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Question ①

The viscous Burger's equation

$$u_t + u u_x = \alpha u_{xx}$$

periodic domain size $= 1.0$, $\Delta t = 0.0004$

$t_{end} = 0.075$

$$u(x,0) = \sin(4\pi x) + \sin(6\pi x) + \sin(10\pi x)$$

ⓐ using Euler and 1st order upwind scheme with $\alpha = 0$,

(i) For $u > 0$, we use backward difference,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \left[\frac{u_i^n - u_{i-1}^n}{\Delta x} \right] = 0$$

$$\Rightarrow u_i^{n+1} = \left(1 - \frac{\Delta t}{\Delta x} u_i^n\right) u_i^n + \left(\frac{\Delta t}{\Delta x} u_i^n\right) u_{i-1}^n$$

(ii) For $u < 0$, we use forward difference

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \left[\frac{u_{i+1}^n - u_i^n}{\Delta x} \right] = 0$$

$$\Rightarrow u_i^{n+1} = \left(1 + \frac{\Delta t}{\Delta x} u_i^n\right) u_i^n - \left(\frac{\Delta t}{\Delta x} u_i^n\right) u_{i+1}^n$$

ⓑ using Euler and 2nd order central difference

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

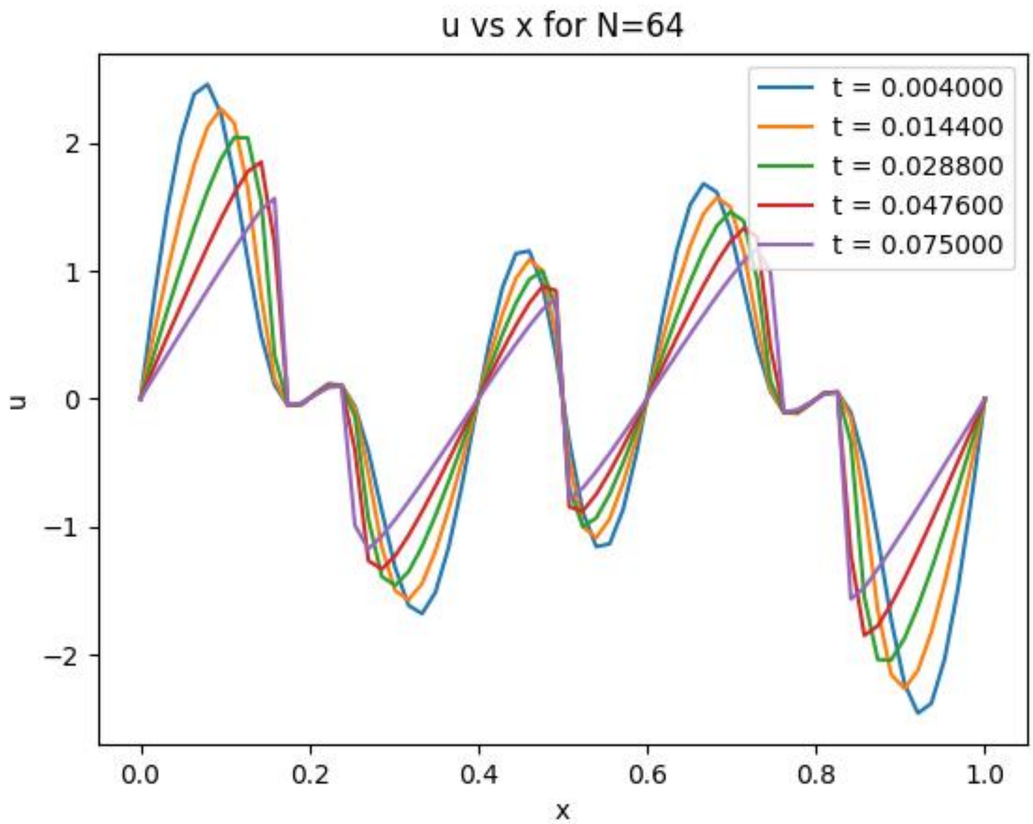
$$\Rightarrow$$

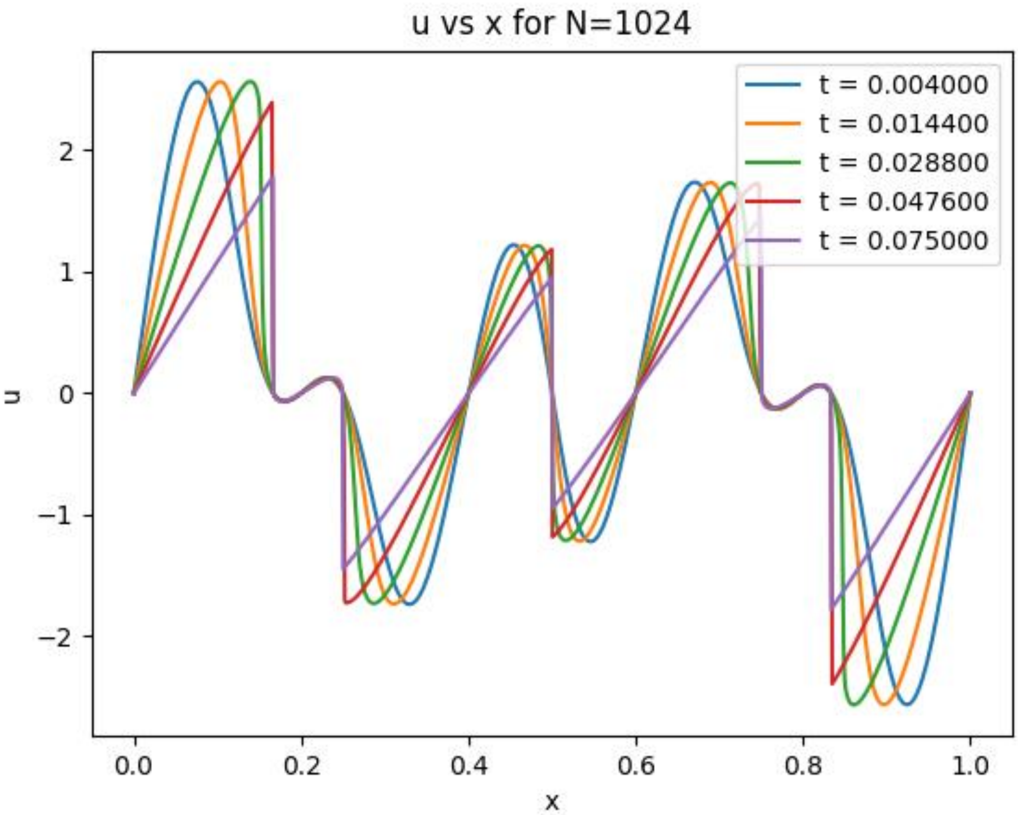
$$u_i^{n+1} = \left(1 - 2\alpha \frac{\Delta t}{\Delta x^2}\right) u_i^n + \left(-\frac{u_i^n \Delta t}{2\Delta x} + \alpha \frac{\Delta t}{\Delta x^2}\right) u_{i+1}^n + \left(\frac{\alpha \Delta t}{\Delta x^2} + \frac{u_i^n \Delta t}{2\Delta x}\right) u_{i-1}^n$$

Question 1(a): The observation is that the amplitude of the solution is decreasing with x for the case of 64 grid points, while the amplitude of the solution remains stable before the actual decay (that is for $t = 0.0476$) for the case of 1024 grid points.

This behavior indicates that numerical dissipation is higher in the case of 64 grid points. The reason for this is that the value of Δx will be higher in the case of 64 grid points, which will result in larger dissipation due to the order of accuracy being $O(\Delta x,\ \Delta t)$. This shows that truncation error which causes dissipation error will be higher for grid points=64 compared to the grid points=1024.In contrast, the higher number of grid points in the case of 1024 grid points results in a smoother curve, and the solution converges faster than 64 grid points.

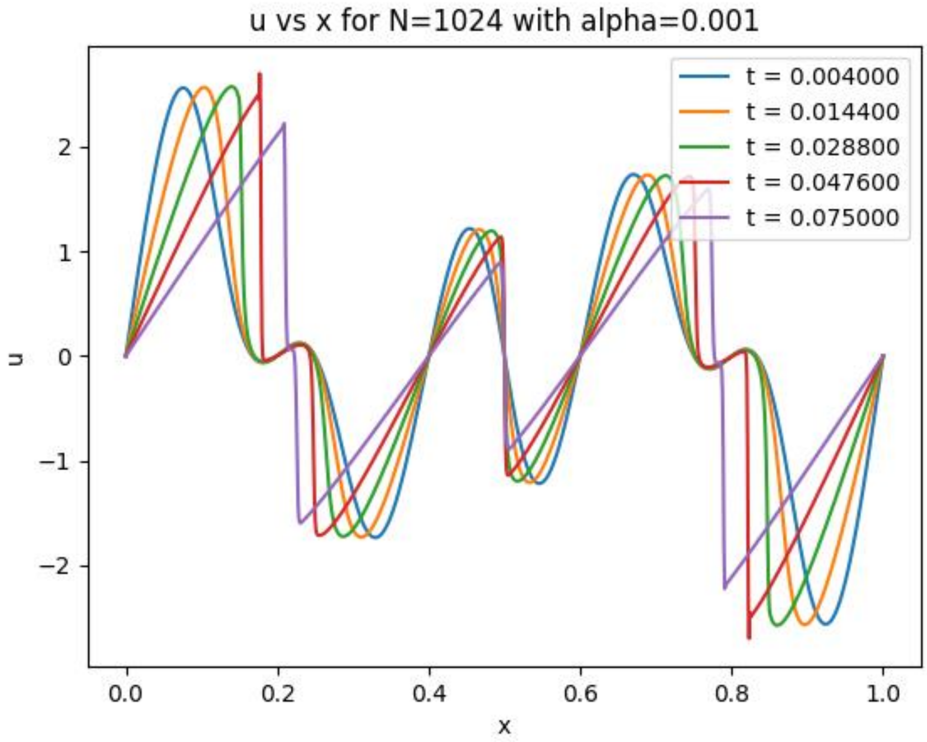
However, the solution exhibits dispersive errors in the form of sharp turns, which is likely due to the equation representing some sort of discontinuity.

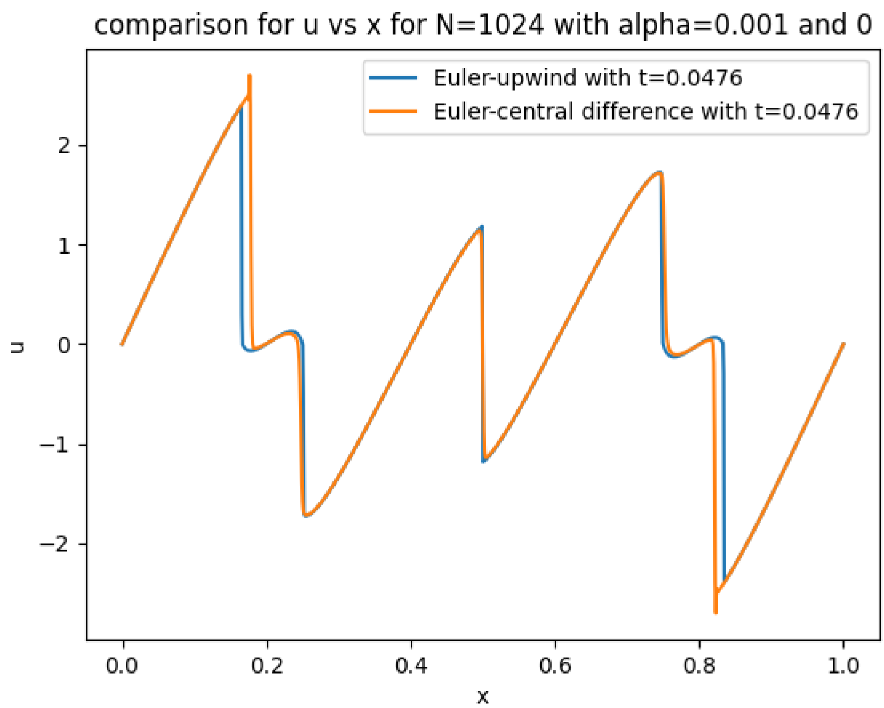


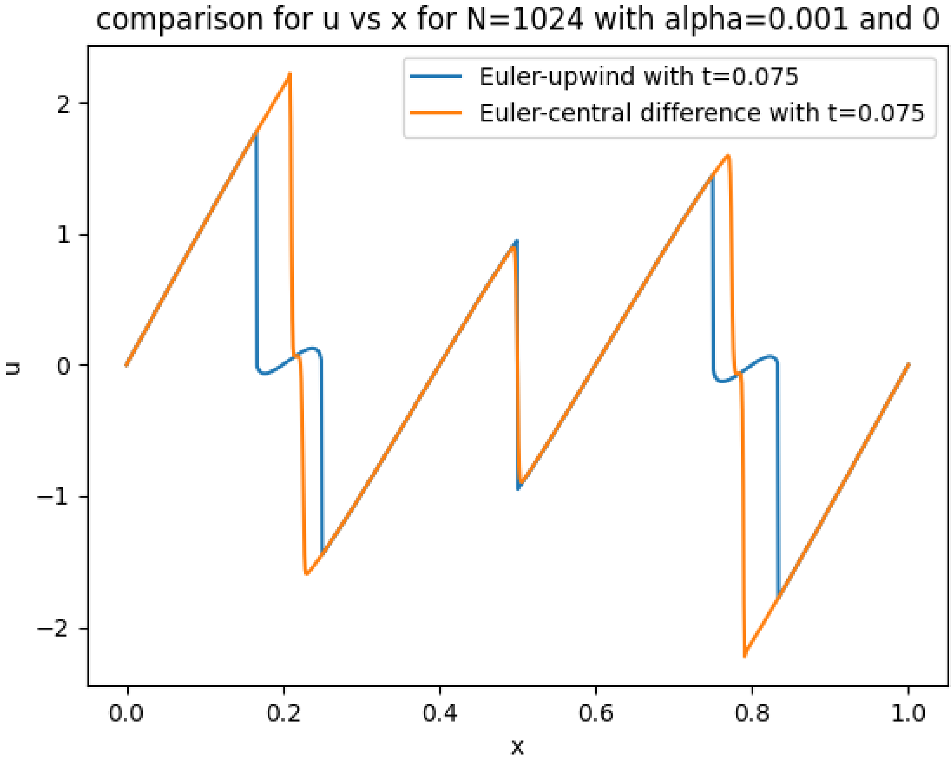


Question 1(b): Here order of accuracy will be $O((\Delta x)^2, \Delta t)$, we expect to have better accuracy than first order methods. When solving the viscous Burgers' equation using the Euler-2nd order central difference scheme, with a viscosity term of $0.001u_{xx}$, it is observed that the solution is less dissipative than the previous case. This is because the viscosity term acts as a regularizing term and dampens out high-frequency oscillations, reducing numerical dissipation. However, the solution still exhibits dispersive errors in the form of sharp turns, which is a consequence of the second-order central difference approximation. Overall, the numerical solution is a balance between dispersive and dissipative errors, with the viscosity term reducing dissipation but increasing dispersion due to the second-order approximation.

For comparison between part(a) and part(b) for grid sizes=1024, we can observe that dissipation error for $\alpha = 0$ is more than dissipation error for $\alpha = 0.001$ and dispersive error for $\alpha = 0.001$ is more than $\alpha = 0$.


u vs x for N=1024 with alpha=0.001
t = 0.004000
t = 0.014400
t = 0.028800
t = 0.047600
t = 0.075000
u
x
0.0
0.2
0.4
0.6
0.8
1.0
2
1
0
−1
−2


comparison for u vs x for N=1024 with alpha=0.001 and 0
Euler-upwind with t=0.0476
Euler-central difference with t=0.0476
u
x
0.0
0.2
0.4
0.6
0.8
1.0
2
1
0
−1
−2


comparison for u vs x for N=1024 with alpha=0.001 and 0
Euler-upwind with t=0.075
Euler-central difference with t=0.075
u
x
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0

Question 2

The linear wave eq^n: $u_t + cu_x = 0$

Here we use Leap frog method for time derivative and 2nd order central difference for space derivative.

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

$$\Rightarrow u_j^{n+1} - u_j^{n-1} + \frac{c\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) = 0$$

Here we apply von-neumann stability

$$u_j^n = v^n e^{ikx_j}$$

Then $v^{n+1} e^{ikx_j} - v^{n-1} e^{ikx_j} + \frac{c\Delta t}{\Delta x}\left(v^n e^{ik(x_j+\Delta x)} - v^n e^{ik(x_j-\Delta x)}\right) = 0$

Dividing $v^{n-1} e^{ikx_j}$

$$\Rightarrow v^2 - 1 + \frac{c\Delta t}{\Delta x} v \left(e^{ik\Delta x} - e^{-ik\Delta x}\right) = 0$$

$$\Rightarrow v^2 - \left(\frac{c\Delta t}{\Delta x}\cdot 2i\sin(k\Delta x)\right) v - 1 = 0$$

Solving this eq^n

$$v = \frac{-\frac{c\Delta t}{\Delta x}\cdot 2i\sin(k\Delta x) \pm \sqrt{-\left(\frac{c\Delta t}{\Delta x}\right)^2 4\sin^2(k\Delta x) + 4}}{2}$$

$$\Rightarrow v = -i\left(\frac{c\Delta t}{\Delta x}\right)\sin(k\Delta x) \pm \sqrt{1 - \left(\frac{c\Delta t}{\Delta x}\sin(k\Delta x)\right)^2}$$

Now we see that when $1 - \left(\frac{c\Delta t}{\Delta x}\sin k\Delta x\right)^2 < 0$

v is purely imaginary.

Now the product of two roots $= -1$

since 'i' is not a root of that eq^n. one of the roots will have magnitude greater than 1 which leads to instability.

Hence $1 - \left(\frac{c\Delta t}{\Delta x}\sin k\Delta x\right)^2 \geq 0$

Then $|v| = 1 - \left(\frac{c\Delta t}{\Delta x}\sin k\Delta x\right)^2 + \left(\frac{c\Delta t}{\Delta x}\sin k\Delta x\right)^2$

$= 1$

This is consider as neutral stability and wave will propagate with constant amplitude.

Now

$$1-\left(\frac{c\Delta t}{\Delta x}\sin k\Delta x\right)^2 \geq 0$$

$$\Rightarrow \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x) \leq 1$$

$$\Rightarrow \left(\frac{c\Delta t}{\Delta x}\right)^2 \leq 1 \qquad (\because \max(\sin k\Delta x)=1)$$

$$\Rightarrow \left|\frac{c\Delta t}{\Delta x}\right| \leq 1$$ This is our stability condition

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Modified equation:

we have

$$u_i^{n+1} - u_i^{n-1} + \frac{c\Delta t}{\Delta x}(u_{i+1}^n - u_{i-1}^n) = 0 \quad \text{—} (1)$$

using Taylor series expansion

[$\dot{u}$ denotes $\frac{\partial u}{\partial t}$, u' denotes $\frac{\partial u}{\partial x}$]

$$u_i^{n+1} = u_i^n + \dot{u}_i^n \Delta t + \ddot{u}_i^n \frac{(\Delta t)^2}{2!} + \dddot{u}_i^n \frac{(\Delta t)^3}{3!} + \cdots$$

$$u_i^{n-1} = u_i^n - \dot{u}_i^n \Delta t + \ddot{u}_i^n \frac{(\Delta t)^2}{2!} - \dddot{u}_i^n \frac{(\Delta t)^3}{3!} + \cdots$$

$$u_{i+1}^n = u_i^n + u_i'^n \Delta x + u_i''^n \frac{(\Delta x)^2}{2!} + u_i'''^n \frac{(\Delta x)^3}{3!} + \cdots$$

$$u_{i-1}^n = u_i^n - u_i'^n \Delta x + u_i''^n \frac{(\Delta x)^2}{2!} - u_i'''^n \frac{(\Delta x)^3}{3!} + \cdots$$

putting this in (1)

$$\left(\dot{u}_i^n \Delta t + \dddot{u}_i^n \frac{(\Delta t)^3}{3!} + \cdots\right) + \frac{c\Delta t}{\Delta x}\left(u_i'^n \Delta x + u_i'''^n \frac{\Delta x^3}{3!} + \cdots\right) = 0$$

$$\Rightarrow \left(\dot{u}_i^n + \dddot{u}_i^n \frac{(\Delta t)^2}{3!} + \cdots\right) + c\left(u_i'^n + u_i'''^n \frac{(\Delta x)^2}{3!} + \cdots\right) = 0$$

$$\Rightarrow \dot{u}_i^n + c u_i'^n = \left(-\dddot{u}_i^n \frac{(\Delta t)^2}{3!} - c\, u_i'''^n \frac{(\Delta x)^2}{3!}\right) - \cdots \quad \text{—} (2)$$

This is the modified eq^n.

Now we change the time derivative into space derivative

Now $$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

$$\Rightarrow \dot{u} = -c u'$$

$$\Rightarrow \ddot{u} = +\frac{\partial}{\partial t}(u')(-c) = c^2 u''$$

$$\Rightarrow \dddot{u} = (-c)^3 u'''$$

From (2), $$\dot{u}_i^n + c u_i''^n = c^3 u_i'''^n \frac{(\Delta t)^2}{3!} - c\, u_i'''^n \frac{(\Delta x)^2}{3!} - \cdots$$

so the leading order term in truncation error

$$= \left(c^3 \frac{(\Delta t)^2}{3!} - c\frac{(\Delta x)^2}{3!}\right) u_i'''^n$$ and it contains odd order derivative so, there is dispersive error.

$$\therefore TE = O(\Delta t^2, \Delta x^2)$$

Question 3 |

ⓐ Beam on elastic foundation

$$\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right)+kw=f \quad \text{for } 0<x<L$$

$$w=b\frac{d^2w}{dx^2}=0 \quad \text{at } x=0,L$$

suppose w is exact solution and w_n is the approximate and z_i is the corresponding weight

Now Residual $R=\frac{d^2}{dx^2}\left(b\frac{d^2w_n}{dx^2}\right)+kw_n-f$

using the method of weighted residuals

$$\langle R, z_i\rangle=0$$

$$\Rightarrow \int_0^L\left(\frac{d^2}{dx^2}\left(b\frac{d^2w_n}{dx^2}\right)+kw_n-f\right)z_i\,dx=0$$

$$\Rightarrow \int_0^L z_i\frac{d^2}{dx^2}\left(b\frac{d^2w_n}{dx^2}\right)dx+\int_0^L(kw_n-f)z_i\,dx=0 \qquad \text{①}$$

using integration by parts for the 1st term,

$$\int_0^L z_i\frac{d^2}{dx^2}\left(b\frac{d^2w_n}{dx^2}\right)dx$$

$$=\left[z_i\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)\right]_0^L-\int_0^L\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)\frac{dz_i}{dx}\,dx$$

$$=\left[z_i\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)\right]_0^L+\int_0^L b\frac{d^2w_n}{dx^2}\frac{d^2z_i}{dx^2}\,dx-\left[b\frac{d^2w_n}{dx^2}\frac{dz_i}{dx}\right]_0^L$$

$$=z_i(L)\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)\Big|_{x=L}-z_i(0)\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)\Big|_{x=0}$$
$$+\int_0^L b\frac{d^2w_n}{dx^2}\frac{d^2z_i}{dx^2}\,dx-b\frac{d^2w_n}{dx^2}\Big|_{x=L}\frac{dz_i}{dx}\Big|_{x=L}+b\frac{d^2w_n}{dx^2}\Big|_{x=0}\frac{dz_i}{dx}\Big|_{x=0}$$

(using boundary condition)

$$=z_i(L)\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)\Big|_{x=L}-z_i(0)\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)\Big|_{x=0}$$
$$+\int_0^L b\frac{d^2w_n}{dx^2}\frac{d^2z_i}{dx^2}\,dx \qquad \text{②}$$

Putting this in eq^n ①

$$z_i(L)\frac{d}{dx}\left(b\frac{d^2w_h}{dx^2}\right)\Big|_{x=L} - z_i(0)\frac{d}{dx}\left(b\frac{d^2w_h}{dx^2}\right)\Big|_{x=0}$$

$$+\int_0^L b\frac{d^2w_h}{dx^2}\frac{d^2z_i}{dx^2}\,dx + \int_0^L (Kw_h - f)z_i\,dx = 0$$

$$\Rightarrow \int_0^L\left[b\frac{d^2w_h}{dx^2}\frac{d^2z_i}{dx^2} + (Kw_h - f)z_i\right]dx + z_i(L)\frac{d}{dx}\left(b\frac{d^2w_h}{dx^2}\right)\Big|_{x=L}$$

$$- z_i(0)\frac{d}{dx}\left(b\frac{d^2w_h}{dx^2}\right)\Big|_{x=0}$$

with the boundary condition

$$w_h = b\frac{d^2w_h}{dx^2} = 0 \quad \text{at } x = 0, L$$

This is the require weak form

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3 (b)

The nonlinear equation

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right) + f = 0 \quad \text{for } 0 < x < 1$$

$$\frac{du}{dx}\Big|_{x=0} = 0 \quad , \quad u(1) = \sqrt{2}$$

Suppose u_h is the approximate solution,

Then Residual $R = -\frac{d}{dx}\left(u_h\frac{du_h}{dx}\right) + f$

Let w_i is the weighting function then by method of weighted residual

$$\int_0^1 R\,w_i\,dx = 0$$

$$\Rightarrow \int_0^1\left[-\frac{d}{dx}\left(u_h\frac{du_h}{dx}\right) + f\right]w_i\,dx = 0$$

$$\Rightarrow \int_0^1 -\frac{d}{dx}\left(u_h\frac{du_h}{dx}\right)w_i\,dx + \int_0^1 f\,w_i\,dx = 0$$

Now integrating by parts,

$$\left[-\left(u_h\frac{du_h}{dx}\right)w_i\right]_0^1 + \int_0^1 u_h\frac{du_h}{dx}\frac{dw_i}{dx}\,dx + \int_0^1 f\,w_i\,dx$$

$$\Rightarrow -u_h(1)\frac{du_h}{dx}\Big|_{x=1}w_i(1) + u_h(0)\frac{du_h}{dx}\Big|_{x=0}w_i(0) + \int_0^1\left(u_h\frac{du_h}{dx}\frac{dw_i}{dx} + f\,w_i\right)dx$$

$$= 0$$

$$\Rightarrow \int_0^1 \left(u_h \frac{du_h}{dx} \frac{dw_i}{dx} + f w_i\right) dx - \sqrt{2} \left.\frac{du_h}{dx}\right|_{x=1} w_i(1) = 0$$

with boundary condition

$$\left.\frac{du_h}{dx}\right|_{x=0} = 0 \quad u(1) = \sqrt{2}$$

This is the required weak form.

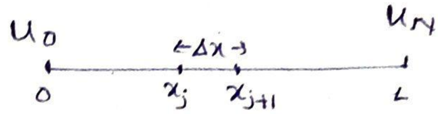
Q.4 Poisson equation that occurs in the following model for vertical deflection of a bar with a distributed load $P(x)$

$$A_c E \frac{d^2u}{dx^2} = P(x) \quad \text{—— ①}$$

Given that $u=0$ at both ends

$A_c = 0.1\ m^2$, $E = 200\times10^9\ N/m^2$, $L = 10\ m$. $P(x) = 100\ N/m$

Then ① becomes



$$0.1\times200\times10^9\ \frac{d^2u}{dx^2} = 100$$

$$\Rightarrow \frac{d^2u}{dx^2} = 5\times10^{-9}$$

suppose u_h be the approximate function then residual $R = \frac{d^2u_h}{dx^2} - 5\times10^{-9}$

and suppose w_i are the weighting function then then for a single element

$$\int_{x_j}^{x_{j+1}} R.\,w_i\,dx = 0$$

$$\Rightarrow \int_{x_j}^{x_{j+1}} \left(\frac{d^2u_h}{dx} - 5\times10^{-9}\right) w_i\,dx = 0$$

$$\Rightarrow \int_{x_j}^{x_{j+1}} w_i \frac{d^2u_h}{dx^2}\,dx = \int_{x_j}^{x_{j+1}} 5\times10^{-9}\,w_i\,dx$$

$$\Rightarrow \left[w_i \frac{du_h}{dx}\right]_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} \frac{dw_i}{dx}\frac{du_h}{dx}\,dx = \int_{x_j}^{x_{j+1}} 5\times10^{-9}\,w_i\,dx$$

Now weak form contains only 1st derivative so any function with non zero derivative can be an interpolant.

suppose $u_h(x) = c_1 + c_2 x$

Then $u_j = c_1 + c_2 x_j$ $\quad u_{j+1} = c_1 + c_2 x_{j+1}$

Gives $u_h(x) = \frac{x - x_{j+1}}{x_j - x_{j+1}} u_j + \frac{x - x_j}{x_{j+1} - x_j} u_{j+1}$

$= L_1(x)\,u_j + L_2(x)\,u_{j+1}$ $\quad$ L_1, L_2 are Lagrangian Polynomial

Now we use Galerkian method i.e $w_i = L_1$ or L_2
$i=1$ or 2

$$\int_{x_j}^{x_{j+1}} \frac{dw_i}{dx}\frac{du_h}{dx}\,dx - \left[w_i \frac{du_h}{dx}\right]_{x_j}^{x_{j+1}} = -\int_{x_j}^{x_{j+1}} 5\times10^{-9}\,\frac{x-x_{j+1}}{x_j-x_{j+1}}\,dx$$

$$= +5\times10^{-9}\,\frac{1}{\Delta x}\left[\frac{1}{2}\left(x_{j+1}^2-x_j^2\right) - x_{j+1}\left(x_{j+1}-x_j\right)\right]$$

$$= +5\times10^{-9}\left[\frac{1}{2}\left(x_{j+1}+x_j\right) - x_{j+1}\right]$$

$$= -\frac{5\times10^{-9}}{2}\times\Delta x = -1.25\times10^{-9}$$

$(\Delta x = 0.5)$

$$\Rightarrow \int_{x_j}^{x_{j+1}} \frac{dw_i}{dx}\frac{du_h}{dx}\,dx = \left[w_i\frac{du_h}{dx}\right]_{x_j}^{x_{j+1}} - 1.25\times10^{-9} \quad \text{--- (2)}$$

Now for $w_i = L_1(x) = \dfrac{x-x_{j+1}}{x_j-x_{j+1}}$ (for Lagrange polynomial)

$$= -\frac{x-x_{j+1}}{\Delta x}$$

From (2),

$$\int_{x_j}^{x_{j+1}} -\frac{1}{\Delta x}\left(-\frac{1}{\Delta x}u_j + \frac{1}{\Delta x}u_{j+1}\right)dx$$

$$= L_1(x_{j+1})\left.\frac{du_h}{dx}\right|_{x_{j+1}} - L_1(x_j)\left.\frac{du_h}{dx}\right|_{x_j} - 1.25\times10^{-9}$$

$$\Rightarrow \boxed{(u_j - u_{j+1})\frac{1}{\Delta x} = -\frac{du_h(x_j)}{dx} - 1.25\times10^{-9}} \quad \text{--- (3)}$$

From (2)
Again putting $w_i = L_2$ for 2nd Lagrange polynomial

$$= \frac{x-x_j}{x_{j+1}-x_j} = \frac{1}{\Delta x}(x-x_j)$$

$$\int_{x_j}^{x_{j+1}} \frac{1}{\Delta x}\left(-\frac{1}{\Delta x}u_j + \frac{1}{\Delta x}u_{j+1}\right)dx = L_2(x_{j+1})\frac{du_h(x_{j+1})}{dx} - L_2(x_j)\frac{du_h(x_j)}{dx} - 1.25\times10^{-9}$$

$$\Rightarrow \boxed{-\frac{1}{\Delta x}u_j + \frac{1}{\Delta x}u_{j+1} = +\frac{du_h(x_{j+1})}{dx} - 1.25\times10^{-9}} \quad \text{--- (4)}$$

Now for a particular jth element the matrix is

$$\begin{pmatrix} \frac{1}{\Delta x} & -\frac{1}{\Delta x} \\ -\frac{1}{\Delta x} & \frac{1}{\Delta x} \end{pmatrix} \begin{pmatrix} u_j \\ u_{j+1} \end{pmatrix} = \begin{pmatrix} -\frac{du_n(x_j)}{dx} - 1.25 \times 10^{-9} \\ +\frac{du_n(x_{j+1})}{dx} - 1.25 \times 10^{-9} \end{pmatrix}$$

similiarly for the next element i.e for (j+1)th

$$\begin{pmatrix} \frac{1}{\Delta x} & -\frac{1}{\Delta x} \\ -\frac{1}{\Delta x} & \frac{1}{\Delta x} \end{pmatrix} \begin{pmatrix} u_{j+1} \\ u_{j+2} \end{pmatrix} = \begin{pmatrix} -\frac{du_n(x_{j+1})}{dx} - 1.25 \times 10^{-9} \\ \frac{du_n}{dx}(x_{j+2}) - 1.25 \times 10^{-9} \end{pmatrix}$$

Combining all the N elements we have

$$\begin{pmatrix} \frac{1}{\Delta x} & 0 & 0 & \cdots & & \cdots & 0 \\ -\frac{1}{\Delta x} & \frac{2}{\Delta x} & -\frac{1}{\Delta x} & \cdots & & \cdots & 0 \\ 0 & -\frac{1}{\Delta x} & \frac{2}{\Delta x} & -\frac{1}{\Delta x} & \cdots & & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & -\frac{1}{\Delta x} & \frac{2}{\Delta x} & & -\frac{1}{\Delta x} \\ 0 & 0 & \cdots & & \cdots & & \frac{1}{\Delta x} \end{pmatrix} \begin{pmatrix} u_0 \\ \vdots \\ \vdots \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ -2.5 \times 10^{-9} \\ \vdots \\ \vdots \\ -2.5 \times 10^{-9} \\ 0 \end{pmatrix}$$

Where the first and last row for $u=0$ at both ends, rest of the part is tri diagonal system

Question 4: To solve this problem using the finite-element method, the bar is divided into discrete elements with a fixed length of$\Delta x = 0.5\ m$. The deflection at each node is then approximated using a trial function that is a linear combination of shape functions defined on each element. The shape functions are chosen to satisfy the boundary conditions and to ensure that the trial function is continuous across element boundaries. The coefficients of the shape functions are determined by minimizing the energy functional of the system, which results in a set of linear equations that can be solved for the deflection at each node.

The resulting plot shows the deflection profile of the bar along its length. The deflection is maximum at the center of the bar, where the load is applied, and decreases towards the endpoints, where it is fixed. Therefore it has a minima at close to $x = 5$ as expected .The deflection curve is smooth and concave downwards, which is consistent with the physical behavior of a loaded bar. The deflection at the endpoints is zero, as expected from the boundary conditions.