ISI CMI OLYMPIAD MATHEMATICS ONLINE MATERIAL

- **OLYMPIAD TOPIC DISCUSSIONS**
- **TOMATO SOLUTION**
- **PREVIOUS YEAR SOLUTIONS**

Also see : http://www.ctanujit.org/isi-bstat-bmath-exam.html

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Number Theory

Congruency :

$a \equiv b \pmod{c}$

It reads a congruent to b modulus c.

It means a, b, c are integers and $a - b$ is divisible by c or if a is divided by c then the remainder is b.

Example :

$7 \equiv 2 \pmod{5}$

Properties :

(a) If $a \equiv b \pmod{c}$ then $a^n \equiv b^n \pmod{c}$

Proof :

Now, $a^n - b^n = (a - b)(a^{n-1} + ba^{n-2} + \ldots + b^{n-2}a + b^{n-1})$

Now, $a - b$ is divisible by c.

- ⇨ $a^n - b^n$ is divisible by c.
- ⇨ $a^n \equiv b^n \pmod{c}$

Proved.

(b) If $a \equiv b \pmod{c}$ then $ad \equiv bd \pmod{c}$.

Proof :

Now, ad – bd = d(a – b)

(a – b) is divisible by c

So, (a – b)d is also divisible by c.

⇨ ad ≡ bd (mod c)

Proved.

(c) If a ≡ b (mod c) then a + d ≡ b + d (mod c).

Proof :

Now, (a + d) – (b + d) = (a – b) divisible by c.

⇨ a + d ≡ b + d (mod c)

Proved.

(d) If a ≡ b (mod c) then a – d ≡ b – d (mod c).

Proof :

Now, (a – d) – (b – d) = (a – b) divisible by c.

⇨ a – d ≡ b – d (mod c)

Proved.

(e) If a ≡ b (mod c) and a ≡ b (mod d) then a ≡ b (mod cd) if gcd(c, d) = 1.

Proof :

Now, a − b is divisible by c and d and gcd(c, d) = 1.

- ⇨ a − b is divisible by cd.
- ⇨ a ≡ b (mod cd)

(f) If a ≡ b (mod c) then a/d ≡ b/d (mod c) where gcd(c, d) = 1.

Proof :

Now, (a/d) − (b/d) = (a − b)/d

Now, c divides a − b so c divides (a − b)/d as c and d are relatively prime.

- ⇨ a/d ≡ b/d (mod c)

Corollary :

If a ≡ b (mod c) and gcd(c, d) = g then a/d ≡ b/d (mod c/g)

Proof :

Now, (a/d) − (b/d) = (a − b)/d = (a − b)/gd_1 where d = gd_1

Now, gcd(d_1, c) = 1

Now, c divides (a − b)

- ⇨ c/g divides (a − b)/g.
- ⇨ a/d ≡ b/d (mod c/g)

Proved.

An example :

Find the remainder when $(n^2 + 1)/2$ is divided by 4 where n is odd.

Solution :

Now, as n is odd so $n \equiv 1$ or $3 \pmod 4$

- ⇨ $n \equiv \pm 1 \pmod 4$
- ⇨ $n^2 \equiv (\pm 1)^2 \pmod 4$
- ⇨ $n^2 \equiv 1 \pmod 4$
- ⇨ $n^2 + 1 \equiv 1 + 1 \pmod 4$
- ⇨ $n^2 + 1 \equiv 2 \pmod 4$
- ⇨ $(n^2 + 1)/2 \equiv 2/2 \pmod 4$
- ⇨ $(n^2 + 1)/2 \equiv 1 \pmod 4$

This is wrong as we have divided by 2 and gcd(2, 4) = 2 and w3e have not changed the modulus accordingly.

This is a common mistake done by lots of students.

Right procedure is,

$n \equiv \pm 1, \pm 3 \pmod 8$ (We are starting from modulus 8 as we have to divide by 2 and the modulus will become 8/2 = 4 and we will get the result)

- ⇨ $n^2 \equiv 1, 9 \pmod 8$
- ⇨ $n^2 \equiv 1 \pmod 8$ (As $9 \equiv 1 \pmod 8$)
- ⇨ $n^2 + 1 \equiv 2 \pmod 8$
- ⇨ $(n^2 + 1)/2 \equiv 2/2 \pmod{8/2}$ (As gcd(8, 2) = 2)
- ⇨ $(n^2 + 1)/2 \equiv 1 \pmod 4$

Note that though the answer was correct but the procedure was wrong.

Tips to solve number theory problems :

1. $(\text{Any odd number})^2 \equiv 1 \pmod 4$

Proof :

Let n is odd.

Then $n \equiv \pm 1 \pmod 4$

- ⇨ $n^2 \equiv (\pm 1)^2 \pmod 4$
- ⇨ $n^2 \equiv 1 \pmod 4$

Proved.

Application :

Problem 1 :

Prove that sum of squares of 8n+4 consecutive positive integers cannot be perfect square.

Solution 1 :

There are 8n + 4 consecutive integers.

- ⇨ There are 4n + 2 odd integers and 4n + 2 even integers.

Now, $(\text{any even integer})^2 \equiv 0 \pmod 4$ (As there must be a factor at least 2 in even integer and when it is squared then $2^2 = 4$ is at least a factor of square of an even number).

So, sum of all even integer square $\equiv 0 \pmod 4$

Now, there are 4n + 2 odd integers.

If they are summed up it will give an even integer as there are even number of odd terms.

So, let us consider the equation, $a_1^2 + a_2^2 + \dots + a_{4n+2}^2 = b^2$ where a_i's are odd i = 1, 2, .., 4n+2 and b is even.

Now, dividing the equation by 4 we get,

$1 + 1 + \dots$ (4n + 2) times $\equiv 0 \pmod 4$ (As b is even)

- ⇨ $4n + 2 \equiv 0 \pmod 4$
- ⇨ $2 \equiv 0 \pmod 4$ (As $4n \equiv 0 \pmod 4$)

Which is impossible.

So, sum of squares of 8n + 4 consecutive positive integers cannot be perfect square.

Proved.

Problem 2 :

Prove that sum of squares of 4n+3 number of odd integers cannot be perfect square where $n \geq 0$.

Solution 2 :

Let us take the equation, $a_1^2 + a_2^2 + \ldots\ldots + a_{4n+3}^2 = b^2$ where a_i's are odd $i = 1, 2, \ldots, 4n+3$

Now, b^2 is sum of odd number (4n + 3) of odd numbers (a_i^2)

- ⇨ b^2 is odd.

Now, dividing the equation by 4 we get,

$1 + 1 + \ldots.$ (4n+3 times) $\equiv 1 \pmod 4$

- ⇨ $4n + 3 \equiv 1 \pmod 4$
- ⇨ $3 \equiv 1 \pmod 4$ (As $4n \equiv 0 \pmod 4$)

Which is impossible.

Proved.

2. (Any odd integer)$^2 \equiv 1 \pmod 8$

Proof :

Let, n is any odd integer.

Now, $n \equiv 1, 3, 5, 7 \pmod 8$

- ⇨ $n^2 \equiv 1, 9, 25, 49 \pmod 8$
- ⇨ $n^2 \equiv 1 \pmod 8$ (As, 9, 25, 49 all $\equiv 1 \pmod 8$)

Proved.

3. (Any odd integer)$^4 \equiv 1 \pmod{16}$

Proof :

Let n is any odd integer.

Now, $n^2 \equiv 1 \pmod 8$

- ⇨ $n^2 \equiv 1, 9 \pmod{16}$
- ⇨ $(n^2)^2 \equiv 1, 81 \pmod{16}$
- ⇨ $n^4 \equiv 1 \pmod{16}$ (As $81 \equiv 1 \pmod{16}$)

Proved.

4. In general, (Any odd integer)^(2^n) $\equiv 1$ (mod $2^{(n+2)}$)

Proof :

We will prove it by induction.

Let m is any odd integer.

For n = 1 we have, $m^2 \equiv 1 \pmod 8$

Let, for n = k this is true i.e. we have, m^(2^k) $\equiv 1$ (mod $2^{(k+2)}$)

Now, m^(2^{k+1}) = {m^(2^k)}^{2}

Now, we have, m^(2^k) $\equiv 1$ (mod 2^{k+2})

- ⇨ m^(2^k) $\equiv 1, (1 + 2^{k+2})$ (mod 2^{k+3})
- ⇨ {m^(2^k)}$^2 \equiv 1, (1 + 2^{k+3} + 2^{2k+4})$ (mod 2^{k+3})
- ⇨ m^(2^{k+1}) $\equiv 1$ (mod 2^{k+3})

Proved.

Application :

Problem 1 :

Prove that the sum of 2^n-th power of any $m*(2^{n+2}) + 2r + 1$ odd integers cannot be 2^n-th power of an integer where m, n ∈ N and r is an integer such that $0 < r < 2^{n+1}$.

Solution 1 :

Let, $k = m*(2^{n+2}) + 2r + 1$.

Let us consider the equation, $a_1\text{^}2^n + a_2\text{^}2^n + ... + a_k\text{^}2^n = b\text{^}2^n$. Clearly b is odd.

Dividing the equation by 2^{n+2} we get,

$1 + 1 + ...$ (k times) $\equiv 1 \pmod{2^{n+2}}$

- ⇨ $k \equiv 1 \pmod{2^{n+2}}$
- ⇨ $m*(2^{n+2}) + 2r + 1 \equiv 1 \pmod{2^{n+2}}$
- ⇨ $2r \equiv 0 \pmod{2^{n+2}}$
- ⇨ $r \equiv 0 \pmod{2^{n+1}}$\

Now, $r < 2^{n+1}$ so the above equation cannot hold true.

Proved.

5. If m is any odd integer not divisible by 3 then $m^2 \equiv 1 \pmod{24}$.

Proof :

As m is not divisible by 3 so, $m \equiv \pm 1 \pmod{3}$

- ⇨ $m^2 \equiv 1 \pmod 3$

And from above we have, $m^2 \equiv 1 \pmod 8$

As gcd(3, 8) = 1 so we have, $m^2 \equiv 1 \pmod{3*8}$

Proved.

Application :

Problem 1 :

Prove that sum of squares of 24n + 17 primes where all the primes > 3 cannot be a square number.

Solution 1 :

This problem cannot be solved by dividing by 4 or 8.

This problem needs to be solved by dividing by 24.

As all the primes > 3 so they don't have a factor of 3.

Consider the equation, $a_1^2 + a_2^2 + \ldots + a_{24n+17}^2 = b^2$ where a_i's are odd primes and b is odd.

Dividing the equation by 24 we get,

$1 + 1 + \ldots$ (24n+17 times) $\equiv 1$ or $9 \pmod{24}$ (If b is not divisible by 3 then 1 if divisible then 9)

- ⇨ $24n + 17 \equiv 1$ or $9 \pmod{24}$
- ⇨ $17 \equiv 1$ or $9 \pmod{24}$

Which is impossible.

Proved.

Note : This problem can also be solved by dividing by 3.

6. If any n consecutive positive integers are divided by n then remainders are 0, 1, 2, 3, ..., n-1 where the loop remains same but starting and ending rotates.

Proof :

Self-explanatory.

Application :

Problem 1 :

Prove that sum of squares of n consecutive positive integers is divisible by n where n is odd and n is not divisible by 3.

Solution :

Now, we have seen that the remainders are 0, 1, 2, ..., n-1 in order but the starting point may be different. So, if we take sum then we are good to go to start with 1.

So, sum of square of remainders = $1^2 + 2^2 + 3^2 + ... + (n-1)^2 = (n-1)n(2n-1)/6$

Now, n is odd so n is not divisible by 2 and also n is not divisible by 3.

- ⇨ 6 doesn't divide n.
- ⇨ $(n - 1)n(2n - 1)/6$ is divisible by n.

Proved.

Theorems :

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1. Fermat's Little Theorem :

If p is a prime then $a^{(p-1)} \equiv 1 \pmod p$ where a and p are relatively prime.

Proof :

Let us consider the integers a, 2a, 3a, ..., (p − 1)a.

None of these divisible by p. Also no two of these are congruent modulo p, because if $ra \equiv sa \pmod p$ for some integers r, s such that $1 \le r < s \le p - 1$, then cancelling a (since a is prime to p) we must have $r \equiv s \pmod p$, a contradiction.

This means the integers a, 2a, 3a, ..., (p − 1)a are congruent to 1, 2, 3, ..., p − 1 modulo p, taken in some order.

Therefore, $a*2a*3a*....(p-1)a \equiv 1*2*3*....*(p-1) \pmod p$

- ⇨ $a^{(p-1)}*(p-1)! \equiv (p-1)! \pmod p$
- ⇨ $a^{(p-1)} \equiv 1 \pmod p$ (since $\gcd(p, (p-1)!) = 1$.

Proved.

Application :

Problem 1 :

There are p number of leaves kept in a circular form where p is prime. The leaves are numbered L_1, L_2, L_3,..., L_p in anti-clockwise direction. A frog first jumps to any leaf say, L_n leaf. Then it jumps to leaf L_{2n}, then to leaf L_{4n}, then L_{8n}, ... Prove that after p^{th} jump the frog will reach the first leaf where he started i.e. L_n.

Solution 1 :

First understand that the frog will stay in the leaf L_i after j^{th} jump if $t_j \equiv i \pmod p$.

Let us take an example to understand this.

Let there are 7 leaves.

Let the frog first jumps to L_3.

The frog will then jump to L_6, then L_{12} . Now what is L_{12}? $12 \equiv 5 \pmod 7$. So the frog will be staying in L_5 after 3^{rd} jump.

Now, let us get back to the general case as depicted in the problem.

Let, the frog first jumps to L_n.

At 2^{nd} jump the term is $n*2$.

At 3^{rd} jump the term is $n*2^2$.

...

...

At p^{th} jump the term is $n*2^{p-1}$.

Now, we need to find the remainder of $n*2^{p-1}$ divided by p.

Now, as per Fermat's little theorem, $2^{p-1} \equiv 1 \pmod p$

⇨ $n*2^{p-1} \equiv n \pmod p$

So, the frog will reach L_n after p^{th} jump.

Proved.

Corollary :

If p and p+2 are twin prime i.e. both p and p+2 are primes then $p*2^p + 1 \equiv 0 \pmod{p+2}$

Proof :

Now, $p \equiv -2 \pmod{p + 2}$

- ⇨ $p*2^p \equiv -2^{(p+1)}$ (mod p + 2)
- ⇨ $p*2^p \equiv -1$ (mod p + 2) (By Fermat's little theorem, $2^{(p+1)} \equiv 1$ (mod p + 2)
- ⇨ $p*2^p + 1 \equiv 0$ (mod p + 2)

Application :

(a) Find the remainder when $41*2^{41} + 1$ is divided by 43.

Solution :

Now, 41 and 43 are both prime, putting p = 41 in the above expression we get,

$41*2^{41} + 1 \equiv 0$ (mod 43)

2. If p is a prime then ${}^{p-1}C_r \equiv (-1)^r$ (mod p)

Proof :

Now, ${}^{(p-1)}C_r = (p-1)!/\{(p-r-1)!*r! = (p-1)(p-2)...(p-r)/r!$

Now, p – 1 ≡ -1 (mod p)

p – 2 ≡ -2 (mod p)

p – 3 ≡ -3 (mod p)

...

...

p – r ≡ -r (mod p)

⇨ (p-1)(p-2)...(p-r) ≡ (-1)(-2)(-3)...(-r) (mod p)
⇨ (p-1)(p-2)...(p-r) ≡ $(-1)^r*r!$ (mod p)
⇨ (p-1)(p-2)...(p-r)/r! ≡ $(-1)^r$ (mod p) (we can divide because p is prime and r! doesn't contain any factor of p)
⇨ ${}^{(p-1)}C_r \equiv (-1)^r$ (mod p)

Proved.

3. ${}^{p}C_{r} \equiv 0$ (mod all p_i) where $p \le p_i < \max(r, p-r)$

Proof :

If $p-r > r$

Now, ${}^{p}C_{r} = p!/\{(p-r)!*r!\} = p(p-1)(p-2)...(p-r+1)/r!$

Now, the primes which are greater than r and less than or equal to p do not have any factor in r! Because they are prime.

If $p-r < r$ then ${}^{p}C_{r} = p!/\{(p-r)!*r!\} = p(p-1)...(r+1)/(p-r)!$

Now, the primes which are greater than r and less than or equal to p do not have a factor in (p-r)! Because they are prime.

⇨ ${}^{p}C_{r} \equiv 0$ (mod all p_i) where $p \le p_i < \max(r, p-r)$

Proved.

4. If p is a prime and $r < p$ then ${}^{(p+r)}C_{r} \equiv 1 \pmod p$

Proof :

Now, ${}^{(p+r)}C_{r} = (p+r)!/\{(p!)(r!) = (p + r)(p+ r - 1)(p + r - 2)....(p + 1)/r!$

Now, $p + r \equiv r \pmod p$

$(p + r - 1) \equiv (r - 1) \pmod p$

..

..

..

$(p + 1) \equiv 1 \pmod p$

⇨ $(p+r)(p+r-1)....(p+1)/r! \equiv r*(r - 1)...*2*1/r! \pmod p$ (We can divide by r! because p is prime and p > r so gcd(p, r!) = 1 i.e. they are relatively prime)
⇨ $(p+r)(p+r-1)....(p+1)/r! \equiv r!/r! \pmod p$
⇨ ${}^{(p+r)}C_r \equiv 1 \pmod p$

Proved.

5. If p is a prime and r > p such that r is not a multiple of p then ${}^{(p+r)}C_r \equiv 2 \pmod p$

Proof :

We will prove this using induction.

Let ${}^{(p+r)}C_r \equiv 2 \pmod p$

Now, ${}^{(p+r+1)}C_{(r+1)} = (p+r+1)!/\{(p!)(r+1)! = \{(p+r+1)/(r+1)\}*{}^{(n+r)}C_r \equiv (r+1)/(r+1)*2 \pmod p$ (we can divide because r + 1 is not a multiple of p.)

⇨ ${}^{(p+r)}C_r \equiv 2 \pmod p$

Proved.

6. Wilson's Theorem :

If p is a prime then $(p - 1)! + 1 \equiv 0 \pmod p$.

Corollary :

$(-1)^{(p-1)/2}[\{(p - 1)/2\}!]^2 + 1 \equiv 0 \pmod p$

Proof :

Now, $(p - 1)! = 1*2*3*.....*\{(p - 1)/2\}*\{(p + 1)/2\}*....*(p - 3)*(p - 2)*(p - 1)$

Now, $p - 1 \equiv -1 \pmod p$

Similarly, $p - 2 \equiv -2 \pmod p$

...

...

...

$(p + 1)/2 \equiv -(p - 1)/2 \pmod p$

- ⇨ $1*2*....*\{(p - 1)/2\}*\{(p + 1)/2\}*....*(p - 2)*(p - 1) \equiv [1*2*...*\{(p - 1)/2\}]*[(-1)*(-2)*....*\{-(p - 1)/2\}] \pmod p$
- ⇨ $(p - 1)! \equiv (-1)^{(p-1)/2}[\{(p - 1)/2\}!]^2 \pmod p$
- ⇨ $(p - 1)! + 1 \equiv (-1)^{(p-1)/2}[\{(p - 1)/2\}!]^2 + 1 \pmod p$
- ⇨ $(-1)^{(p-1)/2}[\{(p - 1)/2\}!]^2 + 1 \equiv 0 \pmod p$

Proved.

7. Euler's Theorem :

$a^{\Phi(n)} \equiv 1 \pmod n$ where n is any natural number and $\Phi(n)$ is Euler's totient function and $\gcd(a, n) = 1$ i.e. a and n are relatively prime.

Factor, GCD :

Factors of a positive integer is defined as the positive integers less than the integer and which divides the integer.

For example, if $p_1, p_2, p_3, p_4 < n$ and divides n then p_1, p_2, p_3, p_4 are called factors of n. For example, the factors of 12 are 1, 2, 3, 4, 6, 12.

Note that when a factor of a positive integer occurs then another factor by default occurs to produce result $p_1*p_2 = n$. So every number has even number of factors but what if $p_1*p_1 = n$ then p_1 comes twice and it is counted as 1 factor. In that case the numbers have odd number of factors. In the above example we see that n is a square number and $n = p_1^2$ i.e. only the square numbers have odd number of factors.

Application :

Problem 1 :

There are 1000 doors named D_1, D_2, D_3, ..., D_{1000}. There are 1000 persons named P_1, P_2, P_3, ..., P_{1000}. At first all the doors are closed. P_1 goes and opens all the doors. Then P_2 goes and closes even numbered doors i.e. D_2, D_4, D_6, ..., D_{1000} and leaves the odd numbered doors i.e. D_1, D_3, D_5, ..., D_{999} as it is. Then P_3 goes and changes the state of the doors (If open then closes and if closed then opens) which are multiple of 3 i.e. the doors D_3, D_6, ... Then P_4 goes and changes the state of the doors which are multiple of 4. Similarly in this way 1000 persons goes and changes the states of the doors accordingly. At the end how many doors will be open and how many doors will be closed?

Solution 1 :

Let us take an example.

Let us see how many times and which persons are operating on D_{28}.

The persons who are operating on D_{28} are P_1, P_2, P_4, P_7, P_{14}, P_{28}.

Now, 1, 2, 4, 7, 14, 28 are factors of 28.

So, any door D_i is getting operated by the persons P_j where j's are the factors of i.

In this case we are not interested to find the factors of every number up to 1000. Rather we are interested in which doors are getting operated odd number of times and which doors are getting operated even number of times i.e. which numbers have odd number of factors and which numbers have even number of factors. If odd number of factors then odd number of operation and the door will be open and if even number of factors then even number of operation and the door will be closed.

We have seen that every number has even number of factors except the square numbers which have odd number of factors.

So, D_1, D_4, D_9, D_{16}, ..., D_{961} these 31 doors will stay open and rest of the doors will stay closed.

Number of factors of an integer :

Let, $n = p_1^r * p_2^s$

Then number of factors of $n = (r_1 + 1)(r_2 + 1)$

Proof :

Now, $p_1, p_1^2, p_1^3, ..., p_1^r$ divides n.

So there are r factors.

Similarly, $p_2, p_2^2, p_2^3, ..., p_2^s$ divides n.

So there are s factors.

Now, $p_1*p_2, p_1*p_2^2, ..., p_1*p_2^s$ divides n.

So, for p_1 number of factors occurring with the set of p_2 is s.

Similarly, for p_1^2 number of factors occurring with the set of p_2 is s.

And s number of factors occurring for every p_1^i with the set of p_2.

So, number of factors in this way = r*s.

And 1 divides n.

So, number of factors of n = r + s + r*s + 1 = (r + 1)(s + 1)

Proved.

This can be extended or generalized for any $n = (p_1\hat{}a_1)(p_2\hat{}a_2)....(p_m\hat{}a_m)$ then,

Number of factors of $n = (a_1 + 1)(a_2 + 1)....(a_m + 1)$

Example :

Find the number of factors of 12.

$12 = 2^2*3$

So, number of factors of 12 = (2 + 1)(1 + 1) = 6

Those are = 1, 2, 3, 4, 6, 12.

Finding number of factors divisible by some factor of the number.

Let $n = (p_1\hat{}a_1)(p_2\hat{}a_2)(p_3\hat{}a_3).....(p_m\hat{}a_m)$

Find number of factors which are divisible by $p_1^2p_2$. ($a_1 \geq 2$, $a_2 > 0$)

We define $r = n/p_1^2p_2 = \{p_1\hat{}(a_1 - 2)\}\{p_2\hat{}(a_2 - 1)\}(p_3\hat{}a_3)....(p_m\hat{}a_m)$

Now, number of factors of $r = (a_1 - 2 + 1)(a_2 - 1 + 1)(a_3 + 1).....(a_m + 1)$

$= (a_1 - 1)a_2(a_3 + 1).....(a_m + 1)$

Now, all these factors are factors of n and if we multiply $p_1^2p_2$ with these factors then also remains factors of n and all these factors are divisible by $p_1^2p_2$.

So, number of factors of n divisible by $p_1^2p_2$ = number of factors of r = $(a_1 - 1)a_2(a_3 + 1).....(a_m + 1)$.

For example,

Let $n = 2^3*3^2*5*7$

Number of factors divisible by 10 = number of factors of 2^2*3^2*7 = (2 + 1)(2 + 1)(1 + 1) = 18.

Factors of 2520 are : 1 2 3 4 5 6 7 8 9 10 12 14 15 18 20 21 24 28 30 35 36 40 42 45 56 60 63 70 72 84 90 105 120 126 140 168 180 210 252 280 315 360 420 504 630 840 1260 2520

Clearly there are 18 factors divisible by 10 viz. 10, 20, 30, 40, 60, 70, 90, 120, 140, 180, 210, 280, 360, 420, 630, 840, 1260, 2520.

Similarly, sum of the factors of n divisible by $p_1^2p_2$ = $p_1^2p_2$*sum of factors of r.

In the above example, sum of factors of 2^3*3^2*5*7 divisible by 10 = $10*\{(2^3 - 1)/(2 - 1)\}\{(3^3 - 1)/(3 - 1)\}\{(7^2 - 1)/(7 - 1)\}$ = 10*7*13*8 = 7280.

Number of even and odd factors of an integer :

Let n = $2^a*(p_1\^a_1)(p_2\^a_2).....(p_m\^a_m)$

Now, we find number of factors of n/2 = r = $2^{(a-1)}*(p_1\^a_1)(p_2\^a_2).....(p_m\^a_m)$

Number of factors of r = $(a - 1 +)(a_1 + 1)(a_2 + 1)........(a_m + 1)$

= $a(a_1 + 1)(a_2 + 1).......(a_m + 1)$

Number of odd factors = number of factors of n – number of factors of r = $(a + 1)(a_1 + 1)....(a_m + 1) - a(a_1 + 1).....(a_m + 1)$

= $(a_1 + 1)(a_2 + 1)......(a_m + 1)$

Sum of the factors :

Let, n = $p_1^r*p_2^s$

Then sum of the factors of n = $\{(p_1^{r+1} - 1)/(p_1 - 1)\}*\{(p_2^{s+1} - 1)/(p_2 - 1)\}$

Proof :

Now, sum of the factors of n = $1 + (p_1 + p_1^2 + ... + p_1^r) + (p_2 + p_2^2 + ... + p_2^s) + \{p_1(p_2 + p_2^2 + ... + p_2^s) + p_1^2(p_2 + p_2^2 + ... + p_2^s) + ... + p_1^r(p_2 + p_2^2 + ... + p_2^s)\}$

= $(1 + p_1 + p_1^2 + ... + p_1^r) + (p_2 + p_2^2 + ... + p_2^s) + (p_2 + p_2^2 + ... + p_2^s)(p_1 + p_1^2 + ... + p_1^r)$

= $(1 + p_1 + p_1^2 + ... + p_1^r) + (p_2 + p_2^2 + ... + p_2^s)(1 + p_1 + p_1^2 + ... + p_1^r)$

$= (1 + p_1 + p_1^2 + ... + p_1^r)(1 + p_2 + p_2^2 + ... + p_2^s)$

$= \{(p_1^{r+1} - 1)/(p_1 - 1)\}\{(p_2^{s+1} - 1)/(p_2 - 1)$

Proved.

This can be extended to general case as,

If $n = (p_1\text{^}a_1)(p_2\text{^}a_2)....(p_m\text{^}a_m)$

Then, sum of factors of $n = [\{p_1\text{^}(a_1+1) - 1\}/(p_1 - 1)][\{p_2\text{^}(a_2 + 1) - 1\}/(p_2 - 1)]....[\{p_m\text{^}(a_m + 1) - 1\}/(p_m - 1)]$

Example :

Find the sum of factors of 12.

$12 = 2^2*3$

Sum of factors of $12 = \{(2^{2+1} - 1)/(2 - 1)\}\{(3^{1+1} - 1)/(3 - 1)\} = (8 - 1)(9 - 1)/2 = 7*8/2 = 7*4 = 28$.

Check, sum of factors of 12 = 1 + 2 + 3 + 4 + 6 + 12 = 28.

If p, q are odd then either of $(p - q)/2$ and $(p + q)/2$ is odd and another is even.

Proof :

Let, $p + q = 2^s*m_1$ where m_1 is odd.

And $p - q = 2^t*m_2$ where m_2 is odd.

Adding the two equations we get,

$2p = 2^s*m_1 + 2^t*m_2$

⇨ $p = 2^{s-1}*m_1 + 2^{t-1}*m_2$

Now, p is odd.

- ⇨ Either s = 1, t > 1 or t = 1 and s > 1.
- ⇨ Either of (p – q)/2 and (p + q)/2 is odd and another is even.

Application :

Problem 1 :

There are two square boards with (p – q)/2 and (p + q)/2 number of rows or columns where p, q are odd. The boards have $\{(p - q)/2\}^2$ and $\{(p + q)/2\}^2$ number of small squares respectively. The small squares are numbered 2, 3, 4, ..., row-wise. Define a function f : N -> N + {0} such that f(m) = m{(m - 1)! - 1}. At first all the small squares are coloured black. One paints white and counts 1, then black counts 2, then white again counts 3, then black again counts 4, ... till he reaches counting f(m) for m^{th} box. Prove that out of the two boards exactly one board will turn into a CHESS board.

Solution 1 :

Now, f(m) = m{(m – 1)! – 1} = m! – m

Now, m! is always even. So m! – m = even if m is even and odd if m is odd.

Now, consecutive integers i.e. 2, 3, 4, ... comes in alternate even and odd.

For m = 2, f(2) = 0 = even.

For m = 3, f(3) = 3 = odd;

For m = 4, f(4) = 20 = even.

...

...

So, if we alternately paint white and black and count to f(m) for every m then we will have white, black, white, black, ... this colours in the small squares of the board row-wise.

Now, if the board has even number of rows then if it starts with black then it will end the row with white and then black again in the second row first column small square. So, all the first column small squares of the board will be black then second column white, then black,... So it will not turn into a CHESS board.

Now, if the board has odd number of rows then if it starts with black in the first row first column then it will end up with black. So, second row first column square will be black. And then continues white, black,... So, it will turn into a CHESS board.

Now, out of $(p - q)/2$ and $(p + q)/2$ one is odd and another is even. The even will not turn into CHESS board and the odd will turn into CHESS board.

- ⇨ Out of the two boards exactly one will turn into a CHESS board.

Proved.

Definition of GCD :

GCD (greatest common divisor) of two positive integers is defined as the greatest common divisor of both the integers.

For example, if $n_1 = p_1^r * p_2^s$ and $n_2 = p_1^t * p_3^u$ where $t < r$.

Then $GCD(n_1, n_2) = p_1^t$

For example, $12 = 2^2*3$ and $28 = 2^2*7$

Then $GCD(12, 28) = 2^2 = 4$.

Relatively prime :

If there is no factor between n_1 and n_2 (n_1, n_2 may be prime or composite) then n_1 and n_2 are said to be relatively prime to each other.

Another definition, if $GCD(n_1, n_2) = 1$ then n_1 and n_2 are called relatively prime.

For example, 12 and 35 are relatively prime though they both are composite number.

Euler's totient function :

Euler's totient function denoted by $\Phi(n)$ of a positive integer n is defined as number of positive integers that are less than n and relatively prime to n.

For example, $\Phi(12) = 4$ (These are 1, 5, 7, 11)

Let, $n = p^k$ where p is prime.

Now, p, 2p, 3p, ..., $p*p = p^2$, $(p+1)*p$, $(p+2)*p$, ...,$p^{k-1}*p = p^k$ are not relatively prime to n.

Clearly, the number of such integers = p^{k-1}

So, number of positive integers that are less than n and relatively prime to n = $p^k - p^{k-1} = p^k(1 - 1/p)$

Now, $\Phi(mn) = \Phi(m)\Phi(n)$ where, m and n are relatively prime.

Let, $t = (p_1\hat{}k_1)(p_2\hat{}k_2)....(p_s\hat{}k_s)$

Then, $\Phi(t) = \Phi\{(p_1\hat{}k_1)(p_2\hat{}k_2)....(p_s\hat{}k_s)\} = \Phi(p_1\hat{}k_1)\Phi(p_2\hat{}k_2)....\Phi(p_s\hat{}k_s) = (p_1\hat{}k_1)(1 - 1/p_1)(p_2\hat{}k_2)(1 - 1/p_2).....(p_s\hat{}k_s)(1 - 1/p_s) = \{p_1\hat{}(k_1 - 1)\}\{p_2\hat{}(k_2 - 1)\}...\{p_s\hat{}(k_s - 1)\}(p_1 - 1)(p_2 - 1)....(p_s - 1)$

Application :

Problem 1 :

There are 1000 doors named D_1, D_2, D_3, ..., D_{1000}. There are 1000 persons named P_1, P_2, P_3, ..., P_{1000}. At first, all the doors are closed. Now, P_1 goes and opens all the doors. Then P_2 goes and closes D_3, D_5,...., D_{999} i.e. the numbers that are relatively prime and leaves other doors as it is. Then P_3 goes and reverses D_i (i.e. if D_i is open then closes and if it is closed then opens) where gcd of (i, 3) = 1 and i > 3. Note that gcd(2, 3) = 1 but P_3 will not reverse D_2 because 2 < 3. In this way P_t goes and changes the doors D_j where gcd(j, t) = 1 and j > t. In this way P_{1000} goes and leaves all the doors unchanged because 1000 is the greatest number here. At the end which door(s) will be open and which door(s) will be closed?

Solution 1:

Let us take an example.

Let us see which persons are operating on D_{12} and how many times D_{12} is getting operated.

Clearly, P_1, P_5, P_7, P_{11} are operating on D_{12}.

Now, 1, 5, 7, 11 are relatively prime to 12 and less than 12.

So, any door D_i is getting operated by P_j where j's are less than i and relatively prime to i.

This is the definition of Euler's totient function.

Now, the expression says $\Phi(n) = \Phi(p_1\hat{}k_1)\Phi(p_2\hat{}k_2)....\Phi(p_s\hat{}k_s) = (p_1\hat{}k_1)(1 - 1/p_1)(p_2\hat{}k_2)(1 - 1/p_2).....(p_s\hat{}k_s)(1 - 1/p_s) = \{p_1\hat{}(k_1 - 1)\}\{p_2\hat{}(k_2 - 1)\}...\{p_s\hat{}(k_s - 1)\}(p_1 - 1)(p_2 - 1)....(p_s - 1)$ where $n = (p_1\hat{}k_1)(p_2\hat{}k_2)....(p_s\hat{}k_s)$.

Now, inspect the expression and it is easy to verify that $\Phi(n)$ is even except for $\Phi(1) = \Phi(2) = 1$.

So, every door will be operated even number of times except D_1 and D_2.

So all the doors will be closed except D_1 and D_2 which will stay opened.

Problem 2 :

There is an nxn board with n^2 small squares where n is odd. All are painted white. They are numbered 3, 4, 5, 6, ..., $n^2 + 2$. There are n^2+1 brushes numbered B(2), B(3), ..., B(n^2), B(n^2+1), B(n^2+2). First B(2) will be picked up and it will paint black on the i numbered square if gcd(i, 2) is not equal to 1 i.e. it will paint all the even numbered squares black. Now, B(3) will be picked up and it will change the colour (if it is painted white then black and if it is painted black then white) of j numbered square if gcd(j, 3) is not equal to 1 and j > 3. In this way B(t) will be picked up and it will change the colour of k numbered square if gcd(t, k) is not equal to 1 and k > t. For example gcd(6, 12) = 6 but B(12) will not change the colour of 6 numbered square as 6 < 12. Prove that after the painting of B(n^2+2) brush i.e. all the brushes the board will turn into a CHESS board.

Solution 2 :

Let us take an example.

Let us see which brushes are painting on 12^{th} square and how many times 12^{th} square is getting painted.

12^{th} square is getting painted by 2, 3, 4, 6, 8, 9, 10, 12.

The set is nothing but $12 - \Phi(12)$.

So, any square i is getting painted by $i - \Phi(i)$.

Now, $\Phi(i)$ is always even as there is no 1 or 2 numbered square.

- ⇨ $i - \Phi(i)$ is even if i is even and odd if i is odd.

Now, if odd number of operation then square will be painted black and if even number of operation then square will be painted white.

So, consecutive squares (odd, even, odd, even, ..) will be black, white, black, white, ...

Now, n is odd.

So, first row will start with black and end with black.

So, second row will start with white and end with white.

Similarly goes the rest of the rows.

So, the board is turning into a CHESS board.

Proved.

$a^4 + 4b^4$

$= (a^2)^2 + (2b^2)^2$

$= (a^2 + 2b^2)^2 - 2*a^2*2b^2$

$= (a^2 + 2b^2)^2 - (2ab)^2$

$= (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$

So, we see that if we get an expression of the form $a^4 + 4b^4$ then it cannot be prime as it can be factored.

Problem :

Prove that $n^4 + 4^n$ cannot be prime where n is natural number.

Solution :

Clearly if n is even then it is even and cannot be prime.

So, n needs to be odd.

Note that if n is odd then n – 1 is even.

Now, $n^4 + 4^n$

$= n^4 + 4*4^{n-1}$

$= n^4 + 4*(2^{(n-1)/2})^4$

$= a^4 + 4b^4$ form.

So cannot be prime as it can be factored.

Induction :

If an positive integer equation is satisfied by n = 1, n = k where k is any natural number and if this is satisfied by n = k+1 given for n = k this is true then the equation is satisfied by any positive integer. This is called law of induction.

Application :

Problem 1 :

Prove that $1^3 + 2^3 + 3^3 + ... + n^3 = \{n(n+1)/2\}^2$

Solution 1 :

Let us check for n = 1.

Now, putting n = 1 in RHS we get, $\{1(1 + 1)/2\}^3 = 1 =$ LHS.

So, let us consider that this is true for $n = k$.

That is, we have, $1^3 + 2^3 + 3^3 + ... + k^3 = \{k(k+1)/2\}^2$

Now, we will see what happens with $n = k + 1$.

LHS = $1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3$

$= \{k(k+!)/2\}^2 + (k+1)^3$

$= \{(k+1)/2\}^2(k^2 + 4k + 4)$

$= \{(k+1)(k+2)/2\}^2$

= RHS for $n = k + 1$.

So, this is true for $n = 1$, k and $k + 1$ given this is true for $n = k$.

So, this will hold for any positive integer n according to the law of induction.

Proved.

Inequality :

First we will discuss about AM (Arithmetic mean), GM (Geometric mean) and HM (Harmonic mean).

AM of any n *positive* real number is defined as = $(x_1 + x_2 + ...+ x_n)/n$.

GM of any n *positive* real number is defined as = $(x_1x_2....x_n)^{1/n}$

HM of any n *positive* real number is defined as = $n/\{(1/x_1) + (1/x_2) + ... + (1/x_n)\}$

Theorem :

AM ≥ GM ≥ HM

Proof :

Let, a and b are two positive real numbers.

Then, $(a - b)^2 \geq 0$

- ⇨ $(a + b)^2 - 4ab \geq 0$
- ⇨ $(a + b)^2 \geq 4ab$
- ⇨ $\{(a + b)/2\}^2 \geq ab$
- ⇨ $(a + b)/2 \geq \sqrt{(ab)}$
- ⇨ AM ≥ GM

This is proved for two positive real numbers. It can be extended to any number of positive real numbers.

Now, we have, $(a + b)/2 \geq \sqrt{(ab)}$

- ⇨ $1/\sqrt{(ab)} \geq 2/(a + b)$
- ⇨ $\sqrt{(ab)}/ab \geq 2/(a + b)$
- ⇨ $\sqrt{(ab)} \geq 2ab/(a + b)$
- ⇨ $\sqrt{(ab)} \geq 2/\{(1/a) + 1/b)\}$
- ⇨ GM ≥ HM

This is proved for two positive real numbers. It can be extended to any number of positive real numbers.

Equality holds when a = b.

Proved.

Note that "positive" is written in *Italic* to emphasize on the word that whenever you will be using AM ≥ GM ≥ HM then all the real numbers must be positive.

Application :

Problem 1 :

Prove that $a^3 + b^3 + c^3 \geq 3abc$ where a, b, c are positive real numbers.

Solution 1 :

Applying AM ≥ GM on a^3, b^3 and c^3 we get,

$(a^3 + b^3 + c^3)/3 \geq (a^3b^3c^3)^{1/3}$

- ⇨ $(a^3 + b^3 + c^3)/3 \geq abc$
- ⇨ $a^3 + b^3 + c^3 \geq 3abc$.

Proved.

Weighted AM, GM, HM :

If $x_1, x_2, \ldots, x_n$ are n real numbers with weights $w_1, w_2, \ldots, w_n$ then weighted AM is defined as,

Weighted AM = $(x_1w_1 + x_2w_2 + \ldots + x_nw_n)/(w_1 + w_2 + \ldots + w_n)$

Weighted GM = $\{(x_1\text{^}w_1)(x_2\text{^}w_2)\ldots(x_n\text{^}w_n)\}\text{^}\{1/(w_1 + w_2 + \ldots + w_n)\}$

Weighted HM = $(w_1 + w_2 + \ldots + w_n)/\{(w_1/x_1) + (w_2/x_2) + \ldots + (w_n/x_n)\}$

Also, Weighted AM ≥ Weighted GM ≥ Weighted HM but remember the word positive whenever applying this.

<u>Permutation and Combination :</u>

Fundamental theorem :

If a work can be done in m ways and another work can be done in n ways then the two works can be done simultaneously in m*n ways.

Note : Emphasize on the word simultaneously because most of the students get confused where to apply multiplication and where to apply addition. When both the works need to be done to complete a set of work then apply multiplication and if the works are disjoint then apply addition.

Permutation :

There are n things and we need to take r things at a time and we need to arrange it with respect to order then the total number of ways is ^nP_r where $^nP_r = n!/(n - r)!$.

For example there are 3 tuples (1, 2, 3)

6 permutations are possible = (1, 2, 3); (1, 3, 2); (2, 1, 3); (2, 3, 1); (3, 1, 2) and (3, 2, 1)

Now, we will check by the formula.

Here number of permutations = $^3P_3 = 3!/(3 - 3)! = 3*2/1 = 6$.

Proof :

We can take 1st thing in n ways, 2nd thing in (n − 1) ways, ..., rth thing in {n − (r − 1)} = (n − r + 1) ways.

By fundamental theorem, total number of ways = n(n − 1)(n − 2)....(n − r + 1)

= n(n − 1)(n − 2)....(n − r + 1)(n − r)(n − r − 1)....2*1/{(n − r)(n − r − 1).....*2*1} (Multiplying numerator and denominator by (n − r)(n − r − 1)....*2*1

= n!/(n − r)!

So, $^nP_r = n!/(n - r)!$

Proved.

Combination :

If there are n things and we need to select r things at a time (order is not important) then total number of ways of doing this = $^nC_r = n!/(n - r)!*r!$.

For example, there are 5 numbers (1, 2, 3, 4, 5). We need to select 3 at a time.

Total number of ways = (1, 2, 3); (1, 2, 4); (1, 2, 5); (1, 3, 4); (1, 3, 5); (1, 4, 5); (2, 3, 4); (2, 3, 5); (2, 4, 5); (3, 4, 5) i.e. 10 number of ways.

Now, we will check by formula.

Here number of combinations = $^5C_3 = 5!/(5 - 3)!*2! = 5*4*3!/3!*2 = 10$.

Note that here (1, 2, 3) is equivalent to (1, 3, 2) etc. as order is not important in combination but this is important in permutation.

Proof :

Now, the order is not important.

Hence the number of ways r things can permutate among themselves is r!.

Therefore, $r!*{}^nC_r = {}^nP_r = n!/(n - r)!$

⇨ ${}^nC_r = n!/\{(n - r)!*r!\}$

Proved.

Number of non-negative solution :

Problem :

a + b + c + d = n where a, b, c, d, n are all integers. Prove that number of non-negative solution of the equation is ${}^{(n + 3)}C_3$.

Solution :

First we fix a = 0.

Now, we fix b = 0.

Number of solutions for c and d are (n + 1) as c is running from 0 to n and d is running from n to 0.

Now, we fix b = 1.

Number of solution for c and d are n as c is running from 0 to (n – 1) and d from (n – 1) to 0.

So, if we keep on fixing b = 2, 3, ..., n we get total number of solution for b, c, d for a = 0 as (n + 1) + n + (n – 1) + ... + 1 = (n + 1)(n + 2)/2

Now, if we fix a = i, then the equation becomes, b + c + d = n – i

Now, if we keep on fixing b = 0, 1, 2, ..., n − i we get number of solution for b, c, d as (n − i + 1) + (n − i) + (n − i − 1) + ... + 1 = (n − i + 1)(n − i + 2)/2

So, total number of solution as a varies from 0 to n is, ∑(n − i + 1)(n − i + 2)/2 (i running from 0 to n)

= (1/2)∑(n + 1)(n + 2) − (1/2)(2n + 3)∑i + (1/2)∑i^2 (i running from 0 to n)

= (1/2)$(n + 1)^2$(n + 2) − (1/4)(2n + 3)n(n + 1) + n(n + 1)(2n + 1)/12

= (1/12)(n + 1)[$6n^2$ + 18n + 12 − $6n^2$ − 9n + $2n^2$ + n]

= (1/12)(n + 1)($2n^2$ + 10n + 12)

= (1/6)(n + 1)((n^2 + 5n + 6)

= (1/6)(n + 1)(n + 2)(n + 3)

= (n + 3)!/(n!*3!)

= $^{(n+3)}C_3$

Proved.

Problem :

a + b + c + d + e = n where a, b, c, d, e, n are all integers. Prove that number of non-negative solution of the equation is $^{(n+4)}C_4$.

Solution :

Let a = i.

Then the equation becomes, b + c + d + e = n − i.

Now, we have from the previous problem that number of solution in that case = (n − i + 1)(n − i + 2)(n − i + 3)/6

So, total number of solution as a varies from 0 o n is ∑(n − i + 1)(n − i + 2)(n − i + 3)/6 (i running from 0 to n)

= (1/6)∑{(n + 1)(n + 2) − i(2n + 3) + i^2}(n − i + 3) (i running from 0 to n)

= (1/6)∑(n + 1)(n + 2)(n + 3) − (1/6){(2n + 3)(n + 3) + (n + 1)(n + 2)}∑i +(1/6){(2n + 3) + (n + 3)}∑i^2 − (1/6)∑i^3 (i running from 0 to n)

$= (1/6)(n + 1)^2(n + 2)(n + 3) - (1/12)(3n^2 + 12n + 11)n(n + 1) + (1/36)(3n + 6)n(n + 1)(2n + 1) - (1/24)n^2(n + 1)^2$

$= (1/24)(n + 1)[4(n^2 + 3n + 2)(n + 3) - 6n^3 - 24n^2 - 22n + (2n + 4)(2n^2 + n) - n^3 - n^2]$

$= (1/24)(n + 1)[4n^3 + 24n^2 + 44n + 24 - 6n^3 - 24n^2 - 22n + 4n^3 + 10n^2 + 4n - n^3 - n^2]$

$= (1/24)(n + 1)(n^3 + 9n^2 + 26n + 24)$

$= (1/24)(n + 1)(n^3 + 2n^2 + 7n^2 + 14n + 12n + 24)$

$= (1/24)(n + 1)\{n^2(n + 2) + 7n(n + 2) + 12(n + 2)\}$

$= (1/24)(n + 1)(n + 2)(n^2 + 7n + 12)$

$= (1/24)(n + 1)(n + 2)(n + 3)(n + 4)$

$= (n + 4)!/(n!*4!)$

$= {}^{(n + 4)}C_4$

Proved.

Problem :

$a_1 + a_2 + ... + a_r = n$ where $a_1, a_2, ..., a_n$, n are all integers. Prove that number of non-negative solution of the equation is ${}^{(n + r - 1)}C_{(r - 1)}$.

Solution :

We will prove it by induction.

Clearly, this is true for r = 1.

Let, this is true for r = k i.e. number of non-negative solution when there are k variables in the LHS is ${}^{(n + k - 1)}C_{(k - 1)}$.

Now, number of non-negative solution for r = k + 1 i.e. when an extra variable gets added in LHS is $\sum({}^{(n - i + k - 1)}C_{(k - 1)})$ (i running from 0 to n)

Now, we have to prove that, $\sum({}^{(n - i + k - 1)}C_{(k - 1)})$ (i running from 0 to n) = ${}^{(n + k)}C_k$

Now, we will prove this by another induction.

For, n = 1, LHS = ${}^{k}C_{(k - 1)} + {}^{(k - 1)}C_{(k - 1)} = k + 1$

RHS = ${}^{(k+1)}C_k = k + 1$.

So, this is true for n = 1.

Let, this is true for n = p i.e. we have, $\sum({}^{(p-i+k-1)}C_{(k-1)})$ (i running from 0 to p) = ${}^{(p+k)}C_k$

For, n = p + 1, LHS = $\sum({}^{(p+1-i+k-1)}C_{(k-1)})$ (i running from 0 to p + 1)

= ${}^{(p+k)}C_{(k-1)} + \sum({}^{(p-i+k-1)}C_{(k-1)})$ (i running from 0 to p)

= ${}^{(p+k)}C_{(k-1)} + {}^{(p+k)}C_k$

= $(p + k)!/\{(p + 1)!*(k - 1)!\} + (p + k)!/\{p!*k!\}$

= $[(p + k)!/\{(p + 1)!*k!\}](k + p + 1)$

= $(p + k + 1)!/\{(p + 1)!*k!\}$

= ${}^{(p+1+k)}C_k$

= RHS for n = p + 1.

Proved.

Number of positive solutions :

Problem :

a + b + c + d = n where a, b, c, d are all integers > 0. Prove that number of positive solution of this equation is ${}^{n-1}C_3$.

Solution :

We fix a = 1, b = 1. Number of solution for c and d is n – 2 as c is running from 1 to n – 2 and d is running from n – 2 to 1.

Now, we fix b = 2. Number of solution for c and d is n – 3 as c is running from 1 to n – 3 and d is running from n – 3 to 1.

Now, if we keep on fixing b = 3, 4, …, n – 2, number of solutions for c and d will be n – 4, n – 5, …, 1.

So, when a = 0, number of solutions for b, c, d is (n – 2) + (n – 3) + … + 1 = (n – 1)(n – 2)/2.

If we fix a = i, then the equation becomes, b + c + d = n − i.

And total number of solution = (n − i − 2) + (n − i − 3) + …. + 1 = (n − i − 1)(n − i − 2)/2

As i runs from 1 to n − 3 number of solutions = ∑(n − i − 1)(n − i − 2)/2 (i running from 1 to n − 3)

= (1/2)∑(n − 1)(n − 2) − {(2n − 3)/2}∑i + (1/2)∑i^2 (i running from 1 to n − 3)

= (1/2)(n − 1)(n − 2)(n − 3) − (2n − 3)(n − 2)(n − 3)/4 + (n − 3)(n − 2)(2n − 5)/12

= {(n − 3)(n − 2)/12}(6n − 6 − 6n + 9 + 2n − 5)

= {(n − 2)(n − 3)/12}(2n − 2)

= (n − 1)(n − 2)(n − 3)/6

= $^{n-1}C_3$

Proved.

Problem :

a + b + c + d + e = n where a, b, c, d, e, n are all integers > 0. Prove that number of positive solutions of this equation is $^{n-1}C_4$.

Solution :

We fix a = i.

The equation becomes, b + c + d + e = n − i.

Number of solutions (from previous problem) = (n − i − 1)(n − i − 2)(n − i − 3)/6.

So, number of solutions as i varies from 1 to n − 4 is ∑(n − i − 1)(n − i − 2)(n − i − 3)/6 (I running from 1 to n − 4)

= (1/6)∑(n − 1)(n − 2)(n − 3) − {($3n^2$ − 12n + 11)/6}∑i + {(3n − 6)/6}∑i^2 − (1/6)∑i^3 (i running from 1 to n − 4)

= (n − 1)(n − 2)(n − 3)(n − 4)/6 − ($3n^2$ − 12n + 11)(n − 3)(n − 4)/12 + (3n − 6)(n − 4)(n − 3)(2n − 7)/36 - {(n − 3)$(n − 4)^2$/24

$= \{(n - 3)(n - 4)/72\}[12n^2 - 36n + 24 - 18n^2 + 72n - 66 + 12n^2 - 66n + 84 - 3n^2 + 21n - 36]$

$= \{(n - 3)(n - 4)/72\}(3n^2 - 9n + 6)$

$= (n - 1)(n - 2)(n - 3)(n - 4)/24$

$= {}^{n-1}C_4$.

Proved.

Problem :

$a_1 + a_2 + \dots + a_r = n$ where $a_1, a_2, \dots, a_r, n$ are all positive integers. Prove that number of positive solutions of this equation is ${}^{n-1}C_{r-1}$.

Solution :

We will prove this by induction.

Clearly this is true for $r = 1$.

Let this is true for $r = k$ i.e. if there are k number of variables in LHS then number of solutions of the equation is ${}^{n-1}C_{k-1}$.

Now, for $r = k + 1$ i.e. if an extra variable gets added then number of solutions of the equation is, $\sum {}^{n-i-1}C_{k-1}$ (i running from 1 to $n - k$)

We have to prove that, $\sum {}^{n-i-1}C_{k-1}$ (i running from 1 to $n - k$) $= {}^{n-1}C_k$.

We will prove this by another induction.

Clearly this is true for $n = 1$.

Let this is true for $n = p$ i.e. $\sum {}^{p-i-1}C_{k-1}$ (i running from 1 to $p - k$) $= {}^{p-1}C_k$.

Now, for $n = p + 1$, LHS $= \sum {}^{p-i}C_{k-1}$ (i running from 1 to $p - k + 1$) $= {}^{p-1}C_{k-1} + {}^{p-1}C_k$ (from above)

$= {}^{p}C_k$ = RHS for $n = p + 1$.

Proved.

Problem :

Find number of terms in the expansion of $(x + y + z + w)^n$.

Solution :

Now, there are 4 variables and any term consists of the 4 terms such that,

$x_1 + y_1 + z_1 + w_1 = n$ where x_1, y_1, z_1, w_1 are powers of x, y, z, w in any term.

Now, x_1, y_1, z_1, w_1 runs from 0 to n.

So, we need to find number of non-negative solution of this equation and we are done with number of terms of this equation.

From previous article we know, number of non-negative solution of this equation = ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$.

In general if there are r variables then number of terms = ${}^{n+r-1}C_{r-1}$.

Problem :

Find number of terms which are independent of x in the expansion of $(x + y + z + w)^n$.

Solution :

Now, $x_1 = 0$. So, we need to find number of non-negative solution of the equation, $y_1 + z_1 + w_1 = n$ and then we need to subtract this from ${}^{n+3}C_3$ and we are done with the number of terms which are independent of x.

It is, ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$.

So, number of terms which are independent of x = ${}^{n+3}C_3 - {}^{n+2}C_2$.

In general if there are r variables then number of terms excluding one variable = ${}^{n+r-1}C_{r-1} - {}^{n+r-2}C_{r-2}$.

Similarly, we can find number of terms independent of x and y and so on.

Problem :

Prove that number of ways of distributing n identical things among r members where every member gets at least 1 thing is ${}^{n-1}C_{r-1}$.

Solution :

Let, first member gets x_1 things, second member gets x_2 things and so on i.e. t^{th} member gets x_t things.

So, $x_1 + x_2 + \ldots + x_r = n$

Now, we need to find number of positive solutions of this equation because $x_1, x_2, \ldots, x_n > 0$.

From above it is ${}^{n-1}C_{r-1}$.

Proved.

Problem :

In an arrangement of m H's and n T's, an uninterrupted sequence of one kind of symbol is called a run. (For example, the arrangement HHHTHHTTTH of 6 H's and 4T's opens with an H-run of length 3, followed successively by a T-run of length 1, an H-run of length 2, a T-run of length 3 and, finally an H-run if length 1.)

Find the number of arrangements of m H's and n T's in which there are exactly k H-runs.

Solution :

Now, m H's can be put in k places with k+1 holes (spaces) between them in ${}^{m-1}C_{k-1}$ ways.

Now, k – 1 spaces between the H's must be filled up by at least one T.

So, number of ways is ${}^{n-1}C_{k-2}$.

So, in this case number of ways = ${}^{m-1}C_{k-1} * {}^{n-1}C_{k-2}$.

Now, if k spaces (i.e. one space from either side first or last) can be filled by n T's where in every space at least one T is there in ${}^{n-1}C_{k-1}*2$ ways.

So, total number of ways in this case = $2 * {}^{m-1}C_{k-1} * {}^{n-1}C_{k-1}$.

Now, if k+1 spaces (i.e. including first and last space) can be filled up by n T's where in every space at least one T is there in ${}^{n-1}C_k$ ways.

So, in this case total number of ways = ${}^{m-1}C_{k-1}*{}^{n-1}C_k$.

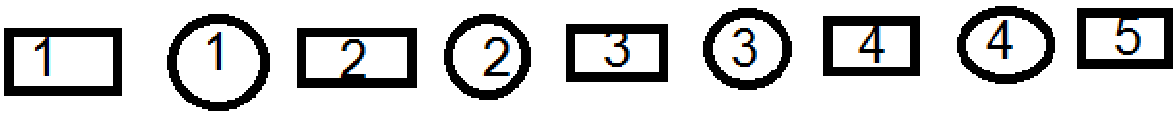
So, total number of ways = ${}^{m-1}C_{k-1}({}^{n-1}C_{k-2} + 2*{}^{n-1}C_{k-1} + {}^{n-1}C_k) = {}^{m-1}C_{k-1}({}^{n-1}C_{k-2} + {}^{n-1}C_{k-1} + {}^{n-1}C_{k-1} + {}^{n-1}C_k)$

$= {}^{m-1}C_{k-1}({}^{n}C_{k-1} + {}^{n}C_k) = {}^{n-1}C_{k-1}*{}^{n+1}C_k$

Problem :

Show that number of ways in which four distinct integers can be chosen from 1, 2, …, n (n ≥ 7) such that no two are consecutive is equal to ${}^{n-3}C_4$.

Solution :



We choose 4 integers as shown in figure by circle.

So, there are maximum 5 spaces between them shown in figure by boxes.

Now, let us say, 2, 3, 4 spaces i.e. boxes are to be filled by other n – 4 integers (4 integers already chosen for 4 circles).

Number of ways = ${}^{n-5}C_2$ (As number of ways is ${}^{n-1}C_{r-1}$ for at least one to be there)

Similarly, for 1, 2, 3, 4, boxes and 2, 3, 4, 5 boxes to be filled by other n – 4 integers number of ways = $2*{}^{n-5}C_3$.

For 5 boxes to be filled by other n – 4 integers number of ways = ${}^{n-5}C_4$.

Total number of ways = ${}^{n-5}C_2 + 2*{}^{n-5}C_3 + {}^{n-5}C_4 = ({}^{n-5}C_2 + {}^{n-5}C_3) + ({}^{n-5}C_3 + {}^{n-5}C_4) = {}^{n-4}C_3 + {}^{n-4}C_4 = {}^{n-3}C_4$.

Proved.

Problem :

Prove that number of ways of distributing n identical things to r members (no condition) is ${}^{n+r-1}C_{r-1}$.

Solution :

Let first member gets x_1 things, second member gets x_2 things and so on i.e. t^{th} member gets x_t things.

We have, $x_1 + x_2 + ... + x_r = n$

We need to find number of non-negative solutions of this equation.

From above it is ${}^{n+r-1}C_{r-1}$.

Proved.

Problem :

Find the number of all possible ordered k-tuples of non-negative integers $(n_1, n_2, ..., n_k)$ such that $\sum n_i$ (i running from 1 to k) = 100.

Solution :

Clearly, it needs number of non-negative solution of the equation, $n_1 + n_2 + ... + n_k = 100$.

It is ${}^{n+k-1}C_{k-1}$.

Problem :

Show that the number of all possible ordered 4-tuples of non-negative integers (n_1, n_2, n_3, n_4) such that $n_1 + n_2 + n_3 + n_4 \leq 100$ is ${}^{104}C_4$.

Solution :

Clearly, required number = ${}^{3}C_3 + {}^{4}C_3 + {}^{5}C_3 + ... + {}^{103}C_3 = {}^{104}C_4$.

Problem :

How many 6-letter words can be formed using the letters A, B and C so that each letter appears at least once in the word?

Solution :

Let x_1 number of A, x_2 number of B and x_3 number of C are chosen where $x_1, x_2, x_3 > 0$

Now, $x_1 + x_2 + x_3 = 6$.

Number of positive solution of this equation is ${}^{6-1}C_{3-1} = {}^{5}C_{2} = 10$.

So, combinations are as follows,

4 A, 1 B, 1 C, number of words = 6!/4! = 30

3 A, 2 B, 1 C, number of words = 6!/(3!*2!) = 60

3 A, 1 B, 2 C, number of words = 6!/(3!*2!) = 60

2 A, 1 B, 3 C, number of words = 6!/(2!*3!) = 60

2 A, 2 B, 2 C, number of words = 6!/(2!*2!*2!) = 90

2 A, 3 B, 1 C, number of words = 6!/(2!*3!) = 60

1 A, 1 B, 4 C, number of words = 6!/4! = 30

1 A, 2 B, 3 C, number of words = 6!/(2!*3!) = 60

1 A, 3 B, 2 C, number of words = 6!/(3!*2!) = 60

1 A, 4 B, 1 C, number of words = 6!/4! = 30

So, total number of words = 30 + 60 + 60 + 60 + 90 + 60 + 30 + 60 + 60 + 30 = 540.

Problem :

All the permutations of the letters a, b, c, d, e are written down and arranged in alphabetical order as in a dictionary. Thus the arrangement abcde is in the first position and abced is in the second position. What is the position of the arrangement debac?

Solution :

Now, first fix a at first place. Number of arrangements = 4!

Now, fix b at first place. Number of arrangements = 4!

Now, fix c at first place. Number of arrangements = 4!

Now comes d the first letter of the required arrangement.

Now fix d at first position and a at second position. Number of arrangement = 3!

Fix b at second place. Number of arrangement = 3!

Fix c at second place. Number of arrangement = 3!

Now, comes e at second place and we have de.

Now, fix a at third place. Number of arrangement = 2!

Now comes b which is required and we have deb.

Then comes a and then c.

So, debac comes after (4! + 4! + 4! + 3! + 3! + 3! + 2!) = 92 arragement.

So, it will take 92 + 1 = 93^{rd} position.

Problem :

x red balls, y black balls and z white balls are to be arranged in a row. Suppose that any two balls of the same color are indistinguishable. Given that x + y + z = 30, show that the number of possible arrangements is the largest for x = y = z = 10.

Solution :

Clearly, number of possible arrangement is (x + y + z)!/{x!*y!*z!} = 30!/{x!*y!*z!}

Now, it will be largest when x!*y!*z! = minimum.

Let us say, x = 12 and y = 8

Now, 12!*8! = 12*11*10!*10!/(10*9) = (12*11/10*9)*$(10!)^2$

- ⇨ 12!*8!/$(10!)^2$ = (12*11)/(10*9) > 1
- ⇨ $(10!)^2$ < 12!*8!
- ⇨ It will be least when x = y = z = 10.

Proved.

Problem :

Find number of arrangements of the letters of the word MISSISIPPI.

Solution :

Number of letters = 10

Number of I's = 4, number of S's = 3, number of P's = 2

Therefore, total number of words that can be formed from the letters of the word is 10!/(4!*3!*2!).

Problem :

Find the number of words (meaningful or non-meaningful) that can be formed from the letters of the word MOTHER.

Solution :

Number of letters = 6.

All are distinct.

Hence total number of words = 6!

Problem :

Show that the number of ways one can choose a set of distinct positive integers, each smaller than or equal to 50, such that their sum is odd, is 2^{49}.

Solution :

The sum is odd.

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⇨ We need to select odd number of integers.

Now, we can select 1 integer from 50 integers in $^{50}C_1$ ways.

We can select 3 integers from 50 integers in $^{50}C_3$ ways.

…

..

We can select 49 integers from 50 integers in $^{50}C_{49}$ ways.

So, number of ways = $^{50}C_1 + {}^{50}C_3 + \ldots.. + {}^{50}C_{49}$.

Now, $^{50}C_0 + {}^{50}C_1 + \ldots.. + {}^{50}C_{49} + {}^{50}C_{50} = 2^{50}$

Now, $^{50}C_0 - {}^{50}C_1 + \ldots.. - {}^{50}C_{49} + {}^{50}C_{50} = 0$

Subtracting the above two equations we get,

$2({}^{50}C_1 + {}^{50}C_3 + \ldots. + {}^{50}C_{49}) = 2^{50}$

⇨ $^{50}C_1 + {}^{50}C_3 + \ldots. + {}^{50}C_{49} = 2^{49}$.

Proved.

Number of ways of distributing n **distinct** things to r persons (r < n) so that every person gets *at least* one thing.

Total number of ways = r^n

Now, let A_i denotes that the i^{th} person doesn't get a gift and B denotes that every person gets at least one gift.

Therefore, $r^n = |B| + |A_1 \cup A_2 \cup A_3 \cup \ldots.. \cup A_r|$

Now, $|A_1 \cup A_2 \cup \ldots. \cup A_r| = \sum A_i - \sum A_i \cap A_j + \sum A_i \cap A_j \cap A_k - \ldots.. + (-1)^{r-1}[A_1 \cap A_2 \cap \ldots \cap A_r]$

Now, $|A_i| = (r-1)^n$, $|A_i \cap A_j| = (r-2)^n$,…., $A_1 \cap A_2 \cap \ldots. \cap A_r = (r-r)^n = 0^n$

So, $|A_1 \cup A_2 \cup \ldots \cup A_r| = {}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \ldots. + (-1)^{r-1}\,{}^rC_r0^n$

Therefore, $|B| = r^n - [{}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \ldots. + (-1)^{r-1}\,{}^rC_r0^n]$

$= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \ldots. + (-1)^{r}\,{}^rC_r0^n$

Solved.

<u>Origin of Fermat numbers and Mersenne numbers :</u>

Definition of Fermat numbers :

$F_n = 2^{(2^n)} + 1$ where n = 0, 1, 2, ... are called Fermat numbers.

Definition of Mersenne numbers :

$M_n = 2^n - 1$ where n = 1, 2, 3, ... are called Mersenne numbers.

$M_p = 2^p - 1$ where p is prime, are called Mersenne number with prime index.

Discussion :

Now, the question is when $a^n + b^n$ is prime.

Let n is odd.

Then $a^n + b^n \equiv (-b)^n + b^n \pmod{a + b}$

- ⇨ $a^n + b^n \equiv -b^n + b^n \pmod{a + b}$ (As n is odd)
- ⇨ $a^n + b^n \equiv 0 \pmod{a + b}$

So, we see that if n is odd then $a^n + b^n$ is divisible by the sum of the bases. So cannot be prime.

Let n is even but not of the form 2^x

Let, $n = 2^x*m$ where m is odd not necessarily prime.

Now, $a^n + b^n = a^{(2^x*m)} + b^{(2^x*m)}$

- ⇨ $a^n + b^n = \{a^{(2^x)}\}^m + \{b^{(2^x)}\}^m$

Now, $\{a^{(2^x)}\}^m + \{b^{(2^x)}\}^m \equiv \{-b^{(2^x)}\}^m + \{b^{(2^x)}\}^m\} \pmod{a^{(2^x)} + b^{(2^x)}}$

- ⇨ $a^n + b^n \equiv -\{b^{(2^x)}\}^m + \{b^{(2^x)}\}^m \pmod{a^{(2^x)} + b^{(2^x)}}$ (As m is odd)
- ⇨ $a^n + b^n \equiv 0 \pmod{a^{(2^x)} + b^{(2^x)}}$
- ⇨ $a^n + b^n$ iss divisible by $a^{(2^x)} + b^{(2^x)}$ and cannot be prime.
- ⇨ $a^n + b^n$ can be prime if n is of the form 2^k.

Now, put a = 2 and b = 1 and 2^n in place of n as n is of the form 2^k we get,

$2^{(2^n)} + 1$ which is called Fermat numbers.

Corollary :

$2^k + 1$ can be prime if k is of the form 2^n.

Proof :

Clearly, if k is odd then $2^k + 1 = 2^k + 1^k$ is divisible by sum of the bases = 2 + 1 = 3.

$2^k + 1 \equiv (-1)^k + 1 \pmod 3$

- ⇨ $2^k + 1 \equiv -1 + 1 \pmod 3$ (As k is odd)
- ⇨ $2^k + 1 \equiv 0 \pmod 3$

Now, if k is even but not of the form 2^x, let $k = 2^x*m$ where m is odd, then $2^k + 1$ is divisible by $2^{(2^x)} + 1$.

$\{2^{(2^x)}\}^m + 1 \equiv (-1)^m + 1 \pmod{2^{(2^x)} + 1}$

- ⇨ $2^k + 1 \equiv -1 + 1 \pmod{2^{(2^x)} + 1}$ (As m is odd)
- ⇨ $2^k + 1 \equiv 0 \pmod{2^{(2^x)} + 1}$
- ⇨ k must be of the form 2^n to be $2^k + 1$ prime.

Discussion :

Now, the question comes when $a^n – b^n$ is prime.

Now, $a^n – b^n = (a – b)(a^{n-1} + ba^{n-2} + ... + b^{n-1})$

Clearly, $a - b$ is a factor of $a^n - b^n$ and cannot be prime whatever be the value of n.

What if, a and b are consecutive integers.

Then, $a - b = 1$.

So, we don't have a trivial factor.

Now, put $a = 2$ and $b = 1$, we get,

$2^n - 1$ which are called Mersenne numbers.

Now, if n is composite, say $n = ab$.

Then, $2^n - 1 = 2^{ab} - 1$

- ⇨ $2^n - 1 = (2^a)^b - 1$

Now, $(2^a)^b - 1 \equiv (1)^b - 1 \pmod{2^a - 1}$

- ⇨ $2^n - 1 \equiv 0 \pmod{2^a - 1}$

Similarly, $2^n - 1 \equiv 0 \pmod{2^b - 1}$

- ⇨ $2^n - 1$ is divisible by $2^a - 1$ and $2^b - 1$ where a and b are factors of n. So, cannot be prime.

Now, if n is prime then we don't have any trivial factor of $2^n - 1$.

From here the concept of Mersenne numbers with prime index has come up.

Prove that $2^m - 1$ cannot divide $2^n + 1$ where $n > m$.

Proof :

Let, $n = mr + s$ where $s < m$

Now, $2^m \equiv 1 \pmod{2^m - 1}$

- ⇨ $2^{mr} \equiv 1 \pmod{2^m - 1}$
- ⇨ $2^{mr+s} \equiv 2^s \pmod{2^m - 1}$
- ⇨ $2^n + 1 \equiv 2^s + 1 \pmod{2^m - 1}$

Now, $2^m - 1$ cannot divide $2^s + 1$ as $s < m$.

- ⇨ $2^m - 1$ cannot divide $2^n + 1$.

Proved.

Application :

Problem 1 :

Find n such that 7 divides $5^n + 1$.

Solution :

Now, $5^n + 1 \equiv (-2)^n + 1 \pmod{2^3 - 1}$ (As $7 = 2^3 - 1$)

- ⇨ $(-2)^n + 1 \equiv 0 \pmod{2^3 - 1}$ (As 7 divides $5^n + 1$)

Now, if n is even then the equation becomes, $2^n + 1 \equiv 0 \pmod{2^3 - 1}$

Which is impossible as from the above proof we have, $2^m - 1$ cannot divide $2^n + 1$.

So, n must be odd.

So, the equation becomes, $2^n - 1 \equiv 0 \pmod{2^3 - 1}$

Now, $2^n - 1$ gets divided by $2^a - 1$ where a is factor of n.

- ⇨ n must be of the form 3m where m is odd.
- ⇨ n = 3(2m + 1) (putting 2m + 1 in place of m as m is odd)
- ⇨ n = 6m + 3 where m = 0, 1, 2, 3, ...

Problem 2 :

Find n such that 63 divides $61^n + 1$.

Solution :

Now, $61^n + 1 \equiv (-2)^n + 1 \pmod{2^6 - 1}$ (as $63 = 2^6 - 1$)

From the same argument as the above problem, n cannot be even and n must be odd.

So, the equation becomes, $2^n - 1 \equiv 0 \pmod{2^6 - 1}$

Now, n = 6m but n cannot be even.

⇨ No solution.

Polynomial

Let, P(x) be a polynomial of degree d, then P(x) can be written as,

$P(x) = a_1x^d + a_2x^{d-1} + a_3x^{d-2} + ... + a_{d-1}x^2 + a_dx + a_{d+1}$

Remainder theorem :

Consider a polynomial of degree d > 1. P(x) gives the remainder P(a) if P(x) is divided by x – a.

Proof :

Let P(x) = (x – a)Q(x) + R where R is constant as the divider is linear so at most degree of R is 0 i.e. free of x or constant.

Q(x) is the quotient and R is remainder.

Putting x = a in the above expression we get,

P(a) = (a – a)Q(a) + R

⇨ R = P(a)

Proved.

Example :

Problem 1 :

Find the remainder when $P(x) = x^2 + x + 1$ is divided by x + 1.

Solution 1 :

From above we have remainder = R = P(-1)

- ⇨ $R = P(-1) = (-1)^2 + (-1) + 1 = 1$.

Solved.

Consider a polynomial P(x) of degree d > 1. We will now find the remainder if there is any repeated root in divider. So, we will find the remainder when P(x) is divided by $(x - a)^2$.

Let, $P(x) = (x - a)^2Q(x) + R(x)$ (Note that this time R(x) is not constant and have degree 1 as divider is quadratic)

Let, R(x) = Ax + B.

Q(x) is the quotient.

Now, putting the value of R(x) in the above equation the equation becomes,

$P(x) = (x - a)^2Q(x) + Ax + B$

Now, putting x = a in the above equation we get,

$P(a) = (a - a)^2Q(a) + Aa + B$

- ⇨ $Aa + B = P(a)$

Now, differentiating the above equation w.r.t. x we get,

$P'(x) = 2(x - a)Q(x) + (x - a)^2Q'(x) + A$

Putting x = a in the above equation we get,

$P'(a) = 2(a - a)Q(a) + (a - a)^2Q'(a) + A$

- ⇨ $A = P'(a)$
- ⇨ $B = P(a) - Aa = P(a) - a*P'(a)$
- ⇨ $R(x) = Ax + B = P'(a)x + P(a) - a*P'(a) = P'(a)(x - a) + P(a)$

Done.

Example :

Problem 1 :

Consider a polynomial P(x) of degree d > 2. Let R(x) be the remainder when P(x) is divided by $(x - 1)^2$. P'(1) = P(1) = 1. Find R(x).

Solution 1 :

From the above result we have,

R(x) = P'(1)(x – 1) + P(1) = 1*(x – 1) + 1 = x.

Solved.

Consider a polynomial P(x) of degree d > 1. Now we will find the remainder when the divider is quadratic and have two distinct roots. Let us find the remainder when P(x) is divided by (x – a)(x – b).

Let, P(x) = (x – a)(x – b)Q(x) + R(x)

Q(x) = quotient and R(x) = remainder = Ax + B.

Putting value of R(x) in the above equation we get,

P(x) = (x – a)(x – b)Q(x) + Ax + B

Putting x = a in the above equation we get,

P(a) = (a – a)(a – b)Q(a) + Aa + B

⇨ Aa + B = P(a) ....(i)

Now, putting x = b in the above equation we get,

P(b) = (b – a)(b – b)Q(b) + Ab + B

⇨ Ab + B = P(b) ......(ii)

Now, from (i) and (ii) we get,

$Aa + B - Ab - B = P(a) - P(b)$

- ⇨ $A = \{P(a) - P(b)\}/(a - b$
- ⇨ $B = \{aP(b) - bP(a)\}/(a - b)$
- ⇨ $R(x) = Ax + B = \{P(a) - P(b)\}x/(a - b) + \{aP(b) - bP(a)\}/(a - b)$

Done.

Consider a polynomial of degree $d > 1$. Now, we will work with quotient. Let $Q(x)$ be the quotient when $P(x)$ is divided by $(x - a)$. Then we will have the relation $Q(a) = P'(a)$.

Proof :

Let, $P(x) = (x - a)Q(x) + R$ where R is remainder and note that R is constant as divider is linear i.e. of degree 1.

Differentiating the above equation w.r.t. x we get,

$P'(x) = Q(x) + (x - a)Q'(x) + R'$

Note that $R' = 0$ as R is constant.

So, we have, $P'(x) = Q(x) + (x - a)Q'(x)$

Putting $x = a$ in the above equation we get,

$P'(a) = Q(a) + (a - a)Q'(a)$

- ⇨ $Q(a) = P'(a)$

Proved.

Example :

Problem 1 :

Consider a polynomial $P(x) = x^3 + 3x^2 + 2x + 1$. $Q(x)$ is the quotient when $P(x)$ is divided by $x - 1$. Find the value of $Q(1)$.

Solution 1 :

From the above we have the result, $Q(1) = P'(1)$

Given $P(x) = x^3 + 3x^2 + 3x + 1$

- ⇨ $P'(x) = 3x^2 + 6x + 3$
- ⇨ $P'(1) = 3*1^2 + 6*1 + 3$
- ⇨ $Q(1) = P'(1) = 12$.

Solved.

Problem 2 :

Consider a polynomial $P(x)$ of degree $d > 1$. $Q(x) = 4x + 3$ is the quotient when $P(x)$ is divided by $x - 7$. Find the slope of $P(x)$ at $x = 7$ or put in other way find $P'(7)$.

Solution 2 :

From above we have, $P'(7) = Q(7) = 4*7 + 3 = 31$.

Solved.

Consider a polynomial $P(x)$ of degree $d > 1$. Now, we will see relation between quotient i.e. $Q(x)$ and $P(x)$ when there is repeated root in the divider. Let, $Q(x)$ is the quotient when $P(x)$ is divided by $(x - a)^2$. Then we will have the relation $Q(a) = P''(a)/2$.

Proof :

Let, $P(x) = (x - a)^2Q(x) + R(x)$ (Note that $R(x)$ is linear here and so $R''(x) = 0$)

Differentiating w.r.t. x we get,

$P'(x) = 2(x - a)Q(x) + (x - a)^2Q'(x) + R'(x)$

Differentiating again w.r.t. x we get,

$P''(x) = 2Q(x) + 2(x - a)Q'(x) + 2(x - a)Q'(x) + (x - a)^2Q''(x) + R''(x)$

Putting x = a in the4 above expression we get,

$P''(a) = 2Q(a) + 2(a - a)Q'(a) + 2(a - a)Q'(a) + (a - a)^2Q''(a) + 0$

- ⇨ $Q(a) = P''(a)/2$.

Proved.

Example :

Problem 1 :

Let $Q(x) = 3x^2 + 2$ is the quotient when P(x) is divided by $(x - 1)^2$. Find the value of $P''(1)$.

Solution 1 :

From the above result we have,

$P''(1) = 2Q(1) = 2(3*1^2 + 2) = 10$.

Solved.

Problem 2 :

Let P(x) be a polynomial of degree $d > 2$. Q(x) is the quotient when P(x) is divided by $(x - 2)^2$. $Q(2) = 4$. Find $P''(2)$.

Solution 2 :

From the above result we have,

$P''(2) = 2Q(2) = 2*4 = 8$.

Solved.

Tips to solve problems :

1. The remainder of P(x) divided by x+ a can be found by putting x = -a i.e. p(-a) will give the remainder when P(x) is divided by x + a.
2. If there is a root between (a, b) then P(a) and P(b) will be of opposite sign.
3. If P(x) is strictly increasing or decreasing then P(x) have at most one real root. P(x) can be proved increasing if $P'(x) > 0$ and decreasing if $P'(x) < 0$.
4. To find number of real roots in P(x) draw the graph of LHS and RHS and count the number of intersection points and that is the answer.
5. If there is any mention of sum of co-efficients then think of P(1) and vice versa.
6. If there is any repeated root think of derivative.
7. If there is any involvement of quotient Q(x) then write the equation P(x) = D(x)Q(x) + R(x) where D(x) is the divider and think of derivative.
8. If there is any mention of becoming the polynomial prime think of P(0) i.e. the constant term.
9. R(x) has at most degree d - 1 where d is the degree of divider.
10. Complex roots come in pair. If a polynomial is of degree d which is odd then the polynomial must have at least one real root.
11. If there is any question/mention of multiplicity of a root then do derivative for m + 1 times where m is at most multiplicity of the root and show that $P^{(m+1)}(x)$ doesn't have the root.

Application :

Problem 1 :

Consider the polynomial $P(x) = 30x^7 - 35x^6 + 42x^5 + 210x^3 - 1470$. Prove that P(x) = 0 have only one real root and the root lies between (1, 2).

Solution 1 : (Tips number 2 and 3)

Now, $P(x) = 30x^7 - 35x^6 + 42x^5 + 210x^3 - 1470$

- ⇨ $P'(x) = 210x^6 - 210x^5 + 210x^4 + 630x^2$
- ⇨ $P'(x) = 210x^4(x^2 - x + 1) + 630x^2$
- ⇨ $P'(x) = 210x^4\{(x - ½)^2 + ¾\} + 630x^2$

Which is always greater than 0.

- ⇨ P(x) is increasing.
- ⇨ P(x) has at most one real root.

Now, $P(1) = 30*1^7 - 35*1^6 + 42*1^5 + 210*1^3 - 1470 < 0$

And, $P(2) = 30*2^7 - 35*2^6 + 42*2^5 + 210*2^3 - 1470 > 0$

- ⇨ There is a root between (1, 2) and this root is the only real root of P(x) = 0.

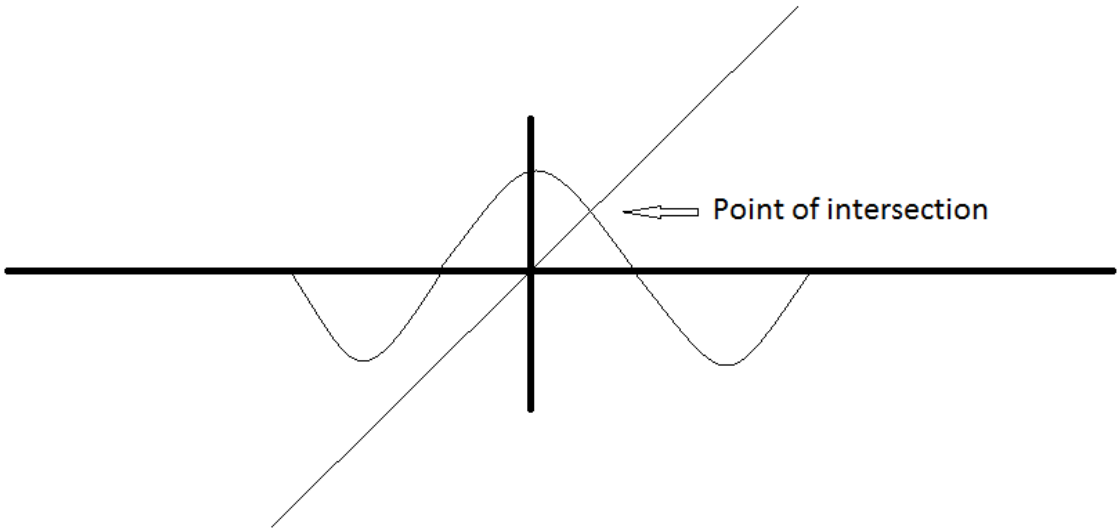
Proved.

Problem 2 :

Prove that x = cosx has only one real root.

Solution 2 : (Tips number 4)

Now, drawing the graph of LHS and RHS i.e. y = x and y = cosx.



Clearly, there is one point of intersection.

⇨ There is one real root.

Proved.

Problem 3 :

P(x) and Q(x) are two polynomials such that the sum of the co-efficient is same for both. Prove that the remainders when P(x) and Q(x) are divided by x - 1 are same.

Solution 3 : (Tips number 5)

Let, $P(x) = a_1x^d + a_2x^{d-1} + ... + a_dx + a_{d+1}$

Putting x = 1 we get,

$P(1) = a_1 + a_2 + ... + a_d + a_{d+1}$ = sum of the co-efficients.

Now, from remainder theorem (also tips number 1) we have the remainder P(x) gives when P(x) is divided by (x – 1) is P(1) = sum of co-efficients.

Similar thing goes for Q(x) i.e. Q(1) = sum of co-efficients.

Now, it is given that sum of co-efficients of P(x) = sum of co-efficients of Q(x)

- ⇨ P(1) = Q(1).

Proved.

Problem 4 :

P(x) is a polynomial of degree $d > 1$ with integer co-efficients. P(1) is divisible by 3. All the co-efficients are placed side by side in any order to make a positive integer. For example if there is 2 co-efficients 20 and 9 then the numbers formed are 209 or 920. Prove that the number thus generated is divisible by 3.

Solution 4 : (Tips number 5)

Now, in the previous example we have seen that P(1) = sum of co-efficients of P(x).

Now, the sum is divided by 3.

As per the rule of divisibility by 3 says a number is divisible by 3 if sum of the digits is divisible by 3.

- ⇨ The sum of the digits of the cp-effiiients of P(x) is divisible by 3.

Now, whatever be the order of placing the digits of the co-efficients of P(x) the sum of their digits must be same.

- ⇨ The sum of the digits is divisible by 3.
- ⇨ Thus the number generated with whatever be the order of placing the digits is divisible by 3.

Proved.

Problem 5 :

P(x) and Q(x) be two polynomials with integer coefficients. Suppose the sum of the coefficients of both the polynomials are equal to S. It is given that $(P(x))^3 - (Q(x))^3 = P(x^3) - Q(x^3)$ for all x.

1. Show that $P(x) - Q(x) = (x - 1)^a R(x)$; $a \geq 1$
2. Show that S^2 = 3^{a-1}

Solution 5 : (Tips number 5 and 6)

This question deals with sum of the coefficients are same means P(1) = Q(1) and repeated roots at x = 1 i.e. we have to go for differentiation and prove that $(a - 1)^{th}$ time derivative have the root at x = 1. Thus we follow the below steps to solve the problem.

Let, F(x) = P(x) – Q(x)

P(1) = Q(1) = sum of coefficients = S. (As per tips 5)

F(1) = P(1) – Q(1) = S – S = 0

We will show that F(x) have repeated roots at x = 1. For this we need to show that F′(1), F′(2), ... have roots (x – 1) (As per tips 6)

We have $(P(x))^3 - (Q(x))^3 = P(x^3) - Q(x^3)$

⇨ $(P(x) - Q(x))^3 - 3P(x)Q(x)(P(x) - Q(x)) = F(x^3)$
⇨ $(F(x))^3 - 3P(x)Q(x)F(x) = F(x^3)$

Differentiating w.r.t. x we get,

$3(F(x))^2F(x) - 3(P'(x)Q(x)F(x) + P(x)Q'(x)F(x) + P(x)Q(x)F'(x)) = 3x^2F'(x^3)$ ......(A)

Putting x = 1 we get,

$3(F(1))^2F'(1) - 3(P'(1)Q(1)F(1) + P(1)Q'(1)F(1) + P(1)Q(1)F'(1)) = 3F'(1)$

⇨ $3*0^2*F'(1) - 3(P'(1)Q(1)*0 + P(1)Q'(1)*0 + S*S*F'(1)) = 3F'(1)$ (As F(1) = 0)
⇨ $3S^2F'(1) = 3F'(1)$

Now, either $S^2 = 1$ or F′(1) = 0

We consider F′(1) = 0

Now, again differentiating equation (A) w.r.t. x we get,\

$6F(x)F'(x) + 3F(x)^2F''(x) - 3(P''(x)Q(x)F(x) + P'(x)Q'(x)F(x) + P'(x)Q(x)F'(x) + P'(x)Q'(x)F(x) + P(x)Q''(x)F(x) + P(x)Q'(x)F'(x) + P'(x)Q(x)F'(x) + P(x)Q'(x)F'(x) + P(x)Q(x)F''(x)) = 6xF'(x^3) + 9x^4F''(x^3)$

Putting x = 1 and F(1) = F′(1) = 0 we get,

$3*S*S*F''(x) = 9*1^4*F''(x)$

- ⇨ $S^2F''(x) = 3F''(x)$

Now, either $S^2 = 3$ or $F''(x) = 0$.

Continuing in this way we can find $S^2 = 3^{a-1}$ and $F^{(a-1)}(x) = 0$ because in each case a 3 will be generated.

- ⇨ F(x) have a times repeated root at x = 1.
- ⇨ $F(x) = P(x) - Q(x) = (x-1)^aR(x)$ and $S^2 = 3^{a-1}$.

Proved.

Problem 6 :

Prove that there is no non-constant polynomial P(x) with integer coefficients such that P(n) is prime number for all positive integers n.

Solution 6 : (Tips number 8)

Let, there is a polynomial P(x) with such property.

Let, $P(x) = a_1x^m + a_2x^{m-1} + ... + a_{m+1}$.

Now, $P(a_{m+1}) = a_1a_{m+1}{}^m + a_2a_{m+1}{}^{m-1} + ... + a_{m+1} = a_{m+1}(a_1a_{m+1}{}^{m-1} + a_2a_{m-1}{}^{m-2} + ... + 1)$ which is clearly not a prime. (Putting $x = a_{m+1} = P(0)$ as per tips 8)

Now, if there is no constant term a_{m+1} then it is divisible by all numbers.

- ⇨ Our assumption was wrong.
- ⇨ There is no non-constant polynomial with such property.

Proved.

Problem 7 :

Show that the equation $x(x - 1)(x - 2).....(x - 2009) = c$ has real roots of multiplicity at most 2.

Solution 7 : (tips number 11)

As per the tips we need to show $P''(x)$ doesn't have the root which $P(x)$ and $P'(x)$ has.

We have, $x(x - 1)(x - 2).....(x - 2009) = c$

Differentiating w.r.t. x we get,

$(x - 1)(x - 2)....(x - 2009) + x(x - 2)(x - 3)....(x - 2009) + x(x - 1)(x - 3)...(x - 2009) + ... + x(x - 1)(x - 2)....(x - 2008) = 0$

- ⇨ $c/x + c/(x - 1) + c/(x - 2) + ... + c/(x - 2009) = 0$ (Putting value from the given equation)
- ⇨ $c\{1/x + 1/(x - 1) + 1/(x - 2) + ... + 1/(x - 2009)\} = 0$

Now, differentiating again w.r.t. x we get,

$-c\{1/x^2 + 1/(x - 1)^2 + 1/(x - 2)^2 + ... + 1/(x - 2009)^2\} = 0$

This equation cannot hold true as sum of squares of real numbers equal to 0 but they are always greater than 0.

- ⇨ The given polynomial cannot have real roots of multiplicity more than 2.

Proved.

Problem 8 :

Prove that the polynomial $P(x) = x^3 + x - 2$ have at least one real root.

Solution 8 : (Tips number 10)

Let all the roots of P(x) is complex.

Complex roots come in pair.

- ⇨ There needs to be 4 roots of P(x)

But P(x) have at most 3 roots as the degree of the polynomial is 3.

- ⇨ Our assumption was wrong. It may have at most 2 complex root.
- ⇨ There is at least one real root of P(x).

Proved.

Solved examples :

1. Consider a polynomial P(x) of degree d > 1. Q(x) is the quotient when P(x) is divided by x - a. Prove that P'(a) = Limiting value of Q(x) as x -> a.

Solution :

We can write, P(x) = (x – a)Q(x) + R(x) where R(x) is remainder when P(x) is divided by (x – a).

- ⇨ R(x) is constant.

Now, P'(a) = {P(x) – P(a)}/(x – a) as x -> a

- ⇨ P'(a) = {(x – a)Q(x) + R(x) – R(a)}/(x – a) as x -> a (P(a) = R(a))
- ⇨ P'(a) = Q(x) as x -> a + {R(x) – R(a)}/(x – a) as x -> a
- ⇨ P'(a) = Q(x) as x -> a + R'(a)
- ⇨ P'(a) = Q(x) as x -> a (as R'(a) = 0 because R(x) is constant)

Proved.

2. Consider a polynomial P(x) of degree d > 2. Let R(x) be the remainder when P(x) is divided by $(x - 1)^2$. P'(1) = P(1) = 1. Find R(x).

Solution :

We can write, $P(x) = (x - 1)^2Q(x) + R(x)$ where Q(x) is the quotient when P(x) is divided by $(x - 1)^2$.

- ⇨ $P'(x) = 2(x - 1)Q(x) + (x - 1)^2Q'(x) + R'(x)$
- ⇨ P'(1) = R'(1) = 1

And, P(1) = R(1) = 1

Now, R(x) is remainder when P(x) is divided by $(x - 1)^2$

⇨ R(x) is linear.

Say, R(x) = ax + b

Now, R'(x) = a

⇨ R'(1) = a = 1

Now, R(1) = a + b = 1

⇨ b = 0.

So, R(x) = x.

3. Let P(x) be a polynomial of degree d > 2. Q(x) is the quotient when P(x) is divided by $(x - 2)^2$. Q(2) = 4. Find P''(2).

Solution :

We can write, $P(x) = (x - 2)^2Q(x) + R(x)$ where R(x) is remainder when P(x) is divided by $(x - 2)^2$.

So, R(x) is linear.

Now, $P'(x) = 2(x - 2)Q(x) + (x - 2)^2Q'(x) + R'(x)$

⇨ $P''(x) = 2Q(x) + 2(x - 2)Q'(x) + 2(x - 2)Q'(x) + (x - 2)^2Q''(x) + R''(x)$

⇨ P''(2) = 2Q(2) + R''(2)

Now, R''(2) = 0 as R9x) is linear.

⇨ P''(2) = 2Q(2) = 2*4 = 8.

4. P(x) and Q(x) are two polynomials such that the sum of the co-efficient is same for both. Prove that the remainders when P(x) and Q(x) are divided by x - 1 are same.

Solution :

Sum of the coefficients of P(x) and Q(x) are same.

⇨ P(1) = Q(1).

Now, as per Remainder theorem P(1) and Q(1) are remainders when P(x) and Q(x) are divided by (x – 1) and they are clearly same.

Proved.

5. P(x) is a polynomial of degree d > 1 with integer coefficients. P(1) is divisible by 3. All the coefficients are placed side by side in any order to make a positive integer. For example if there is 2 coefficients 20 and 9 then the numbers formed are 209 or 920. Prove that the number thus generated is divisible by 3.

Solution :

P(1) is divisible by 3.

- ⇨ Sum of the coefficients is divisible by 3.
- ⇨ If we place the coefficients side by side then the number formed will be divisible by 3 as the sum of the digits is divisible by 3 as per P(1) is divisible by 3.

Proved.

6. Consider a polynomial P(x) of degree d > 2. Q(x) is the quotient when P(x) is divided by D(x). D(x) is quadratic and x = a is a root. P''(a) = b and Q(a) = b/2. Prove that D(x) has repeated root at x = a.

Solution :

Let, D(x) = (x – a)(x – d)

We can write, P(x) = D(x)Q(x) + R(x) where R(x) is remainder when P(x) is divided by D(x). R(x) is linear as D(x) is quadratic.

- ⇨ $P(x) = (x - a)(x - d)Q(x) + R(x)$
- ⇨ $P'(x) = (x - d)Q(x) + (x - a)Q(x) + (x - a)(x - d)Q'(x) + R'(x)$
- ⇨ $P''(x) = Q(x) + (x - d)Q'(x) + Q(x) + (x - a)Q'(x) + (x - d)Q'(x) + (x - a)Q'(x) + (x - c)(x - d)Q''(x) + R''(x)$
- ⇨ $P''(a) = 2Q(a) + 2(a - d)Q'(a) + R''(a)$
- ⇨ $b = 2(b/2) + 2(a - d)Q'(a)$ ($R''(a) = 0$ as R(x) is linear)
- ⇨ $(a - d)Q'(a) = 0$
- ⇨ $d = a$
- ⇨ D(x) has repeated root at x = a.

Proved.

7. Consider two polynomials P(x) and Q(x) of degree $d > 0$ with integer coefficients. $P(0) = Q(0)$. Prove that there exists an integer n which divides both P(n) and Q(n).

Solution :

$P(0) = Q(0)$

⇨ The constant term of P(x) and Q(x) are equal.

Let, $P(x) = a_1x^p + a_2x^{p-1} + \ldots.. + a_px + n$

Let, $Q(x) = b_1x^q + b_2x^{q-1} + \ldots.. + b_qx + n$

Clearly n divides both P(n) and Q(n).

Proved.

8. Let P(x) be a polynomial of degree 3d - 1 where $d > 0$. Let $P^{(i)}(0) = 3*(i!)$ where $P^{(i)}(x)$ is i-th derivative of P(x) w.r.t. x. Prove that P(1) is divisible by 9.

Solution :

Let, $P(x) = a_1x^n + a_2x^{n-1} + \ldots. + a_{n-1}x^2 + a_nx + a_{n+1}$

$P(0) = a_{n+1} = 3$

$P'(0) = a_n = 3$

$P''(0) = (2!)a_{n-1} = 3*2!$

⇨ $a_{n-1} = 3$.

Similarly, $a_{n-2} = a_{n-3} = \ldots. = a_1 = 3$.

Now, $P(1) = a_1 + a_2 + \ldots + a_{n+1} = a_1 + a_2 + \ldots. + a_{3d}$ as $n = 3d - 1$.

⇨ $P(1) = 3 + 3 + \ldots.$ 3d times. $= 3*3d = 9d$
⇨ P(1) is divisible by 9.

Proved.

9. Consider a polynomial P(x) of degree d > 1. Given P(0) = 25. All the roots of P(x) are distinct positive integers. $P^{(d)}(0) = d!$. Find the value of $P^{(d-1)}(0)/(d-1)!$ where $P^{(m)}(x)$ is m-th derivative of P(x) w.r.t. x.

Solution :

Let, $P(x) = a_1x^d + a_2x^{d-1} + \ldots\ldots + a_{d-1}x^2 + a_dx + 25$ (As P(0) = 25)

Now, $P'(0) = a_d = 1$

$P''(0) = (2!)a_{d-1} = 2!$

⇨ $a_{d-1} = 1$

Similarly, $a_{d-2} = a_{d-3} = a_{d-4} = \ldots. = a_2 = a_1 = 1$

Now, $P^{(d-1)}(0) = (d - 1)!*a_2$

⇨ $P^{(d-1)}(0)/(d - 1)! = a_2 = 1$

10. Let P(x) and Q(x) be two polynomials of degree d_1 and d_2 respectively where d_1 and d_2 are both odd. Prove that the sum of the squares of the number of real roots of P(x) and Q(x) cannot be equal to a^n where a and n are positive integers, n > 1.

Solution :

Now, if P(x) have complex roots then they will come in pair (complex + conjugate)

So, number of real roots of P(x) must be odd as degree = d_1 = odd.

Similarly, number of real roots of Q(x) must be odd.

Let, number of real roots of P(x) and Q(x) are u and v respectively.

Now, let, $u^2 + v^2 = a^n$

As u and v are both odd, a is even.

Now, dividing the equation by 4 we get,

$1 + 1 \equiv 0 \pmod 4$ as n > 1

⇨ $2 \equiv 0 \pmod 4$

Which is impossible

⇨ Sum of squares of the number of real roots of P(x) and Q(x) cannot be equal to a^n where a and n are positive integers, n > 1.

Proved.

11. Let a, b, c be three distinct integers, and let P be a polynomial with integer coefficients. Show that in this case the conditions P(a) = b, P(b) = c, P(c) = a cannot be satisfied simultaneously (USO 1974).

Solution :

Suppose the conditions are satisfied. We derive a contradiction.

$P(x) - b = (x - a)P_1(x)$ ……………… (1)

$P(x) - c = (x - b)P_2(x)$ ……………… (2)

$P(x) - a = (x - c)P_3(x)$ ……………… (3)

Among the numbers a, b, c, we choose the pair with maximal absolute difference.

Suppose this is $|a - c|$. Then we have

$|a - b| < |a - c|$……………………. (4)

If we replace x by c in (1), then we get

$a - b = (c - a)P_1(c)$.

Since $P_1(c)$ is an integer, we have $|a - b| \geq |c - a|$, which contradicts (4).

Proved.

12. Let f(x) be a monic polynomial with integral coefficients. If there are four different integers a, b, c, d, so that f(a) = f(b) = f(c) = f(d) = 5, then there is no integer k, so that f(k) = 8.

Solution :

Monic polynomial means the highest degree coefficient is 1.

Now, $f(x) = (x - a)(x - b)(x - c)(x - d)Q(x) + 5$

Now, $f(k) = (k - a)(k - b)(k - c)(k - d)Q(k) + 5$

- ⇨ $(k - a)(k - b)(k - c)(k - d)Q(k) = 3$ (As $f(k) = 8$)
- ⇨ 3 is factor of at least four distinct integers $k - a$, $k - b$, $k - c$, $k - d$ as a, b, c, d are distinct.

But this is impossible as 3 is prime and may have maximum 3 factors viz. 3, 1, -1.

Proved.

Problem (Special type) :

Check whether $n^{68} + n^{37} + 1$ is divisible by $n^2 + n + 1$.

Solution :

As ω is cube root of unity so we have $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$.

Put ω in place of n and check whether the given expression is divisible by $\omega^2 + \omega + 1$ or not. This is possible because $n^3 \equiv 1 \pmod{n^2 + n + 1}$

So, $n^{68} + n^{37} + 1 = \omega^{68} + \omega^{37} + 1 = \omega^2 + \omega + 1 = 0$

Hence, $n^2 + n + 1$ divides $n^{68} + n^{37} + 1$.

Solved.

Symmetric Polynomials :

A polynomial $f(x, y)$ is symmetric, if $f(x, y) = f(y, x)$ for all x, y. Examples,

(a) The elementary symmetric polynomials in x and y
$\sigma_1 = x + y$ and $\sigma_2 = xy$
(b) The power sums, $s_i = x^i + y^i$ $i = 0, 1, 2, \ldots$.

A polynomial symmetric in x, y can be represented as a polynomial in σ_1, σ_2.

$s_n = x^n + y^n = (x + y)(x^{n-1} + y^{n-1}) - xy(x^{n-2} + y^{n-2}) = \sigma_1 s_{n-1} - \sigma_2 s_{n-2}$

Thus, we have the recursion $s_0 = 2$, $s_1 = \sigma_1$ and $s_n = \sigma_1 s_{n-1} - \sigma_2 s_{n-2}$, $n \geq 2$.

Problem 1 :

Solve the system : $x^5 + y^5 = 33$; $x + y = 3$.

Solution :

We set $\sigma_1 = x + y$ and $\sigma_2 = xy$.

Then the system becomes, $\sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 = 33$, $\sigma_1 = 3$.

Substituting $\sigma_1 = 3$ in the first equation we get, $\sigma_2^2 - 9\sigma_2 + 14 = 0$ with two solutions, $\sigma_2 = 2$ and $\sigma_2 = 7$. Now we must solve $x + y = 3$, $xy = 2$; and $x + y = 3$, $xy = 7$ resulting in $(2, 1)$; $(1, 2)$; $(x_3, y_3) = (3/2 + i\sqrt{19}/2, 3/2 - i\sqrt{19}/2)$; $(x_4, y_4) = (y_3, x_3)$.

Solved.

Problem 2 :

Find the real solutions of the equation $\sqrt[4]{(97 - x)} + \sqrt[4]{x} = 5$

Solution :

We set, $\sqrt[4]{x} = y$ and $\sqrt[4]{(97 - x)} = z$

We get, $y^4 + z^4 = x + 97 - x = 97$ and $y + z = 5$.

Setting $\sigma_1 = y + z$ and $\sigma_2 = yz$, we get system of equations $\sigma_1 = 5$ and $\sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 = 97$ resulting in $\sigma_2^2 - 50\sigma_2 + 264 = 0$ with solutions, $\sigma_2 = 6$, $\sigma_2 = 44$.

We must solve the system $y + z = 5$, $yz = 6$ with solutions $(2, 3)$; $(3, 2)$.

Now, $x_1 = 16$, $x_2 = 81$.

The solutions, $y + z = 5$, $yz = 44$ gives complex values.

Solved.

How to solve the questions like “how many real solutions does this equation have?” and you have a four-degree equation.

Steps :

1. First check if you can factorize using Vanishing method with 1, 2, -1, -2, maximum verify by 3.
2. Then check the degree of the equation. If it is odd then it has at least one real solution. If it is even then it may have no real solution at all because complex roots come in pair.
3. Then use Descartes' sign rule to evaluate if there is any positive or negative real roots. Descartes' sign rule says : check number of sign changes of the coefficients from higher degree to lower degree of the polynomial and that says number of maximum possible positive roots of the equation. If it has 4 sign changes then it may have 4 or 2 or 0 number of positive roots i.e. it comes down by an even number 2. Check of negative roots is my same method but of the polynomial P(-x). So, put x = -x and then find number of negative roots of the equation. If there is no then all roots are complex, otherwise it may have real roots.
4. Check whether the polynomial is increasing or decreasing for some value of x. For example, it is a fourth degree equation and we have evaluated that it may have 4 positive roots. And you see the polynomial is increasing for x > 0. Implies the polynomial doesn't meet the x-axis after x > 0. Therefore, all the roots it has negative but from Descartes' sign rule we have zero negative roots. Implies all the roots of the equation are complex.
5. Take any complex root of the equation as a + ib, then a − ib is also a root of the equation. Now, do P(a + ib) − P(a − ib) = 0 and check whether b = 0 for sure. If it comes out to be b = 0, then it has no imaginary roots as the imaginary part of the root is zero.

Problem :

The number of real roots of $x^5 + 2x^3 + x^2 + 2 = 0$

Solution :

First check x = -1 is a solution. So, we will first factorize it by vanishing method.

Now, $x^5 + 2x^3 + x^2 + 2$

$= x^5 + x^4 - x^4 - x^3 + 3x^3 + 3x^2 - 2x^2 - 2x + 2x + 2$

$= x^4(x + 1) - x^3(x + 1) + 3x^2(x + 1) - 2x(x + 1) + 2(x + 1)$

$= (x + 1)(x^4 - x^3 + 3x^2 - 2x + 2)$

Now, it is a fifth degree equation and we have evaluated one real root x = -1.

Now, we have a four degree equation, $x^4 - x^3 + 3x^2 - 2x + 2 = 0$

Number of sign change = 4. Therefore, it may have 4, 2 or 0 positive roots. And it has 0 negative roots.

$P(x) = x^4 - x^3 + 3x^2 - 2x + 2$

P(0) = 2, P(1) = 3, P(2) = 18 and we are seeing that it is increasing with (+)-ve value of x.

So, $P'(x) = 4x^3 - 3x^2 + 3x - 2 = x(4x^2 - 4x + 1) + (x^2 + 2x - 2) = x(2x - 1)^2 + (x - 1)(x + 2) + x > 0$ for $x > 2$

- ⇨ P(x) is increasing for x > 2.
- ⇨ P(x) may have negative real roots but from Descartes' sign rule it has no negative roots.
- ⇨ All the roots of the four degree equation are complex.
- ⇨ The equation has only one real root and that is x = -1.

Solved.

__Test of Mathematics at the 10+2 level Subjective Solution__

1. A function f from set A into set B is a rule which assigns to each element x in A, a unique (one and only one) element (denoted by f(x)) in B. A function f from A into B is called onto function, if for each element y in B there is some element x in A, such that f(x) = y. Now suppose that A = {1, 2, …, n} and B = {1, 2, 3}. Determine the total number of onto functions from A into B.

Solution :

So, we need 3 distinct elements in A to have onto function from A into B.

We can choose any 3 element from A in ${}^nC_3$ ways.

Now, this 3 element will get permutated among themselves in 3! ways. Therefore total number of onto functions from A into B = ${}^nC_3 * 3! = {}^nP_3$.

2. Find the number of ways in which 5 different gifts can be presented to 3 children so that each child receives at least one gift.

Solution :

Label the children 1, 2, 3. Let A_i be the ways to distribute gifts so that child i gets no gift. Let B denote the ways to distribute gifts so that every child gets a gift.

For every way of giving out gifts, either every child got a gift or (at least) one child did not get any gifts, hence $3^5 = |B| + |A_1 \cup A_2 \cup A_3|$

The inclusion exclusion formula tells us that

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$

Now, $|A_1| = 2^5$ (because A_1 is 1st child doesn't get a gift)

$|A_2| = |A_3| = 2^5$

Now, $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_1| = 1^5$

And $|A_1 \cap A_2 \cap A_3| = 0$

Therefore, $|A_1 \cup A_2 \cup A_3| = {}^3C_1|A_i| - {}^3C_2|A_i \cap A_j| + {}^3C_3|A_1 \cap A_2 \cap A_3| = {}^3C_1*2^5 - {}^3C_2*1^5 + {}^3C_3*0$

- ⇨ $|B| = 3^5 - {}^3C_1 2^5 + {}^3C_2 1^5 - {}^3C_3*0 = 243 - 96 + 3 = 150$

So, total number of ways = 150.

3. x red balls, y black balls and z white balls are to be arranged in a row. Suppose that any two balls of the same color are indistinguishable. Given that x + y + z = 30, show that the number of possible arrangements is the largest for x = y = z = 10.

Solution :

Clearly, number of possible arrangement is $(x + y + z)!/\{x!*y!*z!\} = 30!/\{x!*y!*z!\}$

Now, it will be largest when $x!*y!*z!$ = minimum.

Let us say, x = 12 and y = 8

Now, $12!*8! = 12*11*10!*10!/(10*9) = (12*11/10*9)*(10!)^2$

- ⇨ $12!*8!/(10!)^2 = (12*11)/(10*9) > 1$
- ⇨ $(10!)^2 < 12!*8!$
- ⇨ It will be least when x = y = z = 10.

Proved.

4. All the permutations of the letters a, b, c, d, e are written down and arranged in alphabetical order as in a dictionary. Thus the arrangement abcde is in the first position and abced is in the second position. What is the position of the arrangement debac?

Solution :

Now, first fix a at first place. Number of arrangements = 4!

Now, fix b at first place. Number of arrangements = 4!

Now, fix c at first place. Number of arrangements = 4!

Now comes d the first letter of the required arrangement.

Now fix d at first position and a at second position. Number of arrangement = 3!

Fix b at second place. Number of arrangement = 3!

Fix c at second place. Number of arrangement = 3!

Now, comes e at second place and we have de.

Now, fix a at third place. Number of arrangement = 2!

Now comes b which is required and we have deb.

Then comes a and then c.

So, debac comes after (4! + 4! + 4! + 3! + 3! + 3! + 2!) = 92 arragement.

So, it will take 92 + 1 = 93^{rd} position.

5. In an arrangement of m H's and n T's, an uninterrupted sequence of one kind of symbol is called a run. (For example, the arrangement HHHTHHTTTH of 6 H's and 4T's opens with an H-run of length 3, followed successively by a T-run of length 1, an H-run of length 2, a T-run of length 3 and, finally an H-run if length 1.)
Find the number of arrangements of m H's and n T's in which there are exactly k H-runs.

Solution :

Now, m H's can be put in k places with k+1 holes (spaces) between them in ${}^{m-1}C_{k-1}$ ways.

Now, k – 1 spaces between the H's must be filled up by at least one T.

So, number of ways is ${}^{n-1}C_{k-2}$.

So, in this case number of ways = ${}^{m-1}C_{k-1} * {}^{n-1}C_{k-2}$.

Now, if k spaces (i.e. one space from either side first or last) can be filled by n T's where in every space at least one T is there in ${}^{n-1}C_{k-1}*2$ ways.

So, total number of ways in this case = $2*{}^{m-1}C_{k-1}*{}^{n-1}C_{k-1}$.

Now, if k+1 spaces (i.e. including first and last space) can be filled up by n T's where in every space at least one T is there in ${}^{n-1}C_k$ ways.

So, in this case total number of ways = ${}^{m-1}C_{k-1}*{}^{n-1}C_k$.

So, total number of ways = ${}^{m-1}C_{k-1}({}^{n-1}C_{k-2} + 2*{}^{n-1}C_{k-1} + {}^{n-1}C_k) = {}^{m-1}C_{k-1}({}^{n-1}C_{k-2} + {}^{n-1}C_{k-1} + {}^{n-1}C_{k-1} + {}^{n-1}C_k)$

$= {}^{m-1}C_{k-1}({}^{n}C_{k-1} + {}^{n}C_k) = {}^{n-1}C_{k-1}*{}^{n+1}C_k$

6. Show that the number of ways one can choose a set of distinct positive integers, each smaller than or equal to 50, such that their sum is odd, is 2^{49}.

Solution :

We need to select odd number of odd numbers. We can select any number of even number.

So, number of odd numbers below 50 = 25.

Number of even number less than or equal to 50 = 25.

We can select any 1 odd number in ${}^{25}C_1$ ways.

We can select any 3 odd number in ${}^{25}C_3$ ways.

…

…

We can select any 25 odd numbers in ${}^{25}C_{25}$ ways.

So, total number of ways to select odd numbers = ${}^{25}C_1 + {}^{25}C_3 + …. + {}^{25}C_{25} = 2^{24}$.

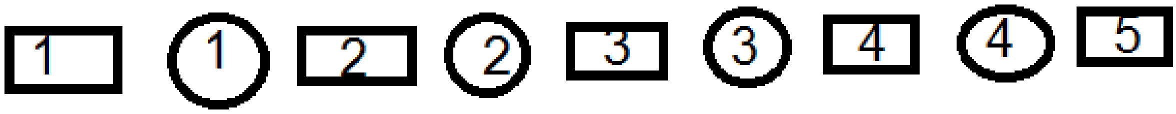
Now, we can select even numbers in $^{25}C_0 + {}^{25}C_1 + {}^{25}C_2 + \ldots. + {}^{25}C_{25} = 2^{25}$ ways.

So, total number of ways = $2^{24}*2^{25} = 2^{49}$.

Proved.

7. Show that number of ways in which four distinct integers can be chosen from 1, 2, …, n (n ≥ 7) such that no two are consecutive is equal to $^{n-3}C_4$.

Solution :



We choose 4 integers as shown in figure by circle.

So, there are maximum 5 spaces between them shown in figure by boxes.

Now, let us say, 2, 3, 4 spaces i.e. boxes are to be filled by other n – 4 integers (4 integers already chosen for 4 circles).

Number of ways = $^{n-5}C_2$ (As number of ways is $^{n-1}C_{r-1}$ for at least one to be there)

Similarly, for 1, 2, 3, 4, boxes and 2, 3, 4, 5 boxes to be filled by other n – 4 integers number of ways = $2*{}^{n-5}C_3$.

For 5 boxes to be filled by other n – 4 integers number of ways = $^{n-5}C_4$.

Total number of ways = $^{n-5}C_2 + 2*{}^{n-5}C_3 + {}^{n-5}C_4 = ({}^{n-5}C_2 + {}^{n-5}C_3) + ({}^{n-5}C_3 + {}^{n-5}C_4) = {}^{n-4}C_3 + {}^{n-4}C_4 = {}^{n-3}C_4$.

Proved.

8. How many 6-letter words can be formed using the letters A, B and C so that each letter appears at least once in the word?

Solution :

Let x_1 number of A, x_2 number of B and x_3 number of C are chosen where $x_1, x_2, x_3 > 0$

Now, $x_1 + x_2 + x_3 = 6$.

Number of positive solution of this equation is ${}^{6-1}C_{3-1} = {}^{5}C_2 = 10$.

So, combinations are as follows,

4 A, 1 B, 1 C, number of words = 6!/4! = 30

3 A, 2 B, 1 C, number of words = 6!/(3!*2!) = 60

3 A, 1 B, 2 C, number of words = 6!/(3!*2!) = 60

2 A, 1 B, 3 C, number of words = 6!/(2!*3!) = 60

2 A, 2 B, 2 C, number of words = 6!/(2!*2!*2!) = 90

2 A, 3 B, 1 C, number of words = 6!/(2!*3!) = 60

1 A, 1 B, 4 C, number of words = 6!/4! = 30

1 A, 2 B, 3 C, number of words = 6!/(2!*3!) = 60

1 A, 3 B, 2 C, number of words = 6!/(3!*2!) = 60

1 A, 4 B, 1 C, number of words = 6!/4! = 30

So, total number of words = 30 + 60 + 60 + 60 + 90 + 60 + 30 + 60 + 60 + 30 = 540.

9. Consider the set S of all integers between and including 1000 and 99999. Call two integers x and y in S to be in the same equivalence class if the digits appearing in x and y are the same. For example, if x = 1010, y = 1000 and z = 1201, then x and y are in same equivalence class, but y and z are not. Find the number of distinct equivalence classes that can be formed out of S.

Solution :

We have 10 digits.

We can choose any 1 digit out of 10 digits in ${}^{10}C_1$ ways. Out of them 0 needs to be omitted as only 0 cannot form any number between 1000 and 99999.

So, we have ${}^{10}C_1 - 1$ number of equivalence classes.

Now, we can choose 2 digits out of 10 digits in ${}^{10}C_2$ ways.

We can choose 3 digits out of 10 digits in ${}^{10}C_3$ ways.

We can choose 4 digits out of 10 digits in ${}^{10}C_4$ ways.

We can choose 5 digits out of 10 digits in $^{10}C_5$ ways.

99999 is the biggest 5 digit number and hence we cannot have 6 digit number.

Therefore, total number of distinct equivalence classes in S = $^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 - 1$.

10. Find the number of all possible ordered k-tuples of non-negative integers $(n_1, n_2, \ldots, n_k)$ such that $\sum n_i$ (i running from 1 to k) = 100.

Solution :

Clearly, it needs number of non-negative solution of the equation, $n_1 + n_2 + \ldots + n_k = 100$.

It is $^{n+k-1}C_{k-1}$.

11. Show that the number of all possible ordered 4-tuples of non-negative integers (n_1, n_2, n_3, n_4) such that $n_1 + n_2 + n_3 + n_4 \le 100$ is $^{104}C_4$.

Solution :

Clearly, required number = $^3C_3 + {}^4C_3 + {}^5C_3 + \ldots + {}^{103}C_3 = {}^{104}C_4$.

12. Solve $6x^2 - 25x + 12 + 25/x + 6/x^2 = 0$.

Solution :

$6x^2 - 25x + 12 + 25/x + 6/x^2 = 0$

- ⇨ $6(x^2 + 1/x^2) - 25(x - 1/x) + 12 = 0$
- ⇨ $6(x - 1/x)^2 + 12 - 25(x - 1/x) + 12 = 0$
- ⇨ $6(x - 1/x)^2 - 25(x - 1/x) + 24 = 0$

Let, $x - 1/x = a$

The equation becomes, $6a^2 - 25a + 24 = 0$

- ⇨ $a = \{25 \pm \sqrt{(625 - 4*6*24)}\}/12$

- ⇨ $a = \{25 \pm 7\}/12$
- ⇨ $a = 8/3, 3/2$
- ⇨ $x - 1/x = 8/3, 3/2$

Let us take first, $x - 1/x = 8/3$

- ⇨ $3x^2 - 8x - 3 = 0$
- ⇨ $x = \{8 \pm \sqrt{(64 + 4*3*3)}\}/6$
- ⇨ $x = \{8 \pm 10\}/6$
- ⇨ $x = 3, -1/3$

Now, $x - 1/x = 3/2$

- ⇨ $2x^2 - 3x - 2 = 0$
- ⇨ $2x^2 - 4x + x - 2 = 0$
- ⇨ $2x(x - 2) + (x - 2) = 0$
- ⇨ $(x - 2)(2x + 1) = 0$
- ⇨ $x = 2, -1/2$

So, $x = -1/2, -1/3, 2, 3$.

13. Consider the system of equations $x + y = 2$, $ax + y = b$. Find conditions on a and b under which
    (i) The system has exactly one solution;
    (ii) The system has no solution;
    (iii) The system has more than one solution.

Solution :

If $a \neq 1$ then the equations have only one solution.

If $a = 1$ and $b \neq 2$ then the equations have no solution.

If $a = 1$ and $b = 2$ then the equations have more than one solution.

14. If any one pair among the straight lines $ax + by = a + b$, $bx - (a + b)y = -a$, $(a + b)x - ay = b$ intersect, then show that three straight lines are concurrent.

Solution :

It is clear that the three equations are not independent.

First + second gives third.

So, follows the statement.

Proved.

15. If $f(x)$ is real-valued function of a real variable x, such that $2f(x) + 3f(-x) = 16 - 4x$ for all x, find the function $f(x)$.

Solution :

$2f(x) + 3f(-x) = 16 - 4x$ ……. (1)

Putting $x = -x$ we get,

$2f(-x) + 3f(x) = 16 + 4x$ ………. (2)

(1)*2 – (2)*3 gives,

$4f(x) + 6f(-x) - 6f(-x) - 9f(x) = 32 - 8x - 48 - 12x$

- ⇨ $-5f(x) = -16 - 20x$
- ⇨ $f(x) = 16/5 + 4x$

16. Show that for all real x, the expression $ax^2 + bx + c$ (where a, b, c are real constants with $a > 0$), has the minimum value $(4ac - b^2)/4a$. Also find the value of x for which this minimum value is attained.

Solution :

Now, $ax^2 + bx + c$

$= a\{x^2 + (b/a)x + (c/a)\}$

$= a\{x^2 + 2*x*(b/2a) + (b/2a)^2\} + (c/a) - (b/(2a)^2$

$= a(x + b/2a)^2 + (4ac - b^2)/4a$

Now, $(x + b/2a)^2 \geq 0$ and minimum value attains when $x = -b/2a$ and minimum value is $(4ac - b^2)/4a$

Proved.

17. Describe the set of all real numbers x which satisfy $2\log_{2x+3}x < 1$.

Solution :

$2\log_{2x+3}x < 1$

Let, $2x + 3 > 0$

- ⇨ $x > -3/2$

Therefore, $x^2 < 2x + 3$

- ⇨ $x^2 - 2x - 3 < 0$
- ⇨ $(x - 1)^2 < 4$
- ⇨ $|x - 1| < 2$
- ⇨ $-2 < x - 1 < 2$
- ⇨ $-1 < x < 3$

So, $-1 < x < 3$ is solution.

Now, let $2x+3 < 0$ i.e. $x < -3/2$

Then, $x^2 > 2x + 3$

- ⇨ $(x - 1)^2 > 4$
- ⇨ Either $x - 1 > 2$ or $x - 1 < -2$
- ⇨ $x > 3$ or $x < -1$
- ⇨ $x < -3/2$ is solution.

18. The sum of squares of the digits of a three-digit positive number is 146, while the sum of the two digits in the unit's and the ten's place is 4 times the digit in the hundred's place. Further, when the number is written in the reverse order, it is increased by 297. Find the number.

Solution :

Let the number is abc i.e. $100a + 10b + c$.

Now, $a^2 + b^2 + c^2 = 146$

Again, $b + c = 4a$

And, $100c + 10b + a - 100a - 10b - c = 297$

- ⇨ $99(c - a) = 297$
- ⇨ $c - a = 3$
- ⇨ $c = a + 3$

Now, $b + c = 4a$

- ⇨ $b + a + 3 = 4a$
- ⇨ $b = 3a - 3$

Putting in $a^2 + b^2 + c^2 = 146$ we get,

$a^2 + (3a - 3)^2 + (a + 3)^2 = 146$

- ⇨ $a^2 + 9a^2 - 18a + 9 + a^2 + 6a + 9 = 146$
- ⇨ $11a^2 - 12a - 128 = 0$
- ⇨ $a = \{12 \pm \sqrt{(144 + 4*11*128)}\}/22$
- ⇨ $a = \{12 \pm 76\}/22$
- ⇨ $a = 4$ (as a is not negative)
- ⇨ $b = 3*4 - 3 = 9$
- ⇨ $c = 4 + 3 = 7$

So, the number is 497.

19. Show that there is at least one real value of x for which $\sqrt[3]{x} + \sqrt{x} = 1$.

Solution :

Let $x = y^6$

The equation becomes, $y^3 + y^2 - 1 = 0$

This is a three degree (odd) equation.

- ⇨ There is at least one real solution of y.
- ⇨ There is at least one real solution of x.

Proved.

20. Let $\theta_1, \theta_2, \ldots, \theta_{10}$ be any values in the closed interval $[0, \pi]$. Show that $(1 + \sin^2\theta_1)(1 + \cos^2\theta_1)(1 + \sin^2\theta_2)(1 + \cos^2\theta_2)\ldots(1 + \sin^2\theta_{10})(1 + \cos^2\theta_{10}) \le (9/4)^{10}$.

Solution :

Now, $(1 + \sin^2\theta_1)(1 + \cos^2\theta_1) = 1 + \sin^2\theta_1 + \cos^2\theta_1 + \sin^2\theta_1\cos^2\theta_1 = 2 + (1/4)\sin^2 2\theta_1$

Now, $\sin^2 2\theta_1 \le 1$

- ⇨ $(1/4)\sin^2 2\theta_1 \le 1/4$
- ⇨ $2 + (1/4)\sin^2 2\theta_1 \le 2 + 1/4 = 9/4$

So, $(1 + \sin^2\theta_1)(1 + \cos^2\theta_1) \le 9/4$

Similarly others.

Hence, multiplying all such inequalities we get the required inequality.

Proved.

21. If a, b, c are positive numbers, then show that $(b^2 + c^2)/(b + c) + (c^2 + a^2)/(c + a) + (a^2 + b^2)/(a + b) \ge a + b + c$

Solution :

Now, $(b^2 + c^2)/2 \ge \{(b + c)/2\}^2$

⇨ $(b^2 + c^2)/(b + c) \ge (b + c)/2$

Similarly, $(c^2 + a^2)/(c + a) \ge (c + a)/2$ and $(a^2 + b^2)/(a + b) \ge (a + b)/2$

Adding the above three inequalities we get the required inequality.

Proved.

22. Let a, b, c, d be positive real numbers such that abcd = 1. Show that $(1 + a)(1 + b)(1 + c)(1 + d) \ge 16$.

Solution :

Now, $(1 + a)/2 \ge \sqrt{a}$

⇨ $(1 + a) \ge 2\sqrt{a}$

Similarly, $(1 + b) \ge 2\sqrt{b}$, $(1 + c) \ge 2\sqrt{c}$, $(1 + d) \ge 2\sqrt{d}$

Multiplying the above inequalities and putting abcd = 1 we get the required inequality.

Proved.

23. If a, b are positive real numbers such that a + b = 1, prove that $(a + 1/a)^2 + (b + 1/b)^2 \ge 25/2$

Solution.

Now, $(a + 1/a)^2 + (b + 1/b)^2 = (a^2 + b^2) + (a^{-2} + b^{-2}) + 4$

Now, $(a^2 + b^2)/2 \geq \{(a + b)/2\}^2 = ¼$

⇨ $a^2 + b^2 \geq ½$

Now, $(a^{-2} + b^{-2})/2 \geq \{(a + b)/2\}^{-2} = 4$

⇨ $a^{-2} + b^{-2} \geq 8$

So, $(a^2 + b^2) + (a^{-2} + b^{-2}) + 4 \geq ½ + 8 + 4 = 25/2$

⇨ $(a + 1/a)^2 + (b + 1/b)^2 \geq 25/2$.

Proved.

24. Show that there is exactly one value of x which satisfies the equation $2\cos^2(x^3 + x) = 2^x + 2^{-x}$.

Solution :

Now, $\cos^2(x^3 + x) = (2^x + 2^{-x})/2 \geq 1$ (AM ≥ GM)

⇨ $\cos^2(x^3 + x) = 1$ (As $\cos\theta \leq 1$)
⇨ $\cos(x^3 + x) = \pm 1$
⇨ $x^3 + x = 0$ and $x^3 + x = \pi$
⇨ $x = 0$ and other solution from the equation $x^3 + x = \pi$

Now, $(2^x + 2^{-x})/2 \geq 1$.

Equality holds when $2^x = 2^{-x}$

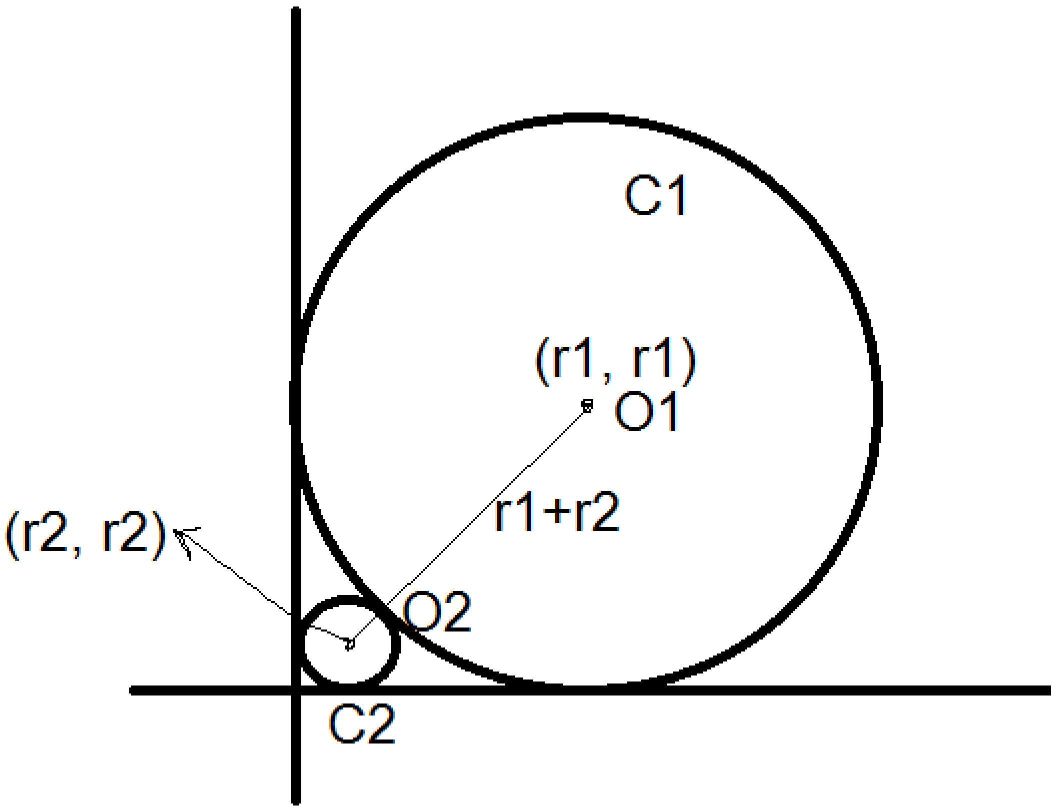
⇨ $x = 0$.

So, only one solution, $x = 0$.

Proved.

25. Let $\{C_n\}$ be an infinite sequence of circles lying in the positive quadrant of the XY-plane, with strictly decreasing radii and satisfying the following conditions. Each C_n touches both the X-axis and the Y-axis. Further, for all $n \geq 1$, the circle C_{n+1} touches circle C_n externally. If C_1 has radius 10 cm, then show that the sum of the areas of all these circles is $25\pi/(3\sqrt{2} - 4)$.

Solution :



Now, let r_n is the center of C_n.

Distance between O_1 and O_2 = $\sqrt{\{(r_1 - r_2)^2 + (r_1 - r_2)^2\}} = \sqrt{2}(r_1 - r_2)$

Also, distance between O_1 and O_2 = $r_1 + r_2$ (as C_1 and C_2 touches each other externally)

- ⇨ $\sqrt{2}(r_1 - r_2) = r_1 + r_2$
- ⇨ $r_2 = \{(\sqrt{2} - 1)/(\sqrt{2} + 1)\}r_1$

Similarly, $r_3 = \{(\sqrt{2} - 1)/(\sqrt{2} + 1)\}r_2 = \{(\sqrt{2} - 1)/(\sqrt{2} + 1)\}^2 r_1$

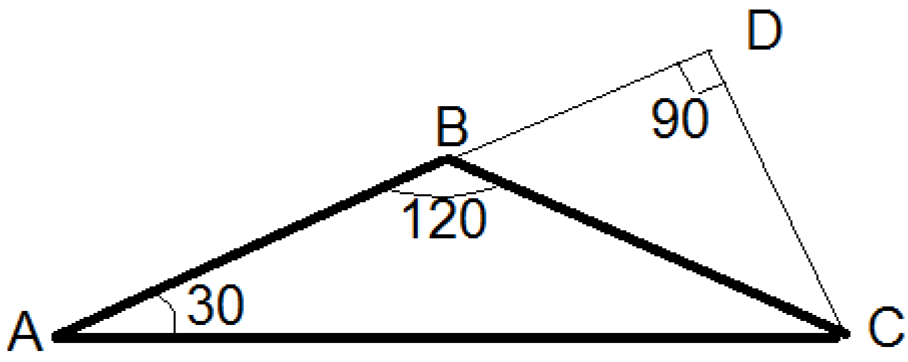
So, sum of areas = $\pi r_1^2[1 + \{(\sqrt{2} - 1)/(\sqrt{2} + 1)\}^2 + \{(\sqrt{2} - 1)/(\sqrt{2} + 1)\}^4 + ...]$

$= \pi r_1^2(1/[1 - \{(\sqrt{2} - 1)/(\sqrt{2} + 1)\}^2]) = 25\pi/(3\sqrt{2} - 4)$ (putting $r_1 = 10$)

Proved.

26. Let ABC be an isosceles triangle with AB = BC = 1 cm and A = 30. Find the volume of the solid obtained by revolving the triangle about line AB.

Solution :



Now, AB = BC = 1.

BC/sin30 = AC/sin120

⇨ AC = $\sqrt{3}$

CD/AC = sin30 (From triangle ADC)

⇨ CD = $\sqrt{3}/2$

AD/AC = cos30 (From triangle ADC)

⇨ AD = 3/2

Now, BD = $\sqrt{(BC^2 - CD^2)} = ½$

Now, if we rotate the triangle ADC about the line AD then a right circular cone will be generated and if we subtract the right circular cone generated by rotating triangle BDC about BD then we will find the volume of the solid generated by rotating triangle ABC about AB.

Now, the radius and height of the right circular cone generated by rotating triangle ADC about AD are CD and AD respectively.

Hence volume = $(1/3)\pi(\sqrt{3}/2)^2*(3/2) = (3/8)\pi$

Now, the radius and height of the right circular cone generated by rotating triangle BDC about BD are CD and BD respectively.

Hence volume = $(1/3)\pi(\sqrt{3}/2)^2*(1/2) = \pi/8$

So, required volume = $(3/8)\pi - \pi/8 = \pi/4$

27. Suppose there are k teams playing round robin tournament; that is, each team plays against all the other teams and no games ends in a draw. Suppose the i^{th} team loses l_i games and wins w_i

games. Show that $\sum l_i^2$ (i running from 1 to k) = $\sum w_i^2$ (i running from 1 to k).

Solution :

Now, $\sum l_i$ (i running from 1 to k) = $\sum w_i$ (i running from 1 to k) = total number of game = ${}^kC_2$

- ⇨ $\sum(l_i - w_i)$ (i running from 1 to k) = 0

Now, $l_i + w_i$ = constant = k – 1.

- ⇨ $\sum(l_i - w_i)(l_i + w_i)$ (i running from 1 to k) = 0 (Multiplying both sides by $l_i + w_i$)
- ⇨ $\sum(l_i^2 - w_i^2)$ (i running from 1 to k) = 0
- ⇨ $\sum l_i^2$ (i running from 1 to k) = $\sum w_i^2$ (i running from 1 to k)

Proved.

28. If $\sin^4x/a + \cos^4x/b = 1/(a + b)$, then show that $\sin^6x/a^2 + \cos^6x/b^2 = 1/(a + b)^2$.

Solution :

Now, $\sin^4x/a + \cos^4x/b = 1/(a + b)$

- ⇨ $\tan^4x/a + 1/b = (1 + \tan^2x)^2/(a + b)$
- ⇨ $\tan^4x/a + 1/b = (1 + 2\tan^2x + \tan^4x)/(a + b)$
- ⇨ $\tan^4x[(1/a) - \{1/(a + b)\}] - \{2/(a + b)\}\tan^2x + [(1/b) - \{1/(a + b)\}] = 0$
- ⇨ $\tan^4x(b/a) - 2\tan^2x + (a/b) = 0$
- ⇨ $\tan^4x - 2(a/b)\tan^2x + (a/b)^2 = 0$
- ⇨ $\{\tan^2x - (a/b)\}^2 = 0$
- ⇨ $\tan^2x = a/b$

Now, LHS = $\sin^6x/a^2 + \cos^6x/b^2$

$= \cos^6x[\tan^6x/a^2 + 1/b^2]$

$= [\tan^6x/a^2 + 1/b^2]/(1 + \tan^2x)^3$

$= [(a/b)^3/a^2 + 1/b^2]/(1 + a/b)^3$

$= [a/b^3 + 1/b^2]/[(a + b)^3/b^3]$

$= [(a + b)/b^3]/[(a + b)^3/b^3]$

$= 1/(a + b)^2$

Proved.

29. If A, B, C are the angles of a triangle, then show that $\sin A + \sin B - \cos C \leq 3/2$.

Solution :

Now, $\sin A + \sin B - \cos C$

$= 2\sin\{(A+B)/2\}\cos\{(A-B)/2\} - \cos C$

Now, $\cos\{(A-B)/2\} \leq 1$

- ⇨ $2\sin\{(A+B)/2\}\cos\{(A-B)/2\} \leq 2\sin\{(A+B)/2\}$
- ⇨ $2\sin\{A+B)/2\}\cos\{(A-B)/2\} - \cos C \leq 2\sin\{(A+B)/2\} - \cos C$
- ⇨ $\sin A + \sin B - \cos C \leq 2\cos(C/2) - 2\cos^2(C/2) + 1 = -2\{\cos(C/2) - ½\}^2 + 3/2$

Now, $\{\cos(C/2) - ½\}^2 \geq 0$

- ⇨ $-2\{\cos(C/2) - ½\}^2 \leq 0$
- ⇨ $-2\{\cos(C/2) - ½\}^2 + 3/2 \leq 3/2$
- ⇨ $\sin A + \sin B - \sin C \leq -2\{\cos(C/2) - ½\}^2 + 3/2 \leq 3/2$.

Proved.

30. If $A + B + C = n\pi$ where n is a positive integer, show that $\sin 2A + \sin 2B + \sin 2C = (-1)^{n-1}4\sin A\sin B\sin C$

Solution :

Now, $\sin 2A + \sin 2B + \sin 2C$

$= 2\sin(A + B)\cos(A - B) + \sin\{2(A + B)\}$ (As $A + B + C = n\pi$)

$= 2\sin(A + B)\cos(A - B) + 2\sin(A + B)\cos(A + B)$

$= 2\sin(A + B)\{\cos(A - B) + \cos(A + B)\}$

$= 2(-1)^{n-1}\sin C*2\sin A\sin B$ (As $A + B + C = n\pi$)

$= (-1)^{n-1}4\sin A\sin B\sin C$

Proved.

31. Let triangles ABC and DEF be inscribed in the same circle. If the triangles are of equal perimeter, then prove that sinA + sinB + sinC = sinD + sinE + sinF.

Solution :

Let radius of the circum-circle is R.

Perimeter is same.

- ⇨ a + b + c = d + e + f (usual notation)
- ⇨ (a/2R) + (b/2R) + (c/2R) = (d/2R) + (e/2R) + (f/2R) (Dividing both sides by 2R)
- ⇨ sinA + sinB + sinC = sinD + sine + sinF.

Proved.

32. Let x and n be positive integers such that $1 + x + x^2 + ... + x^{n-1}$ is prime number. Then show that n is a prime number.

Solution :

Now, $1 + x + x^2 + ... + x^{n-1} = (x^n - 1)/(x - 1)$

Let n is composite and n = ab.

Now, $(x^n - 1)/(x - 1) = (x^{ab} - 1)/(x - 1) = (x^a - 1)*m/(x - 1)$ where m is positive integer.

$= (x - 1)(x^{a-1} + x^{a-2} + ... + 1)*m/(x - 1)$

$= (x^{a-1} + x^{a-2} + ... + 1)*m$

But $(x^n - 1)/(x - 1)$ is prime.

Here is the contradiction.

- ⇨ n is a prime number.

Proved.

33. How many natural numbers less than 10^8 are there, whose sum of digits equals 7?

Solution :

Let the digits are $x_1, x_2, \ldots, x_8$

Now, $x_1 + x_2 + \ldots + x_8 = 7$

If we find number of non-negative solution of this equation then we get out answer.

(Explanation : If $x_1 = 0$ then we get 7 digit numbers, if $x_1 = x_2 = 0$ then we get 6 digit numbers and so on. So everything gets covered here)

Number of non-negative solution = ${}^{7+8-1}C_{8-1} = {}^{14}C_7$.

34. Suppose k, n are integers ≥ 1. Show that (k*n)! is divisible by $(k!)^n$.

Solution :

We will prove this by induction on n.

This is true for n = 1 as k! divides k!.

Now, let this is true for n = m i.e. $(k!)^m$ divides (k*m)!

Now, for n = m + 1,

$\{k*(m + 1)\}! = (km + k)! = (km + k)(km + k - 1)(km + k - 2)\ldots\{km + k - (k - 1)\}(km)!$

Now, $(k!)^m$ divides (km)! as per our assumption.

So, we have to prove that k! divides $(km + k)(km + k - 1)\ldots\{km + k - (k - 1)\}$

This is multiplication of k consecutive positive integers and k! always divides any k number of consecutive positive integers.

So, by the principle of induction this is always true for any positive integer value of n.

Proved.

35. If the coefficients of a quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$), are all odd integers, show that the roots cannot be rational.

Solution :

Now, discriminant = $b^2 - 4ac$ and it has to be a perfect square for the roots to be rational.

Let, $b^2 – 4ac = p^2$

As, all a, b, c are odd implies p is odd.

Dividing the equation by 8 we get,

$1 – 4 \equiv 1 \pmod 8$ (As any odd number square congruent to 1 modulus 8)

⇨ $-3 \equiv 1 \pmod 8$

Which is impossible.

⇨ The roots cannot be rational.

Proved.

36. Let $D = a^2 + b^2 + c^2$, where a and b are successive positive integers and c = ab. Prove that $\sqrt{D}$ is an odd positive integer.

Solution :

Now, $D = a^2 + b^2 + a^2b^2$

$|a – b| = 1$

⇨ $(a – b)^2 = 1$
⇨ $a^2 – 2ab + b^2 = 1$
⇨ $a^2 + b^2 = 1 + 2ab$

Putting in D, we get, $D = 1 + 2ab + a^2b^2 = (1 + ab)^2$

⇨ $\sqrt{D} = 1 + ab$

Now, a, b are consecutive integers. Therefore ab is even and ab + 1 is odd.

Proved.

37. Let, $u_n = (3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ for n = 1, 2, ….
(i) Show that for each n, u_n is an integer.
(ii) Show that $u_{n+1} = 6u_n – 4u_{n-1}$ for all $n \geq 2$.
(iii) Use (ii) above to show that u_n is divisible by 2^n.

Solution :

Now, $6u_n - 4u_{n-1} = 6(3 + \sqrt{5})^n + 6(3 - \sqrt{5})^n - 4(3 + \sqrt{5})^{n-1} - 4(3 - \sqrt{5})^{n-1}$

$= (3 + \sqrt{5})^{n-1}(18 + 6\sqrt{5} - 4) + (3 - \sqrt{5})^{n-1}(18 - 6\sqrt{5} - 4)$

$= (3 + \sqrt{5})^{n-1}(3 + \sqrt{5})^2 + (3 - \sqrt{5})^{n-1}(3 - \sqrt{5})^2$

$= (3 + \sqrt{5})^{n+1} + (3 - \sqrt{5})^{n+1} = u_{n+1}$

Proved. (ii)

Now, $u_1 = 3 + \sqrt{5} + 3 - \sqrt{5} = 6$ = integer.

$u_2 = (3 + \sqrt{5})^2 + (3 - \sqrt{5})^2 = 2(9 + 5) = 28$ = integer.

Therefore by using (ii) and by the principle of induction u_n is always integer.

u_1 is divisible by 2 and u_2 is divisible by 2^2.

Let for n = k and k+1, u_k and u_{k+1} are divisible by 2^k and 2^{k+1} respectively.

Now, $u_{k+2} = 6u_{k+1} - 4u_k = 6*2^{k+1}*m_1 - 4*2^k*m_2 = 2^{k+2}(3m_1 - m_2)$

⇨ u_{k+2} is divisible by 2^{k+2}.

Hence by the principle of induction 2^n divides u_n.

38. Show that $2^{2n} - 3n - 1$ is divisible by 9 for all $n \geq 1$.

Solution :

Clearly this is true for n = 1.

Let this is true for n = k

So, we have, $2^{2k} - 3k - 1 = 9m$

⇨ $2^{2k} = 3k + 1 + 9m$

Now, for n = k + 1 we have, $2^{2k+2} - 3(k + 1) - 1 = 4*2^{2k} - 3k - 4 = 4(3k + 1 + 9m) - 3k - 4 = 36m + 9k = 9(4m+k)$

⇨ This is divisible by 9 for n = k + 1.

Therefore, by the principle of induction this is true for all $n \geq 1$.

39. Suppose that the roots of $x^2 + px + q$ are rational numbers and p, q are integers. Then show that the roots are integers.

Solution :

Now, $x = \{-p \pm \sqrt{(p^2 - 4q)}\}/2$

Now, we don't need to bother about whether $p^2 - 4q$ is square number or not. It is obviously square as it is given that the roots are rational.

Let, p, q both odd.

Then $p^2 - 4q$ = odd and $\sqrt{(p^2 - 4q)}$ = odd.

⇨ Roots are integers as numerator is even.

Let, p, q both even. Then clearly the roots are integers.

Let, p even, q odd.

Then $p^2 - 4q$ = even and the roots are integers as numerator is even.

Let, p odd, q even.

Then $p^2 - 4q$ is odd and again roots are integers as numerator is even.

Hence proved.

40. Show that for every integer n, 7 divides $3^{2n+1} + 2^{n+2}$.

Solution :

Clearly this is true for n = 1.

Let, this is true for n = k.

So, $3^{2k+1} + 2^{k+2} \equiv 0 \pmod 7$

⇨ $3^{2k+1} \equiv -2^{k+2} \pmod 7$

Now, for n = k + 1, we have, $3^{2k+3} + 2^{k+3} = 9*3^{2k+1} + 2^{k+3} \equiv 9(-2^{k+2}) + 2^{k+3} \pmod 7 \equiv -2^{k+2}(9 - 2) \pmod 7 \equiv 0 \pmod 7$

So, by the principle of induction this is true for all n.

41. Show that if n is any odd integer greater than 1, then $n^5 - n$ is divisible by 80.

Solution :

Now, $n^5 - n = n(n^4 - 1)$

Now, $n^4 - 1 \equiv 0 \pmod{16}$ (as n is odd)

If n is not divisible by 5 then $n^4 - 1 \equiv 0 \pmod 5$

Otherwise n is divisible by 5.

So, whatever the case $n^5 - n$ is divisible by 16*5 = 80.

Proved.

42. If k is an odd positive integer, prove that for any integer $n \geq 1$, $1^k + 2^k + \dots + n^k$ is divisible by $n(n+1)/2$.

Solution :

Let n is even.

Then n has two middle terms viz. n/2 and n/2 + 1.

Now, $1^k + n^k \equiv 1^k + (-1)^k \pmod{n + 1} \equiv 0 \pmod{n + 1}$ (as k is odd)

Similarly, $2^k + (n - 1)^k$, $3^k + (n - 2)^k$, …. , $(n/2)^k + (n/2 + 1)^k \equiv 0 \pmod{n + 1}$

Therefore, n + 1 divides the expression.

Now, n/2 divides n^k.

Now, there are $1^k + 2^k + \dots + (n - 1)^k$

Now, $1^k + (n - 1)^k \equiv 1^k + (-1)^k \pmod{n/2} \equiv 0 \pmod{n/2}$ (as k is odd)

Similarly, $2^k + (n - 2)^k$, $3^k + (n - 3)^k$, …., $(n/2 - 1)^k + (n/2 + 1)^k \equiv 0 \pmod{n/2}$

And middle term $(n/2)^k$ is divisible by n/2.

Therefore n/2 divides the expression.

So, n(n+1)/2 divides the expression when n is even.

Similarly we can prove for n = odd.

Proved.

43. Let k be a fixed odd positive integer. Find the minimum value of $x^2 + y^2$, where x, y are non-negative integers and $x + y = k$.

Solution :

We have, $(x^2 + y^2)/2 \geq \{(x + y)/2\}^2$

⇨ $x^2 + y^2 \geq k^2/2$

Now, $x = y = k/2$ is not possible as k is odd. Therefore minimum value is $(k^2 + 1)/2$ and it attains when $x =(k - 1)/2$ and $y = (k + 1)/2$.

44. (a) Prove that for any odd positive integer n, n^4 when divided by 16 always leaves remainder 1.
(b) Hence or otherwise show that we cannot find integers $n_1, n_2, \ldots, n_8$ such that $n_1^4 + n_2^4 + \ldots + n_8^4 = 1993$.

Solution :

(a) $n \equiv \pm1, \pm3, \pm5, \pm7 \pmod{16}$ (as n is odd)
⇨ $n^2 \equiv 1, 9, 25, 49 \pmod{16}$
⇨ $n^2 \equiv 1, 9 \pmod{16}$ (As $25 \equiv 9$ and $49 \equiv 1 \pmod{16}$)
⇨ $n^4 \equiv 1, 81 \pmod{16}$
⇨ $n^4 \equiv 1 \pmod{16}$ (as $81 \equiv 1 \pmod{16}$)

Proved.

(b) Dividing the equation by 16 we get, $x_1 + x_2 + \ldots + x_8 \equiv 9 \pmod{16}$

Now, $x_1, x_2, \ldots, x_8$ are either 1 (if odd) or 0 (if even) so, the sum cannot exceed 8. But RHS is 9.

So, it is impossible.

⇨ The equation cannot hold true.

Proved.

45. Two integers m and n are called relatively prime if the greatest common divisor of m and n is 1. Prove that among any five consecutive positive integers there is one integer which is relatively prime to the other four integers.

Solution :

Let us take 3 integers even. So, they have a common factor 2. One of the odd integer and one of the even integer may get divisible by 3. Let us take the another odd integer. This is clearly relatively prime to the other four as 2 and 3 doesn't divide the integer and if 5 divides or any prime > 5 divides the integer then these four will not be divisible by that prime.

Proved.

46. Show that if a prime number p is divided by 30, then the remainder is either a prime or is 1.

Solution :

Let p = 2*3*5m + r (30 = 2*3*5)

Now, r cannot be any integer which have 2, 3 or 5 as factor because in that case p will not be prime. For example if r = 14 then p = 2(3*5*m + 7) and p can be factored. Here is contradiction.

So, r must be prime. r can be composite number 7^2 as after 2, 3, 5 next prime is 7 and the lowest composite number generated by 7 is 7^2 = 49 which is greater than 30 but r less than 30.

So, r is either prime or 1.

Proved..

47. If k and l are positive integers such that k divides l, show that for every positive integer m, 1 + (k + m)l and 1 + ml are relatively prime.

Solution :

Let p divides both 1 +ml and 1 + (k + m)l

So, $1 + ml \equiv 0 \pmod p$

And, $1 + (k + m)l \equiv 0 \pmod p$

- ⇨ $1 + ml + kl \equiv 0 \pmod p$
- ⇨ $kl \equiv 0 \pmod p$ (from above)

⇨ p divides either k or l. Let p divides l then p cannot divide 1 + ml. Let p divides k then p divides again l as k divides l. Again p cannot divide 1 + ml.

Here is the contradiction.

⇨ They are relatively prime.

Proved.

48. Let f(x) be a polynomial with integer coefficients. Suppose that there exist distinct integers a_1, a_2, a_3, a_4 such that $f(a_1) = f(a_2) = f(a_3) = f(a_4) = 3$. Show that there doesn't exist any integer b with $f(b) = 14$.

Solution :

Clearly, $f(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)Q(x) + 3$ where Q(x) is a polynomial.

Now, $f(b) = (b - a_1)(b - a_2)(b - a_3)(b - a_4)Q(b) + 3 = 14$

⇨ $(b - a_1)(b - a_2)(b - a_3)(b - a_4)Q(b) = 11$

Now, 11 is a prime and 11 is factor of at least 4 distinct integers. As a_1, a_2, a_3, a_4 are distinct hence $b - a_1$, $b - a_2$, $b - a_3$, $b - a_4$ are distinct. But 11 can have maximum 3 factors viz. 11, 1, -1 or -11, 1, -1. But it cannot have 4 factors.

So, $f(b) = 14$ is not possible.

Proved.

49. Show that if $n > 2$, then $(n!)^2 > n^n$.

Solution :

Now, $1*n = n$.

$2*(n - 1) > n$.

$3*(n - 2) > n$.

…

…

(n – 1)*2 > n

n*1 = n.

Multiplying the above inequalities we get, $(n!)^2 > n^n$ for n > 2.

Proved.

50. Let J = {0, 1, 2, 3, 4}. For x, y in J define x(cross)y to be the remainder of the usual sum of x and y after division by 5 and x(dot)y to be the remainder of the usual product of x and y after division by 5. For example, 4(cross)3 = 2 while 4(dot)2 = 3. Find x, y in J, satisfying the following equations simultaneously : (3(dot)x)(cross)(2(dot)y) = 2, (2(dot)x)(cross)(4(dot)y) = 1.

Solution :

To satisfy first equation, 3(dot)x = 4 and 2(dot)y = 3 or 3(dot)x = 3 and 2(dot)y = 4

First case, x = 3, y = 4 but it doesn't satisfy second equation.

Second case, x = 1, y = 2 but it doesn't satisfy second equation.

Now, 3(dot)x = 0 and 2(dot)y = 2 or 3(dot)x = 2 and 2(dot)y = 0

First case, x = 0, y = 1 but it doesn't satisfy second equation.

Second case, x = 4, y = 0 but it doesn't satisfy second equation.

So, no solution.

51. A pair of complex numbers z_1, z_2 is said to have property P if for every complex number z, we can find real numbers r and s such that $z = rz_1 + sz_2$. Show that a pair z_1, z_2 has property P if and only if the points z_1, z_2 and 0 on the complex plane are not collinear.

Solution :

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Let $z = x + iy = r(x_1 + iy_1) + s(x_2 + iy_2) = rx_1 + sx_2 + i(ry_1 + sy_2)$

From this we get, $rx_1 + sx_2 = x$ and $ry_1 + sy_2 = y$

Solving for r and s we get, $r = (xy_2 - yx_2)/(x_1y_2 - y_1x_2)$ and $s = (xy_1 - yx_2)/(x_2y_1 - x_1y_2)$

Now, r and s have real solutions if $x_1y_2 - y_1x_2 \neq 0$ and $x_2y_1 - x_1y_2 \neq 0$ i.e. $y_1/x_1 \neq y_2/x_2$ i.e. they are not collinear with 0 or they are not parallel.

Proved.

52. Show that a necessary and sufficient condition for the line $ax + by + c = 0$ where s, b, c are non-zero real numbers, to pass through the first quadrant is either $ac > 0$ or $bc > 0$.

Solution :

Now, $ax + by + c = 0$

⇨ $x/(-c/a) + y/(-c/b) = 1$

Now, the straight line will not pass through the first quadrant if $-c/a < 0$ and $-c/b < 0$ i.e. $-ac/a^2 < 0$ and $-bc/b^2 < 0$ i.e. $-ac < 0$ and $-bc < 0$ i.e. $ac > 0$ and $bc > 0$

Now, it's NOT statement is either $ac > 0$ or $bc > 0$.

Proved.

53. Let a and b be real numbers such that the equations $2x + 3y = 4$ and $ax - by = 7$ have exactly one solution. Then, show that the equations $12x - 8y = 9$ and $bx + ay = 0$ also have exactly one solution.

Solution :

Clearly the lines $12x - 8y = 9$ i.e. $3x - 2y = 9/4$ and $bx + ay = 0$ are perpendicular to the lines $2x + 3y = 4$ and $ax - by = 7$. So they must meet at one point.

54. Let the circles $x^2 + y^2 - 2cy - a^2 = 0$ and $x^2 + y^2 - 2bx + a^2 = 0$, with centres at A and B intersect at P and Q. Show that the points A, B, P, Q and O = (0, 0) lie on a circle.

Solution :

Any circle passing through P and Q is $(x^2 + y^2 - 2cy - a^2) + k(x^2 + y^2 - 2bx + a^2) = 0$

Now, this passes through O = (0, 0)

- ⇨ $-a^2 + ka^2 = 0$
- ⇨ $k = 1$.

So, the equation of the circle passing through P, Q and O is $x^2 + y^2 - bx - cy = 0$

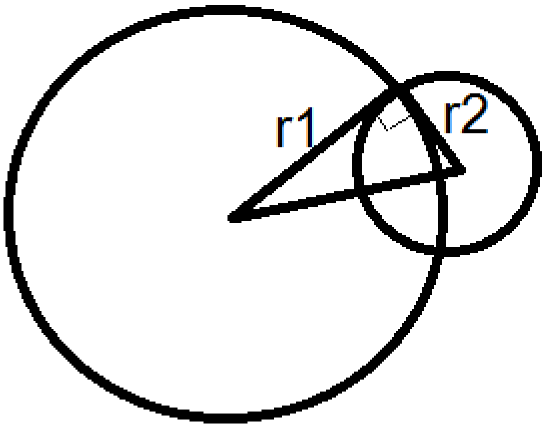
Clearly, A = (0, c) and B = (b, 0)

Clearly, A and B satisfies the above equation of the circle.

- ⇨ P, Q, A, B and O lie on a circle.

55. Two intersecting circles are said to be orthogonal to each other, if the tangents to the two circles at any point of intersection; are perpendicular to each other. Show that every circle passing through the points (2, 0) and (-2, 0) is orthogonal to the circle $x^2 + y^2 - 5x + 4 = 0$

Solution :



Let the equation of any circle passing through (2, 0) and (-2, 0) is $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre = (-g, -f) and radius = $\sqrt{(g^2 + f^2 - c)}$

The given circle is, $(x - 5/2)^2 + y^2 = (3/2)^2$

Now, from the figure, distance between centre^2 = $g^2 + f^2 - c + (3/2)^2$

- ⇨ $(g - 5/2)^2 + f^2 = g^2 + f^2 - c + 9/4$
- ⇨ $5g - c = 4$

Now, (2, 0) and (-2, 0) will satisfy the equation of the circle.

- ⇨ $4 + 4g + c = 0$ and $4 - 4g + c = 0$.

Solving this we get, $g = 0$ and $c = -4$ which satisfies $5g - c = 4$.

Hence proved.

56. Suppose that AB is an arc of a circle with a given radius and centre subtending an angle θ ($0 < \theta < \pi$ is fixed) at the centre. Consider an arbitrary point P on this arc and the product l(AP).l(PB), where l(AP) and l(PB)denote the lengths of the straight lines AP and PB, respectively. Determine possible location(s) of P for which this product will be minimized. Justify your answer.

Solution :

Let radius of the circle = a.

The co-ordinate axes are so chosen that the centre of the circle is the origin (0, 0).

Let co-ordinate of point A = (acosA, asinA) and B = (acos(A + θ), sin(A + θ)) and that of P is (acosΦ, asinΦ).

$l(AP) = a\sqrt{\{(\cos A - \cos\Phi)^2 + (\sin A - \sin\Phi)^2\}} = a\sqrt{\{2 - \cos(A - \Phi)\}}$

$l(BP) = a\sqrt{\{(\cos(A + \theta) - \cos\Phi)^2 + (\sin(A + \theta) - \sin\Phi)^2\}} = a\sqrt{\{2 - \cos(A + \theta - \Phi)\}}$

Let $D_1 = l(AP)*l(BP) = a^2\sqrt{[\{2 - \cos(A - \Phi)\}\{2 - \cos(A + \theta - \Phi)\}]}$

Let $D = D_1^2/a^2 = \{2 - \cos(A - \Phi)\}\{2 - \cos(A + \theta - \Phi)\}$

D_1 will be maximum when D will be maximum.

So, $dD/d\Phi = -\sin(A - \Phi)\{2 - \cos(A + \theta - \Phi)\} - 2\sin(A + \theta - \Phi)\{2 - \cos(A - \Phi)\} = 0$

- ⇨ $2\{\sin(A - \Phi) + \sin(A + \theta - \Phi)\} - \sin(2A + \theta - 2\Phi) = 0$
- ⇨ $4\sin\{(2A + \theta - 2\Phi)/2\}\cos(\theta/2) - 2\sin\{(2A + \theta - 2\Phi)/2\}\cos\{(2A + \theta - 2\Phi)/2\} = 0$
- ⇨ $\sin\{2A + \theta - 2\Phi)/2\} = 0$
- ⇨ $\Phi = A + \theta/2$ i.e. middle point of the arc AB.

57. Let P be the fixed point (3, 4) and Q the point $(x, \sqrt{(25 - x^2)})$. If M(x) is the slope of the line PQ, find lim M(x) as x -> 3.

Solution :

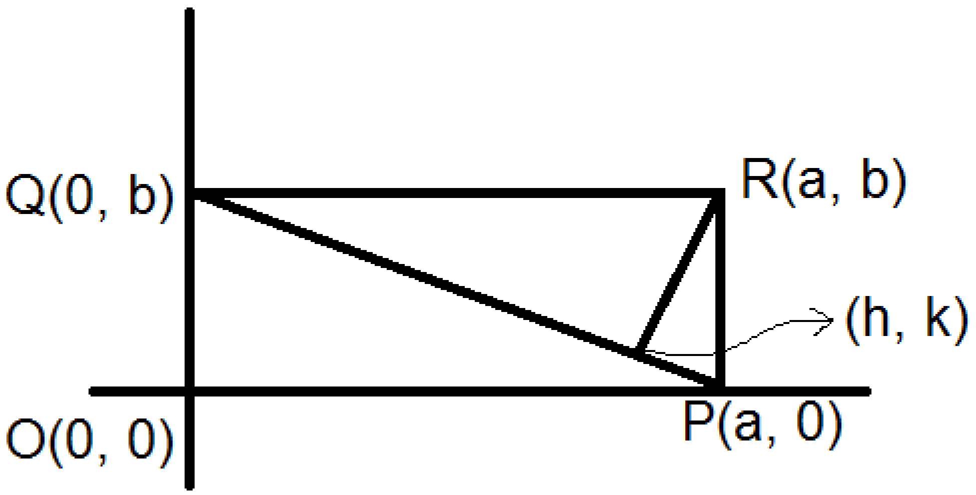
$M(x) = \{\sqrt{(25 - x^2)} - 4\}/(x - 3) = (25 - x^2 - 16)/(x - 3)\{\sqrt{(25 - x^2)} + 4\} = (3 - x)(3 + x)/(x - 3)\{\sqrt{(25 - x^2)} + 4\}$

$= - (3 + x)/\{\sqrt{(25 - x^2)} + 4\}$ (as $x \neq 3$)

Now, lim M(x) as x -> 3 = -6/8 = -3/4

58. Let PQ be a line segment of a fixed length L with its two ends P and Q sliding along the X-axis and Y-axis respectively. Complete the rectangle OPRQ where O is the origin. Show that the locus of the foot of the perpendicular drawn from R on PQ is given by $x^{2/3} + y^{2/3} = L^{2/3}$.

Solution :



Now, $(k - b)/(h - a)(-b/a) = -1$

⇨ $ah - bk = a^2 - b^2$ ……….. (1)

Again, (0, b); (h, k) and (a, 0) are collinear.

⇨ $(1/2)[a(b - k) + 0(k - 0) + h(0 - b)] = 0$
⇨ $hb + ak = ab$ …………………. (2)

$(1)*a + (2)*b \Rightarrow h = a^3/(a^2 + b^2)$ and $k = b^3/(a^2 + b^2)$

$\Rightarrow h^{2/3} + k^{2/3} = (a^2 + b^2)/(a^2 + b^2)^{2/3} = (a^2 + b^2)^{1/3} = L^{2/3}$ (As $a^2 + b^2 = L^2$)

So, locus is $x^{2/3} + y^{2/3} = L^{2/3}$ (Putting x in place of h and y in place of k)

Proved.

59. Let $\{x_n\}$ be a sequence such that $x_1 = 2$, $x_2 = 1$ and $2x_n - 3x_{n-1} + x_{n-2} = 0$ for $n > 2$. Find an expression for x_n.

Solution :

Now, $2x_n - 3x_{n-1} + x_{n-2} = 0$

- $\Rightarrow$ $2x_n - 2x_{n-1} = x_{n-1} - x_{n-2}$

Putting $n = n - 1$ we get, $2x_{n-1} - 2x_{n-2} = x_{n-2} - x_{n-3}$

Putting $n = n - 2$ we get, $2x_{n-2} - 2x_{n-3} = x_{n-3} - x_{n-4}$

….

…

Putting $x = 3$, we get, $2x_3 - 2x_2 = x_2 - x_1$

Adding the above equalities we get,

$2x_n - 2x_2 = x_{n-1} - x_1$

- $\Rightarrow$ $2x_n - x_{n-1} = 2x_2 - x_1 = 0$
- $\Rightarrow$ $2x_n = x_{n-1}$
- $\Rightarrow$ $x_n = (1/2)x_{n-1} = (1/2)(1/2)x_{n-2} = (1/2)^2(1/2)x_{n-3} = \ldots.. = (1/2)^{n-1}x_1 = (1/2)^{n-2}$

60. Find a real number x, let [x] denote the largest integer less than or equal to x and <x> denote x – [x]. Find all the solutions of the equation 13[x] + 25<x> = 271.

Solution :

Now, $13[x] + 25\langle x\rangle = 271$

- $\Rightarrow$ $13[x] + 25(x - [x]) = 271$
- $\Rightarrow$ $25x - 12[x] = 271$

Now, [x] is integer implies 25x is integer.

- ⇨ x has fractional part as r/5 or s/25 where $0 < r < 5$ and $0 < s < 25$, r and s positive integer.

Putting $x = 21$ we get, $25x - 12[x] = 273$

Putting $x = 20$ we get, $25x - 12[x] = 260$

- ⇨ $20 < x < 21$.

Let, $x = 20 + r/5$

Putting this value in the above equation we get, $500 + 5r - 240 = 271$

- ⇨ $r = 11/5$ not an integer. Hence no solution.

Now, putting $x = 20 + s/25$ in the above equation we get,

$500 + s - 240 = 271$

- ⇨ $s = 11$.

So, solution is $x = 20 + 11/25$.

61. Let x be a positive number. A sequence $\{x_n\}$ of real numbers is defined as follows : $x_1 = (1/2)(x + 5/x)$, $x_2 = (1/2)(x_1 + 5/x_1)$, ….., and in general, $x_{n+1} = (1/2)(x_n + 5/x_n)$ for all $n \geq 1$.
    (a) Show that, for all $n \geq 1$, $(x_n - \sqrt{5})/(x_n + \sqrt{5}) = \{(x - \sqrt{5})/(x + \sqrt{5})\}^{\wedge}2^n$.
    (b) Hence find $\lim x_n$ as $n \to \infty$.

Solution :

Now, $(x_n - \sqrt{5})/(x_n + \sqrt{5}) = \{(1/2)(x_{n-1} + 5/x_{n-1}) - \sqrt{5}\}/\{(1/2)(x_{n-1} + 5/x_{n-1}) + \sqrt{5}\} = (x_{n-1}^2 - 2\sqrt{5}x_{n-1} + 5)/(x_{n-1}^2 + 2\sqrt{5}x_{n-1} + 5) = \{(x_{n-1} - \sqrt{5})/(x_{n-1} + \sqrt{5})\}^2 = \{(x_{n-2} - \sqrt{5})/(x_{n-2} + \sqrt{5})\}^{\wedge}2^2 = ….. = \{(x - \sqrt{5})/(x + \sqrt{5})\}^{\wedge}2^n$.

Now, $1 - (x_n - \sqrt{5})/(x_n + \sqrt{5}) = 1 - \{(x - \sqrt{5})/(x + \sqrt{5})\}^{\wedge}2^n$.

- ⇨ $2\sqrt{5}/(x_n + \sqrt{5}) = 1 - \{(x - \sqrt{5})/(x + \sqrt{5})\}^{\wedge}2^n$.
- ⇨ $2\sqrt{5}/(\lim x_n + \sqrt{5}) = 1 - \lim [\{(x - \sqrt{5})/(x + \sqrt{5})\}^{\wedge}2^n]$ as $n \to \infty$
- ⇨ $2\sqrt{5}/(\lim x_n + \sqrt{5}) = 1$ as $n \to \infty$ (as $x > 0$ numerator < denominator and the term vanishes)
- ⇨ $\lim x_n$ as $n \to \infty = \sqrt{5}$.

62. Let P(x) be a polynomial of degree n, such that P(k) = k/(k + 1) for k = 0, 1, 2,…., n. Find the value of P(n + 1).

Solution :

Let Q(x) = (x + 1)P(x) − x. Then the polynomial Q(x) vanishes for k = 0, 1, 2,. . . , n, that is,

$(x + 1)P(x) − x = a \cdot x \cdot (x − 1)(x − 2) \cdot\cdot\cdot (x − n)$.

To find a we set x = −1 and get $1 = a(−1)^{n+1}(n + 1)!$

Thus, $P(x) = \{(−1)^{n+1}x(x − 1) \cdot\cdot\cdot (x − n)/(n + 1)! + x\}/(x + 1)$

P(n + 1) = 1 for odd n,

P(n + 1) = n/(n + 2) for even n.

63. Let $P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ….. + a_1x + a_0$ be a polynomial with integer coefficients, such that, P(0) and P(1) are both odd integers. Show that
(a) P(x) doesn't have any even integer roots.
(b) P(x) doesn't have any odd integers roots.

Solution :

Let P(x) have an integer root k.

So, P(x) = (x – k)Q(x) where Q(x) is a polynomial of degree n – 1.

P(0) = -kQ(0) = odd and P(1) = (1 – k)Q(1) = odd.

Now, -k and 1 – k are consecutive integers and hence one must be even and hence either P(0) or P(1) must be even.

Here is the contradiction.

⇨ P(x) cannot have any integer root.

64. Let $P_1, P_2, …., P_n$ be polynomials in x, each having all integer coefficients, such that $P_1 = P_1^2 + P_2^2 + …. + P_n^2$. Assume that P_1 is not zero polynomial. Show that $P_1 = 1$ and $P_2 = P_3 = …. = P_n = 0$.

Solution :

Putting x = r (any integer) we get, $P_1(r) = P_1^2(r) + P_2^2(r) + …. + P_n^2(r)$.

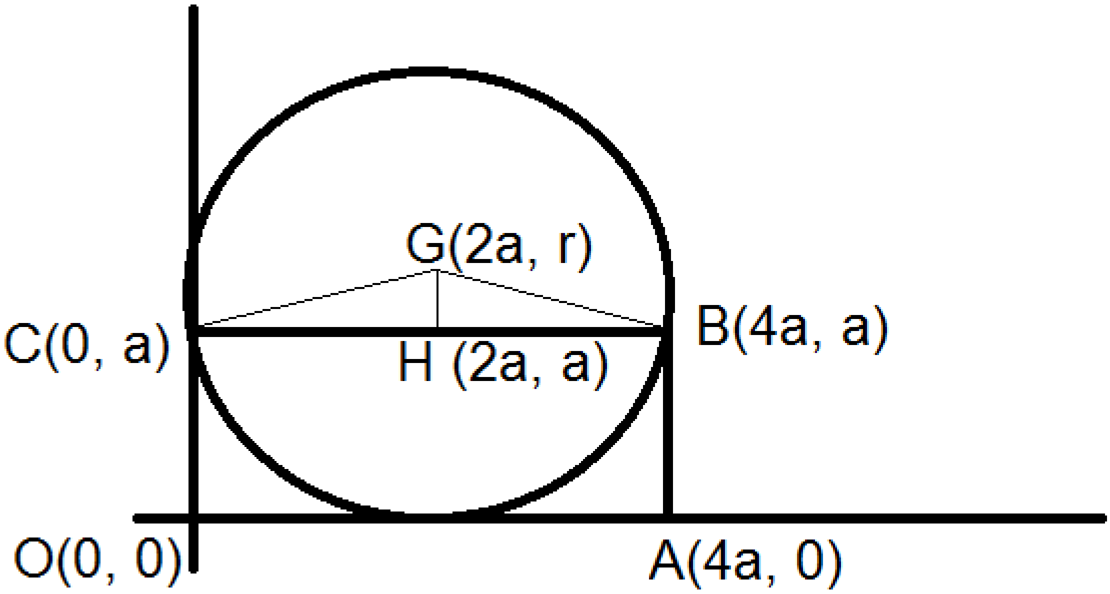
Now, right hand side is sum of square terms and hence all greater than or equal to 0.

Now, $P_1(r) \leq P_1^2(r)$

- ⇨ $P_1 = 1$ as P_1 is not zero polynomial and $P_2 = P_3 = …. = P_n = 0$.

65. A rectangle OACB with the two axes as two sides, the origin O as a vertex is drawn in which the length OA is four times the width OB. A circle is drawn passing through the points B and C and touching OA at its mid-point, thus dividing the rectangle into three parts. Find the ratio of the areas of these three parts.

Solution :



Now, GC = $\sqrt{\{(2a - 0)^2 + (r - a)^2\}} = r$

- ⇨ r = 5a/2.

Let, angle CGB = θ

From triangle CGB, we get, 5a/2sin(90 – θ/2) = 4a/sinθ

- ⇨ sinθ/2 = 4/5
- ⇨ θ = $2\sin^{-1}(4/5)$

GH = (5a/2 – a) = 3a/2

Area of triangle GHC = $(1/2)*(3a/2)*4a = 3a^2$.

Area of joining curve G, C, mid-point of OA and B is $\{\pi(5a/2)^2/2\pi\}*\theta = (25a^2/4)\sin^{-1}(4/5)$

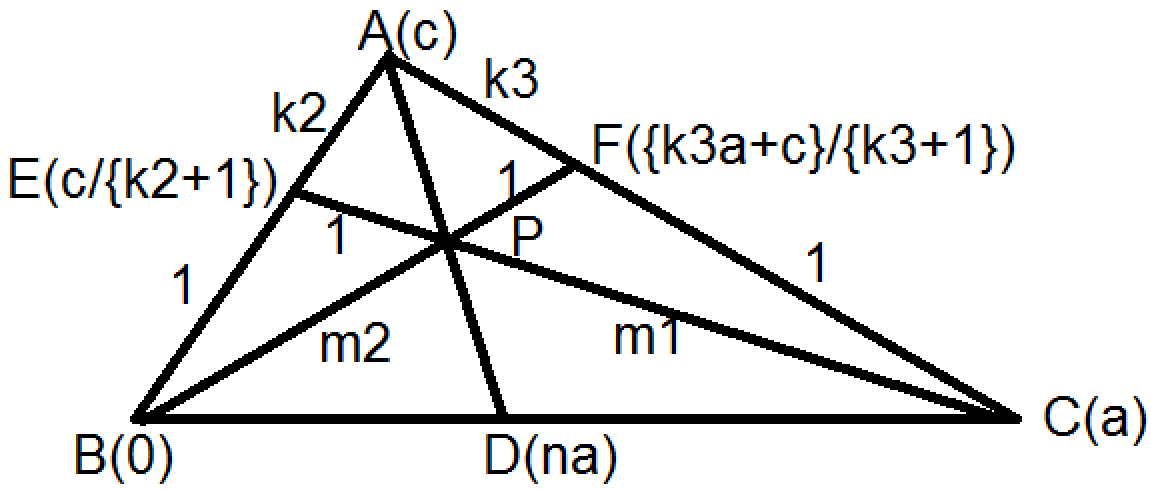
Area of the part BHC(and mid-point of joining OA) is $(25a^2/4)\sin^{-1}(4/5) - 3a^2$

Area of each small part of the rectangle (i.e. point joining O, C and mid-point of OA) = $[4a^2 - \{(25a^2/4)\sin^{-1}(4/5) - 3a^2\}]/2$

Hence we can find the ratio.

66. Let P be an interior point of the triangle ABC. Assume that AP, BP and CP meet the opposite sides BC, CA and AB at D, E and F, respectively. Show that AF/FB + AE/EC = AP/PD.

Solution :



In the figure a, c are vectors. All else are scalars.

Now, $P = \{m_1c/(k_2+1) + a\}/(m_1 + 1) = m_2\{(k_3a + c)/(k_3+1)\}/(m_2 + 1)$

Equating, the coefficients of vectors a and c from both sides we get,

$\{m_1/(m_1+1)\}/(k_2 + 1) = \{m_2/(m_2+1)\}/(k_3+1)$ and $1/(m_1 + 1) = m_2k_3/\{(k_3 + 1)(m_2 + 1)\}$

⇨ $m_1 = (k_2 + 1)/k_3$

Therefore, $P = (c + k_3a)/(k_2 + k_3 + 1)$

Let, $AP/PD = k_4$.

Then, P = $(k_4na + c)/(k_4 + 1)$

Therefore, $(c + k_3a)/(k_2 + k_3 + 1) = (k_4na + c)/(k_4 + 1)$

Equating coefficients of vector c from both sides we get, $1/(k_2 + k_3 + 1) = 1/(k_4 + 1)$

- ⇨ $k_2 + k_3 = k_4$
- ⇨ AF/FB + AE/EC = AP/PD

Proved.

67. Suppose $x_1 = \tan^{-1}2 > x_2 > x_3 > \ldots$.. are positive real numbers satisfying $\sin(x_{n+1} - x_n) + 2^{-(n+1)}\sin x_n \sin x_{n+1} = 0$ for $n \geq 1$. Find $\cot x_n$. Also show that lim x_n as n -> ∞ = π/4.

Solution :

Now, $\sin(x_{n+1} - x_n) + 2^{-(n+1)}\sin x_n \sin x_{n+1} = 0$

- ⇨ $\sin x_{n+1}\cos x_n - \cos x_{n+1}\sin x_n + 2^{-(n+1)}\sin x_n \sin x_{n+1} = 0$
- ⇨ $\cot x_n - \cot x_{n+1} + 2^{-(n+1)} = 0$ (Dividing both sides by $\sin x_n \sin x_{n+1}$)

Putting n = n – 1 we get, $\cot x_{n-1} - \cot x_n + 2^{-n} = 0$

Putting n = n – 2 we get, $\cot x_{n-2} - \cot x_{n-1} + 2^{-(n-1)} = 0$

…

..

Putting n = 1, we get, $\cot x_1 - \cot x_2 + 2^{-2} = 0$

Adding the above equalities we get, $\cot x_1 - \cot x_n + (2^{-n} + 2^{-(n-1)} + \ldots + 2^{-2}) = 0$

- ⇨ $½ - \cot x_n + 2^{-n}(2^{n-1} - 1)/(2 - 1) = 0$
- ⇨ $\cot x_n = ½ + 2^{-n}(2^{n-1} - 1) = 1 - 2^{-n}$
- ⇨ lim $\cot x_n = 1 -$ lim 2^{-n} as n -> ∞
- ⇨ cot(lim x_n) = 1 as n -> ∞
- ⇨ lim x_n as n -> ∞ = π/4.

68. If A = $\int\{(\cos x)/(x + 2)^2\}dx$ (integration running from 0 to π), then show that $\int\{(\sin x\cos x)/(x + 1)\}dx$ (integration running from 0 to π/2) = $(1/2)\{1/2 - 1/(\pi + 2) - A\}$.

Solution :

Now, ∫{(sinxcosx)/(x + 1)}dx (integration running from 0 to π/2)

Putting x = z/2 we get, dx = dz/2

For x = 0, z = 0 and for x = π/2, z = π.

∫{(sinxcosx)/(x + 1)}dx (integration running from 0 to π/2)

= (1/2)∫[{2sin(z/2)cos(z/2)}/(z + 2)]dz (integration running from 0 to π)

= (1/2)∫{sinz/(z + 2)}dz (integration running from 0 to π)

= (1/2)[{1/(z + 2)}(-cosz) - ∫[{-1/$(z + 2)^2$}(-cosz)]dz] (integration running from 0 to π) (doing by parts)

= (1/2)[1/(π + 2) + ½ - ∫{cosz/$(z + 2)^2$}dz] (integration running from 0 to π)

= (1/2)[1/2 + 1/(π + 2) – A]

Proved.

69. Find the value of ∫dx/(1 – x) (integration running from 2 to 11).

Solution :

Now, ∫dx/(1 – x) (integration running from 2 to 11)

= -∫dx/(x – 1) (integration running from 2 to 11)

= --ln|x – 1| (upper limit 11, lower limit 2)

= -ln(10) + ln(1) = -ln(10)

70. Evaluate lim {1/(n + 1) + 1/(n + 2) + ….. + 1/(n + n)} as n -> ∞.

Solution :

lim {1/(n + 1) + 1/(n + 2) + …… + 1/(n + n)} as n -> ∞

= lim ∑{1/(n + r)} as n -> ∞ (summation running from 1 to n)

= lim (1/n)∑{1/(1 + r/n)} as n -> ∞ (summation running from 1 to n)

= ∫dx/(1 + x) (integration running from 0 to 1)

= ln|1 + x| (upper limit 1, lower limit 0)

= ln(2) – ln(1) = ln(2)

71. Prove by induction or otherwise that ∫{sin(2n+1)x/sinx}dx (integration running from 0 to π/2) = π/2.

Solution :

We will prove by induction.

Clearly this is true for n = 0.

Let this is true for n = k.

For n = k + 1, we have, ∫{sin(2k+3)x/sinx}dx (integration running from 0 to π/2)

= ∫[sin{(2k + 1) + 2x}/sinx]dx (integration running from 0 to π/2)

= ∫[{sin(2k+1)xcos2x + cos(2k+1)xsin2x}/sinx]dx (integration running from 0 to π/2)

= ∫[sin{(2k+1)x}(1 – $2\sin^2 x$)/sinx]dx + ∫[cos{(2k+1)x}*2sinxcosx/sinx]dx (integration running from 0 to π/2

= ∫{sin(2k+1)x/sinx}dx - 2∫sin{(2k+1)x}sinxdx + 2∫cos{(2k+1)x}cosxdx

= π/2 + 2∫[cos(2k+1)xcosx – sin(2k+1)xsinx]dx (integration running from 0 to π/2)

= π/2 - 2∫cos(2k+2)xdx (integration running from 0 to π/2)

= $\pi/2 - 2\sin(2k+2)x/(2k+2)|_0^{\pi/2}$

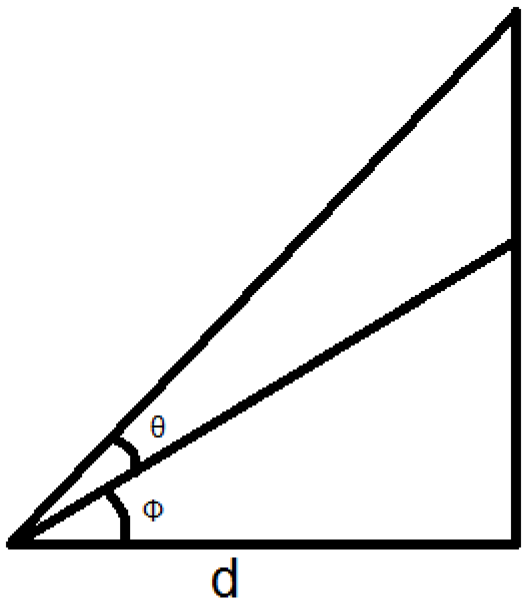
= π/2 – 2[sin(k+1)π – sin0]/(2k+2)

= π/2

Proved.

72. A man walking towards a building, on which a flagstaff is fixed vertically, observes the angle subtended by the flagstaff to be the greatest when he is at a distance d from the building. If θ is the observed greatest angle, show that the length of the flagstaff is 2dsinθ.

Solution :



Now, when θ is maximum then length of the flagstaff is maximum.

Length of the flagstaff = d{tan(θ + Φ) – tanΦ} = d{sin(θ + Φ)/cos(θ + Φ) – sinΦ/cosΦ} = dsinθ/{cos(θ + Φ)cosθ} = 2dsinθ/{cos(θ + 2Φ) + cosθ)}

Now, this will be maximum when cos(θ + 2Φ) will be minimum and hence cos(θ + 2Φ) = 0

Therefore, length of the flagstaff = 2dsinθ/cosθ = 2dtanθ

Proved.

73. Let A and B two fixed points 3 cm apart. Prove that the locus of all points P in the plane such that PA = 2PB is a circle.

Solution :

Let A(0, 0) and B(3, 0) and P(h, k)

Now, PA = 2PB

- ⇨ $(h^2 + k^2) = 4\{(h - 3)^2 + k^2\}$
- ⇨ $h^2 + k^2 = 4h^2 - 24h + 36 + 4k^2$
- ⇨ $3h^2 + 3k^3 - 24h + 36 = 0$
- ⇨ $h^2 + k^2 - 8h + 12 = 0$

Putting h = x and y = b we get the required locus as, $x^2 + y^2 - 8x + 12 = 0$ which is a circle.

Proved.

74. Out of a circular sheet of paper of radius a, a sector with central angle θ is cut out and folded into a shape of a conical funnel. Show that the volume of the funnel is maximum when θ equals $2\pi\sqrt{(2/3)}$.

Solution :

Now, arc length of the sector = base perimeter of the cone = $a\theta$

Let, radius of the base is r. Therefore, $2\pi r = a\theta$

⇨ $r = a\theta/2\pi$.

Now, height of the cone = $h = \sqrt{(a^2 - r^2)} = \sqrt{\{a^2 - (a\theta/2\pi)^2\}} = a\sqrt{\{1 - (\theta/2\pi)^2\}}$

Volume = $(1/3)\pi(a\theta/2\pi)^2 a\sqrt{\{1 - (\theta/2\pi)^2\}}$

Now, volume will be maximum when $D = \theta^4\{1 - (\theta/2\pi)^2\}$ will be maximum because rest are constant.

Now, $dD/d\theta = 4\theta^3\{1 - (\theta/2\pi)^2\} + \theta^4\{-2\theta/(2\pi)^2\} = 0$

⇨ $\theta = 2\pi\sqrt{(2/3)}$

Proved.

75. There are 1000 doors named $D_1, D_2, D_3, \ldots, D_{1000}$. There are 1000 persons named $P_1, P_2, P_3, \ldots, P_{1000}$. At first all the doors are closed. P_1 goes and opens all the doors. Then P_2 goes and closes even numbered doors i.e. $D_2, D_4, D_6, \ldots, D_{1000}$ and leaves the odd numbered doors i.e. $D_1, D_3, D_5, \ldots, D_{999}$ as it is. Then P_3 goes and changes the state of the doors (If open then closes and if closed then opens) which are multiple of 3 i.e. the doors $D_3, D_6, \ldots$ Then P_4 goes and changes the state of the doors which are multiple of 4. Similarly in this way 1000 persons goes and changes the states of the doors accordingly. At the end how many doors will be open and how many doors will be closed?

Solution :

Let us take an example.

Let us see how many times and which persons are operating on D_{28}.

The persons who are operating on D_{28} are P_1, P_2, P_4, P_7, P_{14}, P_{28}.

Now, 1, 2, 4, 7, 14, 28 are factors of 28.

So, any door D_i is getting operated by the persons P_j where j's are the factors of i.

In this case we are not interested to find the factors of every number up to 1000. Rather we are interested in which doors are getting operated odd number of times and which doors are getting operated even number of times i.e. which numbers have odd number of factors and which numbers have even number of factors. If odd number of factors then odd number of operation and the door will be open and if even number of factors then even number of operation and the door will be closed.

We have seen that every number has even number of factors except the square numbers which have odd number of factors.

So, D_1, D_4, D_9, D_{16}, ..., D_{961} these 31 doors will stay open and rest of the doors will stay closed.

76. Evaluate lim $\{(1 + 1/2n)(1 + 3/2n)(1 + 5/2n)\ldots..(1 + (2n - 1)/2n)\}^{1/2n}$ as n -> ∞.

Solution :

Let, $y = \{(1 + 1/2n)(1 + 3/2n)(1 + 5/2n)\ldots..(1 + (2n - 1)/2n)\}^{1/2n}$

- ⇨ ln(y) = (1/2n){ln(1+ 1/2n) + ln(1 + 2/2n) + …. + ln(1 + 2n/2n)} – (1/2n){ln(1 + 2/2n) + ln(1 + 4/2n) + …. + ln(1 + 1n/2n)}
- ⇨ lim ln(y) = lim (1/2n){ln(1+ 1/2n) + ln(1 + 2/2n) + …. + ln(1 + 2n/2n)} – lim (1/2n){ln(1 + 2/2n) + ln(1 + 4/2n) + …. + ln(1 + 1n/2n)} as n -> ∞
- ⇨ ln(lim y) = ∫ln(1 + x)dx – (1/2)∫ln(1 + x)dx (ingration running from 0 to 1) = (1/2)∫ln(1 + x)dx (integration running from 0 to 1) = $(1/2)[\ln(1 + x)x|_0^1 - \int x dx/(1 + x)]$ (integration running from 0 to 1) = (1/2)[ln2 - ∫dx + ∫dx/(1 + x)] (integration running from 0 to 1) = (1/2)[ln2 – 1 + ln2] = ln2 – ½
- ⇨ lim y as n -> ∞ = $e^{\ln 2 - ½}$ = 2/√e.

77. Show that for every integer n, $\sqrt{n}$ is either an integer or an irrational number.

Solution :

Clearly, if n is a perfect square then $\sqrt{n}$ is an integer.

If n is not a perfect square, then n can be written as multiple of primes.

Now, we will prove that $\sqrt{p}$ is irrational where p is a prime.

Let, $\sqrt{p} = m$ an integer.

Then $p = m^2$

- ⇨ p divides m^2 i.e. p divides m (as p is prime) and $p = m$.
- ⇨ $m = m^2$
- ⇨ $m = 0$ or 1.

So, p is not prime.

Here is the contradiction.

- ⇨ $\sqrt{n}$ is either an integer or irrational.

78. If n is a positive integer greater than 1 such that $3n + 1$ is a perfect square, then show that $n + 1$ is the sum of three perfect squares.

Solution :

Let, $3n + 1 = m^2$

- ⇨ $n + 1 = (m^2 + 2)/3$

Now, m^2 is of the form $3m + 1$

- ⇨ m is of the form $3k + 1$ or $3k + 2$.

If m is of the form $3k + 1$ then $m^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 = 3m + 1$. So we are good to go.

If m is of the form $3k + 2$ then $m^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1 = 3m + 1$. So we are good to go.

Now, consider the case m is of the form $3k + 1$.

Then, $n + 1 = (m^2 + 2)/3 = (9k^2 + 6k + 3)/3 = 3k^2 + 2k + 1 = k^2 + k^2 + (k + 1)^2$ = sum of three squares.

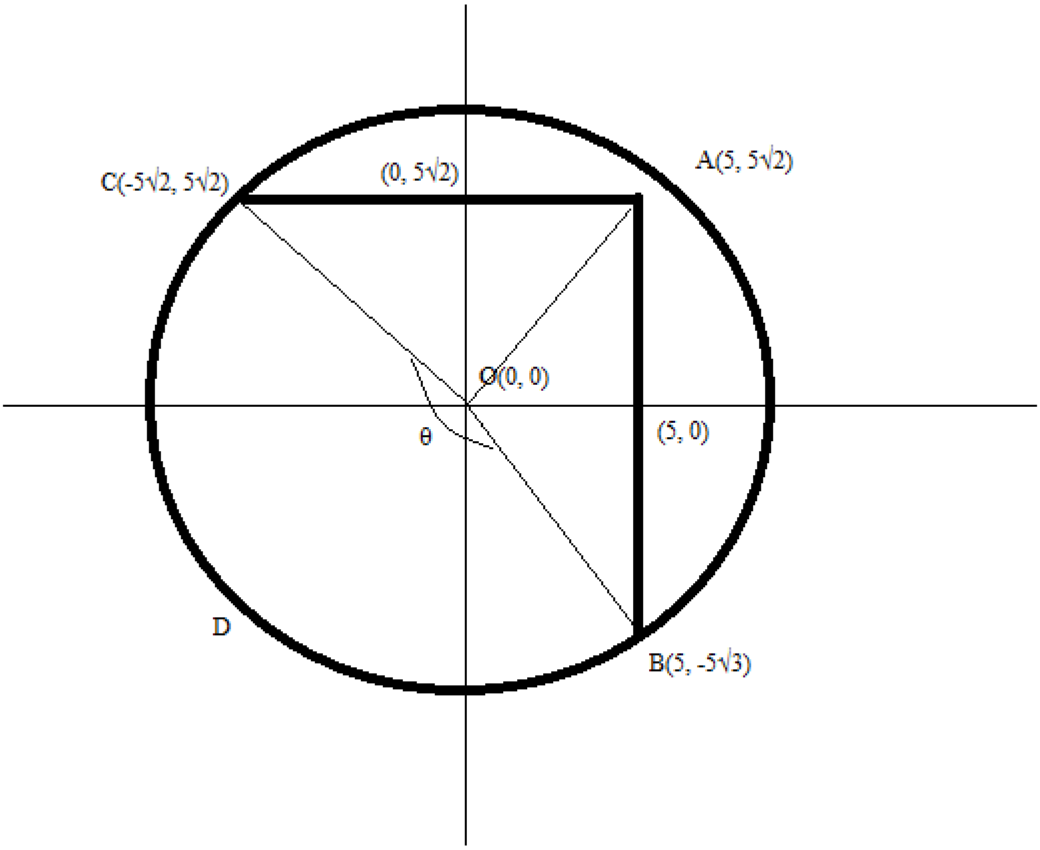
Now, consider the case m is of the form 3k + 2.

Then $n + 1 = (m^2 + 2)/3 = (9k^2 + 12k + 6)/3 = 3k^2 + 4k + 2 = (k + 1)^2 + (k + 1)^2 + k^2$ = sum of three squares.

Proved.

79. A cow is grazing with a rope around her neck and the other end of the rope is tied to a pole. The length of the rope is 10 metres. There are two boundary walls perpendicular to each other, one at a distance of 5 metres to the east of the pole and another at a distance of 5√2 metres to the north of the pole. Find the area the cow can graze on.

Solution :



By solving the equation of the circle $x^2 + y^2 = 10^2$ and the straight lines x = 5 and y = 5√2 we get the co-ordinate of the points A, B, C as given in the figure.

Now, slope of line OB = -5√3/5 = -√3 and slope of line OC = 5√2/(-5√2) = -1.

So, tanθ = (-1 + √3)/(1 + √3) = 2 - √3

⇨ $\theta = \tan^{-1}(2 - \sqrt{3})$

Area of OCDB = $\{(\pi*10^2)/2\pi)\}*\tan^{-1}(2 - \sqrt{3}) = 50\tan^{-1}(2 - \sqrt{3})$

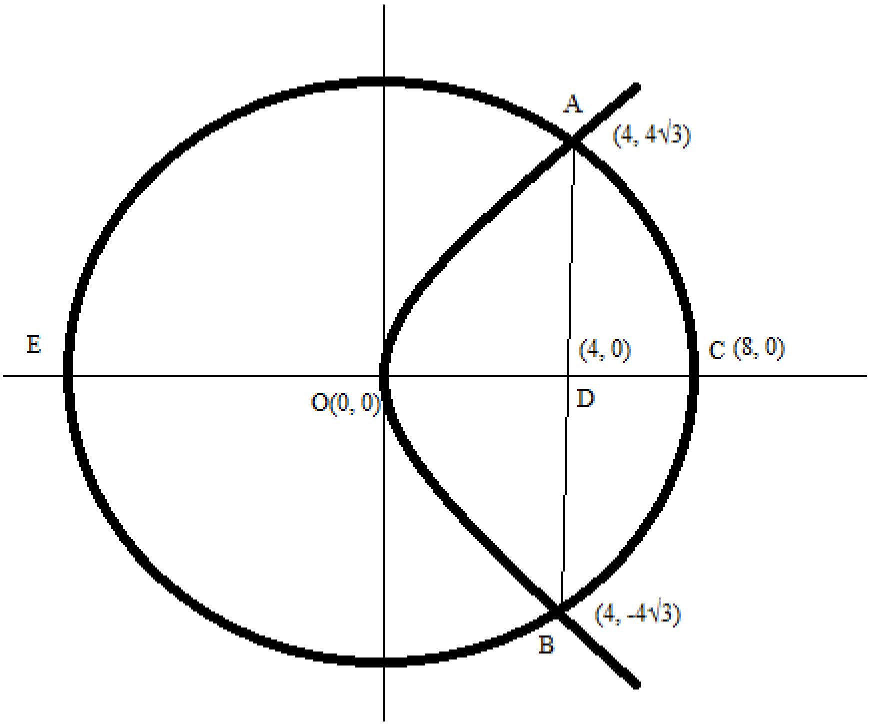
Area of triangle OAB = $(1/2)*5*(5\sqrt{2} + 5\sqrt{3}) = 25(\sqrt{3} + \sqrt{2})/2$.

Area of triangle OAC = $(1/2)*5\sqrt{2}*(5 + 5\sqrt{2}) = 25(\sqrt{2} + 1)/2$.

The cow can graze in the area = area OCDB + area OAB + area OAC.

80. Show that the larger of the two areas into which the circle $x^2 + y^2 = 64$ is divided by the curve $y^2 = 12x$ is $(16/3)(8\pi - \sqrt{3})$.

Solution :



By solving the equations we get the intersection points as given in the picture.

Now, area of ODA = $\int\sqrt{(12x)}dx$ (integration running from 0 to 4) = $32\sqrt{3}/3$

Area of ADBO = $2*32\sqrt{3}/3 = 64\sqrt{3}/3$.

Now, area of ADC = $\int\sqrt{(64 - x^2)}dx$ (integration running from 4 to 8) = $32\pi/3 - 8\sqrt{3}$

Area of ADBC = $2*(32\pi/3 - 8\sqrt{3}) = 64\pi/3 - 16\sqrt{3}$.

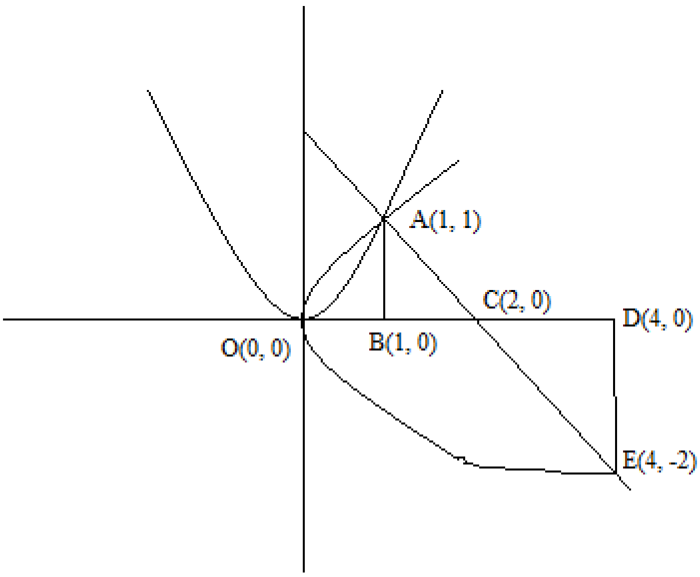
Area of the circle = 64π

Area of the larger portion i.e. AEB = 64π – (64√3/3 + 64π/3 - 16√/3) = (16/3)(8π - √3).

Proved.

81. Find the area of the region in the xy-plane, bounded by the graphs of $y = x^2$, $x + y = 2$ and $y = -\sqrt{x}$.

Solution :



By solving the curves we get the required co-ordinate of the points as shown in the figure.

Now, area of ODEO = |∫-√xdx| (integration running from 0 to 4) = 16/3.

Now, area of triangle CED = (1/2)*2*2 = 2

So, area of the portion OBCEO = 16/3 – 2 = 10/3.

Area of OABO = $\int x^2 dx$ (integration running from 0 to 1) = 1/3.

Area of triangle ABC = (1/2)*1*1 = ½.

Area of OACBO = 1/3 + ½ = 5/6.

Area of the required region = 10/3 + 5/6 = 25/6.

82. Determine m so that the equation $x^4 - (3m + 2)x^2 + m^2 = 0$ has four roots in arithmetic progression.

Solution :

Let, the roots are p – d, p, p + d, p + 2d.

Sum of the roots = 4p + 2d = 0

⇨ d = -2p.

So roots are, 3p, p, -p, -3p.

Now, multiplication of the roots = $9p^4 = m^2$

⇨ $m = 3p^2$.

Sum of the roots taken two at a time = $3p^2 - 3p^2 - 9p^2 - p^2 - 3p^2 + 3p^2 = -10p^2 = -(3m + 2)$

⇨ $10p^2 = 9p^2 + 2$
⇨ $p^2 = 2$
⇨ $m = 3p^2 = 6$

83. Let a and b be two real numbers. If the roots of the equation $x^2 - ax - b = 0$ have absolute value less than one, show that each of the following conditions holds :
(a) $|b| < 1$.
(b) $a + b < 1$.
(c) $b - a < 1$.

Solution :

Let, x_1, x_2 are the roots of the equation.

Therefore, $-b = x_1x_2$

⇨ $|b| = |x_1||x_2| < 1*1 = 1$

Roots are = $\{a \pm \sqrt{(a^2 + 4b)}\}/2$

Now, $\{a + \sqrt{(a^2 + 4b)}\}/2 < 1$

⇨ $\sqrt{(a^2 + 4b)} < 2 - a$
⇨ $a^2 + 4b < 4 - 4a + a^2$
⇨ $a + b < 1$

Now, again, $-1 < \{a - \sqrt{(a^2 + 4b)}\}/2$

- ⇨ $-2 - a < -\sqrt{(a^2 + 4b)}$
- ⇨ $2 + a > \sqrt{(a^2 + 4b)}$
- ⇨ $4 + 4a + a^2 > a^2 + 4b$
- ⇨ $b - a < 1$.

Proved.

84. Suppose that the three equations $ax^2 - 2bx + c = 0$, $bx^2 - 2cx + a = 0$ and $cx^2 - 2ax + b = 0$ all have only positive roots. Show that $a = b = c$.

Solution :

Roots of $ax^2 - 2bx + c = 0$ are $\{b \pm \sqrt{(b^2 - ac)}\}/2a$

Roots of $bx^2 - 2cx + a = 0$ are $\{c \pm \sqrt{(c^2 - ab)}\}/2b$

Roots of $cx^2 - 2ax + b = 0$ are $\{a \pm \sqrt{(a^2 - bc)}\}/2c$

Now, to be all roots positive $a, b, c > 0$ and $b^2 - ac \geq 0$, $c^2 - ab \geq 0$ and $a^2 - bc \geq 0$.

- ⇨ $b^2 \geq ac$, $c^2 \geq ab$ and $a^2 \geq bc$

Multiplying the above three inequalities we get, $a^2b^2c^2 \geq a^2b^2c^2$

- ⇨ The equality holds.
- ⇨ $b^2 - ac = 0$, $c^2 - ab = 0$ and $a^2 - bc = 0$

Adding the above equalities, we get, $a^2 + b^2 + c^2 - ab - bc - ca = 0$

- ⇨ $(1/2)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = 0$
- ⇨ $a = b = c$.

Proved.

85. For $x > 0$, show that $(x^n - 1)/(x - 1) \geq nx^{(n-1)/2}$, where n is a positive integer.

Solution :

Now, $(x^n - 1)/(x - 1) = 1 + x + x^2 + \ldots. + x^{n-1} \geq n(1*x*x^2*\ldots.*x^{n-1})^{1/n}$ (Applying AM ≥ GM) $= nx^{(n-1)/2}$

Proved.

86. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are $\alpha - 1/\beta\gamma$, $\beta - 1/\gamma\alpha$ and $\gamma - 1/\alpha\beta$.

Solution :

Very easy problem. Do it yourself.

87. In a competition six teams A, B, C, D, E< F play each other in the preliminary round – called round robin tournament. Each game either a win or a loss. The winner is awarded two points while the loser is awarded zero points. After the round robin tournament, the three teams with the highest scores move to the final round. Based on the following information, find the score of each team at the end of the round robin tournament.

(i) In the game between E and F, team E won.

(ii) After each team had played four games, team A had 6 points, team B had 8 points and team C had 4 points. The remaining matches yet to be played were

(a) Between A and D.

(b) Between B and E and

(c) Between C and F.

(iii) The teams D, E and F had won their games against A, B and C respectively.

(iv) Teams A, B and D had moved to the final round of the tournament.

Solution :

Clearly, team A's point = 6, Team B's point = 8 and team C's point = 4.

Now, E's point = 4 (as E has won games against B and E and E did not move to final round, hence E's point < 6)

F's point as of now 2 as F as won game against C.

Total number of games = ${}^6C_2$.

Total point = $2 * {}^6C_2 = 30$.

Now, as of now total points found = 6 + 8 + 4 + 4 + 2 = 24.

D's point must be 30 – 24 = 6 as D has moved to final round so D's point > 4.

So, F's point = 2 and D's point = 6.

88. An operation * on a set G is a mapping that associates with every pair of elements of a and b of the set G, a unique element a*b of G. G is said to be a group under the operation *, if the following conditions hold :

(i) (a*b)*c = a*(b*c) for all elements a, b and c in G;

(ii) There is an element e of G such that a*e = e*a = a for all elements a of G; and

(iii) For each element a of G, there is an element a' of G such that a*a' = a'*a = e.

If G is the set whose elements are all subsets of a set X, and if * is the operation on G defined as $A*B = (A \cup B)\backslash(A \cap B)$, show that G is a group under *. (For any two subsets C and D of X C\D denotes the set of all those elements which are in C but not in D).

Solution :

Clearly, $(A \cup B)\backslash(A \cap B) = (A \cap B^c) \cup (A^c \cap B)$

Now, $(A*B)*C = \{(A \cap B^c) \cup (A^c \cap B) \cap C^c\} \cup [\{A \cap B^c) \cup (A^c \cap B)\}^c \cap C] = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C)$ (This can be easily verified by the use of Venn diagram)

Similarly, $A*(B*C) = \{(A \cap B^c) \cup (A^c \cap B) \cap C^c\} \cup [\{A \cap B^c) \cup (A^c \cap B)\}^c \cap C] = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C)$ (This can be easily verified by the use of Venn diagram)

⇨ $(A*B)*C = A*(B*C)$

Now, $e = \Phi$ (the null set) and $A' = A$.

Proved.

89. Let X = {0, 1, 2, 3, …., 99}. For a, b in X, we define a*b to be the remainder obtained by dividing the product ab by 100. For example, 9*18 = 62 and 7*5 = 35. Let x be an element in X. An element y in X is called the inverse of x if x*y = 1. Find which of the elements 1, 2, 3, 4, 5, 6, 7 have inverses and write down their inverses.

Solution :

Inverse of 1 is 1.

2, 4, 5, 6, doesn't have inverses because 2, 4, 6 are even and 5 doesn't divide a number whose last digit is 1.

Inverse of 3 is 67.

Inverse of 7 is 43.

90. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, d are integers. The sums of the pairs of roots of P(x) are given by 1, 2, 5, 6, 9, 10. Find P(1/2).

Solution :

Let, p, q, r, s are the roots of the equation.

Also, let, $p < q < r < s$.

Now, $p + q = 1$, $p + r = 2$, $q + s = 9$, $r + s = 10$ and $p + s$ and $q + r$ take the value either of 5 or 6 another the other.

Now, adding the all equalities, we get, $p + q + r + s = 11$.

Let, $p + s = 6$.

Now, $p + q + p + r = 1 + 2$

- ⇨ $p + (p + q + r) = 3$
- ⇨ $p + 11 - s = 3$
- ⇨ $s - p = 8$ and $s + p = 6$ (as per out assumption)
- ⇨ $s = 7$ and $p = 1$.
- ⇨ $q = 0$

But, $p < q$. Here is the contradiction.

- ⇨ $p + s = 5$ and $q + r = 6$

Which gives, $p = -3/2$, $q = 5/2$, $r = 7/2$ and $s = 13/2$

Now, we have got all the roots. Now put the values in P(x) and you will get 4 equations from that you need to solve 4 unknowns a, b, c, d.

Put back the values of a, b, c, d and you get P(x). Then put $x = ½$ and get P(1/2).

91. Let, $x_n = (1/2)(3/4)(5/6)\ldots\{(2n - 1)/2n\}$. Then show that $x_n \leq 1/\sqrt{(3n + 1)}$ for all n = 1, 2. 3, …..

Solution :

Clearly this is true for $n = 1$.

Let this is true for $n = k$ i.e. $x_k \le 1/\sqrt{(3k + 1)}$

For $n = k + 1$, $x_{k+1} = (1/2)(3/4)(5/6)\ldots.\{(2k - 1)/2k\}\{(2k + 1)/(2k + 2)\} = x_k\{(2k + 1)/(2k + 2)\} \le \{(2k + 1)/(2k + 2)\}/\sqrt{(3k + 1)}$

We have to prove that, $\{(2k + 1)/(2k + 2)\}/\sqrt{(3k + 1)} \le 1/\sqrt{(3k + 4)}$

i.e. to prove, $(2k + 1)^2(3k + 4) \le (2k + 2)^2(3k + 1)$

i.e. to prove $5k + 1 \ge 0$ which is obvious.

Then by the principle of induction this is true for all $n = 1, 2, 3, \ldots$.

Proved.

92. In the identity $n!/\{x(x + 1)(x + 2)\ldots\ldots(x + n)\} = \sum A_k/(x + k)$ (summation running from $k = 0$ to $k = n$), prove that $A_k = (-1)^k{*}^nC_k$.

Solution :

Now, $n! = A_0(x + 1)(x + 2)\ldots.(x + n) + A_1{*}x(x + 2)(x + 3)\ldots(x + n) + \ldots. + A_k{*}x(x + 1)\ldots(x + k - 1)(x + k + 1)\ldots(x + n) + \ldots. + A_n{*}x(x + 1)\ldots(x + n - 1)$

Putting $x = -k$ we get, $A_k{*}(-k)(-k + 1)\ldots.(-1){*}1{*}2{*}\ldots(-k + n) = n!$

- ⇨ $A_k{*}(-1)^k(1{*}2{*}\ldots{*}k)\{1{*}2{*}\ldots.(n - k)\} = n!$
- ⇨ $A_k = (-1)^k{*}n!/\{(n - k)!{*}k!\} = (-1)^k{*}^nC_k$.

Proved.

93. Show that $3/(1{*}2{*}4) + 4/(2{*}3{*}5) + 5/(3{*}4{*}6) + \ldots\ldots + (n + 2)/\{n(n + 1)(n + 3) = (1/6)\{(29/6) - 4/(n + 1) - 1/(n + 2) - 1/(n + 3)\}$.

Solution :

Clearly, this is true for $n = 1$.

Let this is true for $n = k$.

We have to prove that this is true for n = k + 1 i.e. 3/(1*2*4) + 4/(2*3*5) + 5/(3*4*6) + ….. + (k + 2)/{k(k + 1)(k + 3)} + (k + 3)/{(k + 1)(k + 2)(k + 4)} = (1/6){29/6 – 4/(k + 2) – 1/(k + 3) – 1/(k + 4)}

i.e. to prove, (1/6){29/6 – 4/(k + 1) – 1/(k + 2) – 1/(k + 3)} + (k + 3)/{(k + 1)(k + 2)(k + 4)} = (1/6){29/6) – 4/(k + 2) – 1/(k + 3) – 1/(k + 4)}

i.e. to prove, (1/6){4/(k + 1) – 3/(k + 2) – 1/(k + 4)} = (k + 3)/{(k + 1)(k + 2)(k + 4)}

Now, (1/6){4/(k + 1) – 3/(k + 2) – 1/(k + 4)}

= (1/6){4(k + 2)(k + 4) – 3(k + 1)(k + 4) – (k + 1)(k + 2)}/{k+1)(k + 2)(k + 4}

= (1/6)(4k^2 + 24k + 32 – 3k^2 – 15k – 12 – k^2 – 3k – 2)/{(k + 1)(k + 2)((k + 4)}

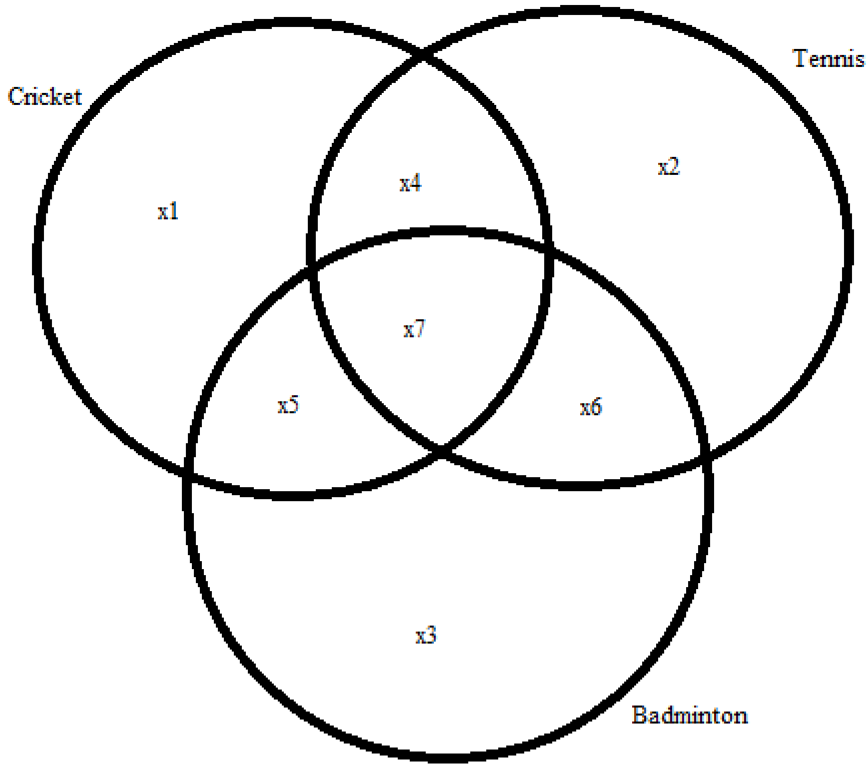
= (1/6)(6k + 18)/{(k + 1)(k + 2)(k + 4)}

= (k + 3)/{(k + 1)(k + 2)(k + 4)}

Proved (By the principle of Induction).

94. In a club of 80 members, 10 members play none of the games Tennis, Badminton and Cricket. 30 members play exactly one of these three games and 30 members play exactly two of these three games. 45 members play at least one of the games among Tennis and Badminton, whereas 18 members play both Tennis and Badminton. Determine the number of Cricket playing members.

Solution :



From the figure, we get, $x_1 + x_2 + x_3 = 30$ and $x_4 + x_5 + x_6 = 30$

Out of 80 members 10 members play none of the games.

⇨ $80 - 10 = 70$ members play games.

Therefore, $x_7 = 70 - (x_1 + x_2 + x_3) - (x_4 + x_5 + x_6) = 70 - 30 - 30 = 10$.

Now, 45 members play at least one of the games among Tennis and Badminton.

⇨ $70 - x_1 = 45$
⇨ $x_1 = 25$.

Now, 18 members play both Tennis and Badminton.

⇨ $x_6 + x_7 = 18$
⇨ $x_6 = 8$.
⇨ $x_4 + x_5 = 30 - 8 = 22$.

So, Cricket playing members = $x_1 + x_7 + x_4 + x_5 = 25 + 10 + 22 = 57$.

95. Let $x = (x_1, x_2, …., x_n)$, $y = (y_1, y_2, …., y_n)$, where $x_1, x_2, …., x_n, y_1, y_2, …., y_n$ are real numbers. We write $x > y$, if for some k, $1 \le k \le (n - 1)$, $x_1 = y_1, x_2 = y_2, ….., x_k = y_k$, but $x_{k+1} > y_{k+1}$. Show that for $u = (u_1, u_2, …, u_n)$, $v = (v_1, v_2, …., v_n)$, $w = (w_1, w_2, …, w_n)$ and $z = (z_1, z_2, ….., z_n)$, if $u > v$ and $w > z$, then $(u + w) > (v + z)$.

Solution :

Let $u_1 = v_1$, $u_2 = v_2$, …., $u_k = v_k$ and $u_{k+1} > v_{k+1}$.

Also, $w_1 = z_1$, $w_2 = z_2$, ….., $w_k = z_k$ and $w_{k+1} > z_{k+1}$.

- ⇨ $u_1 + w_1 = v_1 + z_1$, $u_2 + w_2 = v_2 + z_2$, …., $u_k + w_k = v_k + z_k$ and $u_{k+1} + w_{k+1} > v_{k+1} + z_{k+1}$.
- ⇨ $(u + w) > (v + z)$.

Proved.

96. We say a sequence $\{a_n\}$ has property P, if there exists a positive integer m such that $a_n \le 1$ for every $n \ge m$. For each of the following sequences, determine whether it has the property P or not. [Do not use the results on limits.]
    (i) $a_n = 0.9 + 200/n$ if n is even and $a_n = 1/n$ if n is odd.
    (ii) $a_n = \{1 + \cos(n\pi/2)\}/n$ if n is even $a_n = 1/n$ if n is odd.

Solution :

Now, $a_n \le 1$

- ⇨ $0.9 + 200/n \le 1$
- ⇨ $200/n \le 0.1$
- ⇨ $n \ge 2000$

So, if we set m = 2001 then for every $n \ge m$ $a_n > 1$.

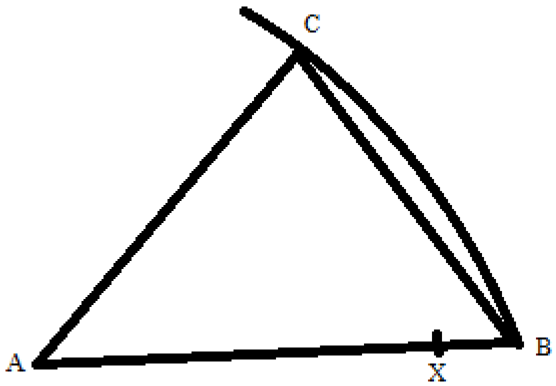
So, (i) has property P.

Now, $\cos(n\pi/2) = 1$ or -1 as n is even.

So, there cannot be any m such that for any $n \ge m$ $a_n \ge 1$.

So, (ii) doesn't have property P.

97. Let X be a point on a straight line segment AB such that $AB*BX = AX^2$. Let C be a point on the circle with centre at A and radius AB such that BC = AX. (See figure.) Show that the angle BAC = 36.



Solution :

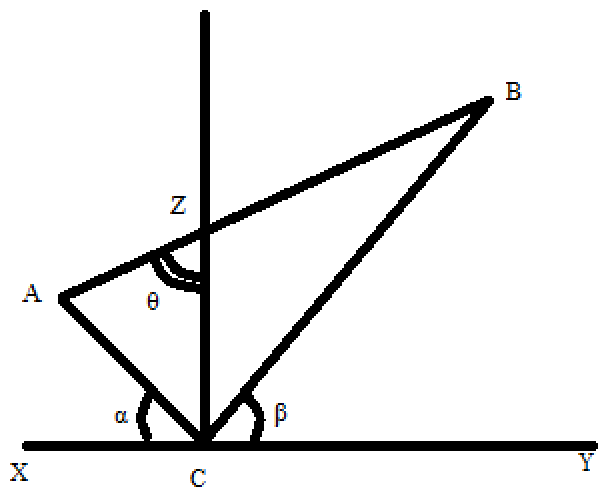
Let angle BAC = θ

From triangle ABC, we get, BC/sinθ = AB/sin(90 – θ/2) (Angle ACB = Angle ABC as AC = BC = radius)

- ⇨ BC/{2sin(θ/2)cos(θ/2)} = AB/cos(θ/2)
- ⇨ 2sinθ/2 = BC/AB = BC*BX/AX^2 = BX/AX (BC = AX)
- ⇨ 2sin(θ/2) + 1 = AB/BC = 1/(2sinθ/2) (From above line)
- ⇨ 4sin^2(θ/2) + 2sin(θ/2) – 1 = 0
- ⇨ sin(θ/2) = {-2 + √(4 + 16)}/8 = (√5 – 1)/4 = sin18
- ⇨ θ/2 = 18
- ⇨ θ = 36.

Proved.

98. In the adjoining figure CZ is perpendicular to XY and the ratio of the lengths AZ to ZB is 1:2. The angle ACX is α and the angle BCY is β. Find an expression for the angle AZC in terms of α and β.



Solution :

Let angle AZC = θ.

Now, angle ZAC = 90 – α

- ⇨ Angle CAZ = 180 – (90 – α) – θ = 90 + α – θ

From triangle, AZC we get, CZ/sin(90 + α – θ) = AZ/sin(90 – α)

- ⇨ CZ/cos(α – θ) = AZ/cosα …… (1)

Again, angle ZCB = 90 – β

- ⇨ Angle CBZ = 180 – (180 – θ) – (90 – β) = -{90 – (θ + β)}

From triangle BZC, we get, CZ/sin[-{90 – (θ + β)}] = BZ/sin(90 – β)

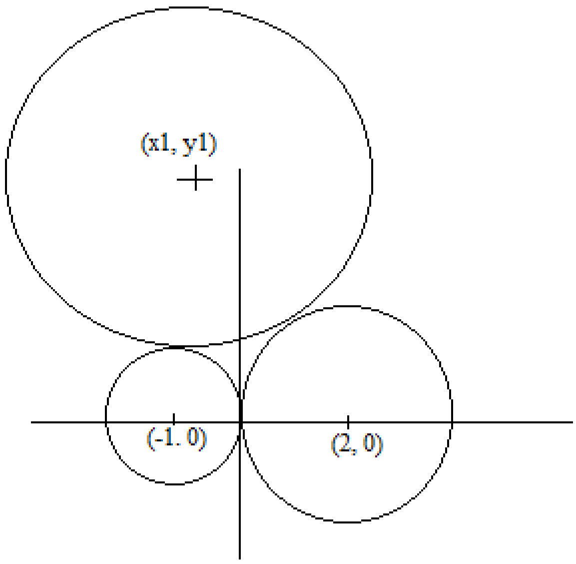
- ⇨ CZ/{-cos(θ + β)} = BZ/cosβ ….. (2)

Dividing equation (1) by (2) we get, {-cos(θ + β)}/cos(α – θ) = (AZ/BZ)(cosβ/cosα)

- ⇨ (sinθsinβ – cosθcosβ)/(cosαcosθ + sinαsinθ) = cosβ/(2cosα) (AZ/BZ = ½)
- ⇨ (tanθsinβ – cosβ)/(cosα + sinαtanθ) = cosβ/(2cosα) (Dividing numerator and denominator of LHS by cosθ)
- ⇨ 2cosαsinβtanθ – 2cosαcosβ = cosαcosβ + sinαcosβtanθ
- ⇨ tanθ(2cosαsinβ – sinαcosβ) = 3cosαcosβ
- ⇨ tanθ = 3cosαcosβ/(2cosαsinβ – sinαcosβ)

99. The circles C_1, C_2 and C_3 with radii 1, 2 and 3, respectively, touch each other externally. The centres of C_1 and C_2 lie on the x-axis, while C_3 touches them from the top. Find the ordinate of the center of the circle that lies in the region enclosed by the circles C_1, C_2 and C_3 and touches all of them.

Solution :



Let, C_1 and C_2 touches each other at origin.

Now, $(x_1 + 1)^2 + y_1^2 = (1 + 3)^2$

⇨ $(x_1 + 1)^2 + y_1^2 = 16$ …… (i)

Again, $(x_1 - 2)^2 + y_1^2 = (2 + 3)^2$

⇨ $(x_1 - 2)^2 + y_1^2 = 25$ ……. (ii)

Doing (ii) – (i) we get, $(x_1 - 2)^2 - (x_1 + 1)^2 = 25 - 16$

⇨ $x_1 = -1$
⇨ $y_1 = 4$

Now, let the co-ordinate of the centre of the required circle is (α, β)

We need to find β.

Again, $(\alpha + 1)^2 + \beta^2 = (1 + r)^2$ (where r is radius of the required circle) …… (1)

Also, $(\alpha - 2)^2 + \beta^2 = (2 + r)^2$ ……… (2)

And, $(\alpha + 1)^2 + (\beta - 4)^2 = (3 + r)^2$ ……. (3)

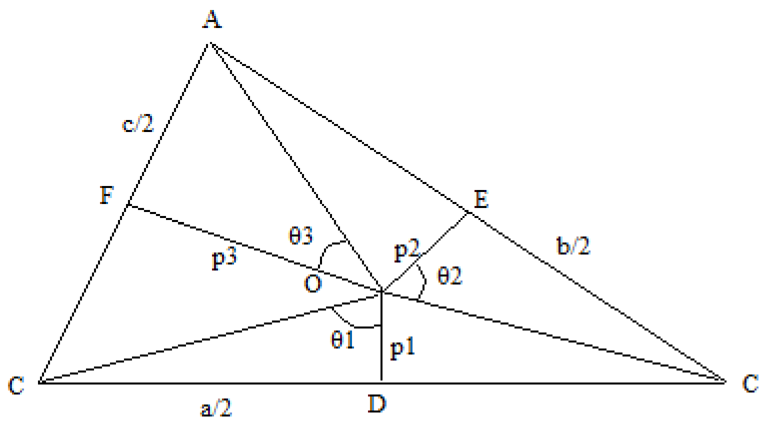
We have 3 independent equations and 3 unknowns α, β, r.

We can solve β from here and that is our answer.

(This problem is based on the theory if two circles touch externally then, $O_1O_2 = r_1 + r_2$ where O_1, O_2 are centres of circles C_1, C_2 with radii r_1 and r_2 respectively)

100. If a, b and c are the lengths of the sides of a triangle ABC and p_1, p_2 and p_3 are the lengths of the perpendiculars drawn from the circumcentre onto the sides BC, CA and AB respectively, then show that $a/p_1 + b/p_2 + c/p_3 = abc/(4p_1p_2p_3)$.

Solution :



Let O is the circumcentre.

From triangle OCD, we get, $a/2p_1 = \tan\theta_1$

Similarly, from triangle OCE we get, $b/2p_2 = \tan\theta_2$ and from triangle OAF, we get, $c/2p_3 = \tan\theta_3$.

So, we have to prove that, $(a/2p_1) + (b/2p_2) + (c/2p_3) = (a/2p_1)(b/2p_2)(c/2p_3)$

i.e. $\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = \tan\theta_1\tan\theta_2\tan\theta_3$ where $2\theta_1 + 2\theta_2 + 2\theta_3 = 360$, i.e. $\theta_1 + \theta_2 + \theta_3 = 180$.

Now, $\tan\theta_1 + \tan\theta_2 + \tan\theta_3$

$= \tan(\theta_1 + \theta_2)(1 - \tan\theta_1\tan\theta_2) + \tan\theta_3$

$= \tan(180 - \theta_3)(1 - \tan\theta_1\tan\theta_2) + \tan\theta_3$

$= -\tan\theta_3(1 - \tan\theta_1\tan\theta_2) + \tan\theta_3$

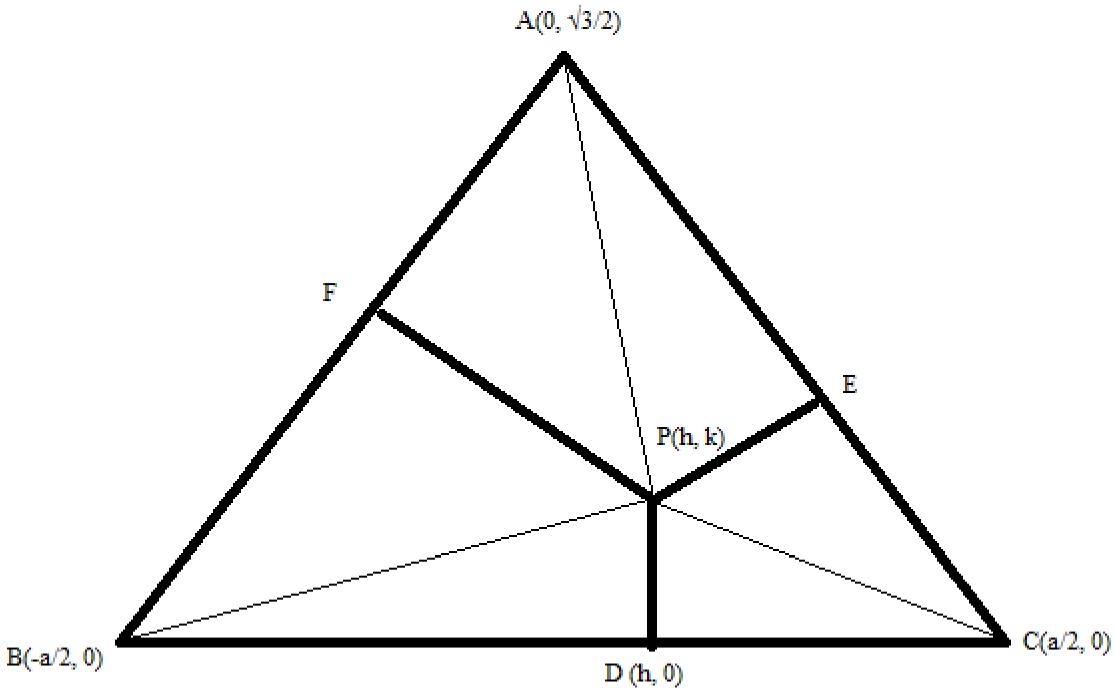
$= -\tan\theta_3 + \tan\theta_1\tan\theta_2\tan\theta_3 + \tan\theta_3$

$= \tan\theta_1\tan\theta_2\tan\theta_3$.

Proved.

101. Inside an equilateral triangle ABC, an arbitrary point P is taken from which the perpendiculars PD, PE, PF are dropped onto the sides BC, CA and AB, respectively. Show that the ratio (PD +PE + PF)/(BD + CE + AF) doesn't depend upon the choice of the point P and find its value.

Solution :



Equation of line AC, $\sqrt{3}x + y - \sqrt{3}a/2 = 0$

Equation of line PE which is perpendicular to AC and passing through (h, k) is, $x - \sqrt{y} - (h - \sqrt{k}) = 0$

Solving these two lines we get, co-ordinate of E.

$x = (1/4)\{3a/2 + (h - \sqrt{3}k)\}$ and $y = (\sqrt{3}/4)\{a/2 - (h - \sqrt{3}k)\}$

$CE^2 = [(1/4)\{3a/2 + (h - \sqrt{3}k)\} - a/2]^2 + [(\sqrt{3}/4)\{(a/2 - (h - \sqrt{3}k)\}]^2$

$= [(1/4)\{a/2 - (h - \sqrt{3}k)\}]^2 + [(\sqrt{3}/4)\{(a/2 - (h - \sqrt{3}k)\}]^2$

$= \{(a/2 - (h - \sqrt{3}k)\}^2(1/16 + 3/16)$

⇨ $CE = (1/2) \{(a/2 - (h - \sqrt{3}k)\}$

Now, equation of AB, $\sqrt{3}x - y + \sqrt{3}a/2 = 0$

Equation of line PF which is perpendicular to AB and passing through (h, k) is, $x + \sqrt{3}y) - (h + \sqrt{k}) = 0$

Solving these two equations we get co-ordinate of F.

$x = (1/4)\{(h + \sqrt{3}k) - 3a/2\}$ and $y = (\sqrt{3}/4)\{a/2 + (h + \sqrt{3}k)\}$

$AF^2 = [(1/4)\{(h + \sqrt{3}k) - 3a/2\}]^2 + [(\sqrt{3}/4)\{a/2 + (h + \sqrt{3}k)\} - \sqrt{3}a/2]^2$

$= [(1/4)\{3a/2 - (h + \sqrt{3}k)\}]^2 + [(\sqrt{3}/4)\{3a/2 - (h + \sqrt{3}k)\}]^2$

$= \{3a/2 - (h + \sqrt{3}k)\}^2(1/16 + 3/16)$

⇨ $AF = (1/2)\{3a/2 - (h + \sqrt{3}k)\}$

Now, $CE + AF = a - h = 3a/2 - (h + a/2)$

⇨ $BD + CE + AF = 3a/2$.

Now, area of triangle BPC = (1/2)*a*PD

Area of triangle ACP = (1/2)*a*PE

Area of triangle ABP = (1/2)*a*PF

Now, Area of triangle BPC + area of triangle ACP + Area of triangle ABP = Area of triangle ABC

⇨ $(1/2)a(PD + PE + PF) = \sqrt{3}a^2/4$
⇨ $PD + PE + PF = \sqrt{3}a/2$

Therefore, $(PD + PE + PF)/(BD + CE + AF) = (\sqrt{3}a/2)/(3a/2) = 1/\sqrt{3}$.

102. Let $f : N \to N$ be the function defined by $f(0) = 9$, $f(1) = 1$ and $f(n) = f(n - 1) + f(n - 2)$ for $n \geq 2$, where N is the set of all non-negative integers. Prove the following results :
   (i) $f(n) < f(n + 1)$ for all $n \geq 2$.
   (ii) There exists precisely four non-negative integers n for which $f(f(n)) = f(n)$.
   (iii) $f(5n)$ is divisible by 5, for all n

Solution :

Now, $f(0) = 1$, $f(1) = 2$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$, $f(5) = 5$, $f(6) = 8$, $f(7) = 13$, …

Let, $f(n) > f(n + 1)$

Now, $f(n + 1) = f(n) + f(n - 1)$

- ⇨ $f(n) - f(n + 1) = -f(n - 1)$
- ⇨ $-f(n - 1) > 0$
- ⇨ $f(n - 1) < 0$ which is impossible for $n \geq 2$.
- ⇨ $f(n) < f(n + 1)$.

Clearly, for $n = 0, 1, 2, 5$ $f(f(n)) = f(n)$.

Now, $f(n) = f(n - 1) + f(n - 2)$

$= 2f(n - 2) + f(n - 3)$

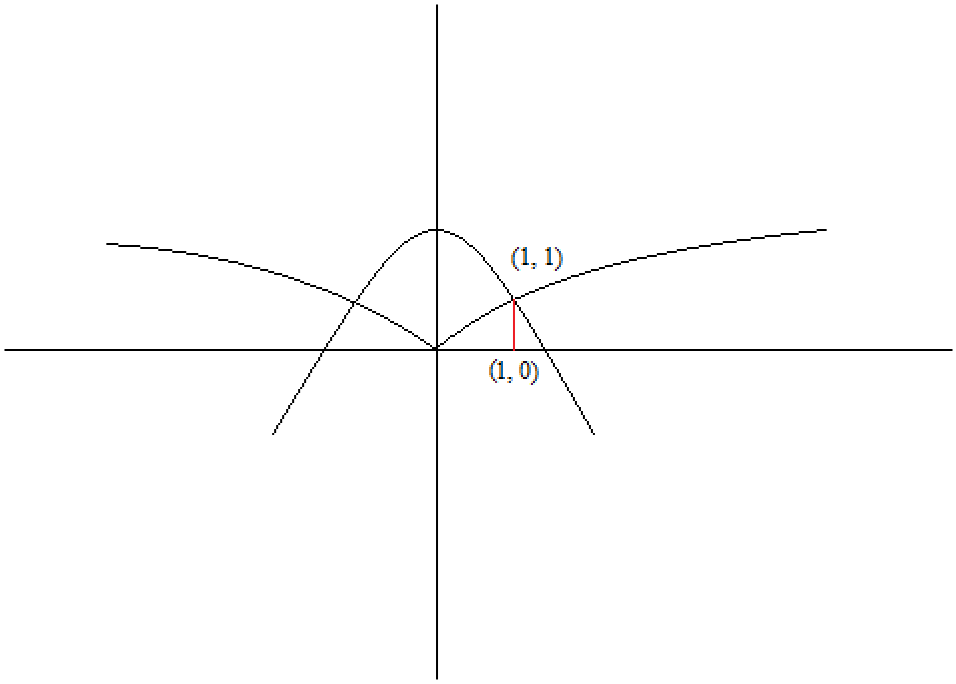
$= 3f(n - 3) + 2f(n - 4)$

$= 5f(n - 4) + 3f(n - 5)$

- ⇨ $f(n) \equiv 3f(n - 5) \pmod 5$
- ⇨ $f(5n) \equiv 3f(5(n - 1)) \pmod 5$ which is true for $n = 1$ and $f(0)$ and $f(5)$ both are divisible by 5.
- ⇨ $f(5n) \equiv 0 \pmod 5$ for all n. (By the principle of Induction)

103. Show that the area of the bounded region enclosed between the curves $y^3 = x^2$ and $y = 2 - x^2$, is $2 + 2/15$.

Solution :



Solving the equations we get the intersection point (1, 1) as shown in the figure.

Area = $2[\int(2 - x^2)dx - \int x^{2/3}dx]$ (integration running from 0 to 1)

$= 2[2x - x^3/3 - (3/5)x^{5/3}]_0^1$

$= 2(2 - 1/3 - 3/5)$

$= 2 + 2/15.$

Proved.

104. Suppose f(x) is a continuous function such that $f(x) = \int f(t)dt$ (integration running from 0 to x). Prove that f(x) is identically equal to zero.

Solution :

Now, $f(x) = \int f(t)dt$ (integration running from 0 to x)

⇨ $f(0) = \int f(t)dt$ (integration running from 0 to 0)

As upper limit and lower limit of integration is same hence, $f(0) = 0$

Now, $f(x) = \int f(t)dt$ (integration running from 0 to t)

⇨ $f'(x) = f(x)$
⇨ $\int\{f'(x)/f(x)\}dx = \int dx$

- ⇨ $\ln(f(x)) = x + c$
- ⇨ $f(x) = e^{x+c}$
- ⇨ $f(x) = ke^{x}$

Now, $f(0) = k*e^{0}$

- ⇨ $k = 0$ (as $f(0) = 0$)
- ⇨ $f(x) = 0$ for all x..

Proved.

105. Consider the function $f(t) = e^{-1/t}$, $t > 0$. Let for each positive integer n, P_n be the polynomial such that $d^n/dt^n(f(t)) = P_n(1/t)e^{-1/t}$ for all $t > 0$. Show that $P_{n+1}(x) = x^2\{P_n(x) - d/dx(P_n(x))\}$.

Solution :

Now, $d^n/dt^n(f(t)) = P_n(1/t)e^{-1/t}$

- ⇨ $d^{n+1}/dt^{n+1}(f(t) = P_{n+1}(1/t)e^{-1/t}$ (putting n + 1 in place of n)
- ⇨ $d^n/dt^n(f(1/t)) = P_n(t)e^{-t}$ …… (1)

Again, $d^n/dt^n(f(t)) = P_n(1/t)e^{-1/t}$

- ⇨ $d^n/dt^n(f(1/t) = P_n(t)e^{-t}$

Differentiating w.r.t. t we get,

$d^{n+1}/dt^{n+1}(f(1/t))(-1/t^2) = d/dt(P_n(t))e^{-t} + P_n(t)e^{-t}(-1)$

- ⇨ $P_{n+1}(t)e^{-t}(-1/t^2) = d/dt(P_n(t))e^{-t} - P_n(t)e^{-t}$ (from (1))
- ⇨ $P_{n+1}(t) = t^t[P_n(t) - d/dt(P_n(t))]$
- ⇨ $P_{n+1}(x) = x^2[P_n(x) - d/dx(P_n(x)]$

Proved.

106. By considering the expression $(1 + \sqrt{2})^n + (1 - \sqrt{2})^n$, where n is positive integer, show that the integers $[(1 + \sqrt{2})^n]$ are alternatively even and odd as n takes values 1, 2, …. Here for any real number x, [x] denotes the greatest integer less than or equal to x.

Solution :

Let $E = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n$

Let $(1 + \sqrt{2})^n = I + f$ where I is integer part and $0 \le f < 1$ is fractional part

So, $I = [(1 + \sqrt{2})^n]$

Also, $\sqrt{2} - 1 = 1/(\sqrt{2} + 1)$ i.e. $0 < \sqrt{2} - 1 < 1$

- ⇨ $0 < (\sqrt{2} - 1)^n < 1$

Let, $F = (\sqrt{2} - 1)^n$

- ⇨ $0 < F < 1$

Case 1 : (n odd)

$I + f - F = (\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n = 2[{}^nC_0 + {}^nC_2(\sqrt{2})^2 + ...]$ = even = 2k (say)

$f - F = 2k - I$ = an integer.

Since $f - F$ is an integer, $f - F = 0$

- ⇨ $f = F$
- ⇨ Therefore, $I - 2k = 0$
- ⇨ $I = 2k = [(1 + \sqrt{2})^n]$ = even when n is odd integer.

Case 2 : n is even

$I + f + F = (\sqrt{2} + 1)^n + (\sqrt{2} - 1)^n = 2k_1$ = even integer

- ⇨ $f + F = 2k_1 - I$ = integer

Sin $0 < f < 1$ and $0 < F < 1$

- ⇨ $0 < f + F < 2$
- ⇨ $f + F = 1$

Therefore, $2k_1 - I = 1$

- ⇨ $I = 2k_1 + 1$
- ⇨ Therefore, $[(1 + \sqrt{2})^n]$ is an odd integer.

Proved.

107. A partition of a set S is formed by disjoint, nonempty subsets of S whose union is S. For example, { {1, 3, 5}, {2}, {4, 6} } is a partition of the set T = {1, 2, 3, 4, 5, 6} consisting of subsets {1, 3, 5}, {2} and {4, 6}. However, { {1, 2, 3, 5}, {3, 4, 6} } is not a partition of T.
If there are k nonempty subsets in a partition, then it is called a partition into k classes. Let S_k^n stand for the number of different partitions of a set with n elements into k classes.

(i) Find S_2^n.
(ii) Show that $S_k^{n+1} = S_{k-1}^n + kS_k^n$.

Solution :

We can choose 1 element from n elements in $^nC_1$ ways.

We can choose 2 elements from n elements in $^nC_2$ ways.

…

…

We can choose n – 1 elements from n elements in $^nC_{n-1}$ ways.

We are choosing for one subset and as it is nonempty so $^nC_0$ and $^nC_{n-1}$ will not come into picture.

So, total number of ways = $^nC_1 + {}^nC_2 + …. + {}^nC_{n-1} = 2^n – 2$.

Therefore, $S_2^n = 2^n – 2$.

Now, S_k^{n+1} : To get this partition either putting $(n + 1)^{th}$ element in one of the existing k classes with n elements or we can form a new class and appending that class to existing k – 1 classes. For the first case, we can choose 1-class out of the k classes in $^kC_1 = k$ ways and number of partition for class is S_k^n ways. So total number of partition kS_k^n. For the second case we can partition n elements into (k – 1) classes in S_{k-1}^n ways and form a new class (k^{th} class) with $(n + 1)^{th}$ element.

Therefore, $S_k^{n+1} = S_{k-1}^n + kS_k^n$.

Proved.

108. In a certain game, 30 balls of k different colors are kept inside a sealed box. You are told only the value of k, but not the number of balls of each color. Based on this, you have to guess whether it is possible to split the balls into 10 groups of 3 each, such that in each group the three balls are of different colors. Your answer is to be a simple YES or NO. You win or lose a point according as your guess is correct or not. For what values of k, you can say YES and be sure of winning? Justify your answer.

Solution :

If $k \le 2$, then "NO" => Sure winning. If $k = 1$ then it is obvious. If $k = 2$ any of the two kinds of ball must be more than 10 in numbers. Hence at least one of the 10 groups of 3 balls must contain 2 similar type of balls.

Whereas, if $k = 3$ then if each type of balls are 10 in numbers then "NO" may not be sure winning.

If $k \ge 21$, then "YES" => Sure winning.

Because let if $k = 21$, if a single type of ball is 11 in numbers resulting in duplicate type of ball in any group, total ball is $(30 - 11) = 19$ and remaining types and $(21 - 1) = 20$ and it is not possible.

Let $k = 20$, we can have 11 balls of single kind and 19 balls of different kinds => not sure winning.

109. If c is a real number with $0 < c < 1$, then show that the values taken by the function $y = (x^2 + 2x + c)/(x^2 + 4x + 3c)$, as x varies over real numbers, range over all real numbers.

Solution :

$f(x) = (x^2 + 2x + c)/(x^2 + 4x + 3c)$

- ⇨ $f'(x) = 2(x^2 + 2cx + c)/(x^2 + 4x + 3c)^2$ (After simplification)

Now, discriminant of $x^2 + 2cx + c$ is, $4c^2 - 4c = 4c(c - 1) < 0$ as $0 < c < 1$ and co-efficient of $x^2 > 0$.

- ⇨ $x^2 + 2cx + c > 0$ for all x.
- ⇨ $f'(x) > 0$
- ⇨ $f(x)$ is strictly increasing and $f(x)$ being ratio of two polynomials is continuous.

Hence, $f(x)$ varies over all real numbers as x varies over all real numbers.

Proved.

110. Find the set of all values of m such that $y = (x^2 - x)/(1 - mx)$ can take all real values.

Solution :

$y = (x^2 - x)/(1 - mx)$

- ⇨ $x^2 - x(1 - my) - y = 0$

Since x is real so discriminant of the quadratic equation is non-negative.

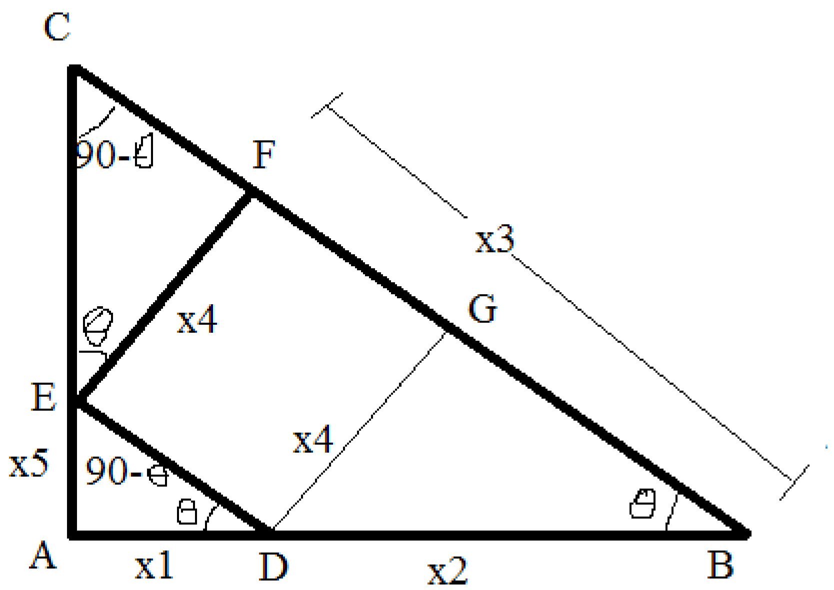
- ⇨ $(1 - my)^2 - 4y \geq =$
- ⇨ $m^2y^2 - 2y(m + 2) + 1 \geq 0$

Since y is real and $m^2 > 0$ so for the equation to hold discriminant < 0

- ⇨ $4(m + 2)^2 - 4m^2 < 0$
- ⇨ $(m + 2 + m)*2 < 0$
- ⇨ $m < -1$

111. Let ABC be any triangle, right-angles at A, with D any point on the side AB. The line DE is drawn parallel to BC to meet the side AC at the point E. F is the foot of the perpendicular drawn from E to BC. If AD = x_1, DB = x_2, BF = x_3, EF = x_4 and AE = x_5, then show that $x_1/x_5 + x_2/x_5 = (x_1x_3 + x_4x_5)/(x_1x_3 - x_1x_4)$.

Solution :



Here it is easily seen that triangle AED and triangle CEF are similar.

Therefore, AE/CF = AD/EF

- ⇨ CF = x_4x_5/x_1

Draw a perpendicular from D to BC. Let it meets BC at G. Since DG perpendicular to AC, DG = EF = x_4

Again triangle AED and triangle PDG are similar.

Therefore, $x_4/x_5 = BG/x_1$

⇨ $BG = x_1x_4/x_5$

Therefore, $GF = ED = x_3 - x_1x_4/x_5$

Again triangle AED and triangle ABC are similar.

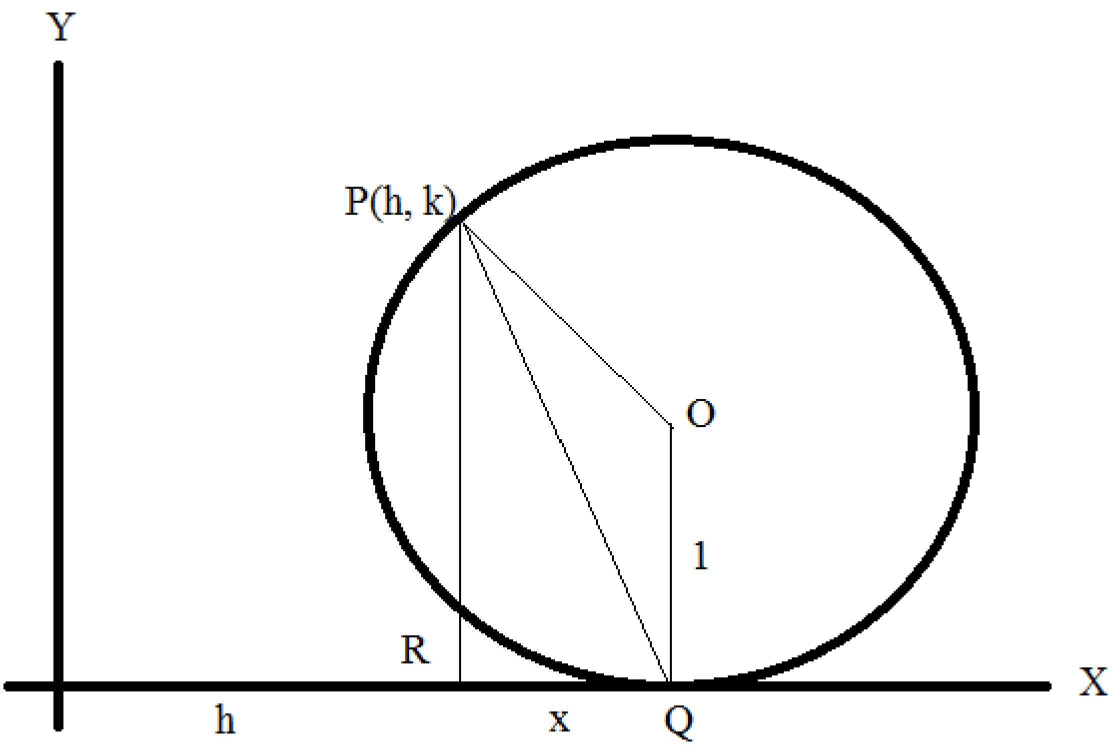
Therefore, $x_1/(x_1 + x_2) = ED/BC = (x_3x_5 - x_1x_4)/x_5(x_3 + CF)$

⇨ $x_1/(x_1 + x_2) = (x_3x_5 - x_1x_4)/x_5(x_3 + x_4x_5/x_1)$
⇨ $x_1/x_5 + x_2/x_5 = (x_1x_3 + x_4x_5)/(x_3x_5 - x_1x_4)$

Proved.

112. Consider the circle of radius 1 with its centre at the point (0, 1). From this initial position, the circle is rolled along the positive x-axis without slipping. Find the locus of the point P in the circumference of the circle which is on the origin at the initial position of the circle.

Solution :



Let at some point of time the point P(0, 0) which was initially at the circumference of the circle moved to point P(h, k).

Let at that point O is the centre of the circle. Draw perpendicular to O to x-axis and P to X-axis and let they meet a t Q and R respectively.

Now, $OP^2 = 1$

- ⇨ $(h + x - h)^2 + (k - 1)^2 = 1$ since co-ordinate of O = (h + x, 1)
- ⇨ $x^2 + k^2 = 2k$ …… (1)

From triangle OPQ, we get, $PQ^2 = 1^2 + 1^2 - 2\cos(POQ) = 2(1 - \cos\theta)$ (angle POQ = θ) (say)

- ⇨ $x^2 + k^2 = 2(1 - \cos\theta)$
- ⇨ $2k = 2(1 - \cos\theta)$
- ⇨ $k = 1 - \cos\theta$
- ⇨ $\cos\theta = 1 - k$ and $\theta = \cos^{-1}(1 - k)$

Also, arc PQ = $h + x = 1*\theta$

- ⇨ $x = \theta - h$

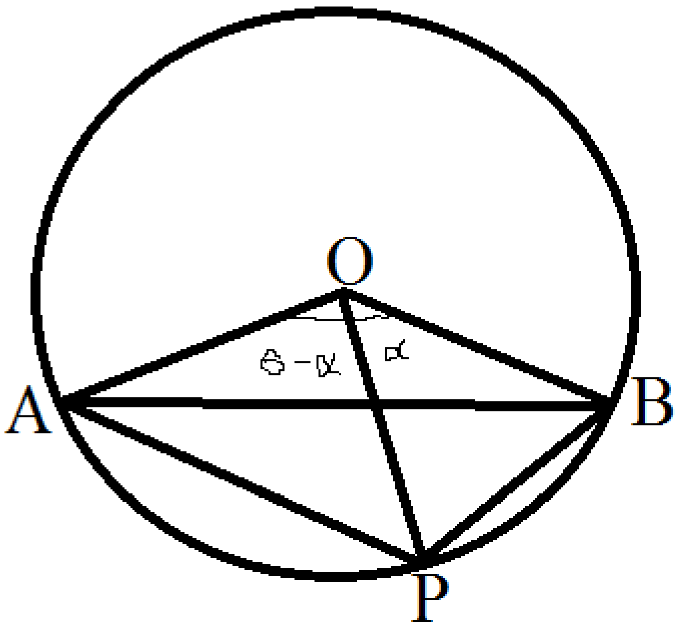
From (1) we get, $x = \pm\sqrt{(2k - k^2)}$

- ⇨ $\theta - h = \pm\sqrt{(2k - k^2)}$
- ⇨ $\cos^{-1}(1 - k) - h = \pm\sqrt{(2k - k^2)}$
- ⇨ $\{\cos^{-1}(1 - k) - h\}^2 = 2k - k^2$

Therefore, locus is, $\{\cos^{-1}(1 - y) - x\}^2 = 2y - y^2$

113. AB is a chord of a circle C.
    (a) Find a point P on the circumference of C such that PA*PB is the maximum.
    (b) Find a point P on the circumference of C which maximizes PA + PB.

Solution :



Let O be the centre of the circle and AB be the chord. Join OA, OB, OP. Let radius is r. Let Angle AOB = θ.

$AB^2 = 2r^2(1 - \cos\theta) = 4r^2\sin^2(\theta/2)$

- ⇨ $AB = 2r\sin(\theta/2)$

Similarly, $PB = 2r\sin(\alpha/2)$ where angle POB = α and $PA = 2r\sin\{(\theta - \alpha)/2\}$ where angle AOP = θ − α

$PA*PB = 4r^2\sin(\alpha/2)\sin\{(\theta - \alpha)/2\} = f(\alpha)$ (say)

- ⇨ $f'(\alpha) = 4r^2[(1/2)\cos(\alpha/2)\sin\{(\theta - \alpha)/2\} - (1/2)\sin(\alpha/2)\cos\{(\theta - \alpha)/2\}] = 2r^2\sin\{(\theta - 2\alpha)/2\}$

So, necessary condition for f(α) has extreme value is f'(α) = 0

- ⇨ $\sin\{(\theta - 2\alpha)/2\} = 0$
- ⇨ $\alpha = \theta/2$

Now, $f''(\alpha) = -2r^2\cos\{(\theta - 2\alpha)/2\} < 0$

So, at α = θ/2, f(α) is maximum. i.e. P lies on the middle point of the arc APB.

$PA + PB = 2r[\sin(\alpha/2) + \sin\{(\theta - \alpha)/2\}] = 4r\sin(\theta/4)\cos\{\alpha/2 - \theta/4\}$

Now, PA + PB will be maximum when $\cos(\alpha/2 - \theta/4) = 1$ i.e.. α = θ/2 i.e. P lies in the middle point of the arc APB.

114. Let f(x) be a real valued function of a variable x such that f′(x) takes both positive and negative values and f′′(x) > 0 for all x. Show that there is a real number p such that f(x) is an increasing function of x for all x ≥ p.

Solution :

Given f′′(x) > 0 for all x. i.e. graph of f(x) lies above any of it's tangent line.

Also, f′(x) < 0 for some x and f′(0) > 0 for some x.

Let g(x) = f′(x)

- ⇨ g′(x) = f′′(x) > 0 for all x.
- ⇨ g(x) is increasing.

Since g(x) can take positive and negative values, so g(x) can have only one root i.e. f′(x) = 0 for only one x

Let f′(p) = 0

So, after p f′(p) > 0 because f′′(x) > 0.

115. Let a_0 and b_0 be any two positive integers. Define a_n, b_n for n ≥ 1 using the relations $a_n = a_{n-1} + 2b_{n-1}$, $b_n = a_{n-1} + b_{n-1}$ and let $c_n = a_n/b_n$, for n = 0, 1, 2, …..

(a) Write $(\sqrt{2} - c_{n+1})$ in terms of $(\sqrt{2} - c_n)$.

(b) Show that $|\sqrt{2} - c_{n+1}| < \{1/(1 + \sqrt{2})\}|\sqrt{2} - c_n|$

(c) Show that lim c_n as n -> ∞ = √2.

Solution :

a) $(\sqrt{2} - c_{n+1}) = (1 - c_{n+1})(\sqrt{2} - c_n)/(\sqrt{2} + 1)$

b) Since $a_0 > 0$, $b_0 > 0$

⇨ a_n, $b_n > 0$ for all n.

Also, $a_n = a_{n-1} + 2b_{n-1}$

- ⇨ $a_n = a_{n-1} + 2(b_n - a_{n-1})$
- ⇨ $a_n - 2b_n = -a_{n-1}$
- ⇨ $c_n - 2 = -a_{n-1}/b_n < 0$
- ⇨ $c_n < 2$ for all n

So, $0 < c_n < 2$ for all n.

Now, $(\sqrt{2} - c_{n+1}) = (1 - c_{n+1})(\sqrt{2} - c_n)/(\sqrt{2} + 1)$

- ⇨ $|\sqrt{2} - c_{n+1}| = |(1 - c_{n+1})(\sqrt{2} - c_n)/(\sqrt{2} + 1)|$
- ⇨ $|\sqrt{2} - c_{n+1}| < |(\sqrt{2} - c_n)/(\sqrt{2} + 1)|$

c) From (b) $|\sqrt{2} - c_{n+1}| < |(\sqrt{2} - c_n)/(\sqrt{2} + 1)|$
$< \{1/(1 + \sqrt{2})^2\}|\sqrt{2} - c_{n-1}|$
..
..
$< \{1/(1 + \sqrt{2})^{n+1}\}|\sqrt{2} - c_0|$
When $n \to \infty$, RHS $\to 0$
Therefore, $\lim |\sqrt{2} - c_{n+1}|$ as $n \to \infty \le 0$ but since $|\sqrt{2} - c_{n+1}| \ge 0$

- ⇨ $\lim |\sqrt{2} - c_{n+1}|$ as $n \to \infty = 0$
- ⇨ $\lim c_n$ as $n \to \infty = \sqrt{2}$.

Proved.

116. Find the maximum and minimum values of the function $f(x) = x^2 - x\sin x$, in the closed interval $[0, \pi/2]$.

Solution :

$f(x) = x^2 - x\sin x$, $x \in [0, \pi/2]$

Now, $f(x)$ is increasing in $[0, \pi/2]$ because $f(x) = x(x - \sin x)$ and $\sin x \le x$ for all $x \in [0, \pi/2]$

Since $f(x)$ is an increasing function minimum value of $f(x)$ is at $x = 0$ i.e. $f_{min}(x) = f(0) = 0$

Maximum value of $f(x)$ is at $x = \pi/2$.

Therefore, $f_{max}(x) = f(\pi/2) = (\pi/2)^2 - \pi/2 = (\pi/2)(\pi/2 - 1)$

117. Show that $\int|\sin x/x|dx \ge (2/\pi)(1 + ½ + ….. + 1/n)$.

Solution :

Let $I = \int\{|\sin nx|/x\}dx$ (integration running from 0 to π) (as $x > 0$, so $|x| = x$)

Put $y = nx$

- ⇨ $dy = ndx$

Therefore, $I = \int\{|\sin y|/y\}dy$ (integration running from 0 to $n\pi$)

= ∑∫{|siny|/y|dy (integration running from (m – 1)π to mπ) (summation running from m = 1 to m = n)

Now, ∫{|siny|/y|}dy (integration running from (m – 1)π to mπ) ≥ (1/mπ)∫|siny|dy (integration running from (m – 1)π to mπ) (Since y ≤ mπ => 1/y ≥ 1/mπ), therefore, |sint|/y ≥ |siny|/mπ)

Therefore, I_m = ∫{|siny|/y|}dy (integration running from (m – 1)π to mπ) ≥ (1/mπ)∫|siny|dy (integration running from (m – 1)π to mπ)

As we know, ∫f(x)dx (integration running from aT to bT) = (b – a)∫f(x)dx (integration running from 0 to T) if period of f(x) is T and we know that period of |siny| is π.

Therefore, I_m ≥ (1/mπ)∫sinydy (integration running from 0 to π) = 2/mπ

So, I ≥ 2/π + 2/2π + ….. + 2/nπ = (2/π)(1 + ½ + ….. + 1/n)

Proved.

118. Show that 2(√251 – 1) < ∑(1/√k) (summation running from k = 1 to k = 250) < 2√250.

Solution :

First we show that 2(√(m + 1) - √m) < 1/√m < 2(√m - √(m – 1)

Now, 2(√(m + 1) - √m) = 2(√(m + 1) - √m)(√(m + 1) + √m)/(√(m + 1) + √m) = 2/(√(m + 1) + √m) < 2/2√m = 1/√m (because √(m + 1) > √m)

Now, 2(√(m - √(m – 1)) = 2/(√m + √(m – 1)) > 2/(√m + √m) = 1/√m

So, 2(√(m + 1) - √m) < 1/√m < 2(√m - √(m – 1)

Now, put m = 1, 2, …., 250 and adding the inequalities we get the required inequality.

Proved.

119. Using the identity logx = ∫dt/t (integration running from 1 to x) x > 0, or otherwise, prove that 1/(n + 1) ≤ log(1 + 1/n) ≤ 1/n. for all integers n ≥ 1.

Solution :

Given logx = ∫dt/t (integration running from 1 to x) x > 0

Therefore, log(1 + 1/n) = ∫dt/t (integration running from 1 to 1 + 1/n)

Now, t ≤ 1 + 1/n

⇨ 1/t ≥ n/(n + 1)

Therefore, ∫dt/t (integration running from 1 to 1 +1/n) ≥ {n/(n + 1)}∫dt (integration running from 1 to 1 + 1/n)

⇨ log(1 + 1/n) ≥ 1/(n + 1)

Again, t ≥ 1

⇨ 1/t ≤ 1

Therefore, ∫dt/t (integration running from 1 to 1 + 1/n) ≤ ∫dt (integration running from 1 to 1 + 1/n) = 1/n

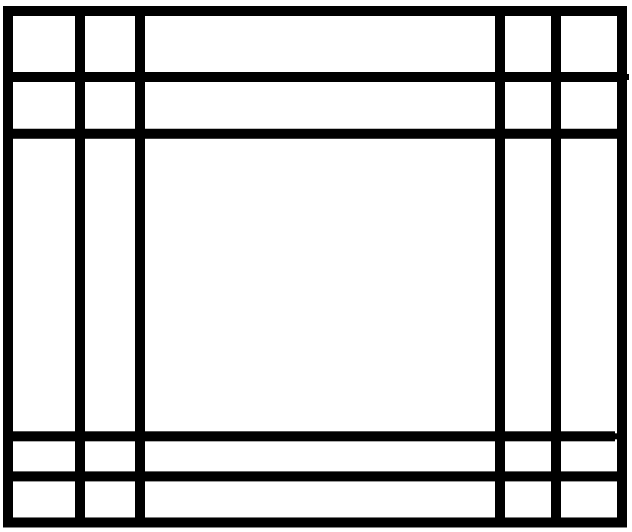
⇨ log(1 + 1/n) ≤ 1/n

So, 1/(n + 1) ≤ log(1 + 1/n) ≤ 1/n

Proved.

120. Consider the set of points S = {(x, y) : x, y are non-negative integers ≤ n}. Find the number of squares that can be formed with vertices belonging to S and sides parallel to the axes.

Solution :



According to the problem the squares can be looked like the picture given where the vertices of the squares belong to S = {(x, y) : x, y non-negative integers ≤ n}.

The number of square with length i unit is (n – i + 1), i = 1, 2, …., n – 1

Total number of squares = $\sum(n - i + 1)^2$ (summation running from 1 to n) = $n^2 + (n-1)^2 + \ldots + 2^2 + 1^2 = n(n+1)(2n+1)/6$.

121. Consider the following simultaneous equations in x and y : x + y + axy = a, $x - 2y - xy^2 = 0$ where a is a real constant. Show that these equations admit real solutions in x and y.

Solution :

Now, $x - 2y - xy^2 = 0$

⇨ $x = 2y/(1 - y^2)$ (x takes real values when $y^2 \neq 1$ i.e. $y \neq \pm 1$)

Putting this value in the second equation we get, $2y/(1 - y^2) + y + ay*2y/(1 - y^2) = a$

⇨ $y^3 - 3ay^2 - 3y + a = 0$

As a is real, so all coefficients of the equation are real.

Hence complex roots will come in pair i.e. if z is a root then z′ (conjugate) is also a root.

And the equation is 3 degree (i.e. odd)

So, there is at least one real root.

So, y attains real root.

If y = ±1 then a = ±1.

So, the equation attains real values of x excluding y = ±1 if a ≠ ±1.

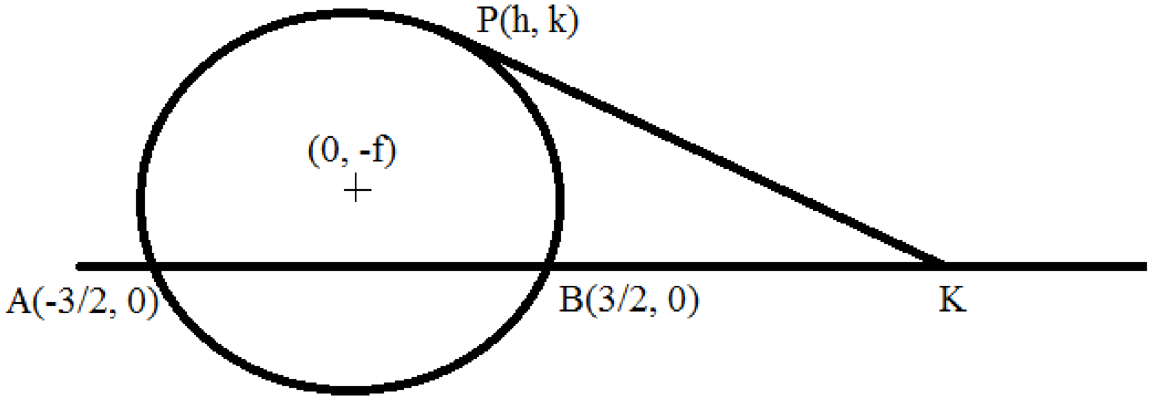
⇨ x also takes real values.

So, the equation attains real solution in x and y except a = ±1.

122. Let A and B be two fixed points 3 cm apart.

Let P be any point not collinear with A and B, such that PA = 2PB. The tangent at P to the circle passing through the points P, A and B meets the extended line AB at the point K. Find the lengths of the segments KB and KP.

Solution :



Let the equation of the circle passing through A, B and P is $x^2 + y^2 + 2gx + 2fy + c = 0$

Now, this circle passes through A(-3/2, 0) and B(3/2, 0)

So, $(-3/2)^2 - 3g + c = 0$ and $(3/2)^2 + 3g + c = 0$

Solving for g and c we get, $g = 0$ and $c = -9/4$

Putting back the values in the circle we get, $x^2 + y^2 + 2fy - 9/4 = 0$

Now, this circle passes through P(h, k)

So, we have, $h^2 + k^2 + 2fk - 9/4 = 0$ ………………… (1)

Also we have, PA = 2PB

- ⇨ $PA^2 = 4PB^2$
- ⇨ $(h + 3/2)^2 + k^2 = 4\{(h - 3/2)^2 + k^2\}$
- ⇨ $h^2 + k^2 - 5h + 9/4 = 0$ ……………….. (2)

Now, slope of the line joining centre and P is $(k + f)/h$

So, slope of PK is $-h/(k + f)$

Equation of KP is, $y - k = \{-h/(k + f)\}(x - h)$

- ⇨ $xh + (k + f)y = h^2 + k^2 + kf$

Putting y = 0 we get, abscissa of K, $x = (h^2 + k^2 + kf)/h$

Co-ordinate of K is $\{(h^2 + k^2 + kf)/h, 0\}$

Now, $(h^2 + k^2 + kf)/h = [h^2 + k^2 + (1/2)\{9/4 - (h^2 + k^2)\}]/h$ (from (1))

$= \{(1/2)(h^2 + k^2) + 9/8\}/h$

$= \{(1/2)(5h - 9/4) + 9/8\}/h$ (from (2))

$= 5h/2h = 5/2$

So, co-ordinate of K is (5/2, 0)

KB = 5/2 – 3/2 = 1.

$KP^2 = (5/2 - h)^2 + k^2$

$= 25/4 - 5h + h^2 + k^2$

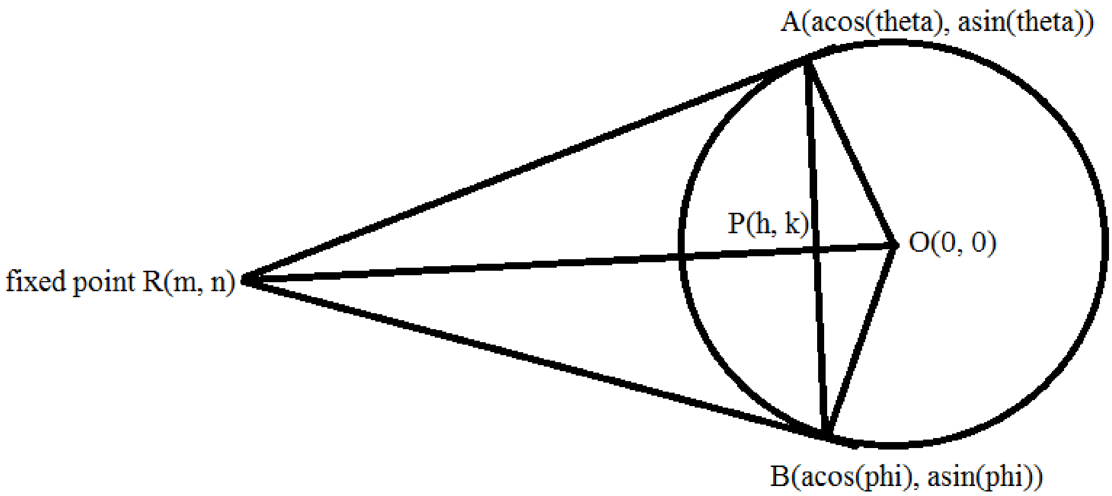
$= 25/4 - 9/4$ (from (2))

$= 4$

⇨ KP = 2.

123. Tangents are drawn to a given circle from a point on a given straight line, which doesn't meet the given circle. Prove that the locus of the mid-point of the chord joining the two points of contact of the tangents with the circle is a circle.

Solution :



We are required to find the locus of P.

Let the radius of the circle is a (fixed).

Now, $h = a(\cos\theta + \cos\Phi)/2 = a\cos\{(\Phi + \theta)/2\}\cos\{(\Phi - \theta)/2\}$

And, $k = a(\sin\Phi + \sin\theta)/2 = a\sin\{(\Phi + \theta)/2\}\cos\{(\Phi - \theta)/2\}$

Now, $h^2 + k^2 = a^2\cos^2\{(\Phi - \theta)/2\}$

Now, angle AOP = $(\Phi - \theta)/2$.

From triangle AOR we get, $\cos\{(\Phi - \theta)/2\} = AO/OR = a/OR$

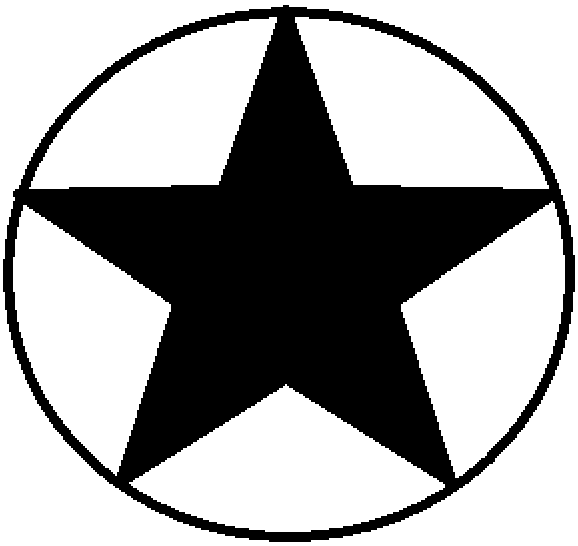
Now, OR is fixed as the point (m, n) is fixed.

Therefore, we get, $h^2 + k^2$ = constant

- ⇨ The locus is $x^2 + y^2$ = constant
- ⇨ The locus is a circle.

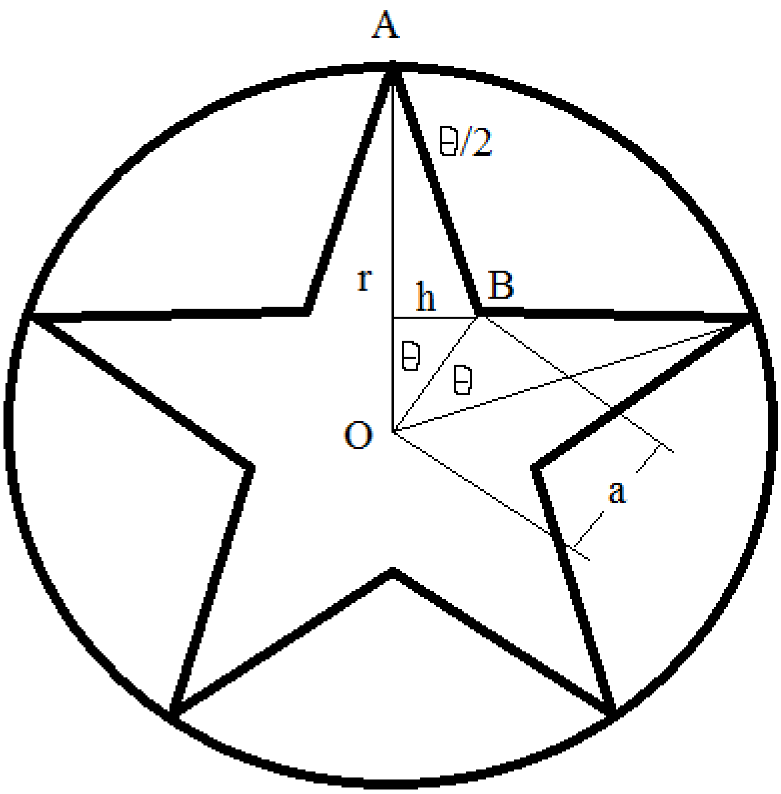
Proved.

124.



A regular five-pointed star is inscribed in a circle if radius r. (See the figure) Show that the area if the region inside the star us $10r^2\tan(\pi/10)/\{3 - \tan^2(\pi/10)\}$.

Solution :



Let, angle AOB = θ, angle OAB = $\theta/2$, angle ABO = $\pi - 3\theta/2$

Now, $2\theta = 2\pi/5$

- ⇨ $\theta = \pi/5$
- ⇨ Angle ABO = $\pi - 3\pi/10 = 7\pi/10$

From triangle, AOB we get, $r/\sin(7\pi/10) = a/\sin(\pi/10)$

- ⇨ $a = r\sin(\pi/10)/\sin(3\pi/10)$
- ⇨ $a = r\sin(\pi/10)/\{3\sin(\pi/10) - 4\sin^3(\pi/10)\}$
- ⇨ $a = r/\{3 - 4\sin^2(\pi/10)\}$
- ⇨ $h = a\sin\theta = a\sin(\pi/5) = 2\sin(\pi/10)\cos(\pi/10)*r/\{3 - 4\sin^2(\pi/10)\}$
- ⇨ $h = 2r\tan(\pi/10)/\{3\sec^2(\pi/10) - 4\tan^2(\pi/10)\}$ (Dividing numerator and denominator by $\cos^2(\pi/10)$)
- ⇨ $h = 2r\tan(\pi/10)/\{3 - \tan^2(\pi/10)\}$

Area of triangle AOB = $(1/2)*r*h = (1/2)*r*2r\tan(\pi/10)/\{3 - \tan^2(\pi/10)\} = r^2\tan(\pi/10)/\{3 - \tan^2(\pi/10)\}$

Area of the star = 10*area of triangle AOB = $10r^2\tan(\pi/10)/\{3 - \tan^2(\pi/10)\}$

Proved.

125. Consider the circle C whose equation is $(x - 2)^2 + (y - 8)^2 = 1$ and the parabola with the equation $y^2 = 4x$. Find the minimum value of the length of the segment AB where A moves on the circle C and B moves on the parabola P.

Solution :

The point on the circumference of the circle, the point on the parabola and the centre of the circle will be collinear in case of minimum distance between the circle and the parabola.

So, we will calculate minimum distance of the centre of the circle from the parabola and then we will subtract the radius to get the required result.

Any point on the parabola = $(t^2, 2t)$ and the centre of the circle = (2, 8)

$D^2 = (t^2 - 2)^2 + (2t - 8)^2$

$dD^2/dt = 2(t^2 - 2)*2t + 2(2t - 8)*2 = 0$

- ⇨ $t^3 - 2t + 2t - 8 = 0$
- ⇨ $t^3 = 8$
- ⇨ $t = 2$

Now, $d^2D^2/dt^2 = 3t^2 > 0$ at $t = 2$

So, D^2 has minimum value at $t = 2$.

Therefore, $D_{min}{}^2 = (2^2 - 2)^2 + (2*2 - 8)^2 = 20$

- ⇨ $D_{min} = 2\sqrt{5}$

Therefore required distance = $(2\sqrt{5} - 1)$ (As radius = 1)

126. Consider the squares of an 8x8 chessboard filled with the numbers 1 to 64 as in the figure below. If we choose 8 squares with the property that there is exactly one from each row and exactly one from each column, and add up the numbers in the chosen squares, show that the sum obtained is always 260.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Solution :

We can write the numbers in first row as 0*8 + 1, 0*8 + 2, …., 0*8 + 8

We can write the numbers in the second row as 1*8 + 1, 1*8 + 2, ….., 1*8 + 8

Similarly, for third rows we can write, 2*8 + 1, 2*8 + 2, …., 2*8 + 8

Similarly, for other rows.

So, if we choose 8 numbers from different rows, then the numbers will be $0*8 + a_1 + 1*8 + a_2$, …., $7*8 + a_8$ where $a_1, a_2, \dots, a_8$ all less than or equal to 8 and distinct as they are from different columns.

So, we have, $a_1 + a_2 + \dots + a_8 = 1 + 2 + \dots + 8 = 8*9/2 = 36$

Now, the sum of the 8 chosen numbers = $0*8 + a_1 + 1*8 + a_2 + \dots + 7*8 + a_8$

$= 8(1 + 2 + \dots + 7) + (a_1 + a_2 + \dots + a_8)$

= 8*7*8/2 + 36

= 224 + 36

= 260

Proved.

127. Let l, b be positive integers. Divide lxb rectangle into lb unit squares in the usual manner. Consider one of the two diagonals of this rectangle. How many of these unit squares contain a segment of positive length of this diagonal?

Solution :

Now, to cross b columns it must have gone through b unit squares.

Now, to go l rows down, it has to go through l – 1 number of unit squares as to go from 1st row to second row it has to go to one additional unit square, to go from second row to third row it has to go to one additional unit square.

So, number of unit squares contains a segment of positive length of this diagonal = b + l – 1.

Now, the bottom right corner has coordinate (b – 1, 0) and top left corner have coordinate (0, l – 1)

The equation of the diagonal is, x/(b – 1) + y/(l – 1) = 1.

Note that, this straight line passes through {(b – 1)/d, (l – 1) – (l – 1)/d}.

Now this point will be integer if d divides both (l – 1) and (d – 1).

Now also note that if {(b – 1)/d, (l – 1) – (l – 1)/d} is a point then {(b – 1) – (b – 1)/d, (l – 1)/d} is also a point through which the straight line passes.

So, twice the number of common divisors of (b – 1) and (l – 1) will be subtracted from b + l – 1 as there will be no additional crossing of any column unit squares to go from one row to another.

So, total number of unit squares = l + b – 1 – 2m where m is number of common divisors of l – 1 and b – 1.

Now, again note that if d = 2 then (b – 1)/d = (b – 1) – (b – 1)/d. So, two points will be counted as one.

So, if (b – 1) and (l – 1) are both even i.e. if they have a common factor 2 then total number of unit squares = l + b – 1 – 2m + 1 = l + b – 2m.

ISI B Stat B Math 2009-2014 year-wise Objective Solution

B. Stat. (Hons.) Admission Test : 2009

Group A (Each of the following questions has exactly one correct option and you have to identify it)

1. If k times the sum of the first n natural numbers is equal to the sum of the squares of the first n natural numbers, then $\cos^{-1}\{(2n - 3k)/2\}$ is
   (a) $5\pi/6$
   (b) $2\pi/3$
   (c) $\pi/3$
   (d) $\pi/6$

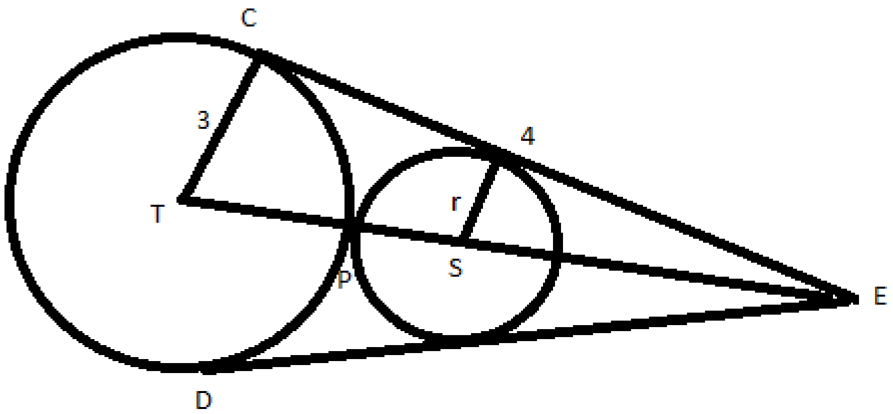
Solution :

We have $kn(n + 1)/2 = n(n + 1)(2n + 1)/6$

- ⇨ $3k = 2n + 1$
- ⇨ $(2n - 3k)/2 = -1/2$
- ⇨ $\cos^{-1}\{(2n - 3k)/2\} = \cos^{-1}(-1/2) = 2\pi/3$.
- ⇨ Option (b) is correct.

2. Two circles touch each other at P. The two common tangents to the circles, none of which passes through P, meet at E. They touch the larger circle at C and D. The larger circle has radius 3 units and CE has length 4 units. Then the radius of the smaller circle is
   (a) 1
   (b) 5/7
   (c) ¾
   (d) ½.

Solution :



Clearly, TE = $\sqrt{(3^2 + 4^2)} = 5$

TS = 3 + r

- ⇨ SE = 5 – 3 – r = 2 – r

Now, r/3 = SE/TE (triangles are similar)

- ⇨ r/3 = (2 – r)/5
- ⇨ 5r = 6 – 3r
- ⇨ r = 6/8 = ¾
- ⇨ Option (c) is correct.

3. Suppose ABCDEFGHIJ is a ten digit number, where the digits are all distinct. Moreover, A > B > C satisfy A + B + C = 9, D > E > F are consecutive even digits and G > H > I > J are consecutive odd digits. Then A is
   (a) 8
   (b) 7
   (c)6
   (d) 5.

Solution :

Now, G > H > I > J are consecutive odd digits.

- ⇨ Either (G, H, I, J) = (7, 5, 3, 1) or (9, 7, 5, 3) i.e. 3, 5, 7 must be there.

Now, A + B + C = 9.

- ⇨ All of A, B, C cannot be even because RHS is 9 i.e. odd.

Now, odd numbers that are available is 1 or 9.

Now, A or B or C cannot be 9 otherwise sum of the rest two will be 0 and all are distinct.

- ⇨ A or B or C is 1.

Now, A cannot be 1 as there is no option 1.

Let C = 1.

- ⇨ A + B = 8

Now, A cannot be 5 or 7 because all the odd digits are occupied.

- ⇨ Option (b) and (d) cannot be true.

Let A = 6, then B = 2

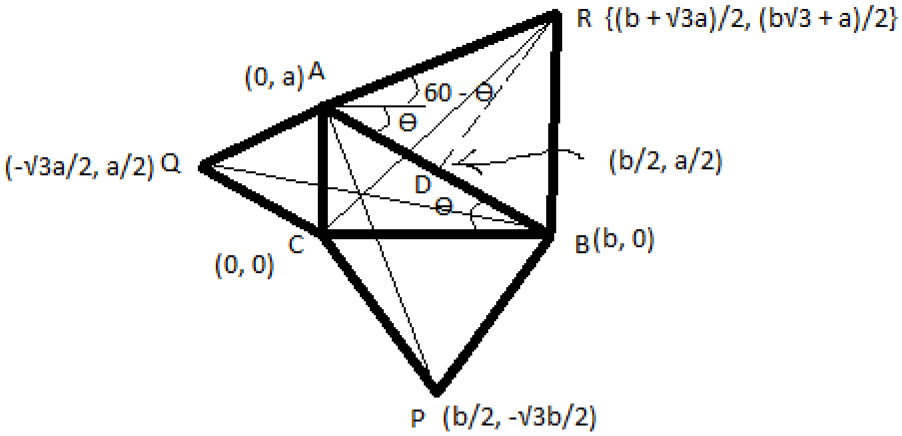
Now, D > E > F are consecutive even digits.

So, the above case cannot hold true.

- ⇨ Option (c) cannot be true.
- ⇨ Option (a) is correct.

4. Let ABC be a right angled triangle with AB > BC > CA. Construct three equilateral triangle BCP, CQA and ARB, so that A and P are on opposite sides of BC; B and Q are on opposite sides of CA; C and R are on opposite sides of AB. Then
   (a) CR > AP > BQ
   (b) CR < AP < BQ
   (c) CR = AP = BQ
   (d) $CR^2 = AP^2 + BQ^2$.

Solution :



We have defined the co-ordinate system as shown in figure.

Now, BCP is an equilateral triangle. So, it's median and perpendicular from P to BC is same. Thus we get co-ordinate of P $(b/2, -\sqrt{3}b/2)$

Now, $AP^2 = (b/2 - 0)^2 + (-\sqrt{3}b/2 - a)^2 = a^2 + b^2 + \sqrt{3}ab$

Similarly, we get, $BQ^2 = (b + \sqrt{3}a/2)^2 + (0 - a/2)^2 = a^2 + b^2 + \sqrt{3}ab$

Let, angle CBA = Θ

- ⇨ Angle BAS = Θ (S is any point on the line passing through A and parallel to x-axis)
- ⇨ Angle RAS = $60° - \Theta$ (As angle RAB = 60°)
- ⇨ The slope of the line AR = $\tan(60° - \Theta) = (\tan 60° - \tan\Theta)/(1 + \tan 60° \tan\Theta)$

Now, $\tan\Theta = a/b$ (From triangle ABC)

- ⇨ The slope of AR = $(a/b - \sqrt{3})/(1 + \sqrt{3}b/a)$
- ⇨ Equation of AR is, $y - a = (a/b - \sqrt{3})/(1 + \sqrt{3}b/a)x$

Now, perpendicular from R to AB is the median also as ARB is equilateral.

- ⇨ Co-ordinate of D = $(b/2, a/2)$

Slope of AB = $-a/b$

- ⇨ Slope of DR = a/b (As DR and AB are perpendicular to each other)
- ⇨ Equation of DR is, $y - a/2 = (a/b)(x - b/2)$

Solving equation of AR and DR we get, co-ordinate of R $\{(b + \sqrt{3}a)/2, (b\sqrt{3} + a)/2\}$

- ⇨ $CR^2 = \{(b + \sqrt{3}a)/2\}^2 + \{(b\sqrt{3} + a)/2\}^2 = a^2 + b^2 + \sqrt{3}ab$
- ⇨ CR = AP = BQ

⇨ Option (c) is correct.

5. The value of (1 + tan1°)(1 + tan2°)....(1 + tan44°) is
   (a) 2
   (b) A multiple of 22
   (c) Not an integer
   (d) A multiple of 4.

Solution :

Now, tan(45° - ϴ) = (tan45° - tanϴ)/(1 + tan45°tanϴ) = (1 - tanϴ)/(1 + tanϴ)

⇨ 1 + tan(45° - ϴ) = 2/(1 + tanϴ)
⇨ {1 + tan(45° - ϴ)}(1 + tanϴ) = 2

When ϴ = 1° we get, (1 + tan44°)(1 + tan1°) = 2

When ϴ = 2°, we get, (1 + tan43°)(1 + tan2°) = 2

...

...

When ϴ = 22°, we get, (1 + tan23°)(1 + tan22°) = 2

So, the expression = 2^{22} (a multiple of 4)

⇨ Option (d) is correct.

6. Let y = x/(1 + x), where x = w^(2009^2009^2009^...2009 times) and w is a complex cube root of 1. Then y is
   (a) w
   (b) −w
   (c) w^2
   (d) $-w^2$

Solution :

Now, 2009 ≡ -1 (mod 3)

⇨ $2009^m \equiv (-1)^m = -1 \pmod 3$ if m is odd.

Now, 2009^2009^.... = odd as 2009 is odd.

⇨ $x = w^{3r-1} = (w^3)^r(1/w) = 1*w^3/w = w^2$

- ⇨ $y = w^2/(1 + w^2) = w^2/(-w) = -w$
- ⇨ Option (b) is correct.

7. The number of solutions of Θ in the interval [0, 2π] satisfying $\{\log_{\sqrt{3}}(\tan\Theta)\sqrt{\{\log_{\tan\Theta}(3) + \log_{\sqrt{3}}(3\sqrt{3})\}} = -1$ is
   (a) 0
   (b) 2
   (c)4
   (d) 6

Solution :

Now, $\{\log_{\sqrt{3}}(\tan\Theta)\sqrt{\{\log_{\tan\Theta}(3) + \log_{\sqrt{3}}(3\sqrt{3})\}} = -1$

- ⇨ $\{\log(\tan\Theta)/\log(\sqrt{3})\}\sqrt{\{\log(3)/\log(\tan\Theta) + \log_{\sqrt{3}}(\sqrt{3})^3\}} = -1$
- ⇨ $\{2\log(\tan\Theta)/\log(3)\}\sqrt{\{\log(3)/\log(\tan\Theta) + 3\}} = -1$

Let, $\log(\tan\Theta) = x$ and $\log(3) = a$

The equation becomes, $(2x/a)\sqrt{\{(a + 3x)/x\}} = -1$

- ⇨ $2\sqrt{\{x(a + 3x)\}} = -a$
- ⇨ $4x(a + 3x) = a^2$
- ⇨ $12x^2 + 4xa - a^2 = 0$
- ⇨ $x = \{-4a \pm \sqrt{(16a^2 + 48a^2)}\}/24$
- ⇨ $x = \{-4a \pm 8a)/24$
- ⇨ $x = -a/2$, $a/6$
- ⇨ $\log(\tan\Theta) = -\log(3)/2$
- ⇨ $\log(\tan\Theta) = \log(1/\sqrt{3})$
- ⇨ $\tan\Theta = 1/\sqrt{3}$
- ⇨ $\Theta = \pi/6$

Now, $x = a/6$

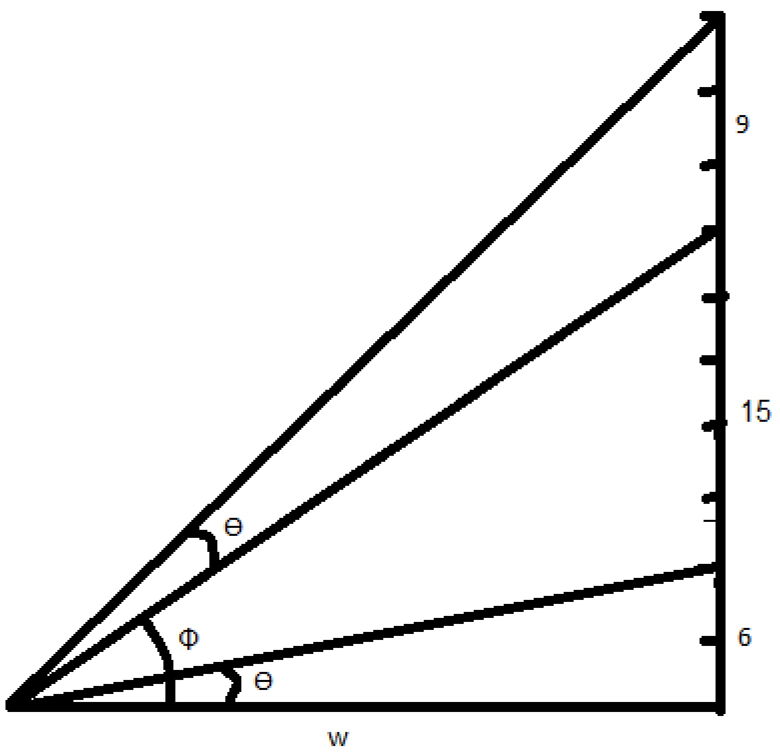
- ⇨ $\log(\tan\Theta) = \log(3)/6$
- ⇨ $\log(\tan\Theta) = \log(3^{1/6})$
- ⇨ $\tan\Theta = 3^{1/6}$
- ⇨ $\Theta = \tan^{-1}(3^{1/6})$
- ⇨ 2 solutions.
- ⇨ Option (b) is correct.

8. A building with ten storeys, each storey of height 3 metres, stands on one side of a wide street. From a point on the other side of the street directly opposite to the building, it is observed that the three

uppermost storeys together subtend an angle equal to that subtended by the two lowest storeys. The width of the street is

(a) $6\sqrt{35}$ metres
(b) $6\sqrt{70}$ metres
(c) 6 metres
(d) $6\sqrt{3}$ metres

Solution :



Let width of the street is w.

Clearly, from figure, $\tan\Theta = 6/w$ and $\tan\Phi = 21/w$

Now, also from figure, $\tan(\Phi + \Theta) = 30/w$

- ⇨ $(\tan\Phi + \tan\Theta)/(1 - \tan\Phi\tan\Theta) = 30/w$
- ⇨ $\{(21/w) + (6/w)\}/\{1 - (21/w)(6/w) = 30/w$
- ⇨ $(27/w)/\{(w^2 - 126)/w^2\} = 30/w$
- ⇨ $9w^2/(w^2 - 126) = 10$
- ⇨ $9w^2 = 10w^2 - 1260$
- ⇨ $w^2 = 1260$
- ⇨ $w = 6\sqrt{35}$
- ⇨ Option (a) is correct.

9. A collection of black and white balls are to be arranged on a straight line, such that each ball has at least one neighbour of different colour. If there are 100 black balls, then the maximum number of white balls that allow such an arrangement is
   (a) 100
   (b) 101
   (c) 202
   (d) 200

Solution :

The hundred black balls are put into the line. Now, we can put maximum 2 white balls in between 2 black balls. Now, there are 100 black balls. So there are 99 gaps between the hundred balls. So number of white balls we can keep in between the 100 black balls = 2*99 = 198. Now, put one white ball before the left most black ball and put another white ball after the right most black ball i.e. the row starts with a white ball and also ends with a white ball.

So maximum number of white balls = 198 + 1 + 1 = 200.

⇨ Option (d) is correct.

10. Let $f(x)$ be a real-valued function satisfying $af(x) + bf(-x) = px^2 + qx + r$, where a and b are distinct real numbers and p, q and r are non-zero real numbers. Then $f(x) = 0$ will have real solution when
   (a) $\{(a + b)/(a - b)^2 \le q^2/4pr$
   (b) $\{(a + b)/(a - b)\}^2 \le 4pr/q^2$
   (c) $\{(a + b)/(a - b)^2 \ge q^2/4pr$
   (d) $\{(a + b)/(a - b)\}^2 \ge 4pr/q^2$

Solution :

Now, $af(x) + bf(-x) = px^2 + qx + r$

Putting $x = -x$ we get, $af(-x) + bf(x) = px^2 - qx + r$

Adding the above two equations, we get, $a\{f(x) + f(-x)\} + b\{f(x) + f(-x) = 2px^2 + 2r$

⇨ $f(x) + f(-x) = (2px^2 + 2r)/(a + b)$

Now, subtracting the above two equations, we get, $a\{f(x) - f(-x)\} - b\{f(x) - f(-x)\} = 2qx$

- ⇨ $f(x) - f(-x) = 2qx/(a - b)$

Now, adding the evaluated two equations we get, $2f(x) = (2px^2 + 2r)/(a + b) + 2qx/(a - b)$

- ⇨ $f(x) = (px^2 + r)/(a + b) + qx/(a - b)$

Now, $f(x) = 0$

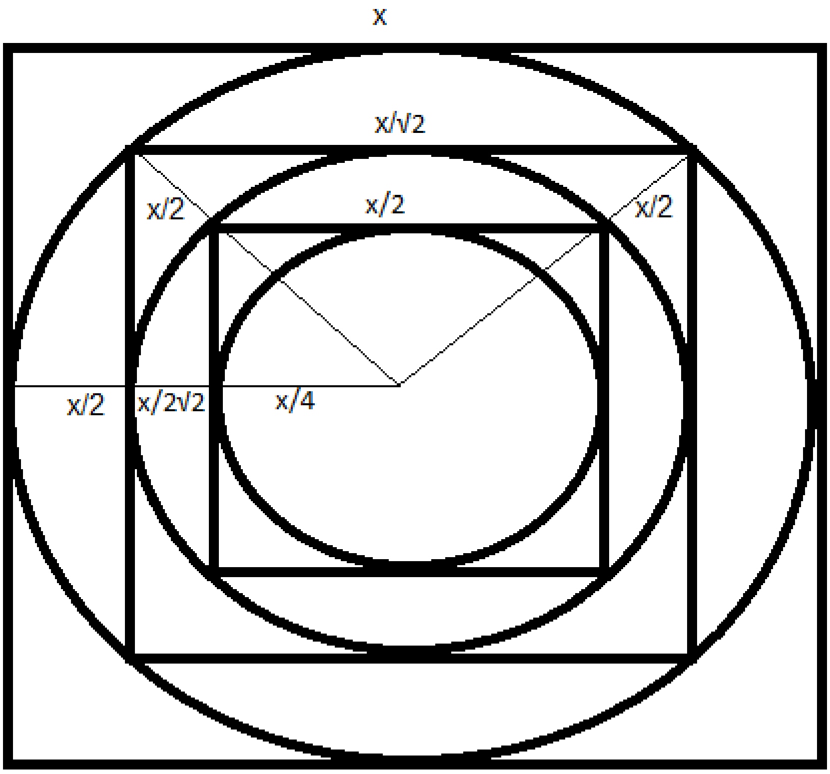
- ⇨ $(px^2 + r)/(a + b) + qx/(a - b) = 0$
- ⇨ $(a - b)px^2 + qx(a + b) + r(a - b) = 0$

Now, this equation has real roots if $\{q(a + b)\}^2 - 4(a - b)p*r(a - b) \geq 0$

- ⇨ $\{(a + b)/(a - b)\}^2 \geq 4pr/q^2$
- ⇨ Option (d) is correct.

11. A circle is inscribed in a square of side x, then a square is inscribed in that circle, a circle is inscribed in the latter square, and so on. If S_n is the sum of the areas of the first n circles so inscribed, then $\lim(S_n)$ as $n \to \infty$ is
    (a) $\pi x^2/4$
    (b) $\pi x^2/3$
    (c) $\pi x^2/2$
    (d) πx^2.

Solution :



Clearly from the picture, radius of first circle is $x/2$, then $x/2^{3/2}$, then $x/2^2$, then $x/2^{5/2}$, ...

The power of 2 makes an arithmetic progression, 1, 3/2, 2, 5/2,...

So,n^{th} term is $1 + (n - 1)(1/2) = (n + 1)/2$

So, radiuses are $x/2$, $x/2^{3/2}$, $x/2^2$, $x/2^{5/2}$, ..., $x/2^{(n+1).2}$.

So areas are $\pi(x/2)^2$, $\pi(x/2^{3/2})^2$, $\pi(x/2^2)^2$, $\pi(x/2^{5/2})^2$, ..., $\pi(x/2^{(n+1)/2})^2$

This is a geometric progression with first term $\pi(x/2)^2$ and common ratio 1/2

- ⇨ $S_n = \pi(x/2)^2\{1 - (1/2)^n\}/(1 - ½)$
- ⇨ $S_n = (\pi x^2/2)\{1 - (1/2)^n\}$
- ⇨ $\lim S_n$ as $n \to \infty = (\pi x^2/2)(1 - 0) = \pi x^2/2$
- ⇨ Option (c) is correct.

12. Let 1, 4, ... and 9, 14, ... be two arithmetic progressions. Then the number of distinct integers in the collection of first 500 terms of each of the progressions is
(a) 833
(b) 835
(c) 837
(d) 901

Solution :

Clearly 19 is the first common term of both the series.

The common term will repeat after 15.

Now, 500^{th} term of the first progression is 1 + (500 – 1)*3 = 1498

Let there are r number of common terms in both the sequence.

Therefore, $19 + (r - 1)*15 \leq 1498$

- ⇨ $(r - 1)*15 \leq 1479$
- ⇨ $(r - 1) \leq 98.6$
- ⇨ $r \leq 99.6$
- ⇨ $r = 99$

So, 99 terms common.

- ⇨ Number of distinct integers = 2*500 – 99 = 901
- ⇨ Option (d) is correct.

13. Consider all the 8-letter words that can be formed by arranging the letters in BACHELOR in all possible ways. Any two such words are called *equivalent* if those two words maintain the same relative order of the letters A, E, O. For example, BACOHELR and CABLROEH are equivalent. How many words are there which are equivalent to BACHELOR?
    (a) $^8C_3*3!$
    (b) $^8C_3*5!$
    (c) $2*(^8C_3)^2$
    (d) 5!*3!*2!

Solution :

Now, there can be 3! Arrangement of the letters A, E and O.

Each combination will have same number of words arranged.

Now, total number of possible cases is 8!

- ⇨ For any particular of arrangement of A, E and O will have 8!/3! Equivalent words.
- ⇨ The answer is $8!/3! = \{8!/(5!*3!)\}*5! = {}^8C_3*5!$
- ⇨ Option (b) is correct.

14. The limit
$\lim \{1/6 + 1/24 + 1/60 + 1/120 + ... + 1/(n^3 - n)\}$
equals
(a) 1
(b) ½
(c) ¼
(d) 1/8

Solution :

Let $S = \{1/6 + 1/24 + 1/60 + 1/120 + ... + 1/(n^3 - n)\}$

$t_n = 1/(n^3 - n) = 1/n(n - 1)(n + 1) = (1/2)\{(n + 1) - (n - 1)\}/n(n - 1)(n + 1) =(1/2)\{1/n(n - 1) - 1/n(n + 1)\}$

- ⇨ $2t_n = 1/n(n - 1) - 1/n(n + 1)$

$2t_2 = 1/(1*2) - 1/(2*3)$

$2t_3 = 1/(2*3) - 1/(3*4)$

$2t_3 = 1/(3*4) - 1/(4*5)$

...

...

$2t_n = 1/n(n - 1) - 1/n(n + 1)$

Adding we get, $2S = 1/(1*2) - 1/n(n + 1)$

- ⇨ $S = (1/4) - 1/2n(n + 1)$
- ⇨ $\lim S$ as $n \to \infty = ¼ - 0 = ¼$
- ⇨ Option (c) is correct.

15. Let a and b be two real numbers satisfying $a^2 + b^2 \neq 0$. Then the set of real numbers c, such that the equations $al + bm = c$ and $l^2 + m^2 = 1$ have real solutions for l and m is
(a) $[-\sqrt{(a^2 + b^2)}, \sqrt{(a^2 + b^2)}]$
(b) $[-|a + b|, |a + b|]$
(c) $[0, a^2 + b^2]$
(d) $(-\infty, \infty)$.

Solution :

Let $l = \sin A$, then $m = \cos A$

Now, al + bm = c

- ⇨ asinA + bcosA = c
- ⇨ $\sqrt{(a^2 + b^2)}[\{a/\sqrt{(a^2 + b^2)}\}\sin A + \{b/\sqrt{(a^2 + b^2)}\}\cos A] = c$

Let, $a/\sqrt{(a^2 + b^2)} = \cos B$, then $b/\sqrt{(a^2 + b^2)} = \sin B$

The equation becomes, $\sqrt{(a^2 + b^2)}[\sin A\cos B + \cos A\sin B] = c$

- ⇨ $\sqrt{(a^2 + b^2)}\sin(A + B) = c$

Now, $- 1 \le \sin(A + B) \le 1$

- ⇨ $-\sqrt{(a^2 + b^2)} \le \sqrt{(a^2 + b^2)}\sin(A + B) \le \sqrt{(a^2 + b^2)}$
- ⇨ $-\sqrt{(a^2 + b^2)} \le c \le \sqrt{(a^2 + b^2)}$
- ⇨ Option (a) is correct.

16. Let f be an onto and differentiable function defined on [0, 1] to [0, T], such that f(0) = 0. Which of the following statements is necessarily true?
    (a) f′(x) is greater than or equal to T for all x
    (b) f′(x) is smaller than T for all x
    (c)f′(x) is greater than or equal to T for some x
    (d) f′(x) is smaller than T for some x.

Solution :

Let us take an example, $f(x) = \sin^{-1}(x)$ for $0 \le x \le 1$

Range is [0, π/2]. Here T = π/2

Now, $f'(x) = 1/\sqrt{(1 - x^2)}$

Clearly for some x ∈ [0, 1], f′(x) > π/2

- ⇨ Option (c) is correct.

17. The area of the region bounded by $|x| + |y| + |x + y| \le 2$ is
    (a) 2
    (b) 3
    (c)4
    (d) 6

Solution :

Let, x > 0, y > 0; then x + y + x + y ≤ 2

⇨ $x + y \leq 1$

Let, $x < 0$, $y < 0$; then $-x - y - (x + y) \leq 2$

⇨ $x + y \geq -1$

Let, $x > 0$, $y < 0$, $x + y > 0$; then $x - y + x + y \leq 2$

⇨ $x \leq 1$

Let, $x > 0$, $y < 0$, $x + y < 0$; then $x - y - x - y \leq 2$

⇨ $y \geq -1$

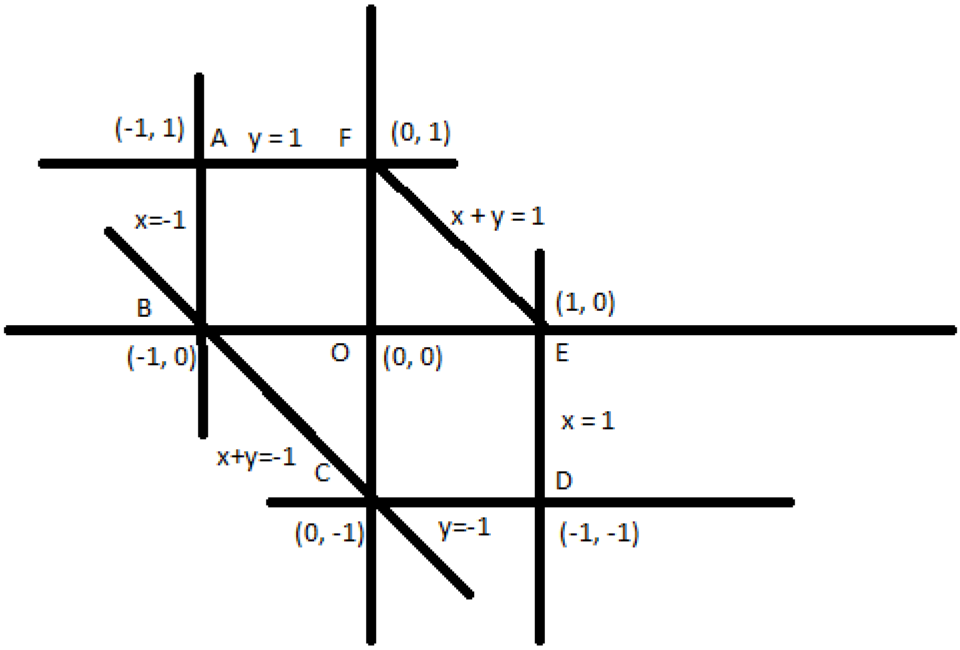
Let, $x < 0$, $y > 0$, $x + y > 0$; then $-x + y + x + y \leq 2$

⇨ $y \leq 1$

Let, $x < 0$, $y > 0$, $x + y < 0$; then $- x + y - x - y \leq 2$

⇨ $x \geq -1$

The enclosed portion is shown in the figure,



Clearly, from figure ABCDEFA is the enclosed region.

Now, area of triangle OFE = $(1/2)*1*1 = ½$

Similarly area of triangle OCB = ½

Area of square ABOF = $1*1 = 1$

Similarly, area of square OCDE = 1

Therefore, area of ABCDEFA = ½ + ½ + 1 + 1 = 3

⇨ Option (b) is correct.

18. Let f and g be two positive valued functions defined on [-1, 1], such that f(-x) =1/f(x) and g is an even function with ∫g(x)dx = 1 (integration running from -1 to 1). Then I = ∫f(x)g(x)dx (integration running from -1 to 1) satisfies
(a) I ≥ 1
(b) I ≤ 1
(c) 1/3 < I < 3
(d) I = 1.

Solution :

Let f(x) = e^x (f(x) is positive valued, f(-x) = 1/f(x) | all criteria satisfied)

Let g(x) = |x|

Then g(x) is even.

g(x) = -x, x < 0

g(x) = x, x > 0

∫g(x)dx (integration running from -1 to 1) = ∫g(x)dx (integration running from -1 to 0) + ∫g(x)dx (integration running from 0 to 1)

= -∫xdx (integration running from -1 to 0) + ∫xdx (integration running from 0 to 1)

$= - x^2/2|_{-1}^{0} + x^2/2|_{0}^{1}$

= 1

So, all criteria are satisfied.

Now, ∫f(x)g(x)dx (integration running from -1 to 1)

= ∫f(x)g(x)dx (integration running from -1 to 0) + ∫f(x)g(x)dx (integration running from 0 to 1)

= -∫xe^xdx (integration running from -1 to 0) + ∫xe^xdx (integration running from 0 to 1)

$= -xe^x + e^x|_{-1}^{0} + (xe^x - e^x)|_{0}^{1}$

$= e^0 - e^{-1} - e^{-1} + e - e + e^0$

$= 2 - 2e^{-1}$

$= 2(1 - e^{-1}) > 1$

⇨ Option (a) is correct.

19. How many possible values of (a, b, c, d) with a, b, c, d real, are there such that abc = d, bcd = a, cda = b, dab = c?
    (a) 1
    (b) 6
    (c) 9
    (d) 17

Solution :

Now, (abc)(bcd)(cda)(dab) = abcd

⇨ $(abcd)^3 - (abcd) = 0$
⇨ $abcd\{(abcd)^2 - 1\} = 0$
⇨ abcd = 0, abcd = 1, abcd = -1

(a, b, c, d) = (0, 0, 0, 0); (-1, -1, 1, 1); (-1, 1, -1, 1); (-1, 1, 1, -1); (1, -1, -1, 1); (1, -1, 1, -1); (1, 1, -1, -1); (-1, -1, -1, -1); (1, 1, 1, 1)

So, 9 values are possible.

⇨ Option (c) is correct.

20. What is the maximum possible value of a positive integer n, such that for any choice of seven distinct elements from {1, 2, ..., n}, there will exist two numbers x and y satisfying $1 < x/y \le 2$?
    (a) 2*7
    (b) $2^7 - 2$
    (c) $7^2 - 2$
    (d) $7^7 - 2$.

Solution :

If we take the below numbers then the condition doesn't hold for minimum n.

1

1*2 + 1 = 3

$3*2 + 1 = 7$

$7*2 + 1 = 15$

$15*2 + 1 = 31$

$31*2 + 1 \;\; 63$

$63*2 + 1 = 127$

So, maximum value of n to hold the condition = $126 = 2^7 - 2$.

- ⇨ Option (b) is correct.

Group B (Each of the following questions has either one or two correct options and you have to identify all the correct options.)

21. Which of the following are roots of the equation $x^7 + 27x = 0$?
    (a) $-\sqrt{3}i$
    (b) $(\sqrt{3}/2(-1 + \sqrt{3}i)$
    (c)$-(\sqrt{3}/2)(1 + i)$
    (d) $(\sqrt{3}/2)(\sqrt{3} - i)$

Solution :

Now, $x^7 + 27x = 0$

- ⇨ $(x^2)^3 + 3^3 = 0$
- ⇨ $(x^2 + 3)(x^4 - 3x^2 + 9) = 0$
- ⇨ $(x^2 + 3) = 0$
- ⇨ $x = -\sqrt{3}i$ is a root

Now, $x^4 - 3x^2 + 9 = 0$

- ⇨ $x^2 = [3 - \sqrt{\{(-3)^2 - 4*1*9\}}]/(2*1)$
- ⇨ $x^2 = (3/2)(1 - i\sqrt{3})$
- ⇨ $x^2 = (3/4)(2 - 2\sqrt{3}i)$
- ⇨ $x^2 = (3/4)\{(\sqrt{3})^2 - 2*\sqrt{3}*i + i^2\}$
- ⇨ $x^2 = (3/4)(\sqrt{3} - i)^2$
- ⇨ $x = (\sqrt{3}/2)(\sqrt{3} - i)$
- ⇨ Option (a) and (d) are correct.

22. The equation $|x^2 - x - 6| = x + 2$ has
    (a) Two positive roots
    (b) Two real roots

(c) Three real roots
(d) None of the above.

Solution :

Now, $|x^2 - x - 6| = (x + 2)$

- ⇨ $(x + 2)^2(x - 3)^2 = (x + 2)^2$
- ⇨ $(x + 2)^2\{(x - 3)^2 - 1\} = 0$
- ⇨ $(x + 2)^2(x - 2)(x - 4) = 0$
- ⇨ Two positive roots, $x = 2, 4$
- ⇨ Three real roots $x = -2, 2, 4$
- ⇨ Option (a) and (c) are correct.

23. If $0 < x < \pi/2$, then
(a) $\cos(\cos x) > \sin x$
(b) $\sin(\sin x) > \sin x$
(c) $\sin(\cos x) > \cos x$
(d) $\cos(\sin x) > \sin x$

Solution :

Clearly (b), (c), (d) cannot be true as they are of the form $\sin(A) > A$, $\cos(A) > A$

- ⇨ Option (a) is correct.

24. Suppose ABCD is a quadrilateral such that the coordinates of A, B and C are (1, 3), (-2, 6) and (5, -8) respectively. For which choices of the coordinates of D will ABCD be a trapezium?
(a) (3, -6)
(b) (6, -9)
(c) (0, 5)
(d) (3, -1)

Solution :

Slope of AB = $(6 - 3)/(-2 - 1) = -1$

If we take D as option (a) then slope of opposite side CD = $(-6 + 8)/(3 - 5) = -1$

So, option (a) is correct.

Now, slope of BC = (-8 – 6)/(5 + 2) = -2

If we take D as option (d) then slope of opposite side AD = (-1 – 3)/(3 – 1) = -2

So, option (d) is correct.

⇨ Options (a) and (d) are correct.

25. Let x and y be two real numbers such that 2log(x – 2y) = log(x) + log(y) holds. Which of the following are possible values of x/y?
(a) 4
(b) 3
(c) 2
(d) 1.

Solution :

Now, 2log(x – 2y) = log(x) + log(y)

⇨ $(x – 2y)^2 = xy$
⇨ $\{(x/y) – 2\}^2 = (x/y)$ (Dividing both sides by y^2)

Clearly x/y = 4 satisfies the equation.

⇨ Option (a) is correct.

26. Let f be a differentiable function satisfying f′(x) = f′(-x) for all x. Then
(a) f is an odd function
(b) f(x) + f(-x) = 2f(0) for all x
(c) (½)f(x) + (½)f(y) = f{(1/2){x + y)} for all x, y
(d) If f(1) = f(2), then f(-1) = f(-2).

Solution.

Now, f′(x) = f′(-x)

⇨ ∫f′(x)dx = ∫f′(-x)dx + c
⇨ f(x) = -f(-x) + c
⇨ f(x) + f(-x) = c

Putting x = 0 we get, c = 2f(0)

⇨ $f(x) + f(-x) = 2f(0)$

Putting $x = 1$ we get, $f(1) + f(-1) = 2f(0)$

Putting $x = 2$ we get, $f(2) + f(-2) = 2f(0)$

⇨ $f(1) + f(-1) = f(2) + f(-2)$

Now, if $f(1) = f(2)$ then $f(-1) = f(-2)$

⇨ Options (b) and (d) are correct.

27. Consider the function $f(x) = \max\{x, 1/x\}/\min\{x, 1/x\}$ when $x \neq 0$; $f(x) = 1$ when $x = 0$. Then
(a) $\lim f(x) = 0$ as $x \to 0+$
(b) $\lim f(x) = 0$ as $x \to 0-$
(c) $f(x)$ is continuous for all $x \neq 0$
(d) f is differentiable for all $x \neq 0$.

Solution :

$f(x) = (1/x)/x = 1/x^2$ when $x < -1$

$f(x) = x/(1/x) = x^2$ when $-1 \leq x < 0$

$f(x) = (1/x)/x = 1/x^2$ when $0 < x \leq 1$

$f(x) = x/(1/x) = x^2$ when $x > 1$

Now, $\lim f(x)$ as $x \to 0+$ $= \lim (1/x^2)$ as $x \to 0+$ $= \infty$

$\lim f(x)$ as $x \to 0-$ $= \lim (x^2)$ as $x \to 0-$ $= 0$

$\lim f(x)$ as $x \to 1+$ $= \lim (x^2)$ as $x \to 1+$ $= 1$

$\lim f(x)$ as $x \to 1-$ $= \lim (1/x^2)$ as $x \to 1-$ $= 1$

$f(1) = 1$

So, $f(x)$ is continuous at $x = 1$.

Now, $\lim f(x)$ as $x \to -1+$ $= \lim (x^2)$ as $x \to -1+$ $= 1$

$\lim f(x)$ as $x \to -1-$ $= \lim (1/x^2)$ as $x \to -1-$ $= 1$

$f(-1) = 1$

⇨ $f(x)$ is continuous for all $x \neq 0$

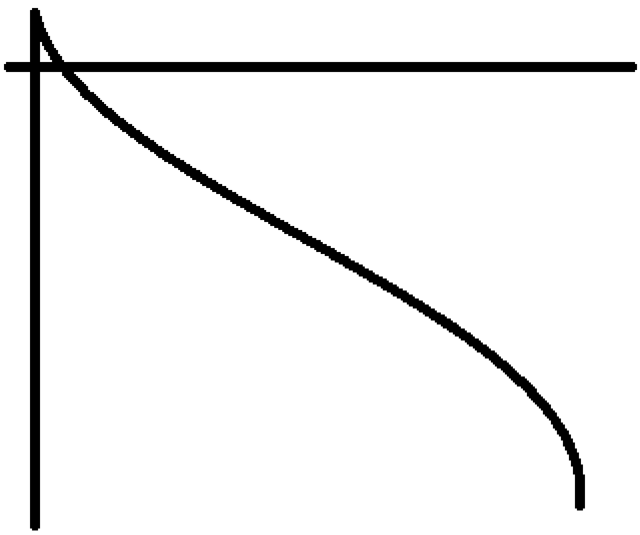
Now, $\lim [\{f(x) - f(1)\}/(x - 1)]$ as $x \to 1+$ $= \lim \{(x^2 - 1)/(x - 1)\}$ as $x \to 1+$ $= \lim (x + 1)$ as $x \to 1+$ $= 2$

lim [{f(x) – f(1)}/(x – 1)} as x -> 1- = lim {(1/x^2 – 1)/(x – 1)} as x -> 1- = lim [(1 – x^2)/{x^2(x – 1)}] as x -> 1- = lim –(1 + x)/x^2 as x -> 1- = -2

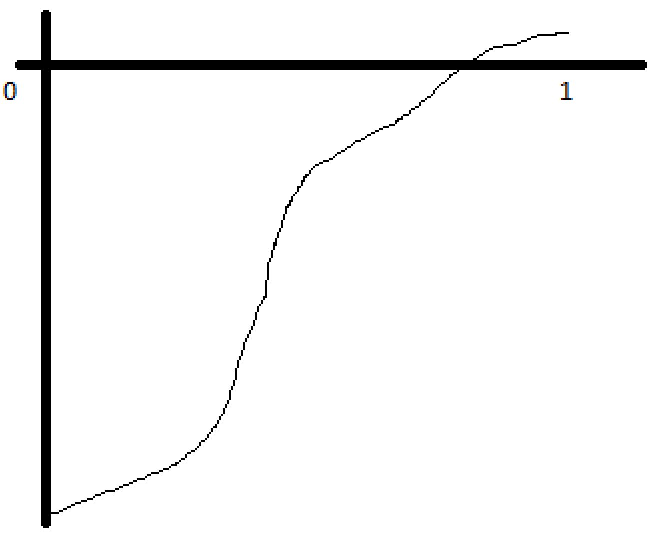
So, f(x) is not differentiable at x = 1.

⇨ Options (b) and (c) are correct.

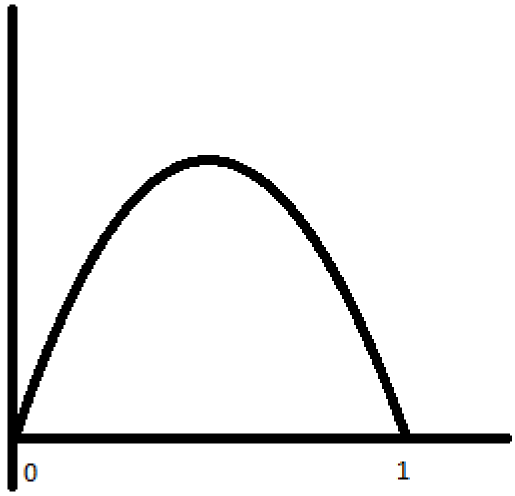
28. Which of the following graphs represent functions whose derivatives have a maximum in the interval (0, 1)?
(a)



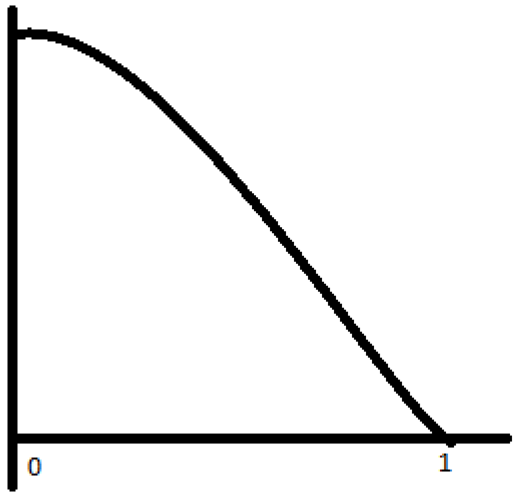
(b)



(c)



(d)



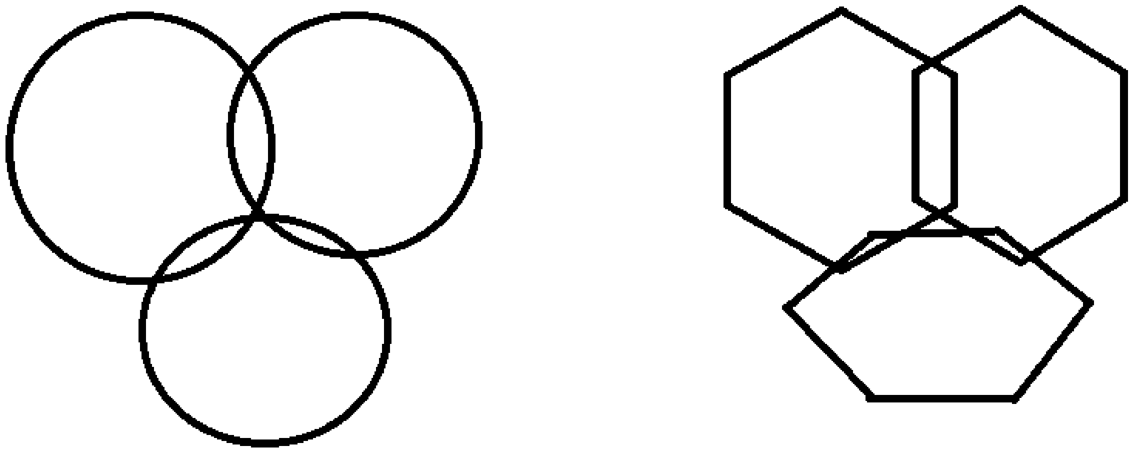
Solution :

Options (a) and (d) are correct.

29. A collection of geometric figures is said to satisfy *Helly property* if the following condition holds :
For any choice of three figures A, B, C from the collection satisfying $A \cap B \neq \Phi$, $B \cap C \neq \Phi$ and $C \cap A \neq \Phi$, one must have $A \cap B \cap C \neq \Phi$.
Which of the following collections satisfy Helly property?

(a) A set of circles
(b) A set of hexagons
(c) A set of squares with sides parallel to the axes
(d) A set of horizontal line segments.

Solution :



Clearly from the above figure option (a) and (b) is not true.

And it is clear that options (c) and (d) are correct.

30. Consider an array of m rows and n columns obtained by arranging the first mn integers in some order. Let b_i be the maximum of the numbers in the i-th row and c_j be the minimum of the numbers in the j-th column. If
$b = \min(b_i)$ where $1 \le i \le m$ and $c = \max(c_j)$ where $1 \le j \le n$,
then which of the following statements are necessarily true?
(a) $m \le c$
(b) $n \ge b$
(c) $c \ge b$
(d) $c \le b$.

Solution :

If we put 1, 2, 3, ..., n in i-th row then b_i has the minimum value at i-th column and it is n

So, b = n and c_n is maximum at c_n and it is equal to n.

So, $c = n$ (if $n < m$ then $c < m$, so option (a) cannot be true)

- ⇨ $b = c$

If we put $mn, mn - 1, mn - 2, \ldots, mn - m$ in j-th column, then c_j has the maximum value and it is equal to $mn - m$.

So, $c = mn - m$ and b_m has the minimum value and it is equal to $mn - m$

So, $b = mn - m$ (option (b) cannot be true)

- ⇨ $b = c$

Now, let us take $m = 4$ and $n = 3$ and put the numbers in random from 1 to 12 as shown in the picture below,

11	6	8
7	1	10
4	9	12
2	5	3

Here, $b_1 = 11$, $b_2 = 10$, $b_3 = 12$, $b_4 = 5$

- ⇨ $b = 5$

$c_1 = 2$, $c_2 = 1$, $c_3 = 3$

- ⇨ $c = 3$
- ⇨ $c < b$
- ⇨ Option (c) cannot be true.
- ⇨ Option (d) is correct.

B. Math. (Hons.) Admission Test : 2009

1. The domain of definition of $f(x) = -\log(x^2 - 2x - 3)$ is :
   (a) $(0, \infty)$
   (b) $(-\infty, -1)$
   (c) $(-\infty, -1)\cup(3, \infty)$
   (d) $(-\infty, -3)\cup(1, \infty)$.

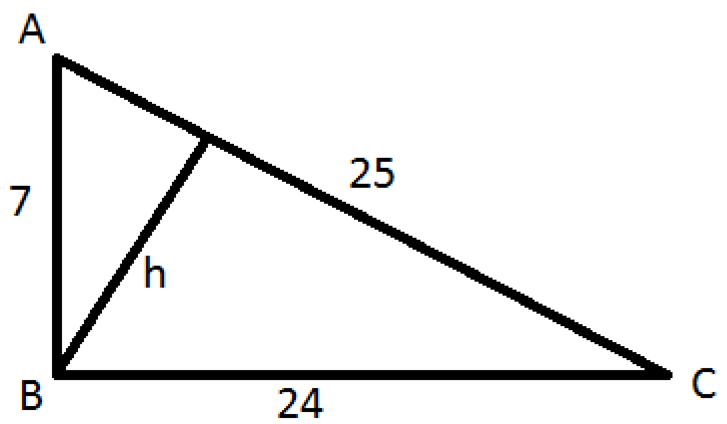
Solution :

$x^2 - 2x - 3 > 0$

- ⇨ $(x - 1)^2 > 4$
- ⇨ $x - 1 > 2$ or $x - 1 < -2$
- ⇨ $x > 3$ or $x < -1$
- ⇨ Option (c) is correct.

2. ABC is a right-angled triangle with the right angle at B. If AB = 7 and BC = 24, then the length of the perpendicular from B to AC is
   (a) 12.2
   (b) 6.72
   (c) 7.2
   (d) 3.36

Solution :



Clearly $AC^2 = AB^2 + BC^2$

- ⇨ AC = 25

Let the length of the perpendicular from B to AC is h.

Area of triangle ABC = (1/2)*7*24 = (1/2)*h*25

- ⇨ h = 6.72.
- ⇨ Option (b) is correct.

3. If the points z_1 and z_2 are on the circles $|z| = 2$ and $|z| = 3$ respectively and the angle included between these vectors is 60°, then $|(z_1 + z_2)/(z_1 - z_2)|$ equals
   (a) $\sqrt{(19/7)}$
   (b) $\sqrt{19}$
   (c) $\sqrt{7}$
   (d) $\sqrt{133}$

Solution :

Clearly, $|z_1| = 2$ and $|z_2| = 3$.

Let, $z_1 = 2e^{i\Theta}$ and $z_2 = 3e^{i(\Theta + \pi/3)}$

Now, $(z_1 + z_2)/(z_1 - z_2) = (2e^{i\Theta} + 3e^{i(\Theta + \pi/3)})/(2e^{i\Theta} - 3e^{i(\Theta + \pi/3)})$

- ⇨ $(z_1 + z_2)/(z_1 - z_2) = (2 + 3e^{i\pi/3})/(2 - 3e^{i\pi/3})$
- ⇨ $|(z_1 + z_2)/(z_1 - z_2)|^2 = \{2 + 3\cos(\pi/3)\}^2 + 3\sin^2(\pi/3)\}/(2 - 3\cos(\pi/3)\}^2 + 3\sin^2(\pi/3)\}$
- ⇨ $|(z_1 + z_2)/(z_1 - z_2)|^2 = \{13 + 12\cos(\pi/3)\}/\{13 - 12\cos(\pi/3)\} = (19/7)$
- ⇨ $|(z_1 + z_2)/(z_1 - z_2)| = \sqrt{(19/7)}$
- ⇨ Option (a) is correct.

4. Let a, b, c and d be positive integers such that $\log_a(b) = 3/2$ and $\log_c(d) = 5/4$. If $a - c = 9$, then $b - d$ equals
   (a) 55
   (b) 23
   (c) 89
   (d) 93.

Solution :

$\log_a(b) = 3/2$

- ⇨ $b = a^{3/2}$

Similarly, $d = c^{5/4}$

b and d are both integers.

- ⇨ a must be a square number and c must be a 4^{th} power number.

Let, $a = x^2$ and $c = y^4$

Now, $a - c = 9$

- ⇨ $x^2 - y^4 = 9$
- ⇨ $(x + y^2)(x - y^2) = 9$

Now, 9 can be divided in two ways viz. 3*3 or 9*1.

Now, 3*3 is not possible otherwise $y = 0$ i.e. $c = 0$. But c is positive integer.

- ⇨ $x + y^2 = 9$ and $x - y^2 = 1$

- ⇨ $x = 5$ and $y^2 = 4$
- ⇨ $a = 25$ and $c = 16$.
- ⇨ $b = 125$ and $d = 32$
- ⇨ $b - d = 125 - 32 = 93$.
- ⇨ Option (d) is correct.

5. $1 - x - e^{-x} > 0$ for :
   (a) All $x \in R$.
   (b) No $x \in R$.
   (c) $x > 0$.
   (d) $x < 0$.

Solution :

Clearly, for $x \geq 1$, $1 - x - e^{-x} < 0$.

So, option (a) and (c) cannot be true.

Let is take $x = -1$.

For $x = -1$, $1 - x - e^{-x} = 1 - (-1) - e = 2 - e < 0$ (As $2 < e < 3$)

- ⇨ Option (d) cannot be true.
- ⇨ Option (b) is correct.

6. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$ where $ac \neq 0$, then the equation $P(x)Q(x) = 0$ has :
   (a) Only real roots.
   (b) No real roots.
   (c) At least two real roots.
   (d) Exactly two real roots.

Solution :

Discriminant of $P(x) = b^2 - 4ac$ and discriminant of $Q(x) = b^2 + 4ac$.

If $4ac > b^2$ then roots of $Q(x)$ are real and roots of $P(x)$ is not real.

If $4ac < b^2$ then roots of $P(x)$ are real but nothing can b said about roots of $Q(x)$ i.e. both roots of $Q(x)$ may be real or both may not be real.

If $4ac = b^2$ then both the roots of $P(x)$ and $Q(x)$ are real.

So, we attend the conclusion that at least two of the roots are real of the equation $P(x)Q(x) = 0$

⇨ Option (c) is correct.

7. lim|√(x^2 + x) – x| as x -> ∞ is equal to
   (a) ½
   (b) 0
   (c) ∞
   (d) 2.

Solution :

Now, √(x^2 + x) – x = x/{√(x^2 + x) + x}

Let, z = 1/x. As x -> ∞, z -> 0.

The above expression becomes, (1/z)/√{(1/z^2 + 1/z) + (1/z)} = 1/{√(1 + z) + 1}

Now, lim[1/{√(1 + z) + 1}] as z -> 0 is ½.

⇨ Option (a) is correct.

8. lim (π/2^n)∑sin(jπ/2^n) where j runs from 1 to 2^n as n -> ∞ is equal to
   (a) 0
   (b) π
   (c) 2
   (d) 1.

Solution :

Option (c) is correct.

9. Let f : R -> R is given by f(x) = x(x – 1)(x + 1). Then,
   (a) f is 1 – 1 and onto
   (b) f is neither 1 – 1 nor onto
   (c) f is 1 – 1 but not onto
   (d) f is onto but not 1 – 1.

Solution :

Clearly f(0) = f(1) = f(-10 = 0

So, f cannot be 1 – 1.

Definition of onto function : A function f from A to B is called onto if for all b in B there is an a in A such that f (a) = b. All elements in B are used.

Clearly f(x) can take any real value.

- ⇨ f(x) is onto.
- ⇨ Option (d) is correct.

10. The last digit of 22^{22} is :
    (a) 2
    (b) 4
    (c) 6
    (d) 0

Solution :

$2^{10} = 1024$

- ⇨ $2^{11} = 2048$
- ⇨ $2^{11} \equiv 8 \pmod{10}$
- ⇨ $2^{22} \equiv 64 \pmod{10}$
- ⇨ $2^{22} \equiv 4 \pmod{10}$
- ⇨ Option (b) is correct.

11. The average scores of 10 students in a test is 25. The lowest score is 20. Then the highest score is at most
    (a) 100
    (b) 30
    (c) 70
    (d) 75.

Solution :

Let all the 9 students except the student who has got highest mark have got 20 each which is the lowest score.

Then sum of their scores = 20*9 = 180

Now, sum of the scores of 10 students = 25*10 = 250.

So, highest mark can be at most 250 – 180 = 70.

- ⇨ Option (c) is correct.

12. The coefficient of t^3 in the expansion of $\{(1 - t^6)/(1 - t)\}^3$ is
(a) 10
(b) 12
(c) 8
(d) 9

Solution :

Now, $1 - t^6 = (1 - t^3)(1 + t^3) = (1 - t)(1 + t + t^2)(1 + t^3)$

- ⇨ $(1 - t^6)/(1 - t) = (1 + t + t^2)(1 + t^3)$
- ⇨ $\{(1 - t^6)(1 - t)\}^3 = (1 + t + t^2)^3*(1 + t^3)^3$
- ⇨ $\{(1 - t^6)(1 - t)\}^3 = \{1 + 3(t + t^2) + 3(t + t^2)^2 + (t + t^2)^3\}(1 + 3t^3 + 3t^6 + t^9)$
- ⇨ $\{(1 - t^6)(1 - t)\}^3 = \{1 + 3t + 3t^2 + 3(t^2 + 2t^3 + t^4) + t^3 + 3t^4 + 3t^5 + t^6\}(1 + 3t^3 + 3t^6 + t^9)$
- ⇨ $\{(1 - t^6)(1 - t)\}^3 = (1 + 3t + 6t^2 + 7^3 + ...)(1 + 3t^3 + ..)$
- ⇨ $\{(1 - t^6)(1 - t)\}^3 = (1 + 3t^3 + 3t + 3t^4 + 6t^2 + 18t^5 + 7t^3 + ...)$

Clearly, coefficient of t^3 is 3 + 7 = 10.

- ⇨ Option (a) is correct.

13. Let $p_n(x)$, $n \geq 0$ be polynomials defined by $p_0(x) = 1$, $p_1(x) = x$ and $p_n(x) = xp_{n-1}(x) - p_{n-2}(x)$ for $n \geq 2$. Then $p_{10}(x)$ equals
(a) 0
(b) 10
(c) 1
(d) -1.

Solution :

$p_{10}(0) = 0*p_9(0) - p_8(0)$

- ⇨ $p_{10}(0) = - p_8(0) = - \{- p_6(0)\} = p_6(0) = - p_4(0) = - \{- p_2(0)\} = p_2(0) = - p_0(0) = -1$
- ⇨ Option (d) is correct.

14. Suppose A, B are matrices satisfying AB + BA = 0. Then A^2B^5 is equal to
(a) 0
(b) B^2A^5

(c) $-B^2A^5$
(d) AB

Solution :

AB + BA = 0

⇨ AB = -BA

Now, $A^2B^5 = (AB)^2B^3 = (-BA)^2B^3 = B^2(AB)^2B = B^2(-BA)^2B = B^4A^2B = B^4A(AB) = B^4A(-BA) = B^4(-AB)A = B^4(BA)A = B^5A^2$

⇨ Option (b) is correct.

15. The number of terms in the expansion of $(x + y + z + w)^{2009}$ is
(a) $^{2009}C_4$
(b) $^{2013}C_4$
(c) $^{2012}C_3$
(d) $(2010)^4$

Solution :

Number of terms = $^{2009 + 4 - 1}C_{4 - 1} = {}^{2012}C_3$.

⇨ Option (c) is correct.

16. If a, b, c are positive real numbers satisfying ab + bc + ca = 12, then the maximum value of abc is
(a) 8
(b) 9
(c) 6
(d) 12

Solution :

Now, ab + bc + ca = 12

⇨ (1/a) + (1/b) + (1/c) = 12/abc

Now, $\{(1/a) + (1/b) + (1/c)\}/3 \geq 1/(abc)^{1/3}$ (As AM ≥ GM)

⇨ $4/abc \geq 1/(abc)^{1/3}$
⇨ $(abc)^{2/3} \leq 4$

- ⇨ $abc \le 8$
- ⇨ Option (a) is correct.

17. If at least 90 percent students in a class are good in sports, and at least 80 percent are good in music and at least 70 percent are good in studies, then the percentage of students who are good in all three is at least
    (a) 25
    (b) 40
    (c) 20
    (d) 50

Solution :

At least 90 percent students are good in sports.

- ⇨ At most 10 percent students are not good in sports.

Similarly, at most 20 percent students are not good in music and at most 30 percent students are not good in studies.

If all the students in the above record are different then at most (10 + 20 + 30) = 60 percent students are not good in sports, music and studies.

- ⇨ At least 100 – 60 = 40 percent students are good in all three.
- ⇨ Option (b) is correct.

18. If $\cot\{\sin^{-1}\sqrt{(13/17)}\} = \sin(\tan^{-1}\theta)$, then θ is
    (a) $2/\sqrt{17}$
    (b) $\sqrt{(13/17)}$
    (c) $\sqrt{(2/\sqrt{13})}$
    (d) 2/3.

Solution :

Let, $\sin^{-1}\sqrt{(13/17)} = A$

- ⇨ $\sin A = \sqrt{(13/17)}$
- ⇨ $\cot A = 2/\sqrt{13}$
- ⇨ $\sin(\tan^{-1}\theta) = 2/\sqrt{13}$

Let, $\tan^{-1}\theta = B$

- ⇨ $\tan B = \theta$
- ⇨ $\sin B = \theta/\sqrt{(\theta^2 + 1)}$

- ⇨ $\Theta/\sqrt{(\Theta^2 + 1)} = 2/\sqrt{13}$
- ⇨ $\Theta^2/(\Theta^2 + 1) = 4/13$
- ⇨ $13\Theta^2 = 4\Theta^2 + 4$
- ⇨ $9\Theta^2 = 4$
- ⇨ $\Theta = 2/3$.
- ⇨ Option (d) is correct.

19. Let $f(t) = (t + 1)/(t - 1)$. Then $f(f(2010))$ equals
    (a) 2011/2009
    (b) 2010
    (c) 2010/2009
    (d) None of the above.

Solution :

$f(2010) = (2010 + 1)/(2010 - 1) = 2011/2009$

$f(f(2010)) = f(2011/2009) = \{(2011/2009) + 1\}/\{)2011/2009) - 1\} = (2011 + 2009)/(2011 - 2009) = 4020/2 = 2010$

- ⇨ Option (b) is correct.

20. If each side of a cube is increased by 60%, then the surface area of the cube increased by
    (a) 156%
    (b) 160%
    (c) 120%
    (d) 240%

Solution :

Surface area = S = $6a^2$ where a is each side of the cube.

Now, each side is increased by 60%

New side length = 1.6a

Let new surface area = $S_1 = (1.6a)^2 = 2.56a^2$

Percentage increase in surface area = $\{(S_1 - S)/S\}*100\% = \{(2.56a^2 - a^2)/a^2\}*100\% = 156\%$

- ⇨ Option (a) is correct.

21. If $a > 2$, then
    (a) $\log_e(a) + \log_a(10) < 0$
    (b) $\log_e(a) + \log_a(10) > 0$
    (c) $e^a < 1$
    (d) None of the above is true.

Solution :

Now, $e^a < 1$

- ⇨ $\log(e^a) < \log(1)$
- ⇨ $a < 0$
- ⇨ Option (c) is not correct.

Now, $\log_e(a) + \log_a(10)$

$= \log_e(a) + \log_e(10)/\log_e(a)$

Now, $\log_e(a)$ and $\log_e(10)$ both > 0

- ⇨ $\text{Log}_e(a) + \log_a(10) > 0$
- ⇨ Option (b) is correct.

22. The number of complex numbers w such that $|w| = 1$ and imaginary part of w^4 is 0, is
    (a) 4
    (b) 2
    (c) 8
    (d) Infinite.

Solution :

Let $w = e^{i\Theta}$

- ⇨ $w^4 = e^{i4\Theta} = \cos4\Theta + i\sin4\Theta$

Now, $\sin4\Theta = 0$

- ⇨ $4\Theta = n\pi$

Now, this will given 8 distinct results for n = 0, 1, ..., 7 and then it will run into loop.

- ⇨ Option (c) is correct.

23. Let $f(x) = c*\sin(x)$ for all $x \in R$. Suppose $f(x) = \sum f(x + k\pi)/2^k$ (summation is running from $k = 1$ to $k = \infty$) for all $x \in R$. Then
(a) $c = 1$
(b) $c = 0$
(c) $c < 0$
(d) $c = -1$

Solution :

Now, $f(x + k\pi) = c*\sin(x + k\pi) = c*(-1)^k*\sin(x)$

We have, $f(x) = \sum f(x + k\pi)/2^k$ (summation is running from $k = 1$ to $k = \infty$)

- ⇨ $c*\sin(x) = c*\sin(x)\sum\{(-1)^k/2^k\}$ (summation is running from $k = 1$ to $k = \infty$)
- ⇨ $c*\sin(x) = c*\sin(x)[(-1/2)/\{1 - (-1/2)\}]$ (infinite GP series with first term -1/2 and common ratio -1/2)
- ⇨ $c*\sin(x) - c*\sin(x)(-1/3) = 0$
- ⇨ $c*\sin(x)*(4/3) = 0$
- ⇨ $c*\sin(x) = 0$
- ⇨ $c = 0$.
- ⇨ Option (b) is correct.

24. The number of points at which the function $f(x) = \max(1 + x, 1 - x)$ if $x < 0$ and $f(x) = \min(1 + x, 1 + x^2)$ if $x \geq 0$ is not differentiable, is
(a) 1
(b) 0
(c) 2
(d) None of the above.

Solution :

Clearly $f(x) = 1 - x$ if $x < 0$

Now, $1 + x^2 \leq 1 + x$

- ⇨ $x(x - 1) \leq 0$
- ⇨ $x < 1$ as $x \geq 0$
- ⇨ $f(x) = 1 + x^2$ if $0 \leq x < 1$

Now, $1 + x < 1 + x^2$

- ⇨ $x(x - 1) > 0$
- ⇨ $x > 1$ as $x \geq 0$

⇨ f(x) = 1 + x if x ≥ 1

Now, f′(0-) = lim(f(x) − f(0))/(x − 0) as x -> 0- = lim(1 − x − 1)/x as x -> 0- = -1

Now, f′(0+) = lim(f(x) − f(0))/(x − 0) as x -> 0+ = lim(1 + x^2 − 1)/x as x -> 0+ = 0

So, the function is not differentiable at x = 0.

Now, f′(1-) = lim(f(x) − f(1))/(x − 1) as x -> 1- = lim(1 + x^2 − 2)/(x − 1) as x -> 1- = lim(x + 1)(x − 1)/(x − 1) as x -> 1- = lim(x + 1) as x -> 1- = 2.

Now, f′(1+) = lim(f(x) − f(1))/(x − 1) as x -> 1+ = lim(1 + x − 2)/(x − 1) as x -> 1+ = lim(x − 1)/(x − 1) as x -> 1+ = 1

⇨ f(x) is not differentiable at x = 1.
⇨ Option (c) is correct.

25. The greatest value of function f(x) = $\sin^2(x)\cos(x)$
(a) 2/3√3
(b) √(2/3)
(c) 2/9
(d) √2/3√3

Solution :

Now, f(x) = $\sin^2(x)\cos(x)$

⇨ f′(x) = 2sin(x)cos(x)cos(x) + $\sin^2(x)$\{-sin(x)\}
⇨ f′(x) = 2sin(x)$\cos^2(x)$ − $\sin^3(x)$

Now, f′(x) = 0 gives, 2sin(x)$\cos^2(x)$ − $\sin^3(x)$ = 0

⇨ sin(x) = 0 or 2$\cos^2(x)$ − $\sin^2(x)$ = 0 i.e. sin(x) = ±√(2/3)

Now, f′′(x) = 2cos(x)$\cos^2(x)$ + 2sin(x)*2cos(x){-sin(x)} − 3$\sin^2(x)$cos(x)

= 2$\cos^3(x)$ − 4$\sin^2(x)$cos(x) − 3$\sin^2(x)$cos(x) = cos(x){2$\cos^2(x)$ − 4$\sin^2(x)$ − 3$\sin^2(x)$} = cos(x){2 − 9$\sin^2(x)$} < 0 for cos(x) = 1/√3 ($\sin^2(x)$ = 2/3 i.e. $\cos^2(x)$ = 1/3 i.e. cos(x) = ±1/√3)

So, $f(x)_{max}$ = (2/3)(1/√3) = 2/3√3

⇨ Option (a) is correct.

26. Let $g(t) = \int(x^2 + 1)^{10}dx$ (integration running from -10 to t) for all $t \geq -10$. Then
(a) g is not differentiable.
(b) g is constant.
(c) g is increasing in $(-10, \infty)$.
(d) g is decreasing in $(-10, \infty)$.

Solution :

Now, $g'(t) = (t^1 + 1)^{10} > 0$

- ⇨ g is increasing.
- ⇨ Option (c) is correct.

27. Let p(x) be a continuous function which is positive for all x and $\int p(x)dx = c\int p\{(x+4)/2\}dx$ (first integration is running from 2 to 3 and second integration running from 0 to 2). Then
(a) c = 4
(b) c = ½
(c) c = ¼
(d) c = 2.

Solution :

Now, $\int p\{(x + 4)/2\}dx$ (integration running from 0 to 2)

Putting $(x + 4)/2 = z$ we get, $dx = 2dz$

Now, $\int p\{(x + 4)/2\}dx$ (integration running from 0 to 2) = $2\int p(z)dz$ (integration running from 2 to 3) = $2\int p(x)dx$ (integration running from 2 to 3) (change of variable)

- ⇨ $\int p(x)dx = 2c\int p(x)dx$ (both integration running from 2 o 3)
- ⇨ $2c = 1$
- ⇨ $c = ½$
- ⇨ Option (b) is correct.

28. Let $f : [0, 1] \to (1, \infty)$ be a continuous function. Let $g(x) = 1/x$ for $x > 0$. Then, the equation $f(x) = g(x)$ has
(a) No solution.
(b) All points in (0, 1] as solutions.
(c) At least one solution.
(d) None of the above.

Solution :

Clearly, the intersection of domain of definition of f(x) and g(x) is (0, 1].

Hence f(x) = g(x) should be defined for all (0, 1]

- ⇨ Option (b) is correct.

29. Let $0 \leq \Theta, \Phi < 2\pi$ be two angles. Then the equation $\sin\Theta + \sin\Phi = \cos\Theta + \cos\Phi$
    (a) Determines Θ uniquely in terms of Φ
    (b) Gives two value of Θ for each value of Φ
    (c) Gives more than two values of Θ for each value of Φ
    (d) None of the above.

Solution :

We have, $\sin\Theta + \sin\Phi = \cos\Theta + \cos\Phi$

- ⇨ $\sin\Theta - \cos\Theta = \cos\Phi - \sin\Phi$
- ⇨ $(\sin\Theta - \cos\Theta)^2 = (\cos\Phi - \sin\Phi)^2$
- ⇨ $\sin^2\Theta + \cos^2\Theta - 2\sin\Theta\cos\Theta = \cos^2\Phi + \sin^2\Phi - 2\cos\Phi\sin\Phi$
- ⇨ $1 - \sin2\Theta = 1 - \sin2\Phi$
- ⇨ $\sin2\Theta = \sin2\Phi$
- ⇨ $2\Theta = 2\Phi$ or $2\Theta = \pi - 2\Phi$ or $2\Theta = 2\pi - 2\Phi$ or $2\Theta = 4\pi - 2\Phi$
- ⇨ $\Theta = \Phi$ or $\Theta = \pi/2 - \Phi$ or $\Theta = \pi - \Phi$ or $\Theta = 2\pi - \Phi$

Now, out of these 4 relations only 2 satisfies the given equation viz. $\Theta = \Phi$ and $\Theta = \pi/2 - \Phi$

- ⇨ We are getting two values of Θ for each Φ.
- ⇨ Option (b) is correct.

30. Ten players are to pay a tennis tournament. The number of pairings for the first round is
    (a) $10!/2^5 5!$
    (b) 2^{10}
    (c) ${}^{10}C_2$
    (d) ${}^{10}P_2$

Solution :

Clearly it is ${}^{10}C_2$.

⇨ Option (c) is correct.

Note : if order is important then answer will be (d).

B. Stat. (Hons.) Admission Test : 2010

1. There are 8 balls numbered 1, 2, ..., 8 and 8 boxes numbered 1, 2, ..., 8. The number of ways one can put these balls in the boxes so that each box gets one ball and exactly 4 balls go in their corresponding numbered boxes is
   (a) $3*{}^8C_4$
   (b) $6*{}^8C_4$
   (c) $9*{}^8C_4$
   (d) $12*{}^8C_4$

Solution :

We can choose 4 balls out of 8 balls in ${}^8C_4$ ways.

Let us say boxes 5, 6, 7, 8 has got the balls numbered 5, 6, 7, 8 respectively. So we are done with exactly 4 balls in their corresponding number. Now we will find the number of ways in which we can put rest of the balls in different numbered boxes.

So, we have 1, 2, 3, 4 numbered boxes and 1, 2, 3, 4 numbered balls.

The sample space looks like below figure,

	1	2	3	4
1.	4	1	2	3
2.	2	1	4	3
3.	3	1	4	2
4.	4	3	1	2
5.	3	4	1	2
6.	2	4	1	3
7.	4	3	2	1
8.	3	4	2	1
9.	2	3	4	1

So, we can put the balls in the boxes with given condition in $9 * {}^8C_4$ ways.

⇨ Option (c) is correct.

2. Let α and β be two positive real numbers. For every integer $n > 0$, define $a_n = \int\{\alpha/u(u^\alpha + 2 + u^{-\alpha})\}du$ (integration runs from β to n). Then $\lim(a_n)$ as $n \to \infty$ is equal to
   (a) $1/(1 + \beta^\alpha)$
   (b) $\beta^\alpha/(1 + \beta^{-\alpha})$
   (c) $\beta^\alpha/(1 + \beta^\alpha)$
   (d) $\beta^{-\alpha}/(1 + \beta^\alpha)$

Solution :

We have, $a_n = \int\{\alpha/u(u^\alpha + 2 + u^{-\alpha})\}du$ (integration runs from β to n)

⇨ $a_n = \int\{\alpha u^\alpha/u(u^{2\alpha} + 2u^\alpha + 1)\}du$ (integration runs from β to n)
⇨ $a_n = \int\{\alpha u^{\alpha-1}/(u^\alpha + 1)^2\}du$ (integration runs from β to n)

Substitute, $u^\alpha + 1 = z$

- ⇨ $\alpha u^{\alpha-1}du = dz$
- ⇨ $a_n = \int dz/z^2$ (integration runs from $1 + \beta^\alpha$ to $1 + n^\alpha$)
- ⇨ $a_n = [-1/z]$ (z running from $1 + \beta^\alpha$ to $1 + n^\alpha$)
- ⇨ $a_n = -1/(1 + n^\alpha) + 1/(1 + \beta^\alpha)$
- ⇨ $\lim(a_n)$ as $n \to \infty = 1/(1 + \beta^\alpha)$
- ⇨ Option (a) is correct.

3. Let $f : R \to R^2$ be a function given by $f(x) = (x^m, x^n)$, where $x \in R$ and m, n are fixed positive integers. Suppose that f is one-one. Then
   (a) Both n and m must be odd
   (b) At least one of m and n must be odd
   (c) Exactly one of m and n must be odd
   (d) Neither m nor n can be odd.

Solution :

Let us take x = 2 and x = -2.

$f(2) = (2^m, 2^n)$ and $f(-2) = \{(-2)^m, (-2)^n\}$

Clearly m, n both cannot be even otherwise f is not one-one.

Clearly if at least one of them is odd then f is one-one.

- ⇨ Option (b) is correct.

4. $\lim[(e^{\wedge}x^2 - e^{2x})/\{(x - 2)e^{2x}\}]$ as $x \to 2$ equals
   (a) 0
   (b) 1
   (c) 2
   (d) 3

Solution :

$\lim[(e^{\wedge}x^2 - e^{2x})/\{(x - 2)e^{2x}\}]$ as $x \to 2$

$= \lim[e^{2x}\{e^{\wedge}(x^2 - 2x) - 1\}/\{(x - 2)e^{2x}\}]$ as $x \to 2$

$= \lim[\{e^{\wedge}(x^2 - 2x) - 1\}/(x - 2)]$ as $x \to 2$

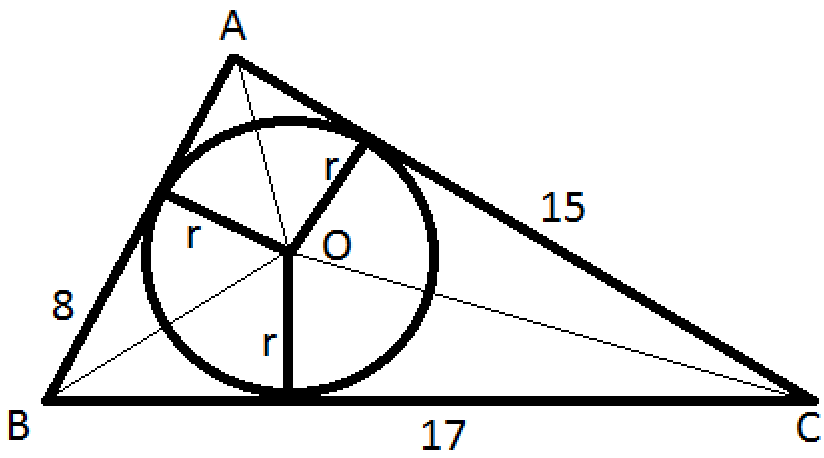
$= \lim\{(2x - 2)e^{\wedge}(x^2 - 2x)\}/1$ as $x \to 2$ (Applying L'Hospital rule)

= $(2*2 - 2)e^0/1 = 2$

⇨ Option (c) is correct.

5. A circle is inscribed in a triangle with sides 8, 15, 17 centimetres. The radius of the circle in centimetres is
   (a) 3
   (b) 22/7
   (c) 4
   (d) None of the above.

Solution :



Area of triangle ABC = √[{(15 + 17 + 8)/2}{(15 + 17 – 8)/2}{(15 + 8 – 17)/2}{(17 + 8 – 15)/2} = √(20*12*3*5) = 5*3*4

Now, area of triangle ABC = area of triangle AOB + area of triangle BOC + area of triangle COA = (1/2)*r*8 + (1/2)*r*17 + (1/2)*r*15 = (1/2)*(8 + 17 + 15)*r = 20*r

Now, 20*r = 5*3*4

⇨ r = 3
⇨ Option (a) is correct.

6. Let α, β and γ be the angles of an acute angled triangle. Then the quantity tanαtanβtanγ
   (a) Can have any real value
   (b) Is ≤ 3√3
   (c) Is ≥ 3√3
   (d) None of the above.

Solution :

Now, $\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$

- ⇨ $\tan(\pi - \gamma) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$ (As $\alpha + \beta + \gamma = \pi$)
- ⇨ $-\tan\gamma(1 - \tan\alpha\tan\beta) = (\tan\alpha + \tan\beta)$

Now, $\tan\alpha\tan\beta\tan\gamma$

$= \tan\gamma - \tan\gamma + \tan\alpha\tan\beta\tan\gamma$

$= \tan\gamma - \tan\gamma(1 - \tan\alpha\tan\beta)$

$= \tan\gamma + \tan\alpha + \tan\beta$ (From above)

So, we have, $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma = x$ (say)

Now, AM ≥ GM

- ⇨ $(\tan\alpha + \tan\beta + \tan\gamma)/3 \geq (\tan\alpha\tan\beta\tan\gamma)^{1/3}$
- ⇨ $x/3 \geq x^{1/3}$
- ⇨ $x^{2/3} \geq 3$
- ⇨ $x \geq 3\sqrt{3}$
- ⇨ Option (c) is correct.

7. Let $f(x) = |x|\sin x + |x - \pi|\cos x$ for $x \in \mathbb{R}$. Then
   (a) f is differentiable at $x = 0$ and $x = \pi$
   (b) f is not differentiable at $x = 0$ and $x = \pi$
   (c) f is differentiable at $x = 0$ but not differentiable at $x = \pi$
   (d) f is not differentiable at $x = 0$ but differentiable at $x = \pi$

Solution :

Now, $f(x) = |x|\sin x + |x - \pi|\cos x$

- ⇨ $f(x) = -x\sin x - (x - \pi)\cos x$ for $x \leq 0$
- ⇨ $f(x) = x\sin x - (x - \pi)\cos x$ for $0 < x \leq \pi$
- ⇨ $f(x) = x\sin x + (x - \pi)\cos x$ for $x > \pi$

Now, $\lim(f(x) - f(0))/(x - 0)$ as $x \to 0-$ $= \lim\{-x\sin x - (x - \pi)\cos x - \pi\}/x$ as $x \to 0-$ $= \lim\{-\sin x - x\cos x - \cos x + (x - \pi)\sin x\}/1$ as $x \to 0-$ (Applying L'Hospital rule) = - 1

Now, $\lim(f(x) - f(0))/(x - 0)$ as $x \to 0+$ $= \lim\{x\sin x - (x - \pi)\cos x - \pi\}/x$ as $x \to 0+$ $= \lim\{\sin x + x\cos x - \cos x + (x - \pi)\sin x\}/1$ as $x \to 0+$ (Applying L'Hospital rule) = -1

So, we have f is differentiable at x = 0 as both the limit are same.

Now, lim(f(x) – f(п))/(x – п) as x -> п- = lim{xsinx – (x – п)cosx – 0}/(x – п) as x -> п- = lim{sinx + xcosx – cosx + (x – п)sinx}/1 as x -> п- (Applying L'Hospital rule) = пcosп – cosп = -п + 1

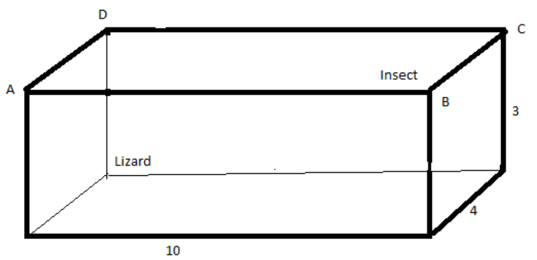
Now, lim(f(x) – f(п))/(x – п) as x -> п+ = lim{xsinx + (x – п)cosx – 0}/(x – п) as x -> п+ = lim{sinx + xcosx + cosx – (x – п)sinx}/1 as x -> п+ (Applying L'Hospital rule) = пcosп + cosп = -п – 1

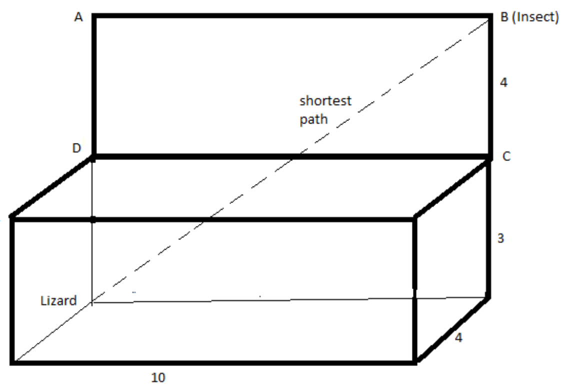
As both the limits are not same so we have, f(x) is not differentiable at x = п.

⇨ Option (c) is correct.

8. Consider a rectangular cardboard box of eight 3, breadth 4 and length 10 units. There is a lizard in one corner A of the box and an insect in the corner B which is farthest from A. The length of the shortest path between the lizard and the insect along the surface of the box is
(a) $\sqrt{(5^2 + 10^2)}$
(b) $\sqrt{(7^2 + 10^2)}$
(c) $4 + \sqrt{(3^2 + 10^2)}$
(d) $3 + \sqrt{(4^2 + 10^2)}$

Solution :





Clearly the shortest path is $\sqrt{(7^2 + 10^2)}$

⇨ Option (b) is correct.

9. Recall that, for any non-zero complex number w which does not lie on the negative real axis, arg(w) denotes the unique real number Ɵ in (-п, п) such that w = |w|(cosƟ + isinƟ). Let z be any complex

number such that its real and imaginary parts are both non-zero. Further, suppose that z satisfies the relations arg(z) > arg(z + 1) and arg(z) > arg(z+ i). Then cos(arg(z)) can take

(a) Any value in the set (-1/2, 0)∪(0, ½) but none from outside
(b) Any value in the interval (-1, 0) but none from outside
(c) Any value in the interval (0, 1) but none from outside
(d) Any value in the set (-1, 0)∪(0, 1) but none from outside.

Solution :

Now, z + 1 = cosΘ + isinΘ + 1 = (1 + cosΘ) + isinΘ

⇨ arg(z + 1) = $\tan^{-1}$\{(sinΘ/(1 + cosΘ)\} = $\tan^{-1}$[\{2sin(Θ/2)cos(Θ/2)\}/2$\cos^2$(Θ/2)] = $\tan^{-1}$\{tan(Θ/2)\} = Θ/2

Now, arg(z) > arg(z + 1)

⇨ Θ > Θ/2
⇨ Θ is positive.

Now, z + i = cosΘ + isinΘ + i = cosΘ + i(1 + sinΘ)

Now, arg(z + i) = $\tan^{-1}$\{(1 + sinΘ)/cosΘ\} = $\tan^{-1}$\{(cosΘ/2 + sinΘ/2)^2/(cosΘ/2 + sinΘ/2)(cosΘ/2 - sinΘ/2)\} = $\tan^{-1}$\{(cosΘ/2 + sinΘ/2)/(cosΘ/2 - sinΘ/2)\} = $\tan^{-1}$\{(1 + tanΘ/2)/(1 - tanΘ/2)\} = $\tan^{-1}$tan(π/4 + Θ/2) = π/4 + Θ/2.

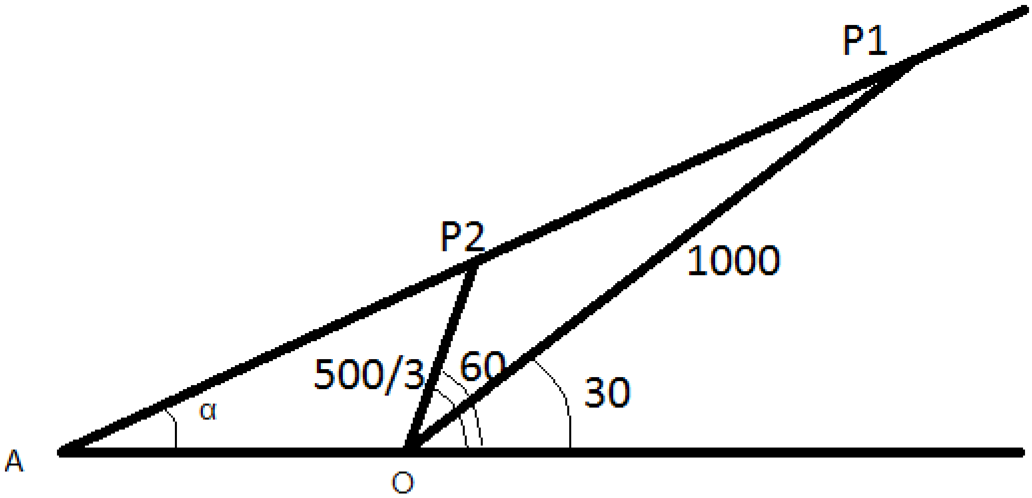
Now, arg(z) > arz(z + i)

⇨ Θ > π/4 + Θ/2
⇨ Θ > π/2

Now, for Θ > π/2 cosΘ is negative and can take any value in the interval (-1, 0)

⇨ Option (b) is correct.

10. An aeroplane P is moving in the air along a straight line path which passes through the points P_1 and P_2, and makes an angle α with the ground. Let O be the position of an observer as shown in the figure below. When the plane is at the position P_1 its angle of elevation is 30° and when it is at P_2its angle of elevation is 60° from the position of the observer. Moreover, the distances of the observer from the points P_1 and P_2 respectively are 100 metres and 500/3 metres.



Then α is equal to

(a) $\tan^{-1}\{(2 - \sqrt{3})/(2\sqrt{3} - 1)\}$
(b) $\tan^{-1}\{(2\sqrt{3} - 3)/(4 - 2\sqrt{3})\}$
(c) $\tan^{-1}\{(2\sqrt{3} - 2)/(5 - \sqrt{3})\}$
(d) $\tan^{-1}\{(6 - \sqrt{3})/(6\sqrt{3} - 1)$

Solution :

Angle $AOP_2 = 120°$

⇨ Angle $OP_2P_1 = 120° + \alpha$
⇨ Angle $OP_1P_2 = 180° - (120° + \alpha + 30°) = 30° - \alpha$ (As angle $P_1OP_2 = 60° - 30° = 30°$)

Now, in triangle OP_1P_2, we have, $POP_1/\sin(OP_2P_1) = OP_2/\sin(OP_1P_2)$

⇨ $1000/\sin(120° + \alpha) = 500/3\sin(30° - \alpha)$
⇨ $2*3\sin(30° - \alpha) = \sin(120° + \alpha)$
⇨ $6(\sin30°\cos\alpha - \cos30°\sin\alpha) = \sin120°\cos\alpha + \cos120°\sin\alpha$
⇨ $3\cos\alpha - 3\sqrt{3}\sin\alpha = (\sqrt{3}/2)\cos\alpha - (1/2)\sin\alpha$
⇨ $(6 - \sqrt{3})\cos\alpha = (6\sqrt{3} - 1)\sin\alpha$
⇨ $\tan\alpha = (6 - \sqrt{3})/(6\sqrt{3} - 1)$
⇨ $\alpha = \tan^{-1}\{(6 - \sqrt{3})/(6\sqrt{3} - 1)\}$
⇨ Option (d) is correct.

11. The sum of all even positive divisors of 1000 is

(a) 2170
(b) 2184
(c) 2325
(d) 2340

Solution :

Now, $1000 = 2^3 5^3$

Sum of all the divisors of 1000 (including 1 and 1000) = $\{(2^4 - 1)/(2 - 1)\}*\{(5^4 - 1)/(5 - 1)\} = 15*156$.

Sum of all the divisors of 5^3 (including 1 and 125) = $(5^4 - 1)/(5 - 1) = 156$.

So, sum of all even positive divisors of 1000 = $15*156 - 156 = 156(15 - 1) = 156*14 = 2184$.

⇨ Option (b) is correct.

12. The equation $x^2 + (b/a)x + (c/a) = 0$ has two real roots α and β. If $a > 0$, then the area under the curve $f(x) = x^2 + (b/a)x + (c/a)$ between α and β is
(a) $(b^2 - 4ac)/2a$
(b) $(b^2 - 4ac)^{3/2}/6a^3$
(c) $-(b^2 - 4ac)^{3/2}/6a^3$
(d) $-(b^2 - 4ac)/2a$

Solution :

Now, α, β are roots of the equation $x^2 + (b/a)x + (c/a)$

⇨ $\alpha + \beta = -(b/a)$ and $\alpha\beta = c/a$
⇨ $\beta - \alpha = \sqrt{\{(\beta + \alpha)^2 - 4\alpha\beta\}} = \sqrt{\{(b^2/a^2) - 4c/a\}} = \sqrt{(b^2 - 4ac)}/a$

Area under curve f(x) between α and β = ∫f(x)dx (integration running from α to β)

$= \int\{x^2 + (b/a)x + (c/a)\}dx$ (integration running from α to β)

$= [\{(x^3/3) + (b/a)(x^2/2) + (c/a)x]$ (from α to β)

$= \{(\beta^3/3) + (b/a)(\beta^2/2) + (c/a)\beta\} - \{(\alpha^3/3) + (b/a)(\alpha^2/2) + (c/a)\alpha\}$

$= (\beta^3 - \alpha^3)/3 + (b/a)(\beta^2 - \alpha^2)/2 + (c/a)(\beta - \alpha)$

$= (\beta - \alpha)(\beta^2 + \alpha\beta + \alpha^2)/3 + (b/a)(\beta - \alpha)(\beta + \alpha)/2 + (c/a)(\beta - \alpha)$

$= (\beta - \alpha)[\{(\beta + \alpha)^2 - \alpha\beta\}/3 + (b/a)(\beta + \alpha)/2 + (c/a)]$

$= \{\sqrt{(b^2 - 4ac)}/a\}[\{(b^2/a^2) - (c/a)\}/3 - (b/a)(b/a)/2 + (c/a)]$

$= \{\sqrt{(b^2 - 4ac)}/a\}\{-(b^2/a^2)/6 + (2/3)(c/a)\}$

$= -\{\sqrt{(b^2 - 4ac)}/a\}(b^2 - 4ac)/6a$

$= -(b^2 - 4ac)^{3/2}/6a^2$

⇨ Option (c) is correct.

13. The minimum value of $x_1^2 + x_2^2 + x_3^2 + x_4^2$ subject to $x_1 + x_2 + x_3 + x_4 = a$ and $x_1 - x_2 + x_3 - x_4 = b$ is
(a) $(a^2 + b^2)/4$
(b) $(a^2 + b^2)/2$
(c) $(a + b)^2/4$
(d) $(a + b)^2/2$

Solution :

Now, $x_1 + x_3 = (a + b)/2$ and $x_2 + x_4 = (a - b)/2$

Now, $(x_1^2 + x_3^2)/2 \geq \{(x_1 + x_3)/2\}^2$

⇨ $(x_1^2 + x_3^2) \geq (a + b)^2/8$

Similarly, $(x_2^2 + x_4^2) \geq (a - b)^2/8$

Adding the above two inequalities we get, $x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq \{(a + b)^2 + (a - b)^2\}/8 = 2(a^2 + b^2)/8 = (a^2 + b^2)/4$

⇨ Option (a) is correct.

14. The value of $\lim[\{\Sigma({}^{2n}C_{2r}*3^r)\}/\{\Sigma({}^{2n}C_{2r+1}*3^r)\}]$ (numerator summation running from r = 0 to r = n, denominator summation is running from r = 0 to r n-1) as $n \to \infty$ is
(a) 0
(b) 1
(c) $\sqrt{3}$
(d) $(\sqrt{3} - 1)/(\sqrt{3} + 1)$

Solution :

Now, $(1 + x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1*x + {}^{2n}C_2*x^2 + {}^{2n}C_3x^3 + \ldots + {}^{2n}C_{2n-1}*x^{2n-1} + {}^{2n}C_{2n}*x^{2n}$

$(1 - x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1*x + {}^{2n}C_2*x^2 - {}^{2n}C_3*x^3 - \ldots - {}^{2n}C_{2n-1}*x^{2n-1} + {}^{2n}C_{2n}*x^{2n}$

Adding we get, $(1 + x)^{2n} + (1 - x)^{2n} = 2[{}^{2n}C_0 + {}^{2n}C_2*x^2 + {}^{2n}C_4*x^4 + \ldots + {}^{2n}C_{2n}x^{2n}$

Subtracting we get, $(1 + x)^{2n} - (1 - x)^{2n} = 2x[{}^{2n}C_1 + {}^{2n}C_3*x^2 + {}^{2n}C_5*x^4 + ... + {}^{2n}C_{2n-1}*x^{2n-2})$

Now, putting $x = \sqrt{3}$ we get, $\{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}\}/2 = {}^{2n}C_0 + {}^{2n}C_2*3 + {}^{2n}C_4*3^2 + ... + {}^{2n}C_{2n}*3^n$ = Numerator.

⇨ Numerator = $\{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}\}/2$

Similarly putting $x = \sqrt{3}$ in the 2nd equation we get,

Denominator = $\{(1 + \sqrt{3})^{2n} - (1 - \sqrt{3})^{2n}\}/2\sqrt{3}$

Therefore, the given expression = $\lim[\{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}\}/2]/[\{(1 + \sqrt{3})^{2n} - (1 - \sqrt{3})^{2n}\}/2\sqrt{3}]$ as $n \to \infty$

$= \sqrt{3} \lim [1 + \{(\sqrt{3} - 1)/(\sqrt{3} + 1)\}^{2n}]/[1 - \{(\sqrt{3} - 1)/(\sqrt{3} + 1)\}^{2n}]$ as $n \to \infty$

$= \sqrt{3}(1 - 0)/(1 - 0)$

$= \sqrt{3}$

⇨ Option (c) is correct.

15. For any real number x l $\tan^{-1}(x)$ denote the unique real number Θ in $(-\pi/2, \pi/2)$ such that $\tan\Theta = x$. Then $\lim[\Sigma\tan^{-1}\{1/(1 + m + m^2)\}]$ where m runs from 1 to n as $n \to \infty$
(a) Is equal to $\pi/2$
(b) Is equal to $\pi/4$
(c) Does not exist
(d) None of the above.

Solution :

Now, $\tan^{-1}\{1/(1 + m + m^2)\} = \tan^{-1}[\{(m + 1) - m\}/\{1 + m(m+1)\}] = \tan^{-1}(m + 1) - \tan^{-1}(m)$

$\tan^{-1}\{1/(1 + 1 + 1^2)\} = \tan^{-1}(2) - \tan^{-1}(1)$

$\tan^{-1}\{1/(1 + 2 + 2^2)\} = \tan^{-1}(3) - \tan^{-1}(2)$

$\tan^{-1}\{1/(1 + 3 + 3^2)\} = \tan^{-1}(4) - \tan^{-1}(3)$

.....

....

$\tan^{-1}\{1/(1 + n + n^2)\} = \tan^{-1}(n + 1) - \tan^{-1}(n)$

- ⇨ $\sum\tan^{-1}\{1/(1 + m + m^2)\} = \tan^{-1}(n + 1) - \tan^{-1}(1)$ where m runs from 1 to n

The given expression becomes, $\lim\{\tan^{-1}(n + 1) - \tan^{-1}(1)\}$ as n -> ∞

$= \tan^{-1}(\infty) - \tan^{-1}(1)$

$= (\pi/2) - (\pi/4)$

$= \pi/4$

- ⇨ Option (b) is correct.

16. Let n be an integer. The number of primes which divide both $n^2 - 1$ and $(n + 1)^2 - 1$ is
(a) At most one.
(b) Exactly one.
(c) Exactly two.
(d) None of the above.

Solution :

Now, $n^2 - 1 = (n + 1)(n - 1)$ and $(n + 1)^2 - 1 = n(n + 2)$

Now, n + 1 is relatively prime with both n and n + 2 as they are consecutive integers.

So, only possible case is a prime may divide n – 1 and n + 2 as n – 1 and n are also relatively prime.

We see that6 if 3 divides n – 1 then 3 divides n + 2.

We also see that if 3 divides n or n +1 then 3 doesn't divide the rest three.

So, at most one prime (3) can divide both $n^2 - 1$ and $(n + 1)^2 - 1$.

- ⇨ Option (a) is correct.

17. The value of $\lim[\sum\{6n/(9n^2 - r^2)\}]$ as n -> ∞ is
(a) 0
(b) log(3/2)
(c) log(2/3)
(d) log(2)

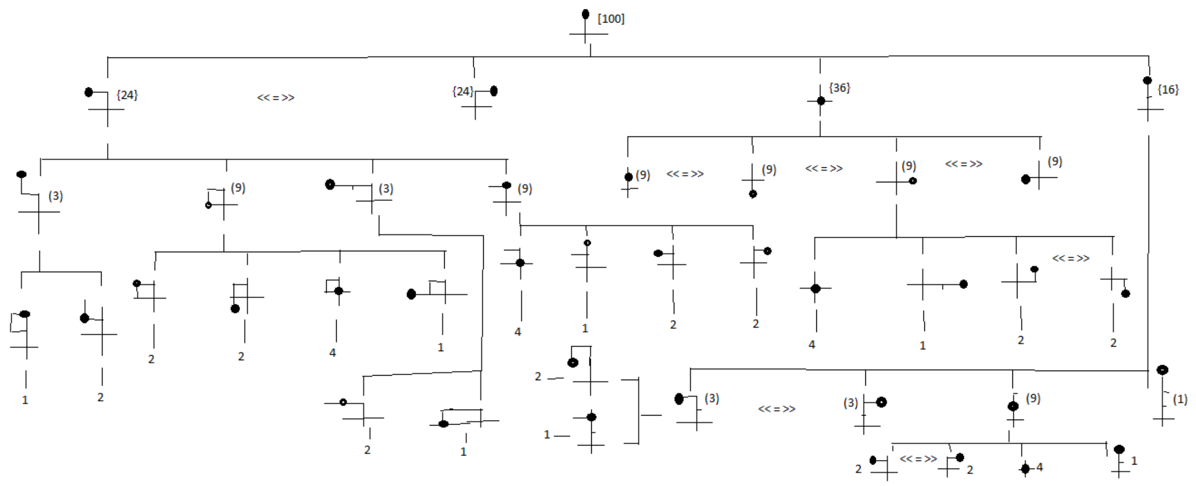
Solution :

Option (d) is correct.

18. A person X standing at a point P on a flat plane starts walking. At each step, he walks exactly 1 foot in one of the directions North, South, East or West. Suppose that after 6 steps X comes to the original position P. Then the number of distinct paths that X can take is
(a) 196
(b) 256
(c) 344
(d) 400.

Solution :

In the picture below we have showed the cases for 1st step towards north. There are total 4 cases for 1st step viz. north, south, east, west. So if we find number of cases for any one then we can get total number of cases by multiplying it by 4. In the diagram equivalent cases are shown by "<< = >>". The cases for 5th and 6th steps are omitted as 6th step can be taken in 1 way always and 5th step we can calculate in mind. The numbers beside the cases written are the number of ways for that particular case which is calculated by summing up the cases under that.



Hence, total number of cases = 100*4 = 400.

⇨ Option (d) is correct.

19. Consider the branch of the rectangular hyperbola xy = 1 in the first quadrant. Let P be a fixed point on this curve. The locus of the mid-point of the line segment joining P and an arbitrary point Q on the curve is part of

(a) A hyperbola
(b) A parabola
(c) An ellipse
(d) None of the above.

Solution :

Let co-ordinate of point P is (a, b)

Let co-ordinate of point Q is (h, k)

Co-ordinate of mid-point of P and Q is {(h + a)/2, (k + b)/2}

Now, P and Q lies on the hyperbola xy = 1

So, we have, ab = 1 and hk = 1.

Putting (h + a)/2 = x and (k + b)/2 = y i.e. h = 2x – a and k = 2y – b we get,

(2x – a)(2y – b) = 1

- ⇨ 4xy – 2ay – 2bx + ab = 1
- ⇨ 4xy – 2ay – 2bx = 0 (As ab = 1)
- ⇨ xy – bx/2 – ay/2 = 0

Which is the equation of a hyperbola.

- ⇨ Option (a) is correct.

20. The digit at the unit place of $(1! - 2! + 3! - \ldots + 25!)^{(1! - 2! + 3! - \ldots + 25!)}$ is
(a) 0
(b) 1
(c) 5
(d) 9

Solution :

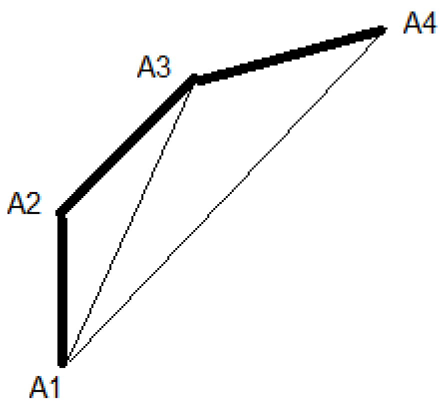
Now, 1! – 2! + 3! – 4! + 5! = 1 – 2 + 6 – 24 + 120 = 101

Now, last digit of - 6! + 7! - .... + 25! Is zero.

- ⇨ 1! – 2! + 3! - .... + 25! Has the last digit as 1.
- ⇨ The last digit of $(1! - 2! + 3! - \ldots + 25!)^{(1! - 2! + 3! - \ldots + 25!)}$ is 1 because 1 to the power anything = 1.
- ⇨ Option (b) is correct.

21. Let A_1, A_2, ..., A_n be the vertices of a regular polygon and A_1A_2, A_2A_3, ..., $A_{n-1}A_n$, A_nA_1 be its n sides. If $(1/A_1A_2) - (1/A_1A_4) = (1/A_1A_3)$, then the value of n is
(a) 5
(b) 6
(c) 7
(d) 8

Solution :



Let, the side of the regular polygon is a.

Now, angle $A_1A_2A_3 = (n - 2)\pi/n$

In triangle $A_1A_2A_3$, $A_1A_2 = A_2A_3$

- ⇨ Angle $A_2A_1A_3$ = Angle $A_2A_3A_1$
- ⇨ Angle $A_2A_3A_1 = (1/2)[\pi - (n - 2)\pi/n] = \pi/n$

In triangle $A_1A_2A_3$, $A_1A_3/\sin\{(n - 2)\pi/n\} = a/\sin(\pi/n)$

- ⇨ $A_1A_3 = a*\sin(\pi - 2\pi/n)/\sin(\pi/n) = a*\sin(2\pi/n)/\sin(\pi/n) = 2a*\cos(\pi/n)$

Let, Angle $A_3A_1A_4 = A$

Now, angle $A_1A_3A_4 = (n - 2)\pi/n - \pi/n = \pi - 3\pi/n$

Angle $A_1A_4A_3 = \pi - A - \pi + 3\pi/n = 3\pi/n - A$

Now, in triangle $A_1A_3A_4$ we have,

$A_1A_4/\sin(\pi - 3\pi/n) = a/\sin A = A_1A_3/\sin(3\pi/n - A)$

From, $a/\sin A = A_1A_3/\sin(3\pi/n - A)$ we get,

$\cot A = \{2\cos(\pi/n) + \cos(3\pi/n)\}/\sin(3\pi/n)$ (using $\sin(3\pi/n - A) = \sin(3\pi/n)\cos A - \cos(3\pi/n)\sin A$)

- ⇨ $\sin A = \sin(3\pi/n)/\sqrt{[1 + 4\cos(\pi/n)\cos(3\pi/n) + 4\cos^2(\pi/n)]}$

Now, from the relation, $A_1A_4/\sin(\pi - 3\pi/n) = a/\sin A$ we get,

$A_1A_4/\sin(3\pi/n) = a*\sqrt{[1 + 4\cos(\pi/n)\cos(3\pi/n) + 4\cos^2(\pi/n)]}/\sin(3\pi/n)$

- ⇨ $A_1A_4 = a*\sqrt{[1 + 4\cos(\pi/n)\{4\cos^3(\pi/n) - 3\cos(\pi/n)\} + 4\cos^2(\pi/n)]}$
- ⇨ $A_1A_4 = a(4\cos^2(\pi/n) - 1)$

Now, given $1/A_1A_2 - 1/A_1A_3 = 1/A_1A_4$

- ⇨ $(1/a) - 1/2a\cos(\pi/n) = 1/a(4\cos^2(\pi/n) - 1)$
- ⇨ $1 - 1/2\cos(\pi/n) = 1/(4\cos^2(\pi/n) - 1)$

This equation is satisfied by $n = 7$.

- ⇨ Option (c) is correct.

22. Suppose that α and β are two distinct numbers in the interval $(0, \pi)$. If $\sin\alpha + \sin\beta = \sqrt{3}(\cos\alpha - \cos\beta)$ then the value of $\sin3\alpha + \sin3\beta$ is
    (a) 0
    (b) $2\sin\{3(\alpha + \beta)/2\}$
    (c) $2\cos\{3(\alpha - \beta)/2\}$
    (d) $\text{Cos}\{3(\alpha - \beta)/2\}$

Solution :

Now, $\sin\alpha + \sin\beta = \sqrt{3}(\cos\alpha - \cos\beta)$

- ⇨ $2\sin\{(\alpha + \beta)/2\}\cos\{(\alpha - \beta)/2\} = \sqrt{3}*2\sin\{(\alpha + \beta)/2\}\sin\{(\beta - \alpha)/2\}$
- ⇨ $\cos\{(\alpha - \beta)/2\} = -\sqrt{3}\sin\{(\alpha - \beta)/2\}$
- ⇨ $\tan\{(\alpha - \beta)/2\} = -1/\sqrt{3}$
- ⇨ $\tan\{(\alpha - \beta)/2\} = \tan(\pi - \pi/6)$
- ⇨ $(\alpha - \beta)/2 = \pi - \pi/6$

Now, $\sin3\alpha + \sin3\beta$

$= 2\sin\{3(\alpha + \beta)/2\}\cos\{3(\alpha - \beta)/2\}$

$= 2\sin\{3(\alpha + \beta)/2\}\cos\{3(\pi - \pi/6)\}$

$= 2\sin\{3(\alpha + \beta)/2\}\cos(3\pi - \pi/2)$

$= 2\sin\{3(\alpha + \beta)/2\}\{-\cos(\pi/2)\}$

$= 0$ (As $\cos(\pi/2) = 0$)

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⇨ Option (a) is correct.

23. Consider the function $h(x) = x^2 - 2x + 2 + 4/(x^2 - 2x + 2)$, $x \in R$. Then $h(x) - 5 = 0$ has
(a) No solution
(b) Only one solution
(c) Exactly two solutions
(d) Exactly three solutions.

Solution :

Now, $h(x) - 5 = 0$

⇨ $x^2 - 2x + 2 + 4/(x^2 - 2x + 2) - 5 = 0$
⇨ $(x^2 - 2x + 2)^2 - 5(x^2 - 2x + 2) + 4 = 0$ (As $x^2 - 2x + 2 \neq 0$ because roots are imaginary)
⇨ $(x^2 - 2x + 2)^2 - 4(x^2 - 2x + 2) - (x^2 - 2x + 2) + 4 = 0$
⇨ $(x^2 - 2x + 2)(x^2 - 2x + 2 - 4) - (x^2 - 2x + 2 - 4) = 0$
⇨ $(x^2 - 2x + 2)(x^2 - 2x - 2) - (x^2 - 2x - 2) = 0$
⇨ $(x^2 - 2x - 2)(x^2 - 2x + 2 - 1) = 0$
⇨ $(x^2 - 2x - 2)(x^2 - 2x + 1) = 0$
⇨ $(x^2 - 2x - 2)(x - 1)^2 = 0$

So, $x = 1$ is a solution and $x^2 - 2x - 2 = 0$

⇨ $x = \{2 \pm \sqrt{(4 + 8)}\}/2$
⇨ $x = 1 \pm \sqrt{3}$

So, the equation has 3 solutions viz. $x = 1, 1 + \sqrt{3}$ and $1 - \sqrt{3}$

⇨ Option (d) is correct.

24. Consider the quadratic equation $x^2 + bx + c = 0$. The number of pairs (b, c) which the equation has solutions of the form $\cos\alpha$ and $\sin\alpha$ for some α is
(a) 0
(b) 1
(c) 2
(d) Infinite.

Solution :

Option (d) is correct.

25. Let $\Theta_1 = 2\pi/3$, $\Theta_2 = 4\pi/7$, $\Theta_3 = 7\pi/12$. Then
(a) $(\sin\Theta_1)^\wedge(\sin\Theta_1) < (\sin\Theta_2)^\wedge(\sin\Theta_2) < (\sin\Theta_3)^\wedge(\sin\Theta_3)$
(b) $(\sin\Theta_2)^\wedge(\sin\Theta_2) < (\sin\Theta_1)^\wedge(\sin\Theta_1) < (\sin\Theta_3)^\wedge(\sin\Theta_3)$
(c) $(\sin\Theta_3)^\wedge(\sin\Theta_3) < (\sin\Theta_1)^\wedge(\sin\Theta_1) < (\sin\Theta_2)^\wedge(\sin\Theta_2)$
(d) $(\sin\Theta_1)^\wedge(\sin\Theta_1) < (\sin\Theta_3)^\wedge(\sin\Theta_3) < (\sin\Theta_2)^\wedge(\sin\Theta_2)$

Solution :

Option (d) is correct.

26. Consider the following two curves on the interval (0, 1) :
$C_1 : y = 1 - x^4$ and $C_2 : y = \sqrt{(1 - x^2)}$.
Then on (0, 1)
(a) C_1 lies above C_2
(b) C_2 lies above C_1
(c) C_1 and C_2 intersect at exactly one point
(d) None of the above.

Solution :

If we look for the point of intersection of C_1 and C_2 we get,

$1 - x^4 = \sqrt{(1 - x^2)}$

- ⇨ $(1 + x^2)(1 - x^2) = \sqrt{(1 - x^2)}$
- ⇨ $(1 + x^2)\sqrt{(1 - x^2)} = 0$
- ⇨ $1 - x^2 = 0$ (As $1 + x^2$ cannot be equal to 0)
- ⇨ x = 1 (As -1 doesn't fall in the interval (0, 1))
- ⇨ C_1 and C_2 intersect at exactly one point.
- ⇨ Option (c) is correct.

27. Let f be a real valued function on R such that f is twice differentiable. Suppose that f′ vanishes only at 0 and f′′ is everywhere negative. Define a function h by $h(x) = (x - a)^2 - f(x)$, where $a > 0$. Then
(a) h has a local minima in (0, a)
(b) h has a local maxima in (0, a)
(c) h is monotonically increasing in (0, a)
(d) h is monotonically decreasing in (0, a).

Solution :

Now, $h(x) = (x - a)^2 - f(x)$

- ⇨ $h'(x) = 2(x - a) - f'(x)$
- ⇨ $h''(x) = 2 - f''(x)$

Now, $f''(x)$ is negative everywhere.

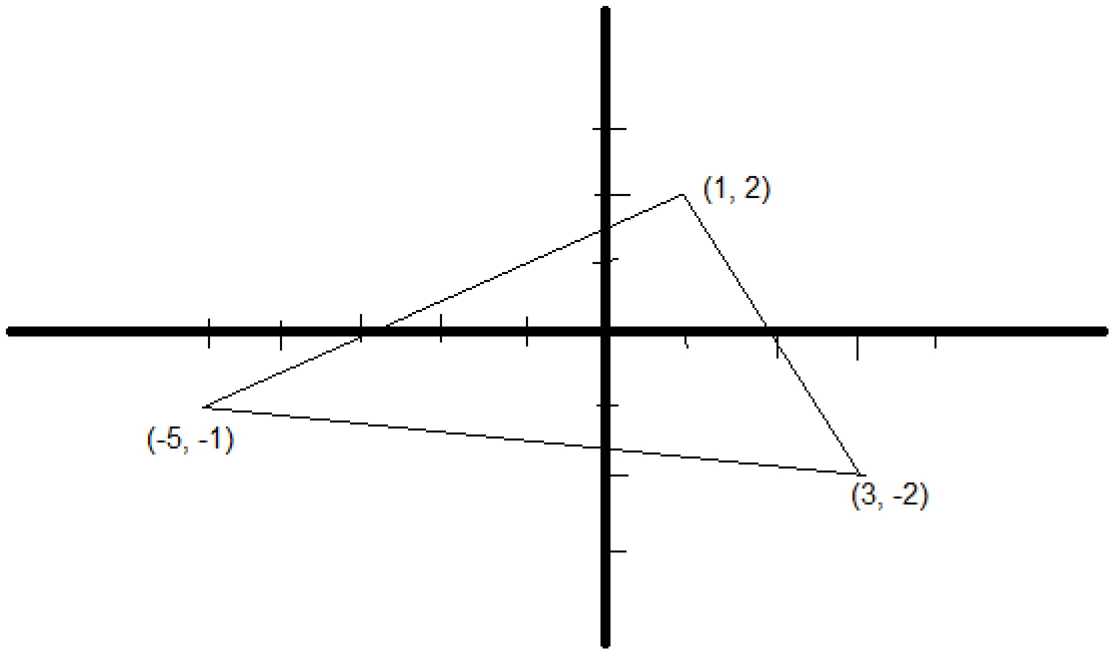
- ⇨ $h''(x) > 0$ everywhere.
- ⇨ $h(x)$ has a local minima.

Now, $h'(x) = 0$ gives, $x = a + f'(x)/2$

- ⇨ Option (a) is correct.

28. Consider the triangle with vertices (1, 2), (-5, -1) and (3, -2). Let Δ denote the region enclosed by the above triangle. Consider the function f : Δ -> R defined by $f(x, y) = |10x - 3y|$. Then the range of f is the interval
(a) [0, 36]
(b) [0, 47]
(c) [4, 47]
(d) [36, 47]

Solution :



Clearly (0, 0) is inside the triangle.

Hence, f(0, 0) = 0 in R. So, 0 is in the rang of f.

- ⇨ Option (c) and (d) cannot be true.

Now, (-5, -1) is in Δ

f(-5, -1) = |10*(-5) – 3*(-1)| = 47

So, 47 is in range of f.

- ⇨ Option (a) cannot be true.
- ⇨ Option (b) is correct.

29. For every positive integer n, let <n> denote the integer closest to √n. Let A_k = {n > 0 : <n> = k}. The number of elements in A_{49} is
 (a) 97
 (b) 98
 (c) 99
 (d) 100

Solution :

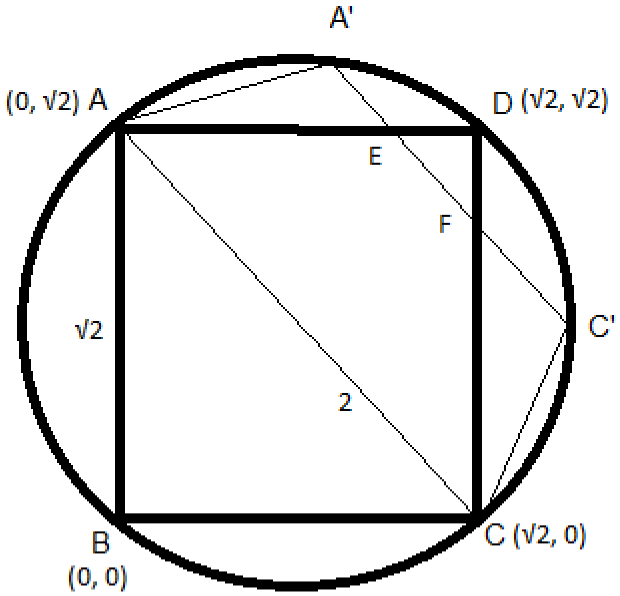
49.5^2 = 2450.25 and 48.5^2 = 2352.25

Number of elements in A_{49} = 2450 – 2353 + 1 = 98.

- ⇨ Option (b) is correct.

30. Consider a square ABCD inscribed in a circle of radius 1. Let A′ and C′ be two points on the (smaller) arcs AD and CD respectively, such that A′ABCC′ is a pentagon in which AA′ = CC′. If P denotes the area of the pentagon A′ABCC′ then
 (a) P cannot be equal to 2.
 (b) P lies in the interval (1, 2].
 (c) P is greater than or equal to 2.
 (d) None of the above.

Solution :



Clearly from the picture none of option (a), (b), (c) always correct because A′ and C′ can move from A to D and C to D respectively. When A′ and C′ very close to D i.e. A′D = C′D -> 0 then area of the trapezium A′ACC′ is near to 1 and when A′ and C′ are close to A and C respectively i.e. AA′ and CC′ both -> 0 then area of trapezium A′ACC′ is near to 0.

Also, clearly, area of triangle AA′E = area of triangle CC′F

If area of triangle AA′E > (area of triangle DEF)/2 then area of the pentagon A′ABCC′ greater than 2 as area of triangle ADC = 1

When A′ is at position such that E is the mid-point of AD.

Then area of triangle DEF = $(1/2)*(1/\sqrt{2})*(1/\sqrt{2}) = 1/4$

Then, AE = $1/\sqrt{2}$ and perpendicular from A′ to AE = $1 - 1/\sqrt{2}$

So, area of triangle A′AE = $(1/2)*(1/\sqrt{2})*(1 - 1/\sqrt{2}) = (\sqrt{2} - 1)/4$

- ⇨ Area of triangle A′AE + area of triangle C′CF = $(\sqrt{2} - 1)/2$

Now, $(\sqrt{2} - 1)/2 - 1/4 = (2\sqrt{2} - 3)/4 > 0$

- ⇨ Area of the pentagon can be > 2 and also from above discussion we have area of pentagon can be < 2 and so it can be equal to 2.
- ⇨ Option (d) is correct.

B. Math. (Hons.) Admission Test : 2010

1. The product of the first 100 positive integers ends with

(a) 21 zeros
(b) 22 zeros
(c) 23 zeros
(d) 24 zeros.

Solution :

Number of zeros = [100/5] + [100/5^2] where [x] denotes the largest integer less than or equal to x.

= 20 + 4 = 24.

⇨ Option (d) is correct.

2. Given four 1-gm stones, four 5-gm stones, four 25-gm stones and four 125-gm stones each, it is possible to weigh material of any integral weight up to
   (a) 600 gms
   (b) 625 gms
   (c) 624 grms
   (d) 524 grms

Solution :

Any integer ≡ 1, 2, 3, 4, 5 (mod 5)

Now, it is clear that we can measure any weight up to 24 with four 1-gm and four 5-gm stones.

We can measure 25.

Similarly, we can measure weights up to 50-gm, 75-gm, 100-gm, 124-gm.

Now, 125-gm is available.

So, we can measure up to 150-gm, 175-gm, 200-gm, 225-gm, 250-gm, 500-gm, 525-gm, 550-gm, 575-gm, 600-gm, 624-gm.

⇨ Option (c) is correct.

3. The function f(x) = |x| + sin(x) + $\cos^3(x)$ is
   (a) Continuous but not differentiable at x = 0
   (b) Differentiable at x = 0
   (c) A bounded function which is not continuous at x = 0

(d) A bounded function which is continuous at $x = 0$.

Solution :

$f(x) = x + \sin(x) + \cos^3(x)$ for $x \geq 0$

$f(x) = -x + \sin(x) + \cos^3(x)$ for $x < 0$

Now, lim f(x) as x -> 0+ = lim $\{x + \sin(x) + \cos^3(x)\}$ as x -> 0+ = 1

And, lim f(x) as x -> 0- = lim $\{-x + \sin(x) + \cos^3(x)\}$ as x -> 0- = 1

And, f(0) = 1

As, lim f(x) as x -> 0+ = lim f(x) as x -> 0- = f(0) = 1, so f(x) is continuous at x = 0

Now, lim $[\{f(x) – f(0)\}/(x – 0)]$ as x -> 0+ = lim $[\{x + \sin(x) + \cos^3(x) – 1\}/x]$ as x -> 0+ = lim $[\{1 + \cos(x) – 3\cos^2(x)\sin(x)\}/1$ as x -> 0+ (Applying L'Hospital rule) = (1 + 1 – 0)/1 = 2

And, lim $[\{f(x) – f(0)\}/(x – 0)]$ as x -> 0- = lim $[\{-x + \sin(x) + \cos^3(x) – 1\}/x]$ as x -> 0- = lim $[\{-1 + \cos(x) – 3\cos^2(x)\sin(x)\}/1\}$ as x -> 0- (Applying L'Hospital rule) = (-1 + 1 – 0)/1 = 0

As, lim $[\{f(x) – f(0)\}/(x – 0)]$ as x -> 0+ ≠ lim $[\{f(x) – f(0)\}/(x – 0)]$ as x -> 0-, so f(x) is not differentiable at x = 0.

Now, f(x) is not bounded above or below as f(x) -> ∞ when x -> ∞ .

- ⇨ Option (a) is correct.

4. The sum of the first n terms of an arithmetic progression whose first term is a (not necessarily positive) integer and common difference is 2, is known to be 153. If n> 1, then number of possible values of n is
   (a) 2
   (b) 3
   (c)4
   (d) 5

Solution :

Let the first term of the arithmetic progression is a.

We have, $(n/2)\{2a + (n – 1)*2\} = 153$

- ⇨ $n(a + n – 1) = 153$

⇨ $n^2 + n(a - 1) - 153 = 0$
⇨ $n = [-(a - 1) \pm \sqrt{(a - 1)^2 + 4*1*153}]/2$

Let, $(a - 1)^2 + 4*1*153 = p^2$

⇨ $p^2 - (a - 1)^2 = 2^2*3^2*17$
⇨ $\{p + (a - 1)\}\{p - (a - 1)\} = 2^2*3^2*17$

Now, $p + (a - 1) = 2*3*17$ and $p - (a - 1) = 2*3$

⇨ $p = 54$ and $a - 1 = 48$ gives $n = 3$.

Let, $p + (a - 1) = 2*3$ and $p - (a - 1) = 2*3*17$

⇨ $p = 54$ and $a - 1 = -48$ gives $n = 51$.

Let, $p + (a - 1) = 2*17$ and $p - (a - 1) = 2*3^2$

⇨ $p = 26$ and $a - 1 = 8$ gives $n = 9$

Let, $p + (a - 1) = 2*3^2$ and $p - (a - 1) = 2*17$

⇨ $p = 26$ and $a - 1 = -8$ gives $n = 17$

Let, $p + (a - 1) = 2*3^2*17$ and $p - (a - 1) = 2$

⇨ $p = 154$ and $a - 1 = 152$ gives $n = 1$ (not possible as $n > 1$)

Let, $p + (a - 1) = 2$ and $p - (a - 1) = 2*3^2*17$

⇨ $p = 154$ and $a - 1 = -152$ gives $n = 153$

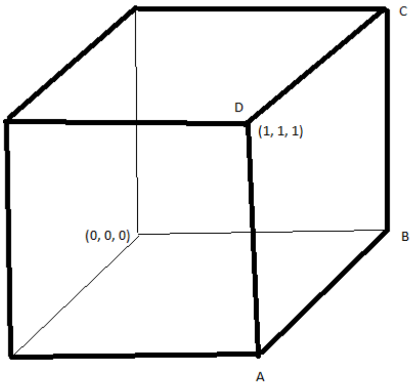
No other combination is possible as $(a - 1)$ and p are either both odd or both even.

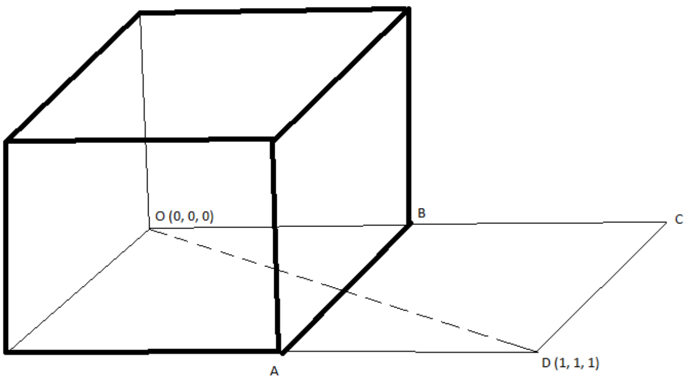
So, possible values of n are 3, 51, 9, 17 and 153.

⇨ Option (d) is correct.

5. Consider a cubical box of 1 m side which has one corner placed at (0, 0, 0) and the opposite corner placed at (1, 1, 1). The least distance that an ant crawling from the point (0, 0, 0) to the point (1, 1, 1) must travel is
(a) $\sqrt{6}$ m
(b) $\sqrt{5}$ m
(c) $2\sqrt{3}$ m
(d) $1 + \sqrt{3}$ m.

Solution :





We have opened the side ABCD of the cube and clearly OD is the shortest path to reach D from O.

Clearly, OD = $\sqrt{(OC^2 + CD^2)} = \sqrt{(2^2 + 1^2)} = \sqrt{5}$.

- ⇨ Option (b) is correct.

6. Let $x_1 < -1$ and $x_{n+1} = x_n/(1 + x_n)$ for all $n \geq 1$. Then
   (a) $\{x_n\} \to -1$ as $n \to \infty$
   (b) $\{x_n\} \to 1$ as $n \to \infty$
   (c) $\{x_n\} \to 0$ as $n \to \infty$
   (d) $\{x_n\}$ diverges.

Solution :

Now, $x_1 < -1$

- ⇨ $1 + x_1 < 0$
- ⇨ $x_2 = x_1/(1 + x_1) > 0$ as both numerator and denominator < 0
- ⇨ All $x_i > 0$ except x_1

Therefore, option (a) cannot be true.

Now, $x_3 = x_2/(1 + x_2) < x_2$

Similarly, $x_4 < x_3$ and in general we can write $x_i < x_j$ if $i > j$

- ⇨ Option (d) cannot be true.

Also, $x_3 = x_2/(1 + x_2) < 1$

Similarly all terms of the sequence $\{x_n\} < 1$ except x_2

So, $0 < \{x_n\} < 1$ for $n > 2$

And $\{x_n\}$ is approaching towards 0 as $n \to \infty$

⇨ Option (c) is correct.

7. The number of perfect cubes among the first 4000 positive integers is
   (a) 16
   (b) 15
   (c) 14
   (d) 13

Solution :

Clearly $15^3 = 3375$ and $16^3 = 4096$

⇨ Option (b) is correct.

8. The roots of the equation $x^4 + x^2 = 1$ are
   (a) All real and positive
   (b) Never real
   (c) 2 positive and 2 negative
   (d) 1 positive, 1 negative and 2 non-real.

Solution :

Now, $x^4 + x^2 - 1 = 0$

⇨ $x^2 = [-1 \pm \sqrt{1^2 - 4*1(-1)}]/2*1 = (-1 \pm \sqrt{5})/2$

Now, for $x^2 = (-1 - \sqrt{5})/2$ two roots are non-real

Now, we consider, $x^2 = (\sqrt{5} - 1)/2$

⇨ $x = \pm\sqrt{(\sqrt{5} - 1)/2}$
⇨ one is positive and another is negative.
⇨ Option (d) is correct.

9. If $\{(x + a)/(x - a)\}^x \to 9$ as $x \to \infty$ then a =
   (a) 3^e
   (b) log(3)
   (c) log(9)
   (d) 3.

Solution :

Now, $\{(x + a)/(x - a)\}^x = (1 + a/x)^x/(1 - a/x)^x$

Now, $\lim (1 + a/x)^x/(1 - a/x)^x$ as $x \to \infty = e^a/(1/e^a) = e^{2a} = 9$ (given)

- ⇨ $2a = \log(9)$
- ⇨ $2a = 2\log(3)$
- ⇨ $a = \log(3)$
- ⇨ Option (b) is correct.

10. The number of multiples of 4 among all 10 digit numbers is
    (a) $25*10^8$
    (b) $25*10^7$
    (c) $225*10^7$
    (d) $234*10^7$

Solution :

It's an arithmetic progression with first term = 10^9 and common difference = 4 and last term = $10^{10} - 4$.

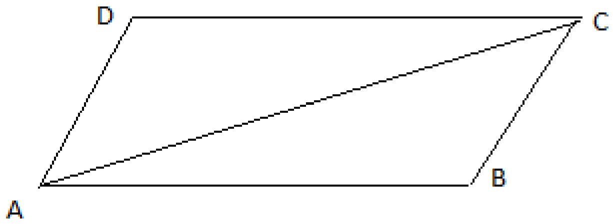
Let number of terms = n

Then, $10^9 + (n - 1)*4 = 10^{10} - 4$

- ⇨ $25*10^7 + n - 1 = 25*10^8 - 1$
- ⇨ $n = 25*10^8 - 25*10^7 = 10^7(250 - 25) = 225*10^7$
- ⇨ Option (c) is correct.

11. The larger diagonal of a parallelogram of area 8 must have length
    (a) At least 4
    (b) Equal to 8
    (c) At most 4
    (d) Equal to $\sqrt{8}$.

Solution :



Area of triangle ABC = 8/2 = 4

Clearly point C can rotate among point B and length of the diagonal gets changed.

So, any particular value cannot be an answer.

- ⇨ Option (b) and (d) cannot be true.

Now, let AB = 4 and perpendicular from C to AB = 2.

Then area of triangle ABC = (1/2)*4*2 = 4

So, we are good.

Now, AC > $\sqrt{(4^2 + 2^2)}$ (As perpendicular from C to AB will be stretched portion of AB)

- ⇨ AC > 4
- ⇨ Option (d) cannot be true.
- ⇨ Option (a) is correct.

12. Let $\{a_n\}$ be a sequence of positive real numbers such that lim $2^n(a_n + a_{n+1})$ as n -> ∞ is finite. Then
    (a) $\{a_n\}$ converges to 1
    (b) $\{a_n\}$ converges to 0
    (c) $\{a_n\}$ converges to ½
    (d) $\{a_n\}$ converges to $1/\sqrt{2}$.

Solution :

Option (b) is correct.

13. Consider the following two statements about a positive integer n and choose the correct option below.
    (I) n is perfect square.
    (II) The number of positive integer divisors of n is odd.
    (a) I and II are equivalent

(b) I implies II but not conversely
(c) II implies I but not conversely
(d) Nithr statement implies the other.

Solution :

Any perfect square number has odd number of factors or divisors.

⇨ Option (a) is correct.

14. For triangles ABC and PQR, it is given that AB = PQ, BC = QR and the angle ACB equals the angle QRP. Then the triangles ABC and PQR
(a) Are equivalent
(b) Cannot be congruent
(c) Need not be congruent but must be similar
(d) Need not be similar, if they are, then they must be congruent

Solution :

If 2 sides of two triangles and the angle between the sides are equal then the two triangles are similar.

Here AB = PQ, BC = QR and the angle between AB and BC is angle ABC and also the angle between PQ and QR is angle PQR. The relation angle ABC = angle PPQR is not given. So, the triangles need not to be similar. If they are they are congruent because the rest sides must be equal.

⇨ Option (d) is correct.

15. A particle starts at the origin and travels along the positive x-axis. For the first one second, its speed is 1 m/sec. Therefore, its speed at any time t is at the most (9/10)-ths of its speed at time t – 1. Then
(a) The particle reaches any point x > 0 at some finite time
(b) The particle must reach x = 10
(c) The particle may or may not reach x = 9 but it will never reach x = 10
(d) Nothing of the above nature can be predicted without knowing the exact speed.

Solution :

So, in infinite time the particle reaches at most $1 + (9/10) + (9/10)^2 + \ldots$

$= 1/(1 - 9/10) = 10$.

So, the particle cannot reach $x = 10$ in finite time.

⇨ Option (c) is correct.

16. Let $f(x) = \min(e^x, e^{-x})$ for any real number x. Then
(a) f has no maximum
(b) f attains its maximum at a point where $f'(x) = 0$
(c) f attains its maximum at a point where it is not differentiable
(d) $M := \max(f(x) : x \text{ real}) < \infty$ but there is no number x_0 such that $f(x_0) = M$.

Solution :

Option (c) is correct.

17. If Θ is an acute angle, the maximum value of $3\sin\Theta + 4\cos\Theta$ is
(a) 4
(b) 5
(c) $5\sqrt{2}$
(d) $3(1 + \sqrt{3}/2)$

Solution :

Now, $3\sin\Theta + 4\cos\Theta = \sqrt{(3^2 + 4^2)}[\{3/\sqrt{(3^2 + 4^2)}\}\sin\Theta + \{4/\sqrt{(3^2 + 4^2)}\}\cos\Theta] = 5\{\cos\alpha\sin\Theta + \sin\alpha\cos\Theta)$ where $\cos\alpha = 3/\sqrt{(3^2 + 4^2)}$ and $\sin\alpha = 4/\sqrt{(3^2 + 4^2)}$

⇨ $3\sin\Theta + 4\cos\Theta = 5(\sin\Theta\cos\alpha + \cos\Theta\sin\alpha) = 5\sin(\Theta + \alpha)$

Now, maximum value of $\sin(\Theta + \alpha) = 1$

⇨ Maximum value of $3\sin\Theta + 4\cos\Theta$ is 5.
⇨ Option (b) is correct.

18. I sold 2 books for Rs. 30 each. My profit on one was 25% and the loss on the other was 25%. Then on whole, I
(a) Lost Rs. 5

(b) Lost Rs. 4
(c) Gained Rs. 4
(d) Neither gained nor lost.

Solution :

Let, the buying prize of first book is Rs. x.

Gain in first book = 25x/100

Selling prize = x + 25x/100 = 125x/100

Now, 125x/100 = 30

⇨ x = 24.

Let, the buying prize of another book is Rs. y.

Loss in that book = 25y/100

Selling prize = y – 25y/100 = 75y/100

Now, 75y/100 = 30

⇨ y = 40.

Gain in first book = (30 – 24) = Rs. 6

Loss in another book = (40 – 30) = Rs. 10

So, I lost (10 – 6) = Rs. 4.

⇨ Option (b) is correct.

19. Suppose x is an irrational number and a, b, c, d are non-zero rational numbers. If (ax + b)/(cx + d) is rational, then we must have
(a) a = c = 0
(b) a = c, b = d
(c) ad = bc
(d) a + d = b + c.

Solution :

(ax + b)/(cx + d) = p/q

⇨ aqx + bq = cpx + dp
⇨ (aq – cp)x = dp – bq

⇨ $x = (dp - bq)/(aq - cp)$

Option (c) is correct.

20. If a, b, c are real numbers so that $x^3 + ax^2 + bx + c = (x^2 + 1)g(x)$ for some polynomial g, then
(a) b = 1, a = c
(b) b = 0 = c
(c) a = 0
(d) None of the above.

Solution :

We have, $x^3 + ax^2 + bx + c = (x^2 + 1)g(x)$

Putting x = i we get, $i^3 + ai^2 + bi + c = (i^2 + 1)g(i)$

⇨ $-i - a + bi + c = (-1 + 1)g(i)$ ($i^2 = -1$)
⇨ $(c - a) + i(b - 1) = 0$

Equating the real and imaginary part of both sides we get, c = a and b = 1

⇨ Option (a) is correct.

21. The average scores of 12 students in a test is 74. The highest score is 79. Then, the minimum possible lowest score must be
(a) 25
(b) 12
(c) 19
(d) 28

Solution :

Let all the students except the student who has got lowest mark has got 79 each.

Sum of scores of 12 students = 12*74

Sum of scores of 11 students = 79*11

Minimum possible lowest score = 12*74 – 79*11 = 19

⇨ Option (c) is correct.

22. If $x > y$ are positive integers such that $3x + 11y$ leaves a remainder 2 when divided by 7 and $9x + 5y$ leaves a remainder 3 when divided by 7, then the remainder when $x - y$ is divided by 7, equals
(a) 3
(b) 4
(c) 5
(d) 6

Solution :

$(9x + 5y) - (3x + 11y) \equiv 3 - 2 \pmod 7$

⇨ $6(x - y) \equiv 1 \pmod 7$

Now, if $x - y \equiv 3 \pmod 7$, then $6(x - y) \equiv 4 \pmod 7$

So, option (a) is not true.

If $x - y \equiv 4$ (mod 7(, then $6(x - y) \equiv 3 \pmod 7$

So, option (b) is not true.

If $x - y \equiv 5 \pmod 7$ then $6(x - y) \equiv 2 \pmod 7$

So, option (c) is not true.

If $(x - y) \equiv 6 \pmod 7$, then $6(x - y) \equiv 1 \pmod 7$

⇨ Option (d) is correct.

23. The set of all real numbers which satisfy $(x^2 - 2x + 3)/\sqrt{(x^2 - 2x + 2)} \geq 2$ is
(a) The set of all integers
(b) The set of all rational numbers
(c) The set of all positive real numbers
(d) The set of all real numbers.

Solution :

Now, $x^2 - 2x + 2 = (x - 1)^2 + 1 > 0$

Let, $x^2 - 2x + 2 = a$

Now, the equation becomes, $(a + 1)/\sqrt{a} \geq 2$

⇨ $(a + 1)/2 \geq \sqrt{a}$

Which is true for any a > 0 as AM ≥ GM.

We have shown that a > 0 always.

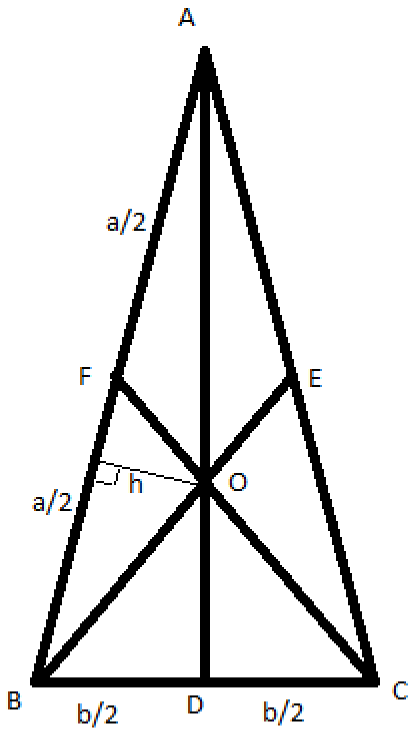
- ⇨ Option (d) is correct.

24. Let ABC be a triangle such that the three medians divide it into six parts of equal area. Then, the triangle
    (a) Cannot exist.
    (b) Can be any triangle
    (c) Must be equilateral
    (d) Need not be equilateral but must be isosceles.

Solution :

If the triangle is equilateral then it must have all the areas equal.

If it is an isosceles triangle as shown below in figure, then also it can be true.



Now, it is clear that area of triangle BOF = area of triangle AOF (as base is same = a/2 and perpendicular from O to the base is same = h) = area of triangle COE = area of triangle AOE

Also, area of triangle BOD = area of triangle COD.

Now, we will see if area of triangle BOD = area of triangle BOF

Then we have the required condition i.e. 6 triangles have same area.

Now, area of triangle BOF = (1/2)*h)*(a/2)

AD = $\{\sqrt{(a^2 - b^2)}\}/2$

- ⇨ OD = $(1/3)\{\sqrt{(a^2 - b^2)}\}/2$
- ⇨ Area of triangle BOD = $(1/2)(1/3)(b/2)\{\sqrt{(a^2 - b^2)}\}/2$

Equating both the area, we get, $h = (b/6a)\sqrt{(a^2 - b^2)}$

Which is very much possible.

- ⇨ It can be true for equilateral as well as isosceles triangle.
- ⇨ Option (a) , (c), (d) cannot be true.
- ⇨ Option (b) is correct.

25. From a bag containing 10 distinct objects, the number of ways one can select an odd number of objects is
   (a) 2^{10}
   (b) 2^9
   (c) 10!
   (d) 5.

Solution :

Total number of selection = ${}^{10}C_0 + {}^{10}C_1 + \ldots + {}^{10}C_{10} = 2^{10}$

Now, number of ways to select an odd number of objects = number of ways to select an even number of objects.

- ⇨ Number of ways to select an odd number of objects = $2^{10}/2 = 2^9$.
- ⇨ Option (b) is correct.

26. Consider the two statements :
   (I) Between any two rational numbers, there is an irrational number.
   (II) Between any two irrational numbers, there is a rational number.
   Then,
   (a) Both (I) and (II) are true.
   (b) (I) is true but (II) is not
   (c) (I) is false but (II) is not
   (d) Both (I) and (II) are false.

Solution :

Option (a) is correct.

27. Let $f(x) = ax^2 + bx + c$ where a, b, c are real numbers. Suppose $f(x) \neq x$ for any real number x. Then the number of solutions of $f(f(x)) = x$ in real numbers x is
(a) 4
(b) 2
(c) 0
(d) Cannot be determined.

Solution :

Now, $f(f(x)) = x$

⇨ $f(ax^2 + bx + c) = x$
⇨ $a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c = x$
⇨ $a(a^2x^4 + b^2x^2 + c^2 + 2abx^3 + 2acx^2 + 2bcx) + b(ax^2 + bx + c) + c = x$
⇨ $a^3x^4 + 2a^2bx^3 + (ab^2 + 2a^2c + ab)x^2 + (2abc + b^2 - 1)x + (ac^2 + bc + c) = 0$

Option (c) is correct.

28. Let, $S = \sum ne^{-n}$ where the summation runs from $n = 1$ to $n = \infty$. Then
(a) $S \leq 1$
(b) $1 < S < \infty$
(c) S is infinite
(d) $S = 0$.

Solution :

$S = 1*e^{-1} + 2*e^{-2} + 3*e^{-3} + 4*e^{-4} + ...$

$Se^{-1} = \quad 1*e^{-2} + 2*e^{-3} + 4*e^{-4} + ...$

Subtracting we get, $S(1 - e^{-1}) = e^{-1} + e^{-2} + e^{-3} + e^{-4} + ...$

⇨ $S(1 - e^{-1}) = e^{-1}/(1 - e^{-1})$

Option (b) is correct.

29. Let $f(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 where a, b, c, d are real. Then
(a) $f(x) \to \infty$ as $x \to \infty$
(b) f is 1-1 as well as onto
(c) The graph of f(x) meets the x-axis in one or three points
(d) f must be onto but need not be 1-1.

Solution :

Definition of onto function : A function f from A to B is called onto if for all b in B there is an a in A such that f (a) = b. All elements in B are used.

Now, f(x) can take any value.

⇨ f is onto.

Now, f(x) is of degree. 3. So it has either 1 or 3 real root. If it has 3 real roots then f(x) = 0 for 3 values of x.

⇨ f(x) need not be 1-1

Further, putting x = 1 we get, $f(1) = a + b + c + d$

Putting x = -1 we get $f(-1) = -a + b - c + d$

Now, $f(1) = f(-1)$ if $a + b + c + d = -a + b - c + d$ i.e. if $a + c = 0$ which is possible.

⇨ f(x) need not be 1 – 1.
⇨ Option (d) is correct.

Note that, option (a) and (c) are also correct.

30. Let $a_1, \ldots, a_n$ be arbitrary integers and suppose $b_1, \ldots, b_n$ is a permutation of the a_i's. Then the value of $|a_1 - b_1| + |a_2 - b_2| + \ldots + |a_n - b_n|$
(a) Is less than or equal to n
(b) Can be an arbitrary positive integer
(c) Can be any even nonnegative integer
(d) Must be 0.

Solution :

Let, $(a_1, a_2, \ldots, a_n) = (1, 2, \ldots, n)$ and $(b_1, b_2, \ldots, b_n) = (n, n-1, \ldots, 1)$

Now, $|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n| = (n - 1) + (n - 2) + ... > n$

⇨ Option (a) and (d) cannot be true.

Let, $(b_1, b_2, ..., b_n) = (2, 3, ..., n, 1)$

Now, $|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n| = 1 + 1 + ...$ (n-1) times $+ (n - 1) = 2(n - 1)$

And in the first case $|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n| > (n - 1) + (n - 2) + (n - 3) = 3(n - 2)$

Now, $3(n - 2) > 2 (n - 1)$

⇨ Option (b) cannot be true.
⇨ Option (c) is correct.

B. Stat. (Hons.) Admission test : 2011

Group A

Each of the following questions has exactly one correct option and you have to identify it.

1. The limit
   lim $[\{1 - \cos(\sin^2\alpha x)\}/x]$ as $x \to 0$
   (a) Equals 1
   (b) Equals α
   (c) Equals 0
   (d) Does not exist.

Solution :

Now, lim $[\{1 - \cos(\sin^2\alpha x)\}/x]$ as $x \to 0$ = lim $[\{\sin(\sin^2\alpha x)*2\sin(\alpha x)\cos(\alpha x)*\alpha\}/1]$ as $x \to 0$ (Applying L'Hospital rule) $= \sin(\sin^2 0)2\sin(0)\cos(0)*\alpha/1 = 0$

⇨ Option (c) is correct.

2. The set of all x for which the function $f(x) = \log_{1/2}(x^2 - 2x - 3)$ is defined and monotone increasing is
   (a) $(-\infty, 1)$
   (b) $(-\infty, -1)$

(c)$(1, \infty)$
(d) $(3, \infty)$

Solution :

Now, $x^2 - 2x - 3 > 0$

- ⇨ $(x - 1)^2 > 4$
- ⇨ $x - 1 > 2$ or $x - 1 < -2$
- ⇨ $x > 3$ or $x < -1$

$f(x) = \{\log(x^2 - 2x - 3)\}/\log(1/2) = -\{\log(x^2 - 2x - 3)\}/\log(2)$

- ⇨ $f'(x) = -(1/\log 2)(2x - 2)/(x^2 - 2x - 3)$
- ⇨ $f'(x) = -(1/\log 2)2(x - 1)/\{(x + 1)(x - 3)\}$

To be monotonically increasing $f'(x)$ must be > 0

Let us take $x = 4$, we get $f'(4) < 0$

- ⇨ Options (c) and (d) cannot be true.

Now, let us put $x = 0$, we get, $f'(0) < 0$

- ⇨ Option (a) cannot be true.
- ⇨ Option (b) is correct.

3. Let a line with slope of 60° be drawn through the focus F of the parabola $y^2 = 8(x + 2)$. If the two points of intersection of the line with the parabola are A and B and the perpendicular bisector of the chord AB intersects the x-axis at the point P, then the length of the segment PF is
(a) $16/3$
(b) $8/3$
(c)$16\sqrt{3}/3$
(d) $8\sqrt{3}$

Solution :

The centre of the parabola is at $(-2, 0)$.

Comparing the equation of parabola with $y^2 = 4a(x - \alpha)$ we get, $a = 2$.

Hence, coordinate of focus F is $(0, 0)$

The equation of the line passing through focus and having slope tan60° is, $y = \sqrt{3}x$

Now, solving the equation of this line with parabola will give A and B.

Now, $(\sqrt{3}x)^2 = 8(x + 2)$

- ⇨ $x = 4, -4/3$

Hence coordinate of A and B are $(4, 4\sqrt{3})$ and $(-4/3, -4\sqrt{3}/3)$

Midpoint of AB = $(4/3, 4\sqrt{3}/3)$ and slope of AB = $\{4\sqrt{3} - (-4\sqrt{3}/3)\}/\{4/3 - (-4/3)\} = \sqrt{3}$

- ⇨ Slope of perpendicular bisector of AB = $-1/\sqrt{3}$
- ⇨ Equation of perpendicular bisector of AB is, $y - 4\sqrt{3}/3 = (-1/\sqrt{3})(x - 4/3)$

Putting y = 0 we get x-coordinate of P, x = 16/3.

- ⇨ Coordinate of P is (16/3, 0)
- ⇨ Now, distance between F (0, 0) and P (16/3, 0) is 16/3.
- ⇨ Option (a) is correct.

4. Suppose z is a complex number with $|z| < 1$. Let $w = (1 + z)/(1 - z)$. Which of the following is always true?
   [Re(w) is the real part of w and Im(w) is the imaginary part of w.]
   (a) $\mathrm{Re}(w) > 0$
   (b) $\mathrm{Im}(w) \geq 0$
   (c) $|w| \leq 1$
   (d) $|w| \geq 1$

Solution :

Let, $z = re^{i\Theta}$

Now, $w = (1 + r\cos\Theta + ir\sin\Theta)/(1 - r\cos\Theta - ir\sin\Theta) = (1 + r\cos\Theta + ir\sin\Theta)(1 - r\cos\Theta + ir\sin\Theta)/\{(1 - \cos\Theta)^2 + \sin^2\Theta\} = (1 - r^2\cos^2\Theta + 2ir\sin\Theta - r^2\sin^2\Theta)/(2 - 2\cos\Theta) = (1 - r^2)/2(1 - \cos\Theta) + ir\sin\Theta/(1 - \cos\Theta)$

$\mathrm{Re}(z) = (1 - r^2)/\{2(1 - \cos\Theta)\}$

Here, $r = |z| < 1$ and $\cos\Theta < 1$

- ⇨ $\mathrm{Re}(z) > 0$
- ⇨ Option (a) is correct.

5. Among all the factors of $4^6 6^7 21^8$, the number of factors which are perfect squares is
   (a) 240

(b) 360
(c)400
(d) 640.

Solution :

Now, $4^6 6^7 21^8 = 2^{19} 3^{15} 7^8$

There are 2, 4, ..., 18 = 9 even numbers we can select from power of 2.

There are 2, 4, ..., 14 = 7 even numbers we can select from power of 3.

There are 2, 4, 6, 8 = 4 even numbers we can select from power of 7.

We can select 1 number from 9 numbers in ${}^9C_1$ ways. We can select 1 number from 7 numbers in ${}^7C_1$ ways. We can select 1 number from 4 numbers in ${}^4C_1$ ways.

So, number of factors which contain all prime factors (2, 3, 7) and square are ${}^9C_1*{}^7C_1*{}^4C_1 = 252$.

Similarly number of factors which contain 2 and 3 prime factors and not 7 and square are ${}^9C_1*{}^7C_1 = 63$.

Similarly, number of factors which contain 3 and 7 prime factors and not 2 and square are ${}^7C_1*{}^4C_1 = 28$.

Number of factors which contain 2 and 7 prime factors and not 3 and square are ${}^9C_1*{}^4C_1 = 36$

Number of factors which contain only factor 2 and not 3 and 7 and square are ${}^9C_1 = 9$

Number of factors which contain only factor 3 and not 2 and 7 and square are ${}^7C_1 = 7$

Number of factors which contain only factor 7 and not 2 and 3 and square are ${}^4C_1 = 4$

And 1.

Therefore total number of factors which are perfect square = 252 + 63 + 28 + 36 + 9 + 7 + 4 + 1 = 400.

⇨ Option (c) is correct.

6. Let A be the set {1, 2, ..., 20{. Fix two disjoint subsets S_1 and S_2 of A, each with exactly three elements. How many 3-element

subsets of A are there, which have exactly one element common with S_1 and at least one element common with S_2?

(a) 51
(b) 102
(c) 135
(d) 153

Solution :

Two cases are there. One, select one element from S_1, select one element from S_2 and select one element from rest 14 numbers. We can do it in $^3C_1 * ^3C_1 * ^{14}C_1 = 126$ ways.

Two, select one element from S_1 and select two elements from S_2. We can do it in $^3C_1 * ^3C_2 = 9$ ways.

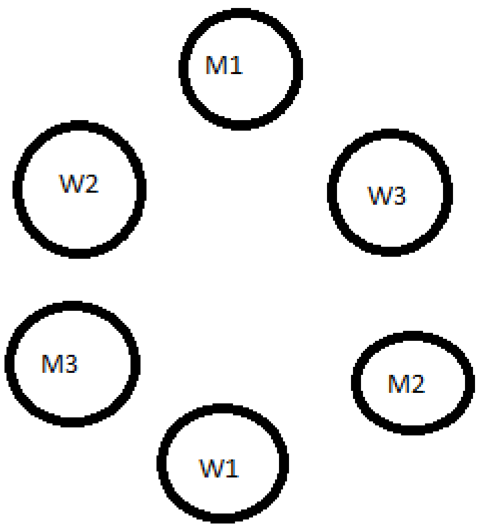
Therefore required number of subsets = 126 + 9 = 135

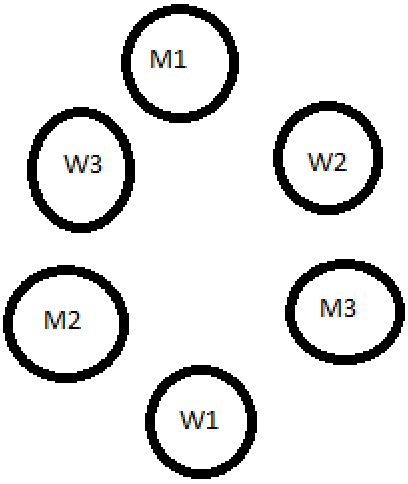
⇨ Option (c) is correct.

7. In how many ways can 3 couples sit around a round table such that men and women alternate and none of the couples sit together?
(a) 1
(b) 2
(c) 5!/3
(d) None of these.

Solution :

Only 2 ways as depicted in figure.





- ⇨ Option (b) is correct.

8. The equation $x^3 + y^3 = xy(1 + xy)$ represents
   (a) Two parabolas intersecting at two points
   (b) Two parabolas touching at one point
   (c) Two non-intersecting hyperbolas
   (d) One parabola passing through the origin.

Solution :

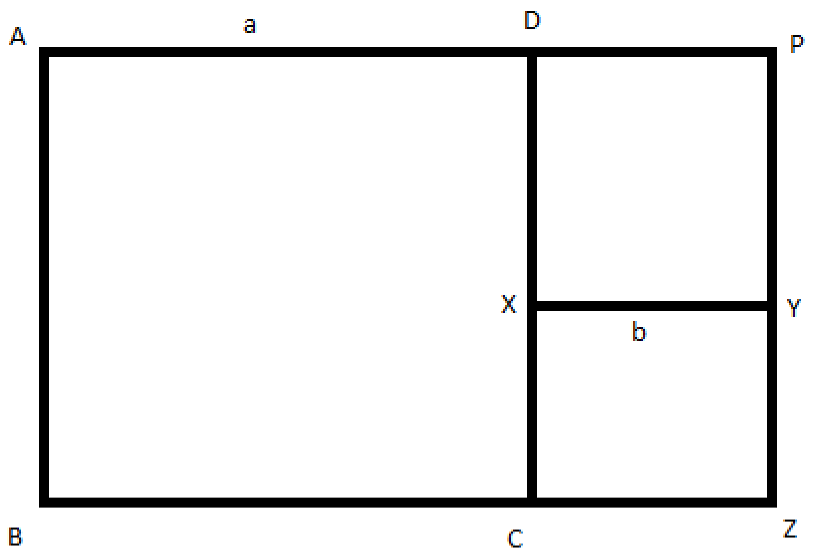
We have, $x^3 + y^3 = xy(1 + xy)$

- ⇨ $x^3 - x^2y^2 + y^3 - xy = 0$
- ⇨ $-x^2(y^2 - x) + y(y^2 - x) = 0$
- ⇨ $(y^2 - x)(-x^2 + y) = 0$
- ⇨ $(y^2 - x)(x^2 - y) = 0$
- ⇨ This is equation of two parabolas $y^2 = x$ and $x^2 = y$.

The two parabolas intersect at (0, 0) and (1, 1).

- ⇨ Option (a) is correct.

9. Consider the diagram below where ABZP is a rectangle and ABCD and CXYZ are squares whose areas add up to 1.



The maximum possible area of the rectangle ABZP is
(a) $1 + 1/\sqrt{2}$
(b) $2 - \sqrt{2}$
(c) $1 + \sqrt{2}$
(d) $(1 + \sqrt{2})/2$

Solution :

Area of square ABCD = a^2 and area of square CXYZ = b^2

We have, $a^2 + b^2 = 1$

Now, length of ABZP rectangle = $(a + b)$ and breadth = a

Area of ABZP rectangle = $a(a + b)$ which we have to maximize.

Let, $a = \cos A$, then $b = \sin A$ (as $a^2 + b^2 = 1$)

- ⇨ $a(a + b) = \cos A(\cos A + \sin A) = \cos^2 A + \sin A\cos A = (1/2)(2\cos^2 A + 2\sin A\cos A) = (1/2)(1 + \cos 2A + \sin 2A) = ½ + (1/\sqrt{2})\{(1/\sqrt{2})\cos 2A + (1/\sqrt{2})\sin 2A\} = ½ + (1/\sqrt{2})(\sin 45°\cos 2A + \cos 45°\sin 2A) = ½ + (1/\sqrt{2})\sin(2A + 45°)$

Now, $\sin(2A + 45°) \le 1$

- ⇨ $½ + (1/\sqrt{2})\sin(2A + 45°) \le ½ + 1/\sqrt{2}$
- ⇨ Maximum possible area of the rectangle ABZP = $½ + 1/\sqrt{2} = (1 + \sqrt{2})/2$
- ⇨ Option (d) is correct.

10. Let A be the set {1, 2, ..., 6}. How many functions f from A to A are there such that the range of f has exactly 5 elements?

(a) 240
(b) 720
(c) 1800
(d) 10800

Solution :

Now, there is nothing mentioned for domain of f.

So we will consider the domain of f consists of exactly 6 elements.

Now, 2 numbers will map to 1 number in range set A.

We can choose 2 numbers out of 6 numbers in $^6C_2 = 15$ ways.

Now, take the 2 numbers which are getting mapped into 1 number as unit.

So, there are 5! Cases the 5 numbers (2 unit + 4 distinct) can map into 5 numbers in range set A.

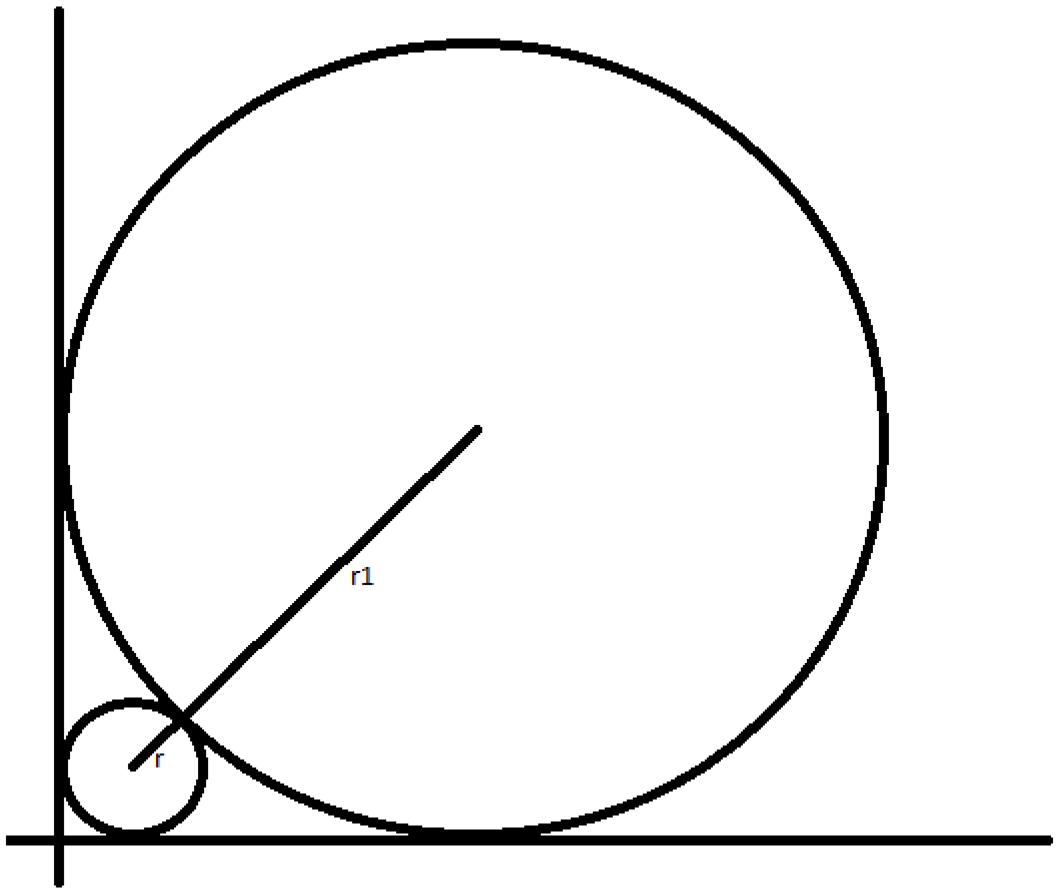
Now, we can choose 5 numbers in range set A from 6 numbers in $^6C_5 = 6$ ways.

So, number of such functions = 15*5!*6 = 10800

⇨ Option (d) is correct.

11. Let C_1, C_2 and C_3 be three circles lying in the same quadrant, each touching both the axes. Suppose also that C_1 touches C_2 and C_2 touches C_3. If the area of the smallest circle is 1 unit, then area of the largest circle is
(a) $\{(\sqrt{2} + 1)/(\sqrt{2} - 1)\}^4$
(b) $(1 + \sqrt{2})^2$
(c) $(2 + \sqrt{2})^2$
(d) 2^4

Solution :



Let the radius of the smallest circle is r, that of larger is r_1 and that of largest is r_2.

Coordinate of the centre of the smallest circle is (r, r); that of larger circle is (r_1, r_1) and that of largest circle is (r_2, r_2)

We have, $r_1 + r = \sqrt{\{(r_1 - r)^2 + (r_1 - r)^2\}}$

- ⇨ $r_1 + r = \sqrt{2}(r_1 - r)$
- ⇨ $(r_1 + r)/(r_1 - r) = \sqrt{2}$
- ⇨ $r_1/r = (\sqrt{2} + 1)/(\sqrt{2} - 1)$

Similarly, $r_2/r_1 = (\sqrt{2} + 1)/(\sqrt{2} - 1)$

Now, multiplying the two equations we get, $r_2/r = \{(\sqrt{2} + 1)/(\sqrt{2} - 1)\}^2$

- ⇨ $(\pi r_2^2) = (\pi r^2)\{(\sqrt{2} + 1)/(\sqrt{2} - 1)\}^4$
- ⇨ Area of largest circle = $\{(\sqrt{2} + 1)/(\sqrt{2} - 1)\}^4$ (As $\pi r^2 = 1$)
- ⇨ Option (a) is correct.

12. Let $[x]$ denote the largest integer less than or equal to to x. Then $\int[x^k + n]dx$ (integration running from 0 to $n^{1/k}$) equals
    (a) $n^2 + \sum i^{1/k}$ summation runs from $i = 1$ to $i = n$

(b) $2n^{(1+k)/k} - \sum i^{1/k}$ summation runs from i = 1 to i = n
(c) $2n^{(1+k)/k} - \sum i^{1/k}$ summation runs from i = 1 to i = n – 1
(d) None of these.

Solution :

Now, $\int[x^k + n]dx$ (integration running from 0 to $n^{1/k}$) = $\int[x^k + n]dx$ (integration running from 0 to $1^{1/k}$) + $\int[x^k + n]dx$ (integration running from $1^{1/k}$ to $2^{1/k}$) = $\int[x^k + n]dx$ (integration running from $2^{1/k}$ to $3^{1/k}$) + $\int[x^k + n]dx$ (integration running from $3^{1/k}$ to $4^{1/k}$) + ... + $\int[x^k + n]dx$ (integration running from $(n-1)^{1/k}$ to $n^{1/k}$)

= $n\int dx$ (integration running from 0 to $1^{1/k}$) + $(n+1)\int dx$ (integration running from $1^{1/k}$ to $2^{1/k}$) + $(n+2)\int dx$ (integration running from $2^{1/k}$ to $3^{1/k}$) + $(n+3)\int dx$ (integration running from $3^{1/k}$ to $4^{1/k}$) + ... + $(2n-1)\int dx$ (integration running from $(n-1)^{1/k}$ to $n^{1/k}$)

$= n*1^{1/k} + (n+1)(2^{1/k} - 1^{1/k}) + (n+2)(3^{1/k} - 2^{1/k}) + (n+3)(4^{1/k} - 3^{1/k}) + ... + (2n-1)\{n^{1/k} - (n-1)^{1/k}\}$

$= \{n*1^{1/k} - (n+1)*1^{1/k}\} + \{(n+1)*2^{1/k} - (n+2)*2^{1/k}\} + \{(n+2)*3^{1/k} - (n+3)*3^{1/k}\} + ... + \{(2n-2)*(n-1)^{1/k} - (2n-1)(n-1)^{1/k}\} + 2n*n^{1/k} - n^{1/k}$

$= 2n*n^{1/k} - 1^{1/k} - 2^{1/k} - 3^{1/k} - ...... - (n-1)^{1/k} - n^{1/k}$

$= 2n^{(1+k)/k} - \sum i^{1/k}$ (summation runs from i = 1 o i = n)

⇨ Option (b) is correct.

13. Consider the function
$f(x) = x(x-1)e^{2x}$ if $x \le 0$
$f(x) = x(1-x)e^{-2x}$ if $x > 0$
Then f(x) attains its maximum value at
(a) $1 - 1/\sqrt{2}$
(b) $1 + 1/\sqrt{2}$
(c) $-1/\sqrt{2}$
(d) $1/\sqrt{2}$

Solution :

Clearly, $f(1 - 1/\sqrt{2}) > 0$ and $f(1 + 1/\sqrt{2}) < 0$

⇨ $f(1 - 1/\sqrt{2}) > f(1 + 1/\sqrt{2})$
⇨ Option (b) cannot be true.

Now, $f(-1/\sqrt{2}) = (-1/\sqrt{2})((-1/\sqrt{2} - 1)e^{\sqrt{2}} = (1/\sqrt{2})(1 + 1/\sqrt{2})e^{\sqrt{2}}$

$f(1/\sqrt{2}) = (1/\sqrt{2})(1 - 1/\sqrt{2})e^{-\sqrt{2}}$

Now, $f(-1/\sqrt{2})/f(1/\sqrt{2}) = \{(1 + 1/\sqrt{2})/(1 - 1/\sqrt{2})\}e^{2\sqrt{2}} > 1$

- ⇨ $f(-1/\sqrt{2}) > f(1/\sqrt{2})$
- ⇨ Option (d) cannot be true.

Now, $f(1 - 1/\sqrt{2})/f(-1/\sqrt{2}) = (1 - 1/\sqrt{2})(1/\sqrt{2})e^{-2(1 - 1/\sqrt{2})}/\{(1/\sqrt{2})(1 + 1/\sqrt{2})e^{\sqrt{2}}\} = \{(\sqrt{2} - 1)/(\sqrt{2} + 1)\}e^{-2} < 1$

- ⇨ $f(-1/\sqrt{2}) > f(1 - /\sqrt{2})$
- ⇨ Option (a) cannot be true.
- ⇨ Option (c) is correct.

14. Consider the function $f(x) = x^n(1 - x)^n/n!$, where $n \geq 1$ is a fixed integer. Let $f^{(k)}$ denote the k-th derivative of f. Which of the following is true for all $k \geq 1$?
    (a) $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers.
    (b) $f^{(k)}(0)$ is an integer, but not $f^{(k)}(1)$
    (c) $f^{(k)}(1)$ is an integer, but not $f^{(k)}(0)$
    (d) Neither $f^{(k)}(1)$ nor $f^{(k)}(0)$ is an integer.

Solution :

Let us take n = 3.

Now, $f(x) = x^3(1 - x)^3/3!$

- ⇨ $f^{(1)}(x) = (1/3!)\{3x^2(1 - x)^3 - 3x^3(1 - x)^2\}$
- ⇨ $f^{(1)}(0)$ and $f^{(1)}(1) = 0$ and integer.
- ⇨ $f^{(2)}(x) = (3/3!)\{2x(1 - x)^3 - 3x^2(1 - x)^2 - 3x^2(1 - x)^2 + 2x^3(1 - x)\}$
- ⇨ $f^{(2)}(0)$ and $f^{(2)}(1) = 0$ and integer
- ⇨ $f^{(3)}(x) = (3/3!)\{2(1 - x)^3 - 6x(1 - x)^2 - 12x(1 - x)^2 + 12x^2(1 - x) + 6x^2(1 - x) - 2x^3\}$
- ⇨ $f^{(3)}(x) = (3*2/3!)\{(1 - x)^3 - 3x(1 - x)^2 - 6x(1 - x)^2 + 6x^2(1 - x) + 3x^2(1 - x) - x^3\}$
- ⇨ $f^{(3)}(x) = (1 - x)^3 - 3x(1 - x)^2 - 6x(1 - x)^2 + 6x^2(1 - x) + 3x^2(1 - x) - x^3$
- ⇨ $f^{(3)}(0) = 1$ and $f^{(3)}(1) = -1$, both are integers.

Now, there is nothing in denominator.

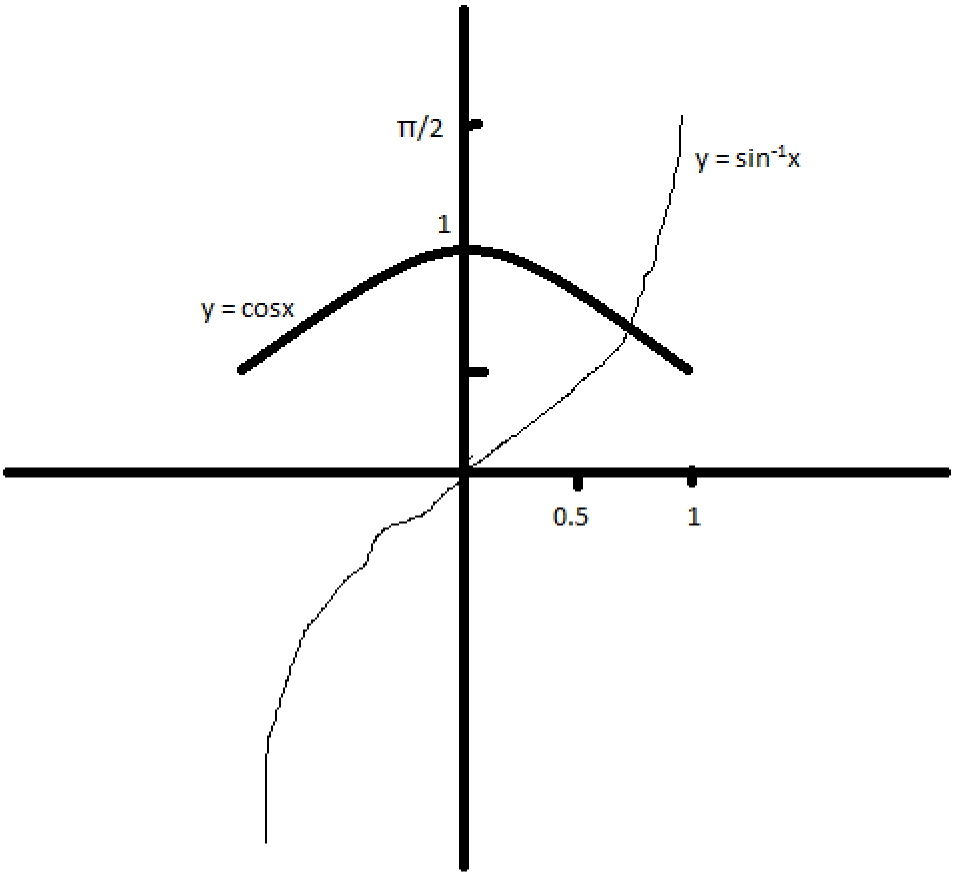
- ⇨ $f^{(k)}(x)$ is always integer.
- ⇨ Option (a) is correct.

15. The number of solutions of the equation $\sin(\cos\Theta) = \Theta$, $-1 \le \Theta \le 1$, is
(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Now, $\sin(\cos\Theta) = \Theta$

⇨ $\cos\Theta = \sin^{-1}\Theta$



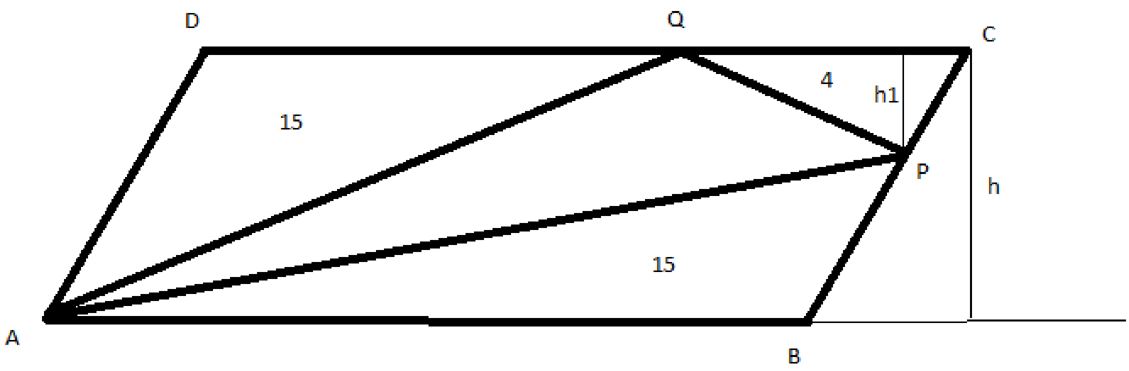
One intersection point.

⇨ One solution
⇨ Option (b) is correct.

16. Suppose ABCD is a parallelogram and P, Q are points on the sides BC and CD respectively, such that $PB = \alpha BC$ and $DQ = \beta DC$. If the area of the triangles ABP, ADQ, PCQ are 15, 15 and 4 respectively, then the area of APQ is

(a) 14
(b) 15
(c) 16
(d) 18.

Solution :



Now, PB = αBC and CQ = βDC

Note that h_1/h = PC/BC as the triangles have equal angles.

⇨ $h_1 = h*(BC - PB)/BC = h*(BC - \alpha BC)/BC = h(1 - \alpha)$

Now, area of triangle PCQ = $(1/2)*CQ*h_1 = 4$

⇨ $(1/2)*(1 - \beta)*DC*(1 - \alpha)*h = 4$ (As CQ = DC – DQ = DC – β*DQ = (1- β)*DQ) ....(1)

Now, area of triangle ADQ = $(1/2)*DQ*h = 15$

⇨ $(1/2)*\beta*DC*h = 15$ .....(2)

Now, area of triangle ABP = $(1/2)*AB*\alpha*h = 15$ ....(3)

Now, dividing (1) by (2) we get, $(1 - \beta)(1 - \alpha)/\beta = 4/15$ ......(4)

Now, dividing (2) by (3) we get,

$\beta/\alpha = 1$ (As AB = DC)

⇨ $\beta = \alpha$

Putting $\beta = \alpha$ in (4) we get,

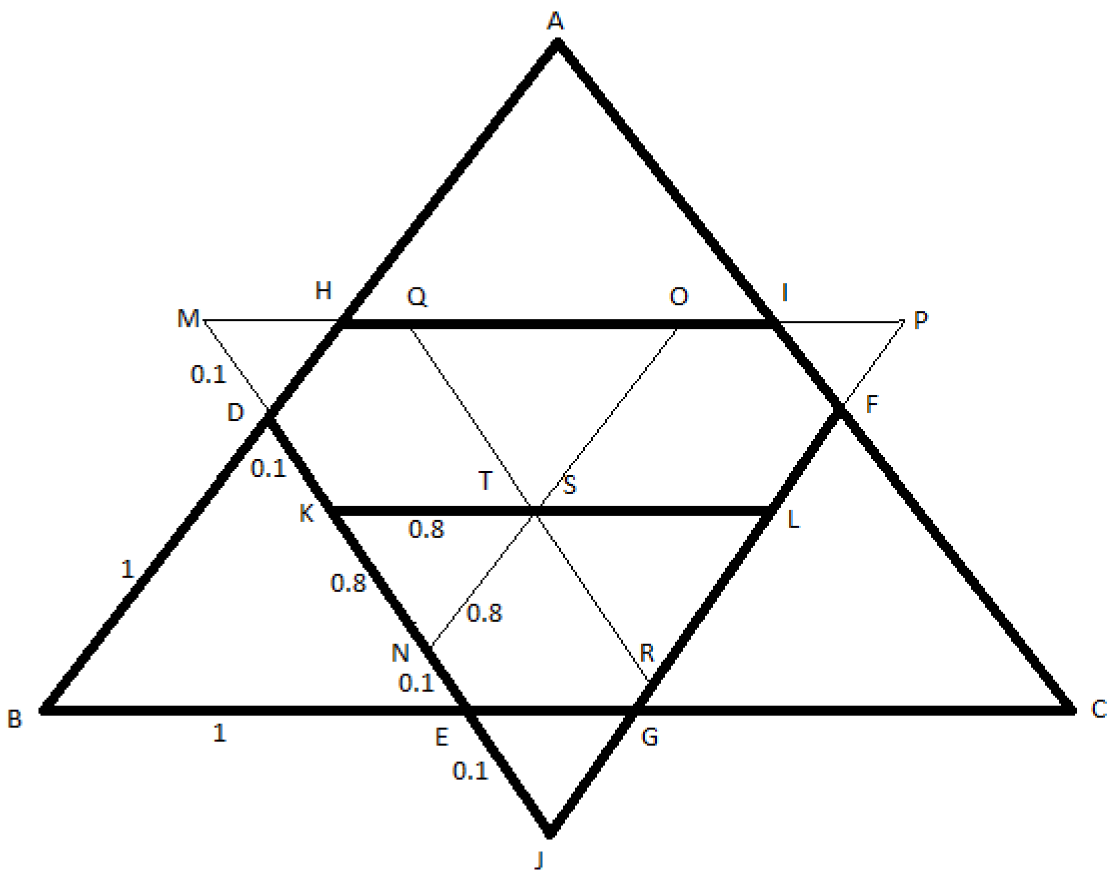
$(1 - \alpha)^2/\alpha = 4/15$

⇨ $15\alpha^2 - 34\alpha + 15 = 0$
⇨ $\alpha = \{34 \pm \sqrt{(34^2 - 4*15*15)}\}/2*15$

- ⇨ $\alpha = \{17 \pm \sqrt{(17^2 - 15^2)}\}/15$
- ⇨ $\alpha = \{17 \pm 8\}/15$
- ⇨ $\alpha = 9/15 = 3/5$ ($\alpha < 1$)
- ⇨ AB*h = 15*2/(3/5) (From (3)
- ⇨ Area of the parallelogram = 50
- ⇨ Area of triangle APQ = 50 – (15 + 15 + 4) = 16
- ⇨ Option (c) is correct.

17. Consider an equilateral triangle ABC with side 2.1 cm. You want to place a number of smaller equilateral triangles, each with side 1 cm, over the triangle ABC, so that the triangle ABC is fully covered. What is the minimum number of smaller triangles that you need?
    (a) 4
    (b) 5
    (c)6
    (d) 7.

Solution :



Please note S and T are different points

At first triangles BDE, CFG, AHI are placed.

Now, triangle JKL is placed.

EG = BC – BE – CG = 2.1 – 1 – 1 = 0.1

- ⇨ EJ = 0.1
- ⇨ KE = KJ – EJ = 1 – 0.1 = 0.9
- ⇨ DK = DE – KE = 1 – 0.9 = 0.1

Now, triangle MNO is placed.

Similarly, MD = NE = 0.1

- ⇨ KN = DE – DK – NE = 1 – 0.1 – 0.1 = 0.8
- ⇨ KS = 0.8
- ⇨ SL = KL – KS = 1 – 0.8 = 0.2

Now, if we place triangle PQR then the rest of the region of triangle ABC will be filled up as, TL = 0.8 and SL = 0.2 (Similarly).

So, triangles required = BDE, CFG, AHI, JKL, MNO, PQR.

So, number of triangles required = 6.

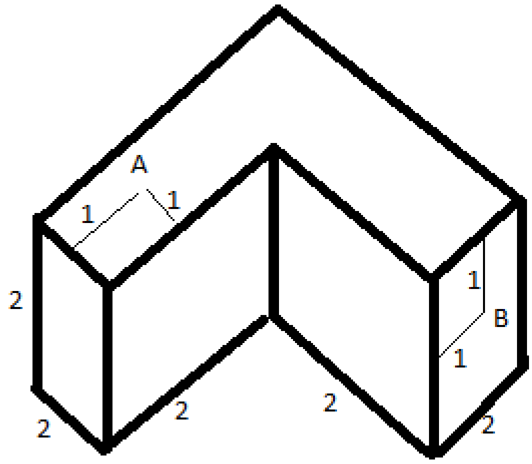
- ⇨ Option (c) is correct.

18. A regular tetrahedron has all its vertices on a sphere of radius R. Then the length of each edge of the tetrahedron is
 (a) $(\sqrt{2}/\sqrt{3})R$
 (b) $(\sqrt{3}/2)R$
 (c) $(4/3)R$
 (d) $(2\sqrt{2}/\sqrt{3})R$

Solution :

Option (d) is correct (Standard formula).

19. Consider the L-shaped brick in the diagram below.



If an ant starts from A, find the minimum distance it has to travel along the surface to reach B.
(a) $\sqrt{5}$
(b) $2\sqrt{5}$
(c) $(3/2)\sqrt{5}$
(d) $3\sqrt{5}$

Solution :

Option (c) is correct.

20. Let $f(x) = (\tan x)^{3/2} - 3\tan x + \sqrt{\tan x}$. Consider the three integrals $I_1 = \int f(x)dx$ (integration running from 0 to 1); $I_2 = \int f(x)dx$ (integration running from 0.3 to 1.3) and $I_3 = \int f(x)dx$ (integration running from 0.5 to 1.5). Then,
(a) $I_1 > I_2 > I_3$
(b) $I_2 > I_1 > I_3$
(c) $I_3 > I_1 > I_2$
(d) $I_1 > I_3 > I_2$

Solution :

Let us first find $\int f(x)dx$ then we will put the different limits and will get to the answer.

Now, $\int f(x)dx$

$= \int\{(\tan x)^{3/2} - 3\tan x + \sqrt{\tan x}\}dx$

$= \int\sqrt{(\tan x)}(1 + \tan x)dx - 3\int\tan x dx$

Let $\tan x = z^2$ in th first integral.

- ⇨ $x = \tan^{-1}(z^2)$
- ⇨ $dx = dz/(1+z^4)$

Putting the value we get,

$\int f(x)dx = 2\int\{(z^2 + z^4)/(1 + z^4\}\}dz + 3\ln|\cos x|$

$= 2\int\{(1 + z^4 + z^2 - 1)/(1 + z^4)\}dz + 3\ln|\cos x|$

$= 2\int\{(1 + z^4)/(1 + z^4)\}dz + 2\int\{(z^2 - 1)/(1 + z^4)\}dz + 3\ln|\cos x|$

$= 2\int dz + 2\int\{(1 - 1/z^2)/(z^2 + 1/z^2)\}dz + 3\ln|\cos x|$

$= 2z + 2\int(1 - 1/z^2)/[\{(z + 1/z)^2 - 2\}]dz + 3\ln|\cos x|$

Let, $z + 1/z = t$

- ⇨ $(1 - 1/z^2)dz = dt$

Putting the value we get,

$\int f(x)dx = 2z + 2\int dt/(t^2 - 2) + 3\ln|\cos x|$

$= 2z + (1/\sqrt{2})[\ln|t - \sqrt{2}| - \ln|t + \sqrt{2}|] + 3\ln|\cos x|$

$= 2\sqrt{\tan x} + (1/\sqrt{2})\ln|(z + 1/z - \sqrt{2})/(z + 1/z + \sqrt{2})| + 3\ln|\cos x|$

$= 2\sqrt{\tan x} + (1/\sqrt{2})\ln|(z^2 - \sqrt{2}z + 1)/(z^2 + \sqrt{2}z + 1)| + 3\ln|\cos x|$

$= 2\sqrt{\tan x} + (1/\sqrt{2})\ln|(\tan x - \sqrt{(2\tan x)} + 1)/(\tan x + \sqrt{(2\tan x)} + 1| + 3\ln|\cos x|$

Now, putting the limits we get, $I_1 > I_3 > I_2$

- ⇨ Option (d) is correct.

Group B

Each of the following questions has either one or two correct options and you have to identify all the correct options.

21. Let $a < b < c$ be three real numbers and w denote a complex cube root of unity. If $(a + bw \;\; cw^2)^3 + (a + bw^2 + cw)^3 = 0$, then which of the following must be true?
(a) $a + b + c = 0$
(b) $abc = 0$
(c) $ab + bc + ca = 0$
(d) $b = (c + a)/2$.

Solution :

Now, $(a + bw + cw^2)^3 + (a + bw^2 + cw)^3 = 0$

- ⇨ $(a + bw + cw^2 + a + bw^2 + cw)\{(a + bw + cw^2)^2 + (a + bw^2 + cw)^2 - (a + bw + cw^2)(a + bw^2 + cw)\} = 0$
- ⇨ $\{2a + b(w + w^2) + c(w + w^2)\}\{(a + bw + cw^2 - a - bw^2 - cw)^2 + 2(a + bw + cw^2)(a + bw^2 + cw) - (a + bw + cw^2)(a + bw^2 + cw)\} = 0$
- ⇨ $(2a - b - c)[\{bw(1 - w) - cw(1 - w)\}^2 + (a + bw + cw^2)(a + bw^2 + cw)] = 0$
- ⇨ $\{w(1 - w)\}^2(b - c)^2 + a^2 + b^2w^3 + c^2w^3 + abw^2 + acw + abw + bcw^2 + caw^2 + bcw^4 = 0$
- ⇨ $w^2(1 - 2w + w^2)(b - c)^2 + a^2 + b^2 + c^2 - ab - ac - bc = 0$
- ⇨ $(w^2 - 2 + w)(b - c)^2 + a^2 + b^2 + c^2 - ab - bc - ca = 0$
- ⇨ $-3(b^2 - 2bc + c^2) + a^2 + b^2 + c^2 - ab - bc - ca = 0$
- ⇨ $a^2 - 2b^2 - 2c^2 - ab + 5bc - ca = 0$

If we put $b = (c + a)/2$ the equation gets satisfied. No other option is correct.

- ⇨ Option (d) is correct.

22. Suppose f is continuously differentiable up to 3[rd] order ad satisfies $\int\{6f(x) + x^3f'''(x)\}dx = f''(1)$ (integration running from x = 0 to x = 1). Which of the following must be true?
    (a) $f(1) = 0$
    (b) $f'(1) = 2f(1)$
    (c) $f'(1) = f(1)$
    (d) $f'(1) = 0$

Solution :

Now, $\int\{6f(x) + x^3f'''(x)\}dx = f''(1)$ (integration running from x = 0 to x = 1)

- ⇨ $6\int f(x)dx + \int x^3f'''(x)dx = f''(1)$ (integration running from x = 0 to x = 1)
- ⇨ $6\int f(x)dx + x^3f''(x)$ (limit 0 to 1) $- \int 3x^2f''(x)dx = f''(1)$ (integration running from x = 0 to x = 1)
- ⇨ $6\int f(x)dx + f''(1) - 3x^2f'(x)$(limit 0 to 1) $+ 3\int 2xf'(x)dx = f''(1)$ (integration running from x = 0 to x = 1)
- ⇨ $6\int f(x)dx - 3f'(1) + 6xf(x)$ (limit 0 to 1) $- 6\int f(x)dx = 0$ (integration running from x = 0 to x = 1)
- ⇨ $-3f'(1) + 6f(1) = 0$

- ⇨ $f'(1) = 2f(1)$
- ⇨ Option (b) is correct.

23. Let $f(x) = ax^2 + bx + c$ for some real numbers a, b, and c. If $f(-5) \geq 10$, $f(-3) < 6$ and $f(2) \geq 7$, then which of the following cannot be true?
    (a) $f(3) = 6$
    (b) $f(3) \geq 16$
    (c) $f(4) = 5$
    (d) $f(4) \geq 6.2$

Solution :

Options (a) and (c) are correct.

24. Consider the sequence x_n, $n \geq 1$, defined as :
    $x_n = \{(1 + 2/n^a)^\wedge(-n^b)\}n^c$
    where a, b and c are real numbers. Which of the following are true?
    (a) If $b < a$, $x_n \to 0$ as $n \to \infty$
    (b) If $a < b$, $x_n \to 0$ as $n \to \infty$
    (c) If $a = b$ and $c > 0$, $x_n \to \infty$ as $n \to \infty$
    (d) If $a = b$ and $c < 0$, $x_n \to \infty$ as $n \to \infty$

Solution :

Options (b) and (c) are correct.

25. The value of $n^{1/n} - 1$
    (a) Tends to 0 as $n \to \infty$
    (b) Is greater than $(\log n)/n$ for all $n \geq 3$
    (c) Is greater than $\log(n)$ for all $n \geq 3$
    (d) Is greater than $1/\sqrt{n}$ for all $n \geq 3$

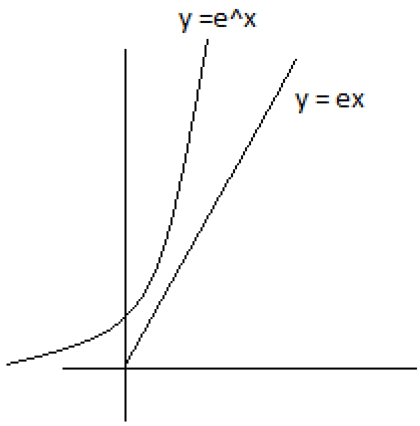
Solution :

Let, $x = n^{1/n}$

- ⇨ $\log x = (1/n)\log n = \{(n - 1)/n\}\{1/(n - 1)\}\log\{1 + (n - 1)\} = (1 - 1/n)\{\log(1 + (n - 1))\}/(n - 1) \to (1 - 0)*1$ as $n \to \infty = 1$
- ⇨ $n^{1/n} - 1 \to 1 - 1 = 0$ as $n \to \infty$

Let us take the inequality, $x - 1 < \log x$

- ⇨ $e^{x - 1} < x$
- ⇨ $ex > e^x$



From the graph it is clear that the inequality doesn't hold true.

- ⇨ $x - 1 > \log x$

Putting $x = n^{1/n}$ we get,

$n^{1/n} - 1 > \log n^{1/n}$

- ⇨ $n^{1/n} - 1 > (\log n)/n$
- ⇨ Options A, B are correct.

26. If the complex numbers $1 + i$ and $5 - 3i$ represent two diagonally opposite vertices of a square, which of the following complex numbers can represent another vertex of the square?
(a) $5 + 2i$
(b) $3 + 2\sqrt{2} - i$
(c) $1 - 3i$
(d) $4 + 2\sqrt{2} + 2\sqrt{2}i$

Solution :

The coordinates are (1, 1) and (5, -3)

Let us take option (a) i.e. (5, 2)

Now, $\sqrt{\{(5 - 1)^2 + (2 - 1)^2\}} = \sqrt{17}$

Now, $\sqrt{\{(5 - 5)^2 + (2 + 3)^2\}} = 5$

The distances are not equal.

- ⇨ Option (a) cannot be true.

Now, let us take option (b) i.e. $(3 + 2\sqrt{2}, -1)$

Now, $\sqrt{\{(3 + 2\sqrt{2} - 1)^2 + (-1 - 1)^2} = \sqrt{(16 + 8\sqrt{2})}$

Now, $\sqrt{\{(3 + 2\sqrt{2} - 5)^2 + (-1 - 3)^2\}} = \sqrt{(28 - 8\sqrt{2})}$

The distances are not equal.

- ⇨ Option (b) cannot be true.

Let us take option (c) i.e. $(1, -3)$

Now, $\sqrt{\{(1 - 1)^2 + (-3 - 1)^2\}} = 4$

Now, $\sqrt{\{(1 - 5)^2 + (-3 + 3)^2\}} = 4$

Distances same but it can be a rhombus.

Now, we will calculate the slope of the two lines and if the product is -1 then it must be a square.

Now, $(-3 - 1)/(1 - 1) = \infty$

- ⇨ The line is perpendicular to x-axis.

Now, $(-3 + 3)/(5 - 1) = 0$

- ⇨ The line is parallel to x-axis.
- ⇨ The angle between the two straight lines is 90°
- ⇨ Option (c) is correct.

Now, let us take option (d) i.e. $(4 + 2\sqrt{2}, 2\sqrt{2})$

Now, $\sqrt{\{(4 + 2\sqrt{2} - 1)^2 + (2\sqrt{2} - 1)^2} = \sqrt{(26 + 8\sqrt{2})}$

Now, $\sqrt{\{4 + 2\sqrt{2} - 5)^2 + (2\sqrt{2} + 3)^2\}} = \sqrt{(26 + 8\sqrt{2})}$

Distances are same but it can be a rhombus.

So, $(2\sqrt{2} - 1)/(4 + 2\sqrt{2} - 1) = (2\sqrt{2} - 1)/(3 + 2\sqrt{2})$

And, $(2\sqrt{2} + 3)/(4 + 2\sqrt{2} - 5) = (2\sqrt{2} + 3)/(2\sqrt{2} - 1)$

So, the multiplication of the slopes is 1 and not -1.

- ⇨ They are not perpendicular to each other.
- ⇨ It cannot be a coordinate of a square.
- ⇨ Option (d) cannot be true.
- ⇨ Only option (c) is correct.

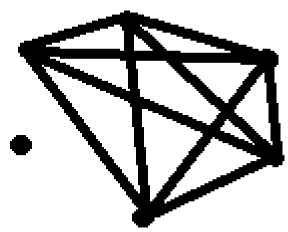
27. Suppose x and y are two positive numbers satisfying the equation $x^y = y^x$. Which of the following are true?
   (a) For all $x > 1$, there always exist a $y > x$ such that the above equation holds.
   (b) For all $x > e$ there is always a $y > x$ such that the above equation holds.
   (c) For all $1 < x < e$ there is always a $y > x$ such that the above equation holds.
   (d) If $x < 1$, the y must be equal to x.

Solution :

Options (c) and (d) are correct.

28. Consider 6 points on the plane no three of which are collinear. An edge is a straight line joining one point to another. Two points are called connected if one can go from one point to another through edges. Suppose you are only told how many edges are there in total, but not where they are. Which of the following are true?
   (a) If you are told there are 7 edges, you cannot be sure that all pairs of points are connected.
   (b) If you are told that there are 9 edges, you can always ensure that all pairs of points are connected.
   (c) If you are told that there are 12 edges, you cannot be sure that all pairs of points are connected.
   (d) If you are told that there are 13 edges, you can always ensure that all pairs of points are connected.

Solution :



There are 6 points. If we leave a point isolated i.e. not connected then we can join rest of the 5 points with ${}^5C_2 = 10$ edges.

So, maximum number of edges required to be sure that all the points are connected is 11.

- ⇨ Option (a) and (d) are correct.

29. Suppose f : R -> R is differentiable everywhere. Which of the following conditions imply that |f(x)| is also differentiable?
 (a) f(x) = 0 whenever f′(x) = 0.
 (b) f′(x) = 0 whenever f(x) = 0.
 (c)f′(x) never takes the value 0.
 (d) f(x) never takes the value 0.

Solution :

Let us take an example.

Let, f(x) = |x – a|

- ⇨ f(x) = x – a if x > a
- ⇨ f(x) = a – x if x < a

And f(a) = 0

Now. lim [{f(x) – f(a)}/(x – a)] as x -> a+ = lim {(x – a – 0)/(x – a)} as x -> a+ = lim (1/1) as x -> a+ (Applying L'Hospital rule) = 1

Now, lim [{f(x) – f(a)}/(x – a)] as x -> a- = lim {(a – x – 0)/(x – a)} as x -> a- = lim (-1/1) as x -> a- (Applying L'Hospital rule) = -1

So, f(x) is not differentiable at x = a.

Now, lim [{f(x) – f(a)}/(x – a)] as x -> a+ or a- = lim f′(x)/1 as x -> a+ or a-

If f′(a) = 0 then it is differentiable.

We will demonstrate this with an example.

Let f(x) = $|(x - a)^3|$

- ⇨ f(x) = $(x - a)^3$ if x > a
- ⇨ f(x) = $(a - x)^3$ if x < a

And f(a) = 0

Now, lim [{f(x) – f(a)}/(x – a)] as x -> a+ = lim $\{(x - a)^3 - 0\}/(x - a)$ as x -> a+ = lim $(x - a)^2$ as x-> a+ = 0

Now, lim [{f(x) – f(a)}/(x – a)] as x -> a- = lim $\{(a - x)^3 - 0\}/(x - a)$ as x -> a- = lim $-(x - a)^2$ as x -> a- = 0

- ⇨ f(x) is differentiable as f′(x) = 0 when f(x) = 0 at x = a.
- ⇨ Options (b) and (d) are correct.

30. Let the coordinates of the centre of a circle be $(-7/10, 2\sqrt{2})$. Then the number of points (x, y) on the circle such that both x and y are rational
(a) Cannot be 3 or more.
(b) At least 1, but at most 2.
(c) At least 2, but infinitely many.
(d) Infinitely many.

Solution :

The equation of the circle is $(x + 7/10)^2 + (y - 2\sqrt{2})^2 = r^2$ (where r is radius)

⇨ $x^2 + y^2 + (7/5)x - 4\sqrt{2}y + (7/5)^2 + 8 - r^2 = 0$

Let $r^2 = 4\sqrt{2}t$ where $t > 0$

Let, $y = -t$.

Then the equation becomes, $x^2 + (7/5)x + (7/5)^2 + 8 + t^2 = 0$

Now, the discriminant of the equation is, $(7/5)^2 - 4\{(7/5)^2 + 8 + t^2\}$. Clearly it is < 0.

⇨ No real solution for x.

Now, let $y = 0$, the equation becomes, $x^2 + (7/5)x + (7/5)^2 + 8 - r^2 = 0$

The disciminant of the equation is, $(7/5)^2 - 4\{(7/5)^2 + 8 - r^2\} = 4r^2 - 3(7/5)^2 - 32$. Depending on the value of r it can be > 0 also it is possible that it is less than 0. Also it can be a square number of some rational number, so x is rational. Also if $4r^2 - 3(7/5)^2 - 32 = 0$ then the roots are equal i.e. only one solution for x.

So, it can have 0 solution, 1 solution or 2 solutions but not more than that.

⇨ Option (a) is correct.

B. Math. (Hons.) Admission Test : 2011

1. The equation of the circle of smallest radius which passes through the points $(-1, 0)$ and $(0, -1)$ is :
(a) $x^2 + y^2 + 2xy = 0$
(b) $x^2 + y^2 + x + y = 0$

(c) $x^2 + y^2 - x - y = 0$
(d) $x^2 + y^2 + x + y + 1/4 = 0$

Solution :

Clearly the circle of option (b) passes through the given two points. No other option satisfies this condition.

- ⇨ Option (b) is correct.

2. The function $f(x) = x^2e^{-|x|}$ defined on entire real line is
   (a) Not continuous at exactly one point
   (b) Continuous everywhere but not differentiable at exactly one point
   (c) Differentiable everywhere
   (d) Differentiable everywhere.

Solution :

Now, $f(x) = x^2e^{-|x|}$

- ⇨ $f(x) = x^2e^{-x}$ if $x > 0$
- ⇨ $f(x) = x^2e^x$ if $x < 0$
- ⇨ $f(x) = 0$ if $x = 0$

Now, lim f(x) as x -> 0+ = lim x^2e^{-x} as x -> 0+ = 0

lim f(x) as x -> 0- = lim x^2e^x as x -> 0- = 0

f(0) = 0

- ⇨ f(x) is continuous everywhere.

Now, lim [{f(x) – f(0)}/(x – 0)] as x -> 0+ = lim {(x^2e^{-x} – 0)/x} as x -> 0+ = lim ($2xe^{-x} - x^2e^{-x}$)/1 as x -> 0+ (Applying L'Hospital rule) = 0

lim [{f(x) – f(0)}/(x – 0)] as x -> 0- = lim {(x^2e^x – 0)/x} as x -> 0- = lim ($2xe^e + x^2e^x$)/1 as x -> 0- (Applying L'Hospital rule) = 0

So, f(x) is differentiable everywhere.

- ⇨ Option (c) is correct.

3. Let c_1 and c_2 be positive real numbers. Consider the function
   $f(x) = c_1x \quad 0 \le x < 1/3$
   $f(x) = c_2(1 - x) \quad 1/3 \le x \le 1$

If f is continuous and $\int f(x)dx = 1$ (integration running from 0 to 1), the value of c_2 is
(a) 2
(b) 1
(c) 3
(d) ½

Solution :

Now, lim f(x) as x -> 1/3- = lim(c_1x) as x -> 1/3- = $c_1/3$

lim f(x) as x -> 1/3+ = lim{$c_2(1 - x)$} as x -> 1/3+ = $2c_2/3$

And, f(1/3) = $2c_2/3$

f is continuous.

- ⇨ $c_1/3 = 2c_2/3$
- ⇨ $c_1 = 2c_2$

Now, $\int f(x)dx = 1$ (integration running from 0 to 1)

- ⇨ $\int f(x)dx$ (integration running from 0 to 1/3) + $\int f(x)dx$ (integration running from 1/3 to 1) = 1
- ⇨ $\int c_1xdx$ (integration running from 0 to 1/3 + $\int c_2(1 - x)dx$ (integration running from 1/3 to 1) = 1
- ⇨ $c_1x^2/2$ (0 to 1/3) – $c_2(1 - x)^2/2$ (1/3 to 1) = 1
- ⇨ $c_1/18 + 4c_2/18 = 1$
- ⇨ $2c_2/18 + 4c_2/18 = 1$ (Putting $c_1 = 2c_2$)
- ⇨ $6c_2/18 = 1$
- ⇨ $c_2 = 3$
- ⇨ Option (c) is correct.

4. Mr. Gala purchased 10 plots of land in the year 2007, all plots costing the same amount. He made a profit of 25 percent on each of the 6 plots which he sold in 2008. He had a loss of 25 percent on each of the remaining plots when he sold them in 2009. If he ended with a total profit of Rs. 2 crores in this project, his total purchase price was
(a) 8 crores
(b) 40 crores
(c) 10 crores
(d) 20 crores.

Solution :

Let purchase price of each plot is x crores.

In 2008 he made profit = x*(25/100)*6 = 3x/2

In 2009 he made loss = x*(25/100)*4 = x

Net profit = 3x/2 – x = x/2

According to question, x/2 = 2 crores

- ⇨ x = 4 crores.
- ⇨ 10x = 40 crores.
- ⇨ Option (b) is correct.

5. Let f(x) = xsin(1/x) for x > 0. Then
   (a) f is unbounded
   (b) f is bounded
   (c) lim f(x) as x -> ∞ = 1
   (d) lim f(x) as x -> ∞ = 0.

Solution.

lim f(x) as x -> ∞ = lim {xsin(1/x)} as x -> ∞

Let, z = 1/x

z -> 0 as x -> ∞

So, the limit is lim (sinz)/z as z -> 0 = 1

- ⇨ Option (c) is correct.

6. Let
   $f(x) = x^2\sin(1/x)$, $x \neq 0$
   f(x) = 0, x = 0
   Then
   (a) f is continuous at x = 0.
   (b) f is differentiable, and f' is continuous.
   (c) f is not differentiable at x = 0.
   (d) f is differentiable at every x but f' is discontinuous at x = 0.

Solution :

Now, lim f(x) as x -> 0+ = lim $x^2\sin(1/x)$ as x -> 0+

Let z = 1/x

$z \to \infty$ as $x \to 0$

The limit is, lim $(\sin z)/z^2$ as $z \to \infty = 0$ (as $-1 \le \sin z \le 1$)

Similarly, lim f(x) as $x \to 0- = 0$

⇨ f(x) is continuous everywhere.

Now, lim $[\{f(x) - f(0)\}/(x - 0)]$ as $x \to 0+ =$ lim $\{x^2\sin(1/x)\}/x$ as $x \to 0+ = x\sin(1/x)$ as $x \to 0+ = 1$

Similarly, lim $[\{f(x) - f(0)\}/(x - 0)]$ as $x \to 0- = 1$

⇨ f(x) is differentiable everywhere.

Now, $f'(x) = 2x\sin(1/x) + x^2\cos(1/x)(-1/x^2) = 2x\sin(1/x) - \cos(1/x)$

Lim $f'(x)$ as $x \to 0 =$ lim $\{2x\sin(1/x) - \cos(1/x)\}$ as $x \to 0 =$ does not exist

⇨ Option (d) is correct.

7. Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial of degree n with real coefficients a_i. Suppose that there is a constant $C > 0$ and an integer $k \ge 0$ such that $|P(x)| < cx^k$ for all $x > 0$. Then
   (a) n must be equal to k
   (b) The given information is not sufficient to say anything about n
   (c) $n \ge k$
   (d) $n \le k$.

Solution :

Let, $n = 1$ and so, $P(x) = a_0 + a_1x$

Now, let $k = 0$

Then $-C < a_0 + a_1x < C$

⇨ $x < (C - a_0)/a_1$
⇨ This is not defined for all $x > 0$
⇨ $k \ge n$
⇨ Option (d) is correct.

8. Let f be a strictly increasing function on R, that is $f(x) < f(y)$ whenever $x < y$. Then
   (a) f is a continuous function
   (b) f is a bounded function

(c) f is an unbounded function
(d) The given information is not sufficient to say anything about continuity or boundedness of f.

Solution :

Option (d) is correct.

9. The minimum value of $x^2 + y^2$ subject to $x + y = 1$ is
(a) 0
(b) ½
(c) ¼
(d) 1

Solution :

Now, $(x^2 + y^2)/2 \geq \{(x + y)/2\}^2$

- ⇨ $x^2 + y^2 \geq ½$
- ⇨ Option (b) is correct.

10. The number 2532645918 is divisible by
(a) 3 but not 11
(b) 11 but not 3
(c) Both 3 and 11
(d) Neither 3 nor 11.

Solution :

(8 + 9 + 4 + 2 + 5) – (1 + 5 + 6 + 3 + 2) = 28 – 17 = 11

So, divisible by 11.

Now, 28 + 17 = 45

So, divisible by 3 as well.

- ⇨ Option (c) is correct.

11. Let p > 3 be a prime number. Which of the following is always false?
(a) p + 2 is a prime number.

(b) $p + 4$ is a prime number.
(c) Both $p + 2$ and $p + 4$ are prime numbers.
(d) Neither $p + 2$ nor $p + 4$ are prime numbers.

Solution :

Now, p is a prime> 3.

- ⇨ $p \equiv 1, -1 \pmod 3$

At first, let $p \equiv 1 \pmod 3$

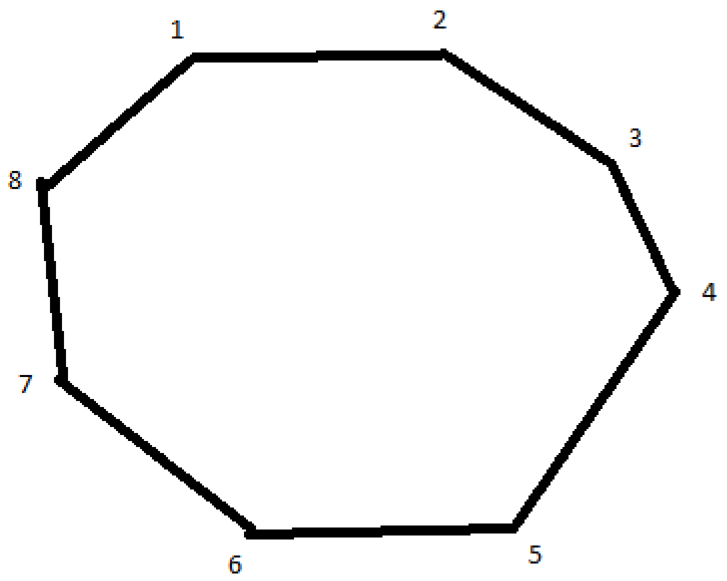
- ⇨ $p + 2 \equiv 1 + 2 = 3 \equiv 0 \pmod 3$

Now, $p \equiv -1 \pmod 3$

- ⇨ $p + 4 \equiv -1 + 1 = 0 \pmod 3$
- ⇨ Either $p + 2$ or $p + 4$ i divisible by 3 and both together cannot be prime when p is prime.
- ⇨ Option (c) is correct.

12. By a diagonal of a convex polygon, we mean a line segment between any two non-consecutive vertices. The number of diagonals of a convex polygon of 8 sides is :
(a) 15
(b) 20
(c) 28
(d) 35.

Solution :



The diagonals are,

From 1, (1, 3); (1, 4); (1, 5); (1, 6); (1, 7) = 5

From 2, (2, 4); (2, 5); (2, 6); (2, 7); (2, 8) = 5

From 3, (3, 5); (3, 6); (3, 7); (3, 8) = 4 (3, 1 is already taken)

From 4, (4, 6); (4, 7); (4, 8) = 3 ((4, 1); (4, 2) are already taken)

From 5, (5, 7); (5, 8) = 2 ((5, 1); (5, 2); (5, 3) are already taken)

From 6, (6, 8) = 1 ((6, 1); (6, 2); (6, 3); (6, 4) are already taken)

Therefore, total number of diagonals = 5 + 5 + 4 + 3 + 2 + 1 = 20.

⇨ Option (b) is correct.

13. The coefficients of three consecutive terms in the expansion of $(1 + t)^n$ are 120, 210, 252. Then, n must be
(a) 10
(b) 12
(c) 14
(d) 16

Solution :

Let, ${}^nC_r = 120$, ${}^nC_{r+1} = 210$ and ${}^nC_{r+2} = 252$

- ⇨ $n!/\{(n - r)!*r!\} = 120$, $n!/\{(n - r - 1)!*(r + 1)!\} = 210$ and $n!/\{(n - r - 2)!*(r + 2)!\} = 252$

Now, $[n!/\{(n - r - 1)!*(r + 1)!\}]/[n!/\{(n - r)!*r!\}] = 210/120$

- ⇨ $(n - r)/(r + 1) = 7/4$
- ⇨ $4n - 4r = 7r + 7$
- ⇨ $4n - 11r = 7$ ......(A)

Now, $[n!/\{(n - r - 2)!*(r + 2)!\}]/[n!/\{(n - r - 1)!*(r + 1)!\}] = 252/210$

- ⇨ $(n - r - 1)/(r + 2) = 126/105$
- ⇨ $105n - 105r - 105 = 126r + 252$
- ⇨ $105n - 231r = 357$ ....(B)

From (A), $84n - 231r = 147$ (multiplying both sides by 11)

Now, subtracting from (B) we get,

$105n - 84n = 357 - 147$

- ⇨ $21n = 210$
- ⇨ $n = 10$
- ⇨ Option (a) is correct.

14. If $2\sec(2\alpha) = \tan(\beta) + \cot(\beta)$, then $\alpha + \beta$ can have the value
    (a) $\pi/2$
    (b) $\pi/3$
    (c) $\pi/4$
    (d) 0.

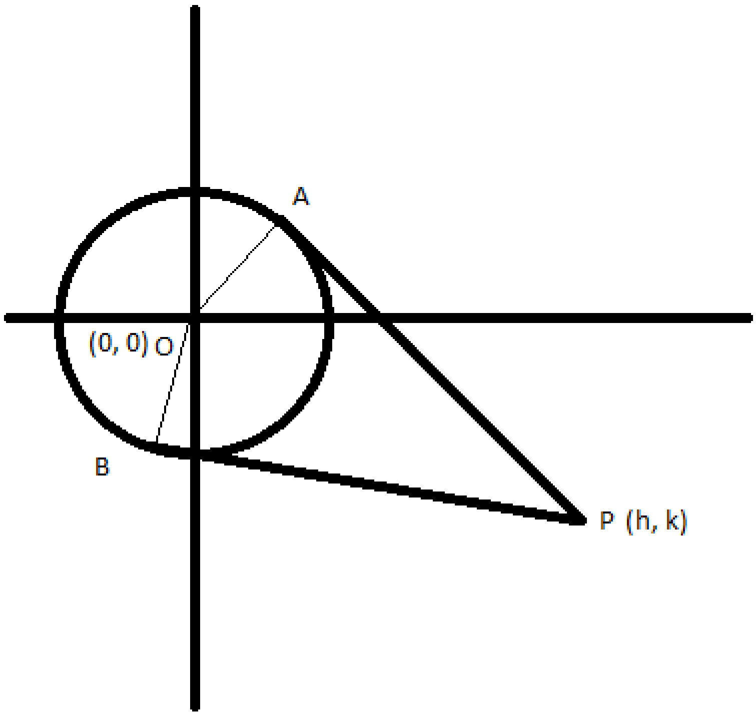
Solution :

Now, $2\sec(2\alpha) = \tan(\beta) + \cot(\beta)$

- ⇨ $2\sec(2\alpha) = \sin(\beta)/\cos(\beta) + \cos(\beta)/\sin(\beta)$
- ⇨ $2\sec(2\alpha) = \{\sin^2(\beta) + \cos^2(\beta)\}/\sin(\beta)\cos(\beta)$
- ⇨ $\sec(2\alpha) = 1/2\sin(\beta)\cos(\beta)$
- ⇨ $1/\cos(2\alpha) = 1/\sin(2\beta)$
- ⇨ $\cos(2\alpha)= \sin(2\beta)$
- ⇨ $2\alpha = 2\beta = \pi/4$
- ⇨ $(\alpha + \beta) = \pi/4$.
- ⇨ Option (c) is correct.

15. Consider the unit circle $x^2 + y^2 = 1$. The locus of a point P such that the tangents PA, PB at the points A, B respectively, of the circle are so that angle AOB = 60°, where O is the origin, is
(a) A circle of radius $2/\sqrt{3}$ with centre O.
(b) A circle of radius $\sqrt{3}$ with centre O.
(c) A circle of radius 2 with centre O.
(d) A pair of straight lines.

Solution :



Angle AOB = 60°

- ⇨ Angle POB = 30°

Now, OB = 1 and OP = $\sqrt{(h^2 + k^2)}$

In triangle BOP, cos30° = OB/OP

- ⇨ $1/\sqrt{(h^2 + k^2)} = \sqrt{3}/2$
- ⇨ $h^2 + k^2 = 4/3$
- ⇨ Locus of point P is, $x^2 + y^2 = 4/3$
- ⇨ Option (a) is correct.

16. Let, x, y be integers. Consider the two statements (I) 10x + y is divisible by 7, and, (II) x + 5y is divisible by 7. Then

(a) (I) implies (II) but not conversely
(b) (II) implies (I) but not conversely
(c) The two statements are equivalent
(d) Neither statement implies the other.

Solution :

Now, $10x + y \equiv 0 \pmod 7$

- ⇨ $3x - 6y \equiv 0 \pmod 7$
- ⇨ $x - 2y \equiv 0 \pmod 7$
- ⇨ $x + 5y \equiv 0 \pmod 7$

Now, $x + 5y \equiv 0 \pmod 7$

- ⇨ $x - 2y \equiv 0 \pmod 7$
- ⇨ $3x - 6y \equiv 0 \pmod 7$
- ⇨ $10x + y \equiv 0 \pmod 7$
- ⇨ Option (c) is correct.

17. The number of solutions of the equation $6m + 15n = 8$ in integers m and n are
(a) Zero
(b) One
(c) More than one but finitely many
(d) Infinitely many.

Solution :

The equation is, $6m + 15n = 8$

Dividing the equation by 3 we get.

$0 + 0 \equiv 2 \pmod 3$

Which is impossible.

- ⇨ No solution.
- ⇨ Option (a) is correct.

18. Let A, B be real numbers both greater than 0. The graph of the function $f(x) = Bx^5 + 2Ax + A\sin(2x)$ passes through the two points $P = (-1, 2)$ and $Q = (0, 1)$ for
(a) Finitely many values of A and infinitely many values of B
(b) Infinitely many values of A and infinitely many values of B

(c) No values of A and B
(d) None of the above.

Solution :

The graph of the function passes through Q = (0, 1)

- ⇨ f(0) = 1
- ⇨ $B*0^5 + 2A*0 + A\sin(2*0) = 1$
- ⇨ 0 = 1

Which is impossible.

So we take option (c).

19. A triangle in the plane has area 1. Then its perimeter (= sum of the lengths of its three sides) p must satisfy
(a) $p < 1$
(b) $p < 2$
(c) $p > 2$
(d) $p = 2$.

Solution :

Now, $\sqrt{\{s(s - a)(s - b)(s - c)} = 1$

Now, $\{s + (s - a) + (s - b) + (s - c)\}/4 > \{s(s - a)(s - b)(s - c)\}^{1/4}$ (AM > GM)

- ⇨ $\{4s - (a + b + c)\}/4 > 1$
- ⇨ $(4s - 2s)/4 > 1$
- ⇨ $s > 2$
- ⇨ $p/2 > 2$
- ⇨ $p > 4$

None of the option matches.

We take option (c) as it is closest.

20. A sequence is defined by $a_1 = 1$ and the inductive formula $a_{n+!} = \sqrt{(1 + a_n^u)}$ where u is a real number greater than 0. If this sequence converges to a finite limit then u must be
(a) > 0
(b) > 2
(c) < 2

(d) = 2.

Solution :

Let u = 2.

Now, $a_2 = \sqrt{2}$, $a_3 = \sqrt{3}$, $a_4 = \sqrt{4}$, ... i.e. in general $a_n = \sqrt{n}$ and it is not a converging sequence.

Now, u > 2, let u = 4

Now, $a_2 = \sqrt{2}$, $a_3 = \sqrt{5}$, $a_4 = \sqrt{26}$, ... which is not a converging sequence.

⇨ Options (b) and (d) cannot be true.
⇨ Option (c) is correct.

21. Let a, b, c be three nonzero real numbers. If $f(x) = ax^2 + bx + c$ has equal roots, then a, b, c are in
(a) Arithmetic progression
(b) Geometric progression
(c) Harmonic progression
(d) None of the above.

Solution :

Roots are equal.

⇨ $b^2 = 4ac$
⇨ Option (d) is correct.

22. Let a, b, c be real numbers such that $3b > a^2$. Then the function g : R -> R given by $g(x) = x^3 + ax^2 + bx + c$ is
(a) One-one an onto
(b) Onto but not one-one
(c) One-one but not onto
(d) Neither one-one nor onto.

Solution :

Let us take any two points x_1 and x_2

Let us see if $g(x_1) = g(x_2)$ or not.

So we have, $x_1^3 + ax_1^2 + bx_1 + c = x_2^3 + ax_2^2 + bx_1 + c$

- ⇨ $(x_1^3 - x_2^3)\ a(x_1^2 - x_2^2) + b(x_1 - x_2) = 0$
- ⇨ $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) + a(x_1 + x_2)(x_1 - x_2) + b(x_1 - x_2) = 0$
- ⇨ $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + ax_1 + ax_2 + b) = 0$

Now, we assume, $x_1^2 + x_1x_2 + x_2^2 + ax_1 + ax_2 + b = 0$

- ⇨ $x_1^2 + x_1(x_2 + a) + (x_2^2 + ax_2 + b) = 0$

Now, this is a quadratic on x_1^2.

Now, the discriminant of the equation is, $(x_2 + a)^2 - 4(x_2^2 + ax_2 + b) \geq 0$ (for real solution of x_1)

- ⇨ $x_2^2 + 2ax_2 + a^2 - 4x_2^2 - 4ax_2 - 4b \geq 0$
- ⇨ $0 \leq a^2 - 2ax_2 - 3x_2^2 - 4b < a^2 - 2ax_2 - 3x_2^2 - 4a^2/3$ (As $3b > a^2$)
- ⇨ $-a^2/3 - 2ax_2 - 3x_2^2 > 0$
- ⇨ $a^2 + 6ax_2 + 9x_2^2 < 0$
- ⇨ $(a + 3x_2)^2 < 0$
- ⇨ Our assumption was wrong.
- ⇨ If $g(x_1) = g(x_2)$ then $x_1 = x_2$.
- ⇨ $g(x)$ is one-one.

Now, $g(x) = x^3 + ax^2 + bx + c$

- ⇨ $g'(x) = 3x^2 + 2ax + b$

Now, the discriminant of the equation is, $4a^2 - 12b = 4(a^2 - 3b) < 0$ (As $3b > a^2$)

- ⇨ $g'(x) = 0$ has no solution.
- ⇨ $g(x)$ doesn't have any extremum point.
- ⇨ $g(x)$ is stretching towards $-\infty$ to ∞.
- ⇨ $g(x)$ is onto.
- ⇨ Option (a) is correct.

23. Express the polynomial $f(x) = (2 + x)^n$ as $f(x) = c_0 + c_1x + c_2x^2 + ... + c_nx^n$, where n is a positive integer. If $\sum c_j$ (summation running from $j = 0$ to $j = n$) = 81, then the largest coefficient c_j of f is
(a) 64
(b) 16
(c) 24
(d) 32.

Solution :

Let n = 6, then the middle term = $2^3 * {}^6C_3$ = 8*6*5*4/3*2 = 160 > 81

Let n = 5, then middle term = $2^2 * {}^5C_2$ = 4*10 = 40

There is another middle term and = 40.

First term i.e. c_0 = 32

- ⇨ The sum > 81

Let n = 4,

Then $c_0 = 16$, $c_1 = 8 * {}^4C_1 = 32$, $c_2 = 4 * {}^4C_2 = 24$, $c_3 = 2 * {}^4C_1 = 8$, $c_4 = 1$

Now, $c_0 + c_1 + c_2 + c_3 + c_4 = 16 + 32 + 24 + 8 + 1 = 81$

- ⇨ n = 4 and largest $c_j = c_1 = 32$
- ⇨ Option (d) is correct.

24. Let l, m, n be any three positive such that $l^2 + m^2 = n^2$. Then,
    (a) 3 always divides mn
    (b) 3 always divides lm
    (c) 3 always divides ln
    (d) 3 does not divide lmn.

Solution :

Now, $l^2 + m^2 = n^2$

Let 3 does not divide any of l, m and n

- ⇨ $l \equiv \pm 1 \pmod 3$
- ⇨ $l^2 \equiv (\pm 1)^2 = 1 \pmod 3$

Similarly, m^2 , $n^2 \equiv 1 \pmod 3$

Now, dividing the equation by 3 we get,

$1 + 1 \equiv 1 \pmod 3$

- ⇨ $2 \equiv 1 \pmod 3$

Which is impossible.

Let 3 divides n.

Now, dividing the equation by 3 we get,

$1 + 1 \equiv 0 \pmod 3$

Which is impossible.

Let 3 divides any one of m or l

Let 3 divides m.

Now, dividing the equation by 3 we get,

$1 + 0 \equiv 1 \pmod 3$

Which is consistent equation.

- ⇨ 3 divides any one of l or m.
- ⇨ Option (b) is correct.

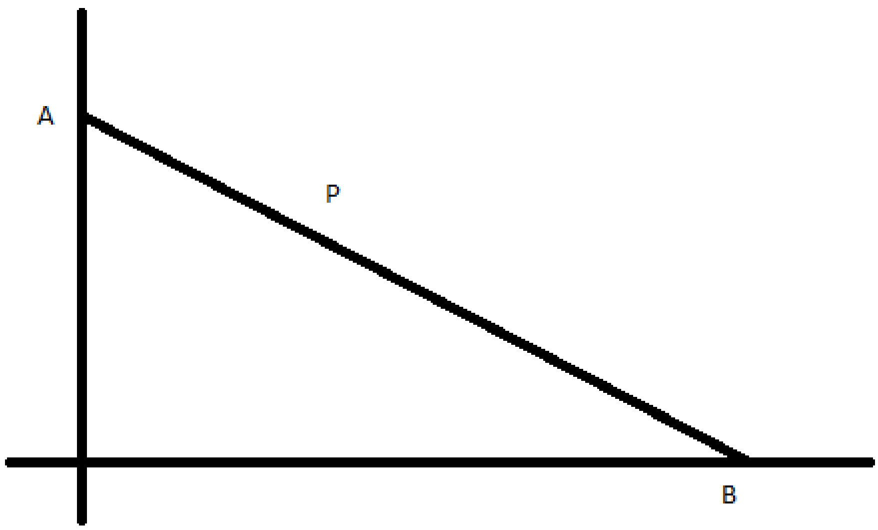
25. Let $a_1 = 10$, $a_2 = 20$ and define $a_{n+!} = a_{n-1} - 4/a_n$ for $n > 1$. The smallest k for which $a_k = 0$
    (a) Does not exist.
    (b) Is 200
    (c) Is 50
    (d) Is 52.

Solution :

Option (d) is correct.

B. Stat. (Hons.) & B. Math. (Hons.) Admission Test : 2012

1. A rod AB of length 3 rests on a wall as follows :



P is a point on AB such that AP : PB = 1 : 2. If the rod slides along the wall, then the locus of P lies on

(a) $2x + y + xy = 2$
(b) $4x^2 + y^2 = 4$
(c) $4x^2 + xy + 4y^2 = 4$
(d) $x^2 + y^2 - x - 2y = 0$

Solution :

Let coordinate of P is (h, k)

Let coordinate of A is (0, a) and coordinate of B is (b, 0)

Now, AP : PB = 1 : 2

- ⇨ $h = (0*2 + 1*b)/3$
- ⇨ $h = b/3$
- ⇨ $k = (a*2 + 0*1)/3$
- ⇨ $k = 2a/3$

Now, length of the rod remains constant as the rod slides on the wall.

- ⇨ $\sqrt{(a^2 + b^2)} = c$
- ⇨ $(3k/2)^2 + (3h)^2 = c^2$
- ⇨ $4h^2 + k^2 = c_1^2$
- ⇨ The locus of P is $4x^2 + y^2 = c_1^2$
- ⇨ Option (b) is correct.

2. Consider the equation $x^2 + y^2 = 2007$. How many solutions (x, y) exist such that x and y are positive integers?

   (a) None

(b) Exactly two
(c) More than two but finitely many
(d) Infinitely many.

Solution :

Now, $x^2 + y^2 = 2007$ = odd.

- ⇨ One of x and y is even and another is odd.
- ⇨ Let x is even and y is odd (without loss of generality)

Now, $x^2 \equiv 0 \pmod 4$ (as x is even)

And, $y \equiv \pm 1 \pmod 4$

- ⇨ $y^2 \equiv (\pm 1)^2 = 1 \pmod 4$

Now, dividing the equation by 4 we get,

$0 + 1 \equiv 3 \pmod 4$

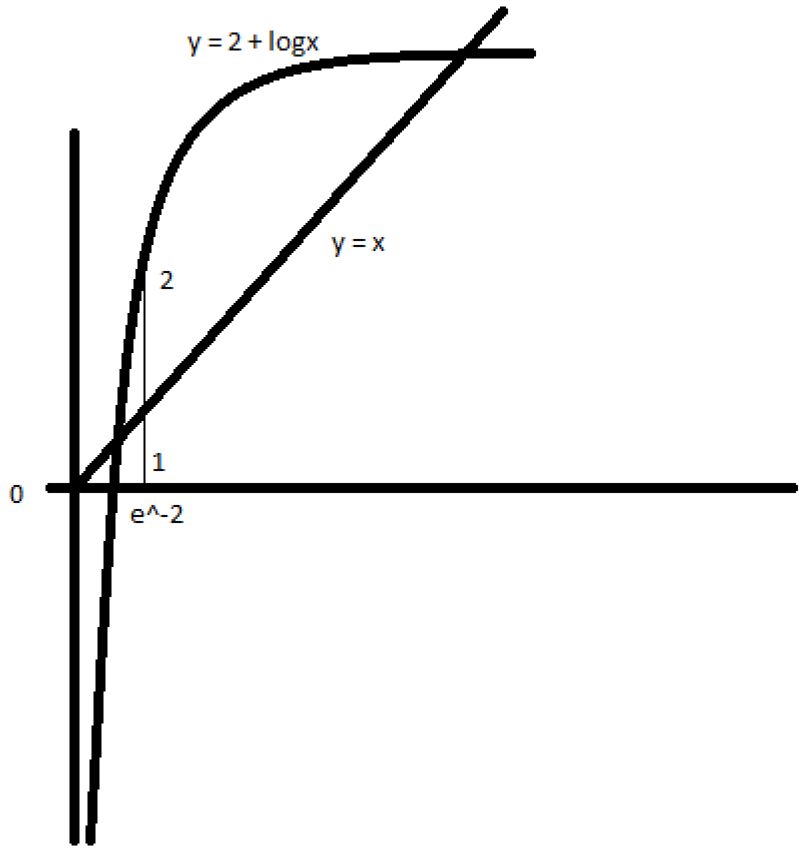
- ⇨ $1 \equiv 3 \pmod 4$

Which is impossible.

- ⇨ No solution.
- ⇨ Option (a) is correct.

3. Consider the functions $f_1(x) = x$, $f_2(x) = 2 + \log_e x$, $x > 0$ (where e is the base of natural logarithm). The graphs of the functions intersect
   (a) Once in (0, 1) and never in (1, ∞)
   (b) Once in (0, 1) and once in (e^2, ∞)
   (c) Once in (0, 1) and once in (e, e^2)
   (d) More than twice in (0, ∞).

Solution :



Clearly they will intersect at to points. One in (0, 1) and another in (e, e^2) as $f_1(e) < f_2(e)$ and $f_1(e^2) > f_2(e^2)$.

⇨ Option (c) is correct.

4. Consider the sequence $u_n = \sum(r/2^r)$ (summation running from r = 1 to r = n), n ≥ 1. Then the limit of u_n as n -> ∞ is
   (a) 1
   (b) 2
   (c) e
   (d) ½

Solution :

Now, $u_n = ½^1 + 2/2^2 + 3/2^3 + 4/2^4 + ... + n/2^n$

⇨ $(1/2)u_n = ½^2 + 2/2^3 + 4/2^4 + ... + (n - 1)/2^n + n/2^{(n + 1)}$

Now, $u_n - (1/2)u_n = ½ + (2/2^2 - ½^2) + (3/2^3 - 2/2^3) + (4/2^4 - 3/2^4) + ... + \{n/2^n - (n - 1)/2^n\} - n/2^{(n + 1)}$

⇨ $(1/2)u_n = (½ + ½^2 + ½^3 + ½^4 + ... + 1/2^n) - n/2^{(n + 1)}$

- ⇨ $u_n = (1 + ½ + ½^2 + ½^3 + \ldots + 1/2^{n-1}) - n/2^n$
- ⇨ $u_n = 1*\{1 - (1/2)^n\}/(1 - ½) - n/2^n$
- ⇨ $u_n = 2(1 - 1/2^n) - n/2^n$
- ⇨ $\lim u_n$ as $n \to \infty = 2$
- ⇨ Option (b) is correct.

5. Suppose that z is a complex number which is not equal to any of $\{3, 3w, 3w^2\}$ where w is a complex cube root of unity. Then $1/(z - 3) + 1/(z - 3w) + 1/(z - 3w^2)$ equals
   (a) $(3z^2 + 3z)/(z - 3)^3$
   (b) $(3z^2 + 3wz)/(z^3 - 27)$
   (c) $3z^2/(z^3 - 3z^2 + 9z - 27)$
   (d) $3z^2/(z^3 - 27)$

Solution :

Now, $1/(z - 3) + 1/(z - 3w) + 1/(z - 3w^2)$

$= \{(z - 3w)(z - 3w^2) + (z - 3)(z - 3w^2) + (z - 3)(z - 3w)\}/(z - 3)(z - 3w)(z - 3w^2)$

$= \{z^2 - 3wz - 3w^2z + 9w^3 + (z - 3)(z - 3w^2 + z - 3w)\}/(z - 3)(z^2 - 3wz - 3w^2z + 9w^3)$

$= \{z^2 + 3z + 9 + (z - 3)(2z + 3)\}/(z - 3)(z^2+ 3z + 9)$

$= (z^2 + 3z + 9 + 2z^2 + 3z - 6z - 9)/(z^3 - 27)$

$= 3z^2/(z^3 - 27)$

- ⇨ Option (d) is correct.

6. Consider all function f : {1, 2, 3, 4} -> {1, 2, 3, 4} which are one-one, onto and satisfy the following property :
   If f(k) is odd then f(k + 1) is even, k = 1, 2, 3.
   The number of such functions is
   (a) 4
   (b) 8
   (c) 12
   (d) 16

Solution :

The sample space looks like below,

{n, f(n)} = {(1, 1); (2, 2); (3, 3); (4, 4)}; {(1, 1); (2, 2); (3, 4); (4, 3)}; {(1, 1); (2, 4); (3, 3); (4, 2)}; {(1, 1); (2, 4); (3, 2); (4, 3)}; {(1, 2); (2, 1); (3, 4); (4, 3}; {(1, 2); (2, 3); (3, 4); (4, 1)}; {(1, 3); (2, 2); (3, 1); (4, 4)}; {(1, 3); (2, 2); (3, 4); (4, 1)}; {(1, 3); (2, 4); (3, 1); (4, 2)}; {(1, 3); (2, 4); (3, 2); (4, 1)}; {(1, 4); (2, 1); (3, 2); (4, 3)}; {(1, 4); (2, 3); (3, 2); (4, 1)}

- ⇨ Number of such functions = 12.
- ⇨ Option (c) is correct.

7. A function f : R -> R is defined by
   $f(x) = e^{-1/x}$, x > 0
   f(x) = 0, x ≤ 0
   Then
   (a) f is not continuous.
   (b) f is differentiable but f′ is not continuous
   (c)f is continuous but f′(0) does not exist
   (d) f is differentiable and f′ is continuous.

Solution :

Now, lim [{f(x) – f(0)}/(x – 0)] as x -> 0+ = lim $e^{-1/x}/x$ as x -> 0+

Let, z = 1/x. z -> ∞ as x -> 0

So, lim (z/e^z) as z -> ∞ = lim $1/e^z$ as z -> ∞ (Applying L'Hospital rule) = 0

Now, lim[{f(x) – f(0)}/(x – 0)} as x -> 0- = 0

- ⇨ f is differentiable.

Now, f′(x) = $e^{-1/x}/x^2$for x > 0 and = 0 for x ≤ 0

Now, lim f(x) as x -> 0+ = lim $e^{-1/x}/x^2$

Let, z = 1/x, z -> ∞ as x -> 0+

So, lim z^2/e^z as z -> ∞ = lim $2z/e^z$ as z -> ∞ (Applying L'hospital rule) = lim $2/e^z$ as z -> ∞ (Again applying L'Hospital rule) = 0

And lim f′(x) as x -> 0- = 0 ans also f′(0) = 0

- ⇨ Option (d) is correct.

8. The last digit of 9! + 3^{9966}
   (a) 3

(b) 9
(c) 7
(d) 1

Solution :

Now, last digit of 9! Is 0.

Now, $3^2 \equiv -1 \pmod{10}$

- ⇨ $(3^2)^{4983} \equiv (-1)^{4983} \pmod{10}$
- ⇨ $3^{9966} \equiv -1 \pmod{10}$
- ⇨ $3^{9966} \equiv 9 \pmod{10}$
- ⇨ Option (b) is correct.

9. Consider the function $f(x) = (2x^2 + 3x + 1)/(2x - 1)$, $2 \le x \le 3$. Then
   (a) Maximum of f is attained inside the interval (2, 3)
   (b) Minimum of f is 28/5
   (c) Maximum of f is 28/5
   (d) f is decreasing function in (2, 3).

Solution :

$f'(x) = \{(4x + 3)(2x - 1) - (2x^2 + 3x + 1)*2\}/(2x - 1)^2 = 0$

- ⇨ $8x^2 - 4x + 6x - 3 - 4x^2 - 6x - 2 = 0$
- ⇨ $4x^2 - 4x - 5 = 0$
- ⇨ $x = \{4 \pm \sqrt{(16 + 4*4*5)}\}/2*4$
- ⇨ $x = (4 \pm 4\sqrt{6})/8$
- ⇨ $x = (1 \pm \sqrt{6})/2$
- ⇨ For no value of x in [2, 3] f(x) attains maximum or minimum value.

Now, $f'(x)$ also is not less than 0 for all values of x in (2, 3).

- ⇨ f(x) is not decreasing in (2, 3).

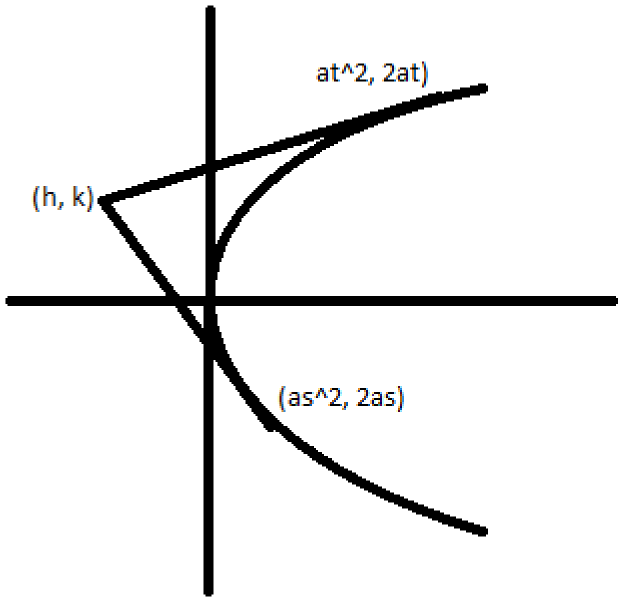
Now, $f(2) = 5$ and $f(3) = 28/5$

- ⇨ Option (c) is correct.

10. A particle P moves in the plane in such a way that the angle between the two tangents drawn from P to the curve $y^2 = 4ax$ is always 90°. The locus of P is
   (a) A parabola

(b) A circle
(c) An ellipse
(d) A straight line.

Solution :



Now, the angle between the two tangents is 90°.

Now, the angle between the two tangents and their respective normal is 90°.

The two normal meets at focus (a, 0)

Now, the quadrilateral of four points (h, k), (at^2, 2at), (as^2, 2at), (a, 0) have total angle 360°

- ⇨ The normals meet at 90° at focus (360° - 90° - 90° - 90° = 90°)
- ⇨ The quadrilateral is a square because the length of the tangents from a point are equal.

So, we have, $(at^2 - a)^2 + (2at - 0)^2 = (as^2 - a)^2 + (2as - 0)^2$ (distances of (at^2, 2at) and (as^2, 2as) from focus are same as it is a square)

- ⇨ $(t^2 - 1)^2 + 4t^2 = (s^2 - 1)^2 + 4s^2$
- ⇨ $(t^2 + 1)^2 = (s^2 + 1)^2$
- ⇨ $t^2 + 1 = s^2 + 1$
- ⇨ $t^2 - s^2 = 0$
- ⇨ $(t + s)(t - s) = 0$
- ⇨ $t + s = 0$ (As $t \neq s$)
- ⇨ $s = -t$

Now, the points are $(at^2, 2at)$ and $(at^2, -2at)$

Now, the distances of these two points from P are equal.

- ⇨ $(at^2 - h)^2 + (2at - k)^2 = (at^2 - h)^2 + (-2at - k)^2$
- ⇨ $(2at + k)^2 = (2at - k)^2$
- ⇨ $k = 0$
- ⇨ The locus of P is $y = 0$
- ⇨ A straight line.
- ⇨ Option (d) is correct.

11. Let, $f : R \to R$ be given by $f(x) = |x^2 - 1|$, $x \in R$. Then
    (a) f has a local minima at $x = \pm 1$ but no local maxima
    (b) f has a local maximum at $x = 0$ but no local minima
    (c) f has a local minima at $x = \pm 1$ and a local maximum at $x = 0$
    (d) None of the above is true.

Solution :

Clearly option (c) is correct.

12. The number of triplets (a, b, c) of positive integers satisfying $2^a - 5^b 7^c = 1$ is
    (a) Infinite
    (b) 2
    (c) 1
    (d) 0

Solution :

Now, $2^a - 5^b 7^c = 1$

a, b, c are positive.

Minimum value of b, c = 1

- ⇨ $a > 5$.

Dividing the equation by 8 we get,

$0 - (5 \text{ or } 1)(-1)^c \equiv 1 \pmod 5$ (If b is odd $5^b \equiv 5 \pmod 8$ and if even then $5^b \equiv 1 \pmod 8$)

Clearly to hold the equation, b must be even and c must be odd.

Now, dividing the equation by 5 we get,

$2^a - 0 \equiv 1 \pmod 5$

- ⇨ a is divisible by 4 because $(2^2)^{2m} \equiv (-1)^{2m} \equiv 1 \pmod 5$

Now, dividing the equation by 3 we get,

$(-1)^a - (-1)^b * 1^c \equiv 1 \pmod 3$

=> $1 - 1 \equiv 1 \pmod 3$ (As a, b both even)

Which is impossible.

- ⇨ The equation has no solution.
- ⇨ Option (d) is correct.

13. Let a be a fixed real number greater than -1. The locus of z ∈ C satisfying $|z - ia| = Im(z) + 1$ is
    (a) Parabola
    (b) Ellipse
    (c) Hyperbola
    (d) Not a conic.

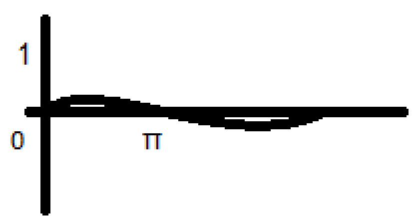
Solution :

Let $z = h + ik$

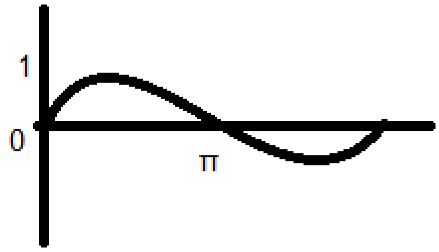
We have, $|h + ik - ia| = k + 1$

- ⇨ $\sqrt{h^2 + (k - a)^2} = k + 1$
- ⇨ $h^2 + k^2 - 2ak + a^2 = k^2 + 2k + 1$
- ⇨ $h^2 - 2k(a + 1) + (a^2 - 1) = 0$
- ⇨ Locus of z is $x^2 - 2y(a + 1) + a^2 - 1 = 0$
- ⇨ It's a parabola
- ⇨ Option (a) is correct.

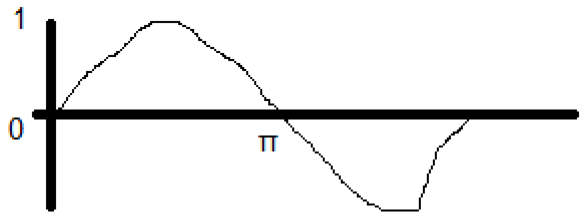
14. Which of the following closest to the graph tan(sinx), x >0
    (a)



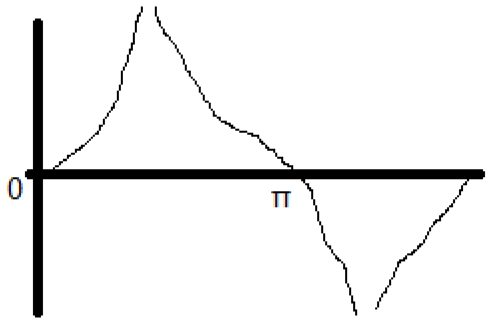
(b)



(c)



(d)



Solution :

Now, tan(sinx) never attains the value 1.

⇨ Option (b) is correct.

15. Consider the function f : R\{1} - > R\{2} given by f(x) = 2x/(x – 1). Then
(a) f is one-one and not onto
(b) f is onto but not one-one
(c)f is neither one-one nor onto
(d) f is both one-one and onto.

Solution :

Now, $f'(x) = \{2(x - 1) - 2x*1\}/(x - 1)^2 = -2/(x - 1)^2 < 0$

- ⇨ f(x) is strictly decreasing.
- ⇨ f(x) is one-one and onto.
- ⇨ Option (d) is correct.

16. Consider a real valued continuous function f satisfying $f(x + 1) = f(x)$ for all $x \in R$. Let $g(t) = \int f(x)dx$ (integration running from 0 to t), $t \in R$. Define $h(t) = \lim g(t + n)/n$ as $n \to \infty$, provided the limit exists. Then
    (a) h(t) is defined only for t = 0
    (b) h(t) is defined only when t is an integer
    (c) h(t) is defined for all $t \in R$ and is independent of t
    (d) None of the above is true.

Solution :

Now, $g(t) = \int f(x)dx$ (integration running from 0 to t) $= \int f(c)dx$ (integration running from 0 to t) (As $f(x + 1) = f(x) = tf(c)$

- ⇨ $g'(t) = f(c)$

Now, $h(t) = \lim g(t + n)/n$ as $n \to \infty = \lim g'(t + n)/1$ as $n \to \infty$ (Applying L'Hospital rule) $= \lim f(c)$ as $n \to \infty = f(c)$.

So, h(t) is defined for all $t \in R$ and independent of t.

- ⇨ Option (c) is correct.

17. Consider the sequence $a_1 = 24^{1/3}$, $a_{n+1} = (a_n + 24)^{1/3}$, $n \geq 1$. Then the integer part of a_{100} equals
    (a) 2
    (b) 10
    (c) 100
    (d) 24

Solution :

Now, $a_1 = 24^{1/3} < 3$

Let $a_n < 3$

Now, $a_{n+1} = (a_n + 24)^{1/3}$

We have, $a_n < 3$

- ⇨ $a_n + 24 < 27$

- ⇨ $(a_n + 24)^{1/3} < 3$
- ⇨ $a_{n+1} < 3$
- ⇨ $a_{100} < 3$
- ⇨ The integer part of a_{100} is 2.
- ⇨ Option (a) is correct.

18. Let x y ∈ (-2, 2) and xy = -1. Then the minimum value of $4/(4 - x^2) + 9/(9 - y^2)$ is
    (a) 8/5
    (b) 12/5
    (c) 12/7
    (d) 15/7

Solution :

Now, $4/(4 - x^2) + 9/(9 - y^2)$

$= 4/(4 - x^2) + 9/(9 - 1/x^2)$

$= 4/(4 - x^2) + 9x^2/(9x^2 - 1)$

$= 4/(4 - x^2) + 1/(9x^2 - 1) + 1$

$= (36x^2 - 4 + 4 - x^2)/(4 - x^2)(9x^2 - 1) + 1$

$= 35x^2/(4 - x^2)(9x^2 - 1) + 1$

Let, $f(x) = 35x^2/(4 - x^2)(9x^2 - 1) + 1$

$f'(x) = 35[\{2x(4 - x^2)(9x^2 - 1) + 2x*x^2(9x^2 - 1) - 18x*x^2(4 - x^2)\}/(4 - x^2)^2(9x^2 - 1)^2 = 0$

- ⇨ $36x^2 - 4 - 9x^4 + x^2 + 9x^4 - x^2 - 36x^2 + 9x^4 = 0$
- ⇨ $9x^4 - 4 = 0$
- ⇨ $x^2 = 2/3$

Now, $f(\pm\sqrt{(2/3)}) = 35*(2/3)/(4 - 2/3)\{9*(2/3) - 1\} + 1 = 12/5$

- ⇨ Option (b) is correct.

19. What is the limit of $\{1 + 1/(n^2 + n)\}$^$(n^2 + \sqrt{n})$ as n -> ∞?
    (a) e
    (b) 1
    (c) 0
    (d) ∞

Solution :

Clearly the limit is e.

- ⇨ Option (a) is correct.

20. Consider the function $f(x) = x^4 + x^2 + x - 1$, $x \in (-\infty, \infty)$. The function
    (a) Is zero at x = -1, but is increasing near x = -1
    (b) Has a zero in $(-\infty, -1)$
    (c) Has two zeros in (-1, 0)
    (d) Has exactly one local minimum in (-1, 0).

Solution :

$f(-1) = 0$

$f(x) = x(x^3 + 1) + (x^2 - 1) = x(x + 1)(x^2 - x + 1) + (x + 1)(x - 1) = (x + 1)(x^3 - x^2 + x + x - 1) = (x + 1)(x^3 - x^2 + 2x - 1)$

Now, $x^3 - x^2 + 2x - 1$ = (-)ve for all $x < 0$

- ⇨ There cannot be any root for $x < 0$
- ⇨ Option (b) and (c) cannot be true.

Now, $f(x) = x^4 + x^2 + x - 1$

- ⇨ $f'(x) = 4x^3 + 2x + 1$

Now, $f'(-1) = -5$ and $f'(0) = 1$

The sign changes.

- ⇨ $f'(x)$ has at least one root in (-1, 0)

Now, $f''(x) = 12x^2 + 2 > 0$

- ⇨ f has a local minimum in (-1, 0)

Now, $f''(x) > 0$ for all x.

- ⇨ $f'(x)$ is strictly increasing.
- ⇨ $f'(x)$ has only one root and other two roots are complex.
- ⇨ f has **exactly** one local minima in (-1, 0)
- ⇨ Option (d) is correct.

21. Consider a sequence of 10 A’s and 8 B’s placed in a row. By a run we mean one or more letters of the same type placed side by

side. Here is an arrangement of 10 A's and 8 B's which contains 4 runs of A and 4 runs of B :
AAABBABBBAABAAAABB
In how many ways can 10 A's and 8 B's be arranged in a row so hat there are 4 runs of A and 4 runs of B?
(a) $2*{}^9C_3*{}^7C_3$
(b) ${}^9C_3*{}^7C_3$
(c) ${}^{10}C_4*{}^8C_4$
(d) ${}^{10}C_5*{}^8C_5$.

Solution :

We take the arrangement A()()B,()()A()()B()()A()B()A()()B()()

Now, we can put rest 6 A's in 4 places in ${}^{6+4-1}C_{4-1}$ ways. = ${}^9C_3$ ways.

Similarly, we can put rest 4 B's in 4 places in ${}^{4+4-1}C_{4-1}$ ways = ${}^7C_3$ ways.

So, this can be done in ${}^9C_3*{}^7C_3$ ways.

Now, we have started with A. We can also start with B. So same another case appear.

- ⇨ Total number of ways = $2*{}^9C_3*{}^7C_3$
- ⇨ Option (a) is correct.

22. Suppose $n \geq 2$ is a fixed positive integer and $f(x) = x^n|x|$, $x \in R$. Then
(a) f is differentiable everywhere only when n is even
(b) f is differentiable everywhere except at 0 if n is odd
(c) f is differentiable everywhere
(d) None of the above is true.

Solution :

$f(x) = x^{n+1}$ if $x > 0$

$f(x) = -x^{n+1}$ if $x < 0$

$f(x) = 0$ if $x = 0$

lim $[\{f(x) - f(0)\}/(x - 0)]$ as x -> o+ = lim $\{(x^{n+1} - 0)/x\}$ as x -> 0+ = lim x^n as x -> 0+ = 0

lim $[\{f(x) - f(0)]/(x - 0)]$ as x -> 0- = lim $\{(-x^{n+1} - 0)/x\}$ as x -> 0- = lim $-x^n$ as x -> 0- = 0

- ⇨ f is differentiable everywhere.
- ⇨ Option (c) is correct.

23. The line $2x + 3y - k = 0$ with $k > 0$ cuts the x axis and y axis at points a and B respectively. Then the equation of the circle having AB as diameter is
    (a) $x^2 + y^2 - kx/2 - ky/3 = k^2$
    (b) $x^2 + y^2 - kx/3 - ky/2 = k^2$
    (c) $x^2 + y^2 - kx/2 - ky/3 = 0$
    (d) $x^2 + y^2 - kx/3 - ky/2 = 0$

Solution :

The equation of the given line is, $2x + 3y - k = 0$

- ⇨ $x/(k/2) + y/(k/3) = 1$
- ⇨ $A = (k/2, 0)$ and $B = (0, k/3)$

Mid-point of AB i.e. centre of the circle = $(k/4, k/6)$

Radius = $\sqrt{\{(k/2 - k/4)^2 + (0 - k/6)^2\}} = \sqrt{(k^2/16 + k^2/36)} = (k/12)\sqrt{(9 + 4)} = (k\sqrt{13})/12$

Equation of the circle is, $(x - k/4)^2 + (y - k/6)^2 = \{(k\sqrt{13})/12\}^2$

i.e. $x^2 + y^2 - kx/2 - ky/3 = 13k^2/144 - k^2/16 - k^2/36$

i.e. $x^2 + y^2 - kx/2 - ky/3 = 0$

=> Option (c) is correct.

24. Let $\alpha > 0$ and consider the sequence
    $x_n = \{(\alpha + 1)^n - (\alpha - 1)^n\}/(2\alpha)^n$, n = 1, 2, ...
    Then lim x_n as $n \to \infty$ is
    (a) 0 for any $\alpha > 0$
    (b) 1 for any $\alpha > 0$
    (c) 0 or 1 depending on what $\alpha > 0$ is
    (d) 0, 1 or ∞ depending on what $\alpha > 0$ is.

Solution :

If $0 < \alpha < ½$

Then, $2\alpha < 1$

- ⇨ $(2\alpha)^n \to 0$ as $n \to \infty$
- ⇨ $x_n \to \infty$

If $\alpha > ½$

Then $2\alpha > 1$

- ⇨ $(2\alpha)^n \to \infty$ as $n \to \infty$
- ⇨ $x_n \to 0$
- ⇨ Option (d) is correct.

25. If $0 < \Theta < \pi/2$ then
    (a) $\Theta < \sin\Theta$
    (b) $\cos(\sin\Theta) < \cos\Theta$
    (c) $\sin(\cos\Theta) < \cos(\sin\Theta)$
    (d) $\cos\Theta < \sin(\cos\Theta)$

Solution :

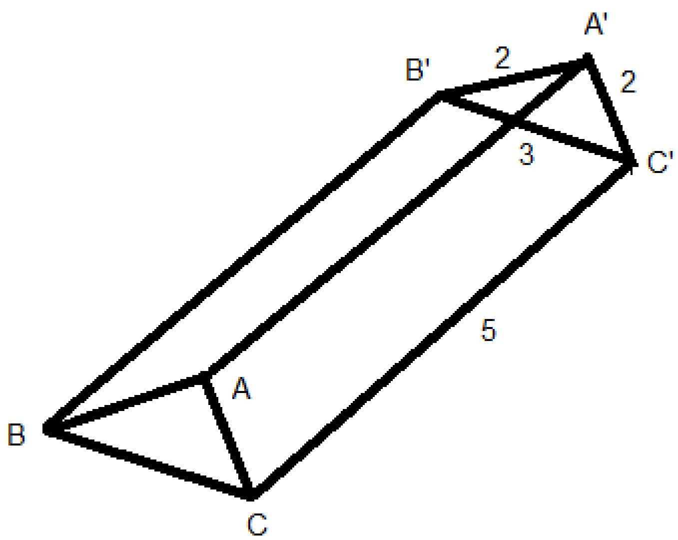
Now, option (a) cannot be true as $\Theta \in (0, \pi/2)$ and $\sin\Theta \in (0, 1)$

Now, option (b) cannot be true. Let $\Theta = \pi/4$, then $\cos(\sin\Theta) > \cos\Theta$

Option (d) is same as option (a). Let $\cos\Theta = x$ then it becomes, $x < \sin x$

- ⇨ Option (c) is correct.

26. Consider a cardboard box in the shape of a prism as shown below.



The lebgth of the prism is 5. The two triangular faces ABC and A'B'C' are congruent and isosceles with side lengths 2, 2, 3. The shortest distance between B and A' along the surface of the prism is
(a) $\sqrt{29}$
(b) $\sqrt{28}$
(c) $\sqrt{(29 - \sqrt{5})}$
(d) $\sqrt{(29 - \sqrt{3})}$

Solution :

Clearly shortest distance between A and B' along the surface is $\sqrt{(5^2 + 2^2)} = \sqrt{29}$.

⇨ Option (a) is correct.

27. Assume the following inequalities for positive integer k :
$1/\{2\sqrt{(k + 1)}\} < \sqrt{(k + 1)} - \sqrt{k} < 1/(2\sqrt{k})$
The integer part of $\Sigma(1/\sqrt{k})$ summation running from k = 2 to k = 9999
(a) 198
(b) 197
(c) 196
(d) 195

Solution :

Let, $\Sigma(1/\sqrt{k})$ summation running from k = 2 to k = 9999 = S

Now, $1/\{2\sqrt{(k + 1)}\} < \sqrt{(k + 1)} - \sqrt{k}$

$1/2\sqrt{2} < \sqrt{2} - \sqrt{1}$

$1/2\sqrt{3} < \sqrt{3} - \sqrt{2}$

$1/2\sqrt{4} < \sqrt{4} - \sqrt{3}$

...

....

$1/2\sqrt{9999} < \sqrt{9999} - \sqrt{9998}$

Adding we get, $(1/2)\Sigma(1/\sqrt{k})$ summation running from k = 2 to k = 9999 $< \sqrt{9999} - \sqrt{1}$

⇨ $S < 2(\sqrt{9999} - 1)$
⇨ $S < 197.99$

Now,

$\sqrt{(k + 1)} - \sqrt{k} < 1/(2\sqrt{k})$

$\sqrt{3} - \sqrt{2} < 1/2\sqrt{2}$

$\sqrt{4} - \sqrt{3} < 1/2\sqrt{3}$

$\sqrt{5} - \sqrt{4} < 1/2\sqrt{4}$

...

...

$\sqrt{10000} - \sqrt{9999} < 1/2\sqrt{9999}$

Adding the inequalities we get,

$\sqrt{10000} - \sqrt{2} < \sum(1/2\sqrt{k})$ summation running from k = 2 to k = 9999

- ⇨ $S > 2(100 - \sqrt{2})$
- ⇨ $S > 197.17$
- ⇨ The integer part of S is 197.
- ⇨ Option (b) is correct.

28. Consider the sets defined by the inequalities
    $A = \{(x, y) \in R^2 : x^4 + y^2 \le 1\}$, $B = \{(x, y) \in R^2 : x^6 + y^4 \le 1\}$.
    Then
    (a) B is subset of A
    (b) A is subset of B
    (c) Each of the sets A – B, B – A and A∩B is non-empty
    (d) None of the above is true.

Solution :

Now, $x^6 + y^4 - (x^4 + y^2)$

$= x^4(x^2 - 1) + y^2(y^2 - 1)$

Now, x and y both less than or equal to 1.

- ⇨ x^2 and y^2 both less than or equal to 1.
- ⇨ $x^6 + y^4 \le x^4 + y^2 \le 1$
- ⇨ All (x, y) that satisfies $x^4 + y^2 \le 1$ also satisfies $x^6 + y^4 \le 1$
- ⇨ A is a subset of B.
- ⇨ Optio0n (b) is correct.

29. The number $(2^{10}/11)^{11}$ is

(a) Strictly larger than $({}^{10}C_1)^2({}^{10}C_2)^2({}^{10}C_3)^2({}^{10}C_4)^2({}^{10}C_5)$
(b) Strictly larger than $({}^{10}C_1)^2({}^{10}C_2)^2({}^{10}C_3)^2({}^{10}C_4)^2$ but strictly smaller than $({}^{10}C_1)^2({}^{10}C_2)^2({}^{10}C_3)^2({}^{10}C_4)^2({}^{10}C_5)$.
(c) Less than or equal to $({}^{10}C_1)^2({}^{10}C_2)^2({}^{10}C_3)^2({}^{10}C_4)^2$
(d) Equal to $({}^{10}C_1)^2({}^{10}C_2)^2({}^{10}C_3)^2({}^{10}C_4)^2({}^{10}C_5)$.

Solution :

Now, AM ≥ GM

$({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + ... + {}^{10}C_{10})/11 > \{({}^{10}C_0)({}^{10}C_1)....({}^{10}C_{10})\}^{1/11}$ (Equality cannot hold as all the numbers are not equal)

- ⇨ $(2^{10}/11)^{11} > \{({}^{10}C_1)({}^{10}C_9\}\{({}^{10}C_2)({}^{10}C_8)\}...\{({}^{10}C_4)({}^{10}C_6)\}({}^{10}C_5) = ({}^{10}C_1)^2({}^{10}C_2)^2({}^{10}C_3)^2({}^{10}C_4)^2({}^{10}C_5)$
- ⇨ Option (a) is correct.

30. If the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ are in geometric progression then
(a) $b^2 = ac$
(b) $a^2 = b$
(c) $a^2b^2 = c^2$
(d) $c^2 = a^2d$

Solution :

Let the roots of the equation are s, sr, sr^2, sr^3

Now, $-a = s + sr + sr^2 + sr^3 = s(r^4 - 1)/(r - 1)$

$b = s*sr + s*sr^2 + s*sr^3 + sr*sr^2+ sr*sr^3 + sr^2*sr^3$

- ⇨ $b = s^2r + s^2r^2 + 2s^2r^3 + s^2r^4 + s^2r^5 = s^2r(r^5 - 1)/(r - 1) + s^2r^3$
- ⇨ $c = s*sr*sr^2 + sr*sr^2*sr^3 + sr^2*sr^3*s + sr^3*s*sr$
- ⇨ $c = s^3r^3 + s^3r^6 + s^3r^5 + s^3r^4$
- ⇨ $c = s^3r^3(r^4 - 1)/(r - 1)$
- ⇨ $d = s*sr*sr^2*sr^3$
- ⇨ $d = s^4r^6$

Now, $c^2 = s^6r^6\{(r^4 - 1)/(r - 1)\}^4 = (s^4r^6)*[s(r^4 - 1)/(r - 1)]^2 = d*a^2$

- ⇨ $c^2 = a^2d$
- ⇨ Option (d) is correct.

B. Stat (Hons.) & B. Math. (Hons.) Admission Test : 2013

1. Let $i = \sqrt{(-1)}$ and $S = \{i + i^2 + ... + i^n : n \geq 1\}$. The number of distinct real numbers in the set S is
   (a) 1
   (b) 2
   (c) 3
   (d) Infinite

Solution :

Let n = 1 then S = {i}

Let, n = 2 then S {i – 1}

Let n = 3, then S = {i – 1 – i} = {-1}

Let n = 4, then S ={i – 1 – i + 1} = {0}

Let n = 5 then S = {0 + i} = {i}

Let n = 6 then S = {i – 1}

Let n = 7 then S = {i – 1 + i} = {-1}

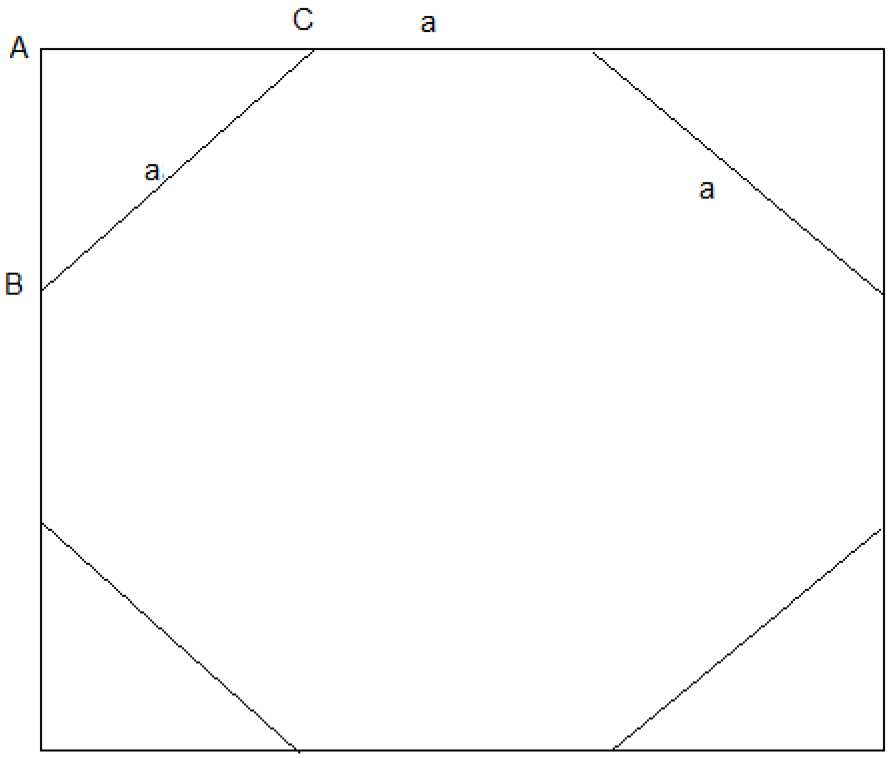
Let n = 8 then S = {-1 + 1} = {0}

And it runs in a loop.

⇨ 2 real numbers viz. -1, 0
⇨ Option (b) is correct.

2. From a square of unit length, pieces from the corners are removed to form a regular octagon. Then, the value of the area removed is
   (a) ½
   (b) $1/\sqrt{2}$
   (c) $\sqrt{2} - 1$
   (d) $(\sqrt{2} - 1)^2$

Solution :



- ⇨ Let BC = a

Now, from triangle ABC we get, $AC^2 + AB^2 = BC^2$

- ⇨ AC = AB = $a/\sqrt{2}$

Now, 2AC + a = 1 (As the length of the side of the square is 1)

- ⇨ $2a/\sqrt{2} + a = 1$
- ⇨ $a = 1/(\sqrt{2} + 1)$
- ⇨ $AC = AB = 1/(2 + \sqrt{2})$

Area of triangle ABC = $(1/2)\{1/(2 + \sqrt{2})\}\{(1/(2 + \sqrt{2})\}$

= $(1/2)\{1/(6 + 4\sqrt{2}) = (1/4)\{1/(3 + 2\sqrt{2})\}$

Area of the removed portion = $4*(1/4)\{1/(3 + 2\sqrt{2})\} = 1/(3 + 2\sqrt{2}) = (3 - 2\sqrt{2}) = (\sqrt{2} - 1)^2$

- ⇨ Option (d) is correct.

3. We define the dual of a line y = mx + c to be the point (m, -c). Consider a set of n non-vertical lines, n > 3, passing through the point (1, 1). Then the duals of these lines will always
   (a) Be the same
   (b) Lie on a circle

(c) Lie on a line
(d) Form the vertices of a polygon with positive area.

Solution.

The equation of the family of non-vertical lines through (1, 1) is, $y - 1 = m(x - 1)$

⇨ $y = mx + (-m + 1)$

$c = -m + 1$

⇨ $m + c = 1$

Putting $m = x$ and $c = -y$ we get the locus as, $x - y = 1$

⇨ Option (c) is correct.

4. Suppose α, β and γ are three real numbers satisfying $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$. Then the value of $\cos(\alpha - \beta)$ is
   (a) -1/2
   (b) -1/4
   (c) ¼
   (d) ½

Solution :

Now, $\cos\alpha + \cos\beta = -\cos\gamma$

⇨ $(\cos\alpha + \cos\beta)^2 = \cos^2\gamma$

Similarly, $(\sin\alpha + \sin\beta)^2 = \sin^2\gamma$

Adding the above two equations we get,

$\cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \cos^2\gamma + \sin^2\gamma$

⇨ $1 + 1 + 2\cos(\alpha - \beta) = 1$
⇨ $\cos(\alpha - \beta) = -1/2$
⇨ Option (a) is correct.

5. The value of $\lim (3^x + 7^x)^{1/x}$ as $x \to \infty$ is
   (a) 7
   (b) 10
   (c) e^7
   (d) ∞

Solution :

Now, lim $(3^x + 7^x)^{1/x}$ as x -> ∞ = lim $7\{1 + (3/7)^x\}^{1/x}$ as x -> ∞

Now, $(3/7)^x$ -> 0 as x -> ∞ and 1/x -> 0 as x -> ∞

So, the limit is $7*(1 + 0)^0 = 7$

⇨ Option (a) is correct.

6. The distance between the two foci of the rectangular hyperbola defined by xy = 2 is
   (a) 2
   (b) 2√2
   (c)4
   (d) 4√2

Solution :

The foci are (2, 2), (-2, -2).

Clearly the distance is 4√2.

⇨ Option (d) is correct.

7. Suppose f is a differentiable and increasing function on [0, 1] such that f(0) < 0 < f(1). Let F(t) = ∫f(x)dx (integration running from 0 to t). Then
   (a) F is an increasing function on [0, 1]
   (b) F is a decreasing function on [0, 1]
   (c)F has a unique maximum in the open interval (0, 1)
   (d) F has a unique minimum in the open interval (0, 1)

Solution :

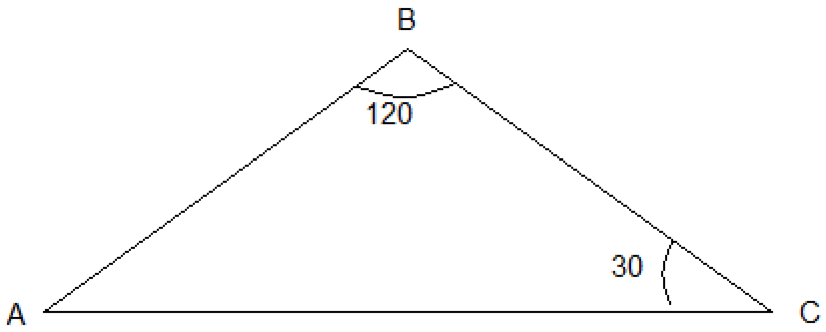
F′(t) = f(t) = 0 has a root in the open interval (0, 1) as f(0) < 0 < f(1)

Now, F′′(t) = f′(t) > 0 (As f is increasing function)

⇨ F(t) has a unique minimum in the open interval (0, 1).
⇨ Option (d) is correct.

8. In an isosceles triangle ABC, he angle ABC = 120°. Then the ratio of two sides AC : AB is
   (a) 2 : 1
   (b) 3 : 1
   (c) $\sqrt{2}$: 1
   (d) $\sqrt{3}$: 1

Solution :



Now, AC/sin120° = AB/sin30°

- ⇨ AC/AB = sin120°/sin30° = $(\sqrt{3}/2)/(1/2) = \sqrt{3}$
- ⇨ AC : AB = $\sqrt{3}$: 1
- ⇨ Option (d) is correct.

9. Let x, y, z be positive real numbers. If the equation $x^2 + y^2 + z^2 = (xy + yz + zx)\sin\Theta$ has a solution for Θ, then x, y, z must satisfy
   (a) $x = y = z$
   (b) $x^2 + y^2 + z^2 \le 1$
   (c) $xy + yz + zx = 1$
   (d) $0 < x, y, z \le 1$

Solution :

Now, $(x - y)^2 \ge 0$

- ⇨ $x^2 + y^2 \ge 2xy$

Similarly, $y^2 + z^2 \ge 2yz$ and $z^2 + x^2 \ge 2zx$

Adding the three inequalities we get,

$x^2 + y^2 + z^2 \ge xy + yz + zx$

⇨ $(x^2 + y^2 + z^2)/(xy + yz + zx) \geq 1$
⇨ $\sin\Theta \geq 1$

The equation holds only if $\sin\Theta = 1$ i.e. $x^2 + y^2 + z^2 = xy + yz + zx$

The equality holds only if $x = y = z$.

⇨ Option (a) is correct.

10. Suppose $\sin\Theta = 4/5$ and $\sec\alpha = 7/4$ where $0 \leq \Theta \leq \pi/2$ and $-\pi/2 \leq \alpha \leq 0$. Then $\sin(\Theta + \alpha)$ is
(a) $3\sqrt{33}/35$
(b) $-3\sqrt{33}/35$
(c) $(16 + 3\sqrt{33})/35$
(d) $(16 - 3\sqrt{33})/35$

Solution :

$\sin\Theta = 4/5$, $\cos\Theta = 3/5$

$\sec\alpha = 7/4$, $\cos\alpha = 4/7$, $\sin\alpha = -\sqrt{33}/7$

$\sin(\Theta + \alpha) = \sin\Theta\cos\alpha + \cos\Theta\sin\alpha = (4/5)(4/7) + (3/5)(-\sqrt{33}/7) = (16 - 3\sqrt{33})/35$

⇨ Option (d) is correct.

11. Let $i = \sqrt{(-1)}$ and $z_1, z_2, \ldots$ be a sequence of complex numbers defined by $z_1 = i$ and $z_{n+1} = z_n^2 + i$ for $n \geq 1$. Then $|z_{2013} - z_1|$ is
(a) 0
(b) 1
(c) 2
(d) $\sqrt{5}$

Solution :

$z_2 = i^2 + i = i - 1$

$z_3 = (i - 1)^2 + i = -i$

$z_4 = (-i)^2 + i = i - 1$

$z_5 = (i - 1)^2 + i = -i$

And it runs in a loop.

If n is odd then z_n = -i

- ⇨ $Z_{2013} = -i$
- ⇨ $|z_{2013} - z_1| = |-i - i| = 2$
- ⇨ Option (c) is correct.

12. The last digit of the number $2^{100} + 5^{100} + 8^{100}$ is
    (a) 1
    (b) 3
    (c) 5
    (d) 7

Solution :

Now, $2^4 \equiv 6 \pmod{10}$

- ⇨ $2^{100} \equiv 6^{25} \equiv 6 \pmod{100}$

Last digit of 5^{100} is 5

Last digit of 8^{100} is 6 as $8^{100} = 2^{300}$ and 300 is divisible by 4.

- ⇨ Last digit of the given expression = $6 + 5 + 6 \equiv 7 \pmod{10}$
- ⇨ Option (d) is correct.

13. The maximum value of $|x - 1|$ subject to the condition $|x^2 - 4| \le 5$
    (a) 2
    (b) 3
    (c) 4
    (d) 5

Solution :

Now, $|x^2 - 4| \le 5$

- ⇨ $-5 \le x^2 - 4 \le 5$
- ⇨ $-1 \le x^2 \le 9$
- ⇨ $|x| \le 3$
- ⇨ $-3 \le x \le 3$
- ⇨ $-4 \le x - 1 \le 2$
- ⇨ Maximum value of $|x - 1|$ is $|-4| = 4$.
- ⇨ Option (c) is correct

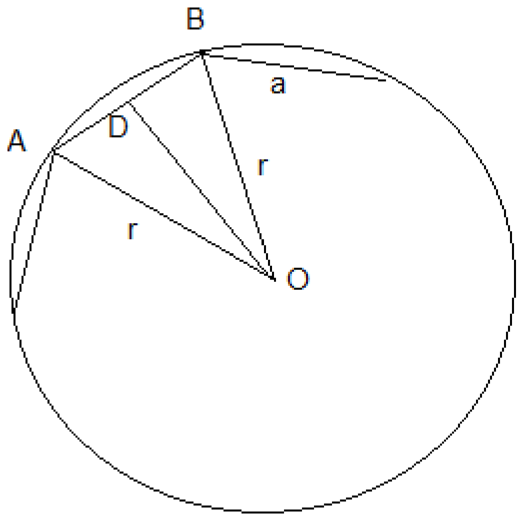
14. Which of the following is correct?
 (a) $ex \le e^x$ for all x
 (b) $ex < e^x$ for $x > 1$ and $ex \ge e^x$ for $x \ge 1$
 (c) $ex \ge e^x$ for all x
 (d) $ex < e^x$ for $x \ge 1$ and $ex \ge e^x$ for $x \le 1$

Solution :

Clearly option (a) is correct.

15. The area of a regular polygon of 12 sides that can be inscribed in a circle $x^2 + y^2 - 6x + 5 = 0$ is
 (a) 6 units
 (b) 9 units
 (c) 12 units
 (d) 15 units

Solution :



Now equation of the circle is, $x^2 + y^2 - 6x + 5 = 0$

⇨ $(x - 3)^2 + y^2 = 2^2$
⇨ $r = 2$

Now, Angle A = (12 – 2)*180°/12 = 150°

⇨ Angle OAB =150°/2 = 75°
⇨ Angle BOD = 180° - 2*75° = 30°

From triangle OAB, we get, r/sin75° = a/sin30°

- ⇨ a = rsin30°/sin75°
- ⇨ AD = a/2 = rsin30°/2sin75°
- ⇨ OD = $\sqrt{\{r^2 - (a/2)^2\}}$ = r√{1 - $\sin^2 30°/4\sin^2 75°$} = r√{1 - (1/4)/(2 + √3)} = r√{(4 − 2 + √3)} = r√(2 + √3)/2 = r√{(4 + 2√3)/2}/2 = r(√3 + 1)/2√2

Area of triangle OAB = (1/2)(rsin30°/sin75°)*r(√3 + 1)/2√2 = $r^2/4$

- ⇨ Area of the polygon = 12*(r^2/4) = $3r^2$ = $3*2^2$ = 12 square units.
- ⇨ Option (c) is correct.

16. Let f(x) = √($\log_2 x$ − 1) + (1/2)$\log_{1/2} x^3$ + 2. The set of all real values of x for which the function f(x) is defined and f(x) < 0 is
    (a) x > 2
    (b) x > 3
    (c)x > e
    (d) x > 4

Solution :

x > 0 and $\log_2 x$ − 1 > 0 i.e. x > 2

So, the function is defined for x > 2

Now, f(x) < 0

- ⇨ √($\log_2 x$ − 1) + (1/2)$\log_{1/2} x^3$ + 2 < 0
- ⇨ √($\log_2 x$ − 1) < (3/2)$\log_2 x$ − 2
- ⇨ $\log_2 x$ - 1 < $\{(3/2)\log_2 x - 2\}^2$
- ⇨ $\log_2 x$ − 1 < (9/4)$(\log_2 x)^2$ − 6$\log_2 x$ + 4
- ⇨ 9$(\log_2 x)^2$ − 28$\log_2 x$ + 20 > 0
- ⇨ $(3\log_2 x - 14/3)^2$ > 196/9 − 20 = 16/9
- ⇨ 3$\log_2 x$ − 14/3 > 4/3
- ⇨ 3$\log_2 x$ > 14/3 + 4/3
- ⇨ 3$\log_2 x$ > 6
- ⇨ $\log_2 x$ > 2
- ⇨ x > 4
- ⇨ Option (d) is correct.

17. Let a be the largest integer *strictly* smaller than 7b/8 where b is also an integer. Consider the following inequalities :
    (1) 7b/8 − a ≤ 1
    (2) 7b/8 − a ≥ 1/8

    And find which of the following is correct.

(a) Only (1) is correct.
(b) Only (2) is correct
(c) Both (1) and (2) are correct
(d) None of them is correct.

Solution :

If 7b/8 is an integer then 7b/8 – a = 1 (strictly less than)

If 7b/8 is not an integer then 7b/8 – a = fraction < 1.\

⇨ (1) is correct.

Now, if (7b/8) is an integer then 7b/8 – a = 1 > 1/8

Now, if we divide 7b by 8 then remainder can be 1, 2, ..., 7.

Now, the least remainder is 1. So if 7b ≡ 1 (mod 8)

Then 7b/8 = a + 1/8

⇨ 7b/8 – a = 1/8

In other cases 7b/8 – a > 1/8

⇨ (2) is also correct.
⇨ Option (c) is correct.

18. The value of $\sum x^k/k!$ Summation running from k = 1 to k = 1000, is
(a) -∞
(b) ∞
(c) 0
(d) e^{-1}

Solution :

Clearly the limit -> ∞

⇨ Option (b) is correct.

19. For integers m and n, let $f_{m,n}$ denote the function from the set of integers to itself, defined by $f_{m,n}(x) = mx + n$.
Let F be the set of all such functions,
F = {$f_{m,n}$: m, n integers}

Call an element f ∈ F *invertible* if there exists an element d ∈ F such that d(f(x)) = f(g(x)) = x for all integers x. Then which of the following is true?

(a) Every element of F is invertible

(b) F has infinitely many invertible and infinitely many non-invertible elements

(c) F has finitely many invertible elements

(d) No element of A is invertible.

Solution :

Now, $f_{m,n}(x) = mx + n$

Now, g(f(x)) = x

⇨ g(mx + n) = x

Putting, x = (x – n)/m we get,

g(x) = (x – n)/m

Also, f(g(x)) = f((x – n)/m) = m(x – n)/m + n = x

So, we see that if there is an element g such that m divides x – n then f is invertible.

Now, depending on values of m, n there can be infinitely many invertible and also infinitely many non-invertible elements in F.

⇨ Option (b) is correct.

20. Consider six players P_1, P_2, P_3, P_4, P_5 and P_6. A team consists of two players. (Thus there are 15 distinct teams.) Two teams play a match exactly once if there is no common player. For example, team $\{P_1, P_2\}$ cannot play with $\{P_2, P_3\}$ but will play with $\{P_4, P_5\}$. Then the total number of possible matches is

(a) 36

(b) 40

(c) 45

(d) 54

Solution :

The teams are $\{P_1, P_2\}$; $\{P_1, P_3\}$; $\{P_1, P_4\}$; $\{P_1, P_5\}$; $\{P_1, P_6\}$

$\{P_2, P_3\}$; $\{P_2, P_4\}$; $\{P_2, P_5\}$; $\{P_2, P_6\}$

$\{P_3, P_4\}$; $\{P_3, P_5\}$; $\{P_3, P_6\}$

$\{P_4, P_5\}$; $\{P_4, P_6\}$

$\{P_5, P_6\}$

Now, the first row teams each can play 6 matches. Thus making 6*5 = 30 matches

The second row teams can play 3 matches each. Thus making 4*3 = 12 matches. (Do not go to upper rows as the matches has already been considered)

The third row team can play 1 math each. Thus making 1*3 = 3 matches.

There are no more combination available for 4th and 5th rows.

Thus, total number of matches = 30 + 12 + 3 = 45.

⇨ Option (c) is correct.

21. The minimum value of $f(\Theta) = 9\cos^2\Theta + 16\sec^2\Theta$ is

(a) 25
(b) 24
(c) 20
(d) 16

Solution :

Now, $9\cos^2\Theta + 16\sec^2\Theta = (3\cos\Theta - 4\sec\Theta)^2 + 24$

Now, $f(\Theta)$ will be minimum when $3\cos\Theta - 4\sec\Theta = 0$ i.e. attains minimum value 0

⇨ $\cos\Theta = 2/\sqrt{3}$

But in that case $\cos\Theta > 1$ which is impossible.

Now, if we take any value of $\cos^2\Theta < 1$ then the ratio by which 9 will be reduced due to multiplication of $\cos^2\Theta$ will be lesser than the ratio by which 16 will be increased due to division by $\cos^2\Theta$.

To make this point clear we take the inequality $16\sec^2\Theta - 16 < 9 - 9\cos^2\Theta$. If the inequality holds true then $\cos^2\Theta$ has to be minimum i.e. 0 because 9 is getting reduced less than 16 is getting increased. If the inequality doesn't hold true then $\cos^2\Theta$ needs to be maximum as we are here to minimize the difference and hence getting the minimum value.

Now, $16\sec^2\theta - 16 < 9 - 9\cos^2\theta$

- ⇨ $16(1 - \cos^2\theta)/\cos^2\theta < 9(1 - \cos^2\theta)$

Now, if $\cos^2\theta < 1$ then we have, $16 < 9\cos^2\theta$ which is impossible.

- ⇨ $\cos^2\theta$ must be equal to 1 and we can see also the difference in that case becomes 0.
- ⇨ The minimum value of $9\cos^2\theta + 16\sec^2\theta = 9*1 + 16/1 = 25$.
- ⇨ Option (a) is correct.

22. The number of 0's at the end of the integer 100! – 101! + ... – 109! + 110! Is
    (a) 24
    (b) 25
    (c) 26
    (d) 27

Solution :

Now, 100! – 101! + ... – 109! + 110! = 100!(1 – 101 + 101*102 - ... – 101*102*..109 + 101*102*...110)

Now, -101*102*..105 + 101*102*...106 - .... + 101*102*...110 has 2 0's at the end.

Now, 1 – 101 + 101*102 – 101*102*103 + 101*102*103*104 last digit is 1 – 1 + 2 – 6 + 4 = 0

Now, 100! Has $[100/5] + [100/5^2] = 20 + 4 = 24$ 0's

So, there are 24 + 1 = 25 0's.

- ⇨ Option (b) is correct.

23. We denote the largest integer less than or equal to z by [z]. Consider the identity
    $(1 + x)(10 + x)(10^2 + x)....(10^{10} + x) = 10^a + 10^b x + a_2x^2 + ... + a_{11}x^{11}$.
    Then
    (a) [a] > [b]
    (b) [a] = [b] and a > b
    (c) [a] < [b]
    (d) [a] = [b] and a < b

Solution :

Clearly, $10^a = 1*10*10^2*....*10^{10} = 10^{(1+2+...+10)} = 10^{10*11/2} = 10^{55}$

- ⇨ a = 55.

Let, f(x) = $(1 + x)(10 + x)(10^2 + x)....(10^{10} + x)$

- ⇨ $f'(x) = (10 + x)(10^2 + x)...(10^{10} + x) + (1 + x)(10^2 + x)...(10^{10} + x) + ... + (1 + x)(10 + x)...(10^9 + x)$

Now, $f'(0) = 10^b$

- ⇨ $10*10^2*...10^{10} + 1*10^2*...10^{10} + ... + 1*10*10^2*...*10^9 = 10^b$
- ⇨ $10^b = (1*10*10^2*...10^{10})(1 + 10^{-1} + 10^{-2} + ... + 10^{-10})$
- ⇨ $10^b = 10^{(1+2+..+10)}*1*(1 - 10^{-11})/(1 - 10^{-1})$
- ⇨ $10^b = 10^{10*11/2}*(1 - 10^{-11})/(1 - 10^{-1})$
- ⇨ $10^b = 10^{55}*(1 - 10^{-11})/(1 - 10^{-1})$
- ⇨ $10^b = 10^a*(1 - 10^{-11})/(1 - 10^{-1})$
- ⇨ $10^{b-a} = (1 - 10^{-11})/(1 - 10^{-1}) > 1$
- ⇨ $10^{b-a} > 10^0$
- ⇨ b- a > 0
- ⇨ b > a

As a and b are both integers, [b] > [a]

- ⇨ Option (c) is correct.

24. The number of four tuples (a, b, c, d) of *positive integers* satisfying all three equations
$a^3 = b^2$
$c^3 = d^2$
$c - a = 64$
is
(a) 0
(b) 1
(c)2
(d) 4

Solution :

Now, $a^3 = b^2$

- ⇨ a must6 be a square number as b is positive integer.

Let a = x^2

Similarly, let c = y^2

Now, $c - a = 64$

- ⇨ $y^2 - x^2 = 64$
- ⇨ $(y + x)(y - x) = 2^6$

Let, $y + x = 2^m$ and $y - x = 2^{6-m}$

- ⇨ $y = 2^{m-1} + 2^{5-m}$ and $x = 2^{m-1} - 2^{5-m}$

Let, $m = 1$, $y = 17$ and $x = -(ve)$. This combination cannot hold true.

Let $m = 2$, $y = 10$ and $x = (-)ve$. This combination cannot hold true.

Let, $m = 3$, $y = 8$, $x = 0$. This combination cannot hold

Let $m = 4$, $y = 10$ and $x = 6$

- ⇨ $a = 6^2$, $c = 10^2$, $b = a^{3/2} = 6^3$, $d = c^{3/2} = 10^3$

Let, $m = 5$, $y = 17$ and $x = 15$

- ⇨ $a = 15^2$, $c = 17^2$, $b = a^{3/2} = 15^3$ and $d = c^{3/2} = 17^3$.

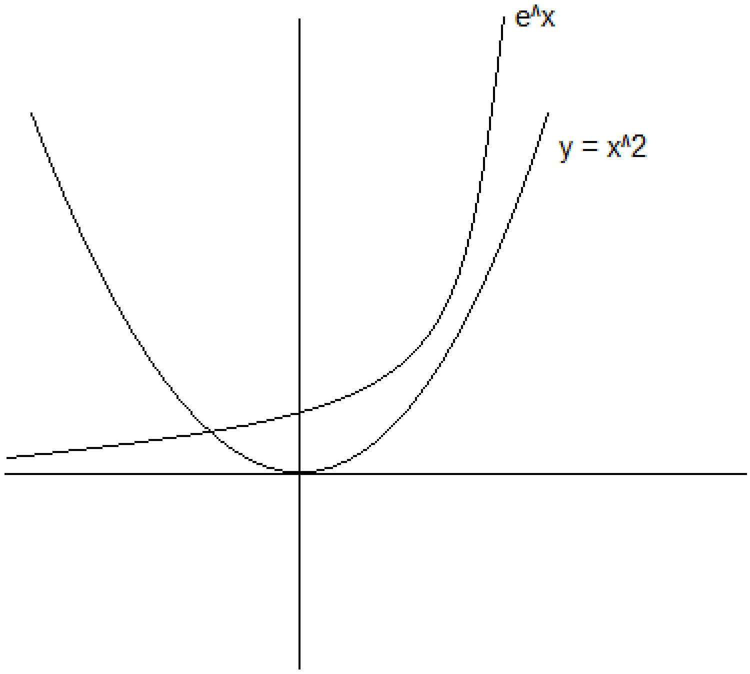
So 2 solutions.

- ⇨ Option (c) is correct.

25. The number of real roots of $e^x = x^2$ is

(a) 0
(b) 1
(c)2
(d) 3

Solution :



- ⇨ Option (b) is correct.

26. Suppose $α_1$, $α_2$, $α_3$ and $α_4$ are the roots of the equation $x^4 + x^2 + 1 = 0$. Then the value of $α_1^4 + α_2^4 + α_3^4 + α_4^4$ is
    (a) -2
    (b) 0
    (c) 2
    (d) 4

Solution :

Now, $x^4 + x^2 + 1 = 0$

- ⇨ $x^2 = \{-1 \pm \sqrt{(1 - 4)}\}/2 = (-1 \pm i\sqrt{3})/2$
- ⇨ $x^2 = (-2 \pm i2\sqrt{3})/4$
- ⇨ $x^2 = (-3 \pm i2\sqrt{3} + 1)/4$
- ⇨ $x^2 = (i\sqrt{3} \pm 1)^2/4$
- ⇨ $x = \pm(i\sqrt{3} \pm 1)/2$
- ⇨ $x = -w, w, w^2, -w^2$ (where w is complex cube root of unity)
- ⇨ $α_1^4 + α_2^4 + α_3^4 + α_4^4 = (-w)^4 + w^4 + (w^2)^4 + (-w^2)^3 = w + w + w^2 + w^2 = -1 - 1 = -2$
- ⇨ Option (a) is correct.

27. Among the four time instances given in the options below, when is the angle between the minute hand and the hour hand the smallest?
(a) 5:25 p.m.
(b) 5:26 p.m.
(c) 5:29 p.m.
(d) 5:30 p.m.

Solution :

Clearly, the answer is 5:26 p.m. as the hour hand has not crossed the mid-point of 5 and 6 so at 5:26 the angle will be smaller than angle at 5:29 p.m.

⇨ Option (b) is correct.

28. Suppose all roots of the polynomial $P(x) = a_{10}x^{10} + a_9x^9 + ... + a_1x + a_0$ are real and smaller than 1. Then, for ny such polynomial, the function
$f(x) = a_{10}(e^{10x}/10) + a_9(e^{9x}/9 + ... + a_1e^x + a_0x, x > 0$
(a) Is increasing
(b) Is either increasing or decreasing
(c) Is decreasing
(d) Is neither increasing nor decreasing.

Solution :

Now, $f(x) = a_{10}(e^{10x}/10) + a_9(e^{9x}/9 + ... + a_1e^x + a_0x$

⇨ $f'(x) = a_{10}e^{10x} + a_9e^{9x} + ... + a_1e^x + a_0$

Now, all the roots of $P(x) < 1$.

⇨ For all real numbers greater than 1 P(x) is either negative or positive.

Because if P(a) is positive and P(b) is negative where a, b > 1 then there is a root between P(a) and P(b) which contradicts the statement that all roots of $P(x) < 1$.

Now, $f'(x) = P(e^x)$ which is either greater than 0 for all values of x or less than 0 for all values of x.

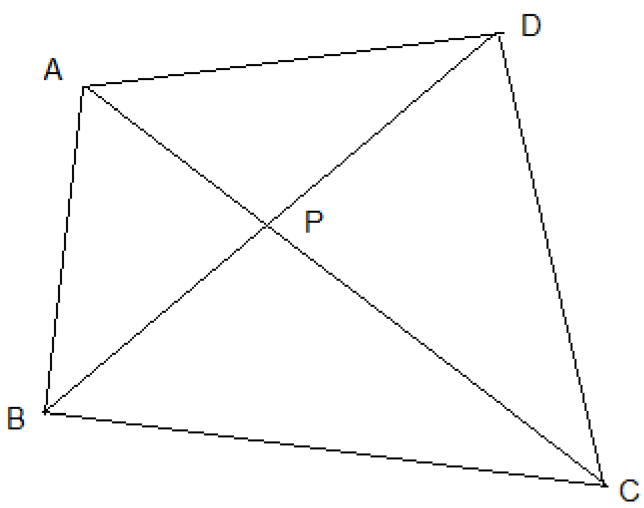
Now, $e^x < 1$ when $x < 0$ but $x > 0$.

⇨ e^x is always greater than 1 for $x > 0$

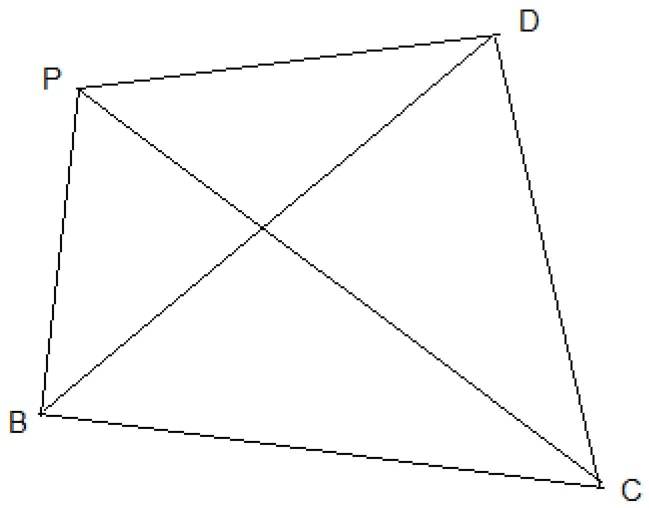
- ⇨ f(x) is either increasing or decreasing.
- ⇨ Option (b) is correct.

29. Consider a quadrilateral ABCD in the XY-plane with all of its angles less than 180°. Let P be an arbitrary point in the plane and consider the six triangles each of which is formed by the point P and two of the points A, B, C, D. Then the point P is
    (a) Outside the quadrilateral
    (b) One of the vertices of the quadrilateral
    (c) Intersection of the diagonals of the quadrilateral
    (d) None of the points given in (a), (b) or (c).

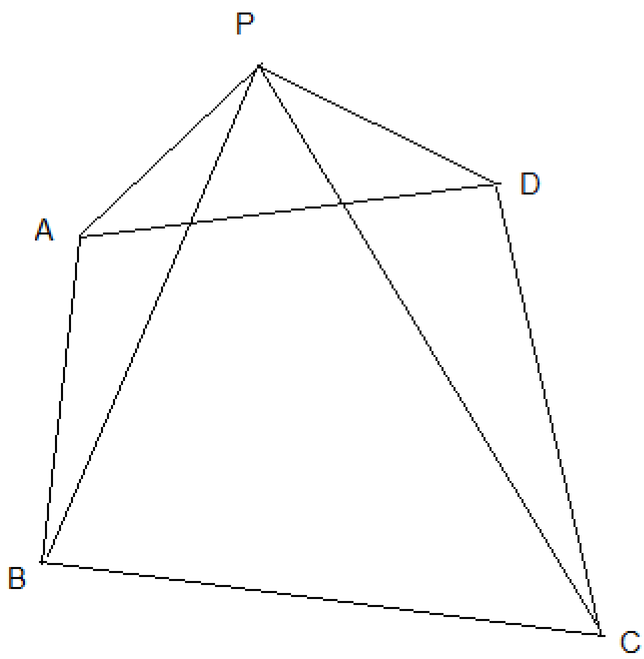
Solution :



Clearly if the point P is on the intersection of the diagonals then the area of the six triangles = area of the quadrilateral.



If the point P is on vertex A then triangle PCD and PBC cover the area of the quadrilateral. Now consider one more triangle PBD and the sum of the area is more.\



If P is outside the quadrilateral then triangles PDC, PBC, PAB cover the quadrilateral and also extra area.

Now, if the point P was inside the quadrilateral then also the area of the 4 triangles PAB, PBC, PCD, PDA cover the area of the quadrilateral but area of triangles PAC and PBD are not zero.

⇨ Option (c) is correct.

30. The graph of the equation $x^3 + 3x^2y + 3xy^2 + y^3 - x^2 + y^2 = 0$ omprises
(a) One point
(b) Union of a line and a parabola
(c) One line
(d) Union of a line and a hyperbola

Solution :

Now, $x^3 + 3x^2y + 3xy^2 + y^3 - x^2 + y^2 = 0$

⇨ $(x + y)^3 - (x^2 - y^2) = 0$
⇨ $(x + y)^3 - (x + y)(x - y) = 0$
⇨ $(x + y)(x^2 + 2xy + y^2 - x + y) = 0$
⇨ Option (b) is correct.

B. Stat. (Hons.) & B. Math. (Hons.) Admission Test : 2014

1. The system of inequalities $a - b^2 \geq ¼$, $b - c^2 \geq ¼$, $c - d^2 \geq ¼$, $d - a^2 \geq ¼$ has
(a) No solution
(b) Exactly one solution
(c) Exactly two solutions
(d) Infinitely many solutions.

Solution :

Now, $a - b^2 \geq ¼$

⇨ $a \geq b^2 + ¼$

Similarly, $b \geq c^2 + ¼$

⇨ $a \geq (c^2 + ¼)^2 + ¼$

Again $c \geq d^2 + ¼$

⇨ $a \geq \{(d^2 + ¼)^2 + ¼\}^2 + ¼$

Again $d \geq a^2 + ¼$

⇨ $a \geq [\{(a^2 + ¼)^2 + ¼\}^2 + ¼]^2 + ¼$

Now, $(a^2 + ¼)/2 \geq \sqrt{\{a^2*(1/4)\}} = a/2$

- ⇨ $a^2 + ¼ \geq a$
- ⇨ $(a^2 + ¼)^2 \geq a^2$ .... (A)
- ⇨ $(a^2 + ¼)^2 + ¼ \geq a^2 + ¼$
- ⇨ $\{(a^2 + ¼)^2 + ¼\}^2 \geq (a^2 + ¼)^2 \geq a^2$ (using (A))

So going forward in this way we will get,

$a \geq [\{(a^2 + ¼)^2 + ¼\}^2 + ¼]^2 + ¼ \geq a^2 + ¼$ (Using (A) again and again)

- ⇨ $a^2 - a + ¼ \leq 0$
- ⇨ $(a - ½)^2 \leq 0$
- ⇨ $a - ½ = 0$
- ⇨ $a = ½$

Similarly, b = ½, c = ½ and d = ½

Putting the values backward, we get, $a - b^2 = ¼$, $b - c^2 = ¼$, $c - d^2 = ¼$ and $d - a^2 = ¼$

- ⇨ Only one solution.
- ⇨ Option (b) is correct.

2. Let $\log_{12}18 = a$. Then $\log_{24}16$ is equal to
   (a) (8 – 4a)/(5 – a)
   (b) 1/(3 + a)
   (c)(4a – 1)/(2 + 3a)
   (d) (8 – 4a)/(5 + a)

Solution :

Now, $\log_{12}18 = a$

- ⇨ log18/log12 = a
- ⇨ $\log2*3^2/\log2^2*3 = a$
- ⇨ (log2 + 2log3)/(2log2 + log3 = a
- ⇨ log2 + 2log3 = 2alog2 + alog3
- ⇨ (2a – 1)log2 = (2 – a)log3
- ⇨ log3/log2 = (2a – 1)/(2 – a)

Now, $\log_{24}16$ = log16/log24

$= \log2^4/\log2^3*3$

= 4log2/(3log2 + log3)

= 4/(3 + log3/log2)

= 4/{3 + (2a – 1)/(2 – a)}

= 4(2 − a)/(6 − 3a + 2a − 1)

= (8 − 4a)/(5 − a)

- ⇨ Option (a) is correct.

3. The number of solutions of the equation tanx + secx = 2cosx, where $0 \le x \le \pi$, is
   (a) 0
   (b) 1
   (c) 2
   (d) 3

Solution :

Now, $\sec^2 x - \tan^2 x = 1$

- ⇨ (secx − tanx)(secx + tanx) = 1
- ⇨ (secx − tanx) = 1/(secx + tanx)
- ⇨ (secx − tanx) = 1/2cosx
- ⇨ (secx − tanx) = secx/2
- ⇨ secx/2 = tanx
- ⇨ sinx = ½
- ⇨ x = π/6, 5π/6
- ⇨ Option (c) is correct.

4. Using only the digits 2, 3, 9 how many six digit numbers can be formed which are divisible by 6?
   (a) 41
   (b) 80
   (c) 81
   (d) 161

Solution :

Now, last digit must be 2.

3 and 9 both are divisible by 3. We need to consider 3 2’s or 6 2’s to get the number divided by 3 and hence 6.

We put 3 2’s and the rest 3 numbers can be two 3 one 9 or two 9 one 3 or three 3 or three 9.

In first case number of numbers = 5!/(2!*2!) = 30 (note that last digit 2 is fixed, it cannot permutate)

In second case number of numbers = 5!/(2!*2!) = 30

In third case number of numbers = 5!/(3!*2!) = 10

In fourth case number of numbers = 5!/(3!*2!) = 10

And 1 number for 6 2's.

So, total number of six digit numbers with the given criteria = 30 + 30 + 10 + 10 + 1 = 81

- ⇨ Option (c) is correct.

5. What is the value of the following integral?
   $\int\{(\tan^{-1}x)/x\}dx$ (integration running from 1/2014 to 2014)
   (a) (π/4)log2014
   (b) (π/2)log2014
   (c)(π)log2014
   (d) (1/2)log2014

Solution :

Let, $I = \int\{(\tan^{-1}x)/x\}dx$ (integration running from 1/2014 to 2014)

- ⇨ $I = \int[\{\cot^{-1}(1/x)\}/x]dx$ (integration running from 1/2014 to 2014)
- ⇨ $I = \int[\{\pi/2 - \tan^{-1}(1/x)\}/x]dx$ (integration running from 1/2014 to 2014)
- ⇨ $I = (\pi/2)\int dx - \int[\{\tan^{-1}(1/x)\}/x]dx$ (integration running from 1/2014 to 2014)
- ⇨ $I = (\pi/2)\log x|_{1/2014}^{2014} - \int[\{\tan^{-1}(1/x)\}/x]dx$ (integration running from 1/2014 to 2014)

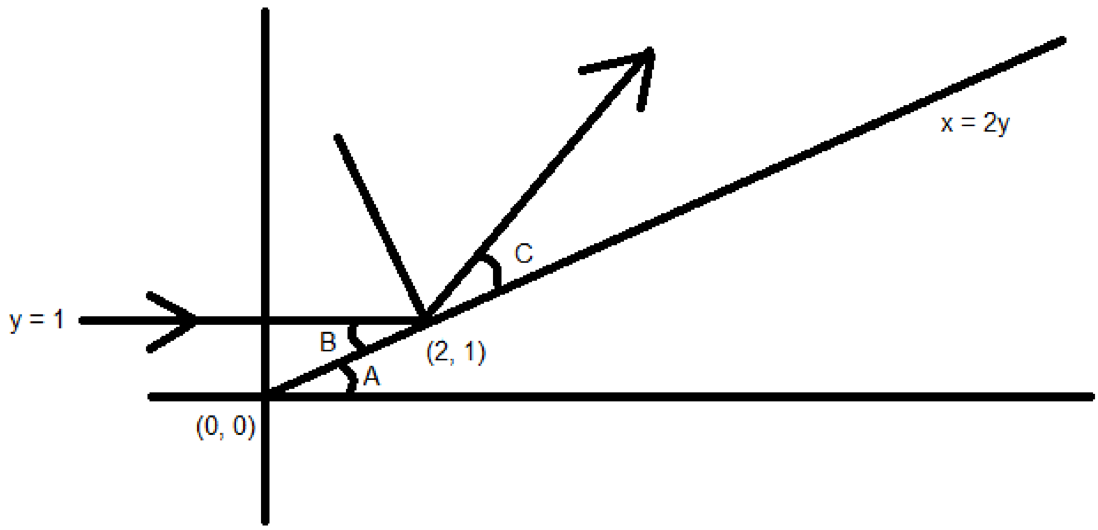
Now, let $x = 1/z$

- ⇨ $dx = -dz/z^2$, at $x = 1/2014$, $z = 2014$ and $x = 2014$, $z = 1/2014$
- ⇨ $I = (\pi/2)\{\log(2014) - \log(1/2014)\} - \int\{(z\tan^{-1}z)/(-z^2)\}dz$ (integration running from 2014 to 1/2014)
- ⇨ $I = (\pi/2)\{\log(2014) - \log(2014)^{-1}\} + \int\{(\tan^{-1}z)/z\}dz$ (integration running from 2014 to 1/2014)
- ⇨ $I = (\pi/2)\{\log(2014) + \log(2014)\} - \int\{(\tan^{-1}z)/z\}dz$ (integration running from 1/2014 to 2014) (note that – sign came up due to upper limit and lower limit change)
- ⇨ $I = (\pi/2)*2\log(2014) - I$
- ⇨ $2I = \pi\log(2014)$
- ⇨ $I = (\pi/2)\log(2014)$

Option (b) is correct.

6. A light ray travelling along the line y = 1, is reflected by a mirror placed along the line x = 2y. The reflected ray travels along the line
   (a) 4x – 3y = 5
   (b) 3x – 4y = 2
   (c) x – y = 1
   (d) 2x – 3y = 1

Solution :



Angle C = Angle B = Angle A = $\tan^{-1}(1/2)$

- ⇨ tanC = ½
- ⇨ (m – ½)/(1 + m/2) = ½ (m is slope of the reflected ray line and ½ is slope of x = 2y)
- ⇨ m – ½ = ½ + m/4
- ⇨ 3m/4 = 1
- ⇨ m = 4/3

The equation of reflected ray line is, y – 1 = (4/3)(x – 2)

- ⇨ 4x – 3y = 5
- ⇨ Option (a) is correct.

7. For a real number x, let [x] denote the greatest integer less than or equal to x. Then the number of real solutions of |2x – [x]| = 4 is
   (a) 1
   (b) 2

(c) 3
(d) 4

Solution :

If $x > 0$, then $2x - [x] = 4$

- ⇨ $2x = 4 + [x]$

RHS is an integer as $[x]$ is an integer.

- ⇨ Either x is (an odd number)/2 or an integer.

First case (x is an odd number/2)

Let $x = p/2$ where p is odd.

Then $[x] = (p - 1)/2$

Putting the values we get,

$p = 4 + (p - 1)/2$

- ⇨ $p = 7$
- ⇨ $x = 3.5$

Second case (x is an integer)

$2x = 4 + x$

- ⇨ $x = 4$

If $x < 0$ then $[x] - 2x = 4$

Similarly 2 cases, x is an odd integer/2 or x is an integer.

Third case (x is an odd integer/2)

Let $x = p/2$ where p is odd integer

- ⇨ $[x] = (p - 1)/2$

Putting the values we get,

$(p - 1)/2 - p = 4$

- ⇨ $p = -9$
- ⇨ $x = -4.5$

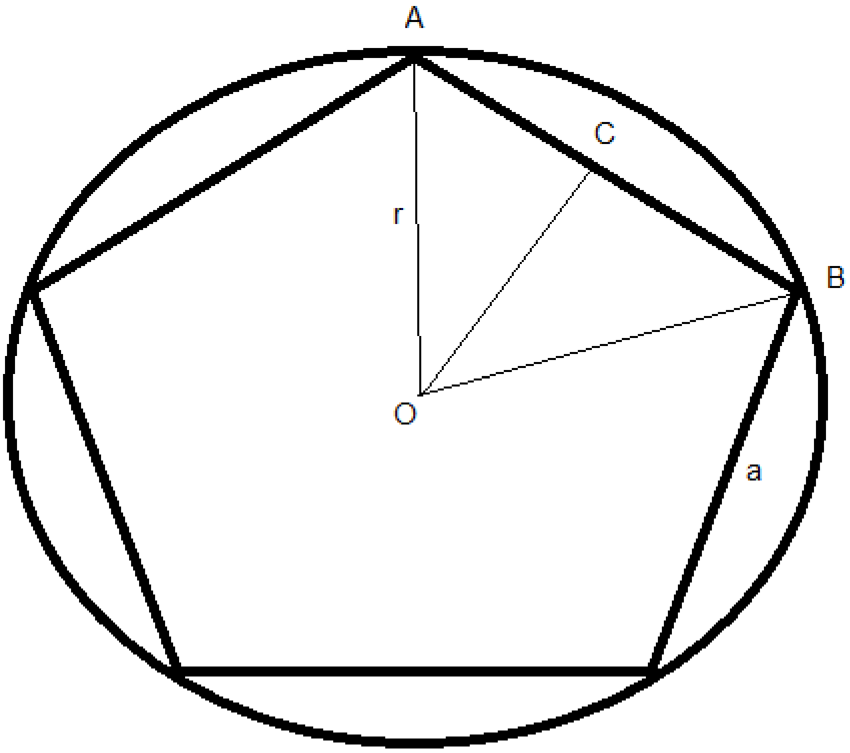
Fourth case (x is integer)

So, $x - 2x = 4$

⇨ x = -4
⇨ Four solutions viz. 3.5, 4, -4.5, -4
⇨ Option (d) is correct.

8. What is the ratio of the areas of the regular pentagons inscribed inside and circumscribed around a given circle?
(a) $\cos 36°$
(b) $\cos^2 36°$
(c) $\cos^2 54°$
(d) $\cos^2 72°$

Solution :



Angle A = (5 – 2)*180°/5 = 108°

⇨ Angle OAB = 108°/2 = 54°

From triangle OAC. Cos54° = AC/OA

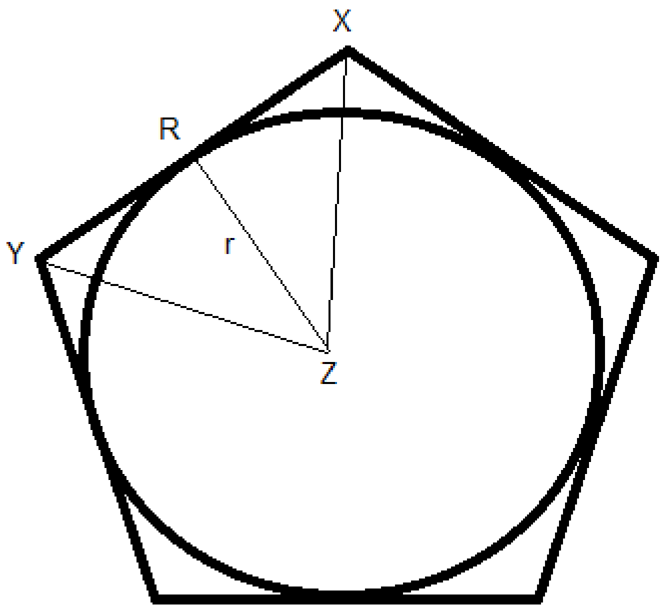
⇨ AC = rcos54°
⇨ AB = 2rcos54°

Now, sin54° = OC/OA

⇨ OC = rsin54°

Area of triangle OAB = (1/2)*OC*AB = (1/2)(rsin54°)(2rcos54°) = $(r^2/2)$sin108°

- ⇨ Area of the inscribed pentagon = 5*$(r^2/2)$sin108°



Angle X = (5 – 2)*180°/5 = 108°

- ⇨ Angle RXZ = 108°/2 = 54°

Now, RX/RZ = cot54°

- ⇨ RX = rcot54°
- ⇨ XY = 2rcot54°

Now, area of triangle XYZ = (1/2)*XY*RZ = (1/2)*(2rcot54°)*r = r^2cot54°

- ⇨ Area of circumscribed pentagon = 5r^2cot54°

Now, ratio of area of inscribed and circumscribed pentagons is {5*$(r^2/2)$sin108°}/(5r^2cot54°)

= (2sin54°cos54°)(sin54°)/2cos54°

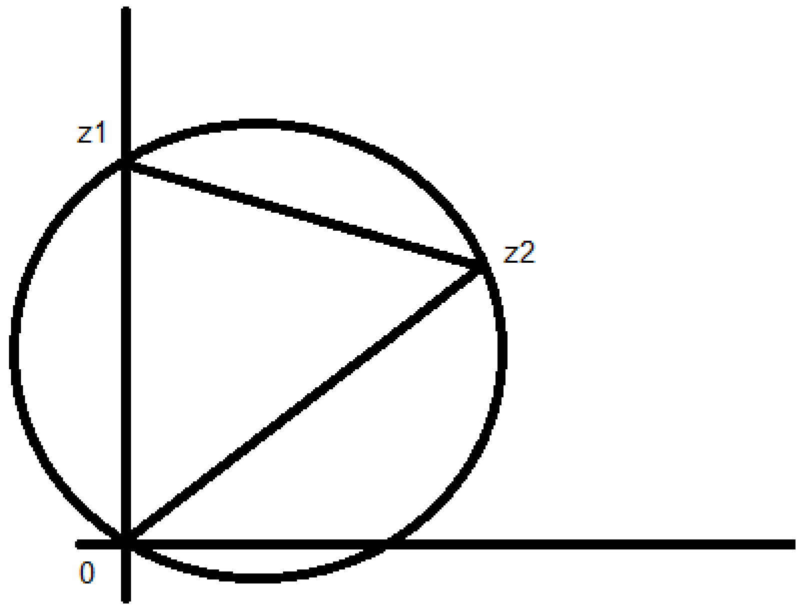
= $\sin^2 54°$

= $\cos^2 36°$

- ⇨ Option (b) is correct.

9. Let z_1, z_2 be two complex numbers satisfying $|z_1 + z_2| = |z_1 - z_2|$. The circumcentre of the triangle with the points z_1, z_2 and origin as its vertices is given by
   (a) $(z_1 - z_2)/2$
   (b) $(z_1 + z_2)/3$
   (c) $(z_1 + z_2)/2$
   (d) $(z_1 - z_2)/3$

Solution :



Let, $z_1 = iy_1$ and $z_2 = x_2 + iy_2$

Now, mid-point of z_1, z_2 = $x_2/2 + i(y_1 + y_2)/2$

Slope of z_1, z_2 = $(y_2 - y_1)/x_2$

Slope of perpendicular bisector of z_1, z_2 = $-x_2/(y_2 - y_1)$

Equation of perpendicular bisector of z_1, z_2 is, $y - (y_1 + y_2)/2 = -\{x_2/(y_2 - y_1)\}(x - x_2/2)$

Equation of perpendicular bisector of z_1 is, $y = y_1/2$

Solving the two equations we get the circumcentre of triangle z_1, z_2, 0 as the intersection of these two lines is circumcentre.

Now, $y_1/2 - (y_1 + y_2)/2 = -(x_2/(y_2 - y_1)\}(x - x_2/2)$

⇨ $y_2(y_2 - y_1)/2x_2 = x - x_2/2$

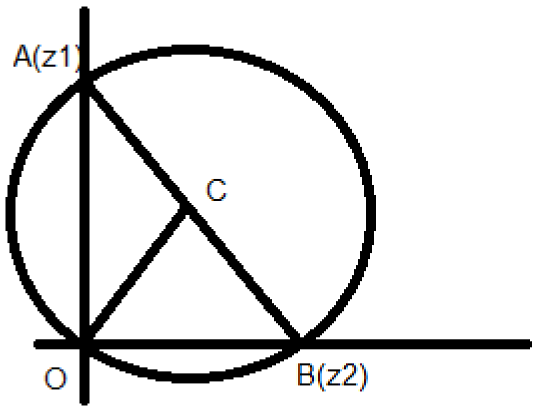
- ⇨ $x = y_2(y_2 - y_1)/2x_2 + x_2/2$

Now, we have, $|z_1 + z_2| = |z_1 - z_2|$

- ⇨ $|iy_1 + x_2 + iy_2| = |iy_1 - x_2 - iy_2|$
- ⇨ $x_2^2 + (y_1 + y_2)^2 = (-x_2)^2 + (y_1 - y_2)^2$
- ⇨ $4y_1y_2 = 0$
- ⇨ $y_2 = 0$
- ⇨ $x = x_2/2$
- ⇨ The circumcentre $z = x_2/2 + iy_1/2 = (z_1 + z_2)/2$

Alternate solution :

We take a special case here that the triangle z_1, z_2, 0 is a right angles triangle.



Now Angle AOB is 90°

- ⇨ AB is diameter of the circle (As semicircular angle is 90°)
- ⇨ The circumcentre C must be mid-point of AB to make sure that |OC| = |AC| = |BC| = radius
- ⇨ The circumcentre is $(z_1 + z_2)/2$.
- ⇨ Option (c) is correct.

10. In how many ways can 20 identical chocolates be distributed among 8 students so that each student gets at least one chocolate and exactly two students get at least two chocolates each?
 (a) 308
 (b) 364
 (c) 616
 (d) $^8C_2 * ^{17}C_7$

Solution :

Every one gets at least one chocolate and exactly 2 students get at least 2 chocolate. Now, the rest 8 – 2 = 6 students get exactly one chocolate because if they get more than one chocolate then there will be more than 2 students who will get at least 2 chocolates and hence contradicting the statement that exactly 2 students get at least 2 chocolates.

Now, we can choose any 2 students out of 8 students in $^8C_2$ ways.

Now, we give one chocolate to 8 students each.

Remaining chocolates = 20 – 8 = 12

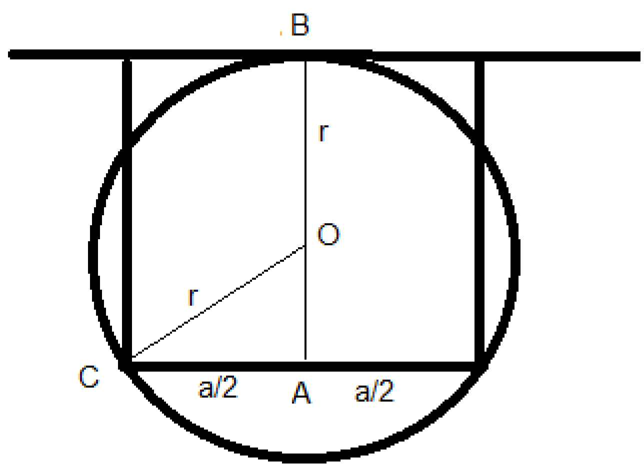
Now, the chosen two students will get at least one chocolate among 12 chocolates and we need to distribute rest 12 chocolates among those 2 students only.

Number of ways = $^{12-1}C_{2-1}$ = $^{11}C_1$

- ⇨ Total number of ways = $^8C_2 * ^{11}C_1$ = (8*7/2)*11 = 308
- ⇨ Option (a) is correct.

11. Two vertices of a square lie on a circle of radius r, and the other two vertices lie on a tangent to this circle. Then each side of the square is
    (a) 3r/2
    (b) 4r/3
    (c) 6r/5
    (d) 8r/5

Solution :



Now, from triangle OAC we get, OA = $\sqrt{(r^2 - a^2/4)}$

Now, AO + OB = a

- ⇨ $\sqrt{(r^2 - a^2/4)} + r = a$
- ⇨ $\sqrt{(r^2 - a^2/4)} = a - r$
- ⇨ $r^2 - a^2/4 = a^2 - 2ar + r^2$
- ⇨ $5a^2/4 = 2ar$
- ⇨ $a = 8r/5$
- ⇨ Option (d) is correct.

12. Let P be the set of all numbers obtained by multiplying five distinct integers between 1 and 100. What is the largest integer n such that 2^n divides at least one element of P?
    (a) 8
    (b) 20
    (c) 24
    (d) 25

Solution :

The element of P which gets divided by 2^n where n is largest is $2^6*2^5*(3*2^5)*2^4*(3*2^4) = 9*2^{24}$

- ⇨ n = 24
- ⇨ Option (c) is correct.

13. Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers with a > 0. If f is strictly increasing, then the function $g(x) = f'(x) - f''(x) + f'''(x)$ is
    (a) Zero for some $x \in R$
    (b) Positive for all $x \in R$
    (c) Negative for all $x \in R$
    (d) Strictly increasing.

Solution :

So, $g(x) = 3ax^2 + 2bx + c - 6ax - 2b + 6a$

Let us put x = -r where r is real and positive.

$g(-r) = 3ar^2 - 2b(r + 1) + c + 6ar + 6a$

Now, $f'(x) = 3ax^2 + 2bx + c$

$f'(-(r + 1)) > 0$ (As f(x) is increasing)

- ⇨ $3a(r + 1)^2 - 2b(r + 1) + c > 0$ ....(A)

Now, $g(-r) = 3a(r^2 + 2r + 1) - 2b(r + 1) + c + 3a = 3a(r + 1)^2 - 2b(r + 1) + c + 3a > 0$ (As $a > 0$ and from (A))

Now, let $r > 0$

$g(r) = 3ar^2 - 6ar + 3a + 2b(r - 1) + c + 3a$

$= 3a(r - 1)^2 + 2b(r - 1) + c + 6a$

Now, $f'(r - 1) = 3a(r - 1)^2 + 2b(r - 1) + c$ ....(B)

- ⇨ $g(r) > 0$ (As $a > 0$ and from (B))

Now, $g(0) = 6a - 2b + c$

Now, $f'(-1) = 3a - 2b + c > 0$ .....(C)

- ⇨ $g(0) = 3a - 2b + c + 3a > 0$ (As $a > 0$ and from (C))
- ⇨ $g(x)$ is always positive.
- ⇨ Option (b) is correct.

14. Let A be the set of all points (h, k) such that the area of the triangle formed by (h, k), (5, 6) and (3, 2) is 12 square units. What is the least possible length of a line segment joining (0, 0) to a point in A?
    (a) $4/\sqrt{5}$
    (b) $8/\sqrt{5}$
    (c) $12/\sqrt{5}$
    (d) $16/\sqrt{5}$

Solution :

We have, $(1/2)\{h(6 - 2) + 5(2 - k) + 3(k - 6)\} = 12$

- ⇨ $4h - 2k = 32$
- ⇨ $k = 2h - 16$

Now, distance of a point in A from (0, 0) is $B = \sqrt{(h^2 + k^2)}$

Let, $C = B^2 = h^2 + k^2$

Now, B will be minimum when C is minimum.

Now, $C = h^2 + (2h - 16)^2$ (Putting value of k from above)

- ⇨ $C = 5h^2 - 64h + 256$
- ⇨ $dC/dh = 10h - 64 = 0$
- ⇨ $h = 32/5$

Now, $d^2C/dh^2 = 10 > 0$

- ⇨ C is minimum when $h = 32/5$
- ⇨ C is minimum when $k = 2*(32/5) - 16 = -16/5$
- ⇨ $C_{min} = (32/5)^2 + (-16/5)^2 = (16/5)^2(4 + 1) = 16^2/5$
- ⇨ $B_{min} = \sqrt{C_{min}} = 16/\sqrt{5}$
- ⇨ Option (d) is correct.

15. Let P = {abc : a, b, c positive integers, $a^2 + b^2 = c^2$, and 3 divides c}. What is the largest integer n such that 3^n divides every element of P?

Solution :

Now, $a^2 + b^2 = c^2$

Let 3 doesn't divide a or b.

Dividing the equation by 4 we get,

$(\pm1)^2 + (\pm1)^2 \equiv 0 \pmod 3$

- ⇨ $1 + 1 \equiv 0 \pmod 3$

Which is impossible.

Now, Let c divides one of a, b.

Dividing the equation by 3 again we get,

$(\pm1)^2 + 0 \equiv 0 \pmod 3$

- ⇨ $1 \equiv 0 \pmod 3$

Which is impossible.

- ⇨ 3 divides both a and b.

Now, we have, $(3a_1)^2 + (3b_1)^2 = (3c_1)^2$ where $a = 3a_1$, $b = 3b_1$ and $c = 3c_1$

- ⇨ $a_1^2 + b_{12}^2 = c_1^2$

Let, 3 doesn't divide any of a_1, b_1, c_1

Dividing the equation by 3 we get,

$1 + 1 \equiv 1 \pmod 3$

Which is impossible

Let, 3 divides both of a_1, b_1

Dividing the equation by 3 we get,

$0 + 0 \equiv 1 \pmod 3$

Which is impossible.

- ⇨ 3 divides exactly one of a_1, b_1. Let 3 divides b_1 (Without loss of generality)

In that case dividing the equation by 3 we get,

$1 + 0 \equiv 1 \pmod 3$

Which is consistent.

- ⇨ 3 divides b_1
- ⇨ 3 divides a, 3^2 divides b and 3 divides c
- ⇨ 3^4 divides abc
- ⇨ Option (d) is correct.

16. Let $A_0 = \Phi$ (the empty set). For each i = 1, 2, 3, ..., define $A_i = A_{i-1} \cup \{A_{i-1}\}$. The set A_3 is
    (a) Φ
    (b) $\{\Phi\}$
    (c) $\{\Phi, \{\Phi\}\}$
    (d) $\{\Phi, \{\Phi\}, \{\Phi, \{\Phi\}\}\}$

Solution :

Option (d) is correct.

17. Let $f(x) = 1/(x - 2)$. The graphs of the functions f and f^{-1} intersect at
    (a) $(1 + \sqrt{2}, 1 + \sqrt{2})$ and $(1 - \sqrt{2}, 1 - \sqrt{2})$
    (b) $(1 + \sqrt{2}, 1 + \sqrt{2})$ and $(\sqrt{2}, -1 - 1/\sqrt{2})$
    (c) $(1 - \sqrt{2}, 1 - \sqrt{2})$ and $(-\sqrt{2}, -1 + 1/\sqrt{2})$
    (d) $(\sqrt{2}, -1 - 1/\sqrt{2})$ and $(-\sqrt{2}, -1 + 1/\sqrt{2})$

Solution :

Now, $f(x) = 1/(x - 2)$

- ⇨ $x - 2 = 1/f$

- ⇨ x = 1/f + 2
- ⇨ f^{-1} = 1/x + 2 = (2x + 1)/x

Now, 1/(x – 2) = (2x + 1)/x

- ⇨ x = $2x^2$ + x – 4x – 2
- ⇨ x^2 – 2x – 1 = 0
- ⇨ x = {2 ± √(4 + 4*1*1)}/2
- ⇨ x = 1 ± √2

When x = 1 + √2, f = 1/(1 + √2 – 2) = 1/(√2 – 1) = (√2 + 1)/{√2 – 1)(√2 + 1) = 1 + √2

When x = 1 - √2, f = 1/(1 - √2 – 2) = 1/(-1 - √2) = (-1 + √2)/{(-1 - √2)(-1 + √2)] = 1 - √2

So the intersection points are (1 + √2, 1 + √2) and (1 - √2, 1 - √2)

- ⇨ Option (a) is correct.

18. Let N be a number such that whenever you take N consecutive positive integers, at least one of them is coprime to 374. What is the smallest possible value of N?
    (a) 4
    (b) 5
    (c)6
    (d) 7

Solution :

374 = 2*11*17

Now, if we take, 119, 120 and 121 then no one is coprime to 374.

If we take 118, 119, 120, 121, 122 then also no one is coprime to 374. Because there are 2 odd prime factors of 374 and consecutive odd numbers are divisible by 11 and 17.

So, if we take 3 odd numbers then one must be coprime to 373.

So, if we take, 117, 118, 119, 120, 121, 122 then 117 is coprime to 374. Also if we take 118, 119, 120, 121, 122, 123 then 123 is coprime to 371.

- ⇨ N = 6
- ⇨ Option (c) is correct.

19. Let, A_1, A_2, ..., A_{18} be the vertices of a regular polygon with 18 sides. How many of the triangle $A_iA_jA_k$, $1 \leq i < j < k \leq 18$, are isosceles but not equilateral?
(a) 63
(b) 70
(c) 126
(d) 144

Solution :

Option (c) is correct.

20. The limit lim $(\sin^{\alpha}x)/x$ as x -> 0 exists only when
(a) $\alpha \geq 1$
(b) $\alpha = 1$
(c) $|\alpha| \leq 1$
(d) α is a positive integer.

Solution :

Clearly the limit holds for $\alpha \geq 1$

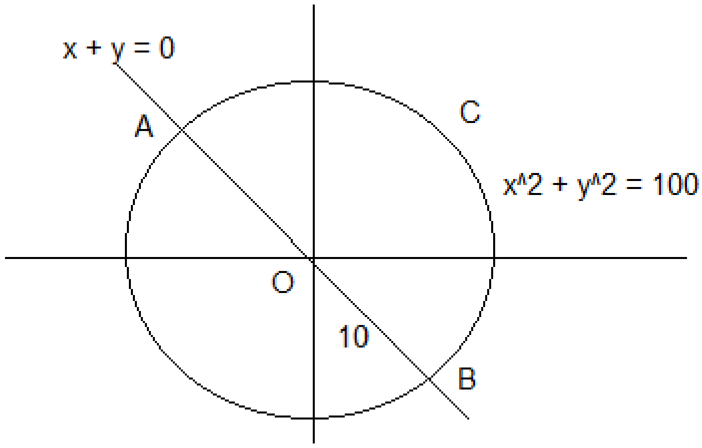
In that case lim $(\sin^{\alpha}x)/x$ as x -> 0 = lim $\{(\sin x)/x\}*\sin^{\alpha-1}x$ as x -> 0 = 1*0 = 0

If $\alpha - 1$ is negative then it makes a 0 in nujmerator.

⇨ Option (a) is correct.

21. Consider the region $R = \{(x, y) : x^2 + y^2 \leq 100, \sin(x + y) > 0\}$. What is the area of R?
(a) 25π
(b) 50π
(c) 50
(d) $100\pi - 50$

Solution :



Now, sin(x + y) > 0

- ⇨ sin(x + y) > sin0
- ⇨ x + y > 0

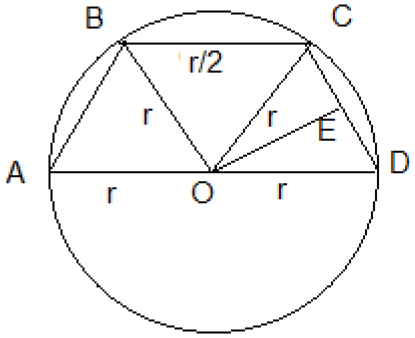
Clearly the area represented by R is the semicircular area AOBCA

Now, radius of the circle = 10

- ⇨ Area of the semicircular region AOBCA = $(\pi * 10^2)/2 = 50\pi$
- ⇨ Option (b) is correct.

22. Consider a cyclic trapezium whose circumcentre is on one of the sides. If the ratio of the two parallel sides is 1:4, what is the ratio of the sum of the two oblique sides to the longer parallel side?
(a) $\sqrt{3} : \sqrt{2}$
(b) 3 : 2
(c) $\sqrt{2} : 1$
(d) $\sqrt{5} : \sqrt{3}$

Solution :



The circumcentre lie on AD.

- ⇨ O, the circumcentre must divide AD in equal halves.
- ⇨ BC = CD

Now, from triangle OBC, let Angle BOC = Θ

- ⇨ $\cos\Theta = (r^2 + r^2 - r^2/4)/2*r*r$
- ⇨ $\cos\Theta = 7/8$
- ⇨ Angle DOC = $(180° - \Theta)/2$
- ⇨ $\cos(DOC) = \cos(90° - \Theta/2) = \sin(\Theta/2)$
- ⇨ $\sin(\Theta/2) = (r^2 + r^2 - a^2)/2*r*r$
- ⇨ $\sin(\Theta/2) = 1 - a^2/2r^2$
- ⇨ $\sin^2(\Theta/2) = (1 - a^2/2r^2)^2$
- ⇨ $(1 - \cos\Theta)/2 = (1 - a^2/2r^2)^2$
- ⇨ $(1 - 7/8)/2 = (1 - a^2/2r^2)^2$
- ⇨ $1/16 = (1 - a^2/2r^2)^2$
- ⇨ $1 - a^2/2r^2 = 1/4$
- ⇨ $a^2/2r^2 = 3/4$
- ⇨ $a^2/r^2 = 3/2$
- ⇨ $a/r = \sqrt{3} : \sqrt{2}$
- ⇨ $2a/2r = \sqrt{3} : \sqrt{2}$
- ⇨ Option (a) is correct.

23. Consider the function $f(x) = [\log_e\{(4 + \sqrt{2}x)/x\}]^2$ for $x > 0$. Then
    (a) f decreases upto some point and increases after that
    (b) f increases upto some point and decreases after that
    (c) f increases initially, then decreases and then again increases
    (d) f decreases initially, then increase and then again decreases.

Solution :

Option (b) is correct.

24. What is the number of ordered triplets (a, b, c) where a, b, c are positive integers (not necessarily distinct), such that abc = 1000?
    (a) 64
    (b) 100
    (c) 200
    (d) 560

Solution :

The combinations are when none of a, b, c is 1.

$(2, 2, 2*5^3) = 3!/2! = 3$

$(2, 2^2, 5^3) = 3! = 6$

$(5, 5, 5*2^3) = 3!/2! = 3$

$(5, 5^2, 2^3) = 3! = 6$

$(2, 2*5, 2*5^2) = 3! = 6$

$(2, 2^2*5, 5^2) = 3! = 6$

$(2, 2^2*5^2, 5) = 3! = 6$

$(2^2, 2*5, 5^2) = 31 = 6$

$(2^2*5, 2*5, 5) = 3! = 6$

$(5, 2*5^2, 2^2) = 3! = 6$

$(2*5, 2*5, 2*5) = 1$

Number of cases = 3 + 6 + 3 + 6 + 6 + 6 + 6 + 6 + 6 + 6 + 1 = 55

Now, when one of a, b, c is 1.

$(1, 2, 2^2*5^2) = 3! = 6$

$(1, 2^2, 2*5^3) = 3! = 6$

$(1, 2^3, 5^3) = 3! = 6$

$(1, 2*5, 2^2*5^2) = 3! = 6$

$(1, 2*5^2, 2^2*5) = 3! = 6$

$(1, 2^3*5, 5^2) = 3! = 6$

$(1, 2^3*5^2, 5) = 3! = 6$

Number of cases = 6 + 6 + 6 + 6 + 6 + 6 + 6 = 42

Now, when 2 of a, b, c are 1.

(1, 1, 1000) = 3!/2! = 3

Therefore, total number of cases = 55 + 42 + 3 = 100

- ⇨ Option (b) is correct.

25. Let, $f : (0, \infty) \to (0, \infty)$ be a function differentiable at 3, and satisfying, $f(3) = 3f'(3) > 0$ Then the limit
$\lim [f(3 + 3/x)/f(3)]^x$ as $x \to \infty$
(a) Exists and is equal to 3
(b) Exists and is equal to e
(c) Exists and is always equal to f(3)
(d) Need not always exist.

Solution :

Let $z = 1/x$, then $z \to 0$ as $x \to \infty$

The limit is $\lim \{f(3 + z)/f(3)\}^{1/z}$ as $z \to 0$

$= \lim [1 + \{f(3 + z) - f(3)\}/f(3)]^{1/z}$

Now, $\{f(z + 3) - f(3)\} /f(3) \to 0$ as $z \to 0$

So, the limit is = e

⇨ Option (b) is correct.

26. Let z be a non-zero complex number such that $|z - 1/z| = 2$. What is the maximum value of $|z|$?
(a) 1
(b) $\sqrt{2}$
(c) 2
(d) $1 + \sqrt{2}$

Solution :

Let $z = re^{i\Theta}$

Now, $z - 1/z = re^{i\Theta} + 1/re^{i\Theta} = (r\cos\Theta + \cos\Theta/r) + i(r\sin\Theta - \sin\Theta/r)$

⇨ $|z - 1/z| = \sqrt{(\ r^2\cos^2\Theta + \cos^2\Theta/r^2 - 2\cos^2\Theta + r^2\sin^2\Theta + \sin^2\Theta/r^2 - 2\sin^2\Theta)}$
⇨ $2 = \sqrt{(\ r^2 + 1/r^2 - 2)}$
⇨ $r^2 + 1/r^2 = 6$
⇨ $r^2 + 2 + 1/r^2 = 8$
⇨ $(r + 1/r)^2 = 8$
⇨ $r + 1/r = 2\sqrt{2}$
⇨ $r^2 - r2\sqrt{2} + 1 = 0$
⇨ $r = \{2\sqrt{2} + \sqrt{(8 - 4)}\}/2$
⇨ $r = (1 + \sqrt{2})$
⇨ Option (d) is correct.

27. The minimum value of $|\sin x + \cos x + \tan x + \operatorname{cosec} x + \sec x + \cot x|$ is
(a) 0
(b) $2\sqrt{2} - 1$
(c) $2\sqrt{2} + 1$
(d) 6

Solution :

Let, $f(x) = \sin x + \cos x + \tan x + \operatorname{cosec} x + \sec x + \cot x$

- ⇨ $f'(x) = \cos x - \sin x + \sec^2 x - \operatorname{cosec} x \cot x + \sec x \tan x - \operatorname{cosec}^2 x$
- ⇨ $f'(x) = (\cos x - \sin x) + (\sec^2 x + \sec x \tan x) - (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x)$
- ⇨ $f'(x) = (\cos x - \sin x) + (1 + \sin x)/\cos^2 x - (1 + \cos x)/\sin^2 x$
- ⇨ $f'(x) = \cos^2 x \sin^2 x(\cos x - \sin x) + \sin^2 x + \sin^3 x - \cos^2 x - \cos^3 x$
- ⇨ $f'(x) = \cos^2 x \sin^2 x(\cos x - \sin x) - (\cos^2 x - \sin^2 x) - (\cos^3 x - \sin^3 x) = 0$
- ⇨ $f'(x) = \cos^2 x \sin^2 x(\cos x - \sin x) - (\cos x + \sin x)(\cos x - \sin x) - (\cos x - \sin x)(\cos^2 x + \sin^2 x + \sin x \cos x)$
- ⇨ $f'(x) = (\cos x - \sin x)(\cos^2 x \sin^2 x - \cos x - \sin x - 1 - \sin x \cos x)$

Now, $f'(x) = 0$ gives,

$\cos^2 x \sin^2 x - \cos x - \sin x - 1 - \sin x \cos x = 0$

- ⇨ $\cos^2 x \sin^2 x - \sin x \cos x - 1 = \cos x + \sin x$
- ⇨ $(\cos^2 x \sin^2 x - \sin x \cos x - 1)^2 = (\cos x + \sin x)^2$
- ⇨ $(\sin^2 2x - 2\sin 2x - 4)^2 = 16(1 + \sin 2x)$ (Multiplying both sides by 16 and putting $2\sin x \cos x = \sin 2x$)
- ⇨ $\sin^4 2x + 4\sin^2 2x + 16 - 4\sin^3 2x - 8\sin^2 2x + 16\sin 2x = 16 + 16\sin 2x$
- ⇨ $\sin^4 2x - 4\sin^3 2x - 4\sin^2 2x = 0$
- ⇨ $\sin^2 2x - 4\sin 2x - 4 = 0$ ($\sin 2x \neq 0$)
- ⇨ $\sin 2x = \{4 \pm \sqrt{(16 + 16)}\}/2$
- ⇨ $\sin 2x = 2 - 2\sqrt{2}$ ($\sin 2x \neq 2 + 2\sqrt{2}$ as $-1 \le \sin 2x \le 1$)

Now, $f(x) = \sin x + \cos x + \tan x + \operatorname{cosec} x + \sec x + \cot x$

$= (\sin^2 x \cos x + \cos^2 x \sin x + \sin^2 x + \cos^2 x + \cos x + \sin x)/\sin x \cos x$

$= \{\sin x \cos x(\cos x + \sin x) + (\cos x + \sin x) + 1\}/\sin x \cos x$

$= \{(\sin x \cos x + 1)(\cos x + \sin x) + 1\}/\sin x \cos x$

$= [(\sin 2x + 2)\{\pm\sqrt{(1 + \sin 2x)} + 2]/\sin 2x$ (Multiplying numerator and denominator by 2)

$= [(2 - 2\sqrt{2} + 2)\{\pm\sqrt{(1 + 2 - 2\sqrt{2})}\} + 2]/(2 - 2\sqrt{2})$

$= [(4 - 2\sqrt{2})\{\pm\sqrt{(\sqrt{2} - 1)^2}\} + 2]/(2 - 2\sqrt{2})$

$= [(4 - 2\sqrt{2})\{\pm(\sqrt{2} - 1)\} + 2]/(2 - 2\sqrt{2})$

Let us take the + sign,

$f(x) = \{(4 - 2\sqrt{2})(\sqrt{2} - 1) + 2\}/(2 - 2\sqrt{2})$

$= (4\sqrt{2} - 4 - 4 + 2\sqrt{2} + 2)/(2 - 2\sqrt{2})$

$= (6\sqrt{2} - 6)/(2 - 2\sqrt{2})$

$= - 3$

Now, taking the – sign we get,

$f(x) = [-(4 - 2\sqrt{2})(\sqrt{2} - 1) + 2]/(2 - 2\sqrt{2})$

$= (-4\sqrt{2}+ 4 + 4 - 2\sqrt{2} + 2)/(2 - 2\sqrt{2})$

$= (10 - 6\sqrt{2})/(2 - 2\sqrt{2})$

$= (5 - 3\sqrt{2})/(1 - \sqrt{2})$

$= (5 - 3\sqrt{2})(1 + \sqrt{2})/(1 - \sqrt{2})(1 + \sqrt{2})$

$= (5 + 5\sqrt{2} - 3\sqrt{2} - 6)/(1 - 2)$

$= (2\sqrt{2} - 1)/(-1)$

$= -(2\sqrt{2} - 1)$

$|f(x)| = 3, 2\sqrt{2} - 1$

Now, $2\sqrt{2} - 1 < 3$

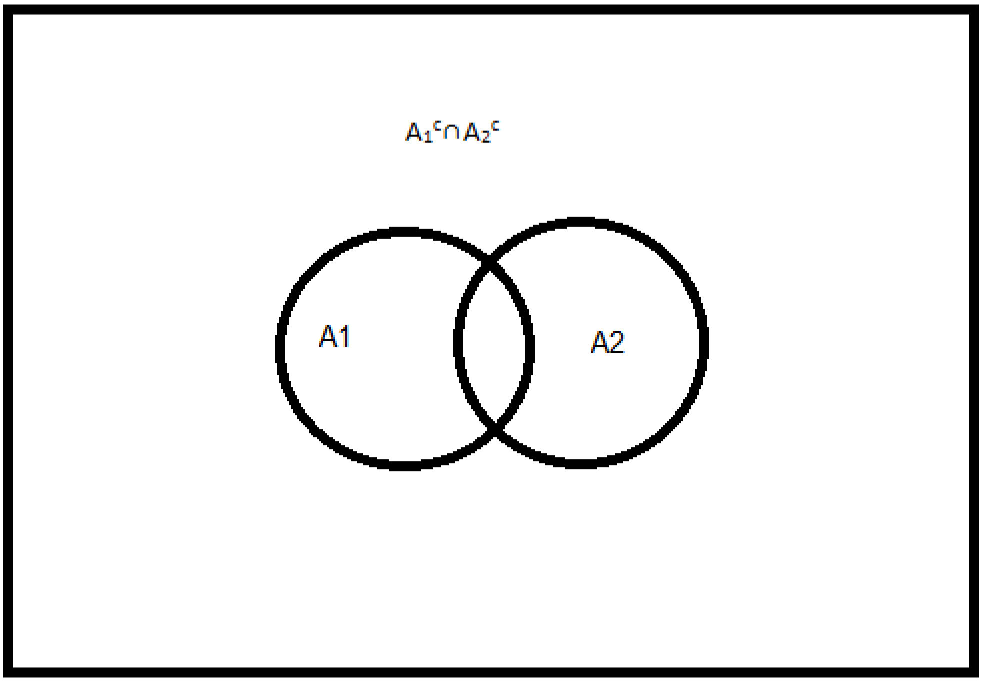
- ⇨ The minimum value of the given expression is $2\sqrt{2} - 1$
- ⇨ Option (b) is correct.

28. For any function $f : X \to Y$ and any subset A of Y, define $f^{-1}(A) = \{x \in X : f(x) \in A\}$.
Let A^c denote the complement of A in Y. For subsets A_1, A_2 of Y, consider the following statements :
(i) $f^{-1}(A_1^c \cap A_2^c) = (f^{-1}(A_1))^c \cup (f^{-1}(A_2))^c$
(ii) If $f^{-1}(A_1) = f^{-1}(A_2)$ then $A_1 = A_2$

Then

(a) Both (i) and (ii) are always true.
(b) (i) is always true, but (ii) may not always be true
(c)(ii) is always true, but (i) may not always be true
(d) Neither (i) nor (ii) is always true.\

Solution :



$(f^{-1}(A_1))^c \cup (f^{-1}(A_2))^c$ includes the whole area except $A_1 \cap A_2$

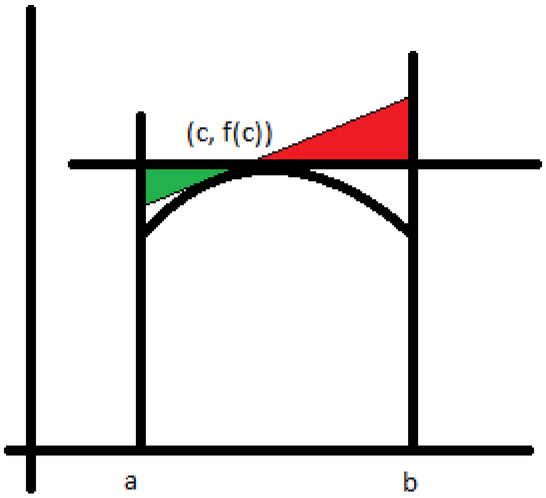
So, this may not be always true.

Second statement holds if the function is one-one and onto.

So, option (d) is correct.

29. Let f be a function such that $f''(x)$ exists, and $f''(x) > 0$ for all $x \in [a, b]$. For any point $c \in [a, b]$, let $A(c)$ denote the area of the region bounded by $y = f(x)$, the tangent to the graph of f at $x = c$ and the lines $x = a$ and $x = b$. Then
   (a) $A(c)$ attains its minimum at $c = (a + b)/2$ for any such f
   (b) $A(c)$ attains its maximum at $c = (a + b)/2$ for any such f
   (c) $A(c)$ attains its minimum at both $c = a$ and $c = b$ for any such f
   (d) The points c where $A(c)$ attains its minimum depend on f.

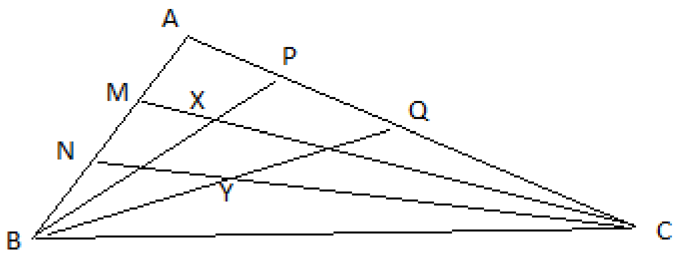
Solution :



Clearly when c is shifted from (a + b)/2 then the red area > green area and so A(c) is minimum when c = (a + b)/2.

⇨ Option (a) is correct.

30. In triangle ABC, the lines BP, BQ trisect angle ABC and the lines CM and CN trisect angle ACB. Let BP and CM intersect at X and BQ and CN intersect at Y. If angle ABC = 45° and angle ACB = 75°, then angle BXY is



(a) 45°
(b) (47 + ½)°
(c) 50°
(d) 55°

Solution :

Option (c) is correct.

__Test of Mathematics at 10+2 level Objective Solution__

1. A worker suffers a 20% cut in wages. He regains his original pay by obtaining a rise of
   (a) 20%
   (b) 22.5%
   (c) 25%
   (d) 27.5%

Solution :

Let his wage is x.

After 20% cut his wage is x – 20x/100 = 80x/100

In 80x/100 he needs to regain 20x/100

In 1 he needs to regain (20x/100)/(80x/100) = ¼

In100 he needs to regain 100*(1/4) = 25

Therefore, he needs a rise of 25% to regain his wage.

Option (c) is correct.

2. If m men can do a job in d days, then the number of days in which m + r men can do the job is
   (a) d + r
   (b) (d/m)(m + r)
   (c) d/(m + r)
   (d) md/(m + r)

Solution :

m men can do a job in d days

1 man can do the job in md days

m + r men can do the job in md/(m + r) days.

Option (d) is correct.

3. A boy walks from his home to school at 6 km per hour (kmph). He walks back at 2 kmph. His average speed, in kmph, is
   (a) 3
   (b) 4

(c) 5
(d) $\sqrt{12}$

Solution :

Let the distance from his home to school is x km.

Therefore, total distance covered = x + x = 2x.

Time taken to go to school = x/6 hours.

Time taken to come home from school = x/2 hours.

Total time = x/6 + x/2

So, average speed = total distance/total time = 2x/(x/6 + x/2) = 2/(1/6 + ½) = 3 kmph

Option (a) is correct.

4. A car travels from P to Q at 30 kilometres per hour (kmph) and returns from Q to P at 40 kmph by the same route. Its average speed, in kmph, is nearest to
(a) 33
(b) 34
(c) 35
(d) 36

Solution :

Let the distance between P and Q is x km.

Total distance covered = x + x = 2x

Time taken to go from P to Q = x/30 hours.

Time taken to go from Q to P = x/40 hours.

Total time = x/30 + x/40

Average speed = total distance/total time = 2x/(x/40 + x/30) = 2/(1/40 + 1/30) = 2*40*30/70 = 240/7 = 34.285 (approx.)

Option (b) is correct.

5. A man invests Rs. 10000 for a year. Of this Rs. 4000 is invested at the interest rate of 5% per year, Rs. 3500 at 4% per year and the rest at α% per year. His total interest for the year is Rs. 500. Then α equals
   (a) 6.2
   (b) 6.3
   (c) 6.4
   (d) 6.5

Solution :

Interest from Rs. 4000 = 5*4000/100 = Rs. 200

Interest from Rs. 3500 = 4*3500/100 = Rs. 140

Rest money = 10000 – (4000 + 3500) = Rs. 2500

Interest from Rs. 2500 = α*2500/100 = 25α

As per the question, 200 + 140 + 25α = 500

⇨ $\alpha = 160/25 = 6.4$

Option (c) is correct.

6. Let $x_1, x_2, \ldots, x_{100}$ be positive integers such that $x_i + x_{i+1} = k$ for all i, where k is constant. If $x_{10} = 1$, then the value of x_1 is
   (a) k
   (b) k – 1
   (c) k + 1
   (d) 1

Solution :

Clearly, $x_9 = k - 1$, $x_8 = 1$, $x_7 = k - 1$, ….., $x_1 = k - 1$

Option (b) is correct.

7. If $a_0 = 1$, $a_1 = 1$ and $a_n = a_{n-1}a_{n-2} + 1$ for n > 1, then
   (a) a_{465} is odd and a_{466} is even
   (b) a_{465} is odd and a_{466} is odd
   (c) a_{465} is even and a_{466} is even
   (d) a_{465} is even and a_{466} is odd.

Solution :

As, a_0 and a_1 both odd so, $a_2 = 2 =$ even.

As a_2 is even, both a_3 and a_4 will be odd because between a_{n-1} and a_{n-2} one is even and hence added to 1 becomes odd.

Then a_5 will be even as a_3 and a_4 are both odd.

So, the sequence will go in the way, a_0, a_1 odd, a_2 even, a_3, a_4 odd, a_5 even, a_6, a_7 odd, a_8 even and so on..

So, the numbers which are congruent to 2 modulus 3 are even and rest are odd.

Now, $465 \equiv 0 \pmod 3$ and $466 \equiv 1 \pmod 3$

⇨ a_{465} and a_{466} are both odd.

Option (b) is correct.

8. Two trains of equal length L, travelling at speeds V_1 and V_2 miles per hour in opposite directions, take T seconds to cross each other. Then L in feet (1 mile 5280 feet) is
   (a) $11T/15(V_1 + V_2)$
   (b) $15T/11(V_1 + V_2)$
   (c) $11(V_1 + V_2)T/15$
   (d) $11(V_1 + V_2)/15T$

Solution :

Speed = V_1 miles per hour = $V_1*5280/3600$ feet/second = $22V_1/15$ feet/second

Relative velocity = $22V_1/15 + 22V_2/15 = 22(V_1 + V_2)/15$ feet/second

Total distance covered = sum of train lengths = L + L = 2L

Therefore, $2L = \{22(V_1 + V_2)/15\}*T$

⇨ $L = 11(V_1 + V_2)T/15$

Option (c) is correct.

9. A salesman sold two pipes at Rs. 12 each. His profit on one was 20% and the loss on the other was 20%. Then on the whole, he
   (a) Lost Re. 1
   (b) Gained Re. 1

(c) Neither gained nor lost
(d) Lost Rs. 2

Solution :

Let the cost price of the pipe on which he made profit = x.

- ⇨ x + 20x/100 = 12
- ⇨ 120x/100 = 12
- ⇨ x = 12*100/120
- ⇨ x = 10

Let the cost price of the pipe on which he lost = y.

- ⇨ y – 20y/100 = 12
- ⇨ 80y/100 = 12
- ⇨ y = 12*100/80
- ⇨ y = 15

Therefore, total cost price = 10 + 15 = 25

Total selling price = 2*12 = 24

So, he lost (25 – 24) = Re. 1

Option (a) is correct.

10. The value of $(256)^{0.16}(16)^{0.18}$ is
(a) 4
(b) 16
(c) 64
(d) 256.25

Solution :

$(256)^{0.16}(16)^{0.18} = 2^{8*0.16}*2^{4*0.18} = 2^{8*0.16 + 4*0.18} = 2^{4(2*0.16 + 0.18)} = 2^{4*0.5} = 2^2 = 4$

Option (a) is correct.

11. The digit in the unit place of the integer 1! + 2! + 3! + ….. + 99! Is
(a) 3
(b) 0
(c) 1

(d) 7

Solution :

Now, after 5! = 120 all the terms end with 0 i.e. unit place digit is 0.

So, the unit place digit of the given integer is the unit place digit of the integer 1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33 i.e. 3

Option (a) is correct.

12. July 3, 1977 was a SUNDAY. Then July 3, 1970 was a
   (a) Wednesday
   (b) Friday
   (c) Sunday
   (d) Tuesday

Solution :

In 1970 after July 3 there are = 28 + 31 + 30 + 31 + 30 + 31 = 181 days.

In 1971 there are 365 days.

In 1972 there are 366 days.

In 1973, 1974, 1975 there are 3*365 = 1095 days.

In 1976 there are 366 days.

In 1977 up to July 3 there are 31 + 28 + 31 + 30 + 31 + 30 + 3 = 184 days.

Therefore total number of days up to July 3, 1970 from July 3, 1977 is 181 + 365 + 366 + 1095 + 366 + 184 = 2557

$2557 \equiv 2 \pmod 7$

- ⇨ July 3, 1970 was a Friday. (Sunday – 2)

Option (b) is correct.

13. June 10, 1979 was a SUNDAY. Then May 10, 1972, was a
   (a) Wednesday
   (b) Thursday
   (c) Tuesday

(d) Friday

Solution :

After May 10, 1972 there are 21 + 30 + 31 + 31 + 30 + 31 + 30 + 31 = 235 days.

In 1973, 1974, 1975 there are 3*365 = 1095 days.

In 1976 there are 366 days.

In 1977, 1978 there are 365*2 = 730 days.

In 1979 up to June 10, there are 31 + 28 + 31 + 30 + 31 + 10 = 161 days.

Therefore, total number of days from May 10, 1972 to June 10, 1979 is 235 + 1095 + 366 + 730 + 161 = 2587 days.

Now, $2587 \equiv 4 \pmod 7$

⇨ May 10, 1972 was a Wednesday. (Sunday – 4)

Option (a) is correct.

14. A man started from home at 14:30 hours and drove to a village, arriving there when the village clock indicated 15:15 hours. After staying for 25 minutes (min), he drove back by a different route of length (5/4) times the first route at a rate twice as fast, reaching home at 16:00 hours. As compared to the clock at home, the village clock is
(a) 10 min slow
(b) 5 min slow
(c)5 min fast
(d) 20 min fast

Solution :

Let the distance from home to the village is x and he drove to the village by v speed.

Therefore, time taken to reach village is x/v.

Now, time taken to come back home = (5x/4)/2v = 5x/8v

Total time = x/v + 5x/8v + 25 = (13/8)(x/v) + 25

⇨ (13/8)(x/v) + 25 = (16:00 – 14:30)*60 = 90

- ⇨ (x/v) = 65*8/13 = 40

So, he should reach village at 14:30 + 40 min = 15:10 hours.

Therefore, the village clock is (15:15 – 15:10) = 5 min fast.

Option (c) is correct.

15. If (a + b)/(b + c) = (c + d)/(d + a), then
    (a) a = c
    (b) either a = c or a + b + c + d = 0
    (c) a + b + c + d = 0
    (d) a = c and b = d

Solution :

(a + b)/(b + c) = (c + d)/(d + a)

- ⇨ (a + b)/(b + c) – 1 = (c + d)/(d + a) – 1
- ⇨ (a – c)/(b + c) = - (a – c)/(d + a)
- ⇨ (a – c)/(b + c) + (a – c)/(d + a) = 0
- ⇨ (a – c){1/(b + c) + 1/(d + a)} = 0
- ⇨ (a – c)(a + b + c + d)/{(b + c)(d + a)} = 0
- ⇨ (a – c)(a + b + c + d) = 0
- ⇨ Either a = c or a + b + c + d = 0

Option (b) is correct.

16. The expression $(1 + q)(1 + q^2)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$, $q \neq 1$, equals
    (a) $(1 – q^{128})/(1 – q)$
    (b) $(1 – q^{64})/(1 – q)$
    (c) $\{1 – q\hat{}(2^{1 + 2 + \ldots + 6})\}/(1 – q)$
    (d) None of the foregoing expressions.

Solution :

Let $E = (1 + q)(1 + q^2)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$

- ⇨ $(1 – q)*E – (1 – q)(1 + q)(1 + q^2)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$ $(q \neq 1)$
- ⇨ $(1 – q)*E = (1 – q^2)(1 + q)(1 + q^2)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$
- ⇨ $(1 – q)*E = (1 – q^4)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$

- ⇨ $(1 - q)*E = (1 - q^8)(1 + q^8)(1 + q1^6)(1 + q^{32})(1 + q^{64})$
- ⇨ $(1 - q)*E = (1 - q^{16})(1 + q^{16})(1 + q^{32})1 + q^{64})$
- ⇨ $(1 - q)*E = (1 - q^{32})(1 + q^{32})(1 + q^{64})$
- ⇨ $(1 - q)*E = (1 - q^{64})(1 + q^{64})$
- ⇨ $(1 - q)*E = (1 - q^{128})$
- ⇨ $E = (1 - q^{128})/(1 - q)$ $(q \neq 1)$

Option (a) is correct.

17. In an election 10% of the voters on the voters' list did not cast their votes and 60 voters cast their ballot paper blank. There were only two candidates. The winner was supported by 47% of all voters in the list and he got 308 votes more than his rival. The number of voters on the list was
    (a) 3600
    (b) 6200
    (c)4575
    (d) 6028

Solution :

Let number of voters on the voters' list is x.

Did not cast their vote = 10x/100

Therefore, total number of voters who cast their vote in favor of a candidate = x – 10x/100 – 60 = 90x/100 – 60

Winner got 47x/100 votes.

Rival got 90x/100 – 60 – 47x/100 = 43x/100 – 60

According to question, 47x/100 – (43x/100 – 60) = 308

- ⇨ 4x/100 + 60 = 308
- ⇨ x/25 = 308 – 60
- ⇨ x/25 = 248
- ⇨ x = 248*25
- ⇨ x = 6200

Option (b) is correct.

18. A student took five papers in an examination, where the full marks were the same for each paper. His marks in these papers were in the proportion of 6 : 7 : 8 : 9 : 10. He obtained (3/5) part

of the total full marks. Then the number of papers in which he got more than 50% marks is
(a) 2
(b) 3
(c)4
(d) 5

Solution :

Let the full mark of each paper is x.

Total full marks = 5x

He got total 5x*(3/5) = 3x marks

In first paper he got 3x*6/(6 + 7 + 8 + 9 + 10) = 9x/20

In second paper he got 3x*7/(6 + 7 + 8 + 9 + 10) = 21x/40

In third paper he got 3x*8/(6 + 7 + 8 + 9 + 10) = 3x/5

In fourth paper he got 3x*9/(6 + 7 + 8 + 9 + 10) = 27x/40

In fifth paper he got 3x*10/(6 + 7 + 8 + 9 + 10) = 3x/4

In first paper percentage of marks = (9x/20)*100/x = 45%

In second paper percentage of marks = (21x/40)*100/x = 52.5%

⇨ He got more than 50% in 4 papers.

Option (c) is correct.

19. Two contestants run in a 3-kilometre race along a circular course of length 300 metres. If their speeds are in the ratio of 4 : 3, how often and where would the winner pass the other? (The initial start-off is not counted as passing.)
(a) 4 times, at the starting point
(b) Twice, at the starting point
(c)Twice, at a distance 225 metres from the starting point
(d) Twice, once at 75 metres and again at 225 metres from the starting point.

Solution :

Let they meet at x metres from the starting point.

Let first runner loops n times and second runner loops m times when they meet for the first time.

Therefore, $(300n + x)/4 = (300m + x)/3$

- ⇨ $900n + 3x = 1200m + 4x$
- ⇨ $x = 900n - 1200m$
- ⇨ $x = 300(3n - 4m)$

Let $x = 225$, then $300(3n - 4m) = 225$

- ⇨ $4(3n - 4m) = 3$ which is impossible as $3n - 4m$ is an integer.

Let, $x = 75$, then $300(3n - 4m) = 75$

- ⇨ $4(3n - 4m) = 1$ which is impossible as $3n - 4m$ is an integer.

Therefore, none of the options (c), (d) are correct.

So, they will meet at starting point.

- ⇨ $x = 0$
- ⇨ $3n = 4m$

Minimum value of n is 4 and m is 3.

Total there are 3000/300 = 10 loops.

Therefore, they will meet for twice as 12^{th} loop is not in course.

Option (b) is correct.

20. If a, b, c and d satisfy the equations $a + 7b + 3c + 5d = 0$, $8a + 4b + 6c + 2d = -16$, $2a + 6b + 4c + 8d = 16$, $5a + 3b + 7c + d = -16$, then $(a + d)(b + c)$ equals
    (a) 16
    (b) -16
    (c)0
    (d) None of the foregoing numbers.

Solution :

$a + 7b + 3c + 5d = 0$ ………… (1)

$8a + 4b + 6c + 2d = -16$ ……. (2)

$2a + 6b + 4c + 8d = 16$ ……… (3)

$5a + 3b + 7c + d = -16$ ………. (4)

Adding equations (2) and (3) we get, $10a + 10b + 10c + 10d = 0$

- ⇨ $a + b + c + d = 0$
- ⇨ $(a + d) = -(b + c)$ …………… (5)

Adding equations (1) and (4) we get, $6a + 10b + 10c + 6d = -16$

- ⇨ $6(a + d) + 10(b + c) = -16$
- ⇨ $-6(b + c) + 10(b + c) = -16$ (from (5))
- ⇨ $4(b + c) = -16$
- ⇨ $(b + c) = -4$
- ⇨ $(a + d) = 4$
- ⇨ $(a + d)(b + c) = 4*(-4) = -16$

Option (b) is correct.

21. Suppose x and y are positive integers, $x > y$, and $3x + 2y$ and $2x + 3y$ when divided by 5, leave remainders 2 and 3 respectively. It follows that when $x - y$ is divided by 5, the remainder necessarily equals
    (a) 2
    (b) 1
    (c) 4
    (d) None of the foregoing numbers.

Solution :

$3x + 2y \equiv 2 \pmod 5$

$2x + 3y \equiv 3 \pmod 5$

- ⇨ $(3x + 2y) - (2x + 3y) \equiv 2 - 3 \pmod 5$
- ⇨ $(x - y) \equiv -1 \pmod 5$
- ⇨ $(x - y) \equiv 4 \pmod 5$

Option (c) is correct.

22. The number of different solutions (x, y, z) of the equation $x + y + z = 10$, where each of x, y and z is a positive integer, is
    (a) 36
    (b) 121
    (c) $10^3 - 10$
    (d) ${}^{10}C_3 - {}^{10}C_2$.

Solution :

Clearly it is $^{10-1}C_{3-1} = {}^9C_2 = 36$ (For reference please see Number Theory book)

Option (a) is correct.

23. The hands of a clock are observed continuously from 12:45 p.m. onwards. They will be observed to point in the same direction some time between
    (a) 1:03 p.m. and 1:04 p.m.
    (b) 1:04 p.m. and 1:05 p.m.
    (c) 1:05 p.m. and 1:06 p.m.
    (d) 1:06 p.m. and 1:07 p.m.

Solution :

Clearly, option (a) and (b) cannot be true.

The hour hand moves $2\pi/12$ angle in 60 minutes

The hour hand moves $2\pi/60$ angle in $60*(2\pi/60)/(2\pi/12) = 12$ minutes.

So, option (d) cannot be true as it takes 12 minutes to move to 1:06 for hour hand.

Option (c) is correct.

24. A, B and C are three commodities. A packet containing 5 pieces of A, 3 of B and 7 of C costs Rs. 24. 50. A packet containing 2, 1 and 3 of A, B and C respectively costs Rs. 17.00. The cost of packet containing 16, 9 and 23 items of A, B and C respectively
    (a) is Rs. 55.00
    (b) is Rs. 75.50
    (c) is Rs. 100.00
    (d) cannot be determined from the given information.

Solution :

Clearly, 2*first packet + 3* second packet gives the answer.

Therefore, required cost = 2*24.50 + 3*17 = Rs. 100

Option (c) is correct.

25. Four statements are given below regarding elements and subsets of the set {1, 2, {1, 2, 3}}. Only one of them is correct. Which one is it?
(a) {1, 2} ε {1, 2, {1, 2, 3}}
(b) {1, 2} is proper subset of {1, 2, {1, 2, 3}}
(c){1, 2, 3} is proper subset of {1. 2, {1, 2, 3}}
(d) 3 ε {1, 2, {1, 2, 3}}

Solution :

{1, 2} is not an element of {1, 2, {1, 2, 3}}. So option (a) cannot be true.

3 is not an element of {1, 2, {1, 2, 3}}. So option (d) cannot be true. (Elements are 1, 2, {1, 2, 3})

{1, 2, 3} is an element of {1, 2, {1, 2, 3}} not a subset, rather {{1, 2, 3}} is a subset containing the element {1, 2, 3}. So, (c) cannot be true.

Option (b) is correct.

26. A collection of non-empty subsets of the set {1, 2, ….., n} is called a *simplex* if, whenever a subset S is included in the collection, any non-empty subset T of S is also included in the collection. Only one of the following collections of subsets of {1, 2, …., n} is a simplex. Which one is it?
(a) The collection of all subsets S with the property that 1 belongs to S.
(b) The collection of all subsets having exactly 4 elements.
(c)The collection of all non-empty subsets which do not contain any even number.
(d) The collection of all non-empty subsets except for the subset {1}.

Solution :

Option (a) cannot be true as {2} is not included in the collection of subsets which is subset of the subset {1, 2}.

Option (b) cannot be true as 3 elements subset are not included in the collection of subsets.

Option (d) cannot be true as {1} is not included which is subset of the subset {1, 2}.

Option (c) is correct.

27. S is the set whose elements are zero and all even integers, positive and negative. Consider the five operations : [1] addition; [2] subtraction; [3] multiplication; [4] division; and [5] finding the arithmetic mean. Which of these operations when applied to any pair of elements of S, yield only elements of S?
(a) [1], [2], [3], [4]
(b) [1], [2], [3], [5]
(c)[1], [3], [5]
(d) [1], [2], [3]

Solution :

If two even integers are added then an even integer is generated. So [1] is true.

If two even integers are subtracted then an even integer is generated. So [2] is true.

If two even integers are multiplied then an even integer is generated. So [3] is true.

If 6/4 then the generated integer doesn't belong to S. So, [4] cannot be true.

(6 + 4)/2 is an odd integer and doesn't belong to S. So, [5] cannot be true.

Option (d) is correct.

Directions for items 28 to 36 :

For sets P, Q of numbers, define

PUQ : the set of all numbers which belong to at least one of P and Q;

P∩Q : the set of all numbers which belong to both P and Q;

P – Q : the set of all numbers which belong to P but not to Q;

PΔQ = (P – Q)U(Q – P) : the set of all numbers which belong to set P alone or set Q alone, but not to both at the same time. For example, if P = {1, 2, 3}, Q = {2, 3, 4} then PUQ = {1, 2, 3, 4}, P∩Q = {2, 3}, P – Q = {1}, PΔQ = {1, 4}.

28. If $X = \{1, 2, 3, 4\}$, $Y = \{2, 3, 5, 7\}$, $Z = \{3, 6, 8, 9\}$, $W = \{2, 4, 8, 10\}$, then $(X\Delta Y)\Delta(Z\Delta W)$ is
    (a) $\{4, 8\}$
    (b) $\{1, 5, 6, 10\}$
    (c) $\{1, 2, 3, 5, 6, 7, 9, 10\}$
    (d) None of the foregoing sets.

Solution :

$(X\Delta Y) = \{1, 4, 5, 7\}$

$(Z\Delta W) = \{2, 3, 4, 6, 9, 10\}$

$(X\Delta Y)\Delta(Z\Delta W) = \{1, 2, 3, 5, 6, 7, 9, 10\}$

Option (c) is correct.

29. If X, Y, Z are any three sets of numbers, then the set of all numbers which belong to exactly two of the sets X, Y, Z is
    (a) $(X\cap Y)\cup(Y\cap Z)\cup(Z\cap X)$
    (b) $[(X\cup Y)\cup Z] - [(X\Delta Y)\Delta Z]$
    (c) $(X\Delta Y)\cup(Y\Delta Z)\cup(Z\Delta X)$
    (d) Not necessarily any of (a) to (c).

Solution :

Option (b) is correct. It can be easily verified by Venn diagram.

30. For any three sets P, Q and R s is an element of $(P\Delta Q)\Delta R$ if s is in
    (a) Exactly one of P, Q and R
    (b) At least one of P, Q and R, but not in all three of them at the same time
    (c) Exactly one of P, Q and R
    (d) Exactly one P, Q and R or all the three of them.

Solution :

Option (d) is correct. It can be easily verified by Venn diagram.

31. Let $X = \{1, 2, 3, \ldots, 10\}$ and $P = \{1, 2, 3, 4, 5\}$. The number of subsets Q of X such that $P\Delta Q = \{3\}$ is
(a) $2^4 - 1$
(b) 2^4
(c) 2^5
(d) 1.

Solution :

The only subset $Q = \{1, 2, 4, 5\}$ then $P\Delta Q = \{3\}$

Option (d) is correct.

32. For each positive integer n, consider the set $P_n = \{1, 2, 3, \ldots, n\}$. Let $Q_1 = P_1$, $Q_2 = P_2\Delta Q_1 = \{2\}$, and in general $Q_{n+1} = P_{n+1}\Delta Q_n$, for $n \geq 1$. Then the number of elements in Q_{2k} is
(a) 1
(b) $2k - 2$
(c) $2k - 3$
(d) k

Solution :

$Q_3 = \{1, 3\}$

$Q_4 = \{2, 4\}$

$Q_5 = \{1, 3, 5\}$

$Q_6 = \{2, 4, 6\}$

$Q_7 = \{1, 3, 5, 7\}$

$Q_8 = \{2, 4, 6, 8\}$

Clearly, option (d) is correct.

33. For any two sets S and T, $S\Delta T$ is defined as the set of all elements that belong to either S or T but not both, that is, $S\Delta T = (S \cup T) - (S \cap T)$. Let A, B and C be sets such that $A \cap B \cap C = \Phi$, and the number of elements in each of $A\Delta B$, $B\Delta C$ and $C\Delta A$ equals 100. Then the number of elements in $A \cup B \cup C$ equals
(a) 150
(b) 300
(c) 230

(d) 210

Solution :

$A \cup B \cup C = A \cup B + C - (A \cup B) \cap C$

$= A + B - A \cap B + C - [A + B - (A \cap B)] \cap C$

$= A + B + C - A \cap B - A \cap C - B \cap C + A \cap B \cap C$

$= A + B + C - A \cap B - B \cap C - C \cap A$

$= (1/2)[2A + 2B + 2C - 2A \cap B - 2B \cap C - 2C \cap A]$

$= (1/2)[(A + B - A \cap B - A \cap B) + (B + C - B \cap C - B \cap C) + (C + A - C \cap A - C \cap A)]$

$= (1/2)[(A \cup B - A \cap B) + (B \cup C - B \cap C) + (C \cup A - C \cap A)]$

$= (1/2)[A \Delta B + B \Delta C + C \Delta A]$

$= (1/2)(100 + 100 + 100)$

$= 150$

Option (a) is correct.

34. Let A, B, C and D be finite sets such that $|A| < |C|$ and $|B| = |D|$, where $|A|$ stands for the number of elements in the set A. Then
(a) $|A \cup B| < |C \cup D|$
(b) $|A \cup B| \leq |C \cup D|$ but $|A \cup B| < |C \cup D|$ need not always be true
(c) $|A \cup B| < 2|C \cup D|$ but $|A \cup B| \leq |C \cup D|$ need not always be true
(d) None of the foregoing statements is true.

Solution :

It is given option (c) is correct but I think option (d) is correct.

Let us take an example.

A is inside B and C and D are disjoint.

Then $|A \cup B| = |B|$ and $|C \cup D| = |C| + |D| = |C| + |B|$

⇨ $|C \cup D| > |A \cup B|$

35. The subsets A and B of a set X, define A*B as
$A*B = (A\cap B)\cup((X - A)\cap(X - B))$.
Then only one of the following statements is true. Which one is it?
(a) $A*(X - B)$ is a subset of $A*B$ and $A*(X - A) \neq A*B$
(b) $A*B = A*(X - B)$
(c) $A*B$ is a subset of $A*(X - B)$ and $A*B \neq A*(X - B)$
(d) $X - (A*B) = A*(X - B)$

Solution :

Option (d) is correct. It can be easily verified by Venn diagram.

36. Suppose that A, B and C are sets satisfying $(A - B)\Delta(B - C) = A\Delta B$. Which of the following statements must be true?
(a) $A = C$
(b) $A\cap B = B\cap C$
(c) $A\cup B = B\cup C$
(d) None of the foregoing statements necessarily follows.

Solution :

Option (b) is correct. This can be easily verified by Venn diagram.

Directions for items 37 to 39 :

A word is a finite string of the two symbols α and β. (An empty string, that is, a string containing no symbols at all, is also considered a word.) Any collection of words is called a language. If P and Q are words, then P.Q is meant the word formed by first writing the string of symbols in P and then following it by that of Q. For example P = αβαα and Q = ββ are words and P.Q = αβααββ. For the languages L_1 and L_2, $L_1.L_2$ denotes the language consisting of all words of the form P.Q with the word P coming from L_1 and Q coming from L_2. We also use abbreviations like α^3 for the word ααα, $\alpha\beta^3\alpha^2$ for αβββαα, $(\alpha^2\beta\alpha)^2$ for $\alpha^2\beta\alpha\alpha^2\beta\alpha$ (= $\alpha^2\beta\alpha^3\beta\alpha$) and α^0 or β^0 for the empty word.

37. If $L_1 = \{\alpha^n : n = 0, 1, 2, \ldots\}$ and $L_2 = \{\beta^n : n = 0, 1, 2, \ldots\}$, then $L_1.L_2$ is
(a) $L_1\cup L_2$
(b) The language consisting of all words
(c) $\{\alpha^n\beta^m : n = 0, 1, 2, \ldots. m = 0, 1, 2, \ldots\}$

(d) $\{\alpha^n\beta^n : n = 0, 1, 2, \ldots\}$

Solution :

Clearly option (c) is correct.

38. Suppose L is a language which contains the empty word and has the property that whenever P is in L, the word α.P.β is also in L. The smallest such L is
(a) $\{\alpha^n\beta^m : n = 0, 1, 2, \ldots\ m = 0, 1, 2, \ldots\}$
(b) $\{\alpha^n\beta^n : n = 0, 1, 2, \ldots\}$
(c) $\{(\alpha\beta)^n : n = 0, 1, 2, \ldots\}$
(d) The language consisting of all possible words.

Solution :

Option (a) is also true but (b) is smallest.

Therefore, option (b) is correct.

39. Suppose L is a language which contains the empty word, the word α and the word β, and has the property that whenever P and Q are in L, the word P.Q is also in L. The smallest such L is
(a) The language consisting of all possible words.
(b) $\{\alpha^n\beta^n : n = 0, 1, 2, \ldots\}$
(c) The language containing precisely the words of the form $(\alpha^{\wedge}n_1)(\beta^{\wedge}n_1)(\alpha^{\wedge}n_2)(\beta^{\wedge}n_2)\ldots(\alpha^{\wedge}n_k)(\beta^{\wedge}n_k)$
Where k is any positive integer and $n_1, n_2, \ldots, n_k$ are nonnegative integers
(d) None of the foregoing languages.

Solution :

Option (b) and (c) cannot be true as α, β are the words of the language.

Option (a) is correct.

40. A relation denoted by <- is defined as follows : For real numbers x, y, z and w, say that "(x, y) <- (z, w)" is either (i) x < z or (ii) x = z and y > w. If (x, y) <- (z, w) and (z, w) <- (r, s) then which one of the following is always true?

(a) (y, x) <- (r, s)
(b) (y, x) <- (s, r)
(c)(x, y) <- (s, r)
(d) (x, y) <- (r, s)

Solution :

(x, y) <- (z, w)

- ⇨ $x \leq z$ and $y > w$

(z, w) <- (r, s)

- ⇨ $z \leq r$ and $w > s$
- ⇨ $x \leq r$ and $y > s$
- ⇨ (x, y) <- (r, s)

Option (d) is correct.

41. A subset W of all real numbers is called a *ring* if the following two conditions are satisfied :
    (i) 1 ε W and
    (ii) If a, b ε W then a – b ε W and ab ε W.

Let $S = \{m/2^n \mid m$ and n are integers$\}$ and $T = \{p/q \mid p$ and q are integers and q is odd$\}$

Then

(a) Neither S nor T is a ring
(b) S is a ring and T is not
(c)T is a ring and S is not
(d) Both S and T are rings.

Solution :

If m = 2 and n = 1 then 1 ε S.

Now, $e/2^x - f/2^y = (e*2^y - f*2^x)/2^{x+y}$ ε S (if $e*2^y - f*2^x$ is negative then also it is ok as m is integer, so positive and negative both)

Now, $(e/2^x)*(f/2^y) = ef/2^{x+y}$ ε S

So, S is a ring.

If p = q then 1 ε S.

Now, $p_1/q_1 - p_2/q_2 = (p_1q_2 - p_2q_1)/q_1q_2$ ε T as q_1q_2 = odd because q_1 and q_2 both odd.

Now, $(p_1/q_1)*(p_2/q_2) = p_1p_2/(q_1q_2)$ ε T

So, T is a ring.

Option (d) is correct.

42. For a real number a, define $a^+ = \max\{a, 0\}$. For example, $2^+ = 2$, $(-3)^+ = 0$. Then, for two real numbers a and b, the equality $(ab)^+ = (a^+)(b^+)$ holds if and only if
(a) Both a and b are positive.
(b) a and b have the same sign
(c) a = b = 0
(d) at least one of a and b is greater than or equal to 0.

Solution :

Clearly, if a and b are both negative then $(ab)^+ = ab$ and $a^+ = 0$, $b^+ = 0$ and the equality doesn't hold.

If one of a and b are negative then $(ab)^+ = 0$ and either of a^+ or $b^+ = 0$ (whichever is negative) and the equality holds.

If both are positive then the equality holds.

So, option (d) is correct.

43. For any real number x, let [x] denote the largest integer less than or equal to x and <x> = x – [x], that is, the fractional part of x. For arbitrary real numbers x, y and z, only one of the following statements is correct. Which one is it?
(a) [x + y + z] = [x] + [y] + [z]
(b) [x + y + z] = [x + y] + [z] = [x] + [y + z] = [x + z] + [y]
(c) <x + y + z> = y + z – [y + z] + <x>
(d) [x + y + z] = [x + y] + [z + <y + x>].

Solution :

Let x = 2.4, y = 2.5 and z = 2.6

Then [x + y + z] = 7 and [x] + [y] + [z] = 6. So option (a) is not true.

Let, [x + y] = 4 and [z] = 2. So option (b) cannot be true.

$<x + y + z> = 0.5$ but $<y + z> + <x> = 0.9 + 0.6 = 1.5$. So, option (c) cannot be true.

Option (d) is correct.

44. Suppose that $x_1, x_2, \ldots, x_n$ $(n > 2)$ are real numbers such that $x_i = x_{n-i+1}$ for $1 \le i \le n$. Consider the sum $S = \sum\sum\sum x_i x_j x_k$, where summations are taken over all $i, j, k : 1 \le i, j, k \le n$ and i, j, k are all distinct. Then S equals
 (a) $n!x_1x_2\ldots x_n$
 (b) $(n - 3)(n - 4)$
 (c) $(n - 3)(n - 4)(n - 5)$
 (d) None of the foregoing expressions.

Solution :

$S = \sum\sum x_i x_j(P - x_i - x_j)$ $i \neq j$ and $P = x_1 + x_2 + \ldots + x_n$

$= P\sum\sum x_i x_j - \sum\sum x_i^2 x_j - \sum\sum x_i x_j^2$

$= P\sum x_i(P - x_i) - \sum x_i^2(P - x_i) - \sum x_i(Q - x_i^2)$ where $Q = x_1^2 + x_2^2 + \ldots + x_n^2$

$= P^3 - 3PQ + 2\sum x_i^3$

If n is even then $P = \sum x_i^3 = 0$

Therefore, $S = 0$

If n is odd, then $P = x_{(n+1)/2}$ and $\sum x_i^3 = (x_{(n+1)/2})^3$

Therefore, $S = (x_{(n+1)/2})^3 - 3x_{(n+1)/2}Q + 2(x_{(n+1)/2})^3 = 3x_{(n+1)/2}\{(x_{(n+1)/2})^2 - Q\}$

Clearly it doesn't match with any of the expressions in (a), (b), (c)

Option (d) is correct.

45. By an *upper bound* for a set A of real numbers, we mean any real number x such that every number a in A is smaller than or equal to x. If x is an upper bound for a set A and no number is strictly smaller than x is an upper bound for a, then x is called sup A.
 Let A and B be two sets of real numbers with x = sup A and y = sup B. Let C be the set of all real numbers of the form a + b where a is in A and b is in B. If z = sup C, then
 (a) $z > x + y$
 (b) $z < x + y$

(c) $z = x + y$
(d) nothing can be said in general about the relation between x, y and z.

Solution :

x = sup A means all the elements of a are equal and equal to x i.e. if a is in x then a = x

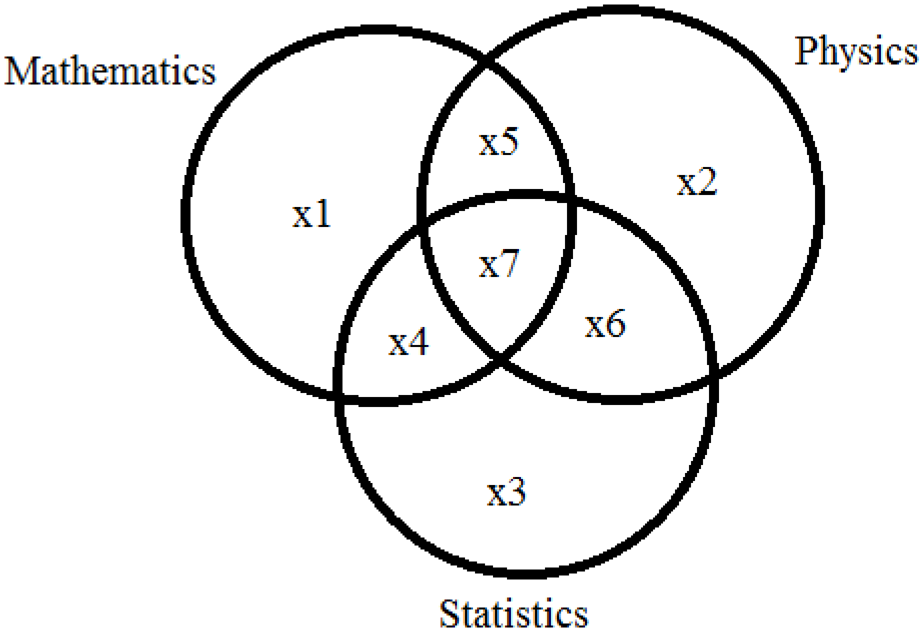
Similarly, if b is in B then b = y

Similarly, z = a + b = x + y

Option (c) is correct.

46. There are 100 students in a class. In an examination, 50 students of them failed in Mathematics, 45 failed in Physics and 40 failed in Statistics, and 32 failed in exactly two of these three subjects. Only one student passed in all the three subjects. The number of students failing all the three subjects
(a) is 12
(b) is 4
(c) is 2
(d) cannot be determined from the given information.

Solution :



Now, $x_1 + x_4 + x_5 + x_7 = 50$ ……….. (1)

$x_2 + x_5 + x_6 + x_7 = 45$ ……………. (2)

$x_3 + x_4 + x_6 + x_7 = 40$ ……………….. (3)

$x_4 + x_5 + x_6 = 32$ ………………………. (4)

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + 1 = 100$

⇨ $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 99$ …………… (5)

Doing (5) – (4) we get, $x_1 + x_2 + x_3 = 67 - x_7$ ………… (6)

Adding (1), (2), (3) we get, $(x_1 + x_2 + x_3) + 2(x_4 + x_5 + x_6) + 3x_7 = 135$

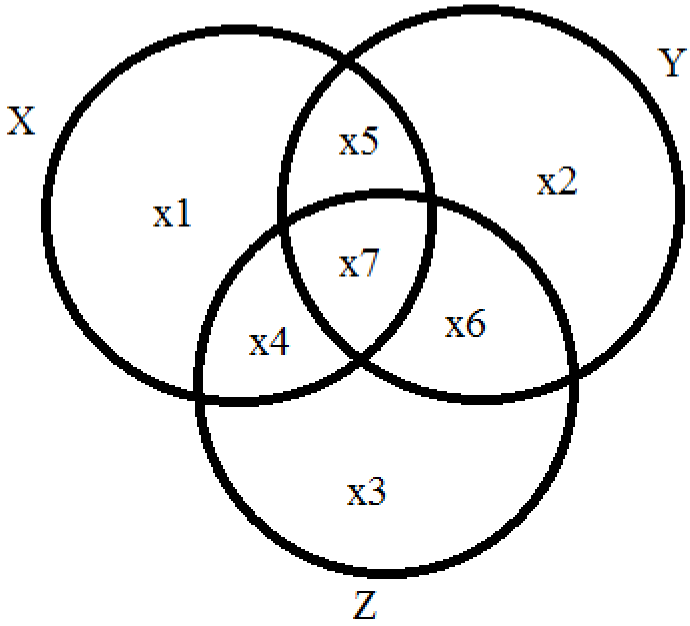
⇨ $67 - x_7 + 2*32 + 3x_7 = 135$ (from (4) and (6))
⇨ $2x_7 = 4$
⇨ $x_7 = 2$

Option (c) is correct.

47. A television station telecasts three types of programs X, Y and Z. A survey gives following data on television viewing. Among the people interviewed 60% watch program X, 50% watch program Y, 50% watch program Z, 30% watch programs X and Y, 20% watch programs Y and Z, 30% watch programs X and Z while 10% do not watch any television program. The percentage of people watching all the three programs X, Y and Z is

(a) 90
(b) 50
(c) 10
(d) 20

Solution :



Now, $x_1 + x_4 + x_5 + x_7 = 60\%$ .... (1)

$x_2 + x_5 + x_6 + x_7 = 50\%$ .... (2)

$x_3 + x_4 + x_6 + x_7 = 50\%$ .... (3)

$x_5 + x_7 = 30\%$ .... (4)

$x_6 + x_7 = 20\%$ .... (5)

$x_4 + x_7 = 30\%$ .... (6)

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + 10\% = 100\%$

⇨ $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 90\%$ .... (7)

Adding (4), (5), (6) we get, $x_4 + x_5 + x_6 = 80\% - 3x_7$ .... (8)

Adding (1), (2), (3) we get, $(x_1 + x_2 + x_3) + 2(x_4 + x_5 + x_6) + 3x_7 = 160\%$

⇨ $(x_1 + x_2 + x_3) = 160\% - 2(80\% - 3x_7) - 3x_7$
⇨ $(x_1 + x_2 + x_3) = 3x_7$ .... (9)

Putting value of (8) and (9) in (7) we get, $3x_7 + 80\% - 3x_7 + x_7 = 90\%$

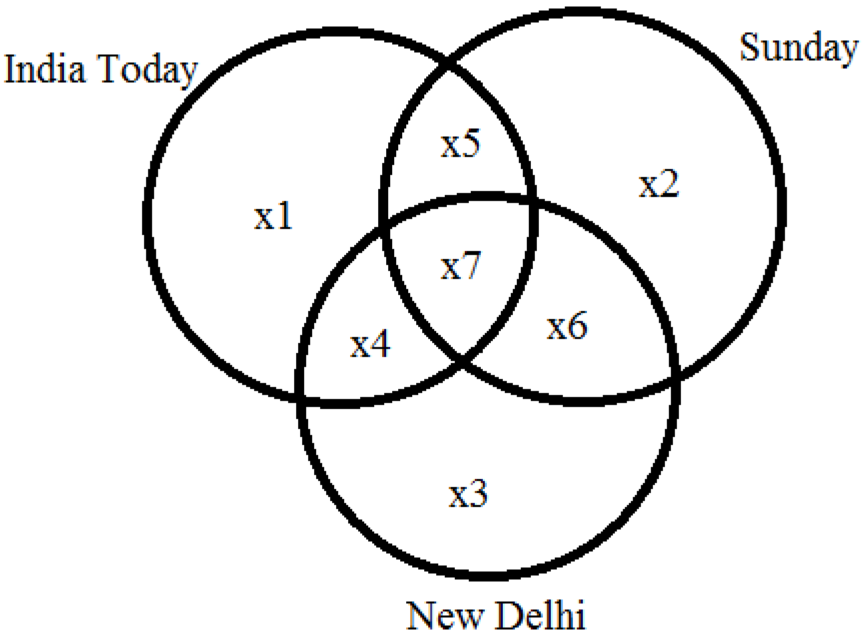
⇨ $x_7 = 10\%$

Option (c) is correct.

48. In a survey of 100 families, the number of families that read the most recent issues of various magazines was found to be : *India Today* 42, *Sunday* 30, *New Delhi* 28, *India Today* and *Sunday* 10, *India Today* and *New Delhi* 5, *Sunday* and *New Delhi* 8, all three magazines 3. Then the number of families that read none of the three magazines is

(a) 30
(b) 26
(c) 23
(d) 20

Solution :



Let, a number of families read none of the three magazines.

Now, $x_1 + x_4 + x_5 + x_7 = 42$ ……….. (1)

$x_2 + x_5 + x_6 + x_7 = 30$ …………. (2)

$x_3 + x_4 + x_6 + x_7 = 28$ ………… (3)

$x_5 + x_7 = 10$ ……….. (4)

$x_4 + x_7 = 5$ ……….. (5)

$x_6 + x_7 = 8$ …………. (6)

$x_7 = 3$

And, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + a = 100$ …….. (7)

Adding (4), (5), (6) we get, $x_4 + x_5 + x_6 + 3x_7 = 23$

⇨ $(x_4 + x_5 + x_6) = 14$ ($x_7 = 3$) …………. (8)

Adding (1), (2), (3) we get, $(x_1 + x_2 + x_3) + 2(x_4 + x_5 + x_6) + 3x_7 = 100$

⇨ $(x_1 + x_2 + x_3) = 63$ (from (8) and $x_7 = 3$) ……………. (9)

Putting the value from (8), (9) and $x_7 = 3$ in (7) we get, $63 + 14 + 3 + a = 100$

⇨ $a = 20$

Option (d) is correct.

49. In a survey of 100 families, the number of families that read the most recent issues of various magazines was found to be : *India Today* 42, *Sunday* 30, *New Delhi* 28, *India Today* and *Sunday* 10, *India Today* and *New Delhi* 5, *Sunday* and *New Delhi* 8, all three magazines 3. Then the number of families that read either both or none of the two magazines *Sunday* and *India Today* is
(a) 48
(b) 38
(c) 72
(d) 58

Solution :

From the previous problem's figure we need to find $x_3 + x_5 + x_7 + a$.

From equation (4) $x_5 = 7$, from equation (5) $x_4 = 2$, from equation (6) $x_6 = 5$.

From equation (3), $x_3 = 18$

So, $x_3 + x_5 + x_7 + a = 18 + 7 + 3 + 20 = 48$.

Option (a) is correct.

50. In a village of 1000 inhabitants, there are three newspapers P, Q and R in circulation. Each of these papers is read by 500 persons. Papers P and Q are read by 250 persons, papers Q and R are read by 250 persons, papers R and P are read by 250 persons. All the three papers are read by 250 persons. Then the number of persons who read no newspaper at all

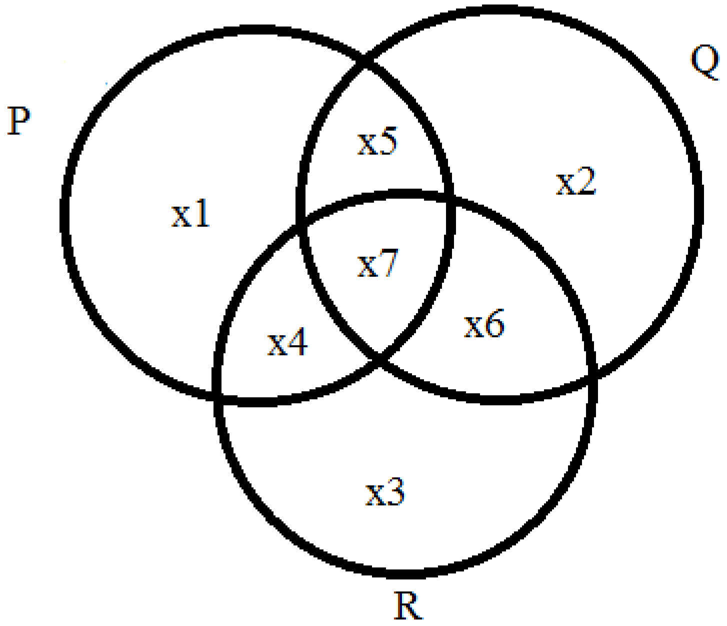
(a) is 500

(b) is 250

(c) is 0

(d) cannot be determined from the given information.

Solution :



Let, a number of people read no newspaper at all.

Now, $x_1 + x_4 + x_5 + x_7 = 500$

$x_2 + x_5 + x_6 + x_7 = 500$

$x_3 + x_4 + x_6 + x_7 = 500$

$x_5 + x_7 = 250$

$x_4 + x_7 = 250$

$x_6 + x_7 = 250$

$x_7 = 250$

And, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + a = 1000$

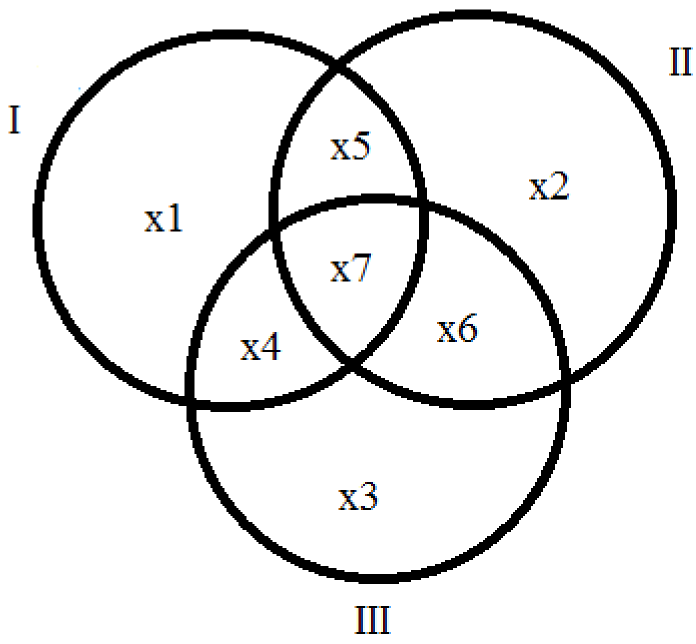
⇨ $x_4 = x_5 = x_6 = 0$
⇨ $x_1 = 250, x_2 = 250, x_3 = 250$
⇨ $a = 1000 - 250 - 250 - 250 - 0 - 0 - 0 - 250 = 0$

Option (c) is correct.

51. Sixty (60) students appeared in a test consisting of three papers I, II and III. Of these students, 25 passed in Paper I, 20 in Paper II and 8 in Paper III. Further, 42 students passed in at least one of Papers I and II, 30 in at least one of Papers I and III, 25 in at least one of Papers II and III. Only one student passed in all the three papers. Then the number of students who failed in all the papers is

(a) 15
(b) 17
(c) 45
(d) 33

Solution :



Let, the number of students who failed in all the three papers is a.

Now, $x_1 + x_4 + x_5 + x_7 = 25$ ……….. (1)

$x_2 + x_5 + x_6 + x_7 = 20$ ……………. (2)

$x_3 + x_4 + x_6 + x_7 = 8$ ………… (3)

$x_1 + x_2 + x_4 + x_5 + x_6 + x_7 = 42$ ……….. (4)

$x_1 + x_3 + x_4 + x_5 + x_6 + x_7 = 30$ ……… (5)

$x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 25$ ………… (6)

$x_7 = 1$

And, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + a = 60$ …………… (7)

(1) + (2) – (4) gives, $x_5 + x_7 = 3 \Rightarrow x_5 = 2$
(2) + (3) – (6) gives, $x_6 + x_7 = 3 \Rightarrow x_6 = 2$
(3) + (1) – (5) gives, $x_4 + x_7 = 3 \Rightarrow x_4 = 2$

From (1), $x_1 - x_7 + (x_4 + x_7) + (x_5 + x_7) = 25$

⇨ $x_1 - x_7 = 19 \Rightarrow x_1 = 20$

Similarly, from (2), $x_2 - x_7 = 14 \Rightarrow x_2 = 15$

And from (3), $x_3 - x_7 = 2 \Rightarrow x_3 = 3$

Putting all the values in (7) we get, $a = 60 - 20 - 15 - 3 - 2 - 2 - 2 - 1 = 15$

Option (a) is correct.

52. A student studying the weather for d days observed that (i) it rained on 7 days, morning or afternoon; (ii) when it rained in the afternoon, it was clear in the morning; (iii) there were five clear afternoons; and (iv) there were six clear mornings. Then d equals
(a) 7
(b) 11
(c) 10
(d) 9

Solution :

There were d – 5 days in which it rained in afternoons.

There were d – 6 days in which it rained in mornings.

No day it rained in morning and afternoon.

Therefore, $d - 5 + d - 6 = 11$

⇨ $d = 9$

Option (d) is correct.

53. A club with x members is organized into four committees according to the following rules :
(i) Each member belongs to exactly two committees.
(ii) Each pair of committees has exactly one member in common.

Then

(a) x = 4
(b) x = 6
(c) x = 8
(d) x cannot be determined from the given information.

Solution :

Let the groups be I, II, III, IV.

So, we need to find number of pair-wise combinations of the group.

I and II, I and III, I and IV, II and III, II and IV, III and IV

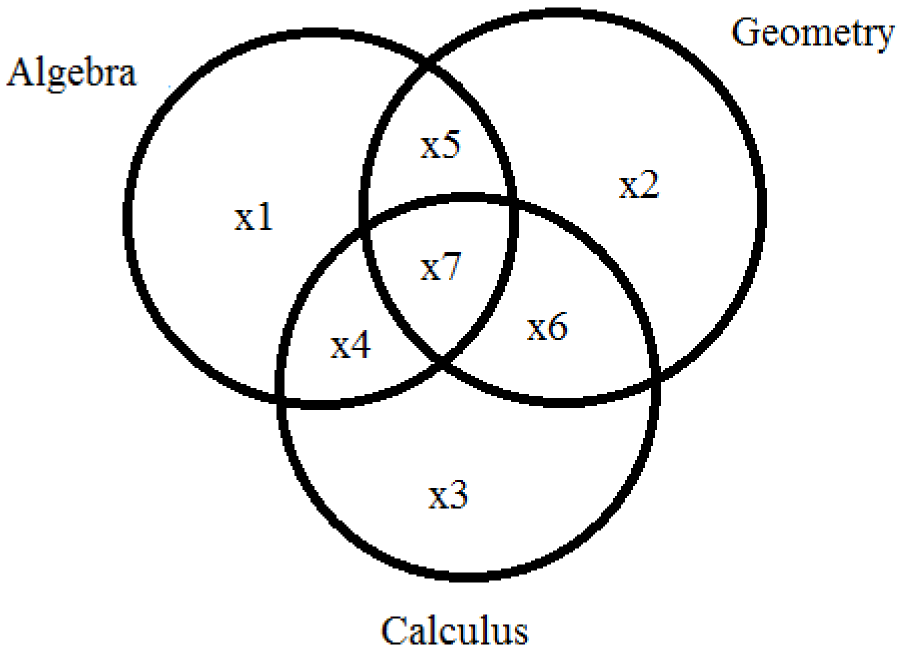
There are 6 pairs.

Therefore, x = 6

Option (b) is correct.

54. There were 41 candidates in an examination and each candidate was examined in Algebra, Geometry and Calculus. It was found that 12 candidates failed in Algebra, 7 failed in Geometry and 8 failed in Calculus, 2 in Geometry and Calculus, 3 in Calculus and Algebra, 6 in Algebra and Geometry, whereas only 1 failed in all three subjects. Then number of candidates who passed in all three subjects
(a) is 24
(b) is 2
(c) is 14
(d) cannot be determined from the given information.

Solution :



Let, number of candidates who passed in all three subjects is a.

Now, $x_1 + x_4 + x_5 + x_7 = 12$

$x_2 + x_5 + x_6 + x_7 = 7$

$x_3 + x_4 + x_6 + x_7 = 8$

$x_6 + x_7 = 2$

$x_4 + x_7 = 3$

$x_5 + x_7 = 6$

$x_7 = 1$

⇨ $x_6 = 1, x_4 = 2, x_5 = 5$ and $x_1 = 4, x_2 = 0, x_3 = 4$

And, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + a = 41$

⇨ $a = 41 - 4 - 0 - 4 - 2 - 5 - 1 - 1 = 24$

Option (a) is correct.

55. In a group of 120 persons there are 70 Bengalis, 35 Gujaratis. Further, 70 persons in the group and Muslims and the remaining Hindus. Then the number of Bengali Muslims in the group is
(a) 30 or more
(b) Exactly 20

(c) Between 15 and 25
(d) Between 20 and 25

Solution :

Let all the Gujaratis are Muslims.

Therefore, 30 Bengali Muslims are there.

Option (a) is correct.

56. In a group of 120 persons there are 70 Bengalis, 35 Gujaratis and 15 Maharashtrians. Further, 75 persons in the group are Muslims and remaining are Hindus. Then the number of Bengali Muslims in the group is
(a) Between 10 and 14
(b) Between 15 and 19
(c) Exactly 20
(d) 25 or more.

Solution :

Let, all the Gujaratis and Maharashtrians are Muslims. Then there are 25 Bengali Muslims.

Option (d) is correct.

57. Four passengers in a compartment of the Delhi-Howrah Rajdhani Express discover that they form an interesting group. Two are lawyers and two are doctors. Two of them speak Bengali and the other two Hindi and no two of the same profession speak the same language. They also discover that two of them are Christians and two Muslims, no two of the same religion are of the same profession and no two of the same religion speak the same language. The Hindi speaking doctor is a Christian. Then only one of the statements below logically follows. Which one is it?
(a) The Bengali-speaking lawyer is a Muslim.
(b) The Christian lawyer speaks Bengali.
(c) The Bengali-speaking doctor is a Christian.
(d) The Bengali-speaking doctor is a Hindu.

Solution :

Clearly, option (d) cannot be true because there is no one whose religion is Hindu.

Also clearly, the another Christian cannot be doctor. (As no two same religion have same profession)

- ⇨ Option (c) cannot be true.

Now, another Christian must speak Bengali and he is a lawyer.

Option (b) is correct.

58. In a football league, a particular team played 60 games in a season. The team never lost three games consecutively and never won five games consecutively in that season. If N is the number of games the team won in that season, then N satisfies
    (a) $24 \le N \le 50$
    (b) $20 \le N \le 48$
    (c) $12 \le N \le 40$
    (d) $18 \le N \le 42$

Solution :

Let, the team lost 2 games consecutively and won 1 game consecutively.

- ⇨ The team won 1 game in 3 games.
- ⇨ The team won (1/3)*60 = 20 games in 60 games.
- ⇨ $N \ge 20$

Let, the team won 4 games consecutively and lost 1 game consecutively.

- ⇨ The team won 4 games in 5 games.
- ⇨ The team won (4/5)*60 = 48 games in 60 games.
- ⇨ $N \le 48$
- ⇨ $20 \le N \le 48$

Option (b) is correct.

59. A box contains 100 balls of different colors : 28 red, 17 blue, 21 green, 10 white, 12 yellow, 12 black. The smallest number n such that any n balls drawn from the box will contain at least 15 balls of the same color, is
    (a) 73
    (b) 77
    (c) 81
    (d) 85

Solution :

Let us take the worst possible scenario.

All the white, yellow and black balls are selected and blue, green and red balls are selected 14 each. If we take 1 more ball then it must be from red, blue of green making any one color at least 15.

Therefore, n = 10 + 12 + 12 + 14*3 + 1 = 77

Option (b) is correct.

60. Let x, y, z, w be positive real numbers, which satisfy the two conditions that
   (i) If $x > y$ then $z > w$; and
   (ii) If $x > z$ then $y < w$.

   Then one of the statements given below is a valid conclusion. Which one is it?

   (a) If $x < y$ then $z < w$
   (b) If $x < z$ then $y > w$
   (c) If $x > y + z$ then $z < y$
   (d) If $x > y + z$ then $z > y$

Solution :

Option (a) and (b) cannot be true because there is no such statement that the vice versa will be true.

Option (c) cannot be true as if $x > y$ and $x > z$ then $x > y + z$ but $z > w > y$

So, option (d) is true.

61. Consider the statement : $x(\alpha - x) < y(\alpha - y)$ for all x, y with $0 < x < y < 1$. The statement is true
   (a) If and only if $\alpha \geq 2$
   (b) If and only if $\alpha > 2$
   (c) If and only if $\alpha < -1$
   (d) For no values of α.

Solution :

Now, $x(a - x) < y(a - y)$

⇨ $ax - x^2 < ay - y^2$
⇨ $y^2 - x^2 - ay + ax < 0$
⇨ $(y - x)(y + x) - a(y - x) < 0$
⇨ $(y - x)(y + x - a) < 0$

Now, $y - x > 0$

⇨ $y + x - a < 0$
⇨ $a > x + y$

Now, maximum value of $x + y$ is 2^-

Therefore, $a \geq 2$.

Option (a) is correct. **(For $a = 0.4$, $x = 0.1$, $y = 0.2$ the equation holds good)*

62. In a village, at least 50% of the people read a newspaper. Among those who read a newspaper at the most 25% read more than one paper. Only one of the following statements follows from the statements we have given. Which one is it?
(a) At the most 25% read exactly one newspaper.
(b) At least 25% real all the newspapers.
(c) At the most 37.5% read exactly one newspaper.
(d) At least 37.5% read exactly one newspaper.

Solution :

Let number of people in the village is x.

Let number of people who read newspaper is y.

Therefore, $y \geq 50x/100 = x/2$

Let, t number of people reads exactly one newspaper.

Therefore, $t \geq y - 25y/100 = 75y/100 \geq (75/100)*(x/2) = 37.5x/100 = 37.5\%$

Option (d) is correct.

63. We consider the relation "a person x shakes hand with a person y". Obviously, if x shakes hand with y, then y shakes hand with x. In a gathering of 99 persons, one of the following statements is *always* true, considering 0 to be an even number. Which one is it?

(a) There is at least one person who shakes hand exactly with an odd number of persons.
(b) There is at least one person who shakes hand exactly with an even number of persons.
(c) There are even number of persons who shake hand exactly with an even number of persons.
(d) None of the foregoing statements.

Solution :

Let there is one handshake with everybody. Then two people shakes hand will never shake hand with others. Therefore it makes pairs of people. 99 is an odd number. So, it is not possible.

Similarly, to do any odd number of handshakes between n number of people n must be even.

But 99 is odd.

Therefore, there will be always at least one people who will shake hand even number of times considering 0 as even number.

Option (b) is correct.

64. Let P, Q, R, S and T be statements such that if P is true then both Q and R are true, and if both R and S are true then T is false. We then have :
(a) If T is true then both P and R must be true.
(b) If T is true then both P and R must be false.
(c) If T is true then at least one of P and R must be true.
(d) If T is true then at least one of P and R must be false.

Solution :

If T is true then at least one of R and S is false.

If P is false then at least one of Q and R is false.

If R is false then P is false.

If R is true and Q is true then P can be true.

It is given option (d) as answer. *(But consider the case P, Q, R, T all true and S is false; no contradiction found)*

65. Let P, Q, R and S be four statements such that if P is true then Q is true, if Q is true then R is true and if S is true then at least one of Q and R is false. Then it follows that
(a) if S is false then both Q and R are true
(b) if at least one of Q and R is true then S is false
(c) if P is true then S is false
(d) if Q is true then S is true.

Solution :

Clearly, if S is false then Q and R both must be true.

Option (a) is correct.

Also, if P is true then Q and R both true. Implies S is false.

Option (c) is correct.

If Q is true then Q and R are both true then S cannot be true. So option (d) cannot be true.

Option (b) is clearly cannot be true.

So, (a), (c) are correct. But it is given answer (c) only.

66. If A, B, C and D are statements such that if at least one of A and B is true, then at least one of C and D must be true. Further, both A and C are false. Then
(a) if D is false then B is false
(b) both B and D are false
(c) both B and D are true
(d) if D is true then B is true.

Solution :

Clearly option (a) is correct as if D is false then C and D both are false.

⇨ A and B both are false. A is already false means B is also false.

Option (a) is correct.

67. P, Q and R are statements such that if P is true then at least one of the following is correct : (i) Q is true, (ii) R is not true. Then
(a) if both P and Q are true then R is true
(b) if both Q and R are true then P is true

(c) if both P and R are true then Q is true
(d) none of the foregoing statements is correct.

Solution :

If, P and R are true, then as P is true but R is also true so (ii) is not satisfied. Implies (i) must be satisfied. Implies Q is true.

Option (c) is correct.

68. It was a hot day and four couples drank together 44 bottles of cold drink. Anita had 2, Biva 3, Chanchala 4, and Dipti 5 bottles. Mr. Panikkar drank just as many bottles as his wife, but each of the other men drank more than his wife – Mr. Dube twice, Mr. Narayan thrice and Mr. Rao four times as many bottles. Then only one of the following is correct. Which one is it?
(a) Mrs. Panikkar is Chanchala.
(b) Anita's husband had 8 bottles.
(c) Mr. Narayan had 12 bottles.
(d) Mrs. Rao is Dipti.

Solution :

If Dipti is Mrs. Rao then Mr. Rao had 20 bottles.

So, Mr. Panikkar, Mr. Dube and Mr. Narayan had 44 – (20 + 2 + 3 + 4 + 5) = 10 bottles.

So, Mr. Narayan can have maximum 2*3 = 6 bottles. Mr. Panikkar can have maximum 3*2 = 6 bottles which crosses 10.

So, option (d) cannot be true.

If Mrs. Panikkar is Chanchala then Mr. Panikkar had 4 bottles.

So, Mr. Dube, Mr. Narayan and Mr. Rao had 44 – (2 + 3 + 4 + 5 + 4) = 26 bottles.

Dipti is not Mrs. Rao.

Therefore, Mr. Rao can have maximum 3*4 = 12 bottles.

Mr. Narayan can have maximum 5*3 = 15 bottles.

In that case answer doesn't match.

Mr. Narayan had 2*3 = 6 bottles.

And Mr. Dube had 5*2 = 10 bottles.

In that case 12 + 6 + 10 = 28 and not 26.

So, option (a) cannot be true.

If Anita's husband had 8 bottles then Mr. Rao is Anita's husband.

So, Mr. Panikkar, Mr. Dube and Mr. Narayan had 44 – (2 + 3 + 4 + 5 + 8) = 22 bottles.

Now, Chanchala is not Mrs. Panikkar.

So, Mr. Panikkar had either 3 bottles or 5 bottles.

Mr. Dube had 6 bottles, 8 bottles or 10 bottles.

Mr. Narayan had 9 bottles, 12 bottles or 15 bottles.

If Mr. Panikkar had 5 bottles (i.e. Dipti is Mrs. Panikkar), Mr. Dube had 8 bottles (i.e. Chanchala is Mrs. Dube) and Mr. Narayan had 9 bottles (i.e. Biva is Mrs. Narayan) then it is true.

Option (b) is correct.

69. Every integer of the form $(n^3 - n)(n - 2)$, (for n = 3, 4, ….) is
   (a) Divisible by 6 but not always divisible by 12
   (b) Divisible by 12 but not always divisible by 24
   (c) Divisible by 24 but not always divisible by 48
   (d) Divisible by 9.

Solution :

$(n^3 - n)(n - 2) = (n + 1)n(n - 1)(n - 2)$ = multiplication of consecutive four integers which is always divisible by 8 and 3

⇨ It is divisible by 24 as gcd(3, 8) = 1

Option (c) is correct.

70. The number of integers n > 1, such that n, n + 2, n + 4 are all prime numbers, is
   (a) Zero
   (b) One
   (c) Infinite
   (d) More than one, but finite

Solution :

Only one (3, 5, 7)

Option (b) is correct.

71. The number of *ordered* pairs of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$ is
    (a) 2
    (b) 4
    (c) 6
    (d) 8

Solution :

If x = 0, then y = ± 2

So, two pairs (0, 2); (0, -2)

x needs to be negative as if x is positive then y^2 = negative which is not possible.

Let x = -1, y = ±3

So, two more pairs (-1, 3); (-1, -3)

We have till now 4 pairs.

Let x = -2 then $y^2 = 12$ not giving integer solution.

Let x = -3 then $y^2 = 13$ not giving integer solution.

Let x = -4, then $y^2 = 12$ not giving integer solution.

Let x = -5, then y = ±3

We have two more pairs viz. (-5, 3); (-5, -3)

So, we have 6 pairs till now.

Let x = -6 then y = ±2.

So, we have 8 pairs.

Option (d) is correct.

72. The number of integer (positive, negative or zero) solutions of $xy - 6(x + y) = 0$ with $x \le y$ is
    (a) 5

(b) 10
(c) 12
(d) 9

Solution :

$xy - 6(x + y) = 0$

⇨ $y = 6x/(x - 6)$

$x = 0, y = 0$

$x > 6$. (as $x \le y$)

$x = 7, y = 42$

$x = 8, y = 24$

$x = 9, y = 27$

$x = 10, y = 15$

$x = 11$ doesn't give integer solution.

$x = 12, y = 12$

$x = 13, y < x$

$x = -1$ doesn't give integer solution.

$x = -2$ doesn't give integer solution.

$x = -3, y = 2$

$x = -4$ doesn't give integer solution.

$x = -5$ doesn't give any integer solution.

$x = -6, y = 3$

$x = -7$ doesn't give any integer solution

$x = -8$ doesn't give any integer solution

$x = -9$ doesn't give any integer solution

$x = -10$, doesn't give any integer solution.

$x = -11$ doesn't give any integer solution.

$x = -12, y = 4$
No other x will give integer solution.

Option (d) is correct.

73. Let P denote the set of all positive integers and S = {(x, y) : x ε P, y ε P and $x^2 - y^2 = 666$}. The number of distinct elements in the set S is
(a) 0
(b) 1
(c) 2
(d) More than 2

Solution :

$666 = 2*3^2*37$

⇨ $(x + y)(x - y) = 2*3^2*37$

Now, x and y are both even or both odd.

Both cannot be even as 666 is divisible by 2 and not 4.

Again x and y both cannot be odd as x + y and x – y both will be even so 666 must have at least 8 as a factor.

Option (a) is correct.

74. If numbers of the form $3^{4n-2} + 2^{6n-3} + 1$, where n is a positive integer, are divided by 17, the set of all possible remainders is
(a) {1}
(b) {0, 1}
(c) {0, 1, 7}
(d) {1, 7}

Solution :

$3^{4n-2} + 2^{6n-3} + 1 = 9^{2n-1} + 8^{2n-1} + 1 \equiv (-8)^{2n-1} + 8^{2n-1} + 1 \pmod{17} \equiv -8^{2n-1} + 8^{2n-1} + 1 \pmod{7}$ (as 2n-1 is odd) $\equiv 1 \pmod{17}$

Option (a) is correct.

75. Consider the sequence : $a_1 = 101$, $a_2 = 10101$, $a_3 = 1010101$, and so on. Then a_k is a composite number (that is, not a prime number)
(a) if and only if $k \geq 2$ and 11 divides $10^{k+1} + 1$

(b) if and only if $k \geq 2$ and 11 divides $10^{k+1} - 1$
(c) if and only if $k \geq 2$ and $k - 2$ is divisible by 3
(d) if and only if $k \geq 2$.

Solution :

$a_k = 101010\ldots k \text{ times}1 = 1*10^{2k} + 1*10^{2k-2} + \ldots. + 1 = 1*[\{(10^2)^{k+1} - 1\}/(10^2 - 1)] = (10^{2k+2} - 1)/99$

Now, if k is odd then $10^{2k+2} = 10^{2(k+1)} = 100^{k+1} \equiv (-1)^{k+1} = 1 \pmod{101}$ (k +1 is even)

⇨ $10^{2k+2} - 1 \equiv 0 \pmod{101}$

If $k = 6m + 2$ form then $2k + 2 = 12m + 6 = 6(2m + 1)$

Now, $10^3 \equiv 1 \pmod{27}$

⇨ $10^{6(2m+1)} - 1 \equiv 0 \pmod{27}$
⇨ They are divisible by 3.

None of the (a), (b), (c) are true.

Option (d) is correct.

76. Let n be a positive integer. Now consider all numbers of the form $3^{2n+1} + 2^{2n+1}$. Only one of the following statements is true regarding the *last digit* of these numbers. Which one is it?
(a) It is 5 for some of these numbers but not for all.
(b) It is 5 for all these numbers.
(c) It is always 5 for $n \leq 10$ and it is 5 for some $n > 10$
(d) It is odd for all of these numbers but not necessarily 5.

Solution :

Last digit of 3^{2n+1} is 3 (for n = 1, 5, 9, …) or 7 (for n = 3, 7, 11, …).

Last digit of 2^{2n+1} is 2 (for n = 1, 5, 9, ….) or 8 (for n = 3, 7, 11, ….).

⇨ Last digit is always 5

Option (b) is correct.

77. Which one of the following numbers can be expressed as the sum of squares of two integers?
(a) 1995

(b) 1999
(c) 2003
(d) None of these integers.

Solution :

Let $x^2 + y^2 = 1995, 1999, 2003$ where x is even (say) and y is odd.

Dividing the equation by 4 we get, $0 + 1 \equiv 3, 3, 3 \pmod 4$

Which is impossible,

So, option (d) is correct.

78. If the product of an odd number odd integers is of the form $4n + 1$, then
(a) An even number of them must be always of the form $4n + 1$
(b) An odd number of them always be of the form $4n + 3$
(c) An odd number of them must always be of the form $4n + 1$
(d) None of the foregoing statements is true.

Solution :

Option (c) is correct.

79. The two sequences of numbers {1, 4, 16, 64, ….} and {3, 12, 48, 192, ….} are mixed as follows : {1, 3, 4, 12, 16, 48, 64, 192, ….}. One of the numbers in the mixed series is 1048576. Then the number immediately preceding it is
(a) 786432
(b) 262144
(c) 814572
(d) 786516

Solution :

1048576 is not divisible by 3.

Hence it is from first sequence.

So, $1*4^{n-1} = 1048576$

⇨ $4^{n-1} = 4^{10}$

- ⇨ n = 11.

Therefore, we need to find 10^{th} term of the second sequence.

It is $3*4^{10-1} = 3*4^9 = 786432$

Option (a) is correct.

80. Let $(a_1, a_2, a_3, \ldots.)$ be a sequence such that $a_1 = 2$ and $a_n - a_{n-1} = 2n$ for all $n \geq 2$. Then $a_1 + a_2 + \ldots. + a_{20}$ is
 (a) 420
 (b) 1750
 (c) 3080
 (d) 3500

Solution :

Now, $a_n - a_{n-1} = 2n$

Putting n = 2, we get, $a_2 - a_1 = 2*2$

Putting n = 3, we get, $a_3 - a_2 = 2*3$

Putting n = 4, we get, $a_4 - a_3 = 2*4$

…

…

Putting n = n, we get, $a_n - a_{n-1} = 2*n$

Adding the above equalities we get, $a_n - a_1 = 2(2 + 3 + \ldots. + n) = 2(1 + 2 + \ldots. + n) - 2 = 2\{ n(n + 1)/2\} - 2$

- ⇨ $a_n = n(n + 1)$ (as $a_1 = 2$)
- ⇨ $\sum a_n$ (n running from 1 to 20) = $\sum n^2 + \sum n = 20*21*41/6 + 20*21/2 = 70*41 + 210 = 3080$

Option (c) is correct.

81. The value of $\sum ij$, where the summation is over all i and j such that $1 \leq i < j \leq 10$, is
 (a) 1320
 (b) 2640
 (c) 3025
 (d) None of the foregoing numbers.

Solution :

$\sum ij$, where the summation is over all i and j such that $1 \le i < j \le 10$

= 1*(2 + 3 + … + 10) + 2(3 + 4 + … + 10) + 3(4 + 5 + … + 10) + 4(5 + 6 +… + 10) + 5(6 + 7 + … + 10) + 6(7 + 8 + 9 + 10) + 7(8 + 9 + 10) + 8(9 + 10) + 9*10

= (9/2){2*2 + (9 – 1)*1} + 2*(8/2){2*3 + (8 – 1)*1} + 3(7/2){2*4 + (7 – 1)*1} + 4(6/2){2*5 + (6 – 1)*1} + 5(5/2){2*6 + (5 – 1)*1} + 6*34 + 7*27 + 8*19 + 90

= 54 + 104 + 147 + 180 + 200 + 204 + 189 + 152 + 90 = 1320

Option (a) is correct.

82. Let $x_1, x_2, \ldots, x_{100}$ be hundred integers such that the sum of any five of them is 20. Then
(a) The largest x_i equals 5
(b) The smallest x_i equals 3
(c) $x_{17} = x_{83}$
(d) none of the foregoing statements is true.

Solution :

$x_i + x_j + x_k + x_l + x_m = 20$

Again, $x_i + x_j + x_k + x_l + x_n = 20$

⇨ $x_m = x_n$
⇨ All the integers are equal.
⇨ $x_{17} = x_{83}$

Option (c) is correct.

83. The smallest positive integer n with 24 divisors (where 1 and n are also considered as divisors of n) is
(a) 420
(b) 240
(c) 360
(d) 480

Solution :

$240 = 2^4 \cdot 3 \cdot 5$, number of divisors = (4 + 1)*(1 + 1)*(1 + 1) = 20

$360 = 2^3*3^2*5$, number of divisors = $(3 + 1)*(2 + 1)*(1 + 1) = 24$

Option (c) is correct.

84. The last digit of 2137^{754} is
(a) 1
(b) 3
(c) 7
(d) 9

Solution :

$2137^2 \equiv -1 \pmod{10}$

- $\Rightarrow (2137^2)^{377} \equiv (-1)^{377} \pmod{10}$
- $\Rightarrow 2137^{754} \equiv -1 \pmod{10} \equiv 9 \pmod{10}$
- $\Rightarrow$ Last digit is 9.

Option (d) is correct.

85. The smallest integer that produces remainders of 2, 4, 6 and 1 when divided by 3, 5, 7 and 11 respectively is
(a) 104
(b) 1154
(c) 419
(d) None of the foregoing numbers.

Solution :

Now, $n = 3t_1 - 1$

$n = 5t_2 - 1$

$n = 7t_3 - 1$

$n = 11t_4 + 1$

- $\Rightarrow n = 3*5*7t_5 - 1 = 105t_5 - 1 = 104 + 105t_6$

Now, $104 + 105t_6 \equiv 1 \pmod{11}$

- $\Rightarrow 5 + 6t_6 \equiv 1 \pmod{11}$
- $\Rightarrow 6t_6 \equiv 7 \pmod{11}$
- $\Rightarrow t_6 = 3 + 11t_7$

Therefore, $n = 104 + 105(3 + 11t_7) = 419 + 1155t_7$

Therefore, least n is 419.

Option (c) is correct.

86. How many integers n are there such that $2 \leq n \leq 1000$ and the highest common factor of n and 36 is 1?
   (a) 166
   (b) 332
   (c) 361
   (d) 416

Solution :

$36 = 2^2*3^2$

Number of positive integers divisible by 2 = 500

Number of positive integers divisible by 3, $3 + (p - 1)*3 = 999$

⇨ $p = 333$

Number of positive integers which are divisible by both 2 and 3, $6 + (m - 1)*6 = 996$

⇨ $m = 166$.

Therefore number of positive integers divisible by 2 and 3 = 500 + 333 – 166 = 667

Therefore number of n = 999 – 667 = 332

Option (b) is correct.

87. The remainder when 3^{37} is divided by 79 is
   (a) 78
   (b) 1
   (c) 2
   (d) 35

Solution :

$3^4 \equiv 2 \pmod{79}$

⇨ $(3^4)^9 \equiv 2^9 \pmod{79}$
⇨ $3^{37} \equiv 3*2^9 \pmod{79} \equiv 17*16 \pmod{79} \equiv (-11)*4 \pmod{79} \equiv -44 \pmod{79} \equiv 35 \pmod{79}$

Option (d) is correct.

88. The remainder when 4^{101} is divided by 101 is
   (a) 4
   (b) 64
   (c) 84
   (d) 36

Solution :

By Fermat's little theorem, $4^{100} \equiv 1 \pmod{101}$ (101 is prime)

⇨ $4^{101} \equiv 4 \pmod{101}$

Option (a) is correct.

89. The 300-digit number with all digits equal to 1 is
   (a) Divisible by neither 37 nor 101
   (b) Divisible by 37 but not by 101
   (c) Divisible by 101 but not by 37
   (d) Divisible by both 37 and 101.

Solution :

As 300 is divisible by 3 so, the number is divisible by 37 as 37 divides 111.

Now, $1*10^{299} + 1*10^{298} + ….. + 1*10 + 1 = 1*(10^{300} - 1)/(10 - 1) = (10^{300} - 1)/9$

Now, $10^2 \equiv - 1 \pmod{101}$

⇨ $(10^2)^{150} \equiv (-1)^{150} \pmod{101}$
⇨ $10^{300} \equiv 1 \pmod{101}$
⇨ $10^{300} - 1 \equiv 0 \pmod{101}$
⇨ The number is divisible by 101.

Option (d) is correct.

90. The remainder when $3^{12} + 5^{12}$ is divided by 13 is
   (a) 1
   (b) 2
   (c) 3

(d) 4

Solution :

By Fermat's little theorem, 3^{12}, $5^{12} \equiv 1 \pmod{13}$

⇨ $3^{12} + 5^{12} \equiv 1 + 1 \pmod{13} \equiv 2 \pmod{13}$

Option (b) is correct.

91. When $3^{2002} + 7^{2002} + 2002$ s divided by 29 the remainder is
(a) 0
(b) 1
(c) 2
(d) 7

Solution :

By Fermat's little theorem, 3^{28}, $7^{28} \equiv 1 \pmod{29}$

Now, $2002 \equiv 14 \pmod{28}$ and $2002 \equiv 1 \pmod{29}$

Therefore, $3^{2002} + 7^{2002} + 2002 \equiv 3^{14} + 7^{14} + 1 \pmod{29} \equiv \pm1 \pm1 + 1 \pmod{29}$ (if $a^{(p-1/2)} \equiv \pm1 \pmod{p}$)

If both are +1 then the remainder is 3 which is not option.

If both are -1 then the remainder is -1 which is not option.

So, one is +1 and another is -1 and hence the remainder is $+1 - 1 + 1 = 1$

Option (b) is correct.

92. Let x = 0.101001000100001…. + 0.272727….. Then x
(a) is irrational.
(b) Is rational but $\sqrt{x}$ is irrational
(c) is a root of $x^2 + 0.27x + 1 = 0$
(d) satisfies none of the above properties.

Solution :

x = 0.101001000100001….. + 0.272727…..

$= 1*10^{-1} + 1*10^{-3} + 1*10^{-6} + 1*10^{-10} + \ldots. + 27*10^{-2} + 27*10^{-4} + 27*10^{-6} + \ldots..$

$= 10^{-1} + 10^{-3} + \ldots\ldots + 27*10^{-2}\{1/(1 - 10^{-2})\}$

$= 10^{-1} + 10^{-3} + 10^{-6} + 10^{-10} + 3/11$

= irrational.

Option (a) is correct.

93. The highest power of 18 contained in ${}^{50}C_{25}$ is
(a) 3
(b) 0
(c) 1
(d) 2

Solution :

${}^{50}C_{25} = 50*49*48*\ldots\ldots*26/(25*24*\ldots.*2*1) = (2^{25}*3^{12}*\ldots.)/(2^{22}*3^{10}*\ldots.) = 2^3*3^2*\ldots.$

Therefore, highest power of 18 contained in ${}^{50}C_{25}$ is 1.

Option (c) is correct.

94. The number of divisors of 2700 including 1 and 2700 equals
(a) 12
(b) 16
(c) 36
(d) 18

Solution :

$2700 = 2^2*3^3*5^2$

Number of divisors = (2 + 1)(3 + 1)(2 + 1) = 36

Option (c) is correct.

95. The number of different factors of 1800 equals
(a) 12
(b) 210
(c) 36

(d) 18

Solution :

$1800 = 2^3*3^2*5^2$

Number of factors = (3 + 1)(2 + 1)(2 + 1) = 36

Option (c) is correct.

96. The number of different factors of 3003 is
(a) 2
(b) 15
(c) 7
(d) 16

Solution :

3003 has 3 as a factor. So, 2 cannot be answer. Only square numbers have odd number of factors. So, option (a), (b), (c) cannot be true.

Option (d) is correct.

97. The number of divisors of 6000, where 1 and 6000 are also considered as divisors of 6000, is
(a) 40
(b) 50
(c) 60
(d) 30

Solution :

$6000 = 2^4*3*5^3$

Number of divisors = (4 + 1)(1 + 1)(3 + 1) = 40

Option (a) is correct.

98. The number of positive integers which divide 240 (where 1 and 240 are considered as divisors) is
(a) 18
(b) 20

(c) 30
(d) 24

Solution :

$240 = 2^4*3*5$

Number of positive integers which divide 240 = (4 + 1)(1 + 1)(1 + 1) = 20

Option (b) is correct.

99. The sum of all the positive divisors of 1800 (including 1 and 1800) is
(a) 7201
(b) 6045
(c) 5040
(d) 4017

Solution :

$1800 = 2^3*3^2*5^2$

Sum of divisors = $\{(2^{3+1} - 1)/(2 - 1)\}\{(3^{2+1} - 1)/(3 - 1)\}\{(5^{2+1} - 1)/(5 - 1)\} = 15*13*31 = 6045$

Option (b) is correct.

100. Let d_1, d_2, …., d_k be all the factors of a positive integer n including 1 and n. Suppose $d_1 + d_2 + …. + d_k = 72$. Then the value of $1/d_1 + 1/d_2 + … + 1/d_k$
(a) is $k^2/72$
(b) is 72/k
(c) is 72/n
(d) cannot be computed from the given information.

Solution :

Now, $1/d_1 + 1/d_2 + …. + 1/d_k = (1/n)(n/d_1 + n/d_2 + …. + n/d_k)$

Now, n/d_1 will give another factor, n/d_2 will give another factor and so on.

⇨ $n/d_1 + n/d_2 + … + n/d_k = d_1 + d_2 + …. + d_k = 72$

⇨ $1/d_1 + 1/d_2 + \dots + 1/d_k = 72/n$

Option (c) is correct.

101. The number of ways of distributing 12 identical things among 4 children so that every child gets at least one and no child more than 4 is
(a) 31
(b) 52
(c) 35
(d) 42

Solution :

Let first child gets x_1 oranges, second child gets x_2 oranges, third child gets x_3 oranges, fourth child gets x_4 oranges.

So, $x_1 + x_2 + x_3 + x_4 = 12$ where $1 \le x_1, x_2, x_3, x_4 \le 4$

We fix $x_1 = 1$, then $x_2 + x_3 + x_4 = 11$

Now, x_2 cannot be equal to 1 or 2. So, $x_2 = 3$.

Then only one solution, $x_3 = 4$, $x_4 = 4$.

Now, let, $x_2 = 4$, two solutions, $x_3 = 3$, $x_4 = 4$ and $x_3 = 4$, $x_3 = 3$.

So, we have, 3 solutions.

Now, $x_1 = 2$, $x_2 = 2$, $x_3 + x_4 = 8$ one solution, $x_3 = 4$, $x_4 = 4$.

$x_2 = 3$, then two solutions, $x_3 = 3$, $x_4 = 4$ and $x_3 = 4$, $x_4 = 3$

$x_2 = 4$, $x_3 + x_4 = 6$, $x_3 = 2$, $x_4 = 4$; $x_3 = 3$, $x_4 = 3$; $x_3 = 4$, $x_4 = 2$

So, we have, 6 solutions.

Now, $x_1 = 3$, $x_2 = 1$, one solution $x_3 = 4$, $x_4 = 4$

$x_2 = 2$, two solutions, $x_3 = 3$, $x_4 = 4$; $x_3 = 4$, $x_4 = 3$

$x_2 = 3$, three solutions.

$x_2 = 4$, then, $x_3 = 1$, $x_4 = 4$; $x_3 = 2$, $x_4 = 3$; $x_3 = 3$, $x_4 = 2$; $x_3 = 4$, $x_4 = 1$

So we have, 10 solutions.

Now, $x_1 = 4$, $x_2 = 1$; two solutions.

$x_2 = 2$, three solutions

$x_2 = 3$, four solutions

$x_2 = 4$, $x_3 = 1$, $x_4 = 3$; $x_3 = 2$, $x_4 = 2$; $x_3 = 3$, $x_4 = 1$ – three solutions

So we have, 12 solutions.

So we have total 3 + 6 + 10 + 12 = 31 solutions.

Therefore, we can distribute the things in 31 ways.

Option (a) is correct.

102. The number of terms in the expansion of $[(a + 3b)^2(a - 3b)^2]^2$, when simplified is
(a) 4
(b) 5
(c) 6
(d) 7

Solution :

$[(a + 3b)^2(a - 3b)^2]^2 = (a^2 - 9b^2)^4$ = 5 terms

Option (b) is correct.

103. The number of ways in which 5 persons P, Q, R, S and T can be seated in a ring so that P sits between Q and R is
(a) 120
(b) 4
(c) 24
(d) 9

Solution :

Let us take (QPR) as unit.

Therefore, total (QPR), S, T – 3 persons will sit in a ring, It can be done in (3 – 1)! = 2 ways.

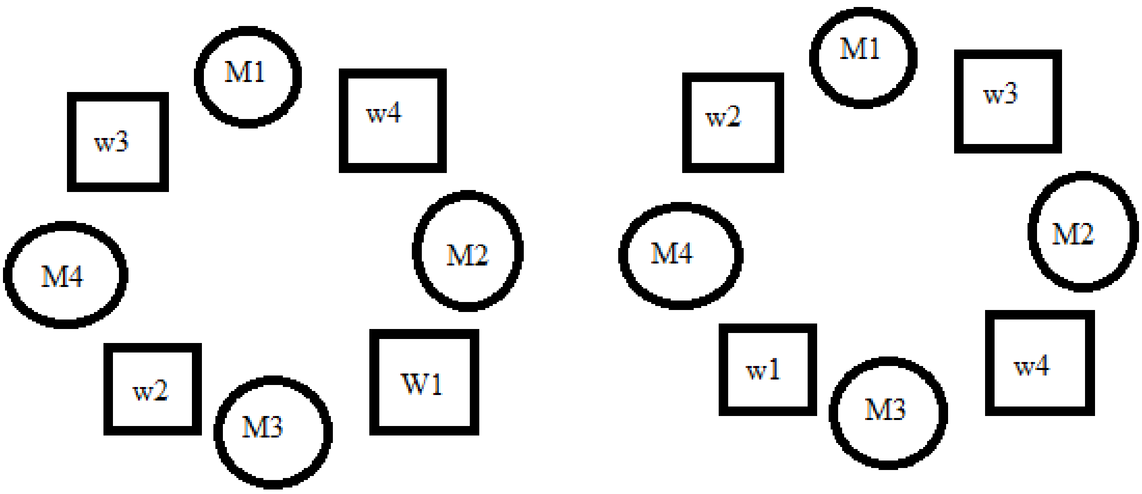
Now, Q and P can move among themselves in 2! = 2 ways.

Therefore, the whole arrangement can be done in 2*2 = 4 ways.

Option (b) is correct.

104. Four married couples are to be seated in a merry-go-round with 8 identical seats. In how many ways can they be seated so that
(i) males and females seat alternatively; and
(ii) no husband seats adjacent to this wife?
(a) 8
(b) 12
(c) 16
(d) 20

Solution :



W_1 can sit in two seats either in the seat in left side figure or in the seat in right side figure. In left side figure when W_1 is given seat then W_4 can sit in one seat only as shown and accordingly W_2 and W_3 can also take only one seat. Similarly, right side figure also reveals one possible way to seat. So there are two ways to seat for every combination of Men. Now, Men can arrange themselves in (4 – 1)! = 6 ways. So number of ways = 2*6 = 12.

Option (b) is correct.

105. For a regular polygon with n sides (n > 5), the number of triangles whose vertices are joining non-adjacent vertices of the polygon is
(a) $n(n - 4)(n - 5)$
(b) $(n - 3)(n - 4)(n - 5)/3$
(c) $2(n - 3)(n - 4)(n - 5)$

(d) $n(n - 4)(n - 5)/6$

Solution :

The total number of triangles that can be formed using any three vertices $= {}^nC_3$.

Now, taking two consecutive vertices and one other vertices total number of triangles that can be formed $= n*(n - 4)$

Now, taking three consecutive vertices total number of triangles that can be formed $= n$

Therefore, number of required triangles $= {}^nC_3 - n(n - 4) - n = n(n - 1)(n - 2)/6 - n(n - 4) - n$

$= (n/6)(n^2 - 3n + 2 - 6n + 24 - 6) = (n/6)(n^2 - 9n + 20) = n(n - 4)(n - 5)/6$

Option (d) is correct.

106. The term that is independent of x in the expansion of $(3x^2/2 - 1/3x)^9$ is
(a) ${}^9C_6(1/3)^3(3/2)^6$
(b) ${}^9C_5(3/2)^5(-1/3)^4$
(c) ${}^9C_3(1/6)^3$
(d) ${}^9C_4(3/2)^4(-1/3)^5$

Solution :

$t_r = {}^9C_r(3x^2/2)^r(-1/3x)^{9 - r} = {}^9C_r(3/2)^r(-1/3)^{9-r}x^{2r - 9 + r}$

Now, $2r - 9 + r = 0$

- ⇨ $r = 3$
- ⇨ The term independent of x is ${}^9C_3(3/2)^3(-1/3)^6 = {}^9C_3(1/6)^3$

Option (c) is correct.

107. The value of $({}^{50}C_0)({}^{50}C_1) + ({}^{50}C_1)({}^{50}C_2) + ….. + ({}^{50}C_{49})({}^{50}C_{50})$ is
(a) ${}^{100}C_{50}$
(b) ${}^{100}C_{51}$
(c) ${}^{50}C_{25}$
(d) $({}^{50}C_{25})^2$

Solution :

Now, $(1 + x)^{50} = {}^{50}C_0 + {}^{50}C_1x + {}^{50}C_2x^2 + ... + {}^{50}C_{49}x^{49} + {}^{50}C_{50}x^{50}$

$(x + 1)^{50} = {}^{50}C_0x^{50} + {}^{50}C_1x^{49} + ... + {}^{50}C_{50}$

Now, $(1 + x)^{50}(x + 1)^{50} = ({}^{50}C_0 + {}^{50}C_1x + {}^{50}C_2x^2 + ... + {}^{50}C_{49}x^{49} + {}^{50}C_{50}x^{50})({}^{50}C_0x^{50} + {}^{50}C_1x^{49} + ... + {}^{50}C_{50})$

$\Rightarrow (1 + x)^{100} = ({}^{50}C_0{}^{50}C_1 + {}^{50}C_1{}^{50}C_2 + ... + {}^{50}C_{49}{}^{50}C_{50})x^{49} + ...$

Now, coefficient of x^{49} in the expansion of $(1 + x)^{100} = {}^{100}C_{49} = {}^{100}C_{51}$

Option (b) is correct.

108. The value of $({}^{50}C_0)^2 + ({}^{50}C_1)^2 + ... + ({}^{50}C_{50})^2$ is
(a) ${}^{100}C_{50}$
(b) 50^{50}
(c) 2^{100}
(d) 2^{50}

Solution :

From the previous question's solution we get, the coefficient of $({}^{50}C_0)^2 + ({}^{50}C_1)^2 + ... + ({}^{50}C_{50})^2$ is x^{50}

Therefore, coefficient of x^{50} in the expansion of $(1 + x)^{100}$ is ${}^{100}C_{50}$.

Option (a) is correct.

109. The value of $({}^{100}C_0)({}^{200}C_{150}) + ({}^{100}C_1)({}^{200}C_{151}) + ... + ({}^{100}C_{50})({}^{200}C_{200})$ is
(a) ${}^{300}C_{50}$
(b) $({}^{100}C_{50})({}^{200}C_{150})$
(c) $({}^{100}C_{50})^2$
(d) None of the foregoing numbers.

Solution :

$(1 + x)^{100} = {}^{100}C_0 + {}^{100}C_1x + {}^{100}C_2x^2 + ... + {}^{100}C_{100}x^{100}$

$(x + 1)^{200} = {}^{200}C_0x^{200} + ... + {}^{200}C_{150}x^{150} + {}^{200}C_{151}x^{49} + ... + {}^{200}C_{200}$

Now, $(1 + x)^{100}(x + 1)^{200} = ({}^{100}C_0{}^{200}C_{150} + {}^{100}C_1{}^{200}C_{151} + \ldots. + {}^{100}C_{50}{}^{200}C_{200})x^{50} + \ldots..$

Now, in LHS i.e. $(1 + x)^{100}(x + 1)^{200} = (1 + x)^{300}$ coefficient of x^{50} is ${}^{300}C_{50}$

Therefore, ${}^{100}C_0{}^{200}C_{150} + {}^{100}C_1{}^{200}C_{151} + \ldots. + {}^{100}C_{50}{}^{200}C_{200} = {}^{300}C_{50}$

Option (a) is correct.

110. The number of four-digit numbers strictly greater than 4321 that can be formed from the digits 0, 1, 2, 3, 4, 5 allowing for repetition of digits is
(a) 310
(b) 360
(c) 288
(d) 300

Solution :

The first digit can be either 4 or 5

Case 1 : first digit is 4.

Second digit can be 3, 4, 5

So, we can choose second digit when 4 or 5 in 2 ways.

Third digit, fourth digit can be anything when second digit is 4 or 5.

So, number of numbers = 2*6*6 = 72

When second digit is 3, third digit can be 2, 3, 4, 5

When third digit is 3, 4, or 5 fourth digit can be anything.

So, number of numbers = 3*6 = 18

When third digit is 2, fourth digit can be 2, 3, 4, 5. So number of numbers = 4

Therefore, total number of numbers when first digit is 4 is 72 + 18 + 4 = 94

When first digit is 5 then number of numbers = 6^3 = 216

So, total number of required numbers = 94 + 216 = 310

Option (a) is correct.

111. The sum of all the distinct four-digit numbers that can be formed using the digits 1, 2, 3, 4 and 5, each digit appearing at most once, is
(a) 399900
(b) 399960
(c) 390000
(d) 360000

Solution :

Now, 1 will appear as the first digit in 4! = 24 numbers.

Similar thing goes for other digits.

Similar case goes for other digit of the numbers i.e. second digit, third digit, fourth digit.

Therefore sum = 24{1000(1 + 2 + 3 + 4 + 5) + 100(1 + 2 + 3 + 4 + 5) + 10(1 + 2 + 3 + 4 + 5) + (1 + 2 + 3 + 4 + 5)}

= 24*(1 + 2 + 3 + 4 + 5)(1000 + 100 + 10 + 1)

= 24*15*1111

= 399960

Option (b) is correct.

112. The number of integers lying between 3000 and 8000 (including 3000 and 8000) which have at least two digits equal is
(a) 2481
(b) 1977
(c) 4384
(d) 2755

Solution :

Let us consider the number from 3000 – 3999

Let the number is 3xyz

x = 3, and y, z ≠ 3, there are 9*9 = 81 numbers.

Similarly, for y = 3, x, z ≠ 3 and z = 3, x, y ≠ 3 there are 81 + 81 = 162 numbers.

Now, x = y there are $^{10}C_1$*10 = 100 numbers.

Now, y = z, there are $^{10}C_1*10 = 100$ numbers.

Now, z = x, there are $^{10}C_1*10 = 100$ numbers.

Now, x = y = z, there are 10 numbers.

Now, x = y = 3, z ≠ 3, there are 9 numbers.

Similarly, y = z = 3, x ≠ 3 there are 9 numbers and x = z = 3, y ≠ 3 there are 9 numbers.

So, number of required numbers between 3000 and 3999 is 81 + 162 + 100 + 100 + 100 – 3*9 – 2*10 = 496.

So, number of required numbers between 3000 and 7999 is 496*5 = 2480

So including 8000 there are 2480 + 1 = 2481 numbers.

Option (a) is correct.

113. The greatest integer which, when dividing the integers 13511, 13903 and 14593 leaves the same remainder is
(a) 98
(b) 56
(c)2
(d) 7

Solution :

Let the remainder is a and the number is x.

Therefore, $13511 - a = xm_1$, $13903 - a = xm_2$, $14593 - a = xm_3$

Now, $13903 - a - (13511 - a) = xm_2 - xm_1$

⇨ $392 = x(m_2 - m_1)$

Now, 392 is not divisible by 56, therefore option (b) cannot be true.

Now, $14953 - a -(13903 - a) = xm_3 - xm_2$

⇨ $1050 = x(m_3 - m_2)$

Now, 1050 is not divisible by 98 therefore option (a) cannot be true.

Now, $14593 - a - (13511 - a) = xm_3 - xm_1$

⇨ $1082 = x(m_3 - m_1)$

Now, 1082 is not divisible by 7 therefore option (d) cannot be true.

Option (c) is correct.

114. An integer has the following property that when divided by 10, 9, 8, …., 2, it leaves remainders 9, 8, 7, …., 1 respectively. A possible value of n is
   (a) 59
   (b) 419
   (c) 1259
   (d) 2519

Solution :

$n = 10m_1 - 1$,

$n = 9m_2 - 1$,

…

..

$n = 2m_9 - 1$

Therefore, $n = 10*9*4*7m_{10} - 1 = 2520m_{11} + 2519$

Option (d) is correct.

115. If n is a positive integer such that 8n + 1 is a perfect square, then
   (a) n must be odd
   (b) n cannot be a perfect square
   (c) n must be a prime number
   (d) 2n cannot be a perfect square

Solution :

Let n = odd.

$8n + 1 = 8(2m + 1) + 1 = 16m + 9$

But, (an odd number)$^2 \equiv 1, 9 \pmod{16}$ so, (a) cannot be true.

If n is prime then 8n + 1 is sometimes perfect square and sometimes not. For example n = 3, n = 5.

Option (c) cannot be true.

If n is perfect square then the statement is always not true because $8m^2 + 1 = p^2$ cannot hold true. If we divide the equation by 9 then we can find contradiction as $-m^2 + 1 \equiv p^2 \pmod 9$ and $m^2, p^2 \equiv 0, 3, 6 \pmod 9$

Option (b) cannot be true.

If, 2n is not a perfect square, 8n + 1 = 4*(2n) + 1 which is always true.

Option (d) is correct.

116. For any two positive integers a and b, define a ≡ b if a − b is divisible by 7. Then (1512 + 121)*(356)*(645) ≡
   (a) 4
   (b) 5
   (c) 3
   (d) 2

Solution :

1512 ≡ 0, 121 ≡ 2, 356 ≡ 6, 645 ≡ 1

Therefore, (1512 + 121)*(356)*(645) ≡ (0 + 2)*6*1 = 12 ≡ 5

Option (b) is correct.

117. The coefficient of x^2 in the binomial expansion of $(1 + x + x^2)^{10}$ is
   (a) ${}^{10}C_1 + {}^{10}C_2$
   (b) ${}^{10}C_2$
   (c) ${}^{10}C_1$
   (d) None of the foregoing numbers.

Solution :

$(1 + x + x^2)^{10} = {}^{10}C_0(1 + x)^{10} + {}^{10}C_1(1 + x)^{10}x^2 + \ldots.. = (1 + {}^{10}C_1x + {}^{10}C_2x^2 + \ldots.) + {}^{10}C_1(1 + \ldots)x^2 + \ldots. = ({}^{10}C_2 + {}^{10}C_1)x^2 + \ldots.$

Option (a) is correct.

118. The coefficient of x^{17} in the expansion of $\log_e(1 + x + x^2)$, where $|x| < 1$, is
   (a) 1/17
   (b) -1/17

(c) 3/17
(d) None of the foregoing quantities.

Solution :

$\log_e\{(1 + x + x^2)(1 - x)/(1 - x)\} = \log_e\{(1 - x^3)/(1 - x) = \log_e(1 - x^3) - \log_e(1 - x)$

$= (-x^3 - x^6/2 - \ldots) - (-x - x^2/2 - \ldots - x^{17}/17 - \ldots)$

$= (1/17)x^{17} + \ldots$

Option (a) is correct.

119. Let $a_1, a_2, \ldots, a_{11}$ be an arbitrary arrangement (i.e. permutation) of the integers 1, 2, …, 11. Then the number $(a_1 - 1)(a_2 - 2)\ldots(a_{11} - 11)$ is
(a) Necessarily ≤ 0
(b) Necessarily 0
(c) Necessarily even
(d) Not necessarily ≤ 0, 0 or even.

Solution :

There are 6 odd numbers and 5 even numbers. If we subtract all the odd numbers from even numbers then also one odd number remains which when subtracted from the odd number given an even number.

So, it is necessarily even

Option (c) is correct.

120. Three boys of class I, 4 boys of class II and 5 boys of class III sit in a row. The number of ways they can sit, so that boys of the same class sit together is
(a) 3!4!5!
(b) 12!/(3!4!5!)
(c) $(3!)^2 4!5!$
(d) 3*4!5!

Solution :

Let us take the boys of each class as an unit.

Therefore there are 3 units which can be permutated in 3! ways.

Now, class I boys can permutate among themselves in 3! ways, class II boys in 4! And class III boys in 5! ways.

Therefore, total number of ways is 3!*3!*4!*5! = $(3!)^2 4!5!$

Option (c) is correct.

121. For each positive integer n consider the set S_n defined as follows : $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, …, and, in general, S_{n+1} consists of n + 1 consecutive integers the smallest of which is one more than the largest integer in S_n. Then the sum of all the integers in S_{21} equals
(a) 1113
(b) 53361
(c) 5082
(d) 4641

Solution :

The largest integer in S_n is the triangular numbers i.e. n(n + 1)/2.

Now, the largest number of S_{20} = 20*21/2 = 210

Therefore, required summation = 211 + 212 + …. 21 terms = (21/2){2*211 + (21 – 1)*1} = 21*(211 + 10) = 4641.

Option (d) is correct.

122. If the constant term in the expansion of $(\sqrt{x} - k/x^2)^{10}$ is 405, then k is
(a) $\pm(3)^{1/4}$
(b) ±2
(c) $\pm(4)^{1/3}$
(d) ±3

Solution :

Any term = ${}^{10}C_r(\sqrt{x})^r(k/x^2)^{10-r} = {}^{10}C_r k^{10-r} x^{r/2 - 20 + 2r}$

Now, r/2 – 20 + 2r = 0

⇨ 5r/2 = 20
⇨ r = 8

So, $^{10}C_8 * k^{10-8} = 405$

- ⇨ $(10*9/2)*k^2 = 405$
- ⇨ $k^2 = 9$
- ⇨ $k = \pm 3$

Option (d) is correct.

123. Consider the equation of the form $x^2 + bx + c = 0$. The number of such equations that have real roots and have coefficients b and c in the set {1, 2, 3, 4, 5, 6}, (b may be equal to c), is
(a) 20
(b) 18
(c) 17
(d) 19

Solution :

Roots are real.

- ⇨ $b^2 - 4c \geq 0$
- ⇨ $b^2 \geq 4c$

b cannot be 1

if b = 2, c = 1

if b = 3, c = 1, 2

if b = 4, c = 1, 2, 3, 4

if b = 5, c = 1, 2, 3, 4, 5, 6

if b = 6, c = 1, 2, 3, 4, 5, 6

Therefore, total number of equations = 1 + 2 + 4 + 6 + 6 = 19

Option (d) is correct.

124. The number of polynomials of the form $x^3 + ax^2 + bx + c$ which is divisible by $x^2 + 1$ and where a, b and c belong to {1, 2, …., 10} is
(a) 1
(b) 10
(c) 11
(d) 100

Solution :

Now, $x^2 \equiv -1 \pmod{x^2 + 1}$

⇨ $x^3 + ax^2 + bx + c \equiv x(1 - b) + (c - a) \pmod{x^2 + 1}$

But remainder must be 0.

⇨ $b = 1$ and $a = c$

Now, a = c can be done in 10 ways and b = 1 can be done in 1 way.

Therefore, total number of polynomials = 10*1 = 10

Option (b) is correct.

125. The number of distinct 6-digit numbers between 1 and 300000 which are divisible by 4 and are obtained by rearranging the digits 112233, is
(a) 12
(b) 15
(c) 18
(d) 90

Solution :

Last digit needs to be 2.

First digit can be 1 or 2.

First digit is 1, last digit is 2 and fifth digit cannot be 2 as the five-digit number up to fifth digit from left then congruent to 2 or 0 modulus 4 and in both the cases the six digit number is not divisible by 4.

So, fifth digit is 1 or 3.

First digit 1, fifth digit 1, last digit 2.

The digits left for 3 digits are 3, 3, 2

Number of numbers = 3!/2! = 3

First digit 1, fifth digit 3, last digit 2.

The digits left are 1, 3, 2.

Number of numbers = 3! = 6

Total numbers in this case = 3 + 6 = 9

First digit 2, fifth digit 1, last digit 2.

Digits left are 1, 3, 3

Number of numbers = 3!/2! = 3

First digit 2, fifth digit 3, last digit 2.

Digits left = 1, 1, 3

Number of numbers = 3!/2! = 3

Total number of numbers in this case = 3 + 3 = 6

So, number of required numbers = 9 + 6 = 15.

Option (b) is correct.

126. The number of odd positive integers smaller than or equal to 10000 which are divisible by neither 3 nor by 5 is
(a) 3332
(b) 2666
(c) 2999
(d) 3665

Solution :

Let n number of odd numbers divisible by 3.

Then 3 + (n – 1)*6 = 9999

- ⇨ 2n – 1 = 3333
- ⇨ n = 1667

Let m number of odd numbers are divisible by 5.

Then, 5 + (m – 1)*10 = 9995

- ⇨ 2m – 1 = 1999
- ⇨ m = 1000

Let p number of odd numbers are divisible by 3*5 = 15

Then 15 + (p – 1)*30 = 9975

- ⇨ 2p – 1 = 665
- ⇨ p = 333
- ⇨ Number of odd numbers divisible by 3 or 5 = 1667 + 1000 – 333 = 2334

⇨ Number of odd numbers which are neither divisible by 3 nor by 5 is 5000 – 2334 = 2666.

Option (b) is correct.

127. The number of ways one can put three balls numbered 1, 2, 3 in three boxes labeled a, b, c such that at the most one box is empty is equal to
(a) 6
(b) 24
(c)42
(d) 18

Solution :

No box is empty – number of ways = 3! = 6

Box a is empty – combinations are – 1, 2 in b, 3 in c; 1, 3 in b, 2 in c; 2, 3 in b; 1 in c; 1 in b, 2, 3 in c; 2 in b, 1, 3 in c; 3 in b, 1, 2 in c – 6 ways

Similarly, box b is empty – 6 ways and box c is empty – 6 ways.

Therefore total number of ways = 6 + 6 + 6 + 6 = 24.

Option (b) is correct.

128. A bag contains colored balls of which at least 90% are red. Balls are drawn from the bag one by one and their color noted. It is found that 49 of the first 50 balls drawn are red. Thereafter 7 out of every 8 balls are red. The number of balls in the bag CAN NOT BE
(a) 170
(b) 210
(c)250
(d) 194

Solution :

Let number of balls in the bag is n.

Let, m number of times 8 balls are drawn.

Therefore, n = 50 + 8m

Red balls = 49 + 7m

Percentage of red balls = $\{(49 + 7m)/(50 + 8m)\}*100 \geq 90$

- ⇨ $49 + 7m \geq 50*0.9 + 8m*0.9$
- ⇨ $49 + 7m \geq 45 + 7.2m$
- ⇨ $0.2m \leq 4$
- ⇨ $m \leq 20$
- ⇨ $n \leq 50 + 8*20 = 210$

Option (c) is correct.

129. There are N boxes, each containing at most r balls. If the number of boxes containing at least i balls is N_i, for i = 1, 2, …., r, then the total number of balls in these boxes
 (a) Cannot be determined from the given information
 (b) Is exactly equal to $N_1 + N_2 + ….. + N_r$
 (c) Is strictly larger than $N_1 + N_2 + …. + N_r$
 (d) Is strictly smaller than $N_1 + N_2 + …. + N_r$

Solution :

Option (b) is correct.

130. For all n, the value of ${}^{2n}C_n$ is equal to
 (a) ${}^{2n}C_0 - {}^{2n}C_1 + {}^{2n}C_2 - {}^{2n}C_3 + …. + {}^{2n}C_{2n}$
 (b) $({}^{2n}C_0)^2 + ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 + ….. + ({}^{2n}C_n)^2$
 (c) $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - ({}^{2n}C_3)^2 + ….. + ({}^{2n}C_{2n})^2$
 (d) None of the foregoing expressions.

Solution :

Now, the expression in option (a) is equal to zero.

The expression is option (b) is obviously greater than ${}^{2n}C_n$.

Let is check the expression in option (c).

$(1 + x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + ….. + {}^{2n}C_{2n}x^{2n}$

$(x - 1)^{2n} = {}^{2n}C_0x^{2n} - {}^{2n}C_1x^{2n-1} + ….. + {}^{2n}C_{2n}$

Now, $(1 + x)^{2n}(x - 1)^{2n} = \{({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + …. + ({}^{2n}C_{2n})^2\}x^{2n} + …..$

Now, $(1 + x)^{2n}(x - 1)^{2n} = (x^2 - 1)^{4n}$ in this coefficient of x^{2n} is $(-1)^n*{}^{4n}C_n$.

So, option (d) is correct.

131. The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are 165, 330 and 462. Then the value of n is
(a) 10
(b) 12
(c) 13
(d) 11

Solution :

Let ${}^nC_{r-1} = 165$, ${}^nC_r = 330$, ${}^nC_{r+1} = 462$

Now, ${}^nC_r/{}^nC_{r-1} = 330/165$

⇨ $[n!/\{(n - r)!r!\}]/[n!/\{(n - r + 1)!(r - 1)!\} = 2$
⇨ $(n - r + 1)/r = 2$
⇨ $n - r + 1 = 2r$
⇨ $n + 1 = 3r$ ……. (i)

Now, ${}^nC_{r+1}/{}^nC_r = 462/330$

⇨ $[n!/\{(n - r - 1)!(r + 1)!]/[n!/\{(n - r)!r!\}] = 7/5$
⇨ $(n - r)/(r + 1) = 7/5$
⇨ $5n - 5r = 7r + 7$
⇨ $5n = 12r + 7$
⇨ $5n = 4(n + 1) + 7$ (From (i))
⇨ $n = 11$

Option (d) is correct.

132. The number of ways in which 4 persons can be divided into two equal groups is
(a) 3
(b) 12
(c) 6
(d) None of the foregoing numbers.

Solution :

Let the persons are A, B, C, D.

We can choose any two persons in ${}^4C_2$ ways = 6 ways. This is the answer as we can place them in 1 way.

Option (c) is correct.

133. The number of ways in which 8 persons numbered 1, 2, ……, 8 can be seated in a ring so that 1 always sits between 2 and 3 is
(a) 240
(b) 360
(c) 72
(d) 120

Solution :

Let us take 213 as unit.

So there are 6 units.

They can sit in $(6 - 1)! = 5!$ ways.

Now, 2 and 3 can interchange their position in 2! ways.

Therefore, total number of sitting arrangement is $5!*2! = 240$.

Option (a) is correct.

134. There are seven greeting cards, each of a different color, and seven envelops of the same seven colors. The number of ways in which the cards can be put in the envelops so that exactly 4 cards go into the envelops into the right colors is
(a) ${}^7C_3$
(b) $2*{}^7C_3$
(c) $(3!){}^4C_3$
(d) $(3!)*{}^7C_3*{}^4C_3$

Solution :

We can choose any 4 greeting card which go to correct color envelop in ${}^7C_4 = {}^7C_3$ ways.

Now, let the color of the rest three envelops are red, green, blue and the greeting card of the same color go to different color envelop in 2 ways, as given below :

Red envelop	green envelop	blue envelop
Green card	blue card	red card
Blue card	red card	green card

So, number of ways = $2*{}^7C_3$.

Option (b) is correct.

135. The number of distinct positive integers that can be formed using 0, 1, 2, 4 where each integer is used at most once is equal to
(a) 48
(b) 84
(c) 64
(d) 36

Solution :

One digit number can be formed = 3

Two digit number can be formed = ${}^4C_2*2! - {}^3C_1*1! = 9$ (we need to subtract the numbers which begin with 0)

Three digit number that can be formed = ${}^4C_3*3! - {}^3C_2*2! = 18$

Four digit number that can be formed = $4! - 3! = 18$

Total number of such numbers = 3 + 9 + 18 + 18 = 48

Option (a) is correct.

136. A class contains three girls and four boys. Every Saturday five students go on a picnic, a different group being sent each week. During the picnic, each girl in the group is given a doll by the accompanying teacher. After all possible groups of five have gone once, the total number of dolls the girls have got is
(a) 27
(b) 11
(c) 21
(d) 45

Solution :

Number of picnics, in which 1 girl, 4 boys went = ${}^4C_4*{}^3C_1 = 3$

Number of picnics in which 2 girls, 3 boys went = ${}^3C_2*{}^4C_3 = 12$

Number of picnics in which 3 girls, 2 boys went = ${}^3C_3*{}^4C_2 = 6$

So, number of dolls = 3*1 + 12*2 + 6*3 = 3 + 24 + 18 = 45

Option (d) is correct.

137. From a group of seven persons, seven committees are formed. Any two committees have exactly one member in common. Each person is in exactly three committees. Then
(a) At least one committee must have more than three members.
(b) Each committee must have exactly three members.
(c) Each committee must have more than three members.
(d) Nothing can be said about the sizes of the committees.

Solution :

First committee contains A_1, A_2, A_3

Second committee contains A_1, A_4, A_5

Third committee contains A_1, A_6, A_7

Fourth committee contains A_2, A_4,A_6

Fifth committee contains A_2, A_5, A_7

Sixth committee contains A_3, A_5, A_6

Seventh committee contains A_3, A_7, A_4

Option (b) is correct.

138. Three ladies have each brought a child for admission to a school. The head of the school wishes to interview the six people one by one, taking care that no child is interviewed before its mother. In how many different ways can the interviews be arranged?
(a) 6
(b) 36
(c) 72
(d) 90

Solution :

$M_1M_2C_1C_2M_3C_3$ – in this combination there are 2!*2! = 4 ways. (mothers can interchange among them in 2! Ways and children can in 2! Ways)

Now, there are 3 such combinations with change of position of mothers and children. So 4*3 = 12 ways.

Now, $M_1C_1M_2M_3C_2C_3$ – in this combination there are again 12 ways.

So, total 12 + 12 = 24 ways.

Now, take $M_1C_1M_2C_2M_3C_3$ – taken the mother child combination as unit then there are 3! = 6 ways.

Now, take the combination $M_1M_2M_3C_1C_2C_3$ – in this combination 3!*3! = 6*6 = 36 ways.

Take this combination $M_1M_2C_1M_3C_2C_3$

We can choose any 2 mother in ${}^3C_2$ ways, any 1 child from 2 children of the two interviewed mother in ${}^2C_1$ ways and the two mother can interchange among themselves in 2! Ways and the two child at the end can interchange among themselves in 2! Ways. So total ${}^3C_2*{}^2C_1*2!*2!$ = 24 ways.

Therefore, total number of ways = 24 + 6 + 36 + 24 = 90.

Option (d) is correct.

139. The coefficient of x^4 in the expansion of $(1 + x - 2x^2)^7$ is
   (a) -81
   (b) -91
   (c) 81
   (d) 91

Solution :

$(1 + x - 2x^2)^7 = {}^7C_0 + {}^7C_1(x - 2x^2) + {}^7C_2(x - 2x^2)^2 + {}^7C_3(x - 2x^2)^3 + {}^7C_4(x - 2x^2)^4 + ….$

$= \{{}^7C_2*2^2 + {}^7C_3*3*1^2*(-2) + {}^7C_4\}x^4 + ….$

So, coefficient of x^4 = (7*6/2)*4 – (7*6*5/3*2)*3*2 + 7*6*5/3*2

= 84 – 35*6 + 35

= 84 – 35*5

= 84 – 175

= - 91

Option (b) is correct.

140. The coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$ is

(a) 12!/(3!4!5!)
(b) ${}^6C_3*3!$
(c) 33
(d) $3*{}^6C_3$

Solution :

$(bc + ca + ab)^6 = {}^6C_0(bc)^6 + {}^6C_1(bc)^5(ca + ab) + {}^6C_2(bc)^4(ca + ab)^2 + {}^6C_3(bc)^3(ca + ab)^3 + \ldots..$ *(power of a is more than 3)*

$= {}^6C_3(a^3b^3c^3)(c + b)^3 + \ldots.$

$= {}^6C_3(a^3b^3c^3)(c^3 + 3c^2b + 3cb^2 + b^3) + \ldots.$

$= 3*{}^6C_3a^3b^4c^5 + \ldots.$

Option (d) is correct.

141. The coefficient of t^3 in the expansion of $\{(1 - t^6)/(1 - t)\}^3$ is
(a) 10
(b) 12
(c) 18
(d) 0

Solution :

$\{(1 - t^6)/(1 - t)\}^3 = (1 + t + t^2 + t^3 + t^4 + t^5)^3 = (1 + t + t^2 + t^3 + t^4 + t^5)(1 + t + t^2 + t^3 + t^4 + t^5)(1 + t + t^2 + t^3 + t^4 + t^5)$

$= \{1 + 2t + 3t^2 + 4t^3 + \ldots\}(1 + t + t^2 + t^3 + t^4 + t^5)$

Coefficient of $t^3 = 1 + 2 + 3 + 4 = 10$.

Option (a) is correct.

142. The value of $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots.. - ({}^{2n}C_{2n-1})^2 + ({}^{2n}C_{2n})^2$ is
(a) ${}^{4n}C_{2n}$
(b) ${}^{2n}C_n$
(c) 0
(d) $(-1)^n*{}^{2n}C_n$

Solution :

The answer I am getting is $(-1)^{n} * {}^{4n}C_n$ (see solution of problem 130)

But, the answer is given as option (d).

143. There are 14 intermediate stations between Dusi and Visakhapatnam on the South Eastern Railway. A train is to be arranged from Dusi to Visakhapatnam so that it halts at exactly three intermediate stations, no two of which are consecutive. Then number of ways of doing this is
(a) ${}^{14}C_3 - ({}^{13}C_1)({}^{12}C_1) + {}^{12}C_1$
(b) 10*11*12/6
(c) ${}^{14}C_3 - {}^{14}C_2 - {}^{14}C_1$
(d) ${}^{14}C_3 - {}^{14}C_2 + {}^{14}C_1$

Solution :

Any three stations can be selected in ${}^{14}C_3$ ways.

Two stations consecutive and one station any station in $({}^{13}C_1)({}^{12}C_1)$ ways.

Now, 3 stations consecutive will appear twice so add $({}^{12}C_1)$ i.e. 3 stations consecutive.

So, total number of ways = ${}^{14}C_3 - ({}^{13}C_1)({}^{12}C_1) + {}^{12}C_1$

Option (a) is correct.

144. The letters of the word "MOTHER" are permutated, and all the permutations so formed are arranged in alphabetical order as in a dictionary. Then the number of permutations which come before the word "MOTHER" is
(a) 503
(b) 93
(c) 6!/2 – 1
(d) 308

Solution :

MOTHER

Alphabetically, E, H, M, O, R, T

First letter will be E, there will be 5! = 120 words.

First letter H, there are 5! = 120 words.

Now, comes M.

Second letter E, there are 4! = 24 words.

Second letter H, there are 4! = 24 words.

Now, comes O.

Third letter E, there are 3! = 6 words.

Third letter H, there are 3! = 6 words.

Third letter R, there are 3! = 6 words.

Now comes T.

Fourth letter E, there are 2! = 2 words.

Now comes H.

Now comes fifth letter H and sixth letter R.

So, there are, 120 + 120 + 24 + 24 + 6 + 6 + 6 + 2 = 308 words before MOTHER.

Option (d) is correct.

145. All the letters of the word PESSIMISTIC are to be arranged so that no two S's occur together, no two I's occur together, and S, I do not occur together. The number of such arrangement is

(a) 2400
(b) 5480
(c) 48000
(d) 50400

Solution :

S – S – S – I – I – I in the five places between S's and I's 5 letters viz. P, E, M, T, C will be placed.

SSSIII will get permutated among themselves in 6!/(3!*3!) = 20 ways.

5 letters in the gaps can get permutated among themselves in 5! ways.

So, number of arrangement = 5!*6!/(3!*3!) = 2400

Option (a) is correct.

146. Suppose that x is irrational number and a, b, c, d are rational numbers such that (ax + b)/(cx + d) is rational. Then it follows that
(a) a = c = 0
(b) a = c and b = d
(c) a + b = c + d
(d) ad = bc

Solution :

Let, (ax + b)/(cx + d) = m where m is rational.

- ⇨ ax + b = mcx + dm
- ⇨ a = mc and b = dm (equating the rational and irrational coefficients from both sides)
- ⇨ a/c = b/d = m
- ⇨ ad = bc

Option (d) is correct.

147. Let p, q and s be integers such that $p^2 = sq^2$. Then it follows that
(a) p is an even number
(b) if s divides p, then s is a perfect square
(c) s divides p
(d) q^2 divides p

Solution :

Now, if s divides p, then p^2/s is a perfect square and p^2 already a perfect square.

- ⇨ s must be a perfect square.

Option (b) is correct.

148. The number of pairs of positive integers (x, y) where x and y are prime numbers and $x^2 - 2y^2 = 1$ is
(a) 0
(b) 1
(c) 2
(d) 8

Solution :

Now, $x^2 - 2y^2 = 1$

If x and y are both odd primes then dividing the equation by 4 we get, $1 - 2*1 \equiv 1 \pmod 4$

Which is impossible so x, y both cannot be odd.

x cannot be even to hold the equality.

Let y is even prime = 2.

Therefore, $x^2 - 2*2^2 = 1$

- ⇨ $x^2 = 9$
- ⇨ $x = 3$
- ⇨ Only one solution x = 3, y = 2.

Option (b) is correct.

149. A point P with coordinates (x, y) is said to be *good* if both x and y are positive integers. The number of good points on the curve $xy = 27027$ is
   (a) 8
   (b) 16
   (c) 32
   (d) 64

Solution :

Now, $27027 = 3^3*7*11*13$

So, number of factors = (3 + 1)(1 + 1)(1 + 1)(1 + 1) = 32

Option (c) is correct.

150. Let p be an odd prime number. Then the number of positive integers k with $1 < k < p$, for which k^2 leaves a remainder of 1 when divided by p, is
   (a) 2
   (b) 1
   (c) $p - 1$
   (d) $(p - 1)/2$

Solution :

$k^2 \equiv 1 \pmod p$

- ⇨ $k^2 - 1 \equiv 0 \pmod p$
- ⇨ $(k - 1)(k + 1) \equiv 0 \pmod p$
- ⇨ $k - 1$ or $k + 1$ is divisible by p as p is prime.
- ⇨ Only one solution $k = p - 1$.

Option (b) is correct.

151. Let $n = 51! + 1$. Then the number of primes among $n + 1$, $n + 2$, …., $n + 50$ is
    (a) 0
    (b) 1
    (c) 2
    (d) More than 2.

Solution :

Now, $n + i = 51! + (i + 1) = (i + 1)(51*50*…(i + 2)*i! + 1)$

This can be factorized in this way when i = 1, 2, …., 50.

Therefore no prime numbers.

Option (a) is correct.

152. If three prime numbers, all greater than 3, are in A.P., then their common difference
    (a) must be divisible by 2 but not necessarily by 3
    (b) must be divisible by 3 but not necessarily by 2
    (c) must be divisible by both 2 and 3
    (d) need not be divisible by any of 2 and 3

Solution :

As primes great than 3 so all are odd. Hence the common difference must be divisible by 2.

Let the primes are p, $p + d$, $p + 2d$

Let $p \equiv 1 \pmod 3$

And $d \equiv 1 \pmod 3$

- ⇨ p + 2d is divisible by 3. Which is not possible as p + 2d is prime.
- ⇨ d ≡ 2 (mod 3)
- ⇨ p + d is divisible by 3. Which is not possible as p + d is prime.

Let, p ≡ 2 (mod 3) and d ≡ 1 (mod 3)

Then p + d is divisible by 3 which is not possible as p + d is prime.

Let, d ≡ 2 (mod 3)

- ⇨ p + 2d is divisible by 3 which is not possible as p +2d is prime.
- ⇨ d is divisible by 3.

Option (c) is correct.

153. Let N be a positive integer not equal to 1. Then note that none of the numbers 2, 3, …., N is a divisor of (N! – 1). From this, we can conclude that
(a) (N! – 1) is a prime number
(b) At least one of the numbers N + 1, N + 2, …., N! – 2 is a divisor of (N! – 1).
(c) The smallest number between N and N! which is a divisor of (N! – 1), is a prime number.
(d) None of the foregoing statements is necessarily correct.

Solution :

(N! – 1) may be a prime or may not be a prime number. So, option (a) and (b) not necessarily correct.

The smallest number between N and N! which divides (N! – 1) must be a prime because if it is not prime then it has a factor of primes between 1 and N. But no primes between 1 and N divides (N! – 1).

Option (c) is correct.

154. The number 1000! = 1*2*3*….*1000 ends exactly with
(a) 249 zeros
(b) 250 zeros
(c) 240 zeros
(d) 200 zeros.

Solution :

Number of zeros at the end of 1000! = $[1000/5] + [1000/5^2] + [1000/5^3] + [1000/5^4]$ where [x] denotes the greatest integer less than or equal to x.

= 200 + 40 + 8 + 1 = 249

Option (a) is correct.

155. Let A denote the set of all prime numbers, B the set of all prime numbers and the number 4, and C the set of all prime numbers and their squares. Let D be the set of positive integers k, for which (k – 1)!/k is not an integer. Then
(a) D = A
(b) D = B
(c) D = C
(d) B subset of D subset of C.

Solution :

Now, (k – 1)!/k is not an integer if k is prime. If k can be factored then the factors will come from 1 to k. Therefore k will divide (k – 1)! except 4.

Therefore, D = B.

Option (b) is correct.

156. Let n be any integer. Then n(n + 1)(2n + 1)
(a) is a perfect square
(b) is an odd number
(c) is an integral multiple of 6
(d) does not necessarily have any foregoing properties.

Solution :

n(n + 1) is divisible by 2 as they are consecutive integers.

Now, let n ≡ 1 (mod 3)

Then 2n + 1 is divisible by 3.

Let n ≡ 2 (mod 3)

Then n + 1 is divisible by 3

Now, if n is divisible by 3, then we can say that n(n + 1)(2n + 1) is always divisible by 2*3 = 6

Option (c) is correct.

157. The numbers 12n + 1 and 30n + 2 are relatively prime for
(a) any positive integer n
(b) infinitely many, but not all, integers n
(c) for finitely many integers n
(d) none of the above.

Solution :

Let p divides both 12n + 1 and 30n + 2

⇨ $12n + 1 \equiv 0 \pmod{p}$
⇨ $12n \equiv -1 \pmod{p}$

Also, $30n + 2 \equiv 0 \pmod{p}$

⇨ $60n + 4 \equiv 0 \pmod{p}$ (p is odd prime as 12n + 1 is odd)
⇨ $5*(12n) + 4 \equiv 0 \pmod{p}$
⇨ $5(-1) + 4 \equiv 0 \pmod{p}$
⇨ $-1 \equiv 0 \pmod{p}$ which is impossible.

Option (a) is correct.

158. The expression $1 + (1/2)(^nC_1) + (1/3)(^nC_2) + \ldots + \{1/(n + 1)\}(^nC_n)$ equals
(a) $(2^{n+1} - 1)/(n + 1)$
(b) $2(2^n - 1)/(n + 1)$
(c) $(2^n - 1)/n$
(d) $2(2^{n+1} - 1)/(n + 1)$

Solution :

Now, $\{1/(r + 1)\}(^nC_r) = \{1/(r + 1)\}*n!/\{(n - r)r!\} = \{1/(n + 1)\}(n + 1)!/\{(n - r)!(r + 1)!\} = \{1/(n + 1)\}(^{n+1}C_{r+1})$

So, the expression becomes, $\{1/(n + 1)\}[^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_{n+1}] = (2^{n+1} - 1)/(n + 1)$

Option (a) is correct.

159. The value of $^{30}C_1/2 + {}^{30}C_3/4 + {}^{30}C_5/6 + \ldots + {}^{30}C_{29}/30$ is
(a) $2^{31}/30$
(b) $2^{30}/31$
(c) $(2^{31} - 1)/31$
(d) $(2^{30} - 1)/31$

Solution :

Now, $^{30}C_{2r-1}/2r = 30!/\{(30 - 2r + 1)!(2r - 1)!2r\} = (1/31)[31!/\{(31 - 2r)!(2r)!\}] = {}^{31}C_{2r}/31$

Now, the expression becomes, $(1/31)[{}^{31}C_2 + {}^{31}C_4 + \ldots + {}^{31}C_{30}] = (2^{30} - {}^{31}C_0)/31 = (2^{30} - 1)/31$.

Option (c) is correct.

160. The value of $[\sum({}^kC_i)({}^{M-k}C_{100-i})\{(k - i)/(M - 100)\}]/{}^MC_{100}$, where $M - k > 100$, $k > 100$ and $^mC_n = m!/\{(m - n)!n!\}$ equals (summation running from i = 0 to i = 100)
(a) k/M
(b) M/k
(c) k/M^2
(d) M/k^2

Solution :

Option (a) is correct.

161. The remainder obtained when 1! + 2! + …. + 95! Is divided by 15 is
(a) 14
(b) 3
(c) 1
(d) None of the foregoing numbers.

Solution :

From 5! all the numbers are divisible by 15.

So, it is required to find the remainder when 1! + 2! + 3! + 4! is divided by 15

$1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33 \equiv 3 \pmod{15}$

Option (b) is correct.

162. Let $x_1, x_2, \ldots, x_{50}$ be fifty integers such that the sum of any six of them is 24. Then
   (a) The largest of x_i equals 6
   (b) The smallest of x_i equals 3
   (c) $x_{16} = x_{34}$
   (d) none of the foregoing statements is correct.

Solution :

All x_i's are equal and = 4. (See solution of problem 82)

Thus, option (c) is correct.

163. Let $x_1, x_2, \ldots, x_{50}$ be fifty nonzero numbers such that $x_i + x_{i+1} = k$ for all i, $1 \le i \le 49$. If $x_{14} = a$, $x_{27} = b$ then $x_{20} + x_{37}$ equals
   (a) $2(a + b) - k$
   (b) $k + a$
   (c) $k + b$
   (d) none of the foregoing expressions.

Solution :

Now, $x_i + x_{i+1} = k$ and $x_{i+1} + x_{i+2} = k$

⇨ $x_i = x_{i+2}$
⇨ $x_{14} = x_{16} = x_{18} = x_{20} = a$

And, $x_{27} = x_{29} = x_{31} = x_{33} = x_{35} = x_{37} = b$

Therefore, $x_{20} + x_{37} = a + b$

Now, $x_{14} + x_{15} = k$

$x_{15} = x_{17} = x_{19} = x_{21} = x_{23} = x_{25} = x_{27} = b$

⇨ $a + b = k$
⇨ $x_{20} + x_{37} = 2(a + b) - k$

Option (a) is correct.

164. Let S be the set of all numbers of the form $4^n - 3n - 1$, where $n = 1, 2, 3, \ldots$. Let T be the set of all numbers of the form $9(n - 1)$,

where n = 1, 2, 3, ….. Only one of the following statements is correct. Which one is it?
(a) Each number in S is also in T
(b) Each number in T is also in S
(c) Every number in S in T and every number in T is in S.
(d) There are numbers in S which are not in T and there are numbers in T which are not in S.

Solution :

Now, $4^n = (1 + 3)^n = 1 + 3n + {}^nC_2(3)^2 + ….. + (3)^n$

⇨ $4^n - 3n - 1 = 9({}^nC_2 + …. + 3^{n-2})$

Clearly, option (a) is correct.

165. The number of four-digit numbers greater than 5000 that can be formed out of the digits 3, 4, 5, 6 and 7, no digit being repeated, is
(a) 52
(b) 61
(c) 72
(d) 80

Solution :

First digit can be 5, 6 or 7.

If first digit is 5 number of such numbers = ${}^4C_3$*3! = 24

Similarly, if first digit is 6 or 7 in each case number of such numbers = 24

Therefore, total number of such numbers = 24*3 = 72

Option (c) is correct.

166. The number of positive integers of 5 digits such that each digit is 1, 2 or 3, and all three of the digits appear at least once, is
(a) 243
(b) 150
(c) 147
(d) 193

Solution :

Number of combinations = ${}^{5-1}C_{3-1} = {}^{4}C_{2} = 6$.

Three 1, one 2, one 3, number of numbers = 5!/3! = 20

Two 1, one 2, two 3, number of numbers = 5!/(2!*2!) = 30

Two 1, two 2, one 3, number of such numbers = 5!/(2!*2!) = 30

One 1, one 2, three 3, number of such numbers = 5!/3! = 20

One 1, two 2, two 3, number of such numbers = 5!/(2!*2!) = 30

One 1, three 2, one 3, number of such numbers = 5!/(3!) = 20

Therefore, total number of such numbers = 20 + 30 + 30 + 20 + 30 + 20 = 150

Option (b) is correct.

167. In a chess tournament, each of the 5 players plays against every other player. No game results in a draw and the winner of each game gets one point and loser gets zero. Then which one of the following sequences *cannot* represent the scores of the five players?
(a) 3, 3, 2, 1, 1
(b) 3, 2, 2, 2, 1
(c)2, 2, 2, 2, 2
(d) 4, 4, 1, 1, 0

Solution :

As in option (d) we see that first two players have won all the games.

It cannot be true because the game in between them one must lose and one must win.

So, it is not possible.

Option (d) is correct.

168. Ten (10) persons numbered 1, 2, …, 10 play a chess tournament, each player playing against every other player exactly one game. Assume that each game results in a win for one of the players (that is, there is no draw). Let w_1, w_2, …., w_{10} be the number of games won by players 1, 2, …, 10 respectively. Also, let

$l_1, l_2, \ldots, l_{10}$ be the number of games lost by players 1, 2, …, 10 respectively. Then

(a) $w_1^2 + w_2^2 + \ldots + w_{10}^2 = 81 - (l_1^2 + l_2^2 + \ldots. + l_{10}^2)$

(b) $w_1^2 + w_2^2 + \ldots. + w_{10}^2 = 81 + (l_1^2 + l_2^2 + \ldots.. + l_{10}^2)$

(c) $w_1^2 + w_2^2 + \ldots. + w_{10}^2 = l_1^2 + l_2^2 + \ldots. + l_{10}^2$

(d) none of the foregoing equalities is necessarily true.

Solution :

Now, $w_1 + w_2 + \ldots. + w_{10} = l_1 + l_2 + \ldots.. + l_{10}$ = number of games.

And, $w_i + l_i$ = constant = one player playing number of games for i = 1, 2, …, 10

- ⇨ $(w_1 - l_1) + (w_2 - l_2) + \ldots + (w_{10} - l_{10}) = 0$
- ⇨ $(w_1 + l_1)(w_1 - l_1) + (w_2 + l_2)(w_2 - l_2) + \ldots. + (w_{10} + l_{10})(w_{10} - l_{10}) = 0$
- ⇨ $w_1^2 - l_1^2 + w_2^2 - l_2^2 + \ldots. + w_{10}^2 - l_{10}^2 = 0$
- ⇨ $w_1^2 + w_2^2 + \ldots + w_{10}^2 = l_1^2 + l_2^2 + \ldots. + l_{10}^2$

Option (c) is correct.

169. A game consisting of 10 rounds is played among three players A, B and C as follows : Two players play in each round and the loser is replaced by the third player in the next round. If the only rounds when A played against B are the first, fourth and tenth rounds, the number of games won by C

(a) is 5

(b) is 6

(c) is 7

(d) cannot be determined from the above information.

Solution :

First round between A and B.

One of them lost and C joined in 2nd round.

C won as A and B did not play in 3rd round. (**So win 1)**

C lost in 3rd round as A and B played fourth round.

C joined in 5th round and won as A and B did not play 6th round (**So win 1**)

6^{th}, 7^{th}, 8^{th} round won by C (**So 3 wins**), in 9^{th} round C lost as A and B played 10^{th} round.

So, number of games won by C = 5.

Option (a) is correct.

170. An nxn chess board is a square of side n units which has been sub-divided into n^2 unit squares by equally-spaced straight lines parallel to the sides. The total number of squares of all sizes on the nxn board is
   (a) $n(n + 1)/2$
   (b) $1^2 + 2^2 + …. + n^2$
   (c) $2*1 + 3*2 + 4*3 + …. + n*(n - 1)$
   (d) Given by none of the foregoing expressions.

Solution :

If we take all the lines then there are n^2 squares.

If we take 2 unit squares together then there are $(n - 1)^2$ squares.

If we take 3 unit squares together then there are $(n - 2)^2$ squares.

…

…

If we take all the n unit squares then there are 1^2 unit squares.

Therefore, total number of squares = $1^2 + 2^2 + …. + n^2$.

Option (b) is correct.

171. Given any five points in the square $I^2 = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$, only one of the following statements is true. Which one is it?
   (a) The five points lie on a circle.
   (b) At least one square can be formed using four of the five points.
   (c) At least three of the five points are collinear.
   (d) There are at least two points such that distance between them does not exceed $1/\sqrt{2}$.

Solution :

The farthest four points may be at the four corners of the square.

So, one of them must be nearer to some point wherever they are located, the distance less than or equal to half of the diagonal. If the fifth point is on the diagonal then the distance is $1/\sqrt{2}$, otherwise it is less.

Option (d) is correct.

172. The quantities l, c, h and m are measured in the units mentioned against each l : centimetre; c : centimetre per second; h : ergs*second; mc^2 : ergs. Of the expressions $\alpha = (ch/ml)^{1/2}$; $\beta = (mc/hl)^3$; $\gamma = h/mcl$, which ones are pure numbers, that is, do not involve any unit?
(a) Only α
(b) Only β
(c) Only γ
(d) None

Solution :

Clearly, option (c) is correct.

173. The number of distinct rearrangements of the letters of the word "MULTIPLE" that can be made preserving the order in which the vowels (U, I, E) occur and not containing the original arrangement is
(a) 6719
(b) 3359
(c) 6720
(d) 3214

Solution :

U, I, E can get permutated among themselves in 3! = 6 ways.

Out of them only one permutation is required.

Therefore in this permutation number of arrangements = total number of arrangement/6 = 8!/{(2!)6} = 3360.

Excluding the original arrangement 3360 – 1 = 3359

Option (b) is correct.

174. The number of terms in the expansion of $(x + y + z + w)^{10}$ is
(a) $^{10}C_4$
(b) $^{13}C_3$
(c) $^{14}C_4$
(d) 11^4

Solution :

Any term = (coefficient)$x^r y^s z^t w^u$ where r + s + t + w = 10 and r, s, t, w are non-negative integers.

Number of solution of this equation is $^{n+r-1}C_{r-1}$ where r = number of variables and n is the sum.

Here n = 10, r = 4. (See number theory note for proof)

Therefore, number of terms = $^{10+4-1}C_{4-1} = {}^{13}C_3$.

Option (b) is correct.

175. The number of ways in which three non-negative integers n_1, n_2, n_3 can be chosen such that $n_1 + n_2 + n_3 = 10$ is
(a) 66
(b) 55
(c) 10^3
(d) 10!/(3!2!1!)

Solution :

Number of solution of this equation is $^{10+3-1}C_{3-1} = {}^{12}C_2$ (see number theory note for proof) = 12*11/2 = 66.

Option (a) is correct.

176. In an examination, the score in each four languages – Bengali, Hindi, Urdu and Telegu – can be integers between 0 and 10. Then the number of ways in which a student can secure a total score of 21 is
(a) 880
(b) 760
(c) 450
(d) 1360

Solution :

Let B be the score in Bengali, H be the score in Hindi, U be the score in Urdu and T be the score in Telegu.

Therefore, B + H + U + T = 21 and B, H, U, T are non-negative integers and less than or equal to 10.

Let B = 0, H = 1, U = 10, T = 10 (one solution)

B = 0, H = 2, two solutions

B = 0, H = 3, three solutions.

…

…

B = 0, H = 10, ten solutions.

Number of solutions for B = 0 is 1 + 2 + … + 10 = 10*11/2 = 55

B = 1, H = 0, one solution

B = 1, H = 1, two solutions

…

..

B = 1, H = 10, eleven solutions

Number of solutions for B = 1 is 1 + 2 + … + 11 = 11*12/2 = 66

B = 2, H = 0, two solutions.

..

..

B = 2, H = 8, ten solutions.

B = 2, H = 9, eleven solutions.

B = 2, H = 10, ten solutions.

Number of solutions for B = 2 is 2 + 3 + … + 10 + 11 + 10 = 75.

B = 3, H = 0, three solutions.

..

..

B = 3, H = 7, ten solutions.

B = 3, H = 8, eleven solutions.

B = 3, H = 9, ten solutions.

B = 3, H = 10, nine solutions.

Number of solutions for B = 3 is 3 + 4 + … + 10 + 11 + 10 + 9 = 82

Number of solutions for B = 4 is 4 + 5 + … + 10 + 11 + 10 + 9 + 8 = 87

Number of solutions for B = 5 is 5 + 6 + … + 10 + 11 + 10 + … + 7 = 90

Number of solutions for B = 6 is, 6 + 7 + .. + 10 + 11 + 10 + … + 6 = 91

Number of solutions for B = 7 is, 7 + 8 + 9 + 10 + 11 + 10 + … + 5 = 90

Number of solutions for B = 8 is, 8 + 9 + 10 + 11 + 10 + … + 4 = 87

Number of solutions for B = 9 is, 9 + 10 + 11 + 9 + …. + 3 = 82

Number of solutions for B = 10 is 10 + 11 + 10 + 9 + … + 2 = 75

Total number of solutions = 55 + 2(75 + 82 + 87 + 90) + 91 + 66 = 880

Option (a) is correct.

177. The number of ordered pairs (x, y) of positive integers such that x + y = 90 and their greatest common divisor is 6 equals
(a) 15
(b) 14
(c)8
(d) 10

Solution :

Let $x = 6x_1$ and $y = 6y_1$

- ⇨ $6(x_1 + y_1) = 90$
- ⇨ $x_1 + y_1 = 15$ ($\gcd(x_1, y_1) = 1$)
- ⇨ (1, 14); (2, 13); (4, 11); (7, 8)

So there are 4*2 = 8 pairs.

Option (c) is correct.

178. How many pairs of positive integers (m, n) are there satisfying $m^3 - n^3 = 21$?

(a) Exactly one
(b) None
(c) Exactly three
(d) Infinitely many

Solution :

Now, $m^3 - n^3 = 21$

⇨ $(m - n)(m^2 + mn + n^2) = 3*7$

So, two case can be possible, $m - n = 3$, $m^2 + mn + n^2 = 7$ and $m - n = 1$, $m^2 + mn + n^2 = 21$

First case, $(3 + n)^2 + (3 + n)n + n^2 = 7$

⇨ $3n^2 + 9n + 2 = 0$
⇨ $n = \{-9 \pm \sqrt{(9^2 - 4*3*2)}\}/6$ = not integer solution.

So this case is not possible.

Second case, $(n + 1)^2 + (n + 1)n + n^2 = 21$

⇨ $3n^2 + 3n - 20 = 0$
⇨ $n = \{-3 \pm \sqrt{(9 + 4*3*20)}\}/6$ = not integer solution.
⇨ Option (b) is correct.

179. The number of ways in which three distinct numbers in A.P. can be selected from 1, 2, …, 24 is
(a) 144
(b) 276
(c) 572
(d) 132

Solution :

With 1 common difference we can select A.P.'s = 22

With 2 common difference we can select A.P.'s = 2*10 = 20

With 3 common difference we can select A.P.'s = 3*6 = 18

With 4 common difference we can select A.P.'s = 4*4 = 16

With 5 common difference we can select A.P.'s = 3*4 + 2 = 14

With 6 common difference we can select A.P.'s = 2*6 = 12

With 7 common difference we can select A.P.'s = 3*2 + 4*1 = 10

With 8 common difference we can select A.P.'s = 1*8 = 8

With 9 common difference we can select A.P.'s = (1, 10, 19); (2, 11, 20); (3, 12, 21); (4, 13, 22); (5, 14, 23); (6, 15, 24) = 6

With 10 common difference we can select A.P.'s = (1, 11, 21); (2, 12, 22); (3, 13, 23); (4, 14, 24) = 4

With 11 common difference we can select A.P.'s = (1, 12, 23); (2, 13, 24) = 2

So, total number of A.P.'s = 22 + 20 + … + 2 = (11/2){2*22 + (11 – 1)(-2)} = 11(22 – 10) = 11*12 = 132

Option (d) is correct.

[In general if we need to select A.P.'s from 1, 2, …, n then with common difference 1 we can select n – 2 A.P.'s , with common difference 2 the largest A.P. with first term will be n – 4, so n – 4 A.P.'s, with common difference 3 the largest A.P. with first term will be n – 6, so n – 6 A.P.'s and so on. So, number of A.P.'s will be (n – 2) + (n – 4) + ….. + up to 2 or 1 according to n is even or odd.]

180. The number of ways you can invite 3 of your friends on 5 consecutive days, exactly one friend a day, such that no friend is invited on more than two days is
(a) 90
(b) 60
(c) 30
(d) 10

Solution :

Let the friends are A, B, C.

We need to distribute A, B, C in 5 places such that A, B, C occurs at least once.

Two A, two B, one C = 5!/(2!2!) = 30

Two A, one B, two C = 5!/(2!2!) = 30

One A, two B, two C = 5!/(2!2!) = 30

Total number of ways = 30 + 30 + 30 = 90

Option (a) is correct.

181. Consider three boxes, each containing 10 balls labeled 1, 2, …, 10. Suppose one ball is drawn from each of the boxes. Denote by n_i, the label of the ball drawn from the i-th box, i = 1, 2, 3. Then the number of ways in which the balls can be chosen such that $n_1 < n_2 < n_3$, is
(a) 120
(b) 130
(c) 150
(d) 160

Solution :

If $n_1 = 1$, $n_2 = 2$, n_3 can be 8

If $n_1 = 1$, $n_2 = 3$, n_3 can be 7

So, if $n_1 = 1$ then the possible ways = 8 + 7 + … + 1 = 8*9/2 = 36

If, $n_1 = 2$, $n_2 = 3$, n_3 can be 7

So, if $n_1 = 2$, then the possible ways = 7 + 6 + … + 1 = 7*8/2 = 28

So, if $n_1 = 3$, then the possible ways = 6 + 5 + … + 1 = 6*7/2 = 21

So, these are the triangular numbers.

Therefore, total possible ways = ∑n(n + 1)/2 (summation running from n = 1 to n = 8) = (1/2)∑n^2 + (1/2)∑n = (1/2)*8*9*17/6 + (1/2)8*9/2 = 102 + 18 = 120

Option (a) is correct.

182. The number of sequences of length five with 0 and 1 as terms which contain at least two consecutive 0's is
(a) $4*2^3$
(b) ${}^5C_2$
(c) 20
(d) 19

Solution :

Two consecutive 0'sin left means third is 1, fourth and fifth can be put in 2*2 = 4 ways.

Two consecutive 0's in the middle then both side is 1 and another one can be put in 2 ways in both the side. Therefore, 2*2 = 4 sequences.

Two consecutive 0's in the right means third from right is 1. Fourth and fifth can be put in 2*2 = 4 ways.

So, for two consecutive 0's number of sequences = 4 + 4 + 4 = 12

Three consecutive zeros in left means fourth is 1 and fifth can be put in 2 ways.

Three consecutive 0's in middle means 1 way.

Three consecutive 0's in right means 2 ways.

For three consecutive 0's number of sequences = 2 + 1 + 2 = 5.

Four consecutive zeros = 1 + 1 = 2 sequences.

Five consecutive 0's – no sequence as there is no 1.

Total number of sequence = 12 + 5 + 2 = 19.

Option (d) is correct.

183. There are 7 identical white balls and 3 identical black balls. The number of distinguishable arrangements in a row of all the balls, so that no two black balls are adjacent, is
(a) 120
(b) 89(8!)
(c) 56
(d) $42*5^4$

Solution :

Total number of arrangements = 10!/(7!3!) = 10*9*8/6 = 120

Now, take the black balls as unit. So there are 8 units.

Therefore, total number of arrangements = 8!/7! = 8

Now, take 2 black balls as unit. There are 9 units.

Total number of arrangements = 9!/7! = 72

So, number of arrangements in which at least 2 black balls will come together = 72 – 8 = 64

So, number of required arrangements = 120 – 64 = 56.

Option (c) is correct.

184. In a multiple-choice test there are 6 questions. Four alternative answers are given for each question, of which only one answer is correct. If a candidate answers all the questions by choosing one answer for each question, then the number of ways to get 4 correct answers is
(a) $4^6 - 4^2$
(b) 135
(c) 9
(d) 120

Solution :

4 questions can be chosen from 6 questions in $^6C_4$ ways = 15 ways.

Now, rest two questions can be answered wrong in 3 ways each (because 1 is correct).

So number of ways of doing this = 3*3 = 9

Therefore, total number of ways = 15*9 = 135.

Option (b) is correct.

185. In a multiple-choice test there are 8 questions. Each question has 4 alternatives, of which only one is correct.. If a candidate answers all the questions by choosing one alternative for each, the number of ways of doing it so that exactly 4 answers are correct is
(a) 70
(b) 2835
(c) 5670
(d) None of the foregoing numbers.

Solution :

Same question as previous one. Total number of ways = $^8C_4*3^4$ = 5670.

Option (c) is correct.

186. Among the 8! Permutations of the digits 1, 2, 3, …, 8, consider those arrangements which have the following property : if you take any five consecutive positions, the product of the digits in

those positions is divisible by 5. The number of such arrangements is

(a) 7!

(b) 2*7!

(c) 8*7!

(d) $4(^7C_4)5!3!4!$

Solution :

So, 5 can be in 4^{th} or 5^{th} place.

In 4^{th} place total number of arrangements = 7!, same goes for 5^{th} place.

Therefore, total number of required permutations = 2*7!

Option (b) is correct.

187. A closet has 5 pairs of shoes. The number of ways in which 4 shoes can be chosen from it so that there will be no complete pair is

(a) 80

(b) 160

(c) 200

(d) None of the foregoing numbers.

Solution :

4 pairs can be chosen from 5 pairs in $^5C_4$ = 5 ways.

Out of these 4 pairs 1 shoe can to be chosen from each pair in $^2C_1 * ^2C_1 * ^2C_1 * ^2C_1$ = 16 ways.

Therefore, total number of ways = 5*16 = 80.

Option (a) is correct.

188. The number of ways in which 4 distinct balls can be put into 4 boxes labeled a, b, c, d so that exactly one box remains empty is

(a) 232

(b) 196

(c) 192

(d) 144

Solution :

Let box d is empty.

Number of ways in which we can put 4 distinct balls into 3 boxes where each box gets at least one ball = $3^4 - {}^3C_1*2^4 + {}^3C_2*1^4 - {}^3C_3*0^4$ (for this formula please see my number theory book)

= 81 – 48 + 3 – 0

= 36

Now, for four boxes there will be 36*4 = 144 ways.

Option (d) is correct.

189. The number of permutations of the letters a, b, c, d such that b does not follow a, and c does not follow b, and d does not follow c, is
(a) 12
(b) 11
(c) 14
(d) 13

Solution :

acbd, adcb, badc, bdac, bdca, cadb, cbad, cbda, dacb, dbac, dcba = 11.

Option (b) is correct.

190. The number of ways of seating three gentlemen and three ladies in a row, such that each gentlemen is adjacent to at least one lady, is
(a) 360
(b) 72
(c) 720
(d) None of the foregoing numbers.

Solution :

Three gentlemen together can seat in 4!*3! = 144 ways.

Now, two gentlemen at left end. Then a lady. Number of arrangement = ${}^3C_2*{}^3C_1*3!*2!$ = 108 ways. (2 gentlemen at left end can be chosen from 3 gentlemen in ${}^3C_2$ ways, they can get permuted among them in 2! ways. One lady from 3 ladies can be chosen in ${}^3C_1$ ways and the rest one lady and two gentlemen can get permutated among themselves in 3! ways)

Similarly, two gentlemen at right end and then one lady, number of arrangement = 108.

Total number of cancelled arrangement = 144 + 108 + 108 = 360.

Six gentlemen and six ladies can seat in 6! = 720 ways.

Therefore, number of arrangements = 720 – 360 = 360.

191. The number of maps f from the set {1, 2, 3} into the set {1, 2, 3, 4, 5} such that $f(i) \le f(j)$, whenever $i < j$, is
(a) 30
(b) 35
(c) 50
(d) 60

Solution :

1 - > 1, 2 - > 1, 3 can map to 5 numbers.

1 - > 1, 2 - > 2, 3 can map into 4 numbers.

So, for 1 - > 1 number of mapping = 5 + 4 + 3 + 2 + 1 = 15

1 - > 2, 2 - > 2, 3 can map to 4 numbers.

So, number of mapping = 4 + 3 + 2 + 1 = 10 numbers.

In general number of mapping = $\sum n(n + 1)/2$ (summation running from n = 1 to n = 5) = $(1/2)\sum n^2 + (1/2)\sum n = (1/2)*5*6*11/6 + (1/2)*5*6/2 = 55/2 + 15/2 = 35$.

Option (b) is correct.

192. For each integer i, $1 \le i \le 100$; ε_i be either +1 or -1. Assume that $\varepsilon_1 = +1$ and $\varepsilon_{100} = -1$. Say that a sign change occurs at $i \ge 2$ if ε_i, ε_{i-1} are of opposite sign. Then the total number of sign changes
(a) is odd
(b) is even
(c) is at most 50
(d) can have 49 distinct values

Solution :

Now, $\varepsilon_1 = +1$, now it will continue till it gets a -1, if any ε is -1 then the next ε will be +1 again because sign change will occur. So, if it gets again -1 then +1 will occur. So, + to + sign change is even, as + to – and then – to +, now, ε_{100} is -1. So, number of sign changes must be odd.

Option (a) is correct.

193. Let S = {1, 2, …, n}. The number of possible pairs of the form (A, B) with A subset of B for subsets A and B of S is
(a) 2^n
(b) 3^n
(c) $\sum({}^nC_k)({}^nC_{n-k})$ (summation running from k = 0 to k = n)
(d) n!

Solution :

We can choose r elements for B from n elements in ${}^nC_r$ ways. Now, for r elements number of subsets = 2^r.

Therefore, number of pairs = ${}^nC_r*2^r$.

Therefore, total number of pairs = $\sum({}^nC_r)*2^r$ (r running from 0 to n) = 3^n.

Option (b) is correct.

194. There are 4 pairs of shoes of different sizes. Each of the shoes can be colored with one of the four colors : black, brown, white and red. In how many ways can one color the shoes so that in at least three pairs, the left and the right shoes do not have the same color?
(a) 12^4
(b) $28*12^3$
(c) $16*12^3$
(d) $4*12^3$

Solution :

3 pairs different color + 4 pairs different color. (at least 3 pairs different color)

3 pairs different color :

We can choose any 3 pairs of shoes out of 4 pairs in ${}^4C_3$ = 4 ways. We can paint first shoe in ${}^4C_1$ and second shoe in ${}^3C_1$ i.e. one pair in ${}^4C_1*{}^3C_1$

=12 ways. So, three pairs in 12^3 ways. And the last have same color and we can choose any one color from 4 colors in $^4C_1$ = 4 ways.

So, total number of ways = $4*12^3*4 = 16*12^3$

4 pairs different color :

In this case clearly, total number of ways = $12*12^3$

So, at least 3 pair of shoes are of different color the number of ways of painting = $12*12^3 + 16*12^3 = 28*12^3$

Option (b) is correct.

195. Let S = {1, 2, …., 100}. The number of nonempty subsets A of S such that the product of elements in A is even is
(a) $2^{50}(2^{50} – 1)$
(b) $2^{100} – 1$
(c) $2^{50} – 1$
(d) None of these numbers.

Solution :

We can select at least one even numbers in $^{50}C_1 + {}^{50}C_2 + {}^{50}C_3 + …. + {}^{50}C_{50} = 2^{50} – 1$

We can select any number of odd numbers in $^{50}C_0 + {}^{50}C_1 + …. + {}^{50}C_{50} = 2^{50}$

So, total number of subsets = $2^{50}(2^{50} – 1)$

Option (a) is correct.

196. The number of functions f from {1, 2, …., 20} onto {1, 2, …, 20} such that f(k) is a multiple of 3 whenever k is a multiple of 4 is
(a) 5!*6!*9!
(b) 5^6*15!
(c) 6^5*14!
(d) 15!*6!

Solution :

{4, 8, 12, 16, 20} - > {3, 6, 9, 12, 15, 18}

We can select any 5 numbers from 6 numbers of the later set in ${}^6C_5$ ways and they will get permutated in 5! ways. So, in this case number of permutation = ${}^6C_5*5! = 6!$

Rest 15 numbers will map to 15 numbers in 15! ways.

Therefore, total number of functions = 6!*15!

Option (d) is correct.

197. Let X = $\{a_1, a_2, …., a_7\}$ be a set of seven elements and Y = $\{b_1, b_2, b_3\}$ a set of three elements. The number of functions f from X to Y such that (i) f is onto and (ii) there are exactly three elements x in X such that $f(x) = b_1$, is
(a) 490
(b) 558
(c) 560
(d) 1680

Solution :

We can select any 3 elements from 7 elements to be mapped to b_1 in ${}^7C_3$ ways.

Now, rest 4 elements needs to be distributed among b_2 and b_3 so that b_2 and b_3 gets at least one element.

Now, we can distribute the 4 elements in b_2 and b_3 in $2*2*2*2 = 2^4$ ways out of which in 2 ways one is for b_1 gets none and b_2 gets every elements and other is b_2 gets none and b_1 gets all elements. So, total number of ways = $2^4 - 2 = 14$.

Therefore, total mapping = ${}^7C_3*14 = 490$.

Option (a) is correct.

198. Consider the quadratic equation of the form $x^2 + bx + c = 0$. The number of such equations that have real roots and coefficients b and c from the set {1, 2, 3, 4, 5} (b and c may be equal) is
(a) 18
(b) 15
(c) 12
(d) None of the foregoing quantities.

Solution :

Now, $b^2 > 4c$

b cannot be equal to 1.

If b = 2, c = 1

If b = 3, c = 1, 2

If b = 4, c = 1, 2, 3, 4

If b = 5, c = 1, 2, 3, 4, 5

Total number of equations = 1 + 2 + 4 + 5 = 12

Option (c) is correct.

199. Let A_1, A_2, A_3 be three points on a straight line. Let B_1, B_2, B_3, B_4, B_5 be five points on a straight line parallel to the first line. Each of the three points on the first line is joined by a straight line to each of the five points on the second straight line. Further, no three or more of these joining lines meet at a point except possibly at the A's or the B's. Then the number of points of intersections of the joining lines lying between the two given straight lines is
(a) 30
(b) 25
(c) 35
(d) 20

Solution :

We first calculate number of intersection points when a straight line from A_1 meets other straight lines from A_2, A_3.

A_2, A_3, -> B_1 – 2 +2 + 2 + 2 = 8 points (A_1 – B_2, B_3, B_4, B_5)

A_2, A_3 -> B_2 – 2 + 2 + 2 = 6 points (A_1 – B_3, B_4, B_5)

A_2, A_3 -> B_3 – 2 + 2 = 4 points (A_1 – B_4, B_5)

A_2, A_3 -> B_4 – 2 points (A_1 – B_5)

A_2 -> B_5 – 4 points (A_3 – B_4, B_3, B_2, B_1)

A_2 -> B_4 – 3 points (A_3 – B_3, B_2, B_1)

A_2 -> B_3 – 2 points (A_3 – B_2, B_1)

A_2 -> B_2 – 1 point (A_3 – B_1)

So, total = 8 + 6 + 4 + 2 + 4 + 3 + 2 + 1 = 30 points.

Option (a) is correct.

200. There are 11 points on a plane with 5 lying on one straight line and another 5 lying on second straight line which is parallel to the first line. The remaining point is not collinear with any two of the previous 10 points. The number of triangles that can be formed with vertices chosen from these 11 points is
(a) 85
(b) 105
(c) 125
(d) 145

Solution :

We can choose 1 point from first straight line, 1 point from second straight line and 1 the single point. Number of ways of doing this ${}^5C_1*{}^5C_1*1 = 25$.

We can choose 2 points from first straight line and 1 single point in ${}^5C_2*1 = 10$, same goes for the second straight line, so number of triangles = 10*2 = 20.

We can choose 1 point from first straight line and 2 points from second straight line and vice versa. Number of triangles = $2*{}^5C_2*{}^5C_1 = 2*10*5 = 100$.

Total number of triangles = 25 + 20 + 100 = 145.

Option (d) is correct.

201. Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers such that lim $a_n = \infty$ as n -> ∞. For any real number x, define an integer-valued function f(x) as the smallest positive integer n for which $a_n \geq x$. Then for any integer n ≥ 1 and any real number x,
(a) $f(a_n) \leq n$ and $a_{f(x)} \geq x$
(b) $f(a_n) \leq n$ and $a_{f(x)} \leq x$
(c) $f(a_n) \geq n$ and $a_{f(x)} \geq x$
(d) $f(a_n) \geq n$ and $a_{f(x)} \leq x$

Solution :

Option (a) is correct.

202. There are 25 points in a plane, of which 10 are on the same line. Of the rest, no three number are collinear and no two are collinear with any of the first ten points. The number of different straight lines that can be formed joining these points is
(a) 256
(b) 106
(c) 255
(d) 105

Solution :

From 15 non-collinear points number of straight lines can be formed = ${}^{15}C_2$ = 105

Taking 1 point from 10 collinear points and taking 1 point from 15 non-collinear points number of straight lines can be formed = ${}^{10}C_1 * {}^{15}C_1$ = 150.

One straight line joining the ten points.

Therefore total number of straight lines = 105 + 150 + 1 = 256.

Option (a) is correct.

203. If $f(x) = \sin(\log_{10}x)$ and $h(x) = \cos(\log_{10}x)$, then $f(x)f(y) – (1/2)[h(x/y) – h(xy)]$ equals
(a) $\sin[\log_{10}(xy)]$
(b) $\cos[\log_{10}(xy)]$
(c) $\sin[\log_{10}(x/y)]$
(d) none of the foregoing expressions.

Solution :

$f(x)f(y) – (1/2)[h(x/y) – h(xy)] = \sin(\log_{10}x)\sin(\log_{10}y) – (1/2)[\cos\log_{10}(x/y) – \cos\log_{10}(xy)]$

$= (1/2)[2\sin(\log_{10}x)\sin(\log_{10}y)] – (1/2)[\cos\log_{10}(x/y) – \cos\log_{10}(xy)]$

$= (1/2)[\cos(\log_{10}x – \log_{10}y) – \cos(\log_{10}x + \log_{10}y) – \cos\log_{10}(x/y) + \cos\log_{10}(xy)]$

$= (1/2)[\cos\log_{10}(x/y) – \cos\log_{10}(xy) – \cos\log_{10}(x/y) + \cos\log_{10}(xy)]$

$= 0$

Option (d) is correct.

204. The value of $\log_5(125)(625)/25$ is
   (a) 725
   (b) 6
   (c) 3125
   (d) 5

Solution :

$\log_5(125)(625)/25 = \log_5 5*5^4 = \log_5 5^5 = 5\log_5 5 = 5$

Option (d) is correct.

205. The value of $\log_2 10 - \log_8 125$ is
   (a) $1 - \log_2 5$
   (b) 1
   (c) 0
   (d) $1 - 2\log_2 5$

Solution :

Now, $\log_2 10 - \log_8 125 = \log_2 10 - (3/3)\log_2{}^5 = \log_2 10 - \log_2 5 = \log_2(10/5) = \log_2 2 = 1$

Option (b) is correct.

206. If $\log_k x*\log_5 k = 3$ then x equals
   (a) k^5
   (b) k^3
   (c) 125
   (d) 245

Solution :

Now, $\log_k x*\log_5 k = 3$

- ⇨ $\{(\log x)/(\log k)\}*\{\log k/\log 5) = 3$
- ⇨ $(\log x)/(\log 5) = 3$
- ⇨ $\log_5 x = 3$
- ⇨ $x = 5^3 = 125$

Option (c) is correct.

207. If $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$, then the number of real x satisfying the equation $(\log_a x)(\log_b x) = \log_a b$ is
(a) 0
(b) 1
(c) 2
(d) Infinite

Solution :

Now, $(\log_a x)(\log_b x) = \log_a b$

- ⇨ $\{(\log x)/(\log a)\}\{(\log x/\log b)\} = \log b/\log a$
- ⇨ $(\log x)^2 = (\log b)^2$
- ⇨ $\log x = \pm\log b$
- ⇨ $\log x = \log(b)^{\pm 1}$
- ⇨ $x = b, 1/b$
- ⇨ 2 solutions.

Option (c) is correct.

208. If $\log_{10}x = 10^{\wedge}(\log_{100}4)$, then x equals
(a) 4^{10}
(b) 100
(c) $\log_{10}4$
(d) none of the foregoing numbers.

Solution :

Now, $\log_{10}x = 10^{\wedge}(\log_{100}4) = 10^{\wedge}\{(2/2)\log_{10}2\} = 10^{\wedge}(\log_{10}2)$

Now, $10^{\wedge}(\log_{10}2) = a$ (say)

- ⇨ $(\log_{10}2)\log_{10}10 = \log_{10}a$
- ⇨ $\log_{10}2 = \log_{10}a$
- ⇨ $a = 2$
- ⇨ $\log_{10}x = 2$
- ⇨ $x = 10^2 = 100$

Option (b) is correct.

209. If $\log_{12}27 = a$, then $\log_6 16$ equals
(a) $(1 + a)/a$

(b) $4(3 - a)/(3 + a)$
(c) $2a/(3 - a)$
(d) $5(2 - a)/(2 + a)$

Solution :

Now, $\log_{12}27 = a$

- ⇨ $\log 27/\log 12 = a$
- ⇨ $\log 3^3/(\log 2^2 + \log 3) = a$
- ⇨ $3\log 3/(2\log 2 + \log 3) = a$
- ⇨ $(2\log 2 + \log 3)/\log 3 = 3/a$
- ⇨ $2\log 2/\log 3 + 1 = 3/a$
- ⇨ $\log 2/\log 3 = (½)(3/a - 1) = (3 - a)/2a$
- ⇨ $\log 3/\log 2 = 2a/(3 - a)$

Now, $\log_6 16 = \log 16/\log 6 = \log 2^4/(\log 2 + \log 3) = 4\log 2/(\log 3 + \log 2) = 4/(\log 3/\log 2 + 1) = 4/(2a/(3 - a) + 1) = 4(3 - a)/(3 + a)$

Option (b) is correct.

210. Consider the number $\log_{10}2$. It is
(a) Rational number less than 1/3 and greater than ¼
(b) A rational number less than ¼
(c) An irrational number less than ½ and greater than ¼
(d) An irrational number less than ¼

Solution :

Let $\log_{10}2 = x$

- ⇨ $10^x = 2$
- ⇨ $x < 1/3$ as $8^{1/3} = 2$, $10 > 8 \Rightarrow 10^{1/3} > 2$
- ⇨ $x > ¼$ as $16^{1/4} = 2$, $10 < 16 \Rightarrow 10^{1/4} < 2$
- ⇨ $¼ < x < 1/3$

Now, let x is rational $= p/q$ where $q \neq 0$ and $\gcd(p, q) = 1$

$10^{p/q} = 2$

- ⇨ $10^p = 2^q$
- ⇨ $5^p = 2^{q-p}$

LHS is odd and RHS is even, only solution p = q which is not possible also q = p = 0. Not possible.

- ⇨ x is irrational.

Option (c) is correct.

211. If $y = a + b\log_e x$, then
(a) $1/(y - a)$ is proportional to x^b
(b) $\log_e y$ is proportional to x
(c) e^y is proportional to x^b
(d) $y - a$ is proportional to x^b

Solution :

$e^y = e^a x^b$

Option (c) is correct.

212. Let $y = \log_a x$ and $a > 1$. Then only one of the following statements is *false*. Which one is it?
(a) If $x = 1$, then $y = 0$
(b) If $x < 1$, then $y < 0$
(c) If $x = ½$ then $y = ½$
(d) If $x = a$, then $y = 1$

Solution :

Clearly, Option (a) is true.

Clearly, option (b) is true.

Option (c) is false and option (d) is true.

Option (c) is correct.

213. If $p = s/(1 + k)^n$ then n equals
(a) $\log[n/\{p(1 + k)\}]$
(b) $\log(s/p)/\log(1 + k)$
(c) $\log s/\log\{p(1 + k)\}$
(d) $\log(1 + k)/\log(s/p)$

Solution :

$\log p = \log s - n\log(1 + k)$

⇨ $n\log(1 + k) = \log s - \log p = \log(s/p)$

⇨ $n = \log(s/p)/\log(1 + k)$

Option (b) is correct.

214. If $(\log_5 x)(\log_x 3x)(\log_{3x} y) = \log_x x^3$, then y equals
(a) 125
(b) 25
(c) 5/3
(d) 243

Solution :

Now, $(\log_5 x)(\log_x 3x)(\log_{3x} y) = \log_x x^3$

⇨ $(\log x/\log 5)(\log 3x/\log x)(\log y/\log 3x) = 3\log_x x$
⇨ $(\log y)/(\log 5) = 3$
⇨ $\log_5 y = 3$
⇨ $y = 5^3 = 125$

Option (a) is correct.

215. If $(\log_5 k)(\log_3 5)(\log_k x) = k$, then the value of x equals
(a) k^3
(b) 5^k
(c) k^5
(d) 3^k

Solution :

Now, $(\log_5 k)(\log_3 5)(\log_k x) = k$

⇨ $(\log k/\log 5)(\log 5/\log 3)(\log x/\log k) = k$
⇨ $(\log x)/(\log 3) = k$
⇨ $\log_3 x = k$
⇨ $x = 3^k$

Option (d) is correct.

216. Given that $\log_p x = \alpha$ and $\log_q x = \beta$, the value of $\log_{p/q} x$ equals
(a) $\alpha\beta/(\beta - \alpha)$
(b) $(\beta - \alpha)/\alpha\beta$
(c) $(\alpha - \beta)/\alpha\beta$
(d) $\alpha\beta/(\alpha - \beta)$

Solution :

$\log_p x = \alpha$

- ⇨ $\log x/\log p = \alpha$
- ⇨ $\log p/\log x = 1/\alpha$

Similarly, $\log q/\log x = 1/\beta$

Subtracting the above equations, we get, $\log p/\log x - \log q/\log p = 1/\alpha - 1/\beta$

- ⇨ $(\log p - \log q)/\log x = (\beta - \alpha)/\alpha\beta$
- ⇨ $\log(p/q)/\log x = (\beta - \alpha)/\alpha\beta$
- ⇨ $\log x/\log(p/q) = \alpha\beta/(\beta - \alpha)$
- ⇨ $\log_{p/q} x = \alpha\beta/(\beta - \alpha)$

Option (a) is correct.

217. If $\log_{30}3 = a$ and $\log_{30}5 = b$, then $\log_{30}8$ is equal to
    (a) $a + b$
    (b) $3(1 - a - b)$
    (c) 12
    (d) 12.5

Solution :

Now, $\log_{30}3 + \log_{30}5 = a + b$

- ⇨ $\log_{30}15 = a + b$
- ⇨ $\log 15/\log 30 = a + b$
- ⇨ $\log 15/(\log 2 + \log 15) = a + b$
- ⇨ $(\log 2 + \log 15)/\log 15) = 1/(a + b)$
- ⇨ $\log 2/\log 15 + 1 = 1/(a + b)$
- ⇨ $\log 2/\log 15 = 1/(a + b) - 1 = (1 - a - b)/(a + b)$
- ⇨ $\log 15/\log 2 = (a + b)/(1 - a - b)$
- ⇨ $\log 15/\log 2 + 1 = (a + b)/(1 - a - b) + 1$
- ⇨ $(\log 15 + \log 2)/\log 2 = (a + b + 1 - a - b)/(1 - a - b)$
- ⇨ $\log 30/\log 2 = 1/(1 - a - b)$
- ⇨ $\log 2/\log 30 = 1 - a - b$
- ⇨ $\log_{30}2 = 1 - a - b$
- ⇨ $3\log_{30}2 = 3(1 - a - b)$
- ⇨ $\log_{30}2^3 = 3(1 - a - b)$
- ⇨ $\log_{30}8 = 3(1 - a - b)$

Option (b) is correct.

218. If $\log_a x = 6$ and $\log_{25a}(8x) = 3$, then a is
   (a) 8.5
   (b) 10
   (c) 12
   (d) 12.5

Solution :

$x = a^6$ and $8x = (25a)^3$

- ⇨ $x/8x = a^6/(25a)^3$
- ⇨ $(1/2)^3 = (a/25)^3$
- ⇨ $a/25 = ½$
- ⇨ $a = 12.5$

Option (d) is correct.

219. Let $a = (\log_{100}10)(\log_2(\log_4 2))(\log_4(\log_2(256)^2))/(\log_4 8 + \log_8 4)$ then the value of a is
   (a) -1/3
   (b) 2
   (c) -6/13
   (d) 2/3

Solution :

$a = (1/2)(\log_2(1/2))(\log_4 16)/(3/2 + 2/3) = -(1/2)*2/(13/6) = -6/13$

Option (c) is correct.

220. If $f(x) = \log\{(1 + x)/(1 - x)\}$, then $f(x) + f(y)$ is
   (a) $f(x + y)$
   (b) $f((x + y)/(1 + xy))$
   (c) $(x + y)f(1/(1 + xy))$
   (d) $f(x) + f(y)/(1 + xy)$

Solution :

$f(x) + f(y) = \log\{(1 + x)/(1 - x)\} + \log\{(1 + y)/(1 - y)\} = \log[(1 + x)(1 + y)/\{(1 - x)(1 - y)\}] = \log[(1 + xy + x + y)/\{1 + xy - (x + y)\}] = \log[\{1 + (x + y)/(1 + xy)\}/\{1 - (x + y)/(1 + xy)\}] = f((x + y)/(1 + xy))$

Option (b) is correct.

221. If $\log_{ab}a = 4$, then the value of $\log_{ab}(\sqrt[3]{a}/\sqrt{b})$ is
   (a) 17/6
   (b) 2
   (c) 3
   (d) 7/6

Solution :

Now, $\log_{ab}a = 4$

- ⇨ loga/(loga + logb) = 4
- ⇨ (loga + logb)/loga = ¼
- ⇨ 1 + logb/loga = ¼
- ⇨ logb/loga = -3/4
- ⇨ loga/logb = -4/3

Now, $\log_{ab}(\sqrt[3]{a}/\sqrt{b})$ = (1/3)$\log_{ab}a$ – (1/2)$\log_{ab}b$ = 4/3 – (1/2)logb/(loga + logb) = 4/3 – (1/2)/(loga/logb + 1) = 4/3 – (1/2)/(-4/3 + 1) = 4/3 + 3/2 = 17/6

Option (a) is correct.

222. The value of √{10^(2 + (1/2)$\log_{10}16$)} is
   (a) 80
   (b) 20√2
   (c) 40
   (d) 20

Solution :

√{10^(2 + (1/2)$\log_{10}16$)} = √[10^{2(1 + $\log_{10}2$)}] = 10^(1 + $\log_{10}2$ = 10*10^($\log_{10}2$) = 10*2 = 20

Option (d) is correct.

223. If $\log_b a = 10$, then $\log_{b5}a^3$ (base is b^5) equals
   (a) 50/3
   (b) 6
   (c) 5/3
   (d) 3/5

Solution :

Now, $\log_{b^5}a^3 = (3/5)\log_b a = (3/5)*10 = 6$

Option (b) is correct.

224. If $(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$, then y equals
(a) 9/2
(b) 9
(c) 18
(d) 27

Solution :

Now, $(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$

⇨ $(\log x/\log 3)(\log 2x/\log x)(\log y/\log 2x) = 2\log_x x$
⇨ $\log y/\log 3 = 2$
⇨ $\log_3 y = 2$
⇨ $y = 3^2 = 9$

Option (b) is correct.

225. The number of real roots of the equation $\log_{2x}(2/x)(\log_2 x)^2 + (\log_2 x)^4 = 1$ for values of $x > 1$ is
(a) 0
(b) 1
(c) 2
(d) None of the foregoing numbers.

Solution :

Now, $\log_{2x}(2/x)(\log_2 x)^2 = 1 - (\log_2 x)^4 = \{1 - (\log_2 x)^2\}\{1 + (\log_2 x)^2\}$

⇨ $(\log_2 x)^2$ divides either $\{1 - (\log_2 x)^2\}$ or $\{1 + (\log_2 x)^2\}$

Now, it can divide only if either of them equal to 0. $1 + (\log_2 x)^2$ cannot be zero as it is sum of positive terms so, $1 - (\text{lox}_2 x)^2 = 0$

⇨ $(\log_2 x)^2 = 1$
⇨ $\log_2 x = \pm 1$
⇨ $x = 2, ½$

Now, $x > 1$

⇨ $x = 2$ which satisfies the equation.

Therefore, one solution.

Option(b) is correct.

226. The equation $\log_3 x - \log_x 3 = 2$ has
(a) no real solution
(b) exactly one real solution
(c) exactly two real solution
(d) infinitely many real solutions.

Solution :

Now, $\log_3 x - \log_x 3 = 2$

⇨ $\log_3 x - 1/\log_3 x = 2$

Let, $\log_3 x = a$

The equation becomes, $a - 1/a = 2$

⇨ $a^2 - 2a - 1 = 0$
⇨ $a = \{2 \pm \sqrt{(4 + 4*1*1)}\}/2 = (2 \pm 2\sqrt{2})/2 = 1 \pm \sqrt{2}$

Now, $\log_3 x = 2 \pm \sqrt{2}$

⇨ two solutions.

Option (c) is correct.

227. If $(\log_3 x)(\log_4 x)(\log_5 x) = (\log_3 x)(\log_4 x) + (\log_4 x)(\log_5 x) + (\log_5 x)(\log_3 x)$ and $x \neq 1$, then x is
(a) 10
(b) 100
(c) 50
(d) 60

Solution :

Now, $(\log_3 x)(\log_4 x)(\log_5 x) = (\log_3 x)(\log_4 x) + (\log_4 x)(\log_5 x) + (\log_5 x)(\log_3 x)$

Let $\log x = y$ and $\log 3 = a$, $\log 4 = b$ and $\log 5 = c$

The equation becomes, $y^3/abc = y^2/ab + y^2/bc + y^2/ca$

- ⇨ $y = abc/ab + abc/bc + abc/ca = a + b + c = \log 3 + \log 4 + \log 5 = \log(3*4*5) = \log 60$
- ⇨ $\log x = \log 60$
- ⇨ $x = 60$

Option (d) is correct.

228. If $\log_2(\log_3(\log_4 x)) = \log_3(\log_4(\log_2 y)) = \log_4(\log_2(\log_3 z)) = 0$ then $x + y + z$ is
 (a) 99
 (b) 50
 (c) 89
 (d) 49

Solution :

Now, $\log_2(\log_3(\log_4 x)) = 0$

- ⇨ $\log_3(\log_4 x) = 2^0$
- ⇨ $\log_3(\log_4 x) = 1$
- ⇨ $\log_4 x = 3^1$
- ⇨ $\log_4 x = 3$
- ⇨ $x = 4^3 = 64$

Similarly, $y = 2^4 = 16$ and $z = 3^2 = 9$

Therefore, $x + y + z = 64 + 16 + 9 = 89$

Option (c) is correct.

229. If x is a positive number different from 1 such that $\log_a x$, $\log_b x$ and $\log_c x$ are in A.P., then
 (a) $c^2 = (ac)^{\wedge}\log_a b$
 (b) $b = (a + c)/2$
 (c) $b = \sqrt{(ac)}$
 (d) none of the foregoing equations is necessarily true.

Solution :

Now, $\log_a x + \log_c x = 2\log_b x$

- ⇨ $(\log x)(1/\log a + 1/\log c) = \log x/\log\sqrt{b}$
- ⇨ $1/\log a + 1/\log c = 2/\log b$

- ⇨ (logc + loga)/logalogc = 2/logb
- ⇨ log(ac) = 2logc/$\log_a$b
- ⇨ ($\log_a$b)log(ac) = $\log c^2$
- ⇨ log{(ac)^($\log_a$b)} = $\log c^2$
- ⇨ c^2 = (ac)^($\log_a$b)

Option (a) is correct.

230. Given that $\log_{10}5 = 0.70$ and $\log_{10}3 = 0.48$, the value of $\log_{30}8$ (correct upto 2 places of decimal) is
   (a) 0.56
   (b) 0.61
   (c) 0.68
   (d) 0.73

Solution :

Now, $\log_{10}5 = 0.70$

- ⇨ $\log_{10}(5*2)/2 = 0.70$
- ⇨ $\log_{10}10 - \log_{10}2 = 0.70$
- ⇨ $1 - \log_{10}2 = 0.70$
- ⇨ $\log_{10}2 = 1 - 0.70 = 0.30$

Now, $\log_{30}8 = \log 8/\log 30 = \log 2^3/(\log 3 + \log 10) = 3\log_{10}2/(\log_{10}3 + 1) = 3*0.3/(0.48 + 1) = 0.9/1.48 = 0.61$

Option (b) is correct.

231. If x is a real number and $y = (1/2)(e^x - e^{-x})$, then
   (a) x can be either $\log(y + \sqrt{(y^2 + 1)})$ or $\log(y - \sqrt{(y^2 + 1)})$
   (b) x can only be $\log(y + \sqrt{(y^2 + 1)})$
   (c) x can be either $\log(y + \sqrt{(y^2 - 1)})$ or $\log(y - \sqrt{(y^2 - 1)})$
   (d) x can only be $\log(y + \sqrt{(y^2 - 1)})$

Solution :

Now, $y = (1/2)(e^x - e^{-x})$

- ⇨ $2y = e^x - 1/e^x$
- ⇨ $e^{2x} - 2ye^x - 1 = 0$
- ⇨ $e^x = \{2y \pm \sqrt{(4y^2 + 4)}\}/2 = y \pm \sqrt{(y^2 + 1)}$

as $\sqrt{(y^2 + 1)} > y$ and e^x cannot be negative so, $e^x = y + \sqrt{(y^2 + 1)}$

⇨ $x = \log(y + \sqrt{(y^2 + 1)})$

Option (b) is correct.

232. A solution to the system of equations $ax + by + cz = 0$ and $a^2x + b^2y + c^2z = 0$ is

(a) $x = a(b - c)$, $y = b(c - a)$, $z = c(a - b)$

(b) $x = k(b - c)/a^2$, $y = k(c - a)/b^2$, $z = k(a - b)/c^2$, where k is an arbitrary constant

(c) $x = (b - c)/bc$, $y = (c - a)/ca$, $z = (a - b)/ab$

(d) $x = k(b - c)/a$, $y = k(c - a)/b$, $z = k(a - b)/c$, where k is an arbitrary constant.

Solution :

Clearly, three variables viz. x, y, z and two equations. So infinitely many solutions.

Therefore, option (b) or (d) can be true.

Clearly, option (d) satisfies both the equations.

Option (d) is correct.

233. $(x + y + z)(yz + zx + xy) - xyz$ equals

(a) $(y + z)(z + x)(x + y)$

(b) $(y - z)(z - x)(x - y)$

(c) $(x + y + z)^2$

(d) None of the foregoing expressions, in general.

Solution :

Now, $(x + y + z)(yz + zx + xy) - xyz$

$= xyz + zx^2 + x^2y + y^2z + xyz + xy^2 + yz^2 + z^2x + xyz - xyz$

$= z(x^2 + y^2 + 2xy) + z^2(x + y) + xy(x + y)$

$= z(x + y)^2 + z^2(x + y) + xy(x + y)$

$= (x + y)(zx + yz + z^2 + xy)$

$= (x + y)\{z(z + x) + y(z + x)$

$= (x + y)(y + z)(z + x)$

Option (a) is correct.

234. The number of points at which the curve $y = x^6 + x^3 - 2$ cuts the x-axis is
(a) 1
(b) 2
(c) 4
(d) 6

Solution :

For, x-axis cut, we put $y = 0$

The equation is, $x^6 + x^3 - 2 = 0$

- ⇨ $(x^3 + 2)(x^3 - 1) = 0$
- ⇨ $x^3 = -2, x^3 = 1$
- ⇨ $x = (-2)^{1/3}, x = 1$
- ⇨ two points.

Option (b) is correct.

235. Suppose $a + b + c$ and $a - b + c$ are positive and $c < 0$. Then the equation $ax^2 + bx + c = 0$
(a) has exactly one root lying between -1 and +1
(b) has both the roots lying between -1 and +1
(c) has no root lying between -1 and +1
(d) nothing definite can be said about the roots without knowing the values of a, b and c.

Solution :

Option (b) is correct.

236. Number of real roots of the equation $8x^3 - 6x + 1 = 0$ lying between -1 and 1 is
(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Let, $f(x) = 8x^3 - 6x + 1$

- ⇨ $f'(x) = 24x^2 - 6 = 6(4x^2 - 1)$ for $x > 1$ it is strictly increasing. For $x < -1$ $4x^2 - 1 > 0$, strictly increasing.

Therefore, only sign change occurs between -1 and 1. $f(1) = 3 > 0$ and $f(-1) = -1 < 0$

So, all the roots are between -1 and 1.

Let, f(x) has a complex root a + ib then another root is a – ib.

$f(a + ib) = 8(a + ib)^3 - 6(a + ib) + 1 = 0$

And, $f(a - ib) = 8(a - ib)^3 - 6(a - ib) + 1 = 0$

Subtracting the two equations, we get, $8\{(a + ib)^3 - (a - ib)^3\} - 6\{(a + ib) - (a - ib)\} = 0$

- ⇨ $8(a^3 + i3a^2b - 3ab^2 - ib^3 - a^3 + i3a^2b + 3ab^2 - ib^3) - 6(a + ib - a + ib) = 0$
- ⇨ $8(2ib)(3a^2 - b^2) - 12ib = 0$
- ⇨ $4ib\{4(3a^2 - b^2) - 3\} = 0$
- ⇨ $b = 0$
- ⇨ Imaginary part = 0
- ⇨ The equation has all roots real.
- ⇨ 3 roots lying between -1 and 1

Option (d) is correct.

237. The equation $(x^3 + 7)/(x^2 + 1) = 5$ has
   (a) no solution in [0, 2]
   (b) exactly two solutions in [0, 2]
   (c) exactly one solution in [0, 2]
   (d) exactly three solution in [0, 2]

Solution :

Let $f(x) = (x^3 + 7)/(x^2 + 1) - 5$

$f(2) = -2 < 0$

$f(0) = 2 > 0$

Now, $f(10) = 1007/101 - 5 > 0$ a sign change between f(10) and f(2).

- ⇨ There is a root between 2 and 10

f(-2) = (-1)/5 – 5 < 0 a sing change between f(-1) and f(0)

- ⇨ There is a root between 0 and -1.
- ⇨ There is one root in [0, 2]

Option (c) is correct.

238. The roots of the equation $2x^2 - 6x - 5\sqrt{(x^2 - 3x - 6)} = 10$ are
(a) $3/2 \pm (½)\sqrt{41}$, $3/2 \pm (1/2)\sqrt{35}$
(b) $3 \pm \sqrt{41}$, $3 \pm \sqrt{35}$
(c) -2, 5, $3/2 \pm (1/2)\sqrt{34}$
(d) -2, 5 $3 \pm \sqrt{34}$

Solution :

Clearly, x = -2 satisfies the equation.

Therefore, option (c) or (d) is correct.

Let us put $x = 3 + \sqrt{34}$

$2(3 + \sqrt{34})^2 - 6(3 + \sqrt{34}) - 5\sqrt{\{(3 + \sqrt{34})^2 - 3(3 + \sqrt{34}) - 6\}}$

$= 2(45 + 6\sqrt{34}) - 18 - 6\sqrt{34} - 5\sqrt{(45 + 6\sqrt{34} - 6 - 3\sqrt{34} - 6)}$

$= 72 + 6\sqrt{34} - 5\sqrt{(33 + 3\sqrt{34})}$

It is not giving any solution.

Therefore, Option (c) is correct.

239. Suppose that the roots of the equation $ax^2 + b\lambda x + \lambda = 0$ (where a and b are given real numbers) are real for all positive values of λ. Then we must have
(a) $a \geq 0$
(b) $a = 0$
(c) $b^2 \geq 4a$
(d) $a \leq 0$

Solution :

Discriminant = $b^2\lambda^2 - 4a\lambda = \lambda(b^2\lambda - 4a)$

Now, $\lambda > 0$ so $b^2\lambda - 4a > 0$

Now, if $a \leq 0$ then the quantity is always > 0

Option (d) is correct.

240. The equations $x^2 + x + a = 0$ and $x^2 + ax + 1 = 0$
(a) cannot have a common real root for any value of a
(b) have a common real root for exactly one value of a
(c) have a common root for exactly two values of a
(d) have a common root for exactly three values of a.

Solution :

Let the equations have a common root α.

Now, $\alpha^2 + \alpha + a = 0$

And, $\alpha^2 + a\alpha + 1 = 0$

⇨ $\alpha^2/(1 - a^2) = \alpha/(a - 1) = 1/(a - 1)$
⇨ $\alpha = (1 - a^2)/(a - 1) = (a - 1)/(a - 1)$
⇨ $1 + a = 1$ $(a \neq 1)$
⇨ $a = 0$

Option (b) is correct.

241. It is given that the expression $ax^2 + bx + c$ takes positive values for all x greater than 5. Then
(a) the equation $ax^2 + bx + c = 0$ has equal roots.
(b) $a > 0$ and $b < 0$
(c) $a > 0$, but b may or may not be negative
(d) $c > 5$

Solution :

Clearly option (c) is correct.

242. The roots of the equation $(1/2)x^2 + bx + c = 0$ are integers if
(a) $b^2 - 2c > 0$
(b) $b^2 - 2c$ is the square of an integer and b is an integer
(c) b and c are integers
(d) b and c are even integers

Solution :

$x = -b + \sqrt{(b^2 - 2c)}$

Clearly, option (b) is correct.

243. Consider the quadratic equation $(a + c - b)x^2 + 2cx + (b + c - a) = 0$, where a, b, c are distinct real numbers and $a + c - b \neq 0$. Suppose that both the roots of the equation are rational. Then
(a) a, b and c are rational
(b) $c/(a - b)$ is rational
(c) $b/(c - a)$ is rational
(d) $a/(b - c)$ is rational

Solution :

Discriminant = $4c^2 - 4(a + c - b)(b + c - a)$

$= 4[c^2 - \{c + (a - b)\}\{c - (a - b)\}]$

$= 4[c^2 - c^2 + (a - b)^2]$

$= 4(a - b)^2$

Roots = $[-2c \pm \sqrt{\{4(a - b)^2\}}]/2(a + c - b)$

$= [-2c \pm 2(a - b)]/2\{c - (a - b)\}$

$= \{-c/(a - b) \pm 1\}/\{c/(a - b) - 1\}$

Option (b) is correct.

244. Let α and β be the roots of the equation $x^2 + x + 1 = 0$. Then the equation whose roots are 1/α and 1/β is
(a) $x^2 + x + 1 = 0$
(b) $x^2 - x + 1 = 0$
(c) $x^2 - x - 1 = 0$
(d) $x^2 + x - 1 = 0$

Solution :

Now, $\alpha + \beta = -1$, $\alpha\beta = 1$.

$1/\alpha + 1/\beta = (\alpha + \beta)/\alpha\beta = -1/1 = -1$

$(1/\alpha)(1/\beta) = 1/\alpha\beta = 1/1 = 1$

The equation is, $x^2 - (-1)x + 1 = 0$

⇨ $x^2 + x + 1 = 0$

Option (a) is correct.

245. If α, β are the roots of $ax^2 + bx + c = 0$, then the equation whose roots are α^2, β^2 is
(a) $a^2x^2 + (b^2 - 2ac)x + c^2 = 0$
(b) $a^2x^2 - (b^2 + 2ac)x + c^2 = 0$
(c) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$
(d) none of the foregoing equations.

Solution :

Now, $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = b^2/a^2 - 2c/a = (b^2 - 2ac)/a^2$

And, $\alpha^2\beta^2 = (\alpha\beta)^2 = c^2/a^2$

The equation is, $x^2 - \{(b^2 - 2ac)/a^2\}x + c^2/a^2 = 0$

⇨ $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$

Option (c) is correct.

246. Suppose that the equation $ax^2 + bx + c = 0$ has roots α and β, both of which different from ½. Then an equation whose roots are $1/(2\alpha - 1)$ and $1/(2\beta - 1)$ is
(a) $(a + 2b + 4c)x^2 + 2(a + b)x + a = 0$
(b) $4cx^2 + 2(b - 2c)x + (a - b + c) = 0$
(c) $cx^2 + 2(a + b)x + (a + 2b + 4c) = 0$
(d) none of the foregoing equations.

Solution :

Now, $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$

Now, $1/(2\alpha - 1) + 1/(2\beta - 1) = (2\alpha - 1 + 2\beta - 1)/(2\alpha - 1)(2\beta - 1) = \{2(\alpha + \beta) - 2\}/\{4\alpha\beta - 2(\alpha + \beta) + 1\} = \{2(-b/a) - 2\}/\{4c/a - 2(-b/a) + 1\} = -2(a + b)/(4c + 2b + a)$

$1/\{(2\alpha - 1)(2\beta - 1)\} = 1/\{4\alpha\beta - 2(\alpha + \beta) + 1\} = 1/\{4c/a - 2(-b/a) + 1\} = a/(a + 2b + 4c)$

Equation is, $x^2 - \{-2(a + b)/(a + 2b + 4c)\}x + a/(a + 2b + 4c) = 0$

⇨ $(a + 2b + 4c)x^2 + 2(a + b)x + a = 0$

Option (a) is correct.

247. If α and β are roots of the equation $x^2 + 5x + 5 = 0$, then $\{1/(\alpha + 1)\}^3 + \{1/(\beta + 1)\}^3$ equals
(a) -322
(b) 4/27
(c) -4/27
(d) $3 + \sqrt{5}$

Solution :

Now, $\alpha + \beta = -5$ and $\alpha\beta = -5$

Now, $\{1/(\alpha + 1)\}^3 + \{1/(\beta + 1)\}^3$

$= \{(\alpha + 1)^3 + (\beta + 1)^3\}/\{(\alpha + 1)(\beta + 1)\}^3$

$= (\alpha + 1 + \beta + 1)\{(\alpha + 1)^2 - (\alpha + 1)(\beta + 1) + (\beta + 1)^2\}/\{\alpha\beta + (\alpha + \beta) + 1\}^3$

$= \{(\alpha + \beta) + 2\}\{\alpha^2 + \beta^2 + 2(\alpha + \beta) + 2 - \alpha\beta - (\alpha + \beta) - 1\}/(-5 - 5 + 1)^3$

$= (-5 + 2)\{(\alpha + \beta)^2 - 2\alpha\beta -10 + 1 + 5 + 5)/(-9^3)$

$= (25 + 10 + 1)/3*81$

$= 36/3*81$

$= 4/27$

Option (b) is correct.

248. If α is a positive integer and the roots of the equation $6x^2 - 11x + \alpha = 0$ are rational numbers, then the smallest value of α is
(a) 4
(b) 5
(c) 6
(d) None of the foregoing numbers

Solution :

Discriminant = $121 - 24\alpha = m^2$ (as the roots are rational)

If $\alpha = 3$, $121 - 24\alpha = 49$ which is a square number.

Therefore, smallest value of ɑ = 3

Option (d) is correct.

249. $P(x)$ is a quadratic polynomial whose values at $x = 1$ and $x = 2$ are equal in magnitude but opposite in sign. If -1 is a root of the equation $P(x) = 0$, then the value of the other root is
(a) 8/5
(b) 7/6
(c) 13/7
(d) None of the foregoing numbers.

Solution :

Let another root is a.

Therefore, $P(x) = (x - a)(x + 1)$

$P(1) = -P(2)$

- ⇨ $(1 - a)*2 = -(2 - a)*3$
- ⇨ $2 - 2a = -6 + 3a$
- ⇨ $5a = 8$
- ⇨ $a = 8/5$

Option (a) is correct.

250. If $4x^{10} - x^9 - 3x^8 + 5x^7 + kx^6 + 2x^5 - x^3 + kx^2 + 5x - 5$, when divided by $(x + 1)$ gives remainder -14, then the value of k equals
(a) 2
(b) 0
(c) 7
(d) -2

Solution :

By Remainder theorem, when $P(x)$ is divided by $(x + 1)$ then the remainder is $P(-1)$

Therefore, remainder = $4 + 1 - 3 - 5 + k - 2 + 1 + k - 5 - 5 = -14$

- ⇨ $k - 14 = -14$
- ⇨ $k = 0$

Option (b) is correct.

251. A polynomial $f(x)$ with real coefficients leaves the remainder 15 when divided by $x - 3$ and the remainder $2x + 1$ when divided by $(x - 1)^2$. Then the remainder when $f(x)$ is divided by $(x - 3)(x - 1)^2$ is

(a) $2x^2 - 2x + 3$
(b) $6x - 3$
(c)$x^2 + 2x$
(d) $3x + 6$

Solution :

$f(x) = (x - 3)Q(x) + 15$ and $f(x) = (x - 1)^2S(x) + 2x + 1$

$f(3) = 15, f(1) = 3$

$f'(x) = 2(x - 1)S(x) + (x - 1)^2S'(x) + 2$

⇨ $f'(1) = 2$

Let, $f(x) = (x - 3)(x - 1)^2D(x) + Ax^2 + Bx + C$

$f(3) = 9A + 3B + C = 15$ ……….. (1)

$f(1) = A + B + C = 3$ …………. (2)

And, $f'(x) = (x - 1)^2D(x) + 2(x - 3)(x - 1)D(x) + (x - 3)(x - 1)^2D'(x) + 2Ax + B$ (remainder is quadratic as the divided is cubic)

$f'(1) = 2A + B = 2$ …………. (3)

(1) $- (2) = 8A + 2B = 12$
⇨ $4A + B = 6$
⇨ $4A + B - 2A - B = 6 - 2$ (subtracting (3))
⇨ $2A = 4$
⇨ $A = 2$

From (3), $B = -2$

From (2), $C = 3$

Remainder $= 2x^2 - 2x + 3$

Option (a) is correct.

252. The remainder obtained when the polynomial $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ is divided by $x^2 - 1$ is

(a) $6x + 1$
(b) $5x + 1$
(c) $4x$
(d) $6x$

Solution :

Let P(x) be the polynomial.

By remainder theorem, when P(x) is divided by $(x - 1)$ remainder is $P(1) = 6$

When P(x) is divided by $(x + 1)$, remainder is $P(-1) = -6$

Let, $P(x) = (x - 1)(x + 1)Q(x) + Ax + B$ (remainder is linear as divider is quadratic)

$P(1) = A + B = 6$ ……. (1)

$P(-1) = -A + B = -6$ ….. (2)

Adding the above equations we get, B = 0 and A = 6

The remainder is 6x.

Option (d) is correct.

253. Let $(1 + x + x^2)^9 = a_0 + a_1x + …. + a_{18}x^{18}$. Then
(a) $a_0 + a_2 + … + a_{18} = a_1 + a_3 + …. + a_{17}$
(b) $a_0 + a_2 + … + a_{18}$ is even
(c) $a_0 + a_2 + … + a_{18}$ is divisible by 9
(d) $a_0 + a_2 + … + a_{18}$ is divisible by 3 but not by 9.

Solution :

Putting x = 1, we get, $3^9 = a_0 + a_1 + a_2 + …. + a_{18}$

Putting x = -1, we get, $1 = a_0 - a_1 + a_2 - …. + a_{18}$

Adding the two equations we get, $2(a_0 + a_2 + …. + a_{18}) = 3^9 + 1$

⇨ $a_0 + a_2 + … + a_{18} = (3^9 + 1)/2$

Now, $3 \equiv - 1 \pmod 4$

⇨ $3^9 \equiv (-1)^9 = -1 \pmod 4$
⇨ $3^9 + 1 \equiv 0 \pmod 4$
⇨ $(3^9 + 1)/2$ is even.

Option (b) is correct.

254. The minimum value of $x^8 - 8x^6 + 19x^4 - 12x^3 + 14x^2 - 8x + 9$ is
   (a) -1
   (b) 9
   (c) 6
   (d) 1

Solution :

f(2) = 1

Option (d) is correct.

255. The cubic expression in x, which takes the value zero when x = 1 and x = -2, and takes values -800 and 28 when x = -7 and x = 2 respectively, is
   (a) $3x^3 + 2x^2 - 7x + 2$
   (b) $3x^3 + 4x^2 - 5x - 2$
   (c) $2x^3 + 3x^2 - 3x - 2$
   (d) $2x^3 + x^2 - 5x + 2$

Solution :

Let the expression is m(x – a)(x – 1)(x + 2)

Now, m(-7 – a)(-8)(-5) = -800

⇨ m(7 + a) = 20

Also, m(2 – a)*1*4 = 28

⇨ m(2 – a) = 7

Dividing the two equations we get, (7 + a)/(2 – a) = 20/7

⇨ 49 + 7a = 40 – 20a
⇨ 27a = -9
⇨ a = -1/3

Putting in above equation we get, m(2 + 1/3) = 7

⇨ m = 7*3/7
⇨ m = 3

Expression is, $3(x + 1/3)(x - 1)(x + 2) = (3x + 1)(x^2 + x - 2) = 3x^3 + 4x^2 - 5x - 2$

Option (b) is correct.

256. If f(x) is a polynomial in x and a, b are distinct real numbers, then the remainder in the division of f(x) by $(x - a)(x - b)$ is
(a) $\{(x - a)f(a) - (x - b)f(b)\}/(a - b)$
(b) $\{(x - a)f(b) - (x - b)f(a)\}/(b - a)$
(c) $\{(x - a)f(b) - (x - b)f(a)\}/(a - b)$
(d) $\{(x - a)f(a) - (x - b)f(b)\}/(b - a)$

Solution :

Upon division of f(x) by $(x - a)$ and $(x - b)$ the remainders are f(a) and f(b) respectively (by remainder theorem)

Let $f(x) = (x - a)(x - b)Q(x) + Ax + B$ (remainder is linear as divider is quadratic)

⇨ $Aa + B = f(a)$ and $Ab + B = f(b)$

Subtracting the above equations we get, $A(a - b) = f(a) - f(b)$

⇨ $A = \{f(a) - f(b)\}/(a - b)$

Putting value of A we get, $a\{f(a) - f(b)\}/(a - b) + B = f(a)$

⇨ $B = f(a) - a\{f(a) - f(b)\}/(a - b) = \{-bf(a) + af(b)\}/(a - b)$

Therefore, remainder $= x\{f(a) - f(b)\}/(a - b) + \{-bf(a) + af(b)\}/(a - b)$

$= \{(x - b)f(a) - (x - a)f(b)\}/(a - b)$

$= \{(x - a)f(b) - (x - b)f(a)\}/(b - a)$

Option (b) is correct.

257. The number of real roots of $x^5 + 2x^3 + x^2 + 2 = 0$ is
(a) 0
(b) 3
(c) 5
(d) 1

Solution :

$x^5 + 2x^3 + x^2 + 2 = 0$

- ⇨ $x^5 + x^4 - x^4 - x^3 + 3x^3 + 3x^2 - 2x^2 - 2x + 2x + 2 = 0$
- ⇨ $x^4(x + 1) - x^3(x + 1) + 3x^2(x + 1) - 2x(x + 1) + 2(x + 1) = 0$
- ⇨ $(x + 1)(x^4 - x^3 + 3x^2 - 2x + 2) = 0$

Let, $f(x) = x^4 - x^3 + 3x^2 - 2x + 2$

By Descartes' sign rule this equation has maximum 4 roots positive and no negative roots. Therefore if it has a real root then it must be positive.

$f(0) = 2 > 0$

$f(1) = 3 > 0$

$f(2) = 18 > 0$

Now, $f(x) = x^3(x - 1) + x(3x - 2) + 2 > 0$ for $x > 3/2$

So, no more real root.

- ⇨ The equation has only one real root $x = -1$.

Option (d) is correct.

258. Let a, b, c be distinct real numbers. Then the number of real solutions of $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$ is
 (a) 1
 (b) 2
 (c)3
 (d) Depends on a, b, c.

Solution :

Let, $f(x) = (x - a)^3 + (x - b)^3 + (x - c)^3$

- ⇨ $f'(x) = 3\{(x - a)^2 + (x - b)^2 + (x - c)^2\} > 0$ for any real x.
- ⇨ f(x) is strictly increasing over all real x.

As f(x) is cubic (odd) it must have at least one real root.

Therefore, f(x) has only one real root.

Option (a) is correct.

259. Let a, b and c be real numbers. Then the fourth degree polynomial in x, $acx^4 + b(a + c)x^3 + (a^2 + b^2 + c^2)x^2 + b(a + c)x + ac$

(a) Has four complex (non-real) roots
(b) Has either four real roots or four complex roots
(c) Has two real roots and two complex roots
(d) Has four real roots.

Solution :

Option (b) is correct.

260. Let $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$, where $ac \neq 0$. Consider the polynomial $P(x)Q(x)$.
(a) All its roots are real.
(b) None of its roots are real
(c) At least two of its roots are real
(d) Exactly two of its roots are real.

Solution :

Now, discriminant = $b^2 - 4ac$ and $b^2 + 4ac$. One of them must be positive. Both may be positive also. So at least two roots are definitely real.

Option (c) is correct.

261. For the roots of the quadratic equation $x^2 + bx - 4 = 0$ to be integers
(a) it is sufficient that $b = 0, \pm 3$
(b) it is sufficient that $b = 0, \pm 2$
(c) it is sufficient that $b = 0, \pm 4$
(d) none of the foregoing conditions is sufficient.

Solution :

Roots = $\{-b \pm \sqrt{(b^2 + 16)}\}/2$

Clearly, option (a) is correct.

262. The smallest positive solution of the equation $(81)^{\wedge}(\sin^2 x) + (81)^{\wedge}(\cos^2 x) = 30$ is
(a) $\pi/12$
(b) $\pi/6$

(c) $\pi/8$
(d) none of the foregoing quantities.

Solution :

Now, $(81)^{(\sin^2 x)} + (81)^{(1 - \sin^2 x)} = 30$

⇨ $(81)^{(\sin^2 x)} + 81/(81)^{(\sin^2 x)} = 30$

Let $(81)^{\sin^2 x} = a$

The equation becomes, $a + 81/a = 30$

⇨ $a^2 - 30a + 81 = 0$
⇨ $(a - 3)(a - 27) = 0$
⇨ $a = 3,\ a = 27$
⇨ $(81)^{(\sin^2 x)} = 3$
⇨ $3^{(4\sin^2 x)} = 3$
⇨ $4\sin^2 x = 1$
⇨ $\sin x = \pm 1/2$
⇨ smallest $x = \pi/6$

Now, $(81)^{(\sin^2 x)} = 27$

⇨ $3^{(4\sin^2 x)} = 3^3$
⇨ $4\sin^2 x = 3$
⇨ $\sin x = \pm\sqrt{3}/2$
⇨ smallest $x = \pi/3$
⇨ smallest $x = \pi/6$

Option (b) is correct.

263. If α and β are the roots of the equation $x^2 + ax + b = 0$, where $b \neq 0$, then the roots of the equation $bx^2 + ax + 1 = 0$ are
(a) $1/\alpha, 1/\beta$
(b) α^2, β^2
(c) $1/\alpha^2, 1/\beta^2$
(d) $\alpha/\beta, \beta/\alpha$

Solution :

Now, $\alpha + \beta = -a$ and $\alpha\beta = b$

Let the roots of the equation $bx^2 + ax + 1 = 0$ are m, n

Therefore, $m + n = -a/b$ and $mn = 1/b$

⇨ $m + n = (\alpha + \beta)/\alpha\beta$ and $mn = 1/\alpha\beta$
⇨ $m + n = 1/\alpha + 1/\beta$ and $mn = (1/\alpha)(1/\beta)$

Option (a) is correct.

264. A necessary and sufficient condition for the quadratic function $ax^2 + bx + c$ to take positive and negative values is
(a) $ab \neq 0$
(b) $b^2 - 4ac > 0$
(c) $b^2 - 4ac \geq 0$
(d) none of the foregoing statements.

Solution :

If $b^2 - 4ac = 0$ then both the roots will be equal. So, $b^2 - 4ac > 0$ for the roots to be real.

Option (b) is correct.

265. The quadratic equation $x^2 + bx + c = 0$ (b, c real numbers) has both roots real and positive, if and only if
(a) $b < 0$ and $c > 0$
(b) $bc < 0$ and $b \geq 2\sqrt{c}$
(c) $bc < 0$ and $b^2 \geq 4c$
(d) $c > 0$ and $b \leq -2\sqrt{c}$

Solution :

Roots are $\{-b \pm \sqrt{(b^2 - 4c)}\}/2$

Now, if $b \leq -2\sqrt{c}$, it means $b < 0$ and hence $-b$ is positive and $b^2 - 4c < -b$ and hence both roots are positive.

Option (d) is correct.

266. If the equation $ax^2 + bx + c = 0$ has a root less than -2 and root greater than 2 and if $a > 0$, then
(a) $4a + 2|b| + c < 0$
(b) $4a + 2|b| + c > 0$
(c) $4a + 2|b| + c = 0$
(d) None of the foregoing statements need always be true.

Solution :

$|\{-b \pm \sqrt{(b^2 - 4ac)}\}/2a| > 2$

- ⇨ $b^2 \pm 2b\sqrt{(b^2 - 4ac)} + b^2 - 4ac > 16a^2$
- ⇨ $b^2 \pm b\sqrt{(b^2 - 4ac)} - 2ac > 8a^2$
- ⇨ $\pm b\sqrt{(b^2 - 4ac)} > 2ac + 8a^2 - b^2$
- ⇨ $b^4 - 4acb^2 < 64a^4 + b^4 + 4a^2c^2 + 32a^3c - 4acb^2 - 16a^2b^2$
- ⇨ $64a^4 + 4a^2c^2 + 32a^3c - 16a^2b^2 > 0$
- ⇨ $16a^2 + c^2 + 8ac - 4b^2 > 0$
- ⇨ $(4a + c)^2 > (2b)^2$
- ⇨ $|4a + c| > 2|b|$
- ⇨ $4a + c < -2|b|$
- ⇨ $4a + c + 2|b| < 0$

Option (a) is correct.

267. Which of the following is a square root of $21 - 4\sqrt{5} + 8\sqrt{3} - 4\sqrt{15}$?
(a) $2\sqrt{3} - 2 - \sqrt{5}$
(b) $\sqrt{5} - 3 + 2\sqrt{3}$
(c) $2\sqrt{3} - 2 + \sqrt{5}$
(d) $2\sqrt{3} + 2 - \sqrt{5}$

Solution :

Option (b) cannot be true as sum of squares of each term is not equal to 21.

Now, out of (a), (c) and (d) only (d) yields the term $-4\sqrt{15}$

Therefore, option (d) is correct.

268. If $x > 1$ and $x + x^{-1} < \sqrt{5}$, then
(a) $2x < \sqrt{5} + 1$, $2x^{-1} > \sqrt{5} - 1$
(b) $2x < \sqrt{5} + 1$, $2x^{-1} < \sqrt{5} - 1$
(c) $2x > \sqrt{5} + 1$, $2x^{-1} < \sqrt{5} + 1$
(d) None of the foregoing pair of inequalities hold.

Solution :

Now, $x + 1/x < \sqrt{5}$

Now, $x > 0$, then $x^2 - x\sqrt{5} + 1 < 0$

- ⇨ $x^2 - 2*x*(\sqrt{5}/2) + (\sqrt{5}/2)^2 < 1/4$
- ⇨ $(x - \sqrt{5}/2)^2 < (1/2)^2$
- ⇨ $|x - \sqrt{5}/2| < 1/2$
- ⇨ $-1/2 < x - \sqrt{5}/2 < 1/2$
- ⇨ $\sqrt{5} - 1 < 2x < \sqrt{5} + 1$
- ⇨ $2x < \sqrt{5} + 1$
- ⇨ $1/2x > 1/(\sqrt{5} + 1)$
- ⇨ $1/2x > (\sqrt{5} - 1)/4$
- ⇨ $2x^{-1} > \sqrt{5} - 1$

Option (a) is correct.

269. If the roots of $1/(x + a) + 1/(x + b) = 1/c$ are equal in magnitude but opposite in sign, then the product of the roots is
 (a) $-(a^2 + b^2)/2$
 (b) $-(a^2 + b^2)/4$
 (c) $(a + b)/2$
 (d) $(a^2 + b^2)/2$

Solution :

Now, $1/(x + a) + 1/(x + b) = 1/c$

- ⇨ $(2x + a + b)c = (x + a)(x + b)$
- ⇨ $x^2 + x(a + b - 2c) + (ab - bc - ca) = 0$

Roots = $[-(a + b - 2c) \pm \sqrt{\{(a + b - 2c)^2 - 4(ab - bc - ca)\}}/2$

Now, $[-(a + b - 2c) + \sqrt{\{(a + b - 2c)^2 - 4(ab - bc - ca)\}}]/2 = -[-(a + b - 2c) - \sqrt{\{(a + b - 2c)^2 - 4(ab - bc - ca)\}}]/2$

- ⇨ $2(a + b - 2c) = 0$
- ⇨ $a + b = 2c$

Product of the roots = $(ab - bc - ca) = ab - c(a + b) = ab - (a + b)^2/2 = -\{(a + b)^2 - 2ab\}/2 = -(a^2 + b^2)/2$

Option (a) is correct.

270. If α, β are the roots of the equation $x^2 + x + 1 = 0$, then the equation whose roots are α^k, β^k where k is a appositive integer not divisible by 3, is
 (a) $x^2 - x + 1 = 0$
 (b) $x^2 + x + 1 = 0$
 (c) $x^2 - x - 1 = 0$
 (d) none of the foregoing equations.

Solution :

The roots are w and w^2 where w is cube root of unity.

Therefore, $w^k + w^{2k} = -1$ and $w^k * w^{2k} = w^{3k} = 1$

The equation is, $x^2 - (-1)x + 1 = 0$

⇨ $x^2 + x + 1 = 0$

Option (b) is correct.

271. If α and β are the roots of the quadratic equation $x^2 + x + 1 = 0$, then the equation whose roots are α^{2000}, β^{2000} is
(a) $x^2 + x - 1 = 0$
(b) $x^2 + x + 1 = 0$
(c) $x^2 - x + 1 = 0$
(d) $x^2 - x - 1 = 0$

Solution :

Now, α, β are nothing but w and w^2 when w, w^2 are complex cube root of unity. Therefore, $w^3 = 1$.

Now, $\alpha^{2000} = w^2$ and $\beta^{2000} = w$. Therefore, roots are w and w^2.

Option (b) is correct.

272. If α, β, γ are the roots of $x^3 + 2x^2 + 3x + 3 = 0$, then the value of $\{\alpha/(\alpha + 1)\}^3 + \{\beta/(\beta + 1)\}^3 + \{\gamma/(\gamma + 1)\}^3$ is
(a) 18
(b) 44
(c) 13
(d) None of the foregoing numbers.

Solution :

Now, α, β, γ will satisfy the equation as they are roots of the equation.

Therefore, $\alpha^3 + 2\alpha^2 + 3\alpha + 3 = 0$

Now, $\{\alpha/(\alpha + 1)\}^3 = \alpha^3/(\alpha^3 + 3\alpha^2 + 3\alpha + 1) = (-2\alpha^2 - 3\alpha - 3)/(\alpha^2 - 2) = (-2\alpha^2 - 3\alpha - 3)/(\alpha^2 - 2) + 2 - 2 = -(3\alpha + 7)/(\alpha^2 - 2) - 2$

Therefore, the expression becomes, $-(3\alpha + 7)/(\alpha^2 - 2) - 2 - (3\beta + 7)/(\beta^2 - 2) - 2 - (3\gamma + 7)/(\gamma^2 - 2) - 2$

$= - \sum(3\alpha + 7)(\beta^2 - 2)(\gamma^2 - 2)/(\alpha^2 - 2)(\beta^2 - 2)(\gamma^2 - 2)$ (summation over cyclic α, β, γ)

Now, $\sum (3\alpha + 7)(\beta^2 - 2)(\gamma^2 - 2)$

$= -[3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha) - 6\alpha\beta(\alpha + \beta) - 6\beta\gamma(\beta + \gamma) - 6\gamma\alpha(\gamma + \alpha) +7(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) - 28(\alpha^2 + \beta^2 + \gamma^2)]$

$= -[-9*3 - 6\alpha\beta(-2 - \gamma) - 6\beta\gamma(-2 - \alpha) - 6\gamma\alpha(-2 - \beta) + 7\{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta\gamma)^2((\alpha^2 + \beta^2 + \gamma^2)\} - 28(\alpha^2 + \beta^2 + \gamma^2)]$

$= -[-27 + 12(\alpha\beta + \beta\gamma + \gamma\alpha) + 18\alpha\beta\gamma + 7*9 - 154(\alpha^2 + \beta^2 + \gamma^2)]$

$= 27 - 12*3 - 18*(-3) - 63 + 154\{(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\}$

$= -18 + 154\{4 - 6)$

$= -326$

Now, $(\alpha^2 - 2)(\beta^2 - 2)(\gamma^2 - 2)$

$= (\alpha\beta\gamma)^2 - 2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) + 4(\alpha^2 + \beta^2 + \gamma^2) - 8$

$= 9 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2 + 4(\alpha\beta\gamma)^2(\alpha^2 + \beta^2 + \gamma^2) + 4(\alpha^2 + \beta^2 + \gamma^2) - 8$

$= 9 - 18 + 40(\alpha^2 + \beta^2 + \gamma^2) - 8$

$= -17 + 40\{(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\}$

$= -17 + 40(4 - 6)$

$= -97$

Therefore, the expression $= (-326)/(-97) - 6$

Option (b) is correct. (*there is some calculation mistake, so it is not coming option (b), whatever the sum is easy but lengthy, you can give it a try. And hope you have got the procedure. So moving to next.*)

273. $a \pm bi$ ($b \neq 0$, $i = \sqrt{(-1)}$) are complex roots of the equation $x^3 + qx + r = 0$, where a, b, q and r are real numbers. Then q in terms of a and b is
(a) $a^2 - b^2$
(b) $b^2 - 3a^2$
(c) $a^2 + b^2$
(d) $b^2 - 2a^2$

Solution :

Now, other root of the equation must be real as it is 3 degree (odd) equation.

Let the other root is α.

Now, a + ib + a – ib + α = 0

⇨ α = -2a

Now, (a + ib)(a – ib) + α(a + ib) + α(a – ib) = q

⇨ $q = a^2 + b^2 + 2a\alpha$
⇨ $q = a^2 + b^2 – 4a^2$ (Putting α = -2a)
⇨ $q = b^2 – 3a^2$

Option (b) is correct.

274. Let α, β, γ be the roots of $x^3 – x – 1 = 0$. Then the equation whose roots are (1 + α)/(1 – α), (1 + β)/(1 – β), (1 + γ)/(1 – γ) is given by
(a) $x^3 + 7x^2 – x – 1 = 0$
(b) $x^3 – 7x^2 – x + 1 = 0$
(c) $x^3 + 7x^2 + x – 1 = 0$
(d) $x^3 + 7x^2 – x – 1 = 0$

Solution :

Now, α + β + γ = 0, αβ + βγ + γα = -1 and αβγ = 1

Now, we have to find the sum, product taken two at a time and the product of the roots and then form the equation. It is easy but lengthy problem. You can give it a try.

Option (a) is correct.

275. Let 1, w and w^2 be the cube roots of unity. The least possible degree of a polynomial with real coefficients, having 2w, 2 + 3w, 2 + $3w^2$ and 2 – w – w^2 as roots is
(a) 4
(b) 5
(c) 6
(d) 8

Solution :

Now, $2 - w - w^2 = 3$ (real root)

Now, $w, w^2 = \{-1 \pm \sqrt{(1 - 4)}\}/2 = -1 \pm i\sqrt{3}/2$

Therefore, $2 + 3w$ and $2 + 3w^2$ are conjugate of each other.

Therefore, total roots is, 2w and it's conjugate, $2 + 3w$, $2 + 3w^2$ and 3 = 5

Option (b) is correct.

276. Let x_1 and x_2 be the roots of the equation $x^2 - 3x + a = 0$, and let x_3 and x_4 be the roots of the equation $x^2 - 12x + b = 0$. If $x_1 < x_2 < x_3 < x_4$ are in G.P., then a*b equals
(a) 5184
(b) 64
(c) -5184
(d) -64

Solution :

$x_2 = x_1r$, $x_3 = x_1r^2$, $x_4 = x_1r^3$, $r > 1$

Now, $x_1 + x_2 = 3$, $x_1(1 + r) = 3$

And, $x_3 + x_4 = 12$, $x_1r^2(1 + r) = 12$

Dividing the second equation by first equation we get, $r^2 = 4$, $r = 2$ ($r > 1$)

Putting the value in first equation we get, $x_1(1 + 2) = 3$, $x_1 = 1$

Therefore, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 8$

$a*b = x_1x_2x_3x_4 = 1*2*4*8 = 64$

Option (b) is correct.

277. If $x = \{3 + 5\sqrt{(-1)}\}/2$ is a root of the equation $2x^3 + ax^2 + bx + 68 = 0$, where a, b are real numbers, then which of the following is also a root?
(a) $\{5 + 3\sqrt{(-1)}\}/2$
(b) -8
(c) -4
(d) Cannot be answered without knowing the values of a and b.

Solution :

(3 + 5i)/2 and (3 − 5i)/2 are roots of the equation (as a, b are real). Let another root is α.

Now, {(3 + 5i)/2}{(3 − 5i)/2}α = -68/2

- ⇨ (17/2)α = -34
- ⇨ α = -4

Option (c) is correct.

278. If the equation $6x^3 - ax^2 + 6x - 1 = 0$ has three real roots α, β and γ such that 1/α, 1/β and 1/γ are in Arithmetic Progression, then the value of a is
(a) 9
(b) 10
(c) 11
(d) 12

Solution :

Now, 1/α + 1/γ = 2/β

- ⇨ (α + γ)/αγ = 2/β
- ⇨ αβ + βγ = 2αγ
- ⇨ αβ + βγ + γα = 3(αβγ)/β
- ⇨ 6/6 = 3*1/6β
- ⇨ β = 1/2

Now, α + γ = a/6 − ½ and αγ = 1/3

Now, αβ + βγ + γα = 6/6

- ⇨ β(α + γ) + 1/3 = 1
- ⇨ (1/2)(a/6 − ½) + 1/3 = 1
- ⇨ a/3 − 1 + 4/3 = 4
- ⇨ a − 3 + 4 = 12
- ⇨ a = 11

Option (c) is correct.

279. Let x, y and z ne real numbers. Then *only* one of the following statements is true. Which one is it?
(a) If x < y, then xz < yz for all values of z.

(b) If $x < y$, then $x/z < y/z$ for all values of z.
(c) If $x < y$, then $(x + z) < (y + z)$ for all values of z.
(d) Id $0 < x < y$, then $xz < yz$ for all values of z.

Solution :

Option (a), (b) and (d) is not true when $z \leq 0$

Option (c) is correct.

280. If $x + y + z = 0$ and $x^3 + y^3 + z^3 - kxyz = 0$, then only one of the following is true. Which one is it?
(a) $k = 3$ whatever be x, y, z
(b) $k = 0$ whatever be x, y, z
(c) k can only of the numbers +1, -1, 0
(d) If none of x, y, z is zero, then $k = 3$.

Solution :

We know, $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$= (1/2)(x + y + z)\{(x - y)^2 + (y - z)^2 + (z - x)^2\}$

Clearly, option (d) is correct.

281. For real numbers x and y, if $x^2 + xy - y^2 + 2x - y + 1 = 0$, then
(a) y cannot be between 0 and 8/5
(b) y cannot be between -8/5 and 8/5
(c) y cannot be between -8/5 and 0
(d) none of the foregoing statements is correct.

Solution :

Let $y = 2$, then $x^2 + 2x - 4 + 2x - 2 + 1 = 0$

⇨ $x^2 + 4x - 5 = 0$

which gives real solution of x. So, option (a) and (b) cannot be true.

Now, $x^2 + x(y + 2) - (y^2 + y - 1) = 0$

⇨ $x = [-(y + 2) \pm \sqrt{\{(y + 2)^2 + 4(y^2 + y - 1)\}}]/2$
⇨ $(y + 2)^2 + 4(y^2 + y - 1) \geq 0$

- ⇨ $5y^2 + 8y \geq 0$
- ⇨ $y(5y + 8) \geq 0$
- ⇨ $y \geq 0$ and $y \leq -8/5$

Option (c) is correct.

282. It is given that the expression $ax^2 + bx + c$ takes negative values for $x < 7$. Then
    (a) the equation $ax^2 + bx + c = 0$ has equal roots.
    (b) a is negative
    (c) a and b both are negative
    (d) none of the foregoing statements is correct.

Solution :

Let $ax^2 + bx + x = (x - a_1)(x - a_2)$ where $a_1, a_2 \geq 7$.

Now, if we take x as any value less than 7 then $(x - a_1)$ and $(x - a_2)$ both negative i.e. $(x - a_1)(x - a_2)$ positive.

So, the factors must be of the form $(x - a_1)(a_2 - x)$

Clearly, option (b) is correct.

283. The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are 45, 120 and 210. Then the value of n is
    (a) 8
    (b) 12
    (c) 10
    (d) None of the foregoing numbers.

Solution :

Let, ${}^nC_{r-1} = 45$, ${}^nC_r = 120$, ${}^nC_{r+1} = 210$.

Now, ${}^nC_r/{}^nC_{r-1} = 120/45$

- ⇨ $[n!/\{(n - r)!r!\}]/[n!/\{(n - r + 1)!(r - 1)!\} = 8/3$
- ⇨ $(n - r + 1)/r = 8/3$
- ⇨ $3n - 3r + 3 = 8r$
- ⇨ $11r = 3n + 3$

Now, ${}^nC_{r+1}/{}^nC_r = 210/120$

- ⇨ $[n!/\{(n - r - 1)!(r + 1)!\}]/[n!/\{(n - r)!r!\}] = 7/4$

- ⇨ $(n - r)/(r + 1) = 7/4$
- ⇨ $4n - 4r = 7r + 7$
- ⇨ $11r = 4n - 7$
- ⇨ $3n + 3 = 4n - 7$ (from above)
- ⇨ $n = 10$

Option (c) is correct.

284. The polynomials $x^5 - 5x^4 + 7x^3 + ax^2 + bx + c$ and $3x^3 - 15x^2 + 18x$ have three common roots. Then the values of a, b and c are
   (a) c = 0 and a and b are arbitrary.
   (b) a = -5, b = 6 and c = 0
   (c) a = -5b/6, b is arbitrary, c = 0
   (d) none of the foregoing statements.

Solution :

Now, $3x^3 - 15x^2 + 18x = 0$

- ⇨ $3x(x^2 - 5x + 6) = 0$
- ⇨ $x(x - 2)(x - 3) = 0$
- ⇨ $x = 0, 2, 3$

Now, c = 0 (putting x = 0)

$2^5 - 5*2^4 + 7*2^3 + a*2^2 + b*2 = 0$ (putting x = 2)

- ⇨ $32 - 80 + 56 + 4a + 2b = 0$
- ⇨ $2a + b + 4 = 0$ ……. (1)

And, $243 - 405 + 189 + 9a + 3b = 0$ (putting x = 3)

- ⇨ $9a + 3b + 27 = 0$
- ⇨ $9a + 3(-2a - 4) + 7 = 0$ (from (1))
- ⇨ $3a = -15$
- ⇨ $a = -5$

Putting a = -5 in (1) we get, $b = -4 - 2*(-5) = 6$.

Option (b) is correct.

285. The equation $x^3 + 2x^2 + 2x + 1 = 0$ and $x^{200} + x^{130} + 1 = 0$ have
   (a) exactly one common root
   (b) no common root
   (c) exactly three common roots

(d) exactly two common roots.

Solution :

Now, $x^3 + 2x^2 + 2x + 1$

$= x^3 + x^2 + x + x + 1$

$= x^2(x + 1) + x(x + 1) + (x + 1)$

$= (x + 1)(x^2 + x + 1)$

Therefore, roots are, -1, w, w^2 where w is cube root of unity and $w^3 = 1$.

Now, x = -1 doesn't satisfy the second equation.

x = w, satisfies the equation and also, $x = w^2$ satisfies the equation.

Option (d) is correct.

286. For any integer $p \geq 3$, the largest integer r, such that $(x - 1)^r$ is a factor of the polynomial $2x^{p+1} - p(p + 1)x^2 + 2(p^2 - 1)x - p(p - 1)$, is
(a) P
(b) 4
(c) 1
(d) 3

Solution :

Let, $P(x) = 2x^{p+1} - p(p + 1)x^2 + 2(p^2 - 1)x - p(p - 1)$

$P(1) = 2 - p(p + 1) + 2(p^2 - 1) - p(p - 1) = 2 - p^2 - p + 2p^2 - 2 - p^2 + p = 0$

$P'(x) = 2(p + 1)x^p - 2p(p + 1)x + 2(p^2 - 1)$

$P'(1) = 2(p + 1) - 2p(p + 1) + 2(p^2 - 1) = 2p + 2 - 2p^2 - 2p + 2p^2 - 2 = 0$

$P''(x) = 2p(p + 1)x^{p-1} - 2p(p + 1)$

$P''(1) = 2p(p + 1) - 2p(p + 1) = 0$

$P'''(X) = 2p(p - 1)(p + 1)x^{p-2}$, $P'''(1) \neq 0$

Option (d) is correct.

287. When $4x^{10} - x^9 + 3x^8 - 5x^7 + cx^6 + 2x^5 - x^4 + x^3 - 4x^2 + 6x - 2$ is divided by $(x - 1)$, the remainder is +2. The value of c is,
(a) +2
(b) +1
(c)0
(d) -1

Solution :

Remainder = 4 – 1 + 3 – 5 + c + 2 – 1 + 1 – 4 + 6 – 2 (By remainder theorem if P(x) is divided by x – 1 then the remainder is P(1))

⇨ $3 + c = 2$
⇨ $c = -1$

Option (d) is correct.

288. The remainder R(x) obtained by dividing the polynomial x^{100} by the polynomial $x^2 - 3x + 2$ is
(a) $2^{100} - 1$
(b) $(2^{100} - 1)x - 2(2^{99} - 1)$
(c)$2^{100}x - 3*2^{100}$
(d) $(2^{100} - 1)x + 2(2^{99} - 1)$

Solution :

Now, $x^2 - 3x + 2 = (x - 1)(x - 2)$

Let, $x^{100} = (x - 1)(x - 2)Q(x) + Ax + B$ (as the divider is quadratic, remainder is linear, Q(x) is quotient)

Putting x = 1 we get, A + B = 1 ………. (1)

Putting x = 2, 23 get, $2A + B = 2^{100}$

⇨ $2A + 1 - A = 2^{100}$ (from (1))
⇨ $A = 2^{100} - 1$

Putting value of A in equation (1) we get, $B = 1 - 2^{100} + 1 = -2(2^{99} - 1)$

Remainder = $(2^{100} - 1)x - 2(2^{99} - 1)$

Option (b) is correct.

289. If $3x^4 - 6x^3 + kx^2 - 8x - 12$ is divided by $x - 3$ then it is also divisible by
   (a) $3x^2 - 4$
   (b) $3x^2 + 4$
   (c) $3x^2 + x$
   (d) $3x^2 - x$

Solution :

Let $P(x) = 3x^4 - 6x^3 + kx^2 - 8x - 12$

By remainder theorem, if we divide it by $x - 3$ then the remainder is $P(3)$.

Therefore, $P(3) = 0$ (as it is divisible by $x - 3$)

- ⇨ $243 - 162 + 9k - 24 - 12 = 0$
- ⇨ $9k + 45 = 0$
- ⇨ $k = -5$

Therefore, $P(x) = 3x^4 - 6x^3 - 5x^2 - 8x - 12$

$= 3x^4 - 9x^3 + 3x^3 - 9x^2 + 4x^2 - 12x + 4x - 12$

$= 3x^3(x - 3) + 3x^2(x - 3) + 4x(x - 3) + 4(x - 3)$

$= (x - 3)(3x^3 + 3x^2 + 4x + 4)$

$= (x - 3)\{3x^2(x + 1) + 4(x + 1)\}$

$= (x - 3)(x + 1)(3x^2 + 4)$

Option (b) is correct.

290. The number of integers x such that $2^{2x} - 3(2^{x+2}) + 2^5 = 0$ is
   (a) 0
   (b) 1
   (c) 2
   (d) None of the foregoing numbers.

Solution :

Let $2^x = a$

The equation becomes, $a^2 - 12a + 32 = 0$

- ⇨ $(a - 4)(a - 8) = 0$
- ⇨ $a = 4, 8$
- ⇨ $2^x = 2^2, 2^3$

⇨ $x = 2, 3$

Option (c) is correct.

291. If the roots of the equation $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$ (where a, b, c are real numbers) are equal, then
(a) $b^2 - 4ac = 0$
(b) $a = b = c$
(c) $a + b + c = 0$
(d) none of the foregoing statements is correct.

Solution :

Now, the equation is, $3x^2 - 2x(a + b + c) + (ab + bc + ca) = 0$

So, $4(a + b + c)^2 - 12(ab + bc + ca) = 0$

⇨ $a^2 + b^2 + c^2 - ab - bc - ca = 0$
⇨ $(1/2)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = 0$
⇨ $a = b = c$

Option (b) is correct.

292. Suppose that a, b, c are three distinct real numbers. The expression $(x - a)(x - b)/\{(c - a)(c - b) + (x - b)(x - c)/\{(a - b)(a - c)\} + (x - c)(x - a)/\{(b - c)(b - a)\} - 1$ takes the value zero for
(a) no real x
(b) exactly two distinct real x
(c) exactly three distinct real x
(d) more than three real x.

Solution :

Option (d) is correct.

293. If $|x^2 - 7x + 12| > x^2 - 7x + 12$, then
(a) $x \leq 3$ or $x \geq 4$
(b) $3 \leq x \leq 4$
(c) $3 < x < 4$
(d) x can take any value except 3 and 4.

Solution :

$|(x - 3)(x - 4)| > (x - 3)(x - 4)$

Clearly, option (c) is correct.

294. The real numbers x such that $x^2 + 4|x| - 4 = 0$ are
   (a) $-2 \pm \sqrt{8}$
   (b) $2 \pm \sqrt{8}$
   (c) $-2 \pm \sqrt{8}, 2 \pm \sqrt{8}$
   (d) $\pm(\sqrt{8} - 2)$

Solution :

$|x|^2 + 4|x| - 4 = 0$

- ⇨ $|x|^2 + 4|x| + 4 = 8$
- ⇨ $(|x| + 2)^2 = 8$
- ⇨ $|x| + 2 = \pm \sqrt{8}$
- ⇨ $|x| = -2 \pm \sqrt{8}$
- ⇨ $x = \pm(-2 \pm \sqrt{8}) = -2 \pm \sqrt{8}, 2 \pm \sqrt{8}$

Option (c) is correct.

295. The number of distinct real roots of the equation $|x^2 + x - 6| - 3x + 7 = 0$
   (a) 0
   (b) 2
   (c) 3
   (d) 4

Solution :

$x^2 + x - 6 - 3x + 7 = 0$

- ⇨ $x^2 - 2x + 1 = 0$
- ⇨ $(x - 1)^2 = 0$
- ⇨ $x = 1$

where, $x^2 + x - 6 > 0$

- ⇨ $(x + 3)(x - 2) > 0$
- ⇨ $x = 1$ is not a solution.

Now, $-x^2 - x + 6 - 3x + 7 = 0$

⇨ $x^2 + 4x - 13 = 0$
⇨ $x = \{-4 \pm \sqrt{(16 + 52)}\}/2$
⇨ $x = -2 \pm \sqrt{17}$

Where, $x^2 + x - 6 < 0$

⇨ $(x + 3)(x - 2) < 0$

Both are no solution.

Option (a) is correct.

296. If a is strictly negative and is not equal to -2, then the equation $x^2 + a|x| + 1 = 0$
(a) cannot have any real roots
(b) must have either four real roots or no real roots
(c) must have exactly two real roots
(d) must have either two real roots or no real roots.

Solution :

Now, $x^2 - ax + 1 = 0$ and $x^2 + ax + 1 = 0$ both have same discriminant = $a^2 - 4$.

So, either both have real roots or both have imaginary roots.

Option (b) is correct.

297. The angles of a triangle are in A.P. and the ratio of the greatest to the smallest angle is 3 : 1. Then the smallest angle is
(a) $\pi/6$
(b) $\pi/3$
(c) $\pi/4$
(d) none of the foregoing angles.

Solution :

Let angles are $A - d$, A, $A + d$.

$A - d + A + A + d = \pi$

⇨ $A = \pi/3$

Now, $(A + d)/(A - d) = 3/1$

⇨ $A + d = 3A - 3d$

⇨ $d = A/2 = \pi/6$

Smallest angle = $\pi/3 - \pi/6 = \pi/6$

Option (a) is correct.

298. Let $x_1, x_2, \ldots$. Be positive integers in A.P., such that $x_1 + x_2 + x_3 = 12$ and $x_4 + x_6 = 14$. Then x_5 is
(a) 7
(b) 1
(c) 4
(d) None of the foregoing numbers.

Solution :

Let common difference is d.

$x_1 + x_1 + d + x_1 + 2d = 12$

⇨ $x_1 + d = 4$ …… (1)

Now, $x_1 + 3d + x_1 + 5d = 14$

⇨ $x_1 + 4d = 7$
⇨ $4 - d + 4d = 7$ (from (1))
⇨ $3d = 3$
⇨ $d = 1$.
⇨ $x_1 = 3$ (from (1))
⇨ $x_5 = x_1 + 4d = 3 + 4 = 7$

Option (a) is correct.

299. The sum of the first m terms of an Arithmetic Progression is n and the sum of the first n terms is m, where $m \neq n$. Then the sum of first m + n terms is
(a) 0
(b) m + n
(c) $-mn$
(d) $-m - n$

Solution :

Let, first term is a and common difference is d.

$(m/2)\{2a + (m - 1)d\} = n$ and $(n/2)\{2a + (n - 1)d\} = m$

- ⇨ $2a + (m - 1)d = 2n/m$ and $2a + (n - 1)d = 2m/n$

Subtracting we get, $(m - 1 - n + 1)d = 2n/m - 2m/n$

- ⇨ $(m - n)d = 2(n - m)(n + m)/mn$
- ⇨ $d = -2(m + n)/mn$
- ⇨ $2a - 2(m - 1)(m + n)/mn = 2n/m$
- ⇨ $a = n/m + (m - 1)(m + n)/mn$

Sum of m + n terms = $\{(m + n)/2\}\{2a + (m + n - 1)d\}$

$= \{(m + n)/2\}\{2n/m + 2(m - 1)(m + n)/mn - 2(m + n - 1)(m + n)/mn\}$

$= (m + n)\{n/m + (m - 1)(m + n)/mn - (m - 1)(m + n)/mn - n(m + n)/mn\}$

$= (m + n)(-1)$

$= -m - n$

Option (d) is correct.

300. In an A.P., suppose that, for some $m \neq n$, the ratio of the sum of the first m terms to the sum of the first n terms is m^2/n^2. If the 13^{th} term of the A.P. is 50, then the 26^{th} term of the A.P. is
(a) 75
(b) 76
(c) 100
(d) 102

Solution :

Now, $(m/2)\{2a + (m - 1)d\}/[(n/2)\{2a + (n - 1)d\} = m^2/n^2$

- ⇨ $\{2a + (m - 1)d\}/\{2a + (n - 1)d\} = m/n$
- ⇨ $\{2a + (m - 1)d\}/\{2a + (n - 1)d\} - 1 = m/n - 1$
- ⇨ $(m - n)d/\{2a + (n - 1)d\} = (m - n)/n$
- ⇨ $2a + (n - 1)d = nd$
- ⇨ $2a = d$

Now, $a + 12d = 50$

- ⇨ $a + 24a = 50$ (from above)
- ⇨ $a = 2, d = 4$
- ⇨ 26^{th} term = $a + 25d = 102$

Option (d) is correct.

301. Let S_n, $n \geq 1$, be the set defined as follows :
$S_1 = \{0\}$, $S_2 = \{3/2, 5/2\}$, $S_3 = \{8/3, 11/3, 14/3\}$, $S_4 = \{15/4, 19/4, 23/4, 27/4\}$, and so on. Then, the sum of the elements of S_{20} is
(a) 589
(b) 609
(c) 189
(d) 209

Solution :

First term of $S_{20} = (20^2 - 1)/20$ and common difference = 1

Therefore, sum = $(20/2)[2*\{(20^2 - 1)/20\} + (20 - 1)*1] = 10(399/10 + 19) = 10(39.9 + 19) = 10*58.9 = 589$

Option (a) is correct.

302. The value of 1*2 + 2*3 + 3*4 + …. + 99*100 equals
(a) 333000
(b) 333300
(c) 30330
(d) 33300

Solution :

$\sum n(n + 1)$ (summation running from n = 1 to n = 99)

$= \sum(n^2 + n) = \sum n^2 + \sum n = 99*100*199/6 + 99*100/2 = 333300$

Option (b) is correct.

303. The value of 1*2*3 + 2*3*4 + 3*4*5 + …. + 20*21*22 equals
(a) 51330
(b) 53130
(c) 53310
(d) 35130

Solution :

$\sum n(n + 1)(n + 2)$ (summation running from n =1 to n = 20)

$= \sum n(n^2 + 3n + 2)$

$= \sum(n^3 + 3n^2 + 2n)$

$= \sum n^3 + 3\sum n^2 + 2\sum n$

$= \{20*21/2\}^2 + 3*20*21*41/6 + 2*20*21/2$

= 53130

Option (b) is correct.

304. Six numbers are in A.P. such that their sum is 3. The first number is four times the third number. The fifth number is equal to
(a) -15
(b) -3
(c) 9
(d) -4

Solution :

a = 4(a + 2d) (first term = a, common difference = d)

⇨ 3a + 8d = 0

(6/2){2a + (6 – 1)d} = 3

⇨ 2a + 5d = 1
⇨ d = -3, a = 8 (solving above two equations)
⇨ 5^{th} term = 8 + 4(-3) = -4

Option (d) is correct.

305. The sum of the first n terms (n > 1) of an A.P. is 153 and the common difference is 2. If the first term is an integer, the number of possible values of n is
(a) 3
(b) 4
(c) 5
(d) 6

Solution :

(n/2){2a + (n – 1)*2} = 153

⇨ $n(a + n - 1) = 3^2*17$

Number of factors of 153 excluding 1 is $(2 + 1)(1 + 1) - 1 = 5$

Option (c) is correct.

306. Six numbers are in G.P. such that their product is 512. If the fourth number is 4, then the second number is
(a) ½
(b) 1
(c) 2
(d) None of the foregoing numbers.

Solution :

$ar^3 = 4$ (a = first term, r = common ratio)

$a*ar*ar^2*ar^3*ar^4*ar^5 = 512$

⇨ $a^6r^{15} = 512$
⇨ $a^2r^5 = 8$
⇨ $(ar^3)^2/r = 8$
⇨ $16/r = 8$ (from above)
⇨ $r = 2$.
⇨ $a = ½$
⇨ $ar = 1$

Option (b) is correct.

307. Let a and b be positive integers with no common factors. Then
(a) a + b and a − b have no common factor other than3
(b) a + b and a − b have no common factor greater than 2, whatever be a and b
(c) a + b and a − b have a common factor, whatever be a and b
(d) none of the foregoing statements is correct.

Solution :

If a and b both odd then a + b and a − b have 2 as common factor. So option (a) cannot be true.

Let us consider 20 and 23. Then 43 and 3 doesn't have any common factor. So, option (c) cannot be true.

Option (b) is correct. (Because 4 cannot be a factor of a + b and a − b)

308. If positive numbers a, b, c, d are in harmonic progression and $a \neq b$, then
(a) $a + d > b + c$ is always true
(b) $a + b > c + d$ is always true
(c) $a + c > b + d$ is always true
(d) none of the foregoing statements is always true.

Solution :

Now, $1/a + 1/d = 1/b + 1/c$

⇨ $(a + d)/ad = (b + c)/bc$
⇨ $(a + d)/(b + c) = ad/bc = (1/b)(1/c)/\{(1/a)(1/d)\} = (1/a + d)(1/a + 2d)/\{(1/a + 3d)(1/a)\}$

Let $1/a = a_1$

⇨ $(a + d)/(b + c) = (a_1 + r)(a_1 + 2r)/\{(a_1 + 3r)a_1\} = (a_1^2 + 3ra_1 + 2dr^2)/(a_1^2 + 3ra_1)$
⇨ $(a + d)/(b + c) – 1 = 2r^2/(a_1^2 + 3ra_1) = 2r^2/(1/a)(1/d) = 2r^2ad > 0$ (as $a, d > 0$ and $r^2 > 0$)
⇨ $(a + d)/(b + c) > 1$
⇨ $(a + d) > (b + c)$

Option (a) is correct.

309. The sum of the series 1 + 11 + 111 + ….. to n terms is
(a) $(1/9)[(10/9)(10^n – 1) + n]$
(b) $(1/9)[(10/9)(10^n -) – n]$
(c) $(10/9)[(1/9)(10^n – 1) – n]$
(d) $(10/9)[(1/9)(10^n – 1) + n]$

Solution :

1 + 11 + 111 + …. To ne terms

= (1/9)(9 + 99 + 999 + … to n terms)

= $(1/9)(10 – 1 + 10^2 – 1 + 10^3 – 1 +$ …. To n terms)

= $(1/9)[(10 + 10^2 + …. + 10^n) – n]$

= $(1/9)[10*(10^n – 1)/(10 – 1) – n]$

= $(1/9)[(10/9)(10^n – 1) – n]$

Option (b) is correct.

310. Two men set out at the same time to walk towards each other from points A and B, 72 km apart. The first man walks at the rate of 4 km per hour. The second man walks 2 km the first hour, 2.5 km the second hour, 3 km the third hour, and so on. Then the men will meet

(a) in 7 hours
(b) nearer A than B
(c) nearer B than A
(d) midway between A and B

Solution :

Let, they meet after n hour.

So, the first man goes = 4n km

Second man goes = 2 + 2.5 + 3 + 3.5 + …. To n terms = (n/2){2*2 + (n – 1)*0.5} = (n/2){4 + (n – 1)0.5}

Now, 4n + (n/2){4 + (n – 1)0.5} = 72

- ⇨ (n/2)(8 + 4 + 0.5n – 0.5) = 72
- ⇨ $0.5n^2 + 11.5n - 144 = 0$
- ⇨ $n^2 + 23n - 288 = 0$
- ⇨ (n + 32)(n – 9) = 0
- ⇨ n = 9

First person goes, 4*9 = 36 km

Therefore they meet at midway between A and B.

Option (d) is correct.

311. The second term of a geometric progression (of positive numbers) is 54and the fourth term is 24. Then the fifth term is

(a) 12
(b) 18
(c) 16
(d) None of the foregoing numbers.

Solution :

ar = 54 and $ar^3 = 24$ (a = first term and r = common ratio)

- ⇨ $ar^3/ar = 24/54$

- ⇨ $r^2 = 4/9$
- ⇨ $r = 2/3$ (as G.P. is of positive terms)
- ⇨ $a = 81$
- ⇨ fifth term = fourth term*r = 24*(2/3) = 16

Option (c) is correct.

312. Consider an arithmetic progression whose first term is 4 and the common difference is -0.1. Let S_n stand for the sum of the first n terms. Suppose r is a number such that $S_n = r$ for some n. Then the number of *other* values of n for which $S_n = r$ is
(a) 0 or 1
(b) 0
(c) 1
(d) > 1

Solution :

Option (a) is correct.

313. The three sides of a right-angles triangle are in G.P. The tangents of the two acute angles are
(a) $(\sqrt{5} + 1)/2$ and $(\sqrt{5} - 1)/2$
(b) $\sqrt{\{(\sqrt{5} + 1)/2\}}$ and $\sqrt{\{(\sqrt{5} - 1)/2\}}$
(c) $\sqrt{5}$ and $1/\sqrt{5}$
(d) None of the foregoing pairs of numbers.

Solution :

Now, $b^2 = ac$ (Right angle at A)

- ⇨ $\sin^2 B = \sin A \sin C$
- ⇨ $\sin^2 B = \sin C$ ($\sin A = 1$)
- ⇨ $\sin^2 B = \sin(90 - B)$ ($C + B = 90$)
- ⇨ $\sin^2 B = \cos B$
- ⇨ $1 - \cos^2 B = \cos B$
- ⇨ $\cos^2 B + \cos B - 1 = 0$
- ⇨ $\cos B = \{-1 + \sqrt{(1 + 4)}\}/2$ (As B is acute)
- ⇨ $\cos B = (\sqrt{5} - 1)/2$
- ⇨ $\tan B = \sqrt{\{2^2 - (\sqrt{5} - 1)^2\}}/(\sqrt{5} - 1) = \sqrt{(4 - 5 - 1 + 2\sqrt{5})}/(\sqrt{5} - 1) = \sqrt{\{2(\sqrt{5} - 1)\}}/(\sqrt{5} - 1)$
- ⇨ $\tan B = \sqrt{\{2/(\sqrt{5} - 1)\}} = \sqrt{\{(\sqrt{5} + 1)/2\}}$
- ⇨ $\tan C = 1/\tan B = \sqrt{\{2/(\sqrt{5} + 1)\}} = \sqrt{\{(\sqrt{5} - 1)/2\}}$

Option (b) is correct.

314. The m^{th} term of an arithmetic progression is x and n^{th} term is y. Then the sum of the first (m + n) teerms is
(a) {(m + n)/2}[(x + y) + (x – y)/(m – n)]
(b) {(m + n)/2}[(x – y) + (x + y)/(m – n)]
(c)(1/2)[(x + y)/(m + n) + (x – y)/(m – n)]
(d) (1/2)[(x + y)/(m + n) – (x – y)/(m – n)]

Solution :

(m/2){2a + (m – 1)d} = x

(n/2){2a + (n – 1)d} = y

Two equations, two unknowns – a, d. Solve them and put in {(m + n)/2}[2a + (m + n – 1)d] and check which answer is correct. It is a long calculation based problem.

Option (a) is correct.

315. The time required for any initial amount of a radioactive substance to decrease to half amount is called the *half-life* of that substance. For example, radium has a half-life of 1620 years. If 1 gm of radium is taken in a capsule, then after 4860 years, the amount of radium left in the capsule will be, in gm,
(a) 1/3
(b) ¼
(c)1/6
(d) 1/8

Solution :

Now, 4860/1620 = 3

Third term of the G.P. = ½ , ¼, 1/8

Option (d) is correct.

316. The sum of all the number s between 200 and 400 which are divisible by 7 is
(a) 9872
(b) 7289

(c)8729
(d) 8279

Solution :

Now, 200 ≡ 4 (mod 7)

First term = 203.

400 ≡ 1 (mod 7)

Last term = 399

Let 203 + (n – 1)7 = 399

- ⇨ (n – 1)*7 = 196
- ⇨ n – 1 = 28
- ⇨ n = 29.

Sum = (29/2)(203 + 399) = (29/2)*602 = 29*301 = 8729

Option (c) is correct.

317. The sum of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... - 100^2$ is
(a) -10100
(b) -5050
(c)-2525
(d) None of the foregoing numbers.

Solution :

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... - 100^2$

$= (1^2 + 2^2 + 3^2 +.... + 99^2 + 100^2) - 2*2^2(1^2 + 2^2 + ... + 50^2)$

= 100*101*201/6 – 8*50*51*101/6

= 101*50(67 – 68)

= - 5050

Option (b) is correct.

318. x_1, x_2, x_3, …. Is an infinite sequence of positive integers in G.P., such that $x_1x_2x_3x_4 = 64$. Then the value of x_5 is

(a) 4
(b) 64
(c) 128
(d) 16

Solution :

Let, common ratio = r

So, $x_1*(x_1r)*(x_1r^2)*(x_1r^3) = 64$

- ⇨ $x_1^4r^6 = 64$
- ⇨ $x_1^2r^3 = 8$
- ⇨ $x_1 = 1$, $r = 2$ (as $r \neq 1$)
- ⇨ $x_5 = 1*2^4 = 16$

Option (d) is correct.

319. The value of 100[1/(1*2) + 1/(2*3) + 1/(3*4) + …. + 1/(99*100)]
(a) is 99
(b) lies between 50 and 98
(c) is 100
(d) is different from values specified in the foregoing statements.

Solution :

100[1 – ½ + ½ - 1/3 + 1/3 – ¼ + …. + 1/99 – 1/100] = 100(1 – 1/100) = 100(99/100) = 99

Option (a) is correct.

320. The value of 1*3*5 + 3*5*7 + 5*7*9 + …. + 17*19*21 equals
(a) 12270
(b) 17220
(c) 12720
(d) 19503

Solution :

∑(2n -1)(2n + 1)(2n + 3) (summation running from n = 1 to n = 9)

$= \sum(4n^2 - 1)(2n + 3)$

$= \sum(8n^3 + 12n^2 - 2n - 3)$

$= 8\sum n^3 + 12\sum n^2 - 2\sum n - \sum 3$

$= 8*(9*10/2)^2 + 12(9*10*19/6) - 2*(9*10/2) - 3*9$

$= 8*45^2 + 12*15*19 - 90 - 27$

$= 4*15(270 + 57) - 117$

= 19503

Option (d) is correct.

321. The sum 1*1! + 2*2! + 3*3! + …. + 50*50! Equals
(a) 51!
(b) 2*51!
(c) 51! – 1
(d) 51! + 1

Solution :

Now, n*n! = (n + 1 – 1)n! = (n + 1)*n! – n! = (n + 1)! – n!

The sum is, 2! – 1! + 3! – 2! + 4! – 3! + …. + 51! – 50!

= 51! – 1! = 51! – 1

Option (c) is correct.

322. The value of 1/(1*3*5) + 1/(3*5*7) + 1/(5*7*9) + 1/(7*9*11) + 1/(9*11*13) equals
(a) 70/249
(b) 53/249
(c) 35/429
(d) 35/249

Solution :

Now, 1/(3*5*7) = (1/2)(5 – 3)/(3*5*7) = (1/2){1/(3*7) – 1/(5*7)} = (1/8)(7 – 3)/(3*7) – (1/4)(7 – 5)/(7*5)

= (1/8)(1/3) – (1/8)(1/7) – (1/4)(1/5) + (1/4)(1/7)

= (1/8)(1/3) – (1/4)(1/5) + (1/8)(1/7)

Similarly doing for other terms the sum becomes,

(1/8)(1/1) – (1/4)(1/3) + (1/8)(1/5) + (1/8)(1/3) – (1/4)(1/5) + (1/8)(1/7) + (1/8)(1/5) – (1/4)(1/7) + (1/8)(1/9) + …. + (1/8)(1/9) – (1/4)(1/11) + (1/8)(1/13)

= 1/8 – 1/(8*3) – 1/(8*11) + 1/(8*13)

= {1/(8*3*11*13)}(3*11*13 – 11*13 – 39 + 33)

= (2*11*13 – 6)/(8*3*11*13)

= (11*13 – 3)/(4*3*11*13)

= 140/(4*3*11*13)

= 35/429

Option (c) is correct.

323. The value of 1/(1*2*3*4) + (1/2*3*4*5) + 1/(3*4*5*6) + …. + 1/(9*10*11*12) is
   (a) 73/1320
   (b) 733/11880
   (c) 73/440
   (d) 1/18

Solution :

Same process as the previous one.

Option (a) is correct.

324. The value of 1/(1*3*5) + 1/(3*5*7) + …. + 1/(11*13*15) equals
   (a) 32/195
   (b) 16/195
   (c) 64/195
   (d) None of the foregoing numbers.

Solution :

Same process as previous one.

Option (b) is correct.

325. The value of $(1*2)/3! + (2*2^2)/4! + (3*2^3)/5! + \ldots. + (15*2^{15})/17!$ Equals
(a) $2 - (16*2^{17})/17!$
(b) $2 - 2^{17}/17!$
(c) $1 - (16*2^{17})/17!$
(d) $1 - 2^{16}/17!$

Solution :

Now, $n*2^n/(n + 2)! = (n + 2 - 2)*2^n/(n+2)! = (n + 2)2^n/(n + 2)! - 2^{n+1}/(n + 2)! = 2^n/(n + 1)! - 2^{n+1}/(n + 2)!$

Putting n = 1, we get, $2^1/2! - 2^2/3!$

Putting n = 2, we get, $2^2/3! - 2^3/4!$

Putting n = 3, we get, $2^3/4! - 2^4/5!$

…

..

Putting n = 15 we get, $2^{15}/16! - 2^{16}/17!$

Adding the above equalities we get, the sum $= 2^1/2! - 2^{16}/17! = 1 - 2^{16}/17!$

Option (d) is correct.

326. The value of $4^2 + 2*5^2 + 3*6^2 + \ldots. + 27*30^2$ is
(a) 187854
(b) 187860
(c) 187868
(d) 187866

Solution :

Now, $\sum n*(n + 3)^2$ (summation running from n = 1 to n = 27)

$= \sum n(n^2 + 6n + 9)$

$= \sum(n^3 + 6n^2 + 9n)$

$= \sum n^3 + 6\sum n^2 + 9\sum n$

$= (27*28/2)^2 + 6*27*28*55/6 + 9*27*28/2$

= 142884 + 41580 + 3402

= 187866

Option (d) is correct.

327. The distances passed over by a pendulum bob in successive swings are 16, 12, 9, 6.75, …. Cm. Then the total distance traversed by the bob before it comes to rest is (in cm)
(a) 60
(b) 64
(c) 65
(d) 67

Solution :

First term = 16, r = ¾ (r = common ratio)

This is sum of an infinite G.P. = a/(1 – r) = 16/(1 – ¾) = 64

Option (b) is correct.

328. In a sequence a_1, a_2, …. of real numbers it is observed that $a_p = \sqrt{2}$, $a_q = \sqrt{3}$ and $a_r = \sqrt{5}$, where $1 \leq p < q < r$ are positive integers. Then a_p, a_q, a_r can be terms of
(a) an arithmetic progression
(b) a harmonic progression
(c) an arithmetic progression if and only if p, q, r are perfect squares
(d) neither an arithmetic progression nor an harmonic progression.

Solution :

Clearly, option (d) is correct.

329. Suppose a, b, c are in G.P. and $a^p = b^q = c^r$. Then
(a) p, q, r are in G.P.
(b) p, q, r are in A.P.
(c) 1/p, 1/q, 1/r are in A.P.
(d) None of the foregoing statements is true.

Solution :

$b^2 = ac$

⇨ $2\log b = \log a + \log c$

Now, $a^p = b^q = c^r = k$

$P\log a = k$

⇨ $\log a = k/p$, similarly, $\log b = k/q$ and $\log c = k/r$

Putting values in above equation we get, $2k/q = k/p + k/r$

⇨ $1/p + 1/r = 2/q$

Option (c) is correct.

330. Three real numbers a, b, c are such that a^2, b^2, c^2 are terms of an arithmetic progression. Then
(a) a, b, c are terms of a geometric progression
(b) (b + c), (c + a), (a + b) are terms of an arithmetic progression
(c)(b + c), (c + a), (a + b) are terms of an harmonic progression
(d) None of the foregoing statements is necessarily true.

Solution :

$2b^2 = c^2 + a^2$

Option (a) cannot be true.

If, option (b) is correct, then $2(c + a) = 2b + c + a$

⇨ $(c + a) = 2b$
⇨ Option (b) cannot be true.

If option (c) is correct then $2/(c + a) = 1/(b + c) + 1/(a + b)$

⇨ $2(b + c)(a + b) = (c + a)(c + a + 2b)$
⇨ $2b^2 + 2ab + 2bc + 2ca = c^2 + a^2 + 2ca + 2ab + 2bc$
⇨ $2b^2 = c^2 + a^2$

Option (c) is correct.

331. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \le 0$, then
(a) a, b, c and d are in H.P.

(b) ab, bc and cd are in A.P.
(c) a, b, c and d are in A.P.
(d) a, b, c and d are in G.P.

Solution :

Now, $(a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2) \leq 0$

⇨ $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 \leq 0$
⇨ Sum of squares less than or equal to zero.
⇨ Individually all equal to zero.
⇨ $a/b = c/b = d/c = p$

Option (d) is correct.

332. Let n quantities be in A.P., d being the common difference. Let the arithmetic mean of the squares of these quantities exceed the square of the arithmetic mean of these quantities by a quantity p. Then o
(a) is always negative
(b) equals $\{(n^2 - 1)/12\}d^2$
(c) equals $d^2/12$
(d) equals $(n^2 - 1)/12$

Solution :

$(a_1^2 + a_2^2 + \ldots. + a_n^2)/n - \{(a_1 + a_2 + \ldots + a_n)/n\}^2 = p$

⇨ $p = (a_1^2 + a_1^2 + 2a_1d + d^2 + a_1^2 + 4a_1d + 4d^2 + a_1^2 + 6a_1d + 9d^2 + \ldots.)/n - [a_1 + \{(n - 1)/2\}d]^2$
⇨ $p = a_1^2 + 2a_1d(n - 1)/2 + d^2(n - 1)(2n - 1)/6 - a_1^2 - (n - 1)a_1d - (n - 1)^2d^2/4$
⇨ $p = (d^2/12)(4n^2 - 6n + 2 - 3n^2 + 6n - 3)$
⇨ $p = \{(n^2 - 1)/12\}d^2$

Option (b) is correct.

333. Suppose that $F(n + 1) = (2F(n) + 1)/2$ for $n = 1, 2, 3, \ldots.$ and $F(1) = 2$. Then $F(101)$ equals
(a) 50
(b) 52
(c) 54
(d) None of the foregoing quantities.

Solution :

Now, $F(n + 1) - F(n) = ½$

Putting n = 1, we get, $F(2) - F(1) = ½$

Putting n = 2, we get, $F(3) - F(2) = ½$

Putting n = 3, we get, $F(4) - F(3) = ½$

…

…

Putting n = 100, we get, $F(101) - F(100) = ½$

Adding the above equalities we get, $F(101) - F(1) = 100*(1/2)$

- ⇨ $F(101) = 52$

Option (b) is correct.

334. Let $\{F_n\}$ be the sequence of numbers defined by $F_1 = 1 = F_2$; $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Let f_n be the remainder left when F_n is divided by 5. Then f_{2000} equals
(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Fibonacci numbers are, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …..

So, F_5, F_{10}, … i.e. whose index is divisible by 5 are divisible by 5.

Therefore, $f_{2000} = 0$ (as 2000 is divisible by 5)

Option (a) is correct.

335. Consider the two arithmetic progressions 3, 7, 11, …., 407 and 2, 9, 16, …, 709. The number of common terms of these two progressions is
(a) 0
(b) 7
(c) 15

(d) 14

Solution :

First A.P.. first term = 3, common difference = 4

Second A.P. first term = 2, common difference = 7.

So, common term will come after 7^{th} term of the first A.P. and 4^{th} term of second A.P.

First common term is 23.

Number of terms = n of first A.P. (say)

- ⇨ $3 + (n - 1)*4 = 407$
- ⇨ $n = 102$.
- ⇨ $23 = 6^{th}$ term.
- ⇨ $6 + (m - 1)*7 \leq 102$
- ⇨ $7(m - 1) \leq 96$
- ⇨ $m - 1 = 13$
- ⇨ $m = 14$

Option (d) is correct.

336. The arithmetic mean of two positive numbers is 18 + ¾ and their geometric mean is 15. The larger of the two numbers is
(a) 24
(b) 25
(c) 20
(d) 30

Solution :

$a > b$

$(a + b)/2 = 18 + ¾$, $a + b = 37.5$

$ab = 15^2 = 225$

$(a - b)^2 = (a + b)^2 - 4ab = (37.5)^2 - 4*225$

- ⇨ $a - b = 22.5$
- ⇨ $a + b = 37.5$
- ⇨ $a = 25$

Option (b) is correct.

337. The difference between roots of the equation $6x^2 + \alpha x + 1 = 0$ is 1/6. Further, α is a positive number. Then the value of α is
(a) 3
(b) 4
(c) 5
(d) 2 + 1/3

Solution :

Let roots are a and b.

$a - b = 1/6$, $a + b = -\alpha/6$ and $ab = 1/6$

$(a + b)^2 = (a - b)^2 + 4ab$

- ⇨ $\alpha^2/36 = 1/36 + 4/6$
- ⇨ $\alpha^2 = 1 + 24$
- ⇨ $\alpha = 5$

Option (c) is correct.

338. If $4^x - 4^{x-1} = 24$, then $(2x)^x$ equals
(a) $5\sqrt{5}$
(b) $25\sqrt{5}$
(c) 125
(d) 25

Solution :

Let, $4^x = a$

The equation becomes, $a - a/4 = 24$

- ⇨ $3a/4 = 24$
- ⇨ $a = 32$
- ⇨ $4^x = 2^5$
- ⇨ $2^{2x} = 2^5$
- ⇨ $x = 5/2$

$(2x)^x = 5^{5/2} = 25\sqrt{5}$

Option (b) is correct.

339. The number of solutions f the simultaneous equations $y = 3\log_e x$, $y = \log_e(3x)$ is
   (a) 0
   (b) 1
   (c) 3
   (d) Infinite

Solution :

Now, $3\log_e x = \log_e(3x)$

⇨ $\log_e x^3 = \log_e(3x)$
⇨ $x^3 = 3x$
⇨ $x^2 = 3$ ($x \neq 0$)
⇨ $x = \sqrt{3}$ (x cannot be negative)

Option (b) is correct.

340. The number of solutions to the system of simultaneous equations $|z + 1 - i| = \sqrt{2}$ and $|z| = 3$ is
   (a) 0
   (b) 1
   (c) 2
   (d) > 2

Solution :

Let $z = x + iy$

Now $|z + 1 - i| = \sqrt{2}$

⇨ $|x + iy + 1 - i| = \sqrt{2}$
⇨ $|(x + 1) + i(y - 1)| = \sqrt{2}$
⇨ $(x + 1)^2 + (y - 1)^2 = 2$ …….. (1)

And, $|z| = 3$

⇨ $x^2 + y^2 = 9$ ……… (2)

Doing (1) – (2) we get, $2x + 1 - 2y + 1 = -7$

⇨ $x - y = -9/2$
⇨ $y = 9/2 + x$ ………. (3)

Putting value of (3) in (1) we get, $x^2 + (9/2 + x)^2 = 9$

⇨ $2x^2 + 9x + 81/4 = 9$

- ⇨ $2x^2 + 9x + 45/4 = 0$
- ⇨ $8x^2 + 36x + 45 = 0$
- ⇨ $x = \{-36 \pm \sqrt{(36^2 - 4*8*45)}\}/16$
- ⇨ $x = (-36 \pm \sqrt{(-144)}/16$

So, no real value of x. So no solution at all.

Option (a) is correct.

341. The number of pairs (x, y) of real numbers that satisfy $2x^2 + y^2 + 2xy - 2y + 2 = 0$ is
(a) 0
(b) 1
(c) 2
(d) None of the foregoing numbers.

Solution :

Now, $2x^2 + 2xy + (y^2 - 2y + 2) = 0$

x is real. Therefore, discriminiant ≥ 0

- ⇨ $4y^2 - 4*2*(y^2 - 2y + 2) \geq 0$
- ⇨ $-4y^2 + 16y - 16 \geq 0$
- ⇨ $y^2 - 4y + 4 \leq 0$
- ⇨ $(y - 2)^2 \leq 0$
- ⇨ $y = 2$

Putting y = 2 we get, $2x^2 + 4x + (4 - 4 + 2) = 0$

- ⇨ $x^2 + 2x + 1 = 0$
- ⇨ $(x + 1)^2 = 0$
- ⇨ $x = -1$

One solution (-1, 2)

Option (b) is correct.

342. Consider the following equation in x and y : $(x - 2y - 1)^2 + (4x + 3y - 4)^2 + (x - 2y - 1)(4x + 3y - 4) = 0$. How many solutions to (x, y) with x, y real, does the equation have?
(a) none
(b) exactly one
(c) exactly two
(d) more than two

Solution :

$(x - 2y - 1)^2 - w(x - 2y - 1)(4x + 3y - 4) - w^2(x - 2y - 1)(4x + 3y - 4) + w^3(4x + 3y - 4)^2 = 0$ (where w is cube root of unity)

- ⇨ $(x - 2y - 1)\{(x - 2y - 1) - w(4x + 3y - 4)\} - w^2(4x + 3y - 4)\{(x - 2y - 1) - w(4x + 3y - 4)\} = 0$
- ⇨ $\{(x - 2y - 1) - w(4x + 3y - 4)\}\{(x - 2y - 1) - w^2(4x + 3y - 4)\} = 0$
- ⇨ $(x - 2y - 1) - w(4x + 3y - 4) = 0$
- ⇨ $(x - 2y - 1) - \{(-1 + i\sqrt{3})/2\}(4x + 3y - 4) = 0$

Equating the real and imaginary parts from both sides we get,

$x - 2y - 1 + (1/2)(4x + 3y - 4) = 0$ and $4x + 3y - 4 = 0$

- ⇨ $x - 2y - 1 = 0$
- ⇨ $4x - 8y - 4 = 0$

Subtracting we get, $3y + 8y - 4 + 4 = 0$

- ⇨ $y = 0, x = 1$

Now, equating the real and imaginary part of the second equation we get same solution.

Therefore, option (b) is correct.

343. Let x and y be positive numbers and let a and b be real numbers, positive or negative. Suppose that $x^a = y^b$ and $y^a = x^b$. Then we can conclude that
(a) $a = b$ and $x = y$
(b) $a = b$ but x need not be equal to y
(c) $x = y$ but a need not be equal to b
(d) $a = b$ if $x \neq y$

Solution :

Dividing the two equations we get, $(x/y)^a = (y/x)^b$

- ⇨ $(x/y)^{(a - b)} = 1$
- ⇨ $a - b = 0$
- ⇨ $a = b$ if $x \neq y$ because if $x = y$ then a may not be equal to b.

Option (d) is correct.

344. On a straight road XY, 100 metres long, 15 heavy stones are placed one metre apart beginning at the end X. A worker, starting at X, has to transport all the stones to Y, by carrying only one stone at a time. The minimum distance he has to travel is (in km)
(a) 1.395
(b) 2.79
(c) 2.69
(d) 1.495

Solution :

First stone carried which is at X, 100 metre distance covered.

Second stone carried, 2*99 metre distance covered.

Third stone carried 2*98 metre distance covered.

…

15th stone carried 2*86 metre distance covered.

Therefore, total distance = 100 + 2(99 + 98 + … + 86)

= 100 + 2*(14/2){2*99 + (14 – 1)*(-1)}

= 100 + 14(198 – 13)

= 100 + 14*185

= 2690 metre = 2.69 km

Option (c) is correct.

345. lim [1/(1*3) + 1/(2*4) + 1/(3*5) + …. + 1/{n(n + 2)}] as n - > ∞ is
(a) 0
(b) 3/2
(c) ½
(d) ¾

Solution :

Now, [1/(1*3) + 1/(2*4) + 1/(3*5) + …. + 1/{n(n + 2)}] = (1/2)[1/1 – 1/3 + ½ - ¼ + 1/3 – 1/5 + … + 1/n – 1/(n + 2)]

= (1/2)[1 + ½ - 1/(n + 1) – 1/(n + 2)]

= (1/2)(1 + ½ - 0 – 0] as n - > ∞

= ¾

Option (d) is correct.

346. lim [$1*3/2n^3 + 3*5/2n^3 + \ldots + (2n - 1)(2n + 1)/2n^3$] as n - > ∞ is
(a) 2/3
(b) 1/3
(c) 0
(d) 2

Solution :

Now, $\sum(2r - 1)(2r + 1)$ (summation running from r = 1 to r = n)

$= \sum(4r^2 - 1)$

$= 4\sum r^2 - \sum 1$

$= 4n(n + 1)(2n + 1)/6 - n$

Now, $\sum(2r - 1)(2r + 1)/2n^3 = 4n(n + 1)(2n + 1)/(6*2n^3) - n/2n^3$

$= (1 + 1/n)(2 + 1/n)/3 - 1/2n^2$

$= (1 + 0)(2 + 0)/3 - 0$ as n - > ∞

= 2/3

Option (a) is correct.

347. The coefficient of x^n in the expansion of $(2 - 3x)/(1 - 3x + 2x^2)$ is
(a) $(-3)^n - (2)^{n/2 - 1}$
(b) $2^n + 1$
(c) $3(2)^{n/2 - 1} - 2(3)^n$
(d) None of the foregoing numbers.

Solution :

Now, $1 - 3x + 2x^2 = (1 - x)(1 - 2x)$

Now, $(1 - x)^{-1} = 1 + (-1)(-x) + \{(-1)(-1 -1)/2!\}(-x)^2 + \{(-1)(-1-1)(-1-2)/3!\}(-x)^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots$

Now, $(1 - 2x)^{-1} = 1 + (-1)(-2x) + \{(-1)(-1 - 1)/2!\}(-2x)^2 + \{(-1)(-1-1)(-1-2)/3!\}(-2x)^3 + \ldots$

$= 1 + 2x + (2x)^2 + (2x)^3 + \ldots..$

Coefficient of x^n in $(1 - x)^{-1}(1 - 2x)^{-1} = 2^n + 2^{n-1} + 2^{n-2} + \ldots. + 2 + 1 = 2^{n+1} - 1$

Coefficient of x^{n-1} in $(1 - x)^{-1}(1 - 2x)^{-1} = 2^n - 1$

Now, coefficient of x^n in $(2 - 3x)(1 - x)^{-1}(1 - 2x)^{-1} = 2^{n+2} - 2 - 3(2^n - 1) = 2^{n+2} - 3(2)^n + 1 = 4*2^n - 3*2^n + 1 = 2^n + 1$.

Option (b) is correct.

348. The infinite sum 1 + 1/3 + (1*3)/(3*6) + (1*3*5)/(3*6*9) + (1*3*5*7)/(3*6*9*12) + …. is
(a) $\sqrt{2}$
(b) $\sqrt{3}$
(c) $\sqrt{(3/2)}$
(d) $\sqrt{(1/3)}$

Solution :

Now, $(1 - x)^{-1/2} = 1 + (-1/2)(-x) + \{(-1/2)(-3/2)/2!\}(-x)^2 + \{(-1/2)(-3/2)(-5/2)/3!\}(-x)^3 + \ldots..$

$= 1 + (1/2)x + [(1*3)/\{2^2(2!)\}]x^2 + [(1*3*5)/\{2^3(3!)\}]x^3 + \ldots..$

Putting x = 2/3 we get,

$(1 - 2/3)^{-1/2} = 1 + 1/3 + (1*3)/(3*6) + (1*3*5)/(3*6*9) + \ldots.$

Therefore, required sum $= (1 - 2/3)^{-1/2} = (1/3)^{-1/2} = \sqrt{3}$

Option (b) is correct.

349. The sum of the infinite series 1 + (1 + 2)/2! + (1 + 2 + 3)/3! + (1 + 2 + 3 + 4)/4! + … is
(a) 3e/2
(b) 3e/4
(c) $3(e + e^{-1})/2$
(d) $e^2 - e$

Solution :

General term = $(1 + 2 + ... + n)/n! = n(n + 1)/2(n!) = (n + 1)/2(n - 1)!$
$= (n - 1 + 2)/2(n - 1)! = 1/\{2(n - 2)!\} + 1/(n - 1)!$

Now, $e^x = \sum(x^n/n!)$ (summation running from $n = 0$ to $n = \infty$)

$e = \sum(1/n!)$ (summation running from $n = 0$ to $n = \infty$)

Therefore, required sum = $e/2 + e = 3e/2$

Option (a) is correct.

350. For a nonzero number x, if $y = 1 - x + x^2/2! - x^3/3! + ...$ and $z = -y - y^2/2 - y^3/3 - .....$ then the value of $\log_e\{1/(1 - e^z)\}$ is
(a) $1 - x$
(b) $1/x$
(c) $1 + x$
(d) x

Solution :

$z = -\sum(y^n/n)$ (summation running from $n = 1$ to $n = \infty$) $= \log_e(1 - y)$

Now, $y = \sum(-x)^n/n!$ (summation running from $n = 0$ to $n = \infty$) $= e^{-x}$

Now, $\log_e\{1/(1 - e^z)\} = \log_e\{1/(1 - (1 - y)\} = \log_e(1/y) = \log_e e^x = x$

Option (d) is correct.

351. For a given real number $\alpha > 0$, define $a_n = (1^\alpha + 2^\alpha + ... + n^\alpha)$ and $b_n = n^n(n!)^\alpha$, for n = 1, 2, … Then
(a) $a_n < b_n$ for all $n > 1$
(b) there exists an integer $n > 1$ such that $a_n < b_n$
(c) $a_n > b_n$ for all $n > 1$
(d) there exists integers n and m both larger than one such that $a_n > b_n$ and $a_m < b_m$.

Solution :

Now, $(1^\alpha + 2^\alpha + ... + n^\alpha)n > \{(1^\alpha)(2^\alpha)…(n^\alpha)\}^{1/n}$ (A.M. > G.M. for unequal quantities)

⇨ $a_n > n^n(n!)^\alpha$
⇨ $a_n > b_n$

Option (c) is correct.

352. Let $a_n = (10^{n+1} + 1)/(10^n + 1)$ for n = 1, 2, ….. Then
(a) for every n, $a_n \geq a_{n+1}$
(b) for every n, $a_n \leq a_{n+1}$
(c) there is an integer k such that $a_{n+k} = a_n$ for all n
(d) none of the above holds.

Solution :

Now, $a_{n+1} - a_n = (10^{n+2} + 1)/(10^{n+1} + 1) - (10^{n+1} + 1)/(10^n + 1)$

$= \{(10^{n+2} + 1)(10^n + 1) - (10^{n+1} + 1)^2\}/(10^{n+1} + 1)(10^n + 1)$

Numerator $= 10^{2n+2} + 10^{n+2} + 10^n + 1 - 10^{2n+2} - 2*10^{n+1} - 1 = 8*10^{n+1} + 10^n$

⇨ $a_{n+1} > a_n$

Option (b) is correct.

353. Let a, b and c be fixed positive real numbers. Let $u_n = na/(b + nc)$ for $n \geq 1$. Then as n increases
(a) u_n increases
(b) u_n decreases
(c) u_n increases first and then decreases
(d) none of the foregoing statements is necessarily true

Solution :

$u_n = na/(b + nc) = a/(b/n + c)$

As n increases b/n decreases and b/n + c decreases, so u_n increases.

Option (a) is correct.

354. Suppose n is a positive integer. Then the least value of N for which $|(n^2 + n + 1)/(3n^2 + 1) - 1/3| < 1/10$, when $n \geq N$, is
(a) 4
(b) 5
(c) 100
(d) 1000

Solution :

$|(n^2 + n + 1)/(3n^2 + 1) - 1/3| < 1/10$

- ⇨ $|(3n^2 + 3n + 3 - 3n^2 - 1)/3(3n^2 + 1)| < 1/10$
- ⇨ $|(3n + 2)|/3(3n^2 + 1) < 1/10$ (as $3n^2 + 1$ is always positive)
- ⇨ $10|3n + 2| < 9n^2 + 3$
- ⇨ $10(3n + 2) < 9n^2 + 3$ (as $n > 0$)
- ⇨ $9n^2 - 30n - 17 > 0$
- ⇨ $9n^2 - 30n + 25 > 25 + 17$
- ⇨ $(3n - 5)^2 > 42$
- ⇨ $3n - 5 > \sqrt{42}$
- ⇨ $3n - 5 > 6 + f$ $(0 < f < 1)$
- ⇨ $3n > 11 + f$
- ⇨ $n > 11/3 + f/3$
- ⇨ $n > 3 + f_1$ $(0 < f_1 < 1)$
- ⇨ $N = 4$

Option (a) is correct.

355. The maximum value of xyz for positive x, y, z, subject to the condition $xy + yz + zx = 12$ is
(a) 9
(b) 6
(c)8
(d) 12

Solution :

Now, $(xy + yz + zx)/3 \geq (xy*yz*zx)^{1/3}$ (A.M. ≥ G.M.)

- ⇨ $12/3 \geq (xyz)^{2/3}$
- ⇨ $xyz \leq (4)^{3/2} = 8$

Option (c) is correct.

356. If a, b are positive real numbers satisfying $a^2 + b^2 = 1$, then the minimum value of a + b +] 1/ab is
(a) 2
(b) $2 + \sqrt{2}$
(c)3
(d) $1 + \sqrt{2}$

Solution :

Now, $(a^2 + b^2)/2 \geq \{(a + b)/2\}^2$

⇨ $(a + b) \leq \sqrt{2}\ (a^2 + b^2 = 1)$

Now, $ab \leq (a^2 + b^2)/2$ (GM ≤ AM)

⇨ $1/ab \geq 2$

Now, $a + b \leq \sqrt{2}$ and $1/ab \geq 2$

The rate of increase of 1/ab is more than rate of decrease of a + b. So minimum value will occur when 1/ab is minimum i.e. $a = b = 1/\sqrt{2}$

So, minimum value of $a + b + 1/ab = 1/\sqrt{2} + 1/\sqrt{2} + 2 = \sqrt{2} + 2$

Option (b) is correct.

357. Let M and m be, respectively the maximum and the minimum of n arbitrary real numbers $x_1, x_2, \ldots, x_n$. Further, let M′ and m′ denote the maximum and the minimum, respectively, of the following numbers :
$x_1, (x_1 + x_2)/2, (x_1 + x_2 + x_3)/3, \ldots, (x_1 + x_2 + \ldots + x_n)/n$
Then
(a) $m \leq m' \leq M \leq M'$
(b) $m \leq m' \leq M' \leq M$
(c) $m' \leq m \leq M' \leq M$
(d) $m' \leq m \leq M \leq M'$

Solution :

The minimum value of $(x_1 + x_2 + \ldots + x_r)/r$ occurs when $x_1 = x_2 = \ldots = x_r$. In that case $m' = m$

Otherwise the minimum value is > the minimum of $x_1, x_2, \ldots, x_r$

⇨ $m' > m$
⇨ $m \leq m'$

Now, maximum value of the above expression ≤ maximum of $x_1, x_2, \ldots, x_r$

⇨ $M' \leq M$
⇨ $m \leq m' \leq M' \leq M$

Option (b) is correct.

358. A stick of length 20 units is to be divided into n parts so that the product of the lengths of the part is greater than unity. The maximum possible value of n is
(a) 18
(b) 20
(c) 19
(d) 21

Solution :

Now, $(x_1x_2....x_n)^{1/n} \leq (x_1 + x_2 + ... + x_n)/n$ (GM ≤ AM) where $x_1, x_2, \ldots, x_n$ are the lengths of n parts.

⇨ $(x_1x_2....x_n) \leq (20/n)^n$
⇨ Maximum n = 19.

Option (c) is correct.

359. It is given that the numbers a ≥ 0, b ≥ 0, c ≥ 0 are such that a + b + c = 4 and (a + b)(b + c)(c + a) = 24. Then only one of the following statements is correct. Which one is it?
(a) More information is needed to determine the values of a, b and c.
(b) Even when a is given to be 1, more information needed to determine the value of b and c.
(c) These two equations are inconsistent.
(d) There exist values of a and b from which value of c could be determined.

Solution :

$(a + b)(4 - a)(4 - b) = 24$

⇨ $(a + b)\{16 - 4(a + b) + ab\} = 24$
⇨ $(4 - c)\{16 - 4(a + b) + ab\} = 24$
⇨ $64 - 16(a + b) + 4ab - 16c + 4c(a + b) - abc = 24$
⇨ $64 - 16(a + b + c) + 4ab - 16c + 4c(a + b) - abc = 24$
⇨ $64 - 64 + 4ab - 16c + 4c(a + b) - abc = 24$
⇨ $4ab - 4c(4 - a - b) - abc = 24$
⇨ $4ab - 4c^2 - abc = 24$
⇨ $ab(4 - c) - 4c^2 = 24$
⇨ $ab(a + b) = 24 + 4c^2$

Now, minimum value of c = 0, maximum value of a + b = 4 and maximum value of ab occurs when a = b = 2.

Therefore, maximum value of ab(a + b) = 2*2*4 = 16 whereas minimum RHS = 24

So, the equations are inconsistent.

Option (c) is correct.

360. Let a, b, c be any real numbers such that $a^2 + b^2 + c^2 = 1$, Then the quantity (ab + bc + ca) satisfies the condition(s)
(a) (ab + bc + ca) is constant
(b) $-1/2 \leq (ab + bc + ca) \leq 1$
(c) $-1/4 \leq (ab + bc + ca) \leq 1$
(d) $-1 \leq (ab + bc + ca) \leq ½$

Solution :

Now, $(a + b + c)^2 \geq 0$

⇨ $(a^2 + b^2 + c^2) + 2(ab + bc + ca) \geq 0$
⇨ $(ab + bc + ca) \geq - (a^2 + b^2 + c^2)/2 = -1/2$

Now, $(a - b)^2 \geq 0$

⇨ $ab \leq (a^2 + b^2)/2$
⇨ $bc \leq (b^2 + c^2)/2$
⇨ $ca \leq (c^2 + a^2)/2$
⇨ $ab + bc + ca \leq a^2 + b^2 + c^2$ (adding the above inequalities) = 1

Option (b) is correct.

361. Let x, y, z be positive numbers. The least value of {x(1 + y) + y(1 + z) + z(1 + x)}/√(xyz) is
(a) 9/√2
(b) 6
(c) 1/√6
(d) None of these numbers.

Solution :

Now, x(1 + y) + y(1 + z) + z(1 + x) = x + y + z + xy + yz + zx

Now, $(x + y + z + xy + yz + zx)/6 \geq \{x*y*z*xy*yz*zx\}^{1/6} = \sqrt{xyz}$ (AM ≥ GM)

⇨ $(x + y + z + xy + yz + zx)/\sqrt{(xyz)} \geq 6$

⇨ Least value is 6

Option (b) is correct.

362. Let a, b and c be such that $a + b + c = 0$ and $l = a^2/(2a^2 + bc) + b^2/(2b^2 + ca) + c^2/(2c^2 + ab)$ is defined. Then the value of l is
(a) 1
(b) -1
(c) 0
(d) None of the foregoing numbers.

Solution :

Now, $a^2/(2a^2 + bc) + b^2/(2b^2 + ca)$

$= a^2/(2a^2 – ab – b^2) + b^2/(2b^2 – ab – a^2)$ (Putting $c = -a – b$)

$= a^2/\{(2a + b)(a – b)\} + b^2/\{(2b + a)(b – a)$

$= \{1/(a – b)\}\{a^2/(2a + b) – b^2/(2b + a)\}$

$= \{1/(a – b)\}[(2a^2b + a^3 – 2ab^2 – b^3)/\{(2b + a)(2a + b)\}]$

$= \{1/(a – b)\}[\{2ab(a – b) + (a – b)(a^2 + ab + b^2)\}/\{(2b + a)(2a + b)\}]$

$= (a^2 + 3ab + b^2)/\{(2b + a)(2a + b)\}$

$= (1/2)\{2a^2 + 6ab + 2b^2 – (2b + a)(2a + b)\}/\{(2b + a)(2a + b) + 1$

$= (1/2)[ab/\{(2b + a)(2a + b)\} + 1$

Now, $(1/2)[ab/\{(2b + a)(2a + b)\}] + 1 + c^2/(2c^2 + ab)$

$= (1/2)[ab/\{(2b + a)(a – c)\}] + 1 + c^2/\{(2c + a)(c – a)\}$

$= \{1/(c – a)\}[(4bc^2 + 2ac^2 – 2abc – a^2b)/\{2(2c + a)(2b + a)\}] + 1$

$= \{1/(c – a)\}[\{b(4c^2 – a^2) + 2ac(c – b)\}/\{2(2c + a)(2b + a)\}] + 1$

$= \{1/(c – a)\}[\{b(2c + a)(2c – a) + 2ac(2c + a)\}/\{2(2c + a)(2b + a)\}] + 1$

$= \{1/(c – a)\}[(2bc – ab + 2ac)/\{2(2b + a)\}] + 1$

$= \{1/(c – a)\}[(-2c^2 – 2ac + ac + a^2 + 2ac)/\{2(2b + a)\}] + 1$

$= \{1/(c – a)\}[(a^2 + ac – 2c^2)/\{2(2b + a)\} + 1$

$= \{1/(c – a)\}[(a – c)(a + 2c)/\{2(2b + a)\}] + 1$

$= -(a + 2c)/\{2(2b + a)\} + 1$

$= -(a + 2c)/\{2(-2c - 2a + a)\} + 1$

$= (a + 2c)/\{2(a + 2c)\} + 1$

$= ½ + 1 = 3/2$

I think there is some calculation mistake. Whatever the problem is easy and evolves just calculation. Nothing else. So, you can give it a try.

Option (a) is correct.

363. Let a, b and c be distinct real numbers such that $a^2 - b = b^2 - c = c^2 - a$. Then $(a + b)(b + c)(c + a)$ equals
(a) 0
(b) 1
(c) -1
(d) None of the foregoing numbers.

Solution :

Now, $a^2 - b = b^2 - c$

⇨ $a^2 - b^2 = b - c$
⇨ $(a + b)(a - b) = (b - c)$
⇨ $(a + b) = (b - c)/(a - b)$

Similarly, $(b + c) = (c - a)/(b - c)$ and $(c + a) = (a - b)/(c - a)$

Multiplying the three above equalities we get, the answer is 1.

Option (b) is correct.

364. Let a and y be real numbers such that $x + y \neq 0$. Then there exists an angle θ such that $\sec^2\theta = 4xy/(x + y)^2$ if and only if
(a) $x + y > 0$
(b) $x + y > 1$
(c) $xy > 0$
(d) $x = y$

Solution :

Now, $(x - y)^2 \geq 0$

⇨ $(x + y)^2 \geq 4xy$
⇨ $4xy/(x + y)^2 \leq 1$

- ⇨ $\sec^2\theta \leq 1$

But, $\sec^2\theta \geq 1$

So, it holds if and only if, $\sec^2\theta = 1$ i.e. $x = y$ (the equality holds when $x = y$)

Option (d) is correct.

365. Consider the equation $\sin\theta = (a^2 + b^2 + c^2)/(ab + bc + ca)$, where a, b, c are fixed non-zero real numbers. The equation has a solution for θ
    (a) whatever be a, b, c
    (b) if and only if $a^2 + b^2 + c^2 < 1$
    (c) if and only if a, b and c all lie in the interval (-1, 1)
    (d) if and only if $a = b = c$

Solution :

Now, $(1/2)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \geq 0$

- ⇨ $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$
- ⇨ $a^2 + b^2 + c^2 \geq ab + bc + ca$
- ⇨ $(a^2 + b^2 + c^2)/(ab + bc + ca) \geq 1$
- ⇨ $\sin\theta \geq 1$

But, $\sin\theta \leq 1$

- ⇨ The equation holds if $\sin\theta = 1$ i.e. $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$ i.e. if and only if $a = b = c$

Option (d) is correct.

366. Consider the real-valued function f. defined over the set of real numbers, as $f(x) = e^{\sin(x^2 + px + q)}$, $-\infty < x < \infty$, where p, q are arbitrary real numbers. The set of real numbers y for which the equation $f(x) = y$ has a solution depends on
    (a) p and not on q
    (b) q and not on p
    (c) both p and q
    (d) neither p nor q

Solution :

Clearly, $-1 \leq \sin(x^2 + px + q) \leq 1$

So, the range of the function is $e^{-1} \leq f(x) \leq e$. So it doesn't depend on p and q.

Option (d) is correct.

367. The equation $x - \log_e(1 + e^x) = c$ has a solution
(a) for every $c \geq 1$
(b) for every $c < 1$
(c) for every $c < 0$
(d) for every $c > -1$

Solution :

Now, $x - \log_e(1 + e^x) = c$

- ⇨ $\log_e(1 + e^x) = x - c$
- ⇨ $1 + e^x = e^{x-c}$
- ⇨ $1 = e^x(e^{-c} - 1)$
- ⇨ $e^{-c} = 1/e^x + 1$
- ⇨ $e^c = e^x/(1 + e^x) < 1$
- ⇨ $c < 0$

Option (c) is correct.

368. A real value of $\log_e(6x^2 - 5x + 1)$ can be determined if and only if x lies in the subset of the real numbers defined by
(a) $\{x : 1/3 < x < ½\}$
(b) $\{x : x < 1/3\} \cup \{x : x > ½\}$
(c) $\{x : x \leq 1/3\} \cup \{x : x \geq ½\}$
(d) all the real numbers.

Solution :

Now, $6x^2 - 5x + 1 > 0$

- ⇨ $(3x - 1)(2x - 1) > 0$
- ⇨ $x > 1/3$ and $x > ½$ or $x < 1/3$ and $x < ½$
- ⇨ $x > ½$ or $x < 1/3$

Option (b) is correct.

369. The domain of definition of the function $f(x) = \sqrt{[\log_{10}\{(3x - x^2)/2\}]}$ is

(a) (1, 2)
(b) (0, 1]U[2, ∞)
(c)[1, 2]
(d) (0, 3)

Solution :

Now, $(3x - x^2)/2 > 0$

- ⇨ $x(3 - x) > 0$
- ⇨ $x > 0$ and $x < 3$ or $x < 0$ and $x > 3$
- ⇨ $0 < x < 3$

Now, $\log_{10}\{(3x - x^2)/2\} \geq 0$

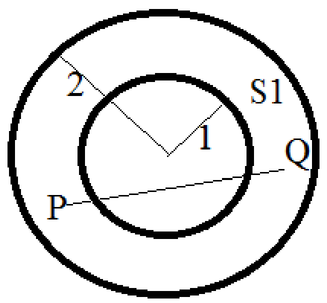
- ⇨ $(3x - x^2)/2 \geq 1$
- ⇨ $3x - x^2 \geq 2$
- ⇨ $x^2 - 3x + 2 \leq 0$
- ⇨ $(x - 1)(x - 2) \leq 0$
- ⇨ $x \leq 1$ and $x \geq 2$ or $x \geq 1$ and $x \leq 2$
- ⇨ $1 \leq x \leq 2$

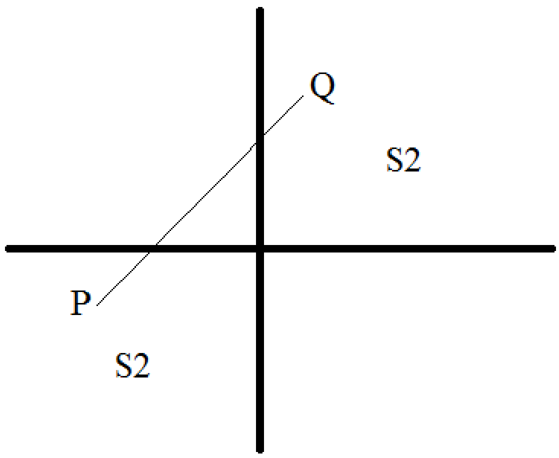
Now, the intersection of $0 < x < 3$ and $1 \leq x \leq 2$ is $1 \leq x \leq 2$

Option (c) is correct.

370. A collection S of points (x, y) of the plane is said to be *convex*, if whenever two points P = (u, v) and Q = (s, t) belong to S, every point on the line segment PQ also belongs to S. Let S_1 be the collection of all points (x, y) for which $1 < x^2 + y^2 < 2$ and let S_2 be the collection of all points (x, y) for which x and y have the same sign. Then
(a) S_1 is convex and S_2 is not convex
(b) S_1 and S_2 are both convex
(c)neither S_1 nor S_2 is convex
(d) S_1 is not convex and S_2 is convex.

Solution :

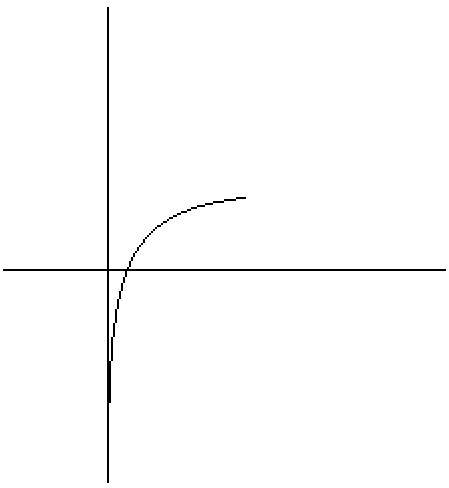




Clearly, option (c) is correct.

371. A function $y = f(x)$ is said to be *convex* if the *line segment* joining any two points $A = (x_1, f(x_1))$ and $B = (x_2, f(x_2))$ on the graph of the function *lies above the graph*. Such a line may also touch the graph at some or all points. Only one of the following four functions is not *convex*. Which one is it?
(a) $f(x) = x^2$
(b) $f(x) = e^x$
(c) $f(x) = \log_e x$
(d) $f(x) = 7 - x$

Solution :



This is graph of $y = \log_e x$

Clearly, option (c) is correct.

372. If S is the set of all real numbers x such that $|1 - x| - x \geq 0$, then
(a) $S = (-\infty, -1/2)$
(b) $S = [-1/2, ½]$
(c) $S = (-\infty, 0]$
(d) $S = (-\infty, ½]$

Solution :

Now, $|1 - x| - x \geq 0$

- ⇨ $|1 - x| \geq x$
- ⇨ $(1 - x)^2 \geq x^2$
- ⇨ $1 - 2x + x^2 \geq x^2$
- ⇨ $1 - 2x \geq 0$
- ⇨ $2x \leq 1$
- ⇨ $x \leq ½$

Option (d) is correct.

373. The inequality $\sqrt{(x + 2)} \geq x$ is satisfied if and only if
(a) $-2 \leq x \leq 2$
(b) $-1 \leq x \leq 2$
(c) $0 \leq x \leq 2$
(d) None of the foregoing conditions.

Solution :

Clearly, Option (a) is correct.

374. If $l^2 + m^2 + n^2 = 1$ and $l'^2 + m'^2 + n'^2 = 1$, then the value of $ll' + mm' + nn'$
(a) is always greater than 2
(b) is always greater than 1, but less than 2
(c) is always less than or equal to 1
(d) doesn't satisfy any of the foregoing conditions

Solution :

Now, $ll' \le (l^2 + l'^2)/2$, $mm' \le (m^2 + m'^2)/2$, $nn' \le (n^2 + n'^2)/2$

⇨ $(ll' + mm' + nn') \le (1/2)\{(l^2 + m^2 + n^2) + (l'^2 + m'^2 + n'^2)\} = 1$

Option (c) is correct.

375. If a and b are positive numbers and c and d are real numbers, positive or negative, then $a^c \le b^d$
(a) if $a \le b$ and $c \le d$
(b) if either $a \le b$ or $c \le d$
(c) if $a \ge 1$, $b \ge 1$, $d \ge c$
(d) is not implied by any of the foregoing conditions.

Solution :

The condition should be, $a \ge 1$, $b \ge 1$, $a \le b$ and $d \ge c$.

Option (d) is correct.

376. For all x such that $1 \le x \le 3$, the inequality $(x - 3a)(x - a - 3) < 0$ holds for
(a) no value of a
(b) all a satisfying $2/3 < a < 1$
(c) all a satisfying $0 < a < 1/3$
(d) all a satisfying $1/3 < a < 2/3$

Solution :

Taking x = 1 and a = ¾ we get, $(1 - 9/4)(1 - 3/4 - 3) > 0$

Option (b) cannot be true.

Taking x = 1, a = ¼ we get, $(1 - 3/4)(1 - 1/4 - 3) < 0$

So, option (a) cannot be true.

Taking x = 1, a = 2/5 we get, $(1 - 6/5)(1 - 2/5 - 3) > 0$

Option (c) is correct.

377. Given that x is a real number satisfying $(3x^2 - 10x + 3)(2x^2 - 5x + 2) < 0$, it follows that
(a) $x < 1/3$
(b) $1/3 < x < 1/2$

(c) $2 < x < 3$
(d) $1/3 < x < ½$ or $2 < x < 3$

Solution :

Putting $x = 0$, we get, $3*2 > 0$

Option (a) cannot be true.

Taking $x = 2/5$, we get, $(12/25 - 4 + 3)(8/25 - 2 + 2) < 0$

Taking $x = 5/2$, we get, $(75/4 - 25 + 3)(25/2 - 25/2 + 2) < 0$

So, option (d) is correct.

378. If x, y, z are arbitrary real numbers satisfying the condition $xy + yz + zx < 0$ and if $u = (x^2 + y^2 + z^2)/(xy + yz + zx)$, then only one of the following is always correct. Which one is it?
(a) $-1 \le u < 0$
(b) u takes all negative real values
(c) $-2 < u < -1$
(d) $u \le -2$

Solution :

$(x + y + z)^2 \ge 0$

⇨ $(x^2 + y^2 + z^2) \ge -2(xy + yz + zx)$
⇨ $(x^2 + y^2 + z^2)/(xy + yz + zx) \le -2$ (as $xy + yz + zx < 0$ so the sing changes)
⇨ $u \le -2$

Option (d) is correct.

379. The inequality $(|x|^2 - |x| - 2)/(2|x| - |x|^2 - 2) > 2$ holds if and only if
(a) $-1 < x < -2/3$ or $2/3 < x < 1$
(b) $-1 < x < 1$
(c) $2/3 < x < 1$
(d) $x > 1$ or $x < -1$ or $-2/3 < x < 2/3$

Solution :

Taking $x = 5/6$ we get, $(25/36 - 5/6 - 2)/(5/3 - 25/36 - 2) = (25 - 30 - 72)/(60 - 25 - 72) = 77/37 > 2$

Taking $x = -5/6$, we get same result.

Option (a) is correct.

380. The set of all real numbers x satisfying the inequality $|x^2 + 3x| + x^2 - 2 \geq 0$ is
(a) all the real numbers x with either $x \leq -3$ or $x \geq 2$
(b) all the real numbers x with either $x \leq -3/2$ or $x \geq ½$
(c) all the real numbers x with either $x \leq -2$ or $x \geq ½$
(d) described by none of the foregoing statements.

Solution :

Now, $x^2 + 3x + x^2 - 2 \geq 0$ where $x^2 + 3x \geq 0$ i.e. $x \geq 0$ or $x \leq -3$

- ⇨ $2x^2 + 3x - 2 \geq 0$
- ⇨ $(2x - 1)(x + 2) \geq 0$
- ⇨ $x \geq ½$ and $x \geq -2$ or $x \leq ½$ and $x \leq -2$
- ⇨ $x \geq ½$ or $x \leq -2$

The intersection is, $x \geq ½$ or $x \leq -3$

Now, $-(x^2 + 3x) + x^2 - 2) \geq 0$ where $x^2 + 3x \leq 0$ i.e. $-3 \leq x \leq 0$

- ⇨ $3x \leq -2$
- ⇨ $x \leq -2/3$

Intersection is $-3 \leq x \leq -2/3$

Now, $x \geq ½$ or $x \leq -3$ U $-3 \leq x \leq -2/3$ = $x \geq ½$ or $x \leq -2/3$

Option (b) is correct.

381. The least value of $1/x + 1/y + 1/z$ for positive a, y, z satisfying the condition $x + y + z = 9$ is
(a) 15/7
(b) 1/9
(c) 3
(d) 1

Solution :

Now, AM ≥ HM on 1/x, 1/y and 1/z we get, (1/x + 1/y + 1/z)/3 ≥ 3/(x + y + z) = 1/3

⇨ 1/x + 1/y + 1/z ≥ 1

Option (d) is correct.

382. The smallest value of ɑ satisfying the conditions that ɑ is a positive integer and that ɑ/540 is a square of a rational number is
(a) 15
(b) 5
(c) 6
(d) 3

Solution :

Now, $540 = 2^2*3^3*5 = (2*3)^2*(3*5)$

So, ɑ = 15 so that the non square term in the denominator cancels.

Option (a) is correct.

383. The set of all values of x satisfying the inequality $(6x^2 + 5x + 3)/(x^2 + 2x + 3) > 2$ is
(a) x > ¾
(b) |x| > 1
(c) either x > ¾ or x < -1
(d) |x| > ¾

Solution :

Taking x = -2, we get, (24 – 10 + 3)/(4 – 4 + 2) = 17/2 > 2

Taking x = 5/6 we get, (25/6 + 25/6 + 3)/(25/36 + 5/3 + 3) = 68*6/193 = 408/193 > 2

Taking x = -5/6 we get, (25/6 – 25/6 + 3)/(25/36 – 5/3 + 3) = 3*36/73 = 108/73 < 2

Option (c) is correct.

384. The set of all x satisfying $|x^2 – 4| > 4x$ is
(a) x < 2(√2 – 1) or x > 2(√2 + 1)

(b) $x > 2(\sqrt{2} + 1)$
(c) $x < -2(\sqrt{2} - 1)$ or $x > 2(\sqrt{2} + 1)$
(d) none of the foregoing sets

Solution :

Now, $x^2 - 4 > 4x$ where $x^2 > 4$ i.e. $|x| > 2$ i.e. $x > 2$ or $x < -2$

- ⇨ $x^2 - 4x + 4 > 8$
- ⇨ $(x - 2)^2 > 8$
- ⇨ $|x - 2| > 2\sqrt{2}$
- ⇨ $x > 2 + 2\sqrt{2}$ or $x < 2 - 2\sqrt{2}$
- ⇨ $x > 2(\sqrt{2} + 1)$ or $x < 2(\sqrt{2} - 1)$

Intersection is $x > 2(\sqrt{2} + 1)$ or $x < -2$

Now, $x^2 - 4 < -4x$ where $x^2 < 4$ i.e. $-2 < x < 2$

- ⇨ $x^2 + 4x + 4 < 8$
- ⇨ $(x + 2)^2 < 8$
- ⇨ $|x + 2| < 2\sqrt{2}$
- ⇨ $-2\sqrt{2} < x + 2 < 2\sqrt{2}$
- ⇨ $-2(\sqrt{2} + 1) < x < 2(\sqrt{2} - 1)$

Intersection is $-2 < x < 2(\sqrt{2} - 1)$

Therefore, required set = $x < 2(\sqrt{2} - 1)$ or $x > 2(\sqrt{2} + 1)$

Option (a) is correct.

385. If a, b, c are positive real numbers and $\alpha = (b^2 + c^2)/(b + c) + (c^2 + a^2)/(c + a) + (a^2 + b^2)/(a + b)$ then only one of the following statements is *always* true. Which one is it?
(a) $0 \le \alpha < a$
(b) $a \le \alpha < a + b$
(c) $a + b \le \alpha < a + b + c$
(d) $a + b + c \le \alpha < 2(a + b + c)$

Solution :

Now, $(b^2 + c^2)/2 \ge \{(b + c)/2\}^2$

- ⇨ $(b^2 + c^2)/(b + c) \ge (b + c)/2$

Similarly, $(c^2 + a^2)/(c + a) \ge (c + a)/2$ and $(a^2 + b^2)/(a + b) \ge (a + b)/2$

Adding the above inequalities we get, $\alpha \ge a + b + c$

Now, $2bc > 0$

- ⇨ $b^2 + c^2 + 2bc > b^2 + c^2$
- ⇨ $(b^2 + c^2) < (b + c)^2$
- ⇨ $(b^2 + c^2)/(b + c) < b + c$

Similarly, $(c^2 + a^2)/(c + a) < c + a$ and $(a^2 + b^2)/(a + b) < a + b$

Adding the above inequalities we get, $\alpha < 2(a + b + c)$

Option (d) is correct.

386. Suppose a, b, c are real numbers such that $a^2b^2 + b^2c^2 + c^2a^2 = k$, where k is constant. Then he set of all possible values of $abc(a + b + c)$ is precisely the interval
 (a) $[-k, k]$
 (b) $[-k/2, k/2]$
 (c) $[-k/2, k]$
 (d) $[-k, k/2]$

Solution :

Now, $(a^2b^2 + b^2c^2 + c^2a^2)/3 \geq (abc)^{4/3}$ (AM ≥ GM)

- ⇨ $abc \leq (k/3)^{3/4}$

Maximum value of abc occurs when $a = b = c = (k/3)^{1/4}$

So, maximum value of $abc(a + b + c) = (k/3)^{3/4}*3(k/3)^{1/4} = 3*(k/3) = k$

Let us take, $abc(a + b + c) = -3k/4$

Now, $abc(a + b + c) = a^2bc + ab^2c + abc^2 \geq 3abc$ (AM ≥ GM)

$(ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2(a^2bc + ab^2c + abc^2) = a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c)$

Now, $(ab + bc + ca)^2 \geq 0$

- ⇨ $k + 2abc(a + b + c) \geq 0$
- ⇨ $abc(a + b + c) \leq -k/2$

Option (c) is correct.

387. If a, b, c, d are real numbers such that $b > 0$, $d > 0$ and $a/b < c/d$, then only one of the following statements is *always* true. Which one is it?
 (a) $a/b < (a - c)/(b - d) < c/d$

(b) $a/b < (a + c)/(b + d) < c/d$
(c) $a/b < (a - c)/(b + d) < c/d$
(d) $a/b < (a + c)/(b - d) < c/d$

Solution :

Now, $a/b < c/d$

- ⇨ $ad < bc$ (as $b > 0$ and $d > 0$)
- ⇨ $ad + cd < bc + cd$
- ⇨ $d(a + c) < c(b + d)$
- ⇨ $(a + c)/(b + d) < c/d$ (as $b > 0$, $d > 0$)

Now, $ad < bc$

- ⇨ $ad + ab < ab + bc$
- ⇨ $a(b + d) < b(a + c)$
- ⇨ $a/b < (a + c)/(b + d)$
- ⇨ $a/b < (a + c)/(b + d) < c/d$

Option (b) is correct.

388. If x, y, z are arbitrary positive numbers satisfying the equation $4xy + 6yz + 8zx = 9$, then the maximum possible value of the product xyz is
(a) $1/2\sqrt{2}$
(b) $\sqrt{3}/4$
(c) $3/8$
(d) None of the foregoing values.

Solution :

Now, $(4xy + 6yz + 8zx)/3 \geq (4xy*6yz*8zx)^{1/3}$

- ⇨ $4*3^{1/3}(xyz)^{2/3} \leq 3$
- ⇨ $64*3(xyz)^2 \leq 3^3$
- ⇨ $(xyz)^2 \leq 3^2/64$
- ⇨ $xyz \leq 3/8$

Option (c) is correct.

389. Let P and Q be the subsets of the X-Y plane defined as : $P = \{(x, y) : x > 0, y > 0 \text{ and } x^2 + y^2 = 1\}$, and $Q = \{(x, y) : x > 0, y > 0 \text{ and } x^8 + y^8 < 1\}$. Then $P \cap Q$ is

(a) The empty set Φ
(b) P
(c) Q
(d) None of the foregoing sets.

Solution :

Clearly, Option (b) is correct.

390. The minimum value of the quantity $(a^2 + 3a + 1)(b^2 + 3b + 1)(c^2 + 3c + 1)/abc$, where a, b and c are positive real numbers, is
(a) $11^3/2^3$
(b) 125
(c) 25
(d) 27

Solution :

Now, $(a^2*1 + a*3 + 1*1)/(1 + 3 + 1) \geq \{(a^2)(a)^3*1\}^{1/(1 + 3 + 1)}$ (weighted AM ≥ weighted GM) = a

⇨ $(a^2 + 3a + 1)/a \geq 5$

Similarly, others, hence minimum value = 5*5*5 = 125

Option (b) is correct.

391. The smallest integer greater than the real number $(\sqrt{5} + \sqrt{3})^{2n}$ (for nonnegative integer n) is
(a) 8^n
(b) 4^{2n}
(c) $(\sqrt{5} + \sqrt{3})^{2n} + (\sqrt{5} - \sqrt{3})^{2n} - 1$
(d) $(\sqrt{5} + \sqrt{3})^{2n} + (\sqrt{5} - \sqrt{3})^{2n}$

Solution :

Option (d) is correct.

392. The set of all values of m for which $mx^2 - 6mx + 5m + 1 > 0$ for all real x is
(a) $0 \leq m \leq ¼$

(b) $m < 1/4$
(c) $m \geq 0$
(d) $0 \leq m < 1/4$

Solution :

Let, $m = 1/4$

$x^2/4 - 3x/2 + 5/4 + 1 > 0$

- ⇨ $x^2 - 6x + 5 + 4 > 0$
- ⇨ $(x - 3)^2 > 0$

This is not true for $x = 3$.

Therefore, $m = 1/4$ is not a solution.

Therefore, option (a) and (c) cannot be true.

Let $m = -1/4$

$-x^2/4 + 3x/2 - 5/4 + 1 > 0$

=> $-x^2 + 6x - 1 > 0$

=> $x^2 - 6x + 1 < 0$

=> $(x - 3)^2 < 8$

This is not true for $x = 10$.

So, option (b) cannot be true.

Option (d) is correct.

393. The value of $(1^r + 2^r + \dots + n^r)^n$, where r is a real number, is
(a) greater than or equal to $n^n*(n!)^r$
(b) less than $n^n*(n!)^{2r}$
(c) less than or equal to $n^{2n}*(n!)^r$
(d) greater than $n^n*(n!)^r$

Solution :

See solution of problem 351.

Option (d) is correct.

394. The value of $(\sqrt{3}/2 + i*1/2)^{165}$ is
(a) -1
(b) $\sqrt{3}/2 - i*1/2$
(c) i
(d) –i

Solution :

Now, $(\sqrt{3}/2 + i*1/2)^{165} = (\cos\pi/6 + i\sin\pi/6)^{165} = e^{(i\pi/6)*165} = e^{i55\pi/2} = \cos(55\pi/2) + i\sin(55\pi/2) = \cos(28\pi - \pi/2) + i\sin(28\pi - \pi/2) = \cos(\pi/2) - i\sin(\pi/2) = -i$

Option (d) is correct.

395. The value of the expression $[\{-1 + \sqrt{(-3)}\}/2]^n + [\{-1 - \sqrt{(-3)}\}/2]^n$ is
(a) 3 when n is positive multiple of 3, and 0 when n is any other positive integer
(b) 2 when n is a positive multiple of 3, and -1 when n is any other positive integer
(c) 1 when n is a positive multiple of 3 and -2 when n is any other positive integer
(d) None of the foregoing numbers.

Solution :

$\{-1 + \sqrt{(-3)}\}/2 = w$ and $\{-1 - \sqrt{(-3)}\}/2 = w^2$ where w is cube root of unity.

Therefore, it is $w^n + w^{2n} = -1$ when n is not a multiple of 3; $= 1 + 1 = 2$ when n is positive multiple of 3.

Option (b) is correct.

396. How many integers k are there for which $(1 - i)^k = 2^k$? (here $i = \sqrt{(-1)}$)
(a) One
(b) None
(c) Two
(d) More than two

Solution :

$(1 - i)^k = 2^k$

- ⇨ $\{(1 - i)/2\}^k = 1$
- ⇨ $k = 0$

Option (a) is correct.

397. If n is a multiple of 4, the sum $S = 1 + 2i + 3i^2 + …. + (n + 1)i^n$, where $i = \sqrt{(-1)}$ is
    (a) $1 - i$
    (b) $(n + 2)/2$
    (c) $(n^2 + 8 - 4ni)/8$
    (d) $(n + 2 - ni)/2$

Solution :

$S = 1 + 2i + 3i^2 + …. + (n + 1)i^n$

$Si = \quad 1 + 2i^2 + …..+ ni^n + (n + 1)i^{n+1}$

- ⇨ $(S - Si) = 1 + i + i^2 + …. + i^n - (n + 1)i^{n+1}$
- ⇨ $S(1 - i) = (i^{n+1} - 1)/(i - 1) - (n + 1)i$ (As n is multiple of 4)
- ⇨ $S(1 - i) = (i - 1)/(i - 1) - (n + 1)i$ (as n is multiple of 4)
- ⇨ $S(1 - i) = 1 - (n + 1)i$
- ⇨ $S = \{1 - (n + 1)i\}/(1 - i)$
- ⇨ $S = (1 + i)\{1 - (n + 1)i\}/2$
- ⇨ $S = \{1 + i - (n + 1)i + (n + 1)\}/2$
- ⇨ $S = (n + 2 - ni)/2$

Option (d) is correct.

398. If $a_0, a_1, …., a_n$ are real numbers such that $(1 + z)^n = a_0 + a_1z + a_2z^2 + …. + a_nz^n$, for all complex numbers z, then the value of $(a_0 - a_2 + a_4 - a_6 + ….)^2 + (a_1 - a_3 + a_5 - a_7 + ….)^2$ equals
    (a) 2^n
    (b) $a_0^2 + a_1^2 + …. + a_n^2$
    (c) $2^\wedge(n^2)$
    (d) $2n^2$

Solution :

Putting z = i we get, $(1 + i)^n = a_0 + a_1i - a_2 - a_3i + a_4 + a_5i - ….. = (a_0 - a_2 + a_4 - …) + i(a_1 - a_3 + a_5 - ….)$

Putting z = -i we gwt, $(1 - i)^n = a_0 - a_1i - a_2 + a_3i + a_4 - a_5i + \ldots. = (a_0 - a_2 + a_4 - \ldots) - i(a_1 - a_3 + a_5 - \ldots)$

Multiplying the above two equations, we get,

$\{(1 + i)(1 - i)\}^n = (a_0 - a_2 + a_4 - \ldots)^2 + (a_1 - a_3 + a_5 - \ldots.)^2 = 2^n$

Option (a) is correct.

399. If $t_k = {}^{100}C_k x^{100-k}$, for k = 0, 1, 2, …, 100, then $(t_0 - t_2 + t_4 - \ldots + t_{100})^2 + (t_1 - t_3 + t_5 - \ldots - t_{99})^2$ equals
(a) $(x^2 - 1)^{100}$
(b) $(x + 1)^{100}$
(c) $(x^2 + 1)^{100}$
(d) $(x - 1)^{100}$

Solution :

$(xi + 1)^{100} = t_0 - it_1 - t_2 + \ldots. + t_{100} = (t_0 - t_2 + t_4 - \ldots. + t_{100}) - i(t_1 - t_3 + t_5 - \ldots. - t_{99})$

$(-xi + 1)^{100} = t_0 + it_1 - t_2 + \ldots. + t_{100} = (t_0 - t_2 + t_4 - \ldots. + t_{100}) + i(t_1 - t_3 + t_5 - \ldots - t_{99})$

Multiplying the above two equalities we get, $(t_0 - t_2 + t_4 - \ldots + t_{100})^2 + (t_1 - t_3 + t_5 - \ldots - t_{99})^2 = (1 + x^2)^{100}$

Option (c) is correct.

400. The expression $(1 + i)^n/(1 - i)^{n-2}$ equals
(a) $-i^{n+1}$
(b) i^{n+1}
(c) $-2i^{n+1}$
(d) 1

Solution :

$(1 + i)^n/(1 - i)^{n-2} = \{(1 + i)^{2n-2}/2^{n-2} = (1 + 2i + i^2)^{n-1}/2^{n-2} = 2^{n-1}i^{n-1}/2^{n-2} = 2i^{n-1} = -2i^{n+1}$

Option (c) is correct.

401. The value of the sum $\cos(\pi/1000) + \cos(2\pi/1000) + ... + \cos(999\pi/1000)$ equals
(a) 0
(b) 1
(c) 1/1000
(d) An irrational number.

Solution :

Now, $\cos(\pi/1000) + \cos(999\pi/1000) = \cos(\pi/2)\cos(499\pi/1000) = 0$ (as $\cos(\pi/2) = 0$)

Similarly, $\cos(2\pi/1000) + \cos(998\pi/1000) = 0$

…

…

$\cos(500\pi/1000) = \cos(\pi/2) = 0$

Therefore, the sum equals 0

Option (a) is correct.

402. The sum $1 + {}^nC_1\cos\theta + {}^nC_2\cos2\theta + ... + {}^nC_n\cos n\theta$ equals
(a) $(2\cos(\theta/2))^n\cos(n\theta/2)$
(b) $(2\cos^2(\theta/2))^n$
(c) $(2\cos^2(n\theta/2))^n$
(d) None of the foregoing quantities.

Solution :

Now, $(1 + x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$

Putting $x = \cos\theta + i\sin\theta$ we get,

$(1 + \cos\theta + i\sin\theta)^n = 1 + {}^nC_1(\cos\theta + i\sin\theta) + {}^nC_2(\cos\theta + i\sin\theta)^2 + ... + {}^nC_n(\cos\theta + i\sin\theta)^n$

- ⇨ $\{2\cos^2(\theta/2) + i*2\sin(\theta/2)\cos(\theta/2)\}^n = 1 + {}^nC_1(\cos\theta + i\sin\theta) + {}^nC_2(\cos2\theta + i\sin2\theta) + ... + {}^nC_n(\cos n\theta + i\sin n\theta)$
- ⇨ $(2\cos(\theta/2))^n\{\cos(\theta/2) + i\sin(\theta/2)\}^n = (1 + {}^nC_1\cos\theta + {}^nC_2\cos2\theta + ... + {}^nC_n\cos n\theta) + i(\sin\theta + ... + {}^nC_n\sin n\theta)$
- ⇨ $(2\cos(\theta/2))^n\{\cos(n\theta/2) + i\sin(n\theta/2)\} = (1 + {}^nC_1\cos\theta + {}^nC_2\cos2\theta + ... + {}^nC_n\cos n\theta) + i(\sin\theta + ... + {}^nC_n\sin n\theta)$

Equating the real part from both sides of the equation we get,

$(1 + {}^nC_1\cos\theta + {}^nC_2\cos2\theta + \dots + {}^nC_n\cos n\theta) = (2\cos(\theta/2))^n\cos(n\theta/2)$

Option (a) is correct.

403. Let $i = \sqrt{(-1)}$. Then
   (a) i and –i each has exactly one square root
   (b) i has two square roots but –i doesn't have any
   (c) neither i nor –i has any square root
   (d) i and –i each has exactly two square roots.

Solution :

Let $z^2 = i = \cos(\pi/2) + i\sin(\pi/2) = e^{i\pi/2}$

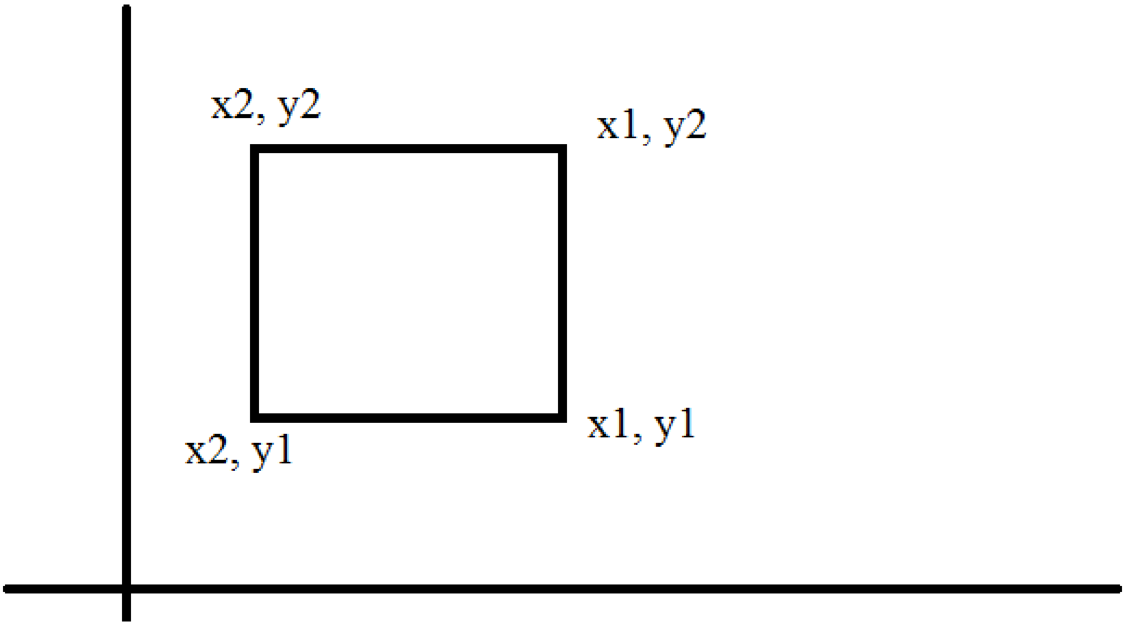
⇨ $z = \pm e^{i\pi/4} = \pm(\cos(\pi/4) + i\sin(\pi/4)) = \pm(1/\sqrt{2} + i/\sqrt{2})$

Similarly, -i has exactly two square roots = $\pm e^{i3\pi/4} = \pm(\cos(3\pi/4) + i\sin(3\pi/4)) = \pm(-1/\sqrt{2} + i/\sqrt{2})$

Option (d) is correct.

404. If the complex numbers w and z represent two diagonally opposite vertices of a square, then the other two vertices are given by the complex numbers
   (a) $w + iz$ and $w - iz$
   (b) $(1/2)(w + iz) + (1/2)(w - iz)$ and $(1/2)(w + z) - (1/2)i(w + z)$
   (c) $(1/2)(w - z) + (1/2)i(w - z)$ and $(1/2)(w - z) - (1/2)i(w - z)$
   (d) $(1/2)(w + z) + (1/2)i(w - z)$ and $(1/2)(w + z) - (1/2)i(w - z)$.

Solution :



Option (d) is correct. (Self-explanatory)

405. Let A = {a + b√(-1) a, b are integers} and U = {x Ɛ A | 1/x Ɛ A}. Then the number of elements of U is

(a) 2
(b) 4
(c) 6
(d) 8

Solution :

a = 0, b = 1 and a = 0, b = -1 i.e. i and –i belong to U.

a = 1, b = 0 i.e. 1 belong to U.

a = -1, b = 0 i.e. -1 belong to U.

Option (b) is correct.

406. Let i = √(-1). Then the number of distinct elements in the set S = {$i^n + i^{-n}$: n an integer} is

(a) 3
(b) 4
(c) greater than 4 but finite
(d) infinite

Solution :

Let, $n = 1$, $i + 1/i = 0$

$n = 2$, $i^2 + 1/i^2 = -1 -1 = -2$

$n = 3$, $i^3 + 1/i^3 = -i + i = 0$

$n = 4$, $i^4 + 1/i^4 = 1 + 1 = 2$

Option (a) is correct.

407. Let $i = \sqrt{(-1)}$ and p be a positive integer. A necessary and sufficient condition for $(-i)^p = i$ is
(a) p is one of 3, 11, 19, 27, ….
(b) p is an odd integer
(c) p is not divisible by 4
(d) none of the foregoing conditions.

Solution :

Clearly, p = 3, 7, 11, 15, 19, 23, 27, ….

Option (d) is correct.

408. Recall that for a complex number $z = x + iy$, where $i = \sqrt{(-1)}$, $z' = x - iy$ and $|z| = (x^2 + y^2)^{1/2}$. The set of all pairs of complex numbers (z_1, z_2) which satisfy $|(z_1 - z_2)/(1 - z_1'z_2)| < 1$ is
(a) all possible pairs (z_1, z_2) of complex numbers
(b) all pairs of complex numbers (z_1, z_2) for which $|z_1| < 1$ and $|z_2| < 1$
(c) all pairs of complex numbers (z_1, z_2) for which at least one of the following statements is true :
(i) $|z_1| < 1$ and $|z_2| > 1$
(ii) $|z_1| > 1$ and $|z_2| < 1$
(d) all pairs of complex numbers (z_1, z_2) for which at least one of the following statements is true :
(i) $|z_1| < 1$ and $|z_2| < 1$
(ii) $|z_1| > 1$ and $|z_2| > 1$

Solution :

$|(z_1 - z_2)/(1 - z_1'z_2)| < 1$

⇨ $|z_1 - z_2| < |1 - z_1'z_2|$
⇨ $|z_1 - z_2|^2 < |1 - z_1'z_2|^2$

- ⇨ $(z_1 - z_2)(z_1' - z_2') < (1 - z_1'z_2)(1 - z_1z_2')$
- ⇨ $|z_1|^2 - z_1z_2' - z_1'z_2 + |z_2|^2 < 1 - z_1z_2' - z_1'z_2 + |z_1|^2|z_2|^2$
- ⇨ $1 - |z_1|^2 - |z_2|^2 + |z_1|^2|z_2|^2 > 0$
- ⇨ $(1 - |z_1|^2)(1 - |z_2|^2) > 0$
- ⇨ $1 - |z_1|^2 > 0$ and $1 - |z_2|^2 > 0$ or $1 - |z_1|^2 < 0$ and $1 - |z_2|^2 < 0$
- ⇨ $|z_1| < 1$ and $|z_2| < 1$ or $|z_1| > 1$ and $|z_2| > 1$

Option (d) is correct.

409. Suppose z_1, z_2 are complex numbers satisfying $z_2 \neq 0$, $z_1 \neq z_2$ and $|(z_1 + z_2)/(z_1 - z_2)| = 1$. Then z_1/z_2 is
(a) real and negative
(b) real and positive
(c) purely imaginary
(d) not necessarily any of these.

Solution :

$|(z_1 + z_2)/(z_1 - z_2)| = 1$

- ⇨ $|z_1 + z_2| = |z_1 - z_2|$
- ⇨ $|z_1 + z_2|^2 = |z_1 - z_2|^2$
- ⇨ $(z_1 + z_2)(z_1' + z_2') = (z_1 - z_2)(z_1' - z_2')$
- ⇨ $|z_1|^2 + z_1z_2' + z_1'z_2 + |z_2|^2 = |z_1|^2 - z_1z_2' - z_1'z_2 + |z_2|^2$
- ⇨ $2z_1z_2' = -2z_1'z_2$
- ⇨ $z_1/z_2 = -z_1'/z_2'$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Therefore, $z_1/z_2 = (x_1 + iy_1)/(x_2 + iy_2) = (-x_1 + iy_1)/(x_2 - iy_2) = (x_1 + iy_1 - x_1 + iy_1)/(x_2 + iy_2 + x_2 - iy_2) = iy_1/x_2$ = purely imaginary.

Option (c) is correct.

410. The modulus of the complex number $\{(2 + i\sqrt{5})/(2 - i\sqrt{5})\}^{10} + \{(2 - i\sqrt{5})/(2 + i\sqrt{5})\}^{10}$ is
(a) $2\cos(20\cos^{-1}2/3)$
(b) $2\sin(10\cos^{-1}2/3)$
(c) $2\cos(10\cos^{-1}2/3)$
(d) $2\sin(20\cos^{-1}2/3)$

Solution :

$\{(2 + i\sqrt{5})/(2 - i\sqrt{5})\}^{10} + \{(2 - i\sqrt{5})/(2 + i\sqrt{5})\}^{10}$

$= \{(2 + i\sqrt{5})^2/9\}^{10} + \{(2 - i\sqrt{5})/9\}^{10}$

$= (2/3 + i\sqrt{5}/3)^{20} + (2/3 - i\sqrt{5}/3)^{20}$

Let, $\cos A = 2/3$, then $\sin A = \sqrt{5}/3$

The expression becomes, $(\cos A + i\sin A)^{20} + (\cos A - i\sin A)^{20}$

$= \cos 20A + i\text{son}20A + \cos 20A - i\sin 20A$

$= 2\cos 20A$

Now, $|2\cos 20A| = 2\cos 20A = 2\cos(20\cos^{-1}2/3)$

Option (a) is correct.

411. For any complex number $z = x + iy$ with x and y real, define $<z> = |x| + |y|$. Let z_1 and z_2 be any two complex numbers. Then

(a) $<z_1 + z_2> \leq <z_1> + <z_2>$

(b) $<z_1 + z_2> = <z_1> + <z_2>$

(c) $<z_1 + z_2> \geq <z_1> + <z_2>$

(d) None of the foregoing statements need always be true.

Solution :

$|x_1 + x_2| \leq |x_1| + |x_2|$ and $|y_1 + y_2| \leq |y_1| + |y_2|$

Option (a) is correct.

412. Recall that for a complex number $z = x + iy$, where $i = \sqrt{(-1)}$, $|z| = (x^2 + y^2)^{1/2}$ and $\arg(z)$ = principal value of $\tan^{-1}(y/x)$. Given complex numbers $z_1 = a + ib$, $z_2 = (a/\sqrt{2})(1 - i) + (b/\sqrt{2})(1 + i)$, $z_3 = (a/\sqrt{2})(i - 1) - (b/\sqrt{2})(i + 1)$, where a and b are real numbers, only one of the following statements is true. Which one is it?

(a) $|z_1| = |z_2|$ and $|z_2| > |z_3|$

(b) $|z_1| = |z_3|$ and $|z_1| < |z_2|$

(c) $\arg(z_1) = \arg(z_2)$ and $\arg(z_1) - \arg(z_3) = \pi/4$

(d) $\arg(z_2) - \arg(z_1) = -\pi/4$ and $\arg(z_3) - \arg(z_2) = \pm\pi$

Solution :

Clearly, option (d) is correct. (*Very easy problem but lengthy)*

413. If $a_0, a_1, \ldots, a_{2n}$ are real numbers such that $(1 + z)^{2n} = a_0 + a_1z + a_2z^2 + \ldots + a_{2n}z^{2n}$, for all complex numbers z, then
(a) $a_0 + a_1 + a_2 + \ldots + a_{2n} = 2^n$
(b) $(a_0 - a_2 + a_4 - \ldots)^2 + (a_1 - a_3 + a_5 - \ldots)^2 = 2^{2n}$
(c) $a_0^2 + a_1^2 + a_2^2 + \ldots + a_{2n}^2 = 2^{2n}$
(d) $(a_0 + a_2 + \ldots)^2 + (a_1 + a_3 + a_5 + \ldots)^2 = 2^{2n}$

Solution :

See solution of problems 398, 399.

Option (b) is correct.

414. If z is a nonzero complex number and z/(1 + z) is purely imaginary, then z
(a) can be neither real nor purely imaginary
(b) is real
(c) is purely imaginary
(d) satisfies none of the above properties

Solution :

Now, z/(1 + z) = purely imaginary

⇨ (x + iy)/{(1 + x) + iy)} = purely imaginary
⇨ (x + iy){(1 + x) – iy}/{$(1 + x)^2 + y^2$} = purely imaginary
⇨ {$x(1 + x) + y^2$}/{$(1 + x)^2 + y^2$} = 0 (real part = 0)
⇨ $x = -(x^2 + y^2)$

Option (a) is correct.

415. Let a and b be any two nonzero real numbers. Then the number of complex numbers z satisfying the equation $|z|^2 + a|z| + b = 0$ is
(a) 0 or 2 and both these values are possible
(b) 0 or 4 and both these values are possible
(c) 0, 2 or 4 and all these values are possible
(d) 0 or infinitely many and both these values are possible

Solution :

$|z| = \{-a \pm \sqrt{(a^2 - 4b)}\}/2$

Obviously option (d) is correct.

416. Let C denote the set of complex numbers and define A and B by A = {(z, w) : z, w Ɛ C and |z| = |w|}; B = {(z, w) : z, w Ɛ C and $z^2 = w^2$}. Then
(a) A = B
(b) A is a subset of B
(c) B is a subset of A
(d) None of the foregoing statements is correct.

Solution :

Let, $z = x_1 + iy_1$ and $w = x_2 + iy_2$

$|z| = |w| \Rightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$

And, $z^2 = w^2 \Rightarrow x_1^2 - y_1^2 + i(2x_1y_1) = x_2^2 - y_2^2 + i(2x_2y_2)$ i.e. $x_1^2 - y_1^2 = x_2^2 - y_2^2$ and $x_1y_1 = x_2y_2$

Now, $(x_1^2 + y_1^2)^2 = (x_1^2 - y_1^2)^2 + (2x_1y_1)^2 = (x_2^2 - y_2^2)^2 + (2x_2y_2)^2 = (x_2^2 + y_2^2)^2$

⇨ $(x_1^2 + y_1^2) = (x_2^2 + y_2^2)$

Option (c) is correct.

417. Among the complex numbers z satisfying $|z - 25i| \le 15$, the number having the least argument is
(a) 10i
(b) -15 + 25i
(c) 12 + 16i
(d) 7 + 12i

Solution :

Put each of the complex numbers of the option in the given equation and see them if satisfy the equation. Those who satisfy, find arguments of them and check the least one.

Option (c) is correct.

418. The minimum possible value of $|z|^2 + |z - 3|^2 + |z - 6i|^2$, where z is a complex number and $i = \sqrt{(-1)}$, is

(a) 15
(b) 45
(c) 30
(d) 20

Solution :

Let $z = x + iy$,

$|z|^2 + |z - 3|^2 + |z - 6i|^2 = x^2 + y^2 + (x - 3)^2 + y^2 + x^2 + (y - 6)^2 = 3(x^2 + y^2) - 6x - 12y + 45 = 3(x^2 + y^2 - 2x - 4y + 15) = 3(x - 1)^2 + 3(y - 2)^2 + 30 \geq 30$

Option (c) is correct.

419. The curve in the complex plane given by the equation $Re(1/z) = ¼$ is a
(a) vertical straight line at a distance of 4 from the imaginary axis
(b) circle with radius unity
(c) circle with radius 2
(d) straight line not passing through the origin

Solution :

Let, $z = x + iy$

$1/z = 1/(x + iy) = (x - iy)/(x^2 + y^2)$

$Re(1/z) = x/(x^2 + y^2) = ¼$

- ⇨ $x^2 + y^2 = 4x$
- ⇨ $x^2 + y^2 - 4x + 4 = 4$
- ⇨ $(x - 2)^2 + y^2 = 2^2$

Option (c) is correct.

420. The set of all complex numbers z such that $\arg\{(z - 2)/(z + 2)\} = \pi/3$ represents
(a) part of a circle
(b) a circle
(c) an ellipse
(d) part of an ellipse

Solution :

Now, $\arg\{(z - 2)/(z + 2)\} = \pi/3$

- ⇨ $\arg[\{(x - 2) + iy\}/\{(x + 2) + iy\}] = \pi/3$
- ⇨ $\arg\{(x - 2) + iy\} - \arg\{(x + 2) + iy\} = \pi/3$
- ⇨ $\tan^{-1}\{y/(x - 2)\} - \tan^{-1}\{y/(x + 2)\} = \pi/3$
- ⇨ $\tan^{-1}[\{y/(x - 2) - y/(x + 2)\}/\{1 + y^2/(x^2 - 4)\}] = \pi/3$
- ⇨ $\{y(x + 2) - y(x - 2)\}/(x^2 - 4 + y^2) = \sqrt{3}$
- ⇨ $4y = \sqrt{3}(x^2 + y^2 - 4)$
- ⇨ $\sqrt{3}(x^2 + y^2) - 4y - 4\sqrt{3} = 0$

Option (a) is correct. (As arg represents principal value)

421. Let $z = x + iy$ where x and y are real and $i = \sqrt{(-1)}$. The points (x, y) in the plane, for which $(z + i)/(z - i)$ is purely imaginary (that is, it is of the form ib when b is a real number), lie on
   (a) a straight line
   (b) a circle
   (c) an ellipse
   (d) a hyperbola

Solution :

$(z + i)/(z - i) = \{x + i(y + 1)\}/\{x + i(y - 1)\} = \{x^2 + (y^2 - 1) - ix(y - 1) + ix(y + 1)\}/\{x^2 + (y - 1)^2\}$

Real part = 0

- ⇨ $\{x^2 + (y^2 - 1)\}/\{x^2 + (y - 1)^2\} = 0$
- ⇨ $x^2 + y^2 = 1$

Option (b) is correct.

422. If the point z in the complex plane describes a circle of radius 2 with centre at the origin, then the point $z + 1/z$ describe
   (a) a circle
   (b) a parabola
   (c) an ellipse
   (d) a hyperbola

Solution :

$|z| = 2 \Rightarrow x^2 + y^2 = 4$

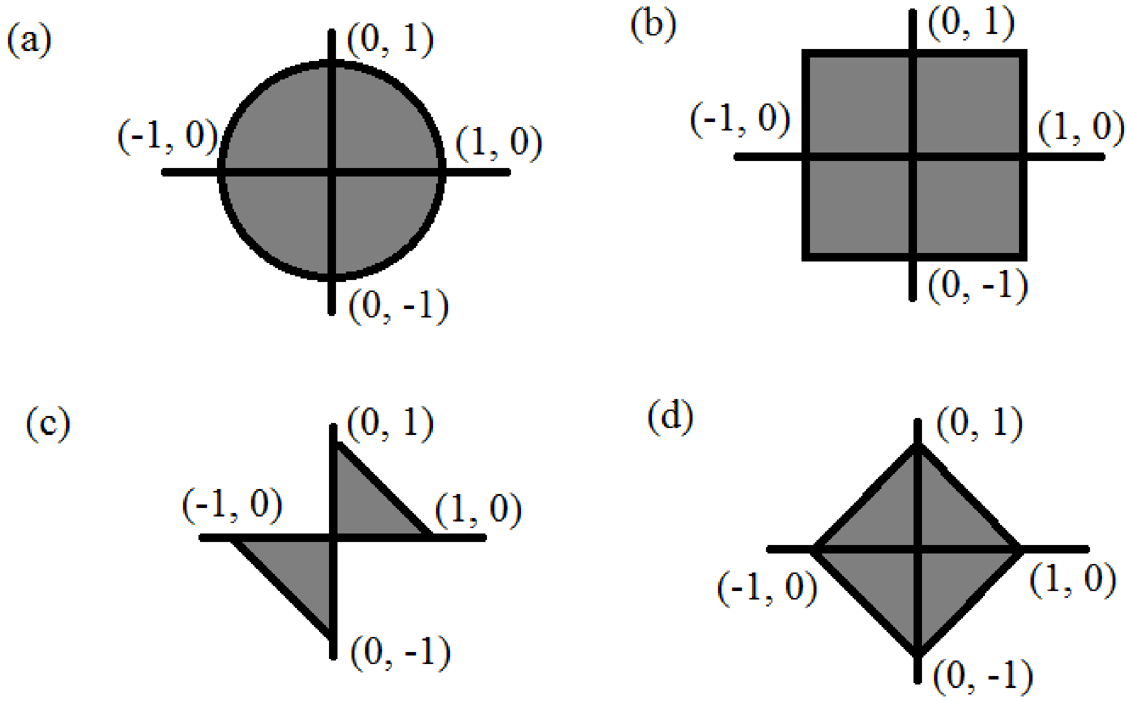
$z + 1/z = x + iy + 1/(x + iy) = x + iy + (x - iy)/(x^2 + y^2) = x + iy + (x - iy)/4 = (5x + 3iy)/4$

$|z + 1/z| = \sqrt{(25x^2 + 9y^2)}/4 = a$

⇨ $25x^2 + 9y^2 = 16a^2$

Option (c) is correct.

423. The set $\{(x, y) : |x| + |y| \leq 1\}$ is represented by the shaded region in one of the four figures. Which one is it?



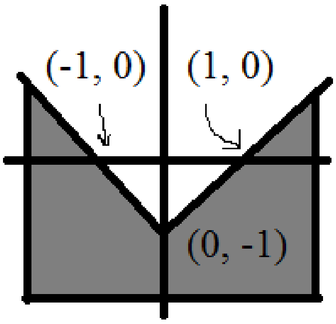
Solution :

The curves are $x + \leq 1$, $- x - y \leq 1$, $x - y \leq 1$ and $-x + y \leq 1$ all are straight lines.

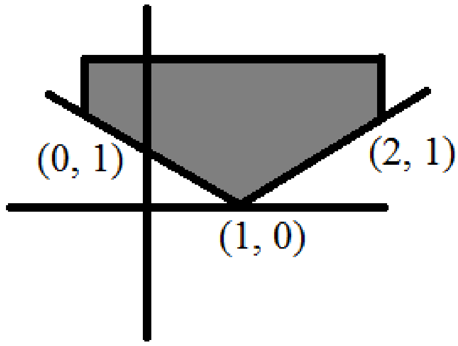
Clearly, option (d) is correct.

424. The sets $\{(x, y) : |y - 1| - x \geq 1\}$; $\{(x, y) : |x| - y \geq 1\}$; $\{(x, y) : x - |y| \leq 1\}$; $\{(x, y) : y - |x - 1| \geq 0\}$ are represented by the shaded regions in the figures given below in some order.

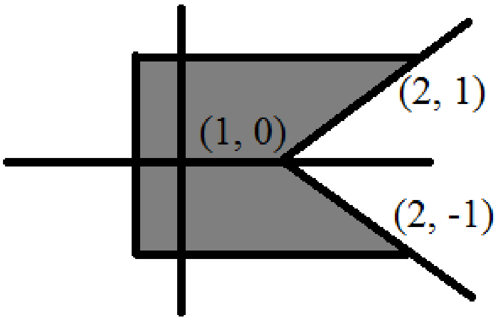
F1



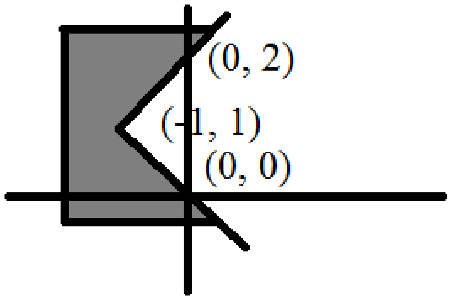
F2



F3



F4

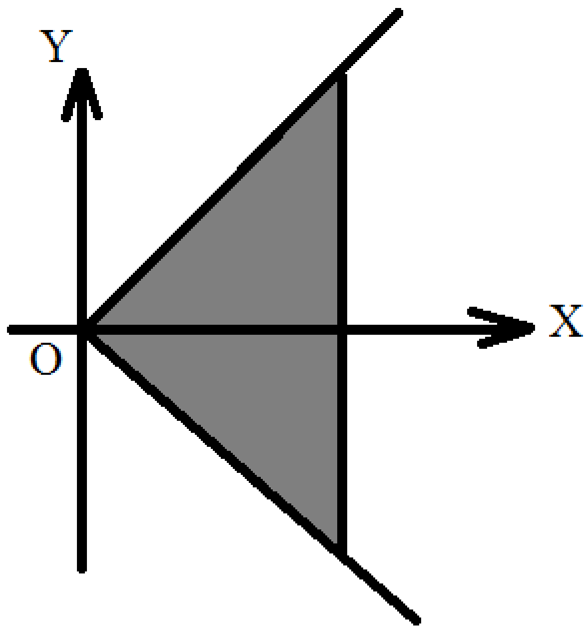


Then the correct order of the figures is
(a) F_4, F_1, F_2, F_3
(b) F_4, F_2, F_3, F_1
(c) F_1, F_4, F_3, F_2
(d) F_4, F_1, F_3, F_2

Solution :

Option (d) is correct.

425. The shaded region in the diagram represents the relation



(a) $y \leq x$
(b) $|y| \leq |x|$
(c) $y \leq |x|$
(d) $|y| \leq x$

Solution :

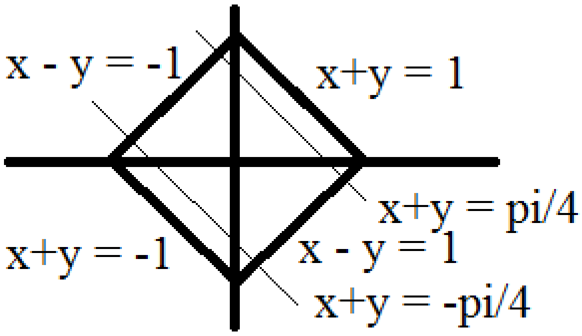
Option (d) is correct.

426. The number of points (x, y) in the plane satisfying the two equations $|x| + |y| = 1$ and $\cos\{2(x + y)\} = 0$ is
(a) 0
(b) 2
(c) 4
(d) Infinitely many

Solution :

Now, $\cos\{2(x + y)\} = 0 = \cos(\pm\pi/2)$

⇨ $x + y = \pm\pi/4$



4 points of intersection.

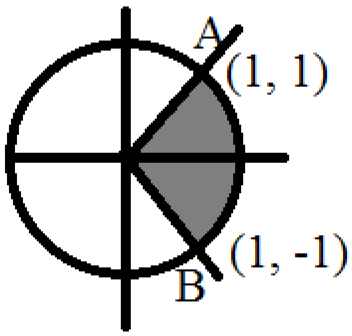
Option (c) is correct.

Directions for Items 427 and 428 :

Let the diameter of a subset S of the plane be defined as the maximum of the distances between arbitrary pairs of points of S.

427. Let S = {(x, y) : (y – x) ≤ 0, (x + y) ≥ 0, $x^2 + y^2 \le 2$}. Then the diameter of S is
(a) 4
(b) 2
(c) $2\sqrt{2}$
(d) $\sqrt{2}$

Solution :



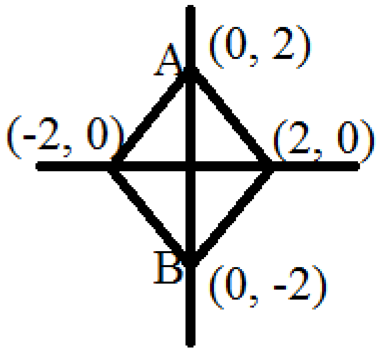
Clearly, diameter = AB = 2

Option (b) is correct.

428. Let S = {(x, y) : |x| + |y| = 2}. Then the diameter of S is
(a) 2

(b) $4\sqrt{2}$
(c) 4
(d) $3\sqrt{2}$

Solution :



Clearly, diameter = AB = 4

Option (c) is correct.

429. The points (2, 1), (8, 5) and (x, 7) lie on a straight line. The value of x is
(a) 10
(b) 11
(c) 12
(d) 11 + 2/3

Solution :

Now, the area formed by the triangle by the points = 0

⇨ $(1/2)\{2(5 - 7) + 8(7 - 1) + x(1 - 5)\} = 0$
⇨ $4x = 44$
⇨ $x = 11$

Option (b) is correct.

430. In a parallelogram PQRS, P is the point (-1, -1), Q is (8, 0) and R is (7, 5). Then S is the point
(a) (-1, 4)
(b) (-2, 4)
(c) (-2, 3.5)

(d) (-1.5, 4)

Solution :

Now, (P + R)/2 = (Q + S)/2 (as in a parallelogram the diagonals bisects each other)

⇨ (Q + S)/2 = (3, 2)

Clearly, option (b) is correct.

431. The equation of the line passing through the point of intersection of the lines x − y + 1 = 0 and 3x + y − 5 = 0 and is perpendicular to the line x + 3y + 1 = 0 us
(a) x + 3y − 1 = 0
(b) x − 3y + 1 = 0
(c) 3x − y + 1 = 0
(d) 3x − y − 1 = 0

Solution :

The equation of the straight line which is perpendicular to the line x + 3y + 1 = 0 is, 3x − y + c = 0

Now, x − y + 1 = 0 and 3x + y − 5 = 0

Solving them we get, x = 1, y = 2

Putting the values in the equation we get, 3 − 2 + c = 0 => c = -1

Option (d) is correct.

432. A rectangle PQRS joins the points P = (2, 3), Q = (x_1, y_1), R = (8, 11), S = (x_2, y_2). The line QS is known to be parallel to the y-axis. Then the coordinates of Q and S respectively
(a) (0, 7) and (10, 7)
(b) (5, 2) and (5, 12)
(c) (7, 6) and (7, 10)
(d) None of the foregoing pairs

Solution :

Now, as QS is parallel to y-axis, so x-coordinate of both Q and S are same.

Option (a) cannot be true.

Now, diagonals of a rectangle bisects each other.

Therefore, (P + R)/2 = (Q + S)/2

⇨ Option (c) cannot be true.

Slope of PQ, (2 – 3)/(5 – 2) = -1/3

Slope of QR = (11 – 2)/(8 – 5) = 3

Therefore, PQ and QR are perpendicular.

So, option (b) is correct.

433. The sum of the interior angles of a polygon is equal to 56 right angles. Then the number of sides of the polygon is
(a) 12
(b) 15
(c) 30
(d) 25

Solution :

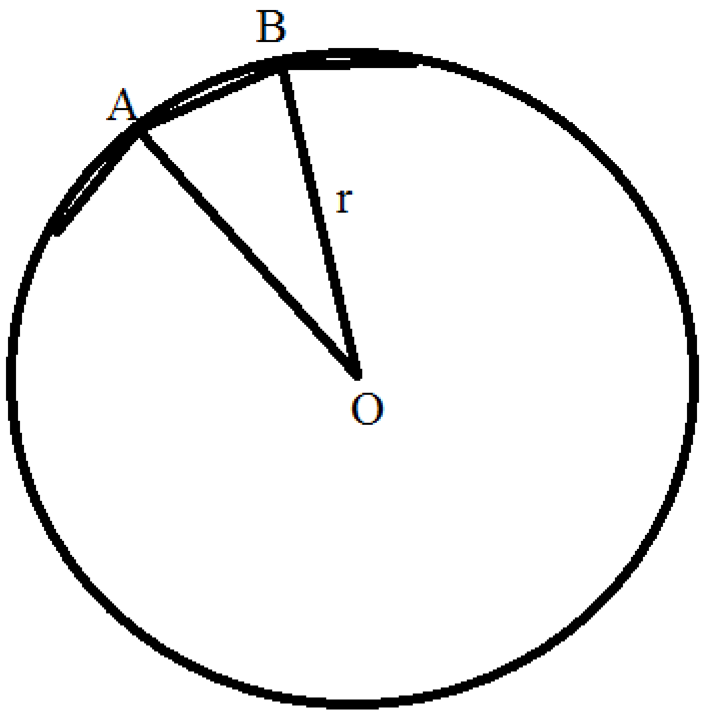
Now, (n – 2)*π = 56*(π/2)

⇨ n = 30

Option (c) is correct.

434. The ration of a circumference of a circle to the perimeter of the inscribed regular polygon with n sides is
(a) 2π : 2nsin(π/n)
(b) 2π : nsin(π/n)
(c) 2π : 2nsin(2π/n)
(d) 2π : nsin(2π/n)

Solution :



Angle OAB = $(1/2)(n - 2)\pi/n$ = Angle OBA

Angle AOB = $\pi - (n - 2)\pi/n = 2\pi/n$

Now, in triangle OAB we get, $OA/\sin\{(n - 2)\pi/2n\} = AB/\sin(2\pi/n)$

- ⇨ $AB = r\sin(2\pi/n)/\sin(\pi/2 - \pi/n)$
- ⇨ $AB = r*2\sin(\pi/n)\cos(\pi/n)/\cos(\pi/n)$
- ⇨ $AB = 2r\sin(\pi/n)$

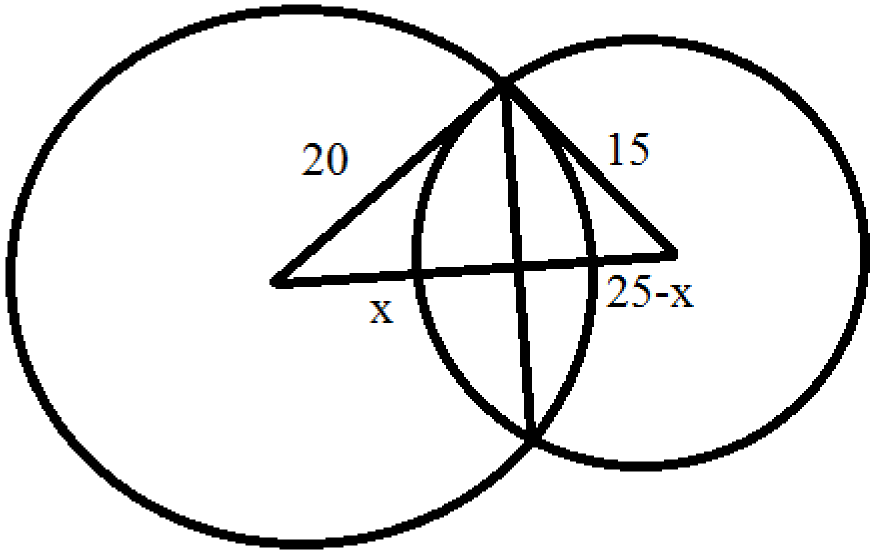
Perimeter = $r*2n\sin(\pi/n)$

Therefore, required ratio = $2\pi r : r*2n\sin(\pi/n) = 2\pi : 2n\sin(\pi/n)$

Option (a) is correct.

435. The length of the common chord of two circles of radii 15 cm and 20 cm, whose centres are 25 cm apart, is (in cm)
   (a) 24
   (b) 25
   (c) 15
   (d) 20

Solution :



Now, $(20^2 - x^2) = 15^2 - (25 - x)^2$

⇨ $400 - x^2 = 225 - 625 + 50x - x^2$
⇨ $50x = 800$
⇨ $x = 16$

Length of the chord = $2\sqrt{(20^2 - 16^2)} = 2*12 = 24$ cm

Option (a) is correct.

436. A circle of radius $\sqrt{3} - 1$ units with both coordinates of the centre negative, touches the straight line $y - \sqrt{3}x = 0$ and $x - \sqrt{3}y = 0$. The equation of the circle is
(a) $x^2 + y^2 + 2(x + y) + (\sqrt{3} - 1)^2 = 0$
(b) $x^2 + y^2 + 2(x + y) + (\sqrt{3} + 1)^2 = 0$
(c) $x^2 + y^2 + 4(x + y) + (\sqrt{3} - 1)^2 = 0$
(d) $x^2 + y^2 + 4(x + y) + (\sqrt{3} + 1)^2 = 0$

Solution :

Let the coordinate of the centre of the circle = (-g, -f).

Now, $|-f + \sqrt{3}g|/2 = \sqrt{3} - 1$

⇨ $\sqrt{3}g - f = 2(\sqrt{3} - 1)$

And, $\sqrt{3}f - g = 2(\sqrt{3} - 1)$

$3f - \sqrt{3}g = 2\sqrt{3}(\sqrt{3} - 1)$

Adding we get, $2f = 2(\sqrt{3} - 1)(\sqrt{3} + 1)$

- ⇨ $f = 2$
- ⇨ $g = 2$

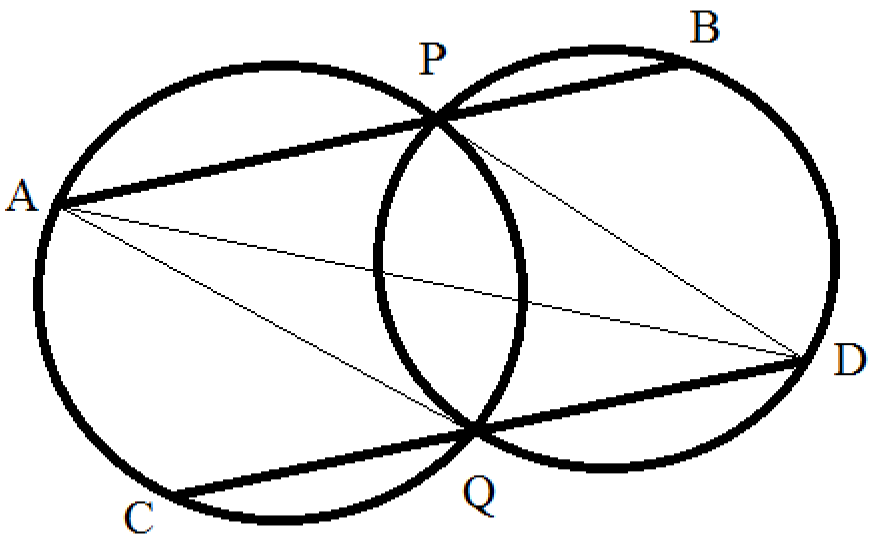
Equation of circle is, $(x + 2)^2 + (y + 2)^2 = (\sqrt{3} - 1)^2$

- ⇨ $x^2 + y^2 + 4(x + y) + 8 = 3 - 2\sqrt{3} + 1$
- ⇨ $x^2 + y^2 + 4(x + y) + (\sqrt{3} + 1)^2 = 0$

Option (d) is correct.

437. Two circles APQC and PBDQ intersect each other at the points P and Q and APB and CQD are two parallel straight lines. Then only one of the following statements is *always* true. Which one is it?
(a) ABDC is a cyclic quadrilateral
(b) AC is parallel to BD
(c) ABDC is a rectangle
(d) Angle ACQ is right angle

Solution :



We join A and Q, P and D, A and D.

Now, Angle PAQ and PDQ are on the same arc PQ. So, Angle PAQ = Angle PDQ.

Now, Angle PDQ = Angle BPD (AB||CD and PD is intersector)

- ⇨ Angle PAQ = Angle BPD
- ⇨ PD||AQ

- ⇨ PAQD is a parallelogram.
- ⇨ PA = DQ

Similarly, PB = CQ

- ⇨ AB = CD

Now, AB = CD and AB||CD

- ⇨ ACDB is a parallelogram.

Therefore, AC||BD

Option (b) is correct.

438. The area of the triangle whose vertices are (a, a), (a + 1, a + 1), (a + 2, a) is
(a) a^2
(b) 2a
(c) 1
(d) √2

Solution :

Area = |(1/2){a(a + 1 − a) + (a + 1)(a − a) + (a + 2)(a − a − 1)}|

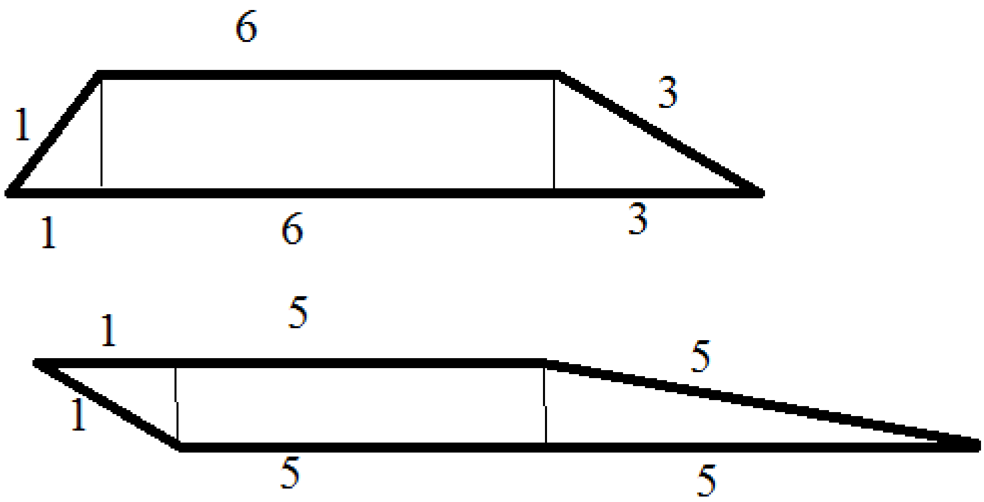
= |(1/2){a − a − 2}|

= |-1| = 1

Option (c) is correct.

439. In a trapezium, the lengths of two parallel sides are 6 and 10 units. If one of the oblique sides has length 1 unit, then the length of the other oblique side must be
(a) greater than 3 units but less than 4 units
(b) greater than 3 units but less than 5 units
(c) less than or equal to 3 units
(d) greater than 5 units but less than 6 units

Solution :



Between 3 and 5 units.

Option (b) is correct.

440. If in a triangle, the radius of the circumcircle is double the radius of the inscribed circle, then the triangle is
(a) equilateral
(b) isosceles
(c) right-angled
(d) not necessarily any of the foregoing types

Solution :

So, $R = 2r$

Distance between circumcentre and incentre $= R^2 - 2Rr = 4r^2 - 4r^2 = 0$

⇨ Incentre and circumcentre is same.
⇨ Triangle is equilateral.

Option (a) is correct.

441. If in a triangle ABC with a, b, c denoting sides opposite to angles A, B and C respectively, $a = 2b$ and $A = 3B$, then the triangle
(a) is isosceles
(b) is right-angled but not isosceles
(c) is right-angled and isosceles
(d) need not necessarily be any of the above types

Solution :

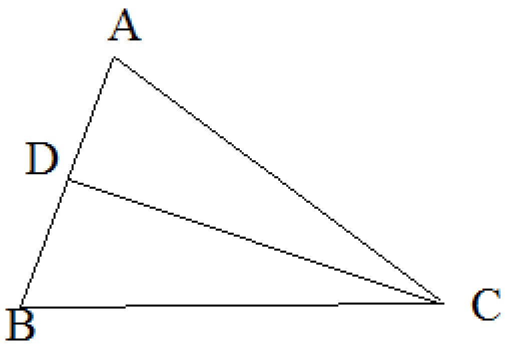
a = 2b

- ⇨ sinA = 2sinB
- ⇨ sin3B = 2sinB
- ⇨ $3\sin B - 4\sin^3 B - 2\sin B = 0$
- ⇨ $1 - 4\sin^2 B = 0$
- ⇨ sinB = ½
- ⇨ B = 30
- ⇨ A = 90
- ⇨ C = 60

Option (b) is correct.

442. Let the bisector of the angle at C of a triangle ABC intersect the side AB in a point D. Then the geometric mean of CA and CB
(a) is less than CD
(b) is equal to CD
(c) is greater than CD
(d) doesn't always satisfy any one of the foregoing properties

Solution :



Angle BDC = 180 – B – C/2 = 180 – B – C + C/2 = A + C/2

Similarly, CDA = B + C/2

In triangle BCD, CB/sin(A + C/2) = CD/sin(C/2)

- ⇨ CB = CDsin(A + C/2)/sinB
- ⇨ CA = CDsin(B + C/2)/sinA
- ⇨ $CA*CB = CD^2\sin(A + C/2)\sin(B + C/2)/\sin A\sin B$
- ⇨ $CA*CB = CD^2\{2\sin(A + C/2)\sin(B + C/2)/2\sin A\sin B$
- ⇨ $CA*CB = CD^2\{\cos(A - B) - \cos(A + B + C)\}/\{\cos(A - B) - \cos(A + B)\}$

- ⇨ $CA*CB = CD^2\{\cos(A - B) + 1\}/\{\cos(A - B) + \cos C\}$

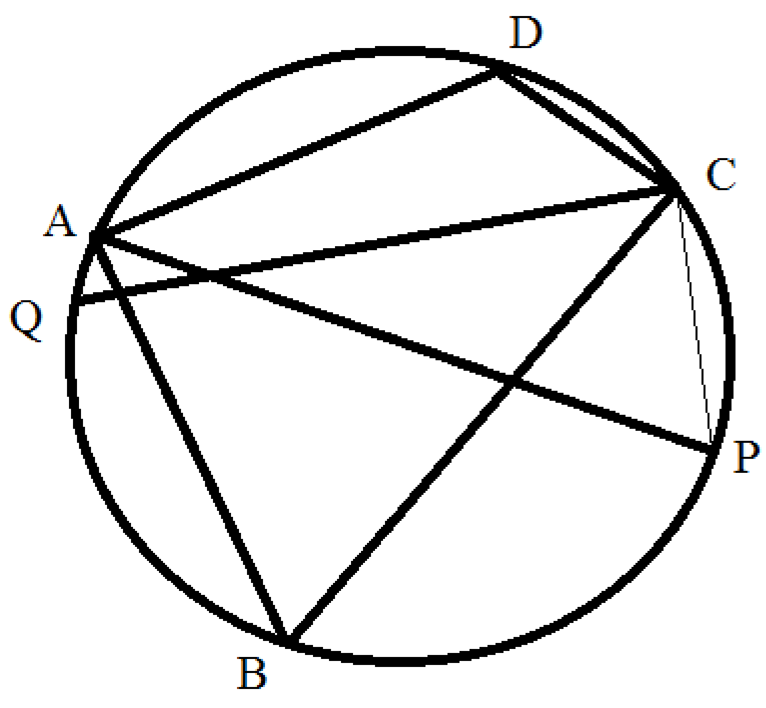
Now, $\cos C < 1$

- ⇨ $\cos(A - B) + \cos C < \cos(A - B) + 1$
- ⇨ $\{\cos(A - B) + 1\}/\{\cos(A - B) + \cos C) > 1$
- ⇨ $CA*CB > CD^2$
- ⇨ $\sqrt{(CA*CB)} > CD$

Option (c) is correct.

443. Suppose ABCD is a cyclic quadrilateral within a circle of radius r. The bisector of the angle A cuts the circle at a point P and the bisector of angle C cuts the circle at point Q. Then
 (a) AP = 2r
 (b) PQ = 2r
 (c) BQ = DP
 (d) PQ = AP

Solution :



Now, Angle A + Angle C = 180

- ⇨ Angle A/2 + Angle C/2 = 90
- ⇨ Angle BAP + Angle BQC = 90

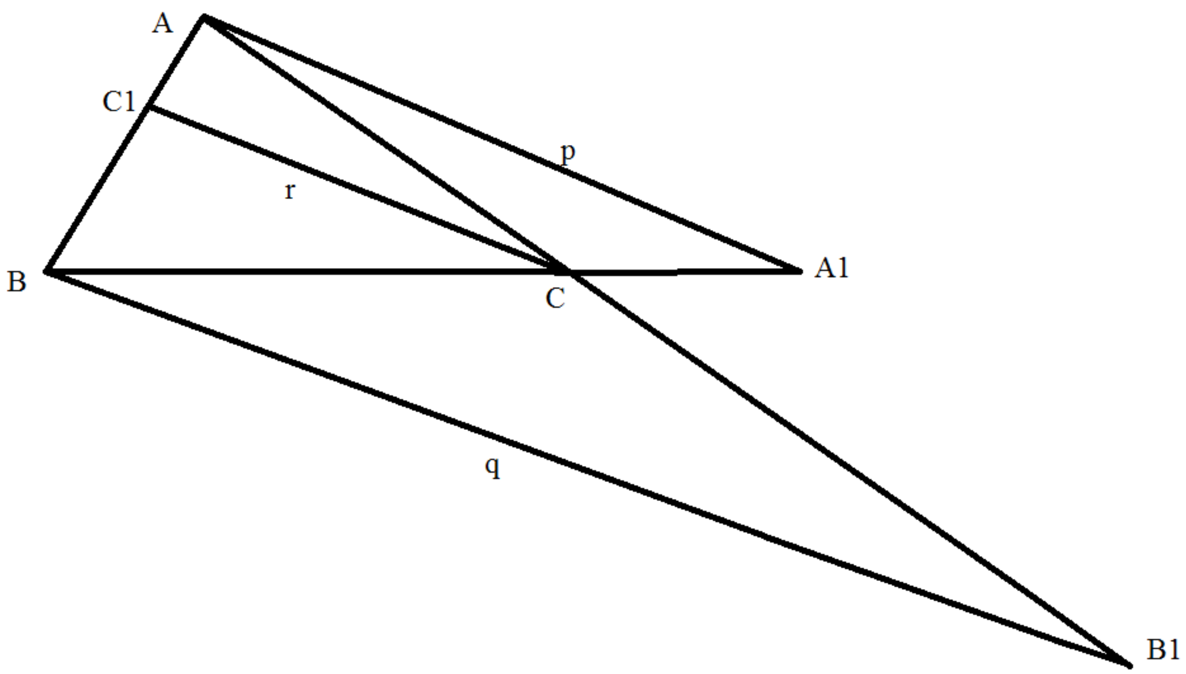
Now, Angle BAP = Angle BCP (On the samw arc BP)

- ⇨ Angle BCP + Angle BQC = 90
- ⇨ Angle PQC = 90
- ⇨ PQ = diameter (As semicircular angle is right-angle)
- ⇨ PQ = 2r

Option (b) is correct.

444. In a triangle ABC, let C_1 be any point on the side AB other than A or B. Join CC_1. The line passing through A and parallel to CC_1 intersects the line BC extended at A_1. The line passing through B and parallel to CC_1 intersects the line AC extended at B_1. The lengths AA_1, BB_1, CC_1 are given to be p, q, r units respectively. Then
(a) r = pq/(p + q)
(b) r = (p + q)/4
(c)r = √(pq)/2
(d) none of the foregoing statements is true.

Solution :



Triangle AC_1C and triangle ABB_1 are similar,

Therefore, r/q = AC_1/AB

Triangle BC_1C and triangle ABA_1 are similar.

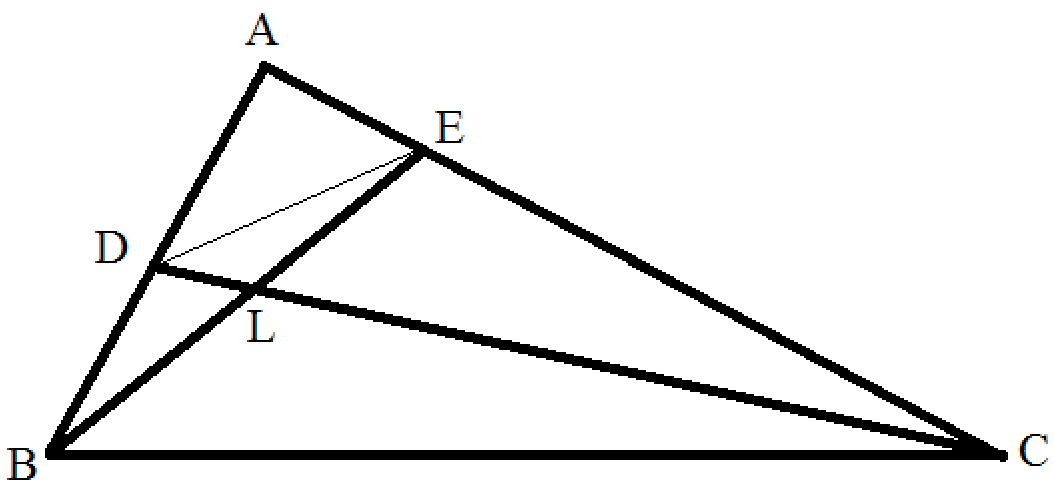
Therefore, r/p = BC_1/AB

- ⇨ $r/q + r/p = (AC_1 + BC_1)/AB$
- ⇨ $r(p + q)/pq = 1$
- ⇨ $r = pq/(p + q)$

Option (a) is correct.

445. In a triangle ABC, D and E are the points on AB and AC respectively such that Angle BDC = Angle BEC. Then
   (a) Angle BED = Angle BCD
   (b) Angle CBE = Angle BED
   (c) Angle BED + Angle CDE = Angle BAC
   (d) Angle BED + Angle BCD = Angle BAC

Solution :



In triangle BDL and triangle LEC, Angle DLB = Angle CLE (opposite angle)

Angle BDL = Angle CEL (given)

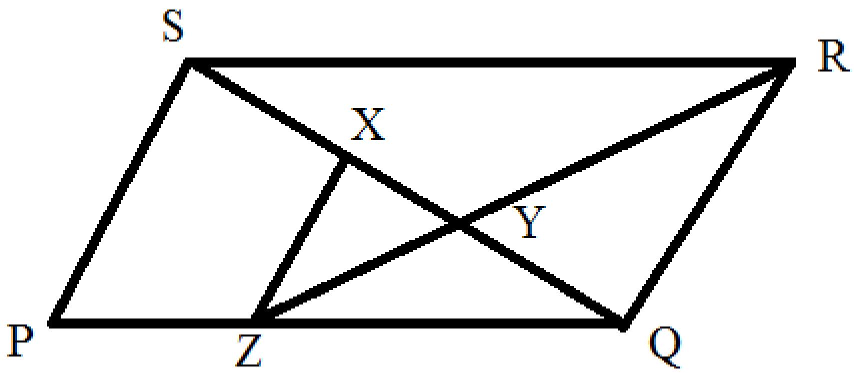
Therefore, Angle DBL = Angle LCE i.e. Angle DBE = Angle DCE

Therefore, BCED is a cyclic quadrilateral (Angle DBE and Angle DCE are on the same arc DE and same)

- ⇨ Angle BED = Angle BCD (on same arc BD)

Option (a) is correct.

446. In the picture, PQRS is a parallelogram. PS is parallel to ZX and PZ/ZQ equals 2/3. Then XY/SQ equals



(a) ¼
(b) 9/40
(c) 1/5
(d) 9/25

Solution :

Triangles QXZ and QSP are similar.

Therefore, QX/SQ = ZX/PS = ZQ/PQ

Now, in triangles XYZ and YRQ, Angle XYZ = Angle RYQ (opposite angle)

Angle YXZ = Angle YQR and Angle YZX = Angle YRQ (ZX||RQ)

Triangles XYZ and YRQ are similar.

Therefore, XY/YQ = ZX/QR = ZX/PS = ZQ/PQ

Let XY/SQ = a.

Now, YQ/XY = PQ/ZQ

- ⇨ (YQ + XY)/XY = PQ/ZQ + 1
- ⇨ QX/XY = PQ/ZQ + 1
- ⇨ (SQ/XY)(QX/SQ) = PQ/ZQ + 1
- ⇨ QX/SQ = a(PQ/ZQ + 1)
- ⇨ ZQ/PQ = a(PQ/ZQ + 1)

Now, PZ/ZQ = 2/3

- ⇨ (PZ + ZQ)/ZQ = (2 + 3)/3
- ⇨ ZQ/PQ = 3/5

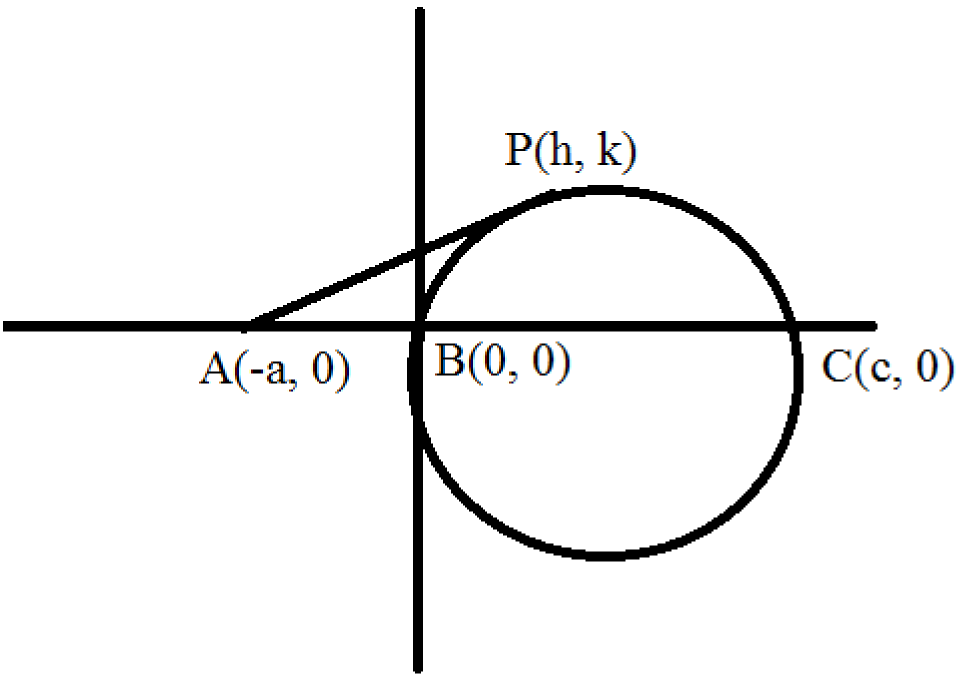
Therefore, the above equation becomes, 3/5 = a(5/3 + 1)

- ⇨ 3/5 = 8a/3
- ⇨ a = 9/40

Option (b) is correct.

447. Let A, B, C be three points on a straight line, B lying between A and C. Consider all circles passing through B and C. The points of contact of the tangents from A to these circles lie on
(a) a straight line
(b) a circle
(c) a parabola
(d) a curve of none of the foregoing types

Solution :



Let, centre of the circle = (-g, -f)

Let equation of the circle = $x^2 + y^2 + 2gx + 2fy + c_1 = 0$

The circle passes through (0, 0), so $c_1 = 0$

Now, the circle passes through (c, 0), so, $c^2 + 2gc = 0$, $g = -c/2$

The circle passes through (h, k), so $h^2 + k^2 + 2(-c/2)h + 2fk = 0$

⇨ $f = -(h^2 + k^2 + ch)/2k$

Now, slope of AP = $(k - 0)/(h + a) = k/(h + a)$

Slope of OP (O being the centre) = $\{k - (h^2 + k^2 + ch)/2k\}/(h - c/2) = (k^2 - h^2 - ch)/(2kh - kc)$

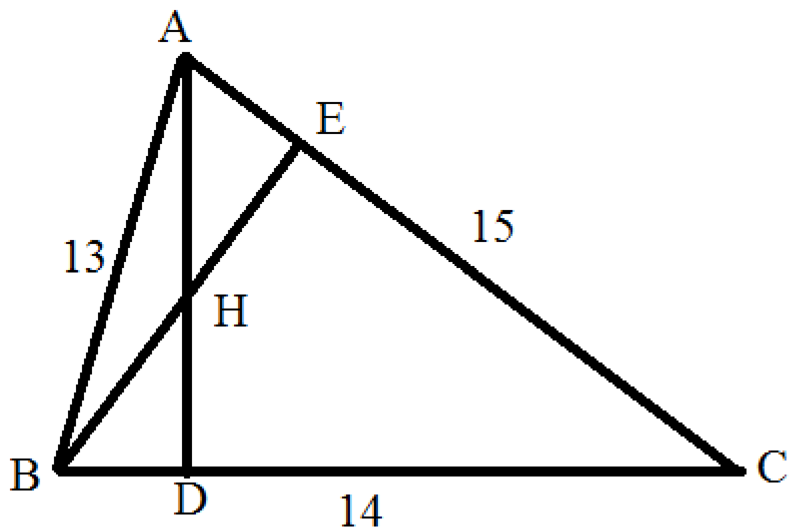
Now, slope of AP*slope of OP = -1 (perpendicular as AP is tangent)

⇨ $\{(k^2 - h^2 - ch)/(2kh - kc)\}\{k/(h + a) = -1$
⇨ $k(k^2 - h^2 - ch) = -2kh^2 + hkc - 2kha + kac$
⇨ $k^3 - h^2k - hkc = -2kh^2 + hkc - 2kha + kac$
⇨ $k^3 + kh^2 - 2khc + 2kha - kac = 0$
⇨ $h^2 + k^2 - 2h(c - a) - ac = 0$
⇨ a circle.

Option (b) is correct.

448. ABC be a triangle with AB = 13; BC = 14 and CA = 15. AD and BE are the altitudes from A and B to BC and AC respectively. H is the point of intersection of AD and BE. Then the ratio HD/HB is
(a) 3/5
(b) 12/13
(c) 4/5
(d) 5/9

Solution :



Now, Angle ABE = 180 – (90 + A) = 90 – A (from triangle ABE)

Angle HBD = B – (90 – A) = A + B – 90

sin(HBD) = sin(A + B – 90) = sin(180 – C – 90) = sin(90 – C) = cosC = $(a^2 + b^2 - c^2)/2ab = (14^2 + 15^2 - 13^2)/(2*14*15) = (196 + 225 - 169)/(2*14*15) = 252/(2*14*15) = 3/5$

⇨ HD/HB = 3/5 (from triangle HBD)

Option (a) is correct.

449. ABC is a triangle such that AB = AC. Let D be the foot of the perpendicular from C to AB and E the foot of the perpendicular from B to AC. Then

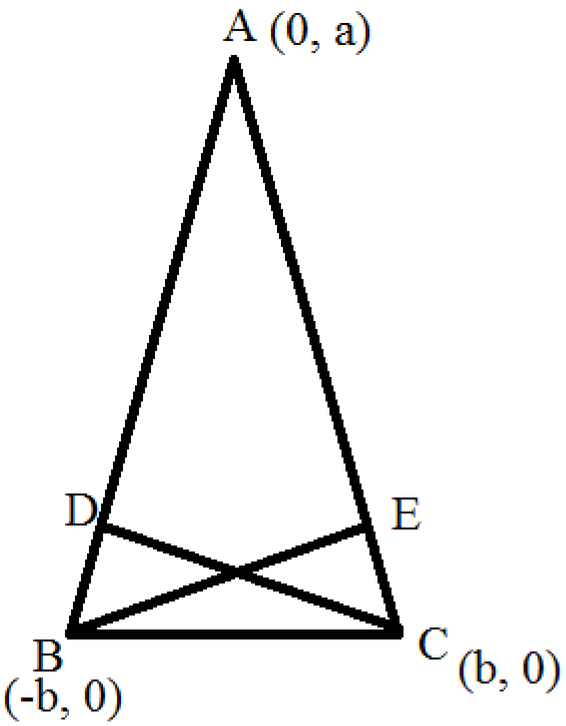
(a) $BC^3 < BD^3 + BE^3$

(b) $BC^3 = BD^3 + BE^3$

(c) $BC^3 > BD^3 + BE^3$

(d) None of the foregoing statements need always be true.

Solution :



Equation of AC is, $x/b + y/a = 1$ i.e. $ax + by = ab$, slope $= - a/b$

Slope of BE $= b/a$ (as perpendicular on AC)

Equation of BE is, $y - 0 = (b/a)(x + b)$ i.e. $bx - ay + b^2 = 0$

Solving them we get, $x = b(a^2 - b^2)/(a^2 + b^2)$, $y = 2ab^2/(a^2 + b^2)$

Therefore, $E = \{b(a^2 - b^2)/(a^2 + b^2), 2ab^2/(a^2 + b^2)\}$

$BE = \sqrt{[\{2ab^2/(a^2 + b^2)\}^2 + \{b(a^2 - b^2)/(a^2 + b^2) + b\}^2]}$

$BE^3 = [\{2ab^2/(a^2 + b^2)\}^2 + \{2a^2b/(a^2 + b^2)\}^2]^{3/2}$

$BE^3 = [\{2ab/(a^2 + b^2)\}^2(a^2 + b^2)]^{3/2}$

$BE^3 = (2ab)^3/(a^2 + b^2)^{3/2}$

$CE = [\{b(a^2 - b^2)/(a^2 + b^2) - b\}^2 + \{2ab^2/(a^2 + b^2)\}^2]^{1/2}$

$CE = [\{2b^3/(a^2 + b^2)\}^2 + \{2ab^2/(a^2 + b^2)\}^2]^{1/2}$

$CE = [\{2b^2/(a^2 + b^2)\}^2(b^2 + a^2)]^{1/2}$

$CE = 2b^2/(a^2 + b^2)^{1/2}$

$CE^3 = (2b^2)^3/(a^2 + b^2)^{3/2}$

$BE^3 + CE^3 = \{(2ab)^3 + (2b^2)^3\}/(a^2 + b^2)^{3/2}$

$= (2b)^3(a^3 + b^3)/(a^2 + b^2)^{3/2}$

$= (BC)^3\{(a^3 + b^3)/(a^2 + b^2)^{3/2}\} < (BC)^3$

Now, let, $(a^3 + b^3)^2 \geq (a^2 + b^2)^3$

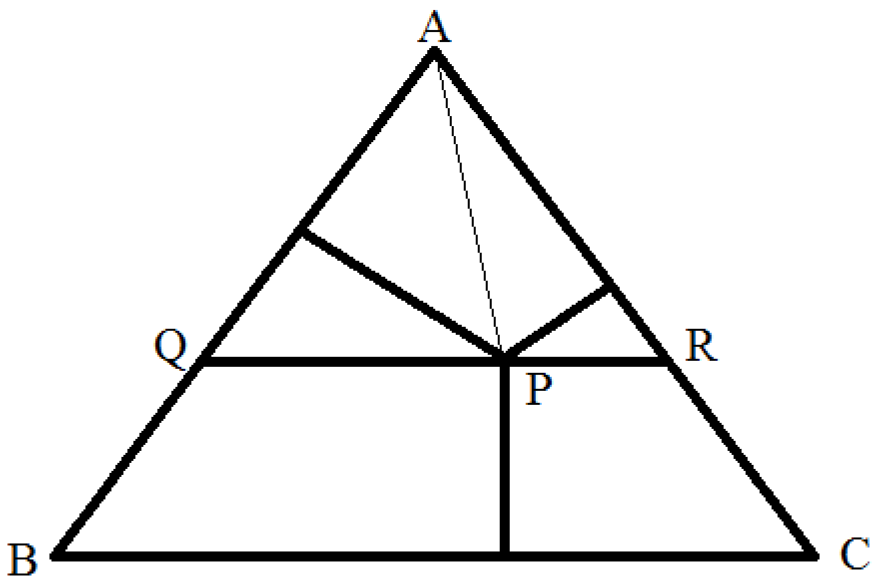
- ⇨ $a^6 + b^6 + 2a^3b^3 \geq a^6 + b^6 + 3a^2b^2(a^2 + b^2)$
- ⇨ $ab/3 \geq (a^2 + b^2)/2$ (But AM ≥ GM says, $(a^2 + b^2)/2 \geq ab > ab/3$)
- ⇨ $(a^3 + b^3) < (a^2 + b^2)^{3/2}$
- ⇨ $BC^3 > BE^3 + BD^3$ (As CE = BD)

Option (c) is correct.

450. Through the centroid of an equilateral triangle, a line parallel to the base is drawn. On this line, an arbitrary point P is taken inside the triangle. Let h denote the distance of P from the base of the triangle. Let h_1 and h_2 be the distances of P from the other two sides of the triangle. Then

(a) $h = (h_1 + h_2)/2$
(b) $h = \sqrt{(h_1h_2)}$
(c) $h = 2h_1h_2/(h_1 + h_2)$
(d) none of the foregoing conditions is necessarily true.

Solution :



AQ = AR = 2a/3 (where a is side of the equilateral triangle)

Area of triangle APQ = $(1/2)h_1*(2a/3) = ah_1/3$

Similarly, area of triangle APR = $ah_2/3$

Area of triangle APQ = area of triangle APQ + area of triangle APR = $(a/3)(h_1 + h_2)$

Now, height of the triangle AQR = 2H/3 where H is height of triangle ABC.

h = H/3

Therefore, height of the triangle AQR = 2h

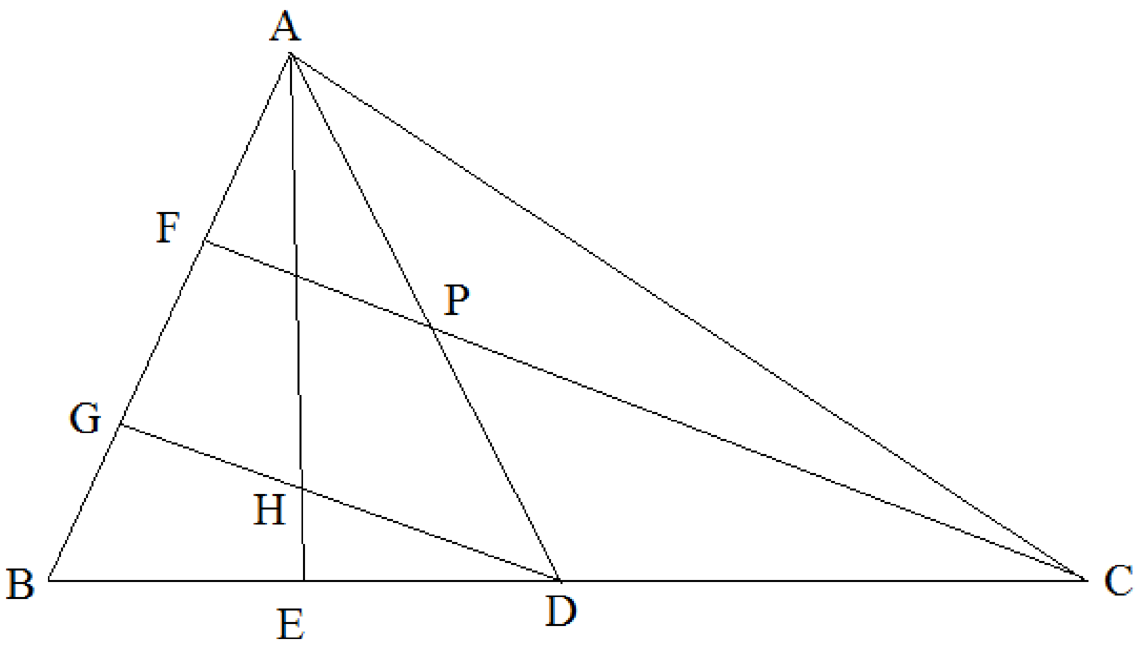
Area of triangle AQR = (1/2)(2a/3)*2h = 2ah/3

Therefore, $2ah/3 = (a/3)(h_1 + h_2)$

⇨ $h = (h_1 + h_2)/2$

Option (a) is correct.

451. In the figure that follows, BD = CD, BE = DE, AP = PD and DG||CF. Then (area of triangle ADH)/(area of triangle ABC) is equal to



(a) 1/6
(b) ¼
(c) 1/3
(d) None of the foregoing quantities.

Solution :

P is mid-point of AD and PF||DG.

Therefore, F is mid-point of AG.

In triangle BCE, GD||CF and D is mid-point of BC.

Therefore, G is mid-point of BF.

Therefore, AF = FG = BG

⇨ DGB = BGF = DFA
⇨ DGB = (1/3)ABD

In triangle AGD, GD||PF and P is mid-point of AD. Therefore, H is

Option (c) is correct.

452. Let A be the fixed point (0, 4) and B be a moving point (2t, 0). Let M be the mid-point of AB and let the perpendicular bisector of AB meet the y-axis at R. The locus of the mid-point P of MR is
(a) $y + x^2 = 2$
(b) $x^2 + (y - 2)^2 = ¼$

(c) $(y - 2)^2 - x^2 = ¼$
(d) None of the foregoing curves.

Solution :

Mid-point of AB = (t, 2)

Slope of AB = $(4 - 0)/(0 - 2t) = -2/t$

Slope of perpendicular bisector of AB = $t/2$

Equation of perpendicular bisector of AB is, $y - 2 = (t/2)(x - t)$

Putting $x = 0$, we get, $y = -t^2/2 + 2$

So, $R = (0, -t^2/2 + 2)$

Mid-point of MR, $P = (t/2, -t^2/4 + 2)$

$h = t/2$ and $k = -t^2/4 + 2$

⇨ $k + h^2 = 2$

Locus is $y + x^2 = 2$

Option (a) is correct.

453. Let l_1 and l_2 be a pair of intersecting lines in the plane. Then the locus of the points P such that the distance of P from l_1 is twice the distance of P from l_2 is
(a) an ellipse
(b) a parabola
(c) a hyperbola
(d) a pair of straight lines

Solution :

Let $l_1 \Rightarrow a_1x + b_1y + c_1 = 0$ and $l_2 \Rightarrow a_2x + b_2y + c_2 = 0$

P = (h, k)

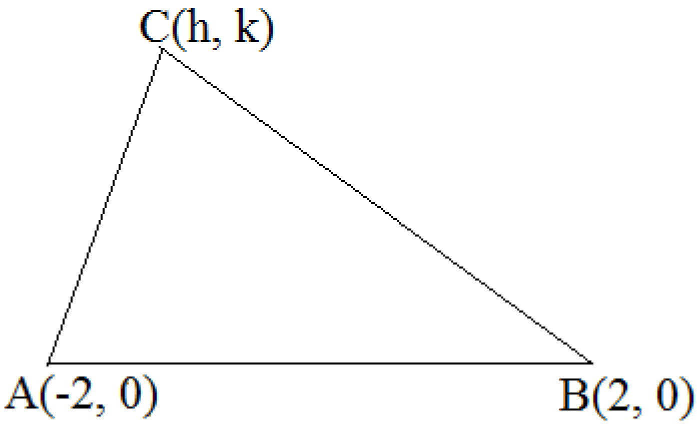
So, $|(a_1h + b_1k + c_1)/\sqrt{(a_1^2 + b_1^2)}| = 2|(a_2h + b_2k + c_2)/\sqrt{(a_2^2 + b_2^2)}|$

⇨ Locus is pair of straight lines. (One for + and one straight line for -)

Option (d) is correct.

454. A triangle ABC has fixed base AB and the ratio of the other two unequal sides is a constant. The locus of the vertex C is
(a) A straight line parallel to AB
(b) A straight line which is perpendicular to AB
(c) A circle with AB as a diameter
(d) A circle with centre on AB

Solution :



Now, CA/CB = constant = c

⇨ $\sqrt{\{(h + 2)^2 + k^2\}}/\sqrt{\{(h - 2)^2 + k^2\}} = c$
⇨ $(h + 2)^2 + k^2 = c^2(h - 2)^2 + c^2k^2$
⇨ $(1 - c^2)(h^2 + k^2) + 4h(1 + c^2) - 4c^2 = 0$

Locus is, $(1 - c^2)(x^2 + y^2) + 4x(1 + c^2) - 4c^2 = 0$

Option (d) is correct.

455. P is a variable point on a circle C and Q is a fixed point outside of C. R is a point on PQ dividing it in the ratio p : q, where p > 0 and q > 0 are fixed. Then the locus of R is
(a) a circle
(b) an ellipse
(c) a circle if p = q and an ellipse otherwise
(d) none of the foregoing curves

Solution :

Let C is $x^2 + y^2 = a^2$.

Let P = (b, c)

Q = (m, n)

Therefore, $b^2 + c^2 = a^2$.

R = (h, k)

So, h = (pm + qb)/(p + q)   and   k = (pn + qc)/(p + q)

- ⇨ qb = (p + q)h – pm   and qc = (p + q)k – pn
- ⇨ $q^2(b^2 + c^2) = (p + q)^2h^2 + (p + q)^2k^2 - 2pm(p + q)h + p^2m^2 - 2pn(p + q)k + p^2n^2$
- ⇨ $q^2a^2 = (p + q)^2(h^2 + k^2) - 2p(p + q)(mh + nk) + p^2(m^2 + n^2)$
- ⇨ $(p + q)^2(h^2 + k^2) - 2p(p + q)(mh + nk) + p^2(m^2 + n^2) - q^2a^2 = 0$

Locus is, $(p + q)^2(x^2 + y^2) - 2p(p + q)(mx + ny) + p^2(m^2 + n^2) - q^2a^2 = 0$

A circle.

Option (a) is correct.

456. Let r be the length of then chord intercepted by the ellipse $9x^2 + 16y^2 = 144$ on the line 3x + 4y = 12. Then
   (a) r = 5
   (b) r > 5
   (c)r = 3
   (d) r = √7

Solution :

Solving the two equations,

3x = 12 – 4y

$(3x)^2 + 16y^2 = 144$

- ⇨ $(12 - 4y)^2 + 16y^2 = 144$
- ⇨ $144 - 96y + 16y^2 + 16y^2 = 144$
- ⇨ 32y(y – 3) = 0
- ⇨ y = 0 , y = 3
- ⇨ x = 4, x = 0

Points are (4, 0) and (0, 3)

Distance = $\sqrt{(4 - 0)^2 + (0 - 3)^2} = 5$

Option (a) is correct.

457. The angles A, B and C of a triangle ABC are in arithmetic progression. AB = 6 and BC = 7. Then AC is
(a) 5
(b) 7
(c)8
(d) None of the foregoing numbers.

Solution :

Let A = B – d and C = B + d

A + B + C = 180

- ⇨ B – d + B + B + d = 180
- ⇨ B = 60

Now, AB/sinC = BC/sinA = AC/sinB

- ⇨ sinA = 7/(AC*√3/2) = 14/AC√3
- ⇨ sinC = 12/AC√3

sin(A + C) = sinAcosC + cosAsinC

- ⇨ sin120 = {14/AC√3)cosC + cosA{12/AC√3}
- ⇨ √3/2 = {14/AC√3}cosC + cosA{12/AC√3}
- ⇨ ¾ = 7cosC/AC + 6cosA/AC
- ⇨ ¾ = $7\sqrt{(3AC^2 - 144)}/AC^2\sqrt{3} + 6\sqrt{(3AC^2 - 196)}/AC^2\sqrt{3}$
- ⇨ $3\sqrt{3}AC^2/4 = 7\sqrt{(3AC^2 - 144)} + 6\sqrt{(AC^2 - 196)}$

Clearly, none of (a), (b), (c) satisfies the equation.

Option (d) is correct.

458. ABC is a triangle. P, Q and R are respectively the mid-points of AB, BC and CA. The area of the triangle ABC is 20. Then the area of the triangle PQR is
(a) 4
(b) 5
(c)6
(d) 8

Solution :

Area of triangle PQR = (1/4)*area of triangle ABC = 5

Option (b) is correct.

459. Let AC and CE be perpendicular line segments, each of length 18. Suppose B and D are the mid-points of AC and CE, respectively. If F is the point of intersection of EB and AD, then the area of the triangle DEF is

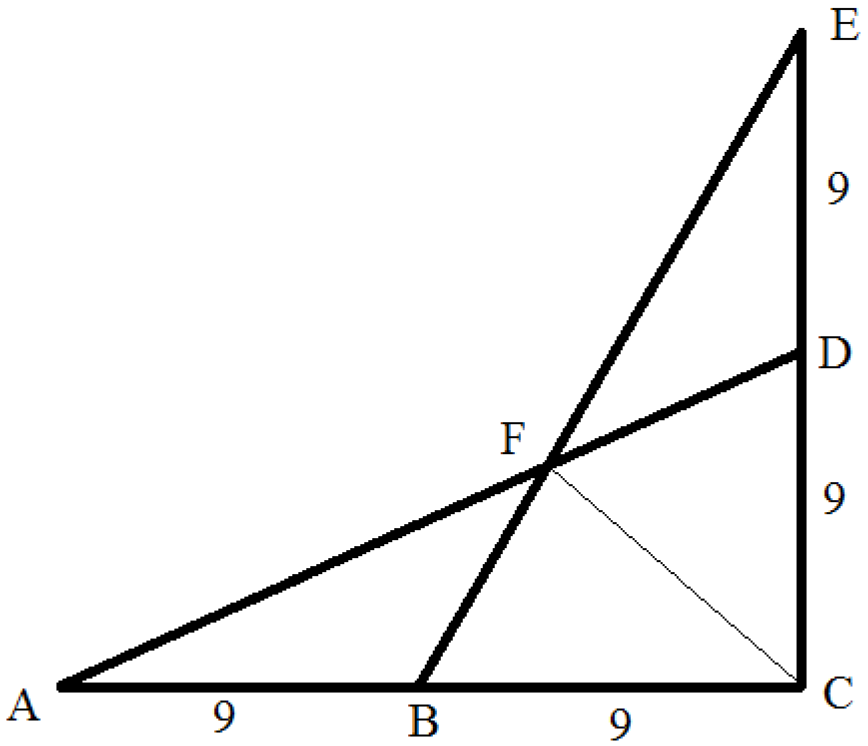
(a) 18

(b) 18√2

(c) 27

(d) 5√85/2

Solution :



Now, area of triangle DEF = area of triangle DFC (as base is same and height is same)

= area of triangle BFC

Therefore, area of triangle DEF = (1/3)*area of triangle EBC

Area of triangle EBC = (1/2)*9*18 = 81

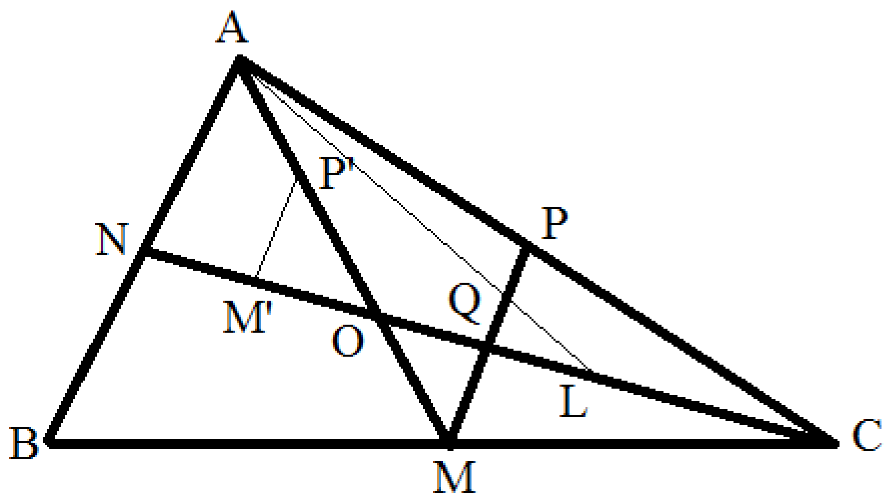
Area of triangle DEF = 81/3 = 27

Option (c) is correct.

460. In a triangle ABC, the medians AM and CN to the sides BC and AB respectively, intersect at the point O. Let P be the mid-point of AC and let MP intersect CN at Q. If the area of the triangle OMQ is s square units, the area of ABC is

(a) 16s
(b) 18s
(c) 21s
(d) 24s

Solution :



In triangles OMQ and ANO, Angle QOM = Angle AON (opposite angle)

Angle OMQ = Angle OAN and Angle OQM = Angle ANO (PM||AN as P and M are mid-points of AC and BC respectively)

Now, OQ/ON = OM/OA = QM/AN = ½ (QM = (1/2)BN (In triangle BNC Q and M are mid-points of CN and BC)

So, we draw P'OM' where OP' = OM = ½ OA and OM' – OQ = ½ ON.

Therefore, OMQ = OM'P'= (1/4)AON

L is mid-point of OC and we know medians intersects at 2 : `1 ratio.

Therefore, ON = OL = LC

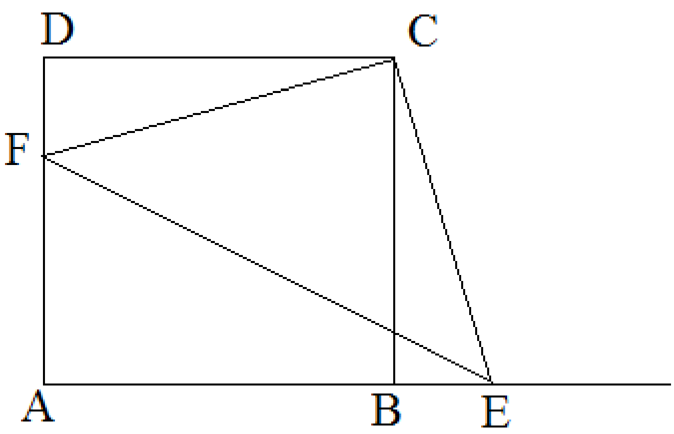
- ⇨ AON = AOL = ALC
- ⇨ AON = (1/3)ANC
- ⇨ OMQ = (1/4)AON = (1/4)(1/3)ANC = (1/12)(1/2)ABC
- ⇨ ABC = 24OMQ = 24s

Option (d) is correct.

461. Let F be a point on the side AD of a square ABCD of area 256. Suppose the perpendicular to the line FC at C meets the line segment AB extended at E. If the area of the triangle CEF is 200, then the length of BE is

(a) 12
(b) 14
(c) 15
(d) 20

Solution :



In triangles CFD and BEC, Angle FDC = Angle CBE (both right angles)

CD = CB (both sides of square ABCD)

Angle DCF = Angle BCE (Angle DCB – Angle FCB = Angle FCE – Angle FCB)

So, CF = CE

Area of triangle CEF = (1/2)*CF*CE = (1/2)CE^2 = 200

⇨ CE = 20

$CB^2 = 256$

⇨ CB = 16

BE = $\sqrt{(20^2 - 16^2)} = 12$

Option (a) is correct.

462. Consider the circle with centre C = (1, 2) which passes through the points P = (1, 7) and Q = (4, -2). If R is the point of intersection of the tangents to the circle drawn at P and Q, then the area of the quadrilateral CPRQ is

(a) 50
(b) $50\sqrt{2}$
(c) 75
(d) 100

Solution :

Slope of CP = (7 − 2)/(1 − 1) = 5/0

Slope of tangent at P = 0

Therefore, equation of tangent at P is, y − 7 = 0(x − 1)

⇨ y = 7

Slope of CQ = (2 + 2)/(1 − 4) = -4/3

Slope of tangent at Q = ¾

Equation of tangent at Q is, y + 2 = (3/4)(x − 4)

⇨ 3x − 4y = 20

Solving them we get, x = 16

R = (16, 7)

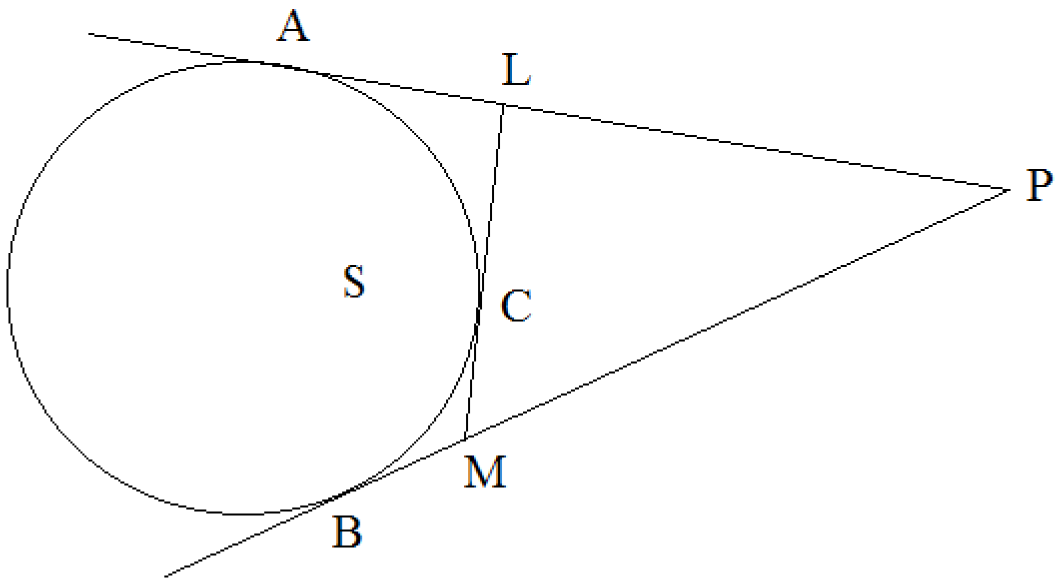
Area of triangle CPR = |(1/2){1(7 − 7) + 1(7 − 2) + 16(2 − 7)}| = 75/2

Area of triangle CRQ = |(1/2){1(7 + 2) + 16(-2 − 2) + 4(2 − 7)| = 75/2

Area of quadrilateral CPRQ = 75/2 + 75/2 = 75

Option (c) is correct.

463. PA and PB are tangents to a circle S touching S at points A and B. C is a point on S in between A and B as shown in the figure. LCM is a tangent to S intersecting PA and PB in points L and M, respectively. Then the perimeter of the triangle PLM depends on



(a) A, B, C and P
(b) P, but not on C
(c) P and C only
(d) the radius of S only

Solution :

Now, LA = LC (tangents from same point L)

PA – PL = LC

⇨ PA = PL + LC

MB = MC

PB – PM = MC

⇨ PB = PM + MC
⇨ PA + PB = PL + LC + PM + MC = PL + PM + LM = constant.
⇨ Therefore, it depends on only P.

Option (b) is correct.

464. A and B are two points lying outside a plane ∏, but on the same side of it. P and Q are, respectively, the feet of perpendiculars from A and B on ∏. Let X be any point on ∏. Then (AX + XB) is minimum when X
(a) lies on PQ and Angle AXP = Angle BXQ
(b) is the mid-point of PQ

(c) is any point of ∏ with Angle AXP = Angle BXQ
(d) is any point on the perpendicular bisector of PQ in ∏

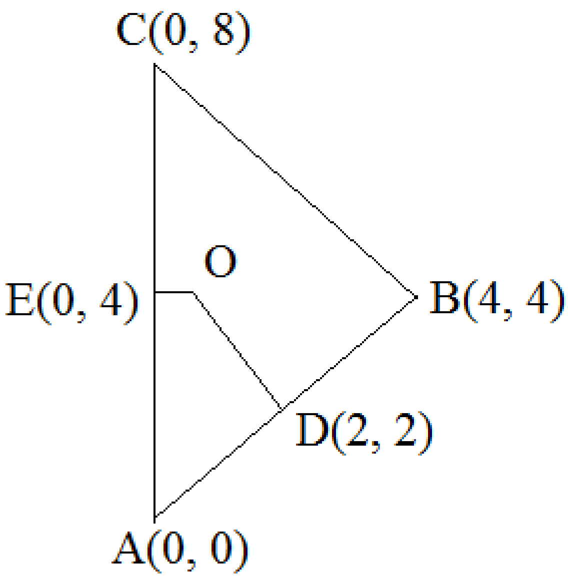
Solution :

This is self-explanatory.

Option (a) is correct.

465. The vertices of a triangle are the points (0, 0), (4, 4) and (0, 8). The radius of the circumcircle of the triangle is
(a) $3\sqrt{2}$
(b) $2\sqrt{2}$
(c) 3
(d) 4

Solution :



Slope of AB = (4 – 0)/(4 – 0) = 1

Slope of OD = -1

Equation of OD is, y – 2 = (-1)(x – 2)

⇨ x + y = 4

Equation of OE is, y = 4

Solving them we get, $x = 0$

Therefore, $O = (0, 4)$

Circumradius = 4

Option (d) is correct.

466. The number of different angles θ satisfying the equation $\cos\theta + \cos 2\theta = -1$ and at the same time satisfying the condition $0 < \theta < 360$ is
(a) 0
(b) 4
(c) 2
(d) 3

Solution :

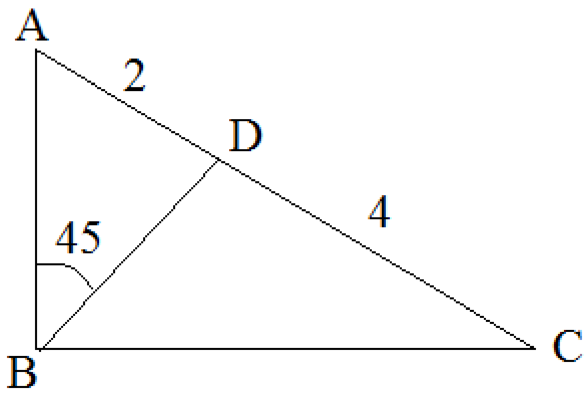
$\cos\theta + 2\cos^2\theta - 1 = -1$

- ⇨ $\cos\theta(2\cos\theta + 1) = 0$
- ⇨ $\cos\theta = 0, \theta = 90, 270$
- ⇨ $2\cos\theta + 1 = 0$
- ⇨ $\cos\theta = -1/2$
- ⇨ $\theta = 120, 240$

Option (b) is correct.

467. ABC is a right-angled triangle with right angle at B. D is a point on AC such that Angle ABD = 45. If AC = 6 cm and AD = 2 cm then AB is
(a) $6/\sqrt{5}$ cm
(b) $3\sqrt{2}$ cm
(c) $12/\sqrt{5}$ cm
(d) 2 cm

Solution :



Now, in triangle ABD, 2/sin45 = AB/sin(ADB) = AB/sin(180 – BDC) = AB/sin(BDC)

In triangle BDC, 4/sin45 = BC/sin(BDC)

Dividing both the equation we get, ½ = AB/BC

⇨ BC = 2AB

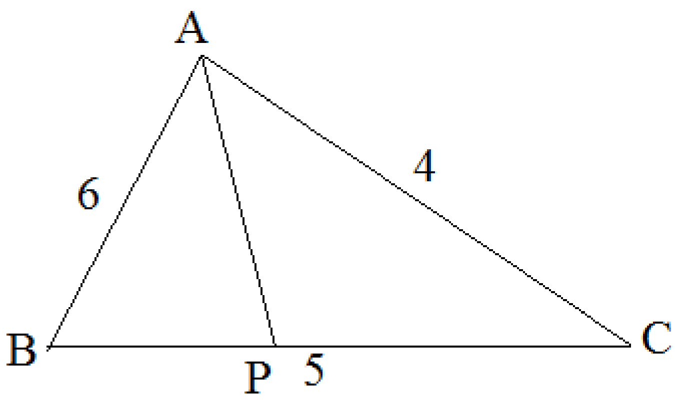
Now, $AB^2 + BC^2 = 6^2$

⇨ $AB^2 + 4AB^2 = 6^2$
⇨ $5AB^2 = 6^2$
⇨ $AB = 6/\sqrt{5}$

Option (a) is correct.

468. In the triangle ABC, AB = 6, BC = 5, CA = 4. AP bisects the angle A and P lies on BC. Then BP equals
(a) 3
(b) 3.1
(c) 2.9
(d) 4.5

Solution :



In triangle ABP, BP/sin(A/2) = 6/sin(APB) = 6/sin(180 – APC) = 6/sin(APC)

In triangle ACP, PC/sin(A/2) = 4/sin(APC)

Dividing the two equations we get, BP/PC = 3/2

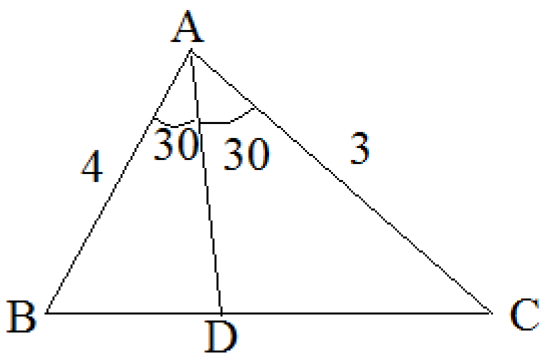
BP = 5*(3/5) = 3

Option (a) is correct.

469. In a triangle ABC, the internal bisector of the angle A meets BC at D. If AB = 4, AC = 3 and A = 60, then length of AD is

(a) $2\sqrt{3}$
(b) $12\sqrt{3}/7$
(c) $15\sqrt{3}/8$
(d) None of these numbers.

Solution :



Now, $\cos 60 = (4^2 + 3^2 - BC^2)/(2*4*3)$

⇨ $BC = \sqrt{13}$

Now, BD/CD = 4/3 (See previous problem)

⇨ $BD = \sqrt{13}(4/7) = 4\sqrt{13}/7$

Now, in triangle ABD, $\cos 30 = \{4^2 + AD^2 - (4\sqrt{13}/7)^2\}/(2*4*AD)$

⇨ $4\sqrt{3}AD = 16 - 16*13/49 + AD^2$
⇨ $AD^2 - 4\sqrt{3}AD + 16*36/49 = 0$
⇨ $AD = \{4\sqrt{3} \pm \sqrt{(48 - 4*16*36/49)}\}/2 = \{4\sqrt{3} \pm 4\sqrt{3}/7\}/2 = 2\sqrt{3}(1 \pm 1/7) = 16\sqrt{3}/7, 12\sqrt{3}/7$
⇨ $AD = 12\sqrt{3}/7$ (as $16\sqrt{3}/7 > 4 + 4\sqrt{13}/7$)

Option (b) is correct.

470. ABC is a triangle with BC = a, CA = b and Angle BCA = 120. CD is the bisector of Angle BCA meeting AB at D. Then length of CD is
(a) $(a + b)/4$
(b) $ab/(a + b)$
(c) $(a^2 + b^2)/2(a + b)$
(d) $(a^2 + ab + b^2)/3(a + b)$

Solution :

Same problem as the previous one.

Option (b) is correct.

471. The diagonal of the square PQRS is a + b. The perimeter of a square with twice the area of PQRS is
(a) $2(a + b)$
(b) $4(a + b)$
(c) $\sqrt{8}(a + b)$
(d) $8ab$

Solution :

Area of PQRS = $(a + b)^2/2$

Area of required square = $(a + b)^2$

Side of the required square = $(a + b)$

Perimeter = $4(a + b)$

Option (b) is correct.

472. A string of length 12 inches is bent first into a square PQRS and then into a right-angled triangle PQT by keeping the side PQ of the square fixed. Then the area of PQRS equals

(a) area of PQT
(b) 2(area + PQT)
(c) 3(area of PQT)/2
(d) None of the foregoing numbers.

Solution :

PQ = 12/4 = 3

QT + TP = 12 – 3 = 9

TP = (9 – QT)

Now, $TP^2 = QT^2 + PQ^2$ (right angle at Q)

⇨ $(9 - QT)^2 = QT^2 + 9$
⇨ $81 - 18QT + QT^2 = QT^2 + 9$
⇨ 18QT = 72
⇨ QT = 4

Area of triangle PQT = (1/2)*PQ*QT = (1/2)*3*4 = 6

Area of PQRS = $3^2 = 9$

Option (c) is correct.

473. Instead of walking along two adjacent sides of a rectangular field, a boy took a short-cut along the diagonal and saved a distance equal to half the longer side. Then the ratio of the shorter side to the longer side is

(a) ½
(b) 2/3
(c) ¼
(d) ¾

Solution :

Longer side = b, shorter side = a.

$a + b - \sqrt{(a^2 + b^2)} = b/2$

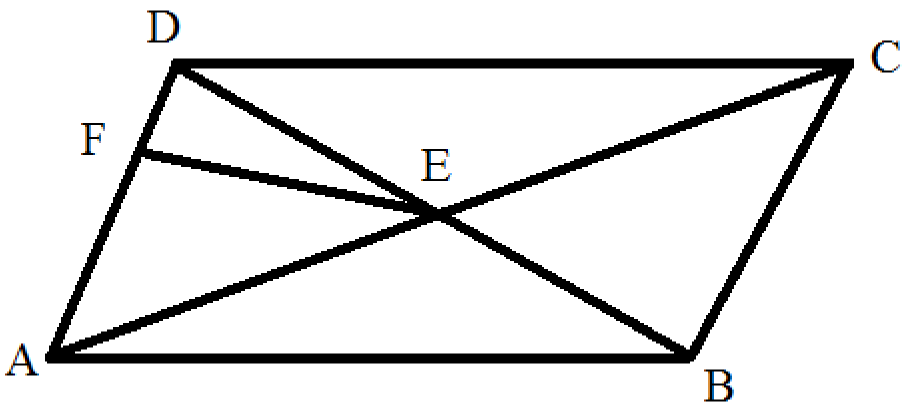
⇨ $\sqrt{(a^2 + b^2)} = a + b/2$

- ⇨ $a^2 + b^2 = a^2 + ab + b^2/4$
- ⇨ $3b^2/4 = ab$
- ⇨ $a/b = 3/4$

Option (d) is correct.

474. Consider a parallelogram ABCD with E as the midpoint of its diagonal BD. The point E is connected to a point F on DA such that DF = (1/3)DA. Then, the ratio of the area of the triangle DEF to the area of the quadrilateral ABEF is
(a) 1 : 2
(b) 1 : 3
(c) 1 : 5
(d) 1 : 4

Solution :



Now, DF = (1/3)DA

- ⇨ Area of DEF = (1/3)(area of AED) = (1/6)(area of ABD) = (1/6)(area of DEF + area of ABEF)
- ⇨ (5/6)(area of DEF) = (1/6)(area of ABEF)
- ⇨ (area of DEF)/(area of ABEF) = 1/5

Option (c) is correct.

475. The external length, breadth and height of a closed box are 10 cm, 9 cm and 7 cm respectively. The total inner surface area of the box is 262 sq cm. If the walls of the box are of uniform thickness d cm, then d equals
(a) 1.5
(b) 2

(c) 2.5
(d) 1

Solution :

Inner length = 10 – 2d, inner breadth = 9 – 2d, inner height = 7 – 2d

Now, $2\{(10 - 2d)(9 - 2d) + (9 - 2d)(7 - 2d) + (10 - 2d)(7 - 2d)\} = 262$

- ⇨ $90 - 38d + 4d^2 + 63 - 32d + 4d^2 + 70 - 34d + 4d^2 = 131$
- ⇨ $12d^2 - 104d + 92 = 0$
- ⇨ $3d^2 - 26d + 23 = 0$
- ⇨ $3d^2 - 3d - 23d + 23 = 0$
- ⇨ $3d(d - 1) - 23(d - 1) = 0$
- ⇨ $(d - 1)(3d - 23) = 0$
- ⇨ $d = 1$ ($d = 23/3 > 7$)

Option (d) is correct.

476. A hollow spherical ball whose inner radius 4 cm is full of water. Half of the water is transferred to a conical cup and it completely fills the cup. If the height of the cup is 2 cm, then the radius of the base of the cone, in cm, is
(a) 4
(b) 8π
(c) 8
(d) 16

Solution :

Volume of the sphere = $(4/3)\pi*4^3 = 256\pi/3$

Volume of the cone = $(1/2)*256\pi/3 = 128\pi/3$

Volume of cone = $(1/3)\pi r^2h = 128\pi/3$

- ⇨ $r^2*2 = 128$
- ⇨ $r = 8$

Option (c) is correct.

477. PQRS is a trapezium with PQ and RS parallel, PQ = 6 cm, QR = 5 cm, RS = 3 cm, PS = 4cm. The area of PQRS
(a) is 27 cm^2
(b) 12 cm^2

(c) $18 cm^2$
(d) cannot be determined from the given information

Solution :

Let, the distance between PQ and RS is h.

Therefore, $\sqrt{(4^2 - h^2)} + \sqrt{(5^2 - h^2)} + 3 = 6$

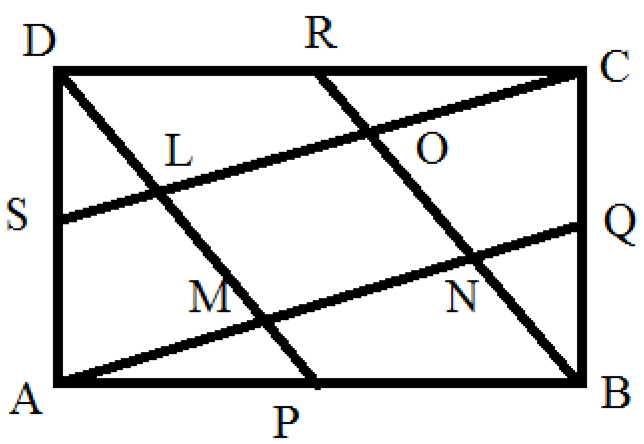
- ⇨ $\sqrt{(16 - h^2)} = 3 - \sqrt{(25 - h^2)}$
- ⇨ $16 - h^2 = 9 - 6\sqrt{(25 - h^2)} + 25 - h^2$
- ⇨ $\sqrt{(25 - h^2)} = 3$
- ⇨ $25 - h^2 = 9$
- ⇨ $h = 4$

Area = $(1/2)(5 + 4)*4 = 18 cm^2$

Option (c) is correct.

478. Suppose P, Q, R and S are the midpoints of the sides AB, BC, CD and DA, respectively, of a rectangle ABCD. If the area of the rectangle is Δ, then the area of the figure bounded by the straight lines AQ, BR, CS and DP is
(a) Δ/4
(b) Δ/5
(c) Δ/8
(d) Δ/2

Solution :



SL = (1/2)AM (S is mid-point of AD and SL||AM) = (1/2)OC = (1/2)OL

Height of triangle DSL = distance between SL and AM

- ⇨ Area of DSL = (1/2)(area of OML) = (1/4)(area of LMNO)
- ⇨ Area of OCR = (1/2)(area of ONM) = (1/4)(area of LMNO)

Now, Area of DSL = (1/4)(area of DAM)

Similarly, area of BOC = (1/4)(area of OCB)

Area of OCR = (1/4)(area of CLD)

Now, area of AMD + area of ANB + area of BOC + area of CLD + area of LMNO = Δ

- ⇨ 4(area of DSL) + 4(area of AMP) + 4(area of BNQ) + 4(area of OCR) + area of LMNO = Δ
- ⇨ 8(area of DSL) + 8(area of OCR) +area of LMNO = Δ
- ⇨ 2(area of LMNO) + 2(area + LMNO) + area of LMNO = Δ
- ⇨ Area of LMNO = Δ/5

Option (b) is correct.

479. The ratio of the area of a triangle ABC to the area of the triangle whose sides are equal to the medians of the triangle is
(a) 2 : 1
(b) 3 : 1
(c)4 : 3
(d) 3 : 2

Solution :

Take an equilateral triangle of side a and calculate the ratio.

Area of ABC = $(\sqrt{3}/4)a^2$

Median = $(\sqrt{3}/2)a$

Area of triangle formed by medians = $(\sqrt{3}/4)(3a^2/4)$

Ratio = $(\sqrt{3}/4)a^2 : (\sqrt{3}/4)(3a^2/4) = 4 : 3$

Option (c) is correct.

480. Let C_1 and C_2 be the inscribed and circumscribed circles of a triangle with sides 3 cm, 4 cm and 5 cm. Then (area of C_1)/(area of C_2) is
(a) 16/25

(b) 4/25
(c) 9/25
(d) 9/16

Solution :

R = abc/4Δ and r = Δ/s

Now, S = (3 + 4 + 5)/2 = 6

Δ = √{6(6 – 3)(6 – 4)(6 – 5)} = 6

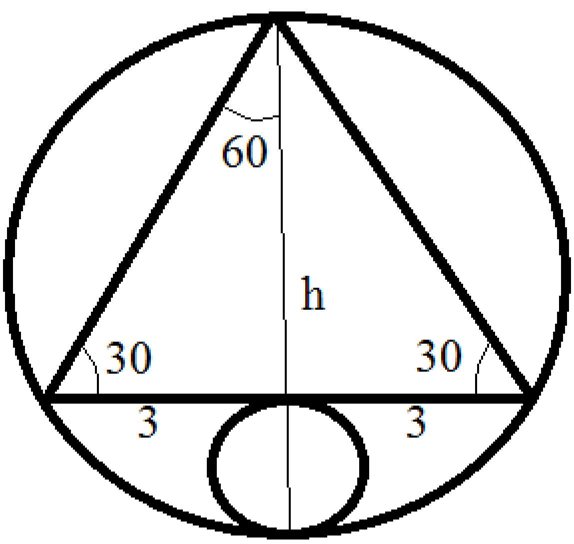
r = 1, R = 3*4*5/(4*6) = 5/2

(area of C_1)/(area of C_2) = (π*1^2)/(π*25/4) = 4/25

Option (b) is correct.

481. An isosceles triangle with base 6 cm and base angles 30 each is inscribed in a circle. A second circle touches the first circle and also touches the base of the triangle at its midpoint. If the second circle is situated outside the triangle, then its radius (in cm) is
(a) 3√3/2
(b) √3/2
(c) √3
(d) 4/√3

Solution :



Now, h/sin30 = 3/sin60

- ⇨ h = 3*(1/2)/(√3/2) = √3
- ⇨ r = radius of big circle = 6/(2sin120) = 3/(√3/2) = 2√3
- ⇨ Radius of small circle = (2r – h)/2 = r – h/2 = 2√3 - √3/2 = 3√3/2

Option (a) is correct.

482. In an isosceles triangle ABC, A = C = π/6 and the radius of its circumcircle is 4. The radius of its incircle is
 (a) 4√3 – 6
 (b) 4√3 + 6
 (c) 2√3 – 2
 (d) 2√3 + 2

Solution :

B = π - 2π/6 = 2π/3

a = 2*4*sin(π/6) = 4

b = 2*4*sin(2π/3) = 4√3

c = 4

S = (a + b + c)/2 = 4 + 2√3

Δ = √(4 + 2√3)(4 + 2√3 – 4)(4 + 2√3 - 4√3)(4 + 2√3 – 4) = (2√3)√(4 + 2√3)(4 - 2√3) = (2√3)√(16 – 12) = 4√3

r = Δ/s = 4√3/(4 + 2√3) = 4√3(4 - 2√3)/4 = 4√3 – 6

Option (a) is correct.

483. PQRS is a quadrilateral in which PQ and SR are parallel (that is, PQRS is a trapezium). Further, PQ = 10, QR = 5, RS = 4, SP = 5. Then area of the quadrilateral is
 (a) 25
 (b) 28
 (c) 20
 (d) 10√10

Solution :

See solution of problem 477. Same problem.

Option (b) is correct.

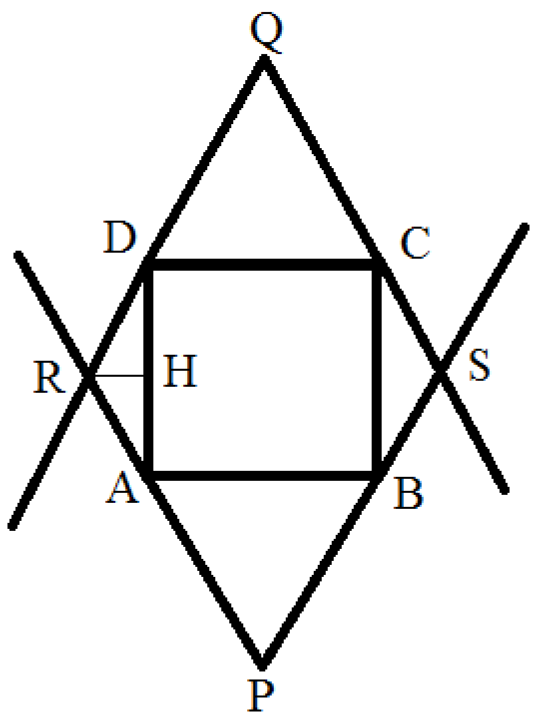
484. The area of quadrilateral ABCD with sides a, b, c, d is given by the formula $\{(s - a)(s - b)(s - c)(s - d) - abcd\cos^2\theta]^{1/2}$, where 2s is the perimeter and 2θ is the sum of opposite angles A and C. Then the area of the quadrilateral circumscribing a circle is given by
(a) $\tan\theta\sqrt{(abcd)}$
(b) $\cos\theta\sqrt{(abcd)}$
(c) $\sin\theta\sqrt{(abcd)}$
(d) none of the foregoing formula

Solution :

Option (c) is correct.

485. Consider a unit square ABCD. Two equilateral triangles PAB and QCD are drawn so that AP, DQ intersect at R, and BP, CQ intersect in S. The area of the quadrilateral PRQS is equal to
(a) $(2 - \sqrt{3})/6$
(b) $(2 - \sqrt{3})/3$
(c) $(2 + \sqrt{3})/6\sqrt{3}$
(d) $(2 - \sqrt{3})/\sqrt{3}$

Solution :



Now, Angle QDC = 60 (equilateral triangle)

Angle CDA = 90

Therefore, Angle RDA = 180 – (90 + 60) = 30

DH = ½

RH/DH = tan30, RH = (1/2)(1/√3) = 1/2√3

Area of RAD = (1/2)(1/2√3)*1= 1/4√3

Area of RAD + Area of CSB = 2(1/4√3) = 1/2√3

Area of equilateral triangles = (√3/4)*1^2*2 = √3/2

Area of the square = 1^2 = 1

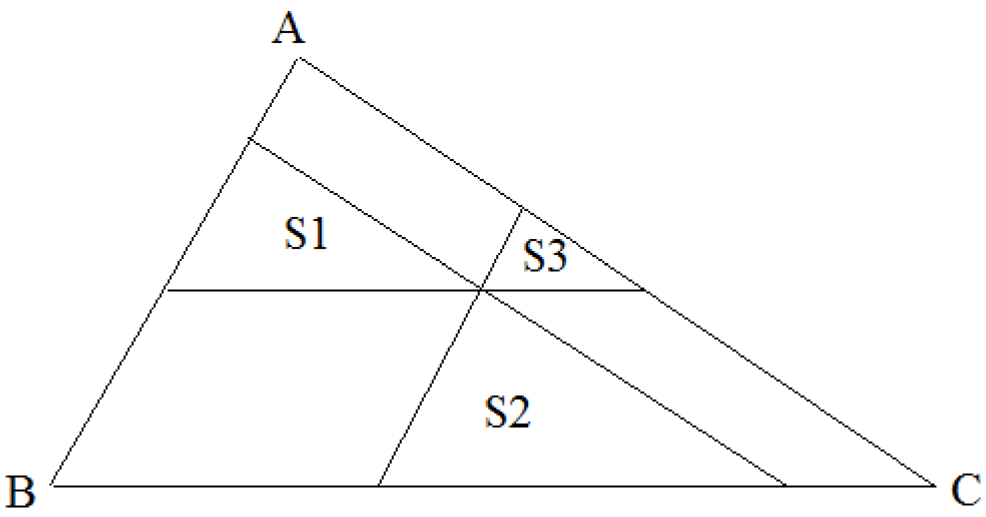
Therefore area of PRQS = 1 + √3/2 + 1/2√3 = 1 + 4/2√3 = (4 + 2√3)/2√3) = (2 + √3)/√3

It is given option (d) is correct.

486. Through an arbitrary point lying inside a triangle, three straight lines parallel to its sides are drawn. These lines divide the triangle into six parts, three of which are triangles. If the area of these triangles are S_1, S_2 and S_3, then the area of the given triangle equals

(a) $3(S_1 + S_2 + S_3)$
(b) $(\sqrt{(S_1S_2)} + \sqrt{(S_2S_3)} + \sqrt{(S_3S_1)})^2$
(c) $(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2$
(d) None of the foregoing quantities.

Solution :



Option (c) is correct.

487. The sides of a triangle are given by $\sqrt{(b^2 + c^2)}$, $\sqrt{(c^2 + a^2)}$ and $\sqrt{(a^2 + b^2)}$, where a, b, c are positive. Then the area of the triangle equals

(a) $(1/2)\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$
(b) $(1/2)\sqrt{(a^4 + b^4 + c^4)}$
(c) $(\sqrt{3}/2)\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$
(d) $(\sqrt{3}/2)(bc + ca + ab)$

Solution :

$\cos A = (c^2 + a^2 + a^2 + b^2 - b^2 - c^2)/2\sqrt{\{(c^2 + a^2)(a^2 + b^2)\}} = a^2/\sqrt{\{(c^2 + a^2)(a^2 + b^2)}$

$\sin A = \sqrt{\{(c^2 + a^2)(a^2 + b^2) - a^4\}}/\sqrt{\{(c^2 + a^2)(a^2 + b^2)\}} = \sqrt{(b^2c^2 + a^2b^2 + c^2a^2)}/\sqrt{\{(c^2 + a^2)(a^2 + b^2)\}}$

Area $= (1/2)\sqrt{(c^2 + a^2)}\sqrt{(a^2 + b^2)}\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}/\sqrt{\{(c^2 + a^2)(a^2 + b^2)\}}$

$= (1/2)\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$

Option (a) is correct.

488. Two sides of a triangle are 4 and 5. Then, for the area of the triangle which one of the following bounds is the sharpest?
(a) < 10
(b) ≤ 10
(c) ≤ 8
(d) > 5

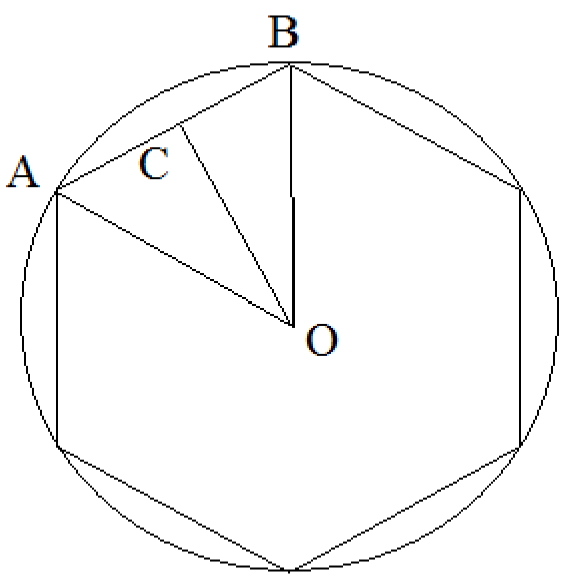
Solution :

Area = (1/2)*4*5*sinA = 10sinA ≤ 10

Option (b) is correct.

489. The area of a regular hexagon (that is, six-sided polygon) inscribed in a circle of radius 1 is
(a) $3\sqrt{3}/2$
(b) 3
(c) 4
(d) $2\sqrt{3}$

Solution :



Angle OAB = {(6 – 2)π/6}/2 = π/3

OC/AO = sin(π/3)

- ⇨ OC = 1*(√3/2) = √3/2
- ⇨ AC = OCcos(π/3) = ½
- ⇨ AB = 1

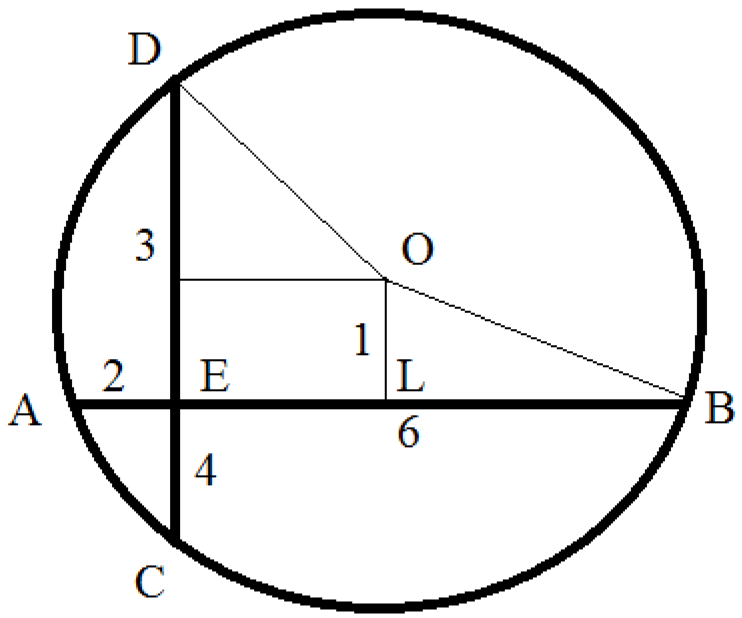
Area of AOB = (1/2)*(√3/2)*1 = √3/4

Area of hexagon = 6*(√3/4) = 3√3/2

Option (a) is correct.

490. Chords AB and CD of a circle intersect at a point E at right angles to each other. If the segments AE, EB and ED are of lengths 2, 6 and 3 units respectively, then the diameter of the circle is
(a) √65
(b) 12
(c) √52
(d) √63

Solution :



Now, if we join B, C and A, D then triangles ADE and BCE are similar. (Angle ECB = Angle EAD; on same arc BD and Angle DEA = Angle CEB (right angles))

So, CE/AE = EB/ED

- ⇨ CE = 4

Now, LB = (6 + 2)/2 = 4

OL = DC/2 – 3 = (3 + 4)/2 – 3 = ½ (figure is not drawn to the scale)

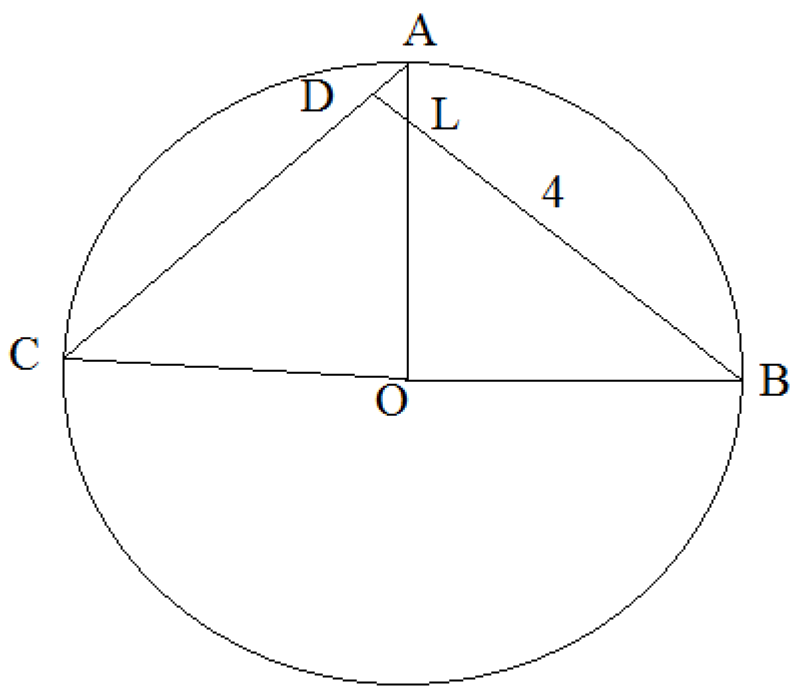
In triangle OLB, $OB^2 = r^2 = (½)^2 + 4^2 = 65/4$

- ⇨ r = √65/2
- ⇨ 2r = √65

Option (a) is correct.

491. In a circle with centre O, OA and OB are two radii perpendicular to each other. Let AC be a chord and D the foot of the perpendicular drawn from B to AC. If the length of BD is 4 cm then the length of CD (in cm) is
 (a) 4
 (b) 2√2
 (c)2√3
 (d) 3√2

Solution :



In triangle ACO, Angle OAC = Angle OCA (OA = OC both radius)

In triangles ADL and OLB, Angle DLA = Angle OLB (opposite angle)

Angle LDA = Angle LOB (both right angles)

- ⇨ Angle DAL = Angle LBO
- ⇨ Angle OCA = Angle DBO

Now, in triangles ODC and OBD,

OD is common, OC = OB (both radius) and Angle OCD = Angle OBD

- ⇨ ODC and OBD are equal triangles.
- ⇨ CD = DB = 4

Option (a) is correct.

492. ABC is a triangle and P is a point inside it such that Angle BPC = Angle CPA = Angle APB. Then P is
(a) the point of intersection of medians
(b) the incentre
(c) the circumcentre
(d) none of the foregoing points

Solution :

Clearly, none of the foregoing points satisfy this.

Therefore, option (d) is correct.

493. Suppose the circumcentre of a triangle ABC lies on BC. Then the orthocentre of the triangle is
(a) the point A
(b) the incentre of the triangle
(c) the mid-point of the line segment joining the mid-points of AB and AC
(d) the centroid of the triangle

Solution :

It means ABC is right-angled triangle with right angle at A.

And circumcentre is the mid-point of BC.

Obviously, orthicentre of the triangle is point A.

Option (a) is correct.

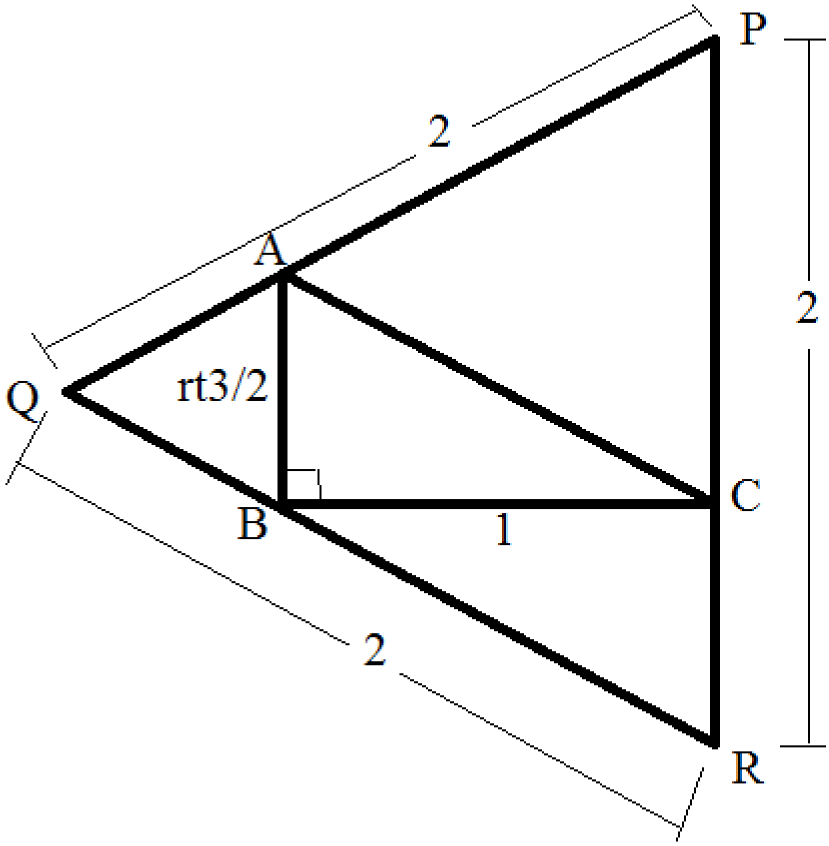
494. ABC is a triangle inscribed in a circle. AD, AE are straight lines drawn from the vertex A to the base BC parallel to the tangents at B and C respectively. If AB = 5 cm, AC = 6 cm, and CE = 9 cm, then the length of BD (in cm) equals

(a) 7.5
(b) 10.8
(c) 7.0
(d) 6.25

Solution :

Option (d) is correct.

495. ABC is a triangle with AB = √3/2, BC = 1 and B = 90. PQR is an equilateral triangle with sides PQ, QR, RP passing through the points A, B, C respectively and each having length 2. Then the length of the segment BR is



(a) (2/√3)sin75
(b) 4/(2 + √3)
(c) either 1 or 15/13
(d) 2 − sin75

Solution :

Option (c) is correct.

496. The equation $x^2y - 2xy + 2y = 0$ represents
(a) a straight line
(b) a circle
(c) a hyperbola
(d) none of the foregoing curves

Solution

Now, $x^2y - 2xy + 2y = 0$

⇨ $y(x^2 - 2x + 2) = 0$
⇨ $y\{(x - 1)^2 + 1\} = 0$

Now, $(x - 1)^2 + 1 > 0$ (always)

⇨ $y = 0$

Option (a) is correct.

497. The equation $r = 2a\cos\theta + 2b\sin\theta$, in polar coordinates, represents
(a) a circle passing through the origin
(b) a circle with the origin lying outside it
(c) a circle with radius $2\sqrt{(a^2 + b^2)}$
(d) a circle with centre at the origin.

Solution :

$r = 2a\cos\theta + 2b\sin\theta = 2ax/r + 2by/r$

⇨ $r^2 = 2ax + 2by$
⇨ $x^2 + y^2 - 2ax - 2by = 0$

Option (a) is correct.

498. The curve whose equation in polar coordinates is $r\sin^2\theta - \sin\theta - r = 0$, is
(a) an ellipse
(b) a parabola

(c) a hyperbola
(d) none of the foregoing curves

Solution :

$r\sin^2\theta - \sin\theta - r = 0$

- ⇨ $r(y^2/r^2) - y/r - r = 0$
- ⇨ $y^2 - y - r^2 = 0$
- ⇨ $y^2 - y - x^2 - y^2 = 0$
- ⇨ $x^2 = -y$
- ⇨ a parabola

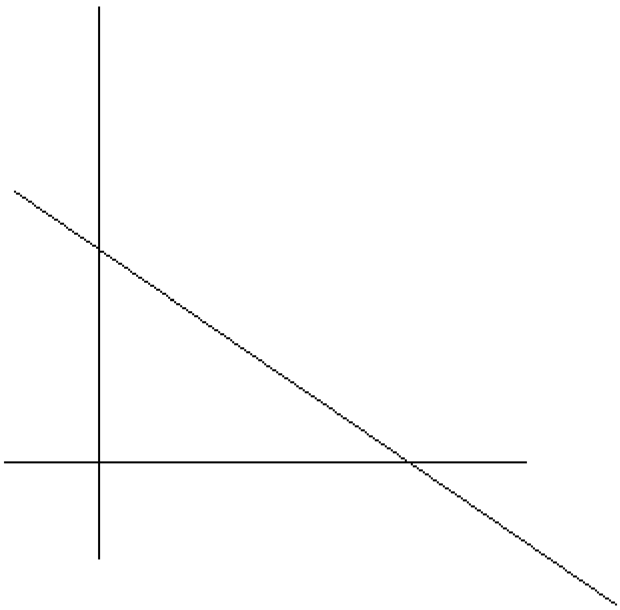
Option (b) is correct.

499. A point P on the line $3x + 5y = 15$ is equidistant from the coordinate axes. P can lie in
(a) quadrant I only
(b) quadrant I or quadrant II
(c) quadrant I or quadrant III
(d) any quadrant

Solution :

$3x + 5y = 15$

- ⇨ $x/5 + y/3 = 5$



Now, P cannot lie on quadrant III as the straight line is not there.

So, option (c) and (d) cannot be true.

Now, P can lie on quadrant II as well as the straight line is bent along x-axis in quadrant II. So, somewhere we will find a point on the line which is equidistant from both the axis.

Let co-ordinate of P is (h, -h) i.e. considering in quadrant IV.

Then $3h - 5h = 15$

- ⇨ $h = -15/2$ but h is positive. So it cannot stay on quadrant IV.

Let coordinate of P is (-h, h) i.e. considering it in quadrant II

Then $-3h + 5h = 15$

- ⇨ $h = 15/2 > 0$ (so possible)

Option (b) is correct.

500. The set of all points (x, y) in the plane satisfying the equation $5x^2y - xy + y = 0$ forms
    (a) a straight line
    (b) a parabola
    (c) a circle
    (d) none of the foregoing curves

Solution :

$5x^2y - xy + y = 0$

- ⇨ $y(5x^2 - x + 1) = 0$
- ⇨ $5y(x^2 - x/5 + 1/5) = 0$
- ⇨ $5y(x^2 - 2*(1/10)*x + 1/100 + 1/5 - 1/100) = 0$
- ⇨ $5y\{(x - 1/10)^2 + 19/100\} = 0$

Now, $(x - 1/10)^2 + 19/100 > 0$ (always)

Therefore, $y = 0$

Option (a) is correct.

501. The equation of the line passing through the intersection of the lines $2x + 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and perpendicular to the line $7x - 5y + 8 = 0$ is
    (a) $5x + 7y - 1 = 0$

(b) $7x + 5y + 1 = 0$
(c) $5x - 7y + 1 = 0$
(d) $7x - 5y - 1 = 0$

Solution :

$2x + 3y + 4 = 0$ …. (1)

$3x + 4y - 5 = 0$ …. (2)

Doing (1)*3 – (2)*2 we get, $6x + 9y + 12 - 6x - 8y + 10 = 0$

- ⇨ $y = -22$
- ⇨ $x = (-4 + 3*22)/2 = 31$

Slope of $7x - 5y + 8 = 0$ is $7/5$

Slope of required straight line is $(-5/7)$

Equation of the required straight line is, $y + 22 = (-5/7)(x - 31)$

- ⇨ $7y + 154 = -5x + 155$
- ⇨ $5x + 7y - 1 = 0$

Option (a) is correct.

502. Two equal sides of an isosceles triangle are given by the equations $y = 7x$ and $y = -x$ and its third side passes through (1, -10). Then the equation of the third side is
(a) $3x + y + 7 = 0$ or $x - 3y - 31 = 0$
(b) $x + 3y + 29 = 0$ or $-3x + y + 13 = 0$
(c) $3x + y + 7 = 0$ or $x + 3y + 29 = 0$
(d) $X = 3y - 31 = 0$ or $-3x + y + 13 = 0$

Solution :

$m_1 = 7$ and $m_2 = -1$

Now, $(7 - m)/(1 + 7m) = (m + 1)/(1 - m)$

- ⇨ $(7 - m)(1 - m) = (1 + 7m)(m + 1)$
- ⇨ $7 - 8m + m^2 = 1 + 8m + 7m^2$
- ⇨ $6m^2 + 16m - 6 = 0$
- ⇨ $3m^2 + 8m - 3 = 0$
- ⇨ $3m^2 + 9m - m - 3 = 0$
- ⇨ $3m(m + 3) - (m + 3) = 0$
- ⇨ $(m + 3)(3m - 1) = 0$

⇨ $m = -3, 1/3$

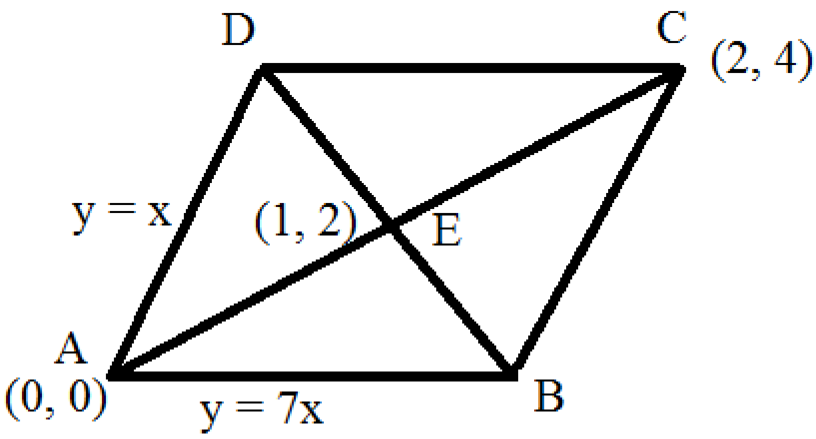
Equation is, $y + 10 = -3(x - 1)$ or $y + 10 = (1/3)(x - 1)$

⇨ $y + 10 = -3x + 3$ or $3y + 30 = x - 1$
⇨ $3x + y + 7 = 0$ or $x - 3y - 31 = 0$

Option (a) is correct.

503. The equation of two adjacent sides of a rhombus are given by $y = x$ and $y = 7x$. The diagonals of the rhombus intersect each other at the point (1, 2). The area of the rhombus is
(a) 10/3
(b) 20/3
(c) 50/3
(d) None of the foregoing quantities

Solution :



Now, Equation of BC is, $y - 4 = 1(x - 2)$

⇨ $x - y + 2 = 0$

Solving $x - y + 2 = 0$ and $y = 7x$, we get, $x = 1/3$, $y = 7/3$

Therefore, B = (1/3, 7/3)

$BE = \sqrt{\{(1 - 1/3)^2 + (2 - 7/3)^2\}} = \sqrt{5}/3$

$AC = \sqrt{\{(2 - 0)^2 + (4 - 0)^2\}} = 2\sqrt{5}$

Area of triangle ABC = $(1/2)(\sqrt{5}/3)(2\sqrt{5}) = 5/3$

Area of rhombus = 2*(5/3) = 10/3

Option (a) is correct.

504. It is given that three distinct points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear. Then a necessary and sufficient condition for (x_2, y_2) to lie on the line segment joining (x_3, y_3) to (x_1, y_1) is

(a) either $x_1 + y_1 < x_2 + y_2 < x_3 + y_3$ or $x_3 + y_3 < x_2 + y_2 < x_1 + y_1$

(b) either $x_1 - y_1 < x_2 - y_2 < x_3 - y_3$ or $x_3 - y_3 < x_2 - y_2 < x_1 - y_1$

(c) either $0 < (x_2 - x_3)/(x_1 - x_3) < 1$ or $0 < (y_2 - y_3)/(y_1 - y_2) < 1$

(d) none of the foregoing statements

Solution :

The ratio $(x_2 - x_3)/(x_1 - x_3)$ says that the distance between x-coordinate between x_2 and x_3 and the distance between the x-coordinate between x_1 and x_3 are of same sign and the modulus of the previous is smaller than the latter i.e. x_2 lie between x_1 and x_3.

Let (x_2, y_2) divides (x_1, y_1) and (x_3, y_3) in the ratio m : n.

Therefore, $x_2 = (mx_1 + nx_3)/(m + n)$

- ⇨ $mx_2 + nx_2 = mx_1 + nx_3$
- ⇨ $m(x_2 - x_1) = n(x_3 - x_2)$
- ⇨ $(x_2 - x_1)/(x_3 - x_2) = n/m$
- ⇨ $(x_2 - x_1 + x_3 - x_2)/(x_3 - x_2) = (n + m)/m$
- ⇨ $(x_3 - x_2)/(x_3 - x_1) = m/(n + m)$
- ⇨ $(x_2 - x_3)/(x_1 - x_3) = m/(n + m)$

Now, $0 < m/(n + m) < 1$

Follows, $0 < (x_2 - x_3)/(x_1 - x_3) < 1$

Option (c) is correct.

505. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$ be four points such that x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 are both in A.P. If Δ denotes the area of the quadrilateral ABCD, then

(a) $\Delta = 0$

(b) $\Delta > 1$

(c) $\Delta < 1$

(d) Δ depends on the coordinates of A, B, C and D.

Solution :

Let the common difference of the A.P. x_1, x_2, x_3, y_3 is d and the common difference of the A.P. y_1, y_2, y_3, y_4 is d_1.

Now, AB = $\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}} = \sqrt{(d^2 + d_1^2)}$ = a (say) > 0

BC = CD = a

DA = $\sqrt{\{(x_4 - x_1)^2 + (y_4 - y_1)^2\}}$ = 3a

AC = $\sqrt{\{(x_3 - x_1)^2 + (y_3 - y_1)^2\}}$ = 2a

Now, AB + BC = AC

⇨ A, B, C are collinear.

Now, AC + CD = DA

⇨ A, C, D are collinear.

Therefore, $\Delta = 0$

Option (a) is correct.

506. The number of points (x, y) satisfying (i) $3y - 4x = 20$ and (ii) $x^2 + y^2 \leq 16$ is
(a) 0
(b) 1
(c)2
(d) Infinite

Solution :

Let us see the intersection point of $3y - 4x = 20$ with the circle $x^2 + y^2 = 16$

Now, y = (4x + 20/3

$\{(4x + 20)/3\}^2 + x^2 = 16$

⇨ $16x^2 + 160x + 400 + 9x^2 = 144$
⇨ $25x^2 + 160x + 256 = 0$
⇨ $(5x)^2 + 2*5x*16 + (16)^2 = 0$
⇨ $(5x + 16)^2 = 0$
⇨ $x = -16/5$

One solution. Therefore it touches the circle.

So, no intersection in the inside of the circle.

Therefore, 1 solution.

Option (b) is correct.

507. The equation of the line parallel to the line $3x + 4y = 0$ and touching the circle $x^2 + y^2 = 9$ in the first quadrant is
(a) $3x + 4y = 9$
(b) $3x + 4y = 45$
(c) $3x + 4y = 15$
(d) None of the foregoing equations

Solution :

Equation of the required line is, $3x + 4y = c$

Distance of the line from (0, 0) is $|-c/\sqrt{(3^2 + 4^2)}|$ = radius = 3

⇨ $c = 15$

Option (c) is correct.

508. The distance between the radii of the largest and smallest circles, which have their centres on the circumference of the circle $x^2 + 2x + y^2 + 4y = 4$ and pass through the point (a, b) lying outside the given circle, is
(a) 6
(b) $\sqrt{\{(a + 1)^2 + (b + 2)^2\}}$
(c) 3
(d) $\sqrt{\{(a + 1)^2 + (b + 2)^2\}} - 3$.

Solution :

Option (a) is correct.

509. The perimeter of the region bounded by $x^2 + y^2 \leq 100$ and $x^2 + y^2 - 10x - 10(2 - \sqrt{3})y \leq 0$ is
(a) $(5\pi/3)(5 + \sqrt{6} - \sqrt{2})$
(b) $(5\pi/3)(1 + \sqrt{6} - \sqrt{2})$
(c) $(5\pi/3)(1 + 2\sqrt{6} - 2\sqrt{2})$
(d) $(5\pi/3)(5 + 2\sqrt{6} - 2\sqrt{2})$

Solution :

Subtracting the equations we get, $10x + 10(2 - \sqrt{3})y = 100$

⇨ $x = 10 - (2 - \sqrt{3})y$

Putting in first equation we get, $\{10 - (2 - \sqrt{3})y\}^2 + y^2 = 100$

⇨ $100 - 20(2 - \sqrt{3})y + y^2 = 0$
⇨ $y = 0, 20(2 - \sqrt{3})$
⇨ $x = 10, 10 - 20(2 - \sqrt{3})^2 = 80\sqrt{3} - 130$

So, the points are $(10, 0)$ and $(80\sqrt{3} - 130, 20(2 - \sqrt{3}))$

So, $m_1 = (0 - 0)/(10 - 0) = 0$ and $m_2 = \{20(2 - \sqrt{3}) - 0\}/(80 - 130\sqrt{3} - 0) = 2(2 - \sqrt{3})/(8 - 13\sqrt{3})$

So, $\theta = \tan^{-1}\{2(2 - \sqrt{3})/(8 - 13\sqrt{3})\}$

Now, $s = r\theta = 10\tan^{-1}\{2(2 - \sqrt{3})/(8 - 13\sqrt{3})\}$

Centre of second circle = $(5, 5(2 - \sqrt{3}))$

Radius = $\sqrt{[5^2 + \{5(2 - \sqrt{3})\}^2]} = 5\sqrt{(1 + 4 + 3 - 4\sqrt{3})} = 5\sqrt{(8 - 4\sqrt{3})} = 5(\sqrt{6} - \sqrt{2})$

Now, $m_3 = (0 - 5)/(10 - 5(2 - \sqrt{3}))$, $m_4 = \{20(2 - \sqrt{3}) - 5\}/(80\sqrt{3} - 130 - 5(2 - \sqrt{3}))$

$\tan\theta = (m_3 - m_4)/(1 + m_3m_4)$

From this, $s_1 = r\theta = 5(\sqrt{6} - \sqrt{2})\tan^{-1}\{(m_3 - m_4)/(1 + m_3m_4)\}$

Perimeter = $s + s_1$

After simplification, option (c) will be the answer.

510. The equation of the circle which has both coordinate axes as its tangents and which touches the circle $x^2 + y^2 = 6x + 6y - 9 - 4\sqrt{2}$ is
(a) $x^2 + y^2 = 2x + 2y + 1$
(b) $x^2 + y^2 = 2x - 2y + 1$
(c) $x^2 + y^2 = 2x - 2y - 1$
(d) $x^2 + y^2 = 2x + 2y - 1$

Solution :

Centre is (r, r) where r is radius.

Centre of second circle = $(3, 3)$ and radius = $2\sqrt{2} - 1$

Distance between centres = sum of radius

- ⇨ $\sqrt{(r - 3)^2 + (r - 3)^2} = r + 2\sqrt{2} - 1$
- ⇨ $|r - 3|\sqrt{2} = r + 2\sqrt{2} - 1$
- ⇨ $(r - 3)\sqrt{2} = r + 2\sqrt{2} - 1$
- ⇨ $r(\sqrt{2} - 1) = 5\sqrt{2} - 1$
- ⇨ $r = (5\sqrt{2} - 1)/(\sqrt{2} - 1)$

Also, $-\sqrt{2}(r - 3) = r + 2\sqrt{2} - 1$

- ⇨ $r(\sqrt{2} + 1) = (\sqrt{2} + 1)$
- ⇨ $r = 1$

Equation is, $(x - 1)^2 + (y - 1)^2 = 1^2$

- ⇨ $x^2 + y^2 = 2x + 2y - 1$

Option (d) is correct.

511. A circle and a square have the same perimeter. Then
   (a) their areas are equal
   (b) the area of the circle is larger
   (c) the area of the square is larger
   (d) the area of the circle is π times the area of the square

Solution :

Now, $2\pi r = 4a$

- ⇨ $\pi r = 2a$
- ⇨ $(\pi r)^2 = 4a^2$
- ⇨ $(\pi r^2)/a^2 = 4/\pi > 1$
- ⇨ area of circle > area of square

Option (b) is correct.

512. The equation $x^2 + y^2 - 2xy - 1 = 0$ represents
   (a) two parallel straight lines
   (b) two perpendicular straight lines
   (c) a circle
   (d) a hyperbola

Solution :

$x^2 + y^2 - 2xy - 1 = 0$

⇨ $(x - y)^2 = 1$
⇨ $x - y = \pm 1$

Pair of parallel straight lines.

Option (a) is correct.

513. The equation $x^3 - yx^2 + x - y = 0$ represents
(a) a straight line
(b) a parabola and two straight lines
(c) a hyperbola and two straight lines
(d) a straight line and a circle

Solution :

Now, $x^3 - yx^2 + x - y = 0$

⇨ $x^2(x - y) + x - y = 0$
⇨ $(x - y)(x^2 + 1) = 0$
⇨ $x - y = 0$ as $x^2 + 1 > 0$ (always)

Option (a) is correct.

514. The equation $x^3y + xy^3 + xy = 0$ represents
(a) a circle
(b) a circle and a pair of straight lines
(c) a rectangular hyperbola
(d) a pair of straight lines

Solution :

Now, $x^3y + xy^3 + xy = 0$

⇨ $xy(x^2 + y^2 + 1) = 0$
⇨ $xy = 0$ as $x^2 + y^2 + 1 > 0$ (always)
⇨ $x = 0, y = 0$

Option (d) is correct.

515. A circle of radius r touches the parabola $x^2 + 4ay = 0$ $(a > 0)$ at the vertex of the parabola. The centre of the circle lies below the vertex and the circle lies entirely within parabola. Then the largest possible value of r is

(a) a
(b) 2a
(c) 4a
(d) None of the foregoing expressions

Solution :

Any point on the parabola $(2at, -at^2)$

Centre of circle is $(0, -r)$

Distance = $\sqrt{(2at - 0)^2 + (-at^2 + r)^2} \geq r$

- ⇨ $4a^2t^2 + a^2t^4 - 2at^2r + r^2 \geq r^2$
- ⇨ $4a^2t^2 + a^2t^4 - 2at^2r \geq 0$
- ⇨ $4a + at^2 - 2r \geq 0$
- ⇨ $r \leq 2a + at^2/2$

Maximum value will occur when $t = 0$ i.e. $r = 2a$

Option (b) is correct.

516. The equation $16x^4 - y^4 = 0$ represents
(a) a pair of straight lines
(b) one straight line
(c) a point
(d) a hyperbola

Solution :

$16x^4 - y^4 = 0$

- ⇨ $(4x^2 - y^2)(4x^2 + y^2) = 0$
- ⇨ $(2x - y)(2x + y(4x^2 + y^2) = 0$

A pair of straight lines as $4x^2 + y^2 > 0$

Option (a) is correct.

517. The equation of the straight line which passes through the point of intersection of the lines $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ and is perpendicular to the straight line $y - x = 8$ is
(a) $6x + 6y - 8 = 0$
(b) $x + y + 2 = 0$
(c) $4x + 8y + 12 = 0$

(d) $3x + 3y - 6 = 0$

Solution :

Solving $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ we get, (-1, -1)

Equation of the required straight line is $x + y + c = 0$

$-1 -1 +c = 0$

=> $c = 2$

$x + y + 2 = -$

Option (b) is correct.

518. Two circles with equal radii are intersecting at the points (0, 1) and (0, -1). The tangent at (0, 1) to one of the circle passes through the centre of the other circle. Then the centres of the two circles are at

(a) (2, 0) and (-2, 0)

(b) (0.75, 0) and (-0.75, 0)

(c)(1, 0) and (-1, 0)

(d) None of the foregoing pairs of points.

Solution :

Let the centre of the two circles are at (a, 0) and (-a, 0)

$(C_1C_2)^2 = r^2 + r^2$

$4a^2 = a^2 + 1 + a^2 + 1$

- ⇨ $2a^2 = 2$
- ⇨ $a = \pm 1$

Option (c) is correct.

519. The number of distinct solutions (x, y) of the system of equations $x^2 = y^2$ and $(x - a)^2 + y^2 = 1$, where a is any real number, can only be

(a) 0, 1, 2, 3, 4 or 5

(b) 0, 1 or 3

(c)0, 1, 2 or 4

(d) 0, 2, 3 or 4

Solution :

Problem incomplete.

520. The number of distinct points common to the curves $x^2 + 4y^2 = 1$ and $4x^2 + y^2 = 4$ is
(a) 0
(b) 1
(c) 2
(d) 4

Solution :

Now, $4(x^2 + 4y^2) - (4x^2 + y^2) = 4*1 - 4$

- ⇨ $y = 0$
- ⇨ $x = \pm 1$

Two points (1, 0), (-1, 0)

Option (c) is correct.

521. The centres of the three circles $x^2 + y^2 - 10x + 9 = 0$, $x^2 + y^2 - 6x + 2y + 1 = 0$ and $x^2 + y^2 - 9x - 4y + 2 = 0$
(a) lie on the straight line $x - 2y = 5$
(b) lie on the straight line $y - 2x = 5$
(c) lie on the straight line $2y - x - 5 = 0$
(d) do not lie on a straight line

Solution :

Centres are (5, 0); (3, -1); (9/2, 2)

Area = $(1/2)[5(-1 - 2) + 3(2 - 0) + (9/2)(0 + 1)] = (1/2)[-15 + 6 + 9/2] \neq 0$

Option (d) is correct.

522. In a parallelogram ABCD, A is the point (1, 3), B is the point (5, 6), C is the point (4, 2). Then D is the point
(a) (0, -1)

(b) (-1, 0)
(c)(-1, 1)
(d) (1, -1)

Solution :

Clearly, (A + C)/2 = (B + D)/2

⇨ {(1, 3) + (4, 2)}/2 = {(5, 6) + D}/2
⇨ D = (0, -1)

Option (a) is correct.

523. A square, whose side is 2 metres, has its corners cut away so as to form a regular octagon. Then area of the octagon, in square metres, equals
(a) 2
(b) 8/(√2 + 1)
(c)4(3 - 2√2)
(d) None of the foregoing numbers.

Solution :

Let the length of the sides which is cut out is x.

The length of the side of the octagon = (2 – 2x)

The hypotenuse of the cut triangle = x√2

Now, 2 – 2x = x√2

⇨ x = 2/(2 + √2)

Therefore, x√2 = 2/(√2 + 1)

Area = 2[(1/2){2/(√2 + 1)}*2*{2/(2 + √2)} + 2*2/(√2 + 1) = 8/(√2 + 1)

Option (b) is correct.

524. The equation of the line passing through the intersection of the lines 3x + 4y = -5, 4x + 6y = 6 and perpendicular to 7x – 5y + 3 = 0 is
(a) 5x + 7y – 2 = 0
(b) 5x – 7y + 2 = 0

(c) $7x - 5y + 2 = 0$
(d) $5x + 7y + 2 = 0$

Solution :

$4(3x + 4y) - 3(4x + 6y) = 4(-5) - 3*6$

⇨ $-2y = -38$
⇨ $y = 19$
⇨ $x = -27$

Equation of the line perpendicular to $7x - 5y + 4 = 0$ is $5x + 7y + c = 0$

Therefore, $5(-27) + 7*19 + c = 0$

⇨ $c = -133 + 135 = 2$

Equation is, $5x + 7y + 2 = 0$

Option (d) is correct.

525. The area of the triangle formed by the straight lines whose equations are $y = 4x + 2$, $2y = x + 3$, $x = 0$, is
(a) $25/7\sqrt{2}$
(b) $\sqrt{2}/28$
(c) $1/28$
(d) $15/7$

Solution :

Solving $y = 4x + 2$ and $2y = x + 3$, $8x + 4 = x + 3$ i.e. $x = -1/7$, $y = 10/7$

Solving $y = 4x + 2$ and $x = 0$, $x = 0$, $y = 2$

Solving $2y = x + 3$ and $x = 0$, $x = 0$, $y = 3/2$

So, vertices are $(-1/7, 10/7)$, $(0, 2)$, $(0, 3/2)$

Area $= |(1/2)[(-1/7)(2 - 3/2) + 0(3/2 - 10/7) + 0(10/7 - 2)]| = 1/28$

Option (c) is correct.

526. A circle is inscribed in an equilateral triangle and a square in inscribed in the circle. The ratio of the area of the triangle to the area of the square is

(a) $\sqrt{3} : \sqrt{2}$
(b) $3\sqrt{3} : \sqrt{2}$
(c) $3 : \sqrt{2}$
(d) $\sqrt{3} : 1$

Solution :

Let the side of the triangle = a.

Area of the triangle = $(\sqrt{3}/4)a^2$

Radius of the circle = $(1/3)(\sqrt{3}/2)a = (1/2\sqrt{3})a$

Diagonal of the square = $2*(1/2\sqrt{3})*a = a/\sqrt{3}$

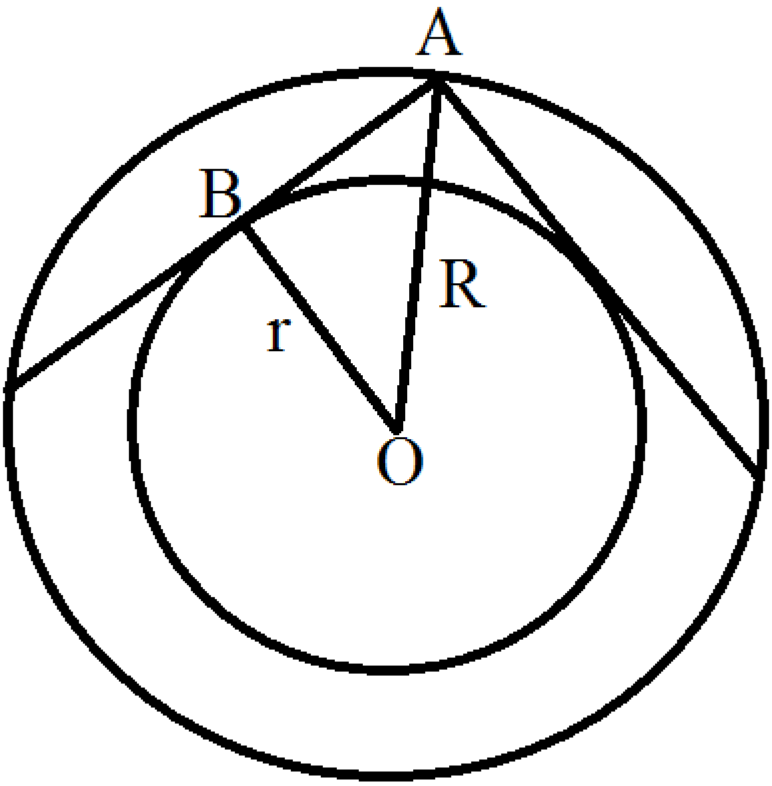
Area of the square = $(1/2)*(a/\sqrt{3})^2 = a^2/6$

Ratio = $(\sqrt{3}/4)a^2 : a^2/6 = 3\sqrt{3} : 2$

Option (b) is correct.

527. If the area of the circumcircle of a regular polygon with n sides is A then the area of the circle inscribed in the polygon is
(a) $A\cos^2(2\pi/n)$
(b) $(A/2)(\cos(2\pi/n) + 1)$
(c) $(A/2)\cos^2(\pi/n)$
(d) $A(\cos(2\pi/n) + 1)$

Solution :



Now, $\pi R^2 = A$

Now, Angle OAB = $\{(n - 2)\pi/n\}/2 = (\pi/2 - \pi/n)$

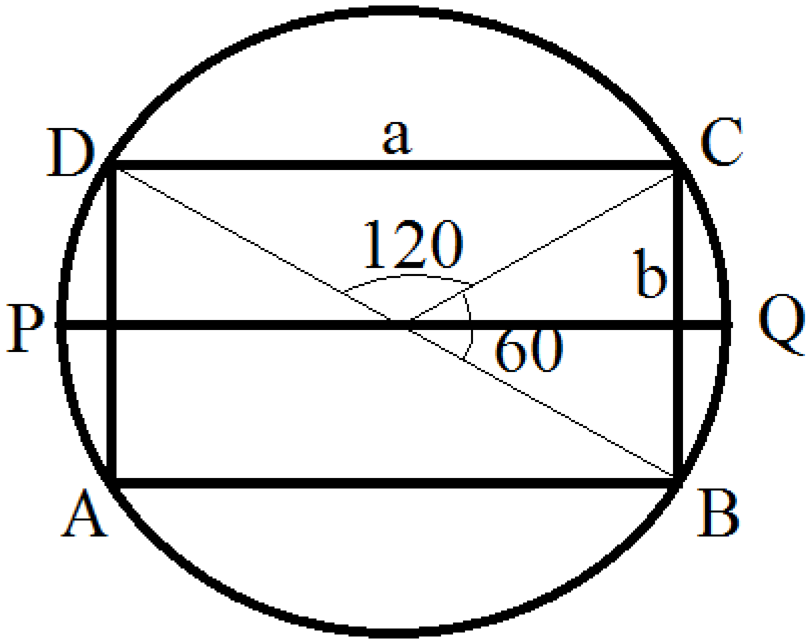
In triangle, OAB, OB/OA = $\sin(\pi/2 - \pi/n)$

- ⇨ $r/R = \cos(\pi/n)$
- ⇨ $r = R\cos(\pi/n)$
- ⇨ $\pi r^2 = (\pi R^2)\cos^2(\pi/n) = A\cos^2(\pi/n) = (A/2)(\cos(2\pi/n) + 1)$

Option (b) is correct.

528. A rectangle ABCD is inscribed in a circle. Let PQ be the diameter of the circle parallel to the side of AB. If Angle BPC = 30, then the ratio of the area of the rectangle to that of the circle is
(a) $\sqrt{3}/\pi$
(b) $\sqrt{3}/2\pi$
(c) $3/\pi$
(d) $\sqrt{3}/9\pi$

Solution :



Angle BOC = 2*(Angle BPC) = 60 (central angle = 2*peripheral angle)

Let the radius of the circle is r.

From triangle BOC we get, r = b

And from triangle DOC we get, a/sin120 = r/sin30

- ⇨ $a = r(\sqrt{3}/2)/(1/2) = r\sqrt{3}$
- ⇨ Area of rectangle = $r*r\sqrt{3} = r^2\sqrt{3}$

Area of circle = πr^2

Ratio = $r^2\sqrt{3}/(\pi r^2) = \sqrt{3}/\pi$

Option (a) is correct.

529. Consider a circle passing through the points (0, 1 – a), (a, 1) and (0, 1 + a). If a parallelogram with two adjacent sides having lengths a and b and an angle 150 between them has the same area as the circle, then b equals
(a) πa
(b) $2\pi a$
(c) $(1/2)\pi a$
(d) None of these numbers

Solution :

Area of parallelogram = (1/2)(a + a)*bsin30 = ab/2

Let, the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

$0^2 + (1 - a)^2 + 2g*0 + 2f(1 - a) + c = 0$

- ⇨ $(1 - a)^2 + 2f(1 - a) + c = 0$

Again, $0^2 + (1 + a)^2 + 2g*0 + 2f(1 + a) + c = 0$

- ⇨ $(1 + a)^2 + 2f(1 + a) + c = 0$

Subtracting we get, $(1 + a)^2 - (1 - a)^2 + 2f(1 + a - 1 + a) = 0$

- ⇨ $(1 + a + 1 - a)(1 + a - 1 + a) + 2f*2a = 0$
- ⇨ $2a*2 + 2f*2a = 0$
- ⇨ $f = -1$
- ⇨ $c = -\{(1 + a)^2 - 2(1 + a)\} = -\{(1 + a)(1 + a - 2)\} = -(a + 1)(a - 1) = 1 - a^2$

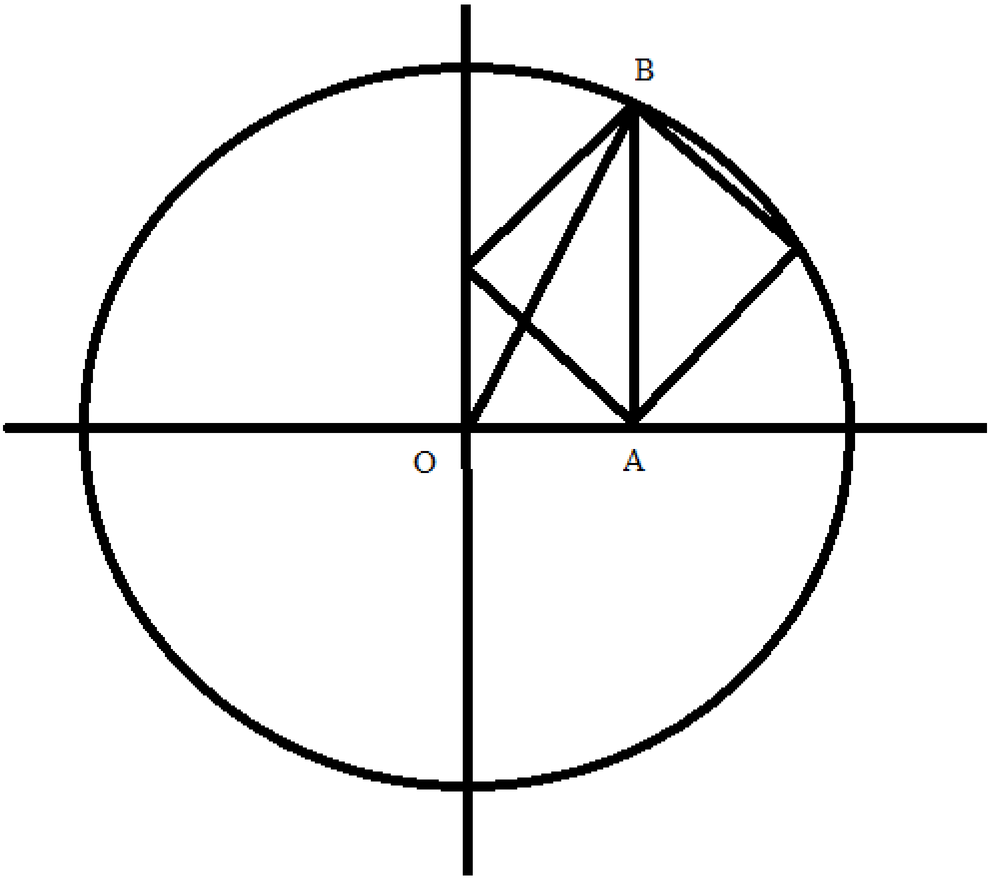
Now, $a^2 + 1^2 + 2g*a + 2f*1 + c = 0$

- ⇨ $a^2 + 1 + 2ga - 2 + 1 - a^2 = 0$
- ⇨ $2ga = 0$
- ⇨ $g = 0$
- ⇨ $r^2 = g^2 + f^2 - c = (-1)^2 - (1 - a^2) = a^2$
- ⇨ $\pi r^2 = \pi a^2 = ab/2$
- ⇨ $b = 2\pi a$

Option (b) is correct.

530. A square is inscribed in a *quarter-circle* in such a manner that two of its adjacent vertices lie on the two radii at an equal distance from the centre, while the other two vertices lie on the circular arc. If the square has sides of length x, then the radius of the circle is
(a) $16x/(\pi + 4)$
(b) $2x/\sqrt{\pi}$
(c) $\sqrt{5}x/\sqrt{2}$
(d) $\sqrt{2}x$

Solution :



Now, OA = $x/\sqrt{2}$ and AB = $x\sqrt{2}$

From triangle OAB we get, $r^2 = OA^2 + AB^2 = 5x^2/2$

⇨ $r = \sqrt{5}x/\sqrt{2}$

Option (c) is correct.

531. Let Q = (x_1, y_1) be an exterior point and P is a point on the circle centred at the origin and with radius r. Let θ be the angle which the line joining P to the centre makes with the positive direction of the x-axis. If the line PQ is tangent to the circle, then $x_1\cos\theta + y_1\sin\theta$ equal to

(a) r
(b) r^2
(c) $1/r$
(d) $1/r^2$

Solution :

Equation of OP is, $y = x\tan\theta$

⇨ $y\cos\theta - x\sin\theta = 0$

Therefore, equation of the tangent at P is, $x\cos\theta + y\sin\theta + c = 0$

It passes through (x_1, y_1)

Therefore, $x_1\cos\theta + y_1\sin\theta + c = 0$

- ⇨ $c = -(x_1\cos\theta + y_1\sin\theta)$

Equation is, $x\cos\theta + y\sin\theta - (x_1\cos\theta + y_1\sin\theta) = 0$

Distance from origin = radius

- ⇨ $|-(x_1\cos\theta + y_1\sin\theta)/\sqrt{(\cos^2\theta + \sin^2\theta)}| = r$
- ⇨ $x_1\cos\theta + y_1\sin\theta = r$

Option (a) is correct.

532. A straight line is drawn through the point (1, 2) making an angle θ $0 \le \theta \le \pi/3$, with the positive direction of the x-axis to intersect the line $x + y = 4$ at a point P so that the distance of P from the point (1, 2) is $\sqrt{6}/3$. Then the value of θ is
   (a) $\pi/18$
   (b) $\pi/12$
   (c) $\pi/10$
   (d) $\pi/3$

Solution :

Equation of the line is, $y - 2 = \tan\theta(x - 1)$

Solving this equation with $x + y = 4$ we get, $x = (2 + \tan\theta)/(1 + \tan\theta)$, $y = (2 + 3\tan\theta)/(1 + \tan\theta)$

Distance of P from (1, 2) is $\sqrt{[\{(2 + \tan\theta)/(1 + \tan\theta) - 1\}^2 + \{(2 + 3\tan\theta)/(1 + \tan\theta) - 2\}^2]}$

$= \sqrt{[\{1/(1 + \tan\theta)\}^2 + \{\tan\theta/(1 + \tan\theta)\}^2]} = \sec\theta/(1 + \tan\theta) = 1/(\sin\theta + \cos\theta) = \sqrt{6}/3$

- ⇨ $\sin\theta + \cos\theta = 3/\sqrt{6}$
- ⇨ $\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 9/6$
- ⇨ $1 + \sin 2\theta = 3/2$
- ⇨ $\sin 2\theta = ½$
- ⇨ $2\theta = \pi/6$
- ⇨ $\theta = \pi/12$

Option (b) is correct.

533. The area of intersection of two circular discs each of radius r and with the boundary of each disc passing through the centre of the other is

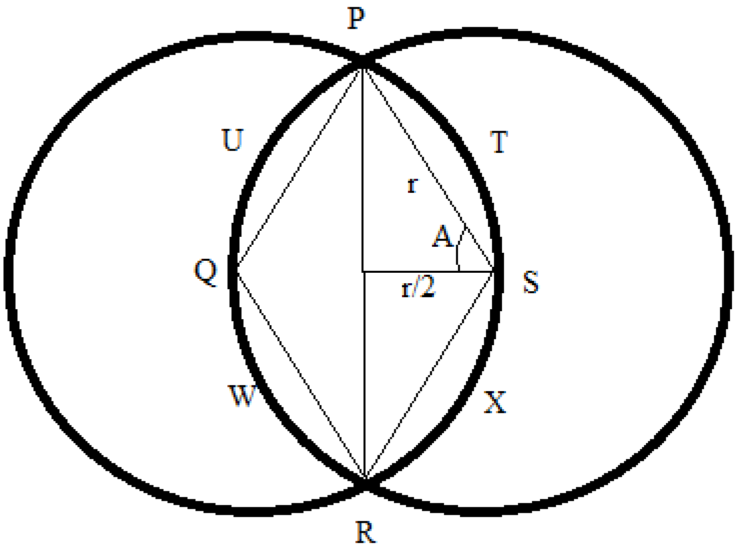
(a) $\pi r^2/3$

(b) $\pi r^2/6$

(c) $(\pi r^2/4)(2\pi - \sqrt{3}/2)$

(d) $(r^2/6)(4\pi - 3\sqrt{3})$

Solution :



From the figure we get, $(r/2)/r = \cos A$

- ⇨ $A = \pi/3$
- ⇨ $2A = 2\pi/3$
- ⇨ Area of SRWQUPS = $(\pi r^2)(2\pi/3)/2\pi = (\pi/3)r^2$

Similarly, area of QRXSTPQ = $(\pi/3)r^2$

PR = $2*\sqrt{(r^2 - r^2/4)} = r\sqrt{3}$

Now, area of PQRS = $(1/2)r*(r\sqrt{3}) = \sqrt{3}r^2/2$

Area of SRWQUPS + area of QRXSTPQ = $(2\pi/3)r^2$

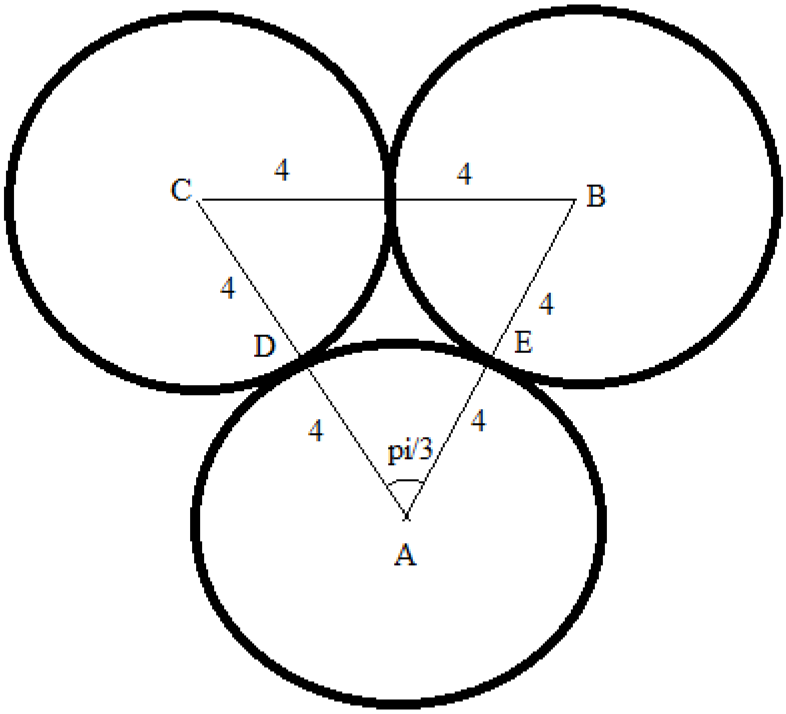
- ⇨ Area of RWQR + area of PUQP + area of PQRS + area of SXRS + area of STPS + area of PQRS = $(2\pi/3)r^2$
- ⇨ Area of RQQR + area of PUQP + area of SXRS + area of STPS + area of PQRS + $\sqrt{3}r^2/2 = (2\pi/3)r^2$
- ⇨ Required area = $(2\pi/3 - \sqrt{3}/2)r^2 = (r^2/6)(4\pi - 3\sqrt{3})$

Option (d) is correct.

534. Three cylinders each of height 16 cm and radius 4 cm are placed on a plane so that each cylinder touches the other two. Then the volume of the region between the three cylinders is, in cm^3,

(a) $98(4\sqrt{3} - \pi)$
(b) $98(2\sqrt{3} - \pi)$
(c) $98(\sqrt{3} - \pi)$
(d) $128(2\sqrt{3} - \pi)$

Solution :



Area of ADE = $(\pi 4^2)(\pi/3)/2\pi = 8\pi/3$

Therefore, area of same three portions = $3*(8\pi/3) = 8\pi$

Area of equilateral triangle ABC = $(\sqrt{3}/4)8^2 = 16\sqrt{3}$

Base area = $16\sqrt{3} - 8\pi = 8(2\sqrt{3} - \pi)$

Volume = $16*8(2\sqrt{3} - \pi) = 128(2\sqrt{3} - \pi)$

Option (d) is correct.

535. From a solid right circular cone made of iron with base of radius 2 cm and height 5 cm, a hemisphere of diameter 2 cm and centre coinciding with the centre of the base of the cone is scooped out. The resultant object is then dropped in a right circular cylinder

whose inner diameter is 6 cm and inner height is 10 cm. Water is then poured into the cylinder to fill it up to brim. The volume of water required is

(a) 80π cm^3
(b) $250\pi/3$ cm^3
(c) $270\pi/4$ cm^3
(d) 84π cm^3

Solution :

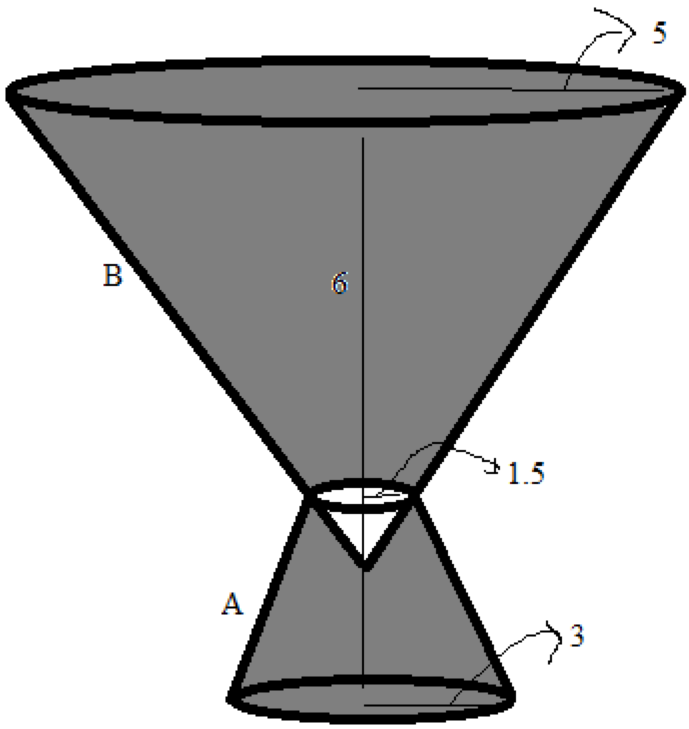
Volume of resultant object after cutting the hemisphere = $(1/3)\pi*2^2*5 - (1/2)(4/3)\pi*1^3 = 20\pi/3 - 2\pi/3 = 6\pi$.

Volume of cylinder = $\pi*3^2*10 = 90\pi$

Volume of water required = $90\pi - 6\pi = 84\pi$

Option (d) is correct.

536. A right-circular cone A with base radius 3 units and height 5 units is truncated in such a way that the radius of the circle at the top is 1.5 units and the top parallel to the base. A second right-circular cone B with base radius 5 units and height 6 units is placed vertically inside the cone A as shown in the diagram. The total volume of the portion of the cone B that is outside cone A and the portion of the cone A excluding the portion of cone B that is inside A (that is, the total volume of the shaded portion in the diagram) is



(a) 1867π/40
(b) 1913π/40
(c) 2417π/40
(d) 2153π/40

Solution :

Let the height of the portion of cone B that is inside A is h

Therefore, h/6 = 1.5/5

⇨ h = 9/5

Volume of the portion of cone B that is inside A is $(1/3)\pi(1.5)^2*(9/5) = 27\pi/20$

Volume of the portion of cone B that is outside cone A = $(1/3)\pi*5^2*6 - 27\pi/20 = 50\pi - 27\pi/20 = 973\pi/20$

Let the height of the portion of cone A that is excluded is h_1.

Therefore, $h_1/5 = 1.5/3$

⇨ $h_1 = 5/2$

Volume of the portion of cone A that is excluded = $(1/3)\pi*(1.5)^2*(5/2) = 15\pi/8$

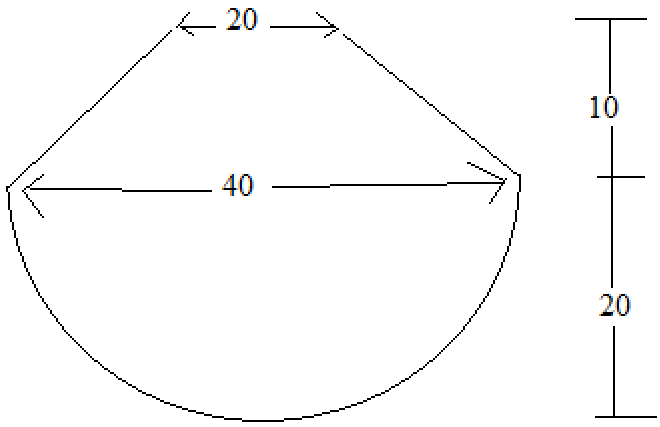
Volume of the portion of cone A that is there (including volume of portion B that is inside A) = $(1/3)\pi(3)^2*5 - 15\pi/8 = 15\pi - 15\pi/8 = 105\pi/8$

Volume of cone A excluding portion of cone B that is inside A = $105\pi/8 - 27\pi/20 = 471\pi/40$

Required volume = $973\pi/20 + 471\pi/40 = (1946\pi + 471\pi)/40 = 2417\pi/40$

Option (c) is correct.

537. A cooking pot has a spherical bottom, while the upper part is a truncated cone. Its vertical cross-section is shown in the figure. If the volume of food increases by 15% during cooking, the maximum initial volume of food that can be cooked without spilling is, in, cc,



(a) $14450(\pi/3)$
(b) $19550(\pi/3)$
(c) $(340000/23)(\pi/3)$
(d) $20000(\pi/3)$

Solution :

Let the height of the portion of the cone that is truncated is h

Therefore, $h/(h + 10) = 20/40$

- ⇨ $2h = h + 10$
- ⇨ $h = 10$

Volume of the truncated portion = $(1/3)\pi*10^2*10 = 1000(\pi/3)$

Volume of the cone = $(1/3)\pi*20^2*20 = 8000(\pi/3)$

Therefore, volume of the cone portion of the pot = 8000(π/3) – 1000(π/3) = 7000(π/3)

Now, volume of the hemispherical portion = $(1/2)(4/3)\pi*20^3$ = 16000(π/3)

Total volume of the pot = 16000(π/3) + 7000(π/3) = 23000(π/3)

Let the volume of the initial food = x

Volume during cooking = 115x/100

Now, 115x/100 = 23000(π/3)

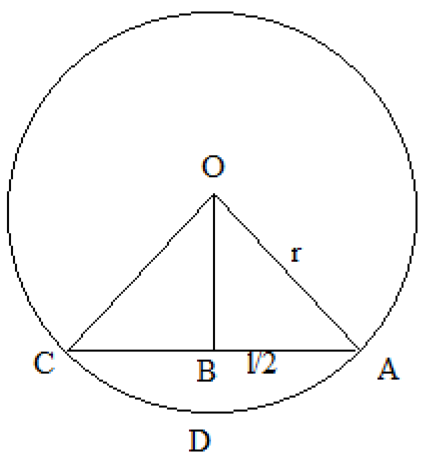
⇨ x = 20000(π/3)

Option (d) is correct.

538. A sealed cylinder drum of radius r is 90% filled with paint. If the drum is tilted to rest on its side, the fraction of its *curved* surface area (not counting the flat sides) that will be under the paint is
(a) less than 1/12
(b) between 1/12 and 1/6
(c) between 1/6 and ¼
(d) greater than ¼

Solution :

Let the length of the upper surface of paint is l.



Angle AOB = $\sin^{-1}(l/2r)$

Angle COA = $2\sin^{-1}(l/2r)$

Now, Area of OADCA = $(\pi r^2)\{2\sin^{-1}(l/2r)\}/2\pi = r^2\sin^{-1}(l/2r)$

Area of triangle OAB = $(1/2)*(l/2)*\sqrt{(r^2 - l^2/4)}$

Area of triangle OAC = $l\sqrt{(r^2 - l^2/4)}/2$

Area of ACDA = $r^2\sin^{-1}(l/2r) - (l/2)\sqrt{(r^2 - l^2/4)}$

Volume of paint = $\{ r^2\sin^{-1}(l/2r) - (l/2)\sqrt{(r^2 - l^2/4)}\}*h$ (where h is height of the cylindrical drum)

Now, $\{r^2\sin^{-1}(l/2r) - (l/2)\sqrt{(r^2 - l^2/4)}\}*h = (9/100)*(\pi r^2 h)$ ……. (1)

Now, $s = r*\sin^{-1}(l/2r)$

Curved surface area that is under paint = $r*\sin^{-1}(l/2r)*h$

Total curved surface area = $2\pi rh$

We have to find the ratio = $r*\sin^{-1}(l/2r)*h/2\pi rh = (1/2\pi)\sin^{-1}(l/2r)$

Manipulating equation (1) we have to find a range of the ratio.

Option (b) is correct.

539. The number of tangents that can be drawn from the point (2, 3) to the parabola $y^2 = 8x$ is
(a) 1
(b) 2
(c) 0
(d) 3

Solution :

Now, $y^2 - 8x = 0$

$3^2 - 8*2 < 0$

⇨ The point is within parabola.

Option (c) is correct.

540. A ray of light passing through the point (1, 2) is reflected on the x-axis at a point P, and then passes through the point (5, 3). Then the abscissa of the point P is
(a) 2 + 1/5
(b) 2 + 2/5
(c) 2 + 3/5

(d) 2 + 4/5

Solution :

Let co-ordinate of point P is (x, 0)

Now, slope of the incident ray is, $m_1 = (2 - 0)/(1 - x) = 2/(1 - x)$

So, $\theta_1 = \tan^{-1}\{2/(1 - x)\}$

And, $\theta_2 = \tan^{-1}\{(3 - 0)/(5 - x)\} = \tan^{-1}\{3/(5 - x)\}$

Now, $\tan^{-1}\{2/(1 - x)\} - \pi/2 = \pi/2 - \tan^{-1}\{3/(5 - x)\}$

- ⇨ $\tan^{-1}\{2/(1 - x)\} + \tan^{-1}\{3/(5 - x)\} = \pi$
- ⇨ $\{2/(1 - x) + 3/(5 - x)\}/[1 - \{2/(1 - x)\}\{3/(5 - x)\} = 0$
- ⇨ $2(5 - x) + 3(1 - x) = 0$
- ⇨ $10 - 2x + 3 - 3x = 0$
- ⇨ $x = 13/5 = 2 + 3/5$

Option (c) is correct.

541. If P, Q and R are three points with coordinates (1, 4), (4, 2) and (m, 2m – 1) respectively, then the value of m for which PR + RQ is minimum is
(a) 17/8
(b) 5/2
(c)7/2
(d) 3/2

Solution :

$PR = \sqrt{\{(m - 1)^2 + (2m - 1 - 4)^2\}} = \sqrt{(5m^2 - 22m + 26)}$

$RQ = \sqrt{\{(m - 4)^2 + (2m - 1 - 2)^2\}} = \sqrt{(5m^2 - 20m + 25)}$

Let $S = PR + RQ = \sqrt{(5m^2 - 22m + 26)} + \sqrt{(5m^2 - 20m + 25)}$

- ⇨ $dS/dm = (10m - 22)/2\sqrt{(5m^2 - 22m + 26)} + (10m - 20)/2\sqrt{(5m^2 - 20m + 25)} = 0$
- ⇨ $(5m - 11)\sqrt{(5m^2 - 20m + 25)} = -(5m - 10)\sqrt{(5m^2 - 22m + 26)}$
- ⇨ $(5m - 11)^2(5m^2 - 20m + 25) = (5m - 10)^2(5m^2 - 22m + 26)$
- ⇨ $(25m^2 - 110m + 121)/(25m^2 - 100m + 100) = (5m^2 - 22m + 26)/(5m^2 - 20m + 25)$
- ⇨ $(-10m + 21)/(25m^2 - 100m + 100) = (-2m + 1)/(5m^2 - 20m + 25)$
- ⇨ $(-10m + 21)/(-2m + 1) = (25m^2 - 100m + 100)/(5m^2 - 20m + 25)$
- ⇨ $16/(-2m + 1) = -25/(5m^2 - 20m + 25)$

- ⇨ $(5m^2 - 20m + 25)/(2m - 1) = 25/16$
- ⇨ $(m^2 - 4m + 5)/(2m - 1) = 5/16$
- ⇨ $16m^2 - 64m + 80 = 10m - 5$
- ⇨ $16m^2 - 74m + 85 = 0$
- ⇨ $m = \{74 \pm \sqrt{(74^2 - 4*16*85)}\}/2*16 = (74 \pm 6)/32 = 5/2, 17/8$

Now, we have to find d^2S/dm^2 and check that for $m = 17/8$ it is > 0.

Option (a) is correct.

542. Let A be the point (1, 2) and L be the line $x + y = 4$. Let M be the line passing through A such that the distance between A and the point of intersection of L and M is $\sqrt{(2/3)}$. Then the angle which M makes with L is
(a) 45
(b) 60
(c) 75
(d) 30

Solution :

M is, $y - 2 = m(x - 1)$

- ⇨ $y = 2 + m(x - 1)$

Now, $x + y = 4$

- ⇨ $x + 2 + m(x - 1) = 4$
- ⇨ $x(m + 1) = 2 + m$
- ⇨ $x = (m + 2)/(m + 1)$
- ⇨ $y = 2 + m\{(m + 2)/(m + 1) - 1\} = 2 + m/(m + 1) = (3m + 2)/(m + 1)$

Now, $\sqrt{[\{(m + 2)/(m + 1) - 1\}^2 + \{(3m + 2)/(m + 1) - 2\}^2]} = \sqrt{(2/3)}$

- ⇨ $1/(m + 1)^2 + m^2/(m + 1)^2 = 2/3$
- ⇨ $3(m^2 + 1) = 2(m^2 + 2m + 1)$
- ⇨ $m^2 - 4m + 1 = 0$
- ⇨ $m = \{4 \pm \sqrt{(16 - 4)}\}/2 = 2 \pm \sqrt{3}$

Angle which M makes with L = $\tan^{-1}|\{(2 + \sqrt{3} + 1)/1 - (2 + \sqrt{3})\}| = \tan^{-1}|\{(3 + \sqrt{3})/(-\sqrt{3} - 1)\}| = \tan^{-1}|(-\sqrt{3})| = -\tan^{-1}(\sqrt{3}) = 60$

Option (b) is correct.

543. The equation $x^2 + y^2 - 2x - 4y + 5 = 0$ represents

(a) a circle
(b) a pair of straight lines
(c) an ellipse
(d) a point

Solution :

Now, $x^2 + y^2 - 2x - 4y + 5 = 0$

⇨ $(x - 1)^2 + (y - 2)^2 = 0$
⇨ $x = 1, y = 2$
⇨ a point

Option (d) is correct.

544. The line $x = y$ is tangent at $(0, 0)$ to a circle of radius 1. The centre of the circle is
(a) $(1, 0)$
(b) either $(1/\sqrt{2}, 1/\sqrt{2})$ or $(-1/\sqrt{2}, -1/\sqrt{2})$
(c) either $(1/\sqrt{2}, -1/\sqrt{2})$ or $(-1/\sqrt{2}, 1/\sqrt{2})$
(d) none of the foregoing points

Solution :

Let centre = (h, k)

Therefore, $h^2 + k^2 = 1$

$|(h - k)/\sqrt{2}| = 1$

⇨ $(h - k)^2 = 2$
⇨ $h^2 + k^2 - 2hk = 2$
⇨ $2hk = -1$
⇨ $hk = -1/2$
⇨ $k = -1/2h$

Putting in first equation, $h^2 + 1/4h^2 = 1$

⇨ $4h^4 - 4h^2 + 1 = 0$
⇨ $(2h^2 - 1) = 0$
⇨ $h = \pm 1/\sqrt{2}$

Therefore centre is either $(1/\sqrt{2}, -1/\sqrt{2})$ or $(-1/\sqrt{2}, 1/\sqrt{2})$

Option (c) is correct.

545. Let C be the circle $x^2 + y^2 + 4x + 6y + 9 = 0$. The point (-1, -2) is
(a) inside C but not the centre of C
(b) outside C
(c) on C
(d) the centre of C

Solution :

$(x + 2)^2 + (y + 3)^2 = 2^2$

C is not centre.

Now, $(-1)^2 + (-2)^2 + 4(-1) + 6(-2) + 9 = 1 + 4 - 4 - 12 + 9 < 0$

Inside circle but not centre.

Option (a) is correct.

546. The equation of the circle circumscribing the triangle formed by the points (0, 0), (1, 0), (0, 1) is
(a) $x^2 + y^2 + x + y = 0$
(b) $x^2 + y^2 + x - y + 2 = 0$
(c) $x^2 + y^2 + x - y - 2 = 0$
(d) $x^2 + y^2 - x - y = 0$

Solution :

Triangle is right-angled.

Therefore, centre = (1/2, ½)

Radius = $\sqrt{(1 - ½)^2 + (0 - ½)^2} = 1/\sqrt{2}$

Equation is, $(x - ½)^2 + (y - ½)^2 = ½$

⇨ $x^2 + y^2 - x - y = 0$

Option (d) is correct.

547. The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy = 0$ at the origin is
(a) $fx + gy = 0$
(b) $gx + fy = 0$
(c) $x = 0$

(d) $y = 0$

Solution :

Centre = (-g, -f)

Slope of the normal at (0, 0) = f/g

Hence slope of the tangent at (0, 0) = -g/f

Equation is $y = (-g/f)x$ i.e. $gx + fy = 0$

Option (b) is correct.

548. The equation of the circle circumscribing the triangle formed by the points (3, 4), (1, 4) and (3, 2) is
(a) $x^2 - 4x + y^2 - 6y + 11 = 0$
(b) $x^2 + y^2 - 4x - 4y + 3 = 0$
(c) $8x^2 + 8y^2 - 16x - 13y = 0$
(d) None of the foregoing equations.

Solution :

Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Now, $3^2 + 4^2 + 6g + 8f + c = 0$ i.e. $6g + 8f + c + 25 = 0$ …. (1)

$1 + 16 + 2g + 8f + c = 0$ i.e. $2g + 8f + c + 17 = 0$ …… (2)

Doing (1) – (2) we get, $4g + 8 = 0$, i.e. $g = -2$

$9 + 4 + 6g + 4f + c = 0$, i.e. $6g + 4f + c + 13 = 0$ …. (3)

Doing (1) – (3) we get, $4f + 12 = 0$, i.e. $f = -3$

Putting these values in (3) we get, $-12 - 12 + c + 13 = 0$, i.e. $c = 11$

Equation is, $x^2 + y^2 -4x - 6y + 11 = 0$

Option (a) is correct.

549. The equation of the diameter of the circle $x^2 + y^2 + 2x - 4y + 4 = 0$ that is parallel to $3x + 5y = 4$ is
(a) $3x + 5y = 7$
(b) $3x - 5y = 7$
(c) $3x + 5y = -7$

(d) $3x - 5y = -7$

Solution :

Equation of line which is parallel to $3x + 5y = 4$ is $3x + 5y = c$

Now, centre = (-1, 2)

The diameter passes through centre. Thus, $-3 + 10 = c$, i.e. $c = 7$

Option (a) is correct.

550. Let C_1 and C_2 be the circles given by the equations $x^2 + y^2 - 4x - 5 = 0$ and $x^2 + y^2 + 8y + 7 = 0$. Then the circle having the common chord of C_1 and C_2 as its diameter has
(a) centre at (-1, 1) and radius 2
(b) centre at (1. -2) and radius $2\sqrt{3}$
(c) centre at (1, -2) and radius 2
(d) centre at (3, -3) and radius 2

Solution :

Centre of first circle = (2, 0), centre of second circle = (0, -4)

Mid-point is centre of the circle.

So, centre = (1, -2)

Common chord, $8y + 7 + 4x + 5 = 0$

⇨ $4x + 8y + 12 = 0$
⇨ $x + 2y + 3 = 0$

Distance from centre of C_1 is $(2 + 3)/\sqrt{(1^2 + 2^2)} = \sqrt{5}$

Radius of $C_1 = \sqrt{(4 + 5)} = 3$

Radius of required circle is, $r = \sqrt{(3^2 - 5)} = 2$

Option (c) is correct.

551. The equation of a circle which passes through the origin, whose radius is a and for which $y = mx$ is a tangent is
(a) $\sqrt{(1 + m^2)}(x^2 + y^2) + 2max + 2ay = 0$
(b) $\sqrt{(1 + m^2)}(x^2 + y^2) + 2ax - 2may = 0$
(c) $\sqrt{(1 + m^2)}(x^2 + y^2) - 2max + 2ay = 0$

(d) $\sqrt{(1 + m^2)}(x^2 + y^2) + 2ax + 2may = 0$

Solution :

Let, the centre of the circle is (-g, -f)

Therefore, $|(-f + gm)/\sqrt{(1 + m^2)}| = a$

- ⇨ $gm - f = a\sqrt{(1 + m^2)}$

Now, $g^2 + f^2 = a^2$

$g^2 + \{gm - a\sqrt{(1 + m^2)}\}^2 = a^2$

- ⇨ $g^2(1 + m^2) - 2gam\sqrt{(1 + m^2)} + a^2m^2 = 0$
- ⇨ $g = 2am\sqrt{(1 + m^2)} + \sqrt{\{4a^2m^2(1 + m^2) - 4a^2m^2(1 + m^2)\}}/2(1 + m^2) = am/\sqrt{(1 + m^2)}$
- ⇨ $f = am^2/\sqrt{(1 + m^2)} - a\sqrt{(1 + m^2)} = -a/\sqrt{(1 + m^2)}$

Equation is, $x^2 + y^2 + 2amx/\sqrt{(1 + m^2)} - 2ay/\sqrt{(1 + m^2)} = 0$

- ⇨ $\sqrt{(1 + m^2)}(x^2 + y^2) + 2amx - 2ay = 0$

There is no such option. So, the previous equation should be $f - gm = a\sqrt{(1 + m^2)}$

And it will come out to be option (c).

552. The circles $x^2 + y^2 + 4x + 2y + 4 = 0$ and $x^2 + y^2 - 2x = 0$
(a) intersect at two points
(b) touch at one point
(c) do not intersect
(d) satisfy none of the foregoing properties.

Solution :

Subtracting we get, $4x + 2y + 4 + 2x = 0$

- ⇨ $6x + 2y + 4 = 0$
- ⇨ $2x + y + 2 = 0$
- ⇨ $y = -(2x + 2)$

Putting in second equation we get, $x^2 + \{-(2x + 2)\}^2 - 2x = 0$

- ⇨ $5x^2 + 6x + 4 = 0$

Now, discriminant = $6^2 - 4*4*5 < 0$

They do not intersect or touch.

Option (c) is correct.

553. Let P be the point of intersection of the lines $ax + by - a = 0$ and $bx - ay + b = 0$. A circle with centre (1, 0) passes through P. The tangent to this circle at P meets the x-axis at the point (d, 0). Then the value of d is
(a) $2ab/(a^2 + b^2)$
(b) 0
(c) -1
(d) None of the foregoing values.

Solution :

Now, $ax + by - a = 0$ ….. (1)

And, $bx - ay + b = 0$ ……. (2)

Doing (1)*a + (2)*b we get, $(a^2 + b^2)x = a^2 - b^2$

⇨ $x = (a^2 - b^2)/(a^2 + b^2)$

$ax + by - a = 0$

⇨ $a(a^2 - b^2)/(a^2 + b^2) - a + by = 0$
⇨ $by = -2ab^2/(a^2 + b^2)$
⇨ $y = -2ab/(a^2 + b^2)$

Slope of normal at P is, $-2ab/(a^2 + b^2)/\{(a^2 - b^2)/(a^2 + b^2) - 1\} = -2ab/(-2b^2) = a/b$

Slope of tangent at P is $-(b/a)$

Equation of tangent at P is $y + 2ab/(a^2 + b^2) = -(b/a)\{x - (a^2 - b^2)/(a^2 + b^2)\}$

Putting y = 0, we get, $x - (a^2 - b^2)/(a^2 + b^2) = -2a^2/(a^2 + b^2)$

⇨ $x = -1$

Option (c) is correct.

554. The circles $x^2 + y^2 = 1$ and $x^2 + y^2 - 8x - 6y + c = 0$ touch each other externally. That is, the circles are mutually tangential and they lie outside each other. Then value of c is
(a) 9

(b) 8
(c)6
(d) 4

Solution :

Subtracting we get, $8x + 6y - c = 1$

⇨ $y = \{(c + 1) - 8x\}/6$

Putting in first equation we get, $x^2 + [\{(c + 1) - 8x\}/6]^2 = 1$

⇨ $36x^2 + 64x^2 - 16(c + 1)x + (c + 1)^2 - 36 = 0$
⇨ $100x^2 - 16(c + 1)x + \{(c + 1)^2 - 36\} = 0$

As the circles touch each other so, roots are equal. Therefore, discriminant = 0

⇨ $256(c + 1)^2 - 4*100\{c + 1)^2 - 36\} = 0$
⇨ $16(c + 1)^2 - 25(c + 1)^2 = -25*36$
⇨ $9(c + 1)^2 = (5*6)^2$
⇨ $3(c + 1) = \pm 30$
⇨ $(c + 1) = \pm 10$
⇨ $c = 9, -11$

To check for what value of c the circles touch internally and externally apply $C_1C_2 = r_1 - r_2$ and $C_1C_2 = r_1 + r_2$ respectively.

Option (a) is correct.

555. The circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ will touch if
(a) $1/a^2 + 1/b^2 = 1/c^2$
(b) $a^2 + b^2 = c^2$
(c)$a + b = c$
(d) $1/a + 1/b = 1/c$

Solution :

Subtracting we get, $2ax - 2by = 0$

⇨ $y = ax/b$

Putting in first equation we get, $x^2 + a^2x^2/b^2 + 2ax + c^2 = 0$

⇨ $x^2(1 + a^2/b^2) + 2ax + c^2 = 0$

The circles will touch if discriminant = 0

- ⇨ $4a^2 - 4c^2(1 + a^2/b^2) = 0$
- ⇨ $a^2/c^2 = 1 + a^2/b^2$
- ⇨ $1/a^2 + 1/b^2 = 1/c^2$

Option (a) is correct.

556. Two circles are said to cut *each other orthogonally* if the tangents at a point of intersection are perpendicular to each other. The locus of the center of a circle that cuts the circle $x^2 + y^2 = 1$ orthogonally and touch the line $x = 2$ is
(a) a pair of straight lines
(b) an ellipse
(c) a hyperbola
(d) a parabola

Solution :

Let the centre of the circle is (-g, -f)

Equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Therefore, 2*g*0 + 2*f*0 = c – 1

- ⇨ $c = 1$

Now, putting $x = 2$ in the equation of the circle we get, $4 + y^2 + 4g + 2fy + 1 = 0$

- ⇨ $y^2 + 2fy + (4g + 5) = 0$

As the circle touches the line $x = 2$, so roots are equal i.e. discriminant = 0

- ⇨ $4f^2 - 4(4g + 5) = 0$
- ⇨ $f^2 - 4g - 5 = 0$
- ⇨ $y^2 + 4x + 5 = 0$
- ⇨ a parabola

Option (d) is correct.

557. The equation of the circle circumscribing the triangle formed by the lines $y = 0$, $y = x$ and $2x + 3y = 10$ is
(a) $x^2 + y^2 + 5x - y = 0$
(b) $x^2 + y^2 - 5x - y = 0$
(c) $x^2 + y^2 - 5x + y = 0$

(d) $x^2 + y^2 - x + 5y = 0$

Solution :

Vertex are (0, 0), (5, 0), (2, 2)

Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy = 0$ (c = 0 as passes through (0, 0))

Now, $25 + 10g = 0$

⇨ $g = -5/2$

Now, $4 + 4 + 4g + 4f = 0$

⇨ $f = -2 - g = -2 + 5/2 = 1/2$

Equation is $x^2 + y^2 - 5x + y = 0$

Option (c) is correct.

558. Two gas companies X and Y, where X is situated at (40, 0) and Y at (0, 30) (unit = 1 km), offer to install equally priced gas furnaces in buyers' houses. Company X adds a charge of Rs. 40 per km of distance (measured along a straight line) between its location and the buyers' house, while company Y charges Rs. 60 per km of distance in the same way. Then the region where it is cheaper to have furnace installed by company X is
(a) the inside of circle $(x - 54)^2 + (y + 30)^2 = 3600$
(b) the inside of circle $(x - 24)^2 + (y + 30)^2 = 2500$
(c) the outside of the circle $(x + 32)^2 + (y - 54)^2 = 3600$
(d) the outside of the circle $(x + 24)^2 + (y - 12)^2 = 2500$

Solution :

Distance between X and Y = 50.

Let at a distance x from company X it is same to have any company's furnace installed.

So, $40*x = 60(50 - x)$

⇨ $2x = 150 - 3x$
⇨ $x = 30$

So, it divides the line joining company X and Y in 30 : 20 = 3 : 2

The coordinate = (2*40 + 3*0)/(2 + 3) = 16 and (2*0 + 3*30)/(2 + 3) = 18 i.e. (16, 18)

Let u be the distance from y in the far end from X such that there both the company's cost is same.

40*(50 + u) = 60*u

- ⇨ 100 + 2u = 3u
- ⇨ u = 100
- ⇨ 50 + u = 150
- ⇨ Diameter = 150 – 30 = 120
- ⇨ Radius = 120/2 = 60

Let the other side of the diameter is (x_1, y_1)

(0, 30) divides the line joining (x_1, y_1) and (40, 0) in 100 : 50 = 2 : 1

Therefore, $0 = (x_1 + 2*40)/(1 + 2)$

- ⇨ $x_1 = -80$

And, $30 = (y_1 + 2*0)/(1 + 2)$

- ⇨ $y_1 = 90$

Centre = (-80 + 16)/2 = -32 and (90 + 18/2) = 54 i.e. (-32, 54)

So, outside the circle $(x + 32)^2 + (y – 54)^2 = 3600$

Option (c) is correct.

559. Let C be the circle $x^2 + y^2 – 4x – 4y – 1 = 0$. The number of points common to C and the sides of the rectangle by the lines x = 2, x = 5, y = -1 and y = 5, equals
   (a) 5
   (b) 1
   (c)2
   (d) 3

Solution :

Put x = 2, $4 + y^2 – 8 – 4y – 1 = 0$

- ⇨ $y^2 – 4y – 5 = 0$
- ⇨ $(y – 2)^2 = 9$
- ⇨ y = 5, -1

Points are (2, 5), (2, -1)

Put x = 5 we get, $25 + y^2 - 20 - 4y - 1 = 0$

- ⇨ $y^2 - 4y + 4 = 0$
- ⇨ $(y - 2)^2 = 0$
- ⇨ $y = 2$
- ⇨ point is (5, 2)

Put y = -1, $x^2 + 1 - 4x + 4 - 1 = 0$

- ⇨ $(x - 2)^2 = 0$
- ⇨ $x = 2$
- ⇨ point is (2, -1) which is evaluated earlier.

Put y = 5, $x^2 + 25 - 4x - 20 - 1 = 0$

- ⇨ $x^2 - 4x + 4 = 0$
- ⇨ $(x - 2)^2 = 0$
- ⇨ $x = 2$
- ⇨ point is (2, 5) which is evaluated earlier.
- ⇨ Therefore, 3 points.

Option (d) is correct.

560. A circle of radius a with both coordinates of centre positive, touches the x-axis and also the line 3y = 4x. Then its equation is
   (a) $x^2 + y^2 - 2ax - 2ay + a^2 = 0$
   (b) $x^2 + y^2 -6ax - 4ay + 12a^2 = 0$
   (c) $x^2 + y^2 - 4ax - 2ay + 4a^2 = 0$
   (d) none of the foregoing equations

Solution :

Centre = (h, a)

Now, $(4h - 3a)/\sqrt{(4^2 + 3^2)} = a$

- ⇨ h = 2a

Equation is $(x - 2a)^2 + (y - a)^2 = a^2$

- ⇨ $x^2 + y^2 - 4ax - 2ay + 4a^2 = 0$

Option (c) is correct.

561. The equation of the circle with centre in the first quadrant and radius ½ such that the line 15y = 8x and the X-axis are both tangents to the circle, is

(a) $x^2 + y^2 - 8x - y + 16 = 0$
(b) $x^2 + y^2 - 4x - y + 4 = 0$
(c) $x^2 + y^2 - x - 4y + 4 = 0$
(d) $x^2 + y^2 - x - 8y + 16 = 0$

Solution :

Centre = $(h, ½)$

Now, $(8h - 15/2)/\sqrt{(8^2 + 15^2)} = ½$

⇨ $h = 2$

Equation is, $(x - 2)^2 + (y - ½)^2 = (1/2)^2$

⇨ $x^2 + y^2 - 4x - y + 4 = 0$

Option (b) is correct.

562. The centre of the circle $x^2 + y^2 - 8x - 2fy - 11 = 0$ lies on the straight line which passes through the point (0, -1) and makes an angle of 45 with the positive direction of the horizontal axis. The circle

(a) touches the vertical axis
(b) touches the horizontal axis
(c) passes through origin
(d) meets the axes at four points

Solution :

Equation of straight line is, $y + 1 = 1(x - 0)$

⇨ $x - y - 1 = 0$

Centre of the circle is $(4, -f)$

So, $4 + f - 1 = 0$

⇨ $f = -3$

Radius = $\sqrt{(4^2 + 3^2 + 11)} = 6$

Therefore, option (d) is correct.

563. Let P and Q be any two points on the circles $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 - 8x - 8y + 28 = 0$, respectively. If d is the distance between P and Q, then the set of all possible values of d is

(a) $0 \le d \le 9$
(b) $0 \le d \le 8$
(c) $1 \le d \le 8$
(d) $1 \le d \le 9$

Solution :

Subtracting we get, $-2x - 3 + 8x + 8y - 28 = 0$

- ⇨ $6x + 8y = 31$
- ⇨ $y = (31 - 6x)/8$

Putting in first equation we get, $x^2 + \{(31 - 6x)/8\}^2 - 2x - 3 = 0$

- ⇨ $64x^2 + 36x^2 - 372x + 961 - 128x - 192 = 0$
- ⇨ $100x^2 - 500x + 769 = 0$

Discriminant = $500^2 - 4*100*769 = 400(125 - 769) < 0$ so both the circle does not meet.

Centres = (1, 0) and (4, 4)

Now, we need to find the equation of the line joining the centres and then solve with the two circles, you will get 4 points, then calculate minimum and maximum distance.

But, here we will go by short-cut method. According to options minimum distance cannot be 0 and hence minimum distance = 1.

Now, radius of the circles = $\sqrt{(1^1 + 3)} = 2$ and $\sqrt{(4^2 + 4^2 - 28)} = 2$

Therefore, maximum distance = $1 + 2(2 + 2) = 9$

Option (d) is correct.

564. All points whose distance from the nearest point on the circle $(x - 1)^2 + y^2 = 1$ is half the distance from the line $x = 5$ lie on

(a) an ellipse
(b) a pair of straight lines
(c) a parabola
(d) a circle

Solution :

Let the point is (h, k).

Centre of the circle (1, 0) and radius = 1

Nearest distance from circle = $\sqrt{(h-1)^2 + k^2} - 1$

Distance from the line is $|h - 5|$

So, $\sqrt{(h-1)^2 + k^2} - 1 = (1/2)|h - 5|$

- ⇨ $4(h-1)^2 + 4k^2 = (h-5)^2 + 2|h-5| + 1$

Option (a) is correct.

565. If P = (0, 0), Q = (1, 0) and R = $(1/2, \sqrt{3}/2)$, then the centre of the circle for which the lines PQ, QR and RP are tangents, is
(a) (1/2, ¼)
(b) $(1/2, \sqrt{3}/4)$
(c)$(1/2, 1/2\sqrt{3})$
(d) $(1/2, -1/\sqrt{3})$

Solution :

PQ is x-axis.

So, centre = (h, r)

Equation of RP is, $y = \sqrt{3}x$

So, $|(r - \sqrt{3}h)/2| = r$

- ⇨ $(\sqrt{3}h - r) = 2r$ (otherwise r and h will be of opposite sign but the centre is in first quadrant)
- ⇨ $h = \sqrt{3}r$

Equation of QR is, $(y - 0)/(\sqrt{3}/2 - 0) = (x - 1)/(1/2 - 1)$

- ⇨ $2y/\sqrt{3} = -2(x - 1)$
- ⇨ $\sqrt{3}x + y - \sqrt{3} = 0$

So, $|(\sqrt{3}h + r - \sqrt{3})/2| = r$

- ⇨ $-4r + \sqrt{3} = 2r$
- ⇨ $r = \sqrt{3}/6 = 1/2\sqrt{3}$ (because $r < \sqrt{3}/2$ which you will get if you take $4r - \sqrt{3} = 2r$)
- ⇨ h = ½

Option (c) is correct.

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566. The equations of the pair of straight lines parallel to the x-axis and tangent to the curve $9x^2 + 4y^2 = 36$ are

(a) $y = -3, y = 9$
(b) $y = 3, y = -6$
(c) $y = \pm 6$
(d) $y = \pm 3$

Solution :

Let us say, the equation of the tangent is $y = a$

So, putting $y = a$ in the equation of ellipse we get, $9x^2 + 4a^2 = 36$

⇨ $9x^2 = 4(9 - a^2) = 0$ (because x must have one solution)
⇨ $a = \pm 3$

Option (d) is correct.

567. If the parabola $y = x^2 + bx + c$ is tangent to the straight line $x = y$ at the point (1, 1) then

(a) $b = -1, c = +1$
(b) $b = +1, c = -1$
(c) $b = -1$, c arbitrary
(d) $b = 0, c = -1$

Solution :

Putting $y = x$ we get, $x^2 + x(b - 1) + c = 0$

⇨ $(b - 1)^2 - 4c = 0$ …… (1) (roots are equal as tangent)

Now, the parabola passes through (1, 1)

⇨ $1 = 1^2 + 1*b + c$
⇨ $b + c = 0$
⇨ $c = -b$
⇨ $(b - 1)^2 + 4b = 0$ (from (1))
⇨ $(b + 1)^2 = 0$
⇨ $b = -1, c = +1$

Option (a) is correct.

568. The condition that the line $x/a + y/b = 1$ be a tangent to the curve $x^{2/3} + y^{2/3} = 1$ is

(a) $a^2 + b^2 = 2$

(b) $a^2 + b^2 = 1$
(c) $1/a^2 + 1/b^2 = 1$
(d) $a^2 + b^2 = 2/3$

Solution :

Any point on the curve is $(\cos^3\theta, \sin^3\theta)$

Now, $x^{2/3} + y^{2/3} = 1$

- ⇨ $(2/3)x^{-1/3} + (2/3)y^{-1/3}(dy/dx) = 0$
- ⇨ $dy/dx = -y^{1/3}/x^{1/3}$
- ⇨ (dy/dx) at $(\cos^3\theta, \sin^3\theta) = -\tan\theta$

Now, $-\tan\theta = -b/a$

- ⇨ $a\sin\theta = b\cos\theta$

Now, the line passes through $(\cos^3\theta, \sin^3\theta)$

So, $\cos^3\theta/a + \sin^3\theta/b = 1$

- ⇨ $\cos^2\theta\sin\theta/b + \sin^3\theta/b = 1$
- ⇨ $\sin\theta/b = 1$
- ⇨ $\sin\theta = b$
- ⇨ $\cos\theta = a$
- ⇨ $a^2 + b^2 = 1$

Option (b) is correct.

569. If the two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is
(a) $x - 1 = 0$
(b) $2x + 1 = 0$
(c) $x + 1 = 0$
(d) $2x - 1 = 0$

Solution :

Vertex = (0, 0) and a = 1

Therefore, equation of directrix is, $x = -1$ i.e. $x + 1 = 0$

If two tangents to a parabola from a given point are at right angles then the point lies on the directrix.

Option (c) is correct.

570. Let A be the point (0, 0) and let B be the point (1, 0). A point P moves so that the angle APB measures $\pi/6$. The locus of P is
(a) a parabola
(b) arcs of two circles with centres $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$
(c) arcs of two circles each of radius 1
(d) a pair of straight lines

Solution :

Let P = (h, k)

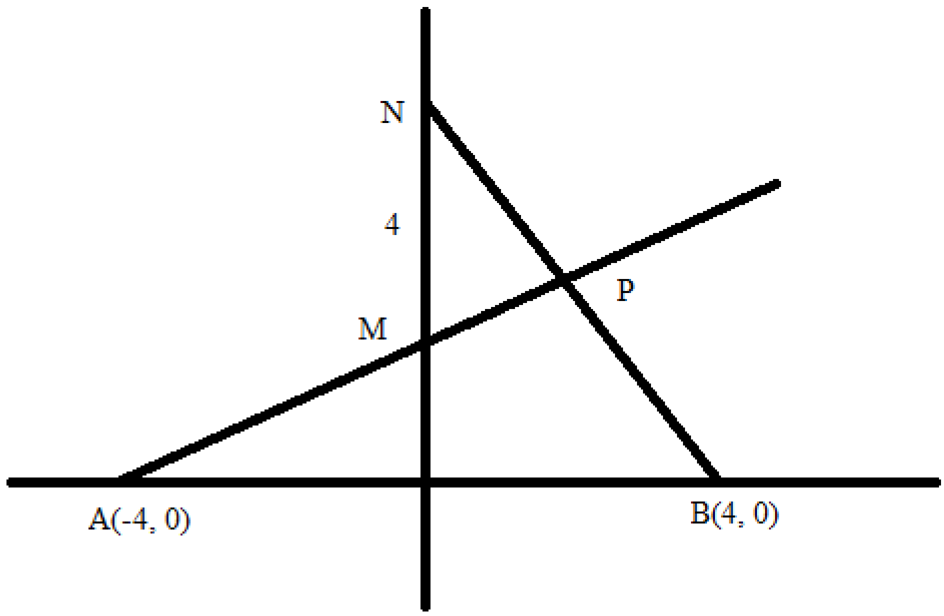
Slope of AP = k/h and slope of BP = k/(h – 1)

$\tan(APB) = |\{k/h - k/(h-1)\}/\{1 + k^2/h(h-1)\}|$

- ⇨ $1/\sqrt{3} = |k(h - 1 - h)/(h^2 + k^2 - h)|$
- ⇨ $k/(h^2 + k^2 - h) = \pm 1/\sqrt{3}$
- ⇨ $h^2 + k^2 - h \pm \sqrt{3}k = 0$
- ⇨ Therefore, radius = $\sqrt{\{(1/2)^2 + (\sqrt{3}/2)^2\}} = 1$

Option (c) is correct.

571. Let A = (-4, 0) and B = (4, 0). Let M and N be points on the y-axis, with MN = 4. Let P be the point of intersection of AM and BN. This is illustrated in the figure. Then the locus of P is



(a) $x^2 - 2xy = 16$

(b) $x^2 + 2xy = 16$
(c) $x^2 + 2xy + y^2 = 64$
(d) $x^2 - 2xy + y^2 = 64$

Solution :

Let, M = (0, a), N = (0, 4 + a)

Equation of AM is, $(y - 0)/(a - 0) = (x + 4)/(0 + 4)$

⇨ $4y = ax + 4a$

Equation of BN is, $(y - 0)/(4 + a - 0) = (x - 4)/(0 - 4)$

⇨ $-4y = (4 + a)x - 4(4 + a)$

Adding the equations we get, $0 = ax + 4a + (4 + a)x - 4(4 + a)$

⇨ $x(4 + 2a) = 16$
⇨ $x = 8/(2 + a)$
⇨ $4y = 8a/(2 + a) + 4a = (16a + 4a^2)/(2 + a)$
⇨ $y = a(4 + a)/(2 + a)$

Let, P = (h, k)

So, $h = 8/(2 + a)$ and $k = a(4 + a)/(2 + a)$

$k/h = a(4 + a)/8$

Now, $2 + a = 8/h$

⇨ $a = 8/h - 2 = (8 - 2h)/h$

$k/h = \{(8 - 2h)/h\}(4 + (8 - 2h)/h)/8$

⇨ $8k/h = (64 - 4h^2)/h^2$
⇨ $2kh = 16 - h^2$
⇨ $h^2 + 2kh = 16$
⇨ $x^2 + 2xy = 16$

Option (b) is correct.

572. Consider a circle in the XY plane with diameter 1, passing through the origin O and through the point A(1, 0). For any point B on the circle, let C be the point of intersection of the line OB with the vertical line through A. If M is the point on the line OBC such that OM and BC are of equal length, then the locus of the point M as B varies is given by the equation
(a) $y = \sqrt{x(x^2 + y^2)}$

(b) $y^2 = x$
(c) $(x^2 + y^2)x - y^2 = 0$
(d) $y = x\sqrt{(x^2 + y^2)}$

Solution :

Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy = 0$ ($c = 0$ as passes through origin)

Now, $1^2 + 0 + 2g*1 + 0 = 0$

- ⇨ $g = -1/2$

Now, $g^2 + f^2 = (1/2)^2$ (radius = ½)

- ⇨ $f = 0$

Equation of the circle is, $x^2 + y^2 - x = 0$

Let $B = (x_1, y_1)$

So, $x_1^2 + y_1^2 - x_1 = 0$ …………….. (1)

Equation of OB is, $y = (y_1/x_1)x$

Equation of vertical line through A is, $x = 1$.

Putting $x = 1$, we get, $y = y_1/x_1$

So, $C = (1, y_1/x_1)$

Now, $BC^2 = (x_1 - 1)^2 + (y_1 - y_1/x_1)^2 = (x_1 - 1)^2 + y_1^2(x_1 - 1)^2/x_1^2 = (x_1 - 1)^2(1 + y_1^2/x_1^2) = (x_1 - 1)^2(x_1^2 + y_1^2)/x_1^2 = (x_1 - 1)^2/x_1$ (from (1))

Let $M = (h, k)$

$OM^2 = h^2 + k^2$

Now, $h^2 + k^2 = (x_1 - 1)^2/x_1$

Now, $k = (y_1/x_1)h$ (as M lies on OB)

- ⇨ $k^2/h^2 = y_1^2/x_1^2$
- ⇨ $(h^2 + k^2)/h^2 = (x_1^2 + y_1^2)/x_1^2 = 1/x_1$ (from (1))
- ⇨ $x_1 = h^2/(h^2 + k^2)$

Putting value in above equation we get, $h^2 + k^2 = \{(h^2/(h^2 + k^2) - 1)^2/h^2\}(h^2 + k^2)$

- ⇨ $h^2 = \{k^2/(h^2 + k^2)\}^2$
- ⇨ $h = k^2/(h^2 + k^2)$
- ⇨ $h(h^2 + k^2) = k^2$

⇨ $y^2 = x(x^2 + y^2)$

Option (c) is correct.

573. The locus of the foot of the perpendicular from any focus upon any tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is
(a) $x^2/b^2 + y^2/a^2 = 1$
(b) $x^2 + y^2 = a^2 + b^2$
(c) $x^2 + y^2 = a^2$
(d) none of the foregoing curves

Solution :

Let the foot of the perpendicular from focus = (h, k)

Focus = (ae, 0)

Slope of focus joining (h, k) = (k – 0)/(h – ae) = k/(h – ae)

Therefore, slope of tangent = -(h – ae)/k

Equation of tangent is, y – k = -{(h – ae)/k}(x – h)

⇨ y = k – (h – ae)(x – h)/k

Putting in the equation of ellipse we get, $x^2/a^2 + \{k - (h - ae)(x - h)/k\}^2/b^2 = 1$

Now, equate the discriminant of this equation to zero and use $(a^2 - b^2)/a^2 = e^2$ and reduce the equation of (h, k) and then put (x, y) in place of (h, k) and you get the locus.

Option (c) is correct.

574. The area of the triangle formed by a tangent of slope m to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the two coordinate axes is
(a) $\{|m|/2\}(a^2 + b^2)$
(b) $\{1/2|m|\}(a^2 + b^2)$
(c) $\{|m|/2\}(a^2m^2 + b^2)$
(d) $\{1/2|m|\}(a^2m^2 + b^2)$

Solution :

Let the tangent is at the point (acosθ, bsinθ)

Now, $x^2/a^2 + y^2/b^2 = 1$

- ⇨ $2x/a^2 + (2y/b^2)(dy/dx) = 0$
- ⇨ (dy/dx) at $(a\cos\theta, b\sin\theta) = -(a\cos\theta/a^2)/(b\sin\theta/b^2) = -b\cos\theta/a\sin\theta = m$
- ⇨ $\tan\theta = -b/am$

Now, equation of the tangent is, $y - b\sin\theta = m(x - a\cos\theta)$

- ⇨ $mx - y - am\cos\theta + b\sin\theta = 0$

Putting $x = 0$ we get, $y = am\cos\theta - b\sin\theta$ and putting $y = 0$ we get, $x = (am\cos\theta - b\sin\theta)/m$

Area $= |(1/2)(am\cos\theta - b\sin\theta)^2/m| = \{1/2|m|\}(a^2m^2\cos^2\theta + b^2\sin^2\theta - 2amb\cos\theta\sin\theta)$

$= \{1/2|m|\}(a^2m^2 - a^2m^2\sin^2\theta + b^2\sin^2\theta - 2amb\cos\theta\sin\theta)$

$\{1/2|m|\}(a^2m^2 - b^2\cos^2\theta + b^2\sin^2\theta + 2b^2\cos^2\theta)$ (from $\tan\theta = -b/am$)

$= \{1/2|m|\}(a^2m^2 + b^2)$

Option (d) is correct.

575. Consider the locus of a moving point $P = (x, y)$ in the plane which satisfies the law $2x^2 = r^2 + r^4$, where $r^2 = x^2 + y^2$. Then only one of the following statements is true. Which one is it?
   (a) For every positive real number d, there is a point (x, y) on the locus such that $r = d$.
   (b) For every value d, $0 < d < 1$, there are exactly four points on the locus, each of which is at a distance d from the origin.
   (c) The point P always lies in the first quadrant.
   (d) The locus of P is an ellipse.

Solution :

Clearly, option (b) is correct.

Because let $r = 50$, $r^2 = 2500$, $r^4 = 6250000$

So, $2x^2 = 2500 + 6250000$

- ⇨ $x^2 > r^2$

So, option (a) cannot be true. And option (c) cannot be true because P may be anywhere. It doesn't matter if x or y is negative. And (d) is not true because it is not the equation of an ellipse.

576. Let A be any variable point on the X-axis and B the point (2, 3). The perpendicular at A to the line AB meets the Y-axis at C. Then the locus of the mid-point of the segment AC as A moves is given by the equation

(a) $2x^2 - 2x + 3y = 0$
(b) $3x^2 - 3x + 2y = 0$
(c) $3x^2 - 3x - 2y = 0$
(d) $2x - 2x^2 + 3y = 0$

Solution :

Let A = (a, 0)

Now, slope of AB = $(3 - 0)/(2 - a) = 3/(2 - a)$

Slope of perpendicular on AB = $(a - 2)/3$

Equation of perpendicular to AB at A is, $y - 0 = \{(a - 2)/3\}(x - a)$

Putting x = 0, we get, $y = a(2 - a)/3$

C = $(0, a(2 - a)/3)$

Let, mid-point of AC = (h, k)

Therefore, $h = a/2$, $k = a(2 - a)/6$

- ⇨ $k = h(2 - 2h)/3$
- ⇨ $3k = 2h - 2h^2$
- ⇨ $2h^2 - 2h + 3k = 0$

Locus is , $2x^2 - 2x + 3y = 0$

Option (a) is correct.

577. A straight line segment AB of length a moves with its ends on the axes. Then the locus of the point P such that AP : BP = 2 : 1 is

(a) $9(x^2 + y^2) = 4a^2$
(b) $9(x^2 + 4y^2) = 4a^2$
(c) $9(y^2 + 4x^2) = 4a^2$
(d) $9x^2 + 4y^2 = a^2$

Solution :

Let A = (0, q) and B = (p, 0)

Let P = (h, k)

Therefore, h = 2p/3 and k = q/3

- ⇨ p = 3h/2 an q = 3k

Now, $p^2 + q^2 = a^2$

- ⇨ $9h^2/4 + 9k^2 = a^2$
- ⇨ $9(h^2 + 4k^2) = 4a^2$

Locus is, $9(x^2 + 4y^2) = 4a^2$

Option (b) is correct.

578. Let P be a point moving on the straight line √3x + y = 2. Denote the origin by O. Suppose now that the line-segment OP is rotated. With O fixed, by an angle 30 in anti-clockwise direction, to get OQ. The locus of Q is
(a) √3x + 2y = 2
(b) 2x + √3y = 2
(c)√3x + 2y = 1
(d) x + √3y = 2

Solution :

Let coordinate of P = (a, b)

So, √3a + b = 2

Now, slope of OP = b/a

Let Q = (h, k)

Slope of OQ = k/h

Now, tan30 = {(k/h) – (b/a)}/(1 + (k/h)(b/a))

- ⇨ 1/√3 = (ak – bh)/(ah + bk)
- ⇨ ah + bk = √3ak - √3bh
- ⇨ a(√3k –h) = b(k + √3h)

Now, √3a + b = 2

- ⇨ √3(k + √3h)b/(√3k – h) + b = 2
- ⇨ b{√3k + 3h + √3k – h) = 2(√3k – h)
- ⇨ b = 2(√3k – h)/2(√3k + h) = (√3k – h)/(√3k + h)
- ⇨ a = (k + √3h)/(√3k + h)

And we have, $h^2 + k^2 = a^2 + b^2$

- ⇨ $h^2 + k^2 = \{(k + \sqrt{3}h)/(\sqrt{3}k + h)\}^2 + \{(\sqrt{3}k - h)/(\sqrt{3}k + h)\}^2 = (k^2 + 3h^2 + 2\sqrt{3}hk + 3k^2 + h^2 - 2\sqrt{3}hk)/(\sqrt{3}k + h)^2$
- ⇨ $h^2 + k^2 = 4(h^2 + k^2)/(\sqrt{3}k + h)^2$
- ⇨ $(\sqrt{3}k + h)^2 = 4$
- ⇨ $\sqrt{3}k + h = 2$

Locus is, $x + \sqrt{3}y = 2$

Option (d) is correct.

579. Consider an ellipse with centre at the origin. From any arbitrary point P on the ellipse, perpendiculars PA and PB are dropped on the axes of the ellipse. Then the locus of point Q that divides AB in the fixed ratio m : n is
(a) a circle
(b) an ellipse
(c) a hyperbola
(d) none of the foregoing curves

Solution :

Let the equation of the ellipse is $x^2/a^2 + y^2/b^2 = 1$

Any point on the ellipse = $(a\cos\theta, b\sin\theta)$

Now, A = $(a\cos\theta, 0)$ and B = $(0, b\sin\theta)$

Let Q = (h, k)

Therefore, $h = na\cos\theta/(m + n)$ and $k = mb\sin\theta/(m + n)$

- ⇨ $h/na = \cos\theta/(m + n)$ and $k/mb = \sin\theta/(m + n)$
- ⇨ $(h/na)^2 + (k/mb)^2 = 1/(m + n)^2$

Locus is, $x^2/n^2a^2 + y^2/m^2b^2 = 1/(m + n)^2$

- ⇨ An ellipse.

Option (b) is correct.

580. Let A and C be two distinct points in the plane and B a point on the line segment AC such that AB = 2BC. Then, the locus of the point P lying in the plane and satisfying $AP^2 + CP^2 = 2BP^2$ is
(a) a straight line parallel to the line AC
(b) a straight line perpendicular to the line AC
(c) a circle passing through A and C
(d) none of the foregoing curves

Solution :

Let A = (a, 0) and C = (c, 0)

B = ((2c + a)/3, 0)

Let P = (h, k)

Therefore, $(h - a)^2 + k^2 + (h - c)^2 + k^2 = 2\{h - (2c + a)/3\}^2 + 2k^2$

- ⇨ $2h^2 - 2h(a + c) + a^2 + c^2 = 2h^2 - 2h(2c + a)/3 + \{(2c + a)/3\}^2$
- ⇨ $2h(2c + a - 3a - 3c)/3 = (4c^2 + a^2 - 9c^2 - 9a^2)/9$
- ⇨ $2h(2a + c)/3 = (8a^2 + 5c^2)/9$
- ⇨ $h = (8a^2 + 5c^2)/\{6(2a + c)\}$

Locus is, x = b

So, straight line perpendicular to AC.

Option (b) is correct.

581. Let C be a circle and L is a line on the same plane such that C and L do not intersect. Let P be a moving point such that the circle drawn with centre at P to touch L also couches C. Then the locus of P is
    (a) A straight line parallel to L not intersecting C
    (b) A circle concentric with C
    (c) A parabola whose focus is centre of C and whose directrix is L
    (d) A parabola whose focus is the centre of C and whose directrix is a straight line parallel to L.

Solution :

Let P = (h, k)

Let C is $x^2 + y^2 = 4$ and L is y = 3.

Let the radius of the circle is r.

Therefore, (3 – k) = r

And $\sqrt{(h^2 + k^2)} = 2 + r$ (as the circle touches the circle C) = 2 + 3 – k = 4 – k

- ⇨ $h^2 + k^2 = 16 - 8k + k^2$
- ⇨ $h^2 = -8(k - 2)$

  Locus is $x^2 = -4*2(y - 2)$

So, vertex = (0, 2) focus = (0, 0)

Option (d) is correct.

582. A right triangle with sides 3, 4 and 5 lies inside the circle $2x^2 + 2y^2 = 25$. The triangle is moved inside the circle in such a way that its hypotenuse always forms a chord of the circle. The locus of the vertex opposite to the hypotenuse is
(a) $2x^2 + 2y^2 = 1$
(b) $x^2 + y^2 = 1$
(c) $x^2 + y^2 = 2$
(d) $2x^2 + 2y^2 = 5$

Solution :

Option (a) is correct.

583. Let P be the point (-3, 0) and Q be a moving point (0, 3t). Let PQ be trisected to R so that R is nearer to Q. RN is drawn perpendicular to PQ meeting the x-axis at N. The locus of the mid-point of RN is
(a) $(x + 3)^2 – 3y = 0$
(b) $(y + 3)^2 – 3x = 0$
(c) $x^2 – y = 1$
(d) $y^2 – x = 1$

Solution :

PR : RQ = 2 : 1

R = (-3/3, 6t/3) = (-1, 2t)

Slope of PQ = (3t – 0)/(0 + 3) = t

Slope of perpendicular to PQ = -1/t

Equation of RN is, y – 2t = (-1/t)(x + 1)

Putting y = 0, we get, $x = 2t^2 – 1$

So, $N = (2t^2 – 1, 0)$

Let mid-point of RN = (h, k)

Therefore, $h = (2t^2 – 1 – 1)/2$ and k = 2t/2

- ⇨ $h = t^2 - 1$ and $k = t$
- ⇨ $h = k^2 - 1$

Locus is, $y^2 - x = 1$

Option (d) is correct.

584. The maximum distance between two points of the unit cube is
   (a) $\sqrt{2} + 1$
   (b) $\sqrt{2}$
   (c) $\sqrt{3}$
   (d) $\sqrt{2} + \sqrt{3}$

Solution :

Maximum distance = $\sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}$ (between two opposite vertex along space diagonal)

Option (c) is correct.

585. Each side of a cube is increased by 50%. Then the surface area of the cube is increased by
   (a) 50%
   (b) 100%
   (c) 125%
   (d) 150%

Solution :

Let side of cube = a.

Surface area = $6a^2$

New side = $3a/2$

New surface area = $6(3a/2)^2 = 27a^2/2$

Increase = $27a^2/2 - 6a^2 = 15a^2/2$

% increase = $\{(15a^2/2)/6a^2\}*100 = 125$

Option (c) is correct.

586. A variable plane passes through a fixed point (a, b, c) and cuts the coordinate axes at P, Q, R. Then the coordinates (x, y, z) of the centre of the sphere passing through P, Q, R and the origin satisfy the equation
(a) $a/x + b/y + c/z = 2$
(b) $x/a + y/b + z/c = 3$
(c) $ax + by + cz = 1$
(d) $ax + by + cz = a^2 + b^2 + c^2$

Solution :

Option (a) is correct.

587. Let A = (0, 10) and B = (30, 20) be two points in the plane and let P = (x, 0) be a moving point on the x-axis. The value of x for which the sum of the distances of P from A and B is minimum equals
(a) 0
(b) 10
(c) 15
(d) 20

Solution :

$D = \sqrt{(x^2 + 100)} + \sqrt{\{(x - 30)^2 + 400\}}$

$dD/dx = 2x/2\sqrt{(x^2 + 100)} + 2(x - 30)/2\sqrt{\{(x - 30)^2 + 400\}} = 0$

- ⇨ $x\sqrt{\{(x - 30)^2 + 400\}} = -(x - 30)\sqrt{(x^2 + 100)}$
- ⇨ $x^2(x - 30)^2 + 400x^2 = (x - 30)^2x^2 + 100(x - 30)^2$
- ⇨ $4x^2 = x^2 - 60x + 900$
- ⇨ $3x^2 + 60x - 900 = 0$
- ⇨ $x^2 + 20x - 300 = 0$
- ⇨ $(x + 30)(x - 10) = 0$
- ⇨ $x = 10$

Option (b) is correct.

588. The number of solutions to the pair of equations $\sin\{(x + y)/2\} = 0$ and $|x| + |y| = 1$ is
(a) 2
(b) 3
(c) 4

(d) 1

Solution :

sin{(x + y)/2} = 0

- ⇨ (x + y)/2 = 0
- ⇨ x + y = 0
- ⇨ x = ½ and y = -1/2 and x = -1/2 and y = ½

Two solutions.

Option (a) is correct.

589. The equation $r^2\cos\theta + 2ar\sin^2(\theta/2) - a^2 = 0$ (a positive) represents
(a) a circle
(b) a circle and a straight line
(c) two straight lines
(d) none of the foregoing curves

Solution :

Now, $r^2 x\cos\theta + 2ar\sin^2(\theta/2) - a^2 = 0$

- ⇨ $rx + ar(1 - \cos\theta) - a^2 = 0$
- ⇨ $rx + ar - ax - a^2 = 0$
- ⇨ $r(x + a) - a(x + a) = 0$
- ⇨ $(x + a)(r - a) = 0$
- ⇨ $x + a = 0$, $r = a$ i.e. $x^2 + y^2 = a^2$
- ⇨ a circle and a straight line

Option (b) is correct.

590. The number of distinct solutions of $\sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta$, in the interval $0 \le \theta \le \pi/2$ is
(a) 5
(b) 4
(c) 8
(d) 9

Solution :

Now, $\sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta$

- ⇨ $2\sin 5\theta \cos 3\theta = 2\sin 9\theta \cos 7\theta$
- ⇨ $\sin 8\theta + \sin 2\theta = \sin 16\theta + \sin 2\theta$
- ⇨ $\sin 8\theta = \sin 16\theta$
- ⇨ $\sin 8\theta - 2\sin 8\theta \cos 8\theta = 0$
- ⇨ $\sin 8\theta(1 - 2\cos 8\theta) = 0$
- ⇨ $\sin 8\theta = 0$ or $\cos 8\theta = \frac{1}{2}$
- ⇨ $8\theta = 0, \pi, 2\pi, 3\pi, 4\pi$
- ⇨ $\theta = 0, \pi/8, \pi/4, 3\pi/8, \pi/2$

Now, $\cos 8\theta = \frac{1}{2}$

$8\theta = \pi/3, 2\pi - \pi/3, 2\pi + \pi/3, 4\pi - \pi/3$

- ⇨ $\theta = \pi/24, \pi/4 - \pi/24, \pi/4 + \pi/24, \pi/2 - \pi/24$
- ⇨ 9 solutions.

Option (d) is correct.

591. The value of sin15 is
   - (a) $(\sqrt{6} - \sqrt{2})/4$
   - (b) $(\sqrt{6} + \sqrt{2})/4$
   - (c) $(\sqrt{5} + 1)/2$
   - (d) $(\sqrt{5} - 1)/2$

Solution :

$\cos 30 = \sqrt{3}/2$

$1 - 2\sin^2 15 = \sqrt{3}/2$

$\sin^2 15 = (2 - \sqrt{3})/4 = (4 - 2\sqrt{3})/8 = \{(\sqrt{3} - 1)/2\sqrt{2}\}^2$

- ⇨ $\sin 15 = (\sqrt{3} - 1)/2\sqrt{2} = (\sqrt{6} - \sqrt{2})/4$

Option (a) is correct.

592. The value of sin25sin35sin85 is equal to
   - (a) $\sqrt{3}/4$
   - (b) $\sqrt{(2 - \sqrt{3})}/4$
   - (c) $5\sqrt{3}/9$
   - (d) $\sqrt{(1/2 + \sqrt{3}/4)}/4$

Solution :

sin25sin35sin85 = (1/2)(2sin25sin35)sin85 = (1/2)(-cos60 + cos10)sin85 = (1/2)(-sin85/2 + cos10sin85) = (1/2){-sin85/2 + (1/2)2cos10cos5)} = (1/2){-cos5/2 + (1/2)cos15 + cos5/2} = (1/4)cos15 = -(1/4)√{(1 + cos30)/2} = (1/4)√(1/2 + √3/4)

Option (d) is correct.

593. The angle made by the complex number $1/(\sqrt{3} + i)^{100}$ with the positive real axis is
(a) 135
(b) 120
(c) 240
(d) 180

Solution :

$1/(\sqrt{3} + i)^{100} = (\sqrt{3} - i)^{100}/2^{100} = \{\sqrt{3}/2 - i(1/2)\}^{100} = \{\cos(\pi/6) - i\sin(\pi/6)\}^{100}$ = cos(100π/6) – isin(100π/6) = cos(16π + 4π/6) – isin(16π + 4π/6) = cos(2π/3) - isin(2π/3) = cos(2π - 2π/3) + isin(2π - 2π/3) = cos(4π/3) + isin(4π/3)

Option (c) is correct.

594. The value of $\tan\{(\pi/4)\sin^2 x\}$, -∞ < x < ∞, lies between
(a) -1 and +1
(b) 0 and 1
(c) 0 and ∞
(d) -∞ and +∞

Solution :

$0 \le \sin^2 x \le 1$

- ⇨ $0 \le (\pi/4)\sin^2 x \le (\pi/4)$
- ⇨ $0 \le \tan\{(\pi/4)\sin^2 x\} \le 1$

Option (b) is correct.

595. If tan(πcosθ) = cot(πsinθ), then the value of cos(θ – π/4) is
(a) ±1/2√2
(b) ±1/2
(c) ±1/√2

(d) 0

Solution :

Now, $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$

- ⇨ $\pi\cos\theta = \pi/2 - \pi\sin\theta$
- ⇨ $\cos\theta + \sin\theta = 1/2$
- ⇨ $(1/\sqrt{2})\cos\theta + (1/\sqrt{2})\sin\theta = 1/2\sqrt{2}$
- ⇨ $\cos(\pi/4)\cos\theta + \sin(\pi/4)\sin\theta = 1/2\sqrt{2}$
- ⇨ $\cos(\theta - \pi/4) = 1/2\sqrt{2}$

Option (a) is correct.

596. If $f(x) = (1 - x)/(1 + x)$. then $f(f(\cos x))$ equals
   (a) x
   (b) $\cos x$
   (c) $\tan^2(x/2)$
   (d) none of the foregoing expressions

Solution :

$f(\cos x) = (1 - \cos x)/(1 + \cos x) = \tan^2(x/2)$

$f(f\cos x) = \{1 - \tan^2(x/2)\}/\{1 + \tan^2(x/2)\} = \cos x$

Option (b) is correct.

597. If $\cos x/\cos y = a/b$, then $a\tan x + b\tan y$ equals
   (a) $(a + b)\cot\{(x + y)/2\}$
   (b) $(a + b)\tan\{(x + y)/2\}$
   (c) $(a + b)\{\tan(x/2) + \tan(y/2)\}$
   (d) $(a + b)\{\cot(x/2) + \cot(y/2)\}$

Solution :

As there is a factor $(a + b)$ in every option so we start with

$(a\tan x + b\tan y)/(a + b)$

$= (b\sin x/\cos y + b\sin y/\cos y)/(b\cos x/\cos y + b)$

$= (\sin x + \sin y)/(\cos x + \cos y)$

= 2sin{x + y)/2}cos{(x − y)/2}/[2cos{(x + y)/2}cos{(x − y)/2}]

= tan{(x + y)/2}

Option (b) is correct.

598. Let θ be an angle in the second quadrant (that is 90 ≤ θ ≤ 180) with tanθ = -2/3. Then the value of {tan(90 + θ) + cos(180 + θ)}/{sin(270 − θ) − cot(-θ)} is
(a) (2 + √13)/(2 - √13)
(b) (2 - √13)/(2 + √13)
(c)(2 + √39)/(2 - √39)
(d) 2 + 3√13

Solution :

Now, {tan(90 + θ) + cos(180 + θ)}/{sin(270 − θ) − cot(-θ)}

= (-cotθ − cosθ)/(-cosθ + cotθ)

= (cosθ + cotθ)/(cosθ − cotθ)

= (-3/√13 − 3/2)/(-3/√13 + 3/2)

= (2 + √13)/(2 - √13)

Option (a) is correct.

599. Let P be a moving point such that if PA and PB are the two tangents drawn from P to the circle $x^2 + y^2 = 1$ (A, B being the points of contact), then Angle AOB = 60, where O is origin. Then the locus of P is
(a) a circle of radius 2/√3
(b) a circle of radius 2
(c)a circle of radius √3
(d) none of the foregoing curves

Solution :

P = (h, k)

Let A = (cosA, sinA) and B = (cosB, sinB)

Now, (cosA/sinA){(cosA − k)/(sinA − h)} = -1

⇨ $\cos^2 A$ − kcosA = $-\sin^2 A$ + hsinA

- ⇨ hsinA + kcosA = 1
- ⇨ htanA + k = secA
- ⇨ $(htanA + k)^2 = sec^2A$
- ⇨ $h^2tan^2A + 2hktanA + k^2 = 1 + tan^2A$
- ⇨ $tan^2A(h^2 – 1) + 2hktanA + (k^2 – 1) = 0$
- ⇨ $tanA + tanB = -2hk/(h^2 – 1)$ and $tanAtanB = (k^2 – 1)/(h^2 – 1)$

Now, tan60 = (tanA – tanB)/(1 + tanAtanB)

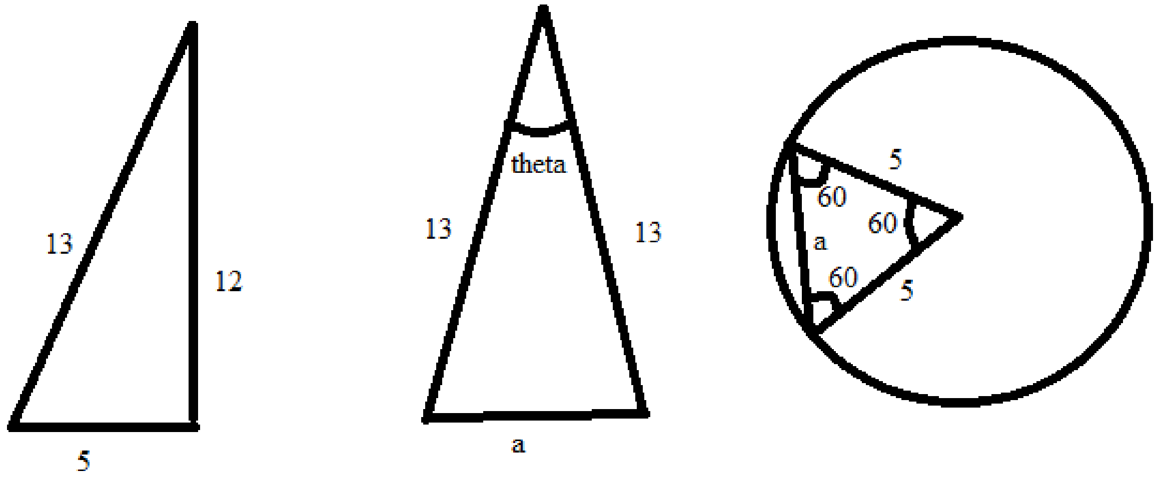
- ⇨ $3 = \{(tanA + tanB)^2 – 4tanAtanB\}/(1 + tanAtanB)^2$
- ⇨ $3\{1 + (k^2 – 1)/(h^2 – 1)\}^2 = \{4h^2k^2/(h^2 – 1)^2 – 4(k^2 – 1)/(h^2 – 1)\}$
- ⇨ $3(h^1 + k^2 – 2)^2 = 4\{h^2k^2 – (h^2 – 1)(k^2 – 1)\}$
- ⇨ $3(h^2 + k^2 – 2)^2 = 4(h^2k^2 – h^2k^2 + h^2 + k^2 – 1\}$
- ⇨ $3(h^2 + k^2 – 2)^2 = 4(h^2 + k^2 – 1)$
- ⇨ $3(h^2 + k^2)^2 – 12(h^2 + k^2) + 12 = 4(h^2 + k^2) – 4$
- ⇨ $3(h^2 + k^2)^2 – 16(h^2 + k^2) + 16 = 0$
- ⇨ $(h^2 + k^2) = \{16 \pm \sqrt{(256 – 4*3*16)}\}/6 = (16 \pm 8)/6 = 4, 4/3$
- ⇨ $h^2 + k^2 = 4/3$
- ⇨ Locus is $x^2 + y^2 = (2/\sqrt{3})^2$

Option (a) is correct.

600. A ring of 10 cm in diameter is suspended from a point 12 cm vertically above the centre by six equal strings. The strings are attached to the circumference of the ring at equal intervals, thus keeping the ring in a horizontal plane. The cosine of the angle between two adjacent strings is

(a) $2/\sqrt{13}$
(b) 313/338
(c) $5/\sqrt{26}$
(d) $5\sqrt{651}/338$

Solution :

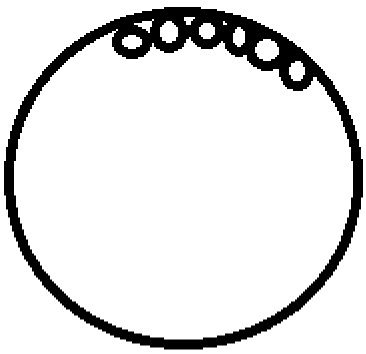


Now, from third figure, a = 5

From second figure, $\cos\theta = (13^2 + 13^2 - 5^2)/2*13*13 = 313/338$

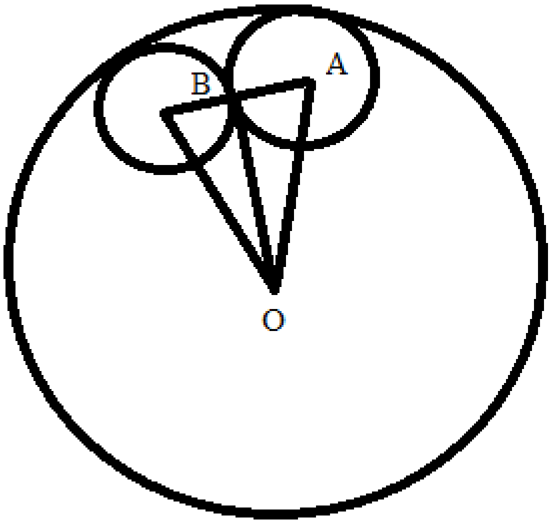
Option (b) is correct.

601. If, inside a big circle, exactly n (n ≥ 3) small circles, each of radius r, can be drawn in such a way that each small circle touches the big circle and also touches both its adjacent circles (as shown in the picture), then the radius of the big circle is



(a) $r\operatorname{cosec}(\pi/n)$
(b) $r(1 + \operatorname{cosec}(2\pi/n))$
(c) $r(1 + \operatorname{cosec}(\pi/2n))$
(d) $r(1 + \operatorname{cosec}(\pi/n))$

Solution :



Angle AOB = $(2\pi/n)/2 = (\pi/n)$

Now, $\sin(\pi/n) = AB/OA$

- ⇨ $OA = AB\text{cosec}(\pi/n) = r\text{cosec}(\pi/n)$
- ⇨ Radius = $r + r\text{cosec}(\pi/n) = r\{1 + \text{cosec}(\pi/n)\}$

Option (d) is correct.

602. The range of values taken by $4\cos^3 A - 3\cos A$ is
   (a) all negative values
   (b) all positive and negative values between -4/3 and +4/3
   (c) all positive and negative values between -1 and +1
   (d) all positive values

Solution :

$4\cos^3 A - 3\cos A = \cos 3A$

Option (c) is correct.

603. If $-\pi/4 < \theta < \pi/4$ then $\cos\theta - \sin\theta$ is
   (a) always negative
   (b) sometimes zero
   (c) always positive
   (d) sometimes positive, sometimes negative

Solution :

$\cos\theta - \sin\theta = \sqrt{2}\{(1/\sqrt{2})\cos\theta - (1/\sqrt{2})\sin\theta)\} = \sqrt{2}\{\sin(\pi/4)\cos\theta - \cos(\pi/4)\sin\theta\} = \sqrt{2}\sin(\pi/4 - \theta) > 0$ as $(\pi/4 - \theta) > 0$

Option (c) is correct.

604. For all angles A $\sin 2A\cos A/(1 + \cos 2A)(1 + \cos A)$ equals
(a) $\sin A/2$
(b) $\cos A/2$
(c) $\tan A/2$
(d) $\sin A$

Solution :

$\sin 2A\cos A/(1 + \cos 2A)(1 + \cos A)$

$= 2\sin A\cos^2 A/\{(2\cos^2 A)(2\cos^2 A/2)\}$

$= 2\sin(A/2)\cos(A/2)/2\cos^2(A/2)$

$= \tan A/2$

Option (c) is correct.

605. If the angle θ with $0 < \theta < \pi/2$ is measured in radians, then $\cos\theta$ always lies between
(a) 0 and $1 - \theta^2/2$
(b) $1 - \theta^2/2 + \theta$ and 1
(c) $1 - \theta^2/3$ and 1
(d) $1 - \theta^2/2$ and 1

Solution :

$\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \ldots\ldots$

For small values of θ neglecting the higher power terms we get, $\cos\theta = 1 - \theta^2/2$

So, $\cos\theta$ always lies between $1 - \theta^2/2$ and 1.

Option (d) is correct.

606. All possible values of x in $[0, 2\pi]$ satisfying the inequality $\sin 2x < \sin x$, are given by
(a) $\pi/3 < x < 5\pi/3$

(b) $\pi/3 < x < 2\pi/3$ and $4\pi/3 < x < 5\pi/3$
(c) $\pi/3 < x < \pi$ and $4\pi/3 < x < 2\pi$
(d) $\pi/3 < x < \pi$ and $5\pi/3 < x < 2\pi$

Solution :

$\sin 2x - \sin x < 0$

⇨ $2\cos(3x/2)\sin(x/2) < 0$

Now, $\sin(x/2) > 0$ (always)

Therefore, we need to find the values of x for which $\cos(3x/2) < 0$

⇨ $\pi/2 < 3x/2 < 3\pi/2$ and $5\pi/2 < 3x/2 < 3\pi$
⇨ $\pi/3 < x < \pi$ and $5\pi/3 < x < 2\pi$

Option (d) is correct.

607. If $0 \le \alpha \le \pi/2$, then which of the following is true?
(a) $\sin(\cos\alpha) < \cos(\sin\alpha)$
(b) $\sin(\cos\alpha) \le \cos(\sin\alpha)$ and equality holds for some $\alpha \in [0, \pi/2]$
(c) $\sin(\cos\alpha) > \cos(\sin\alpha)$
(d) $\sin(\cos\alpha) \ge \cos(\sin\alpha)$ and equality holds for some $\alpha \in [0, \pi/2]$

Solution :

Clearly, option (a) is correct. Because equality will never hold. To hold the equality $\cos\alpha = \pi/4$ and $\sin\alpha = \pi/4$ and $\cos^2\alpha + \sin^2\alpha \ne 1$. Here is the contradiction.

608. The value of $\cos^4(\pi/8) + \cos^4(3\pi/8) + \cos^4(5\pi/8) + \cos^4(7\pi/8)$ is
(a) ¾
(b) $1/\sqrt{2}$
(c) $3/2$
(d) $\sqrt{3}/2$

Solution :

Now, $\cos^4(\pi/8) + \cos^4(3\pi/8) + \cos^4(5\pi/8) + \cos^4(7\pi/8) = 2\{\cos^4(\pi/8) + \cos^4(3\pi/8)\} = 2\{\cos^4(\pi/8) + \sin^4(\pi/8)\}$ (as $\pi/2 - \pi/8 = 3\pi/8$) $= 2\{1 - 2\cos^2(\pi/8)\sin^2(\pi/8)\} = 2\{1 - (1/2)\sin^2(\pi/4)\} = 2\{1 - ¼) = 2(3/4) = 3/2$

Option (c) is correct.

609. The expression $\tan\theta + 2\tan(2\theta) + 2^2\tan(2^2\theta) + \ldots\ldots + 2^{14}\tan(2^{14}\theta) + 2^{15}\cot(2^{15}\theta)$ is equal to
(a) $2^{16}\tan(2^{16}\theta)$
(b) $\tan\theta$
(c) $\cot\theta$
(d) $2^{16}[\tan(2^{16}\theta) + \cot(2^{16}\theta)]$

Solution :

Now, $2^{14}\tan(2^{14}\theta) + 2^{15}\cot(2^{15}\theta)$

$= 2^{14}\{\tan(2^{14}\theta) + 2/\tan(2^{15}\theta)\}$

$= 2^{14}[\tan(2^{14}\theta) + \{1 - \tan^2(2^{14}\theta)\}/\tan(2^{14}\theta)]$ (writing $\tan 2\theta = 2\tan\theta/(1 - \tan^2\theta)$

$= 2^{14}[1/\tan(2^{14}\theta)]$

$= 2^{14}\cot(2^{14}\theta)$

So, again $2^{13}\tan(2^{13}\theta) + 2^{14}\cot(2^{14}\theta) = 2^{13c}\cot(2^{13}\theta)$

..

..

The expression becomes, $\tan\theta + 2\cot 2\theta = \tan\theta + 2/\tan 2\theta = \tan\theta + (1 - \tan^2\theta)/\tan\theta = 1/\tan\theta = \cot\theta$

Option (c) is correct.

610. If α and β are two different solutions, lying between $-\pi/2$ and $+\pi/2$, of the equation $2\tan\theta + \sec\theta = 2$, then $\tan\alpha + \tan\beta$ is
(a) 0
(b) 1
(c) 4/3
(d) 8/3

Solution :

Now, $\sec\theta = 2(1 - \tan\theta)$

⇨ $\sec^2\theta = 2(1 - \tan\theta)^2$
⇨ $1 + \tan^2\theta = 4(1 - 2\tan\theta + \tan^2\theta)$

- ⇨ $3\tan^2\theta - 8\tan\theta + 3 = 0$
- ⇨ $\tan\alpha + \tan\beta = -(-8/3) = 8/3$ (sum of roots $= -b/a$)

Option (d) is correct.

611. Given that $\tan\theta = b/a$, the value of $a\cos2\theta + b\sin2\theta$ is
   (a) $a^2(1 - b^2/a^2) + 2b^2$
   (b) $(a^2 + b^2)/a$
   (c) a
   (d) $(a^2 + b^2)/a^2$

Solution :

$a\cos2\theta + b\sin2\theta$

$= a(1 - \tan^2\theta)/(1 + \tan^2\theta) + b2\tan\theta/(1 + \tan^2\theta)$

$= a(1 - b^2/a^2)/(1+ b^2/a^2) + b*2(b/a)/(1 + b^2/a^2)$

$= a(a^2 - b^2)/(a^2 + b^2) + 2b^2a/(a^2 + b^2)$

$= \{a/(a^2 + b^2)\}(a^2 - b^2 + 2b^2)$

$= \{a/(a^2 + b^2)\}(a^2 + b^2)$

$= a$

Option (c) is correct.

612. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the value of $\cos(\theta - \pi/4)$ is
   (a) ½
   (b) $\pm1/2\sqrt{2}$
   (c) $-1/2\sqrt{2}$
   (d) $1/2\sqrt{2}$

Solution :

See solution of problem 595.

Option (b) is correct.

613. If $\tan x = 2/5$, then $\sin2x$ equals
   (a) 20/29
   (b) $\pm10/\sqrt{29}$

(c) -20/29
(d) None of the foregoing numbers

Solution :

$\sin 2x = 2\tan x/(1 + \tan^2 x) = 2(2/5)/\{1 + (2/5)^2\} = 4*5/(5^2 + 2^2) = 20/29$

Option (a) is correct.

614. If x = tan15, then
(a) $x^2 + 2\sqrt{3}x - 1 = 0$
(b) $x^2 + 2\sqrt{3}x + 1 = 0$
(c) $x = 1/2\sqrt{3}$
(d) $x = 2/\sqrt{3}$

Solution :

Now, $\tan 30 = 2\tan 15/(1 - \tan^2 15)$

⇨ $1 - \tan^2 15 = 2\sqrt{3}\tan 15$
⇨ $\tan^2 15 + 2\sqrt{3}\tan 15 - 1 = 0$

Option (a) is correct.

615. The value of $2\sin(\theta/2)\cos(3\theta/2) + 4\sin\theta\sin^2(\theta/2)$ equals
(a) $\sin(\theta/2)$
(b) $\sin(\theta/2)\cos\theta$
(c) $\sin\theta$
(d) $\cos\theta$

Solution :

$2\sin(\theta/2)\cos(3\theta/2) + 4\sin\theta\sin^2(\theta/2)$

$= \sin 2\theta - \sin\theta + 2\sin\theta(1 - \cos\theta)$

$= \sin 2\theta - \sin\theta + 2\sin\theta - 2\sin\theta\cos\theta$

$= \sin 2\theta + \sin\theta - \sin 2\theta$

$= \sin\theta$

Option (c) is correct.

616. If a and b are given positive numbers, then the values of c and θ with $0 \le \theta \le \pi$ for which asinx + bcosx = csin(x + θ) is true for all x are given by
(a) $c = \sqrt{(a^2 + b^2)}$ and tanθ = a/b
(b) $c = -\sqrt{(a^2 + b^2)}$ and tanθ = b/a
(c)$c = a^2 + b^2$ and tanθ = b/a
(d) $c = \sqrt{(a^2 + b^2)}$ and tanθ = b/a

Solution :

Now, asinx + bcosx = $[\sqrt{(a^2 + b^2)}][\{a/\sqrt{(a^2 + b^2)}\}\sin x + \{b/\sqrt{(a^2 + b^2)}\}\cos x]$ = $[\sqrt{(a^2 + b^2)}][\cos\theta \sin x + \sin\theta \cos x]$ (where $\cos\theta = a/\sqrt{(a^2 + b^2)}$ and $\sin\theta = b/\sqrt{(a^2 + b^2)}$ i.e. tanθ = b/a)

$= \sqrt{(a^2 + b^2)}\sin(x + \theta)$

⇨ $c = \sqrt{(a^2 + b^2)}$

Option (d) is correct.

617. The value of $\sin 330 + \tan 45 - 4\sin^2 120 + 2\cos^2 135 + \sec^2 180$ is
(a) ½
(b) √3/2
(c)-√3/2
(d) -1/2

Solution :

$\sin 330 + \tan 45 - 4\sin^2 120 + 2\cos^2 135 + \sec^2 180$

= -sin30 + 1 – 4(3/4) + 2(1/2) + 1

= -1/2 + 1 – 3 + 1 + 1

= -½

Option (d) is correct.

618. Given that sin(π/4) = cos(π/4) = 1/√2, then the value of tan(5π/8) is
(a) −(√2 + 1)
(b) -1/(√2 + 1)

(c) $1 - \sqrt{2}$
(d) $1/(\sqrt{2} - 1)$

Solution :

$\tan(5\pi/8) = \tan(\pi/2 + \pi/8) = -\cot(\pi/8) = -\cos(\pi/8)/\sin(\pi/8) = -2\cos^2(\pi/8)/\{2\sin(\pi/8)\cos(\pi/8)\} = -(1 + \cos(\pi/4))/\sin(\pi/4) = -(1 + 1/\sqrt{2})/(1/\sqrt{2}) = -(\sqrt{2} + 1)$

Option (a) is correct.

619. $\sin^6(\pi/49) + \cos^6(\pi/49) - 1 + 3\sin^2(\pi/49)\cos^2(\pi/49)$ equals
(a) 0
(b) $\tan^6(\pi/49)$
(c) ½
(d) None of the foregoing numbers

Solution :

Now, $\sin^6(\pi/49) + \cos^6(\pi/49) - 1 + 3\sin^2(\pi/49)\cos^2(\pi/49)$

$= \{\sin^2(\pi/49)\}^3 + \{\cos^2(\pi/49)\}^3 + 3\sin^2(\pi/49)\cos^2(\pi/49)\{\sin^2(\pi/49) + \cos^2(\pi/49)\} - 1$

$= \{\sin^2(\pi/49) + \cos^2(\pi/49)\}^3 - 1$

$= 1^3 - 1 = 0$

Option (a) is correct.

620. If $a\sin\theta = b\cos\theta$, then the value of $\sqrt{\{(a - b)/(a + b)\}} + \sqrt{\{(a + b)/(a - b)\}}$ equals
(a) $2\cos\theta$
(b) $2\cos\theta/\sqrt{\cos 2\theta}$
(c) $2\sin\theta/\sqrt{\cos 2\theta}$
(d) $2/\sqrt{\cos 2\theta}$

Solution :

$a\sin\theta = b\cos\theta$

⇨ $\tan\theta = b/a$

Now, $\sqrt{\{(a - b)/(a + b)} + \sqrt{\{(a + b)/(a - b)\}}$

$= \sqrt{(1 - b/a)/(1 + b/a)} + \sqrt{(1 + b/a)/(1 - b/a)}$

$= \sqrt{(1 - \tan\theta)/(1 + \tan\theta)} + \sqrt{(1 + \tan\theta)/(1 - \tan\theta)}$

$= (1 - \tan\theta + 1 + \tan\theta)/\sqrt{(1 - \tan\theta)(1 + \tan\theta)}$

$= 2/\sqrt{(1 - \tan^2\theta)}$

$= 2\cos\theta/\sqrt{\cos 2\theta}$

Option (b) is correct.

621. The sides of a triangle are given to be $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Then the largest of the three angles of the triangle is
(a) 75
(b) $\{x/(1 + x)\}\pi$
(c) 120
(d) 135

Solution :

$\cos A = \{(x^2 - 1)^2 + (2x + 1)^2 - (x^2 + x + 1)^2\}/\{2(x^2 - 1)(2x + 1)\}$

$= \{x^4 - 2x^2 + 1 + 4x^2 + 4x + 1 - x^4 - x^2 - 1 - 2x^3 - 2x^2 - 2x\}/2\{(x^2 - 1)(2x + 1)\}$

$= -(2x^3 + x^2 - 2x - 1)/2\{(x^2 - 1)(2x + 1)\}$

$= -(2x + 1)(x^2 - 1)/2\{(x^2 - 1)(2x + 1)\}$

$= -1/2$

⇨ $A = 120$

Option (c) is correct.

622. If A, B, C are angles of a triangle, then the value of $1 - \{\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2)\}$ equals
(a) $2\sin A\sin B\sin C$
(b) $2\sin(A/2)\sin(B/2)\sin(C/2)$
(c) $4\sin(A/2)\sin(B/2)\sin(C/2)$
(d) $4\sin A\sin B\sin C$

Solution :

$1 - \{\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2)\}$

$= 1 - (1/2)\{1 - \cos A + 1 - \cos B + 1 - \cos C\}$

$= -1/2 + (1/2)[2\cos\{(A + B)/2\}\cos\{(A - B)/2\} + \cos C]$

$= -1/2 + (1/2)[2\sin(C/2)\cos\{(A - B)/2\} + 1 - 2\sin^2(C/2)]$

$= \{\sin(C/2)\}[\cos\{(A - B)/2\} - \cos\{(A + B)/2\}]$

$= \{\sin(C/2)\}*2\sin(A/2)\sin(B/2)$

$= 2\sin(A/2)\sin(B/2)\sin(C/2)$

Option (b) is correct.

623. In any triangle if $\tan(A/2) = 5/6$, $\tan(B/2) = 20/37$, and $\tan(C/2) = 2/5$, then
(a) $a + c = 2b$
(b) $a + b = 2c$
(c) $b + c = 2a$
(d) none of these holds

Solution :

$\sin A = 2\tan(A/2)/\{1 + \tan^2(A/2)\} = 2(5/6)/\{1 + 25/36\} = 60/61$

$\sin B = 2(20/37)/\{1 + 400/1369\} = 1480/1769$

$\sin C = 2(2/5)/(1 + 4/25) = 20/29$

Now, $\sin A + \sin C = (60*29 + 61*20)/(29*61) = 2960/1769 = 2(1480/1769) = 2\sin B$

⇨ $a + c = 2b$

Option (a) is correct.

624. Let $\cos(\alpha - \beta) = -1$. Then only one of the following statements is *always* true. Which one is it?
(a) α is not less than β
(b) $\sin\alpha + \sin\beta = 0$ and $\cos\alpha + \cos\beta = 0$
(c) Angles α and β are both positive
(d) $\sin\alpha + \sin\beta = 0$ but $\cos\alpha + \cos\beta$ may not be zero

Solution :

$\sin\alpha + \sin\beta = 2\sin\{(\alpha + \beta)/2\}\cos\{(\alpha - \beta)/2\} = 2\sin\{(\alpha + \beta)/2\}\sqrt{\{1 + \cos(\alpha - \beta)\}} = 0$

$\cos\alpha + \cos\beta = 2\cos\{(\alpha + \beta)/2\}\cos\{(\alpha - \beta)/2\} = 0$

Option (b) is correct.

625. If the trigonometric equation $1 + \sin^2 x\theta = \cos\theta$ has a nonzero solution in θ, then x must be
   (a) an integer
   (b) a rational number
   (c) an irrational number
   (d) strictly between 0 and 1

Solution :

Now, $1 + \sin^2 x\theta = \cos\theta$

- ⇨ $(1 - \cos\theta) + \sin^2 x\theta = 0$
- ⇨ $2\sin^2(\theta/2) + \sin^2 x\theta = 0$
- ⇨ $\sin(\theta/2) = 0$ and $\sin x\theta = 0$
- ⇨ $\theta = 2n\pi$ and $\theta = m\pi/x$
- ⇨ $m\pi/x = 2n\pi$
- ⇨ $x = m/2n$

Option (b) is correct.

626. It is given that $\tan A$ and $\tan B$ are the roots of the equation $x^2 - bx + c = 0$. Then value of $\sin^2(A + B)$ is
   (a) $b^2/\{b^2 + (1 - c)^2\}$
   (b) $b^2/(b^2 + c^2)$
   (c) $b^2/(b + c)^2$
   (d) $b^2/\{c^2 + (1 - b)^2\}$

Solution :

Now, $\tan A + \tan B = b$ and $\tan A\tan B = c$

$\sin^2(A + B) = (\sin A\cos B + \cos A\sin B)^2 = \cos^2 A\cos^2 B(\tan A + \tan B)^2 = b^2/\sec^2 A\sec^2 B = b^2/(1 + \tan^2 A)(1 + \tan^2 B) = b^2/(1 + \tan^2 A + \tan^2 B + \tan^2 A\tan^2 B) = b^2/\{1 + (\tan A + \tan B)^2 - 2\tan A\tan B + c^2\} = b^2/\{1 + b^2 - 2c + c^2) = b^2/\{b^2 + (1 - c)^2\}$

Option (a) is correct.

627. If $\cos x + \cos y + \cos z = 0$, $\sin x + \sin y + \sin z = 0$, then $\cos\{(x - y)/2\}$ is
(a) $\pm\sqrt{3}/2$
(b) $\pm 1/2$
(c) $\pm 1/\sqrt{2}$
(d) 0

Solution :

Now, $\cos x + \cos y = -\cos z$

⇨ $(\cos x + \cos y)^2 = \cos^2 z$ and $(\sin x + \sin y)^2 = \sin^2 z$

Adding we get, $\cos^2 x + \cos^2 y + 2\cos x\cos y + \sin^2 x + \sin^2 y + 2\sin x\sin y = \cos^2 z + \sin^2 z$

⇨ $1 + 1 + 2\cos(x - y) = 1$
⇨ $2\{1 + \cos(x - y)\} = 1$
⇨ $2\cos^2\{(x - y)/2\} = ½$
⇨ $\cos^2\{(x - y)/2\} = ¼$
⇨ $\cos\{(x - y)/2\} = \pm 1/2$

Option (b) is correct.

628. If x, y, z are in G.P. and $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are in A.P., then
(a) $x = y = z$ or $y = \pm 1$
(b) $z = 1/x$
(c) $x = y = z$ but their common value is not necessarily zero
(d) $x = y = z = 0$

Solution :

$\tan^{-1}x + \tan^{-1}z = 2\tan^{-1}y$

⇨ $\tan^{-1}\{(x + z)/(1 - zx)\} = \tan^{-1}\{2y/(1 - y^2)\}$
⇨ $(x + z)/(1 - y^2) = 2y/(1 - y^2)$ $(zx = y^2)$
⇨ $x + z = 2y$ or $y = \pm 1$ (if $y = \pm 1$ then both sides are undefined mean $\tan^{-1}(\text{undefined}) = \pi/2$)
⇨ $(x + z)^2 = 4y^2$
⇨ $(x + z)^2 - 4zx = 0$ $(y^2 = zx)$
⇨ $(z - x)^2 = 0$
⇨ $z = x$
⇨ $x = y$
⇨ $x = y = z$

Option (a) is correct.

629. If α and β satisfy the equation $\sin\alpha + \sin\beta = \sqrt{3}(\cos\alpha - \cos\beta)$, then
(a) $\sin 3\alpha + \sin 3\beta = 1$
(b) $\sin 3\alpha + \sin 3\beta = 0$
(c) $\sin 3\alpha - \sin 3\beta = 0$
(d) $\sin 3\alpha - \sin 3\beta = 1$

Solution :

$\sin\alpha + \sin\beta = \sqrt{3}(\cos\alpha - \cos\beta)$

- ⇨ $2\sin\{(\alpha + \beta)/2\}\cos\{(\alpha - \beta)/2\} = 2\sqrt{3}\sin\{(\alpha + \beta)/2\}\sin\{(\beta - \alpha)/2\}$
- ⇨ $\sin\{(\alpha + \beta)/2\} = 0$ or $\tan\{(\beta - \alpha)/2\} = 1/\sqrt{3}$
- ⇨ $\alpha + \beta = 0$ or $\beta - \alpha = \pi/3$

Now, $\sin 3\alpha + \sin 3\beta$

$= 2\sin\{3(\alpha + \beta)/2\}\cos\{3(\alpha - \beta)/2\}$

If $\alpha + \beta = 0$ then it is equal to 0. Also if $\beta - \alpha = \pi/3$, then $\cos\{3(\beta - \alpha)/2\} = \cos(\pi/2) = 0$ ($\cos(-x) = \cos x$)

Option (b) is correct.

630. If $\cos 2\theta = \sqrt{2}(\cos\theta - \sin\theta)$, then $\tan\theta$ is
(a) $1/\sqrt{2}$, $-1/\sqrt{2}$ or 1
(b) 1
(c) 1 or -1
(d) None of the foregoing values

Solution :

Now, $\cos 2\theta = \sqrt{2}(\cos\theta - \sin\theta)$

- ⇨ $\cos^2 2\theta = 2(\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta)$
- ⇨ $1 - \sin^2 2\theta = 2(1 - \sin 2\theta)$
- ⇨ $\sin^2 2\theta - 2\sin 2\theta + 1 = 0$
- ⇨ $(\sin 2\theta - 1)^2 = 0$
- ⇨ $\sin 2\theta = 1$
- ⇨ $2\tan\theta/(1 + \tan^2\theta) = 1$
- ⇨ $\tan^2\theta - 2\tan\theta + 1 = 0$
- ⇨ $(\tan\theta - 1)^2 = 0$

⇨ $\tan\theta = 1$

Option (b) is correct.

631. The number of roots between 0 and π of the equation $2\sin^2x + 1 = 3\sin x$ equals
(a) 2
(b) 4
(c) 1
(d) 3

Solution :

Now, $2\sin^2x + 1 = 3\sin x$

⇨ $2\sin^2x - 3\sin x + 1 = 0$
⇨ $(2\sin x - 1)(\sin x - 1) = 0$
⇨ $\sin x = ½$ or $\sin x = 1$
⇨ $x = \pi/6, \pi - \pi/6, x = \pi/2$

Option (d) is correct.

632. The equation in θ given by $\mathrm{cosec}^2\theta - (2\sqrt{3}/3)\mathrm{cosec}\theta\sec\theta - \sec^2\theta = 0$ has solutions
(a) only in the first and third quadrants
(b) only in the second and fourth quadrants
(c) only in the third quadrant
(d) in all the four quadrants

Solution :

Now, $\mathrm{cosec}^2\theta - (2\sqrt{3}/3)\mathrm{cosec}\theta\sec\theta - \sec^2\theta = 0$

⇨ $3\mathrm{cosec}^2\theta/\sec^2\theta - 2\sqrt{3}\mathrm{cosec}\theta/\sec\theta - 3 = 0$
⇨ $3\cot^2\theta - 2\sqrt{3}\cot\theta - 3 = 0$
⇨ $\cot\theta = \{2\sqrt{3} \pm \sqrt{(12 + 36)}\}/6 = \sqrt{3}, -1/\sqrt{3}$

Option (d) is correct.

633. If $\tan\theta + \cot\theta = 4$, then θ, for some integer n, is
(a) $n\pi/2 + (-1)^n(\pi/12)$
(b) $n\pi + (-1)^n(\pi/12)$
(c) $n\pi + \pi/12$

(d) $n\pi - \pi/12$

Solution :

Now, $\tan\theta + \cot\theta = 4$

- ⇨ $(\sin^2\theta + \cos^2\theta)/(\sin\theta\cos\theta) = 4$
- ⇨ $1/(2\sin\theta\cos\theta) = 2$
- ⇨ $\sin2\theta = ½$
- ⇨ $\sin2\theta = \sin(\pi/6)$
- ⇨ $2\theta = n\pi + (-1)^n(\pi/6)$
- ⇨ $\theta = n\theta/2 + (-1)^n(\pi/12)$

Option (a) is correct.

634. The equation $\sin x(\sin x + \cos x) = k$ has real solutions if and only if k is a real number such that

(a) $0 \le k \le (1 + \sqrt{2})/2$
(b) $2 - \sqrt{3} \le k \le 2 + \sqrt{3}$
(c) $0 \le k \le 2 - \sqrt{3}$
(d) $(1 - \sqrt{2})/2 \le k \le (1 + \sqrt{2})/2$

Solution :

Now, $\sin x(\sin x + \cos x) = k$

- ⇨ $2\sin^2x + 2\sin x \text{coxs} = 2k$
- ⇨ $1 - \cos2x + \sin2x = 2k$
- ⇨ $\sin2x - \cos2x = 2k - 1$
- ⇨ $\sin^22x + \cos^22x - 2\sin2x\cos2x = (2k - 1)^2$
- ⇨ $1 - \sin4x = 4k^2 - 4k + 1$
- ⇨ $\sin4x = 4k - 4k^2$

  Now, $\sin4x \le 1$
- ⇨ $4k - 4k^2 \le 1$
- ⇨ $4k^2 - 4k + 1 \ge 0$
- ⇨ $(2k - 1)^2 \ge 0$ which is obvious

Now, $-1 \le \sin4x$

- ⇨ $-1 \le 4k - 4k^2$
- ⇨ $4k^2 - 4k - 1 \le 0$
- ⇨ $(2k - 1)^2 \le 2$
- ⇨ $|2k - 1| \le \sqrt{2}$
- ⇨ $-\sqrt{2} \le 2k - 1 \le \sqrt{2}$
- ⇨ $(1 - \sqrt{2})/2 \le k \le (1 + \sqrt{2})/2$

Option (d) is correct.

635. The number of solutions of the equation $2\sin\theta + 3\cos\theta = 4$ for $0 \le \theta \le 2\pi$ is
- (a) 0
- (b) 1
- (c) 2
- (d) More than 2

Solution :

$2\sin\theta + 3\cos\theta = 4$

- ⇨ $2\tan\theta + 3 = 4\sec\theta$
- ⇨ $(2\tan\theta + 3)^2 = 16\sec^2\theta$
- ⇨ $4\tan^2\theta + 12\tan\theta + 9 = 16 + 16\tan^2\theta$
- ⇨ $12\tan^2\theta - 12\tan\theta + 7 = 0$

Now, discriminate = 144 – 4*12*7 < 0

- ⇨ No real solution.

Option (a) is correct.

636. The number of values of x satisfying the equation $\sqrt{\sin x} - 1/\sqrt{\sin x} = \cos x$ is
- (a) 1
- (b) 2
- (c) 3
- (d) More than 3

Solution :

Now, $\sqrt{\sin x} - 1/\sqrt{\sin x} = \cos x$

- ⇨ $\sin x - 1 = \cos x\sqrt{\sin x}$
- ⇨ $\sin^2x - 2\sin x + 1 = \cos^2x\sin x$
- ⇨ $\sin^2x - 2\sin x + 1 = \sin x - \sin^3x$
- ⇨ $\sin^3x + \sin^2x - 3\sin x + 1 = 0$
- ⇨ $(\sin x - 1)(\sin^2x + 2\sin x - 1) = 0$
- ⇨ $\sin x = 1$ or $\sin^2x + 2\sin x - 1 = 0$
- ⇨ $\sin x = \{-2 \pm \sqrt{(4 + 4)}\}/2 = -1 \pm \sqrt{2}$
- ⇨ $\sin x = \sqrt{2} - 1$, $\sin x \ne -1 - \sqrt{2}$ as $\sin x > -1$
- ⇨ 2 values in $0 < x \le \pi/2$

Option (d) is correct. (as there is no boundary for x specified)

637. The number of times the function f(x) = |minimum{sinx, cosx}| takes the value 0.8 between 20π/3 and 43π/6 is
   (a) 2
   (b) More than 2
   (c) 0
   (d) 1

Solution :

It never can happen because if sinx > 0.5 then cosx < 0.5 or if cosx > 0.5, then sinx < 0.5

Option (a) is correct.

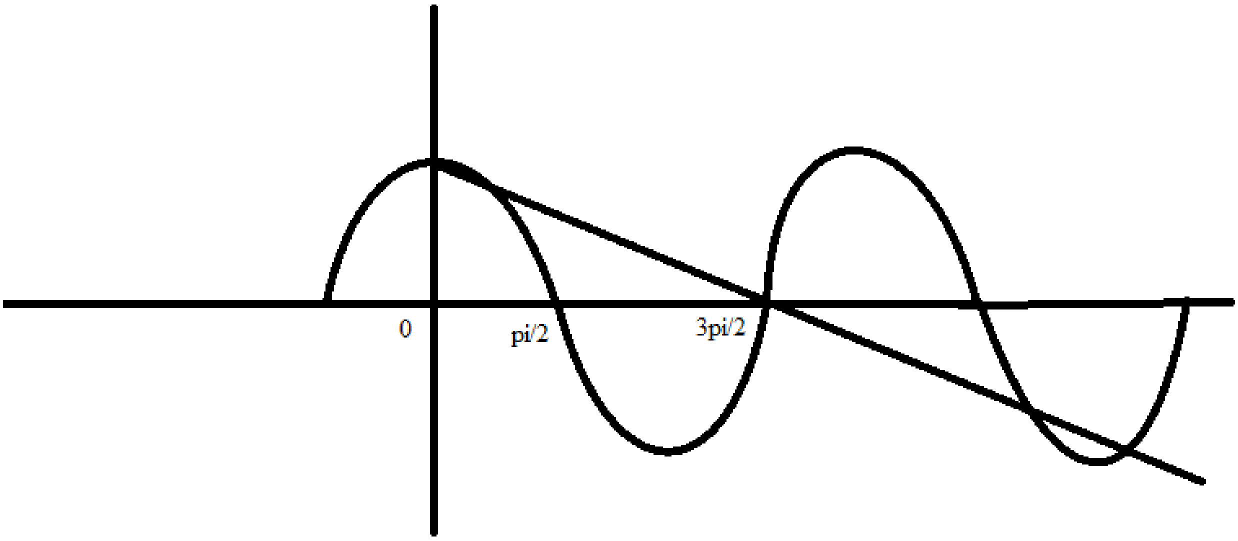
638. The number of roots of the equation 2x = 3π(1 – cosx), where x is measured in radians, is
   (a) 3
   (b) 5
   (c) 4
   (d) 2

Solution :

Now, 2x = 3π(1 – cosx)

⇨ cosx = 1 – 2x/3π

Now, we will draw the graph of y = cosx and y = 1 – 2x/3π and see the number of intersection point. That will give number of solutions.



Option (b) is correct.

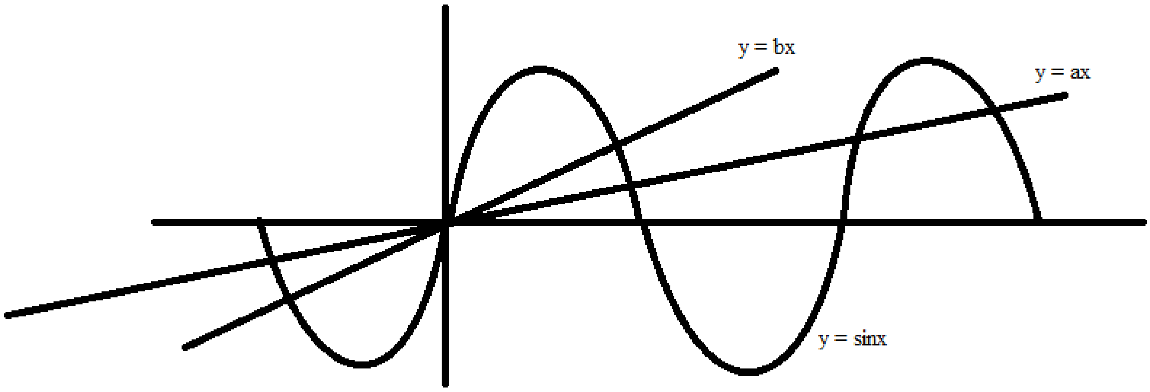
639. Let f(x) = sinx – ax and g(x) = sinx – bx, where 0 < a, b < 1. Suppose that the number of real roots of f(x) = 0 is greater than that of g(x) = 0. Then
(a) a < b
(b) a > b
(c) ab = π/6
(d) none of the foregoing relations hold

Solution :

f(x) = 0

⇨ sinx = ax

Now to see number of real roots of this equation we will draw curves of y = sinx and y = ax and see number of intersection point that will give number of solutions.



Now, this must be the scenario to have f(x) = 0 more roots than g(x) = 0. So a < b.

Option (a) is correct.

640. The number of solutions $0 < \theta < \pi/2$ of the equation $\sin 7\theta - \sin\theta = \sin 3\theta$ is
(a) 1
(b) 2
(c) 3
(d) 4

Solution :

Now, $\sin 7\theta - \sin\theta = \sin 3\theta$

- ⇨ $2\cos 4\theta \sin 3\theta - \sin 3\theta = 0$
- ⇨ $\sin 3\theta(2\cos 4\theta - 1) = 0$
- ⇨ $\sin 3\theta = 0$ or $\cos 4\theta = ½$
- ⇨ $3\theta = \pi$ or $4\theta = \pi/6, 2\pi - \pi/6$
- ⇨ $\theta = \pi/3$ or $\theta = \pi/24, \pi/2 - \pi/24$
- ⇨ 3 solutions

Option (c) is correct.

641. The number of solutions of the equation $\tan 5\theta = \cot 2\theta$ such that $0 \le \theta \le \pi/2$ is
(a) 1
(b) 2
(c) 3
(d) 4

Solution :

tan5θ = cot2θ

- ⇨ tan5θ = tan{(2n – 1)π/2 - 2θ}
- ⇨ 5θ = (2n – 1)π/2 - 2θ
- ⇨ 7θ = (2n – 1)π/2
- ⇨ θ = (2n – 1)π/14
- ⇨ θ = π/14, 3π/14, 5π/14, 7π/14
- ⇨ 4 solutions.

Option (d) is correct.

642. If $\sin^{-1}(1/\sqrt{5})$ and $\cos^{-1}(3/\sqrt{10})$ are angles in [0, π/2], then their sum is equal to
(a) π/6
(b) π/4
(c) π/3
(d) $\sin^{-1}(1/\sqrt{50})$

Solution :

Let $\sin^{-1}(1/\sqrt{5})$ = A

- ⇨ sinA = 1/√5
- ⇨ cosA = 2/√5

Let, $\cos^{-1}(3/\sqrt{10})$ = B

- ⇨ cosB = 3/√10
- ⇨ sinB = 1/√10

Now, sin(A + B) = sinAcosB + cosAsinB = (1/√5)(3/√10) + (2/√5)(1/√10) = 5/√50 = 1/√2

- ⇨ A + B = π/4

Option (b) is correct.

643. If $\cot(\sin^{-1}\sqrt{(13/17)}) = \sin(\tan^{-1}\alpha)$, then α is
(a) 4/17
(b) $\sqrt{\{(17^2 - 13^2)/(17*13)\}}$
(c) $\sqrt{\{(17^2 - 13^2)/(17^2 + 13^2)\}}$
(d) 2/3

Solution :

Now, $\cot(\sin^{-1}\sqrt{(13/17)}) = \sin(\tan^{-1}\alpha)$

- ⇨ $\cot(\cot^{-1}2/\sqrt{13}) = \sin[\sin^{-1}\{\alpha/\sqrt{(1 + \alpha^2)}\}]$
- ⇨ $2/\sqrt{13} = \alpha/\sqrt{(1 + \alpha^2)}$
- ⇨ $4/13 = \alpha^2/(1 + \alpha^2)$
- ⇨ $1 - 4/13 = 1 - \alpha^2/(1 + \alpha^2)$
- ⇨ $9/13 = 1/(1 + \alpha^2)$
- ⇨ $1 + \alpha^2 = 13/9$
- ⇨ $\alpha^2 = 4/9$
- ⇨ $\alpha = 2/3$

Option (d) is correct.

644. The minimum value of $\sin2\theta - \theta$ for $-\pi/2 \le \theta \le \pi/2$ is
  (a) $-\sqrt{3}/2 + \pi/6$
  (b) $-\pi$
  (c) $\sqrt{3}/2 - \pi/6$
  (d) $-\pi/2$

Solution :

Let $f(\theta) = \sin2\theta - \theta$

- ⇨ $f'(\theta) = 2\cos2\theta - 1 = 0$
- ⇨ $\cos2\theta = ½$
- ⇨ $2\theta = \pi/3, -\pi/3$
- ⇨ $\theta = \pi/6, -\pi/6$
- ⇨ $f''(\theta) = -4\sin2\theta > 0$ for $\theta = -\pi/6$

Minimum value of $f(\theta) = f(-\pi/6) = -\sin(\pi/3) + \pi/6 = -\sqrt{3}/2 + \pi/6$

Option (a) is correct.

645. The number of solutions θ in the range $-\pi/2 < \theta < \pi/2$ and satisfying the equation $\sin^3\theta + \sin^2\theta + \sin\theta - \sin\theta\sin2\theta - \sin2\theta - 2\cos\theta = 0$ is
  (a) 0
  (b) 1
  (c) 2
  (d) 3

Solution :

Now, $\sin^3\theta + \sin^2\theta + \sin\theta - \sin\theta\sin2\theta - \sin2\theta - 2\cos\theta = 0$

- ⇨ $\sin\theta(\sin^2\theta + \sin\theta + 1) - 2\cos\theta(\sin^2\theta + \sin\theta + 1) = 0$
- ⇨ $(\sin^2\theta + \sin\theta + 1)(\sin\theta - 2\cos\theta) = 0$
- ⇨ $\sin\theta - 2\cos\theta = 0$ (as $\sin^2\theta + \sin\theta + 1 = 0$ has imaginary roots)
- ⇨ $\tan\theta = 2$
- ⇨ 1 solution.

Option (b) is correct.

646. The number of roots of the equation $\cos^8\theta - \sin^8\theta = 1$ in the interval $[0, 2\pi]$ is
(a) 4
(b) 8
(c) 3
(d) 6

Solution :

Now, $\cos^8\theta - \sin^8\theta = 1$

- ⇨ $(\cos^4\theta - \sin^4\theta)(\sin^4\theta + \cos^4\theta) = 1$
- ⇨ $(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)\{(\cos^2\theta + \sin^2\theta)^2 - 2\cos^2\theta\sin^2\theta) = 1$
- ⇨ $\cos2\theta(1 - \sin^2 2\theta/2) = 1$
- ⇨ $\cos2\theta(1 + \cos^2 2\theta) = 2$
- ⇨ $\cos^3 2\theta + \cos2\theta - 2 = 0$
- ⇨ $(\cos2\theta - 1)(\cos^2 2\theta + \cos2\theta + 2) = 0$
- ⇨ $\cos2\theta = 1$ (as $\cos^2 2\theta + \cos2\theta + 2 = 0$ gives imaginary root)
- ⇨ $2\theta = 0, 2\pi, 4\pi$
- ⇨ $\theta = 0, \pi, 2\pi$
- ⇨ 3 solutions.

Option (c) is correct.

647. If $\sin6\theta = \sin4\theta - \sin2\theta$, then θ must be, for some integer n, equal to
(a) $n\pi/4$
(b) $n\pi \pm \pi/6$
(c) $n\pi/4$ or $n\pi \pm \pi/6$
(d) $n\pi/2$

Solution :

Now, sin6θ = sin4θ − sin2θ

- ⇨ sin6θ + sin2θ = sin4θ
- ⇨ 2sin4θcos2θ − sin4θ = 0
- ⇨ sin4θ(2cos2θ − 1) = 0
- ⇨ sin4θ = 0 or cos2θ = ½
- ⇨ 4θ = nπ or 2θ = 2nπ ± π/3
- ⇨ θ = nπ/4 or θ = nπ ± π/6

Option (c) is correct.

648. Consider the solutions of the equation $\sqrt{2}\tan^2 x - \sqrt{10}\tan x + \sqrt{2} = 0$ in the range $0 \le x \le \pi/2$. Then only one of the following statements is true. Which one is it?
   (a) No solutions for x exist in the given range
   (b) Two solutions x_1 and x_2 exist with $x_1 + x_2 = \pi/4$
   (c) Two solutions x_1 and x_2 exist with $x_1 - x_2 = \pi/4$
   (d) Two solutions x_1 and x_2 exist with $x_1 + x_2 = \pi/2$

Solution :

Now, $\tan x = \{\sqrt{10} \pm \sqrt{(10 - 8)}\}/2\sqrt{2} = (\sqrt{5} \pm 1)/2$

- ⇨ Two solutions exist.

Now, $\tan x_1 + \tan x_2 = \sqrt{5}$ and $\tan x_1 \tan x_2 = 1$

Now, $\tan(x_1 + x_2) = (\tan x_1 + \tan x_2)/(1 - \tan x_1 \tan x_2) = \sqrt{5}/(1 - 1)$

- ⇨ $x_1 + x_2 = \pi/2$

Option (d) is correct.

649. The set of all values of θ which satisfy the equation cos2θ = sinθ + cosθ is
   (a) θ = 0
   (b) θ = nπ + π/2, where n is any integer
   (c) θ = 2nπ or θ = 2nπ − π/2 or θ = nπ − π/4, where n is any integer
   (d) θ = 2nπ or θ = nπ + π/4, where n is any integer

Solution :

cos2θ = sinθ + cosθ

Clearly the values of option (c) satisfies the equation.

Therefore, option (c) is correct.

650. The equation $2x = (2n + 1)\pi(1 - \cos x)$, where n is a positive integer, has
 (a) infinitely many real solutions
 (b) exactly $2n + 1$ real roots
 (c) exactly one real root
 (d) exactly $2n + 3$ real roots

Solution :

If we take $x = (2n + 1)\pi$ then the equation gets satisfied where n is any positive integer.

So, it should have infinitely many real solutions.

But option (d) is given as correct.

651. The number of roots of the equation $\sin 2x + 2\sin x - \cos x - 1 = 0$ in the range $0 \le x \le 2\pi$ is
 (a) 1
 (b) 2
 (c) 3
 (d) 4

Solution :

$\sin 2x + 2\sin x - \cos x - 1 = 0$

- ⇨ $2\sin x(\cos x + 1) + (\cos x + 1) = 0$
- ⇨ $(\cos x + 1)(2\sin x + 1) = 0$
- ⇨ $\cos x = -1$ or $\sin x = -1/2$
- ⇨ $x = \pi$ or $x = \pi + \pi/6,\ 2\pi - \pi/6$
- ⇨ 3 solutions

Option (c) is correct.

652. If $2\sec 2\alpha = \tan\beta + \cot\beta$, then one possible value of $\alpha + \beta$ is
 (a) $\pi/2$
 (b) $\pi/4$
 (c) $\pi/3$

(d) 0

Solution :

$2\sec 2\alpha = \tan\beta + \cot\beta$

- ⇨ $2\sec 2\alpha = \tan\beta + 1/\tan\beta = (1 + \tan^2\beta)/\tan\beta$
- ⇨ $\sec 2\alpha = 1/\{2\tan\beta/(1 + \tan^2\beta)\}$
- ⇨ $\sec 2\alpha = 1/\sin 2\beta$
- ⇨ $\sin 2\beta = \cos 2\alpha$
- ⇨ $\sin 2\beta - \sin(\pi/2 - 2\alpha) = 0$
- ⇨ $2\cos(\beta - \alpha + \pi/4)\sin(\alpha + \beta - \pi/4) = 0$
- ⇨ $\alpha + \beta = \pi/4$

Option (b) is correct.

653. The equation $[3\sin^4\theta - 2\cos^6\theta + y - 2\sin^6\theta + 3\cos^4\theta]^2 = 9$ is true
(a) for any value of θ and $y = 2$ or -4
(b) only for $\theta = \pi/4$ or π and $y = -2$ or 4
(c) only for $\theta = \pi/2$ or π and $y = 2$ or -4
(d) only for $\theta = 0$ or $\pi/2$ and $y = 2$ or -2

Solution :

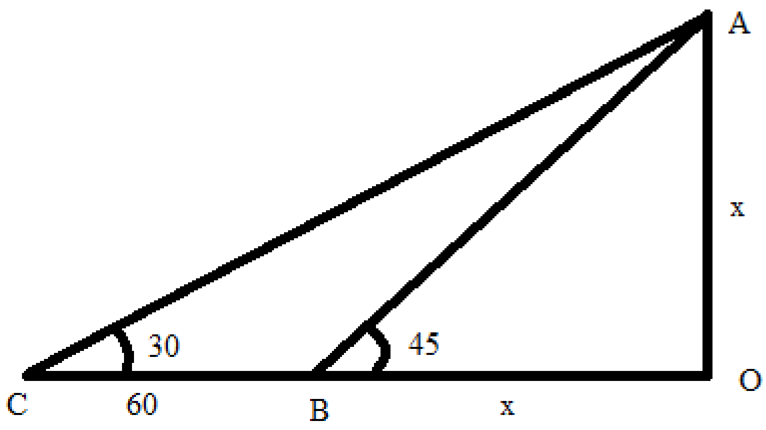
$[3\{(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta\} + y - 2\{(\cos^2\theta + \sin^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta)\}]^2 = 9$

- ⇨ $[3(1 - \sin^2 2\theta/2) + y - 2(1 - 3\sin^2 2\theta/4]^2 = 9$
- ⇨ $(1 + y)^2 = 9$
- ⇨ For any value of θ and $y = 2$ or -4

Option (a) is correct.

654. If the shadow of a tower standing on the level plane is found to be 60 feet (ft) longer when the sun's altitude is 30 than when it is 45, then the height of the tower is, in ft,
(a) $30(1 + \sqrt{3}/2)$
(b) 45
(c) $30(1 + \sqrt{3})$
(d) 30

Solution :



From triangle OAB, we get, $x/(60 + x) = \tan 30 = 1/\sqrt{3}$

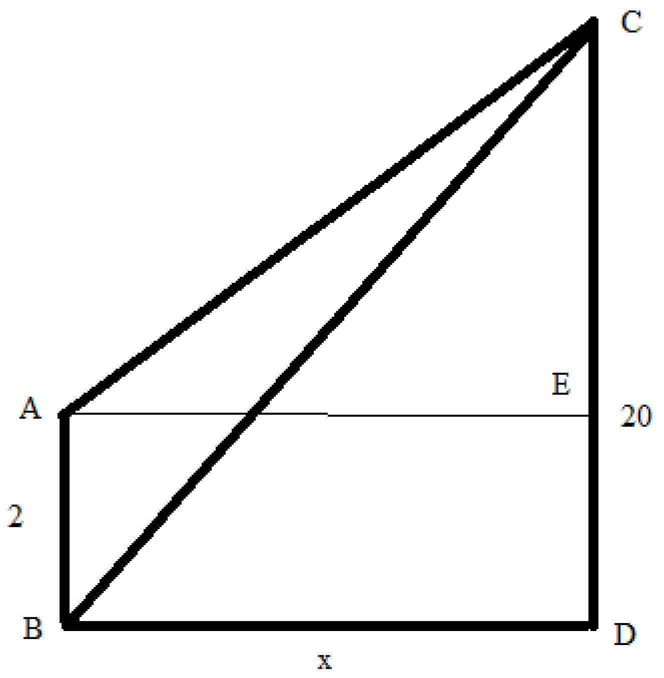
- ⇨ $\sqrt{3}x = 60 + x$
- ⇨ $x(\sqrt{3} - 1) = 60$
- ⇨ $x = 60/(\sqrt{3} - 1) = 60(\sqrt{3} + 1)/2 = 30(\sqrt{3} + 1)$

Option (c) is correct.

655. Two poles, AB of length 2 metres and CD of length 20 metres are erected vertically with bases at B and D. The two poles are at a distance not less than twenty metres. It is observed that tan(ACB) = 2/77. The distance between the two poles, in metres, is

(a) 72
(b) 68
(c) 24
(d) 24.27

Solution :



From triangle BCD we get, tan(BCD) = x/20

From triangle AEC we get, tan(ACE) = x/18

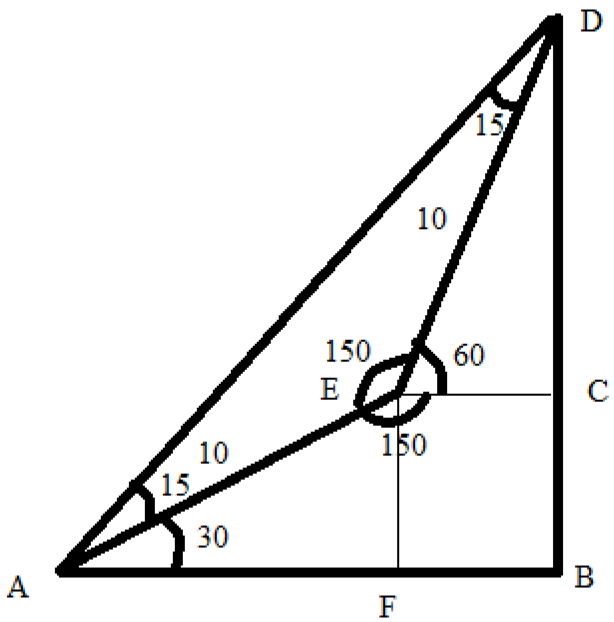
Now, tan(ACB) = tan(ACE – BCD) = {tan(ACE) – tan(BCD)}/{1 + tan(ACE)tan(BCD)}

- ⇨ 2/77 = (x/18 – x/20)/{1 + (x/18)(x/20)}
- ⇨ 2/77 = 2x/(360 + x^2)
- ⇨ 720 + $2x^2$ = 154x
- ⇨ $2x^2$ – 154x + 720 = 0
- ⇨ x^2 – 77x + 360 = 0
- ⇨ (x – 72)(x – 5) = 0
- ⇨ x = 72 (x ≠ 5 as distance between the poles greater than 20 metres)

Option (a) is correct.

656. The elevation of the top of a tower from a point A is 45. From A, a man walks 10 metres up a path sloping at an angle of 30. After this the slope becomes steeper and after walking up another 10 metres the man reaches the top. Then the distance of A from the foot of the tower is
(a) 5(√3 + 1) metres
(b) 5 metres
(c) 10√2 metres
(d) 5√2 metres

Solution :



From quadrilateral ABCE, Angle E = 150

From triangle AED, Angle E = 150

Therefore, Angle DEC = 360 – (150 + 150) = 60

Now, from triangle DCE, we get, CE/DE = cos60

⇨ CE = 10*(1/2) = 5

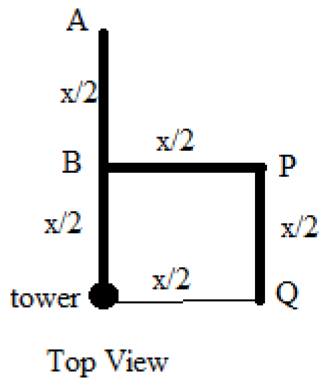
From triangle AEF, we get, AF/AE = cos30

⇨ AF = 10(√3/2) = 5√3
⇨ AB = AF + BF = AF + CE = 5√3 + 5 = 5(√3 + 1)

Option (a) is correct.

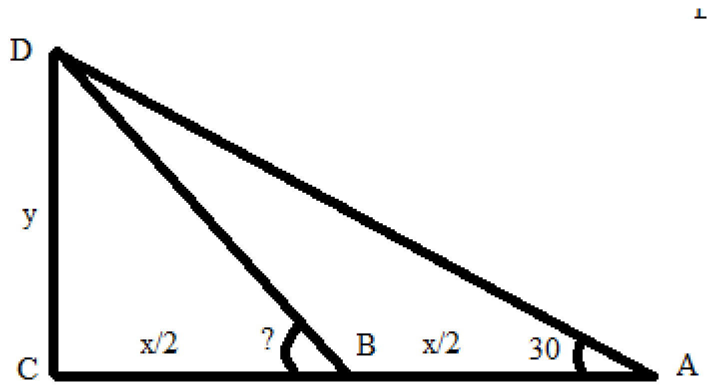
657. A man standing x metres to the north of a tower finds the angle of elevation of its top to be 30. He then starts walking towards the tower. After walking a distance of x/2 metres, he turns east and walks x/2 metres. Then again he turns south and walks x/2 metres. The angle of elevation of the top of the tower from his new position is
(a) 30
(b) $\tan^{-1}\sqrt{(2/3)}$
(c) $\tan^{-1}(2/\sqrt{3})$
(d) none of the foregoing quantities

Solution :



Clearly, from the above figure the angle of elevation at point Q = angle of elevation at point B.

Therefore, we draw the following picture.



From triangle ACD we get, CD/AC = tan30

⇨ $y/x = 1/\sqrt{3}$

Now, from triangle BCD we get, CD/BC = $\tan\theta$

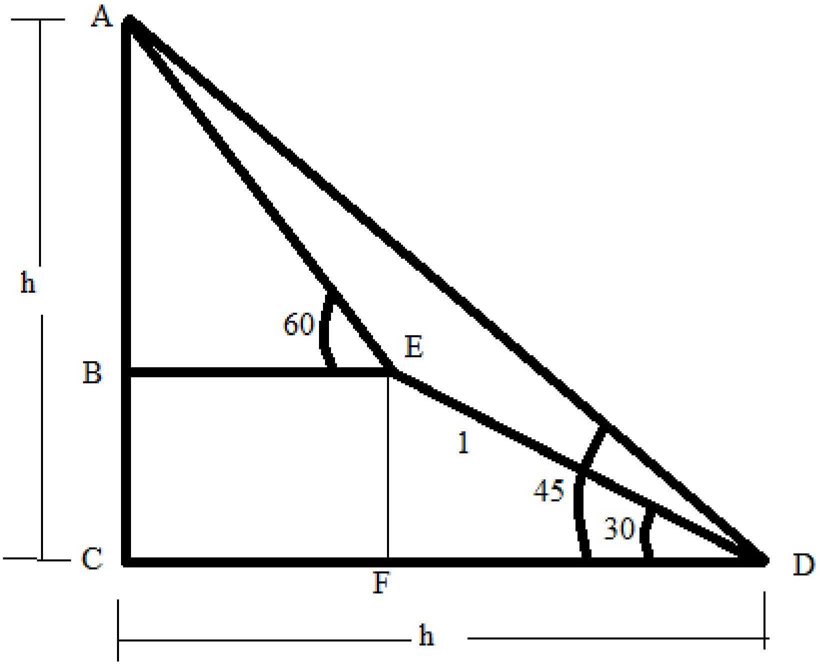
⇨ $\tan\theta = y/(x/2) = 2(y/x) = 2/\sqrt{3}$
⇨ $\theta = \tan^{-1}(2/\sqrt{3})$

Option (c) is correct.

658. The elevation of the summit of mountain is found to be 45. After ascending one km the summit up a slope of 30 inclination, the elevation is found to be 60. Then the height of the mountain is, in km,
(a) $(\sqrt{3} + 1)/(\sqrt{3} - 1)$
(b) $(\sqrt{3} - 1)/(\sqrt{3} + 1)$
(c) $1/(\sqrt{3} - 1)$

(d) $1/(\sqrt{3} + 1)$

Solution :



From triangle DEF we get, EF/DE = sin30

- ⇨ $EF = 1*(1/2) = ½$
- ⇨ $DF = DE\cos 30 = 1*(\sqrt{3}/2) = \sqrt{3}/2$
- ⇨ $CF = (h - \sqrt{3}/2)$ and $AB = h – ½$

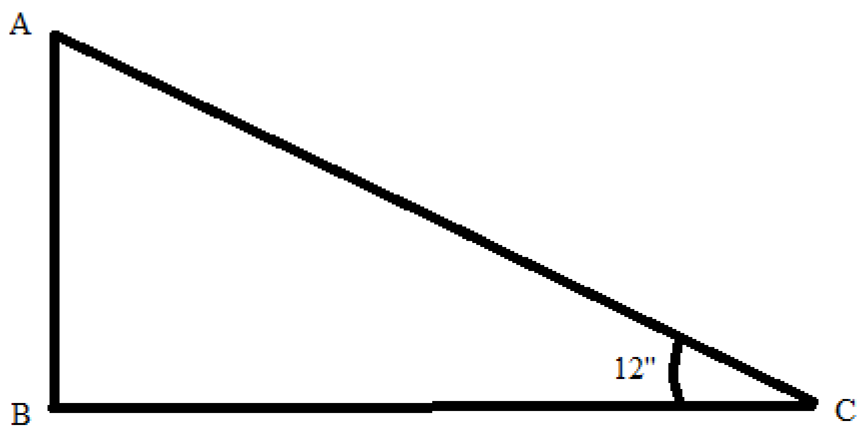
From triangle, ABE we get, AB/BE = tan60

- ⇨ $(h – ½)/(h - \sqrt{3}/2) = \sqrt{3}$
- ⇨ $h – ½ = \sqrt{3}h – 3/2$
- ⇨ $h(\sqrt{3} – 1) = 1$
- ⇨ $h = 1/(\sqrt{3} – 1)$

Option (c) is correct.

659. The distance at which a vertical pillar, of height 33 feet, subtends an angle of 12’’ (that is, 12 seconds) is, approximately in yards (1 yard = 3 feet),
(a) 11000000/6π
(b) 864000/11π
(c) 594000/π
(d) 864000/π

Solution :



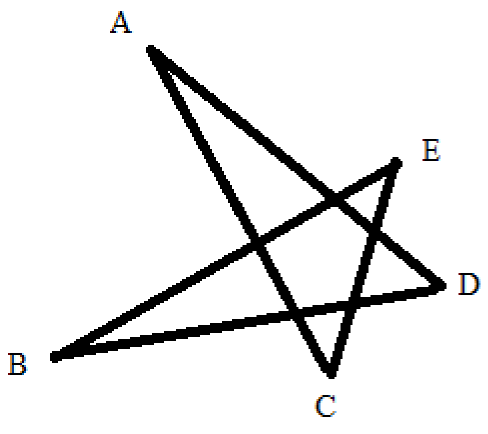
12'' = (12/3600)*(п/180) = п/54000 radian

AB/BC = tan(п/54000)

⇨ BC = AB/tan(п/54000) = 33/(п/54000) (approx.) = 33*54000/п feet = (1/3)(33*54000/п) yard = 584000/п

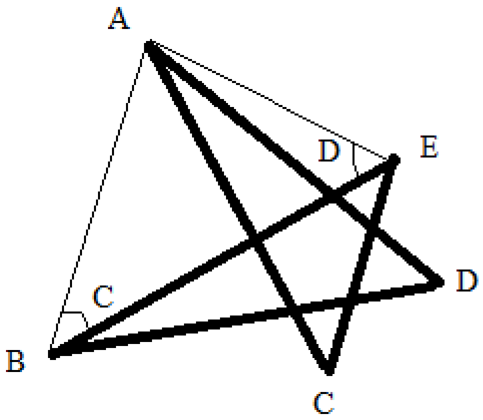
Option (c) is correct.

660. If the points A, B, C, D and E in the figure lie on a circle, then AD/BE



(a) equals sin(A + D)/sin(B + E)
(b) equals sinB/sinD
(c) equals sin(B + C)/sin(C + D)
(d) cannot be found unless the radius of the circle is given

Solution :



Now, angle ABE = C (both are on same arc AE)

Similarly, Angle AEB = D (both are on same arc AB)

From triangle ABE we get, BE/sin{180 – (C + D)} = AC/sinD

- ⇨ BE/sin(C + D) = AC/sinD

From triangle ABD we get, AD/sin(B + C) = AC/sinD

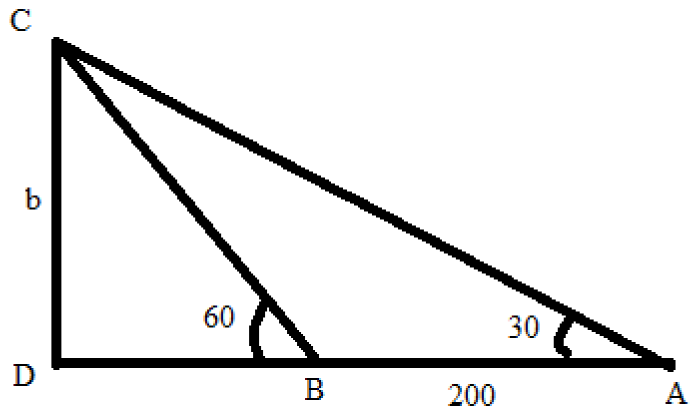
- ⇨ BE/sin(C + D) = AD/sin(B + C)
- ⇨ AD/BE = sin(B + C)/sin(C + D)

Option (c) is correct.

661. A man stands at a point A on the bank AB of a straight river and observes that the line joining A to a post C on the opposite bank makes with AB an angle of 30. He then goes 200 metres along the bank to B, finds that BC makes an angle of 60 with the bank. If b is breadth of the river, then

(a) 50√3 is the only possible value of b
(b) 100√3 is the only possible value of b
(c) 50√3 and 100√3 are the only possible values of b
(d) None of the foregoing statements is correct.

Solution :



From triangle ADC we get, CD/AD = tan30

- ⇨ AD = b√3

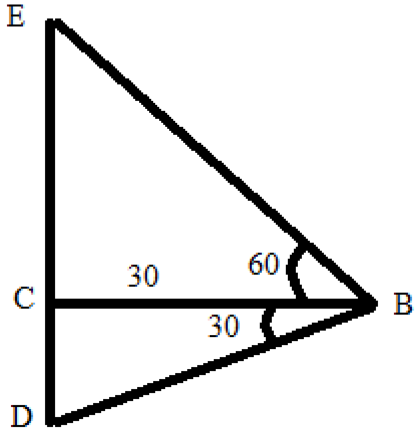
From triangle BDC we get, CD/BD = tan60

- ⇨ BD = b/√3
- ⇨ AD – BD = b√3 – b/√3
- ⇨ AB = b(2/√3)
- ⇨ 200 = b(2/√3)
- ⇨ b = 100√3

Option (b) is correct.

662. A straight pole A subtends a right angle at a point B of another pole at a distance of 30 metres from A, the top of A being 60 above the horizontal line joining the point B to the point A. The length of the pole A is, in metres,
(a) 20√3
(b) 40√3
(c)60√3
(d) 40/√3

Solution :



From triangle BCE we get, EC/BC = tan60

⇨ EC = $30\sqrt{3}$

From triangle BCD we get, CD/BC = tan30

⇨ CD = $30/\sqrt{3} = 10\sqrt{3}$

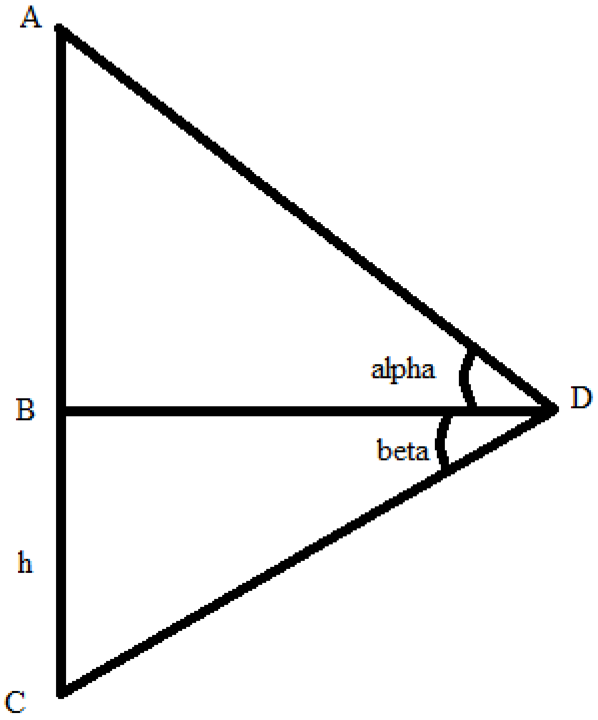
Therefore, length of the pole A = DE = CD + EC = $10\sqrt{3} + 30\sqrt{3} = 40\sqrt{3}$

Option (b) is correct.

663. The angle of elevation of a bird from a point h metres above a lake is α and the angle of depression of its image in the lake from the same point is β. The height of the bird above the lake is, in metres,

(a) $h\sin(\beta - \alpha)/(\sin\beta\cos\alpha)$
(b) $h\sin(\beta + \alpha)/(\sin\alpha\cos\beta)$
(c) $h\sin(\beta - \alpha)/\sin(\alpha + \beta)$
(d) $h\sin(\beta + \alpha)/\sin(\beta - \alpha)$

Solution :



From triangle BCD we get, BC/BD = tanβ

⇨ BD = h/tanβ

From triangle ABD we get, AB/BD = tanα

⇨ AB = htanα/tanβ

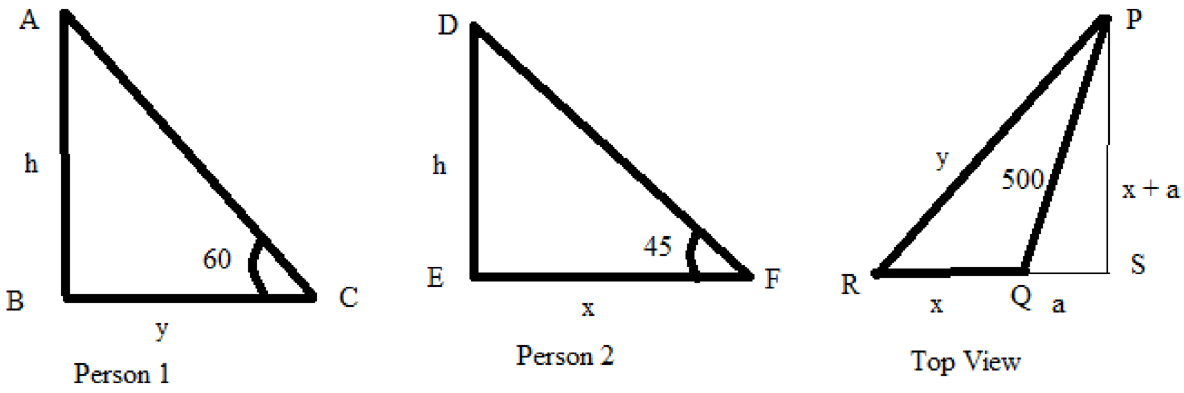
Height of the bird from lake = AC = AB + BC = htanα/tanβ + h = h{tanα/tanβ + 1) = h{sinαcosβ/cosαsinβ + 1) = h{(sinαcosβ + cosαsinβ)/(cosαsinβ)} = hsin(β + α)/(cosαsinβ)

It is given that option (d) is correct.

664. Two persons who are 500 metres apart, observe the direction and the angular elevation of a balloon at the same instant. One finds the elevation to be 60 and the direction South-West, while the other the elevation to be 45 and the direction West. Then the height of the balloon is, in metres,

(a) $500\sqrt{(12 + 3\sqrt{6})/10}$
(b) $500\sqrt{(12 - 3\sqrt{6})/10}$
(c) $250\sqrt{3}$
(d) None of the foregoing numbers.

Solution :



From triangle ABC we get, AB/BC = tan60

⇨ $y = h/\sqrt{3}$

From triangle DEF we get, $x = h$

From triangle SQR we get, $a^2 = 500^2 – (x + a)^2$

⇨ $a^2 = 500^2 – x^2 – 2ax – a^2$
⇨ $2a^2 + 2ax = 500^2 – x^2$
⇨ $2a(x + a) = 500^2 – x^2$

From triangle PSR we get, $y = \sqrt{2}(x + a)$

⇨ $a = (y - \sqrt{2}x)/\sqrt{2}$

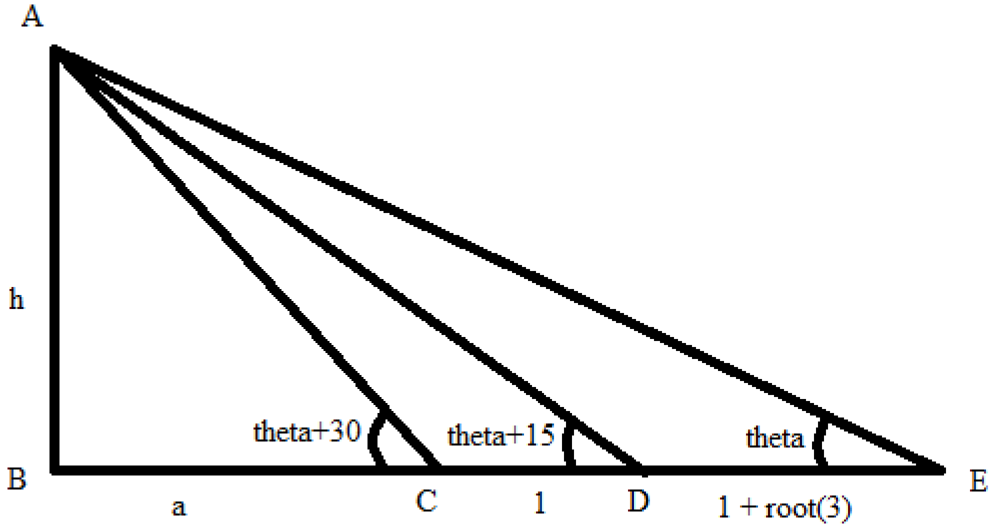
Putting in above equation we get, $2\{(y - \sqrt{2}x)/\sqrt{2}\}(y/\sqrt{2}) = 500^2 – h^2$

⇨ $y^2 - \sqrt{2}xy = 500^2$
⇨ $h^2/3 - \sqrt{2}*h*(h/\sqrt{3}) + h^2 = 500^2$
⇨ $h^2(4 - \sqrt{6})/3 = 500^2$
⇨ $h = 500\sqrt{3}/\sqrt{(4 - \sqrt{6})}$
⇨ $h = 500\sqrt{3}(\sqrt{(4 + \sqrt{6})}/\sqrt{10}$
⇨ $h = 500\sqrt{\{(12 + 3\sqrt{6})/10\}}$

Option (a) is correct.

665. Standing far from a hill, an observer records its elevation. The elevation increases 15 as he walks $1 + \sqrt{3}$ miles towards the hill, and by a further 15 as he walks another mile in the same direction. Then, the height of the hill is
(a) $(\sqrt{3} + 1)/2$ miles
(b) $(\sqrt{3} – 1)/(\sqrt{2} – 1)$ miles
(c) $(\sqrt{3} – 1)/2$ miles
(d) none of these

Solution :



From triangle ABE we get, $h/(a + 2 + \sqrt{3}) = \tan\theta$

From triangle ABD we get, $h/(a + 1) = \tan(\theta + 15)$

From triangle ABC we get, $h/a = \tan(\theta + 30)$

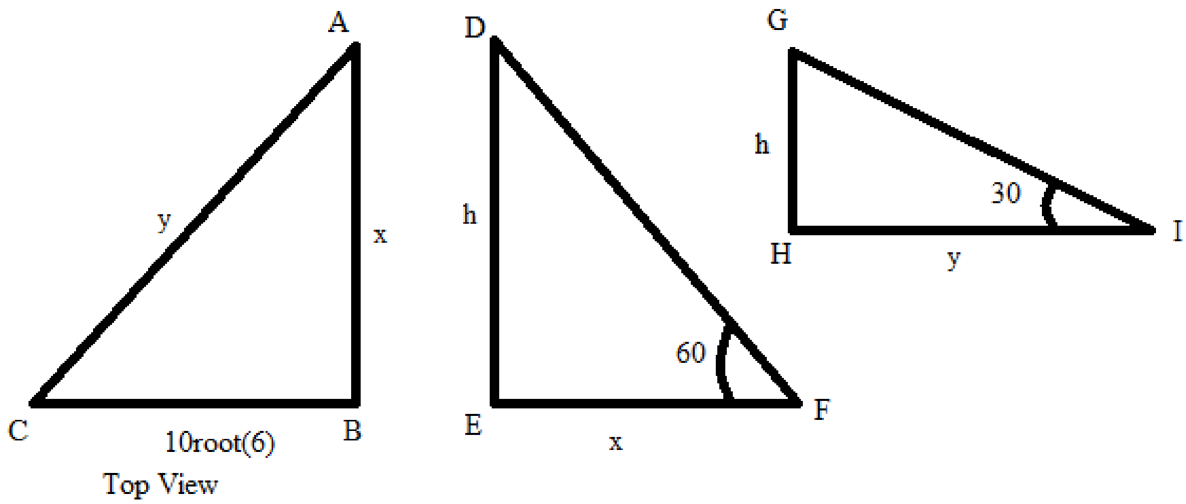
Now, there are three unknowns a, h, θ and three equations so we can solve h.

Option (c) is correct.

666. A man finds that at a point due south of a tower the angle of elevation of the tower is 60. He then walks due west $10\sqrt{6}$ metres on a horizontal plane and finds that the angle of elevation of the tower at that point is 30. Then the original distance of the man from the tower is, in metres,

(a) $5\sqrt{3}$
(b) $15\sqrt{3}$
(c) 15
(d) 180

Solution :



From triangle DEF we get, h/x = tan60

⇨ h = x√3

From triangle GHI we get, h/y = tan30

⇨ h = y/√3

Dividing the two equations we get, 1 = x√3/(y/√3)

⇨ y = 3x

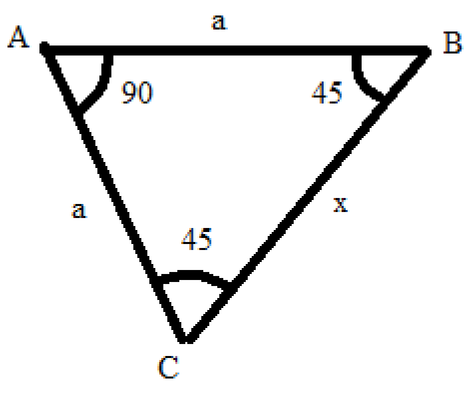
Now, from triangle ABC, we get, $y^2 = (10\sqrt{6})^2 + x^2$

⇨ $(3x)^2 = 600 + x^2$
⇨ $8x^2 = 600$
⇨ $x^2 = 75$
⇨ x = 5√3

Option (a) is correct.

667. A man stands a mteres due east of a tower and finds the angle of elevation of the top of the tower to be θ. He then walks x metres north west and finds the angle of elevation to be θ again. Then the value of x is
(a) a
(b) √2a
(c)a/√2
(d) none of the foregoing expressions

Solution :



Top View

From the data it is clear that AC = a (otherwise at B and C angle of elevation cannot be same)

In triangle ABC, AB = a and AC = a, implies Angle ACB = Angle ABC = 45 (as per data)

From the triangle ABC, x = √2a

Option (b) is correct.

668. The angle of elevation of the top of a hill from a point is α. After walking a distance d towards the top, up a slope inclined to the horizon at an angle θ, which is less than α, the angle of elevation is β. The height of the hill equals

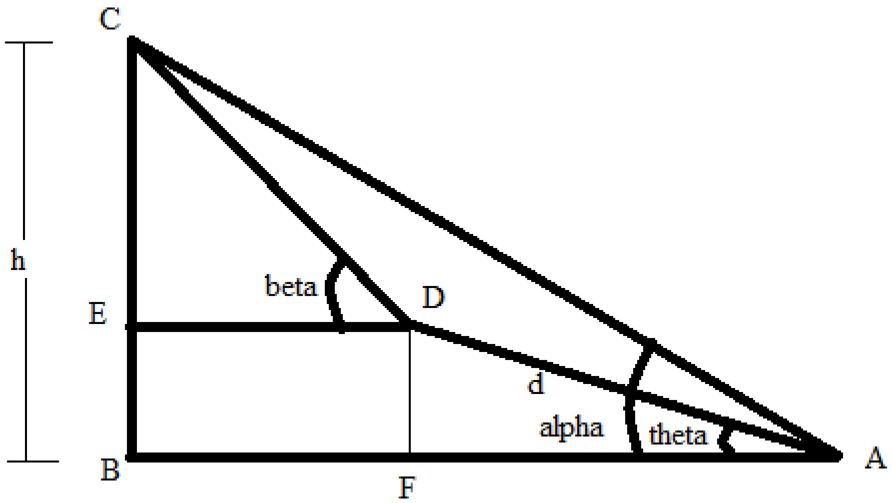
(a) dsinαsinθ/sin(β – α)

(b) dsin(β – α)sinθ/sinαsinβ

(c) dsin(α – θ)sin(β – α)/sin(α – θ)

(d) dsinαsin(β – θ)/sin(β – α)

Solution :



From triangle ADF we get, DF/AD = $\sin\theta$

⇨ DF = $d\sin\theta$

Again, AF/AD = $\cos\theta$

⇨ AF = $d\cos\theta$

CE = $h - d\sin\theta$

Now, from triangle ABC we get, BC/AB = $\tan\alpha$

⇨ AB = $h/\tan\alpha$
⇨ BF = $h/\tan\alpha - d\cos\theta$ = DE

From triangle BDE we get, CE/DE = $\tan\beta$

⇨ $(h - d\sin\theta)/(h/\tan\alpha - d\cos\theta) = \tan\beta$
⇨ $h - d\sin\theta = h\tan\beta/\tan\alpha - d\cos\theta\tan\beta$
⇨ $h(1 - \tan\beta/\tan\alpha) = d(\sin\theta - \cos\theta\sin\beta/\cos\beta)$
⇨ $h(\sin\alpha\cos\beta - \cos\alpha\sin\beta)/(\sin\alpha\cos\beta) = d(\sin\theta\cos\beta - \cos\theta\sin\beta)/\cos\beta$
⇨ $h\sin(\alpha - \beta) = d\sin(\theta - \beta)\sin\alpha$
⇨ $h = d\sin\alpha\sin(\beta - \theta)/\sin(\beta - \alpha)$

Option (d) is correct.

669. A person observes the angle of elevation of a peak from a point A on the ground to be α. He goes up an incline of inclination β, where $\beta < \alpha$, to the horizontal level towards the top of the peak and observes that the angle of elevation of the peak now is γ. If B is the second place of observation and AB = y metres, then height of the peak above the ground is

(a) $y\sin\beta + y\sin(\alpha - \beta)\operatorname{cosec}(\gamma - \alpha)\sin\gamma$

(b) ysinβ + ysin(β − α)sec(γ − α)sinγ
(c) ysinβ + ysin(α − β)sec(α − γ)sinγ
(d) ysinβ + ysin(α − β)cosec(α − γ)sinγ

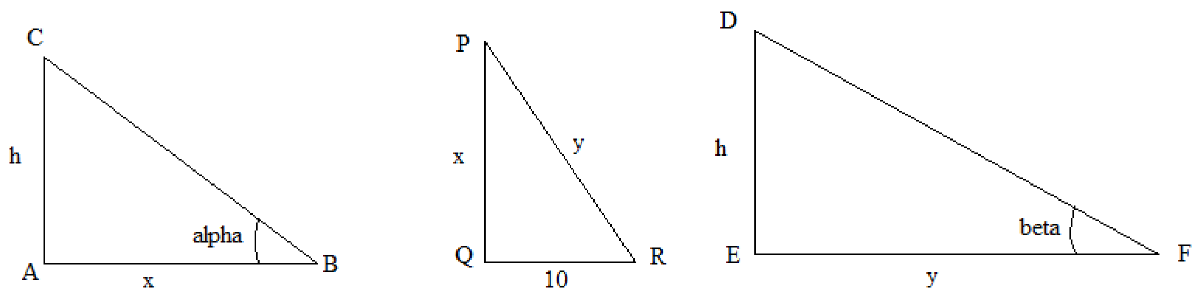
Solution :

Same problem as the previous one.

Option (a) is correct.

670. Standing on one side of a 10 metre wide straight road, a man finds that the angle of elevation of a statue located on the same side of the road is α. After crossing the road by the shortest possible distance, the angle reduces to β. The height of the statue is
(a) $10\tan\alpha\tan\beta/\sqrt{(\tan^2\alpha - \tan^2\beta)}$
(b) $10\sqrt{(\tan^2\alpha - \tan^2\beta)}/(\tan\alpha\tan\beta)$
(c) $10\sqrt{(\tan^2\alpha - \tan^2\beta)}$
(d) $10/\sqrt{(\tan^2\alpha - \tan^2\beta)}$

Solution :



Top View

From triangle ABC we get, $h/x = \tan\alpha$ i.e. $x = h/\tan\alpha$

From triangle DEF we get, $h/y = \tan\beta$ i.e. $y = h/\tan\beta$

Now, from triangle PQR we get, $y^2 = x^2 + 10^2$

⇨ $(h/\tan\beta)^2 - (h/\tan\alpha)^2 = 10^2$
⇨ $h^2(\tan^2\alpha - \tan^2\beta)/(\tan^2\alpha\tan^2\beta = 10$
⇨ $h = 10\tan\alpha\tan\beta/\sqrt{(\tan^2\alpha - \tan^2\beta)}$

Option (a) is correct.

671. The complete set of solutions of the equation $\sin^{-1}x = 2\tan^{-1}x$ is

(a) ± 1
(b) 0
(c) ± 1, 0
(d) $\pm 1/2$, ± 1, 0

Solution :

Now, $\sin^{-1}x = 2\tan^{-1}x$

Let $2\tan^{-1}x = A$

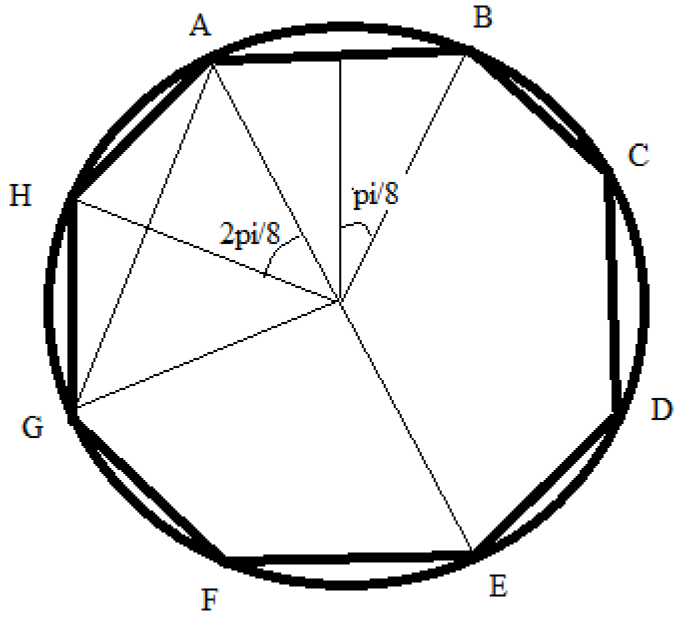
- ⇨ $\tan(A/2) = x$
- ⇨ $\sin A = 2\tan(A/2)/(1 + \tan^2(A/2)) = 2x/(1 + x^2)$
- ⇨ $A = \sin^{-1}\{2x/(1 + x^2)\} = \sin^{-1}x$
- ⇨ $2x/(1 + x^2) = x$
- ⇨ $x\{2/(1 + x^2) - 1\} = 0$
- ⇨ $x = 0$ or $2/(1 + x^2) - 1 = 0$
- ⇨ $2 = 1 + x^2$
- ⇨ $x = \pm 1$

Therefore, $x = 0, \pm 1$

Option (c) is correct.

672. For a regular octagon (a polygon with 8 equal sides) inscribed in a circle of radius 1, the product of the distances from a fixed vertex to the other seven vertices is
(a) 4
(b) 8
(c) 12
(d) 16

Solution :



From the figure it is clear that the product of the distances from a fixed vertes to other vertex is

= {2rsin(π/8)}{2rsin(2π/8)}(2rsin(3π/8)}^{2}{2rsin(4π/8)}

= $2^6\sin^2(\pi/8)\sin^2(3\pi/8)$

= 2^4\{1 – cos(π/4)\{1 – cos(3π/4)\}

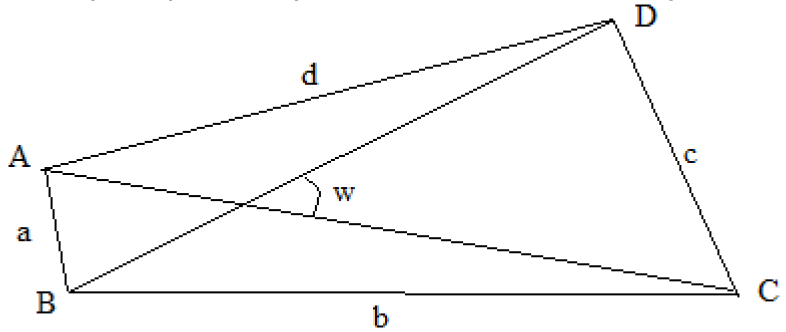
= $2^4(1 - 1/\sqrt{2})(1 + 1/\sqrt{2})$

= $2^4(1 - ½)$

= $2^3 = 8$

Option (b) is correct.

673. In the quadrilateral in the in figure, the lengths of AC and BD are x and y respectively. Then the value of 2xycosw equals



(a) $b^2 + d^2 - a^2 - c^2$
(b) $b^2 + a^2 - c^2 - d^2$

(c) $a^2 + c^2 - b^2 - d^2$
(d) $a^2 + d^2 - b^2 - c^2$

Solution :

Option (a) is correct.

674. In a triangle ABC with sides a = 5, b = 3, c = 7, the value of 3cosC + 7cosB is
(a) 3
(b) 7
(c) 10
(d) 5

Solution :

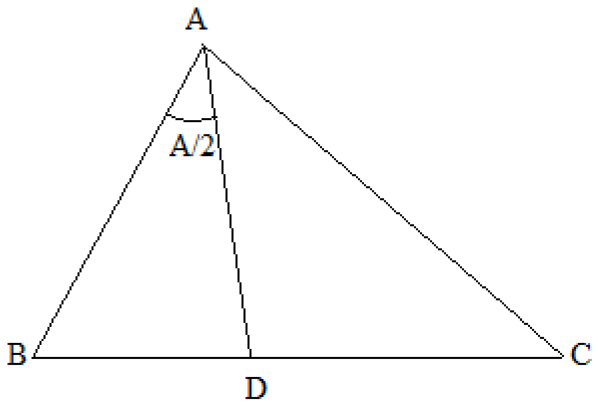
We know, a = bcosC + ccosB

⇨ 3cosC + 7cosB = 5

Option (d) is correct.

675. If in a triangle ABC, the bisector of the angle A meets the side BC at the point D, then the length of AD equals
(a) 2bcos(A/2)/(b + c)
(b) bccos(A/2)/(b + c)
(c) bccosA/(b + c)
(d) 2bcsin(A/2)/(b + c)

Solution :



From triangle ABD we get, AD/sinB = BD/sin(A/2)

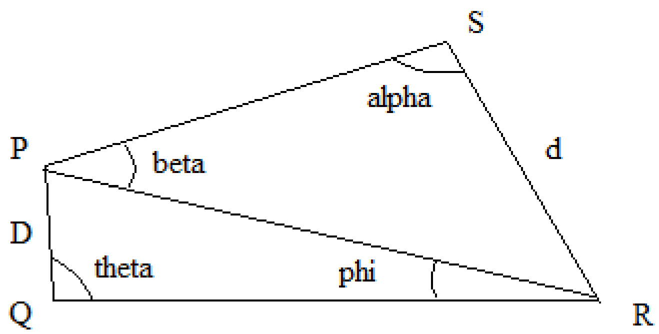
From triangle ACD we get, AD/sinC = CD/sin(A/2)

Adding we get, AD(1/sinB + 1/sinC) = (BD + CD)/sin(A/2)

- ⇨ AD(sinB + sinC)/(sinBsinC) = a/sin(A/2)
- ⇨ AD = absinC/(b + c)sinA/2
- ⇨ AD = 2absinCcos(A/2)/{(b + c)(2sin(A/2)coa(A/2)}
- ⇨ AD = 2absinCcos(A/2)/{(b + c)sinA)
- ⇨ AD = 2abccos(A/2)/{a(b + c)}
- ⇨ AD = 2bccos(A/2)/(b + c)

Option (a) is correct.

676. In an arbitrary quadrilateral with sides and angles as marked in the figure, the value of d is equal to



(a) Dsinθsinα/(sinΦsinβ)
(b) DsinΦsinβ/(sinθsinα)
(c) Dsinθsinβ/(sinΦsinα)
(d) DsinθsinΦ/(sinαsinβ)

Solution :

From triangle PQR we get, $D/\sin\Phi = PR/\sin\theta$

From triangle PRS we get, $d/\sin\beta = PR/\sin\alpha$

Dividing the equations we get, $(d/\sin\beta)/(D/\sin\Phi) = (PR/\sin\alpha)/(PR/\sin\theta)$

- ⇨ $d\sin\Phi/D\sin\beta = \sin\theta/\sin\alpha$
- ⇨ $d = D\sin\theta\sin\beta/(\sin\Phi\sin\alpha)$

Option (c) is correct.

677. Suppose the internal bisectors of the angles of a quadrilateral form another quadrilateral. Then the sum of the cosines of the angles of the second quadrilateral
   (a) is a constant independent of the first quadrilateral
   (b) always equals the sum of the sines of the angles of the first quadrilateral
   (c) always equals the sum of the cosines of the angles of the first quadrilateral
   (d) depends on the angles as well as the sides of the first quadrilateral

Solution :

$S = \pi - (A/2 + D/2)$

$\cos S = -\cos(A/2 + D/2)$

Similarly, $\cos P = -\cos(C/2 + D/2)$, $\cos Q = -\cos(B/2 + C/2)$ and $\cos R = -\cos(A/2 + B/2)$

- ⇨ $\cos P + \cos Q + \cos R + \cos S = -[\cos(C/2 + D/2) + \cos(B/2 + C/2) + \cos(A/2 + B/2) + \cos(D/2 + A/2)]$

$= -[2\cos\{(A + B + C + D)/4\}\cos\{(C + D - A - B)/4\} + 2\cos\{(A + B + C + D)/4\}\cos\{(B + C - D - A)/4\}]$

$= -2\cos\{(A + B + C + D)/4\}[\cos\{(C + D - A - B)/4\} + \cos\{(B + C - A - D)/4\}]$

$= -2\cos(\pi/2)[\cos\{(C + D - A - B)/4\} + \cos\{(B + C - A - D)/4\}]$ $(A + B + C + D = 2\pi)$

$= 0$

Option (a) is correct.

678. Consider the following two statements :
P : all cyclic quadrilaterals ABCD satisfy tan(A/2)tan(B/2)tan(C/2)tan(D/2) = 1.
Q : all trapeziums ABCD satisfy tan(A/2)tan(B/2)tan(C/2)tan(D/2) = 1.
Then
(a) both P and Q are true
(b) P is true but Q is not true
(c) P is not true and Q is true
(d) Neither P nor Q is true

Solution :

In a cyclic quadrilateral, A + C = B + D = 180

⇨ A = 180 – C
⇨ A/2 = 90 – C/2
⇨ tan(A/2) = tan(90 – C/2) = cot(C/2)
⇨ tan(A/2)tan(C/2) = 1

Similarly, tan(B/2)tan(d/2) = 1

Therefore, tan(A/2)tan(B/2)tan(C/2)tan(D/2) = 1 for cyclic quadrilateral

In trapezium with AB||CD, A + D = B + C = 180 (i.e. sum of adjacent angles is 180)

⇨ A/2 = 90 – D/2
⇨ tan(A/2) = cot(D/2)
⇨ tan(A/2)tan(D/2) = 1

Similarly, tan(B/2)tan(C/2) = 1

Therefore, tan(A/2)tan(B/2)tan(C/2)tan(D/2) = 1 for trapezium

Option (a) is correct.

679. Let a, b, c denote the three sides of a triangle and A, B, C the corresponding opposite angles. Only one of the expressions below has the same value for all triangles. Which one is it?
(a) sinA + sinB + sinC
(b) tanAtanB + tanBtanC + tanCtanA
(c) (a + b + c)/(sinA + sinB + sinC)
(d) cotAcotB + cotBcotC + cotCcotA

Solution :

Option (a) and (c) cannot be true because those are function of R (radius of circumcircle)

Let us try cotAcotB + cotBcotC + cotCcotA

= cotB(1/tanA + 1/tanC) + cotCcotA

= cotB(tanA + tanC)/(tanAtanC) + cotCcotA

= cotBtan(A + C)(1 – tanAtanC)/(tanAtanC) + cotCcotA

= cotB(-tanB)(1 – tanAtanC)cotAcotC + cotCcotA

= -(1 – tanAtanC)cotAcotC + cotCcotA

= -cotAcotC + 1 + cotAcotC

= 1

Option (d) is correct.

680. In a triangle ABC, 2sinCcosB = sinA holds. Then one of the following statements is correct. Which one is it?
(a) The triangle must be equilateral.
(b) The triangle must be isosceles but not necessarily equilateral
(c) C must be an obtuse angle
(d) None of the foregoing statements is necessarily true.

Solution :

Now, 2sinCcosB = sinA

- ⇨ $2c(c^2 + a^2 - b^2)/(2ac) = a$
- ⇨ $c^2 + a^2 - b^2 = a^2$
- ⇨ $c^2 = b^2$
- ⇨ $c = b$

Option (b) is correct.

681. If A, B, C are the angles of a triangle and $\sin^2 A + \sin^2 B = \sin^2 C$, then C equals
(a) 30 degree
(b) 90 degree
(c) 45 degree
(d) None of the foregoing angles

Solution :

$\sin^2A + \sin^2B = \sin^2C$

- ⇨ $a^2 + b^2 = c^2$
- ⇨ c is the hypotenuse of the right-angled triangle ABC
- ⇨ C is 90 degree.

Option (b) is correct.

682. The value of (cos37 + sin37)/(cos37 – sin37) equals
(a) tan8
(b) cot8
(c) sec8
(d) cosec8

Solution :

(cos37 + sin37)/(cos37 – sin37)

= (1 + tan37)/(1 – tan37)

= (tan45 + tan37)/(1 – tan45tan37)

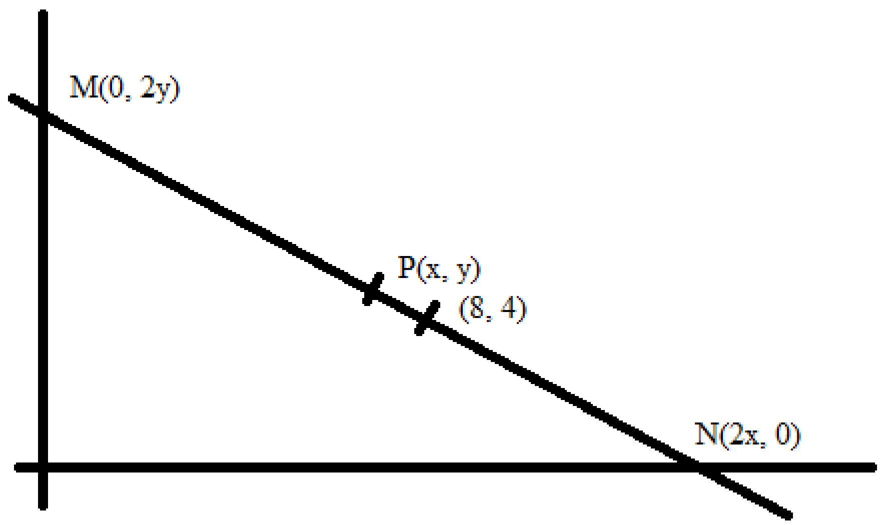
= tan(45 + 37)

= tan82

= cot8

Option (b) is correct.

683. A straight line passes through the fixed point (8, 4) and cuts the y-axis at M and the x-axis at N as in figure. Then the locus of the middle point P of MN is



(a) $xy - 4x - 2y + 8 = 0$
(b) $xy - 2x - 4y = 0$
(c) $xy + 2x + 4y = 64$
(d) $xy + 4x + 2y = 72$

Solution :

Let the slope of the line is m.

Therefore, equation of the line is $y - 4 = m(x - 8)$

- ⇨ $mx - y = (8m - 4)$
- ⇨ $x/\{(8m - 4)/m\} + y/(4 - 8m) = 1$

Therefore, $2x = (8m - 4)/m$ and $2y = 4 - 8m$

$2y/2x = -m$

- ⇨ $m = -y/x$
- ⇨ $2y = 4 + 8y/x$
- ⇨ $xy = 2x + 4y$
- ⇨ $xy - 2x - 4y = 0$

Option (b) is correct.

684. In a triangle ABC, a, b and c denote the sides opposite to angles A, B and C respectively. If $\sin A = 2\sin C\cos B$, then
(a) $b = c$
(b) $c = a$
(c) $a = b$
(d) none of the foregoing statements is true.

Solution :

Now, sinA = 2sinCcosB

- ⇨ $a = 2c(c^2 + a^2 - b^2)/(2ca)$
- ⇨ $a^2 = c^2 + a^2 - b^2$
- ⇨ $c^2 = b^2$
- ⇨ $c = b$

Option (a) is correct.

685. The lengths of the sides CB and CA of a triangle ABC are given by a and b, and the angle C is 2π/3. The line CD bisects the angle C and meets AB at D. Then the length of CD is
   (a) $1/(a + b)$
   (b) $(a^2 + b^2)/(a + b)$
   (c) $ab/\{2(a + b)\}$
   (d) $ab/(a + b)$

Solution :

See solution of problem 675.

Option (d) is correct.

686. Suppose in a triangle ABC, bcosB = ccosC. Then the triangle
   (a) is right-angled
   (b) is isosceles
   (c) is equilateral
   (d) need not necessarily be any of the above types

Solution :

Now, bcosB = ccosC

- ⇨ $b(a^2 + c^2 - b^2)/(2ac) = c(a^2 + b^2 - c^2)/(2ab)$
- ⇨ $b^2(a^2 + c^2 - b^2) = c^2(a^2 + b^2 - c^2)$
- ⇨ $a^2b^2 + b^2c^2 - b^4 = c^2a^2 + b^2c^2 - c^4$
- ⇨ $b^4 - c^4 - a^2b^2 + c^2a^2 = 0$
- ⇨ $(b^2 - c^2)(b^2 + c^2) - a^2(b^2 - c^2) = 0$
- ⇨ $(b^2 - c^2)(b^2 + c^2 - a^2) = 0$
- ⇨ $b = c$ or $b^2 + c^2 = a^2$

i.e. it may be isosceles or right-angled.

Option (d) is correct.

687. Let $V_0 = 2$, $V_1 = 3$ and for any natural number $k \geq 1$, let $V_{k+1} = 3V_k - 2V_{k-1}$. Then for any $n \geq 0$, V_n equals
(a) $(1/2)(n^2 + n + 4)$
(b) $(1/6)(n^3 + 5n + 12)$
(c) $2^n + 1$
(d) None of the foregoing expressions.

Solution :

Now, $V_{k+1} = 3V_k - 2V_{k-1}$

⇨ $V_{k+1} - V_k = 2V_k - 2V_{k-1}$

Putting k = 1 we get, $V_2 - V_1 = 2V_1 - 2V_0$

Putting k = 2, we get, $V_3 - V_2 = 2V_2 - 2V_1$

…

..

Putting k = n -1 we get, $V_n - V_{n-1} = 2V_{n-1} - 2V_{n-2}$

Summing over we get, $V_n - V_1 = 2V_{n-1} - 2V_0$

⇨ $V_n - 3 = 2V_{n-1} - 4$
⇨ $V_n = 2V_{n-1} - 1$
⇨ $V_n = 2(2V_{n-2} - 1) - 1 = 2^2V_{n-2} - 1 - 2 = 2^2(2V_{n-3} - 1) - 1 - 2 = 2^3V_{n-3} - 1 - 2 - 2^2 = … = 2^nV_0 - (1 + 2 + 2^2 + … + 2^{n-1})$
⇨ $V_n = 2^{n+1} - 1(2^n - 1)/(2 - 1) = 2^{n+1} - 2^n + 1 = 2^n + 1$

Option (c) is correct.

688. If $a_n = 1000^n/n!$, for n = 1, 2, 3, …., then the sequence $\{a_n\}$
(a) doesn't have a maximum
(b) attains maximum at exactly one value of n
(c) attains maximum at exactly two values of n
(d) attains maximum for infinitely many values of n

Solution :

Let, $a_n = a_k$

- ⇨ $1000^n/n! = 1000^k/k!$
- ⇨ $1000^{n-k} = n(n - 1)\dots(k + 1)$ $(n > k)$

It can be only true if n = 1000 and k = 999.

Now, $1000^{1000}/1000! - 1000^{998}/998! = 1000^{998}/(1000!)(1000^2 - 999*1000) > 0$

Now, $1000^{1000}/1000! - 1000^{1001}/1001! = 1000^{1000}/1001!(1001 - 1000) > 0$

- ⇨ For n = 1000 it is maximum.
- ⇨ For n = 999 it is maximum.

Option (c) is correct.

689. Let f be a function of a real variable such that it satisfies f(r + s) = f(r) + f(s), for all r, s. Let m and n be integers. Then f(m/n) equals
 (a) m/n
 (b) f(m)/f(n)
 (c)(m/n)f(1)
 (d) None of the foregoing expressions, in general.

Solution :

$f(r + s) = f(r) + f(s)$

Putting s = 0 we get, $f(r) = f(r) + f(0)$ i.e. $f(0) = 0$

$f(r + s) = f(r) + f(s)$

- ⇨ $f(r + (m - 1)r) = f(r) + f((m - 1)r) = f(r) + f(r) + f((m - 2)r) = \dots = mf(r) + f(0) = mf(r)$
- ⇨ $f(mr) = mf(r)$
- ⇨ $f((m/n)r) = (m/n)f(r)$
- ⇨ $f(m/n) = (m/n)f(1)$

Option (c) is correct.

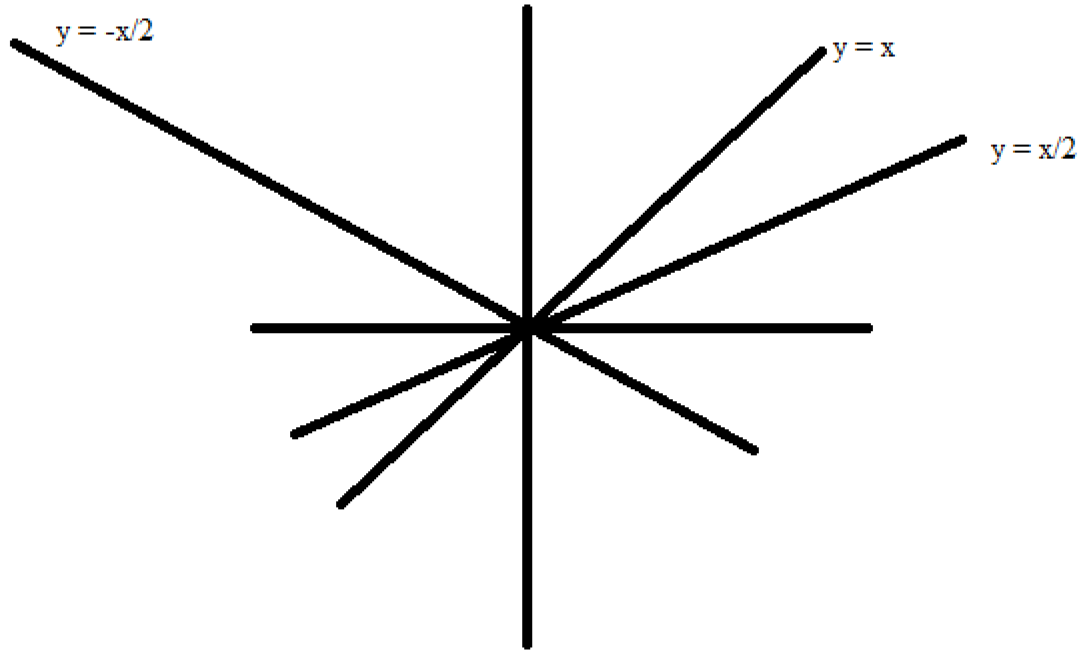
690. Let f(x) be a real-valued function defined for all real numbers x such that $|f(x) - f(y)| \le (1/2)|x - y|$ for all x, y. Then the number of points of intersection of the graph of y = f(x) and the line y = x is
 (a) 0
 (b) 1
 (c)2

(d) None of the foregoing numbers.

Solution :

|f(x) – f(y)| ≤ (1/2)|x – y|

- ⇨ lim |{f(x) – f(y)}/(x – y)| as x -> y ≤ lim (1/2) as x - > y
- ⇨ |f′(y)| ≤ ½
- ⇨ -1/2 ≤ f′(y) ≤1/2
- ⇨ -y/2 ≤ f(y) ≤ y/2 (integrating)
- ⇨ -x/2 ≤ f(x) ≤ x/2



From the figure it is clear that intersection point is 1.

Option (b) is correct.

691. The limit of $(1/n^4)\sum k(k + 2)(k + 4)$ (summation running from k = 1 to k = n) as n -> ∞
(a) exists and equals ¼
(b) exists and equals 0
(c) exists and equals 1/8
(d) does not exist

Solution :

Now, $(1/n^4)\sum k(k + 2)(k + 4)$ (summation running from k = 1 to k = n)

$= (1/n^4)\sum(k^3 + 6k^2 + 8k)$ (summation running from k = 1 to k = n)

$= (1/n^4)[\sum k^3 + 6\sum k^2 + 8\sum k]$ (summation running from k = 1 to k = n)

$= (1/n^4)[\{n(n + 1)/2\}^2 + 6n(n + 1)(2n + 1)/6 + 8n(n + 1)/2]$

$= (1 + 1/n)^2/4 + (1 + 1/n)(2 + 1/n)(1/n) + 4(1 + 1/n)(1/n^2)$

Now, lim of this as $n \to \infty = 1/4$

Option (a) is correct.

692. The limit of the sequence $\sqrt{2}$, $\sqrt{(2\sqrt{2})}$, $\sqrt{\{2\sqrt{(2\sqrt{2})}\}}$, …. Is
(a) 1
(b) 2
(c) $2\sqrt{2}$
(d) ∞

Solution :

Now, $a_n^2 = 2a_{n-1}$

⇨ $\lim a_n^2 = 2\lim a_{n-1}$ as $n \to \infty$

Let $\lim a_n$ as $n \to \infty = a$

⇨ $\lim a_{n-1}$ as $n \to \infty = a$
⇨ $a^2 = 2a$
⇨ $a = 2$ $(a \neq 0)$

Option (b) is correct.

693. Let $P_n = \{(2^3 - 1)/(2^3 + 1)\}\{(3^3 - 1)/(3^3 + 1)\}\ldots\{(n^3 - 1)/(n^3 + 1)\}$; n = 2, 3, ….. $\lim P_n$ as $n \to \infty$ is
(a) ¾
(b) 7/11
(c) 2/3
(d) ½

Solution :

Option (c) is correct.

694. Let $a_1 = 1$ and $a_n = n(a_{n-1} + 1)$ for n = 2, 3, …. Define $P_n = (1 + 1/a_1)(1 + 1/a_2)….(1 + 1/a_n)$. Then lim P_n as n - > ∞ is

(a) 1 + e

(b) e

(c) 1

(d) ∞

Solution :

Option (b) is correct.

695. Let x be a real number. Let $a_0 = x$, $a_1 = \sin x$ and, in general, $a_n = \sin a_{n-1}$. Then the sequence $\{a_n\}$

(a) oscillates between -1 and +1, unless x is a multiple of π

(b) converges to 0 whatever be x

(c) converges to 0 if and only if x is a multiple of π

(d) sometimes converges and sometimes oscillates depending on x

Solution :

Now, for bigger x, sinx < x

⇨ $a_2 < a_1$, $a_3 < a_2$, …., $a_n < a_{n-1}$

So, lim a_n as n -> ∞ = small number = b (say)

Now, lim a_{n-1} as n - > ∞ = b (if converges)

⇨ b = sinb which is true for small b
⇨ The sequence converges.

Option (b) is correct.

696. If k is an integer such that lim $\{\cos^n(k\pi/4) – \cos^n(k\pi/6)\} = 0$, then

(a) k is divisible neither by 4 nor by 6

(b) k must be divisible by 12, but not necessarily by 24

(c) k must be divisible by 24

(d) either k is divisible by 24 or k is divisible neither by 4 not by 6

Solution :

If k is divisible by 24 then $\cos(k\pi/4) = \cos(k\pi/6) = 1$

- ⇨ The limit exists and equal to RHS i.e. 0

If k is not divisible by 4 or 6 then $\cos(k\pi/4)$, $\cos(k\pi/6)$ both < 1

- ⇨ $\lim \cos^n(k\pi/4), \cos^n(k\pi/6) = 0$
- ⇨ The equation holds.

Option (d) is correct.

697. The limit of $\sqrt{x}\{\sqrt{(x + 4)} - \sqrt{x}\}$ as $x \to \infty$
  (a) does not exist
  (b) exists and equals 0
  (c) exists and equals ½
  (d) exists and equals 2

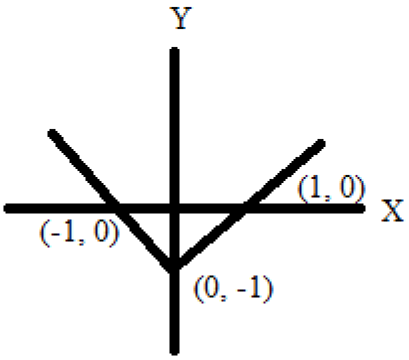
Solution :

Now, $\sqrt{x}\{\sqrt{(x + 4)} - \sqrt{x}\} = \sqrt{x}\{\sqrt{(x + 4)} - \sqrt{x}\}\{\sqrt{(x + 4)} + \sqrt{x}\}/\{\sqrt{(x + 4)} + \sqrt{x}\} = \sqrt{x}(x + 4 - x)/\{\sqrt{(x + 4)} + \sqrt{x}) = 4/\{\sqrt{(1 + 4/x)} + 1\}$

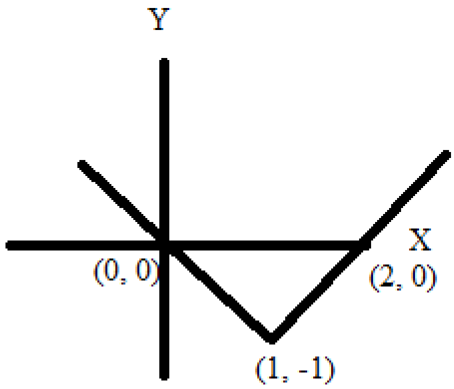
Now, $\lim x \to \infty$ this $= 4/(1 + 1) = 2$

Option (d) is correct.

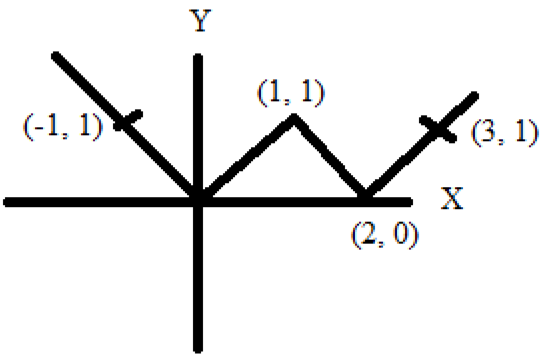
698. Four graphs marked G_1, G_2, G_3 and G_4 are given in the figure which are graphs of the four functions $f_1(x) = |x - 1| - 1$, $f_2(x) = ||x - 1| - 1|$, $f_3(x) = |x| - 1$, $f_4(x) = 1 - |x|$, not necessarily in the correct order.



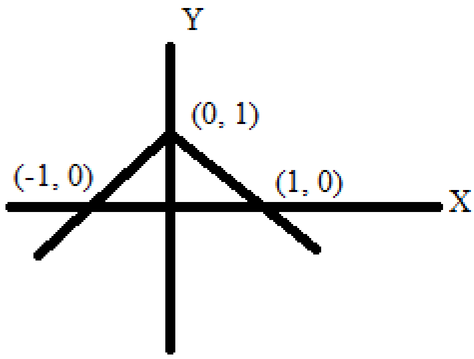
G1



G2



G3



G4

The correct order is
(a) G_2, G_1, G_3, G_4
(b) G_3, G_4, G_1, G_2
(c) G_2, G_3, G_1, G_4
(d) G_4, G_3, G_1, G_2

Solution :

Take the function $f_3(x) = |x| - 1$

If $x > 0$ $y = x - 1$, i.e. $x/1 + y/(-1) = 1$

If $x < 0$ $y = -x - 1$, i.e. $x/(-1) + y/(-1) = 1$

Clearly, G_1 is the graph.

Now, take the function $f_4(x) = 1 - |x|$

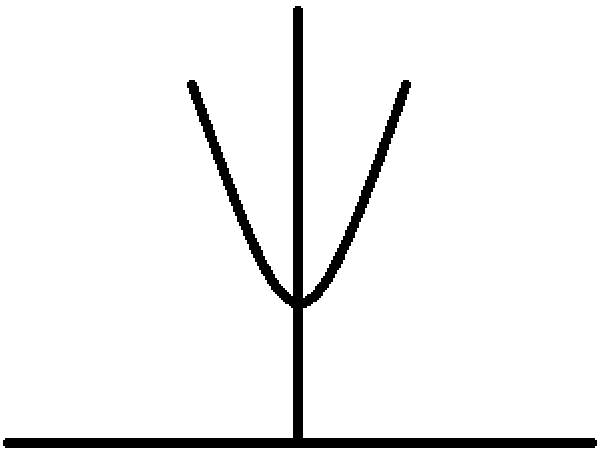
If $x > 0$, $y = 1 - x$ i.e. $x/1 + y/1 = 1$

If $x < 0$ $y = 1 + x$ i.e. $x/(-1) + y/1 = 1$

Clearly, G_4 is the graph.

Hence, option (c) is correct.

699. The adjoining figure is the graph of



(a) $y = 2e^x$
(b) $y = 2e^{-x}$
(c) $y = e^x + e^{-x}$
(d) $y = e^x - e^{-x} + 2$

Solution :

Option (c) is correct.

700. Suppose that the three distinct real numbers a, b, c are in G.P. and a + b + c = xb. Then
(a) $-3 < x < 1$
(b) $x > 1$ or $x < -3$
(c) $x < -1$ or $x > 3$
(d) $-1 < x < 3$

Solution :

Now, a + b + c = xb

- ⇨ $(a/b) + 1 + (c/b) = x$
- ⇨ $x = r + 1/r + 1$ (where r = common ration of the G.P.)

Let $r > 0$

$(r + 1/r) > 2$ (AM > GM)

- ⇨ $x > 3$

Let $r < 0$

$(r + 1)^2 > 0$

- ⇨ $r^2 + 2r + 1 > 0$
- ⇨ $r + 1/r + 2 < 0$ (as $r < 0$)
- ⇨ $r + 1/r + 1 < -1$
- ⇨ $x < -1$

Option (c) is correct.

701. The maximum value attained by the function $y = 10 - |x - 10|$ in the range $-9 \leq x \leq 9$ is
(a) 10
(b) 9
(c) $+\infty$
(d) 1

Solution :

Clearly, $|x - 10|$ is minimum when $x = 9$

- ⇨ Maximum value of $y = 10 - 1 = 9$

Option (b) is correct.

702. Let $f(x)$ be a real-valued function of a real variable. Then the function is said to be 'one-to-one' if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. The function is said to be 'onto' if it takes all real values. Suppose now $f(x) = x^3 - 3x^2 + 6x - 5$. Then
(a) f is one-to-one and onto
(b) f is one-to-one but not onto
(c) f is onto but not one-to-one
(d) f is neither one-to-one nor onto

Solution :

Now, $f(x) = x^3 - 3x^2 + 6x - 5$

- ⇨ $f'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2) = 3\{(x - 1)^2 + 1\} > 0$ for all real x.
- ⇨ f(x) is increasing

And f(x) is a polynomial so f(x) is continuous everywhere.

⇨ f(x) is one-to-one and onto.

Option (a) is correct.

703. Let f be a function from a set X to X such that f(f(x)) = x for all x Ɛ X. Then

(a) f is one-to-one but need not be onto
(b) f is onto but need not be one-to-one
(c) f is both one-to-one and onto
(d) none of the foregoing statements is true

Solution :

Let, $f(x_1) = x_2$

Now, f(f(x) = x

Putting $x = x_1$ we get, $f(f(x_1)) = x_1$

⇨ $f(x_2) = x_1$

So, x_1 maps to x_2 and x_2 maps to x_1.

Let $f(x_3) = x_2$

Putting $x = x_3$ we get, $f(f(x_3)) = x_3$

⇨ $f(x_2) = x_3$
⇨ $x_3 = x_1$
⇨ if $x_1 \neq x_3$ then $f(x_1) \neq f(x_3)$
⇨ f(x) is one-to-one

And f(x) is onto also because the mapping is from X to X.

Option (c) is correct.

Directions for Items 704 to 706 :

A real-valued function f(x) of a real variable x is said to be periodic if there is a strictly positive number p such that f(x + p) = f(x) for every x. The smallest p satisfying the above property is called the period of f.

704. Only one of the following is not periodic. Which one is it?

(a) $e^{\sin x}$
(b) 1/(10 + sinx + cosx)
(c) $\log_e(\cos x)$
(d) $\sin(e^x)$

Solution :

Now sinx and cosx are periodic. So, option (a), (b) and (c) are periodic.

Option (d) is correct.

705. Suppose f is periodic with period greater than h. Then
(a) for all h' > h and for all x, f(x + h') = f(x)
(b) for all x, f(x + h) ≠ f(x)
(c) for some x, f(x + h) ≠ f(x)
(d) none of the foregoing statements is true

Solution :

Clearly, option (c) is correct.

706. Suppose f is a function with period a and g is a function with period b. Then the function h(x) = f(g(x))
(a) may not have any period
(b) has period a
(c) has period b
(d) has period ab

Solution :

Now, h(x + b) = f(g(x + b)) = f(g(x) = h(x)

Option (c) is correct.

707. A function f is said to be odd if f(-x) = -f(x) for all x. Which of the following is not odd?
(a) A function f such that f(x + y) = f(x) + f(y) for all x, y
(b) $f(x) = xe^{x/2}/(1 + e^x)$
(c) f(x) = x – [x]
(d) $f(x) = x^2\sin x + x^3\cos x$

Solution :

Putting y = -x in option (a) we get, f(0) = f(x) + f(-x)

Now, putting y = 0 in option (a) we get, f(x) = f(x) + f(0) i.e. f(0) = 0

- ⇨ f(x) + f(-x) = 0
- ⇨ f(-x) = -f(x)
- ⇨ odd

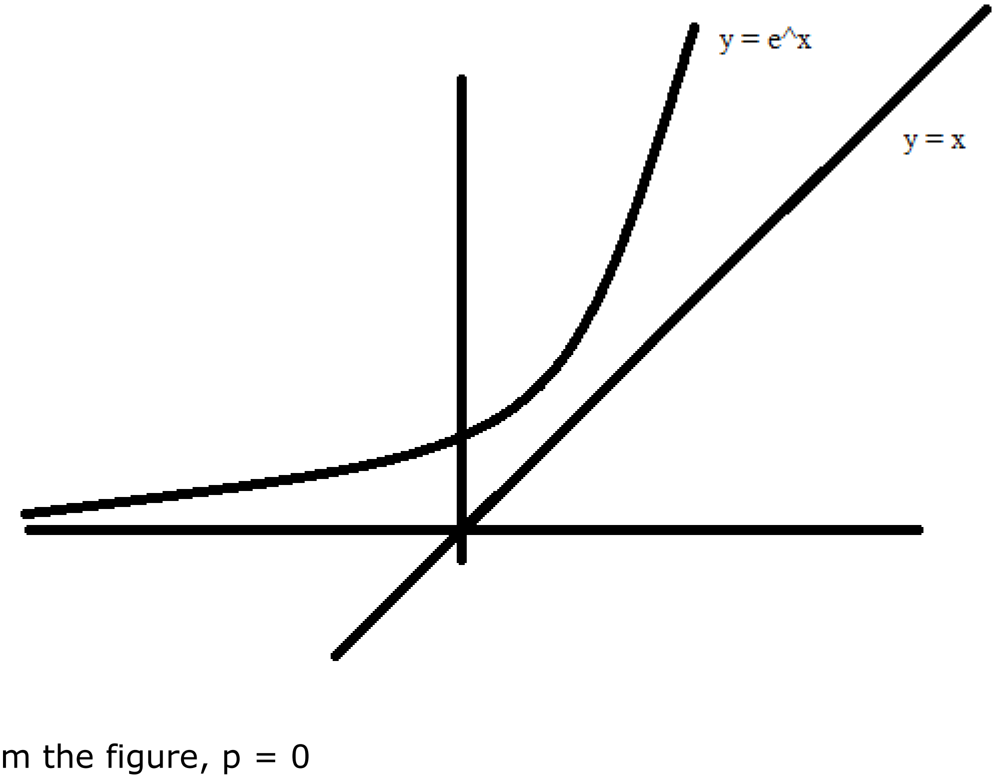
Option (b) and (d) can be proved odd easily.

Option (c) is correct.

708. If n stands for the number of negative roots and p for the number of positive roots of the equation $e^x = x$, then
   (a) n = 1, p = 0
   (b) n = 0, p = 1
   (c) n = 0, p > 1
   (d) n = 0, p = 0

Solution :

As $e^x > 0$ (always for all x) so n = 0



From the figure, p = 0

Option (d) is correct.

709. In the interval (-2π, 0) the function $f(x) = \sin(1/x^3)$

(a) never changes sign
(b) changes sign only once
(c) changes sign more than once, but a finite number of times
(d) changes sign infinite number of times

Solution :

As x becomes < 1 and tends to zero then it crosses π, 2π, 3π, ….

So, number of sign changes is infinite.

Option (d) is correct.

710. If $f(x) = a_0 + a_1\cos x + a_2\cos 2x + …. + a_n\cos nx$, where $a_0, a_1, …., a_n$ are nonzero real numbers and $a_n > |a_0| + |a_1| + …. + |a_{n-1}|$, then the number of roots of $f(x) = 0$ in $0 \le x \le 2\pi$, is
(a) at most n
(b) more than n but less than 2n
(c) at least 2n
(d) zero

Solution :

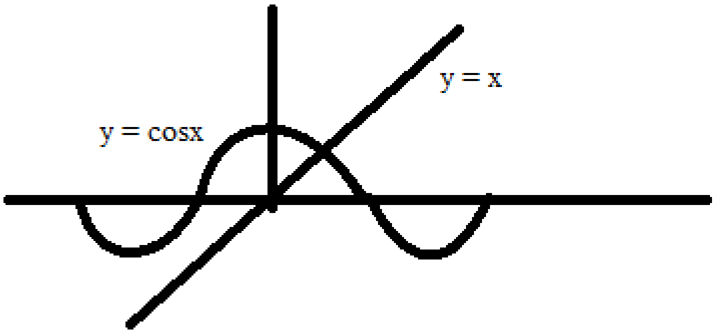
Option (c) is correct.

711. The number of roots of the equation $x^2 + \sin^2 x = 1$ in the closed interval $[0, \pi/2]$ is
(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Now, $x^2 + \sin^2 x = 1$

⇨ $x^2 = 1 - \sin^2 x$
⇨ $x^2 = \cos^2 x$
⇨ $x = \cos x$ (as $\cos x > 0$ in the interval 0 to $\pi/2$)



So, one intersecting point.

⇨ One root

Option (b) is correct.

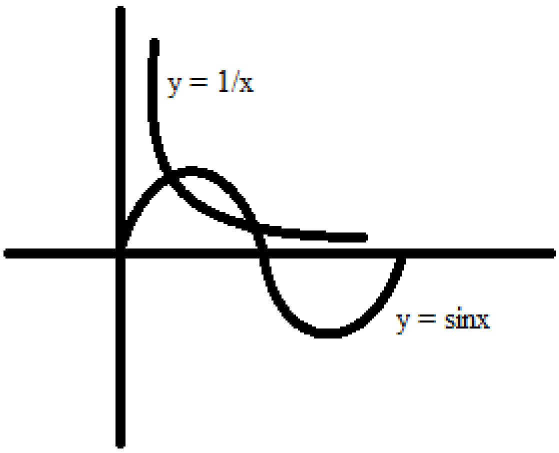
712. The number of roots of the equation xsinx = 1 in the interval $0 < x \leq 2\pi$ is

(a) 0
(b) 1
(c) 2
(d) 4

Solution :

Now, xsinx = 1

⇨ sinx = 1/x



Two intersecting points.

⇨ Two roots.

Option (c) is correct.

713. The number of points in the rectangle $\{(x, y) \mid -10 \le x \le 10$ and $-3 \le y \le 3\}$ which lie on the curve $y^2 = x + \sin x$ and at which the tangent to the curve is parallel to the x-axis, is

(a) 0
(b) 2
(c) 4
(d) 5

Solution :

Now, $y^2 = x + \sin x$

- ⇨ $2y(dy/dx) = 1 + \cos x$
- ⇨ $(dy/dx) = (1 + \cos x)/y = 0$
- ⇨ $\cos x = -1$
- ⇨ $x = \pm\pi, \pm 3\pi$

For $x = \pi$, $y^2 = \pi + \sin\pi = \pi$

- ⇨ $y = \pm\sqrt{\pi}$
- ⇨ $(\pi, \sqrt{\pi})$ and $(\pi, -\sqrt{\pi})$ both inside the rectangle.

Now, $x = -\pi, -3\pi$ doesn't give any solution.

Now, $x = 3\pi$, $y^2 = 3\pi + \sin 3\pi = 3\pi$

- ⇨ $y = \pm\sqrt{(3\pi)} > 3$
- ⇨ 2 points.

Option (b) is correct.

714. The set of all real numbers x satisfying the inequality $x^3(x + 1)(x - 2) \ge 0$ can be written

(a) as $2 \le x \le \infty$
(b) as $0 \le x \le \infty$
(c) as $-1 \le x \le \infty$
(d) in none of the foregoing forms

Solution :

$x > 0, x < -1, x < 2$ => no intersection point. So no solution.

$x < 0, x < -1, x > 2$ => no intersection point, So no solution.

$x < 0, x > -1, x < 2$ => $-1 \le x \le 0$

$x > 0$, $x > -1$, $x > 2 \Rightarrow 2 \le x \le \infty$

Therefore, option (d) is correct.

715. A set S is said to *have a minimum* if there is an element a in S such that $a \le y$ for all y in S. Similarly, S is said to *have a maximum* if there is an element b in S such that $b \ge y$ for all y in S. If $S = \{y : y = (2x + 3)/(x + 2), x \ge 0\}$, which one of the following statements is correct?
(a) S has both a maximum and a minimum
(b) S has neither a maximum nor a minimum
(c) S has a maximum but no minimum
(d) S has a minimum but no maximum

Solution :

Let, $f(x) = (2x + 3)/(x + 2)$

- ⇨ $f'(x) = \{2(x + 2) - (2x + 3)\}/(x + 2)^2 = 1/(x + 2)^2 > 0$
- ⇨ f(x) is increasing.
- ⇨ f(x) doesn't have a maximum
- ⇨ f(x) is minimum at x = 0.
- ⇨ $y_{min} = 3/2$

Option (d) is correct.

716. $\lim (20 + 2\sqrt{x} + 3\sqrt[3]{x})/\{2 + \sqrt{(4x - 3)} + \sqrt[3]{(8x - 4)}\}$ as $x \to \infty$ is
(a) 10
(b) 3/2
(c) 1
(d) 0

Solution :

$\lim (20/\sqrt{x} + 2 + 3/x^{1/6})/\{2/\sqrt{x} + \sqrt{(4 - 3/x)} + 1/(8x - 4)^{1/6}\}$ as $x \to \infty$ (dividing numerator and denominator by $\sqrt{x}$)

$= 2/\sqrt{4} = 1$

Option (c) is correct.

717. $\lim [x\sqrt{(x^2 + a^2)} - \sqrt{(x^4 + a^4)}]$ as $x \to \infty$ is

(a) ∞
(b) $a^2/2$
(c) a^2
(d) 0

Solution :

$\lim [x\sqrt{(x^2 + a^2)} - \sqrt{(x^4 + a^4)}][x\sqrt{(x^2 + a^2)} + \sqrt{(x^4 + a^4)}]/ [x\sqrt{(x^2 + a^2)} + \sqrt{(x^4 + a^4)}]$ as $x \to \infty$

$= \lim (x^4 + a^2x^2 - x^4 - a^4)/ [x\sqrt{(x^2 + a^2)} + \sqrt{(x^4 + a^4)}]$ as $x \to \infty$

$= \lim (a^2 - a^4/x^2)/[\sqrt{(1 + a^2/x^2)} + \sqrt{(1 + a^4/x^4)}]$ as $x \to \infty$

$= a^2/2$

Option (b) is correct.

718. The limit of $x^3[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} - x\sqrt{2}]$ as $x \to \infty$
(a) exists and equals $1/2\sqrt{2}$
(b) exists and equals $1/4\sqrt{2}$
(c) does not exist
(d) exists and equals $3/4\sqrt{2}$

Solution :

$\lim x^3[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} - x\sqrt{2}][\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}]/[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}]$ as $x \to \infty$

$= x^3[x^2 + \sqrt{(x^4 + 1)} - 2x^2]/[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}]$ as $x \to \infty$

$= x^3[\sqrt{(x^4 + 1)} - x^2]/[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}]$ as $x \to \infty$

$= x^3[x^4 + 1 - x^4]/[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}][\sqrt{(x^4 + 1)} + x^2]$ as $x \to \infty$

$= 1/[\sqrt{\{1 + \sqrt{(1 + 1/x^4)}\}} + \sqrt{2}][\sqrt{(1 + 1/x^4)} + 1]$ as $x \to \infty$

$= 1/\{\sqrt{(1 + 1)} + \sqrt{2}\}(1 + 1)$

$= 1/4\sqrt{2}$

Option (b) is correct.

719. If $f(x) = \sqrt{\{(x - \cos^2x)/(x + \sin x)\}}$, then the limit of $f(x)$ as $x \to \infty$ is
(a) 0

(b) 1
(c) ∞
(d) None of 0, 1 or ∞

Solution :

Now, f(x) = √{(1 – $\cos^2$x/x)/(1 + sinx/x)}

Limit of this as x -> ∞ = √(1/1) = 1

Option (b) is correct.

720. Consider the function f(x) = $\tan^{-1}${2tan(x/2)}, where –π/2 ≤ f(x) ≤ π/2. (lim x -> π-0 means limit from the left at π amd lim x -> π+0 means limit from the right.) Then
(a) lim f(x) as x -> π-0 = π/2, lim f(x) as x -> π+0 = -π/2
(b) lim f(x) as x -> π-0 = -π/2, lim f(x) as x -> π+0 = π/2
(c) lim f(x) as x -> π = π/2
(d) lim f(x) as x-> π = -π/2

Solution :

lim f(x) as x-> π-0 = lim $\tan^{-1}${2tan(x/2)} as x-> π-0 = π/2

Now, lim f(x) as x-> π+0 = lim $\tan^{-1}${2tan(x/2)} = - π/2 as tan(π/2 + small value) = -tan(π/2)

Option (a) is correct.

721. The value of lim {(xsina – asinx)/(x – a)} as x -> a is
(a) non-existent
(b) sina + acosa
(c) asina – cosa
(d) sina – acosa

Solution :

Now, lim {(xsina – asinx)/(x – a) as x -> a = lim {(sina – acosx)/1} as x -> a (applying L'Hospital rule) = sina – acosa

Option (d) is correct.

722. The limit lim $[(\cos x - \sec x)/\{x^2(x + 1)\}]$ as $x \to 0$
(a) is 0
(b) is 1
(c) is -1
(d) does not exist

Solution :

Now, lim $[(\cos x - \sec x)/\{x^2(x + 1)\}]$ as $x \to 0$

$= \lim -(\sin^2 x/x^2)[1/\{\cos x(x + 1)\}]$ as $x \to 0$

$= -1*[1/\{1(0 + 1)\}]$

$= -1$

Option (c) is correct.

723. The limit lim $\{(\tan x - x)/(x - \sin x)\}$ as $x \to 0+$ equals
(a) -1
(b) 0
(c) 1
(d) 2

Solution :

Now, lim $\{(\tan x - x)/(x - \sin x)\}$ as $x \to 0+$ = lim $\{(\sec^2 x - 1)/(1 - \cos x)\}$ as $x \to 0+$ (Applying L'Hospital rule) = lim $[(1 - \cos x)(1 + \cos x)/\{\cos^2 x(1 - \cos x)\}]$ as $x \to 0+$ = lim $\{(1 + \cos x)/\cos^2 x\}$ as $x \to 0+$ = 2

Option (d) is correct.

724. lim $[\{(1 + x)^{1/2} - 1\}/\{(1 + x)^{1/3} - 1\}]$ as $x \to 0$ is
(a) 1
(b) 0
(c) 3/2
(d) ∞

Solution :

Now, lim $[\{(1 + x)^{1/2} - 1\}/\{(1 + x)^{1/3} - 1\}]$

= lim [{(1 + x) − 1}{$(1 + x)^{2/3}$ + $(1 + x)^{1/3}$ + 1}/{(1 + x) − 1}{$(1 + x)^{1/2}$ + 1}] as x-> 0

= lim [{$(1 + x)^{2/3}$ + $(1 + x)^{1/3}$ + 1}/{$(1 + x)^{1/2}$ + 1}] as x -> 0

= (1 + 1 + 1)/(1 + 1)

= 3/2

Option (c) is correct.

725. A right circular cylinder container closed on both sides is to contain a fixed volume of motor oil. Suppose its base has diameter d and its height is h. The overall surface area of the container is minimum when
(a) h = (4/3)πd
(b) h = 2d
(c) h = d
(d) conditions other that the foregoing are satisfied

Solution :

Surface area = S = $2\pi(d/2)^2 + 2\pi(d/2)h = \pi d^2/2 + \pi dh$

Now, dS/dd = 2πd/2 + πh = 0 => d = -h

Now, $d^2S/dd^2 = \pi > 0$ so minimum.

Condition is, h = d

Option (c) is correct.

726. lim (logx − x) as x -> ∞
(a) equals +∞
(b) equals e
(c) equals -∞
(d) does not exist

Solution :

lim (logx − x) as x -> ∞ = -∞ (clearly)

Option (c) is correct.

727. lim $x\tan(1/x)$ as $x \to 0$
(a) equals 0
(b) equals 1
(c) equals ∞
(d) does not exist

Solution :

Option (d) is correct.

728. The limit lim ∫ $\{h/(h^2 + x^2)\}dx$ (integration running from x =-1 to x = 1) as $h \to 0$
(a) equals 0
(b) equals π
(c) equals –π
(d) does not exist

Solution :

Now, $\int\{h/(h^2 + x^2)dx$ (integration running from x = -1 to x = 1)

Let, $x = h\tan y$

⇨ $dx = h\sec^2 y dy$
⇨ $x = -1$, $y = -\tan^{-1}(1/h)$ ansd $x = 1$, $y = \tan^{-1}(1/h)$

$\int\{h/(h^2 + x^2)\}dx = \int h(h\sec^2 y dy)/h^2\sec^2 y$ (integration running from $y = -\tan^{-1}(1/h)$ to $y = \tan^{-1}(1/h)$)

= y (upper limit = $\tan^{-1}(1/h)$ and lower limit = $-\tan^{-1}(1/h)$

= $2\tan^{-1}(1/h)$

Now, lim $2\tan^{-}(1/h)$ as $h \to 0$ doesn't exist

Option (d) is correct.

729. If the area of an expanding circular region increases at a constant rate (with respect to time), then the rate of increase of the perimeter with respect to time
(a) varies inversely as the radius
(b) varies directly as the radius
(c) varies directly as the square of the radius
(d) remains constant

Solution :

$A = \pi r^2$

- ⇨ $dA/dt = 2\pi r(dr/dt)$ = constant = k
- ⇨ $dr/dt = k/(2\pi r)$

Now, perimeter = $P = 2\pi r$

- ⇨ $dP/dt = 2\pi(dr/dt) = 2\pi k/(2\pi r) = k/r$
- ⇨ varies inversely as the radius

Option (a) is correct.

730. Let $y = \tan^{-1}[\{\sqrt{(1 + x^2)} - 1\}/x]$. Then dy/dx equals
    (a) $1/\{2(1 + x^2)\}$
    (b) $2/(1 + x^2)$
    (c) $(-1/2)\{1/(1 + x^2)\}$
    (d) $-2/(1 + x^2)$

Solution :

$dy/dx = 1/[1 + \{\sqrt{(1 + x^2)} - 1\}^2/x^2][x^2/\sqrt{(1 + x^2)} - \{\sqrt{(1 + x^2)} - 1\}]/x^2$

$= [1/\{x^2 + 1 + x^2 - 2\sqrt{(1 + x^2)} + 1\}][x^2 - 1 - x^2 + \sqrt{(1 + x^2)}]$

$= (1/[\{2\sqrt{(1 + x^2)}\}\{\sqrt{(1 + x^2)} - 1\}])\{\sqrt{(1 + x^2)} - 1\}$

$= 1/\{2\sqrt{(1 + x^2)}\}$

Option (a) is correct.

731. If θ is an acute angle then the largest value of $3\sin\theta + 4\cos\theta$ is
    (a) 4
    (b) $3(1 + \sqrt{3}/2)$
    (c) $5\sqrt{2}$
    (d) 5

Solution :

Now, $3\sin\theta + 4\cos\theta$

$= 5\{(3/5)\sin\theta + (4/5)\cos\theta\}$

= 5{sinθcosα + cosθsinα) where cosα = 3/5 i.e. sinα = 4/5

= 5sin(θ + α)

Maximum value = 5

Option (d) is correct.

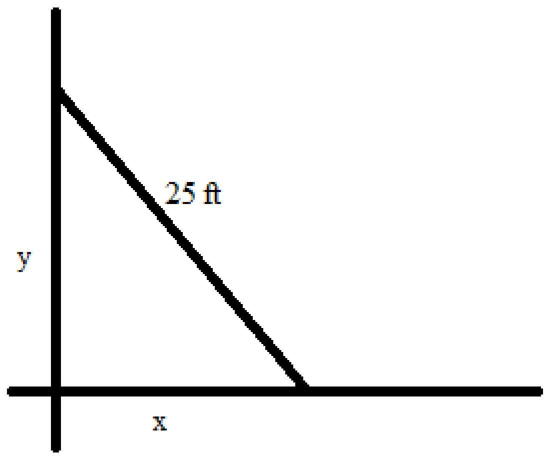
732. Let $f(x) = (x - 1)e^x + 1$. Then
(a) $f(x) \geq 0$ for all $x \geq 0$ and $f(x) < 0$ for all $x < 0$
(b) $f(x) \geq 0$ for all $x \geq 1$ and $f(x) < 0$ for all $x < 1$
(c) $f(x) \geq 0$ for all x
(d) none of the foregoing statements is true

Solution :

Clearly option (c) is correct. You can check by considering values > 0 and < 0 or by drawing graph of f(x).

733. A ladder AB, 25 feet (ft) (1 ft = 12 inches (in)) long leans against a vertical wall. The lower end A, which is at a distance of 7 ft from the bottom of the wall, is being moved away along the ground from the wall at the rate of 2 ft/sec. Then the upper end B will start moving towards the bottom of the wall at the rate of (in in/sec)
(a) 10
(b) 17
(c) 7
(d) 5

Solution :



Now, $x^2 + y^2 = (25*12)^2$

- ⇨ $2x(dx/dt) + 2y(dy/dt) = 0$
- ⇨ $(dy/dt) = -(x/y)(dx/dt)$

At $x = 7*12$, $y = \sqrt{(25*12)^2 - (7*12)^2} = 24*7$

$(dy/dt) = -(7*12/24*12)*(2*12) = -7$ in/sec

(-) occurred due to the opposite motion of x and y.

Option (c) is correct.

734. Let $f(x) = ||x - 1| - 1|$ if $x < 1$ and $f(x) = [x]$ if $x \geq 1$, where, for any real number x, [x] denotes the largest integer $\leq x$ and |y| denotes absolute value of y. Then, the set of discontinuity-points of the function f consists of
(a) all integers ≥ 0
(b) all integers ≥ 1
(c) all integers > 1
(d) the integer 1

Solution :

Let us first check at x = 1.

lim f(x) as x -> 1- = lim ||x – 1| - 1| as x -> 1- = 1

lim f(x) as x-> 1+ = lim [x] as x-> 1+ = 1

So, continuous at x = 1.

Let us now check at x = 0.

lim f(x) as x -> 0- = lim ||x – 1| - 1| as x -> 0- = 0

lim f(x) as x-> 0+ = lim ||x – 1| - 1| as x-> 0+ = 0

So continuous at x = 0.

Let us now check at x= 2.

lim f(x) as x -> 2- = lim [x] as x -> 2- = 1

lim f(x) as x-> 2+ = lim [x] as x -> 2+ = 2

Discontinuous at x = 2.

Option (c) is correct.

735. Let f and g be two functions defined on an interval I such that f(x) ≥ 0 and g(x) ≤ 0 for all x Ɛ I, and f is strictly decreasing on I while g is strictly increasing on I. Then
(a) the product function fg is strictly increasing on I
(b) the product function fg is strictly decreasing on I
(c) the product function fg is increasing but not necessarily strictly increasing on I
(d) nothing can be said about the monotonicity of the product function fg

Solution :

Now, f' < 0 and g ≤ 0 => f'g ≥ 0

And, g' > 0 and f ≥ 0 => fg' ≥ 0

- ⇨ f'g + fg' ≥ 0
- ⇨ (fg)' ≥ 0

Now the equality holds if and only if f and g are zero at same point. But f(x) is decreasing that means f = 0 at final value of I if we order the set I from decreasing to increasing value and g = 0 at first value of I as g is increasing. So they cannot be equal to zero together.

- ⇨ fg is strictly increasing

Option (a) is correct.

736. Given that f is a real-valued differentiable function such that f(x)f'(x) < 0 for all real x, it follows that
(a) f(x) is an increasing function

(b) $f(x)$ is a decreasing function
(c) $|f(x)|$ is an increasing function
(d) $|f(x)|$ is a decreasing function

Solution :

Now, $f(x)$ is differentiable means $f(x)$ is continuous. Now $f(x) \neq 0$ for any x.

Therefore, $f(x)$ either > 0 or < 0

If $f(x) > 0$ then $f'(x) < 0$ and if $f(x) < 0$ then $f'(x) > 0$

- ⇨ $f(x)$ is either increasing or decreasing function.

Now, $|f(x)| > 0$ so, $|f(x)|' < 0$

- ⇨ $|f(x)$ is decreasing function.

Option (d) is correct.

737. Let x and y be positive numbers. Which of the following always implies $x^y \geq y^x$?
(a) $x \leq e \leq y$
(b) $y \leq e \leq x$
(c) $x \leq y \leq e$ or $e \leq y \leq x$
(d) $y \leq x \leq e$ or $e \leq x \leq y$

Solution :

Let us take, $x, y < e$, say $x = 2$, $y = 1$

Now, $2^1 > 1^2$

- ⇨ $x^y > y^x$
- ⇨ So if $x, y < e$ then $x > y$

It is with option (d).

Now, let us check the other part of option (d).

Let, $x, y > e$, $x = 4$ and $y = 5$

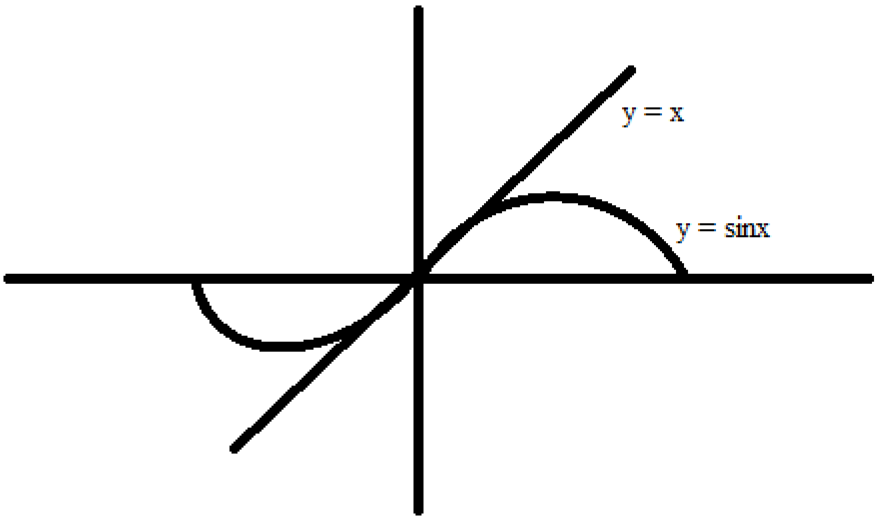
Now, $4^5 > 5^4$

- ⇨ $x^y > y^x$
- ⇨ Option (d) is correct.

738. Let f be a function $f(x) = \cos x - 1 + x^2/2$. Then
(a) f(x) is an increasing function on the real line
(b) f(x) is a decreasing function on the real line
(c) f(x) is an increasing function in the interval $-\infty < x \leq 0$ and decreasing in the interval $0 \leq x < \infty$
(d) f(x) is a decreasing function in the interval $-\infty < x \leq 0$ and increasing in the interval $0 \leq x < \infty$

Solution :

Now, $f'(x) = x - \sin x$



From the figure it is clear that for $x > 0$, $x - \sin x > 0$ and for $x < 0$ $x - \sin x < 0$

Option (d) is correct.

739. Consider the function f(n) defined for all positive integers as follows :
$f(n) = n + 1$ if n is odd, and $f(n) = n/2$ if n is even.
Let $f^{(k)}$ denote f applied k times; e.g., $f^{(1)}(n) = f(n)$, $f^{(2)}(n) = f(f(n))$ and so on. Then
(a) there exists one integer k_0 such that $f^{(k0)}(n) = 1$
(b) for each $n \geq 2$, there exists an integer k (depending on n) such that $f^{(k)}(n) = 1$
(c) for each $n \geq 2$, there exists an integer k (depending on n) such that $f^{(k)}(n)$ is a multiple of 4
(d) for each n, $f^{(k)}(n)$, $f^{(k)}(n)$ is a decreasing function of k

Solution :

Let us take an odd integer m.

f(m) = m + 1

f((m + 1)) = (m + 1)/2

if it is odd, then f((m + 1)/2) = (m + 1)/2 + 1 = (m + 3)/2 which is even

f((m + 3)/2) = (m + 3)/4

Now, (m + 1)/2 – (m + 1) = -(m + 1)/2

(m + 3)/4 – (m + 1)/2 = -(m + 1)/2

- ⇨ We are getting always a decreased number when the function is applied after two operation when it is odd and when it is even then it is getting halved.
- ⇨ All the function values must come to 1 (the minimum positive integer) but number of application of f may differ.
- ⇨ Option (b) is correct.

740. Let $p_n(x)$, n = 0, 1, …. be a polynomial defined by $p_0(x) = 1$, $p_1(x) = x$ and $p_n(x) = xp_{(n-1)}(x) - p_{(n-2)}(x)$ for $n \geq 2$. Then $p_{10}(0)$ equals
(a) 0
(b) 10
(c) 1
(d) -1

Solution :

Now, $p_n(x) = xp_{(n-1)}(x) - p_{(n-2)}(x)$

- ⇨ $p_{(n)}(0) = -p_{(n-2)}(0)$
- ⇨ $p_{10}(0) = -p_8(0) = p_6(0) = -p_4(0) = p_2(0) = -p_0(0) = -1$

Option (d) is correct.

741. Consider the function f(x) = x(x – 1)(x + 1) from **R** to **R**, where **R** is the set of all real numbers. Then,
(a) f is one-one and onto
(b) f is neither one-one nor onto
(c) f is one-one but not onto

(d) f is not one-one but onto

Solution :

Now, $f(x) = x(x - 1)(x + 1) = 0$ for $x = 0, -1, 1$

So, $f(x)$ is not one-one.

As $f(x)$ is a polynomial function so $f(x)$ is continuous everywhere.

And $f(\infty) = \infty$ and $f(-\infty) = -\infty$

So, f in onto

Option (d) is correct.

742. For all integers $n \geq 2$, define $f_n(x) = (x + 1)^{1/n} - x^{1/n}$, where $x > 0$. Then, as a function of x

(a) f_n is increasing for all n

(b) f_n is decreasing for all n

(c) f_n is increasing for n odd and f_n is decreasing for n even

(d) f_n is decreasing for n odd and f_n is increasing for n even

Solution :

$f_n'(x) = (1/n)(x + 1)^{1/n - 1} - (1/n)x^{1/n - 1}$

⇨ $f_n'(x) = (1/n)[1/(1 + x)^{(n - 1)/n} - 1/x^{(n - 1)/n}] = (1/n)[x^{(n - 1)/n} - (1 + x)^{(n - 1)/n}]/\{x(1 + x)\}^{(n - 1)/n} < 0$

⇨ f_n is decreasing for all n.

Option (b) is correct.

743. Let $g(x) = \int tf'(t)dt$ (integration running from $t = -10$ to $t = x$) for $x \geq -10$, where f is an increasing function. Then

(a) $g(x)$ is an increasing function of x

(b) $g(x)$ is a decreasing function of x

(c) $g(x)$ is increasing for $x > 0$ and decreasing for $-10 < x < 0$

(d) none of the foregoing conclusions is necessarily true

Solution :

Now, $g'(x) = xf'(x) > 0$ for $x > 0$ as $f(x)$ is increasing and < 0 for $x < 0$

Option (c) is correct.

744. Let $f(x) = x^3 - x + 3$ for $0 < x \leq 1$, $f(x) = 2x + 1$ for $1 < x \leq 2$, $f(x) = x^2 + 1$ for $2 < x < 3$. Then
   (a) f(x) is differentiable at x = 1 and at x = 2
   (b) f(x) is differentiable at x = 1 but not at x = 2
   (c) f(x) is differentiable at x = 2 but not at x = 1
   (d) f(x) is differentiable neither at x = 1 nor at x = 2

Solution :

lim {f(x) – f(1)}/(x – 1) as x -> 1- = lim $\{x^3 - x + 3 - 3\}/(x - 1)$ as x -> 1- = lim x(x – 1)(x + 1)/(x – 1) as x -> 1- = lim x(x + 1) as x -> 1- = 2

lim {f(x) – f(1)}/(x – 1) as x -> 1+ = lim (2x + 1 – 3)/(x – 1) as x -> 1+ = lim 2(x – 1)/(x – 1) as x -> 1+ = lim 2 as x -> 1+ = 2

f(x) is differentiable at x = 1.

lim {f(x) – f(2)}/(x – 2) as x -> 2- = lim {2x + 1 – 5}/(x – 2) as x-> 2- = lim 2(x – 2)/(x – 2) as x -> 2- = lim 2 as x-> 2- = 2

lim {f(x) – f(2)}/(x – 2) as x -> 2+ = lim $\{x^2 + 1 - 5\}/(x - 2)$ as x-> 2+ = lim (x – 2)(x + 2)/(x – 2) as x->2+ = lim (x + 2) as x -> 2+ = 4

The two limits are not same. Hence f(x) is not differentiable at x = 2.

Option (b) is correct.

745. If the function $f(x) = (x^2 - 2x + A)/\sin x$ when $x \neq 0$, f(x) = B when x = 0, is continuous at x = 0, then
   (a) A = 0, B = 0
   (b) A = 0, B = -2
   (c) A = 1, B = 1
   (d) A = 1, B = 0

Solution :

lim f(x) as x-> 0 = lim $(x^2 - 2x + A)/\sin x$ as x -> 0

To exist the limit A = 0 (must be)

Therefore, lim $(x^2 - 2x)/\sin x$ as x -> 0 = lim $(2x - 2)/\cos x$ as x-> 0 (Applying L'Hospital rule) = -2

$B = -2$

Option (b) is correct.

746. The function $f(x) = (1 - \cos 4x)/x^2$ if $x < 0$, $f(x) = a$ if $x = 0$, $f(x) = 2\sqrt{x}/\{\sqrt{(16 + \sqrt{x})} - 4\}$ if $x > 0$, is continuous at $x = 0$ for
(a) $a = 8$
(b) $a = 4$
(c) $a = 16$
(d) no value of a

Solution :

lim f(x) as $x \to 0-$ = lim $(1 - \cos 4x)/x^2$ as $x \to 0-$ = lim $4\sin 4x/2x$ as $x \to 0-$ (Applying L'Hospital rule) = lim $16\cos 4x/2$ as $x \to 0-$ (Again applying L'Hospital rule) = 8

lim f(x) as $x \to 0+$ = lim $2\sqrt{x}/\{\sqrt{(16 + \sqrt{x})} - 4\}$ as $x \to 0+$ = lim $2\sqrt{x}\{\sqrt{(16 + \sqrt{x})} + 4\}/(16 + \sqrt{x} - 16)$ as $x \to 0+$ = lim $2\sqrt{x}\{\sqrt{(16 + \sqrt{x})} + 4\}/\sqrt{x}$ as $x \to 0+$ = lim $2\{\sqrt{(16 + \sqrt{x})} + 4\}$ as $x \to 0+$ = 16

Therefore, the limit doesn't exist.

Option (d) is correct.

747. Consider the function $f(x) = 0$ if x is rational, $f(x) = x^2$ if x is irrational. Then only one of the following statements is true. Which one is it?
(a) f is differentiable at $x = 0$ but not continuous at any other point
(b) f is not continuous anywhere
(c) f is continuous but not differentiable at $x = 0$
(d) None of the foregoing statements is true.

Solution :

Clearly, option (a) is correct.

748. Let $f(x) = x\sin(1/x)$ if $x \neq 0$, and let $f(x) = 0$ if $x = 0$. Then f is
(a) not continuous at 0
(b) continuous but not differentiable at 0

(c) differentiable at 0 and f′(0) = 1
(d) differentiable at 0 and f′(0) = 0

Solution :

lim f(x) as x->0 = lim xsin(1/x) as x->0 = 0

f(0) = 0

So, continuous at x = 0

Now, lim {f(x) – f(0)}/(x – 0) as x -> 0 = lim xsin(1/x)/x as x->0 = lim sin(1/x) as x -> 0 doesn't exist.

So, not differentiable.

Option (b) is correct.

749. Let f(x) be the function defined on the interval (0, 1) by f(x) = x if x is rational, f(x) = 1 – x otherwise. Then f is continuous
(a) at no point in (0, 1)
(b) at exactly one point in (0, 1)
(c) at more than one point, but finitely many points in (0, 1)
(d) at infinitely many points in (0, 1)

Solution :

Clearly, f is continuous at x = ½

Option (b) is correct.

750. The function f(x) = [x] + √(x – [x]), where [x] denotes the largest integer smaller than or equal to x, is
(a) continuous at every real number x
(b) continuous at every real number x except at negative integer values
(c) continuous at every real number x except at integer values
(d) continuous at every real number x except x = 0

Solution :

Let, x = -n where n > 0 i.e. x is a negative integer.

x – [x] = -n – [-n] = - n –(-n) = -n + n = 0

Therefore, f(x) = -n.

lim f(x) as x-> -n- = lim [x] + √(x – [x]) as x -> -n- = -n + √(-n – (-n)) = -n

lim f(x) as x-> -n+ = lim [x] + √(x – [x]) as x -> -n+ = -(n + 1) + √(-n – (-(n + 1))) = -(n + 1) + √(-n + n + 1) = -n – 1 + 1 = -n

So, f(x) is continuous at negative integer values.

So, option (b) and (c) cannot be true.

At x = 0,

lim f(x) as x-> 0- = lim [x] + √(x – [x]) as x-> 0- = -1 + √(0 – (-1)) = -1 + 1 = 0

lim f(x) as x-> 0+ = lim [x] + √(x – [x]) as x-> 0+ = 0 + √(0 – 0) = 0

f(x) is continuous at x = 0

Option (d) cannot be true.

Option (a) is correct.

751. For any real number x and any positive integer n, we can uniquely write x = mn + r, where m is an integer (positive, negative or zero) and 0 ≤ r < n. With this notation we define x mod n = r. For example, 13.2 mod 3 = 1.2. The number of discontinuity points of the function f(x) = $(x \text{ mod } 2)^2$ + (x mod 4) in the interval 0 < x < 9 is
(a) 0
(b) 2
(c)4
(d) 6

Solution :

Now, at x = 2,

lim f(x) as x-> 2- = lim $(x \text{ mod} 2)^2$ + (x mod 4) as x-> 2- = 2^2 + 2 = 6

lim f(x) as x-> 2+ = lim $(x \text{ mod } 2)^2$ + (x mod 4) as x -> 2+ = 0^2 + 2 = 2

discontinuous at x = 2.

Similarly, discontinuous at x = 4, 6, 8 i.e. all even numbers.

Option (c) is correct.

752. Let f(x) and g(x) be defined as follows :
$f(x) = x$ if $x \geq 0$, $f(x) = 0$ if $x < 0$
$g(x) = x^2$ if $x \geq 0$, $g(x) = 0$ if $x < 0$
Then
(a) f and g both differentiable at x = 0
(b) f is differentiable at x = 0 but g is not
(c) g is differentiable at x = 0 but f is not
(d) neither f nor g is differentiable at x = 0

Solution :

lim (f(x) – f(0)}/(x – 0) as x -> 0- = lim (0 – 0)/(x – 0) as x-> 0- = 0

lim {f(x) – f(0)}/(x – 0) as x-> 0+ lim (x – 0)/x as x ->0+ = lim x/x as x -> 0+ = lim 1 as x-> 0+ = 1

f is not differentiable at x = 0

lim {g(x) – g(0)}/(x – 0) x -> 0- = lim (0 - 0)/(x – 0) as x-> 0- = 0

lim {g(x) – g(0)}/(x – 0) as x-> 0+ = lim (x^2 – 0)/x as x-> 0+ = lim x as x -> 0+ = 0

g(0) = 0

g is differentiable at x = 0

Option (c) is correct.

753. The number of points at which the function f(x) = min{|x|, x^2} if $-\infty < x < 1$, f(x) = min{2x – 1, x^2} otherwise, is not differentiable is
(a) 0
(b) 1
(c) 2
(d) More than 1

Solution :

At x = 1,

lim {f(x) – f(1)}/(x – 1) as x -> 1- = lim (x^2 – 1)/(x – 1) as x-> 1- = lim (x + 1) as x-> 1- = 2 lim $\{f(x) - f(1)\}/(x - 1)$ as $x \to 1+$ = lim $(2x - 1 - 1)/(x - 1)$ as $x \to 1+$ = lim 2 as $x \to 1+$ = 2

at $x = 1$ $f(x)$ is differentiable.

Now, $x^2 > 2x - 1$ i.e. $x^2 - 2x + 1 > 0$ i.e. $(x - 1)^2 > 0$ i.e. $x > 1$.

So, for $x > 1$ $2x - 1$ is always minimum.

So, at every point > 1 $f(x)$ is differentiable.

$f(x) = x^2$ if $x > -1$ and $f(x) = |x|$ if $x < -1$

So, we check differentiability at $x = -1$.

lim $\{f(x) - f(-1)\}/(x + 1)$ as $x \to -1-$ = lim $(|x| - 1)/(x + 1)$ as $x \to -1-$ = lim $-(x + 1)/(x + 1)$ as $x \to -1-$ = -1

lim $\{f(x) - f(-1)\}/(x + 1)$ as $x \to -1+$ = lim $(x^2 - 1)/(x + 1)$ as $x \to -1+$ = lim $(x - 1)$ as $x \to -1+$ = - 2

Not differentiable at $x = -1$

Option (b) is correct.

754. The function $f(x)$ is defined as $f(x) = 1/|x|$, for $|x| > 2$, $f(x) = a + bx^2$ for $|x| \le 2$, where a and b are known constants. Then, only one of the following statements is true. Which one is it?

(a) $f(x)$ is differentiable at $x = -2$ if and only if $a = ¾$ and $b = -1/16$

(b) $f(x)$ is differentiable at $x = -2$, whatever be the values of a and b

(c) $f(x)$ is differentiable at $x = -2$, if $b = -1/16$ whatever be the value of a

(d) $f(x)$ is differentiable at $x = -2$, if $b = 1/16$ whatever be the value of a

Solution :

lim $\{f(x) - f(-2)\}/(x + 2)$ as $x \to -2-$ = lim $\{-1/x - a - 4b)/(x + 2)$ as $x \to -2-$

To exist the limit, $a + 4b = 1/2$

Therefore, the limit is, lim $\{-1/x - ½\}/(x + 2)$ as $x \to -2-$ = lim $-1/(2x)$ as $x \to -2-$ = 1/4 lim {f(x) − f(-2)}/(x + 2) as x -> -2+ = lim {a + bx^2 − a − 4b}/(x + 2) as x -> -2+ = lim b(x − 2)(x + 2)/(x + 2) as x -> -2+ = lim b(x − 2) as x -> -2+ = -4b

So, -4b = 1/4

- ⇨ b = -1/16
- ⇨ a = 1/2 + 4/16 = 1/2 + ¼ = ¾

Option (a) is correct.

755. The function f(x) = $\sin x^2/x$ if x ≠ 0, f(x) = 0 if x = 0
(a) is continuous, but not differentiable at x = 0
(b) is differentiable at x = 0 but the derivative is not continuous at x = 0
(c) is differentiable at x = 0 and the derivative is continuous at x = 0
(d) is not continuous at x = 0

Solution :

lim f(x) as x -> 0 = lim $\sin x^2/x$ as x -> 0 = lim $x(\sin x^2/x^2)$ as x -> 0 = 0*1 = 0

And f(0) = 0.

f(x) is continuous at x = 0.

lim {f(x) − f(0)}/(x − 0) as x -> 0 = lim {($\sin x^2/x$)/x} as x -> 0 = lim $\sin x^2/x^2$ as x^2 -> 0 = 1

Differentiable at x = 0.

f′(x) = ($2x\sin x^2 * x - 2x\sin x^2$)/$x^2$ = $2\sin x^2 - 2\sin x^2/x$

lim f′(x) as x -> 0 = lim $2\sin x^2 - 2\sin x^2/x$ as x -> 0 = lim − $x(\sin x^2/x^2)$ as x -> 0 = -0*1 = 0

f′(0) = 0

So, f′(x)is continuous at x = 0.

Option (c) is correct.

756. Let f(x) = x[x] where [x] denotes the greatest integer smaller then or equal to x. When x is not an integer, what is f′(x)?
(a) 2x

(b) [x]
(c) 2[x]
(d) It doesn't exist.

Solution :

f(x) = x[x]

⇨ f'(x) = [x] + x*d/dx([x])

Now, d/dx[x] = 0 as it is constant.

Therefore, f(x) = [x]

Option (b) is correct.

757. If f(x) = (sinx)(sin2x)….(sinnx), then f'(x) is
(a) ∑kcoskxf(x) (summation running from k = 1 to k = n)
(b) (cosx)(2cos2x)(3cos3x)…(ncosnx)
(c) ∑(kcoskx)(sinkx) (summation running from k = 1 to k = n)
(d) ∑(kcotkx)f(x) (summation running from k = 1 to k = n)

Solution :

f'(x) = (cosx)(sin2x)(sin3x)….(sinnx) + (sinx)(2cos2x)(sin3x)(sin4x)…(sinnx) + …. + (sinx)(sin2x)…(sin(n – 1)x)(ncosnx)

= (cotx)f(x) + (2cot2x)f(x) + …. + (ncotnx)f(x)

= ∑(kcotkx)f(x) (summation running from k = 1 to k = n)

Option (d) is correct.

758. Let $f(x) = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3$ where a_0, a_1, a_2 and a_3 are constants. Then only one of the following statements is correct. Which one is it?
(a) f(x) is differentiable at x = 0 whatever be a_0, a_1, a_2, a_3
(b) f(x) is not differentiable at x = 0 whatever be a_0, a_1, a_2, a_3
(c) If f(x) is differentiable at x = 0, then $a_1 = 0$
(d) If f(x) is differentiable at x = 0, then $a_1 = 0$ and $a_3 = 0$

Solution :

lim $\{f(x) - f(0)\}/(x - 0)$ as $x \to 0-$ = lim $\{a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3 - a_0\}/x$ as lim $x \to 0-$ = lim $(-a_1x + a_2x^2 - a_3x^3)/x$ as $x \to 0-$ = lim $(-a_1 + a_2x - a_3x^2)$ as $x \to 0-$ = $-a_1$

lim $\{f(x) - f(0)\}/(x - 0)$ as $x \to 0+$ = lim $\{a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3\}/x$ as $x \to 0+$ = lim $(a_1x + a_2x^2 + a_3x^3)/x$ as $x \to 0+$ = lim $(a_1 + a_2x + a_3x^2)$ as $x \to 0+$ = a_1

Now, if f(x) is differentiable at x = 0 then $-a_1 = a_1$ i.e. $a_1 = 0$

Option (c) is correct.

759. Consider the function f(x) = |sinx| + |cosx| defined for x in the interval (0, 2π). Then
   (a) f(x) is differentiable everywhere
   (b) f(x) is not differentiable at x = π/2 and 3π/2 and differentiable everywhere else
   (c) f(x) is not differentiable at x = π/2, π and 3π/2 and differentiable everywhere else
   (d) none of the foregoing statements is true

Solution :

At x = π/2

lim $\{f(x) - f(\pi/2)\}/(x - \pi/2)$ as $x \to \pi/2-$ = lim $\{|\sin x| + |\cos x| - 1\}/(x - \pi/2)$ as $x \to \pi/2-$ = lim $(\sin x + \cos x - 1)/(x - \pi/2)$ as $x \to \pi/2-$ = lim $(\cos x - \sin x)/1$ as $x \to \pi/2-$ (Applying L'Hospital rule) = -1

lim $\{f(x) - f(\pi/2)\}/(x - \pi/2)$ as $x \to \pi/2+$ = lim $\{|\sin x| + |\cos x| - 1\}/(x - \pi/2)$ as $x \to \pi/2+$ = lim $(\sin x - \cos x - 1)/(x - \pi/2)$ as $x \to \pi/2+$ = lim $(\cos x + \sin x)/1$ as $x \to \pi/2+$ (Applying L'Hispital rule) = 1

Not differentiable at x = π/2

At x = 3π/2

lim $\{f(x) - f(3\pi/2)\}/(x - 3\pi/2)$ as $x \to 3\pi/2-$ = lim $\{|\sin x| + |\cos x| - 1\}/(x - 3\pi/2)$ as $x \to 3\pi/2-$ = lim $(-\sin x - \cos x - 1)/(x - 3\pi/2)$ as $x \to 3\pi/2-$ (both sinx and cosx are in 3rd quadrant where only tan is positive) = lim $(-\cos x + \sin x)/1$ as $x \to 3\pi/2-$ (Applying L'Hospital rule) = -1

lim $\{f(x) - f(3\pi/2)\}/(x - 3\pi/2)$ as $x \to 3\pi/2+$ = lim $\{|\sin x| + |\cos x| - 1\}/(x - 3\pi/2)$ as $x \to 3\pi/2+$ = lim $(-\sin x + \cos x - 1)/(x - 3\pi/2)$ as $x \to 3\pi/2+$ (sinx is negative and cosx is positive in 4th quadrant) = lim $(-\cos x - \sin x)/1$ as $x \to 3\pi/2+$ = 1

Not differentiable at x = 3π/2

At x = π

lim {f(x) − f(π)}/(x − π) as x -> π- = lim {|sinx| + |cosx| - 1}/(x − π) as x -> π- = lim (sinx − cosx − 1)/(x − π) as x -> π- (sinx is positive and cosx is nrgative in 2^{nd} quadrant) = lim (cosx + sinx)/1 as x -> π- (Applying L'Hospital rule) = -1

lim {f(x) − f(π)}/(x − π) as x -> π+ = lim {|sinx| + |cosx| - 1}/(x − π) as x -> π+ = lim (-sinx − cosx − 1)/(x − π) as x -> π+ = lim (-cosx + sinx)/1 as x -> π+ (Applying L'Hospital rule) = 1

Not differentiable at x = π

Option (c) is correct.

760. A curve in the XY plane is given by the parametric equations $x = t^2 + t + 1$, $y = t^2 - t + 1$, where the parameter t varies over all nonnegative real numbers. The number of straight line passing through the point (1, 1) which are tangent to the curve, is
(a) 2
(b) 0
(c) 1
(d) 3

Solution :

Now, x − y = 2t

- ⇨ t = (x − y)/2
- ⇨ $x = (x - y)^2/4 + (x - y)/2 + 1$
- ⇨ $4x = x^2 - 2xy + y^2 + 2x - 2y + 4$
- ⇨ $x^2 + y^2 - 2xy - 2x - 2y + 4 = 0$

Equation of the tangent is y − 1 = m(x − 1)

- ⇨ y = mx − (m − 1)

Putting the value of y in the equation of curve we get,

$x^2 + \{mx - (m - 1)\}^2 - 2x\{mx - (m - 1)\} - 2x - 2\{mx - (m - 1)\} + 4 = 0$

- ⇨ $x^2 + m^2x^2 - 2m(m - 1)x + (m - 1)^2 - 2mx^2 + 2(m - 1)x - 2x - 2mx + 2(m - 1) + 4 = 0$
- ⇨ $x^2(m^2 - 2m + 1) - 2x(m^2 - m - 2m + 2 + 2 + 2m) + \{(m - 1)^2 + 2(m - 1) + 4\} = 0$
- ⇨ $x^2(m - 1)^2 - 2x(m^2 - m + 4) + (m^2 + 3) = 0$

Now, discriminant = 0

- ⇨ $4(m^2 - m + 4)^2 - 4(m - 1)^2(m^2 + 3) = 0$
- ⇨ $m^4 + m^2 + 16 - 2m^3 - 8m + 8m^2 - (m^2 - 2m + 1)(m^2 + 3) = 0$
- ⇨ $m^4 - 2m^3 + 9m^2 - 8m + 16 - m^4 - 3m^2 + 6m + 2m^3 - m^2 - 3 = 0$
- ⇨ $5m^2 - 2m + 13 = 0$
- ⇨ No solution as discriminant < 0
- ⇨ No tangents can be drawn.

Option (b) is correct.

761. If $f(x) = \{(a + x)/(b + x)\}^{a + b + 2x}$, then $f'(0)$ equals
    (a) $\{(b^2 - a^2)/b^2\}(a/b)^{a + b - 1}$
    (b) $\{2\log(a/b) + (b^2 - a^2)/ab\}(a/b)^{a + b}$
    (c) $2\log(a/b) + (b^2 - a^2)/ab$
    (d) None of the foregoing expressions

Solution :

$\log f(x) = (a + b + 2x)[\log(a + x) - \log(b + x)]$

$f'(x)/f(x) = 2[\log(a + x) - \log(b + x)] + (a + b + 2x)[1/(a + x) - 1/(b + x)$

- ⇨ $f'(0)/f(0) = 2[\log a - \log b] + (a + b)(1/a - 1/b)$
- ⇨ $f'(0) = \{(a/b)^{a + b}\}\{2\log(a/b) + (b^2 - a^2)/ab\}$

Option (b) is correct.

762. If $y = 2\sin^{-1}\sqrt{(1 - x)} + \sin^{-1}[2\sqrt{\{x(1 - x)\}}]$ for $0 < x < ½$ then dy/dx equals
    (a) $2/\sqrt{\{x(1 - x)\}}$
    (b) $\sqrt{\{(1 - x)/x\}}$
    (c) $-1/\sqrt{\{x(1 - x)\}}$
    (d) 0

Solution :

Let $x = \cos^2 A$

Now,$2 \sin^{-1}\sqrt{(1 - \cos^2 A)} = 2\sin^{-1}\sin A = 2A$

Now, $\sin^{-1}[2\cos A\sin A] = \sin^{-1}(\sin 2A) = 2A$

Therefore, $y = 2A + 2A = 4A = 4\cos^{-1}\sqrt{x}$

$dy/dx = -4/\sqrt{(1 - x)}$

It is given option (d) is correct.

763. If $y = \sin^{-1}(3x - 4x^3)$ then dy/dx equals
   (a) $3x$
   (b) 3
   (c) $3/\sqrt{(1 - x^2)}$
   (d) None of the foregoing expressions.

Solution :

Let $x = \sin A$

$3x - 3x^3 = 3\sin A - \sin^3 A = \sin 3A$

$\sin^{-1}(3x - 3x^3) = \sin^{-1}(\sin 3A) = 3A = 3\sin^{-1}x$

$dy/dx = 3/\sqrt{(1 - x^2)}$

Option (c) is correct.

764. If $y = 3^{\sin ax/\cos bx}$, then dy/dx is
   (a) $3\text{^}\{(a\cos ax\cos bx + b\sin ax\sin bx)/\text{cox}^2 bx\}$
   (b) $3^{\sin ax/\cos bx}\{(a\cos ax\cos bx + b\sin ax\sin bx)/\cos^2 bx\}\log 3$
   (c) $3^{\sin ax/\cos bx}\{(a\cos ax\cos bx - b\sin ax\sin bx)/\cos^2 bx\}\log 3$
   (d) $3^{\sin ax/\text{cobx}}\log 3$

Solution :

$\log y = (\sin ax/\cos bx)\log 3$

$(dy/dx)/y = \{(a\cos ax\cos bx + b\sin bx\sin ax)/\cos^2 bx\}\log 3$

Option (b) is correct.

765. $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, then the value of d^2y/dx^2 at $\theta = \pi/2$ equals
   (a) $-1/a$
   (b) $-1/4a$
   (c) $-a$
   (d) None of the foregoing numbers.

Solution :

Now, $d^2y/dx^2 = d/dx(dy/dx) = d/dx\{(dy/d\theta)/(dx/d\theta)\} = d/d\theta\{(dy/d\theta)/(dx/d\theta)\}(d\theta/dx)$

$= \{(d^2y/d\theta^2)(dx/d\theta) - (d^2x/d\theta^2)(dy/d\theta)\}/(dx/d\theta)^3$

Now put the values here and get the answer.

Option (a) is correct.

766. Let $F(x) = e^x$, $G(x) = e^{-x}$ and $H(x) = G(F(x))$, where x is a real number. Then dH/dx at x = 0 is
(a) 1
(b) -1
(c) -1/e
(d) –e

Solution :

$H(x) = e^\wedge(-e^x)$

$\log H(x) = -e^x$

$H'(x)/H(x) = -e^x$

$H'(x) = -H(x)e^x$

$H'(0) = -H(0)*1 = -H(0)$

$H(0) = e^\wedge(-e^0) = e^{-1} = 1/e$

$H'(0) = -1/e$

Option (c) is correct.

767. Let $f(x) = |\sin^3 x|$ and $g(x) = \sin^3 x$, both being defined for x in the interval (-π/2, π/2). Then
(a) $f'(x) = g'(x)$ for all x
(b) $f'(x) = -g'(x)$ for all x
(c) $f'(x) = |g'(x)|$ for all x
(d) $g'(x) = |f'(x)|$ for all x

Solution :

$f(x) = \sin^3 x$ for $0 \le x < \pi/2$

$f(x) = -\sin^3 x$ for $-\pi/2 < x < 0$

$g(x) = f(x)$ for $0 \le x < \pi/2$ and $g(x) = -f(x)$ for $-\pi/2 < x < 0$

$g'(x) = f'(x)$ and $g'(x) = -f'(x)$

$g'(x) = |f'(x)|$

Option (d) is correct.

768. Consider the functional equation $f(x - y) = f(x)/f(y)$. If $f'(0) = p$ and $f'(5) = q$, then $f'(-5)$ is

(a) p^2/q
(b) q/p
(c) p/q
(d) q

Solution :

Putting $y = 0$ we get, $f(0) = 1$

Putting $y = 5$ we get, $f(x - 5) = f(x)/f(5)$

⇨ $f'(x - 5) = f'(x)/f(5)$

Putting $x = 0$ we get, $f'(-5) = f'(0)/f(5) = p/f(5)$

Putting $x = 5$ and $y = x$ we get, $f(5 - x) = f(5)/f(x)$

⇨ $f'(5 - x)(-1) = -\{f(5)/(f(x))^2\}f'(x)$

Putting $x = 0$ we get,- $f'(5) = -f(5)f'(0)/(f(0))^2 = -f(5)p$

⇨ $f(5) = q/p$
⇨ $f'(-5) = p/(q/p) = p^2/q$

Option (a) is correct.

769. Let f be a polynomial. Then the second derivative of $f(e^x)$ is

(a) $f''(e^x)*e^x + f'(e^x)$
(b) $f''(e^x)*e^{2x} + f'(x)*e^x$
(c) $f''(e^x)$
(d) $f''(e^x)*e^{2x} + f'(e^x)*e^x$

Solution :

Let $g(x) = f(e^x)$

$g'(x) = f'(e^x)*e^x$

$g''(x) = f''(e^x)*e^x*e^x + f'(e^x)*e^x = f''(e^x)*e^{2x} + f'(e^x)e^x$

Option (d) is correct.

770. If A(t) is the area of the region enclosed by the curve $y = e^{-|x|}$ and portion of the x-axis between –t and +t, then lim A(t) as t -> ∞
(a) is 1
(b) is ∞
(c) is 2
(d) doesn't exist

Solution :

$A(t) = \int e^{-x}dx$ (integration running from 0 to t) + $\int e^x dx$ (integration running from –t to 0)

$= -e^{-x}$ (upper limit = t and lower limit = 0) + e^x (upper limit = 0 and lower limit = -t)

$= -e^{-t} + 1 + 1 - e^{-t}$

$= 2(1 - e^{-t})$

Lim A(t) as t -> ∞ = 2

Option (c) is correct.

771. lim $\{(e^x - 1)\tan^2x/x^3\}$ as x -> 0
(a) doesn't exist
(b) exists and equals 0
(c) exists and equals 2/3
(d) exists and equals 1

Solution :

lim $\{(e^x - 1)/x\}(\tan x/x)^2$ as x -> 0 = 1*1 = 1

Option (d) is correct.

772. If $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log_e x$, and if $F(x) = h(g(f(x)))$, then d^2F/dx^2 equals
(a) $-2\operatorname{cosec}^2 x$
(b) $2\operatorname{cosec}^3 x$
(c) $2\cot(x^2) - 4x^2\operatorname{cosec}^2(x^2)$
(d) $2x\cot(x^2)$

Solution :

$F(x) = h(g(\sin x)) = h(\sin^2 x) = \log_e \sin^2 x = 2\log_e \sin x$

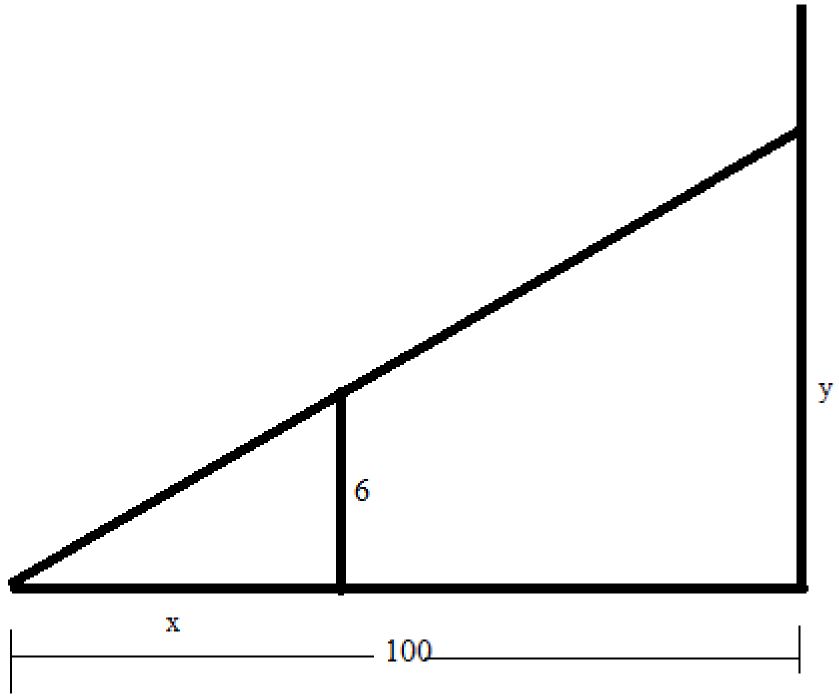
$dF/dx = (2/\sin x)\cos x$

- ⇨ $dF/dx = 2\cot x$
- ⇨ $d^2F/dx^2 = -2\operatorname{cosec}^2 x$

Option (a) is correct.

773. A lamp is placed on the ground 100 feet (ft) away from a wall. A man six ft tall is walking at a speed of 10 ft/sec from the lamp to the nearest point on the wall. When he is midway between the lamp and the wall, the rate of change in the length of his shadow is (in ft/sec)
(a) 2.4
(b) 3
(c) 12
(d) 3.6

Solution :



From the figure it is clear that. 6/x = y/100

- ⇨ xy = 600
- ⇨ (dx/dt)y + x(dy/dt) = 0
- ⇨ (dy/dt) = -(y/x)(dx/dt)

When x = 50, y = 12

(dy/dt) = -(12/50)*10 = - 2.4

Option (a) is correct.

774. A water tank has the shape of a right-circular cone with its vertex down. The radius of the top of the tank is 15 ft and the height is 10 ft. Water is poured into the tank at a constant rate of C cubic feet per second. Water leaks out from the bottom at a constant rate of one cubic foot per second. The value of C for which the water level will be rising at the rate of four ft per second at the time point when the water is two ft deep, is given by
(a) C = 1 + 36π
(b) C = 1 + 9π
(c) C = 1 + 4π
(d) C = 1 + 18π

Solution :

V = (1/3)πr^2h

Now, r/h = constant.

- ⇨ r = kh
- ⇨ 15 = k*10
- ⇨ k = 3/2
- ⇨ r = 3h/2
- ⇨ V = (1/3)π($9h^2$/4)h
- ⇨ V = (3/4)πh^3
- ⇨ dV/dt = (9/4)πh^2(dh/dt)
- ⇨ C – 1 = (9/4)π*2^2*4
- ⇨ C = 1 + 36π

Option (a) is correct.

775. Let f(x) = a|sinx| + $be^{|x|}$ + $c|x|^3$. If f(x) is differentiable at x = 0, then
(a) a = b = c = 0
(b) a = b = 0 and c can be any real value
(c) b = c = 0 and a can be any real value
(d) c = a = 0 and b can be any real value

Solution :

lim {f(x) – f(0)}/(x – 0) as x -> 0- = lim {a|sinx| + $be^{|x|}$ + $c|x|^3$ – b}/x as x -> 0- = lim {-asinx + be^{-x} – cx^3 – b}/x as x -> 0- = lim (–acosx – be^{-x} – $3cx^2$)/1 as x -> 0- (Applying L'Hospital rule) = -a – b

lim {f(x) – f(0)}/(x – 0) as x -< 0+ = lim {a|sinx| + $be^{|x|}$ + $c|x|^3$ – b}/x as x -> 0+ = lim {asinx + be^x + cx^3 – b)/x as x -> 0+ = lim (acosx + be^x + $3cx^2$)/1 as x -> 0+ (Applying L'Hospital rule) = a + b

Now, -a – b = a + b

- ⇨ a + b = 0

Option (b) is correct.

776. A necessary and sufficient condition for the function f(x) defined by f(x) = x^2 + 2x if x ≤ 0, f(x) = ax + b if x > 0 to be differentiable at the point x = 0 is that
(a) a = 0 and b = 0
(b) a = 0 while b can be arbitrary
(c) a = 2 while b can be arbitrary
(d) a = 2 and b = 0

Solution :

lim {f(x) – f(0)}/(x – 0) as x -> 0- = lim {x^2 + 2x – 0}/x as x -> 0- = lim x + 2 as x -> 0- = 2

lim {f(x) – f(0)}/(x – 0) as x -> 0+ = lim {ax + b – 0}/x as x -> 0+ = lim (ax + b)/x as x -> 0+

Now, to hold the limit b = 0.

lim ax/x as x -> 0+ = lim a as x -> 0+ = a

And, a = 2

Option (d) is correct.

777. If f(x) = $\log_{x2}(e^x)$ defined for x > 1, then the derivative f′(x) of f(x) is
(a) (logx – 1)/2$(\log x)^2$
(b) (logx – 1)/$(\log x)^2$
(c)(logx + 1)/2$(\log x)^2$
(d) (logx + 1)/$(\log x)^2$

Solution :

f(x) = $\log_{x2}(e^x)$ = $\log(e^x)/(\log x^2)$ = x/2logx

f′(x) = {1*(2logx) – x(2/x)}/$(2\log x)^2$ = (logx – 1)/2$(\log x)^2$

Option (a) is correct.

778. For x > 0, if g(x) = $x^{\log x}$ and f(x) = $e^{g(x)}$, then f′(x) equals
(a) {$2x^{(\log x - 1)}$logx}f(x)
(b) {$x^{(2\log x - 1)}$logx}f(x)
(c)(1 + x)e^x
(d) None of the foregoing expressions

Solution :

g(x) = $x^{\log x}$

logg(x) = $(\log x)^2$

g′(x)/g(x) = 2logx(1/x)

g′(x) = (2/x)$x^{\log x}$logx = $2x^{(\log x - 1)}$logx

$f(x) = e^{g(x)}$

$\log f(x) = g(x)$

- ⇨ $f'(x)/f(x) = g'(x)$
- ⇨ $f'(x) = \{2x^{(\log x - 1)}\log x\}f(x)$

Option (a) is correct.

779. Suppose f and g are functions having second derivatives f'' and g'' everywhere. If f(x)g(x) = 1 for all x and f' and g' are never zero, then $f''(x)/f'(x) - g''(x)/g'(x)$ equals
   (a) $-2f'(x)/f(x)$
   (b) 0
   (c) $-f'(x)/f(x)$
   (d) $2f'(x)/f(x)$

Solution :

Now, $f(x)g(x) = 1$

- ⇨ $f'(x)g(x) + f(x)g'(x) = 0$
- ⇨ $f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x) = 0$
- ⇨ $\{f''(x)/f'(x)\}\{g(x)/g'(x)\} + 2 + \{g''(x)/g'(x)\}\{f(x)/f'(x)\} = 0$ (dividing by $f'(x)g'(x)$)

Now, $f'(x)g(x) + f(x)g'(x) = 0$

- ⇨ $g(x)/g'(x) + f(x)/f'(x) = 0$ (dividing by $f'(x)g'(x)$)
- ⇨ $g(x)/g'(x) = -f(x)/f'(x)$
- ⇨ $\{f''(x)/f'(x)\}\{-f(x)/f'(x)\} + \{g''(x)/g'(x)\}\{f(x)/f'(x)\} = -2$
- ⇨ $-(f(x)/f'(x))\{f''(x)/f'(x) - g''(x)/g'(x)\} = -2$
- ⇨ $f''(x)/f'(x) - g''(x)/g'(x) = 2f'(x)/f(x)$

Option (d) is correct.

780. If $f(x) = a_1e^{|x|} + a_2|x|^5$, where a_1, a_2 are constants, is differentiable at x = 0, then
   (a) $a_1 = a_2$
   (b) $a_1 = a_2 = 0$
   (c) $a_1 = 0$
   (d) $a_2 = 0$

Solution :

lim $\{f(x) - f(0)\}/(x - 0)$ as $x \to 0-$ = lim $\{a_1e^{|x|} + a_2|x|^5 - a_1\}/x$ as $x \to 0-$ = lim $\{a_1e^{-x} - a_2x^5 - a_1\}/x$ as $x \to 0-$ = lim $(-a_1e^{-x} - 5a_2x^4)/1$ as $x \to 0-$ (Applying L'Hospital rule) = $-a_1$

lim $\{f(x) - f(0)\}/(x - 0)$ as $x \to 0+$ = lim $\{a_1e^{|x|} + a_2|x|^5 - a_1\}/x$ as $x \to 0+$ = lim $\{a_1e^{x} + a_2x^5 - a_1\}/x$ as $x \to 0+$ = lim $(a_1e^{x} + 5a_2x^4)/1$ as $x \to 0+$ (Applying L'Hospital rule) = a_1

So, $-a_1 = a_1$

⇨ $a_1 = 0$

Option (c) is correct.

781. If $y = (\cos^{-1}x)^2$, then the value of $(1 - x^2)d^2y/dx^2 - xdy/dx$ is
   (a) -1
   (b) -2
   (c) 1
   (d) 2

Solution :

$y = (\cos^{-1}x)^2$

$dy/dx = 2(\cos^{-1}x)\{-1/\sqrt{(1 - x^2)}\} = -2\cos^{-1}x/\sqrt{(1 - x^2)}$

$d^2y/dx^2 = [\{2/\sqrt{(1 - x^2)}\}\sqrt{(1 - x^2)} - 2\cos^{-1}x\{2x/2\sqrt{(1 - x^2)}\}]/(1 - x^2)$

$(1 - x^2)d^2y/dx^2 = 2 + xdy/dx$

⇨ $(1 - x^2)d^{2y}/dx^2 - xdy/dx = 2$

Option (d) is correct.

782. The n^{th} derivative of the function $f(x) = 1/(1 - x^2)$ at the point $x = 0$, where n is even, is
   (a) n^nC_2
   (b) 0
   (c) n!
   (d) none of the foregoing quantities

Solution :

$f'(x) = -(-2x)/(1 - x^2)^2 = 2x/(1 - x^2)^2$

$f''(x) = \{2(1 - x^2)^2 - 2x*2(1 - x^2)(-2x)\}/(1 - x^2)^4 = 2(1 - x^2)(1 - x^2 + 4x^2)/(1 - x^2)^4$

$f''(0) = 2$

$f''(x) = 2(1 + 3x^2)/(1 - x^2)^3$

$f'''(x) = [2(6x)(1 - x^2)^3 - 2(1 + 3x^2)*3(1 - x^2)^2(-2x)]/(1 - x^2)^6 = 2(1 - x^2)^2[6x - 6x^3 + 6x(1 + 3x^2)]/(1 - x^2)^6$

$= 2(12x + 12x^3)/(1 - x^2)^4 = 24x(1 + x^2)/(1 - x^2)^4$

$f^{(4)}(x) = [24(1 + 3x^2)(1 - x^2)^4 - 4(1 - x^2)^3(-2x)24x(1 + x^2)]/(1 - x^2)^8 = 24(1 - x^2)^3[(1 + 3x^2)(1 - x^2) + 8x(1 + x^2)]/(1 - x^2)^8$

$f^{(4)}(0) = 24 = 4!$

Option (c) is correct.

783. Let $f(x) = x^n(1 - x)^n/n!$. Then for any integer $k \geq 0$, the k-th derivative $f^{(k)}(0)$ and $f^{(k)}(1)$
(a) are both integers
(b) are both rational numbers but not necessarily integers
(c) are both integers
(d) do not satisfy any of the foregoing properties

Solution :

$f'(x) = [nx^{n-1}(1 - x)^n + x^n n(1 - x)^{n-1}(-1)]/n!$

$f'(0) = 0, f'(1) = 0$

$f'(x) = x^{n-1}(1 - x)^{n-1}(1 - x + x)/(n - 1)! = x^{n-1}(1 - x)^{n-1}/(n - 1)!$

It is obvious that $f^{(k)}(0)$ and $f^{(k)}(1) = 0$ till $k = n$, after that it is an integer.

So, option (c) is correct.

784. Let $f_1(x) = e^x$, $f_2(x) = e^\wedge(f_1(x))$, $f_3(x) = e^\wedge(f_2(x))$ …. and, in general $f_{n+1}(x) = e^\wedge(f_n(x))$ for any $n \geq 1$. Then for any fixed n, the value of $d/dx(f_n(x))$ equals
(a) $f_n(x)$
(b) $f_n(x)f_{n-1}(x)$
(c) $f_n(x)f_{n-1}(x)\ldots.f_2(x)f_1(x)$
(d) $f_n(x)f_{n-1}(x)\ldots.f_1(x)e^x$

Solution :

$f_n'(x) = e^{(f_{n-1}(x))}f'_{n-1}(x) = f_n(x)e^{(f_{n-2}(x))}f_{n-2}'(x) = f_n(x)f_{n-1}(x)f_{n-2}'(x) = \dots$
$= f_n(x)f_{n-1}(x) \dots f_2(x)f_1'(x) = f_n(x)f_{n-1}(x) \dots f_2(x)f_1(x)$ (as $f_1'(x) = f_1(x)$)

Option (c) is correct.

785. The maximum value of $5\sin\theta + 12\cos\theta$ is
(a) 5
(b) 12
(c) 13
(d) 17

Solution :

Now, $5\sin\theta + 12\cos\theta = 13\{(5/13)\sin\theta + (12/13)\cos\theta\} = 13(\cos\alpha\sin\theta + \sin\alpha\cos\theta)$ where $\cos\alpha = 5/13$ and $\sin\alpha = 12/13$

$= 13\sin(\alpha + \theta)$

Option (c) is correct.

786. Let A and B be the points (1, 0) and (3, 0) respectively. Let P be a variable point on the y-axis. Then the maximum value of the angle APB is
(a) 22.5 degree
(b) 30 degree
(c) 45 degree
(d) None of the foregoing quantities

Solution :

Let $p = (0, t)$

Slope of AP = $(t - 0)/(0 - 1) = -t$

Slope of BP = $(t - 0)/(0 - 3) = -t/3$

$\tan(APB) = (-t/3 + t)/(1 + t^2/3)$

Let $F = (2t/3)/(1 + t^2/3) = 2t/(3 + t^2)$

$dF/dt = \{2(3 + t^2) - 2t*2t\}/(3 + t^2)^2 = 0$

- ⇨ $6 + 2t^2 - 4t^2 = 0$
- ⇨ $t^2 = 3$

⇨ $t = \pm\sqrt{3}$

$dF/dt = (6 - 2t^2)/(3 + t^2)^2$

$d^2F/dt^2 = \{-4t(3 + t^2) - 2t(3 + t^2)(6 - 2t^2)\}/(3 + t^2)^4 = (-4t - 12t + 4t^3)/(3 + t^2)^3 = 4t(t^2 - 4)/(3 + t^2)^3 < 0$ at $t = \sqrt{3}$

So, maximum value $= 2\sqrt{3}/(3 + 3) = 1/\sqrt{3}$

Tan(APB) $= 1/\sqrt{3}$

⇨ APB = 30 degree

Option (b) is correct.

787. The least value of the expression $(1 + x^2)/(1 + x)$, for values $x \geq 0$, is
(a) $\sqrt{2}$
(b) 1
(c) $2\sqrt{2} - 2$
(d) None of the foregoing numbers.

Solution :

Let $f(x) = (1 + x^2)/(1 + x)$

$f'(x) = \{2x(1 + x) - (1 + x^2)\}/(1 + x)^2 = (x^2 + 2x - 1)/(1 + x)^2 = 0$

⇨ $x = \{-2 \pm \sqrt{(4 + 4)}\}/2 = -1 \pm \sqrt{2}$

$f''(x) = \{(2x + 2)(1 + x)^2 - 2(1 + x)(x^2 + 2x - 1)\}/(1 + x)^4 = 2\{1 + 2x + x^2 - x^2 - 2x + 1)/(1 + x)^3 = 4/(1 + x)^3 > 0$ at $x = \sqrt{2} - 1$

Minimum value $= \{1 + (\sqrt{2} - 1)^2\}/(1 + \sqrt{2} - 1) = \{1 + 2 + 1 - 2\sqrt{2}\}/\sqrt{2} = 2\sqrt{2} - 2$

Option (c) is correct.

788. The maximum value of $3x + 4y$ subject to the condition $x^2y^3 = 6$ and x and y are positive, is
(a) 10
(b) 14
(c) 7
(d) 13

Solution :

Weighted A.M. ≥ Weighted G.M.

- ⇨ $\{2(3x/2) + 3(4y/3)\}/(2 + 3) \geq \{(3x/2)^2(4y/3)^3\}^{1/5} = \{(16/3)x^2y^3\}^{1/5} = (16*6/3)^{1/5} = 2$
- ⇨ $3x + 4y \geq 10$

Option (a) is correct.

789. A window is in the form of a rectangle with a semicircular band on the top. If the perimeter of the window is 10 metres, the radius, in metres, of the semicircular band that maximizes the amount of light admitted is
 (a) $20/(4 + \pi)$
 (b) $10/(4 + \pi)$
 (c) $10 - 2\pi$
 (d) None of the foregoing numbers.

Solution :

$2r + 2y + \pi r = 10$ where y is height of the rectangular portion and r is the radius of the semicircular portion.

- ⇨ $y = (10 - 2r - \pi r)/2$

$A = 2ry + \pi r^2/2 = r(10 - 2r - \pi r) + \pi r^2/2 = 10r - 2r^2 - \pi r^2/2$

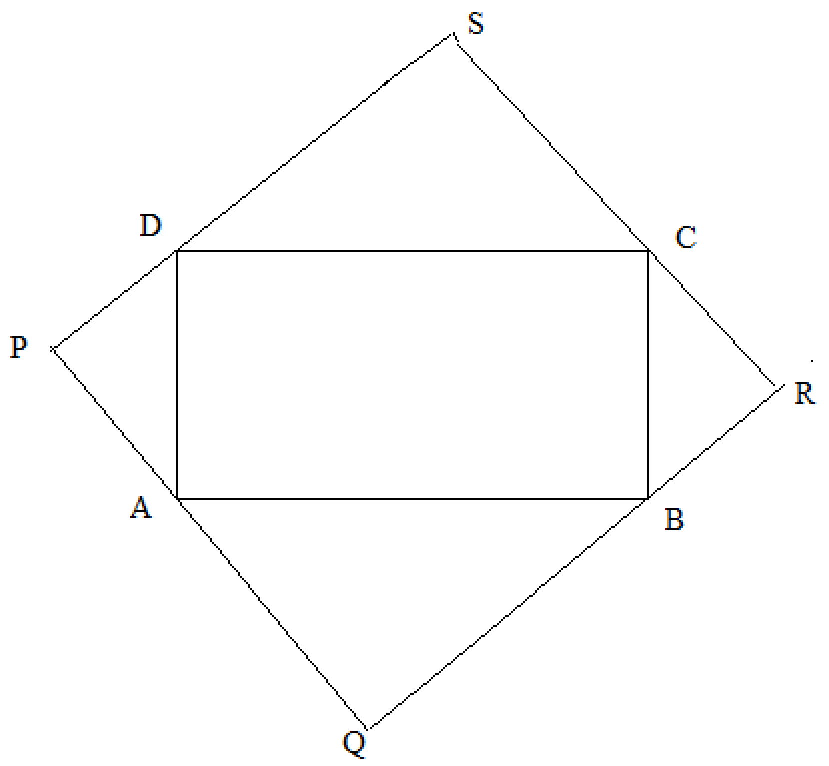
$dA/dr = 10 - 4r - \pi r = 0$

- ⇨ $r = 10/(4 + \pi)$

Option (b) is correct.

790. ABCD is a fixed rectangle with AB = 2cm and BC = 4 cm. PQRS is a rectangle such that A, B, C and D lie on PQ, QR, RS and SP respectively. Then the maximum possible area of PQRS is
 (a) 16 cm^2
 (b) 18 cm^2
 (c) 20 cm^2
 (d) 22 cm^2

Solution :



PQRS rectangle will be maximum when it will be a square.

In that case, $SD^2 + SC^2 = 4^2$ (SD = SC)

⇨ $SD = 4/\sqrt{2}$

Similarly, $PD = 2/\sqrt{2}$

$PS = 4/\sqrt{2} + 2/\sqrt{2} = 6/\sqrt{2} = 3\sqrt{2}$

Area = $PS^2 = 18$

Option (b) is correct.

791. The curve $y = 2x/(1 + x^2)$ has
- (a) exactly three points of inflection separated by a point of maximum and a point of minimum
- (b) exactly two points of inflection with a point of maximum lying between them
- (c) exactly two points of inflection with a point of minimum lying between them
- (d) exactly three points of inflection separated by two points of maximum

Solution :

$dy/dx = \{2(1 + x^2) - 2x*2x\}/(1 + x^2)^2 = 2(1 - x^2)/(1 + x^2)^2$

$d^2y/dx^2 = \{2(-2x)(1 + x^2)^2 - 2(1 + x^2)*2x*2(1 - x^2)\}/(1 + x^2)^2 = -4x(1 + x^2 + 2 - 2x^2)/(1 + x^2)^2 = 4x(x^2 - 3)/(1 + x^2)^2$

Now, $d^2y/dx^2 = 0$ gives, three solutions, $x = 0$, $x = \pm \sqrt{3}$

$dy/dx = 0$ gices, $x = \pm 1$ at which $d^2y/dx^2 < 0$ for $x = 1$ and > 0 for $x = -1$ i.e. maximum and minimum points

So, option (a) is correct.

792. As x varies all real numbers, the range of function $f(x) = (x^2 - 3x + 4)/(x^2 + 3x + 4)$ is
   (a) [1/7, 7]
   (b) [-1/7, 7]
   (c) [-7, 7]
   (d) (-∞, 1/7)U(7, ∞)

Solution :

Now, $(x^2 - 3x + 4)/(x^2 + 3x + 4) = f(x)$

- ⇨ $x^2 - 3x + 4 = f(x)x^2 + 3f(x)x + 4f(x)$ where $x^2 + 3x + 4 > 0$
- ⇨ $x^2(1 - f(x)) - 3x(1 + f(x)) + 4(1 - f(x)) = 0$ where $(x + 3/2)^2 + 7/4 > 0$
- ⇨ $9(1 + f(x))^2 - 16(1 - f(x))^2 \geq 0$
- ⇨ $9 + 18f(x) + 9f(x)^2 - 16 - 16f(x)^2 + 32f(x) \geq 0$
- ⇨ $7f(x)^2 - 50f(x) + 7 \leq 0$
- ⇨ $(7f(x) - 1)(f(x) - 7) \leq 0$
- ⇨ $f(x) \leq 1/7$ and $f(x) \geq 1/7$ or $f(x) \geq 1/7$ and $f(x) \leq 7$
- ⇨ $1/7 \leq f(x) \leq 7$

Option (a) is correct.

793. The minimum value of $f(x) = x^8 + x^6 - x^4 - 2x^3 - x^2 - 2x + 9$ is
   (a) 5
   (b) 1
   (c) 0
   (d) 9

Solution :

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$f'(x) = 8x^7 + 6x^5 - 4x^3 - 6x^2 - 2x - 2 = 0$

- ⇨ $(x - 1)(8x^6 + 8x^5 + 14x^4 + 10x^3 + 4x + 2) = 0$
- ⇨ $x = 1$

$f''(x) = 56x^6 + 30x^4 - 12x^2 - 12x - 2 > 0$ at $x = 1$ hence minimum.

Therefore, minimum value = $f(1) = 5$

Option (a) is correct.

794. The number of minima of the polynomial $10x^6 - 24x^5 + 15x^4 + 40x^2 + 108$ is
(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Let $P(x) = 10x^6 - 24x^5 + 15x^4 + 40x^2 + 108$

- ⇨ $P'(x) = 60x^5 - 120x^4 + 60x^3 + 80x = 0$
- ⇨ $3x^5 - 4x^4 + 3x^3 + 2x = 0$
- ⇨ $3x^4 - 4x^3 + 3x^2 + 2 = 0$ or $x = 0$

Clearly, it has no negative roots. For $x < 0$ $3x^4 - 4x^3 + 3x^2 + 2 > 0$

Now, $d/dx(3x^4 - 4x^3 + 3x^2 + 2) = 12x^3 - 12x^2 + 6x = 6x(2x^2 - 2x + 1) = 6x\{x^2 + (x - 1)^2\} > 0$ for $x > 0$

Therefore, $3x^4 - 4x^3 + 3x^2 + 2$ is increasing for $x > 0$ and at $x = 0$ it is 2 and at $x = 1$, it is 4.

So, $P'(x)$ has only one root $x = 0$.

$P''(x) = 20(15x^4 - 16x^3 + 9x^2 + 2) > 0$ at $x = 0$

- ⇨ At $x = 0$ $P(x)$ is minimum.
- ⇨ One minimum

Option (b) is correct.

795. The number of local maxima of the function $f(x) = x + \sin x$ is
(a) 1
(b) 2
(c) Infinite

(d) 0

Solution :

$f'(x) = 1 + \cos x = 0$

⇨ $\cos x = -1$

$f''(x) = -\sin x = 0$ for $\cos x = -1$

So, no local maxima.

Option (d) is correct.

796. The maximum value of $\log_{10}(4x^3 - 12x^2 + 11x - 3)$ in the interval [2, 3] is
(a) $\log_{10}3$
(b) $1 + \log_{10}5$
(c) $-(3/2)\log_{10}3$
(d) None of these.

Solution :

Let $f(x) = 4x^3 - 12x^2 + 11x - 3$

$f'(x) = 12x^2 - 24x + 11 = 0$

⇨ $x = (24 \pm 4\sqrt{3})/24 = 1 \pm \sqrt{3}/6$

$f''(x) = 24x - 24 = 24(x - 1) < 0$ at $x = (24 - 4\sqrt{3})/24$

Therefore, maximum but $(24 - 4\sqrt{})/24 < 1$

We need to find maximum value in [2, 3]

⇨ Maximum value of $f(x) = f(3) = 4*27 - 12*9 + 11*3 - 3 = 30$
⇨ $\log_{10}30 = \log_{10}3 + 1$

Option (d) is correct.

797. The maximum value of the function $f(x) = (1 + x)^{0.3}/(1 + x^{0.3})$ in the interval $0 \le x \le 1$ is
(a) 1
(b) $2^{0.7}$
(c) $2^{-0.7}$
(d) None of these.

Solution :

$f'(x) = \{0.3(1 + x)^{-0.7}(1 + x^{0.3}) - 0.3x^{-0.7}(1 + x)^{0.3}\}/(1 + x^{0.3})^2 = 0.3\{x^{0.7}(1 + x^{0,3}) - (1 + x)\}/\{x^{0.7}(1 + x)^{0.7}(1 + x^{0.3})\} = 0.3(x^{0.7} - 1)/\{x^{0.7}(1 + x)^{0.7}(1 + x^{0.3})\} = 0$ gives $x = 1$

$f''(x) = 0.3[0.7x^{-0.3}x^{0.7}(1 + x)^{0.7}(1 + x^{0,3}) - (x^{0.7} - 1)\{0.7x^{-0.3}(1 + x)^{0.7}(1 + x^{0.3}) + 0.7x^{0.7}(1 + x)^{-0.3}(1 + x^{0.3}) + 0.3x^{-0.7}x^{0.7}(1 + x)^{0.7}\}]/\{x^{0.7}(1 + x)^{0.7}(1 + x^{0.3})\}^2 > 0$ for $x = 1$

Therefore, no local maximum value here.

Now, $f(0) = 1$ and $f(1) = 2^{-0.7}$

So, maximum value = $f(0) = 1$

Option (a) is correct.

798. The number of local maxima of the function $f(x) = x - \sin x$ is
(a) Infinitely many
(b) Two
(c) One
(d) Zero

Solution :

$f'(x) = 1 - \cos x = 0$ given $\cos x = 1$

$f''(x) = \sin x = 0$ for $\cos x = 0$

Hence, no local maxima.

Option (d) is correct.

799. From a square tin sheet of side 12 feet (ft) a box with its top open is made by cutting away equal squares at the four corners and then bending the tin sheet so as to form the sides of the box. The side of the removed square for which the box has the maximum possible volume is, in ft,
(a) 3
(b) 1
(c) 2
(d) None of the foregoing numbers.

Solution :

Let the side of the cut square = x.

So, height of the box is x and base area = $(12 - 2x)^2$

Volume = V = $x(12 - 2x)^2$

Now, $dV/dx = (12 - 2x)^2 + x*2(12 - 2x)(-2) = 0 \Rightarrow x = 6, 12 - 2x - 4x = 0, x = 2$

$d^2V/dx^2 = 2(12 - 2x)(-2) + 4(12 - 2x) + 4x(-2) < 0$ as x = 2.

Therefore, maximum value.

Option (c) is correct.

800. A rectangular box of volume 48 cu ft is to be constructed, so that its length is twice its width. The material to be used for the top and the four sides is three times costlier per sq ft than that used for the bottom. Then, the box that minimizes the cost has height equal to (in ft)
(a) 8/27
(b) $8\sqrt[3]{4/3}$
(c) 4/27
(d) 8/3

Solution :

Let the height is h and width is x

Therefore, length = 2x

Volume = $x*2x*h = 2x^2h = 48$

- ⇨ $x^2 = 24/h$

Cost = $c*x*2x + 3c(x*2x + 2*x*h + 2*2x*h)$ where c is cost per sq ft

Cost = C = $c(48/h) + 3c(48/h + 6h\sqrt{(24/h)}) = 4*48c/h + 36c\sqrt{(6h)}$

$dC/dh = -4*48c/h^2 + 36c\sqrt{6}/2\sqrt{h} = 0$

- ⇨ $h^{3/2} = 4*48*2/36\sqrt{6} = 16*\sqrt{2}/3^{3/2}$
- ⇨ $h^{3/2} = (8/3)^{3/2}$
- ⇨ $h = 8/3$

Option (d) is correct.

801. A truck is to be driven 300 km on a highway at a constant speed of x kmph. Speed rules for highway require that $30 \le x \le 60$. The fuel costs Rs. 10 per litre and is consumed at the rate of $2 + x^2/600$ litres per hour. The wages of the driver are Rs. 200 per hour. The most economical speed to drive the truck, in kmph, is
(a) 30
(b) 60
(c) $30\sqrt{3.3}$
(d) $20\sqrt{33}$

Solution :

Time = 300/x

Cost = $(300/x)*200 + (300/x)(2 + x^2/600)*10 = 60000/x + 6000/x + 5x = 66000/x + 5x$

Now, we have to minimize cost.

Let $C = 66000/x + 5x$

$dC/dx = -66000/x^2 + 5 = 0$

- ⇨ $x^2 = 66000/5 = 13200$
- ⇨ $x = 20\sqrt{33}$

$d^2C/dx^2 = -66000*2/x^3 < 0$ at $x = 20\sqrt{33}$, therefore maximum.

So, C at x = 30 is 2200 + 150 = 2350

C at x = 60 is 1100 + 300 = 1400

Therefore, it is economical to drive at x = 60 kmph

Option (b) is correct.

802. Let P be a point in the first quadrant lying on the ellipse $x^2/8 + y^2/18 = 1$. Let AB be the tangent at P to the ellipse meeting the x-axis at A and y-axis at B. If O is the origin, then minimum possible area of the triangle OAB is
(a) 4π
(b) 9π
(c) 9
(d) 12

Solution :

Let P = $(2\sqrt{2}\cos\theta, 3\sqrt{2}\sin\theta)$

Now, $x^2/8 + y^2/18 = 1$

- ⇨ $x/4 + (y/9)(dy/dx) = 0$
- ⇨ $dy/dx = - 9x/4y$
- ⇨ $(dy/dx)_P = -9*2\sqrt{2}\cos\theta/(4*3\sqrt{2}\sin\theta) = -3\cot\theta/2$

Equation of AB is, $y - 3\sqrt{2}\sin\theta = (-3\cot\theta/2)(x - 2\sqrt{2}\cos\theta)$

Putting y = 0 we get, $x = (2\sqrt{2}\cos^2\theta + 2\sqrt{}\sin^2\theta)/\cos\theta = 2\sqrt{2}/\cos\theta$

Putting x = 0 we get, $y = 3\sqrt{2}/\sin\theta$

Therefore, A = $(2\sqrt{2}/\cos\theta, 0)$ and B = $(0, 3\sqrt{2}/\sin\theta)$

Therefore, area of triangle OAB = S = $(1/2)*(2\sqrt{2}/\cos\theta)(3\sqrt{2}/\sin\theta)$ = $12/\sin 2\theta$

$dS/d\theta = (-12/\sin^2 2\theta)(2\cos 2\theta) = 0 \Rightarrow \theta = \pi/4$

Area = $12/\sin(\pi/2) = 12$

Option (d) is correct.

803. Consider the parabola $y^2 = 4x$. Let P and Q be the points (4, -4) and (9, 6) of the parabola. Let R be a moving point on the arc of the parabola between P and Q. Then area of the triangle RPQ is largest when
(a) Angle PRQ = 90 degree
(b) R = (4, 4)
(c) R = (1/4, 1)
(d) Condition other than the foregoing conditions is satisfied.

Solution :

Let R = $(t^2, 2t)$

Area of triangle RPQ = A = $(1/2)[4(2t - 6) + t^2(6 + 4) + 9(-4 - 2t)] = 5t^2 - 5t - 30$

$dA/dt = 10t - 5 = 0$

- ⇨ t = ½

So, R = (1/4, 1)

Option (c) is correct.

804. Out of a circular sheet of paper of radius a, a sector with central angle θ is cut out and folded into the shape of a conical funnel. The volume of this funnel is maximum when θ equals
(a) $2\pi/\sqrt{2}$
(b) $2\pi\sqrt{(2/3)}$
(c) $\pi/2$
(d) π

Solution :

Length of the arc = $a\theta$

- ⇨ $2\pi r = a\theta$ (where r is the radius of the base of the funnel)
- ⇨ $r = a\theta/2\pi$
- ⇨ h = height of the funnel = $\sqrt{(a^2 - r^2)} = a\sqrt{\{1 - (\theta/2\pi)^2\}}$

Volume = $V = (1/3)\pi(a\theta/2\pi)^2 a\sqrt{\{1 - (\theta/2\pi)^2\}}$

$dV/d\theta = (1/3)(a^2/4\pi)[2\theta\sqrt{\{1 - (\theta/2\pi)^2\}} + \theta^2(-\theta/2\pi^2)/2\sqrt{\{1 - (\theta/2\pi)^2\}} = 0$

- ⇨ $2 - 2(\theta/2\pi)^2 - (\theta/2\pi)^2 = 0$
- ⇨ $\theta/2\pi = \sqrt{(2/3)}$
- ⇨ $\theta = 2\pi\sqrt{(2/3)}$

Option (b) is correct.

805. Let $f(x) = 5 - 4(\sqrt[3]{(x - 2)})^2$. Then at x = 2, the function f(x)
(a) attains a minimum value
(b) attains a maximum value
(c) attains neither a minimum value nor a maximum value
(d) is undefined

Solution :

$x > 2$.

For $x > 2$, $\{\sqrt[3]{(x - 2)}\}^2 > 0$

- ⇨ f(x) attains maximum value at x = 2.

Option (b) is correct.

806. A given circular cone has a volume p, and the largest right circular cylinder that can be inscribed in the given cone has a volume q. Then the ratio p : q equals
(a) 9 : 4
(b) 8 : 3
(c) 7 : 2
(d) None of the foregoing ratios.

Solution :

Let, the radius of the base of the cone is R and height is H.

Let the height of cylinder is h and base radius is r.

We have, r/(H – h) = R/H

⇨ r = R(H – h)/H

Now, V = volume of the cylinder = $\pi r^2 h = \pi(R/H)^2(H - h)^2 h$

$dV/dh = \pi(R/H)^2[2(H - h)(-1)h + 1(H - h)^2] = 0$

⇨ -2h + H – h = 0
⇨ h = H/3
⇨ r = R(2H/3)/H = 2R/3
⇨ $q = \pi(2R/3)^2*(H/3) = (1/3)\pi R^2 H*(4/9) = p*(4/9)$
⇨ p/q = 9/4
⇨ p : q = 9 : 4

Option (a) is correct.

807. If [x] stands for the largest integer not exceeding x, then the integral ∫[x]dx (integration running from x = -1 to x = 2) is
(a) 3
(b) 0
(c) 1
(d) 2

Solution :

∫[x]dx (integration running from x = -1 to x = 2)

= ∫[x]dx (integration running from x = -1 to x = 0) + ∫[x]dx (integration running from x = 1 to x = 2) + ∫[x]dx (integration running from x = 1 to x = 2)

= -1(0 + 1) + 0(1 – 0) + 1(2 – 1) = 0

Option (b) is correct.

808. For any real number x, let [x] denote the greatest integer m such that $m \leq x$. Then $\int[x^2 - 1]dx$ (integration running from -2 to 2) equals
(a) $2(3 - \sqrt{3} - \sqrt{2})$
(b) $2(5 - \sqrt{3} - \sqrt{2})$
(c) $2(1 - \sqrt{3} - \sqrt{2})$
(d) None of these.

Solution :

Now, $\int[x^2 - 1]dx$ (integration running from -2 to 2) = $2\int[x^2 - 1]dx$ (integration running from 0 to 2) (As $[x^2 - 1]$ is even function)

= $2[\int[x^2 - 1]dx$ (integration running from 0 to 1) + $\int[x^2 - 1]dx$ (integration running from 1 to $\sqrt{2}$) + $\int[x^2 - 1]dx$ (integration running from $\sqrt{2}$ to $\sqrt{3}$) + $\int[x^2 - 1]dx$ (integration running from $\sqrt{3}$ to 2)

= $2[(-1)(1 - 0) + 0(\sqrt{2} - 1) + 1(\sqrt{3} - \sqrt{2}) + 2(2 - \sqrt{3})]$

= $2(-1 + \sqrt{3} - \sqrt{2} + 4 - 2\sqrt{3})$

= $2(3 - \sqrt{3} - \sqrt{2})$

Option (a) is correct.

809. Let f(x) be a continuous function such that its first two derivatives are continuous. The tangents to the graph of f(x) at the points with abscissa x = a and x = b make with X-axis angles $\pi/3$ and $\pi/4$ respectively. Then the value of the integral $\int f'(x)f''(x)dx$ (integration running from x = a to x = b) equals
(a) $1 - \sqrt{3}$
(b) 0
(c) 1
(d) -1

Solution :

Let $f'(x) = z$

⇨ $f''(x)dx = dz$

Therefore, $\int f'(x)f''(x)dx = \int z dz = z^2/2 = \{f'(x)\}^2/2|_a^b = [\{f'(b)\}^2 - \{f'(a)\}^2]/2 = \{\tan^2(\pi/4) - \tan^2(\pi/3)\}/2 = (1 - 3)/2 = -1$

Option (d) is correct.

810. The integral $\int e^{x - [x]}dx$ (integration running from 0 to 100) is
(a) $(e^{100} - 1)/100$
(b) $(e^{100} - 1)/(e - 1)$
(c) $100(e - 1)$
(d) $(e - 1)/100$

Solution :

$\int e^{x - [x]}dx$ (integration running from 0 to 100) = $\sum\int e^{x - [x]}dx$ (integration running from i tp i + 1) (Summation running from i = 0 to i = 99)

Now, $\int e^{x - [x]}dx$ (integration running from i to i + 1)

= $\int e^{x - i}dx$ (integration running from i to i + 1)

= $e^{x - i}|_i^{i+1} = e - 1$

Now, $\sum(e - 1)$ (summation running from i = 0 to i = 99) = $100(e - 1)$

Option (c) is correct.

811. If $S = \int\{e^t/(t + 1\}dt$ (integration running from 0 to 1) then $\int\{e^{-t}/(t - a - 1)\}dt$ (integration running from a – 1 to a) is
(a) Se^a
(b) Se^{-a}
(c) $-Se^{-a}$
(d) $-Se^a$

Solution :

Now, $\int\{e^{-t}/(t - a - 1)\}dt$ (integration running from a – 1 to a)

= $\int\{e^{-(2a - 1 - t)}/(2a - 1 - t - a - 1)\}dt$ (integration running from a – 1 to a) (As $\int f(x)dx = \int f(a + b - x)dx$ when integration running from a to b)

= $e^{-(2a-1)}\int\{e^t/(a - 2 - t)\}dt$

Let $t = z + a - 1$

$dt = dz$ and when $t = a - 1, z = 0$; $t = a, z = 1$

= $e^{-(2a - 1)}\int\{e^{z + a - 1}/(a - 2 - z - a + 1)\}dz = -e^{-(2a - 1) + a - 1}\int\{e^z/(z + 1)\}dz = -e^{-a}S$

Option (c) is correct.

812. If the value of the integral $\int\{e^\wedge(x^2)\}dx$ (integration running from 1 to 2) is ɑ, then the value of $\int\sqrt{}(\log x)dx$ (integration running from e to e^4) is
(a) $e^4 - e - ɑ$
(b) $2e^4 - e - ɑ$
(c) $2(e^4 - e) - ɑ$
(d) None of the foregoing quantities.

Solution :

$\int\sqrt{}(\log x)dx$ (integration running from e to e^4)

$= \sqrt{}(\log x)x$(upper limit = e^4, lower limit = e) – $(1/2)\int\{x/x\sqrt{}(\log x)\}dx$ (integration running from e to e^4)

$= 2e^4 - e - (1/2)\int\{1/\sqrt{}(\log x)\}dx$ (integration running from e to e^4)

Let $\sqrt{}(\log x) = z$

- ⇨ $(½)dx/x\sqrt{}(\log x) = dz$
- ⇨ $(1/2)dx/\sqrt{}(\log x)dx = \{e^\wedge(z^2)\}dz$

When $x = e$, $z = 1$; $x = e^4$, $z = 2$

$= 2e^4 - e - \int\{e^\wedge(z^2)\}dz$

$= 2e^4 - e - ɑ$

Option (b) is correct.

813. The value of the integral $\int|1 + 2\cos x|dx$ (integration running from 0 to π) is
(a) $π/3 + \sqrt{3}$
(b) $π/3 + 2\sqrt{3}$
(c) $π/3 + 4\sqrt{3}$
(d) $2π/3 + 4\sqrt{3}$

Solution :

$\int|1 + 2\cos x|dx$ (integration running from 0 to π

$= \int(1 + 2\cos x)dx$ (integration running from 0 to 2π/3) + $\int-(1 + 2\cos x)dx$ (integration running from 2π/3 to π)

$= x + 2\sin x|_0^{2\pi/3} - (x + 2\sin x)|_{2\pi/3}^{\pi}$

$= 2\pi/3 + 2(\sqrt{3}/2) - (\pi + 2\sin\pi) + (2\pi/3 + 2(\sqrt{3}/2))$

$= 2\pi/3 + \sqrt{3} - \pi + 2\pi/3 + \sqrt{3}$

$= \pi/3 + 2\sqrt{3}$

Option (b) is correct.

814. The value of the integral $\int\sqrt{(1 + \sin(x/2))}dx$ (integration running from 0 to u), where $0 \le u \le \pi$, is
(a) $4 + 4\{\sin(u/4) - \cos(u/4)\}$
(b) $4 + 4\{\cos(u/4) - \sin(u/4)\}$
(c) $4 + (1/4)(\cos(u/4) - \sin(u/4))$
(d) $4 + (1/4)\{\sin(u/4) - \cos(u/4)\}$

Solution :

$\int\sqrt{(1 + \sin(x/2))}dx$ (integration running from 0 to u)

$= \int(\cos(x/4) + \sin(x/4))dx$ (integration running from 0 to u)

$= 4[\sin(x/4) - \cos(x/4)]|_0^u$

$= 4\{\sin(u/4) - \cos(u/4)\} - (-4)$

$= 4 + 4\{\sin(u/4) - \cos(u/4)\}$

Option (a) is correct.

815. The definite integral $\int dx/(1 + \tan^{101}x)$ (integration running from 0 to $\pi/2$) equals
(a) π
(b) $\pi/2$
(c) 0
(d) $\pi/4$

Solution :

Let $I = \int dx/(1 + \tan^{101}x)$ (integration running from 0 to $\pi/2$)

$= \int dx/(1 + \cot^{101}x)$ (integration running from 0 to $\pi/2$) (As $\int f(x)dx = \int f(a - x)dx$ when integration is running from 0 to a)

$= \int \tan^{101}x dx/(1 + \tan^{101}x)$ (integration running from 0 to $\pi/2$)

$I + I = \int dx$ (integration running from 0 to π/2) = π/2

- ⇨ $2I = \pi/2$
- ⇨ $I = \pi/4$

Option (d) is correct.

816. If f(x) is a nonnegative continuous function such that f(x) + f(1/2 + x) = 1 for all x. 0 ≤ x ≤ ½, then ∫f(x)dx (integration running from 0 to 1) is equal to
    (a) ½
    (b) ¼
    (c) 1
    (d) 2

Solution :

∫f(x)dx (integration running from 0 to 1)

= ∫f(x)dx (integration running from 0 to ½) + ∫f(x)dx (integration running from ½ to 1)

= I + J

J = ∫f(x)dx integration running from ½ to 1)

Let x = z + ½

- ⇨ dx = dz and x = ½, z = 0 ; x = 1, z = ½

J = ∫f(z + ½)dz (integration running from 0 to ½)

= ∫{1 – f(z)}dz (integration running from 0 to ½) (From the given relation)

$= z|_0^{1/2} - I$

- ⇨ I + J = ½

Option (a) is correct.

817. The value of the integral $\int \log_e(1 + \tan\theta)d\theta$ (integration running from 0 to π/4) is
    (a) π/8
    (b) $(\pi/8)\log_e 2$
    (c) 1
    (d) $2\log_e 2 - 1$

Solution :

Let I = $\int \log_e(1 + \tan\theta)d\theta$ (integration running from 0 to π/4)

= $\int \log_e\{1 + \tan(\pi/4 - \theta)\}d\theta$ (integration running from 0 to π/4)

= $\int \log_e\{1 + (1 - \tan\theta)/(1 + \tan\theta)\}d\theta$ (integration running from 0 to π/4)

= $\int [\log_e\{2/(1 + \tan\theta)\}]d\theta$ (integration running from 0 to π/4)

= $\log_e 2\int d\theta - \int \log_e(1 + \tan\theta)d\theta$ (integration running from 0 to π/4)

= $\log_e 2(\pi/4 - 0) - I$

⇨ $2I = (\pi/4)\log_e 2$
⇨ $I = (\pi/8)\log_e 2$

Option (b) is correct.

818. Define the real-valued function f on the set of real numbers by $f(x) = \int\{(x^2 + t^2)/(2 - t)\}dt$ (integration running from 0 to 1). Consider the curve y = f(x). It represents
(a) a straight line
(b) a parabola
(c) a hyperbola
(d) an ellipse

Solution :

Let 2 – t = z

⇨ dt = -dz and t = 0, z = 2; t = 1, z = 1

$-\int[\{x^2 + (2 - z)^2\}/z]dz$ (integration running from 2 to 1)

= $\int[\{x^2 + 4 - 4z + z^2\}/z]dz$ (integration running from 1 to 2)

= $(x^2 + 4)\int dz/z - 4\int dz + \int z dz$ (integration running from 1 to 2)

= $(x^2 + 4)\log 2 - 4 + 3/2$

⇨ It is a parabola

Option (b) is correct.

819. lim $(1/n)\Sigma\cos(r\pi/2n)$ (summation running from 0 to n – 1) as n -> ∞

(a) is 1
(b) is 0
(c) is $2/\pi$
(d) does not exist

Solution :

lim $(1/n)\sum\cos(r\pi/2n)$ (summation running from 0 to n – 1) as $n \to \infty$

$= \int\cos(\pi x/2)dx$ (integration running from 0 to 1)

$= (2/\pi)\sin(\pi x/2)|_0^1$

$= 2/\pi$

Option (c) is correct.

820. lim $(\sqrt{1} + \sqrt{2} + \ldots + \sqrt{(n-1)})/n\sqrt{n}$ as $n \to \infty$ is equal to
(a) ½
(b) 1/3
(c) 2/3
(d) 0

Solution :

lim $(1/n)\sum\sqrt{(r/n)}$ (summation running from 0 to n – 1) as $n \to \infty$

$= \int\sqrt{x}dx$ (integration running from 0 to 1)

Let $x = z^2$

⇨ $dx = 2zdz$ and $x = 0, z = 0; x= 1, z = 1$

$= \int 2z^2dz$ (integration running from 0 to 1)

$= 2/3$

Option (c) is correct.

821. The value of lim $\sum(1/n)[\sqrt{(4i/n)}]$ (summation running from i = 1 to i = n) as $n \to \infty$, where [x] is the largest integer smaller than or equal to x, is
(a) 3
(b) ¾
(c) 4/3
(d) None of the foregoing numbers.

Solution :

lim $(1/n)\Sigma[\sqrt{(4i/n)}]$ (summation running from 1 to n) as $n \to \infty$

= $\int[\sqrt{(4x)}]dx$ (integration running from 0 to 1)

= $\int[\sqrt{(4x)}]dx$ (integration running from 0 to ¼) + $\int[\sqrt{(4x)}]dx$ (integration running from ¼ to 1)

= 0(1/4 – 0) + 1(1 – ¼) = ¾

Option (b) is correct.

822. Let $\alpha = \lim (1^2 + 2^2 + \dots + n^2)/n^3$ as $n \to \infty$ and $\beta = \lim \{(1^3 - 1^2) + (2^3 - 2^2) + \dots + (n^3 - n^2)\}/n^4$ as $n \to \infty$. Then
(a) $\alpha = \beta$
(b) $\alpha < \beta$
(c) $4\alpha - 3\beta = 0$
(d) $3\alpha - 4\beta = 0$

Solution :

$\alpha = \lim (1/n)\Sigma(r/n)^2$ (summation running from 1 to n) as $n \to \infty$

= $\int x^2dx$ (integration running from 0 to 1)

= 1/3

$\beta = \lim (1/n)\Sigma\{(r^3 - r^2)/n^3\}$ (summation running from 1 to n) as $n \to \infty$

= $\int x^3dx$ (integration running from 0 to 1)

= ¼

⇨ $3\alpha - 4\beta = 0$

Option (d) is correct.

823. The value of the integral $\int|x - 3|dx$ (integration running from -4 to 4) is
(a) 13
(b) 8
(c) 25
(d) 24

Solution :

$\int|x - 3|dx$ (integration running from -4 to 4)

$= \int(3 - x)dx$ (integration running from -4 to 3) + $\int(x - 3)dx$ (integration running from 3 to 4)

$= 3x - x^2/2|_{-4}^{3} + (x^2/2 - 3x)|_3^4$

$= 9 - 9/2 - (-12 - 8) + 8 - 12 - (9/2 - 9)$

$= 9/2 + 20 - 4 + 9/2$

$= 25$

Option (c) is correct.

824. The value of $\int|x(x - 1)|dx$ (integration running from -2 to 2) is
(a) 11/3
(b) 13/3
(c) 16/3
(d) 17/3

Solution :

$\int|x(x - 1)|dx$ (integration running from -2 to 2)

$= \int(x^2 - x)dx$ (integration running from -2 to 0) + $\int(x - x^2)dx$ (integration running from 0 to 1) + $\int(x^2 - x)dx$ (integration running from 1 to 2)

$= (x^3/3 - x^2/2)|_{-2}^{0} + (x^2/2 - x^3/3)|_0^1 + (x^3/3 - x^2/2)|_1^2$

$= -(-8/3 - 2) + (1/2 - 1/3) + (8/3 - 2) - (1/3 - ½)$

$= 16/3 + 1 - 2/3$

$= 14/3 + 1$

$= 17/3$

Option (d) is correct.

825. $\int|x\sin\pi x|dx$ (integration running from -1 to 3/2) is equal to
(a) $(3\pi + 1)/\pi^2$
(b) $(\pi + 1)/\pi^2$
(c) $1/\pi^2$

(d) $(3\pi - 1)/\pi^2$

Solution :

$\int|x\sin\pi x|dx$ (integration running from -1 to 3/2)

$= \int x\sin\pi x dx$ (integration running from -1 to 1) + $\int -x\sin\pi x dx$ (integration running from 1 to 3/2)

$= \int x\sin\pi x dx$ (integration running from -1 to 1) - $\int x\sin\pi x dx$ (integration running from 1 to 3/2)

Now, $\int x\sin\pi x dx$

$= x\text{-}\cos\pi x/\pi - \int 1*\{(-\cos\pi x)/\pi\}dx$

$= -x\cos\pi x/\pi + \sin\pi x/\pi^2$

Now, $-x\cos\pi x/\pi + \sin\pi x/\pi^2|_{-1}^{1} = 2/\pi$

And, $-x\cos\pi x/\pi + \sin\pi x/\pi^2|_{1}^{3/2} = -1/\pi^2 - 1/\pi$

So, $(2/\pi) - (-1/\pi^2 - 1/\pi) = (3\pi + 1)/\pi^2$

Option (d) is correct.

826. The set of values of a for which the integral $\int(|x - a| - |x - 1|)dx$ (integration running from 0 to 2) is nonnegative, is

(a) all numbers $a \geq 1$
(b) all real numbers
(c) all numbers a with $0 \leq a \leq 2$
(d) all numbers ≤ 1

Solution :

$\int(|x - a| - |x - 1|)dx$ (integration running from 0 to 2)

$= \int|x - a|dx - \int|x - 1|dx$ (integration running from 0 to 2)

$= \int|x - a|dx$ (integration running from 0 to 2) $- \int(1 - x)dx$ (integration running from 0 to 1) - $\int(x - 1)dx$ (integration running from 1 to 2)

$= \int|x - a|dx$ (integration running from 0 to 2) $- (x - x^2/2)|_0^1 - (x^2/2 - x)|_1^2$

$= \int|x - a|dx$(integration running from 0 to 2) $- ½ - ½$

$= \int|x - a|dx$ (integration running from 0 to 2) $- 1$

Let $a \le 0$

Therefore, $\int|x - a|dx$ (integration running from 0 to 2) = $(x^2/2 - ax)|_0^2$ = $2 - 2a$

Which shows that the given integration is positive.

Let $a \ge 2$, therefore $\int|x - a|dx$ (integration running from 0 to 2) = $(ax - x^2/2)|_0^2 = 2a - 2$

Which shows the given integration is positive.

So, option (a), (c), (d) cannot be true.

- ⇨ Option (b) is correct.

827. The maximum value of a for which the integral $\int e^{-(x - 1)^2}dx$ (integration running from a – 1 to a + 1), where a is a real number, is attained at
    (a) a = 0
    (b) a = 1
    (c) a = -1
    (d) a = 2

Solution :

Let $f(a) = \int e^{-(x - 1)^2}dx$ (integration running from a – 1 to a + 1)

- ⇨ $f'(a) = e^{(-a^2)} - e^{-(a - 2)^2} = 0$
- ⇨ $e^{a^2 - (a - 2)^2} = 1$
- ⇨ $a^2 - (a - 2)^2 = 0$
- ⇨ $a = 1$

Option (b) is correct.

828. Let $f(x) = \int\{5 + |1 - y|\}dy$ (integration running from 0 to x) if $x > 2$, $f(x) = 5x + 1$ if $x \le 2$. Then
    (a) f(x) is continuous but not differentiable at x = 2
    (b) f(x) is not continuous at x = 2
    (c) f(x) is differentiable everywhere
    (d) the tight derivative of f(x) at x = 2 does not exist

Solution :

$f(x) = \int\{5 + |1 - y|\}dy$ (integration running from 0 to x) $x > 2$

= ∫{5 + 1 − y}dy (integration running from 0 to 1) + ∫{5 + y − 1}dy (integration running from 1 to x)

= $6y - y^2/2|_0^1 + (4y + y^2/2)|_1^x$

= $11/2 + 4x + x^2/2 - 4 - ½$

= $x^2/2 + 4x + 1$ for x > 2

lim f(x) as x -> 2- = lim ($x^2/2 + 4x + 1$) as x -> 2- = 11

lim f(x) as x -> 2+ = lim (5x + 1) as x -> 2+ = 11

f(2) = 11

So, f(x) is continuous at x = 2.

lim {f(x) − f(2)}/(x − 2) as x -> 2- = lim {$x^2/2 + 4x + 1 - 11$)/(x − 2) as x -> 2- = lim (x + 4)/1 as x -> 2- (Applying L'Hospital rule) = 6

lim {f(x) − f(2)}/(x − 2) as x -> 2+ = lim {5x + 1 − 11}/(x − 2) as x -> 2+ = lim 5(x − 2)/(x − 2) as x -> 2+ = 5

f(x) is not differentiable at x = 2.

Option (a) is correct.

829. Consider the function f(x) = ∫[t]dt (integration running from 0 to x) where x > 0 and [t] denotes the largest integer less than or equal to t. Then

(a) f(x) is not defined for x = 1, 2, 3, ….

(b) f(x) is defined for all x > 0 but is not continuous at x = 1, 2, 3, …..

(c) f(x) is continuous at all x > 0 but is not differentiable at x = 1, 2, 3, ….

(d) f(x) is differentiable at all x > 0

Solution :

f(I) = ∫[t]dt (integration running from 0 to I) where I is any positive integer

= ∑∫[t]dt (summation running from 0 to I − 1) (integration running from r to r + 1)

= ∑r(r + 1 − r) (summation running from 0 to I − 1)

= I(I − 1)/2

So, f(x) is defined for x = 1, 2, 3, ….

lim f(x) as x -> 1- = lim ∫[t]dt (integration running from 0 to x) x -> 1- = lim 0 x -> 1- = 0

lim f(x) as x ->1 + = lim ∫[t]dt (integration running frm 0 to x) x -> 1+ = lim 0 x -> 1+ = 0

f(1) = ∫[t]dt (integration running from 0 to 1) = 0

f(x) is continuous at x = 1, Similarly, f(x) is continuous at x = 2, 3, ….

lim {f(x) – f(1)}/(x – 1) as x -> 1- = lim {∫[t]dt – 0}/(x – 1) (integration running from 0 to x) as x -> 1- = lim 0/(x – 1) as x -> 1- = 0

lim {f(x) – f(1)}/(x – 1) as x -> 1+ = lim {∫[t]dt – 0}/(x – 1) (integration running from 0 to x) as x -> 1+ = lim {∫[t]dt (integration running from 0 to 1) + ∫[t]dt (integration running from 1 to x)}/(x – 1) as x -> 1+ = lim (x – 1)/(x – 1) as x -> 1+ = 1

So, f(x) is not differentiable at x = 1. Similarly, f(x) is not differentiable at x = 2, 3, ….

Option (c) is correct.

830. Let f(x) = 2 if 0 ≤ x ≤ 1, f(x) = 3 if 1 < x ≤ 2. Define g(x) = ∫f(t)dt (integration running from 0 to x), for 0 ≤ x ≤ 2. Then
(a) g is not differentiable at x = 1
(b) g'(1) = 2
(c)g'(1) = 3
(d) none of the above holds

Solution :

g(x) = ∫f(t)dt (integration running from 0 to x) for 0 ≤ x ≤ 2

lim {g(x) – g(1)}/(x – 1)} as x -> 1- = lim {∫f(t)dt – 2}/(x – 1) (integration running from 0 to x) as x -> 1- = lim (2 – 2)/(x – 2) as x -> 2- = 0

lim {g(x) – g(1)}/(x – 1) aas x ->1+ = lim {∫f(t)dt – 2}/(x – 1) (integration running from 0 to x) as x -> 1+ = lim {∫f(t)dt (integration running from 0 to 1) + ∫f(t)dt (integration running from 1 to x) – 2}/(x – 1) as x -> 1+ = lim {2 + (x – 1) – 2}/(x – 1) as x -> 1+ = 1

g is not differentiable at x = 1.

Option (a) is correct.

831. Let [x] denote the greatest integer which is less than or equal to x. Then the value of the integral $\int[3\tan^2 x]dx$ (integration running from 0 to π/4) is

(a) $\pi/3 - \tan^{-1}\sqrt{(2/3)}$
(b) $\pi/4 - \tan^{-1}\sqrt{(2/3)}$
(c) $3 - [3\pi/4]$
(d) $[3 - 3\pi/4]$

Solution :

$\int[3\tan^2 x]dx$ (integration running from 0 to π/4)

$= \int[3\tan^2 x]dx$ (integration running from 0 to π/6) + $\int[3\tan^2 x]dx$ (integration running from π/6 to $\tan^{-1}\sqrt{(2/3)}$) + $\int[3\tan^2 x]dx$ (integration running from $\tan^{-1}\sqrt{(2/3)}$ to π/4)

$= 0(\pi/6 - 0) + 1(\tan^{-1}\sqrt{(2/3)} - \pi/6) + 2(\pi/4 - \tan^{-1}\sqrt{(2/3)})$

$= \pi/3 - \tan^{-1}\sqrt{(2/3)}$

Option (a) is correct.

832. Consider continuous functions f on the interval [0, 1] which satisfy the following two conditions :

(i) $f(x) \leq \sqrt{5}$ for all x Ɛ [0, 1]
(ii) $f(x) \leq 2/x$ for all x Ɛ [1/2, 1].

Then, the smallest real number α such that inequality $\int f(x)dx$ (integration running from 0 to 1) ≤ α holds for any such f is

(a) $\sqrt{5}$
(b) $\sqrt{5}/2 + 2\log 2$
(c) $2 + 2\log(\sqrt{5}/2)$
(d) $2 + \log(\sqrt{5}/2)$

Solution :

Option (c) is correct.

833. Let $f(x) = \int e^{(-t^2)}dt$ (integration running from 0 to x) for all x > 0. Then for all x > 0,

(a) $xe^{(-x^2)} < f(x)$
(b) $x < f(x)$
(c) $1 < f(x)$

(d) None of the foregoing statements is necessarily true.

Solution :

Integration means sum of the values from lower limit to upper limit.

$f(x) > (x - 0)e^{(-x^2)} = xe^{(-x^2)}$

Option (a) is correct.

834. Let $f(x) = \int \cos\{(t^2 + 2t + 1)/5\}dt$ (integration running from 0 to x), where $0 \le x \le 2$. Then

(a) f(x) increases monotonically as x increases from 0 to 2
(b) f(x) decreases monotonically as x increases from 0 to 2
(c) f(x) has a maximum at $x = \alpha$ such that $2\alpha^2 + 4\alpha = 5\pi - 2$
(d) f(x) has a minimum at $x = \alpha$ such that $2\alpha^2 + 4\alpha = 5\pi - 2$

Solution :

$f'(x) = \cos\{(x^2 + 2x + 1)/5\} = 0$

⇨ $(x^2 + 2x + 1)/5 = \pi/2$
⇨ $2x^2 + 2x = 5\pi - 2$

$f''(x) = -\sin\{(x^2 + 2x + 1)/5\}(2x + 2) < 0$ at $x = \alpha$ which satisfies the equation $x^2 + 2x = 5\pi - 2$

Option (c) is correct.

835. The maximum value of the integral $\int\{1/(1 + x^8)\}dx$ (integration running from a – 1 to a + 1) is attained

(a) exactly at two values of a
(b) only at one value of a which is positive
(c) only at one value of a which is negative
(d) only at a = 0

Solution :

Let $f(a) = \int\{1/(1 + x^8)\}dx$ (integration running from a -1 to a + 1)

$f'(a) = 1/\{1 + (a + 1)^8\} - 1/\{1 + (a - 1)^8\}$

$= \{1 + (a - 1)^8 - 1 - (a + 1)^8\}/\{(a + 1)(a - 1)\}^8 = 0$

Gives, $(a - 1)^8 = (a + 1)^8$

Clearly, this equation is satisfied only when a = 0

Option (d) is correct.

836. The value of the integral ∫coslogxdx is
(a) x[coslogx + sinlogs]
(b) (x/2)[coslogx + sinlogx]
(c)(x/2)[sinlogx – coslogx]
(d) (x/2)[coslogx + sinlogx]

Solution :

∫coslogxdx

Let logx = z

⇨ $x = e^z$
⇨ $dx = e^z dz$

$I = \int e^z \cos z dz$

$= e^z \sin z - \int e^z \sin z dx$

$= e^z \sin z - e^z(-\cos z) + \int e^z(-\cos z)dz$

$= e^z(\sin + \cos z) - I$

⇨ $2I = e^z(\sin z + \cos z)$
⇨ $I = (e^z/2)(\sin z + \cos z) = (x/2)(\cos\log x + \sin\log x)$

Option (d) is correct.

837. If $u_n = \int \tan^n x dx$ (integration running from 0 to π/4) for n ≥ 2, then $u_n + u_{n-2}$ equals
(a) 1/(n – 1)
(b) 1/n
(c)1/(n + 1)
(d) 1/n + 1/(n – 2)

Solution :

Now, $u_n + u_{n-2} = \int \tan^{n-2}x(1 + \tan^2 x)dx = \int \tan^{n-2}x\sec^2 x dx$ (integration running from 0 to π/4)

Let tanx = z, $\sec^2 x dx = dz$ and x = 0, z = 0, x = π/4, z =1

Therefore, $u_n + u_{n-2} = \int z^{n-2}dz$ (integration running from 0 to 1) = $z^{n-1}/(n - 1)$ (upper limit = 1, lower limit = 0) = 1/(n – 1)

Option (a) is correct.

838. $\int \tan^{-1}x dx$ (integration running from 0 to 1) is equal to
  (a) $\pi/4 - \log_e\sqrt{2}$
  (b) $\pi/4 + \log_e\sqrt{2}$
  (c) $\pi/4$
  (d) $\log_e\sqrt{2}$

Solution :

$\int \tan^{-1}x dx$ (integration running from 0 to 1)

$= \tan^{-1}x*x|_0^1 - \int\{1/(1 + x^2)\}xdx$ (integration running from 0 to 1)

$= \pi/4 - (1/2)\int 2xdx/(1 + x^2)$ (integration running from 0 to 1)

Let $1 + x^2 = z$

⇨ 2xdx = dz and x = 0, z = 1; x= 1, z = 2

$= \pi/4 - (1/2)\int dz/z$ (integration running from 1 to 2)

$= \pi/4 - (1/2)\log z|_1^2$

$= \pi/4 - \log_e\sqrt{2}$

Option (a) is correct.

839. $\int\{\sin^{100}x/(\sin^{100}x + \cos^{100}x)\}dx$ (integration running from 0 to π/2) equals
  (a) π/4
  (b) π/2
  (c) 3π/4
  (d) π/3

Solution :

Let, $I = \int\{\sin^{100}x/(\sin^{100}x + \cos^{100}x)dx$ (integration running from 0 to π/2)

$= \int\{\cos^{100}/(\cos^{100}x + \sin^{100}x)dx$ (integration running from 0 to $\pi/2$) (Using the property $\int f(x)dx = \int f(a - x)dx$ when integration is running fro 0 to a)

- ⇨ $I + I = \int\{(\sin^{100}x + \cos^{100}x)/(\sin^{100}x + \cos^{100}x)dx$ (integration running from 0 to $\pi/2$)
- ⇨ $2I = \int dx$ (integration running from 0 to $\pi/2$)
- ⇨ $2I = \pi/2 - 0$
- ⇨ $I = \pi/4$

Option (a) is correct.

840. The indefinite integral $\int\{\sqrt{x}/\sqrt{(a^3 - x^3)}\}dx$ equals
    (a) $(2/3)\sin^{-1}(x/a)^{3/2} + C$ where C is constant
    (b) $\cos^{-1}(x/a)^{3/2} + C$ where C is constant
    (c) $(2/3)\cos^{-1}(x/a)^{3/2} + C$ where C is constant
    (d) None of the foregoing functions.

Solution :

$\int\{\sqrt{x}/\sqrt{(a^3 - x^3)}\}dx$

Let $x = a\sin^{2/3}z$

- ⇨ $dx = (2/3)a\cos z/\sin^{1/3}z dz$

$= \int\{a^{1/2}\sin^{1/3}z(2/3)a\cos z/\sin^{1/3}z\}dz/a^{3/2}\cos z$

$= (2/3)\int dz$

$= (2/3)z + C$

$= (2/3)\sin^{-1}(x/a)^{3/2} + C$

Option (a) is correct.

841. The value of the integral $\int\{e^x\sqrt{(e^x - 1)}/(e^x + 3)\}dx$ (integration running from 0 to log5) is
    (a) 4π
    (b) 4
    (c) $\pi/2$
    (d) $4 - \pi$

Solution :

$\int\{e^x\sqrt{(e^x - 1)/(e^x + 3)}\}dx$ (integration running from 0 to log5)

Let $e^x - 1 = z^2$

⇨ $e^x dx = 2zdz$, $x = 0, z = 0; x = \log 5, z = 2$

$= \int z*2zdz/(z^2 + 4)$ (integration running from 0 to 2)

$= \int(2z^2 + 8 - 8)dz/(z^2 + 4)$ (integration running from 0 to 2)

$= 2\int\{(z^2 + 4)/(z^2 + 4)\}dz - 8\int dz/(z^2 + 4)$ (integration running from 0 to 2)

$= 2z|_0^2 - (8/2)\tan^{-1}(z/2)|_0^2$

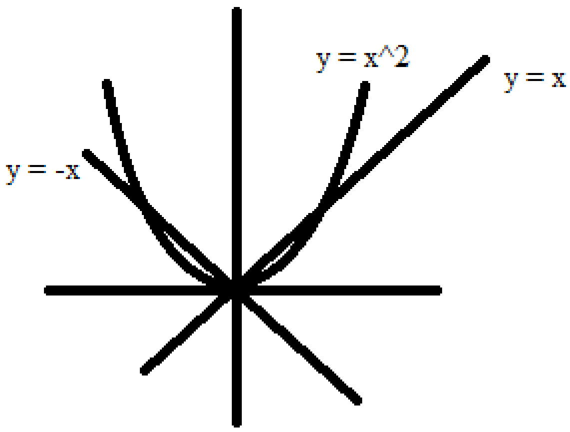
$= 4 - 4(\pi/4 - 0)$

$= 4 - \pi$

Option (d) is correct.

842. The area of the region $\{(x, y) : x^2 \le y \le |x|\}$ is
(a) 1/3
(b) 1/6
(c) ½
(d) 1

Solution :



$x^2 = y$ and $y = x$ solving this we get, $x = 0, x = 1$ and $y = 0, y = 1$

So, the intersection point is (1, 1)

Area $=2[\int xdx - \int x^2dx]$ (integration running from 0 to 1)

$= 2*(x^2/2 - x^3/3)|_0^1$

$= 2(1/2 - 1/3)$

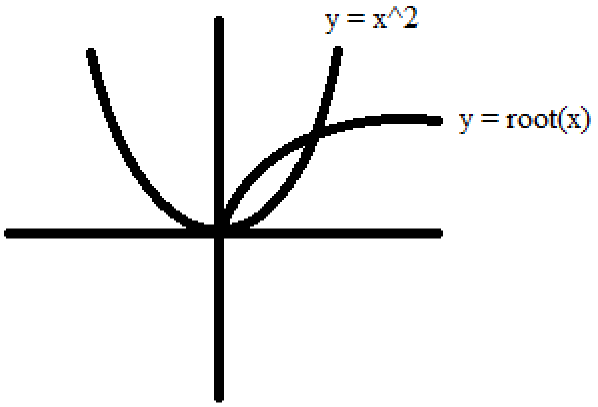
$= 2(1/6)$

$= 1/3$

Option (a) is correct.

843. The area bounded by the curves $y = \sqrt{x}$ and $y = x^2$ is
(a) 1/3
(b) 1
(c) 2/3
(d) None of the foregoing numbers.

Solution :



Solving $y = x^2$ and $y = \sqrt{x}$ we get, $x = 0$, $x = 1$ and $y = 0$, $y = 1$

So, the intersection point is (1, 1)

Area $= \int(\sqrt{x} - x^2)dx$ (integration running from 0 to 1)

$= (2/3)x^{3/2} - x^3/3|_0^1$

$= (2/3) - 1/3$

$= 1/3$

Option (a) is correct.

844. The area bounded by the curve $y = \log_e x$, the x-axis and the straight line $x = e$ equals

(a) e
(b) 1
(c) $1 - 1/e$
(d) None of the foregoing numbers.

Solution :

Solving $y = 0$ and $y = \log_e x$ we get, $x = 1$, $y = 0$

So, the intersection point is (1, 0)

Solving $y = \log_e x$ and $x = e$, we get, $x = e$, $y = 1$

So, the intersection point is (e, 1)

Area = $\int \log_e x dx$ (integration running fro m1 to e)

$= \log_e x * x|_1^e - \int (1/x)*x dx$ (integration running from 1 to e)

$= e - (e - 1)$

$= 1$

Option (b) is correct.

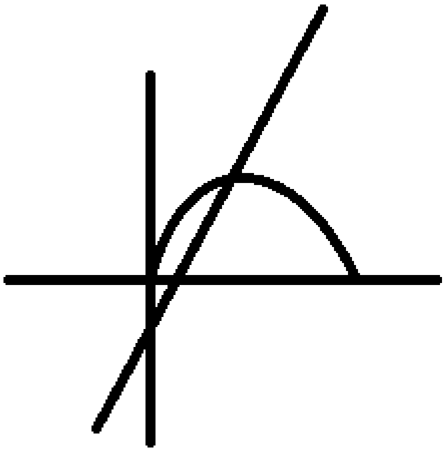
845. The area of the region in the first quadrant bounded by $y = \sin x$ and $(2y - 1)/(\sqrt{3} - 1) = (6x - \pi)/\pi$ equals
(a) $(\sqrt{3} - 1)/2 - (\pi/24)(\sqrt{3} + 1)$
(b) $(\sqrt{3} + 1)/2 - (\pi/24)(\sqrt{3} - 1)$
(c) $\{(\sqrt{3} - 1)/2\}(1 - \pi/12)$
(d) None of the above quantities.

Solution :

Now, $(2y - 1)/(\sqrt{3} - 1) = (6x - \pi)/\pi$

⇨ $2y - 1 = (\sqrt{3} - 1)(6x/\pi) - (\sqrt{3} - 1)$
⇨ $y = \{3(\sqrt{3} - 1)/\pi\}x - (\sqrt{3} - 2)/2$

Solving the two equations we get, $(\pi/3, \sqrt{3}/2)$



When $y = 0$, the straight line intersects the x-axis at, $x = (\pi/6)(\sqrt{3} - 2)/(\sqrt{3} - 1) = \alpha$

Area = $\int[\{3(\sqrt{3} - 1)/\pi\}x - (\sqrt{3} - 2)/2]dx$ (integration running from α to $\pi/3$) + $\int \sin x dx$ (integration running from $\pi/3$ to π)

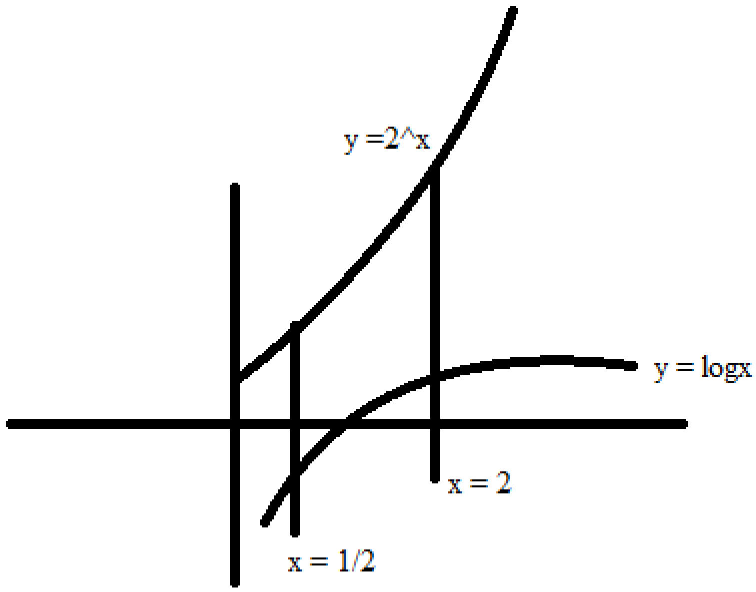
Solving this integration you will find the area.

Option (a) is correct.

846. The area of the region bounded by the straight lines $x = ½$ and $x = 2$, and the curves given by the equations $y = \log_e x$ and $y = 2^x$ is

(a) $(1/\log_e 2)(4 + \sqrt{2}) - (5/2)\log_e 2 + 3/2$
(b) $(1/\log_e 2)(4 - \sqrt{2}) - (5/2)\log_e 2$
(c) $(1/\log_e 2)(4 - \sqrt{2}) - (5/2)\log_e 2 + 3/2$
(d) is not equal to any of the foregoing expressions

Solution :

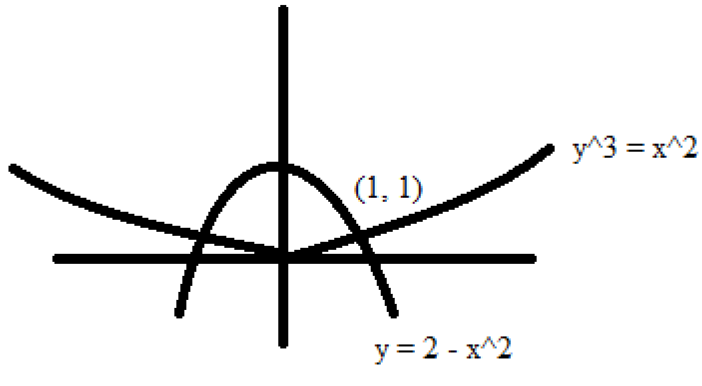


Find the intersection points and write the integrations accordingly and solve them. You will get the area.

Option (c) is correct.

847. The area of the bounded region between the curves $y^3 = x^2$ and $y = 2 - x^2$ is
(a) $2 + 4/15$
(b) $1 + 1/15$
(c) $2 + 2/15$
(d) $2 + 14/15$

Solution :



Area = $2\int(2 - x^2 - x^{2/3})dx$ (integration running from 0 to 1)

$= 2(2x - x^3/3 - (3/5)x^{5/3})|_0^1$

$= 2(2 - 1/3 - 3/5)$

$= 2(2 - 14/15)$

$= 2(16/15)$

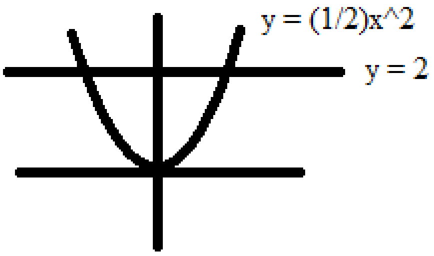
$= 32/15$

$= 2 + 2/15$

Option (c) is correct.

848. The area of the region enclosed between the curve $y = (1/2)x^2$ and the straight line $y = 2$ equals (in sq. units)
(a) 4/3
(b) 8/3
(c) 16/3
(d) 32/3

Solution :



Solving $y = (1/2)x^2$ and $y = 2$ we get, $y = 2$, $x = \pm 2$

Area = $2[2*2 - \int(1/2)x^2dx]$ (integration running from 0 to 2)

$= 2[4 - (1/2)(8/3)]$

$= 2(4 - 4/3)$

$= 8(1 - 1/3)$

$= 16/3$

Option (c) is correct.

849. The value of the integral $\int\{e^{(-x^2/2)}\}\sin x dx$ (integration running from $-\pi/2$ to $\pi/2$) is

(a) π/2 − 1
(b) π/3
(c) √(2π)
(d) None of the foregoing numbers.

Solution :

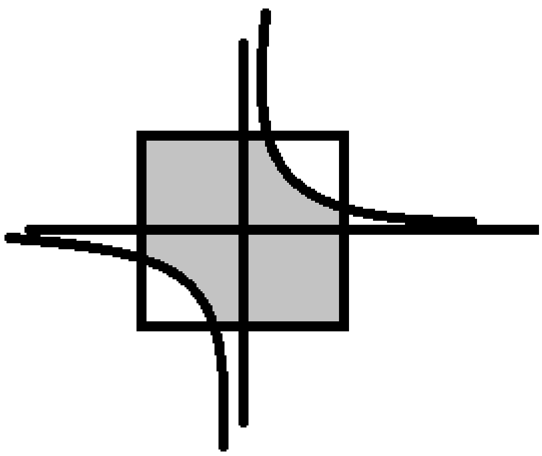
f(x) = {e^(-x^2/2)}sinx = odd function.

So, the integral is zero.

Option (d) is correct.

850. The area of the region of the plane bounded by max(|x|, |y|) ≤ 1 and xy ≤ ½ is
(a) ½ + log2
(b) 3 + log2
(c) 7 + ¾
(d) None of the foregoing numbers.

Solution :



2∫(1/2x)dx (integration running from ½ to 1)

= -log(1/2)

= log2

Area = 1 + 1 + (1/2)log2 + 1*(1/2)*2 = 3 + log2

Option (b) is correct.

851. The largest area of a rectangle which has one side on the x-axis and two vertices on the curve $y = e^{(-x^2)}$ is
(a) $(1/\sqrt{2})e^{-1/2}$
(b) $(1/2)e^{-2}$
(c) $\sqrt{2}e^{-1/2}$
(d) $\sqrt{2}e^{-2}$

Solution :

Let one point is (a, y_1) and another is $(-a, y_1)$

Therefore, area of the rectangle = $2ay_1 = 2a\{e^{(-a^2)}\}$ (as (a, y_1) lies on the curve $e^{(-x^2)}$)

Let, $A = 2ae^{(-a^2)}$

$dA/da = 2[e^{(-a^2)} + ae^{(-a^2)}*(-2a)] = 0$

- ⇨ $1 - 2a^2 = 0$
- ⇨ $a = \pm 1/\sqrt{2}$

Therefore, largest area = $2(1/\sqrt{2})e^{(-1/2)}$

$= \sqrt{2}e^{-1/2}$

Option (c) is correct.

852. Approximate value of the integral $I(x) = \int(\cos t)\{e^{(-t^2/10)}\}dt$ (integration running from 0 to x) are given in the following table.

x	π/2	π	3π/2	2π
I(x)	0.95	0.44	0.18	0.22

Which of the following numbers best approximates the value of the integral $\int(\cos t)\{e^{(-t^2/10)}\}dt$ (integration running from 0 to 5π/4)?
(a) 0.16
(b) 0.23
(c) 0.32
(d) 0.40

Solution :

Option (b) is correct.

853. The maximum of the areas of the isosceles triangles lying between the curve $y = e^{-x}$ and the x-axis, with the base on the positive x-axis, is

(a) 1/e
(b) 1
(c) ½
(d) e

Solution :

Let the coordinate of the vertex which lies on the curve is (a, y_1)

Therefore area of the triangle = $(1/2)2a*y_1 = ae^{-a}$

Let, $A = ae^{-a}$

$dA/da = e^{-a} + ae^{-a}(-1) = 0$

⇨ $a = 1$

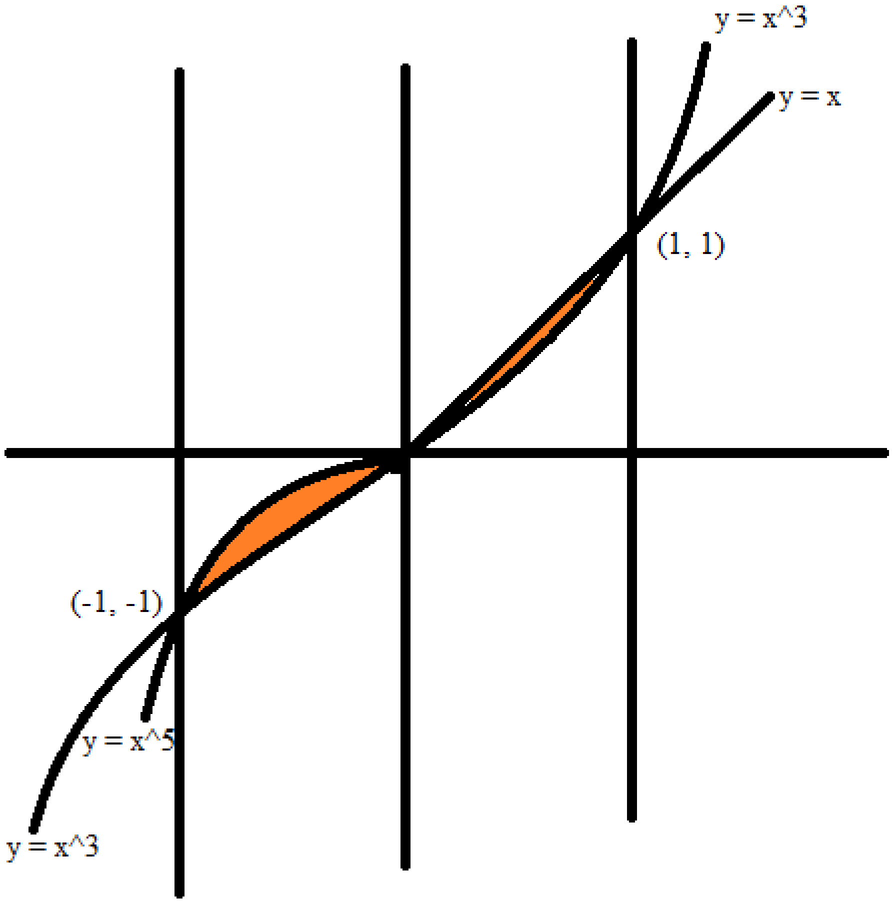
$A_{max} = 1/e$

Option (a) is correct.

854. The area bounded by the straight lines x = -1 and x = 1 and the graphs of f(x) and g(x), where $f(x) = x^3$ and $g(x) = x^5$ if $-1 \le x \le 0$, $g(x) = x$ if $0 \le x \le 1$ is

(a) 1/3
(b) 1/8
(c) ½
(d) ¼

Solution :



Area = $\int(x - x^3)dx$ (integration running from 0 to 1) + $|\int(x^3 - x^5)dx|$ (integration running from -1 to 0)

= (1/2 – ¼) + |(1/4 – 1/6)|

= ¼ + 1/12

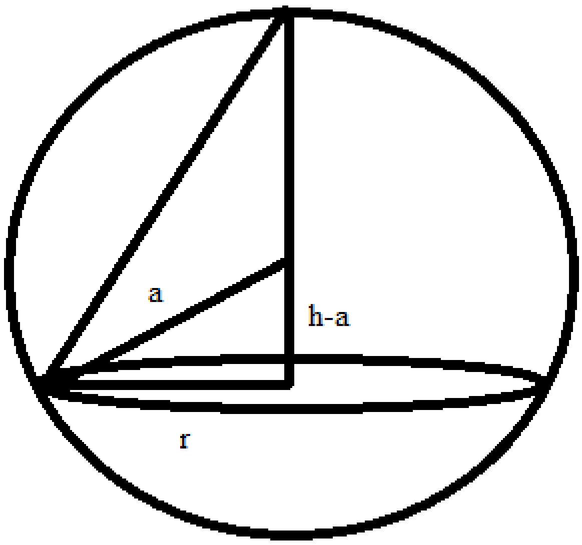
= 1/3

Option (a) is correct.

855. A right circular cone is cut from a solid sphere of radius a, the vertex and the circumference of the base lying on the surface of the sphere. The height of the cone when its volume is maximum is

(a) 4a/3
(b) 3a/2
(c) a
(d) 6a/5

Solution :



Let the radius of the cone is r and height is h.

Then we have, $a^2 = (h - a)^2 + r^2$

⇨ $r^2 = a^2 - (h - a)^2 = a^2 - h^2 + 2ha - a^2 = 2ha - h^2$

Volume $= V = (1/3)\pi r^2 h = (1/3)\pi(2ha - h^2)h = (1/3)\pi(2ah^2 - h^3)$

$dV/dh = (1/3)\pi(4ah - 3h^2) = 0$

⇨ $h = 4a/3$

Option (a) is correct.

856. For any choice of *five* distinct points in the unit square (that is, a square with side 1 unit), we can assert that there is a number c such that there are at least two points whose distance is less than or equal to c. The smallest value c for which such an assertion can be made is
(a) $1/\sqrt{2}$
(b) 2/3
(c) ½
(d) None of the foregoing numbers.

Solution :

When the four points are the vertex and the fifth point is the centre of the square.

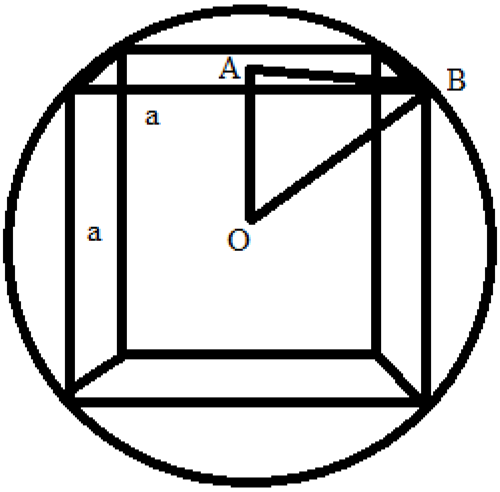
So, $c^2 + c^2 = 1$

⇨ $c = 1/\sqrt{2}$

Option (a) is correct.

857. The largest volume of a cube that can be enclosed in a sphere of diameter 2 cm, is, in cm^3,
(a) 1
(b) $2\sqrt{2}$
(c) π
(d) $8/3\sqrt{3}$

Solution :



$AB^2 = (a/2)^2 + (a/2)^2 = a^2/2$

$OA^2 = (a/2)^2 = a^2/4$

$OB = 2/2 = 1$

Now, $OB^2 = AB^2 + OA^2$

⇨ $1^2 = a^2/2 + a^2/4 = 3a^2/4$
⇨ $a = 2/\sqrt{3}$
⇨ $V = a^3 = 8/3\sqrt{3}$

Option (d) is correct.

858. A lane runs perpendicular to a road 64 feet wide. If it is just possible to carry a pole 125 feet long from the road into the lane, keeping it horizontal, then the minimum width of the lane must be (in feet)

(a) $(125/\sqrt{2} - 64)$
(b) 61
(c) 27
(d) 36

Solution :

Option (c) is correct.

INMO Solution of 20 selected Problems on Number Theory

1. If x, y are integers, and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides $xy - 12x + 15y$.

Solution :

Now, 17 divides $x^2 - 3xy + 2y^2 + x - y$

⇨ 17 divides $(x - y)((x - 2y + 1)$

Adding the given two expressions we get, 17 divides $2x^2 - 5xy + 3y^2 - 4x + 6y$

⇨ 17 divides $(2x - 3y)(x - y - 2)$

Now, 3 cases arises.

(i) 17 divides $x - y$ and $2x - 3y$
(ii) 17 divides $x - 2y + 1$ and $2x - 3y$.
(iii) 17 divides $x - y - 2$ and $x - 2y + 1$.

Case (i) :

$x, y \equiv 0 \pmod{17}$

⇨ 17 divides $2x^2 - 5xy + 15y$.

Case (ii) :

$x \equiv 3$ and $y \equiv 2 \pmod{17}$

$xy - 12x + 15y \equiv 3*2 - 12*3 + 15*2 = 0 \pmod{17}$

Case (iii) :

$x \equiv 5$ and $y \equiv 3 \pmod{17}$

xy – 12x + 15y ≡ 5*3 – 12*5 + 15*3 = 0 (mod 17)

Proved.

2. Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (For example, 31545, 34155 are two such numbers)

Solution :

Let us fix 15 at the right end i.e. the numbers are |||15.

Now, if a digit is divided by 3 then remainders can be 0, 1, -1.

So, we divide the digits as (0, 3, 6, 9); (1, 4, 7); (2, 5, 8).

Now, 1+5 = 6 is divisible by 3.

So, we need to take 3 numbers as 3 from any set or 1 from each set.

So, number of numbers = $4^3 + 3^3 + 3^3 + 4*3*3 = 154$.

Now, there the numbers with left most digit 0 are included. We need to opt out those.

So, we fix 0 in the left most position i.e. the numbers are 0||15.

We can take 2 numbers from 0 remainder set or 1 from each 1 remainder and -1 remainder set.

Number of numbers = $4^2 + 3*3 = 25$.

Therefore number of numbers when block is at the right most end is 154 – 25 = 129.

Now, we push the block 1 digit left i.e. the numbers are ||155.

We can take 2 numbers from -1 remainder set or 1 from each 0 and 1 remainder set.

So, number of numbers = $3^2 + 3*4 = 21$.

Now, we need to opt out the numbers with 0 at the left most position.

So, we fix 0 at the left most digit i.e. the numbers are 0|155.

We can choose 1 number from 1 remainder set.

So, the number of such numbers = 3.

Therefore, number of numbers with block at one digit left is 21 – 3 = 18.

Now, we push the block one more digit left i.e. the numbers are |15|5.

We will get same result as just the previous one.

So, in this case number of numbers = 18.

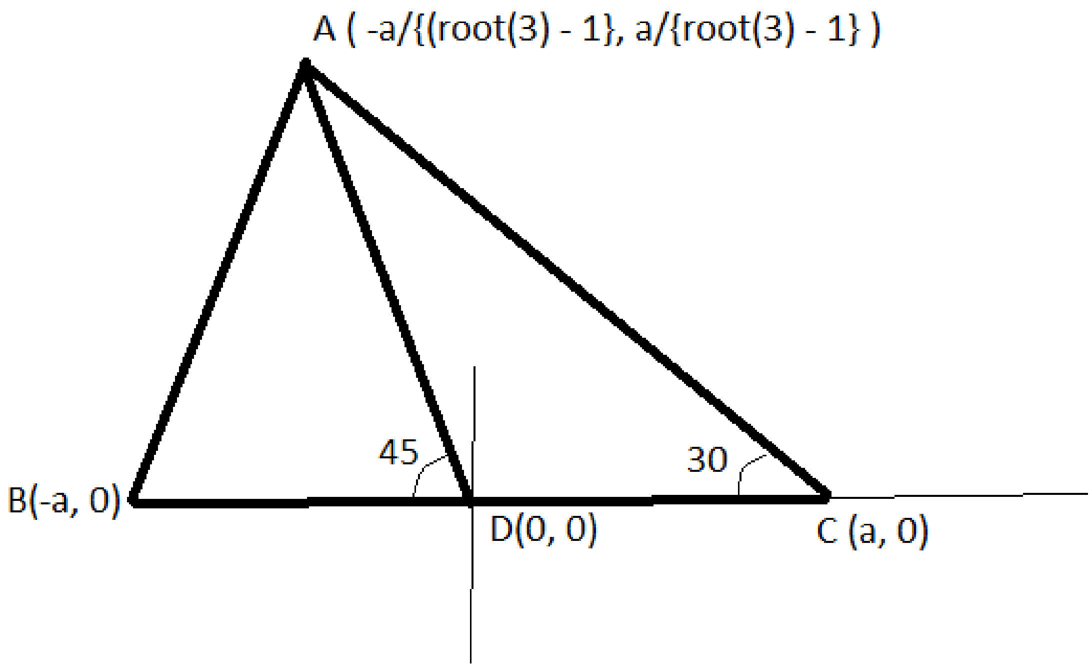
Now, we push the block at left most position i.e. the numbers are 15||5.

Clearly, number of such numbers = 21 as derived earlier. (No case of left most digit becoming 0)

So, total number of numbers = 129 + 18 + 18 + 21 = 186.

3. In triangle ABC, let D be the midpoint of BC. If angle(ADB) = 45 and angle(ACD) = 30, determine angle(BAD).

Solution :



Equation of AC => y – 0 = tan150(x – a)

⇨ $y = (-1/\sqrt{3})(x - a)$

Equation of AD => y – 0 = tan(180 – 45)(x – 0)

⇨ y = -x.

Solving the equation of AC and AD we get co-ordinate of A $(-a/(\sqrt{3} - 1), z/(\sqrt{3} - 1))$

Slope of line AB = $\{a/(\sqrt{3} - 1) - 0\}/\{-a/(\sqrt{3} - 1) + a\} = 1/(\sqrt{3} - 2)$

tan(Angle between AB and AD) = $\{ - 1 - 1/(\sqrt{3} - 2)\}/\{1 - 1/(\sqrt{3} - 2)\}$

- ⇨ $\tan(BAD) = 1/\sqrt{3}$
- ⇨ Angle(BAD) = 30

4. Find the number of positive integers n for which (i) $n \leq 1991$ and (ii) 6 is a factor of $(n^2 + 3n + 2)$.

Solution :

Now, $n^2 + 3n + 2 = (n+1)(n+2)$

As this is factor of two consecutive integer so it is always divided by 2.

Now, it will be divided by 3 if n is not divisible by 3.

So, n cannot be multiple of 3 i.e. n cannot be 3, 6, 9, ..., 1989

Now, $1989 = 3 + (m - 1)*3$

- ⇨ m = 663.

Therefore n cannot take 663 values.

So, n can be 1991 – 663 = 1328.

So, number of required positive integers is 1328.

5. Given a triangle ABC, define the quantities x, y, z as follows :

$x = \tan\{(B - C)/2\}\tan(A/2)$

$y = \tan\{(C - A)/2\}\tan(B/2)$

$z = \tan\{(A - B)/2\}\tan(C/2)$

Prove that $x + y + z + xyz = 0$.

Solution :

We have to prove, $x + y + z + xyz = 0$

i.e. to prove, $(x + y)/(1 + xy) = -z$

Now, LHS = $(x + y)/(1 + xy)$

= [tan{(B – C)/2}tan(A/2) + tan{(C – A)/2}tan(B/2)]/[1 + tan{(B – C)/2}tan(A/2)tan{(C – A)/2}tan(B/2)]

= [tan{(B – C)/2}cot{(B + C)/2} + tan{(C – A)/2}cot{(C + A)/2}]/[1 + tan{(B – C)/2}cot({(B + C)/2}tan(C – A)/2}cot{(C + A)/2}]   (As A + B + C = π)

= [sin{(B – C)/2}cos{(B + C)/2}cos{(C – A)/2}sin{(C + A)/2} + sin{(C – A)/2}cos{(C + A)/2}cos{(B – C)/2}sin{(B + C)/2}]/[cos{(B – C)/2}sin{(B + C)/2}cos{(C – A)/2}sin{(C + A)/2} + sin{(B – C)/2}cos{(B + C)/2}sin{(C – A)/2}cos{(C + A)/2}]

= [(2sin{(B – C)/2}cos{(B + C)/2})(2cos{(C – A)/2}sin{(C + A)/2}) + (2sin{(C – A)/2}cos{(C + A)/2})(2cos{(B – C)/2}sin{(B + C)/2})]/[(2cos{(B – C)/2}sin{(B + C)/2})(2cos{(C – A)/2}sin{(C + A)/2}) + (2sin{(B – C)/2}cos{(B + C)/2})(2sin{(C – A)/2}cos{(C + A)/2})]

= [(sinB – sinC)(sinC + sinA) + (sinC – sinA)(sinB + sinC)]/[(sinB + sinC)(sinC + sinA) + (sinB – sinC)(sinC – sinA)

= (sinBsinC + sinAsinB – $\sin^2$C – sinCsinA + sinBsinC + $\sin^2$C – sinAsinB – sinCsinA)/(sinBsinC + sinAsinB + $\sin^2$C + sinCsinA + sinBsinC – sinAsinB – $\sin^2$C + sinCsinA)

= [2sinC(sinB – sinC)]/[2sinC(sinA + sinB)

= (sinB – sinC)/(sinA + sinB)

= 2cos{(B + A)/2}sin{(B – A)/2}/2sin{(A + B)/2}cos{(A – B)/2}

= -cos{(A + B)/2}sin{(A –B)/2}/sin{(A + B)/2}cos{(A – B)/2}

= - cot{(A + B)/2}tan{(A – B)/2}

= - tan{(A – B)/2}tanC/2 (As A + B + C = π)

= - z

= RHS.

Proved.

6. (i) Determine the set of all positive integers n for which 3^{n+1} divides $2^{\wedge}(3^n) + 1$.

(ii) Prove that 3^{n+2} does not divide $2^{(3^n)} + 1$ for any positive integer n.

Solution :

(i) Putting n = 1 we see that 3^{n+1} divides $2^{(3^n)} + 1$.
Putting n = 2 we see that 3^{n+1} doesn't divide $2^{(3^n)} + 1$.
Putting n = 3 we see that 3^{n+1} divides $2^{(3^n)} + 1$.
Putting n = 4 we see that 3^{n+1} doesn't divide $2^{(3^n)} + 1$.

So, we see that for odd n 3^{n+1} divides $2^{(3^n)} + 1$.

Now, we will prove it by induction.

Putting 2n+1 in place of n we get,

$2^{(3^n)} = 2^{(3^{2n+1})} = (2^3)^{(3^{2n})} = 8^{(9^n)}$

And 3^{n+1} becomes $3^{2n+2} = 9^{n+1}$.

Let, for n = k, 9^{n+1} divides $8^{(9^n)} + 1$.

So, we have, $8^{(9^k)} + 1 = m*9^{k+1}$ .... (A)

We have to prove that $8^{(9^{k+1})} + 1 = m_1 9^{k+2}$

Now, $8^{(9^{k+1})} + 1$

$= \{8^{(9^k)}\}^9 + 1$

$= (m*9^{k+1} - 1)^9 + 1$ (From (A))

$= m^9*\{9^{(k+1)}\}^9 - 9*m^8*(9^{k+1})^8 + {}^9C_2 m^7*(9^{k+1})^7 - .... + 9*m*(9^{k+1}) - 1 + 1$

$= 9^{k+2}*m_1$

So, 3^{n+1} divides $2^{(3^n)}$ when n is odd.

(ii) Now, $2^{(3^n)} + 1$

$= (3 - 1)^{(3^n)} + 1$

$= 3^{(3^n)} - 3^n(3)^{(3^n - 1)} + ... - \{3^n(3^n - 1)/2\}3^2 + 3^n*3 - 1 + 1$

Now, all the terms before 3^n*3 have the factor 3^{n+2} but $3^n*3 = 3^{n+1}$ is not divided by 3^{n+2}.

So, the above expression becomes $m*3^{n+2} + 3^{n+1}$

Clearly, it is not divisible by 3^{n+2}.

Proved.

7. In a triangle ABC, angle A is twice angle B. Show that $a^2 = b(b+c)$.

Solution :

Now, A = 2B

- ⇨ sinA = sin2B
- ⇨ sinA = 2sinBcosB
- ⇨ $a = 2b(c^2 + a^2 - b^2)/2ca$
- ⇨ $ca^2 - ba^2 = b(c^2 - b^2)$
- ⇨ $a^2(c - b) = b(c + b)(c - b)$
- ⇨ $a^2 = b(b + c)$

Proved.

8. Find the remainder when 19^{92} is divided by 92.

Solution :

Now, $19^2 \equiv -7 \pmod{92}$

- ⇨ $19^{92} \equiv (-7)^{46} = 7^{46} \pmod{92}$

$7^4 \equiv 9 \pmod{92}$

- ⇨ $7^{44} \equiv 9^{11} \pmod{92}$
- ⇨ $7^{46} \equiv 7^2*9^{11} \pmod{92}$

Now, $9^2 \equiv -11 \pmod{92}$

- ⇨ $9^{10} \equiv -11^5 \pmod{92}$
- ⇨ $7^2*9^{11} \equiv 7^2*9*(-11^5) \pmod{92}$
- ⇨ $7^2*9^{11} \equiv 19*11^5 \pmod{92}$ (As $7^2*9 \equiv -19 \pmod{92}$

Now, $11^2 \equiv 29 \pmod{92}$

- ⇨ $11^4 \equiv 29^2 \pmod{92}$
- ⇨ $19*11^5 \equiv 19*11*29^2 \pmod{92} \equiv 25*13 \pmod{92} \equiv 49 \pmod{92}$

9. Determine all pairs (m, n) of positive integers for which $2^m + 3^n$ is a perfect square.

Solution :

Now, $2^m + 3^n = (\text{odd number})^2$

Now, if we divide any $(\text{odd number})^2$ by 4 then the remainder is 1.

Let, $2^m + 3^n = p^2$

Now, dividing the equation by 4 we get,

$0 + (-1)^n \equiv 1 \pmod 4$ (m > 10

To hold the equation true n must be even where m can be even or odd.

Case 1 : m is odd.

$2^{2m+1} + 3^{2n} = p^2$ (putting 2m+1 in place of m as it is odd and 2n in place of n as it is even)

Now, dividing the equation by 3 we get,

$(-1)^{2m+1} + 0 \equiv 1 \pmod 3$ (If we divide any square number by 3 then remainder is 0 or 1. Here p^2 cannot be congruent to 0 modulus 3 otherwise the equation will not hold)

=> $-1 \equiv 1 \pmod 3$

Which is impossible.

- ⇨ m cannot be odd.
- ⇨ m is even.

SO, the equation becomes, $2^{2m} + 3^{2n} = p^2$

Now, $2^{2m} + 3^{2n} = (2^m)^2 + (3^n)^2 = (2^m + 3^n)^2 - 2*2^m*3^n = (2^m + 3^n)^2 - 2^{m+1}*3^n$

The equation becomes, $(2^m + 3^n)^2 - 2^{m+1}*3^n = p^2$

- ⇨ $(2^m + 3^n)^2 - p^2 = 2^{m+1}3^n$
- ⇨ $(2^m + 3^n + p)(2^m + 3^n - p) = 2^{m+1}*3^n$
- ⇨ $2^m + 3^n + p = 2^m*3^r$ and $2^m + 3^n - p = 2*3^{n-r}$ (Other combinations of 2 will give p as even but p is odd)
- ⇨ $p = 3^{(n-r)} + 2^{m-1}*3^r$
- ⇨ p is divisible by 3 which is not possible.
- ⇨ $2^m + 3^n + p = 2^m$ and $2^m + 3^n - p = 2*3^n$
- ⇨ $p = 2^{m-1} + 3^n$ and $2^m + 3^n = 2^{m-1} - 3^n$

Second equation is impossible.

⇨ There no pair of positive integers (m, n) exists for which $2^m + 3^n$ is a perfect square.

10. Determine all functions f satisfying the functional relation f(x) + f{1/(1 – x)} = 2(1 – 2x)/{x(1 – x)}, where x is real number x ≠ 0, x ≠ 1. [Here f : R – [0, 1] -> R.]

Solution :

Now, f(x) + f{1/(1 – x)} = 2(1 – 2x)/{x(1 – x)}

Putting 1/(1 – x) in place of x we get,

f{1/(1 – x)} + f(1 – 1/x) = 2(1 – x^2)/x

Now, subtracting the above two equations we get,

f(x) – f(1 – 1/x) = 2(1 – 2x)/{x(1 – x)} – 2(1 – x^2)/x

Now, putting 1/x in place of x we get,

f(1/x) – f(1 – x) = 2(x – 2)x/(x – 1) – 2(x^2 – 1)/x

Now, putting 1 – x in place of x we get,

f(1/(1 – x)) – f(x) = 2(1 + x)(1 –x)/x – 2x(2 – x)/(1 – x)

We have f(x) + f{1/(1 – x)} = 2(1 – 2x)/{x(1 – x)}

Subtracting the above two equations we get,

2f(x) = 2(1 – 2x)/{x(x – 1)} – 2(1 –x^2)/x + 2x(2 – x)/(1 – x)

⇨ f(x) = -(1+x)/(1 – x)

11. Let A = {1, 2, 3, ..., 100} and B be a subset of A having 53 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.

Solution :

If we go on dividing consecutive integers by 11 then we find the remainders 0, 1, 2, 3, 4, 5, -5, -4, -3, -2, -1.

As of now we drop the case of 0.

Now, if I select integers with remainders 1, 2, 3, 4, 5 only then no two integer sum will be divisible by 11.

So, there are total 5*9 = 45 integers we can select e.g. the integers are 1, 2, 3, 4, 5, 12, 13, 14, 15, 16, 23, 24, 25, 26, 27, ..., 89, 90, 91, 92, 93.

Now, we will select 100 as it gives remainder 1 when divided by 11 and we have selected 1 as the remainder. So 1 is necessary to be selected in this case.

So, we have selected 45 + 1 = 46 integers.

Now, we can select 1 integer with 0 remainder when divided by 11 because if we select 2 then sum of them will be divisible be 11.

So, let us say we have selected 11.

So, as of now we have selected 46 + 1 = 47 integers.

Now, if we select any integer from the rest then it will give a remainder of 0 or -1 or -2 or -3 or -4 or -5.

So, one can find two distinct integers x and y whose sum is divisible by 11.

So, if we select 47 + 1 = 48 integers then the desired result is obtained.

Now, 53 > 48 so it is an obvious case.

Proved.

12. In any set of 181 square integers, prove that one can always find a subset of 19 numbers, sum of which is divisible by 19.

Solution :

If we divide any natural number by 19 then remainder can be 0, ±1, ±2, ±3, ±4, ±5, ±6, ±7, ±8, ±9.

Now their squares will leave the remainders as, 0, 1, 4, 9, -3, 6, -2, 11, 7, 5.

We see 10 distinct remainders.

If we take 18 square numbers of each remainders then we have selected 18*10 = 180 square numbers and there no 19 numbers' sum is divisible by 19. Now, if we take one number then square of it will have a remainder from the above 10 remainder set. So, e.g. if it is 1 then we have taken 19 numbers each of remainder 1 and their sum is divisible by 19 as sum of the remainders when divided by 19 is 1*19 = 19. Similar argument goes for other remainders.

So, when we have picked one after selecting 180 square numbers i.e. we have selected 181 square numbers then there one can always find a subset of 19 numbers, sum of which is divisible by 19.

Proved.

13. If $f : R \to R$ is a function satisfying the properties (i) $f(-x) = -f(x)$, (ii) $f(x + 1) = f(x) + 1$, (iii) $f(1/x) = f(x)/x^2$ for $x \neq 0$, prove that $f(x) = x$ for all real values of x. Here R denotes the set of all real numbers.

Solution :

$f(x + 1) - f(x) = 1$

$f(x) - f(x - 1) = 1$

...

...

$f(2) - f(1) = 1$

$f(1) - f(0) = 1$

Adding the above equations we get, $f(x + 1) - f(0) = x + 1$

Now, we have, $f(-x) = -f(x)$

Putting $x = 0$ we get, $f(0) = 0$

Putting value of $f(0)$ in the above equation we get, $f(x +1) = x + 1$

Putting $x - 1$ in place of x we get,

$f(x) = x$.

Proved.

14. Show that there do not exist positive integers m and n such that $(m/n) + (n + 1)/m = 4$.

Solution :

Now, $(m/n) + (n + 1)/m = 4$

⇨ $m^2 + n^2 + n = 4mn$

Now, dividing the equation by n we get,

$m^2 \equiv 0 \pmod{n}$

Now, dividing the equation by m we get,

$n(n + 1) \equiv 0 \pmod{m}$

Now, n divides m^2 and s m cannot divide n. (If m = n then it is possible but doesn't satisfy the equation when m = n).

So, m divides n + 1 as n and n+1 are relatively prime except when n = 1. (If n = 1 then $m + 2/m = 4$ and this doesn't give integer solution of m).

So, we have, $m^2 = k_1 n$ and $n + 1 = k_2 m$

⇨ $m^2 + k_1 = k_1(n + 1) = k_1 k_2 m$

Now, dividing the equation by m we get, $k_1 \equiv 0 \pmod{m}$

⇨ m divides k_1.

The equation is, $m^2 = k_1 n$

⇨ $m = (k_1/m)*n$
⇨ n divides m.

Let, $m = k_3 n$

⇨ $m + k_3 = k_3(n + 1) = k_2 k_3 m$
⇨ $m = k_3/(k_2 k_3 - 1)$

Clearly, the denominator is greater than numerator except when $k_2 = 1$ and it gives m as integer only for one value of k_3 i.e. $k_3 = 2$.

In that case, m = 2, n = 1 and $(m/n) + (n+1)/m = 3$.

Hence, no solution.

Proved.

15. If a, b, c are real numbers such that $abc \neq 0$ and $\{xb + (1 - x)c\}/a = \{xc + (1 - x)a\}/b = \{xa + (1 - x)b\}/c$ then prove that $a = b = c$.

Solution :

Let, $\{xb + (1 - x)c\}/a = \{xc + (1 - x)a\}/b = \{xa + (1 - x)b\}/c = t$

⇨ $xb + (1 - x)c = at$, $xc + (1 - x)a = bt$ and $xa + (1 - x)b = ct$

Now, adding the equations we get, $x(a + b + c) + (1 - x)(a + b + c) = (a + b + c)t$

⇨ $x + 1 - x = t$ $(a + b + c \neq 0)$
⇨ $t = 1$.
⇨ $xb + (1 - x)c = a$, $xc + (1 - x)a = b$ and $xa + (1 - x)b = c$
⇨ $x = (a - c)/(b - c) = (b - a)/(c - a) = (c - b)/(a - b)$

Now, $(a - c)/(b - c) = (b - a)/(c - a)$

⇨ $-(c - a)^2 = -(a - b)(b - c)$
⇨ $c^2 - 2ca + a^2 = ab - b^2 - ca + bc$
⇨ $a^2 + b^2 + c^2 - ab - bc - ca = 0$
⇨ $(1/2)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = 0$
⇨ $a = b = c$.

Proved.

16. If a, b, c are positive real numbers such that $abc = 1$, prove that $a^{(b+c)}b^{(c+a)}c^{(a+b)} \leq 1$.

Solution :

We have to prove, $a^{(b+c)}b^{(c+a)}c^{(a+b)} \leq 1$

i.e. to prove, $a^{(a+b+c)}b^{(a+b+c)}c^{(a+b+c)} \leq a^a b^b c^c$

i.e. to prove, $a^a b^b c^c \geq (abc)^{(a+b+c)}$

i.e. to prove, $a^a b^b c^c \geq 1$

Now, we will apply GM ≥ HM on a, b, c with weights a, b, c respectively.

$(a^a b^b c^c)^{1/(a+b+c)} \geq (a + b + c)/(a/a + b/b + c/c)$

⇨ $a^a b^b c^c \geq \{(a + b + c)/3\}^{(a+b+c)}$

Now, $(a + b + c)/3 \geq (abc)^{1/3}$ (AM ≥ GM)

⇨ $(a + b + c)/3 \geq 1$
⇨ $\{(a + b + c)/3\}^{(a+b+c)} \geq 1$
⇨ $a^a b^b c^c \geq 1$

Proved.

17. Determine the least positive value taken by the expression $a^3 + b^3 + c^3 - 3abc$ as a, b, c vary over all positive integers. Find also all triplets (a, b, c) for which this least value is attained.

Solution :

Now, applying AM ≥ GM we can prove the expression will take minimum value 0 when a = b = c. But we need to find out minimum positive value.

Let, a < b, c.

Let, b = a + k and c = a + r where r and k are non-negative integers.

Now, putting the value in $a^3 + b^3 + c^3 - 3abc$ we get,

$a^3 + (a + k)^3 + (a + r)^3 - 3a(a + k)(a + r)$

$= 3a^3 + 3a^2k + 3ak^2 + k^3 + 3a^2r + 3ar^2 + r^3 - 3a^3 - 3a^2r - 3a^2k - 3akr$

$= 3a(k^2 + r^2 - rk) + k^3 + r^3$

Now, $k^2 + r^2 \geq 2kr$ (AM ≥ GM)

⇨ $k^2 + r^2 \geq rk$
⇨ $k^2 + r^2 - rk$ is positive.

So, the value will be minimum when r and k will assume minimum values. Note that r and k both cannot take 0 value because in that case a = b = c and we will get 0 as the minimum value.

So, we put k = 0 and r = 1.

The expression becomes, 3a + 1.

Now, this will assume minimum value when a is minimum.

Now, a > 0 and an integer.

⇨ a = 1.

So, $(a^3 + b^3 + c^3 - 3abc)_{min} = 3*1 + 1 = 4$.

The triplets for minimum value are (1, 1, 2); (1, 2, 1); (2, 1, 1).

18. Let x, y be positive reals such that x + y = 2. Prove that $x^3y^3(x^3 + y^3) \le 2$.

Solution :

Now, $x^3y^3(x^3 + y^3) = x^3y^3\{((x + y)^3 - 3xy(x + y)\} = (xy)^3(8 - 6xy) = 2z^3(4 - 3z)$ (Putting xy = z)

We have to prove, $2z^3(4 - 3z) \le 2$

i.e. to prove, $z^3(4 - 3z) \le 1$

Now, GM ≤ AM

- ⇨ $xy \le \{(x + y)/2\}^2 = 1$
- ⇨ $4 - 3z = 4 - 3xy \ge 4 - 3*1 = 1$

i.e. $4 - 3z$ is positive.

Applying GM ≤ AM on $4 - 3z$, z, z, z we get,

$\{z^3(4 - 3z)\}^{1/4} \le (4 - 3z + z + z + z)/4$

- ⇨ $z^3(4 - 3z) \le 1$

Proved.

19. Do there exist three distinct positive real numbers a, b, c such that the numbers, a, b, c, b + c − a, c +a − b, a + b- c, a +b + c form a 7-term arithmetic progression in some order?

Solution :

Let, a > b > c

We will arrange the terms in increasing order i.e. we will form an arithmetic progression with common difference > 0.

As of now, we have, c, b, a.

Now, b + c − a < c as (a − b > 0)

So, as of now we have, $b + c - a, c, b, a$

Now, $c + a - b > c$ and $< a$ as $a - b > 0$ and $c - b < 0$. But may be greater than b or less than b.

First we will consider, $c + a - b > b$.

So, as of now we have, $b + c - a, c, b, c + a - b, a$.

Now, $a + b - c > a$ as $b - c > 0$.

So, as of now we have, $b + c - a, c, b, c + a - b, a, a + b - c$.

Now, $a + b + c$ is the biggest term.

So, we have, $b + c - a, c, b, c + a - b, a, a + b - c, a + b + c$.

Now, we will see the difference between each consecutive terms. It must be same as it will form an arithmetic progression.

So, the differences are, $a - b, b - c, c + a - 2b, b - c, b - c, 2c$

So, we have, $b - c = 2c$ and $a - b = 2c$ which give $b = 3c$ and $a = 5c$.

Now, $c + a - 2b = c + 5c - 2*3c = 0$.

Here is the contradiction.

Now, we will take the case $c + a - b < b$.

So, we have, $b + c - a, c, c + a - b, b, a, a + b - c, a + b + c$.

The differences are, $a - b, a - b, 2b - c - a, a - b, b - c, 2c$.

Again we get, $a - b = 2c$ and $b - c = 2c$.

⇨ $2b - c - a = 0$.

Here is the contradiction.

So, there doesn't exist such three real numbers a, b, c.

Proved.

20. Let ABC be a triangle with sides a, b, c. Consider a triangle $A_1B_1C_1$ with sides equal to $a + b/2$, $b + c/2$, $c + a/2$. Show that $[A_1B_1C_1] \geq (9/4)[ABC\}$ where [XYZ] denotes the area of triangle XYZ.

Solution :

$S_1 = [(a + b/2) + (b + c/2) + (c + a/2)]/2 = (3/4)(a + b + c)$

$S_1 - (a + b/2) = (3c + b - a)/4$

$S_1 - (b + c/2) = (3a + c - b)/4$

$S_1 - (c + a/2) = (3b + a - c)/4$

$[A_1B_1C_1] = \sqrt{[S_1\{S_1 - (a + b/2)\}\{S_1 - (b + c/2)\}\{S_1 - (c + a/2)\}]}$

$= \sqrt{[(3/4)(a + b + c)\{(3c + b - a)/4\}\{(3a + c - b)/4\}\{3b + a - c)/4\}]}$

$[ABC] = \sqrt{[(1/2)(a + b + c)\{(b + c - a)/2\}\{(c + a - b)/2\}\{(a + b - c)/2\}]}$

We have to prove, $[A_1B_1C_1] \geq (9/4)[ABC]$

i.e. to prove, $\sqrt{[(3/4)(a + b + c)\{(3c + b - a)/4\}\{(3a + c - b)/4\}\{3b + a - c)/4\}]} \geq 1/2)(a + b + c)\{(b + c - a)/2\}\{(c + a - b)/2\}\{(a + b - c)/2\}]$

i.e. to prove, $27(b + c - a)(c + a - b)(a + b - c) \leq (3c + b - a)(3a + c - b)(3b + a - c)$

Now, applying GM ≤ AM on $(3c + b - a)$, $(3a + c - b)$ and $(3b + a - c)$ we get,

$(3c + b - a)(3a + c - b)(3b + a - c) \leq \{(3c + b - a + 3b + a - c + 3b + a - c)/3\}^3$

⇨ $(3c + b - a)(3a + c - b)(3b + a - c) \leq (a + b + c)^3$

So, we have to prove, $27(b + c - a)(c + a - b)(a + b - c) \leq (a + b + c)^3$

i.e. to prove, $\{(a + b + c)/3\}^3 \geq (b + c - a)(c + a - b)(a + b - c)$

Now, applying AM ≥ GM on $(b + c - a)$, $(c + a - b)$ and $(a + b - c)$ we get,

$\{(b + c - a + c + a - b + a + b - c)/3\}^3 \geq (b + c - a)(c + a - b)(a + b - c)$

⇨ $\{(a + b + c)/3\}^3 \geq (b + c - a)(c + a - b)(a + b - c)$

Proved.

<u>Some Solution of selected Problems from ISI B Stat B Math papers</u>

Problem 1 :

Let n be a positive integer. If n has odd number of divisors (other than 1 and n), then show that n is a perfect square.

Solution 1 :

Any number when divided by a number then another number generated. For example is a is a factor of n i.e. a divides n then another integer b born such that ab = n.

Now, if a and b are distinct then the number has even number of divisors including or excluding 1 and n and the number is not a perfect square. If the number is a perfect square then a*a = n.

In that case only 1 integer i.e. a counts because a appears twice.

But for other divisors of the number always 2 distinct integer comes.

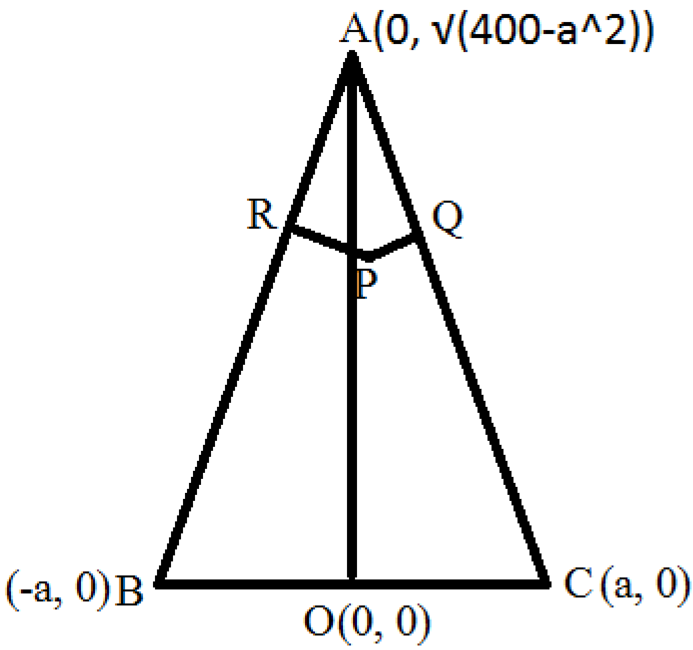
- ⇨ For other divisors number of divisor is even.
- ⇨ For a perfect square number of divisors = even + 1 = odd.

Proved.

Problem 2 :

Let ABC be an isosceles triangle with AB = AC = 20. Let P be a point inside the triangle ABC such that the sum of the distances of P to AB and AC = 1. Describe the locus of all such points.

Solution 2 :



Let the co-ordinate of P is (x_1, y_1)

Equation of AB is, $x/(-a) + y/(\sqrt{(400-a^2)}) = 1$

⇨ $\{\sqrt{(400-a^2)}\}x - ay + a = 0$

Distance of AB from P is $[\{\sqrt{(400 - a^2)}\}x_1 - ay_1 + a]/\sqrt{(400 - a^2 + a^2)} = |[\{\sqrt{(400 - a^2)}\}x_1 - ay_1 + a]/20|$

Similarly, distance of AC from P is $|[\{\sqrt{(400 - a^2)}\}x_1 + ay_1 - a]/20|$

Now, adding we get, $\{\sqrt{(400 - a^2)}\}x_1/10$ or $\pm a(y_1 - 1)/10$

According to question, $\{\sqrt{(400 - a^2)}\}x_1/10$ or $\pm a(y_1 - 1)/10 = 1$

⇨ The locus is either $\{\sqrt{(400 - a^2)}\}x = 10$ or $\pm a(y - 1) = 10$ (Putting (x, y) in place of (x_1, y_1)).

Solved.

Problem 3 :

Let P(X) be a polynomial with integer coefficients of degree $d > 0$. If a and b are two integers such that $P(a) = 1$ and $P(b) = -1$, then prove that $|b - a|$ divides 2.

Solution 3 :

Let $P(X) = a_1X^d + a_2X^{d-1} + ... + a_{d+1} = 0$

Now, $P(a) - P(b) = 1 - (-1) = 2$

- ⇨ $a_1a^d + a_2a^{d-1} + ... + a_{d+1} - (a_1b^d + a_2b^{d-1} + ... + a_{d+1}) = 2$
- ⇨ $a_1(a^d - b^d) + a_2(a^{d-1} - b^{d-1}) + ... + a_d(a - b) = 2$
- ⇨ $a_1(a - b)(a^{d-1} + ... + b^{d-1}) + a_2(a - b)(a^{d-2} + ... + b^{d-2}) + ... + a_d(a - b) = 2$
- ⇨ $(a - b)*m = 2$
- ⇨ $|a - b|$ divides 2.

Proved.

Problem 4 :

In ISI Club each member is on two committees and any two committees have exactly one member in common. There are five committees. How many members does ISI Club have?

Solution 4 :

Consider first committee.

There are 4 persons connected to rest 4 committees.

Now, consider second committee.

Now, with first committee the committee member is already counted. So there are 3 members connected to rest 3 committees.

So, in this way going forward we find the required number of members = 4 + 3 + 2 + 1 = 10.

Solved.

Problem 5 :

In a group of five people any two are either friends or enemies, no three of them are friends of each other and no three of them are enemies of each other. Prove that every person in this group has exactly two friends.

Solution 5 :

Let P_1 person have only 1 friend and he is friend with P_2.

Then that person have 4 enemies.

Now, P_2 must have all friends because otherwise if P_2 is enemy with P_3 then P_1, P_2, P_3 are enemies which contradicts the statement no three of them are enemies of each other.

Now, if P_3 have any friend with P_4 or P_5 then P_2, P_3, P_4 or P_5 are friends which contradicts the statement no three of them are friends.

- ⇨ P_3, P_4 and P_3, P_5 are enemies.
- ⇨ P_1, P_3, P_4 are enemies which contradicts the statement no three of them can be enemies.
- ⇨ Our assumption was wrong.

Similarly, we can prove no person can have 1 enemy.

- ⇨ There are 2 friends or 2 enemies.

Proved.

Problem 6 :

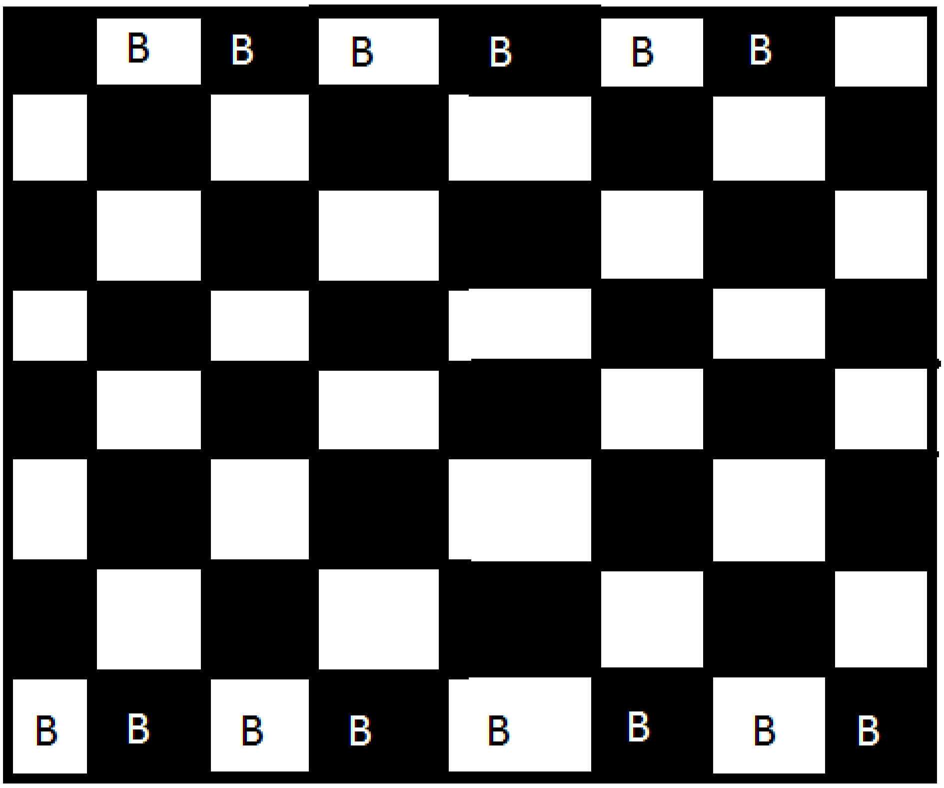
Bishops on a chessboard move along diagonals (i.e. on lines parallel to the two main diagonals). Prove that the maximum number of non-attacking bishops on an nxn chessboard is 2n – 2. (Two bishops are said to be attacking if they are on a common diagonal.)

Solution 6 :

There n bishops can be put in a line. All lines except the n – 2 i.e. excluding the two corners of the opposite line are non-attacking.

So, number of bishops is n + n – 2 = 2n – 2.

This can be easily visualized for 8x8 chessboard as shown in the below figure.



Solved.

Problem 7 :

Find the number of ways in which three numbers can be selected from the set {1, 2, ..., 4n}, such that the sum of the three numbers is divisible by 4.

Solution 7 :

Now, the residue combination of the three selected numbers modulus 4 can be as follows,

(1, 1, 2); (2, 2, 0); (1, 3, 0); (0, 0, 0).

For (1, 1, 2) number of combinations are ${}^{(n-1)}C_2 * {}^{(n-1)}C_1$ as there are n-1 number of numbers, in each category, which are congruent to 1 and 2 modulus 4.

Similarly, number of combinations for (2, 2, 0) are ${}^{(n-1)}C_2 * {}^{n}C_1$ as there are n number of numbers which are divisible by 4.

Similarly, for (1, 3, 0) number of combinations are ${}^{(n-1)}C_1 * {}^{(n-1)}C_1 * {}^{n}C_1$

For (0, 0, 0) number of combinations are ${}^{n}C_3$.

Therefore, total number of combinations are = ${}^{(n-1)}C_2 * {}^{(n-1)}C_1 + {}^{(n-1)}C_2 * {}^{n}C_1 + {}^{(n-1)}C_1 * {}^{(n-1)}C_1 * {}^{n}C_1 + {}^{n}C_3$.

Solved.

Problem 8 :

Prove that there is no non-constant polynomial P(x) with integer coefficients such that P(n) is prime number for all positive integers n.

Solution 8 :

Let, there is a polynomial P(x) with such property.

Let, $P(x) = a_1x^m + a_2x^{m-1} + ... + a_{m+1}$.

Now, $P(a_{m+1}) = a_1a_{m+1}{}^m + a_2a_{m+1}{}^{m-1} + ... + a_{m+1} = a_{m+1}(a_1a_{m+1}{}^{m-1} + a_2a_{m-1}{}^{m-2} + ... + 1)$ which is clearly not a prime.

Now, if there is no constant term a_{m+1} then it is divisible by all numbers.

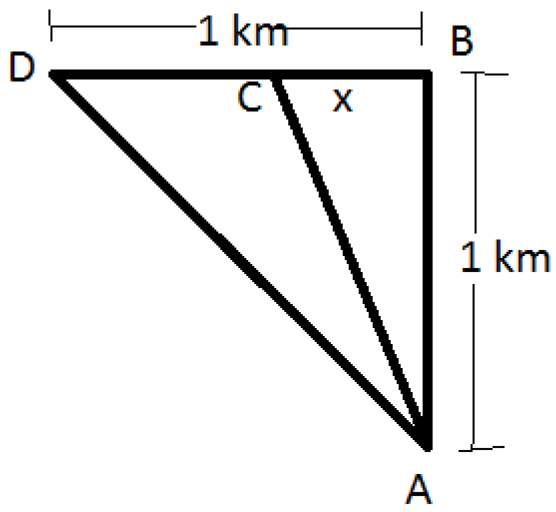
- ⇨ Our assumption was wrong.
- ⇨ There is no non-constant polynomial with such property.

Proved.

Problem 9 :

You are standing at the edge of a river which is one km wide. You have to go to your camp on the opposite bank. The distance of the camp from the point on the opposite bank directly across you is 1 km. You can swim at 2 km/h and walk at 3 km/h. What is the shortest time you will take to reach your camp? (Ignore the speed of the river and assume that the river banks are straight and parallel.)

Solution 9 :



Let the camp is at D and you are standing at point A.

Let, you have swum the river through the path AC.

Let, BC = x km.

Now, from triangle ABC we get,

AC = $\sqrt{(x^2+1)}$ and CD = 1 − x

Now, time taken to swim across the river via path AC = $\{\sqrt{(x^2+1)}\}/2$

Time taken to walk the CD = $(1 - x)/3$

Total time taken = t = $\{\sqrt{(x^2+1)}\}/2 + (1 - x)/3$

Now, differentiating w.r.t. x we get,

$dt/dx = x/2\sqrt{(x^2+1)} - 1/3 = 0$

- ⇨ $3x = 2\sqrt{(x^2+1)}$
- ⇨ $9x^2 = 4x^2 + 4$
- ⇨ $x = 2/\sqrt{5}$

Now, putting $x = 2/\sqrt{5}$ in the above equation we get,

$t = (9 - 2\sqrt{5})/6\sqrt{5}$.

Solved.

Problem 10 :

There are 100 people in a queue waiting to enter a hall. The hall has exactly 100 seats numbered from 1 to 100. The first person in the queue enters the hall, chooses any seat and sits there. The n^{th} person in the queue, where n can be 2,...., 100, enters the hall after $(n-1)^{th}$ person is seated. He sits in the seat number n if he finds it vacant; otherwise he takes any unoccupied seat. Find the total number of ways in which 100 seats can be filled up, provided the 100^{th} person occupies seat number 100.

Solution 10 :

All persons can sit in their sit number. Number of ways is 1.

Now, the first person let seat in seat number p.

p^{th} person seats in seat 1.

Then all other persons occupy their respective seat number.

So, there can be total 98 cases as 100^{th} person seats in 100^{th} seat.

Now, let first person seats in p^{th} seat.

p^{th} person seats in q^{th} seat and q^{th} person seats in first seat.

Then all other person occupies their numbered seat.

2 persons can be chosen in ${}^{98}C_2$ ways.

Similarly, we will consider the case for 3 persons, 4 persons, ..., 98 persons as 100^{th} person seats in 100^{th} position.

So, total number of ways = $1 + {}^{98}C_1 + {}^{98}C_2 + {}^{98}C_3 + ... + {}^{98}C_{98} = 2^{98}$.

Solved.

Problem 11 :

Find a four digit number M such that the number N = 4M has the following properties.

(a) N is a four digit number.
(b) N has the same digits as in M but in reverse order.

Solution 11 :

Let M = 1000d + 100c + 10b + a

Then, N = 1000a + 100b + 10c + d

Now, N = 4M

⇨ 1000a + 100b +10c + d = 4(1000d +100c + 10b + a)
⇨ 3999d + 390c = 60b + 996a

From the above equation it is clear that d is even.

Now, d cannot be 4, 6 or 8 otherwise 4M would not be four digit number.

⇨ d = 2.

Putting d = 2 in the above equation we get,

3999*2 + 390c = 60b + 996a

⇨ 3999 + 195c = 30b + 498a

From the above equation it is clear that c is odd.

Now, c cannot be 5, 7 or 9 because in that case M would not be a 4 digit number.

Let c = 3.

Putting c = 3 in the above equation we get,

3999 + 195*3 = 30b + 498a

- ⇨ 4584 = 30b + 498a

Now, maximum value of a is 9.

In that case the equation becomes, 30b = 102 which doesn't give b as integer.

Now, putting a = 8 we get,

30b = 600

- ⇨ b = 20

But maximum value of b is 9.

- ⇨ c cannot be 3
- ⇨ c = 1.

Now, putting c = 1 in the above equation we get,

3999 + 195*1 = 30b + 498a

- ⇨ 4194 = 30b + 498a

Now, putting a = 9 we get,

30b = -288 which is ambiguous.

Putting a = 8 we get,

30b = 210

- ⇨ b = 7.

Now, putting a = 7 we get,

30b = 708

- ⇨ b > 9.

So, only solution is d = 2, c = 1, b = 7, a = 8.

So, the number is 2178.

Solved.

Problem 12 :

For any integer n greater than 1, show that $2^n < {}^{2n}C_n < 2^n/(1 - 1/n)(1 - 2/n)...\{1 - 1/(n-1)\}$

Solution 12 :

Now, ${}^{2n}C_n$ = 2n(2n-1)(2n-2)(2n-3)...(n+1)n/n*(n-1)...2*1

Now, 2n/n = 2

(2n-1)/(n-1) > 2

(2n-2)/(n-2) > 2

....

....

n+1 >2

n > 2

Now, multiplying the above equations we get, 2n(2n-1)...(n+1)n/n*(n-1)...*2*1 > 2^n

⇨ ${}^{2n}C_n > 2^n$

Now, (1 – 1/n)(1 – 2/n)...{1 – (n-1)/n}*2n(2n-1)(2n-2)...(n+1)n/n*(n-1)...*2*1

= [{(n-1)/n]{(n-2)/n}....(1/n)]* 2n(2n-1)(2n-2)...(n+1)n/n*(n-1)...*2*1

= 2n(2n-1)(2n-2)...(n+1)n/n^n

Now, we have to prove, 2n(2n-1)(2n-2)...(n+1)n/$n^n < 2^n$

i.e. to prove, 2n(2n-1)(2n-2)...(n+1)n < $(2n)^n$

Now, 2n = 2n.

2n > 2n-1.

2n > 2n-2.

...

...

2n > n

Multiplying the above inequality we get,

$(2n)^n > 2n(2n-1)(2n-2)...(n+1)n$

- ⇨ $2^n/(1 - 1/n)(1 - 2/n)....\{1 - (n-1)/n\} > {}^{2n}C_n$

Proved.

Problem 13 :

Let, $a^2 + b^2 = 1$, $c^2 + d^2 = 1$, $ac + bd = 0$

Prove that $a^2 + c^2 = 1$, $b^2 + d^2 = 1$, $ab + cd = 0$.

Solution 13 :

Let, a = SinA, b = Cos A and c = SinB , d = CosB.

Now, ac + bd = 0

- ⇨ SinASinB + CosACosB = 0
- ⇨ Cos(A – B) = 0
- ⇨ A – B = 90

Now, $a^2 + c^2 = \text{Sin}^2A + \text{Sin}^2B = \text{Sin}^2(90+B) + \text{Sin}^2B = \text{Cos}^2B + \text{Sin}^2B = 1$.

Now, $b^2 + d^2 = \text{Cos}^2A + \text{Cos}^2B = \text{Cos}^2(90+B) + \text{Cos}^2B = (-\text{SinB})^2 + \text{Cos}^2B = \text{Sin}^2B + \text{Cos}^2B = 1$.

Now, ab + cd = SinACosA + SinBCosB = (Sin2A+ Sin2B)/2 = Sin(A+B)Cos(A-B) = {Sin(A+B)Cos90} = 0.

Proved.

Problem 14 :

If p is a prime number and a > 1 is a natural number, then show that the greatest common divisor of the two numbers a – 1 and $(a^p - 1)/(a - 1)$ is either 1 or p.

Solution 14 :

First we will prove p can be the greatest common divisor of the two numbers then we will prove that no other prime can be greatest common divisor of the two numbers.

Now, $(a^p - 1)/(a - 1) = 1 + a + a^2 + ... + a^{p-1}$

Now, $a - 1 \equiv 0 \pmod{p}$ (As p is a divisor of a – 1)

Now, $1 + a + a^2 + ... + a^{p-1} \equiv 1 + 1 + ..$ p times (mod p)

- ⇨ $1 + a + a^2 + ... + a^{p-1} \equiv p \equiv 0 \pmod{p}$
- ⇨ p can be divisor of both the numbers.

Let another prime q divides both the numbers.

So, we have, $a \equiv 1 \pmod{q}$

Now, $1 + a + a^2 + ... + a^{p-1} \equiv 1 + 1 +. ...$ p times (mod q)

- ⇨ $1 + a + a^2 + ... + a^{p-1} \equiv p \pmod{q}$

Now, as p is prime so q cannot divide p

- ⇨ No other prime can divide both the numbers.
- ⇨ 1 or p can be the greatest common divisor of the two numbers.

Proved.

Problem 15 :

Show that, for any positive integer n, the sum of 8n+4 composite positive integers cannot be a perfect square.

Solution 15 :

Let us take the sum from m+1.

The sum is (1+2+...+m) + (m+1)+{(m+2)+...+(m+8n+4)} – (1+2+...+m)

$= (m+8n+4)(m+8n+5)/2 - m(m+1)/2$

$= \{m^2 + m(16n+9) + (8n+4)(8n+5) - m^2 - m\}/2$

$= \{m(16n+8) + (8n+4)(8n+5)\}/2$

$= 2m(4n+2) + (4n+2)(8n+5)$

$= (4n+2)(2m+8n+5)$

Now, in the above expression $4n+2$ cannot be equal to $2m+8n+5$.

So, the sum cannot be a square number.

Proved.

Problem 16 :

Prove that in each year, the 13th day of some month occurs on a Friday.

Solution 16 :

Let, 13th January is Monday.

Then 13th February is Wednesday.

If the year is not leap year then 13th March is Tuesday.

13th April is Friday.

In this way considering all possible cases it can be shown.

Or,

$13 \equiv 6 \pmod 7$

$6 + 31 \equiv 2 \pmod 7$ (January is of 31 days)

$2 + 28 \equiv 2 \pmod 7$ (Not leap year so February is of 28 days)

$2 + 31 \equiv 5 \pmod 7$ (March is of 31 days)

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$5 + 30 \equiv 0 \pmod 7$ (April is of 30 days)

$0 + 31 \equiv 3 \pmod 7$ (May is of 31 days)

$3 + 30 \equiv 5 \pmod 7$ (June is of 30 days)

$5 + 31 \equiv 1 \pmod 7$ (July is of 31 days)

$1 + 31 \equiv 4 \pmod 7$ (August is of 31 days)

$4 + 30 \equiv 6 \pmod 7$ (September is of 30 days)

We see that every remainder of 7 has occurred from 0 to 6.

- ⇨ At least one day will be Friday.

Similarly, we need to consider the case of leap year.

Solved.

<u>Some extra problems</u>

1. Find number of 4-tuples (a, b, c, d) of positive integers satisfying the equations, $b^2 = a^3$, $c^2 = d^3$ and $c - a = 64$.

a) 0.
b) 1.
c) 2.
d) 4.

Solution :

Now, $b^2 = a^3$

- ⇨ $b = a^{3/2}$
- ⇨ a have to be a square number because b is integer.

Similarly, from equation $c^2 = d^3$ we get c have to be square number.

Let, $c = x^2$ and $a = y^2$

Now, $c - a = 64$

- ⇨ $x^2 - y^2 = 64$
- ⇨ $(x + y)(x - y) = 2^6$

Let, $x + y = 2^a$ and $x - y = 2^{6 - a}$

⇨ $x = 2^{a-1} + 2^{6-a-1}$
⇨ $x = 2^{a-1} + 2^{5-a}$ and $y = 2^{a-1} - 2^{5-a}$

Putting a = 5 we get, x = 17 and y = 15

Putting a = 4 we get, x = 10 and y = 6

If a = 3 then y = 0 but a is positive integer.

For a < 3 y is negative.

So, we have two solutions, $c = x^2 = 17^2$, $a = y^2 = 15^2$ and $c = 10^2$ and $a = 6^2$

Case 1 : ($c = 17^2$, $a = 15^2$)

Now, $b^2 = a^3$

⇨ $b = (15^2)^{3/2} = 15^3$ and $d = (17^2)^{3/2} = 17^3$

Case 2 : ($c = 10^2$, $a = 6^2$)

Again, $b^2 = a^3$

⇨ $b = (6^2)^{3/2} = 6^3$ and $d = (10^2)^{3/2} = 10^3$

So, 4-tuples are $(15^2, 15^3, 17^2, 17^3)$ and $(6^2, 6^3, 10^2, 10^3)$

Number of 4-tuples satisfying the equations is 2.

Option (c) is correct.

2. Find the n^{th} term of the series 4, 6, 13, 27, 50, 84.

Solution :

Let, $S = 4 + 6 + 13 + 27 + 50 + 84 + \dots\dots\dots + t_n$

$S = \quad 4 + \quad 6 + 13 + 27 + 50 + 84 + \dots + t_{n-1} + t_n$

Subtracting the above equations we get,

$0 = 4 + (2 + 7 + 14 + 23 + 34 + \dots.\ (n-1)^{th}\text{ term}) - t_n$

⇨ $t_n = 4 + (2 + 7 + 14 + 23 + 34 + \dots. + t'_{n-1})$ ....(A)
⇨ $t_n = \quad 4 + 2 + \quad 7 + 14 + 23 + \dots..+ t'_{n-2} + t'_{n-1}$

Subtracting the above equations we get,

$0 = (4 + 2 - 4) + (5 + 7 + 9 + 11 + \dots..\ (n-2)^{th}\text{ term}) - t'_{n-1}$

- ⇨ $t'_{n-1} = 2 + \{(n - 2)/2\}\{2*5 + (n - 3)*2\}$
- ⇨ $t'_{n-1} = 2 + (n - 2)(n + 2)$
- ⇨ $t'_{n-1} = 2 + n^2 - 4$
- ⇨ $t'_{n-1} = n^2 - 2$

Now, from equation (A) we get, $t_n = 4 + \sum t'_{n-1} = 4 + \sum(n^2 - 2)$ (n running from 2 to n – 1)

- ⇨ $t_n = 4 + \sum n^2 - \sum 2 - 1$ (Adding and subtracting 1^2 to make the $\sum n^2$ to run from 1 to n – 1)
- ⇨ $t_n = 4 + (n - 1)(n - 1 + 1)\{2(n - 1) + 1\}/6 - 2(n - 2) - 1$
- ⇨ $t_n = 3 + n(n - 1)(2n - 1)/6 - 2(n - 2)$

Putting n = 2 we get, $t_2 = 4$.

Putting n = 3 we get, $t_3 = 6$.

Putting n = 4 we get, $t_4 = 13$.

Putting n = 5 we get, $t_5 = 27$.

Putting n = 6 we get, $t_6 = 50$.

Putting n = 7 we get, $t_7 = 84$.

3. Write a_n for the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, ... formed by natural numbers not containing the digit 0.

Solution :

Now, we have, 1^{st} to 9^{th} term $f_1(n) = n$.

At 10^{th} term it is 11.

At 19^{th} term it is 21.

At 28^{th} term it is 31.

At 37^{th} term it is 41....and so on for 2-digit numbers.

We see that 10, 19, 28, 37 are in AP.

So we find the n^{th} term here = $10 + (n - 1)*9 = 1 + 9n$

So, we have for n = 1, 1+9n = 10.

For n = 2, 1 + 9n = 19.

For n = 3, 1 + 9n = 28.

Let, $f_2(1 + 9n) = n$

Putting n/9 in place of n we get,

$f_2(1 + 9(n/9) = n/9$

⇨ $f_2(1 + n) = n/9$

Now, putting n – 1 in place of n we get, $f_2(1 + n - 1) = (n - 1)/9$

⇨ $f_2(n) = (n - 1)/9$
⇨ $f_2(10) = 1$
⇨ $f_2(19) = 2$
⇨ $f_2(28) = 3$

Now, we need to add these numbers on 10^{th} , 19^{th}, 28^{th}, ... terms. Rest of the terms will go with the increment from these numbers. So, we can add $[(n - 1)/9]$ with $f_1(n)$ to feed the 2 digit numbers, where [x] denotes the largest integer less than or equal to x.

So, as of now we have $a_n = f_1(n) + f_2(n) = n + [(n - 1)/9]$

Now, we will have 90^{th} term = 90 + [(90 – 1)/9] = 90 + 9 = 99.

Now, 91^{st} term = 91 + [(91 – 1)/9] = 91 + 10 = 101.

We see that it neglects 100 but it cannot neglect 101, 102, 103, ...

But 91^{st} term should be 111.

So, we need to add 10 to 99^{th} term.

172^{nd} term = 172 + [(172 – 1)/9] = 172 + 19 = 191.

If 10 is added to 91^{st} term then 172^{nd} term would be 191 + 10 = 201.

But 172^{nd} term should be 211.

So, we need to add 10 more at 172^{nd} term.

Now for n = 1, we need to have 91.

For n = 2, we need to have 172.

So, we will find n^{th} term of 91, 172, ... and then we will define $f_3(n)$.

By inspection we can write, $f_3(n) = [(|\{(n - 1)/9\} - 1|)/9]*10$

From n = 1 to 8 (n – 1)/9 – 1 is negative so we have put mod |x| there where |x| = x if x ≥ 0 and |x| = - x if x < 0.

From 9 to 90 it is 0.

At 91 it becomes 1*10 = 10 which we need to add to 91^{st} term to make it 111.

From 92^{nd} to 171^{st} term it will remain 10.

At 172^{nd} term it will give 2*10 = 20 i.e. 10 more to make 172^{nd} term 211.

In this way it will take care of 3 digit numbers.

So, as of now, $a_n = f_1(n) + f_2(n) + f_3(n)$

⇨ $a_n = n + [(n - 1)/9] + [(|\{(n - 1)/9\} - 1|)/9]*10$

In this way going forward we can write,

$a_n = f_1(n) + f_2(n) + f_3(n) + f_4(n) + f_5(n) + ...$

⇨ $a_n = n + [(n - 1)/9] + [(|\{(n - 1)/9\} - 1|)/9]*10 + [(|\{(|\{(n - 1)/9\} - 1|)/9\} - 1|)/9]*100 + ...$

4. Consider n (> 1) lotus leaves placed around a circle. A frog jumps from one leaf to another in the following manner. It starts from some selected leaf. From there, it skips one leaf in the clockwise direction and jumps to the next one. Then it skips exactly two leaves in clockwise direction and jumps to the next one. Then it skips three leaves in clockwise direction and jumps to the next one, and so on, Notice that the frog may visit the same leaf more than once, Suppose it turns out that if the frog continues this way, then all the leaves are visited by the frog sometime or the other. Show that n *cannot* be odd.

Solution :

Clearly the remainders are 1, 2, 3, 4, 5, ..., n – 1, 0, 1, 2, ... i.e. the remainders will move into a circle.

For example, if we choose n = 5 then first it will be at 1 i.e. 1^{st} leaf. Then the remainder is 2 and it will be added to the last position i.e. 1. So it will go to 1 + 2 = 3^{rd} leaf. Then remainder is 3 and it will go to $3 + 3 = 6 \equiv 1$ i.e. 1^{st} leaf. Then the remainder is 4 and it will go to 1 + 4 = 5^{th} leaf. Then the remainder is 5 i.e. 0 and it will again go to 5 + 0 = 5^{th} leaf. Then the remainder is 1 again and it will continue the same motion in the leaves as again remainder 1 is at position 1.

Note that, if we want the position of the frog after 3^{rd} step, say, then we can take the 3^{rd} term i.e. $3(3 + 1)/2 = 6 \equiv 1 \pmod 5$. So after 3 step it will move to 1^{st} leaf.

So, $(n - 1)^{th}$ term = $(n - 1)n/2$ and as n is odd so n – 1 is divisible by 2 and $(n - 1)n/2$ is divisible by n.

So, if n is odd then after $(n - 1)^{th}$ step the frog will stay at n^{th} leaf.

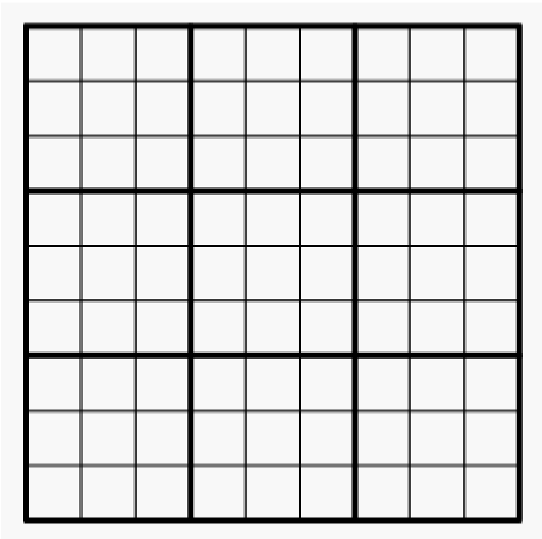
And n^{th} term is n(n+1)/2 and again as n is odd it is divisible by n and the frog will stay again at n^{th} leaf.

Now, note that again the remainder will be 1, 2, 3, ... so it will move into same leaves again.

Now, the frog has stayed in n^{th} leaf for twice so it has not gone to every leaves.

So, it is proved that the frog cannot go to every leaf if n is odd.

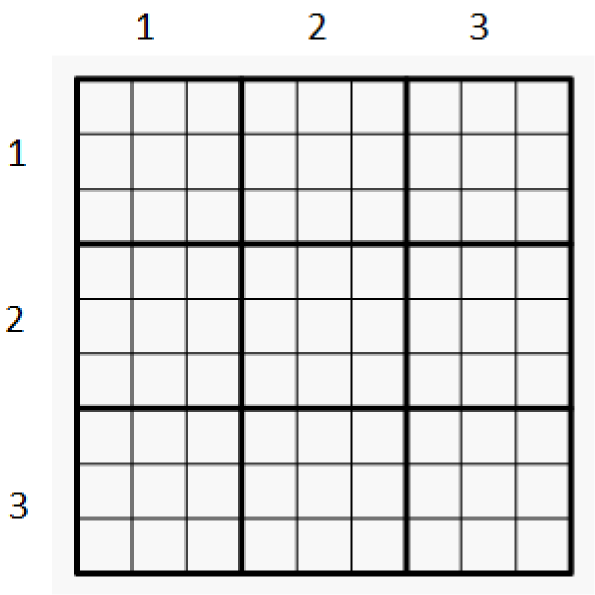
5. The following figure shows a 3^2x3^2 grid divided into 3^2 subgrids of size 3x3. This grid has 81 cells, 9 in each subgrid.



Now, consider an n^2xn^2 grid divided into n^2 subgrids of size nxn. Find the number of ways in which you can select n^2 cells from this grid such that there is exactly one cell coming from each subgrid, one from each row and one from each column.

Solution :

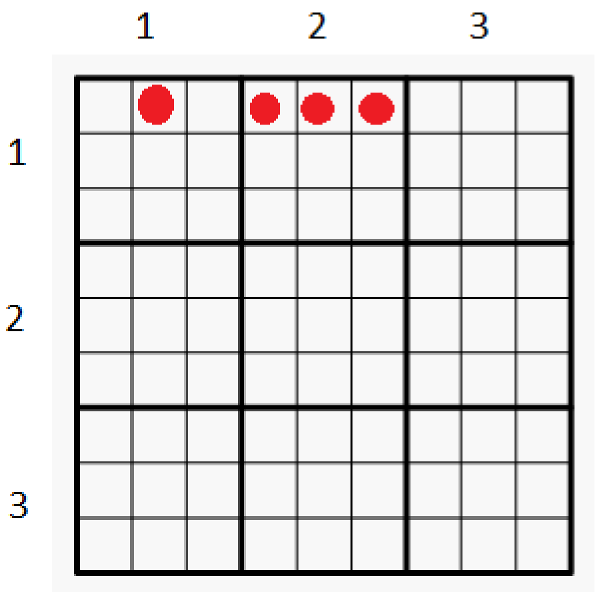
Now consider the (1, 1) subgrib i.e. the first subgrid as shown in the figure below,



There are n^2 numbers of cells. We can select any one cell in n^2 ways.

So, up to this point total number ways is = n^2

Let the cell shown with red dot is selected as shown in below figure.

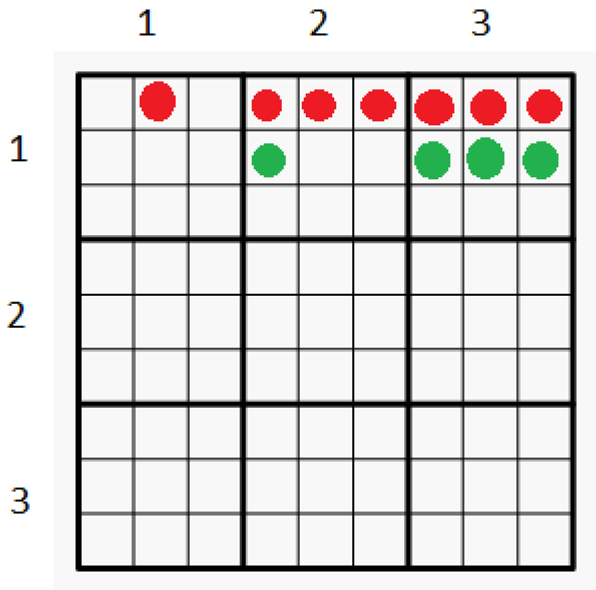


Now, to select the cell from (1, 2) subgrid i.e. second subgrid we can't select the n cells with red dot as shown in the above figure because they are from the same row (one from each row). So, number of cells remaining = $n^2 - n$ as n cells are there in a row.

We can select any one cell from $n^2 - n$ cells in $n^2 - n$ ways.

So, up to this point total number ways is = $n^2*(n^2 - n)$

Now, let the green dot cell is selected from the second subgrid as shown in the below figure,



Now, to select the cell from third subgrid i.e. from the subgrid (1, 3) we need to opt out 2 rows because one due to first subgrid selection marked with red dot and one due to second subgrid selection marked with green dot.

So, we can't select n cells in a row due to red dot and n cells due to green dot.

So, we have to leave n + n = 2n cells when selecting a cell from third subgris i.e. subgrid (1, 3).

So, we are left with $(n^2 - 2n)$ cells.

We can choose a cell from $n^2 - 2n$ cells in $n^2 - 2n$ ways.

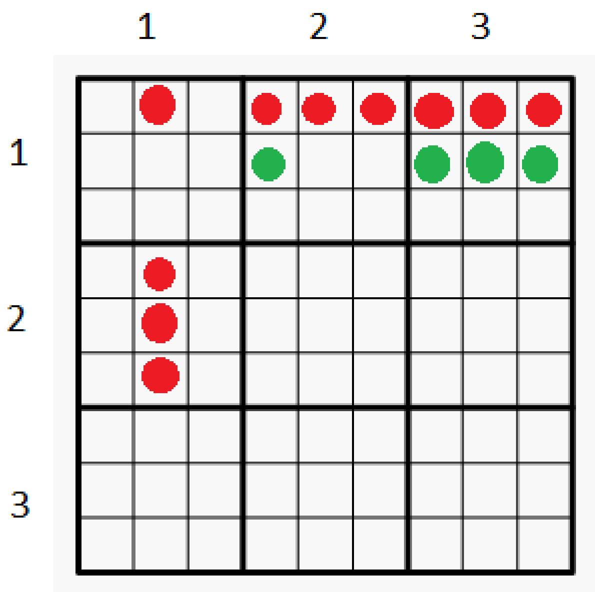
So, up to this point total number of ways is = $n^2*(n^2 - n)*(n^2 - 2n)$

Just following the same rule it is clear that up to n subgrids in first subgrid row total number of ways to choose cells without taking more than one cell from each row will be $n^2*(n^2 - n)*(n^2 - 2n)*(n^2 - 3n)*(n^2 - 5n)*......\{n^2 - (n-1)n\}$

$= n^2*(n^2 - n)*(n^2 - 2n)*(n^2 - 3n)*(n^2 - 5n)*......n.$

Now, we will start selecting cells from second subgrid rows i.e. we will select a cell from the subgrids (2, 1) then (2, 2) and then (2, 3) ....

Now, when we are selecting cell from (2, 1) then the column of red dot needs to be opted out as we can select one cell from each column as shown below.

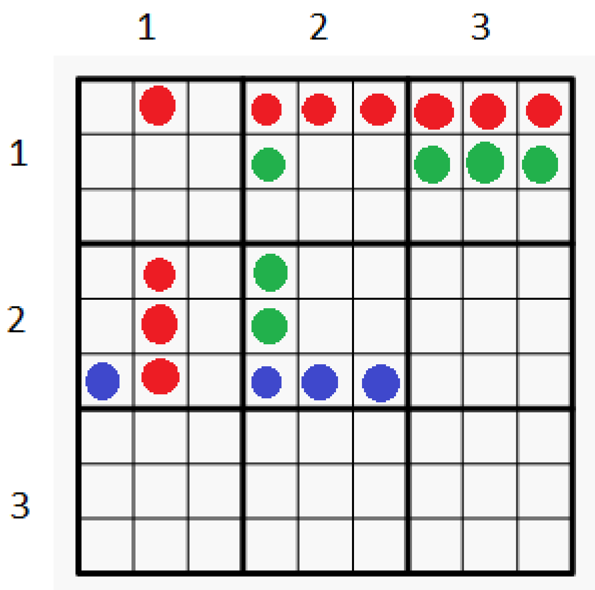


So, number of cells remaining = $n^2 - n$.

We can select a cell from $n^2 - n$ cells in $n^2 - n$ ways.

So, up to this point total number of ways = $\{n^2(n^2 - n)(n^2 - 2n)(n^2 - 3n)(n^2 - 4n)*....*n\}*\{(n^2 - n)\}$

Now, let the cell indicated by blue dot is selected from (2, 1) subgrid as shown in the figure below,



Now, to select cell from the subgrid (2, 2) we need to opt out the green dotted cells due to selection of blue cell in (2, 1) subgrid (We can select one cell from each row) and we also need to opt out the green cells in (2, 1) subgrid due to the selection of green dotted cell in subgrid (1, 2).

Now, note that if we opt out n + n = 2n cells then we are considering one cell twice i.e. focus on the cell which is intersection of green column and blue row in subgrid (2, 2).

So, we need to add 1.

Therefore, total cells remaining = $n^2 - 2n + 1$.

We can select one cell from $n^2 - 2n + 1$ cells in $n^2 - 2n + 1$ ways.

So, up to this point total number of ways = $\{n^2(n^2 - n)(n^2 - 2n)(n^2 - 3n)(n^2 - 4n)*....*n\}*\{(n^2 - n)*(n^2 - 2n + 1)\}$

Following these rules for other subgrids we can arrive at the result i.e. total number of ways = $\{n^2(n^2 - n)(n^2 - 2n)(n^2 - 3n)(n^2 - 4n)*....*n\}*\{(n^2 - n)(n^2 - 2n + 1)(n^2 - 3n + 2)....(n - 1)\}*\{(n^2 - n)(n^2 - 2n + 1)(n^2 - 3n + 2)(n^2 - 4n + 3)....(n - 1)\}*\{(n^2 - 2n)*(n^2 - 3n + 2)*(n^2 - 4n + 4)*(n^2 - 5n + 6).....(n - 2)\}*\{n^2 - 3n)*(n^2 - 4n + 3)*(n^2 - 5n + 6)*(n^2 - 6n + 9).....(n - 3)\}*......*\{(n - 1)*(n - 2)*(n - 3)*....3*2*1\}$.

6. Let $x_1, x_2, ... , x_n$ be positive real numbers with $x_1 + ... + x_n = 1$. Then show that $\{x_1/(2 - x_1)\} + \{x_2/(2 - x_2)\} + ... + \{x_n/(2 - x_n)\} \geq n/(2n - 1)$.

Solution :

Now, $\{x_1/(2 - x_1)\} + \{x_2/(2 - x_2)\} + ... + \{x_n/(2 - x_n)\} = \{x_1/(2 - x_1) + 1\} + \{x_2/(2 - x_2) + 1\} + ... + \{x_n/(2 - x_n) + 1\} - n$ (Adding and subtracting n)

$= \{2/(2 - x_1)\} + \{2/(2 - x_2)\} + ... + \{2/(2 - x_n)\} - n$

Now, AM ≥ HM

- ⇨ $[\{2/(2 - x_1)\} + \{2/(2 - x_2)\} + ... + \{2/(2 - x_n)\}]/n \geq n/[\{(2 - x_1)/2\} + \{(2 - x_2)/2\} + ... + \{(2 - x_n)/2]$
- ⇨ $\{2/(2 - x_1)\} + \{2/(2 - x_2)\} + ... + \{2/(2 - x_n)\} \geq 2n^2/[2n - (x_1 + x_2 + ... + x_n)]$
- ⇨ $\{2/(2 - x_1)\} + \{2/(2 - x_2)\} + ... + \{2/(2 - x_n)\} - n \geq 2n^2/(2n - 1) - n$ (As $x_1 + x_2 + ... + x_n = 1$)
- ⇨ $\{x_1/(2 - x_1)\} + \{x_2/(2 - x_2)\} + ... + \{x_n/(2 - x_n)\} \geq (2n^2 - 2n^2 + n)/(2n - 1)$
- ⇨ $\{x_1/(2 - x_1)\} + \{x_2/(2 - x_2)\} + ... + \{x_n/(2 - x_n)\} \geq n/(2n - 1)$

Proved.

7. Consider three positive real numbers a, b and c. Show that there cannot exist two distinct positive integers m and n such that both $a^m + b^m = c^m$ and $a^n + b^n = c^n$ hold.

Solution :

Let, there are two integer exists m and n such that $a^m + b^m = c^m$ and $a^n + b^n = c^n$

Let, m > n (Without loss of generality)

Now, $(a^m + b^m)/(a^n + b^n) = c^{(m-n)}$

- ⇨ $a^m + b^m = a^n c^{(m-n)} + b^n c^{(m-n)}$
- ⇨ $a^m – a^n c^{(m-n)} = b^n c^{(m-n)} - b^m$
- ⇨ $a^n(a^{(m-n)} – c^{(m-n)}) = b^n(c^{(m-n)} – b^{(m-n)})$

Now, c > a and b which gives LHS negative and RHS positive.

So, the above equation cannot hold true.

- ⇨ Our assumption was wrong.

So, such integers m and n cannot exist simultaneously.

Proved.

8. (a) Show that there cannot exist three prime numbers, each greater than 3, which are in arithmetic progression with a common difference less than 5.

(b) Let k > 3 be an integer. Show that it is not possible for k prime numbers, each greater than k, to be in arithmetic progression with a common difference less than or equal to k+1.

Solution :

(a) The common difference less than 5 can be either 4 or 2. Otherwise at least one of the three numbers will be even and hence not prime.

So, let us consider the case p, p+2, p+4 i.e. common difference 2.

Now, let, $p \equiv 1 \pmod 3$

⇨ $p + 2 \equiv 0 \pmod 3$

Hence not prime.

Let, $p \equiv -1 \pmod 3$

⇨ $p + 4 \equiv 0 \pmod 3$

Hence not prime.

Now, let us consider the case p, p+4, p+8 i.e. common difference 4.

Let, $p \equiv 1 \pmod 3$

⇨ $p + 8 \equiv 0 \pmod 3$

Hence not prime.

Let, $p \equiv -1 \pmod 3$

⇨ $p + 4 \equiv 0 \pmod 3$

Hence not prime.

Proved.

(b) Same rule applies.

9. Consider all non-empty subsets of the set {1, 2, ..., n}. For every such subset we find the product of the reciprocals of each of its elements. Denote the sum of all these products as S_n. For example, $S_3 = (1/1) + (1/2) + (1/3) + (1/1*2) + (1/1*3) + 1/(2*3) + 1/(1*2*3)$

(a) Show that $S_n = (1/n) + \{1 + (1/n)\}s_{(n-1)}$.

(b) Hence or otherwise, deduce that $S_n = n$.

Solution :

(a) Now, $S_{(n-1)} = (1/1) + (1/2) + ... + (1/n-1) + (1/1*2) + ...$ $(1/(n-1)*n)$ + taking three at a time + taking 4 at a time + ... + $(1/(1*2*...(n-1))$

Now, $S_n = (1/1) + (1/2) + ... + (1/(n-1)) + (1/n)$ + taking 2 at a time + $1/n[(1/1) + (1/2) + .. + 1/(n-1)$ + taking 3 at a time + $1/n[1/(1*2) + 1/(1*3) + ... + (1/(n-1))$ + taking 4 at a time + $1/n*$(taking 3 at a time) + ... + $(1/n)*(1/1*2*..(n-1))$

Clearly, $S_n = 1/n + S_{(n-1)} + (1/n)S_{(n-1)}$

- ⇨ $S_n = (1/n) + (1 + 1/n)S_{(n-1)}$

Proved.

(b) We will apply induction theorem to prove $S_n = n$.

Now, $S_n = 1/n + (1 + 1/n)S_{(n-1)}$

- ⇨ $S_2 = ½ + (1 + ½)S_1$
- ⇨ $S_2 = ½ + 3/2$ (As $S_1 = 1$)
- ⇨ $S_2 = 2$.

Let, $S_k = k$

Now, $S_{(k+1)} = 1/(k+1) + \{1 + 1/(k+1)\}S_k$

- ⇨ $S_{(k+1)} = 1/(k+1) + (k+2)*k/(k+1)$
- ⇨ $S_{(k+1)} = (1 + k^2 + 2k)/(k+1)$
- ⇨ $S_{(k+1)} = (k+1)^2/(k+1)$
- ⇨ $S_{(k+1)} = k + 1$

Proved.

10. Show that the triangle whose angles satisfy the equality $(\sin^2A + \sin^2B + \sin^2C)/(\cos^2A + \cos^2B + \cos^2C) = 2$ is right angled.

Solution :

We have, $(\sin^2A + \sin^2B + \sin^2C)/(\cos^2A + \cos^2B + \cos^2C) = 2$

- ⇨ $\cos^2A + \cos^2B + \cos^2C = 1$ (Putting $\sin^2\theta = 1 - \cos^2\theta$
- ⇨ $\cos2A + \cos2B + \cos2C = -1$ (Putting $2\cos^2\theta = 1 + \cos2\theta$)
- ⇨ $\cos(A+B)\cos(A-B) + (1+\cos2C) = 0$
- ⇨ $\cos C\cos(A-B) + \cos^2C = 0$ ($A + B = \pi - C$)
- ⇨ $\cos C\{\cos(A-B) + \cos C) = 0$
- ⇨ $\cos C = 0$
- ⇨ C is a right angle.

Proved.

11. Let X, Y, Z be the angles of a triangle.

(i) Prove that $\tan(X/2)\tan(Y/2) + \tan(X/2)\tan(Z/2) + \tan(Z/2)\tan(Y/2) = 1$.

(ii) Using (i) or otherwise prove that $\tan(X/2)\tan(Y/2)\tan(Z/2) \le 1/3\sqrt{3}$

Solution :

(i) We have to prove $\tan(X/2)\tan(Y/2) + \tan(X/2)\tan(Z/2) + \tan(Z/2)\tan(Y/2) = 1$
i.e. to prove $\sin(X/2)\sin(Y/2)\cos(Z/2) + \sin(X/2)\sin(Z/2)\cos(Y/2) + \sin(Z/2)\sin(Y/2)\cos(X/2) = \cos(X/2)\cos(Y/2)\cos(Z/2)$

Now, LHS $= \sin(X/2)\sin(Y/2)\cos(Z/2) + \sin(X/2)\sin(Z/2)\cos(Y/2) + \sin(Z/2)\sin(Y/2)\cos(X/2)$

$= \sin(x/2)\{\sin(Y/2)\cos(Z/2) + \cos(Y/2)\sin(Z/2)\} + \sin(Z/2)\sin(Y/2)\cos(X/2)$

$= \sin(X/2)\sin\{(Y+Z)/2\} + \sin(Z/2)\sin(Y/2)\cos(X/2)$

$= \sin(X/2)\sin(90 - X/2) + \sin(Z/2)\sin(Y/2)\cos(X/2)$ (As $X+Y+Z = 180$)

$= \sin(X/2)\cos(X/2) + \sin(Z/2)\sin(Y/2)\cos(X/2)$

$= \cos(X/2)\{\sin(X/2) + \sin(Z/2)\sin(Y/2)\}$

$= \cos(X/2)[\sin\{90 - (Y/2 + Z/2)\} + \sin(Z/2)\sin(Y/2)\}$ (As $X+Y+Z = 180$)

$= \cos(X/2)[\cos(Y/2 + Z/2) + \sin(Z/2)\sin(Y/2)]$

$= \cos(x/2)[\cos(Y/2)\cos(Z/2) - \sin(Y/2)\sin(Z/2) + \sin(Y/2)\sin(Z/2)]$

$= \cos(x/2)\cos(Y/2)\cos(Z/2)$

= RHS

Proved.

(ii) Now, AM ≥ GM

⇨ $[\tan(X/2)\tan(Y/2) + \tan(X/2)\tan(Z/2) + \tan(Z/2)\tan(Y/2)]/3 \ge [\tan(X/2)\tan(Y/2)\tan(X/2)\tan(Z/2)\tan(Z/2)\tan(Y/2)]^{1/3}$

⇨ $[\tan(X/2)\tan(Y/2)\tan(Z/2)]^{2/3} \le 1/3$ (From (i))

⇨ $\tan(X/2)\tan(Y/2)\tan(Z/2) \le 1/3\sqrt{3}$

Proved.

12. Write the set of all positive integers in triangular array as

1 3 6 10 15 ....

2 5 9 14 ....

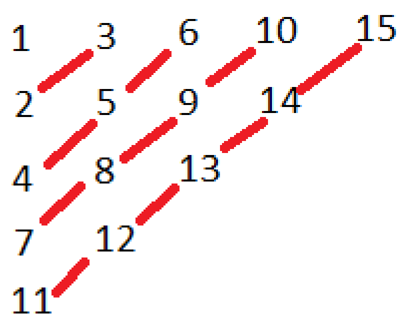
4 8 13 ....

7 12 ....

11 ....

Find the row number and column number where 20096 occurs. For example 8 appears in the third row and second column.

Solution :



Now, take the diagonals.

Diagonal 1 consists of 1.

Diagonal 2 consists of 2, 3 i.e. 2 numbers.

Diagonal 3 consists of 4, 5, 6 i.e. 3 numbers and so on.

So, n^{th} diagonal consists of n numbers.

Up to n diagonal or diagonal n will end with $1 + 2 + 3 + ... + n = n(n+1)/2$ i.e. in n^{th} diagonal 1^{st} row will be $n(n+1)/2$.

Now, we have to find the row and column number of 20096.

So, we form the equation, $n(n+1)/2 < 20096$

- ⇨ $n^2 + n - 40192 < 0$
- ⇨ $(n + ½)^2 < 160769/4$
- ⇨ $|n + 1/2| < 400.96/2$

- ⇨ $n < 199.98$
- ⇨ $n = 199$.

Now, 199^{th} diagonal will end with the number 199(199+1)/2 = 19900.

Now, 200^{th} diagonal will contain 20096 as 20096 – 19900 = 196

So, column number is 196.

Now, 200^{th} diagonal will end with 200*(200+1)/2 = 20100.

Now, 20100 – 20096 = 4.

So, row number is 4+1=5.

So, 20096 will appear in 5^{th} row and 196^{th} column.

13. Let m be a natural number with digits consisting entirely of 6's and 0's. Prove that m is not the square of a natural number.

Solution :

We can put always even number of 0's after end of 6's in the right side and that will be a perfect square if up to end of 6 is perfect square.

If odd number of 0's are put then it will contain $\sqrt{10}$ and not a perfect square.

So, we will omit the cases where m ends with 0's.

So, we will consider the case where m ends with 6.

Now, last digit of 4^2 is 6 and last digit of 6^2 is 6.

First we will consider 4^2 case.

Now, $4^2 = 16$ and we put 6 and in hand we have 1 which is an odd number.

Now, whatever be the previous digit of the square root number i.e. $\sqrt{m}$ we will always find an even number as in the 10^{th} place and adding it to 1 it will give an odd number.

Now, if we multiply by 10^{th} digit of $\sqrt{m}$ then it will always generate an even number as the unit place digit is 4 i.e. an even number and it we add the odd + even = odd but 6 and 0 both are even.

So, m is not consists of only 6's and 0's.

Here is the contradiction.

To understand this we take an example say m is 194^2.

Now, we will multiply 194 with 194 to obtain m and put concentration on the 10^{th} digit of resulting number i.e. m.

194

194

76

6x

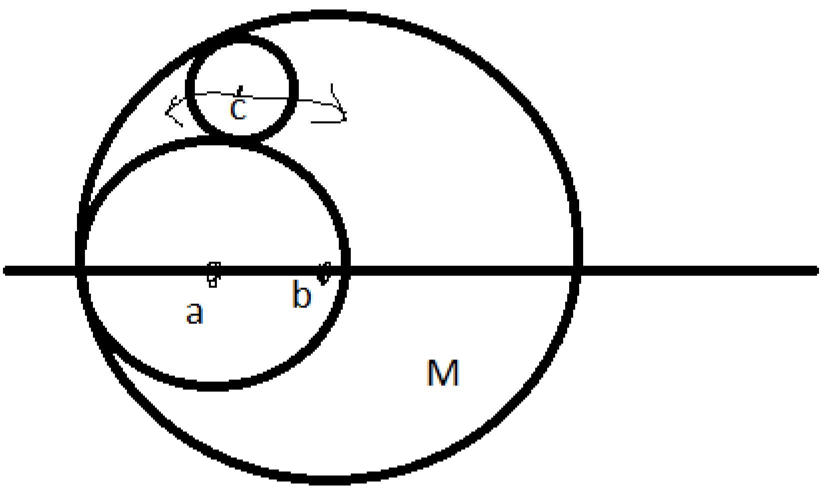
36

3 is odd and it never can be an even so cannot be 6 or 0.

Same contradiction we will find if we work with 6^2.

Proved.

14. Let, $0 < a < b$. Consider two circles with radii a and b and centers $(a, 0)$ and $(0, b)$ respectively with $0 < a < b$. Let c be the center of any circle in the crescent shaped region M between the two circles and tangent to both (See figure below). Determine the locus of c as its circle traverses through region M maintaining tangency.



Solution :

Let co-ordinate of c is (x_1, y_1).

Now, the equation of the circle centered at a is $(x - a)^2 + y^2 = a^2$

The equation of the circle centered at b is $x^2 + (y - b)^2 = b^2$

Now, the touching point co-ordinate of circle a and c is,

$\{(ar + ax_1)/(r + a) , y_1a/(r + a)\}$ where r is radius of circle c.

This will satisfy the equation of circle a.

So, we will find an equation with x_1, y_1 and r.

Let, the touching point co-ordinate of circles b and c is (x_2, y_2)

Then, $x_1 = x_2(b - r)/b$ and $y_1 = \{y_2(b - r) + rb\}/b$

⇨ $x_2 = bx_1/(b - r)$ and $y_2 = (y_1b - rb)/(b - r)$

Now, (x_2, y_2) will satisfy the equation of circle b.

So, we will have another equation with x_1, y_1 and r.

Now, eliminate r from both the equation, you will find an equation with x_1, y_1. Now put (x, y) in place of (x_1, y_1) and you will get the locus of c.

15. For any positive integer n, show that $(1/2)(3/4)(5/6)....(2n-1)/2n < 1/\sqrt{(2n+1)}$

Solution :

We will apply theorem of induction to solve the problem.

Now, put n = 1.

Clearly, $(1/2) < 1/\sqrt{3}$

Put n = 2.

Clearly, $3/8 < 1/\sqrt{5}$

Now, let this is true for n = k.

So, we have, $(1/2)(3/4)(5/6)....(2k - 1)/2k < 1/\sqrt{(2k+1)}$

Now, we will prove for n = k+1 then done.

We have to prove, $(1/2)(3/4)(5/6).....\{(2k-1)/2k\}\{(2k+1)/2(k+1)\} < 1/\sqrt{(2k+3)}$

Now, $(1/2)(3/4)(5/6)....\{(2k-1)/2k\}\{(2k+1)/2(k+1)\} < \{1/\sqrt{(2k+1)}\}\{(2k+1)/2(k+1)\}$

$= \sqrt{(2k+1)}/2(k+1)$

Now, we have to prove that,

$\sqrt{(2k+1)}/2(k+1) < 1/\sqrt{(2k+3)}$

i.e. to prove, $(2k+1)(2k+3) < 4(k+1)^2$

i.e. to prove, $4k^2 + 8k + 3 < 4k^2 + 8k + 4$

i.e. to prove, $3 < 4$

Which is obvious.

- ⇨ The inequality is true for any positive integer n.

Proved.

16. Let R and S be two cubes with sides r and s respectively, where r and s are positive integers. Show that the difference of their volumes equals the difference of their surface areas, if and only if r = s.

Solution :

According to the question we have,

$(4/3)\pi r^3 - (4/3)\pi s^3 = 4\pi r^2 - 4\pi s^2$

- ⇨ $r^3 - s^3 = 3(r^2 - s^2)$
- ⇨ $(r - s)(r^2 + rs + s^2) - 3(r - s)(r + s) = 0$
- ⇨ $(r - s)(r^2 + rs + s^2 - 3r - 3s) = 0$

Let, $r - s \neq 0$

Then, $r^2 + rs + s^2 - 3r - 3s = 0$

- ⇨ $(r - 3/2)^2 + (s - 3/2)^2 + rs = 9/2$

Now, r and s are positive integers.

If r or s > 2 then LHS > RHS.

If r = 2, s = 1, then LHS < RHS.

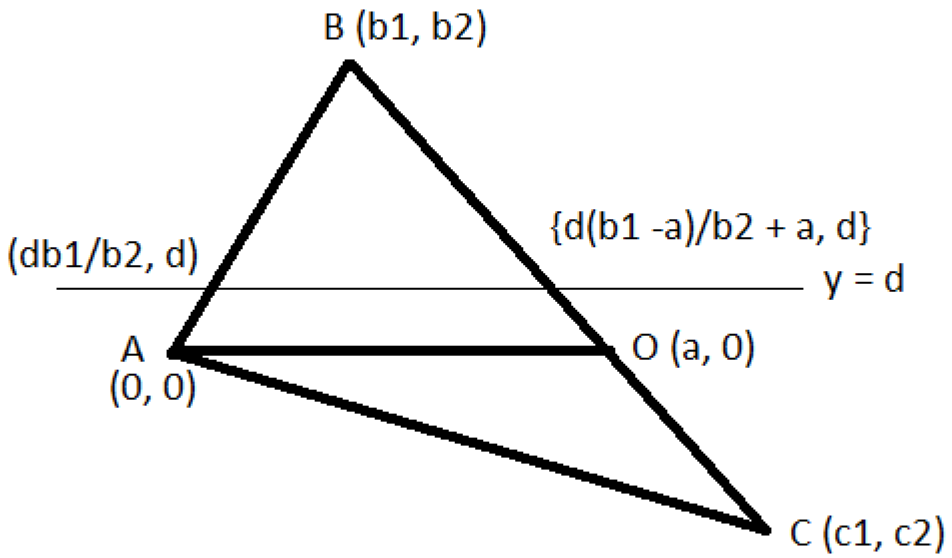
The equality doesn't hold.

- ⇨ Our assumption was wrong.
- ⇨ r = s.

Proved.

17. Let ABC be any triangle and let O be a point on the line segment BC. Show that there exists a line parallel to AO which divides the triangle ABC into two equal parts of equal area.

Solution :



Let, the co-ordinate of point A is (0, 0).

Let co-ordinate of point B is (b_1, b_2).

Let co-ordinate of point C is (c_1, c_2).

Let co-ordinate of O is (a, 0).

Let equation of the line parallel to AO is y = d.

Now, equation of AB is $(x - 0)/(b_1 - 0) = (y - 0)/(b_2 - 0)$

- ⇨ $xb_2 = yb_1$

Intersection of AB and line parallel to AO is $(db_1/b_2, d)$ as shown in the figure.

Equation of line BO is $(x - a)/(b_1 - a) = (y - 0)/(b_2 - 0)$

- ⇨ $b_2(x - a) = y(b_1 - a)$

Now, solving $y = d$ and the above equation we get co-ordinate of intersection point of BO and line parallel to AO as $(d(b_1 - a)/b_2 + a, d)$ as shown in the figure.

Now, area of upper triangle of line parallel to AO is,

$(1/2)[b_1(d - d) + (db_1/b_2)(d - b_2) + \{d(b_1 - a)/b_2 + a\}(b_2 - d)]$

$= (1/2)(a/b_2)(b_2 - d)^2$

Now, area of triangle AOC is,

$(1/2)[0(c_2 - 0) + c_1(0 - 0) + a(0 - c_2) = -(1/2)ac_2$ (Note that c_2 is negative)

Area of trapezium below the parallel line of AO and above AO is,

$(1/2)d(a + a - ad/b_2)$

Now, adding the two areas we get the area of the below portion of line parallel to AO as,\

$-(1/2)ac_2 + (1/2)d(a + a - ad/b_2)$

Now, equating the two areas as per question we get,

$d(2b_2 - d) - b_2c_2 = (b_2 - d)^2$

- ⇨ $2d^2 - 4db_2 + b_2(b_2 + c_2) = 0$

Now, this is a quadratic on d.

Discriminant $= 16b_2^2 - 8b_2(b_2 + c_2) = 8b_2(2b_2 - b_2 - c_2) = 8b_2(b_2 - c_2) > 0$

- ⇨ The above equation has real solution for d.
- ⇨ There exists a parallel line of AO which cuts the triangle ABC into two equal parts of equal area.

Proved.

18. Let $t_1 < t_2 < ..... < t_{99}$ be real numbers, and consider the function f : R -> R be given by $f(x) = |x - t_1| + |x - t_2| + ... + |x - t_{99}|$. Show that $\min_{x \in R} = f(t_{50})$.

Solution :

Now, $f(t_1) = |t_1 - t_2| + |t_1 - t_3| + ... + |t_1 - t_{99}|$

$f(t_2) = |t_2 - t_1| + |t_2 - t_3| + ... + |t_2 - t_{99}|$

Now, $|t_1 - t_2| = |t_2 - t_1|$

Now, $|t_1 - t_3| > |t_2 - t_3|$ as $t_1 < t_2$

Similar arguments go for other terms also.

So, clearly $f(t_2) < f(t_1)$

If we come back from t_{99} to t_{98} we will find same result with same arguments that $f(t_{98}) < f(t_{99})$.

So, minimum value will occur at the middle term which is t_{50}.

- ⇨ $\text{Min}_{x \in R} f(x) = f(t_{50})$

Proved.

Theory behind the problem :

If there are n terms of an AP with common difference d and if p is a positive integer such that gcd(d, p) = 1 then p divides at least one term of the AP where $p \leq n$.

Proof :

Let the first term of the AP is a.

Let p gives same remainder when r^{th} term and s^{th} term are divided by p where r and s both less than p.

Then, $a + (r - 1)d \equiv a + (s - 1)d \pmod p$

- ⇨ $(r - 1)d \equiv (s - 1)d \pmod p$
- ⇨ $r - 1 \equiv s - 1 \pmod p$ (We can divide both sides by d as gcd(d, p) = 1)
- ⇨ $r \equiv s \pmod p$

Which is impossible as r and s both less than p and r and s are distinct.

- ⇨ p gives different remainders when first p terms of the AP is divided by p.

Now, p can give the remainders 0, 1, 2, ..., p – 1

- ⇨ There must be exactly one term of the AP which will generate remainder 0.
- ⇨ That term is divisible by p.

As p ≤ n so, p can divide more terms of the AP if n ≥ 2p, 3p, ...

- ⇨ The statement.

Proved.

Problem :

19. (a) Show that there cannot exist three prime numbers, each greater than 3, which are in arithmetic progression with a common difference less than 5.

(b) Let k > 3 be an integer. Show that it is not possible for k prime numbers, each greater than k, to be in arithmetic progression with a common difference less than or equal to k+1.

Solution :

(c) The common difference less than 5 can be either 4 or 2. Otherwise at least one of the three numbers will be even and hence not prime.

So, let us consider the case p, p+2, p+4 i.e. common difference 2.

Now, let, $p \equiv 1 \pmod 3$

- ⇨ $p + 2 \equiv 0 \pmod 3$

Hence not prime.

Let, $p \equiv -1 \pmod 3$

- ⇨ $p + 4 \equiv 0 \pmod 3$

Hence not prime.

Now, let us consider the case p, p+4, p+8 i.e. common difference 4.

Let, $p \equiv 1 \pmod 3$

- ⇨ $p + 8 \equiv 0 \pmod 3$

Hence not prime.

Let, p ≡ -1 (mod 3)

⇨ p + 4 ≡ 0 (mod 3)

Hence not prime.

Proved.

(d) Now, according to above theory, if k + 1 is not divisible by 3 then one of the 3 consecutive terms of the AP will be divisible by 3.

⇨ (k + 1) must be divisible by 3.

Similarly, if k + 1 is not divisible by 5 then one of the 5 consecutive terms of the AP will be divisible by 5.

⇨ (k + 1) must be divisible by 5.

In this way k + 1 must be divisible by all primes less than k + 1.

Let, there are n number of primes before k + 1.

Then k + 1 = $p_1*p_2*...*p_n$ where p_1, p_2, ..., p_n are consecutive n primes less than k + 1.

Next prime is $p_{(n+1)}$.

Now, p_{n+1} must be greater than k + 1 otherwise p_{n+1} will divide one of the p_{n+1} consecutive terms of the AP (As number of terms is k).

Now, $p_1*p_2*....*p_n - 1$ is not divisible by any of the prime p_1, p_2, ..., p_n.

⇨ This has to be prime.
⇨ There is a prime p_{n+1} less than or equal to k + 1.
⇨ Any one of the term must be divisible by p_{n+1}.
⇨ All the terms cannot be prime.

Proved.

20. Let p be a prime bigger than 5. Suppose, the decimal expansion of 1/p looks like $0.\overline{a_1a_2\cdots a_r}$ where the line denotes a recurring decimal. Prove that 10^r leaves a re4mainder of 1 on dividing by p.

Solution :

Now, 1/p = $0.\overline{a_1 a_2 \cdots a_r}$

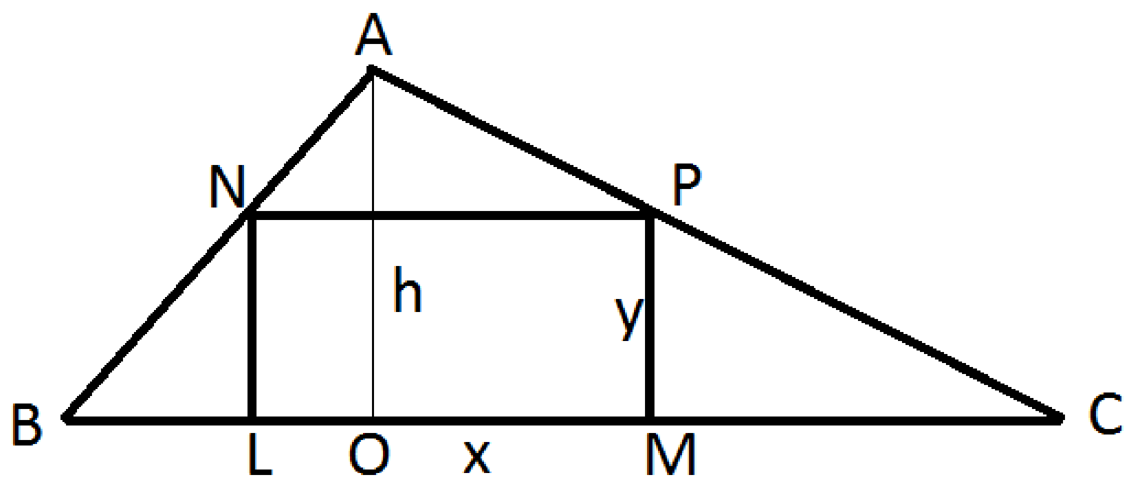
- ⇨ 1/p = $(a_1a_2...a_r – 0)$/9999...r times
- ⇨ (9999...r times)/p = $a_1a_2....a_n$
- ⇨ 10^r – 1 = p*($a_1a_2...a_n$)
- ⇨ 10^r = 1 + p*9$a_1a_2....a_n$)

Clearly on division by p 10^r leaves a remainder of 1.

Proved.

21. Compute the maximum area of a rectangle which can be inscribed in a triangle of area M.

Solution :



The area of the triangle BNL = (1/2)**BL***y

The area of the triangle CMP = (1/2)CM*y

Area of rectangle NLMP = xy

Area of triangle ANP = (1/2)*x*(h – y)

Now, area of triangle ABC = area of triangle BNL + area of triangle CMP + area of rectangle NLMP + area of triangle ANP

- ⇨ M = (1/2)BL*y + (1/2)CM*y + xy + (1/2)*x*(h – y)
- ⇨ (1/2)(BL + CM)*y + xy + (1/2)*x*(h – y) = M
- ⇨ (1/2)(a – x)*y + xy + (1/2)*x*(h – y) = M (BC = a)
- ⇨ (1/2)ay – (1/2)xy + xy + (1/2)xh – (1/2)xy = M
- ⇨ ay + xh = 2M
- ⇨ y = (2M – xh)/a

Now, A = xy

- ⇨ A = x(2M – xh)/a
- ⇨ dA/dx = (1/a){(2M – xh) + x(-h)}
- ⇨ dA/dx = (2/a)(M – xh) = 0
- ⇨ x = M/h

Now, dA/dx = (2/a)(M – xh)

d^2A/dx^2 = (2/a)(-h) < 0

- ⇨ A is maximum at x = M/h)

So, maximum area of the rectangle = (M/h){2M – (M/h)*h}/a = M^2/ah

Now, area of the triangle ABC = (1/2)ah = M

- ⇨ ah = 2M

So, maximum area of t6he rectangle = M^2/2M = M/2.

22. Given odd integers a, b, c prove that the equation $ax^2 + bx + c = 0$ cannot have a solution x which is a rational number.

Solution :

Now, the discriminant of the equation = $b^2 – 4ac$

Dividing the discriminant by 8 we get, $b^2 – 4ac \equiv 1 – 4 \pmod 8$ (as any odd number$^2 \equiv 1 \pmod 8$)

- ⇨ The discriminant ≡ -3 (mod 8)

But any odd number$^2 \equiv 1 \pmod 8$

So, the discriminant cannot be a perfect square.

- ⇨ The solution of x is of the form p + √q.
- ⇨ The roots of the equation cannot be rational.

Proved.

23. Let f(x) be a polynomial with integer coefficients. Suppose that there exists distinct integers a_1, a_2, a_3, a_4 such that $f(a_1) = f(a_2) = f(a_3) = f(a_4) = 3$.Show that there does not exist any integer b with $f(b) = 14$.

Solution :

Let $g(x) = f(x) - 3$

Then, a_1, a_2, a_3, a_4 are roots of g(x).

We can write, $g(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)h(x)$ where h(x) is another polynomial with integer coefficients.

Putting x = b we get,

$g(b) = (b - a_1)(b - a_2)(b - a_3)(b - a_4)h(b)$

- ⇨ $(b - a_1)(b - a_2)(b - a_3)(b - a_4)h(b) = f(b) - 3$
- ⇨ $(b - a_1)(b - a_2)(b - a_3)(b - a_4)h(b) = 14 - 3$
- ⇨ $(b - a_1)(b - a_2)(b - a_3)(b - a_4)h(b) = 11$
- ⇨ 11 is product of at least 4 distinct integers as $b - a_1$, $b - a_2$, $b - a_3$, $b - a_4$ are all distinct as a_1, a_2, a_3, a_4 are distinct.

But 11 is a prime and we can find maximum 3 distinct factors as -1, 1, -11.

Here is the contradiction.

- ⇨ There does not exist any integer b with f(b) = 14.

24. Show that the equation $x^3 + 7x - 14(n^2 + 1) = 0$ has no integral root for any integer n.

Solution :

Dividing the equation by 7 we get, $x^3 \equiv 0 \pmod 7$

- ⇨ $x \equiv 0 \pmod 7$ (As 7 is prime)

Now, putting $x = 7x_1$ we get,

$7^3x_1^3 + 7^2x_1 - 14(n^2 + 1) = 0$

- ⇨ $49x_1^3 + 7x_1 - 2(n^2 + 1) = 0$

Dividing the equation by 7 we get,

$-2(n^2 + 1) \equiv 0 \pmod 7$

$\Rightarrow n^2 + 1 \equiv 0 \pmod 7$ (As gcd(2, 7) = 1)

But, $n^2 + 1$ is never divisible by 7 as 7 gives the remainders 0, 1, 4, 2 when a square number is divided by it.

- ⇨ The equation doesn't have any integral solution.

25. Show that the number 11....1 with 3^n digits is divisible by 3^n.

Solution :

We will prove it by induction.

For n = 1, 111 is divisible by 3.

So, this is true for n = 1.

Let this is true for n = k i.e. 11...1 with 3^k digits is divisible by 3^k.

For n = k + 1, 11....1 with 3^{k+1} digits = 11....1(with 3^k digits)*10^(2*3^k) + 11...1(with 3^k digits)*10^(3^k) + 11...1(with 3^k digit) = 11...1(with 3^k digits)[10^(2*3^k) + 10^(3^k) + 1]

Now, 11...1(with 3^k digits) is divisible by 3^k

And, 10^(2*3^k) + 10^(3^k) + 1 ≡ 1 + 1 + 1 (mod 3) ≡ 0 (mod 3)

- ⇨ 11...1(with 3^{k+1} digits) is divisible by $3^k*3 = 3^{k+1}$.

Proved.

26. Suppose p is a prime number such that (p – 1)/4 and (p + 1)/2 are also primes. Show that p = 13.

Solution :

Let (p – 1)/4 = t (prime)

- ⇨ p = 4t + 1

⇨ $(p + 1)/2 = 2t + 1$

Now, t, 4t + 1 and 2t + 1 are all primes.

Let $t \equiv 1 \pmod 3$

Then, $2t + 1 \equiv 0 \pmod 3$ and not prime.

Contradiction.

Let $t \equiv 2 \pmod 3$

Then, $4t + 1 \equiv 0 \pmod 3$ and not prime.

Contradiction.

So, t must be divisible by 3.

But t is also prime, so cannot be divisible by 3.

⇨ t = 3.

p = 4t + 1 = 4*3 + 1 = 13.

Proved.

27. Show that if a prime number p is divided by 30, the remainder is either prime or 1.

Solution :

Let, p < 30.

Then any prime when divided by 30 then that prime is the remainder.

Now, if p > 30

Then, p = 30q + r where q and r are quotient and remainder when p is divided y 30 respectively.

⇨ p = 2*3*5q + r

Now, r cannot be a factor of 2, 3 or 5.

⇨ If r is composite then min(r) = 7*7 (As after 2, 3, 5 next prime is 7)

But, 7*7 = 49 > 30 but the remainder must be less than 30.

⇨ The remainder cannot be a composite number.

Proved.

28. How many natural numbers less than 10^8 are there, whose sum of digits equals 7?

Solution :

Let the digits are $x_1, x_2, \ldots, x_8$

If $x_1 = 0$ then the numbers formed will be 7 digit, if x_1 and $x_2 = 0$ then the numbers formed will be of 6 digits and so on.

So, we see that $x_1, x_2, \ldots, x_8$ can take any non negative values.

Consider the equation, $x_1 + x_2 + \ldots + x_8 = 7$ (As sum of digits is 7)

Number of non-negative solution of this equation is ${}^{(7+8-1)}C_{(8-1)} = {}^{14}C_7$.

29. Find out the number of ways in which three number can be chosen from the set of {1,2,3,…..4n} such that the sum of three selected number is divisible by 4.

Solution :

The choice of remainders of the three numbers modulus 4 can be,

(0, 0, 0); (0, 1, 3); (0, 2, 2); (1, 1, 2)

Now, the numbers that give remainder 0 upon divided by 4 are (4, 8, 12, ..., 4n) i.e. n numbers.

The numbers that give remainder 1 upon divided by 4 are (1, 5, 9, ..., $4(n-1)+1$} i.e. n numbers.

The numbers that give remainder 2 upon divided by 4 are (2, 6, 10, ..., $4(n-1)+2$) i.e. n numbers.

The numbers that give remainder 3 upon divided by 4 are (3, 7, 11, ..., $4(n-1)+3$ i.e. n numbers.

Now, number of ways to select 3 numbers such that the remainder set is (0, 0, 0) is ${}^nC_3$.

Number of ways to select 3 numbers such that remainder set is (0, 1, 3) is ${}^nC_1 * {}^nC_1 * {}^nC_1 = n^3$.

Number of ways to select 3 numbers such that remainder set is (0, 2, 2) is ${}^nC_1 * {}^nC_2 = n^2(n-1)/2$.

Number of ways to select 3 numbers such that remainder set is (1, 1, 2) is $^nC_2*^nC_1 = n^2(n - 1)/2$.

Total number of ways $= ^nC_3 + n^3 + n^2(n - 1)/2 + n^2(n - 1)/2 = ^nC_3 + 3n^3 - 2n^2$.

30. In how many can you divide a set of eight no. (2,3,….9) into four pair such that no pair of the number has the g.c.d. equal to 2.

Solution :

According to the question, no pair can contain both even number.

So, we divide the numbers into two pairs (2, 4, 6, 8) and (3, 5, 7, 9)

Now we have to take one number from the even set and another from odd set.

So, number of ways $= ^4C_1*^4C_1 + ^3C_1*^3C_1 + ^2C_1*^2C_1 + ^1C_1*^1C_1 = 4^2 + 3^2 + 2^2 + 1^2 = 30$.

Now, we can take also (4, 8) together.

In that case we can select any odd number with 6 and with 2 another odd number.

So, number of ways $= ^4C_1*^3C_1 = 12$. ($^4C_1$ is choosing any odd number for 6 and $^3C_1$ is choosing any odd number with 2)

So, total number of ways = 30 + 12 = 4.

31. Number 3 problem is incomplete.

32. Let A is a complex number such that $a^2 +(1/a^2)+a+(1/a)+1=0$. If m is a positive integer then $a^{2m} +(1/a^{2m})+a^m+(1/a^m)= ?$

Solution :

Now, we have, $a^2 + (1/a^2) + a + (1/a) + 1 = 0$

⇨ $(a + 1/a)^2 + (a + 1/a) - 1 = 0$
⇨ $(a + 1/a) = \{-1 + \sqrt{(1 + 4)}\}/2 = (\sqrt{5} - 1)/2 = \text{real}$

Let, $a = re^{iA}$

Now, $Im(a + 1/a) = 0$

- ⇨ $Im(re^{iA} + e^{-iA}/r) = 0$
- ⇨ $r\sin A - \sin A/r = 0$
- ⇨ $r = 1$
- ⇨ $a = e^{iA}$
- ⇨ $a + 1/a = e^{iA} + e^{-iA} = 2\cos A$

Putting this value in above equation we get, $2\cos A = (\sqrt{5} - 1)/2$

- ⇨ $\cos A = \cos 72$
- ⇨ $A = 72$

Now, $(a^{2m} + 1/a^{2m}) + (a^m + 1/a^m)$

$= e^{i2mA} + e^{-i2mA} + e^{imA} + e^{-imA}$

$= 2(\cos 2mA + \cos mA)$ where $A = 72$

This is answer.

Now, putting m = 2 we get, $(a^{2m} + 1/a^{2m}) + (a^m + 1/a^m) = -1$

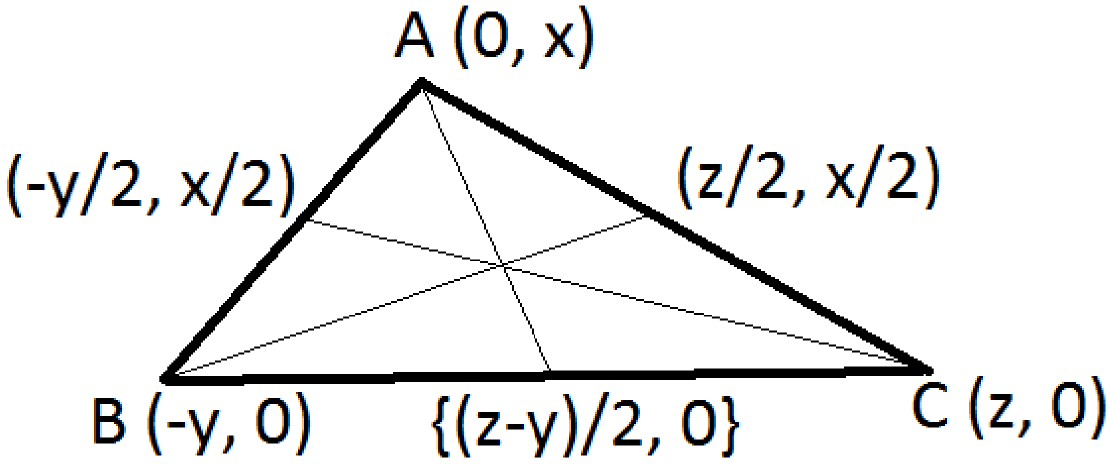
Putting m = 3 we get, $(a^{2m} + 1/a^{2m}) + (a^m + 1/a^m) = -1$

Putting m = 4 we get, $(a^{2m} + 1/a^{2m}) + (a^m + 1/a^m) = -1$

Putting m = 5 we get, $(a^{2m} + 1/a^{2m}) + (a^m + 1/a^m) = 4$

33. Show that it is not possible to have a triangle with sides a,b,c whose median is $\frac{2}{3}a$ $\frac{2}{3}b$ $\frac{4}{5}c$.

Solution :



ONLY THE METHOD IS DISCUSSED (NOT COMPLETE SOLUTION)

Now, BC = z + y = a

$AC^2 = z^2 + x^2 = b^2$

$AB^2 = y^2 + x^2 = c^2$

From the last two equations we get, $z^2 - y^2 = c^2 - b^2$

- ⇨ $(z + y)(z - y) = c^2 - b^2$
- ⇨ $z - y = (c^2 - b^2)/a$ (Putting value of z + y from first equation)

Now, z + y = a

From this we get, $z = (c^2 - b^2 + a^2)/a$ and $y = (a^2 + b^2 - c^2)/a$

Putting value of either z or y, we get, $x^2 = b^2 - \{(c^2 - b^2 + a^2)/a\}^2$

Now, square of length of median from A = $\{(z - y)/2\}^2 + x^2 = (2a/3)^2$

Similarly, we get, $(z/2 + y)^2 + (x/2)^2 = (2b/3)^2$

And $(z + y/2)^2 + (x/2)^2 = (4c/5)^2$

We have got 3 equations and we have x, y, z in terms of a, b, c so we have 3 equations and 3 unknown.

From this we need to either calculate a, b, c or doing some trigonometric substitution (like z/2c = cosB etc.) we need to find a contradiction.

34. Let x,y,z be the non zero real number. Let $|\alpha| = |\beta| = |\gamma|$, if $x+y+z = 0 = \alpha x+\beta y+\gamma z$ then $\alpha=\beta=\gamma$.

Solution :

We have, $\alpha x + \beta y + \gamma z = x + y + z$

- ⇨ $\alpha(x - 1) + \beta(y - 1) + \gamma(z - 1) = 0$

And we have, $\alpha x + \beta y + \gamma z = 0$

Now, subtracting the above two equations we get, $\alpha + \beta + \gamma = 0$

Now, we have, again, $\alpha x + \beta y + \gamma z = 0$

- ⇨ $\alpha x + \beta y - \alpha z - \beta z = 0$ (As $\alpha + \beta + \gamma = 0$)
- ⇨ $\alpha(x - z) = \beta(z - y)$
- ⇨ $\alpha/\beta = (z - y)/(x - z)$
- ⇨ α/β = real.

Let, $\alpha = re^{iA}$, $\beta = re^{iB}$ and $\gamma = re^{iC}$

Now, re^{iA}/re^{iB} = real

- ⇨ $e^{i(A - B)}$ = real
- ⇨ $\sin(A - B) = 0$
- ⇨ $A = B$

Similarly we will get, $B = C$ and $C = A$

So, we have, $\alpha = \beta = \gamma$.

Proved.

ISI and CMI 2015

ISI B. Stat B. Math entrance exam 2015

1. Let $y = x^2 + ax + b$ be a parabola that cuts the coordinate axes at three distinct points. Show that the circle passing through these three points also passes through (0, 1)

Solution :

Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Now, putting $y = 0$ in the equation of parabola to get the points on x-axis we get, $x^2 + ax + b = 0$

Now, putting y = 0 in the equation of circle we get, $x^2 + 2gx + c = 0$

Now, both the above two equations have same roots as the circle passes through the points where the parabola cuts the x-axis.

So, comparing the co-efficients we get, $2g = a$ and $c = b$

So, the equation of the circle becomes, $x^2 + y^2 + ax + 2fy + b = 0$

Now, putting x = 0 in the equation of the parabola we get the point where it cuts the y-axis.

⇨ $y = b$

Now, putting x = 0 in the equation of circle we get, $y^2 + 2fy + b = 0$

Now, y = b is a root of the equation as the circle passes through the point where the parabola cuts the y-axis.

⇨ $b^2 + 2fb + b = 0$
⇨ $2f = -(b + 1)$

Now, the equation of circle becomes, $x^2 + y^2 + ax - (b + 1)y + b = 0$

Clearly, (0, 1) satisfies the equation of the circle.

⇨ The circle passes through the point (0, 1).

Proved.

2. Find all such natural number n such that 7 divides $5^n + 1$.

Solution 1 :

Now, $5^n + 1 \equiv 0 \pmod 7$

⇨ $(-2)^n + 1 \equiv 0 \pmod 7$
⇨ $(-1)^n*2^n + 1 \equiv 0 \pmod 7$

We must have $2^n \equiv 1 \pmod 7$ and n must be odd.

Then the above equation is true.

Now, $2^3 \equiv 1 \pmod 7$

And we need to have, $2^n \equiv 1 \pmod 7$

⇨ n must be equal to 3m

And also m is odd as n needs to be odd.

Putting 2m+1 in place of m we get, n = 3(2m + 1) = 6m + 3 (Answer)

Solution 2 :

Now, $5^1 \equiv 5 \pmod 7$; $5^2 \equiv 4 \pmod 7$; $5^3 \equiv -1 \pmod 7$; $5^4 \equiv 2 \pmod 7$; $5^5 \equiv 3 \pmod 7$; $5^6 \equiv 1 \pmod 7$

Now, again $5^7 \equiv 5 \pmod 7$ and it will continue running in a loop.

So, we see that $5^3 \equiv -1 \pmod 7$

We need to have $5^n \equiv -1 \pmod 7$

- ⇨ n must be equal to 3m

So, we get, $5^3 \equiv -1 \pmod 7$

- ⇨ $5^{3m} \equiv (-1)^m \pmod 7$ (Raising both sides to the power m)
- ⇨ m must be odd because then only $(-1)^m = -1$.
- ⇨ n = 3(2m + 1) (Putting 2m+1 in place of m as m is odd)
- ⇨ n = 6m + 3 (Answer)

3. Find all functions f, such that $|f(x) - f(y)| = 2|x - y|$.

Solution :

We have, $|f(x) - f(y)| = 2|x - y|$

- ⇨ $|\{f(x) - f(y)\}/(x - y)| = 2$
- ⇨ $\lim |\{f(x) - f(y)]/(x - y)|$ as $x \to y$ = $\lim (2)$ as $x \to y$
- ⇨ $|f'(y)| = 2$
- ⇨ $|f'(x)| = 2$
- ⇨ $f'(x) = 2, -2$
- ⇨ $\int d\{f(x)\} = 2\int dx, -2\int dx$
- ⇨ $f(x) = 2x + c_1, -2x + c_2$ (Answer)

4. Say $0 < a_1 < a_2 < \ldots.. < a_n$ be n real numbers. Show that the equation $a_1/(a_1 - x) + a_2/(a_2 - x) + \ldots.. + a_n/(a_n - x) = 2015$ has n real solutions.

Solution :

This is a polynomial of degree n on x.

So, there are n number of roots.

We have to prove that n roots are real.

⇨ There is no complex root of the equation.

Let, there are two complex roots s + ir and s – ir (As complex roots come in conjugate pair)

Now, s + ir will satisfy the equation.

⇨ $a_1/(a_1 - s - ir) + a_2/(a_2 - s - ir) + ... + a_n/(a_n - s - ir) = 2015$

Similarly, $a_1/(a_1 - s + ir) + a_2/(a_2 - s + ir) + ... + a_n/(a_n - s + ir) = 2015$

Now, subtracting the equations we get,

$\{a_1/(a_1 - s - ir) - a_1/(a_1 - s + ir)\} + \{a_2/(a_2 - s - ir) - a_2/(a_2 - s + ir)\} + ... + \{a_n/(a_n - s - ir) - a_n/(a_n - s + ir)\} = 2015 - 2015$

⇨ $a_1(2ir)/\{(a_1 - s)^2 + r^2\} + a_2(2ir)/\{(a_2 - s)^2 + r^2\} + ... + a_n(2ir)/\{(a_n - s)^2 + r^2\} = 0$
⇨ $(2ir)[a_1/\{(a_1 - s)^2 + r^2\} + a_2/(a_2 - s)^2 + r^2\} + ... + a_n/\{(a_n - s)^2 + r^2\}] = 0$

Now, $a_1, a_2, ..., a_n$ all are greater than 0.

⇨ The expression inside the square bracket is greater than 0 as denominators are sum of square numbers and greater than 0.
⇨ r has to be 0.
⇨ The imaginary part of the roots are zero.
⇨ The roots are no more complex.
⇨ Our assumption was wrong.
⇨ There is no complex root of the equation.
⇨ All roots are real.
⇨ There are n number of real roots.

Proved.

5. Consider the set S = {1, 2, 3, ..., j}. In a subset P of S, Max P be the maximum element of that subset. Show that the sum of all Max P (over all subsets of the set) is $(j - 1)*2^j + 1$.

Solution :

Let us take any element i of S i.e. i ∈ S.

Now, i will be maximum element of all the subsets if we choose element from the set A = {1, 2, 3, ..., i – 1}.

Now, we can choose 0 element from A in $^{(i-1)}C_0$ way.

We can choose 1 element from A in $^{(i-1)}C_1$ way.

We can choose 2 element from A in $^{(i-1)}C_2$ ways.

...

....

We can choose (i – 1) element from A in $^{(i-1)}C_{(i-1)}$ way.

So, total number of subset where i is maximum element is $^{(i-1)}C_0 + {}^{(i-1)}C_1 + {}^{(i-1)}C_2 + \ldots.. + {}^{(i-1)}C_{(i-1)} = 2^{(i-1)}$.

So, sum of the maximum elements in those sets = $i*2^{(i-1)}$

- ⇨ We have to prove that, $\sum i*2^{(i-1)}$ (summation running over i = 1 to i = j) = $(j-1)*2^j + 1$

We will prove it by induction.

For j = 1, LHS = $1*2^{(1-1)} = 1$ and RHS = $(1-1)*2^1 + 1 = 1$

So, this is true for j = 1.

Let this is true for j = n i.e. we have, $\sum i*2^{(i-1)}$ (summation running over i = 1 to i = n) = $(n-1)*2^n + 1$

Now, for j = n + 1,

LHS = $\sum i*2^{(i-1)}$ (summation running over i = 1 to i = n + 1)

= $\sum i*2^{(i-1)}$ (summation running over i = 1 to i = n) + $(n+1)*2^{(n+1-1)}$

= $(n-1)*2^n + 1 + (n+1)*2^n$

= $2^n(n-1+n+1) + 1$

= $2n*2^n + 1$

= $n*2^{(n+1)} + 1$

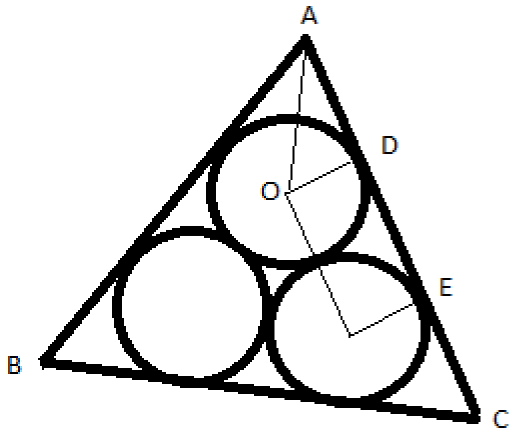
For j = n + 1, RHS = $(n+1-1)*2^{(n+1)} + 1 = n*2^{(n+1)} + 1$

By the principle of induction this is true.

Proved.

6. There are three unit circles each of which is tangential to the other two. A triangle is drawn such that each side of the triangle is tangential to exactly two of the circle. Find the length of sides of this triangle.

Solution :



Now, ABC must be equilateral.

Angle OAD = 30°

From triangle OAD we have, cot30° = AD/OD

- ⇨ AD = 1*√3 (OD = radius of the circle = 1 given)
- ⇨ AD = √3
- ⇨ CE = AD = √3

Now, DE = distance between two centres of the circles = 1 + 1 = 2 (As they are tangential)

- ⇨ AC = side of the triangle = AD + DE + CE = √3 + 2 + √3 = 2(1 + √3) (Answer)

7. Question still not found.
8. Let $P(x) = x^7 + x^6 + b_5x^5 + b_4x^4 + \ldots\ldots + b_0$ and $Q(x) = x^5 + c_4x^4 + c_3x^3 + \ldots\ldots + c_0$, $P(i) = Q(i)$, $i = 1, 2, \ldots, 6$. Show that there exists a negative integer r such that $P(r) = Q(r)$.

Solution :

Let, F(x) = P(x) – Q(x)

- ⇨ 1, 2, ..., 6 are roots of F(x).

Now, F(x) is of degree 7.

So, there are 7 roots of F(x).

Let the unknown root be r.

Now, F(x) = $x^7 + x^6 + (b_5 - 1)x^5 + (b_4 - c_4)x^4 + (b_3 - c_3)x^3 + ... + (b_0 - c_0)$

Clearly, sum of the roots of F(x) = -1/1

- ⇨ 1 + 2 + ... + 6 + r = -1
- ⇨ 6*7/2 + r = -1
- ⇨ 21 + r = -1
- ⇨ r = -22 = negative.

Proved.

CMI 2015

1. A geometry problem on Straight edge construction.
2. Let a be a positive integer from set {2, 3, ..., 9999}. Show that there are exactly two positive integers in that set such that 10000 divides a*(a – 1).

Solution :

Now, 10000 = 2^4*5^4

Now, a and a – 1 are relatively prime.

- ⇨ 5^4 divides any one of them and another is divisible by 2^4.

So, we need to have $5^4*k \pm 1 \equiv 0 \pmod{16}$

- ⇨ $k \pm 1 \equiv 0 \pmod{16}$ (As $5^4 \equiv 1 \pmod{16}$)

So, k can be 1, 15, 17.

But, $5^4*17 = 10625 > 9999$.

- ⇨ There can be only 2 values of k viz. 1, 15.

If k = 1, then $5^4*k - 1 = 624$ and it is divisible by 16.

If k = 15, then $5^4*k + 1 = 5^4*15 + 1 = 9376$ and it is divisible by 16.

So, there are exactly two integers in the set which satisfies the given criteria viz. 625 and 9376.

Proved.

3. P(x) is a polynomial. Show that lim $P(t)/e^t$ as t -> ∞ exists. Also show that the limit does not depend on the polynomial.

Solution :

Let, $P(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$

Now, lim $(a_nt^n + a_{n-1}t^{n-1} + ... + a_1t + a_0)/e^t$ as t -> ∞

This is ∞/∞ form. So, we can use L'Hospital rule.

And we will apply it till $P^{(n)}(t)$ where $P^{(n)}(t)$ is n-th derivative of P(t) w.r.t. t.

Then we will have the limit as, lim $(a_n*n!)/e^t$ as t -> ∞ = 0.

Therefore, the limit exists and also it is independent of the polynomial.

Proved.

4. We define function $f(x) = e^{-1/x}/x$ when x < 0; f(x) = 0 if x = 0 and $f(x) = e^{-1/x}/x$ when x > 0. Show that the function is continuous and differentiable. Find limit at x = 0.

Solution :

Now, lim f(x) as x -> 0- = lim f(x) x -> 0+ as the function is same in both the cases.

So, lim f(x) as x ->0- = lim $e^{-1/x}/x$ as x -> 0-

Let, x = 1/z as x -> 0, z -> ∞

The limit becomes, lim z/e^z as z -> ∞

We can apply L'Hospital rule as it is ∞/∞ form.

So, lim $1/e^z$ as z -> ∞ = 0.

So, lim f(x) as x -> 0- = lim f(x) as x -> 0+ = f(0)

⇨ The function is continuous.

Now, lim {f(x) – f(0)}/(x – 0) as x -> 0- = lim $e^{-1/x}/x^2$ as x -> 0

Let, x = 1/z, x -> 0, z -> ∞

So, lim z^2/e^z as z -> ∞ = lim $2z/e^z$ as z -> ∞ (Applying L'Hospital rule) = lim $2/e^z$ as z -> ∞ = 0

Similarly, lim {f(x) – f(0)}/(x – 0) as x -> 0 = 0.

⇨ The function is differentiable.

Proved.

And already discovered above that lim f(x) as x -> 0 = 0. (Answer)

5. p, q, r any real number such that $p^2 + q^2 + r^2 = 1$.
   (a) Show that $3(p^2q + p^2r) + 2(r^3 + q^3) \le 2$
   (b) Suppose $f(p, q, r) = 3(p^2q + p^2r)+ 2(r^3 + q^3)$. At what values of (p, q, r) does f(p, q, r) maximizes and minimizes?

Solution :

(a) Now, $3(p^2q + p^2r) + 2(r^3 + q^3) = 3p^2(r + q) + 2(r + q)(r^2 – rq + q^2)$
$= (r + q)(3p^2 + 2r^2 – 2rq + 2q^2)$
$= (r + q)(3 – r^2 – 2rq – q^2)$ (As $p^2 = 1 – r^2 – q^2$)
$= (r + q)\{3 – (r + q)^2\}$
So, we have to prove, $(r + q)\{3 – (r + q)^2\} \le 2$
i.e. to prove, $(r + q)^3 – 3(r + q) + 2 \ge 0$
i.e. to prove, $(r + q – 1)^2(r + q + 2) \ge 0$ (Apply vanishing method and factorize)
Now, $(r + q – 1)^2$ always greater than 0.
So, we have to prove, $r + q + 2 > 0$
Now, $p^2 + q^2 + r^2 = 1$
⇨ $p^2 + q^2 = 1 – r^2 > 0$ (As $p^2 + q^2$ cannot be negative)
⇨ $r^2 < 1$
⇨ $-1 < r < 1$
Similarly, $-1 < q < 1$
Adding the two inequalities we get, $-2 < r + q < 2$
⇨ $r + q + 2 > 0$

Proved.

(b) From (a) it is clear that maximum value of f(p, q, r) is 2 and it attains when q + r = 1.

Now, $p^2 + q^2 + r^2 = 1$

⇨ $p^2 + (q + r)^2 - 2qr = 1$

⇨ $p^2 = 2qr$ (As q + r = 1)

⇨ $p^2 = 2(1 - r)r$

⇨ $p^2 = 2r - 2r^2$

⇨ $2r^2 - 2r + p^2 = 0$

⇨ $r = \{2 \pm \sqrt{(4 - 4*2*p^2)}\}/2*2$

⇨ $r = \{1 \pm \sqrt{(1 - 2p^2)}\}/2$

Now, $1 - 2p^2 > 0$

⇨ $p^2 < ½$

⇨ $-1/\sqrt{2} < p < 1/\sqrt{2}$

Now, q + r = 1

Let, r is negative, then q > 1. But q < 1.

⇨ r and q both must be positive.

So the solution is,

$-1/\sqrt{2} < p < 1/\sqrt{2}$

$0 < r < 1$

$0 < q < 1$

$r = \{1 \pm \sqrt{(1 - 2p^2)}\}/2$

$q = 1 - r$.

(Answer)

The minimization part is remaining.

6. Let g(n) is GCD of (2n + 9) and $6n^2 + 11n - 2$ then find the greatest value of g(n).

Solution :

Now, $6n^2 + 11n - 2$

$= 6n^2 + 12n - n - 2$

$= 6n(n + 2) - (n + 2)$

$= (n + 2)(6n - 1)$

Now, let p divides both (n + 2) and (2n + 9)

⇨ $n + 2 \equiv 0 \pmod{p}$

⇨ $n \equiv -2 \pmod{p}$

Also, $2n + 9 \equiv 0 \pmod{p}$

⇨ $2*(-2) + 9 \equiv 0 \pmod{p}$

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- ⇨ $5 \equiv 0 \pmod{p}$
- ⇨ $p = 5$.

So, maximum common factor between (n + 2) and (2n + 9) is 5.

Now, let q divides both (2n + 9) and (6n – 1)

- ⇨ $2n + 9 \equiv 0 \pmod{q}$
- ⇨ $2n \equiv -9 \pmod{q}$

Also, $6n - 1 \equiv 0 \pmod{q}$

- ⇨ $3*(-9) - 1 \equiv 0 \pmod{q}$
- ⇨ $-28 \equiv 0 \pmod{q}$
- ⇨ $28 \equiv 0 \pmod{q}$
- ⇨ $2^2*7 \equiv 0 \pmod{q}$
- ⇨ $7 \equiv 0 \pmod{q}$ (As q cannot have a factor 2 because 2n + 9 and 6n – 1 both odd)
- ⇨ $q = 7$.

So, maximum common factor between (2n + 9) and (6n – 1) is 7.

So, maximum value of g(n) = 5*7 = 35 (Answer). *{occurs at n = 13m+35, m ∈ N}*

End.

☺