DETERMINANT

1. INTRODUCTION :

If the equations $a_1x + b_1 = 0$, $a_2x + b_2 = 0$ are satisfied by the same value of x, then $a_1b_2 - a_2b_1 = 0$. The expression $a_1b_2 - a_2b_1$ is called a determinant of the second order, and is denoted by :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

A determinant of second order consists of two rows and two columns.

Next consider the system of equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$

If these equations are satisfied by the same values of x and y, then on eliminating x and y we get.

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

The expression on the left is called a determinant of the third order, and is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

A determinant of third order consists of three rows and three columns.

Illustration 1 : Eliminate ℓ, m, n from the equations $a\ell + cm + bn = 0$, $c\ell + bm + an = 0$, $b\ell + am + cn = 0$ and express the result in the simplest form.

Solution : The given set of equations can also be written as (if $n \neq 0$) :

$$a\left(\frac{\ell}{n}\right) + c\left(\frac{m}{n}\right) + b = 0\ ; \quad c\left(\frac{\ell}{n}\right) + b\left(\frac{m}{n}\right) + a = 0\ ; \quad b\left(\frac{\ell}{n}\right) + a\left(\frac{m}{n}\right) + c = 0$$

Then, let $\frac{\ell}{n} = x$; $\frac{m}{n} = y$

$\Rightarrow$ System of equations :

$ax + cy + b = 0$ …(i)

$cx + by + a = 0$ …(ii)

$bx + ay + c = 0$ …(iii)

We have to eliminate x & y from these simultaneous linear equations.

Since these equations are satisfied by the same values of x and y, then eliminating x and y we get,

$$\begin{vmatrix} a & c & b \\ c & b & a \\ b & a & c \end{vmatrix} = 0$$

2. VALUE OF A DETERMINANT :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Note : Sarrus diagram to get the value of determinant of order three :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{array}{ccc|cc} & & -ve & -ve & -ve \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ & & +ve & +ve & +ve \end{array} = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$$

Note that the product of the terms in first bracket (i.e. $a_1a_2a_3b_1b_2b_3c_1c_2c_3$) is same as the product of the terms in second bracket.

Illustration 2 : The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ is -

(A) 213 (B) – 231 (C) 231 (D) 39

Solution : $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = 1\begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ 2 & -7 \end{vmatrix}$

$= (27 + 42) - 2(-36 - 12) + 3(28 - 6) = 231$

Alternative : By sarrus diagram

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{matrix}\right|\begin{matrix} 1 & 2 \\ -4 & 3 \\ 2 & -7 \end{matrix}$$

$= (27 + 24 + 84) - (18 - 42 - 72) = 135 - (18 - 114) = 231$ **Ans. (C)**

3. MINORS & COFACTORS :

The minor of a given element of determinant is the determinant obtained by deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have "9 minors".

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element is given by : $C_{ij} = (-1)^{i+j}.\ M_{ij}$

Illustration 3 : Find the minors and cofactors of elements '–3', '5', '–1' & '7' in the determinant $\begin{vmatrix} 2 & -3 & 1 \\ 4 & 0 & 5 \\ -1 & 6 & 7 \end{vmatrix}$

Solution : Minor of $-3 = \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix} = 33$; Cofactor of $-3 = -33$

Minor of $5 = \begin{vmatrix} 2 & -3 \\ -1 & 6 \end{vmatrix} = 9$; Cofactor of $5 = -9$

Minor of $-1 = \begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix} = -15$; Cofactor of $-1 = -15$

Minor of $7 = \begin{vmatrix} 2 & -3 \\ 4 & 0 \end{vmatrix} = 12$; Cofactor of $7 = 12$

4. EXPANSION OF A DETERMINANT IN TERMS OF THE ELEMENTS OF ANY ROW OR COLUMN:

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(i) The sum of the product of elements of any row (column) with their corresponding cofactors is always equal to the value of the determinant.

D can be expressed in any of the six forms :

$a_1A_1 + b_1B_1 + c_1C_1$, $a_1A_1 + a_2A_2 + a_3A_3$,

$a_2A_2 + b_2B_2 + c_2C_2$, $b_1B_1 + b_2B_2 + b_3B_3$,

$a_3A_3 + b_3B_3 + c_3C_3$, $c_1C_1 + c_2C_2 + c_3C_3$,

where A_i, B_i & C_i $(i = 1,2,3)$ denote cofactors of a_i, b_i & c_i respectively.

(ii) The sum of the product of elements of any row (column) with the cofactors of other row (column) is always equal to zero.

Hence,

$a_2A_1 + b_2B_1 + c_2C_1 = 0$,

$b_1A_1 + b_2A_2 + b_3A_3 = 0$ and so on.

where A_i, B_i & C_i $(i = 1,2,3)$ denote cofactors of a_i, b_i & c_i respectively.

Do yourself -1 :

(i) Find minors & cofactors of elements '6', '5', '0' & '4' of the determinant $\begin{vmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \\ 3 & 0 & 4 \end{vmatrix}$.

(ii) Calculate the value of the determinant $\begin{vmatrix} 5 & -3 & 7 \\ -2 & 4 & -8 \\ 9 & 3 & -10 \end{vmatrix}$

(iii) The value of the determinant $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix}$ is equal to -

(A) $a^3 - b^3$ (B) $a^3 + b^3$ (C) 0 (D) none of these

(iv) Find the value of 'k', if $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & k & 2 \end{vmatrix} = 4$

(v) Prove that $\begin{vmatrix} 1 & z & -y \\ -z & 1 & x \\ y & -x & 1 \end{vmatrix} = 1 + x^2 + y^2 + z^2$

5. PROPERTIES OF DETERMINANTS :

(a) The value of a determinant remains unaltered, if the rows & columns are inter-changed,

e.g. if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D_1 = -D$.

(c) If all the elements of a row (or column) are zero, then the value of the determinant is zero.

(d) If all the elements of any row (or column) are multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D_1 = KD$

(e) If all the elements of a row (or column) are proportional (or identical) to the element of any other row, then the determinant vanishes, i.e. its value is zero.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D = 0$; If $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D_1 = 0$

Illustration 4 : Prove that $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

Solution :

$$D = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} \quad \text{(By interchanging rows \& columns)}$$

$$= -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} \quad (C_1 \leftrightarrow C_2)$$

$$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} \quad (R_1 \leftrightarrow R_2)$$

Illustration 5 : Find the value of the determinant $\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix}$

Solution :

$$D = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = a\begin{vmatrix} a & b & c \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = abc\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0$$

Since all rows are same, hence value of the determinant is zero.

Do yourself -2 :

(i) Without expanding the determinant prove that $\begin{vmatrix} a & p & \ell \\ b & q & m \\ c & r & n \end{vmatrix} + \begin{vmatrix} r & n & c \\ q & m & b \\ p & \ell & a \end{vmatrix} = 0$

(ii) If $D = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$, then $\begin{vmatrix} 2\alpha & 2\beta \\ 2\gamma & 2\delta \end{vmatrix}$ is equal to -

(A) D (B) 2D (C) 4D (D) 16D

(iii) If $D = \begin{vmatrix} p & q & r \\ x & y & z \\ \ell & m & n \end{vmatrix}$, then KD is equal to -

(A) $\begin{vmatrix} Kp & q & r \\ x & Ky & z \\ \ell & m & Kn \end{vmatrix}$ (B) $\begin{vmatrix} p & q & r \\ x & y & z \\ K\ell & Km & Kn \end{vmatrix}$ (C) $\begin{vmatrix} p & Kx & \ell \\ q & Ky & m \\ r & Kz & n \end{vmatrix}$ (D) $\begin{vmatrix} Kp & Kx & K\ell \\ Kq & Ky & Km \\ Kr & Kz & Kn \end{vmatrix}$

(f) If each element of any row (or column) is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

e.g. $\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note that : If $D_r = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

where $r \in N$ and a, b, c, a_1, b_1, c_1 are constants, then

$$\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(g) **Row - column operation :** The value of a determinant remains unaltered under a column (C_i) operation of the form $C_i \rightarrow C_i + \alpha C_j + \beta C_k$ $(j, k \neq i)$ or row (R_i) operation of the form $R_i \rightarrow R_i + \alpha R_j + \beta R_k$ $(j, k \neq i)$. In other words, the value of a determinant is not altered by adding the elements of any row (or column) to the same multiples of the corresponding elements of any other row (or column)

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D = \begin{vmatrix} a_1 + \alpha a_2 & b_1 + \alpha b_2 & c_1 + \alpha c_2 \\ a_2 & b_2 & c_2 \\ a_3 + \beta a_1 & b_3 + \beta b_1 & c_3 + \beta c_1 \end{vmatrix} \quad (R_1 \rightarrow R_1 + \alpha R_2;\ R_3 \rightarrow R_3 + \beta R_2)$$

Note : (i) By using the operation $R_i \rightarrow xR_i + yR_j + zR_k$ $(j, k \neq i)$, the value of the determinant becomes x times the original one.

(ii) While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

Illustration 6 : If $D_r = \begin{vmatrix} r & r^3 & 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix}$, find $\sum_{r=0}^{n} D_r$.

Solution :

$$\sum_{r=0}^{n} D_r = \begin{vmatrix} \sum_{r=0}^{n} r & \sum_{r=0}^{n} r^3 & \sum_{r=0}^{n} 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix} = \begin{vmatrix} \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix} = 0$$ **Ans.**

Illustration 7 : Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution :

$$D = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$D = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad (R_1 \to R_1 + R_2 + R_3)$$

$$D = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$D = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} \quad (C_3 \to C_3 - C_1 ;\ C_2 \to C_2 - C_1)$$

$D = (a + b + c)^3$

Illustration 8 : Determinant $\begin{vmatrix} a+b+nc & (n-1)a & (n-1)b \\ (n-1)c & b+c+na & (n-1)b \\ (n-1)c & (n-1)a & c+a+nb \end{vmatrix}$ is equal to -

(A) $(a + b + c)^3$ (B) $n(a + b + c)^3$ (C) $(n - 1)(a + b + c)^3$ (D) none of these

Solution : Applying $C_1 \to C_1 + (C_2 + C_3)$

$$D = n(a+b+c)\begin{vmatrix} 1 & (n-1)a & (n-1)b \\ 1 & b+c+na & (n-1)b \\ 1 & (n-1)a & c+a+nb \end{vmatrix}$$

$$D = n(a+b+c)\begin{vmatrix} 1 & (n-1)a & (n-1)b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix} \quad \begin{Bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{Bmatrix}$$

$= n(a + b + c)^3$ **Ans. (B)**

Illustration 9 : If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then the value of k is-

(A) 2 (B) 1 (C) –1 (D) 0

Solution : Applying ($C_3 \to C_3 - C_1$)

$$D = \begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0 \qquad (R_3 \to R_3 - R_2;\ R_2 \to R_2 - R_1)$$

$\Rightarrow \quad k - 1 = 0 \Rightarrow k = 1$ **Ans. (B)**

Do yourself - 3 :

(i) Find the value of $\begin{vmatrix} 53 & 106 & 159 \\ 52 & 65 & 91 \\ 102 & 153 & 221 \end{vmatrix}$.

(ii) Solve for x : $\begin{vmatrix} x & 2 & 0 \\ 2+x & 5 & -1 \\ 5-x & 1 & 2 \end{vmatrix} = 0$

(iii) Using row-column operations prove that

(a) $\begin{vmatrix} x+a & x+b & x+c \\ y+a & y+b & y+c \\ z+a & z+b & z+c \end{vmatrix} = 0$

(b) $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

(iv) If $D_r = \begin{vmatrix} 2r & 1 & n \\ 1 & -2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$, then find the value of $\sum_{r=1}^{n} D_r$.

(h) **Factor theorem :** If the elements of a determinant D are rational integral functions of x and two rows (or columns) become identical when x = a then (x – a) is a factor of D.

Note that if r rows become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of D.

Illustration 10 : Prove that $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = m(x-a)(x-b)$

Solution : Using factor theorem,

Put x = a

$$D = \begin{vmatrix} a & a & a \\ m & m & m \\ b & a & b \end{vmatrix} = 0$$

Since R_1 and R_2 are proportional which makes D = 0, therefore (x – a) is a factor of D.

Similarly, by putting x = b, D becomes zero, therefore (x – b) is a factor of D.

$$D = \begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = \lambda(x-a)(x-b) \qquad \ldots\ldots\ldots(i)$$

To get the value of λ, put $x = 0$ in equation (i)

$$\begin{vmatrix} a & a & 0 \\ m & m & m \\ b & 0 & b \end{vmatrix} = \lambda ab$$

$amb = \lambda ab \quad \Rightarrow \quad \lambda = m$

$\therefore \qquad D = m(x-a)(x-b)$

Illustration 11 : Prove that $\begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix} = 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$

Solution : $D = \begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix}$

Using factor theorem,

Put $x = y$

$$D = \begin{vmatrix} (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix}$$

R_1 and R_2 are identical which makes $D = 0$. Therefore, $(x-y)$ is a factor of D.

Similarly $(y-z)$ & $(z-x)$ are factors of D

Now put $a = b$

$$D = \begin{vmatrix} (x-b)^2 & (x-b)^2 & (x-c)^2 \\ (y-b)^2 & (y-b)^2 & (y-c)^2 \\ (z-b)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix}$$

C_1 and C_2 become identical which makes $D = 0$. Therefore, $(a-b)$ is a factor of D.

Similarly $(b-c)$ and $(c-a)$ are factors of D.

Therefore, $D = \lambda(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$

To get the value of λ put $x = -1 = a$, $y = 0 = b$ and $z = 1 = c$

$$D = \begin{vmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{vmatrix} = \lambda(-1)(-1)(2)(-1)(-1)(2)$$

$\Rightarrow 4\lambda = 8 \Rightarrow \lambda = 2$

$\therefore \qquad D = 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$

Do yourself - 4 :

(i) Without expanding the determinant prove that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) Using factor theorem, find the solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

6. MULTIPLICATION OF TWO DETERMINANTS :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D_1 is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D_1 = D^{n-1}$

Illustration 12 : If a, b, c x, y, z $\in$ R, then prove that $\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}$

Solution :

$$\text{L.H.S.} = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2az+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix} \quad \text{(Row by Row)}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times (-1) \begin{vmatrix} a^2 & 2a & 1 \\ b^2 & 2b & 1 \\ c^2 & 2c & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times (-1)(-1) \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \quad (C_1 \leftrightarrow C_3)$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix}$$

Multiplying row by row

$$= \begin{vmatrix} 1+2ax+a^2x^2 & 1+2bx+b^2x^2 & 1+2cx+c^2x^2 \\ 1+2ay+a^2y^2 & 1+2by+b^2y^2 & 1+2cy+c^2y^2 \\ 1+2az+a^2z^2 & 1+2bz+b^2z^2 & 1+2cz+c^2z^2 \end{vmatrix}$$

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}$$

= R.H.S.

Illustration 13 : Let α & β be the roots of equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ for $n \geq 1$. Evaluate the value of the determinant $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

Solution :

$$D = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix} = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}^2 = [(1-\alpha)(1-\beta)(\alpha-\beta)]^2$$

$D = (\alpha - \beta)^2 (\alpha + \beta - \alpha\beta - 1)^2$

$\because$ α & β are roots of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad \alpha+\beta = \frac{-b}{a} \quad \& \quad \alpha\beta = \frac{c}{a} \quad \Rightarrow \quad |\alpha-\beta| = \frac{\sqrt{b^2-4ac}}{|a|}$$

$$D = \frac{(b^2-4ac)}{a^2}\left(\frac{a+b+c}{a}\right)^2 = \frac{(b^2-4ac)(a+b+c)^2}{a^4}$$

Ans.

Illustration 14 : If $D_1 = \begin{vmatrix} y^5z^6(z^3-y^3) & x^4z^6(x^3-z^3) & x^4y^5(y^3-x^3) \\ y^2z^3(y^6-z^6) & xz^3(z^6-x^6) & xy^2(x^6-y^6) \\ y^2z^3(z^3-y^3) & xz^3(x^3-z^3) & xy^2(y^3-x^3) \end{vmatrix}$ and $D_2 = \begin{vmatrix} x & y^2 & z^3 \\ x^4 & y^5 & z^6 \\ x^7 & y^8 & z^9 \end{vmatrix}$. Then D_1D_2 is equal to -

(A) D_2^3 (B) D_2^2 (C) D_2^4 (D) none of these

Solution : The given determinant D_1 is obtained by corresponding cofactors of determinant D_2.

Hence $D_1 = D_2^{\,2}$ $\Rightarrow D_1D_2 = D_2^2D_2 = D_2^3$ **Ans. (A)**

Do yourself - 5 :

(i) If the determinant $D = \begin{vmatrix} 1 & 1 & 1 \\ \alpha+\beta & \alpha^2+\beta^2 & 2\alpha\beta \\ \alpha+\beta & 2\alpha\beta & \alpha^2+\beta^2 \end{vmatrix}$ and $D_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \alpha \end{vmatrix}$, then find the determinant D_2 such that $D_2 = \frac{D}{D_1}$.

(ii) If $D_1 = \begin{vmatrix} ab^2-ac^2 & bc^2-a^2b & a^2c-b^2c \\ ac-ab & ab-bc & bc-ac \\ c-b & a-c & b-a \end{vmatrix}$ & $D_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix}$, then D_1D_2 is equal to -

(A) 0 (B) D_1^2 (C) D_2^2 (D) D_2^3

7. SPECIAL DETERMINANTS :

(a) Cyclic Determinant :

The elements of the rows (or columns) are in cyclic arrangement.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) = -(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

$$= -\frac{1}{2}(a+b+c) \times \{(a-b)^2+(b-c)^2+(c-a)^2\}$$

$= -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$, where ω, ω^2 are cube roots of unity

(b) Other Important Determinants :

(i) $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

Illustration 15 : Prove that $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1-\alpha^3)^2$.

Solution : This is a cyclic determinant.

$$\Rightarrow \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1+\alpha+\alpha^2)(1+\alpha^2+\alpha^4-\alpha-\alpha^2-\alpha^3)$$

$$= -(1+\alpha+\alpha^2)(-\alpha+1-\alpha^3+\alpha^4) = -(1+\alpha+\alpha^2)(1-\alpha)^2(1+\alpha+\alpha^2)$$

$$= -(1-\alpha)^2(1+\alpha+\alpha^2)^2 = -(1-\alpha^3)^2$$

Do yourself - 6 :

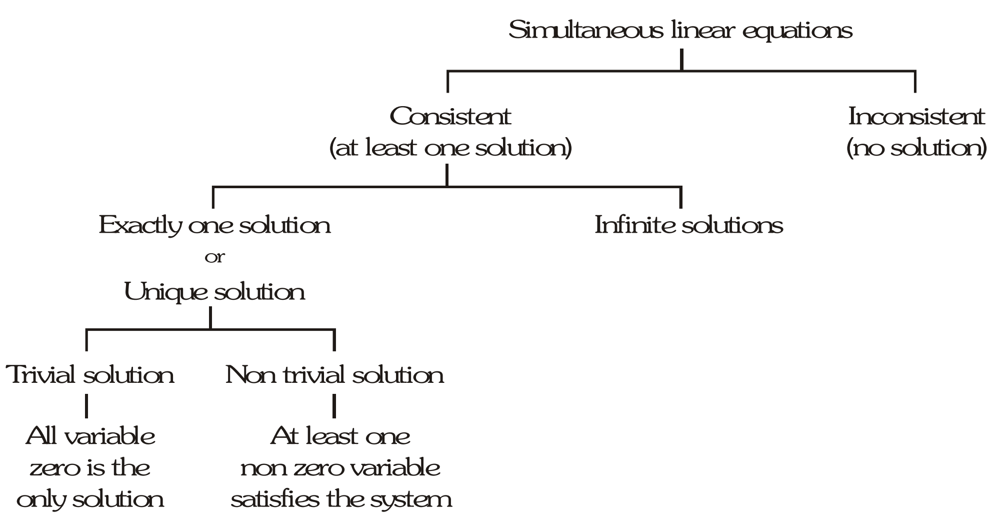
(i) The value of the determinant $\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix}$ is

(A) $k(a+b)(b+c)(c+a)$
(B) $kabc(a^2+b^2+c^2)$
(C) $k(a-b)(b-c)(c-a)$
(D) $k(a+b-c)(b+c-a)(c+a-b)$

(ii) Find the value of the determinant $\begin{vmatrix} a^2+b^2 & a^2-c^2 & a^2-c^2 \\ -a^2 & 0 & c^2-a^2 \\ b^2 & -c^2 & b^2 \end{vmatrix}$.

(iii) Prove that $\begin{vmatrix} a & b & c \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

8. CRAMER'S RULE (SYSTEM OF LINEAR EQUATIONS) :



(a) Equations involving two variables :

(i) Consistent Equations : Definite & unique solution (Intersecting lines)

(ii) Inconsistent Equations : No solution (Parallel lines)

(iii) Dependent Equations : Infinite solutions (Identical lines)

Let, $a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$ then :

(1) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ Given equations are consistent with unique solution

(2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent

(3) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are consistent with infinite solutions

(b) Equations Involving Three variables :

Let $a_1x + b_1y + c_1z = d_1$ .... (i)

$a_2x + b_2y + c_2z = d_2$ .... (ii)

$a_3x + b_3y + c_3z = d_3$ .... (iii)

Then, $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Note :

(i) If $D \neq 0$ and atleast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations is consistent and has unique non trivial solution.

(ii) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has trivial solution only.

(iii) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has infinite solutions.

Note that In case $\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_1x + b_1y + c_1z &= d_2 \\ a_1x + b_1y + c_1z &= d_3 \end{aligned}\right\}$ (Atleast two of d_1 , d_2 & d_3 are not equal)

$D = D_1 = D_2 = D_3 = 0$. But these three equations represent three parallel planes. Hence the system is inconsistent.

(iv) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(c) Homogeneous system of linear equations :

Let $a_1x + b_1y + c_1z = 0$ .... (i)

$a_2x + b_2y + c_2z = 0$ .... (ii)

$a_3x + b_3y + c_3z = 0$ .... (iii)

$\Rightarrow$ $D_1 = D_2 = D_3 = 0$

$\therefore$ The system always possesses atleast one solution $x = 0$, $y = 0$, $z = 0$, which is called **Trivial** solution, i.e. this system is always **consistent.**

Check value of D

$D \neq 0$ → **Unique Trivial** solution

$D = 0$ → **Trivial & Non-Trivial** solutions (infinite solutions)

Note that if a given system of linear equations has **Only Zero** solutions for all its variables then the given equations are said to have **TRIVIAL SOLUTION.**

Also, note that if the system of equations $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$; $a_3x + b_3y + c_3 = 0$

is always consistent then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ but converse is **NOT** true.

9. APPLICATION OF DETERMINANTS IN GEOMETRY :

(a) The lines : $a_1x + b_1y + c_1 = 0$ .... (i)

$a_2x + b_2y + c_2 = 0$ .... (ii)

$a_3x + b_3y + c_3 = 0$ .... (iii)

are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

This is the condition for consistency of three simultaneous linear equations in 2 variables.

(b) Equation $ax + 2hxy + by + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$abc + 2fgh - af - bg - ch = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(c) Area of a triangle whose vertices are (x_r, y_r) ; $r = 1, 2, 3$ is $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

If $D = 0$ then the three points are collinear.

(d) Equation of a straight line passing through points (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Illustration 16 : Find the nature of solution for the given system of equations :

$x + 2y + 3z = 1$; $2x + 3y + 4z = 3$; $3x + 4y + 5z = 0$

Solution :

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$$\text{Now, } D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 0 & 4 & 5 \end{vmatrix} = 5$$

$\because$ $D = 0$ but $D_1 \neq 0$

Hence no solution. **Ans.**

Illustration 17 : Find the value of λ, if the following equations are consistent :

$x + y - 3 = 0;\ (1 + \lambda)x + (2 + \lambda)y - 8 = 0;\ x - (1 + \lambda)y + (2 + \lambda) = 0$

Solution : The given equations in two unknowns are consistent, then $\Delta = 0$

i.e. $$\begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$

$$\therefore \quad \begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & 3\lambda-5 \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (5+\lambda) - (3\lambda-5)(-2-\lambda) = 0 \qquad \Rightarrow \qquad 3\lambda^2 + 2\lambda - 5 = 0$

$\therefore \quad \lambda = 1, -5/3$

Illustration 18 : If the system of equations $x + \lambda y + 1 = 0$, $\lambda x + y + 1 = 0$ & $x + y + \lambda = 0$. is consistent then find the value of λ.

Solution : For consistency of the given system of equations

$$D = \begin{vmatrix} 1 & \lambda & 1 \\ \lambda & 1 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 3\lambda = 1 + 1 + \lambda^3$ or $\lambda^3 - 3\lambda + 2 = 0 \quad \Rightarrow \quad (\lambda-1)^2 (\lambda + 2) = 0 \Rightarrow \lambda = 1$ or $\lambda = -2$ **Ans.**

Illustration 19 : If x, y, z are not all simultaneously equal to zero, satisfying the system of equations

$\sin(3\theta)\, x - y + z = 0;\ \cos(2\theta)x + 4y + 3z = 0;\ 2x + 7y + 7z = 0$, then find the values of $\theta (0 \le \theta \le 2\pi)$.

Solution : Given system of equations is a system of homogeneous linear equations which posses non-zero solution set, therefore $D = 0$.

$$\Rightarrow \quad D = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} \quad \Rightarrow \quad D = \begin{vmatrix} \sin 3\theta & -1 & 0 \\ \cos 2\theta & 4 & 7 \\ 2 & 7 & 14 \end{vmatrix} \qquad (C_3 \to C_3 + C_2)$$

$$D = \begin{vmatrix} \sin 3\theta & -1 & 0 \\ \cos 2\theta - 1 & 0.5 & 0 \\ 2 & 7 & 14 \end{vmatrix} \qquad (R_2 \to R_2 - \frac{R_3}{2})$$

$$D = 14\left(\frac{\sin 3\theta}{2} + \cos 2\theta - 1\right)$$

$\because \quad D = 0$

$\therefore \quad \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow \quad 3\sin\theta - 4\sin^3\theta = 4\sin^2\theta \quad \Rightarrow \quad (\sin\theta)(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow \quad (\sin\theta)(2\sin\theta - 1)(2\sin\theta + 3) = 0 \quad \Rightarrow \quad \sin\theta = 0\ ;\ \sin\theta = \frac{1}{2}\ ;\ \sin\theta = -\frac{3}{2}$

$\sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi; \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}; \quad \sin\theta = -\frac{3}{2} \Rightarrow$ no solution.

$\Rightarrow \quad \theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$ **Ans.**

Do yourself -7 :

(i) Find nature of solution for given system of equations

$2x + y + z = 3; \quad x + 2y + z = 4\ ; \quad 3x + z = 2$

(ii) If the system of equations $x + y + z = 2$, $2x + y - z = 3$ & $3x + 2y + kz = 4$ has a unique solution then

(A) $k \neq 0$ (B) $-1 < k < 1$ (C) $-2 < k < 1$ (D) $k = 0$

(iii) The system of equations $\lambda x + y + z = 0$, $-x + \lambda y + z = 0 = 0$ & $-x - y + \lambda z = 0$ has a non-trivial solution, then possible values of λ are -

(A) 0 (B) 1 (C) –3 (D) $\sqrt{3}$

ANSWERS FOR DO YOURSELF

1. (i) minors : 4, –1, –4, 4 ; cofactors : –4, –1, 4, 4 (ii) –98 (iii) B (iv) 0

2. (ii) C (iii) B,C

3. (i) 0 (ii) 2 (iv) 0

4. (ii) x = –1, 2

5. (i) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \beta & \alpha \end{vmatrix}$ (ii) D

6. (i) C (ii) 0

7. (i) infinite solutions (ii) A (iii) A