0 Claims

I claim that the complexity of evac's value $evac(i, \epsilon)$ is $O(n(i+1)\log(\frac{n(i+1)}{\min\{\epsilon, \frac{1}{2}\}}))$ and that there is an algorithm which can compute the value of $evac(i, \epsilon)$ in $O(\log(n)+\log(i+1)+\log(\log(\frac{1}{\min\{\epsilon, \frac{1}{2}\}})))$. To prove this, one of the ways is finding an upper bound like K. I also claim that K could be $16n(i+1)\log(\frac{n(i+1)}{\min\{\epsilon, \frac{1}{2}\}})$, and I showed that this is true in Section 1. Then, using the upper bound achieved, I proved the correctness of the first two claims ($\log(x) = \log_2(x)$).

1 Finding an upper bound

In this section, I showed in a recursive way that there is a constant c_1 such that in any case $evac(i, \epsilon) < c_1 n(i+1)\log(\frac{n(i+1)}{\min\{\epsilon, \frac{1}{2}\}})$ due to the definition of $evac(i, \epsilon)$. For any $c_2 \geq c_1$, if $k := c_2 n(i+1)\log(\frac{n(i+1)}{\min\{\epsilon, \frac{1}{2}\}})$ then $e(i) \leq \epsilon$. And it's obvious that for $\epsilon_1 < \epsilon \rightarrow evac(i, \epsilon) \leq evac(i, \epsilon_1)$. So it's OK to set ϵ equal to $\min\{\epsilon, \frac{1}{2}\}$, and I split the problem into two general cases. First, I solve the problem for $K \in R_+ \cup \{0\}$, then I solve it for non-negative integer values of K.

Using the gamma function, which is a generalization of the factorial, the factorial product of a non-negative real number can be calculated.

Note: Since it's a recursive solution, all the states we go to are reversible, but not necessarily walkable. That is, they may not be arguable from our previous state in the solution, but if we go from the end to the beginning, all the arguments will be correct.

1.1 Lemmas and definitions

Here are some definitions and four lemmas that will help to make the proof easier:

$$\epsilon := \min\{\epsilon, \frac{1}{2}\},\ K := c_1 n(i+1)\log(\frac{n(i+1)}{\epsilon}),\ k := c_2 n(i+1)\log(\frac{n(i+1)}{\epsilon})$$

In every step, I showed why it's possible to apply that step using these four lemmas by writing four symbols: L1, L2, L3, and L4, and the argument.

1. $(x \leq y \wedge y \leq z) \rightarrow x \leq z$
2. $\forall x > 0 : \ \log(x) < x$
3. $\forall x > 1 : \ \frac{1}{2x} < \log(x) - \log(x-1)$
4. $\forall x \geq 15 : \ 0 < \frac{x}{2} - \log(x) - 2$

Given that the proof of these four lemmas is relatively simple and the last three can be proven using derivatives, I did not prove them.

1.2 Case $i = 0$:

$$\frac{(n-1)^k}{n^{k-1}} \leq \epsilon \leftrightarrow klg(n-1) - (k-1)lg(n) \leq lg(\epsilon) \leftrightarrow$$
$$0 \leq lg(\epsilon) + (k-1)lg(n) - klg(n-1) \leftrightarrow 0 \leq lg(\frac{\epsilon}{n}) + k(lg(n) - lg(n-1))$$

Using L1 and L3 it's OK to make right side smaller like this:

$$\frac{k}{2n} = \frac{c_2}{2} lg(\frac{n}{\epsilon}) \leq k(lg(n) - lg(n-1))$$

$$\xleftarrow[L1,L3]{\text{explained above}} 0 \leq lg(\frac{\epsilon}{n}) + \frac{c_2}{2} lg(\frac{n}{\epsilon}) \leftrightarrow 0 \leq (\frac{c_2}{2} - 1)lg(\frac{n}{\epsilon})$$
$$\xleftarrow[L1]{1 \leq lg(\frac{n}{\epsilon})} 0 \leq \frac{c_2}{2} - 1 \leftarrow 15 \leq c_2 \leftrightarrow \mathbf{c_1 := 15}$$

1.3 Case $0 < i$:

$$C(k,i)\frac{(n-1)^{k-i}}{n^{k-1}} \leq \epsilon \leftrightarrow lg(C(k,i)) + (k-i)lg(n-1) - (k-1)lg(n) \leq lg(\epsilon)$$

Using L1 it's OK to make the left side bigger like this:

$$C(k,i) \leq k^i \leftrightarrow lg(C(k,i)) \leq ilg(k)$$

$$\xleftarrow[L1]{\text{explained above}} ilg(k) + (k-i)lg(n-1) - (k-1)lg(n) \leq lg(\epsilon)$$
$$\leftrightarrow ilg(k) \leq lg(\epsilon) + ilg(n-1) - lg(n) + k(lg(n) - lg(n-1))$$

Using L1 and L3 it's OK to make right side smaller like this:

$$\frac{k}{2n} = \frac{c_2}{2}(i+1)lg(\frac{n(i+1)}{\epsilon}) \leq k(lg(n) - lg(n-1))$$

$$\xleftarrow[L1,L3]{\text{explained above}} ilg(k) \leq lg(\frac{\epsilon}{n}) + \frac{c_2}{2}(i+1)lg(\frac{n(i+1)}{\epsilon}) + ilg(n-1)$$
$$\leftrightarrow ilg(k) \leq (\frac{c_2}{2}(i+1) - 1)lg(\frac{n}{\epsilon}) + \frac{c_2}{2}(i+1)lg(i+1) + ilg(n-1)$$

Using L1 and L2 it's OK to make left side bigger like this:

$$c_2 n(i+1)lg(\frac{n(i+1)}{\epsilon}) \leq c_2 n(i+1)\frac{n(i+1)}{\epsilon} \leq c_2 \frac{n^2(i+1)^2}{\epsilon^2}$$

$$\xleftarrow[L1,L2]{\text{explained above}} 2ilg(\frac{n(i+1)}{\epsilon}) + ilg(c_2) \leq (\frac{c_2}{2}(i+1) - 1)lg(\frac{n(i+1)}{\epsilon}) + lg(i+1) + ilg(n-1)$$

$$\leftrightarrow ilg(c_2) \leq (\frac{c_2}{2}(i+1) - 2i - 1)lg(\frac{n(i+1)}{\epsilon}) + lg(i+1) + ilg(n-1)$$

$$\leftrightarrow ilg(c_2) \leq (\frac{c_2}{2}i + \frac{c_2}{2} - 2i - 1)lg(\frac{n(i+1)}{\epsilon}) + lg(i+1) + ilg(n-1)$$

$$\leftrightarrow ilg(c_2) \leq i(\frac{c_2}{2} - 2)lg(\frac{n(i+1)}{\epsilon}) + (\frac{c_2}{2} - 1)lg(\frac{n(i+1)}{\epsilon}) + lg(i+1) + ilg(n-1)$$

$$\xleftarrow[L1]{0 \leq lg(i+1) + ilg(n-1) + (\frac{c_2}{2} - 1)lg(\frac{n(i+1)}{\epsilon})} ilg(c_2) \leq i(\frac{c_2}{2} - 2)lg(\frac{n(i+1)}{\epsilon})$$

$$\xleftarrow[L1]{1 \leq lg(\frac{n(i+1)}{\epsilon}) \wedge 0 < i} lg(c_2) \leq (\frac{c_2}{2} - 2) \leftrightarrow 0 \leq \frac{c_2}{2} - lg(c_2) - 2$$

$$\xleftarrow{L4} c_1 = 15 \leq c_2 \leftarrow \mathbf{c_1 := 15}$$

1.4 Upper bound for non-negative integer values of K

$$evac(i, \epsilon) \leq ceil(K) < K + 1 \text{ so if } c_1 := 16$$

$$\xrightarrow{\frac{K}{c_1} > 1} evac(i, \epsilon) < K$$

2 Proof of evac's value complexity:

$$evac(i, \epsilon) < K$$

$$\rightarrow evac(i, \epsilon) \in O(n(i+1)lg(\frac{n(i+1)}{min\{\epsilon, \frac{1}{2}\}}))$$

3 Proof of calculator's time complexity:

Given that $c_1 := 16$, then $evac(i, \epsilon) < K$ (it has an upper bound), the value of $evac(i, \epsilon)$ is binary searchable. This can be achieved by performing two binary searches: one to find the value of k that maximizes $e(i)$ and the second to find the $evac(i, \epsilon)$ in a bounded sequence with decreasing $e(i)$ for k in sequence. The complexity of binary search is $O(\log(K))$, thus:

$$O(lg(n(i+1)lg(\frac{n(i+1)}{min\{\epsilon, \frac{1}{2}\}})))$$

$$= O(lg(n) + lg(i+1) + lg(lg(n) + lg(i+1) + lg(\frac{1}{min\{\epsilon, \frac{1}{2}\}})))$$

$$= O(lg(n) + lg(i+1) + lg(lg(\frac{1}{min\{\epsilon, \frac{1}{2}\}})))$$