

Chapter 3: Linear Maps

Linear Algebra Done Right, by Sheldon Axler

A: The Vector Space of Linear Maps

Problem 1

Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxy).$$

Show that T is linear if and only if $b = c = 0$.

Proof. (\Leftarrow) Suppose $b = c = 0$. Then

$$T(x, y, z) = (2x - 4y + 3z, 6x).$$

Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$. Then

$$\begin{aligned} T((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2), 6(x_1 + x_2)) \\ &= (2x_1 + 2x_2 - 4y_1 - 4y_2 + 3z_1 + 3z_2, 6x_1 + 6x_2) \\ &= (2x_1 - 4y_1 + 3z_1, 6x_1) + (2x_2 - 4y_2 + 3z_2, 6x_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2). \end{aligned}$$

Now, for $\lambda \in \mathbb{F}$ and $(x, y, z) \in \mathbb{R}^3$, we have

$$\begin{aligned} T(\lambda(x, y, z)) &= T(\lambda x, \lambda y, \lambda z) \\ &= (2(\lambda x) - 4(\lambda y) + 3(\lambda z), 6(\lambda x)) \\ &= (\lambda(2x - 4y + 3z), \lambda(6x)) \\ &= \lambda(2x - 4y + 3z, 6x) \\ &= \lambda T(x, y, z), \end{aligned}$$

and thus T is a linear map.

(\Rightarrow) Suppose T is a linear map. Then

$$T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \quad (\dagger)$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$. In particular, by applying the definition of T and comparing first coordinates of both sides of (\dagger) , we have

$$\begin{aligned} 2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + b &= \\ (2x_1 - 4y_1 + 3z_1 + b) + (2x_2 - 4y_2 + 3z_2 + b), \end{aligned}$$

and after simplifying, we have $b = 2b$, and hence $b = 0$. Now by applying the definition of T and comparing second coordinates of both sides of (\dagger) , we have

$$\begin{aligned} 6(x_1 + x_2) + c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2) &= 6x_1 + c(x_1 y_1 z_1) + 6x_2 + c(x_2 y_2 z_2) \\ &= 6(x_1 + x_2) + c(x_1 y_1 z_1 + x_2 y_2 z_2), \end{aligned}$$

which implies

$$c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2) = c(x_1 y_1 z_1 + x_2 y_2 z_2).$$

Now suppose $c \neq 0$. Then choosing $(x_1, y_1, z_1) = (x_2, y_2, z_2) = (1, 1, 1)$, the equation above implies $8 = 2$, a contradiction. Thus $c = 0$, completing the proof. \square

Problem 3

Suppose $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \dots, m$ and $k = 1, \dots, n$ such that

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

for every $(x_1, \dots, x_n) \in \mathbb{F}^n$.

Proof. Given $x \in \mathbb{F}^n$, we may write

$$x = x_1 e_1 + \dots + x_n e_n,$$

where e_1, \dots, e_n is the standard basis of \mathbb{F}^n . Since T is linear, we have

$$Tx = T(x_1 e_1 + \dots + x_n e_n) = x_1 T e_1 + \dots + x_n T e_n.$$

Now for each $T e_k \in \mathbb{F}^m$, where $k = 1, \dots, n$, there exist $A_{1,k}, \dots, A_{m,k} \in \mathbb{F}$ such that

$$\begin{aligned} T e_k &= A_{1,k} e_1 + \dots + A_{m,k} e_m \\ &= (A_{1,k}, \dots, A_{m,k}) \end{aligned}$$

and thus

$$x_k T e_k = (A_{1,k} x_k, \dots, A_{m,k} x_k).$$

Therefore, we have

$$\begin{aligned} Tx &= \sum_{k=1}^n (A_{1,k} x_k, \dots, A_{m,k} x_k) \\ &= \left(\sum_{k=1}^n A_{1,k} x_k, \dots, \sum_{k=1}^n A_{m,k} x_k \right), \end{aligned}$$

and thus there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \dots, m$ and $k = 1, \dots, n$ of the desired form. \square

Problem 5

Prove that $\mathcal{L}(V, W)$ is a vector space.

Proof. We check each property in turn.

Commutative: Given $S, T \in \mathcal{L}(V, W)$ and $v \in V$, we have

$$(T + S)(v) = Tv + Sv = Sv + Tv = (S + T)(v)$$

and so addition is commutative.

Associative: Given $R, S, T \in \mathcal{L}(V, W)$ and $v \in V$, we have

$$\begin{aligned} ((R + S) + T)(v) &= (R + S)(v) + Tv = Rv + Sv + Tv \\ &= R + (S + T)(v) = (R + (S + T))(v) \end{aligned}$$

and so addition is associative. And given $a, b \in \mathbb{F}$, we have

$$((ab)T)(v) = (ab)(Tv) = a(b(Tv)) = (a(bT))(v)$$

and so scalar multiplication is associative as well.

Additive identity: Let $0 \in \mathcal{L}(V, W)$ denote the zero map, let $T \in \mathcal{L}(V, W)$, and let $v \in V$. Then

$$(T + 0)(v) = Tv + 0v = Tv + 0 = Tv$$

and so the zero map is the additive identity.

Additive inverse: Let $T \in \mathcal{L}(V, W)$ and define $(-T) \in \mathcal{L}(V, W)$ by $(-T)v = -Tv$. Then

$$(T + (-T))(v) = Tv + (-T)v = Tv - Tv = 0,$$

and so $(-T)$ is the additive inverse for each $T \in \mathcal{L}(V, W)$.

Multiplicative identity: Let $T \in \mathcal{L}(V, W)$. Then

$$(1T)(v) = 1(Tv) = Tv$$

and so the multiplicative identity of \mathbb{F} is the multiplicative identity of scalar multiplication.

Distributive properties: Let $S, T \in \mathcal{L}(V, W)$, $a, b \in \mathbb{F}$, and $v \in V$. Then

$$\begin{aligned} (a(S + T))(v) &= a((S + T)(v)) = a(Sv + Tv) = a(Sv) + a(Tv) \\ &= (aS)(v) + (aT)(v) \end{aligned}$$

and

$$((a + b)T)(v) = (a + b)(Tv) = a(Tv) + b(Tv) = (aT)(v) + (bT)(v)$$

and so the distributive properties hold.

Since all properties of a vector space hold, we see $\mathcal{L}(V, W)$ is in fact a vector space, as desired. \square

Problem 7

Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Proof. Since $\dim V = 1$, a basis of V consists of a single vector. So let $w \in V$ be such a basis. Then there exists $\alpha \in \mathbb{F}$ such that $v = \alpha w$ and $\lambda \in \mathbb{F}$ such that $Tw = \lambda w$. It follows

$$Tv = T(\alpha w) = \alpha Tw = \alpha \lambda w = \lambda(\alpha w) = \lambda v,$$

as desired. \square

Problem 9

Give an example of a function $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbb{C}$ but φ is not linear. (Here \mathbb{C} is thought of as a complex vector space.)

Proof. Define

$$\begin{aligned} \varphi : \mathbb{C} &\rightarrow \mathbb{C} \\ x + yi &\mapsto x - yi. \end{aligned}$$

Then for $x_1 + y_1i, x_2 + y_2i \in \mathbb{C}$, it follows

$$\begin{aligned} \varphi((x_1 + y_1i) + (x_2 + y_2i)) &= \varphi((x_1 + x_2) + (y_1 + y_2)i) \\ &= (x_1 + x_2) - (y_1 + y_2)i \\ &= (x_1 - y_1)i + (x_2 - y_2)i \\ &= \varphi(x_1 + y_1i) + \varphi(x_2 + y_2i) \end{aligned}$$

and so φ satisfies the additivity requirement. However, we have

$$\varphi(i \cdot i) = \varphi(-1) = -1$$

and

$$i \cdot \varphi(i) = i(-i) = 1$$

and hence φ fails the homogeneity requirement of a linear map. \square

Problem 11

Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V . In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Suppose U is a subspace of V and $S \in \mathcal{L}(U, W)$. Let v_1, \dots, v_m be a basis of U and let $v_1, \dots, v_m, v_{m+1}, \dots, v_n$ be an extension of this basis to V . For any $z \in V$, there exist $a_1, \dots, a_n \in \mathbb{F}$ such that $z = \sum_{k=1}^n a_k v_k$, and so we define

$$\begin{aligned} T : V &\rightarrow W \\ \sum_{k=1}^n a_k v_k &\mapsto \sum_{k=1}^m a_k S v_k + \sum_{k=m+1}^n a_k v_k. \end{aligned}$$

Since every $v \in V$ has a unique representation as a linear combination of elements of our basis, the map is well-defined. We first show T is a linear map. So suppose $z_1, z_2 \in V$. Then there exist $a_1, \dots, a_n \in \mathbb{F}$ and $b_1, \dots, b_n \in \mathbb{F}$ such that

$$z_1 = a_1 v_1 + \dots + a_n v_n \quad \text{and} \quad z_2 = b_1 v_1 + \dots + b_n v_n.$$

It follows

$$\begin{aligned} T(z_1 + z_2) &= T\left(\sum_{k=1}^n a_k v_k + \sum_{k=1}^n b_k v_k\right) \\ &= T\left(\sum_{k=1}^n (a_k + b_k) v_k\right) \\ &= \sum_{k=1}^m (a_k + b_k) S v_k + \sum_{k=m+1}^n (a_k + b_k) v_k \\ &= \left(\sum_{k=1}^m a_k S v_k + \sum_{k=m+1}^n a_k v_k\right) + \left(\sum_{k=1}^m b_k S v_k + \sum_{k=m+1}^n b_k v_k\right) \\ &= T\left(\sum_{k=1}^n a_k v_k\right) + T\left(\sum_{k=1}^n b_k v_k\right) \\ &= T z_1 + T z_2 \end{aligned}$$

and so T is additive. To see that T is homogeneous, let $\lambda \in \mathbb{F}$ and $z \in V$, so

that we may write $z = \sum_{k=1}^n a_k v_k$ for some $a_1, \dots, a_n \in \mathbb{F}$. We have

$$\begin{aligned}
T(\lambda z) &= T\left(\lambda \sum_{k=1}^n a_k v_k\right) \\
&= T\left(\sum_{k=1}^n (\lambda a_k) v_k\right) \\
&= S\left(\sum_{k=1}^m (\lambda a_k) v_k\right) + \sum_{k=m+1}^n (\lambda a_k) v_k \\
&= \lambda S\left(\sum_{k=1}^m a_k v_k\right) + \lambda \sum_{k=m+1}^n a_k v_k \\
&= \lambda \left(S\left(\sum_{k=1}^m a_k v_k\right) + \sum_{k=m+1}^n \lambda a_k v_k \right) \\
&= \lambda T\left(\sum_{k=1}^n a_k v_k\right) \\
&= \lambda Tz
\end{aligned}$$

and so T is homogeneous as well hence $T \in \mathcal{L}(V, W)$. Lastly, to see that $T|_U = S$, let $u \in U$. Then there exist $a_1, \dots, a_m \in \mathbb{F}$ such that $u = \sum_{k=1}^m a_k v_k$, and hence

$$\begin{aligned}
Tu &= T\left(\sum_{k=1}^m a_k v_k\right) \\
&= S\left(\sum_{k=1}^m a_k v_k\right) \\
&= Su,
\end{aligned}$$

and so indeed T agrees with S on U , completing the proof. \square

Problem 13

Suppose v_1, \dots, v_m is a linearly dependent list of vectors in V . Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \dots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \dots, m$.

Proof. Since v_1, \dots, v_m is linearly dependent, one of them may be written as a linear combination of the others. Without loss of generality, suppose this is v_m .

Then there exist $a_1, \dots, a_{m-1} \in \mathbb{F}$ such that

$$v_m = a_1 v_1 + \dots + a_{m-1} v_{m-1}.$$

Since $W \neq \{0\}$, there exists some nonzero $z \in W$. Define $w_1, \dots, w_m \in W$ by

$$w_k = \begin{cases} z & \text{if } k = m \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose there exists $T \in \mathcal{L}(V, W)$ such that $Tv_k = w_k$ for $k = 1, \dots, m$. It follows

$$\begin{aligned} T(0) &= T(v_m - a_1 v_1 - \dots - a_{m-1} v_{m-1}) \\ &= Tv_m - a_1 Tv_1 - \dots - a_{m-1} Tv_{m-1} \\ &= z. \end{aligned}$$

But $z \neq 0$, and thus $T(0) \neq 0$, a contradiction, since linear maps take 0 to 0. Therefore, no such linear map can exist. \square

B: Null Spaces and Ranges

Problem 1

Give an example of a linear map T such that $\dim \text{null } T = 3$ and $\dim \text{range } T = 2$.

Proof. Define the map

$$\begin{aligned} T : \mathbb{R}^5 &\rightarrow \mathbb{R}^5 \\ (x_1, x_2, x_3, x_4, x_5) &\mapsto (0, 0, 0, x_4, x_5). \end{aligned}$$

First we show T is a linear map. Suppose $x, y \in \mathbb{R}^5$. Then

$$\begin{aligned} T(x + y) &= T((x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5) \\ &= (0, 0, 0, x_4 + y_4, x_5 + y_5) \\ &= (0, 0, 0, x_4, x_5) + (0, 0, 0, y_4, y_5) \\ &= T(x) + T(y). \end{aligned}$$

Next let $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} T(\lambda x) &= T(\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) \\ &= (0, 0, 0, \lambda x_4, \lambda x_5) \\ &= \lambda(0, 0, 0, x_4, x_5) \\ &= \lambda T(x), \end{aligned}$$

and so T is in fact a linear map. Now notice that

$$\text{null } T = \{(x_1, x_2, x_3, 0, 0) \in \mathbb{R}^5 \mid x_1, x_2, x_3 \in \mathbb{R}\}.$$

This space clearly has as a basis $e_1, e_2, e_3 \in \mathbb{R}^5$ and hence has dimension 3. Now, by the Fundamental Theorem of Linear Maps, we have

$$\dim \mathbb{R}^5 = 3 + \dim \text{range } T$$

and hence $\dim \text{range } T = 2$, as desired. \square

Problem 3

Suppose v_1, \dots, v_m is a list of vectors in V . Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m.$$

- (a) What property of T corresponds to v_1, \dots, v_m spanning V ?
- (b) What property of T corresponds to v_1, \dots, v_m being linearly independent?

Proof. (a) We claim surjectivity of T corresponds to v_1, \dots, v_m spanning V . To see this, suppose T is surjective, and let $w \in V$. Then there exists $z \in \mathbb{F}^m$ such that $Tz = w$. This yields

$$z_1 v_1 + \dots + z_m v_m = w,$$

and hence every $w \in V$ can be expressed as a linear combination of v_1, \dots, v_m . That is, $\text{span}(v_1, \dots, v_m) = V$.

- (b) We claim injectivity of T corresponds to v_1, \dots, v_m being linearly independent. To see this, suppose T is injective, and let $a_1, \dots, a_m \in \mathbb{F}$ such that

$$a_1 v_1 + \dots + a_m v_m = 0.$$

Then

$$T(a) = T(a_1, \dots, a_m) = a_1 v_1 + \dots + a_m v_m = 0$$

which is true iff $a = 0$ since T is injective. That is, $a_1 = \dots = a_m = 0$ and hence v_1, \dots, v_m is linearly independent. \square

Problem 5

Give an example of a linear map $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that

$$\text{range } T = \text{null } T.$$

Proof. Define

$$\begin{aligned} T : \mathbb{R}^4 &\rightarrow \mathbb{R}^4 \\ (x_1, x_2, x_3, x_4) &\mapsto (x_3, x_4, 0, 0). \end{aligned}$$

Clearly T is a linear map, and we have

$$\text{null } T = \{(x_1, x_2, x_3, x_4) \mid x_3 = x_4 = 0 \in \mathbb{R}\} = \mathbb{R}^2 \times \{0\}^2$$

and

$$\text{range } T = \{(x, y, 0, 0) \mid x, y \in \mathbb{R}\} = \mathbb{R}^2 \times \{0\}^2.$$

Hence $\text{range } T = \text{null } T$, as desired. \square

Problem 7

Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) \mid T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Proof. Let $Z = \{T \in \mathcal{L}(V, W) \mid T \text{ is not injective}\}$, let v_1, \dots, v_m be a basis of V , where $m \geq 2$, and let w_1, \dots, w_n be a basis of W , where $n \geq m$. We define $T \in \mathcal{L}(V, W)$ by its behavior on the basis

$$Tv_k := \begin{cases} 0 & \text{if } k = 1 \\ w_2 & \text{if } k = 2 \\ \frac{1}{2}w_k & \text{otherwise} \end{cases}$$

so that clearly T is not injective since $Tv_1 = 0 = T(0)$, and hence $T \in Z$. Similarly, define $S \in \mathcal{L}(V, W)$ by its behavior on the basis

$$Sv_k := \begin{cases} w_1 & \text{if } k = 1 \\ 0 & \text{if } k = 2 \\ \frac{1}{2}w_k & \text{otherwise} \end{cases}$$

and note that S is not injective either since $Sv_2 = 0 = S(0)$, and hence $S \in Z$. However, notice

$$(S + T)(v_k) = w_k \text{ for } k = 1, \dots, n$$

and hence $\text{null}(S + T) = \{0\}$ since it takes the basis of V to the basis of W , so that $S + T$ is in fact injective. Therefore $S + T \notin Z$, and Z is not closed under addition. Thus Z is not a subspace of $\mathcal{L}(V, W)$. \square

Problem 9

Suppose $T \in \mathcal{L}(V, W)$ is injective and v_1, \dots, v_n is linearly independent in V . Prove that Tv_1, \dots, Tv_n is linearly independent in W .

Proof. Suppose $a_1, \dots, a_n \in \mathbb{F}$ are such that

$$a_1Tv_1 + \dots + a_nTv_n = 0.$$

Since T is a linear map, it follows

$$T(a_1v_1 + \dots + a_nv_n) = 0.$$

But since $\text{null } T = \{0\}$ (by virtue of T being a linear map), this implies $a_1v_1 + \dots + a_nv_n = 0$. And since v_1, \dots, v_n are linearly independent, we must have $a_1 = \dots = a_n = 0$, which in turn implies Tv_1, \dots, Tv_n is indeed linearly independent in W . \square

Problem 11

Suppose S_1, \dots, S_n are injective linear maps such that $S_1S_2 \dots S_n$ makes sense. Prove that $S_1S_2 \dots S_n$ is injective.

Proof. For $n \in \mathbb{Z}_{\geq 2}$, let $P(n)$ be the statement: S_1, \dots, S_n are injective linear maps such that $S_1S_2 \dots S_n$ makes sense, and the product $S_1S_2 \dots S_n$ is injective. We induct on n .

Base case: Suppose $n = 2$, and assume $S_1 \in \mathcal{L}(V_0, V_1)$ and $S_2 \in \mathcal{L}(V_1, V_2)$, so that the product S_1S_2 is defined, and assume that both S_1 and S_2 are injective. Suppose $v_1, v_2 \in V_0$ are such that $v_1 \neq v_2$, and let $w_1 = S_2v_1$ and $w_2 = S_2v_2$. Since S_2 is injective, $w_1 \neq w_2$. And since S_1 is injective, this in turn implies that $S_1(w_1) \neq S_1(w_2)$. In other words, $S_1(S_2(v_1)) \neq S_1(S_2(v_2))$, so that S_1S_2 is injective as well, and hence $P(2)$ is true.

Inductive step: Suppose $P(k)$ is true for some $k \in \mathbb{Z}^+$, and consider the product $(S_1S_2 \dots S_k)S_{k+1}$. The term in parentheses is injective by hypothesis, and the product of this term with S_{k+1} is injective by our base case. Thus $P(k+1)$ is true.

By the principle of mathematical induction, the statement $P(n)$ is true for all $n \in \mathbb{Z}_{\geq 2}$, as was to be shown. \square

Problem 13

Suppose T is a linear map from \mathbb{F}^4 to \mathbb{F}^2 such that

$$\text{null } T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Proof. We claim the list

$$(5, 1, 0, 0), (0, 0, 7, 1)$$

is a basis of $\text{null } T$. This implies

$$\begin{aligned}\dim \text{range } T &= \dim \mathbb{F}^4 - \dim \text{null } T \\ &= 4 - 2 \\ &= 2,\end{aligned}$$

and hence T is surjective (since the only 2-dimensional subspace of \mathbb{F}^2 is the space itself). So let's prove our claim that this list is a basis.

Clearly the list is linearly independent. To see that it spans $\text{null } T$, suppose $x = (x_1, x_2, x_3, x_4) \in \text{null } T$, so that $x_1 = 5x_2$ and $x_3 = 7x_4$. We may write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5x_2 \\ x_2 \\ 7x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 7 \\ 1 \end{pmatrix},$$

and indeed x is in the span of our list, so that our list is in fact a basis, completing the proof. \square

Problem 15

Prove that there does not exist a linear map from \mathbb{F}^5 to \mathbb{F}^2 whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 \mid x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Proof. Suppose such a $T \in \mathcal{L}(\mathbb{F}^5, \mathbb{F}^2)$ did exist. We claim

$$(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$$

is a basis of $\text{null } T$. This implies

$$\begin{aligned}\dim \text{range } T &= \dim \mathbb{F}^5 - \dim \text{null } T \\ &= 5 - 2 \\ &= 3,\end{aligned}$$

which is absurd, since the codomain of T has dimension 2. Hence such a T cannot exist. So, let's prove our claim that this list is a basis.

Clearly $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$ is linearly independent. To see that it spans $\text{null } T$, suppose $x = (x_1, \dots, x_5) \in \text{null } T$, so that $x_1 = 3x_2$ and $x_3 = x_4 = x_5$. We may write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

and indeed x is in the span of our list, so that our list is in fact a basis, completing the proof. \square

Problem 17

Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Proof. (\Rightarrow) Suppose $T \in \mathcal{L}(V, W)$ is injective. If $\dim V > \dim W$, Theorem 3.23 tells us that no map from V to W would be injective, a contradiction, and so we must have $\dim V \leq \dim W$.

(\Leftarrow) Suppose $\dim V \leq \dim W$. Then the inclusion map $\iota : V \rightarrow W$ is both a linear map and injective. \square

Problem 19

Suppose V and W are finite-dimensional and that U is a subspace of V . Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\text{null } T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Proof. (\Leftarrow) Suppose $\dim U \geq \dim V - \dim W$. Since U is a subspace of V , there exists a subspace U' of V such that

$$V = U \oplus U'.$$

Let u_1, \dots, u_m be a basis for U , let u'_1, \dots, u'_n be a basis for U' , and let w_1, \dots, w_p be a basis for W . By hypothesis, we have

$$m \geq (m + n) - p,$$

which implies $p \geq n$. Thus we may define a linear map $T \in \mathcal{L}(V, W)$ by its values on the basis of $V = U \oplus U'$ by taking $Tu_k = 0$ for $k = 1, \dots, m$ and $Tu'_j = w_j$ for $j = 1, \dots, n$ (since $p \geq n$, there is a w_j for each u'_j). The map is linear by Theorem 3.5, and its null space is U by construction.

(\Rightarrow) Suppose U is a subspace of V , $T \in \mathcal{L}(V, W)$, and $\text{null } T = U$. Then, since $\text{range } T$ is a subspace of W , we have $\dim \text{range } T \leq \dim W$. Combining this inequality with the Fundamental Theorem of Linear Maps yields

$$\begin{aligned} \dim \text{null } T &= \dim V - \dim \text{range } T \\ &\geq \dim V - \dim W. \end{aligned}$$

Since $\dim \text{null } T = \dim U$, we have the desired inequality. \square

Problem 21

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W .

Proof. (\Rightarrow) Suppose $T \in \mathcal{L}(V, W)$ is surjective, so that W is necessarily finite-dimensional as well. Let v_1, \dots, v_m be a basis of V and let $n = \dim W$, where $m \geq n$ by surjectivity of T . Note that

$$Tv_1, \dots, Tv_m$$

span W . Thus we may reduce this list to a basis by removing some elements (possibly none, if $n = m$). Suppose this reduced list were $Tv_{i_1}, \dots, Tv_{i_n}$ for some $i_1, \dots, i_n \in \{1, \dots, m\}$. We define $S \in \mathcal{L}(W, V)$ by its behavior on this basis

$$S(Tv_{i_k}) := v_{i_k} \text{ for } k = 1, \dots, n.$$

Suppose $w \in W$. Then there exist $a_1, \dots, a_n \in \mathbb{F}$ such that

$$w = a_1Tv_{i_1} + \dots + a_nTv_{i_n}$$

and thus

$$\begin{aligned} TS(w) &= TS(a_1Tv_{i_1} + \dots + a_nTv_{i_n}) \\ &= T(S(a_1Tv_{i_1} + \dots + a_nTv_{i_n})) \\ &= T(a_1S(Tv_{i_1}) + \dots + a_nS(Tv_{i_n})) \\ &= T(a_1v_{i_1} + \dots + a_nv_{i_n}) \\ &= a_1Tv_{i_1} + \dots + a_nTv_{i_n} \\ &= w, \end{aligned}$$

and so TS is the identity map on W .

(\Leftarrow) Suppose there exists $S \in \mathcal{L}(W, V)$ such that $TS \in \mathcal{L}(W, W)$ is the identity map, and suppose by way of contradiction that T is not surjective, so that $\dim \text{range } TS < \dim W$. By the Fundamental Theorem of Linear Maps, this implies

$$\begin{aligned} \dim W &= \dim \text{null } TS + \dim \text{range } TS \\ &< \dim \text{null } TS + \dim W \end{aligned}$$

and hence $\dim \text{null } TS > 0$, a contradiction, since the identity map can only have trivial null space. Thus T is surjective, as desired. \square

Problem 23

Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}.$$

Proof. We will show that both $\dim \text{range } ST \leq \dim \text{range } S$ and $\dim \text{range } ST \leq \dim \text{range } T$, since this implies the desired inequality.

We first show that $\dim \text{range } ST \leq \dim \text{range } S$. Suppose $w \in \text{range } ST$. Then there exists $u \in U$ such that $ST(u) = w$. But this implies that $w \in \text{range } S$ as well, since $Tu \in S^{-1}(w)$. Thus $\text{range } ST \subseteq \text{range } S$, which implies $\dim \text{range } ST \leq \dim \text{range } S$.

We now show that $\dim \text{range } ST \leq \dim \text{range } T$. Note that if $v \in \text{null } T$, so that $Tv = 0$, then $ST(v) = 0$ (since linear maps take zero to zero). Thus we have $\text{null } T \subseteq \text{null } ST$, which implies $\dim \text{null } T \leq \dim \text{null } ST$. Combining this inequality with the Fundamental Theorem of Linear Maps applied to T yields

$$\dim U \leq \dim \text{null } ST + \dim \text{range } T. \quad (1)$$

Similarly, we have

$$\dim U = \dim \text{null } ST + \dim \text{range } ST. \quad (2)$$

Combining (1) and (2) yields

$$\dim \text{null } ST + \dim \text{range } ST \leq \dim \text{null } ST + \dim \text{range } T$$

and hence $\dim \text{range } ST \leq \dim \text{range } T$, completing the proof. \square

Problem 25

Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{range } T_1 \subseteq \text{range } T_2$ if and only if there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2 S$.

Proof. (\Leftarrow) Suppose there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2 S$, and let $w \in \text{range } T_1$. Then there exists $v \in V$ such that $T_1 v = w$, and hence $T_2 S(v) = w$. But then $w \in \text{range } T_2$ as well, and hence $\text{range } T_1 \subseteq \text{range } T_2$.

(\Rightarrow) Suppose $\text{range } T_1 \subseteq \text{range } T_2$, and let v_1, \dots, v_n be a basis of V . Let $w_k = T_1 v_k$ for $k = 1, \dots, n$. Then there exist $u_1, \dots, u_n \in V$ such that $T_2 u_k = w_k$ for $k = 1, \dots, n$ (since $w_k \in \text{range } T_1$ implies $w_k \in \text{range } T_2$). Define $S \in \mathcal{L}(V, V)$ by its behavior on the basis

$$Sv_k := u_k \text{ for } k = 1, \dots, n.$$

It follows that $T_2 S(v_k) = T_2 u_k = w_k = T_1 v_k$. Since $T_2 S$ and T_1 are equal on the basis, they are equal as linear maps, as was to be shown. \square

Problem 27

Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.

Proof. Suppose $\deg p = n$, and consider the linear map

$$\begin{aligned} D : \mathcal{P}_{n+1}(\mathbb{R}) &\rightarrow \mathcal{P}_n(\mathbb{R}) \\ q &\mapsto 5q'' + 3q'. \end{aligned}$$

If we can show D is surjective, we're done, since this implies that there exists some $q \in \mathcal{P}_{n+1}(\mathbb{R})$ such that $Dq = 5q'' + 3q' = p$. To that end, suppose $r \in \text{null } D$. Then we must have $r'' = 0$ and $r' = 0$, which is true if and only if r is constant. Thus any $\alpha \in \mathbb{R}^\times$ is a basis of $\text{null } D$, and so $\dim \text{null } D = 1$. By the Fundamental Theorem of Linear Maps, we have

$$\dim \text{range } D = \dim \mathcal{P}_{n+1}(\mathbb{R}) - \dim \text{null } D,$$

and hence

$$\dim \text{range } D = (n + 2) - 1 = n + 1.$$

Since the only subspace of $\mathcal{P}_n(\mathbb{R})$ with dimension $n + 1$ is the space itself, D is surjective, as desired. \square

Problem 29

Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in $\text{null } \varphi$. Prove that

$$V = \text{null } \varphi \oplus \{au \mid a \in \mathbb{F}\}.$$

Proof. First note that since $u \in V - \text{null } \varphi$, there exists some nonzero $\varphi(u) \in \text{range } \varphi$ and hence $\dim \text{range } \varphi \geq 1$. But since $\text{range } \varphi \subseteq \mathbb{F}$, and $\dim \mathbb{F} = 1$, we must have $\dim \text{range } \varphi = 1$. Thus, letting $n = \dim V$, it follows

$$\begin{aligned} \dim \text{null } \varphi &= \dim V - \dim \text{range } \varphi \\ &= n - 1. \end{aligned}$$

Let v_1, \dots, v_{n-1} be a basis for $\text{null } \varphi$. We claim v_1, \dots, v_{n-1}, u is an extension of this basis to a basis of V , which would then imply $V = \text{null } \varphi \oplus \{au \mid a \in \mathbb{F}\}$, as desired.

To show v_1, \dots, v_{n-1}, u is a basis of V , it suffices to show linearly independence (since it has length $n = \dim V$). So suppose $a_1, \dots, a_n \in \mathbb{F}$ are such that

$$a_1v_1 + \dots + a_{n-1}v_{n-1} + a_nu = 0.$$

We may write

$$a_nu = -a_1v_1 - \dots - a_{n-1}v_{n-1},$$

which implies $a_nu \in \text{null } \varphi$. By hypothesis, $u \notin \text{null } \varphi$, and thus we must have $a_n = 0$. But now each of the a_1, \dots, a_{n-1} must be 0 as well (since v_1, \dots, v_{n-1} form a basis of $\text{null } \varphi$ and thus are linearly independent). Therefore, v_1, \dots, v_{n-1}, u is indeed linearly independent, proving our claim. \square

Problem 31

Give an example of two linear maps T_1 and T_2 from \mathbb{R}^5 to \mathbb{R}^2 that have the same null space but are such that T_1 is not a scalar multiple of T_2 .

Proof. Let e_1, \dots, e_5 be the standard basis of \mathbb{R}^5 . We define $T_1, T_2 \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^2)$ by their behavior on the basis (using the standard basis for \mathbb{R}^2 as well)

$$\begin{aligned} T_1 e_1 &:= e_2 \\ T_1 e_2 &:= e_1 \\ T_1 e_k &:= 0 \text{ for } k = 3, 4, 5 \end{aligned}$$

and

$$\begin{aligned} T_2 e_1 &:= e_1 \\ T_2 e_2 &:= e_2 \\ T_2 e_k &:= 0 \text{ for } k = 3, 4, 5. \end{aligned}$$

Clearly $\text{null } T_1 = \text{null } T_2$. We claim T_2 is not a scalar multiple of T_1 . To see this, suppose not. Then there exists $\alpha \in \mathbb{R}$ such that $T_1 = \alpha T_2$. In particular, this implies $T_1 e_1 = \alpha T_2 e_1$. But this is absurd, since $T_1 e_1 = e_2$ and $T_2 e_1 = e_1$, and of course e_1, e_2 is linearly independent. Thus no such α can exist, and T_1, T_2 are as desired. \square

C: Matrices**Problem 1**

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W , the matrix of T has at least $\dim \text{range } T$ nonzero entries.

Proof. Let v_1, \dots, v_n be a basis of V , let w_1, \dots, w_m be a basis of W , let $r = \dim \text{range } T$, and let $s = \dim \text{null } T$. Then there are s basis vectors of V which map to zero and r basis vectors of V with nontrivial representation as linear combinations of w_1, \dots, w_m . That is, suppose $Tv_k \neq 0$, where $k \in \{1, \dots, n\}$. Then there exist $a_1, \dots, a_m \in \mathbb{F}$, not all zero, such that

$$Tv_k = a_1 w_1 + \dots + a_m w_m.$$

The coefficients form column k of $\mathcal{M}(T)$, and there are r such vectors in the basis of V . Hence there are r columns of $\mathcal{M}(T)$ with at least one nonzero entry, as was to be shown. \square

Problem 3

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except the entries in row j , column j , equal 1 for $1 \leq j \leq \dim \text{range } T$.

Proof. Let R be the subspace of V such that

$$V = R \oplus \text{null } T,$$

let r_1, \dots, r_m be a basis of R (where $m = \dim \text{range } T$), and let v_1, \dots, v_n be a basis of $\text{null } T$ (where $n = \dim \text{null } T$). Then $r_1, \dots, r_m, v_1, \dots, v_n$ is a basis of V . It follows that Tr_1, \dots, Tr_m is a basis of $\text{range } T$, and hence there is an extension of this list to a basis of W . Suppose $Tr_1, \dots, Tr_m, w_1, \dots, w_p$ is such an extension (where $p = \dim W - m$). Then, for $j = 1, \dots, m$, we have

$$Tr_j = \left(\sum_{i=1}^m \delta_{i,j} \cdot Tr_i \right) + \left(\sum_{k=1}^p 0 \cdot w_k \right),$$

where $\delta_{i,j}$ is the Kronecker delta function. Thus, column j of $\mathcal{M}(T)$ has an entry of 1 in row j and 0's elsewhere, where j ranges over 1 to $m = \dim \text{range } T$. Since $Tv_1 = \dots = Tv_n = 0$, the remaining columns of $\mathcal{M}(T)$ are all zero. Thus $\mathcal{M}(T)$ has the desired form. \square

Problem 5

Suppose w_1, \dots, w_n is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis v_1, \dots, v_m of V such that all the entries in the first row of $\mathcal{M}(T)$ (with respect to the bases v_1, \dots, v_m and w_1, \dots, w_n) are 0 except for possibly a 1 in the first row, first column.

Proof. First note that if $\text{range } T \subseteq \text{span}(w_2, \dots, w_n)$, the first row of $\mathcal{M}(T)$ will be all zeros regardless of choice of basis for V .

So suppose $\text{range } T \not\subseteq \text{span}(w_2, \dots, w_n)$ and let $u_1 \in V$ be such that $Tu_1 \notin \text{span}(w_2, \dots, w_n)$. There exist $a_1, \dots, a_n \in \mathbb{F}$ such that

$$Tu_1 = a_1 w_1 + \dots + a_n w_n,$$

and notice $a_1 \neq 0$ since $Tu_1 \notin \text{span}(w_2, \dots, w_n)$. Hence we may define

$$z_1 := \frac{1}{a_1} u_1.$$

It follows

$$Tz_1 = w_1 + \frac{a_2}{a_1} w_2 + \dots + \frac{a_n}{a_1} w_n. \quad (3)$$

Now extend z_1 to a basis z_1, \dots, z_m of V . Then for $k = 2, \dots, m$, there exist $A_{1,k}, \dots, A_{n,k} \in \mathbb{F}$ such that

$$Tz_k = A_{1,k}w_1 + \dots + A_{n,k}w_n,$$

and notice

$$\begin{aligned} T(z_k - A_{1,k}z_1) &= Tz_k - A_{1,k}Tz_1 \\ &= (A_{1,k}w_1 + \dots + A_{n,k}w_n) - A_{1,k} \left(w_1 + \frac{a_2}{a_1}w_2 + \dots + \frac{a_n}{a_1}w_n \right) \\ &= (A_{1,k} - A_{1,k})\frac{a_1}{a_1}w_1 + (A_{2,k} - A_{1,k}\frac{a_2}{a_1})w_2 + \dots + (A_{n,k} - A_{1,k}\frac{a_n}{a_1})w_n. \end{aligned} \quad (4)$$

Now we define a new list in V by

$$v_k := \begin{cases} z_1 & \text{if } k = 1 \\ z_k - A_{1,k}z_1 & \text{otherwise} \end{cases}$$

for $k = 1, \dots, m$. We claim v_1, \dots, v_m is a basis. To see this, it suffices to prove the list is linearly independent, since its length equals $\dim V$. So suppose $b_1, \dots, b_m \in \mathbb{F}$ are such that

$$b_1v_1 + \dots + b_mv_m = 0.$$

By definition of the v_k , it follows

$$b_1z_1 + b_2(z_2 - A_{1,2}z_1) + \dots + b_m(z_m - A_{1,m}z_1) = 0.$$

But since z_1, \dots, z_m is a basis of V , the expression on the LHS above is simply a linear combination of vectors in a basis. Thus we must have $b_1 = \dots = b_m = 0$, and indeed v_1, \dots, v_m are linearly independent, as claimed.

Finally, notice (3) tells us the first column of $\mathcal{M}(T, v_k, w_k)$ is all 0's except a 1 in the first entry, and (4) tells us the remaining columns have a 0 in the first entry. Thus $\mathcal{M}(T, v_k, w_k)$ has the desired form, completing the proof. \square

Problem 7

Suppose $S, T \in \mathcal{L}(V, W)$. Prove that $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Proof. Let v_1, \dots, v_m be a basis of V and let w_1, \dots, w_n be a basis of W . Also, let $A = \mathcal{M}(S)$ and $B = \mathcal{M}(T)$ be the matrices of these linear transformations with respect to these bases. It follows

$$\begin{aligned} (S+T)v_k &= Sv_k + Tv_k \\ &= (A_{1,k}w_1 + \dots + A_{n,k}w_n) + (B_{1,k}w_1 + \dots + B_{n,k}w_n) \\ &= (A_{1,k} + B_{1,k})w_1 + \dots + (A_{n,k} + B_{n,k})w_n. \end{aligned}$$

Hence $\mathcal{M}(S+T)_{j,k} = A_{j,k} + B_{j,k}$, and indeed we have $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$, as desired. \square

Problem 9

Suppose A is an m -by- n matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an n -by-1 matrix.

Prove that

$$Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}.$$

Proof. By definition, it follows

$$\begin{aligned} Ac &= \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= \begin{pmatrix} A_{1,1}c_1 + A_{1,2}c_2 + \cdots + A_{1,n}c_n \\ A_{2,1}c_1 + A_{2,2}c_2 + \cdots + A_{2,n}c_n \\ \vdots \\ A_{m,1}c_1 + A_{m,2}c_2 + \cdots + A_{m,n}c_n \end{pmatrix} \\ &= c_1 \begin{pmatrix} A_{1,1} \\ A_{2,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + c_2 \begin{pmatrix} A_{1,2} \\ A_{2,2} \\ \vdots \\ A_{m,2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} A_{1,n} \\ A_{2,n} \\ \vdots \\ A_{m,n} \end{pmatrix} \\ &= c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}, \end{aligned}$$

as desired. \square

Problem 11

Suppose $a = (a_1, \dots, a_n)$ is a 1-by- n matrix and C is an n -by- p matrix. Prove that

$$aC = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot}.$$

Proof. By definition, it follows

$$\begin{aligned}
aC &= (a_1, \dots, a_n) \begin{pmatrix} C_{1,1} & C_{1,2} & \dots & C_{1,p} \\ C_{2,1} & C_{2,2} & \dots & C_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \dots & C_{n,p} \end{pmatrix} \\
&= \left(\sum_{k=1}^n a_k C_{k,1}, \sum_{k=1}^n a_k C_{k,2}, \dots, \sum_{k=1}^n a_k C_{k,p} \right) \\
&= \sum_{k=1}^n (a_k C_{k,1}, \dots, a_k C_{k,p}) \\
&= \sum_{k=1}^n a_k (C_{k,1}, \dots, C_{k,p}) \\
&= \sum_{k=1}^n a_k C_{k,\cdot},
\end{aligned}$$

as desired. \square

Problem 13

Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E , and F are matrices whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense. Prove that $AB+AC$ and $DF+EF$ both make sense and that $A(B+C) = AB+AC$ and $(D+E)F = DF+EF$.

Proof. First note that if $A(B+C)$ makes sense, then the number of columns of A must equal the number of rows of $B+C$. But the sum of two matrices is only defined if their dimensions are equal, and hence the number of rows of both B and C must equal the number of columns of A . Thus $AB+AC$ makes sense. So suppose $A \in \mathbb{F}^{m,n}$ and $B, C \in \mathbb{F}^{n,p}$. It follows

$$\begin{aligned}
(A(B+C))_{j,k} &= \sum_{r=1}^n A_{j,r}(B+C)_{r,k} \\
&= \sum_{r=1}^n A_{j,r}(B_{r,k} + C_{r,k}) \\
&= \sum_{r=1}^n (A_{j,r}B_{r,k} + A_{j,r}C_{r,k}) \\
&= \sum_{r=1}^n A_{j,r}B_{r,k} + \sum_{r=1}^n A_{j,r}C_{r,k} \\
&= (AB)_{j,k} + (AC)_{j,k},
\end{aligned}$$

proving the first distributive property.

Now note that if $(D + E)F$ makes sense, then the number of columns of $D + E$ must equal the number of rows of F . Hence the number of columns of both D and E must equal the number of rows of F , and thus $DF + EF$ makes sense as well. So suppose $D, E \in \mathbb{F}^{m,n}$ and $F \in \mathbb{F}^{n,p}$. It follows

$$\begin{aligned} ((D + E)F)_{j,k} &= \sum_{r=1}^n (D + E)_{j,r} F_{r,k} \\ &= \sum_{r=1}^n (D_{j,r} + E_{j,r}) F_{r,k} \\ &= \sum_{r=1}^n D_{j,r} F_{r,k} + \sum_{r=1}^n E_{j,r} F_{r,k} \\ &= \sum_{r=1}^n D_{j,r} F_{r,k} + \sum_{r=1}^n E_{j,r} F_{r,k} \\ &= (DF)_{j,k} + (EF)_{j,k}, \end{aligned}$$

proving the second distributive property. \square

Problem 15

Suppose A is an n -by- n matrix and $1 \leq j, k \leq n$. show that the entry in row j , column k , of A^3 (which is defined to mean AAA) is

$$\sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}.$$

Proof. For $1 \leq p, k \leq n$, we have

$$(A^2)_{p,k} = \sum_{r=1}^n A_{p,r} A_{r,k}.$$

Thus, for $1 \leq j, k \leq n$, it follows

$$\begin{aligned} (A^3)_{j,k} &= \sum_{p=1}^n A_{j,p} (A^2)_{p,k} \\ &= \sum_{p=1}^n A_{j,p} \sum_{r=1}^n A_{p,r} A_{r,k} \\ &= \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}, \end{aligned}$$

as desired. \square

D: Invertibility and Isomorphic Vector Spaces

Problem 1

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Proof. For all $u \in U$, we have

$$\begin{aligned} (T^{-1}S^{-1}ST)(u) &= T^{-1}(S^{-1}(S(T(u)))) \\ &= T^{-1}(I(T(u))) \\ &= T^{-1}(T(u)) \\ &= u \end{aligned}$$

and hence $T^{-1}S^{-1}$ is a left inverse of ST . Similarly, for all $w \in W$, we have

$$\begin{aligned} (STT^{-1}S^{-1})(w) &= S(T(T^{-1}(S^{-1}(w)))) \\ &= S(I(S^{-1}(w))) \\ &= S(S^{-1}(w)) \\ &= w \end{aligned}$$

and hence $T^{-1}S^{-1}$ is a right inverse of ST . Therefore, ST is invertible, as desired. \square

Problem 3

Suppose V is finite-dimensional, U is a subspace of V , and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Proof. (\Leftarrow) Suppose S is injective, and let W be the subspace of V such that $V = U \oplus W$. Let u_1, \dots, u_m be a basis of U and let w_1, \dots, w_n be a basis of W , so that $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of V . Define $T \in \mathcal{L}(V)$ by its behavior on this basis of V

$$\begin{aligned} Tu_k &:= Su_k \\ Tw_j &:= w_j \end{aligned}$$

for $k = 1, \dots, m$ and $j = 1, \dots, n$. Since S is injective, so too is T . And since V is finite-dimensional, this implies that T is invertible, as desired.

(\Rightarrow) Suppose there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$. Since T is invertible, it is also injective. And since T is injective, so too is $S = T|_U$, completing the proof. \square

Problem 5

Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{range } T_1 = \text{range } T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$.

Proof. (\Rightarrow) Suppose $\text{range } T_1 = \text{range } T_2 := R$, so that $\text{null } T_1 = \text{null } T_2 := N$ as well. Let Q be the unique subspace of V such that

$$V = N \oplus Q,$$

and let u_1, \dots, u_m be a basis of N and v_1, \dots, v_n be a basis of Q . We claim there exists a unique $q_k \in Q$ such that $T_2 q_k = T_1 v_k$ for $k = 1, \dots, n$. To see this, suppose $q_k, q'_k \in Q$ are such that $T_2 q_k = T_2 q'_k = T_1 v_k$. Then $T_2(q_k - q'_k) = 0$, and hence $q_k - q'_k \in N$. But since $N \cap Q = \{0\}$, this implies $q_k - q'_k = 0$ and thus $q_k = q'_k$. And so the choice of q_k is indeed unique. We now define $S \in \mathcal{L}(V)$ by its behavior on the basis

$$\begin{aligned} Su_k &= u_k \text{ for } k = 1, \dots, m \\ Sv_j &= q_j \text{ for } j = 1, \dots, n. \end{aligned}$$

Let $v \in V$, so that there exist $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that

$$v = a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n.$$

It follows

$$\begin{aligned} (T_2 S)(v) &= T_2(S(a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n)) \\ &= T_2(a_1 S u_1 + \dots + a_m S u_m + b_1 S v_1 + \dots + b_n S v_n) \\ &= T_2(a_1 u_1 + \dots + a_m u_m + b_1 q_1 + \dots + b_n q_n) \\ &= a_1 T_2 u_1 + \dots + a_m T_2 u_m + b_1 T_2 q_1 + \dots + b_n T_2 q_n \\ &= b_1 T_1 v_1 + \dots + b_n T_1 v_n. \end{aligned}$$

Similarly, we have

$$\begin{aligned} T_1 v &= T_1(a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n) \\ &= a_1 T_1 u_1 + \dots + a_m T_1 u_m + b_1 T_1 v_1 + \dots + b_n T_1 v_n \\ &= b_1 T_1 v_1 + \dots + b_n T_1 v_n, \end{aligned}$$

and so indeed $T_1 = T_2 S$. To see that S is invertible, it suffices to prove it is injective. So let $v \in V$ be as before, and suppose $Sv = 0$. It follows

$$\begin{aligned} Sv &= S(a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n) \\ &= (a_1 u_1 + \dots + a_m u_m) + (b_1 S v_1 + \dots + b_n S v_n) \\ &= 0. \end{aligned}$$

By the proof of Theorem 3.22, sv_1, \dots, sv_n is a basis of R , and thus the list $u_1, \dots, u_m, sv_1, \dots, sv_n$ is a basis of V , and each of the a 's and b 's must be zero. Therefore S is indeed injective, completing the proof in this direction.

(\Leftarrow) Suppose there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2S$. If $w \in \text{range } T_1$, then there exists $v \in V$ such that $T_1v = w$, and hence $(T_2S)(v) = T_2(S(v)) = w$, so that $w \in \text{range } T_2$ and we have $\text{range } T_1 \subseteq \text{range } T_2$. Conversely, suppose $w' \in \text{range } T_2$, so that there exists $v' \in V$ such that $T_2v' = w'$. Then, since $T_2 = T_1S^{-1}$, we have $(T_1S^{-1})(v') = T_1(S^{-1}(v')) = w'$, so that $w' \in \text{range } T_1$. Thus $\text{range } T_2 \subseteq \text{range } T_1$, and we have shown $\text{range } T_1 = \text{range } T_2$, as desired. \square

Problem 7

Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) \mid Tv = 0\}.$$

- (a) Show that E is a subspace of $\mathcal{L}(V, W)$.
- (b) Suppose $v \neq 0$. What is $\dim E$?

Proof. (a) First note that the zero map is clearly an element of E , and hence E contains the additive identity of $\mathcal{L}(V, W)$. Now suppose $T_1, T_2 \in E$. Then

$$(T_1 + T_2)(v) = T_1v + T_2v = 0$$

and hence $T_1 + T_2 \in E$, so that E is closed under addition. Finally, suppose $T \in E$ and $\lambda \in \mathbb{F}$. Then

$$(\lambda T)(v) = \lambda T v = \lambda 0 = 0,$$

and so E is closed under scalar multiplication as well. Thus E is indeed a subspace of $\mathcal{L}(V, W)$.

- (b) Suppose $v \neq 0$, and let $\dim V = m$ and $\dim W = n$. Extend v to a basis v, v_2, \dots, v_m of V , and endow W with any basis. Let \mathcal{E} denote the subspace of $\mathbb{F}^{m,n}$ of matrices whose first column is all zero.

We claim $T \in E$ if and only if $\mathcal{M}(T) \in \mathcal{E}$, so that $\mathcal{M} : E \rightarrow \mathcal{E}$ is an isomorphism. Clearly if $T \in E$ (so that $Tv = 0$), then $\mathcal{M}(T)_{\cdot,1}$ is all zero,

and hence $T \in \mathcal{E}$. Conversely, suppose $\mathcal{M}(T) \in \mathcal{E}$. It follows

$$\begin{aligned}\mathcal{M}(Tv) &= \mathcal{M}(T)\mathcal{M}(v) \\ &= \begin{pmatrix} 0 & A_{1,2} & \cdots & A_{1,n} \\ 0 & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & A_{m,2} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\end{aligned}$$

and thus we must have $Tv = 0$ so that $T \in E$, proving our claim. So indeed $E \cong \mathcal{E}$.

Now note that \mathcal{E} has as a basis the set of all matrices with a single 1 in a column besides the first, and zeros everywhere else. There are $mn - n$ such matrices, and hence $\dim \mathcal{E} = mn - n$. Thus we have $\dim E = mn - n$ as well, as desired. \square

Problem 9

Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Proof. (\Leftarrow) Suppose S and T are both invertible. Then by Problem 1, ST is invertible.

(\Rightarrow) Suppose ST is invertible. We will show T is injective and S is surjective. Since V is finite-dimensional, this implies that both S and T are invertible. So suppose $v_1, v_2 \in V$ are such that $Tv_1 = Tv_2$. Then $(ST)(v_1) = (ST)(v_2)$, and since ST is invertible (and hence injective), we must have $v_1 = v_2$, so that T is injective. Next, suppose $v \in V$. Since T^{-1} is surjective, there exists $w \in V$ such that $T^{-1}w = v$. And since ST is surjective, there exists $p \in V$ such that $(ST)(p) = w$. It follows that $(STT^{-1})(p) = T^{-1}(w)$, and hence $Sp = v$. Thus S is surjective, completing the proof. \square

Problem 11

Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Proof. Notice STU is invertible since $STU = I$ and I is invertible. By Problem 9, we have that $(ST)U$ is invertible if and only if ST and U are invertible. By

a second application of the result, ST is invertible if and only if S and T are invertible. Thus S, T , and U are all invertible. To see that $T^{-1} = US$, notice

$$\begin{aligned} US &= (T^{-1}T)US \\ &= T^{-1}(S^{-1}S)TUS \\ &= T^{-1}S^{-1}(STU)S \\ &= T^{-1}S^{-1}S \\ &= T^{-1}, \end{aligned}$$

as desired. \square

Problem 13

Suppose V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective. Prove that S is injective.

Proof. Since V is finite-dimensional and RST is surjective, RST is also invertible. By Problem 9, we have that $(RS)T$ is invertible if and only if RS and T are invertible. By a second application of the result, RS is invertible if and only if R and S are invertible. Thus R, S , and T are all invertible, and hence injective. In particular, S is injective, as desired. \square

Problem 15

Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an m -by- n matrix A such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$.

Proof. Endow both $\mathbb{F}^{n,1}$ and $\mathbb{F}^{m,1}$ with the standard basis, and let $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$. Let $A = \mathcal{M}(T)$ with respect to these bases, and note that if $x \in \mathbb{F}^{n,1}$, then $\mathcal{M}(x) = x$ (and similarly if $y \in \mathbb{F}^{m,1}$, then $\mathcal{M}(y) = y$). Hence

$$\begin{aligned} Tx &= \mathcal{M}(Tx) \\ &= \mathcal{M}(T)\mathcal{M}(x) \\ &= Ax, \end{aligned}$$

as desired. \square

Problem 16

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

Proof. (\Rightarrow) Suppose $T = \lambda I$ for some $\lambda \in \mathbb{F}$, and let $S \in \mathcal{L}(V)$ be arbitrary. For any $v \in V$, we have $STv = S(\lambda I)v = \lambda Sv$ and $TSv = (\lambda I)Sv = \lambda Sv$, and hence $ST = TS$. Since S was arbitrary, we have the desired result.

(\Leftarrow) Suppose $ST = TS$ for every $S \in \mathcal{L}(V)$, and let $v \in V$ be arbitrary. Consider the list v, Tv . We claim it is linearly dependent. To see this, suppose not. Then v, Tv can be extended to a basis v, Tv, u_1, \dots, u_n of V . Define $S \in \mathcal{L}(V)$ by

$$S(\alpha v + \beta Tv + \gamma_1 u_1 + \dots + \gamma_n u_n) = \beta v,$$

where $\alpha, \beta, \gamma_1, \dots, \gamma_n$ are the unique coefficients of our basis for the given input vector. In particular, notice $S(Tv) = v$ and $Sv = 0$. Thus $STv = TSv$ implies $v = T(0) = 0$, contradicting our assumption that v, Tv is linearly independent. So v, Tv must be linearly dependent, and so for all $v \in V$ there exists $\lambda_v \in \mathbb{F}$ such that $Tv = \lambda_v v$ (where λ_0 can be any nonzero element of \mathbb{F} , since $T0 = 0$). We claim λ_v is independent of the choice of v for $v \in V - \{0\}$, hence $Tv = \lambda v$ for all $v \in V$ (including $v = 0$) and some $\lambda \in \mathbb{F}$, and thus $T = \lambda I$.

So suppose $w, z \in V - \{0\}$ are arbitrary. We want to show $\lambda_w = \lambda_z$. If w and z are linearly dependent, then there exists $\alpha \in \mathbb{F}$ such that $w = \alpha z$. It follows

$$\begin{aligned} \lambda_w w &= Tw \\ &= T(\alpha z) \\ &= \alpha Tz \\ &= \alpha \lambda_z z \\ &= \lambda_z(\alpha z) \\ &= \lambda_z w. \end{aligned}$$

Since $w \neq 0$, this implies $\lambda_w = \lambda_z$. Next suppose w and z are linearly independent. Then we have

$$\begin{aligned} \lambda_{w+z}(w+z) &= T(w+z) \\ &= Tw + Tz \\ &= \lambda_w w + \lambda_z z, \end{aligned}$$

and hence

$$(\lambda_{w+z} - \lambda_w)w + (\lambda_{w+z} - \lambda_z)z = 0.$$

Since w and z are assumed to be linearly independent, we have $\lambda_{w+z} = \lambda_w$ and $\lambda_{w+z} = \lambda_z$, and hence again we have $\lambda_w = \lambda_z$, completing the proof. \square

Problem 17

Suppose V is finite-dimensional and \mathcal{E} is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ and $TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$.

Proof. If $\mathcal{E} = \{0\}$, we're done. So suppose $\mathcal{E} \neq \{0\}$. If $\dim V = n$, then $\mathcal{L}(V) \cong \mathbb{F}^{n,n}$, and so there exists an isomorphic subspace $\mathfrak{E} := \mathcal{M}(\mathcal{E}) \subseteq \mathbb{F}^{n,n}$ with the property that $AB \in \mathfrak{E}$ and $BA \in \mathfrak{E}$ for all $A \in \mathbb{F}^{n,n}$ and all $B \in \mathfrak{E}$. It suffices to show $\mathfrak{E} = \mathbb{F}^{n,n}$.

Define $E^{i,j}$ to be the matrix which is 1 in row i and column j and 0 everywhere else, and let $A \in \mathbb{F}^{n,n}$ be nonzero. Then there exists some $1 \leq j, k \leq n$ such that $A_{j,k} \neq 0$. Notice for $1 \leq i, j, r, s \leq n$, we have $E^{i,j}A \in \mathfrak{E}$, and hence $E^{i,j}AE^{r,s} \in \mathfrak{E}$. This product has the form

$$E^{i,j}AE^{k,\ell} = A_{j,k} \cdot E^{i,\ell}.$$

In other words, $E^{i,j}AE^{k,\ell}$ takes $A_{j,k}$ and puts it in the i^{th} row and ℓ^{th} column of a matrix which is 0 everywhere else. Since \mathfrak{E} is closed under addition, this implies

$$E^{1,j}AE^{k,1} + E^{2,j}AE^{k,2} + \dots + E^{n,j}AE^{k,n} = A_{j,k} \cdot I \in \mathfrak{E}.$$

But since \mathfrak{E} is closed under scalar multiplication, and $A_{j,k} \neq 0$, we have

$$\left(\frac{1}{A_{j,k}} \cdot A_{j,k} \right) \cdot I = I \in \mathfrak{E}.$$

Since \mathfrak{E} contains I , by our characterization of \mathfrak{E} it must also contain every element of $\mathbb{F}^{n,n}$. Thus $\mathfrak{E} = \mathbb{F}^{n,n}$, and since $\mathfrak{E} \cong \mathcal{E}$, we must have $\mathcal{E} = \mathcal{L}(V)$, as desired. \square

Problem 19

Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is such that T is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$.

- (a) Prove that T is surjective.
- (b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbb{R})$.

Proof. (a) Let $q \in \mathcal{P}(\mathbb{R})$, and suppose $\deg q = n$. Let $T_n = T|_{\mathcal{P}_n(\mathbb{R})}$, so that T_n is the restriction of T to a linear operator on $\mathcal{P}_n(\mathbb{R})$. Since T is injective, so is T_n . And since T_n is an injective linear operator over a finite-dimensional vector space, T_n is surjective as well. Thus there exists $r \in \mathcal{P}_n(\mathbb{R})$ such that $T_n r = q$, and so we have $Tr = q$ as well. Therefore T is surjective.

- (b) We induct on the degree of the restriction maps $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbb{R}))$, each of which is bijective by (a). Let $P(k)$ be the statement: $\deg T_k p = k$ for every nonzero $p \in \mathcal{P}_k(\mathbb{R})$.

Base case: Suppose $p \in \mathcal{P}_0(\mathbb{R})$ is nonzero. Since T_0 is a bijective, $T_0 p = 0$ iff $p = 0$ (the zero polynomial), which is the only polynomial with degree

< 0 . Since p is nonzero by hypothesis, we must have $\deg T_0 p = 0$. Hence $P(0)$ is true.

Inductive step: Let $n \in \mathbb{Z}^+$, and suppose $P(k)$ is true for all $0 \leq k < n$. Let $p \in \mathcal{P}_n(\mathbb{R})$ be nonzero. If $\deg T_n p < n$, then for some $k < n$ there exists $q \in \mathcal{P}_k(\mathbb{R})$ and $T_k \in \mathcal{P}(\mathbb{R})$ such that $T_k q = p$ (since T_k is surjective). Hence $Tq = Tp$, a contradiction since $\deg p \neq \deg q$ and T is injective. Thus we must have $\deg T_n p = n$, and $P(n)$ is true.

By the principle of mathematical induction, $P(k)$ is true for all $k \in \mathbb{Z}_{\geq 0}$. Hence $\deg Tp = \deg p$ for all nonzero $p \in \mathcal{P}(\mathbb{R})$, since $Tp = T_k p$ for $k = \deg p$. \square

E: Products and Quotients of Vector Spaces

Problem 1

Suppose T is a function from V to W . The **graph** of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W \mid v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Proof. Define $G := \{(v, Tv) \in V \times W \mid v \in V\}$.

(\Rightarrow) Suppose T is a linear map. Since T is linear, $T0 = 0$, and hence $(0, 0) \in G$, so that G contains the additive identity. Next, let $(v_1, Tv_1), (v_2, Tv_2) \in G$. Then

$$(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, Tv_1 + Tv_2) = (v_1 + v_2, T(v_1 + v_2)) \in G,$$

and G is closed under addition. Lastly, let $\lambda \in \mathbb{F}$ and $(v, Tv) \in G$. It follows

$$\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v)) \in G,$$

and G is closed under scalar multiplication. Thus G is a subspace of $V \times W$.

(\Leftarrow) Suppose G is a subspace of $V \times W$, and let $(v_1, Tv_1), (v_2, Tv_2) \in G$. Since G is closed under addition, it follows

$$(v_1 + v_2, Tv_1 + Tv_2) \in G,$$

and hence we must have $Tv_1 + Tv_2 = T(v_1 + v_2)$, so that T is additive. And since G is closed under scalar multiplication, for $\lambda \in \mathbb{F}$ and $(v, Tv) \in G$, it follows

$$(\lambda v, \lambda Tv) \in G,$$

and hence we must have $\lambda Tv = T(\lambda v)$, so that T is homogeneous. Therefore, T is a linear map, as desired. \square

Problem 3

Give an example of a vector space V and subspaces U_1, U_2 of V such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ but $U_1 + U_2$ is not a direct sum.

Proof. Define the following two subspaces of $\mathcal{P}(\mathbb{R})$

$$U_1 := \mathcal{P}(\mathbb{R})$$

$$U_2 := \mathbb{R},$$

so that $U_1 \cap U_2 = \mathbb{R}$ and the sum $U_1 + U_2 = \mathcal{P}(\mathbb{R})$ is not direct. Endow $\mathcal{P}(\mathbb{R})$ and \mathbb{R} with their standard bases, and define φ by its behavior on the basis of $U_1 \times U_2$

$$\begin{aligned} \varphi : U_1 \times U_2 &\rightarrow U_1 + U_2 \\ (X^k, 0) &\mapsto X^{k+1} \\ (0, 1) &\mapsto 1. \end{aligned}$$

We claim φ is an isomorphism. To see that φ is injective, suppose

$$(a_0 + a_1X + \cdots + a_mX^m, \alpha), (b_0 + b_1X + \cdots + b_nX^n, \beta) \in U_1 \times U_2$$

and

$$(a_0 + a_1X + \cdots + a_mX^m, \alpha) \neq (b_0 + b_1X + \cdots + b_nX^n, \beta).$$

We have

$$\varphi(a_0 + a_1X + \cdots + a_mX^m, \alpha) = \alpha + a_0X + a_1X^2 + \cdots + a_mX^{m+1} \quad (5)$$

and

$$\varphi(b_0 + b_1X + \cdots + b_nX^n, \beta) = \beta + b_0X + b_1X^2 + \cdots + b_nX^{n+1}. \quad (6)$$

Since $\alpha \neq \beta$, this implies (5) does not equal (6) and hence φ is injective. To see that φ is surjective, suppose $c_0 + c_1X + \cdots + c_pX^p \in U_1 + U_2$. Then

$$\varphi(c_1 + c_2X + \cdots + c_pX^{p-1}, c_0) = c_0 + c_1X + \cdots + c_pX^p$$

and φ is indeed surjective.

Since φ an injective and surjective linear map, it is an isomorphism. Thus $U_1 \times U_2 \cong U_1 + U_2$, as was to be shown. \square

Problem 5

Suppose W_1, \dots, W_m are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic vector spaces.

Proof. Define the projection map π_k for $k = 1, \dots, m$ by

$$\begin{aligned}\pi_k : W_1 \times \dots \times W_m &\rightarrow W_k \\ (w_1, \dots, w_m) &\mapsto w_k.\end{aligned}$$

Clearly π_k is linear. Now define

$$\begin{aligned}\varphi : \mathcal{L}(V, W_1 \times \dots \times W_m) &\rightarrow \mathcal{L}(V, W_1) \times \dots \times \mathcal{L}(V, W_m) \\ T &\mapsto (\pi_1 T, \dots, \pi_m T).\end{aligned}$$

To see that φ is linear, let $T_1, T_2 \in \mathcal{L}(V, W_1 \times \dots \times W_m)$. It follows

$$\begin{aligned}\varphi(T_1 + T_2) &= (\pi_1(T_1 + T_2), \dots, \pi_m(T_1 + T_2)) \\ &= (\pi_1 T_1 + \pi_1 T_2, \dots, \pi_m T_1 + \pi_m T_2) \\ &= (\pi_1 T_1, \dots, \pi_m T_1) + (\pi_1 T_2, \dots, \pi_m T_2) \\ &= \varphi(T_1) + \varphi(T_2),\end{aligned}$$

and hence φ is additive. Now for $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W_1 \times \dots \times W_m)$, we have

$$\begin{aligned}\varphi(\lambda T) &= (\pi_1(\lambda T), \dots, \pi_m(\lambda T)) \\ &= (\lambda(\pi_1 T), \dots, \lambda(\pi_m T)) \\ &= \lambda(\pi_1 T, \dots, \pi_m T),\end{aligned}$$

and thus φ is homogenous. Therefore, φ is linear.

We now show φ is an isomorphism. To see that it is injective, suppose $T \in \mathcal{L}(V, W_1 \times \dots \times W_m)$ and $\varphi(T) = 0$. Then

$$(\pi_1 T, \dots, \pi_m T) = (0, \dots, 0)$$

which is true iff T is the zero map. Thus φ is injective. To see that φ is surjective, suppose $(S_1, \dots, S_m) \in \mathcal{L}(V, W_1) \times \dots \times \mathcal{L}(V, W_m)$. Define

$$\begin{aligned}S : V &\rightarrow W_1 \times \dots \times W_m \\ v &\mapsto (S_1 v, \dots, S_m v),\end{aligned}$$

so that $\varphi_k S = S_k$ for $k = 1, \dots, m$. Then

$$\begin{aligned}\varphi(S) &= (\pi_1 S, \dots, \pi_m S) \\ &= (S_1, \dots, S_m)\end{aligned}$$

and S is indeed surjective. Therefore, φ is an isomorphism, and we have

$$\mathcal{L}(V, W_1 \times \dots \times W_m) \cong \mathcal{L}(V, W_1) \times \dots \times \mathcal{L}(V, W_m),$$

as desired. □

Problem 7

Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Proof. First note that since $v + 0 = v \in v + U$, there exists $w_0 \in W$ such that $v = x + w_0$, and hence $v - x = w_0 \in W$. Similarly, there exists $u_0 \in U$ such that $x - v = u_0 \in U$.

Suppose $u \in U$. Then there exists $w \in W$ such that $v + u = x + w$, and hence

$$u = (x - v) + w = -w_0 + w \in W,$$

and we have $U \subseteq W$. Conversely, suppose $w' \in W$. Then there exists $u' \in U$ such that $x + w' = v + u'$, and hence

$$w' = (v - x) + u' = -u_0 + u' \in U,$$

and we have $W \subseteq U$. Therefore $U = W$, as desired. \square

Problem 8

Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.

Proof. (\Rightarrow) Suppose $A \subseteq V$ is an affine subset of V . Then there exists $x \in V$ and a subspace $U \subseteq V$ such that $A = x + U$. Suppose $v, w \in A$. Then there exist $u_1, u_2 \in U$ such that $v = x + u_1$ and $w = x + u_2$. Thus, for all $\lambda \in \mathbb{F}$, we have

$$\begin{aligned} \lambda v + (1 - \lambda)w &= \lambda(x + u_1) + (1 - \lambda)(x + u_2) \\ &= x + \lambda u_1 + (1 - \lambda)u_2. \end{aligned}$$

Since $\lambda u_1 + (1 - \lambda)u_2 \in U$, this implies $\lambda v + (1 - \lambda)w \in x + U = A$, as desired.

(\Leftarrow) Suppose $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$. Choose $a \in A$ and define

$$U := -a + A.$$

We claim U is a subspace of V . Clearly $0 \in U$ since $a \in A$. Let $x \in U$, so that $x = -a + a_0$ for some $a_0 \in A$, and let $\lambda \in \mathbb{F}$. It follows

$$\lambda a_0 + (1 - \lambda)a \in A \Rightarrow -\lambda a + \lambda a_0 + a \in A \Rightarrow \lambda(-a + a_0) \in -a + A = U,$$

and thus $\lambda x = \lambda(-a + a_0) \in U$, and U is closed under scalar multiplication. Now let $x, y \in U$. Then there exist $a_1, a_2 \in A$ such that $x = -a + a_1$ and $y = -a + a_2$. Notice

$$\frac{1}{2}a_1 + \left(1 - \frac{1}{2}\right)a_2 = \frac{1}{2}a_1 + \frac{1}{2}a_2 \in A,$$

and hence

$$-a + \frac{1}{2}a_1 + \frac{1}{2}a_2 \in U.$$

It follows

$$\begin{aligned} x + y &= -2a + a_1 + a_2 \\ &= 2 \left(-a + \frac{1}{2}a_1 + \frac{1}{2}a_2 \right) \in U, \end{aligned}$$

using the fact that U has already been shown to be closed under scalar multiplication. Thus U is also closed under addition, and so U is a subspace of V . Now, since $A = a + U$, we have that A is indeed an affine subset of V , as desired. \square

Problem 9

Suppose A_1 and A_2 are affine subsets of V . Prove that the intersection $A_1 \cap A_2$ is either an affine subset of V or the empty set.

Proof. If $A_1 \cap A_2 = \emptyset$, we're done, so suppose $A_1 \cap A_2$ is nonempty and let $v \in A_1 \cap A_2$. Then we may write

$$A_1 = v + U_1 \quad \text{and} \quad A_2 = v + U_2$$

for some subspaces $U_1, U_2 \subseteq V$.

We claim $A_1 \cap A_2 = v + (U_1 \cap U_2)$, which is an affine subset of V . To see this, suppose $x \in v + (U_1 \cap U_2)$. Then there exists $u \in U_1 \cap U_2$ such that $x = v + u$. Since $u \in U_1$, we have $x \in v + U_1 = A_1$. And since $u \in U_2$, we have $x \in v + U_2 = A_2$. Thus $x \in A_1 \cap A_2$ and $v + (U_1 \cap U_2) \subseteq A_1 \cap A_2$. Conversely, suppose $y \in A_1 \cap A_2$. Then there exist $u_1 \in U_1$ and $u_2 \in U_2$ such that $y = v + u_1$ and $y = v + u_2$. But this implies $u_1 = u_2$, and hence $u_1 = u_2 \in U_1 \cap U_2$, thus $y \in v + (U_1 \cap U_2)$. Therefore $A_1 \cap A_2 \subseteq v + (U_1 \cap U_2)$, and hence we have $A_1 \cap A_2 = v + (U_1 \cap U_2)$, as claimed. \square

Problem 11

Suppose $v_1, \dots, v_m \in V$. Let

$$A = \{\lambda_1 v_1 + \dots + \lambda_m v_m \mid \lambda_1, \dots, \lambda_m \in \mathbb{F} \text{ and } \lambda_1 + \dots + \lambda_m = 1\}.$$

- Prove that A is an affine subset of V .
- Prove that every affine subset of V that contains v_1, \dots, v_m also contains A .
- Prove that $A = v + U$ for some $v \in V$ and some subspace U of V with $\dim U \leq m - 1$.

Proof. (a) Let $v, w \in A$, so that there exist $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ and $\beta_1, \dots, \beta_m \in \mathbb{F}$ such that

$$\begin{aligned} v &= \alpha_1 v_1 + \dots + \alpha_m v_m \\ w &= \beta_1 v_1 + \dots + \beta_m v_m, \end{aligned}$$

where $\sum \alpha_k = 1$ and $\sum \beta_k = 1$. Given $\lambda \in \mathbb{F}$, it follows

$$\begin{aligned} \lambda v + (1 - \lambda)w &= \lambda \sum_{k=1}^m \alpha_k v_k + (1 - \lambda) \sum_{k=1}^m \beta_k v_k \\ &= \sum_{k=1}^m [\lambda \alpha_k + (1 - \lambda) \beta_k] v_k. \end{aligned}$$

But notice

$$\sum_{k=1}^m [\lambda \alpha_k + (1 - \lambda) \beta_k] = \lambda + (1 - \lambda) = 1,$$

and hence $\lambda v + (1 - \lambda)w \in A$ by the way we defined A . By Problem 8, this implies that A is an affine subset of V , as was to be shown.

(b) We induct on m .

Base case: When $m = 1$, the statement is trivially true, since $A = \{v_1\}$, and hence any affine subset of V that contains v_1 of course contains A .

Inductive step: Let $k \in \mathbb{Z}^+$, and suppose the statement is true for $m = k$. Suppose A' is an affine subset of V that contains v_1, \dots, v_{k+1} , and let $x \in A$. Then there exist $\lambda_1, \dots, \lambda_{k+1} \in \mathbb{F}$ such that $\sum_j \lambda_j = 1$ and

$$x = \lambda_1 v_1 + \dots + \lambda_{k+1} v_{k+1}.$$

Now, if $\lambda_{k+1} = 1$, then $x = v_{k+1} \in A'$. Otherwise, we have

$$\frac{\lambda_1}{1 - \lambda_{k+1}} + \dots + \frac{\lambda_k}{1 - \lambda_{k+1}} = 1,$$

and hence by our inductive hypothesis, this implies

$$\frac{\lambda_1}{1 - \lambda_{k+1}} v_1 + \dots + \frac{\lambda_k}{1 - \lambda_{k+1}} v_k \in A'.$$

By Problem 8, we know

$$(1 - \lambda_{k+1}) \left(\frac{\lambda_1}{1 - \lambda_{k+1}} v_1 + \dots + \frac{\lambda_k}{1 - \lambda_{k+1}} v_k \right) + \lambda_{k+1} v_{k+1} \in A'.$$

But after simplifying, this tells us

$$\lambda_1 v_1 + \dots + \lambda_{k+1} v_{k+1} = x \in A'.$$

Hence $A \subseteq A'$, and the statement is true for $m = k + 1$.

By the principal of mathematical induction, the statement is true for all $m \in \mathbb{Z}^+$. Thus any affine subset of V that contains v_1, \dots, v_m also contains A , as was to be shown.

- (c) Define $U := \text{span}(v_2 - v_1, \dots, v_m - v_1)$. Let $x \in A$, so that there exist $\lambda_1, \dots, \lambda_m \in \mathbb{F}$ with $\sum_k \lambda_k = 1$ such that

$$x = \lambda_1 v_1 + \dots + \lambda_m v_m.$$

Notice

$$\begin{aligned} v_1 + \lambda_2(v_2 - v_1) + \dots + \lambda_m(v_m - v_1) &= \left(1 - \sum_{k=2}^m \lambda_k\right) v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m \\ &= \lambda_1 v_1 + \dots + \lambda_m v_m \\ &= x, \end{aligned}$$

and hence $x \in v_1 + U$, so that $A \subseteq v_1 + U$. Next suppose $y \in v_1 + U$, so that there exist $\alpha_1, \dots, \alpha_{m-1} \in \mathbb{F}$ such that

$$y = v_1 + \alpha_1(v_2 - v_1) + \dots + \alpha_{m-1}(v_m - v_1).$$

Expanding the RHS yields

$$y = \left(1 - \sum_{k=1}^{m-1} \alpha_k\right) v_1 + \alpha_1 v_2 + \dots + \alpha_{m-1} v_m.$$

But since

$$\left(1 - \sum_{k=1}^{m-1} \alpha_k\right) + \sum_{k=1}^{m-1} \alpha_k = 1,$$

this implies $y \in A$, and hence $v_1 + U \subseteq A$. Therefore $A = v_1 + U$, and since $\dim U \leq m - 1$, we have the desired result. \square

Problem 13

Suppose U is a subspace of V and $v_1 + U, \dots, v_m + U$ is a basis of V/U and u_1, \dots, u_n is a basis of U . Prove that $v_1, \dots, v_m, u_1, \dots, u_n$ is a basis of V .

Proof. Since

$$\begin{aligned} \dim V &= \dim V/U + \dim U \\ &= m + n, \end{aligned}$$

it suffices to show $v_1, \dots, v_m, u_1, \dots, u_n$ spans V . Suppose $v \in V$. Then there exist $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ such that

$$v + U = \alpha_1(v_1 + U) + \dots + \alpha_m(v_m + U).$$

But then

$$v + U = (\alpha_1 v_1 + \dots + \alpha_m v_m) + U$$

and hence

$$v - (\alpha_1 v_1 + \dots + \alpha_m v_m) \in U.$$

Thus there exist $\beta_1, \dots, \beta_n \in \mathbb{F}$ such that

$$v - (\alpha_1 v_1 + \dots + \alpha_m v_m) = \beta_1 u_1 + \dots + \beta_n u_n,$$

and we have

$$v = \alpha_1 v_1 + \dots + \alpha_m v_m + \beta_1 u_1 + \dots + \beta_n u_n,$$

so that indeed $v_1, \dots, v_m, u_1, \dots, u_n$ spans V . \square

Problem 15

Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $\varphi \neq 0$. Prove that $\dim V/(\text{null } \varphi) = 1$.

Proof. Since $\varphi \neq 0$, we must have $\dim \text{range } \varphi = 1$, so that $\text{range } \varphi = \mathbb{F}$. Since $V/(\text{null } \varphi) \cong \text{range } \varphi$, this implies $V/(\text{null } \varphi) \cong \mathbb{F}$, and hence $\dim V/(\text{null } \varphi) = 1$, as desired. \square

Problem 17

Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that there exists a subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

Proof. Suppose $\dim V/U = n$, and let $v_1 + U, \dots, v_n + U$ be a basis of V/U . Define $W := \text{span}(v_1, \dots, v_n)$. We claim v_1, \dots, v_n must be linearly independent, so that v_1, \dots, v_n is a basis of W . To see this, suppose $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ are such that

$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0.$$

Then

$$(\alpha_1 v_1 + \dots + \alpha_n v_n) + U = \alpha_1(v_1 + U) + \dots + \alpha_n(v_n + U),$$

and hence we must have $\alpha_1 = \dots = \alpha_n = 0$. Thus v_1, \dots, v_n is indeed linearly independent, as claimed.

We now claim $V = U \oplus W$. To see that $V = U + W$, suppose $v \in V$. Then there exist $\beta_1, \dots, \beta_n \in \mathbb{F}$ such that

$$v + U = \beta_1(v_1 + U) + \dots + \beta_n(v_n + U).$$

It follows

$$v - \sum_{k=1}^n \beta_k v_k \in U,$$

and hence

$$v = \left(v - \sum_{k=1}^n \beta_k v_k \right) + \left(\sum_{k=1}^n \beta_k v_k \right).$$

Since first term in parentheses is in U and the second term in parentheses is in W , we have $v \in U + W$, and hence $V \subseteq U + W$. Clearly $U + W \subseteq V$, since U and W are each subspaces of V , and hence $V = U + W$. To see that the sum is direct, suppose $w \in U \cap W$. Since $w \in W$, there exist $\lambda_1, \dots, \lambda_n$ such that $w = \lambda_1 v_1 + \dots + \lambda_n v_n$, and hence

$$\begin{aligned} w + U &= (\lambda_1 v_1 + \dots + \lambda_n v_n) + U \\ &= \lambda_1(v_1 + U) + \dots + \lambda_n(v_n + U). \end{aligned}$$

Since $w \in U$, we have $w + U = 0 + U$. Thus $\lambda_1 = \dots = \lambda_n = 0$, which implies $w = 0$. Since $U \cap W = \{0\}$, the sum is indeed direct. Thus $V = U \oplus W$, with $\dim W = n = \dim V/U$, as desired. \square

Problem 19

Find a correct statement analogous to 3.78 that is applicable to finite sets, with unions analogous to sums of subspaces and disjoint unions analogous to direct sums.

Theorem. Suppose $|V| < \infty$ and $U_1, \dots, U_n \subseteq V$. Then U_1, \dots, U_n are pairwise disjoint if and only if

$$|U_1 \cup \dots \cup U_n| = |U_1| + \dots + |U_n|.$$

Proof. We induct on n .

Base case: Let $n = 2$. Since $|U_1 \cup U_2| = |U_1| + |U_2| - |U_1 \cap U_2|$, we have $U_1 \cap U_2 = \emptyset$ iff $|U_1 \cup U_2| = |U_1| + |U_2|$.

Inductive hypothesis: Let $k \in \mathbb{Z}_{\geq 2}$, and suppose the statement is true for $n = k$. Let $U_{k+1} \subseteq V$. Then

$$|U_1 \cup \dots \cup U_{k+1}| = |U_1 \cup \dots \cup U_k| + |U_{k+1}|$$

iff $U_{k+1} \cap (U_1 \cup \dots \cup U_k) = \emptyset$ by our base case. Combining this with our inductive hypothesis, we have

$$|U_1 \cup \dots \cup U_{k+1}| = |U_1| + \dots + |U_k| + |U_{k+1}|$$

iff U_1, \dots, U_{k+1} are pairwise disjoint, and the statement is true for $n = k + 1$.

By the principal of mathematical induction, the statement is true for all $n \in \mathbb{Z}_{\geq 2}$. \square

F: Duality

Problem 1

Explain why every linear functional is either surjective or the zero map.

Proof. Since $\dim \mathbb{F} = 1$, the only subspaces of \mathbb{F} are \mathbb{F} itself and $\{0\}$. Let V be a vector space (not necessarily finite-dimensional) and suppose $\varphi \in V'$. Since $\text{range } \varphi$ is a subspace of \mathbb{F} , it must be either \mathbb{F} itself (in which case φ is surjective) or $\{0\}$ (in which case φ is the zero map). \square

Problem 3

Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

Proof. Suppose $\dim U = m$ and $\dim V = n$ for some $m, n \in \mathbb{Z}^+$ such that $m < n$. Let u_1, \dots, u_m be a basis of U . Expand this to a basis $u_1, \dots, u_m, u_{m+1}, \dots, u_n$ of V , and let $\varphi_1, \dots, \varphi_n$ be the corresponding dual basis of V' . For any $u \in U$, there exist $\alpha_1, \dots, \alpha_m$ such that $u = \alpha_1 u_1 + \dots + \alpha_m u_m$. Now notice

$$\begin{aligned}\varphi_{m+1}(u) &= \varphi_{m+1}(\alpha_1 u_1 + \dots + \alpha_m u_m) \\ &= \alpha_1 \varphi_{m+1}(u_1) + \dots + \alpha_m \varphi_{m+1}(u_m) \\ &= 0,\end{aligned}$$

but $\varphi_{m+1}(u_{m+1}) = 1$. Thus $\varphi_{m+1}(u) = 0$ for every $u \in U$ but $\varphi_{m+1} \neq 0$, as desired. \square

Problem 5

Suppose V_1, \dots, V_m are vector spaces. Prove that $(V_1 \times \dots \times V_m)'$ and $V_1' \times \dots \times V_m'$ are isomorphic vector spaces.

Proof. For $i = 1, \dots, m$, let

$$\begin{aligned}\xi_i : V_i &\rightarrow V_1 \times \dots \times V_m \\ v_i &\mapsto (0, \dots, v_i, \dots, 0).\end{aligned}$$

Now define

$$\begin{aligned}T : (V_1 \times \dots \times V_m)' &\rightarrow V_1' \times \dots \times V_m' \\ \varphi &\mapsto (\varphi \circ \xi_1, \dots, \varphi \circ \xi_m).\end{aligned}$$

We claim T is an isomorphism. We must show three things: (1) that T is a linear map; (2) that T is injective; and (3) that T is surjective.

To see that T is a linear map, first suppose $\varphi_1, \varphi_2 \in (V_1 \times \cdots \times V_m)'$. It follows

$$\begin{aligned} T(\varphi_1 + \varphi_2) &= ((\varphi_1 + \varphi_2) \circ \xi_1, \dots, (\varphi_1 + \varphi_2) \circ \xi_m) \\ &= (\varphi_1 \circ \xi_1 + \varphi_2 \circ \xi_1, \dots, \varphi_1 \circ \xi_m + \varphi_2 \circ \xi_m) \\ &= (\varphi_1 \circ \xi_1, \dots, \varphi_1 \circ \xi_m) + (\varphi_2 \circ \xi_1, \dots, \varphi_2 \circ \xi_m) \\ &= T(\varphi_1) + T(\varphi_2), \end{aligned}$$

thus T is additive. To see that it is also homogeneous, suppose $\lambda \in \mathbb{F}$ and $\varphi \in (V_1 \times \cdots \times V_m)'$. We have

$$\begin{aligned} T(\lambda\varphi) &= ((\lambda\varphi) \circ \xi_1, \dots, (\lambda\varphi) \circ \xi_m) \\ &= (\lambda(\varphi \circ \xi_1), \dots, \lambda(\varphi \circ \xi_m)) \\ &= \lambda(\varphi \circ \xi_1, \dots, \varphi \circ \xi_m) \\ &= \lambda T(\varphi), \end{aligned}$$

and thus T is homogeneous as well and therefore it is a linear map.

To see that T is injective, suppose $\varphi, \psi \in (V_1 \times \cdots \times V_m)'$ but $\varphi \neq \psi$. Then there exists some $(v_1, \dots, v_m) \in V_1 \times \cdots \times V_m$ such that $\varphi(v_1, \dots, v_m) \neq \psi(v_1, \dots, v_m)$. Since φ and ψ are linear, this means that there exists some index $k \in \{1, \dots, m\}$ such that $\varphi(0, \dots, v_k, \dots, 0) \neq \psi(0, \dots, v_k, \dots, 0)$. But then $\varphi \circ \xi_k \neq \psi \circ \xi_k$, and hence $T(\varphi) \neq T(\psi)$, so that T is injective.

To see that T is surjective, suppose $(\varphi_1, \dots, \varphi_m) \in V_1' \times \cdots \times V_m'$ and define

$$\begin{aligned} \theta : V_1 \times \cdots \times V_m &\rightarrow \mathbb{F} \\ (v_1, \dots, v_m) &\mapsto \sum_{k=1}^m \varphi_k(v_k). \end{aligned}$$

We claim $T(\theta) = (\varphi_1, \dots, \varphi_m)$. To see this, let $k \in \{1, \dots, m\}$. We will show that the map in the k -th component of $T(\theta)$ is equal to φ_k . Given $v_k \in V_k$, we have

$$\begin{aligned} T(\theta)_k(v_k) &= (\theta \circ \xi_k)(v_k) \\ &= \theta(\xi_k(v_k)) \\ &= \theta(0, \dots, v_k, \dots, 0) \\ &= \varphi_1(0) + \cdots + \varphi_k(v_k) + \cdots + \varphi_m(0) \\ &= \varphi_k(v_k), \end{aligned}$$

as desired. Thus $T(\theta) = (\varphi_1, \dots, \varphi_m)$, and T is indeed surjective. Since T is both injective and surjective, it's an isomorphism. \square

Problem 7

Suppose m is a positive integer. Show that the dual basis of the basis $1, \dots, x^m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where $\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$. Here $p^{(j)}$ denotes the j^{th} derivative of p , with the understanding that the 0^{th} derivative of p is p .

Proof. For $j = 0, \dots, m$, we have by direct computation of the j -th derivative

$$\varphi_j(x^k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

so that $\varphi_0, \varphi_1, \dots, \varphi_m$ is indeed the dual basis of $1, \dots, x^m$. Note the uniqueness of the dual basis follows by uniqueness of a linear map (including the linear functionals in the dual basis) whose values on a basis are specified. \square

Problem 9

Suppose v_1, \dots, v_n is a basis of V and $\varphi_1, \dots, \varphi_n$ is the corresponding dual basis of V' . Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n.$$

Proof. Let $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ be such that

$$\psi = \alpha_1\varphi_1 + \dots + \alpha_n\varphi_n.$$

For $k = 1, \dots, n$, we have

$$\begin{aligned} \psi(v_k) &= \alpha_1\varphi_1(v_k) + \dots + \alpha_k\varphi_k(v_k) + \dots + \alpha_n\varphi_n(v_k) \\ &= \alpha_1 \cdot 0 + \dots + \alpha_k \cdot 1 + \dots + \alpha_n \cdot 0 \\ &= \alpha_k. \end{aligned}$$

Thus we have

$$\psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n,$$

as desired \square

Problem 11

Suppose A is an m -by- n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbb{F}^m$ and $(d_1, \dots, d_n) \in \mathbb{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

Proof. (\Rightarrow) Suppose the rank of A is 1. By the assumption that $A \neq 0$, there exists a nonzero entry $A_{i,j}$ for some $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. Thus $\text{span}\{A_{\cdot,1}, \dots, A_{\cdot,n}\} = \text{span}\{A_{\cdot,j}\}$, and hence there exist $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $A_{\cdot,c} = \alpha_c A_{\cdot,j}$ for $c = 1, \dots, n$. Expanding out each of these columns, we have

$$A_{r,c} = \alpha_c A_{r,j} \quad (7)$$

for $r = 1, \dots, m$. Similarly for the rows, we have $\text{span}\{A_{1,\cdot}, \dots, A_{m,\cdot}\} = \text{span}\{A_{i,\cdot}\}$, and hence there exist $\beta_1, \dots, \beta_m \in \mathbb{F}$ such that $A_{r',\cdot} = \beta_{r'} A_{i,\cdot}$ for $r' = 1, \dots, m$. Expanding out each of these rows, we have

$$A_{r',c'} = \beta_{r'} A_{i,c'} \quad (8)$$

for $c' = 1, \dots, n$. Now by replacing the $A_{r,j}$ term in (7) according to (8), we have $A_{r,j} = \beta_r A_{i,j}$, and hence (7) may be rewritten

$$A_{r,c} = \alpha_c \beta_r A_{i,j},$$

and the result follows by defining $c_r = \beta_r A_{i,j}$ and $d_c = \alpha_c$ for $r = 1, \dots, m$ and $c = 1, \dots, n$.

(\Leftarrow) Suppose there exist $(c_1, \dots, c_m) \in \mathbb{F}^m$ and $(d_1, \dots, d_n) \in \mathbb{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$. Then each of the columns is a scalar multiple of $(d_1, \dots, d_n)^t \in \mathbb{F}^{n,1}$ and the column rank is 1. Since the rank of a matrix equals its column rank, the rank of A is 1 as well. \square

Problem 13

Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Suppose φ_1, φ_2 denotes the dual basis of the standard basis of \mathbb{R}^2 and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of \mathbb{R}^3 .

- (a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.
- (b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as a linear combination of ψ_1, ψ_2, ψ_3 .

Proof. (a) Endowing \mathbb{R}^3 and \mathbb{R}^2 with their respective standard basis, we have

$$\begin{aligned} (T'(\varphi_1))(x, y, z) &= (\varphi_1 \circ T)(x, y, z) \\ &= \varphi_1(T(x, y, z)) \\ &= \varphi_1(4x + 5y + 6z, 7x + 8y + 9z) \\ &= 4x + 5y + 6z \end{aligned}$$

and similarly

$$\begin{aligned} (T'(\varphi_2))(x, y, z) &= \varphi_2(4x + 5y + 6z, 7x + 8y + 9z) \\ &= 7x + 8y + 9z. \end{aligned}$$

(b) Notice

$$\begin{aligned}(4\psi_1 + 5\psi_2 + 6\psi_3)(x, y, z) &= 4\psi_1(x, y, z) + 5\psi_2(x, y, z) + 6\psi_3(x, y, z) \\ &= 4x + 5y + 6z \\ &= T'(\varphi_1)(x, y, z)\end{aligned}$$

and

$$\begin{aligned}(7\psi_1 + 8\psi_2 + 9\psi_3)(x, y, z) &= 7\psi_1(x, y, z) + 8\psi_2(x, y, z) + 9\psi_3(x, y, z) \\ &= 7x + 8y + 9z \\ &= T'(\varphi_2)(x, y, z),\end{aligned}$$

as desired. \square

Problem 15

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

Proof. (\Rightarrow) Suppose $T' = 0$. Let $\varphi \in W'$ and $v \in V$ be arbitrary. We have

$$0 = (T'(\varphi))(v) = \varphi(Tv).$$

Since φ is arbitrary, we must have $Tv = 0$. But now since v is arbitrary, this implies $T = 0$ as well.

(\Leftarrow) Suppose $T = 0$. Again let $\varphi \in W'$ and $v \in V$ be arbitrary. We have

$$(T'(\varphi))(v) = \varphi(Tv) = \varphi(0) = 0,$$

and hence $T' = 0$ as well. \square

Problem 17

Suppose $U \subseteq V$. Explain why $U^0 = \{\varphi \in V' \mid U \subseteq \text{null } \varphi\}$.

Proof. It suffices to show that, for arbitrary $\varphi \in V'$, we have $U \subseteq \text{null } \varphi$ if and only if $\varphi(u) = 0$ for all $u \in U$. So suppose $U \subseteq \text{null } \varphi$. Then for all $u \in U$, we have $\varphi(u) = 0$ (simply by definition of $\text{null } \varphi$). Conversely, suppose $\varphi(u) = 0$ for all $u \in U$. Then if $u' \in U$, we must have $u' \in \text{null } \varphi$. That is, $U \subseteq \text{null } \varphi$, completing the proof. \square

Problem 19

Suppose V is finite-dimensional and U is a subspace of V . Show that $U = V$ if and only if $U^0 = \{0\}$.

Proof. (\Rightarrow) Suppose $U = V$. Then

$$\begin{aligned} U^0 &= \{\varphi \in V' \mid U \subseteq \text{null } \varphi\} \\ &= \{\varphi \in V' \mid V \subseteq \text{null } \varphi\} \\ &= \{0\}, \end{aligned}$$

since only the zero functional can have all of V in its null space.

(\Leftarrow) Suppose $U^0 = \{0\}$. It follows

$$\begin{aligned} \dim V &= \dim U + \dim U^0 \\ &= \dim U + 0 \\ &= \dim U. \end{aligned}$$

Since the only subspace of V with dimension $\dim V$ is V itself, we have $U = V$, as desired. \square

Problem 20

Suppose U and W are subsets of V with $U \subseteq W$. Prove that $W^0 \subseteq U^0$.

Proof. Suppose $\varphi \in W^0$. Then $\varphi(w) = 0$ for all $w \in W$. If $\varphi \notin U^0$, then there exists some $u \in U$ such that $\varphi(u) \neq 0$. But since $U \subseteq W$, $u \in W$. This is absurd, hence we must have $\varphi \in U^0$. Thus $W^0 \subseteq U^0$, as desired. \square

Problem 21

Suppose V is finite-dimensional and U and W are subspaces of V with $W^0 \subseteq U^0$. Prove that $U \subseteq W$.

Proof. Suppose not. Then there exists a nonzero vector $u \in U$ such that $u \notin W$. There exists some basis of U containing u . Define $\varphi \in V'$ such that, for any vector v in this basis, we have

$$\varphi(v) = \begin{cases} 1 & \text{if } v = u \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\varphi \in W^0$, and hence $\varphi \in U^0$. But this implies $\varphi(u) = 0$, a contradiction. \square

Problem 22

Suppose U, W are subspaces of V . Show that $(U + W)^0 = U^0 \cap W^0$.

Proof. Since $U \subseteq U + W$ and $W \subseteq U + W$, Problem 20 tells us that $(U + W)^0 \subseteq U^0$ and $(U + W)^0 \subseteq W^0$. Thus $(U + W)^0 \subseteq U^0 \cap W^0$. Conversely, suppose $\varphi \in U^0 \cap W^0$. Let $x \in U + W$. Then there exist $u \in U$ and $w \in W$ such that $x = u + w$. Then

$$\begin{aligned}\varphi(x) &= \varphi(u + w) \\ &= \varphi(u) + \varphi(w) \\ &= 0,\end{aligned}$$

where the second equality follows since $\varphi \in U^0$ and $\varphi \in W^0$ by assumption. Hence $\varphi \in (U + W)^0$ and we have $U^0 + W^0 \subseteq (U + W)^0$. Thus $(U + W)^0 = U^0 \cap W^0$, as desired. \square

Problem 23

Suppose V is finite-dimensional and U and W are subspaces of V . Prove that $(U \cap W)^0 = U^0 + W^0$.

Proof. Since $U \cap W \subseteq U$ and $U \cap W \subseteq W$, Problem 20 tells us that $U^0 \subseteq (U \cap W)^0$ and $W^0 \subseteq (U \cap W)^0$. Thus $U^0 + W^0 \subseteq (U \cap W)^0$. Now, notice (using Problem 22 to deduce the second equality)

$$\begin{aligned}\dim(U^0 + W^0) &= \dim(U^0) + \dim(W^0) - \dim(U^0 \cap W^0) \\ &= \dim(U^0) + \dim(W^0) - \dim((U + W)^0) \\ &= (\dim V - \dim U) + (\dim V - \dim W) - [\dim V - \dim(U + W)] \\ &= \dim V - \dim U - \dim W + \dim(U + W) \\ &= \dim V - [\dim U + \dim W - \dim(U + W)] \\ &= \dim V - \dim(U \cap W) \\ &= \dim((U \cap W)^0).\end{aligned}$$

Hence we must have $U^0 + W^0 = (U \cap W)^0$, as desired. \square

Problem 25

Suppose V is finite-dimensional and U is a subspace of V . Show that

$$U = \{v \in V \mid \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

Proof. Let $A = \{v \in V \mid \varphi(v) = 0 \text{ for every } \varphi \in U^0\}$. Suppose $u \in U$. Then $\varphi(u) = 0$ for all $\varphi \in U^0$, and hence $u \in A$, showing $U \subseteq A$.

Conversely, suppose $v \in A$ but $v \notin U$. Since $0 \in U$, we must have $v \neq 0$. Thus there exists a basis $u_1, \dots, u_m, v, v_1, \dots, v_n$ of V such that u_1, \dots, u_m is a basis of U . Let $\psi_1, \dots, \psi_m, \varphi, \varphi_1, \dots, \varphi_n$ be the dual basis of V' , and consider for a moment the functional φ . Clearly we have both $\varphi \in U^0$ and $\varphi(v) = 1$ by

construction, but this is a contradiction, since we assumed $v \in A$. Thus $A \subseteq U$, and we conclude $U = A$, as was to be shown. \square

Problem 27

Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_5(\mathbb{R}))$ and $\text{null } T' = \text{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbb{R})$ defined by $\varphi(p) = p(8)$. Prove that $\text{range } T = \{p \in \mathcal{P}_5(\mathbb{R}) \mid p(8) = 0\}$.

Proof. By Theorem 3.107, we know $\text{null } T' = (\text{range } T)^0$, and hence $(\text{range } T)^0 = \{\alpha\varphi \mid \alpha \in \mathbb{R}\}$. It follows by Problem 25

$$\begin{aligned} \text{range } T &= \{p \in \mathcal{P}_5(\mathbb{R}) \mid \psi(p) = 0 \text{ for all } \psi \in (\text{range } T)^0\} \\ &= \{p \in \mathcal{P}_5(\mathbb{R}) \mid (\alpha\varphi)(p) = 0 \text{ for all } \alpha \in \mathbb{R}\} \\ &= \{p \in \mathcal{P}_5(\mathbb{R}) \mid \varphi(p) = 0\} \\ &= \{p \in \mathcal{P}_5(\mathbb{R}) \mid p(8) = 0\}, \end{aligned}$$

as desired. \square

Problem 29

Suppose V and W are finite-dimensional, $T \in \mathcal{L}(V, W)$, and there exists $\varphi \in V'$ such that $\text{range } T' = \text{span}(\varphi)$. Prove that $\text{null } T = \text{null } \varphi$.

Proof. By Theorem 3.107, we know $\text{range } T' = (\text{null } T)^0$, and hence $(\text{null } T)^0 = \{\alpha\varphi \mid \alpha \in \mathbb{R}\}$. It follows by Problem 25

$$\begin{aligned} \text{null } T &= \{v \in V \mid \psi(v) = 0 \text{ for all } \psi \in (\text{null } T)^0\} \\ &= \{v \in V \mid \alpha\varphi(v) = 0 \text{ for all } \alpha \in \mathbb{F}\} \\ &= \{v \in V \mid \varphi(v) = 0\} \\ &= \text{null } \varphi, \end{aligned}$$

as desired. \square

Problem 31

Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_n$ is a basis of V' . Show that there exists a basis of V whose dual basis is $\varphi_1, \dots, \varphi_n$.

Proof. To prove this, we will first show $V \cong V''$. We will then take an existing basis of V' , map it to its dual basis in V'' , and then use the inverse of the isomorphism to take this basis of V'' to a basis in V . This basis of V will have the known basis of V' as its dual.

So, for any $v \in V$, define $E_v \in V''$ by $E_v(\varphi) = \varphi(v)$. We claim the map $\hat{\cdot} : V \rightarrow V''$ given by $\hat{v} = E_v$ is an isomorphism. To do so, it suffices to show it to

be both linear and injective, since $\dim(V'') = \dim((V')') = \dim(V') = \dim(V)$.

We first show $\hat{\cdot}$ is linear. So suppose $u, v \in V$. Then for any $\varphi \in V'$, we have

$$\begin{aligned} (\widehat{u+v})(\varphi) &= E_{u+v}(\varphi) \\ &= \varphi(u+v) \\ &= \varphi(u) + \varphi(v) \\ &= E_u(\varphi) + E_v(\varphi) \\ &= \hat{u}(\varphi) + \hat{v}(\varphi) \end{aligned}$$

so that $\hat{\cdot}$ is indeed linear. Next we show it to be homogeneous. So suppose $\lambda \in \mathbb{F}$, and again let $v \in V$. Then for any $\varphi \in V'$, we have

$$\begin{aligned} (\widehat{\lambda v})(\varphi) &= E_{\lambda v}(\varphi) \\ &= \varphi(\lambda v) \\ &= \lambda \varphi(v) \\ &= \lambda E_v(\varphi) \\ &= \lambda \hat{v}, \end{aligned}$$

so that $\hat{\cdot}$ is homogenous as well. Being both linear and homogenous, it is a linear map.

Next we show $\hat{\cdot}$ is injective. So suppose $\hat{v} = 0$ for some $v \in V$. We want to show $v = 0$. Let v_1, \dots, v_n be a basis of V . Then there exist $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $v = \alpha_1 v_1 + \dots + \alpha_n v_n$. Then, for all $\varphi \in V'$, we have

$$\begin{aligned} \hat{v} = 0 &\implies (\alpha_1 v_1 + \dots + \alpha_n v_n)^\wedge = 0 \\ &\implies \alpha_1 \hat{v}_1 + \dots + \alpha_n \hat{v}_n = 0 \\ &\implies (\alpha_1 \hat{v}_1 + \dots + \alpha_n \hat{v}_n)(\varphi) = 0 \\ &\implies \alpha_1 \hat{v}_1(\varphi) + \dots + \alpha_n \hat{v}_n(\varphi) = 0 \\ &\implies \alpha_1 \varphi(v_1) + \dots + \alpha_n \varphi(v_n) = 0. \end{aligned}$$

Since this last equation holds for all $\varphi \in V'$, it holds in particular for each element of the dual basis $\varphi_1, \dots, \varphi_n$. That is, for $k = 1, \dots, n$, we have

$$\alpha_1 \varphi_k(v_1) + \dots + \alpha_k \varphi_k(v_k) + \dots + \alpha_n \varphi_k(v_n) = 0 \implies \alpha_k = 0,$$

and therefore $v = 0 \cdot v_1 + \dots + 0 \cdot v_n = 0$, as desired. Thus $\hat{\cdot}$ is indeed an isomorphism.

We now prove the main result. Suppose $\varphi_1, \dots, \varphi_n$ is a basis of V' , and let Φ_1, \dots, Φ_n be the dual basis in V'' . For each Φ_k , let v_k be the inverse of Φ_k under the isomorphism $\hat{\cdot}$. Since the inverse of an isomorphism is an isomorphism, and isomorphisms take bases to bases, v_1, \dots, v_n is a basis of V . Let us now

check that its dual basis is $\varphi_1, \dots, \varphi_n$. For $j, k = 1, \dots, n$, we have

$$\begin{aligned}\varphi_j(v_k) &= \widehat{v}_k(\varphi_j) \\ &= \Phi_k(\varphi_j) \\ &= \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

so indeed there exists a basis of V whose dual basis is $\varphi_1, \dots, \varphi_n$, as was to be shown. \square

Problem 32

Suppose $T \in \mathcal{L}(V)$ and u_1, \dots, u_n and v_1, \dots, v_n are bases of V . Prove that the following are equivalent:

- (a) T is invertible.
- (b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$.
- (c) The columns of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$.
- (d) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{1,n}$.
- (e) The rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.

Proof. We prove the following: (a) \iff (b) \iff (c) \iff (e) \iff (d).

(a) \iff (b). Suppose T is invertible. That is, for any $w \in V$, there exists a unique $x \in V$ such that $w = Tx$. It follows

$$\begin{aligned}\mathcal{M}(w) &= \mathcal{M}(Tx) \\ &= \mathcal{M}(T)\mathcal{M}(x) \\ &= \mathcal{M}(x)_1\mathcal{M}(T)_{\cdot,1} + \dots + \mathcal{M}(x)_n\mathcal{M}(T)_{\cdot,n}.\end{aligned}$$

That is, every vector in $\mathbb{F}^{1,n}$ can be exhibited as a unique linear combination of the columns of $\mathcal{M}(T)$. This is true if and only if the columns of $\mathcal{M}(T)$ are linearly independent.

(b) \iff (c). Suppose the columns of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$. Since they form a linearly independent list of length $\dim(\mathbb{F}^{n,1})$, they are a basis. But this is true if and only if they span $\mathbb{F}^{n,1}$ as well.

(c) \iff (e). Suppose the columns of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$, so that the column rank is n . Since the row rank equals the column rank, so too must the rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$.

(e) \iff (d). Suppose the rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$. Since they form a spanning list of length $\dim(\mathbb{F}^{1,n})$, they are a basis. But this is true if and only if they are linearly independent in $\mathbb{F}^{1,n}$ as well. \square

Problem 33

Suppose m and n are positive integers. Prove that the function that takes A to A^t is a linear map from $F^{m,n}$ to $F^{n,m}$. Furthermore, prove that this linear map is invertible.

Proof. We first show taking the transpose is linear. So suppose $A, B \in F^{m,n}$ and let $j = 1, \dots, n$ and $k = 1, \dots, m$. It follows

$$\begin{aligned}(A + B)_{j,k}^t &= (A + B)_{k,j} \\ &= A_{k,j} + B_{k,j} \\ &= A_{j,k}^t + B_{j,k}^t,\end{aligned}$$

so that taking the transpose is additive. Next, let $\lambda \in F$. It follows

$$\begin{aligned}(\lambda A)_{j,k}^t &= (\lambda A)_{k,j} \\ &= \lambda A_{k,j} \\ &= \lambda A_{j,k}^t,\end{aligned}$$

so that taking the transpose is homogenous. Since it is both additive and homogeneous, it is a linear map. To see that taking the transpose is invertible, note that $(A^t)^t = A$, so that the inverse of the transpose is the transpose itself. \square

Problem 34

The **double dual space** of V , denoted V'' , is defined to be the dual space of V' . In other words, $V'' = (V')'$. Define $\Lambda : V \rightarrow V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V'$.

- (a) Show that Λ is a linear map from V to V'' .
- (b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.
- (c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V'' .

Proof. We proved (a) and (c) in Problem 31 (where we defined $\hat{\cdot}$ in precisely the same way as Λ). So it only remains to prove (b). So suppose $v \in V$ and $\varphi \in V'$ are arbitrary. Evaluating $T'' \circ \Lambda$, notice

$$\begin{aligned}((T'' \circ \Lambda)(v))(\varphi) &= (T''(\Lambda v))(\varphi) \\ &= (\Lambda v)(T'\varphi) \\ &= (T'\varphi)(v) \\ &= \varphi(Tv),\end{aligned}$$

where the second and fourth equalities follow by definition of the dual map, and the third equality follows by definition of Λ . And evaluating $\Lambda \circ T$, we have

$$\begin{aligned} ((\Lambda \circ T)(v))(\varphi) &= (\Lambda(Tv))(\varphi) \\ &= \varphi(Tv), \end{aligned}$$

so that the two expressions evaluate to the same thing. Since the choice of both v and φ was arbitrary, we have $T'' \circ \Lambda = \Lambda \circ T$, as desired. \square

Problem 35

Show that $(\mathcal{P}(\mathbb{R}))'$ and \mathbb{R}^∞ are isomorphic.

Proof. For any sequence $\alpha = (\alpha_0, \alpha_1, \dots) \in \mathbb{R}^\infty$, let φ_α be the unique linear functional in $(\mathcal{P}(\mathbb{R}))'$ such that $\varphi_\alpha(X^k) = \alpha_k$ for all $k \in \mathbb{Z}^+$ (note that since the list $1, X, X^2, \dots$ is a basis of $\mathcal{P}(\mathbb{R})$, this description of φ_α is sufficient). We claim

$$\begin{aligned} \Phi : \mathbb{R}^\infty &\rightarrow (\mathcal{P}(\mathbb{R}))' \\ \alpha &\mapsto \varphi_\alpha \end{aligned}$$

is an isomorphism. There are three things to show: that Φ is a linear map, that it's injective, and that it's surjective.

We first show Φ is linear. Suppose $\alpha, \beta \in \mathbb{R}^\infty$. For any $k \in \mathbb{Z}^+$, it follows

$$\begin{aligned} (\Phi(\alpha + \beta))(X^k) &= \varphi_{\alpha+\beta}(X^k) \\ &= (\alpha + \beta)_k \\ &= \alpha_k + \beta_k \\ &= \varphi_\alpha(X^k) + \varphi_\beta(X^k) \\ &= (\Phi(\alpha))(X^k) + (\Phi(\beta))(X^k), \end{aligned}$$

so that Φ is additive. Next suppose $\lambda \in \mathbb{R}$. Then we have

$$\begin{aligned} \Phi(\lambda\alpha)(X^k) &= \varphi_{\lambda\alpha}(X^k) \\ &= (\lambda\alpha)_k \\ &= \lambda\alpha_k \\ &= \lambda\Phi(\alpha), \end{aligned}$$

so that Φ is homogenous. Being both additive and homogeneous, Φ is indeed linear.

Next, to see that Φ is injective, suppose $\Phi(\alpha) = 0$ for some $\alpha \in \mathbb{R}^\infty$. Then $\varphi_\alpha(X^k) = \alpha_k = 0$ for all $k \in \mathbb{Z}^+$, and hence $\alpha = 0$. Thus Φ is injective.

Lastly, to see that Φ is surjective, suppose $\varphi \in (\mathcal{P}(\mathbb{R}))'$. Define $\alpha_k = \varphi(X^k)$ for all $k \in \mathbb{Z}^+$ and let $\alpha = (\alpha_0, \alpha_1, \dots)$. By construction, we have $(\Phi(\alpha))(X^k) = \alpha_k$ for all $k \in \mathbb{Z}^+$, and hence $\Phi(\alpha) = \varphi$. Thus Φ is surjective.

Since Φ is linear, injective, and surjective, it's an isomorphism, as desired. \square

Problem 37

Suppose U is a subspace of V . Let $\pi : V \rightarrow V/U$ be the usual quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

- (a) Show that π' is injective.
- (b) Show that $\text{range } \pi' = U^0$.
- (c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0 .

Proof. (a) Let $\varphi \in (V/U)'$, and suppose $\pi'(\varphi) = 0$. Then $(\varphi \circ \pi)(v) = \varphi(v + U) = 0$ for all $v \in V$. This is true only if $\varphi = 0$, and hence π' is indeed injective.

- (b) First, suppose $\varphi \in \text{range } \pi'$. Then there exists $\psi \in (V/U)'$ such that $\pi'(\psi) = \varphi$. So for all $u \in U$, we have

$$\begin{aligned}\varphi(u) &= (\pi'(\psi))(u) \\ &= \psi(\pi(u)) \\ &= \psi(u + U) \\ &= \psi(0 + U) \\ &= 0,\end{aligned}$$

and thus $\varphi \in U^0$, showing $\text{range } \pi' \subseteq U^0$. Conversely, suppose $\varphi \in U^0$, so that $\varphi(u) = 0$ for all $u \in U$. Define $\psi \in (V/U)'$ by $\psi(v + U) = \varphi(v)$ for all $v \in V$. Then $(\pi'(\psi))(v) = \psi(\pi(v)) = \psi(v + U) = \varphi(v)$, and so indeed $\varphi \in \text{range } \pi'$, showing $U^0 \subseteq \text{range } \pi'$. Therefore, we have $\text{range } \pi' = U^0$, as desired.

- (c) Notice that (b) may be interpreted as saying $\pi' : (V/U)' \rightarrow U^0$ is surjective. Since π' was shown to be injective in (a), we conclude π' is an isomorphism from $(V/U)'$ onto U^0 , as desired. \square