Department of Computer Science

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Optimization for ML –BDA 2023

Problem Set 2: Convexity of Functions

1. *Inverse of an increasing convex function*:
Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse function, i.e., the function with domain $(f(a), f(b))$, and $g(f(x)) = x$ for $a < x < b$. Is g convex/concave?

2. For each of the following functions determine whether it is convex, concave, or neither.
(a) $f(x) = e^x - 1$ on $\mathbb{R}$.
(b) $f(x_1, x_2) = x_1x_2$ on $\mathbb{R}^2_{++}$.
(c) $f(x_1, x_2) = 1/(x_1x_2)$ on $\mathbb{R}^2_{++}$.
(d) $f(x_1, x_2) = x_1 / x_2$ on $\mathbb{R}^2_{++}$.
(e) $f(x_1, x_2) = x^2_1 / x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$
(f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where, $0 \le \alpha \le 1$, on $\mathbb{R}^2_{++}$.

3. *Products and ratios of convex functions:* Show that
(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.
(b) If f, g are concave, positive, with one nondecreasing and the other nonincreasing, then fg is concave.
(c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then f/g is convex.

4. *Strong, strict convexity of functions:* For each function below, determine whether it is convex, strictly convex, strongly convex or none of the above.
(a) $f(x) = (x_1 - 3x_2)^2$
(b) $f(x) = (x_1 - 3x_2)^2 + (x_1 - 2x_2)^2$
(c) $f(x) = (x_1 - 3x_2)^2 + (x_1 - 2x_2)^2 + x_1^3$
(d) $f(x) = |x|$, $x \in \mathbb{R}$.
(e) $f(x) = ||x||$, $x \in \mathbb{R}^n$

Exercise problems of Chapter 3 of Nonlinear Programming by Bazaraa et al.

[3.1] Which of the following functions is convex, concave, or neither? Why?

a. $f(x_1, x_2) = 2x_1^2 - 4x_1x_2 - 8x_1 + 3x_2$

b. $f(x_1, x_2) = x_1 e^{-(x_1+3x_2)}$

c. $f(x_1, x_2) = -x_1^2 - 3x_2^2 + 4x_1x_2 + 10x_1 - 10x_2$

d. $f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1^2 + x_2^2 + 2x_3^2 - 5x_1x_3$

[3.2] Over what subset of $\{x : x > 0\}$ is the univariate function $f(x) = e^{-ax^b}$ convex, where $a > 0$ and $b \ge 1$?

[3.3] Prove or disprove concavity of the following function defined over $S = \{(x_1, x_2) : -1 \le x_1 \le 1, -1 \le x_2 \le 1\}$:

$$f(x_1, x_2) = 10 - 3(x_2 - x_1^2)^2.$$

Repeat for a convex set $S \subseteq \{(x_1, x_2) : x_1^2 \ge x_2\}$.

[3.4] Over what domain is the function $f(x) = x^2(x^2 - 1)$ convex? Is it strictly convex over the region(s) specified? Justify your answer.

[3.5] Show that a function f: $R^n \to R$ is affine if and only if f is both convex and concave. [A function f is *affine* if it is of the form $f(\mathbf{x}) = \alpha + \mathbf{c}^t\mathbf{x}$, where a is a scalar and **c** is an n-vector.]

[3.37] Let $f(x_1, x_2) = e^{2x_1^2 - x_2^2} - 3x_1 + 5x_2$. Give the linear and quadratic approximations of f at (1, 1). Are these approximations convex, concave, or neither? Why?

Problems on convexity preserving operations

[3.8] Let $f_1, f_2, \ldots, f_k$: $R^n \to R$ be convex functions. Consider the function f defined by $f(\mathbf{x}) = \sum_{j=1}^{k} \alpha_j f_j(\mathbf{x})$, where $\alpha_j > 0$ for j = 1, 2,..., k. Show that f is convex. State and prove a similar result for concave functions.

[3.10] Let h: $R^n \to R$ be a convex function, and let g: $R \to R$ be a nondecreasing convex function. Consider the composite function f: $R^n \to R$ defined by $f(\mathbf{x})$ = $g[h(\mathbf{x})]$. Show that f is convex.

[3.16] Let g: $R^m \to R$ be a convex function, and let $\mathbf{h}: R^n \to R^m$ be an affine function of the form $\mathbf{h}(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}$, where **A** is an $m \times n$ matrix and **b** is an $m \times 1$ vector. Then show that the composite function f: $R^n \to R$ defined as $f(\mathbf{x})$ = $g[\mathbf{h}(\mathbf{x})]$ is a convex function. Also, assuming twice differentiability of g, derive an expression for the Hessian of f.