DS 289 : NSDE

Assignment - 1

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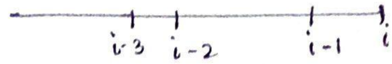
Problem 1 :



(a) Here we assume the grid points are $x_i, x_{i-1}, x_{i-2}, x_{i-3}$.

and $x_i - x_{i-1} = \Delta x_{i-1}$, $\quad u(x_i) = u_i$

$x_i - x_{i-2} = \Delta x_{i-2}$, $\quad u(x_{i-1}) = u_{i-1}$

$x_i - x_{i-3} = \Delta x_{i-3}$. $\quad u(x_{i-2}) = u_{i-2}$

$u(x_{i-3}) = u_{i-3}$

We seek approximation of the form

$$a_0 u_i + a_1 u_{i-1} + a_2 u_{i-2} + a_3 u_{i-3} \simeq u_i''$$

Now using Taylor series expansion.

$$u(x_i) = u_i$$

$$u(x_{i-1}) = u_{i-1} = u_i - \Delta x_{i-1} u_i' + \frac{\Delta x_{i-1}^2}{2} u_i'' - \frac{\Delta x_{i-1}^3}{3!} u_i''' + \frac{\Delta x_{i-1}^4}{4!} u_i^{(iv)} + \cdots$$

$$u(x_{i-2}) = u_{i-2} = u_i - \Delta x_{i-2} u_i' + \frac{\Delta x_{i-2}^2}{2} u_i'' - \frac{\Delta x_{i-2}^3}{3!} u_i''' + \frac{\Delta x_{i-2}^4}{4!} u_i^{(iv)} + \cdots$$

$$u(x_{i-3}) = u_{i-3} = u_i - \Delta x_{i-3} u_i' + \frac{\Delta x_{i-3}^2}{2} u_i'' - \frac{\Delta x_{i-3}^3}{3!} u_i''' + \frac{\Delta x_{i-3}^4}{4!} u_i^{(iv)} + \cdots$$

Then we get the equation in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -\Delta x_{i-1} & -\Delta x_{i-2} & -\Delta x_{i-3} \\ 0 & \frac{(\Delta x_{i-1})^2}{2} & \frac{(\Delta x_{i-2})^2}{2} & \frac{(\Delta x_{i-3})^2}{2} \\ 0 & -\frac{(\Delta x_{i-1})^3}{6} & -\frac{(\Delta x_{i-2})^3}{6} & -\frac{(\Delta x_{i-3})^3}{6} \end{pmatrix} \begin{pmatrix} a_0 \\ a_{-1} \\ a_{-2} \\ a_{-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

We solve this equation in symbolic Matlab

Suppose $\Delta x_{i-1} = a$, $\Delta x_{i-2} = b$, $\Delta x_{i-3} = c$

Then $a_0 = \dfrac{2(a+b+c)}{abc}$

$$a_{-1} = \frac{-2(b+c)}{a(a-b)(a-c)}$$

$$a_{-2} = \frac{2(a+c)}{(b-c)(-b^2+ab)}$$

$$a_{-3} = \frac{2(a+b)}{(ac)^2+(bc)^2 - c^3 - abc}$$

Then $\left.\dfrac{d^2u}{dx^2}\right|_{x=x_i} = a_0 u_i + a_1 u_{i-1} + a_{-2} u_{i-2} + a_{-3} u_{i-3}$

1 (b)

Here we have to derive 2nd order accurate finite difference approximation to compute the cross derivative $\dfrac{\partial^2 u}{\partial x \partial y}$.

using central difference

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{ij} = \frac{\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j}}{2\Delta x} + O(\Delta x^2) \quad \text{--- (1)}$$

For this we need to find

$$\left(\frac{\partial u}{\partial y}\right)_{i-1,j} = \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\Delta y} + O(\Delta y^2) \quad \text{--- (2)}$$

$$\left(\frac{\partial u}{\partial y}\right)_{i+1,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\Delta y} + O(\Delta y^2) \quad \text{--- (3)}$$

combining ①, ②, ③

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}}{4\Delta x \,\Delta y} + O(\Delta x^2) + O(\Delta y^2)$$

$$= A + O(\Delta x^2) + O(\Delta y^2) \quad — ④$$

For finding truncation error, we will take take 4th order of u's.

Let $\frac{\Delta x^4}{4!} u_{xxxx}(x_i, y_i) = a$

$\frac{\Delta x^3 \Delta y}{3!} u_{xxxy}(x_i, y_i) = b$

$\frac{\Delta x^2 \Delta y^2}{2!\,2!} u_{xx} u_{yy}(x_i, y_i) = c$

$\frac{\Delta x \Delta y^3}{3!} u_{xyyy}(x_i, y_i) = d$

$\frac{\Delta y^4}{4!} u_{yyyy}(x_i, y_i) = e$.

Now The 4th order term of
$u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}$

$= (a+b+c+d+e) + (a+b+c+d+e)$
$-(a-b+c-d+e) - (a-b+c-d+e) = 4b + 4d$

$$= 4\frac{\Delta x^3 \Delta y}{3!} u_{xxxy}(x_i, y_i) + 4\frac{\Delta x \Delta y^3}{3!} u_{xyyy}(x_i, y_i)$$

Hence leading order term in error is

$$\boxed{\frac{\Delta x^2}{3!} u_{xxxy}(x_i, y_i) + \frac{\Delta y^2}{3!} u_{xyyy}(x_i, y_i)}$$ (Dividing $4\Delta x \Delta y$)

As we know TE is from eqn (4) $O(\Delta x^2) + O(\Delta y^2)$, this is the leading term of LTE.

Thus Truncation error is of $O(\Delta x^2) + O(\Delta y^2)$

NSDE Assignment1

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Problem 2:.

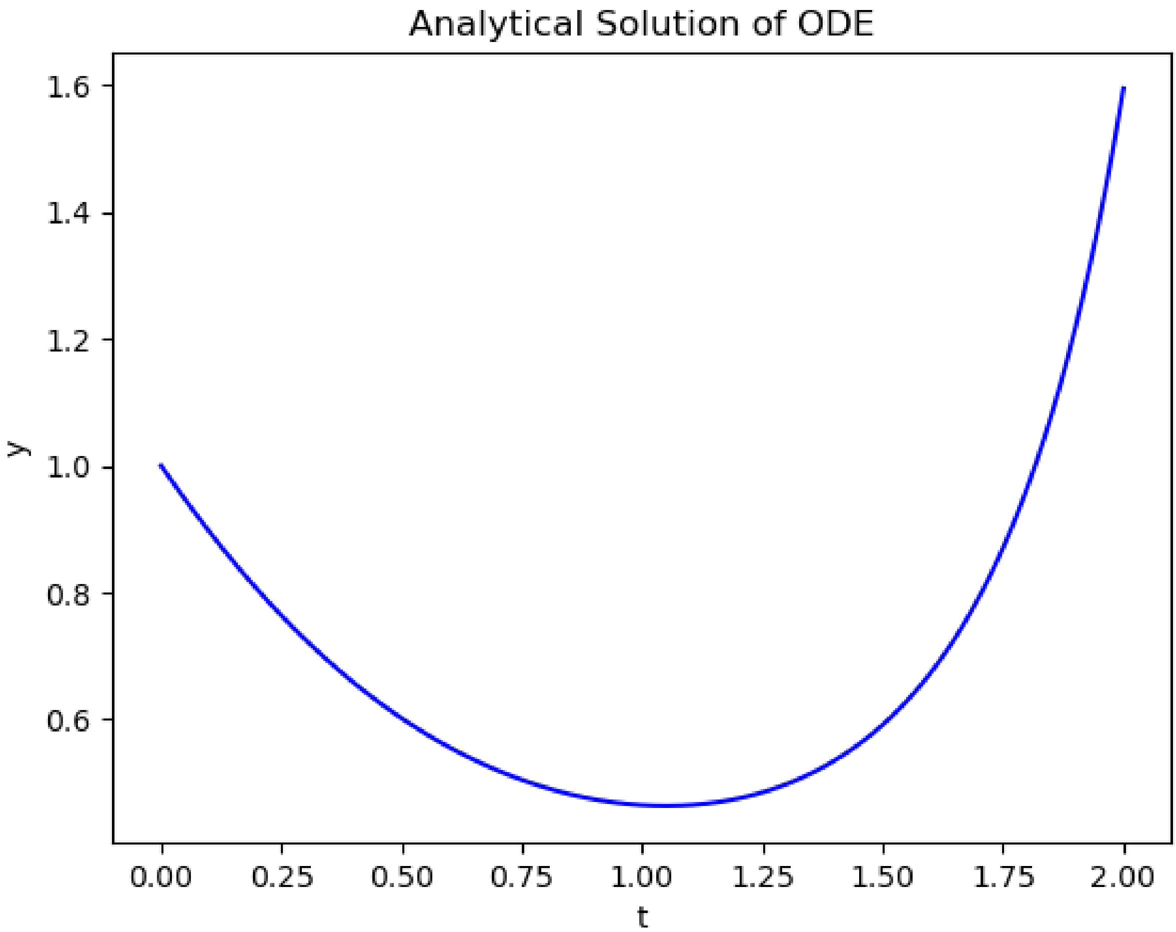
Solution:

2(a):

The given initial value problem ODE

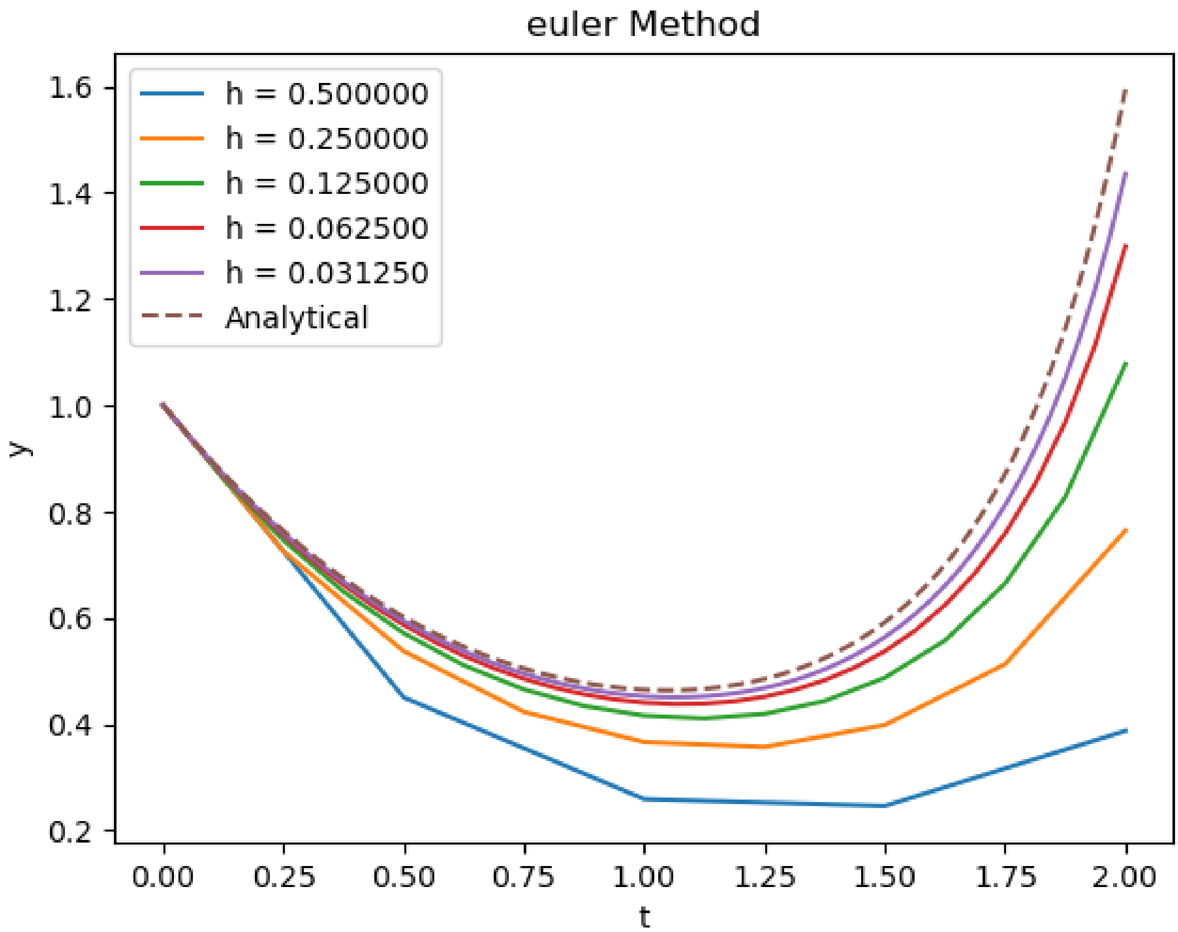
$$\frac{dy}{dt} = yt^2 - 1.1y \quad ; y(0) = 1$$

has an analytical solution (which is obtained using integration) is $e^{(\frac{t^3}{3} - 1.1t)}$.The plot for the analytical solution is given below.

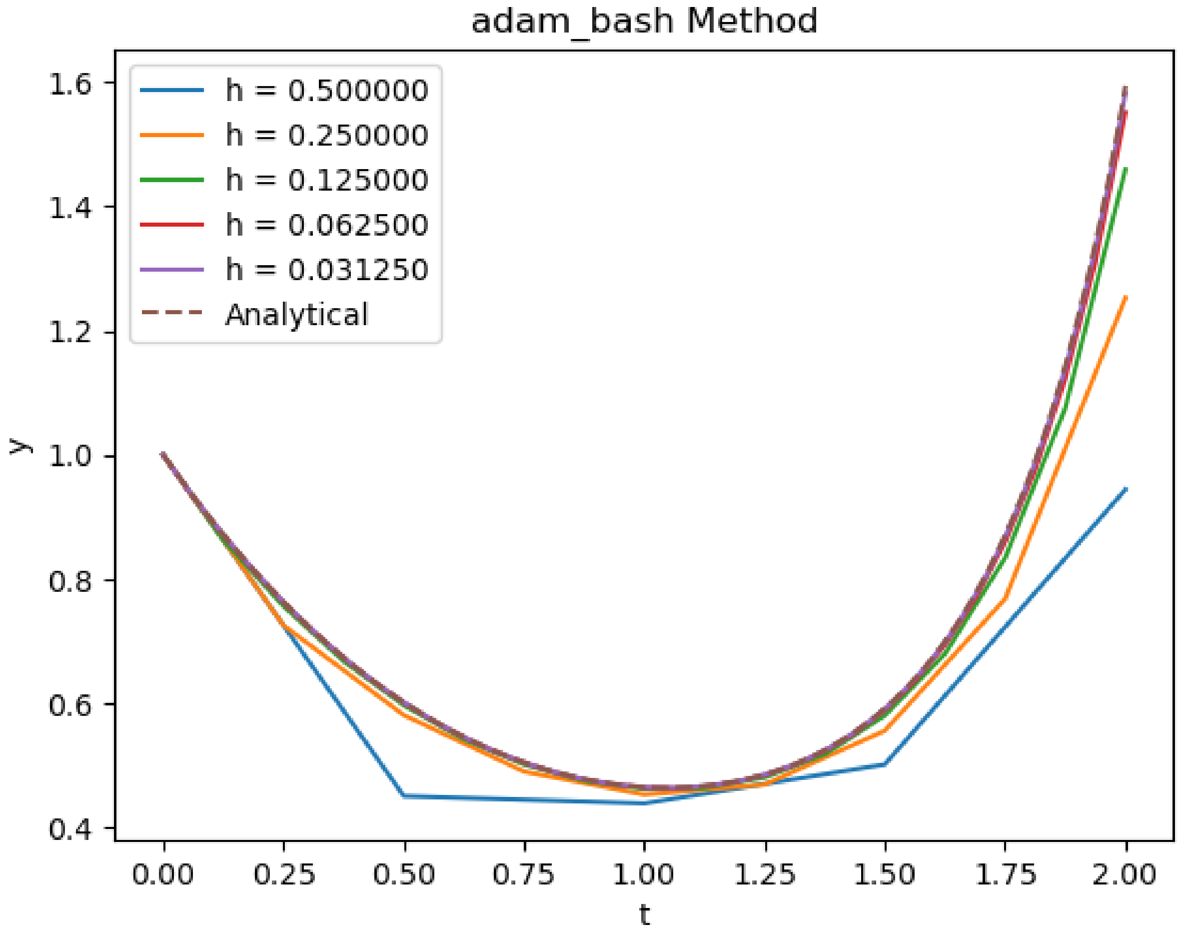


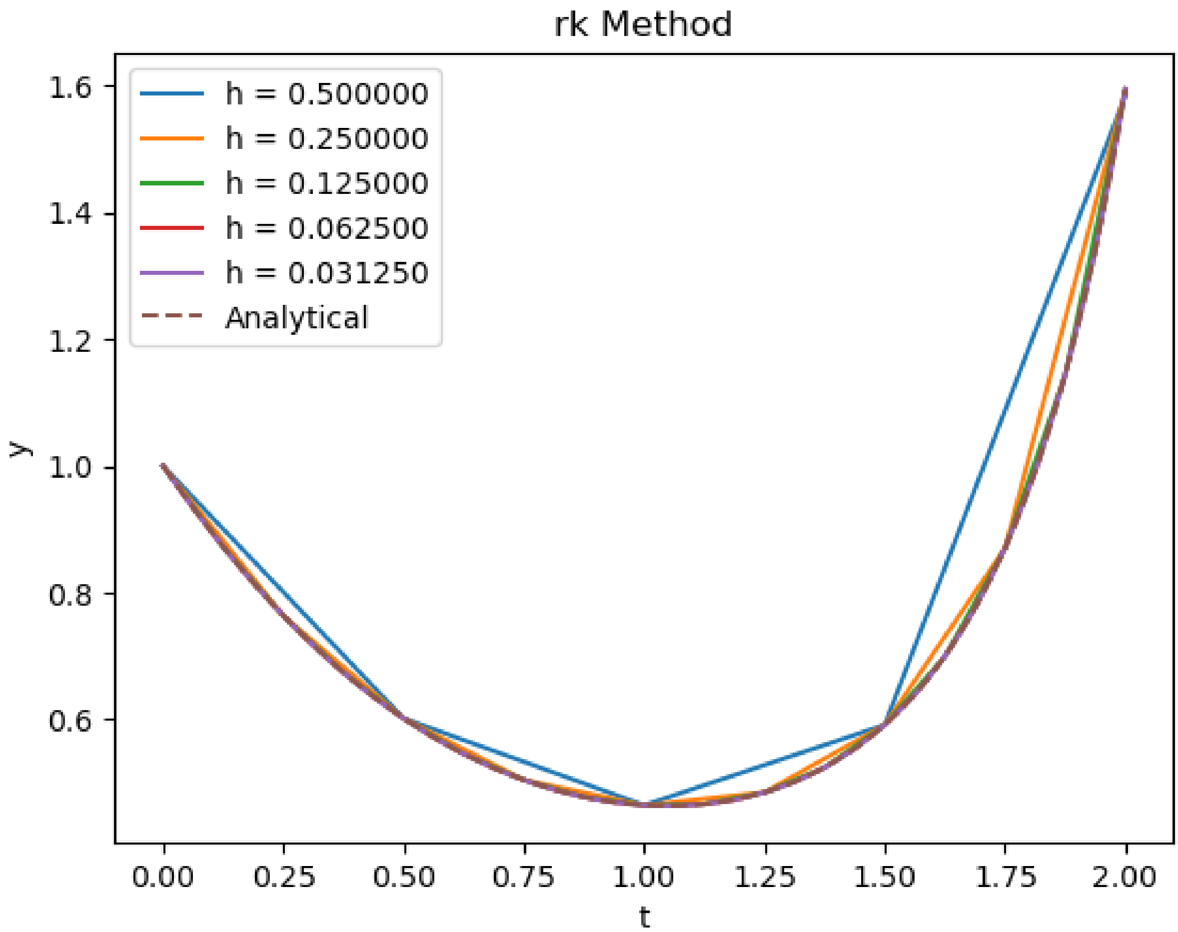
2(b):

Now we plotted the solution obtained from three different method with stepsize(h) =0.5, 0.25, 0.125, 0.0625 and 0.03125.



Now we see that with stepsize(h) decreases, the solution obtained from the numerical method converges.This is expected as we know small stepsize will give better approximate to true solution than using large stepsize(where the solution may diverge).We know that Global Truncation Error for Explicit Euler, Adam-Bashforth of order 2 step method and RK-4 methods are $O(h)$, $O(h^2)$ and $O(h^4)$ respectively.For this reason, when **when h is relatively large RK-4 method gives the better result as with the step size lower than 1, the error term will be much smaller than others.**

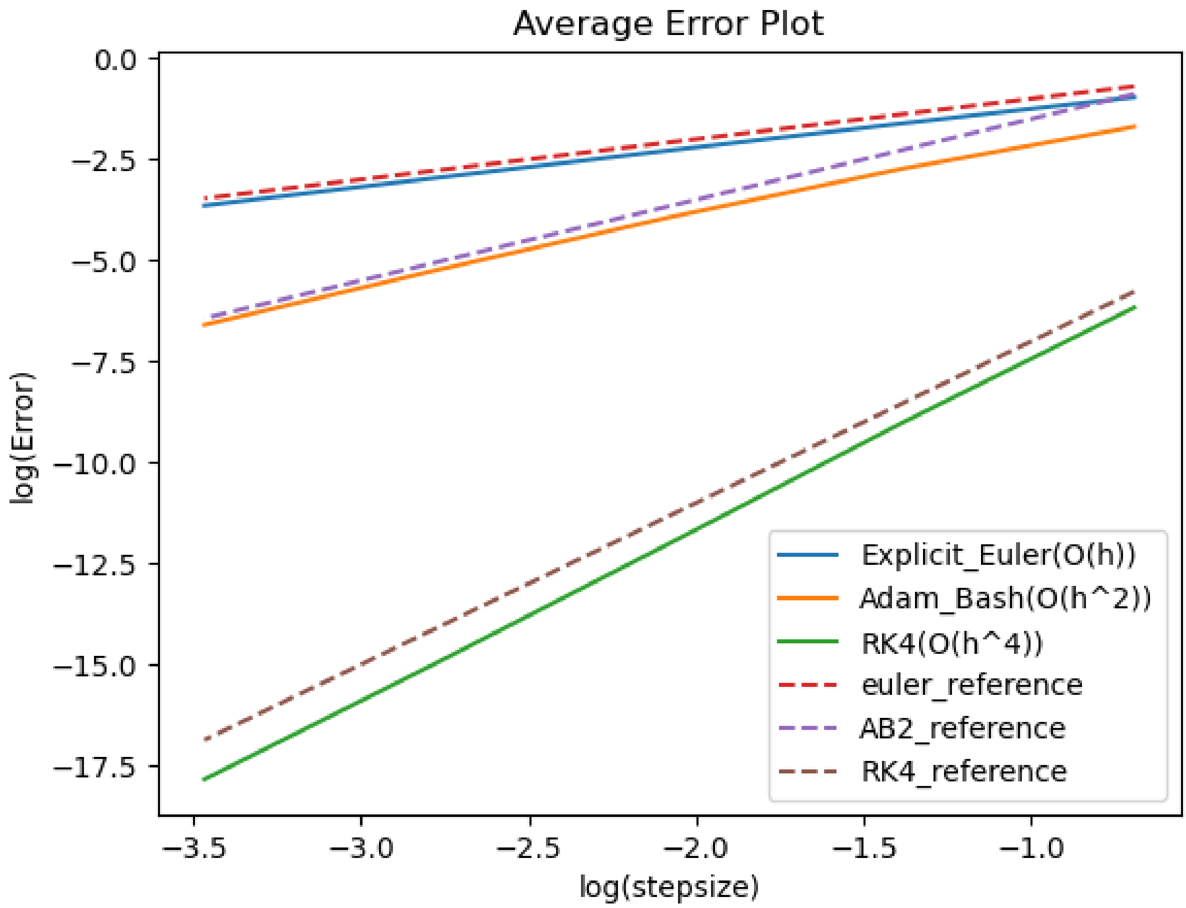

adam_bash Method
h = 0.500000
h = 0.250000
h = 0.125000
h = 0.062500
h = 0.031250
Analytical
y
t
0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75 2.00
0.4 0.6 0.8 1.0 1.2 1.4 1.6

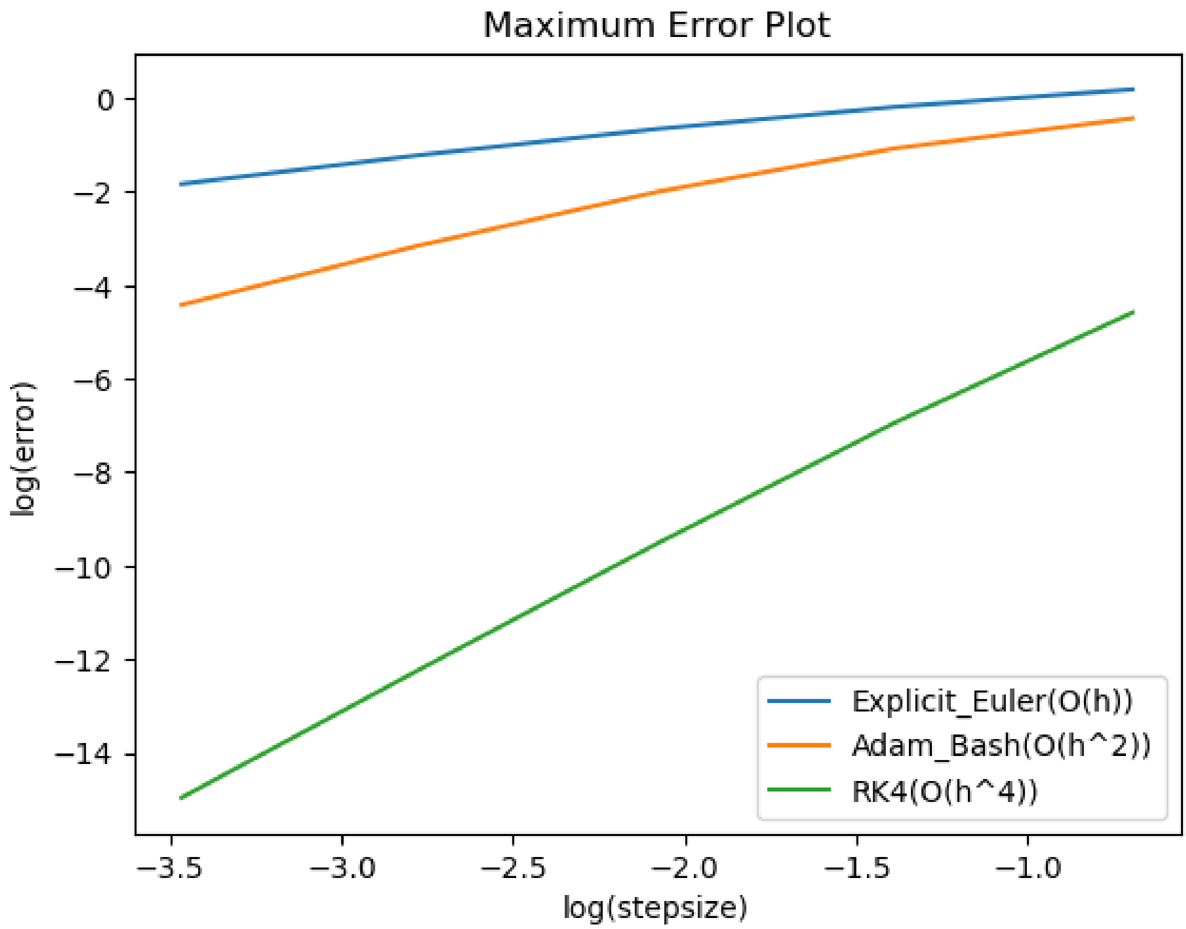

rk Method
h = 0.500000
h = 0.250000
h = 0.125000
h = 0.062500
h = 0.031250
Analytical
y
t
0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75 2.00
0.6 0.8 1.0 1.2 1.4 1.6

2(c):

Now we plot the average error vs stepsize in $\log -$scale and the result suggest that there is almost a linear relationship between $\log$(error) and $\log$(stepsize).Also both average error plot and maximum error plot gives almost same straight line.

The slope in the average error plot corresponding to Euler explicit method = 0.9673404412792722 , Adambash method of order 2 =1.775402354676909, RK4= 4.223620755602876 respectively.As we know if error is proportional to h^a, then slope of log(error) and log(h) will turn out to be 'a'.**The obtained slopes are close to these theoretical values, demonstrating the correctness of our implementation.**





2(d):

1. We implemented Explicit Euler, Adam-Bashforth 2 step method and RK-4 methods and observed that solutions converge for small values of h and are approximate for large values of h. This is to be expected because a smaller h value leads to a better approximation to the continuous problem.

2. For each method, error vs. step size plots were created. The slopes obtained for the Explicit Euler, Adam Bashforth 2 step method, and RK-4 methods, respectively, were 0.9673404412792722 ,1.775402354676909 , and 4.223620755602876, which were close to the theoretical slope values of 1, 2, and 4 corresponding to GTE.We can observe similar behavior of average error as seen with maximum error. It is least for RK4 and large for Explicit Euler and average error decreases with decrease in step size.

3. In conclusion, using a small step size allows for better approximations to the solution of differential equations. Furthermore, it is expected that by experimenting with smaller step sizes, the slope of log(error) vs log(step size) will approach theoretical values. However, one disadvantage of small step size is that more computation is required.

Problem 3:

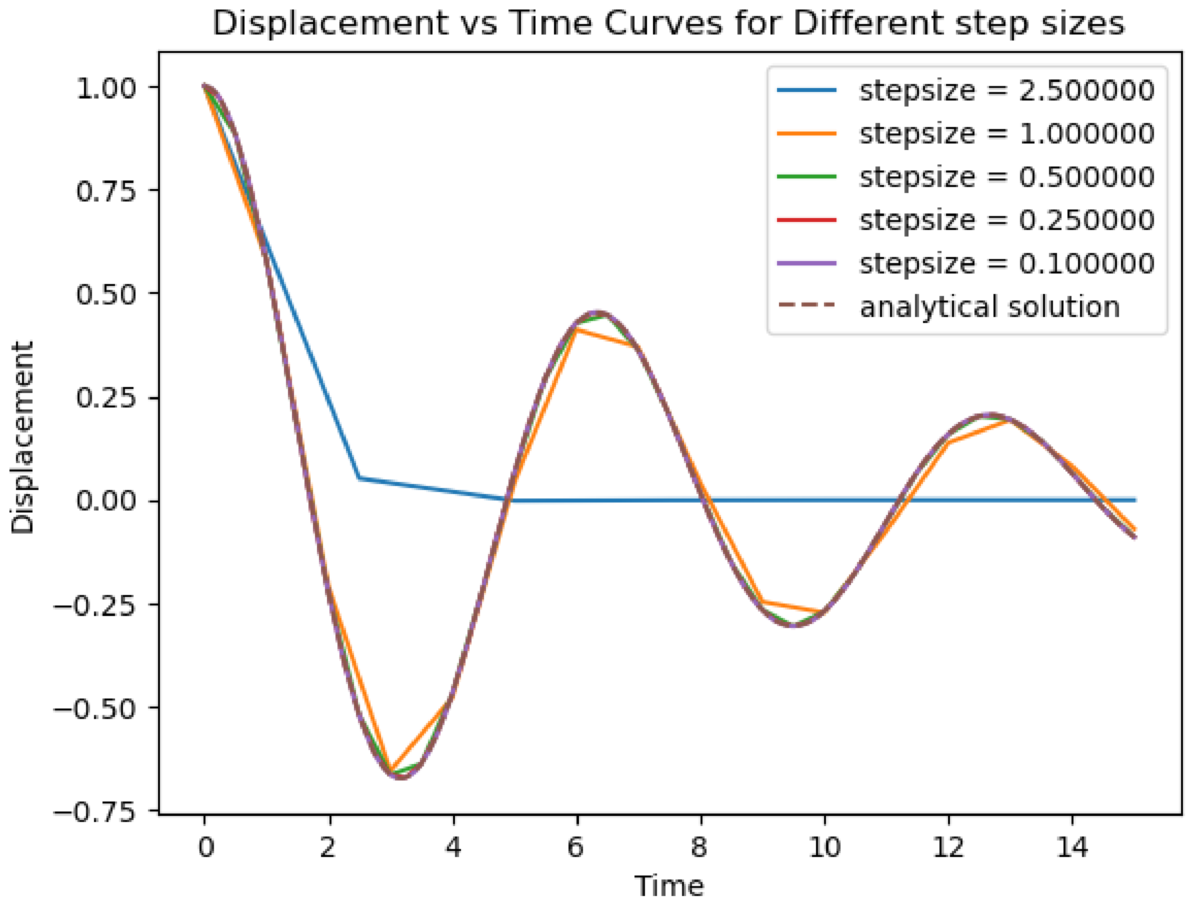
Solution:

The motion of a damped spring-mass system is described by the given ordinary differential equation can be describe as the system of ODEs

$$\frac{dv}{dt} = \frac{(-cv - kx)}{m}; \quad v(0) = 0 \tag{1}$$

$$\frac{dx}{dt} = v; \quad x(0) = 1 \tag{2}$$

Here x denotes displacement and v denotes velocity. The idea was to find optimal step size for convergence of solutions and comparing behaviour of system under different damping coefficient.



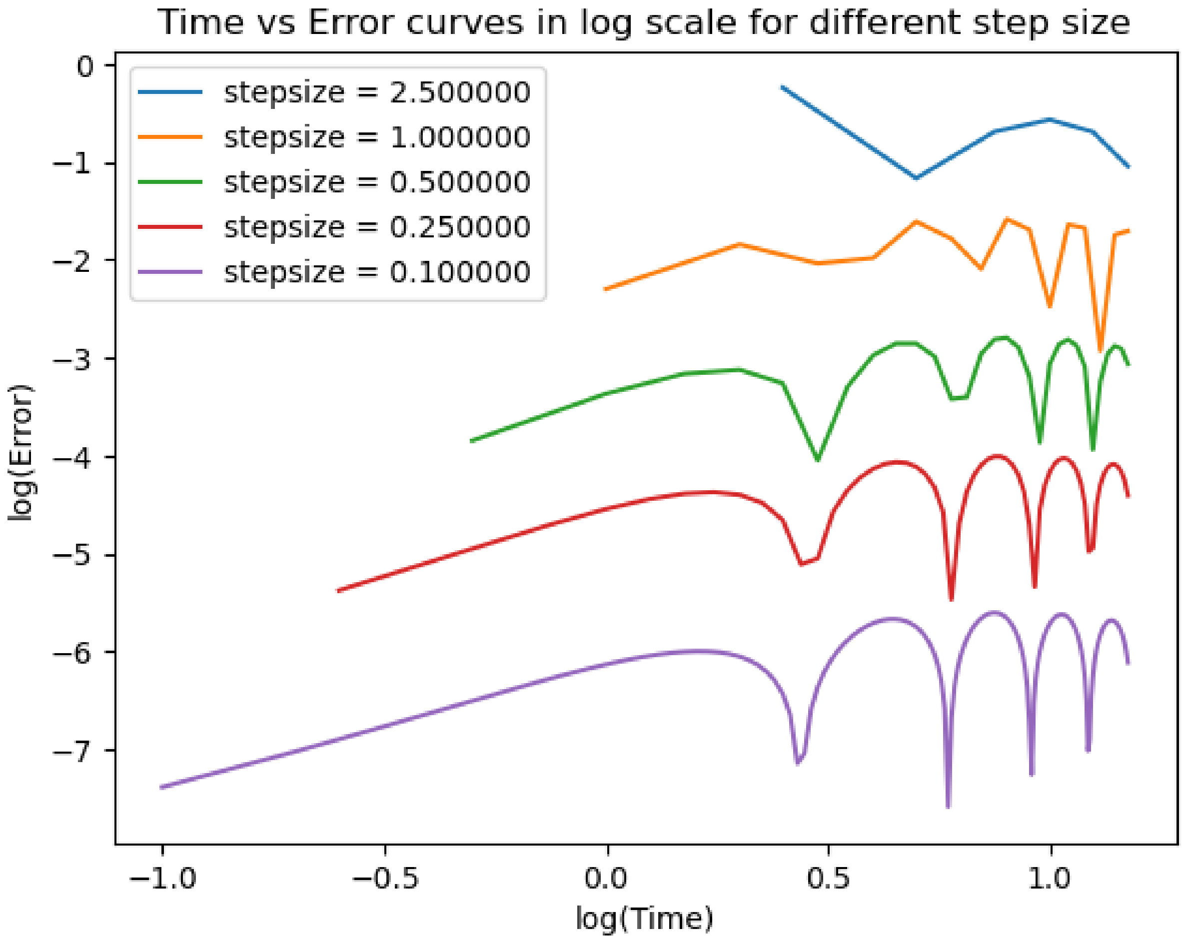
For a constant damping coefficient c=5 and for step sizes =2.5,1, 0.5, 0.25 , 0.1 . The plot we obtained is given above.Here for these varying step sizes the plot obtained is similar and overlapping but for larger step sizes the graph can be seen to be diverging. The results show that with a step size of 0.25 or less, displacement vs time curves are nearly indistinguishable. As a result, **the optimal step size is 0.25, or, to be more conservative, we can choose 0.10**.
Taking $m = k = 20$, the 2nd order ODE: $20\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 20x = 0$ with $x'(0) = 0$ and $x(0) = 1$

has analytical solution

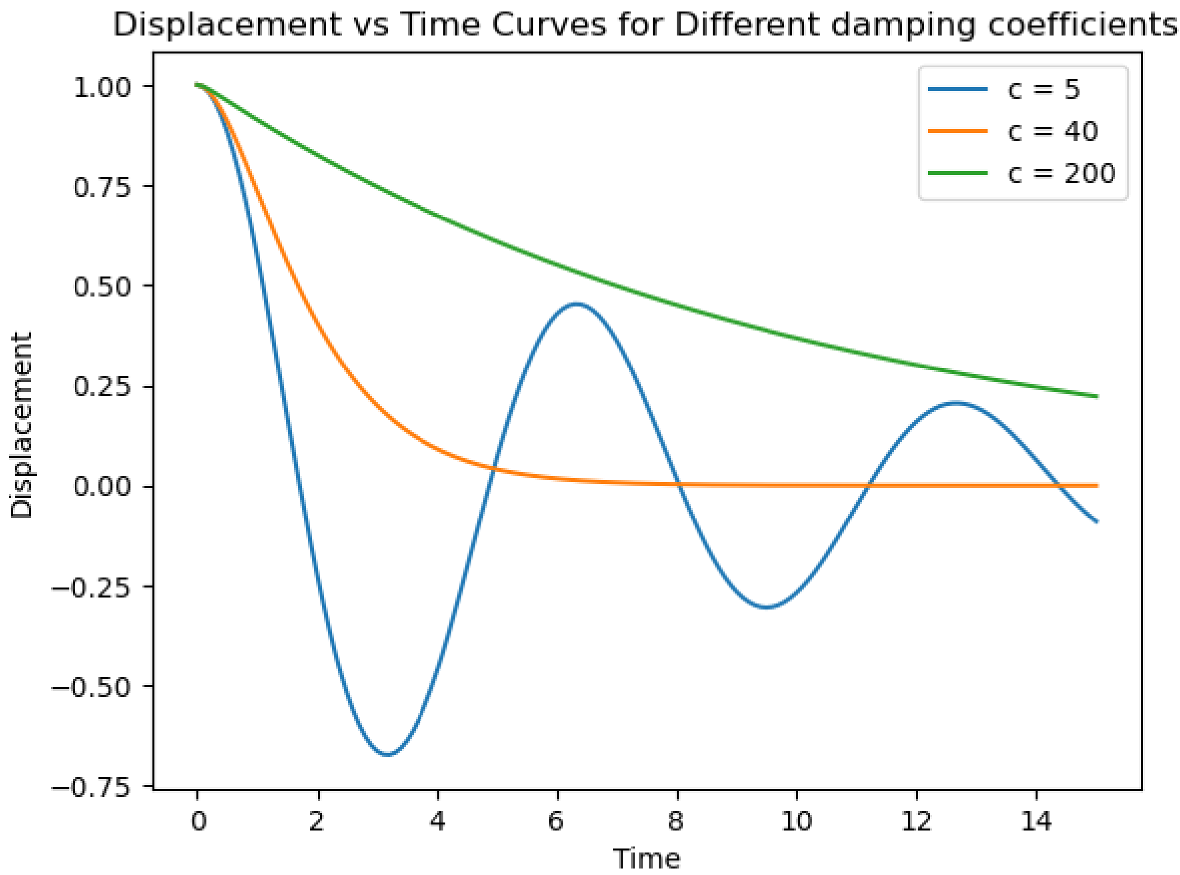
$$x(t) = e^{-\frac{t}{5}}\left(\cos\frac{3\sqrt{7}t}{8} + \frac{1}{3\sqrt{7}}\sin\frac{3\sqrt{7}t}{8}\right)$$

The obtained plot for stepsize=0.10 and plot for analytical solution demonstrates the correctness of our implementation.

Also the error vs time plot in log scale for different stepsize given below



Result obtained from Different Damping Coefficients:



These are 3 plots for underdamping, critically damped and overdamped system So **underdamped system(c=5) oscillates about mean position and amplitude goes on decreasing**.A decaying exponential term that is weighted by an oscillating component is present in the continuous general solution to this problem. This explains the under-damping case's damped oscillatory behaviour.

A critically damped system moves as quickly as possible toward equilibrium without oscillating about the equilibrium.This problem's analytical solution is a linear function in t weighted by an exponentially decaying term. We know from the physics of the problem that critical damping causes the system to return to its 0 displacement equilibrium state the fastest.

Overdamped system does not oscillate but very slowly moves towards equilibrium.This problem's analytical solution is a sum of two exponentially decaying functions. As a result, it is expected to continue decreasing monotonically until it reaches 0.

Problem 4:

4(a):

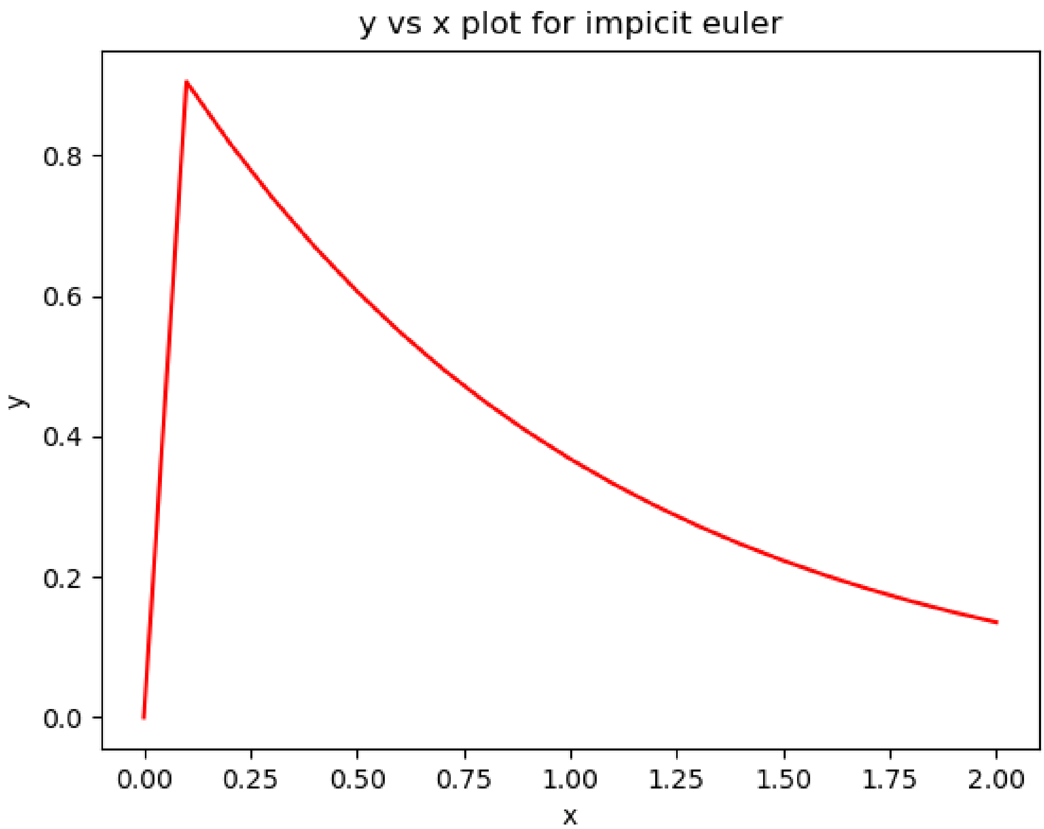
Stability of the solution can be gained by examining the homogeneous part of given equation. The Homogeneous part of the equation is $\frac{dy}{dx} = -200000y$.
Now consider the explicit formulae for the Euler Method with step size h

$$y_i = y_{i-1} - h \times 200000 \times y_{i-1}$$

After the end of n steps this give $y_n = (1 - 200000h)y_{n-1} = y_n = (1 - 200000h)^n y_0$.
For solution to be stable

$$|(1 - 200000h)^n| \leq 1$$
$$\implies |(1 - 200000h)| \leq 1$$
$$\implies h \leq 10^{-5}$$

4(b)



As can be seen, the solution is converging but not blowing up due to instability. This implies that, whereas the explicit Euler method requires a step size of less than 10^{-5} for stability, the implicit Euler method is stable even with a very coarse step size of 0.1. We know that implicit Euler uses information from (x_{i+1}, y_{i+1}) when calculating y_{i+1}, whereas explicit Euler only uses information from previous time steps. As a result, implicit Euler is more stable than explicit Euler.As a result, even when using a large step size, implicit Euler solutions remained stable.

Problem 5:

4(a)

Question 5:

Here equation given as

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + S = 0$$

using central difference scheme for $\frac{d^2T}{dr^2}$, $\frac{dT}{dr}$ we have,

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta r)^2} + \frac{1}{r}\,\frac{T_{i+1} - T_{i-1}}{2\Delta r} + S = 0$$

$$\Rightarrow (2r - \Delta r)\,T_{i-1} + (-4r)\,T_i + (2r + \Delta r)\,T_{i+1} = -2Sr(\Delta r$$

For boundary condition we use forward difference such that

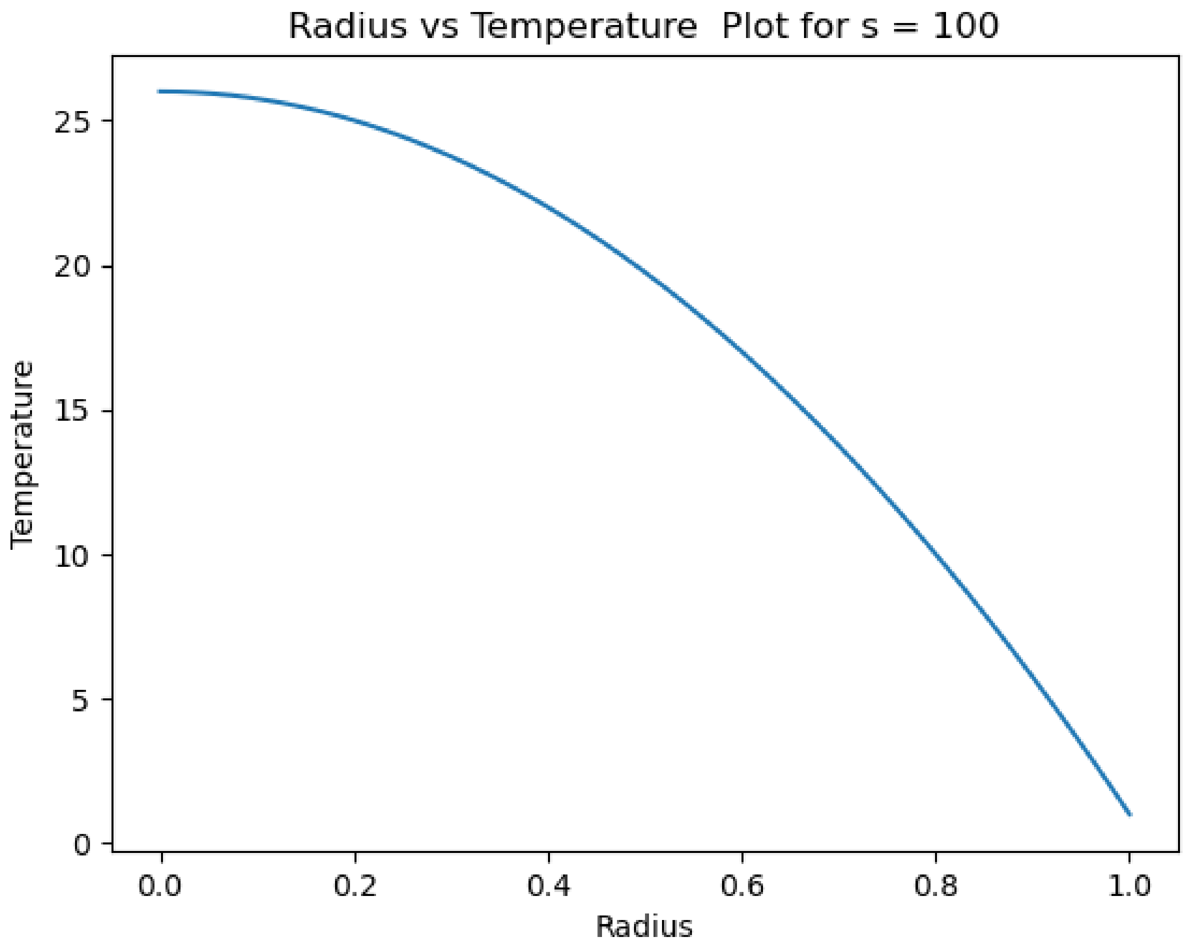
$$\left.\frac{dT}{dr}\right|_{r=0} \Rightarrow \frac{T_1 - T_0}{\Delta r} = 0$$

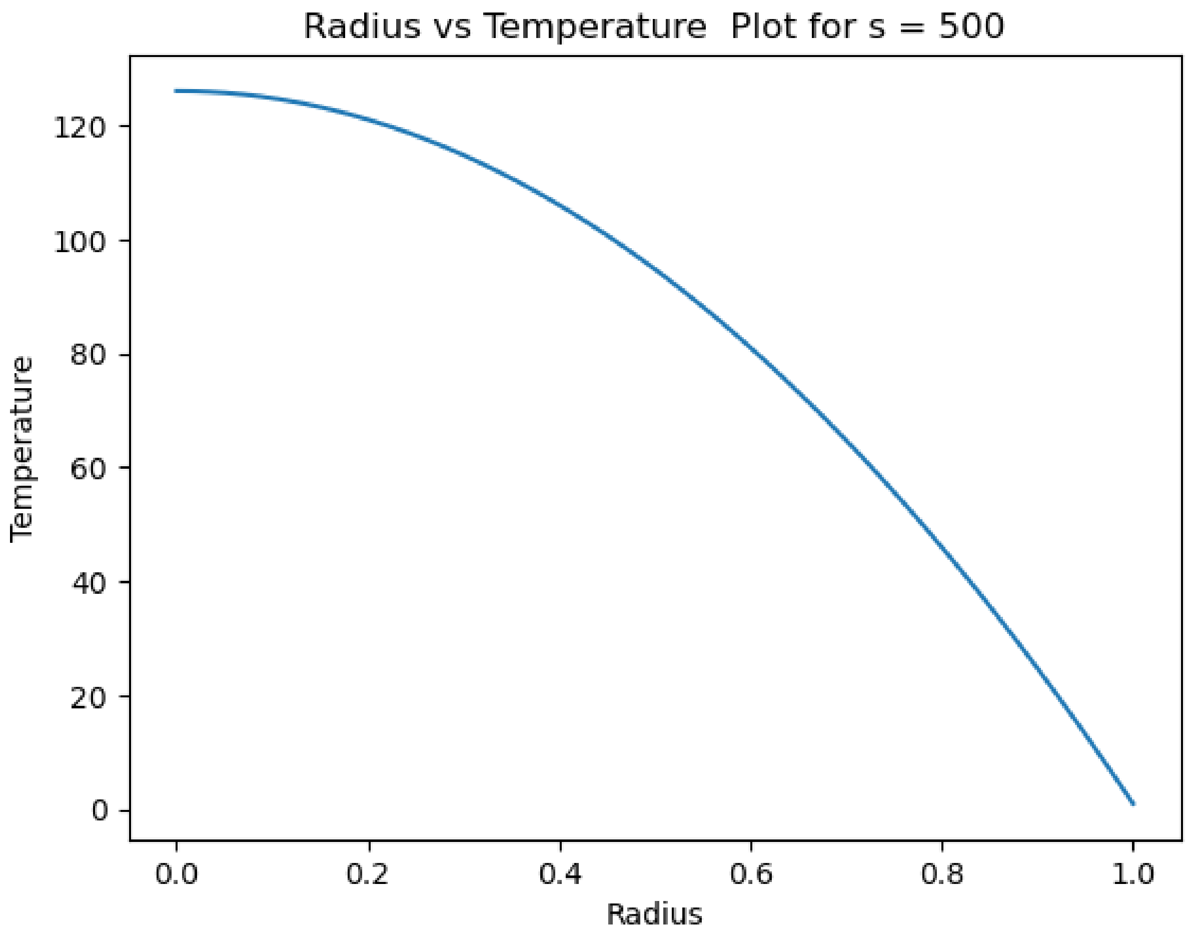
Then we get the system of equation.

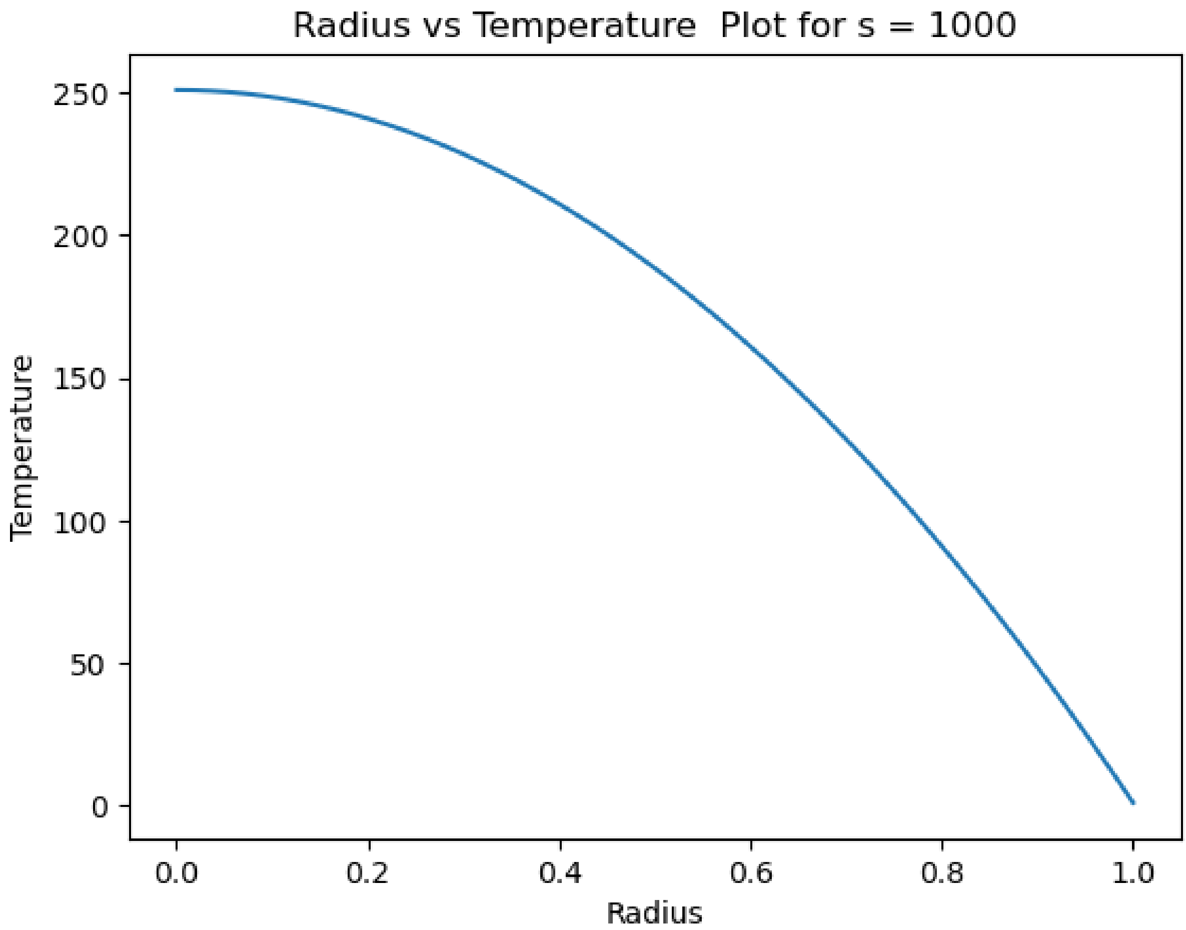
$$\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 2r-\Delta r & -4r & 2r+\Delta r & \cdots & & 0 \\ 0 & 2r-\Delta r & -4r & 2r+\Delta r & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & 2r-\Delta r & -4r & 2r+\Delta r \\ 0 & \cdots & & & & 1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ \vdots \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} 0 \\ -2Sr(\Delta r)^2 \\ \vdots \\ \vdots \\ -2Sr(\Delta r)^2 \\ 1 \end{bmatrix}$$

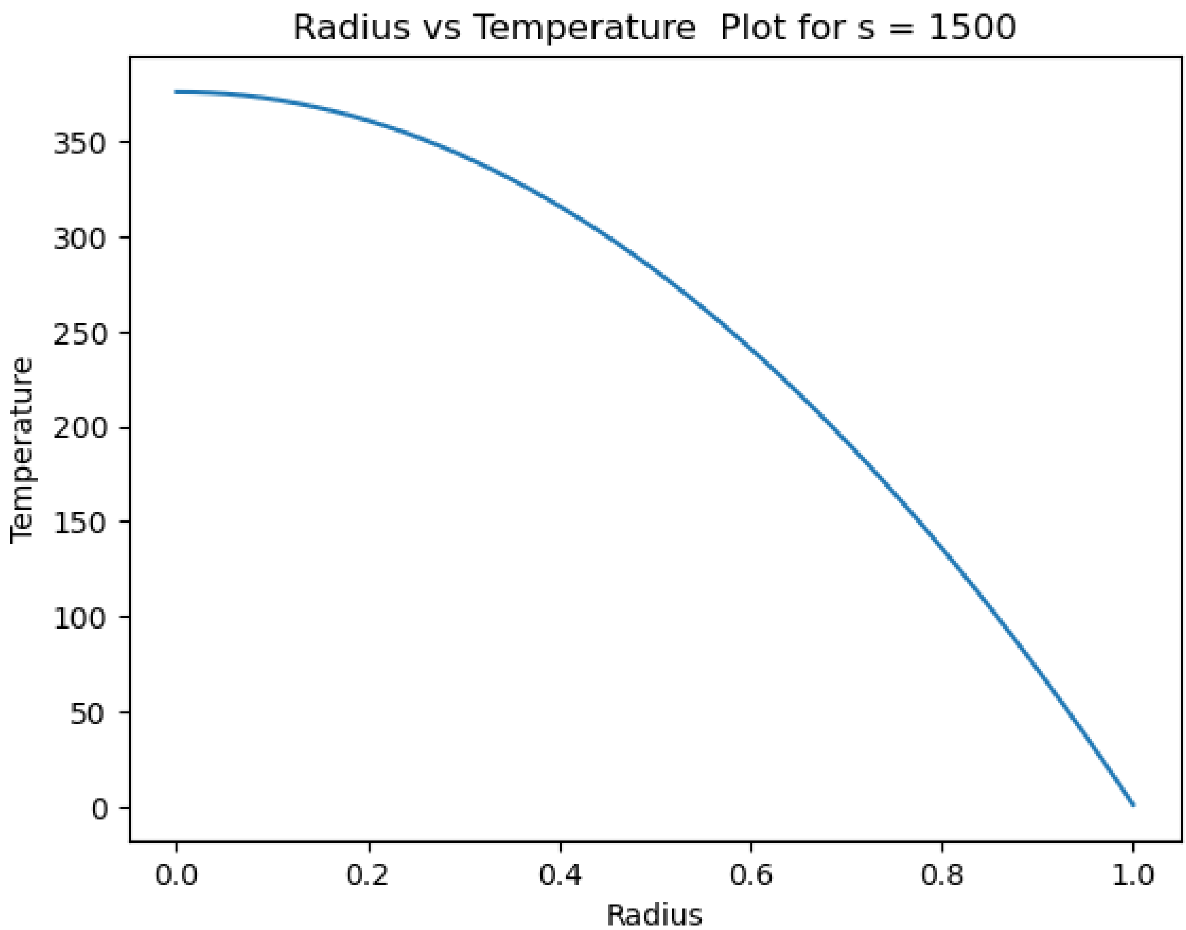
By solving this equation we got temperature distribution.

We solved this equation using the second order central difference scheme and find temperature distribution along the radial direction for $S = 100, 500, 1000, 1500.$ using a grid size of 1024 points. The initial boundary condition $(dT/dr = 0)$ at $r = 0$ was expressed with forward difference to avoid ghost points.


Radius vs Temperature Plot for s = 100
Temperature
Radius
0
5
10
15
20
25
0.0
0.2
0.4
0.6
0.8
1.0

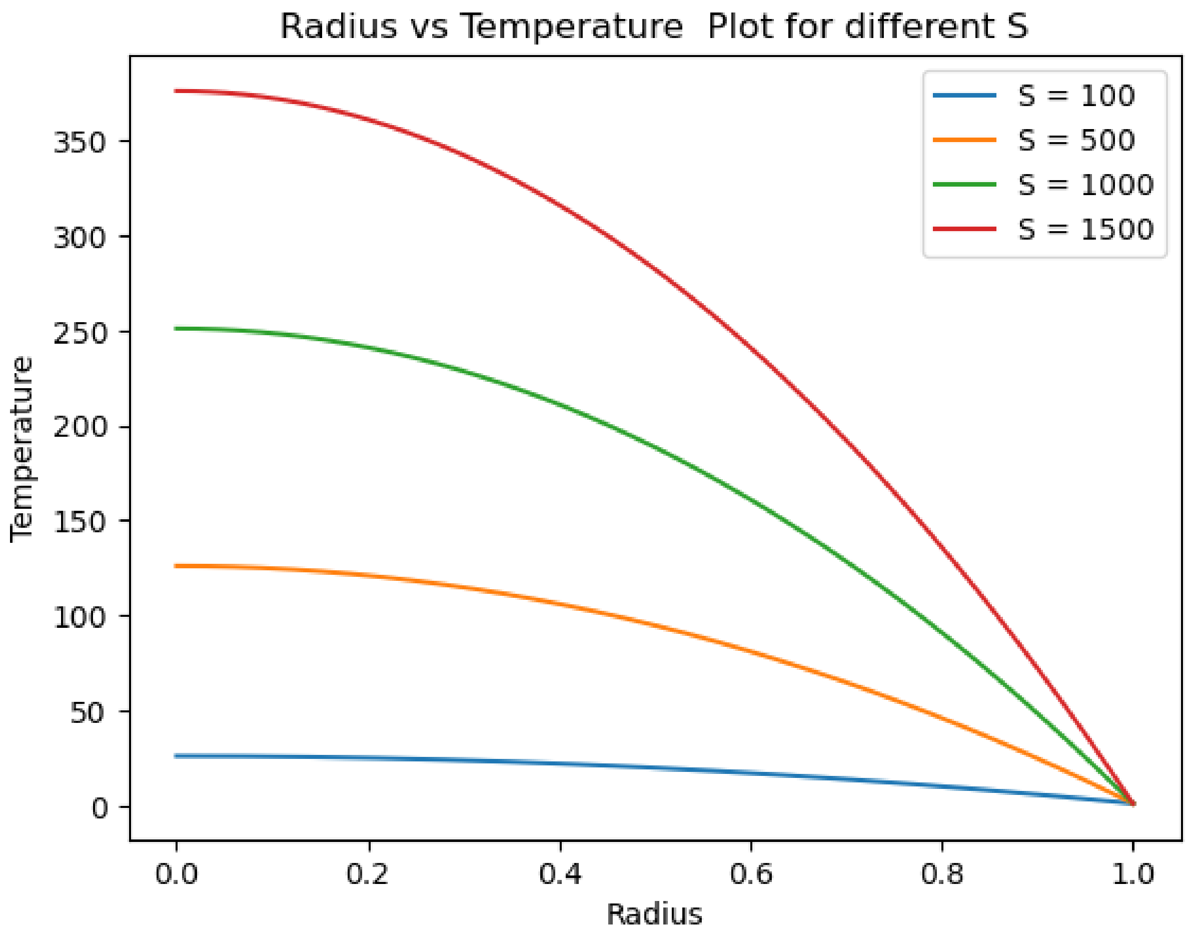

Radius vs Temperature Plot for s = 500
Temperature
Radius
0
20
40
60
80
100
120
0.0
0.2
0.4
0.6
0.8
1.0


Radius vs Temperature Plot for s = 1000
Temperature
Radius
250
200
150
100
50
0
0.0
0.2
0.4
0.6
0.8
1.0


Radius vs Temperature Plot for s = 1500
Temperature
Radius
350
300
250
200
150
100
50
0
0.0
0.2
0.4
0.6
0.8
1.0

The temperature decreases monotonically from the centre of the circular rod to the periphery as given that the heat source is in the centre. Furthermore, while there are no significant differences in the shapes of these plots, the initial values change significantly with the value of S. This is expected because a **higher S value indicates a better heat source, and thus temperature is expected to be higher in that case.**

4(b,c):



For the different S values (heat source strength) the temperature at the centre of the rod is different. However, because $T(r = 1) = 1$, temperature profiles monotonically decay to satisfy the boundary condition. Furthermore, we observe that out of the four S values, $S = 100$ is the highest value such that the peak temperature in the domain does not exceed 100.