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MSc Big Data Analytics : Batch 2023-25
DA109: Linear Algebra and Matrix Computation
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Problem set: 2

Please try to solve all the problems [1] alone. If you are stuck, I encourage you first consult with your batch mates and then TA. If you still need help then come to Instructor.

1. Prove that any two finite dimensional vector space are isomorphic if and only if they have equal dimension.
2. Consider the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (3x, 2y)$. Let $S = \{(x, y) : x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$.
   (a) Find $T(S)$. What is the geometric interpretation of it.
   (b) Find $T^{-1}(S)$. What is the geometric interpretation of it.
3. A linear transformation T rotates each vector in $\mathbb{R}^2$ clockwise through an angle $\theta = \frac{\pi}{2}$.
   (a) Find the matrix of T w.r.t the standard ordered basis.
   (b) Compute the matrix of T^2 and T^4 w.r.t the standard ordered basis.
   (c) How T^2 and T^4 changes the vectors of $\mathbb{R}^2$. Also give the geometric interpretation of these two.
4. Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a linear transformation such that $T(A) = 0$ whenever A is symmetric or skew symmetric. Find $Rank(T)$.
5. Let $T : V \to W$ be a linear transformation. Then prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of w.
6. Let T be the linear operator on $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$. Is T invertible? If so, find a rule for T^{-1} like the one which defines T.
7. Prove the followings:
   (a) The subspaces $\{0\}$, V, R(T), and N(T) are all T-invariant.
   (b) If W is T-invariant, prove that T_W is linear.
8. Is $\mathbb{R}^2 \times \mathbb{R}^3 = \mathbb{R}^5$? Is $\mathbb{R}^2 \times \mathbb{R}^3 \simeq \mathbb{R}^5$?
9. In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where
   (a) T is the reflection of $\mathbb{R}^2$ about L.
   (b) T is the projection on L along the line perpendicular to L.
10. Let V and W be finite-dimensional vector spaces over the field $\mathbb{R}$. Then V is isomorphic to W if and only if $dim(V) = dim(W)$.
11. Let V be a vector space. Determine all linear transformations $T : V \to V$ such that $T = T^2$.

[1] All the problems have been selected by the TA.

12. Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let $T: V \to W$ be linear. Show that for each $u \in V$, we have $[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$.

13. Let $T: \mathbf{P}_3(\mathbb{R}) \to \mathbf{P}_2(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f'(x)$, and let β and γ be the standard ordered bases for $\mathbf{P}_3(\mathbb{R})$ and $\mathbf{P}_2(\mathbb{R})$, respectively. If $A = [T]_\beta^\gamma$ where,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Verify that $[T(p(x))]_\gamma = [T]_\beta^\gamma [p(x)]_\beta$.

14. Let V be a finite-dimensional vector space and $T: V \to V$ be linear. Then if $V = R(T) + N(T)$ prove that $V = R(T) \bigoplus N(T)$.

15. A function $T: V \to W$ between vector spaces V and W is called **additive** if $T(x+y) = T(x) + T(y)$ for all $x, y \in V$. Prove that if V and W are vector spaces over the field of rational numbers, then any additive function from V into W is a linear transformation.

16. Let V be a vector space and W be a subspace of V. Define the mapping $\eta: V \to V/W$ by $\eta(v) = v + W$ for $v \in V$.
    (a) Prove that η is a linear. Find $N(\eta)$.
    (b) Suppose that V is finite-dimensional. Then find the relation between $dim(V)$, $dim(W)$, and $dim(V/W)$.

17. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

18. Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β, and γ, respectively. Let $T: V \to W$ and $U: W \to Z$ be linear transformations. Then prove that $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.

19. Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let $T: V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

20. Let $g(x) = 3 + x$. Let $T: \mathbf{P}_2(\mathbb{R}) \to \mathbf{P}_2(\mathbb{R})$ and $U: \mathbf{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by $T(f(x)) = f'(x)g(x) + 2f(x)$ and $U(a + bx + cx^2) = (a + b, c, a - b)$. Let β and γ be the standard ordered bases of $\mathbf{P}_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively. Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$ directly. Also verify that $[U]_\beta^\gamma [T]_\beta = [UT]_\beta^\gamma$.

21. Let T be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W. Let β and β' be ordered bases for V, and γ and γ' be ordered bases for W. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates i.e. $Q = [I]_{\beta'}^\beta$ and P changes γ'-coordinates into γ-coordinates i.e. $P = [I]_{\gamma'}^\gamma$. Then show that $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q$.

22. Let T be the linear operator on $\mathbb{R}^2$ defined by $T(a, b) = (3a - b, a + 3b)$ and let $\beta = \{(1, 1), (1, -1)\}$ and $\beta' = \{(2, 4), (3, 1)\}$ be the ordered bases. Find $[T]_\beta$, $[T]_{\beta'}$, and coordinate matrix Q. Verify that $[T]_{\beta'} = Q^{-1}[T]_\beta Q$.

23. Let T be the linear operator on $\mathbf{P}_1(\mathbb{R})$ defined by $T(p(x)) = p'(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1 + x, 1 - x\}$. Calculate $[T]_{\beta'}$ by only using $[T]_\beta$ and coordinate matrix Q.

24. Let $T: V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping $\bar{T}: V/N(T) \to Z$ by $\bar{T}(v + N(T)) = T(v)$ for any coset $v + N(T)$ in $V/N(T)$. Then prove that $\bar{T}$ is well-defined, linear and an isomorphism.

25. Show that $W = \{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : Nullity(T) > 2\}$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

26. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap N(T) = \{0\}$ and $R(T) = \{Tu : u \in U\}$.

27. Suppose V is finite dimensional and $v_1, v_2, ..., v_n \in V$. Define a linear map $\Psi : V' \to \mathbb{R}^n$ by $\Psi(\phi) = (\phi(v_1), ..., \phi(v_n))$. Prove that:

    (a) $v_1, v_2, ..., v_n$ spans V if and only if Ψ is injective.

    (b) $\{v_1, v_2, ..., v_n\}$ is linearly independent if and only if Ψ is surjective.

28. For any positive integer m :

    (a) Prove that $\{1, (x-5), ..., (x-5)^m\}$ is a basis for $\mathcal{P}_m(\mathbb{R})$

    (b) What is the dual basis of the basis given in (a)?

29. Prove that two affine subspaces are equal or disjoint.

30. Let $V = \mathbb{R}^3$, and define $f_1, f_2, f_3 \in V^*$ as follows: $f_1(x, y, z) = x - 2y$, $f_2(x, y, z) = x + y + z$, $f_3(x, y, z) = y - 3z$. Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual basis.

31. Let V and W be finite-dimensional vector spaces over $\mathbb{F}$ with ordered bases β and γ, respectively. Let β^* and γ^* be the dual bases for β and γ respectively. Show that for any linear transformation $T : V \to W$, the mapping $T^t : W^* \to V^*$ defined by $T^t(g) = gT$ for all $g \in W^*$ is a linear transformation with the property that $[T^t]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$.

32. Let $\mathbb{F}$ be a field and let f be the linear functional on $\mathbb{F}^2$ defined by $f(x_1, x_2) = ax_1 + bx_2$. For each of the following linear operators T, let $g = T^t(f)$, and find $g(x_1, x_2)$.

    (a) $T(x_1, x_2) = (x_1, 0)$

    (b) $T(x_1, x_2) = (-x_2, x_1)$

    (c) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$

33. In $\mathbb{R}^3$, let $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (0, 1, -2)$, $\alpha_3 = (-1, -1, 0)$.

    (a) If f is a linear functional on $\mathbb{R}^3$ such that $f(\alpha_1) = 1$, $f(\alpha_2) = -1$, $f(\alpha_3) = 3$, and if $\alpha = (a, b, c)$, find $f(\alpha)$.

    (b) Describe explicitly a linear functional f on $\mathbb{R}^3$ such that $f(\alpha_1) = 0$, $f(\alpha_2) = 0$ but $f(\alpha_3) \neq 0$.

34. Let $\beta = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $\mathbb{R}^3$ defined by $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (2, 2, 0)$. Find the dual basis of β.