End Semester Examination, Spring, 2021–22

Full marks: **50** Exam duration: **3 Hours**

Answer **all** questions. Figures next to each question in square bracket indicate marks.
All Parts of a question should be answered at one place.
This question paper contains TWO pages.

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1. State the conditions when a function would be said linear map. What are zero map and identity map? Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $b, c \in \mathbb{R}$
   $T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$
   Show that T is linear if and only if $b = c = 0$. [5]

2. Test the linearity of the following maps. [5]

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $\quad T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $\quad T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 \\ y \end{pmatrix}$

3. Answer the following using the fundamental theorem of linear maps: [10]

   (a) Let V and W are finite dimensional vector spaces such that $\dim V > \dim W$. Then no linear maps $T : V \to W$ is injective.

   (b) Let V and W are finite dimensional vector spaces such that $\dim V < \dim W$. Then no linear maps $T : V \to W$ is surjective.

   (c) If $T : V \to W$ is injective and $\{v_1, v_2, \cdots, v_n\}$ is linearly independent in V then prove that $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is linearly independent in W.

   (d) Give a few examples of $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\text{Im}(T) = \text{Null}(T)$

4. Let $T : V \to W$ be a linear map. If $\{v_1, v_2, \cdots, v_k\}$ spans V then $\{T(v_1), T(v_2), \cdots, T(v_k)\}$ spans $\text{Im}(T) \subseteq W$. [4]

5. Find the Image and Kernel of the following linear maps along with their geometric interpretations: [6]

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $\quad T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y\cos\theta - z\sin\theta \\ y\sin\theta + z\cos\theta \end{pmatrix}$

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $\quad T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $\quad T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$

6. Let $T : V \to V$. A subspace U of V is called invariant under T if $u \in U \Rightarrow T(u) \in U$. Show that the following subspace of V are invariant under T. [4]

   (a) $\{0\}$

   (b) V

   (c) Null(T)

   (d) Im(T)

7. Find x so that $(-1, x, 3)$ is orthogonal to $(1, 2, 3)$. [2]

8. Given a matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}$
   and its characteristic polynomial as $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.
   Write the values of a, b, and c. [2]

9. Find the values of x and y if $A = \begin{pmatrix} x & 1 \\ -2 & y \end{pmatrix}$ and $B = \frac{1}{25}\begin{pmatrix} 114 & 48 \\ 48 & 86 \end{pmatrix}$
   have the same eigenvalues. [2]

10. A discrete random variable X has the probability function — [4]
    $$P(X = x) = \begin{cases} kx & \text{if } x = 2, 4, 6 \\ k(x-2) & \text{if } x = 8 \\ 0 & \text{otherwise} \end{cases}$$
    where k is constant.

    (a) what is the value of k?

    (b) find CDF $F(5)$.

    (c) find expectation $E(X)$.

    (d) find $E(X^2)$

11. Let X be a discrete random variable with the following PMF [3]

x	0.2	0.4	0.5	0.8	1	otherwise
$P(X = x)$	0.1	0.1	0.2	0.3	0.3	0

    (a) find the range of the random variable X.

    (b) find $P(X \leq 0.5)$.

    (c) find $P(0.25 < X < 0.75)$.

12. You take an exam that contains 20 multiple-choice questions. Each question has 4 possible options. You know the answer to 10 questions, but you have no idea about the other 10 questions so you choose answers randomly. Your score X on the exam is the total number of correct answers. Find the PMF of X. What is $P(X > 15)$? [3]

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