Univariate

R.Va: $X_{1\times 1} \Rightarrow f_X(x)$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$E[X] = \int x f(x)\,dx$$

$$V(X) = \int (x - E(X))^2 f(x)\,dx$$

Multivariate

R.Ve: $\underset{\sim}{X}_{p\times 1} \Rightarrow f_{\underset{\sim}{X}}(x_1, \dots, x_p)$

$$F_{\underset{\sim}{X}}(\underset{\sim}{x}) = \int_{-\infty}^{x_p} \cdots \int_{-\infty}^{x_1} f_{\underset{\sim}{X}}(t_1, \dots, t_p)\, dt_1 \dots dt_p$$

$$E[\underset{\sim}{X}] = \begin{bmatrix} EX_1 \\ \vdots \\ EX_p \end{bmatrix}$$

$$= \left[\int_{x_p} \cdots \int_{x_1} x_j f(x_1 \cdots x_p)\, dx_1 \cdots dx_p \right]$$

$$= \left[\int_{x_j} x_j f(x_j)\, dx_j \right]$$

$$V(\underset{\sim}{X}) = \int_{x_i} \int_{x_j} (x_i - E(X_i))(x_j - E(X_j))\, f(x_i, x_j)\, dx_i\, dx_j$$

$$= \left[Cov(X_i, X_j) \right]_{p\times p}$$

$$V[X \pm Y] = \left[Cov(X_i \pm Y_i,\; X_j \pm Y_j) \right]$$

$$= Cov(x_i\, x_j) + Cov(y_i\, y_j) \pm Cov(x_i, y_j) \pm Cov(x_j, y_i)$$

$$= V(x) + V(y) \pm 2\,Cov(x, y)$$

$$V[X_{p\times 1}] = \begin{bmatrix} V(X_1) & Cov(X_1, X_2) \\ Cov(X_2, X_1) & V(X_2) \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$\leq (b'b)(d'd)$$

$$(b'd)^2 \leq (b'Bb)(d'B^{-1}d)$$

$$\sigma_{11} = 0.69, \quad \sigma_{22} = (0-0.2)^2(0.8) + (1-0.2)^2(0.2)$$

$$= (0.2)^2 0.8 + (0.8)^2(0.2)$$

$$= 0.16$$

$$\sigma_{12} = -0.06 - 0.1 \times 0.2 = -0.08$$

MLE

$$\sum_{j=1}^{n} (x_j - \mu)' \Sigma^{-1} (x_j - \mu) \longrightarrow \text{minimize}$$

$$= \sum_{j=1}^{n} (x_j - \bar{x} + \bar{x} - \mu)' \Sigma^{-1} (x_j - \bar{x} + \bar{x} - \mu)$$

$$= \sum_{j=1}^{n} (x_j - \bar{x})' \Sigma^{-1} (x_j - \bar{x}) + \sum_{j=1}^{n} (\bar{x} - \mu)' \Sigma^{-1} (\bar{x} - \mu)$$

$\hat{\mu} = \bar{x}$ (at that summ. is 0)

$$L(\mu) \propto \frac{1}{|\Sigma|^{n/2}} e^{-\frac{1}{2} \sum_{j=1}^{n} (x_j - \bar{x})' \Sigma^{-1} (x_j - \bar{x})}$$

$$= \frac{1}{|\Sigma|^{n/2}} e^{-\frac{1}{2} tr\left(\Sigma^{-1} \underbrace{\sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})'}_{B} \right)}$$

$$= \frac{1}{|\Sigma|^{n/2}} e^{-\frac{1}{2} tr\left[B^{1/2} \Sigma^{-1} B^{1/2} \right]}$$

eigenvalues of $B^{1/2} \Sigma^{-1} B^{1/2}$ as η_j, $j = 1(1)n$.

$$= \frac{|B^{1/2} \Sigma^{-1} B^{1/2}|^{n/2}}{|B|^{n/2}} e^{-\frac{1}{2} \sum_{j=1}^{p} \eta_j}$$

$$= \frac{\left(\prod_{j=1}^{p} \eta_j \right)^{n/2}}{|B|^{n/2}} e^{-\frac{1}{2} \sum_{j=1}^{p} \eta_j}$$

$$= \frac{\prod_{j=1}^{p} \eta_j^{n/2} e^{-\eta_j/2}}{|B|^{n/2}}$$

$$\frac{d\, x^{n/2} e^{-x/2}}{x} = \frac{n}{\cancel{2}} x^{\frac{n}{2}-1} \cancel{e^{-x/2}} - \frac{1}{\cancel{2}} x^{n/2} \cancel{e^{-x/2}} = 0$$

$$\Rightarrow \quad x = n$$

$$\arg\left[\max(\ )\right] = \frac{\prod_{j=1}^{n} n^{n/2} e^{-n/2}}{|B|^{n/2}} = \frac{n^{np/2} e^{-np/2}}{|B|^{n/2}}$$

$$L(\hat{\mu}, \hat{\Sigma}) = \frac{1}{(2\pi)^{np/2}} e^{-np/2} \frac{1}{|\hat{\Sigma}|^{n/2}}$$

$$\propto (\text{Generalized Var})^{-n/2}$$

<u>Likelihood</u>

$$\Lambda = \frac{\max_{\Sigma} L(\mu_0, \Sigma)}{\max_{\mu,\Sigma} L(\mu, \Sigma)} = \left[\frac{e^{-np/2}}{(2\pi)^{np/2} |\hat{\Sigma}_0|^{n/2}}\right]\left[\frac{(2\pi)^{np/2} |\hat{\Sigma}|^{n/2}}{e^{-np/2}}\right]$$

$$= \left[\frac{|\hat{\Sigma}|}{|\hat{\Sigma}_0|}\right]^{n/2}$$

$$\hat{\Sigma}_0 = \frac{1}{n}\sum_{j=1}^{n} (X_j - \mu_0)(X_j - \mu_0)', \quad \hat{\Sigma} = \frac{1}{n}\sum_{j=1}^{n} (X_j - \bar{X})(X_j - \bar{X})'$$

when n is large, under null hyp,

$$-2\ln\Lambda \sim \chi^2_{\nu - \nu_0 = p}$$

unrestricted dof : $\nu = p + p(p+1)/2$

under H_0 : $\nu = p(p+1)/2$

$$\Lambda^{2/n} = \frac{|\hat{\Sigma}|}{|\hat{\Sigma}_0|} = \left(1 + \frac{T^2}{n+1}\right)^{-1}$$

→ Hotelling T^2 stat.

#) Contours of the bivariate Normal density

We shall obtain the axes of a constant prob. density contours for a biv normal distb^n when $\sigma_{11} = \sigma_{22}$. The axes are given by the eigen values and eigenvectors of Σ.

$$|\Sigma - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} \\ \sigma_{21} & \sigma_{22}-I \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - \sigma_{11} - \sigma_{12})(\lambda - \sigma_{12} + \sigma_{12}) = 0$$

So the eigen values are -

$$\lambda_1 = \sigma_{11} + \sigma_{12} \quad , \lambda_2 = \sigma_{11} - \sigma_{12}$$

The eigenvector e_1 is determined by-

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = (\sigma_{11} + \sigma_{12}) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\Rightarrow \sigma_{11} e_1 + \sigma_{12} e_2 = \sigma_{11} e_1 + \sigma_{12} e_2$$

$$\cdot \quad \cdot \quad \cdot$$

These eqns implies that $e_1 = e_2$. and after first normalization, the eigenvalue-eigenvector pair is

$$\lambda_1 = \sigma_{11} + \sigma_{12}, \quad e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

and, $\lambda_2 = \sigma_{11} - \sigma_{12}, \quad e_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

Midsem 21

1) $$f(x,y) = \frac{1}{8\sqrt{3}\pi} \exp\left(-\frac{x^2}{6} - \frac{y^2}{24} + \frac{xy}{12} + \frac{x}{12} + \frac{y}{6} - 7/24\right)$$

$$= \frac{1}{(\sqrt{2\pi})^2 4\sqrt{3}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{3} + \frac{y^2}{12} - \frac{xy}{6} - \frac{x}{6} - \frac{y}{3} + \frac{7}{12}\right)\right]$$

$|\Sigma| = 16 \times 3 = 48$,

$$(x-\mu_1, y-\mu_2)\begin{bmatrix} \frac{\sigma_{22}}{48} & \frac{-\sigma_{21}}{48} \\ -\frac{\sigma_{12}}{48} & \frac{\sigma_{11}}{48} \end{bmatrix}\begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}$$

$$= \left[\frac{(x-\mu_1)\sigma_{22} - (y-\mu_2)\sigma_{12}}{48}, \frac{-\sigma_{21}(x-\mu_1) + (y-\mu_2)\sigma_{11}}{48}\right]\begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}$$

$$= \frac{(x-\mu_1)^2\sigma_{22} - (x-\mu_1)(y-\mu_2)(\sigma_{12}+\sigma_{21}) + (y-\mu_2)^2\sigma_{11}}{48}$$

$$= \frac{(x^2 - 2\mu_1 x + \mu_1^2)\sigma_{22} - (xy - \mu_2 x - \mu_1 y + \mu_1\mu_2)(\sigma_{12}+\sigma_{21}) + (y^2 - 2\mu_2 y + \mu_2^2)\sigma_{11}}{}$$

48

$$= \frac{x^2(\sigma_{22}) + y^2(\sigma_{11}) - xy(\sigma_{12}+\sigma_{21}) + x(\mu_2\sigma_{12}+\mu_2\sigma_{21}-2\mu_1\sigma_{22}) + y(\mu_1\sigma_{12}+\mu_1\sigma_{21}-2\mu_2\sigma_{11}) + (\mu_1^2\sigma_{22}-\mu_1\mu_2\sigma_{12}-\mu_1\mu_2\sigma_{21}+\mu_2^2\sigma_{11})}{48}$$

Now comparing,

$$\frac{x^2\sigma_{22}}{48} = \frac{x^2}{3} \Rightarrow \boxed{\sigma_{22} = 16}$$

$$\frac{y^2\sigma_{11}}{48} = \frac{y^2}{12} \Rightarrow \boxed{\sigma_{11} = 4}$$

$$\frac{xy(\sigma_{12}+\sigma_{21})}{48} = \frac{xy}{6}$$

$$\Rightarrow \sigma_{12}+\sigma_{21} = 8$$

$$|\Sigma| = 48$$

$$\Rightarrow \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} = 48$$

$$\Rightarrow \sigma_{12}\sigma_{21} = 16$$

$$\boxed{\sigma_{12} = \sigma_{21} = 4}$$

$$\therefore \quad \Sigma = \begin{bmatrix} 4 & 4 \\ 4 & 16 \end{bmatrix}$$

3) <u>Bonferroni:-</u>

$$\bar{x}_i \pm t_{n-1}\left(\alpha/2p\right)\sqrt{\frac{S_{ii}}{n}}$$

$$S = \begin{bmatrix} 0.14 & 0.10 \\ 0.10 & 0.16 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} 0.64 \\ 0.52 \end{bmatrix}$$

1) $0.64 \pm t_{39}\left(\frac{0.05}{2\times 2}\right)\sqrt{\frac{0.14}{40}}$

$= 0.64 \pm t_{39}(0.0125)\sqrt{\frac{0.14}{40}}$

$= 0.64 \pm 2.331\sqrt{\frac{0.14}{40}} = (0.502, 0.777)$

<u>T^2</u>

$$\bar{x}_i \pm \sqrt{\frac{p(n-1)}{(n-p)} F_{p,n-p}(\alpha)\, \frac{S_{ii}}{n}}$$

1) $0.64 \pm \sqrt{\frac{2(40-1)}{(40-2)} F_{2,38}(0.05)\, \frac{0.14}{40}}$

$= 0.64 \pm \sqrt{\frac{39}{19} \cdot 3.245 \cdot \frac{0.14}{40}}$

$= (0.487, 0.792)$

Mid-20

1) $$A X^{(1)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

$$B X^{(2)} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 - x_4 + x_5 \\ x_3 + x_4 - 2x_5 \end{bmatrix}$$

$$\Sigma_{11} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \quad \Sigma_{12} = \begin{bmatrix} 0.5 & -0.5 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\Sigma_{21} = \begin{bmatrix} 0.5 & 1 \\ -0.5 & -1 \\ 1 & 0 \end{bmatrix} \quad \Sigma_{22} = \begin{bmatrix} 5 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\operatorname{Cov}(\underset{\sim}{Y_1}, \underset{\sim}{Y_2}) = E\left[(Y_1 - \mu_1)(Y_2 - \mu_2)^T\right]$$

$$E(AX^{(1)}) = E\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$E(BX^{(2)}) = E\begin{bmatrix} x_3 - x_4 + x_5 \\ x_3 + x_4 - 2x_5 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\operatorname{Cov}(AX^{(1)}, BX^{(2)})$$

$$= \operatorname{Cov}\left(\begin{bmatrix} x_1 - x_2 + 2 \\ x_1 + x_2 - 6 \end{bmatrix}, \begin{bmatrix} x_3 - x_4 + x_5 + 4 \\ x_3 + x_4 - 2x_5 - 2 \end{bmatrix}\right)$$

$$= E\left[\begin{bmatrix} x_1 - x_2 + 2 \\ x_4 + x_2 - 6 \end{bmatrix}\begin{bmatrix} x_3 - x_4 + x_5 + 4, & x_3 + x_4 - 2x_5 - 2 \end{bmatrix}\right)$$

$$= E\left[\begin{bmatrix} x_4 x_3 - x_2 x_3 + 2x_3 - x_4 x_4 + x_2 x_4 + 2x_5 \\ \end{bmatrix}\right.$$

2) $N_3(\mu, \Sigma)$, $\mu' = [1, -1, 2]$

$$\Sigma = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

a) $$\rho_{23} = \frac{\sigma_{23}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{33}}} = \frac{1}{\sqrt{4}\sqrt{9}} = \frac{1}{6}$$

b) $$\mathrm{Corr}\left(X_2, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_3\right)$$

$$= \frac{1}{2}\mathrm{Corr}(X_1, X_2) + \frac{1}{2}\mathrm{Corr}(X_2, x_3)$$

$$= \frac{1}{2}\left(\frac{-2}{10} + \frac{1}{6}\right) = \frac{1}{2}\left(\frac{1}{6} - \frac{1}{5}\right) = \frac{-1}{30 \times 2}$$

c) $X_2 \mid X_3 = 2$

Mid-21

4) $$P = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}$$

$$|P - \lambda I| = \begin{vmatrix} \lambda-1 & a & a \\ a & \lambda-1 & a \\ a & a & \lambda-1 \end{vmatrix} = 0$$

eigen values are $1+2a,\ 1-a,\ 1-a$.

$$PX = (1+2a)X$$

$$\Rightarrow \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (1+2a) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \cancel{x_1} + ax_2 + ax_3 = \cancel{x_1} + 2ax_1$$

$$\Rightarrow 2x_1 = x_2 + x_3 \quad \text{—— (i)}$$

$$ax_1 + \cancel{x_2} + ax_3 = \cancel{x_2} + 2ax_2$$

$$\Rightarrow x_1 + x_3 = 2x_2 \quad \text{—— (ii)}$$

$$x_1 + x_2 = 2x_3 \quad \text{—— (iii)}$$

$$2(2x_2 - x_3) = x_2 + x_3$$

$$\Rightarrow 4x_2 - 2x_3 = x_2 + x_3$$

$\Rightarrow \quad x_2 = x_3$

$\therefore \quad x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (1-a) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow \quad \cancel{x_1} + ax_2 + ax_3 = \cancel{x_1} - ax_1$

$\Rightarrow \quad x_1 + x_2 + x_3 = 0$

$\therefore \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Total variance $= \lambda_1 + \lambda_2 + \lambda_3$

$= 1 + 2a + 2(1-a) = 3$

Proportion of variance explained by first component $= \frac{1+2a}{3}$

and for 2nd & 3rd component $= \frac{1-a}{3}$

Mid-19

1.) a) $d(O,P) = \left(|x_1|^a + |x_2|^a\right)^{1/a}$

b) The angle between the two components is related to the correlation coeff. r.

$$r = \frac{Cov(x_1, x_2)}{\sqrt{Var(x_1)}\sqrt{Var(x_2)}}$$

$$= \frac{-6}{\sqrt{8}\sqrt{9}} = -\frac{1}{\sqrt{2}}$$

The correlation coeff is related to the cosine of the angle θ between two vectors

$$r = \cos(\theta)$$

$$\Rightarrow \theta = \cos^{-1}(r) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135^\circ$$

2) a) $\bar{X} = \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix}$ $\Rightarrow \bar{X}_{tot} = 1.873$

$$Var_{tot} = \sum_i \sum_j s_{ij} = 3.913$$

b) Excess of petroleum consumption over nat. gas

$$\bar{X}_{excess} = \bar{X}_1 - \bar{X}_2 = 0.258$$

$$Var(X_{excess}) = Var(X_1 - X_2)$$

$$= Var(X_1) + Var(X_2) - 2\,cov(X_1, X_2)$$

$$= 0.856 + 0.568 - 2(0.635)$$

3)

a) $X = \begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$ $\quad \bar{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$$\hat{\Sigma} = \frac{1}{4} \sum (x_i - \bar{x})(x_i - \bar{x})'$$

$$= \frac{1}{4}\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix}\begin{pmatrix} 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \right]$$

$$= \frac{1}{4}\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 3/2 \end{bmatrix}$$

b) (X_1, X_3) & X_2

$$\Sigma = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$ $\Rightarrow$ Independent

X_1 & $X_1 + 3X_2 - 2X_3$ $\Rightarrow$ dependent

$$\text{Cov}(X_1, X_1) + 3\,\text{Cov}(X_1, X_2) - 2\,\text{Cov}(X_1, X_3) = 6$$

4)

$$Y = C_{p\times p} X + d_{p\times 1}$$

$$\bar{y} = C\bar{x} + d \,, \quad S_y = \frac{1}{n-1}\sum (y_i - \bar{y})(y_i - \bar{y})'$$

$$= \frac{1}{n-1}\sum (Cx + d - C\bar{x} - d)(Cx + d - C\bar{x} - d)'$$

$$= \frac{1}{n-1}\sum C(x - \bar{x})(x - \bar{x})'C'$$

$$= C S_x C'$$

$$\mu_y = E(Y) = E(CX + d) = C\mu_x + d$$

$$\therefore T_y^2 = n(\bar{y} - \mu_{y_0})' S_y^{-1} (\bar{y} - \mu_{y_0})$$

$$= n(C\bar{x} + d - C\mu_{x_0} - d)' (C S_x C')^{-1} (C\bar{x} + d - C\mu_{x_0} - d)$$

$$= n(\bar{x} - \mu_{x_0})' C' (C')^{-1} S_x^{-1} (C)^{-1} C (\bar{x} - \mu_{x_0})$$

$$= n(\bar{x} - \mu_{x_0})' S_x^{-1} (\bar{x} - \mu_{x_0})$$

$$= T_x^2 \quad \text{Proved}$$

5) $\bar{x} = \begin{bmatrix} 0.564 \\ 0.603 \end{bmatrix}$ $\quad S = \begin{bmatrix} 0.0144 & 0.0117 \\ 0.0117 & 0.0146 \end{bmatrix}$

$n = 42$

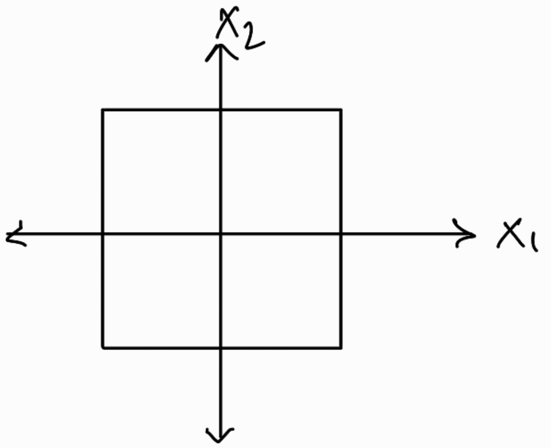
95% T^2 simultaneous CI:

$$\mu_1 \in \left(\bar{X}_1 \pm \sqrt{\frac{p(n-1)}{n-p} F_{p,(n-p)}(\alpha)} \sqrt{\frac{S_{11}}{n}} \right)$$

$$= \left(0.564 \pm \sqrt{\frac{2(41)}{40} F_{2,40}(0.05)} \sqrt{\frac{0.0144}{42}} \right.$$

Mid-18

2)



3) A matrix $X^T A X$ is positive definite if for every non-zero vector $x \in \mathbb{R}^K$, the quadratic form $x^T A x$ is strictly positive:

$$x^T A x > 0 \quad \forall x \neq 0$$

Suppose v is a eigenvector of A corresponding to the eigenvalue λ ie $\quad Av = \lambda v$

The quadratic form

$$v^T A v = v^T \lambda v = \lambda (v^T v)$$

Here $v^T v$ is the dot product of v with itself which is the same as $\|v\|^2$.

Since $v \neq 0$, $\|v\|^2 > 0$ —— (i)

$$\therefore v^T A v = \lambda \|v\|^2$$

Since A is positive definite, for any nonzero vector v, we know that $v^T A v > 0$

$$\therefore \lambda \|v\|^2 > 0 \quad \text{—— (ii)}$$

So from (i) & (ii) we get $\lambda > 0$

4) $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = 0$

$\Rightarrow (9-\lambda)(6-\lambda) - 4 = 0$

$\Rightarrow 54 - 15\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 - 15\lambda + 50 = 0 \quad \Rightarrow (\lambda-5)(\lambda-10) = 0$

$\Rightarrow \lambda = 10, 5$

$$X^T A X = \lambda \|X\|^2$$

$\Rightarrow X^T A X = 10 \times 1 = 10$ Ans.

6) $X_1 \sim N(0,1)$, $X_2 = \begin{cases} -X_1 & , -1 \le X_1 \le 1 \\ X_1 & \text{o.w.} \end{cases}$

$P[-1 \le X_1 \le x] = P[-x \le X_1 \le 1]$ (by symmetry)

$$\begin{aligned} P[X_2 \le x_2] &= P[X_2 \le -1] + P[-1 \le X_2 \le x_2] \\ &= P[x_1 \le -1] + P[-1 \le -X_1 \le x_2] \\ &= P[X_1 \le -1] + P[-x_2 \le X_1 \le 1] \\ &= P[X_1 \le -1] + P[-1 \le X_1 \le x_2] \\ &= P[X_1 \le x_2] \end{aligned}$$

Now suppose (X_1, X_2) follows Bivariate Normal

So any linear combination of X_1 & X_2 will be a Normal r.v.

Now, let $X_1 - X_2$ is a r.v.

$$X_1 - X_2 = \begin{cases} 2X_1 & , -1 \le X_1 \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$P[X_1 - X_2 = 0]$

$= 1 - P[-1 \le X_1 \le 1] = 1 - 0.68 = 0.32$

Mid-17

4) $A = \begin{bmatrix} A_{11} & A_{12} \\ \end{bmatrix}$ A_{22} is invertible

$$\begin{bmatrix} A_{21} & A_{22} \end{bmatrix}$$