Randomized Algorithms

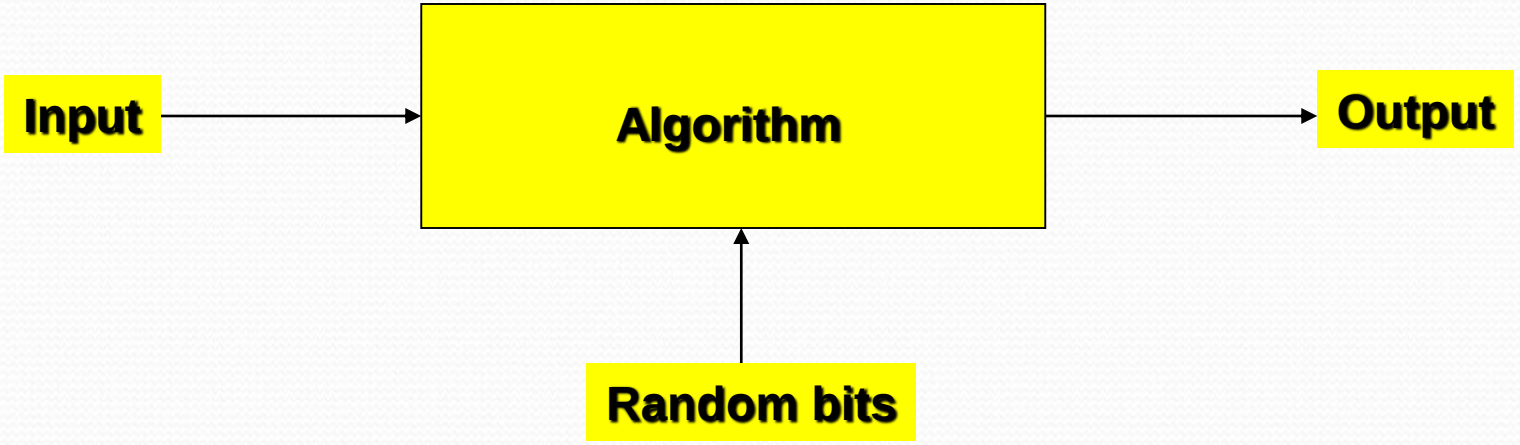
Part-II

Bibhudatta Sahoo

National Institute of Technology Rourkela

Randomized Algorithms

A randomized algorithm is defined as an algorithm that is allowed to access a source of independent, unbiased random bits, and it is then allowed to use these random bits to influence its computation.



Probabilistic or Randomized algorithm

- At least once during the algorithm, a random number is used to make a decision instead of spending time to work out which alternative is best.
- The worst-case running time of a randomized algorithm is almost always the same as the worst-case running time of the non-randomized algorithm.
- A good randomized algorithm has no bad input, but only bad random numbers.
- The random numbers are important, and we can get an expected running time, where we now average over all possible random numbers instead of over all possible inputs, or the mean time that it would take to solve the same instance over and over again

Probabilistic or Randomized algorithm

- A randomized algorithm runs quickly but occasionally makes an error. The probability of error can, however, be make negligibly small. Any purported solution can be verified efficiently for correctness.
- A randomized algorithm may give probabilistic answers which are not necessarily exact.
- An expected running time bound is somewhat stronger than an average-case bound, but is weaker than the corresponding worst-case bound.

Probabilistic or Randomized algorithm

- The same algorithm may behave differently when it is applied twice to the same instance. Its execution time, and even the result obtained, may vary considerable from one use to the next. If the algorithm gets stuck (e.g., core dump), simply restart it on the same instance for a fresh chance of success. If there is more than one correct answer, several different ones may be obtained by running the probabilistic algorithm more than once.

Non-deterministic sorting algorithm

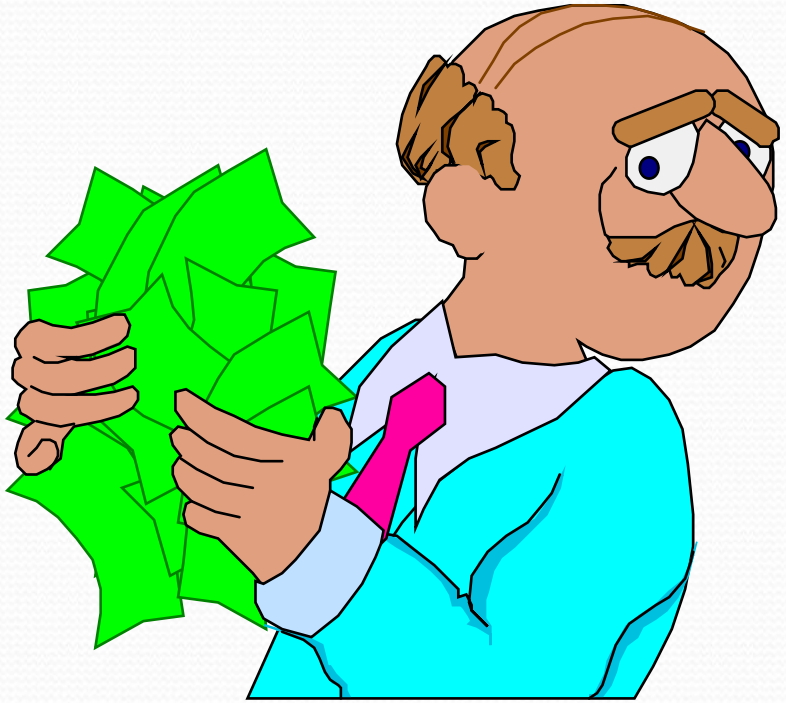
- Let A[i], 1< i < n, be an un-sorted array of positive integers. The non-deterministic algorithm **NSort(A, n)** sorts the numbers into non-decreasing order and then outputs them in this order.
- An auxiliary array **B[1 : n]** is used for convenience.
- The assignment statement **j :=Choice(1,n)** could result in j being assigned any one of the integers in the range[1, n].
- Its complexity is O(n), where as the all deterministic sorting algorithms must have a complexity O(n log n)

Replace Choice(1,n) with Uniform(1,n)

```
1   Algorithm NSort(A, n)
2   // Sort n positive integers.
3   {
4       for i := 1 to n do B[i] := 0; // Initialize B[ ].
5       for i := 1 to n do
6       {
7           j := Choice(1, n);
8           if B[j] ≠ 0 then Failure();
9           B[j] := A[i];
10      }
11      for i := 1 to n − 1 do  // Verify order.
12          if B[i] > B[i + 1] then Failure();
13      write (B[1 : n]);
14      Success();
15  }
```

Classification of Randomized Algorithm

1. Numerical Probabilistic Algorithm
2. Monte-carlo Algorithm
3. Las-Vegas Algorithm
4. Sherwood Algorithm



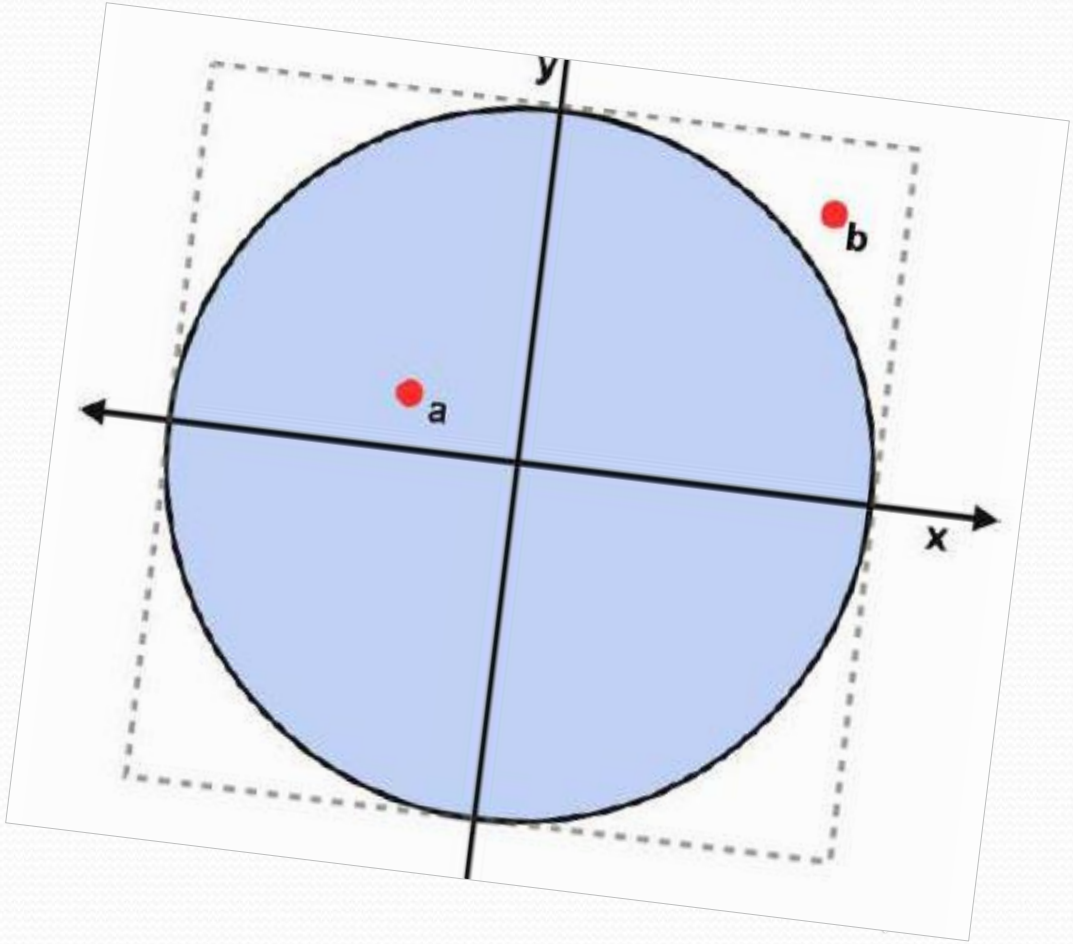
Numerical Probabilistic Algorithm

Here the random element allows us to get approximate numerical results, often much faster than direct methods, and multiple runs will provide increasing approximation.

Numerical Probabilistic Algorithm

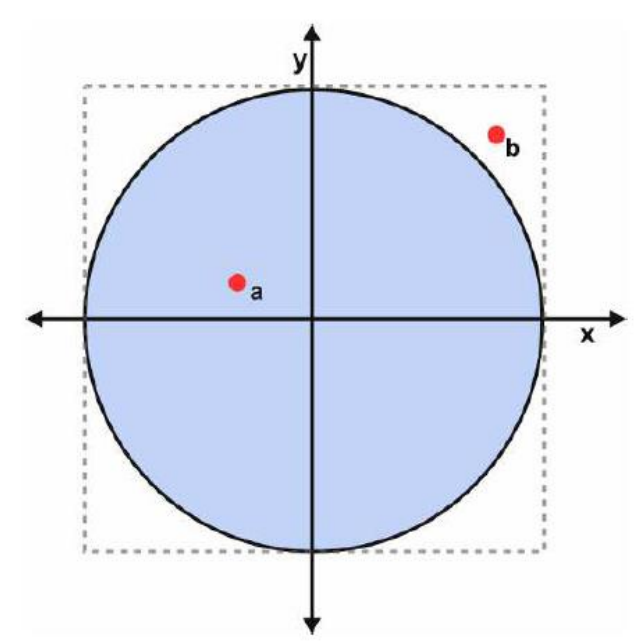
- For certain real-life problems, computation of an exact solution is not possible even in principle, e.g., uncertainties in the experimental data, digital computers handle only binary values, etc.
- For other problems, a precise answer exists but it would take too long to figure it out exactly.
- Numerical algorithms yield a confidence interval, and the expected precision improves as the time available to the algorithm increase.
- The error is usually inversely proportional to the square root of the amount of work performed

Computing Π



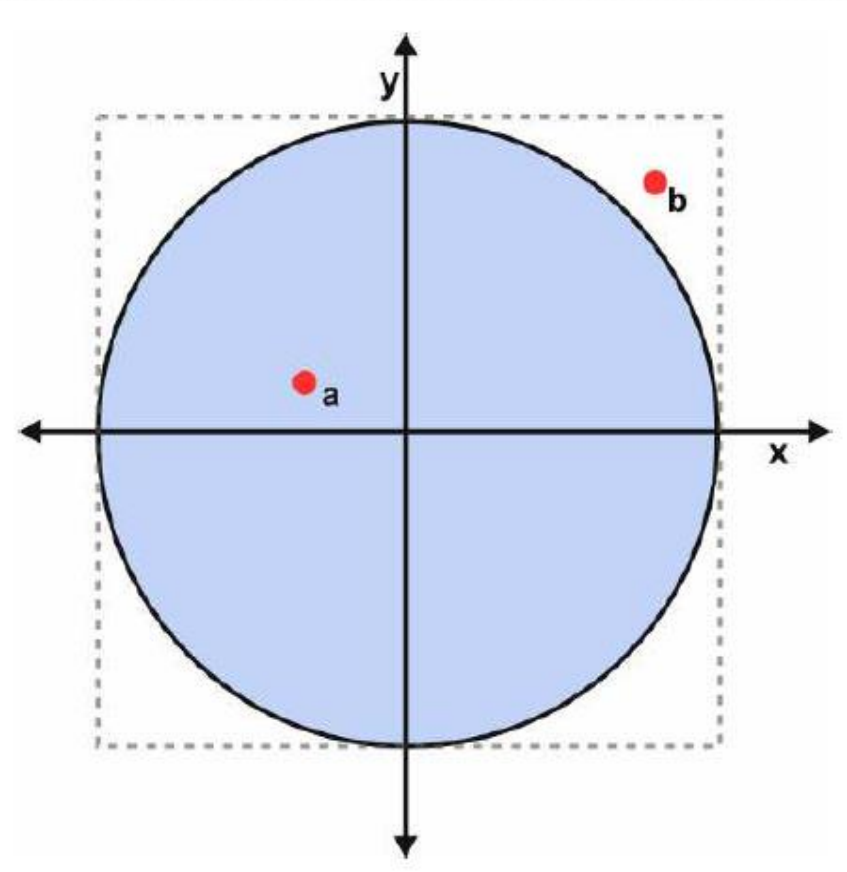
Computing Π

- One way to estimate the value of PI is to randomly throw darts at a unit circle (with radius of 1). Count how many of the darts fall inside of the circle.
- For example, in the figure below, point a is inside the circle and point b is outside the circle. Let n be the total number of darts thrown and let k be the number of darts falling inside the circle.

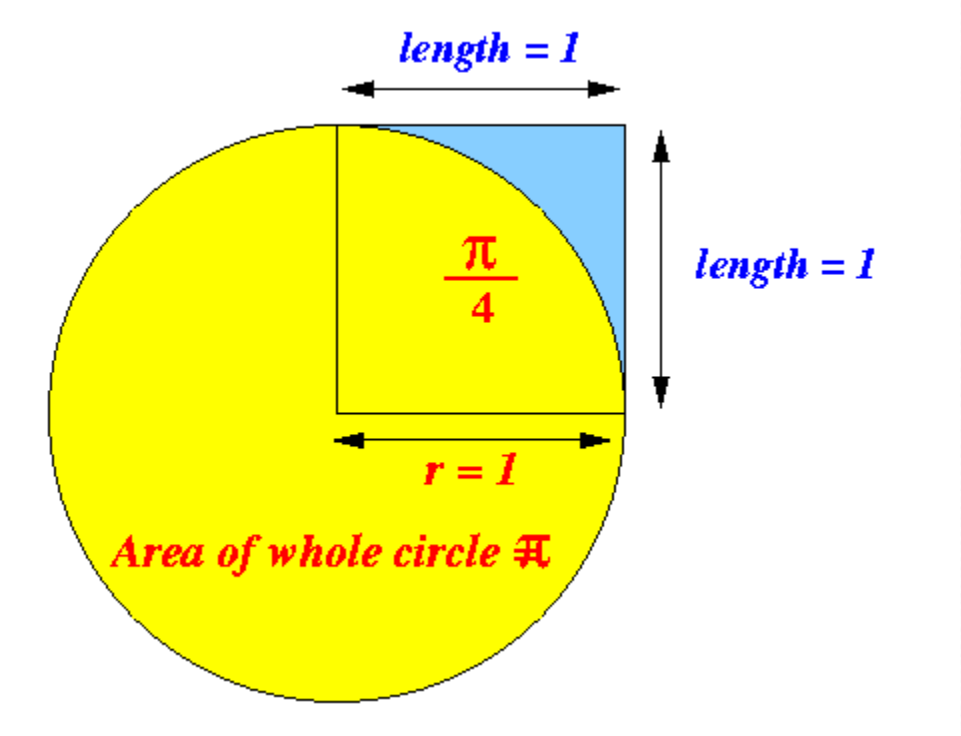


Basis for estimating a value of π

- With number of trials n/k will be approaching the ratio of the area of the square and area of the circle.
- With sufficiently large n, $n/k = (4r^2/\pi r^2)$, implies $\pi = 4k/n$



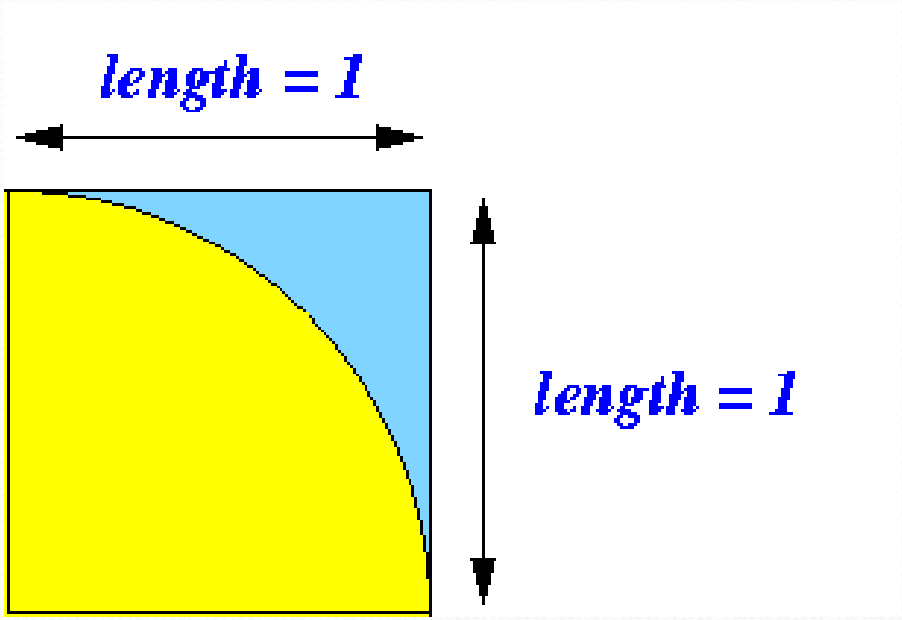
The **area** of the **quarter circle**



- The **area** of the **full circle** = $\pi \times 1^2 = \pi$
- The **area** of the **quarter circle** = $\pi / 4$

Computing the probability that a dart hits the yellow portion of the dart board

- If You through n darts and count the number of



Computing Π

1. Algorithm **Phi**
2. // Numerical probabilistic algorithm to compute Π
3. {
4. k:=0;
5. for j := 1 to n do
6. {
7. x := Uniform(0,1);
8. // Uniform(0,1) returns a uniformly random number between 0 and 1
9. y:= Uniform(0,1);
10. If ($x^2 + y^2 <= 1$) then k:= k+1;
11. }
12. write ("Value of Π = ", 4k/n)
13. }

Program design to compute π

1. Simulate the throw of a dart by generating random numbers between 0 and 1 for the x and y coordinates of a point. Note that the rand() function generates integer numbers. So you'll have to find a way to map those numbers to the range of 0 and 1. Since these are real numbers, you'll have to declare both x and y as float and do floating arithmetic. One way to do this is **x = rand()%1000/1000.0**
2. if the point (dart) falls inside or outside the circle. This is done by calculating the distance d from the origin of the circle to the point by using the Pythagoras Theorem which is sqrt(x * x + y * y), where sqrt is the square root function.

Program design to compute π

3. If the distance d calculated in step 2 is less than or equal to 1 then it is inside the circle and so you want to count it by executing inside++;
4. 4) Repeat steps 1 to 3 for n = 100 times.
5. 5) At the end of the loop, calculate PI as 4.0*inside/n. We need to multiply by 4 because we are generating the coordinates between 0 and 1. So our points are only in the first quadrant (or ¼) of the circle as shown in the figure below

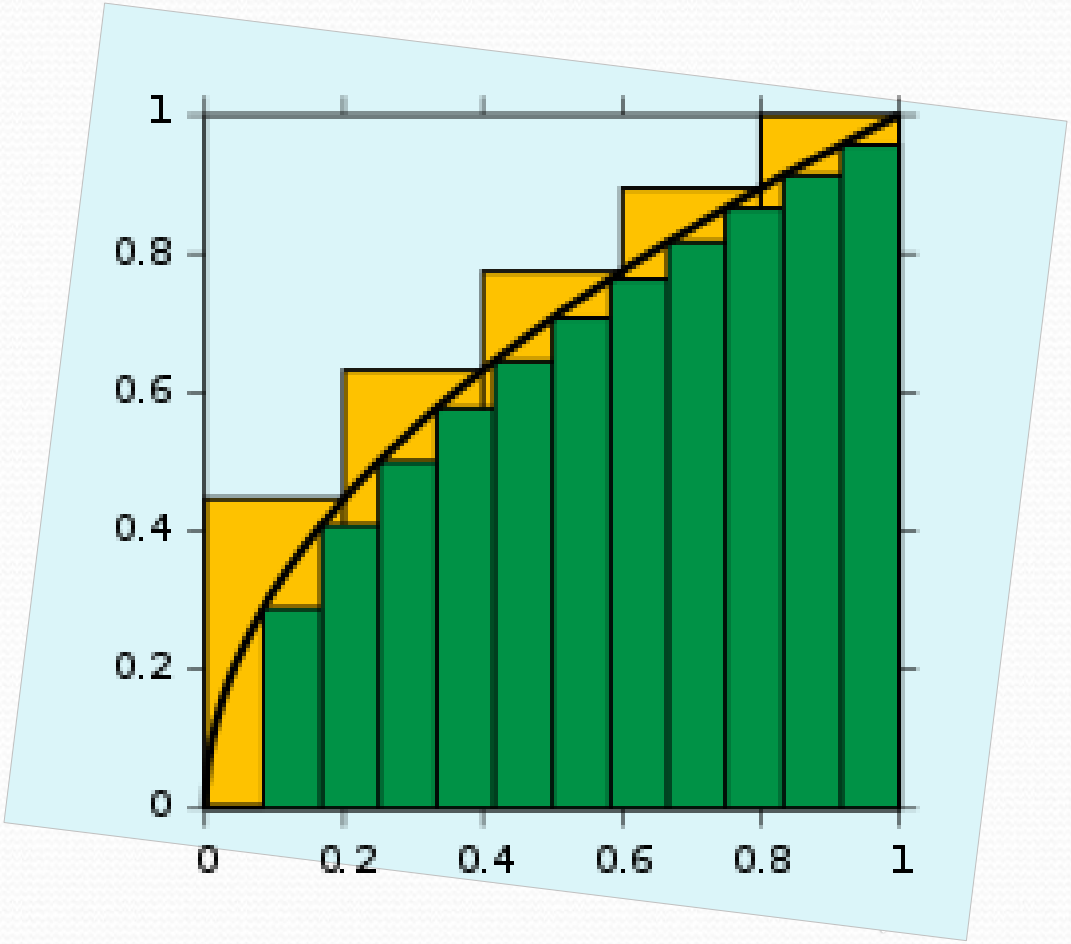
Matlab code

Simulation results

Questions on simulation experiments

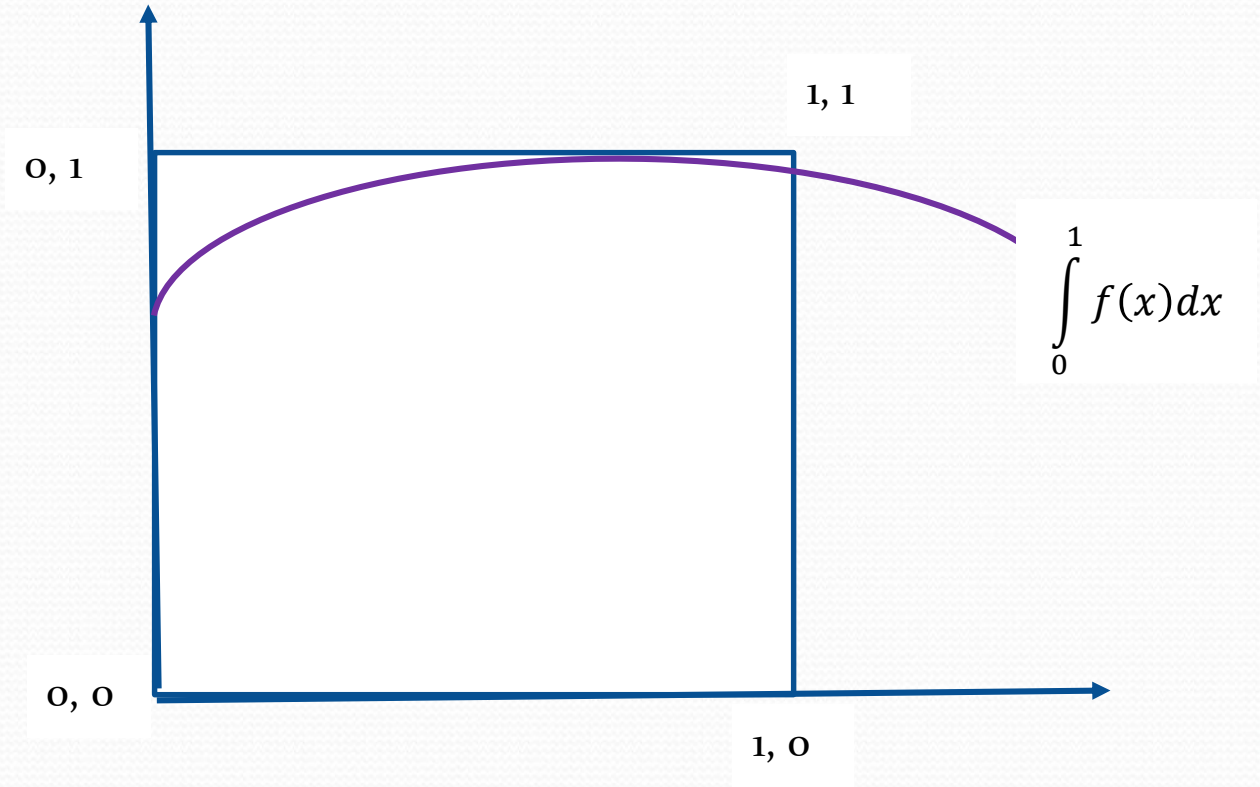
1. How accurate is your value of PI?
2. Do you get the same value when you run your program again?
3. Is your PI more accurate if you make n larger in step 4?
4. In step 1, what happens if you use integer calculations?
5. In step 1, you can also make PI more accurate if you increase the number.
6. How accurate can you make PI to be?

Numerical Integration



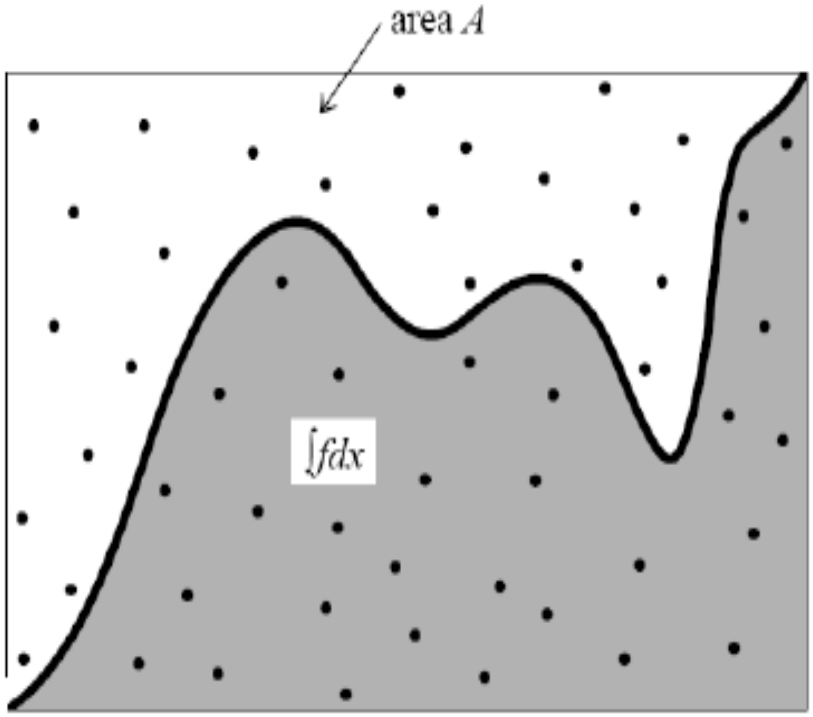
Finding integral of a continuous function

- Let f be continuous function from [0, 1] to [0, 1].
- The size of the area bounded by the curve y = f(x) in Cartesian coordinate is given by $\int_0^1 f(x)dx$



Finding integral of a continuous function

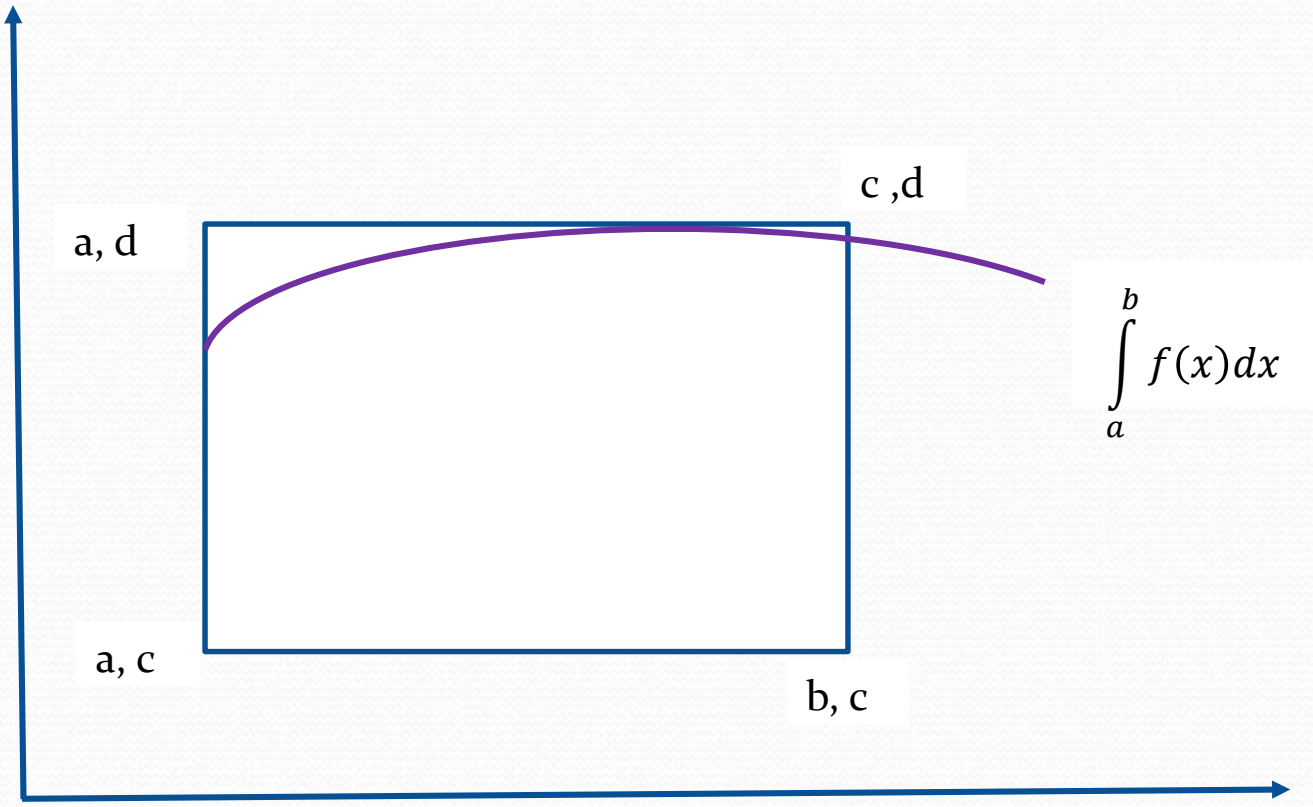
- The value of the integral can be estimated by throwing large number of darts(n) at the unit square. Let k be the darts that hits the square below the curve *f(x)*.
- The ratio **k/n** gives the value of the integral when n is sufficiently large.



Numerical Integration

```
1.  Algorithm Integral1
2.  // Numerical probabilistic algorithm for Integral f(x)
3.  {
4.  k:=0;
5.  for  j := 1 to n do
6.  {
7.  x := Uniform(0,1);
8.  y := Uniform(0,1);
9.  If ( y <= f(x))  then k:= k+1;
10. }
11. write ("Value of  the Integral = ", k/n)
12. }
```

Numerical Integration f(x) in [a,b] to [c,d]



Numerical Integration f(x) in [a,b] to [c,d]

```
1.  Algorithm Integral2
2.  // Numerical probabilistic algorithm for Integral f(x)
3.  {
4.  k:=0;
5.  for  j := 1 to n do
6.  {
7.  x := Uniform(a, b);
8.  y := Uniform(c, d);
9.  If ( y <= f(x))  then k:= k+1;
10. }
11. result := (b-a)(c+(d-c)k/n);
12. write ("value of  the integral = ", result)
13. }
```

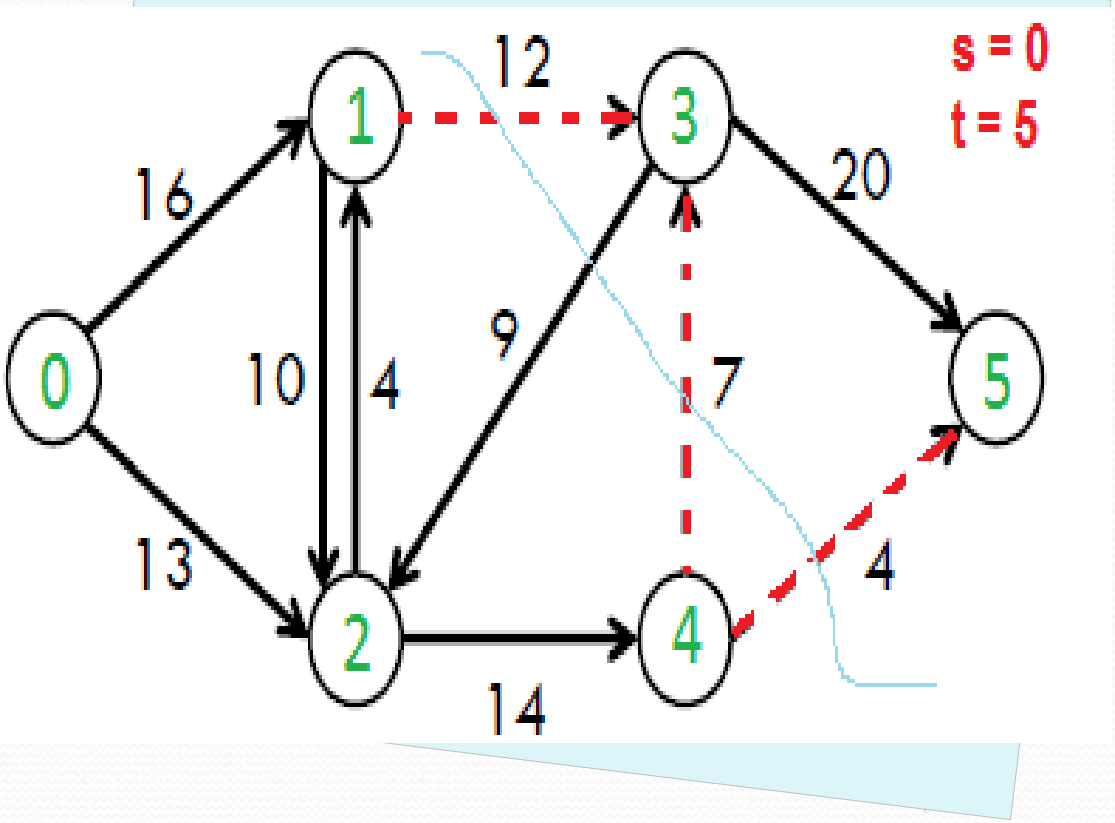
Simulation results

Monte-carlo Algorithm

Algorithms which always return a result, but the result may not always be correct. We attempt to minimise the probability of an incorrect result, and using the random element, multiple runs of the algorithm will reduce the probability of incorrect results.

- **Randomized Min-Cut**
- **Primality Testing**
- **Finding Majority Element**

Randomized
MIN_CUT

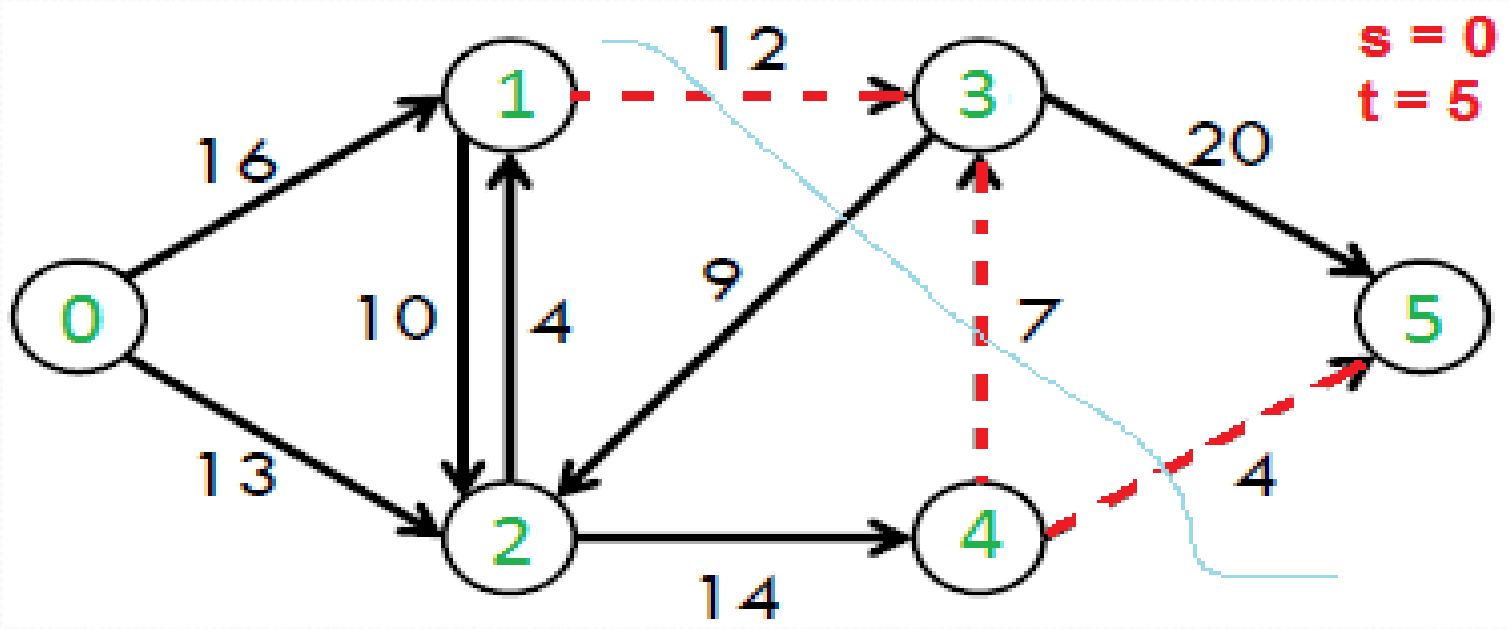


Cut and cut-set

- In graph theory, a **cut** is a partition of the vertices of a graph into two disjoint subsets.
- Any cut determines a **cut-set**, the set of edges that have one endpoint in each subset of the partition.
- These edges are said to **cross** the cut. In a connected graph, each cut-set determines a unique cut, and in some cases cuts are identified with their cut-sets rather than with their vertex partitions.
- In a flow network, an **s–t cut** is a cut that requires the *source* and the *sink* to be in different subsets, and its *cut-set* only consists of edges going from the source's side to the sink's side.
- The *capacity* of an s–t cut is defined as the sum of **capacity** of each edge in the *cut-set*.

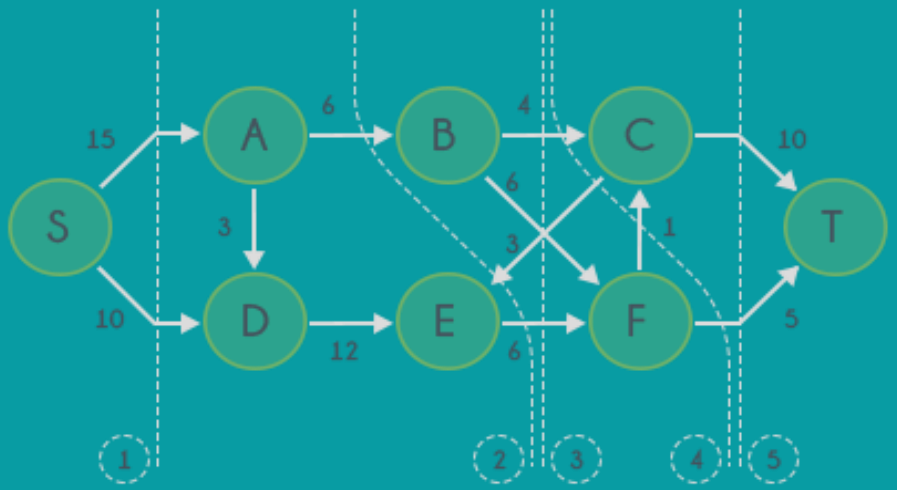
Minimum cut

- A cut is *minimum* if the size or weight of the cut is not larger than the size of any other cut. The illustration on the right shows a minimum cut: the size of this cut is 2, and there is no cut of size 1 because the graph is bridgeless.



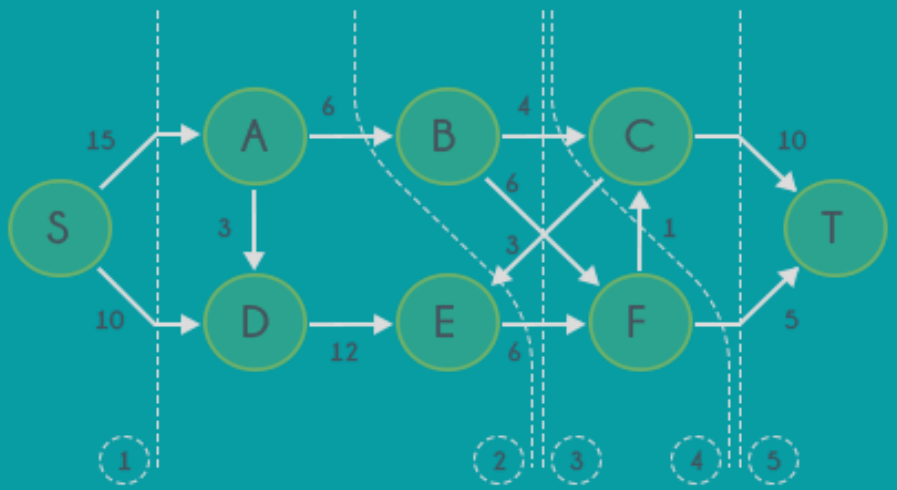
Min-Cut of a weighted graph

- Min-Cut of a weighted graph is defined as the minimum sum of weights of (at least one)edges that when removed from the graph divides the graph into two groups.



Min-Cut of a weighted graph

- Few possible cuts in the graph are shown in the graph and weights of each cut are as Follows: Cut1:25, Cut2:12, Cut3:16, Cut4:10, Cut5:15Cut1:25, Cut2:12, Cut3:16, Cut4:10, Cut5:15.



Graph division

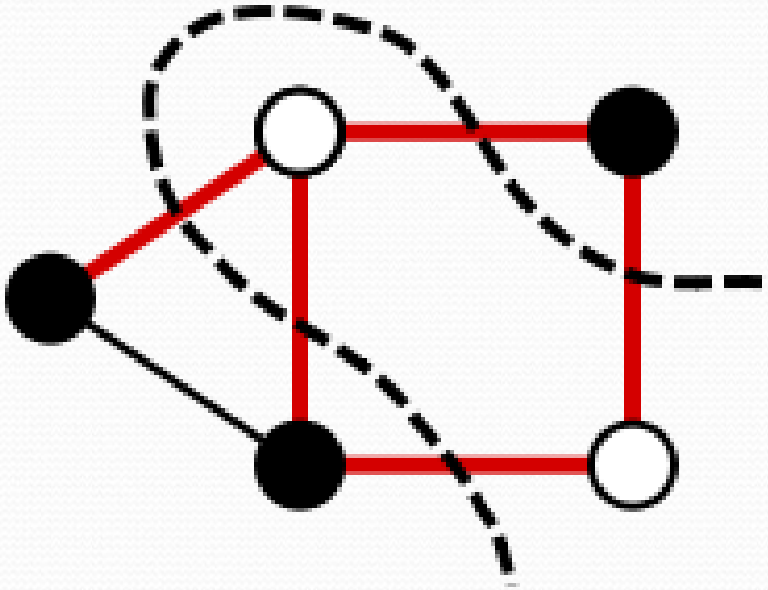
- You are given a graph with the following information:
- The graph contains N bidirectional edges. Each edge is denoted as (U, V), where U and V represent nodes of the graph. Each edge (U, V) has a Value Val associated with it.
- You are also given two nodes X and Y. You have to divide the graph by deleting some edges such that:
- Each part of the divided graph contains either X or Y.
- The loss of the values from the edges is minimal.
- Write an algorithm to find the value that is lost while dividing this graph.

The Problem

- Input: Undirected graph $G=(V,E)$

  Edges have non-negative weights

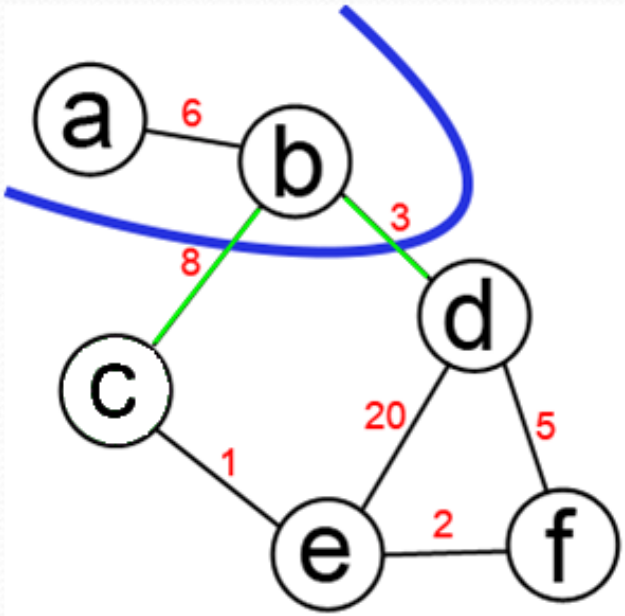
- Output: A minimum cut of G



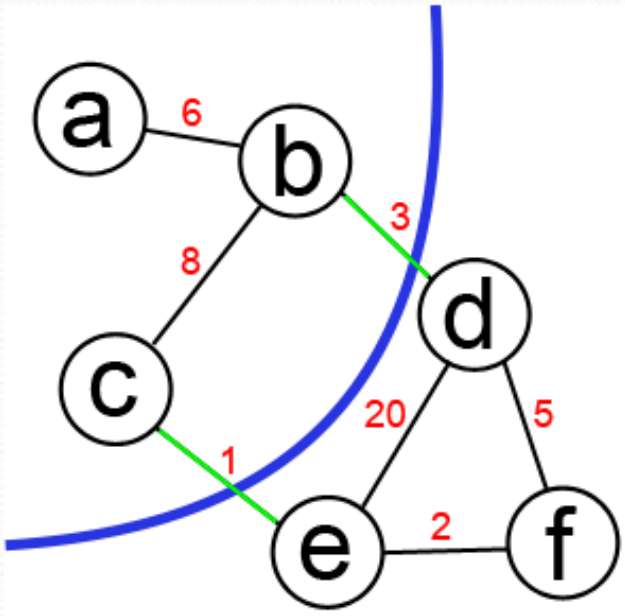
Cut Example

Cut: set of edges whose removal disconnects *G*

Min-Cut: a cut in *G* of minimum cost



Weight of this cut: 11

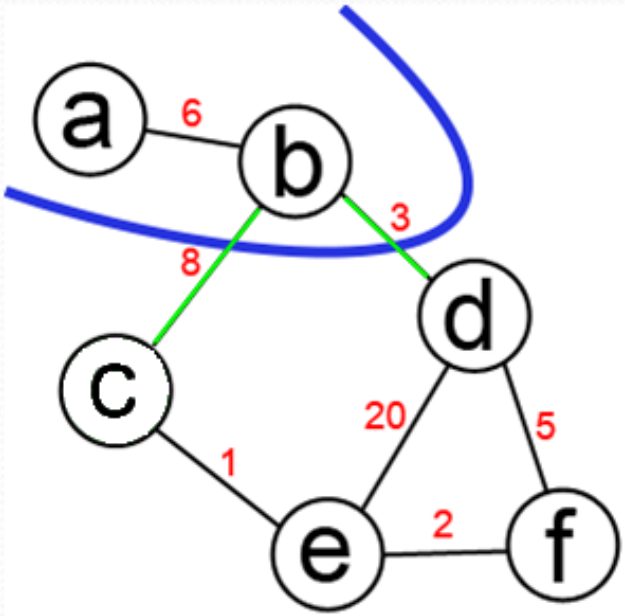


Weight of min cut: 4

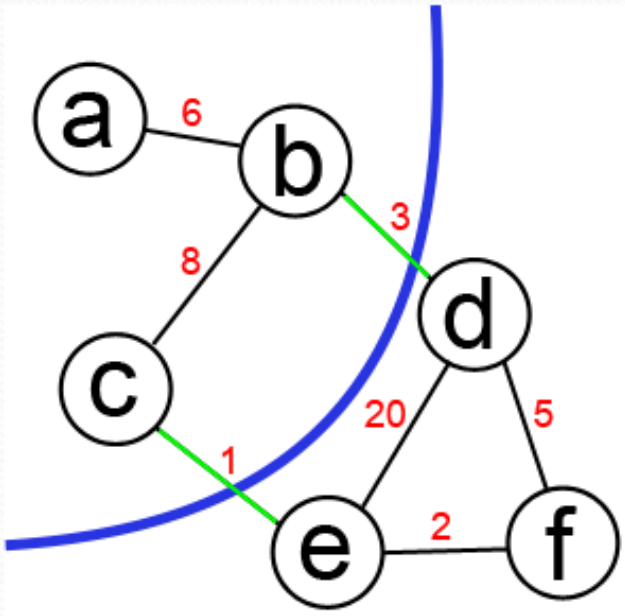
s-t Cut Example

s-t cut: cut with *s* and *t* in different partitions

s = *a* and *t* = *d*



Weight of this *a-d* cut: 11



Weight of min *a-d* cut: 4

Naive Solution

- Check every possible cut
- Take the minimum
- A complete graph has an edge between any two vertices. You can get an edge by picking any two vertices. So if there are n vertices, there are n(n–1)/**2 edges**.
- Running time: $O(2^n)$

Randomized Min-cut

- Given an undirected, connected multi-graph $G(V,E)$, we want to find a cut (V_1,V_2) such that the number of edges between V_1 and V_2 is minimum.
- This problem can be solved optimally by applying the max-flow min-cut algorithm $O(n^2)$ time by trying all pairs of source and destination.

Karger's algorithm: 1/3

- Karger's algorithm is a randomized algorithm to compute a minimum cut of a connected **Graph**.
- It was invented by **David Karger** and first published in 1993.
- The algorithm is a **Monte carlo algorithm**, i.e. its running time is deterministic, but it isn't guaranteed that at every iteration the best solution will be found.

Karger algorithm: 2/3

- **Karger algorithm** builds a cut of the graph by randomly creating this partitions, and in particular by choosing at each iteration a random edge, and contracting the graph around it:
- basically, merging its two endpoints in a single vertex, and updating the remaining edges, such that the self-loops introduced (like the chosen edge itself) are removed from the new Graph
- storing parallel-edges (if the algorithm chooses an edge (u,v) and both u and v have edges to a third vertex w, then the new Graph will have two edges between the new vertex z and w)
- After n-2 iterations, only two macro-vertex will be left, and the parallel edges between them will form the cut.

Karger's algorithm:3/3

- Actually the probability of finding the minimum cut in one run of the algorithm is pretty low, with an upper bound of $1/(n^2)$, where n is the number of vertices in the Graph.
- Nonetheless, by running the algorithm multiple times and storing the best result found, the probability that none of the runs founds the minimum cut becomes very small: $1/e$(Neper) for n squared runs, and $1/n$ for $n^2 * \log(n)$ runs - for large values of n, i.e. for large Graphs, that's a negligible probability.

Simple randomized algorithm: D. Karger

- The simple algorithm (view #1)

1. Pick an edge (x, y) at random in G.
2. CONTRACT the edge, keeping multi-edges, but removing self-loops. (i.e., if there were edges (x, v) and (y, v), we keep both of them.)
3. If there are more than 2 nodes, go back to 1. Else, output the edges remaining as your cut.

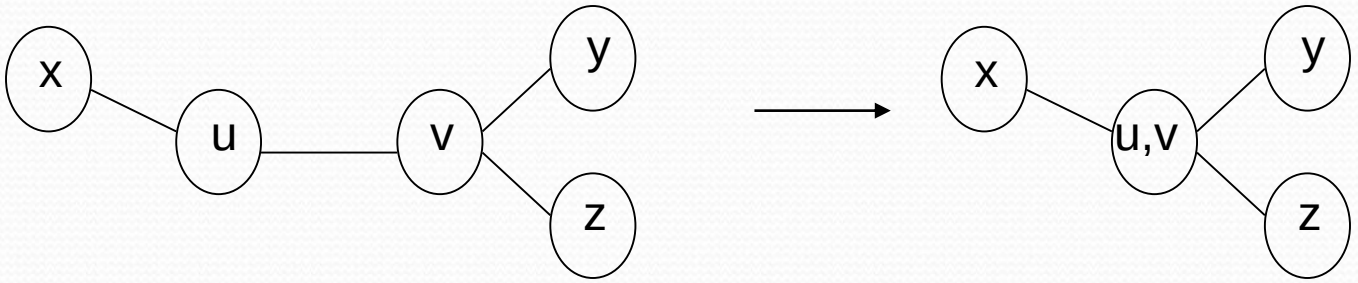
Randomized algorithm for min-cut

1. Algorithm Min-cut(G = (V, E))
2. {
3. While (|V| > 2) do
4. {
5. Select an edge (x, y) $\in$ E uniformly at random;
6. Contract the edge (x, y);
7. }
8. return (|E|);
9. }

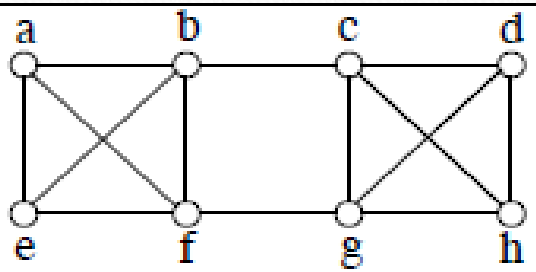
Randomized Min-cut

In randomized Min-cut, we repeatedly do the following:

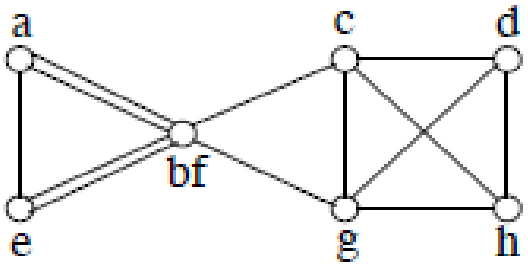
- Pick randomly an edge $e(u,v)$. Merge u and v, and remove all the edges between u and v. For example:
- until there are only 2 vertices left. We will report the cut between these 2 vertices as the min-cut.



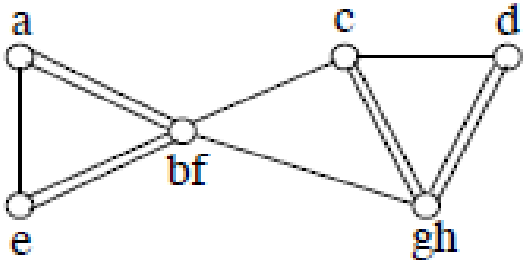
Working of Karger's algorithm for Min-cut



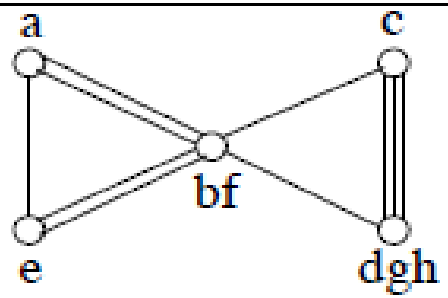
14 edges to choose from
Pick $b - f$ (probability 1/14)



13 edges to choose from
Pick $g - h$ (probability 1/13)

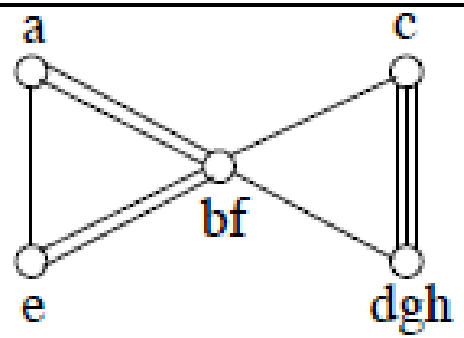


12 edges to choose from
Pick $d - gh$ (probability 1/6)

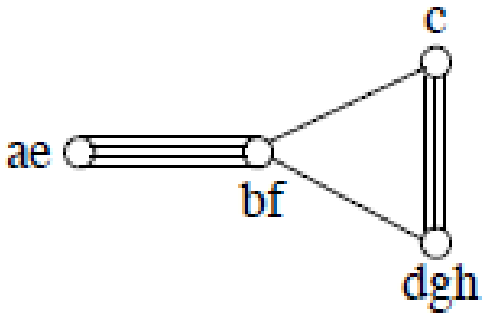


10 edges to choose from
Pick $a - e$ (probability 1/10)

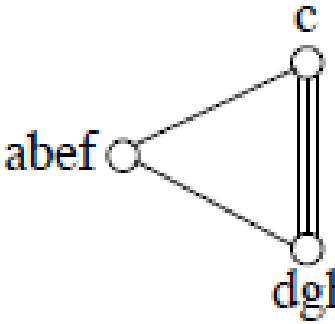
Working of Karger's algorithm for Min-cut



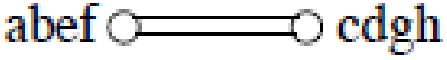
10 edges to choose from
Pick $a - e$ (probability 1/10)



9 edges to choose from
Pick $ab - ef$ (probability 4/9)



5 edges to choose from
Pick $c - dgh$ (probability 3/5)



Done: just two nodes remain

Analysis of Randomized Min-cut

- Let k be the min-cut of the given graph $G(E,V)$ where $|V|=n$.
- Then $|E| \geq kn/2$.
- The probability q_1 of picking one of those k edges in the first merging step $\leq 2/n$
- The probability p_1 of not picking any of those k edges in the first merging step $\geq (1-2/n)$
- Repeat the same argument for the first n-2 merging steps.
- Probability p of not picking any of those k edges in all the merging steps $\geq (1-2/n)(1-2/(n-1))(1-2/(n-2))\ldots(1-2/3)$

Analysis of Randomized Min-cut

- Therefore, the probability of finding the min-cut:

$$\begin{aligned} p &\geq (1-\frac{2}{n})(1-\frac{2}{n-1})\ldots(1-\frac{2}{3}) \\ &= \frac{(n-2)(n-3)\ldots(1)}{n(n-1)\ldots 3} \\ &= \frac{2}{n(n-1)} \end{aligned}$$

- If we repeat the whole procedure $n^2/2$ times, the probability of not finding the min-cut is at most

$$(1-2/n^2)^{n^2/2} \cong 1/e$$

- This expression is less than 1/e , where e = 1 + 1 +1/2! + 1/3! + … which is 2.71828 approximately.

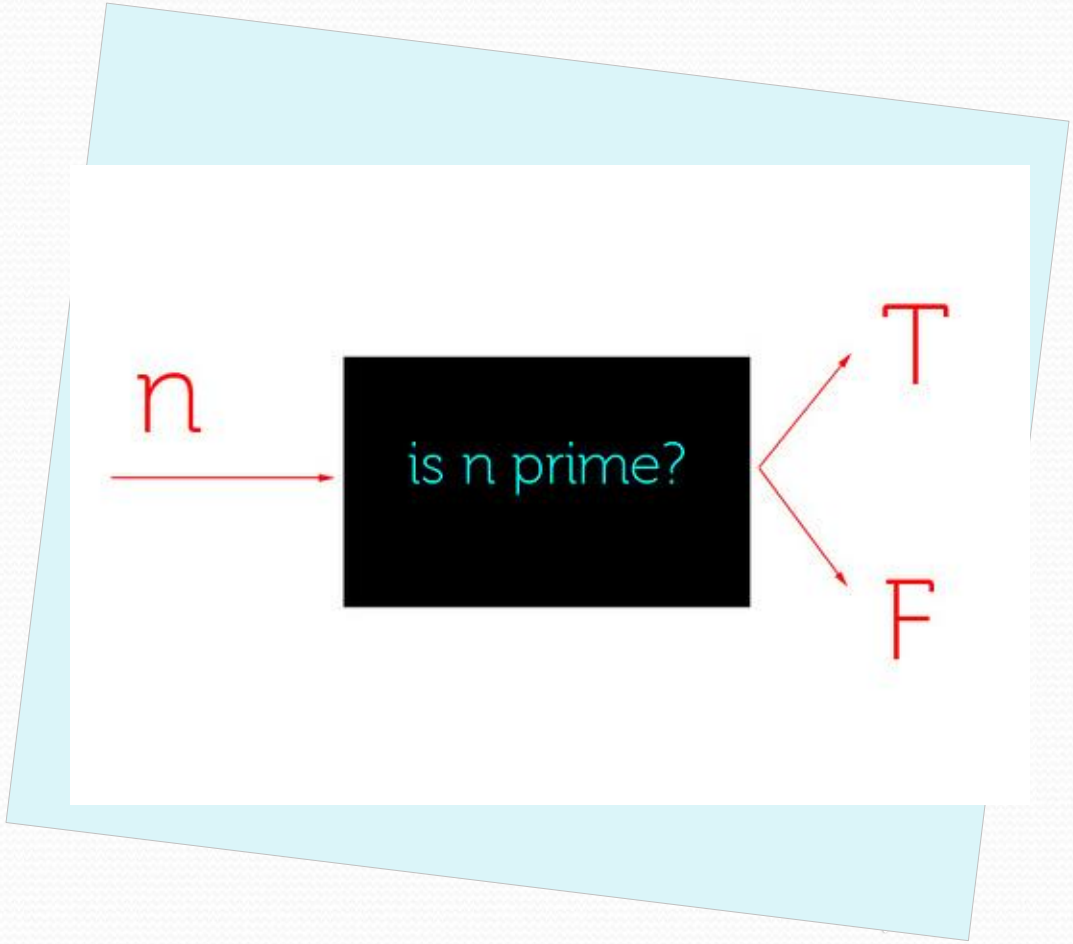
Analysis of Randomized Min-cut

Hence the probability of failure has been reduced from

$$(1-2/n^2)^{n^2/2} \text{ to } 1/e$$

through repetition process.

Primality Testing



Primality Testing

- Primality testing is an important algorithmic problem.
- In addition to being a fundamental mathematical question, the problem of how to determine whether a given number is prime has tremendous practical importance.
- Every time someone uses the RSA public-key cryptosystem, they need to generate a private key consisting of two large prime numbers and a public key consisting of their product.
- To do this, one needs to be able to check rapidly whether a number is prime.

Primality Testing

- In 1980, Michael Rabin discovered a randomized polynomial-time algorithm to test whether a number is prime.
- It is called the Miller-Rabin primality test because it is closely related to a deterministic algorithm studied by Gary Miller in 1976.
- This is still the most practical known primality testing algorithm, and is widely used in software libraries that rely on RSA encryption

Primality test

Definition:

An integer $p \geq 2$ is prime **iff** ($a \mid p \rightarrow a = 1$ **or** $a = p$).

Algorithm: deterministic primality test (naive)
Input: integer $n \geq 2$
Output: answer to the question: Is n prime?

```
if n = 2 then return true
if n even then return false
for i = 1 to √n/2 do
        if 2i + 1 divides n
        then return false
return true
```

Complexity: $\Theta(\sqrt{n})$

Primality test

Goal:

Randomized method

- Polynomial time complexity (in the length of the input)
- If answer is “not prime”, then n is not prime
- If answer is “prime”, then the probability that n is not prime is at most $p>0$

k iterations: probability that n is not prime is at most p^k

Primality test

Observation:
Each odd prime number p divides $2^{p-1} - 1$.

Examples: $p = 17$, $2^{16} - 1 = 65535 = 17 * 3855$
$p = 23$, $2^{22} - 1 = 4194303 = 23 * 182361$

Simple primality test:
1 Calculate $z = 2^{n-1} \bmod n$
2 if $z = 1$
3 then n is possibly prime
4 else n is definitely not prime

Advantage: This only takes polynomial time

Simple primality test

Definition:

n is called pseudoprime to base 2, if n is not prime and

$$2^{n-1} \bmod n = 1.$$

Example: $n = 11 * 31 = 341$

$$2^{340} \bmod 341 = 1$$

Randomized primality test

Theorem: (Fermat's little theorem)

If p prime and $0 < a < p$, then

$$a^{p-1} \bmod p = 1.$$

Definition:

n is pseudoprime to base a, if n not prime and

$$a^{n-1} \bmod n = 1.$$

Example: $n = 341$, $a = 3$

$$3^{340} \bmod 341 = 56 \neq 1$$

Randomized primality test

Algorithm: 1 Randomized primality test

1 Randomly choose $a \in [2, n\text{-}1]$
2 Calculate $a^{n-1} \bmod n$
3 **if** $a^{n-1} \bmod n = 1$
4 **then** n is possibly prime
5 **else** n is definitely not prime

Prob(n is not prim, but $a^{n-1} \bmod n = 1$) ?

Monte-Carlo algorithm for primality test

1. Algorithm Ptesting(N,k)
2. // k is the number of attempts to test N for primality
3. {
4. i:=1;
5. R := Uniform(1, N-1);
6. Remainder := MOD(N, r);
7. If (Remainder = 0) Goto **12**;
8. i :=i+1;
9. if(i ≤ k) then Goto **5**;
10. Print(‘N is prime”);
11. Exit;
12. Print (‘N is Composite’);
13. Exit;
14. }

Monte-Carlo algorithm for primality test

- The parameter k controls number of repetitions.
- By increasing the value of k, the probability of the error that the number is composite but is reported as prime by the algorithm may be reduced.

Majority element

Majority element

- An array A of N natural numbers has a majority element n, if the number of occurrences of n in A is greater than N/2.
- Question: How would you decide whether an array has a majority element using a deterministic algorithm, and what is its complexity?
- By choosing an element at random and testing to see if it is a majority element we get a probabilistic algorithm.
- The complexity of the algorithm is O(Nk), where k is the number of times the algorithm **Majority()** is invoked.

Majority element

```
1.  Algorithm Main(A, N)
2.  {
3.  Counter :=0;
4.  For i:= 1 to k do
5.  {
6.  Call Majority(A, N, R, E);
7.  If R ='True' then Counter = Counter + 1;
8.  If  Counter > 0 then goto 10;
9.  }
10. If Counter > 0 then
11. {
12. Print('Majority Element =' E);
13. Else
14. Print('No Majority Element ");
15. }
16. }
```

Algorithm Majority(A, N, R, E)

```
1.  Algorithm Majority(A, N, R, E)
2.  // A is an array of N element
3.  {
4.  R := 'False';
5.  choice := Uniform(1,N);
6.  count := 0;
7.  for i:= 1 to N do
8.  {
9.  If A(i) = A(choice) then count := count +1;
10. }
11. If (count > N/2) then
12. {
13. R := 'True';
14. E := A(choice);
15. }
16. }
```

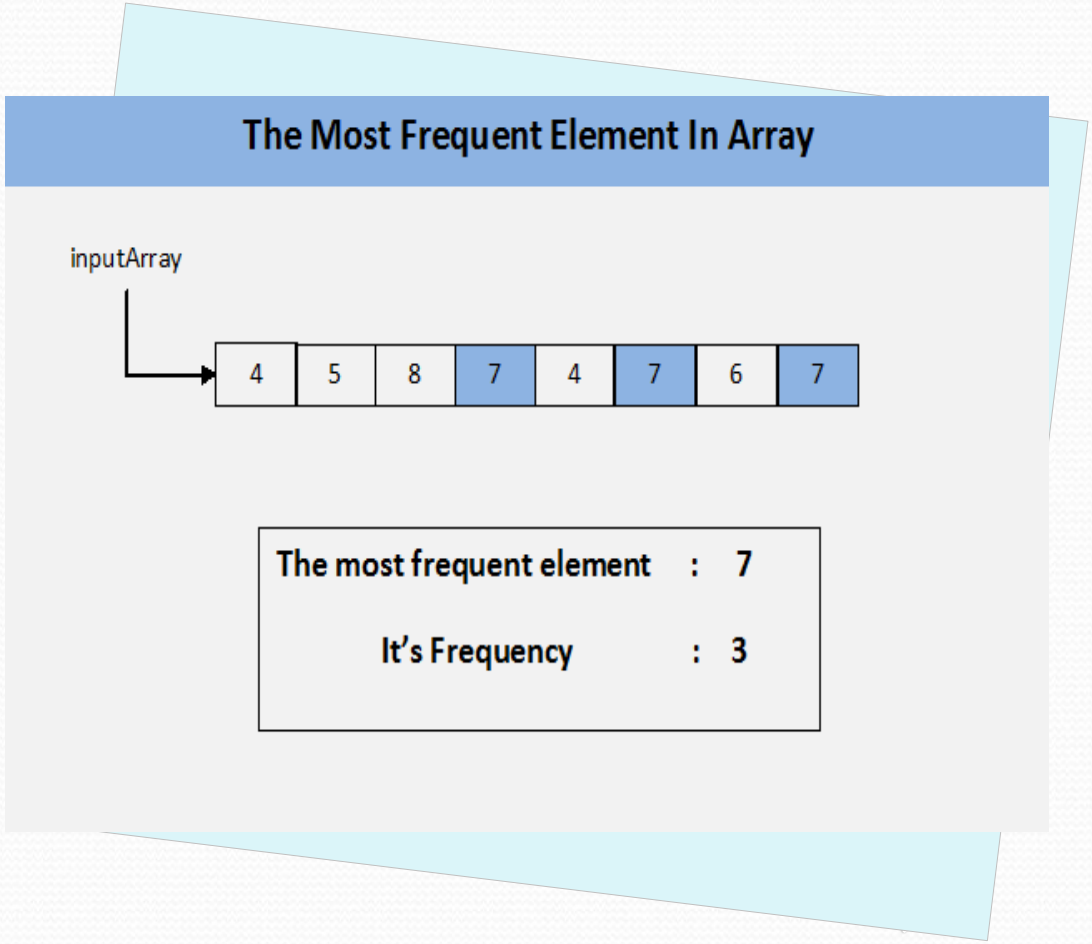
Las-Vegas Algorithm

- These are randomized algorithms which never produce incorrect results, but whose execution time may vary from one run to another .
- Random choices made within the algorithm are used to establish an expected running time for the algorithm that is, essentially, independent of the input.

Identifying the Repeated Element

8 Queen Problem

Identifying the Repeated Element



Identifying the Repeated Element

- Consider an array a[ ] of n numbers that has n/2 distinct elements and n/2 copies of another element. The problem is to identify the repeated element.
- Any deterministic algorithm for solving this problem will need at least (n/2) +2 steps in the worst case.

Identifying the repeated array number

- a simple and elegant randomized Las Vegas algorithm that takes only O(log n) time.
- The algorithm returns the array index of one of the copies of the repeated element

```
1   RepeatedElement(a, n)
2   // Finds the repeated element from a[1 : n].
3   {
4       while (true) do
5       {
6           i := Random() mod n + 1; j := Random() mod n + 1;
7           // i and j are random numbers in the range [1, n].
8           if ((i ≠ j) and (a[i] = a[j])) then return i;
9       }
10  }
```

Proving the run time RepeatedElement

- Any iteration of the while loop will be successful in identifying the repeated number if i is any one the n/2 array indices corresponding to the repeated element and j is any one of the same n/2 indices other than **i.**
- The probability that the algorithm quits in any given iteration of the while loop is $P = \{ (n/2(n/2-1))/n^2 \}$, which is $\geq 1/5$ for all all $n \geq 10$.
- Hence the probability that the algorithm does not quit in a given iteration is < 4/5.
- There fore the probability that the algorithm does not quit in 10 iterations is $< (4/5)^{10} < 0.1074$.
- So, Algorithm will terminate in 10 iterations or less with probability ≥ 0.8926

Proving the run time RepeatedElement

- The probability that the algorithm does not terminate in 100 iterations is $< (4/5)^{100} < 2.04 * 10^{-10}$. That is, almost Certainly the algorithm will quit in 100 iterations or less.
- If n equals $2 * 10^6$, for example, any deterministic algorithm will have to spend at least one million time steps as opposed to the 100 iterations.
- In general, the probability that the algorithm does not quit in the first **c α logn** (c is a constant to be fixed) iterations is

$$< (4/5)^{c \alpha \log n} = n^{c \alpha \log n (5/4)}$$

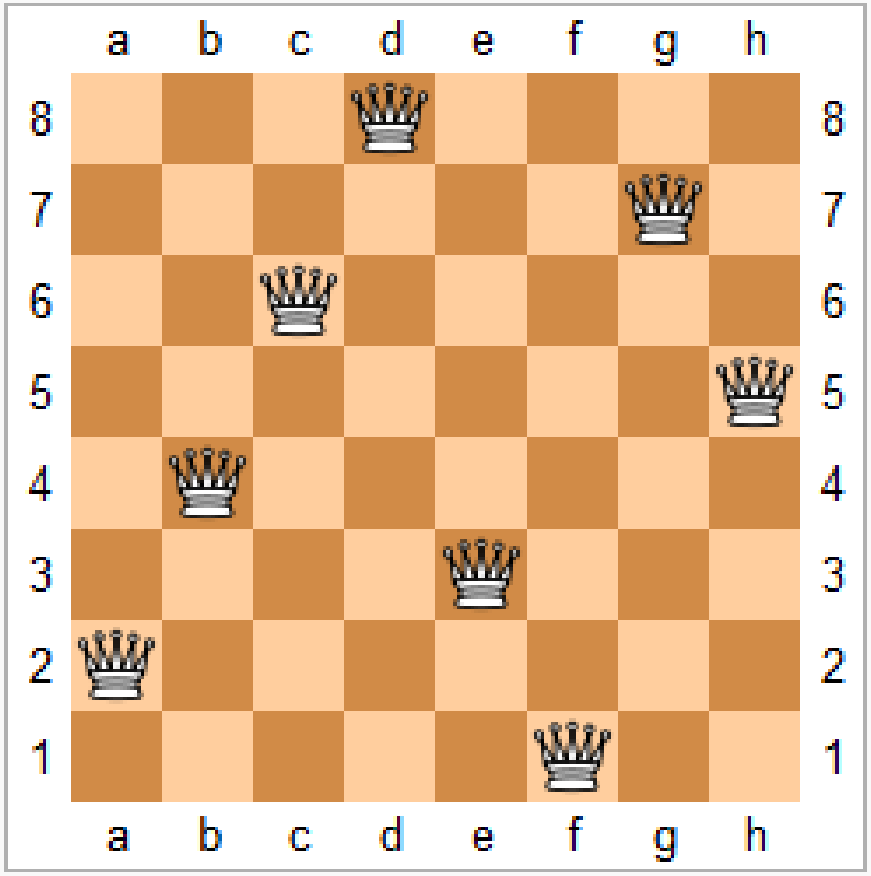
which will be $< n^{-\alpha}$ if we pick $c \geq (1/(\log 5/4))$

- Thus the algorithm terminates in $(1/(\log 5/4))$ **α logn** iterations or less with probability $\geq (1 - n^{-\alpha})$.

- Since each iteration of the while loop takes o(1) time, the run time of the algorithm is O(logn)

- Note that this algorithm, if it terminates, will always output the correct answer and hence is of the Las Vegas type. The above analysis shows that the algorithm will terminate quickly with high probability

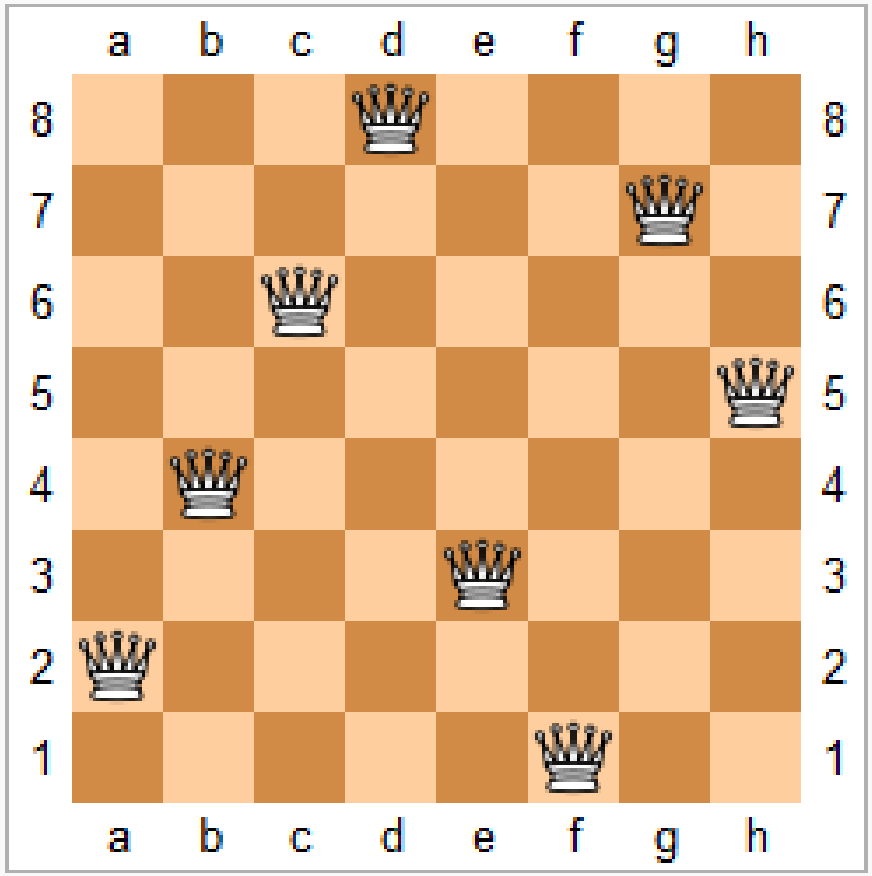
8 Queen Problem



One solution to the eight queens puzzle

The eight queens puzzle

- The **eight queens puzzle** is the **problem** of placing **eight** chess **queens** on an **8×8**chessboard so that no two **queens** threaten each other. Thus, a **solution** requires that no two **queens** share the same **row**, **column**, or diagonal.



One solution to the eight queens puzzle

The eight queens puzzle

- The classic solution is via backtracking. Place the first queen in the top-left corner.
- Now with some non-attacking queens in place, place a queen in the next row, if it attacks previous queens move it across the row, square by square.
- If there is no suitable position in this row then move the queen in the immediately previous row on one space if possible.
- If neither row leads to a required arrangement, move the queen in the previous row by one space, etc. In fact this returns a solution after examining only 114 of the 2,057 states.

A Las Vegas solution to the 8 Queen problem:

- Suppose that k rows ($0 \leq k \leq 8$) have been successfully occupied (initially $k = 0$).
- If $k = 8$ then stop with success. If not then we wish to occupy row $k + 1$.
- Calculate all the possible positions on this row i.e. those which are not attacked by already placed queens.
- If there are no such places, then fail. Otherwise choose one such place at random, increment k, and repeat

A Las Vegas solution to the 8 Queen problem:

- It can be calculated that the probability of success of this random placement is 0.1293...
- This may seem low, but it means that a solution is obtained 1 time out of 8 simply by guessing! .
- The expected number of states explored is approximately 56, compared to 114 for the deterministic backtracking solution.

Example

Finding a Closest Pair

A Randomized Algorithm for Finding a Closest Pair

- **Algorithm**
- **Input:** A set S consisting of n elements $x_1, x_2, \ldots, x_n$, where S R^2 .
- **Output:** The closest pair in S.
- **Step 1.** Randomly choose a set S_1={ } where $m=n^{2/3}$. Find the closest pair of S_1 and let the distance between this pair of points be denoted as .
- **Step 2.** Construct a set of squares T with mesh-size .

A Randomized Algorithm for Finding a Closest Pair

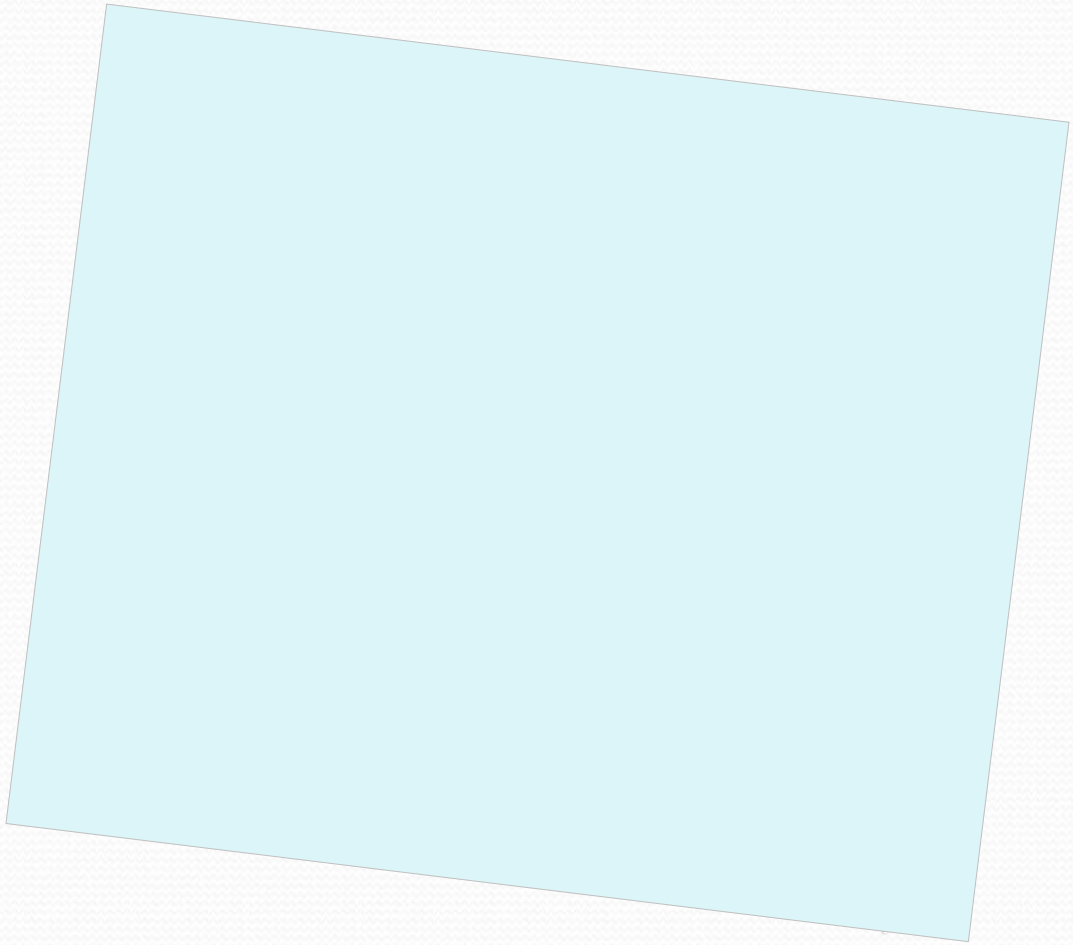
- **Step 3.** Construct four sets of squares T_1, T_2, T_3 and T_4 derived from T by doubling the mesh-size to 2 .
- **Step 4.** For each T_i, find the induced decomposition $S=S_1^{(i)}$ $S_2^{(i)}$ … , 1 i 4, where $S_j^{(i)}$ is a non-empty intersection of S with a square of T_i.
- **Step 5.** For each x_p, $x_q \in S_j^{(i)}$, compute $d(x_p, x_q)$. Let x_a and x_b be the pair of points with the shortest distance among these pairs. Return x_a and x_b as the closest pair.

Sherwood Algorithm

Algorithms which always return a result and the correct result, but where a random element increases the efficiency, by avoiding or reducing the probability of worst-case behaviour.

This is useful for algorithms which have a poor worst-case behaviour but a good average-case behaviour, and in particular can be used where embedding an algorithm in an application may lead to increased worst-case behaviour.

Randomized Quick Sort



Example

Selection Problem

The Selection Problem

- **Input:** Array A[1...n] of the elements in the an arbitrary order, and an index k
- **Output:** the kth smallest element in A[1..n].
- If k = 1, we are asking for the smallest element
- If k = n, we are asking for the largest element
- If k = n/2, we are asking for the **median**

Quick-Selection

- Choose a random element, split the array, eliminate the fraction that does not contain the desired element and recursively apply the procedure.
- Quick-Selection($A,1,n,k$)
  - More generally, Quick-Selection(A,p,r,k)
  - A procedure that selects the kth smallest element of elements in sub-array $A[p..r]$

Quick-Selection(*A, p, r, k*)

Quick-Selection(*A, p,r,k*)

if *p* =*r* and *k* = *1*, **do return** A(p)

if *p*< *r* **then**

q ← RANDOM (p, r)

t = Partition(A, p,r,q)

if t = k **do return** A[t]

else if t > k **do return** Quick-Selection(A,p,t-1,k)

else if t < k **do return** Quick-Selection(A,t+1,r,k-t)

Partition(A,p,r,q)

Suppose there are t-1 elements in A that is smaller than s = A[q]. Then return t and reorder A so that A[p..p+t-1] < A[t] = s <= A[t+1..r]

- Partition(A,p,r,q)
  - [redacted] SWAP(A[q],A[r])
  - $i \leftarrow p-1$
  - **for** $j \leftarrow p$ **to** $r-1$
    - **do if**
      - **then** $A[j] \le x$
  - $i \leftarrow i+1; SWAP(A[i], A[j])$
  - $SWAP(A[i+1], A[r])$
  - **return** $i+1$

Expected Time Complexity

$$E(T(n)) \le \frac{\sum_{i=1}^{n-1} E(T(\max(i, n-i)))}{n} + 3n$$

$$\le \frac{2\sum_{i=n/2}^{n-1} E(T(i))}{n} + 3n$$

$$= \frac{2\sum_{i=n/2}^{3n/4} E(T(i))}{n} + \frac{2\sum_{i=3n/4+1}^{n-1} E(T(i))}{n} + 3n$$

$$\le \frac{(n/2)E(T(3/4n))}{n} + \frac{(n/2)E(T(n-1))}{n} + 3n$$

Time Complexity

$$E(T(n)) \le \frac{E(T(3/4n))}{2} + \frac{E(T(n-1))}{2} + 3n$$

- Let G(n) = E(T(n)), we have

$$G(n) \le \frac{G(3/4n)}{2} + \frac{G(n-1)}{2} + 3n$$

Time Complexity

$$G(n) \le \frac{G(3/4n)}{2} + \frac{G(n-1)}{2} + 3n$$

- **Conjecture:** G(n) = O(n).
- **Use the substitution method**
  - Assuming for all m < n,
  - Need to show

$$G(m) \le cm$$

$$G(n) \le cn$$

Time Complexity (if c>=24)

$$G(n) \le \frac{G(3/4n)}{2} + \frac{G(n-1)}{2} + 3n$$
$$\le \frac{c(3/4n)}{2} + \frac{c(n-1)}{2} + 3n$$
$$\le \frac{7cn}{8} - \frac{c}{2} + 3n$$
$$\le cn$$

Quick-Sort

- Quick-Sort($A,1,n$)
  - Divide
    - $q \leftarrow RANDOM(1, n)$
    - t = Partition(A,1,n,q)
  - Conquer
    - Quick-Sort(A,1,t-1)
    - Quick-Sort(A,t+1,n)

Quick-Sort(*A,p,r*)

- Quick-Sort(*A,p,r*)
  - **if** $p < r$ **then**
    - $q \leftarrow RANDOM(p, r)$
    - t = Partition(A,p,r,q)
    - Quick-Sort(A,p,t-1)
    - Quick-Sort(A,t+1,r)

- Theorem: Worst Case: $\Omega(n^2)$
- Theorem: Expected Case: $O(n \lg n)$

Partition(A,p,r,q)

Suppose there are t-1 elements in A that is smaller than s = A[q]. Then return t and reorder A so that A[p..p+t-1] < A[t] = s <= A[t+1..r]

- Partition(A,p,r,q)
  - $x = A[q]; A[q] \leftarrow A[r]; A[r] \leftarrow x$ SWAP(A[q],A[r])
  - $i \leftarrow p-1$
  - **for** j p **to** *r-1*
    - **do if**
      - **then** $A[j] \le x$

        $i \leftarrow i+1; SWAP(A[i], A[j])$
  - $SWAP(A[i+1], A[r])$
  - **return** *i+1*

Complexity of Quick-Sort

- The Element chosen by RANDOM is called **the pivot element**
- Each number can be a pivot element at most once
- So totally, at most *n* calls to Partition procedure
- So the total steps is bounded by a constant factor of the number of comparisons in Partition.

Compute the total number of comparison in calls to Partition

- When does the algorithm compare two elements?
- When does not the algorithm compare two elements?
- Suppose ($Z[1],Z[2],\ldots,Z[n]$) are the sorted array of elements in A
- that is, $Z[k]$ is the kth smallest element of A.

Compute the total number of comparison in calls to Partition

- The reason to use Z rather than A directly is that is it hard to locate elements in A during Quick-Sort because elements are moving around.
- But it is easier to identify $Z[k]$ because they are in a sorted order.
- We call this type of the analysis scheme: **the backward analysis.**

Compute the total number of comparisons in calls to Partition

- Under what condition does Quick-Sort compare $Z[i]$ and $Z[j]$?
- What is the probability of comparison?
- First: $Z[i]$ and $Z[j]$ are compared at most once!!!
- Let E_{ij} be the random event that Z[i] is compared to Z[j].
- Let X_{ij} be the indicator random variable of E_{ij}.
  - X_{ij} = I{Z[i] is compared to Z[j]}

Compute the total number of comparisons in calls to Partition

- So the total number of comparisons is

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

- We are interested in

$$E(X) = E\left(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right)$$

Compute the total number of comparisons in calls to Partition

- By linearity of expectation, we have

$$E(X) = E\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}\right)$$
$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E\left(X_{ij}\right)$$
$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \Pr\left(Z[i] \text{ is compared to } Z[j]\right)$$

- So what is $\Pr(Z[i]$ is compared to $Z[j])$?

Compute the total number of comparisons in calls to Partition

- So what is Pr($Z[i]$ is compared to $Z[j]$)?
- What is the condition that
  - $Z[i]$ is compared to $Z[j]$?
- What is the condition that
  - $Z[i]$ is not compared to $Z[j]$?
  - Answer: no element is chosen from $Z[i+1]$ … $Z[j-1]$ before $Z[i]$ or $Z[j]$ is chosen as a pivot in Quick-Sort
- therefore …

Compute the total number of comparisons in calls to Partition

- **Therefore**

$$
\begin{aligned}
&\Pr\{Z[i] \text{ is compared to } Z[j]\} \\
&= \Pr\{Z[i] \text{ or } Z[j] \text{ is first pivot from } Z[i] \ldots Z[j]\} \\
&= \Pr\{Z[i] \text{ is first pivot from } Z[i] \ldots Z[j]\} \\
&\quad + \Pr\{Z[j] \text{ is first pivot from } Z[i] \ldots Z[j]\} \\
&= \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}
\end{aligned}
$$

Compute the total number of comparisons in calls to Partition

- By linearity of expectation, we have

$$E(X) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr(Z[i] \text{ is compared to } Z[j])$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k}$$

$$= \sum_{i=1}^{n-1} O(\lg n) = O(n \lg n)$$

Original Quick-Sort (Tony Hoare)

- Partition with the first element
- **Average-Case Complexity**:
  - Assume inputs come from uniform permutations.
- Our analysis of the Expected time analysis of Random Quick-Sort extends directly.
- Notice the difference of randomized algorithm and average-case complexity of a deterministic algorithm

Example

0-1 Knapsack problem

The 0-1 knapsack problem

- Even if we don't use a randomized algorithm to *find* a solution, we might use one to *improve* a solution
- The 0-1 knapsack problem can be expressed as follows:
  - A thief breaks into a house, carrying a knapsack…
    - He can carry up to 25 kg of loot
    - He has to choose which of N items to steal
      - Each item has some weight and some value
      - "0-1" because each item is stolen (1) or not stolen (0)
    - He has to select the items to steal in order to maximize the value of his loot, but cannot exceed 25 pounds
- A greedy algorithm does not find an optimal solution…but…
- **We could use a greedy algorithm to find an initial solution, then use a randomized algorithm to try to improve that solution**

Improving the knapsack solution

- We can employ a greedy algorithm to fill the knapsack
- Then--
  - Remove a randomly chosen item from the knapsack
  - Replace it with other items (possibly using the same greedy algorithm to choose those items)
  - If the result is a better solution, keep it, else go back to the previous solution
  - Repeat this procedure as many times as desired
    - You might, for example, repeat it until there have been no improvements in the last k trials, for some value of k
- You probably won't get an optimal solution this way, but you might get a better one than you started with

Example

Quick Sort

Randomized Quick Sort

- In traditional Quick Sort, we will always pick the first element as the pivot for partitioning.
- The worst case runtime is $O(n^2)$ while the expected runtime is $O(n\log n)$ over the set of all input.
- Therefore, some input are born to have long runtime, e.g., an inversely sorted list.

Randomized Quick Sort

- In randomized Quick Sort, we will pick randomly an element as the pivot for partitioning.
- The expected runtime of any input is $O(n \log n)$.

Analysis of Randomized QS

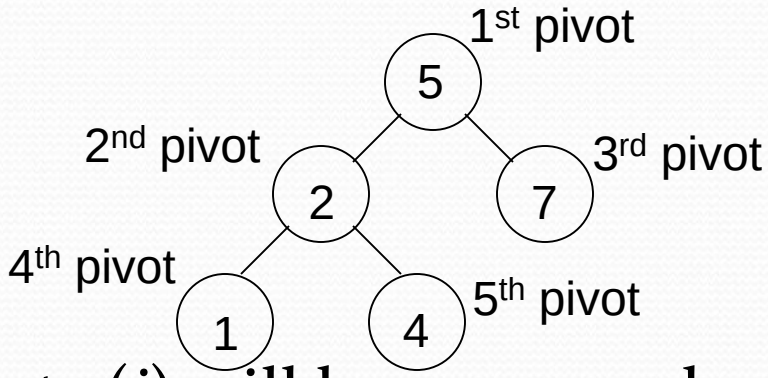
- Let $s(i)$ be the i^{th} smallest element in the input list S.
- X_{ij} is a random variable such that $X_{ij} = 1$ if $s(i)$ is compared with $s(j)$; $X_{ij} = 0$ otherwise.
- Expected runtime t of randomized QS is:

$$t = E[\sum_{i=1}^{n} \sum_{j \geq i} X_{ij}] = \sum_{i=1}^{n} \sum_{j \geq i} E[X_{ij}]$$

- $E[X_{ij}]$ is the expected value of X_{ij} over the set of all random choices of the pivots, which is equal to the probability p_{ij} that $s(i)$ will be compared with $s(j)$.

Analysis of Randomized QS

- We can represent the whole sorting process by a binary tree T:



- Notice that $s(i)$ will be compared with $s(j)$ where i<j if and only if $s(i)$ or $s(j)$ is the first one among the set $\{s(i), s(i+1), \ldots, s(j)\}$ to be selected as the pivot.
- Note that $p_{ij} = 2/(j-i+1)$. Why?

Analysis of Randomized QS

Therefore, the expected runtime t:

$$t = \sum_{i=1}^{n} \sum_{j \geq i} E[X_{ij}] = \sum_{i=1}^{n} \sum_{j \geq i} p_{ij}$$

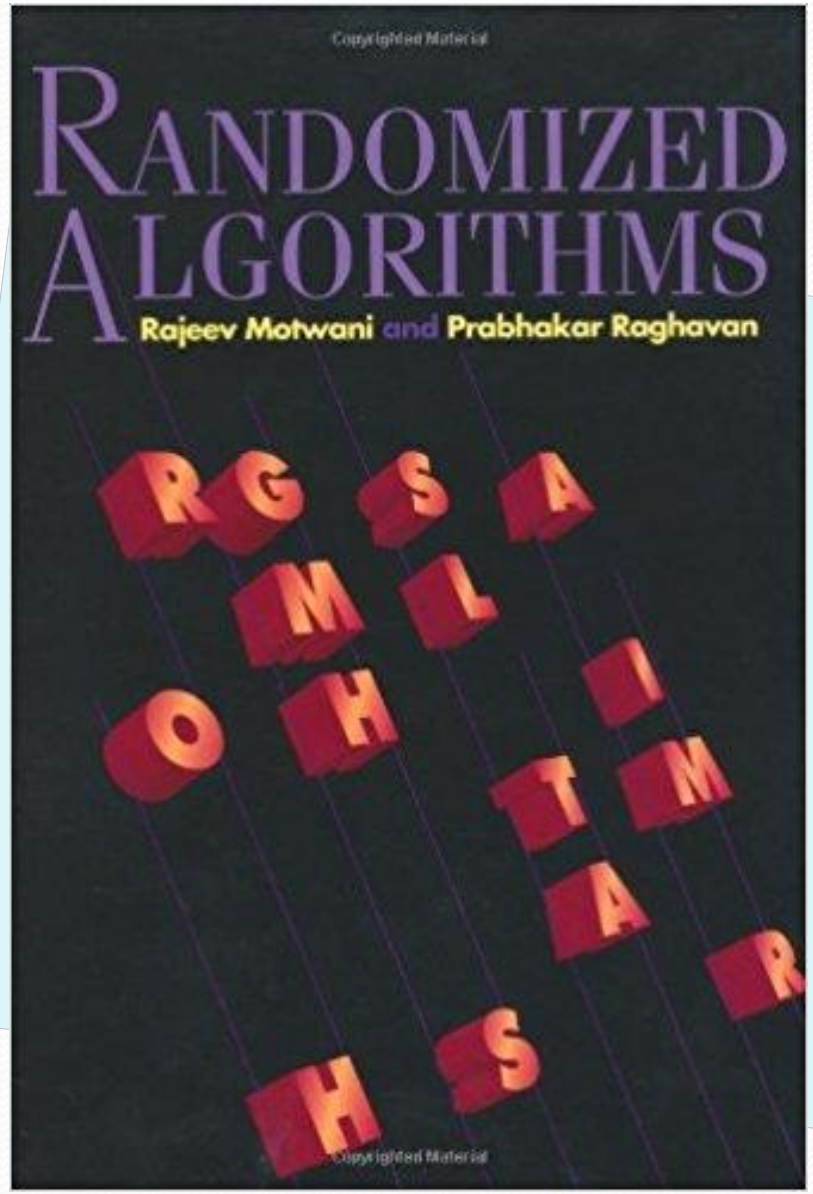
$$= \sum_{i=1}^{n} \sum_{j \geq i} \frac{2}{j-i+1} \leq n \sum_{j=1}^{n} \frac{2}{j}$$

$$\cong 2n \ln n$$

Note that $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \cong \ln n$

Randomized QS is a Las Vegas algorithm.

Randomized Algorithm Classes



Randomized Algorithm Classes

- **RP (Randomized Polynomial time)**
  - Monte Carlo algorithm running in worst-case polynomial time
  - If input part of language, P(accept) >= 0.5
  - If input *not* part of language, P(accept) = 0
  - "one-sided error" – could err on 'YES'
- **co-RP (complement of Randomized Polynomial time)**
  - Monte Carlo algorithm running in worst-case polynomial time
  - If input part of language, P(accept) = 1
  - If input *not* part of language, P(accept) <= 0.5
  - "one-sided error" – could err on 'NO'
- **ZPP (Zero-error Probabilistic Polynomial time)**
  - Las Vegas algorithm running in expected polynomial time
  - If input part of language, P(accept) = 1
  - If input *not* part of language, P(accept) = 0
  - "neither-sided error"

Randomized Algorithm Classes

- **PP (Probabilistic Polynomial time)**
  - Monte Carlo algorithm running in worst-case polynomial time
  - If input part of language, P(accept) > 0.5
  - If input *not* part of language, P(accept) < 0.5
  - "two-sided error" – could err on 'YES' and 'NO' – not so reliable
- **BPP (Bounded-error Probabilistic Polynomial time)**
  - Monte Carlo algorithm running in worst-case polynomial time
  - If input part of language, P(accept) >= 0.75
  - If input *not* part of language, P(accept) <= 0.25
  - "two-sided error" – could err on 'YES' and 'NO', but more useful than PP

Randomized Algorithms: Recap

- **Faster and simpler**: Even when a fast deterministic algorithm exists, it is more complex
- **Lots of theory** usually needed in the specific application area to prove either:
  - a reasonable one-time successful probability, such that repeating would lead to good overall probability (***Monte Carlo***)
  - a reasonable expected running time (***Las Vegas***)
- **Just as good**: High probability of success just as good as deterministic success: $P = 0.5^{100}$ is smaller than probability of your hard drive being hit by a meteor…while running the algorithm
- **Problem classes:** Another set to consider

Other Applications

- Expander graphs
- Convex hulls
- Linear programming
- Parallel and distributed algorithms
- Online algorithms
- Quadratic residues
- Approximation algorithms (for NP-complete problems)

References

Algorithmics: Theory and Practice, G. Brassard and P. Bratley. Prentice-Hall. 1988. Introductory chapter on the topic.

Randomized Algorithms, R. Motwani and P. Raghavan. Cambridge University Press, 1995. Very good and detailed development of the subject.

Algorithms in Java, R. Sedgewick, Addison Wesley. Chapter on random number generators.


Thank you for your attention