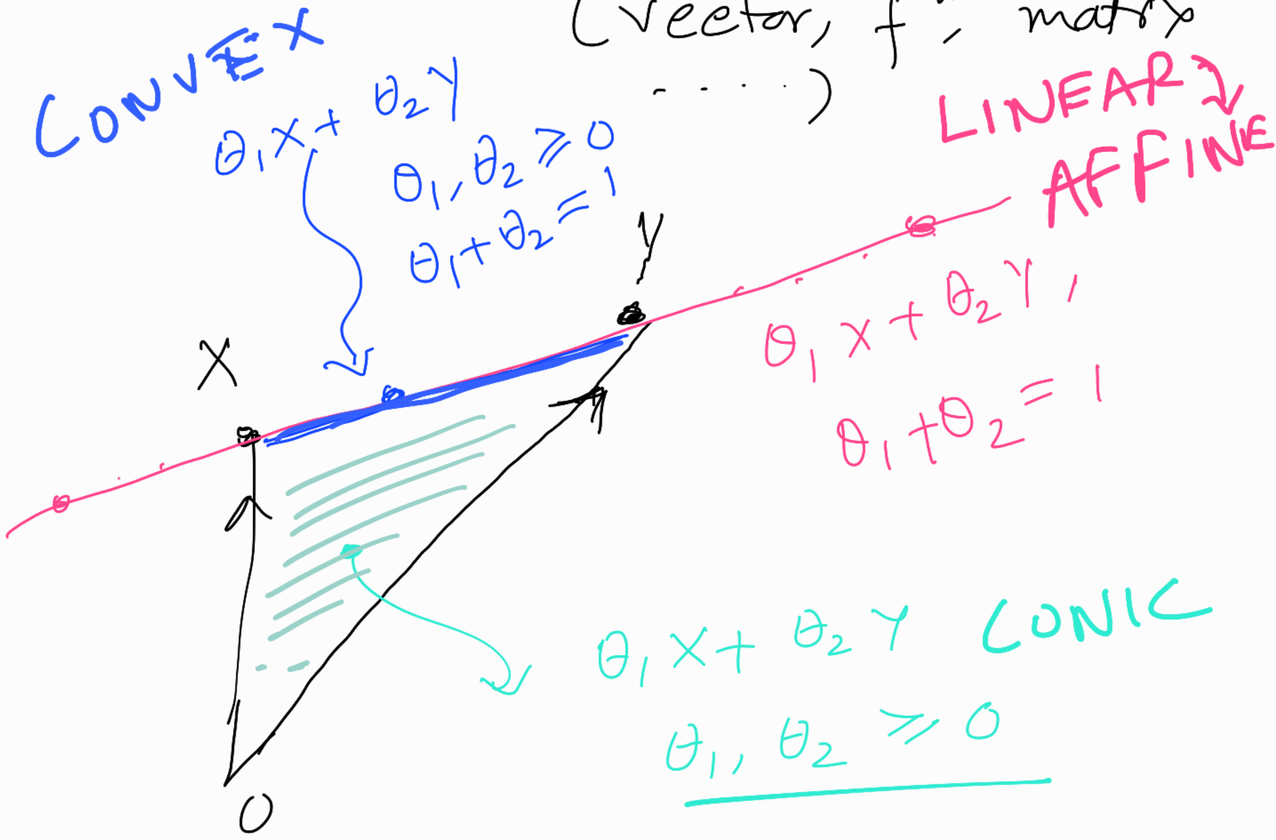
Combination of mathematical objects

(vector, f^n, matrix .....)



Affine hull :

The affine hull of a set C is the set of all affine combinations of points in C,

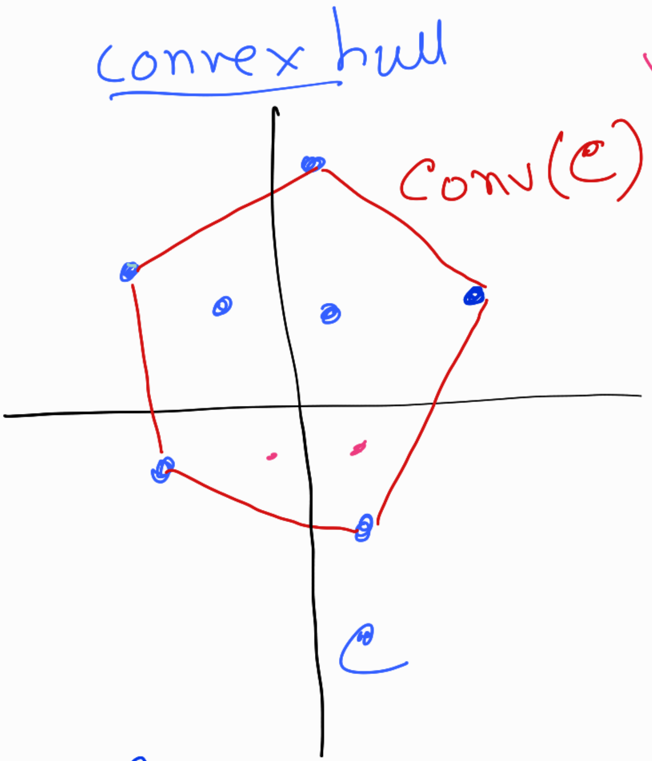
$$\operatorname{aff} C = \left\{ \sum \theta_i x_i \,\middle|\, \sum \theta_i = 1 \right\}$$

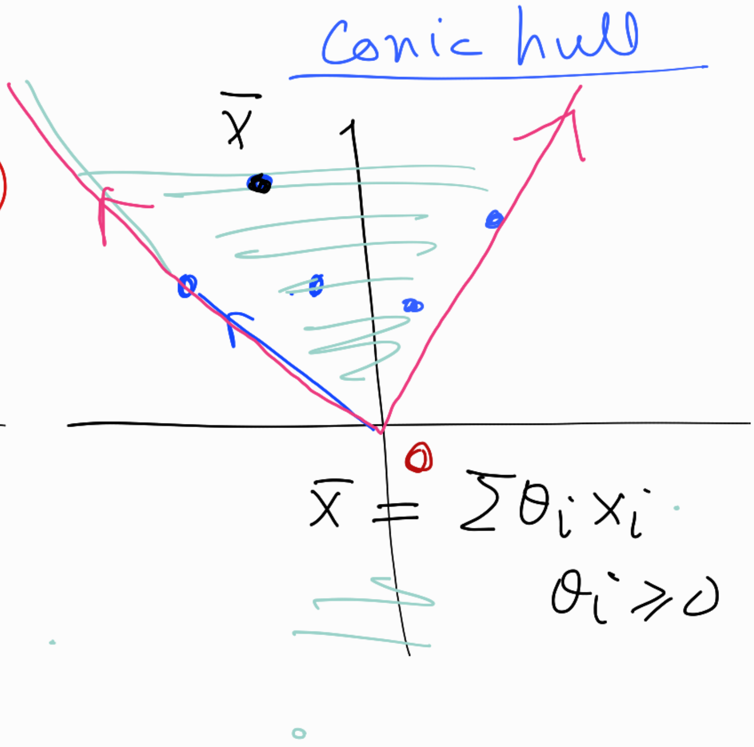
Convex hull of $C \subseteq \mathbb{R}^n$ is defined as

$$\operatorname{Conv}(C) = \left\{ \sum \theta_i x_i ,\ \underline{x_i \in C} \,\middle|\, \sum \theta_i = 1,\ \theta_i \geq 0 \right\}$$

<u>Conic hull</u>: of a set C is defined as

$$\left\{ \sum \theta_i x_i \;\middle|\; \theta_i \geq 0 \right\}$$

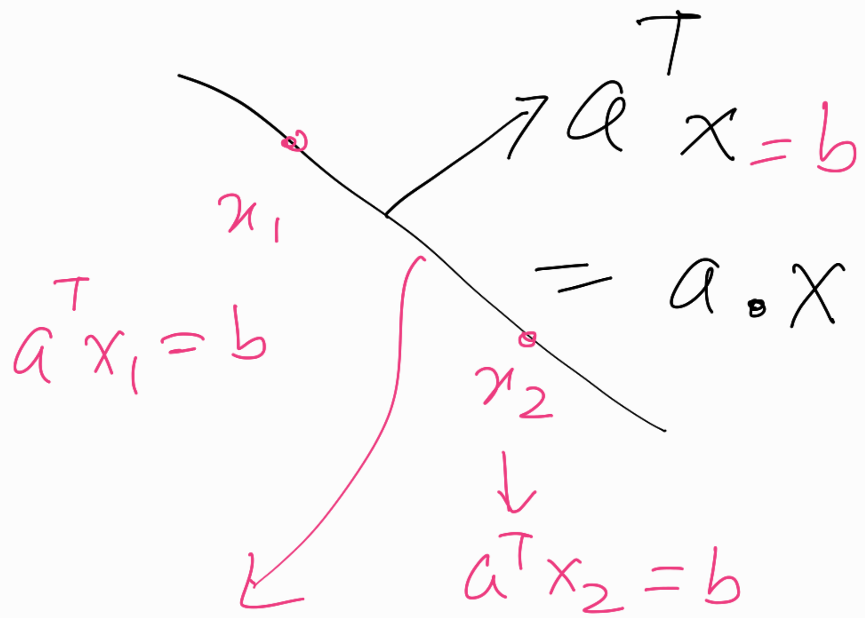




Convex set

Examples of convex set

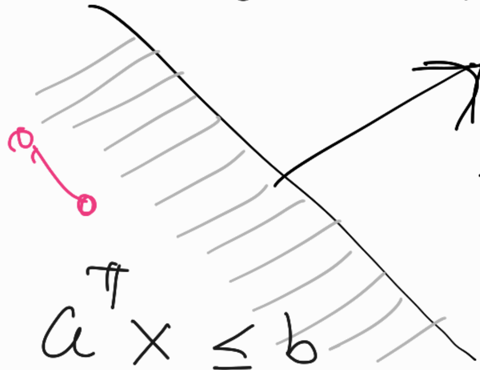
1) Hyperplane



$$a^T(\theta_1 x_1 + \theta_2 x_2) = \theta_1 b + \theta_2 b = b$$

2) Half Space

$H^+ : a^T x \geq b$



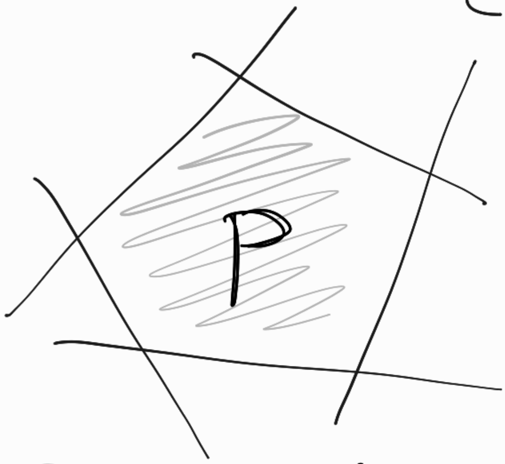
$a^T x = b$

$H^- : a^T x \leq b$

3) Polyhedron $P = \{ x \mid a_i^T x \leq b_i, i = 1 \cdots m \}$

$\Downarrow$

Intersection of halfspaces

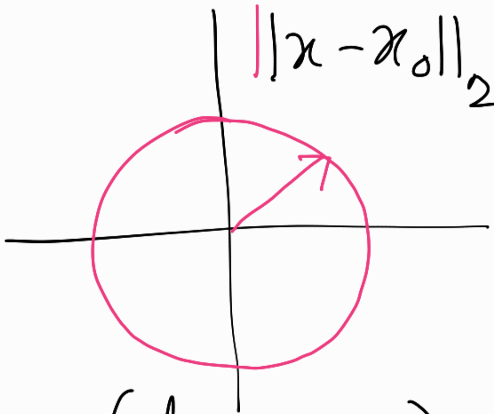


<u>Note</u>: Intersection of convex sets is also convex

4) <u>Norm ball</u> : $\{ x \mid \|x - x_0\|_p \leq r \}$

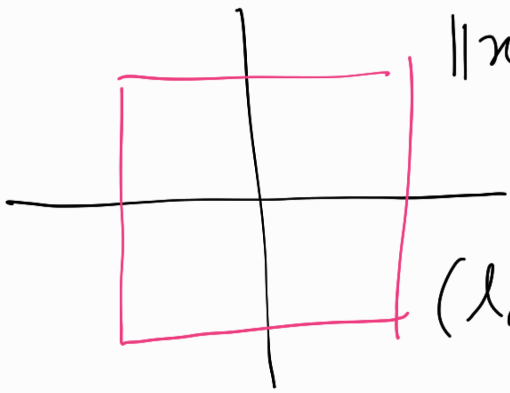
$r > 0, \; p \geq 1$

$\|x - x_0\|_2 \leq r$



(l_2 norm)

$\|x - x_0\|_\infty < r$



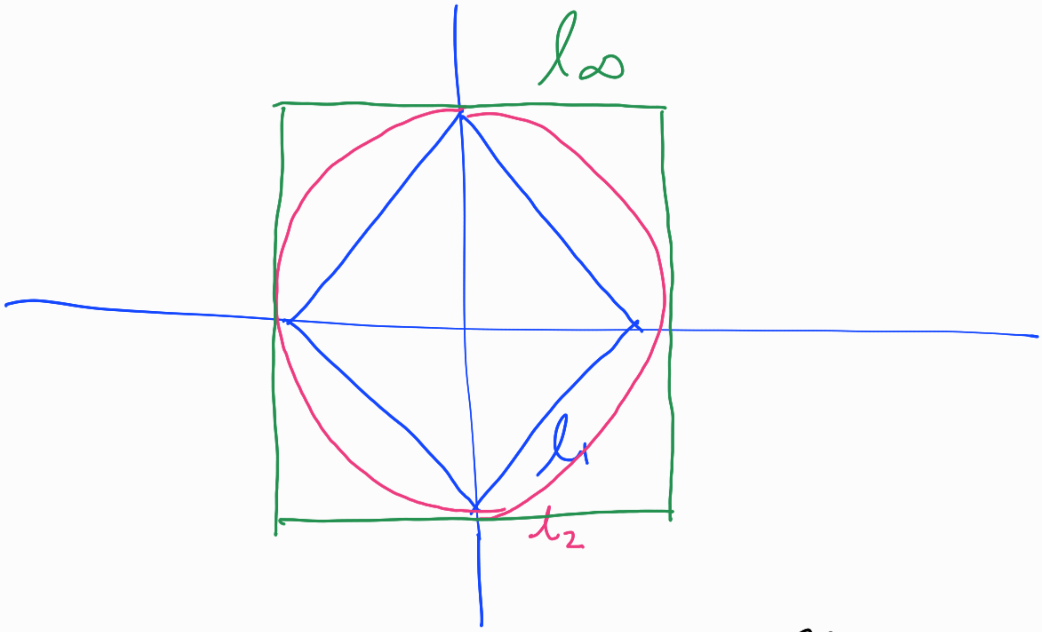
(l_∞ norm)

l_1 norm : $\|x-y\|_1 = \sum |x_i - y_i|$

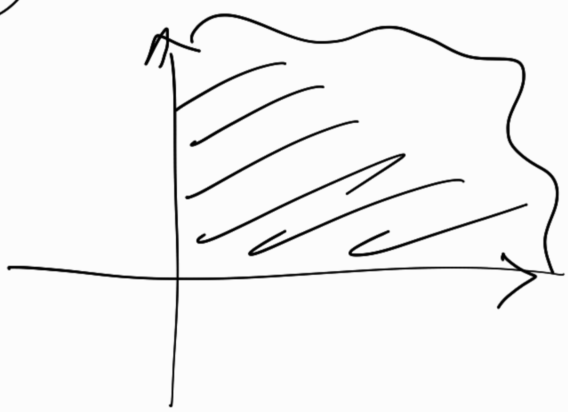
l_2-norm : $\|x-y\|_2 \doteq \sqrt{\sum (x_i - y_i)^2}$

l_p norm : $\|x-y\|_p = \sqrt[p]{\sum (x_i - y_i)^p}$

l_∞ norm : $\|x-y\|_\infty = \max\{|x_i - y_i|, i=1\ldots\}$



5) Positive orthant $\mathbb{R}^n_+$



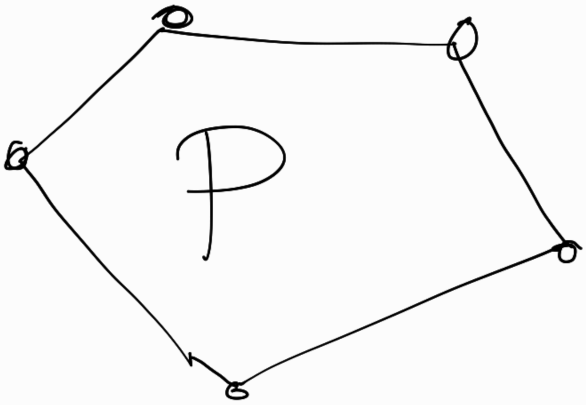
(6) $\mathbb{R}^n$ is Convex

Try to show convex hull of $x_1 \dots x_n$ is convex and smallest set that contains $\{x_1 \dots x_n\}$

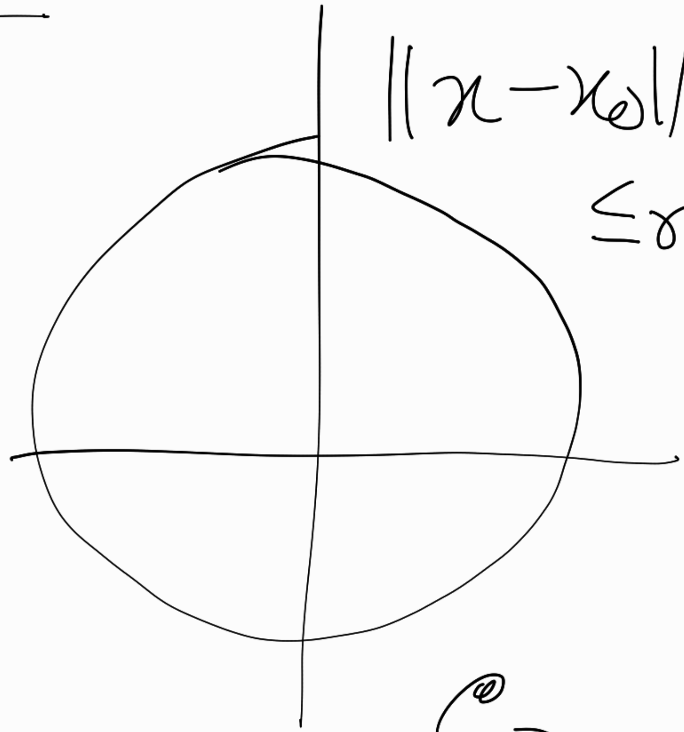
Convex sets

→ Separated?

→ Supported



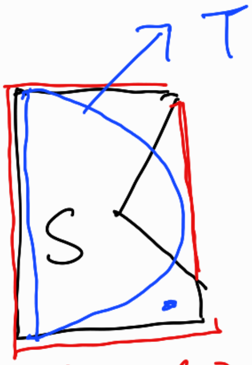
C_1



$\|x - x_0\| \leq r$

C_2

Theorem: $\operatorname{Conv}(S)$ is the smallest convex set that contains S.



$\operatorname{conv}(s) \supseteq S$

By definition it is convex.

To show it is smallest

assume T is convex, contains S but is smaller

$(T \supseteq S)$

i.e., $\operatorname{conv}(S) \supseteq T$

Let $\boxed{\bar{x} \notin T}$ but $\bar{x} \in \operatorname{conv}(S)$

$$\bar{x} \in \operatorname{conv}(s) \Rightarrow \bar{x} = \sum \theta_i x_i, \ \sum \theta_i = 1, \ \theta_i \geq 0$$

$$x_i \in S \quad \forall i$$

$$\Rightarrow x_i \in T, \quad \forall i$$

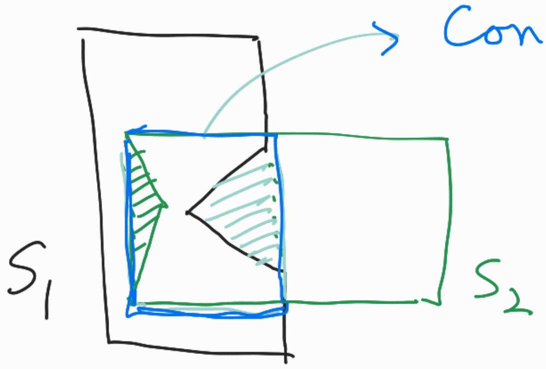
$$\Rightarrow \boxed{\bar{x} \in T} \quad (\because T \text{ is convex})$$

CONTRADICTION

Example: Show $\operatorname{Conv}(S_1 \cap S_2) \subseteq \operatorname{Conv}(S_1) \cap \operatorname{conv}(S_2)$

Show $x \in \operatorname{Conv}(S_1 \cap S_2) \Rightarrow$

$$x \in \operatorname{Conv}(S_1) \cap \operatorname{Conv}(S_2)$$



$\text{conv}(S_1) \cap \text{conv}(S_2)$

$= \text{conv}(S_1 \cap S_2)$

Let $x \in \text{conv}(S_1 \cap S_2) \Rightarrow x = \sum x_i \theta_i$

$x_i \in S_1 \cap S_2$

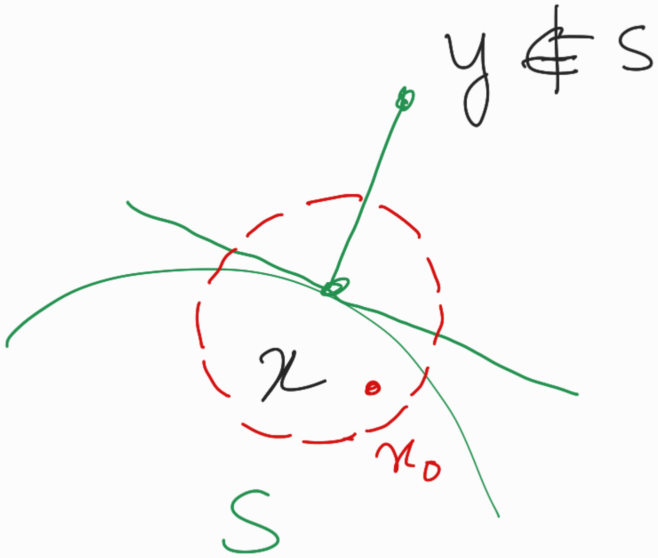
$\Rightarrow x_i \in S_1 \;\&\; x_i \in S_2$

$\Rightarrow x \in \text{conv}(S_1) \;\&\; x \in \text{conv}(S_2)$

$\because \text{conv}(S_1) \supseteq S_1$ etc.

$\Rightarrow x \in \underline{\text{conv}(S_1) \cap \text{conv}(S_2)}$

PROJECTIONS



$x = \Pi_S\, y$

y is projection of x on S convex

What is x ?

Ans :
$$\min_{x \in S} \|y - x\|$$

$$\equiv \min \|y - x\|$$
$$\text{s.t. } x \in S \cap \boxed{\{x \in S \mid \underbrace{\|y - x\| \leq \|y - x_0\|}\}}$$

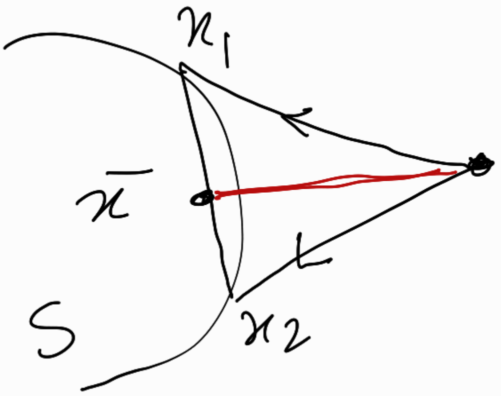
distance of x_0 and y is more, $x_0 \in S$

$\underbrace{\text{bounded, closed}}_{\equiv \text{ compact}}$ and convex

because it is norm ball

According to Wierstrass theorem a continuous function has an extremum on a compact set.

Is x unique? Suppose x_1 and x_2 are equidistant from y



$$x_1, x_2 \in S \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S$$

Take $\lambda = 1/2$ so, $\bar{x} = \frac{1}{2} x_1 + \frac{1}{2} x_2 \in S$

Check that $\|y - \bar{x}\|$

$$= \|y - \tfrac{1}{2}(x_1 + x_2)\|$$

$$\cancel{\leq} \; \tfrac{1}{2}\|y - x_1\| + \tfrac{1}{2}\|y - x_2\|$$

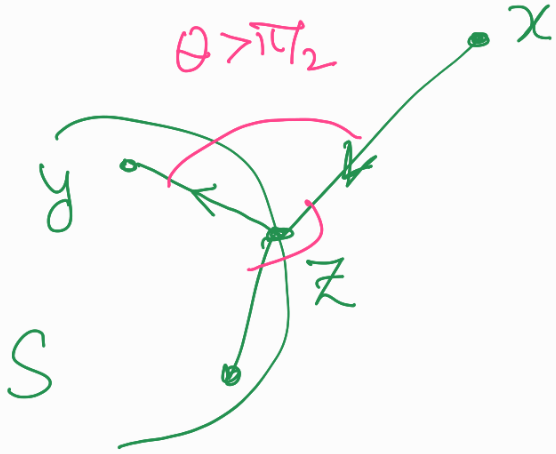
$$= \|y - x_1\|$$

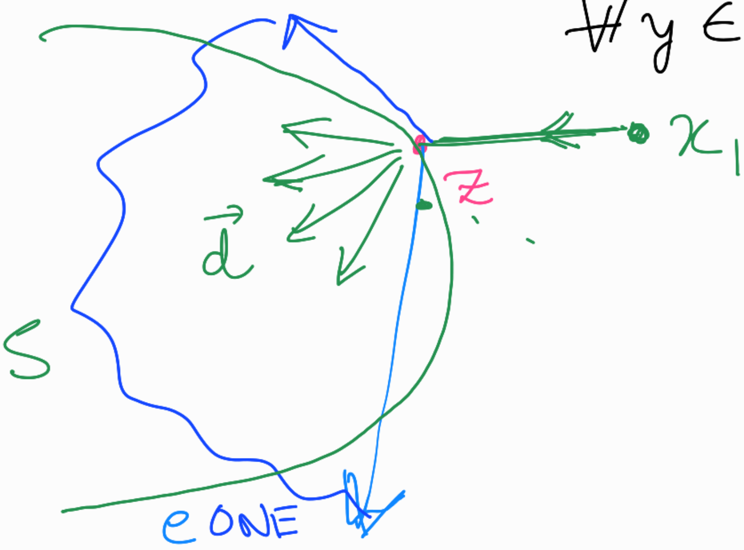
This leads to contradiction. Therefore there are no two points x_1 and x_2 which are both projections. So, projection point is always unique.

Theorem : S convex, nonempty, closed

Then $z = \Pi_S(x)$, for some $x \notin S$

iff $\langle x-z, y-z \rangle \leq 0$
$\forall y \in S$





$\langle x-z, y-z \rangle \leq 0$
$\forall y \in S$

" $\Rightarrow$ "

Suppose $z = \Pi_S(x)$ is the projection of x on S (convex)

For any $y \in S$, let

$$g(t) = \| x - z - t(y-z) \|^2$$

$$= \|x-z\|^2 - 2t\langle x-z, y-z\rangle + t^2 \|y-z\|^2$$

put $t=0$, $g(0) = \|x-z\|^2 < g(t)$
$\forall t \in (0,1]$

So, $2\langle x-z, y-z \rangle < t\|y-z\|^2$
$t \in (0,1]$

Therefore $\langle x-z, y-z \rangle \leq 0$

Now, suppose,

$$\langle x-z,\ y-z\rangle \le 0 \quad \forall y \in S$$

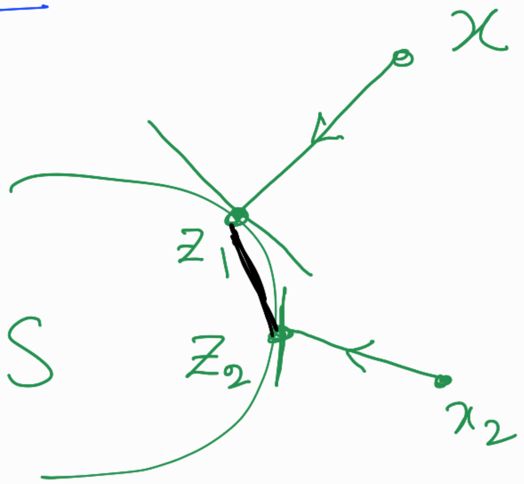
Since $g(0) < g(t) \quad \forall t \in (0,1]$

we have $g(0) < g(1)$, for $y \neq z$

$$\Rightarrow \quad \|x-z\| < \|x-y\| \quad \forall y \in S \text{ and } y \neq z$$

So, z must be projection point.

Theorem:



$$\Pi_S(x_1) - \Pi_S(x_2) \le \|x_1 - x_2\|$$

$$\|z_1 - z_2\| \le \|x_1 - x_2\|$$

$$\langle a, b\rangle + \langle c, b\rangle = \langle a+c, b\rangle$$

Proof: $\langle x_1 - z_1,\ z_1 - z_2\rangle \le 0$

and $\langle x_2 - z_2,\ z_1 - z_2\rangle \le 0$

Add both

$$\|z_1 - z_2\|^2 + \langle z_1 - z_2,\ x_1 - x_2\rangle \le 0$$

$$\Rightarrow \|z_1 - z_2\|^2 \leq \langle z_1 - z_2, x_2 - x_1 \rangle$$

$$= \|z_1 - z_2\| \cdot \|x_2 - x_1\|$$

$$\Rightarrow \|z_1 - z_2\| < \|x_2 - x_1\| \quad \underline{\text{QED.}}$$

<u>Homework</u>

(convex) Image set

1) Let $S \subseteq \mathbb{R}^n$, $A_{m\times n}$ and $a \in \mathbb{R}$

show $AS = \{y \mid y = AX, X \in S\}$ is convex

$\in \mathbb{R}^m$

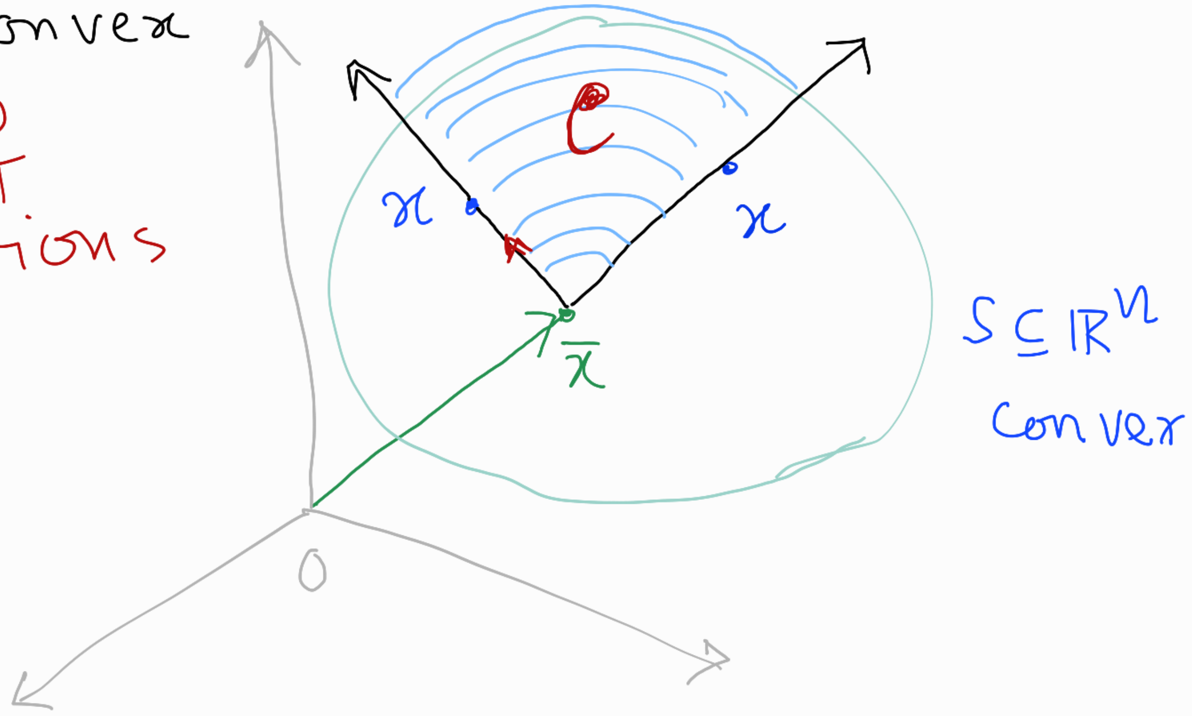
(convex)

2) Let $S \subseteq \mathbb{R}^n$, $S \neq \phi$ and $\bar{x} \in S$,

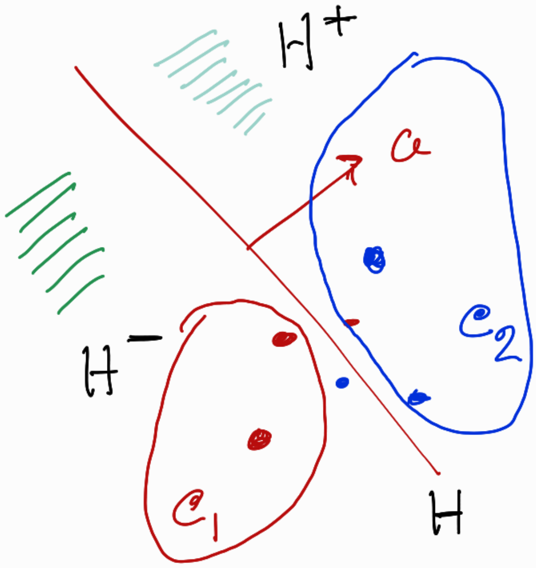
Show $C = \{y \mid y = \lambda(x - \bar{x}), \lambda \geq 0, \underline{x \in S}\}$

is convex

Cone of directions



Separation of Hyperplanes



Hyperplane is defined as

$$H = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle = \alpha \}$$

$a \neq 0$

$a \in \mathbb{R}^n$

$\alpha \in \mathbb{R}$

$$H^- = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle \leq \alpha \}$$

$$H^+ = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle \geq \alpha \}$$

Closed Half spaces

$$H^{--} = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle < \alpha \}$$

$$H^{++} = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle > \alpha \}$$

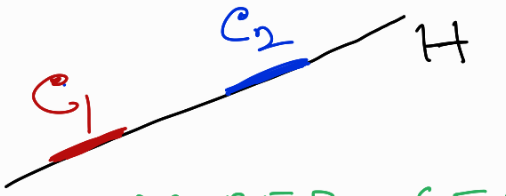
Open Halfspace

Let C_1 and C_2 be two non-empty sets.

① H is a separating hyperplane of C_1 & C_2 if $C_1 \subseteq H^+$ and $C_2 \subseteq H^-$

② .... strictly separating ..... if $C_1 \subseteq H^{++}$ and $C_2 \subseteq H^{--}$

(3) H is called properly separating hyperplane if H separates C_1 and C_2 and those are not both contained in H

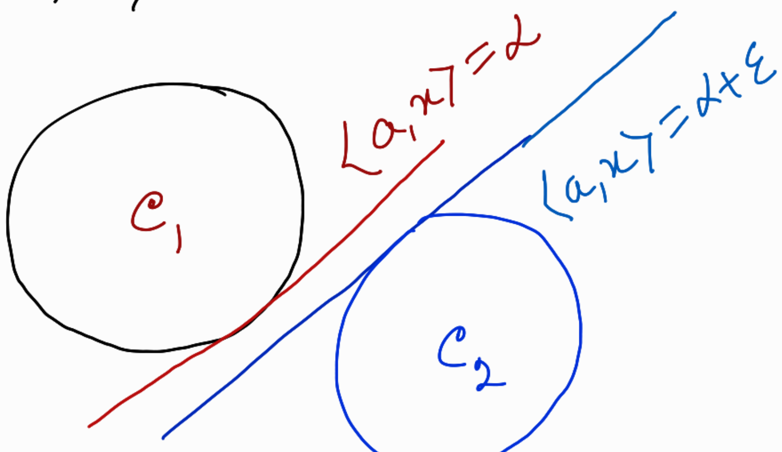


IMPROPER SEPARATION

(4) H is called strongly separating hyperplane of C_1 and C_2 if:

$$\langle a, x \rangle = \alpha \quad \text{for } x \in C_1 \quad \text{and}$$

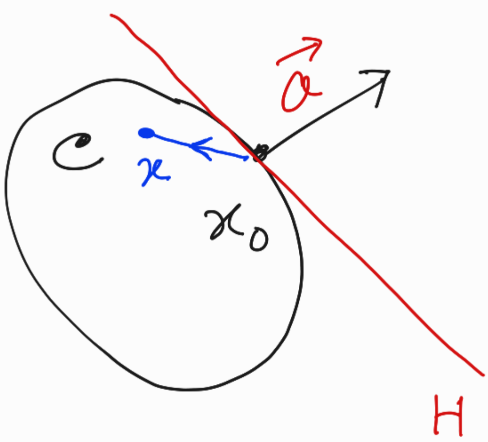
$$\langle a, x \rangle = \alpha + \varepsilon \quad \text{for } x \in C_2, \quad \varepsilon > 0$$



Theorem

Supporting hyperplane: If $C \subseteq \mathbb{R}^n$ is a closed convex set and $x_0 \in \partial C$ (boundary point), then there exists a hyperplane to C at x_0 s.t.

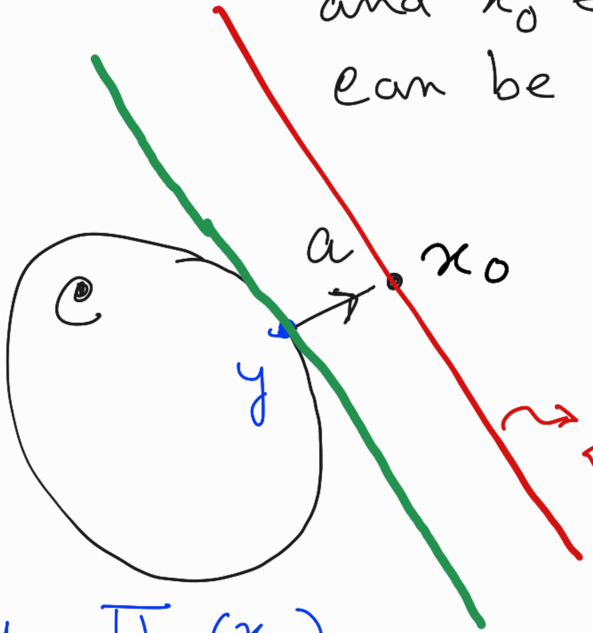
$$\langle a, x \rangle \le \langle a, x_0 \rangle, \forall x \in C$$



Separation Theorems

(Separation between a point and set)

Theorem: Let $C \subseteq \mathbb{R}^n$ be closed convex and $x_0 \notin C$. Then C and x_0 can be strongly separated



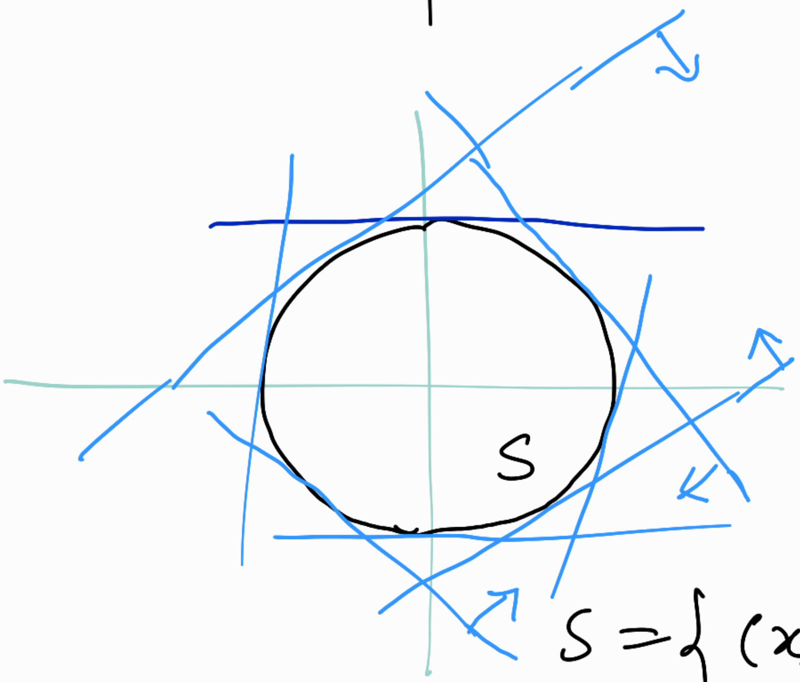
$\langle a, x\rangle = \alpha + \varepsilon, \; x \notin C$

$y = \Pi_C(x_0)$

Supporting hyperplane of C at point $y = \Pi_C(x_0)$
$\langle a, x\rangle = \alpha, \; x \in C$

Proof:

Note: A closed convex set is the intersection of closed halfspaces



$S = \{ (x,y) \mid x^2 + y^2 \leq r^2 \}$

$$S = \bigcap_{x \in S} H_i \,, \quad H_i = \{ x \mid \langle a, x \rangle \leq \alpha \}$$

Separating hyperplane theorem

Let C_1 and C_2 be non-empty convex sets, if $C_1 \cap C_2 = \phi$ then there exists hyperplane that separates them.

Proof $C = C_1 - C_2$

$$= \{x = x_1 - x_2 \mid x_1 \in C_1, x_2 \in C_2\}$$

As, $C_1 \cap C_2 = \phi$, so, $0 \notin C$

so, by separation of set and point not in set, we can say C and 0 can be separated, ie, $\exists\, a \neq 0$,

$$\langle a, x \rangle \geq 0 \quad \forall x \in C$$

$$\Rightarrow \quad \langle a, x_1 \rangle \geq \langle a, x_2 \rangle, \text{ for } x_1 \in C_1, \; x_2 \in C_2$$