# DS 298: WORK ASSIGNMENT - 3

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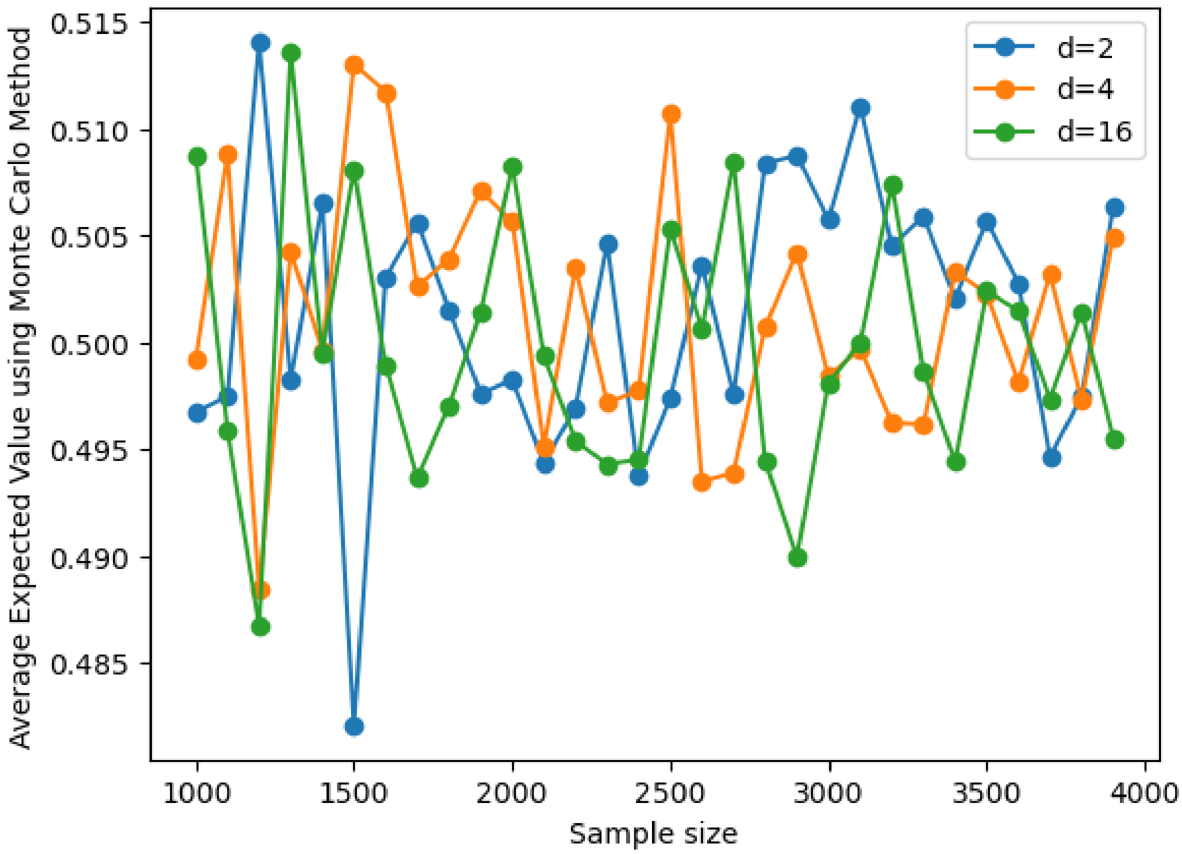
For estimating the expected value of a function over a domain, convergence refers to the behavior of the estimated value as the number of samples increases. As more samples are taken, the estimated value should approach the true value of the expected value.

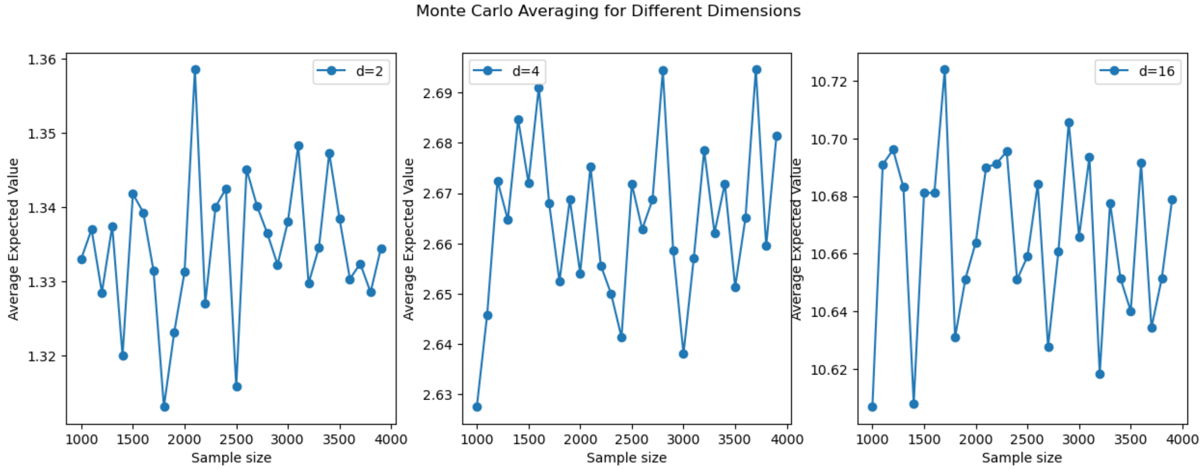
The convergence of Monte Carlo methods depends on the number of samples and the variance of the function being estimated. The variance of the function can be reduced by using importance sampling, which means that samples are drawn from a distribution that better matches the function being estimated, rather than a uniform distribution. Importance sampling can reduce the variance of the function and improve the convergence rate of Monte Carlo methods.

In addition to the variance, the convergence rate of Monte Carlo methods can also depend on the dimensionality of the space being sampled. As the dimensionality increases, the number of samples required to achieve a certain level of accuracy grows exponentially. This is known as the "curse of dimensionality". To mitigate this effect, methods such as importance sampling can be used to reduce the effective dimensionality of the problem and improve convergence.

## 1 Uniform sampling

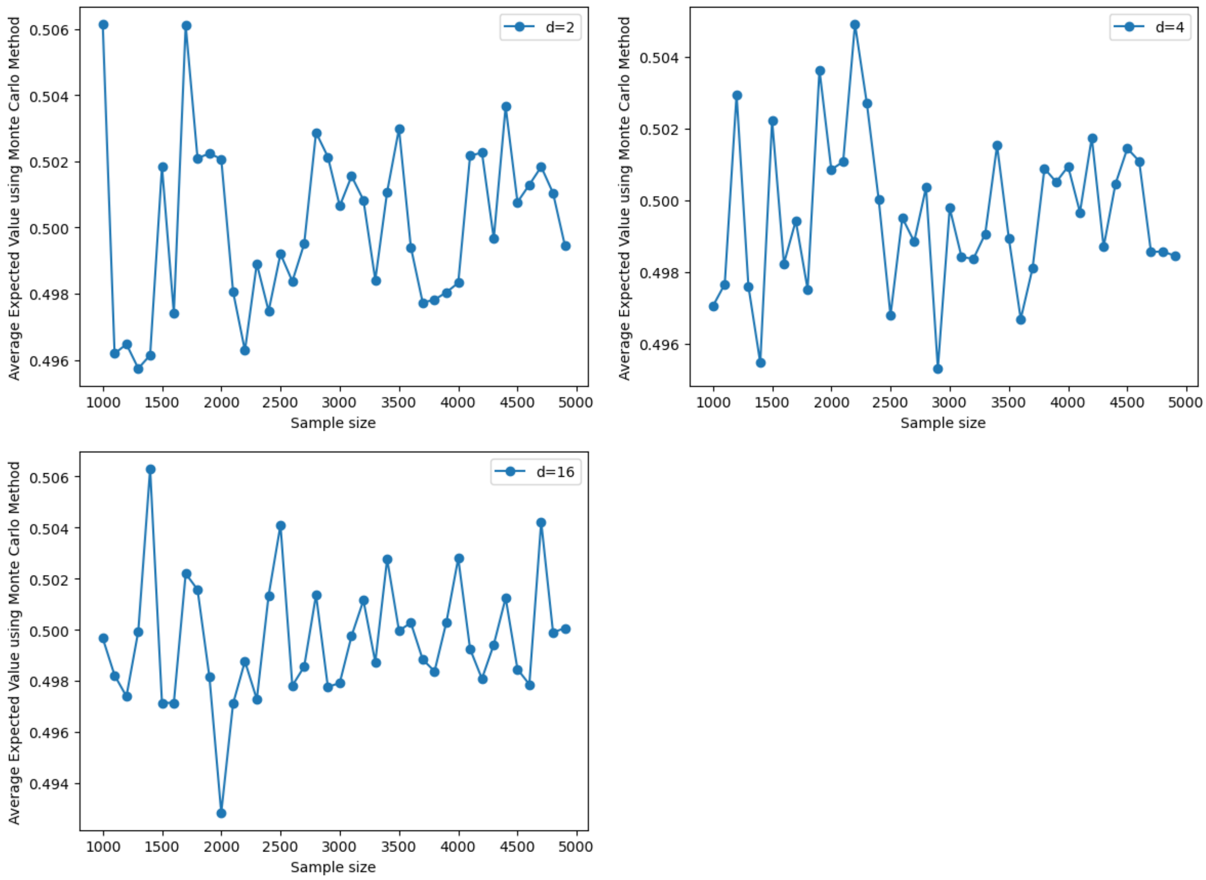



here we can see that with $g(x) = |x_1|$ the value of $E[g(x)]$ is near 0.5 for all the dimensions.

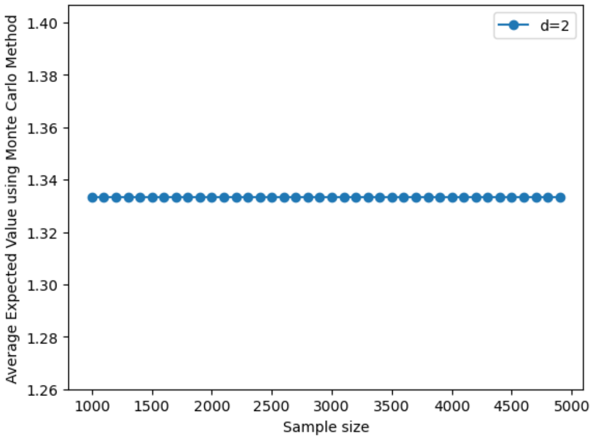


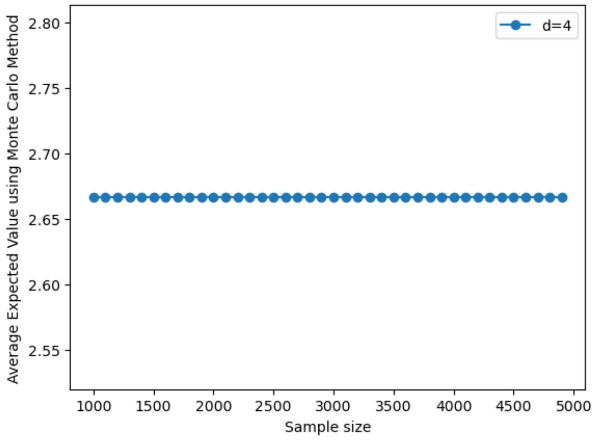

Here we can see that with $g(x) = d - r^2$ the value of $E[g(x)]$ is near 1.33 for the dimension=2, 2.67 for dimensions =4, 10.67 for dimension=16.

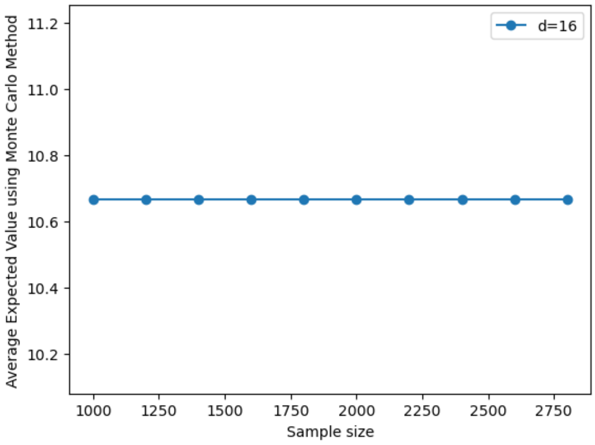
## 2 Importance sampling with $q(x) = c||x||_1$



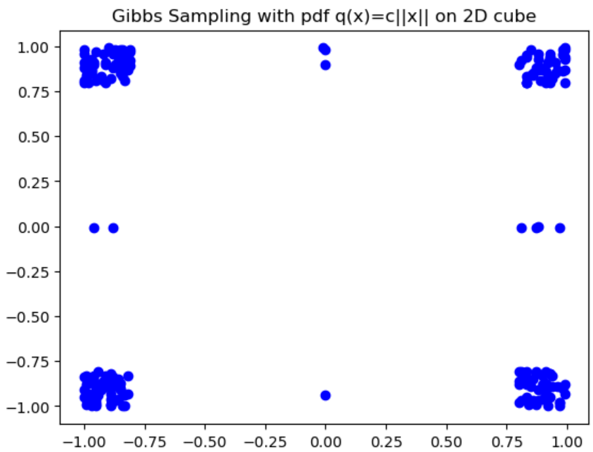
# 3 Importance sampling with $q(x) = c|g(x)|$

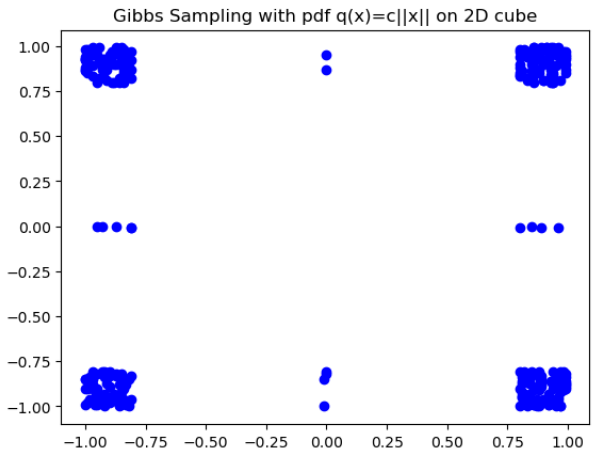



# 4 Gibbs sampling on 2d cube



[E] The main goal of this problem is to evaluate $E[g(x)]$ on a cubic domain. where dimension $d = 2, 4, 16$.

The $g(x)$ is given as (a) $g(x) = |x_i|_2$ where $X = [x_1, \dots x_d]$ is any random point in domain.

(b) $g(x) = (d - r^2)$ where $r = \sqrt{x_1^2 + \dots x_d^2}$

Here we use gibbs sampling:

Initiaze $x_0, y_0$

for $j = 1, 2, 3, \dots$ do

sample $x_j \sim P(x | y_{j-1})$

sample $y_j \sim P(y | x_{j-1})$

end for.

① Uniform sampling:

We want to evaluate $E[g(x)]$, with N samples.

Now $E[g(x)] = \int g(x) \, P(x) \, dx \simeq \frac{1}{N} \sum_{i=1}^{N} g(x_i)$ (using Law of large number)

where $x_i$'s are uniformly distributed with pdf $P(x)$.

and $g(x)$ are given above.

② Importance ~~Uniform~~ sampling: To estimate the expected value of $g(x)$ using the importance sampling, we generate random points from the cubic domain using different pdf $q(x)$ and reweight the function as $g(x)/q(x)$

We want to find $c$ for (a) $q(x) = c \, \|x\|_1$

As we know $\int_{d\text{-cube}} q(x) \, dx = 1$

$$\Rightarrow c \underbrace{\int_{-1}^{1} \cdots \int_{-1}^{1}}_{\leftarrow d \text{ times} \rightarrow} \|x\|_1 \, dx = 1 \Rightarrow c \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} (|x_1| + \dots + |x_d|) \, dx_1 \dots dx_d = 1$$

$$\Rightarrow c \cdot d \int_{-1}^{1} \cdots \int_{-1}^{1} |x_1| \, dx_1 \dots dx_d = 1$$

$$\Rightarrow c \cdot d \cdot 2^{d-1} \cdot \int_{-1}^{1} |x_1| \, dx_1 = 1$$

$$\Rightarrow \boxed{c = \frac{1}{d \, 2^{d-1}}}$$

(b) For $q(x) = c\,|g(x)|$

$$\int_{d\text{-cube}} q(x)\,dx = 1 \Rightarrow \int_{-1}^{1}\cdots\int_{-1}^{1} c(d-r^2)\,dx_1 \cdots dx_d = 1$$

$$\Rightarrow c\int_{-1}^{1}\cdots\int_{-1}^{1}(d - x_1^2 - x_2^2 - \cdots - x_d^2)\,dx_1 \cdots dx_d = 1$$

$$\Rightarrow c(d \cdot 2^d) - c \cdot d\int_{-1}^{1}\cdots\int_{-1}^{1} x_1^2\,dx_1 \cdots dx_d = 1$$

$$\Rightarrow c \cdot d \cdot 2^d - c \cdot d \cdot 2^{d-1} \cdot \frac{2}{3} = 1 \Rightarrow c = \frac{3}{d\,2^{d+1}}$$

Now $$E[g(x)] = \int_{-1}^{1} g(x)\frac{1}{2^d}\,dx = \int_{-1}^{1} g(x)\frac{1}{2^d}\,dx = \int_{-1}^{1}\frac{g(x)}{2^d q(x)}\,q(x)\,dx$$

$$\simeq \frac{1}{N}\sum_{i=1}^{N}\frac{g(x_i)}{2^d q(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{g(x)}{2^d \times \frac{1}{d\,2^{d-1}}\|x\|_1}$$

$$\boxed{\simeq \frac{1}{N}\sum_{i=1}^{N}\frac{g(x)}{\|x\|_1}\,\frac{d}{2}}$$ when $q(x) = c\|x\|_1$

When $q(x) = c\,|g(x)|$

$$E[g(x)] \simeq \frac{1}{N}\sum_{i=1}^{N}\frac{g(x_i)}{2^d q(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{g(x_i)}{2^d \cdot \frac{3}{d\,2^{d+1}}}$$

$$= \frac{1}{N}\sum_{i=1}^{N}\frac{2d}{3}\,g(x_i)$$

Both the cases pdf of $q(x)$ is chosen in such a way that ~~the~~ $\frac{g(x)}{2^d q(x)}$ has high variance.