End Semester Examination, Spring–2020-21

Full marks: **50** Exam duration: **2 Hours**

Answer **all** questions. Figures next to each question in square bracket indicate marks.
All Parts of a question should be answered at one place.
This question paper contains TWO pages.

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1. Given a vector $\vec{v} = (1, -2, \sqrt{3}, 4, 5)^\top$. Find its length. [2]

2. Given a vector $\vec{u} = (\frac{1}{2}, -\frac{1}{4}, p, \frac{1}{3})^\top$ and $|\vec{u}| = 1$.
   How many values p may have and what are they? [2]

3. Given two vectors $\vec{a} = (1, -2, 0, \frac{1}{2})^\top$ and $\vec{b} = (-3, 0, 5, 7)^\top$.
   Find the angle between them. [2]

4. Given four vectors:
   $\vec{a} = (1, 0, -4, \frac{1}{2})^\top$, $\vec{b} = (3, -\frac{1}{2}, 0, 0)^\top$, $\vec{c} = (0, 2, 3, 4)^\top$, and $\vec{d} = (11, -\frac{1}{2}, \frac{13}{2}, 3)^\top$
   Test whether the above four vectors are linearly dependent or not. [2]

5. Find x so that $(-1, x, 3)$ is orthogonal to $(1, 2, 3)$. [2]

6. Is A symmetric, if row space of A = column space of A, and $\mathbf{N}(A) = \mathbf{N}(A^{\mathrm{T}})$? [2]

7. Decompose the vector $\vec{x} = (2, 7)^\top$ into the sum of two vectors, one of which is parallel to $\vec{a} = (-3, 4)^\top$ and the other one is perpendicular to $\vec{a}$. [2]

8. If $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors, show that $\vec{u} + \vec{v}$ is orthogonal to $\vec{u} - \vec{v}$. What are the lengths of those vectors? [2]

9. Find LU factorization of following matrices and solve the linear system $A\vec{x} = \vec{b}$, where $\vec{b}$ is the column vector with all elements equal to 1.
   $$\begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$ [2]

10. Test the positive definiteness of $\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$. [2]

11. Test the positive definiteness of $\begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix}$. [2]

12. Given the following three vectors:
    $\vec{u} = (2, 0, -1)$, $\vec{v} = (3, 1, 0)$, and $\vec{w} = (1, -1, c)$
    where $c \in \mathbb{R}$
    The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$ provided that $c \neq$ __?__. [2]

13. Given a matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}$
and its characteristic polynomial as $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.
Write the values of a, b, and c. [2]

14. Find the values of x and y if $A = \begin{pmatrix} x & 1 \\ -2 & y \end{pmatrix}$ and $B = \frac{1}{25}\begin{pmatrix} 114 & 48 \\ 48 & 86 \end{pmatrix}$
have the same eigenvalues. [2]

15. If P is a projection matrix, then $(I - P)^2 = I - P$. (True/False) [2]

16. The linear combinations of $\vec{u} = (1, 1, 0)^\top$ and $\vec{v} = (0, 1, 1)^\top$ fill a plane in $\mathbb{R}^3$. Find a vector $\vec{z}$ that is perpendicular to $\vec{u}$ and $\vec{v}$. Show that $\vec{z}$ is perpendicular to every vector $c\vec{u} + d\vec{v}$ on the plane. Find a vector $\vec{w}$ that is not on the plane. [4]

17. Find the matrices C_1 and C_2 containing independent columns of A_1 and A_2.
$A_1 = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{pmatrix}$ $\quad A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
Factor each of the above matrices into $A = CR$.
Produce a basis for the column spaces of A_1 and A_2. What are the dimensions of those column spaces — the number of independent vectors? What are the ranks of A_1 and A_2? How many independent rows in A_1 and A_2? [4]

18. How is the null space of C related to the null spaces of A and B, if $C = \begin{pmatrix} A \\ B \end{pmatrix}$? [4]

19. Four possibilities for the rank r and size m, n match four possibilities for $Ax = b$. Find four matrices A_1 to A_4 that demonstrate those possibilities. [4]
    (a) $r = m = n$ $\quad A_1x = b$ has 1 solution for every b
    (b) $r = m < n$ $\quad A_2x = b$ has 1 or ∞ solutions
    (c) $r = n < m$ $\quad A_3x = b$ has 0 or 1 solution
    (d) $r < m, r < n$ $\quad A_4x = b$ has 0 or ∞ solutions

20. Show that AA^{T} has the same null space as that of A. [4]

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