MATH 262/CME 372: Applied Fourier Analysis and Elements of Modern Signal Processing	Winter 2016

Lecture 17 — March 1, 2016

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1 Outline

Agenda: Lenses

1. Thin lenses
2. Fourier transform properties of thin lenses
3. Examples

Last Time: Using Rayleigh-Sommerfeld's diffraction formula, we studied two regimes of diffraction. Fresnel diffraction occurs when the aperture plane and the screen are far apart relative to the wavelength, but we are still able to observe near-field effects. In this case, the diffraction pattern is, modulo a factor, the Fourier transform of the wave field on the aperture, *after* being modulated by a quadratic phase term. On the other hand, Fraunhofer diffraction occurs when the aperture plane and the screen are much further apart, and only far-field effects are observed. In this case the quadratic phase term can be neglected, and the interference pattern is, modulo a modulation by a quadratic phase term, the Fourier transform of *just* the wave field. To distinguish these two regimes, we introduced the Fresnel number F, which separates Fresnel and Fraunhofer regimes by $F \gg 1$ and $F \ll 1$, repsectively.

2 Thin lenses

Lenses are optical devices that transmit and refract light. In the most basic sense, the refraction they induce is needed for *selective* transmission (see Fig. 1). The refraction is due to the change in speed of propagation. For example,

$$n = 1 \text{ in vacuum (and } \approx 1 \text{ air), and } n = 1.5 \text{ in glass.}$$

In wave optics, the change in the speed of propagation through a glass lens introduces a phase-shift to the wave field. This phase-shift will be proportional to the length of the path traversed by light. In addition, if we have a **thin lens** the effects of propagation *within* the lens can be neglected. Therefore, the wave field is only multiplied by a phase factor. This will be our working assumption.

To study this effect, we can enclose the lens on a box of width Δ_0, the thickness of the lens. If we denote[1] by $\Delta = \Delta(\underline{\boldsymbol{x}})$ the thickness of the lens at $\underline{\boldsymbol{x}}$ (see Fig. 2a), then the phase delay experienced

[1] See Figure 1 in Lecture 16 for the coordinate system used.

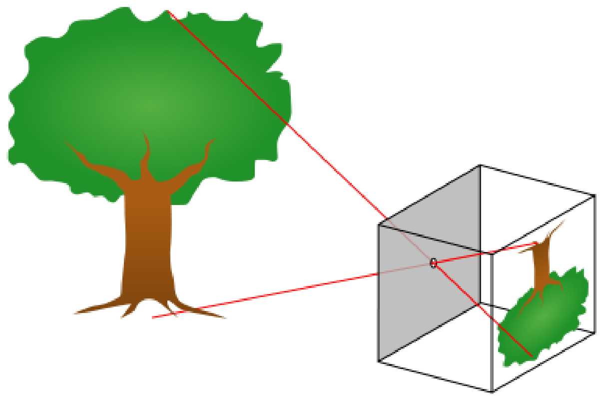


Figure 1: If a sensing screen were placed away from the tree, the contributing optical wave fields would come from all directions. To isolate just the light waves from the part of the environment containing the tree, an optical system of lenses is placed an appropriate distance away, and the screen behind it. In this lecture, we will discuss what the correct distance is, as well as why the image detected is rotated.

by the wave after traversing this box is

$$\phi(\underline{\boldsymbol{x}}) = \underbrace{kn\Delta(\underline{\boldsymbol{x}})}_{\text{glass}} + \underbrace{k(\Delta_0 - \Delta(\underline{\boldsymbol{x}}))}_{\text{vacuum}}. \tag{1}$$

Consequently, the difference between the incoming and outgoing wave fields u_{in} and u_{out}, respectively, is

$$u_{\text{out}}(\underline{\boldsymbol{x}}) = T(\underline{\boldsymbol{x}})u_{\text{in}}(\underline{\boldsymbol{x}}), \qquad \text{where} \qquad T(\underline{\boldsymbol{x}}) = e^{ik\Delta_0}e^{ik(n-1)\Delta(\underline{\boldsymbol{x}})}$$

is known as the **transmission function**.

Of course, to understand the concrete effect of this we need to determine the **thickness function** Δ of the lens. Here we will focus on circular lenses. In this case, each one of its surfaces is a circular sector of a sphere. We will call the line joining the two centers the **lens axis**. We follow the conventions that

1. If a ray encounters a convex surface (as in Fig. 2b), then radius of the corresponding circle is taken to be positive.

2. If a ray encounters a concave surface (as in Fig. 2c), then the radius of the corresponding circle is taken to be negative.

A lens such as the one shown in Fig. 2a is a **converging**, or **doubly convex**, lens. In this case,

1. The thickness of the left component of the lens is

$$\Delta_1(\underline{\boldsymbol{x}}) = \Delta_{0,1} - R_1\left(1 - \sqrt{1 - \frac{\|\underline{\boldsymbol{x}}\|^2}{R_1^2}}\right),$$

where $R_1 > 0$, and $\Delta_{0,1} > 0$ is the maximum thickness of this component.

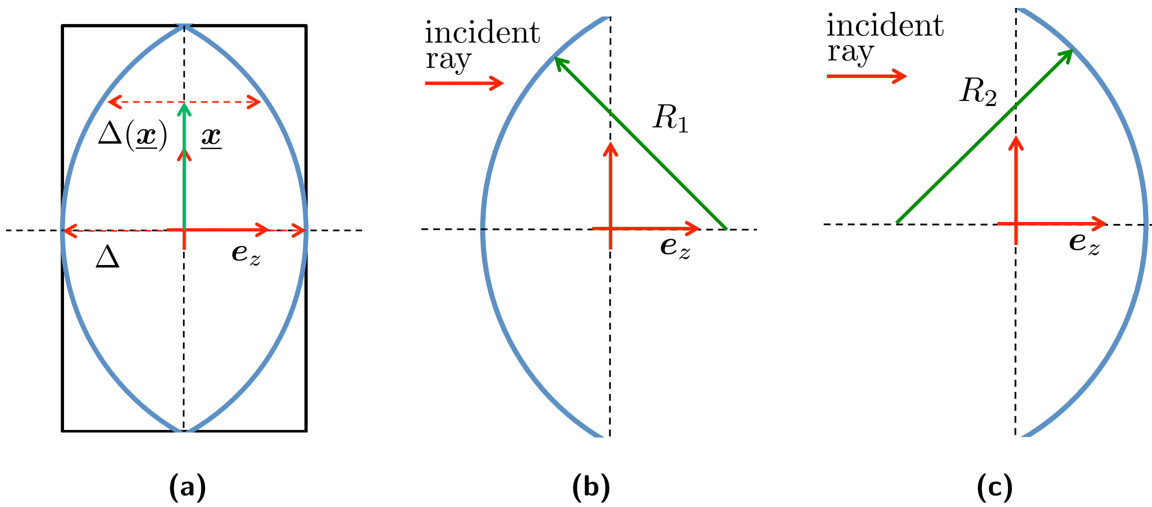


Figure 2: **(a)** Coordinate system used to describe the thickness of the lens. **(b)** Convex surface with $R_1 > 0$. **(c)** Concave surface with $R_2 < 0$.

2. The thickness of the right component of the lens is

$$\Delta_2(\underline{\boldsymbol{x}}) = \Delta_{0,2} + R_2 \left(1 - \sqrt{1 - \frac{\|\underline{\boldsymbol{x}}\|^2}{R_2^2}}\right),$$

where $R_2 < 0$, and $\Delta_{0,2} > 0$ is the maximum thickness of this component.

Consequently, the thickness function is given by

$$\Delta(\underline{\boldsymbol{x}}) = \Delta_1(\underline{\boldsymbol{x}}) + \Delta_2(\underline{\boldsymbol{x}}) = \Delta_0 - R_1 \left(1 - \sqrt{1 - \frac{\|\underline{\boldsymbol{x}}\|^2}{R_1^2}}\right) + R_2 \left(1 - \sqrt{1 - \frac{\|\underline{\boldsymbol{x}}\|^2}{R_2^2}}\right).$$

As we are interested in the paraxial case (i.e. the light field *near* the lens axis), we may think of $\|\underline{\boldsymbol{x}}\|$ as small relative to the radii R_1 and R_2. In this case, the Taylor expression

$$\sqrt{1 - x^2} = 1 - \frac{x^2}{2} + o\left(x^4\right)$$

suggests the following **paraxial approximations**:

$$\sqrt{1 - \frac{\|\underline{\boldsymbol{x}}\|^2}{R_1^2}} \approx 1 - \frac{\|\underline{\boldsymbol{x}}\|^2}{2R_1^2} \quad \text{and} \quad \sqrt{1 - \frac{\|\underline{\boldsymbol{x}}\|^2}{R_2^2}} \approx 1 - \frac{\|\underline{\boldsymbol{x}}\|^2}{2R_2^2},$$

These simplifications are accurate whenever

$$\frac{\|\underline{\boldsymbol{x}}\|}{R_1} \ll 1 \quad \text{and} \quad \frac{\|\underline{\boldsymbol{x}}\|}{R_2} \ll 1,$$

Using our approximations, we write the lens thickness as

$$\Delta(\underline{\boldsymbol{x}}) = \Delta_0 - \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \|\underline{\boldsymbol{x}}\|^2.$$

The physical properties of the lens can then be encoded by a single number f, called the lens **focal length**:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

Under the paraxial approximation, the phase-shift (1) induced by the lens is now written

$$\begin{aligned}\phi(\underline{\boldsymbol{x}}) &= kn\Delta(\underline{\boldsymbol{x}}) + k(\Delta_0 - \Delta(\underline{\boldsymbol{x}})) \\ &= k\Delta_0 + k(n-1)\Delta(\underline{\boldsymbol{x}}) \\ &= k\Delta_0 + k(n-1)\left(\Delta_0 - \frac{1}{2f(n-1)}\|\underline{\boldsymbol{x}}\|^2\right) \\ &= kn\Delta_0 - \frac{k}{2f}\|\underline{\boldsymbol{x}}\|^2.\end{aligned}$$

The transmission function, which introduces this phase delay, is then

$$T(\underline{\boldsymbol{x}}) = e^{ikn\Delta_0} \underbrace{e^{-\frac{ik}{2f}\|\underline{\boldsymbol{x}}\|^2}}_{\text{important part}}.$$

For instance, if the incident signal u_{in} is a unit amplitude plane wave, then the outgoing wave, on the other side of the lens, is

$$u_{\text{out}}(\underline{\boldsymbol{x}}) \propto e^{-\frac{ik}{2f}\|\underline{\boldsymbol{x}}\|^2}, \tag{2}$$

which is a quadratic approximation to a spherical wave.

The sign of f indicates the position of the lens relative to the source of this wave, and so (2) corresponds to the time-reversal of an expanding spherical wave with source at the origin, distance f away. That is, the converging lens will focus a plane wave as an *incoming* spherical wave converging in *front* of the lens, on the lens axis and at the focal distance $f > 0$. A similar result holds for a **diverging**, or **doubly concave** lens, for which the focal length is negative. In that case, the lens will focus a plane wave as an *outgoing* spherical wave with source *behind* the lens, again on the lens axis, at the focal length $f < 0$. Fig. 3 illustrates the difference between these cases.

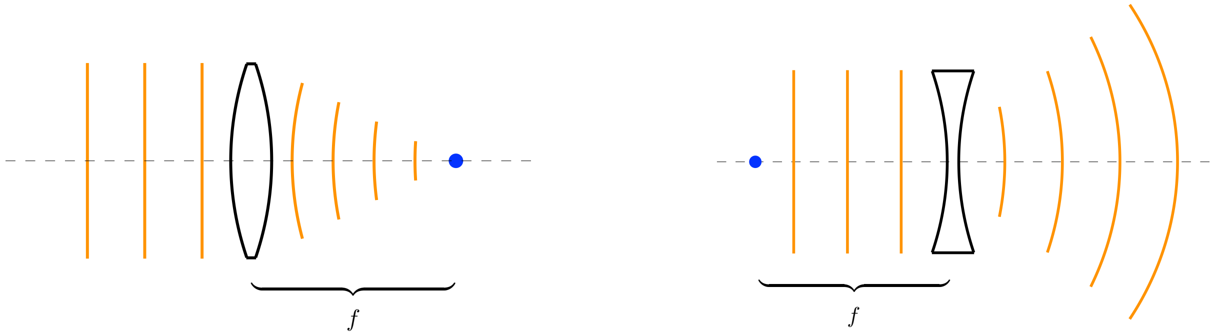
3 Fourier transform properties of thin lenses

Suppose we place a converging lens on the aperture. In our previous study we did not account for the fact that the lens is finite. So here we define the **pupil function** P, which is simply the indicator function of the shape of the lens. If we denote the wave field just behind the lens aperture as u_{in}, then the wave field immediately in front of the lens aperture is

$$u_{\text{out}}(\underline{\boldsymbol{x}}) = P(\underline{\boldsymbol{x}})T(\underline{\boldsymbol{x}})u_{\text{in}}(\underline{\boldsymbol{x}}).$$

After traversing the lens, the wave field experiences diffraction. If we are interested in near-field effects, Fresnel's diffraction integral yields

$$\begin{aligned}u(\underline{\boldsymbol{x}}_0) &= \frac{1}{i\lambda z}e^{ikz}e^{\frac{ik}{2z}\|\underline{\boldsymbol{x}}_0\|^2}\int_A u_{\text{out}}(\underline{\boldsymbol{x}})e^{\frac{ik}{2z}\|\underline{\boldsymbol{x}}\|^2}e^{-\frac{ik}{z}\langle\underline{\boldsymbol{x}},\underline{\boldsymbol{x}}_0\rangle}\,d\underline{\boldsymbol{x}} \\ &= \frac{1}{i\lambda z}e^{ikz}e^{\frac{ik}{2z}\|\underline{\boldsymbol{x}}_0\|^2}\int u_{\text{in}}(\underline{\boldsymbol{x}})P(\underline{\boldsymbol{x}})T(\underline{\boldsymbol{x}})e^{\frac{ik}{2z}\|\underline{\boldsymbol{x}}\|^2}e^{-\frac{ik}{z}\langle\underline{\boldsymbol{x}},\underline{\boldsymbol{x}}_0\rangle}\,d\underline{\boldsymbol{x}}.\end{aligned}$$



(a) converging lens

(b) diverging lens

Figure 3: Plane wave encountering a converging (doubly convex) or diverging (doubly concave) lens. The focal point (blue dot) is along the lens axis (dashed line). The wave front (orange) is focused inward in **(a)** and outward in **(b)**, in either case as a spherical wave with source at the focal point. The thin lens approximation has been made, and the finite spatial extension of the lens has been ignored.

Using the paraxial approximation yields

$$\begin{aligned} T(\underline{\boldsymbol{x}})e^{\frac{ik}{2z}\|\underline{\boldsymbol{x}}\|^2}e^{-\frac{ik}{z}\langle\underline{\boldsymbol{x}},\underline{\boldsymbol{x}}_0\rangle} &= e^{ikn\Delta_0}e^{-\frac{ik}{2f}\|\underline{\boldsymbol{x}}\|^2}e^{\frac{ik}{2z}\|\underline{\boldsymbol{x}}\|^2}e^{-\frac{ik}{z}\langle\underline{\boldsymbol{x}},\underline{\boldsymbol{x}}_0\rangle} \\ &= e^{ikn\Delta_0}e^{-\frac{ik}{2f}\left(1-\frac{f}{z}\right)\|\underline{\boldsymbol{x}}\|^2}e^{-\frac{ik}{z}\langle\underline{\boldsymbol{x}},\underline{\boldsymbol{x}}_0\rangle}. \end{aligned}$$

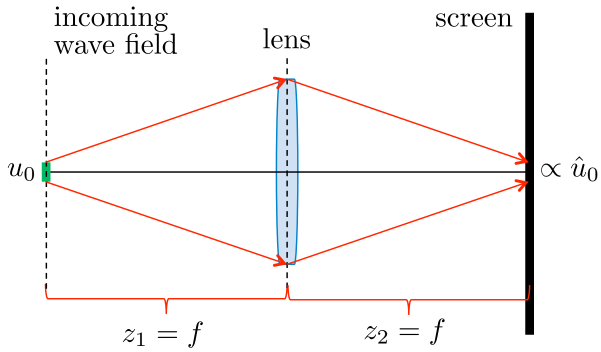
Therefore, if the screen is placed at the focal distance $z = f$, exponential terms exactly cancel, and the diffraction pattern becomes

$$u(\underline{\boldsymbol{x}}_0) = \frac{1}{i\lambda f}e^{ik(f+n\Delta_0)}e^{\frac{ik}{2f}\|\underline{\boldsymbol{x}}_0\|^2}\int P(\underline{\boldsymbol{x}})u_{\text{in}}(\underline{\boldsymbol{x}})e^{-\frac{ik}{f}\langle\underline{\boldsymbol{x}},\underline{\boldsymbol{x}}_0\rangle}\,d\underline{\boldsymbol{x}}.$$

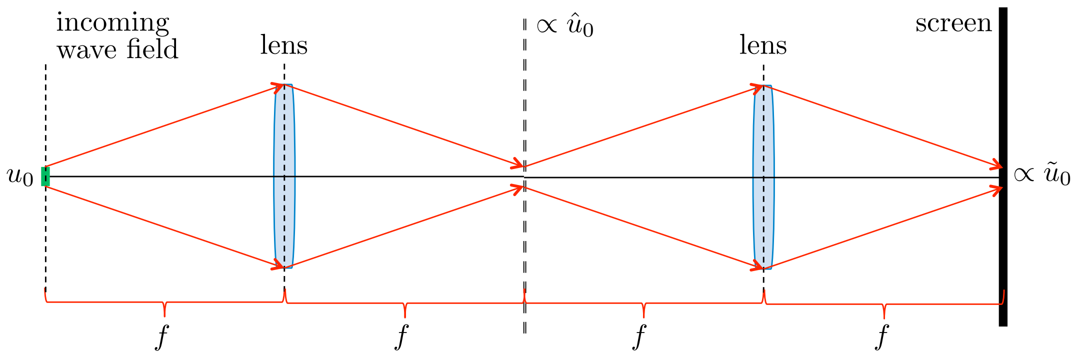
Therefore, if the incident wave field is localized in the lens (that is, the physical extent of the input is much smaller than the lens aperture, such as in a microscope or a telescope examining stars), then we will observe its Fourier transform, up to modulation by a quadratic phase factor. Note that this modulation is experimentally unimportant since we see only intensity and not phase. Otherwise, we will observe the Fourier transform of the incoming wave convolved with the Fourier transform of the pupil function. Recall that multiplication in the spatial domain is equivalent to convolution in the frequency domain. This convolution kernel, the Fourier transform of the pupil function, is called the **point-spread function** (PSF). In the case of a circular lens, this is the Airy disk. The key feature of thin lenses is that they allow us to see the effects of diffraction at much shorter distances.

We can consider more general arrays including lenses. Suppose we have an array as the one shown in Fig. 4a with z_1 and z_2 assumed to be arbitrary at first. Once again, we will neglect the finite extent of the apertures so that the pupil function is not needed. If the incoming wave field is u_0, then the wave field just before the lens is

$$u_{\text{in}}(\underline{\boldsymbol{x}}_0) = \frac{1}{i\lambda z_1}e^{ikz_1}\int u_0(\underline{\boldsymbol{x}})e^{\frac{ik}{2z_1}\|\underline{\boldsymbol{x}}-\underline{\boldsymbol{x}}_0\|^2}\,d\underline{\boldsymbol{x}},$$



(a) 2-f optical system



(b) 4-f optical system

Figure 4: Example of optical systems. **(a)** This system allows us to observe the Fourier transform of the incoming wave field up to a multiplicative factor. **(b)** By replicating the 2-f system, we can obtain the incoming wave field up to a multiplicative factor and a rotation about the origin. Here $\tilde{u}_0$ represents a rotation of u_0 with respect to the origin.

and the wave field just after the lens is

$$u_{\text{out}}(\underline{\boldsymbol{x}}) = T(\underline{\boldsymbol{x}})u_{\text{in}}(\underline{\boldsymbol{x}}).$$

And then the wave field observed on the screen is

$$\begin{aligned} u(\underline{\boldsymbol{x}}_0) &= \frac{1}{i\lambda z_2} e^{ikz_2} \int u_{\text{out}}(\underline{\boldsymbol{x}}) e^{\frac{ik}{2z_2}\|\underline{\boldsymbol{x}}-\underline{\boldsymbol{x}}_0\|^2} \, d\underline{\boldsymbol{x}} \\ &= -\frac{1}{\lambda^2 z_1 z_2} e^{ik(z_1+z_2+n\Delta_0)} \iint u_0(\underline{\boldsymbol{x}}') e^{-\frac{ik}{2f}\|\underline{\boldsymbol{x}}\|^2} e^{\frac{ik}{2z_1}\|\underline{\boldsymbol{x}}-\underline{\boldsymbol{x}}'\|^2} e^{\frac{ik}{2z_2}\|\underline{\boldsymbol{x}}-\underline{\boldsymbol{x}}_0\|^2} \, d\underline{\boldsymbol{x}} d\underline{\boldsymbol{x}}'. \end{aligned}$$

Notice that the phase term in the integral can be written as

$$\frac{k}{2}\left[-\left(\frac{1}{f} - \frac{1}{z_1} - \frac{1}{z_2}\right)\|\underline{\boldsymbol{x}}\|^2 - 2\left\langle \underline{\boldsymbol{x}},\, \frac{1}{z_1}\underline{\boldsymbol{x}}' + \frac{1}{z_2}\underline{\boldsymbol{x}}_0 \right\rangle + \frac{1}{z_1}\|\underline{\boldsymbol{x}}'\|^2 + \frac{1}{z_2}\|\underline{\boldsymbol{x}}_0\|^2\right].$$

For simplicity, suppose $z_1 = z_2 = f$. (One can deduce this, but this assumption greatly simplifies the computations.) Then we can write the above phase term as

$$\frac{k}{2f}\left[\|\underline{\boldsymbol{x}}\|^2 - 2\langle \underline{\boldsymbol{x}},\, \underline{\boldsymbol{x}}' + \underline{\boldsymbol{x}}_0\rangle + \|\underline{\boldsymbol{x}}'\|^2 + \|\underline{\boldsymbol{x}}_0\|^2\right] = \frac{k}{2f}\left[\|\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}' - \underline{\boldsymbol{x}}_0\|^2 - 2\langle \underline{\boldsymbol{x}}',\, \underline{\boldsymbol{x}}_0\rangle\right],$$

and therefore we observe

$$u(\underline{\boldsymbol{x}}_0) = -\frac{1}{\lambda^2 f^2} e^{ik(2f+n\Delta_0)} \int \left(\int e^{-\frac{ik}{2f}\|\underline{\boldsymbol{x}}-\underline{\boldsymbol{x}}'-\underline{\boldsymbol{x}}_0\|^2} \, d\underline{\boldsymbol{x}} \right) u_0(\underline{\boldsymbol{x}}') e^{-\frac{ik}{f}\langle \underline{\boldsymbol{x}}', \underline{\boldsymbol{x}}_0 \rangle} \, d\underline{\boldsymbol{x}}'.$$

However, since we are ignoring the finite extent of the apertures, we can perform a change of variables to obtain

$$\int e^{-\frac{ik}{2f}\|\underline{\boldsymbol{x}}-\underline{\boldsymbol{x}}'-\underline{\boldsymbol{x}}_0\|^2} \, d\underline{\boldsymbol{x}} = \int e^{-\frac{ik}{2f}\|\underline{\boldsymbol{x}}\|^2} \, d\underline{\boldsymbol{x}} = e^{-\frac{i\pi}{4}} \sqrt{\frac{\pi f}{k}},$$

and deduce

$$\begin{aligned} u(\underline{\boldsymbol{x}}_0) &= -\sqrt{\frac{\pi f}{k}} \frac{1}{\lambda^2 f^2} e^{-\frac{i\pi}{4}} e^{ik(2f+n\Delta_0)} \int u_0(\underline{\boldsymbol{x}}') e^{-\frac{ik}{f}\langle \underline{\boldsymbol{x}}', \underline{\boldsymbol{x}}_0 \rangle} \, d\underline{\boldsymbol{x}}' \\ &= -\sqrt{\frac{\pi f}{k}} \frac{1}{\lambda^2 f^2} e^{-\frac{i\pi}{4}} e^{ik(2f+n\Delta_0)} \hat{u}_0 \left(\frac{k}{f} \underline{\boldsymbol{x}}_0 \right), \end{aligned}$$

which is yet another Fourier transform!

Unfortunately, in practice we have only the physical means of measuring the intensity of a wave field; phase information is lost. This is not a problem if we are observing the desired, final image (since our eyes only detect intensity), but thus far we have only shown how to observe its Fourier transform. Our rescue is the fact the Fourier transform is nearly its own inverse. Indeed, in two dimensions the Fourier transform of $\hat{f}$ is the scaled rotation $(2\pi)^2 f(-\boldsymbol{x})$. So we consider the extended **4-f system**, which replicates the setup of the 2-f system to take a second Fourier transform. The resulting output is a scaled and rotated version of the input. In the next lecture we will study simpler ways of achieving the same result.