Network scheduling by PERT/CPM

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August 26, 2016

Outline

Introduction

- **Network Scheduling** is a technique used for planning and scheduling large projects in the fields of construction, maintenance, fabrication, purchasing, research and development designs etc.
- The technique is a method of minimizing trouble spots such as production bottlenecks, delays and interruptions, by determining critical factors and coordinating various parts of overall job.

There are 2 basic planning and control techniques that complete a predetermined project or schedule. These are

1. **PERT :** Programme Evaluation and Review Technique.
2. **CPM :** Critical Path Method

Network

Definition

A **network** is a graphic representation of a project's operations and is composed of activities and events that must be completed to reach the end objective of a project, showing the planning sequence of their accomplishments, their dependence and their inter-relationships.

Definition

Events

An **event** or **node** is a specific physical or intellectual accomplishment in a program or project plan . It is recognizable as a particular instant in time and does not consume time or resource. An event is generally represented on the network by a circle, rectangle, hexagon or some other geometric shape.

Definition

Activity

An **activity** is a task, or item of work to be done, that consumes time effort, money and other resources. It lies between two events, called the 'preceding' and 'succeeding' ones. An activity is represented on the network by an arrow with its head indicating the sequence in which the events are to occur.

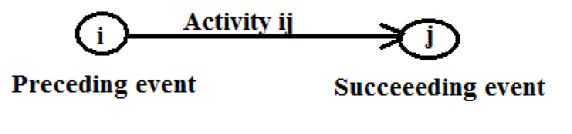


Figure 1: A specimen of an event and an activity

Definition

Dummy Activity

Certain activities which neither consumes time nor resources but are used simply to represent a connection between events, are known as dummies. A dummy (or zero-time) activity is generally shown by a chain of dotted arrows. Dummies are generally serve the following purpose:

1. To maintain uniqueness in the number system, as every activity may have a distinct set of events by which the activity can be defined.
2. To maintain the proper logic of the network.

Definition

Dummy Activity

Consider the network in fig. 2, where activities B and C have activity A as their immediate predecessor. The drawback is removed when activities B and C are identified by a distinct set of events by using a dummy as shown.

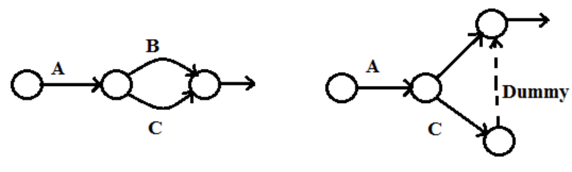


Figure 2: Dummy activity insertion

Definition

Common Errors

There are three types of errors which are most common in network drawing, viz.,

1. Formation of Loop
2. Dangling
3. Redundancy

Definition

Formation of loop

If an activity were represented as going back in time, a closed loop would occur. This is showing in fig 3, which is simply structure of fig 2 with activity B reversed in direction. Cycling in a network can result through a simple error or when while developing the activity plans, one tries to show the repetition of an activity plans, one tries to show the repetition of an activity before beginning the next activity.

Definition

Formation of loop

A closed loop would produce an endless cycle in computer programmes without a built in routine for detection and identification of the cycle. This one property of a correctly constructed network diagram is that it is "non- cyclic".

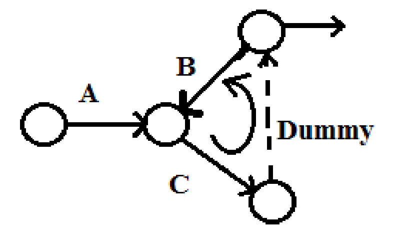
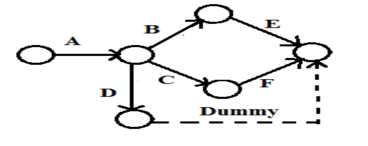


Figure 3

Definition

Dangling

No activity should end without being joined to the end event. If it is not so, a dummy activity is introduced in order to maintain the continuity of the system. Such end events other than the end of the project as a whole are called dangling events.



In the above network , activity D leads to Dangling. A dummy activity is therefore introduced to avoid this dangling.

Definition

Redundancy

If a dummy activity is the only activity emanating from an event, it can be eliminated . For example, in the network shown in the below fig. the dummy activity is redundant and can be eliminated, and the network redrawn.

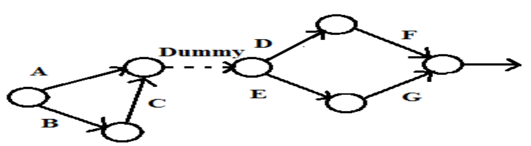


Figure 4

Rules of network contruction

Rules:

There are a number of ground rules in connection with the handling of events and activities of a project network that should be followed:

1. No event can occur until every activity preceding it has been completed.
2. An activity succeeding an event cannot be started until that event has occurred.
3. An event cannot occur twice, i.e., a path of activities cannot form a loop that returns to any event previously accomplished. Thus, no event can depend for its completion upon the completion of a succeeding event.

Rules of network contruction[Contd..]

Rules:

4. Each activity must start from and terminate in an event.
5. Time flows from left to right.
6. An activity must be completed in order to reach the end-event.
7. Dummy activities should only be introduced if absolutely necessary.

Numbering the events

Rules:

In event numbering, the following rules should be observed:

1. Event numbers should be unique.
2. Event numbering should be carried out an sequential basis from left to right.
3. The initial event which has all out going arrows with no incoming arrow is numbered 0 or 1.
4. The head of an arrow should always bear a number higher than the one assigned at the tail of the arrow.
5. Gaps Should be left in the sequence of event numbering to accommodate subsequent inclusion of activities, if necessary.

Time calculations in networks

Rules:

Let zero be the starting time of the project. For each activity an estimate must be made of time that will be spend on the actual accomplishment of the of that activity. Time estimates are usually written in the network immediately above the arrow. These calculations are done in following ways:

1. Let zero be the starting time of the project. then for each activity their is an **Earliest starting time (ES)** relative to the project starting time, which is the earliest possible time when the activity can begin, assuming that all of the predecessor are also started at their ES. then for that activity, the **Earliest finish time (EF)** is simply the ES activity time.

Time calculations in networks

ES and EF Rules:

ES_i denotes the earliest start time of all the activities emanating from event i: and t_{ij} is the estimated time for activity (i, j), then

$$EF_i \text{ or } ES_j = max.(ES_i + t_{ij}) \tag{1}$$

For all the defined (i, j) activities , with $ES_i = 0$ being the earliest start time of the beginning event of the project.

Time calculations in networks

Rules:

2. Let suppose we have a target time for completing the project.Then this time is called **Latest finish time (LF)** for the final activity. The **latest start time (LS)** is the latest time at which the activity can start if the target is to be maintained. It means that final activity, its LS is simply LF - activity time.

Time calculations in networks

LF Rules:

LF_i denotes the latest finish time of all the activities emanating from event i: and t_{ij} is the estimated time for activity (i, j), then

$$LF_i \;= min.(LF_j + t_{ij}) \qquad (2)$$

For all the defined (i, j) activities.

Critical Path Calculation

Rules:

An activity is said to be critical if a delay in its start will cause a further delay in the completion of entire project. on the other hand a non- critical activity is such that the time between its ES and LF is longer than its actual duration. In this case, the non-critical activity is said to have a slack or float time. It means that the difference between the latest finish and earliest start time is defined as slack.

Critical Path Calculation

CP Rules:

For the activity (i, j) to lie on the critical path, following conditions must satisfied:

1. $ES_i = LF_i$
2. $ES_j = LF_j$
3. $ES_j - ES_i = LF_j - LF_i = t_{ij}$

All the three conditions indicate that there is no float or slack time between ES and LS of the activity. Hence, activity must be critical.

Float and Slack values

Float and Slack values

There are many activities where the maximum time to finish the activity is more than the time required to complete it. the difference between this two is known as **Total Float** available for the activity.

- **Total Float:** This is calculated for any activity by using following rules.
  1. Determine the difference between early start time of tail event and latest finish time of head of the activity.
  2. Subtract duration time of the activity from the value obtained in step 1. to get the required total float for the activity.

$$TF_{ij} = LF_j - ES_i - t_{ij} \tag{3}$$

TF_{ij} is the total float.

Float and Slack values

Float and Slack values

- **Free Float:** It is defined by assuming that all the activities start as early as possible. The free float for the activity (i, j) is the excess available time over its duration.

$$FF_{ij} = ES_j - ES_i - t_{ij} \tag{4}$$

- **Interference Float:** The difference between total float and free float is known as Interference Float.

Float and Slack values

Float and Slack values

- **Independent Float:** The time by which an activity can be rescheduled without affecting the preceding or the succeeding activities is known as Independent Float.

$$IndependentFloat = FreeFloat - TaileventSlack \tag{5}$$

Remark

- The basic deiiference between slack and float times is that slack is used for events only, whereas float is applied fro activities.
- An activity is critical is its total float is zero.

Critical Path Method (CPM)

CPM Procedure:

The iterative procedure of determining the critical path is as follows.

1. List all the jobs and then draw network diagram. Each job is indicated by an arrow with the direction of the arrow showing the sequence of jobs. The length of the arrows has no significance. Place the jobs on the diagram one by one keeping in mind that what precedes and follows each job as well as what job can be done simultaneously.
2. Consider the job's time can be deterministic . Indicate them above the arrow representing the task.
3. Calculate EST, EFT, LFT and LST.

Critical Path Method (CPM)

CPM Procedure:[conti..]

4. Tabulate various times, i.e., activity normal times, earliest and latest times, and mark EST and LFT on the arrow diagram.
5. Determine the total float (slack) for each activity by taking difference between EST and LFT.
6. Identify the critical activities and connect them with the beginning node and the ending node in the network diagram by the double line arrows. this gives the Critical path.
7. Calculate the total project duration.

Question

A project consists of a series of tasks labelled A, B,...., H, I with the following relationships ($W < X$,Y means X and Y cannot start until W is completed ;$X, Y < W$ means W cannot start until both X and Y are completed). With this notations construct the network diagram having the following constraints:
$A < D, E$; $B, D < F$; $C < G$; $B, G < H$; $F, G < I$;
Find also the minimum time completion of the project, when the time (in days) of completion of each task is as follows:

Task:	A	B	C	D	E	F	G	H	I
Time:	23	8	20	16	24	18	19	4	10

Question

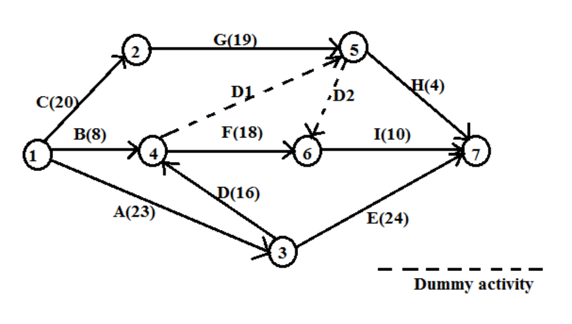


Figure 5: Resulting Network

Calculation Table

Task	Normal time tij	Earliest time		Latest time		Total float
		Start	Finish	Start	Finish	
(1,2)	20	0	20	18	38	18
(1,3)	23	0	23	0	23	0
(1,4)	8	0	8	31	39	31
(2,5)	19	20	39	38	57	18
(3,4)	16	23	39	23	39	0
(3,7)	24	23	47	43	67	20
(4,5)	0	39	39	57	57	18
(4,6)	18	39	57	39	57	0
(5,6)	0	39	39	57	57	18
(5,7)	4	39	43	63	67	24
(6,7)	10	57	67	57	67	0

Critical events: (1,3), (3,4), (4,6), (6,7)
Critical tasks: A, D, F, I
Total duration: 67 days

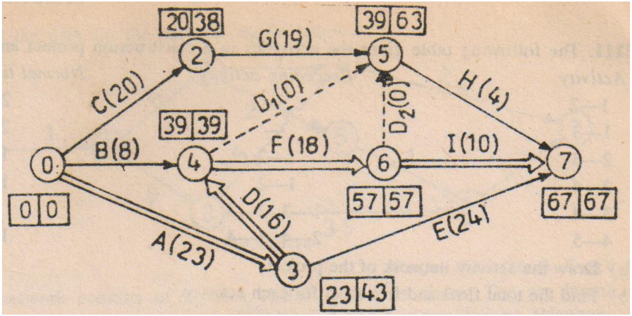


Figure 6: Resulting network with critical path

PERT

- The CPM network methods may be termed as *deterministic*, since estimated activity times are assumed to be the expected values.
- But no recognition is given to the fact that expected activity times is the mean of a distribution of possible values which could occur.
- Deterministic network methods assume that the expected time is the actual time taken.
- Probabilistic methods, on the other hand, assume the reverse, more realistic situation, where activity times are represented by a probability distribution.

Time estimates

The probability distribution of activity time is based upon three different time estimates made for each activity.

- t_o: the *optimistic time* is the shortest possible time to complete the activity if all goes well. That is, there is very little chance that activity can be done in the time less than t_o.
- t_p: the *pessimistic time* is the longest time that an activity could take if every thing goes wrong. That is, there is little chance that the activity would take time longer than t_p.
- t_m: the *most likely time* is the estimate of the normal time an activity would take. If only one were available, this would be it. Otherwise it is the mode of the probability distribution.

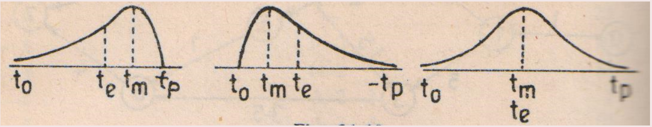
Time estimates

From these three values, the *expected time*, t_e, commonly referred to as the *average* or *mean*, is accomplished as

$$t_e = \frac{1}{3}[2t_m + \frac{1}{2}(t_o + t_p)] = \frac{1}{6}[4t_m + t_o + t_p] \tag{6}$$

Time estimates

- The probability distribution of the time involved in performing the activity might be characterized by any one of the curves illustrated below.
- It is assumed that the distribution is unimodal and the peak will represent the value of t_m.
- There is very little chance that the activity can be completed before t_o or affter t_p.
- The point, t_m, is free to move between these two.



Time estimates

Another assumption concerning the distribution is that, for unimodal distributions, the standard deviation can be estimated roughly as one-sixth of range, *i.e.*,

$$\sigma = \frac{t_p - t_o}{6} \tag{7}$$

and therefore the variance is

$$(\sigma)^2 = (\frac{t_p - t_o}{6})^2 \tag{8}$$

PERT Calculations

- The PERT computations are essentially the same as those used in CPM.
- The main difference is that instead of activity duration, we use expected time for the activity.
- With each event, we also associate the variance.
- Thus, the variation of the project is the expected time.

Procedure for obtaining mean and variance

- Let μ_i be the earliest expected time of an event i.
- Then, since the times of the activities summing up to i are random variables, μ_i will also be a random variable.
- Assuming all the activities in the network to be statistically independent, if there is only one path leading from the starting i, $E\{\mu_i\}$ will be given by the sum of the expected times 't' for the activities along this path.
- Also, $Var\{\mu_i\}$, in this case, will be the sum of the variances of the same activities.
- When more than one event leads to the same event, it becomes difficult to evaluate $E\{\mu_i\}$ and $Var\{\mu_i\}$ exactly. In such cases, first of all the statistical distribution of the longest path is developed and then $E\{\mu_i\}$ and $Var\{\mu_i\}$ are evaluated.

Procedure for obtaining mean and variance

Thus, the expected value and variance of μ_i are given by

$$T_e \equiv E\{\mu_i\} \tag{9}$$

and

$$\sigma_i^2 \equiv Var\{\mu_i\} = \sum_k V_k \tag{10}$$

where k defines the activities along the longest path leading to i and V_k is the variance of activity at node k.

Probability of Meeting the Schedule Time

- With PERT, it is possible to determine the probability of completing a contract on schedule.
- The scheduled dates are expressed as a number of time units from the present time.
- Initially, they may be the latest times, T_L, for each event, and after a project is strated we shall know how far it has progressed at any given date, and the scheduled times will be the latest time if the project is to be completed on its original schedule.
- Since, μ_i is the sum of independent random variables, using *Central Limit Theorem*, it is approximately normally distributed with mean $E\{\mu_i\}$ and variance, $Var\{\mu_i\}$.

Probability of Meeting the Schedule Time

- Now, because the earliest expected time is μ_i, therefore the event i will meet a certain schedule time ST_i (specified by the analysis) with probability

$$P\{\mu_i \leq ST_i\} = P\{\frac{\mu_i - E\{\mu_i\}}{\sqrt{Var\{\mu_i\}}} \leq \frac{ST_i - E\{\mu_i\}}{\sqrt{Var\{\mu_i\}}}\} = P\{z \leq z_0\} \tag{11}$$

where z is the standard normal variate with mean zero and variance unity and

$$z_0 = \frac{ST_i - E\{\mu_i\}}{\sqrt{Var\{\mu_i\}}} \tag{12}$$

Pert Algorithm

1. Make a list of activities that make up the project including immediate predecessors.
2. Making use of *step 1* sketch the required network.
3. Denote the most likely time by t_m the optimistic time by t_o and pessimistic time by t_p.
4. Using beta distribution for the activity duration, the expected time t_e, for each activity is computed by using the formula

$$t_e = (t_p + 4t_m + t_o)/6 \tag{13}$$

5. Tabulate various times, *i.e.*, expected activity times, earliest and latest times and mark the *ES* and *LF* on the arrow diagram.

Pert Algorithm

1. Determine the total float for each activity by taking the difference between EF and LF.
2. Identify the critical activities and connect them with the beginning node and the ending node, in the network diagram by double line arrows. This gives the critical path and the expected date of completion of the project.
3. Using the values of t_p and t_o, compute the variance (σ^2) of each activity's time estimates by using the formula:

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

Pert Algorithm

1. Compute the standard normal deviate:

$$z_0 = \frac{DueDate - ExpectedDateOfCompletion}{\sqrt{ProjectVariance}}$$

2. Use standard normal tables to find the probability $\mathrm{P}(z \leq z_0)$ of completing the project within the scheduled time, where $z \sim N(0, 1)$.

Sample Problem

A small project is composed of seven activities whose time estimates are listed in the table as follow.

Activity		Estimated duration (weeks)		
i	j	Optimistic	Most likely	Pessimistic
1	2	1	1	7
1	3	1	4	7
1	4	2	2	8
2	5	1	1	1
3	5	2	5	14
4	6	2	5	8
5	6	3	6	15

Sample Problem

- Draw the project network.
- Find the expected duration and variance of each activity.
- Calculate early and late occurrence times for each event. What is the expected project length.
- Calculate the variance and standard deviation of project length. What is the probability that the project will be completed:
  - at least 4 weeks earlier than expected?
  - no more than 4 weeks later than exepcted.
- If the project due date is 19 weeks, what is the probability of meeting the due date?

  Given

z	0.50	0.67	1.00	1.33	2.00
p	0.3085	0.2514	0.1587	0.0918	0.0228

Solution

The expected time and variance of each activity is computed in table below.

Activity $(i–j)$	a	m	b	$t_e = \frac{a+4m+b}{6}$	$(\frac{b-a}{6})^2$
1–2	1	1	7	2	1
1–3	1	4	7	4	1
1–4	2	2	8	3	1
2–5	1	1	1	1	0
3–5	2	5	14	6	4
4–6	2	5	8	5	1
5–6	3	6	15	7	4

Solution

- Earliest expected times $E\{\mu_i\}$ for each event is obtained by taking the sum of expected times for all the activities leading to an event i.
- When more than one activity leads to event i, the greatest of $E\{\mu_i\}$ is chosen.
- Thus,
  $E\{\mu_1\} = 0.$
  $E\{\mu_2\} = 0 + 2 = 2.$
  $E\{\mu_3\} = 0 + 4 = 4.$
  $E\{\mu_4\} = 0 + 3 = 3.$
  $E\{\mu_5\} = max.\{4 + 6, 2 + 1\} = 10.$
  $E\{\mu_6\} = max.\{10 + 7, 3 + 5\} = 17.$

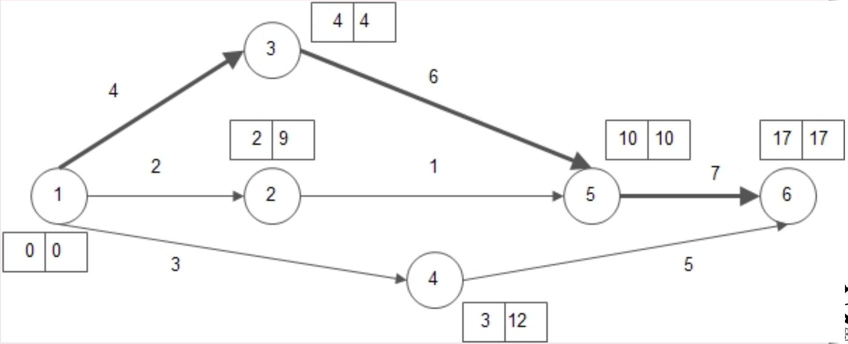
Solution

- For the latest expected times we start with $E\{\mu_i\}$ for the last event and move backwards, subtracting "t_e" for each activity link.
- Thus,
  $E\{L_6\} = 17.$
  $E\{L_5\} = 17 - 7 = 10.$
  $E\{L_4\} = 17 - 5 = 12.$
  $E\{L_3\} = 10 - 6 = 4.$
  $E\{L_2\} = 10 - 1 = 9.$
  $E\{L_1\} = min.\{9 - 2, 4 - 4, 12 - 3\} = 0.$

Solution

Resultant network

The critical path is shown by thick line arrows.



Solution

- **Critical path:** 1→3→5→6.
- **Other paths:** 1→2→5→6 and 1→4→6.
- The expected duration of the project is 17 weeks.
- Variance of the project length is given by $\sigma^2 = 1 + 4 + 4 = 9$.
- The standard normal deviate is:

$$z_0 = \frac{DueDate - ExpectedDateOfCompletion}{\sqrt{Variance}}$$

Solution

- $$z_0 = \frac{13-17}{3} = -\frac{4}{3} = -1.33$$

$$P(z \leq -1.33) = 0.5 - \Phi(1.33) = 0.5 - 0.4082 = 0.0918 \equiv 9.18\%$$

or

$$P(z \leq -1.33) = \Phi(-1.33) = 1 - \Phi(1.33) = 1 - 0.9082 = 0.0918 \equiv 9.$$

Solution

- The interpretation of the above is that if the project is performed 100 times under the same conditions, there will be 9 occasions when this job will be completed 4 weeks earlier than expected.
- $$z = \frac{21-17}{3} = \frac{4}{3} = 1.33$$
- The probability of meeting the due date (4 weeks later than expected) is

$$P(z \leq 1.33) = 0.5+\Phi(1.33) = 0.5+0.4082 = 0.9082 \equiv 90.82\%$$

Solution

- When the due date is 19 weeks

$$z = \frac{19 - 17}{3} = \frac{2}{3} = 0.67$$

The probability of meeting the due date is

$$P(z \leq 0.67) = 0.5 + \Phi(0.67) = 0.7486 \equiv 74.86\%$$

- Thus, the probability of not meeting the due date is 1-0.7486, i.e., 0.2514 or 25.14%.

Normal Table

TABLE A1 *Standard Normal Distribution Function:* $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998