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Department of Computer Science

MSc Big Data Analytics and MSc Computer Science : Batch 2023-25
DA109: Linear Algebra and Matrix Computation
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Problem set: 1

Please try to solve all the problems [1] alone. If you are stuck, I encourage you first consult with your batch mates and then TA. If you still need help then come to Instructor.

1. Answer the followings :

   i) By which property of a vector space V you will explain that $0 + 0 = 0$ where 0 is the zero vector of V.

   ii) Prove that :
   a) $0.x = 0$, where $x, 0 \in V$ and 0 is the zero vector.
   b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$.

2. What is the Parallelogram Law of Vector Addition? Consider the vector space $\mathbb{R}^2$, add the vectors $(3, 1)$ and $(2, 1)$ in $\mathbb{R}^2$ by this law and explain it by a picture.

3. Prove that the set of all $m \times n$ matrices over a field $\mathbb{F}$ denoted by $M_{m\times n}(\mathbb{F})$ is a vector space under usual matrix addition and matrix multiplication.

4. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and $(b_1,\, b_2)$ are elements of V and c is an element of F. Define, $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$. Is V is a vector space under this operations? Justify your answer.

5. Consider the set of all polynomials with coefficients from a field $\mathbb{F}$ with usual polynomial addition and multiplication denoted by $P(\mathbb{F})$. Is $P(\mathbb{F})$ a vector space over $\mathbb{F}$? Justify your answer.

6. Define the Subspace of a Vector Space. Consider $\mathbb{R}^2$, prove that $W = \{(a, a) : a \in \mathbb{R}^2\}$ is a subspace of $\mathbb{R}^2$. Draw the subspace in $\mathbb{R}^2$. What are the basis and dimension of W?

7. Which of the following sets are the subspaces of $\mathbb{R}^3$?
   (a) $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) : \lambda, \mu \in \mathbb{R}^2\}$
   (b) $B = \{(\lambda^2, -\lambda^3, 0) : \lambda \in \mathbb{R}\}$
   (c) Let $\gamma \neq 0$ be in $\mathbb{R}$. $C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$
   (d) $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_2 \in \mathbb{Z}\}$

8. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

9. The set of all differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ where $\mathbb{R}^{(0,3)}$ is the set of real-valued functions on the interval (0, 3) if and only if b = 0.

10. Let V be a vector space and $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

11. Answer the following:

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[1] All the problems have been selected by the TA.

(a) What are the standard bases and dimensions for the following vector spaces: $\{0\}$, $\mathbb{R}^n$, $P_n(\mathbb{R})$ (set of polynomials of degree $<= n$), $M_{m\times n}(\mathbb{R})$ (set of matrices of of order $m \times n$).

(b) How do you relate the bases of $M_{m\times n}(\mathbb{R})$ and $\mathbb{R}^k$ for any k?

12. Find the bases and dimensions of the following subspaces of $\mathbb{R}^n$:
    (a) $W_1 = \{(a_1, a_2, ..., a_n) \in \mathbb{R}^n : a_1 + a_2 - a_3 = 0\}$
    (b) $W_2 = \{(a_1, a_2, ..., a_n) \in \mathbb{R}^n : a_{11} + 2a_{21} = 0, a_{27} = a_4 = a_9\}$
    (c) $W_3 = \{(a_1, a_2, ..., a_n) \in \mathbb{R}^n : m$ constraints are given, $m \le n\}$. What happens if we assume $m \ge n$?

13. Let $S = \{x_1, x_2, ..., x_n\}$ be a linearly independent subset of a vector space V over the field $\mathbb{Z}_2$. How many elements are there in $span(S)$? Justify your answer.

14. If V and W are vector spaces over $\mathbb{F}$ for which $|V| = |W|$ then does it mean that $dim(V) = dim(W)$?

15. Choose $x = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including x itself, span a subspace S. Find specific vectors x so that the dimension of S is: (a) zero, (b) one, (c) three, (d) four.

16. Let V be a real vector space of all polynomial functions from $\mathbb{R}$ into $\mathbb{R}$ of degree 2 or less. Let t be a fixed real number and define $g_1(x) = 1$, $g_2(x) = x + t$, $g_3(x) = (x + t)^2$. Prove that $B = \{g_1, g_2, g_3\}$ is a basis for V. If $f(x) = c_0 + c_1x + c_2x^2$ then what are the coordinates of f in this ordered basis B?

17. Let $v = (x_1, x_2)$ and $w = (y_1, y_2)$ be vectors in $\mathbb{R}^2$ such that $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1$ and $x_1y_1 + x_2y_2 = 0$. Prove that $B = \{v, w\}$ is a basis for $\mathbb{R}^2$. Find the coordinates of the vector (a, b) in the basis B . Can you interpret the conditions geometrically?

18. Let W_1 and W_2 be two finite-dimensional subspaces of a vector space V. Then prove the following:
    (a) $W_1 + W_2$ is the smallest subspace of V that contains both W_1 and W_2.
    (b) $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$.
    (c) If $V = W_1 + W_2$ then $V = W_1 \oplus W_2$ if and only if $dim(V) = dim(W_1) + dim(W_2)$, where $\oplus$ denote the *direct sum*.

19. Suppose V is a vector space over $\mathbb{R}$. Prove that V cannot be written as the set-theoretic union of a finite number of proper subspaces.

20. Give an example of distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$. Where $R(T)$ and $N(T)$ are the Range of T and Null space of T respectively.

21. Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^4$ such that $U \bigoplus V = \mathbb{F}^4$, where $\oplus$ denote the *direct sum*.

22. Let W_1, W_2, W_3 be three distinct subspaces of $\mathbb{R}^{10}$ with each W_i has dimension 9. If $W = W_1 \bigcap W_2 \bigcap W_3$ then prove that $7 \le dim(W) \le 8$.

23. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.
    (a) Prove that if $dim(V) < dim(W)$, then T cannot be onto.
    (b) Prove that if $dim(V) > dim(W)$, then T cannot be one-to-one.

24. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ and $S : \mathbb{R}^4 \to \mathbb{R}^3$ be two linear transformations such that $S \circ T = I$ on $\mathbb{R}^3$. Then show that $T \circ S$ is neither one-one nor onto.

25. T and S be two linear operators on $\mathbb{R}^n$ such that $ST = TS = 0$ and $T + S$ is invertible. Then show that:
    (a) $Rank(T) + Rank(S) = n$
    (b) $Nullity(T) + Nullity(S) = n$

[Hint : Use $Rank(T + S) \leq Rank(T) + Rank(S) \leq Rank(TS) + n$]

26. A linear transformation T rotates each vector in $\mathbb{R}^2$ clockwise through an angle θ. Find the matrix of T w.r.t the standard ordered basis.

27. Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ such that $Tx = 0$ iff $x = 0$. Find $Rank(T)$.

28. Let V be a vector space and $T : V \to V$ be a linear transformation. Prove that the following two statements about T are equivalent:

    (a) The intersection of the range of T and the null space of T is the zero subspace of V.

    (b) If $T(Ta) = 0$, then $Ta = 0$.

29. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = O$ but $UT \neq O$.

30. If W is a k-dimensional subspace of an n-dimensional vector space V, then prove that W is the intersection of n - k hyperspaces in V.

31. Let V be a finite-dimensional vector space and $T : V \to V$ be linear. Then if $V = R(T) + N(T)$ prove that $V = R(T) \bigoplus N(T)$, where $\bigoplus$ denotes the *direct sum*.

32. A function $T : V \to W$ between vector spaces V and W is called additive if $T(x + y) = T(x) + T(y)$ for all $x, y \in V$. Prove that if V and W are vector spaces over the field of rational numbers, then any additive function from V into W is a linear transformation.

33. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

34. Show that $W = \{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : Nullity(T) > 2\}$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

35. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap N(T) = \{0\}$ and $R(T) = \{Tu : u \in U\}$.

36. Suppose U, V, and W are arbitrary subspaces of a finite-dimensional vector space. Then prove that:

    (a) $U \cap (V + W) \supset (U \cap V) + (U \cap W)$.

    (b) $(U \cap V) + W \subset (U + W) \cap (V + W)$.