28/9 FAIR PRICING.

Let $\underset{\sim}{S_0} = (S_{00}, S_{01}, S_{02}, \ldots, S_{0N})^T \longrightarrow$ price vector or N+1 assets.

at time 0. closing price

the 1st one is the risky asset.

& $\underset{\sim}{\phi_0} = (\phi_{00}, \ldots, \phi_{0N})$.

No's representing how many one wishes to buy at time '0'

Thus $V_0 = \underset{\sim}{\phi_0}' \underset{\sim}{S_0}$

initial value of the pf.

$V_n = \underset{\sim}{\phi_n}' \underset{\sim}{S_n}$ value of pf at time n.

A pf is called self-financing if

$\underset{\sim}{\phi_i}' \underset{\sim}{S_{i-1}} = \underset{\sim}{\phi_{i-1}}' \underset{\sim}{S_{i-1}} - \underbrace{C_{i-1}}_{\text{consumption}} \quad \forall i$

Simple Random walk

$S_n = S_0 + X_1 + \ldots + X_n$

where $X_i = \begin{cases} +1 & \text{wp } p \\ -1 & \text{wp } 1-p. \end{cases}$ (S_0 is fixed). indept wrt $\{X_i\}$.

X_i iid

$E(X_i) = p(1) + (-1)(1-p) = 2p-1$

$\{S_n\}$ is said to be an martingale if

i) $E|S_n| < \infty$

*** ii) $E(S_{n+1} | \mathcal{F}_n) = S_n \quad \forall n$

$\mathcal{F}_n$: inf abt past all S_i's

$\geq \rightarrow$ submartingale

$\leq \rightarrow$ supermartingale.

$$E(S_{n+1} \mid \mathcal{F}_n) = E(S_n + X_{n+1} \mid \mathcal{F}_n)$$

$$= E(S_n \mid \mathcal{F}_n) + E(X_{n+1} \mid \mathcal{F}_n).$$

$$= S_n + E(X_{n+1}). \quad \text{indept.}$$

$$= \begin{cases} S_n + 2p - 1 \\ S_n & \text{if } p = \frac{1}{2} \end{cases}$$

From martingale: $E(S_{n+1} \mid \mathcal{F}_n) = S_n.$

$$\Rightarrow \underbrace{E(E(S_{n+1} \mid \mathcal{F}_n))}_{= E(S_{n+1})} = E(S_n).$$

Also, $$\underbrace{E(E(S_{n+1} \mid \mathcal{F}_n) \mid \mathcal{F}_{n-1})}_{= E(S_{n+1} \mid \mathcal{F}_{n-1})} = E(S_n \mid \mathcal{F}_{n-1}) = S_{n-1}$$

$$E(E(X \mid Y)) = E(X)$$

$$E(E(X \mid Y, Z) \mid Y) = E(X \mid Y)$$

Rough Proof: $= \sum_z \sum_x x \cdot P(X = x \mid Y = y, Z = z) \cdot P(Z = z \mid Y = y).$

$$= \sum_z \sum_x x \cdot \frac{P(x, y, z)}{P(y, z)} \cdot \times P(Z = z \mid Y = y)$$

$$= \sum \sum x \frac{P(X = x, Y = y, Z = z)}{\cancel{P(Y = y, Z = z)}} \cdot \frac{\cancel{P(Y = y, Z = z)}}{P(Y = y)}.$$

$$= \sum_z \sum_x x \cdot \frac{P(X = x, Y = y, Z = z)}{P(Y = y)}$$

$$= \sum_x x \sum_z \frac{P(X=x, Y=y, Z=z)}{P(Y=y)}.$$

$$= \sum_x x \cdot \frac{P(X=x, Y=y)}{P(Y=y)} = \sum x \cdot P(X=x \mid Y=y).$$
$$= E(X \mid Y).$$

Generalizing,

$$E(S_{n+1} \mid \mathcal{F}_i) = S_i \quad i \leq n$$

? $\therefore E\left[E\{E(S_{n+1} \mid \mathcal{F}_n) \mid \mathcal{F}_{n-1}\} \mid \mathcal{F}_{n-2}\right] =$
$$= E(S_{n+1} \mid \mathcal{F}_{n-2}).$$

Let τ_a be the time $S_n = a$ for the first time

Define $\tau_a \wedge \tau_b = \tau$ for some $a < c < b$

$$c \downarrow S_0.$$

If i) $P(\tau < \infty) = 1$

ii) $E|S_\tau| < \infty$

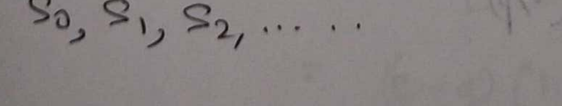
iii) $E|S_m| \mathbb{1}_{\{\tau > m\}} \longrightarrow 0$ as $m \to \infty$.

$$\tau = \begin{cases} \tau_a & \text{if } \tau_a < \tau_b \\ \tau_b & \text{if } \tau_b < \tau_a. \end{cases}$$

$|S_m| \mathbb{1}_{(\tau \geq m)}$
$\leq \max(|a|, |b|) E(\mathbb{1}_{\{\tau \geq m\}}).$

$S_0, S_1, S_2, \ldots\ldots$

$S_{\tau_a} < S_m < S_{\tau_b}.$



$|S_i| \quad i = 0, .., 999$
for $\tau = 1000$.

$$|S_i| \leq |a| \vee |b| \quad \forall i \leq \tau$$

Then $E(\tau_c \mid \mathcal{F}_0) = S_0 = c$

$$S_\tau = \begin{cases} a & \text{if } \tau_a < \tau_b. \\ b & \text{if } \tau_b < \tau_a. \end{cases}$$

$$a \cdot \underbrace{P(\tau_a < \tau_b \mid \mathcal{F}_0)}_{P_{a,b}} + b \cdot \underbrace{P(\tau_b < \tau_a \mid \mathcal{F}_0)}_{P_{b,a}.}$$

$$\boxed{\begin{aligned} &\Rightarrow a \cdot P_{a,b} + b \cdot P_{b,a} = c. \\ &\Rightarrow P_{a,b}(a-b) = c \Rightarrow \end{aligned}}$$

$\Rightarrow a \cdot P_{a,b} + b(1 - P_{b,a}) = c$

$\Rightarrow P_{a,b}(a-b) = c - b.$

$\Rightarrow P(\tau_a < \tau_b \mid S_0 = c) = \dfrac{c-b}{a-b} = \dfrac{b-c}{b-a}.$

$Y_n = S_n - (p-q)n$

$E(Y_\tau \mid \mathcal{F}_0) = Y_0.$ product/ exponential martingale

$Z_n = (q/p)^{S_n}$

$Z_{n+1} = (q/p)^{S_{n+1}} = (q/p)^{S_n} \times (q/p)^{X_{n+1}}$

$= Z_n (q/p)^{X_{n+1}}$

$$E(Z_\tau | \mathcal{F}_0) = Z_0 = (q/p)^c$$

$$\Rightarrow (q/p)^a P(\tau_a < \tau_b | S_0 = c).$$

$$+ \left(\frac{q}{p}\right)^b P(\tau_b < \tau_a | S_0 = c) = (q/p)^c$$

$$\Rightarrow \left(\frac{q}{p}\right)^a P_{a,b} + \left(\frac{q}{p}\right)^b (1 - P_{a,b}) = \left(\frac{q}{p}\right)^c$$

$$\Rightarrow P_{a,b} = \frac{\left(\frac{q}{p}\right)^c - \left(\frac{q}{p}\right)^b}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b} = \frac{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^c}{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^a}$$

For finding $E(\tau)$ use $\{Y_n\}$ for $p \neq q$

and use $W_n = S_n^2 - n$ for $p = q$

(Doob-Meyer Decomposition)

Submartingale can be expressed as sum of martingle + increasing fn. (rv).

$$E(S_\tau^2 - \tau | S_0 = c) = c^2$$

$$\Rightarrow E(\tau | S_0 = c) = E(S_\tau^2 | S_0 = c) - c^2.$$

$$= -c^2 + \left[a^2 P(\tau_a < \tau_b | S_0 = c) + b^2 P(\tau_b < \tau_c | S_0 = c)\right]$$

$$= -c^2 + \left[a^2\left(\frac{b-c}{b-a}\right) + b^2\left(\frac{c-a}{b-a}\right)\right]$$

$$= (c-a)(b-c).$$

Suppose:

$\{S_n\}$ is a martingale (componentwise)

Then $\{V_n\}$ is a martingale

$$V_n = \phi_n' S_n = \sum_{j=1}^{n} (\phi_j' S_j - \phi_{j-1}' S_{j-1}) + \overbrace{V_0}^{=\phi_0' S_0}$$

$$= \sum_{j=1}^{n} (\phi_j' S_j - \phi_j' S_{j-1})$$

$$+ V_0 - \sum_{j=1}^{n} c_{j-1}$$

$\phi_j' S_{j-1}$
$= \phi_{j-1} S_{j-1} - c_{j-1}$

$$V_n = V_{n-1} + (\phi_n' S_n - \phi_{n-1}' S_{n-1}).$$

Check (i) $E|V_n| < \infty$ if $|S_n| < \infty$
& $|\phi_n|$ is bounded.

To check

ii) $E(V_n \mid \mathcal{F}_{n-1})$

$= V_{n-1} + E(\phi_n' (S_n - S_{n-1}) - c_{n-1} \mid \mathcal{F}_{n-1})$

$= V_{n-1} + \phi_n' E(\underbrace{E(S_n \mid \mathcal{F}_{n-1}) - S_{n-1}}_{=0}) - c_{n-1}$

~~$= V_{n-1}$ if $c_n = 0$.~~

~~if $c_n >$~~

$= V_{n-1} - c_{n-1}$

$$\begin{cases} = V_{n-1} & \text{if } c_{n-1} = 0. \\ \leq V_{n-1} & \text{if } c_{n-1} \geq 0. \\ \geq V_{n-1} & \text{if } c_{n-1} \leq 0. \end{cases}$$

Note:

When consumptions are 0 then ~~$V_n = V_0 + \sum \phi_j'(S_j$~~

$$V_n = V_0 + \sum \underbrace{\phi_j'}_{\text{predictable}} \underbrace{(S_j - S_{j-1})}_{\text{martingale difference}}.$$

is a martingale whenever $\{S_n\}$ is a martingale.

Real Market Model:

$$E(\underset{\sim}{S_j} \mid \mathcal{F}_{j-1}) = S_{j-1}(1+r)$$

r ↳ risk free rate of growth

ARBITRAGE

$$\phi_1' S_0 = \phi_0' S_0$$

If i) $\underset{\sim}{\phi}' S_0 = 0$. & $\phi' S_1 = \phi' D_j \geq 0$ for some ϕ.

& $\phi' D_{j_0} > 0$ for some j_0 & ϕ

then market is said to have arbitrage

m-scenarios.

$$S_1 = \begin{pmatrix} S_{10} \\ S_{11} \\ \vdots \\ S_{N1} \end{pmatrix} = \left[\begin{pmatrix} D_{10} \\ D_{11} \\ \vdots \\ D_{1N} \end{pmatrix} \cdots \begin{pmatrix} D_{M0} \\ D_{M1} \\ \vdots \\ D_{MN} \end{pmatrix} \right]$$

ii) $\phi' S_0 < 0$ & $\phi' D_j \geq 0 \; \forall j$

$\phi' D_j \geq \phi' S_0 \; \forall j$ & $\phi D_{j_0} > \phi' S_0$ for some j_0.

If $\exists$ any such ϕ then market is said to ~~be~~ have no arbitrage.

finance

Example:-

Let $M=2$

$N=1$

$$\begin{array}{ccc} 0 & & 1 \\ & \nearrow & S_{1up} \\ S_{10} & & \\ & \searrow & S_{1Down} \end{array}$$

$r_t \rightarrow$ Risk free rate of growth

$r_f = 6\%$, $S_{10} = 100$, $S_{1u} = 103$, $S_{1D} = 98$

$\phi_0 \quad \phi_1 = 10$

100×10

$\phi_1 S_{10} + \phi_0 \times 1 = 0$

$\phi_0 = 1000(1+0.06) = 1060$

Assume $S_{0u} = 1$

for Price $S_{1u} \times 10 = 1030$

Balance $= 1060 - 1030 = 30$ — Arbitrage System.

for price $S_{1D} \times 10 = 980$

Balance $= 1060 - 980 = 80$

$$D_1 = \begin{pmatrix} 1+r_f \\ S_{1u} \end{pmatrix} \quad D_2 = \begin{pmatrix} 1+r_f \\ S_{1D} \end{pmatrix} \quad \phi_1' D_1 \geq 0 \quad \phi_2' D_2 \geq 0$$

$$S_{10} \rightarrow S_{1u}, \quad S_{10} \rightarrow S_{1D}, \quad S_{10} \rightarrow S_{10}(1+r_f) \quad \leftarrow \boxed{\text{Arbitrage}}$$

$S_{10} = 100$, $S_{1u} = 110$, $S_{1D} = 107$ | short sale risk free asset investing in the risky asset

$$S_{10} \rightarrow S_{1u}, \quad S_{10} \rightarrow S_{10}(1+r_f), \quad S_{10} \rightarrow S_{1D} \quad \leftarrow \boxed{\text{No Arbitrage}} \Rightarrow \nexists\, \phi \text{ st } \phi' S_0 = 0 \text{ and } \phi' D_j \geq 0 \ \forall j$$

and $\phi' D_{j_0} > 0$ for some j_0

First fundamental theorem (math finance):-

A market has no arbitrage iff $\exists$ a strictly positive martingale measure

Does there exist a $\underset{\sim}{p}$

$$\begin{array}{ccc} & \text{Col 1} & \text{Col 2} \\ D = & \begin{bmatrix} 1+r & 1+r \\ S_{1u} & S_{1D} \end{bmatrix} \end{array}$$

$\underset{\sim}{S_0}(1+r) = D\underset{\sim}{p}$

$\underset{\sim}{S_0} = \hat{\underset{\sim}{S_0}} = E_{\underset{\sim}{p}}(\tilde{S}_1 \mid \mathcal{F}_0)$

$\hat{\underset{\sim}{S_1}} = \underset{\sim}{S_1}(1+r)^{-1}$

$\Leftrightarrow \underset{\sim}{S_0}(1+r) = E_p(\underset{\sim}{S_1} \mid \mathcal{F}_0)$

Have to solve for p_u & p_d

$$\begin{pmatrix} 1 \\ S_{10} \end{pmatrix}(1+r) = \begin{bmatrix} 1+r & 1+r \\ S_{1u} & S_{1d} \end{bmatrix}\begin{pmatrix} p_u \\ p_d \end{pmatrix}$$

$\Rightarrow (p_u + p_d)(1+r) = (1+r) \Rightarrow p_u + p_d = 1$

$S_{1u} p_u + S_{1d} p_d = S_{10}(1+r)$

$\Rightarrow S_{1u} p_u + S_{1d}(1-p_u) = S_{10}(1+r)$

$\Rightarrow p_u(S_{1u} - S_{1d}) + S_{1d} = S_{10}(1+r)$

$\Rightarrow p_u = \dfrac{S_{10}(1+r) - S_{1d}}{S_{1u} - S_{1d}}$

Note

$0 < p_u < 1$

$\Leftrightarrow S_{1u} > S_{10}(1+r_f) > S_{1d} \longrightarrow$ $\boxed{\text{No arbitrage.}}$

Pf

Suppose $\exists\ \underset{\sim}{p}$ (strictly positive) vector st $S_0(1+r) = D\underset{\sim}{p}$

If possible let $\underset{\sim}{\phi}$ be a (non-zero) vector st

(i) $\underset{\sim}{\phi}' \underset{\sim}{S_0} = 0$ and $\underset{\sim}{\phi}' D_j \geq 0\ \forall j$ and $\phi' D_{j_0} > 0$ for some j_0

Then $\boxed{\underset{\sim}{\phi}' \underset{\sim}{S_0}(1+r) = \underset{\sim}{\phi}' DP}$ (*) . Note LHS $= 0$ $\quad \left[D\underset{\sim}{p} = \sum_{j=1}^{M} D_j p_j\right]$

and RHS $\phi' DP = \sum_{j=1}^{M} \phi' D_j p_j \geq \phi' D_{j_0} p_{j_0} + 0 > 0$

Hence contradiction i.e (i) cannot happen

for (ii) $\phi' S_0 < 0$ and $\phi' D_j \geq 0\ \forall j$.

In that case LHS of (*) < 0 but RHS $\sum \phi' D_j p_j \geq 0$

Hence contradicting (ii)

Thus there cannot be any arbitrage in the market.

Pf

Suppose the market has no arbitrage. Let $L = \{(-\underset{\sim}{\phi}' S_0, \underset{\sim}{\phi}' \underset{\sim}{D_1}, \ldots, \underset{\sim}{\phi}' \underset{\sim}{D_M}) : \phi \in \mathbb{R}^{M+1}\}$

Then,

$\mathbb{R}_+^{M+1} = \{\underset{\sim}{x} : x_i \geq 0\ \forall\ i = 1, \ldots, M+1\}$ $\quad \boxed{L \cap \mathbb{R}_+^{M+1} = \{\underset{\sim}{0}\}}$

under no arbitrage

Note:- L is a subspace (Closed under addition & scalar multiplication)

In particular L is a convex set and so is $\mathbb{R}_+^{M+1}$

Therefore by seperating hyperplane theorem there exists

a hyperplane $H = \left\{ \underset{\sim}{x} \in \mathbb{R}^{M+1} ;\ \sum_{j\geq 1}^{M+1} a_j x_j = 0 \right\}$ passing through

the origin, s.t. $\underset{\sim}{a}'\underset{\sim}{y} < \underset{\sim}{a}'\underset{\sim}{z} \quad \forall\ y \in L \ \&\ z \in \mathbb{R}_+^{M+1},\ y,z \neq \{0\}$

$\boxed{\text{can } a'y < 0 \ ?}$

if possible let $\underset{\sim}{a}'\underset{\sim}{y} < 0$

$\left\{ \underset{\sim}{a}'\underset{\sim}{z} : z \in \mathbb{R}_+^{M+1} \right\} \supseteq (0, \infty)$

then $-y \in L$ and $a'(-y) > 0$ and that's a contradiction.

$\therefore\ a'y \not> 0$ for any $y \in L$.

Therefore $\underset{\sim}{a}'\underset{\sim}{y} = 0 \quad \forall\ \underset{\sim}{y} \in L \Rightarrow L \subseteq H$

$\Rightarrow \underset{\sim}{a}'\underset{\sim}{z} > 0 \quad \forall\ \underset{\sim}{z} \neq \underset{\sim}{0}$. take $z_j = (0, 0, \ldots, \underset{j\text{th position}}{1}, 0, \ldots 0)$

$\underset{\sim}{a}' z_j = a_j > 0\ \forall j$, then $\sum_{j\geq 1}^{M} a_j \phi D_j = a_0 \phi' S_0 \quad \forall\ \phi \in \mathbb{R}^{M+1}$

$\Rightarrow \sum a_j D_j = a_0 S_0$ choosing $\phi_j = (0, 0, \ldots, \underset{i\text{th position}}{1}, 0, \ldots 0)$

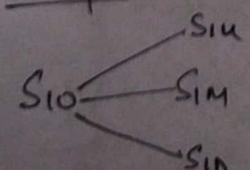
$\Rightarrow \sum \frac{a_j D_j}{a_0} = S_0 \Rightarrow \sum \pi_j D_j = S_0$

Observing the 1st row to get, $\sum \pi_j (1+r) = 1 \Rightarrow \sum \pi_j = \frac{1}{1+r}$.

Define $p_j = \frac{\pi_j}{\sum \pi_j} > 0 \Rightarrow \sum_{j\geq 1}^{M} p_j D_j = \frac{S_0}{\sum \pi_i} = S_0(1+r)$

Thus $\underset{\sim}{p}$ strictly is the martingale measure.

Example



$$D\underset{\sim}{p} = \begin{pmatrix} 1+r & 1+r & 1+r \\ S_{1u} & S_{1M} & S_{1d} \end{pmatrix} \begin{pmatrix} p_u \\ p_m \\ p_d \end{pmatrix} = \begin{pmatrix} 1 \\ S_0 \end{pmatrix} (1+r)$$

$$p_u^{(1)} = \frac{S_{10}(1+r) - S_{1m}}{S_{1u} - S_{1m}}, \quad p_m^{(1)} = 1 - p_u^{(1)}, \quad p_d^{(1)} = 0$$

$$p_u^{(2)} = \frac{S_{10}(1+r) - S_{2m}}{S_{2u} - S_{2m}}, \quad p_d^{(2)} = 1 - p_d^{(2)}, \quad p_m^{(2)} = 0$$

$$p^{(\alpha)} = \alpha p^{(1)} + (1-\alpha) p^{(2)} > 0 \qquad 0 < \alpha < 1$$

Check that,

$$E_{p^{(\alpha)}}\left(S_1 \,\middle|\, \mathcal{F}_0\right) = S_0 (1+r)$$

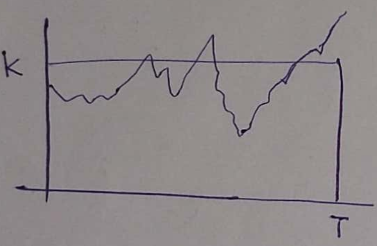
19/10/24

Replicability.

A contigent clause $\{f\}$ is said to be replicable (at time 1) if $\exists\ \underset{\sim}{\Phi}$ st,

$$\underset{\sim}{\Phi}' S_1 = f_1 \quad \begin{bmatrix} \text{only 2 time steps} \\ 0 \ \& \ 1 \end{bmatrix}$$

Call option: A right to buy certain asset or commodity at a prefered price (say K) or before a maturity time (say T).



$$f_T = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K. \end{cases}$$

$$= \max(S_T - K, 0).$$

$$= (S_T - K)^+$$

$M = 2$

$N = 1$

$T = 1$

$$S_0 \begin{cases} S_u \\ S_d \end{cases}$$

$$f_0 \begin{cases} f_u = f(S_u) \\ f_d = f(S_d) \end{cases}$$

$$\phi_0(1+r) + \phi_1 S_u = f_u$$

$$\phi_0(1+r) + \phi_1 S_d = f_d$$

$$\Rightarrow \phi_1(S_u - S_d) = f_u - f_d$$

$$\Rightarrow \phi_1 = \frac{f_u - f_d}{S_u - S_d}$$

$$(\phi_0 \ \ \phi_1) \begin{pmatrix} 1+r & 1+r \\ S_u & S_d \end{pmatrix} = (f_u \ \ f_d)$$

& solve ϕ_0.

$$(\phi_0, \phi_1)\begin{pmatrix} 1+r & 1+r \\ S_u & S_d \end{pmatrix} p = (f_u, f_d) p$$

LHS: $\underbrace{(\phi_0, \phi_1)\begin{pmatrix} 1 \\ S_0 \end{pmatrix}(1+r)}$ where $p_u = \dfrac{S_0(1+r) - S_d}{S_u - S_d}$

$= f_0(1+r)$

$p_d = 1 - p_u$

$$E_p(S_T \mid \mathcal{F}_0) = p_u S_u + p_d S_d$$

$$= \frac{(S_u S_0(1+r) - S_u S_d) + (S_u S_d - S_d S_0(1+r))}{S_u - S_d}$$

$$= \frac{S_0(1+r)(S_u - S_d)}{S_u - S_d} = S_0(1+r).$$

RHS: $E_p[f_1 \mid \mathcal{F}_0]$

~~2nd Fundamental Thm of Math Fin~~
~~A market is said to be complete iff the market martingale mean measure is unique~~

— x —

A market is said to be complete if all (contigent) claims in the market are replicable. A marked is said to be incomplete if $\exists$ a claim which is not replicable.

2ND Fundamental Thm. of Math Fin :-

A NA market is complete iff the martingale measure is unique.

$$\begin{cases} \phi_0(1+r) + \phi_1 S_u = f_u. \\ \phi_0(1+r) + \phi_1 S_d = f_d \end{cases}$$

$$\Rightarrow \underbrace{f_u - \phi_1 S_u} = f_d - \phi_1 S_d$$

long	short
f	S
1	ϕ_1

time 0: $f_0 - \phi_1 S_0$ — if same then

time 1: $f_u - \phi_1 S_u$

or $f_d - \phi_1 S_d$ — risk neutral

Under no arbitrage, a risk neutral pf can only grow at the same rate as that of risk free asset.

$$(f_0 - \phi S_0)(1+r) = f_u - \phi S_u - f_d - \phi_1 S_d$$

$$E_P[f_1 | \mathcal{F}_0] = f_u p_u + f_d p_d$$

$$= \frac{(f_u \cdot S_0(1+r) - f_u S_d) + (f_d S_u - f_d S_0(1+r))}{S_u - S_d}$$

$$= \frac{S_0(1+r)(f_u - f_d)}{S_u - S_d} + \frac{f_d S_u - f_u S_d}{S_u - S_d}$$

$$= \phi_1 S_0 (1+r) + \frac{f_d(S_u - S_d) - S_d(f_u - S_d)}{S_u - S_d}$$

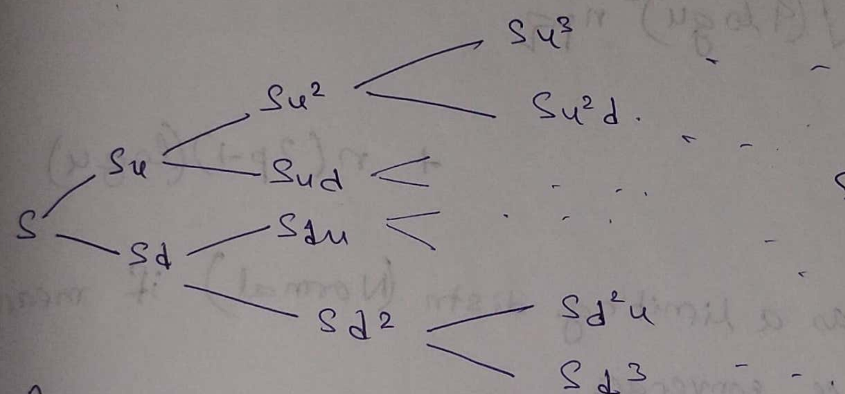
$= \phi_1 S_0(1+r) + f_d - \phi_1 S_d$

$= \phi_1 S_0(1+r) + (f_0 - \phi_1 S_0)(1+r)$

$= f_0(1+r)$.

$\Rightarrow$ f_0 can be found using martingale pricing uniquely.

COX – ROSS – RUBINSTEINE (1978).



$\cancel{S_T = S_0 u \; S_0 u^k d^{n-1}}$

$\boxed{S_T = S_0 u^k d^{n-k}}$.

Assume: proportional growth.

$\cancel{S_T = S_0 u^k d^{n-k}}$

$\Rightarrow \log\left(\frac{S_T}{S_0}\right) = \log(u^k d^{n-k})$.

$= k \log u + (n-k)\log d$.

Assun: Symmetric jumps $d = 1/u$

H/w: Find the distn for assumption jumps.

Define $X_T = \log\left(\frac{S_T}{S_0}\right) = (\log u)(2B - n)$

$B \sim Bin(n,p)$.

When $S_u = S_0 u$ & $S_d = S_0 d$

$$p_u = \frac{S_0(1+r) - S_d}{S_u - S_d} = \frac{S_0(1+r-d)}{S_0(u-d)} = \frac{1+r-d}{u-d}$$

$$P_{uu} = \frac{S_u(1+r) - \overbrace{S_{ud}}^{= S_0 ud}}{S_{uu} - S_{ud}} = \frac{S_0 u[(1+r) - d]}{S_0 u[u-d]}$$

$$= \frac{1+r-d}{u-d}.$$

$$\frac{X_T - E(X_T)}{Var(X_T)} \sim N(0,1). \quad \text{as } n \to \infty$$

$$\Rightarrow X_T = \frac{X_T - (\log u)\, n(2p-1)}{\sqrt{(4\log u)^2 npq}} * (2\log u)\sqrt{npq}$$

$$+ n(2p-1)(\log u)$$

So X_T has a limiting distn (Normal) if mean & variance converges

$$Var(X_T) = 4(\log u)^2 npq \qquad \cancel{p_n < 1/2 \text{ or } q < 1/2}$$

$$p = 1/2 + \delta_n. \qquad \delta_n \to 0.$$

$$q = 1/2 - \delta_n$$

$$pq = 1/4 - \delta_n^2$$

$2n\delta_n \log u \rightarrow$ must converge

$$\Rightarrow n\, 4(\log u)^2 (1/4 - \delta_n^2)$$

$$= n(\log u)^2 - \underbrace{n\delta_n(\log u)^2}_{\to 0}.$$

$= n(\log u)^2 - n\,\delta_n(\log u)^2$

$n(\log u)^2 \propto \sigma^2$

$n(\log u)^2 = c\,\sigma^2$

$\Rightarrow \cancel{(\log u)^2}\,(\log u)^2 = c\,\frac{\sigma^2}{n}$

$\Rightarrow \log u = \sigma\sqrt{\frac{c}{n}}$

or $u = e^{\sigma\sqrt{\frac{c}{n}}}$

$d = e^{-\sigma\sqrt{\frac{c}{n}}}$

$$P_n = \frac{1 + \frac{rT}{n} - e^{-\sigma\sqrt{\frac{c}{n}}}}{e^{\sigma\sqrt{c/n}} - e^{-\sigma\sqrt{c/n}}}$$

$$= \frac{\text{Num}}{\text{Denom}} = \frac{\frac{rT}{n} + \sigma\sqrt{\frac{c}{n}} - \frac{\sigma^2}{2}\frac{c}{n} + O\left(\frac{1}{n^2}\right)}{2\sigma\sqrt{\frac{c}{n}} + O\left(\frac{1}{n^2}\right)}$$

$= \frac{1}{2} + rT -$

deg
deb
leo
Rakesh
Ayan
Sandy
dipankar
Ankit
Sunmitra
Saynik

(take notes from here)

Midterm

Q1. $S_n^{(i)} = X_1^{(i)} + \dots + X_n^{(i)}, \; i = 1,2$

$$P(S_n^{(1)} - n > c) \gtrless P(S_n^{(2)} - n > c)$$

$$\Rightarrow P(S_n^{(1)} > c) \gtrless P(S_n^{(2)} > c)$$

$$P\left(\frac{S_n^{(i)} - n\mu_{(i)}}{\sigma_i\sqrt{n}} > \frac{c - n\mu_{(i)}}{\sigma_i\sqrt{n}}\right)$$

$$\simeq 1 - \Phi\left(\frac{c - n\mu_1}{\sigma_1\sqrt{n}}\right)$$

Game 1 is better than game 2 if

$$\Phi\left(\frac{c - n\mu_{(1)}}{\sigma_1\sqrt{n}}\right) < \Phi\left(\frac{c - n\mu_2}{\sigma_2\sqrt{n}}\right)$$

i.e, to check

$$\frac{c - n\mu}{\sigma_1\sqrt{n}} < \frac{c - n\mu}{\sigma_2\sqrt{n}}$$

$$\Leftrightarrow \sigma_1 > \sigma_2 \quad \text{if } c - n\mu > 0$$

$$\sigma_1 < \sigma_2 \quad \text{if } c - n\mu < 0$$

2. Mortage 1 $\Rightarrow$ 2 yr fixed at 1% rate, rest of 8 yrs at 6%
Mortage 2 $\Rightarrow$ 5 yr " " 4% rate " " 5 yrs at 6%

$$\left(P\left(1+\frac{r}{12}\right) - x\right)\left(1+\frac{r}{12}\right) - x\Big)\dots = 0$$

$$\Rightarrow P\left(1+\frac{r}{12}\right)^{12\times10} - x\left[\left(1+\frac{r}{12}\right)^{m-1} + \dots + \left(1+\frac{r}{12}\right)^0\right] = 0$$

$$\Rightarrow \frac{x\left(\left(1+\frac{r}{12}\right)^m - 1\right)}{\left(1+\frac{r}{12} - 1\right)} = P\left(1+\frac{r}{12}\right)^m$$

$$\Rightarrow x_0 = \frac{P\left(1+\frac{r}{12}\right)^m \times \frac{r}{12}}{\left(1+\frac{r}{12}\right)^m - 1}$$

$$P\left(1+\frac{r}{12}\right)^{2\times12=n} - x\left(\left(1+\frac{r}{12}\right)^{n-1} + \dots + 1\right)$$

$$P\left(1+\frac{r}{12}\right)^{n} - x_0\left(\frac{\left(1+\frac{r}{12}\right)^{n}-1}{\left(1+\frac{r}{12}\right)-1}\right) = \text{New } P_1$$

$$x_1 = \frac{P_1\left(1+\frac{r_1}{12}\right)^{8\times12}\frac{r_1}{12}}{\left(1+\frac{r}{12}\right)^{8\times12}-1}$$

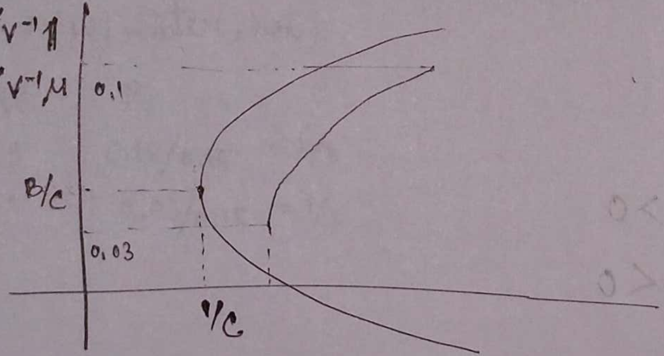
Total premium paid for mortage 1.

$= 24\, x_0^{(1)} + 96\, x_1^{(1)}$

Q3. $w_{mvp} = \dfrac{V^{-1}\mathbb{1}}{\mathbb{1}'V^{-1}\mathbb{1}}$ (allowing short sale)

$C = \mathbb{1}'V^{-1}\mathbb{1}$

$B = \mathbb{1}'V^{-1}\mu$



asset 1 & 2

" 1 & 3

" 2 & 3

$$\text{submatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\text{inverse}} \frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{ad-bc}$$

$$V = \begin{pmatrix} 0.2 & 0.45 & 0.36 \\ 0.45 & 1.5 & 0.75 \\ 0.36 & 0.75 & 1.00 \end{pmatrix}$$

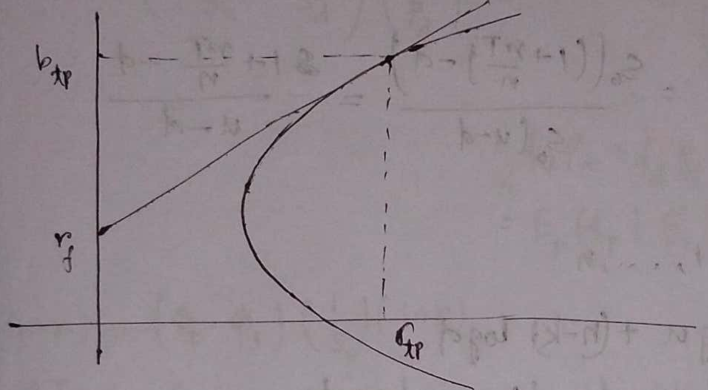
$$\begin{pmatrix} 1.5 & 0.75 \\ 0.75 & 1.00 \end{pmatrix}^{-1} = \begin{pmatrix} 16/15 & -4/5 \\ -4/5 & 8/5 \end{pmatrix}$$

$$\mathbb{1}'V^{-1}\mathbb{1} = (1 \;\; 1)\begin{pmatrix} 16/15 & -4/5 \\ -4/5 & 8/5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= 16/5 = C$$

$\text{Var} = \frac{1}{C} = \frac{5}{16} > 0.2$ where all investment is done in asset 1.

Q4.



$$E(r_q) = r_f(1-\beta) + \beta E(r_{tp})$$

$$r_q = r_f(1-\beta) + \beta r_{tp} + \epsilon$$

$$\beta = \frac{\text{Cov}(r_q, r_{tp})}{\text{Var}(r_{tp})}$$

$$\text{Var (the pf)} = \beta^2 \sigma_{tp}^2$$

$$E(r_q) = \beta E(r_{tp}) + (1-\beta)\, E(r_{zc(tp)})$$

$$r_q = \beta r_{tp} + (1-\beta)\, r_{zc(tp)} + \epsilon$$

$$\text{Var (investment pf)} = \beta^2 \sigma_{tp}^2 + (1-\beta)^2 \sigma_{zc(tp)}^2$$

Class

Dist of $\log\left(\frac{S_T}{S_0}\right)$ given $S_0 \sim N\left(\left(r-\frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$

$S_u = S_0 u$

$S_{uu} = S_0 u^2$

$S_d = S_0 d$

$S_{dd} = S_0 d^2$

$$p = \frac{S_0\left(1+\frac{rT}{n}\right) - S_0 d}{S_0 u - S_0 d} = \frac{S_0\left(\left(1+\frac{rT}{n}\right)-d\right)}{S_0(u-d)} = \frac{1+\frac{rT}{n}-d}{u-d}$$

$S_T = S_0 u^k d^{n-k}, \quad k=0,1,\dots,n$

$$X_T = \log\left(\frac{S_T}{S_0}\right) = k\log u + (n-k)\log d$$
$$= k(\log u - \log d) + n\log d$$

$K \sim \text{Bin}(n,p)$

$$X_T = \frac{X_T - E(X_T)}{\sqrt{Var(X_T)}} \times \sqrt{Var(X_T)} + E(X_T)$$

$\rightarrow N(0,1)$

$Var(X_T) = npq(\log u - \log d)^2$

$E(X_T) = np(\log u - \log d) + n\log d$

For $d = \frac{1}{u}$ then $Var(X_T) = n(4pq)(\log u)^2 \rightarrow \sigma^2 T$

$E(X_T) = n\log u(2p-1) \rightarrow \left(r-\frac{\sigma^2}{2}\right)T$ $\Big\}$ as $n\to\infty$

$2p-1 \neq 0$, $\not< 0$ in the limit as it is unlikely financial scenario

$$(\log u)^2 = \frac{\sigma^2 T}{n}$$

as $n\to\infty$ $\quad \log u = \sigma\sqrt{T/n}$

$u = \exp(\sigma\sqrt{T/n})$

$\Rightarrow d = \frac{1}{u}\exp(-\sigma\sqrt{T/n})$

$E(S_T \mid \mathcal{F}_0)$

$= E(S_0 e^{X_T} \mid \mathcal{F}_0)$

(Hedge amount)

Project submit date - 27th Nov
& presentation - 30th Nov

$$E_P (\phi_0 \;\; \phi_1) \begin{pmatrix} 1+r & 1+r \\ S_u & S_d \end{pmatrix} \begin{pmatrix} P_u \\ P_d \end{pmatrix} = (f_u \;\; f_d) \begin{pmatrix} P_u \\ P_d \end{pmatrix}$$

$$= f_u P_u + f_d P_d$$

$$= E_P (f_T \mid \mathcal{F}_0)$$

$$f_0 (1+r) = (\phi_0 \;\; \phi_1) \begin{pmatrix} 1 \\ S_0 \end{pmatrix} (1+r)$$

$$f_0 e^{KT} = E(f(S_T) \mid \mathcal{F}_0)$$

Finance

$$E_{LN}(S_T \mid \mathcal{F}_0) = S_0 e^{rT}$$

where $\log\left(\frac{S_t}{S_0}\right) \mid S_0 \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$

$0 \qquad T$

$$\begin{tikzcd}[row sep=small] & S_u \\ S_0 \arrow[ur] \arrow[dr] & \\ & S_d \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & f(S_u) = f_u \\ f_0 \arrow[ur] \arrow[dr] & \\ & f(S_d) = f_d \end{tikzcd}$$

$E_P(S_T \mid \mathcal{F}_0) = S_0(1 + rT)$,

where $P = \frac{S_0(1+rT) - S_d}{S_u - S_d}$

$E_P(f_T \mid \mathcal{F}_0) = f_0(1 + rT)$

$\exists (\varphi_0, \varphi_1)$

s.t. $(\varphi_0 \;\; \varphi_1)\begin{pmatrix} 1+rT & 1+rT \\ S_u & S_d \end{pmatrix} = (f_u \;\; f_d)$

Then it is checked that

$E_P(f_T \mid \mathcal{F}_0) = P_u f_u + P_d f_d$

$= f_0(1 + rT)$

$f_T = f(S_T) = f(S_0 u^k d^{n-k})$

$S_T = S_{n(\frac{T}{n})}$

$S_{(n-1)\times(\frac{T}{n})}$

$E_P(S_{n(\frac{T}{n})} \mid \mathcal{F}_{(n-1)\times\frac{T}{n}})$

$= S_{(n-1)\frac{T}{n}} \times \left(1 + r\frac{T}{n}\right)$

$E_P f(S_{n(\frac{T}{n})} \mid \mathcal{F}_{(n-1)\frac{T}{n}})$

$= f(S_{(n-1)\frac{T}{n}})\left(1 + r\frac{T}{n}\right)$

$$\begin{tikzcd}[row sep=small] & f(S_{(n-1)\frac{T}{n}} \times u) \\ f(S_{(n-1)\frac{T}{n}}) \arrow[ur] \arrow[dr] & \\ & f(S_{(n-1)\frac{T}{n}} \times d) \end{tikzcd}$$

Using replicable pf idea (i.e) using (φ_0, φ_1). we can show the same way. $Pf(S_{(n-1)\frac{T}{n}} \times u) + (1-P) f(S_{(n-1)\frac{T}{n}} \times d) = f(S_{(n-1)\frac{T}{n}})\left(1 + r\frac{T}{n}\right)$

Iterating this process we can show

$$E\left(f(S_{n\frac{T}{n}}) \mid \mathcal{F}_{(n-2)\frac{T}{n}}\right) =$$

$$\text{R.H.S} = f\left(S_{(n-2)\frac{T}{n}}\right)\left(1+\frac{\gamma T}{n}\right)^2$$

$$= E\left(E\left(f(S_{n\frac{T}{n}}) \mid \mathcal{F}_{(n-1)\frac{T}{n}}\right) \mid \mathcal{F}_{(n-2)\frac{T}{n}}\right) = \left[f\left(S_{(n-1)\frac{T}{n}}\right)\left(1+\frac{\gamma T}{n}\right)\right]$$

~~Using replicable~~

Iterating the process n times we get.

$$E_P\left(f(S_{n\frac{T}{n}}) \mid \mathcal{F}_0\right) = f_0\left(1+\frac{\gamma T}{n}\right)^n$$

$$\Rightarrow E_{LN}\left(f(S_T) \mid \mathcal{F}_0\right) = f_0 e^{\gamma T}$$

Given $\mathcal{F}_0$, Distn of $S_{n\frac{T}{n}}$ (through binomial paths)

converges LogNormal with parameters

$\mu = (\gamma - \frac{\sigma^2}{2})T$, $v = \sigma^2 T$

Let f be price fn for call option.

Then $f(S_T) = \begin{cases} S_T - k & \text{if } S_T > k \\ 0 & \text{if } S_T \le k \end{cases}$

Then $E\left(f(S_T) \mid \mathcal{F}_0\right) = \int_k^{\infty} (S_T - k)\, dF_{S_T}$ given $S_0 = \int_{\log\frac{k}{S_0}}^{\infty} (S_0 e^{x_T} - k)\, dF_{X_T}$

$\left[S_T = S_0 e^{X_T} > k \iff e^{X_T} > \frac{k}{S_0} \right.$

where $X_T \sim N\left((\gamma - \frac{\sigma^2}{2}); \sigma^2 T\right)$

(See notes of others)

$$= \int_{\log \frac{K}{S_0}}^{\infty} S_0 e^{x} \frac{e^{-\frac{(x-\mu)^2}{2v^2}}}{v\sqrt{2\pi}} dx$$

where $\mu = (r - \frac{\sigma^2}{2})T$

$v^2 = \sigma^2 T$

$$- \int_{\log(\frac{K}{S_0})}^{\infty} K \frac{e^{-\frac{(x-\mu)}{2v^2}}}{v\sqrt{2\pi}} dx = I - II$$

$$II = K\left(1 - \Phi\left(\frac{\log\left(\frac{K}{S_0}\right) - \mu}{v}\right)\right)$$

$$= K \Phi\left(\frac{\log \frac{S_0}{K} + \mu}{v}\right)$$

$$= K \Phi\left(\frac{\log \frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$e^{x - \frac{(x-\mu)^2}{2v^2}}$$

$$\text{Exponent} = x - \frac{(x-\mu)^2}{2v^2}$$

$$= \frac{2v^2 x - (x^2 - 2\mu x + \mu^2)}{2v^2}$$

$$= \left[-(x^2 - 2x(\mu + v^2)) + (\mu + v^2)^2 - \mu^2 - (\mu + v^2)^2\right] / 2v^2$$

$$= \frac{-(x - (\mu + v^2))^2}{2v^2} + \frac{[\mu^2 + v^4 + 2\mu v^2 - \mu^2]}{2v^2}$$

$$= -\frac{(x-(\mu+v^2))^2}{2v^2} + \frac{v^2}{2} + \mu$$

$$\Rightarrow e^{x-\frac{(x-\mu)^2}{2v^2}} = e^{-\frac{(x-(r+\frac{\sigma^2}{2})T)^2}{2\sigma^2 T}} \cdot e^{rT}$$

$$I = S_0 \cdot e^{rT} \int_{\log\frac{K}{S_0}}^{\infty} \frac{e^{-\frac{(x-(r+\frac{\sigma^2}{2})T)^2}{2\sigma^2 T}}}{\sigma\sqrt{T}\sqrt{2\pi}}\, dx$$

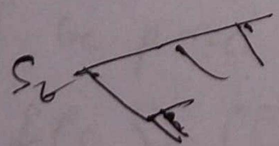
$$= S_0 e^{rT}\left(1-\Phi\left(\frac{\log\frac{K}{S_0} - (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right)$$

$$= S_0 e^{rT}\,\Phi\left(\frac{\log\frac{S_0}{K} + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

. . . . (see notes of others)

$0 < t < T$

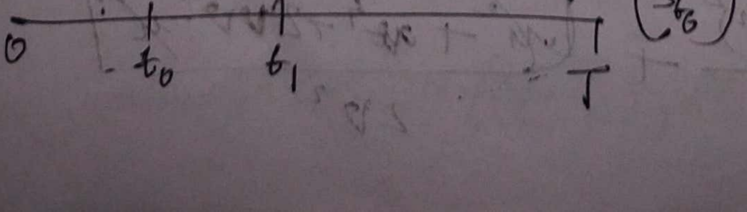
$E(f(S_T)|\mathcal{F}_t)$



$E(S_T|\mathcal{F}_t)$

$\log\left(\frac{S_T}{S_t}\right)$

$E(S_T|S_t) = S_t e^{r(T-t)}$



$$\log\left(\frac{S_{t_1}}{S_{t_0}}\right) \sim N\left(\left(r - \sigma^2\right)(t_1 - t_0), \sigma^2 (t_1 - t_0)\right)$$

$$p = \frac{1 + r\frac{t_1 - t_0}{n} - d}{u - d}$$

$$u = e^{\sigma\sqrt{\frac{t_1 - t_0}{n}}}$$

$$(\log u)^2 \times \frac{t_1 - t_0}{n} \to \frac{t}{n}$$

$$(\log u)^2 = \sigma^2 \frac{t_1 - t_0}{n}$$

$$\log u = \sigma\sqrt{\frac{t_1 - t_0}{n}}$$

$$u = e^{\sigma\sqrt{\frac{t_1 - t_0}{n}}}$$

$$d = \frac{1}{u}$$

$$E(f(S_T) | \mathcal{F}_t) = f_t e^{r(T-t)}$$

$$\Rightarrow f_t = S_t \Phi(d_{1t}) - K e^{-r(T-t)} \Phi(d_{2t})$$

$$\text{where } d_{1t} = \frac{\log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_{2t} = d_{1t} - \sigma\sqrt{T - t}$$

$$\Phi_1 = \frac{f_u - f_d}{S_u - S_d} \rightarrow \frac{\partial f}{\partial S} = \Delta \text{ (Delta hedging parameter)}$$

$$f_t = S_t \Phi(d_{1t}) - Ke^{-r(T-t)} \Phi(d_{2t})$$

$$\frac{\partial f_1}{\partial S_t} = \Phi(d_{1t}) + S_1 \phi(d_{1t}) \frac{\partial d_{1t}}{\partial S_1}$$

$$- Ke^{-r(T-t)} \phi(d_{2t}) \frac{\partial d_{2t}}{\partial S}$$

$$\phi(d_{1t}) = \phi(d_{2t} + \sigma\sqrt{T-t}) = \frac{e^{-\frac{(d_{2t} + \sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}}$$

Note: $(d_{2t} + \sigma\sqrt{T-t})^2 = d_{2t}^2 + 2\sigma\sqrt{T-t}\cdot d_{2t} + \sigma^2(T-t)$

$$= d_{2t}^2 + 2\left[\log\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] + \sigma^2(T-t)$$

$$= d_{2t}^2 + 2\left[\log\frac{S_t}{K} + r(T-t)\right]$$

$$\Rightarrow e^{-\frac{d_{1t}^2}{2}} = e^{-\left[\frac{d_{2t}^2}{2} + \log\frac{S_t}{K} + r(T-t)\right]}$$

$$= e^{-\frac{d_{2t}^2}{2}} \times \frac{K}{S_t} \times e^{-r(T-t)}$$

$$\Rightarrow S_t \phi(d_{1t}) = S_t \phi(d_{2t}) \frac{Ke^{-r(T-t)}}{S_t}$$

$$= Ke^{-r(T-t)} \phi(d_{2t})$$

$\Rightarrow \dfrac{\partial f_t}{\partial S_t} = \Phi(d_{1t}) > 0$

$\Rightarrow t \to T,$

$S_T < K.$

Case 1:

$$d_{1t} = \frac{\left(r + \frac{\sigma^2}{2}\right)\sqrt{T-t}}{\sigma} + \frac{\log \frac{S_{T-t}}{K}}{\sigma\sqrt{T-t}} < 0$$

$\to -\infty$

$\Rightarrow \Phi(d_{1t}) \to 0$

Case 2:

$$\frac{\log \frac{S_{T-t}}{K}}{\sigma\sqrt{T-t}} > 0$$

$\to \infty$

$\Phi(d_{1t}) \to 1$

Calculating

$$\frac{\partial f}{\partial K}, \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \sigma}, \frac{\partial f}{\partial T}$$

Put-Call Parity

C_E

~~European~~

~~Call options~~
~~Gives the holder~~
~~Right to buy~~

European

Call options

gives the holder
the right to buy certain at a
pre-fixed price K at a maturity time, T.

$$\text{Payoff for call} = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T \le K \end{cases}$$

European

Put options : . .
. . . right to sell .
. . . .

~~Put payoff~~

$$\text{Payoff of Put} = \begin{cases} K - S_T & , \text{if } K > S_T \\ 0 & , \text{if } K \le S_T \end{cases}$$

Put-call parity :- $(0 \le t \le T)$

$$C_E^{(t)} + K e^{-r(T-t)} = P_E^{(t)} + S_t$$

Put - Call Parity for European - Option:
under No arbitrage:

$$C_E + K e^{-rT} = P_E + S_0$$

Pf I

~~(C_E) + cash amount Ke^{-rT}~~

A Call $\xrightarrow{}$ price $C_E(0)$ → time 0 + cash amount Ke^{-rT}

Pf II

A Put $\xrightarrow{}$ price $P_E(0)$ → time 0 & Underlying asset with price S_0.

at time T

Value of Pf I $= C_E(T) + (K e^{-rT}) e^{rT}$

Now, $C_E(T) = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T \le K \end{cases}$

" Payoff

$$\text{Value (Pf I)} = \begin{cases} S_T - K + K & \text{if } S_T > K \\ \Rightarrow S_T & \text{if } S_T > K \\ \Rightarrow 0 + K & \text{if } S_T \le K \end{cases}$$

$$= \max(S_T, K)$$

Holder of the put option has the right to sell at ~~a pre-fixed price K~~ the underlying asset at a pre-fixed price (say K) at a maturity time (say T).

/ Holder of a call option has the right to sell the underlying asset at the pre-fixed price at a maturity time (say T).

P.T.O

Value (Pf II) = $P_E(T) + S_T$

Now $P_E(T) \stackrel{\text{Payoff}}{=}$
$$= \begin{cases} K - S_T & \text{if } S_T < K \\ 0 & \text{if } S_T \geq K \end{cases}$$

HW
Calculate the Put Price using LN density of the asset price

$$\Rightarrow P_E(T) + S_T = \begin{cases} K - S_T + S_T & \text{if } S_T < K \\ 0 + S_T & \text{if } S_T \geq K \end{cases}$$

$$= \begin{cases} K, & \text{if } S_T < K \\ S_T, & \text{if } S_T \geq K \end{cases}$$

$= \max(S_T, K)$

!!! (In exam allways use ln not log (use base e not 10)

B. $K - S_0 > 0$

$P_E(0)$

$C_E(0) =$

$T = 3$ months (but in accounted as years $= \frac{1}{4}$)

$\sigma = 0.25$

$r = 6\%$

$S_0 > K$

$S_0 = 60$, $K = 55$

$$d_1 = \frac{\log \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} = \frac{\log \frac{60}{55} + \left(0.06 + \frac{(0.25)^2}{2}\right) \frac{1}{4}}{0.25 \sqrt{\frac{1}{4}}}$$

$= \cancel{0.03778} + \cancel{0.4286}\ 0.878$

$d_2 = d_1 - \sigma \sqrt{T} = \cancel{0.4286}\ \cancel{0.3886}\ \cancel{0.875}\ 0.75$

so, $S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$

$C_E = S_0\, 0.81003 - 55 \times e^{-\frac{.06}{4}} \times 0.7733$

$C_E = 60 \times 0.81003 - 41.89$

$= 6.70$

$\therefore P_E = C_E - (S_0 - Ke^{-rT})$ will be less.

---

this cannot happen under no arbitrage.

Can value of (Pf I (0)) > Value (Pf II (0))
(or <)

if $(C_E(0) + Ke^{-rT} > P_E(0) + S_0)$

Example :- $2 + 50 = 60 > 2 + 55 = 57$

$\therefore$ Cash at hand $= (60 - 57) \longrightarrow 3 \times e^{+rT} > 0$

Risk-free gain

$S_T < K$ &

$S_T > K$

(no arbitrage)

Given NA this cannot happen.

Similarly argue Value Pf I (0) < Value Pf II (0) cannot happen under NA

$\therefore\ C_E(0) + Ke^{-\delta T} = P_E(0) + S_0$; under NA

Prove that

$$C_E(t) + Ke^{-\delta(T-t)} = P_E(t) + S_t$$ (H/W)

for any $0 \le t \le T$..

~~American Call type of~~

American Options

Call option: Holder has the right to buy an underlying asset at a prefixed price (say K) on or before maturity time (say T).

Put option: Holder has the right to sell an ... .. (say K) on or before maturity time (say T).

~~Inequa~~

Inequalities for the American options,

$$C_A^{(0)} + Ke^{-\delta T} \le P_A^{(0)} + S_0 \le C_A^{(0)} + K$$

(H/W) see the RS

$>$ not possible under NA (Bigger not possible).

1.02
50
050
51.0

Un Pt:-

HW $C_E(t) + Ke^{-r(T-t)} = P_E(t) + S_t$

Book: John Hull
options, futures & ~~other~~ other ~~derative~~ derivatives)

16/11/24 (Finance)

American Options

Put Options

$S_0 = 50$

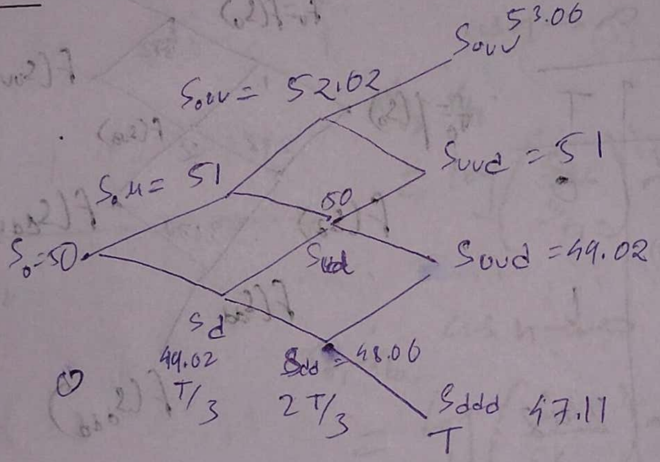
$K = 52$

$r = 6\%$

$u = 1.02$

$d = 1/u = 0.98$

$T = 3$ months



$S_u = S_0 u$

$S_d = S_0 d$

$S_{uu} = S_0 u^2$

$S_{uuu} = S_0 u^3$

$S_{ud} = S_0 ud$ for $d = \frac{1}{u}$

Under NA,

$$p = \frac{S_0\left(1 + r\frac{T}{3}\right) - S_d}{S_u - S_d} = \frac{S_0\left[\left(1 + r\frac{T}{3}\right) - d\right]}{S_0[u - d]}$$

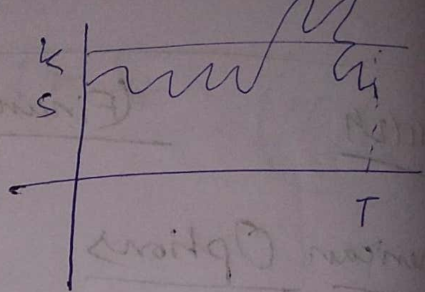
$$= \frac{1 + 0.6 \times \frac{3}{3 \times 12} - 0.98}{1.02 - 0.98}$$

$$= \cancel{0.875}\ \frac{0.025}{0.04} = 0.625$$

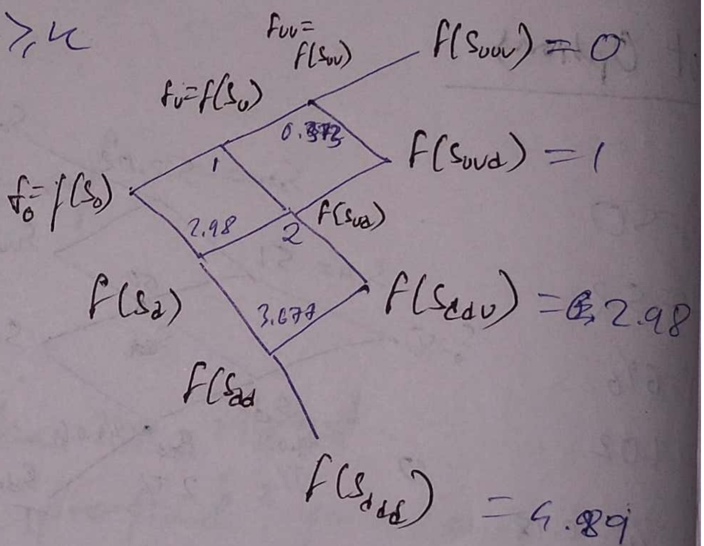
$$\Rightarrow q = \cancel{0.125}\ 0.375$$

Putt option payoff at time T $= \begin{cases} K - S_T & \text{if } S_T < K \\ 0 & \text{o.w} \end{cases}$

American Put option intrinsic value at time $t \leq T$.



$= \begin{cases} K - S_t & \text{if } S_t < K \\ 0 & \text{if } S_t \geq K \end{cases}$



0		$T/3$		$2T/3$		T
EV	IV	EV	IV	EV	IV	
				max (0.373	0)	0
		max (0.978	1)			1
max (1.733	2)			max (1.733	2)	
		max (2.616	2.98)			2.98
				max (3.677	2.98)	
						4.89

Carry forward the max at nodal value

expected value $f(S_{dd}) = (p \times 2.98 + q \times 4.89) \times \left(1 - \frac{rT}{3}\right)$

$= 3.646\left(1 - \frac{rT}{3}\right)$

$= 3.69125$

Intrinsic value at $f(S_{dd}) = 3.94$

$EV\ f(S_0) = (p \times 0.373 + q \times 2) \times (1 - rT/3)$

$= 0.978$

$E(f(S_{ud})) = (p \times 1 + q \times 2.98) \times \left(1 - \frac{rT}{3}\right) = 1.733$

$EV(f(S_d)) = (p \times 2 + q \times 3.677) \times \left(1 - \frac{rT}{3}\right)$

$= 2.616$

$E(f(S_{uu})) = 0.373$

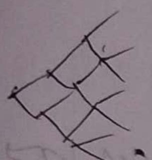
$\therefore E(S_0) = (p \times 1 + q \times 2.98)(1 - rT/3)$

0.998

~~Check the American~~

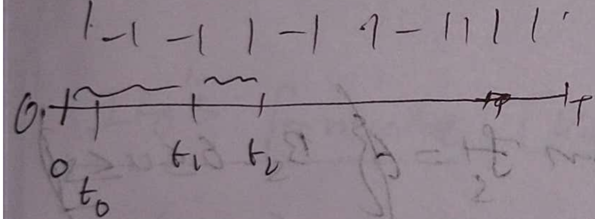
Compute the American Call option price & European Put & Call option prices with the same data.

CRR model & $dist^n$ of S_T



$$\log\left(\frac{S_T}{S_0}\right) \sim N\left(\left(r-\frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

as $n \to \infty$



$$X_{t_2} - X_{t_1} = \log\left(\frac{S_{t_2}}{S_{t_1}}\right) \sim N\left(\left(r-\frac{\sigma^2}{2}\right)(t_2-t_1), \sigma^2(t_2-t_1)\right)$$

as $n \to \infty$

$$X_{t_1} - X_{t_0} = \log\left(\frac{S_{t_1}}{S_{t_0}}\right) \sim N\left(\left(r-\frac{\sigma^2}{2}\right)(t_1-t_0), \sigma^2(t_1-t_0)\right)$$

as $n \to \infty$

$\therefore$ $(X_{t_1} - X_{t_0})$, $(X_{t_2} - X_{t_1})$ are independent from the assumption CRR model.

More generally, for any $0 \le t_0 \le t_1 < \cdots < t_n \le T$

$(X_{t_1} - X_{t_0})$, $(X_{t_2} - X_{t_1})$, $\ldots$, $(X_{t_n} - X_{t_{n-1}})$ are independent & $(X_{t_i} - X_{t_{i-1}}) \sim N\left(\left(r-\frac{\sigma^2}{2}\right)\cdot(t_i - t_{i-1}), \sigma^2(t_i - t_{i-1})\right)$

Standard Brownian motion in a stochastic process $\{B_t : t \geq 0\}$ defined on the experiment with following properties

i) $B_0 = 0$, $t \to B_t$ is const for almost all particles.
(Continuous sample path property)

ii) For $0 \leq t_0 < t_1 < \cdots < t_n$, $k \geq 1$
$(B_{t_1} - B_{t_0}), (B_{t_2} - B_{t_1}), \ldots, (B_{t_n} - B_{t_{n-1}})$ as $n \to \infty$
(Indp increment prop)
are independent.

iii) For $s < t$, $(B_t - B_s)$ given $\mathcal{F}_s = \sigma\{B_u : 0 \leq u \leq s\}$
(Conditional dis^n property) follows $N(0, t-s)$ thus

Thus $X_t = \mu t + \sigma B_t$, where $\mu = r - \frac{\sigma^2}{2}$
in the risk-neutral case, & hence it is Brownian motion.

where (i) & (ii) are satisfied by $\{X_t ; t \geq 0\}$
& (iii) $\to$ (iii) for $s < t$ $(X_t - X_s)$ given $\mathcal{F}_s$
follows $N(\mu(t-s), \sigma^2(t-s))$

$\{B_t\}$ is nowhere differentiable as a fn of t (a.s)

(Stochastic Calculus)

(i) $(dB_t)^2 = dt$

(ii) $dB_t \times dt = 0$

(iii) $(dt)^2 = 0$

(Understanding): $E(B_{t+h} - B_t)^2$

$= E\left[(B_{t+h} - B_t)(t+h-t)\right]$

$= \dfrac{(B_{t+h} - B_t)\cancel{x}}{\cancel{x}} \to 0$

Itô's formula (Kiyosi Itô) [Itô Calculus]

$f(t, B_t)$

$\log\left(\dfrac{S_t}{S_0}\right) = X_t = \mu t + \sigma B_t$

$df(t, B_t) \simeq$

$\Rightarrow S_t = S_0 e^{X_t}$

$= f(t+h, B_{t+h}) - f(t, B_t)$

$\Rightarrow S_t = S_0 e^{\mu t + \sigma B_t}$

(Taylor's Expansion)

$= f(t, B_t)$

$= f_t (t+h-t) + f_x (B_{t+h} - B_t)$

$+ \frac{1}{2} f_{tt} (t+h-t)^2 + \frac{1}{2} f_{xx} (B_{t+h} - B_t)^2$

$+ f_{tx} (t+h-t)(B_{t+h} - B_t) + \ldots\ldots$

$= f_t dt + f_x dB_t + \frac{1}{2} \cancel{f_{tt} (dt)^2}^{=0} + \frac{1}{2} f_{xx} (dB_t)^2$

$+ f_{tx} \cancel{dt\, dB_t}^{=0} + \ldots$ (all other terms are zero (i, ii, iii))

$= f_t dt + \frac{1}{2} f_{xx} dt + f_x dB_t$

P.T.O

$\because$ If $X_t = \mu t + \sigma B_t$

$$S_t = S_0 e^{\mu t + \sigma B_t}$$

$$= f(t, B_t)$$

$\Rightarrow dS_t = df(t, B_t) = \mu f\, dt + \sigma f\, dB_t + \frac{1}{2}\sigma^2 f\, dt$

$$= \left(\mu + \frac{\sigma^2}{2}\right) S_t\, dt + \sigma S_t\, dB_t$$

$$= \gamma S_t\, dt + \sigma S_t\, dB_t$$

for $\mu = \gamma - \frac{\sigma^2}{2}$

H/W

\# Compute i) $d(\log S_t)$

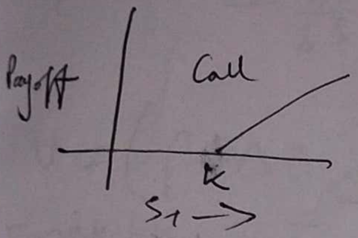
ii) $df(t, S_t)$

23/11/24

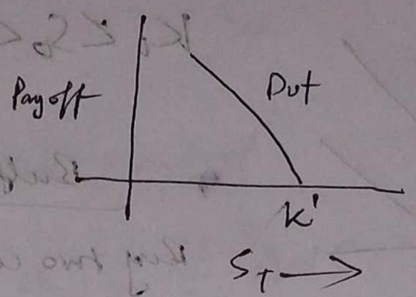
Problem
Model
Algo
environment
Data

Trading Strategies

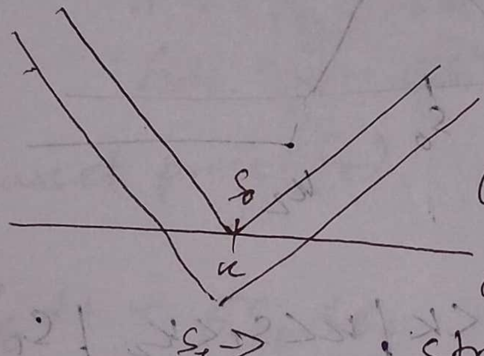
① Bull market



② Bear Market

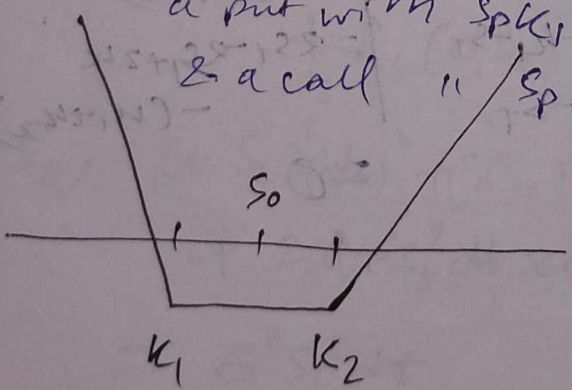


③ Volatile Market



Holding Call & Put at the same strike price K & same maturity T.

Buy
a put with Sp K_1
& a call " Sp $K_2 > K_1$



$K_1 < S_T < K_2$

If $S_T > K_2$
Exercise Call

If $S_T < K_1$
Exercise Put.

4. Stagnant Market

Buy a call at K_1

& " " put at $K_2 > K_1$

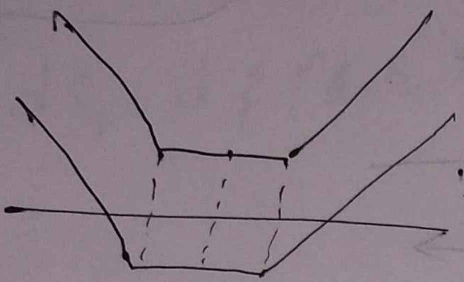
HW

see $C(K_1) + C(K_2) \geq 2C(K)$

where $K_1 < K < K_2$

& $K_1 + K_2 = 2K$

$K_1 < S_0 < K_2$



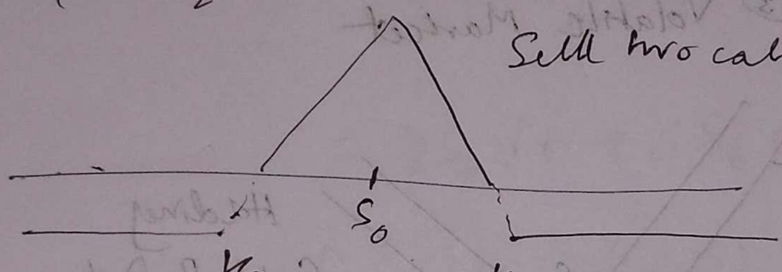
Butterfly

Buy two calls one at K_1

& another at $K_2 > K_1$.

Sell two calls at K

$= \frac{K_1 + K_2}{2}$



Payoff Table:-

$S_T < K_1$	$K_1 < S_T < K$	$K < S_T < K_2$	$S_T > K_2$
0	$S_T - K_1$	$S_T - K_1 - 2(S_T - K)$ $= 2K - K_1 - S_T$ $= K_2 - S_T$	$(S_T - K_1) + (S_T - K_2) - 2(S_T - K)$ $= 2S_T - 2S_T + 2K - (K_1 + K_2)$ $= 0$

B-S-M eqn :-

HW

Find Δ for Put option (Euro).

$$\frac{f \qquad S}{1 \qquad \Delta = \frac{\delta f}{\delta S}} \qquad \frac{f_u - f_d}{S_u - S_d} \to \frac{\delta f}{\delta S}$$

$V = (f - \Delta S) \to$ risk-neutral

For risk-free

$\frac{dV_t}{dt} = dV_t = r V_t dt$

$V_t = V_0 e^{rt}$

$\frac{dV_0}{dt} = V_0 e^{rt} r$

$= rV$

$dV = df - \Delta dS$

$df(t,S) = f_t dt + f_s dS_t + \frac{1}{2} f_{ss} \sigma^2 S_t^2 dt$

$- f_s dS = \left(f_t + \frac{\sigma^2 S_t^2}{2} f_{ss}\right) dt$

① $(dB_t)^2 = dt$

② $dB_t \times dt = 0$

③ $(dt)^2 = 0$

Model the asset price (S_t)

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

$$df(t, S_t) = f_t^{(S_t)} dt + \cancel{\frac{1}{2} f_{tt} (dt)^2} + f_s dS_t + \frac{1}{2} f_{ss} (dS_t)^2 + f_{st} dt \times dS_t$$

$$(dS_t)^2 = (\mu S_t)^2 (dt)^2 + (\sigma S_t)^2 (dB_t)^2 + 2\mu \sigma S_t^2 dt \times dB_t$$

$$= \sigma^2 S_t^2 dt$$

$f(t+h, S_{t+h}) = f(t, S_t)$

$= f_t(t+h-t) + \frac{1}{2} f_{tt} (t+h-t)^2 (dt)^2 \ldots$

$+ f_s(S_{t+h} - S_t) + \frac{1}{2} f_{ss} (S_{t+h} - S_t)^2 + f_{st}(t+h-t)(S_{t+h} - S_t) + \ldots$

$dt \times dS_t = \mu S_t (dt)^2 \to 0 + \sigma S_t (dt\, dB_t) \to 0$

$\Rightarrow df(t, S_t) = f_t dt + f_s dt + f_s dS + \frac{1}{2} f_{ss} (dS_t)^2$

$= f_t dt + f_s dS + \frac{1}{2} f_{ss} \sigma^2 S_t^2 dt$

~~$dV_t = r f_s S$~~

$dV_t = r(f - f_s S)\,dt$

$\rightarrow \left(f_t + \frac{1}{2}\sigma^2 S^2 f_{ss}\right) dt = r(f - f_s S)\,dt$

$\rightarrow \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$

$\hookrightarrow$ Black Scholes Markov Eqn (BSM) for risk-neutral portfolio

~~Discretize~~

<u>Discretization</u>

$$\frac{\partial f}{\partial t}(i,j) \approx \frac{f(i+1,j) - f(i,j)}{\Delta t}$$

$$\frac{\partial f}{\partial S}(i,j) \approx \frac{f(i,j+1) - f(i,j)}{\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2}(i,j) \approx \frac{\frac{(f(i,j+1) - f(i,j))}{\Delta S} - \frac{(f(i,j) - f(i,j-1))}{\Delta S}}{\Delta S}$$

$$= \frac{f(i,j+1) + f(i,j-1) - 2f(i,j)}{(\Delta S)^2}$$