$$\sum \binom{n}{r} p^{r}(1-p)^{n-r} = \int_0^p \frac{n!}{(i-1)!(n-r)!} t^{i-1}(1-t)^{n-i}\, dt$$

$0 < p < 1$

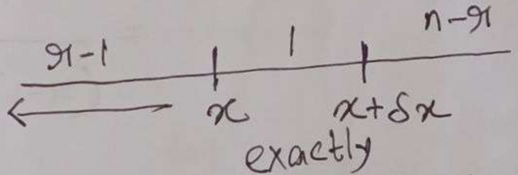
Order stat.

$X_1, X_2, \dots, X_n$ are iid r.v's Y_r denote the r.v of rank r

$$g_r(x) = \lim_{\delta x \to 0} \frac{P(x < Y_r < x+\delta x)}{\delta x}$$

Random variables of rank $\leq r-1$ takes values $\leq x$

· Random variable of rank $> r$ takes values $> x+\delta x$



Prob. that among n r.v's r-1 r.v's takes values $\leq x$. (event A)

$$= \binom{n}{r-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}$$

B: ∃ exactly one r.v that lies in $(x, x+\delta x]$

$$P(B) = \binom{n-r+1}{1}[F(x+\delta x) - F(x)]$$

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{\binom{n}{r-1}[F(x)]^{r-1}\binom{n-r+1}{1}[F(x+\delta x)-F(x)]\binom{n-r}{n-r}\{1-F(x+\delta x)\}^{n-r}}{\binom{n}{r-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}}$$

$P(C|AB) = 1$

C: exactly $n-r$ x_i's are greater than $x+\delta x$

$$P(C) = \binom{n-r}{n-r} [1-F(x+\delta x)]^{n-r}$$

$$P(x < Y_r \leq x+\delta x) = P(A \cap B \cap C)$$
$$= P(A) \, P(B) \cdot P(C)$$
$$= \binom{n}{r-1} \binom{n-r+1}{1} \binom{n-r}{n-r} \cdot$$
$$[F(x)]^{r-1} [F(x+\delta x) - F(x)]$$
$$[1-F(x+\delta x)]^{n-r}$$

$$\lim_{\delta x \to 0} \frac{P(x < Y_r \leq x+\delta x)}{\delta x} = \frac{\quad}{\delta x}$$

$$= \frac{n!}{(r-1)!\,(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x)$$

density fun of min $(X_1, X_2, \ldots, X_n)$

$$g_1(x) = n [1-F(x)]^{n-1} f(x)$$

$$G_1(x) = \sum_{K=1}^{n-1} \binom{n}{K} [F(y)]^K [1-F(y)]^{n-K} + [F(y)]^n$$

?

$$= 1 - (1-F(x))^n$$

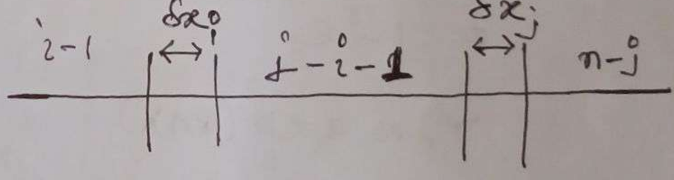
$$P\{\min(X_1, X_2, \dots, X_n) \le y\}^c$$
$$= P\{Y \le y\}^c = P\{Y > y\}$$
$$= P\big\{\{X_1 > y\} \cap \{X_2 > y\} \cap \dots \cap \{X_n > y\}\big\}$$
$$= [1 - F(y)]^n$$

$$G_1(y) = P(Y \le y)$$
$$= 1 - [1 - F(y)]^n$$

$$G_n(x) = P\{Y_n \le x\}$$
$$= P\{X_1 \le x, X_2 \le x, \dots, X_n \le x\}$$
$$= [F(x)]^n \quad [\max(X_1, X_2, \dots, X_n) \le x]$$

Joint density of Y_i, Y_j, $i < j$

$$g_{i,j}(x_i, x_j)$$



$$= \frac{n!}{(i-1)!\,1!\,(j-i-1)!\,1!\,(n-j)!} [F(x_i)]^{i-1} f(x_i) [F(x_j) - F(x_i)]^{j-i-1} f(x_j) [1 - F(x_j)]^{n-j}$$

$$\mathcal{C} := \{(S_A, S_B, S_C) \mid A \subseteq [n], |A| = r-1, B \subseteq [n], B = 1, C \subseteq [n], |C| = n-r, A \cup B \cup C = [n]\}$$

let S denote partition $= (S_A, S_B, S_C)$

Joint density of $Y_1, Y_2, \ldots, Y_n$

$$g(x_1, x_2, \ldots, x_n) = n!\, f(x_1) f(x_2) \cdots f(x_n)$$

$$-\infty < x_1 < x_2 < \cdots < x_n < \infty.$$

Problem.

Ex 1: let $X_1, X_2, \ldots, X_n$ be a random ~~variable~~ sample from a population with continious density show that $Y_1 = \min(X_1, \ldots, X_n)$ is exponential with parameter $n\lambda$ iff each X_i is exponential with paremeter λ.

pdf $\lambda e^{-\lambda x}$

cdf $1 - e^{-\lambda x} = P(X \leq x)$

→ Distribution funⁿ of Y_1

$$= 1 - (1 - F(x))^n$$

$$= 1 - (1 - (1 - e^{-\lambda x}))^n$$

$$= 1 - e^{-\lambda n x}.$$

$Y_1 \sim \exp(n\lambda)$

$X_i \sim \exp(\lambda) \Rightarrow Y_1 \sim \exp(n\lambda).$

Now, $Y_1 \sim \exp(n\lambda).$

$$\Rightarrow P(Y_1 \leq x) = 1 - e^{-\lambda n x}$$

$$\Rightarrow P(Y_1 > x) = e^{-\lambda n x}$$

$$\Rightarrow P(\{X_1 > x\} \cap \{X_2 > x\} \cap \cdots \cap \{X_n > x\}) = e^{-\lambda n x}$$

$$\Rightarrow \prod_{i=1}^{n} P(X_i > x) = e^{-\lambda n x}$$

$$\left[P(X_1 > x)\right]^n = e^{-\lambda n x}$$

$$\Rightarrow P(X_1 > x) = e^{-\lambda x}$$

$$\Rightarrow P(X_1 < x) = 1 - e^{-\lambda x}$$

Ex 2: ~~Suppose X ~ exp(1). Given $X_1, X_2, \ldots, X_n$~~

suppose $X_1, X_2, \ldots, X_n$ are iid and $X_i \sim exp(1)$
compute cdf $Y_n - \log n$ as $n \to \infty$. (P-708) G Kapoor

$$\boxed{\begin{array}{l} 1 - e^{-x} \\ \lim_{n\to\infty}\left(1 + \frac{m}{n}\right)^n = e^m \end{array}}$$

$\rightarrow X_i \sim exp(1)$

$$G_n(x) = P(Y_n \leq x)$$

~~$P(Y_n - \log n \leq x)$~~ $P(Y_n - \log n \leq y)$

$$\Rightarrow P(Y_n \leq y + \log n)$$

$$\Rightarrow G_n(y + \log n) = \left[F(y + \log n)\right]^n$$

$$= \left(1 - e^{-y - \log n}\right)^n = \left(1 - \frac{e^{-y}}{e^{\log n}}\right)^n$$

$$\lim_{n\to\infty}\left(1 - \frac{e^{-y}}{e^{\log n}}\right)^n = e^{-e^{-y}} = \exp\left(-e^{-y}\right)$$

Ex 3: Show that for a random sample of size 2 from $N(0, \sigma^2)$ population

$$E(Y_1) = -\sigma/\sqrt{\pi}$$

$\rightarrow$

$$\text{pdf } f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad -\infty < x < \infty$$

$$E[Y_1] = \int_{-\infty}^{\infty} x\, g_1(x)\, dx$$

$$g_1(x) = \frac{2}{1!\ 1!}(1-F(x))^{2-1} f(x)$$

$$= 2(1-F(x)) f(x)$$

$$\cancel{= \int_{-\infty}^{\infty} x \cdot \frac{2!}{\quad}}$$

$$= \int_{-\infty}^{\infty} x \cdot 2(1-F(x)) f(x)\, dx$$

$$\log(f(x)) = \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \left(-\frac{x^2}{2\sigma^2}\right)$$

$$\frac{1}{f(x)} f'(x) = -\frac{2}{2\sigma^2} \cdot x = -\frac{x}{\sigma^2}$$

$$\cancel{f'(x) = f(x) - \frac{x}{\sigma^2}}$$

$$\Rightarrow f(x) = -f'(x)\frac{\sigma^2}{x}$$

$$\cancel{= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \left(-\frac{x}{\sigma^2}\right)}$$

$$\cancel{f(x) = -f'(x)\frac{\sigma^2}{x}}$$

$$\therefore \cancel{E[Y_1] = \int_{-\infty}^{\infty} x \cdot 2(1-F(x)) - f'(x)\frac{\sigma^2}{x}\, dx}$$

$$\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\, dx = \frac{\sqrt{\pi}}{\sigma} \quad \text{Am.}$$

$$= \int_{-\infty}^{\infty} 2\,(1-F(x))\,(-f'(x))\,\sigma^2\,dx$$

$$= -2\sigma^2 \int_{-\infty}^{\infty} (1-F(x))\, f'(x)\,dx$$

$$= -2\sigma^2 \left[(1-F(x)) \int_{-\infty}^{\infty} f'(x)\,dx - \int \left(-F'(x)\, f'(x)\right) dx \right]$$

$$= -2\sigma^2 \left[(1-F(x))\, f(x) - \int_{-\infty}^{\infty} \right.$$

$$E[Y_1] = (1-F(x)) \int_{-\infty}^{\infty} x f(x)\,dx - \int_{-\infty}^{\infty} \left(\frac{d}{dx}(1-F(x)) \int x f(x)\,dx \right) dx$$

$$= (1-F(x)) \int_{-\infty}^{\infty} -\cancel{x} f'(x)\,\frac{\sigma^2}{\cancel{x}}\,dx - \int_{-\infty}^{\infty} -f(x) \int \cancel{x} \left(-\frac{f'(x)\,\sigma^2}{x}\right) dx\; dx$$

$$= -\sigma^2 (1-F(x)) \int_{-\infty}^{\infty} f'(x)\,dx - \int_{-\infty}^{\infty} +f(x) \int \left(\sigma^2 f'(x)\,dx\right) dx$$

complete it !!

β integral. (Beta)

$$B(z_1, z_2) = \int_0^1 t^{z_1 - 1} (1-t)^{z_2 - 1} \, dt$$

$$z_1 > 0, \quad z_2 > 0$$

Beta function

Gamma integrals:

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} \, du$$

$$\Gamma(z) = (z-1)\,\Gamma(z-1)$$

If z is a positive integer

$$\Gamma(z) = (z-1)$$

Relation between Beta & Gamma

$$B(z_1, z_2) = \frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1 + z_2)}$$

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

$$g_r(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x)$$

$$= \frac{1}{B(r, n-r+1)}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x).$$

Ex 4: Show that in odd samples of size n from $U[0,1]$ population, the mean and variance of the distribution of median are $1/2$ and $\frac{1}{4(n+2)}$ respectively.

→

$n = 2m+1$

$(m+1)^{th}$ order statistics is the median.

$E[Y_{m+1}] \leftarrow$ to compute

$$g_{m+1}(x) = \frac{1}{B(m+1, m+1)}[F(x)]^{m}[1-F(x)]^{m} f(x)$$

$$= \frac{1}{B(m+1, m+1)} \cdot x^{m} \cdot (1-x)^{m}$$

$$E[Y_{m+1}] = \frac{1}{B(m+1, m+1)} \int_0^1 x \cdot x^{m}(1-x)^{m}\, dx$$

$$= \frac{1}{B(m+1, m+1)} \int_0^1 x^{m+1}(1-x)^{m}\, dx$$

$$= \frac{1}{B(m+1, m+1)} \int_0^1 x^{(m+2)-1}(1-x)^{(m+1)-1}\, dx$$

$$= \frac{B(m+2, m+1)}{B(m+1, m+1)}$$

$$= \frac{\Gamma(m+2)\,\cancel{\Gamma(m+1)}}{\Gamma(2m+3)} \cdot \frac{\Gamma(2m+2)}{\Gamma(m+1)\,\cancel{\Gamma(m+1)}}$$

$$= \frac{\Gamma(m+2)\,\Gamma(2m+2)}{\Gamma(2m+3)\,\Gamma(m+1)} = \frac{(m+1)\,\cancel{\Gamma(m+1)}\,\cancel{\Gamma(2m+2)}}{\cancel{\Gamma(2m+2)}\,(2m+2)\,\cancel{\Gamma(m+1)}}$$

$$= \frac{m+1}{2m+2} = \frac{1}{2}$$

$$E\left[Y^2_{m+1}\right] = \frac{1}{B(m+1, m+1)} \int_0^1 x^2 x^m (1-x)^m \, dx$$

$$= \frac{1}{B(m+1, m+1)} \int_0^1 x^{m+2} (1-x)^m \, dx$$

$$= \frac{1}{B(m+1, m+1)} \int_0^1 x^{(m+3)-1} (1-x)^{(m+1)-1} \, dx$$

$$= \frac{B(m+3, m+1)}{B(m+1, m+1)} = \frac{\Gamma(m+3)\,\cancel{\Gamma(m+1)}\,\Gamma(2m+2)}{\cancel{\Gamma(m+1)}\,\Gamma(m+1)\,\Gamma(2m+4)}$$

$$= \frac{\Gamma(m+3)\,\Gamma(2m+2)}{\Gamma(m+1)\,\Gamma(2m+4)} = \frac{(m+2)\,\Gamma(m+2)\,\Gamma(2m+2)}{\Gamma(m+1)\,(2m+3)\,\Gamma(2m+3)}$$

$$= \frac{(m+2)(m+1)\,\cancel{\Gamma(m+1)}\,\cancel{\Gamma(2m+2)}}{\cancel{\Gamma(m+1)}\,(2m+3)(2m+2)\,\cancel{\Gamma(2m+2)}} = \frac{m+2}{2(2m+3)}$$

Variance =

$$E(Y_{m+1}^2) - [E(Y_{m+1})]^2$$

$$= \frac{m+2}{2(2m+3)} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4(n+2)}$$

Distribution of the range

$$R = Y_{(n)} - Y_{(1)}$$

$$P(R \leq x) = P(Y_{(n)} - Y_{(1)} \leq x)$$

$$= \iint_{x_n - x_1 \leq x} g_{1,n}(x_1, x_n)\, dx_1\, dx_n$$

$$= \iint_{x_n - x_1 \leq x} \frac{n!}{(n-2)!\, \cancel{(\ldots)!\,(\ldots)!}} f(x_1) f(x_n) [F(x_n) - F(x_1)]^{n-2} dx_1\, dx_n$$

$$= c \int_{-\infty}^{\infty} \int_{x_1}^{x_1 + x} [F(x_n) - F(x_1)]^{n-2} f(x_n)\, dx_n\, f(x_1)\, dx_1$$

$$y = F(x_n) - F(x_1)$$

$$dy = f(x_n)\,dx_n$$

$$\int_{x_1}^{x_1+x} \left[F(x_n) - F(x_1)\right]^{n-2} f(x_n)\,dx_n$$

$$= \int_0^{F(x_1+x)-F(x_1)} y^{n-2}\,dy = \frac{1}{n-1}\, y^{n-1}\Big|_0^{F(x_1+x)-F(x_1)}$$

$$= \frac{1}{n-1}\left[F(x_1+x) - F(x_1)\right]^{n-1},$$

$\therefore$ $P(R \le x)$

$$= c\int_{-\infty}^{\infty} \frac{n!}{(n-2)!\,(n-1)}\left[F(x_1+x) - F(x_1)\right]^{n-1} f(x_1)\,dx_1$$

Joint Probability Distribution of Functions of random variable

Suppose $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$ where X_1, X_2 are jointly continuous.

We assume g_1, g_2 satisfies the following Equations

- $y_1 = g_1(x_1, x_2)$, $y_2 = g_2(x_1, x_2)$ can be uniquely solved to give $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$
- g_1, g_2 have continuous partial derivatives. $\forall\ x_1, x_2$

We can write

$$\boxed{f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(h_1(y_1, y_2), h_2(y_1, y_2)) \times |J|^{-1}}$$

$$J = \det \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}$$

Ex 1: X_1, X_2 are two r.v's with joint pdf f_{X_1, X_2}. let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. we want to compute joint pdf of f_{Y_1, Y_2}

→

$$y_1 = x_1 + x_2 \qquad x_1 = \frac{y_1 + y_2}{2}$$
$$y_2 = x_1 - x_2$$
$$x_2 = \frac{y_1 - y_2}{2}$$

$$g_1(x_1, x_2) = x_1 + x_2$$

$$g_2(x_1, x_2) = x_1 - x_2$$

$$\begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ +1 & -1 \end{vmatrix} = -2$$

$$\therefore |J|^{-1} = \frac{1}{2}$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2} \right) \times \frac{1}{2}$$

let X_1, X_2 are two r.v's s.t $X_1 \sim \exp(\lambda_1)$, $X_2 \sim \exp(\lambda_2)$

$$y_1 = \lambda_1 e^{-\lambda_1 x_1}$$

$$y_2 = \lambda_2 e^{-\lambda_2 x_2}$$

$$\Rightarrow \cancel{e^{-\lambda_1 x_1} = \frac{y_1}{\lambda_1}} \Rightarrow \cancel{-\lambda_1 x_1 = \log\left(\frac{y_1}{\lambda_1}\right)}$$

$$\Rightarrow \cancel{x_1 = -\frac{1}{\lambda_1} \log_e\left(\frac{y_1}{\lambda_1}\right)}$$

$$f_{Y_1, Y_2} = \begin{cases} \frac{1}{2} \lambda_1 \lambda_2 \exp\left(-\frac{\lambda_1}{2}(y_1 + y_2)\right) \exp\left(-\frac{\lambda_2}{2}(y_1 - y_2)\right) & \cancel{y_1 + y_2 \geq 0,\ y_1 - y_2 \geq 0.}\ \ y_1 > |y_2| \\ 0 & \text{, otherwise.} \end{cases}$$

Ex 2: X_1, X_2 are two r.v's with joint pdf

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} 4 x_1 x_2 e^{-(x_1^2 + x_2^2)} & x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

We want to compute density of $Y_1 = \sqrt{X_1^2 + X_2^2}$

$y_1 = \sqrt{x_1^2 + x_2^2}\ , \quad y_2 = x_1$

$y_1^2 = x_1^2 + x_2^2\ , \quad y_2^2 = x_1^2$

$x_2^2 = y_1^2 - y_2^2\ , \quad x_1^2 = y_2^2$

we have $x_2^2 > 0 \Rightarrow y_1^2 > y_2^2$

as $x_1 > 0$, Then $y_1 > y_2 > 0$.

$$\left| \begin{matrix} \frac{\partial y_1}{\partial x_1} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} & \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \\ 1 & 0 \end{matrix} \right|$$

$$= \frac{-x_2}{\sqrt{x_1^2 + x_2^2}}$$

$$|J|^{-1} = \frac{\sqrt{x_1^2 + x_2^2}}{x_2}$$

$$f_{Y_1, Y_2} = \begin{cases} \frac{\sqrt{x_1^2 + x_2^2}}{x_2} \cdot 4 y_2 \sqrt{y_1^2 - y_2^2}\, e^{-\left(y_2^2 + y_1^2 - y_2^2\right)} & \text{where } y_1, y_2 > 0 \\ 0\ , & \text{o/w} \end{cases}$$

$$= \begin{cases} \frac{\sqrt{x_1^2 + x_2^2}}{x_2} \cdot 4 y_2 \sqrt{y_1^2 - y_2^2}\, e^{-y_1^2} & , y_1, y_2 > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

$$= \left\{ \frac{y_1}{\cancel{\sqrt{y_1^2 - y_2^2}}} \, 4 y_2 \cancel{\sqrt{y_1^2 - y_2^2}}\, e^{-y_1^2} = 4 y_1 y_2 e^{-y_1^2} \right.$$

$$f_{Y_1}(y_1) = \int_0^\infty 4y_1 y_2 e^{-y_1^2} dy_2$$

$$= \int_{y_2=0}^{y_1} 4y_1 y_2 e^{-y_1^2} dy_2$$

$$= 4y_1 e^{-y_1^2} \int_0^{y_1} y_2 \, dy_2 = 2y_1 e^{-y_1^2} y_1^2$$

$$= 2y_1^3 e^{-y_1^2}$$

<u>Ex 3</u>: X_1, X_2 are two non-neg indep r.v's we want to compute density of $Y_1 = X_1 + X_2$

$\rightarrow$ $y_1 = x_1 + x_2$, $y_2 = x_1$ $x_1, x_2 > 0$

$x_2 = y_1 - y_2$, $x_1 = y_2$

$$\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

$|J|^{-1} = 1$

$$f_{Y_1, Y_2} = \begin{cases} f_{X_1, X_2}(y_2, y_1 - y_2) & y_1, y_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} f_{X_1}(y_2) \cdot f_{X_2}(y_1 - y_2) & y_1 > y_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^{y_1} f_{x_1}(y_2) \cdot f_{x_2}(y_1 - y_2)\, dy_2.$$

$$f(x_1, x_2) = \begin{cases} \alpha^{-2} e^{-(x_1+x_2)/\alpha} & , x_1 > 0, x_2 > 0, \alpha > 0 \\ 0 . & , \text{otherwise} . \end{cases}$$

compute pdf of $\frac{1}{2}(x_1 - x_2)$.

$g_2(x_1, x_2)$

$y_1 = \frac{1}{2}(x_1 - x_2)$, $y_2 = x_2$

$\therefore \quad y_1 = \frac{1}{2}(x_1 - y_2)$

$2y_1 = x_1 - y_2 \Rightarrow x_1 = 2y_1 + y_2$

$x_2 = y_2$.

$x_2 > 0$.

$\Rightarrow y_2 > 0$.

$x_1 > 0 \Rightarrow 2y_1 + y_2 > 0$

$\Rightarrow y_2 > -2y_1$

$$J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

$|J^{-1}| = 2$

$$f_{Y_1, Y_2}(y_1, y_2) = 2 f_{X_1, X_2}(2y_1 + y_2, y_2)$$

$$= 2 \times \frac{1}{\alpha^2} e^{-\frac{2(y_1 + y_2)}{\alpha}}$$

$-\infty < y_1 < \infty$

$-2y_1 < y_2$ if $y_1 < 0$

$0 < y_2 < \infty$ if $y_1 \geq 0$

$y_1 > 0$,

$$f_{Y_1}(y) = \int_0^\infty \frac{2}{\alpha^2} e^{-\frac{2(y_1 + y_2)}{\alpha}} dy_2.$$

$$= \frac{2}{\alpha^2} \int_0^\infty e^{-\frac{2(y_1 + y_2)}{\alpha}} dy_2$$

$$= \frac{1}{\alpha} e^{-2y_1/\alpha}$$

$y_1 < 0$

$$\int_{-2y_1}^{\infty} \frac{2}{\alpha^2} e^{-\frac{2(y_1+y_2)}{\alpha}} \, dy_2 = \frac{2}{\alpha^2} \int_{-2y_1}^{\infty} e^{-\frac{2(y_1+y_2)}{\alpha}} \, dy_2$$

$$= \frac{2}{\alpha^2} \left. \frac{e^{-\frac{2(y_1+y_2)}{\alpha}}}{-2/\alpha} \right|_{-2y_1}^{\infty}$$

$$= \frac{-1}{\alpha} \left. e^{-\frac{2(y_1+y_2)}{\alpha}} \right|_{-2y_1}^{\infty} = -\frac{1}{\alpha}\left(0 - e^{-\frac{2(y_1-2y_1)}{\alpha}}\right)$$

$$= \frac{1}{\alpha} e^{+2y_1/\alpha} = \frac{1}{\alpha} e^{2y_1/\alpha}$$

$$\therefore f_{Y_1}(y) = \frac{1}{\alpha} e^{-\frac{2|y|}{\alpha}} \quad \text{where } -\infty < y < \infty$$

<u>Ex 5</u>: let $X_1, X_2, \ldots, X_n$ be iid exponentially distributed r.v's with parameter/rate λ

let $Y_i = \sum_{j=1}^{i} X_j \quad , i = 1, 2, \ldots, n$

a) Compute joint density Y_i's.

b) Compute density Y_n using result of (a)

c) Compute conditional density of $Y_1, Y_2, \ldots, Y_{n-1}$ given $Y_n = t$

$\rightarrow$ $\cancel{Y_i = \sum_{j=1}^{i} \lambda e^{-\lambda x_j}}$

$\therefore Y_1 = X_1, \; y_2 = Y_2 = X_1 + X_2, \; y_3 = Y_3 = X_1 + X_2 + X_3, \ldots$

$x_1 = Y_1, \; X_2 = Y_2 - Y_1, \; X_3 = Y_3 - Y_2, \ldots$

$$0 < y_1 < y_2 < \cdots < y_n < \infty$$

$$J = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 1 & 1 & \cdots & \cdots & 1 \end{vmatrix} = 1.$$

$|J|^{-1} = 1$

$f_{Y_1, Y_2, \ldots, Y_n}(y_1, \ldots, y_n)$

$= f_{X_1, X_2, \ldots, X_n}(Y_1, Y_2 - Y_1, Y_3 - Y_2, \ldots, Y_n - Y_{n-1})$

$\left[f_{X_1, X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = \lambda^n e^{-\lambda \sum x_i} \right]$

$= \lambda^n e^{-\lambda y_n}$

(b)

$$f_{Y_2, \ldots, Y_n}(y_2, y_3, \ldots, y_n) = \int_0^{y_2} f_{Y_1, Y_2, \ldots, Y_n}(y_1, y_2, \ldots, y_n)\, dy_1$$

$$= \lambda^n y_2 e^{-\lambda y_n}$$

$$f_{Y_3, \ldots, Y_n}(y_3, \ldots, y_n) = \int_0^{y_3} f_{Y_1, Y_2, \ldots, Y_n}(y_1, \ldots, y_n)\, dy_2$$

$$= \int_0^{y_3} \lambda^n y_2 e^{-\lambda y_n}\, dy_2$$

$$= \frac{\lambda^n y_3^2}{2!} e^{-\lambda y_n}$$

$\cancel{2 \lambda^n \frac{e^{-\lambda y_n}}{-y_n}\Big|_0^{y_3}} \quad \cancel{-\lambda^n \left(\frac{e^{-\lambda y_3}}{y_3} \right)}$

$\vdots$

$$f_{Y_n}(y_n) = \int_0^{y_n} \lambda^n e^{-\lambda y_n} = \lambda^n \frac{y_n^{n-1}}{(n-1)!} e^{-\lambda y_n}$$

$$f_{U|V}(u|v) = \lim_{\substack{du \to 0 \\ dv \to 0}} \frac{f_{U,V}(u,v)\,du\,dv}{f_V(v)\,dv} = \frac{f_{U,V}(u,v)}{f_V(v)}$$

c)
$$(Y_1, Y_2, \ldots, Y_{n-1} \mid Y_n) = \frac{f_{Y_1, Y_2, \ldots, Y_{n-1}, Y_n}(y_1, y_2, \ldots, y_{n-1}, y_n)}{f_{Y_n}(y_n)}$$

$$= \frac{\lambda^n e^{-\lambda y_n}}{\lambda^n \frac{y_n^{n-1}}{(n-1)!} e^{-\lambda y_n}} = \frac{(n-1)!}{y_n^{n-1}}$$

$$= \frac{(n-1)!}{t^{n-1}}$$

$(\because y_n = t)$

Covariance.

The covariance b/w X and Y denoted by COV(X, Y) is defined by

$$COV(X,Y) = E\left([X - E(X)][Y - E(Y)]\right)$$

$$= E\left(XY - XE(Y) - YE(X) + E(X)E(Y)\right)$$

$$= \cancel{E(XY) + E(X)}$$

$$= \cancel{E[XY] + E[X] \cdot E[Y]}$$

$$= E[XY] - 2E(X) \cdot E(Y) + E(X)E(Y)$$

$$= E[XY] - E(X) \cdot E(Y).$$

If X and Y are independent

Then $E[X] \cdot E[Y] = E[XY]$

Hence $COV(X,Y) = 0$.

But converse is not True.

$$P\{X=0\} = P\{X=1\} = P\{X=-1\} = \frac{1}{3}$$

$$Y = \begin{cases} 0 & X \neq 0 \\ 1 & X = 0 \end{cases}$$

$E[X] = 0$

$XY = 0$

$E[XY] = 0$

$Cov(X, Y)$

$= 0 - 0 = 0.$

X and Y are independent ? No.

$$P\{X \in A\} P\{Y \in B\} = P\{X \in A, Y \in B\}.$$

$A = \{0\}$, $B = \{0\}$

$P\{X \in A, Y \in B\} = 0.$

$P\{X \in A\} = \frac{1}{3}$ $\quad P\{Y \in B\} = \frac{2}{3}$

$\therefore P\{X \in A\} P\{Y \in B\} \neq P\{X \in A, Y \in B\}$

$\therefore$ Not independent.

P1. $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

P2 $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

P3 $\mathrm{Cov}(aX,Y) = a\,\mathrm{Cov}(X,Y)$

P4 $\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$.

Proof:

P4: $\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right)$

$$= \cancel{E\left[\left(\sum_{i=1}^{n} x_i \cdot \sum_{j=1}^{m} y_j\right)\right]}$$

$$= E\left[\left(\sum_{i=1}^{n} X_i - E\left(\sum X_i\right)\right)\left(\sum_{j=1}^{m} Y_j - E\left(\sum Y_j\right)\right)\right]$$

$$= E\left[\left(\sum X_i - \sum E(X_i)\right)\left(\sum Y_j - \sum E(Y_j)\right)\right]$$

$$= E\left[\left(\sum (X_i - E(X_i))\right)\left(\sum (Y_j - E(Y_j))\right)\right]$$

$$= E\left(\sum\sum (X_i - E(X_i))(Y_j - E(Y_j))\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(x_i, Y_j)$$

P5. $\mathrm{Var}\left(\sum X_i\right) = \sum \mathrm{Var}(X_i) + 2\sum\sum_{i<j} \mathrm{Cov}(X_i, X_j)$

If X_i and X_j are independent then

$\mathrm{Var}\left(\sum X_i\right) = \sum \mathrm{Var}(X_i)$

Proof:

$$var(\sum X_i) = Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right) \text{ [by P2]}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, X_j) \text{ [by P4]}$$

$$= \sum_{i=1}^{n} var(X_i) + \sum\sum_{i \neq j} cov(X_i, X_j) \text{ [by P2]}$$

$$= \sum_{i=1}^{n} var(X_i) + 2\sum\sum_{i<j} cov(X_i, X_j) \text{ [By P1]}$$

Variance of Binomial Distribution.

$$Y = \sum X_i$$

$Y \sim Bin(n,p)$, $X_i \sim Ber(p)$

X_i's are independent.

$$var(Y)$$
$$= var\left(\sum X_i\right)$$
$$= \sum_{i=1}^{n} var(X_i) = \sum_{i=1}^{n} p(1-p)$$
$$= np(1-p).$$

Correlation b/w two r.v's X & Y

denoted by $\rho(X,Y)$ is defined as long as $var(X)\, var(Y)$ is positive.

$$var(X) \cdot var(Y) \neq 0.$$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}$$

Proof: $-1 \le \rho(X,Y) \le 1$

$$\operatorname{Var}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) \ge 0. \text{ where}$$

$$\sigma_X^2 = \operatorname{Var}(X)$$
$$\sigma_Y^2 = \operatorname{Var}(Y)$$

$$\Rightarrow \frac{1}{\sigma_X^2}\operatorname{Var}(X) + \frac{1}{\sigma_Y^2}\operatorname{Var}(Y) + 2\frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y} \ge 0$$

$$\Rightarrow \frac{1}{\sigma_X^2}\operatorname{Var}(X) + \frac{1}{\sigma_Y^2}\operatorname{Var}(Y) + 2\rho(X,Y) \ge 0.$$

$$\Rightarrow \rho(X,Y) \ge -1 \quad \text{— (i)}$$

Now,

$$\operatorname{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) \ge 0.$$

$$\Rightarrow \frac{1}{\sigma_X^2}\operatorname{Var}(X) + \frac{1}{\sigma_Y^2}\operatorname{Var}(Y) - 2\frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y} \ge 0.$$

$$\Rightarrow 2\frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y} \le 2$$

$$\Rightarrow \rho(X,Y) \le 1 \quad \text{— (ii)}$$

From (i) & (ii)

$$-1 \le \rho(X,Y) \le 1 \quad \text{(Proved).}$$

Ex-1 let $X_1, X_2, \ldots, X_n$ be iid r.v s.t.

$E[X_i] = \mu$ and $Var(X_i) = \sigma^2$

let $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ denote the sample mean.

The random variable

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1} \text{ is called}$$

sample variance.

Compute a) $Var(\bar{X})$. (b) $E[S^2]$

a) $$Var(\bar{X}) = Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n^2}\left(Var \sum_{i=1}^{n} x_i\right)$$

$$= \frac{\sum Var(X_i)}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

b) $$(n-1)S^2 = \sum (X_i - \bar{X})^2$$

$$= \sum (X_i - \mu + \mu - \bar{X})^2$$