DS 298: Work Assignment - 2

Due Feb 24, 2023

The goal of this work is to analyze the errors E in the matrix-multiplication Algorithm-1 for various values of the dimension n and number of samples c, in each case of the three different classes of matrices described.

Let $M^{m\times p} \approx A^{m\times n}B^{n\times p}$ be the evaluation considered where either $m = p = \frac{n}{5}$, or $m = p = n$, as fixed by you. Use values of n (x-axis in plots) as 100, 200, 500, 1000. The relative error $E = \|M - AB\|_F / \|AB\|_F$ is to be traced (in the y-axis) for three different sampling fractions $c/n = 0.01, 0.05, 0.2$ in a single plot with appropriate scaling. Average over 10 runs of the algorithms for each data point in the plot. This plot is to be replicated for three matrix classes I, II and III.

Class-I matrices are defined by a distribution of norms of the columns of A and rows of B i.e. $\|A^{(k)}\|$ and $\|B_{(k)}\|$, given by random variables X_1 and X_2 respectively, both with an exponential probability density $ae^{-\frac{5x}{x_{max}}}$. Use appropriate values for constants a where the minimum (x_{min}) and maximum (x_{max}) values of X are 1 and $\sqrt{m}$, respectively.

Class-II matrices similarly have a density $ae^{\frac{5x}{x_{max}}}$ for the norms of columns and rows i.e. X_1 and X_2, where appropriate constants a are to be chosen again with the minimum (x_{min}) and maximum (x_{max}) values as 1 and $\sqrt{m}$, respectively. Class-III matrices have norms given by X_1 and X_2 uniformly distributed as $U[1, \sqrt{m}]$. Include plots of the three probability densities of X (for matrix classes I, II and III) used for the dimension $n = 1000$, for a verification. Note that you may need the cumulative distribution function as well for the algorithms, to evaluate variables X_1 and X_2.

Algorithm-1 for the matrix multiplication, and Algorithm-2 for generating the input matrices, are briefly described in the next page.

Note : Submit the responses, plots, and the code as separate files, all zipped into a single folder identified by your name in full, to *abhijeetj@iisc.ac.in*

Algorithm 1 : Matrix multiplication $M \approx AB$

Inputs : c, p_k, $A^{(k)}$, $B_{(k)}$ for $k \in \{1, 2, \dots n\}$.
Outputs : E, $M \longleftarrow [\ ]$; an approximation of product AB, and its error.
While trials $t \leq c$ do {
Random sampling :
For a uniformly distributed random integer $k \in \{1, 2, \dots n\}$
If $max\{p_k\}U < p_k$
Accept k and $t = t + 1$; Sample indices k with a probability mass p_k using rejection sampling.
One-rank products : $M \longleftarrow M + A^{(k)}B_{(k)}/(cp_k)$; evaluate products and update sum for c samples
} break while loop
Evaluate error : $E \longleftarrow \|M - AB\|_F/\|AB\|_F$; evaluate relative error in the Frobenius norm.

Algorithm 2 : Generation of matrices in a class

Inputs : m, n, p, F_X; dimensions of matrix and distribution function of row and column norms.
Initialize : $A \longleftarrow rand[m, n]$ and $B \longleftarrow rand[n, p]$; Initialize A and B as random matrices.
For $k \in \{1, 2, \dots n\}$ do {
Generate random norms : $X_1 \longleftarrow F_X^{-1}(U)$ and $X_2 \longleftarrow F_X^{-1}(U)$; generate X_1 and X_2 with two independent uniformly distributed random variables.
Re-weight columns/rows : $A^{(k)} \longleftarrow X_1 A^{(k)}/\|A^{(k)}\|$ and $B_{(k)} \longleftarrow X_2 B_{(k)}/\|B_{(k)}\|$; generate matrices of required class.
Evaluate relative probability: $p_k \longleftarrow X_1 X_2$; using product of norms.
$S \longleftarrow S + p_k$; update the sum for use in normalization.
} end for
Normalize probability mass of samples : $p \longleftarrow \frac{p}{S}$.