FINANCE

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- Louis Bachelier.
- Brownian Motion
- Random walk.
- Martingale
- Norbert Wiener.
- An Kolmogorov
- JL Doob.
- LTCM (Hedge Fund failed in 1998 due to global crisis)
- Bid Ask Price.
- Option Pricing Theory (Robert Black
- ~~III types of people in stock market~~ : Investor, Hedges, Trader
- ~~LTCM (1A~~
- Roulette.

1, 2, ..., 36, $\boxed{0}\boxed{00}$.

Game 1.

Odd	Even
1	2
3	4
5	⋮
⋮	⋮
35	36

Bet 1 Re on ~~odd~~ even

X = payoff. $= +1$ or -1

$P(X=1) = \frac{18}{38}$.

$P(X=-1) = \frac{20}{38}$.

$$E(X) = 1 \cdot P(X=1) + (-1)P(X=-1) = \frac{18}{38} - \frac{20}{38} = \frac{-2}{38} = \frac{-1}{19}$$

$$V(X) = E(X^2) - (E(X))^2 = 1 - \frac{1}{361} = \frac{360}{361}$$

Game 2

Col 1	Col 2	Col 3
1	13	25
⋮	⋮	⋮
12	24	26

Bet 1 Re on Col 1.

X = payoff

$= +2$ or -1

$P(X=2) = \frac{12}{38}$ $\qquad$ $P(X=-1) = \frac{26}{38}$

$E(X) = 2 \cdot \frac{12}{38} + (-1)\frac{26}{38} = \frac{-1}{19}$

$V(X) = E(X^2) - (E(X))^2 = \frac{74}{38} - \frac{1}{361} = \frac{702}{361}$

→ Utility Functions.

To decide which of the 2 games one should play:

i) Define your target

ii) Decide a game to play

iii) Prove that your choice is best for your target

→ Here $Var_1(X) < Var_2(X)$.

Bernoulli Dist.

$X = \begin{cases} 1 & \text{wp } p \\ 0 & \text{wp } 1-p \end{cases}$ $\qquad 0 \le p \le 1.$

Let $y = b$ or a ; $b > a$ $\qquad$ $P(y=b) = p$

$P(y=a) = 1-a.$

$\exists$ c, d st $cY + d = X$

$cE(Y) + d = p$, $c = \frac{1}{b-a}$, $d = \frac{-a}{-b}$.

Assignment

Name : Debayan Datta.
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1) define your target.

↳ The target in these games is to make a choice based on risk ~~prese~~ such that it is minimum.

2) Decide a game to play.

↳ The 1st Game

3) Prove that your choice is best for your target.

↳ Both the games have the same Expected value = −0.0526. but,
since we desire an outcome with less risk and a bit consistent value, the game with lower variance is chosen. (lower variance 0.997). Hence 1st game is prefered over 2nd game which indeed has some higher risk to it.

—— x ——

03/8

Roulette

Game 1: Betting on even/odd. 1, 2, 3, .., 36, 0, 00

odd	even
1	2
3	4
5	⋮
⋮	⋮
35	36

X = payoff
$= +1$ or -1

$P(X = +1) = 18/38$

$P(X = -1) = 20/38$

$E(X) = 1 \cdot \frac{18}{38} + (-1)\frac{20}{38} = \frac{-1}{19}$.

$V(X) = E(X^2) - E^2(X) = 1 - \left(\frac{-1}{19}\right)^2 = \frac{360}{361}$

Game 2: Betting on cols with Re 1

Col 1	Col 2	Col 3.
1	13	25
2	14	26
⋮	⋮	⋮
12	24	36

X = payoff $= +2$ or -1

$P(X = +2) = 12/38 \quad P(X = -1) = \frac{26}{38}$

$E(X) = 2\left(\frac{12}{38}\right) + (-1)\left(\frac{26}{38}\right) = \frac{-1}{19}$

$V(X) = 4\left(\frac{12}{38}\right) + 1\left(\frac{26}{38}\right) = \frac{702}{361}$

To decide on a game, let us look at

$$S_n^{(1)} = S_0^{(1)} + X_1^{(1)} + \dots + X_n^{(1)}$$

where $X_i^{(1)}$ = payoff from ith game when you play game 1

& $S_0^{(1)}$ = initial capital that you start with to play game 1.

Similarly define,

$$S_n^{(2)} = S_0^{(2)} + X_1^{(2)} + X_2^{(2)} + \dots + X_n^{(2)}$$

where $X_i^{(2)}$ = payoff from ith game for game 2.

& $S_0^{(2)}$ = initial capital to start with for game 2

<u>Objective</u> is to get $S_n^{(j)} > c$ after playing n^{th} game

$(S_n^{(j)} - S_0^{(j)} > c)$ $j = 1, 2$

assuming $S_0^{(j)} = 0$.

WLOG let us assume $S_0' = 0$, where c is your chosen no.

<u>Calculate</u> (perhaps as that of Demovire Th.)

$P(S_n^{(j)} > c)$ for $j = 1, 2$

$$= P\left(\frac{S_n^{(j)} - E(S_n^{(j)})}{\sqrt{Var(S_n^{(j)})}} > \frac{c - E(S_n^{(j)})}{\sqrt{Var(S_n^{(j)})}} \right)$$

$$\approx P\left(Z > \frac{c - E(S_n^{(j)})}{\sqrt{V(S_n^{(j)})}} \right)$$

$$= 1 - \Phi\left(\frac{c - E(S_n^{(j)})}{\sqrt{V(S_n^{(j)})}} \right)$$

Now for $j=1$,

$$E(S_n^{(1)}) = E(X_1^{(1)}) + E(X_2^{(1)}) + \dots + E(X_n^{(1)})$$
$$= n\left(-\frac{1}{19}\right) = -\frac{n}{19}.$$

$$Var(S_n^{(1)}) = n \cdot Var(X_i^{(1)}) \qquad [\because X_i\text{'s indept}]$$
$$= n\,\frac{360}{361}.$$

The Central Limit Th:

$X_1, \dots, X_n$ be a seq. of iid r.v with finite mean $E(X_i) = \mu$ & finite variance $V(X_i) = \sigma^2$

Let $S_n = X_1 + X_2 + \dots + X_n$

then the distn of $\frac{S_n - E(S_n)}{\sqrt{Var(S_n)}} = Z_n$ (say). is approximately $N(0,1)$

$$P(Z_n < y) \approx P(Z < y) \longrightarrow \int_{-\infty}^{y} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$
$$= \Phi(y) \quad \text{where } Z \sim N(0,1).$$

$$V(X+Y) = V(X) + V(Y) + 2\,Cov(X,Y)$$
if X, Y are indept $\Rightarrow$ $Cov(X,Y) = 0$

Take $n = 50$ & $c = 0$,

$$P\left(Z > \frac{c - E(S_n^{(j)})}{\sqrt{V(S_n^{(j)})}}\right) = P\left(Z > \frac{0 - \frac{n}{19}}{\sqrt{n \cdot 360/361}}\right)$$

$$= P\left(Z > -\frac{50}{19}\sqrt{\frac{361}{50 \times 360}}\right) = P(Z > -0.373)$$
$$= 0.645$$

similarly for $j = 2$,

$$P(S_n^{(2)} > c) = 1 - 0.397 = 0.603.$$

Take $n=50$, $c \geq 50$

$$P(S_n^{(1)} > c) < P(S_n^{(2)} > c) \quad \forall c > -\frac{n}{19}.$$

$$P(S_n^{(1)} > c) > P(S_n^{(2)} > c) \quad \forall c < -\frac{n}{19}$$

$\cancel{P(S_n^{(1)} > S_n^{(2)})}$.

$$P(S_n^{(1)} - S_n^{(2)} > c).$$

$$= P(Y_n > c) = P\left(\frac{Y_n - E(Y_n)}{\sqrt{V(Y_n)}} > \frac{c - E(Y_n)}{\sqrt{V(Y_n)}}\right).$$

$\{S_n^{(1)}$ & $S_n^{(2)}$ are indept$\}$

$$\left\{E(Y_n) = 0 \; ; \; Var(Y_n) = n\left(\frac{360}{361} + \frac{702}{361}\right)\right\}.$$

$$\simeq 1 - \phi\left(\frac{1 - E(Y_n)}{\sqrt{V(Y_n)}}\right).$$

X is called stochastically larger than Y if $P(X \geq c) \geq P(Y \geq c)$ $\forall c$

also denoted: $X \overset{FSD}{>} Y$

FSD = First order stochastic Dominance

Problems

① Find an optimal strategy combined Game I & II to maximize your profit subject a loss of c amt for a given no. of games.

② a) $P(S_n^{(j)} > c \mid S_K^{(j)} > 0, \; \forall K \leq n)$.

b) $P(S_n^{(1)} > S_n^{(2)} \mid S_K^{(1)} \geq S_K^{(2)}, \; \forall K \leq n)$.

(Gambler's Ruin / Investor's problem).

Qn 1) $P(S_n = b$ for some n before hitting $a \mid S_0 = c)$

for $a < c < b$

R_i = Return at the end of the ith day of some underlying asset

$$= \frac{P_i - P_{i-1}}{P_{i-1}} \quad ; \quad P_i = \text{Price at the end of the } i\text{th day of some underlying asset}$$

$$\Rightarrow 1 + R_i = \frac{P_i}{P_{i-1}} \Rightarrow \log(1+R_i) = \log\left(\frac{P_i}{P_{i-1}}\right)$$

Q2) $E(\text{min time to reach } a \text{ or } b \mid S_0 = c)$

(starting at c)

Distn of min time :

$S_i - S_{i-1} = X_i$ where $S_i = \log P_i$

& $X_i = \log(1+R_i)$

$\Rightarrow S_i = S_{i-1} + X_i$

$= S_0 + X_1 + X_2 + \dots + X_i$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1 \quad \text{Thus } X_1 = \log(1+R_1)$$

$\simeq R_1$

where R_1 is small.

Assignment

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1) Combined strategy:-

Since both games have -ve Expectation individually, we need to balance them.

Strategy:=

i) Bet size: of game I & II be B_1, B_2 respectively. (Amount to bet).

ii) No. of games: N games, then total expected loss is $N \times \{E(\text{Game I}) B_1 + E(\text{Game 2}) B_2\}$.

iii) Loss constraint: shouldnot exceed 'c'.

$$N \times [-0.0526 B_1 - 0.0526 B_2] \leq c$$

Distribute bets equally := $B_1 = B_2$

Let B be bet per game for each of the 2 games.

∴ Expected Total Loss $= N(-0.0526 \times 2B)$

$= -0.1052 NB$

To stay within loss limit 'c':

$$-0.1052 NB \leq c$$

$$\Rightarrow B \leq \frac{c}{0.1052 N}$$

Finally,

Bet $B_1 = B_2 = \dfrac{c}{0.1052 N}$

this Bet size will maximise the playtime.

2) $P(S_n^{(j)} > c \mid S_k^{(j)} > 0) \quad \forall k \leq n$

$$= \frac{P(S_n^{(j)} > c \cap S_k^{(j)} > 0)}{P(S_k^{(j)} > 0)}$$

Define $Y = S_n^{(j)} - S_k^{(j)}$

$$= X_{k+1}^{(j)} + X_{k+2}^{(j)} + \cdots + X_n^{(j)}$$

So, $S_n^{(j)} = S_k^{(j)} + Y$

So, $P(S_n^{(j)} > c \mid S_k^{(j)} > 0) = P(S_k^{(j)} + Y > c \mid S_k^{(j)} > 0)$

Assume, $S_k^{(j)}$ is the initial capital & $S_k^{(j)} = C_0$.

So, $P(S_k^{(j)} + Y > c \mid S_k^{(j)} = C_0) = P(Y > c - C_0)$.

$$= P\left(\frac{Y - E(Y)}{\sqrt{V(Y)}} > \frac{(c - C_0) - E(Y)}{\sqrt{V(Y)}}\right).$$

$$= P\left(Z > \frac{(c - C_0) - E(Y)}{\sqrt{V(Y)}}\right)$$

$$= 1 - \Phi\left(\frac{c - C_0 - E(Y)}{\sqrt{Var(Y)}}\right)$$

10/8 **Recap:**

Reason for Random walk model:

$$R_i = \frac{P_i - P_{i-1}}{P_{i-1}} = \frac{P_i}{P_{i-1}} - 1$$

$\Rightarrow$ ~~$\log R_i$~~ $\log(R_i + 1) = \log P_i - \log P_{i-1}$

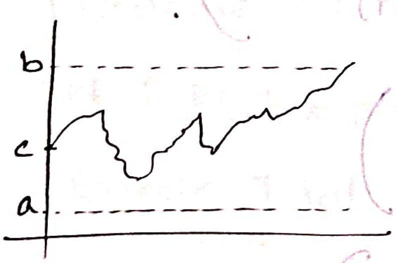
& $S_i = S_{i-1} + X_i \Rightarrow S_n = S_0 + X_1 + \dots + X_n$

where $S_i = \log P_i \ \forall i$

& $X_i = \log(1+R_i) \simeq R_i$ for R_i small

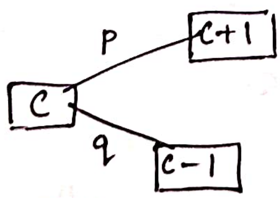
Simple Random Walk.

$$X_i = \begin{cases} +1 \ ; P(X=+1) = p \\ -1 \ ; P(X=-1) = 1-p = q \end{cases}$$



Q1.) $P(S_n = b$ for some n before hitting $a \mid S_0 = c)$?

$\varphi_c = P(S_n = b$ for some n before hitting $a \mid S_0 = c)$



$$\boxed{Q_c = p Q_{c+1} + q Q_{c-1}}$$

$$P(\overbrace{S_1 = c+1}^{A}, \overbrace{S_n = b \text{ some } n \text{ before hitting } a}^{B} \mid \overbrace{S_0 = c}^{D})$$

$$+ \ P(\underbrace{S_1 = c-1}_{A}, S_n = b \ \dots\dots \mid S_0 = c)$$

$$P(A \cap B \mid D) = P(B \mid D \cap A) \, P(A \mid D).$$

$$(p+q)Q_c = pQ_{c+1} + qQ_{c-1}$$

$$\Rightarrow p(Q_{c+1} - Q_c) = q(Q_c - Q_{c-1})$$

$$\Rightarrow Q_{c+1} - Q_c = \frac{q}{p}(Q_c - Q_{c-1}).$$

Boundary conditions :-

$\varphi_b = 1$ P(hitting b before hitting a) (already at b)

$\varphi_a = 0$ (already at a)

Case 1: $p = q$. $\boxed{\varphi \leftarrow Q}$ $\varphi = Q$

~~$Q_{c+1} - Q_c = Q_c - Q_{c-1} = \cdots = Q_{a+1} - Q_a$~~

$$\varphi_{c+1} - \varphi_c = \varphi_c - \varphi_{c-1} = \cdots = \varphi_{a+1} - \varphi_a$$

$$\varphi_c - \varphi_{c-1} = \cdots\cdots = \varphi_{a+1} - \varphi_a$$

$$\vdots \qquad\qquad \vdots$$

$$\varphi_{a+1} - \varphi_a = \cdots\cdots = \varphi_{a+1} - \varphi_a$$

for c=b-1

$$\varphi_{c+1} - \varphi_a = (c-a+1)(\varphi_{a+1} - \varphi_a), \quad a \le c < b$$

$$\Rightarrow 1 = \varphi_b - \varphi_a = (b-a)(\varphi_{a+1} - \varphi_a)$$

$$= (b-a)\varphi_{a+1} \quad \{\because \varphi_a = 0\}$$

$$\Rightarrow \varphi_{a+1} = \frac{1}{b-a}$$

$$\Rightarrow \varphi_c = \frac{c-a}{b-a}.$$

Case 2: $p \neq q$

$$\phi_{c+1} - \phi_c = \left(\frac{q}{p}\right)(\phi_c - \phi_{c-1}) = \left(\frac{q}{p}\right)^2(Q_{c-1} - \phi_{c-2})$$

$$= \cdots = \left(\frac{q}{p}\right)^{c-a}(\phi_{a+1} - \phi_a)$$

$$\phi_{c+1} - \phi_c = \left(\frac{q}{p}\right)(\phi_c - \phi_{c-1}) = \ldots\ldots = \left(\frac{q}{p}\right)^{c-a}(\phi_{a+1} - \phi_a)$$

$$\phi_c - \phi_{c-1} = \ldots\ldots\ldots\ldots = \left(\frac{q}{p}\right)^{c-a-1}(\phi_{a+1} - \phi_a)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\phi_{a+1} - \phi_a = \ldots\ldots\ldots = \left(\frac{q}{p}\right)^{0}(\phi_{a+1} - \phi_a)$$

(+)

$$\phi_{c+1} - \phi_a = \left[1 + \left(\frac{q}{p}\right) + \ldots + \left(\frac{q}{p}\right)^{c-a}\right](\phi_{a+1} - \phi_a)$$

$$= \frac{1 - (q/p)^{c-a+1}}{1 - (q/p)}(\phi_{a+1} - \phi_a)$$

$\{$for $c = b-1\}$

$$1 = \phi_b - \phi_a = \frac{1 - (q/p)^{b-a}}{1 - (q/p)}\phi_{a+1}$$

$$\Rightarrow \phi_{a+1} = \frac{1 - (q/p)}{1 - (q/p)^{b-a}}$$

~~for $c = 30$, $b = 100$, $a = 0$~~

~~$\phi_c \ldots = 1 -$~~

Game I: $p = \frac{18}{38}$ $\quad q = \frac{20}{38}$

$$q/p = \frac{20}{18} = \frac{10}{9}$$

$c = 30$, $b = 100$, $a = 0$

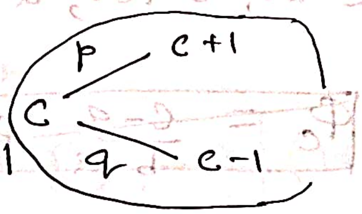
$$\phi_c = \frac{1 - \left(\frac{10}{9}\right)^{30}}{1 - \left(\frac{10}{9}\right)^{100}} = \frac{-22.59}{-37647.62} = 0.0006.$$

for $b = 50$, $c = 30$, $a = 0$, $\Phi_{30} \approx 0.117$.

for $p = q$: $b = 50$, $c = 30$, $a = 0$, $\Phi_{30} = 0.6$.

Q2: $\Psi_c = E(\overbrace{\text{min time to reach } a \text{ and } b}^{\Psi \text{ is a RV.}} \mid S_0 = c)$

$$(p+q)\Psi_c = p\Psi_{c+1} + q\Psi_{c-1} + 1$$



case 1: $p = q$

$$p(\Psi_{c+1} - \Psi_c) = q(\Psi_c - \Psi_{c-1}) - 1$$

$$\Rightarrow \Psi_{c+1} - \Psi_c = \frac{q}{p}(\Psi_c - \Psi_{c-1}) - \frac{1}{p}$$

$$\Rightarrow \Psi_{c+1} - \Psi_c = \Psi_c - \Psi_{c-1} - 2 \qquad \{p = q = 1/2\}$$

{ Boundary conditions: $\Psi_b = 0$, $\Psi_a = 0$ }.

Now,

$$\Psi_{c+1} - \Psi_c = \Psi_c - \Psi_{c-1} - 2 = \dots = \Psi_{a+1} - \Psi_a - 2(c-a)$$

$$\Psi_c - \Psi_{c-1} = \dots = \Psi_{a+1} - \Psi_a - 2(c-a-1)$$

$$\vdots$$

$$\Psi_{a+1} - \Psi_a = \dots = \Psi_{a+1} - \Psi_a - (2 \times 0)$$

(+)

$$\Psi_{c+1} - \Psi_a = (\Psi_{a+1} - \Psi_a)(c-a+1) - 2[1 + \dots + (c-a)]$$

$$= (\Psi_{a+1} - \Psi_a)(c-a+1) - \frac{2(c-a)(c-a+1)}{2}$$

for $c = b-1$,

$$\Rightarrow 0 = \Psi_b - \Psi_a = \Psi_{a+1}(b-a) - (b-a)(b-a-1)$$

$a \le c < b$

$$\Rightarrow \psi_{a+1} = \frac{(b-a-1)(b-a)}{b-a} = b-a-1$$

$$\Rightarrow \psi_c = (b-a-1)(c-a) - (c-a-1)(c-a)$$
$$= (c-a)\left(b-a-1-(c-a-1)\right)$$
$$= (c-a)(b-c)$$

$\{c=30, b=100, a=0\} = 30 \times 70 = 2100$

$\cancel{\boxed{\phi_c = \frac{c-a}{b-a} \text{ for } p=q}}$

Case 2 : $p \neq q$:

Do yourself.

H/W:

a) Calculate $E(\tau \mid S_0 = c)$ for $p \neq q$

b) Find out the value of the same for $c=30$, $b=100$, $a=0$ where $p = \frac{18}{38}$, $q = \frac{20}{38}$

Q3 $P(\tau = n \mid S_0 = c)$. Read Bhattocharya & Wayasize (Chp 1).

Mortgage & other loans :-

P = Initial Amount to repay = 10 Lakh (say). (Principal)

m = no. of years to repay

fixed rate r = rate of interest on the amt to repay per annum.

reverse repo rate < repo rate.

~~repo rate < reverse~~

x = Fixed premium for every month

$$\left[\left[\left[P\left(1+\frac{r}{12}\right)-x\right]\left(1+\frac{r}{12}\right)-x\right]\left(1+\frac{r}{12}\right)-x\right]\left(1+\frac{r}{12}\right)\cdots = 0$$

$m = n\times 12$

$$\Rightarrow P\left(1+\frac{r}{12}\right)^{n\times 12} - x\left[\left(1+\frac{r}{12}\right)^{m-1}+\left(1+\frac{r}{12}\right)^{m-2}+\cdots\cdots 1\right]=0$$

$$\Rightarrow P\left(1+\frac{r}{12}\right)^{m} - x\left[\frac{\left(1+\frac{r}{12}\right)^{m}-1}{1+\frac{r}{12}-1}\right]=0$$

$$\Rightarrow x = \frac{P\left(1+\frac{r}{12}\right)^{m}\left(\frac{r}{12}\right)}{\left(1+\frac{r}{12}\right)^{m}-1}$$

$r = 12\%$ p.a

$m = 20\times 12 = 240$

$P = 10^6$

$\cancel{x = \frac{10^6(1+1)^{240}}{(1+1)^{240}-1}}$ $\cancel{10^6(2)^{240}-1}$

$\cancel{\frac{10^6\cdot 2^{240}}{2^{240}-1}}$

$$x = \frac{10^6\left(1+\frac{0.12}{12}\right)^{240}\cdot\left(\frac{0.12}{12}\right)}{\left(1+\frac{0.12}{12}\right)^{240}-1} = \frac{10^6(10.89)(0.01)}{10.89-1}$$

$$= \frac{108900}{9.89} = 11{,}011.12$$

; $m\times x = 2642012$ (quite high)

Total payment is 2.6 times higher than Principal.

H/W : Go to ~~a~~ Banks (two atleast).
and take their rates (various offers)
ask for EMI. Calculate ~~yourself th~~ yourself
those EMI and compare them.

17/8

Portfolio Investment.

A market has ~~a~~ N-risky assets.

$$\underset{\sim}{R_t} = \begin{pmatrix} R_{1t} \\ \vdots \\ R_{Nt} \end{pmatrix} \qquad R_{it} = \frac{P_{it} - P_{i(t-1)}}{P_{i(t-1)}}.$$

Return

$$\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} = E(R_t \mid \mathcal{F}_{t-1}).$$

past history

$$V = CoVar(\underset{\sim}{R_t} \mid \mathcal{F}_{t-1}).$$

$$V_{N\times N} = ((var_{ij})).$$

where $var_{ij} = (Cov(R_{it}, R_{jt} \mid \mathcal{F}_{t-1}))$.

$$\underset{\sim}{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}$$

initial weights.

w_i = proportion of wealth invested in ith stock initially.

~~Problem:~~

$\cancel{\min_{\underset{\sim}{w}} Var(\underset{\sim}{w}'\underset{\sim}{R})}$ $\cancel{\text{subj to } E(\underset{\sim}{w}'\underset{\sim}{R}) \geq b}$

$\cancel{\& \sum w_i = 1}$

~~$\Leftrightarrow$ alternatively,~~

$$\begin{bmatrix} \frac{\partial f}{\partial w_1} \\ \vdots \\ \frac{\partial f}{\partial w_N} \end{bmatrix} = \frac{\partial f}{\partial \underset{\sim}{w}}$$

<u>Problem</u>:

$$\min_{w} Var(\underset{\sim}{w}'\underset{\sim}{R}_{t_0} \mid \mathcal{F}_{t_0-1}) \quad \text{subj to} \quad E(\underset{\sim}{w}'\underset{\sim}{R}_t \mid \mathcal{F}_{t_0-1}) \geq b$$

$$\& \quad \sum w_i = 1$$

<u>alternatively</u>,

maximise $E(\underset{\sim}{w}'\underset{\sim}{R}_{t_0} \mid \mathcal{F}_{t_0-1})$ subj to

$Var(\underset{\sim}{w}'\underset{\sim}{R}_{t_0} \mid \mathcal{F}_{t_0-1}) \leq a$

$\&$ $\sum w_i = 1$

$\{ t_0$ is the initial time $\}$

To solve, we drop t & solve the static problem

<u>Note</u>: $E(\underset{\sim}{w}'\underset{\sim}{R}_t \mid \mathcal{F}_{t-1}) = \underset{\sim}{w}'\underset{\sim}{\mu}$

$\& \ Var(\underset{\sim}{w}'\underset{\sim}{R}_t \mid \mathcal{F}_{t-1}) = \underset{\sim}{w}' Cov(\underset{\sim}{R}_t \mid \mathcal{F}_{t-1}) \underset{\sim}{w}$

$= \underset{\sim}{w}'V\underset{\sim}{w}$

Let $f(\underset{\sim}{w}, \delta_1, \delta_2) = \frac{1}{2}\underset{\sim}{w}'V\underset{\sim}{w} - \delta_1(\underset{\sim}{w}'\underset{\sim}{\mu} - b)$

$- \delta_2(\underset{\sim}{w}'\underset{\sim}{1} - 1)$

[$\underset{\sim}{w}'\underset{\sim}{1}$ coming from $\sum w_i = 1$

$\& \ \underset{\sim}{1}^T = (1, \ldots, 1)$

$\underset{\sim}{1} \in \mathbb{R}^{N \times 1}$]

$$\frac{\partial f}{\partial \underset{\sim}{\omega}} = \begin{bmatrix} \frac{\partial f}{\partial \omega_1} \\ \vdots \\ \frac{\partial f}{\partial \omega_N} \end{bmatrix} = V\underset{\sim}{\omega} - \delta_1 \underset{\sim}{\mu} - \delta_2 \underset{\sim}{1} \quad \text{①}$$

If V is not p.d. then $\exists\ \underset{\sim}{\omega}$ st $\underset{\sim}{\omega}_0' V_0 \underset{\sim}{\omega}_0 = 0$
(as covariance is always psd)

$\underset{\sim}{\omega}_0' V_0 \underset{\sim}{\omega}_0 = 0$

$\Rightarrow \text{Var}(\underset{\sim}{\omega}_0' \underset{\sim}{R}) = 0$. $\left\{ \begin{array}{l} \text{Var}(RV) = 0 \\ \Rightarrow \text{degenerate } RV \end{array} \right\}$

$\Rightarrow$
$P(\underset{\sim}{\omega}_0' \underset{\sim}{R} = c) = 1$ for some const c

$c \to \underset{\sim}{\omega}_0' \mu$

$\Rightarrow \omega_{01} R_1 + \omega_{02} R_2 + \ldots + \omega_{0N} R_N = c$

(∴ dim lowered from N to $N-1$)

Assuming $\omega_{0i} \neq 0$

& $R_1 = \frac{1}{\omega_{01}}\left(c - \sum_{i \neq 1} \omega_{0i} * R_i\right)$

$\boxed{\therefore \text{ To have a unique minimum one needs } V \text{ to be pd.}}$

Then drop 1st asset for optimization problem & carry on with $(N-1)$ risky assets & continue.

$$\frac{\partial f}{\partial \delta_1} = \underset{\sim}{w}'\underset{\sim}{\mu} - b = 0 \quad \text{------------} \; (2).$$

$$\frac{\partial f}{\partial \delta_2} = \underset{\sim}{w}'\underset{\sim}{1} - 1 = 0 \quad \text{------------} \; (3).$$

$$(1) \Rightarrow \underset{\sim}{w} = \delta_1 V^{-1}\underset{\sim}{\mu} + \delta_2 V^{-1}\underset{\sim}{1}$$

$$(2) \,\&\, (1) \Rightarrow b = \underset{\sim}{\mu}'\underset{\sim}{w} = \delta_1 \underset{\sim}{\mu}'V^{-1}\underset{\sim}{\mu} + \delta_2 \underset{\sim}{\mu}'V^{-1}\underset{\sim}{1} \quad (4).$$

$$(3) \,\&\, (1) \Rightarrow 1 = \underset{\sim}{1}'\underset{\sim}{w} = \delta_1 \underset{\sim}{1}'V^{-1}\underset{\sim}{\mu} + \delta_2 \underset{\sim}{1}'V^{-1}\underset{\sim}{1} \quad (5).$$

$$(4) \,\&\, (5) \Rightarrow \underbrace{\begin{pmatrix} A & B \\ B & C \end{pmatrix}}_{G} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \underbrace{\begin{pmatrix} b \\ 1 \end{pmatrix}}_{d}$$

where $A = \underset{\sim}{\mu}'V^{-1}\underset{\sim}{\mu}$, $B = \underset{\sim}{\mu}'V^{-1}\underset{\sim}{1}$

& $C = \underset{\sim}{1}'V^{-1}\underset{\sim}{1}$, $D = AC - B^2 \neq 0$.

C-S inequality:
$(\underset{\sim}{x}'\underset{\sim}{x})(\underset{\sim}{y}'\underset{\sim}{y}) - (\underset{\sim}{x}'\underset{\sim}{y})^2 \leq 0$

$$(\underset{\sim}{\mu}'V^{-1}\underset{\sim}{\mu})(\underset{\sim}{\mu}'V^{-1}\underset{\sim}{1}) - (\underset{\sim}{\mu}'V^{-1}\underset{\sim}{1})^2$$

∴ Here $x = V^{-1/2}\underset{\sim}{\mu}$
$y = V^{-1/2}\underset{\sim}{1}$

Equality holds iff $\underset{\sim}{x} = c_0 \underset{\sim}{y}$ for some const c_0

ie, $V^{-1/2}\underset{\sim}{\mu} = c_0 V^{-1/2}\underset{\sim}{\mathbb{1}}$

$\Rightarrow \underset{\sim}{\mu} = c_0 \underset{\sim}{\mathbb{1}}.$

① if $c_0 < 0$ then choose there with max variance of it's return & invest everything there

ie, suppose $i_0 = \text{argmax}(v_{ii})$ ~~then $w_{i_0}=1$~~

↑ diagonal elements of V

then $w_{i_0} = 1$

& $w_j = 0 \;\forall\, j \neq i_0$

$D \neq 0 \Rightarrow$ unique soln. for δ_1 & δ_2

$$\Rightarrow \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = \frac{1}{D}\begin{pmatrix}C & -B\\ -B & A\end{pmatrix}\begin{pmatrix}b\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}\dfrac{Cb - B}{D}\\[2ex] \dfrac{A - Bb}{D}\end{pmatrix}$$

Putting this in Eq

$$\underset{\sim}{w} = V^{-1}[\underset{\sim}{\mu} : \underset{\sim}{\mathbb{1}}]\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = [V^{-1}\underset{\sim}{\mu} : V^{-1}\underset{\sim}{\mathbb{1}}]\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}$$

$$\Rightarrow \underset{\sim}{w}'V\underset{\sim}{w} = \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}'\begin{bmatrix}\mu' V^{-1}\\ \mathbb{1}' V^{-1}\end{bmatrix} V [V^{-1}\mu : V^{-1}\mathbb{1}]\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}$$

$$= \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}'\underbrace{\begin{pmatrix}\mu'\\ \mathbb{1}'\end{pmatrix} V^{-1}[\mu : \mathbb{1}]}_{= G}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}$$

$$\& \; d = \begin{pmatrix} b \\ 1 \end{pmatrix}$$

$$\left[\text{where } G = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \; \therefore \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = G^{-1} \underset{\sim}{d} \right]$$

$$= \underset{\sim}{d}' G^{-1} \cdot G \cdot G^{-1} \underset{\sim}{d} = \underset{\sim}{d}^{-1} G^{-1} \underset{\sim}{d}.$$

$$G = \begin{bmatrix} \mu' V^{-1} \mu & \mu' V^{-1} \mathbb{1} \\ \mathbb{1}' V^{-1} \mu & \mathbb{1}' V^{-1} \mathbb{1} \end{bmatrix}$$

Since we know,

$\underset{\sim}{w}' V \underset{\sim}{w}$ is variance.

$\therefore$ ~~$\sigma^2_{min} = \frac{1}{D} (b^2$~~

$$\sigma^2_{min} = \frac{1}{D} (b^2 C - 2Bb + A)$$

$$\boxed{= (b, 1) \frac{\begin{pmatrix} C & -B \\ -B & A \end{pmatrix}}{D} \begin{pmatrix} b \\ 1 \end{pmatrix}. \quad \text{also}}$$

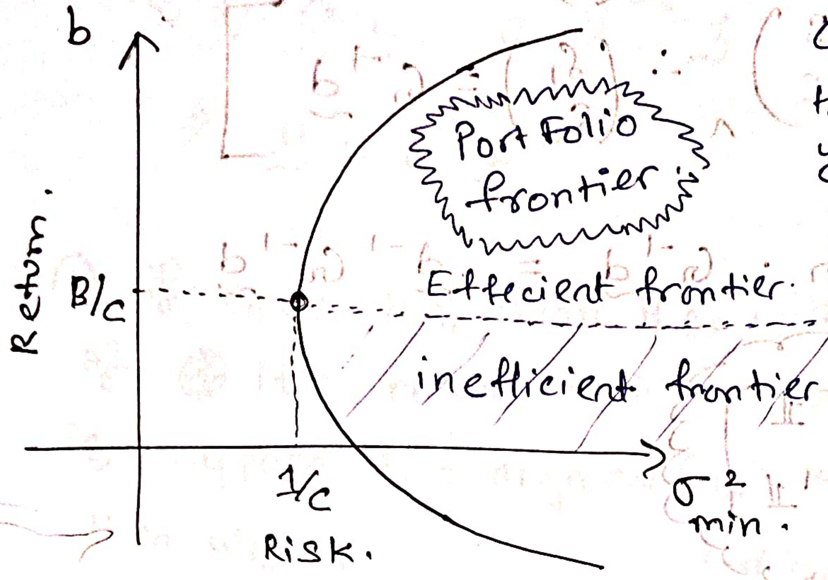
$$= \frac{C}{D} \left(b^2 - 2b\frac{B}{C} + \frac{A}{C}\right).$$

$$= \frac{C}{D} \left[\left(b - \frac{B}{C}\right)^2 + \frac{A}{C} - \frac{B^2}{C^2} \right]$$

$$= \frac{C}{D} \left[\left(b - \frac{B}{C}\right)^2 + \frac{AC - B^2}{C^2} \right]$$

$$= \frac{C}{D} \left(b - \frac{B}{C}\right)^2 + \frac{1}{C} \quad \leftarrow \text{Quad}.$$

$\{$min when $b = \frac{B}{C}$ ie σ^2_{min} is min$\}$.



Given the return, this frontier gives you min possible variance.

$$\frac{B}{c} = \frac{\underset{\sim}{\mu}' V^{-1} \underset{\sim}{\mathbb{1}}}{\underset{\sim}{\mathbb{1}}' V^{-1} \underset{\sim}{\mathbb{1}}}, \quad \frac{1}{c} = \frac{1}{\underset{\sim}{\mathbb{1}}' V^{-1} \underset{\sim}{\mathbb{1}}}.$$

$$\boxed{\omega_{MVP} = \frac{V^{-1} \underset{\sim}{\mathbb{1}}}{\underset{\sim}{\mathbb{1}}' V^{-1} \underset{\sim}{\mathbb{1}}}}$$

H/w collect 10 stocks, 2 each from 5 sectors. collect data for 2 yrs; find Return.

Use 1.5 yrs of data to estimate $\underset{\sim}{\mu}$ & V.

compute $\underset{\sim}{\omega}$ as done here & compare your performance wrt next 6 months actual returns.

change your $\underset{\sim}{\omega}$ for every two months and compare.

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$A = \underset{\sim}{\mu}' V^{-1} \underset{\sim}{\mu}$, $B = \underset{\sim}{\mu}' V^{-1} \underset{\sim}{1}$, $C = \underset{\sim}{1}' V^{-1} \underset{\sim}{1}$

$D = AC - B^2$.

$\cancel{w_{op}' V w_{op} = \sigma_{op}^2 =}$

$$w_{mvp} = \frac{V^{-1} \underset{\sim}{1}}{\underset{\sim}{1}' V^{-1} \underset{\sim}{1}}$$

$$w_{op}' V w_{op} = \sigma_{op}^2 = \frac{C}{D}\left(b - \frac{B}{C}\right)^2 + \frac{1}{C}$$

$$w_{op} = \underset{\sim}{g}\, \underset{\substack{\uparrow \\ \text{return}}}{b} + \underset{\sim}{h}$$ $\Leftarrow$ Form of w_{op}.

Let us see, how.

~~where g =~~

$$\cancel{w_{op} = [V^{-1}\underset{\sim}{\mu} \cdot V^{-1}\underset{\sim}{1}]\, G^{-1} \begin{pmatrix} b \\ 1 \end{pmatrix}}$$

$$\cancel{= (V^{-1}\underset{\sim}{\mu}, V^{-1}\underset{\sim}{1}) \begin{pmatrix} C & -B \\ B & A \end{pmatrix} \begin{pmatrix} b \\ 1 \end{pmatrix}}$$ $(N\times 2)(2\times 2)(2\times 1)$

$$\cancel{= \frac{1}{D}\left[(V^{-1}\underset{\sim}{\mu}, V^{-1}\underset{\sim}{1})\right.}$$

$\boxed{\text{Frontier Portfolio}}$

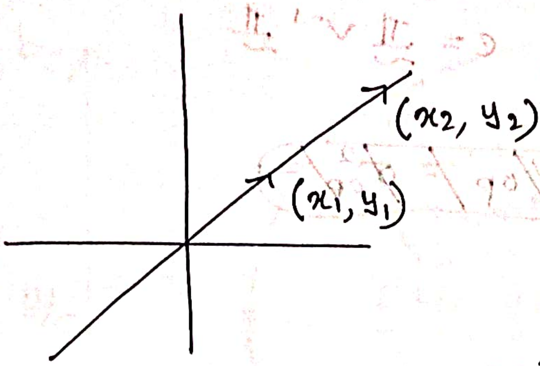
$$w_{op} = (V^{-1}\underset{\sim}{\mu}, V^{-1}\underset{\sim}{1})\, G^{-1} \begin{pmatrix} b \\ 1 \end{pmatrix}. \qquad G^{-1} = \begin{pmatrix} C & -B \\ B & A \end{pmatrix} \frac{1}{D}.$$

$$= (V^{-1}\underset{\sim}{\mu}, V^{-1}\underset{\sim}{1}) \frac{\begin{pmatrix} C & -B \\ B & A \end{pmatrix}}{D} \begin{pmatrix} b \\ 1 \end{pmatrix} \qquad (N\times 2)(2\times 2)(2\times 1).$$

$$= \frac{1}{D}\left[(V^{-1}\underset{\sim}{\mu})C - (V'\underset{\sim}{1})B \;:\; (V^{-1}\underset{\sim}{1})A - (V^{-1}\underset{\sim}{\mu})B\right] \begin{pmatrix} b \\ 1 \end{pmatrix}$$

$$= b\left(\frac{C}{D} V^{-1}\underset{\sim}{\mu} - \frac{B}{D} V^{-1}\underset{\sim}{1}\right) + \left(\frac{A}{D} V^{-1}\underset{\sim}{1} - \frac{B}{D} V^{-1}\underset{\sim}{\mu}\right)$$

$$= \underset{\sim}{g}\, b + \underset{\sim}{h}$$



$$a_1\begin{pmatrix}x_1\\y_1\end{pmatrix} + a_2\begin{pmatrix}x_2\\y_2\end{pmatrix}$$

$$= a_1\begin{pmatrix}x_1\\mx_1\end{pmatrix} + a_2\begin{pmatrix}x_2\\mx_2\end{pmatrix}$$

$$= \begin{pmatrix}a_1 x_1\\m \cdot a_1 x_1\end{pmatrix} + \begin{pmatrix}a_2 \cdot x_2\\m \cdot a_2 x_2\end{pmatrix}$$

$$= \begin{pmatrix}a_1 x_1 + a_2 x_2\\m(a_1 x_1 + a_2 x_2)\end{pmatrix}$$

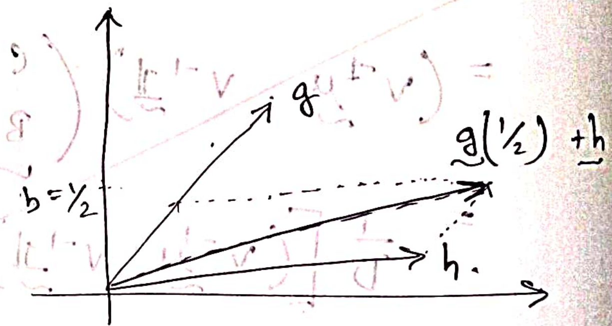
$$= \begin{pmatrix}x_3\\y_3\end{pmatrix}$$

$$\lambda \overset{Fp}{\vec{\omega}_1} + (1-\lambda)\overset{Fp}{\vec{\omega}_2}$$

$$= \underset{\sim}{g}\,(\lambda b_1 + (1-\lambda) b_2)$$

$$+ \underset{\sim}{h}\,(\lambda + (1-\lambda))$$

$$= \underset{\sim}{g}\, b_3 + \underset{\sim}{h} \rightarrow Fp$$



Relation b/w 2 PortFolios (PF) return where both are Frontier P. (FP).

$$\mathrm{Cov}(\omega_1' R, \omega_2' R) = \omega_1' V \omega_2$$

$$= \frac{\mathbb{1}' \omega_2}{\underset{\sim}{\mathbb{1}}' V^{-1} \underset{\sim}{\mathbb{1}}} = \frac{1}{\underset{\sim}{\mathbb{1}}' V^{-1} \underset{\sim}{\mathbb{1}}}$$

$$= (\underset{\sim}{g} b + \underset{\sim}{h})' V (\underset{\sim}{g} b + \underset{\sim}{h})$$

$$= (\underset{\sim}{g}' V \underset{\sim}{g}) b^2 + 2(\underset{\sim}{g}' V \underset{\sim}{h}) b + (\underset{\sim}{h}' V \underset{\sim}{h})$$

$$= (\underset{\sim}{g}'V\underset{\sim}{g})\left[b^2 + \frac{2\underset{\sim}{g}'V\underset{\sim}{h}}{(\underset{\sim}{g}'V\underset{\sim}{g})}b + \left(\frac{\underset{\sim}{g}'V\underset{\sim}{h}}{\underset{\sim}{g}'V\underset{\sim}{g}}\right)^2\right]$$

$$+\left(\underset{\sim}{h}'V\underset{\sim}{h} - \frac{(\underset{\sim}{g}'V\underset{\sim}{h})^2}{(\underset{\sim}{g}'V\underset{\sim}{g})}\right)$$

$$= (\underset{\sim}{g}'V\underset{\sim}{g})\left(b + \frac{\underset{\sim}{g}'V\underset{\sim}{h}}{\underset{\sim}{g}'V\underset{\sim}{g}}\right)^2 + \Bigg(\underset{\sim}{h}'V\underset{\sim}{h} - \underbrace{\frac{(\underset{\sim}{g}'V\underset{\sim}{h})^2}{(\underset{\sim}{g}'V\underset{\sim}{g})}}_{B/C}\Bigg).$$

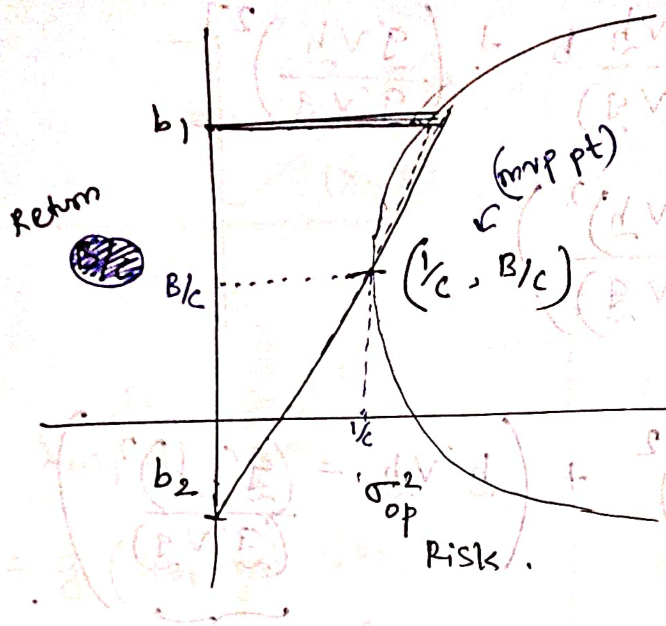
$$\begin{aligned}
\omega_1'V\omega_2 &= (\underset{\sim}{g}b_1 + \underset{\sim}{h})V(\underset{\sim}{g}b_2 + \underset{\sim}{h}) \\
&= \underset{\sim}{g}'V\underset{\sim}{g}\, b_1b_2 + (b_1+b_2)\underset{\sim}{g}'V\underset{\sim}{h} + \underset{\sim}{h}'V\underset{\sim}{h} \\
&= \underbrace{(\underset{\sim}{g}'V\underset{\sim}{g})}_{C/D.}\left(b_1 - \frac{B}{C}\right)(b_2 - B/C) \\
&\qquad + \underbrace{h'Vh - \frac{(\underset{\sim}{g}'V\underset{\sim}{h})^2}{(\underset{\sim}{g}'V\underset{\sim}{g})}}_{1/C}.
\end{aligned}$$

$$= \frac{C}{D}\left(b_1 - \frac{B}{C}\right)\left(b_2 - \frac{B}{C}\right) + \frac{1}{C} = 0.$$

For a given b_1,

$$b_2 - \frac{B}{C} = \frac{1}{C} \times \frac{D}{C}\frac{1}{\left(b_1 - \frac{B}{C}\right)}$$

$$\Rightarrow b_2 = \frac{B}{C} - \frac{D}{C^2(b_1 - B/C)}.$$



Line goes through these pts,

$$y - \frac{B}{c} = \frac{b_1 - \frac{B}{c}}{\left(\sigma_1^2 - \frac{1}{c}\right)}\left(x - \frac{1}{c}\right)$$

$$= \frac{\left(b_1 - \frac{B}{c}\right)}{\frac{c}{D}\left(b_1 - \frac{B}{c}\right)^2}\left(x - \frac{1}{c}\right).$$

$$= \frac{D}{c}\frac{(x - 1/c)}{(b_1 - B/c)}$$

$$\therefore\ y\Big|_{x=0} = \frac{B}{c} - \frac{D}{c^2(b_1 - B/c)}$$

$$= b_2.$$

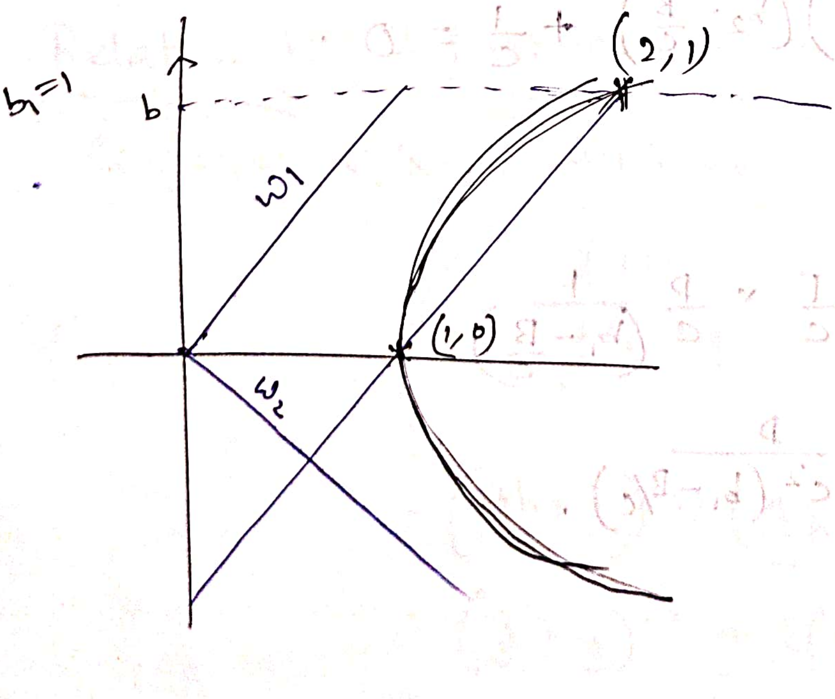
Example:

$\underset{\sim}{g} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. | $\underset{\sim}{w} = \underset{\sim}{g} b + h$.

$\underset{\sim}{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. | $= \begin{pmatrix} 1 \\ b \end{pmatrix}$

$\sigma^2 = w' V w$

$= \begin{pmatrix} 1 \\ b \end{pmatrix}' \begin{pmatrix} 1 \\ b \end{pmatrix} = b^2 + 1$



$$w = \lambda w_1 + \cancel{8}(1-\lambda) w_2$$

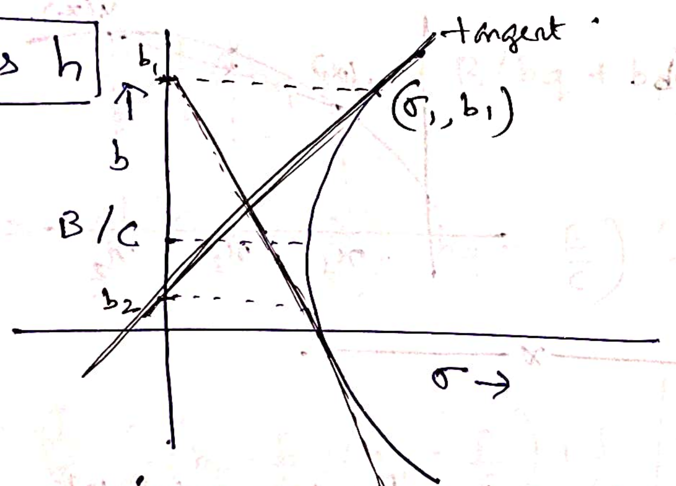
$$E(w'R) = \lambda w_1'\mu + (1-\lambda) w_2'\mu = w'\mu$$

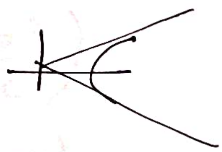
$$\text{Var}(\underset{\sim}{w}'R) = w'Vw$$
$$= \lambda^2 \,\text{var}(\underset{\sim}{w}'R) + (1-\lambda)^2 \,\text{var}(\underset{\sim}{w}'R).$$
$$+ \, 2\lambda(1-\lambda)\, \text{cov}(w_1'R, w_2'R).$$

$$w_1 V w_2 = (\underset{\sim}{g} b_1 + \underset{\sim}{h}) V (\underset{\sim}{g} b_2 + \underset{\sim}{h})$$

$$\sigma^2 = \frac{C}{D}\left(b - \frac{B}{C}\right)^2 + \frac{1}{C} \leftarrow \text{(hyperbola)}$$

$\boxed{\sigma \text{ vs } b}$





$$x^2 = \frac{C}{D}\left(y - \frac{B}{C}\right)^2 + \frac{1}{C}$$

$$\Rightarrow 2x\,dx = \frac{C}{D}\, 2\left(y - \frac{B}{C}\right) dy$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(\sigma_1, b_1)} = \frac{D}{C(y - B/C)} = \frac{D\sigma_1}{C(b_1 - B/C)}$$

now,

$$y - b_1 = \left.\frac{dy}{dx}\right|_{(\sigma_1, b_1)} \cdot (x - b_1).$$

$$\Rightarrow y\Big|_{x=0} = \frac{D\sigma_1(\cancel{8} - \sigma_1)}{C(b_1 - B/C)} = \frac{-D\left[\frac{C}{D}(b_1 - B/C)^2 + 1/C\right]}{C(b_1 - B/C)} + b_1$$

$$= \frac{-D\left(\frac{C}{D}\left(b_1 - B/C\right)^2 + 1/C\right)}{C\left(b_1 - B/C\right)} + b_1$$

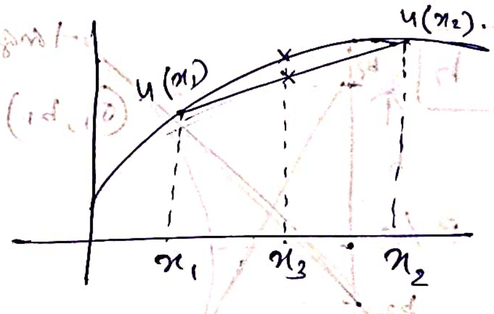
$$= -\left(b_1 - B/C\right) - \frac{D}{C\left(b_1 - B/C\right)} + b_1$$

$$= \frac{B}{C} - \frac{D}{C^2\left(b_1 - B/C\right)} = b_2$$

For concave;

$$u(\lambda x_1 + (1-\lambda) x_2) \geq \lambda u(x_1) + (1-\lambda) u(x_2).$$

λ & $(1-\lambda)$ are prob.



$0 < \lambda < 1.$

———x———

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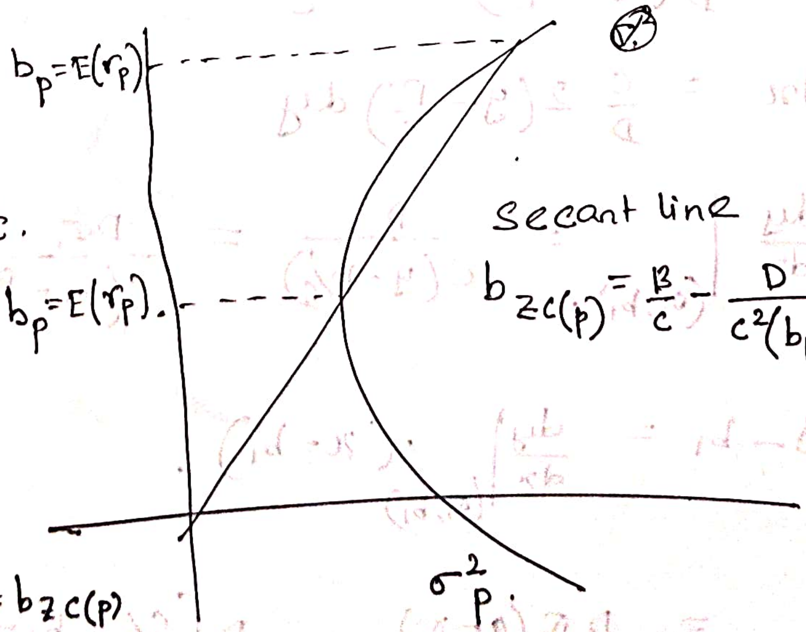
For any other point w_q and a frontier point w_p.

$$\mathrm{Cov}(\underset{\sim}{w_q'}\underset{\sim}{R}, \underset{\sim}{w_p'}\underset{\sim}{R}) = w_q' V w_p$$



$$b_{zc(p)} = \frac{B}{C} - \frac{D}{C^2\left(b_p - \frac{B}{C}\right)}$$

where $r_{zc(p)} = \underset{\sim}{w}'_{zc(p)}\underset{\sim}{R}$

where $\sigma_p^2 = \frac{C}{D}\left(b_p - \frac{B}{C}\right)^2 + \frac{1}{C}$

$$\mathrm{Cov}(\underset{\sim}{\omega}_q'\underset{\sim}{R}, \underset{\sim}{\omega}_p'\underset{\sim}{R}) = \omega_q' V \omega_p \text{ ———— } (1)$$

$$= \underset{\sim}{\omega}_q' [\underset{\sim}{\mu} : \underset{\sim}{\mathbb{1}}] \begin{pmatrix} A & B \\ B & C \end{pmatrix}^{-1} \begin{pmatrix} b_p \\ 1 \end{pmatrix}$$

$$= [\underset{\sim}{\omega}_q'\underset{\sim}{\mu} : \underset{\sim}{\omega}_q'\underset{\sim}{\mathbb{1}}] \begin{pmatrix} A & B \\ B & C \end{pmatrix}^{-1} \begin{pmatrix} b_p \\ 1 \end{pmatrix}.$$

$$= (b_q : 1) \left(\quad " \quad \right) \begin{pmatrix} b_p \\ 1 \end{pmatrix}.$$

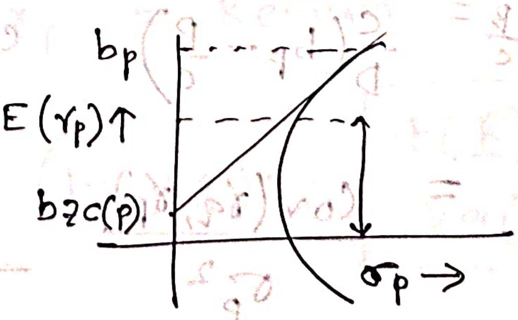
$$= (b_q : 1) \underbrace{\begin{pmatrix} C & -B \\ -B & A \end{pmatrix}}_{D} \begin{pmatrix} b_p \\ 1 \end{pmatrix}. \qquad D = \begin{vmatrix} A & B \\ B & C \end{vmatrix}.$$

$$= \frac{1}{D} \left[C b_q b_p - B(b_q + b_p) + A \right]$$

$$= \frac{C}{D}\left(b_p - \frac{B}{C}\right)\left(b_q - \frac{B}{C}\right) + \frac{1}{D}\left(A - \frac{B^2}{C}\right)$$

$$= \frac{C}{D}\left(b_p - \frac{B}{C}\right)\left(b_q - \frac{B}{C}\right) + \frac{1}{C} \qquad \left| = \frac{1}{D}\left[\frac{AC - B^2}{C}\right] = \frac{1}{D}\frac{D}{C} = \frac{1}{C} \right.$$

$$= \frac{1}{C} = \sigma_{mvp}. \quad \text{iff } E(r_p) = b_p = B/C.$$



$$(1) \Rightarrow \mathrm{Cov}(\underset{\sim}{\omega}_q'\underset{\sim}{R}, \underset{\sim}{\omega}_p'\underset{\sim}{R}) = \mathrm{Cov}(r_q, r_p).$$

$$= \omega_q' V \omega_p = \frac{C}{D}\left(b_p - \frac{B}{C}\right)\left(b_q - \frac{B}{C}\right) + \frac{1}{C}$$

$$\Rightarrow b_q - \frac{B}{c} = \frac{Cov(r_q, r_p)}{\frac{C}{D}\left(b_p - \frac{B}{C}\right)} - \frac{1}{c\left(\frac{C}{D}\left(b_p - \frac{B}{C}\right)\right)}$$

$$\boxed{\beta_{qp} = \frac{Cov(r_q, r_p)}{Var(r_p)}.}$$

(*) $\cancel{= \beta_{qp}\left(b_p - \frac{B}{C}\right) + 1 - \beta}$

$$= \beta_{qp}\left(b_p - \frac{B}{C}\right) + (1 - \beta_{qp})\left(b_{zc(p)} - \frac{B}{C}\right).$$

$$\Rightarrow E(r_q) = \beta_{qp} E(r_p) + (1 - \beta_{qp}) E(r_{zc(p)})$$

Show that: $1 - \beta_{qp} = \frac{Cov(r_q, r_{zc(p)})}{Var(r_{zc(p)})}.$

$$r_q \stackrel{?}{=} \beta_{qp} r_p + \beta_{q\,zc(p)} r_{zc(p)}$$

where $E(q \mid r_p, r_{zc(p)}) = 0.$

(*) $\cancel{Cov}$

$$b_q - \frac{B}{c} = \frac{Cov(r_p, r_q)}{\frac{C}{D}\left(b_p - \frac{B}{C}\right)} - \frac{1}{c\left(\frac{C}{D}\left(b_p - \frac{B}{C}\right)\right)}.$$

$$= \frac{Cov(r_q, r_p)}{\sigma_p^2} \cdot \frac{\frac{C}{D}\left(b_p - \frac{B}{C}\right)^2 + \frac{1}{C}}{\frac{C}{D}\left(b_p - \frac{B}{C}\right)}$$

$$- \frac{1}{c\left(\frac{C}{D}\left(b_p - \frac{B}{C}\right)\right)}$$

$$= \beta_{qp}\left(b_p - \frac{B}{C}\right) - (1-\beta_{qp})\left(b_{zc(p)} - B/C\right)$$

$$= \beta_{qp} b_p + (1-\beta_{qp})\, b_{zc(p)} - B/C$$

$$= \beta_{qp}(b_p - B/C) + (1-\beta_{qp})(b_{zc(p)} - B/C).$$

$$\Rightarrow E(r_q) = \beta_{qp} E(r_p) + (1-\beta_{qp}) E(z_{rc(p)}).$$

$$? \left[\Rightarrow r_q = \beta_{qp} r_p + \beta_{qzc(p)} r_{zc(p)} + q\right]$$

———— x ————

Market with (N+1) assets
(N risky & 1 risk-free).

Let r_f be the return of the risk-free asset.

$$\underset{\sim}{w}'\underset{\sim}{R} + \left(1 - \sum_{i=1}^{N} w_i\right) r_f = \text{Return}.$$

$$E(\text{return}) = \underset{\sim}{w}'\underset{\sim}{\mu} + (1 - \underset{\sim}{w}'\underset{\sim}{1})\, r_f$$

$$\boxed{\begin{aligned} E(\underset{\sim}{R}) &= \mu \\ Var(\underset{\sim}{R}) &= V. \end{aligned}}$$

$$Var(\text{return}) = \underset{\sim}{w}' V \underset{\sim}{w}$$

$$\text{Problem} := \min_{w} \; w'Vw \;\; \text{st} \;\; E(\text{return}) \geq b.$$

$$\underbrace{f(\underset{\sim}{w}, \lambda)} = \frac{1}{2} w' V w - \lambda \left(w'\mu + r_f (1 - \underset{\sim}{w}' \underset{\sim}{\mathbb{1}}) - b \right)$$

min this
wrt w, λ.

$$\frac{\partial f}{\partial w} = V w - \lambda (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}}) = 0 \quad \text{--------} \; (1)$$

$$\frac{\partial f}{\partial \lambda} = w' (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}}) - (b - r_f) = 0 \quad \text{-------} \; (2)$$

$$(1) \Rightarrow V w = \lambda (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}}) \quad \text{------} \; (3)$$

$$(2) \Rightarrow w' (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}}) = b - r_f . \quad \text{------} \; (4)$$

$$(3) \,\&\, (4) \Rightarrow \quad w = \lambda V^{-1} (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}}) \quad \text{------------} \; (5)$$

$$\& \; (b - r_f) = w' (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}})$$

put w.

$$(\because \text{scalar}) \text{ or } (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}})' w$$

$$= \lambda \underbrace{(\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}}) V^{-1} (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}})}_{H}$$

$$\Rightarrow \lambda = \frac{b - r_f}{H}$$

Put this value in (5),

$$w_{op} = \left(\frac{b - r_f}{H} \right) V^{-1} (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}}) .$$

now,

$$\sigma^2_{op} = w'_{op} V w_{op}$$

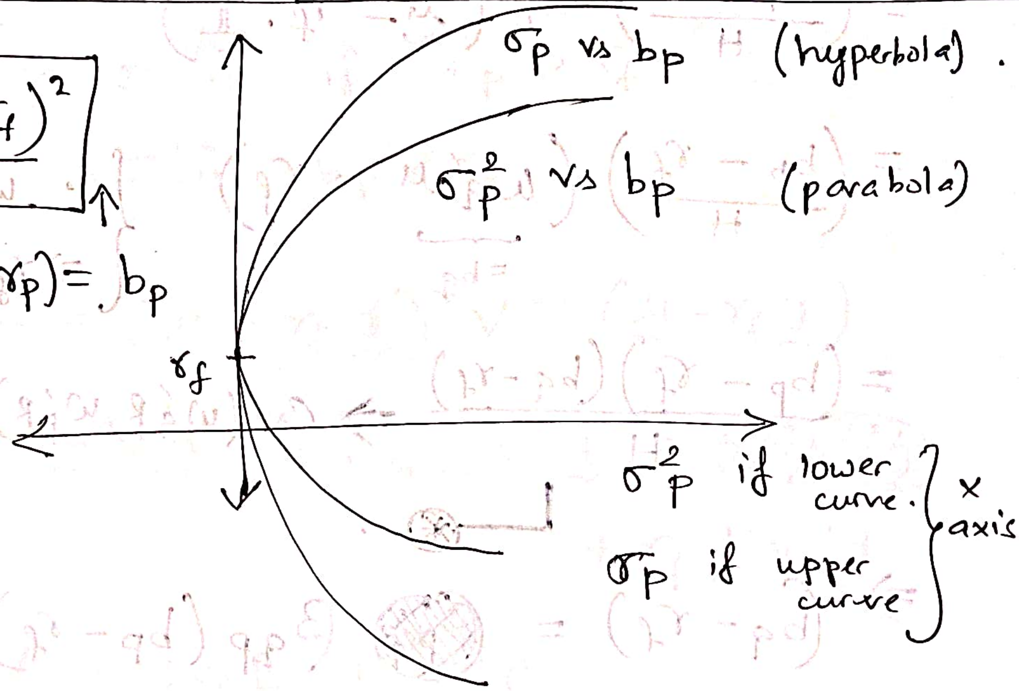
$$= \frac{(b - r_f)^2}{H^2} (\underset{\sim}{\mu} - r_f \underset{\sim}{1})' V^{-1} (\underset{\sim}{\mu} - r_f \underset{\sim}{1})$$

$$= \frac{(b - r_f)^2}{H^2} \times H = \frac{(b - r_f)^2}{H}$$

graph

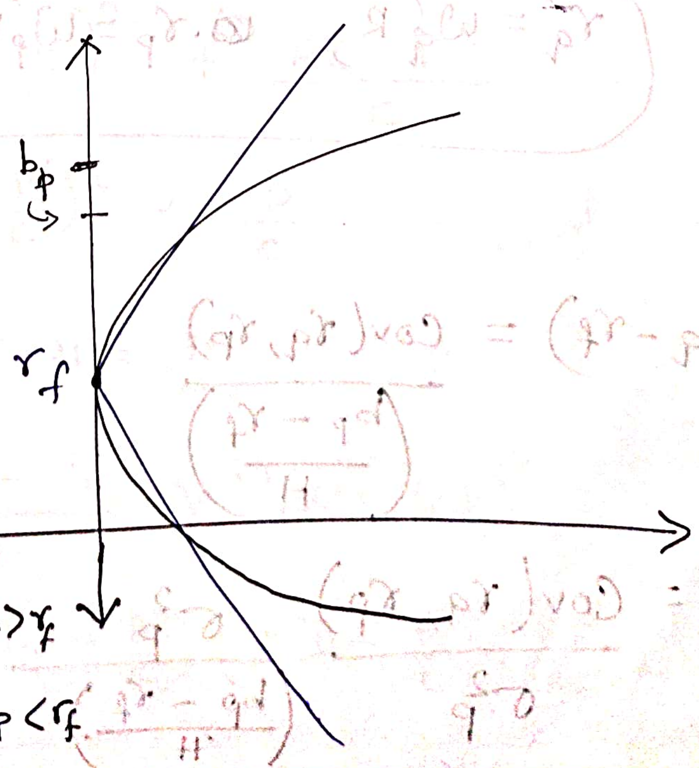
$$\boxed{x^2 = \frac{(y - r_f)^2}{H}}$$

$E(r_p) = b_p$



$$x^2 = \frac{(y - r_f)^2}{H}$$

$$x = \pm \left(\frac{y - r_f}{\sqrt{H}}\right)$$



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$b_p = r_f + \sigma_p \sqrt{H}$, $b_p > r_f$

$b_p = r_f - \sigma_p \sqrt{H}$, $b_p < r_f$

$$\mathrm{Cov}(w_q' \underset{\sim}{R}, w_p' \underset{\sim}{R}) = w_q' V w_p$$

where w_p is a frontier pf invested in $(N+1)$ assets. & w_q : pf (portfolio) for risky assets st, $w_q' \underset{\sim}{\mathbb{1}} = 1$

$$= \left(\frac{b_p - r_f}{H}\right) w_q' (\underset{\sim}{\mu} - r_f \cdot \underset{\sim}{\mathbb{1}})$$

$$= \left(\frac{b_p - r_f}{H}\right) (\underbrace{w_q' \mu}_{= b_q} - r_f)$$

$$\left\{ \begin{array}{l} \because\ w_q' \underset{\sim}{\mathbb{1}} = 1 \\ \Rightarrow r_f w_q' \underset{\sim}{\mathbb{1}} = r_f . \end{array} \right\}$$

$$= \frac{(b_p - r_f)(b_q - r_f)}{H} \Rightarrow \mathrm{Cov}(w_q' \underset{\sim}{R}, w_p' \underset{\sim}{R}) = \frac{(b_p - r_f)(b_q - r_f)}{H}$$

$\circledast$

$$\Rightarrow (b_q - r_f) = \beta_{qp} (b_p - r_f).$$

$$\Rightarrow E(r_q) = \beta_{qp} E(r_p) + (1 - \beta_{qp}) r_f$$

$$r_q = w_q' R, \quad r_p = w_p' R, \quad \beta_{qp} = \frac{\mathrm{Cov}(r_q, r_p)}{\mathrm{Var}(r_p)}.$$

$$\circledast \cdot (b_q - r_f) = \frac{\mathrm{cov}(r_q, r_p)}{\left(\frac{b_p - r_f}{H}\right)}$$

$$= \frac{\mathrm{Cov}(r_q, r_p)}{\sigma_p^2} \quad \frac{\sigma_p^2}{\left(\frac{b_p - r_f}{H}\right)}$$

$$= \beta_{qp} \frac{(b_p - r_f)^2}{H} \times \frac{H}{(b_p - r_f)}$$

$$= \beta_{qp}(b_p - r_f).$$

$$\Rightarrow E(r_q) = \beta_{qp} E(r_p) + (1 - \beta_{qp}) r_f$$

$$b_p = r_f + \sigma\sqrt{H} \quad \text{if } b_p > r_f$$

$$b_p = r_f - \sigma\sqrt{H} \; ; b_p < r_f.$$

Observe $H = (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}})' V^{-1} (\underset{\sim}{\mu} - r_f \underset{\sim}{\mathbb{1}})$

$$= \underset{\sim}{\mu}' V^{-1} \underset{\sim}{\mu} - 2 r_f \underset{\sim}{\mu}' V^{-1} \underset{\sim}{\mathbb{1}} + \cancel{r_f^2 \mathbb{1}' V^{-1} \mathbb{1}}$$

$$+ r_f \underset{\sim}{\mathbb{1}}' V^{-1} \underset{\sim}{\mathbb{1}}$$

$$= A - 2 r_f B + r_f^2 C.$$

$$= C\left(r_f^2 - 2 r_f \frac{B}{C} + \left(\frac{B}{C}\right)^2\right) + A - \frac{B^2}{C}$$

$$= C\left(r_f - \frac{B}{C}\right)^2 + \frac{AC - B^2}{C}$$

$$= C\left(r_f - \frac{B}{C}\right)^2 + \frac{D}{C}$$

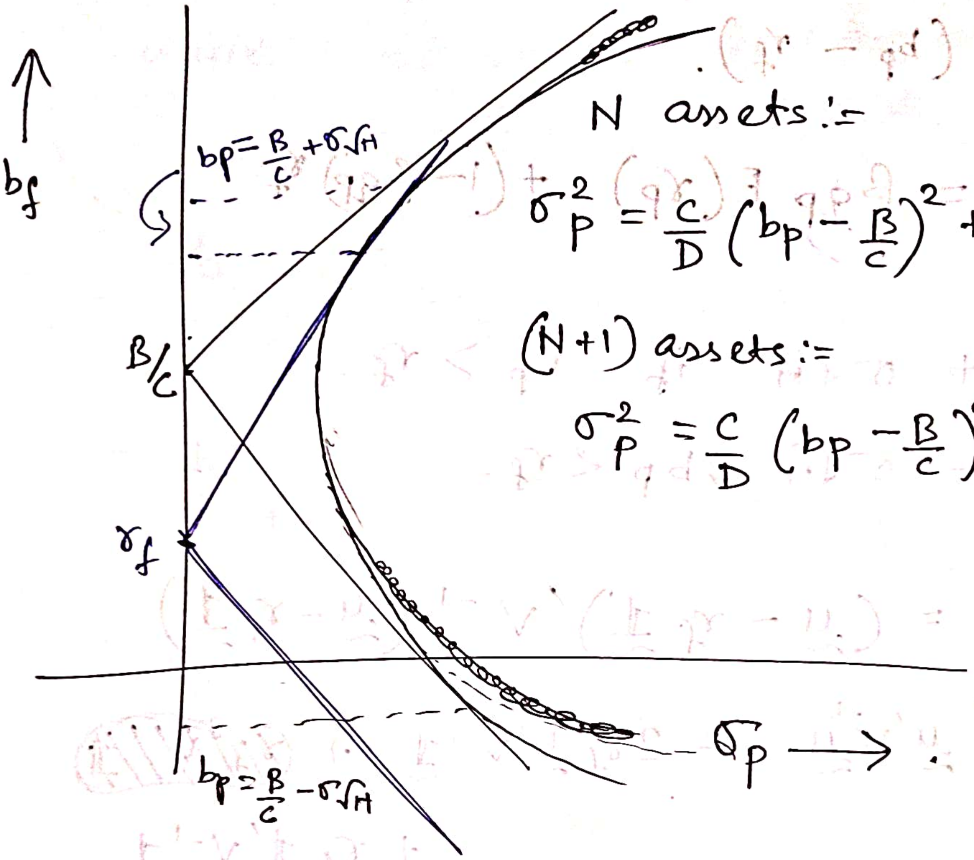
now

if $r_f = \frac{B}{C} \Rightarrow H = \frac{D}{C}$

& $w_p = \frac{b_p - r_f}{H} V^{-1}(\mu - r_f b)$

$$= \frac{C}{D}\left(b_p - \frac{B}{C}\right)\left(V^{-1}\mu - \frac{B}{C} V^{-1} \underset{\sim}{\mathbb{1}}\right).$$

$$\Rightarrow \underset{\sim}{1}' w_p = \frac{C}{D}\left(b_p - \frac{B}{C}\right)\left(B - \frac{B}{C}C\right).$$



N assets :=

$$\sigma_p^2 = \frac{C}{D}\left(b_p - \frac{B}{C}\right)^2 + \frac{1}{C}$$

(N+1) assets :=

$$\sigma_p^2 = \frac{C}{D}\left(b_p - \frac{B}{C}\right)^2, \quad r_f = \frac{B}{C}$$

Case 1: $r_f = B/C$

Case 2: $r_f < B/C$. := $b_p = r_f + \sigma\sqrt{H}$.

Case 3: $r_f > B/C$:= $b_p = r_f - \sigma\sqrt{H}$.

$H = D/C$ if $r_f = B/C$.

$q > D/C$ if $r_f \neq B/C$.

$\cancel{\sigma_p^{2(N+1)} \leq \frac{C}{D}\left(b_p - \frac{B}{C}\right)}$

H/w: Combine r_p & $r_{z(p)}$ to form ~~[illegible]~~ pf and compare

For N+1 asset, find tangent pf & combine with

r_f to ~~form~~ form r_f & compare performance.

N+1 assets (including a risk-free asset with return r_f)

$$\sigma_p^2 = \frac{(b_p - r_p)^2}{H}$$

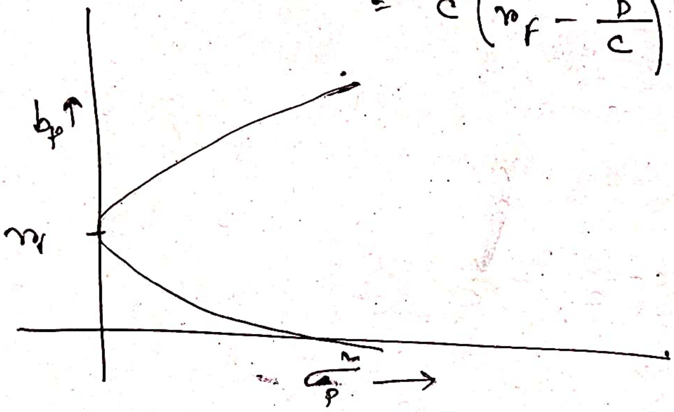
$$H = (\mu - r_f \mathbf{1})' V^{-1} (\mu - r_f \mathbf{1})$$

and $b_p = E(r_p)$ and $\sigma_p^2 = var(r_p)$

$$= \mu' V^{-1} \mu - 2 r_f \mu' V^{-1} \mathbf{1} + r_f^2 \mathbf{1}' V^{-1} \mathbf{1}$$

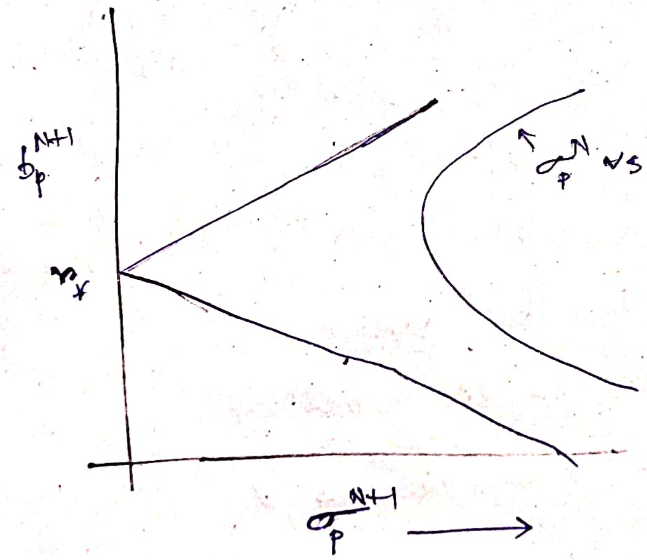
$$= A - 2 r_f B + r_f^2 C$$

$$= C\left(r_f - \frac{B}{C}\right)^2 + \frac{B}{C}$$



Note

$$\sigma_p = \pm \frac{b_p - r_f}{\sqrt{H}}$$



$$\sigma_p^2 = \frac{C}{D}\left(b_p - B/C\right)^2 + 1/C$$

Case 1

$$H = \frac{D}{C} \quad r_f = B/C$$

$$\sigma_p \sqrt{H} = \pm (b_p - r_f)$$

$$\frac{D}{C} \sigma_p^2 = (b_p - B/C)^2 + \frac{D}{C^2}$$

$$\therefore \sigma_p^{2N} = \frac{(b_p^N - r_f)^2}{H} + 1/C$$

$$\sigma_p^{2N+1} = \frac{(b_p^{N+1} - r_f)^2}{H}$$

If $r_f = b_{zc(p)} = E(r_{zc(p)})$

then,

$$\frac{b_{tp} - r_f}{\sigma_{tp}} = \sqrt{H}$$

Note,

$$\frac{b_{tp} - r_f}{\sigma_{tp}}$$

$$= \frac{\left(\frac{B}{C} - \frac{D}{C^2(r_f - B/C)} - r_f\right)}{\sigma_{tp}}$$

$$\sigma_{tp}^2 = \frac{C}{D}\left(b_{tp} - \frac{B}{C}\right)^2 + 1/C = \frac{C}{D}\left(\frac{D}{C^2(r_p - B/C)}\right)^2 + 1/C$$

$$= \frac{\frac{D}{C^2}}{(r_f - B/C)^2} + 1/C$$

$$= \frac{D/C + C(r_f - B/C)^2}{C^2(r_f - B/C)^2}$$

$$= \frac{H}{C^2(r_f - B/C)^2}$$

$$\sigma_{tp} = \frac{\sqrt{H}}{C(B/C - r_f)}$$

$$\frac{b_{tp} - r_f}{\sigma_{tp}} = \frac{\left(B/C - \frac{D}{C^2(r_f - B/C)} - r_f\right)}{\sigma_{tp}} = \frac{Num}{\frac{\sqrt{H}}{C(B/C - r_f)}}$$

$$Num = \left(\frac{B}{C} - r_f\right) + \frac{D/C}{C(B/C - r_f)}$$

$$= \frac{C(B/C - r_f)^2 + D/C}{C(B/C - r_f)} = \frac{H}{C(B/C - r_f)}$$

$$\frac{b_{tp} - r_f}{\sigma_{tp}} = \sqrt{H}$$

Two fund Separation

A market admits two fund separation if for pf with return r_q. $\exists$ two pfs with r_{p_1} and r_{p_2} and a $\lambda \in \mathbb{R}$ s.t.

$$E(v(r_q)) \leq E(v(\lambda r_{p_1} + (1-\lambda) r_{p_2}))$$

$\forall$ concave v

Note for two r.v. X, Y

if $E(U(X)) \leq E(U(Y))$

$\forall$ U concave

then (i) $E(X) = E(Y)$

(ii) $var(X) \geq var(Y)$

Converse does not hold in general.

---

take $U(x) = x$

then $E[X] \leq E[Y]$

Again for $U(x) = -x$

$E[-X] \leq E[-Y] \Rightarrow E[X] \geq E[Y]$

$\therefore$ $E[X] = E[Y]$

For variance, take $U(x) = x - \frac{1}{2}x^2$

$E(U(X)) = E(X) - \frac{1}{2}E(X^2)$

$= E(X) - \frac{1}{2}\left[[E(X)]^2 + var(X)\right]$

$= U(E(X)) - \frac{1}{2}var(X)$

and $E[U(Y)] = U(E(Y)) - \frac{1}{2}var(Y)$

Now $E(U(X)) \leq E(U(Y))$

$-\frac{1}{2}var(X) \leq -\frac{1}{2}var(Y)$

$\Rightarrow$ $var(X) \geq var(Y)$

$$E[r_q) = \lambda E(r_{p_1}) + (1-\lambda)E(r_{p_2}) \Rightarrow \lambda = \frac{E(r_q) - E(r_{p_2})}{E(r_{p_1}) - E(r_{p_2})}$$

$$r_q = \lambda r_{p_1} + (1-\lambda) r_{p_2} + \varepsilon$$

SSD in equivalence to $x \doteq y + e$

$x \leq y$ where $E(\varepsilon) = 0$

and $E(\varepsilon | Y) = 0$

Note if model holds then

$$E(U(x)) = E(E(U(Y+\varepsilon)) | Y)) \leq E(U(E(Y+\varepsilon | Y)))$$

Jensen's ineq.

$$= E(U(E(Y|\varepsilon) + E(\varepsilon|Y))) = E(U(Y+\varepsilon))$$

$$\text{var}(\lambda r_{p_1} + (1-\lambda) r_{p_2})) = \lambda^2 \text{var}(r_{p_1}) + (1-\lambda)^2 \text{var}(r_{p_2}) + 2\lambda(1-\lambda)\text{cov}(r_{p_1}, r_{p_2})$$

when there is + p.f.s. then w_{p_1} and w_{p_2} must be frontier

when $b_{p_2} = b_{z\,c(p_1)}$ and $P_1 = p$

then $\lambda = \beta_{qp} = \dfrac{\text{Cov}(r_q, r_p)}{\text{Var}(r_p)}$

$1 - \lambda = \beta_{q\,z\,c(p)} = 1 - \beta_{qp}$

$$r_q = \lambda \lambda_p + (1-\lambda) r_p + e$$