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MSc Big Data Analytics and MSc Computer Science : Batch 2023-25
DA109: Linear Algebra and Matrix Computation
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Problem set: 4

Please try to solve all the problems [1] alone. If you are stuck, I encourage you first consult with your batch mates and then TA. If you still need help then come to Instructor.

Notations:

- $\cdot$: scalar multiplication
- $\mathbf{Tr(A)}$: *Trace* of a matrix $\mathbf{A}$
- $\mathcal{R}(\mathbf{A})$: *row-space* of a matrix $\mathbf{A}$
- $\mathcal{C}(\mathbf{A})$: *column-space* of a matrix $\mathbf{A}$
- $\mathcal{N}(\mathbf{A})$: *null-space* of a matrix $\mathbf{A}$
- $\rho(\mathbf{A})$: *Rank* of a matrix $\mathbf{A}$
- *g-inverse*: Generalised inverse
- $\mathbf{A}^{-}$: *g-inverse* of a matrix $\mathbf{A}$

1. Prove that Postmultiplication of the permutation $(j_1, j_2, ..., j_n)$ by $[k, l]$ interchanges j_k and j_l while premultiplication interchanges k and l wherever they may occur in the permutation.

2. Suppose $\mathbf{B}$ is obtained from a $\mathbf{A}$ by interchanging two rows or two columns then using the definition of determinant prove that $|\mathbf{A}| = -|\mathbf{B}|$.

3. Show that:

   a. $|\mathbf{E}_{ij}| = -1$ b. $|\mathbf{E}_i(\alpha)| = \alpha$ c. $|\mathbf{E}_{ij}(\beta)| = 1$

4. Prove that
$$\begin{vmatrix} 1+a_1 & a_2 & \dots & a_n \\ a_1 & 1+a_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & 1+a_n \end{vmatrix} = 1 + a_1 + a_2 + ... + a_n$$

5. If $\mathbf{A}$ is the real $n \times n$ matrix where $a_{ij} = \rho^{|i-j|}$, show that $|\mathbf{A}| = (1-\rho^2)^{n-1}$.

[1] All the problems have been selected by the TA from the Rao & Bhimasankaram book [1].

6. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points in $\mathbb{R}^2$. Show that the equation of the line passing through P and Q is
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$
by showing that this equation represents a line, i.e., the coefficients of x and y are not both zero, and that it contains P and Q.

7. If $\mathbf{A}$ is a $n \times n$ matrix such that $a_{ij} = i + j - 2$ for all i and j. Show that $|\mathbf{A}| = 0$ whenever $n \geq 4$.

8. Suppose $\mathbf{A}_n$ be the following $n \times n$ tridiagonal matrix
$$\begin{pmatrix} a & b & 0 & \dots & 0 & 0 \\ c & a & b & \dots & 0 & 0 \\ 0 & c & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & b \\ 0 & 0 & 0 & \dots & c & a \end{pmatrix}$$
Show that $|\mathbf{A}_n| = a|\mathbf{A}_{n-1}| - bc|\mathbf{A}_{n-2}|$ for $n \geq 3$.

9. Without expanding the determinant prove that
$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & xz & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

10. Show that
   (a) $\begin{vmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{vmatrix} = -2abc$
   (b) Using the result in (a) show that
$$\begin{vmatrix} b^2 + c^2 & ab & ac \\ ab & a^2 + c^2 & bc \\ ac & bc & a^2 + b^2 \end{vmatrix} = 4a^2b^2c^2$$

11. Prove that
$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2$$

12. Solve the following system using Cramer's rule:
$$2x_1 - x_2 + x_3 = -3$$
$$x_1 + x_2 - 3x_3 = 17$$
$$5x_1 - 2x_2 - 4x_3 = 20$$

13. Show that the intersection of the two distinct planes $a_1x + b_1y + c_1z = 0$ and $a_2x + b_2y + c_2z = 0$ in $\mathbb{R}^3$ is
$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

14. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ be $m \times m$ matrices such that $\mathbf{A}$ is non-singular and $\mathbf{A}$ commutes with $\mathbf{C}$. Then show that
$$\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = \mathbf{AD} - \mathbf{CB}$$

15. Suppose $\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

   (a) Determine all the eigenvalues of $\mathbf{A}$.

   (b) For each eigenvalue λ of $\mathbf{A}$, find the set of eigenvectors corresponding to λ.

   (c) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of $\mathbf{A}$.

   (d) Determine an invertible matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{Q}^{\mathbf{-1}}\mathbf{AQ} = \mathbf{D}$.

16. Let T be a linear operator on a finite-dimensional vector space $\mathbf{V}$, and let β be an ordered basis for $\mathbf{V}$. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$.

17. (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

   (b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1} .

18. Let T be a linear operator on a vector space $\mathbf{V}$ over the field $\mathbb{F}$, and let $g(t)$ be a polynomial with coeicients from $\mathbb{F}$. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then x is an eigenvector of $g(T)$ with corresponding eigenvalue $g(\lambda)$.

19. For each of the following matrices $\mathbf{A} \in M_{n\times n}(\mathbb{R})$, test $\mathbf{A}$ for diagonalizability, and if $\mathbf{A}$ is diagonalizable, find an invertible matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{Q}^{-1}\mathbf{AQ} = \mathbf{D}$:

   a. $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ b. $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$

20. Find a rank-factorization of the matrix

$$\begin{pmatrix} 2 & 4 & 2 & 4 & 4 \\ 1 & 2 & 1 & 2 & 2 \\ 3 & 0 & 3 & 3 & 0 \\ 0 & -4 & 0 & -2 & -4 \\ 5 & 2 & 5 & 6 & 2 \end{pmatrix}$$

   and hence find the characteristics roots.

21. Let $\mathbf{A}$ be a 2×2 matrix. Then show that $|\mathbf{I} + \mathbf{A}| = 1 + |\mathbf{A}|$ iff $tr(\mathbf{A}) = 0$.

22. For any eigenvalue α of $\mathbf{A}$, the algebraic multiplicity of α with respect to $\mathbf{A}$ is not less than the geometric multiplicity of α with respect to $\mathbf{A}$.

23. If $\mathbf{A}$ is an $n \times n$ singular matrix with k distinct eigenvalues, show that $k - 1 \leq \rho(\mathbf{A}) \leq n - 1$. Also show by construction that $\rho(\mathbf{A})$ can take any value between $k - 1$ and $n - 1$.

24. Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$. Using characteristics polynomial find A^7.

25. Prove that for any two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ the characteristic polynomials of $\mathbf{AB}$ and $\mathbf{BA}$ are the same.

26. Prove that the minimal polynomial of a matrix always divides it's characteristics polynomial.

27. Suppose

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

   Using Cayley-Hamilton theorem, find the inverse of the matrix $\mathbf{B}$.

28. When the minimal polynomial coincides with the characteristic polynomial for a matrix? Show that for the following companion matrix this holds:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}$$

29. Find the minimal polynomial of the following $n \times n$ matrices:

    (a) $\alpha\mathbf{A}$, where the minimal polynomial of $\mathbf{A}$ is $f(x)$.

    (b) $\mathbf{J} = \mathbf{1}\mathbf{1}^T$, where $\mathbf{1}$ is the $n \times 1$ matrix of ones.

30. Show that the vector space $\mathbf{V}$ of all real-valued continuous functions on an interval $[a, b]$ forms a inner product space where the inner product is defined as
$$< f, g >= \int_a^b f(x)g(x)\, dx.$$

31. Show that $< \mathbf{A}, \mathbf{B} >= tr(\mathbf{B}^*\mathbf{A})$ is an inner product on $\mathbb{C}^{m\times n}$.

32. Let $\mathbf{V}$ be an inner product space over $\mathbb{F}$. Prove the polar identities For all $x, y \in V$

    (a) $< x, y >= \frac{1}{4}||x + y||^2 - \frac{1}{4}||x - y||^2$ if $\mathbb{F} = \mathbb{R}$.

    (b) $< x, y >= \frac{1}{4}\sum_{k=1}^{4} i^k||x + i^k y||^2$ if $\mathbb{F} = \mathbb{C}$ where $i^2 = -1$

33. Show that $< x, y >= 0$ for all y iff $x = 0$.

34. Let $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ form an orthonormal set.

    (a) Show that $||\sum_{i=1}^{k} \alpha_i\mathbf{x}_i||^2 = \sum_{i=1}^{k} |\alpha_i|^2$.

    (b) Prove that $||\mathbf{x}||^2 \geq \sum_{i=1}^{k} | < \mathbf{x}, \mathbf{x}_i > |^2$.

35. Let $\mathbf{x}_1 = (1, 1, 1, 1)$, $\mathbf{x}_2 = (0, 1, 1, 1)$, $\mathbf{x}_3 = (0, 0, 1, 1)$ and $\mathbf{x}_4 = (0, 0, 0, 1)$ in $\mathbb{R}^4$. Starting from $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ obtain an orthonormal basis of $\mathbb{R}^4$. If you use $\{\mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1\}$ what is the orthonormal basis obtained?

36. Find an orthonormal basis of the subspace of $\mathbb{R}^4$ spanned by $(2, -1, 0, 1)$, $(6, 1, 4, -5)$ and $(4, 1, 3, -4)$.

37. Consider the inner product $< x, y >= y^T Ax$ on $\mathbb{R}^3$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

    Find an orthonormal basis B of $S := \{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) : \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = 0\}$ and then extend it to an orthonormal basis C of $\mathbb{R}^3$.

38. Consider the subspace $S = \{(\xi_1, \xi_2, \xi_3, \xi_4) : \xi_1 = \xi_2 = \xi_3\}$ and $T = \{(\xi_1, \xi_2, \xi_3, \xi_4) : \xi_1 = \xi_2$ and $\xi_4 = 0\}$ of $\mathbb{R}^4$. Find $S + T$, $S^\perp + T^\perp$ and verify that $(S + T)^\perp = S^\perp \cap T^\perp$.

39. Let $\mathbf{A}$ be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ with corresponding multiplicities $m_1, m_2, ..., m_k$. Prove the following statements:

    (a) $tr(\mathbf{A}) = \sum_{i=1}^{k} m_i\lambda_i$

    (b) $|\mathbf{A}| = \lambda_1^{m_1}\lambda_2^{m_2}....\lambda_k^{m_k}$

40. (a) Using the standard ordered basis of $\mathbf{P}_2(\mathbb{R})$ find a orthonormal basis of it w.r.t the inner product defined as:
$$< f(x), g(x) >= \int_{-1}^{1} f(x)g(x)\, dx \qquad \text{for all } f(x), g(x) \in \mathbf{V}$$

(b) Let $\mathbf{V} = \mathbf{P}_3(\mathbb{R})$ with the inner product defined above. Find the orthogonal projection of $f(x) = x^3$ on $\mathbf{P}_2(\mathbb{R})$.

41. Reduce each of the following to a matrix in reduced echelon form by elementary row operations and find the rank, a row basis, a column basis and a rank-factorization.

(a) $\begin{pmatrix} 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 3 & 0 & 2 \\ 5 & 7 & -9 & 2 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 1 & 2 & 4 \\ 9 & 3 & 6 & 12 \\ 2 & 0 & 0 & 2 \\ 5 & 1 & 2 & 6 \end{pmatrix}$

References

[1] A. Ramachandra Rao and P Bhimasankaram. *Linear Algebra*. Hindustan Book Agency, 2nd edition, 2000.