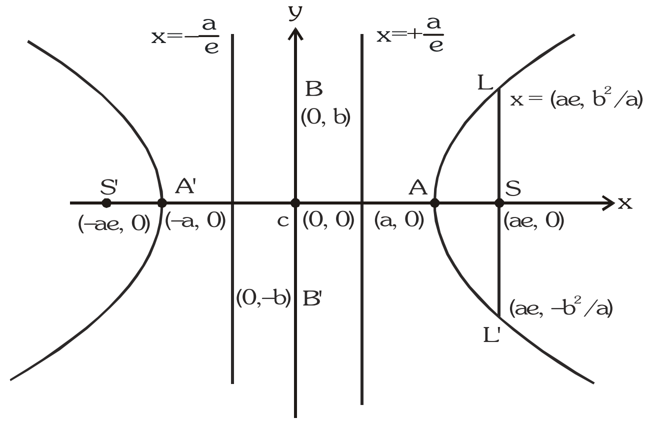
HYPERBOLA

The **Hyperbola** is a conic whose eccentricity is greater than unity. **($e > 1$)**.

1. STANDARD EQUATION & DEFINITION(S) :



Standard equation of the hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$

or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(a) Foci :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(b) Equations of directrices :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

(c) Vertices :

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(d) Latus rectum :

(i) Equation : $x = \pm ae$

(ii) Length $= \frac{2b^2}{a} = \frac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a(e^2 - 1) = 2e$ (distance from focus to directrix)

(iii) Ends : $\left(ae, \frac{b^2}{a}\right), \left(ae, \frac{-b^2}{a}\right)$; $\left(-ae, \frac{b^2}{a}\right), \left(-ae, \frac{-b^2}{a}\right)$

(e) (i) Transverse Axis :

The line segment A′A of length 2a in which the foci S′ & S both lie is called the **Transverse Axis of the Hyperbola.**

(ii) Conjugate Axis :

The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the **Conjugate Axis of the Hyperbola.**

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the **Principal axes of the hyperbola.**

(f) Focal Property :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left||PS| - |PS'|\right| = 2a$. The distance SS′ = focal length.

(g) Focal distance :

Distance of any point P(x, y) on Hyperbola from foci **$PS = ex - a$ & $PS' = ex + a$.**

Illustration 1 : Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Solution : Let $P(x, y)$ be any point on the hyperbola and PM is perpendicular from P on the directrix.

Then by definition $SP = e\ PM$

$\Rightarrow \quad (SP)^2 = e^2\ (PM)^2 \quad \Rightarrow \quad (x-1)^2 + (y-2)^2 = 3\left\{\dfrac{2x+y-1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow \quad 5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow \quad 7x^2 - 2y^2 + 12\ xy - 2x + 14y - 22 = 0$

which is the required hyperbola.

Illustration 2 : The eccentricity of the hyperbola $4x^2 - 9y^2 - 8x = 32$ is -

(A) $\dfrac{\sqrt{5}}{3}$ (B) $\dfrac{\sqrt{13}}{3}$ (C) $\dfrac{\sqrt{13}}{2}$ (D) $\dfrac{3}{2}$

Solution : $4x^2 - 9y^2 - 8x = 32 \Rightarrow 4(x-1)^2 - 9y^2 = 36 \Rightarrow \dfrac{(x-1)^2}{9} - \dfrac{y^2}{4} = 1$

Here $a^2 = 9$, $b^2 = 4$

$\therefore \quad$ eccentricity $e = \sqrt{1+\dfrac{b^2}{a^2}} = \sqrt{1+\dfrac{4}{9}} = \dfrac{\sqrt{13}}{3}$ **Ans.(B)**

Illustration 3 : If foci of a hyperbola are foci of the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$. If the eccentricity of the hyperbola be 2, then its equation is -

(A) $\dfrac{x^2}{4}-\dfrac{y^2}{12}=1$ (B) $\dfrac{x^2}{12}-\dfrac{y^2}{4}=1$ (C) $\dfrac{x^2}{12}+\dfrac{y^2}{4}=1$ (D) none of these

Solution : For ellipse $e = \dfrac{4}{5}$, so foci $= (\pm 4, 0)$

For hyperbola $e = 2$, so $a = \dfrac{ae}{e} = \dfrac{4}{2} = 2$, $b = 2\sqrt{4-1} = 2\sqrt{3}$

Hence equation of the hyperbola is $\dfrac{x^2}{4}-\dfrac{y^2}{12}=1$ **Ans.(A)**

Illustration 4 : Find the coordinates of foci, the eccentricity and latus-rectum, equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

Solution : Equation can be rewritten as $\dfrac{(x-4)^2}{4^2}-\dfrac{(y-3)^2}{3^2}=1$ so $a = 4$, $b = 3$

$b^2 = a^2(e^2 - 1)$ given $e = \dfrac{5}{4}$

Foci : $X = \pm\ ae$, $Y = 0$ gives the foci as $(9, 3)$, $(-1, 3)$

Centre : $X = 0$, $Y = 0$ i.e. $(4, 3)$

Directrices : $X = \pm\dfrac{a}{e}$ i.e. $x - 4 = \pm\dfrac{16}{5}$ $\quad\therefore\quad$ directrices are $5x - 36 = 0$; $5x - 4 = 0$

Latus-rectum $= \dfrac{2b^2}{a} = 2.\dfrac{9}{4} = \dfrac{9}{2}$

Do yourself - 1 :

(i) Find the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which passes through (4, 0) & $(3\sqrt{2}, 2)$

(ii) Find the equation to the hyperbola, whose eccentricity is $\frac{5}{4}$, focus is (a, 0) and whose directrix is 4x − 3y = a.

(iii) In the hyperbola $4x^2 - 9y^2 = 36$, find length of the axes, the co-ordinates of the foci, the eccentricity, and the latus rectum.

(iv) Find the equation to the hyperbola, the distance between whose foci is 16 and whose eccentricity is $\sqrt{2}$.

2. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **Conjugate Hyperbolas** of each other. eg. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ & $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are conjugate hyperbolas of each other.

Note that :

(i) If e_1 & e_2 are the eccentricities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(ii) The foci of a **hyperbola** and its **conjugate** are **concyclic and form the vertices of a square.**

(iii) Two hyperbolas are said to be **similar** if they have the **same eccentricity**.

Illustration 5 : The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is -

(A) 2 (B) $2/\sqrt{3}$ (C) 4 (D) 4/3

Solution : Equation of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is

$-x^2 + 3y^2 = 1 \quad \Rightarrow \quad -\frac{x^2}{1}+\frac{y^2}{1/3}=1$

Here $a^2 = 1$, $b^2 = 1/3$

$\therefore$ eccentricity $e = \sqrt{1+a^2/b^2} = \sqrt{1+3} = 2$ **Ans. (A)**

Do yourself - 2 :

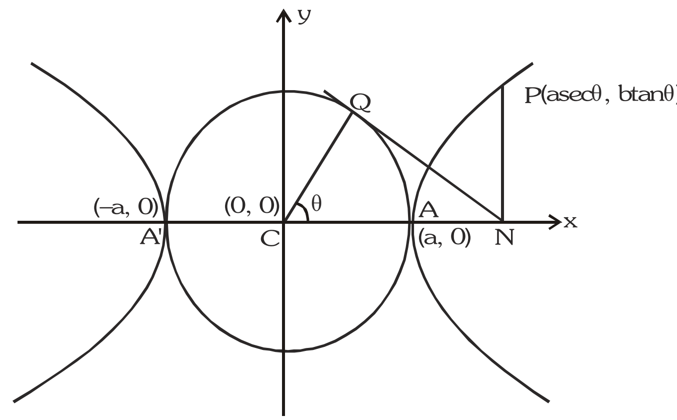
(i) Find eccentricity of conjugate hyperbola of hyperbola $4x^2 - 16y^2 = 64$, also find area of quadrilateral formed by foci of hyperbola & its conjugate hyperbola

3. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and **the length of it's latus rectum is equal to it's transverse or conjugate axis.**

4. AUXILIARY CIRCLE :

A circle drawn with centre C & T.A. as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.



Note from the figure that P & Q are called the **"Corresponding Points"** on the hyperbola & the auxiliary circle. 'θ' is called the **eccentric angle** of the point 'P' on the hyperbola. $(0 \le \theta < 2\pi)$.

Parametric Equation :

The equations $x = a \sec\theta$ & $y = b\tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter. The parametric equations ; $x = a \cosh\phi,\ y = b\sinh\phi$ also represents the same hyperbola.

General Note :

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2 .

5. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is **positive , zero** or **negative** according as the point (x_1, y_1) lies within , upon or outside the curve.

6. LINE AND A HYPERBOLA :

The straight line $y = mx + c$ is a secant , a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 > = < a^2m^2 - b^2$.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining its two points $P(\alpha)$ & $Q(\beta)$ is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

Illustration 6 : Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

Solution : The given line is $x\cos\alpha + y\sin\alpha = p \Rightarrow y\sin\alpha = -x\cos\alpha + p$

$\Rightarrow\ y = -x\cot\alpha + p\,\mathrm{cosec}\,\alpha$

Comparing this line with $y = mx + c$

$m = -\cot\alpha,\ c = p\,\mathrm{cosec}\,\alpha$

Since the given line touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then

$c^2 = a^2m^2 - b^2 \Rightarrow p^2\,\mathrm{cosec}^2\alpha = a^2\cot^2\alpha - b^2$ or $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

Illustration 7 : If (a secθ, b tanθ) and (asecϕ, b tanϕ) are the ends of a focal chord of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equal to -

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{1+e}{1-e}$ (D) $\frac{e+1}{e-1}$

Solution : Equation of chord connecting the points (asecθ, b tanθ) and (asecϕ, b tanϕ) is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right) \quad \text{........ (i)}$$

If it passes through (ae, 0); we have, $e\cos\left(\frac{\theta-\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$

$$\Rightarrow \quad e=\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}=\frac{1-\tan\frac{\theta}{2}.\tan\frac{\phi}{2}}{1+\tan\frac{\theta}{2}.\tan\frac{\phi}{2}} \Rightarrow \tan\frac{\theta}{2}.\tan\frac{\phi}{2}=\frac{1-e}{1+e}$$

Similarly if (i) passes through (–ae, 0), $\tan\frac{\theta}{2}.\tan\frac{\phi}{2}=\frac{1+e}{1-e}$ **Ans. (B, C)**

Do yourself - 3 :

(i) Find the condition for the line $\ell x + my + n = 0$ to touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(ii) If the line y = 5x +1 touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{b^2}=1$ {b > 4}, then -

(A) $b^2=\frac{1}{5}$ (B) $b^2 = 99$ (C) $b^2 = 4$ (D) $b^2 = 100$

7. TANGENT TO THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(a) **Point form :** Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

Note : In general two tangents can be drawn from an external point $(x_1\, y_1)$ to the hyperbola and they are $y - y_1 = m_1 (x - x_1)$ & $y - y_1 = m_2 (x - x_1)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2) m^2 - 2 x_1y_1 m + y_1^2 + b^2 = 0$. If $D < 0$, then **no tangent** can be drawn from $(x_1\, y_1)$ to **the hyperbola.**

(b) **Slope form :** The equation of tangents of slope m to the given hyperbola is $y = mx \pm \sqrt{a^2m^2-b^2}$.

Point of contact are $\left(\mp\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2-b^2}}\right)$

Note that there **are two parallel tangents having the same slope m.**

(c) **Parametric form :** Equation of the tangent to the given hyperbola at the point **(a sec θ , b tan θ)** is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

Note : Point of intersection of the tangents at θ_1 & θ_2 is $x=a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}$, $y=b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$

Illustration 8 : Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Solution : Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore \quad m \quad 1 = -1 \Rightarrow m = -1$

Since $x^2 - 4y^2 = 36$ or $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$

Comparing this with $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\therefore \quad a^2 = 36$ and $b^2 = 9$

So the equation of tangents are $y = (-1)x \pm \sqrt{36 \times (-1)^2 - 9}$

$y = -x \pm \sqrt{27} \Rightarrow x + y \pm 3\sqrt{3} = 0$ **Ans.**

Illustration 9 : The locus of the point of intersection of two tangents of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ if the product of their slopes is c^2, will be -

(A) $y^2 - b^2 = c^2(x^2 + a^2)$
(B) $y^2 + b^2 = c^2(x^2 - a^2)$
(C) $y^2 + a^2 = c^2(x^2 - b^2)$
(D) $y^2 - a^2 = c^2(x^2 + b^2)$

Solution : Equation of any tangent of the hyperbola with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$

If it passes through (x_1, y_1) then

$(y_1 - mx_1)^2 = a^2m^2 - b^2 \Rightarrow (x_1^2 - a^2)\, m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0$

If $m = m_1, m_2$ then as given $m_1m_2 = c^2 \Rightarrow \dfrac{y_1^2 + b^2}{x_1^2 - a^2} = c^2$

Hence required locus will be : $y^2 + b^2 = c^2(x^2 - a^2)$ **Ans.(B)**

Illustration 10 : A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$ is -

(A) $y = 3\sqrt{\dfrac{2}{7}}x - \dfrac{15}{\sqrt{7}}$ (B) $y = 3\sqrt{\dfrac{2}{7}}x + \dfrac{15}{\sqrt{7}}$ (C) $y = -3\sqrt{\dfrac{2}{7}}x + \dfrac{15}{\sqrt{7}}$ (D) $y = -3\sqrt{\dfrac{2}{7}}x - \dfrac{15}{\sqrt{7}}$

Solution : $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, $x^2 + y^2 = 9$

Equation of tangent $y = mx + \sqrt{16m^2 - 9}$ (for hyperbola)

Equation of tangent $y = m'x + 3\sqrt{1 + m'^2}$ (circle)

For common tangent $m = m'$ and $3\sqrt{1 + m'^2} = \sqrt{16m^2 - 9}$

or $\quad 9 + 9m^2 = 16m^2 - 9$

or $\quad 7m^2 = 18 \Rightarrow m = \pm 3\sqrt{\dfrac{2}{7}}$

$\therefore$ required equation is $y = \pm 3\sqrt{\dfrac{2}{7}}\, x \pm 3\sqrt{1 + \dfrac{18}{7}}$

or $\quad y = \pm 3\sqrt{\dfrac{2}{7}}x \pm \dfrac{15}{\sqrt{7}}$ **Ans. (A,B,C,D)**

Do yourself - 4 :

(i) Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$, which is parallel to the line $4y = 5x + 7$.

(ii) Find the equation of the tangent to the hyperbola $16x^2 - 9y^2 = 144$ at $\left(5, \dfrac{16}{3}\right)$.

(iii) Find the common tangent to the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ and an ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$.

8. NORMAL TO THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(a) **Point form** : The equation of the normal to the given hyperbola at the point P (x_1, y_1) on it is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2 = a^2e^2$.

(b) **Slope form** : The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$

foot of normal are $\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$

(c) **Parametric form** : The equation of the normal at the point **P (a secθ , b tanθ)** to the given hyperbola is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2 = a^2e^2$.

Illustration 11 : Line x cos α + y sin α = p is a normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if -

(A) $a^2\sec^2\alpha - b^2\operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$ (C) $a^2\sec^2\alpha + b^2\operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$

(C) $a^2\cos^2\alpha - b^2\sin^2\alpha = \frac{(a^2+b^2)^2}{p^2}$ (D) $a^2\cos^2\alpha + b^2\sin^2\alpha = \frac{(a^2+b^2)^2}{p^2}$

Solution : Equation of a normal to the hyperbola is $ax\cos\theta + by\cot\theta = a^2+b^2$

comparing it with the given line equation

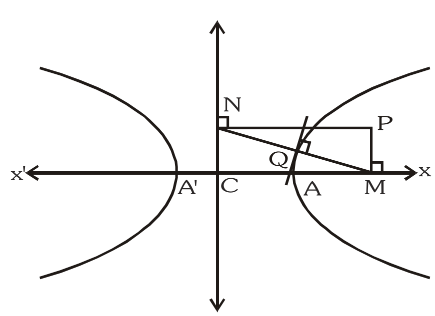
$$\frac{a\cos\theta}{\cos\alpha}=\frac{b\cot\theta}{\sin\alpha}=\frac{a^2+b^2}{p} \Rightarrow \sec\theta=\frac{ap}{\cos\alpha(a^2+b^2)},\ \tan\theta=\frac{bp}{\sin\alpha(a^2+b^2)}$$

Eliminating θ, we get

$$\frac{a^2p^2}{\cos^2\alpha(a^2+b^2)^2}-\frac{b^2p^2}{\sin^2\alpha(a^2+b^2)^2}=1 \Rightarrow a^2\sec^2\alpha - b^2\operatorname{cosec}^2\alpha=\frac{(a^2+b^2)^2}{p^2}$$ **Ans.(A)**

Illustration 12 : The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is hyperbola $(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

Solution : Equation of normal at any point Q is $ax\cos\theta + by\cot\theta = a^2+b^2$



$\therefore$ $M \equiv \left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$, $N \equiv \left(0, \frac{a^2+b^2}{b}\tan\theta\right)$

$\therefore$ Let P ≡ (h, k)

$\Rightarrow$ $h=\frac{a^2+b^2}{a}\sec\theta, \quad k=\frac{a^2+b^2}{b}\tan\theta$

$\Rightarrow$ $\frac{a^2h^2}{(a^2+b^2)}-\frac{b^2k^2}{(a^2+b^2)^2}=\sec^2\theta-\tan^2\theta=1$

$\therefore$ locus of P is $(a^2x^2 - b^2y^2) = (a^2 + b^2)$.

Do yourself - 5 :

(i) Find the equation of normal to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ at (5, 0).

(ii) Find the equation of normal to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at the point $\left(6, \frac{3}{2}\sqrt{5}\right)$.

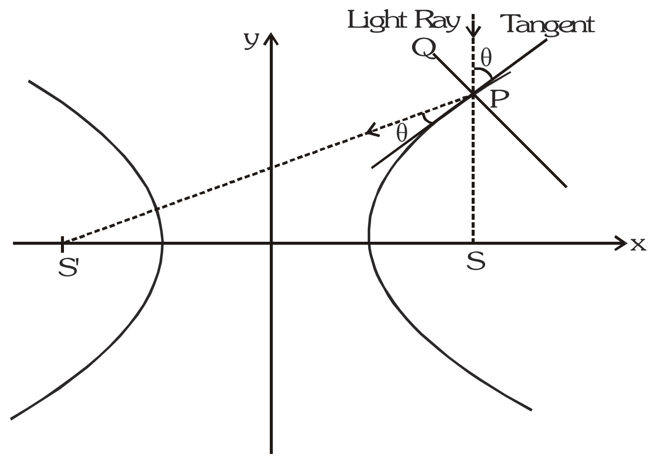
(iii) Find the condition for the line $\ell x + my + n = 0$ is normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

9. HIGHLIGHTS ON TANGENT AND NORMAL :

(a) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of lengths to these perpendiculars is b^2 **(semi Conjugate Axis)^2**

(b) The portion of the tangent between the **point of contact** & **the directrix** subtends a **right angle** at the corresponding **focus**.

(c) The tangent & normal at any point of a hyperbola **bisect** the angle between the **focal radii**. This spells the **reflection property of the hyperbola** as "**An incoming light ray** " aimed towards one **focus is reflected from the outer surface of the hyperbola towards the other focus.** It follows that if an ellipse and a hyperbola have the same foci, they cut at **right angles** at any of their **common point.**



Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

(d) The **foci of the hyperbola** and the **points P** and **Q** in which any tangent meets the tangents at the vertices are **concyclic** with **PQ** as **diameter of the circle.**

10. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at **right angles** is known as the **Director Circle** of the hyperbola. The equation to the **director circle is** : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$, this circle is **real** ; if $b^2 = a^2$ the **radius** of the **circle is zero** & it reduces to a **point circle at the origin.** In this case the **centre is the only point** from which the **tangents at right angles** can be drawn to the **curve**.

If $b^2 > a^2$, the **radius** of the **circle is imaginary**, so that there is **no such circle** & so **no tangents at right angle** can be drawn to the curve.

Note : Equations of chord of contact, chord with a given middle point, pair of tangents from an external point are to be interpreted in the similar way as in ellipse.

11. ASYMPTOTES :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the **Asymptote of the Hyperbola**.

To find the asymptote of the hyperbola :

Let $y = mx + c$ is the **asymptote** of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two we get the quadratic as $(b^2 - a^2m^2)\, x^2 - 2a^2 mcx - a^2 (b^2 + c^2) = 0$ ....(1)

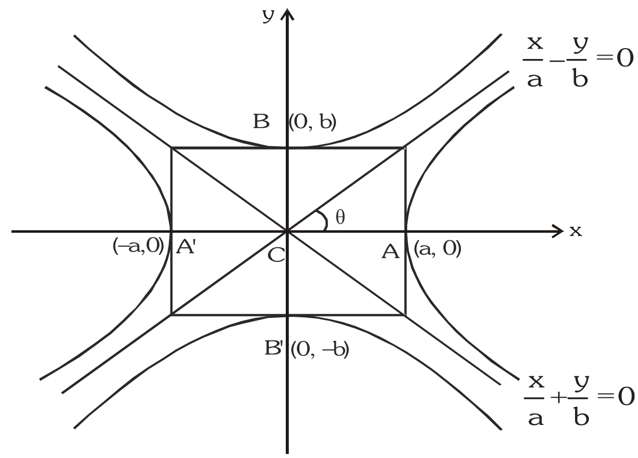
In order that $y = mx + c$ **be an asymptote**, both **roots** of equation (1) must approach infinity, the conditions for which are : **coefficient of $x^2 = 0$** & **coefficient of $x = 0$.**

$\Rightarrow$ $b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ &

$a^2 mc = 0 \Rightarrow c = 0$.

$\therefore$ equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$

and $\frac{x}{a} - \frac{y}{b} = 0$.



combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Particular Case :

When $b = a$ the asymptotes of the rectangular hyperbola.

$x^2 - y^2 = a^2$ are $y = \pm x$ which are at **right angles**.

Note :

(i) **Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.**

(ii) If a **hyperbola is equilateral** then the **conjugate hyperbola is also equilateral**.

(iii) A **hyperbola** and its **conjugate** have the **same asymptote.**

(iv) The equation of the **pair of asymptotes** differ the **hyperbola** & **the conjugate hyperbola** by the **same constant** only.

(v) The asymptotes pass through the **centre of the hyperbola** & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the co–ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as : Let $f(x, y) = 0$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ gives the **centre** of the hyperbola.

Illustration 13 : Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find also the general equation of all the hyperbolas having the same set of asymptotes.

Solution : Let $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ be asymptotes. This will represent two straight line so

$$4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0$$

$\Rightarrow$ $\lambda = 2$

$\Rightarrow$ $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$ are asymptotes

$\Rightarrow$ $(2x + y + 2) = 0$ and $(x + 2y + 1) = 0$ are asymptotes

and $2x^2 + 5xy + 2y^2 + 4x + 5y + c = 0$ is general equation of hyperbola.

Illustration 14 : Find the hyperbola whose asymptotes are $2x - y = 3$ and $3x + y - 7 = 0$ and which passes through the point (1, 1).

Solution : The equation of the hyperbola differs from the equation of the asymptotes by a constant

$\Rightarrow$ The equation of the hyperbola with asymptotes $3x + y - 7 = 0$ and $2x - y = 3$ is $(3x + y - 7)(2x - y - 3) + k = 0$

It passes through (1, 1)

$\Rightarrow$ $k = -6$.

Hence the equation of the hyperbola is $(2x - y - 3)(3x + y - 7) = 6$.

Do yourself - 6 :

(i) Find the equation to the chords of the hyperbola $x^2 - y^2 = 9$ which is bisected at (5, –3)

(ii) If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ which pass through the point (6, 2), then find the value of $11m_1m_2$ and $11(m_1 + m_2)$.

(iii) Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$ which are tangents to the hyperbola $9x^2 - 16y^2 = 144$.

(iv) The asymptotes of a hyperbola are parallel to lines $2x + 3y = 0$ and $3x + 2y = 0$. The hyperbola has its centre at (1, 2) and it passes through (5, 3). Find its equation.

12. HIGHLIGHTS ON ASYMPTOTES

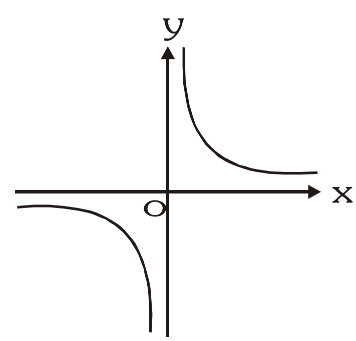
(a) If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

(b) Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

(c) The tangent at any point P on a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with **centre C**, meets the **asymptotes in Q** and **R** and cuts off a **Δ CQR** of **constant area** equal to **ab** from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact . This implies that locus of the centre of the circle circumscribing the **Δ CQR** in case of a rectangular hyperbola is the **hyperbola** itself.

(d) If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ then the eccentricity of the **hyperbola** is **sec θ**.

13. RECTANGULAR HYPERBOLA :



Rectangular hyperbola referred to its asymptotes as axis of coordinates.

(a) Equation is $xy = c^2$ with parametric representation $x = ct$, $y = c/t$, $t \in R - \{0\}$.

(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is

$x + t_1 t_2 y = c(t_1 + t_2)$ with slope, $m = \frac{-1}{t_1 t_2}$

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$

& at P(t) is $\frac{x}{t} + ty = 2c$.

(d) Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as **(h, k)** is **$kx + hy = 2hk$**.

Note :

For the hyperbola, $xy = c^2$

(i) Vertices : (c, c) & $(-c, -c)$.

(ii) Foci : $(\sqrt{2}c, \sqrt{2}c)$ & $(-\sqrt{2}c, -\sqrt{2}c)$

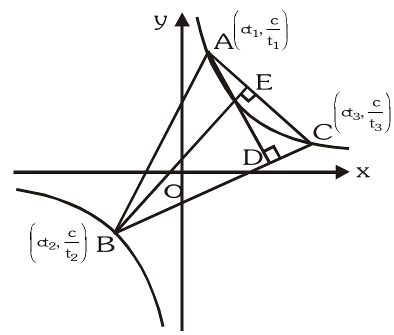
(iii) Directrices : $x + y = \pm\sqrt{2}c$

(iv) Latus rectum : $\ell = 2\sqrt{2}c$ = T . A = C . A

Illustration 15 : A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Solution : Let t_1, t_2 and t_3 are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ co-ordinates of A, B and C are $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively



Now slope of BC is $\dfrac{\frac{c}{t_3} - \frac{c}{t_2}}{ct_3 - ct_2} = -\dfrac{1}{t_2 t_3}$

$\therefore$ Slope of AD is $t_2 t_3$

Equation of altitude AD is $y - \dfrac{c}{t_1} = t_2 t_3 (x - ct_1)$

or $t_1 y - c = x t_1 t_2 t_3 - c t_1^2 t_2 t_3$ .... (i)

Similarly equation of altitude BE is

$t_2 y - c = x t_1 t_2 t_3 - c t_1 t_2^2 t_3$ .... (ii)

Solving (i) and (ii), we get the orthocentre $\left(-\dfrac{c}{t_1 t_2 t_3}, -c t_1 t_2 t_3\right)$ which lies on $xy = c^2$.

Do yourself - 7 :

(i) If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola then write required conditions.

(ii) Find the equation of tangent at the point (1, 2) to the rectangular hyperbola $xy = 2$.

(iii) Prove that the locus of point, tangents from where to hyperbola $x^2 - y^2 = a^2$ inclined at an angle α & β with x-axis such that $\tan\alpha \tan\beta = 2$ is also a hyperbola. Find the eccentricity of this hyperbola.

Miscellaneous Illustrations :

Illustration 16 : Chords of the circle $x^2 + y^2 = a^2$ touch the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Prove that locus of their middle point is the curve $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$.

Solution : Let (h, k) be the mid-point of the chord of the circle $x^2 + y^2 = a^2$,
so that its equation by $T = S_1$ is $hx + ky = h^2 + k^2$

or $y = -\dfrac{h}{k}x + \dfrac{h^2 + k^2}{k}$ i.e. of the form $y = mx + c$

It will touch the hyperbola if $c^2 = a^2m^2 - b^2$

$\therefore$ $\left(\dfrac{h^2 + k^2}{k}\right)^2 = a^2\left(-\dfrac{h}{k}\right)^2 - b^2$ or $(h^2 + k^2)^2 = a^2h^2 - b^2k^2$

Generalising, the locus of mid-point (h, k) is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$

Illustration 17 : C is the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The tangent at any point P on this hyperbola meets the straight lines bx − ay = 0 and bx + ay = 0 in the points Q and R respectively. Show that CQ . CR = a^2 + b^2.

Solution : P is (a secθ, b tanθ)

Tangent at P is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$

It meets bx − ay = 0 i.e. $\frac{x}{a}=\frac{y}{b}$ in Q

$\therefore$ Q is $\left(\frac{a}{\sec\theta-\tan\theta},\frac{b}{\sec\theta-\tan\theta}\right)$

It meets bx + ay = 0 i.e. $\frac{x}{a}=-\frac{y}{b}$ in R.

$\therefore$ R is $\left(\frac{a}{\sec\theta+\tan\theta},\frac{-b}{\sec\theta+\tan\theta}\right)$

$\therefore$ CQ.CR = $\frac{\sqrt{(a^2+b^2)}}{\sec\theta-\tan\theta}.\frac{\sqrt{(a^2+b^2)}}{\sec\theta+\tan\theta}$ = a^2 + b^2 ($\because$ $\sec^2\theta-\tan^2\theta=1$) **Ans.**

Illustration 18 : A circle of variable radius cuts the rectangular hyperbola $x^2-y^2=9a^2$ in points P, Q, R and S. Determine the equation of the locus of the centroid of triangle PQR.

Solution : Let the circle be $(x-h)^2+(y-k)^2=r^2$ where r is variable. Its intersection with $x^2-y^2=9a^2$ is obtained by putting $y^2=x^2-9a^2$.

$x^2+x^2-9a^2-2hx+h^2+k^2-r^2=2k\sqrt{(x^2-9a^2)}$

or $[2x^2-2hx+(h^2+k^2-r^2-9a^2)]^2=4k^2(x^2-9a^2)$

or $4x^4-8hx^3+\ldots\ldots=0$

$\therefore$ Above gives the abscissas of the four points of intersection.

$\therefore$ $\Sigma x_1=\frac{8h}{4}=2h$

$x_1+x_2+x_3+x_4=2h$

Similarly $y_1+y_2+y_3+y_4=2k$.

Now if (α, β) be the centroid of ΔPQR, then $3\alpha=x_1+x_2+x_3$, $3\beta=y_1+y_2+y_3$

$\therefore$ $x_4=2h-3\alpha$, $y_4=2k-3\beta$

But (x_4, y_4) lies on $x^2-y^2=9a^2$

$\therefore$ $(2h-3\alpha)^2+(2k-3\beta)^2=9a^2$

Hence the locus of centroid (α, β) is $(2h-3x)^2+(2k-3y)^2=9a^2$

or $\left(x-\frac{2h}{3}\right)^2+\left(y-\frac{2k}{3}\right)^2=a^2$

Illustration 19 : If a circle cuts a rectangular hyperbola $xy=c^2$ in A, B, C, D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively, then prove that :

(a) $t_1t_2t_3t_4=1$

(b) The centre of mean position of the four points bisects the distance between the centres of the two curves.

Solution : (a) Let the equation of the hyperbola referred to rectangular asymptotes as axes be $xy = c^2$ or its parametric equation be

$x = ct,\ y = c/t$ .... (i)

and that of the circle be

$x^2 + y^2 + 2gx + 2fy + k = 0$ .... (ii)

Solving (i) and (ii), we get

$$c^2t^2 + \frac{c^2}{t^2} + 2gct + 2f\frac{c}{t} + k = 0$$

or $c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$ .... (iii)

Above equation being of fourth degree in t gives us the four parameters t_1, t_2, t_3, t_4 of the points of intersection.

$\therefore\quad t_1 + t_2 + t_3 + t_4 = -\frac{2gc}{c^2} = -\frac{2g}{c}$ .... (iv)

$t_1t_2t_3 + t_1t_2t_4 + t_3t_4t_1 + t_3t_4t_2$

$= -\frac{2fc}{c^2} = -\frac{2f}{c}$ .... (v)

$t_1t_2t_3t_4 = \frac{c^2}{c^2} = 1$. It proves (a) .... (vi)

Dividing (v) by (vi), we get

$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = -\frac{2f}{c}$ .... (vii)

(b) The centre of mean position of the four points of intersection is

$$\left[\frac{c}{4}(t_1 + t_2 + t_3 + t_4),\ \frac{c}{4}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)\right] = \left[\frac{c}{4}\left(-\frac{2g}{c}\right),\ \frac{c}{4}\left(-\frac{2f}{c}\right)\right],$$ by (iv) and (vii)

$= (-g/2,\ -f/2)$

Above is clearly the mid-point of (0, 0) and $(-g,\ -f)$ i.e. the join of the centres of the two curves.

ANSWERS FOR DO YOURSELF

1 : **(i)** $\sqrt{3}$ **(ii)** $7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0$

(iii) 6, 4; $(\pm\sqrt{13}, 0)$; $\sqrt{13}/3$; 8/3 **(iv)** $x^2 - y^2 = 32$

2 : **(i)** $\sqrt{5}$ & 40 sq. units

3 : **(i)** $n^2 = a^2\ell^2 - b^2m^2$ **(ii)** B

4 : **(i)** $24y = 30x \pm\sqrt{161}$ **(ii)** $5x - 3y = 9$ **(iii)** $y = \pm x \pm\sqrt{7}$

5 : **(i)** $y = 0$; **(ii)** $8\sqrt{5}x + 18y = 75\sqrt{5}$ **(iii)** $\frac{a^2}{\ell^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$

6 : **(i)** $5x + 3y = 16$ **(ii)** 20 & 24 **(iii)** $(x^2 + y^2)^2 = 16x^2 - 9y^2$

(iv) $(2x + 3y - 8)(3x + 2y - 7) = 154$

7 : **(i)** $\Delta \neq 0,\ h^2 > ab,\ a + b = 0$ **(ii)** $2x + y = 4$ **(iii)** $e = \sqrt{3}$

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. The eccentricity of the hyperbola $4x^2 - 9y^2 - 8x = 32$ is -

(A) $\frac{\sqrt{5}}{3}$ (B) $\frac{\sqrt{13}}{3}$ (C) $\frac{4}{3}$ (D) $\frac{3}{2}$

2. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different values of k is -

(A) ellipse (B) parabola (C) circle (D) hyperbola

3. If the latus rectum of an hyperbola be 8 and eccentricity be $\frac{3}{\sqrt{5}}$ then the equation of the hyperbola can be -

(A) $4x^2 - 5y^2 = 100$ (B) $5x^2 - 4y^2 = 100$ (C) $4x^2 + 5y^2 = 100$ (D) $5x^2 + 4y^2 = 100$

4. If the centre, vertex and focus of a hyperbola be (0,0), (4, 0) and (6,0) respectively, then the equation of the hyperbola is –

(A) $4x^2 - 5y^2 = 8$ (B) $4x^2 - 5y^2 = 80$ (C) $5x^2 - 4y^2 = 80$ (D) $5x^2 - 4y^2 = 8$

5. The equation of the hyperbola whose foci are (6,5), (–4, 5) and eccentricity 5/4 is–

(A) $\frac{(x-1)^2}{16} - \frac{(y-5)^2}{9} = 1$ (B) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (C) $\frac{(x-1)^2}{16} - \frac{(y-5)^2}{9} = -1$ (D) $\frac{(x-1)^2}{4} - \frac{(y-5)^2}{9} = 1$

6. The vertices of a hyperbola are at (0, 0) and (10,0) and one of its foci is at (18,0). The possible equation of the hyperbola is –

(A) $\frac{x^2}{25} - \frac{y^2}{144} = 1$ (B) $\frac{(x-5)^2}{25} - \frac{y^2}{144} = 1$ (C) $\frac{x^2}{25} - \frac{(y-5)^2}{144} = 1$ (D) $\frac{(x-5)^2}{25} - \frac{(y-5)^2}{144} = 1$

7. The length of the transverse axis of a hyperbola is 7 and it passes through the point (5, –2). The equation of the hyperbola is –

(A) $\frac{4}{49}x^2 - \frac{196}{51}y^2 = 1$ (B) $\frac{49}{4}x^2 - \frac{51}{196}y^2 = 1$ (C) $\frac{4}{49}x^2 - \frac{51}{196}y^2 = 1$ (D) none of these

8. AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies -

(A) $e > \sqrt{3}$ (B) $1 < e < \frac{2}{\sqrt{3}}$ (C) $e = \frac{2}{\sqrt{3}}$ (D) $e > \frac{2}{\sqrt{3}}$

9. The equation of the tangent lines to the hyperbola $x^2 - 2y^2 = 18$ which are perpendicular to the line $y = x$ are -

(A) $y = x \pm 3$ (B) $y = -x \pm 3$ (C) $2x + 3y + 4 = 0$ (D) none of these

10. The equations to the common tangents to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ are -

(A) $y = \pm x \pm \sqrt{b^2 - a^2}$ (B) $y = \pm x \pm (a^2 - b^2)$ (C) $y = \pm x \pm \sqrt{a^2 - b^2}$ (D) $y = \pm x \pm \sqrt{a^2 + b^2}$

11. Locus of the feet of the perpendiculars drawn from either foci on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is -

(A) $x^2 + y^2 = 9$ (B) $x^2 + y^2 = 1/9$ (C) $x^2 + y^2 = 7/144$ (D) $x^2 + y^2 = 1/16$

12. The ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ have the same foci and they intersect at right angles then the equation of the circle through the points of intersection of two conics is -

(A) $x^2 + y^2 = 5$ (B) $\sqrt{5}(x^2 + y^2) - 3x - 4y = 0$

(C) $\sqrt{5}(x^2 + y^2) + 3x + 4y = 0$ (D) $x^2 + y^2 = 25$

13. The equation of the common tangent to the parabola $y^2 = 8x$ and the hyperbola $3x^2 - y^2 = 3$ is -

(A) $2x \pm y + 1 = 0$ (B) $x \pm y + 1 = 0$ (C) $x \pm 2y + 1 = 0$ (D) $x \pm y + 2 = 0$

14. Equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which is bisected at the point (6, 2) is -
(A) $16x-75y=418$ (B) $75x-16y=418$ (C) $25x-4y = 400$ (D) none of these

15. The asymptotes of the hyperbola $xy - 3x - 2y = 0$ are-
(A) $x - 2 = 0$ and $y - 3 = 0$ (B) $x - 3 = 0$ and $y - 2 = 0$
(C) $x + 2 = 0$ and $y + 3 = 0$ (D) $x + 3 = 0$ and $y + 2 = 0$

16. If the product of the perpendicular distances from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of eccentricity $e = \sqrt{3}$ on its asymptotes is equal to 6, then the length of the transverse axis of the hyperbola is -
(A) 3 (B) 6 (C) 8 (D) 12

17. If the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at 't_1' then t^3t_1 has the value equal to -
(A) 1 (B) -1 (C) 0 (D) none

18. Area of triangle formed by tangent to the hyperbola $xy = 16$ at (16, 1) and co-ordinate axes equals -
(A) 8 (B) 16 (C) 32 (D) 64

19. Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is -
(A) $y + mx = 0$ (B) $y - mx = 0$ (C) $my - x = 0$ (D) $my + x = 0$

20. Let P ($a\sec\theta$, $b\tan\theta$) and Q ($a\sec\phi$, $b\tan\phi$), where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h,k) is the point of intersection of the normals at P & Q, then k is equal to - **[JEE 99]**
(A) $\frac{a^2+b^2}{a}$ (B) $-\left(\frac{a^2+b^2}{a}\right)$ (C) $\frac{a^2+b^2}{b}$ (D) $-\left(\frac{a^2+b^2}{b}\right)$

21. If $x = 9$ is the chord of contact of the hyperbola $x^2-y^2 = 9$, then the equation of the corresponding pair of tangents, is - **[JEE 99]**
(A) $9x^2-8y^2 + 18x -9 = 0$ (B) $9x^2-8y^2 -18x +9 = 0$ (C) $9x^2-8y^2 -18x -9 = 0$ (C) $9x^2-8y^2 + 18x +9 = 0$

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

22. Consider the hyperbola $3x^2 - y^2 - 24x + 4y - 4 = 0$ -
(A) its centre is (4, 2) (B) its centre is (2, 4)
(C) length of latus rectum = 24 (D) length of latus rectum = 12

23. Let an incident ray $L_1 = 0$ gets reflected at point A(–2, 3) on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & passes through focus S(2, 0), then -
(A) equation of incident ray is $x + 2 = 0$ (B) equation of reflected ray is $3x + 4y = 6$
(C) eccentricity, $e = 2$ (D) length of latus rectum = 6

24. For the curve $5(x - 1)^2 + 5(y - 2)^2 = 3(2x + y - 1)^2$ which of the following is true -
(A) a hyperbola with eccentricity $\sqrt{3}$ (B) a hyperbola with directrix $2x + y - 1 = 0$
(C) a hyperbola with focus (1, 2) (D) a hyperbola with focus (2, 1)

25. The equation of common tangent of hyperbola $9x^2 - 9y^2 = 8$ and the parabola $y^2 = 32x$ is/are -
(A) $9x + 3y - 8 = 0$ (B) $9x - 3y + 8 = 0$ (C) $9x + 3y + 8 = 0$ (D) $9x - 3y - 8 = 0$

CHECK YOUR GRASP **ANSWER KEY** EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	D	A	C	A	B	C	D	B	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	A	A	B	A	B	B	C	A	D
Que.	21	22	23	24	25					
Ans.	B	A,C	A,B,C,D	A,B,C	B,C					

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. Variable circles are drawn touching two fixed circles externally, then locus of centre of variable circle is -

(A) parabola (B) ellipse (C) hyperbola (D) circle

2. The locus of the mid points of the chords passing through a fixed point (α, β) of the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is -

(A) a circle with centre $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$ (B) an ellipse with centre $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$

(C) a hyperbola with centre $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$ (D) straight line through $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$

3. The locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent is :

(A) $(x^2 - y^2)^2 = 4c^2 xy$ (B) $(x^2 + y^2)^2 = 2c^2 xy$ (C) $(x^2 - y^2) = 4c^2xy$ (D) $(x^2 + y^2)^2 = 4c^2 xy$

4. The equation to the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is -

(A) $\frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$ (B) $\frac{x}{x_1-x_2}+\frac{y}{y_1-y_2}=1$

(C) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$ (D) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$

5. The equation $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ represent a hyperbola -

(A) The length of the transverse axes is 4 (B) Length of latus rectum is 9

(C) Equation of directrix is $x = \frac{21}{5}$ and $x = -\frac{11}{5}$ (D) none of these

6. From the points of the circle $x^2 + y^2 = a^2$, tangents are drawn to the hyperbola $x^2 - y^2 = a^2$; then the locus of the middle points of the chords of contact is -

(A) $(x^2 - y^2)^2 = a^2 (x^2 + y^2)$ (B) $(x^2 - y^2)^2 = 2a^2 (x^2 + y^2)$

(C) $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$ (D) $2(x^2 - y^2)^2 = 3a^2 (x^2 + y^2)$

7. The tangent to the hyperbola $xy = c^2$ at the point P intersects the x-axis at T and the y-axis at T'. The normal to the hyperbola at P intersects the x-axis at N and the y-axis at N'. The areas of the triangles PNT and PN'T' are Δ and Δ' respectively, then $\frac{1}{\Delta}+\frac{1}{\Delta'}$ is -

(A) equal to 1 (B) depends on t (C) depends on c (D) equal to 2

8. The tangent to the hyperbola, $x^2 - 3y^2 = 3$ at the point $(\sqrt{3}, 0)$ when associated with two asymptotes constitutes -

(A) isosceles triangle (B) an equilateral triangle

(C) a triangles whose area is $\sqrt{3}$ sq. units (D) a right isosceles triangle.

9. The asymptote of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ form with any tangent to the hyperbola a triangle whose area is $a^2 \tan \lambda$ in magnitude then its eccentricity is -

(A) $\sec \lambda$ (B) $\operatorname{cosec} \lambda$ (C) $\sec^2 \lambda$ (D) $\operatorname{cosec}^2 \lambda$

10. From any point on the hyperbola $H_1 : (x^2/a^2) - (y^2/b^2) = 1$ tangents are drawn to the hyperbola $H_2 : (x^2/a^2) - (y^2/b^2) = 2$. The area cut-off by the chord of contact on the asymptotes of H_2 is equal to -

(A) ab/2 (B) ab (C) 2 ab (D) 4 ab

11. The tangent at P on the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ meets the asymptote $\frac{x}{a}-\frac{y}{b}=0$ at Q. If the locus of the mid point of PQ has the equation $(x^2/a^2) - (y^2/b^2) = k$, then k has the value equal to -

(A) 1/2 (B) 2 (C) 3/4 (D) 4/3

12. If θ is the angle between the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with eccentricity e, then $\sec\frac{\theta}{2}$ can be -

(A) e (B) e/2 (C) e/3 (D) $\frac{e}{\sqrt{e^2-1}}$

13. If (5, 12) and (24, 7) are the focii of a conic passing through the origin, then the eccentricity of conic is -

(A) $\sqrt{386}/12$ (B) $\sqrt{386}/13$ (C) $\sqrt{386}/25$ (D) $\sqrt{386}/38$

14. The point of contact of line $5x + 12y = 9$ and hyperbola $x^2 - 9y^2 = 9$ will lie on

(A) $4x + 15y = 0$ (B) $7x + 12y = 19$ (C) $4x + 15y + 1 = 0$ (D) $7x - 12y = 19$

15. Equation $(2 + \lambda)x^2 - 2\lambda xy + (\lambda - 1)y^2 - 4x - 2 = 0$ represents a hyperbola if -

(A) $\lambda = 4$ (B) $\lambda = 1$ (C) $\lambda = 4/3$ (D) $\lambda = -1$

16. If a real circle will pass through the points of intersection of hyperbola $x^2 - y^2 = a^2$ & parabola $y = x^2$, then -

(A) $a \in (-1, 1)$ (B) $a \in \left[-\frac{1}{2}, \frac{1}{2}\right] - \{0\}$

(C) area of circle $= \pi - \pi a^2$; $a \in \left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$ (D) area of circle $= \pi - 4\pi a^2$

17. If least numerical value of slope of line which is tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{(a^3+a^2+a)^2}=1$ is $\frac{3}{4}$, $a \in R_0$ is obtained at a = k. For this value of 'a', which of the following is/are true -

(A) $a = -\frac{1}{2}$ (B) $a = \frac{1}{2}$ (C) $LR = \frac{9}{16}$ (D) $e = \frac{5}{4}$

18. If the normal at point P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the transverse and conjugate axes at A and B respectively and C is the centre of the hyperbola, then -

(A) PA = PC (B) PA = PB (C) PB = PC (D) AB = 2PC

19. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y4)$, then - **[JEE 98]**

(A) $x_1 + x_2 + x_3 + x_4 = 0$ (B) $y_1 + y_2 + y_3 + y_4 = 0$ (C) $x_1 x_2 x_3 x_4 = c^4$ (D) $y_1 y_2 y_3 y_4 = c^4$

20. The curve described parametrically by, $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents - **[JEE 99]**

(A) a pair of straight lines (B) an ellipse (C) a parabola (D) a hyperbola

BRAIN TEASERS — ANSWER KEY — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C	D	A	C	A	C	A,B,C	A	D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	C	A,D	A,D	A,B	B,D	B,C	A,C,D	A,B,C,D	A,B,C,D	C

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

MATCH THE COLUMN

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1. Consider the hyperbola $9x^2 - 16y^2 - 36x + 96y + 36 = 0$.

	Column - I		Column - II
(A)	If directrices of the hyperbola are $y = k_1$ & $y = k_2$ then $k_1 + k_2$ is equal to	(p)	16
(B)	If foci of hyperbola are (a, b) & (a, c) then a + b + c is equal to	(q)	10
(C)	Product of the perpendiculars drawn from the foci upon its any tangent is	(r)	6
(D)	Distance between foci of the hyperbola is	(s)	8

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

2.

	Column - I		Column - II
(A)	A tangent drawn to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $p\left(\frac{\pi}{6}\right)$ forms a triangle of area $3a^2$ square units, with coordinate axes, then the square of its eccentricity is equal to	(p)	17
(B)	If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\theta = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\theta + y^2 = 25$ then smallest positive value of θ is $\frac{6\pi}{p}$, value of 'p' is	(q)	8
(C)	For the hyperbola $\frac{x^2}{3}-y^2=3$, angle between its asymptotes is $\frac{\ell\pi}{24}$ then value of 'ℓ' is	(r)	16
(D)	For the hyperbola xy = 8 any tangent of it at P meets co-ordinate axes at Q and R then area of triangle CQR where 'c' is centre of the hyperbola is	(s)	24

ASSERTION & REASON

These questions contain, Statement-I (assertion) and Statement-II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for Statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : Ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and $12x^2 - 4y^2 = 27$ intersect each other at right angle.

Because

Statement-II : Whenever confocal ellipse & hyperbola intersect, they intersect each other orthogonally.

(A) A (B) B (C) C (D) D

2. **Statement-I** : $\frac{5}{3}$ and $\frac{5}{4}$ are the eccentricities of two hyperbola which are conjugate to each other.

Because

Statement-II : If e and e_1 are the eccentricities of two conjugate hyperbolas than $ee_1 > 1$.

(A) A (B) B (C) C (D) D

3. **Statement-I** : A bullet is fired and hit a target. An observer in the same plane heard two sounds the crack of the riffle and the thud of the ball striking the target at the same instant, then locus of the observer is hyperbola where velocity of sound is smaller than velocity of bullet.

Because

Statement-II : If difference of distances of a point 'P' from the two fixed points is constant and less than the distance between the fixed points then locus of 'P' is a hyperbola.

(A) A (B) B (C) C (D) D

4. **Statement-I** : If a circle S = 0 intersects a hyperbola xy = 4 at four points. Three of them are (2, 2)(4, 1) and (6, 2/3) then co-ordinates of the fourth point are (1/4, 16).

Because

Statement-II : If a circle S = 0 intersects a hyperbola $xy = c^2$ at t_1, t_2, t_3, t_4 then $t_1 . t_2 . t_3 . t_4 = 1$.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1 :

If we rotate the axes of the rectangular hyperbola $x^2 - y^2 = a^2$ through an angle $\pi/4$ in the clockwise direction then the equation $x^2 - y^2 = a^2$ reduces to $xy = \frac{a^2}{2} = \left(\frac{a}{\sqrt{2}}\right)^2 = c^2$ (say). Since $x = ct$, $y = \frac{c}{t}$ satisfies $xy = c^2$.

$\therefore \quad (x, y) = \left(ct, \frac{c}{t}\right) (t \neq c)$ is called a 't' point on the rectangular hyperbola.

On the basis of above information, answer the following questions :

1. If t_1 and t_2 are the roots of the equation $x^2 - 4x + 2 = 0$, then the point of intersection of tangents at 't_1' and 't_2' on $xy = c^2$ is -

(A) $\left(\frac{c}{2}, 2c\right)$ (B) $\left(2c, \frac{c}{2}\right)$ (C) $\left(\frac{c}{2}, c\right)$ (D) $\left(c, \frac{c}{2}\right)$

2. If e_1 and e_2 are the eccentricities of the hyperbolas $xy = 9$ and $x^2 - y^2 = 25$, then (e_1, e_2) lie on a circle C_1 with centre origin then the $(\text{radius})^2$ of the director circle of C_1 is -

(A) 2 (B) 4 (C) 8 (D) 16

3. If the normal at the point 't_1' to the rectangular hyperbola $xy = c^2$ meets it again at the point 't_2' then the value of $t_1 t_2$ is -

(A) $-\ t_1^{-1}$ (B) $-\ t_1^{-2}$ (C) $-\ t_1^{-3}$ (D) $-\ t_1^{-4}$

MISCELLANEOUS TYPE QUESTION **ANSWER KEY** EXERCISE-3

- ***Match the Column***

1. (A) → (r) ; (B) → (s); (C) → (p); (D) → (q) 2. (A) → (p); (B) → (s); (C) → (q,r); (D) → (r)

- ***Assertion & Reason***

1. A 2. B 3. A 4. D

- ***Comprehension Based Questions***

Comprehension # 1 : 1. D 2. C 3. B

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

1. The hyperbola $x^2/a^2 - y^2/b^2 = 1$ (a,b > 0) passes through the point of intersection of the lines, $7x + 13y - 87 = 0$ & $5x - 8y + 7 = 0$ and the latus rectum is $32\sqrt{2}/5$. Find 'a' & 'b'.
2. Find the eccentricity of the hyperbola whose latus rectum is half its transverse axis.
3. Find the centre, the foci, the directrices, the length of the latus rectum, the length & the equations of the axes $16x^2 - 9y^2 + 32x + 36y - 164 = 0$.
4. For the hyperbola $x^2/100 - y^2/25 = 1$, prove that

   (a) eccentricity $= \sqrt{5}/2$ (b) SA . S'A = 25, where S & S' are the foci & A is the vertex.
5. Find the eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$.
6. If C is the centre of a hyperbola $x^2/a^2 - y^2/b^2 = 1$, S, S′ its foci and P a point on it. Prove that SP. S′P = $CP^2 - a^2 + b^2$.
7. For what value of λ does the line $y = 3x + \lambda$ touch the hyperbola $9x^2 - 5y^2 = 45$.
8. Find the equation of the tangent to the hyperbola $x - 4y = 36$ which is perpendicular to the line $x - y + 4 = 0$.
9. Tangents are drawn to the hyperbola $3x - 2y = 25$ from the point (0, 5/2). Find their equations.
10. If the normal at a point P to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets the x − axis at G, show that SG = e. SP, S being the focus of the hyperbola.
11. The tangents and normal at a point on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cut the y-axis at A & B. Prove that the circle on AB as diameter passes through the foci of the hyperbola.
12. Show that the locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.
13. The variable chords of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ (b > a) whose equation is $x\cos\alpha + y\sin\alpha = p$ subtends a right angle at the centre. Prove that it always touches a circle.
14. Find the asymptotes of the hyperbola $2x^2 - 3xy - 2y^2 + 3x - y + 8 = 0$. Also find the equation to the conjugate hyperbola & the equation of the principal axes of the curve.
15. Find the equation of the standard hyperbola passing through the point $(-\sqrt{3}, 3)$ and having the asymptotes as straight lines $x\sqrt{5} \pm y = 0$.

CONCEPTUAL SUBJECTIVE EXERCISE **ANSWER KEY** EXERCISE-4(A)

1. $a^2 = 25/2$; $b^2 = 16$ **2.** $\sqrt{\frac{3}{2}}$ **3.** (−1, 2); (4, 2) & (−6, 2); $5x - 4 = 0$ & $5x + 14 = 0$; $\frac{32}{3}$; 6; 8; $y - 2 = 0$; $x + 1 = 0$ **5.** $\sqrt{2}$ **7.** $\lambda = \pm 6$ **8.** $x + y \pm 3\sqrt{3} = 0$
9. $3x + 2y - 5 = 0$; $3x - 2y + 5 = 0$ **14.** $x - 2y + 1 = 0$; $2x + y + 1 = 0$; $2x^2 - 3xy - 2y^2 + 3x - y - 6 = 0$; $3x - y + 2 = 0$; $x + 3y = 0$ **15.** $5x^2 - y^2 = 6$

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. If θ_1 & θ_2 are the parameters of the extremities of a chord through (ae, 0) of a hyperbola $x^2/a^2 - y^2/b^2 = 1$, then show that $\tan\frac{\theta_1}{2} . \tan\frac{\theta_2}{2} + \frac{e-1}{e+1} = 0$.
2. If the tangent at the point (h, k) to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ cuts the auxiliary circle in points whose ordinates are y_1 and y_2 then prove that $1/y_1 + 1/y_2 = 2/k$.
3. Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ & ϕ to the x –axis. If $\tan\theta . \tan\phi = 2$, prove that $\beta^2 = 2\alpha^2 - 7$.
4. The perpendicular from the centre upon the normal on any point of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets at R . Find the locus of R.
5. If the normal to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the point P meets the transverse axis in G & the conjugate axis in g & CF be perpendicular to the normal from the centre C , then prove that $|PF . PG| = b$ & $PF . Pg = a$ where a & b are the semi transverse & semi-conjugate axes of the hyperbola.
6. If a rectangular hyperbola have the equation , $x y = c^2$, prove that the locus of the middle points of the chords of constant length 2 d is $(x^2 + y^2) (x y - c^2) = d^2 x y$.
7. Prove that the locus of the middle point of the chord of contact of tangents from any point of the circle $x^2 + y^2 = r^2$ to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is given by the equation $(x^2/a^2 - y^2/b^2)^2 = (x^2 + y^2) / r^2$.
8. Find the equations of the tangents to the hyperbola $x^2 - 9 y^2 = 9$ that are drawn from (3, 2) . Find the area of the triangle that these tangents form with their chord of contact.
9. A tangent to the parabola $x^2 = 4 ay$ meets the hyperbola $xy = k^2$ in two points P & Q . Prove that the middle point of PQ lies on a parabola .
10. Given the base of a triangle and the ratio of the tangent of half the base angles. Show that the vertex moves on a hyperbola whose foci are the extremities of the base.

BRAIN STORMING SUBJECTIVE EXERCISE	ANSWER KEY	EXERCISE-4(B)
4. $(x^2 + y^2)^2 (a^2y^2 - b^2x^2) = x^2y^2 (a^2 + b^2)^2$		8. $y = \frac{5}{12}x + \frac{3}{4}$; $x - 3 = 0$; 8 sq. unit

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. The latus rectum of the hyperbola $16x^2 - 9y^2 = 144$ is- [AIEEE-2002]

(1) 16/3 (2) 32/3 (3) 8/3 (4) 4/3

2. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is- [AIEEE-2003]

(1) 9 (2) 1 (3) 5 (4) 7

3. The locus of a point P(α, β) moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is- [AIEEE-2005]

(1) a hyperbola (2) a parabola (3) a circle (4) an ellipse

4. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant when α varies ? [AIEEE-2007, IIT-2003]

(1) Abscissae of vertices (2) Abscissae of foci (3) Eccentricity (4) Directrix

5. The equation of the hyperbola whose foci are (–2,0) and (2, 0) and eccentricity is 2 is given by : [AIEEE-2011]

(1) $-3x^2 + y^2 = 3$ (2) $x^2 - 3y^2 = 3$ (3) $3x^2 - y^2 = 3$ (4) $-x^2 + 3y^2 = 3$

PREVIOUS YEARS QUESTIONS — ANSWER KEY — EXERCISE-5 [A]

Que.	1	2	3	4	5
Ans	2	4	1	2	3

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. The equation of the common tangent to the curve $y^2 = 8x$ and $xy = -1$ is - [JEE 2002 Screening]

(A) $3y = 9x + 2$ (B) $y = 2x + 1$ (C) $2y = x + 8$ (D) $y = x + 2$

2. For hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ which of the following remains constant with change in α - [JEE 2003 Screening]

(A) abscissae of vertices (B) abscissae of foci (C) eccentricity (D) directrix

3. The point of contact of the line $2x + \sqrt{6}\ y = 2$ and the hyperbola $x^2 - 2y^2 = 4$ is - [JEE 2004 Screening]

(A) $(4, -\sqrt{6})$ (B) $(\sqrt{6}, 1)$ (C) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (D) $\left(\frac{1}{6}, \frac{3}{2}\right)$

4. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of mid-point of the chord of contact. [JEE 2005 Mains 4M out of 60]

5. If a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and its transverse and conjugate axis coincides with the major and minor axis of the ellipse and product of their eccentricities is 1, then - [JEE 2006, 5M)]

(A) equation of hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(B) equation of hyperbola $\frac{x^2}{9} - \frac{y^2}{25} = 1$

(C) focus of hyperbola (5, 0)

(D) focus of hyperbola $(5\sqrt{3}, 3)$

6. A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is - [JEE 2007, 3M]

(A) $x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$ (B) $x^2\sec^2\theta - y^2\text{cosec}^2\theta = 1$

(C) $x^2\sin^2\theta - y^2\cos^2\theta = 1$ (D) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

7. Match the column - [2007, 6M]

	Column I		Column II
(A)	Two intersecting circles	(p)	have a common tangent
(B)	Two mutually external circles	(q)	have a common normal
(C)	Two circles, one strictly inside the other	(r)	do not have a common tangent
(D)	Two branches of a hyperbola	(s)	do not have a common normal

8. Let a and b be non-zero real numbers. Then, the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents -

(A) four straight lines, when c = 0 and a, b are of the same sign

(B) two straight lines and a circle, when a = b, and c is of sign opposite to that of a

(C) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

(D) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

[JEE 2008, 3M, –1M]

9. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is - **[JEE 2008, 3M, –1M]**

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$ (C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}} + 1$

10. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then :- **[JEE 2009, 4M, –1M]**

(A) equation of ellipse is $x^2 + 2y^2 = 2$ (B) the foci of ellipse are $(\pm 1, 0)$

(C) equation of ellipse is $x^2 + 2y^2 = 4$ (D) the foci of ellipse are $\left(\pm\sqrt{2}, 0\right)$

Paragraph for Question 11 and 12 **[JEE 10, (3M each), –1M]**

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

11. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is -

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$ (C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

12. Equation of the circle with AB as its diameter is -

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

13. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **[JEE 10, 3M]**

14. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then - **[JEE 2011, 4M]**

(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$ (B) a focus of the hyperbola is (2,0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$ (D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

15. Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is - **[JEE 2011, 3M]**

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

16. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are **[JEE 2012, 4M]**

(A) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ (C) $\left(3\sqrt{3}, -2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [B]

1. D 2. B 3. A 4. $\frac{x^2}{9} - \frac{y^2}{4} = \left(\frac{x^2 + y^2}{9}\right)^2$ 5. A,C 6. A

7. (A) → (p, q) ; (B) → (p, q) ; (C) → (q, r) ; (D) → (q, r) 8. B 9. B 10. A, B

11. B 12. A 13. 2 14. B,D 15. B 16. A,B