

Lecture 16 — February 25, 2016

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1 Outline

Agenda: Fourier/wave optics

1. Fresnel-Kirchhoff theory
2. Rayleigh-Sommerfeld theory
3. Fresnel diffraction
4. Fraunhofer diffraction
5. Examples

Last Time: We began the study of diffraction with Maxwell's equations with no sources, from which we deduced that the electric field satisfies the wave equation. Considering a single component and assuming the wave is monochromatic, we reduced the task of solving the wave equation to solving the Helmholtz equation. Using the Green's function, which is an outgoing spherical wave with source at the origin, and Green's theorem, we deduced that the wave field at any point can be obtained from its values over any surface enclosing the point, provided we also know the values of its normal derivative on the surface.

2 Fresnel-Kirchhoff formulation of diffraction

Recall from last time the Green's function to the Helmholtz equation, called the **free-space Green's function**:

$$G(\mathbf{x}) = \frac{e^{ik\|\mathbf{x}\|}}{\|\mathbf{x}\|},$$

where $k = \frac{2\pi}{\lambda}$ is the wavenumber, and λ is the wavelength. For any u that is another solution to the Helmholtz equation, we established the Fresnel-Kirchhoff integral formula:

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \int_{\partial\Omega} \left(u(\mathbf{x}) \frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}=\mathbf{x}_0} - G(\mathbf{x} - \mathbf{x}_0) \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} \right) dS(\mathbf{x}), \quad (1)$$

where $\Omega \subset \mathbb{R}^3$, and $\boldsymbol{\nu}$ is the outward unit normal vector to $\partial\Omega$.

We now use (1) to explain the diffraction of the wave field caused by an aperture. Consider a reference frame with origin on the aperture, and with the z -axis orthogonal to the aperture plane. For $R > 0$, consider the domain

$$\Omega_R = \{\mathbf{x} : \|\mathbf{x}\| \leq R, \langle \mathbf{e}_z, \mathbf{x} \rangle \geq 0\}.$$

Ω_R is the upper half of the ball of radius R centered at the aperture, and its boundary decomposes as $\partial\Omega_R = \Sigma_R \cup \Gamma_R$ (see Fig. 1). Here

$$\Sigma_R = \{\mathbf{x} : \|\mathbf{x}\| \leq R, \langle \mathbf{e}_z, \mathbf{x} \rangle = 0\} \quad \text{and} \quad \Gamma_R = \{\mathbf{x} : \|\mathbf{x}\| = R, \langle \mathbf{e}_z, \mathbf{x} \rangle > 0\}$$

are a disk and a hemisphere, respectively, of radius R and again centered at the origin. We assume the aperture is infinitely thin so that it is a subset of Σ_R .

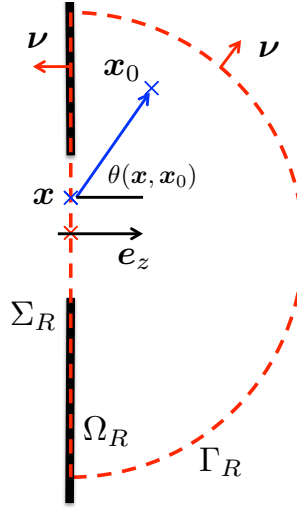


Figure 1: Description of the contour of integration, and the quantities used to deduce Fresnel-Kirchhoff's diffraction formula (4). Here \mathbf{x} is shown as a point on the aperture, which is a subset of the aperture plane.

Applying (1) with $\Omega = \Omega_R$, we obtain

$$\begin{aligned} u(\mathbf{x}_0) &= \frac{1}{4\pi} \int_{\partial\Omega_R} \left(u(\mathbf{x}) \frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}-\mathbf{x}_0} - G(\mathbf{x}-\mathbf{x}_0) \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} \right) dS(\mathbf{x}) \\ &= \frac{1}{4\pi} \int_{\Sigma_R} \left(u(\mathbf{x}) \frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}-\mathbf{x}_0} - G(\mathbf{x}-\mathbf{x}_0) \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} \right) dS(\mathbf{x}) \\ &\quad + \frac{1}{4\pi} \int_{\Gamma_R} \left(u(\mathbf{x}) \frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}-\mathbf{x}_0} - G(\mathbf{x}-\mathbf{x}_0) \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} \right) dS(\mathbf{x}). \end{aligned}$$

We are interested in the interaction of the wave front exactly at the aperture, and so we would like to remove the contribution from Γ_R by focusing on the limiting case $R \rightarrow \infty$. Indeed, for $R \gg 0$, we have for $\mathbf{x} \in \Gamma_R$ that $\|\mathbf{x} - \mathbf{x}_0\| \approx \|\mathbf{x}\|$, and $\boldsymbol{\nu}(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$. Additionally, we can use the approximation

$$\frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}-\mathbf{x}_0} \approx \frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} = \left(ik - \frac{1}{\|\mathbf{x}\|} \right) G(\mathbf{x}),$$

and thus the contribution to (1) from Γ_R can be approximated by

$$\begin{aligned} \frac{1}{4\pi} \int_{\Gamma_R} \left(u(\mathbf{x}) \frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}-\mathbf{x}_0} - G(\mathbf{x}-\mathbf{x}_0) \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} \right) dS(\mathbf{x}) \\ \approx \frac{1}{4\pi} \int_{\Gamma_1} \left((ikR-1) u(R\mathbf{x}) - R \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{R\mathbf{x}} \right) e^{ikR} dS(\mathbf{x}). \end{aligned}$$

Unfortunately, it is not clear whether this last integral over Γ_1 converges as R tends to infinity. Therefore, we impose an additional condition on the field, namely **Sommerfeld's radiation condition**

$$\lim_{\|\mathbf{x}\| \rightarrow \infty} \|\mathbf{x}\| \left(ik u(\mathbf{x}) - \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} \right) = 0.$$

Roughly speaking, this condition guarantees that we are only dealing with *outgoing* waves, that is, there are no sources at infinity. Assuming the radiation condition holds, it is clear that the contribution from Γ_R vanishes as $R \rightarrow \infty$. If we let Σ be the (infinite) aperture plane, we now conclude

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \int_{\Sigma} \left(u(\mathbf{x}) \frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}-\mathbf{x}_0} - G(\mathbf{x}-\mathbf{x}_0) \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} \right) dS(\mathbf{x}).$$

Recall that in order to determine the wave field, we need to know both u and $\frac{\partial u}{\partial \boldsymbol{\nu}}$ over the aperture plane. The choice

$$u(\mathbf{x}) = 0 \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} = 0 \quad \text{for } \mathbf{x} \in \Sigma \setminus A, \quad (2)$$

yields **Fresnel-Kirchhoff's diffraction formula**

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \int_A \left(u(\mathbf{x}) \frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}-\mathbf{x}_0} - G(\mathbf{x}-\mathbf{x}_0) \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} \right) dS(\mathbf{x}). \quad (3)$$

Note that

$$\frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}-\mathbf{x}_0} = - \left(ik - \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|} \right) G(\mathbf{x}-\mathbf{x}_0) \cos(\theta(\mathbf{x}, \mathbf{x}_0)),$$

where

$$\cos(\theta(\mathbf{x}, \mathbf{x}_0)) = \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|} \langle \boldsymbol{\nu}(\mathbf{x}), \mathbf{x}-\mathbf{x}_0 \rangle$$

is the cosine of the angle between the z -axis and the line connecting \mathbf{x} and \mathbf{x}_0 .

When \mathbf{x}_0 is far away from the aperture, so that $\|\mathbf{x}-\mathbf{x}_0\|$ is much larger than the wavelength λ , we can make the approximation

$$\frac{\partial G}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}-\mathbf{x}_0} \approx -ikG(\mathbf{x}-\mathbf{x}_0) \cos(\theta(\mathbf{x}, \mathbf{x}_0)),$$

to obtain the most general form of Fresnel-Kirchhoff's diffraction formula:

$$u(\mathbf{x}_0) = -\frac{1}{4\pi} \int_A \left(ik u(\mathbf{x}) \cos(\theta(\mathbf{x}, \mathbf{x}_0)) + \frac{\partial u}{\partial \boldsymbol{\nu}} \Big|_{\mathbf{x}} \right) G(\mathbf{x}-\mathbf{x}_0) dS(\mathbf{x}). \quad (4)$$

This formula enables us to predict diffraction effects, particularly when the wavelength is very small and observations are made away from the aperture. Otherwise, though, the predictions of this formula can be inaccurate. From an experimental point of view, Fresnel-Kirchhoff's formula does not correctly explain Poisson's spot¹. And from a mathematical perspective, Sommerfeld noticed that if u is analytic, then the choices made in (2) actually force u to be identically zero.

These inconsistencies do not mean the Fresnel-Kirchhoff formula has no predictive power. Instead, we should suspect that it may not be predictive in *all* regimes. A modification to the previous reasoning is required in order to obtain a consistent formulation of diffraction.

3 Rayleigh-Sommerfeld formulation of diffraction

In order to overcome the issues outlined in the previous section, we need to deduce an identity that does not depend on the values of the normal derivative of u on the aperture plane. Note from (3) that we can accomplish this if we use a Green's function that vanishes on the aperture plane. In this section we discuss the clever means by which Rayleigh and Sommerfeld did just that.

Let \mathbf{x}_0 be any point behind the aperture, which we will consider fixed, and let $\bar{\mathbf{x}}_0$ be its reflection with respect to the aperture plane. Define

$$g_{\mathbf{x}_0}(\mathbf{x}) = G(\mathbf{x} - \mathbf{x}_0) - G(\mathbf{x} - \bar{\mathbf{x}}_0) = \frac{e^{ik\|\mathbf{x} - \mathbf{x}_0\|}}{\|\mathbf{x} - \mathbf{x}_0\|} - \frac{e^{ik\|\mathbf{x} - \bar{\mathbf{x}}_0\|}}{\|\mathbf{x} - \bar{\mathbf{x}}_0\|}.$$

By linearity, g is easily seen to be a solution to the Helmholtz equation away from \mathbf{x}_0 and $\bar{\mathbf{x}}_0$. Note that if \mathbf{x} lies on the aperture plane, then $\|\mathbf{x} - \mathbf{x}_0\| = \|\mathbf{x} - \bar{\mathbf{x}}_0\|$, and so $g_{\mathbf{x}_0}(\mathbf{x}) = 0$ (see Fig. 2). We can repeat the same computations as in Lecture 15, using $v(\mathbf{x}) = g_{\mathbf{x}_0}(\mathbf{x})$ and assuming Sommerfeld's radiation condition, to obtain

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \int_{\Sigma} u(\mathbf{x}) \left. \frac{\partial g_{\mathbf{x}_0}}{\partial \boldsymbol{\nu}} \right|_{\mathbf{x}} dS(\mathbf{x}).$$

Now the dependence on the normal derivative of the wave field on the aperture has disappeared! Consequently, we can write

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \int_A u(\mathbf{x}) \left. \frac{\partial g_{\mathbf{x}_0}}{\partial \boldsymbol{\nu}} \right|_{\mathbf{x}} dS(\mathbf{x}).$$

Happily, this expression does not exhibit the inconsistencies of Fresnel-Kirchhoff's formula, even providing a correct description of Poisson's spot.

Note that for \mathbf{x} on the aperture plane A , we have

$$\begin{aligned} \nabla g_{\mathbf{x}_0}(\mathbf{x}) &= \left(\frac{ik}{\|\mathbf{x} - \mathbf{x}_0\|} - \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|^2} \right) G(\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\ &\quad - \left(\frac{ik}{\|\mathbf{x} - \bar{\mathbf{x}}_0\|} - \frac{1}{\|\mathbf{x} - \bar{\mathbf{x}}_0\|^2} \right) G(\mathbf{x} - \bar{\mathbf{x}}_0)(\mathbf{x} - \bar{\mathbf{x}}_0) \\ &= 2 \left(\frac{ik}{\|\mathbf{x} - \mathbf{x}_0\|} - \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|^2} \right) G(\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0), \end{aligned}$$

¹R. L. Lucke, "Rayleigh-Sommerfeld diffraction and Poisson's spot", European Journal of Physics, 27 (2006), pg. 193–204.

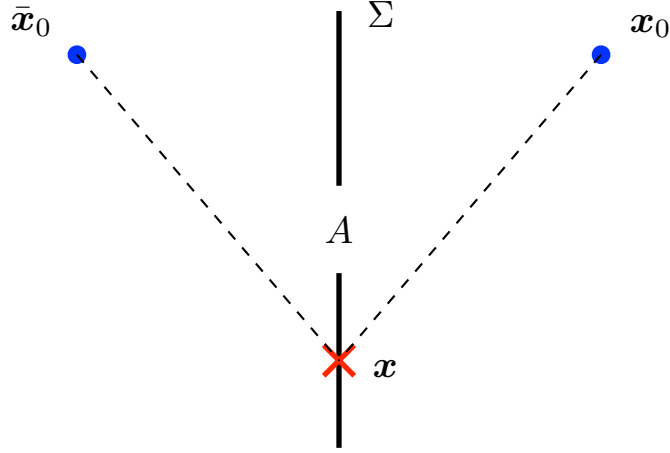


Figure 2: Rayleigh-Sommerfeld's placement of a reflected secondary source \bar{x}_0 . Any $x \in \Sigma$ is equidistance from the two sources, implying the modified Green's function g_{x_0} vanishes on the aperture plane.

and

$$\left. \frac{\partial g_{x_0}}{\partial \nu} \right|_x = -2 \left(ik - \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|} \right) G(\mathbf{x} - \mathbf{x}_0) \cos(\theta(\mathbf{x}, \mathbf{x}_0)).$$

Since we are interested in the regime where the observation point \mathbf{x}_0 is far from the aperture, so that $\|\mathbf{x} - \mathbf{x}_0\|$ is much larger than the wavelength λ , we can use the approximation (similar to before)

$$\left. \frac{\partial g_{x_0}}{\partial \nu} \right|_x \approx -2ikG(\mathbf{x} - \mathbf{x}_0) \cos(\theta(\mathbf{x}, \mathbf{x}_0)).$$

We are led to

$$u(\mathbf{x}_0) = \frac{1}{i\lambda} \int_A u(\mathbf{x}) \cos(\theta(\mathbf{x}, \mathbf{x}_0)) G(\mathbf{x} - \mathbf{x}_0) dS(\mathbf{x}), \quad (5)$$

known as the **Rayleigh-Sommerfeld diffraction formula**.

Here we see an analogy with the Huygens-Fresnel principle. Namely, the Green's function G represents an outgoing spherical wave with source at the origin. Therefore, the integral (5) is a superposition of spherical waves with sources on the aperture. However, the phase and amplitude of these sources are *not* those of the wave field *at* the aperture. Instead we observe a phase shift of $\pi/2$ from the factor of $-i$, and an amplitude attenuation given by the **inclination factor** $\cos(\theta(\mathbf{x}, \mathbf{x}_0))$ (recall that the envelope $u(\mathbf{x})$ is the intensity of the incoming wave). The absence of this factor was the main shortcoming of Huygens and Fresnel.

Note also that (5) is a convolution, as required for us to have constructed a proper Green's function. Indeed, writing the impulse-response

$$\frac{1}{i\lambda} \cos(\theta(\mathbf{x}, \mathbf{x}_0)) G(\mathbf{x} - \mathbf{x}_0) = \frac{1}{i\lambda} \left(\frac{\langle -\mathbf{e}_z, \mathbf{x} - \mathbf{x}_0 \rangle}{\|\mathbf{x} - \mathbf{x}_0\|} \right) G(\mathbf{x} - \mathbf{x}_0) = h(\mathbf{x} - \mathbf{x}_0),$$

we have

$$u(\mathbf{x}_0) = \int_A u(\mathbf{x}) h(\mathbf{x} - \mathbf{x}_0) dS(\mathbf{x}).$$

The linearity of this convolution operation corresponds to the principle of superposition, while the spatial invariance (with respect to translation) models homogeneity in the dielectric medium.

4 Fresnel diffraction

We now explore some regimes where we observe the wave field on a screen parallel to the aperture plane, but placed far from the aperture (relative to the wavelength). We use the coordinate system of Fig. 3, denoting by \underline{x} the orthogonal projection of $\mathbf{x} \in \mathbb{R}^3$ onto the aperture plane Σ (i.e. the xy -axis). When \mathbf{x} is itself in Σ , we simply have $\mathbf{x} = \underline{x}$. Furthermore, in the integral over Σ , the differential surface element reduces to $dS(\mathbf{x}) = d\underline{x}$.

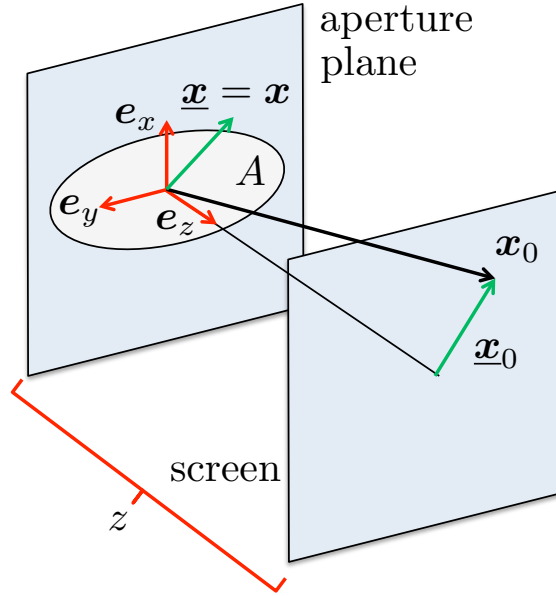


Figure 3: Coordinate system used to analyze the different diffraction regimes. The underscore indicates the component of a vector that lies on the aperture plane. If \mathbf{x} is already in the aperture plane, then $\underline{x} = \mathbf{x}$. When a point of interest \mathbf{x}_0 is far away from the aperture plane but well aligned with \mathbf{x} , $\|\underline{x} - \underline{x}_0\|$ is small relative to $z = \langle \mathbf{e}_z, \mathbf{x}_0 \rangle$.

Let \mathbf{x} be an element of the aperture $A \subset \Sigma$, and \mathbf{x}_0 a point of interest behind the aperture plane with vertical coordinate $z = \langle \mathbf{e}_z, \mathbf{x}_0 \rangle > 0$. Since the observation plane is parallel to the aperture, z is constant in the discussion that follows. Taylor expansion of $\sqrt{1 + x^2}$ gives

$$\begin{aligned} \|\mathbf{x} - \mathbf{x}_0\| &= \sqrt{z^2 + \|\underline{x} - \underline{x}_0\|^2} = z \sqrt{1 + \left(\frac{\|\underline{x} - \underline{x}_0\|}{z} \right)^2} \\ &= z \left[1 + \frac{1}{2} \left(\frac{\|\underline{x} - \underline{x}_0\|}{z} \right)^2 + O \left(\left(\frac{\|\underline{x} - \underline{x}_0\|}{z} \right)^4 \right) \right], \end{aligned} \quad (6)$$

We will neglect higher order terms on the assumption that $\|\underline{x} - \underline{x}_0\| \ll z$, known as the **paraxial approximation** (“near axis”). In doing so, we can make a zeroth order approximation of the

inclination factor

$$\cos(\theta(\mathbf{x}, \mathbf{x}_0)) = \frac{z}{\|\mathbf{x} - \mathbf{x}_0\|} = 1 + O\left(\left(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{z}\right)^2\right) \approx 1.$$

We must be slightly more careful when approximating the phase, as the exponential function is more sensitive to deviations. So we take the first order **Fresnel approximation**

$$\begin{aligned} e^{ik\|\mathbf{x} - \mathbf{x}_0\|} &= e^{ikz \left[1 + \frac{1}{2} \left(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{z}\right)^2 + O\left(\left(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{z}\right)^4\right)\right]} \\ &\approx e^{ikz} e^{\frac{ik}{2z} \|\mathbf{x} - \mathbf{x}_0\|^2}. \end{aligned}$$

These estimates give

$$G(\mathbf{x} - \mathbf{x}_0) = \frac{e^{ik\|\mathbf{x} - \mathbf{x}_0\|}}{\|\mathbf{x} - \mathbf{x}_0\|} \approx \frac{1}{z} e^{ikz} e^{\frac{ik}{2z} \|\mathbf{x} - \mathbf{x}_0\|^2}.$$

If we make the second assumption that our scalar diffraction theory holds, then the Rayleigh-Sommerfeld formula gives

$$u(\mathbf{x}_0) \approx \frac{1}{i\lambda z} e^{ikz} \int_A u(\mathbf{x}) e^{\frac{ik}{2z} \|\mathbf{x} - \mathbf{x}_0\|^2} d\mathbf{x}.$$

We see this formula once again yields a convolution of the wave field on the aperture with the kernel

$$h(\mathbf{x}) = \frac{1}{i\lambda z} e^{ikz} e^{\frac{ik}{2z} \|\mathbf{x}\|^2}.$$

By expanding the square, we deduce **Fresnel's diffraction integral**

$$u(\mathbf{x}_0) = \frac{1}{i\lambda z} e^{ikz} e^{\frac{ik}{2z} \|\mathbf{x}_0\|^2} \int_A u(\mathbf{x}) e^{\frac{ik}{2z} \|\mathbf{x}\|^2} e^{-\frac{ik}{z} \langle \mathbf{x}, \mathbf{x}_0 \rangle} d\mathbf{x}.$$

This is a Fourier transform! It says the diffraction pattern will be the Fourier transform of the wave field on the aperture, after being modulated by a quadratic phase term and scaled by a multiplicative factor.

When is the diffraction integral accurate? The main source of errors will be the approximation used on the phase term. We have used the Taylor expression

$$\sqrt{1 + \varepsilon} = 1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \dots \quad \text{with} \quad \varepsilon = \left(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{z}\right)^2.$$

The Fresnel approximation proposed $e^{ikz\varepsilon^2/8} \approx 1$, which requires

$$\frac{kz\varepsilon^2}{8} = \frac{kz}{8} \frac{\|\mathbf{x} - \mathbf{x}_0\|^4}{z^4} \ll \pi \quad \implies \quad \frac{1}{4\lambda z} \|\mathbf{x} - \mathbf{x}_0\|^2 \left(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{z}\right)^2 \ll 1. \quad (7)$$

We introduce the **characteristic size of the aperture** a^2 , where a is taken such that

$$\|\mathbf{x} - \mathbf{x}_0\| \leq a$$

for any \underline{x} in the aperture and any paraxial point of interest \underline{x}_0 behind the aperture. Now consider that for θ close to zero, we have $\sin \theta \approx \theta$. Hence

$$\theta(\underline{x}, \underline{x}_0) \approx \sin(\theta(\underline{x}, \underline{x}_0)) = \frac{\|\underline{x} - \underline{x}_0\|}{\|\underline{x} - \underline{x}_0\|} \approx \frac{\|\underline{x} - \underline{x}_0\|}{z} \leq \frac{a}{z}.$$

Then defining

$$\theta_{\max} = \frac{a}{z}$$

and **Fresnel's number**

$$F = \frac{a^2}{\lambda z},$$

we obtain that the condition (7) is equivalent to

$$F \frac{\theta_{\max}^2}{4} \ll 1. \quad (8)$$

In practice, the Fresnel number helps determine the regime in which diffraction occurs. Note that in order to obtain Fresnel's integral formula, the quadratic phase term needs to be relevant, that is,

$$\pi \ll \frac{k}{2z} \|\underline{x} - \underline{x}_0\|^2 \implies 1 \ll \frac{a^2}{\lambda z} = F$$

We say there is **Fresnel diffraction**, or **near-field diffraction**, when $F \gg 1$. On the other hand, we have argued that Fresnel's diffraction integral will be accurate if (8) is satisfied. For instance, consider a circular aperture with a radius of 1cm, and light with wavelength $\lambda = 0.5\mu\text{m}$. Then we can take

$$a^2 = \pi \text{cm}^2 \implies a = \sqrt{\pi} \text{cm},$$

and in order to have a good approximation, we need

$$F \frac{\theta_{\max}^2}{4} = \left(\frac{(\pi \times 10^{-4})}{(0.5 \times 10^{-6})} \frac{1}{z} \right) \left(\frac{(\pi \times 10^{-4})}{4} \frac{1}{z^2} \right) = \frac{\pi^2}{200} \frac{1}{z^3} \ll 1 \implies z \gg 37\text{cm},$$

For $z = 37\text{cm}$, we have

$$F = \frac{(\pi \times 10^{-4})}{(0.5 \times 10^{-6})(37 \times 10^{-2})} \approx 15 \quad \text{and} \quad \theta_{\max} \approx 2.74^\circ.$$

In reality, Fresnel's diffraction integral can be accurate even outside the regimes described. In particular, the higher order terms in (6) need not be small, but rather just not change the value of the integral significantly.

5 Fraunhofer diffraction

In Fresnel diffraction, we have a quadratic phase term modulating the wave field on the aperture. This quadratic term is relevant because $F \gg 1$. However, the quadratic phase term can be neglected when $F \ll 1$. In this **Fraunhofer regime**, we say **Fraunhofer diffraction**, or **far-field diffraction**, occurs. In this case, we obtain

$$u(\underline{x}_0) = \frac{1}{i\lambda z} e^{ikz} e^{\frac{ik}{2z} \|\underline{x}_0\|^2} \int_A u(\underline{x}) e^{-\frac{ik}{z} \langle \underline{x}, \underline{x}_0 \rangle} d\underline{x}.$$

Up to modulation by a quadratic phase factor, this is simply the Fourier transform of the wave field at the aperture. That is,

$$u(\underline{\mathbf{x}}_0) = \frac{1}{i\lambda z} e^{ikz} e^{\frac{ik}{2z} \|\underline{\mathbf{x}}_0\|^2} \hat{u}\left(\frac{k}{z} \underline{\mathbf{x}}_0\right).$$

We have assumed $u = 0$ on the part of the aperture plane away from the aperture itself. Continuing our previous example, we require

$$F = \frac{\pi \times 10^{-4}}{0.5 \times 10^{-6}} \frac{1}{z} \ll 1 \quad \implies \quad z \gg 628\text{m}.$$

While this distance seems excessive, we can actually observe the effects of Fraunhofer diffraction at much shorter distances by using lenses². We will discuss this in the next lecture.

6 Examples

In this section we compute two Fraunhofer diffraction patterns, of which we saw pictures in Lecture 15.

6.1 Rectangular aperture

Suppose we have a rectangular aperture aligned with the coordinate axis and with side lengths a_x and a_y (see, for instance, Fig. 4a). If we assume the wave field has constant intensity u_0 across the aperture, we see that in the Fraunhofer regime

$$u(\underline{\mathbf{x}}_0) = \frac{u_0 a_x a_y}{i\lambda z} e^{ikz} e^{\frac{ik}{2z} \|\underline{\mathbf{x}}_0\|^2} \text{sinc}\left(\frac{ka_x}{2\pi z} \langle \mathbf{e}_x, \underline{\mathbf{x}}_0 \rangle\right) \text{sinc}\left(\frac{ka_y}{2\pi z} \langle \mathbf{e}_y, \underline{\mathbf{x}}_0 \rangle\right).$$

On the screen we will observe the **intensity** (*not* phase) of the wave field:

$$I(\underline{\mathbf{x}}_0) = |u(\underline{\mathbf{x}}_0)|^2 = \left(\frac{u_0 |A|}{4\lambda z}\right)^2 \text{sinc}^2\left(\frac{ka_x}{2\pi z} \langle \mathbf{e}_x, \underline{\mathbf{x}}_0 \rangle\right) \text{sinc}^2\left(\frac{ka_y}{2\pi z} \langle \mathbf{e}_y, \underline{\mathbf{x}}_0 \rangle\right),$$

where $|A|$ is the area of the aperture (see Fig. 4b). Amazingly, we observe dark lines separating illuminated regions. These lines occur whenever

$$\langle \mathbf{e}_x, \underline{\mathbf{x}}_0 \rangle = \frac{2\lambda z}{a_x} n \quad \text{or} \quad \langle \mathbf{e}_y, \underline{\mathbf{x}}_0 \rangle = \frac{2\lambda z}{a_y} n \quad \text{for an integer } n.$$

6.2 Circular apertures

Suppose now we have a circular aperture of radius a (see Fig. 4c). We have that

$$\int_{\|\underline{\mathbf{x}}\| \leq 1} d\underline{\mathbf{x}} = \frac{J_1(\|\underline{\mathbf{x}}\|)}{\|\underline{\mathbf{x}}\|},$$

²Another method is to have waves converging at the aperture.

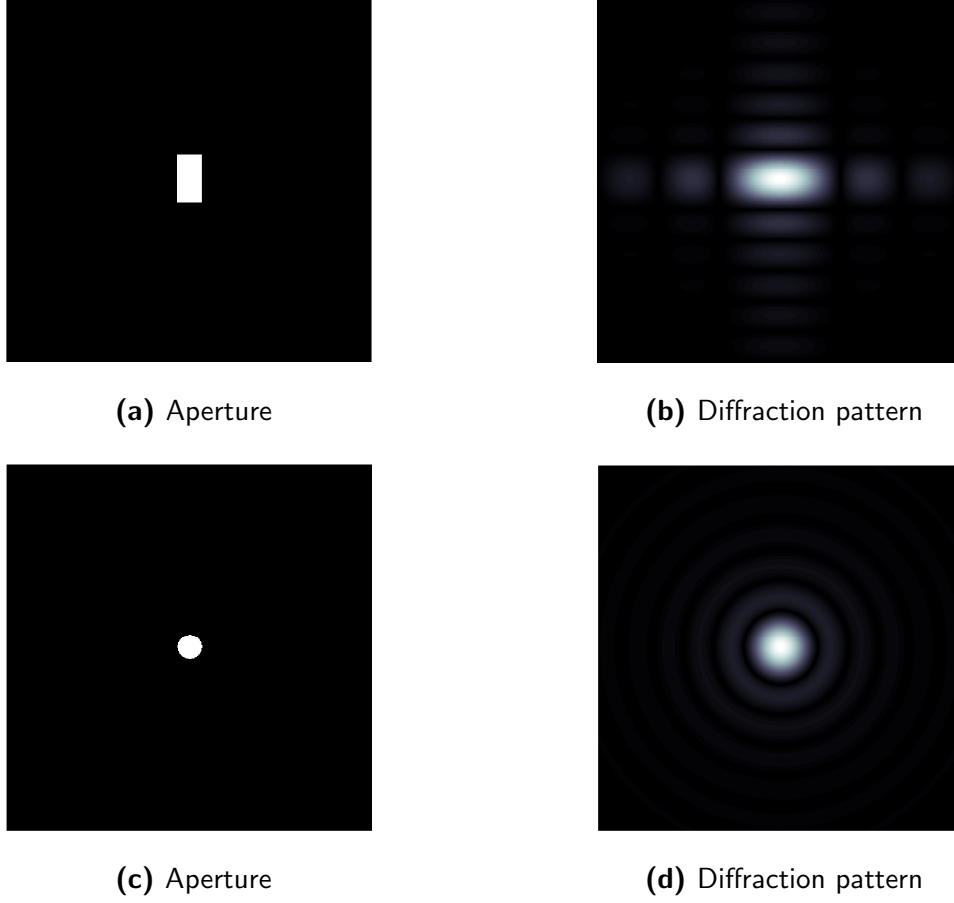


Figure 4: Diffraction patterns for some apertures in the Fraunhofer regime. **(a)** Rectangular aperture with $a_x = 1$, $a_y = 2$. **(b)** Diffraction pattern of the rectangular aperture. Note the rectangular pattern and regions of dark lines. **(c)** Circular aperture with $a = 1$. **(d)** Diffraction pattern of the circular aperture. Note the circular pattern and the dark, concentric circles.

where J_1 is the first Bessel function of the first kind. The pattern we observe in the Fraunhofer regime is (see Fig. 4d)

$$u(\underline{x}_0) = \frac{u_0 a}{i2\pi} e^{ikz} e^{\frac{ik}{z} \|\underline{x}_0\|^2} \frac{J_1\left(\frac{ak}{z} \|\underline{x}_0\|\right)}{\|\underline{x}_0\|},$$

and its intensity becomes

$$I(\underline{x}_0) = |u(\underline{x}_0)|^2 = \left(\frac{u_0 a}{2\pi}\right)^2 \frac{|J_1\left(\frac{ak}{z} \|\underline{x}_0\|\right)|^2}{\|\underline{x}_0\|^2}.$$

In this case we observe dark *circles*, occurring concentrically at radii

$$\|\underline{x}_0\| = \frac{z}{ak} j_{1,n} \quad \text{where } j_{1,n} \text{ is the } n\text{th positive root of } J_1.$$

In particular, the first circular dark spot occurs at

$$\frac{z}{ak} j_{1,1} = z \frac{\lambda}{2a} \frac{j_{1,1}}{\pi} \approx 1.22 \frac{\lambda}{d} z,$$

where d is the diameter of the circular aperture. The central bright region it encloses is called the **Airy disk**.

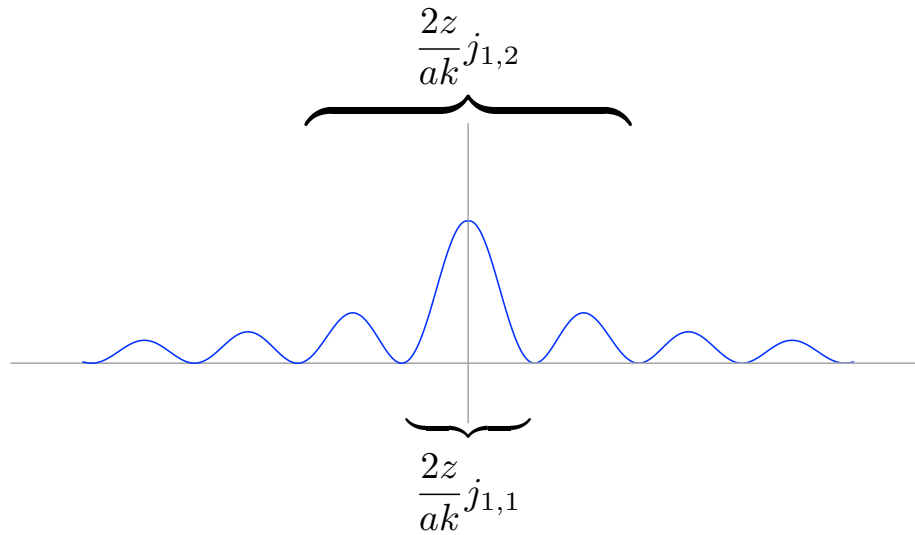


Figure 5: First Bessel function of the first kind, shown on the real axis. This corresponds to a radial cross section of the intensity function for Fraunhofer diffraction with a circular aperture. The widths of the first and second lobes are indicated.

6.3 Other apertures

Fig. 6 and Fig. 7 show diffraction patterns created by other shapes of apertures.

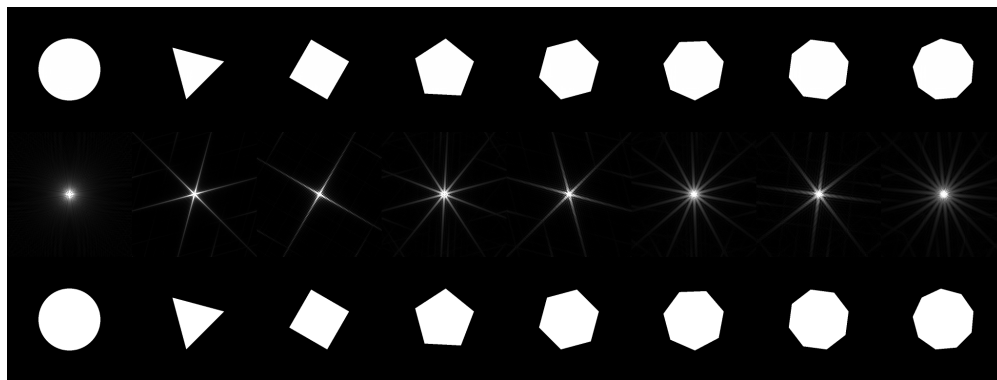


Figure 6: Numerical predictions of Fraunhofer diffraction from various rotated, regular apertures

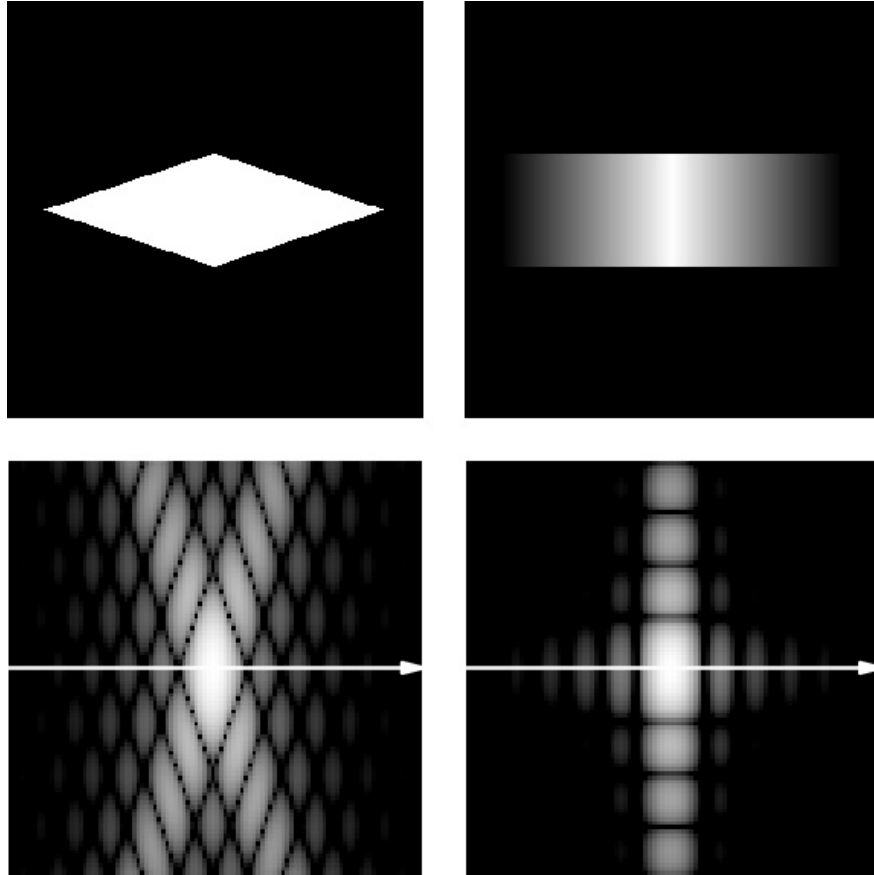


Figure 7: Calculated Fraunhofer diffraction patterns for diamond and rectangular apertures. Note the effect of rotation for the rectangular aperture by comparing the right two images to Fig. 4(a) and 4(b).