MAXIMA-MINIMA

1. INTRODUCTION :

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter

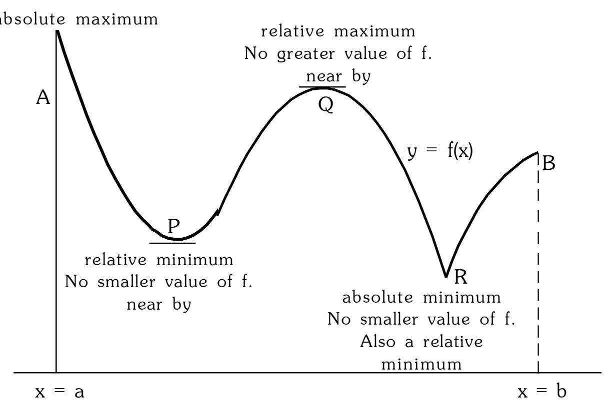
- What is the shape of a vessel that can with-stand maximum pressure ?
- What is the maximum acceleration of a space shuttle ? (This is an important question to the astronauts who have to withstand the effects of acceleration)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maxima and minima.

(a) Maxima (Local/Relative maxima) :

A function f(x) is said to have a maximum at $x = a$ if there exist a neighbourhood $(a - h, a + h) - \{a\}$ such that

$f(a) > f(x) \ \forall \ x \in (a - h, a + h) - \{a\}$



(b) Minima (Local/Relative minima):

A function f(x) is said to have a minimum at $x = a$ if there exist a neighbourhood $(a - h, a + h) - \{a\}$ such that

$f(a) < f(x) \ \forall \ x \in (a - h, a + h) - \{a\}$

(c) Absolute maximum (Global maximum) :

A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

(d) Absolute minimum (Global minimum) :

A function f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D and the number f(c) is called the minimum value of f on D. The maximum and minimum values of f are called the **extreme values** of f.

Note that :

(i) the maximum & minimum values of a function are also known as **local/relative maxima or local/relative minima** as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) the term 'extremum' (or extremal) or 'turning value' is used both for maximum or a minimum value.

(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.

(v) local maximum & local minimum values of a continuous function occur alternately & between two consecutive local maximum values there is a local minimum value & vice versa.

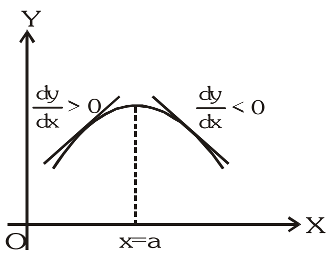
2. DERIVATIVE TEST FOR ASCERTAINING MAXIMA/MINIMA :

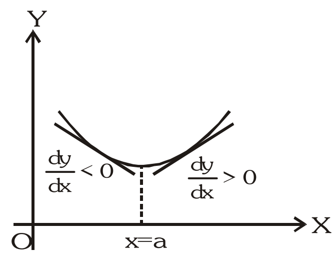
(a) **First derivative test :**

If $f'(x) = 0$ at a point (say $x = a$) and

(i) If $f'(x)$ changes sign from positive to negative while graph of the function passes through $x = a$ then $x = a$ is said to be a point **local maxima.**

(ii) If $f'(x)$ changes sign from negative to positive while graph of the function passes through $x = a$ then $x = a$ is said to be a point **local minima.**





Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then $f(x)$ is either strictly increasing or decreasing throughout this neighbourhood implying that $f(a)$ is not an extreme value of f.

Illustration 1 : Let $f(x) = x + \frac{1}{x}$; $x \neq 0$. Discuss the local maximum and local minimum values of $f(x)$.

Solution : Here, $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$

$$\underset{-1}{\overset{+\quad\quad -}{\longleftarrow\!\!\!+}}\;\underset{1}{\overset{+}{+\!\!\!\longrightarrow}}$$

Using number line rule, $f(x)$ will have local maximum at $x = -1$ and local minimum at $x = 1$

$\therefore$ local maximum value of $f(x) = -2$ at $x = -1$

and local minimum value of $f(x) = 2$ at $x = 1$ **Ans.**

Illustration 2 : If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3, \end{cases}$, then

(A) $f(x)$ is increasing on $[-1, 2)$
(B) $f(x)$ is continuous on $[-1, 3]$
(C) $f'(x)$ does not exist at $x = 2$
(D) $f(x)$ has the maximum value at $x = 2$

Solution : Given, $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$

$\Rightarrow$ $f'(x) = \begin{cases} 6x + 12, & -1 \le x \le 2 \\ -1, & 2 < x \le 3 \end{cases}$

(A) which shows $f'(x) > 0$ for $x \in [-1, 2)$
So, $f(x)$ is increasing on $[-1, 2)$
Hence, (A) is correct.

(B) for continuity of $f(x)$. (check at $x = 2$)
RHL = 35, LHL = 35 and $f(2) = 35$
So, (B) is correct

(C) $Rf'(2) = -1$ and $Lf'(2) = 24$
so, not differentiable at $x = 2$.
Hence, (C) is correct.

(D) we know f(x) is increasing on [–1, 2) and decreasing on (2, 3],

Thus maximum at x = 2,

Hence, (D) is correct.

$\therefore$ (A), (B), (C), (D) all are correct. **Ans.**

Do yourself - 1 :

(i) Find local maxima and local minima for the function $f(x) = x^3 - 3x$.

(ii) If function $f(x) = x^3 - 62x^2 + ax + 9$ has local maxima at x = 1, then find the value of a.

(b) Second derivative test :

If f(x) is continuous and differentiable at x = a where f'(a) = 0 and f''(a) also exists then for ascertaining maxima/minima at x = a, 2^{nd} derivative test can be used -

(i) If $f''(a) > 0 \Rightarrow$ x = a is a point of local minima

(ii) If $f''(a) < 0 \Rightarrow$ x = a is a point of local maxima

(iii) If $f''(a) = 0 \Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

Illustration 3 : If $f(x) = 2x^3 - 3x^2 - 36x + 6$ has local maximum and minimum at x = a and x = b respectively, then ordered pair (a, b) is -

(A) (3, –2) (B) (2, –3) (C) (–2, 3) (D) (–3, 2)

Solution : $f(x) = 2x^3 - 3x^2 - 36x + 6$

$f'(x) = 6x^2 - 6x - 36$ & $f''(x) = 12x - 6$

Now $f'(x) = 0 \Rightarrow 6(x^2 - x - 6) = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = -2, 3$

$f''(-2) = -30$

$\therefore$ x = –2 is a point of local maximum

$f''(3) = 30$

$\therefore$ x = 3 is a point of local minimum

Hence, (–2, 3) is the required ordered pair. **Ans. (C)**

Illustration 4 : Find the point of local maxima of $f(x) = \sin x\,(1+\cos x)$ in $x \in (0, \pi/2)$.

Solution : Let $f(x) = \sin x\,(1 + \cos x) = \sin x + \frac{1}{2}\sin 2x$

$\Rightarrow f'(x) = \cos x + \cos 2x$

$f''(x) = -\sin x - 2\sin 2x$

Now $f'(x) = 0 \Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow \cos 2x = \cos(\pi - x) \Rightarrow x = \pi/3$

Also $f''(\pi/3) = -\sqrt{3}/2 - \sqrt{3} < 0$ $\therefore$ f(x) has a maxima at $x = \pi/3$ **Ans.**

Illustration 5 : Find the global maximum and global minimum of $f(x) = \frac{e^x + e^{-x}}{2}$ in $[-\log_e 2, \log_e 7]$.

Solution : $f(x) = \frac{e^x + e^{-x}}{2}$ is differentiable at all x in its domain.

Then $f'(x) = \frac{e^x - e^{-x}}{2}$, $f''(x) = \frac{e^x + e^{-x}}{2}$

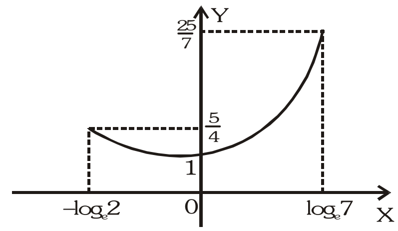
$f'(x) = 0 \Rightarrow \frac{e^x - e^{-x}}{2} = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$

$f''(0) = 1 \quad \therefore \quad x = 0$ is a point of local minimum

Points $x = -\log_e 2$ and $x = \log_e 7$ are extreme points.

Now, check the value of f(x) at all these three points $x = -\log_e 2,\ 0,\ \log_e 7$

$$\Rightarrow \ f(-\log_e 2) = \frac{e^{-\log_e 2} + e^{+\log_e 2}}{2} = \frac{5}{4}$$



$$f(0) = \frac{e^{0} + e^{-0}}{2} = 1$$

$$f(\log_e 7) = \frac{e^{\log_e 7} + e^{-\log_e 7}}{2} = \frac{25}{7}$$

$\therefore \quad x = 0$ is absolute minima & $x = \log_e 7$ is absolute maxima

Hence, absolute/global minimum value of f(x) is 1 at x = 0

and absolute/global maximum value of f(x) is $\frac{25}{7}$ at $x = \log_e 7$

Ans.

Do yourself - 2 :

(i) Find local maximum value of function $f(x) = \frac{\ell nx}{x}$

(ii) If $f(x) = x^2 e^{-2x}$ $(x > 0)$, then find the local maximum value of f(x).

(b) n^{th} derivative test :

Let f(x) function such that $f'(a) = f''(a) = f'''(a) = \ldots\ldots\ldots = f^{n-1}(a) = 0$ & $f^n(a) \neq 0$, then

(i) n = even

(1) $f^n(a) > 0 \quad \Rightarrow \quad$ Minima

(2) $f^n(a) < 0 \quad \Rightarrow \quad$ Maxima

(ii) n = odd

Neither maxima nor minima at x = a

Illustration 6 : Identify a point of maxima/minima in $f(x) = (x + 1)^4$.

Solution :

$f(x) = (x + 1)^4$

$f'(x) = 4(x + 1)^3$

$f''(x) = 12(x + 1)^2$

$f'''(x) = 24(x + 1)$

$f''''(x) = 24$

Now $f'(x) = 0 \Rightarrow x = -1$

$f''(-1) = 0,\ f'''(-1) = 0,\ f''''(-1) = 24 > 0$

$\therefore$ at x = 1 f(x) has point of minima.

Illustration 7 : Find point of local maxima and minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution :

$f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3)$

$= 5x^2(x - 1)(x - 3)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 3$

$f''(x) = 10x(2x^2 - 6x + 3)$

Now $f''(1) < 0 \quad \Rightarrow \quad$ Maxima at x = 1

$f''(3) > 0 \quad \Rightarrow \quad$ Minima at x = 3 and $f''(0) = 0 \Rightarrow$ IInd derivative test fails

so, $f'''(x) = 30(2x^2 - 4x + 1)$

$f'''(0) = 30$

$\Rightarrow$ Neither maxima nor minima at $x = 0$.

Do yourself - 3 :

(i) Identify the point of local maxima/minima in $f(x) = (x - 3)^{10}$.

3. SUMMARY OF WORKING RULE FOR SOLVING REAL LIFE OPTIMIZATION PROBLEM :

First : When possible, draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

Second : Write an equation for the quantity that is to be maximized or minimized. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

Third : If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

Fourth : Using derivative test, test each value of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither.

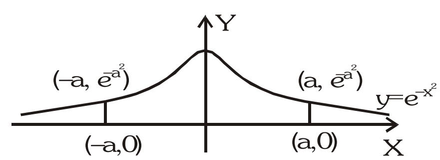
Fifth : If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Sixth : If the function $y = f(x)$ is defined only for $x \in [a,b]$ then examine $x = a$ & $x = b$ for possible extreme values.

Illustration 8 : Determine the largest area of the rectangle whose base is on the x-axis and two of its vertices lie on the curve $y = e^{-x^2}$.

Solution : Area of the rectangle will be $A = 2a.\ e^{-a^2}$

For max. area, $\frac{dA}{da} = \frac{d}{da}(2ae^{-a^2}) = e^{-a^2}[2 - 4a^2]$



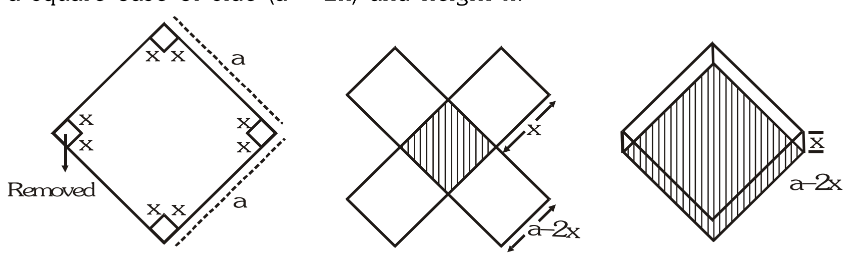
$\frac{dA}{da} = 0 \Rightarrow a = \frac{1}{\sqrt{2}}$

sign of $\frac{dA}{da}$ changes from positive to negative

$\Rightarrow$ $x = \frac{1}{\sqrt{2}}$ is a point of maxima $\Rightarrow A_{max} = \frac{2}{\sqrt{2}} . e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{e^{1/2}}$ sq units . **Ans.**

Illustration 9 : A box of maximum volume with top open is to be made by cutting out four equal squares from four corners of a square tin sheet of side length a ft, and then folding up the flaps. Find the side of the square base cut off.

Solution : Volume of the box is, $V = x(a - 2x)^2$ i.e., squares of side x are cut out then we will get a box with a square base of side $(a - 2x)$ and height x.



$$\therefore \quad \frac{dV}{dx} = (a - 2x)^2 + x\, 2(a - 2x)(-2)$$

$$\frac{dV}{dx} = (a - 2x)(a - 6x)$$

For V to be extremum $\frac{dV}{dx} = 0 \quad \Rightarrow \quad x = a/2,\ a/6$

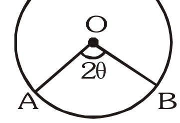
But when $x = a/2$; $V = 0$ (minimum) and we know minimum and maximum occurs alternately in a continuous function.

Hence, V is maximum when $x = a/6$. **Ans.**

Illustration 10 : A Conical vessel is to be prepared out of a circular sheet of gold of unit radius. How much sectorial area is to be removed from the sheet so that vessel has maximum volume.

Solution : Lateral height of cone = Radius of circle = 1

Lateral area of cone = Area of circle with sector removed



i.e. $\pi r(1) = \frac{\pi(1)^2}{2\pi}(2\pi - 2\theta)$

i.e. $r = \frac{\pi - \theta}{\pi}$ (here r is radius of cone)

Height 'h' of cone = $\sqrt{1^2 - r^2}$

Volume of cone $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{\pi-\theta}{\pi}\right)^2 \times \sqrt{1^2 - \left(\frac{\pi-\theta}{\pi}\right)^2}$

upon maximizing V, we get $\frac{\pi-\theta}{\pi} = \sqrt{\frac{2}{3}} \Rightarrow \theta = \pi\left(1 - \sqrt{\frac{2}{3}}\right)$

Area of sector removed $= \frac{1}{2}(1)^2(2\theta) = \pi\left(1 - \sqrt{\frac{2}{3}}\right)$ **Ans.**

Do yourself - 4 :

(i) Find the two positive numbers x & y such that their sum is 60 and xy^3 is maximum.

(ii) If from a wire of length 36 metre, a rectangle of greatest area is made, then find its two adjacent sides in metre.

Important note :

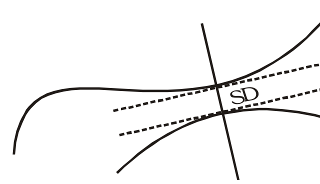
(i) If the sum of two real numbers x and y is constant then their product is maximum if they are equal.

i.e. $xy = \frac{1}{4}[(x + y)^2 - (x - y)^2]$

(ii) If the product of two positive numbers is constant then their sum is least if they are equal.

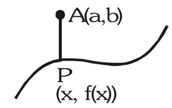
i.e. $(x + y)^2 = (x - y)^2 + 4xy$

4. SHORTEST DISTANCE BETWEEN TWO CURVES :



Shortest distance between two non-intersecting curves always lies along the common normal. (Wherever defined)

Note : Given a fixed point A(a, b) and a moving point P(x, f(x)) on the curve y = f(x). Then AP will be maximum or minimum if it is normal to the curve at P.

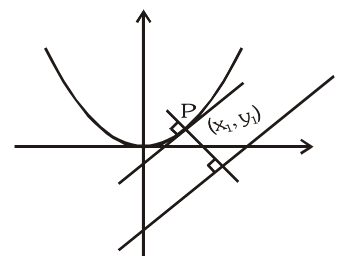


Proof : $F(x) = (x - a)^2 + (f(x) - b)^2 \Rightarrow F'(x) = 2(x - a) + 2(f(x) - b) \,.\, f'(x)$

$$\therefore \quad f'(x) = -\left(\frac{x-a}{f(x)-b}\right). \text{ Also } m_{AP} = \frac{f(x)-b}{x-a}. \text{ Hence } f'(x) \,.\, m_{AP} = -1.$$

Illustration 11 **:** Find the co-ordinates of the point on the curve $x^2 = 4y$, which is at least distance from the line $y = x - 4$.

Solution **:** Let $P(x_1 y_1)$ be a point on the curve $x^2 = 4y$ at which normal is also a perpendicular to the line $y = x - 4$.



Slope of the tangent at (x_1, y_1) is $2x = 4\frac{dy}{dx} \Rightarrow \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{x_1}{2}$

$\therefore \quad \frac{x_1}{2} = 1 \Rightarrow x_1 = 2$

$\because \quad x_1^2 = 4y_1 \Rightarrow y_1 = 1$

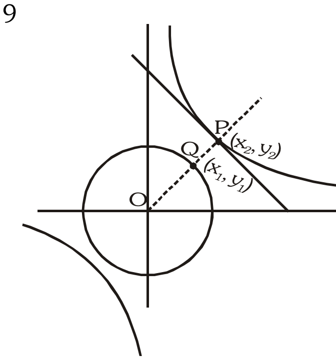
Hence required point is (2, 1)

Illustration 12 : Find the minimum value of $(x_1 - x_2)^2 + \left(\sqrt{2 - x_1^2} - \frac{9}{x_2}\right)^2$ where $x_1 \in \left(0, \sqrt{2}\right)$ and $x_2 \in R^+$

Solution **:** $d^2 = (x_1 - x_2)^2 + \left(\sqrt{2 - x_1^2} - \frac{9}{x_2}\right)^2$

The above expression is the square of the distance between the points $\left(x_1, \sqrt{2 - x_1^2}\right), \left(x_2, \frac{9}{x_2}\right)$ which lie on the curves $x^2 + y^2 = 2$ and $xy = 9$ respectively.

Now, the minimum value of the expression means square of the shortest distance between the two curves. Slope of the normal at $P(x_2, y_2)$ on the curve $xy = 9$



$\frac{dy}{dx} = \frac{-9}{x^2}$

Slope of OP $= \frac{x_2^2}{9} = \frac{y_2}{x_2} \quad \because \quad x_2 y_2 = 9$

$x_2^4 = 81 \quad \Rightarrow \quad x_2 = \pm 3$

$\therefore \quad y_2 = \pm 3$

$(x_2, y_2) = (3, 3)$

Now, shortest distance = PQ = OP − OQ = $3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$

$\therefore \quad d^2 = 8$

Do yourself - 5 :

(i) Find the coordinates of point on the curve $y^2 = 8x$, which is at minimum distance from the line $x + y = -2$.

5. USEFUL FORMULAE OF MENSURATION TO REMEMBER :

(a) Volume of a cuboid = ℓbh.

(b) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(c) Volume of a prism = area of the base   height.

(d) Lateral surface area of prism = perimeter of the base   height.

(e) Total surface area of a prism = lateral surface area + 2 area of the base (Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid $=\frac{1}{3}$ area of the base x height.

(g) Curved surface area of a pyramid $=\frac{1}{2}$ (perimeter of the base) x slant height. (Note that slant surfaces of a pyramid are triangles).

(h) Volume of a cone = $\frac{1}{3}\pi r^2h$.

(i) Curved surface area of a cylinder = $2\pi rh$.

(j) Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.

(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.

(l) Surface area of a sphere = $4\pi r^2$.

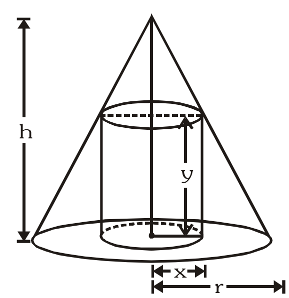
(m) Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.

Illustration 13 : If a right circular cylinder is inscribed in a given cone. Find the dimension of the cylinder such that its volume is maximum.

Solution : Let x be the radius of cylinder and y be its height

$V = \pi x^2 y$

x, y can be related by using similar triangles



$$\frac{y}{r-x}=\frac{h}{r} \Rightarrow y = \frac{h}{r}(r-x)$$

$$\Rightarrow V(x) = \pi x^2\frac{h}{r}(r-x) \Rightarrow V(x) = \frac{\pi h}{r}(rx^2 - x^3)$$

$$V'(x) = \frac{\pi h}{r}(2rx - 3x^2)$$

$$V'(x) = 0 \Rightarrow x = 0,\ \frac{2r}{3}$$

$$V''(x) = \frac{\pi h}{r}\left(2r-6x\right)$$

$V''(0) = 2\pi h \Rightarrow x = 0$ is point of minima

$V''\left(\frac{2r}{3}\right) = -2\pi h \Rightarrow x = \frac{2r}{3}$ is point of maxima

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

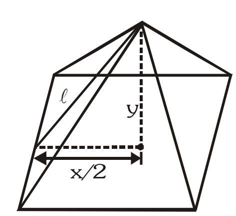
Illustration 14 : Among all regular square pyramids of volume $36\sqrt{2}$ cm^3. Find dimensions of the pyramid having least lateral surface area.

Solution : Let the length of a side of base be x cm and y cm be the perpendicular height of the pyramid

$$V = \frac{1}{3}\text{ area of base height}$$

$$\Rightarrow V = \frac{1}{3}x^2y = 36\sqrt{2}$$

$$\Rightarrow y = \frac{108\sqrt{2}}{x^2}$$



and $S = \frac{1}{2}$ perimeter of base slant height

$$= \frac{1}{2}(Ax)\ .\ \ell$$

but $\ell = \sqrt{\frac{x^2}{4}+y^2} \Rightarrow S = 2x\sqrt{\frac{x^2}{4}+y^2} = \sqrt{x^4+4x^2y^2}$

$\Rightarrow \quad S = \sqrt{x^4+4x^2\left(\frac{108\sqrt{2}}{x^2}\right)^2} \qquad \Rightarrow \quad S(x) = \sqrt{x^4+\frac{8.(108)^2}{x^2}}$

Let $f(x) = x^4+\frac{8.(108)^2}{x^2}$

for minimizing f(x)

$f'(x) = 4x^3 - \frac{16(108)^2}{x^3} = 0 \qquad \Rightarrow \quad f'(x) = 4\frac{(x^6-6^6)}{x^3} = 0$

$\Rightarrow$ $x = 6$ which a point of minima

Hence $x = 6$ cm and $y = 3\sqrt{2}$.

Do yourself - 6 :

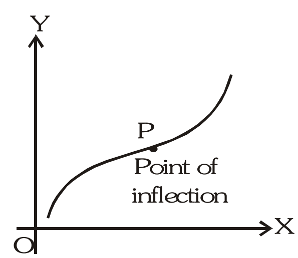
(i) If $ab = 2a + 3b$ where $a > 0$, $b > 0$, then find the minimum value of ab.

(ii) Of all closed right circular cylinders of a given volume of 100 cubic centimetres, find the dimensions of cylinder which has minimum surface area.

6. POINT OF INFLECTION :

A point where the graph of a function has a tangent line and where the concavity changes is called a point of inflection.

If function $y = f(x)$ is double differentiable then the point at which $\frac{d^2y}{dx^2} = 0$ & changes its sign is the point of inflection.



Note : If at any point $\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point then it is also called point of inflection. e.g. for $y = x^{1/3}$, $x = 0$ is point of inflection.

For a given function f(x) graphs of f'(x) and f''(x) can be drawn as shown in the adjacent figure.

Here point C & E are point of inflection.

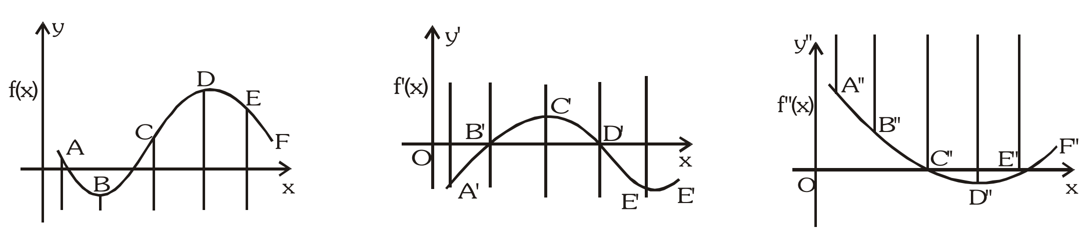


Illustration 15 : The point of inflexion for the curve $y = x^{\frac{5}{3}}$ is -

(A) (1, 1) (B) (0, 0) (C) (1, 0) (D) (0, 1)

Solution : Here $\frac{d^2y}{dx^2} = \frac{10}{9x^{1/3}}$

From the given points we find that (0, 0) is the point of the curve where

$\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point.

$\therefore$ (0, 0) is the required point **Ans. (B)**

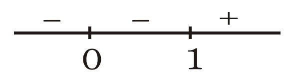
Illustration 16 : Find the inflection point of $f(x) = 3x^4 - 4x^3$. Also draw the graph of f(x) giving due importance to maxima, minima and concavity.

Solution : $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

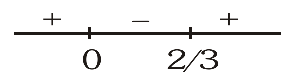
$f'(x) = 12x^2(x - 1)$

$f'(x) = 0 \Rightarrow x = 0, 1$



examining sign change of f'(x)

thus x = 1 is a point of local minima



$f''(x) = 12(3x^2 - 2x)$

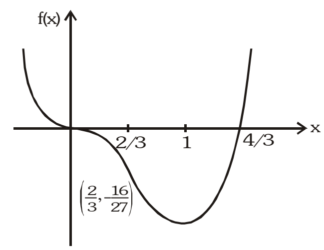
$f''(x) = 12x(3x - 2)$

$f''(x) = 0 \Rightarrow x = 0, \frac{2}{3}$

Again examining sign of f''(x)

thus $x = 0, \frac{2}{3}$ are the inflection points

Hence the graph of f(x) is



Do yourself - 7 :

(i) Find the point of inflection for the curve $y = x^3 - 6x^2 + 12x + 5$

(ii) Find the intervals for $f(x) = \frac{x^4}{12} - \frac{5x^3}{6} + 3x^2 + 7$ in which it is (a) concave upward (b) concave downward.

Miscellaneous Illustrations :

Illustration 17 : Find all the values of a for which the function $f(x) = (a^2 - 3a + 2)\cos\left(\frac{x}{2}\right) + (a - 1)x$ possesses critical points.

Solution : Given $f(x) = (a^2 - 3a + 2)\cos\left(\frac{x}{2}\right) + (a - 1)x$

$$\therefore \quad f'(x) = -\frac{(a^2 - 3a + 2)}{2}\sin\left(\frac{x}{2}\right) + (a - 1) = (a-1)\left\{1 - \frac{1}{2}(a-2)\sin\left(\frac{x}{2}\right)\right\}$$

Put $f'(x) = 0$ then $a = 1$ and $\sin\left(\frac{x}{2}\right) = \frac{2}{a-2}$

but $-1 \le \sin\left(\frac{x}{2}\right) \le 1$

or $\left|\sin\left(\frac{x}{2}\right)\right| \le 1 \Rightarrow \left|\frac{2}{a-2}\right| \le 1 \Rightarrow |a - 2| \ge 2$

$\Rightarrow a - 2 \ge 2$ and $a - 2 \le -2$

$\therefore a \ge 4$ and $a \le 0 \Rightarrow a \in (-\infty, 0] \cup [4, \infty)$

Hence $a \in (-\infty, 0] \cup \{1\} \cup [4, \infty)$

Illustration 18 **:** Divide 64 into two parts such that sum of the cubes of two parts is minimum.

Solution **:** Let x and y be two positive number such that

$x + y = 64$ .... (i)

Let $u = x^3 + y^3$ .... (ii)

Eliminate x from (ii) with the help of (i), then $u = (64 - y)^3 + y^3$

$\therefore \quad \frac{du}{dy} = -3(64-y)^2 + 3y^2$ .... (iii)

and $\frac{d^2u}{dy^2} = 6(64-y) + 6y = 384 > 0$ .... (iv)

For maximum or minimum of u, $\frac{du}{dy} = 0$

Then $3(64)(2y - 64) = 0$

$\therefore \quad y = 32$

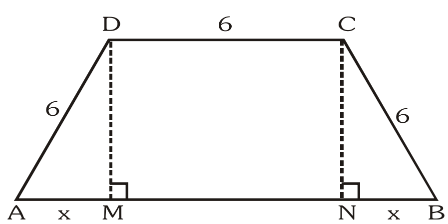
From (i), $x = 32$

It is clear from (iv), u is minimum.

Hence $x = 32$, $y = 32$.

Illustration 19 **:** The three sides of a trapezium are equal each being 6cm long; find the area of trapezium when it is maximum.

Solution **:** Let ABCD be the given trapezium.



Let $AM = BN = x$ cm

then $DM = CN = \sqrt{(36-x^2)}$

$\therefore$ Area of trapezium ABCD is

$S = \frac{1}{2}(6 + x + 6 + x) \quad \sqrt{(36-x^2)}$

$= (6 + x)\sqrt{(36-x^2)}$

or $S^2 = (6 + x)^2(36 - x^2)$

Let $y = (6 + x)^2(36 - x^2)$

$\therefore \quad \frac{dy}{dx} = (6 + x)^2(-2x) + (36 - x^2).2(6 + x)$

$= 2(6 + x)^2(6 - 2x) = 4(3 - x)(6 + x)^2$

and $\frac{d^2y}{dx^2} = -12x(6 + x)$

For max. or min. of y, $\frac{dy}{dx} = 0$ then $x = 3$ $(x \neq -6)$

$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{x=3} = -324 < 0$

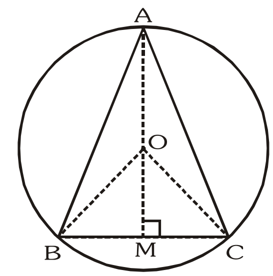
$\therefore$ y is maximum at $x = 3$ then S is also maximum at $x = 3$

$\therefore \quad S = (6 + 3)\sqrt{(36-9)} = 27\sqrt{3}\ \text{cm}^2$

Illustration 20 : Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.

Solution : $\because$ Area = $\frac{1}{2}$ (Base) (Height)

For maximum area height must be maximum. Height will be maximum if triangle is an isosceles triangle.



Let ΔABC is isosceles.

Let $AB = AC$

Let $\angle B = \angle C = \theta$ then $\angle A = \pi - 2\theta$

$\therefore$ $\angle COM = \angle BOM = \pi - 2\theta$

If r be the radius of circle

$\therefore$ $OM = r\cos(\pi - 2\theta)$ and $MC = r\sin(\pi - 2\theta)$

$\therefore$ Area of $\Delta ABC = \frac{1}{2} \times BC \times AM$

$$= \frac{1}{2} \times 2r\sin(\pi - 2\theta) \times \{r + r\cos(\pi - 2\theta)\}$$

$$= r^2 \sin 2\theta(1 - \cos 2\theta)$$

Let $S = r^2 \left\{\sin 2\theta - \frac{1}{2}\sin 4\theta\right\}$

$\therefore$ $\frac{dS}{d\theta} = r^2\{2\cos 2\theta - 2\cos 4\theta\}$ and $\frac{d^2S}{d\theta^2} = r^2\{-4\sin 2\theta + 8\sin 4\theta\}$

For max. or min. of S, $\frac{dS}{d\theta} = 0$

or $\cos 2\theta = \cos 4\theta$ or $4\theta = 2\pi - 2\theta$

$\Rightarrow$ $\theta = \frac{\pi}{3}$ and $\left.\frac{d^2S}{d\theta^2}\right|_{\theta=\pi/3} = -6\sqrt{3}r^2 < 0$

$\therefore$ $\theta = \frac{\pi}{3}$ is point of maxima

$\therefore$ $\angle B = \angle C = \frac{\pi}{3}$ and $\angle A = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

$\therefore$ Area of triangle is maximum if triangle is equilateral.

ANSWERS FOR DO YOURSELF

1 : (i) local max. at x = –1, local min. at x = 1 (ii) 121

2 : (i) $\frac{1}{e}$ (ii) $\frac{1}{e^2}$

3 : (i) local minima at x = 3

4 : (i) 15 & 45 (ii) 9 & 9

5 : (i) (2,–4)

6 : (i) 24 (ii) $r = \left(\frac{50}{\pi}\right)^{1/3}$ cm. & $h = 2\left(\frac{50}{\pi}\right)^{1/3}$ cm.

7 : (i) x = 2 (ii) (a) $(-\infty, 2) \cup (3, \infty)$ (b) (2, 3)

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. On the interval [0,1] the function $x^{25}(1-x)^{75}$ takes its maximum value at the point -
(A) 0 (B) 1/3 (C) 1/2 (D) 1/4

2. The value of 'a' so that the sum of the squares of the roots of the equation $x^2 - (a-2)x - a + 1 = 0$ assume the least value is -
(A) 2 (B) 0 (C) 3 (D) 1

3. The slope of the tangent to the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum when x equals -
(A) 1 (B) 3 (C) 1/2 (D) –1/2

4. The real number x when added to it's reciprocal gives the minimum value of the sum at x equal to -
(A) 1 (B) – 1 (C) – 2 (D) 2

5. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ where $a > 0$, attains it's maximum and minimum at p and q respectively such that $p^2 = q$ then 'a' equals -
(A) 1 (B) 2 (C) 1/2 (D) 3

6. If $f(x) = 1 + 2x^2 + 4x^4 + 6x^6 + ...+ 100x^{100}$ is a polynomial in a real variable x, then f(x) has -
(A) neither a maximum nor a minimum (B) only one maximum
(C) only one minimum (D) none

7. For all $a, b \in R$ the function $f(x) = 3x^4 - 4x^3 + 6x^2 + ax + b$ has -
(A) no extremum (B) exactly one extremum
(C) exactly two extremum (D) three extremum

8. Let $f(x) = \begin{cases} \sin\frac{\pi x}{2}, & 0 \le x < 1 \\ 3-2x, & x \ge 1 \end{cases}$ then :
(A) f(x) has local maxima at x = 1 (B) f(x) has local minima at x = 1
(C) f(x) does not have any local extrema at x = 1 (D) f(x) has a global minima at x = 1

9. Two sides of a triangle are to have lengths 'a' cm & 'b' cm. If the triangle is to have the maximum area, then the length of the median from the vertex containing the sides 'a' and 'b' is -
(A) $\frac{1}{2}\sqrt{a^2+b^2}$ (B) $\frac{2a+b}{3}$ (C) $\sqrt{\frac{a^2+b^2}{2}}$ (D) $\frac{a+2b}{3}$

10. The difference between the greatest and the least value of $f(x) = \cos^2\frac{x}{2}\sin x,\ x \in [0,\pi]$ is -
(A) $\frac{3\sqrt{3}}{8}$ (B) $\frac{\sqrt{3}}{8}$ (C) $\frac{3}{8}$ (D) $\frac{1}{2\sqrt{2}}$

11. Equation of a straight line passing through (1,4) if the sum of its positive intercepts on the coordinate axis is the smallest is -
(A) $2x + y - 6 = 0$ (B) $x + 2y - 9 = 0$ (C) $y + 2x + 6 = 0$ (D) none

12. A rectangle has one side on the positive y-axis and one side on the positive x-axis. The upper right hand vertex on the curve $y = \frac{\ell nx}{x^2}$. The maximum area of the rectangle is -
(A) e^{-1} (B) $e^{-1/2}$ (C) 1 (D) $e^{1/2}$

13. A solid rectangular brick is to be made from 1 cu feet of clay. The brick must be 3 times as long as it is wide. The width of brick for which it will have minimum surface area is a. Then a^3 is -

(A) $\left(\frac{2}{9}\right)^{1/3}$ (B) $\frac{2}{9}$ (C) $\frac{8}{729}$ (D) $\frac{3}{2}$

14. Let h be a twice continuously differentiable positive function on an open interval J. Let

$g(x) = \ell n(h(x))$ for each $x \in J$

Suppose $(h'(x))^2 > h''(x)h(x)$ for each $x \in J$. Then

(A) g is increasing on J (B) g is decreasing on J

(C) g is concave up on J (D) g is concave down on J

15. Function f(x), g(x) are defined on [–1, 3] and $f''(x) > 0$, $g''(x) > 0$ for all $x \in [-1, 3]$, then which of the following is always true ?

(A) f(x) – g(x) is concave upwards on (–1, 3) (B) f(x) g(x) is concave upwards on (–1, 3)

(C) f(x) g(x) does not have a critical point on (–1, 3) (D) f(x) + g(x) is concave upwards on (–1, 3)

16. If the point (1,3) serves as the point of inflection of the curve $y = ax^3 + bx^2$ then the value of 'a' and 'b' are -

(A) $a = 3/2$ & $b = -9/2$ (B) $a = 3/2$ & $b = 9/2$

(C) $a = -3/2$ & $b = -9/2$ (D) $a = -3/2$ & $b = 9/2$

17. The set of value (s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)\ x + 2$ possess a negative point of inflection -

(A) $(-\infty, -2) \cup (0, \infty)$ (B) $\{-4/5\}$ (C) $(-2,0)$ (D) empty set

18. Which of the following statements is true for the general cubic function $f(x) = ax^3 + bx^2 + cx + d$ $(a \neq 0)$

I. If the derivative $f'(x)$ has two distinct real roots then cubic has one local maxima and one local minima.

II. If the derivative $f'(x)$ has exactly one real root then the cubic has exactly one relative extremum.

III. If the derivative $f'(x)$ has no real roots, then the cubic has no relative extrema

(A) only I & II (B) only II and III (C) only I and III (D) all I, II, III are correct.

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THEN ONE CORRECT ANSWERS)

19. If $y = a\ell n|x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then :-

(A) $a = 2, b = -1$ (B) $a = 2, b = -\frac{1}{2}$ (C) $a = -2, b = \frac{1}{2}$ (D) none of these

20. Let S be the set of real values of parameter λ for which the function $f(x) = 2x^3 - 3(2 + \lambda)x^2 + 12\lambda x$ has exactly one local maxima and exactly one local minima. Then the subset of S is -

(A) $(5, \infty)$ (B) $(-4, 4)$ (C) $(3, 8)$ (D) $(-\infty, -1)$

21. The value of 'a' for which the function $f(x) = \begin{cases} -x^3 + \cos^{-1} a, & 0 < x < 1 \\ x^2, & x \geq 1 \end{cases}$ has a local minimum at x = 1, is -

(A) –1 (B) 1 (C) 0 (D) $-\frac{1}{2}$

22. If a continuous function $f(x)$ has a local maximum at x = a, then -

(A) $f'(a^+)$ may be 0 (B) $f'(a^+)$ may be $-\infty$

(C) $f'(a^+)$ may be non-zero finite real number (D) $f'(a^-)$ may be $-\infty$

CHECK YOUR GRASP — ANSWER KEY — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	D	D	A	A	B	C	B	A	A	A	A	A	B	D	D
Que.	16	17	18	19	20	21	22								
Ans.	D	A	C	B	A,C,D	A,D	A,B,C								

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THEN ONE CORRECT ANSWERS)

1. The set of values of p for which the points of extremum of the function, $f(x) = x^3 - 3px^2 + 3(p^2 - 1)x + 1$ lie in the interval $(-2,4)$ is -

(A) $(-3,5)$ (B) $(-3,3)$ (C) $(-1,3)$ (D) $(-1,5)$

2. If $f(x) = 4x^3 - x^2 - 2x + 1$ and $g(x) = \begin{cases} \text{Min } \{f(t) : 0 \le t \le x\}; & 0 \le x \le 1 \\ 3 - x; & 1 < x \le 2 \end{cases}$ then $g\left(\frac{1}{4}\right) + g\left(\frac{3}{4}\right) + g\left(\frac{5}{4}\right)$ has the value equal to :

(A) $\frac{7}{4}$ (B) $\frac{9}{4}$ (C) $\frac{13}{4}$ (D) $\frac{5}{2}$

3. The function 'f' is defined by $f(x) = x^p(1-x)^q$ for all $x \in R$, where p, q are positive integers, has a maximum value, for x equal to :

(A) $\frac{pq}{p+q}$ (B) 1 (C) $\frac{p}{p+q}$ (D) 0

4. If the point of minima of the function, $f(x) = 1 + a^2x - x^3$ satisfy the inequality $\frac{x^2+x+2}{x^2+5x+6} < 0$, then 'a' must lie in the interval :

(A) $\left(-3\sqrt{3},\ 3\sqrt{3}\right)$ (C) $\left(-2\sqrt{3},\ -3\sqrt{3}\right)$

(C) $\left(2\sqrt{3},\ 3\sqrt{3}\right)$ (D) $\left(-3\sqrt{3},\ -2\sqrt{3}\right) \cup \left(2\sqrt{3},\ 3\sqrt{3}\right)$

5. The function $f(x) = \int_0^x \sqrt{1-t^4}\ dt$ is such that :

(A) it is defined on the interval $[-1,1]$ (B) it is an increasing function

(C) it is an odd function (D) the point (0,0) is the point of inflection

6. The function $\frac{\sin(x+a)}{\sin(x+b)}$ has no maxima or minima if -

(A) $b - a = n\pi, \quad n \in I$ (B) $b - a = (2n+1)\pi \quad n \in I$

(C) $b - a = 2n\pi, \quad n \in I$ (D) none of these.

7. The coordinates of the point P on the graph of the function $y = e^{-|x|}$, where area of triangle made by tangent and the coordinate axis has the greatest area, is -

(A) $\left(1, \frac{1}{e}\right)$ (B) $\left(-1, \frac{1}{e}\right)$ (C) (e, e^{-e}) (D) none

8. The least value of 'a' for which the equation $\frac{4}{\sin x} + \frac{1}{1-\sin x} = a$ has atleast one solution on the interval $(0, \pi/2)$ is -

(A) 3 (B) 5 (C) 7 (D) 9

9. Read of the following mathematical statements carefully :

I. A differentiable function 'f' with maximum at $x = c \Rightarrow f''(c) < 0$.

II. Antiderivative of a periodic function is also a periodic function.

III. If f has a period T then for any $a \in R$, $\int_0^T f(x)dx = \int_0^T f(x+a)dx$

IV. If f(x) has a maxima at x = c, then 'f' is increasing in $(c - h, c)$ and decreasing in $(c, c + h)$ as $h \to 0$ for $h > 0$

Now indicate the correct alternative.

(A) exactly one statement is correct
(B) exactly two statements are correct
(C) exactly three statements are correct
(D) all the four statements are correct

10. The lateral edge of a regular rectangular pyramid is 'a' cm long. The lateral edge makes an angle α with the plane of the base. The value of α for which the volume of the pyramid is greatest, is -

(A) $\frac{\pi}{4}$ (B) $\sin^{-1}\sqrt{\frac{2}{3}}$ (C) $\cot^{-1}\sqrt{2}$ (D) $\frac{\pi}{3}$

11. P and Q are two points on a circle of centre C and radius α, the angle PCQ being 2θ then the radius of the circle inscribed the triangle CPQ is maximum when -

(A) $\sin\theta = \frac{\sqrt{3}-1}{2\sqrt{2}}$ (B) $\sin\theta = \frac{\sqrt{5}-1}{2}$ (C) $\sin\theta = \frac{\sqrt{5}+1}{2}$ (D) $\sin\theta = \frac{\sqrt{5}-1}{4}$

12. In a regular triangular prism the distance from the centre of one base to one of the vertices of the other base is ℓ. The altitude of the prism for which the volume is greatest -

(A) $\ell/2$ (B) $\ell/\sqrt{3}$ (C) $\ell/3$ (D) $\ell/4$

13. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots\ldots\ldots + a_nx^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < \ldots. < a_n$. The function P(x) has- **[JEE 1986]**

(A) neither a maximum nor a minimum
(B) only one maximum
(C) only one minimum
(D) only one maximum and only one minimum

14. If $g(x) = 7x^2e^{-x^2}\ \forall\ x \in R$, then g(x) has -

(A) local maximum at x = 0
(B) local minima at x = 0
(C) local maximum at x = –1
(D) two local maxima and one local minima

15. The coordinates of the point on the parabola $y^2 = 8x$, which is at minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are -

(A) (2, –4) (B) (18, –2) (C) (2, 4) (D) none of these

BRAIN TEASERS — **ANSWER KEY** — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	D	C	D	A,B,C,D	A,B,C	A,B	D	A	C
Que.	11	12	13	14	15					
Ans.	B	B	C	B,C,D	A					

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1. Four points A, B, C and D lie in the order on the parabola $y = ax^2 + bx + c$ and the coordinates of A, B and D are known A(–2, 3); B(–1, 1); D(2, 7).

On the basis of above information match the following :

	Column-I		Column-II
(A)	The value of $a + b + c =$	(p)	–1
(B)	If roots of the equation $ax^2 + bx + c = 0$ are α & β then $\alpha^{19} + \beta^7 =$	(q)	8
(C)	If the value of function $(a + 2)x^2 + 2\dfrac{(b+2)}{x} + c$ at minima is L then L – 3 is equal to	(r)	3
(D)	If area of quadrilateral ABCD is greatest and co-ordinates of C are (p, q) then 2p + 4q =	(s)	7

2. For the function $f(x) = x^4(12\ell nx - 7)$ match the following :

	Column-I		Column-II
(A)	If (a, b) is the point of inflection then a – b is equal to	(p)	3
(B)	If e^t is a point of minima then 12t is equal to	(q)	1
(C)	If graph is concave downward in (d, e) then d + 3e is equal to	(r)	4
(D)	If graph is concave upward in (p, ∞) then the least value of p is equal to	(s)	8

ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.
(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.
(C) Statement-I is true, Statement-II is false.
(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : Consider the function, $f(x)=\begin{cases} -\dfrac{x}{2}, & x<0 \\ 7x+8 & x \ge 0 \end{cases}$; f(x) has local minima at x = 0

Because

Statement-II : If $f(a) < f(a - h)$ & $f(a) < f(a + h)$ where 'h' is sufficiently small; then f(x) has local minima at x = a.

(A) A (B) B (C) C (D) D

2. **Statement-I** : The largest term in the sequence $a_n = \dfrac{n^2}{n^3+200}$, $n \in N$ is the 7^{th} term.

Because

Statement-II : The function $f(x) = \dfrac{x^2}{x^3+200}$ attains local maxima at x = 7.

(A) A (B) B (C) C (D) D

3. **Statement-I** : $e^{\pi} > \pi^{e}$.
Because
Statement-II : The function $f(x) = x^{1/x}$ attains global maxima at $x = e$.
(A) A (B) B (C) C (D) D

4. Consider an acute angled triangle ABC.
Statement-I : Minimum value of $\sec A + \sec B + \sec C$ is 6.
Because
Statement-II : If a continuous curve is concave upward then centroid of the triangle inscribed in the curve always lies above the curve.
(A) A (B) B (C) C (D) D

5. Let $y = f(x)$ be a thrice derivable function such that $f(a)\,f(b) < 0$, $f(b)\,f(c) < 0$, $f(c)\,f(d) < 0$ where $a < b < c < d$. Also the equations $f(x) = 0$ & $f''(x) = 0$ have no common roots.
Statement-I : The equation $f(x)(f''(x))^2 + f(x)\,f'(x)\,f'''(x) + (f'(x))^2 f''(x) = 0$ has atleast 5 real roots.
Because
Statement-II : The equation $f(x) = 0$ has atleast 3 real distinct roots & if $f(x) = 0$ has k real distinct roots, then $f'(x) = 0$ has atleast $k - 1$ distinct roots.
(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

Suppose f(x) is a real valued polynomial function of degree 6 satisfying the following condition
(a) 'f' has minimum value at x = 0 & 2
(b) 'f' has maximum value at x = 1
(c) for all x, $\lim\limits_{x\to 0} \frac{1}{x} \ell n \begin{vmatrix} f(x)/x & 1 & 0 \\ 0 & 1/x & 1 \\ 1 & 0 & 1/x \end{vmatrix} = 2$

On the basis of above information, answer the following questions :

1. Number of solutions of the equation $8f(x) - 1 = 0$ is -
(A) one (B) two (C) three (D) four

2. Range of f(x) is -
(A) $\left[-\frac{32}{15}, \infty\right)$ (B) $\left[-\frac{4}{15}, \infty\right)$ (C) $\left(-\infty, \frac{2}{15}\right]$ (D) none of these

3. If the area bounded by $y = f(x)$, x-axis, $x = \pm 1$; is $\frac{a}{b}$, where a & b are relatively prime then the value of $\tan^{-1}(a - b)$ is -
(A) $\pi/4$ (B) $-\pi/4$ (C) $\pi/3$ (D) $\pi/6$

MISCELLANEOUS TYPE QUESTION **ANSWER KEY** EXERCISE-3

- *Match the Column*
1. (A) → (r), (B) → (p), (C) → (s), (D) → (q) 2. (A) → (s), (B) → (r), (C) → (p), (D) → (q)
- *Assertion & Reason*
1. D 2. C 3. A 4. A 5. A
- *Comprehension Based Questions*
Comprehension # 1 : 1. D 2. A 3. B

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

1. Find the points of local maxima/minima of following functions

(a) $f(x) = 2x^3 - 21x^2 + 36x - 20$ (b) $f(x) = -(x - 1)^3(x + 1)^2$

(c) $f(x) = x\ell nx$ (d) $y = \dfrac{3x^2+4x+4}{x^2+x+1}$

2. Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3-b^2+b-1)}{(b^2+3b+2)} & , \ 0 \le x < 1 \\ 2x-3 & , \ 1 \le x \le 3 \end{cases}$. Find all possible real values of b such that f(x) has the smallest value at x = 1. **[JEE 1993]**

3. A cubic f(x) vanishes at x = –2 & has relative minimum/maximum at x = –1 and x = 1/3. If $\int_{-1}^{1} f(x)dx = \dfrac{14}{3}$, then find the cubic f(x).

4. Find the absolute maxima/minima value of following functions

(a) $f(x) = 4x - \dfrac{x^2}{2}$; $x \in \left[-2, \dfrac{9}{2}\right]$ (b) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$; $x \in [0, 3]$

(c) $f(x) = \sin x + \dfrac{1}{2}\cos 2x$; $x \in \left[0, \dfrac{\pi}{2}\right]$

5. Let x and y be two real variables such that x > 0 and xy = 1. Find the minimum value of x + y. **[JEE 1981]**

6. A rectangular sheet of poster has its area 18 m . The margin at the top & bottom are 75 cms and at the sides 50 cms. What are the dimensions of the poster if the area of the printed space is maximum ?

7. If $y = \dfrac{ax+b}{(x-1)(x-4)}$ has a turning value at (2, –1) find a & b and show that the turning value is a maximum.

8. The flower bed is to be in the shape of a circular sector of radius r & central angle θ. If the area is fixed & perimeter is minimum, find θ.

9. What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft ? Assume that one side of the rectangle lies on the base of the triangle.

10. For a given curved surface of a right circular cone when the volume is maximum, prove that the semi vertical angle is $\sin^{-1}\dfrac{1}{\sqrt{3}}$.

11. Of all the line tangent to the graph of the curve $y = \dfrac{6}{x^2+3}$, find the equations of the tangent lines of minimum and maximum slope.

12. Suppose f(x) is a function satisfying the following conditions -

(i) f(0) = 2, f(1) = 1 (ii) If f(x) has a minimum value at $x = \dfrac{5}{2}$ and

(iii) for all x $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Where a, b are some constants. Determine the constants a, b, & the function f(x). **[JEE 1998]**

13. Consider the function, $F(x) = \int_{-1}^{x}(t^2 - t)dt,\ x \in R$. Find

(a) The x and y intercept of F if they exist.

(b) Derivatives F'(x) and F''(x).

(c) The interval on which F is an increasing and the intervals on which F is decreasing.

(d) Relative maximum and minimum points.

(e) Any inflection point.

14. The function f(x) defined for all real numbers x has the following properties

$f(0) = 0$, $f(2) = 2$ and $f'(x) = k(2x - x^2)e^{-x}$ for some constant $k > 0$. Find

(a) the intervals on which f is increasing and decreasing and any local maximum or minimum values.

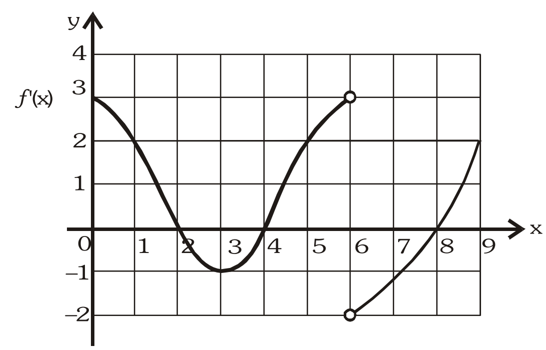
(b) the intervals on which the graph f is concave down and concave up

(c) the function f(x) and plot its graph.

15. The circle $x^2 + y^2 = 1$ cuts the x-axis at P & Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis & the line segment PQ at S. Find the maximum area of the triangle QSR.

16. Investigate for maxima & minima for the function, $f(x) = \int_{1}^{x}[2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]dt$.

17. The graph of the derivative $f\,'$ of a continuous function f is shown with $f(0) = 0$. If



(i) f is monotonic increasing in the interval $[a, b) \cup (c, d) \cup (e, f]$ and decreasing in $(p, q) \cup (r, s)$.

(ii) f has a local minima at $x = x_1$ and $x = x_2$.

(iii) f is concave up in $(\ell, m) \cup (n, t]$

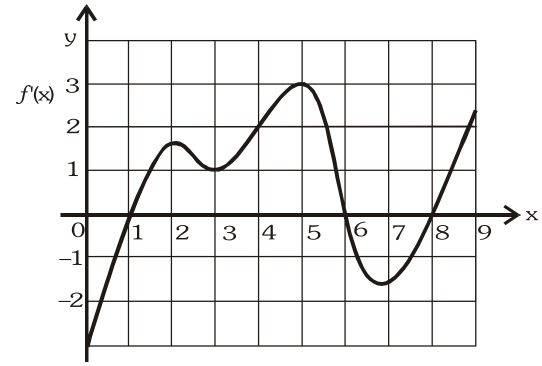
(iv) f has inflection point at $x = k$

(v) number of critical points of $y = f(x)$ is 'w'

Find the value of $(a + b + c + d + e) + (p + q + r + s) + (\ell + m + n) + (x_1 + x_2) + (k + w)$.

18. The graph of the derivative $f\,'$ of a continuous function f

is shown with $f(0) = 0$



(i) On what intervals is f increasing or decreasing ?

(ii) At what values of x does f have a local maximum or minimum ?

(iii) On what intervals is f concave upward or downward ?

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that $f(0) = 0$, sketch a graph of f.

19. A window of perimeter P(including the base of the arch) is in the form of a rectangle surmounded by a semi circle. The semi-circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass. The clear glass transmits three times as much light per square meter as the coloured glass does. What is the ratio for the sides of the rectangle so that the window transmits the maximum light ? [JEE 1991]

20. Find the points on the curve $ax^2+2bxy+ay^2 = c$; $c>b>a>0$, whose distance from the origin is minimum. [JEE 1998]

CONCEPTUAL SUBJECTIVE EXERCISE | **ANSWER KEY** | EXERCISE-4(A)

1. (a) local max. at x = 1, local min. at x = 6 (b) local max. at $x = -\frac{1}{5}$, local min. at x = –1

(c) local min. at $x = \frac{1}{e}$, No local max. (d) $y_{max} = 4$ at x = 0, $y_{min} = \frac{8}{3}$ at x = –2

2. $b \in (-2, -1) \cup [1, \infty)$ **3.** $f(x) = x^3 + x^2 - x + 2$

4. (a) max. = 8, min. = –10; (b) max. = 25, min. = –39;
(c) max. = 3/4 at x = π/6, min. = 1/2 at x = 0 & π/2

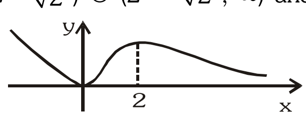
5. 2 **6.** width $2\sqrt{3}$ m, length $3\sqrt{3}$ m **7.** a = 1, b = 0

8. θ = 2 radians **9.** 6' x 18' **11.** 3x + 4y – 9 = 0 ; 3x – 4y + 9 = 0

12. $a = \frac{1}{4}; b = -\frac{5}{4}$

13. (a) (–1, 0), (0, 5/6); (b) $F'(x) = (x^2 - x)$, $F''(x) = 2x - 1$, (c) increasing $(-\infty, 0) \cup (1, \infty)$, decreasing (0, 1);
(d) (0, 5/6); (1, 2/3); (e) x = 1/2

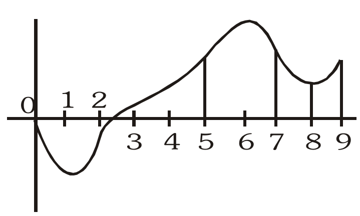
14. (a) increasing in (0, 2) and decreasing in $(-\infty, 0) \cup (2, \infty)$, local min. value = 0 and local max. value = 2
(b) concave up for $(-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$ and concave down in $(2-\sqrt{2}, 2+\sqrt{2})$
(c) $f(x) = \frac{1}{2}e^{2-x}.x^2$



15. $\frac{4}{3\sqrt{3}}$ **16.** max. at x = 1, f(1) = 0, min. at x = 7/5; f(7/5) = –108/3125 **17.** 74

18. (i) I in $(1, 6) \cup (8, 9)$ and D in $(0, 1) \cup (6, 8)$; (ii) L. Min. at x = 1 and x = 8; L. Max. x = 6
(iii) CU in $(0, 2) \cup (3, 5) \cup (7, 9)$ and CD in $(2, 3) \cup (5, 7)$; (iv) x = 2, 3, 5, 7

(v) Graph is



19. 6 + π : 6 **20.** $\left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$ & $\left(-\sqrt{\frac{c}{2(a+b)}}, -\sqrt{\frac{c}{2(a+b)}}\right)$

EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. Consider the function $f(x)=\begin{cases}\sqrt{x}\,\ell nx & \text{when} \quad x>0\\ 0 & \text{for} \quad x=0\end{cases}$

(a) Find whether f is continuous at x = 0 or not.

(b) Find the minima and maxima if they exist.

(c) Does f'(0) exist ? Find $\lim\limits_{x\to 0} f'(x)$.

(d) Find the inflection points of the graph of y = f(x).

2. Consider the function $y = f(x) = \ell n(1 + \sin x)$ with $-2\pi \le x \le 2\pi$. Find

(a) the zeroes of f(x)

(b) inflection points if any on the graph

(c) local maxima and minima of f(x)

(d) asymptotes of the graph

(e) sketch the graph of f(x) and compute the value of the definite integral $\int\limits_{-\pi/2}^{\pi/2} f(x)dx$.

3. Given two points A (– 2 , 0) & B (0 , 4) and a line y = x. Find the co-ordinates of a point M on this line so that the perimeter of the Δ AMB is least.

4. Find the set of values of m for the cubic $x^3 - \frac{3}{2}x^2 + \frac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

5. If the sum of the lengths of the hypotenuse and another side of a right angled triangle is given, show that the area of the triangle is a maximum when the angle between these sides is $\pi/3$.

6. Prove that among all triangles with a given perimeter, the equilateral triangle has the maximum area.

7. The value of 'a' for which $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$ have a positive point of maximum lies in the interval $(a_1, a_2)\cup(a_3, a_4)$. Find the value of $a_2 + 11a_3 + 70a_4$.

8. Use calculus to prove the inequality, $\sin x \ge 2x/\pi$ in $0 \le x \le \pi/2$.

You may use the inequality to prove that, $\cos x \le 1 - x^2/\pi$ in $0 \le x \le \pi/2$

9. Find the maximum perimeter of a triangle on a given base 'a' and having the given vertical angle α.

10. What is the radius of the smallest circular disk large enough to cover every acute isosceles triangle of a given perimeter L ?

11. A swimmer S is in the sea at a distance d km from the closest point A on a straight shore. The house of the swimmer is on the shore at a distance L km from A. He can swim at a speed of u km/hr and walk at a speed of v km/hr (v > u). At what point on the shore should he land so that he reaches his house in the shortest possible time ? **[JEE 1983]**

12. Let $f(x) = \sin^3 x + \lambda \sin^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lie in order that f(x) has exactly one minimum and exactly one maximum. **[JEE 1985]**

13. Let $A(p^2, -p)$, $B(q^2, q)$, $C(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram AFDE is drawn with vertices D, E and F on the line segments BC, CA and AB respectively. Using calculus, show that maximum area of such a parallelogram is $\frac{1}{4}(p + q)(q + r)(p - r)$ **[JEE 1986]**

14. Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point (0, –2).

[JEE 1987]

15. Determine the points of maxima and minima of the function $f(x) = \frac{1}{8}\ell nx - bx + x^2$, $x > 0$ where $b \geq 0$ is a constant.

[JEE 1996]

16. Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c and d denote the length of the sides of the quadrilateral, prove that $2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$.

[JEE 1997]

17. Find the co-ordinates of all the points P on the ellipse $(x^2/a^2)+(y^2/b^2) = 1$ for which the area of the triangle PON is maximum, where O denotes the origin and N the foot of the perpendicular from O to the tangent at P.

[JEE 1999]

BRAIN STORMING SUBJECTIVE EXERCISE **ANSWER KEY** EXERCISE-4(B)

1. (a) f is continuous at x = 0; (b) e^{-2}; (c) does not exist, does not exist; (d) pt. of inflection x = 1

2. (a) $x = -2\pi, -\pi, 0, \pi, 2\pi$, (b) no inflection point, (c) maxima at $x = \frac{\pi}{2}$ and $-\frac{3\pi}{2}$ and no minima, (d) $x = \frac{3\pi}{2}$ and $x = -\frac{\pi}{2}$, (e) $-\pi\ell n2$

3. (0 , 0)

4. $m \in \left(\frac{1}{32}, \frac{1}{16}\right)$

7. 320

9. $P_{max} = a\left(1 + \text{cosec}\frac{\alpha}{2}\right)$

10. L/4

11. $\frac{ud}{\sqrt{v^2 - u^2}}$

12. $\lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$

14. (0, 2)

15. Min. at $x = \frac{1}{4}(b + \sqrt{b^2 - 1})$, max. at $x = \frac{1}{4}(b - \sqrt{b^2 - 1})$

17. $\pm\frac{a^2}{\sqrt{a^2 + b^2}}$, $\pm\frac{b^2}{\sqrt{a^2 + b^2}}$

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals- [AIEEE-2003]

(1) 1/2 (2) 3 (3) 1 (4) 2

2. The real number x when added to its inverse gives the minimum value of the sum at x equal to- [AIEEE-2003]

(1) –2 (2) 2 (3) 1 (4) –1

3. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ then the difference between the maximum and minimum values of u^2 is given by- [AIEEE-2004]

(1) $2(a^2 + b^2)$ (2) $2\sqrt{a^2+b^2}$ (3) $(a + b)^2$ (4) $(a - b)^2$

4. The function $f(x) = \frac{x}{2}+\frac{2}{x}$ has a local minimum at- [AIEEE-2006]

(1) x = –2 (2) x = 0 (3) x = 1 (4) x = 2

5. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is- [AIEEE-2006]

(1) $\sqrt{\frac{x^3}{8}}$ (2) $\frac{1}{2}x^2$ (3) πx^2 (4) $\frac{3}{2}x^2$

6. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of (p + q) is- [AIEEE-2007]

(1) 2 (2) $\frac{1}{2}$ (3) $\frac{1}{\sqrt{2}}$ (4) $\sqrt{2}$

7. Suppose the cubic $x^3 - px + q$ has three real roots where $p > 0$ and $q > 0$. Then which of the following holds ? [AIEEE-2008]

(1) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$ (2) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(3) The cubic has minima at both $-\sqrt{\frac{p}{3}}$ & $\sqrt{\frac{p}{3}}$ (4) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ & $-\sqrt{\frac{p}{3}}$

8. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that x=0 is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval [–1, 1] :-

(1) P(– 1) is the minimum but P(1) is not the maximum of P. [AIEEE-2009]

(2) Neither P(– 1) is the minimum nor P(1) is the maximum of P

(3) P(– 1) is the minimum and P(1) is the maximum of P

(4) P(– 1) is not minimum but P(1) is the maximum of P

9. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is :- [AIEEE-2009]

(1) $\frac{3\sqrt{2}}{5}$ (2) $\frac{\sqrt{3}}{4}$ (3) $\frac{3\sqrt{2}}{8}$ (4) $\frac{2\sqrt{3}}{8}$

10. Let f : R → R be defined by $f(x) = \begin{cases} k-2x, & \text{if } x \le -1 \\ 2x+3, & \text{if } x > -1 \end{cases}$ [AIEEE-2010]

If f has a local minimum at x = – 1, then a possible value of k is :

(1) 1 (2) 0 (3) $-\frac{1}{2}$ (4) –1

11. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\sin t\,dt$. Then f has :- **[AIEEE-2011]**

(1) local minimum at π and local maximum at 2π

(2) local maximum at π and local minimum at 2π

(3) local maximum at π and 2π

(4) local minimum at π and 2π

12. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is :- **[AIEEE-2011]**

(1) $\frac{8}{3\sqrt{2}}$ (2) $\frac{4}{\sqrt{3}}$ (3) $\frac{\sqrt{3}}{4}$ (4) $\frac{3\sqrt{2}}{8}$

13. Let f be a function defined by $f(x) = \begin{cases} \frac{\tan x}{x} & , \quad x \neq 0 \\ 1 \quad , & \quad x = 0 \end{cases}$

Statement - 1: x = 0 is point of minima of f.

Statement - 2: f'(0) = 0. **[AIEEE-2011]**

(1) Statement-1 is false, statement-2 is true.

(2) Statement-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.

(3) Statement-1 is true, statement-2 is true; Statement-2 is not a correct explanation for statement-1.

(4) Statement-1 is true, statement-2 is false.

14. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[AIEEE-2012]**

(1) 9/2 (2) 9/7 (3) 7/9 (4) 2/9

15. Let a, b $\in$ R be such that the function f given by $f(x) = \ln|x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement–1 : f has local maximum at $x = -1$ and at $x = 2$.

Statement–2 : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$. **[AIEEE-2012]**

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

16. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]

[JEE-MAIN 2013]

(1) lies between 1 and 2. (2) lies between 2 and 3.

(3) lies between –1 and 0 (4) does not exist

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

1. 4 **2.** 3 **3.** 4 **4.** 4 **5.** 2 **6.** 4 **7.** 1 **8.** 4 **9.** 3 **10.** 4 **11.** 2
12. 4 **13.** 3 **14.** 4 **15.** 3 **16.** 4

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. Let $f(x) = \begin{cases} |x| & \text{for } 0 < |x| \le 2 \\ 1 & \text{for } x = 0 \end{cases}$. Then at x = 0, 'f' has - **[JEE 2000 Screening, 1M out of 35]**

(A) a local maximum (B) no local maximum (C) a local minimum (D) no extremum.

2. Let $f(x) = (1+b^2)x^2+2bx+1$ and let m (b) the minimum value of f(x). As b varies, the range of m(b) is - **[JEE 01 Screening, 1M, out of 35]**

(A) [0, 1] (B) $\left(0,\frac{1}{2}\right]$ (C) $\left[\frac{1}{2},1\right]$ (D) $(0, 1]$

3. A straight line L with negative slope passes through the point (8,2) and cuts the positive coordinates axes at points P and Q. Find the absolute minimum value of OP + OQ, as L varies, where O is the origin. **[JEE 02 Mains, 5M out of 60]**

4. The minimum value of $f(x) = x^2 + 2bx + 2c^2$ is more than the maximum value of $g(x) = -x^2 - 2cx + b^2$, x being real, for - **[JEE 03, (Scr.) 3M out of 84]**

(A) $|c| > |b|\sqrt{2}$ (B) $0 < c < b\sqrt{2}$ (C) $b\sqrt{2} < c < 0$ (D) no values of b and c

5. For every $\alpha \in \left(0,\frac{\pi}{2}\right)$, the value of $\sqrt{x^2+x}+\frac{\tan^2\alpha}{\sqrt{x^2+x}}, x > 0$ is greater than or equal to - **[JEE 03, (Scr.) 3M out of 84]**

(A) 2 (B) $2\tan\alpha$ (C) $\frac{5}{2}$ (D) $\sec\alpha$

6. For a circle $x^2+y^2 = r^2$, find the value of 'r' for which the area enclosed by the tangents drawn from the point P (6, 8) to the circle and the chord of contact is maximum. **[JEE 03, Mains 2M out of 60]**

7. If p(x) be a polynomial of degree 3 satisfying p(–1) = 10, p(1) = –6 and p(x) has maximum at x = –1and p'(x) has minima at x = 1. Find the distance between the local maximum and local minimum of the curve. **[JEE 05, Mains 4M out of 60]**

8. f(x) is cubic polynomial which has local maximum at x = –1. If f(2) = 18, f(1) = –1 and f'(x) has local minima at x = 0 then - **[JEE 06, (5M, –1M) out of 184]**

(A) the distance between (–1, 2) and (a, f(a)), where x = a is the point of local minima is $2\sqrt{5}$

(B) f(x) is increasing for $x \in (1, 2\sqrt{5}]$

(C) f(x) has local minima at x = 1

(D) the value of f(0) = 5

9. $f(x) = \begin{cases} x & , \quad 0 \le x \le 1 \\ 2-e^{x-1} & , \quad 1 < x \le 2 \\ x-e & , \quad 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$, $x \in [0, 3]$ then g(x) has

(A) local maxima at x = 1 + *l*n2 and local minima at x = e **[JEE 06, (3M, –1M) out of 184]**

(B) local maxima at x = 1 and local minima at x = 2

(C) no local maxima

(D) no local minima

10. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3} & -1 < x < 2 \end{cases}$ is :-

(A) 0 (B) 1 (C) 2 (D) 3

[JEE 08, (3M, –1M)]

11. Match the column : **[JEE 08, 6M]**

	Column I		Column II
(A)	The minimum value of $\frac{x^2+2x+4}{x+2}$ is	(p)	0
(B)	Let A and B be 3 3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and (A+B)(A–B) = (A–B)(A+B). If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible value of k are	(q)	1
(C)	Let a = $\log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than	(r)	2
(D)	If sinθ = cosφ, then the possible values of $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ are	(s)	3

Paragraph for Question 12 to 14

Consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, \ 0 < a < 2.$$

12. Which of the following is true ? **[JEE 2008, (4M, –1M)]**

(A) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

(B) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$

(C) $f'(1) f'(-1) = (2 - a)^2$

(D) $f'(1) f'(-1) = -(2 + a)^2$

13. Which of the following is true ? **[JEE 2008, (4M, –1M)]**

(A) $f(x)$ is decreasing on (–1, 1) and has a local minimum at x = 1

(B) $f(x)$ is increasing on (–1, 1) and has a local maximum at x = 1

(C) $f(x)$ is increasing on (–1, 1) but has neither a local maximum nor a local minimum at x = 1

(D) $f(x)$ is decreasing on (–1, 1) but has neither a local maximum nor a local minimum at x = 1

14. Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2} dt$ **[JEE 2008, (4M, –1M)]**

Which of the following is true ?

(A) g'(x) is positive on (–∞, 0) and negative on (0, ∞)

(B) g'(x) is negative on (–∞, 0) and positive on (0, ∞)

(C) g'(x) charges sign on both (–∞, 0) and (0, ∞)

(D) g'(x) does not change sign on (–∞, ∞)

15. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is **[JEE 2009, 4M, –1M]**

16. Let p(x) be a polynomial of degree 4 having extremum at x=1, 2 and $\lim_{x\to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2$, then the value of p(2) is **[JEE 2009, 4M, –1M]**

17. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then **[JEE 10, 3M, –1M]**

(A) a = b and c ≠ b (B) a = c and a ≠ b (C) a ≠ b and c ≠ b (D) a = b = c

18. Let f be a function defined on **R** (the set of all real numbers) such that $f'(x) = 2010\,(x-2009)\,(x-2010)^2\,(x-2011)^3\,(x-2012)^4$, for all $x \in \mathbf{R}$.
If g is a function defined on **R** with values in the interval $(0,\infty)$ such that
$f(x) = \ell n\,(g(x))$, for all $x \in \mathbf{R}$,
then the number of points in **R** at which g has a local maximum is - **[JEE 10, 3M]**

19. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[JEE 2012, 4M]**

20. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, then p'(0) is **[JEE 2012, 4M]**

21. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio of 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are **[JEE 2013, 4M,–1M]**

(A) 24 (B) 32 (C) 45 (C) 60

22. The function $f(x) = 2|x| + |x+2| - ||x+2| - 2|x||$ has a local minimum or a local maximum at x = **[JEE 2013, 3M,–1M]**

(A) –2 (B) $\frac{-2}{3}$ (C) 2 (D) $\frac{2}{3}$

PREVIOUS YEARS QUESTIONS	ANSWER KEY	EXERCISE-5[B]

1. A **2.** D **3.** 18 **4.** A **5.** B **6.** 5 **7.** Distance between (–1, 10) and (3, –22) is $4\sqrt{65}$ units

8. B, C **9.** A **10.** C **11.** (A) – (r), (B) – (q, s), (C) – (r, s), (D) – (p, r) **12.** A **13.** A

14. B **15.** 7 **16.** 0 **17.** D **18.** 1 **19.** 5 **20.** 9 **21.** A,C **22.** A,B