(2) $\beta_T(j)=1$. $\quad \sum_j \beta_T(j) = \sum_j \quad C_T = \frac{1}{\sum_j \beta_T(j)} = \frac{1}{\sum_j} \quad \hat{\beta}_T(j) = C_T \cdot 1$

If $t = T-1$

$$\hat{\beta}_{T-1}(i) = C_{T-1} \cdot \tilde{\beta}_{T-1}(i)$$
$$= C_{T-1} \cdot \sum_{j=1}^{N} a_{ij} b_j(O_T) \cdot \hat{\beta}_T(j)$$
$$= C_{T-1} \cdot \sum_{j=1}^{N} a_{ij} b_j(O_T) \cdot \beta_T(j) \cdot C_T$$
$$= C_T \cdot C_{T-1} \sum_{j=1}^{N} a_{ij} b_j(O_T) \cdot \beta_T(j)$$
$$= C_T \cdot C_{T-1} \beta_{T-1}(i)$$
$$= \prod_{s=T-1}^{T} C_s \beta_{T-1}(i)$$

$$\therefore \hat{\beta}_t(j) = \prod_{s=t}^{T} C_s \beta_t(j)$$

(3). According to (1)

$$\hat{\alpha}_t(i) = \prod_{s=1}^{t} C_s \alpha_t(i) = \frac{1}{\sum_i \alpha_t(i)} \cdot \alpha_t(i) = \frac{\alpha_t(i)}{\sum_{i=1}^{N} \alpha_t(i)}$$

According to (2)

$$\hat{\beta}_{t+1}(j) = \prod_{s=t+1}^{T} C_s \beta_{t+1}(j) = \frac{1}{\sum a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)} \cdot \beta_{t+1}(j)$$

$$\therefore \hat{\alpha}_t(i) \cdot \hat{\beta}_{t+1}(j) \cdot a_{ij} \cdot b_j(O_{t+1}) = \frac{\alpha_t(i) \cdot \beta_{t+1}(i) \cdot a_{ij} \cdot b_j(O_{t+1})}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}$$
$$= \xi_t(i,j)$$