Problem Set 1. (Rajdeep Da)

1.

i) The property that explains the equation $0+0=0$ in a vector space V is the additive identity property.

There exists an element $0 \in V$ such that $0+v = v \quad \forall\, v \in V$.

ii)

(b) $(a+(-a))x = ax + (-a)x$

$\Rightarrow 0.x = ax + (-a)x$

$\Rightarrow -(ax) = (-a)x$ —— (i)

$\because$ $(-a)x$ is the additive inverse of ax

$a(x+(-x)) = ax + a(-x)$

$\Rightarrow 0 = ax + a(-x)$.

$a(-x)$ is the additive inverses of ax

$\therefore -(ax) = a(-x)$ —— (ii)

$\therefore$ From (i) and (ii)

$(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$

a) $0.(x+0) = 0.x + 0.x \quad \forall\, 0, x \in V$

Thus

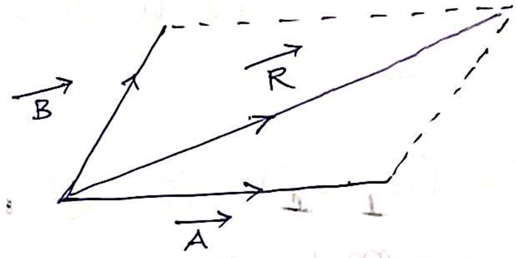
$0 = 0+0$

This implies that $0.x = 0$ for any vector x in the space V where 0 is a zero vector

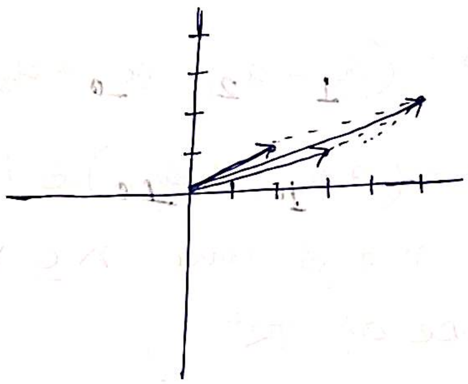
2) Parallelogram law of vector addition.

If two vector are consider to be the adjacent sides of a parallelogram, then the resultant of the two vector is given by the vector that is diagonal passing through the point of contact of two vectors.

$\vec{R}$ is the resultant vector of $\vec{A} + \vec{B}$



(3,1) (2,1)



4. $(1,2) + (x,y) = (1,2)$
$(1+x, 2y) = (1,2)$
$x=0, y=1$

$(1,2) + (x,y) = (0,0)$
$(x=-1)$
$(y=-2)$

In any vector space $1.\alpha = \alpha \ \forall \alpha \in V$.

$((a_1, a_2) + (b_1, b_2)) + (c_1, c_2)$
$= (a_1+b_1, a_2 b_2) + (c_1, c_2)$
$= (a_1 + b_1 + c_1, a_2 b_2 c_2)$
$(a_1, a_2) + (b_1 + c_1, b_2 c_2)$
$(a_1 + b_1 + c_1, a_2 b_2 c_2)$

Not same

$\therefore$ Not a vector space.

$(c+d)(a_1, a_2)$
$= ((c+d)a_1, (c+d)a_2)$
$= (ca_1 + da_1, ca_2 + da_2)$

$c(a_1, a_2) + d(a_1 + a_2)$
$(ca_1, ca_2) + (da_1, da_2)$
$(ca_1 + da_1, ca_2 da_2)$

6. <u>Subspace</u>:

Let V be a vector space. We say that W is a subspace of V if $W \subseteq V$ and if the following condition are satisfied.

i) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$

ii) If c is a scaler and $w \in W$ then $cw \in W$

$\alpha = (a_1, a_1)$, $\beta = (a_2, a_2) \in W$

and $a \in \mathbb{R}$

then $\alpha + \beta = (a_1 + a_2, a_1 + a_2) \in W$.

$a(a_1, a_1) = (aa_1, aa_1) \in W$

also clearly $W \neq \phi$ and $W \subseteq V_2(R)$. Hence W is a subspace of $\mathbb{R}^2$.

$a(1,1)$ for any $a \in \mathbb{R}$ where $(1,1) \in W$

~~Since (1,1) and (0,0) is linearly independent it follows that~~ $B = \{(1,1)\}$ is a basis for W, i.e the dimension of W is 1.

The subspace W consist of all vectors (a,a) which means it lies along the diagonal line that passes through the origin and has slope $= 1$.

7.

a) $(0,0,0) \in A$

$\therefore$ Non-empty

let $w_1 = (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3)$

and $w_2 = (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3)$

$$\therefore w_1 + w_2 = (\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \mu_1^3 + \mu_2^3, \lambda_1 + \lambda_2 - (\mu_1^3 + \mu_2^3))$$

let c be a scaler.

$$cw_0 = (c\lambda_0, c(\lambda_0 + \mu_0^3), c(\lambda_0 - \mu_0^3))$$

$\therefore$ Closed under scalar multiplication.

b) $(0,0,0) \in A$

$w_1 = (\lambda_1^2, -\lambda_1^3, 0)$

$w_2 = (\lambda_2^2, -\lambda_2^3, 0)$

$w_1 + w_2 = (\lambda_1^2 + \lambda_2^2, -(\lambda_1^3 + \lambda_2^3), 0)$

$cw = (c\lambda^2, -c\lambda^3, 0)$

8. $x = \frac{1}{3}(w + (-v))$

$$v + 3x = v + 3\left(\frac{1}{3}(w + (-v))\right) = v + w + (-v) = (v + (-v)) + w = 0 + w = w$$

This completes our prove that $x \in V$ exists and satisfies $v + 3x = w$.

let $\exists\, y \in V$ satisfy $v + 3y = w$.

$0 = w - w$

$= (v + 3x) - (v + 3y)$

$= v - v + 3x - 3y = 3(x-y)$

either $3 = 0$ or $x - y = 0$.

↳ false

$x - y = 0$ or $x = y$ which means $x \in V$ satisfying $v + 3x = w$ is unique.

11.

a. The vector space containing only the zero vector has no non-trivial basis since there is no linearly independent vectors. So basis is empty.

Dimension $= 0$

$e_1 = (1, 0, 0 \ldots 0)$ $e_2 = (0, 1, 0, \ldots 0), \ldots$

$e_n = (0, 0, \ldots, 0, 1)$

Dim $= n$

$a_1 = -(a_3) + (-a_5)$

$= -(a_3 + a_5)$

Polynomial:

$1, x, x^2, \ldots, x^n$

Dim $= n+1$ $a_n \neq 0$

$W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^n,\ a_1 + a_2 = a_3,\ a_1 = a_2,$

$a_2 = a_3,\ \cancel{a_2 + a_3 = a_1}\}$

$a_1 = a_3$

$a_1, a_2, a_1 + a_2$

13. $\beta = \sum_{i=1}^{n} a_i u_i$ Z_2 means modulo 2
element are $\{0, 1\}$

Now you should have 2 choice for a_i namely 0 or 1. Since there are n element with 2 choices, you have 2^n elements.

16. $$x^2 = \{(x+t) - t\}^2$$

$$= (x+t)^2 - 2 \cdot t(x+t) + t^2$$

$$\therefore f(x) = c_0 + c_1 x + c_2 (x+t)^2 - 2c_2 t (x+t) + c_2 t^2$$

$$= c_2 (x+t)^2 + (c_1 - 2c_2 t)(x+t) + (c_0 - tc_1 + c_2 t^2)$$

$$\cancel{c_2 (x+t)^2 + c_1 x + c_1 t - 2c_2 x t - 2c_2 t^2} + c_0$$

The co-ordinates of f relative to B is

$$(c_2, c_1 - 2c_2 t, c_0 - tc_1 + c_2 t^2)$$

18) (a) Here $0 = 0 + 0 \in W_1 + W_2$. So it is non-empty.
let $\alpha, \in F$,

Now, $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$

$$\therefore \alpha w_1 + w_1' \in W_1$$

and $\alpha w_2 + w_2' \in W_2$

$$\alpha (w_1 + w_2) + (w_1' + w_2') = (\alpha w_1 + w_1') + (\alpha w_2 + w_2') \in W_1 + W_2$$

We need to show that if U is any subspace of V such that

$$W_1 \subseteq U \text{ and } W_2 \subseteq U$$

then

$$W_1 + W_2 \subseteq U$$

let $w_1 + w_2 \in W_1 + W_2$ where $w_1 \in W$ and $w_2 \in W$

Since $W_1 \subseteq U$ then must $w_1 \in U$.

Similarly $W_2 \subseteq U$ then must $w_2 \in U$.

and since U is a subspace we must also have that $w_1 + w_2 \in U$. Since our choice $w_1 + w_2 \in W_1 + W_2$ is arbitary we conclude $W_1 + W_2 \subseteq U$

(b) let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$. thus $\dim(U_1 \cap U_2) = m$. Because $u_1, \ldots, u_m$ is a basis of $U_1 \cap U_2$. It is linearly independent in U_1. Hence this list can be extended to a basis $u_1, \ldots, u_m, v_1, \ldots, v_j$ of U_1. $\dim(U_1) = m+j$.

Also it is linearly independent in U_2. Hence Also extend $u_1, \ldots, u_m, w_1, \ldots, w_K$ of U_2. Then $\dim(U_2) = m+K$

∴ We will now show that $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_K$ is a basis of $U_1 + U_2$.

$$\dim(U_1 + U_2) = m + j + K$$

$$= (m+j) + (m+K) - m$$

$$= \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$$

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_K w_K = 0. \qquad (*)$$

$c_1 w_1 + \cdots + c_K w_K = -a_1 u_1 - \cdots - a_m u_m - b_1 v_1 - \cdots - b_j v_j$

$\therefore c_1 w_1 + \cdots + c_K w_K \in U_1$

All w's are in U_2. so,

$c_1 w_1 + \cdots + c_K w_K \in U_1 \cap U_2$

Because $u_1, \ldots u_m$ is a basis of $U_1 \cap U_2$

$\therefore c_1 w_1 + \cdots + c_K w_K = d_1 u_1 + \cdots + d_m u_m$

for some scalars $d_1, \ldots, d_m$.

But $u_1, \ldots, u_m, w_1, \ldots, w_K$ are linearly dependent. so the last equation implies that all c's and d's are equal to 0.

Thus (*) becomes

$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j = 0$.

Because the list $u_1, \ldots, u_m, v_1 \ldots v_j$ are linearly independent, same way we can conclude a's and b's are zero. Now all a's, b's and c's are equal to zero.

20. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x, y) = (0, x)$

and $U: \mathbb{R}^2 \to \mathbb{R}^2$ $U(x,y) = (y,0)$

$N(T)$ = The null space of T consist of all vector such that $T(x,y) = (0,x)$

which means $x = 0$. So $N(T)$ is the set of all vectors of the form $(0,y)$ where y can be any real no.

$R(T)$: Consist of all vector of the form $(0,x)$

Set of all vector lies on x axis.

$N(U)$: $U(x,y) = (y,0)$

$y = 0$. $N(U)$ is the set of all vectors of the form $(x,0)$. where x can be any real no.

$R(U)$: all vector of the form $(y,0)$.
set of all vector lie on y axis.

22. $\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2)$ — dim V

Arbitary intersection of subspace is again a subspace.

$$\max\{0, \dim(W_1)+\dim(W_2) - \dim(V)\} \le \dim(W_1 \cap W_2) \le \min\{\dim(W_1), \dim(W_2)\}$$

↳ common L.I vector

$$\max\{0, 9+9-10\} \le \dim(W_1 \cap W_2) \le \min\{9, 9\}$$

$$8 \le \dim(W_1 \cap W_2) \le 9$$

Possible $\dim(W_1 \cap W_2) = 8$ or 9.
If 9 then basis contain 9 L.I. Or W_1 contain 9 L.I vector as well as W_2. So they are not distinct.

$$W' = \dim(W_1 \cap W_2) = 8.$$

$$W = W' \cap W_3$$

$$\max\{\dim W' + \dim W_3 - \dim V, 0\} \le \dim(W' \cap W_3) \le \min\{\dim W', \dim W_3\}$$

$$\max(8+9-10, 0) \leq \dim(W' \cap W_3) \leq \min\{8, 9\}$$

$$7 \leq \dim(W' \cap W_3) \leq 8$$

$$7 \leq \dim(W) \leq 8$$

23.

a. Suppose $\dim(V) < \dim(W)$ and assume that T is onto. Then $\text{image}(T) = W$.

$\text{Rank}(T) = \dim(W)$.

$$\dim(V) = \text{Rank}(T) + \text{nullity}(T)$$
$$= \dim(W) + \text{nullity}(T)$$

$$\text{nullity}(T) = \dim(V) - \dim(W) < 0$$
$$(\because \dim(V) < \dim(W))$$

but nullity can't be < 0. which is a contradiction.

Then T can't be onto.

b. Suppose $\dim(V) > \dim(W)$ and assume that T is one-one.

Then $\text{null } T = \{0\}$

$$\text{nullity}(T) = \dim(N(T)) = 0$$

$$\dim(V) = \text{Rank}(T) + \text{nullity}(T)$$
$$\dim(V) = \text{Rank}(T)$$
$$\dim(W) < \text{Rank}(T)$$

This is impossible because $R(T)$ is a subspace of W. and therefore always has dimension less than or equal to dimension W. That our contradiction; $\therefore T$ is never injective

18.

C. $V = W_1 + W_2$

$\dim(V) = \dim(W_1) + \dim(W_2)$

$V = W_1 + W_2$

$V = W_1 + W_2$

$\dim(V) = \dim(W_1 + W_2)$

$= \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$= \dim(V) - \dim(W_1 \cap W_2)$

$\therefore \dim(W_1 \cap W_2) = 0$.

$\Rightarrow W_1 \cap W_2 = \{0\}$

$\therefore V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$ then

$V = W_1 \oplus W_2$ ——— (1)

Conversely :

~~dim (V) = dim (W1) + dim (W2)~~

$V = W_1 \oplus W_2$

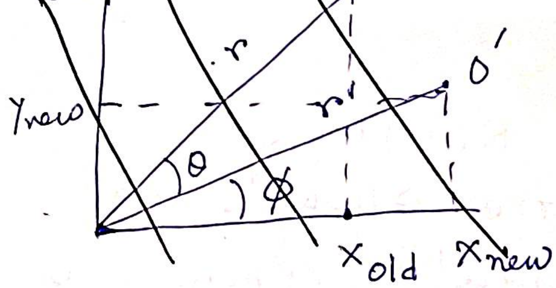
then $W_1 \cup W_2 = V$ & $W_1 \cap W_2 = \{0\}$

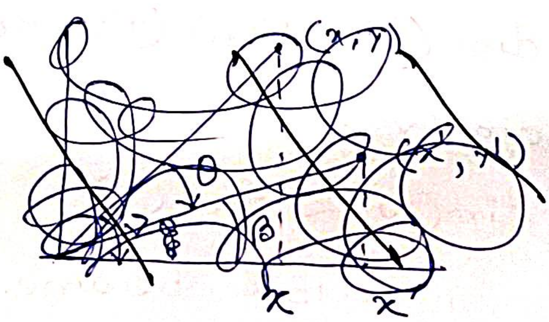
$\dim V = \dim(W_1 + W_2)$

$= \dim(W_1) + \dim(W_2) - \underbrace{\dim(W_1 \cap W_2)}_{=0}$

$= \dim(W_1) + \dim(W_2)$ ——— (ii)

26.





$x_{old} = r\cos(\theta+\phi)$

$y_{old} = r\sin(\theta+\phi)$

$x_{new} = r'\cos\phi$

$y_{new} = r'\sin\phi$

$T: R^4 \to R^5$

$x_{old} =$

27.

$T: R^4 \to R^5$

$T(x,y,z,u) = (0,0,0,0,0)$.

Null T

$N(T) = \{(0,0,0,0)\}$

$\dim(N(T)) = 0$

$\equiv$ Nullity $= 0$.

$\dim(V) = 4$

$\dim(V) = \text{Rank}(T) + \text{nullity}(T)$

$4 = \text{Rank}(T) + 0$

$\therefore \text{Rank}(T) = 4$

28. a) Range $(T) \cap$ Nullspace $(T) = \{0\}$

$T(Ta) = 0$

$Ta \in$ Nullspace (T)

but, Ta already in Range (T)

$\Rightarrow Ta \in$ Range $(T) \cap$ Nullspace (T)

$\Rightarrow Ta = 0$

b) $T(Ta) = 0$

$\Rightarrow Ta = 0$

$x \in$ Null space $(T) \cap$ Range (T)

$x \in$ Range (T) and $x \in$ Nullspace (T)

$\exists\, y \in V$ such that $T(y) = x$

$$T(T(y)) = T(x) = 0 \quad \because x \in \text{Null space}(T)$$

$$\Rightarrow T(T(y)) = 0.$$

$$T(y) = 0$$

$$x = 0.$$

$$\Rightarrow \text{Range}(T) \cap \text{Null}(T) \subseteq \{0\}$$

clearly

$$\{0\} \subseteq \text{Range}(T) \cap \text{Null}(T)$$

$$\therefore \boxed{\text{Range}(T) \cap \text{Null}(T) = \{0\}}$$

29. $T(x,y) = \cancel{(x,0)}\ \cancel{(0,y)}\ (0,0)$

$U(x,y) = (2x, 2y)$

$TU(x,y) = T(2x,2y) = (0,0)$

$UT(x,y) = U(0,0) = (0,0)$

$T\left(n \cdot \frac{1}{n} v\right)$

$nT\left(\frac{1}{n} v\right)$

$T(\alpha v) = \alpha T(v)$ for $\alpha \in \mathbb{Q}$

32.

$T(0) + T(0) = T(0+0) = T(0)$

So $T(0) = 0$.

- $T(n \cdot v) = T(v + \dots + v) = T(v) + \dots + T(v) = n \cdot T(v)$
- $T(-n \cdot v) + T(nv) = T(-nv + nv) = T(0) = 0.$

So $T(-nv) = -T(nv) = -nT(v)$

$$T(v) = T\left(n \cdot \frac{1}{n} \cdot v\right) = n \cdot T\left(\frac{1}{n} v\right)$$

$$\Rightarrow T\left(\frac{1}{n} v\right) = \frac{1}{n} T(v)$$

$$T\left(\frac{m}{n} v\right) = T\left(m \cdot \frac{1}{n} v\right) = m\, T\left(\frac{1}{n} v\right) = \frac{m}{n} T(v)$$

i.e $T(\alpha v) = \alpha T(v)$ for each $\alpha \in \mathbb{Q}$

35.
If V is a finite dimensional and W is a subspace of V then we can find a subspace U of V for which ~~$V = W \oplus U$~~ $V = W \oplus U$

null T is a subspace of V.

setting W = null T.

V = null $T \oplus U$

We want to prove for any subspace U for which V = null $T \oplus U$ satisfy property.

V = null $T \oplus U$, we already have null $T \cap U = \{0\}$

so we just need to show

range $T = \{Tu \mid u \in V\}$

First range $T \subset \{Tu \mid u \in V\}$

let $w \in$ range T. $\therefore \exists$ some $v \in V$ for which $T(v) = w$. Since $v \in V$ and V = null $T \oplus U$ we can find vectors $n \in$ null T and $u \in U$ for which $v = n + u$ $\therefore$ Thus

$$T(v) = T(n) + T(u)$$
$$= 0 + T(u)$$

$w = T(u)$ for some $u \in U$

That means $w \in \{Tu \mid u \in U\}$ $\therefore$ range $T \subset \{Tu \mid u \in U\}$

Now we need to show

$\{Tu \mid u \in U\} \subset \text{range } T$

But for any element $u \in U$, u is also in V as $U \subset V$. Thus Tu is in the image of T by defn. $\therefore \{Tu \mid u \in U\} \subset \text{range } T$

So we have shown that $\text{range } T = \{Tu \mid u \in U\}$

Thus there exists a subspace U of V s.t

$V = \text{null } T \oplus U$ and $\text{range } T = \{Tu \mid u \in U\}$

~~36. If W_1 and W_2 are two subspace of a vector space V then $W_1 \cap W_2$ is the largest subspace of V contained in W_1 and W_2 & $W_1 + W_2$ is the smallest subspace contained in both W_1 and W_2.~~

34. $T : \mathbb{R}^5 \to \mathbb{R}^4$

$T_1(x, y, z, w, t) = (x, 0, 0, 0)$

$T_2(\text{———}) = (0, y, 0, 0)$

$\vdots$

$T_4(x, y, z, w, t) = (0, 0, 0, w)$

$(0, 0, 0, 0, 1) \in \ker(T_1 + T_2 + T_3 + T_4)$

$N(T_1 + T_2 + T_3 + T_4) = 1 < 2$

$(T_1 + T_2 + T_3 + T_4) \notin W$

29. $T(x) = Ax$ $U(x) = Bx$

$$x = \begin{pmatrix} x \\ y \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$TU(x) = T(U(x)) = \cancel{A} ABx$$
$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$UT(x) = U(T(x)) = BAx$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$UT \neq 0$.

NPTEL.

15. V is a vector space over a field F iff for every $x, y \in V$ and $\alpha, \beta \in F$ vector $\alpha x + \beta y$ is the element of V.

$$V = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R} \ \forall i = 1, 2, \ldots n\}$$

$x = (1, 1, \ldots 1)$ $y = (2, 2, \ldots 2)$

$a = i$, $b = 2i$ $5i \notin R$.

$ax + by = (5i, 5i \ldots 5i)$

$(c+d)(a_1 + a_2)$

$c(a_1 + a_2) + d(a_1 + a_2)$

$(ca_1 + da_1, 0)$

$(c+d)(a_1) + (c+d)a_2$

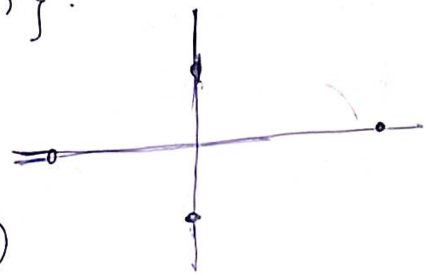
$ca_1 + da_1 + ca_2 + \cancel{a} d\cancel{a}a_2$

Problem 2.

2. $\cancel{S} = \{(1,0), (0,1), (-1,0), (0,-1)\}$.

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (3x, 2y)$



$T(1,0) = (3,0)$ $\quad$ $T(-1,0) = (-3,0)$

$T(0,1) = (0,2)$ $\quad$ $T(0,-1) = (0,-2)$

14. Given that $V = R(T) + N(T)$ — (I)

we have to prove that $R(T) \cap N(T) = \{0\}$.

$$\dim(R(T) \cap N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) + N(T))$$

$$= \underbrace{\dim(R(T)) + \dim(N(T))}_{=\dim(V)} - \dim(V)$$

$= 0.$ (by dimension Thm.) — (II)

from (I) and (II) $V = R(T) \oplus N(T)$

8. $\mathbb{R}^2 \times \mathbb{R}^3$ is not equal to $\mathbb{R}^5$.

The element of $\mathbb{R}^5$ looks like $(x_1, x_2, \ldots, x_5)$ but in $\mathbb{R}^2 \times \mathbb{R}^3$ the elements looks like

$((x_1, x_2), (x_3, x_4, x_5))$

let us define a Linear transformation.

$$T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$$

We have to prove that ~~is il~~ it is both one-one & onto.

$$T((x_1, x_2), (x_3, x_4, x_5)) = T((x_1', x_2'), (x_3', x_4', x_5'))$$

$$\Rightarrow (x_1, x_2, x_3, x_4, x_5) = (x_1', x_2', x_3', x_4', x_5')$$

$$\therefore x_i = x_i' \quad \forall i = 1, 2, \ldots, 5.$$

$$\therefore ((x_1, x_2), (x_3, x_4, x_5)) = ((x_1', x_2'), (x_3', x_4', x_5'))$$

$\therefore$ This transformation is one-one

Now we have to show that it is on-to.

let

$w \equiv (x_1'', x_2'', x_3'', x_4'', x_5'') \in \mathbb{R}^5$.

Now $T((x_1'', x_2''), (x_3'', x_4'', x_5''))$

$$= (x_1'', x_2'', x_3'', x_4'', x_5'').$$

$\therefore \exists ((x_1'', x_2''), (x_3'', x_4'', x_5'')) \in \mathbb{R}^2 \times \mathbb{R}^3$

Such That $T_v = w$ where

$v = ((x_1'', x_2''), (x_3'', x_4'', x_5''))$

$\therefore T$ is on-to.

$\therefore \mathbb{R}^2 \times \mathbb{R}^3 \simeq \mathbb{R}^5$

6. $T(x_1, x_2, x_3)$

$= (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$

Yes T is invertible. Because $T: \mathbb{R}^3 \to \mathbb{R}^3$
They have same dimension.

$F(x, y, z) = (a, b, c)$

$F^{-1}(a, b, c) = (x, y, z)$

$(3x_1, x_1 - x_2, 2x_1 + x_2 + x_3) = (a, b, c)$

$3x_1 = a$

$x_1 - x_2 = b$

$2x_1 + x_2 + x_3 = c$

$3x_1 = a$

$x_1 = a/3$

$\frac{a}{3} - x_2 = b$

$x_2 = \frac{a}{3} - b$

$2\left(\frac{a}{3}\right) + \frac{a}{3} - b + x_3 = c$

$x_3 = -\frac{2a}{3} - \frac{a}{3} + b + c$

$x_3 = -a + b + c$

~~$T^{-1}(a_1, a_2, a_3) =$~~

$T^{-1}(a, b, c) = \left(\frac{a}{3}, \frac{a}{3} - b, -a + b + c\right)$

12.

Fix $u \in V$ and define a transformations $f: F \to V$ by $f(a) = au$ and $g: F \to W$ by $g(a) = aT(u)$ $\forall a \in F$. Let $\alpha = \{1\}$ be the standard ordered ~~basis~~ basis for F. Notice that $g = Tf$.

$$[T(u)]_\gamma = [g(1)]_\gamma = [g]_\alpha^\gamma = [Tf]_\alpha^\gamma$$
$$= [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta = [T]_\beta [u]_\beta$$

13. $p(x) \in P_3(\mathbb{R})$ is a polynomial.

$p(x) =$ ~~00000~~ $9x^3 + 2x^2 + 4x + 6$.

let $q(x) = T(p(x))$. then

$q(x) = p'(x) = 27x^2 + 4x + 4$ Hence

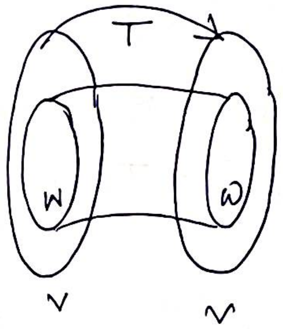
$$[T(p(x))]_\gamma = [q(x)]_\gamma = \begin{pmatrix} 4 \\ 4 \\ 27 \end{pmatrix}$$

but also,

$$[T]_\beta^\gamma [p(x)]_\beta = A [p(x)]_\beta$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 2 \\ 9 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 4 \\ 27 \end{pmatrix}$$

7. <u>T - Invariant.</u>

let $T : V \to V$ be a linear operator



let W be subspace of V then W ~~be~~ is said to be T- invariant Subspace.

$$T(x) \in W \quad \forall x \in W$$

a) $T : V \to V$

(i) $W = \{0\}$ $\quad T0 = 0 \in W$

$\therefore W$ is T invariant subspace.

(ii) $W = \{v\}$

$$Tv = v \in W$$

$\therefore W$ is T invariant

(III) $R(T) = \{Tv : v \in V\}$

$W =$ Range T

$Tw \subseteq$ Range T

let $y \in$ Range T $\quad x \in V$. s.t $T(x) = y$

$T(y) = T(T(x)) \in$ Range T

$T(R(T)) \in$ Range T

$\therefore R(T)$ is T invariant,

(IV) $W = Null(T) = \{x \in V,\ T(x) = 0\}$

let $x \in Null(T) \Rightarrow Tx = 0 \in Null\ T$

$T(Null(T)) \subseteq Null(T)$

$\therefore Null(T)$ is T invariant

b) For $x, y \in W$ we have $x + cy \in W$ since it's a subspace and $T(x)$ and $T(y) \in W$ since ~~T is inv~~ its T-invariant and finally

$$T(x+cy) = T(x) + cT(y).$$

15. From additivity $T(0) + T(0) = T(0+0) = T(0)$

$$\text{so, } T(0) = 0.$$

For any positive integer n.

- $T(nv) = T(\underbrace{v + \cdots + v}_{n \text{ times}}) = \underbrace{T(v) + \cdots + T(v)}_{n \text{ times}} = n \cdot T(v)$

- $T(-n \cdot v) + T(nv) = T(-nv + nv) = T(0) = 0.$

  So $T(-n.v) = -T(nv) = -nT(v)$

Thus $T(Kv) = K \cdot T(v)$ for every integer K.

$$T(v) = T\left(n \cdot \frac{1}{n} v\right) = nT\left(\frac{1}{n} v\right)$$

$$\therefore T\left(\frac{1}{n} v\right) = \frac{1}{n} T(v)$$

$$T\left(\frac{m}{n} \cdot v\right) = \frac{1}{n} T(mv) = \frac{m}{n} T(v)$$

i.e $T(\alpha v) = \alpha T(v)$ for each $\alpha \in \mathbb{Q}$

16. a) $\eta(u+v) = (u+v) + W = (u+W) + (v+W)$
$= \eta(u) + \eta(v)$

and

$$\eta(cv) = cv + W = c(v+W) = c\,\eta(v)$$

$\forall$ element in $v+W$ in V/W we have $\eta(v) = v+W$ and hence its on-to.

Finally if $\eta(v) = u + W = 0 + W$

we have $v - 0 = v \in W$. Hence $N(\eta) = W$

b) Since it's onto we have $R(T) = V/W$

And also we have ~~$\eta(v) = W$~~ $N(\eta) = W$

$$\dim(V) = \dim(N(T)) + \dim(\text{Range}(T))$$

$$\dim(V) = \dim(W) + \dim(V/W)$$

18. let $\alpha = \{v_1, v_2, \ldots, v_n\}$

$\beta = \{w_1, w_2, \ldots, w_m\}$

and $\gamma = \{\varphi_1, \varphi_2, \ldots, \varphi_l\}$

let $[T]_\beta^\gamma = [a_{ij}]$ and $[S]_\alpha^\beta = [b_{pq}]$

$(T \circ S)(v_i) = T(S(v_i))$.

$$= T\left(\sum_{k=1}^{m} b_{ki} w_k\right) = \sum_{k=1}^{m} b_{ki} T(w_k)$$

$$= \sum_{k=1}^{m} b_{ki}\left(\sum_{j=1}^{l} a_{jk} \varphi_j\right) = \sum_{j=1}^{l}\left(\sum a_{jk} b_{ki}\right)\varphi_i$$

This shows,

$$[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta \text{ (Proved)}$$

19. $\beta = \{v_1, v_2, \dots, v_n\}$

$\gamma = \{w_1, w_2, \dots, w_m\}$

Since T is invertible then dim V = dim W and $[T]_\beta^\gamma$ and $[T^{-1}]_\gamma^\beta$ are square matrix of same size.

$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [T \circ T^{-1}]_\gamma = [id]_\gamma$$ is the identity matrix. Hence

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$$

20. β be the standard ordered basis of $P_2(R) = \{1, x, x^2\}$

γ be the standard ordered basis of $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$U : P_2(R) \to R^3$

$U(1) = (1,0,1)$ $U(x^2) = (0,1,0)$

$U(x) = (1,0,-1)$

So $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

$T(1) = 2$

$T(x) = 1(3+x) + 2x = 3 + 3x$

$T(x^2) = 2x(3+x) + 2x^2 = 4x^2 + 6x$

So $[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$

$U(T(1)) = U(2) = (2, 0, 2)$

$U(T(x)) = U(3+3x) = (6, 0, 0)$

$U(T(x^2)) = U(4x^2+6x) = (6, 4, -6)$

So,

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$$

$$[U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix} = [UT]_\beta^\gamma$$

21. let I_V be the identity transformation of V.

and I_W be the identity transformation of W

Now, $[T]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'} [T]_\beta^\gamma [I_V]_{\beta'}^\beta$

or, $[T]_{\beta'}^{\gamma'} = P^{-1} [T]_\beta Q$. (Proved)

22. $T(a,b) = (3a-b,\ a+3b)$

$(2,4) = a(1,1) + b(1,-1)$

$a+b=2$

$a-b=4$ $\qquad b=-1$

$2a=6 \Rightarrow a=3$

$(3,1) = a(1,1) + b(1,-1)$

or, $a+b=3$

$a-b=1$

$2a=4 \Rightarrow a=2,\ b=1$

$\therefore\ Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

$Q^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$

$[T]_{\beta'} = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$

$T(3,1) = (8,6)$

$= a(2,4) + b(3,1)$

$2a+3b=8 \ (\times 2)$

$4a+b=6$

$5b=10$

$b=2$

$a=1$

$T(2,4) = (2,14)$

$= a(2,4) + b(3,1)$

or, $2a+3b=2 \ (\times 2)$

$4a+b=14$

$5b=-10$

$b=-2$

$4a-2=14$

$a=4$

$$[T]_\beta = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

$$T(1,1) = (2,4) = a(1,1) + b(1,-1)$$

$$a + b = 2$$
$$a - b = 4$$

$$2a = 6 \qquad b = -1$$
$$a = 3$$

$$T(1,-1) = (4,-2) = a(1,1) + b(1,-1)$$

$$a + \cancel{b} = 4$$
$$a - \cancel{b} = -2$$

$$2a = 2 \qquad b = 3$$
$$a = 1$$

Now

$$Q^{-1} [T]_\beta Q$$

$$= \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 3+2 & 1-6 \\ 3-3 & 1+9 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3+1 & 2-1 \\ 0-2 & 0+2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$$

$$= [T]_{\beta'}$$

(Proved)

23.

$P_1(R) = V$ $\quad \beta = \{1, x\}$

$\beta' = \{1+x, 1-x\}$

We know, $[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$T(1) = 0 = 0 \cdot 1 + 0 \cdot x$

$T(x) = 1 = 1 \cdot 1 + 0 \cdot x$

$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$1 + x = 1 \cdot 1 + 1 \cdot x$

$1 - x = 1 \cdot 1 - 1 \cdot x$

$\therefore Q^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$Q^{-1} [T]_\beta Q = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

26. If V is finite dimensional and W is a subspace of V then we can find a subspace U of V for which $V = W \oplus U$

null T is a subspace of V.

setting $W = \text{null } T$

$$V = \text{null } T \oplus U$$

Since $V = \text{null } T \oplus U$

$$\therefore \text{null } T \cap U = \{0\}.$$

So we just need to show that

$$\text{range } T = \{Tu \mid u \in U\}.$$

First we show that $\text{Range } T \subset \{Tu \mid u \in U\}$

let $w \in \text{range } T$. That means $\exists$ some $v \in V$ for which $Tv = w$. Since $v \in V$ and we have

$$V = \text{null } T \oplus U$$

we can find vector $n \in \text{null } T$ and $u \in U$ for which

$$\cancel{v = n \oplus u} \quad v = n + u$$

Thus
$$Tv = Tn + Tu$$
$$w = 0 + Tu \quad \because n \in \text{null } T$$
$$w = Tu$$

~~So~~ That means $w \in \{Tu \mid u \in U\}$. Thus

$$\text{range } T \subset \{Tu \mid u \in U\}$$

Now we need to show that $\{Tu \mid u \in U\} \subset$ range T
But for any element $u \in U$, u is also in V as $U \subset V$. Thus Tu is in image of T by definition.
Therefore $\{Tu \mid u \in U\} \subset$ range T.

$$\therefore \{Tu \mid u \in U\} = \text{range } T$$

Thus, there exists a subspace U of V s.t
$V = \text{null } T \oplus U$ and range $T = \{Tu \mid u \in U\}$