Chapter 7: Operators on Inner Product Spaces

Linear Algebra Done Right, by Sheldon Axler

A: Self-Adjoint and Normal Operators

Problem 1

Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, \dots, z_n) = (0, z_1, \dots, z_{n-1}).$$

Find a formula for $T^*(z_1, \dots, z_n)$.

Proof. Fix $(y_1, \dots, y_n) \in \mathbb{F}^n$. Then for all $(z_1, \dots, z_n) \in \mathbb{F}^n$, we have

$$\begin{aligned}
\langle (z_1, \dots, z_n), T^*(y_1, \dots, y_n) \rangle &= \langle T(z_1, \dots, z_n), (y_1, \dots, y_n) \rangle \\
&= \langle (0, z_1, \dots, z_{n-1}), (y_1, \dots, y_n) \rangle \\
&= z_1 y_2 + z_2 y_3 + \dots + z_{n-1} y_n \\
&= \langle (z_1, \dots, z_{n-1}, z_n), (y_2, \dots, y_n, 0) \rangle.
\end{aligned}$$

Thus T^* is the left-shift operator. That is, for all $(z_1, \dots, z_n) \in \mathbb{F}^n$, we have

$$T^*(z_1, \dots, z_n) = (z_2, \dots, z_n, 0),$$

as desired. □

Problem 2

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of T if and only if $\bar{\lambda}$ is an eigenvalue of T^*.

Proof. Suppose λ is an eigenvalue of T. Then there exists $v \in V$ such that $Tv = \lambda v$. It follows

$$\begin{aligned}
\lambda \text{ is not an eigenvalue of } T &\iff T - \lambda I \text{ is invertible} \\
&\iff S(T - \lambda I) = (T - \lambda I)S = I \\
&\qquad \text{for some } S \in \mathcal{L}(V) \\
&\iff S^*(T^* - \lambda I)^* = (T - \lambda I)^* S^* = I^* \\
&\qquad \text{for some } S^* \in \mathcal{L}(V) \\
&\iff (T - \lambda I)^* \text{ is invertible} \\
&\iff T^* - \bar{\lambda} I \text{ is invertible} \\
&\iff \bar{\lambda} \text{ is not an eigenvalue of } T^*.
\end{aligned}$$

Since the first statement and the last statement are equivalent, so too are their contrapositives. Hence λ is an eigenvalue of T if and only if $\bar{\lambda}$ is an eigenvalue of T^*, as was to be shown. $\square$

Problem 3

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

Proof. ($\Rightarrow$) First suppose U is invariant under T, and let $x \in U^\perp$. For any $u \in U$, it follows

$$\begin{aligned}\langle T^* x, u\rangle &= \langle x, Tu\rangle \\ &= 0,\end{aligned}$$

where the second equality follows since $Tu \in U$ (by hypothesis). Thus $T^* x \in U^\perp$ for all $x \in U^\perp$. That is, $U^\perp$ is invariant under T^*.

($\Leftarrow$) Now suppose $U^\perp$ is invariant under T^*, and let $y \in U$. For any $u' \in U^\perp$, it follows

$$\begin{aligned}\langle Ty, u'\rangle &= \langle y, T^* u'\rangle \\ &= 0,\end{aligned}$$

where the second equality follows since $T^* u' \in U^\perp$ (by hypothesis). Thus $Ty \in U$ for all $y \in U$. That is, U is invariant under T, completing the proof. $\square$

Problem 5

Prove that

$$\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$$

and

$$\dim \operatorname{range} T^* = \dim \operatorname{range} T$$

for every $T \in \mathcal{L}(V, W)$.

Proof. Let $T \in \mathcal{L}(V, W)$. Notice

$$\begin{aligned}\dim \operatorname{null} T^* &= \dim (\operatorname{range} T)^\perp \\ &= \dim W - \dim \operatorname{range} T \\ &= \dim W + \dim \operatorname{null} T - \dim V,\end{aligned}$$

where the first equality follows by 7.7(a), the second equality follows by 6.50, and the third equality follows by the Fundamental Theorem of Linear Maps. Next notice

$$\begin{aligned}\dim \operatorname{range} T^* &= \dim\left(\operatorname{null} T\right)^\perp \\ &= \dim V - \dim \operatorname{null} T \\ &= \dim \operatorname{range} T,\end{aligned}$$

where the first equality follows by 7.7(b), and the second and third equalities follow again by the same theorems above. ☐

Problem 7

Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Proof. ($\Rightarrow$) Suppose ST is self-adjoint. We have

$$\begin{aligned}ST &= (ST)^* \\ &= T^*S^* \\ &= TS,\end{aligned}$$

where the second equality follows by 7.6(e).

($\Leftarrow$) Conversely, suppose $ST = TS$. It follows

$$\begin{aligned}(ST)^* &= (TS)^* \\ &= S^*T^*,\end{aligned}$$

where the second equality again follows by 7.6(e), completing the proof. ☐

Problem 9

Suppose V is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Proof. Let $\mathcal{A}$ denote the set of self-adjoint operators on V, and suppose $T \in \mathcal{A}$. By 7.6(b), notice $(iT)^* = -iT^*$, so that $\mathcal{A}$ is not closed under scalar multiplication. Thus $\mathcal{A}$ is not a subspace of $\mathcal{L}(V)$. ☐

Problem 11

Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that there is a subspace U of V such that $P = P_U$ if and only if P is self-adjoint.

Proof. ($\Rightarrow$) First suppose there is a subspace $U \subseteq V$ such that $P = P_U$, and let $v_1, v_2 \in V$. It follows

$$\begin{aligned}
\langle Pv_1, v_2 \rangle &= \langle u_1, u_2 + w_2 \rangle \\
&= \langle u_1, u_2 \rangle + \langle u_1, w_2 \rangle \\
&= \langle u_1, u_2 \rangle \\
&= \langle u_1, u_2 \rangle + \langle w_1, u_2 \rangle \\
&= \langle u_1 + w_1, u_2 \rangle \\
&= \langle v_1, Pv_2 \rangle,
\end{aligned}$$

and thus $P = P^*$.

($\Leftarrow$) Conversely, suppose $P = P^*$. Let $v \in V$. Notice $P(v - Pv) = Pv - P^2 v = 0$, and hence $v - Pv \in \operatorname{null} P$. By 7.7(c), we know $\operatorname{null} P = (\operatorname{range} T^*)^\perp$. By hypothesis, P is self-adjoint, and hence we have $v - Pv \in (\operatorname{range} T)^\perp$. Notice we may write

$$v = Pv + (v - Pv),$$

where $Pv \in \operatorname{range} P$ and $v - Pv \in (\operatorname{range} T)^\perp$. Let $U = \operatorname{range} P$. Since the above holds for all $v \in V$, we conclude $P = P_U$, and the proof is complete. $\square$

Problem 13

Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^4)$ such that T is normal but not self-adjoint.

Proof. Let T be the operator on $\mathbb{C}^4$ whose matrix with respect to the standard basis is

$$\begin{bmatrix} 2 & -3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We claim T is normal and not self-adjoint. To see that T is not self-adjoint, notice that the entry in row 2, column 1 does not equal the complex conjugate of the entry in row 1 column 2.

Next, notice

$$\mathcal{M}(TT^*) = \begin{bmatrix} 2 & -3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathcal{M}(T^*T) = \begin{bmatrix} 2 & 3 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and hence TT^* and T^*T have the same matrix. Thus $TT^* = T^*T$, and T is normal. ☐

Problem 15

Fix $u, x \in V$. Define $T \in \mathcal{L}(V)$ by

$$Tv = \langle v, u \rangle x$$

for every $v \in V$.

(a) Suppose $\mathbb{F} = \mathbb{R}$. Prove that T is self-adjoint if and only if u, x is linearly dependent.

(a) Prove that T is normal if and only if u, x is linearly dependent.

Proof. We first derive a useful formula for T^* which we'll use in both (a) and (b). Let $w_1, w_2 \in V$ and notice

$$\begin{aligned} \langle w_1, T^* w_2 \rangle &= \langle T w_1, w_2 \rangle \\ &= \langle \langle w_1, u \rangle x, w_2 \rangle \\ &= \langle w_1, u \rangle \langle x, w_2 \rangle \\ &= \langle w_1, \overline{\langle x, w_2 \rangle}\, u \rangle \\ &= \langle w_1, \langle w_2, x \rangle u \rangle, \end{aligned}$$

and thus $T^* w_2 = \langle w_2, x \rangle u$. Since w_2 was arbitrary, we may rewrite this as $T^* v = \langle v, x \rangle u$ for all $v \in V$.

(a) ($\Rightarrow$) Suppose T is self-adjoint. Then we have

$$\langle v, u \rangle x - \langle v, x \rangle u = Tv - T^* v = 0$$

for all $v \in V$. In particular, we have

$$\langle u, u \rangle x - \langle u, x \rangle u = 0.$$

We may assume both u and x are nonzero, for otherwise there is nothing to prove. Hence $\langle u, u \rangle \neq 0$, which forces $\langle u, x \rangle$ to be nonzero as well, and thus the equation above shows u, x is linearly dependent.

($\Leftarrow$) Conversely, suppose u, x is linearly dependent. We may again assume both u and x are nonzero, for otherwise $T = 0$, which is self-adjoint. Thus there exists a nonzero $\alpha \in \mathbb{R}$ such that $u = \alpha x$. It follows

$$\begin{aligned} Tv &= \langle v, u \rangle x \\ &= \langle v, \alpha x \rangle \frac{1}{\alpha} u \\ &= \langle v, x \rangle u \\ &= T^*, \end{aligned}$$

and thus T is self-adjoint, completing the proof.

(b) ($\Rightarrow$) Suppose T is normal and let $v \in V$. It follows

$$\begin{aligned} \langle \langle v, u \rangle x, x \rangle u &= T^*(\langle v, u \rangle x) \\ &= T^*Tv \\ &= TT^*v \\ &= T(\langle v, x \rangle u) \\ &= \langle \langle v, x \rangle u, u \rangle x. \end{aligned}$$

We may assume both u and x are nonzero, for otherwise there is nothing to prove. Since the above holds for $v = u$, we may conclude $\langle \langle v, u \rangle x, x \rangle \neq 0$, which also forces $\langle \langle v, x \rangle u, u \rangle \neq 0$. Thus u, x is linearly dependent.

($\Leftarrow$) Conversely, suppose u, x is linearly dependent. We may again assume both u and x are nonzero, for otherwise $T = 0$, which is normal. Thus there exists a nonzero $\alpha \in \mathbb{R}$ such that $u = \alpha x$. It follows

$$\begin{aligned} T^*Tv &= T^*(\langle v, u \rangle x) \\ &= \langle \langle v, u \rangle x, x \rangle u \\ &= \left\langle \langle v, \alpha x \rangle \frac{1}{\alpha} u, \frac{1}{\alpha} u \right\rangle \alpha x \\ &= \langle \langle v, x \rangle u, u \rangle x \\ &= T(\langle v, x \rangle u) \\ &= TT^*v, \end{aligned}$$

and thus T is normal, completing the proof. $\square$

Problem 16

Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\operatorname{range} T = \operatorname{range} T^*.$$

Proof. Suppose $T \in \mathcal{L}(V)$ is normal. We first prove $\operatorname{null} T = \operatorname{null} T^*$. It follows

$$\begin{aligned} v \in \operatorname{null} T &\iff Tv = 0 \\ &\iff \|Tv\| = 0 \\ &\iff \|T^*v\| = 0 \\ &\iff T^*v = 0 \\ &\iff v \in \operatorname{null} T^*, \end{aligned}$$

where the third equivalence follows by 7.20, and indeed we have $\operatorname{null} T = \operatorname{null} T^*$. This implies $(\operatorname{null} T)^\perp = (\operatorname{null} T^*)^\perp$, and by 7.7(b) and 7.7(c), this is equivalent to $\operatorname{range} T^* = \operatorname{range} T$, as desired. □

Problem 17

Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\operatorname{null} T^k = \operatorname{null} T \quad \text{and} \quad \operatorname{range} T^k = \operatorname{range} T$$

for every positive integer k.

Proof. To show $\operatorname{null} T^k = \operatorname{null} T$, we first prove $\operatorname{null} T^k = \operatorname{null} T^{k+1}$ for all $k \in \mathbb{Z}^+$. Let $m \in \mathbb{Z}^+$. If $m = 1$, there's nothing to prove, so we may assume $m > 1$. Clearly, if $v \in \operatorname{null} T^m$, then $v \in \operatorname{null} T^{m+1}$, and hence $\operatorname{null} T^m \subseteq \operatorname{null} T^{m+1}$. Next, suppose $v \in \operatorname{null} T^{m+1}$. Then $T(T^m v) = 0$, and hence $T^m v \in \operatorname{null} T$. By Problem 16, this implies $T^m v \in \operatorname{null} T^*$, and by 7.7(a) we have $T^m \in (\operatorname{range} T)^\perp$. Since of course $T^m v \in \operatorname{range} T$ as well, we must have $T^m v = 0$. Thus $v \in \operatorname{null} T^m$, and therefore $\operatorname{null} T^{m+1} \subseteq \operatorname{null} T^m$. Thus $\operatorname{null} T^m = \operatorname{null} T^{m+1}$. Since m was arbitrary, this implies $\operatorname{null} T^k = \operatorname{null} T$ for all $k \in \mathbb{Z}^+$, as desired.

Now we will show $\operatorname{range} T^k = \operatorname{range} T$ for all $k \in \mathbb{Z}^+$. Let $n \in \mathbb{Z}^+$. If $n = 1$, there's nothing to prove, so we may assume $n > 1$. Suppose $w \in \operatorname{range} T^n$. Then there exists $v \in V$ such that $T^n v = w$, and hence $T(T^{n-1}v) = w$, so that $w \in \operatorname{range} T$ as well and we have $\operatorname{range} T^n \subseteq \operatorname{range} T$. Next, notice

$$\begin{aligned} \dim \operatorname{range} T^n &= \dim V - \dim \operatorname{null} T^n \\ &= \dim V - \dim \operatorname{null} T \\ &= \dim \operatorname{range} T, \end{aligned}$$

where the second equality follows from the previous paragraph. Since $\operatorname{range} T^n$ is a subspace of $\operatorname{range} T$ of the same dimension, it must equal $\operatorname{range} T$. And since n was arbitrary, we conclude $\operatorname{range} T^k = \operatorname{range} T$ for all $k \in \mathbb{Z}^+$, completing the proof. □

Problem 19

Suppose $T \in \mathcal{L}(\mathbb{C}^3)$ is normal and $T(1,1,1) = (2,2,2)$. Suppose $(z_1, z_2, z_3) \in \operatorname{null} T$. Prove that $z_1 + z_2 + z_3 = 0$.

Proof. By Problem 16, $\operatorname{null} T = \operatorname{null} T^*$, hence $T^*(z_1, z_2, z_3) = 0$. Therefore, we have

$$\begin{aligned}
2(z_1 + z_2 + z_3) &= \langle (2, 2, 2), (z_1, z_2, z_3) \rangle \\
&= \langle T(1, 1, 1), (z_1, z_2, z_3) \rangle \\
&= \langle (1, 1, 1), T^*(z_1, z_2, z_3) \rangle \\
&= \langle (1, 1, 1), (0, 0, 0) \rangle \\
&= 0,
\end{aligned}$$

and so $z_1 + z_2 + z_3 = 0$, as was to be shown. □

Problem 21

Fix a positive integer n. In the inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx,$$

let

$$V = \operatorname{span}(1, \cos x, \cos 2x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx).$$

(a) Define $D \in \mathcal{L}(V)$ by $Df = f'$. Show that $D^* = -D$. Conclude that D is normal but not self-adjoint.

(b) Define $T \in \mathcal{L}(V)$ by $Tf = f''$. Show that T is self-adjoint.

Proof. From Problem 4 of 6B, recall that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list, and hence it is an orthonormal basis of V.

(a) For $k = 1, \dots, n$, define

$$e_k = \frac{\cos(kx)}{\sqrt{\pi}} \quad \text{and} \quad f_k = \frac{\sin(kx)}{\sqrt{\pi}}.$$

Notice

$$De_k = -\frac{k\sin(kx)}{\sqrt{\pi}} = -kf_k \quad \text{and} \quad Df_k = \frac{k\cos(kx)}{\sqrt{\pi}} = ke_k,$$

and thus, for any $v, w \in V$, it follows

$$\begin{aligned}
\langle v, D^*w\rangle &= \langle Dv, w\rangle \\
&= \left\langle D\left(\left\langle v, \frac{1}{\sqrt{2\pi}}\right\rangle \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n} \left(\langle v, e_k\rangle e_k + \langle v, f_k\rangle f_k\right)\right), w\right\rangle \\
&= \left\langle \sum_{k=1}^{n} \left(-k\langle v, e_k\rangle f_k + k\langle v, f_k\rangle e_k\right), w\right\rangle \\
&= -\sum_{k=1}^{n} k\langle v, e_k\rangle\langle f_k, w\rangle + \sum_{k=1}^{n} k\langle v, f_k\rangle\langle e_k, w\rangle \\
&= -\sum_{k=1}^{n} k\langle w, f_k\rangle\langle v, e_k\rangle + \sum_{k=1}^{n} k\langle w, e_k\rangle\langle v, f_k\rangle \\
&= \sum_{k=1}^{n} k\langle w, e_k\rangle\langle v, f_k\rangle - \sum_{k=1}^{n} k\langle w, f_k\rangle\langle v, e_k\rangle \\
&= \left\langle v, \sum_{k=1}^{n} k\langle w, e_k\rangle f_k\right\rangle - \left\langle v, \sum_{k=1}^{n} k\langle w, f_k\rangle e_k\right\rangle \\
&= \left\langle v, \sum_{k=1}^{n} \left(k\langle w, e_k\rangle f_k - k\langle w, f_k\rangle e_k\right)\right\rangle \\
&= \left\langle v, -D\left(\left\langle w, \frac{1}{\sqrt{2\pi}}\right\rangle \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n} \left(\langle w, e_k\rangle e_k + \langle w, f_k\rangle f_k\right)\right)\right\rangle \\
&= \langle v, -Dw\rangle,
\end{aligned}$$

and thus $D^* = -D$, showing that D is not self-adjoint. Moreover, notice that this implies

$$DD^* = D(-D) = -DD = (D^*)D = D^*D,$$

so that D is normal, completing the proof.

(b) Notice $T = D^2$, and hence

$$T^* = (DD)^* = D^*D^* = (-D)(-D) = D^2 = T.$$

Thus T is self-adjoint. □

B: The Spectral Theorem

Problem 1

True or false (and give a proof of your answer): There exists $T \in \mathcal{L}(\mathbb{R}^3)$ such that T is not self-adjoint (with respect to the usual inner product) and such that there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of T.

Proof. The statement is true. To see this, consider the linear operator T defined by its action on the basis $(1,0,0),(0,1,0),(0,1,1)$:

$$\begin{aligned} T(1,0,0) &= (0,0,0) \\ T(0,1,0) &= (0,0,0) \\ T(0,1,1) &= (0,1,1). \end{aligned}$$

Notice $T(1,0,0) = 0 \cdot (1,0,0)$ and $T(0,1,0) = 0 \cdot (0,1,0)$, so that $(1,0,0)$ and $(0,1,0)$ are eigenvectors with eigenvalue 0. Also, $(0,1,1)$ is an eigenvector with eigenvalue 1. Thus $(1,0,0),(0,1,0),(0,1,1)$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors of T. That T is not self-adjoint follows from the contrapositive of 7.22, since $(0,1,0)$ and $(0,1,1)$ correspond to distinct eigenvalues yet they are not orthogonal. $\square$

Problem 3

Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^3)$ such that 2 and 3 are the only eigenvalues of T and $T^2 - 5T + 6I \neq 0$.

Proof. Define $T \in \mathcal{L}(\mathbb{C}^3)$ by its action on the standard basis:

$$\begin{aligned} Te_1 &= 2e_2 \\ Te_2 &= e_1 + 2e_2 \\ Te_3 &= 3e_3. \end{aligned}$$

Then

$$\mathcal{M}(T) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

By 5.32, the only eigenvalues of T are the entries on the diagonal: 2 and 3. Now notice

$$
\begin{aligned}
(T^2 - 5T + 6I)e_2 &= (T - 3I)(T - 2I)e_2 \\
&= (T - 3I)(Te_2 - 2e_2) \\
&= (T - 3I)(e_1 + 2e_2 - 2e_2) \\
&= (T - 3I)e_1 \\
&= Te_1 - 3e_1 \\
&= -e_1,
\end{aligned}
$$

so that $T^2 - 5T + 6I \neq 0$. Thus T is an operator of the desired form. $\square$

Problem 5

Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T.

Proof. ($\Leftarrow$) Suppose all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. By 5.41, V has a basis consisting of eigenvectors of T. Dividing each element of the basis by its norm produces an orthonormal basis consisting of eigenvectors of T. By the Real Spectral Theorem, T is self-adjoint, as desired.

($\Rightarrow$) Conversely, suppose T is self-adjoint as suppose $v_1, v_2 \in V$ are eigenvectors of T corresponding to eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \neq \lambda_2$. It follows

$$
\begin{aligned}
0 &= \langle Tv_1, v_2\rangle - \langle v_1, Tv_2\rangle \\
&= \langle \lambda_1 v_1, v_2\rangle - \langle v_1, \lambda_2 v_2\rangle \\
&= \lambda_1\langle v_1, v_2\rangle - \overline{\lambda_2}\langle v_1, v_2\rangle \\
&= \lambda_1\langle v_1, v_2\rangle - \lambda_2\langle v_1, v_2\rangle \\
&= (\lambda_1 - \lambda_2)\langle v_1, v_2\rangle.
\end{aligned}
$$

Since $\lambda_1 \neq \lambda_2$, it must be that $\langle v_1, v_2\rangle = 0$. Thus all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal. By the Real Spectral Theorem, since T is self-adjoint, T is diagonalizable. And by 5.34, this implies

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T, completing the proof. $\square$

Problem 6

Prove that a normal operator on a complex vector space is self-adjoint if and only if all its eigenvalues are real.

Proof. Let T be a normal operator on a complex vector space, V.

($\Rightarrow$) Suppose T is self-adjoint. Then by 7.13, all eigenvalues of T are real.

($\Leftarrow$) Conversely, suppose all eigenvalues of T are real. By the Complex Spectral Theorem, there exists an orthonormal basis $v_1, \ldots, v_n$ of V consisting of eigenvectors of T. Thus there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $Tv_k = \lambda_k v_k$ for $k = 1, \ldots, n$. Thus $\mathcal{M}(T)$ is diagonal, and all entries along the diagonal are real. Therefore $\mathcal{M}(T)$ equals the conjugate transpose of $\mathcal{M}(T)$. By 7.10, this implies $\mathcal{M}(T) = \mathcal{M}(T^*)$, and we conclude $T = T^*$, so that T is indeed self-adjoint. $\square$

Problem 7

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Proof. By the Complex Spectral Theorem, since T is normal, V has an orthonormal basis $v_1, \ldots, v_n$ consisting of eigenvectors of T. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the corresponding eigenvalues, so that

$$Tv_k = \lambda_k v_k$$

for $k = 1, \ldots, n$. Repeatedly applying T to both sides of the equation above 8 times yields

$$T^9 v_k = (\lambda_k)^9 v_k \quad \text{and} \quad T^8 v_k = (\lambda_k)^8 v_k.$$

Since $T^9 = T^8$, we conclude $(\lambda_k)^9 = (\lambda_k)^8$ and thus $\lambda_k \in \{0, 1\}$. In particular, all eigenvalues of T are real, hence by Problem 6 we have that T is self-adjoint.

To see that $T^2 = T$, notice

$$\begin{aligned} T^2 v_k &= (\lambda_k)^2 v_k \\ &= \lambda_k v_k \\ &= T v_k, \end{aligned}$$

where the second equality follows from the fact that $\lambda_k \in \{0, 1\}$, and the proof is complete. $\square$

Problem 9

Suppose V is a complex inner product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V)$ is called a ***square root*** of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

Proof. Suppose $T \in \mathcal{L}(V)$ is normal. By the Complex Spectral Theorem, V has an orthonormal basis $v_1, \ldots, v_n$ consisting of eigenvectors of T. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the corresponding eigenvalues, so that

$$Tv_k = \lambda_k v_k$$

for $k = 1, \ldots, n$. Define $S \in \mathcal{L}(V)$ by its action on this basis:

$$Sv_k = \sqrt{\lambda_k} v_k,$$

choosing the complex square root $\sqrt{\lambda_k}$ by some definite rule. Let $v \in V$. Then there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. It follows

$$\begin{aligned}
S^2 v &= S^2(\alpha_1 v_1 + \cdots + \alpha_n v_n) \\
&= S\left(\alpha_1 \sqrt{\lambda_1} v_1 + \cdots + \alpha_n \sqrt{\lambda_n} v_n\right) \\
&= \alpha_1 \lambda_1 v_1 + \cdots + \alpha_n \lambda_n v_n \\
&= \alpha_1 T v_1 + \cdots + \alpha_n T v_n \\
&= T(\alpha_1 v_1 + \cdots + \alpha_n v_n) \\
&= Tv.
\end{aligned}$$

Thus $S^2 = T$, and indeed T has a square root, as was to be shown. $\square$

Problem 11

Prove or give a counterexample: every self-adjoint operator on V has a cube root. (An operator $T \in \mathcal{L}(V)$ is called a ***cube root*** of $T \in \mathcal{L}(V)$ if $S^3 = T$.)

Proof. Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Regardless of whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{R}$, both Spectral Theorems imply that V has an orthonormal basis $v_1, \ldots, v_n$ consisting of eigenvectors of T. By 7.13, all eigenvalues of T are real. So let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be the eigenvalues corresponding to $v_1, \ldots, v_n$, so that

$$Tv_k = \lambda_k v_k$$

for $k = 1, \ldots, n$. Define $S \in \mathcal{L}(V)$ by its action on this basis:

$$Sv_k = (\lambda_k)^{\frac{1}{3}} v_k,$$

Let $v \in V$. Then there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. It follows

$$\begin{aligned}
S^3 v &= S^3(\alpha_1 v_1 + \cdots + \alpha_n v_n) \\
&= S^2\left(\alpha_1 (\lambda_1)^{\frac{1}{3}} v_1 + \cdots + \alpha_n (\lambda_n)^{\frac{1}{3}} v_n\right) \\
&= S\left(\alpha_1 (\lambda_1)^{\frac{2}{3}} v_1 + \cdots + \alpha_n (\lambda_n)^{\frac{2}{3}} v_n\right) \\
&= \alpha_1 \lambda_1 v_1 + \cdots + \alpha_n \lambda_n v_n \\
&= \alpha_1 T v_1 + \cdots + \alpha_n T v_n \\
&= T(\alpha_1 v_1 + \cdots + \alpha_n v_n) \\
&= Tv.
\end{aligned}$$

Thus $S^3 = T$, and indeed T has a cube root. Thus, all self-adjoint operators on a finite-dimensional inner product space have a cube root. $\square$

Problem 13

Give an alternative proof of the Complex Spectral Theorem that avoids Schur's Theorem and instead follows the pattern of the proof of the Real Spectral Theorem.

Proof. Suppose (c) holds, so that T has a diagonal matrix with respect to some orthonormal basis of V. The matrix of T^* (with respect to the same basis) is obtained by taking the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix. Any two diagonal matrices commute; thus T commutes with T^*, which means that T is normal. That is, (a) holds.

We will prove that (a) implies (b) by induction on $\dim V$. For our base case, suppose $\dim V = 1$. Since 5.21 guarantees the existence of an eigenvector of T, clearly (b) is true in this case. Next assume that $\dim V > 1$ and that (a) implies (b) for all complex inner product spaces of smaller dimension.

Suppose (a) holds, so that T is normal. Let u be an eigenvector of T with $\|u\| = 1$, and set $U = \operatorname{span}(u)$. Clearly U is invariant under T. By Problem 3 of 7A, this implies that $U^\perp$ is invariant under T^* as well. But of course T^* is also normal, and since $\dim U^\perp = \dim V - 1$, our inductive hypothesis implies that there exists an orthonormal basis of $U^\perp$ consisting of eigenvectors of $T\mid_{U^\perp}$. Adjoining u to this basis gives an orthonormal basis of V consisting of eigenvectors of T, completing the proof that (a) implies (b).

We have proved that (c) implies (a) and that (a) implies (b). Clearly (b) implies (c), and the proof is complete. $\square$

Problem 15

Find the value of x such that the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & x \end{bmatrix}$$

is normal.

Proof. Let M be the above matrix. We wish to find $x \in \mathbb{F}$ such that $MM^* = M^*M$. Notice

$$\begin{aligned} MM^* &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & x \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & x \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & x \\ 1 & x & 1+x^2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} M^*M &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & x \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & x \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & x \\ 1 & 2 & 1 \\ x & 1 & 1+x^2 \end{bmatrix}. \end{aligned}$$

Thus it must be that $x = 1$. $\square$

C: Positive Operators and Isometries

Problem 1

Prove or give a counterexample: If $T \in \mathcal{L}(V)$ is self-adjoint and there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $\langle Te_j, e_j \rangle \geq 0$ for each j, then T is a positive operator.

Proof. The statement is false. To see this, let $e_1, e_2 \in \mathbb{R}^2$ be the standard basis and consider $T \in \mathcal{L}(\mathbb{R}^2)$ defined by

$$\begin{aligned} Te_1 &= e_1 \\ Te_2 &= -e_2. \end{aligned}$$

Then

$$\mathcal{M}(T) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and since $\mathcal{M}(T)$ is diagonal, T must be self-adjoint by the Real Spectral Theorem. But notice that the basis

$$v_1 = \frac{1}{\sqrt{2}}(e_1 + e_2)$$
$$v_2 = \frac{1}{\sqrt{2}}(e_1 - e_2)$$

is orthonormal and that

$$\langle Tv_1, v_1 \rangle = \langle v_2, v_1 \rangle = 0$$

and

$$\langle Tv_2, v_2 \rangle = \langle v_1, v_2 \rangle = 0.$$

Thus T is of the desired form, but T is not a positive operator, since

$$\langle Te_2, e_2 \rangle = \langle -e_2, e_2 \rangle = -1,$$

completing the proof. □

Problem 3

Suppose T is a positive operator on V and U is a subspace of V invariant under T. Prove that $T\,|_U \in \mathcal{L}(U)$ is a positive operator on U.

Proof. That $T\,|_U$ is self-adjoint follows by 7.28. Let $u \in U$. Then, since

$$\langle T\,|_U\,(u), u \rangle = \langle Tu, u \rangle > 0,$$

$T\,|_U$ is a positive operator on U, as was to be shown. □

Problem 5

Prove that the sum of two positive operators on V is positive.

Proof. Let $S, T \in \mathcal{L}(V)$ be positive operators. Notice

$$(S+T)^* = S^* + T^* = S + T,$$

hence $S+T$ is self-adjoint. Next, let $v \in V$. It follows

$$\begin{aligned} \langle (S+T)v, v \rangle &= \langle Sv + Tv, v \rangle \\ &= \langle Sv, v \rangle + \langle Tv, v \rangle \\ &\geq 0, \end{aligned}$$

and thus $S+T$ is a positive operator as well. □

Problem 7

Suppose T is a positive operator on V. Prove that T is invertible if and only if

$$\langle Tv, v\rangle > 0$$

for every $v \in V$ with $v \neq 0$.

Proof. Let T be a positive operator on V.

($\Rightarrow$) Suppose T is invertible and let $v \in V \setminus \{0\}$. Since T is a positive operator, by 7.35(e) there exists $R \in \mathcal{L}(V)$ such that $T = R^2$. Since T is invertible, so is R. In particular, R is injective, and thus $Rv \neq 0$. It follows

$$\begin{aligned}
\langle Tv, v\rangle &= \langle R^2, v\rangle \\
&= \langle Rv, R^* v\rangle \\
&= \langle Rv, Rv\rangle \\
&= \|Rv\|^2 \\
&> 0,
\end{aligned}$$

completing the proof in one direction.

($\Leftarrow$) Now suppose $\langle Tv, v\rangle > 0$ for every $v \in V \setminus \{0\}$. Assume by way of contraction that T is not invertible, so that there exists $w \in V \setminus \{0\}$ such that $Tw = 0$. But then $\langle Tw, w\rangle = \langle 0, w\rangle = 0$, a contradiction. Thus T must be invertible, completing the proof. $\square$

Problem 9

Prove or disprove: the identity operator on $\mathbb{F}^2$ has infinitely many self-adjoint square roots.