Line segment intersection for map overlay

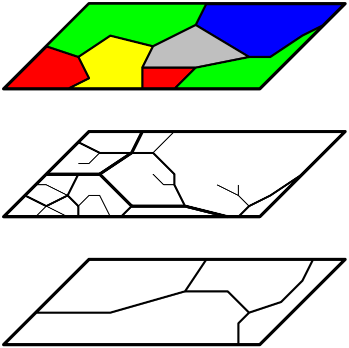
Computational Geometry

Lecture 2: Line segment intersection for map overlay

Map layers

In a geographic information system (GIS) data is stored in separate layers

A layer stores the geometric information about some theme, like land cover, road network, municipality boundaries, red fox habitat, ...

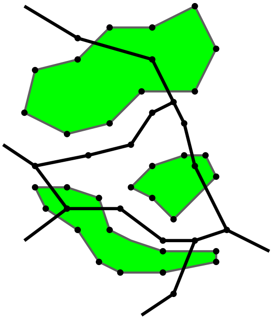


Map overlay

Map overlay is the combination of two (or more) map layers

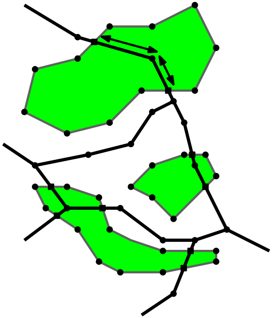
It is needed to answer questions like:

- What is the total length of roads through forests?
- What is the total area of corn fields within 1 km from a river?
- What area of all lakes occurs at the geological soil type “rock”?



Map overlay

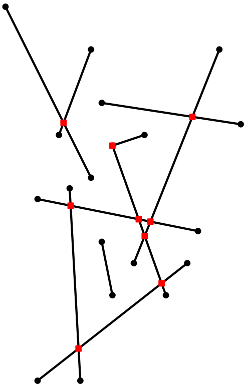
To solve map overlay questions, we need (at the least) intersection points from two sets of line segments (possibly, boundaries of regions)



The (easy) problem

Let's first look at the easiest version of the problem:

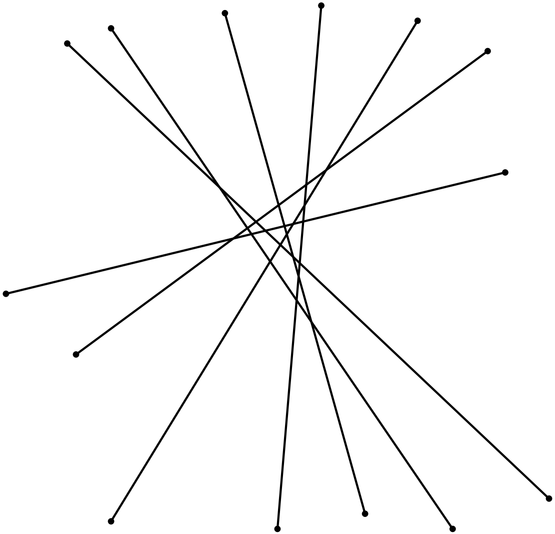
Given a set of of n line segments in the plane, find all intersection points efficiently



An easy, optimal algorithm?

Algorithm FINDINTERSECTIONS(S)
Input. A set S of line segments in the plane.
Output. The set of intersection points among the segments in S.
1. **for** each pair of line segments $e_i, e_j \in S$
2. **do if** e_i and e_j intersect
3. **then** report their intersection point

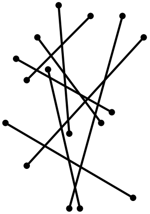
Question: Why can we say that this algorithm is optimal?

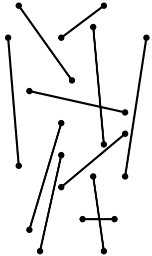


Output-sensitive algorithm

The asymptotic running time of an algorithm is always input-sensitive (depends on n)

We may also want the running time to be output-sensitive: if the output is large, it is fine to spend a lot of time, but if the output is small, we want a fast algorithm

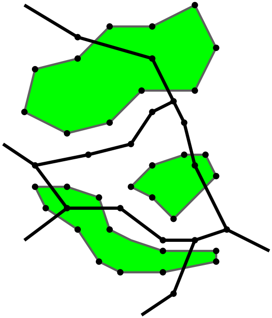




Intersection points in practice

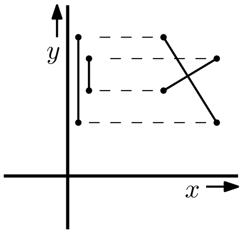
Question: How many intersection points do we typically expect in our application?

If this number is k, and if $k = O(n)$, it would be nice if the algorithm runs in $O(n \log n)$ time



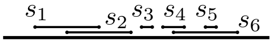
First attempt

Observation: Two line segments can only intersect if their y-spans have an overlap



So, how about only testing pairs of line segments that intersect in the y-projection?

1D problem: Given a set of intervals on the real line, find all partly overlapping pairs



(s_1, s_2), (s_4, s_6), (s_5, s_6)

First attempt

1D problem: Given a set of intervals on the real line, find all partly overlapping pairs

Sort the endpoints and handle them from left to right; maintain currently intersected intervals in a balanced search tree $\mathcal{T}$

- Left endpoint of s_i: for each s_j in $\mathcal{T}$, report the pair s_i, s_j. Then insert s_i in $\mathcal{T}$
- Right endpoint of s_i: delete s_i from $\mathcal{T}$

Question: Is this algorithm output-sensitive for 1D interval intersection?

First attempt

Back to the 2D problem:

Determine the y-intervals of the 2D line segments

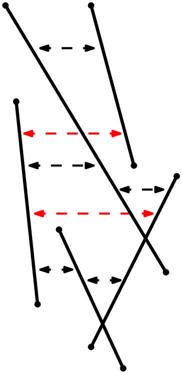
Find the intersecting pairs of intervals with the 1D solution

For every pair of intersecting intervals, test whether the corresponding line segments intersect, and if so, report

Question: Is this algorithm output-sensitive for 2D line segment intersection?

Second attempt

Refined observation: Two line segments can only intersect if their y-spans have an overlap, and they are adjacent in the x-order at that y-coordinate (they are *horizontal neighbors*)

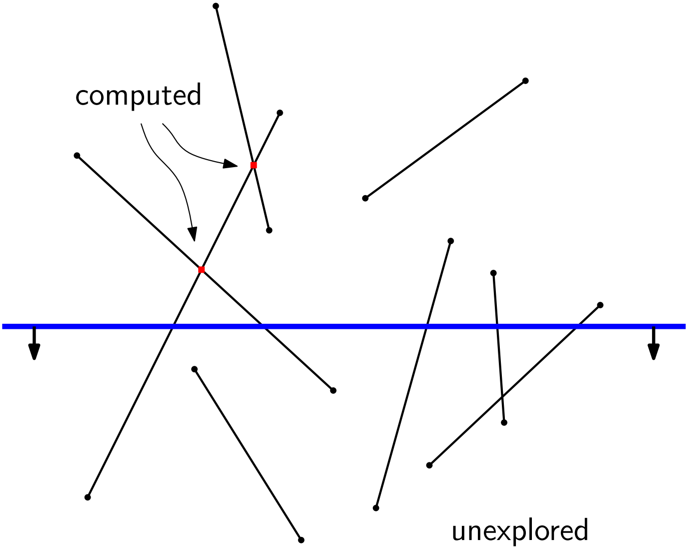


Plane sweep

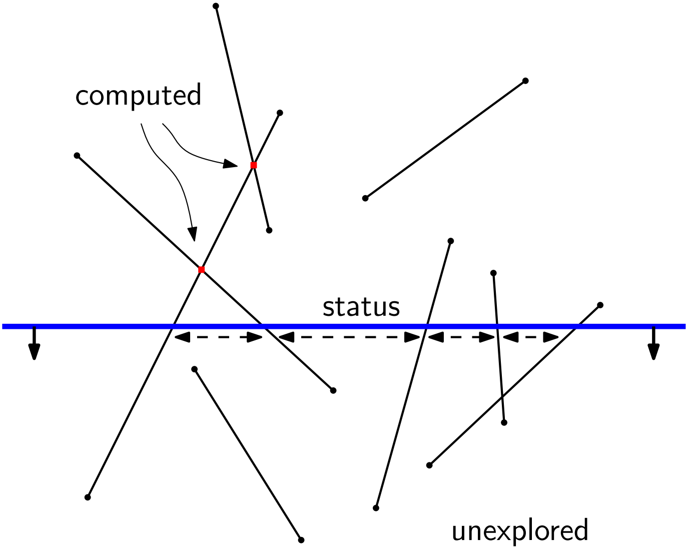
The **plane sweep technique**: Imagine a horizontal line passing over the plane from top to bottom, solving the problem as it moves

- The sweep line stops and the algorithm computes at certain positions ⇒ events
- The algorithm stores the relevant situation at the current position of the sweep line ⇒ status
- The algorithm knows everything it needs to know above the sweep line, and found all intersection points

Sweep



Sweep and status

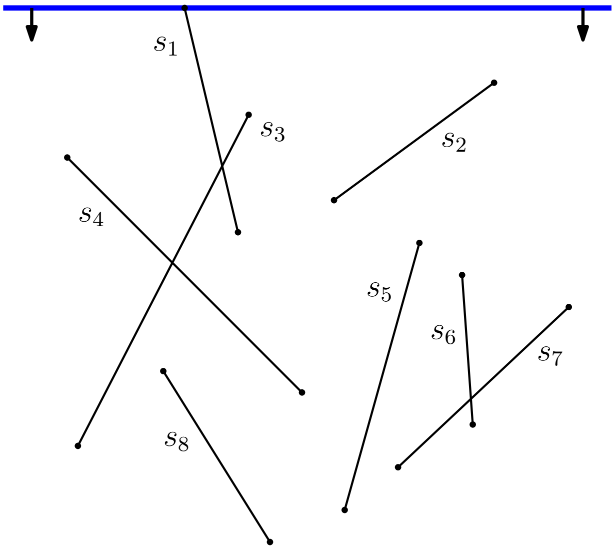


Status and events

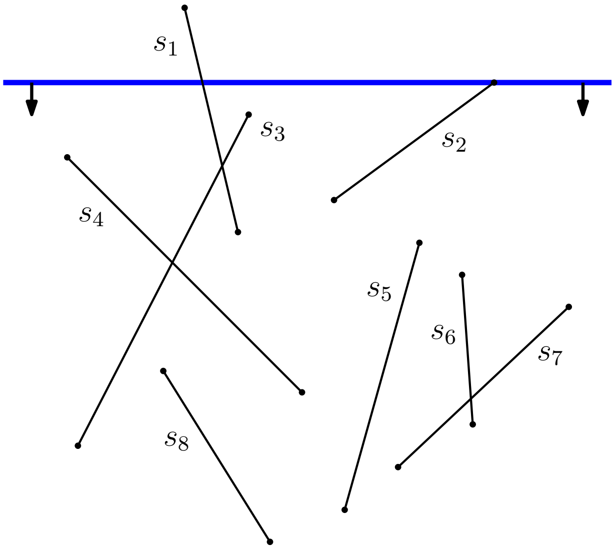
The status of this particular plane sweep algorithm, at the current position of the sweep line, is the set of line segments intersecting the sweep line, ordered from left to right

The events occur when the *status changes*, and when *output is generated*

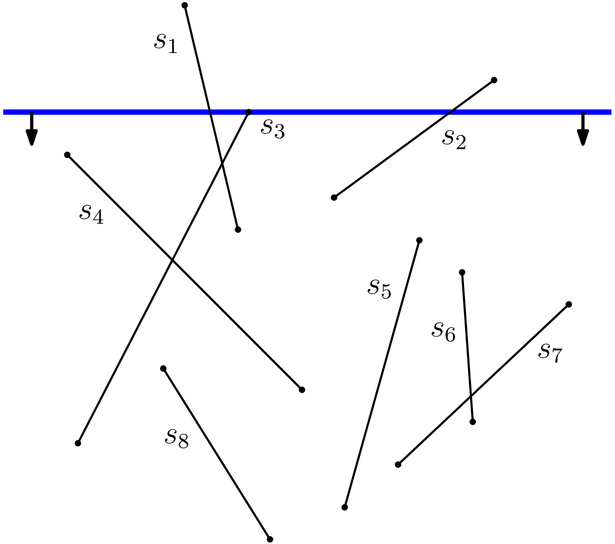
event $\approx$ interesting y-coordinate



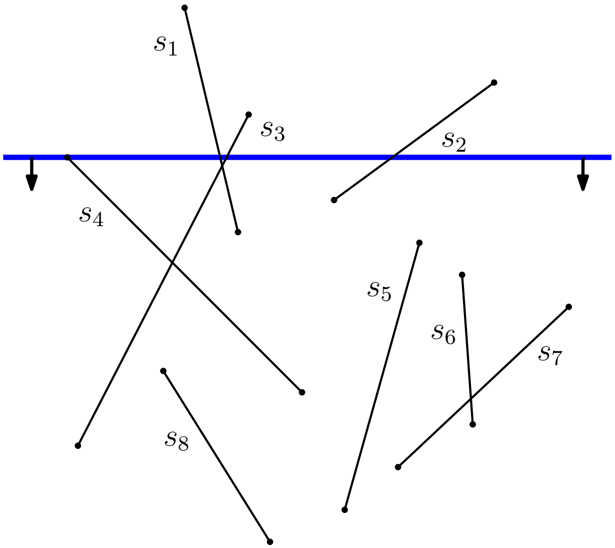
add s_1



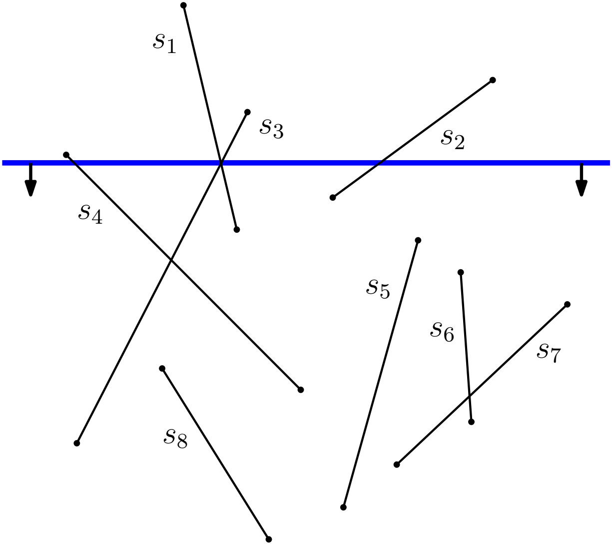
add s_2 after s_1



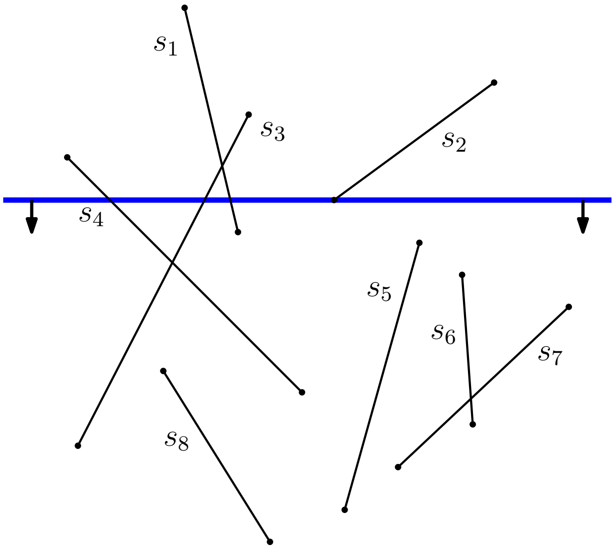
add s_3 between s_1 and s_2



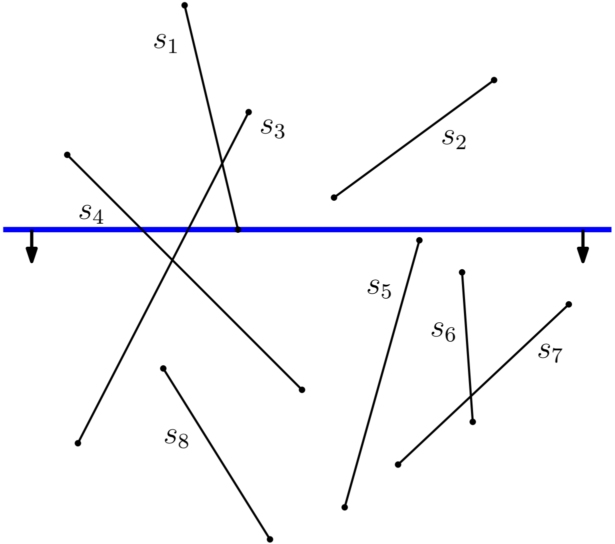
add s_4 before s_1



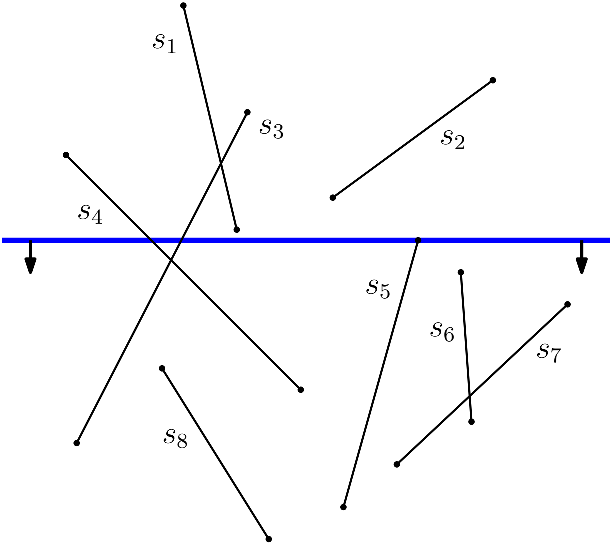
report intersection (s_1, s_2); swap s_1 and s_3



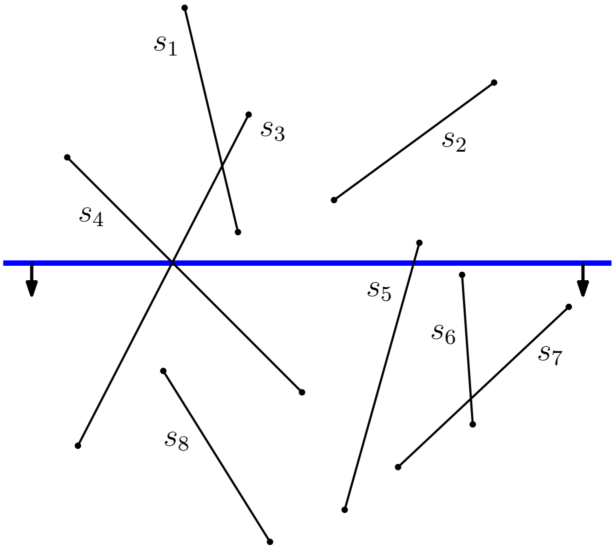
remove s_2



remove s_1



add s_5 after s_3



report intersection (s_3, s_4); swap s_3 and s_4

... and so on ...

The events

When do the events happen? When the sweep line is at

- an upper endpoint of a line segment
- a lower endpoint of a line segment
- an intersection point of a line segment

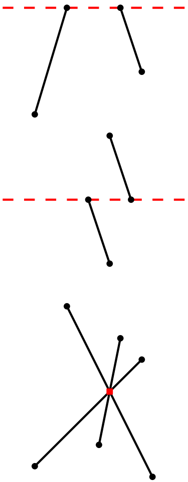
At each type, the **status** changes; at the third type **output** is found too

Assume no degenerate cases

We will at first exclude degenerate cases:

- No two endpoints have the same y-coordinate
- No more than two line segments intersect in a point
- ...

Question: Are there more degenerate cases?

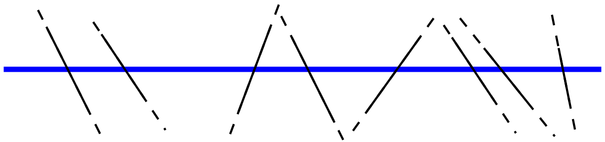


Event list and status structure

The event list is an abstract data structure that stores all events in the order in which they occur

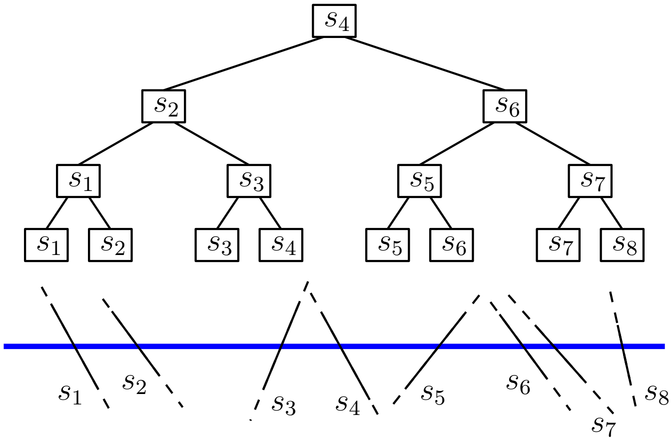
The status structure is an abstract data structure that maintains the current status

Here: The status is the subset of currently intersected line segments in the order of intersection by the sweep line

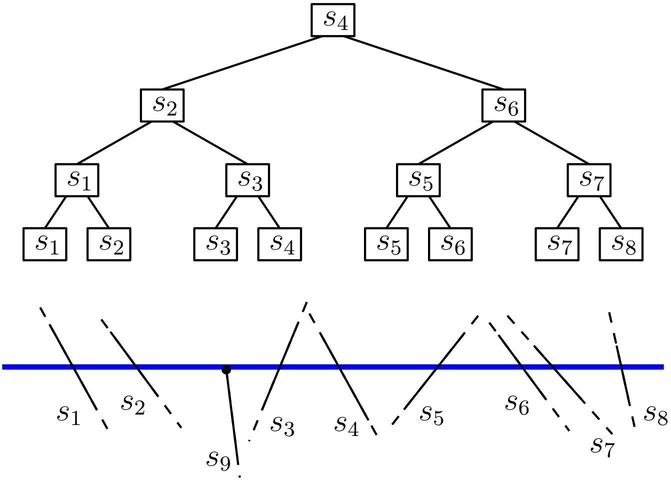


Status structure

We use a balanced binary search tree with the line segments in the leaves as the status structure

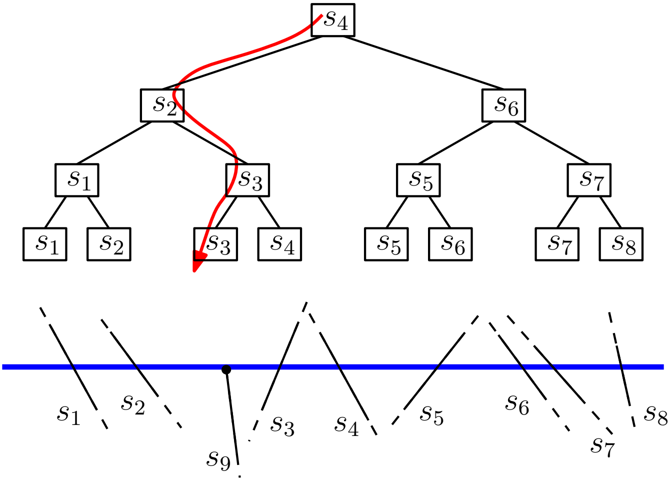


Status structure



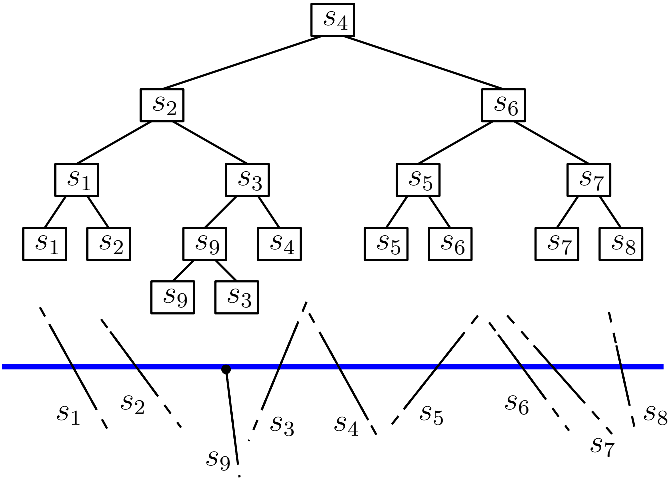
Upper endpoint: search, and insert

Status structure



Upper endpoint: search, and insert

Status structure



Upper endpoint: search, and insert

Status structure

Sweep line reaches lower endpoint of a line segment: delete from the status structure

Sweep line reaches intersection point: swap two leaves in the status structure (and update information on the search paths)

Finding events

Before the sweep algorithm starts, we know all upper endpoint events and all lower endpoint events

But: How do we know intersection point events???
(those we were trying to find ...)

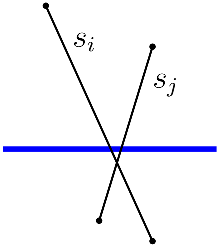
Recall: Two line segments can only intersect if they are horizontal neighbors

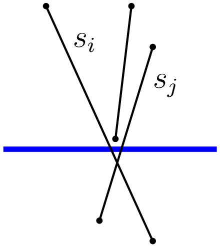
Finding events

Lemma: Two line segments s_i and s_j can only intersect after (= below) they have become horizontal neighbors

Proof: Just imagine that the sweep line is ever so slightly above the intersection point of s_i and s_j, but below any other event $\square$

Also: some earlier (= higher) event made s_i and s_j horizontally adjacent!!!





Event list

The event list must be a balanced binary search tree, because during the sweep, we discover new events that will happen later

We know upper endpoint events and lower endpoint events beforehand; we find intersection point events when the involved line segments become horizontal neighbors

Structure of sweep algorithm

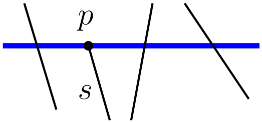
Algorithm FINDINTERSECTIONS(S)
Input. A set S of line segments in the plane.
Output. The intersection points of the segments in S, with for each intersection point the segments that contain it.

1. Initialize an empty event queue Q. Insert the segment endpoints into Q; when an upper endpoint is inserted, the corresponding segment should be stored with it
2. Initialize an empty status structure T
3. **while** Q is not empty
4. **do** Determine next event point p in Q and delete it
5. HANDLEEVENTPOINT(p)

Event handling

If the event is an upper endpoint event, and s is the line segment that starts at p:

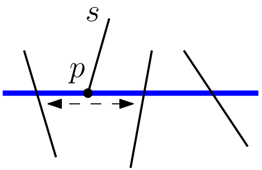
1. Search with p in T, and insert s
2. If s intersects its left neighbor in T, then determine the intersection point and insert in Q
3. If s intersects its right neighbor in T, then determine the intersection point and insert in Q



Event handling

If the event is a lower endpoint event, and s is the line segment that ends at p:

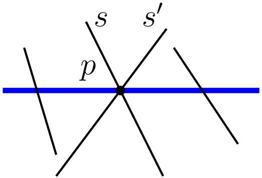
1. Search with p in T, and delete s
2. Let s_l and s_r be the left and right neighbors of s in T (before deletion). If they intersect *below the sweep line*, then insert their intersection point as an event in Q



Event handling

If the event is an intersection point event where s and s' intersect at p:

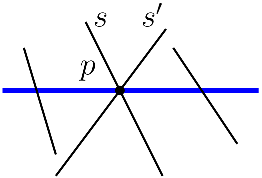
1. ...
2. ...
3. ...
4. ...



Event handling

If the event is an intersection point event where s and s' intersect at p:

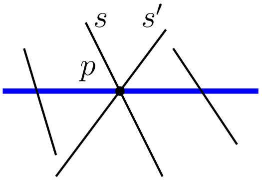
1. Exchange s and s' in T
2. ...
3. ...
4. ...



Event handling

If the event is an intersection point event where s and s' intersect at p:

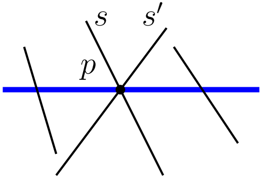
1. Exchange s and s' in T
2. If s' and its new left neighbor in T intersect below the sweep line, then insert this intersection point in Q
3. ...
4. ...



Event handling

If the event is an intersection point event where s and s' intersect at p:

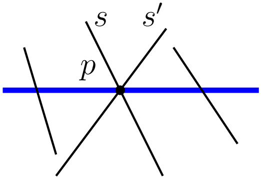
1. Exchange s and s' in T
2. If s' and its new left neighbor in T intersect below the sweep line, then insert this intersection point in Q
3. If s and its new right neighbor in T intersect below the sweep line, then insert this intersection point in Q
4. ...



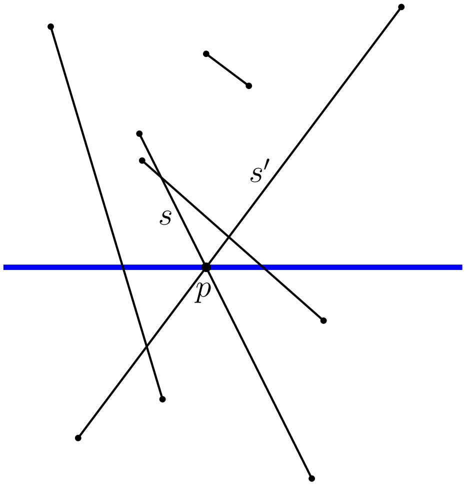
Event handling

If the event is an intersection point event where s and s' intersect at p:

1. Exchange s and s' in T
2. If s' and its new left neighbor in T intersect below the sweep line, then insert this intersection point in Q
3. If s and its new right neighbor in T intersect below the sweep line, then insert this intersection point in Q
4. Report the intersection point



Event handling



Can it be that new horizontal neighbors already intersected above the sweep line?

Can it be that we insert a newly detected intersection point event, but it already occurs in Q?

Efficiency

How much time to handle an event?

At most one search in T and/or one insertion, deletion, or swap

At most twice finding a neighbor in T

At most one deletion from and two insertions in Q

Since T and Q are balanced binary search trees, handling an event takes only $O(\log n)$ time

Efficiency

How many events?

- $2n$ for the upper and lower endpoints
- k for the intersection points, if there are k of them

In total: $O(n+k)$ events

Efficiency

Initialization takes $O(n \log n)$ time (to put all upper and lower endpoint events in Q)

Each of the $O(n+k)$ events takes $O(\log n)$ time

The algorithm takes $O(n \log n + k \log n)$ time

If $k = O(n)$, then this is $O(n \log n)$

Note that if k is really large, the brute force $O(n^2)$ time algorithm is more efficient

Efficiency

Question: How much storage does the algorithm take?

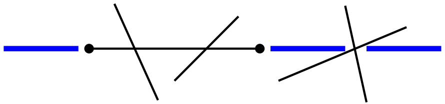
Efficiency

Question: Given that the event list is a binary tree that may store $O(k) = O(n^2)$ events, is the efficiency in jeopardy?

Degenerate cases

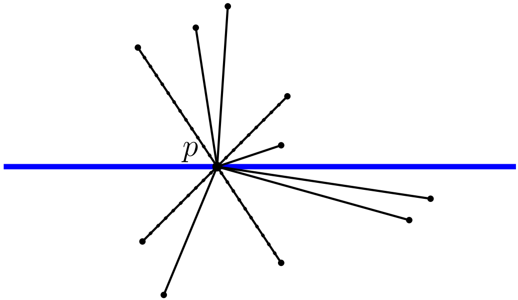
How do we deal with degenerate cases?

For two different events with the same y-coordinate, we treat them from left to right $\Rightarrow$ the “upper” endpoint of a horizontal line segment is its left endpoint



Degenerate cases

How about multiply coinciding event points?



Let $U(p)$ and $L(p)$ be the line segments that have p as upper and lower endpoint, and $C(p)$ the ones that contain p

Question: How do we handle this multi-event?

Degenerate cases

How efficient is a multiply coinciding event point handled?

If $|U(p)|+|L(p)|+|C(p)|=m$, then the event takes $O(m\log n)$ time

What do we report?

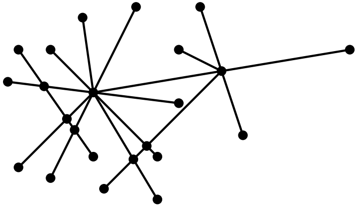
- The intersection point itself
- Every pair of intersecting line segments
- The intersection point and every line segment involved

$\Rightarrow$ the output size for this one event is then $O(1)$, $O(m^2)$, or $O(m)$, respectively

Degenerate cases

Since $m = O(n)$, does this imply that the whole algorithm takes $O(k) \cdot O(m \log n) = O(k) \cdot O(n \log n) = O(nk \log n)$ time?

No, we can bound $\sum m$ over all intersections by the number of edges that arise in the subdivision: $\sum m \leq 2E$. With Euler's formula: $V - E + F \geq 2$, $V = 2n + I$ with I intersections, and $2F \leq 3E$, we get $m \leq 12n + 6I - 12 = O(n + I)$



Result

For any set of n line segments in the plane, all I intersections can be computed in $O(n \log n + I \log n)$ time, and within this time bound, we can report for every intersection which line segments are involved

Conclusion

For every sweep algorithm:

- Define the status
- Choose the status structure and the event list
- Figure out how events must be handled (with sketches!)
- To analyze, determine the number of events and how much time they take

Then deal with degeneracies