
Finance Sem 3 Post
Midsem

A pf is called self-financing if,

$$\varphi_j' S_{j-1} = \varphi_{j-1}' S_{j-1} - C_{i-1} \rightarrow \text{consumption}.$$

Simple Random walk..

$$S_n = S_0 + \cancel{X_1} X_1 + X_2 + \dots + X_n.$$

where $X_i = \pm 1$ w.p. p.

i.i.d. S_0, is fixed & ind r.v.

$$E(X) = E(E(X|Y)).$$

$$E(X|Y) = E(E(X|Y,Z)|Y).$$

$\cancel{E(S_{n+1})}$ $E(S_{n+1}|\mathcal{F}_i) = S_i$

$E(S_n|\mathcal{F}_0) = S_0.$

Let τ_n be the time

$S_n = a$ for the first

Define $\tau_a \wedge \tau_b = \tau$ for some $a < c < b$.

If τ is finite (i) $P(\tau < \infty) = 1$

(ii) $E[S_\tau] < \infty$

(iii) $E(|S_m| \mathbb{1}_{\{\tau \geq m\}}) \to 0$ as $m \to \infty$.

$S_m = a$ for the first time

$\tau = \tau_a$ if $\tau_a < \tau_b$

τ_b if $\tau_b < \tau_a$.   if $m \leq \tau_a \wedge \tau_m$

Proof : (iii)

If $\mathbb{1}_{\{\tau \geq m\}} = 1$ then

$|S_m| \leq \max(|a|, |b|)$.

$\Rightarrow E(|S_m| \mathbb{1}_{\{\tau \geq m\}}) \leq \max(|a|,|b|)\, P(\tau \geq m)$

as $m \to \infty$

$E(|S_m| \mathbb{1}_{\{\tau \geq m\}}) \leq 0$

$\Rightarrow E(|S_m| \mathbb{1}_{\{\tau \geq m\}}) \to 0$.

$E(S_\tau | \mathcal{F}_0) = a P(\tau_a < \tau_b | \mathcal{F}_0) + b P(\tau_b < \tau_a | \mathcal{F}_0)$

$\Rightarrow a\, p_{a,b} + b(1 - p_{a,b}) = c$

$\Rightarrow p_{a,b}(a-b) = c-b$

$\Rightarrow P(\tau_a < \tau_b | S_0 = c) = \dfrac{c-b}{a-b} = \dfrac{b-c}{b-a}$

$Z_n = \left(\frac{q}{p}\right)^{S_n}$

$E[Z_{n+1} | \mathcal{F}_n] = E\left[\left(\frac{q}{p}\right)^{S_{n+1}} \Big| \mathcal{F}_n\right]$

$= E\left[\left(\frac{q}{p}\right)^{X_{n+1}} \Big| \mathcal{F}_n\right] \times \left(\frac{q}{p}\right)^{S_n}$

$= E\left(\left(\frac{q}{p}\right)^{X_{n+1}}\right) \times \left(\frac{q}{p}\right)^{S_n}$

$\boxed{\cancel{= \sum \left(\frac{q}{p}\right)^x}}$

$= \sum \left(\frac{q}{p}\right) p \times \left(\frac{q}{p}\right)^{S_n}$

$= \sum q \times \left(\frac{q}{p}\right)^{S_n}$

$= \left(\frac{q}{p}\right)^{S_n}$

$E(Z_\tau | \mathcal{F}_0) = Z_0 = \left(\frac{q}{p}\right)^c$

$\Rightarrow \left(\frac{q}{p}\right)^a P(\tau_a < \tau_b | S_0 = c) + \left(\frac{q}{p}\right)^b P(\tau_b < \tau_a | S_0 = c) = \left(\frac{q}{p}\right)^c$

$\Rightarrow p_{a,b} = \dfrac{\left(\frac{q}{p}\right)^c - \left(\frac{q}{p}\right)^b}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b} = \dfrac{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^c}{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^a}$.

For find $E(\tau)$ we use $Y_n = S_n - (p-q)n$.

Use $\{Y_n\}$ for $p \neq q$ & we $W_n = S_n^2 - n$.

~~$E(W_n | \mathcal{F}_0) =$~~ $E(W_\tau | \mathcal{F}_0) = W_0 = c^2$

$\Rightarrow$ Doob-Meyer decomposition

$\Rightarrow E(\tau | S_0 = c) = E(S_\tau^2 | S_0 = c) - c^2$

$= -c^2 + [a^2 P(\tau_a < \tau_b | S_0 = c) + b^2 P(\tau_b < \tau_a | S_0 = c)$

$= -c^2 [a^2(\frac{b-c}{b-a}) + b^2(\frac{c-a}{b-a})]$

$= (c-a)(b-c)$

Suppose: $\{\underset{\sim}{S_n}\}$ is a martingale then (component wise).

$\{V_n\}$ is a martingale

$$V_n = \varphi_n' S_n - \sum_{j=1}^{n}(\varphi_j' S_j - \varphi_{j-1}' S_{j-1}) + \underbrace{\varphi_0' S_0}_{V_0}.$$

If $\varphi' S$ is self-financing

$$V_n = \sum_{j=1}^{n}(\varphi_j' S_j - \varphi_{j-1}' S_{j-1}) + V_0.$$

$$= \sum_{j=1}^{n}(\varphi_j' S_j - \varphi_j' S_{j-1}) + V_0 + \sum_{j=1}^{n} C_{j-1}.$$

$$\therefore V_n = V_{n-1} + (\varphi_n' S_n - \varphi_n' S_{n-1}).$$

If S_n is a martingale & φ_n is bounded then V_n is a martingale.

$$E[V_{n+1}|\mathcal{F}_n] = E[V_n + \varphi_{n+1}' S_{n+1} - \varphi_n' S_n | \mathcal{F}_n]$$

$$= V_n + \varphi_{n+1}' \cancel{E(S_{n+1}|\mathcal{F}_n)} - \varphi_n' S_n$$

(X)

$= V_n + \varphi_{n+1}' S_n - \varphi_n' S_n.$

$= \varphi_{n+1}' S_n.$

$$E[V_n|\mathcal{F}_{n-1}] = V_{n-1} + E(\varphi_n'(S_n - S_{n-1}) - C_{n-1} | \mathcal{F}_{n-1})$$

$$= V_{n-1} + \varphi_n'(E(S_n|\mathcal{F}_{n-1}) - S_{n-1}) - C_{n-1}$$

$= V_{n-1} - C_{n-1}$ (If S_n is martingale)

$= V_{n-1}$ (If $C_{n-1} = 0$)

Sub & Super correspondingly

Real Market Model

$E(S_j|\mathcal{F}_{j-1}) = S_{j-1}(1+r)$

Arbitrage $\qquad \varphi_1' S_0 = \varphi_0' S_0.$

If

(i) $\underset{\sim}{\varphi}' S_0 = 0$

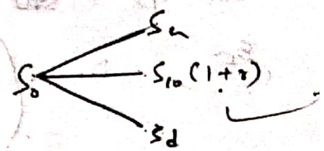
and $\varphi' S_1 = \cancel{\varphi' D_{j0}}\ \varphi' D_{j0} \geq 0$

& $\varphi' D_{j0} > 0$.

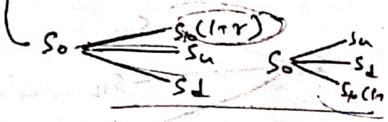
Martingale E

or (ii) $\varphi' S_0 < 0$

& $\varphi' D_j \geq 0 \ \forall j$.



← No arbitrage.



$\varphi' D_j \geq \varphi' S_0$

$\forall j$ & $\varphi' D_j > \varphi' S_0$

If for some φ then the market is said to have arbitrage.

$\tilde{S}_j = S_j (1+r)^{-j}$.

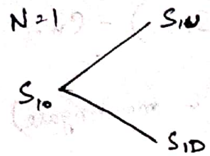
$\boxed{\text{Avellaneda}}$ → Book.

Quant. Fin.

Then $\{\tilde{S}_j\}$ is a martingale.

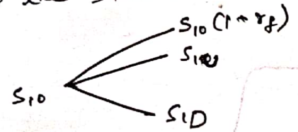
Example: Let $M=2$

$N=1$



$r_f \rightarrow$ risk free rate of growth.

For the situation where,



We will short-sell the risky asset & invest in the risk-free asset (Arbitrage present).

$\varphi_1 = -10.$

$\varphi_1 S_{10} + \varphi_0 \times 1 = 0$, (Assuming $S_{00}=1$).

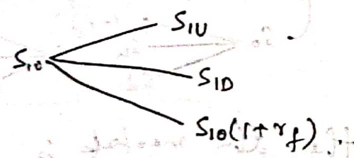
$\varphi_0 = 1000$

For price $S_{10} \times 10$,

Balance $= 1030$

$D_1 = S_0(1+r_f)$

$$D_1 = \begin{pmatrix} 1+r_f \\ S_{1U} \end{pmatrix} \qquad D_2 = \begin{pmatrix} 1+r_f \\ S_{1D} \end{pmatrix}$$



$S_{10} = 100$

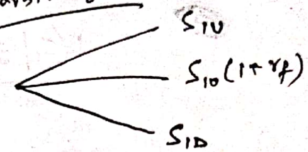
$S_{1U} = 110$

$S_{1D} = 107.$

$r_f = 6\%.$

Short-sell the ~~risky~~ risk-free asset & invest in risky ones.

No-arbitrage



$\nexists \varphi$ s.t., $\varphi' S_0 = 0$ & ~~for all~~ $\varphi' D_j \geq 0$ $\forall j$ & $\varphi' D_{j_0} > 0$ for some j_0

Result 1:

First Fundamental Thm. of Mathematical Finance.

A market has no arbitrage iff $\exists$ a strictly positive martingale measure.

$$D = \begin{array}{cc} \text{Column 1} & \text{Column 2} \\ \left(\begin{pmatrix} 1+r \\ S_{1U} \end{pmatrix} \right. & \left. \begin{pmatrix} 1+r \\ S_{1D} \end{pmatrix}\right) \end{array}$$

S_1

$\exists p$ s.t., $Dp = S_0(1+r)$.

Does there exist a p vector $E_p(\tilde{S}_1 | \mathcal{F}_0) = \tilde{S}_0$ (Discounted Martingale)

$\tilde{S}_1 = S_1(1+r)^{-1}$.

$\Leftrightarrow S_0(1+r) = E_p(\tilde{S}_1 | \mathcal{F}_0)$

$$\begin{pmatrix} 1+r & 1+r \\ S_{1U} & S_{1D} \end{pmatrix} \begin{pmatrix} p_U \\ p_D \end{pmatrix} = \begin{pmatrix} 1 \\ S_{10} \end{pmatrix} (1+r).$$

$\Rightarrow p_U + p_d = 1$

$S_{1U} p_U + S_{1D} p_D = S_{10}(1+r).$

~~$\Rightarrow S_{1U}p_U + S_{1D}p_D = S_{10}$~~ (

$\Rightarrow S_{1U}p_U + S_{1d}(1-p_u) = S_{10}(1+r)$, $\Rightarrow p_U(S_{1U} - S_{1d}) = S_{10}(1+r) - S$

$\Rightarrow p_u = \dfrac{S_{10}(1+r) - S_{1d}}{S_{1u} - S_{1d}}$ $E(S_1 | \mathcal{F}_0)$

$\Rightarrow p_d = 1 - p_u$.

Note $0 < p_u < 1$

$\Rightarrow S_{1u} > S_{10}(1+r) > S_{1d}$.

$m = 2$ & $n = 1$

Pf:- Suppose $\exists\ \underset{\sim}{p}$ (strictly +ve) vector s.t., $S_0(1+r) = D\underset{\sim}{p}$,

($\Rightarrow$) If possible let $\underset{\sim}{\varphi}$ be a non-zero vector s.t., $\underset{\sim}{\varphi}'\underset{\sim}{S_0} = 0$

& $\varphi' D_j \geq 0\ \forall j$ & $\varphi' D_{j_0} \gneq 0$ for some j_0.

Then $\varphi' S_0 (1+r) = \varphi' D p$. L.H.S $= 0$

& R.H.S. $\varphi' D p = \sum_{j=1}^{m} \varphi' D_j p_j \geq \varphi' D_{j_0} p_{j_0} > 0$

Hence, contradiction i.e., (i) cannot happen

For (ii) $\varphi' S_0 < 0$ & $\varphi' D_j \geq 0\ \forall j$ In that case

($\Leftarrow$) L.H.S

Suppose the market has no arbitrage.

Let $L = \{ \underbrace{(-\underset{\sim}{\varphi}' S_0, \underset{\sim}{\varphi}' D_1, \ldots \underset{\sim}{\varphi}' D_n)}_{M+1} : \varphi \in \mathbb{R}^{N+1} \}$,

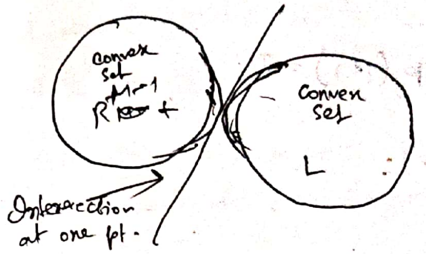
$R_+^{M+1} = \{ \underset{\sim}{x} : x_i \geq 0$ for $i = 1, \ldots, M+1 \}$.

Let $L = \{ (-\underset{\sim}{\varphi}' S_0, \underset{\sim}{\varphi}' D_1, \ldots, \underset{\sim}{\varphi}' D_M) : \varphi \in \mathbb{R}^{M+1} \}$.

Under no-arbitrage $L \cap R_+^{M+1} = \{ \underset{\sim}{0} \}$.

Note: L is a subspace (closed under addition).

In particular, L is a convex set & so is $\mathbb{R}_+^{M+1}$.



By Seperating Hyperplane

$H = a_1 x_1 + a_2 x_2 + \cdots + a_{M+1} x_{M+1} + a_0 = 0$.

Since, it is passing through $\{0\}$, so, $a_0 = 0$.

$H = a_1 x_1 + \cdots + a_{M+1} x_{M+1}$

$\Rightarrow \sum_{j=1}^{M+1} a_j x_j = 0$ passing through the origin.

s.t., $\underset{\sim}{a}'\underset{\sim}{y} \leq \underset{\sim}{a}'\underset{\sim}{z}$. $H = \{ x \in \mathbb{R}^{M+1} : \sum a_j x_j = 0 \}$.

Under no-arbitrage $a' y \not> 0$.

If possible let $\underset{\sim}{a}' y < 0$ then $-y \in L$ & $a'(-y) > 0$.

That means $a' y = 0$.

Therefore, $a' y = 0\ \forall\ y \in L \Rightarrow L \subseteq H$.

$\Rightarrow a' z > 0\ \forall\ z \neq 0$.

All a_j's are positive. (Taking standard basic for z values).

Then $j = 0, 1, \ldots, M$.

$\sum_{j=1}^{M} a_j \varphi' D_j = a_0 \varphi' S_0$.

$\Rightarrow \sum a_j D_j = a_0 S_0$. for taking φ as standard basis.

$\Rightarrow \sum \frac{a_j}{a_0} D_j = S_0$.

$\Rightarrow \sum \pi_j D_j = S_0$. $\pi_j = \frac{a_j}{a_0}$.

Now lets compare the first element, P.T.O.

$\sum \pi_j (1+r) = 1 \Rightarrow \sum \pi_j = \frac{1}{1+r}$.

Define, $p_j = \frac{\pi_j}{\sum \pi_j} \Rightarrow p_j(1+r) = \pi_j$;

Hence $\sum p_j D_j = S_0(1+r)$.

Due to Arrow-Debrieu.

Thus $\underset{\sim}{p}$ strictly is the martingale measure.

If there are 2 assets (1 risky + 1 risk-free) then we can have more than one martingale due to multiple sol^{n}s of the equations $Dp = S_0(1+r)$.

E

$$Dp = \begin{pmatrix} 1+r & 1+r & 1+r \\ S_{1u} & S_{1m} & S_{1d} \end{pmatrix} \begin{pmatrix} p_u \\ p_m \\ p_d \end{pmatrix} = \begin{pmatrix} 1 \\ S_{10} \end{pmatrix} (1+r).$$

(Not strictly +ve) $p_u^{(1)} = \frac{S_{10}(1+r) - S_{1m}}{S_{1u} - S_{1m}}$, $p_m^{(1)} = 1 - p_u$, $p_d^{(1)} = 0$

$p_u^{(2)} = \frac{S_{10}(1+r) - S_{1d}}{S_{1u} - S_{1d}}$, $p_d^{(2)} = 1 - p_u^{(2)}$, $p_m^{(2)} = 0$.

$p^{(\alpha)} = \alpha p_u^{(1)} + (1-\alpha) p_u^{(2)} > 0$ $\quad 0 < \alpha < 1$.

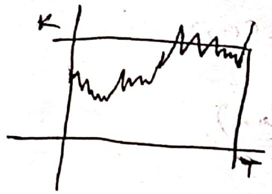
Check that under p: $\boxed{E_{p(\alpha)}(\underset{\sim}{S}_1 \mid \mathcal{F}_0) = S_0(1+r)}$

$\boxed{\text{Discounted Martingale.}}$

Replicability:

A ~~claim~~ (contingent) claim $\{f\}$ is said to be replicable ~~iff~~ if $\exists\, \underset{\sim}{\varphi}$ s.t. $\underset{\sim}{\varphi}'\underset{\sim}{S} = f_1$ [only two time steps 0 & 1].

Call option: A right to buy a certain asset or commodity at a prefixed price (say K) on or before a maturity time. (say T).



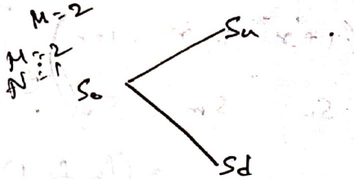
$f_T = S_T - K$, if $S_T > K$

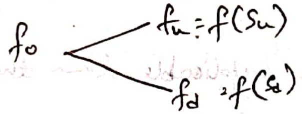
$= 0$, o.w.

$f_T := \max(S_T - K, 0)$

$= (S_T - K)^+$.

Rs. 30.
2 months
40
25





$$(\varphi_0 \;\; \varphi_1) \begin{pmatrix} 1+r & 1+r \\ S_u & S_d \end{pmatrix} = (f_u \;\; f_d).$$

1×2 $\quad 2\times 2$ $\quad 1\times 2$.

$\varphi_0(1+r) + \varphi_1 S_u = f_u$

$\varphi_0(1+r) + \varphi_1 S_d = f_d$.

$\Rightarrow \varphi_1(S_u - S_d) = f_u - f_d$.

$\Rightarrow \varphi = \frac{f_u - f_d}{S_u - S_d}$

$\varphi_0 = \frac{f_u - \frac{S_u(f_u - f_d)}{S_u - S_d}}{1+r} = \frac{f_u(S_u - S_d) - S_u(f_u - f_d)}{(1+r)(S_u - S_d)}$.

$(\varphi_0\ \varphi_1)\begin{pmatrix}1+r & 1+r\\ S_u & S_d\end{pmatrix} \underline{k} = (f_u\ f_d)\,\underline{k}$ $\quad (\hat{x}-\mu)^T \quad = 0$

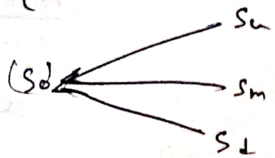
where $p_u = \dfrac{S_0(1+r)-S_d}{S_u-S_d}$, $p_d = 1-p_u$.

$$E_p(S_1|\mathcal{F}_0) = p_u S_u + p_d S_d = \frac{\cancel{S_uS_d}\,(S_uS_0(1+r) - S_uS_d) + (S_uS_d - S_dS_0(1+r))}{S_u-S_d}$$

$$= \frac{S_0(1+r)(S_u-S_d)}{S_u-S_d} = S_0(1+r).$$

R.H.S = $E_p[f_1|\mathcal{F}_0] = f_0(1+r)$. (Claim)

~~3 assets~~ $M=3,\ N=1$



$$(\varphi_0\ \varphi_1)\begin{pmatrix}1+r & 1+r & 1+r\\ S_u & S_m & S_d\end{pmatrix}\underline{k} = (f_u\ f_m\ f_d)\,\underline{k}.$$

If all (contingent) claims ~~if market~~ are replicable then the market is complete.

A market is said to be incomplete if $\exists$ a claim which is not replicable.

2nd Fundamental Thm of Math. Fin.

A NA market is complete iff the martingale measure is unique.

$$\varphi_0(1+r) + \varphi_1 S_u = f_u$$
$$\varphi_0(1+r) + \varphi_1 S_d = f_d$$
$$\Rightarrow f_u - \varphi_1 S_u = f_d - \varphi_1 S_d$$

long f, short S

1, φ_1

$f_0 - \varphi_1 S_0$: time 0 = (should be equal)

$f_u - \varphi_1 S_u$: time 1

OR — risk-neutral (if equal)

$f_d - \varphi_1 S_d$: time 1.

Thm 2: Under no arbitrage a risk neutral portfolio ~~cannot grow~~ can grow at the same rate as the risk-free asset.

$$(f_0 - \varphi_1 S_0)(1+r) = f_u - \varphi_1 S_u = f_d - \varphi_1 S_d.$$

$$E_p[f|\mathcal{F}_0] = f_u p_u + f_d p_d$$

$$= \frac{(f_u S_0(1+r) - f_u S_d) + f_d S_u - f_d S_0(1+r)}{S_u - S_d}$$

$$= \frac{S_0(1+r)(f_u-f_d)}{S_u-S_d} + \frac{f_d S_u - f_u S_d}{S_u - S_d}$$

$$= \varphi_1 S_0(1+r) + \frac{f_d(S_u-S_d) - S_d(f_u-f_d)}{S_u-S_d}$$

$$\cancel{\varphi_1 S_0(1+r)} = \varphi_1 S_0(1+r) + f_d - \varphi_1 S_d$$

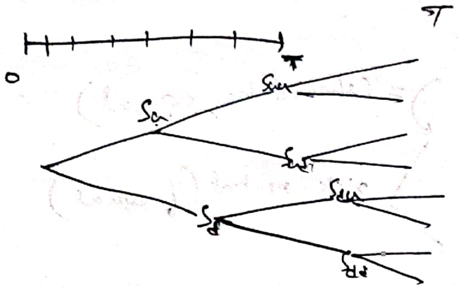
~~$\varphi_1 S_0(1+r) + \varphi_0(1+r)$~~

$$= \varphi_1 S_0(1+r) + (f_0 - \varphi_1 S_0)(1+r) \quad \text{(Risk-neutrality)}$$

$$= (1+r)(\cancel{\varphi_1 S_0} + f_0 - \cancel{\varphi_1 S_0})$$

$$= f_0(1+r)$$

Cox - Ross - Rubinsteine (1978) :



$\Rightarrow \log \frac{S_T}{S_0} = \log(u^k d^{n-k}) = \underline{k \log u + (n-k)\log d} = \log u (2k-n)$

$d = \frac{1}{u}$ · symmetric jumps.

<u>Homework:</u> Find the dist. of ~~non-sy~~ asymmetric jumps.

<u>Define:</u> $X_T = \log\left(\frac{S_T}{S_0}\right) = (\log u)(2B - n)$.

$B \sim Bin(n, p)$.

When $S_u = S_0 u$ & $S_d = S_0 d$

$$p_u = \frac{S_0(1+r) - S_d}{S_u - S_d} = \frac{S_0(1+r-d)}{S_0(u-d)} = \frac{1+r-d}{u-d}.$$

$$p_{uu} = \frac{S_u(1+r) - S_{ud}}{S_{uu} - S_{ud}} = \frac{\cancel{S_0[u(1+r) -}}{S_0 u[(1+r) - d]} = \frac{1+r-d}{u-d}.$$

$$\frac{X_T - E(X_T)}{\sqrt{Var(X_T)}} \sim N(0,1).$$

as $n \to \infty$.

$\Rightarrow X_T^* := \dfrac{X_T - (\log u) n (2p-1)}{\sqrt{(4 \log u)^2 npq}}$

$\boxed{\cancel{P(X_T \le x) - p}}$

So, X_T has a limiting dist^n (Normal) if

$E(X_T) = (\log u) n (2p-1)$

$= 2(\log u)\sqrt{n}\, pq + n(2p-1)(\log u)$

$p = \frac{1}{2} + \delta_n$

$q = \frac{1}{2} - \delta_n$

$pq = \frac{1}{4} - \delta_n^2$

$2n\delta_n \log u \to$ must converge.

$Var(X_T) = 4(\log u)^2 npq$

$= n\, 4(\log u)^2 \left(\frac{1}{4} - \delta_n^2\right)$

$= n(\log u)^2 + n\cancel{\delta_n (\log u)^2} \to 0$

$n(\log u)^2 \propto \sigma^2$.

$n(\log u)^2 = c\sigma^2$

$\Rightarrow (\log u)^2 = \frac{c\sigma^2}{n}$

$\Rightarrow \log u = \sigma\sqrt{\frac{c}{n}} \Rightarrow u = \exp\left(\sigma\sqrt{\frac{c}{n}}\right)$. $\Rightarrow d = \exp\left(-\sigma\sqrt{\frac{c}{n}}\right)$

$$p_n = \frac{1 + \frac{r_T}{n} - e^{-\sigma\sqrt{\frac{c}{n}}}}{e^{\sigma\sqrt{\frac{c}{n}}} - e^{-\sigma\sqrt{\frac{c}{n}}}}.$$

$e^{\sigma\sqrt{\frac{c}{n}}} = 1 + \sigma\sqrt{\frac{c}{n}} + \frac{\sigma^2 c/n}{2!} + O\left(\frac{1}{n^3}\right)$

Denom $= u_n - d_n = 2\sigma\sqrt{c/n} + O\left(\frac{1}{n^{3/2}}\right)$.

$$\text{Num} = \frac{rT}{n} + \sigma\sqrt{\frac{c}{n}} - \frac{\sigma^2}{2}\frac{c}{n} + o\left(\frac{1}{n^{3/2}}\right).$$

$$\therefore p_n = \frac{\text{Num}}{\text{Den}} = \frac{\frac{rT}{n} + \sigma\sqrt{\frac{c}{n}} - \frac{\sigma^2}{2}\frac{c}{n} + o(1/n^{3/2})}{2\sigma\sqrt{c/n} + o(1/n^{3/2})}$$

$$= \frac{1}{2} + \frac{(rT - \sigma^2 c)}{2\sigma/\sqrt{n}} + o\left(1/n^{3/2}\right).$$

$$\text{Var}(X_T) \to c\sigma^2$$

$$E(X_T) = 2(p_n - 1)\log u_n = \frac{n(rT - \frac{\sigma^2}{2}c)}{\sigma\sqrt{cn}} \times \sigma\sqrt{\frac{c}{n}}.$$

$$= \left(rT - \frac{\sigma^2}{2}c\right)\frac{n}{n} + o\left(\frac{1}{n^{1/2}}\right)$$

$$\longrightarrow rT - \frac{\sigma^2}{2}c.$$

$p + (1-p)$.

As $c = T$ as the growth of the variance is proportional to time elapsed,

$$X_T \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right).$$

r - is yearly

$\frac{r}{n}$ is monthly

$\frac{rT}{n}$ - till horizon T.

Next Time on Finance with GKB:

$\{X_T\}$ is ~~geometri~~ Brownian Motion

& $\{S_T\}$ is Geometric Brownian Motion.

$$E(S_T|\mathcal{F}_0) = S_0 e^{rT}$$

where $\log\left(\frac{S_T}{S_0}\right)\Big|S_0 \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$.

$$S_0 \begin{cases} \nearrow S_u \\ \searrow S_d \end{cases} \qquad f_0 \begin{cases} \nearrow f(S_u) = f_u \\ \searrow f(S_d) = f_d \end{cases}$$

$$E_p(S_T|\mathcal{F}_0) = \cancel{S_0 e}\ S_0(1 + rT).$$

where $p = \frac{S_0(1+rT) - S_d}{S_u - S_d}$.

$$E_p(f_T|\mathcal{F}_0) = f_0(1 + r).$$

Assuming: $S_u = S_0 u$, $S_d = S_0 d$. $ud = 1$.

$$f_T = f(S_T) = f(S_0 u^k d^{n-k})$$

$$S_T = S_{n(\frac{T}{n})}$$

$$E_p = \left(S_{n(\frac{T}{n})} \,\Big|\, \mathcal{F}_{(n-1)\times\frac{T}{n}}\right)$$

$$= S_{(n-1)\times(\frac{T}{n})}\left(1 + \frac{rT}{n}\right).$$

$$E_p\left(f(S_{n(\frac{T}{n})} | \mathcal{F}_{(n-1)\frac{T}{n}})\right) = f(S_{(n-1)\frac{T}{n}})\left(1 + \frac{rT}{n}\right).$$

$$f(S_{(n-1)\frac{T}{n}}) \begin{cases} \nearrow f(S_{(n-1)\frac{T}{n}} \times u) \\ \searrow f(S_{(n-1)\frac{T}{n}} \times d) \end{cases}$$

Using replicable pf using (φ_0, φ_1). We can show the same way,

$$p f(S_{(n-1)\frac{T}{n}} \times u) + (1-p) f(S_{(n-1)\frac{T}{n}} \times d)$$

$$= f(S_{(n-1)\frac{T}{n}})\left(1 + \frac{rT}{n}\right),$$

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Iterating this process we can show.

$$E_p\left(f(S_{n(\frac{T}{n})})' \mid \mathcal{F}_{(n-2)\frac{T}{n}}\right) = f(S_{(n-1)\frac{T}{n}})\left(1+\frac{r}{n}\right).$$

$$E\left(E\left(f(S_{n(\frac{T}{n})}) \mid \mathcal{F}_{(n-1)(\frac{T}{n})}\right) \mid \mathcal{F}_{(n-2)\frac{T}{n}}\right) = f(S_{(n-2)(\frac{T}{n})})\left(1+\frac{rT}{n}\right)^2$$

$$\boxed{E(E(X \mid Y, Z) \mid Y) = E(X \mid Y)}$$

Iterating further we get

$$E_p\left(f(S_{n\frac{T}{n}}) \mid \mathcal{F}_0\right) = f_0\left(1+\frac{rT}{n}\right)^n$$

$\cancel{S_T = S_0 e^{rT}}$

$$\Rightarrow E_N\left(f(S_T) \mid \mathcal{F}_0\right) = f_0 e^{rT}.$$

Distⁿ of $S_{n(\frac{T}{n})}$ (log-Normal Binom. paths), converges log-normal with parameter $\mu = (r - \frac{\sigma^2}{2})T$, $v = \sigma^2 T$.

$\therefore$ Let f be the price f^n for call option then

$$f(S_T) = \begin{cases} S_T - K & \text{if } S_T > K. \\ 0 & \text{if } S_T \le K. \end{cases}$$

Then $E(f(S_T) \mid \mathcal{F}_0) = \int_K^\infty (S_T - K)\, dF_{S_T}$ given S_0

$$S_T = S_0 e^{X_T}$$

$$\therefore E(f(S_T) \mid \mathcal{F}_0) = \int_K^\infty S_0 e^{X_T} - K \, dF_{S_T}$$

$$= \int_K^\infty S_0 e^{X_T} dF_{S_T} - \int_K^\infty K \, dF_{S_T}.$$

$$= \int_{\log(\frac{K}{S_0})}^\infty S_0 e^{X_T} dF_{X_T} - \int_{\log(\frac{K}{S_0})}^\infty K \, dF_{X_T}$$

$$= \int_{\log(\frac{K}{S_0})}^\infty \frac{S_0 e^x e^{-\frac{(x-\mu)^2}{2v^2}}}{v\sqrt{2\pi}} dx$$

$$\cancel{- K\left[1 - \Phi\left(\log\left(\frac{K}{S_0}\right)\right)\right]}.$$

$$- K\left[1 - \Phi\left(\frac{\log(\frac{K}{S_0}) - \mu}{v}\right)\right].$$

$$= \int_{\log(\frac{K}{S_0})}^\infty \frac{S_0 e^x e^{-\frac{(x-\mu)^2}{2v^2}}}{v\sqrt{2\pi}} dx - K\,\Phi\left(\frac{\log(\frac{S_0}{K}) + \mu}{v}\right)$$

$$= \int_{\log(\frac{K}{S_0})}^\infty \frac{S_0 e^x e^{-\frac{(x-\mu)^2}{2v^2}}}{v\sqrt{2\pi}} dx - K\,\Phi\left(\frac{\log\frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$B(e^{X})$

$\cancel{\mu + \frac{1}{2}\sigma^2}$.

$= e^{\mu}$

$= e^{\mu + \frac{1}{2}\sigma^2}$.

S_0

$X_T \sim N((r - \frac{\sigma^2}{2})T, \sigma^2 T)$

$e^{r - \frac{\sigma^2}{2} + \frac{1}{2}\sigma^2 T}$.

$S_0 e^{r + \frac{\sigma^2}{2}(T-1)} - K.$

$f(S_0)(1 + \frac{rT}{n})$

Exponent $= x - \dfrac{(x-\mu)^2}{2v^2}$.

$$= \frac{2v^2 x - (x^2 - 2\mu x + \mu^2)}{2v^2}$$

$$= -\frac{x^2 - 2x(\mu + v^2) + \mu^2}{2v^2}$$

$$= -\frac{x^2 - 2x(\mu+v^2) + (\mu+v^2)^2}{2v^2} + \frac{\quad}{\quad}$$

$$= -\frac{(x - (\mu+v^2))^2}{2v^2} + \frac{\cancel{\mu^2} + v^4 + 2\mu v^2 - \cancel{\mu^2}}{2v^2}$$

$$= -\frac{(x - (\mu+v^2))^2}{2v^2} + \frac{\cancel{v^2}(v^2+2\mu)}{2\cancel{v^2}}$$

$\Rightarrow \mu + v^2 = (r + \frac{\sigma^2}{2})T.$

$\mu + \frac{v^2}{2} = rT.$

$$e^{x - \frac{(x-\mu)^2}{2v^2}} = e^{-\frac{(x-(r+\frac{\sigma^2}{2})T)^2}{2\sigma^2 T}} \cdot e^{rT}.$$

Hence,

$$= \frac{S_0 e^{rT}}{\cancel{\sigma\sqrt{2\pi}}} \int_{\log(\frac{K}{S_0})}^{\infty} \frac{e^{-(x-(r+\frac{\sigma^2}{2})T)^2}}{2\sigma^2 T} \times \frac{1}{\sigma^2 T\sqrt{2\pi}}\, dx$$

$$- K\,\Phi\left(\frac{\log(\frac{S_0}{K}) + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= S_0 e^{rT}\left[1 - \Phi\left(\frac{\log(\frac{K}{S_0}) - (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right]$$

$$- K\Phi\left(\frac{\log(\frac{S_0}{K}) + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$\cancel{= S_0 e^{rT}[}$$

$$= S_0 e^{rT}\,\Phi\left(\frac{-\log(\frac{K}{S_0}) + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log(\frac{S_0}{K}) + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= S_0 e^{rT}\,\Phi\left(\frac{\log(\frac{S_0}{K}) + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) -$$

$$K\Phi\left(\frac{\log(\frac{S_0}{K}) + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right).$$

$\Rightarrow f_0 e^{rT} = S_0 e^{rT}\Phi(d_1) - K\Phi(d_2)$

$\cancel{f_0 e^{rT} =} \Rightarrow f_0 = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2).$

where $d_1 = \frac{\log(\frac{S_0}{K}) + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ & $d_2 = d_1 - \sigma\sqrt{T}.$

At any time pt: $0 < t < T$.

$E(f(S_T)\mid \mathcal{F}_t) = f_t e^{r(T-t)}.$

$E(S_T \mid S_t) = S_t e^{r(T-t)}.$

$\log\left(\frac{S_{t_1}}{S_{t_0}}\right) \sim N\left((r-\sigma^2)(t_1 - t_0),\ \sigma^2(t_1 - t_0)\right).$

$p = \frac{1 + r\frac{t-t_0}{n} - d}{u - d} \qquad u = e^{\sigma\sqrt{\frac{t_1 - t_0}{n}}}.$

$(\log u)^2 \propto \frac{t_1 - t_0}{n} \rightarrow \frac{T}{n}.$

$(\log u)^2 = \sigma^2\frac{(t_1 - t_0)}{n}.$

$\log u = \sigma\sqrt{\frac{t_1 - t_0}{n}}.$

$\Rightarrow u = \exp\left(\sigma\sqrt{\frac{t_1 - t_0}{n}}\right).$

$f_t = S_t\Phi(d_{1t}) - Ke^{-r(T-t)}\Phi(d_{2t}).$

$d_{1t} = \frac{\log(\frac{S_t}{K}) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$

& $d_{2t} = d_{1t} - \sigma\sqrt{T-t}.$

Hedging parameter $\varphi_1 = \frac{f_u - f_d}{S_u - S_d} \longrightarrow \frac{\partial f}{\partial S} = \Delta$

Now, the hedging parameter Δ can be found by using the relation between f_t & S_t.

$$\therefore \Delta = \frac{\partial f}{\partial S}$$

$$\Delta = \frac{\partial f}{\partial S} = \Phi(d_{1t}) + S\varphi(d_{1t})\frac{\partial d_{1t}}{\partial S} - Ke^{-r(T-t)}\varphi(d_{2t})\frac{\partial d_{2t}}{\partial S}$$

$$\varphi(d_{1t}) = \varphi(d_{2t} + \sigma\sqrt{T-t}) = \frac{e^{-\frac{(d_{2t}+\sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}}$$

$$(d_{2t} + \sigma\sqrt{T-t})^2 = d_{2t}^2 + 2\sigma\sqrt{T-t} + \sigma^2(T-t)$$

$$= d_{2t}^2 + 2\left[\log\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] + \sigma^2(T-t)$$

$$= d_{2t}^2 + 2\left[\log\frac{S_t}{K} + r(T-t)\right].$$

$$\Rightarrow \frac{d_{1t}^2}{2} = \frac{d_{2t}^2}{2} + \log\frac{S_t}{K} + r(T-t)$$

$$\Rightarrow e^{-\frac{d_{1t}^2}{2}} = e^{-\left[\frac{d_{2t}^2}{2} + \log\frac{S_t}{K} + r(T-t)\right]}.$$

$$= e^{-\frac{d_{2t}^2}{2}} \times \frac{K}{S_t} \times e^{-r(T-t)}$$

$$\Rightarrow S_t\varphi(d_{1t}) = S_t\varphi(d_{2t})\frac{Ke^{-r(T-t)}}{S_t}$$

$$= Ke^{-r(T-t)}\varphi(d_{2t})$$

$$\therefore \Delta = \frac{\partial f}{\partial S} = \Phi(d_{1t}) + Ke^{-r(T-t)}\varphi(d_{2t})\left[\frac{\partial d_{1t}}{\partial S} - \frac{\partial d_{2t}}{\partial S}\right]$$

$$= \Phi(d_{1t}) + Ke^{-r(T-t)}\varphi(d_{2t})\left[\cancel{\frac{\partial d_{2t}}{\partial S}} + \frac{\partial \sigma\sqrt{T-t}}{\partial S} - \cancel{\frac{\partial d_{2t}}{\partial S}}\right]$$

$$= \Phi(d_{1t}).$$

case 1:

$$\frac{\partial f_t}{\partial S_t} = \Phi(d_{1t}) \to 0$$

as
$$d_{1t} = \cancel{\frac{\log\frac{S_T}{K}}{\sigma\sqrt{T-t}}} \to -\infty \; + \; \cancel{\frac{(r+\frac{\sigma^2}{2})(T-t)}{\sigma}} \to 0,$$

case 2:

$$\frac{\log\frac{S_T}{K} > 0}{\sigma\sqrt{T-t}} \to \infty.$$

Calculate $\frac{\partial f}{\partial K}, \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \sigma}, \frac{\partial f}{\partial T}$.

Put - Call Parity,

$$C_E + Ke^{-rT} = P_E + S_0$$

on the same asset
Same ~~stove~~ strike price
in maturity.

→ At any pt. of time

European call option gives the holder right to buy certain at a pre-fixed price, K at a maturity time T.

European put option . . . . . right to sell

$$\text{Payoff Put} = \begin{cases} K - S_T, & \text{if } K > S_T \\ 0, & \text{if } K \le S_T \end{cases}$$

Put-call Parity for No arbitrage:

$c_E + Ke^{-rT} = P_E + S_0$.

$c_E(0) + Ke^{-rT} = P_E(0) + S_0$

$c_E(t) + Ke^{-r(T-t)} = P_E(t) + S_t$

Pf I	Pf II
A call + c_E + Cash amount Ke^{-rT}	A put & underlying asset with price S_0.

at time T, Value of Pf·I $= c_E(T) + (Ke^{-rT})e^{rT}$.

$$c_E(T) = \begin{cases} S_T - K, & \text{if } S_T > K \\ 0, & \text{o.w.} \end{cases}$$

$$\text{Value of PF I} = \begin{cases} S_T - K + K, & \text{if } S_T > K \\ \hookrightarrow S_T, & \text{if } S_T > K \\ 0 + K, & \text{if } S_T \le K. \end{cases}$$

Value of (PF II) $= P_E(T) + S_T$

Now, $$P_E(T) = \begin{cases} K - S_T, & \text{if } S_T < K, \\ 0, & S_T \ge K. \end{cases}$$

Holder of the Put-option has the right to sell the underlying asset at a ~~price~~ pre-fixed price (say K) at a maturity time (say T).

Holder of the Call-option has the right to ~~sell~~ buy the underlying asset at the maturity time (T).

H/W:- Calculate the Put Price using LN density of the asset price.

$$d_1 = \frac{\log\frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

$= \cancel{0.484}\ 0.878$

$d_2 = d_1 - \sigma\sqrt{T} = \cancel{0.484} \cancel{0.359}$.

$= 0.753$.

$S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$

$= \cancel{60}\ \Phi(0.878) - \cancel{600}\ 55 \times e^{-0.06\times 1/4}\Phi(0.753)$

$= 60 \times \Phi(0.878) - 55 \times 0.985\ \Phi(0.753)$

$= 60 \times 0.81 - 55 \times 0.985 \times 0.7743$

~~= 17.15~~

$= 48.6 - 41.95 = 6.65$.

Can Value(PF I(0)) > Value(PF II(0)).

Value(PF I(0)) < Value(PF II(0)).

Under NA,

$c_E(0) + Ke^{-rT} = P_E(0) + S_0$.

American Options:

Call Option: Holder has the right buy the underlying assets at a pre-fixed price (say K). On or before maturity time say T

Put option: Holder has the right to sell the underlying asset at a pre-fixed price (say K) on or before maturity, T.

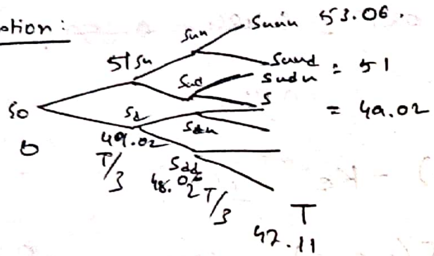
Inea.s for

$$C_A^{(0)} + Ke^{-rT} \le P_A^{(0)} + S_0 \le C_A^{(0)} + K.$$

John Hull : Options future and other derivatives.

American Option :

Put Options :



$S_0 = 50$

$K = 52.$

$r = 6\%$

$u = 1.02, \; d = \frac{1}{u} = 0.98.$

$T = 3$ months.

Under No-Arbitrage

$$p = \frac{S_0(1+\frac{rT}{3}) - S_d}{S_u - S_d} = \frac{\cancel{S_0}\left[(1+\frac{rT}{3}) - d\right]}{\cancel{S_0}[u-d]}$$

$$= \frac{1+\frac{rT}{3} - d}{u-d}$$

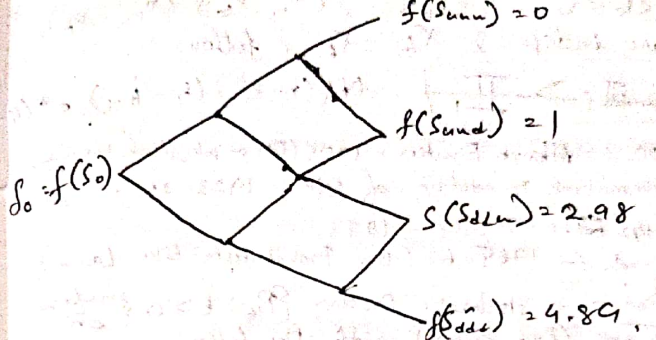
$$= \frac{1+.06\times\frac{3}{12\times3} - 0.98}{1.02 - 0.98}$$

$$= \cancel{0.875}\; 0.625.$$

$$q = 1-p = \cancel{0.125}\; 0.375.$$

Put option Payoff at time $T = \begin{cases} K - S_T, & \text{if } S_T < K. \\ 0, & \text{o.w.} \end{cases}$

American Put option intrinsic value at time $t \le T$ $= \begin{cases} K - S_t, & \text{if } S_t \le K. \\ 0, & \text{if } S_t \ge K. \end{cases}$



Expected $f(S_{dd}) = (p\times2.98 + q\times4.89)\times(1-\frac{rT}{3})$

$$= \cancel{1.824}\;4.82\times\left(1 - \frac{0.06\times\cancel{3}}{12\times\cancel{3}}\right).$$

$$= \cancel{1.8245}\;\cancel{3.96}.\;3.677.$$

0		T/3		2T/3		T
EV	IV	EV	IV	EV	IV	
				max(0.373,	0)	0
max(1.733	2)	max(0.978	1)	max(1.733,	2)	1
		max(3.616,	2.98)	max(3.677,	2.94)	2.98
						4.89

$f(S_{ud}) = (p\times1 + q\times2.98)\times(1-\frac{rT}{3})$, $f(S_d) = 2.618$

~~$f(S_d) = (p\times2$~~ $f(S_u) = 0.978.$

H/W : Check the American Call Option Price & European Put & Call option prices with the same data.

CRR (Cox-Ross-Rubinstein) model & distn of S_T,

0 —— t_1 —— t_2 —— T

$$\log\left(\frac{S_T}{S_0}\right) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right).$$

$$X_{t_2} - X_{t_1} = \log\left(\frac{S_{t_2}}{S_{t_1}}\right) \sim N\left(\left(r-\frac{\sigma^2}{2}\right)(t_2 - t_1), \sigma^2(t_2 - t_1)\right).$$

$$X_{t_1} - X_{t_0} = \log\left(\frac{S_{t_1}}{S_{t_0}}\right) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)(t_1 - t_0), \sigma^2(t_1 - t_0)\right).$$

$(X_{t_2} - X_{t_1})$ & $(X_{t_1} - X_{t_0})$ are independent from the assumption of CRR Model.

More generally, $0 \le t_0 \le t_1 \dots \le t_k \le T$, $(X_{t_1} - X_{t_0}), (X_{t_2} - X_{t_1}), \dots (X_{t_k} - X_{t_{k-1}})$ are indep. & $X_{t_i} - X_{t_{i-1}}$ follows $N\left(\left(r - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}), \sigma^2 (t_i - t_{i-1})\right)$

$\mathcal{F}_s$ $\boxed{\cancel{T_i(n) \dots \Pi_i > \Pi_T}}$

Louis Bachelier – 1900 Albert Einstein – 1905 (BM as physical model)

Norbert Weiner – Mathematical Foundation of BM, – 1923–25.

~~Prishar~~ Myrona.

Black – Scholes – 1971/73 Robert Merton – 1973.

Bachelier thesis discovered in 1965–67 (By Probabilists like Leave)

CRR Model – 1978.

Standard Brownian Motion is a stochastic Process $\{B_t : t \ge 0\}$ defined on ~~the same~~ the same exp. (Prob. space) with the following properties:

(i) $B_0 = 0$, $t \to B_t$ is continuous for allmost all particles

(ii) For $0 \le t_0 < t_1 < \dots < t_k$, $k \ge 1$, $(B_{t_1} - B_{t_0}), (B_{t_2} - B_{t_1}), \dots, (B_{t_k} - B_{t_{k-1}})$ are indep.

(iii) $(B_t - B_s | \mathcal{F}_s) \sim N(0, t-s)$, $s < t$

Thus $X_t = \mu t + \sigma B_t$, where $\mu = r - \frac{\sigma^2}{2}$ & hence X_t is a Brownian Motion

where (i) & (ii) satisfied by X_t & (iii) → (iii)′ as for $s < t$ $(X_t - X_s)$ given $\mathcal{F}_s$ follows $N(\mu(t-s), \sigma^2(t-s))$.

~~Brownian~~ $\{B_t\}$ is no where diff'ble.

Stochastic Calculus: (i) $(dB_t)^2 = dt$ (ii) $(dB_t)dt = 0$ (iii) $(dt)^2 = 0$

Understanding: (i) $E(B_{t+h} - B_t)^2 = h$ (Variance).

(ii) $E\left[\frac{(B_{t+h} - B_t)(t+h-t)}{h}\right] = \frac{(B_{t+h} - B_t)\cancel{h}}{\cancel{h}} \to 0$.

(iii) $\lim_{h \to 0} \frac{h^2}{h} = 0$, $(dB_t) = \lim_{h \to 0} (B_{t+h} - B_t) = 0$.

Ito's formula: Kyosi $f(t, B_t)$ ::

$$df(t, B_t) \approx f(t+h, B_{t+h}) - f(t, B_t) = f_t(t+h-t) + f_x(B_{t+h} - B_t) + \frac{1}{2} f_{tt}(t+h-t)^2 + \frac{1}{2} f_{xx}(B_{t+h} - B_t)^2 + f_{tx}(t+h-t)(B_{t+h} - B_t) + \dots$$

$$= f_t dt + f_x dB_t + \frac{1}{2} f_{tt} \cancel{(dt)^2}^{0} + \frac{1}{2} f_{xx} (dB_t)^2 + \cancel{f_{tx} dt dB_t}^{0} + \dots$$

$$= f_t dt + \frac{1}{2} f_{xx} dt + f_x dB_t.$$

H/W: Complete $d(\log S_t)$, $df(S_t)$.

$$\therefore dS_t = \mu f dt + \sigma f dB_t + \frac{1}{2} \sigma^2 f dt.$$

$$= \left(\mu + \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dB_t = r S_t dt + \sigma S_t dB_t$$

for $\mu = r - \frac{\sigma^2}{2}$

Chernoff

Left tail: let $X_1, \dots, X_n$ be indep. Bern. vars,

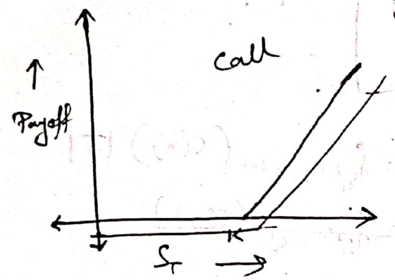
$X_i \sim B_1(p_i)$, $\mu = p_1 + \dots + p_n$.

$X = X_1 + \dots + X_n$)

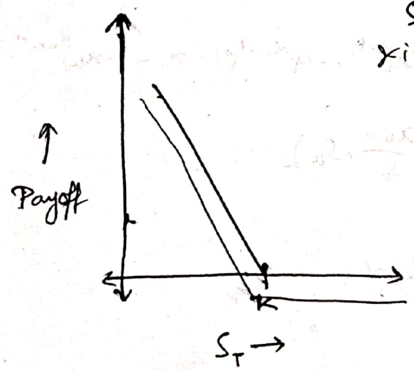
$\cancel{P(|X-\mu|}$

Trading strategies

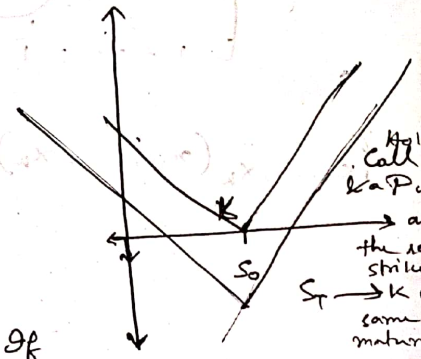
① **Bull market**



② **Bear Market**



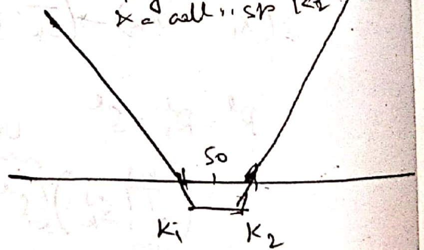
③ **Volatile Market**



Holding Call & a Put at the same strike price $S_T \to K$ & same maturity T.

If $S_T > K$ exercise call & if $S_T < K$ exercise Put.

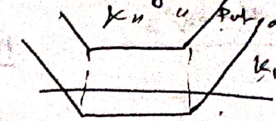
Buy a put with sp K_1 & a call " sp $K_2 > K_1$.



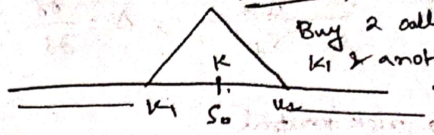
$K_1 \leq S_T < K_2$

④ **Stagnant Market**

Buy a call at K_1 & put at K_2; $K_1 < S_0 < K_2$



Butterfly



Buy 2 calls one at K_1, less K_2, K_1 & another at $K_2 > K_1$; sell 2 calls at $K = \frac{K_1 + K_2}{2}$.

Butterfly: Buy a call at K_1 & put at K_2

$S_T < K_1$	$K_1 < S_T < K_2$	$S_T > K_2$
$K_2 - S_T$	$S_T - K_1 + K_2 - S_T = K_2 - K_1$	$S_T - K_1$

$S_T < K_1$	$K_1 < S_T < K$	$K < S_T < K_2$	$S_T > K_2$
0	$S_T - K$	$S_T - K_1 - 2(S_T - K) = 2K - K_1 - S_T = K_2 - S_T$	$(S_T - K_1) + (S_T - K_2) - 2(S_T - K) = 0$

H/W :- $C(K_1) + C(K_2) \gtrless 2C(K)$.

where $K_1 < K < K_2$ & $K_1 + K_2 = 2K$.

B-S-M eq^n's

$f \quad S$

H/W: Find Δ for Put Option (Eu)

$\phi_{Eu} = \frac{f_u - f_d}{S_u - S_d}$, $\Delta = \frac{\partial f}{\partial S}$.

$V_0 = (f - \Delta S) \rightarrow$ risk neutral

$dS_t = \mu$

$dV = df - \Delta dS$

$df(t,S) = f_t(t,S)\,dt + \frac{1}{2} f_{tt} (dt)^2 + f_s\, dS_t$

$+ \frac{1}{2} f_{ss} (dS_t)^2 + f_{st}\, dt \times dS_t$

$(dS_t)^2 = (\mu S_t)^2 (dt)^2 + (\sigma S_t)^2 (dB_t)^2 + \cancel{2\mu\sigma S_t^2\, dt \times dB_t}^{\,0}$

$= \sigma^2 S_t^2\, dt$

$f(t+h, S_{t+h}) - f(t, S_t) = f_t h + \frac{1}{2} f_{tt} (t+h-t)^2$

$+ f_s (S_{t+h} - S_t) + \frac{1}{2} f_{ss} (S_{t+h} - S_t)$

$+ \underbrace{f_{st} (t+h-t)(S_{t+h} - S_t)} + \ldots$

$dt \times dS_t = \cancel{\mu S_t (dt)^2}^{\,0} + \sigma S_t \cancel{(dt \times dB_t)}^{\,0}$

$df(t, S_t) = f_t\, dt + f_s\, dS + \frac{1}{2} f_{ss} \sigma^2 S_t^2\, dt$.

$\therefore df(S, dS) = f_t\, dt + \cancel{f_s\, dS_t} + \frac{1}{2} f_{ss} \sigma^2 S_t^2\, dt - \cancel{f_s\, dS}$.

$= \left(f_t + \frac{\sigma^2 S_t^2}{2} f_{ss}\right) dt$.

~~dV~~

For risk-neutral pf.

$dV_t = r V_t\, dt$.

$V_t = V_0 e^{rt}$

$\frac{dV_t}{dt} = V_0 e^{rt} = r V_t$.

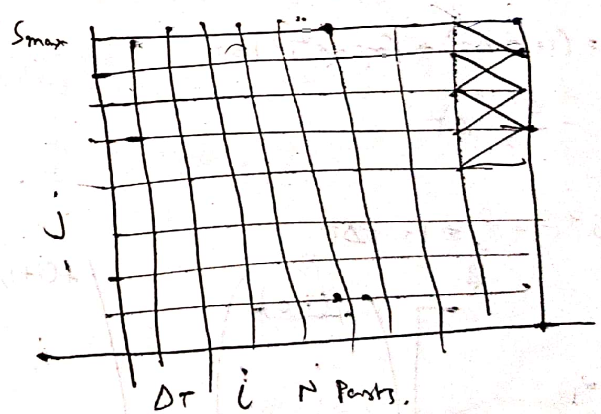
$dV = r(f - f_s S)\, dt$.

$\Rightarrow \left(f_t + \frac{1}{2} \sigma^2 S^2 f_{ss}\right) dt = r(f - f_s S)\, dt$.

$\cancel{\boxed{\Rightarrow \frac{\partial f}{\partial t} + f_s =}}$

$\Rightarrow \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r f$.

Discretization, $(i\Delta T, j\Delta S) = (i, j)$



$\frac{\partial f(i,j)}{\partial t} \approx \frac{f(i+1, j) - f(i,j)}{\Delta T}$

$\frac{\partial f(i,j)}{\partial S} \approx \frac{f(i, j+1) - f(i,j)}{\Delta S}$

$$\frac{\partial f}{\partial t}(i,j) \approx \frac{\dfrac{f(i,j+1)-f(i,j)}{\Delta S} - \dfrac{f(i,j)-f(i,j-1)}{\Delta S}}{\Delta S}$$

$$= \frac{f(i,j+1)+f(i,j-1)-2(f(i,j))}{(\Delta S)^2}$$

$$\Delta T\left(\frac{f(i+1,j)-f(i,j)}{\Delta T} + r(j\Delta S)\frac{f(i,j+1)-f(i,j)}{\Delta S} + \frac{\sigma^2(j\Delta S)^2}{2}\left(\frac{f(i,j+1)+f(i,j-1)-2f(i,j)}{(\Delta S)^2}\right)\right) = r f(i,j)\Delta T$$

$\Rightarrow \alpha_j f(i,j+1) + \beta_j f(i,j) + \gamma_j f(i,j-1) = f(i+1,j)$

where $\alpha_j = -\left(r_j + \frac{\sigma^2 j^2}{2}\right)\Delta T$

$\beta_j = (1+r\Delta T) + (r_j + \sigma^2 j^2)\Delta T$

$\gamma_j = -\frac{\sigma^2 j^2}{2}\Delta T$

AND $\alpha_j + \beta_j + \gamma_j = 1 + r\Delta T$

Tri-diagonal

$$\underset{A}{\begin{bmatrix} \beta_0 & \alpha_0 & & \\ \gamma_1 & \beta_1 & \alpha_1 & \\ & \ddots & \ddots & \ddots \\ & & & \alpha_{M-1} \\ & & \gamma_M & \beta_M \end{bmatrix}} \underset{X_i}{\begin{pmatrix} f(i,0) \\ \vdots \\ f(i,M) \end{pmatrix}} = \begin{pmatrix} f(i+1,0) \\ \vdots \\ f(i+1,M) \end{pmatrix}$$

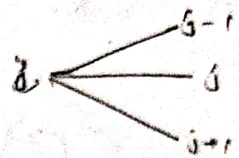
$A X_{N-1} = X_N$

$A X_{N-2} = X_{N-1}$

$\vdots$

$A X_0 = X_1$

Hence $A^N X_0 = X_N$



Stable But computationally Intensive.

To make computationally less expensive,

$$\Delta T\left(\frac{f(i,j)-f(i-1,j)}{\Delta T} + r(j\Delta S)\frac{f(i,j+1)-f(i,j)}{\Delta S} + \frac{\sigma^2(j\Delta S)^2}{2}\frac{f(i,j+1)+f(i,j-1)-2f(i,j)}{(\Delta S)^2}\right) = r f(i,j)\Delta T$$

$\Rightarrow \tilde{\alpha}_j f(i,j+1) + \tilde{\beta}_j f(i,j) + \tilde{\gamma}_j f(i,j-1) = f(i-1,j)$

$\tilde{\alpha}_j = \left(r_j + \frac{\sigma^2 j^2}{2}\right)\Delta T$

$\tilde{\beta}_j = (1-r\Delta T) - (r_j + \sigma^2 j^2)\Delta T$

$\tilde{\gamma}_j = \frac{\sigma^2 j^2}{2}\Delta T$

& $\tilde{\alpha}_j + \tilde{\beta}_j + \tilde{\gamma}_j = 1 - r\Delta T$

ONE eqⁿ one unknown but computationally far less expensive.

Transformation

$Z = \log S \Rightarrow S = e^Z$

$\frac{\partial f}{\partial Z} = \frac{\partial f}{\partial S}e^Z \Rightarrow \frac{\partial^2 f}{\partial Z^2} = \frac{\partial}{\partial Z}\left(\frac{\partial f}{\partial Z}\right) = \frac{\partial}{\partial S}\left(\frac{\partial f}{\partial Z}\right)$

$= \frac{\partial}{\partial S}\left(\frac{\partial f}{\partial S}e^Z\right)\left(\frac{\partial S}{\partial Z}\right) = \left(\frac{\partial^2 f}{\partial S^2}S + \frac{\partial f}{\partial S}\right)$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} = rf$$

Transform

$\Rightarrow$ $$\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial z} + \frac{\sigma^2}{2}\left(\frac{\partial^2 f}{\partial z^2} - S\frac{\partial f}{\partial S}\right) = rf.$$

$\Rightarrow$ $$\frac{\partial f}{\partial t} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial z^2} = rf.$$

$$\tilde{\alpha}_z f(i, j+1) + \tilde{\beta}_z f(i, j) + \tilde{\gamma}_z f(i, j-1) = f(i-1, j).$$

where $\tilde{\alpha}_z = \left(\left(r - \frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2}\right)\Delta T$

$$\tilde{\beta}_z = (1 - r\Delta T) - \left(\frac{\left(1 - \frac{\sigma^2}{2}\right)}{\Delta S} + \frac{\sigma^2}{(\Delta S)^2}\right)$$

$$\tilde{\gamma}_z = \frac{\sigma^2}{2(\Delta S)^2}\Delta T > 0.$$

By the choice of $\frac{\Delta T}{\Delta S} \approx C$ one can make this a trinomial model.