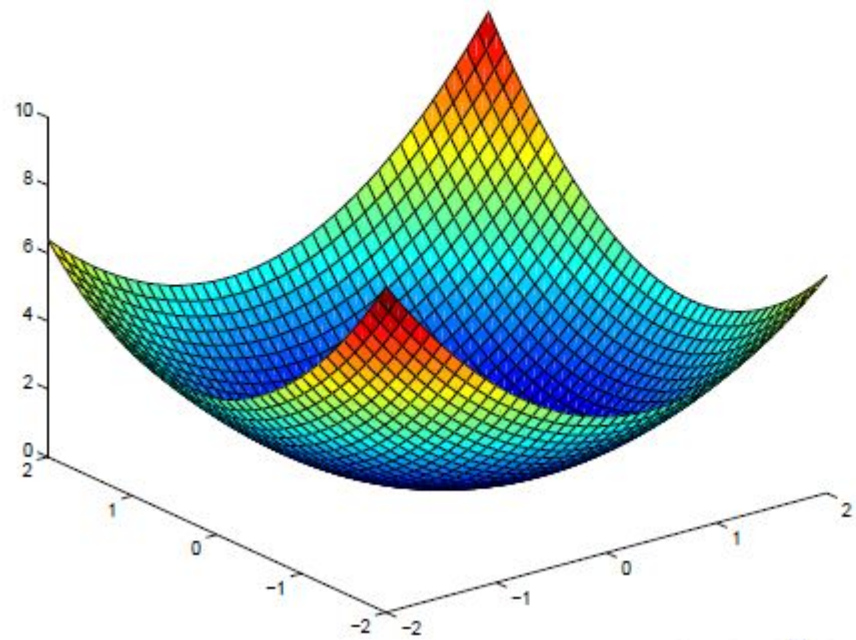
# Introduction to Gradients





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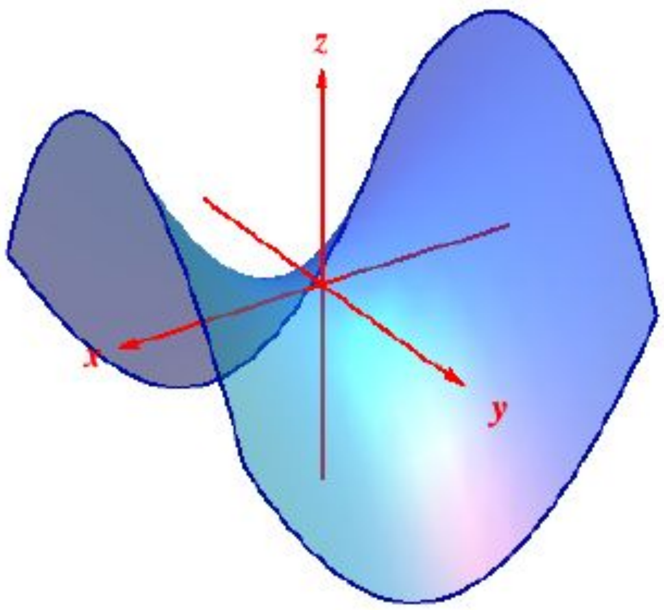
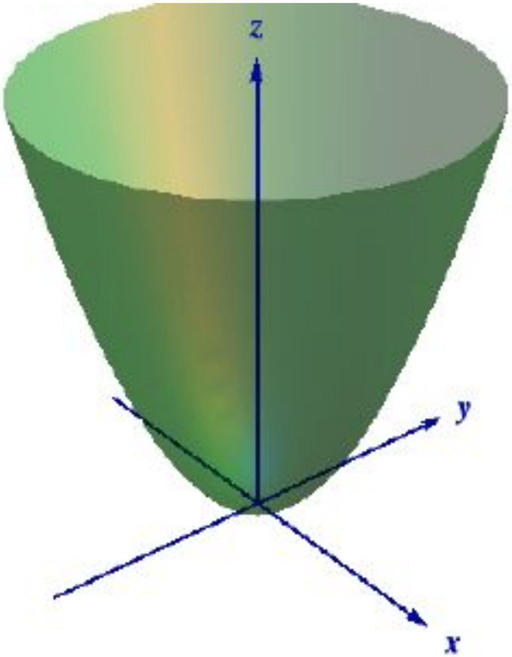
# Graph

real-valued function $z = f(x, y) : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ of two variables

The graph of $z = f(x, y)$ is the surface $S = \{(x, y, f(x, y)) : (x, y) \text{ in } U\}$




$z = f(x, y) = x^2 - y^2$




$z = f(x, y) = x^2 + y^2$

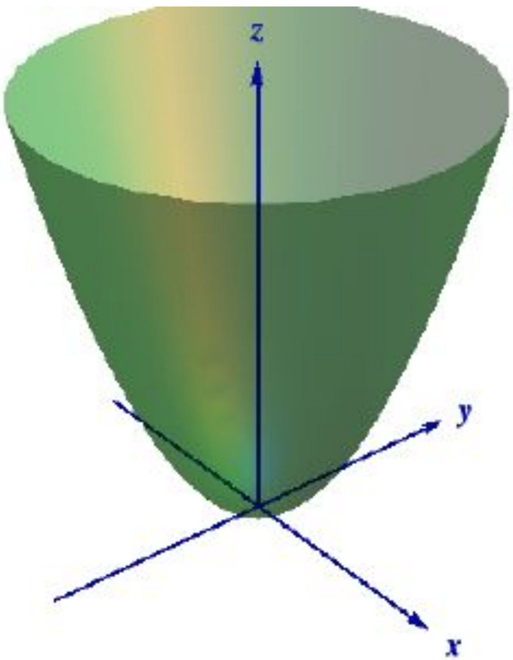
How do we know the surfaces look like that?

# Graph

The basic idea is to take cross-sections of the surface by plane slices.

Because a plane intersects the surface in a curve that also lies in the plane, this curve is often referred to as the **trace of the surface** on the plane.

Identifying traces gives us one way of 'picturing' the surface




$$z = f(x, y) = x^2 + y^2$$

# Graph

- the *trace* on a vertical plane $y = mx + b$ is the curve consisting of all points

$$\{(x, mx+b, f(x, mx+b)) \; : \; (x, mx+b) \text{ in } D\},$$

in the plane $y = mx + b$,

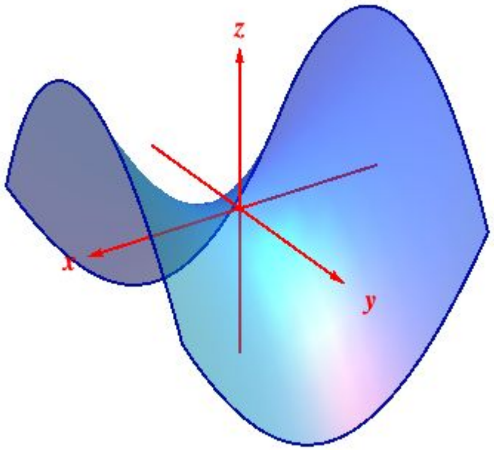
- the *trace* on a horizontal plane $z = c$ is the curve

$$\{(x, y, c) : (x, y) \text{ in } D, \; f(x, y) = c\}$$

in the plane $z = c$.

# Graph

In the case $z = x^2 - y^2$ slicing vertically by $y = b$ means fixing $y = b$ and graphing

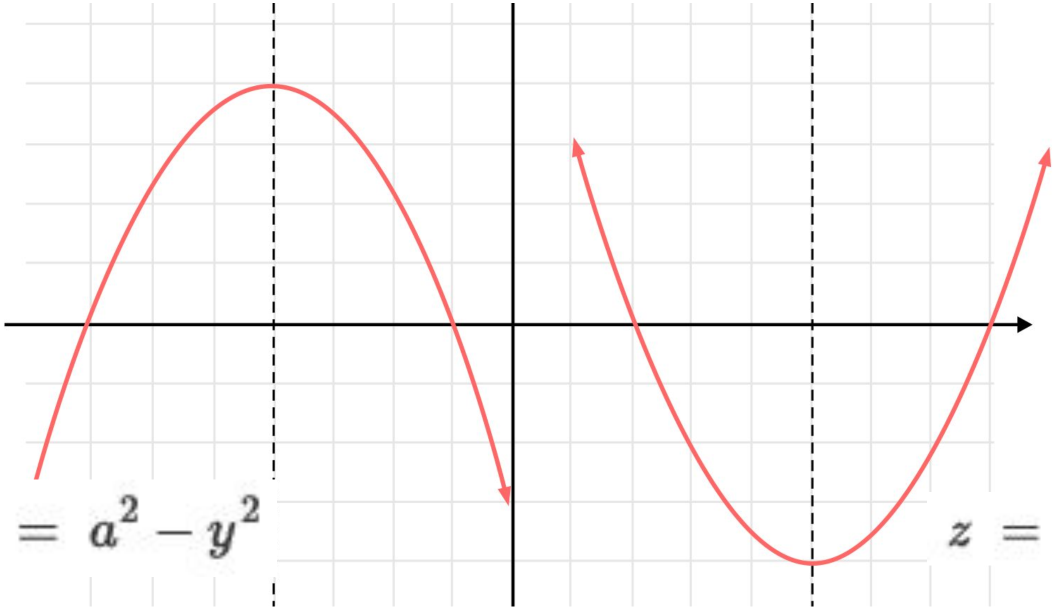



$$z = f(x, b) = x^2 - b^2,$$

while slicing vertically by the plane $x = a$ gives

$$z = f(a, y) = a^2 - y^2,$$

*i.e.*, parabolas opening up and down respectively.



$z = f(a, y) = a^2 - y^2$

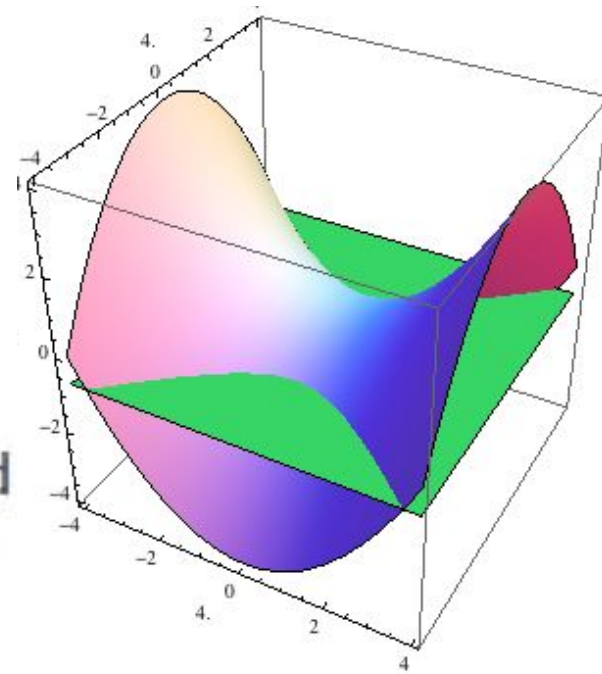
$z = f(x, b) = x^2 - b^2$

# Graph

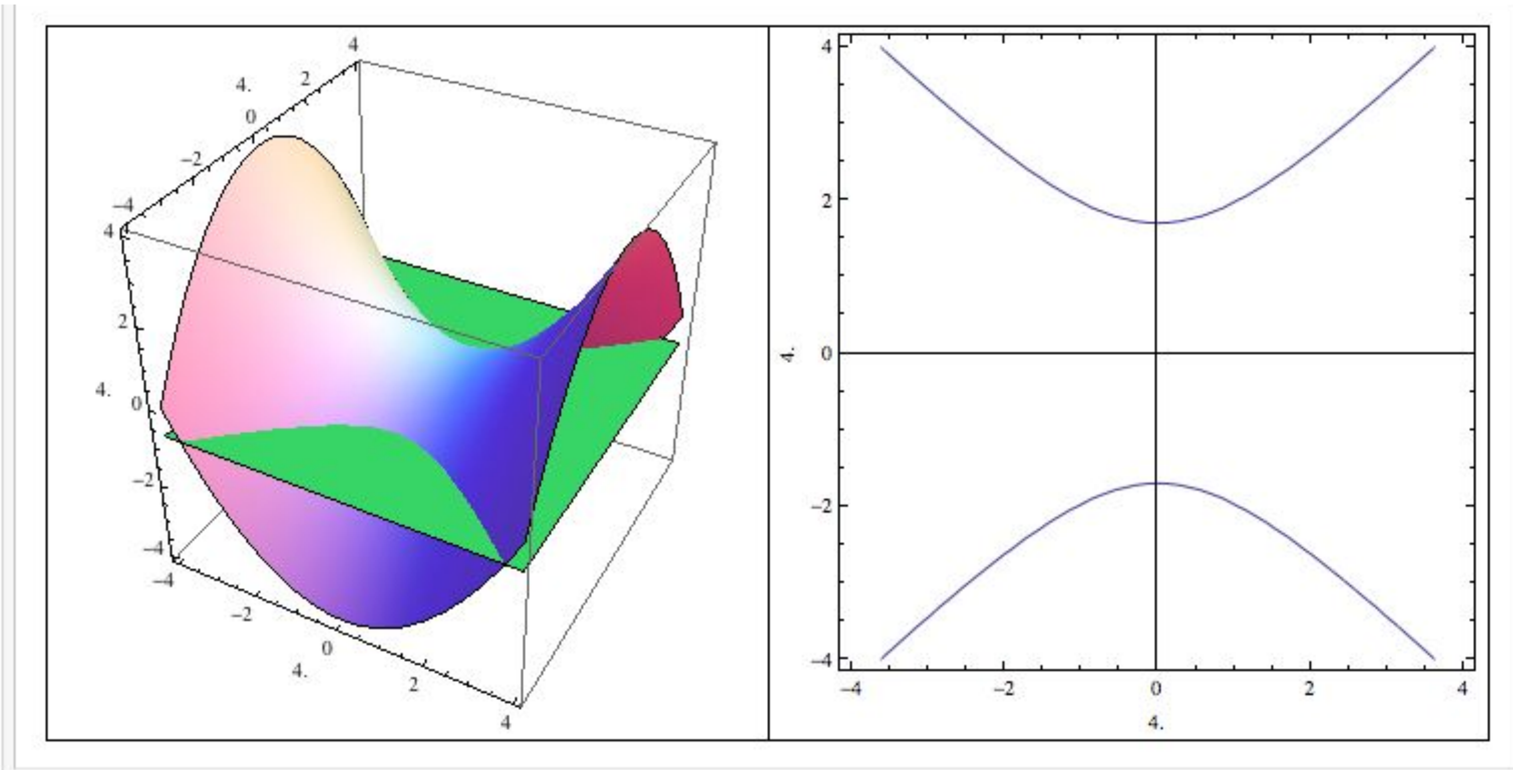
On the other hand, slicing horizontally by $z = c$ gives

$$f(x, y) = c = x^2 - y^2,$$

*i.e.*, hyperbolas opening in the $x$-direction if $c > 0$ and in the $y$-direction if $c < 0$. So the **cross-sections** are parabolas or hyperbolas, and the surface is called a **hyperbolic paraboloid**. You can think of it as a **saddle** or as a *Pringle*!

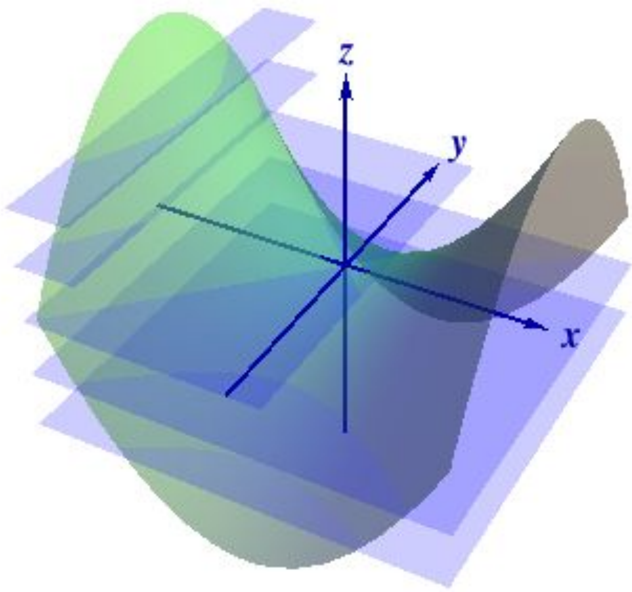


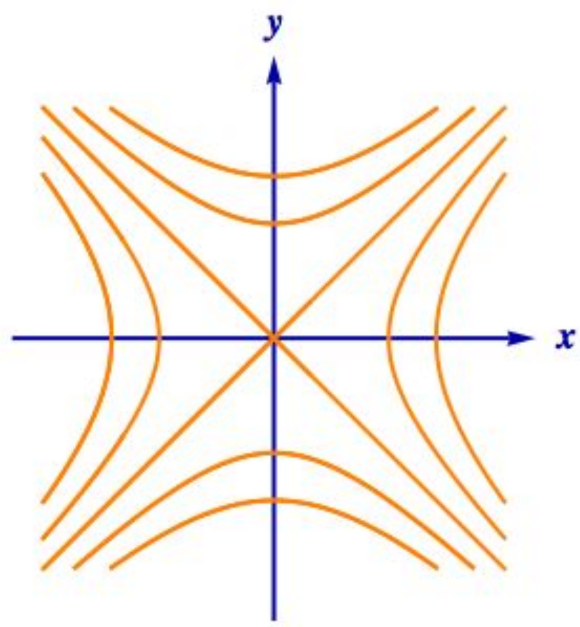
# Graph



# Level curve

**Level curves:** for a function $z = f(x, y) : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ the *level curve of value* $c$ is the curve $C$ in $D \subseteq \mathbb{R}^2$ on which $f\big|_C = c$.

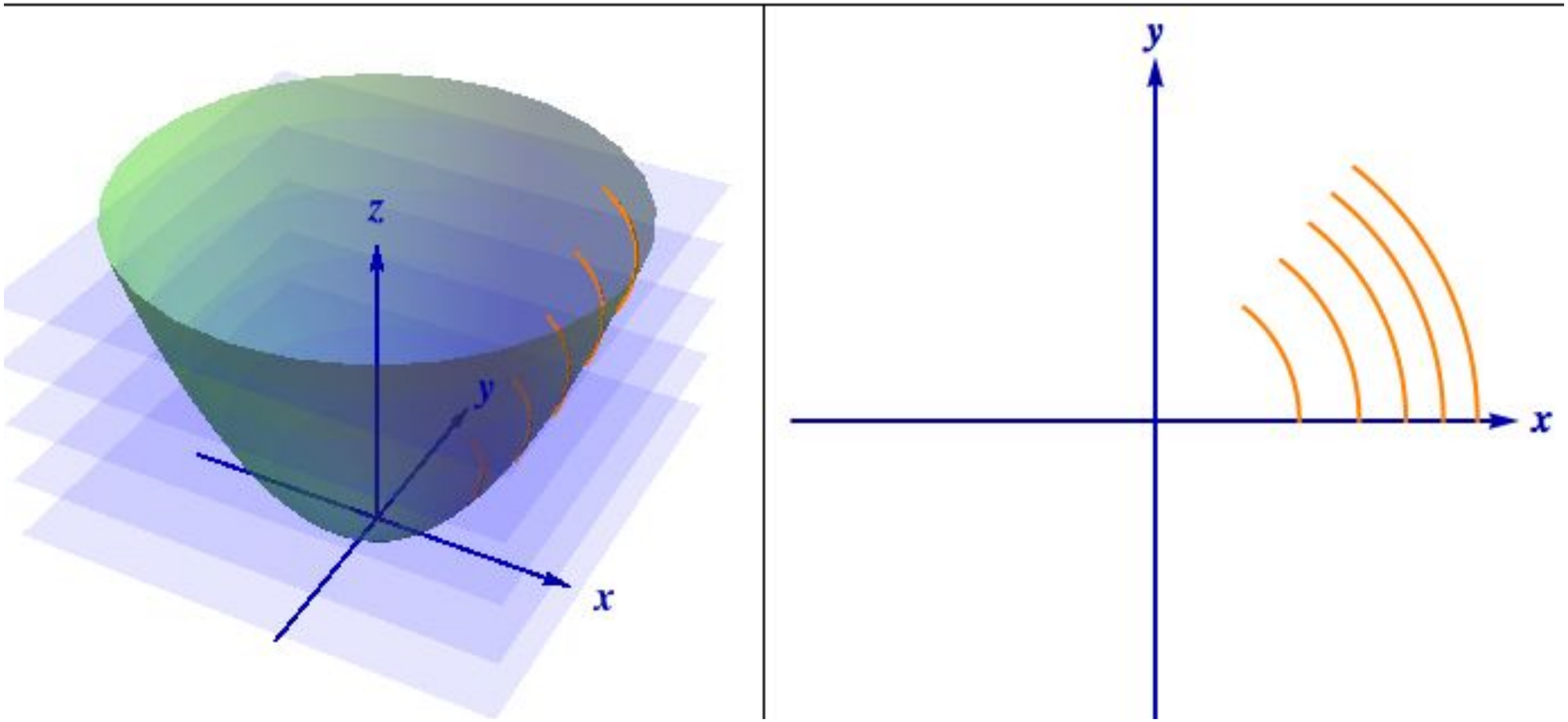



By combining the level curves $f(x,y)=c$ for equally spaced values of $c$ into one figure, say $c=-1,0,1,2,\ldots$, in the $x$-$y$ plane, we obtain a **contour map** of the graph of $z=f(x,y)$

# Level curve

**Level curves:** for a function $z = f(x, y) : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ the *level curve of value* $c$ is the curve $C$ in $D \subseteq \mathbb{R}^2$ on which $f\Big|_C = c$.



# Level curve

**Problem:** *Describe the contour map of a plane in 3-space.*
**Solution:** The equation of a plane in 3-space is

$$Ax + By + Cz = D,$$

so the horizontal plane $z = c$ intersects the plane when

$$Ax + By + Cc = D.$$

For each $c$, this is a line with slope $-A/B$ and $y$-intercept $y = (D - Cc)/B$. Since the slope does not depend on $c$, the level curves are parallel lines, and as $c$ runs over equally spaced values these lines will be a constant distance apart.

# Level curve

**Level curves:** for a function $z = f(x, y) : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ the *level curve of value* $c$ is the curve $C$ in $D \subseteq \mathbb{R}^2$ on which $f\big|_C = c$.

Notice the critical difference between a level curve of value $c$ and the trace on the plane $z = c$,

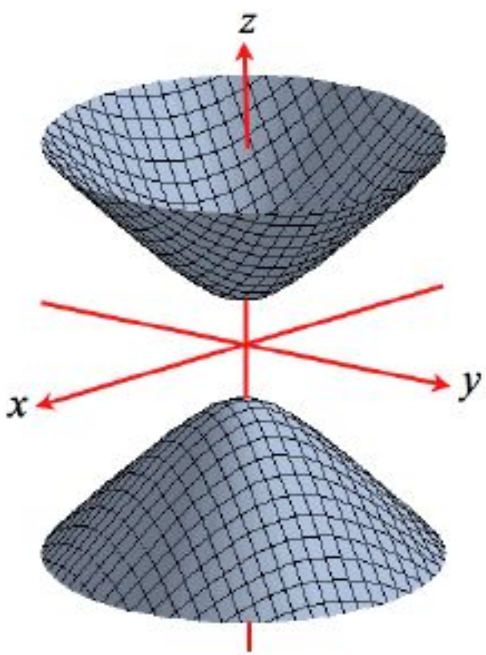
- A level curve ***C*** always lies in the $(x,y)$-plane, and is the set ***C*** of points in the $(x,y)$-plane on which $f(x,y) = c$

- The trace lies in the plane $z=c$, and is the set of points with $(x,y,c)$ with $(x,y)$ in ***C***

# Level Surface

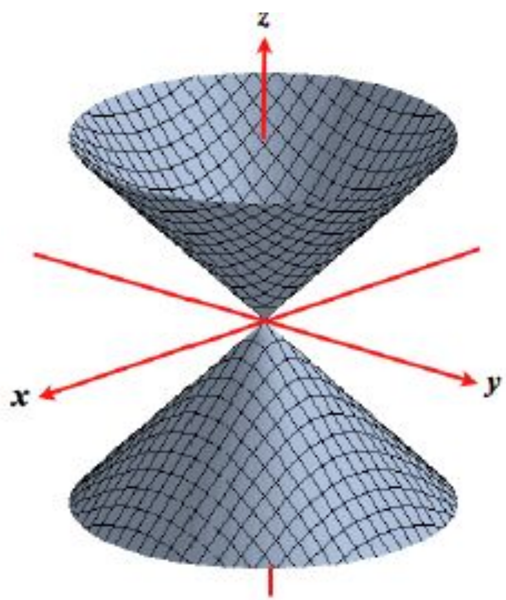
**Level surfaces:** For a function $w = f(x, y, z) : U \subseteq \mathbb{R}^3 \to \mathbb{R}$ the *level surface of value c* is the surface $S$ in $U \subseteq \mathbb{R}^3$ on which $f\big|_S = c$.

Example: $w = f(x, y, z) = x^2 + y^2 - z^2$.



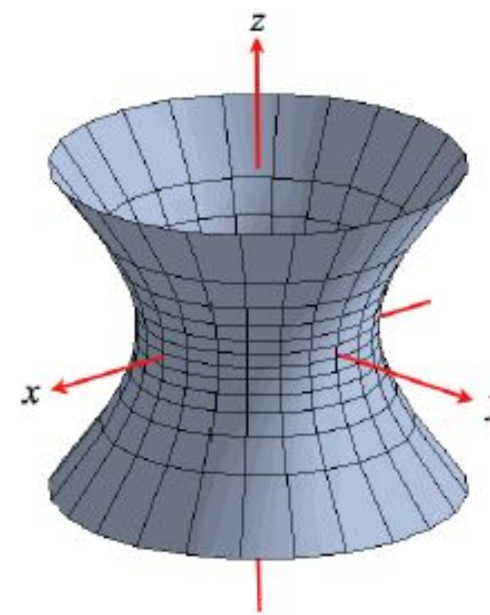

$x^2 + y^2 - z^2 = -1$

Two-sheeted Hyperboloid




$x^2 + y^2 - z^2 = 0$

Double Cone




$x^2 + y^2 - z^2 = 1$

Single-sheeted Hyperboloid

# Level Surface

Example 1: *Spheres* $x^2 + y^2 + z^2 = r^2$

*level surfaces* $w = r^2$ *of* $w = x^2 + y^2 + z^2$

Example 2: *The graph of* $z = f(x, y)$ *as a surface in* $3$*-space*

*the level surface* $w = 0$ *of* $w(x, y, z) = z - f(x, y)$

# Derivative Matrix

derivative of a function $f : \mathbf{R} \to \mathbf{R}$

the derivative of $f(x)$ at $x = a$ $\quad Df(a) = \left[\frac{\mathrm{d}f}{\mathrm{d}x}(a)\right]$ 1 x 1 matrix

For $f : \mathbf{R}^n \to \mathbf{R}$, viewed as a $f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$

derivatives at $\mathbf{x} = \mathbf{a}$: $\quad Df(\mathbf{a}) = \left[\frac{\partial f}{\partial x_1}(\mathbf{a}) \ \frac{\partial f}{\partial x_2}(\mathbf{a}) \ \ldots \ \frac{\partial f}{\partial x_n}(\mathbf{a})\right]$ 1 x n matrix

vector-valued functions, $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_m(\mathbf{x})) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix} \quad D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \frac{\partial f_m}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

**All are matrices of partial derivatives of the function**

m x n matrix

# Gradient as Vector

The matrix of partial derivatives of a scalar-valued function is called gradient

$$\nabla f(\mathbf{x}) = \left( \frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \right)$$

we can think of the gradient as a function $\nabla f : \mathbf{R}^n \to \mathbf{R}^n$,

which can be viewed as a special type of vector field

.

Gradient is a vector **operator** denoted by ∇ and called del or nabla.

$$\nabla f \equiv \text{grad}\,(f).$$

- Let ∅ be a real function of three variables, then in Cartesian coordinates,

$$\nabla \phi\,(x, y, z) = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{z}}$$

# Directional derivative

directional derivative of $f$ in the direction $\mathbf{u}$ at the point $\mathbf{a}$

$$D_{\mathbf{u}} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}.$$

$D_{\mathbf{u}} f$ is a generalization of the partial derivative to the slope of $f$

in a direction of aı arbitrary unit vector $\mathbf{u}$,

$D_{\mathbf{u}} f(\mathbf{a})$ is the slope of $f(x, y)$ when standing at the point $\mathbf{a}$

. For example, if $\mathbf{u} = (1, 0)$, then $D_{\mathbf{u}} f(\mathbf{a}) = \frac{\partial f}{\partial x}(\mathbf{a})$.

$\mathbf{u} = (0, 1)$, then $D_{\mathbf{u}} f(\mathbf{a}) = \frac{\partial f}{\partial y}(\mathbf{a})$

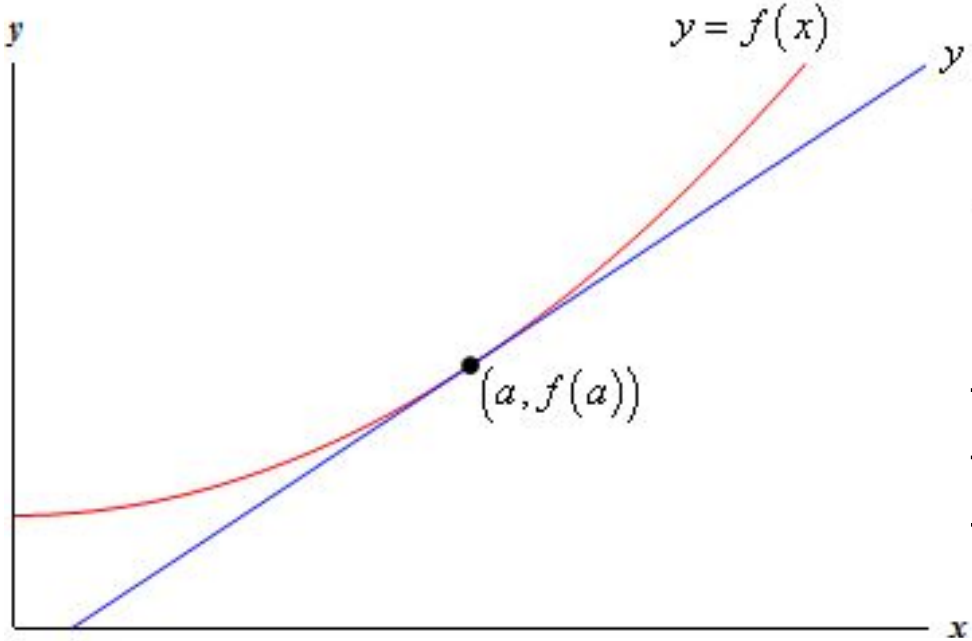
**partial derivatives of $f(x, y)$ w.r.t. $x$ and $y$**

**Example:** Directional derivative on a mountain

https://mathinsight.org/applet/directional_derivative_mountain

# Directional derivative

What does it mean for a function $f(\mathbf{x})$ to be differentiable at the point $\mathbf{x} = \mathbf{a}$?




(linear approximation)

$$L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

Differentiability means that, for all **u** (directions moving from **a**), $f(\mathbf{x})$ and $L(\mathbf{x})$ have the same slope.

$$D_{\mathbf{u}} f(\mathbf{a}) = D_{\mathbf{u}} L(\mathbf{a}) = \lim_{h \to 0} \frac{L(\mathbf{a} + h\mathbf{u}) - L(\mathbf{a})}{h}$$

$$= \lim_{h \to 0} \frac{hDf(\mathbf{a})\mathbf{u}}{h} = \lim_{h \to 0} Df(\mathbf{a})\mathbf{u} = Df(\mathbf{a})\mathbf{u}.$$

$1 \times n$ row vector

$n \times 1$ column vector,

# Gradient & directional derivative

$$
\begin{aligned}
D_{\mathbf{u}} f(\mathbf{a}) &= \nabla f(\mathbf{a}) \cdot \mathbf{u} \\
&= \|\nabla f(\mathbf{a})\| \, \|\mathbf{u}\| \cos\theta \\
&= \|\nabla f(\mathbf{a})\| \cos\theta
\end{aligned}
$$

$\theta$ is the angle between $\mathbf{u}$ and the gradient.

$\mathbf{u}$ is a unit vector, meaning that $\|\mathbf{u}\| = 1$

$$\boxed{-\|\nabla f(\mathbf{a})\| \leq D_{\mathbf{u}} f(\mathbf{a}) \leq \|\nabla f(\mathbf{a})\|}$$

$\theta = \pi$

$\theta = 0$ or $\theta = 2\pi$

- The direction of $\nabla f$ is the orientation in which the directional derivative has the largest value [ put **u** = $\nabla f$ in $D_{\mathbf{u}} f(\mathbf{a})$ ]

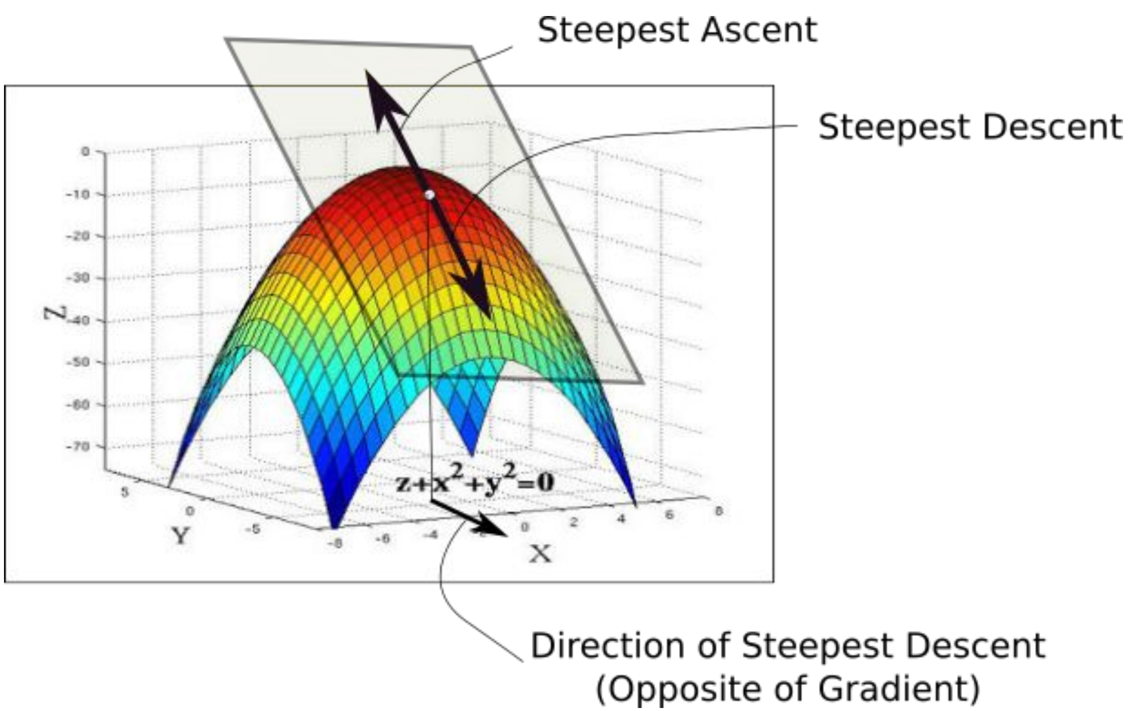
- The value of directional derivative along the direction $\nabla f$ is $\| \nabla f \|$

# Gradient=maximal slope

- There is a direction of maximal slope: $$\frac{\nabla f(\mathbf{a})}{\|\nabla f(\mathbf{a})\|} = \mathbf{m}$$

At $\mathbf{x} = \mathbf{a}$ the gradient is a vector that points in the direction of $\mathbf{m}$ and

$$\|\nabla f(\mathbf{a})\| = D_{\mathbf{m}} f(\mathbf{a}).$$

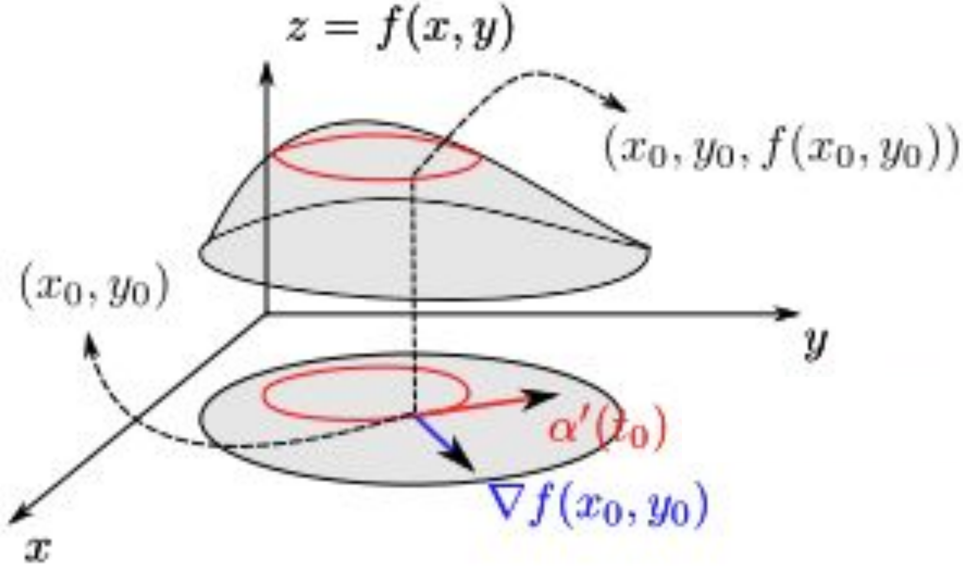


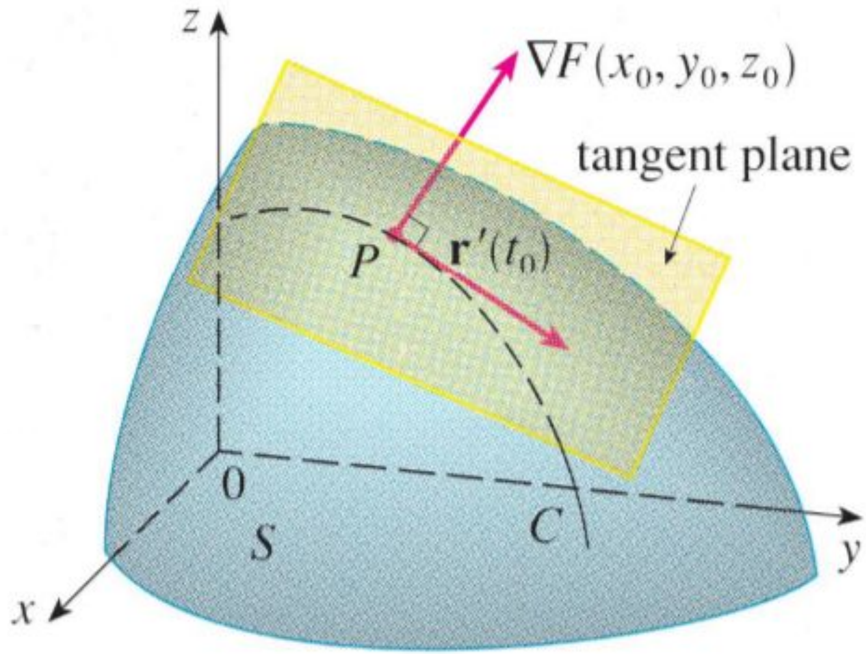
# Gradient and level curve/surface

If $\nabla f \neq 0$ at point **x**, then

- the gradient is perpendicular to the level curve through **x** $=(x_1, ..., x_n)$

- the gradient is perpendicular to the level surface through (**x**, $z$), given by $F(\mathbf{x}, z)=0$.



# Homework

Let $f(x, y) = x^2 y$.

(a) Find $\nabla f(3, 2)$

. (b) Find the derivative of $f$ in the direction of (1,2) at the point (3,2).

c) find the directional derivative of $f$ at the point (3,2) in the direction of $(2, 1)$

d) at the point (3,2), (a) in which direction is the directional derivative maximal,

what is the directional derivative in that direction?

e) at the point (3,2), what is the directional derivative in the direction (-3,4)

f) at the point (3,2) what is the directional derivative in the direction (-4,-3)

# multidimensional differentiability theorem

If, for a function $\mathbf{f}: \mathbf{R}^n \to \mathbf{R}^m$ all the partial derivatives of its matrix of partial derivatives exist and are continuous in a neighborhood of the point $\mathbf{x} = \mathbf{a}$, then $\mathbf{f}(\mathbf{x})$ is differentiable at $\mathbf{x} = \mathbf{a}$.

The function $\mathbf{f}: \mathbf{R}^n \to \mathbf{R}^m$ is *differentiable* at the point $\mathbf{a}$ if there exists a linear transformation $\mathbf{T}: \mathbf{R}^n \to \mathbf{R}^m$ that satisfies the condition

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - \mathbf{T}(\mathbf{x} - \mathbf{a})\|}{\|\mathbf{x} - \mathbf{a}\|} = 0.$$

The $m \times n$ matrix associated with the linear transformation $\mathbf{T}$ is the matrix of partial derivatives, which we denote by $Df(\mathbf{a})$. We can refer to $Df(\mathbf{a})$ as the *total derivative* (or simply the *derivative*) of $\mathbf{f}$.

In order for this limit to exist, we have to get the same result, no matter the route takes on its way to. If we find two routes that give different values for this limit, then we can conclude the limit does not exist.

# multidimensional differentiability theorem

The following is an example of a function that has partial derivatives at the origin but is not differentiable

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0.0) \end{cases}$$

See

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h}.$$

Since $f(0,0)=0$, $f(0+h,0)=f(h,0)=0$, and $f(0,0+h)=f(0,h)=0$,

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0.\end{aligned}$$

# Examples of multidimensional differentiability

Example 1

Let $f(x, y) = x^2 + y^2$. Find $Df(1, 2)$ and the equation for the tangent plane at $(x, y) = (1, 2)$.

Find linear approximation to $f(x, y)$ at $(x, y) = (1, 2)$.

at the point $(2, 3)$, what changes?

Example 2: Find the derivative of

$$\mathbf{f}(x, y, z) = (x^2 y^2 z, y + \sin z) \quad \text{at the point } (1, 2, 0).$$

calculate the linear approximation to $\mathbf{f}$ at the point $(1, 2, 0)$.