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CONCENTRATED CIRCULAR MARKET MAKER

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SUMMARY: An AMM where the price can cross the zero bound into the negative price domain with applications in electricity, energy, and derivatives markets is presented. A unique feature of such an AMM is the ability to swap both negatively and positively priced assets between one another, which unlike traditional markets requires a numeraire in the form of a currency. The LP payoff, invariant, and liquidity distribution of a negative price AMM is compared to a Replicating Market Maker.

Keywords: negative price, negative liquidity, automated market maker, concentrated liquidity, liquidity distribution, power law, heavy tails.

1. INTRODUCTION

Negative prices are an unusual phenomenon in traditional financial markets. As the set of Real World Assets (RWA) is tokenized and becomes tradeable on the blockchain, the probability of encountering a negative price approaches one hundred percent. Currently, we do not have the ability to provide liquidity or trade an asset on the blockchain with a negative price on an AMM. This paper:

(1) Outlines the phenomenon of negative prices in financial literature.
(2) Reviews the limitations of current AMMs and how to enable negative prices.
(3) Compares the payoffs and proposes unique applications of a negative price AMM.

2. DISCUSSION

The phenomenon of negative prices has been observed empirically, especially in commodity markets. Examples being oil in 2020 [1], onions in 1955 [2], electricity in Nebraska in 2024 [3], fat-finger trading mistakes in Finland in 2023 pushing prices negative [4], and synthetic financial products consisting of derivatives of highly liquid US treasury bonds [5].

In the world of traditional finance, price manipulation has gone so far as to encourage Congress to ban commodities that have gone negative in price as was the case in 1958 with the passage of the Onion Futures Act, effectively banning the trading of onions to this day due to the bankruptcies caused by onion price manipulation [6]. Unlike traditional finance, the permissionless nature of decentralized ledgers allows for the exchange of any token, whether banned or not, but the feature of a negative price is not possible with current AMM invariants.

2.1. AMM Invariant

Uniswap v2 introduced a continuous uniform liquidity distribution [7] allowing for the trading of a token in the price range of zero to infinity. Yet prices are not infinite nor bound by the zero barrier, to address part of this problem Uniswap v3 was introduced to allow for the concentration of liquidity based on an LP's preferred viewpoint on the range of price movement between zero and infinity [8]. The constrained nature of a discrete uniform liquidity distribution of a Uniswap v3 position creates a problem of capital inefficiency in the tails where an LP can wind up out of range while earning no fees [Figure1]. To deal with being out of range, multiple Uniswap v3 positions can be customized along curves or, alternatively, AMM invariants can be made that match particular

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price behavior such as Geometric Brownian Motion with Replicating Market Makers (RMM) [9].This approach is extended by making the empirical observation that real world financial assets, as they continue to migrate onto the blockchain, can exhibit power law behavior [10][11] as well as enter the negative price domain [12].

It should be noted that Uniswap's Constant Product Market Maker (CPMM) invariant does have a negative liquidity domain (see Appendix A), but accessing it is limited due to the nature of the hyperbolic equation $xy = k$. We adjust the Uniswap v3 AMM invariant equation below by allowing the curve to fold in on itself to access negative prices.

$$(x-a)^2 + (y-b)^2 = k^2 \tag{1}$$

where a and b are offsets equal to k to concentrate liquidity. The liquidity distribution, derived in Appendix B, happens to exhibit the heavy tails of a power law in price space. The AMM is compared to other invariants in Figure 4 and the RMM directly in desmos: https://www.desmos.com/calculator/o74x097731

This Concentrated Circular Market Maker (CCMM) has a unique feature that allows one to trade two negatively priced assets between one another. While appearing bizarre, it is normal to be able to exchange a barrel of oil costing negative four dollars for two natural gas costing a negative two dollars each. This is at the present moment not doable in centralized exchanges, which in the middle of COVID-19 had difficulty accounting for negative oil prices by having to switch to Arithmetic Brownian Motion from Geometric Brownian Motion via the Bachelier Model [13]. Furthermore, for an AMM where both x and y can go negative, the value of the LP position also loops on itself and, in the event of borrowing such an LP position, can exhibit double convexity [14].

2.2. Applications

The main application happens to be commodities that exhibit holding costs or create deadweight loss if not consumed in time such as electricity, natural gas, oil, and agricultural commodities. For example, excessive sunlight during the day or excessive wind during the night can push electricity prices into the negative domain [15] and may be purchased by blockchain miners and data centers positioned near such exchanges managing input costs.

Additionally, the purpose does not necessarily have to apply to the underlying price going negative. It could apply to an offset representation of the price. For example, an asset can be worth \$100 and yet its offset price token can be \$0 and now a negative offset price of \$ -1 would mean a reduction in the price of the underlying to \$99, creating a variety of intriguing non-linear payoffs resembling options shown in Figure 2 and desmos: https://www.desmos.com/calculator/qkexksh7vf.

Another application of negative prices is to a token that represents the reputation of an individual or an association (DAO, corporation, non-profit, lobbying group). A negative price would signal a negative reputation.

Consider a scenario where a reputation token does go beyond the zero bound after a large sale, maybe an insider was simply interested in making a quick profit or an anonymous team was aiming to simply deceive uninformed people. By having a zero bound the insider would be able to extract value due to information asymmetry versus the uninformed token purchasers since the price can only crash to zero. If a reputation token LP pair creator does not allow for the entry into the negative domain, then it would look incredibly suspicious in a world where reputations can go negative.

This can help deal with a lot of the anonymous rug pulls one sees where LP pair tokens with a zero bound are created in Uniswap [16]. Any swapper would instantly ask why such a token has elected to include a zero bound. If a company/person/anonymous team/DAO truly is committed to their goal, then it should not worry if one's token price can go negative. The vulnerability of the possibility of a negative price acts as a kind of skin in the game for the original token creator.

3. Appendix

3.1. A - Negative Liquidity

Taking the Uniswap invariant [16] with liquidity L

$$x * y = L^2 \tag{2}$$

introducing price p as $p = y/x$ and $y = px$

$$x * p * x = L^2 \tag{3}$$

solving for x

$$x = \sqrt{\frac{L^2}{p}}, \tag{4}$$

note that a sign appears in front of liquidity

$$x = \pm\frac{L}{\sqrt{p}}. \tag{5}$$

3.2. B - Liquidity Distribution Derivation

Start with the full invariant

$$(x-a)^2 + (y-b)^2 = k^2. \tag{6}$$

Rewrite it as a function of the numeraire

$$y_n = \pm\sqrt{k^2 - x^2 + 2ax - a^2} + b. \tag{7}$$

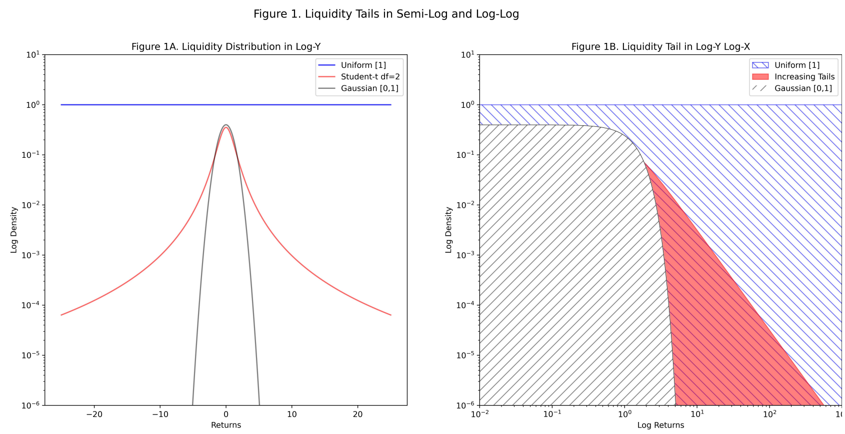

Fig. 1: A CPMM follows a uniform distribution (blue). The liquidity distribution of an RMM [18] follows a Gaussian distribution (black), yet financial assets have the tendency to exhibit power law behavior which can cause an LP to underallocate liquidity in the tails (red area). A CCMM invariant happens to have power law tails in price space, allowing LPs to allocate not just in rapid power law moves up, but also down in price, as well as into the negative price domain.

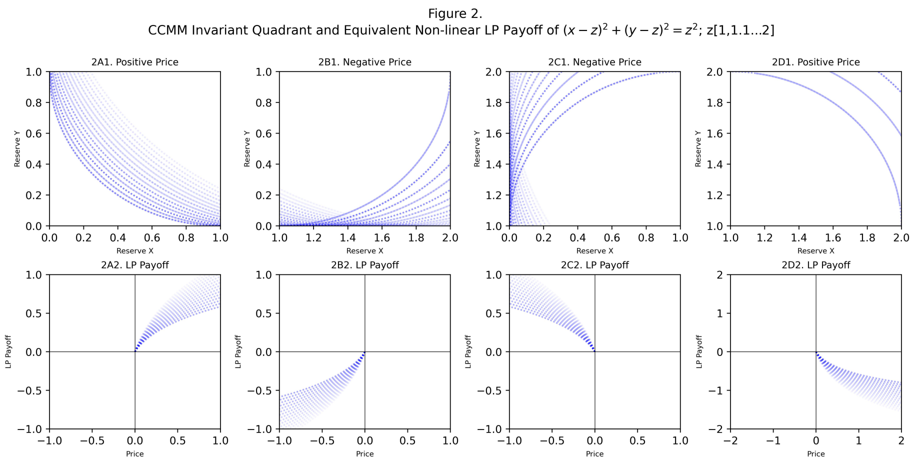

Fig. 2: LP Payoffs of CCMM. Note that an LP Position with negative prices can exhibit convexity.

Taking the derivative and negating it

$$y_x = -\frac{dy_n}{dx} = \frac{-x+a}{\sqrt{k^2 - x^2 + 2ax - a^2}}. \tag{8}$$

We invert to solve for the price of x due to the symmetric nature of the invariant

$$y_y = \frac{\sqrt{k^2 - x^2 + 2ax - a^2}}{-x+a}. \tag{9}$$

Rearranging the equation for x

$$x = \frac{ay_y^2 + a - k\sqrt{y_y^2+1}}{y_y^2+1}. \tag{10}$$

Rewrite it as a function of price $(\sqrt{p})$ by raising x^2, rewrite it as $l(x)$, take its derivative to get the liquidity distribution

$$l(x) = \frac{ax^4 + a - k\sqrt{x^4+1}}{x^4+1}. \tag{11}$$

The distribution happens to have tail of the Student's-T distribution of degree of freedom $df = 2$ - a unique statistical tail where applying the law of large numbers gives no predictability to the variance. It is also a heavy tailed distribution with a Pareto tail index of $\alpha = 3$

$$L(x) = \frac{dl(x)}{dx} = \frac{2kx^3}{(1+x^4)^{\frac{3}{2}}}. \tag{12}$$

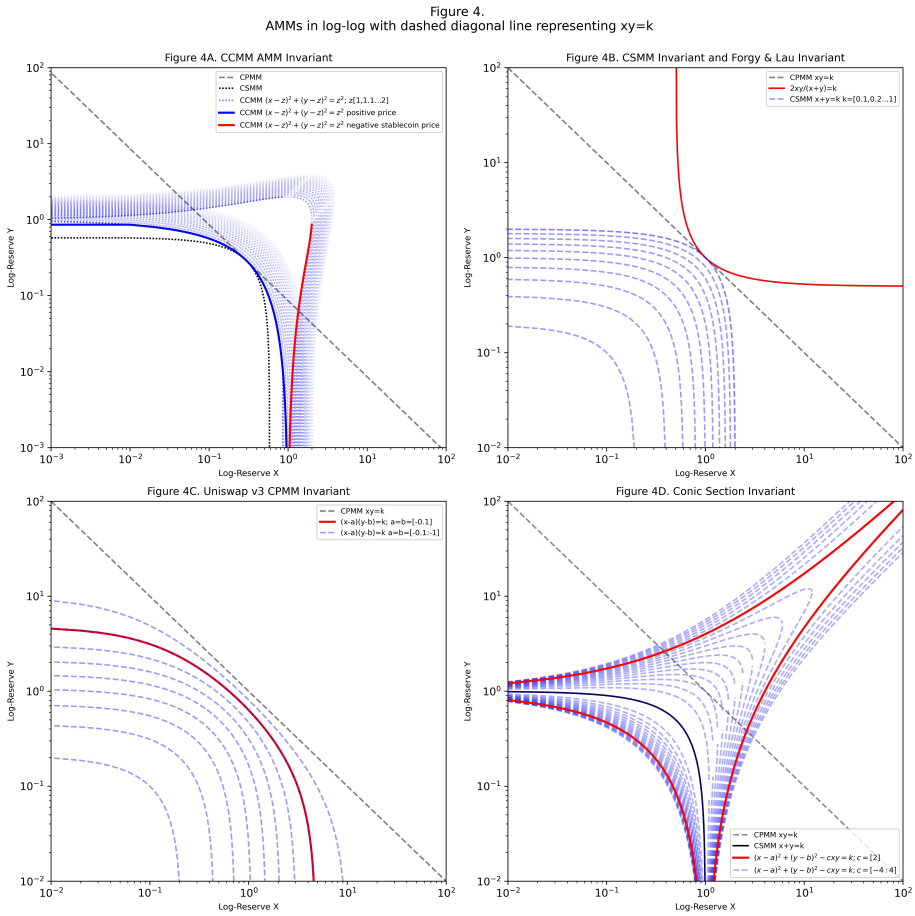

Fig. 3: 4A. CCMM invariant with red line representing a negative price for an asset priced in a currency with a zero bound such as a stablecoin. 4B: AMM invariant of no impermanent loss and CSMM. 4C: Concentrated liquidity in Uniswap v3. 4D: Concentration of liquidity in a negatively priced AMM resembles a Uniswap v3 invariant as long as price does not enter negative territory.

Converting it to tick space t we get a distribution with thicker tails than the RMM's Gaussian distribution:

$$L(t) = \frac{2ke^{3t}}{(1+e^{4t})^{\frac{3}{2}}}. \tag{13}$$

The value function [19] of an LP payoff in the positive domain is given by

$$V(p) = 2p\frac{a+bp-\sqrt{k^2p^2-a^2p^2+2abp+k^2-b^2}}{1+p^2} \tag{14}$$

with the Greeks for Delta=Δ_c, Gamma=Γ_c, and Theta=Θ_c being available in desmos and includes a comparison to a Replicating Market Maker at https://www.desmos.com/calculator/ctyi5leimp. Expected theta is derived from the relationship of concavity to yield as originally outlined by Louis Bachelier in 1900 [19]

$$\Theta_c(x) = E\langle -\frac{\sigma_{iv}^2}{2}\Gamma_c(x)\rangle. \tag{15}$$

Here implied volatility σ_{iv} can be found by looking at the implied volatility at the current ATM strike price or from the relationship between volatility and liquidity at the current tick in Uniswap v3 [20].

4. Disclaimer

This paper is for general information purposes only. It does not constitute investment advice or a recommendation or solicitation to buy or sell any investment and should not be used in the evaluation of the merits of making any investment decision. It should not be relied upon for accounting, legal or tax advice or investment recommendations. This paper reflects current opinions of the author and is not made on behalf of any individuals associated with them. The opinions reflected herein are subject to change without being updated.

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