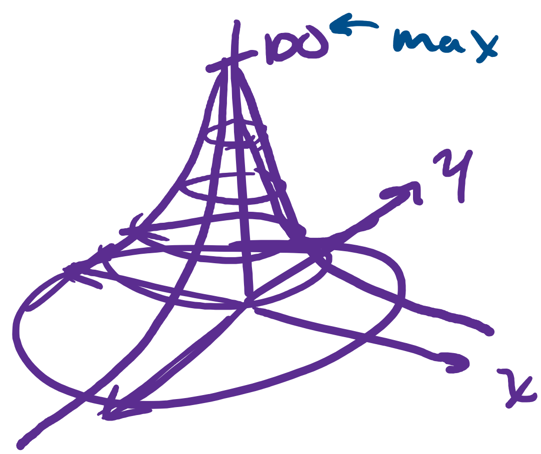
P1

Sunday, April 21, 2024 12:33 AM

A) $T(x,y) = \dfrac{100}{x^2+y^2+1}$

- Denominator can never be <1, so z is never >100.
- x^2+y^2 is a circle around the origin.
- As x and y increase radially, the denominator increases and will make $T(x,y)$ asymptote at 0.

Overally the shape should resemble



with circular level curves.

B)

$T(x,y)$ has its maximum at $T(0,0)$

C)

$$T(x,y) = \frac{100}{x^2+y^2+1}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial x}\left[\frac{100}{x^2+y^2+1}\right] \qquad a = y^2+1$$

$$\underbrace{100 \cdot \frac{\partial T}{\partial x}\left[\frac{1}{x^2+a^2}\right]}_{\displaystyle \downarrow}$$

$$\frac{\partial T}{\partial x}\left[(x^2+a^2)^{-1}\right]$$

$$-1\cdot(x^2+a^2)^{-2}\cdot\frac{\partial T}{\partial x}[x^2+a^2)$$

$$\left(-1\cdot(x^2+a^2)^{-2}\cdot(2x)\right)$$

$$\frac{\partial T}{\partial x}=100\cdot-1\cdot(x^2+a^2)^{-2}\cdot 2x$$

$$=\frac{-200x}{(x^2+y^2+1)^2}$$

B/c symmetry,

$$\frac{\partial T}{\partial y}=\frac{-200y}{(x^2+y^2+1)^2}$$

$$\nabla T=-200\begin{bmatrix}\frac{x}{(x^2+y^2+1)^2}\\ \frac{y}{(x^2+y^2+1)^2}\end{bmatrix}$$

$$\nabla T(3,2)=-200\left[\;3/(3^2+2^2+1)^2\right.$$

$$\nabla T(3,2) = -200\begin{bmatrix} 3/(3^2+2^2+1)^2 \\ 2/(3^2+2^2+1)^2 \end{bmatrix}$$

$$= -200\begin{bmatrix} 3/196 \\ 2/196 \end{bmatrix}$$

$$\nabla T(x,y) = \frac{-200}{196}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\nabla T(x,y) \sim -\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\max\left(\begin{bmatrix} -3 \\ -2 \end{bmatrix} \cdot \vec{v}\right)$$

$$|\vec{v}| = \sqrt{3^2+2^2}$$
$$= \sqrt{9+4}$$
$$= \sqrt{13}$$

$$\vec{v} = \frac{1}{\sqrt{13}}\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \frac{-1}{\sqrt{13}}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\frac{-1}{\sqrt{13}}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \frac{-200}{196}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\frac{200}{196\sqrt{13}} \cdot (3^2+2^2)$$

$$\frac{14\cdot 200}{196\cdot\sqrt{13}} = \frac{100}{91}\cdot\sqrt{13} \approx 3.96$$

Direction of maximum increase will always be $\nabla T(a,b)$, so the vector, $\vec{v}$, that maximizes temperature increase is

$\vec{v} = \frac{-1}{\sqrt{13}}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. The magnitude of increase is $\hat{v}\cdot\nabla T(3,2)$ which equals $\frac{100}{91}\cdot\sqrt{13}$ or ≈ 3.962.

D) Conceptually, the opposite direction must be the greatest decrease, so $-\vec{v}$.

The direction $\frac{1}{\sqrt{13}}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is opposite to the gradient, so it will be the greatest decrease.

This decrease is $\frac{-100}{91}\cdot\sqrt{13}$ or ≈ -3.96

E)

$$\begin{bmatrix} -3 \\ -2 \end{bmatrix}\cdot\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$-3a-2b=0 \qquad a^2+b^2=1$

$-3a=2b$

$-\frac{3}{2}a=b$

$a^2+\left(-\frac{3}{2}a\right)^2=1$

$a^2+\frac{9}{4}a^2=1$

$\frac{13}{4}a^2 = 1$

$a^2 = \frac{4}{13}$

$a = \pm \frac{2}{\sqrt{13}}$

$b = \frac{-3}{2}a$

$b = \frac{-3}{2}\left(\frac{2}{\sqrt{13}}\right)$ $\qquad$ $b = \frac{-3}{2}\left(\frac{-2}{\sqrt{13}}\right)$

$b = \frac{-3}{\sqrt{13}}$ $\qquad$ $b = \frac{3}{\sqrt{13}}$

so,

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

The direction, $\vec{v}$, that satisfies $\vec{v} \cdot \nabla T = 0$ is perpendicular to gradient and will be unaffected by the gradient.

$$\vec{v} = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 \\ -3 \end{bmatrix} \text{ or } \frac{1}{\sqrt{13}}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$