	Meaning
3	= 1, if region is above the top edge; = 0, otherwise.	if $y > y_{max}$ if $y \leq y_{max}$
2	= 1, if region is below the bottom edge; = 0, otherwise.	if $y < y_{min}$ if $y \geq y_{min}$
1	= 1, if region is right to the right edge; = 0, otherwise.	if $x > x_{max}$ if $x \leq x_{max}$
0	= 1, if region is left to the left edge; = 0, otherwise.	if $x < x_{min}$ if $x \geq x_{min}$

If a left-handed coordinate system is used, modify the previous table so that each region keeps its outcode (e.g., the upper left region is assigned 1001, the lower right region is assigned 0110, etc.).

Solution [16 marks; 2 marks each entry]



Bit	Value	Meaning
3	= 1, if region is above the top edge; = 0, otherwise.	if $y < y_{min}$ if $y \geq y_{min}$
2	= 1, if region is below the bottom edge; = 0, otherwise.	if $y > y_{max}$ if $y \leq y_{max}$
1	= 1, if region is right to the right edge; = 0, otherwise.	if $x > x_{max}$ if $x \leq x_{max}$
0	= 1, if region is left to the left edge; = 0, otherwise.	if $x < x_{min}$ if $x \geq x_{min}$

b) [9 marks] Given a 2D polygon specified by the vertices $[-3, 0]^T$, $[3, -1]^T$, $[1, 0]^T$ and $[4, 2]^T$, test whether it is convex or concave using two different methods.

<u>Solution</u>

Using cross product:

Consider $v_{i-1}= [3, -1]^T$, $v_i= [1, 0]^T$ and $v_{i+1}= [4, 2]^T$: [2 marks]

$$\mathbf{e_{i-1}} \times \mathbf{e_i} = (\mathbf{v_i} - \mathbf{v_{i-1}}) \times (\mathbf{v_{i+1}} - \mathbf{v_i}) = \begin{bmatrix} x_i - x_{i-1} \\ y_i - y_{i-1} \end{bmatrix} \times \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix}$$
$$= (x_i - x_{i-1}) * (y_{i+1} - y_i) - (y_i - y_{i-1}) * (x_{i+1} - x_i)$$
$$\Rightarrow (1-3)*(2-0)-(4-1)*(0+1) \Rightarrow -7$$

Consider $v_{i-1}= [1, 0]^T$, $v_i= [4, 2]^T$ and $v_{i+1}= [-3, 0]^T$: [2 marks]

$$\mathbf{e_{i-1}} \times \mathbf{e_i} = (\mathbf{v_i} - \mathbf{v_{i-1}}) \times (\mathbf{v_{i+1}} - \mathbf{v_i}) = \begin{bmatrix} x_i - x_{i-1} \\ y_i - y_{i-1} \end{bmatrix} \times \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix}$$
$$= (x_i - x_{i-1}) * (y_{i+1} - y_i) - (y_i - y_{i-1}) * (x_{i+1} - x_i)$$
$$\Rightarrow (4-1)*(0-2)-(-3-4)*(2-0) \Rightarrow 20$$

Different signs ➔ Concave

Using linear equations:

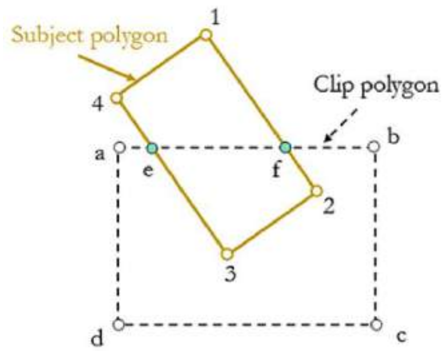
Consider the linear equation given by the line segment <$[1, 0]^T$, $[4,2]^T$>
[2 marks]

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$
$$y - 0 = \frac{2}{3}(x-1)$$
$$3y - 2x + 2 = 0$$

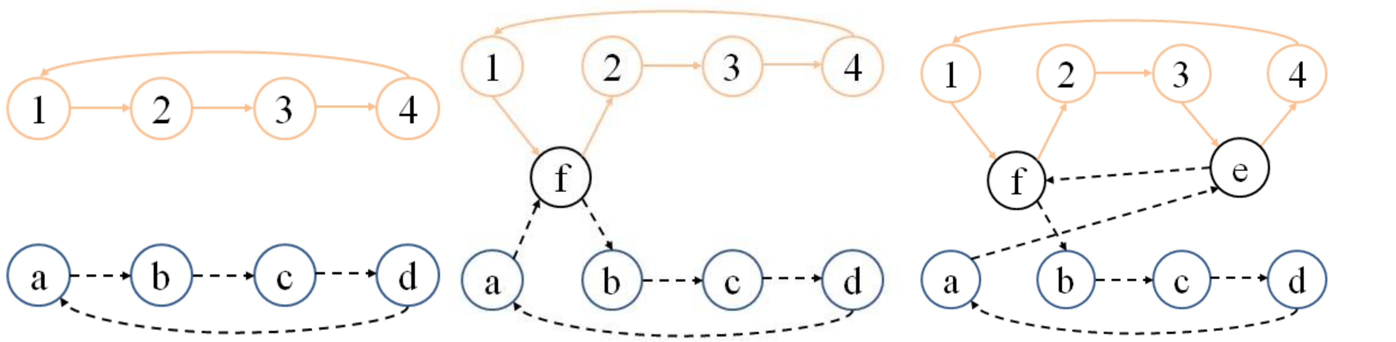
Apply the other two points ($[3,-1]^T$ and $[-3,0]^T$) to the previous equation
3*-1 -2*3 + 2 ➔ -7 [1 mark]
3*0 -2*-3 + 2 ➔ 8 [1 mark]
Different signs ➔ Concave [1 mark]

Question 1: [2D Graphics]

a) [8 marks] Consider the subject and clip polygons shown. Draw the steps that should be followed to populate lists of Weiler-Atherton clipping algorithm for both the subject and clip polygons.



Answers to Question 1a):



[1+2+4 marks]

b) [10 marks] Given a 2D polygon specified by the vertices $[3, 3]^T$, $[6, 3]^T$, $[8, 2]^T$ and $[6, 6]^T$, test whether it is convex or concave using two different methods.

Answers to Question 1b):

Using cross product:

Consider $v_{i-1}= [3, 3]^T$, $v_i= [6, 3]^T$ and $v_{i+1}= [8, 2]^T$: [2 marks]

$$\mathbf{e}_{i-1} \times \mathbf{e}_i = (\mathbf{v}_i - \mathbf{v}_{i-1}) \times (\mathbf{v}_{i+1} - \mathbf{v}_i) = \begin{bmatrix} x_i - x_{i-1} \\ y_i - y_{i-1} \end{bmatrix} \times \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix}$$
$$= (x_i - x_{i-1}) * (y_{i+1} - y_i) - (y_i - y_{i-1}) * (x_{i+1} - x_i)$$
$$\Rightarrow (6 - 3) * (2 - 3) - (0 - 0) * (8 - 6) \Rightarrow -3$$

Consider $v_{i-1}= [6, 3]^T$, $v_i= [8, 2]^T$ and $v_{i+1}= [6, 6]^T$: [2 marks]

$$\mathbf{e}_{i-1} \times \mathbf{e}_i = (\mathbf{v}_i - \mathbf{v}_{i-1}) \times (\mathbf{v}_{i+1} - \mathbf{v}_i) = \begin{bmatrix} x_i - x_{i-1} \\ y_i - y_{i-1} \end{bmatrix} \times \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix}$$
$$= (x_i - x_{i-1}) * (y_{i+1} - y_i) - (y_i - y_{i-1}) * (x_{i+1} - x_i)$$
$$\Rightarrow (8 - 6) * (6 - 2) - (2 - 3) * (6 - 8) \Rightarrow +6$$

Different signs ➔ Concave [1 mark]

Using linear equations:

Consider the linear equation given by the line segment <$[3, 3]^T$, $[6,3]^T$>

[2 marks]

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$
$$y - 3 = 0$$
$$y = 3$$

Apply the other two points ($[6,6]^T$ and $[8,2]^T$) to the previous equation

6-3 ➔ 3 ➔ +ve [1 marks]

2-3 ➔ -1 ➔ -ve [1 marks]

Different signs ➔ Concave [1 marks]

Question 1 [2D Graphics]

a) [8 marks] Consider a clip rectangle spanning from $[5, 3]^T$ to $[15, 15]^T$. This clip rectangle is used to clip a group of lines; among them is a line that extends from $[7, 18]^T$ to $[10, 10]^T$. Determine the intersection point between this line and the top border of the clip rectangle.

Answers to Question 1 a):

This problem has different solutions; any of them is acceptable. The following is from Solution 2.13.

The top border = $[x_{min}, y_{max}]^T \leftrightarrow [x_{max}, y_{max}]^T == [5,15]^T \leftrightarrow [15,15]^T$

The line = $[x_0, y_0]^T \leftrightarrow [x_1, y_1]^T == [7,18]^T \leftrightarrow [10,10]^T$

$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{10-18}{10-7} = -\frac{8}{3} = -2.6667$ [2 marks]

$x = x_0 + \frac{y_{max} - y_0}{m} = 7 + \frac{15-18}{-\frac{8}{3}} = 8.125$ [4 marks]

$y = y_{max} = 15$ [2 marks]

Thus, the intersection point is $[8.125,15]^T$

b) [10 marks] Below is the Bresenham's algorithm used to draw a line on a computer screen between two pixels located at $[x0,y0]^T$ and $[x1,y1]^T$.

```
function line(x0, x1, y0, y1)
  boolean steep := abs(y1 - y0) > abs(x1 - x0)
  if steep then swap(x0, y0) swap(x1, y1)
  if x0 > x1 then swap(x0, x1) swap(y0, y1)
  int deltax := x1 - x0
  int deltay := abs(y1 - y0)
  int error := 0
  int ystep
  int y := y0
  if y0 < y1 then ystep := 1 else ystep := -1
  for x from x0 to x1
    if steep then plot(y,x) else plot(x,y)
    error := error + deltay
    if 2×error >= deltax
      y := y + ystep
      error := error - deltax
```

Modify the algorithm to draw a line using its linear equation

$$ax + by + c = 0 \qquad where\ a \neq 0\ \ and\ \ \ b \neq 0$$

The inputs to your algorithm are the coefficients a, b and c as well as the starting and ending *x*-coordinates (i.e., x0 and x1). The header of the new algorithm would be

`function myLine (a, b, c, x0, x1)`

Answers to Question 1 b):

$$y = \frac{-c - ax}{b}$$

```
function myLine (a, b, c, x0, x1)
  y0=(-c-a*x0)/b [4 marks]
  y1=(-c-a*x1)/b [4 marks]
  line(x0, x1, y0, y1) [2 marks]
```

Question 2 [2D Graphics]:

[12 marks] The listed is the digital differential analyzer (DDA) algorithm used to draw lines when the absolute magnitude of slope (i.e., $|m|$) is less than or equal to 1.

Without swapping coordinates or reflecting about the line y=x, modify it to draw lines when $|m|>1$.

Algorithm

Input: $x_0,\ y_0,\ x_1,\ y_1$

1: $m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$

2: $y = y_0$

3: **for** $(x = x_0 \text{ to } x_1)$ **do**

4: Plot $[x, \lfloor y + 0.5 \rfloor]^T$

5: $y = y + m$

6: **end for**

end

Answers to Question 2:

It can be written that

$$x_i = \frac{1}{m}(y_i - B)$$

Consequently, x_{i+1} is calculated as by incrementing the value of y by 1; so, we can write

$$x_{i+1} = \frac{1}{m}(y_{i+1} - B) = \frac{1}{m}(y_i + 1 - B) = \underbrace{\frac{1}{m}(y_i - B)}_{x_i} + \frac{1}{m} = x_i + \frac{1}{m}$$

Correct Solution:

2: $y = y_0$

3: $for\left(x = x_0\ to\ x_1\ step\ \frac{1}{m}\right)$

4: $Plot\ [\lfloor x + 0.5 \rfloor, y]^T$

5: $y = y + 1$

May be accepted.

Thus, the roles of x and y are reversed by assigning an increment of 1 to y and an increment of $1/m$ to x.

Algorithm

Input: $x_0,\ y_0,\ x_1,\ y_1$

1: $m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$

2: $x = x_0$

3: **for** $(y = y_0 \text{ to } y_1)$ **do**

4: Plot $[\lfloor x + 0.5 \rfloor, y]^T$

5: $x = x + \frac{1}{m}$

6: **end for**

end

[The idea: 4 marks; lines from 2 thru 5: 8 marks]

Question 1 [2D Graphics]:

a) [10 marks] In Cohen-Sutherland Algorithm (listed in the formula sheet) for line clipping, if the clip polygon used is a triangle instead of a rectangle, determine the minimum length of each outcode (i.e., the number of binary digits). Determine all the outcodes in this case.

Answers to Question 1 a):

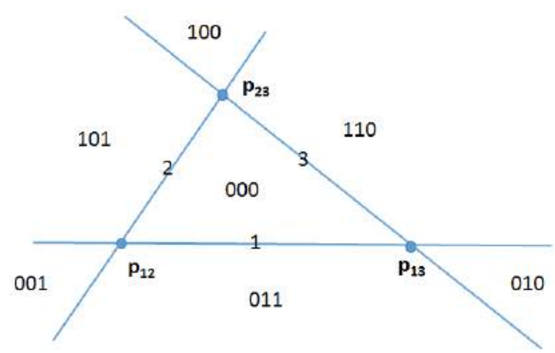
In the original algorithm, the clip polygon was a rectangle; thus, the minimal space required corresponds to the number of borders (i.e., 4 bits).

Notice that the number of regions = 4 (number of borders) * 2 (sides per border) + 1 (the clip rectangle) = 9 regions.

Applying the same idea to the triangle, the minimal space required corresponds to the number of borders (i.e., 3 bits). [3 marks]

The number of regions = 3 (number of borders) * 2 (sides per border) + 1 (the clip region) = 7 regions.

Consider the figure and the outcodes. The borders of the triangle are numbered. An outcode consists of 3 binary digits. The left binary digit corresponds to border 1. It is 1 in front of border 1 and 0 otherwise. The same goes for border 2 (the middle digit) and border 3 (the right digit). [1 mark each outcode = 7 marks]



b) [7 marks] Propose an algorithm to generate these outcodes.

Answers to Question 1 b):

1. Determine the linear equations for each of the sides.
2. Determine the vertices by intersecting lines; $\mathbf{p}_{12}$, $\mathbf{p}_{13}$, $\mathbf{p}_{23}$
3. Outcode = 000

4. Apply vertex $\mathbf{p}_{23}$ to the border line 1. For each region on the same side of border 1 as $\mathbf{p}_{23}$. Outcode OR 100
5. Apply vertex $\mathbf{p}_{13}$ to the border line 2. For each region on the same side of border 2 as $\mathbf{p}_{13}$. Outcode OR 010
6. Apply vertex $\mathbf{p}_{12}$ to the border line 3. For each region on the same side of border 3 as $\mathbf{p}_{12}$. Outcode OR 001

7. Return outcode

[1 mark each step = 7 marks]

c) [7 marks] Modify the Cohen-Sutherland Algorithm to work with clip triangle.

Answers to Question 1 c):

1. Determine outcode for each endpoint. [1 mark]
2. Dealing with the two outcodes of a border: [1 mark]
   a. Bitwise-OR the bits. If this results in 000, trivially accept. [1 mark]
   b. **Otherwise, if both outcodes are equal, trivially reject.** . [2 marks]
   c. Otherwise, segment the line. The outpoint is replaced by the intersection point. Go to Step 2. [1 mark]
3. If trivially accepted, draw the line. [1 mark]

Any other logical alternative can be considered

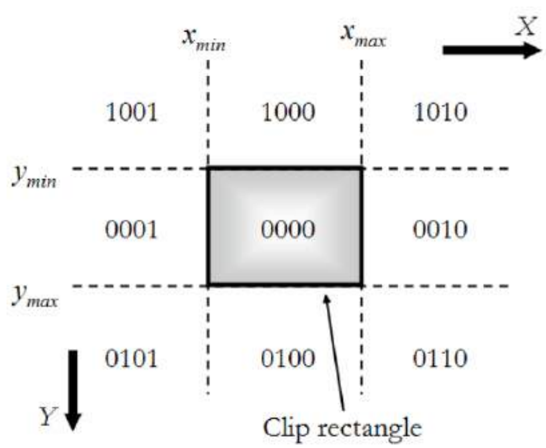
<u>Question 2:</u> (16 Marks)

Assuming a right-handed coordinate system with Cohen-Sutherland clipping algorithm, the following table may be used to assign the outcode values for the 9 regions where bit 3 represents the most-significant bit while bit 0 represents the least-significant bit.

Bit	Value	Meaning
3	$= 1$, if region is above the top edge; $= 0$, otherwise.	if $y > y_{max}$ if $y \le y_{max}$
2	$= 1$, if region is below the bottom edge; $= 0$, otherwise.	if $y < y_{min}$ if $y \ge y_{min}$
1	$= 1$, if region is right to the right edge; $= 0$, otherwise.	if $x > x_{max}$ if $x \le x_{max}$
0	$= 1$, if region is left to the left edge; $= 0$, otherwise.	if $x < x_{min}$ if $x \ge x_{min}$

If a left-handed coordinate system is used, modify the previous table so that each region keeps its outcode (e.g., the upper left region is assigned 1001, the lower right region is assigned 0110, etc.).

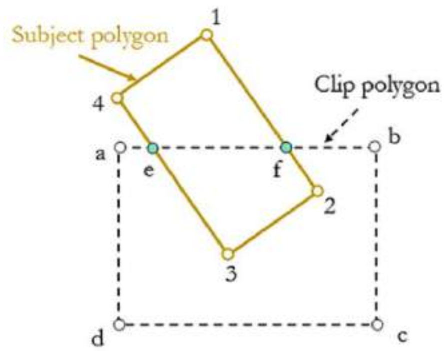
<u>Solution</u> [16 marks; 2 marks each entry]



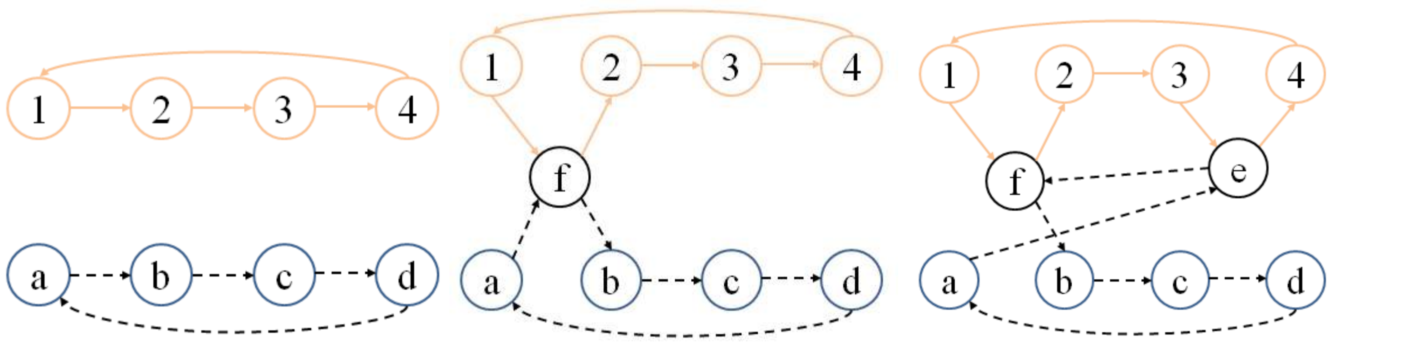
Bit	Value	Meaning
3	$= 1$, if region is above the top edge; $= 0$, otherwise.	if $y < y_{min}$ if $y \ge y_{min}$
2	$= 1$, if region is below the bottom edge; $= 0$, otherwise.	if $y > y_{max}$ if $y \le y_{max}$
1	$= 1$, if region is right to the right edge; $= 0$, otherwise.	if $x > x_{max}$ if $x \le x_{max}$
0	$= 1$, if region is left to the left edge; $= 0$, otherwise.	if $x < x_{min}$ if $x \ge x_{min}$

Question 1: (2D Graphics)

a) [8 marks] Consider the subject and clip polygons shown. Draw the steps that should be followed to populate lists of Weiler-Atherton clipping algorithm for both the subject and clip polygons.



Solutions:



[1+3+4 marks]

b) [10 marks] Given a 2D polygon specified by the vertices $[3, 3]^T$, $[6, 3]^T$, $[8, 2]^T$ and $[6, 6]^T$, test whether it is convex or concave.

Solutions:

Using cross product:

Consider $v_{i-1}= [3, 3]^T$, $v_i= [6, 3]^T$ and $v_{i+1}= [8, 2]^T$: [4 marks]

$$\mathbf{e_{i-1}} \times \mathbf{e_i} = (\mathbf{v_i} - \mathbf{v_{i-1}}) \times (\mathbf{v_{i+1}} - \mathbf{v_i}) = \begin{bmatrix} x_i - x_{i-1} \\ y_i - y_{i-1} \end{bmatrix} \times \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix}$$
$$= (x_i - x_{i-1}) * (y_{i+1} - y_i) - (y_i - y_{i-1}) * (x_{i+1} - x_i)$$
$$\Rightarrow (6 - 3) * (2 - 3) - (0 - 0) * (8 - 6) \Rightarrow -3$$

Consider $v_{i-1}= [6, 3]^T$, $v_i= [8, 2]^T$ and $v_{i+1}= [6, 6]^T$: [4 marks]

$$\mathbf{e_{i-1}} \times \mathbf{e_i} = (\mathbf{v_i} - \mathbf{v_{i-1}}) \times (\mathbf{v_{i+1}} - \mathbf{v_i}) = \begin{bmatrix} x_i - x_{i-1} \\ y_i - y_{i-1} \end{bmatrix} \times \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix}$$
$$= (x_i - x_{i-1}) * (y_{i+1} - y_i) - (y_i - y_{i-1}) * (x_{i+1} - x_i)$$
$$\Rightarrow (8 - 6) * (6 - 2) - (2 - 3) * (6 - 8) \Rightarrow +6$$

Different signs ➔ Concave [2 mark]

Using linear equations:

Consider the linear equation given by the line segment <$[3, 3]^T$, $[6,3]^T$>

[4 marks]

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$
$$y - 3 = 0$$
$$y = 3$$

Apply the other two points ($[6,6]^T$ and $[8,2]^T$) to the previous equation

6-3 ➔ 3 ➔ +ve [2 marks]

2-3 ➔ -1 ➔ -ve [2 marks]

Different signs ➔ Concave [2 marks]

<u>Question 2:</u> [2D Graphics]

[12 marks] The listed is the digital differential analyzer (DDA) algorithm used to draw lines when the absolute magnitude of slope (i.e., $|m|$) is less than or equal to 1.

Without swapping coordinates or reflecting about the line y=x, modify it to draw lines when $|m|>1$.

Algorithm

Input: x_0, y_0, x_1, y_1

1: $m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$

2: $y = y_0$

3: **for** $(x = x_0$ to $x_1)$ **do**

4: Plot $[x, \lfloor y + 0.5 \rfloor]^T$

5: $y = y + m$

6: **end for**

end

Answers to Question 2:

It can be written that

$$x_i = \frac{1}{m}(y_i - B)$$

Consequently, x_{i+1} is calculated as by incrementing the value of y by 1; so, we can write

$$x_{i+1} = \frac{1}{m}(y_{i+1} - B) = \frac{1}{m}(y_i + 1 - B) = \underbrace{\frac{1}{m}(y_i - B)}_{x_i} + \frac{1}{m} = x_i + \frac{1}{m}$$

Thus, the roles of x and y are reversed by assigning an increment of 1 to y and an increment of $1/m$ to x.

Algorithm

Input: x_0, y_0, x_1, y_1

1: $m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$

2: $x = x_0$

3: **for** $(y = y_0$ to $y_1)$ **do**

4: Plot $[\lfloor x + 0.5 \rfloor, y]^T$

5: $x = x + \frac{1}{m}$

6: **end for**

end

[The idea: 4 marks; lines from 2 thru 5: 8 marks]

Question 1 [2D Graphics]:

[5 marks] Given a 2D polygon specified by the vertices $[-3, 0]^T$, $[3, -1]^T$, $[1, 0]^T$ and $[4, 2]^T$, test whether it is convex or concave using linear equations.

Answers to Question 1:

Consider the linear equation given by the line segment <$[1, 0]^T$, $[4,2]^T$>

[2 marks]

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$y - 0 = \frac{2}{3}(x - 1)$$

$$3y - 2x + 2 = 0$$

Apply the other two points ($[3,-1]^T$ and $[-3,0]^T$) to the previous equation

3*-1 -2*3 + 2 ➔ -7 [1 mark]

3*0 -2*-3 + 2 ➔ 8 [1 mark]

Different signs ➔ Concave [1 mark]

Question 1 [2D Graphics]:

[20 marks] The Cohen-Sutherland Algorithm is used for line clipping if the clip polygon is a rectangle. If a circle is used instead, propose an algorithm to clip lines (i.e., to keep what is inside the circle and remove any lines or parts of lines outside the circle). You have to cover **all** possibilities that may occur.

Answers to Question 1:

1. Assume that the center point of the circle is $[x, y]^T$ and the radius is r. [1 mark]
2. Calculate the distances from each endpoints $[x_1, y_1]^T$ and $[x_2, y_2]^T$ to the center $[x, y]^T$. Let us call these d_1 and d_2 respectively. [1 mark]
3. There are 4 possibilities
   a. If $d_1 <= r$ and $d_2 <= r$, [1 mark]
      i. The whole line segment is included inside the circle and should be accepted
   b. If $d_1 > r$ and $d_2 <= r$ [1 mark]
      i. Split the line segment $<[x_1, y_1]^T[x_2, y_2]^T>$ into 2 segments $<[x_1, y_1]^T[x_3, y_3]^T$ and $<[x_3, y_3]^T[x_2, y_2]^T>$ where $[x_3, y_3]^T$ is the intersection point between and line and the circle. [1 mark]
      ii. The line segment $<[x_1, y_1]^T[x_3, y_3]^T>$ will be rejected [1 mark]
      iii. The line segment $<[x_3, y_3]^T[x_2, y_2]^T>$ will be accepted [1 mark]
   c. If $d_1 <= r$ and $d_2 > r$ [1 mark]
      i. Split the line segment $<[x_1, y_1]^T[x_2, y_2]^T>$ into 2 segments $<[x_1, y_1]^T[x_3, y_3]^T$ and $<[x_3, y_3]^T[x_2, y_2]^T>$ where $[x_3, y_3]^T$ is the intersection point between and line and the circle. [1 mark]
      ii. The line segment $<[x_3, y_3]^T[x_2, y_2]^T>$ will be rejected [1 mark]
      iii. The line segment $<[x_1, y_1]^T[x_3, y_3]^T>$ will be accepted [1 mark]
   d. If $d_1 > r$ and $d_2 > r$ ← both endpoints are outside the circle. [1 mark]
      i. Determine the perpendicular distance d_3 from the center $[x, y]^T$ to the line $<[x_1, y_1]^T[x_2, y_2]^T>$ [1 mark]
      ii. If d_3 is $>$ r then the line segment $<[x_1, y_1]^T[x_2, y_2]^T>$ is outside the circle and should be rejected [1 mark]
      iii. If d_3 is $<$ r then the line segment $<[x_1, y_1]^T[x_2, y_2]^T>$ intersects the circle in 2 points $[x_3, y_3]^T$ and $[x_4, y_4]^T$ [1 mark]
      iv. The line segment $<[x_3, y_3]^T[x_4, y_4]^T>$ will be rejected [1 mark]
      v. The line segment $<[x_1, y_1]^T[x_3, y_3]^T>$ will be accepted [1 mark]
      vi. The line segment $<[x_4, y_4]^T[x_1, y_1]^T>$ will be accepted [1 mark]
      vii. If d_3 is = r then the line touches the circle in 1 point which should be accepted [1 mark]

Note that the rejected segments are mentioned before the accepted ones to make sure that the intersection points are included with the accepted segments. [1 mark]

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Department of Digital Media Engineering and Technology
DMET 502 - Computer Graphics Quiz 1.1

DMET 502 – COMPUTER GRAPHICS

QUIZ 1

Quiz Duration: 15 minutes

Name	
ID	
Tutorial	

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DMET 502 - Computer Graphics Quiz 1.1

Q1)

Shown in Figure 1 is the effect of a transformation operation applied to the object on the left hand side to produce the transformed version on the right hand side such that vertices a, b, c, and d on the left are transformed to vertices a`, b`, c` and d` on the right. Construct one possible sequence of primitive transformation matrices in homogenous coordinates that can achieve such a transformation.

Construct a single composite transformation matrix that combines the sequence you found.

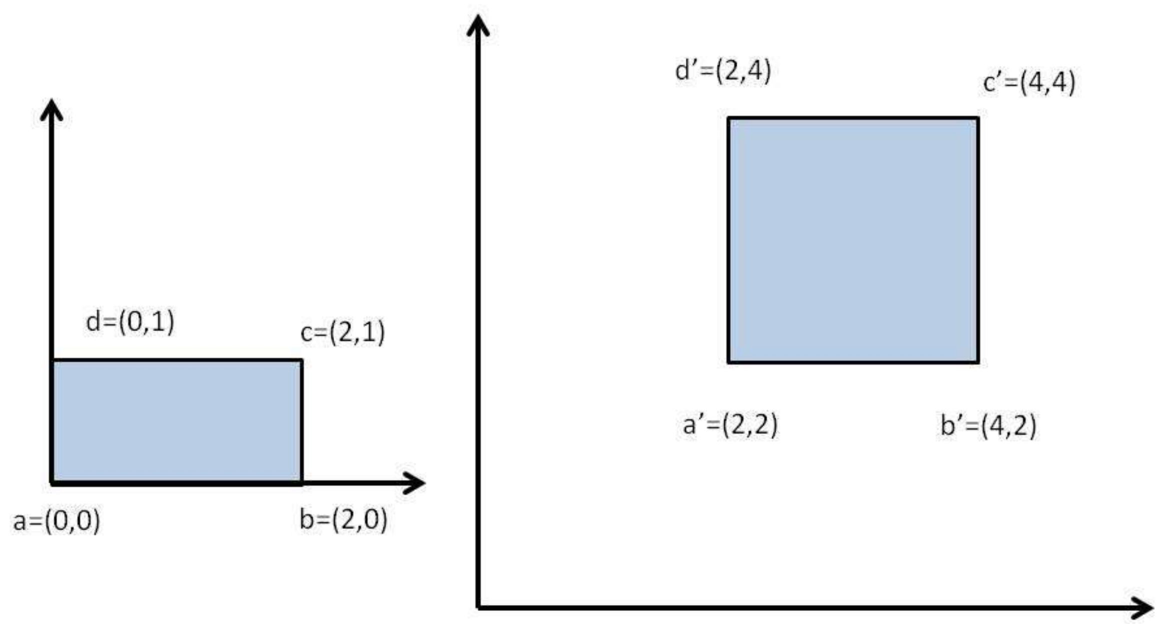

Figure 1: 2D Transformation

Sol1

M1 = Translate (-1,-0.5)

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

M2 = Scale (1, 2)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

M3 = Translate (3,3)

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix multiplication order M = M3*M2*M1

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol2

M1 = Scale (1, 2)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

M2 = Translate (2,2)

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$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix multiplication order M = M2*M1

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

DMET 502 – COMPUTER GRAPHICS

QUIZ 1

Quiz Duration: 15 minutes

Name	
ID	
Tutorial	

Q1)

Shown in Figure 1 is the effect of a transformation operation applied to the object on the left hand side to produce the transformed version on the right hand side such that vertices a, b, c, and d on the left are transformed to vertices a`, b`, c` and d` on the right. Construct one possible sequence of primitive transformation matrices in homogenous coordinates that can achieve such a transformation.

Construct a single composite transformation matrix that combines the sequence you found.

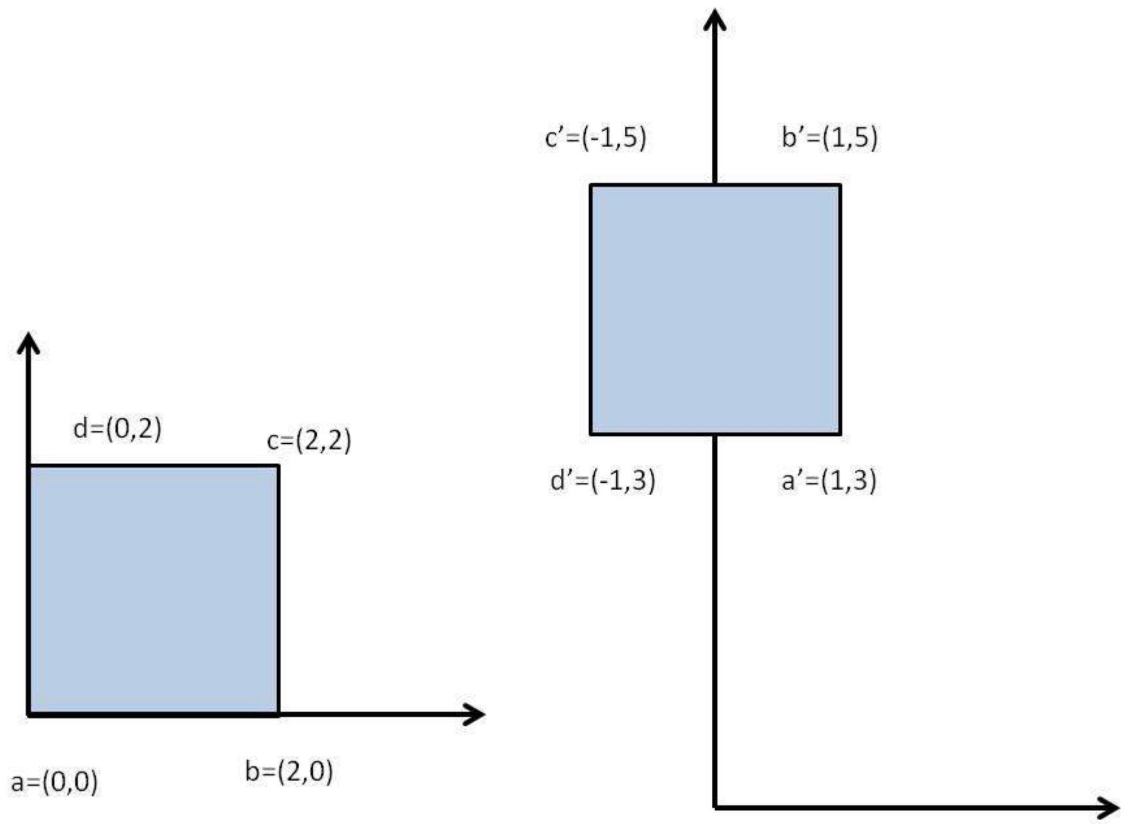

Figure 1: 2D Transformation

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DMET 502 - Computer Graphics Quiz 1.2

Sol1

M1 = Translate (-1,-1)

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

M2 = Rotation (+90)

$$\begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

M3 = Translate (0,4)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix multiplication order M = M3*M2*M1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol2

M1 = Rotation (+90)

$$\begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

M2 = Translate (1,3)

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix multiplication order M = M2*M1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

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DMET 502 - Computer Graphics Quiz 1.1

DMET 502 – Computer Graphics

Quiz 1

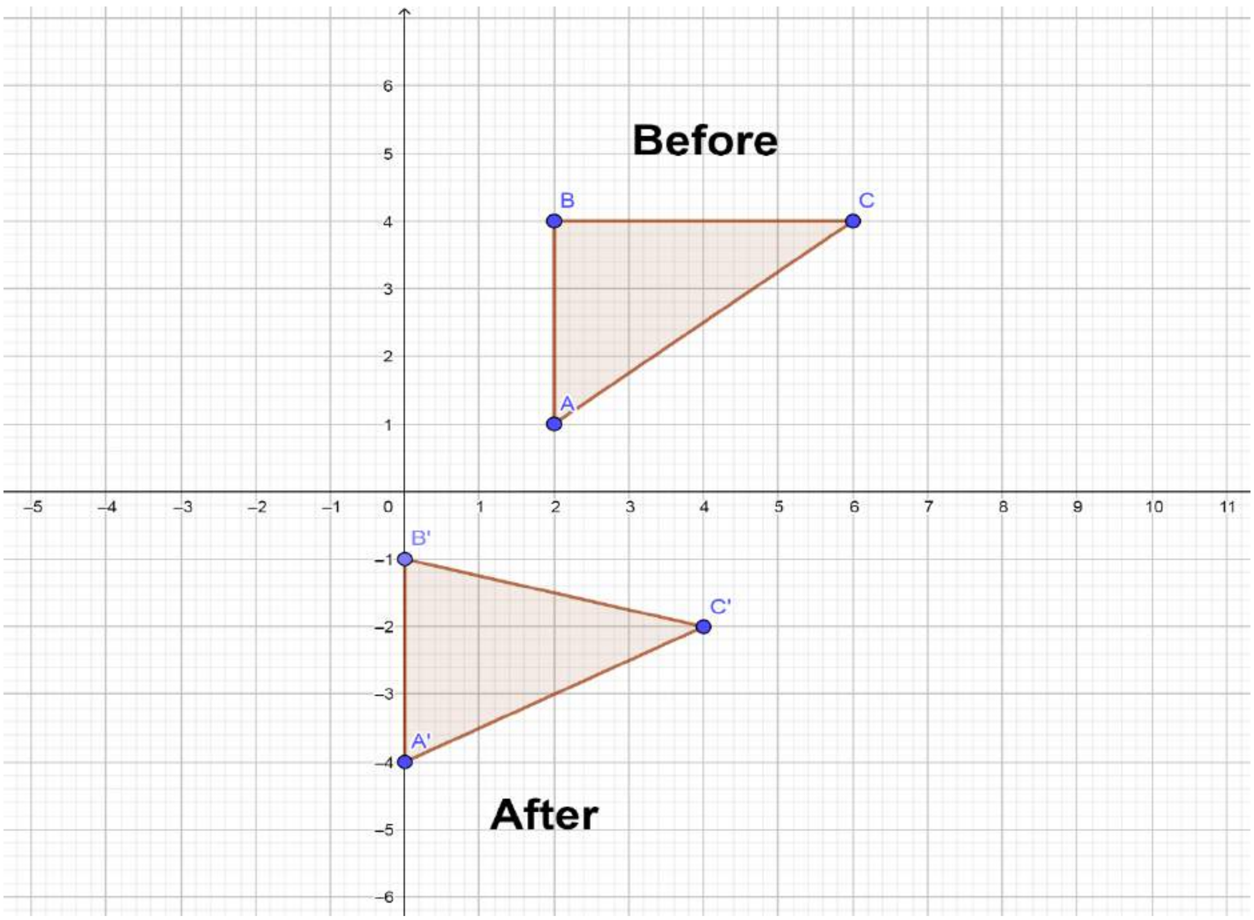
Quiz Duration: 20 minutes

Name:	
ID:	
Tutorial:	
TA Name:	

Question:

Shown in the following Figure the effect of a transformation operation applied to the object to produce the transformed version such that vertices A, B, and C are transformed to vertices A`, B` and C` respectively. Construct one possible sequence of primitive transformation matrices in homogenous coordinates that can achieve such a transformation.

Compute a single composite transformation matrix that combines the sequence you found.



Solution [1]

1. Translation by (-2,-5)

$$m1 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Shearing in Y by -0.25

$$m2 = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$$m = m2 * m1$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ -0.25 & 1 & -4.5 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution [2]

1. Translation by (-2,0)

$$m1 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Shearing in Y by -0.25

$$m2 = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Translation by (0,-5)

$$m3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$$m = m3 * m2 * m1$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ -0.25 & 1 & -4.5 \\ 0 & 0 & 1 \end{bmatrix}$$

Input: x_0, y_0, x_1, y_1

- steep = $|y_1 - y_0| > |x_1 - x_0|$
- if (steep = TRUE) then
  - swap (x_0,y_0)
  - swap (x_1,y_1)
- end if
- if ($x_0 > x_1$) then
  - swap (x_0,x_1)
  - swap (y_0,y_1)
- end if
- If ($y_0 > y_1$) then
  - $\delta y = -1$
- else
  - $\delta y = 1$
- end if
- $\Delta x = x_1 - x_0$
- $\Delta y = |y_1 - y_0|$
- $y = y_0$
- error = 0
- for ($x = x_0$ to x_1) do
  - if (steep = TRUE) then
    - Plot $[y,x]^T$
  - else
    - Plot $[x,y]^T$
  - end if
  - error = error + Δy
  - if (2 x error $\geq \Delta x$) then
    - $y = y + \delta y$
    - error = error – Δx
  - end if
- end for

end

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & SHx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ SHy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

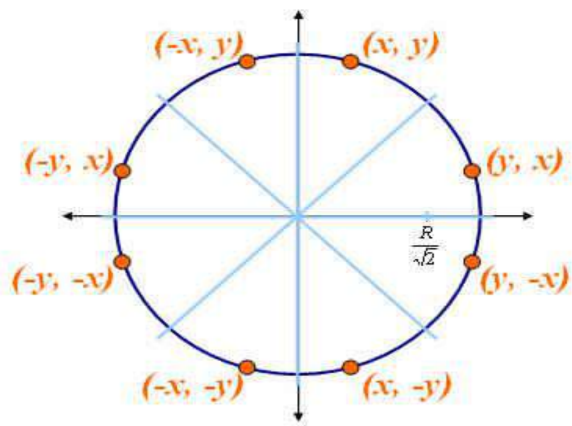
$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} Sx & 0 & 0 \\ 0 & \text{Sy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$



$$outcode = \begin{cases} outcode \text{ OR } 1000, & \text{if } y > y_{max}; \\ outcode \text{ OR } 0100, & \text{if } y < y_{min}, \end{cases}$$

then

$$outcode = \begin{cases} outcode \text{ OR } 0010, & \text{if } x > x_{max}; \\ outcode \text{ OR } 0001, & \text{if } x < x_{min}. \end{cases}$$

$$\mathbf{\sin(\propto + \theta) = \sin(\propto)\cos(\theta) + \cos(\propto)\sin(\theta)}$$

$$\mathbf{\cos(\propto + \theta) = \cos(\propto)\cos(\theta) - \sin(\propto)\sin(\theta)}$$

$$\mathbf{u} \times \mathbf{v} = [\mathbf{u}]_\times \mathbf{v} =$$

$$\begin{bmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

German University in Cairo

Department of Digital Media Engineering and Technology

DMET 502 - Computer Graphics Quiz 1.2

DMET 502 – Computer Graphics

Quiz 1

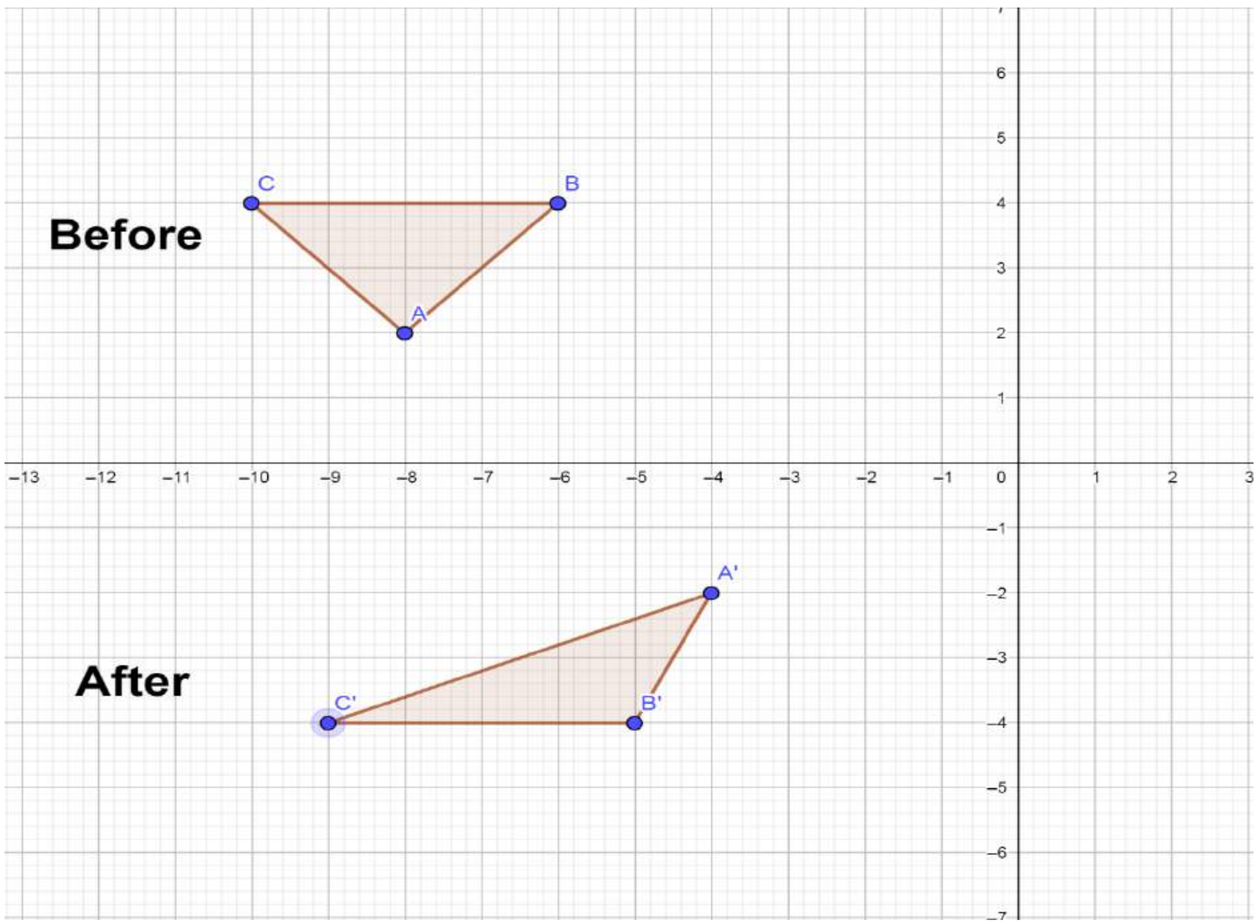
Quiz Duration: 20 minutes

Name:	
ID:	
Tutorial:	
TA Name:	

Question:

Shown in the following Figure the effect of a transformation operation applied to the object to produce the transformed version such that vertices A, B, and C are transformed to vertices A`, B` and C` respectively. Construct one possible sequence of primitive transformation matrices in homogenous coordinates that can achieve such a transformation.

Compute a single composite transformation matrix that combines the sequence you found.



Solution [1]

1. Reflection about x-axis
$$m1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Translation by (0,4)
$$m2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$
3. Shearing in X by factor 1.5
$$m3 = \begin{bmatrix} 1 & 1.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
4. Translation by (1,-4)
$$m4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$$m = m4 * m3 * m2 * m1$$

$$= \begin{bmatrix} 1 & -1.5 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution [2]

1. Translation by (0,-2)
$$m1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Reflection about x-axis
$$m2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
3. Translation by (1,2)
$$m3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Shearing in X by factor 1.5

$$m4 = \begin{bmatrix} 1 & 1.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Translation by (0,-4)

$$m5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$$m = m5 * m4 * m3 * m2 * m1$$

$$= \begin{bmatrix} 1 & -1.5 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Input: x_0, y_0, x_1, y_1

- steep = $|y_1 - y_0| > |x_1 - x_0|$
- if (steep = TRUE) then
  - swap (x_0,y_0)
  - swap (x_1,y_1)
- end if
-
- if ($x_0 > x_1$) then
  - swap (x_0,x_1)
  - swap (y_0,y_1)
- end if
-

- If ($y_0 > y_1$) then
  - $\delta y = -1$
- else
  - $\delta y = 1$
- end if
-
- $\Delta x = x_1 - x_0$
- $\Delta y = |y_1 - y_0|$
- $y = y_0$
- error = 0
-

- for ($x = x_0$ to x_1) do
  - if (steep = TRUE) then
    - Plot $[y,x]^T$
  - else
    - Plot $[x,y]^T$
  - end if
  - error = error + Δy
  - if (2 x error $\geq \Delta x$) then
    - $y = y + \delta y$
    - error = error – Δx
  - end if
- end for

end

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & SHx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ SHy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

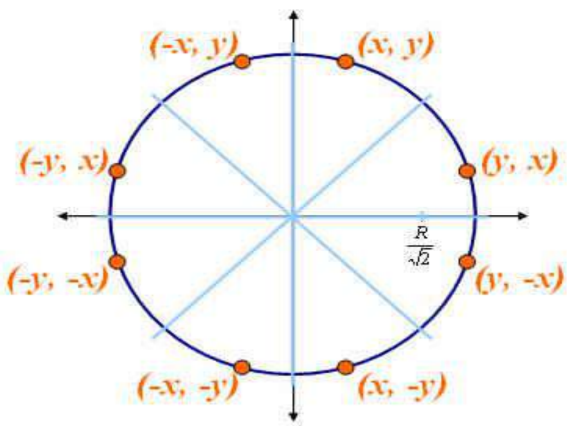
$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} Sx & 0 & 0 \\ 0 & \text{Sy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x2 \\ y2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ 1 \end{bmatrix}$$



$$outcode = \begin{cases} outcode \text{ OR } 1000, & \text{if } y > y_{max}; \\ outcode \text{ OR } 0100, & \text{if } y < y_{min}, \end{cases}$$

then

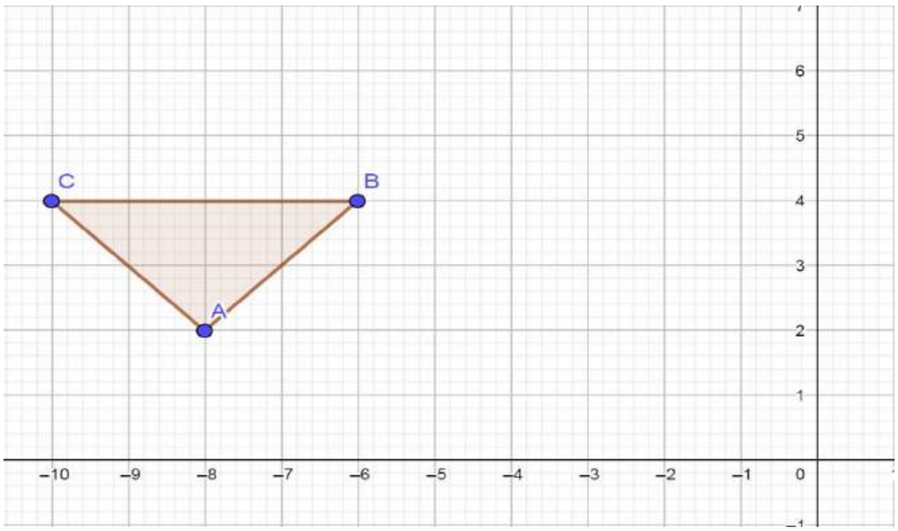
$$outcode = \begin{cases} outcode \text{ OR } 0010, & \text{if } x > x_{max}; \\ outcode \text{ OR } 0001, & \text{if } x < x_{min}. \end{cases}$$

$$\sin(\propto + \boldsymbol{\theta}) = \sin(\propto)\cos(\boldsymbol{\theta}) + \cos(\propto)\sin(\boldsymbol{\theta})$$

$$\cos(\propto + \boldsymbol{\theta}) = \cos(\propto)\cos(\boldsymbol{\theta}) - \sin(\propto)\sin(\boldsymbol{\theta})$$

$$\mathbf{u} \times \mathbf{v} = [\mathbf{u}]_\times \mathbf{v} = \begin{bmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

2. Reflect the following triangle about a line having an equation of y=x+4. Derive the transformation matrix required. (Total 6 marks)



Solution [1]

1. Translation by (0,-4) 0.5 marks

$$m1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Rotation by -45 0.5 for rotation 0.5 for angle

$$m2 = \begin{bmatrix} \text{Cos}(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Reflect about x 1 mark
4. Rotate back by 45 1 mark (0.5 rotation , 0.5 angle)
5. Translate back by 4 in y 0.5 marks

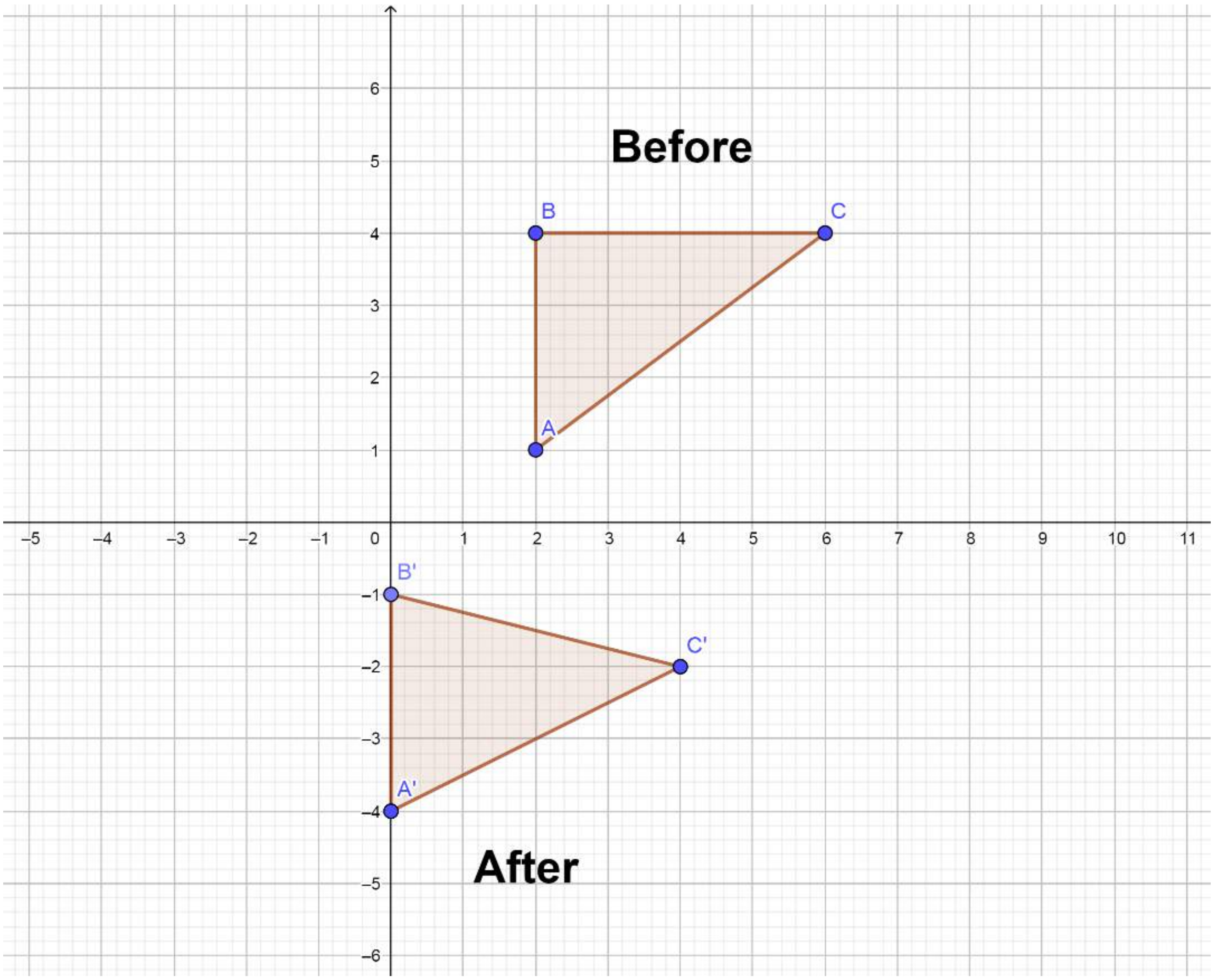
The final matrix is:

$m = m5 * \mathrm{m4} * \mathrm{m3} * \mathrm{m2} * \mathrm{m1}$ 1 mark

$$= \begin{bmatrix} 0 & 1 & -4 \\ 1 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$ 1 mark

please note that rotating the line then reflecting and rotating back does not give the same output

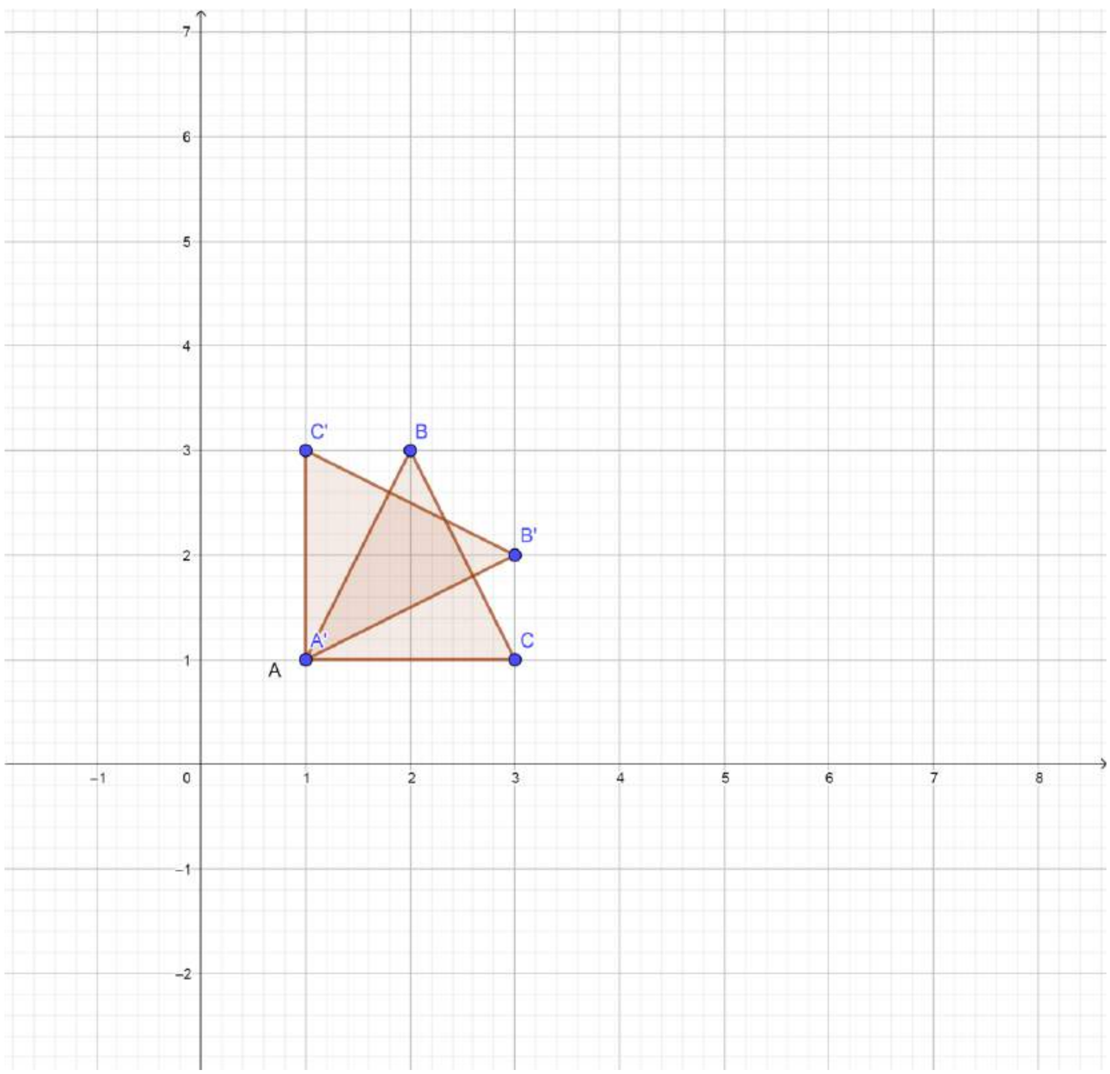
2. Shown in the following Figure the effect of a transformation operation applied to the object to produce the transformed version such that vertices A, B, and C are transformed to vertices A`, B` and C`. Construct one possible sequence of primitive transformation matrices in homogenous coordinates that can achieve such a transformation. Compute a single composite transformation matrix that combines the sequence you found. (Total 6 marks)



German University in Cairo
Faculty of Media Engineering and Technology
Winter 2020
Duration: 30 minutes

Name: ID: Tutorial:

Find the transformation matrix that maps the following shape ABC into the new shape A'B'C'.



Solution [1]

1. Rotation with angle (45)
$$m1 = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Reflection about the y-axis
$$m2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
3. Rotation with angle (-45)
$$m3 = \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$m = m3 * m2 * m1$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution [2]

1. Rotation with angle (45)
$$m1 = \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Reflection about the x-axis
$$m2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
3. Rotation with angle (-45)
$$m3 = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$m = m3 * m2 * m1$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution [3]

1. Translation by (-1, -1)
$$m1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Rotation with angle (45)
$$m2 = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
3. Reflection about the y-axis
$$m3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
4. Rotation with angle (-45)
$$m4 = \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
5. Translation by (1,1)
$$m5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:
$$m = m5 * m4 * m3 * m2 * m1$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution [4]

1. Translation by (-1, -1)
$$m1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Rotation with angle (-45)
$$m2 = \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Reflection about the x-axis

$$m3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Rotation with angle (45)

$$m4 = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Translation by (1,1)

$$m5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

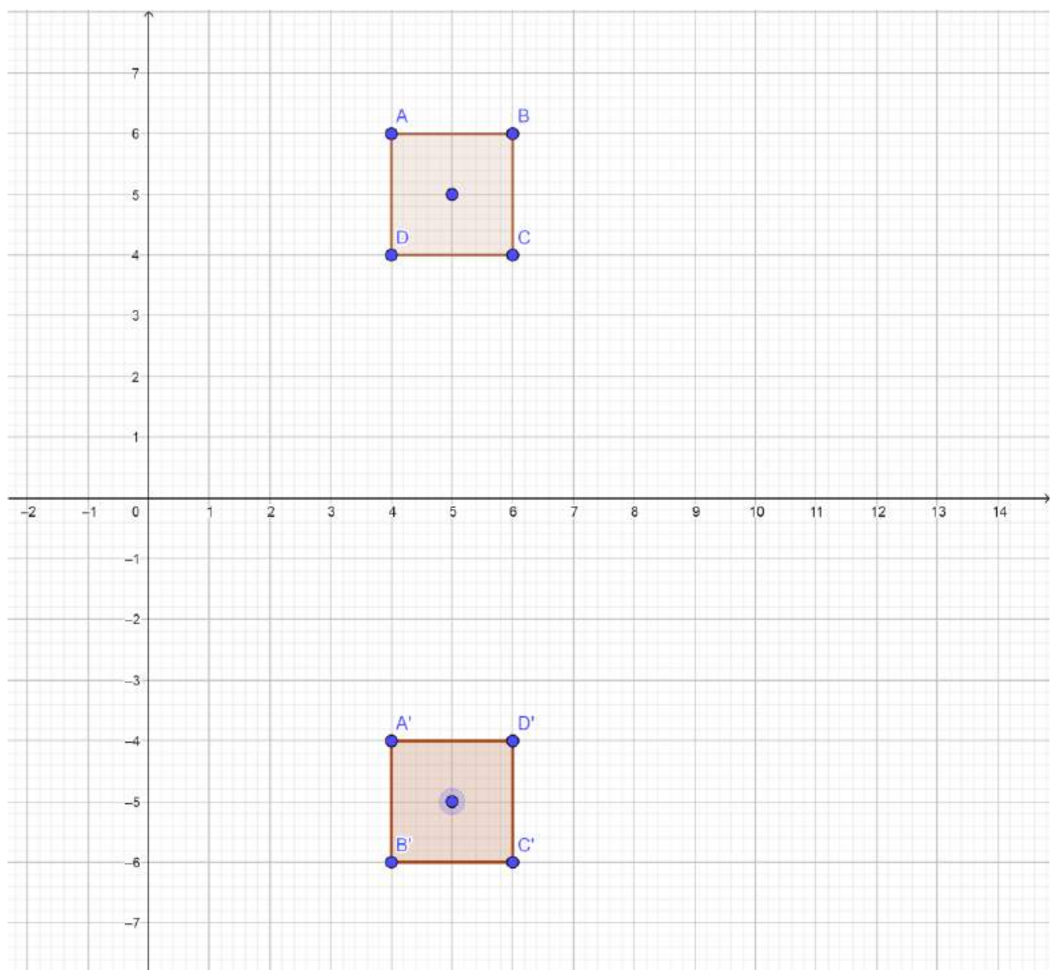
$$m = m5 * m4 * m3 * m2 * m1$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

German University in Cairo **Faculty of Media Engineering and Technology** **Winter 2020** **Duration: 30 minutes**	

Name: ID: Tutorial:

Find the transformation matrix that maps the following shape ABCD into the new shape A’B’C’D’.



Solution [1]

1. Translation by (-5,-5)
   $$m1 = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Rotation with angle (90)
   $$m2 = \begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
3. Translation back by (5,5)
   $$m4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$
4. Reflection about the x-axis
   $$m3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$$m = m4 * m3 * m2 * m1$$
$$= \begin{bmatrix} 0 & -1 & 10 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution [2]

1. Reflection about the x-axis
   $$m1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Translation by (-5,5)
   $$m2 = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$
3. Rotation with angle (-90)
   $$m3 = \begin{bmatrix} \cos(-90) & -\sin(-90) & 0 \\ \sin(-90) & \cos(-90) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
4. Translation back by (5,-5)

$$m4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$m = m4 * m3 * m2 * m1$

$$= \begin{bmatrix} 0 & -1 & 10 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution [3]

1. Translation by (-5,-5)

$$m1 = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Rotation with angle (90)

$$m2 = \begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Reflection about the x-axis

$$m3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Translate by (5,-5)

$$m4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$m = m4 * m3 * m2 * m1$

$$= \begin{bmatrix} 0 & -1 & 10 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution [4]

1. Translation by (-5,-5)

   $$m1 = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Rotation with angle (45)

   $$m2 = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Reflection about the x-axis

   $$m3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Rotation back with angle (-45)

   $$m4 = \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Translation back by (5,-5)

   $$m5 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is:

$$m = m5 * m4 * m3 * m2 * m1$$

$$= \begin{bmatrix} 0 & -1 & 10 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Question 2:</u> (13 Marks)

The equation of a 2D line segment is expressed in $x'y'$-coordinates as

$$y' = -3x' + 11\sqrt{2}.$$

If the x'- and y'-axes are obtained as the x- and y-axes are rotated through an angle of 45° about the origin, express the same linear equation in the old xy-coordinate system.

<u>Solution</u>

$$\begin{aligned} x' &= x\cos(\theta) + y\sin(\theta) \\ &= x\cos(45) + y\sin(45) = \tfrac{x+y}{\sqrt{2}}. \end{aligned}$$

$$\begin{aligned} y' &= y\cos(\theta) - x\sin(\theta) \\ &= y\cos(45) - x\sin(45) = \tfrac{y-x}{\sqrt{2}}. \end{aligned}$$

[5+5 marks]

Substituting the values of x' and y' in the given linear equation, we get [3 marks]

$$\begin{aligned} y' &= -3x' + 11\sqrt{2} \\ \tfrac{y-x}{\sqrt{2}} &= \tfrac{-3(x+y)}{\sqrt{2}} + 11\sqrt{2} \end{aligned}$$

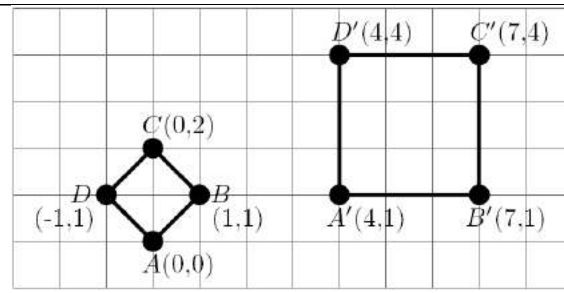
$$y = 5.5 - 0.5x$$

Question 2: [2D Transformations]

[20 points] It is required that you transform the square **abcd** to have the coordinates **a′b′c′d′**. The coordinates of the vertices are **a** = $[0,0]^T$, **b** = $[1,1]^T$, **c** = $[0,2]^T$, **d** = $[-1,1]^T$, **a′** = $[4,1]^T$, **b′** = $[7,1]^T$, **c′** = $[7,4]^T$ and **d′** = $[4,4]^T$. Write down the inhomogeneous equation for that transformation in terms of $[x, y]^T$ and $[x', y']^T$.

Repeat the solution if the square **a′b′c′d′** is transformed to **abcd**.

Solution



Steps: **[6 marks]**

1. Rotate -45
2. Scale using a factor of 3/sqrt(2)
3. Translate $[4,1]^T$

Steps 1 and 2 can be swapped

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(-45) & -\sin(-45) \\ \sin(-45) & \cos(-45) \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{2}} & 0 \\ 0 & \frac{3}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{2}} & 0 \\ 0 & \frac{3}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 & 1.5 \\ -1.5 & 1.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Correct **inhomogeneous** matrices/vector **[3 marks]**
Final matrix **[1 mark]**

Steps: **[6 marks]**

1. Translate $[-4,-1]^T$
2. Scale using a factor of sqrt(2)/3
3. Rotate 45

Steps 2 and 3 can be swapped

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{3} & 0 \\ 0 & \frac{\sqrt{2}}{3} \end{bmatrix} \begin{bmatrix} x'-4 \\ y'-1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{3} & 0 \\ 0 & \frac{\sqrt{2}}{3} \end{bmatrix} \begin{bmatrix} x'-4 \\ y'-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x'-4 \\ y'-1 \end{bmatrix}$$

Correct **inhomogeneous** matrices/vector **[3 marks]**
Final matrix **[1 mark]**

Question 2: [2D Transformations]

[20 marks] The equation of a 2D line segment is expressed in *xy*-coordinates as

$$y = 5.5 - 0.5x.$$

If the *x*- and *y*-axes are rotated through an angle of 45° about the origin to get the *x*′- and *y*′-axes, express the same linear equation in the *x*′ *y*′-coordinate system.

Answers to Question 2:

Problem 3.20: Get two points on the line [2 marks + 2 marks]

When y=0 ➔ 0.5x = 5.5 ➔ x =11 ➔ Point = $[11,0]^T$

When x=0 ➔ y = 5.5 ➔ Point = $[0,5.5]^T$

Apply the rotation

$x_1' = x_1 \cos(\theta) + y_1 \sin(\theta) = \frac{11}{\sqrt{2}} + \frac{0}{\sqrt{2}} = \frac{11}{\sqrt{2}}$ [3 marks]

$y_1' = y_1 \cos(\theta) - x_1 \sin(\theta) = \frac{0}{\sqrt{2}} - \frac{11}{\sqrt{2}} = -\frac{11}{\sqrt{2}}$ [3 marks]

$x_2' = x_2 \cos(\theta) + y_2 \sin(\theta) = \frac{0}{\sqrt{2}} + \frac{5.5}{\sqrt{2}} = \frac{11}{2\sqrt{2}}$ [3 marks]

$y_2' = y_2 \cos(\theta) - x_2 \sin(\theta) = \frac{5.5}{\sqrt{2}} - \frac{0}{\sqrt{2}} = \frac{11}{2\sqrt{2}}$ [3 marks]

The line equation is expressed in *x*′ *y*′-coordinates as

$y' = \frac{y_2'-y_1'}{x_2'-x_1'}(x' - x_1') + y_1'$

$y' = \frac{\frac{11}{2\sqrt{2}}+\frac{11}{\sqrt{2}}}{\frac{11}{2\sqrt{2}}-\frac{11}{\sqrt{2}}}\left(x' - \frac{11}{\sqrt{2}}\right) - \frac{11}{\sqrt{2}}$ [2 marks]

$y' = -3x' + 11\sqrt{2}$ [2 marks]

Question 2 [2D Transformations]:

a) [18 marks] A triangle is to be reflected about an axis inclined at an angle θ with respect to the x-axis and intersecting the y-axis at $[0,y_0]^T$. Prove that the homogeneous transformation matrix is:

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) & -y_0\sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) & y_0(\cos(2\theta)+1) \\ 0 & 0 & 1 \end{bmatrix}$$

Answers to Question 2 a):

Steps: [10 marks]

1. Translate the triangle such that the point of intersection between the axis of reflection and the y-axis is moved to the origin
2. Rotate the triangle through an angle $-\theta$ such that the axis of reflection coincides with the x-axis.
3. Reflect the triangle about the x-axis.
4. Rotate back through an angle θ.
5. Translate back using the same translation vector of Step 1 but along the opposite

$$\mathtt{M}_1 = \mathtt{T}([0, -y_0]^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathtt{M}_2 = \mathtt{R}(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathtt{M}_4 = \mathtt{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{M}_3 = \mathtt{Ref}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathtt{M}_5 = \mathtt{T}([0, y_0]^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrices [5 marks]

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$
Refer to Slide 8 (2D Transformations)

$$\mathtt{M} = \mathtt{M}_5\mathtt{M}_4\mathtt{M}_3\mathtt{M}_2\mathtt{M}_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) & \sin(2\theta) & -y_0\sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) & y_0(\cos(2\theta)+1) \\ 0 & 0 & 1 \end{bmatrix}.$$

Order of multiplication [1 mark]
Final matrix [2 marks]

b) [14 marks] In the previous question, if the same axis is expressed using its slope *m* rather than its inclination angle θ, re-express the transformation matrix above in terms of *m* instead of θ.

Answers to Question 2 b):

We know that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$ Refer to Slide 8 (2D Transformations)

Hence, the transformation matrix can be expressed as [4 marks]

$$\mathtt{M} = \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\sin(\theta)\cos(\theta) & -2y_0\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & -\cos^2(\theta) + \sin^2(\theta) & y_0(\cos^2(\theta) - \sin^2(\theta) + 1) \\ 0 & 0 & 1 \end{bmatrix}$$

Since the slope $m = \tan(\theta)$; [2 marks]

hence, $\sin(\theta) = m/\text{sqrt}(m^2+1)$ [3 marks]

and $\cos(\theta) = 1/\text{sqrt}(m^2+1)$ [3 marks]

Substituting the values of $\sin(\theta)$ and $\cos(\theta)$ in the previous matrix, we get

$$\mathtt{M} = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} & \frac{-2y_0m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} & \frac{2y_0}{m^2+1} \\ 0 & 0 & 1 \end{bmatrix}$$ [2 marks]

Question 3 [2D Transformations]:

a) [10 marks] Consider a 2D line segment whose endpoints are $[x_1, y_1]^T$ and $[x_2, y_2]^T$. If the x- and y-axes are rotated through an angle of 45° about the origin to get the x'- and y'-axes, express the same line in the $x'y'$-coordinate system in terms of x_1, y_1, x_2 and y_2.

If x_1, y_1, x_2 and y_2 are 3, 4, 9 and 1 respectively, write the line equation in the $x'y'$-coordinate system.

Answers to Question 3 a):

Problem 3.21:

$$x_1' = x_1\cos(45) + y_1\sin(45) = \frac{x_1+y_1}{\sqrt{2}} \text{ [2 marks]}$$
$$y_1' = y_1\cos(45) - x_1\sin(45) = \frac{y_1-x_1}{\sqrt{2}} \text{ [2 marks]}$$

$$x_2' = x_2\cos(45) + y_2\sin(45) = \frac{x_2+y_2}{\sqrt{2}} \text{ [2 marks]}$$
$$y_2' = y_2\cos(45) - x_2\sin(45) = \frac{y_2-x_2}{\sqrt{2}} \text{ [2 marks]}$$

The equation of a line

$$y' = \frac{y'_2 - y'_1}{x'_2 - x'_1}(x' - x'_1) + y'_1$$

$$y' = \frac{\frac{y_2-x_2}{\sqrt{2}} - \frac{y_1-x_1}{\sqrt{2}}}{\frac{x_2+y_2}{\sqrt{2}} - \frac{x_1+y_1}{\sqrt{2}}}\left(x' - \frac{x_1+y_1}{\sqrt{2}}\right) + \frac{y_1-x_1}{\sqrt{2}} \text{ [1 mark]}$$

$$y' = \frac{y_2 - x_2 - y_1 + x_1}{x_2 + y_2 - x_1 - y_1}\left(x' - \frac{x_1+y_1}{\sqrt{2}}\right) + \frac{y_1-x_1}{\sqrt{2}}$$

$$y' = \frac{1-9-4+3}{9+1-3-4}\left(x' - \frac{3+4}{\sqrt{2}}\right) + \frac{4-3}{\sqrt{2}}$$

$$y' = \frac{-9}{3}\left(x' - \frac{7}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}$$

$$y' = -3x' + 11\sqrt{2} \text{ [1 mark]}$$

b) [13 points] The equation of a circle having a radius of r and centered at the origin is expressed in xy-coordinates as

$$x^2 + y^2 = r^2$$

If the x- and y-axes are scaled with respect to the origin to get the x'- and y'-axes such that every r units along the x-direction become 1 unit and every r units along the y-direction become 2 units in the new coordinate system, express the same equation in the new $x'y'$-coordinate system. Will it remain a circle? Justify your answer.

Answers to Question 3 b):

Scaling factors are 1/r and 2/r
$x' = (1/r)\, x$ [2 marks]
$y' = (2/r)\, y$ [2 marks]

Consequently,

$x = rx'$ [2 marks]
$y = r\,y'/2$ [2 marks]

Substituting in the equation [3 marks]

$$x^2 + y^2 = r^2$$
$$[rx']^2 + [r\,y'/2]^2 = r^2$$
$$x'^2 + y'^2/4 = 1$$

No, it will not remain a circle as the scaling factors along both directions are different. [2 marks]

Question 2 [2D Transformations]:

a) [4 marks] Assume that the *x*-axis is sheared to the right using a factor δ. Write down the appropriate axes shearing matrices for inhomogeneous and homogeneous coordinates.

Answers to Question 2 a):

It is the same shearing matrix along the x-axis with a factor of –δ. This is as done with all axes transformation opreations.

$$\begin{bmatrix} 1 & -\delta \\ 0 & 1 \end{bmatrix}\text{[3 marks]}$$

$$\begin{bmatrix} 1 & -\delta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\text{[1 mark]}$$

b) [4 marks] Consider a square with vertices $[1, 1]^T$, $[1, -1]^T$, $[-1, -1]^T$ and $[-1, 1]^T$. If the *x*-axis is sheared to the right using a factor 1, determine the new coordinates of the square.

Answers to Question 2 b):

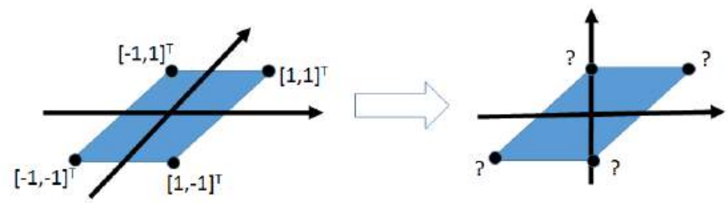
$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\text{[1 mark]}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\text{[1 mark]}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}\text{[1 mark]}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\text{[1 mark]}$$

c) [8 marks] Consider the coordinate system shown on the left where the angle between the axes is not 90° assuming that the horizontal axis is sheared to the right using a factor 1. (along x-axis which affects the vertical axis)



If the axes are sheared again to get back to the Cartesian coordinate system, estimate an inhomogeneous matrix (and a homogeneous matrix) that can be used to obtain the new coordinates of the vertices. Determine these new coordinates.

Answers to Question 2 c):

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{[3 marks]}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{[1 mark]}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{[1 mark]}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{[1 mark]}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \text{[1 mark]}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{[1 mark]}$$

d) [17 marks] Consider a square with vertices $[1, 1]^T$, $[1, -1]^T$, $[-1, -1]^T$ and $[-1, 1]^T$. If this square is sheared along the *y*-axis using a factor 1, determine the new coordinates of the vertices.

If this is followed by *y*-axis shearing using a factor 1, determine the new coordinates in this case.

Determine a single matrix defining these two consecutive operations (i.e., object shearing followed by axes shearing).

Answers to Question 2 d):

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ [3 +1 marks]

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ [1 mark]

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$ [1 mark]

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$ [1 mark]

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ [3 +1 marks]

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ [1 mark]

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$ [1 mark]

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ [1 mark]

We get the identity matrix

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ [3 marks]

OR, in general

$$\begin{bmatrix} 1 & 0 \\ -\beta & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha - \beta & 1 \end{bmatrix}$$

Question 2 [2D Transformations]:

[18 points] The point $[5,5]^T$ is reflected about a line passing through the point $[2,0]^T$ with an inclination angle of 35º. Determine the new location of the point. You **must** use **inhomogeneous** coordinates in all your calculations.

Answers to Question 2:

Steps:

1. Translate $[-2,0]^T$
2. Rotate by -35º.
3. Reflect about the x-axis
4. Rotate by 35º.
5. Translate $[2,0]^T$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(35) & -\sin(35) \\ \sin(35) & \cos(35) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-35) & -\sin(-35) \\ \sin(-35) & \cos(-35) \end{bmatrix} \begin{bmatrix} 5-2 \\ 5-0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7.725 \\ 1.109 \end{bmatrix}$$

Steps with right parameters ➔ 5 points
Right **inhomogeneous** matrices, vectors ➔ 5 points
Right order ➔ 4 points
Final result ➔ 4 points

Question 2 (2D Transformations):

[10 points] Consider the triangle abc where a=$[0,0]^T$, b=$[1,1]^T$, c=$[5,2]^T$. If this triangle is magnified in both directions to twice its size while keeping the point c fixed, estimate the new positions of a and b. You must use inhomogeneous coordinates in all your calculations.

Answers to Question 2:

Steps:

1. Translate $[-5,-2]^T$
2. Scale using a factor of 2
3. Translate $[5,2]^T$

$$\begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0-5 \\ 0-2 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_b \\ y_b \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1-5 \\ 1-2 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Steps with right parameters ➔ 3 points (granted even if omitted if calculations are correct)
Right inhomogeneous matrices, vectors ➔ 3 points
Final result ➔ 2 points for each (=4 points)

Question 2: (2D Transformations)

a) [13 points] The equation of a circle having a radius of *r* and centered at the origin is expressed in *xy*-coordinates as

$$x^2 + y^2 = r^2$$

If the *x*- and *y*-axes are scaled with respect to the origin to get the *x*'- and *y'*-axes such that every *r* units along the *x*-direction become 1 unit and every *r* units along the *y*-direction become 2 units in the new coordinate system, express the same equation in the new *x*'*y*'-coordinate system. Will it remain a circle?

Solutions:

Scaling factors are 1/r and 2/r

$x' = (1/r)\, x$ [2 marks]
$y' = (2/r)\, y$ [2 marks]

Consequently,

$x = rx'$ [2 marks]
$y = r\, y'/2$ [2 marks]

Substituting in the equation [4 marks]

$$x^2 + y^2 = r^2$$
$$[rx']^2 + [r\, y'/2]^2 = r^2$$
$$x'^2 + y'^2/4 = 1$$

No, it will not remain a circle. [1 mark]

b) [20 marks] Derive the reflection matrix about a line having a slope of 0.5 and *y*-intercept of 3.

Solutions:

Either a homogeneous matrix or an inhomogeneous equation is acceptable.

Angle of inclination = $\tan^{-1}(0.5) = 26.565$ [2 marks]

Steps:

1. Translate using $[0,-3]^T$ [2 marks]
2. Rotate through -26.565 ➔ R(-26.565) [2 marks]
3. Reflect about the x-axis [2 marks]
4. Rotate back R(26.565) [1 mark]
5. Translate back $[0,3]^T$ [1 mark]

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$ [1 mark]

$$R_2 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta)1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(-26.565) & -\sin(-26.565) & 0 \\ \sin(-26.565) & \cos(-26.565) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ [2 marks]

$$Ref_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ [1 mark]

$$R_4 = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta)1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(26.565) & -\sin(26.565) & 0 \\ \sin(26.565) & \cos(26.565) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ [1 mark]

$$T_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$ [1 mark]

$$M = T_5 R_4 Ref_x R_2 T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$ [2 marks]

$$M = \begin{bmatrix} 0.6 & 0.8 & -2.4 \\ 0.8 & -0.6 & 4.8 \\ 0 & 0 & 1 \end{bmatrix}$$

[2 marks]

Question 3: [2D Transformations]

[10 marks] Consider a 2D line segment whose endpoints are $[x_1, y_1]^T$ and $[x_2, y_2]^T$. If the x- and y-axes are rotated through an angle of 45° about the origin to get the x'- and y'-axes, express the same line in the x' y'-coordinate system in terms of x_1, y_1, x_2 and y_2.

If x_1, y_1, x_2 and y_2 are 3, 4, 9 and 1 respectively, write the line equation in the x' y'-coordinate system.

Answers to Question 3:

Problem 3.21:

$$x_1' = x_1 \cos(45) + y_1 \sin(45) = \frac{x_1+y_1}{\sqrt{2}} \text{ [2 marks]}$$
$$y_1' = y_1 \cos(45) - x_1 \sin(45) = \frac{y_1 - x_1}{\sqrt{2}} \text{ [2 marks]}$$

$$x_2' = x_2 \cos(45) + y_2 \sin(45) = \frac{x_2 + y_2}{\sqrt{2}} \text{ [2 marks]}$$
$$y_2' = y_2 \cos(45) - x_2 \sin(45) = \frac{y_2 - x_2}{\sqrt{2}} \text{ [2 marks]}$$

The equation of a line

$$y' = \frac{y'_2 - y'_1}{x'_2 - x'_1}(x' - x'_1) + y'_1$$

$$y' = \frac{\frac{y_2 - x_2}{\sqrt{2}} - \frac{y_1 - x_1}{\sqrt{2}}}{\frac{x_2 + y_2}{\sqrt{2}} - \frac{x_1 + y_1}{\sqrt{2}}}\left(x' - \frac{x_1 + y_1}{\sqrt{2}}\right) + \frac{y_1 - x_1}{\sqrt{2}} \text{ [1 mark]}$$

$$y' = \frac{y_2 - x_2 - y_1 + x_1}{x_2 + y_2 - x_1 - y_1}\left(x' - \frac{x_1 + y_1}{\sqrt{2}}\right) + \frac{y_1 - x_1}{\sqrt{2}}$$

$$y' = \frac{1 - 9 - 4 + 3}{9 + 1 - 3 - 4}\left(x' - \frac{3 + 4}{\sqrt{2}}\right) + \frac{4 - 3}{\sqrt{2}}$$

$$y' = \frac{-9}{3}\left(x' - \frac{7}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}$$

$$y' = -3x' + 11\sqrt{2} \text{ [1 mark]}$$

Question 1 [2D Transformations]:

[20 marks] Consider a square whose side length is 2 units. Assume that this square is centered at the origin, where its sides are parallel to the axes of the 2D Cartesian coordinates. Consider that "axes shearing**"** along the *x*-axis is performed so that the angle between the *x*- and *y*-axes becomes 45º instead of 90º.

a) [6 marks] Write down a single shearing matrix that performs axes shearing along the *x*-axis.

Answers to Question 1 a):

As all axes transformation matrices, the shearing matrix along the *x*-axis is used with negative shearing factor *shx*. Since the angle θ between the axes is 45º after shearing, *shx* = 1 (where tan (θ)= 1) [4 marks]

$$M = \begin{bmatrix} 1 & -shx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ [2 marks]

b) [4 marks] Using this matrix, determine the coordinates of the vertices of the square.

Answers to Question 1 b):

The corners of the square before shearing are $[1,1]^T$, $[-1,1]^T$, $[1,-1]^T$ and $[-1,-1]^T$.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$ [1 mark]

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$ [1 mark]

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$ [1 mark]

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$ [1 mark]

c) [4 marks] Write down a single shearing matrix that restores the 2D Cartesian coordinates.

Answers to Question 1 c):

Shearing using a factor +1 instead of -1 is used to restore the coordinate system

$$M = \begin{bmatrix} 1 & shx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ [value of shx 2 marks] [matrix 2 marks]

This is also the inverse of the initial shearing matrix.

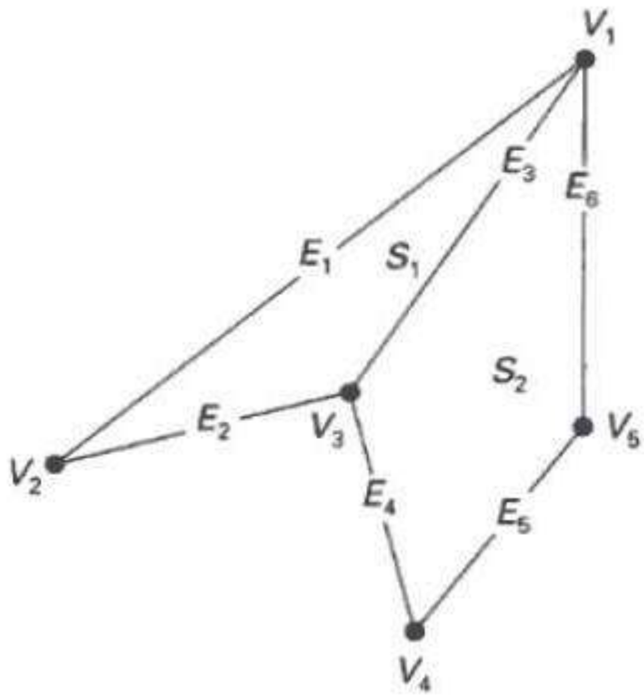
d) [6 marks] If the angle between the *x*- and *y*-axes was 60º instead of 45º, write down a single shearing matrix that performs axes shearing along the *x*-axis.

Answers to Question 1 d):

Since the angle θ between the axes is 60º after shearing, *shx* = 1/sqtr(3) (where tan (90-θ)= 1/sqtr(3)) [4 marks]

$$M = \begin{bmatrix} 1 & -shx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ [2 marks]

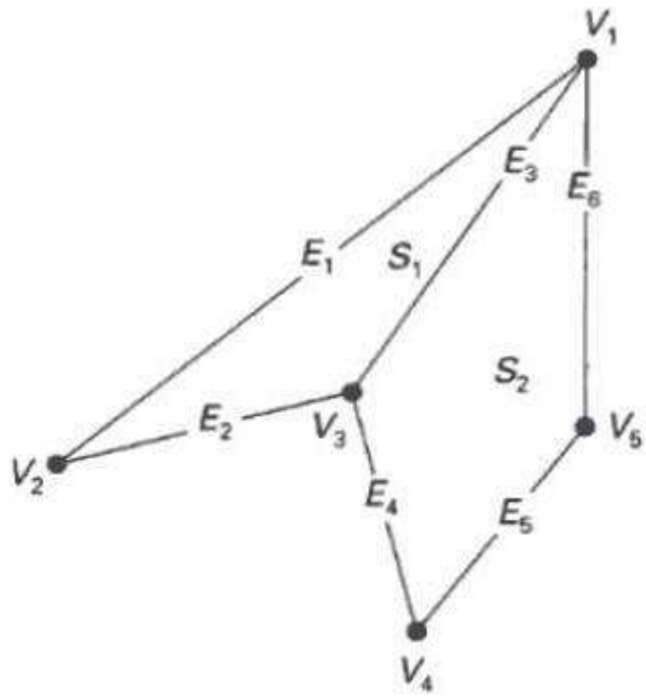
3. Write down the entries of the edge table for the figure below if it is to be represented as a wireframe. (mention start vertex and end vertex of each edge only) (2 marks)



1 mark for 6 edges

1 mark for correct start and end vertices

3. Write down the entries of the edge table for the figure below if it is to be represented as a wireframe. (mention start vertex and end vertex of each edge only)

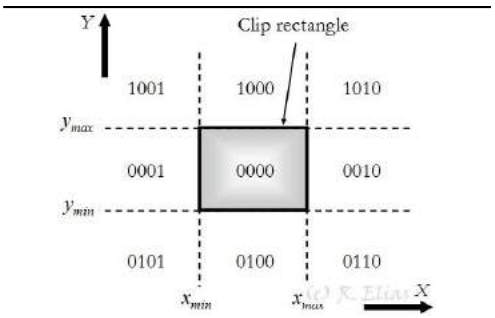


Formula Sheet

Input: x_0, y_0, x_1, y_1

```
steep = |y1 − y0| > |x1 − x0|
if (steep = TRUE) then
    swap (x0, y0)
    swap (x1, y1)
end if

if (x0 > x1) then
    swap (x0, x1)
    swap (y0, y1)
end if

if (y0 > y1) then
    δy = −1
else
    δy = 1
end if

Δx = x1 − x0
Δy = |y1 − y0|
y = y0
error = 0

for (x = x0 to x1) do
    if (steep = TRUE) then
        Plot [y, x]^T
    else
        Plot [x, y]^T
    end if
    error = error + Δy
    if (2 × error ≥ Δx) then
        y = y + δy
        error = error − Δx
    end if
end for
```

end



$j = |||$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\mathbf{t}}.$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathrm{T}([t_x, t_y]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\dot{\mathrm{R}}(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\dot{\mathrm{S}}(s_x, s_y)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\dot{\mathrm{Ref}}_x} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\dot{\mathrm{Ref}}_y} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix}}_{\dot{\mathrm{Sh}}_x(sh_x)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} 1 & 0 \\ sh_y & 1 \end{bmatrix}}_{\dot{\mathrm{Sh}}_y(sh_y)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -t_x \\ -t_y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{s_x} & 0 \\ 0 & \frac{1}{s_y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

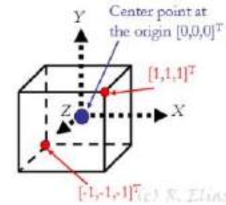
Vertex #	x	y	z

Edge #	Start vertex	End vertex

Edge	Vertices		Faces		Left traverse		Right traverse	
Name	Start	End	Left	Right	Pred	Succ	Pred	Succ

Vertex	edge

Face	edge

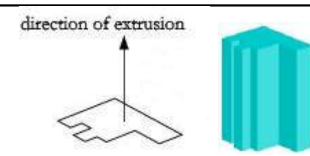


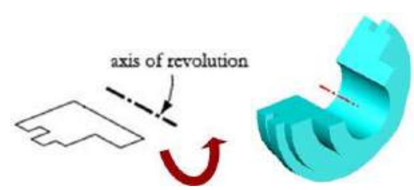
translate(scale(Block, < 1, 1.5, 1.5 >), < 1, 2, 3 >)

Face	Vertices
A	$[x_1, y_1, z_1]^T$, $[x_2, y_2, z_2]^T$, $[x_3, y_3, z_3]^T$
B	$[x_2, y_2, z_2]^T$, $[x_4, y_4, z_4]^T$, $[x_3, y_3, z_3]^T$
⋮	⋮

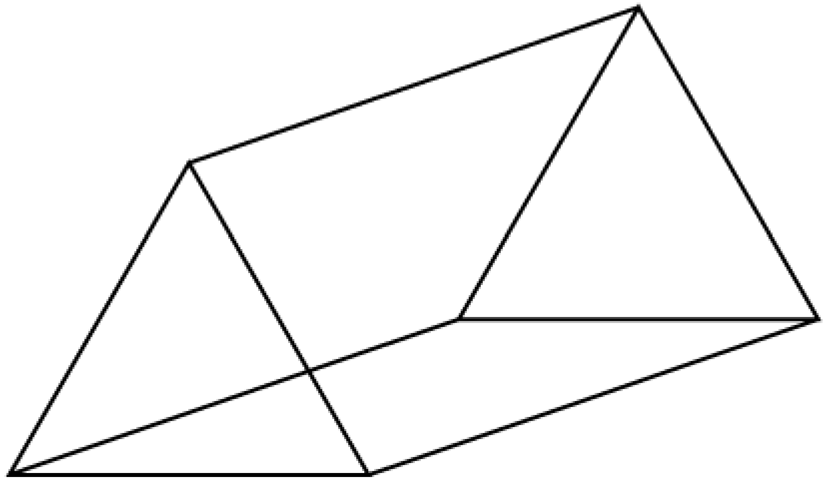
Vertex	Coordinates
1	$[x_1, y_1, z_1]^T$
2	$[x_2, y_2, z_2]^T$
⋮	⋮

Face	Vertices
A	1, 2, 3
B	2, 4, 3
⋮	⋮





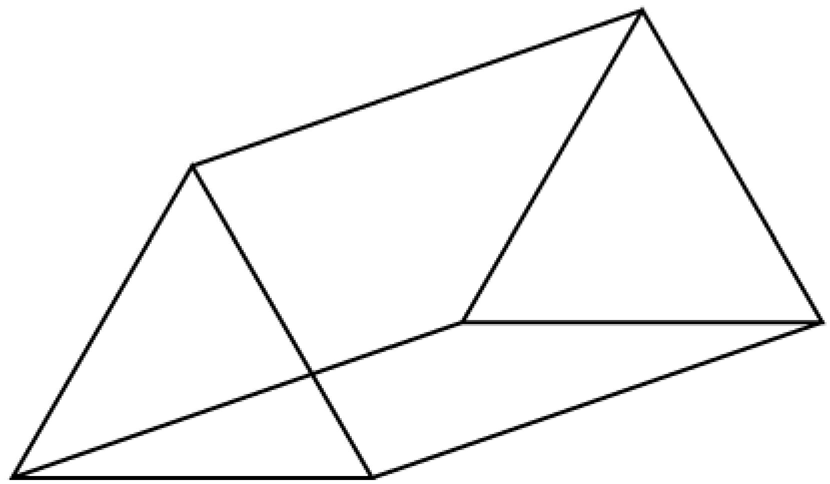
3. Write down the entries of the edge table for the figure below if it is to be represented as a wireframe. (mention start vertex and end vertex of each edge only) (2 marks)



1 mark for 9 edges

1 mark for correct start and end vertices

3. Write down the entries of the edge table for the figure below if it is to be represented as a wireframe.
(mention start vertex and end vertex of each edge only)

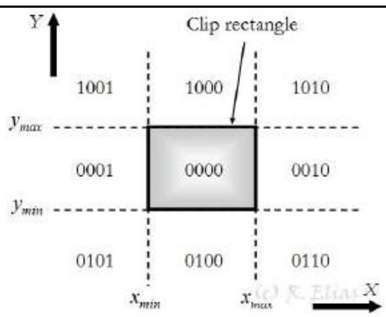


Formula Sheet

Input: x_0, y_0, x_1, y_1

```
steep = |y1 − y0| > |x1 − x0|
if (steep = TRUE) then
    swap (x0, y0)
    swap (x1, y1)
end if

if (x0 > x1) then
    swap (x0, x1)
    swap (y0, y1)
end if

if (y0 > y1) then
    δy = −1
else
    δy = 1
end if

Δx = x1 − x0
Δy = |y1 − y0|
y = y0
error = 0

for (x = x0 to x1) do
    if (steep = TRUE) then
        Plot [y, x]^T
    else
        Plot [x, y]^T
    end if
    error = error + Δy
    if (2 × error ≥ Δx) then
        y = y + δy
        error = error − Δx
    end if
end for
```

end



$j =$ |||

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{p}_2} = \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{p}_1} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{t}.$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{p_2} = \underbrace{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{T([t_x,t_y]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{p_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{p}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\dot{R}(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{p}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{p}_2} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\dot{S}(s_x,s_y)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{p}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{p}_2} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\dot{Ref}_x} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{p}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{p}_2} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\dot{Ref}_y} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{p}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{p}_2} = \underbrace{\begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix}}_{\dot{Sh}_x(sh_x)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{p}_1}$$

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{p}_2} = \underbrace{\begin{bmatrix} 1 & 0 \\ sh_y & 1 \end{bmatrix}}_{\dot{Sh}_y(sh_y)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{p}_1}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -t_x \\ -t_y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{s_x} & 0 \\ 0 & \frac{1}{s_y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

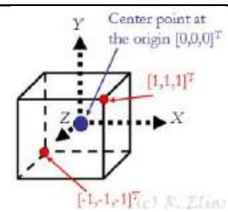
Vertex #	x	y	z

Edge #	Start vertex	End vertex

Edge	Vertices		Faces		Left traverse		Right traverse	
Name	Start	End	Left	Right	Pred	Succ	Pred	Succ

Vertex	edge

Face	edge

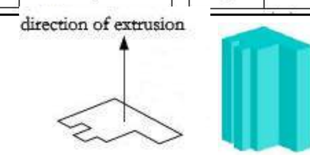


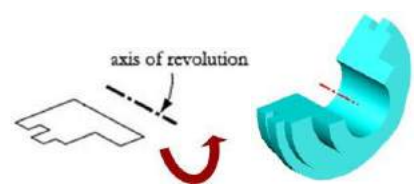
translate(scale(Block, < 1, 1.5, 1.5 >), < 1, 2, 3 >)

Face	Vertices
A	$[x_1, y_1, z_1]^T$, $[x_2, y_2, z_2]^T$, $[x_3, y_3, z_3]^T$
B	$[x_2, y_2, z_2]^T$, $[x_4, y_4, z_4]^T$, $[x_3, y_3, z_3]^T$
⋮	⋮

Vertex	Coordinates
1	$[x_1, y_1, z_1]^T$
2	$[x_2, y_2, z_2]^T$
⋮	⋮

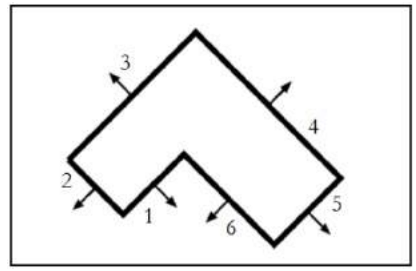
Face	Vertices
A	1, 2, 3
B	2, 4, 3
⋮	⋮





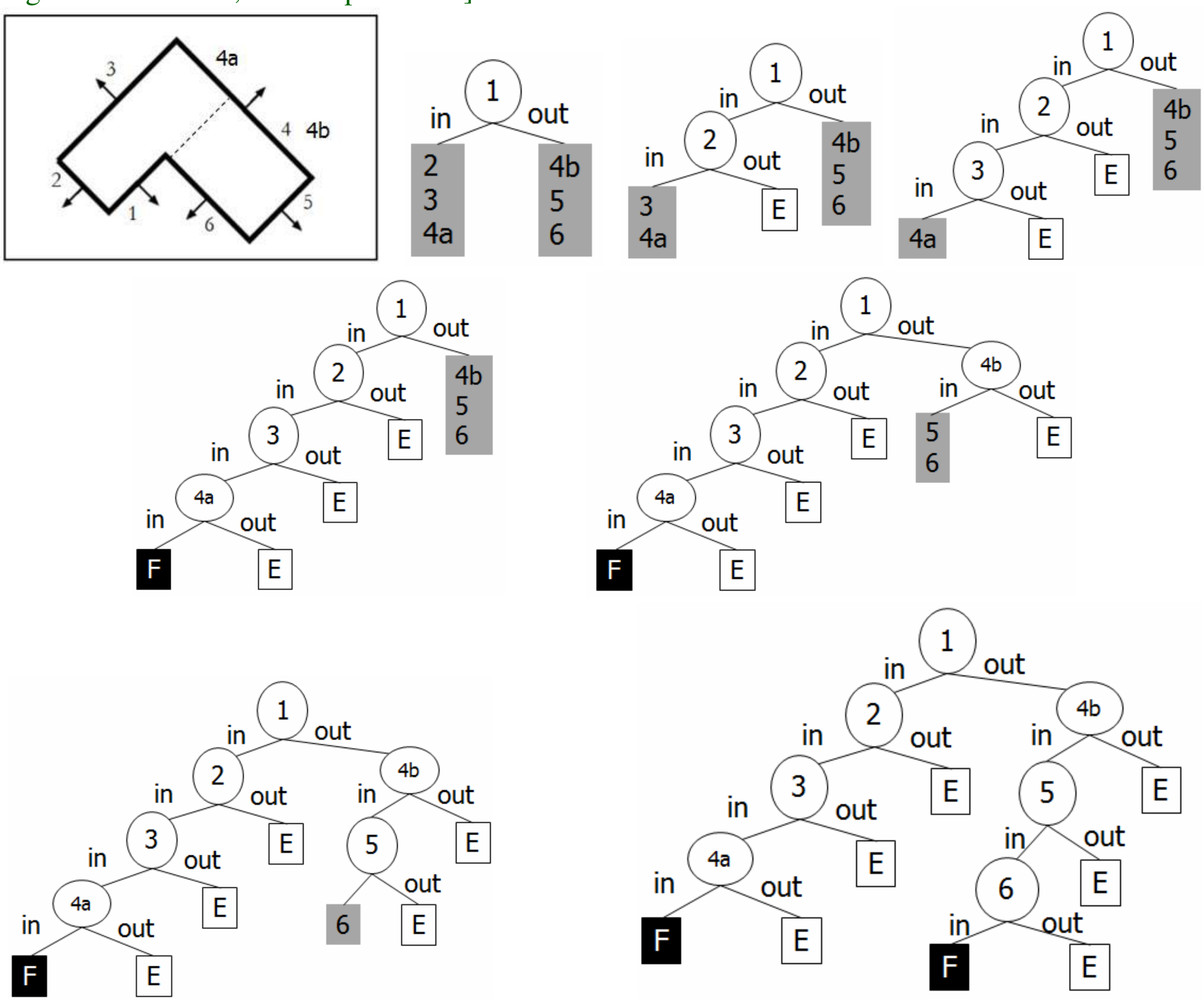
Question 3: (23 Marks)

The top view of a 3D model is shown. The arrows shown indicate the outside directions of the faces. Considering only the top view, represent this model using a BSP tree. Start with face "1."



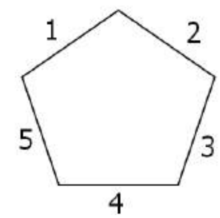
Solution

[Splitting face 4: 2 marks; each step 3 marks]



Question 3: [Solids]

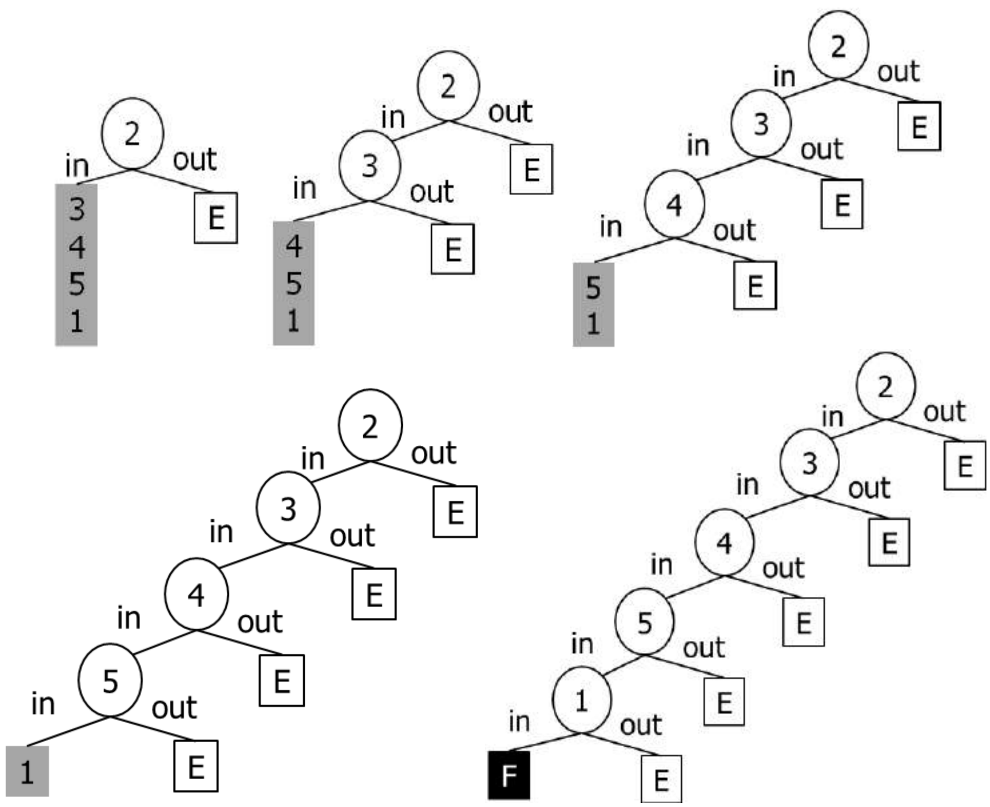
a) [20 marks] A BSP tree is constructed to represent a model whose top view is shown. Ignore the top and bottom surfaces of the model and assume that all normals are pointing outwards. **Draw the steps** of constructing the tree. Start with face 2.



N.B. *Drawing the last state of the tree without the steps will result in severe deduction.*

Solution

[4 marks each step]



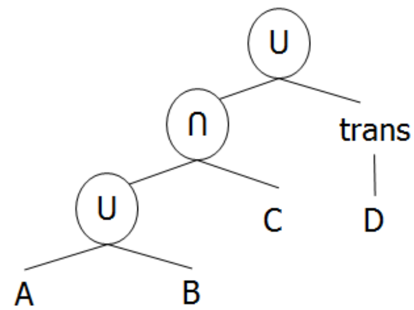
b) [10 marks] Draw the CSG tree for the following CSG expression:

$$A \bigcup B \bigcap C \bigcup trans(D)$$

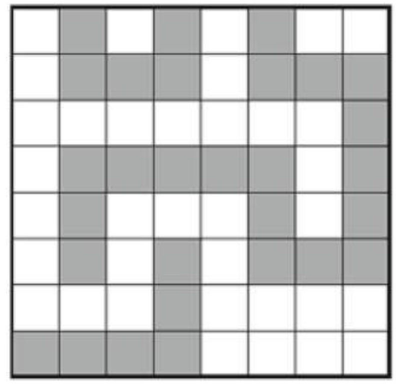
where *A*, *B*, *C* and *D* are solid primitives; ∪ and ∩ represent the union and intersection operations; and *trans*(.) indicates a transformation node.

Solution

[4 marks for leaf primitives; 1.5 marks for trans; 4.5 marks for Boolean operators]



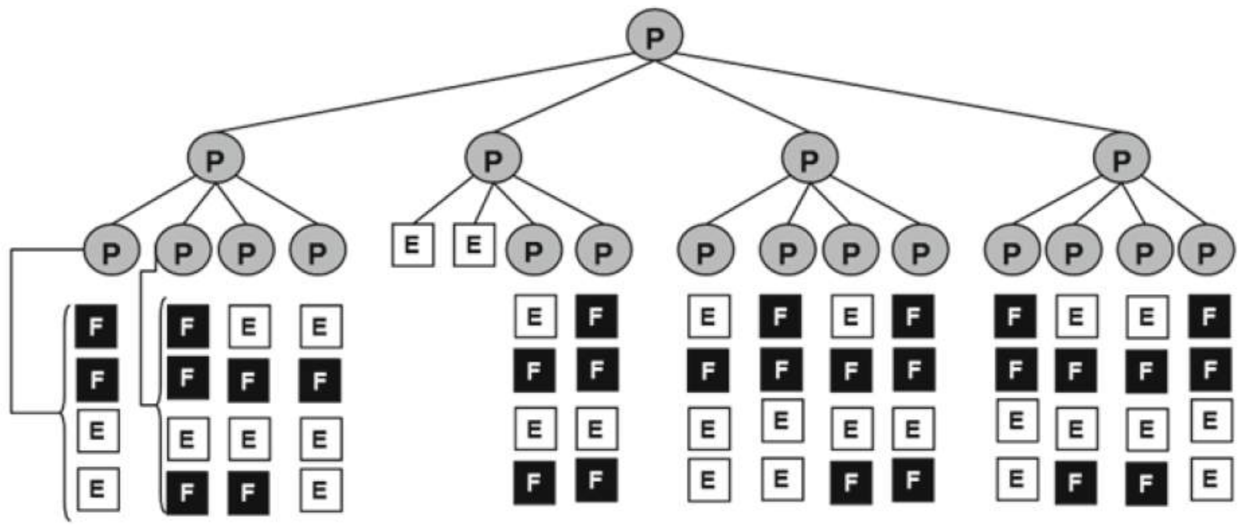
Question 3: [Solids]



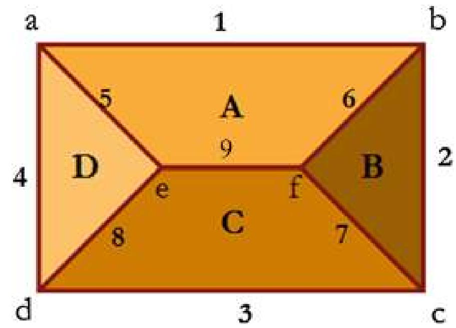
a) [16 marks] Construct a quadtree to partition the pattern shown. The order of constructing nodes is given by

2	3
0	1

Answers to Question 3a):



b) [17 marks] The roof of a house shown is represented as a wireframe where the vertices are indicated by lowercase letters (i.e., “a,” “b,” “c,” . . .), the faces by uppercase letters (i.e., “A,” “B,” “C,” . . .) and the edges by digits (i.e., “1,” “2,” “3,” . . .). The coordinates of the vertices are $[x_a, y_a, z_a]^T$, $[x_b, y_b, z_b]^T$, etc. Write down the entries of the required tables.



Write down the entries of the edge table of Baumgart's Winged-Edge Data Structure. Consider only edges 5, 6, 7, 8 and 9.

Answers to Question 3b):

[3 marks + 4 marks + 10 marks]

Vertex	Coordinates
a	$[x_a, y_a, z_a]^T$
b	$[x_b, y_b, z_b]^T$
c	$[x_c, y_c, z_c]^T$
d	$[x_d, y_d, z_d]^T$
e	$[x_e, y_e, z_e]^T$
f	$[x_f, y_f, z_f]^T$

Edge	Start vertex	End vertex
1	a	b
2	b	c
3	c	d
4	d	a
5	a	e
6	f	b
7	c	f
8	e	d
9	f	e

Edges	Vertices		Faces		Left traverse		Right traverse	
	Start	End	Left	Right	Pred.	Succ.	Pred.	Succ.
5	a	e	A	D	9	1	4	8
6	b	f	B	A	7	2	1	9
7	c	f	C	B	9	3	2	6
8	d	e	D	C	5	4	3	9
9	e	f	A	C	6	5	8	7

Question 3 [Solids]:

a) [9 marks] A sphere whose radius is 5 units and centered at the origin is to be represented as voxels. The voxel size is 1×1×1 units. The center point of each voxel has integer coordinates (e.g., $[0, 0, 0]^T$, $[1, 2, 0]^T$, $[2, 4, 3]^T$, ...). A voxel is declared occupied if its center point is inside the sphere. Determine whether the following voxels are declared vacant or occupied: $[2, 4, 2]^T$, $[2, 3, 3]^T$ and $[0, 5, 1]^T$ where these coordinates represent the center points of the voxels.

Answers to Question 3 a):

A voxel is occupied if the distance from its center to the origin (i.e., the center of the sphere) is less than or equal to the radius of the sphere.

$$d(\dot{\mathbf{P}}_1, \dot{\mathbf{P}}_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2};$$

thus,

$$\begin{aligned} d([0,0,0]^T, [2,4,2]^T) &= \sqrt{(2-0)^2 + (4-0)^2 + (2-0)^2} \\ &= 2\sqrt{6} = 4.899 \le 5 \Longrightarrow \text{occupied}, \end{aligned}$$

$$\begin{aligned} d([0,0,0]^T, [2,3,3]^T) &= \sqrt{(2-0)^2 + (3-0)^2 + (3-0)^2} \\ &= \sqrt{22} = 4.6904 \le 5 \Longrightarrow \text{occupied}, \end{aligned}$$

$$\begin{aligned} d([0,0,0]^T, [0,5,1]^T) &= \sqrt{(0-0)^2 + (5-0)^2 + (1-0)^2} \\ &= \sqrt{26} = 5.099 > 5 \Longrightarrow \text{vacant}. \end{aligned}$$

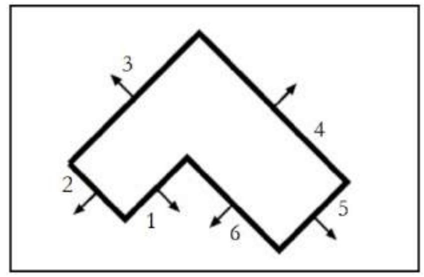
Alternatively, a voxel may be declared occupied or vacant when applying its coordinates in the surface equation of a sphere centered at the origin, which is given by

$$x^2 + y^2 + z^2 - r^2 = 0$$

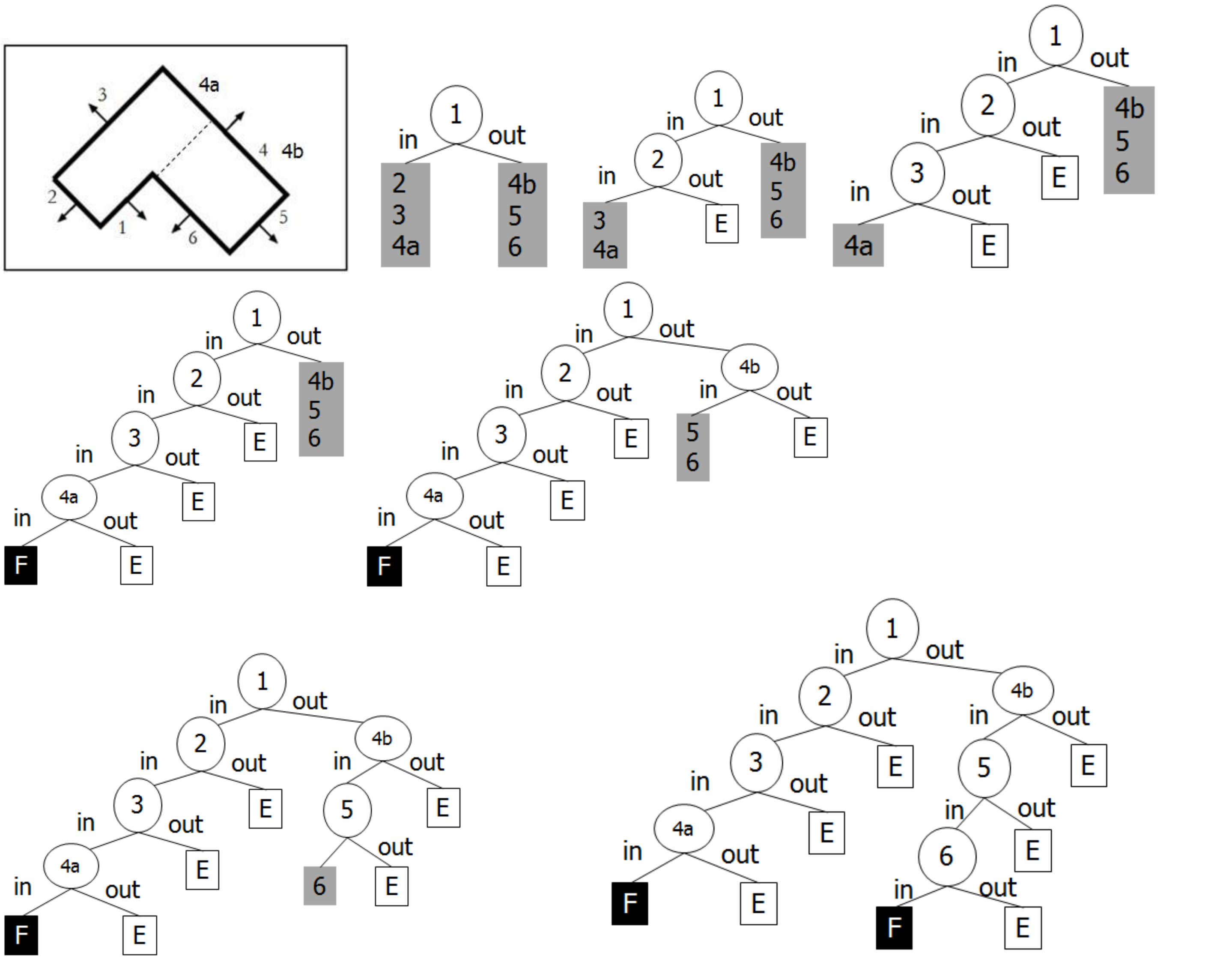
where $[x, y, z]^T$ and r represent the surface point coordinates and the radius of the sphere respectively. Hence,

$$\begin{aligned} 2^2 + 4^2 + 2^2 - 5^2 &= -1 \le 0 \Longrightarrow \text{occupied}, \\ 2^2 + 3^2 + 3^2 - 5^2 &= -3 \le 0 \Longrightarrow \text{occupied}, \\ 0^2 + 5^2 + 1^2 - 5^2 &= +1 > 0 \Longrightarrow \text{vacant}. \end{aligned}$$

b) [16 marks] The top view of a 3D model is shown. The arrows shown indicate the outside directions of the faces. Considering only the top view, represent this model using a BSP tree. Start with face “1.” **<u>Show the steps of constructing this tree (i.e., draw the tree at each step).</u>**



Answers to Question 3 b):



[Splitting face 4: 2 marks; each step 2 marks]

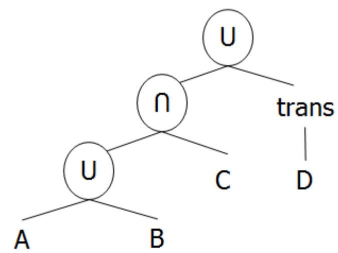
c) [8 marks] Draw the CSG tree for the following CSG expression:

$$A \bigcup B \bigcap C \bigcup trans(D)$$

where A, B, C and D are solid primitives; U and ∩ represent the union and intersection operations; and *trans*(.) indicates a transformation node.

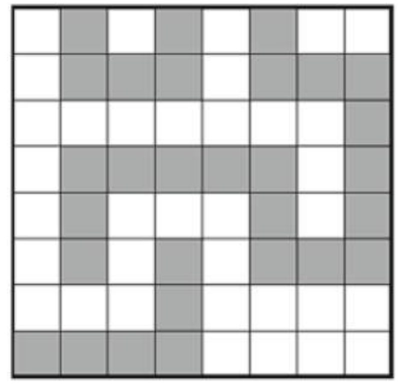
Answers to Question 3 c):

[4 marks for leaf primitives; 1 mark for trans; 3 marks for Boolean operators]



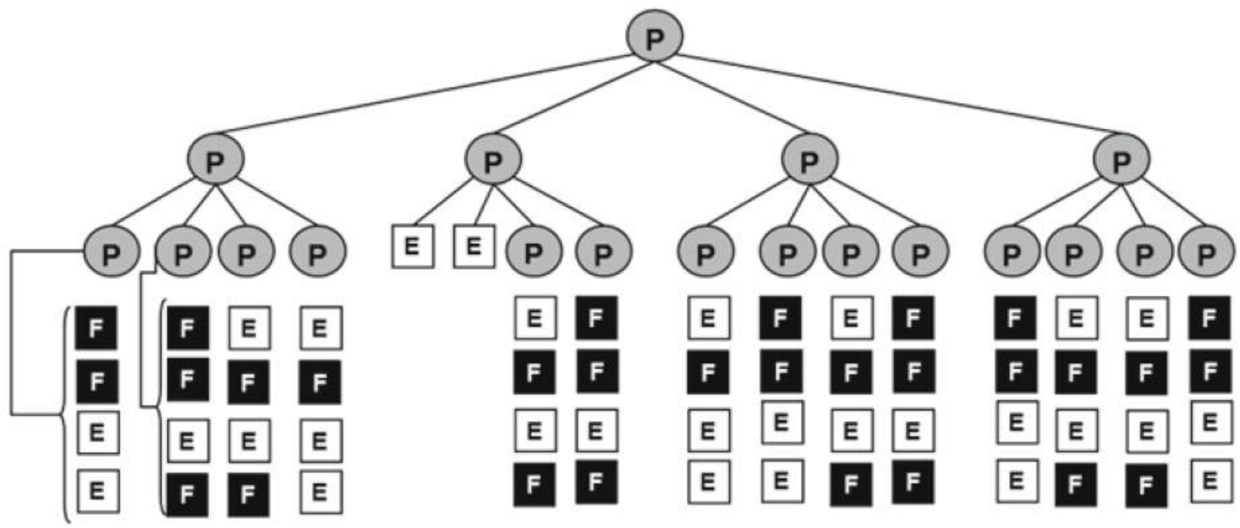
Question 4 [Solids]:

[16 marks] Construct a quadtree to partition the pattern shown. The order of constructing nodes is given by



2	3
0	1

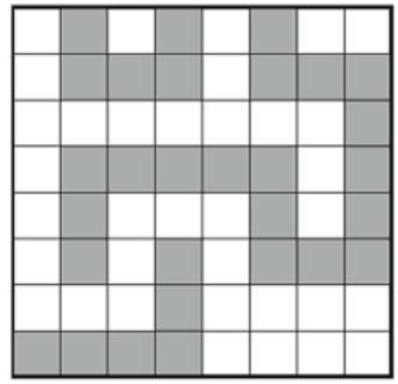
Answers to Question 4:



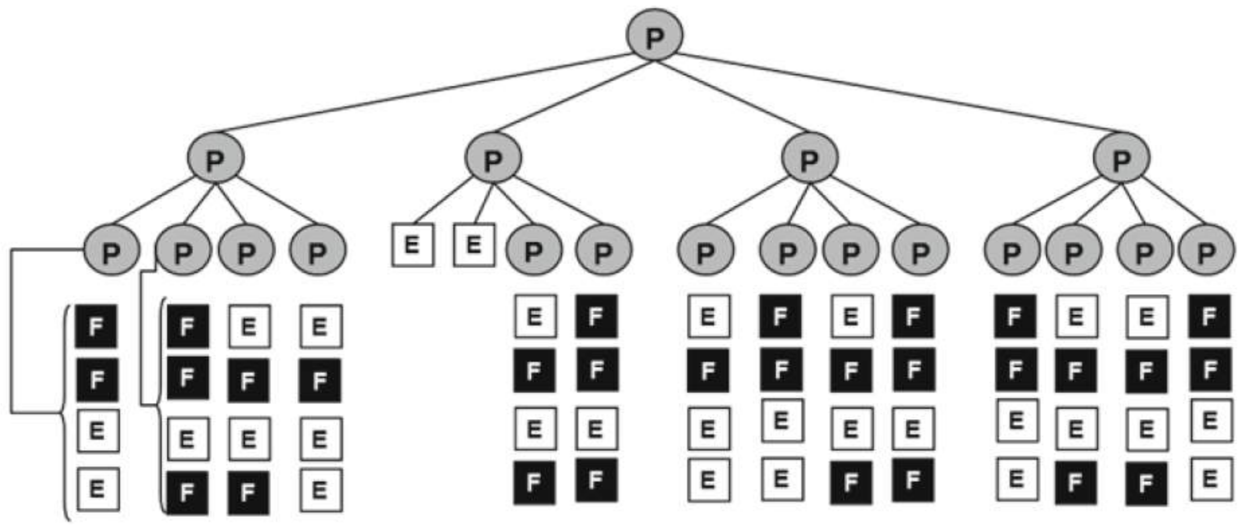
Question 3 [Solids]:

[16 marks] Construct a quadtree to partition the pattern shown. The order of constructing nodes is given by

2	3
0	1

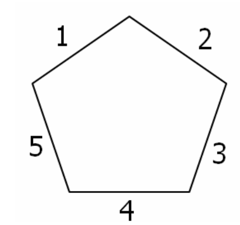


Answers to Question 3:



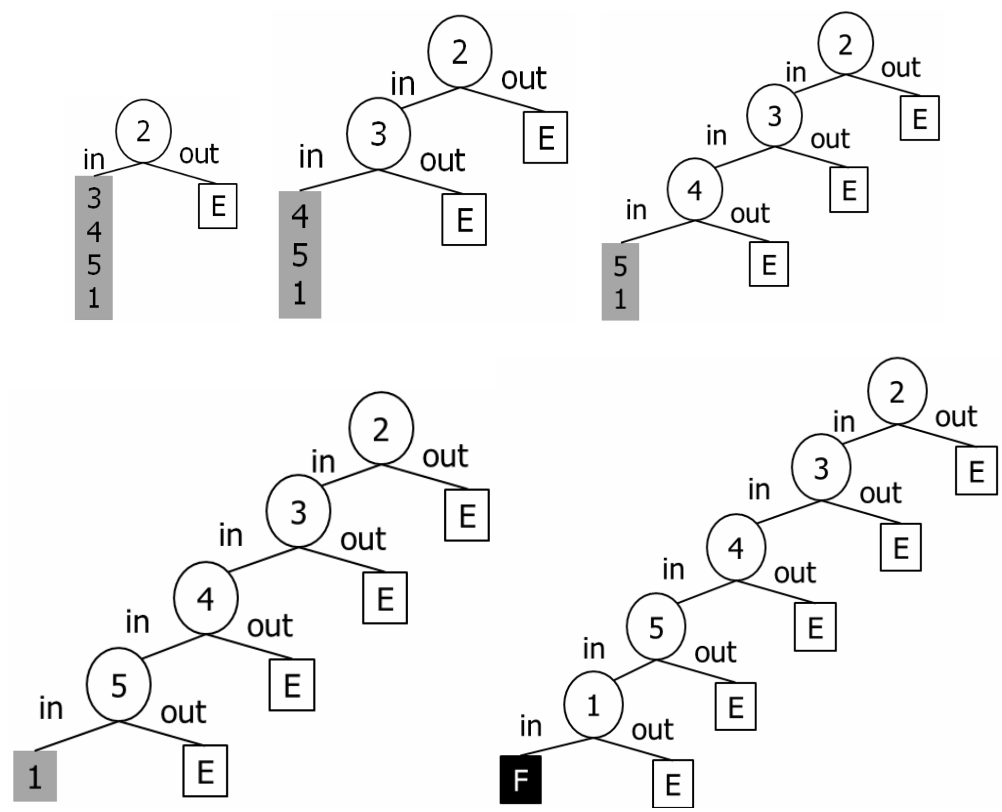
Question 5: (16 Marks)

a) [10 marks] A BSP tree is constructed to represent a model whose top view shown. Ignore the top and bottom surfaces of the model and assume that all normals are pointing outwards. Draw the steps of constructing the tree. Start with face 2.



Solution

[2 marks each step]



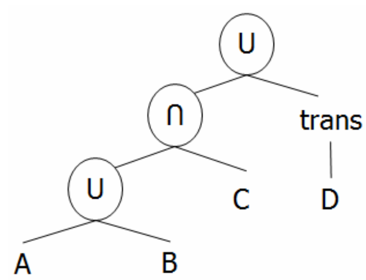
b) [6 marks] Draw the CSG tree for the following CSG expression:

$$A \bigcup B \bigcap C \bigcup trans(D)$$

where A, B, C and D are solid primitives; ∪ and ∩ represent the union and intersection operations; and *trans*(.) indicates a transformation node.

Solution

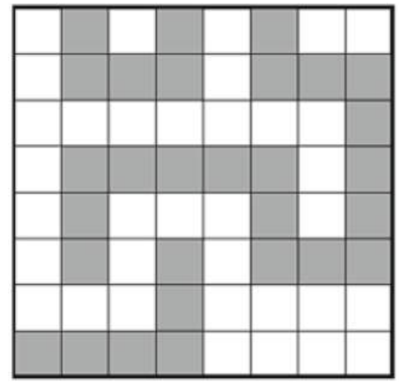
[2 marks for leaf primitives; 1mark for trans; 3 marks for Boolean operators]



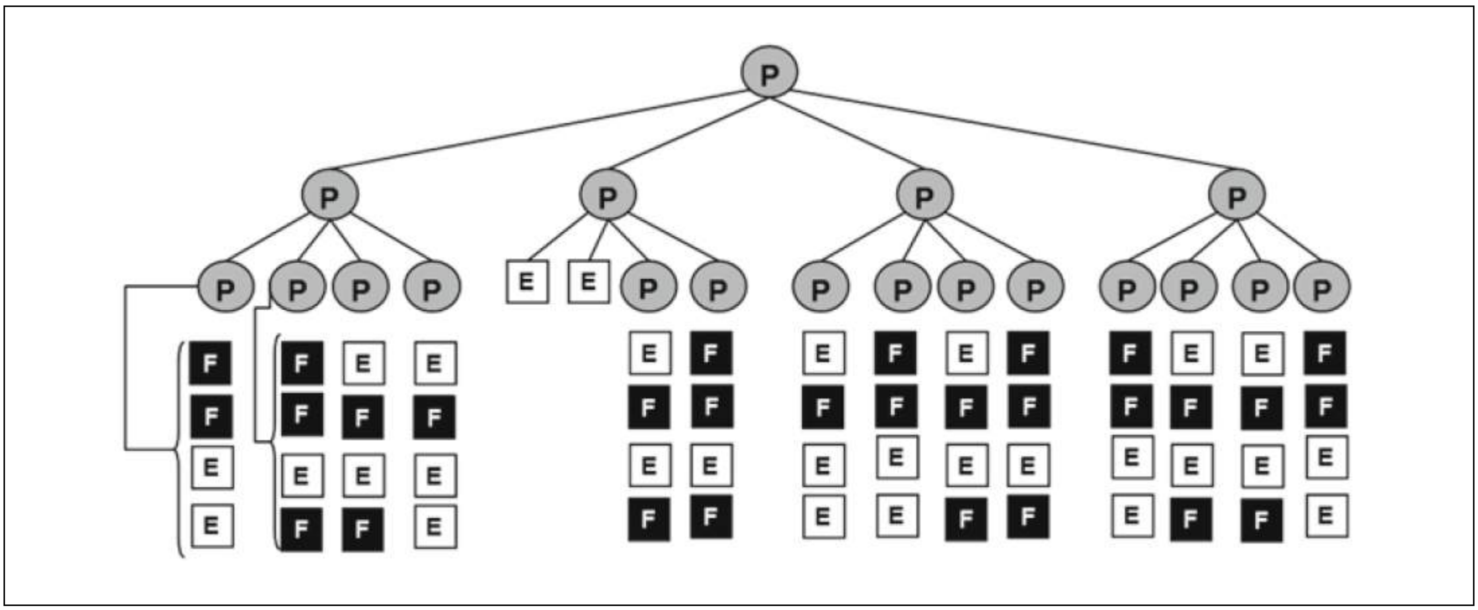
Question 3: (Solids)

[16 marks] Construct a quadtree to partition the pattern shown. The order of constructing nodes is given by

2	3
0	1

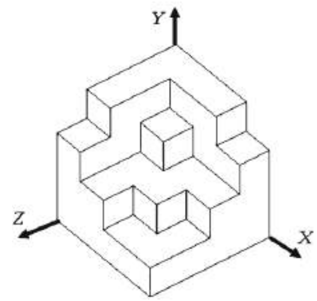


Solutions

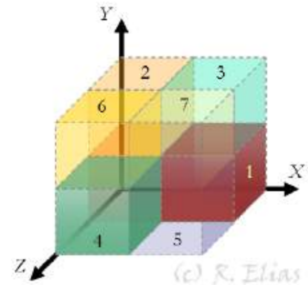


Question 2 [Solids]:

[10 marks] Construct an octree to represent the 3D model shown. The numbering of octants is shown as well.



A model

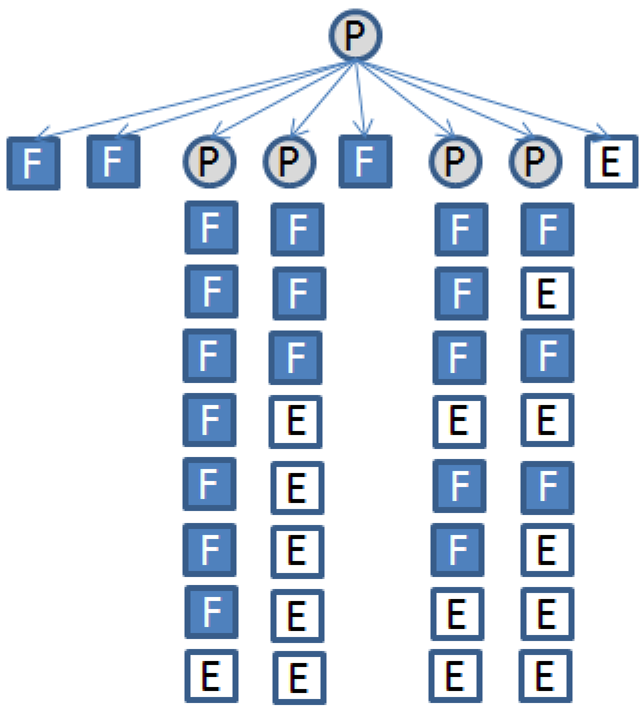


Ordering

Answers to Question 2:

Prob 4.6

[10 marks]



German University in Cairo

Department of Digital Media Engineering and Technology

DMET 502 - Computer Graphics Quiz 2.1

DMET 502 – Computer Graphics

Quiz 2

Quiz Duration: 20 minutes

Name:	
ID:	
Tutorial:	
TA Name:	

Question:

A unit cube is to be rotated about a line passing through the origin and the point $[6,20,4]^T$. Determine a **sequence** to derive the required 3D transformation matrix. **You must avoid rotation about the y-axis.**

Solution [1]:

M1 = Rotation about z-axis by angle $+\theta$ where $\theta = \tan^{-1}\frac{6}{20} = 16.699°$

M2 = Rotation about x-axis by angle $+(90 - \varphi)$ where $\varphi = \tan^{-1}\frac{4}{\sqrt{6^2+20^2}} =$ 10.845°

M3 = Rotation about z-axis by an angle.

M4 = Rotation back about x-axis by angle $-(90 - \varphi)$

M5 = Rotation back about z-axis by angle $-\theta$

Solution [2]:

M1 = Rotation about x-axis by angle $-\theta$ where $\theta = \tan^{-1}\frac{4}{20} = 11.3099°$

M2 = Rotation about z-axis by angle $-(90 - \varphi)$ where $\varphi = \tan^{-1}\frac{6}{\sqrt{4^2+20^2}} =$ 16.3925°

M3 = Rotation about x-axis by an angle.

M4 = Rotation back about z-axis by angle $+(90 - \varphi)$

M5 = Rotation back about x-axis by angle $+\theta$

German University in Cairo

Department of Digital Media Engineering and Technology

DMET 502 - Computer Graphics Quiz 2.2

DMET 502 – Computer Graphics

Quiz 2

Quiz Duration: <u>20 minutes</u>

Name:	
ID:	
Tutorial:	
TA Name:	

Question:

A unit cube is to be reflected about a plane described by these points (12, -4, 0), (12, -4, 11) and (12, 11, 11). Determine a **sequence** to derive the required 3D transformation matrix.

Solution [1]:

M1 = Translate by (-12,0,0) // It has to be -12 in x and any values in y and z

M2 = Reflect about yz-plane

M3 = Translate back by (12,0,0)

Solution [2]:

M1 = Translate by (-12,0,0)

M2 = Rotate about y with angle 90°

M3 = Reflect about xy-plane

M4 = Rotate back about y with angle -90°

M5 = Translate back by (12,0,0)

Solution [3]:

M1 = Translate by (-12,0,0)

M2 = Rotate about z with angle -90°

M3 = Reflect about zx-plane

M4 = Rotate back about z with angle 90°

M5 = Translate back by (12,0,0)

<u>Question 4:</u> (28 Marks)

If a point is to be rotated through an angle of 45º about a line passing through $[0,1,0]^T$ and having the direction $[0,1,1]^T$, derive the transformation matrix required.

Suggest another series of transformations to get the same matrix.

<u>Solution</u>

Steps: [10 marks; for each step:1 mark for transformation type and 1 mark for parameters]

1. Translate using the vector $[0,-1, 0]^T$ so that point $[0,1,0]^T$ coincides with the origin.
2. Rotate through an angle of 45º about the *x*-axis. This enforces the rotation axis to coincide with the *z*-axis.
3. Rotate through an angle of 45º about the *z*-axis.
4. Rotate through an angle of –45º about the *x*-axis to reverse the rotation done in Step 2.
5. Translate the pyramid using the vector $[0, 1, 0]^T$ to reverse the translation done in Step 1.

Matrices: [10 marks; 2 marks each]

$$\mathtt{M}_1 = \mathtt{T}(\,[0,-1,0]^T\,) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{M}_2 = \mathtt{R}_x(45) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(45) & -\sin(45) & 0 \\ 0 & \sin(45) & \cos(45) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{M}_3 = \mathtt{R}_z(45) = \begin{bmatrix} \cos(45) & -\sin(45) & 0 & 0 \\ \sin(45) & \cos(45) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{M}_4 = \mathtt{R}_x(-45) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-45) & -\sin(-45) & 0 \\ 0 & \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_5 = T([0,1,0]^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The overall transformation [3 marks]

$$M = M_5 M_4 M_3 M_2 M_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{2+\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} \\ -\frac{1}{2} & \frac{2-\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & \frac{-2+\sqrt{2}}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Alternatively, the steps become: [5 marks; 1 mark each]

1. Translate using the vector $[0,-1,0]^T$ so that point $[0,1,0]^T$ coincides with the origin.
2. Rotate through an angle of -45º about the *x*-axis. This enforces the rotation axis to coincide with the *y*-axis.
3. Rotate through an angle of 45º about the *y*-axis.
4. Rotate through an angle of 45º about the *x*-axis to reverse the rotation done in Step 2.
5. Translate the pyramid using the vector $[0, 1, 0]^T$ to reverse the translation done in Step 1.

Question 4: [3D Transformations]

[20 points] Consider a pyramid **ABCD** where **A**=$[0,0,0]^T$, **B**=$[\delta,0,0]^T$, **C**=$[0,\delta,0]^T$ and **D**=$[0,0,\delta]^T$ such that δ is a real number. The pyramid is to be rotated through 45° about a line passing through **B** and having the direction $[0,\delta,\delta]^T$. Specify the transformation steps that should be carried out along with the factors used. Use inhomogeneous coordinates to determine the location of **A** (in terms of δ) after transformation.

Solution

Steps

1. Translate $T(-\delta,0,0)$ [2 marks]
2. Rotate $R_x(45)$ [2 marks]
3. Rotate $R_z(45)$ [2 marks]
4. Rotate $R_x(-45)$ [1 mark]
5. Translate $T(\delta,0,0)$ [1 mark]

Or

1. Translate $T(-\delta,0,0)$ [2 marks]
2. Rotate $R_x(-45)$ [2 marks]
3. Rotate $R_y(45)$ [2 marks]
4. Rotate $R_x(45)$ [1 mark]
5. Translate $T(\delta,0,0)$ [1 mark]

$R_x(-45)\, R_z(45)\, R_x(45)$ =

$$\mathbf{P}' = R_x(-45)R_z(45)R_x(45)\begin{bmatrix} x-\delta \\ y \\ z \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{P}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(45) & -\sin(45) \\ 0 & \sin(45) & \cos(45) \end{bmatrix}\begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-45) & -\sin(-45) \\ 0 & \sin(-45) & \cos(-45) \end{bmatrix}\begin{bmatrix} x-\delta \\ y \\ z \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{P}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x-\delta \\ y \\ z \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \\ 0 \end{bmatrix}$$

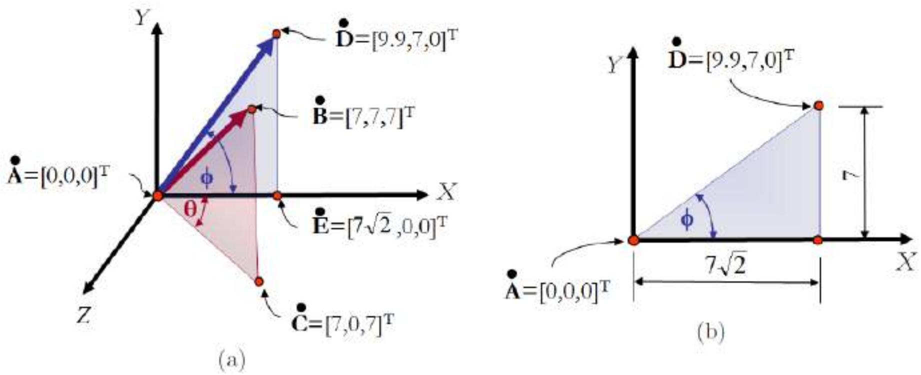
$$\mathbf{P}' = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2\sqrt{2}}+\frac{1}{2} & \frac{1}{2\sqrt{2}}-\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2\sqrt{2}}-\frac{1}{2} & \frac{1}{2\sqrt{2}}+\frac{1}{2} \end{bmatrix}\begin{bmatrix} 0-\delta \\ 0-0 \\ 0-0 \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}-1}{\sqrt{2}}\delta \\ -\frac{1}{2}\delta \\ -\frac{1}{2}\delta \end{bmatrix}$$ [2 marks]

Each matrix/vector [2 marks = 8 marks]

Question 4: [3D Transformations]

[30 marks] A point is rotated through an angle of 30º about a line passing through the origin $\dot{\mathbf{A}} = [0, 0, 0]^T$ and a point $\dot{\mathbf{B}} = [7, 7, 7]^T$. Get the required rotation angles and rotation sequence. There are 12 different rotation sequences to solve this problem. Mention them all. (You do not have to get the final transformation matrix.)

Answers to Question 4:



Steps:

1. Rotate the vector **AB** through an angle θ about the *y*-axis so that we get the vector **AD** on the *xy*-plane. Since **A** is the origin, its position will not change. We only need to get the position of point **D** where

$$\mathbf{D} = \mathrm{R}_y(45)\mathbf{B} = \begin{bmatrix} \cos(45) & 0 & \sin(45) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(45) & 0 & \cos(45) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7\sqrt{2} \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.8995 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

   [Rotation about y: 2 marks; θ =45: 3 marks; matrix: 2 marks; point: 2 marks]

2. Rotate the vector **AD** about the *z*-axis through an angle $-\phi$ to coincide with the *x*-axis.

$$\phi = \tan^{-1}\left(\frac{7}{7\sqrt{2}}\right) = 35.26°$$

   [Rotation about z: 2 marks; angle: 3 marks]

3. Rotate through an angle of 30˚ about the *x*-axis. [2 marks]
4. Reverse rotation of step 2 (i.e., $\mathbf{R}_z$(35.26)). [2 marks]
5. Reverse rotation of step 1 (i.e., $\mathbf{R}_y$(-45)). [2 marks]

Answers to Question 4 (cont.):

Any of the following sequences is correct: [10 marks]

$$R_y(45°) \rightarrow R_z(-35.26°) \rightarrow R_x(30°) \rightarrow R_z(35.26°) \rightarrow R_y(-45°)$$
$$R_y(45°) \rightarrow R_z(54.74°) \rightarrow R_y(30°) \rightarrow R_z(-54.74°) \rightarrow R_y(-45°)$$
$$R_y(-45°) \rightarrow R_x(35.26°) \rightarrow R_z(30°) \rightarrow R_x(-35.26°) \rightarrow R_y(45°);$$
$$R_y(-45°) \rightarrow R_x(-54.74°) \rightarrow R_y(30°) \rightarrow R_x(54.74°) \rightarrow R_y(45°)$$

$\mathbf{R}_z(45) \rightarrow \mathbf{R}_x(-35.26) \rightarrow \mathbf{R}_y(30) \rightarrow \mathbf{R}_x(35.26) \rightarrow \mathbf{R}_z(-45)$
$\mathbf{R}_z(45) \rightarrow \mathbf{R}_x(54.74) \rightarrow \mathbf{R}_z(30) \rightarrow \mathbf{R}_x(-54.74) \rightarrow \mathbf{R}_z(-45)$
$\mathbf{R}_z(-45) \rightarrow \mathbf{R}_y(35.26) \rightarrow \mathbf{R}_x(30) \rightarrow \mathbf{R}_y(-35.26) \rightarrow \mathbf{R}_z(45)$
$\mathbf{R}_z(-45) \rightarrow \mathbf{R}_y(-54.74) \rightarrow \mathbf{R}_z(30) \rightarrow \mathbf{R}_y(54.74) \rightarrow \mathbf{R}_z(45)$

$\mathbf{R}_x(45) \rightarrow \mathbf{R}_y(-35.26) \rightarrow \mathbf{R}_z(30) \rightarrow \mathbf{R}_y(35.26) \rightarrow \mathbf{R}_x(-45)$
$\mathbf{R}_x(45) \rightarrow \mathbf{R}_y(54.74) \rightarrow \mathbf{R}_x(30) \rightarrow \mathbf{R}_y(-54.74) \rightarrow \mathbf{R}_x(-45)$
$\mathbf{R}_x(-45) \rightarrow \mathbf{R}_z(35.26) \rightarrow \mathbf{R}_y(30) \rightarrow \mathbf{R}_z(-35.26) \rightarrow \mathbf{R}_x(45)$
$\mathbf{R}_x(-45) \rightarrow \mathbf{R}_z(-54.74) \rightarrow \mathbf{R}_x(30) \rightarrow \mathbf{R}_z(54.74) \rightarrow \mathbf{R}_x(45)$

Question 4 [3D Transformations]:

[15 points] Check if each of the following inhomogeneous matrices can be a rotation matrix. Mention at least one reason if it cannot be a rotation matrix and at least two reasons if it can.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9 & -0.2 \\ 0 & 0.2 & 0.9 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{10}} & 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{1} & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answers to Question 4:

- R is normalized: the squares of the elements in any row or column sum to 1.
- R is orthogonal: the dot product of any pair of rows or any pair of columns is 0.

(3 points each)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9 & -0.2 \\ 0 & 0.2 & 0.9 \end{bmatrix}$$ No, not normalized

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$ Yes, normalized, orthogonal, det = +1, transpose = inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{10}} & 0 \end{bmatrix}$$ No, not normalized

$$\begin{bmatrix} \frac{\sqrt{3}}{1} & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$$ No, not normalized

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ Yes, normalized, orthogonal, det = +1, transpose = inverse

Question 7 (3D Transformations):

[30 points] Consider the triangle abc where $a=[0,0,0]^T$, $b=[1,1,1]^T$ and $c=[-2,-3,3]^T$. The triangle is to be rotated so that:

- The vertex a remains at the origin
- The edge ab coincides with the *x*-axis and
- The triangle abc coincides with the *xy*-plane.

Estimate the transformation matrix used. Note that no translation, scaling or shearing is performed. (Write down the steps before you start your calculations.)

Answers to Question 7:

Steps:

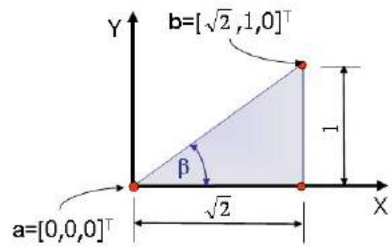
1. Adjust the edge ab to coincide with the *x*-axis
   1.1. Project the edge ab onto the *zx*-plane and get the inclination α with the *x*-axis
   1.2. Rotate about the *y*-axis using angle α (using the matrix $R_y(\alpha)$). Get b after rotation
   1.3. The edge ab now is on the *xy*-plane. Get its inclination β with the *x*-axis
   1.4. Rotate about the *z*-axis using angle β ($R_{zy}=R_z(\beta)\, R_y(\alpha)$)
2. Get the point c after rotation R_{zy}
3. Get the inclination γ of ac with the *y*-axis
4. Rotate about the *x*-axis using angle γ (using the matrix $R_x(\gamma)$)
5. Hence the overall rotation= $R_x(\gamma)\, R_z(\beta)\, R_y(\alpha)$

1.1 $\alpha = 45$

1.2 $$\mathbf{R_y(45)b} = \begin{bmatrix} \cos 45 & 0 & \sin 45 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 45 & 0 & \cos 45 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \\ 1 \end{bmatrix} \Leftarrow \mathbf{b} \text{ after rotation } R_y(45)$$

1.3 $$\beta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 35.26$$

-35.26 should be used as rotation is from y to x.



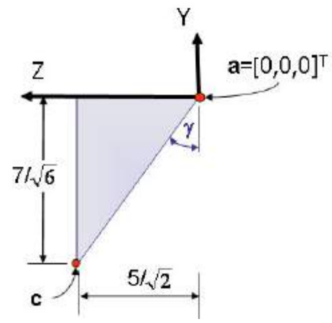
1.4

$$\mathbf{R}_z(\beta)\mathbf{R}_y(\alpha) = \begin{bmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. $\mathbf{R}_z(\beta)\mathbf{R}_y(\alpha)\mathbf{c} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{3}} \\ -\frac{7}{\sqrt{6}} \\ \frac{5}{\sqrt{2}} \\ 1 \end{bmatrix} \Leftarrow \mathbf{c}$ after rotation

$R_z(\beta)R_y(\alpha)$

3. $\gamma = \tan^{-1}\left(\frac{5\sqrt{6}}{7\sqrt{2}}\right) = 51.05$



$$\mathbf{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma & 0 \\ 0 & \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 51.05 & -\sin 51.05 & 0 \\ 0 & \sin 51.05 & \cos 51.05 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{R}_x(51.05)\mathbf{R}_z(-35.26)\mathbf{R}_y(45)$$

5.

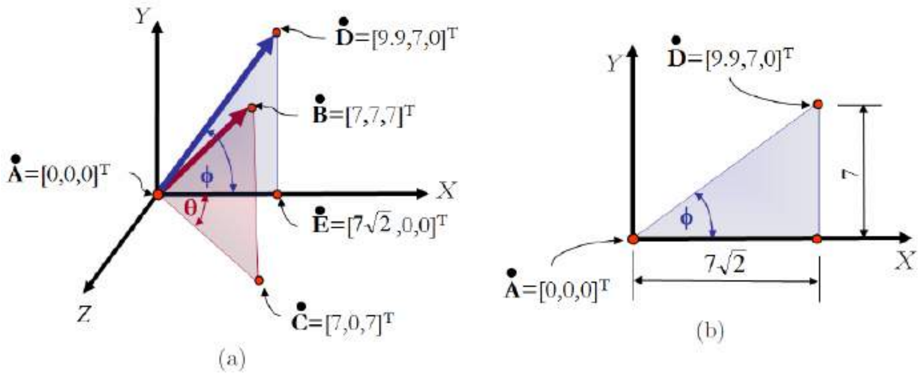
$$\mathbf{R} = \begin{bmatrix} 0.5774 & 0.5774 & 0.5774 & 0 \\ 0.2933 & 0.5133 & -0.8065 & 0 \\ -0.7619 & 0.635 & 0.127 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Steps (5 points)
- α (2 points)
- Ry(α) (1 point) and b after rotation (2 points)
- β (2 points)
- Rz(β)(1 point), RzRy (2 points), right order (2 points), result (2 points)
- c after rotation (2 points)
- γ (2 points)
- Rx(γ)(1 point)
- RxRzRy (2 points), right order (2 points), result (2 points)

<u>Question 4:</u> (20 Marks)

A point is rotated through an angle of 30° about a line passing through the origin $\dot{\mathbf{A}} = [0,\ 0,\ 0]^T$ and a point $\dot{\mathbf{B}} = [7,\ 7,\ 7]T$. Get the required rotation angles and rotation sequence. (You do not have to get the final transformation matrix.)

<u>Solution</u>



Steps:

1. Rotate the vector **AB** through an angle θ about the *y*-axis so that we get the vector **AD** on the *xy*-plane. Since **A** is the origin, its position will not change. We only need to get the position of point **D** where

$$\mathbf{D} = \mathrm{R}_y(45)\mathbf{B} = \begin{bmatrix} \cos(45) & 0 & \sin(45) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(45) & 0 & \cos(45) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7\sqrt{2} \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.8995 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

   [Rotation about y: 2 marks; θ =45: 3 marks; matrix: 2 marks; point: 2 marks]

2. Rotate the vector **AD** about the *z*-axis through an angle $-\phi$ to coincide with the *x*-axis.

$$\phi = \tan^{-1}\left(\frac{7}{7\sqrt{2}}\right) = 35.26^\circ$$

   [Rotation about z: 2 marks; angle: 3 marks]

3. Rotate through an angle of 30° about the *x*-axis. [2 marks]
4. Reverse rotation of step 2 (i.e., $\mathbf{R}_z(35.26)$). [2 marks]
5. Reverse rotation of step 1 (i.e., $\mathbf{R}_y(-45)$). [2 marks]

   Any of the following sequences is correct:

$$\mathrm{R}_y(45^\circ) \to \mathrm{R}_z(-35.26^\circ) \to \mathrm{R}_x(30^\circ) \to \mathrm{R}_z(35.26^\circ) \to \mathrm{R}_y(-45^\circ)$$
$$\mathrm{R}_y(45^\circ) \to \mathrm{R}_z(54.74^\circ) \to \mathrm{R}_y(30^\circ) \to \mathrm{R}_z(-54.74^\circ) \to \mathrm{R}_y(-45^\circ)$$
$$\mathrm{R}_y(-45^\circ) \to \mathrm{R}_x(35.26^\circ) \to \mathrm{R}_z(30^\circ) \to \mathrm{R}_x(-35.26^\circ) \to \mathrm{R}_y(45^\circ);$$
$$\mathrm{R}_y(-45^\circ) \to \mathrm{R}_x(-54.74^\circ) \to \mathrm{R}_y(30^\circ) \to \mathrm{R}_x(54.74^\circ) \to \mathrm{R}_y(45^\circ)$$

<u>Question 4:</u> [3D Transformations]

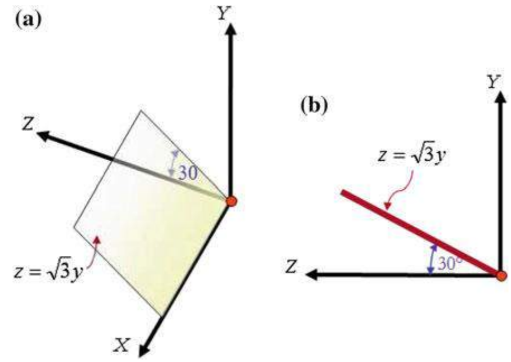
a) [10 marks] Derive a matrix that reflects a point $[x, y, z]^T$ about the plane $z = \sqrt{3}y$.

Answers to Question 4a):

Example 5.14:

The plane is drawn.

1. Calculate the angle between the plane $z = \sqrt{3}y$ and the zx-plane. It is easy to show that this angle is 30º. [2 marks]
2. Rotate through an angle of 30º about the x-axis. [1 mark]
3. Reflect about the zx-plane. [1 mark]
4. Rotate through an angle of −30º about the x-axis. [1 mark]



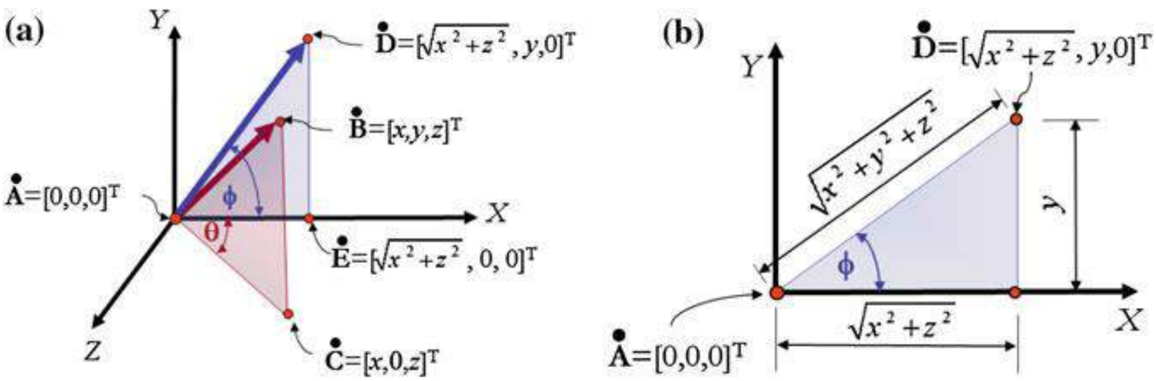
$$
\begin{aligned}
M &= M_3M_2M_1 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-30) & -\sin(-30) & 0 \\ 0 & \sin(-30) & \cos(-30) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(30) & -\sin(30) & 0 \\ 0 & \sin(30) & \cos(30) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Matrices [3 marks]
Order of multiplication [1 mark]
Final matrix [1 mark]

b) [14 marks] Derive a matrix that rotates a vector along the x-axis to coincide with another vector $[x, y, z]^T$.

Answers to Question 4b):

Example 5.7



1. Rotate through an angle φ about the z-axis [3 marks]
2. Rotate through an angle $-\theta$ about the y-axis [3 marks]

$$M_1 = R_z(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{x^2+z^2}}{\sqrt{x^2+y^2+z^2}} & -\frac{y}{\sqrt{x^2+y^2+z^2}} & 0 & 0 \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{\sqrt{x^2+z^2}}{\sqrt{x^2+y^2+z^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_2 = R_y(-\theta) = \begin{bmatrix} \cos(-\theta) & 0 & \sin(-\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-\theta) & 0 & \cos(-\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+z^2}} & 0 & -\frac{z}{\sqrt{x^2+z^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{z}{\sqrt{x^2+z^2}} & 0 & \frac{x}{\sqrt{x^2+z^2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = M_2 M_1 = R_{zy}^{-1}([x, y, z]^T) = R_y(-\theta) R_z(\phi) = \begin{bmatrix} \frac{x}{\sqrt{x^2+z^2}} & 0 & -\frac{z}{\sqrt{x^2+z^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{z}{\sqrt{x^2+z^2}} & 0 & \frac{x}{\sqrt{x^2+z^2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{x^2+z^2}}{\sqrt{x^2+y^2+z^2}} & -\frac{y}{\sqrt{x^2+y^2+z^2}} & 0 & 0 \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{\sqrt{x^2+z^2}}{\sqrt{x^2+y^2+z^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrices [6 marks]
Order of multiplication [1 mark]
Final matrix [1 mark]

Question 3 [3D Transformations]:

[17 marks] We want to rotate a point through an angle of 34.87º about a line passing through the origin and the point $[1, 1, \sqrt{3}]^T$. Determine the series of transformations (with their parameters) required. You **must start** with rotation about the *z*-axis. Also, you **must avoid** rotating about the *y*-axis.

Answers to Question 3:

Prob 5.4

Steps:

1. Rotate through 45º about the *z*-axis [Step 1 mark; parameter = 1 mark]
2. Rotate through $\tan^{-1}$(sqrt(3)/sqrt(2))º (90-50.768=39.23) about the *x*-axis[Step 1 mark; parameter = 2 marks]
3. Rotate through 34.87º about the *z*-axis [Step 1 mark; parameter = 1 mark]1
4. Rotate through -39.23 about the *x*-axis [Step, parameter = 1 mark]
5. Rotate through -45º about the *z*-axis [Step, parameter = 1 mark]

$$M_1 = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(39.23) & -\sin(39.23) \\ 0 & \sin(39.23) & \cos(39.23) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7746 & -0.6324 \\ 0 & 0.6324 & 0.7746 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \cos(34.87) & -\sin(34.87) & 0 \\ \sin(34.87) & \cos(34.87) & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.8205 & -0.5717 & 0 \\ 0.5717 & 0.8205 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(39.23) & \sin(39.23) \\ 0 & -\sin(39.23) & \cos(39.23) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7746 & 0.6324 \\ 0 & -0.6324 & 0.7746 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} \cos(45) & \sin(45) & 0 \\ -\sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrices [5 marks]

Order [2 marks] $M_5 M_4 M_3 M_2 M_1$

Final Matrix [1 mark]

$$M = \begin{bmatrix} 0.8564 & -0.4069 & 0.3179 \\ 0.4788 & 0.8564 & -0.1935 \\ -0.1935 & 0.3179 & 0.9282 \end{bmatrix}$$

b) [15 marks] Check if each of the following inhomogeneous matrices can be a rotation matrix. Mention at least one reason if a matrix cannot be a rotation matrix and at least two reasons if it may. The matrices are:

$$\mathtt{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9 & -0.2 \\ 0 & 0.2 & 0.9 \end{bmatrix} \qquad \mathtt{M}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \qquad \mathtt{M}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{10}} & 0 \end{bmatrix}$$

$$\mathtt{M}_4 = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad \mathtt{M}_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answers to Question 2 b):

Example 5.2:

1. $\mathtt{M}_1$ cannot be a rotation matrix as it is not normalized. In addition, $\det(\mathtt{M}_1) \neq +1$ and $\mathtt{M}_1^{-1} \neq \mathtt{M}_1^T$.
2. $\mathtt{M}_2$ can be a rotation matrix as it is normalized and orthogonal. In addition, $\det(\mathtt{M}_2) = +1$ and $\mathtt{M}_2^{-1} = \mathtt{M}_2^T$.
3. $\mathtt{M}_3$ cannot be a rotation matrix as it is not normalized. In addition, $\det(\mathtt{M}_3) \neq +1$ and $\mathtt{M}_3^{-1} \neq \mathtt{M}_3^T$.
4. $\mathtt{M}_4$ can be a rotation matrix as it is normalized and orthogonal. In addition, $\det(\mathtt{M}_4) = +1$ and $\mathtt{M}_4^{-1} = \mathtt{M}_4^T$.
5. $\mathtt{M}_5$ can be a rotation matrix as it is normalized and orthogonal. In addition, $\det(\mathtt{M}_5) = +1$ and $\mathtt{M}_5^{-1} = \mathtt{M}_5^T$. $\square$

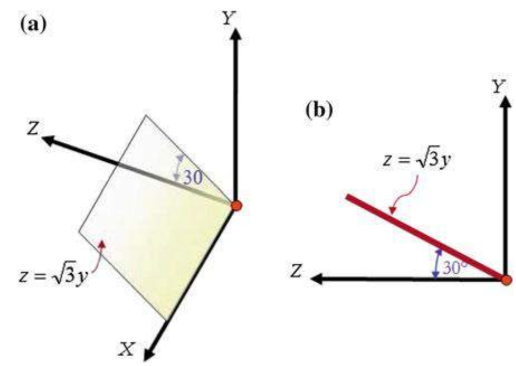
b) [10 marks] Derive a matrix that reflects a point $[x, y, z]^T$ about the plane $z = \sqrt{3}y$.

Answers to Question 2 b):

Example 5.14:

The plane is drawn.

1. Calculate the angle between the plane $z = \sqrt{3}y$ and the zx-plane. It is easy to show that this angle is 30º. [2 marks]
2. Rotate through an angle of 30º about the x-axis. [1 mark]
3. Reflect about the zx-plane. [1 mark]
4. Rotate through an angle of −30º about the x-axis. [1 mark]



$$
\begin{aligned}
M &= M_3M_2M_1 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-30) & -\sin(-30) & 0 \\ 0 & \sin(-30) & \cos(-30) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(30) & -\sin(30) & 0 \\ 0 & \sin(30) & \cos(30) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Matrices [3 marks]
Order of multiplication [1 mark]
Final matrix [1 mark]

German University in Cairo **Faculty of Media Engineering and Technology** **Winter 2020** **Duration: 40 minutes**	GUC German University in Cairo

Name: ID: Tutorial:

Question 1

Given a view plane with a normal vector [4, -3, -5] ^{T}, determine the homogenous trimetric projection matrix (onto the **yz-plane**) that can be used with it.

Solution [1]

1. Rotation about 'y'-axis with angle (-tan^{-1} (5/4) or -51.3402) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m1 = \begin{bmatrix} \cos(-tan^{-1}(\frac{5}{4})) & 0 & \sin(-tan^{-1}(\frac{5}{4})) \\ 0 & 1 & 0 \\ -\sin(-tan^{-1}(\frac{5}{4})) & 0 & \cos(-tan^{-1}(\frac{5}{4})) \end{bmatrix}$$

2. Rotation about 'z'-axis with angle ($tan^{-1}(3/\sqrt{25+16})$ or 25.1041) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m2 = \begin{bmatrix} \cos(tan^{-1}(3/\sqrt{25+16})) & -\sin(tan^{-1}(3/\sqrt{25+16})) & 0 \\ \sin(tan^{-1}(3/\sqrt{25+16})) & \cos(tan^{-1}(3/\sqrt{25+16})) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Side View Projection (1 for the side view projection)

$$m3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is: (1 for the correct sequence and 1 for the final matrix)

$$m = m3 * m2 * m1$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0.2650 & 0.9055 & -0.3313 \\ 0.7809 & 0 & 0.6247 \end{bmatrix}$$

Solution [2]

1. Rotation about 'x'-axis with angle (-tan^{-1} (5/3) or -59.0362) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m1 = \begin{bmatrix} \cos(-tan^{-1}(\frac{5}{3})) & 0 & \sin(-tan^{-1}(\frac{5}{3})) \\ 0 & 1 & 0 \\ -\sin(-tan^{-1}(\frac{5}{3})) & 0 & \cos(-tan^{-1}(\frac{5}{3})) \end{bmatrix}$$

2. Rotation about 'z'-axis with angle ($tan^{-1}(\sqrt{25+9}/4)$ or 55.5501) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m2 = \begin{bmatrix} \cos(tan^{-1}(\sqrt{25+9}/4)) & -\sin\ (tan^{-1}(\sqrt{25+9}/4)) & 0 \\ \sin(tan^{-1}(\sqrt{25+9}/4)) & \cos(tan^{-1}(\sqrt{25+9}/4)) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Side View Projection (1 for the side view projection)

$$m3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is: (1 for the correct sequence and 1 for the final matrix)

$$m = m3 * m2 * m1$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0.8246 & 0.2910 & 0.4851 \\ 0 & -0.8575 & 0.5145 \end{bmatrix}$$

Solution [3]

1. Rotation about 'x'-axis with angle (tan^{-1} (3/5) or 30.9638) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(tan^{-1}(3/5)) & -\sin\ (tan^{-1}(3/5)) \\ 0 & \sin\ (tan^{-1}(3/5)) & \cos(tan^{-1}(3/5)) \end{bmatrix}$$

2. Rotation about 'y'-axis with angle ($-tan^{-1}(\sqrt{9+25}/4)$ or -55.5501) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m2 = \begin{bmatrix} \cos(-tan^{-1}(\sqrt{9+25}/4)) & 0 & \sin\ (-tan^{-1}(\sqrt{9+25}/4)) \\ 0 & 1 & 0 \\ -\sin(-tan^{-1}(\sqrt{9+25}/4)) & 0 & \cos(-tan^{-1}(\sqrt{9+25}/4)) \end{bmatrix}$$

4. Side View Projection (1 for the side view projection)

$$m3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is: (1 for the correct sequence and 1 for the final matrix)

$m = m3 * m2 * m1$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.8575 & -0.5145 \\ 0.8246 & 0.2910 & 0.4851 \end{bmatrix}$$ M1=

Solution [4]

1. Rotation about 'z'-axis with angle (tan^{-1} (3/4) or 36.8699) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

   $$m1 = \begin{bmatrix} \cos(tan^{-1}(3/4)) & -\sin\ (tan^{-1}(3/4)) & 0 \\ \sin(tan^{-1}(3/4)) & \cos(tan^{-1}(3/4)) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Rotation about 'y'-axis with angle ($-tan^{-1}(5/\sqrt{16+9})$ or -45) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

3. $$m2 = \begin{bmatrix} \cos(-tan^{-1}(5/\sqrt{16+9})\ )) & 0 & \sin\ (-tan^{-1}(5/\sqrt{16+9})\ )\ ) \\ 0 & 1 & 0 \\ -\sin(-tan^{-1}(5/\sqrt{16+9})\ )\ ) & 0 & \cos(-tan^{-1}(5/\sqrt{16+9})\ )) \end{bmatrix}$$

5. Side View Projection (1 for the side view projection)

   $$m3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is: (1 for the correct sequence and 1 for the final matrix)

$m = m3 * m2 * m1$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0.6000 & 0.8000 & 0 \\ 0.5657 & -0.4243 & 0.7071 \end{bmatrix}$$

German University in Cairo **Faculty of Media Engineering and Technology** **Winter 2020** **Duration: 40 minutes**	GUC German University in Cairo

Name: ID: Tutorial:

Question 1

Given a view plane with a normal vector [-5, -3, 4] ^{T}, determine the homogenous trimetric projection matrix (onto the **zx-plane**) that can be used with it.

Solution [1]

1. Rotation about 'y'-axis with angle (tan^{-1} (5/4) or 51.3402) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m1 = \begin{bmatrix} \cos(tan^{-1}(\frac{5}{4})) & 0 & \sin(tan^{-1}(\frac{5}{4})) \\ 0 & 1 & 0 \\ -\sin(tan^{-1}(\frac{5}{4})) & 0 & \cos(tan^{-1}(\frac{5}{4})) \end{bmatrix}$$

2. Rotation about 'x'-axis with angle ($tan^{-1}(\sqrt{25+16}/3)$ or 64.8959) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(tan^{-1}(\frac{\sqrt{25+16}}{3})) & -\sin(tan^{-1}(\frac{\sqrt{25+16}}{3})) \\ 0 & \sin(tan^{-1}(\frac{\sqrt{25+16}}{3})) & \cos(tan^{-1}(\frac{\sqrt{25+16}}{3})) \end{bmatrix}$$

3. Top View Projection (1 for the top view projection)

$$m3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is: (1 for the correct sequence and 1 for the final matrix)

$$m = m3 * m2 * m1$$

$$= \begin{bmatrix} 0.6247 & 0 & 0.7809 \\ 0 & 0 & 0 \\ -0.3313 & 0.9055 & 0.2650 \end{bmatrix}$$

Solution [2]

1. Rotation about 'y'-axis with angle (-tan^{-1} (4/5) or - 38.6598) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m1 = \begin{bmatrix} \cos(-tan^{-1}(\frac{4}{5})) & 0 & \sin(-tan^{-1}(\frac{4}{5})) \\ 0 & 1 & 0 \\ -\sin(-tan^{-1}(\frac{4}{5})) & 0 & \cos(-tan^{-1}(\frac{4}{5})) \end{bmatrix}$$

2. Rotation about 'z'-axis with angle ($tan^{-1}(\sqrt{25+16}/3)$ or 64.8959) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m2 = \begin{bmatrix} \cos(tan^{-1}(\frac{\sqrt{25+16}}{3})) & -\sin\ (tan^{-1}(\frac{\sqrt{25+16}}{3})) & 0 \\ \sin(tan^{-1}(\frac{\sqrt{25+16}}{3})) & \cos(tan^{-1}(\frac{\sqrt{25+16}}{3})) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Top View Projection (1 for the top view projection)

$$m3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is: (1 for the correct sequence and 1 for the final matrix)

$m = m3 * m2 * m1$

$$= \begin{bmatrix} 0.3313 & -0.9055 & -0.2650 \\ 0 & 0 & 0 \\ 0.6247 & 0 & 0.7809 \end{bmatrix}$$

Solution [3]

1. Rotation about 'x'-axis with angle (tan^{-1} (4/3) or 53.1301) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(tan^{-1}(\frac{4}{3})) & -\sin\ (tan^{-1}(\frac{4}{3})) \\ 0 & \sin\ (tan^{-1}(\frac{4}{3})) & \cos(tan^{-1}(\frac{4}{3})) \end{bmatrix}$$

2. Rotation about 'z'-axis with angle ($tan^{-1}(5/\sqrt{9+16})$ or 45) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m2 = \begin{bmatrix} \cos(tan^{-1}(5/\sqrt{9+16})) & -\sin\ (tan^{-1}(5/\sqrt{9+16})) & 0 \\ \sin(tan^{-1}(5/\sqrt{9+16})) & \cos(tan^{-1}(5/\sqrt{9+16})) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Top View Projection (1 for the top view projection)

$$m3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is: (1 for the correct sequence and 1 for the final matrix)

$m = m3 * m2 * m1$

$$= \begin{bmatrix} 0.7071 & -0.4243 & 0.5657 \\ 0 & 0 & 0 \\ 0 & 0.8000 & 0.6000 \end{bmatrix}$$

Solution [4]

1. Rotation about 'z'-axis with angle (tan^{-1} (5/3) or 59.0362) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m1 = \begin{bmatrix} \cos(tan^{-1}(\frac{5}{3})) & -\sin\ (tan^{-1}(\frac{5}{3})) & 0 \\ \sin(tan^{-1}(\frac{5}{3})) & \cos(tan^{-1}(\frac{5}{3})) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Rotation about 'x'-axis with angle ($tan^{-1}(4/\sqrt{25+9})$ or 34.4499) (1 for the rotation, 1 for the axis, 0.5 for the angle and 0.5 for the sign)

$$m2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(tan^{-1}(4/\sqrt{25+9})) & -\sin\ (tan^{-1}(4/\sqrt{25+9})) \\ 0 & \sin(tan^{-1}(4/\sqrt{25+9})) & \cos(tan^{-1}(4/\sqrt{25+9})) \end{bmatrix}$$

3. Top View Projection (1 for the top view projection)

$$m3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final matrix is: (1 for the correct sequence and 1 for the final matrix)

$m = m3 * m2 * m1$

$$= \begin{bmatrix} 0.5145 & -0.8575 & 0 \\ 0 & 0 & 0 \\ 0.4851 & 0.2910 & 0.8246 \end{bmatrix}$$

Question 5 [Projections]:

[21 points] A 3D world coordinate system is defined by:

1. An origin at $[0,0,0]^T$.
2. The x-axis, which is indicated by the unit vector $[1,0,0]^T$.
3. The y-axis, which is indicated by the unit vector $[0,1,0]^T$.
4. The z-axis, which is indicated by the unit vector $[0,0,1]^T$.

A new coordinate system, having the axes u, v and n, is to be defined in terms of the old one. The information available is:

1. The new origin is located at $[4,5,6]^T$ with respect to the old system.
2. The n-axis is indicated by the vector $[5,7,-5]^T$.

If there is a 3D point located at $[5,6,7]^T$ in terms of the old system, specify its location in terms of the new coordinate system.

Answers to Question 5:

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} \frac{x}{\|\mathbf{n}\|} \\ \frac{y}{\|\mathbf{n}\|} \\ \frac{z}{\|\mathbf{n}\|} \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{25+49+25}} \\ \frac{7}{\sqrt{25+49+25}} \\ \frac{-5}{\sqrt{x^2+y^2+z^2}} \end{bmatrix} = \begin{bmatrix} \frac{5}{9.95} \\ \frac{7}{9.95} \\ \frac{-5}{9.95} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.7 \\ -0.5 \end{bmatrix}$$ (3 points)

$$\mathbf{up'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0.5 \\ 0.7 \\ -0.5 \end{bmatrix} \right) \begin{bmatrix} 0.5 \\ 0.7 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0.7 \begin{bmatrix} 0.5 \\ 0.7 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -0.35 \\ 0.51 \\ 0.35 \end{bmatrix}$$ (3 points)

$$\mathbf{v} = \begin{bmatrix} \frac{-0.35}{\sqrt{0.1225+0.2601+0.1225}} \\ \frac{0.51}{\sqrt{0.1225+0.2601+0.1225}} \\ \frac{0.35}{\sqrt{0.1225+0.2601+0.1225}} \end{bmatrix} = \begin{bmatrix} \frac{-0.35}{0.71} \\ \frac{0.51}{0.71} \\ \frac{0.35}{0.71} \end{bmatrix} = \begin{bmatrix} -0.49 \\ 0.72 \\ 0.49 \end{bmatrix}$$ (3 points)

$$\mathbf{u} = \mathbf{v} \times \mathbf{n} = \begin{bmatrix} 0 & -0.49 & 0.72 \\ 0.49 & 0 & 0.49 \\ -0.72 & -0.49 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.7 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -0.7 \\ 0 \\ -0.7 \end{bmatrix}$$ (3 points)

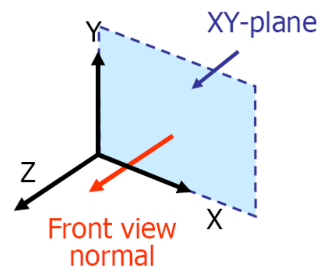
$$u = ([5,6,7]^T - [4,5,6]^T) \bullet [-0.7,0,-0.7]^T = -1.4$$ (3 points)

$$v = ([5,6,7]^T - [4,5,6]^T) \bullet [-0.49,0.72,0.49]^T = 0.72$$ (3 points)

$$n = ([5,6,7]^T - [4,5,6]^T) \bullet [0.5,0.7,-0.5]^T = 0.7$$ (3 points)

Question 4 (Projections):

[15 points] The front view is an orthographic projection onto the *xy*-plane. In this projection, the projectors are parallel to the *z*-axis.



a) An image whose origin is its lower left corner can be created for this front view. Assume that each unit in the world coordinate system (XYZ system) can be represented by one pixel in the image produced. Also, assume that all points in 3D space have positive *x*- and *y*-coordinates. Write down a front view projection matrix that is used to find the projection of any 3D point (e.g., $P=[1,2,3]^T$) as a 2D image point (e.g., $p=[1,2]^T$).

b) If a 3D point (e.g., $P=[1,2,3]^T$) is to be projected onto the *xy*-plane as another 3D point whose z-coordinate is 0 (i.e., $P'=[1,2,0]^T$), write down a matrix that can be used to perform this transformation.

c) If the projectors are not parallel to the *z*-axis, oblique projection can be created. In Cavalier projection with $\alpha=45°$, the lines parallel to the z-axis maintain their actual lengths. In case of Cabinet projection with $\alpha=63.4°$, those lines are displayed at a ratio of 0.5 of their lengths. What would be the ratio if $\alpha=30°$?

Answers to Question 4:

a) Projection as a 2D point (6 points)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

b) Projection as a 3D point (6 points)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Or

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

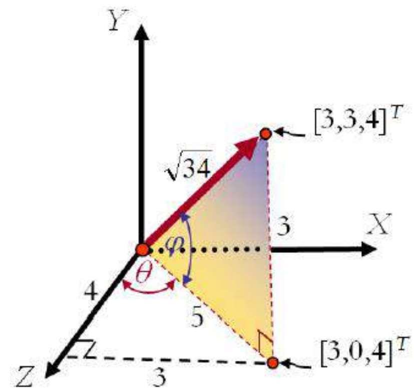
c) Ratio: cot (30) = sqrt(3) (3 points)

Question 5: [Projections]

[12 marks] Given a view plane with a normal vector $[3, 3, 4]^T$, determine the homogeneous dimetric projection matrix (onto the *xy*-plane) that can be used with it.

Answers to Question 5:

Example 8.6



The azimuth angle [4 marks]

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.8699°$$

The elevation angle [4 marks]

$$\varphi = \tan^{-1}\left(\frac{3}{5}\right) = 30.9638°$$

The matrix [4 marks]

$$\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ -\sin(\varphi)\sin(\theta) & \cos(\varphi) & -\sin(\varphi)\cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} & 0 \\ -\frac{9}{5\sqrt{34}} & \frac{5}{\sqrt{34}} & -\frac{12}{5\sqrt{34}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 4 [Projections]:

[12 marks] If the center of projection is placed at an arbitrary point $[x, y, z]^T$, the normal to the view plane, which is perpendicular to the y-axis, makes an angle θ with the z-axis and the distance from the origin to the view plane is d, derive the two-point projection matrix in this case.

Answers to Question 4:

Example 8.18: The matrix is obtained in three steps.

1. Translate the center of projection using the translation vector $[-x,-y,-z]^T$ so it coincides with the origin. Let us apply the matrix M1 where

$$\mathrm{M}_1 = \mathrm{T}([-x,-y,-z]^T) = \begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [3 marks]

At this point, make sure that the view plane does not flip to another octant and the perpendicular distance d remains unchanged.

2. Perform a two-point perspective projection. Let us apply the matrix M2 where

$$\mathrm{M}_2 = \mathrm{P}_{per2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & 0 \end{bmatrix}$$ [3 marks]

3. Translate back using the translation vector $[x, y, z]T$. Let us apply the matrix M3 where

$$\mathrm{M}_3 = \mathrm{T}([x,y,z]^T) = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [3 marks]

Thus, the overall projection is estimated as

$$\mathrm{P}_{per2} = \mathrm{M}_3\mathrm{M}_2\mathrm{M}_1$$

$$= \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\frac{x\sin(\theta)}{d} & 0 & \frac{x\cos(\theta)}{d} & \frac{-x^2\sin(\theta)-zx\cos(\theta)}{d} - x \\ \frac{y\sin(\theta)}{d} & 1 & \frac{y\cos(\theta)}{d} & \frac{-xy\sin(\theta)-zy\cos(\theta)}{d} - y \\ \frac{z\sin(\theta)}{d} & 0 & 1+\frac{z\cos(\theta)}{d} & \frac{-xz\sin(\theta)-z^2\cos(\theta)}{d} - z \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & \frac{-x\sin(\theta)-z\cos(\theta)}{d} \end{bmatrix}$$ [order and final matrix 3 marks]

Question 3 [Projections]:

a) [20 marks] Consider a unit cube with two of its corners placed at $[0, 0, 0]^T$ and $[1, 1, 1]^T$. If a one-point perspective projection is to be performed onto the *xy*-plane where the COP is at $[0.5, 0.5, -1]^T$, derive the projection matrix. Use this matrix to get the projected location of the corner $[1, 0, 1]^T$.

Answers to Question 3 a):

Example 8.15: Modified

1. Translate using a translation vector $\left[-\frac{1}{2}, -\frac{1}{2}, 0\right]^T$. This is using the matrix M_1 where

$$M_1 = T\left(\left[-\frac{1}{2}, -\frac{1}{2}, 0\right]^T\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

[Step: 1 mark, parameters: 2 marks, matrix: 1 mark]

2. Use the one-point projection matrix where d=1 (COP = $[0,0,-1]^T$) to project onto the *xy*-plane

$$M_2 = P'_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \rightarrow \quad M_2 = P'_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

[Step: 1 mark, matrix: 2 marks]

3. Translate using a translation vector $\left[\frac{1}{2}, \frac{1}{2}, 0\right]^T$. This is using the matrix M_3 where

$$M_3 = T\left(\left[\frac{1}{2}, \frac{1}{2}, 0\right]^T\right) = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

[Step: 1 mark, parameters: 2 marks, matrix: 1 mark]

$$M = M_3 M_2 M_1 = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

[Step: 1 mark, final matrix: 2 marks]

$$P = M\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0.75 \\ 0.25 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0.75 \\ 0.25 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

[Homogeneous point: 2 marks]
[Division by 2: 2 marks]
[Inhomogeneous point: 1 mark]
[x, y when z=0: 1 mark]

b) [14 marks] The following is a one-point perspective projection matrix that is used to project onto the *xy*-plane.

$$M_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Use M_{per} to find the location of the vanishing point on the *xy*-plane.

Answers to Question 3 b):

Alternative 1: We can use a point at infinity on the z-axis $[0, 0, 1, 0]^T$. If we multiply this point by the matrix above, it will give the location of the vanishing point.

$$P = M_{per} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Alternative 2: We should find two projected lines that are parallel to the *z*-axis. Then the intersection between those two lines will be the location of the vanishing point.

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$ [2 marks]

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ [2 marks]

The first line ➔ $\begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix}$ [2 marks]

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$ [2 marks]

$$P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ [2 marks]

The second line ➔ $\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0 \\ 0 \end{bmatrix}$ [2 marks]

The vanishing point ➔ $\begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix} \times \begin{bmatrix} -0.5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.25 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ [2 marks]

Question 3 [Projections]:

[15 marks] If the center of projection is placed at an arbitrary point $[x, y, z]^T$, the normal to the view plane, which is perpendicular to the y-axis, makes an angle θ with the z-axis and the distance from the origin to the view plane is d, derive the two-point projection matrix in this case.

Answers to Question 3:

Example 8.18:

1. Translate the center of projection using the translation vector $[-x,-y,-z]^T$

$$\mathrm{M}_1 = \mathrm{T}([-x,-y,-z]^T) = \begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[Step: 1 mark; factors: 2 marks; matrix: 1 mark]

2. Perform a two-point perspective projection

$$\mathrm{M}_2 = \mathrm{P}_{per2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & 0 \end{bmatrix}$$

[Step: 1 mark; factors: 2 marks; matrix: 1 mark]

3. Translate back using the translation vector $[x, y, z]^T$

$$\mathrm{M}_3 = \mathrm{T}([x,y,z]^T) = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[Step: 1 mark; factors: 2 marks; matrix: 1 mark]

$$\begin{aligned}
\mathrm{P}_{per2} &= \mathrm{M}_3\mathrm{M}_2\mathrm{M}_1 \\
&= \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1+\frac{x\sin(\theta)}{d} & 0 & \frac{x\cos(\theta)}{d} & \frac{-x^2\sin(\theta)-zx\cos(\theta)}{d} - x \\ \frac{y\sin(\theta)}{d} & 1 & \frac{y\cos(\theta)}{d} & \frac{-xy\sin(\theta)-zy\cos(\theta)}{d} - y \\ \frac{z\sin(\theta)}{d} & 0 & 1+\frac{z\cos(\theta)}{d} & \frac{-xz\sin(\theta)-z^2\cos(\theta)}{d} - z \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & \frac{-x\sin(\theta)-z\cos(\theta)}{d} \end{bmatrix}
\end{aligned}$$

[Step: 1 mark; order: 1 mark; final matrix: 1 mark]

Question 2

Using left-handed view reference coordinate system and the right-handed world coordinate. Assume that $[-1.4142, 0.7107, 0.7035]^T$ is the location of a point in the world coordinates, if the VRP is located at $[1, 4, 5],^T$ the n-axis is indicated by the vector $[-0.7071, 0, -0.7071]^T$ and the u-axis is indicated by the vector $[5, 7, -5]^T$, express the location of a point defined in view reference coordinate system.

1. Normalize u-axis (1 the rule of normalization and 1 mark for the result)
   $[0.5025, 0.7035, -0.5025]^T$
2. Get v-axis using the following formula (1 the rule of cross product and 1 mark for the result)
   $v = u \times n$
   $v = [-0.4975, 0.7107, 0.4975]^T$
3. Get "Q" using the following formula(1 the rule of Q and 1 mark for the result)
   $$Q = \begin{bmatrix} u^T \\ v^T \\ n^T \end{bmatrix} [P - R]$$
   $Q = [-1.3682, -3.2741, 4.7451]^T$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} 0 \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} x' \\ 0 \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ -\sin(\varphi)\sin(\theta) & \cos(\varphi) & -\sin(\varphi)\cos(\theta) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cos(\phi) \\ 0 & 1 & \sin(\phi) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2}\cos(\phi) \\ 0 & 1 & \frac{1}{2}\sin(\phi) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ \frac{z}{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ 0 \\ \frac{z+d}{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & 0 \end{bmatrix} \qquad \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$|proj_n\, up|$ = **up ● n** v = $\dfrac{up'}{\|up'\|}$

$$\dot{\mathbf{Q}} = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{n}^T \end{bmatrix} \left[\dot{\mathbf{P}} - \dot{\mathbf{R}}\right].$$

up' = up – (up ● n) n u = v **x** n (RHCS)

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$\mathbf{u} \times \mathbf{v} = [\mathbf{u}]_\times \mathbf{v} =$$

$$\begin{bmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

Question 2

Using left-handed view reference coordinate system and the right-handed world coordinate system. Assume that $[-1.4142, 0.7107, 0.7035]^T$ is the location of a point defined in view reference coordinate system. In world coordinate system, if the VRP is located at $[1, 4, 5]$,^{T} the n-axis is indicated by the vector $[-0.3333, \quad 0.6667, \quad -0.3333]^T$ and the u-axis is indicated by the vector $[0.7071, 0, -0.7071]^T$, express the location "P" of the point in the world coordinates.

1. Normalize n-axis (1 the rule of normalization and 1 mark for the result)
   $[-0.4082, 0.8165, -0.4082]^T$
2. Get v-axis using the following formula (1 the rule of cross product and 1 mark for the result)
   $v = u \times n$
   $v = [0.5774, 0.5774, 0.5774]^T$
3. Get "P" using the following formula(1 the rule of p and 1 mark for the result)
   $$p = \left[\begin{bmatrix} u^T \\ v^T \\ n^T \end{bmatrix}^{-1} . Q \right] + R$$
   $p = [0.1232, 4.9848, 6.1232]^T$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} 0 \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} x' \\ 0 \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ -\sin(\varphi)\sin(\theta) & \cos(\varphi) & -\sin(\varphi)\cos(\theta) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cos(\phi) \\ 0 & 1 & \sin(\phi) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2}\cos(\phi) \\ 0 & 1 & \frac{1}{2}\sin(\phi) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ \frac{z}{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ 0 \\ \frac{z+d}{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & 0 \end{bmatrix} \qquad \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$|proj_n\, up|$ = **up ● n** v = $\frac{up'}{\|up'\|}$

$$\dot{\mathbf{Q}} = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{n}^T \end{bmatrix} \left[\dot{\mathbf{P}} - \dot{\mathbf{R}}\right].$$

up' = up – (up ● n) n u = v **x** n (RHCS)

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \qquad \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$\mathbf{u} \times \mathbf{v} = [\mathbf{u}]_\times \mathbf{v} =$$

$$\begin{bmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

<u>Question 6:</u> (22 Marks)

a) [12 marks] The view reference coordinate system is determined by the view reference point and the unit vectors $\mathbf{u}=[x_u, y_u, z_u]^T$, $\mathbf{v}=[x_v, y_v, z_v]^T$ and $\mathbf{n}=[x_n, y_n, z_n]^T$. Show that the homogeneous transformation matrix from the world coordinate to the view reference coordinate system is given by

$$\begin{bmatrix} & \mathbf{u}^T & & -\mathbf{u}\bullet\dot{\mathbf{R}} \\ & \mathbf{v}^T & & -\mathbf{v}\bullet\dot{\mathbf{R}} \\ & \mathbf{n}^T & & -\mathbf{n}\bullet\dot{\mathbf{R}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ where $\dot{\mathbf{R}}$ is the view reference point (VRP) and $\bullet$ is the dot product.

<u>Solution</u>

Steps:

1. Translate so that the origin coincides with the VRP. [step: 2 marks]

$$\mathtt{M}_1 = \mathtt{T}(-\dot{\mathbf{R}}) = \begin{bmatrix} 1 & 0 & 0 & -x_r \\ 0 & 1 & 0 & -y_r \\ 0 & 0 & 1 & -z_r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [matrix: 2 marks]

2. Transform to the VRC system using the matrix [step: 2 marks]

$$\mathtt{M}_2 = \mathtt{R}_{VRC} = \begin{bmatrix} & & & 0 \\ & \dot{\mathtt{R}}_{VRC} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ x_n & y_n & z_n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [matrix: 2 marks]

The overall transformation can then be applied as [order of multiplication: 2 marks]

$$\begin{aligned} \mathtt{M} &= \mathtt{M}_2\mathtt{M}_1 \\ &= \begin{bmatrix} x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ x_n & y_n & z_n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_r \\ 0 & 1 & 0 & -y_r \\ 0 & 0 & 1 & -z_r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_u & y_u & z_u & -x_u x_r - y_u y_r - z_u z_r \\ x_v & y_v & z_v & -x_v x_r - y_v y_r - z_v z_r \\ x_n & y_n & z_n & -x_n x_r - y_n y_r - z_n z_r \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} & \mathbf{u}^T & & -\mathbf{u}\bullet\dot{\mathbf{R}} \\ & \mathbf{v}^T & & -\mathbf{v}\bullet\dot{\mathbf{R}} \\ & \mathbf{n}^T & & -\mathbf{n}\bullet\dot{\mathbf{R}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

[final matrix: 2 marks]

b) [10 marks] The homogeneous matrix to transform a homogeneous point **Q** (expressed in the view reference coordinate system) to a homogeneous point **P** (expressed in the world coordinate) is given by

$$\mathbf{P} = \begin{bmatrix} & \mathbf{u}^T & & -\mathbf{u} \bullet \dot{\mathbf{R}} \\ & \mathbf{v}^T & & -\mathbf{v} \bullet \dot{\mathbf{R}} \\ & \mathbf{n}^T & & -\mathbf{n} \bullet \dot{\mathbf{R}} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \mathbf{Q}$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{n}$, $\dot{\mathbf{R}}$ and $\bullet$ are as defined in part a). Write a single equation to express the same operation in inhomogeneous coordinates.

Solution

Steps (reverse the previous steps):

1. Transform using the matrix $\begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{n}^T \end{bmatrix}^{-1}$ [step: 3 marks]

2. Translate using the vector $\dot{\mathbf{R}}$. [step: 3 marks]

The overall operation is expressed as [4 marks]

$$\dot{\mathrm{P}} = \left[\begin{bmatrix} \mathrm{u}^T \\ \mathrm{v}^T \\ \mathrm{n}^T \end{bmatrix}^{-1} \dot{\mathrm{Q}} \right] + \dot{\mathrm{R}}$$

<u>Question 6:</u> [View Transformations]

a) [10 marks] The unit vectors **u**, **v** and **n** represent the axes of the view reference coordinate system. The sequence of obtaining these vectors is **n** followed by **v** (using the **up** vector) and finally **u**. Explain how to obtain the vectors **n** followed by **u** (using the **up** vector) and finally **v**.

Answers to Question 6a):

Example 9.6

1. Get the vector **n** and normalize it. [2 marks]

2. Get the vector **u**: We know that the vectors **up** = $[0, 1, 0]^T$, **v** and **n** originate from the VRP and reside on the same plane that is perpendicular to the vector **u**. Thus, normalized **u** vector is determined as

$$\mathbf{u} = \frac{\mathbf{up} \times \mathbf{n}}{\|\mathbf{up} \times \mathbf{n}\|}$$

where × and ‖ .‖ denote the cross product and the norm respectively.
[Idea 3 marks; cross product 2 marks; normalization 1 mark]

3. Get the vector **v** as the cross product of the previous two vectors. Thus, [2 marks]

$$\mathbf{v} = \mathbf{n} \times \mathbf{u}$$

b) [10 marks] A 3D point $\dot{\mathrm{P}} = [x, y, z]^T$ previously expressed in the world coordinate system can be expressed in the view reference coordinate system as $[u, v, n]^T$ using the following inhomogeneous equation.

$$\begin{bmatrix} u \\ v \\ n \end{bmatrix} = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{n}^T \end{bmatrix} \left[\dot{\mathrm{P}} - \dot{\mathrm{R}}\right] = \underbrace{\begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_n & y_n & z_n \end{bmatrix}}_{\dot{\mathrm{R}}_{VRC}} \left[\dot{\mathrm{P}} - \dot{\mathrm{R}}\right]$$

where

- $\dot{\mathrm{R}}$ is the view reference point;
- $\dot{\mathrm{R}}_{VRC}$ is a 3×3 matrix composed of **u**, **v** and **n**; and
- **u**, **v** and **n** are unit vectors along the *u*-, *v*- and *n*-directions.

Answers to Question 6b):

Example 9.12

Steps 4 marks
Matrices 2 marks
Multiplication order 2 marks
Result 2 marks

1. Translate point $\dot{\mathbf{P}}$ so that the origin coincides with $\dot{\mathbf{R}}$. This is done using the translation matrix M_1 where

$$\mathrm{M}_1 = \mathrm{T}(-\dot{\mathbf{R}}) = \begin{bmatrix} 1 & 0 & 0 & -x_r \\ 0 & 1 & 0 & -y_r \\ 0 & 0 & 1 & -z_r \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

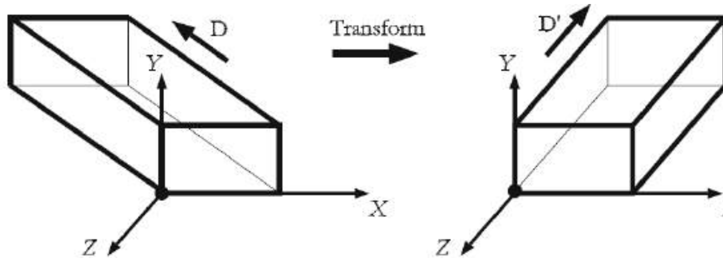
2. Transform to the new coordinate system using the matrix

$$\mathrm{M}_2 = \mathrm{R}_{VRC} = \begin{bmatrix} & & & 0 \\ & \dot{\mathrm{R}}_{VRC} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ x_n & y_n & z_n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (9.11)$$

$$\mathrm{M} = \mathrm{M}_2\mathrm{M}_1 = \begin{bmatrix} x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ x_n & y_n & z_n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_r \\ 0 & 1 & 0 & -y_r \\ 0 & 0 & 1 & -z_r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_u & y_u & z_u & -x_ux_r - y_uy_r - z_uz_r \\ x_v & y_v & z_v & -x_vx_r - y_vy_r - z_vz_r \\ x_n & y_n & z_n & -x_nx_r - y_ny_r - z_nz_r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{u}^T & & -\mathbf{u}\cdot\dot{\mathbf{R}} \\ & \mathbf{v}^T & & -\mathbf{v}\cdot\dot{\mathbf{R}} \\ & \mathbf{n}^T & & -\mathbf{n}\cdot\dot{\mathbf{R}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 5 [View Transformations]:

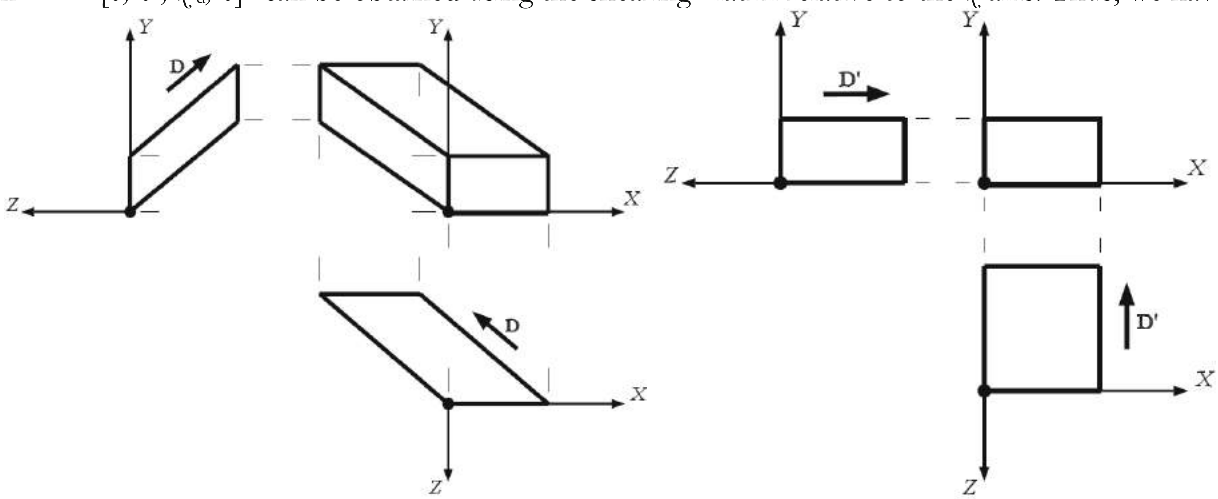
[16 marks] The oblique (left) view volume shown with the direction of projection **D** = $[x_d, y_d, z_d, 0]^T$ is to be sheared so that it takes the orthographic shape (right), which has the direction of projection **D'** = $[0, 0, z'_d, 0]^T$.



a) [10 marks] Derive the transformation matrix required.

Answers to Question 5 a):

Example 9.22:
The left figure shows the oblique view volume as multi-view projections where the direction of projection is given by the vector **D** = $[x_d, y_d, z_d, 0]^T$. The orthographic view volume (the right figure) and the new direction of projection **D'** = $[0, 0, z'_d, 0]^T$ can be obtained using the shearing matrix relative to the z-axis. Thus, we have



$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ z'_d \\ 0 \end{bmatrix}}_{\mathbf{D}'} = \underbrace{\begin{bmatrix} 1 & 0 & sh_{zx} & 0 \\ 0 & 1 & sh_{zy} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathsf{Sh}_z(sh_{zx}, sh_{zy})} \underbrace{\begin{bmatrix} x_d \\ y_d \\ z_d \\ 0 \end{bmatrix}}_{\mathbf{D}}$$ [3 marks]

Note that $z'_d = z_d$. Also

$$x_d + sh_{zx} z_d = 0,$$
$$y_d + sh_{zy} z_d = 0.$$ [4 marks]

Therefore,

$$sh_{zx} = -\frac{x_d}{z_d},$$
$$sh_{zy} = -\frac{y_d}{z_d}.$$ [2 marks]

$$\mathsf{Sh}_z\left(-\frac{x_d}{z_d}, -\frac{y_d}{z_d}\right) = \begin{bmatrix} 1 & 0 & -\frac{x_d}{z_d} & 0 \\ 0 & 1 & -\frac{y_d}{z_d} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [1 mark]

b) [6 marks] If the original direction of projection is $\mathbf{D} = [0, 0, z_d, 0]^T$, determine the change in the previous transformation matrix.

Answers to Question 5 b):

Example 9.23:
In this case, the original situation is an orthographic view volume and the parameters of the shearing matrix are

$$\frac{x_d}{z_d} = \frac{y_d}{z_d} = 0$$ [4 marks]

Thus, the shearing matrix turns to an identity matrix:

$$\mathtt{Sh}_z(0, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathtt{I}$$ [2 marks]

This means that the shearing operation has no effect on the view volume in orthographic case.

Question 4 [View Transformations]:

[10 marks] The unit vectors **u**, **v** and **n** represent the axes of the view reference coordinate system. The sequence of obtaining these vectors is **n** followed by **v** (using the **up** vector) and finally **u**. Explain how to obtain the vectors **n** followed by **u** (using the **up** vector) and finally **v**.

Answers to Question 4:

Example 9.6

1. Get the vector **n** and normalize it. [2 marks]

2. Get the vector **u**: We know that the vectors **up** $= [0, 1, 0]^T$, **v** and **n** originate from the VRP and reside on the same plane that is perpendicular to the vector **u**. Thus, normalized **u** vector is determined as

$$\mathbf{u} = \frac{\mathbf{up} \times \mathbf{n}}{\|\mathbf{up} \times \mathbf{n}\|}$$

where $\times$ and $\|.\|$ denote the cross product and the norm respectively.
[Idea 3 marks; cross product 2 marks; normalization 1 mark]

3. Get the vector **v** as the cross product of the previous two vectors. Thus, [2 marks]

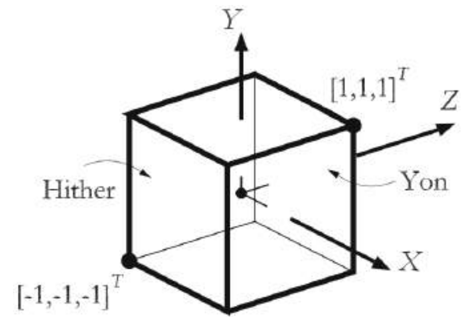
$$\mathbf{v} = \mathbf{n} \times \mathbf{u}$$

Question 4 [View Transformation]:

[15 marks] Assuming that the view reference coordinate system coincides with the right-handed world coordinate system, consider a view volume with planes parallel to the principal planes and passing through the points $[x_{min}, y_{min}, F]^T$ and $[x_{max}, y_{max}, B]^T$.

Derive a matrix to transform this view volume into the shown **left-handed** canonical parallel-projection view volume expressed by the following equations:

$$x = -1$$
$$x = 1$$
$$y = -1$$
$$y = 1$$
$$z = -1$$
$$z = 1$$



Answers to Question 4:

Example 9.21

1. Switch to left-handed coordinate system. This is done by switching the z-component

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [Step: 1 mark; matrix: 1 mark]

2. Translate the volume so that its center is at the origin. The coordinates of the center point is $\left[\frac{x_{max}+x_{min}}{2}, \frac{y_{max}+y_{min}}{2}, \frac{F+B}{2}\right]^T$ Thus, the translation is done using the translation vector $\left[-\frac{x_{max}+x_{min}}{2}, -\frac{y_{max}+y_{min}}{2}, -\frac{F+B}{2}\right]^T$ and the translation matrix M_2 where

$$M_2 = T\left(\left[-\frac{x_{max}+x_{min}}{2}, -\frac{y_{max}+y_{min}}{2}, -\frac{F+B}{2}\right]^T\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{x_{max}+x_{min}}{2} \\ 0 & 1 & 0 & -\frac{y_{max}+y_{min}}{2} \\ 0 & 0 & 1 & -\frac{F+B}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[Step: 1 mark; factors: 3 marks; matrix: 1 mark]

3. Scale the volume using the scaling factors $\frac{2}{x_{max}-x_{min}}, \frac{2}{y_{max}-y_{min}}$ and $\frac{2}{F-B}$

$$M_3 = S\left(\frac{2}{x_{max}-x_{min}}, \frac{2}{y_{max}-y_{min}}, \frac{2}{F-B}\right) = \begin{bmatrix} \frac{2}{x_{max}-x_{min}} & 0 & 0 & 0 \\ 0 & \frac{2}{y_{max}-y_{min}} & 0 & 0 \\ 0 & 0 & \frac{2}{F-B} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[Step: 1 mark; factors: 3 marks; matrix: 1 mark]

Answers to Question 4 (cont.):

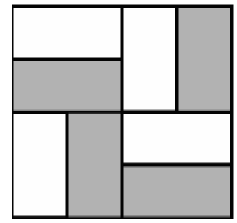
Therefore, the overall transformation is calculated as

$$\mathtt{M} = \mathtt{M}_3\mathtt{M}_2\mathtt{M}_1 = \begin{bmatrix} \frac{2}{x_{max} - x_{min}} & 0 & 0 & -\frac{x_{max} + x_{min}}{x_{max} - x_{min}} \\ 0 & \frac{2}{y_{max} - y_{min}} & 0 & -\frac{y_{max} + y_{min}}{y_{max} - y_{min}} \\ 0 & 0 & -\frac{2}{F - B} & -\frac{F + B}{F - B} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[Step: 1 mark; order: 1 mark; matrix: 1 mark]

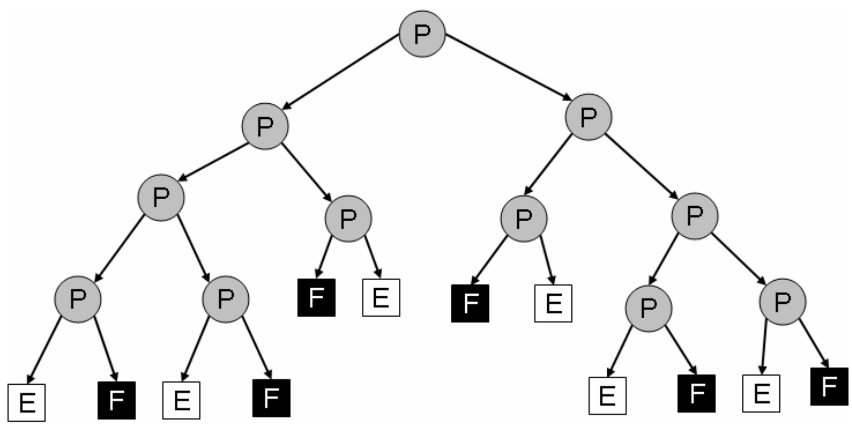
Question 6 [Hidden Surface Removal]:

[11 points] Construct the complete BSP tree for this 2D pattern. The splitting is to be done horizontally as $\boxed{\begin{array}{c}1\\0\end{array}}$ then vertically as $\boxed{0\,|\,1}$.



Answers to Question 6:

BSP tree



5 levels (5 points)
12 leaves (6 points)

Question 5 (Hidden Surface Removal):

[10 points] Assume that a virtual camera is located at $[2,4,6]^T$ and pointed towards the point [7,8,9]. Determine whether or not the planar surface with a surface normal $n=[-2,5,5]^T$ is backfacing with respect to this virtual camera.

Answers to Question 5:

$$\mathbf{v} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

$$\mathbf{n} \bullet \mathbf{v} = \begin{bmatrix} -2 \\ 5 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = +25$$

Positive result ➔ backfacing

(4+4+2)

Question 4: (Visibility)

[6 marks] The algorithm below assumes that the coordinate system used is right-handed. What change(s) should be made to the algorithm to accommodate left-handed coordinate systems?

Algorithm

Input: $x_{min}, x_{max}, y_{min}, y_{max}, polygon\_list$
Output: $zBuffer$
1: **for** $(x = x_{min}$ to $x_{max})$ **do**
2:     **for** $(x = y_{min}$ to $y_{max})$ **do**
3:         $zBuffer(x,y) = 0$
4:     **end for**
5: **end for**
6: **for** (each polygon in $polygon\_list$) **do**
7:     **for** (each pixel $[x,y]^T$ in the polygon) **do**
8:         $z = z$-value at $[x,y]^T$
9:         **if** $(z \geq zBuffer(x,y))$ **then**
10:             $zBuffer(x,y) = z$
11:         **end if**
12:     **end for**
13: **end for**

end

Solutions

In a right-handed coordinate system, the depth values (i.e., the *z*-values) increases outwards the screen (if represented by the *xy*-plane). In case of using a left-handed system, the depth values (i.e., the *z*-values) should increase inwards the screen. Consequently, the condition appearing on Line 9 should change to

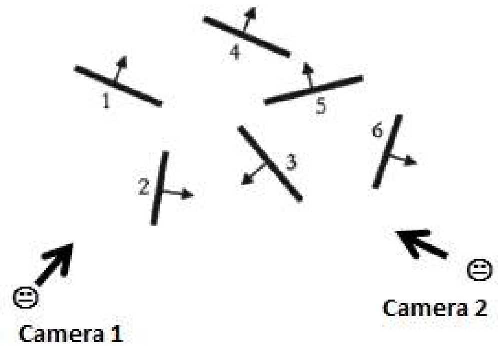
$z \leq zBuffer(x,y)$. [3 marks]

Notice that in case of a left-handed system, the *z*-buffer should be initialized to the largest possible value rather than 0 on Line 3. [3 marks]

Question 6 [Visibility]:

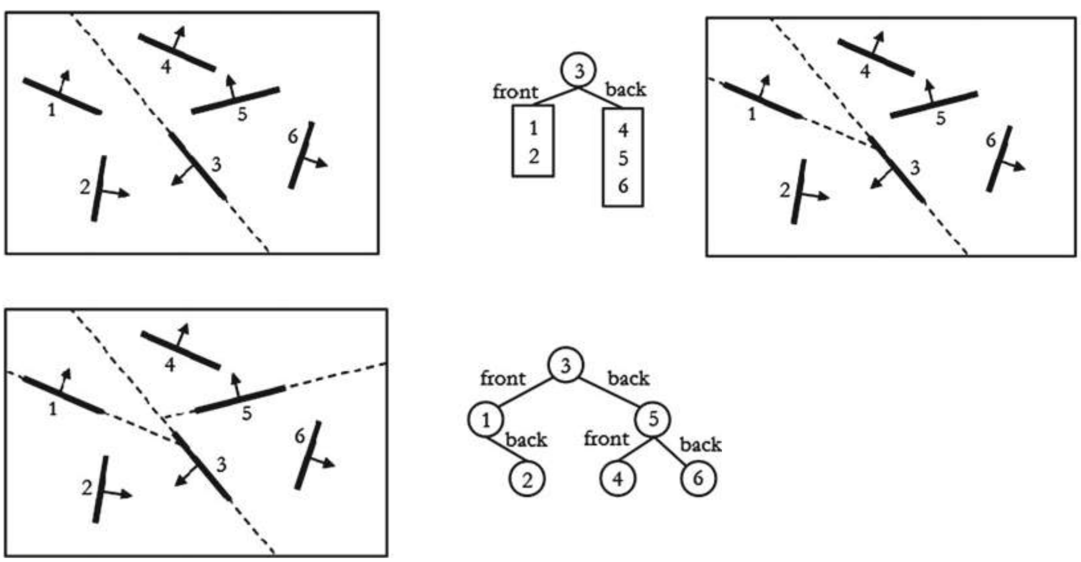
[15 marks]

a) [9 marks] The figure shows a top view of a 3D scene that contains six polygons/surfaces where the normal vectors to those surfaces are shown as arrows. It is required to partition this scene using a BSP tree. Use polygon "3" as a root for the tree. **Draw all the steps** of constructing the tree.



Answers to Question 6 a):

Example 10.7:



Each step [3 marks]

b) [6 marks] Using the BSP tree, determine the order of displaying the polygons of the scene according to the location of the two cameras as well as the viewing direction as indicated.

Answers to Question 6 b):

Camera 1: 4, 5, 6, 3, 1, 2 [3 marks]

Camera 2: 1, 2, 3, 4, 5, 6 [3 marks]

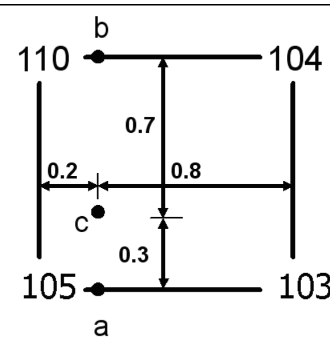
Question 6 (Mapping Techniques):

[10 points] Shown are the intensities of the pixels at the upper left corner of an image. For example, the intensity at $[0,0]^T$ is 100 and at $[3,2]^T$ is 109. Use bilinear interpolation to get the intensity at $[1.2,3.7]^T$.

Y \ X	0	1	2	3	4
0	100	102	104	102	106
1	102	105	107	106	106
2	104	108	108	109	110
3	105	110	104	106	109
4	107	105	103	107	105

Answers to Question 6:

$$I_a = 105 * 0.8 + 103 * 0.2 = 104.6$$
$$I_b = 110 * 0.8 + 104 * 0.2 = 108.8$$
$$I_c = 104.6 * 0.7 + 108.8 * 0.3 = 105.86 \approx 106$$



Question 2 [Transformations]:

a) [14 marks] Assume that a triangle $\mathbf{p_1p_2p_3}$ is transformed into another triangle $\mathbf{p'_1p'_2p'_3}$.

From			To	
$\mathbf{p_1}$	$[3,5]^T$	➔	$\mathbf{p'_1}$	$[13, 2]^T$
$\mathbf{p_2}$	$[9,7]^T$	➔	$\mathbf{p'_2}$	$[12, 7]^T$
$\mathbf{p_3}$	$[11,3]^T$	➔	$\mathbf{p'_3}$	$[1, 13]^T$

Derive the transformation matrix.

Answers to Question 2 a):

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 1 \\ 9 & 7 & 1 \\ 11 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 13 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.1429 & 0.0714 & 0.0714 \\ -0.0714 & 0.2857 & -0.2143 \\ 1.7857 & -1.6429 & 0.8571 \end{bmatrix} \begin{bmatrix} 13 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.9286 \\ 2.2857 \\ 4.3571 \end{bmatrix}$$

[2+1+2+2 marks]

$$\begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 1 \\ 9 & 7 & 1 \\ 11 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 7 \\ 13 \end{bmatrix} = \begin{bmatrix} -0.1429 & 0.0714 & 0.0714 \\ -0.0714 & 0.2857 & -0.2143 \\ 1.7857 & -1.6429 & 0.8571 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ 13 \end{bmatrix} = \begin{bmatrix} 1.1429 \\ -0.9286 \\ 3.2143 \end{bmatrix}$$

[2+1+2 marks]

$$A = \begin{bmatrix} -0.9286 & 2.2857 & 4.3571 \\ 1.1429 & -0.9286 & 3.2143 \\ 0 & 0 & 1 \end{bmatrix}$$

[2 marks]

Question 5 [Mapping]:

[24 marks] In texture mapping, you are dealing with different spaces and coordinates systems:

1. A 3D modeling space using right-handed system.
2. A 2D map that contains the texels where the *y*-axis is pointing downwards.
3. A 2D output image where the *y*-axis is pointing downwards.

Consider the following assumptions:

1. You have a cylinder (e.g., a can of Coke) with lower base residing on the *zx*-plane. This lower base, whose radius *r* is 100 units, is centered at $[100, 0, -100]^T$. The height *h* of the cylinder is 350 units. Note that a point on the surface of the cylinder can be represented in cylindrical coordinates as $[r, \theta, h]^T$ where θ is the polar angle or the angular coordinate.

2. You have a map of size 480 × 350 texels. For simplicity, the intensities can be obtained as
$$I(x_u, y_v) = \left\lceil \frac{3}{2} x_u \% y_v \right\rceil$$
where $I(x_u, y_v)$ is the intensity at $[x_u, y_v]^T$; $\lceil . \rceil$ represents the ceiling operation; and % represents the modulus.

3. The whole map is wrapped onto the curved surface of the cylinder so that this curved surface is fully covered by the map and the whole map appears on the cylinder. The wrapping starts and ends where the cylinder touches the *yz*-plane.

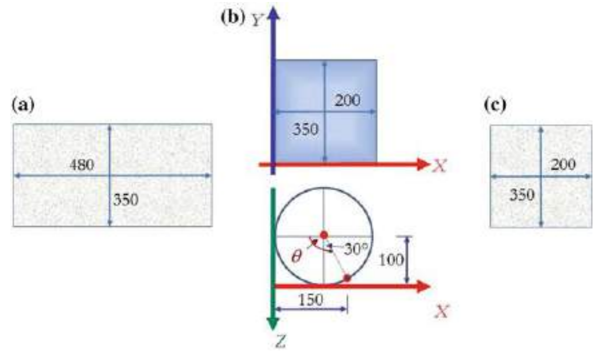
4. A front view whose normal is $[0, 0, 1]^T$ is obtained for the cylinder and an output image of 200 × 350 is created.

Get the intensity of the output pixel $[150, 25]^T$.

Answers to Question 5:

Example 12.9:

Notice that the height of the cylinder is equal to the height of the map (as well as the output image) and the diameter of the cylinder is equal to the width of the output image. This figure shows the three coordinate systems considered and also shows how the angle θ is estimated. We can work with the cylindrical coordinates to get the map coordinates u and v and use those coordinates to get the intensity of the corresponding pixel. The steps of the solution are the following:



1. Get the location of the point in 3D space (in cylindrical coordinates). Thus,

$$r = 100,$$
$$\theta = 90 + \sin^{-1}\left(\frac{50}{100}\right) = 120°,$$
$$y = h - 25 = 350 - 25 = 325.$$

[3 + 3 +3 marks = 9 marks]

Pay attention that the height of the cylinder in this example is along the y-axis. Also, notice that the angle θ is measured starting from the negative x-direction to simplify the current case of front projection.

2. Get the location on the map as $[u, v]^T$ as

$$u = \frac{\theta}{2\pi} = \frac{120}{360} = \frac{1}{3},$$
$$v = \frac{h-y}{h} = \frac{350-325}{350} = \frac{1}{14}.$$

[3 + 3 marks = 6 marks]

Note that we used $h-y$ as the y-axis of the map is pointing downwards while the y-axis of the 3D coordinate system is pointing upwards (as it is a right-handed coordinate system).

3. Get the location of $[u, v]^T$ in pixels $[x_u, y_v]^T$ as

$$x_u = 480u = \frac{480}{3} = 160,$$
$$y_v = 350v = \frac{350}{14} = 25.$$

[3 + 3 marks = 6 marks]

4. Get the intensity of the source point as

$$I(x_u, y_v) = \left\lceil \frac{3}{2} x_u \% y_v \right\rceil = \left\lceil \frac{3}{2} 160 \% 25 \right\rceil = 15.$$

[3 marks]

Thus, the intensity of the output pixel $[150, 25]^T$ is 15.

Question 2 [Transformations]:

a) [14 marks] Assume that a triangle $\mathbf{p_1p_2p_3}$ is transformed into another triangle $\mathbf{p'_1p'_2p'_3}$.

From			To	
$\mathbf{p_1}$	$[3,5]^T$	➔	$\mathbf{p'_1}$	$[13, 2]^T$
$\mathbf{p_2}$	$[9,7]^T$	➔	$\mathbf{p'_2}$	$[12, 7]^T$
$\mathbf{p_3}$	$[11,3]^T$	➔	$\mathbf{p'_3}$	$[1, 13]^T$

Derive the transformation matrix.

Answers to Question 2 a):

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 1 \\ 9 & 7 & 1 \\ 11 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 13 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.1429 & 0.0714 & 0.0714 \\ -0.0714 & 0.2857 & -0.2143 \\ 1.7857 & -1.6429 & 0.8571 \end{bmatrix} \begin{bmatrix} 13 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.9286 \\ 2.2857 \\ 4.3571 \end{bmatrix}$$

[2+1+2+2 marks]

$$\begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 1 \\ 9 & 7 & 1 \\ 11 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 7 \\ 13 \end{bmatrix} = \begin{bmatrix} -0.1429 & 0.0714 & 0.0714 \\ -0.0714 & 0.2857 & -0.2143 \\ 1.7857 & -1.6429 & 0.8571 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ 13 \end{bmatrix} = \begin{bmatrix} 1.1429 \\ -0.9286 \\ 3.2143 \end{bmatrix}$$

[2+1+2 marks]

$$A = \begin{bmatrix} -0.9286 & 2.2857 & 4.3571 \\ 1.1429 & -0.9286 & 3.2143 \\ 0 & 0 & 1 \end{bmatrix}$$

[2 marks]

Question 7 [Shading]:

[13 points] The vertices of a triangle are placed at the positions **a**=$[0,0,0]^T$, **b**=$[5,0,0]^T$ and **c**=$[0,5,0]^T$. The normal vectors at the vertices **a**, **b** and **c** are $[3,4,5]^T$, $[7,8,6]^T$, $[10,10,7]^T$ respectively. If Phong Shading is to be applied to this triangle, calculate the normal vector at the position $[2,2,0]^T$.

Answers to Question 7:

$$A = \frac{5 * 5}{2} = 12.5$$

$$A_b = \frac{2 * 5}{2} = 5$$

$$A_c = \frac{2 * 5}{2} = 5$$

$$A_a = 12.5 - 10 = 2.5$$

Ratios are $\frac{2.5}{12.5}, \frac{5}{12.5}, \frac{5}{12.5} = 0.2, 0.4, 0.4$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 * 0.2 + 7 * 0.4 + 10 * 0.4 \\ 4 * 0.2 + 8 * 0.4 + 10 * 0.4 \\ 5 * 0.2 + 6 * 0.4 + 7 * 0.4 \end{bmatrix} = \begin{bmatrix} 7.4 \\ 8 \\ 6.2 \end{bmatrix}$$

(1 point for each A ➔ 4)
(1 point for each ratio ➔ 3)
(2 points for each component ➔ 6)

<u>Question 5:</u> (Shading)

[8 marks] Consider a triangle whose vertices are $\dot{\mathbf{a}} = [1,1]^T$, $\dot{\mathbf{b}} = [10,2]^T$ and $\dot{\mathbf{c}} = [6,5]^T$. Also, consider a point $\dot{\mathbf{p}} = [7,3]^T$ inside the triangle. Express $\dot{\mathbf{p}}$ in parametric coordinates if the *u*- and *v*-axes go along $\overline{\dot{\mathbf{a}}\dot{\mathbf{b}}}$ and $\overline{\dot{\mathbf{a}}\dot{\mathbf{c}}}$ respectively.

<u>Solutions</u>

$$\mathbf{u} = \dot{\mathbf{b}} - \dot{\mathbf{a}} = \begin{bmatrix} 10 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$ [2 marks]

$$\mathbf{v} = \dot{\mathbf{c}} - \dot{\mathbf{a}} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$ [2 marks]

$$\mathbf{w} = \dot{\mathbf{p}} - \dot{\mathbf{a}} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$ [2 marks]

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{(\mathbf{u}\bullet\mathbf{v})(\mathbf{w}\bullet\mathbf{v})-(\mathbf{v}\bullet\mathbf{v})(\mathbf{w}\bullet\mathbf{u})}{(\mathbf{u}\bullet\mathbf{v})^2-(\mathbf{u}\bullet\mathbf{u})(\mathbf{v}\bullet\mathbf{v})} \\ \frac{(\mathbf{u}\bullet\mathbf{v})(\mathbf{w}\bullet\mathbf{u})-(\mathbf{u}\bullet\mathbf{u})(\mathbf{w}\bullet\mathbf{v})}{(\mathbf{u}\bullet\mathbf{v})^2-(\mathbf{u}\bullet\mathbf{u})(\mathbf{v}\bullet\mathbf{v})} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(\begin{bmatrix} 9 \\ 1 \end{bmatrix}\bullet\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix}\bullet\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)-\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\bullet\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix}\bullet\begin{bmatrix} 9 \\ 1 \end{bmatrix}\right)}{\left(\begin{bmatrix} 9 \\ 1 \end{bmatrix}\bullet\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)^2-\left(\begin{bmatrix} 9 \\ 1 \end{bmatrix}\bullet\begin{bmatrix} 9 \\ 1 \end{bmatrix}\right)\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\bullet\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)} \\ \frac{\left(\begin{bmatrix} 9 \\ 1 \end{bmatrix}\bullet\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix}\bullet\begin{bmatrix} 9 \\ 1 \end{bmatrix}\right)-\left(\begin{bmatrix} 9 \\ 1 \end{bmatrix}\bullet\begin{bmatrix} 9 \\ 1 \end{bmatrix}\right)\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix}\bullet\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)}{\left(\begin{bmatrix} 9 \\ 1 \end{bmatrix}\bullet\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)^2-\left(\begin{bmatrix} 9 \\ 1 \end{bmatrix}\bullet\begin{bmatrix} 9 \\ 1 \end{bmatrix}\right)\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\bullet\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)} \end{bmatrix}$$

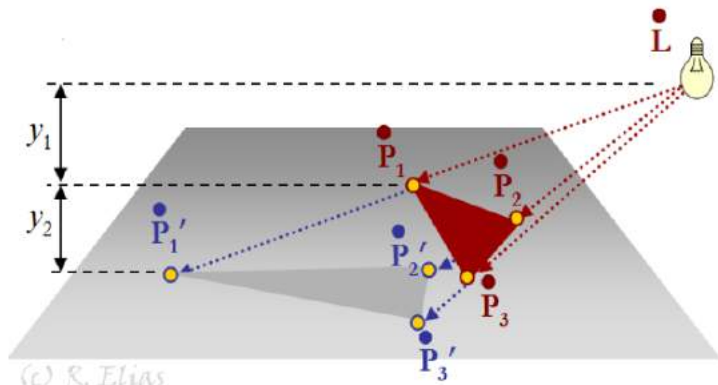
$$= \begin{bmatrix} 0.4516 \\ 0.3871 \end{bmatrix}$$

[2 marks]

Question 5 [Shadowing]:

[10 marks] In class, we saw how to detect the shadow of a triangle on a horizontal infinite plane (shown in the opposite figure) using the following equation:

$$\dot{\mathbf{P}}'_1 = \dot{\mathbf{L}} + \left(\frac{y_1 + y_2}{y_1}\right)\left[\dot{\mathbf{P}}_1 - \dot{\mathbf{L}}\right]$$



Given the location of a light source $\dot{\mathbf{L}}$ and a vertex $\dot{\mathbf{P}}_1$, propose an equation (or equations) to determine the shadow location $\dot{\mathbf{P}}'_1$ if the plane, receiving the shadow, has the general equation $ax + by + cz = d$.

Hint: The normal to the plane is $[a, b, c]^T$.

Answers to Question 5:

It is a line-plane intersection

$\mathbf{n} = [a, b, c]^T$

$\mathbf{v} = \mathbf{P_1} - \mathbf{L}$ (1) [2 marks]

Substituting $\mathbf{P'_1}$ in the plane equation
$\mathbf{P'_1} \bullet \mathbf{n} = d$ (2) [2 marks]

Since $\mathbf{P'_1} = \mathbf{L} + t\mathbf{v}$ (3) [2 marks]

Substitute $\mathbf{P'_1}$ from (3) in (2)

$[\mathbf{L} + t\mathbf{v}] \bullet \mathbf{n} = d$

$\mathbf{L} \bullet \mathbf{n} + t\mathbf{v} \bullet \mathbf{n} = d$ [2 marks]

$t = (-(\mathbf{L} \bullet \mathbf{n}) + d) / (\mathbf{v} \bullet \mathbf{n})$ (4) [2 marks]

Using (1) and (4)
Hence $\mathbf{P'_1} = \mathbf{L} + t\mathbf{v}$

Question 3 [Curves and 2D Transformations]:

[15 points] Consider a cubic Bezier curve passing through the points $[0,0]^T$, $[3,4]^T$, $[7,8]^T$ and $[10,11]^T$. If this curve is rotated through 45° about the point $[3,3]^T$, where would be the location of t=0.3 after rotation?

Answers to Question 3:

$$\mathbf{q}(t) = (1-t)^3\mathbf{p_0} + 3t(1-t)^2\mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + +t^3\mathbf{p}_3$$

$$\mathbf{q}(t) = (1-0.3)\begin{bmatrix}0\\0\end{bmatrix} + 3 * 0.3 * (1-0.3)^2\begin{bmatrix}3\\4\end{bmatrix} +$$

$$3 * 0.3^2 * (1-0.3)\begin{bmatrix}7\\8\end{bmatrix} + 0.3^3\begin{bmatrix}10\\11\end{bmatrix}$$

$$\mathbf{q}(t) = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}1.323\\1.764\end{bmatrix} + \begin{bmatrix}1.323\\1.512\end{bmatrix} + \begin{bmatrix}0.27\\0.297\end{bmatrix} = \begin{bmatrix}2.916\\3.573\end{bmatrix}$$

q(t) ➔ (5 points)

To rotate about $[3,3]^T$:

1. Translate using $[-3,-3]^T$.
2. Rotate through 45°
3. Translate using $[-3,-3]^T$.

$$\mathbf{M}_1 = \mathbf{T(-3,-3)} = \begin{bmatrix}1 & 0 & -3\\0 & 1 & -3\\0 & 0 & 1\end{bmatrix}$$

$$\mathbf{M_2} = \mathbf{R(45^\circ)} = \begin{bmatrix}\cos(45) & -\sin(45) & 0\\\sin(45) & \cos(45) & 0\\0 & 0 & 1\end{bmatrix}$$

$$\mathbf{M}_3 = \mathbf{T(3,3)} = \begin{bmatrix}1 & 0 & 3\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix}$$

Right matrices ➔ (3 points)

Answers to Question 3 (cont.):

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.71 & -0.71 & 3 \\ 0.71 & 0.71 & -1.26 \\ 0 & 0 & 1 \end{bmatrix}$$

Right order of multiplication ➔ (3 points)
Right **M** ➔ (2 point)

$$\mathbf{p} = \begin{bmatrix} 0.71 & -0.71 & 3 \\ 0.71 & 0.71 & -1.26 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2.916 \\ 3.573 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.534 \\ 3.347 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2.534 \\ 3.347 \end{bmatrix}$$

Right **p** ➔ (2 point)

Question 3 (Curves and 2D Transformations):

[15 points] Consider a quadratic curve passing through the points $[0,0]^T$, $[2,3]^T$, $[8,6]^T$ where $[2,3]^T$ is at t=0.3. Assume that the curve is represented parametrically. If this curve is rotated through 45º about the point $[3,3]^T$, where would be the location of t=0.5 after rotation?

Answers to Question 3:

$$\mathbf{a}_0 = \mathbf{p}_0$$

$$\mathbf{a}_2 = \frac{t_1(\mathbf{p}_2 - \mathbf{p}_0) + (\mathbf{p}_0 - \mathbf{p}_1)}{t_1(1 - t_1)}$$

$$\mathbf{a}_1 = \mathbf{p}_2 - \mathbf{p}_0 - \mathbf{a}_2$$

$$\mathbf{a}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{a}_2 = \frac{0.3\left(\begin{bmatrix} 8 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) + \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)}{0.3 * (1 - 0.3)} = \frac{0.3\begin{bmatrix} 8 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \end{bmatrix}}{0.21} = \begin{bmatrix} 1.9 \\ -5.7 \end{bmatrix}$$

$$\mathbf{a}_1 = \mathbf{p}_2 - \mathbf{p}_0 - \mathbf{a}_2 = \begin{bmatrix} 8 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1.9 \\ -5.7 \end{bmatrix} = \begin{bmatrix} 6.1 \\ 11.7 \end{bmatrix}$$

$$\mathbf{p}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2$$

$$\mathbf{p}(0.5) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6.1 \\ 11.7 \end{bmatrix} * 0.5 + \begin{bmatrix} 1.9 \\ -5.7 \end{bmatrix} * 0.25 = \begin{bmatrix} 3.525 \\ 4.425 \end{bmatrix}$$

a_1, a_2, a_3 ➔ (2*3=6 points)

p(0.5) ➔ (2 points)

To rotate about $[3,3]^T$:

1. Translate using $[-3,-3]^T$.
2. Rotate through 45º
3. Translate using $[3,3]^T$.

Answer to Question 3 (cont.):

$$\mathbf{M}_1 = \mathbf{T(-3,-3)} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_2 = \mathbf{R(45^\circ)} = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_3 = \mathbf{T(3,3)} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Right matrices ➔ (3 points)

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.71 & -0.71 & 3 \\ 0.71 & 0.71 & -1.26 \\ 0 & 0 & 1 \end{bmatrix}$$

Right order of multiplication ➔ (2 points)
Right M ➔ (1 point)

$$\mathbf{p} = \begin{bmatrix} 0.71 & -0.71 & 3 \\ 0.71 & 0.71 & -1.26 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.525 \\ 4.425 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.361 \\ 4.3845 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2.361 \\ 4.3845 \end{bmatrix}$$

Right p ➔ (1 point)

Question 3: **(20 Marks)**

Consider a quadratic curve that is represented parametrically. This curve is passing through the points $[0, 0]^T$, $[2, 3]^T$ and $[8, 6]^T$ that are at $t = 0$, $t = 0.3$ and $t = 1$ respectively. If this curve is rotated through an angle of 45° about the point $[3, 3]^T$, estimate the location of the curve at $t = 0.5$ after rotation. Use **inhomogeneous** points.

Solution

The coefficients are

$$\mathbf{c} = \dot{\mathbf{p}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ [1 mark]

$$\mathbf{a} = \frac{t(\dot{\mathbf{p}}_2-\dot{\mathbf{p}}_0)+(\dot{\mathbf{p}}_0-\dot{\mathbf{p}}_1)}{t(1-t)} = \frac{0.3\times\left(\begin{bmatrix} 8 \\ 6 \end{bmatrix}-\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)+\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}-\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)}{0.3\times(1-0.3)} = \begin{bmatrix} 1.9048 \\ -5.7143 \end{bmatrix}$$ [3 marks]

$$\mathbf{b} = \dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_0 - \mathbf{a} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1.9048 \\ -5.7143 \end{bmatrix} = \begin{bmatrix} 6.0952 \\ 11.7143 \end{bmatrix}$$ [3 marks]

Thus, the point at $t = 0.5$ [2 marks]

$$\begin{aligned}\dot{\mathbf{p}}(0.5) &= \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c} \\ &= \begin{bmatrix} 1.9048 \\ -5.7143 \end{bmatrix} \times 0.5^2 + \begin{bmatrix} 6.0952 \\ 11.7143 \end{bmatrix} \times 0.5 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5238 \\ 4.4286 \end{bmatrix}\end{aligned}$$

Steps: [operations: 3 marks; parameters: 3 marks]

1. Translate using the vector $[-3,-3]^T$ so that the center of rotation coincides with the origin.
2. Rotate through an angle of 45° about the origin.
3. Translate back using the vector $[3,3]^T$.

$$\begin{aligned}\dot{\mathbf{p}}' &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_p - x_r \\ y_p - y_r \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix} \\ &= \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix} \begin{bmatrix} 3.5238 - 3 \\ 4.4286 - 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.3602 \\ 4.3805 \end{bmatrix}\end{aligned}$$

[multiplication/addition order: 1 mark]
[matrix + 2 vectors: 3 marks]
[final answer: 1 mark]

The same result would be achieved if the transformation is applied first to the given points and the curve point is estimated afterwards using the transformed points.

$$\dot{\mathbf{p}}'_0 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_p - x_r \\ y_p - y_r \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$

$$= \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix} \begin{bmatrix} 0-3 \\ 0-3 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1.2426 \end{bmatrix}$$

$$\dot{\mathbf{p}}'_1 = \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix} \begin{bmatrix} 2-3 \\ 3-3 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.2929 \\ 2.2929 \end{bmatrix}$$

$$\dot{\mathbf{p}}'_2 = \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix} \begin{bmatrix} 8-3 \\ 6-3 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4.4142 \\ 8.6569 \end{bmatrix}$$

$$\mathbf{c}' = \dot{\mathbf{p}}'_0 = \begin{bmatrix} 3 \\ -1.2426 \end{bmatrix}$$

$$\mathbf{a}' = \frac{t(\dot{\mathbf{p}}'_2 - \dot{\mathbf{p}}'_0) + (\dot{\mathbf{p}}'_0 - \dot{\mathbf{p}}'_1)}{t(1-t)}$$

$$= \frac{0.3 \times \left(\begin{bmatrix} 4.4142 \\ 8.6569 \end{bmatrix} - \begin{bmatrix} 3.0 \\ -1.2426 \end{bmatrix} \right) + \left(\begin{bmatrix} 3 \\ -1.2426 \end{bmatrix} - \begin{bmatrix} 2.2929 \\ 2.2929 \end{bmatrix} \right)}{0.3 \times (1-0.3)} = \begin{bmatrix} 5.3875 \\ -2.6937 \end{bmatrix}$$

$$\mathbf{b}' = \dot{\mathbf{p}}'_2 - \dot{\mathbf{p}}'_0 - \mathbf{a}' = \begin{bmatrix} 4.4142 \\ 8.6569 \end{bmatrix} - \begin{bmatrix} 3 \\ -1.2426 \end{bmatrix} - \begin{bmatrix} 5.3875 \\ -2.6937 \end{bmatrix} = \begin{bmatrix} -3.9733 \\ 12.5932 \end{bmatrix}$$

$$\dot{\mathbf{p}}'(0.5) = \mathbf{a}'t^2 + \mathbf{b}'t + \mathbf{c}' = \begin{bmatrix} 5.3875 \\ -2.6937 \end{bmatrix} \times 0.5^2 + \begin{bmatrix} -3.9733 \\ 12.5932 \end{bmatrix} \times 0.5 + \begin{bmatrix} 3 \\ -1.2426 \end{bmatrix} = \begin{bmatrix} 2.3602 \\ 4.3805 \end{bmatrix}$$

Question 6: (Curves/2D Transformations)

[14 marks] Consider a cubic Bezier curve whose control points are $[0,0]^T$, $[3,4]^T$, $[7,8]^T$ and $[10,11]^T$. If this curve is rotated through an angle of $45°$ about the point $[3,3]^T$, estimate the curve location at $t = 0.3$ after rotation. Use **inhomogeneous** coordinates.

Solutions

The location of the curve point at $t = 0.3$ before rotation [equation: 2 marks; answer: 1 mark]

$$\dot{\mathbf{p}}(t) = (1-t)^3\dot{\mathbf{p}}_0 + 3t(1-t)^2\dot{\mathbf{p}}_1 + 3t^2(1-t)\dot{\mathbf{p}}_2 + t^3\dot{\mathbf{p}}_3$$

$$\dot{\mathbf{p}}(0.3) = (1-0.3)^3\begin{bmatrix}0\\0\end{bmatrix} + 3\times 0.3(1-0.3)^2\begin{bmatrix}3\\4\end{bmatrix} + 3\times 0.3^2(1-0.3)\begin{bmatrix}7\\8\end{bmatrix} + 0.3^3\begin{bmatrix}10\\11\end{bmatrix} = \begin{bmatrix}2.916\\3.573\end{bmatrix}$$

Steps: [operations: 3 marks; parameters: 3 marks]

1. Translate using the vector $[-3,-3]^T$ so that the center of rotation coincides with the origin.
2. Rotate through an angle of $45°$ about the origin.
3. Translate back using the vector $[3,3]^T$.

$$\dot{\mathbf{p}}' = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}x_{\dot{\mathbf{p}}} - x_{\dot{\mathbf{r}}}\\ y_{\dot{\mathbf{p}}} - y_{\dot{\mathbf{r}}}\end{bmatrix} + \begin{bmatrix}x_{\dot{\mathbf{r}}}\\ y_{\dot{\mathbf{r}}}\end{bmatrix}$$

$$= \begin{bmatrix}\cos(45) & -\sin(45)\\ \sin(45) & \cos(45)\end{bmatrix}\begin{bmatrix}2.916-3\\ 3.573-3\end{bmatrix} + \begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}2.5354\\3.3458\end{bmatrix}$$

[multiplication/addition order: 1 mark]
[matrix + 2 vectors: 3 marks]
[final answer: 1 mark]

The same result would be achieved if the transformation is applied first to the given control points and the curve point is estimated afterwards using the transformed points.

Question 6 [Curves]:

[10 marks] Given two 2D endpoints $\mathbf{p}_0$ and $\mathbf{p}_3$ of a cubic parametric curve in addition to two intermediate 2D points; $\mathbf{p}_1$ and $\mathbf{p}_2$ whose parameters are t_1 and t_2, the coefficients of the cubic parametric equation can be estimated.

a. In this case, four equations should be solved simultaneously for different values of the parameter t. Write down these equations. (You do not need to solve them simultaneously; just write down the equations.)
b. Write down the interpolation matrix that can be used with this curve.
c. Determine an equation that uses the interpolation matrix above to calculate the position of a point on that curve.

Answers to Question 6:

The equation
$\dot{\mathbf{p}}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$

When t=0 $\dot{\mathbf{p}}_0(0) = \mathbf{d}$ [1 mark]
When t=1 $\dot{\mathbf{p}}_3(1) = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$ [1 mark]
At the intermediate points $\mathbf{p}_1$ and $\mathbf{p}_2$ whose parameters are t_1 and t_2, we have

$\dot{\mathbf{p}}_1(t_1) = \mathbf{a}t_1^3 + \mathbf{b}t_1^2 + \mathbf{c}t_1 + \mathbf{d}$ [1 mark]
and
$\dot{\mathbf{p}}_2(t_2) = \mathbf{a}t_2^3 + \mathbf{b}t_2^2 + \mathbf{c}t_2 + \mathbf{d}$ [1 mark]

The interpolation matrix

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ t_1^3 & t_1^2 & t_1 & 1 \\ t_2^3 & t_2^2 & t_2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}}_{\mathrm{M}} \begin{bmatrix} \dot{\mathbf{p}}_0 \\ \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \end{bmatrix}$$

[matrix: 2 marks; inverse: 1 mark]

$$\dot{\mathbf{p}}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ t_1^3 & t_1^2 & t_1 & 1 \\ t_2^3 & t_2^2 & t_2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dot{\mathbf{p}}_0 \\ \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \end{bmatrix}}_{\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{bmatrix}^T}$$

[3 marks]