Under no ~~attribute~~ arbitrage a risk neutral pf can only grow at the same rate as that of risk free assets.

$$(f_0 - \varphi S_0)(1+r) = f_u - \varphi_1 S_u - f_d - \varphi_1 S_d$$

$$E_p[f_1 | \mathcal{F}_0] = f_u P_u + f_d P_d$$

$$= \frac{(f_u S_0(1+r) - f_u S_d) + (f_d S_u - f_d S_0(1+r))}{S_u - S_d}$$

$$= \frac{S_0(1+r)(f_u - f_d)}{S_u - S_d} + \frac{f_d S_u - f_u S_d}{S_u - S_d}$$

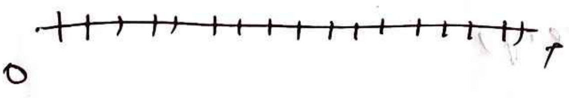
$$= \varphi_1 S_0 (1+r) + \frac{f_d(S_u - S_d) - S_d(f_u - f_d)}{S_u - S_d}$$

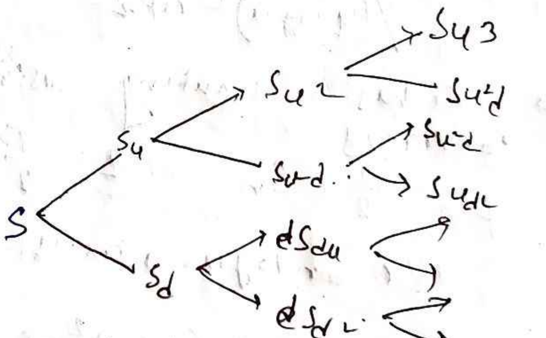
$$= \varphi_1 S_0(1+r) + f_d - \varphi_1 S_d$$

$$= \varphi_1 S_0(1+r) + (f_0 - \varphi_1 S_0)(1+r)$$

$$= f_0(1+r)$$

<u>Cox-Ross-Rubinsteine</u> (1978)





$$S_T = S_0 u^k d^{n-k}$$

<u>Assm</u> proportional growth

$$= \log\left(\frac{S_T}{S_0}\right) = \log\left(u^k d^{n-k}\right)$$

$$= k\log(u) + (n-k)\log d$$

$$= \log u\,(2k-n)$$

<u>Assn</u> Symmetric Jumps $d = \frac{1}{u}$

HW 2 Find the distribution of asymmetric jumps.

Define

$$X_T = \log\left(\frac{S_T}{S_0}\right) = (\log u)(2B - n)$$

$$B \sim Bin(n, P).$$

Where $S_u = S_0 u$ and $S_d = S_0 d$.

$$P_u = \frac{S_0(1+r) - S_d}{S_u - S_d} = \frac{S_0(1+r-d)}{S_0(u-d)} = \frac{1+r-d}{u-d}$$

$$P_{uu} = \frac{S_u(1+r) - S_{ud}}{S_{uu} - S_{ud}} = \frac{S_0 u[(1+r) - d]}{S_0 u(u-d)}$$

$$= \frac{1+r-d}{u-d}$$

$$\frac{X_T - E(X_T)}{var(X_T)} \sim N(0,1) \quad \text{as } n \to \infty.$$

$$\therefore X_T = \frac{X_T - (\log u)n(2p-1)}{\sqrt{(4\log u)^2 npq}} : 2\log(u)\sqrt{npq} + n(2p-1)(\log u)$$

So X_T has a limiting distribution (normal) and variance converges

$$p = \frac{1}{2} + \delta_n \quad q = \frac{1}{2} - \delta_n \quad pq = \frac{1}{4} - \delta_n^2$$

$2n\,\delta_n \log u \to$ must converge

$$var(X_T) = 4(\log u)^2 npq$$
$$= 4n(\log u)^2\left(\frac{1}{4} - \delta_n^2\right)$$
$$= n(\log u)^2 - n\delta_n(\log u)^2 \to 0$$

$$n(\log u)^2 \propto \sigma^2$$
$$n(\log u)^2 = c\sigma^2$$
$$(\log u)^2 = \frac{c\sigma^2}{n}$$
$$\log u = \sigma\sqrt{\frac{c}{n}}$$
$$u = \exp\left(\sigma\sqrt{\frac{c}{n}}\right)$$

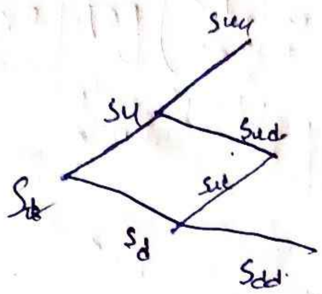
$$d = e^{-\sigma\sqrt{\frac{c}{n}}}$$

$$p_n = \frac{1 + \frac{r_T}{n} - e^{-\sigma\sqrt{\frac{c}{n}}}}{e^{\sigma\sqrt{\frac{c}{n}}} - e^{-\sigma\sqrt{\frac{c}{n}}}}$$

$$\frac{\text{num}}{\text{Dem}} \quad \frac{\frac{r_T}{n} + \sigma\sqrt{\frac{c}{n}} - \frac{\sigma^2}{2}\frac{c}{n} + O\left(\frac{1}{n^{3/2}}\right)}{2\sigma\sqrt{\frac{c}{n}} + O\left(\frac{1}{n^{3/2}}\right)}$$

Distribution of $\log\left(\frac{S_T}{S_0}\right)$

given $S_0 \sim N\left(\left(r-\frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$



$S_u = S_0 u$

$S_{uu} = S_0 u^2$

$S_d = S_0 d$

$S_{dd} = S_0 d^2$

$S_T = S_0 u^k d^{n-k}$ $\quad k=0,1,2,\dots n$

$\log\left(\frac{S_T}{S_0}\right) = k\log u + (n-k)\log d$

$= k(\log u - \log d) + n\log d$

$ud=1 \rightarrow$ Symmetric Jump

$$p = \frac{S_0\left(1+\frac{rT}{n}\right) - S_0 d}{S_0 u - S_0 d} = \frac{S_0\left(\left(1+\frac{rT}{n}\right)-d\right)}{S_0(u-d)} = \frac{1+\frac{rT}{n}-d}{u-d}$$

$$X_T = \frac{X_T - E(X_T)}{\sqrt{var(X_T)}} \sqrt{var(X_T)} + E(X_T)$$

$\hookrightarrow N(0,1)$

$S_T = S_0 u^Y d^{n-Y}$ $\quad k=0,1,2\dots n$

$X_T = \log\left(\frac{S_T}{S_0}\right) = Y\log u + (n-Y)\log d$

$= Y(\log u - \log d) + n\log d$

$Y \sim Bin(n,p)$

<u>Properties of binomial distribution</u>

(1) Fixed no. of trial, say n.

(2) Trials are independent

(3) Each trial has two outcomes S and F.

(4) Prob of S and F remain same throughout the trials. Say $p = P(S)$.

$var(X_T) = npq\,(\log u - \log d)^2$

$E(X_T) = np\,(\log u - \log d) + n\log d$

for $d = \frac{1}{u}$ then $var(X_T) = n(4pq)(\log u)^2 \to \sigma^2 T$

$E(X_T) = n\log u(2p-1) \to (r - \frac{\sigma^2}{2})T$ as $n \to \infty$

$2p - 1 \not\to 0$ in the limit as unlikely financial scenario

$(\log u)^2 = \frac{\sigma^2 T}{n} \Rightarrow \log u = \sigma\sqrt{\frac{T}{n}}$

$u = e^{\sigma\sqrt{T/n}}$

$d = e^{-\sigma\sqrt{T/n}}$

$E(S_T \mid \mathcal{F}_0)$

$E(S_0 e^{X_T} \mid \mathcal{F}_0)$

$= S_0\, E(e^{X_T} \mid \mathcal{F}_0)$

$= S_0 \left\{ e^{(r - \frac{\sigma^2}{2})T + \frac{\sigma^2 T}{2}} \right\}$

$= S_0\, e^{rT}$

$(\varphi_0 \;\; \varphi_1) \begin{pmatrix} 1+r & 1+r \\ Su & Sd \end{pmatrix} \begin{pmatrix} p_u \\ p_d \end{pmatrix} = \begin{pmatrix} f_u & f_d \end{pmatrix} \begin{pmatrix} p_u \\ p_d \end{pmatrix}$

$= f_u p_u + f_d p_d$

$= E_p(f_T \mid \mathcal{F}_0)$

$f_0 (1+r) = (\varphi_0, \varphi_1) \begin{pmatrix} 1 \\ S_0 \end{pmatrix} (1+r)$

$f_0\, e^{rT} = E(f(S_T) \mid \mathcal{F}_0)$

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$$E_{LN}(S_T \mid \mathcal{F}_0) = S_0 e^{r \cdot T}$$

where $\log\left(\frac{S_T}{S_0}\right) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$

$$\begin{tikzcd} & S_u \\ S_0 \arrow[ur] \arrow[dr] & \\ & S_d \end{tikzcd}$$

$$\begin{tikzcd} & f(S_u) = f_u \\ f_0 \arrow[ur] \arrow[dr] & \\ & f(S_d) = f_d \end{tikzcd}$$

$E_p(S_T \mid \mathcal{F}_0) = S_0(1+rT)$

$E_p(f_T \mid \mathcal{F}_0) = f_0(1+rT)$

where $p = \dfrac{S_0(1+rT) - S_d}{S_u - S_d}$

$\exists\, (\varphi_0, \varphi_1)$ s.t.

$$(\varphi_0, \varphi_1)\begin{pmatrix} 1+rT & 1+rT \\ S_u & S_d \end{pmatrix} = (f_u, f_d)$$

Then it is checked that

$$E_p(f_T \mid \mathcal{F}_0) = p_u f_u + p_d f_d = f_0(1+rT).$$

$S_u = S_0 \cdot u$

$S_d = S_0 d$

$ud = 1$
$\neq 1$

$$f_T = f(S_T) = f(S_0 u^k d^{n-k})$$

$$S_T = \cancel{S_0(1+rT)}\, S_{n(T/n)}$$

$$S_{(n-1)\left(\frac{T}{n}\right)}$$

$$= E_p\left(S_{n\left(\frac{T}{n}\right)} \,\middle|\, \mathcal{F}_{(n-1)\times\frac{T}{n}}\right)$$

$$= S_{(n-1)\left(\frac{T}{n}\right)} \times \left(1 + \frac{rT}{n}\right)$$

$$E_p f\left(S_{n\left(\frac{T}{n}\right)} \,\middle|\, \mathcal{F}_{(n-1)\frac{T}{n}}\right)$$

$$= f\left(S_{(n-1)\left(\frac{T}{n}\right)}\right)\left(1 + \frac{rT}{n}\right)$$

$$\begin{tikzcd} & f\left(S_{(n-1)\frac{T}{n}} \times u\right) \\ f\left(S_{(n-1)\frac{T}{n}}\right) \arrow[ur] \arrow[dr] & \\ & f\left(S_{(n-1)\left(\frac{T}{n}\right)} \times d\right) \end{tikzcd}$$

using replicable pf idea i.e using (ϕ_0, ϕ_1) we can show the same way $p f\left(S_{(n-1)(\frac{T}{n})} \times u\right) + (1-p) f\left(S_{(n-1)(\frac{T}{n})} \times d\right)$

$$= f\left(S_{(n-1)(\frac{T}{n})}\right)\left(1+\frac{rT}{n}\right).$$

Iterating this process we can show.

$$E_P\left(f\left(S_{n(\frac{T}{n})}\right) \mid \mathcal{F}_{(n-2)\frac{T}{n}}\right) = \text{- - - - - - - -} \quad (i)$$

$$E\left(E\left(S_{n\frac{T}{n}}\right) \mid \mathcal{F}_{(n-1)\frac{T}{n}} \mid \mathcal{F}_{n-2\frac{T}{n}}\right) = E\left(f\left(S_{n\frac{T}{n}}\right) \mid \mathcal{F}_{n-2\frac{T}{n}}\right).$$

$$E\left(E\left(f\left(S_{n\times\frac{T}{n}}\right)\right) \mid \mathcal{F}_{(n-1)\frac{T}{n}}\right) = f\left(S_{(n-1)\frac{T}{n}}\right)\left(1+\frac{rT}{n}\right).$$

$$R.H.S = f\left(S_{(n-2)\frac{T}{n}}\right)\left(1+\frac{rT}{n}\right)^2 \quad \text{of } (i)$$

Iterating the process n times we get,

$$E_P\left(f\left(S_{n\frac{T}{n}}\right) \mid \mathcal{F}_0\right) = f_0\left(1+\frac{rT}{n}\right)^n$$

$$\Rightarrow E_{LN}\left(f(S_T) \mid \mathcal{F}_0\right) = f_0 e^{rT}$$

Given $\mathcal{F}_0$, distribution of $S_{n\frac{T}{n}}$ converges log normal with parameter $\mu = \left(r - \frac{\sigma^2}{2}\right)T$ and $\sigma^2 = \sigma^2 T$.

Let f be price function for each option.

$$\text{Then } f(S_T) = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T < K \end{cases}$$

$$\text{Then } E\left(f(S_T) \mid \mathcal{F}_0\right) = \int_K^\infty (S_T - K)\, dF_{S_T}$$

$$= \int_K^\infty \left(S_0 e^{X_T} - K\right) dF_{X_T}.$$

$$= S_0 e^{rT} =$$

$$S_T = S_0 e^{X_T} > K \iff e^{X_T} > \frac{K}{S_0}$$

where $X_T \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$

$Y = f(X)$

$E(Y) = \int y \, dF_Y$

$$S_T = S_0 e^{X_T} > K \iff e^{X_T} > \frac{K}{S_0}$$

$$\iff X_T > \log\left(\frac{K}{S_0}\right)$$

$$\int_{\log\left(\frac{K}{S_0}\right)}^{\infty} \left(S_0 e^{X_T} - K\right) dF_{X_T}$$

$$= \int_{\log\frac{K}{S_0}}^{\infty} S_0 e^{x} \frac{e^{-\frac{(x-\mu)^2}{2\nu^2}}}{\nu\sqrt{2\pi}} dx$$

where $\mu = \left(r - \frac{\sigma^2}{2}\right)T$, $\nu^2 = \sigma^2 T$.

$$- \int_{\log\left(\frac{K}{S_0}\right)}^{\infty} K \frac{e^{-\frac{(x-\mu)^2}{2\nu^2}}}{\nu\sqrt{2\pi}} dx = \mathrm{I} - \mathrm{II}$$

$$\mathrm{II} = K\left(1 - \Phi\left(\frac{\log\left(\frac{K}{S_0}\right) - \mu}{\nu}\right)\right)$$

$$= K\,\Phi\left(\frac{\log\frac{S_0}{K} + \mu}{\nu}\right)$$

$$= K\,\Phi\left(\frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$f_0 e^{rT} = S_0 e^{rT}\Phi(d_1) - K\Phi(d_2)$

$f_0 = S_0\Phi(d_1) - K e^{-rT}\Phi(d_2)$

where $d_1 = \frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

and $d_2 = d_1 - \sigma\sqrt{T}$

$$e^{x - \frac{(x-\mu)^2}{2\nu^2}}$$

Expand $x - \frac{(x-\mu)^2}{2\nu^2}$

$$= \frac{2\nu^2 x - (x^2 - 2\mu x + \mu^2)}{2\nu^2}$$

$$= \frac{\left[-\left(x^2 - 2x(\mu+\nu^2) + (\mu+\nu^2)^2\right) - \mu^2 + (\mu+\nu^2)^2\right]}{2\nu^2}$$

$$= -\frac{\left(x - (\mu+\nu^2)\right)^2}{2\nu^2} + \frac{\cancel{\mu^2} + \nu^4 + 2\mu\nu^2 - \cancel{\mu^2}}{2\nu^2}$$

$$= -\frac{\left(x - (\mu+\nu^2)\right)^2}{2\nu^2} + \frac{\nu^2}{2} + \mu.$$

$\mu + \nu^2 = \left(r - \frac{\sigma^2}{2}\right)T + \sigma^2 T$

$= \left(r + \frac{\sigma^2}{2}\right)T.$

$\mu + \frac{\nu^2}{2} = \left(r - \frac{\sigma^2}{2}\right)T + \frac{\sigma^2 T}{2}$

$= rT$

$$e^{x - \frac{(x-\mu)^2}{2\nu^2}}$$

$$= e^{-\frac{\left(x - \left(r + \frac{\sigma^2}{2}\right)T\right)^2}{2\sigma^2 T}} \cdot e^{rT}.$$

$$I = S_0 e^{rT} \int_{\log \frac{K}{S_0}}^{\infty} \frac{e^{-\frac{\left(x - \left(r+\frac{\sigma^2}{2}\right)T\right)^2}{2\sigma^2 T}}}{\sigma\sqrt{T}\sqrt{2\pi}}\, dx.$$

$$= S_0 e^{rT}\left(1 - \Phi\left(\frac{\log\frac{K}{S_0} - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right)$$

$$= S_0 e^{rT}\,\Phi\left(\frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

finally

$$f_0 e^{rT} = S_0 e^{rT}\,\phi(d_1) - K\,\phi(d_2).$$

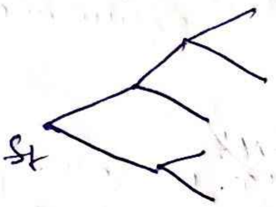
$$f_0 = S_0\,\Phi(d_1) - K e^{-rT}\,\Phi(d_2)$$

where $d_1 = \dfrac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

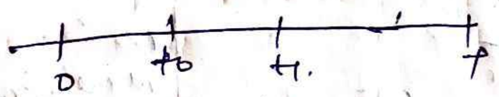
& $d_2 = d_1 - \sigma\sqrt{T}$

At time point $0 < t < T$

$E\left(f(S_T) \mid \mathcal{F}_t\right) = \cancel{\text{of}}$



$E\left(S_T \mid \mathcal{F}_t\right) = S_t e^{r(T-t)}$



$\log\left(\dfrac{S_{t_1}}{S_{t_0}}\right) \sim N\left((r-\sigma^2)(t_1-t_0),\ \sigma^2(t_1-t_0)\right)$

$p = \dfrac{1 + r\frac{t_1-t_0}{n} - d}{u-d}$ $\quad u = e^{\sigma\sqrt{\frac{t_1-t_0}{n}}}$

$(\log u)^2 \propto \dfrac{t_1-t_0}{n} \to T/n$

$(\log u)^2 = \sigma^2 \dfrac{t_1-t_0}{n}$

$\log u = \sigma\sqrt{\dfrac{t_1-t_0}{n}}$

$u = \exp\left(\sigma\sqrt{\dfrac{t_1-t_0}{n}}\right)$

$d = \dfrac{1}{u}$

$d = e^{-\sigma\sqrt{\frac{t_1-t_0}{n}}}$

$E\left(f(S_T) \mid \mathcal{F}_t\right) = f_t e^{r(T-t)}$

$f_t = S_t \Phi(d_{1t}) - K e^{-r(T-t)} \Phi(d_{2t})$

$d_{1t} = \dfrac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

& $d_{2t} = d_{1t} - \sigma\sqrt{(T-t)}$

Hedging parameter $\phi_1 = \dfrac{f_u - f_d}{S_u - S_d} \to \dfrac{\partial f}{\partial S} = \Delta.$

now the hedging parameter Δ can be found by using the relation between f_t & S_t

$$\Delta = \frac{\partial f}{\partial S} = \Phi(d_{1t}) + S_t \phi(d_{1t}) - K e^{-r(T-t)} \phi(d_{2t}) \frac{\partial d_{2t}}{\partial S}$$

$$\phi(d_{1t}) = \phi(d_{2t} + \sigma\sqrt{T-t}) = \frac{e^{-\frac{(d_{2t}+\sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}}$$

$$(d_{2t} + \sigma\sqrt{T-t})^2 = d_{2t}^2 + 2\sigma\sqrt{T-t}\, d_{2t} + \sigma^2(T-t)$$

$$= d_{2t}^2 + 2\left[\log \frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma^2(T-t)\right]$$

$$= d_{2t}^2 + 2\left[\log \frac{S_t}{K} + r(T-t)\right]$$

$$\Rightarrow \frac{d_{1t}^2}{2} = \frac{d_{2t}^2}{2} + \log \frac{S_t}{K} + r(T-t)$$

$$e^{-\frac{d_{1t}^2}{2}} = e^{-\left[\frac{d_{2t}^2}{2} + \log \frac{S_t}{K} + r(T-t)\right]}$$

$$= e^{-\frac{d_{2t}^2}{2}} \times \frac{S_t}{K} \times e^{-r(T-t)}$$

$$S_t \phi(d_{1t}) = S_t \phi(d_{2t}) \frac{K}{S_t} e^{-r(T-t)}$$

$$= K e^{-r(T-t)} \phi(d_{2t})$$

$$\Delta = \frac{\partial f}{\partial S} = \Phi(d_{1t}) + K e^{-r(T-t)} \phi(d_{2t}) \left[\frac{\partial d_{1t}}{\partial S} - \frac{\partial d_{2t}}{\partial S}\right]$$

$$= \Phi(d_{1t}) + K e^{-r(T-t)} \phi(d_{2t}) \left[\cancel{\frac{\partial d_{2t}}{\partial S}} + \frac{\partial \sigma\sqrt{T-t}}{\partial S} - \cancel{\frac{\partial d_{2t}}{\partial S}}\right]$$

$$= \Phi(d_{1t}) > 0.$$

As $t \to T$

$S_T < K$

Case-I

$$d_{1t} = \frac{\log \frac{S_{T-t}}{K}}{\sigma\sqrt{T-t}} \to -\infty + \frac{\left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma} \to 0$$

$\Phi(d_{1t}) \to 0$

Case-2

$$\frac{\log \frac{S_{T-t}}{K}}{\sigma\sqrt{T-t}} > 0 \to \infty$$

$\Phi(d_{1t}) \to 1.$

calculate

$$\frac{\partial f}{\partial K}, \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \sigma}, \frac{\partial f}{\partial t}.$$

Put call parity for european option under no arbitrage

$$C_E + Ke^{-r T} = P_E + S_0$$

$$C_E(0) + Ke^{-rT} = P_E(0) + S_0$$
$$C_E(t) + Ke^{-r(T-t)} = P_E(t) + S_T$$

Pf I

A call price $C_E(0)$ & cash amount Ke^{-rT}.

Pf II

A put price $P_E(0)$ and and underlying asset with price S_0.

Holder of the put option has right to sell the underlying asset at a prefixed price (say K) at a maturity time (say T).

at time T.

Value of Pf I $= C_E(T) + (Ke^{-rT})e^{rT}$

Now $C_E(T) = \text{pay off} \Rightarrow \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T < K \end{cases}$

$$\text{Value of (Pf I)} = \begin{cases} S_T - K + K & \text{if } S_T > K \\ S_T & \text{if } S_T > K \\ 0 + K & \text{if } S_T < K \end{cases}$$

$= \max(S_T, K)$.

Holder of the call option has the right to buy the underlying asset at the prefined price (K) at the maturity time (T).

Holder the put option has right to sell the underlying asset at a pre fixed price (say K) at a maturity time (say T).

Value of Pf (II) $= P_E(T) + S_T$

now $P_E(T)$ -payoff $= \begin{cases} K - S_T & \text{if } S_T < K \\ 0 & \text{if } S_T > K \end{cases}$

$P_E(T) + S_T = \begin{cases} K - S_T + S_T & \text{if } S_T < K \\ 0 + S_T & \text{if } S_T > K \end{cases}$

$= \begin{cases} K & \text{if } S_T < K \\ S_T & \text{if } S_T > K \end{cases}$

$= \max(S_T, K)$.

0.06

HW Calculate the put price using the LN density of the asset price

$T = 3$ months $= \frac{1}{4}$ yrs. $\sigma = 0.25$ $S_0 = 60$ $K = 55$
$r = 6\%$

$$d_1 = \frac{\log \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$= \frac{\log \frac{60}{55} + \left(0.06 + \frac{(0.25)^2}{2}\right)\frac{1}{4}}{0.25 \times \frac{1}{2}} \quad \cancel{0.489}\ 0.878$$

$d_2 = d_1 - \sigma\sqrt{T} = 0.75$

$c_E = S_0\,\Phi(d_1) - K e^{-rT}\,\Phi(d_2)$

$= 60 \times 0.81 - 55 e^{-\frac{0.06}{4}} \times 0.774$

$= 6.65$

call value (Pf I(0)) > value (Pf II (0))
value (Pf I(0)) < value (Pf II (0)).
Therefore under no arbitrage

$$C_E(0) + Ke^{-rT} = P_E(0) + S_0$$

(John C Hull)
Options, futures and derivatives

Prove that

$$C_E(t) + Ke^{-r(T-t)} = P_E(t) + S_t$$

for any $0 \le t \le T$.

American options:

call option: Holder has the right buy the underlying assets at a prefixed price (K) on or before maturity time (say T)

Inequality for american option

$$C_A(0) + Ke^{-rT} \le P_A(0) + S_0 \le C_A(0) + K.$$ HW

$\not=$

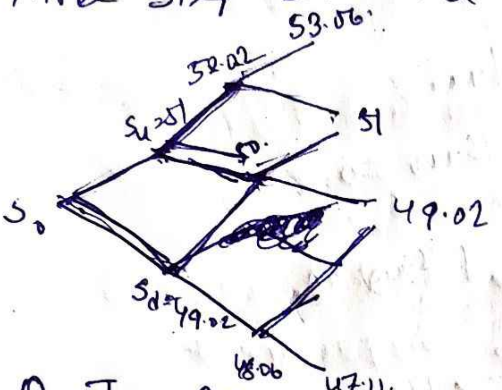
Finance 16/11/24

American option

Put option:

~~Let two step bi~~

Let three step binomial



$0, \frac{T}{3}, \frac{2T}{3}, T$

$S_0 = 50$
$K = 52$
$r = 6\%$
$u = 1.02$
$d = \frac{1}{u}$
$T = 3$ months.

$S_u = S_0 u$
$S_d = S_0 d$

(Symmetric jump).
$S_{ud} = S_0 ud = S_0$
for $d = \frac{1}{u}$.

Under NA

$$p = \frac{S_0\left(1+\frac{rT}{3}\right) - S_d}{S_u - S_d}$$

$$= \frac{\cancel{S_0}\left[\left(1+\frac{rT}{3}\right) - d\right]}{\cancel{S_0}(u-d)}$$

$$= \frac{1+\frac{rT}{3} - d}{u-d}$$

$$= \frac{1+0.06\times\frac{\cancel{3}}{12\times\cancel{3}} - 0.98}{1.02 - 0.98}$$

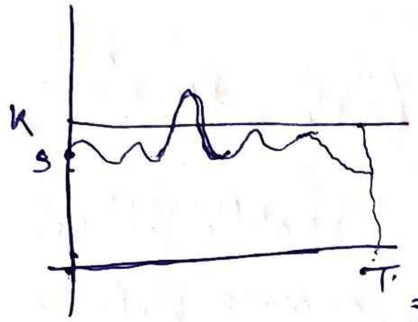
$= \cancel{0.875}$ 0.625

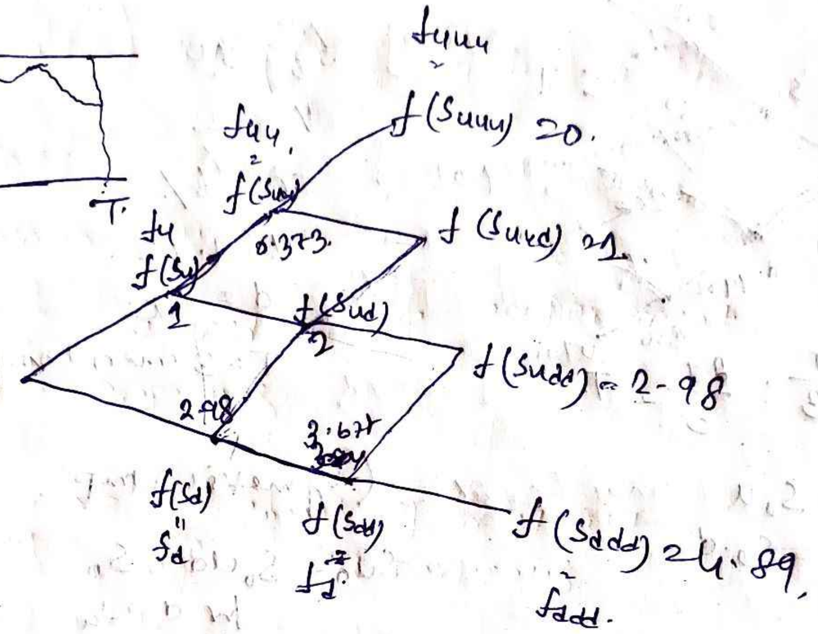
$q = \cancel{0.125}$ 0.325

Put option payoff $= \begin{cases} K - S_T & \text{if } S_T < K \\ 0 & \text{ow} \end{cases}$

American put option intrinsic value at time $t \le T$

$$= \begin{cases} K - S_t & \text{if } S_t < K \\ 0 & \text{if } S_t > K \end{cases}$$





I expected value $f(S_{dd}) = (P \times 2.98 + Q \times 4.89) \times \left(1 - \frac{rT}{3}\right)$

Intrinsic value at $f(S_{dd}) = 3.94$

0		$\frac{T}{3}$		$\frac{2T}{3}$		T
EV	IV	EV	IV	EV	IV	
						0
						1
						2.98
						4.89

expected value $f(S_{ud}) = (P \times 1 + Q \times 2.98)\left(1 - \frac{rT}{3}\right) = 1.68$

Intrinsic value at $f(S_{ud}) = 52 - 50 = 2$

node value $= \max(EV, IV)$

$\left(T - \frac{2T}{3}\right) = \frac{T}{3}$ $\quad \left(\frac{2T}{3} - \frac{T}{3}\right) = \frac{T}{3}$ time gap.

Expected value of $f(S_u) = (p\times 0.373 + q\times 2)\left(1-\frac{r_T}{3}\right)$

$IV = 1$ $\quad = 0.978$

node value $= \max(0.978, 1) = 1$.

EV $\quad f(S_d) = (p\times 2 + q\times 3.677)\left(1-\frac{r_T}{3}\right) = 2.61$

$IV = 2.98$

node value $= \max(2.61, 2.98)$
$= 2.98$

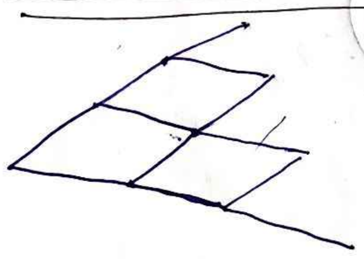
expected $f(S_0) = (p\times 1 + q\times 2.98)\left(1-\frac{r_T}{3}\right)$
$= \cancel{10}\ 1.733$

~~node value~~ $f(S$ Intrinsic value $= 2$

node value $= \max(1.733, 2) = 2$

compute the american call option price and European put and call option price with the same data.

CRR model and distribution of S_T



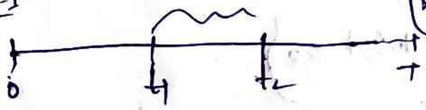
$$\log\left(\frac{S_T}{S_0}\right) \sim N\left(\left(r-\frac{\sigma^2}{2}\right)T, \sigma^2 T\right) \text{ as } n\to\infty$$

Time stamp

0 | | | | ... | T.

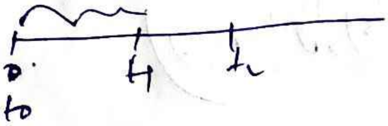
$$(X_{t_2}-X_{t_1}) = \log\left(\frac{S_{t_2}}{S_{t_1}}\right) \sim N\left(\left(r-\frac{\sigma^2}{2}\right)(t_2-t_1), \sigma^2(t_2-t_1)\right) \text{ as } n\to\infty$$

case-I



$$(X_{t_1}-X_{t_0}) = \log\left(\frac{S_{t_1}}{S_{t_0}}\right) \sim N\left(\left(r-\frac{\sigma^2}{2}\right)(t_1-t_0), \sigma^2(t_1-t_0)\right) \text{ as } n\to\infty.$$

case-II



$$X_t = \log\left(\frac{S_t}{S_0}\right).$$

$(X_{t_1}-X_{t_0})$, $(X_{t_2}-X_{t_1})$ are independent from the assumption of CRR model.

More generally, for any

$0 \le t_0 \le t_1 < t_2 < \cdots < t_k \le T$.

$(X_{t_1} - X_{t_0})(X_{t_2} - X_{t_1}) \cdots (X_{t_k} - X_{t_{k-1}})$ are indipendent

and $(X_{t_i} - X_{t_{i-1}}) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}), \sigma^2(t_i - t_{i-1})\right)$

Brownian motion

Standard Brownian motion is a stochastic process $\{B_t : t \geq 0\}$ defined on the experiment with following properties.

(i) $B_0 = 0$, $t \to B_t$ is ~~contis of for almost~~ (continous sample path property). all particales.

(ii) For $0 \le t_0 < t_1 < t_2 < \cdots < t_k$, $k \geq 1$.

$(B_{t_1} - B_{t_0}), (B_{t_2} - B_{t_1}) \cdots (B_{t_k} - B_{t_{k-1}})$

are indipendent (Indipendent ~~Incremant~~ property).

(iii) $\cancel{(B_t - B_s)}$ for $s < t$

$(B_t - B_s)$ given $\mathcal{F}_s = \sigma(\{B_s : 0 \le u \le s\})$

follow $N(0, t-s)$

(conditional distribution property).

Thus $X_t = \mu t + \sigma B_t$ where $\mu = r - \frac{\sigma^2}{2}$ is the risk neutral case and hence it is brownian motion.

where (i) and (ii) and satisfied by $\{X_t : t \geq 0\}$

and (iii) $\to$ (iii) for $s < t$ $(X_t - X_s)$ given $\mathcal{F}_s$

follow $N(\mu(t-s), \sigma^2(t-s))$

$\{B_t\}$ is nowhere differentiable as a function of t

Stocaastic

calculay

(i) $(dB_t)^2 = dt$

(ii) $dB_t \times dt = 0$

(iii) $(dt)^2 = 0$

Understanding $E(B_{t+h} - B_t)^2 = h$

$E[(B_{t+h} - B_t)(t+h-t)]$

$= \frac{(B_{t+h} - B_t)\cancel{h}}{\cancel{h}} \to 0$

$n(n-1)^2$

Itô's formula (Itô calculas)

$f(t, B_t)$ $\quad\quad \log\left(\frac{S_t}{S_0}\right) = \cancel{Y_t} = \mu t + \sigma B_t$

$S_t = S_0 e^{Y_t}$

$df(t, B_t)$ $\quad\quad = S_0 e^{\mu t + \sigma B_t}$

$\approx f(t+h, B_{t+h}) - f(t, B_t)$ $\quad\quad = f(t, B_t)$

by Taylor expansion $\quad\quad$ diff by t

$f_t(t+h-t) + f_x(B_{t+h} - B_t) + \frac{1}{2}f_{tt}(t+h-t)^2 + \frac{1}{2}f_{xx}(B_{t+h} - B_t)^2$
$+ f_{tx}(t+h-t)(B_{t+h} - B_t) + \cdots$

$= f_t\,dt + f_x\,dB_t + \frac{1}{2}f_{tt}\,(dt)^2 \xrightarrow{} 0 + \frac{1}{2}f_{xx}(dB_t)^2$
$+ f_{tx}\,\cancel{dt\,dB_t} \to 0 \cdots$

$$\boxed{\begin{aligned}(dB_t)^3 &= (dB_t)^2\,dB_t \\ &= dt \times dB_t = 0\end{aligned}}$$

$= f_t\,dt + f_x\,dB_t + \frac{1}{2}f_{xx}\,dt$

$dS_t = df(t, B_t)$

$= \mu f\,dt + \sigma f\,dB_t + \frac{1}{2}\sigma^2 f\,dt$

$dS_t = (\mu + \frac{\sigma^2}{2})S_t\,dt + \sigma S_t\,dB_t$

$= r S_t \, dt + \sigma S_t \, dB_t.$

for $\mu = r - \frac{\sigma^2}{2}.$

Ito

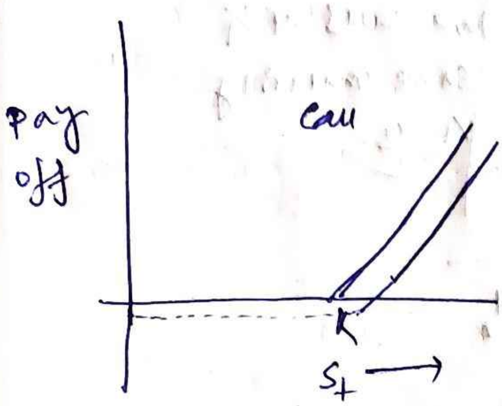
i) compute $d(\log S_t)$

ii) compute $df(t, S_t)$

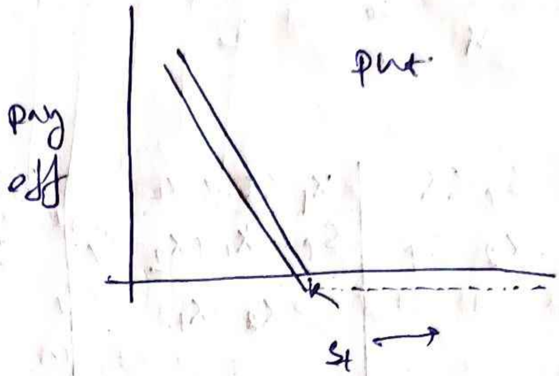
Finance 23/11/24

Trading strategies

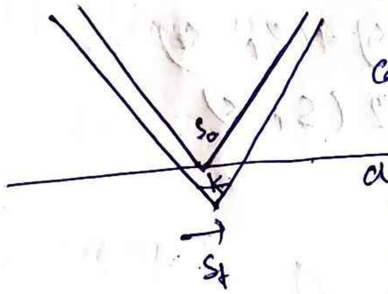
① Bull market.



② Bear market



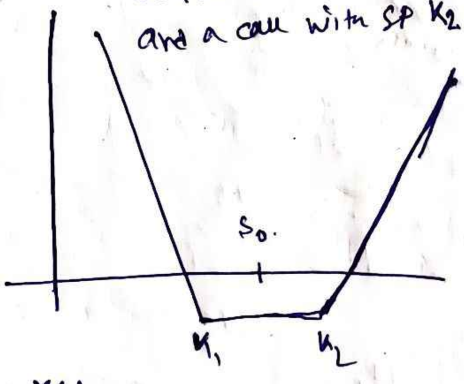
(3) volatile market



Holding call and put at the same strike price K and same maturity K.

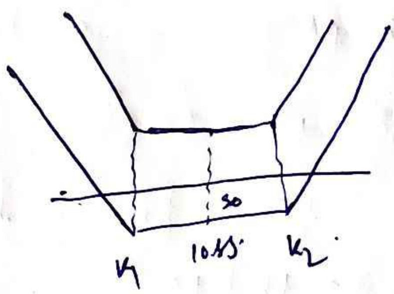
If $S_T > K$ exercise call.
and if $S_T < K$
exercise put.

Buy
a put with SP K_1
and a call with SP $K_2 > K_1$.



$K_1 < S_T < K_2$
If $S_T > K_2$ Exercise call
If $S_T < K_1$.
exercise put.

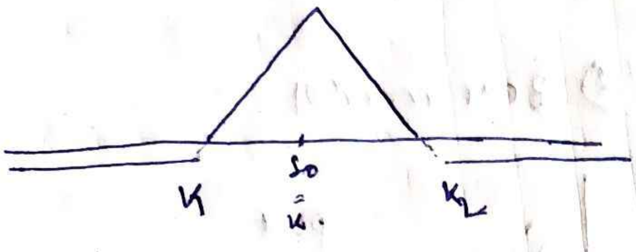
④ Stagnant market



Buy a call at K_1.
Buy a put $K_2 > K_1$

$K_1 < S_0 < K_2$.

Butterfly



Buy two calls one at K_1 and another at K_2, $K_2 > K_1$.

Sell two calls at K with same maturity

$= \frac{K_1 + K_2}{2}$

$S_T < K$	$K_1 < S_T < K_2$	$S_T > K_2$
$K_2 - S_T$	$S_T - K_1 + K_2 - S_T$ $= K_2 - K_1$.	$S_T - K_1$

$S_T < K_1$	$K_1 < S_T < K$	$K < S_T < K_2$	$S_T > K_2$
0	$S_T - K$	$S_T - K_1 - 2(S_T - K)$ $= 2K - K_1 - S_T$ $= K_2 - S_T$	$(S_T - K_1) + (S_T - K_2) - 2(S_T - K)$ $= 0$

H.W.

$$C(K_1) + C(K_2) \geq 2C(K).$$

Where $K_1 < K < K_2$ & $K_1 + K_2 = 2K$.

B-S-M equation

HW Find Δ for put option (European).

f	s
1	$\Delta = \frac{\partial f}{\partial s} = f_s$

$\frac{f_u - f_d}{S_u - S_d}$

$V_0 = (f - \Delta S) \to$ risk neutral

$V_{dt} =$

model of the asset price (S_t).

$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$.

$df(t, S_t) = f_t(t,s)dt + \frac{1}{2} f_{tt} \cancel{(dt)^2} \to 0$

$+ f_s\,dS_t + \frac{1}{2} f_{ss} (dS_t)^2$

$+ f_{st}\,dt \times dS_t$

$dv = df - \Delta\,ds$.

$df(t, s) =$

$(dS_t)^2 = (\mu S_t)^2 \cancel{(dt)^2} \to 0 + (\sigma S_t)^2 (dB_t)^2$

$+ 2\mu\sigma S_t^2 \cancel{dt\,dB_t} \to 0$

$= \sigma^2 S_t^2\,dt$.

$f(t+h, S_{t+h}) - f(t, S_t)$.

$= f_t(t+h-t) + \frac{1}{2} f_{tt} \cancel{(t+h-t)^2} \to 0 \;\; (dt)^2$

$+ f_s(S_{t+h} - S_t) + \frac{1}{2} f_{ss} (S_{t+h} - S_t)^2$

$+ \underbrace{f_{st}(t+h-t)(S_{t+h} - S_t)}_{dt \times dS_t = \mu S_t \cancel{(dt)^2} \to 0 + \sigma S_t \cancel{dB_t\,dt} \to 0} + \cdots$

$$df(t, S_t) = f_t\,dt + f_s\,ds + \frac{1}{2} f_{ss}\,\sigma^2 S_t^2\,dt.$$

$dv = df - \Delta\,ds$.

$df(t, s) = f_t\,dt + f_s \cancel{dS_t} + \frac{1}{2} f_{ss}\,\sigma^2 S_t^2\,dt - f_s \cancel{ds}$.

$= \left(f_t + \frac{\sigma^2 S_t^2}{2} f_{ss}\right) dt$.

for risk neutral pf

$dv = rv\,dt$.

$v_t = v_0 e^{rt}$

$\frac{dv_t}{dt} = v_0 e^{rt} \cdot r$.

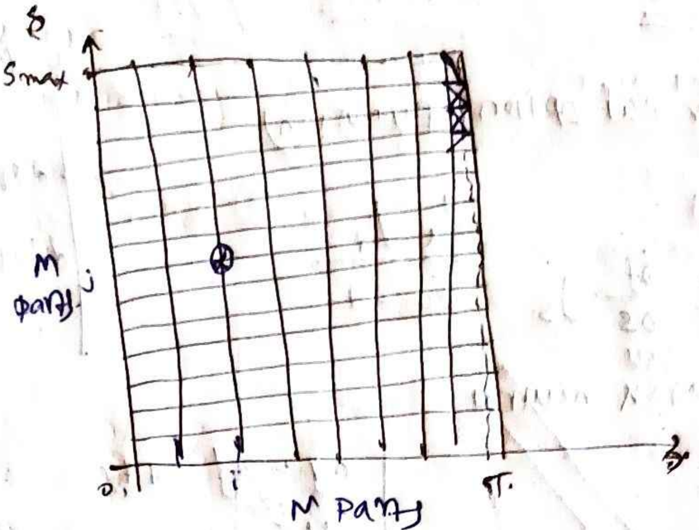
$= r v_t$.

$dv = r(f - f_s S)\,dt$.

$= \left(f_t + \frac{1}{2}\sigma^2 S^2 f_{ss}\right) dt = r(f - f_s S)\,dt$.

$$\Rightarrow \underbrace{\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf}_{\text{B.S.M equation}}$$

Discretization



$(i,j) = (i\Delta t, j\Delta s)$ $\quad i = 0,1,2 \ldots N$

$\quad j = 0,1,2 \ldots M$

$N\Delta t = T$

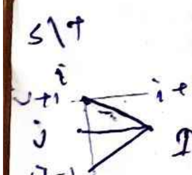
$M\Delta S = S_{max}$

$$\frac{\partial f}{\partial t}(i,j) \approx \frac{f(i+1,j) - f(i,j)}{\Delta t}$$

$$\frac{\partial f}{\partial s}(i,j) \approx \frac{f(i,j+1) - f(i,j)}{\Delta S}$$

$$\frac{\partial^2 f}{\partial s^2}(i,j) \approx \frac{\frac{(f(i,j+1) - f(i,j))}{\Delta S} - \frac{(f(i,j) - f(i,j-1))}{\Delta S}}{\Delta S}$$

$$= \frac{f(i,j+1) + f(i,j-1) - 2f(i,j)}{(\Delta S)^2}$$



Implicit difference

$$\frac{f(i+1,j) - f(i,j)}{\Delta t} + r(j\Delta s)\frac{f(i,j+1) - f(i,j)}{\Delta S}$$

$$+ \frac{\sigma^2 (j\Delta s)^2}{2} \frac{f(i,j+1) - f(i,j-1) - 2f(i,j)}{(\Delta S)^2} = rf(i,j)$$

If i multiply ΔT both sides

$\Delta t \; (\qquad\qquad) = rf(i,j)\Delta T$

$\alpha_j f(i,j+1) + \beta_j f(i,j) + \gamma_j f(i,j-1) = f(i+1,j)$

where $\alpha_j = \left(rj + \frac{\sigma^2 j^2}{2}\right)\Delta T$

$$\beta_j = (1 + r\Delta T) + (rj + \sigma^2 j^2)\Delta T$$

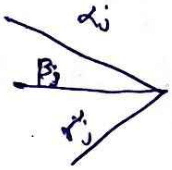
$$\gamma_j = -\frac{\sigma^2 j^2}{2}\Delta T$$

And $\alpha_j + \beta_j + \gamma_j = 1 + r\Delta T$

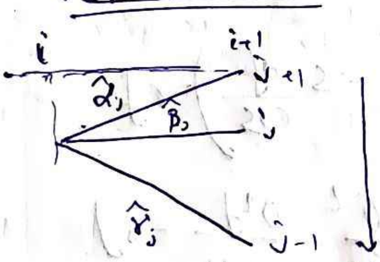
$$\begin{array}{ccc} A & x_i & x_{i+1} \\ \begin{bmatrix} \beta_0 & \alpha_0 & & & \\ \gamma_1 & \beta_1 & \alpha_1 & & \\ 0 & \gamma_2 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \alpha_{m-1} \\ & & & \gamma_m & \beta_m \end{bmatrix} & \begin{pmatrix} f(i,0) \\ f(i,1) \\ f(i,2) \\ \vdots \\ f(i,m-1) \\ f(i,m) \end{pmatrix} & = \begin{pmatrix} f(i+1,0) \\ f(i+1,1) \\ f(i+1,2) \\ \vdots \\ f(i+1,m) \end{pmatrix} \end{array}$$

$$A x_{N-1} = x_N$$
$$A x_{N-2} = x_{N-1}$$
$$\vdots$$
$$A x_0 = x_1$$
$$A^N x_0 = x_N$$

Stable but computationally intensive.



Explicit diff



$$\Delta T\left(\frac{f(i,j) - f(i-1,j)}{\Delta T} + r(j\Delta S)\frac{f(i,j+1) - f(i,j)}{\Delta S} + \frac{\sigma^2 (j\Delta S)^2}{2}\frac{f(i,j+1) + f(i,j-1) - 2f(i,j)}{(\Delta S)^2}\right)$$

$$= \hat{\alpha}_j f(i,j+1) + \hat{\beta}_j f(i,j) + \hat{\gamma}_j f(i,j-1) - f(i-1,j) = r f(i,j)\Delta T$$

where $\hat{\alpha}_j = \left(rj + \frac{\sigma^2 j^2}{2}\right)\Delta T$

$$\hat{\beta}_j = (1 - r\Delta T) - (rj + \sigma^2 j^2)\Delta T$$

$$\hat{\gamma}_j = \frac{\sigma^2 j^2}{2}\Delta T$$

$$\hat{\alpha}_j + \hat{\beta}_j + \hat{\gamma}_j = 1 - r\Delta T$$

one eqn one unknown
computationally much less.
but not as stable.

<u>Transformation</u>

$$z = \log s \Rightarrow s = e^{z}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial s} \frac{\partial s}{\partial z} = \frac{\partial f}{\partial s} \cdot e^{z}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial z}\right)\frac{\partial s}{\partial z}$$

$$= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s} e^{z}\right)\frac{\partial s}{\partial z}$$

$$= \left(\frac{\partial^2 f}{\partial s^2} \cdot s + \frac{\partial f}{\partial s}\right) \cdot s$$

$$= s^2 \frac{\partial^2 f}{\partial s^2} + s \frac{\partial f}{\partial s}$$

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 f}{\partial s^2} = rf$$

Transformed one
~~Transformation~~

$$\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial z} + \frac{\sigma^2}{2}\left(\frac{\partial^2 f}{\partial z^2} - s\frac{\partial f}{\partial s}\right) = rf$$

$$\frac{\partial f}{\partial t} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial z^2} = rf$$

$$\hat{\alpha}_z f(i, j+1) + \hat{\beta}_z f(i, j) + \hat{\gamma}_z f(i, j-1) = f(i-1, j)$$

where $\hat{\alpha}_z = \left(\frac{\left(r - \frac{\sigma^2}{2}\right)}{\Delta s} + \frac{\sigma^2}{2(\Delta s)^2}\right)\Delta t > 0$

$$\hat{\beta}_z = (1 - r\Delta t) - \left(\frac{\left(r - \frac{\sigma^2}{2}\right)}{\Delta s} + \frac{\sigma^2}{(\Delta s)^2}\right)\Delta t$$

$$\hat{\gamma}_z = \frac{\sigma^2}{2(\Delta s)^2}\Delta t > 0$$

and $\hat{\alpha}_z + \hat{\beta}_z + \hat{\gamma}_z = 1 - r\Delta t$

Finance 05/10/24

Ex

Let $M=2$

$N=1$

$$\begin{array}{ccc} 0 & & 1 \\ S_{10} & \nearrow & S_{1up} \\ & \searrow & S_{1Down} \end{array}$$

$r_f \rightarrow$ Risk free rate of growth

$r_f = 6\%$. $S_{10} = 100$, $S_{1u} = 103$, $S_{1D} = 98$.

$$S_{10} \rightarrow S_{10}(1+r_f), \quad S_{10} \rightarrow S_{1u}, \quad S_{10} \rightarrow S_{1d}.$$

$\phi_1 = 10$

Share sell 10 unit — $100 \times 10 = 1000$.

$\phi_0 \rightarrow$ riskfree $\qquad \phi_1 \rightarrow$ risky $= -10$.

$\phi_1 S_{10} + \phi_0 S_{00} = 0$ $\qquad$ [Assuming $S_{00} = 1$]

$\phi_0 = \cancel{10}\ 100 \times 10$ $\qquad$ (0 money investment)

$= 1000 \rightarrow 1000(1+r_f)$

$= 1000(1+\frac{6}{100})$

$= 1060$

Assume $S_{0u} = 1$.

for price $S_{1u} \times 10 = 1030$.

Balance $= 1060 - 1030$

$= 30$

for price $S_{1D} \times 10 = 980$

Balance $= 1060 - 980$

$= 80$

} arbitrage system.

$$D_1 = \begin{pmatrix} 1+r_f \\ S_{1u} \end{pmatrix} \qquad D_2 = \begin{pmatrix} 1+r_f \\ S_{1D} \end{pmatrix}$$

If $S_{1U} = S_{10}(1+r_f)$ then $\varphi' D_j \geq 0$

$$S_{10} \begin{cases} \nearrow S_{1U} \\ \rightarrow S_{1D} \\ \searrow S_{10}(1+r_f) \end{cases}$$

arbitrage.

$S_{10} = 100 \quad S_{U} = 110 \quad S_{1D} = 107$

Short sale risk free asset investment in the risky asset.

$$S_{10} \begin{cases} \nearrow S_{1U} \\ \rightarrow S_0(1+r_f) \\ \searrow S_{1D} \end{cases}$$

no arbitrage.

$\nexists\ \varphi_a$ s.t

$\varphi' S_0 = 0$ and $\varphi_j' D_j \geq 0 \ \forall j$

and $\varphi' D_{j_0} > 0$ for some j_0.

First fundamental theorem

A market has no arbitrage iff $\exists$ a strictly positive martingale

$$\begin{array}{cc} \text{col1} & \text{col2} \end{array}$$
$$D = \left(\begin{pmatrix} 1+r \\ S_{1U} \end{pmatrix} \begin{pmatrix} 1+r \\ S_{1D} \end{pmatrix} \right)$$

Does there exist a p.

$$\underset{\sim}{S_0}(1+r) = D\underset{\sim}{p}$$

$$\underset{\sim}{S_0} = \underset{\sim}{\hat{S_0}} = E_p(\tilde{S}_1 \mid \mathcal{F}_0)$$

$$\tilde{S}_1 = \underset{\sim}{S}(1+r)^{-1}.$$

$$\underset{\sim}{S_0}(1+r) = E_p(S \mid \mathcal{F}_0)$$

Have to solve for P_U and P_d.

$$\begin{pmatrix} 1 \\ S_{10} \end{pmatrix}(1+r) = \begin{pmatrix} 1+r & 1+r \\ S_{1U} & S_{1D} \end{pmatrix} \begin{pmatrix} P_U \\ P_d \end{pmatrix}.$$

$(p_u + p_d)(1+r) = (1+r) \Rightarrow p_u + p_d = 1.$

$S_{1u} p_u + S_{1d} p_d = S_{10}(1+r)$

$S_{1u} p_u + S_{1d} p_d = S_0(1+r)$

$S_{1u} p_u + S_{1d}(1-p_u) = S_0(1+r)$

$p_u(S_{1u} - S_{1d}) + S_{1d} = S_0(1+r)$

$$p_u = \frac{S_0(1+r) - S_{1d}}{S_{1u} - S_{1d}}.$$

<u>note</u>

$0 \le p_u \le 1$

$S_{1u} > S_{10}(1+r_f) > S_{1d} \rightarrow$ no arbitrarage.

<u>Pf</u>

suppose $\exists\ \underset{\sim}{p}$ (strictly positive vector) s.t $S_0(1+r) = D\underset{\sim}{p}$. If possible let $\underset{\sim}{\phi}$ be a (non zero) vector s.t

(i) $\underset{\sim}{\phi}' S_0 = 0$ and $\underset{\sim}{\phi}' D_j \geq 0\ \forall j$ and $\phi' D_{j_0} > 0$ for some j_0.

Then $\boxed{\underset{\sim}{\phi}' S_0(1+r) = \underset{\sim}{\phi}' D_p}$ (*) note L.H.S $= 0$.

and R.H.S $\phi' D_p = \sum_{j=1}^{m} \phi' D_j P_j \geq \phi' D_{j_0} P_{j_0} > 0$

Hence contradiction i.e (i) can not happen.

for (ii) $\phi' S_0 < 0$ & $\phi' D_j \geq 0\ \forall j$ in that case L.H.S of (*) < 0 but R.H.S $\sum \phi' D_j P_j \geq 0$

Hence contradiction (ii)

Then there cannot be any arbitrage in the market.

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Suppose the market has no arbitrage.

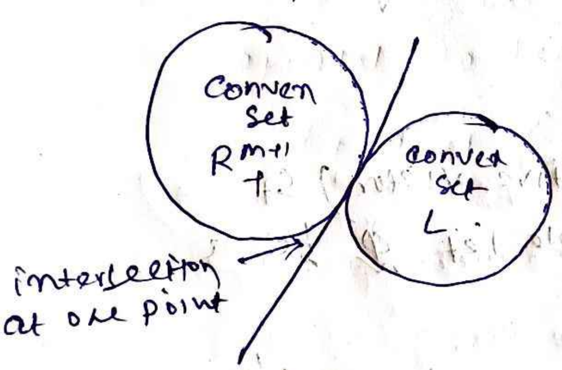
Let $L = \{(-\underset{\sim}{\phi}' S_0, \underset{\sim}{\phi}' \underset{\sim}{D}_1, \cdots \underset{\sim}{\phi}' \underset{\sim}{D}_M) : \phi \in \mathbb{R}^{M+1}\}$

$\mathbb{R}_+^{M+1} = \{\underset{\sim}{x} : x_i \geq 0 \quad \forall i = 1, 2 \cdots M+1\}$

Then $L \cap \mathbb{R}_+^{M+1} = \{\underset{\sim}{0}\}$ under no arbitrage

note L is a subspace (closed under addition)

In particular L is a convex set & So is R_+^{M+1}



By separating hyperplane.

Therefore by separating hyperplane theorem there exists a hyperplane $H = \{\underset{\sim}{x} \in \mathbb{R}^{M+1} : \sum_{j=1}^{M+1} a_j x_j = 0\}$ passing through the origin s.t

$$\underset{\sim}{a}' \underset{\sim}{y} \leq \underset{\sim}{a}' \underset{\sim}{z} \quad \forall y \in L \text{ and } z \in R_+^{M+1}$$

Can $a'y < 0$

If possible let $\underset{\sim}{a}' \underset{\sim}{y}' < 0$

$\{\underset{\sim}{a}' \underset{\sim}{z} : z \in \mathbb{R}_+^{M+1}\} \supseteq (0, \infty)$

Then $-y \in L$ and $a'(-y) > 0$ and that's a contradiction

$\therefore a'y \not> 0$ for any $y \in L$.

Therefore $\underset{\sim}{a}' \underset{\sim}{y} = 0 \ \forall \ \underset{\sim}{y} \in L \Rightarrow L \subseteq H$

$\Rightarrow \underset{\sim}{a}' \underset{\sim}{z} > 0 \ \forall \ \underset{\sim}{z} \neq 0$ take $z_j = (0, 0, 0 \cdots \underset{\uparrow}{1}, 0 \cdots 0)$
jth position

$\underset{\sim}{a}' \underset{\sim}{z_j} = a_j > 0 \ \forall j$

then $\sum_{j=1}^{M} a_j \phi D_j = a_0 \phi S_0 \quad \forall \phi \in \mathbb{R}^{m+1}$

$\Rightarrow \sum a_j D_j = a_0 S_0$ choosing $\phi_j = (0, 0 \cdots 0, \cdots \underset{\uparrow}{1}, 0 \cdots 0)$
jth position

$\Rightarrow \sum \frac{a_j D_j}{a_0} = S_0 \Rightarrow \sum \pi_j D_j = S_0$.

Observing the 1st row target $\sum \pi_j (1+r) = 1$

$\Rightarrow \sum \pi_j = \frac{1}{1+r}$

Define $p_j = \frac{\pi_j}{\sum \pi_j} > 0 \Rightarrow \sum_{j=1}^{M} p_j D_j = \frac{S_0}{\sum \pi_j} = S_0 (1+r)$

Thus $\underset{\sim}{p}$ is the strictly martiangle measure.

If there are 2 assets (1 risky + 1 risk free) then we can have more than one martiangle due to martiangle from the equation $Dp = S_0(1+r)$

Ex

$$\begin{tikzcd}[column sep=small, row sep=small] & S_{1u} \\ S_{10} \arrow[ur] \arrow[r] \arrow[dr] & S_{1m} \\ & S_{1D} \end{tikzcd}$$

$$Dp = \begin{pmatrix} 1+r & 1+r & 1+r \\ S_{1u} & S_{1M} & S_{1D} \end{pmatrix} \begin{pmatrix} p_u \\ p_M \\ p_d \end{pmatrix} = \begin{pmatrix} 1 \\ S_0 \end{pmatrix} (1+r)$$

$$p_u^{(1)} = \frac{S_{10}(1+r) - S_{1M}}{S_{1u} - S_{1M}} \qquad p_M^{(1)} = 1 - p_u. \quad p_d^{(1)} = 0$$

$$p_u^{(2)} = \frac{S_0(1+r) - S_{1d}}{S_{1u} - S_{1d}} \qquad p_d^{(2)} = 1 - p_u^{(2)} \quad p_M^{(2)} = 0.$$

$$\underset{\sim}{p}^{(\alpha)} = \alpha \underset{\sim}{p}^{(1)} + (1-\alpha) \underset{\sim}{p}^{(2)} > 0 \qquad 0 < \alpha < 1$$

Check that $E_{p(\alpha)}(\underset{\sim}{S_1} \mid \mathcal{F}_0) = \underset{\sim}{S_0}(1+r)$.

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$var(X_T) = c\sigma^2$

$E(X_T) = 2(P_n - 1)\log u_n$

$$= \frac{n\left(r_T - \frac{\sigma^2}{2}c\right)}{\sigma\sqrt{cn}} + \sigma\sqrt{\frac{c}{n}}$$

$$= \left(r_T - \frac{\sigma^2}{2}c\right)\frac{n}{n} + o\left(\frac{1}{n^{1/2}}\right)$$

$$\rightarrow r_T - \frac{\sigma^2}{2}c$$

As $c = T$ as the growth of the variance is the proportial to the time elapsed.

$$X_T \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$