CHAPTER

04

Circle

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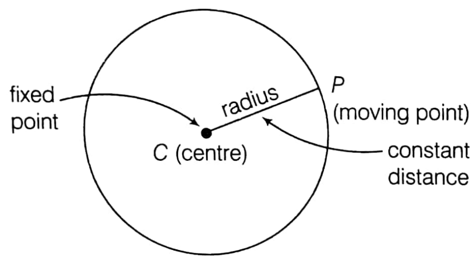
Session 1

Definition, Equation of Circle in Different Forms, Locus of the Mid-point of the Chords of the Circle that Subtends an Angle of 2θ at its Centre

Definition

A circle is the locus of a point which moves in a plane, so that its distance from a *fixed point* in the plane is always *constant.*

The fixed point is called the *centre* of the circle and the constant distance is called its *radius.*



i.e. $\quad CP =$ constant distance = Radius

Equation of a Circle : The curve traced by the moving point is called its *circumference.* i.e. the equation of any circle is satisfied by co-ordinates of all points on its circumference.

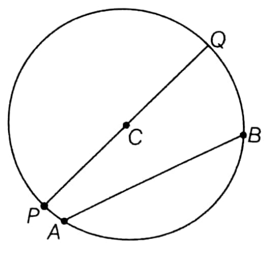
OR

The equation of the circle is meant the equation of the circumference.

OR

It is the set of all points lying on the circumference of the circle.

Chord and Diameter : The line joining any two points on the circumference is called a chord. If any chord passing through its centre is called its diameter.



$AB =$ Chord, $\quad PQ =$ Diameter

where, C is centre of the circle.

Equation of Circle in Different Forms

1. Centre-radius form

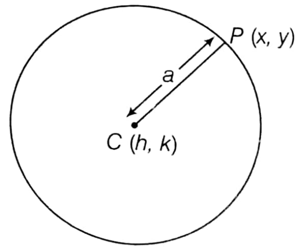
Let a be the *radius* and $C(h, k)$ be the *centre* of any circle.

If $P(x, y)$ be any point on the circumference.

Then, $\quad CP = a \Rightarrow (CP)^2 = a^2$

$\Rightarrow \quad (x-h)^2 + (y-k)^2 = a^2$

This equation is known as the central form of the equation of a circle.



Remark

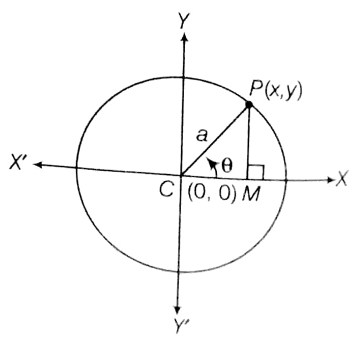
When, $C(h, k) = C(0, 0)$, then equation of circle becomes $x^2 + y^2 = a^2$ which is known as **standard form** of the circle.

2. Parametric form

If the radius of a circle whose centre is at $C(0, 0)$ makes an angle θ with the positive direction of X-axis, then θ is called the parameter.

Let $\quad CP = a$

$\therefore \quad CM = x, PM = y \quad \Rightarrow x = a\cos\theta, y = a\sin\theta$



Hence, $(a\cos\theta, a\sin\theta)$ or 'θ' are the parametric coordinates of the circle $x^2+y^2=a^2$ and $x=a\cos\theta$ and $y=a\sin\theta$ are called parametric equations of the circle $x^2+y^2=a^2$ with parameters a and θ. $(0\le\theta<2\pi)$.

Remarks

1. The parametric coordinates of any point on the circle $(x-h)^2+(y-k)^2=a^2$ are given by $(h+a\cos\theta, k+a\sin\theta)$ $(0\le\theta<2\pi)$ and parametric equations of the circle $(x-h)^2+(y-k)^2=a^2$ are $x=h+a\cos\theta,\ y=k+a\sin\theta$.
2. Equation of the chord of the circle $x^2+y^2=a^2$ joining $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ is
$$x\cos\left(\frac{\alpha+\beta}{2}\right)+y\sin\left(\frac{\alpha+\beta}{2}\right)=a\cos\left(\frac{\alpha-\beta}{2}\right).$$

General form The equation of the circle with centre (h,k) and radius a is $(x-h)^2+(y-k)^2=a^2$

or $$x^2+y^2-2hx-2ky+h^2+k^2-a^2=0 \quad \text{...(i)}$$

which is of the form

$$\boxed{x^2+y^2+2gx+2fy+c=0} \quad \text{...(ii)}$$

This is known as the **general equation** of a circle comparing Eqs. (i) and (ii), we get

$$h=-g, k=-f \quad \text{and} \quad a=\sqrt{(g^2+f^2-c)}$$

$\therefore$ Coordinates of the centre are $(-g,-f)$ and

$$\text{Radius}=\sqrt{(g^2+f^2-c)}\ (g^2+f^2\ge c)$$

Remarks

1. **Rule for finding the centre and radius of a circle**
   (i) Make the coefficients of x^2 and y^2 equal to 1 and right hand side equal to zero.
   (ii) Then, coordinates of centre will be (α,β),
   where, $\alpha=-\frac{1}{2}$ (coefficient of x) and $\beta=-\frac{1}{2}$ (coefficient of y)
   (iii) Radius $=\sqrt{\alpha^2+\beta^2-(\text{constant term})}$
2. **Conditions for a circle** A general equation of second degree
$$ax^2+2hxy+by^2+2gx+2fy+c=0$$
   in x, y represent a circle, if
   (i) coefficient of x^2 = coefficient of y^2
   i.e. $\boxed{a=b}$
   (ii) coefficient of xy is zero
   i.e. $\boxed{h=0}$
3. **Nature of the circle** Radius of the circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{(g^2+f^2-c)}$
   Now, the following cases are possible :
   (i) If $g^2+f^2-c>0$, then the radius of circle will be real. Hence in this case, real circle is possible.
   (ii) If $g^2+f^2-c=0$, then the radius of circle will be real. Hence in this case, circle is called a point circle.
   (iii) If $g^2+f^2-c<0$, then the radius of circle will be imaginary number. Hence in this case, circle is called a virtual circle or imaginary circle.
4. **Concentric circle** Two circles having the same centre $C(h,k)$ but different radii r_1 and r_2 respectively are called concentric circles. Thus, the circles $(x-h)^2+(y-k)^2=r_1^2$ and $(x-h)^2+(y-k)^2=r_2^2, r_1\ne r_2$ are concentric circles. Therefore, the equations of concentric circles differ only in constant terms.

Example 1. Find the centre and radius of the circle $2x^2+2y^2=3x-5y+7$

By partial diff.

Sol. The given equation of circle is
$$2x^2+2y^2=3x-5y+7$$
or $$x^2+y^2-\frac{3}{2}x+\frac{5}{2}y-\frac{7}{2}=0$$
If centre is (α,β), then
$$\alpha=-\frac{1}{2}\left(-\frac{3}{2}\right)=\frac{3}{4}$$
and $$\beta=-\frac{1}{2}\left(\frac{5}{2}\right)=-\frac{5}{4}$$
$\therefore$ Centre of circle is (α,β) i.e. $\left(\frac{3}{4},-\frac{5}{4}\right)$

and radius of the circle
$$=\sqrt{\alpha^2+\beta^2-(\text{constant term})}$$
$$=\sqrt{\frac{9}{16}+\frac{25}{16}+\frac{7}{2}}=\sqrt{\frac{9+25+56}{16}}=\frac{3\sqrt{10}}{4}$$

Example 2. Prove that the radii of the circles $x^2+y^2=1, x^2+y^2-2x-6y=6$ and $x^2+y^2-4x-12y=9$ are in AP.

Sol. Given circles are $x^2+y^2=1$ ...(i)
$$x^2+y^2-2x-6y-6=0 \quad \text{...(ii)}$$
and $$x^2+y^2-4x-12y-9=0 \quad \text{...(iii)}$$
Let r_1, r_2 and r_3 be the radii of the circles Eqs. (i), (ii) and (iii), respectively.

Then, $r_1=1$
$$r_2=\sqrt{(-1)^2+(-3)^2+6}=4$$
and $$r_3=\sqrt{(-2)^2+(-6)^2+9}=7$$
Clearly, $r_2-r_1=4-1=3=r_3-r_2$

Hence, r_1, r_2, r_3 are in AP.

Example 3. Find the equation of the circle whose centre is the point of intersection of the lines $2x-3y+4=0$ and $3x+4y-5=0$ and passes through the origin.

Sol. The point of intersection of the lines $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ is $\left(-\frac{1}{17}, \frac{22}{17}\right)$.

Therefore, the centre of the circle is at $\left(-\frac{1}{17}, \frac{22}{17}\right)$.

Since, the origin lies on the circle, its distance from the centre of the circle is radius of the circle, therefore,

$$r = \sqrt{\left(-\frac{1}{17} - 0\right)^2 + \left(\frac{22}{17} - 0\right)^2} = \sqrt{\frac{485}{289}}$$

$\therefore$ The equation of the circle becomes

$$\left(x + \frac{1}{17}\right)^2 + \left(y - \frac{22}{17}\right)^2 = \frac{485}{289}$$

or $$17(x^2 + y^2) + 2x - 44y = 0$$

Aliter : $\because$ Point of intersection of the lines $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ is $\left(-\frac{1}{17}, \frac{22}{17}\right)$.

Therefore, the centre of the circle is at $\left(-\frac{1}{17}, \frac{22}{17}\right)$

Let required circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Here, $-g = -\frac{1}{17}, -f = \frac{22}{17}, c = 0$

[$\because$ Circle passes through origin]

From Eq. (i), $$x^2 + y^2 + \frac{2x}{17} - \frac{44}{17}y = 0$$

or $$17(x^2 + y^2) + 2x - 44y = 0$$

I Example 4. Find the equation of the circle concentric with the circle $x^2 + y^2 - 8x + 6y - 5 = 0$ and passing through the point $(-2, -7)$.

Sol. The given equation of circle is

$$x^2 + y^2 - 8x + 6y - 5 = 0$$

Therefore, the centre of the circle is at $(4, -3)$. Since, the required circle is concentric with this circle, therefore, the centre of the required circle is also at $(4, -3)$. Since, the point $(-2, -7)$ lies on the circle, the distance of the centre from this point is the radius of the circle. Therefore, we get

$$r = \sqrt{(4+2)^2 + (-3+7)^2} = \sqrt{52}$$

Hence, the equation of the circle becomes

$$(x - 4)^2 + (y + 3)^2 = 52$$

or $$x^2 + y^2 - 8x + 6y - 27 = 0$$

Aliter : Equation of concentric circle is

$$x^2 + y^2 - 8x + 6y + \lambda = 0 \quad \text{...(i)}$$

which pass through $(-2, -7)$, then

Use aliter

$$4 + 49 + 16 - 42 + \lambda = 0$$

$\therefore$ $$\lambda = -27$$

From Eq. (i), required circle is

$$x^2 + y^2 - 8x + 6y - 27 = 0$$

I Example 5. A circle has radius 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through $(7, 3)$.

Sol. Let the centre of the circle be (h, k). Since, the centre lies on $y = x - 1$, we get

$$k = h - 1 \quad \text{...(i)}$$

Since, the circle passes through the point $(7, 3)$, therefore the distance of the centre from this point is the radius r of the circle. We have,

$$r = \sqrt{(h-7)^2 + (k-3)^2}$$

or $$3 = \sqrt{(h-7)^2 + (h-1-3)^2} \quad \text{[from Eq. (i)]}$$

$\Rightarrow$ $$9 = (h-7)^2 + (h-4)^2$$

$\Rightarrow$ $$h^2 - 11h + 28 = 0$$

or $$(h-7)(h-4) = 0$$

or $$h = 7 \text{ and } h = 4$$

For $h = 7$, we get $k = 6$ from Eq. (i)

and for $h = 4$, we get $k = 3$, from Eq. (i).

Hence, there are two circles which satisfy the given conditions. They are

$$(x-7)^2 + (y-6)^2 = 9$$

or $$x^2 + y^2 - 14x - 12y + 76 = 0$$

and $$(x-4)^2 + (y-3)^2 = 9$$

or $$x^2 + y^2 - 8x - 6y + 16 = 0$$

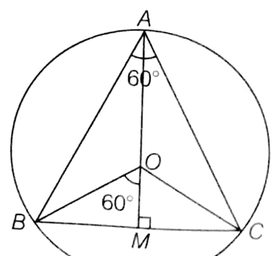
I Example 6. Find the area of an equilateral triangle inscribed in the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Sol. Given circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Let O be the centre and ABC be an equilateral triangle inscribed in the circle Eq. (i).



$$O \equiv (-g, -f)$$

and $$OA = OB = OC = \sqrt{g^2 + f^2 - c} \quad \text{...(ii)}$$

In ΔBOM, $\sin 60° = \dfrac{BM}{OB}$

$$\Rightarrow \quad BM = OB \sin 60° = (OB)\frac{\sqrt{3}}{2}$$

$$\therefore \quad BC = 2BM = \sqrt{3}\,(OB) \quad \text{...(iii)}$$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{\sqrt{3}}{4}(BC)^2$$

$$= \frac{\sqrt{3}}{4} 3\,(OB)^2 \quad \text{[from Eq. (iii)]}$$

$$= \frac{3\sqrt{3}}{4}(g^2 + f^2 - c) \text{ sq units.}$$

| Example 7. Find the parametric form of the equation of the circle

$$x^2 + y^2 + px + py = 0.$$

Sol. Equation of the circle can be re-written in the form

$$\left(x + \frac{p}{2}\right)^2 + \left(y + \frac{p}{2}\right)^2 = \frac{p^2}{2}$$

Therefore, the parametric form of the equation of the given circle is

$$x = -\frac{p}{2} + \frac{p}{\sqrt{2}}\cos\theta = \frac{p}{2}(-1 + \sqrt{2}\cos\theta)$$

and $$y = -\frac{p}{2} + \frac{p}{\sqrt{2}}\sin\theta = \frac{p}{2}(-1 + \sqrt{2}\sin\theta)$$

where, $0 \le \theta < 2\pi$.

| Example 8. If the parametric of form of a circle is given by

(a) $x = -4 + 5\cos\theta$ and $y = -3 + 5\sin\theta$

(b) $x = a\cos\alpha + b\sin\alpha$ and $y = a\sin\alpha - b\cos\alpha$

find its cartesian form.

Sol. (a) The given equations are

$$x = -4 + 5\cos\theta$$

and $$y = -3 + 5\sin\theta$$

or $$(x + 4) = 5\cos\theta \quad \text{...(i)}$$

and $$(y + 3) = 5\sin\theta \quad \text{...(ii)}$$

Squaring and adding Eqs. (i) and (ii), then

$$(x + 4)^2 + (y + 3)^2 = 5^2$$

or $$(x + 4)^2 + (y + 3)^2 = 25$$

(b) The given equations are

$$x = a\cos\alpha + b\sin\alpha \quad \text{...(iii)}$$

$$y = a\sin\alpha - b\cos\alpha \quad \text{...(iv)}$$

Squaring and adding Eqs. (iii) and (iv), then

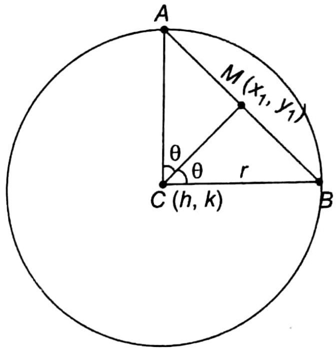
$$x^2 + y^2 = (a\cos\alpha + b\sin\alpha)^2 + (a\sin\alpha - b\cos\alpha)^2$$

$$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$$

Locus of the Mid-point of the Chords of the Circle that Subtends an Angle of 2θ at its Centre

Let mid-point $M(x_1, y_1)$ and centre, radius of circle are (h, k), r respectively, then

$$\cos\theta = \frac{CM}{r} = \frac{\sqrt{(x_1 - h)^2 + (y_1 - k)^2}}{r}$$



$\therefore$ Required locus is $\dfrac{(x - h)^2 + (y - k)^2 - r^2}{r^2} = -\sin^2\theta$

Remembering Method :

First make coefficient of x^2 = coefficient of y^2 = 1

and RHS of circle is zero, then $\boxed{\dfrac{\text{LHS of circle}}{(\text{radius})^2} = -\sin^2\theta}$

| Example 9. Find the locus of mid-points of the chords of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $\dfrac{2\pi}{3}$ at its centre.

Sol. Here, $2\theta = \dfrac{2\pi}{3} \Rightarrow \theta = \dfrac{\pi}{3}$

Equation of circle can be written as

$$x^2 + y^2 - 3x + y + \frac{1}{4} = 0$$

$\therefore$ Required locus is

$$\frac{x^2 + y^2 - 3x + y + \frac{1}{4}}{\left(\sqrt{\frac{9}{4} + \frac{1}{4} - \frac{1}{4}}\right)^2} = -\sin^2\left(\frac{\pi}{3}\right) = -\frac{3}{4}$$

$$\Rightarrow \quad x^2 + y^2 - 3x + y + \frac{1}{4} = -\frac{27}{16}$$

or $$16(x^2 + y^2) - 48x + 16y + 31 = 0$$

Exercise for Session 1

1. If $x^2 + y^2 - 2x + 2ay + a + 3 = 0$ represents a real circle with non-zero radius, then most appropriate is

(a) $a \in (-\infty, -1)$
(b) $a \in (-1, 2)$
(c) $a \in (2, \infty)$
(d) $a \in (-\infty, -1) \cup (2, \infty)$

2. If the equation $ax^2 + (2-b)xy + 3y^2 - 6bx + 30y + 6b = 0$ represents a circle, then $a^2 + b^2$ is

(a) 5
(b) 13
(c) 25
(d) 41

3. The equation of the circle passing through (4, 5) having the centre at (2, 2) is

(a) $x^2 + y^2 + 4x + 4y - 5 = 0$
(b) $x^2 + y^2 - 4x - 4y - 5 = 0$
(c) $x^2 + y^2 - 4x - 13 = 0$
(d) $x^2 + y^2 - 4x - 4y + 5 = 0$

4. Equation of the diameter of the circle is given by $x^2 + y^2 - 12x + 4y + 6 = 0$ is given by

(a) $x + y = 0$
(b) $x + 3y = 0$
(c) $x = y$
(d) $3x + 2y = 0$

5. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the diameter of the circle is

(a) $\frac{3}{2}$
(b) 3
(c) $\frac{5}{2}$
(d) 5

6. Area of a circle in which a chord of length $\sqrt{2}$ makes an angle $\frac{\pi}{2}$ at the centre is

(a) $\frac{\pi}{4}$
(b) $\frac{\pi}{2}$
(c) π
(d) 2π

7. The lines $2x - 3y - 5 = 0$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq units, then the equation of the circle is :

(a) $x^2 + y^2 + 2x - 2y - 62 = 0$
(b) $x^2 + y^2 + 2x - 2y - 47 = 0$
(c) $x^2 + y^2 - 2x + 2y - 62 = 0$
(d) $x^2 + y^2 - 2x + 2y - 47 = 0$

8. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is

(a) $x^2 + y^2 - 2x + 2y - 23 = 0$
(b) $x^2 + y^2 - 2x - 2y - 23 = 0$
(c) $x^2 + y^2 + 2x + 2y - 23 = 0$
(d) $x^2 + y^2 + 2x - 2y - 23 = 0$

9. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (−4,3) respectively, then $\angle QPR$ is equal to

(a) $\frac{\pi}{2}$
(b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$
(d) $\frac{\pi}{6}$

10. If a circle is concentric with the circle $x^2 + y^2 - 4x - 6y + 9 = 0$ and passes through the point (−4, −5), then its equation is

(a) $x^2 + y^2 + 4x + 6y - 87 = 0$
(b) $x^2 + y^2 - 4x + 6y + 87 = 0$
(c) $x^2 + y^2 - 4x - 6y - 87 = 0$
(d) $x^2 + y^2 + 4x + 6y + 87 = 0$

11. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then, the locus of the centroid of the ΔPAB as P moves on the circle is

(a) a parabola
(b) a circle
(c) an ellipse
(d) a pair of straight lines

12. Let PQ and RS be tangents extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals

(a) $\sqrt{PQ \cdot RS}$
(b) $\dfrac{PQ + RS}{2}$
(c) $\dfrac{2PQ.RS}{PQ + RS}$
(d) $\sqrt{\dfrac{(PQ)^2 + (RS)^2}{2}}$

13. Find the centre and radius of the circle $5x^2 + 5y^2 + 4x - 8y = 16$.

14. Prove that the centres of the circles $x^2 + y^2 = 1, x^2 + y^2 + 6x - 2y - 1 = 0$ and $x^2 + y^2 - 12x + 4y = 1$ are collinear.

15. Find the equation of the circle whose centre is $(1, 2)$ and which passes through the point of intersection of $3x + y = 14$ and $2x + 5y = 18$.

16. Find the equation of the circle passing through the centre of the circle $x^2 + y^2 - 4x - 6y = 8$ and being concentric with the circle $x^2 + y^2 - 2x - 8y = 5$.

17. Prove that the locus of the centre of the circle $\frac{1}{2}(x^2 + y^2) + x\cos\theta + y\sin\theta - 4 = 0$ is $x^2 + y^2 = 1$.

18. Find the equation of the following curves in cartesian form. If the curve is a circle, then find its centre and radius

$x = -1 + 2\cos\alpha, y = 3 + 2\sin\alpha \qquad (0 \le \alpha < 2\pi)$

Session 2

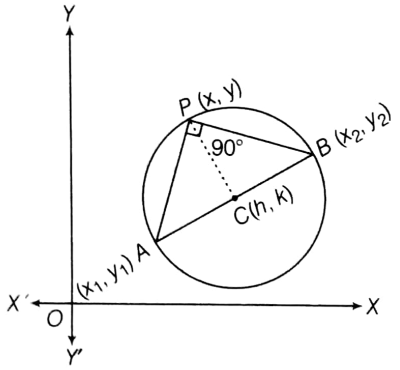
Diametric Form of a Circle, Equation of Circle Passing Through Three Non-Collinear Points

Diametric Form of a Circle

Theorem : The equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

Proof : Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter and let $P(x, y)$ be any point on the circle



Now, Slope of $AP = \dfrac{y - y_1}{x - x_1}$

and Slope of $BP = \dfrac{y - y_2}{x - x_2}$

Since, $\angle APB = 90°$

$\therefore$ Slope of $AP \times$ Slope of $BP = -1$

$$\Rightarrow \quad \frac{(y - y_1)}{(x - x_1)} \times \frac{(y - y_2)}{(x - x_2)} = -1$$

$$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Remark

The diameteric form of a circle can also be written as

$$x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0$$

or $\quad x^2 + y^2 - x$ (sum of abscissae) $- y$ (sum of ordinates) + product of abscissae + product of ordinates $= 0$

| Example 10. Find the equation of the circle the end points of whose diameter are the centres of the circles $x^2 + y^2 + 6x - 14y = 1$ and $x^2 + y^2 - 4x + 10y = 2$.

Sol. The centres of the given circles

$$x^2 + y^2 + 6x - 14y - 1 = 0$$

and $x^2 + y^2 - 4x + 10y - 2 = 0$ are $(-3, 7)$ and $(2, -5)$, respectively.

According to the question, the points $(-3, 7)$ and $(2, -5)$ are the extremities of the diameter of required circle.

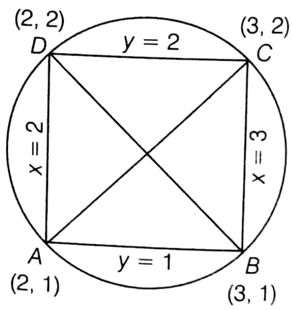
Hence, equation of circle is

$$(x + 3)(x - 2) + (y - 7)(y + 5) = 0$$

$$\Rightarrow \quad x^2 + y^2 + x - 2y - 41 = 0$$

| Example 11. The sides of a square are $x = 2, x = 3, y = 1$ and $y = 2$. Find the equation of the circle drawn on the diagonals of the square as its diameter.

Sol. Let $ABCD$ be a square and equation of its sides AB, BC, CD and DA are $y = 1$, $x = 3$, $y = 2$, and $x = 2$, respectively.



Then, $A \equiv (2, 1)$, $B \equiv (3, 1)$, $C \equiv (3, 2)$ and $D \equiv (2, 2)$

Since, diagonals of squares are the diameters of the circle, then equation of circle is

$$(x - 2)(x - 3) + (y - 1)(y - 2) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 5x - 3y + 8 = 0 \text{ (If } AC \text{ as diameter).}$$

| Example 12. The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter.

Sol. Given equations are

$$x^2 + 2ax - b^2 = 0 \qquad \text{...(i)}$$

and $$x^2 + 2px - q^2 = 0 \qquad \text{...(ii)}$$

Let the roots of the Eq. (i) be α and β and those of Eq. (ii) be γ and δ. Then,

$$\left.\begin{aligned}\alpha + \beta &= -2a\\ \alpha\beta &= -b^2\end{aligned}\right\} \text{ and } \left.\begin{aligned}\gamma + \delta &= -2p\\ \gamma\delta &= -q^2\end{aligned}\right\}$$

Let $A \equiv (\alpha, \gamma)$ and $B \equiv (\beta, \delta)$.

Now, equation of circle whose diameter is AB will be

$$(x-\alpha)(x-\beta) + (y-\gamma)(y-\delta) = 0$$

$$\Rightarrow \quad x^2 + y^2 - (\alpha+\beta)x - (\gamma+\delta)y + \alpha\beta + \gamma\delta = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$$

and $$\text{radius} = \sqrt{(a^2 + p^2 + b^2 + q^2)}$$

Equation of Circle Passing Through Three Non-Collinear Points

Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

If three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ lie on the circle Eq. (i), their coordinates must satisfy its equation. Hence, solving equations

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0, \qquad \text{...(ii)}$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0, \qquad \text{...(iii)}$$

and $$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0, \qquad \text{...(iv)}$$

g, f, c are obtained from Eqs. (ii), (iii) and (iv). Then, to find the circle Eq. (i).

Aliter : Eliminate g, f, c from Eqs. (i), (ii), (iii) and (iv) with the help of determinant

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

which is the required equation of circle

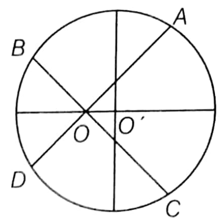
Applying $R_1 \to R_1 - R_4, R_2 \to R_2 - R_4$ and $R_3 \to R_3 - R_4$ then, we get

$$\begin{vmatrix} x^2 + y^2 - x_3^2 - y_3^2 & x - x_3 & y - y_3 & 0 \\ x_1^2 + y_1^2 - x_3^2 - y_3^2 & x_1 - x_3 & y_1 - y_3 & 0 \\ x_2^2 + y_2^2 - x_3^2 - y_3^2 & x_2 - x_3 & y_2 - y_3 & 0 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

i.e. $$\begin{vmatrix} x^2 + y^2 - x_3^2 - y_3^2 & x - x_3 & y - y_3 \\ x_1^2 + y_1^2 - x_3^2 - y_3^2 & x_1 - x_3 & y_1 - y_3 \\ x_2^2 + y_2^2 - x_3^2 - y_3^2 & x_2 - x_3 & y_2 - y_3 \end{vmatrix} = 0$$

Remarks

1. **Cyclic quadrilateral** If all four vertices of a quadrilateral lie on a circle, then the quadrilateral is called a cyclic quadrilateral. The four vertices are said to be concyclic.
2. **Concyclic points** If A, B, C, D are concyclic, then
   $$\boxed{OA \cdot OD = OB \cdot OC}$$
   where, O' be the centre of the circle.



Example 13. Find the equation of the circle which passes through the points $(4, 1), (6, 5)$ and has its centre on the line $4x + y = 16$.

Sol. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

Since, the centre of Eq. (i) i.e. $(-g, -f)$ lies on $4x + y = 16$

then, $$-4g - f = 16$$

or $$4g + f + 16 = 0 \qquad \text{...(ii)}$$

Since, the points $(4, 1)$ and $(6, 5)$ lie on circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we get the equations

$$16 + 1 + 8g + 2f + c = 0$$

or $$17 + 8g + 2f + c = 0 \qquad \text{...(iii)}$$

and $$36 + 25 + 12g + 10f + c = 0$$

or $$61 + 12g + 10f + c = 0 \qquad \text{...(iv)}$$

Subtracting Eq. (iii) from Eq. (iv), then

$$44 + 4g + 8f = 0 \qquad \text{...(v)}$$

Solving Eqs. (ii) and (v), we get

$$f = -4 \quad \text{and} \quad g = -3$$

Now, from Eq. (iii), $17 - 24 - 8 + c = 0$

$$\Rightarrow \quad c = 15$$

Hence, the equation of circle becomes

$$x^2 + y^2 - 6x - 8y + 15 = 0$$

Example 14. Find the equation of the circle passing through the three non-collinear points $(1, 1), (2, -1)$ and $(3, 2)$.

Sol. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

Since, the three given points lie on circle Eq. (i), we get

$$1 + 1 + 2g + 2f + c = 0$$

or $\quad 2g + 2f + c + 2 = 0 \quad$ ...(ii)

$\Rightarrow \quad 4 + 1 + 4g - 2f + c = 0$

or $\quad 4g - 2f + c + 5 = 0 \quad$ ...(iii)

$\Rightarrow \quad 9 + 4 + 6g + 4f + c = 0$

or $\quad 6g + 4f + c + 13 = 0 \quad$ ...(iv)

Subtracting Eq. (ii) from Eq. (iii) and subtracting Eq. (iii) from Eq. (iv), then

$$2g - 4f + 3 = 0 \quad \text{...(v)}$$

and $$2g + 6f + 8 = 0 \quad \text{...(vi)}$$

Solving Eq. (v) and Eq. (vi), we get

$$f = -\frac{1}{2} \text{ and } g = -\frac{5}{2}$$

Now, from Eq. (ii), $-5 - 1 + c + 2 = 0$

$\therefore \quad c = 4$

Hence, from Eq. (i), equation of circle is

$$x^2 + y^2 - 5x - y + 4 = 0$$

Aliter I Equation of circle passing through three points $(1, 1), (2, -1)$ and $(3, 2)$ is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 1^2 + 1^2 & 1 & 1 & 1 \\ 2^2 + (-1)^2 & 2 & -1 & 1 \\ 3^2 + 2^2 & 3 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 2 & 1 & 1 & 1 \\ 5 & 2 & -1 & 1 \\ 13 & 3 & 2 & 1 \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2, R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_2$, then

$$\Rightarrow \begin{vmatrix} x^2 + y^2 - 2 & x - 1 & y - 1 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & -2 & 0 \\ 11 & 2 & 1 & 0 \end{vmatrix} = 0$$

Expand with respect to fourth column, then

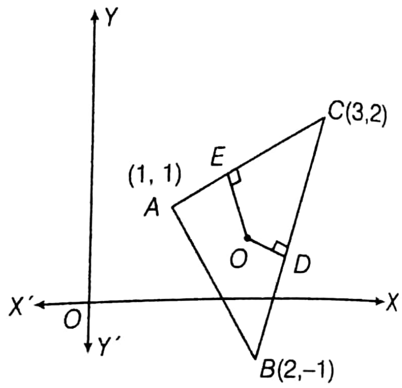
$$\begin{vmatrix} x^2 + y^2 - 2 & x - 1 & y - 1 \\ 3 & 1 & -2 \\ 11 & 2 & 1 \end{vmatrix} = 0$$

Expand with respect to first now, then

For circumcentre $(x^2 + y^2 - 2)(5) - (x - 1)(25) + (y - 1)(-5) = 0$

or $$x^2 + y^2 - 5x - y + 4 = 0$$

Aliter II The centre of the circumcircle is the point of intersection of the right bisectors of the sides of the triangle and the radius is the distance of the circumcentre from any of the vertices of the triangle.



Let D and E are the mid-points of BC and CA, then

$$D \equiv \left(\frac{5}{2}, \frac{1}{2}\right) \text{ and } E \equiv \left(2, \frac{3}{2}\right)$$

Slope of $BC = \dfrac{2 - (-1)}{3 - 2} = 3$

$\therefore \quad$ Slope of $OD = -\dfrac{1}{3}$

$\therefore$ Equation of OD, $y - \dfrac{1}{2} = -\dfrac{1}{3}\left(x - \dfrac{5}{2}\right)$

$\Rightarrow \quad 6y - 3 = -2x + 5$

$\therefore \quad 2x + 6y - 8 = 0 \quad$ ...(i)

or $\quad x + 3y - 4 = 0$

and $\quad$ Slope of $CA = \dfrac{1 - 2}{1 - 3} = \dfrac{1}{2}$

$\therefore \quad$ Slope of $OE = -2$

$\therefore$ Equation of OE,

$$y - \frac{3}{2} = -2(x - 2)$$

$\Rightarrow \quad 2y - 3 = -4x + 8$

$\Rightarrow \quad 4x + 2y - 11 = 0 \quad$ ...(ii)

Solving Eq. (i) and Eq. (ii), we get $x = \dfrac{5}{2}$ and $y = \dfrac{1}{2}$

$\therefore$ Circumcentre is $\left(\dfrac{5}{2}, \dfrac{1}{2}\right)$ and radius

$$OC = \sqrt{\left(3 - \frac{5}{2}\right)^2 + \left(2 - \frac{1}{2}\right)^2} = \sqrt{\frac{5}{2}}$$

$\therefore$ Equation of circle is

$$(x - 5/2)^2 + (y - 1/2)^2 = 5/2$$

$\Rightarrow \quad x^2 + y^2 - 5x - y + 4 = 0$

Example 15. Show that the four points $(1, 0), (2, -7), (8, 1)$ and $(9, -6)$ are concyclic.

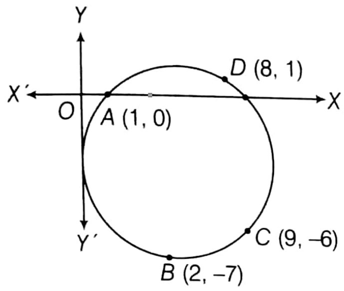
Sol. Since, the given four points are concyclic, we are to show that they lie on a circle. Let the general equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

has three parameters, it is sufficient to obtain the equation of the circle passing through any three of these points. For concyclic, the fourth point should lie on this circle.

Let three points $A(1, 0)$, $B(2, -7)$ and $D(8, 1)$ lie on Eq. (i), then

$$1 + 0 + 2g + 0 + c = 0 \quad \text{or} \quad 1 + 2g + c = 0 \qquad \text{...(ii)}$$



$$(2)^2 + (-7)^2 + 2g(2) + 2f(-7) + c = 0$$

or $$53 + 4g - 14f + c = 0 \qquad \text{...(iii)}$$

and $$(8)^2 + (1)^2 + 2g(8) + 2f(1) + c = 0$$

$\Rightarrow$ $$65 + 16g + 2f + c = 0 \qquad \text{...(iv)}$$

Now, subtracting Eq. (ii) from Eq. (iii), we get

$$52 + 2g - 14f = 0$$

or $$26 + g - 7f = 0 \qquad \text{...(v)}$$

and subtracting Eq. (iii) from Eq. (iv), we get

$$12 + 12g + 16f = 0$$

$\Rightarrow$ $$3 + 3g + 4f = 0 \qquad \text{...(vi)}$$

Solving Eq. (v) and Eq. (vi), we get

$$g = -5 \quad \text{and} \quad f = 3$$

From Eq. (ii), $1 - 10 + c = 0$

$\therefore$ $$c = 9$$

Therefore, equation of circle passing through these points is

$$x^2 + y^2 - 10x + 6y + 9 = 0$$

Substituting the fourth point in the equation of this circle, we get

$$(9)^2 + (-6)^2 - 10(9) + 6(-6) + 9 = 0$$

Hence, the point $C(9, -6)$ lies on the circle, that is, the four points are concyclic.

Exercise for Session 2

1. If the line $x + 2\lambda y + 7 = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$, then the value of λ is

(a) 1 (b) 3
(c) 5 (d) 7

2. If one end of a diameter of the circle $2x^2 + 2y^2 - 4x - 8y + 2 = 0$ is $(-1, 2)$, then the other end of the diameter is

(a) (2, 1) (b) (3, 2)
(c) (4, 3) (d) (5, 4)

3. If a circle passes through the points (0,0), $(a, 0)$ and $(0, b)$, then centre of the circle is

(a) (a, b) (b) $\left(\frac{a}{2}, \frac{b}{2}\right)$
(c) $\left(\frac{a}{2}, \frac{b}{4}\right)$ (d) $\left(\frac{a}{4}, \frac{b}{2}\right)$

4. A circle passes through the points $(-1, 3)$ and $(5, 11)$ and its radius is 5. Then, its centre is

(a) $(-5, 0)$ (b) $(-5, 7)$
(c) $(2, 7)$ (d) $(5, 0)$

5. The radius of the circle, having centre at $(2, 1)$ whose one of the chord is a diameter of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is

(a) 3 (b) 2
(c) 1 (d) $\sqrt{3}$

6. The centre of the circle inscribed in the square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$ is

(a) (4, 7) (b) (7, 4)
(c) (9, 4) (d) (4, 9)

7. $ABCD$ is a square whose side is a. The equation of the circle circumscribing the square, taking AB and AD as the axes of reference is

(a) $x^2 + y^2 + ax + ay = 0$
(b) $x^2 + y^2 - ax + ay = 0$
(c) $x^2 + y^2 - ax - ay = 0$
(d) $x^2 + y^2 + ax - ay = 0$

8. The locus of the centre of the circle for which one end of the diameter is (3, 3) while the other end lies on the line $x + y = 4$ is

(a) $x + y = 3$
(b) $x + y = 5$
(c) $x + y = 7$
(d) $x + y = 9$

9. The equation of the circle which passes through (1, 0) and (0, 1) and has its radius as small as possible is

(a) $x^2 + y^2 + x + y = 0$
(b) $x^2 + y^2 - x + y = 0$
(c) $x^2 + y^2 + x - y = 0$
(d) $x^2 + y^2 - x - y = 0$

10. If the points $(2, 0), (0, 1), (4, 5)$ and $(0, c)$ are concyclic, then the value of c is

(a) 1
(b) -1
(c) $\frac{14}{3}$
(d) $-\frac{14}{3}$

11. The point on a circle nearest to the point $P(2,1)$ is at a distance of 4 units and farthest point is (6, 5), then the centre of the circle is

(a) $(3+\sqrt{2}, 2+\sqrt{2})$
(b) $(2+\sqrt{2}, 3+\sqrt{2})$
(c) $(4+\sqrt{2}, 3+\sqrt{2})$
(d) $(3+\sqrt{2}, 4+\sqrt{2})$

12. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle on AB as a diameter is

(a) $x^2 + y^2 - x - y = 0$
(b) $x^2 + y^2 - x + y = 0$
(c) $x^2 + y^2 + x + y = 0$
(d) $x^2 + y^2 + x - y = 0$

13. Find the equation of the circle, the end points of whose diameter are $(2, -3)$ and $(-2, 4)$. Find the centre and radius.

14. If (4, 1) be an extremity of a diameter of the circle $x^2 + y^2 - 2x + 6y - 15 = 0$, find the coordinates of the other extremity of the diameter.

15. Find the equation of the circle drawn on the diagonal of the rectangle as its diameter whose sides are $x = 4$, $x = -2$, $y = 5$ and $y = -2$.

16. Find the equation of the circle which passes through the points $(1, 1), (2, 2)$ and whose radius is 1.

17. Find the equation of the circle which passes through the points $(3, 4), (3, -6)$ and $(1, 2)$.

Session 3

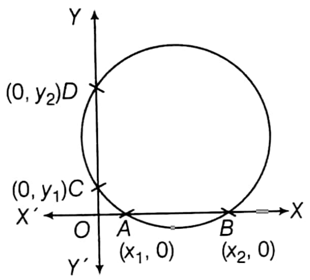
Intercepts Made on the Axes by a Circle, Different Forms of the Equations of a Circle, Position of a Point with Respect to Circle, Maximum and Minimum Distance of a Point from the Circle

Intercepts Made on the Axes by a Circle

Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

Length of intercepts on X-axis and Y-axis are

$|AB| = |x_2 - x_1|$ and $|CD| = |y_2 - y_1|$ respectively.



The circle intersects the X-axis, when $y = 0$

then $$x^2 + 2gx + c = 0$$

Since, the circle intersects the X-axis at $A(x_1, 0)$ and $B(x_2, 0)$

then, $$x_1 + x_2 = -2g, x_1x_2 = c$$

$\therefore$ $$|AB| = |x_2 - x_1| = \sqrt{(x_2 + x_1)^2 - 4x_1x_2}$$
$$= 2\sqrt{(g^2 - c)}$$

and the circle intersects the Y-axis, when $x = 0$, then

$$y^2 + 2fy + c = 0$$

Since, the circle intersects the Y-axis at $C(0, y_1)$ and $D(0, y_2)$

then, $$y_1 + y_2 = -2f, y_1y_2 = c$$

$\therefore$ $$|CD| = |y_2 - y_1| = \sqrt{(y_2 + y_1)^2 - 4y_2y_1}$$
$$= 2\sqrt{(f^2 - c)}$$

Remarks

1. Intercepts are always positive.
2. If circle touches X-axis, then $|AB| = 0$

   $\therefore$ $\boxed{c = g^2}$

   and if circle touches Y-axis, then $|CD| = 0$

   $\therefore$ $c = f^2$
3. If circle touches both axes, then $|AB| = 0 = |CD|$

   $\therefore$ $c = g^2 = f^2$

Example 16. Find the equation of the circle whose diameter is the line joining the points $(-4, 3)$ and $(12, -1)$. Find also the intercept made by it on Y-axis.

Sol. Equation of circle having $(-4, 3)$ and $(12, -1)$ as the ends of a diameter is

$$(x + 4)(x - 12) + (y - 3)(y + 1) = 0$$

$\Rightarrow$ $$x^2 + y^2 - 8x - 2y - 51 = 0 \quad ...(i)$$

Comparing Eq. (i) with standard equation of circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

then, $$g = -4, f = -1, c = -51$$

$\therefore$ Intercept on Y-axis $= 2\sqrt{(f^2 - c)} = 2\sqrt{(1 + 51)} = 4\sqrt{13}$.

Different Forms of the Equations of a Circle

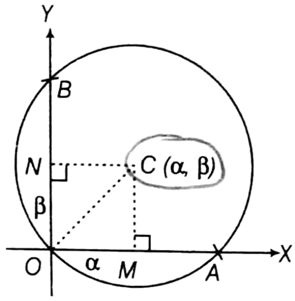
(i) When the circle passes through the origin $(0, 0)$ and has intercepts 2α and 2β on the X-axis and Y-axis, respectively

Here, $OA = 2\alpha, OB = 2\beta$

then, $OM = \alpha$ and $ON = \beta$

Centre of the circle is $C(\alpha, \beta)$ and radius

$$OC = \sqrt{(\alpha^2 + \beta^2)}$$



then, equation of circle is

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2$$

or $$x^2+y^2-2\alpha x-2\beta y=0$$

Remark

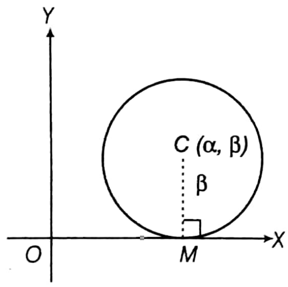
If a circle is passing through origin, then constant term is absent i.e. $x^2+y^2+2gx+2fy=0$

(ii) When the circle touches *X*-axis

Let (α, β) be the centre of the circle, then radius $=|\beta|$

$\therefore$ Equation of circle is

$$(x-\alpha)^2+(y-\beta)^2=\beta^2$$



$$\Rightarrow \quad x^2+y^2-2\alpha x-2\beta y+\alpha^2=0$$

Remark

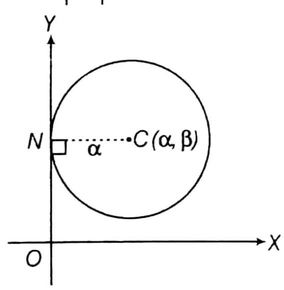
If the circle $x^2+y^2+2gx+2fy+c=0$ touches the X-axis, then

$|-f|=\sqrt{g^2+f^2-c}$ or $c=g^2$

(iii) When the circle touches *Y*-axis

Let (α, β) be the centre of the circle, then

radius $=|\alpha|$



$\therefore$ Equation of circle is

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2$$

$$\Rightarrow \quad x^2+y^2-2\alpha x-2\beta y+\beta^2=0$$

Remark

If the circle $x^2+y^2+2gx+2fy+c=0$ touches the Y-axis, then

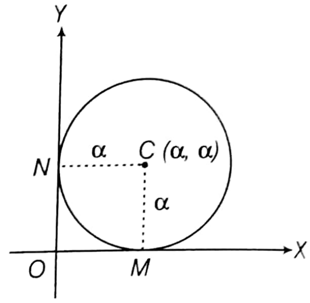
$$|-g|=\sqrt{g^2+f^2-c}$$

or $$c=f^2$$

(iv) When the circle touches both axes

Here, $|OM|=|ON|$

Since, length of tangents are equal from any point on circle.



$\therefore$ Let centre is (α, α) also radius $=\alpha$

$\therefore$ Equation of circle is $(x-\alpha)^2+(y-\alpha)^2=\alpha^2$

$$\Rightarrow \quad x^2+y^2-2\alpha x-2\alpha y+\alpha^2=0$$

Remarks

1. If the circle $x^2+y^2+2gx+2fy+c=0$ touches both the axes then $|-g|=|-f|=\sqrt{g^2+f^2-c}$

   $\therefore \quad c=g^2=f^2$

   $\therefore \quad \boxed{g=f=\pm\sqrt{c}}$

   $\therefore$ Equation of circle is

   $$x^2+y^2\pm 2\sqrt{c}\,x\pm 2\sqrt{c}\,y+c=0$$

   $$\Rightarrow \quad (x\pm\sqrt{c})^2+(y\pm\sqrt{c})^2=c^2$$

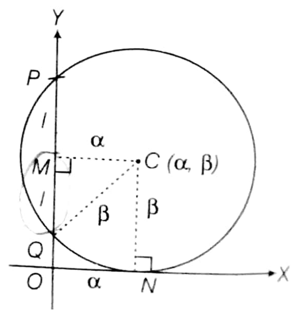
2. If $\boxed{\alpha>0}$ then centres for I, II, III and IV quadrants are $(\alpha, \alpha), (-\alpha, \alpha), (-\alpha, -\alpha)$ and $(\alpha, -\alpha)$, respectively.

   Then, equation of circles in these quadrants are

   $(x-\alpha)^2+(y-\alpha)^2=\alpha^2$, $(x+\alpha)^2+(y-\alpha)^2=\alpha^2$, $(x+\alpha)^2+(y+\alpha)^2=\alpha^2$ and $(x-\alpha)^2+(y+\alpha)^2=\alpha^2$, respectively.

(v) When the circle touches *X*-axis and cut-off intercepts on *Y*-axis of length 2*l*

Let centre be (α, β)



$\therefore$ radius $= \beta$

$CQ = CN = \beta$

In ΔCMQ, $\beta^2 = \alpha^2 + l^2$, $\boxed{\alpha = \sqrt{(\beta^2 - l^2)}}$ (for I quadrant)

$\therefore$ Equation of circle is

$$[x - \sqrt{(\beta^2 - l^2)}]^2 + (y - \beta)^2 = \beta^2$$

Remark

$\because$ Length of intercepts on Y-axis of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $2l = 2\sqrt{(f^2 - c)}$ ☆

i.e. $l^2 = f^2 - c$

and also circle touches X-axis

then, $c = g^2$

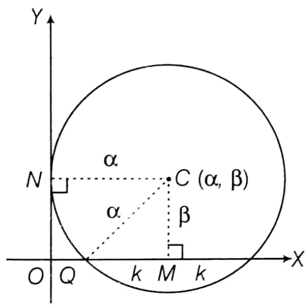
Remember

$\therefore$ $l^2 = f^2 - g^2$ or $l^2 = (-f)^2 - (-g)^2$ ☆

$\therefore$ Locus of centre is $y^2 - x^2 = l^2$ (rectangular hyperbola)

(vi) When the circle touches Y-axis and cut-off intercept on X-axis of length $2k$

Let centre be (α, β)



$\therefore$ radius $= \alpha$

$CN = CQ = \alpha$

In ΔCMQ, $\alpha^2 = \beta^2 + k^2$

$\beta = \sqrt{(\alpha^2 - k^2)}$ (for I quadrant)

$\therefore$ Equation of circle is

$$(x - \alpha)^2 + (y - \sqrt{\alpha^2 - k^2})^2 = \alpha^2$$

Remarks

$\because$ Length of intercept on X-axis of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $2k = 2\sqrt{(g^2 - c)}$

i.e. $k^2 = g^2 - c$

and also circle touches Y-axis

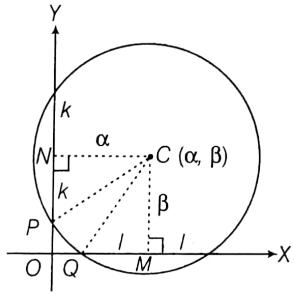
then, $c = f^2$

$\therefore$ $k^2 = g^2 - f^2 = (-g)^2 - (-f)^2$

$\therefore$ Locus of centre is $x^2 - y^2 = k^2$ (rectangular hyperbola)

(vii) When the circle cut-off intercepts on X-axis and Y-axis of lengths $2l$ and $2k$ and not passing through origin

Let centre be (α, β)



$\therefore$ radius $= CP = CQ = \lambda$ (say)

$(CP)^2 = (CQ)^2 = \lambda^2$

$\alpha^2 + k^2 = \beta^2 + l^2 = \lambda^2$

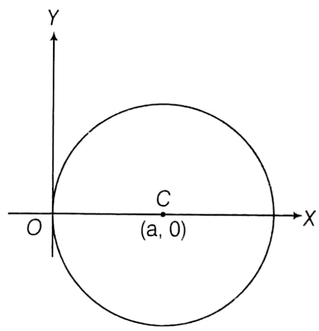
$\therefore$ $\alpha = \sqrt{\lambda^2 - k^2}$ and $\beta = \sqrt{\lambda^2 - l^2}$

$\therefore$ Equation of circle is (for I quadrant)

$$(x - \sqrt{\lambda^2 - k^2})^2 + (y - \sqrt{\lambda^2 - l^2})^2 = \lambda^2$$

(viii) When the circle passes through the origin and centre lies on X-axis

Let centre of circle be $C(a, 0)$



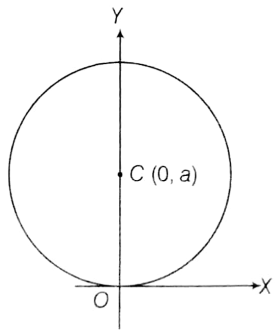
$\therefore$ radius $= a$

$\therefore$ Equation of circle is

$$(x - a)^2 + (y - 0)^2 = a^2 \text{ or } x^2 + y^2 - 2ax = 0$$

(ix) When the circle passes through origin and centre lies on Y-axis

Let centre of circle be $C(0, a)$



$\therefore$ radius $= a$

$\therefore$ Equation of circle is

$$(x-0)^2+(y-a)^2=a^2$$

or $$x^2+y^2-2ay=0$$

Example 17. Find the equation of the circle which touches the axis of y at a distance of 4 units from the origin and cuts the intercept of 6 units from the axis of x.

Sol. $\because$ $CM = NO = 4$

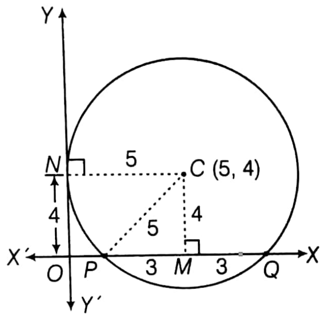
In ΔPCM, $(PC)^2 = (3)^2 + (4)^2$

$\therefore$ $PC = 5$

radius of circle $= 5$

$\therefore$ $NC = 5$

Centre of circle is $(5, 4)$.



$\therefore$ Equation of circle, if centre in I quadrant

$$(x-5)^2+(y-4)^2=25$$

If centre in II, III and IV quadrant, then equations are

$$(x+5)^2+(y-4)^2=25,$$

$$(x+5)^2+(y+4)^2=25$$

and $$(x-5)^2+(y+4)^2=25$$

Hence, there are 4 circles which satisfy the given conditions. They are

$$(x\pm 5)^2+(y\pm 4)^2=25$$

or $$x^2+y^2\pm 10x\pm 8y+16=0$$

Use aliter

Aliter : Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{...(i)}$$

Since, the circle touches the Y-axis

$\therefore$ $c = f^2$

or $f = \pm\sqrt{c}$

Also given the circle makes an intercept of 6 units along X-axis. Therefore,

$$2\sqrt{g^2-c}=6$$

or $$g^2-c=9$$

or $$g=\pm\sqrt{(c+9)}$$

From Eq. (i), the equation of circle can be written as

$$x^2+y^2\pm 2\sqrt{(c+9)}\,x\pm 2\sqrt{c}y+c=0$$

The circle touches the Y-axis

$\therefore$ $x = 0$

$\therefore$ $y^2\pm 2\sqrt{c}y+c=0$

or $(y\pm\sqrt{c})^2=0$

$\therefore$ $y\pm\sqrt{c}=0$

$y=\mp\sqrt{c}$

Since, the circle touches the Y-axis at a distance of 4 units from the origin, we have

$$y=\mp\sqrt{c}=4$$

or $c = 16$

therefore, $f=\pm\sqrt{c}=\pm 4$

and $g=\pm\sqrt{c+9}=\pm\sqrt{16+9}=\pm 5$

Hence, there are 4 circles which satisfy the given conditions. They are

$$x^2+y^2\pm 10x\pm 8y+16=0$$

Example 18. Find the equation of the circle which passes through the origin and makes intercepts of length a and b on the X and Y axes, respectively.

Sol. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{...(i)}$$

Since, the circle passes through the origin, we get $c = 0$ and given the intercepts on X and Y axes are a and b

then, $2\sqrt{(g^2-c)}=a$

or $2\sqrt{(g^2-0)}=a$

$\therefore$ $g=\pm a/2$

and $2\sqrt{(f^2-c)}=b$

or $2\sqrt{(f^2-0)}=b$

$\therefore$ $f=\pm b/2$

Hence, the equation of circle from Eq. (i) becomes

$$x^2+y^2\pm ax\pm by=0$$

Example 19. Find the equation of the circle which touches the axes and whose centre lies on the line $x-2y=3$.

Sol. Since, the circle touches both the axes, let the radius of the circle by a, then

Case I If centre (a, a) but given centre lies on

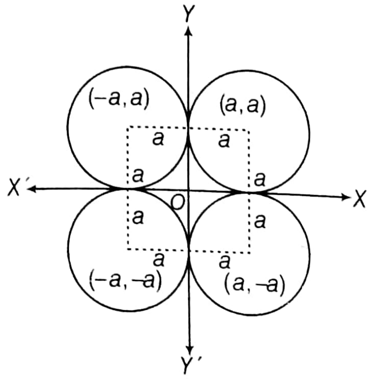
$$x-2y=3$$

$\therefore$ $a-2a=3$

$\therefore$ $a=-3$

$\therefore$ Centre $= (-3, -3)$

and radius $= |-3| = 3$



$\therefore$ Equation of circle is

$$(x+3)^2 + (y+3)^2 = 3^2$$

and $\quad x^2 + y^2 + 6x + 6y + 9 = 0$

Case II If centre $(-a, a)$ but centre lies on $x - 2y = 3$

$\therefore \quad -a - 2a = 3$

$\therefore \quad a = -1$

then, centre $= (1, -1)$ and radius $= |-1| = 1$

$\therefore$ Equation of circle is $\quad (x-1)^2 + (y+1)^2 = 1$

or $\quad x^2 + y^2 - 2x + 2y + 1 = 0$

Case III If the centre $= (-a, -a)$

but centre lies on $x - 2y = 3$

$\therefore \quad -a + 2a = 3$

$\therefore \quad a = 3$

then centre $(-3, -3)$ and radius $= |3| = 3$

$\therefore$ Equation of circle is

$$(x+3)^2 + (y+3)^2 = 3^2$$

or $\quad x^2 + y^2 + 6x + 6y + 9 = 0$

Case IV If centre $= (a, -a)$ but centre lies on $x - 2y = 3$

or $\quad a + 2a = 3$

$\therefore \quad a = 1$

then centre $= (1, -1)$ and radius $= 1$

$\therefore$ Equation of circle is

$$(x-1)^2 + (y+1)^2 = 1$$

or $\quad x^2 + y^2 - 2x + 2y + 1 = 0$

Aliter I : Since, the circle touches both the axes, therefore its centre will be $(a, \pm a)$ and radius will be $|a|$, where a is positive or negative number.

Case I If centre $= (a, a)$

Since, centre lies on $x - 2y = 3$

$\therefore \quad a - 2a = 3$

$a = -3$

$\therefore$ Centre of circle is $(-3, -3)$ and radius $= |-3| = 3$.

Hence, equation of circle will be

$$(x+3)^2 + (y+3)^2 = 3^2$$

or $\quad x^2 + y^2 + 6x + 6y + 9 = 0$

Case II If centre $=(a, -a)$

Since, centre lies on $x - 2y = 3$

$\therefore \quad a + 2a = 3$

$\therefore \quad a = 1$

$\therefore$ Centre of circle is $(1, -1)$ and radius $= |1| = 1$

Hence, equation of circle will be

$$(x-1)^2 + (y+1)^2 = 1$$

or $\quad x^2 + y^2 - 2x + 2y + 1 = 0$

Aliter II : Let the equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

centre $= (-g, -f)$

Since, centre $(-g, -f)$ lies on $x - 2y = 3$

or $\quad -g + 2f = 3 \qquad$ ...(ii)

Since, circle touches both axes

$\therefore \quad g^2 = f^2 = c$ or $g = \pm f$

if $g = f$, then from Eq. (ii), $-f + 2f = 3$

$\therefore \quad f = 3$ and $g = 3$

but $\quad c = f^2 = g^2 = 9$

$\therefore$ Equation of circle from Eq. (i) is

$$x^2 + y^2 + 6x + 6y + 9 = 0$$

and if $g = -f$, then from Eq. (ii)

$f + 2f = 3$

$\therefore \quad f = 1$ and $g = -1$

but $\quad c = g^2 = f^2 = 1$

$\therefore$ Equation of circle from Eq. (i) is

$$x^2 + y^2 - 2x + 2y + 1 = 0$$

Aliter III : Since, centre of circle lies on $x - 2y = 3$, also since circle touches the axes, therefore, its centre will lie on the line $y = x$ or $y = -x$

Case I When the centre lies on the line $y = x$

but $\quad x - 2y = 3$

or $\quad x - 2x = 3$

$\therefore \quad x = -3 = y$

Hence, the centre $= (-3, -3)$ and radius $= |-3| = 3$

Therefore, the equation of circle in this case will be

$$(x+3)^2 + (y+3)^2 = 3^2$$

or $\quad x^2 + y^2 + 6x + 6y + 9 = 0$

Case II When the centre lies on the line $y = -x$

but $\quad x - 2y = 3$

or $\quad x + 2x = 3$

$\therefore \quad x = 1$ then $y = -1$

$\therefore$ Centre of circle $(1, -1)$ and radius is $|1| = 1$

Hence, equation of circle will be

$$(x-1)^2 + (y+1)^2 = 1$$

or $\quad x^2 + y^2 - 2x + 2y + 1 = 0$

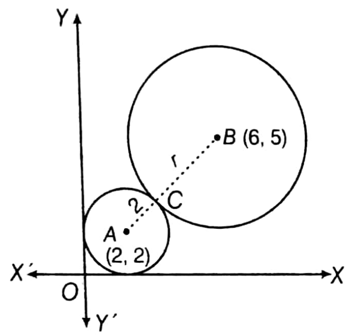
Example 20. A circle of radius 2 lies in the first quadrant and touches both the axes of coordinates. Find the equation of the circle with centre at (6, 5) and touching the above circle externally.

Sol. Given, $AC = 2$ units

and $A \equiv (2, 2), B \equiv (6, 5)$

then $AB = \sqrt{(2-6)^2 + (2-5)^2}$

$= \sqrt{16+9} = 5$



Since $AC + CB = AB$

$\therefore$ $2 + CB = 5$

$\therefore$ $CB = 3$

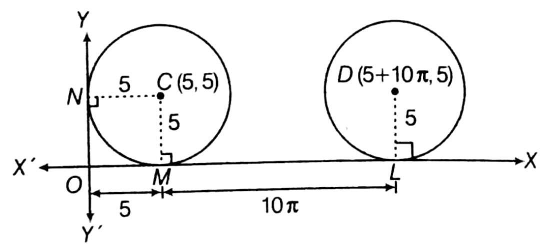
Hence, equation of required circle with centre at (6, 5) and radius 3 is

$$(x-6)^2 + (y-5)^2 = 3^2$$

or $$x^2 + y^2 - 12x - 10y + 52 = 0$$

Example 21. A circle of radius 5 units touches the coordinate axes in first quadrant. If the circle makes one complete roll on X-axis along the positive direction of X-axis, find its equation in the new position.

Sol. Let C be the centre of the circle in its initial position and D be its centre in the new position.



Since, the circle touches the coordinates axes in first quadrant and the radius of circle be 5 units.

$\therefore$ Centre of circle is (5, 5)

Moving length of circle = circumference of the circle

$= 2\pi r = 2\pi(5) = 10\pi$

Now, centre of circle in new position is $(5 + 10\pi, 5)$ and radius is 5 units, therefore, its equation will be

$$(x - 5 - 10\pi)^2 + (y-5)^2 = 5^2$$

or $$x^2 + y^2 - 10(1+2\pi)x - 10y + 100\pi^2 + 100\pi + 25 = 0$$

Position of a Point with Respect to Circle

Theorem : A point (x_1, y_1) lies outside, on or inside a circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

according as $\boxed{S_1 >, =, \text{or} < 0}$

where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

Proof : Let $P(x_1, y_1)$ be the given point and let C be the centre of the circle

Then, $C \equiv (-g, -f)$

$\therefore$ $CP = \sqrt{(x_1+g)^2 + (y_1+f)^2}$

If r be the radius of the circle, then

$$r = \sqrt{(g^2 + f^2 - c)}$$

The point P lies outside, on or inside the circle according as

$\boxed{CP >, =, \text{ or } < r}$

$\Rightarrow$ $(CP)^2 >, =, \text{ or } < r^2$

$\Rightarrow$ $(x_1+g)^2 + (y_1+f)^2 >, =, \text{ or } < g^2 + f^2 - c$

$\Rightarrow$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, \text{ or } < 0$

$\Rightarrow$ $S_1 >, =, \text{ or } < 0$

where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

Example 22. Discuss the position of the points (1, 2) and (6, 0) with respect to the circle

$$x^2 + y^2 - 4x + 2y - 11 = 0.$$

Sol. Let $S \equiv x^2 + y^2 - 4x + 2y - 11 = 0$ for the point (1, 2)

$S_1 = 1^2 + 2^2 - 4 \cdot 1 + 2 \cdot 2 - 11 = -6$

$\therefore$ $S_1 < 0$

and for the point (6, 0)

$S_2 = 6^2 + 0 - 4 \cdot 6 + 2 \cdot 0 - 11$

$= 36 - 24 - 11$

$= 36 - 35 = 1$

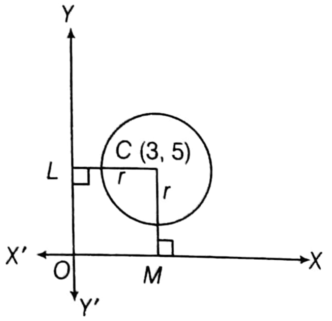
$\therefore$ $S_2 > 0$

Hence, the point (1, 2) lies inside the circle and the point (6, 0) lies outside the circle.

Example 23. The circle $x^2 + y^2 - 6x - 10y + \lambda = 0$ does not touch or intersect the coordinate axes and the point (1, 4) is inside the circle. Find the range of values of λ.

Sol. Let $S \equiv x^2 + y^2 - 6x - 10y + \lambda = 0$

$\because$ Point (1, 4) is inside the circle, then $S_1 < 0$



$$1 + 16 - 6 - 40 + \lambda < 0$$

$$\Rightarrow \quad \lambda < 29 \quad \text{...(i)}$$

Centre and radius of the circle are (3, 5) and $\sqrt{(34 - \lambda)}$, respectively.

$\because$ Circle does not touch or intersect the coordinate axes.

$\therefore \quad 5 > r$ and $3 > r$

or $\quad 5 > \sqrt{(34 - \lambda)}$ and $3 > \sqrt{(34 - \lambda)}$

$\Rightarrow \quad 25 > 34 - \lambda$ and $9 > 34 - \lambda$

$\Rightarrow \quad \lambda > 9$ and $\lambda > 25$

$\therefore \quad \lambda > 25 \quad$ ...(ii)

Also, $\quad 34 - \lambda > 0$

$\therefore \quad \lambda < 34 \quad$ ...(iii)

From Eqs. (i), (ii) and (iii), we get $25 < \lambda < 29$

Maximum and Minimum Distance of a Point from the Circle

Let any point $P(x_1, y_1)$ and circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

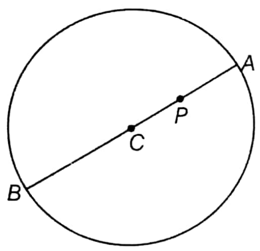
The centre and radius of the circle are

$C(-g, -f)$ and $\sqrt{(g^2 + f^2 - c)}$ respectively

Case I If P inside the circle

In this case $S_1 < 0$

$$\because \quad r = \sqrt{(g^2 + f^2 - c)} = CA = CB$$



The **minimum distance** of P from circle $= PA = CA - CP$

$$\boxed{= r - CP}$$

and the **maximum distance** of P from circle $= PB$

$$= CB + CP = \boxed{r + CP}$$

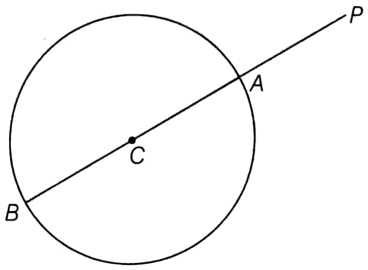
Case II If P outside the circle

In this case $S_1 > 0$ the **minimum distance** of P from circle

$$= PA = CP - CA = \boxed{CP - r}$$

and the **maximum distance** of P from the circle

$$= PB = CP + CB = \boxed{r + CP}$$



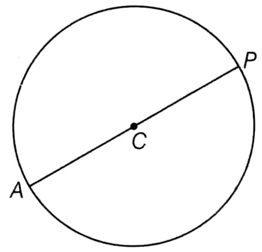
Case III If P on the circle

In this case $S_1 = 0$

the **minimum distance** of P from the circle = 0

and the **maximum distance** of P from the circle

$$= PA = \boxed{2r}$$



Remark

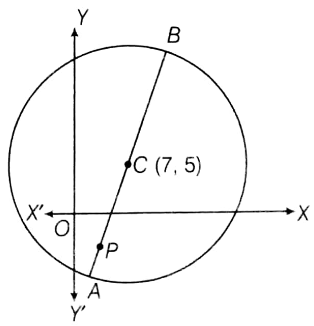
If point P inside or outside or on the circle and centre of circle at C and radius r, then minimum distance of P from the circle $= |CP - r|$ and maximum distance of P from the circle $= CP + r$

Example 24. Find the shortest and largest distance from the point (2, −7) to the circle

$$x^2 + y^2 - 14x - 10y - 151 = 0$$

Sol. Let $\quad S \equiv x^2 + y^2 - 14x - 10y - 151 = 0$

$\therefore \quad S_1 = (2)^2 + (-7)^2 - 14(2) - 10(-7) - 151 = -56 < 0$



$\therefore$ $P(2,-7)$ inside the circle

radius of the circle, $r = \sqrt{(-7)^2 + (-5)^2 + 151} = 15$

$\because$ Centre of circle $C \equiv (7, 5)$

$\therefore$ $$CP = \sqrt{(7-2)^2 + (5+7)^2} = 13$$

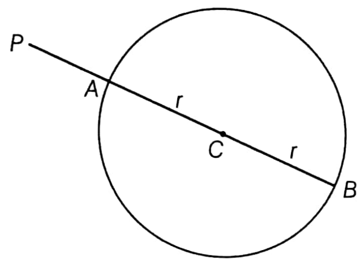
$\therefore$ **Shortest distance** $= PA = r - CP = 15 - 13 = 2$

and **Largest distance** $= PB = r + CP = 15 + 13 = 28$

Example 25. Find the points on the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ which are farthest and nearest to the point $(-5,6)$.

Sol. The given circle is $S \equiv x^2 + y^2 - 2x + 4y - 20 = 0$

Let $P \equiv (-5,6)$



For the point P

$$S_1 = 25 + 36 + 10 + 24 - 20$$
$$= 75 > 0$$

$\therefore$ Point $P(-5,6)$ lies outside the circle.

The centre and radius of the circle are $(1,-2)$ and 5, respectively.

$\because$ $$CP = \sqrt{(1+5)^2 + (-2-6)^2} = 10$$

Now, point A divides CP in the ratio

$$\frac{AP}{AC} = \frac{CP - r}{r} = \frac{10-5}{5} = 1$$

$\therefore$ A is mid-point of CP.

$\therefore$ $$A \equiv \left(\frac{1-5}{2}, \frac{-2+6}{2}\right)$$

or $A \equiv (-2,2)$

and C is the mid-point of AB.

$\therefore$ $$B \equiv (2 \times 1 - (-2), 2 \times -2 - 2)$$

or $B \equiv (4,-6)$

Hence, point $A(-2,2)$ is nearest to P and $B(4,-6)$ is farthest from P.

Exercise for Session 3

1. The length of intercept, the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ makes on the X-axis is

(a) 2 (b) 4 (c) 6 (d) 8

2. The circle $x^2 + y^2 + 4x - 7y + 12 = 0$ cuts an intercept on Y-axis is of length

(a) 1 (b) 3 (c) 5 (d) 7

3. The locus of the centre of a circle which passes through the origin and cuts-off a length $2b$ from the line $x = c$ is

(a) $y^2 + 2cx = b^2 + c^2$ (b) $x^2 + cx = b^2 + c^2$ (c) $y^2 + 2cy = b^2 + c^2$ (d) $x^2 + cy = b^2 + c^2$

4. If a straight line through $C(-\sqrt{8}, \sqrt{8})$ making an angle of 135° with the X-axis cuts the circle $x = 5\cos\theta$, $y = 5\sin\theta$ at points A and B, then the length of AB is

(a) 3 (b) 5 (c) 8 (d) 10

5. If a circle of constant radius $3k$ passes through the origin and meets the axes at A and B, the locus of the centroid of ΔOAB is

(a) $x^2 + y^2 = k^2$ (b) $x^2 + y^2 = 2k^2$ (c) $x^2 + y^2 = 3k^2$ (d) $x^2 + y^2 = 4k^2$

6. The centre of the circle touching Y-axis at $(0, 3)$ and making an intercept of 2 units on positive X-axis is

(a) $(10, \sqrt{3})$ (b) $(\sqrt{3}, 10)$ (c) $(\sqrt{10}, 3)$ (d) $(3, \sqrt{10})$

7. A circle passes through the points $A(1, 0)$ and $B(5,0)$ and touches the Y-axis at $C(0,\lambda)$. If $\angle ACB$ is maximum, then

(a) $|\lambda| = \sqrt{5}$ (b) $|\lambda| = 2\sqrt{5}$ (c) $|\lambda| = 3\sqrt{5}$ (d) $|\lambda| = 4\sqrt{5}$

8. The equation of a circle whose centre is $(3, -1)$ and which intercept chord of 6 units length on straight line $2x - 5y + 18 = 0$ is

(a) $x^2 + y^2 - 6x + 2y - 28 = 0$
(b) $x^2 + y^2 + 6x - 2y - 28 = 0$
(c) $x^2 + y^2 + 4x - 2y + 24 = 0$
(d) $x^2 + y^2 + 2x - 2y - 12 = 0$

9. The locus of the centre of a circle which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the Y-axis, is given by the equation

(a) $x^2 - 6x - 10y + 14 = 0$
(b) $x^2 - 10x - 6y + 14 = 0$
(c) $y^2 - 6x - 10y + 14 = 0$
(d) $y^2 - 10x - 6y + 14 = 0$

10. The locus of the centre of a circle of radius 2 which rolls on the outside of circle $x^2 + y^2 + 3x - 6y - 9 = 0$ is

(a) $x^2 + y^2 + 3x - 6y + 5 = 0$
(b) $x^2 + y^2 + 3x - 6y - 31 = 0$
(c) $x^2 + y^2 + 3x - 6y + 11 = 0$
(d) $x^2 + y^2 + 3x - 6y - 36 = 0$

11. The point $([\lambda + 1], [\lambda])$ is lying inside the circle $x^2 + y^2 - 2x - 15 = 0$. Then, the set of all values of λ is (where [.] represents the greatest integer function)

(a) $[-2, 3]$
(b) $(-2, 3)$
(c) $[-2, 0) \cup (0, 3)$
(d) $[0, 3)$

12. The greatest distance of the point $(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is

(a) 5
(b) 10
(c) 15
(d) 20

13. Find equations to the circles touching Y-axis at $(0, 3)$ and making intercept of 8 units on the X-axis.

14. Show that the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches both the coordinate axes.

15. If the point $(\lambda, -\lambda)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$, then find range of λ.

16. Find the equation of the circle which passes through the origin and cuts-off chords of lengths 4 and 6 on the positive side of the X-axis and Y-axis, respectively.

Session 4

Intersection of a Line and a Circle, Product of the Algebraical Distances *PA* and *PB* is Constant when from *P*, A Secant be Drawn to Cut the Circle in the Points *A* and *B*, The Length of Intercept Cut-off from a Line by a Circle, Tangent to a Circle at a Given Point, Normal to a Circle at a Given Point

Intersection of a Line and a Circle

Let the equation of the circle be

$$x^2 + y^2 = a^2 \qquad \text{...(i)}$$

and the equation of the line be

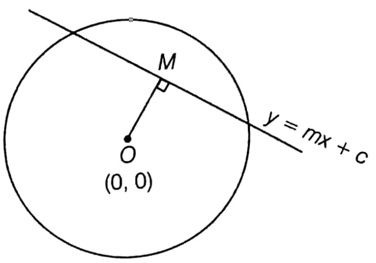
$$y = mx + c \qquad \text{...(ii)}$$

From Eq. (i) and Eq. (ii)

$$x^2 + (mx + c)^2 = a^2$$

or $$(1 + m^2)\,x^2 + 2mcx + c^2 - a^2 = 0 \qquad \text{...(iii)}$$

Case I When points of intersection are real and distinct, then Eq. (iii) has two distinct roots.



$\therefore$ $$B^2 - 4AC > 0$$

or $$4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0$$

or $$a^2 > \frac{c^2}{1 + m^2}$$

or $$\boxed{a > \frac{|c|}{\sqrt{(1 + m^2)}}} = \text{length of perpendicular}$$

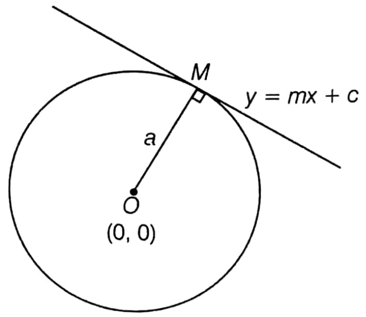
→ radius > ⊥ distance.

from $(0, 0)$ to $y = mx + c$

$\Rightarrow$ $a >$ length of perpendicular from $(0, 0)$ to $y = mx + c$

Thus, a line intersects a given circle at two distinct points if radius of circle is greater than the length of perpendicular from centre of the circle to the line.

Case II When the points of intersection are coincident, then Eq. (iii) has two equal roots



$\therefore$ $$B^2 - 4AC = 0$$

$\Rightarrow$ $$4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0$$

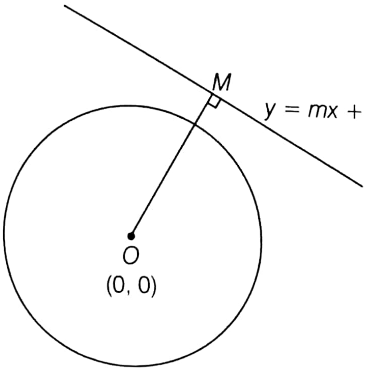
$\therefore$ $$a^2 = \frac{c^2}{(1 + m^2)}$$

or $$a = \frac{|c|}{\sqrt{(1 + m^2)}}$$

$a =$ length of the perpendicular from the point $(0, 0)$ to $y = mx + c$

Thus, a line touches the circle if radius of circle is equal to the length of perpendicular from centre of the circle to the line.

Case III When the points of intersection are imaginary.
In this case (iii) has imaginary roots



$$\therefore \quad B^2 - 4AC < 0$$

$$4m^2c^2 - 4(1+m^2)(c^2 - a^2) = 0$$

$$\therefore \quad a^2 < \frac{c^2}{1+m^2}$$

or $a < \dfrac{|c|}{\sqrt{1+m^2}}$ = length of perpendicular from $(0, 0)$ to $y = mx + c$

or $a <$ length of perpendicular from $(0, 0)$ to $y = mx + c$

Thus, a line does not intersect a circle if the radius of circle is less than the length of perpendicular from centre of the circle to the line.

| Example 26. Find the points of intersection of the line $2x + 3y = 18$ and the circle $x^2 + y^2 = 25$.

Sol. We have, $\quad 2x + 3y = 18 \quad \text{....(i)}$

and $\quad x^2 + y^2 = 25 \quad \text{...(ii)}$

From Eq. (i), $\quad y = \dfrac{18 - 2x}{3}$

Substituting in Eq. (ii), then $x^2 + \left(\dfrac{18-2x}{3}\right)^2 = 25$

$$\Rightarrow \quad 9x^2 + 4(9 - x)^2 = 225$$

$$\Rightarrow \quad 9x^2 + 4(81 - 18x + x^2) = 225$$

$$\Rightarrow \quad 13x^2 - 72x + 324 - 225 = 0$$

$$\Rightarrow \quad 13x^2 - 72x + 99 = 0$$

$$\Rightarrow \quad (x - 3)(13x - 33) = 0$$

$$\Rightarrow \quad x = 3 \quad \text{or} \quad x = \frac{33}{13}$$

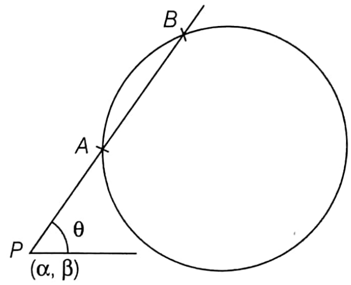
From Eq. (i), $\quad y = 4 \quad \text{or} \quad y = \dfrac{56}{13}$

Hence, the points of intersection of the given line and the given circle are $(3, 4)$ and $\left(\dfrac{33}{13}, \dfrac{56}{13}\right)$.

Product of the Algebraical Distances *PA* and *PB* is Constant when from *P*, A Secant be Drawn to Cut the Circle in the Points *A* and *B*

If a straight line through $P(\alpha, \beta)$ makes an angle θ with the positive direction of X-axis, then its equation is

$$\frac{x - \alpha}{\cos\theta} = \frac{y - \beta}{\sin\theta} = r$$



where, r is the algebraical distance of the point (x, y) from the point $P(\alpha, \beta)$.

$$\therefore \quad (x, y) = (\alpha + r\cos\theta, \beta + r\sin\theta)$$

If this point lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

or $\quad (\alpha + r\cos\theta)^2 + (\beta + r\sin\theta)^2 + 2g(\alpha + r\cos\theta) + 2f(\beta + r\sin\theta) + c = 0$

$$\Rightarrow \quad r^2 + 2r(\alpha\cos\theta + \beta\sin\theta + g\cos\theta + f\sin\theta) + (\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c) = 0$$

This is quadratic equation in r, then PA and PB are the roots of this equation.

$$\therefore \quad \boxed{PA \cdot PB = \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c = \text{constant}}$$

Since, RHS is independent of θ.

Remark

Secants are drawn from a given point A to cut a given circle at the pairs of points $P_1, Q_1; P_2, Q_2; \ldots; P_n, Q_n$, then Imp

$$AP_1 \cdot AQ_1 = AP_2 \cdot AQ_2 = \ldots = AP_n \cdot AQ_n$$

The Length of Intercept Cut-off from a Line by a Circle

Theorem : The length of the intercept cut-off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is

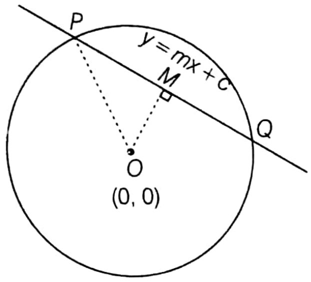
$$2\sqrt{\left\{\frac{a^2(1+m^2) - c^2}{(1+m^2)}\right\}}$$

Proof : Draw OM perpendicular to PQ

Now, OM = length of perpendicular from $O(0,0)$ to

$$(y = mx + c) = \frac{|c|}{\sqrt{(1+m^2)}}$$

and OP = radius of the circle $= a$



In ΔOPM, $PM = \sqrt{(OP)^2 - (OM)^2}$

$$= \sqrt{a^2 - \frac{c^2}{(1+m^2)}} = \sqrt{\left\{\frac{a^2(1+m^2) - c^2}{1+m^2}\right\}}$$

$\therefore$ $$PQ = 2PM = 2\sqrt{\left\{\frac{a^2(1+m^2) - c^2}{1+m^2}\right\}}$$

Remarks

1. If the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$, then intercepted length is zero

i.e. $PQ = 0 \Rightarrow 2\sqrt{\left\{\frac{a^2(1+m^2) - c^2}{1+m^2}\right\}} = 0$

$\therefore$ $\boxed{c^2 = a^2(1+m^2)}$ Remember

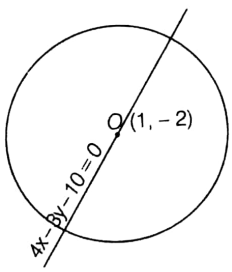
which is the required condition for tangency.

2. If a line touches the circle, then length of perpendicular from the centre upon the line is equal to the radius of the circle.

Example 27. Find the length of the intercept on the straight line $4x - 3y - 10 = 0$ by the circle $x^2 + y^2 - 2x + 4y - 20 = 0$.

MUST

Sol. Centre and radius of the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are $(1, -2)$ and $\sqrt{1 + 4 + 20} = 5$ respectively.



Let OM be the perpendicular from O on the line

$$4x - 3y - 10 = 0$$

then $$OM = \frac{|4 \times 1 - 3 \times (-2) - 10|}{\sqrt{4^2 + (-3)^2}} = 0$$

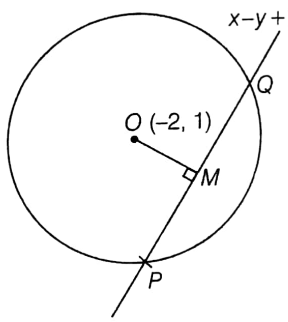
Hence, line $4x - 3y - 10 = 0$ passes through the centre of the circle.

Hence, intercepted length = diameter of the circle

$$= 2 \times 5 = 10$$

Example 28. Find the coordinates of the middle point of the chord which the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ cuts-off the line $x - y + 2 = 0$.

Sol. Centre and radius of the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ are $(-2, 1)$ and $\sqrt{4 + 1 + 3} = 2\sqrt{2}$ respectively.



Draw perpendicular from O upon $x - y + 2 = 0$ is OM.

Equation of OM which is perpendicular to $x - y + 2 = 0$ is $x + y = \lambda$, it passes through $(-2, 1)$

Then, $-2 + 1 = \lambda$

$\therefore$ $\lambda = -1$

then equation of OM is $x + y + 1 = 0$

Since, M is the mid-point of PQ which is point of intersection of $x - y + 2 = 0$ and $x + y + 1 = 0$, coordinates of M is $\left(-\frac{3}{2}, \frac{1}{2}\right)$.

Aliter : Let $M \equiv (\alpha, \beta)$, then

$$\frac{\alpha + 2}{1} = \frac{\beta - 1}{-1} = -\frac{(-2 - 1 + 2)}{1 + 1}$$

(Here, M is foot of perpendicular)

$\Rightarrow$ $$\frac{\alpha + 2}{1} = \frac{\beta - 1}{-1} = \frac{1}{2}$$

or $$\alpha = -\frac{3}{2} \text{ and } \beta = \frac{1}{2}$$

$\therefore$ $$M \equiv \left(-\frac{3}{2}, \frac{1}{2}\right)$$

Example 29. For what value of λ will the line $y = 2x + \lambda$ be a tangent to the circle $x^2 + y^2 = 5$?

Sol. Comparing the given line with $y = mx + c$, we get $m = 2$, $c = \lambda$ and given circle with $x^2 + y^2 = a^2$

then $a^2 = 5$

$\because$ Condition for tangency is

$$c^2 = a^2(1+m^2)$$

$$\Rightarrow \quad \lambda^2 = 5(1+4)$$

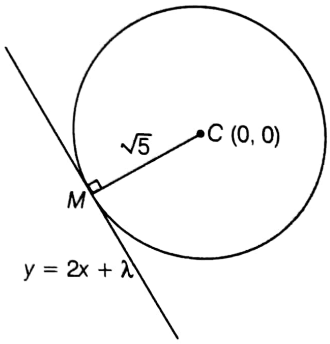
$$\lambda^2 = 25$$

$$\therefore \quad \lambda = \pm 5$$

Aliter : Since, line $y = 2x + \lambda$

or $\quad 2x - y + \lambda = 0$

is the tangent to the circle $x^2 + y^2 = 5$ then length of perpendicular from centre upon the line is equal to the radius of the circle



$$\therefore \quad |CM| = \sqrt{5}$$

$$\text{or} \quad \frac{|0 - 0 + \lambda|}{\sqrt{4+1}} = \sqrt{5}$$

$$\Rightarrow \quad \frac{|\lambda|}{\sqrt{5}} = \sqrt{5}$$

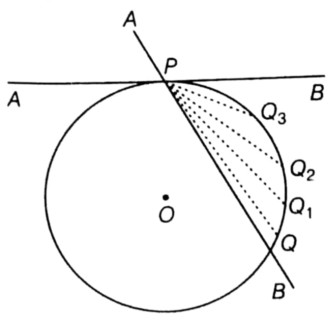
$$\Rightarrow \quad |\lambda| = 5$$

$$\text{or} \quad \lambda = \pm 5$$

Tangent to a Circle at a Given Point

Line cuts a circle atleast two points

Let PQ be a chord and AB be a secant passing through P. Let P be the fixed point and move along the circle towards P, then the secant PQ turns about P. In the limit, when Q coincides with P, then the secant AB becomes a tangent to the circle at the point P.



Different forms of the equations of tangents

1. Point form :

Theorem : The equation of tangent at the point $P(x_1, y_1)$ to a circle

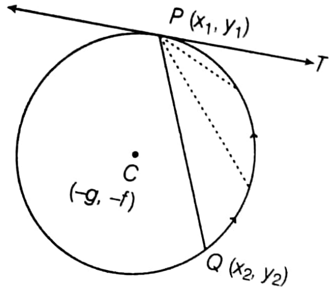
$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Proof : Since, $P(x_1, y_1)$ be a point on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Let $Q(x_2, y_2)$ be any other point on the circle Eq. (i). Since, points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the circle, therefore



$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{...(ii)}$$

$$\text{and} \quad x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \text{...(iii)}$$

On subtracting Eq. (ii) from Eq. (iii), we have

$$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

$$\Rightarrow \quad (x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0$$

$$\Rightarrow \quad \left(\frac{y_2 - y_1}{x_2 - x_1}\right) = -\left(\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}\right) \quad \text{...(iv)}$$

Now, the equation of the chord PQ is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \quad \text{...(v)}$$

Putting the value of $\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)$ from Eq. (iv) in Eq. (v), then equation PQ becomes

$$y - y_1 = -\left(\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}\right)(x - x_1) \quad \text{...(vi)}$$

Now, when $Q \to P$ (along the circle), line PQ becomes tangent at P, we have $x_2 \to x_1, y_2 \to y_1$. So, the equation of tangent at $P(x_1, y_1)$ is :

$$y - y_1 = -\left(\frac{x_1 + x_1 + 2g}{y_1 + y_1 + 2f}\right)(x - x_1)$$

$$\Rightarrow \quad y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

$$\Rightarrow \quad (y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$$

$$\Rightarrow \quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

On adding $gx_1 + fy_1 + c$ to both sides, we get

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{[from Eq. (ii)]}$$

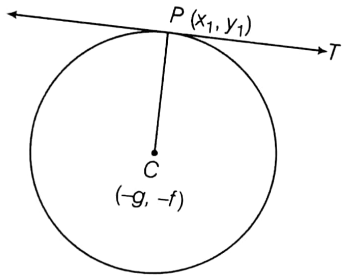
$$\Rightarrow \quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

This is the required equation of the tangent PT to the circle at the point (x_1, y_1).

Aliter : Since, circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

$P(x_1, y_1)$ lie on the circle

$$\therefore \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{...(i)}$$



Its centre is $C(-g, -f)$

$$\therefore \quad \text{Slope of } CP = \frac{y_1 - (-f)}{x_1 - (-g)} = \frac{y_1 + f}{x_1 + g}$$

Since, tangent PT is perpendicular to CP.

$$\therefore \quad \text{Slope of tangent} = -\left(\frac{x_1 + g}{y_1 + f}\right)$$

$\therefore$ Equation of tangent at $P(x_1, y_1)$ is

$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

$$\Rightarrow \quad (y - y_1)(y_1 + f) + (x_1 + g)(x - x_1) = 0$$

$$\Rightarrow \quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

On adding $gx_1 + fy_1 + c$ to both sides, we get

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{[from Eq. (i)]}$$

or $\quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

This is the required equation of the tangent PT to the circle at the point $P(x_1, y_1)$.

Remarks

1. For equation of tangent of circle at (x_1, y_1), substitute xx_1 for x^2, yy_1 for y^2, $\frac{x + x_1}{2}$ for x, $\frac{y + y_1}{2}$ for y and $\frac{xy_1 + x_1y}{2}$ for xy and keep the constant as such.
2. This method of tangent at (x_1, y_1) is applied only for any conics of second degree. i.e. equation of tangent of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Wrong process : Mostly students use wrong process

Suppose any curve

$$ax^3 + by^3 = c$$

or $\quad a(x)(x^2) + b(y)(y^2) = c$

Equation of tangent at (x_1, y_1)

$$\Rightarrow \quad a\left(\frac{x + x_1}{2}\right)xx_1 + b\left(\frac{y + y_1}{2}\right)yy_1 = c^2$$

which is a second degree conic not the equation of tangent.

Reason : This method is applicable only for second degree conic, its a third degree conic. (find its tangent only by calculus)

Example 30. Prove that the tangents to the circle $x^2 + y^2 = 25$ at $(3, 4)$ and $(4, -3)$ are perpendicular to each other.

Sol. The equations of tangents to $x^2 + y^2 = 25$ at $(3, 4)$ and $(4, -3)$ are

$$3x + 4y = 25$$

and $\quad 4x - 3y = 25$

respectively.

Now, $\quad$ slope of Eq. (i) $= -\frac{3}{4} = m_1$

and $\quad$ slope of Eq. (ii) $= \frac{4}{3} = m_2$

Clearly, $\quad m_1m_2 = -1$

Hence, Eq. (i) and Eq. (ii) are perpendicular to each other.

Example 31. Find the equation of tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $[a(1 + \cos\alpha), a\sin\alpha]$.

Sol. The equation of tangent of $x^2 + y^2 - 2ax = 0$ at $[a(1 + \cos\alpha), a\sin\alpha]$ is

$$x \cdot a(1 + \cos\alpha) + y \cdot a\sin\alpha - a[x + a(1 + \cos\alpha)] = 0$$

$$\Rightarrow \quad ax\cos\alpha + ay\sin\alpha - a^2(1 + \cos\alpha) = 0$$

or $\quad x\cos\alpha + y\sin\alpha = a(1 + \cos\alpha)$

Must

Example 32. Show that the circles $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x-6y+14=0$ touch at $(3,-1)$.

Sol. Equation of tangent at $(3,-1)$ of the circle $x^2+y^2-4x+6y+8=0$ is

$$3x+(-1)y-2(x+3)+3(y-1)+8=0$$

or $$x+2y-1=0 \quad \text{...(i)}$$

and equation of tangent at $(3,-1)$ of the circle $x^2+y^2-10x-6y+14=0$ is

$$3\cdot x+(-1)\cdot y-5(x+3)-3(y-1)+14=0$$

or $$-2x-4y+2=0$$

or $$x+2y-1=0 \quad \text{...(ii)}$$

which is the same as Eq (i).

Hence, the given circles touch at $(3,-1)$.

2. Parametric form :

Theorem : The equation of tangent to the circle $x^2+y^2=a^2$ at the point $(a\cos\theta, a\sin\theta)$ is $x\cos\theta+y\sin\theta=a$

Proof : The equation of tangent of $x^2+y^2=a^2$ at (x_1,y_1) is $xx_1+yy_1=a^2$ (using point form of the tangent)

Putting $x_1=a\cos\theta,\ y_1=a\sin\theta$

then, we get $x\cos\theta+y\sin\theta=a$

Corollary 1 : Equation of chord joining $(a\cos\theta, a\sin\theta)$ and $(a\cos\phi, a\sin\phi)$ is

$$x\cos\left(\frac{\theta+\phi}{2}\right)+y\sin\left(\frac{\theta+\phi}{2}\right)=a\cos\left(\frac{\theta-\phi}{2}\right)$$

Corollary 2 : Point of intersection of tangents at $(a\cos\theta, a\sin\theta)$ and $(a\cos\phi, a\sin\phi)$ is

$$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)},\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$$

Remembering method :

$\because$ $$x\cos\left(\frac{\theta+\phi}{2}\right)+y\sin\left(\frac{\theta+\phi}{2}\right)=a\cos\left(\frac{\theta-\phi}{2}\right)$$

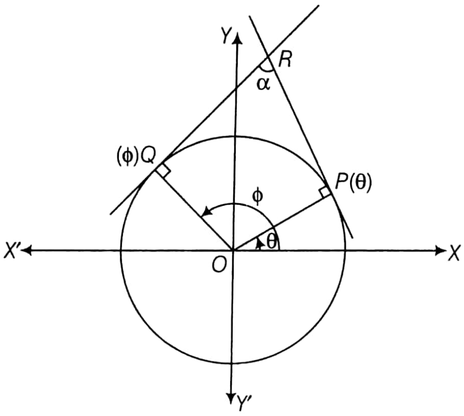
or $$x\left\{\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}+y\left\{\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}=a^2$$

We get $$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)},\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$$

Corollary 3 : The angle between a pair of tangents from a point R to the circle $x^2+y^2=a^2$ is α. Then, the locus of the point R is

$$x^2+y^2=\frac{a^2}{\sin^2\left(\frac{\alpha}{2}\right)}$$

Proof



$\because$ $$\phi-\theta+\alpha=180^\circ$$

$\therefore$ $$\frac{\theta-\phi}{2}=-\left(90^\circ-\frac{\alpha}{2}\right)$$

or $$\cos\left(\frac{\theta-\phi}{2}\right)=\sin\left(\frac{\alpha}{2}\right)$$

Now, point of intersection is

$$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)},\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\right)$$

Let $$x=\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \text{ and } y=\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$\therefore$ $$x^2+y^2=\frac{a^2}{\sin^2\left(\frac{\alpha}{2}\right)}$$

Remarks

Radius ka square

1. The angle between a pair of tangents from a point P to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is 2θ, then the locus of P is $x^2 + y^2 + 2gx + 2fy + c = (g^2 + f^2 - c)\cot^2\theta$
2. If angle between a pair of tangents from a point P to the circle $x^2 + y^2 = a^2$ is $\frac{\pi}{2}$, then the locus of P is

Must

$$\boxed{x^2 + y^2 = 2a^2} \qquad \left(\text{Here, } \alpha = \frac{\pi}{2}\right)$$

which is **director circle** of $x^2 + y^2 = a^2$.

($\because$ locus of point of intersection of perpendicular tangents is **director circle**)

3. The equation of the tangent to the circle $(x-a)^2 + (y-b)^2 = r^2$ at the point $(a + r\cos\theta, b + r\sin\theta)$ is

$$(x-a)\cos\theta + (y-b)\sin\theta = r.$$

Example 33. The angle between a pair of tangents from a point P to the circle $x^2 + y^2 = 25$ is $\frac{\pi}{3}$. Find the equation of the locus of the point P.

Sol. Here, $\alpha = \frac{\pi}{3}$

$\therefore$ Required locus is $x^2 + y^2 = \dfrac{25}{\sin^2\left(\frac{\pi}{6}\right)} = 100$

Example 34. The angle between a pair of tangents from a point P to the circle $x^2 + y^2 - 6x - 8y + 9 = 0$ is $\frac{\pi}{3}$. Find the equation of the locus of the point P.

Sol. Here, $2\theta = \frac{\pi}{3}$ or $\theta = \frac{\pi}{6}$

$\therefore$ Required locus is

$$x^2 + y^2 - 6x - 8y + 9 = (9 + 16 - 9)\cot^2\frac{\pi}{6}$$

or $$x^2 + y^2 - 6x - 8y + 9 = 16 \times 3$$

or $$x^2 + y^2 - 6x - 8y - 39 = 0$$

3. Slope form :

Theorem : The equation of a tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{(1+m^2)}$ and the coordinates of the point of contact are

$$\left(\pm\frac{am}{\sqrt{(1+m^2)}}, \mp\frac{a}{\sqrt{(1+m^2)}}\right)$$

Proof : Let $y = mx + c$ is the tangent of the circle $x^2 + y^2 = a^2$.

$\therefore$ Length of perpendicular from centre of circle $(0, 0)$ on $(y = mx + c)$ = radius of circle

$$\therefore \quad \frac{|c|}{\sqrt{(1+m^2)}} = a \Rightarrow c = \pm a\sqrt{(1+m^2)}$$

On substituting this value of c in $y = mx + c$, we get

$$y = mx \pm a\sqrt{(1+m^2)}$$

which are the required equations of tangents.

Also, let (x_1, y_1) be the point of contact, then equation of tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2$$

On comparing Eq. (i) and Eq. (ii), we get

$$\frac{x_1}{m} = \frac{y_1}{-1} = \frac{a^2}{\pm a\sqrt{(1+m^2)}}$$

$$\Rightarrow \quad \frac{x_1}{m} = -\frac{y_1}{1} = \pm\frac{a}{\sqrt{(1+m^2)}}$$

$$\Rightarrow \quad x_1 = \pm\frac{am}{\sqrt{1+m^2}} \quad \text{and} \quad y_1 = \mp\frac{a}{\sqrt{(1+m^2)}}$$

Hence, $(x_1, y_1) = \left(\pm\dfrac{am}{\sqrt{(1+m^2)}}, \mp\dfrac{a}{\sqrt{(1+m^2)}}\right)$

Corollary : It also follows that $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$, if $c^2 = a^2(1+m^2)$ which is condition of tangency.

Remarks

1. The reason why there are two equations $y = mx \pm a\sqrt{1+m^2}$, there are two tangents, both are parallel and at the ends of diameter.
2. The line $ax + by + c = 0$ is tangent to the circle $x^2 + y^2 = r^2$ if and only if $c^2 = r^2(a^2 + b^2)$.
3. If the line $y = mx + c$ is the tangent to the circle $x^2 + y^2 = r^2$ then point of contact is given by $\left(-\frac{mr^2}{c}, \frac{r^2}{c}\right)$
4. If the line $ax + by + c = 0$ is the tangent to the circle $x^2 + y^2 = r^2$, then point of contact is given by $\left(-\frac{ar^2}{c}, -\frac{br^2}{c}\right)$
5. The condition that the line $lx + my + n = 0$ touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(lg + mf - n)^2 = (l^2 + m^2)(g^2 + f^2 - c)$.
6. Equation of tangent of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of slope is $y + f = m(x + g) \pm \sqrt{(g^2 + f^2 - c)(1 + m^2)}$
7. The equation of tangents of slope m to the circle $(x-a)^2 + (y-b)^2 = r^2$ are given by $(y - b) = m(x - a) \pm r\sqrt{(1+m^2)}$ and the coordinates of the points of contact are

$$\left(a \pm \frac{mr}{\sqrt{(1+m^2)}}, b \mp \frac{r}{\sqrt{(1+m^2)}}\right)$$

| Example 35. Find the equations of the tangents to the circle $x^2 + y^2 = 9$, which

(i) are parallel to the line $3x+4y-5=0$

(ii) are perpendicular to the line $2x+3y+7=0$

(iii) *make an angle of 60° with the X-axis*

Sol. (i) Slope of $3x + 4y - 5 = 0$ is $-\frac{3}{4}$

$$\text{Let } m = -\frac{3}{4}$$

and equation of circle is $x^2 + y^2 = 9$

$\therefore$ Equations of tangents

$$y = -\frac{3}{4}x \pm 3\sqrt{\left(1+\left(-\frac{3}{4}\right)^2\right)}$$

$\Rightarrow \quad 4y = -3x \pm 15$ or $3x + 4y \pm 15 = 0$

(ii) Slope of $2x + 3y + 7 = 0$ is $-\frac{2}{3}$

$\therefore$ Slope of perpendicular to $2x + 3y + 7 = 0$ is $\frac{3}{2} = m$ (say)

and given circle is $x^2 + y^2 = 9$

$\therefore$ Equations of tangents perpendicular to $2x + 3y + 7 = 0$ is

$$y = \frac{3}{2}x \pm 3\sqrt{1+\left(\frac{3}{2}\right)^2}$$

$\Rightarrow \quad 2y = 3x \pm 3\sqrt{13}$

or $\quad 3x - 2y \pm 3\sqrt{13} = 0$

(iii) Since, tangent make an angle 60° with the X-axis

$\therefore \quad m = \tan 60° = \sqrt{3}$

and given circle $\quad x^2 + y^2 = 9$

$\therefore$ Equation of tangents $y = \sqrt{3}x \pm 3\sqrt{1+(\sqrt{3})^2}$

or $\quad \sqrt{3}x - y \pm 6 = 0$

Aliter :

(i) Let tangent parallel to $3x + 4y - 5 = 0$ is

$$3x + 4y + \lambda = 0 \quad \text{...(i)}$$

and circle $\quad x^2 + y^2 = 9$

then perpendicular distance from (0, 0) on Eq. (i) = radius

$$\frac{|\lambda|}{\sqrt{(3^2+4^2)}} = 3$$

or $\quad |\lambda| = 15$

$\therefore \quad \lambda = \pm 15$

From Eq. (i), equations of tangents are

$$3x + 4y \pm 15 = 0$$

(ii) Let tangent perpendicular to $2x + 3y + 7 = 0$ is

$$3x - 2y + \lambda = 0 \quad \text{...(ii)}$$

and circle $\quad x^2 + y^2 = 9$

then, perpendicular distance from (0, 0) on Eq. (ii) = radius

$$\frac{|\lambda|}{\sqrt{3^2+(-2)^2}} = 3$$

or $\quad |\lambda| = 3\sqrt{13}$

or $\quad \lambda = \pm 3\sqrt{13}$

From Eq. (ii), equations of tangents are

$$3x - 2y \pm 3\sqrt{13} = 0$$

(iii) Let equation of tangent which makes an angle of 60° with the X-axis is

$$y = \sqrt{3}x + c \quad \text{...(iii)}$$

or $\quad \sqrt{3}x - y + c = 0$

and circle $\quad x^2 + y^2 = 9$

then, perpendicular distance from (0, 0) to Eq. (iii) = radius

$$\frac{|c|}{\sqrt{(\sqrt{3})^2+(-1)^2}} = 3$$

or $\quad |c| = 6$

or $\quad c = \pm 6$

From Eq. (iii), equations of tangents are

$$\sqrt{3}x - y \pm 6 = 0$$

| Example 36. Prove that the line $lx + my + n = 0$ touches the circle $(x-a)^2 + (y-b)^2 = r^2$ if $(al + bm + n)^2 = r^2(l^2 + m^2)$.

Sol. If the line $lx + my + n = 0$ touches the circle $(x-a)^2 + (y-b)^2 = r^2$, then length of the perpendicular from the centre = radius

$$\frac{|la + mb + n|}{\sqrt{(l^2+m^2)}} = r$$

$$\Rightarrow \quad (la + mb + n)^2 = r^2(l^2 + m^2)$$

Aliter :

Here, line is $lx + my + n = 0$ and circle is $(x-a)^2 + (y-b)^2 = r^2$. Here, centre of circle (a, b) shift at (0, 0), then replacing x by $x + a$ and y by $y + b$ in the equation of straight line $lx + my + n = 0$ and circle $(x-a)^2 + (y-b)^2 = r^2$, the new form of straight line and circle are

$$l(x+a) + m(y+b) + n = 0$$

or $\quad lx + my + (al + mb + n) = 0 \quad \text{...(i)}$

and $\quad x^2 + y^2 = r^2 \quad \text{...(ii)}$

respectively.

On comparing Eq. (i) with $y = Mx + C$

then $\quad M = -\frac{l}{m}$

and $\quad C = -\frac{(al + bm + n)}{m}$

Since, Eq. (i) is the tangent of Eq. (ii), then

$$C^2 = r^2 (1 + M^2)$$

or $$\frac{(al + bm + n)^2}{m^2} = r^2 \left(1 + \frac{l^2}{m^2}\right)$$

or $$(al + bm + n)^2 = r^2 (l^2 + m^2)$$

Example 37. Show that the line $3x - 4y = 1$ touches the circle $x^2 + y^2 - 2x + 4y + 1 = 0$. Find the coordinates of the point of contact.

Sol. The centre and radius of the circle

$$x^2 + y^2 - 2x + 4y + 1 = 0 \text{ are } (1, -2)$$

and $\sqrt{(-1) + (2)^2 - 1} = 2$ respectively.

Since, length of perpendicular from centre $(1, -2)$ on $3x - 4y = 1$ is

$$\frac{|3 \times 1 - 4 \times (-2) - 1|}{\sqrt{(3)^2 + (-4)^2}} = \frac{10}{5}$$

$$= 2 = \text{radius of the circle}$$

Hence, $3x - 4y = 1$ touches the circle

$$x^2 + y^2 - 2x + 4y + 1 = 0$$

Second part : Let point of contact is (x_1, y_1), then tangent at (x_1, y_1) on $x^2 + y^2 - 2x + 4y + 1 = 0$ is

$$xx_1 + yy_1 - (x + x_1) + 2(y + y_1) + 1 = 0$$

$$\Rightarrow \quad x(x_1 - 1) + y(y_1 + 2) - x_1 + 2y_1 + 1 = 0 \quad \text{...(i)}$$

and given line $\quad 3x - 4y - 1 = 0 \quad$ ...(ii)

Since, Eq. (i) and Eq. (ii) are identical, then comparing Eq. (i) and Eq. (ii), we get

$$\frac{x_1 - 1}{3} = \frac{y_1 + 2}{-4} = \frac{-x_1 + 2y_1 + 1}{-1}$$

or $$x_1 = -\frac{1}{5} \text{ and } y_1 = -\frac{2}{5}$$

$\therefore$ Point of contact is $\left(-\frac{1}{5}, -\frac{2}{5}\right)$.

Aliter for second part : Since, perpendicular line to tangent always passes through the centre of the circle, perpendicular line to

$$3x - 4y = 1 \quad \text{...(i)}$$

is $$4x + 3y = \lambda \quad \text{...(ii)}$$

which passes through $(1, -2)$, then

$$4 - 6 = \lambda$$

$\therefore$ $$\lambda = -2$$

From Eq. (ii), $\quad 4x + 3y = -2 \quad$ ...(iii)

Solving Eq. (i) and Eq. (iii), we get the point of contact i.e.

$x = -\frac{1}{5}$ and $y = -\frac{2}{5}$

Hence, point of contact is $\left(-\frac{1}{5}, -\frac{2}{5}\right)$.

Example 38. If $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$, prove that the point (l, m) lies on the circle $x^2 + y^2 = a^{-2}$.

Sol. Since, $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$.

Then, length of perpendicular from $(0, 0)$ on $lx + my = 1$ is equal to radius

then, $\frac{|-1|}{\sqrt{l^2 + m^2}} = a$ or $l^2 + m^2 = a^{-2}$

Hence, locus of (l, m) is $x^2 + y^2 = a^{-2}$

Aliter : Let the point of contact of line $lx + my = 1$ and circle $x^2 + y^2 = a^2$ is (x_1, y_1), then tangent of circle at (x_1, y_1) is $xx_1 + yy_1 = a^2$

Since, $\quad xx_1 + yy_1 = a^2 \quad$ and $\quad lx + my = 1$

are identical, then $\frac{x_1}{l} = \frac{y_1}{m} = \frac{a^2}{1}$

$\therefore$ $$x_1 = la^2, y_1 = ma^2$$

but (x_1, y_1) lie on $x^2 + y^2 = a^2$

then, $$l^2a^4 + m^2a^4 = a^2$$

$\therefore$ $$l^2 + m^2 = a^{-2}$$

$\therefore$ Locus of (l, m) is $x^2 + y^2 = a^{-2}$

Must

Example 39. Show that the line $(x - 2)\cos\theta + (y - 2)\sin\theta = 1$ touches a circle for all values of θ. Find the circle.

Sol. Given line is $(x - 2)\cos\theta + (y - 2)\sin\theta$

$$1 = \cos^2\theta + \sin^2\theta$$

On comparing

$$x - 2 = \cos\theta$$

and $$y - 2 = \sin\theta$$

Squaring and adding Eq. (i) and Eq. (ii), then

$$(x - 2)^2 + (y - 2)^2 = \cos^2\theta + \sin^2\theta$$

$$\Rightarrow \quad (x - 2)^2 + (y - 2)^2 = 1$$

or $$x^2 + y^2 - 4x - 4y + 7 = 0$$

Aliter : Since, tangent at $(\cos\theta, \sin\theta)$ of

$$x^2 + y^2 = 1$$

is $$x\cos\theta + y\sin\theta = 1$$

replacing x by $x - 2$ and y by $y - 2$ in Eqs. (i) and (ii), then

$$(x - 2)^2 + (y - 2)^2 = 1$$

$$(x - 2)\cos\theta + (y - 2)\sin\theta = 1$$

Hence, Eq. (iv) touches the circle Eq. (iii).

$\therefore$ Equation of circle is

$$(x - 2)^2 + (y - 2)^2 = 1$$

or $$x^2 + y^2 - 4x - 4y + 7 = 0$$

Normal to a Circle at a Given Point

The normal of a circle at any point is a straight line which is perpendicular to the tangent at the point and always passes through the centre of the circle.

Different form of the Equation of Normals

1. Point form :

Theorem : The equation of normal at the point $P(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is

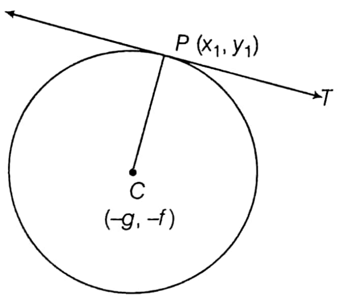
$$\frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f}$$

Proof :

Equation of the given circle is

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots(\text{i})$$

Its centre C is $(-g, -f)$



Let $P(x_1, y_1)$ be the given point.

$\because$ Normal of the circle at $P(x_1, y_1)$ passes through centre $C(-g, -f)$ of the circle.

Then, equation of normal CP passes through the points $C(-g, -f)$ and $P(x_1, y_1)$ is

$$y-y_1=\frac{(y_1+f)}{(x_1+g)}(x-x_1)$$

or

$$\frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f}$$

This is the required equation of normal at $P(x_1, y_1)$ of the given circle.

Remark

Easy method to find normal at (x_1, y_1) of second degree conics

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \ldots(\text{i})$$

then, according to determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

write first two rows as ax_1+hy_1+g and hx_1+by_1+f

Then, normal at (x_1, y_1) of conic (i)

$$\frac{x-x_1}{ax_1+hy_1+g}=\frac{y-y_1}{hx_1+by_1+f}$$

***Corollary* 1 :** Equation of normal of $x^2+y^2=a^2$ at (x_1, y_1) is

$$\frac{x-x_1}{1\cdot x_1+0+0}=\frac{y-y_1}{0+1\cdot y_1+0}$$

(Here, $g, f=0$ and $a=b=1$)

$$\Rightarrow \quad \frac{x-x_1}{x_1}=\frac{y-y_1}{y_1}$$

or

$$\frac{x}{x_1}=\frac{y}{y_1}$$

***Corollary* 2 :** Equation of normal of $x^2+y^2+2gx+2fy+c=0$ at (x_1, y_1) is

$$\frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f} \qquad (\text{Here, } a=b=1 \text{ and } h=0)$$

Remarks

1. Normal always passes through the centre of the circle. Just write the equation of the line joining (x_1, y_1) and the centre of the circle.
2. The equations of the normals show that they pass through the centre i.e. the normals are the radii which we know from *Euclidean geometry*.

Example 40. Find the equation of the normal to the circle $x^2+y^2=2x$, which is parallel to the line $x+2y=3$.

Sol. Given circle is $x^2+y^2-2x=0$

Centre of given circle is $(1, 0)$

Since, normal is parallel to $x+2y=3$

let the equation of normal is $x+2y=\lambda$

Since, normal passes through the centre of the circle i.e. $(1, 0)$

then $\quad 1+0=\lambda$

$\therefore \quad \lambda=1$

then, equation of normal is $x+2y=1$

or $\quad x+2y-1=0$

Aliter Equation of normal at (x_1, y_1) of $x^2+y^2-2x=0$ is

$$\frac{x-x_1}{x_1-1}=\frac{y-y_1}{y_1-0}$$

or $\quad$ Slope $=\dfrac{y_1}{x_1-1}=m_1 \quad$ (say)

Since normal is parallel to $x + 2y = 3$

$\therefore$ Slope $= -\frac{1}{2} = m_2$ (say)

but given $m_1 = m_2$

$$\frac{y_1}{x_1 - 1} = -\frac{1}{2} \text{ or } x_1 + 2y_1 - 1 = 0$$

$\therefore$ Locus of (x_1, y_1) is $x + 2y - 1 = 0$

Example 41. Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point $(5, 6)$.

Sol. Equation of the normal at (5, 6) is

$$\frac{x-5}{5-\frac{5}{2}} = \frac{y-6}{6+1} \Rightarrow \frac{x-5}{\frac{5}{2}} = \frac{y-6}{7} \Rightarrow \frac{2x-10}{5} = \frac{y-6}{7}$$

$\Rightarrow$ $14x - 70 = 5y - 30$

$\therefore$ $14x - 5y - 40 = 0$

Aliter I : Since, centre of the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ is $\left(\frac{5}{2}, -1\right)$, normal at (5,6) is the equation of a line, which passes through $\left(\frac{5}{2}, -1\right)$ and (5, 6) is

$$y + 1 = \frac{6+1}{5-\frac{5}{2}}\left(x - \frac{5}{2}\right) \Rightarrow y + 1 = \frac{14}{5}\left(x - \frac{5}{2}\right)$$

$\Rightarrow$ $y + 1 = \frac{7}{5}(2x - 5)$

$\Rightarrow$ $5y + 5 = 14x - 35$ or $14x - 5y - 40 = 0$

Aliter II : Equation of tangent at (5, 6) is

$$5 \cdot x + 6 \cdot y - \frac{5}{2}(x + 5) + (y + 6) - 48 = 0$$

$\Rightarrow$ $10x + 12y - 5x - 25 + 2y + 12 - 96 = 0$

$\Rightarrow$ $5x + 14y - 109 = 0$

Slope of tangent $= -\frac{5}{14}$

$\therefore$ Slope of normal $= \frac{14}{5}$

$\therefore$ Equation of normal at (5, 6) with slope $\frac{14}{5}$ is

$$y - 6 = \frac{14}{5}(x - 5)$$

$\Rightarrow$ $5y - 30 = 14x - 70$

or $14x - 5y - 40 = 0$

2. Parametric form

Since, parametric coordinates of circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$.

$\therefore$ Equation of normal at $(a\cos\theta, a\sin\theta)$ is

$$\frac{x}{a\cos\theta} = \frac{y}{a\sin\theta}$$

or $\frac{x}{\cos\theta} = \frac{y}{\sin\theta}$ or $y = x\tan\theta$

or $y = mx$, where $m = \tan\theta$

which is slope form of normal.

Exercise for Session 4

1. The length of the chord cut-off by $y = 2x + 1$ from the circle $x^2 + y^2 = 2$ is

(a) $\frac{5}{6}$ (b) $\frac{6}{5}$ (c) $\frac{6}{\sqrt{5}}$ (d) $\frac{\sqrt{5}}{6}$

2. Circle $x^2 + y^2 - 4x - 8y - 5 = 0$ will intersect the line $3x - 4y = \lambda$ in two distinct points, if

(a) $-10 < \lambda < 5$ (b) $9 < \lambda < 20$ (c) $-35 < \lambda < 15$ (d) $-16 < \lambda < 30$

3. If the line $3x - 4y + \lambda = 0$, $(\lambda > 0)$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b), then $\lambda + a + b$ is equal to

(a) -22 (b) -20 (c) 20 (d) 22

4. Tangent which is parallel to the line $x - 3y - 2 = 0$ of the circle $x^2 + y^2 - 4x + 2y - 5 = 0$, has point/points of contact

(a) $(1, -2)$ (b) $(-1, 2)$ (c) $(3, 4)$ (d) $(3, -4)$

5. If a circle, whose centre is $(-1, 1)$ touches the straight line $x + 2y = 12$, then the co-ordinates of the point of contact are

(a) $\left(-\frac{7}{2}, -4\right)$ (b) $\left(-\frac{18}{5}, -\frac{21}{5}\right)$ (c) $(2, -7)$ (d) $(-2, -5)$

6. The area of the triangle formed by the tangent at the point (a, b) to the circle $x^2 + y^2 = r^2$ and the coordinate axes is

(a) $\frac{r^4}{2ab}$ (b) $\frac{r^4}{2|ab|}$ (c) $\frac{r^4}{ab}$ (d) $\frac{r^4}{|ab|}$

7. The equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which make equal intercepts on the positive coordinate axes is

(a) $x + y = 2$ (b) $x + y = 2\sqrt{2}$ (c) $x + y = 4$ (d) $x + y = 8$

8. If $a > 2b > 0$, then the positive value of m for which $y = mx - b\sqrt{(1+m^2)}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x-a)^2 + y^2 = b^2$ is

(a) $\frac{2b}{\sqrt{(a^2 - 4b^2)}}$ (b) $\frac{\sqrt{(a^2 - 4b^2)}}{2b}$ (c) $\frac{2b}{a - 2b}$ (d) $\frac{b}{a - 2b}$

9. The angle between a pair of tangents from a point P to the circle $x^2 + y^2 = 16$ is $\frac{\pi}{3}$ and locus of P is $x^2 + y^2 = r^2$, then value of r is

(a) 5 (b) 6 (c) 7 (d) 8

10. The normal at the point $(3, 4)$ on a circle cuts the circle at the point $(-1, -2)$. Then, the equation of the circle is

(a) $x^2 + y^2 + 2x - 2y - 13 = 0$ (b) $x^2 + y^2 - 2x - 2y - 11 = 0$

(c) $x^2 + y^2 - 2x + 2y + 12 = 0$ (d) $x^2 + y^2 - 2x - 2y + 14 = 0$

11. The line $ax + by + c = 0$ is a normal to the circle $x^2 + y^2 = r^2$. The portion of the line $ax + by + c = 0$ intercepted by this circle is of length

(a) $\sqrt{r}$ (b) r (c) r^2 (d) $2r$

12. If the line $ax + by + c = 0$ touches the circle $x^2 + y^2 - 2x = \frac{3}{5}$ and is normal to the circle $x^2 + y^2 + 2x - 4y + 1 = 0$, then (a, b) are

(a) (1, 3) (b) (3, 1) (c) (1, 2) (d) (2, 1)

13. Show that for all values of θ, $x \sin\theta - y \cos\theta = a$ touches the circle $x^2 + y^2 = a^2$.

14. Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are (i) parallel (ii) perpendicular to the line $3x - 4y - 1 = 0$.

15. Find the equation of the family of circle which touch the pair of straight lines $x^2 - y^2 + 2y - 1 = 0$.

16. Find the value of λ so that the line $3x - 4y = \lambda$ may touch the circle $x^2 + y^2 - 4x - 8y - 5 = 0$.

17. Show that the area of the triangle formed by the positive X-axis, the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is $2\sqrt{3}$.

Session 5

Tangents from a Point to the Circle, Length of the Tangent from a Point to a Circle, Power of a Point with Respect to a Circle, Chord of Contact, Chord Bisected at a Given Point, Pair of Tangents, Director Circle

Tangent from a Point to the Circle

Theorem : From a given point two tangents can be drawn to a circle which are real, coincident or imaginary according as the given point lies outside,on or inside the circle .

Proof : If circle is $x^2 + y^2 = a^2$ ...(i)

any tangent to the circle Eq. (i) is

$$y = mx + a\sqrt{(1+m^2)} \quad \text{...(ii)}$$

If outside point is (x_1, y_1)

then, $y_1 = mx_1 + a\sqrt{1+m^2}$

or $(y_1 - mx_1)^2 = a^2(1+m^2)$

or $y_1^2 + m^2x_1^2 - 2mx_1y_1 = a^2 + a^2m^2$

$\Rightarrow$ $m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0$ ...(iii)

which is quadratic in m which gives two values of m. (real coincident or imaginary) corresponding to any value of x_1 and y_1.

The tangents are real, coincident or imaginary according as the values of m obtained from Eq. (iii) are real, coincident or imaginary.

or Discriminant >, =, or <0

$\Rightarrow$ $4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2)$ >, =, or <0

$\Rightarrow$ $\boxed{(x_1^2 + y_1^2 - a^2) >, = \text{ or } < 0}$

i.e. $P(x_1, y_1)$ lies outside, on or inside the circle $x^2 + y^2 = a^2$.

If P outside the circle, then substituting these values of m in Eq. (ii), we get the equation of tangents.

Aliter :

First write equation of line through (x_1, y_1) say

$$y - y_1 = m(x - x_1) \quad \text{...(i)}$$

which is tangent of the circle $x^2 + y^2 = a^2$, then length of perpendicular from centre (0, 0) to Eq. (i) = radius of the circle

then, $\dfrac{|mx_1 - y_1|}{\sqrt{(1+m^2)}} = a$

or $(mx_1 - y_1)^2 = a^2(1+m^2)$

or $m^2x_1^2 - 2mx_1y_1 + y_1^2 = a^2 + a^2m^2$

$\Rightarrow$ $m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0$

which is quadratic in m which gives two values of m.

Example 42. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ drawn from the point (1, 4).

Sol. Given circle is

$$x^2 + y^2 = 16 \quad \text{...(i)}$$

Any tangent of Eq. (i) in terms of slope is

$$y = mx + 4\sqrt{(1+m^2)} \quad \text{...(ii)}$$

which passes through (1, 4)

then, $4 = m + 4\sqrt{(1+m^2)}$

$\Rightarrow$ $(4 - m)^2 = 16(1+m^2)$

$\Rightarrow$ $15m^2 + 8m = 0$

$\therefore$ $m = 0, -\dfrac{8}{15}$

From Eq. (ii), equations of tangents drawn from (1, 4) are

$$y = 4$$

and $$y = -\frac{8}{15}x + 4\sqrt{\left(1 + \frac{64}{225}\right)}$$

or $8x + 15y = 68$ respectively.

Aliter : Equation of line through (1, 4) is $y - 4 = m(x - 1)$

$$\Rightarrow \quad mx - y + 4 - m = 0 \quad \text{...(i)}$$

Then, perpendicular length from centre (0, 0) to $mx - y + 4 - m = 0$ is equal to radius

then, $$\frac{|4 - m|}{\sqrt{m^2 + 1}} = 4$$

or $$(4 - m)^2 = 16(m^2 + 1)$$

$$\Rightarrow \quad 15m^2 + 8m = 0$$

$$\therefore \quad m = 0, -\frac{8}{15}$$

From Eq. (i), equation of tangents from (1, 4) are $y = 4$ and $8x + 15y = 68$, respectively.

| Example 43. The angle between a pair of tangents from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. Find the equation of the locus of the point P.

Sol. Let coordinates of P be (x_1, y_1) and given circle is

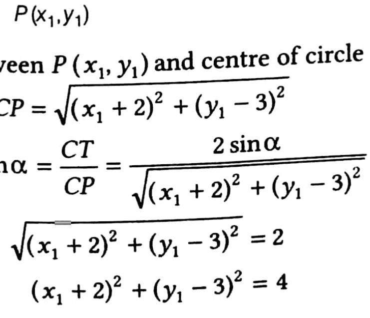
$$x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$$

or $$(x+2)^2 + (y-3)^2 - 4 - 9 + 9\sin^2\alpha + 13\cos^2\alpha = 0$$

$$\Rightarrow \quad (x+2)^2 + (y-3)^2 + 9\sin^2\alpha - 13(1 - \cos^2\alpha) = 0$$

$$\Rightarrow \quad (x+2)^2 + (y-3)^2 = 4\sin^2\alpha$$

$\therefore$ Centre and radius are $(-2, 3)$ and $2\sin\alpha$, respectively.



Distance between $P(x_1, y_1)$ and centre of circle $C(-2, 3)$ is

$$CP = \sqrt{(x_1 + 2)^2 + (y_1 - 3)^2}$$

In ΔPCT, $$\sin\alpha = \frac{CT}{CP} = \frac{2\sin\alpha}{\sqrt{(x_1+2)^2 + (y_1-3)^2}}$$

or $$\sqrt{(x_1+2)^2 + (y_1-3)^2} = 2$$

or $$(x_1+2)^2 + (y_1-3)^2 = 4$$

The required locus of $P(x_1, y_1)$ is

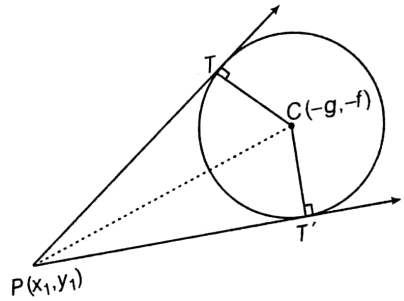
$$(x+2)^2 + (y-3)^2 = 4$$

Length of the Tangent from a Point to a Circle

Theorem : The length of tangent from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)} = \boxed{\sqrt{S_1}}$$

Proof : Let PT and PT' be two tangents from the given point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then, the centre and radius are $C(-g, -f)$ and $\sqrt{(g^2 + f^2 - c)}$ $(= CT = CT')$ respectively.



In ΔPCT,

$$PT = \sqrt{(PC)^2 - (CT)^2}$$

$$= \sqrt{(x_1 + g)^2 + (y_1 + f)^2 - g^2 - f^2 + c}$$

$$= \sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)} = \sqrt{S_1} = PT'$$

where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Remarks

1. To find length of tangent
   let $S = x^2 + y^2 + 2gx + 2fy + c$
   then, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
   where, $P(x_1, y_1)$
   $\therefore$ length of tangent $= \sqrt{S_1}$
2. For S_1 first write the equation of circle in general form i.e. coefficient of x^2 = coefficient of $y^2 = 1$ and making RHS of circle is zero, then let LHS by S.

| Example 44. Find the length of tangents drawn from the point $(3, -4)$ to the circle $2x^2 + 2y^2 - 7x - 9y - 13 = 0$.

Sol. The equation of the given circle is

$$2x^2 + 2y^2 - 7x - 9y - 13 = 0$$

Re-writing the given equation of the circle

i.e. $$x^2 + y^2 - \frac{7}{2}x - \frac{9}{2}y - \frac{13}{2} = 0$$

Let $S = x^2 + y^2 - \frac{7}{2}x - \frac{9}{2}y - \frac{13}{2}$

$\therefore \quad S_1 = (3)^2 + (-4)^2 - \frac{7}{2} \times 3 - \frac{9}{2} \times (-4) - \frac{13}{2}$

$= 25 - \frac{21}{2} + 18 - \frac{13}{2} = 43 - 17 = 26$

$\therefore$ Length of tangent $= \sqrt{S_1} = \sqrt{26}$

Example 45. If the length of tangent from (f, g) to the circle $x^2 + y^2 = 6$ be twice the length of the tangent from (f, g) to circle $x^2 + y^2 + 3x + 3y = 0$, then find the value of $f^2 + g^2 + 4f + 4g$.

Sol. According to the question

$$\sqrt{(g^2 + f^2 - 6)} = 2\sqrt{(f^2 + g^2 + 3f + 3g)}$$

On squaring $g^2 + f^2 - 6 = 4f^2 + 4g^2 + 12f + 12g$

or $\quad 3f^2 + 3g^2 + 12f + 12g + 6 = 0$

or $\quad f^2 + g^2 + 4f + 4g + 2 = 0$

or $\quad f^2 + g^2 + 4f + 4g = -2$

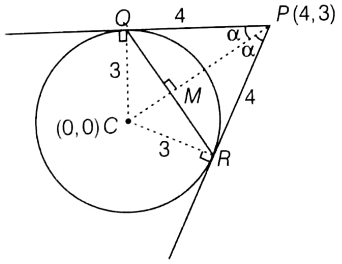
Must

Example 46. Show that the area of the triangle formed by tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line segment joining their points of contact is $7\frac{17}{25}$ square units in length.

Sol. Since, $PQ = PR = \sqrt{4^2 + 3^2 - 9} = 4$ units

$\therefore \quad \angle CPQ = \angle CPR = \alpha$ (let)

$\therefore \quad PC = \sqrt{(4-0)^2 + (3-0)^2} = 5$ units



$\therefore$ In ΔPQC, $\quad \tan\alpha = \frac{3}{4}$,

$\therefore \quad \sin\alpha = \frac{3}{5}$

and $\quad \cos\alpha = \frac{4}{5}$

In ΔPMQ, $\quad \cos\alpha = \frac{PM}{4} = \frac{4}{5}$

$\therefore \quad PM = \frac{16}{5}$

and $\quad \sin\alpha = \frac{QM}{4} = \frac{3}{5}$

$\therefore \quad QM = \frac{12}{5}$

$\therefore$ Area of $\Delta PQR = \frac{1}{2} \cdot QR \cdot PM$

$= \frac{1}{2}(2QM) \cdot PM = (QM)(PM)$

$= \left(\frac{12}{5}\right)\left(\frac{16}{5}\right) = \frac{192}{25}$

$= 7\frac{17}{25}$ sq units

Example 47. Show that the length of the tangent from any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is $\sqrt{(c_1 - c)}$.

Sol. Let (x_1, y_1) be any point on

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

then $\quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad$ ...(i)

$\therefore$ Length of tangent from (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is

$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_1}$

$= \sqrt{(-c + c_1)} = \sqrt{(c_1 - c)}$ [From Eq. (i)]

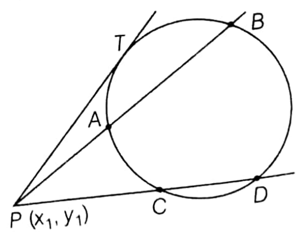
Power of a Point With Respect to a Circle

Theorem : The power of a point $P(x_1, y_1)$ with respect to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is } S_1$$

where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Proof : Let $P(x_1, y_1)$ be a point outside the circle and PAB and PCD drawn two secants. The power of $P(x_1, y_1)$ with respect to



$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

is equal to $PA \cdot PB$ which is

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = S_1$$

$\therefore$ Power remains constant for the circle i.e. independent of A and B.

$\therefore PA \cdot PB = PC \cdot PD = (PT)^2 = S_1 = (\sqrt{S_1})^2$

$\therefore PA \cdot PB = (\sqrt{S_1})^2$ = square of the length of tangent.

Remark

If P outside, inside or on the circle, then $PA \cdot PB$ is + ve, − ve or zero, respectively.

| Example 48. Find the power of point (2, 4) with respect to the circle $x^2 + y^2 - 6x + 4y - 8 = 0$

Sol. The power of the point (2, 4) with respect to the circle

$x^2 + y^2 - 6x + 4y - 8 = 0$ is $(\sqrt{S_1})^2$ or S_1

where, $S = x^2 + y^2 - 6x + 4y - 8$

$\therefore \quad S_1 = (2)^2 + (4)^2 - 6 \times 2 + 4 \times 4 - 8$

$= 4 + 16 - 12 + 16 - 8 = 16$

[$\because$ (2, 4) is outside from the circle $x^2 + y^2 - 6x + 4y - 8 = 0$]

| Example 49. Show that the locus of the point, the powers of which with respect to two given circles are equal, is a straight line.

Sol. Let the given circles be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

and
$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{...(ii)}$$

Let $P(x_1, y_1)$ be a point, the powers of which with respect to the circles Eqs. (i) and (ii) are equal. Then,

$$\therefore \quad [\sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)}]^2$$
$$= [\sqrt{(x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1)}]^2$$

or
$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$
$$= x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$$

$$\Rightarrow \quad 2(g - g_1)x_1 + 2(f - f_1)y_1 + c - c_1 = 0$$

then, locus of $P(x_1, y_1)$ is

$$2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$$

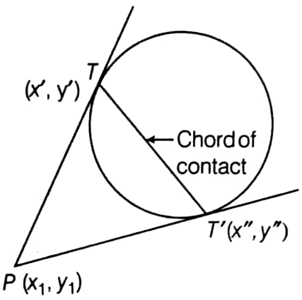
which is a straight line.

Chord of Contact

From any external point, two tangents can be drawn to a given circle. The chord joining the points of contact of the two tangents is called the chord of contact of tangents.

Theorem : The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

Sol. Let $T(x', y')$ and $T'(x'', y'')$ be the points of contact of tangents drawn from $P(x_1, y_1)$ to $x^2 + y^2 = a^2$.



Then, equations of tangents PT and PT' are

$$xx' + yy' = a^2 \text{ and } xx'' + yy'' = a^2 \text{ respectively.}$$

Since, both tangents pass through $P(x_1, y_1)$, then

$$x_1x' + y_1y' = a^2$$

and
$$x_1x'' + y_1y'' = a^2$$

$\because$ Points $T(x', y')$ and $T'(x'', y'')$ lie on

$$xx_1 + yy_1 = a^2$$

$\therefore$ Equation of **chord of contact** TT' is $xx_1 + yy_1 = a^2$

Remark

Equation of chord of contact like as equation of tangent at that point but point different.

Now, for chord of contact at (x_1, y_1), replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + x_1y}{2}$.

***Corollary* 1 :** If R is the radius of the circle and L is the length of the tangent from $P(x_1, y_1)$ on $S = 0$.

Here, $L = \sqrt{S_1}$, then By example 46

(a) Length of chord of contact $TT' = \frac{2LR}{\sqrt{(R^2 + L^2)}}$

(b) Area of triangle formed by the pair of tangents and its chord of contact $= \frac{RL^3}{R^2 + L^2}$

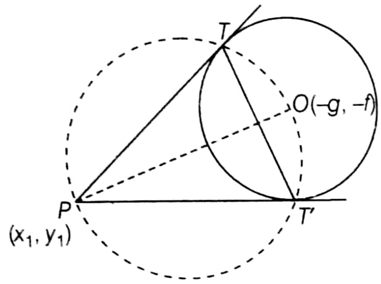
(c) Angle between the pair of tangents from $P(x_1, y_1)$
$= \tan^{-1}\left(\frac{2RL}{L^2 - R^2}\right)$

***Corollary* 2 :** Equation of the circle circumscribing the triangle PTT' is Must

$$(x - x_1)(x + g) + (y - y_1)(y + f) = 0,$$

where, $O(-g, -f)$ is the centre of the circle

$$x^2 + y^2 + 2gx + 2fx + c = 0$$



i.e. Required circle always passes through the centre of the given circle (Here, OP is the diameter of the required circle).

Example 50. If the pair of tangents are drawn from the point (4,5) to the circle $x^2+y^2-4x-2y-11=0$, then,

(i) Find the length of chord of contact.
(ii) Find the area of the triangle formed by a pair of tangents and their chord of contact.
(iii) Find the angle between the pair of tangents.

Sol. Here, $P \equiv (4,5)$,

$$R=\sqrt{((2)^2+(1)^2+11)}=4$$

and $L=\sqrt{S_1}=\sqrt{((4)^2+(5)^2-4\times4-2\times5-11)}=2$

(i) Length of chord of contact

$$=\frac{2LR}{\sqrt{(R^2+L^2)}}=\frac{2\times2\times4}{\sqrt{(4)^2+(2)^2}}=\frac{8}{\sqrt{5}}\text{ unit}$$

(ii) Area of triangle $=\dfrac{RL^3}{R^2+L^2}=\dfrac{4\times8}{16+4}=\dfrac{8}{5}$ sq units

(iii) Angle between the pair of tangents

$$=\pi+\tan^{-1}\left(\frac{2\times4\times2}{2^2-4^2}\right)$$

$$=\pi-\tan^{-1}\left(\frac{4}{3}\right) \qquad (\because L<R)$$

Example 51. Tangents PQ, PR are drawn to the circle $x^2+y^2=36$ from the point $P(-8,2)$ touching the circle at Q, R respectively. Find the equation of the circumcircle of ΔPQR.

Sol. Here, $P\equiv(-8,2)$ and $O\equiv(0,0)$

$\therefore$ Equation of the required circle is

$$(x-(-8))(x-0)+(y-2)(y-0)=0$$

or $\quad x^2+y^2+8x-2y=0 \qquad (\because OP$ is the diameter)

Example 52. Find the condition that chord of contact of any external point (h,k) to the circle $x^2+y^2=a^2$ should subtend right angle at the centre of the circle.

Sol. Equation of chord of contact AB is $hx+ky=a^2$

For equation of pair of tangents of OA and OB, make homogeneous $x^2+y^2=a^2$ with the help of $hx+ky=a^2$ or

$$\frac{hx+ky}{a^2}=1$$

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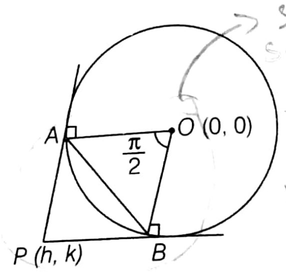


square so, $\sqrt{S_1}=R$ or $L=R$

then, $\quad x^2+y^2=a^2\left(\dfrac{hx+ky}{a^2}\right)^2$

or $\quad a^2(x^2+y^2)=(hx+ky)^2$

or $\quad x^2(a^2-h^2)-2hkxy+y^2(a^2-k^2)=0$

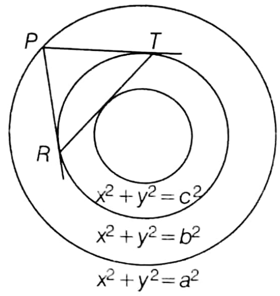
but $\quad \angle AOB=\dfrac{\pi}{2}$

$\therefore$ Coefficient of x^2 + Coefficient of $y^2=0$

$\Rightarrow \quad a^2-h^2+a^2-k^2=0$ or $h^2+k^2=2a^2$

Must **Example 53.** The chord of contact of tangents drawn from a point on the circle $x^2+y^2=a^2$ to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$. Show that a,b,c are in GP.

Sol. Let $P(a\cos\theta, a\sin\theta)$ be a point on the circle $x^2+y^2=a^2$. Then, equation of chord of contact of tangents drawn from $P(a\cos\theta, a\sin\theta)$ to the circle $x^2+y^2=b^2$ is



$$ax\cos\theta+ay\sin\theta=b^2$$

This touches the circle $\quad x^2+y^2=c^2$

$\therefore$ Length of perpendicular from (0, 0) to Eq. (i) = radius of Eq. (ii)

$\therefore \quad \dfrac{|0+0-b^2|}{\sqrt{(a^2\cos^2\theta+a^2\sin^2\theta)}}=c$

or $\quad b^2=ac$

$\Rightarrow \quad a, b, c$ are in GP.

Chord Bisected at a Given Point

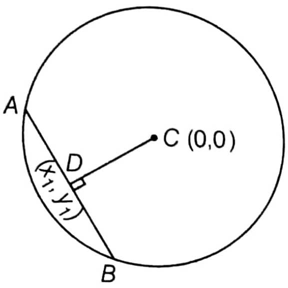
Theorem : The equation of the chord of the circle $x^2 + y^2 = a^2$ bisected at the point (x_1, y_1) is given by

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$$

or $$\boxed{T = S_1}$$

Proof : Let any chord AB of the circle $x^2 + y^2 = a^2$ be bisected at $D(x_1, y_1)$.

If centre of circle is represented by C



then, slope of $DC = \dfrac{0 - y_1}{0 - x_1} = \dfrac{y_1}{x_1}$

$\therefore$ Slope of the chord AB is $-\dfrac{x_1}{y_1}$

then, equation of AB is $y - y_1 = -\dfrac{x_1}{y_1}(x - x_1)$

or $$yy_1 - y_1^2 = -xx_1 + x_1^2$$

or $$xx_1 + yy_1 = x_1^2 + y_1^2$$

or $$\mathbf{xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2 \text{ or } T = S_1}$$

Remarks

1. The equation of chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, which is bisected at (x_1, y_1) ; is $T = S_1$
   where, $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
   and $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
2. The chord bisected at point (x_1, y_1) is the farthest from the centre among all the chords passing through the point (x_1, y_1). Also, for such chord, the length of the chord is minimum. Must

| Example 54. Find the equation of the chord of $x^2 + y^2 - 6x + 10y - 9 = 0$ which is bisected at $(-2, 4)$.

Sol. The equation of the required chord is

$$-2x + 4y - 3(x - 2) + 5(y + 4) - 9$$
$$= 4 + 16 + 12 + 40 - 9$$
$$\Rightarrow \quad -5x + 9y - 46 = 0$$
or $$5x - 9y + 46 = 0$$

| Example 55. Find the middle point of the chord intercepted on line $lx + my + n = 0$ by the circle $x^2 + y^2 = a^2$.

Sol. Let (x_1, y_1) be the middle point of the chord intercepted by the circle $x^2 + y^2 = a^2$ on the line $lx + my + n = 0$. Then, equation of the chord of the circle $x^2 + y^2 = a^2$, whose middle points is (x_1, y_1), is

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$$

or $$xx_1 + yy_1 = x_1^2 + y_1^2 \quad \text{...(i)}$$

Clearly, $lx + my + n = 0$ and Eq. (i) represented the same line,

$$\frac{x_1}{l} = \frac{y_1}{m} = \frac{x_1^2 + y_1^2}{-n} = \lambda \quad \text{(say)}$$

$$\therefore \quad \left.\begin{array}{l} x_1 = l\lambda \\ y_1 = m\lambda \end{array}\right\} \quad \text{...(ii)}$$

and $$x_1^2 + y_1^2 = -n\lambda$$

or $$l^2\lambda^2 + m^2\lambda^2 = -n\lambda \quad \text{[from (ii)]}$$

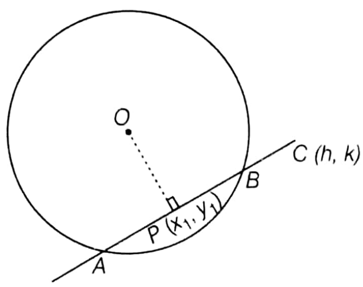
$$\therefore \quad \lambda = -\frac{n}{l^2 + m^2}$$

So, from Eq. (ii), $x_1 = -\dfrac{nl}{l^2 + m^2}, y_1 = -\dfrac{mn}{l^2 + m^2}$

Hence, the required point is $\left(-\dfrac{nl}{l^2 + m^2}, -\dfrac{nm}{l^2 + m^2}\right)$.

| Example 56. Through a fixed point (h, k), secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of mid-point of the portions of secants intercepted by the circle is $x^2 + y^2 = hx + ky$.

Sol. Let $P(x_1, y_1)$ be the middle point of any chord AB, which passes through the point $C(h, k)$.



Equation of chord AB is $T = S_1$

$$\therefore \quad xx_1 + yy_1 - r^2 = x_1^2 + y_1^2 - r^2$$

or $$x_1^2 + y_1^2 = xx_1 + yy_1$$

But since AB passes through $C(h, k)$, then

$$x_1^2 + y_1^2 = hx_1 + ky_1$$

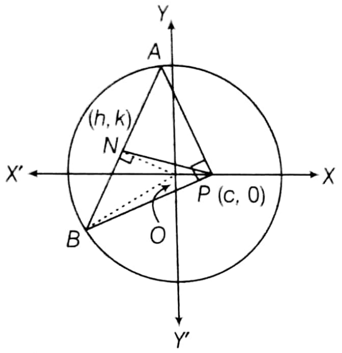
$\therefore$ Locus of $P(x_1, y_1)$ is $x^2 + y^2 = hx + ky$

Example 57. Find the locus of middle points of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point $(c, 0)$.

Sol. Let $N(h, k)$ be the middle point of any chord AB, which subtend a right angle at $P(c, 0)$.

Since, $\angle APB = 90^\circ$

$\therefore \quad NA = NB = NP$



(since distance of the vertices from middle point of the hypotenuse are equal)

or $\quad (NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2 \quad \ldots$(i)

But also $\quad \angle BNO = 90^\circ$

$\therefore \quad (OB)^2 = (ON)^2 + (NB)^2$

$\Rightarrow \quad -(NB)^2 = (ON)^2 - (OB)^2$

$\Rightarrow \quad -[(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$

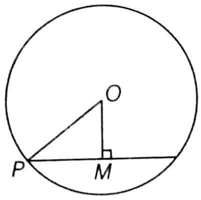
or $\quad 2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$

$\therefore$ Locus of $N(h, k)$ is

$$2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$$

Example 58. Find the equation of the chord of the circle $x^2 + y^2 = r^2$ passing through the point $(2,3)$ farthest from the centre.

Sol. Let $P \equiv (2,3)$ be the given point and M be the middle point of chord of circle $x^2 + y^2 = r^2$ through P.



must

Then, $\quad (OM)^2 = (OP)^2 - (PM)^2$

If OM maximum, then PM is minimum. i.e. P coincides with M, which is middle point of the chord.

Hence, the equation of the chord is

$$T = S_1$$

i.e. $\quad 2x + 3y - r^2 = 2^2 + 3^2 - r^2$

or $\quad 2x + 3y = 13$

Pair of Tangents

Theorem : The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is

$$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

or $\quad \boxed{SS_1 = T^2}$

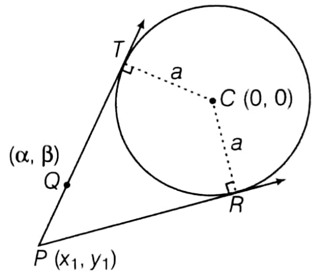
where, $\quad S = x^2 + y^2 - a^2, S_1 = x_1^2 + y_1^2 - a^2$

and $\quad T = xx_1 + yy_1 - a^2$

Proof : The given circle is $x^2 + y^2 = a^2$

Its centre and radius are $C(0, 0)$ and a respectively. Given external point be $P(x_1, y_1)$.

From point $P(x_1, y_1)$ two tangents PT and PR be drawn to the circle, touching circle at T and R respectively.



Let $Q(\alpha, \beta)$ on PT, then equation of PQ is

$$y - y_1 = \frac{\beta - y_1}{\alpha - x_1}(x - x_1)$$

or $\quad y(\alpha - x_1) - x(\beta - y_1) - \alpha y_1 + \beta x_1 = 0$

Length of perpendicular from $C(0, 0)$ on $PT = a$ (radius)

$\Rightarrow \quad \dfrac{|\beta x_1 - \alpha y_1|}{\sqrt{(\alpha - x_1)^2 + (\beta - y_1)^2}} = a$

or $\quad (\beta x_1 - \alpha y_1)^2 = a^2\{(\alpha - x_1)^2 + (\beta - y_1)^2\}$

$\therefore$ Locus of $Q(\alpha, \beta)$ is

$$(yx_1 - xy_1)^2 = a^2\{(x - x_1)^2 + (y - y_1)^2\}$$

$\Rightarrow \quad y^2x_1^2 + x^2y_1^2 - 2xyx_1y_1 = a^2\{x^2 + x_1^2 - 2xx_1 + y^2 + y_1^2 - 2yy_1\}$

$\Rightarrow y^2(x_1^2 - a^2) + x^2(y_1^2 - a^2) - a^2(x_1^2 + y_1^2)$
$= 2xyx_1y_1 - 2a^2xx_1 - 2a^2yy_1$

On adding both sides, $(x^2x_1^2 + y^2y_1^2 + a^4)$, then

$$y^2(x_1^2 + y_1^2 - a^2) + x^2(x_1^2 + y_1^2 - a^2) - a^2(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

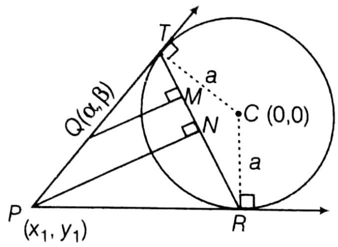
$$\Rightarrow (x^2+y^2-a^2)(x_1^2+y_1^2-a^2)=(xx_1+yy_1-a^2)^2$$

This is the required equation of pair of tangents drawn from (x_1, y_1) to circle $x^2+y^2=a^2$.

Aliter : Let circle be $x^2+y^2=a^2$ with centre $C(0,0)$ and radius a. Length of tangents from $P(x_1, y_1)$ and $Q(\alpha, \beta)$ are

$$PT=\sqrt{x_1^2+y_1^2-a^2}$$

and $$QT=\sqrt{\alpha^2+\beta^2-a^2}$$



Now, equation of TR (chord of contact is)

$$xx_1+yy_1-a^2=0$$

$$\therefore \quad PN=\frac{|x_1^2+y_1^2-a^2|}{\sqrt{(x_1^2+y_1^2)}} \text{ and } QM=\frac{|\alpha x_1+\beta y_1-a^2|}{\sqrt{(x_1^2+y_1^2)}}$$

But from similar Δ^s PNT and QMT,

$$\frac{PN}{QM}=\frac{PT}{QT} \Rightarrow \frac{(PN)^2}{(QM)^2}=\frac{(PT)^2}{(QT)^2}$$

$$\Rightarrow \quad \frac{\dfrac{(x_1^2+y_1^2-a^2)^2}{(x_1^2+y_1^2)}}{\dfrac{(\alpha x_1+\beta y_1-a^2)^2}{(x_1^2+y_1^2)}}=\frac{x_1^2+y_1^2-a^2}{\alpha^2+\beta^2-a^2}$$

$$\Rightarrow \quad (x_1^2+y_1^2-a^2)(\alpha^2+\beta^2-a^2)=(\alpha x_1+\beta y_1-a^2)^2$$

$\therefore$ Locus of $Q(\alpha, \beta)$ is

$$(x^2+y^2-a^2)(x_1^2+y_1^2-a^2)$$
$$=(xx_1+yy_1-a^2)^2$$

This is the required equation of pair of tangents drawn from (x_1, y_1) to circle $x^2+y^2=a^2$.

Corollary : The angle between the two tangents from (x_1, y_1) to the circle $x^2+y^2=a^2$ is $2\tan^{-1}\left(\dfrac{a}{\sqrt{S_1}}\right)$, where $S_1=x_1^2+y_1^2-a^2$.

Remarks

1. Equation of pair of tangents in notation form is $SS_1=T^2$

   where, $S \equiv x^2+y^2-a^2$

   $S_1 \equiv x_1^2+y_1^2-a^2, T \equiv xx_1+yy_1-a^2$

2. When circle is $x^2+y^2+2gx+2fy+c=0$ and tangents are drawn from (x_1, y_1), then pair of tangents is

   $(x^2+y^2+2gx+2fy+c)(x_1^2+y_1^2+2gx_1+2fy_1+c)$

   $=[xx_1+yy_1+g(x+x_1)+f(y+y_1)+c]^2$

   where, $S \equiv x^2+y^2+2gx+2fy+c,$

   $S_1 \equiv x_1^2+y_1^2+2gx_1+2fy_1+c,$

   and $T \equiv xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$

Advised for Students

Students are advised that, if they do not want to use the formula $(SS_1=T^2)$, then use the following method :

Let $y-y_1=m(x-x_1)$ be any line through $P(x_1, y_1)$.

Then, use condition of tangency $p=r$ i.e.

Length of perpendicular from the centre of circle or this line = radius of the circle.

Gives the values of m. In such away, we can find the equations of tangents from P.

Example 59. Find the equations of the tangents from the point $A(3,2)$ to the circle $x^2+y^2+4x+6y+8=0$.

Sol. Combined equation of the pair of tangents drawn from $A(3,2)$ to the given circle $x^2+y^2+4x+6y+8=0$ can be written in the usual notation.

$$T^2=SS_1 \text{ namely}$$

$$\Rightarrow [3x+2y+2(x+3)+3(y+2)+8]^2$$
$$=[x^2+y^2+4x+6y+8][9+4+12+12+8]$$
$$\Rightarrow (5x+5y+20)^2=45(x^2+y^2+4x+6y+8)$$
$$\Rightarrow 5(x+y+4)^2=9(x^2+y^2+4x+6y+8)$$
$$\Rightarrow 5(x^2+y^2+2xy+8x+8y+16)$$
$$=9(x^2+y^2+4x+6y+8)$$
$$\Rightarrow 4x^2+4y^2-10xy-4x+14y-8=0$$
$$\text{or} \quad 2x^2+2y^2-5xy-2x+7y-4=0$$
$$\text{or} \quad (2x-y-4)(x-2y+1)=0$$

Hence, the required tangents to the circle from $A(3,2)$ are

$$2x-y-4=0 \quad \text{and} \quad x-2y+1=0$$

Aliter : Let $S \equiv x^2+y^2+4x+6y+8=0$

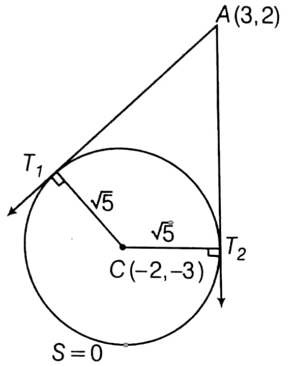
Centre $C(-2,-3)$ and radius $=\sqrt{5}$

Let the slope of a tangent from A to '$S=0$' be m, then equation of tangent is

$$y - 2 = m(x - 3)$$

or $$mx - y + 2 - 3m = 0 \quad \text{...(i)}$$

Length of perpendicular from $C(-2, -3)$ on Eq. (i) = radius of circle.



$\Rightarrow$ $$\frac{|-2m + 3 + 2 - 3m|}{\sqrt{(m^2 + 1)}} = \sqrt{5}$$

or $$(5m - 5)^2 = 5(m^2 + 1)$$

$\Rightarrow$ $$25m^2 - 50m + 25 = 5m^2 + 5$$

$\Rightarrow$ $$20m^2 - 50m + 20 = 0$$

or $$2m^2 - 5m + 2 = 0$$

or $$(2m - 1)(m - 2) = 0$$

$\therefore$ $$m = \frac{1}{2} \text{ or } m = 2$$

Substituting these values of m in Eq. (i), we get the equations of two tangents are $x - 2y + 1 = 0$ and $2x - y - 4 = 0$.

Director Circle

Director circle : The locus of the point of intersection of two perpendicular tangents to a given circle is known as its director circle.

Theorem : The equation of the director circle of the circle $x^2 + y^2 = a^2$ is

$$\boxed{x^2 + y^2 = 2a^2}$$

Proof : The equation of any tangent to the circle $x^2 + y^2 = a^2$

is $$y = mx + a\sqrt{(1 + m^2)} \quad \text{...(i)}$$

Let $P(h, k)$ be the point of intersection of tangents, then $P(h, k)$ lies on Eq. (i)

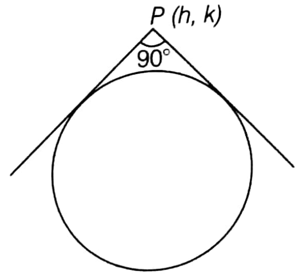
$\therefore$ $$k = mh + a\sqrt{(1 + m^2)}$$

or $$(k - mh)^2 = a^2(1 + m^2)q$$

or $$m^2(h^2 - a^2) - 2mkh + k^2 - a^2 = 0$$

This is quadratic equation in m, let two roots are m_1 and m_2.

But tangents are perpendiculars, then



$$m_1 m_2 = -1$$

$\Rightarrow$ $$\frac{k^2 - a^2}{h^2 - a^2} = -1 \text{ or } k^2 - a^2 = -h^2 + a^2$$

or $$h^2 + k^2 = 2a^2$$

Hence, locus of $P(h, k)$ is $\boldsymbol{x^2 + y^2 = 2a^2}$

Aliter : The combined equation of the pair of tangents drawn from (h, k) to $x^2 + y^2 = a^2$ is

$$SS_1 = T^2$$

where, $$S = x^2 + y^2 - a^2$$

$$S_1 = h^2 + k^2 - a^2$$

and $$T = hx + ky - a^2$$

$\therefore$ $$(x^2 + y^2 - a^2)(h^2 + k^2 - a^2) = (hx + ky - a^2)^2$$

This equation will represent a pair of perpendicular lines if, coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow$ $$h^2 + k^2 - a^2 - h^2 + h^2 + k^2 - a^2 - k^2 = 0$$

$\Rightarrow$ $$h^2 + k^2 - 2a^2 = 0 \text{ or } h^2 + k^2 = 2a^2$$

Hence, the locus of (h, k) is $\boldsymbol{x^2 + y^2 = 2a^2}$

Remarks

1. The equation of the director circle of the circle $(x - h)^2 + (y - k)^2 = a^2$ is $(x - h)^2 + (y - k)^2 = 2a^2$
2. The equation of the director circle of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + c = g^2 + f^2 - c$
3. If two tangents are drawn from a point on the director circle to the circle, then angle between tangents is 90°.

Example 60. If two tangents are drawn from a point on the circle $x^2 + y^2 = 50$ to the circle $x^2 + y^2 = 25$, then find the angle between the tangents.

Sol. $\because$ $x^2 + y^2 = 50$ is the director circle of $x^2 + y^2 = 25$

Hence, angle between tangents $= 90°$

Exercise for Session 5

1. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the Y-axis, then the length PQ is

(a) 4 (b) $2\sqrt{5}$ (c) 5 (d) $3\sqrt{5}$

2. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is touched by $y = x$ at P such that $OP = 6\sqrt{2}$, where O is origin, then the value of c is

(a) 36 (b) 72 (c) 144 (d) 288

3. The chord of contact of tangents from a point P to a circle passes through Q. If l_1 and l_2 are the lengths of tangents from P and Q to the circle, then PQ is equal to

(a) $\frac{l_1 + l_2}{2}$ (b) $\frac{l_1 - l_2}{2}$ (c) $\sqrt{(l_1^2 + l_2^2)}$ (d) $\sqrt{(l_1^2 - l_2^2)}$

4. If the chord of contact of tangents from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ touches the circle $(x - a)^2 + y^2 = a^2$, then the locus of (x_1, y_1) is

(a) a circle (b) a parabola (c) an ellipse (d) a hyperbola

5. The locus of the mid-points of a chord of the circle $x^2 + y^2 = 4$, which subtends a right angle at the origin is

(a) $x + y = 1$ (b) $x^2 + y^2 = 1$ (c) $x + y = 2$ (d) $x^2 + y^2 = 2$

6. The length of tangents from $P(1, -1)$ and $Q(3, 3)$ to a circle are $\sqrt{2}$ and $\sqrt{6}$ respectively, then the length of tangent from $R(-2, -7)$ to the same circle is

(a) $\sqrt{41}$ (b) $\sqrt{51}$ (c) $\sqrt{61}$ (d) $\sqrt{71}$

7. If the angle between the tangents drawn to $x^2 + y^2 + 2gx + 2fy + c = 0$ from $(0,0)$ is $\frac{\pi}{2}$, then

(a) $g^2 + f^2 = 3c$ (b) $g^2 + f^2 = 2c$ (c) $g^2 + f^2 = 5c$ (d) $g^2 + f^2 = 4c$

8. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through a fixed point

(a) $(2, 4)$ (b) $\left(-\frac{1}{2}, -\frac{1}{4}\right)$ (c) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (d) $(-2, -4)$

9. The length of tangent from $(0, 0)$ to the circle $2(x^2 + y^2) + x - y + 5 = 0$ is

(a) $\sqrt{5}$ (b) $\sqrt{\left(\frac{5}{2}\right)}$ (c) $\frac{\sqrt{5}}{2}$ (d) $\sqrt{2}$

10. The perpendicular tangents to the circle $x^2 + y^2 = a^2$ meet at P. Then, the locus of P has the equation

(a) $x^2 + y^2 = 2a^2$ (b) $x^2 + y^2 = 3a^2$ (c) $x^2 + y^2 = 4a^2$ (d) $x^2 + y^2 = 5a^2$

11. The tangents to $x^2 + y^2 = a^2$ having inclinations α and β intersect at P. If $\cot\alpha + \cot\beta = 0$, then the locus of P is

(a) $x + y = 0$ (b) $x - y = 0$ (c) $xy = 0$ (d) $xy = 1$

12. The exhaustive range of values of a such that the angle between the pair of tangents drawn from (a, a) to the circle $x^2 + y^2 - 2x - 2y - 6 = 0$ lies in the range $\left(\frac{\pi}{3}, \pi\right)$ is

(a) $(-1, 3)$ (b) $(-5, -3) \cup (3, 5)$ (c) $(-3, 5)$ (d) $(-3, -1) \cup (3, 5)$

13. Distances from the origin to the centres of the three circles $x^2 + y^2 - 2\lambda x = c^2$, where c is a constant and λ is available, are in GP. Prove that the lengths of tangents drawn from any point on the circle $x^2 + y^2 = c^2$ to the three circles are also in GP.

14. Find the area of the quadrilateral formed by a pair of tangents from the point $(4, 5)$ to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and a pair of its radii.

15. If the length of the tangent from a point (f, g) to the circle $x^2 + y^2 = 4$ be four times the length of the tangent from it to the circle $x^2 + y^2 = 4x$, show that $15f^2 + 15g^2 - 64f + 4 = 0$.

16. Find the equation of that chord of the circle $x^2 + y^2 = 15$ which is bisected at $(3, 2)$.

17. The chords of contact of the pair of tangents to the circle $x^2 + y^2 = 1$ drawn from any point on the line $2x + y = 4$ pass through the point (α, β), then find $\alpha^2 + \beta^2$.

Session 6

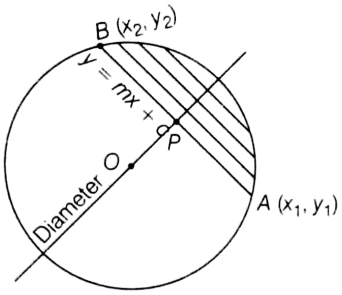
Diameter of a Circle, Two Circles Touching Each Other, Common Tangents to Two Circles, Common Chord of Two Circles, Family of Circles

Diameter of a Circle

The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.

Let the circle be $x^2 + y^2 = a^2$ and equation of parallel chord is

$$y = mx + c$$



Let $P(h, k)$ be the middle point of the chord $y = mx + c$. Since, P is the mid-point of $A(x_1\ y_1)$ and $B(x_2, y_2)$, then

$$\frac{x_1 + x_2}{2} = h \quad \text{and} \quad \frac{y_1 + y_2}{2} = k$$

or $\quad x_1 + x_2 = 2h \quad$ and $\quad y_1 + y_2 = 2k \quad$ ...(i)

$\because \quad P(h, k)$ lie on $y = mx + c$

then, $\quad k = mh + c$

or $\quad k - mh = c \quad$ ...(ii)

Substituting $y = mx + c$ in $x^2 + y^2 = a^2$

then, $\quad x^2 + (mx + c)^2 = a^2$

or $\quad (1 + m^2)\, x^2 + 2mcx + c^2 - a^2 = 0 \quad$ ...(iii)

Let x_1, x_2 are roots of Eq. (iii), then

$$x_1 + x_2 = -\frac{2mc}{1 + m^2}$$

$\Rightarrow \quad 2h = -\dfrac{2m}{(1 + m^2)}(k - mh) \quad$ [from Eq. (i) and Eq. (ii)]

$\Rightarrow \quad h + m^2 h = -mk + m^2 h \quad \Rightarrow \quad \boxed{h + mk = 0}$

Hence, locus of (h, k) is $x + my = 0$

Aliter : Let (h, k) be the middle point of the chord $y = mx + c$ of the circle $x^2 + y^2 = a^2$

then, $\quad T = S_1 \quad \Rightarrow \quad xh + ky = h^2 + k^2$

$$\text{slope} = -\frac{h}{k} = m \quad \Rightarrow \quad h + mk = 0$$

Hence, locus of mid-point is $x + my = 0$.

> ***Remark***
>
> The diameter of circle always passes through the centre of the circle and perpendicular to the parallel chords.
>
> Let circle is $x^2 + y^2 = a^2$ and parallel chord be $y = mx + c$, then equation of line $\perp$ to $y = mx + c$ is
>
> $$my + x + \lambda = 0$$
>
> which passes through origin (centre)
>
> then, $\quad 0 + 0 + \lambda = 0 \quad \therefore \quad \lambda = 0$
>
> Then, equation of diameter from Eq. (i) is $x + my = 0$.

Example 61. Find the equation of the diameter of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which corresponds to the chord $ax + by + d = 0$.

Sol. The diameter of circle passes through the centre of the circle and perpendicular to the chord $ax + by + d = 0$ is

$$bx - ay + \lambda = 0$$

which passes through centre of circle i.e. $(-g, -f)$

Then, $\quad -bg + af + \lambda = 0$

$\therefore \quad \lambda = bg - af$

From Eq. (i), the equation of the diameter is

$$bx - ay + bg - af = 0$$

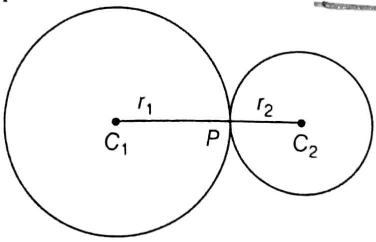
Two Circles Touching Each Other

1. When two circles touch each other externally

Then, distance between their centres = sum of their radii

i.e. $\quad |C_1C_2| = r_1 + r_2$

In such cases, the point of contact P divides the line joining C_1 and C_2 internally in the ratio $r_1 : r_2$



$$\Rightarrow \quad \frac{C_1P}{C_2P} = \frac{r_1}{r_2}$$

If $\quad C_1 \equiv (x_1, y_1)$ and $C_2 \equiv (x_2, y_2)$

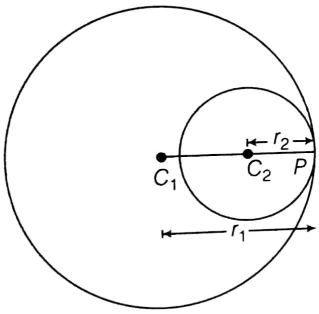
then, coordinate of P is $\left(\dfrac{r_1x_2 + r_2x_1}{r_1 + r_2}, \dfrac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$

2. When two Circles Touch each other Internally

Then, distance between their centres =Difference of their radii

i.e. $\quad |C_1C_2| = |r_1 - r_2|$

In such cases, the point of contact P divides the line joining C_1 and C_2 externally in the ratio $r_1 : r_2$



$$\Rightarrow \quad \frac{C_1P}{C_2P} = \frac{r_1}{r_2}$$

If $\quad C_1 \equiv (x_1, y_1)$ and $C_2 \equiv (x_2, y_2)$

then, coordinates of P is $\left(\dfrac{r_1x_2 - r_2x_1}{r_1 - r_2}, \dfrac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$

| Example 62. Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.

Sol. Given circles are

$$x^2 + y^2 - 2x - 4y = 0 \quad \text{...(i)}$$

and

$$x^2 + y^2 - 8y - 4 = 0 \quad \text{...(ii)}$$

Let centres and radii of circles Eqs. (i) and (ii) are represented by C_1, r_1 and C_2, r_2, respectively.

$\therefore \quad C_1 \equiv (1, 2), r_1 = \sqrt{(1 + 4)}$ or $r_1 = \sqrt{5}$

and $\quad C_2 \equiv (0, 4), r_2 = \sqrt{0 + 16 + 4}$ or $r_2 = 2\sqrt{5}$

Now, $C_1C_2 = \sqrt{(1 - 0)^2 + (2 - 4)^2}$

$$C_1C_2 = \sqrt{5} = r_2 - r_1$$

Hence, the two circles touch each other internally.

| Example 63. Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$.

Sol. Given circles are

$$x^2 + y^2 + 2ax + c^2 = 0 \quad \text{...(i)}$$

and

$$x^2 + y^2 + 2by + c^2 = 0 \quad \text{...(ii)}$$

Let C_1 and C_2 be the centres of circles Eqs. (i) and (ii), respectively and r_1 and r_2 be their radii, then

$$C_1 = (-a, 0), C_2 = (0, -b),$$

$$r_1 = \sqrt{(a^2 - c^2)}, r_2 = \sqrt{(b^2 - c^2)}$$

Here, we do not find the two circles touch each other internally or externally.

For touch, $|C_1C_2| = |r_1 \pm r_2|$

or $\quad \sqrt{(a^2 + b^2)} = |\sqrt{(a^2 - c^2)} \pm \sqrt{(b^2 - c^2)}|$

On squaring

$$a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2 - c^2)}\sqrt{(b^2 - c^2)}$$

or $\quad c^2 = \pm\sqrt{a^2b^2 - c^2(a^2 + b^2) + c^4}$

Again, squaring,

$$c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$$

or $\quad c^2(a^2 + b^2) = a^2b^2$ or $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$

Common Tangents to Two Circles

Different Cases of Intersection of Two Circles :

Let the two circles be

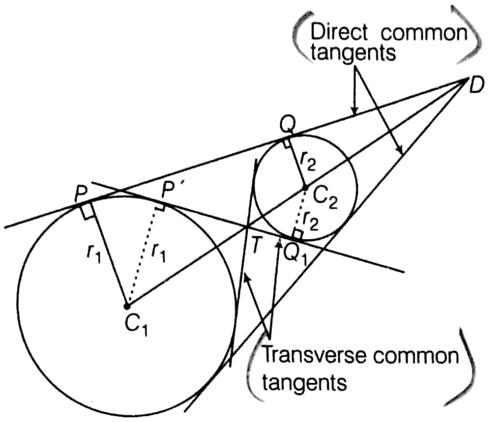
$$(x - x_1)^2 + (y - y_1)^2 = r_1^2 \quad \text{...(i)}$$

and

$$(x - x_2)^2 + (y - y_2)^2 = r_2^2 \quad \text{...(ii)}$$

with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2 respectively. Then following cases may arise :

Case I : When $|C_1C_2| > r_1 + r_2$ i.e. the distance between the centres is greater than the sum of their radii.



$\because \quad C_1 \equiv (x_1, y_1), C_2 \equiv (x_2, y_2)$

$$\therefore \quad D \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2} \right) \equiv (\alpha, \beta) \text{ (say)}$$

$$\text{and} \quad T \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2} \right)$$

$$\equiv (\gamma, \delta) \text{ (say)}$$

How to find direct common tangents Let equation of common tangent through $D(\alpha, \beta)$ is

$$y - \beta = m(x - \alpha) \qquad \text{...(i)}$$

Now, length of $\perp$ from C_1 or C_2 on Eq. (i) $= r_1$ or r_2

Then, we get two values of m.

Substituting the values of m in Eq. (i), we get two direct common tangents.

How to find transverse common tangents Let equation of common tangent through $T(\gamma, \delta)$ is

$$y - \delta = M(x - \gamma) \qquad \text{...(ii)}$$

Now, length of $\perp$ from C_1 or C_2 on Eq. (i) $= r_1$ or r_2

then, we get two values of M.

Substituting the values of M in Eq. (i), we get two transverse common tangents.

Remark

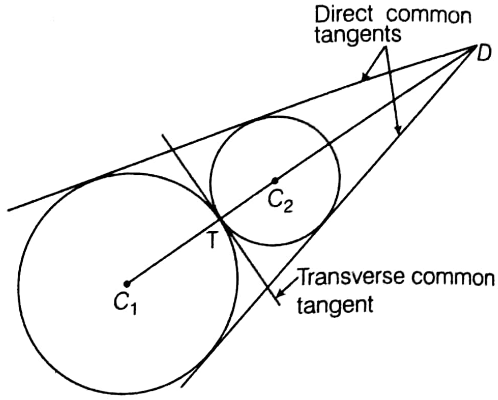
In this case circles neither cut nor touch i.e. Number of solutions of two circles is zero.

Case II : When $|C_1C_2| = r_1 + r_2$

i.e. the distance between the centres is equal to the sum of their radii.

In this case two direct common tangents are real and distinct while the transverse tangents are coincident.

How to find transverse common tangent



$\because$ Equation of circles are

$$S_1 \equiv (x - x_1)^2 + (y - y_1)^2 - r_1^2 = 0$$

$$\text{and} \quad S_2 \equiv (x - x_2)^2 + (y - y_2)^2 - r_2^2 = 0$$

then, equation of common tangent is

$$S_1 - S_2 = 0$$

which is same as equation of common chord.

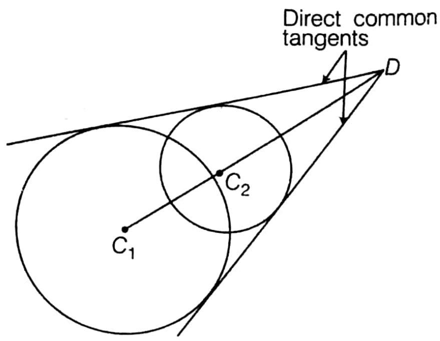
Remark

In this case circles touch at one point i.e. Number of solutions of two circles is one.

Case III : When $|r_1 - r_2| < |C_1C_2| < r_1 + r_2$

i.e. the distance between the centres is less than sum of their radii and greater than difference of their radii.

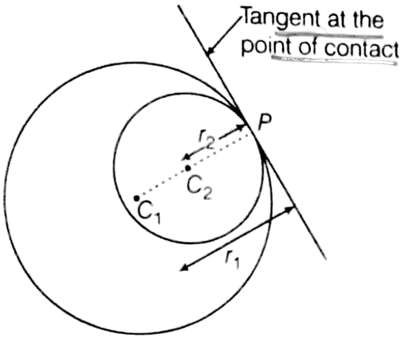
In this case two direct common tangents are real and distinct while the transverse tangents are imaginary.



Remark

In this case circles cuts at two points i.e. Number of solutions of two circles is two.

Case IV : When $|C_1C_2| = |r_1 - r_2|$, i.e. the distance between the centres is equal to the difference of their radii.



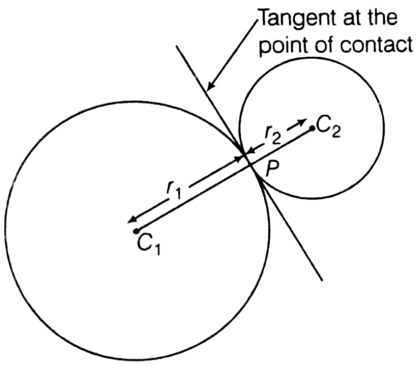
In this case two tangents are real and coincident while the other two tangents are imaginary.

If circles are represented by $S_1 = 0$ and $S_2 = 0$, then equation of common tangent is $S_1 - S_2 = 0$.

Remark

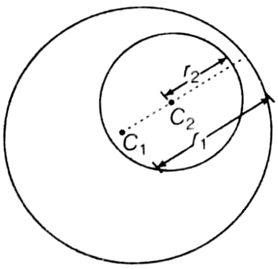
If circles touch each other externally, i.e. $|C_1C_2| = r_1 + r_2$, then equation of tangent at the point of contact is

$$S_1 - S_2 = 0$$



In this case circles touch at one point.
i.e. Number of solutions of two circles is one.

Case V : When $|C_1C_2| < |r_1 - r_2|$, i.e. the distance between the centres is less than the difference of their radii.



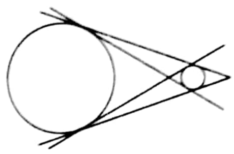
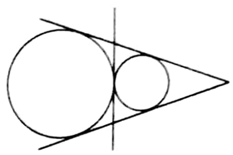
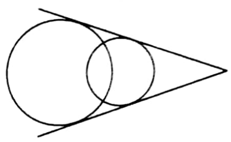
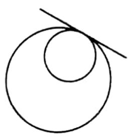
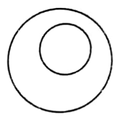
In this case, all the four common tangents are imaginary.

Remark

In this case circles neither cut nor touch each other i.e. Number of solution of two circles is zero.

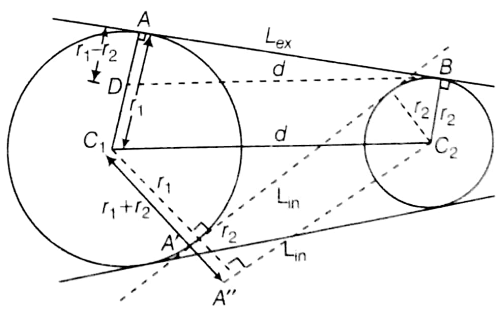
Common Tangents to two Circles

If two circles with centres C_1 and C_2 and their radii are r_1 and r_2, then

	Condition	Figure	Number of common tangents
(i)	$\|C_1C_2\| > r_1 + r_2$		4
(ii)	$\|C_1C_2\| = r_1 + r_2$		3
(iii)	$\|r_1 - r_2\| < \|C_1C_2\| < r_1 + r_2$		2
(iv)	$\|C_1C_2\| = \|r_1 - r_2\|$		1
(v)	$\|C_1C_2\| < \|r_1 - r_2\|$		0

Length of External Common Tangent and Internal Common Tangent to Two Circles

Length of external common tangent $L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2}$ and length of internal common tangent



$L_{in} = \sqrt{d^2 - (r_1 + r_2)^2}$ (Applicable only when $d > r_1 + r_2$)

where, d is the distance between the centres of two circles and r_1, r_2 are the radii of two circles, when $|C_1C_2| = d$.

Angle between Direct Common Tangents (DCT) and Transverse Common Tangents (TCT)

Case I : If $d > r_1 + r_2$, then

Angle between DCT $= 2\sin^{-1}\left(\frac{|r_1 - r_2|}{d}\right)$

And angle between TCT $= 2\sin^{-1}\left(\frac{r_1 + r_2}{d}\right)$

Case II : If $d = r_1 + r_2$, then

angle between DCT $= 2\sin^{-1}\left(\frac{|r_1 - r_2|}{r_1 + r_2}\right)$

and angle between $\boxed{\text{TCT} = \pi}$

Case III : If $|r_1 - r_2| < d < r_1 + r_2$, then

angle between DCT $= 2\sin^{-1}\left(\frac{|r_1 - r_2|}{d}\right)$

Here, transverse common tangents are not possible .

Case IV : If $d = |r_1 - r_2|$

Angle between $\boxed{\text{DCT} = \pi}$

Here, transverse common tangents are not possible.

Case V : If $d < |r_1 - r_2|$

Here, tangents are not possible.

Example 64. Find all the common tangents to the circles
$x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.

Sol. The given circles are

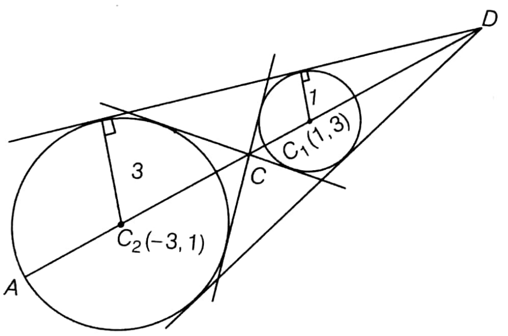
$$x^2 + y^2 - 2x - 6x + 9 = 0$$

$$\Rightarrow \quad (x-1)^2 + (y-3)^2 = 1 \quad \dots(i)$$

and $$x^2 + y^2 + 6x - 2y + 1 = 0$$

$$\Rightarrow \quad (x+3)^2 + (y-1)^2 = 9 \quad \dots(ii)$$

Centres and radii of circles Eq. (i) and Eq. (ii) are



$C_1(1, 3), r_1 = 1$

and $C_2(-3, 1), r_2 = 3$ respectively.

$\therefore \quad C_1C_2 = \sqrt{(16 + 4)} = 2\sqrt{5}$

$\therefore \quad C_1C_2 > r_1 + r_2$

Hence, the circles do not intersect to each other.

The direct common tangents meet AB produced at D, then point D will divide C_2C_1 in the ratio $3:1$ (externally).

Coordinates of D are $\left(\frac{3(1) - 1(-3)}{3-1}, \frac{3(3) - 1(1)}{3-1}\right)$ or $(3, 4)$

and the point C divide C_2C_1 in the ratio $3:1$ (internally)

then coordinates of C are $\left(\frac{3(1) + 1(-3)}{3+1}, \frac{3(3) + 1(1)}{3+1}\right)$ or $(0, 5/2)$

Direct tangents : Any line through $(3, 4)$ is

$$y - 4 = m(x - 3)$$

$$\Rightarrow \quad mx - y + 4 - 3m = 0 \quad \dots(i)$$

Apply the usual condition of tangency to any of the circle

$$\frac{m - 3 + 4 - 3m}{\sqrt{m^2 + 1}} = \pm 1$$

$$\Rightarrow \quad (-2m + 1)^2 = m^2 + 1$$

$$\Rightarrow \quad 3m^2 - 4m = 0$$

$$\Rightarrow \quad m = 0, m = 4/3$$

$\therefore$ Equations of direct common tangents are

$$y = 4 \quad \text{and} \quad 4x - 3y = 0$$

Transverse tangents : Any line through C $(0, 5/2)$ is

$$y - 5/2 = mx$$

or $$mx - y + 5/2 = 0 \quad \dots(ii)$$

Apply the usual condition of tangency to any of the circle

$$\therefore \quad \frac{m \cdot 1 - 3 + 5/2}{\sqrt{m^2 + 1}} = \pm 1$$

$$\Rightarrow \quad m^2 + \frac{1}{4} - m = m^2 + 1$$

$$\Rightarrow \quad 0 \cdot m^2 - m - \frac{3}{4} = 0$$

$$\therefore \quad \boxed{m = \infty} \quad \text{and} \quad m = -3/4$$

Hence, equations of transverse tangents are

$$x = 0 \quad \text{and} \quad 3x + 4y - 10 = 0$$

Example 65. Show that the common tangents to the circles $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 + 2x = 0$ form an equilateral triangle.

Sol. The given circles are

$$x^2 + y^2 - 6x = 0$$

or $$(x-3)^2 + (y-0)^2 = 9 \quad \dots(i)$$

and $$x^2 + y^2 + 2x = 0$$

or $$(x+1)^2 + (y-0)^2 = 1 \quad \dots(ii)$$

Centres and radii of circles Eqs. (i) and (ii) are $C_1(3, 0), r_1 = 3$ and $C_2(-1, 0), r_2 = 1$, respectively.

$\because \quad C_1C_2 = \sqrt{[3 - (-1)]^2 + 0} = 4$

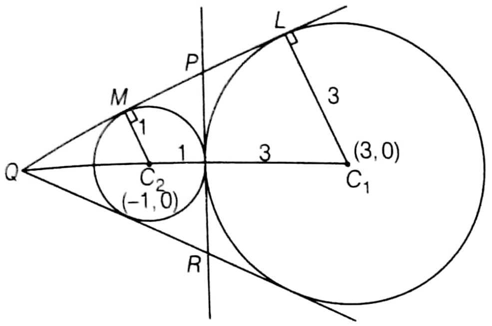
$\therefore \quad C_1C_2 = r_1 + r_2$

Hence, the two circles touch each other externally, therefore, there will be three common tangents. Equation of the common tangent at the point of contact is $S_1 - S_2 = 0$

$\Rightarrow \quad (x^2 + y^2 - 6x) - (x^2 + y^2 + 2x) = 0$

$\Rightarrow \quad -8x = 0$

$\therefore \quad x = 0$



Let the coordinates of Q be (h, k), then

$$\frac{QC_2}{QC_1} = \frac{C_2M}{C_1L} = \frac{1}{3}$$

$\therefore \quad QC_2 : QC_1 = 1:3$

$\therefore \quad h = \frac{1\cdot(3) - 3\cdot(-1)}{1-3} = -3$ and $k = 0$

$\therefore \quad Q \equiv (-3, 0)$

Equation of line passing through $Q(-3, 0)$ is

$$y - 0 = m(x+3)$$

or $\quad mx - y + 3m = 0 \quad$ ...(iii)

where, m is the slope of direct tangents since Eq. (iii) is the common tangent (direct) of the circles Eqs. (i) and (ii), then Length of perpendicular from centre of Eq. (ii) i.e. $(-1, 0)$ to the Eq. (iii) = radius of circle Eq. (ii)

$\Rightarrow \quad \frac{|-m - 0 + 3m|}{\sqrt{m^2+1}} = 1$ or $4m^2 = m^2 + 1$

$\Rightarrow \quad 3m^2 = 1$

$\therefore \quad m = \pm \frac{1}{\sqrt{3}}$

From Eq. (iii), common tangents are (direct)

$$y = \frac{x}{\sqrt{3}} + \sqrt{3} \quad \text{and} \quad y = -\frac{x}{\sqrt{3}} - \sqrt{3}$$

Hence, all common tangents are $\quad x = 0 \quad$ ...(iv)

$$y = \frac{x}{\sqrt{3}} + \sqrt{3} \quad \text{...(v)}$$

and $\quad y = -\frac{x}{\sqrt{3}} - \sqrt{3} \quad$ ...(vi)

Let P, Q, R be the point of intersection of lines Eqs.(iv), (v); (v), (vi) and (iv), (vi) respectively, then

$$P \equiv (0, \sqrt{3});\ Q \equiv (-3, 0) \quad \text{and} \quad R \equiv (0, -\sqrt{3})$$

Now, $PQ = QR = RP = 2\sqrt{3}$

Hence, ΔPQR is an equilateral triangle thus common tangents form an equilateral triangle.

| Example 66. Find the number of common tangents to the circles $x^2+y^2-8x+2y+8=0$ and $x^2+y^2-2x-6y-15=0$.

Sol. For $x^2 + y^2 - 8x + 2y + 8 = 0$

$$C_1 \equiv (4, -1), r_1 = \sqrt{(16+1-8)} = 3$$

and for $x^2 + y^2 - 2x - 6y - 15 = 0$

$$C_2 \equiv (1, 3), r_2 = \sqrt{(1+9+15)} = 5$$

Now, $|C_1C_2|$ = Distance between centres

$$= \sqrt{(4-1)^2 + (-1-3)^2} = 5$$

and $\quad r_1 + r_2 = 3 + 5 = 8$

$\quad |r_1 - r_2| = |3 - 5| = 2$

or $\quad |r_1 - r_2| < |C_1C_2| < r_1 + r_2$

Hence, the two circles intersect at two distinct points. Therefore, two tangents can be drawn.

| Example 67. Find the lengths of external and internal common tangents and also find the angle between external common tangents and internal common tangents of the circles $x^2+y^2+2x-8y+13=0$ and $x^2+y^2-8x-2y+8=0$.

Sol. The given circles are $x^2 + y^2 + 2x - 8y + 13 = 0$

$\Rightarrow \quad (x+1)^2 + (y-4)^2 = 2^2 \quad$ ...(i)

and $\quad x^2 + y^2 - 8x - 2y + 8 = 0$

$\Rightarrow \quad (x-4)^2 + (y-1)^2 = 3^2 \quad$ ...(ii)

Centres and radii of circles Eqs. (i) and (ii) are $C_1(-1, 4)$, $r_1 = 2$ and $C_2(4, 1)$, $r_2 = 3$ respectively.

$\therefore \quad |C_1C_2| = d = \sqrt{(25+9)} = \sqrt{34}$

$\Rightarrow \quad d > r_1 + r_2$

Hence, the circles do not intersect to each other.

$\therefore \quad L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2} = \sqrt{34 - 1} = \sqrt{33}$

and $\quad L_{in} = \sqrt{d^2 - (r_1 + r_2)^2} = \sqrt{(34 - 25)} = 3$

Angle between external common tangents

$$= 2\sin^{-1}\left(\frac{|r_1 - r_2|}{d}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{33}}\right)$$

and angle between internal common tangent

$$= 2\sin^{-1}\left(\frac{r_1 + r_2}{d}\right) = 2\sin^{-1}\left(\frac{5}{\sqrt{33}}\right)$$

Common Chord of Two Circles

The chord joining the points of intersection of two given circles is called their common chord.

Theorem : The equation of common chord of two circles

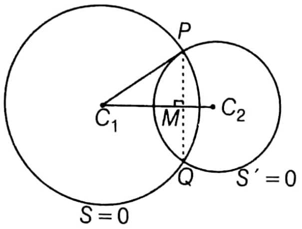
$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$

is $2x(g - g') + 2y(f - f') + c - c' = 0$

i.e. $\boxed{S - S' = 0}$

Proof : $\because$ $S = 0$ and $S' = 0$ be two intersecting circles.



Then, $S - S' = 0$ or

$2x(g - g') + 2y(f - f') + c - c' = 0$

is a first degree equation in x and y.

So, it represent a straight line. Also, this equation satisfied by the intersecting points of two given circles $S = 0$ and $S' = 0$.

Hence, $S - S' = 0$ represents the common chord of circles $S = 0$ and $S' = 0$

Length of common chord :

We have, $PQ = 2(PM)$ $\quad(\because M$ is mid-point of $PQ)$

$$= 2\sqrt{\{(C_1P)^2 - (C_1M)^2\}}$$

where, C_1P = radius of the circle $(S = 0)$

and C_1M = length of perpendicular from C_1 on common chord PQ.

Corollary 1 : The common chord PQ of two circles becomes of the maximum length when it is a diameter of the smaller one between them.

Corollary 2 : Circle on the common chord a diameter, then centre of the circle passing through P and Q lie on the common chord of two circles i.e.

$$S - S' = 0$$

Corollary 3 : If the length of common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.

| Example 68. Prove that the length of the common chord of the two circles :

$$(x - a)^2 + (y - b)^2 = c^2$$

and $(x - b)^2 + (y - a)^2 = c^2$ is $\sqrt{4c^2 - 2(a - b)^2}$.

Find also the condition when the given circles touch.

Sol. The equation of circles are

$$S_1 \equiv (x - a)^2 + (y - b)^2 - c^2 = 0 \quad \text{...(i)}$$

and $$S_2 \equiv (x - b)^2 + (y - a)^2 - c^2 = 0 \quad \text{...(ii)}$$

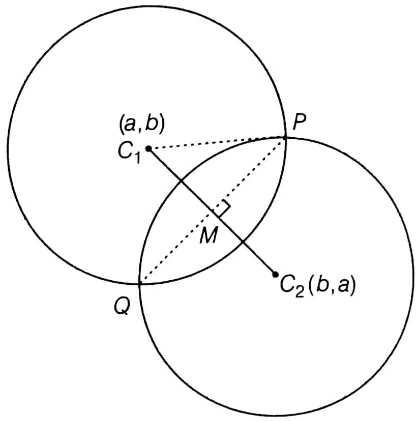
then equation of common chord is $S_1 - S_2 = 0$

$\Rightarrow \quad (x - a)^2 - (x - b)^2 + (y - b)^2 - (y - a)^2 = 0$

or $\quad (2x - a - b)(-a + b) + (2y - b - a)(-b + a) = 0$

$\Rightarrow \quad 2x - a - b - 2y + b + a = 0$

$\Rightarrow \quad x - y = 0$



Now, C_1M = Length of perpendicular from $C_1\,(a, b)$ on $PQ(x - y = 0) = \dfrac{|a - b|}{\sqrt{2}}$

and C_1P = radius of the circle Eq. (i) = c

$\therefore$ In ΔPC_1M, $PM = \sqrt{(PC_1)^2 - (C_1M)^2}$

$$= \sqrt{c^2 - \frac{(a - b)^2}{2}}$$

$\therefore$ $$PQ = 2PM = 2\sqrt{c^2 - \frac{(a - b)^2}{2}}$$

$$= \sqrt{4c^2 - 2(a - b)^2}$$

Also, when the circles touch, then chord PQ becomes the tangent and $PQ = 0$.

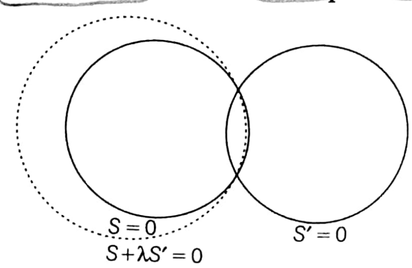
$\therefore$ The condition of tangency is $4c^2 - 2(a - b)^2 = 0$.

i.e. $2c^2 = (a - b)^2$

Family of Circles

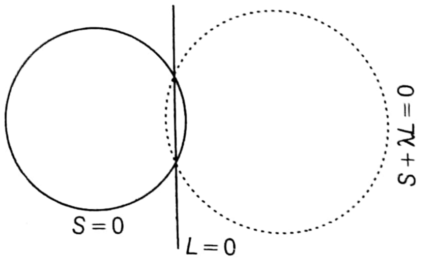
1. The equation of the family of circles passing through the point of intersection of two given circles $S = 0$ and $S' = 0$ is given as

$\boxed{S + \lambda S' = 0}$ (where, λ is a parameter, $\boxed{\lambda \neq -1}$)



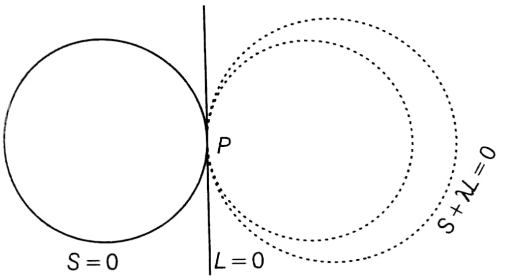
2. The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given as

$$\boxed{S + \lambda L = 0} \quad \text{(where, } \lambda \text{ is a parameter)}$$

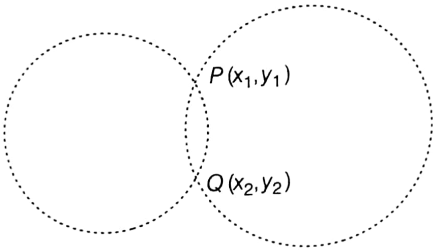


3. The equation of the family of circles touching the circle $S = 0$ and the line $L = 0$ at their point of contact P is

$$\boxed{S + \lambda L = 0} \quad \text{(where, } \lambda \text{ is a parameter)}$$



4. The equation of a family of circles passing through two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form



$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(where, λ is a parameter)

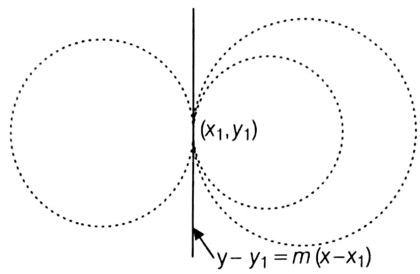
5. The equation of family of circles which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m is

$$(x - x_1)^2 + (y - y_1)^2 + \lambda \{(y - y_1) - m(x - x_1)\} = 0$$

and if m is infinite, the family of circles is

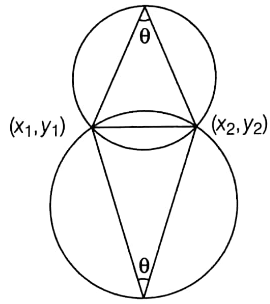
$$(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$$

(where, λ is a parameter)



6. Equation of the circles given in diagram are

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) \pm \cot\theta \{(x - x_1)(y - y_2) - (x - x_2)(y - y_1)\} = 0$$



Example 69. Find the equation of the circle passing through (1, 1) and the points of intersection of the circles $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$.

Sol. The given circles are

$$x^2 + y^2 + 13x - 3y = 0 \quad \text{...(i)}$$

and $$2x^2 + 2y^2 + 4x - 7y - 25 = 0$$

or $$x^2 + y^2 + 2x - \frac{7}{2}y - \frac{25}{2} = 0 \quad \text{...(ii)}$$

Equation of any circle passing through the point of intersection of the circles Eqs. (i) and (ii) is

$$(x^2 + y^2 + 13x - 3y) + \lambda\left(x^2 + y^2 + 2x - \frac{7}{2}y - \frac{25}{2}\right) = 0 \quad \text{...(iii)}$$

Its passes through (1, 1), then

$$(1 + 1 + 13 - 3) + \lambda\left(1 + 1 + 2 - \frac{7}{2} - \frac{25}{2}\right) = 0$$

$\Rightarrow \quad 12 + \lambda(-12) = 0 \therefore \lambda = 1$

Substituting the value of λ in Eq. (iii), the required equation is

$$x^2 + y^2 + 13x - 3y + x^2 + y^2 + 2x - \frac{7}{2}y - \frac{25}{2} = 0$$

$$\Rightarrow \quad 2x^2 + 2y^2 + 15x - \frac{13}{2}y - \frac{25}{2} = 0$$

$$\Rightarrow \quad 4x^2 + 4y^2 + 30x - 13y - 25 = 0$$

Must

Example 70. Find the equation of the circle passing through the point of intersection of the circles $x^2+y^2-6x+2y+4=0$, $x^2+y^2+2x-4y-6=0$ and with its centre on the line $y=x$.

Sol. Equation of any circle through the points of intersection of given circles is

$$(x^2+y^2-6x+2y+4)+\lambda(x^2+y^2+2x-4y-6)=0$$

$$\Rightarrow \quad x^2(1+\lambda)+y^2(1+\lambda)-2x(3-\lambda)+2y(1-2\lambda)+(4-6\lambda)=0$$

or $$x^2+y^2-\frac{2x(3-\lambda)}{(1+\lambda)}+\frac{2y(1-2\lambda)}{(1+\lambda)}+\frac{(4-6\lambda)}{(1+\lambda)}=0 \quad \text{...(i)}$$

Its centre $\left\{\frac{3-\lambda}{1+\lambda}, \frac{2\lambda-1}{1+\lambda}\right\}$ lies on the line $y=x$

then $$\frac{2\lambda-1}{1+\lambda}=\frac{3-\lambda}{1+\lambda} \Rightarrow \lambda \neq -1$$

$\therefore \quad 2\lambda-1=3-\lambda$ or $3\lambda=4$

$\therefore \quad \lambda=4/3$

$\therefore$ Substituting the value of $\lambda=4/3$ in Eq. (i), we get the required equation is

$$7x^2+7y^2-10x-10y-12=0$$

Example 71. Find the equation of the circle passing through the points of intersection of the circles $x^2+y^2-2x-4y-4=0$ and $x^2+y^2-10x-12y+40=0$ and whose radius is 4.

Sol. Equation of the any circle through the points of intersection of given circles is

$$(x^2+y^2-2x-4y-4)+\lambda(x^2+y^2-10x-12y+40)=0$$

$$\Rightarrow \quad x^2(1+\lambda)+y^2(1+\lambda)-2x(1+5\lambda)-2y(2+6\lambda)-4+40\lambda=0$$

or $$x^2+y^2-2x\frac{(1+5\lambda)}{(1+\lambda)}-2y\frac{(2+6\lambda)}{(1+\lambda)}+\frac{(40\lambda-4)}{(1+\lambda)}=0 \quad \text{...(i)}$$

Its radius

$$\sqrt{\left(\frac{1+5\lambda}{1+\lambda}\right)^2+\left(\frac{2+6\lambda}{1+\lambda}\right)^2-\left(\frac{40\lambda-4}{1+\lambda}\right)}=4 \quad \text{(given)}$$

$$\Rightarrow \quad \frac{(1+5\lambda)^2+(2+6\lambda)^2-(40\lambda-4)(1+\lambda)}{(1+\lambda)^2}=16$$

$$\Rightarrow \quad 5\lambda^2-34\lambda-7=0$$

or $$(\lambda-7)(5\lambda+1)=0$$

$\therefore \quad \lambda=7$ or $\lambda=-\frac{1}{5}$

Substituting the values of λ in Eq. (i), the required circles are

$$2x^2+2y^2-18x-22y+69=0$$

and $$x^2+y^2-2y-15=0$$

Example 72. Find the equation of the circle passing through points of intersection of the circle $x^2+y^2-2x-4y+4=0$ and the line $x+2y=4$ which touches the line $x+2y=0$.

Sol. Equation of any circle through points of intersection of the given circle and the line is

$$(x^2+y^2-2x-4y+4)+\lambda(x+2y-4)=0$$

or $$x^2+y^2+(\lambda-2)x+(2\lambda-4)y+4(1-\lambda)=0 \quad \text{...(i)}$$

It will touch the line $x+2y=0$, then solution of Eq. (i) and $x=-2y$ be unique.

Hence, the roots of the equation

$$(-2y)^2+y^2+(\lambda-2)(-2y)+(2\lambda-4)y+4(1-\lambda)=0$$

or $$5y^2+4(1-\lambda)=0$$

must be equal.

Then, $0-4\cdot5\cdot4(1-\lambda)=0$ or $1-\lambda=0$ or $\lambda=1$

From Eq. (i), the required circle is $x^2+y^2-x-2y=0$

Must

Example 73. Find the circle whose diameter is the common chord of the circles $x^2+y^2+2x+3y+1=0$ and $x^2+y^2+4x+3y+2=0$.

Sol. Given circles are

$$S \equiv x^2+y^2+2x+3y+1=0$$

and $$S' \equiv x^2+y^2+4x+3y+2=0$$

Hence, their common chord is $S-S'=0$

$$\Rightarrow \quad -2x-1=0 \quad \text{or} \quad 2x+1=0 \quad \text{...(i)}$$

Now, the required circle must pass through the point of intersection of S and S'.

Hence, its equation is $\quad S+\lambda S'=0$

$$\Rightarrow \quad (x^2+y^2+2x+3y+1)+\lambda(x^2+y^2+4x+3y+2)=0$$

$$\Rightarrow \quad x^2(1+\lambda)+y^2(1+\lambda)+2x(1+2\lambda)+3y(1+\lambda)+(1+2\lambda)=0$$

or $$x^2+y^2+2x\frac{(1+2\lambda)}{(1+\lambda)}+3y+\frac{(1+2\lambda)}{(1+\lambda)}=0 \quad \text{...(ii)}$$

Its centre is $\left(-\frac{1+2\lambda}{1+\lambda}, -\frac{3}{2}\right)$

But from Eq. (i), $2x+1=0$ is a diameter of this circle. Hence, its centre must lie on this line

$$\therefore \quad -2\left(\frac{1+2\lambda}{1+\lambda}\right)+1=0$$

$$\Rightarrow \quad -2-4\lambda+1+\lambda=0$$

$$\Rightarrow \quad -1-3\lambda=0$$

$$\therefore \quad \lambda=-\frac{1}{3}$$

Hence, from Eq. (ii), the required circle is

$$2x^2+2y^2+2x+6y+1=0$$

Must

| Example 74. If two curves, whose equations are $ax^2+2hxy+by^2+2gx+2fy+c=0$ and $a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$ intersect in four concyclic points, prove that $\dfrac{a-b}{h}=\dfrac{a'-b'}{h'}$

Sol. The equation of family of curves passing through the points of intersection of two curves is

$$(ax^2+2hxy+by^2+2gx+2fy+c)$$
$$+\lambda(a'x^2+2h'xy+b'y^2+2g'x+2f'y+c')=0$$

or
$$(a+\lambda a')x^2+2xy(h+h'\lambda)+(b+\lambda b')y^2$$
$$+2x(g+\lambda g')+2y(f+\lambda f')+(c+\lambda c')=0 \quad \text{...(i)}$$

Four concyclic points lie on a circle, then Eq. (i) represents a circle. Then,

coefficient of x^2 = coefficient of y^2 and coefficient of $xy=0$

$\Rightarrow$ $$a+\lambda a'=b+\lambda b'$$

or $$(a-b)=-\lambda(a'-b') \quad \text{...(ii)}$$

and $$2(h+h'\lambda)=0$$

or $$\lambda=-\frac{h}{h'} \quad \text{...(iii)}$$

Substituting the value of λ from Eq. (iii) in Eq. (ii), then

$$\frac{a-b}{h}=\frac{a'-b'}{h'}$$

Exercise for Session 6

1. Circles $x^2+y^2-2x-4y=0$ and $x^2+y^2-8y-4=0$

(a) touch each other internally
(b) touch each other externally
(c) cuts each other at two points
(d) None of these

2. The number of common tangents that can be drawn to the circles $x^2+y^2-4x-6y-3=0$ and $x^2+y^2+2x+2y+1=0$ is

(a) 1 (b) 2 (c) 3 (d) 4

3. If one of the circles $x^2+y^2+2ax+c=0$ and $x^2+y^2+2bx+c=0$ lies within the other, then

(a) $ab>0, c>0$
(b) $ab>0, c<0$
(c) $ab<0, c>0$
(d) $ab<0, c<0$

4. The condition that the circle $(x-3)^2+(y-4)^2=r^2$ lies entirely within the circle $x^2+y^2=R^2$ is

(a) $R+r\le 7$
(b) $R^2+r^2<49$
(c) $R^2-r^2<25$
(d) $R-r>5$

5. The circles whose equations are $x^2+y^2+c^2=2ax$ and $x^2+y^2+c^2-2by=0$ will touch one another externally, if

(a) $\dfrac{1}{b^2}+\dfrac{1}{c^2}=\dfrac{1}{a^2}$
(b) $\dfrac{1}{c^2}+\dfrac{1}{a^2}=\dfrac{1}{b^2}$
(c) $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{c^2}$
(d) $\dfrac{1}{b^2}+\dfrac{1}{c^2}=\dfrac{2}{a^2}$

6. Two circles with radii r_1 and r_2, $r_1>r_2\ge 2$, touch each other externally. If θ be the angle between the direct common tangents, then

(a) $\theta=\sin^{-1}\left(\dfrac{r_1+r_2}{r_1-r_2}\right)$
(b) $\theta=2\sin^{-1}\left(\dfrac{r_1-r_2}{r_1+r_2}\right)$
(c) $\theta=\sin^{-1}\left(\dfrac{r_1-r_2}{r_1+r_2}\right)$
(d) None of these

7. The circles $x^2+y^2-10x+16=0$ and $x^2+y^2=r^2$ intersect each other in two distinct points if

(a) $r<2$ (b) $r>8$ (c) $2<r<8$ (d) $2\le r\le 8$

8. If the circle $x^2+y^2+4x+22y+c=0$ bisects the circumference of the circle $x^2+y^2-2x+8y-d=0$, then $c+d$ is equal to

(a) 40 (b) 50 (c) 60 (d) 70

9. Two circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ are given. Then, the equation of the circle through their points of intersection and the point (1, 1) is

(a) $x^2 + y^2 - 6x + 4 = 0$
(b) $x^2 + y^2 - 3x + 1 = 0$
(c) $x^2 + y^2 - 4x + 2 = 0$
(d) $x^2 + y^2 - 2x + 1 = 0$

10. The equation of the circle described on the common chord of the circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 + 2y = 0$ as diameter is

(a) $x^2 + y^2 + x - y = 0$
(b) $x^2 + y^2 - x + y = 0$
(c) $x^2 + y^2 - x - y = 0$
(d) $x^2 + y^2 + x + y = 0$

11. The equation of the diameter of the circle $3(x^2 + y^2) - 2x + 6y - 9 = 0$ which is perpendicular to the line $2x + 3y = 12$ is

(a) $3x - 2y + 3 = 0$
(b) $3x - 2y - 3 = 0$
(c) $3x - 2y + 1 = 0$
(d) $3x - 2y - 1 = 0$

12. If the curves $ax^2 + 4xy + 2y^2 + x + y + 5 = 0$ and $ax^2 + 6xy + 5y^2 + 2x + 3y + 8 = 0$ intersect at four concyclic points, then the value of a is

(a) -6
(b) -4
(c) 4
(d) 6

13. Find the equation of the circle passing through the points of intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ and the point (1, 1).

14. Show that the common chord of the circles $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 8x - 6y + 23 = 0$ pass through the centre of the second circle and find its length.

15. Prove that the circles $x^2 + y^2 + 2ax + 2by = 0$ and $x^2 + y^2 + 2a_1x + 2b_1y = 0$ touch each other, if $ab_1 = a_1b$.

16. Find the equations of common tangents to the circles $x^2 + y^2 - 24x + 2y + 120 = 0$ and $x^2 + y^2 + 20x - 6y - 116 = 0$.

Session 7

Angle of Intersection of Two Circles, Radical Axis, Radical Centre, Co-axial System of Circles, Limiting Point, Image of the Circle by the Line Mirror

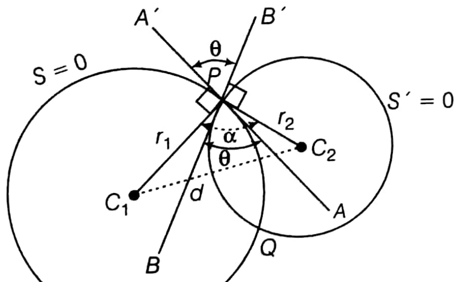
Angle of Intersection of Two Circles

Let the two circles

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

and $\quad S' \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

intersect each other at the points P and Q. The angle θ between two circles $S = 0$ and $S' = 0$ is defined as the angle between the tangents to the two circles at the point of intersection.



C_1 and C_2 are the centres of circles

$S = 0$ and $S' = 0$, then

$$C_1 \equiv (-g, -f) \quad \text{and} \quad C_2 \equiv (-g_1, -f_1)$$

and radii of circles $S = 0$ and $S' = 0$ are

$$r_1 = \sqrt{(g^2 + f^2 - c)} \text{ and } r_2 = \sqrt{(g_1^2 + f_1^2 - c_1)}$$

Let $\quad d = |C_1C_2| =$ Distance between their centres

$$= \sqrt{(-g + g_1)^2 + (-f + f_1)^2}$$

$$= \sqrt{(g^2 + f^2 + g_1^2 + f_1^2 - 2gg_1 - 2ff_1)}$$

Now, in ΔC_1PC_2, $\cos\alpha = \left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right) \left(\because \alpha + \theta + 90° + 90° = 360°\right)$

or $\quad \cos(180° - \theta) = \left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right) \quad (\therefore \alpha = 180° - \theta)$

mod

$\therefore \quad \cos\theta = \left|\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right|$ remember ...(i)

Orthogonal Intersection of Circles

If the angle between the circles is 90°, i.e. $\theta = 90°$, then the circles are said to be **orthogonal circles** or we say that the circles cut each other **orthogonally**.

Then, from Eq. (i), $0 = \dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$ pythagoras

or $\quad r_1^2 + r_2^2 - d^2 = 0$ or $\boxed{r_1^2 + r_2^2 = d^2}$

$\Rightarrow \quad g^2 + f^2 - c + g_1^2 + f_1^2 - c_1 = g^2 + f^2 + g_1^2 + f_1^2 - 2gg_1 - 2ff_1$

or $\quad \boxed{2gg_1 + 2ff_1 = c + c_1}$ remember.

Remark

Equation of a circle cutting the three circles $x^2 + y^2 + 2g_ix + 2f_iy + c_i = 0$ $(i = 1, 2, 3)$ orthogonally is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ -c_1 & g_1 & f_1 & -1 \\ -c_2 & g_2 & f_2 & -1 \\ -c_3 & g_3 & f_3 & -1 \end{vmatrix}$$

| Example 75. Find the angle between the circles

$$S: x^2 + y^2 - 4x + 6y + 11 = 0$$

and $\quad S': x^2 + y^2 - 2x + 8y + 13 = 0$

Sol. Centres and radii of circles S and S' are

$$C_1(2, -3),\ r_1 = \sqrt{2},\ C_2(1, -4),\ r_2 = 2.$$

Distance between centres, $d = |C_1C_2|$

$$= \sqrt{(2-1)^2 + (-3+4)^2} = \sqrt{2}$$

If angle between the circles is θ, then

$$\cos\theta = \left|\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right|$$

$$\cos\theta = \left|\frac{2+4-2}{2\cdot\sqrt{2}\cdot 2}\right| = \frac{1}{\sqrt{2}}$$

$\therefore \quad \theta = 45°$

| Example 76. Show that the circles $x^2+y^2-6x+4y+4=0$ and $x^2+y^2+x+4y+1=0$ cut orthogonally.

Sol. Comparing the given circles by general equation of circles

$$x^2+y^2+2gx+2fy+c=0$$

and $\quad x^2+y^2+2g_1x+2f_1y+c_1=0$

then, $\quad g=-3,\ f=2,\ c=4$

and $\quad g_1=\frac{1}{2},\ f_1=2,\ c_1=1$

Then, given circles cut orthogonally, if

$$2gg_1+2ff_1=c+c_1$$

We have, $\quad 2\times(-3)\times\frac{1}{2}+2\times2\times2=4+1$

$\Rightarrow \quad -3+8=5$ or $5=5$.

Hence, the given circles cut each other orthogonally.

| Example 77. Find the equation of the circle which cuts the circle $x^2+y^2+5x+7y-4=0$ orthogonally, has its centre on the line $x=2$ and passes through the point $(4,-1)$.

Sol. Let the required circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{...(i)}$$

Since, $(4,-1)$ lie on Eq. (i), then

$$17+8g-2f+c=0 \quad \text{...(ii)}$$

Centre of Eq. (i) is $(-g,-f)$

Since, centre lie on $x=2$ then $-g=2$

$\therefore \quad g=-2 \quad$ ...(iii)

From Eq. (ii), $1-2f+c=0 \quad$ ...(iv)

and given circle is

$$x^2+y^2+5x+7y-4=0 \quad \text{...(v)}$$

Given the circles Eqs. (i) and (v) cut each other orthogonally,

$\therefore \quad 2g\times\frac{5}{2}+2f\times\frac{7}{2}=c-4$

or $\quad 5g+7f=c-4$

$-10+7f=c-4 \quad$ [from Eq. (iii)]

or $\quad -6+7f-c=0 \quad$ ...(vi)

Solving Eqs. (iv) and (vi), we get

$$f=1 \text{ and } c=1$$

Substituting the values of g, f, c in Eq. (i), we get

$$x^2+y^2-4x+2y+1=0$$

Must

| Example 78. Find the equations of the two circles which intersect the circles $x^2+y^2-6y+1=0$ and $x^2+y^2-4y+1=0$ orthogonally and touch the line $3x+4y+5=0$.

Sol. Let the required circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{...(i)}$$

and given circles are $x^2+y^2-6y+1=0 \quad$ ...(ii)

and $\quad x^2+y^2-4y+1=0 \quad$ ...(iii)

Since, Eq. (i) cuts Eq. (ii) and Eq. (iii) orthogonally

$\therefore \quad 2g\times0+2f\times(-3)=c+1$

or $\quad -6f=c+1 \quad$ ...(iv)

and $\quad 2g\times0+2f\times(-2)=c+1$

or $\quad -4f=c+1 \quad$ ...(v)

Solving, Eqs. (iv) and (v), we get

$$f=0 \text{ and } c=-1$$

From Eq. (i), $\quad x^2+y^2+2gx-1=0 \quad$ ...(vi)

centre and radius of Eq. (vi) are $(-g,0)$ and $\sqrt{(g^2+1)}$, respectively.

Since, $3x+4y+5=0$ is tangent of Eq. (vi), then length of perpendicular from $(-g,0)$ to this line = radius of circle

or $\quad \frac{|-3g+0+5|}{\sqrt{(9+16)}}=\sqrt{(g^2+1)}$

$$|-3g+5|=5\sqrt{(g^2+1)}$$

or $\quad (-3g+5)^2=25(g^2+1)$

or $\quad 9g^2+25-30g=25g^2+25$

or $\quad 16g^2+30g=0$

$\therefore \quad g=0$ and $g=-\frac{15}{8}$

Equations of circles are from Eq. (vi),

$$x^2+y^2-1=0 \text{ and } x^2+y^2-\frac{15}{4}x-1=0$$

or $\quad x^2+y^2-1=0$ and $4x^2+4y^2-15x-4=0$.

| Example 79. Prove that the two circles, which pass through $(0,a)$ and $(0,-a)$ and touch the line $y=mx+c$, will cut orthogonally, if $c^2=a^2(2+m^2)$

Sol. Let the equation of the circles be

$$x^2+y^2+2gx+2fy+d=0 \quad \text{...(i)}$$

Since, these circles pass through $(0,a)$ and $(0,-a)$, then

$$a^2+2fa+d=0 \quad \text{...(ii)}$$

and $\quad a^2-2fa+d=0 \quad$ ...(iii)

Solving, Eq. (ii) and Eq. (iii), we get $f=0$ and $d=-a^2$.

Substituting these values of f and d in Eq. (i), we obtain

$$x^2+y^2+2gx-a^2=0 \quad \text{...(iv)}$$

Now, $y = mx + c$ touch this circle, therefore, length of the perpendicular from the centre = radius

$$\frac{|-mg - 0 + c|}{\sqrt{1 + m^2}} = \sqrt{(g^2 + a^2)}$$

$$(c - mg)^2 = (1 + m^2)(g^2 + a^2)$$

or $$g^2 + 2mcg + a^2 (1 + m^2) - c^2 = 0$$

Let g_1, g_2 are the roots of this equation

$\therefore \quad g_1 g_2 = a^2(1 + m^2) - c^2 \qquad \text{...(v)}$

Now, the equations of the two circles represented by Eq. (iv) are

$$x^2 + y^2 + 2g_1 x - a^2 = 0$$

and $$x^2 + y^2 + 2g_2 x - a^2 = 0.$$

These two circles will be cuts orthogonal, if

$$2g_1 g_2 + 0 = -a^2 - a^2$$

or $g_1 g_2 = -a^2 \qquad \text{...(vi)}$

From Eqs. (v) and (vi),

$$-a^2 = a^2(1 + m^2) - c^2$$

or $$c^2 = a^2(2 + m^2)$$

which is the required condition.

| Example 80. Find the equation of the circle which cuts orthogonally each of the three circles given below :

$x^2 + y^2 - 2x + 3y - 7 = 0$, $x^2 + y^2 + 5x - 5y + 9 = 0$ and $x^2 + y^2 + 7x - 9x + 29 = 0$.

Sol. Let the required circle be

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$

Since, it is orthogonal to three given circles respectively, therefore

$$2g \times (-1) + 2f \times \frac{3}{2} = c - 7$$

or $-2g + 3f = c - 7 \qquad \text{...(ii)}$

$$2g \times \frac{5}{2} + 2f \times \left(-\frac{5}{2}\right) = c + 9$$

or $5g - 5f = c + 9 \qquad \text{...(iii)}$

and $$2g \times \frac{7}{2} + 2f \times \left(-\frac{9}{2}\right) = c + 29$$

or $7g - 9f = c + 29 \qquad \text{...(iv)}$

Subtracting, Eq. (ii) from Eq. (iii),

$7g - 8f = 16 \qquad \text{...(v)}$

and subtracting Eq. (iii) from Eq. (iv),

$2g - 4f = 20 \qquad \text{...(vi)}$

Solving Eq. (v) and Eq. (vi), we get

$$g = -8 \quad \text{and} \quad f = -9$$

Putting the values of g and f in Eq. (iii)

$$-40 + 45 = c + 9$$

$\Rightarrow \quad 5 = c + 9$

or $\quad c = -4$

Substituting the values of g, f, c in Eq. (i), then required circle is

$$x^2 + y^2 - 16x - 18y - 4 = 0$$

Radical Axis

The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

Consider, $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$

and $\quad S' \equiv x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0 \qquad \text{...(ii)}$

Let $P(x_1, y_1)$ be a point such that

$$|PA| = |PB|$$

$$\Rightarrow \sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)} = \sqrt{(x_1^2 + y_1^2 + 2g_1 x_1 + 2f_1 y_1 + c_1)}$$

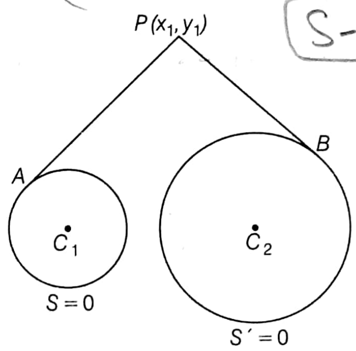
On squaring, $$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = x_1^2 + y_1^2 + 2g_1 x_1 + 2f_1 y_1 + c_1$$

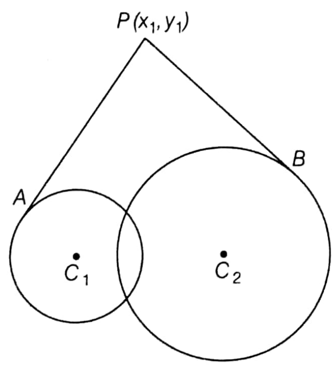
$$\Rightarrow 2(g - g_1)x_1 + 2(f - f_1)y_1 + c - c_1 = 0$$

$\therefore$ Locus of $P(x_1, y_1)$ is

$$2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$$

$\boxed{S - S' = 0}$

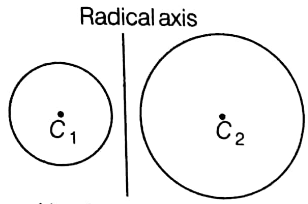




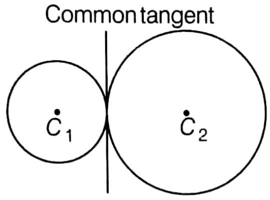
which is the required equation of radical axis of the given circles. Clearly this is a straight line.

Some Properties of the Radical Axis

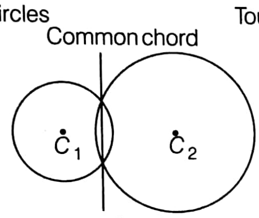
(i) **The radical axis and common chord are identical :** Since, the radical axis and common chord of two circles $S=0$ and $S'=0$ are the same straight line $S-S'=0$, they are identical. The only difference is that the common chord exists only if the circles intersect in two real points, while the radical axis exists for all pair of circles irrespective of their position.



Non-intersecting circles

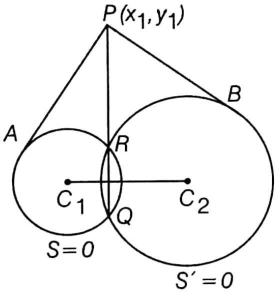


Touching circles



Intersecting circles

The position of the radical axis of the two circles geometrically is shown below:



From Euclidian geometry

$$(PA)^2 = PR \cdot PQ = (PB)^2$$

(ii) **The radical axis is perpendicular to the straight line which joins the centres of the circles :**

Consider, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

and $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(ii)

Since, $C_1 \equiv (-g, -f)$ and $C_2 \equiv (-g_1, -f_1)$ are the centres of the circles Eqs. (i) and (ii), then slope of

$$C_1C_2 = \frac{-f_1 + f}{-g_1 + g} = \frac{f - f_1}{g - g_1} = m_1 \quad \text{(say)}$$

Equation of the radical axis is

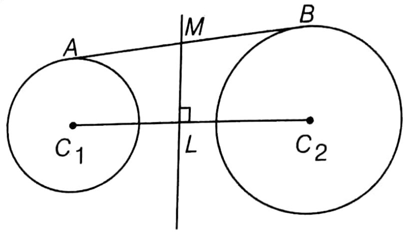
$$2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$$

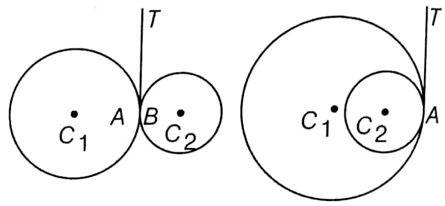
Slope of radical axis is $-\dfrac{(g - g_1)}{f - f_1} = m_2$ (say)

$\therefore \quad m_1 m_2 = -1$

Hence, C_1C_2 and radical axis are perpendicular to each other.

(iii) **The radical axis bisects common tangents of two circles :** Let AB be the common tangent. If it meets the radical axis LM in M, then MA and MB are two tangents to the circles. Hence, $MA = MB$ since lengths of tangents are equal from any point on radical axis. Hence, radical axis bisects the common tangent AB.





If the two circles touch each other externally or internally, then A and B coincide. In this case the common tangent itself becomes the radical axis.

(iv) **The radical axis of three circles taken in pairs are concurrent :** Let the equations of three circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{...(i)}$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \text{...(ii)}$$

$$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0 \quad \text{...(iii)}$$

The radical axis of the above three circles taken in pairs are given by

$$S_1 - S_2 \equiv 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0 \quad \text{...(iv)}$$

$$S_2 - S_3 \equiv 2x(g_2 - g_3) + 2y(f_2 - f_3) + c_2 - c_3 = 0 \quad \text{...(v)}$$

$$S_3 - S_1 \equiv 2x(g_3 - g_1) + 2y(f_3 - f_1) + c_3 - c_1 = 0 \quad \text{...(vi)}$$

Adding Eqs. (iv), (v) and (vi), we find LHS vanished identically. Thus, the three lines are concurrent.

(v) **If two circles cut a third circle orthogonally, the radical axis of the two circles will pass through the centre of the third circle.**

OR

The locus of the centre of a circle cutting two given circles orthogonally is the radical axis of the two circles.

Let $\quad S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad$ ...(i)

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad$ ...(ii)

$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0 \quad$ ...(iii)

Since, Eqs. (i) and (ii) both cut Eq. (iii) orthogonally

$\therefore \quad 2g_1g_3 + 2f_1f_3 = c_1 + c_3$

and $\quad 2g_2g_3 + 2f_2f_3 = c_2 + c_3$

Subtracting, we get

$2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2 \quad$ ...(iv)

Now, radical axis of Eqs. (i) and (ii) is

$$S_1 - S_2 = 0$$

or $\quad 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

Since, it will pass through the centre of Eq. (iii) circle

$\therefore -2g_3(g_1 - g_2) - 2f_3(f_1 - f_2) + c_1 - c_2 = 0$

or $2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2 \quad$ ...(v)

which is true by Eq. (iv),

Remark

Radical axis need not always pass through the mid-point of the line joining the centres of the two circles.

| Example 81. If two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other, then $f'g = fg'$.

Sol. If two circles touch each other, then their radical axis is their common tangent.

$\therefore$ Radical axis of two circles is

$$(x^2 + y^2 + 2gx + 2fy) - (x^2 + y^2 + 2g'x + 2f'y) = 0$$

or $\quad 2x(g - g') + 2y(f - f') = 0$

or $\quad x(g - g') + y(f - f') = 0 \quad$ ...(i)

If this touches the circle $x^2 + y^2 + 2gx + 2fy = 0$, then the length of perpendicular from its centre $(-g, -f)$ to (i) $=$radius $\sqrt{(g^2 + f^2)}$ of the circle

i.e. $\quad \dfrac{|-g(g - g') - f(f - f')|}{\sqrt{(g - g')^2 + (f - f')^2}} = \sqrt{g^2 + f^2}$

or $\quad \{(-(g^2 + f^2) + gg' + ff')\}^2$

$$= (g^2 + f^2)\{(g - g')^2 + (f + f')^2\}$$

or $\quad (g^2 + f^2)^2 + (gg' + ff')^2 - 2(g^2 + f^2)(gg' + ff')$

$$= (g^2 + f^2)\{(g^2 + f^2) + (g'^2 + f'^2) - 2(gg' + ff')\}$$

or $(gg' + ff')^2 = (g^2 + f^2)(g'^2 + f'^2)$

On simplifying, $\quad 2gg'ff' = g^2f'^2 + f^2g'^2$

or $\quad (gf' - g'f)^2 = 0$

or $\quad gf' = g'f$

Aliter : If two circles touch each other, then distance between their centres = sum or difference of their radii

$$\sqrt{(g - g')^2 + (f - f')^2} = \sqrt{(g^2 + f^2)} \pm \sqrt{(g'^2 + f'^2)}$$

or $\quad \sqrt{(g^2 + f^2 + g'^2 + f'^2 - 2gg' - 2ff')}$

$$= \sqrt{(g^2 + f^2)} \pm \sqrt{(g'^2 + f'^2)}$$

On squaring, we have

$g^2 + f^2 + g'^2 + f'^2 - 2gg' - 2ff'$

$$= g^2 + f^2 + g'^2 + f'^2 \pm 2\sqrt{(g^2 + f^2)}\sqrt{(g'^2 + f'^2)}$$

or $(gg' + ff') = \pm\sqrt{(g^2 + f^2)(g'^2 + f'^2)}$

Again, on squaring both sides, we get

$$g^2g'^2 + f^2f'^2 + 2gg'ff' = g^2g'^2 + g^2f'^2 + f^2g'^2 + f^2f'^2$$

or $\quad g^2f'^2 + f^2g'^2 - 2gg'ff' = 0$

or $\quad (gf' - g'f)^2 = 0$

or $\quad gf' - g'f = 0$

or $\quad gf' = g'f$

must

| Example 82. A and B are two fixed points and P moves so that $PA = nPB$. Show that locus of P is a circle and for different values of n all the circles have a common radical axis.

Sol. Let $A \equiv (a, 0)$, $B \equiv (-a, 0)$ and $P \equiv (h, k)$

$\therefore \quad PA = \sqrt{(h - a)^2 + k^2}$

$\quad PB = \sqrt{(h + a)^2 + k^2}$

Since, $\quad PA = nPB$

or $\quad (PA)^2 = n^2(PB)^2$

$\Rightarrow \quad \{(h - a)^2 + k^2\} = n^2\{(h + a)^2 + k^2\}$

$\Rightarrow \quad (h^2 + k^2 - 2ah + a^2) = n^2(h^2 + k^2 + 2ah + a^2)$

$\Rightarrow \quad (1 - n^2)h^2 + (1 - n^2)k^2 - 2ah(1 + n^2) + (1 - n^2)a^2 = 0$

or $\quad h^2 + k^2 - 2ah\dfrac{(1 + n^2)}{(1 - n^2)} + a^2 = 0$

$\therefore$ Locus of P is $\quad x^2 + y^2 - \left(\dfrac{1 + n^2}{1 - n^2}\right)2ax + a^2 = 0$

which is a circle. For different values of n.

If two different values of n are n_1 and n_2, then circles are

$x^2 + y^2 - \left(\dfrac{1 + n_1^2}{1 - n_1^2}\right)2ax + a^2 = 0 \quad$ ...(i)

and $\quad x^2 + y^2 - \left(\dfrac{1 + n_2^2}{1 - n_2^2}\right)2ax + a^2 = 0 \quad$ ...(ii)

∴ Radical axis of Eqs. (i) and (ii) is

$$2ax\left\{\frac{1+n_2^2}{1-n_2^2}-\frac{1+n_1^2}{1-n_1^2}\right\}=0$$

or $\boxed{x=0}$ or Y-axis.

Hence, for different values of n the circles have a common radical axis.

Example 83. Show that the difference of the squares of the tangents to two coplanar circles from any point P in the plane of the circles varies as the perpendicular from P on their radical axis. Also, prove that the locus of a point such that the difference of the squares of the tangents from it to two given circles is constant is a line parallel to their radical axis.

Sol. Let the two circles be

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0 \quad \text{...(i)}$$

and $$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0 \quad \text{...(ii)}$$

and let $P=(h,k)$

∴ Radical axis of Eqs. (i) and (ii) is

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0 \quad \text{...(iii)}$$

Let length of tangents from $P(h,k)$ on Eqs. (i) and (ii) are l_1 and l_2, then

$$l_1=\sqrt{S_1}=\sqrt{(h^2+k^2+2g_1h+2f_1k+c_1)}$$

and $$l_2=\sqrt{S_2}=\sqrt{(h^2+k^2+2g_2h+2f_2k+c_2)}$$

According to the question,

$$l_1^2-l_2^2=2(g_1-g_2)h+2(f_1-f_2)k+c_1-c_2 \quad \text{...(iv)}$$

Let p be the perpendicular distance from $P(h,k)$ on Eq. (iii),

$$\therefore \quad p=\frac{|2(g_1-g_2)h+2(f_1-f_2)k+c_1-c_2|}{\sqrt{4(g_1-g_2)^2+4(f_1-f_2)^2}} \quad \text{...(v)}$$

From Eqs. (iv) and (v), we get

$$p=\frac{|l_1^2-l_2^2|}{2\sqrt{(g_1-g_2)^2+(f_1-f_2)^2}}$$

or $$\frac{|l_1^2-l_2^2|}{p}=2\sqrt{(g_1-g_2)^2+(f_1-f_2)^2}=\text{constant}$$

$$\therefore \quad \boxed{|l_1^2-l_2^2|\propto p}$$

Locus of $P(h,k)$ in Eq. (iv) is

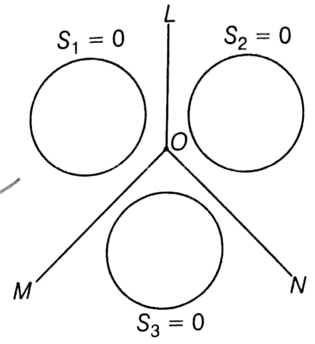
$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=(l_1^2-l_2^2)$$

a line which is parallel to Eq. (iii).

Radical Centre

The radical axes of three circles, taken in pairs, meet in a point, which is called their radical centre. Let the three circles be

$$S_1=0 \quad \text{...(i)}$$
$$S_2=0 \quad \text{...(ii)}$$
$$S_3=0 \quad \text{...(iii)}$$



Let OL, OM and ON be radical axes of the pair sets of circles

$$\{S_1=0, S_2=0\}, \{S_3=0, S_1=0\}$$

and $$\{S_2=0, S_3=0\} \text{ respectively.}$$

Equations of OL, OM and ON are respectively

$$S_1-S_2=0 \quad \text{...(iv)}$$
$$S_3-S_1=0 \quad \text{...(v)}$$
$$S_2-S_3=0 \quad \text{...(vi)}$$

Let the straight lines Eqs. (iv) and (v) i.e. OL and OM meet in O. The equation of any straight line passing through O is

$$(S_1-S_2)+\lambda(S_3-S_1)=0$$

where λ is any constant.

For $\lambda=1$ this equation becomes

$$S_2-S_3=0$$

which is, by Eq. (vi), equation of ON.

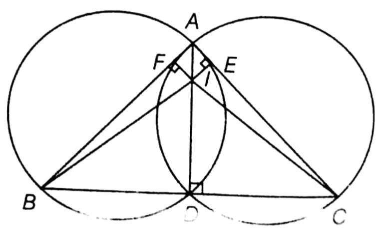
Thus, the third radical axis also passes through the point where the Eqs. (iv) and (v) meet. In the above figure O is the **radical centre.**

Properties of Radical Centre

1. Coordinates of radical centre can be found by solving the equations

$$\boxed{S_1=S_2=S_3=0}$$

2. **The radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle :**



Draw perpendicular from A on BC.

$\therefore \quad \angle ADB = \angle ADC = \pi/2$

Therefore, the circles whose diameters are AB and AC passes through D and A. Hence, AD is their radical axis. Similarly, the radical axis of the circles on AB and BC as diameters is the perpendicular line from B on CA and radical axis of the circles on BC and CA as diameters is the perpendicular line from C on AB. Hence, the radial axis of three circles meet in a point. This point I is radical centre but here radical centre is the point of intersection of altitudes i.e. AD, BE and CF. Hence, radical centre = orthocentre.

3. The radical centre of three given circles will be the centre of a fourth circle which cuts all the three circles orthogonally and the radius of the fourth circle is the length of tangent drawn from radical centre of the three given circles to any of these circles.

Let the fourth circle be $(x-h)^2 + (y-k)^2 = r^2$, where (h, k) is centre of this circle and r be the radius. The centre of circle is the radical centre of the given circles and r is the length of tangent from (h, k) to any of the given three circles.

| Example 84. Find the radical centre of circles $x^2+y^2+3x+2y+1=0$, $x^2+y^2-x+6y+5=0$ and $x^2+y^2+5x-8y+15=0$. Also, find the equation of the circle cutting them orthogonally.

Sol. : Given circles are

$$S_1 \equiv x^2+y^2+3x+2y+1=0$$
$$S_2 \equiv x^2+y^2-x+6y+5=0$$
$$S_3 \equiv x^2+y^2+5x-8y+15=0$$

Equations of two radical axes are

$S_1 - S_2 \equiv 4x-4y-4=0$ or $x-y-1=0$

and $S_2 - S_3 \equiv -6x+14y-10=0$ or $3x-7x+5=0$

Solving them the radical centre is (3, 2) also, if r is the length of the tangent drawn from the radical centre (3, 2) to any one of the given circles, say S_1, we have

$$r = \sqrt{S_1} = \sqrt{3^2+2^2+3\cdot 3+2\cdot 2+1} = \sqrt{27}$$

Hence, (3, 2) is the centre and $\sqrt{27}$ is the radius of the circle intersecting them orthogonally.

$\therefore$ Its equation is

$$(x-3)^2+(y-2)^2 = r^2 = 27 \text{ or } x^2+y^2-6x-4y-14=0$$

Aliter : Let $x^2+y^2+2gx+2fy+c=0$ be the equation of the circle cutting the given circles orthogonally.

$$\therefore \quad 2g\left(\frac{3}{2}\right)+2f(1)=c+1$$

or $\quad 3g+2f=c+1 \quad$ ...(i)

$$2g\left(-\frac{1}{2}\right)+2f(3)=c+5$$

or $\quad -g+6f=c+5 \quad$ ...(ii)

and $\quad 2g\left(\frac{5}{2}\right)+2f(-4)=c+15$

or $\quad 5g-8f=c+15 \quad$ ...(iii)

Solving, Eqs. (i), (ii) and (iii), we get

$$g=-3, f=-2 \text{ and } c=-14$$

$\therefore$ Equation of required circle is

$$x^2+y^2-6x-4y-14=0$$

| Example 85. Find the radical centre of three circles described on the three sides $4x-7y+10=0$, $x+y-5=0$ and $7x+4y-15=0$ of a triangle as diameters.

Sol. Since, the radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle.

$\therefore \quad$ Radical centre = orthocentre

Given sides are $\quad 4x-7y+10=0 \quad$ ...(i)

$x+y-5=0 \quad$ ...(ii)

$7x+4y-15=0 \quad$ ...(iii)

Since, lines Eqs. (i) and (iii) are perpendiculars the point of intersection of Eqs. (i) and (iii) is (1, 2), the orthocentre of the triangle. Hence, radical centre is (1, 2).

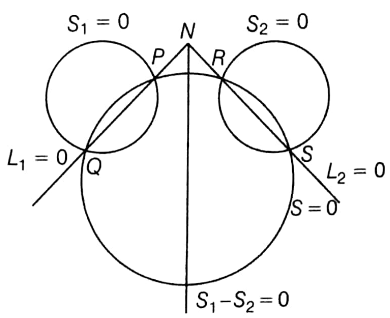
| Example 86. Prove that if the four points of intersection of the circles $x^2+y^2+ax+by+c=0$ and $x^2+y^2+a'x+b'y+c'=0$ by the lines $Ax+By+C=0$ and $A'x+B'y+C'=0$ respectively are concyclic, then

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

Charo points ek circle par lie karte hain.

Sol. The given circles and given lines are

$$S_1 \equiv x^2+y^2+ax+by+c=0$$
$$S_2 \equiv x^2+y^2+a'x+b'y+c'=0$$
$$L_1 \equiv Ax+By+C=0$$
$$L_2 \equiv A'x+B'y+C'=0$$



Let $S_1 = 0$ meet $L_1 = 0$ at two points P and Q and $S_2 = 0$ meet $L_2 = 0$ at two points R and S.

Further P, Q, R and S are given to be concyclic. Let the circle through them is

$$x^2 + y^2 + 2gx + 2fy + \lambda = 0 \qquad \text{...(i)}$$

Radical axis of $S_1 = 0$ and $S_2 = 0$ is

$$S_1 - S_2 = 0$$

$$\Rightarrow \quad (a - a')x + (b - b')y + c - c' = 0 \qquad \text{...(ii)}$$

The radical axis of $S_1 = 0$ and $S = 0$ is $L_1 = 0$

or $\quad Ax + By + C = 0 \qquad$ ...(iii)

and $\quad$ radical axis of $S_2 = 0$ and $S = 0$ is $L_2 = 0$

or $\quad A'x + B'y + C' = 0 \qquad$ ...(iv)

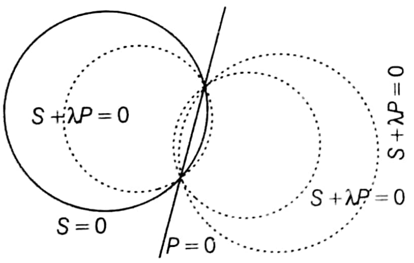
Since, the radical axes of any three circles taken in pairs are concurrent. (i.e. lines Eqs. (ii), (iii) and (iv) are concurrent).

we have $\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$

Co-axial System of Circles

A system (or a family) of circles, every pair of which have the same radical axis, are called co-axial circles.

(1) The equation of a system of co-axial circles, when the equation of the radical axis and of one circle of the system are



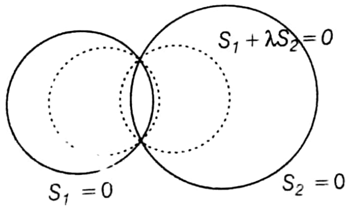
$$P \equiv lx + my + n = 0$$

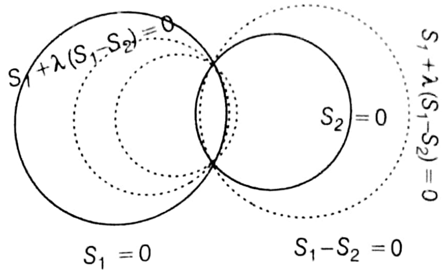
and $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

respectively, is

$\boxed{S + \lambda P = 0}$ (λ is an arbitrary constant)

(2) The equation of a co-axial system of circles, where the equation of any two circles of the system are





$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $\quad S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

respectively is

$$S_1 + \lambda(S_1 - S_2) = 0 \qquad (\lambda \neq -1)$$

or $\quad S_2 + \lambda_1(S_1 - S_2) = 0 \qquad (\lambda_1 \neq -1)$

Other form $\boxed{S_1 + \lambda S_2 = 0} \qquad (\lambda \neq -1)$

(3) The equation of a system of co-axial circles in the simplest form is

$$x^2 + y^2 + 2gx + c = 0$$

where, g is variable and c, a constant.

The common radical axis is the Y-axis (since centre on X-axis) and the equation of a system of other co-axial circles in the simplest form is

$$x^2 + y^2 + 2fy + c = 0$$

where, f is variable and c, a constant (since centre on Y-axis). The common radical axis is the X-axis.

Example 87. Find the equation of the system of circles co-axial with the circles

$$x^2 + y^2 + 4x + 2y + 1 = 0$$

and $x^2 + y^2 - 2x + 6y - 6 = 0$

Also, find the equation of that particular circle whose centre lies on the radical axis.

Sol. Given circles are

$$S_1 \equiv x^2 + y^2 + 4x + 2y + 1 = 0$$

and $\quad S_2 \equiv x^2 + y^2 - 2x + 6y - 6 = 0$

$\therefore$ Radical axis is $\quad S_1 - S_2 = 0$

i.e. $\quad 6x - 4y + 7 = 0 \qquad$ ...(i)

Now, system of co-axial circle is

$$S_1 + \lambda(S_1 - S_2) = 0$$

$$\Rightarrow \quad (x^2 + y^2 + 4x + 2y + 1) + \lambda(6x - 4y + 7) = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2x(2 + 3\lambda) + 2y(1 - 2\lambda) + 1 + 7\lambda = 0 \qquad \text{...(ii)}$$

Its centre $[-(2 + 3\lambda), -(1 - 2\lambda)]$ lies on Eq. (i)

$\therefore \quad 6 \times -(2 + 3\lambda) - 4 \times -(1 - 2\lambda) + 7 = 0$

or $\quad -12 - 18\lambda + 4 - 8\lambda + 7 = 0$

or $\quad -26\lambda - 1 = 0$

$$\therefore \quad \lambda = -\frac{1}{26}$$

Substituting the value of λ in Eq. (ii), the equation of circle is

$$x^2 + y^2 + 2x\left(2 - \frac{3}{26}\right) + 2y\left(1 + \frac{2}{26}\right) + 1 - \frac{7}{26} = 0$$

$$\Rightarrow \quad 26(x^2 + y^2) + 98x + 56y + 9 = 0$$

| Example 88. Prove that the tangents from any point of a fixed circle of co-axial system to two other fixed circles of the system are in a constant ratio.

Sol. Let the equations of the circles be $x^2 + y^2 + 2g_i x + c = 0$, $i = 1, 2, 3$. Since, all the three circles are fixed g_1, g_2 and g_3 are constants.

Let $P(h, k)$ be any point on the first circle, so that

$$h^2 + k^2 + 2g_1 h + c = 0 \quad \text{...(i)}$$

Let PQ and PR be the tangents from P on the other two circles

$$\therefore \quad PQ = \sqrt{(h^2 + k^2 + 2g_2 h + c)}$$

and $$PR = \sqrt{(h^2 + k^2 + 2g_3 h + c)}$$

$$\therefore \quad \frac{(PQ)^2}{(PR)^2} = \frac{h^2 + k^2 + 2g_2 h + c}{h^2 + k^2 + 2g_3 h + c}$$

$$= \frac{-2g_1 h + 2g_2 h}{-2g_1 h + 2g_3 h} \quad \text{[from Eq. (i)]}$$

$$= \frac{g_2 - g_1}{g_3 - g_1} = \text{constant}$$

because g_1, g_2, g_3 are constants.

| Example 89. If A, B, C be the centres of three co-axial circles and t_1, t_2, t_3 be the lengths of the tangents to them from any point, prove that

$$\overline{BC} \cdot t_1^2 + \overline{CA} \cdot t_2^2 + \overline{AB}\ t_3^2 = 0$$

Sol. Let the equations of three circles are $x^2 + y^2 + 2g_i x + c = 0$, $i = 1, 2, 3$, .

According to the question

$$A \equiv (-g_1, 0), B \equiv (-g_2, 0), C \equiv (-g_3, 0)$$

Let any point be $P(h, k)$

$$\therefore \quad t_1 = \sqrt{h^2 + k^2 + 2g_1 h + c}$$

$$t_2 = \sqrt{h^2 + k^2 + 2g_2 h + c}$$

$$t_3 = \sqrt{h^2 + k^2 + 2g_3 h + c}$$

and $$\overline{AB} = (g_1 - g_2)$$

$$\overline{BC} = (g_2 - g_3)$$

and $$\overline{CA} = (g_3 - g_1)$$

Now, $\overline{BC} \cdot t_1^2 + \overline{CA} \cdot t_2^2 + \overline{AB}\ t_3^2$

$$= \Sigma (g_2 - g_3)(h^2 + k^2 + 2g_1 h + c)$$

$$= (h^2 + k^2 + c)\ \Sigma (g_2 - g_3) + 2h \Sigma g_1 (g_2 - g_3)$$

$$= (h^2 + k^2 + c)(g_2 - g_3 + g_3 - g_1 + g_1 - g_2)$$
$$+ 2h\{g_1(g_2 - g_3) + g_2(g_3 - g_1) + g_3(g_1 - g_2)\}$$

$$= (h^2 + k^2 + c)(0) + 2h(0) = 0$$

which proves the result.

Limiting Point

Limiting points of system of co-axial circles are the centres of the point circles belonging to the family (Circles whose radii are zero are called **point circles**).

1. Limiting points of the co-axial system

Let the circle is

$$x^2 + y^2 + 2gx + c = 0 \quad \text{...(i)}$$

where, g is variable and c is constant.

$\therefore$ Centre and the radius of Eq. (i) are $(-g, 0)$ and $\sqrt{(g^2 - c)}$, respectively. Let

$$\sqrt{g^2 - c} = 0$$

$$\therefore \quad g = \pm\sqrt{c}$$

Thus, we get the two limiting points of the given co-axial system as

$$(\sqrt{c}, 0) \text{ and } (-\sqrt{c}, 0)$$

Clearly the above limiting points are real and distinct, real and coincident or imaginary according as $c >, =, < 0$

2. System of co-axial circles whose two limiting point are given :

Let (α, β) and (γ, δ) be the two given limiting points. Then, the corresponding point circles with zero radii are

$$(x - \alpha)^2 + (y - \beta)^2 = 0$$

and $$(x - \gamma)^2 + (y - \delta)^2 = 0$$

or $$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 = 0$$

and $$x^2 + y^2 - 2\gamma x - 2\delta y + \gamma^2 + \delta^2 = 0$$

The equation of co-axial system is

$$(x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2)$$
$$+ \lambda(x^2 + y^2 - 2\gamma x - 2\delta y + \gamma^2 + \delta^2) = 0$$

where, $\lambda \neq -1$ is a variable parameter.

$$\Rightarrow x^2(1+\lambda)+y^2(1+\lambda)-2x(\alpha+\gamma\lambda)$$
$$-2y(\beta+\delta\lambda)+(\alpha^2+\beta^2)+\lambda(\gamma^2+\delta^2)=0$$

or $$x^2+y^2-\frac{2(\alpha+\gamma\lambda)}{(1+\lambda)}x-2\frac{(\beta+\delta\lambda)}{(1+\lambda)}y+\frac{(\alpha^2+\beta^2)+\lambda(\gamma^2+\delta^2)}{(1+\lambda)}=0$$

Centre of this circle is $\left(\frac{(\alpha+\gamma\lambda)}{(1+\lambda)},\frac{(\beta+\delta\lambda)}{(1+\lambda)}\right)$ ...(i)

For limiting point,

Radius
$$=\sqrt{\frac{(\alpha+\gamma\lambda)^2}{(1+\lambda)^2}+\frac{(\beta+\delta\lambda)^2}{(1+\lambda)^2}-\frac{(\alpha^2+\beta^2)+\lambda(\gamma^2+\delta^2)}{(1+\lambda)}}=0$$

After solving, find λ. Substituting value of λ in Eq. (i), we get the limiting point of co-axial system.

Example 90. Find the coordinates of the limiting points of the system of circles determined by the two circles
$$x^2+y^2+5x+y+4=0 \text{ and } x^2+y^2+10x-4y-1=0$$

Sol. The given circles are
$$S_1 \equiv x^2+y^2+5x+y+4=0$$
and $$S_2 \equiv x^2+y^2+10x-4y-1=0$$

$\therefore$ Equation of the co-axial system of circles is $S_1+\lambda S_2=0$

or $$(x^2+y^2+5x+y+4)+\lambda(x^2+y^2+10x-4y-1)=0$$

or $$x^2(1+\lambda)+y^2(1+\lambda)+5x(1+2\lambda)+y(1-4\lambda)+(4-\lambda)=0$$

or $$x^2+y^2+\frac{5(1+2\lambda)}{(1+\lambda)}x+\frac{(1-4\lambda)}{(1+\lambda)}y+\frac{(4-\lambda)}{(1+\lambda)}=0$$

The centre of this circles is
$$\left(\frac{-5(1+2\lambda)}{2(1+\lambda)},-\frac{(1-4\lambda)}{2(1+\lambda)}\right) \quad \text{...(i)}$$

$$\text{Radius}=\sqrt{\frac{25(1+2\lambda)^2}{4(1+\lambda)^2}+\frac{(1-4\lambda)^2}{4(1+\lambda)^2}-\frac{(4-\lambda)}{(1+\lambda)}}=0$$

or $25(1+2\lambda)^2+(1-4\lambda)^2-4(4-\lambda)(1+\lambda)=0$

or $25(4\lambda^2+4\lambda+1)+(16\lambda^2-8\lambda+1)-4(-\lambda^2+3\lambda+4)=0$

or $120\lambda^2+80\lambda+10=0$ or $12\lambda^2+8\lambda+1=0$

or $(6\lambda+1)(2\lambda+1)=0$

i.e. $\lambda=-\frac{1}{6}$ and $-\frac{1}{2}$

Substituting these values of λ in Eq. (i), we get the points $(-2,-1)$ and $(0,-3)$ which are the required limiting points.

Example 91. If the origin be one limiting point of a system of co-axial circles of which $x^2+y^2+3x+4y+25=0$ is a member, find the other limiting point.

Sol. Equation of circle with origin as limiting point is
$$(x-0)^2+(y-0)^2=0 \quad \text{or} \quad x^2+y^2=0$$
belongs to the system of co-axial circles of which one member is
$$x^2+y^2+3x+4y+25=0$$
Hence, the equation of the whole system is
$$(x^2+y^2+3x+4y+25)+\lambda(x^2+y^2)=0$$
or $$x^2(1+\lambda)+y^2(1+\lambda)+3x+4y+25=0$$
or $$x^2+y^2+\frac{3}{(1+\lambda)}x+\frac{4}{(1+\lambda)}y+\frac{25}{(1+\lambda)}=0 \quad \text{...(i)}$$
$\therefore$ Its centre $=\left(-\frac{3}{2(1+\lambda)},-\frac{2}{(1+\lambda)}\right)$ ...(ii)

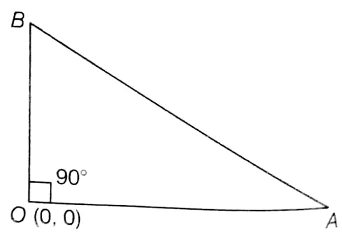
Radius of Eq. (i) can be zero for limiting point, then
$$\frac{9}{4(1+\lambda)^2}+\frac{4}{(1+\lambda)^2}-\frac{25}{(1+\lambda)}=0$$
$$9+16-100(1+\lambda)=0$$
$\Rightarrow$ $1+\lambda=\frac{1}{4}$ or $\lambda=-\frac{3}{4}$

From Eq. (ii), $\left(\frac{-3}{2(1-3/4)},\frac{-2}{(1-3/4)}\right)$

or $(-6,-8)$ is the other limiting point of the system.

Example 92. Prove that the limiting points of the system
$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fx+k)=0$$
subtend a right angle at the origin, if $\frac{c}{g^2}+\frac{k}{f^2}=2$.

Sol. The given circle is
$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+k)=0$$
or $$(1+\lambda)x^2+(1+\lambda)y^2+2gx+2fy\lambda+c+k\lambda=0$$
or $$x^2+y^2+\frac{2g}{(1+\lambda)}x+\frac{2f\lambda}{(1+\lambda)}y+\frac{c+k\lambda}{(1+\lambda)}=0 \quad \text{...(i)}$$
Its centre is $\left(\frac{-g}{1+\lambda},\frac{-f\lambda}{1+\lambda}\right)$ ...(ii)



Radius of circle Eq. (i) is =0

$$\Rightarrow \sqrt{\frac{g^2}{(1+\lambda)^2}+\frac{f^2\lambda^2}{(1+\lambda)^2}-\frac{(c+k\lambda)}{(1+\lambda)}}=0$$

or $\lambda^2(f^2-k)-\lambda(k+c)+g^2-c=0$

which is a quadratic in λ. Let roots be λ_1 and λ_2.

$$\therefore \quad \lambda_1+\lambda_2=\frac{k+c}{f^2-k} \quad \text{and} \quad \lambda_1\lambda_2=\frac{g^2-c}{f^2-k}$$

then limiting points are [from Eq. (ii)]

$$A\left(\frac{-g}{1+\lambda_1},\frac{-f\lambda_1}{1+\lambda_1}\right) \quad \text{and} \quad B\left(\frac{-g}{1+\lambda_2},\frac{-f\lambda_2}{1+\lambda_2}\right)$$

But given that AB subtend a right angle at the origin.

$\therefore$ Slope of OA $\times$ Slope of $OB = -1$

$$\Rightarrow \left(\frac{\frac{-f\lambda_1}{1+\lambda_1}}{\frac{-g}{1+\lambda_1}}\right)\times\left(\frac{\frac{-f\lambda_2}{1+\lambda_2}}{\frac{-g}{1+\lambda_2}}\right)=-1$$

or $$\frac{f\lambda_1}{g}\times\frac{f\lambda_2}{g}=-1$$

or $$f^2\lambda_1\lambda_2+g^2=0$$

or $$f^2\frac{(g^2-c)}{(f^2-k)}+g^2=0$$

or $$2g^2f^2-cf^2-kg^2=0$$

or $$2=\frac{c}{g^2}+\frac{k}{f^2}$$

| Example 93. Find the radical axis of co-axial system of circles whose limiting points are (–1,2) and (2, 3).

Sol. Equations of circles with limiting points are (–1, 2) and (2,3) are

$$(x+1)^2+(y-2)^2=0$$

or $$x^2+y^2+2x-4y+5=0 \quad \ldots(\text{i})$$

and $$(x-2)^2+(y-3)^2=0$$

or $$x^2+y^2-4x-6y+13=0 \quad \ldots(\text{ii})$$

respectively.

$\therefore$ Radical axis of circles Eqs. (i) and (ii) is

$$(x^2+y^2+2x-4y+5)-(x^2+y^2-4x-6y+13)=0$$

or $$6x+2y-8=0$$

or $$3x+y-4=0$$

| Example 94. Find the equation of the circle which passes through the origin and belongs to the co-axial of circles whose limiting points are (1,2) and (4,3).

Sol. Equations of circles whose limiting points are (1,2) and (4,3) are

$$(x-1)^2+(y-2)^2=0$$

or $$x^2+y^2-2x-4y+5=0 \quad \ldots(\text{i})$$

and $$(x-4)^2+(y-3)^2=0$$

or $$x^2+y^2-8x-6y+25=0 \quad \ldots(\text{ii})$$

Therefore, the corresponding system of co-axial circles is

$$(x^2+y^2-2x-4y+5)+\lambda(x^2+y^2-8x-6y+25)=0 \quad \ldots(\text{iii})$$

It passes through origin, then

$$5+25\lambda=0$$

$$\therefore \quad \lambda=-\frac{1}{5}$$

Substituting the value of λ in Eq. (iii), the required circle is

$$5(x^2+y^2-2x-4y+5)-(x^2+y^2-8x-6y+25)=0$$

or $$4x^2+4y^2-2x-14y=0$$

or $$2x^2+2y^2-x-7y=0$$

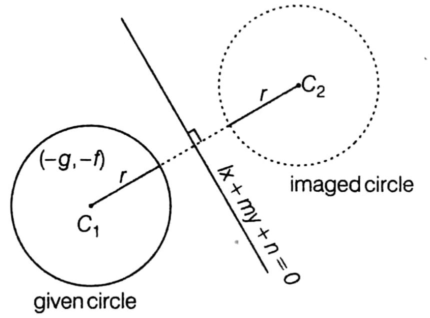
Image of the Circle by the Line Mirror

Let the circle be $x^2+y^2+2gx+2fy+c=0$ and line mirror is $lx+my+n=0$ in this condition, radius of circle remains unchanged but centres changes. Let the centre of imaged circle be (x_1, y_1).

Then, $$\frac{x_1-(-g)}{l}=\frac{y_1-(-f)}{m}=\frac{-2(-lg-mf+n)}{(l^2+m^2)}$$

we get, $$x_1=\frac{(l^2g-m^2g+2mlf-2nl)}{(l^2+m^2)}$$

and $$y_1=\frac{(m^2f-l^2f+2mlg-2mn)}{(l^2+m^2)}$$



$\therefore$ Required imaged circle is $(x-x_1)^2+(y-y_1)^2=r^2$

where, $r=\sqrt{(g^2+f^2-c)}$.

Example 95. Find the equation of the image of the circle $x^2+y^2+16x-24y+183=0$ by the line mirror $4x+7y+13=0$.

Sol. The given circle and line are

$$x^2+y^2+16x-24y+183=0 \quad \ldots(i)$$

and

$$4x+7y+13=0 \quad \ldots(ii)$$

Centre and radius of circle Eq. (i) are $(-8,12)$ and 5, respectively. Let the centre of the imaged circle be (x_1,y_1). Hence, (x_1,y_1) be the image of the point $(-8, 12)$ with respect to the line $4x+7y+13=0$, then

$$\frac{x_1-(-8)}{4}=\frac{y_1-12}{7}=\frac{-2(4(-8)+7(12)+13)}{(4^2+7^2)}$$

$$\Rightarrow \quad \frac{x_1+8}{4}=\frac{y_1-12}{7}=-2$$

$\therefore \; x_1=-16,\, y_1=-2$

$\therefore$ Equation of the imaged circle is $(x+16)^2+(y+2)^2=5^2$

or $\quad x^2+y^2+32x+4y+235=0$

Exercise for Session 7

1. The circles $x^2+y^2+x+y=0$ and $x^2+y^2+x-y=0$ intersect at an angle of

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

2. If the circles of same radius a and centres at (2, 3) and (5, 6) cut orthogonally, then a equals to

(a) 1 (b) 2 (c) 3 (d) 4

3. If the circles $x^2+y^2+2x+2ky+6=0$ and $x^2+y^2+2ky+k=0$ intersect orthogonally, k is

(a) 2 or $-\frac{3}{2}$ (b) -2 or $-\frac{3}{2}$ (c) 2 or $\frac{3}{2}$ (d) -2 or $\frac{3}{2}$

4. If a circle passes through the point (a,b) and cuts the circle $x^2+y^2=4$ orthogonally, then the locus of its centre is

(a) $2ax+2by+(a^2+b^2+4)=0$
(b) $2ax+2by-(a^2+b^2+4)=0$
(c) $2ax-2by+(a^2+b^2+4)=0$
(d) $2ax-2by-(a^2+b^2+4)=0$

5. The locus of the centre of the circle which cuts orthogonally the circle $x^2+y^2-20x+4=0$ and which touches $x=2$ is

(a) $x^2=16y$
(b) $x^2=16y+4$
(c) $y^2=16x$
(d) $y^2=16x+4$

6. The equation of a circle which cuts the three circles $x^2+y^2-3x-6y+14=0$, $x^2+y^2-x-4y+8=0$ and $x^2+y^2+2x-6y+9=0$ orthogonally is

(a) $x^2+y^2-2x-4y+1=0$
(b) $x^2+y^2+2x+4y+1=0$
(c) $x^2+y^2-2x+4y+1=0$
(d) $x^2+y^2-2x-4y-1=0$

7. The equation of radical axis of the circles $x^2+y^2+x-y+2=0$ and $3x^2+3y^2-4x-12=0$ is

(a) $2x^2+2y^2-5x+y-14=0$
(b) $7x-3y+18=0$
(c) $5x-y+14=0$
(d) None of these

8. The radical centre of the circles $x^2+y^2=1$, $x^2+y^2+10y+24=0$ and $x^2+y^2-8x+15=0$ is

(a) $(2,5/2)$
(b) $(-2,5/2)$
(c) $(-2,-5/2)$
(d) $(2,-5/2)$

9. If (1,2) is a limiting point of the co-axial system of circles containing the circle $x^2+y^2+x-5y+9=0$, then the equation of the radical axis is

(a) $x-9y+4=0$
(b) $3x-y+4=0$
(c) $x+3y-4=0$
(d) $9x+y-4=0$

10. The limiting points of the system of circles represented by the equation $2(x^2+y^2)+\lambda x+\frac{9}{2}=0$ are

(a) $\left(\pm\frac{3}{2},0\right)$ (b) $(0,0)$ and $\left(\frac{9}{2},0\right)$
(c) $\left(\pm\frac{9}{2},0\right)$ (d) $(\pm 3, 0)$

11. One of the limiting points of the co-axial system of circles containing the circles $x^2+y^2-4=0$ and $x^2+y^2-x-y=0$ is

(a) $(\sqrt{2},\sqrt{2})$ (b) $(-\sqrt{2},\sqrt{2})$
(c) $(-\sqrt{2},-\sqrt{2})$ (d) None of these

12. The point (2, 3) is a limiting point of a co-axial system of circles of which $x^2+y^2=9$ is a member. The coordinates of the other limiting point is given by

(a) $\left(\frac{18}{13},\frac{27}{13}\right)$ (b) $\left(\frac{9}{13},\frac{6}{13}\right)$
(c) $\left(\frac{18}{13},-\frac{27}{13}\right)$ (d) $\left(-\frac{18}{13},-\frac{9}{13}\right)$

13. Two circles are drawn through the points $(a, 5a)$ and $(4a, a)$ to touch the Y-axis. Prove that they intersect at angle $\tan^{-1}\left(\frac{40}{9}\right)$.

14. Find the equation of the circle which cuts orthogonally the circle $x^2+y^2-6x+4y-3=0$, passes through (3,0) and touches the axis of y.

15. Tangents are drawn to the circles $x^2+y^2+4x+6y-19=0$, $x^2+y^2=9$ from any point on the line $2x+3y=5$. Prove that their lengths are equal.

16. Find the coordinates of the point from which the lengths of the tangents to the following three circles be equal
$3x^2+3y^2+4x-6y-1=0$, $2x^2+2y^2-3x-2y-4=0$ and $2x^2+2y^2-x+y-1=0$

17. Find the equation of a circle which is co-axial with the circles $x^2+y^2+4x+2y+1=0$ and $x^2+y^2-x+3y-\frac{3}{2}=0$ and having its centre on the radical axis of these circles.

18. Find the radical axis of a co-axial system of circles whose limiting points are (1, 2) and (3 , 4).

Shortcuts and Important Results to Remember

1 If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the X-axis and Y-axis in four concyclic points, then $\boxed{a_1a_2 = b_1b_2}$.

2 If two conic sections
$a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$ and
$a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0$ will intersect each other in four concyclic points, if $\dfrac{a_1 - b_1}{a_2 - b_2} = \dfrac{h_1}{h_2}$.

3 If the circle $S_1 = 0$, bisects the circumference of the circle $S_2 = 0$, then their common chord will be the diameter of the circle $S_2 = 0$.

4 The radius of the director circle of a given circle is $\sqrt{2}$ times the radius of the given circle.

5 The point of intersection of the tangents at the points $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$ on the circle $x^2 + y^2 = a^2$ is

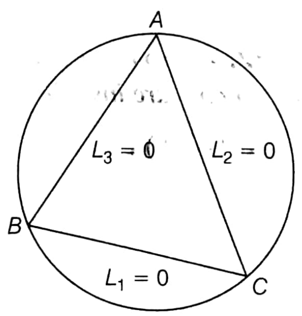
$$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{a\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

6 If the tangent to the circle $x^2 + y^2 = r^2$ at the point (a, b) meets the coordinates axes at the points A and B and O is the origin, then the area of the ΔOAB is $\dfrac{r^4}{2ab}$.

7 The length of the common chord of the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$ is $\sqrt{\dfrac{1}{2}(a+b)^2 - 4c}$.

8 The length of the common chord of the circles $(x-a)^2 + y^2 = a^2$ and $x^2 + (y-b)^2 = b^2$ is $\dfrac{2ab}{\sqrt{a^2 + b^2}}$.

9 Family of circles circumscribing a triangle whose sides are given by $L_1 = 0, L_2 = 0$ and $L_3 = 0$ is given by $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2.

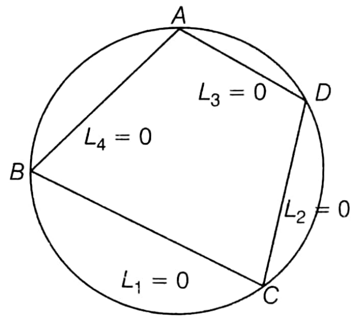


Remark

Equation of the circle circumscribing the triangle formed by the lines $a_rx + b_ry + c_r = 0$, where $r = 1, 2, 3$, is :

$$\begin{vmatrix} \dfrac{a_1^2 + b_1^2}{a_1x + b_1y + c_1} & a_1 & b_1 \\ \dfrac{a_2^2 + b_2^2}{a_2x + b_2y + c_2} & a_2 & b_2 \\ \dfrac{a_3^2 + b_3^2}{a_3x + b_3y + c_3} & a_3 & b_3 \end{vmatrix} = 0$$

10 Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0, L_2 = 0, L_3 = 0$ and $L_4 = 0$ is given by $L_1L_3 + \lambda L_2L_4 = 0$



provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

11 The locus of the middle point of a chord of a circle subtending a right angle at a given point will be a circle.

12 The length of an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$ is $a\sqrt{3}$.

13 The distance between the chord of contact of tangents to $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is $\dfrac{|g^2 + f^2 - c|}{2\sqrt{(g^2 + f^2)}}$.

14 The shortest chord of a circle passing through a point P inside the circle is the chord whose middle point is P.

15 The length of transverse common tangent < the length of direct common tangent.

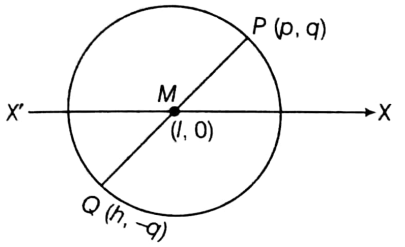
JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1 *Two distinct chords drawn from the point (p,q) on the circle $x^2+y^2=px+qy$, where $pq\neq 0$, are bisected by the X-axis. Then,*

(a) $|p|=|q|$ (b) $p^2=8q^2$ (c) $p^2<8q^2$ (d) $p^2>8q^2$

Sol. (d)



Suppose chord bisect at $M(\lambda,0)$, then other end point of chord is $(h,-q)$

where, $\lambda=\dfrac{p+h}{2}$

which lie on $x^2+y^2=px+qy$

or $h^2+q^2=ph-q^2$

$\Rightarrow$ $h^2-ph+2q^2=0$

for two distinct chords, $\boxed{B^2-4AC>0}$

or $p^2-4\cdot 1\cdot 2q^2>0$

or $p^2>8q^2$

• Ex. 2 *The values of λ for which the circle $x^2+y^2+6x+5+\lambda(x^2+y^2-8x+7)=0$ dwindles into a point are*

(a) $1\pm\dfrac{\sqrt{2}}{3}$ (b) $2\pm\dfrac{2\sqrt{2}}{3}$ (c) $2\pm\dfrac{4\sqrt{2}}{3}$ (d) $1\pm\dfrac{4\sqrt{2}}{3}$

Sol. (c) The given circle is

$$x^2+y^2+6x+5+\lambda(x^2+y^2-8x+7)=0$$

or $$x^2(1+\lambda)+y^2(1-\lambda)+(6-8\lambda)x+(5+7\lambda)=0$$

$\Rightarrow$ $$x^2+y^2+\left(\frac{6-8\lambda}{1+\lambda}\right)x+\left(\frac{5+7\lambda}{1+\lambda}\right)=0$$

This will dwindle into a point circle, then radius of the circle $=0$

$$\sqrt{\left(\frac{3-4\lambda}{1+\lambda}\right)^2+0-\left(\frac{5+7\lambda}{1+\lambda}\right)}=0$$

$\Rightarrow$ $(3-4\lambda)^2-(5+7\lambda)(1+\lambda)=0$

$\Rightarrow$ $9-16\lambda^2-24\lambda-5-5\lambda-7\lambda-7\lambda^2=0$

$\Rightarrow$ $9\lambda^2-36\lambda+4=0$

$$\lambda=\frac{36\pm\sqrt{(36)^2-4.9.4}}{2.9}$$

$\therefore$ $$\lambda=2\pm\frac{4\sqrt{2}}{3}$$

• Ex. 3 *If $f(x+y)=f(x)\cdot f(y)$ for all x and y, $f(1)=2$ and $\alpha_n=f(n), n\in N$, then the equation of the circle having (α_1,α_2) and (α_3,α_4) as the ends of its one diameter is*

(a) $(x-2)(x-8)+(y-4)(y-16)=0$

(b) $(x-4)(x-8)+(y-2)(y-16)=0$

(c) $(x-2)(x-16)+(y-4)(y-8)=0$

(d) $(x-6)(x-8)+(y-5)(y-6)=0$

Sol. (a) $\because$ $f(x+y)=f(x).f(y)$ ...(i)

$\because$ $f(1)=2$

In Eq. (i), Put $x=y=1$,

then $f(2)=f(1).f(1)=2^2$

Now, in Eq. (i), $x=1, y=2$, then

$f(3)=f(1)f(2)=2.2^2=2^3$

Hence, $f(n)=2^n$

$\therefore$ $\alpha_n=f(n)=2^n\,\forall\, n\in N$

$(\alpha_1,\alpha_2)\equiv(2,4)$

and $(\alpha_3,\alpha_4)\equiv(8,16)$

Equation of circle in diametric form is

$(x-2)(x-8)+(y-4)(y-16)=0$

• Ex. 4 *Two circles of radii a and b touching each other externally, are inscribed in the area bounded by $y=\sqrt{(1-x^2)}$ and the X-axis. If $b=\dfrac{1}{2}$, then a is equal to*

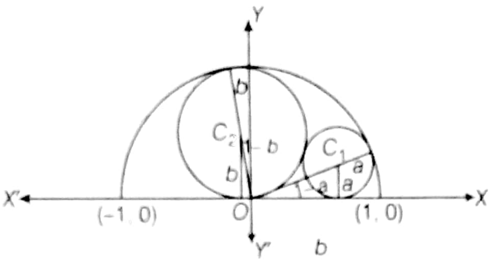
(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{8}$

(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{\sqrt{2}}$

Sol. (a) Let the centres of circles be C_1 and C_2, then

$C_1\equiv(\sqrt{(1-2a)},a)$

and $C_2\equiv(\sqrt{(1-2b)},b)$



Now, $C_1C_2 = a + b$

$$\Rightarrow \quad \left((\sqrt{(1-2a)})^2 + \left(a-\frac{1}{2}\right)^2\right) = \left(a+\frac{1}{2}\right)^2 \qquad \left[\because b = \frac{1}{2}\right]$$

$$\text{or} \quad 1 - 2a + \left(a-\frac{1}{2}\right)^2 = \left(a+\frac{1}{2}\right)^2$$

$$\text{or} \quad 1 - 2a + a^2 + \frac{1}{4} - a = a^2 + \frac{1}{4} + a$$

$$\text{or} \quad a = \frac{1}{4}$$

• **Ex. 5** *There are two circles whose equations are $x^2 + y^2 = 9$ and $x^2 + y^2 - 8x - 6y + n^2 = 0, n \in I$. If the two circles having exactly two common tangents, then the number of possible values of n is*

(a) 2 (b) 7 (c) 8 (d) 9

Sol. (d) Given circles are $S_1 : x^2 + y^2 - 9 = 0$

Its centre $C_1:(0,0)$ and radius $r_1 = 3$

and $S_2 : x^2 + y^2 - 8x - 6y + n^2 = 0$

Its centre $C_2:(4,3)$ and radius $r_2 = \sqrt{(25 - n^2)}$

Here, $25 - n^2 > 0 \Rightarrow -5 < n < 5$ ...(i)

For exactly two common tangents,

$$r_1 + r_2 > C_1C_2$$

$$\Rightarrow \quad 3 + \sqrt{(25-n^2)} > \sqrt{(4^2+3^2)}$$

$$\Rightarrow \quad \sqrt{(25-n^2)} > 2$$

$$\Rightarrow \quad 25 - n^2 > 4$$

$$\text{or} \quad n^2 < 21$$

$$\text{or} \quad -\sqrt{21} < n < \sqrt{21} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$-\sqrt{21} < n < \sqrt{21}$$

But $n \in I$. So, $n = -4, -3, -2, -1, 0, 1, 2, 3, 4$

Hence, number of possible values of n is 9.

• **Ex. 6** *Suppose $f(x, y) = 0$ is the equation of a circle such that $f(x, 1) = 0$ has equal roots (each equal to 2) and $f(1, x) = 0$ also has equal roots (each equal to zero). The equation of circle is*

(a) $x^2 + y^2 + 4x + 3 = 0$ (b) $x^2 + y^2 + 4y + 3 = 0$

(c) $x^2 + y^2 + 4x - 3 = 0$ (d) $x^2 + y^2 - 4x + 3 = 0$

Sol. (d) Let $f(x, y) = x^2 + y^2 + 2gx + 2fy + c$

$\Rightarrow \quad f(x, 1) = x^2 + 1 + 2gx + 2f + c \equiv (x-2)^2$ (given)

then, $g = -2,\ 2f + c = 3$ ...(i)

Also, $f(1, x) = 1 + x^2 + 2g + 2fx + c \equiv (x-0)^2$ (given)

then, $f = 0,\ 2g + c = -1$ ...(ii)

From Eqs. (i) and (ii), we get

$$g = -2, f = 0, c = 3$$

Thus, equation of circle is

$$x^2 + y^2 - 4x + 3 = 0$$

Must

• **Ex. 7** *A variable circle C has the equation $x^2 + y^2 - 2(t^2 - 3t + 1)x - 2(t^2 + 2t)y + t = 0$, where t is a parameter. If the power of point (a, b) w.r.t. the circle C is constant, then the ordered pair (a, b) is*

(a) $\left(\frac{1}{10}, -\frac{1}{10}\right)$ (b) $\left(\frac{1}{10}, \frac{1}{10}\right)$

(c) $\left(-\frac{1}{10}, \frac{1}{10}\right)$ (d) $\left(-\frac{1}{10}, -\frac{1}{10}\right)$

Sol. (c) $\because C : x^2 + y^2 - 2(t^2 - 3t + 1)x - 2(t^2 + 2t)y + t = 0$

given power of circle = constant

$$\therefore \quad a^2 + b^2 - 2(t^2 - 3t + 1)a - 2(t^2 + 2t)b + t = \text{constant}$$

$$\Rightarrow \quad -2(a+b)t^2 + (6a - 4b + 1)t + (a^2 + b^2 - 2a) = \text{constant}$$

$\because$ Power of circle is constant, then

$a + b = 0$ and $6a - 4b + 1 = 0$

or $b = -a$, then $6a + 4a + 1 = 0$

$$\therefore \quad a = -\frac{1}{10}, b = \frac{1}{10}$$

Hence, required ordered pair is $\left(-\frac{1}{10}, \frac{1}{10}\right)$

N • **Ex. 8** *If the radii of the circles $(x-1)^2 + (y-2)^2 = 1$ and $(x-7)^2 + (y-10)^2 = 4$ are increasing uniformly w.r.t. time as 0.3 unit/s and 0.4 unit/s respectively, then they will touch each other at t equals to*

(a) 45 s (b) 90 s

(c) 11 s (d) 135 s

Sol. (b) Given circles are $S_1 : (x-1)^2 + (y-2)^2 = 1$

Its centre $C_1:(1,2)$ and radius $r_1 = 1$

and $S_2 : (x-7)^2 + (y-10)^2 = 4$

Its centre $C_2:(7,10)$ and radius $r_2 = 2$

$\because \quad C_1C_2 = 10 > r_1 + r_2$

Hence, the two circles are separated.

The radii of the two circles at time t are $(1 + 0.3t)$ and $(2 + 0.4t)$

Use vector method.

$0.3 + 0.4 = 0.7$; $0.3 - 0.4 = 0.1$

For the two circles touch each other, then

$$C_1C_2 = |(1+0.3t) \pm (2+0.4t)|$$

$\Rightarrow$ $10 = |3 + 0.7t|$ or $10 = |-1 - 0.1t|$

$\Rightarrow$ $0.7t + 3 = \pm 10$ or $-1 - 0.1t = \pm 10$

$\Rightarrow$ $t = 10$ or $t = 90$ $\quad [\because t > 0]$

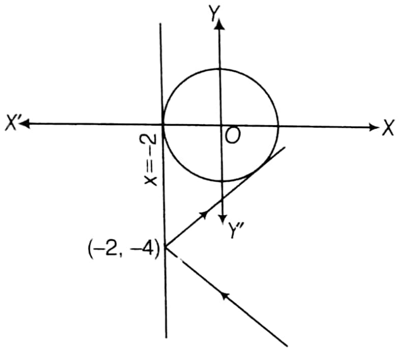
• **Ex. 9** *A light ray gets reflected from $x = -2$. If the reflected ray touches the circle $x^2 + y^2 = 4$ and the point of incident is $(-2, -4)$, then the equation of the incident ray is*

(a) $4y + 3x + 22 = 0$ (b) $3y + 4x + 20 = 0$

(c) $4y + 2x + 20 = 0$ (d) $y + x + 6 = 0$

Sol. (a) Any tangent of $x^2 + y^2 = 4$ is $y = mx \pm 2\sqrt{(1+m^2)}$.

If it passes through $(-2, -4)$, then $-4 = -2m \pm 2\sqrt{(1+m^2)}$



or $\quad (m-2)^2 = 1 + m^2$

or $\quad m = \infty, m = 3/4$

Hence, the slope of the reflected ray is 3/4.

Thus, the equation of the incident ray is

$$y + 4 = -\frac{3}{4}(x + 2)$$

i.e. $\quad 4y + 3x + 22 = 0$

• **Ex. 10** *If a circle having centre at (α, β) radius r completely lies with in two lines $x + y = 2$ and $x + y = -2$, then, $\min.(|\alpha + \beta + 2|, |\alpha + \beta - 2|)$ is*

(a) greater than $\sqrt{2}r$

(b) less than $\sqrt{2}r$

(c) greater than $2r$

(d) less than $2r$

Sol. (a) Minimum distance of the centre from line > radius of circle i.e. $\min.\left\{\frac{|\alpha+\beta+2|}{\sqrt{2}}, \frac{|\alpha+\beta-2|}{\sqrt{2}}\right\} > r$

or $\min.\{|\alpha+\beta+2|, |\alpha+\beta-2|\} > \sqrt{2}r$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which MORE THAN ONE may be correct.

• **Ex. 11** *If point $P(x, y)$ is called a lattice point, if $x, y \in I$. Then, the total number of lattice points in the interior of the circle $x^2 + y^2 = a^2, a \neq 0$ cannot be*

(a) 202 (b) 203 (c) 204 (d) 205

Sol. (a, b, c) Given circle is $x^2 + y^2 = a^2$ ...(i)

Clearly (0, 0) will belong the interior of circle Eq. (i). Also, other points interior to circle Eq. (i) will have the coordinates of the form

$$(\pm\lambda, 0), (0, \pm\lambda), \text{ where } \lambda^2 < a^2$$

and $(\pm\lambda, \pm\mu)$ and $(\pm\mu, \pm\lambda)$, where $\lambda^2 + \mu^2 < a^2$ and $\lambda, \mu \in I$

$\therefore$ Number of lattice points in the interior of the circle will be of the form $1 + 4r + 8t$, where $r, t = 0, 1, 2, \ldots$

$\therefore$ Number of such points must be of the form $4n + 1$, where $n = 0, 1, 2, \ldots$

• **Ex. 12** *Let x, y be real variable satisfying $x^2 + y^2 + 8x - 10y - 40 = 0$. Let $a = \max.\{\sqrt{(x+2)^2 + (y-3)^2}\}$ and $b = \min.\{\sqrt{(x+2)^2 + (y-3)^2}\}$, then*

(a) $a + b = 18$ (b) $a - b = 4\sqrt{2}$

(c) $a + b = 4\sqrt{2}$ (d) $a \cdot b = 73$

Sol. (a, b, d) Given circle is

$$x^2 + y^2 + 8x - 10y - 40 = 0$$

The centre and radius of the circle are $(-4, 5)$ and 9, respectively.

Distance of the centre $(-4, 5)$ from $(-2, 3)$ is

$$\sqrt{(4+4)} = 2\sqrt{2}.$$

Therefore, $\quad a = 2\sqrt{2} + 9$

and $\quad b = -2\sqrt{2} + 9$

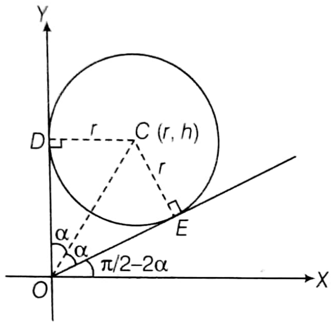
$\therefore \quad a + b = 18, a - b = 4\sqrt{2}, ab = 73$

Ex. 13 *The equation of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, are*

(a) $x = 0$
(b) $y = 0$
(c) $(h^2 - r^2)x - 2rhy = 0$
(d) $(h^2 - r^2)x + 2rhy = 0$

Sol. (a, c) The given equation is $(x-r)^2 + (y-h)^2 = r^2$

tangents are $x = 0$



and $$y = x\tan\left(\frac{\pi}{2} - 2\alpha\right) = x\cot 2\alpha$$

$$= \frac{x(1-\tan^2\alpha)}{2\tan\alpha}$$

$$y = \frac{x\left(1 - \frac{r^2}{h^2}\right)}{2\left(\frac{r}{h}\right)} \qquad \left(\because \text{in } \Delta ODC, \tan\alpha = \frac{r}{h}\right)$$

or $(h^2 - r^2)x - 2rhy = 0$

Ex. 14 *Point M moved on the circle $(x-4)^2 + (y-8)^2 = 20$. Then it broke away from it and moving along a tangent to the circle cut the X-axis at point $(-2, 0)$. The coordinates of the point on the circle at which the moving point broke away is*

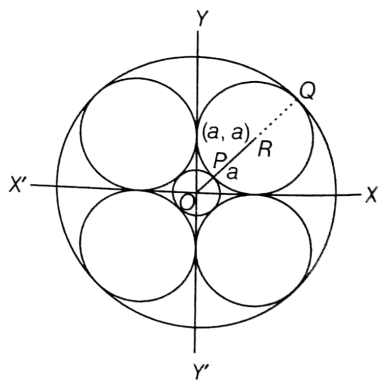
(a) $\left(\frac{42}{5}, \frac{36}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{44}{5}\right)$
(c) $(6, 4)$ (d) $(2, 4)$

Sol. (b, c) Given circle is
$$(x-4)^2 + (y-8)^2 = 20$$
or $$x^2 + y^2 - 8x - 16y + 60 = 0 \quad \text{...(i)}$$
Equation of chord of contact from $(-2, 0)$ is
$$-2\cdot x + 0\cdot y - 4(x-2) - 8(y+0) + 60 = 0$$
or $$3x + 4y - 34 = 0 \quad \text{...(ii)}$$
Solving Eqs. (i) and (ii), we get
$$x^2 + \left(\frac{34-3x}{4}\right)^2 - 8x - 16\left(\frac{34-3x}{4}\right) + 60 = 0$$
or $$5x^2 - 28x - 12 = 0$$
or $$(x-6)(5x+2) = 0$$
or $$x = 6, -\frac{2}{5}$$
Therefore, the points are $(6, 4)$ and $\left(-\frac{2}{5}, \frac{44}{5}\right)$.

Ex. 15 *The equations of four circles are $(x \pm a)^2 + (y \pm a)^2 = a^2$. The radius of a circle touching all the four circles is*

(a) $(\sqrt{2} - 1)a$ (b) $2\sqrt{2}a$
(c) $(\sqrt{2} + 1)a$ (d) $(2 + \sqrt{2})a$

Sol. (a, c) Radius of inner circle $= OR - a$
$$= \sqrt{(a^2 + a^2)} - a$$
$$= a(\sqrt{2} - 1)$$
Radius of outer circle $= OR + RQ$
$$= a\sqrt{2} + a = a(\sqrt{2} + 1)$$



JEE Type Solved Examples : Paragraph Based Questions

- This section contains **2 solved paragraphs** based upon each of the paragraph **3 multiple choice** questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to 18)

Consider the relation $4l^2 - 5m^2 + 6l + 1 = 0$, *where* $l, m \in R$.

16. The line $lx + my + 1 = 0$ touches a fixed circle whose equation is

(a) $x^2 + y^2 - 4x - 5 = 0$ (b) $x^2 + y^2 + 6x + 6 = 0$

(c) $x^2 + y^2 - 6x + 4 = 0$ (d) $x^2 + y^2 + 4x - 4 = 0$

17. Tangents PA and PB are drawn to the above fixed circle from the point P on the line $x + y - 1 = 0$. Then, the chord of contact AB passes through the fixed point

(a) $\left(\frac{1}{2}, -\frac{5}{2}\right)$ (b) $\left(\frac{1}{3}, \frac{4}{3}\right)$ (c) $\left(-\frac{1}{2}, \frac{3}{2}\right)$ (d) $\left(\frac{1}{2}, \frac{5}{2}\right)$

18. The number of tangents which can be drawn from the point $(2, -3)$ are

(a) 0 (b) 1 (c) 2 (d) 1 or 2

Sol.

16. (c) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

The line $lx + my + 1 = 0$ touch circle Eq. (i), then

$$\frac{|-lg - mf + 1|}{\sqrt{(l^2 + m^2)}} = \sqrt{(g^2 + f^2 - c)}$$

$$\Rightarrow \quad (lg + mf - 1)^2 = (l^2 + m^2)(g^2 + f^2 - c)$$

or $(f^2 - c)l^2 + (g^2 - c)m^2 - 2gflm + 2gl + 2fm - 1 = 0$ ...(ii)

But the given condition is

$$4l^2 - 5m^2 + 6l + 1 = 0 \quad \text{...(iii)}$$

Comparing Eqs. (ii) and (iii), we get

$$\frac{f^2 - c}{4} = \frac{g^2 - c}{-5} = \frac{-2gf}{0} = \frac{g}{3} = \frac{2f}{0} = \frac{-1}{1}$$

Then, we get $g = -3$, $f = 0$, $c = 4$

Substituting these values in Eq. (i), the equation of the circle is

$$x^2 + y^2 - 6x + 4 = 0$$

17. (a) Let any point on the line $x + y - 1 = 0$ is

$$P(\lambda, 1 - \lambda), \lambda \in R.$$

Then, equation of AB is

$$\lambda x + (1 - \lambda)y - 3(x + \lambda) + 4 = 0$$

$$\Rightarrow \quad (-3x + y + 4) + \lambda(x - y - 3) = 0$$

for fixed point $-3x + y + 4 = 0$, $x - y - 3 = 0$

$$\therefore \quad x = \frac{1}{2}, y = -\frac{5}{2}$$

$\therefore$ Fixed point is $\left(\frac{1}{2}, \frac{-5}{2}\right)$

18. (c) Let $S \equiv x^2 + y^2 - 6x + 4 = 0$.

$$\therefore \quad S_1 = (2)^2 + (-3)^2 - 6(2) + 4$$

$$= 4 + 9 - 12 - 4$$

$$= 5 > 0$$

Therefore, point $(2, -3)$ lies outside the circle from which two tangents can be drawn.

Paragraph II

(Q. Nos. 19 to 21)

If α*- chord of a circle be that chord which subtends an angle* α *at the centre of the circle.*

19. If $x + y = 1$ is α-chord of $x^2 + y^2 = 1$, then α is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

20. If slope of a $\frac{\pi}{3}$-chord of $x^2 + y^2 = 4$ is 1, then its equation is

(a) $x - y + \sqrt{6} = 0$ (b) $x - y + \sqrt{3} = 0$

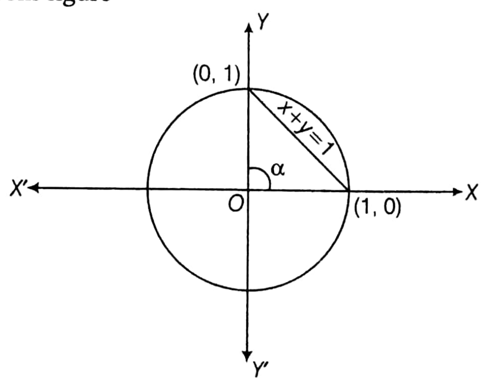
(c) $x - y - \sqrt{3} = 0$ (d) $x - y - 2\sqrt{3} = 0$

21. Distance of $\frac{2\pi}{3}$ — chord of $x^2 + y^2 + 2x + 4y + 1 = 0$ from the centre is

(a) $\frac{1}{\sqrt{2}}$ (b) 1 (c) $\sqrt{2}$ (d) 2

Sol.

19. (c) From figure

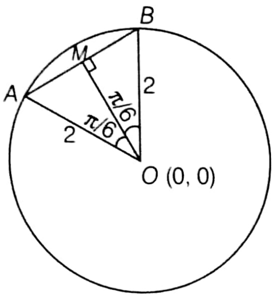


$$\alpha = \frac{\pi}{2}$$

20. (a) $\because$ Slope of chord is 1.

Let the equation of chord be $x - y + \lambda = 0$.

$$\because \qquad OM = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$$



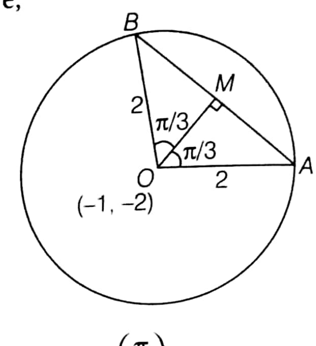
$$\therefore \qquad \frac{|0 - 0 + \lambda|}{\sqrt{2}} = \sqrt{3}$$

$$\Rightarrow \qquad \lambda = \pm\sqrt{6}$$

Hence, equation of chords are

$$x - y \pm \sqrt{6} = 0.$$

21. (b) From figure,



$$OM = 2\cos\left(\frac{\pi}{3}\right) = 1$$

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer,** ranging from 0 to 9 (both inclusive).

• Ex. 22 *A circle with centre in the first quadrant is tangent to $y = x + 10$, $y = x - 6$ and the Y-axis. Let (p, q) be the centre of the circle. If the value of $(p + q) = a + b\sqrt{a}$, when $a, b \in Q$, then the value of $|a - b|$ is*

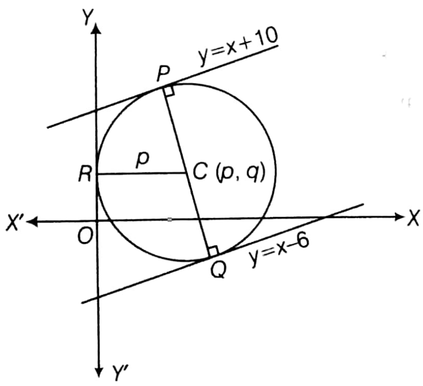
Sol. (6) $\because \qquad CP = CR$

$$\Rightarrow \qquad \frac{|p - q + 10|}{\sqrt{2}} = p$$

or $\qquad p - q + 10 = p\sqrt{2} \qquad$ ...(i)

and $\qquad CP = CQ$

$$\frac{p - q + 10}{\sqrt{2}} = -\left(\frac{p - q - 6}{\sqrt{2}}\right) \text{ or } p - q = -2 \qquad \text{...(ii)}$$



From Eqs. (i) and (ii), we get

$$p = 4\sqrt{2} \text{ and } q = 4\sqrt{2} + 2$$

Now, $\qquad p + q = 2 + 8\sqrt{2} = a + b\sqrt{2} \qquad$ (given)

$\therefore \qquad a = 2, b = 8$

Hence, $|a - b| = |2 - 8| = 6$

• Ex. 23 *If the circles $x^2 + y^2 + (3 + \sin\theta)x + 2\cos\phi y = 0$ and $x^2 + y^2 + (2\cos\phi)x + 2\lambda y = 0$ touch each other, then the maximum value of λ is*

Sol. (1) Since, both the circles are passing through the origin (0, 0), the equation of tangent at (0,0) of first circle will be same as that of the tangent at (0, 0) of second circle.

Equation of tangent at (0, 0) of first circle is

$$(3 + \sin\theta)x + (2\cos\phi)y = 0 \qquad \text{...(i)}$$

Equaton of tangent at (0, 0) of second circle is

$$(2\cos\phi)x + 2\lambda y = 0 \qquad \text{...(ii)}$$

Therefore, Eqs. (i) and (ii) must be identical, then

$$\frac{3 + \sin\theta}{2\cos\phi} = \frac{2\cos\phi}{2\lambda}$$

or $$\lambda = \frac{2\cos^2\phi}{(3 + \sin\theta)}$$

or $\qquad \lambda_{max} = 1 \qquad$ (when $\sin\theta = -1$ and $\cos\phi = 1$)

JEE Type Solved Examples : Matching Type Questions

- This section contains **2 examples.** Examples 24 and 25 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

• Ex. 24. *Consider the circles C_1 of radius a and C_2 of radius b, b > a both lying the first quadrant and touching the coordinate axes.*

Column I	Column II
(A) C_1 and C_2 touch each other and $\frac{b}{a}=\lambda+\sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(p) $\lambda+\mu$ is a prime number
(B) C_1 and C_2 cut orthogonally and $\frac{b}{a}=\lambda+\sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(q) $\lambda+\mu$ is a composite number
(C) C_1 and C_2 intersect so that the common chord is longest and $\frac{b}{a}=\lambda+\sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(r) $2\lambda+\mu$ is a perfect number
(D) C_2 passes through the centre of C_1 and $\frac{b}{a}=\lambda+\sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(s) $\lvert\lambda-\mu\rvert$ is a prime number

Sol. (A)→ (p, s); (B) → (p); (C) → (p, r, s); (D) → (q, r)

$$\because \quad C_1: x^2+y^2-2ax-2ay+a^2=0$$

Centre : (a,a) and radius : a

and $$C_2: x^2+y^2-2bx-2by+b^2=0$$

Centre : (b,b) and radius : b

(A) $\because C_1$ and C_2 touch each other, then

$$\sqrt{2}(b-a)=b+a \Rightarrow \frac{b}{a}=(\sqrt{2}+1)^2=3+\sqrt{8}$$

$$\Rightarrow \quad \lambda=3, \mu=8$$

(B) $\because C_1$ and C_2 intersect orthogonally, then

$$2(b-a)^2=b^2+a^2$$

$$\Rightarrow \quad a^2+b^2-4ab=0$$

or $$\left(\frac{b}{a}\right)^2-4\left(\frac{b}{a}\right)+1=0$$

$$\therefore \quad \frac{b}{a}=\frac{4\pm\sqrt{(16-4)}}{2}=2+\sqrt{3}$$

$$\Rightarrow \quad \lambda=2, \mu=3]$$

(C) $\because C_1$ and C_2 intersect, the common chord is

$$2(b-a)(x+y)=b^2-a^2$$

given common chord is longest, then passes through (a,a)

$$\Rightarrow \quad 2(b-a)(2a)=b^2-a^2$$

or $$(b-3a)(b-a)=0$$

$$\because \quad b-a\neq 0 \quad [b>a]$$

$$\therefore \quad b-3a=0$$

or $$\frac{b}{a}=3 \Rightarrow \lambda=3, \mu=0$$

(D) $\because C_2$ passes through (a,a), then $a^2+a^2-2ab-2ab+b^2=0$

or $b^2-4ab+2a^2=0$

or $$\left(\frac{b}{a}\right)^2-4\left(\frac{b}{a}\right)+2=0$$

or $$\frac{b}{a}=\frac{4\pm\sqrt{(16-8)}}{2}=2+\sqrt{2}$$

$$\Rightarrow \quad \lambda=2, \mu=2$$

• Ex. 25. *Match the following*

Column I	Column II
(A) The circles $x^2+y^2+2x+c=0$ $(c>0)$ and $x^2+y^2+2y+c=0$ touch each other, then the value of $2c$ is	(p) 1
(B) The circles $x^2+y^2+2x+3y+c=0$ $(c>0)$ and $x^2+y^2-x+2y+c=0$ intersect orthogonally, then the value of $2c$ is	(q) 2
(C) The circle $x^2+y^2=9$ is contains the circle $x^2+y^2-2x+1-c^2=0$ $(c>0)$, then $2c$ can be	(r) 3
(D) The circle $x^2+y^2=9$ is contains in the circle $x^2+y^2-2x+1-\frac{c^2}{4}=0$ $(c>0)$, then $(c-6)$ can be	(s) 4

Sol. (A) → (p); (B) → (q); (C) → (p, q, r); (D) → (r, s)

(A) The circles

$$S_1: (x+1)^2+y^2=(\sqrt{(1-c)})^2$$

Centre $C_1: (-1,0)$, radius $r_1: \sqrt{(1-c)}$

and $$S_2: x^2+(y+1)^2=(\sqrt{(1-c)})^2$$

Centre $C_2: (0,-1)$, radius : $r_2=\sqrt{(1-c)}$

Now, $C_1C_2=\sqrt{2}$ and $r_1=r_2$

$\therefore$ The circles will touch externally only and $C_1C_2=r_1+r_2$

$$\Rightarrow \quad \sqrt{2}=2\sqrt{(1-c)} \text{ or } 2c=1$$

(B) The circles $S_1:(x+1)^2+\left(y+\frac{3}{2}\right)^2=\left(\sqrt{\left(\frac{13}{4}-c\right)}\right)^2$

Centre $C_1:\left(-1,-\frac{3}{2}\right)$, radius $r_1:\sqrt{\left(\frac{13}{4}-c\right)}$

and $\quad S_2:\left(x-\frac{1}{2}\right)^2+(y+1)^2=\left(\sqrt{\left(\frac{5}{4}-c\right)}\right)^2$

Centre $C_2:\left(\frac{1}{2},-1\right)$, radius $r_2:\sqrt{\left(\frac{5}{4}-c\right)}$

For intersect orthogonally

$$(C_1C_2)^2=r_1^2+r_2^2$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2=\frac{13}{4}-c+\frac{5}{4}-c$$

or $\quad 2c=2$

(C) The circles

$$S_1:x^2+y^2=3^2$$

Centre $C_1:(0,0)$, radius $r_1:3$

and $\quad S_2:(x-1)^2+y^2=c^2$

Centre $C_2:(1,0)$, radius $r_2:c$

Now, S_2 will be contained in S_1, then

$$C_1C_2<r_1-r_2$$

or $\quad 1<3-c$ or $c<2 \Rightarrow 2c<4$

(D) The circles

$$S_1:x^2+y^2=9$$

Centre $C_1:(0,0)$, radius $r_1:3$ and

$$S_2:(x-1)^2+y^2=\left(\frac{c}{2}\right)^2$$

Centre $C_2:(1,0)$, radius $r_2:\frac{c}{2}$

Now, S_1 will be contained in S_2,

then, $\quad r_2-r_1>C_1C_2$

$\Rightarrow \quad \frac{c}{2}-3>1$ or $c>8$

$\therefore \quad (c-6)>2$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** (Ex. Nos. 26 and 27) are Assertion-Reason Type examples. Each of these examples contains two statements :
  Statement I (Assertion) and **Statement II** (Reason)

  Each of these examples also has four alternative choices only one of which is the correct answer. You have to select the correct choice as given below :

  (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
  (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
  (c) Statement I is true, Statement II is false
  (d) Statement I is false, Statement II is true

- ***Ex. 26*** *C_1 is a circle of radius 2 touching X-axis and Y-axis. C_2 is another circle of radius greater than 2 and touching the axes as well as the circle C_1.*

Statement I *Radius of Circle $C_2=\sqrt{2}(\sqrt{2}+1)(\sqrt{2}+2)$*

Statement II *Centres of both circles always lie on the line $y=x$.*

Sol. (c) $C_1:(x-2)^2+(y-2)^2=2^2$

$$C_2:(x-r)^2+(y-r)^2=r^2 \qquad (r>2)$$

According to question,

$$\sqrt{(r-2)^2+(r-2)^2}=r+2$$

$$(r-2)^2+(r-2)^2=(r+2)^2$$

$$r^2-12r+4=0$$

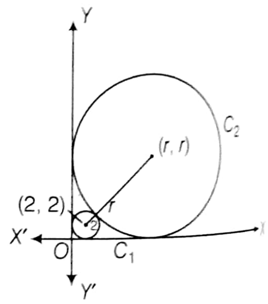
$$r=\frac{12\pm\sqrt{(144-16)}}{2}$$

$$=6\pm4\sqrt{2}$$

$\therefore \quad r=6+4\sqrt{2} \qquad [\because r>2]$

$$=2(\sqrt{2}+1)^2$$

$$=\sqrt{2}(\sqrt{2}+1)(2+\sqrt{2})$$



$\therefore$Statement I is true and Statement II is always not true (where circles in II or IV quadrants)

- ***Ex. 27*** *From the point $P(\sqrt{2},\sqrt{6})$ tangents PA and PB are drawn to the circle $x^2+y^2=4$*

Statement I *Area of the quadrilateral OAPB (O being origin) is 4.*

Statement II *Tangents PA and PB are perpendicular to each other and therefore quadrilateral OAPB is a square.*

Sol. (a) Clearly, $P(\sqrt{2},\sqrt{6})$ lies on $x^2+y^2=8$, which is the director circle of $x^2+y^2=4$.

Therefore, tangents PA and PB are perpendicular to each other So, $OAPB$ is a square.

Hence, area of $OAPB=(\sqrt{S_1})^2=S_1$

$$=(\sqrt{2})^2+(\sqrt{6})^2-4=4$$

$\therefore$Both statements are true and statement II is correct explanation of statement I.

Subjective Type Examples

- In this section, there are **16 subjective solved examples.**

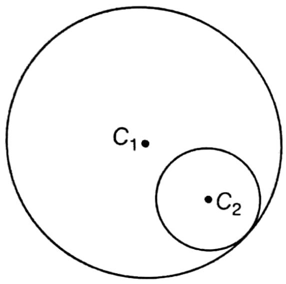
• Ex. 28 *Find the equation of a circle having the lines $x^2+2xy+3x+6y=0$ as its normals and having size just sufficient to contain the circle*

$$x(x-4)+y(y-3)=0.$$

Sol. Given pair of normals is $x^2+2xy+3x+6y=0$

or $$(x+2y)(x+3)=0$$

$\therefore$ Normals are $x+2y=0$ and $x+3=0$ the point of intersection of normals $x+2y=0$ and $x+3=0$ is the centre of required circle, we get centre $C_1 \equiv (-3, 3/2)$ and other circle is



$$x(x-4)+y(y-3)=0$$

or $$x^2+y^2-4x-3y=0 \quad \text{...(i)}$$

its centre $C_2 \equiv (2, 3/2)$ and radius $r=\sqrt{4+\frac{9}{4}}=\frac{5}{2}$

Since, the required circle just contains the given circle(i), the given circle should touch the required circle internally from inside.

$\Rightarrow$ radius of the required circle $=|C_1-C_2|+r$

$$=\sqrt{(-3-2)^2+\left(\frac{3}{2}-\frac{3}{2}\right)^2}+\frac{5}{2}$$

$$=5+\frac{5}{2}=\frac{15}{2}$$

Hence, equation of required circle is

$$(x+3)^2+(y-3/2)^2=\left(\frac{15}{2}\right)^2$$

or $$x^2+y^2+6x-3y-54=0$$

• Ex. 29 *Let a circle be given by*

$$2x(x-a)+y(2y-b)=0 \qquad (a\neq 0, b\neq 0)$$

Find the condition on a and b if two chords, each bisected by the X-axis, can be drawn to the circle from $(a, b/2)$.

Sol. The given circle is $2x(x-a)+y(2y-b)=0$

or $$x^2+y^2-ax-by/2=0$$

Let AB be the chord which is bisected by X-axis at a point M. Let its coordinates be $M(h, 0)$

and let $S \equiv x^2+y^2-ax-by/2=0$

$\therefore$ Equation of chord AB is $T=S_1$

$$hx+0-\frac{a}{2}(x+h)-\frac{b}{4}(y+0)=h^2+0-ah-0$$

Since, its passes through $(a, b/2)$ we have

$$ah-\frac{a}{2}(a+h)-\frac{b^2}{8}=h^2-ah$$

$$\Rightarrow \quad h^2-\frac{3ah}{2}+\frac{a^2}{2}+\frac{b^2}{8}=0$$

Now, there are two chords bisected by the X-axis, so there must be two distinct real roots of h.

$$\therefore \quad B^2-4AC>0$$

$$\Rightarrow \quad \left(\frac{-3a}{2}\right)^2-4\cdot 1\cdot\left(\frac{a^2}{2}+\frac{b^2}{8}\right)>0$$

$$\Rightarrow \quad a^2>2b^2.$$

Aliter : Given circle is

$$2x(x-a)+y(2y-b)=0$$

or $$x^2+y^2-ax-\frac{by}{2}=0 \quad \text{...(i)}$$

Let chords bisected at $M(h, 0)$ but given chords can be drawn $A\left(a, \frac{b}{2}\right)$ then chord cut the circle at $B(\lambda, -b/2)$

$\because$ Mid-point of ordinates of A and B is origin.

$\therefore$ $B(\lambda, b/2)$ lies on Eq. (i)

$$\therefore \quad \lambda^2+\frac{b^2}{4}-a\lambda+\frac{b^2}{4}=0$$

or $$\lambda^2-a\lambda+\frac{b^2}{2}=0$$

$\because$ λ is real

$$\therefore \quad B^2-4AC>0 \quad \text{or} \quad a^2-4\cdot\frac{b^2}{2}>0 \quad \text{or} \quad a^2>2b^2$$

• Ex. 30 *Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the centre of C.*

Sol. Let the given circles C_1 and C_2 have centres O_1 and O_2 with radii r_1 and r_2, respectively. Let centre of circle C is at O radius is r.

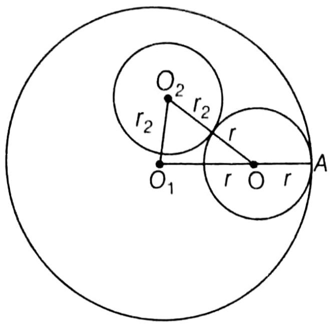
$$\because \quad OO_2=r+r_2$$

$$OO_1=r_1-r$$

$$\Rightarrow \quad OO_1+OO_2=r_1+r_2$$

which is greater than O_1O_2 as $O_1O_2<r_1+r_2$.

$\therefore$ Locus of O is an ellipse with foci O_1 and O_2.



Aliter :

Let $O_1 \equiv (0, 0)$, $O_2 \equiv (a, b)$ and $O \equiv (h, k)$

$$\therefore \quad C_1 : x^2 + y^2 = r_1^2$$
$$C_2 : (x - a)^2 + (y - b)^2 = r_2^2$$
$$C : (x - h)^2 + (y - k)^2 = r^2$$
$$\Rightarrow \quad OO_2 = r + r_2$$
$$\Rightarrow \quad \sqrt{(h - a)^2 + (k - b)^2} = r + r_2 \qquad \text{...(i)}$$
$$\text{and} \quad OO_1 = r_1 - r$$
$$\Rightarrow \quad \sqrt{(h^2 + k^2)} = r_1 - r \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii) we get

$$\sqrt{(h - a)^2 + (k - b)^2} + \sqrt{(h^2 + k^2)} = r_1 + r_2$$

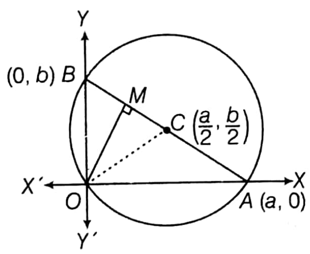
$\therefore$ Locus of O is $\sqrt{(x - a)^2 + (y - b)^2} + \sqrt{(x^2 + y^2)} = r_1 + r_2$

which represents an ellipse with foci are at (a,b) and $(0, 0)$.

• **Ex. 31** *A circle of constant radius r passes through the origin O, and cuts the axes at A and B. Show that the locus of the foot of the perpendicular from O to AB is*

$$(x^2 + y^2)^2 (x^{-2} + y^{-2}) = 4r^2$$

Sol. Let the coordinates of A and B are $(a, 0)$ and $(0, b)$.



$\therefore$ Equation of AB is $\quad \dfrac{x}{a} + \dfrac{y}{b} = 1 \qquad \text{...(i)}$

Centre of circle lie on line AB, since AB is diameter of the circle ($\because \angle AOB = \pi/2$)

$\therefore$ Coordinate of centre C is $C \equiv \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$

Since, the radius of circle $= r$

$$\therefore \quad r = AC = CB = OC$$
$$= \sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2} = \sqrt{\frac{a^2 + b^2}{4}}$$

$$\therefore \quad a^2 + b^2 = 4r^2$$

Equation of OM which is $\perp$ to AB is

$$ax - by = \lambda$$

It passes through $(0, 0)$

$$\therefore \quad 0 = \lambda$$

$\therefore$ Equation of OM is

$$ax - by = 0$$

On solving Eq. (i) and Eq. (iii), we get

$$a = \frac{x^2 + y^2}{x} \text{ and } b = \frac{x^2 + y^2}{y}$$

Substituting the values of a and b in Eq. (ii), we get

$$(x^2 + y^2)^2 \left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4r^2$$

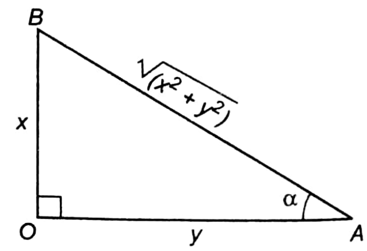
$$\text{or} \quad (x^2 + y^2)^2 (x^{-2} + y^{-2}) = 4r^2$$

which is the required locus.

Aliter :

$\because$ AB is the diameter of circle. If $\angle OAB = \alpha$, then

$$OA = 2r \cos \alpha, \; OB = 2r \sin \alpha$$



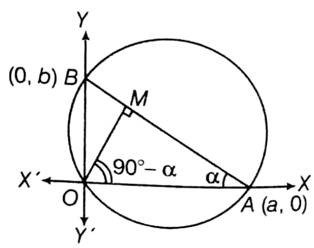
Equation of AB is

$$\frac{x}{2r \cos \alpha} + \frac{y}{2r \sin \alpha} = 1$$

$$\Rightarrow \quad \frac{x}{\cos \alpha} + \frac{y}{\sin \alpha} = 2r$$

and equation of OM is $y = x \tan (90° - \alpha)$

$$\Rightarrow \quad \cot \alpha = \frac{y}{x}$$



$$\therefore \quad \sin \alpha = \frac{x}{\sqrt{(x^2 + y^2)}}$$

$$\text{and} \quad \cos \alpha = \frac{y}{\sqrt{(x^2 + y^2)}}$$

Then, from Eq. (i),

$$\frac{x}{y}\sqrt{(x^2+y^2)}+\frac{y}{x}\sqrt{(x^2+y^2)}=2r$$

$$\Rightarrow \quad \frac{(x^2+y^2)\sqrt{(x^2+y^2)}}{xy}=2r$$

On squaring, we have $(x^2+y^2)^2\,\dfrac{(x^2+y^2)}{x^2y^2}=4r^2$

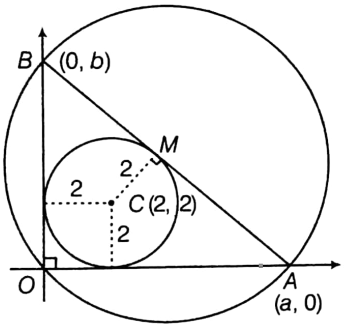
$$\Rightarrow \quad (x^2+y^2)^2(x^{-2}+y^{-2})=4r^2$$

• ***Ex. 32*** *The circle $x^2+y^2-4x-4y+4=0$ is inscribed in a triangle which has two of its sides along the coordinate axes. The locus of the circumcentre of the triangle is $x+y-xy+k(x^2+y^2)^{1/2}=0$. Find k.*

Sol. The given circle is $x^2+y^2-4x-4y+4=0$. This can be re-written as $(x-2)^2+(y-2)^2=4$ which has centre $C(2,2)$ and radius 2.

Let the equation of third side is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \text{(equation of } AB)$$



Length of perpendicular from (2, 2) on AB = radius = CM

$$\therefore \quad \frac{\left|\frac{2}{a}+\frac{2}{b}-1\right|}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}}=2$$

Since, origin and (2, 2) lie on the same side of AB

$$\therefore \quad -\frac{\left(\frac{2}{a}+\frac{2}{b}-1\right)}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}}=2$$

or $$\frac{2}{a}+\frac{2}{b}-1=-2\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)} \qquad \text{...(i)}$$

Since, $\angle AOB=\dfrac{\pi}{2}$

Hence, AB is the diameter of the circle passing through ΔOAB, mid-point of AB is the centre of the circle i.e. $\left(\dfrac{a}{2},\dfrac{b}{2}\right)$.

Let centre be $(h,k)\equiv\left(\dfrac{a}{2},\dfrac{b}{2}\right)$ then $a=2h$ and $b=2k$.

Substituting the values of a and b in Eq. (i), then

$$\frac{2}{2h}+\frac{2}{2k}-1=-2\sqrt{\left(\frac{1}{4h^2}+\frac{1}{4k^2}\right)}$$

$$\Rightarrow \quad \frac{1}{h}+\frac{1}{k}-1=-\sqrt{\left(\frac{1}{h^2}+\frac{1}{k^2}\right)}$$

or $$h+k-hk+\sqrt{(h^2+k^2)}=0$$

∴ Locus of $M(h,k)$ is

$$x+y-xy+\sqrt{(x^2+y^2)}=0$$

Hence, the required value of k is 1.

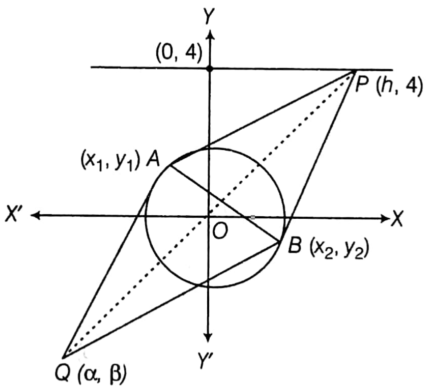
• ***Ex. 33*** *P is a variable on the line $y=4$. Tangents are drawn to the circle $x^2+y^2=4$ from P to touch it at A and B. The parallelogram PAQB is completed. Find the equation of the locus of Q.*

Sol. Let $P(h,4)$ be a variable point. Given circle is

$$x^2+y^2=4 \qquad \text{...(i)}$$

Draw tangents from $P(h,4)$ and complete parallelogram $PAQB$.

Equation of the diagonal AB which is chord of contact of $x^2+y^2=4$ is $hx+4y=4$ ...(ii)



Let coordinates of A and B are (x_1,y_1) and (x_2,y_2), respectively.

Since, $A(x_1,y_1)$ and $B(x_2,y_2)$ lies on Eq. (ii)

$$\therefore \quad hx_1+4y_1=4 \text{ and } hx_2+4y_2=4$$

$$\therefore \quad h(x_1+x_2)+4(y_1+y_2)=8 \qquad \text{...(iii)}$$

Since, $PAQB$ is parallelogram

∴ Mid-point of AB = Mid-point of PQ

$$\Rightarrow \quad \frac{x_1+x_2}{2}=\frac{\alpha+h}{2}$$

and $$\frac{y_1+y_2}{2}=\frac{\beta+4}{2} \qquad \text{...(iv)}$$

Eliminating x from Eqs. (i) and (ii), then

$$\left(\frac{4-4y}{h}\right)^2 + y^2 = 4$$

$$\Rightarrow \quad 16 + 16y^2 - 32y + h^2y^2 = 4h^2$$

$$\Rightarrow \quad (16 + h^2)y^2 - 32y + 16 - 4h^2 = 0$$

$$\therefore \quad y_1 + y_2 = \frac{32}{16 + h^2} \quad \text{...(v)}$$

From Eqs. (iii) and (v), we get

$$x_1 + x_2 = \frac{8h}{16 + h^2} \quad \text{...(vi)}$$

From Eqs. (iv) and (vi)

$$\beta + 4 = \frac{32}{16 + h^2}$$

or $$(16 + h^2)(\beta + 4) = 32 \quad \text{...(vii)}$$

From Eqs. (iv) and (vi)

$$\alpha + h = \frac{8h}{16 + h^2}$$

or $$(16 + h^2)(\alpha + h) = 8h \quad \text{...(viii)}$$

Dividing Eq. (viii) by Eq. (vii), then

$$\frac{\alpha + h}{\beta + 4} = \frac{h}{4} \quad \text{or} \quad h = \frac{4\alpha}{\beta}$$

Substituting the value of h in Eq. (vii) then

$$\left(16 + \frac{16\alpha^2}{\beta^2}\right)(\beta + 4) = 32$$

$$\Rightarrow \quad (\alpha^2 + \beta^2)(\beta + 4) = 2\beta^2$$

Hence, locus of $Q(\alpha, \beta)$ is $(x^2 + y^2)(y + 4) = 2y^2$

• **Ex. 34** *Show that the circumcircle of the triangle formed by the lines $ax + by + c = 0$; $bx + cy + a = 0$ and $cx + ay + b = 0$ passes through the origin if $(b^2 + c^2)(c^2 + a^2)(a^2 + b^2) = abc(b + c)(c + a)(a + b)$.*

Sol. Equation of conic is

$$(bx + cy + a)(cx + ay + b) + \lambda(cx + ay + b)(ax + by + c) + \mu(ax + by + c)(bx + cy + a) = 0 \quad \text{...(i)}$$

where, λ and μ are constants.

Eq. (i) represents a circle if the coefficient of x^2 and y^2 are equal and the coefficient of xy is zero such that

$$bc + \lambda ca + \mu ab = ca + \lambda ab + \mu bc$$

or $$(a - b)c + \lambda(b - c)a + \mu(c - a)b = 0 \quad \text{...(ii)}$$

and $$(c^2 + ab) + \lambda(a^2 + bc) + \mu(b^2 + ac) = 0 \quad \text{...(iii)}$$

on solving Eq. (ii) and Eq. (iii) by cross multiplication rule, we get

$$\frac{1}{(c^2 - ab)(a^2 + b^2)} = \frac{\lambda}{(a^2 - bc)(b^2 + c^2)} = \frac{\mu}{(b^2 - ac)(c^2 + a^2)}$$

$$\therefore \quad \lambda = \frac{(a^2 - bc)(b^2 + c^2)}{(c^2 - ab)(a^2 + b^2)}$$

and $$\mu = \frac{(b^2 - ac)(c^2 + a^2)}{(c^2 - ab)(a^2 + b^2)} \quad \text{...(iv)}$$

and given, Eq. (i) passes through the origin then

$$ab + bc\lambda + ca\mu = 0 \quad \text{...(v)}$$

From Eqs. (iv) and (v), we get

$$ab + \frac{bc(a^2 - bc)(b^2 + c^2)}{(c^2 - ab)(a^2 + b^2)} + \frac{ca(b^2 - ac)(c^2 + a^2)}{(c^2 - ab)(a^2 + b^2)} = 0$$

$$\Rightarrow \quad (c^2 - ab)(a^2 + b^2)ab + (a^2 - bc)(b^2 + c^2)bc + (b^2 - ca)(c^2 + a^2)ca = 0$$

$$\Rightarrow \quad abc^2(a^2 + b^2) + a^2bc(b^2 + c^2) + b^2ca(c^2 + a^2) = a^2b^2(a^2 + b^2) + b^2c^2(b^2 + c^2) + c^2a^2(c^2 + a^2)$$

$$\Rightarrow \quad abc\{c(a^2 + b^2) + a(b^2 + c^2) + b(c^2 + a^2)\} = a^2b^2(a^2 + b^2) + b^2c^2(b^2 + c^2) + c^2a^2(c^2 + a^2)$$

$$\Rightarrow \quad abc\{(a + b)(b + c)(c + a) - 2abc\} = a^2b^2(a^2 + b^2) + b^2c^2(b^2 + c^2) + c^2a^2(c^2 + a^2)$$

$$\Rightarrow \quad abc(a + b)(b + c)(c + a) = 2a^2b^2c^2 + a^2b^2(a^2 + b^2) + b^2c^2(b^2 + c^2) + c^2a^2(c^2 + a^2)$$

$$\Rightarrow \quad abc(a + b)(b + c)(c + a) = (a^2 + b^2)(b^2 + c^2)(c^2 + a^2)$$

Hence, $(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) = abc(a + b)(b + c)(c + a)$

• **Ex. 35** *If four points P, Q, R, S in the plane be taken and the square of the length of the tangents from P to the circle on QR as diameter be denoted by $\{P, QR\}$, show that*

$$\{P, RS\} - \{P, QS\} + \{Q, PR\} - \{Q, RS\} = 0$$

Sol. Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$, $R \equiv (x_3, y_3)$ and $S \equiv (x_4, y_4)$.

Equation of circle with RS as diameter is

$$(x - x_3)(x - x_4) + (y - y_3)(y - y_4) = 0$$

$$\therefore \quad \{P, RS\} = (x_1 - x_3)(x_1 - x_4) + (y_1 - y_3)(y_1 - y_4)$$

Now, equation of circle with QS as diameter is

$$(x - x_2)(x - x_4) + (y - y_2)(y - y_4) = 0$$

$$\therefore \quad \{P, QS\} = (x_1 - x_2)(x_1 - x_4) + (y_1 - y_2)(y_1 - y_4)$$

Equation of circle with PR as diameter is

$$(x - x_1)(x - x_3) + (y - y_1)(y - y_3) = 0$$

$$\therefore \quad \{Q, PR\} = (x_2 - x_1)(x_2 - x_3) + (y_2 - y_1)(y_2 - y_3)$$

Equation of circle with RS as diameter is

$$(x - x_3)(x - x_4) + (y - y_3)(y - y_4) = 0$$

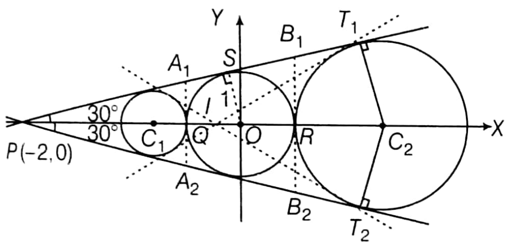
$$\therefore \quad \{Q, RS\} = (x_2 - x_3)(x_2 - x_4) + (y_2 - y_3)(y_2 - y_4)$$

Hence, $\{P, RS\} - \{P, QS\} + \{Q, PR\} - \{Q, RS\} = 0$

• ***Ex. 36** Let T_1, T_2 be two tangents drawn from $(-2, 0)$ on the circle $C : x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles when taken two at a time.*

Sol. In figure $OS = 1$, $OP = 2$

$$\therefore \quad \sin \angle SPO = \frac{1}{2} = \sin 30°$$



$\therefore \quad \angle SPO = 30°$

$\because \quad PA_1 = PA_2 \Rightarrow \angle PA_1A_2 = \angle PA_2A_1$

$\Rightarrow \quad \Delta PA_1A_2$ is an equilateral triangle.

Therefore, centre C_1 is centroid of ΔPA_1A_2, C_1 divides PQ in the ratio 2 : 1.

$$\therefore \quad C_1 \equiv \left(-\frac{4}{3}, 0\right) \text{ and its radius} = C_1Q = \frac{1}{3}$$

$$\Rightarrow \quad C_1 : (x + 4/3)^2 + y^2 = \left(\frac{1}{3}\right)^2 \qquad \text{...(i)}$$

The other circle C_2 touches the equilateral triangle PB_1B_2 externally.

its radius is given by $= \dfrac{\Delta}{s - a}$, where $B_1B_2 = a$

$$= \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3a}{2} - a} = \frac{\sqrt{3}}{2}a$$

but
$$\tan 30° = \frac{a/2}{3} \Rightarrow a = \frac{6}{\sqrt{3}}$$

$$\therefore \quad \text{Radius} = \frac{\sqrt{3}}{2} \cdot \frac{6}{\sqrt{3}} = 3$$

$\Rightarrow$ coordinates of C_2 are $(4, 0)$

$$\therefore \text{ Equation of } C_2 : (x - 4)^2 + y^2 = 3^2 \qquad \text{...(ii)}$$

Equations of common tangents to circle (i) and circle C are

$$x = -1, y = \pm \frac{1}{\sqrt{3}}(x + 2), \{T_1 \text{ and } T_2\}$$

and equations of common tangents to circle (ii) and circle C are

$$x = 1, y = \pm \frac{1}{\sqrt{3}}(x + 2) (\{T_1 \text{ and } T_2\}$$

To find the remaining two transverse common tangents to Eqs. (i) and (ii). If I divides C_1 and C_2 in the ratio $r_1 : r_2 = 1/3 : 3 = 1 : 9$.

Therefore coordinates of I are $(-4/5, 0)$.

Equation of any line through I is $y - 0 = m(x + 4/5)$. If it will touch Eq. (ii)

$$\text{then} \quad \frac{|m(4 + 4/5) - 0|}{\sqrt{(1 + m^2)}} = 3$$

$$\Rightarrow \quad \left(\frac{24}{5}\right)^2 m^2 = 9(1 + m^2)$$

$$\Rightarrow \quad 64m^2 = 25 + 25m^2$$

$$\Rightarrow \quad 39m^2 = 25 \Rightarrow m = \pm \frac{5}{\sqrt{39}}$$

Therefore, equations of transverse common tangents are

$$y = \pm \frac{5}{\sqrt{39}}(x + 4/5)$$

• ***Ex. 37** Find the equation of the circle of minimum radius which contains the three circles*

$$x^2 - y^2 - 4y - 5 = 0$$

$$x^2 + y^2 + 12x + 4y + 31 = 0$$

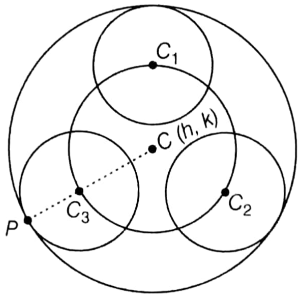
and $$x^2 + y^2 + 6x + 12y + 36 = 0$$

Sol. The coordinates of the centres and radii of three given circles are as given below :

$$C_1 \equiv (0, 2); r_1 = 3$$

$$C_2 \equiv (-6, -2); r_2 = 3$$

and $$C_3 \equiv (-3, -6); r_3 = 3$$



Let $C \equiv (h, k)$ be the centre of the circle passing through the centres $C_1(0, 2)$, $C_2(-6, -2)$ and $C_3(-3, -6)$.

Then, $$CC_1 = CC_2 = CC_3$$

$$\Rightarrow \quad (CC_1)^2 = (CC_2)^2 = (CC_3)^2$$

$$\Rightarrow \quad (h-0)^2+(k-2)^2=(h+6)^2+(k+2)^2$$
$$=(h+3)^2+(k+6)^2$$
$$\Rightarrow \quad -4k+4=12h+4k+40=6h+12k+45$$
$$\Rightarrow \quad 12h+8k+36=0$$
or $$3h+2k+9=0 \quad \text{...(i)}$$
and $$6h-8k-5=0 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get $h=-\frac{31}{18}, k=-\frac{23}{12}$

Now, $CP = CC_3 + C_3P = CC_3 + 3$

$$=\sqrt{\left(-3+\frac{31}{18}\right)^2+\left(-6+\frac{23}{12}\right)^2}+3=\left(\frac{5}{36}\sqrt{(949)}+3\right)$$

Hence, equation of required circle is

$$\left(x+\frac{31}{18}\right)^2+\left(y+\frac{23}{12}\right)^2=\left(3+\frac{5}{\sqrt{36}}\sqrt{949}\right)^2$$

Remark

If radii of three given circles are distinct say $r_1 < r_2 < r_3$ then the radius of the required circle will be equal to (CC_1 or CC_2 or CC_3) $+ r_3$ ($\because CC_1 = CC_2 = CC_3$)

• Ex. 38 *Find the point P on the circle $x^2+y^2-4x-6y+9=0$ such that*

(i) $\angle POX$ is minimum,

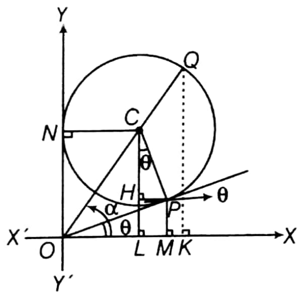
(ii) *OP is maximum, when O is the origin and OX is the X-axis.*

Sol. Given circle is

$$x^2+y^2-4x-6y+9=0$$
or $$(x-2)^2+(y-3)^2=2^2 \quad \text{.... (i)}$$

Its centre is $C \equiv (2,3)$ and radius $r=2$

Eq. (i) Let OP and ON be the two tangents from O to the circle Eq. (i), then $\quad OP = ON = 3$



then $\angle POX$ is minimum when OP is tangent to the circle Eq. (i) at P

Let $\angle POX = \theta$

$\therefore \quad P \equiv (OP\cos\theta, OP\sin\theta)$

i.e. $\quad P \equiv (3\cos\theta, 3\sin\theta) \quad \text{...(ii)}$

From figure, $OM = OL + LM = NC + HP = NC + CP\sin\theta$

$\Rightarrow \quad OP\cos\theta = NC + CP\sin\theta$

$\Rightarrow \quad 3\cos\theta = 2+2\sin\theta$

$\Rightarrow \quad 9(1-\sin^2\theta) = 4(1+\sin\theta)^2$

$\Rightarrow \quad 9(1-\sin\theta) = 4(1+\sin\theta) \quad (\because \sin\theta \neq -1)$

$\therefore \quad \sin\theta = \frac{5}{13}$ and $\cos\theta = \frac{12}{13}$

From Eq. (ii), $P \equiv \left(3\times\frac{12}{13}, 3\times\frac{5}{13}\right)$ i.e. $P \equiv \left(\frac{36}{13}, \frac{15}{13}\right)$

Eq. (ii) OP will be maximum, if P becomes the point extended part of OC cuts the circle. Let this point be Q

then maximum value of $OP = OQ = OC + CQ = (\sqrt{13}+2)$

Let $\quad \angle COX = \alpha$

then, $\quad Q \equiv (OQ\cos\alpha, OQ\sin\alpha)$

$\equiv ((2+\sqrt{13})\cos\alpha, (2+\sqrt{13})\sin\alpha) \quad \text{...(iii)}$

Now, in ΔCOL, $\quad \cos\alpha = \frac{OL}{OC} = \frac{NC}{OC} = \frac{2}{\sqrt{13}}$

$\therefore \quad \sin\alpha = \frac{3}{\sqrt{13}}$

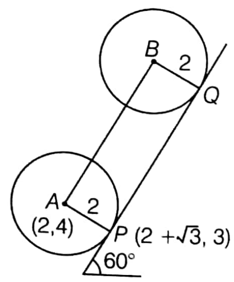
Now, from Eq. (iii), $Q \equiv \left(2+\frac{4}{\sqrt{13}}, 3+\frac{6}{\sqrt{13}}\right)$

• Ex. 39 *The circle $x^2+y^2-4x-8y+16=0$ rolls up the tangent to it at $(2+\sqrt{3}, 3)$ by 2 units, assuming the X-axis as horizontal, find the equation of the circle in the new position.*

Sol. Given circle is

$$x^2+y^2-4x-8y+16=0 \quad \text{...(i)}$$

Let $\quad P \equiv (2+\sqrt{3}, 3)$



Equation of tangent to the circle Eq. (i) at $P(2+\sqrt{3}, 3)$ is

$$(2+\sqrt{3})x+3y-2(x+2+\sqrt{3})-4(y+3)+16=0$$
or $$\sqrt{3}x-y-2\sqrt{3}=0 \quad \text{...(ii)}$$

Let A and B be the centres of the circles in old and new positions, then

$$B \equiv (2+2\cos 60^\circ, 4+2\sin 60^\circ)$$
($\because AB$ makes an angle 60° with X-axis)

or $\quad B \equiv (3, 4+\sqrt{3})$

and radius $= \sqrt{2^2 + 4^2 - 16} = 2$

$\therefore$ Equation of the required circle is

$$(x-3)^2 + (y - 4 - \sqrt{3})^2 = 2^2$$

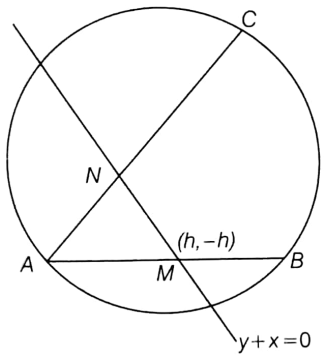
or
$$x^2 + y^2 - 6x - 2(4+\sqrt{3})\,y + 24 + 8\sqrt{3} = 0$$

• Ex. 40 *Find the intervals of the values of 'a' for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle*

$$2x^2 + 2y^2 - (1+\sqrt{2}\,a)x - (1-\sqrt{2}a)y = 0.$$

Sol. The point $A\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ lies on the given circle as its coordinate satisfy the equation of the circle. Let AB and AC are two chords drawn from A. Let M and N are the mid-points of AB and AC.

Let coordinate of M be $(h, -h)$ and coordinate of B is (α, β), then



$$h = \frac{\alpha + \frac{1+\sqrt{2}a}{2}}{2}$$

and
$$-h = \frac{\beta + \frac{1-\sqrt{2}a}{2}}{2}$$

$\therefore$
$$\alpha = 2h - \frac{1-\sqrt{2}a}{2}$$

and
$$\beta = -2h - \frac{1-\sqrt{2}a}{2}$$

Since, $B(\alpha, \beta)$ lies on the given circle, we have

$$\Rightarrow \quad 2\left[2h - \frac{1+\sqrt{2}a}{2}\right]^2 + 2\left[-2h - \frac{1-\sqrt{2}a}{2}\right]^2 - (1+\sqrt{2}a)\left[2h - \frac{1+\sqrt{2}a}{2}\right] - (1-\sqrt{2}a)\left[-2h - \frac{1-\sqrt{2}a}{2}\right] = 0$$

$$\Rightarrow \quad 16h^2 - 4h(1+\sqrt{2}a) + 4h(1-\sqrt{2}a) + \frac{(1+\sqrt{2}a)^2}{2} + \frac{(1-\sqrt{2}a)^2}{2} - 2h(1+\sqrt{2}a) + \frac{(1+\sqrt{2}a)^2}{2} + 2h(1-\sqrt{2}a) + \frac{(1-\sqrt{2}a)^2}{2} = 0$$

$$\Rightarrow \quad 16h^2 - 12\sqrt{2}\,ah + (1+\sqrt{2}a)^2 + (1-\sqrt{2}a)^2 = 0$$

$$\Rightarrow \quad 16h^2 - 12\sqrt{2}\,ah + 2 + 4a^2 = 0$$

or
$$8h^2 - 6\sqrt{2}\,ah + 1 + 2a^2 = 0$$

Hence, for two real and different values of h, we must have

$$(-6\sqrt{2}a)^2 - 4\cdot 8(1+2a^2) > 0$$

or
$$72a^2 - 32(1+2a^2) > 0$$

$$\Rightarrow \quad 8a^2 - 32 > 0$$

$$\Rightarrow \quad a^2 - 4 > 0$$

$$(a+2)(a-2) > 0$$

Hence, the required value of a (from wavy curve)

$$a \in (-\infty, -2) \cup (2, \infty)$$

Aliter : Equation of chord AB whose mid-point is $(h, -h)$ is

$$T = S_1$$

$$2xh - 2yh - (1+\sqrt{2}a)\left(\frac{x+h}{2}\right) - (1-\sqrt{2}a)\left(\frac{y-h}{2}\right) = 2h^2 + 2h^2 - (1+\sqrt{2}a)\,h + (1-\sqrt{2}a)\,h$$

$$\Rightarrow \quad 4xh - 4yh - (1+\sqrt{2}a)(x+h) - (1-\sqrt{2}a)(y-h) = 8h^2 - 2(1+\sqrt{2}a)\,h + 2(1-\sqrt{2}a)\,h$$

$$\Rightarrow \quad x[4h - (1+\sqrt{2}a)] - y[4h + (1-\sqrt{2}a)] - h(1+\sqrt{2}a) + h(1-\sqrt{2}a) = 8h^2 - 2(1+\sqrt{2}a)\,h + 2(1-\sqrt{2}a)\,h$$

or
$$8h^2 - (1+\sqrt{2}a)\,h + (1-\sqrt{2}a)\,h - x[4h - (1+\sqrt{2}a)] + y[4h + (1-\sqrt{2}a)] = 0$$

It passes through $A\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$, then

$$8h^2 - 2\sqrt{2}ah - \left(\frac{1+\sqrt{2}a}{2}\right)[4h - (1+\sqrt{2}a)] + \left(\frac{1-\sqrt{2}a}{2}\right)[4h + (1-\sqrt{2}a)] = 0$$

or
$$8h^2 - 2\sqrt{2}ah - 2h(1+\sqrt{2}a) + \frac{(1+\sqrt{2}a)^2}{2} + 2h(1-\sqrt{2}a) + \frac{(1-\sqrt{2}a)^2}{2} = 0$$

or
$$8h^2 - 6\sqrt{2}ah + 1 + 2a^2 = 0$$

Hence, for two real and different values of h, we must have

$$(-6\sqrt{2}a)^2 - 4\cdot 8\cdot(1+2a^2) > 0$$

or $\quad a^2 - 4 > 0$

$\therefore \quad (a+2)(a-2) > 0$

$\therefore \quad a \in (-\infty, -2) \cup (2, \infty)$

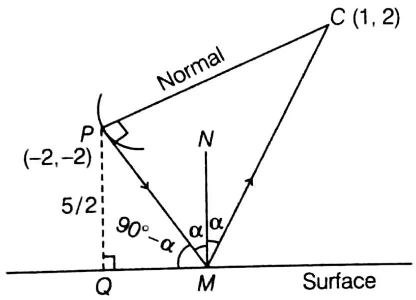
• **Ex. 41** *A ball moving around the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ in anti-clockwise direction leaves it tangentially at the point $P(-2, -2)$. After getting reflected from a straight line, it passes through the centre of the circle. Find the equation of the straight line if its perpendicular distance from P is $5/2$. You can assume that the angle of incidence is equal to the angle of reflection.*

Sol. Radius of the circle $= CP = \sqrt{9+16} = 5$

Let the equation is of surface is $y = mx + c$

given $\quad PQ = \dfrac{5}{2}$

$$\therefore \quad \frac{-2m+2+c}{\sqrt{(1+m^2)}} = \pm\frac{5}{2} \quad \text{...(i)}$$



Tangent at P strikes it at the point M and after reflection passes through the centre $C(1, 2)$.

Let MN be the normal at M.

$$\angle PMN = \angle NMC = \alpha$$

In ΔPCM, $\quad \tan 2\alpha = \dfrac{PC}{PM}$

$\Rightarrow \quad \tan 2\alpha = \dfrac{5}{PM}$

$\Rightarrow \quad PM = 5\cot 2\alpha \quad$ ...(ii)

and in ΔPQM

$$\sin(90°-\alpha) = \frac{5/2}{PM}$$

$\therefore \quad PM = \dfrac{5}{2\cos\alpha} \quad$ ...(iii)

From Eqs. (ii) and (iii), $5\cot 2\alpha = \dfrac{5}{2\cos\alpha}$

$\Rightarrow \quad 2\cot 2\alpha \cos\alpha = 1$

$\Rightarrow \quad \dfrac{2\cos 2\alpha}{\sin 2\alpha}.\cos\alpha = 1$

$\Rightarrow \quad \dfrac{2(1-2\sin^2\alpha)\cos\alpha}{2\sin\alpha\cos\alpha} = 1$

$\Rightarrow \quad 1 - 2\sin^2\alpha = \sin\alpha$

$\Rightarrow \quad 2\sin^2\alpha + \sin\alpha - 1 = 0$

$\Rightarrow \quad (2\sin\alpha - 1)(\sin\alpha + 1) = 0$

$\Rightarrow \quad \sin\alpha \neq -1$

$\therefore \quad \sin\alpha = \dfrac{1}{2}$

$\therefore \quad \alpha = 30°$

Tangent at $P(-2, -2)$ is

$$-2x - 2y - (x-2) - 2(y-2) - 20 = 0$$

$\Rightarrow \quad 3x + 4y + 14 = 0$

Slope of $PM = -3/4$

$\because \quad \angle PMQ = 90° - \alpha = 90° - 30° = 60°$

$\therefore \quad \tan 60° = \left|\dfrac{m + 3/4}{1 - 3m/4}\right|, \quad \sqrt{3} = \dfrac{4m+3}{4-3m}$

$\therefore \quad m = \dfrac{4\sqrt{3}-3}{4+3\sqrt{3}}$

From Eq. (i) $\quad \pm\dfrac{5}{2} = \dfrac{2(1-m)+c}{\sqrt{1+m^2}}$

we get $\quad c = \dfrac{11+2\sqrt{3}}{4+3\sqrt{3}}$ or $\dfrac{-39+2\sqrt{3}}{4+3\sqrt{3}}$

c being intercept on Y-axis made by surface is clearly-ve.
Hence, the required line is

$$y = \left(\frac{4\sqrt{3}-3}{4+3\sqrt{3}}\right)x + \left(\frac{-39+2\sqrt{3}}{4+3\sqrt{3}}\right)$$

$\Rightarrow \quad (4\sqrt{3}-3)x - (4+3\sqrt{3})y - (39 - 2\sqrt{3}) = 0.$

• **Ex. 42** *Find the limiting points of the circles $(x^2 + y^2 + 2gx + c) + \lambda(x^2 + y^2 + 2fy + d) = 0$ and show that the square of the distance between them is*

$$\frac{(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2}{f^2 + g^2}$$

Sol. The given circles are

$$(x^2 + y^2 + 2gx + c) + \lambda(x^2 + y^2 + 2fy + d) = 0$$

$\Rightarrow \quad x^2 + y^2 + \dfrac{2g}{1+\lambda}x + \dfrac{2f\lambda}{1+\lambda}y + \dfrac{(c+\lambda d)}{1+\lambda} = 0$

Centre of the circle $\left(\dfrac{-g}{1+\lambda}, \dfrac{-f\lambda}{1+\lambda}\right)$

Equating the radius of this circle to zero, we get

$$\frac{g^2}{(1+\lambda)^2} + \frac{f^2\lambda^2}{(1+\lambda)^2} - \frac{(c+\lambda d)}{(1+\lambda)} = 0$$

$$\Rightarrow \quad g^2 + f^2\lambda^2 - (c + \lambda d)(1 + \lambda) = 0$$

$$\Rightarrow \quad (f^2 - d)\lambda^2 - (c + d)\lambda + g^2 - c = 0$$

Let the roots be λ_1 and λ_2

then $\lambda_1 + \lambda_2 = \dfrac{(c + d)}{(f^2 - d)}, \lambda_1\lambda_2 = \dfrac{g^2 - c}{f^2 - d}$

$$\therefore \quad (\lambda_1 - \lambda_2) = \sqrt{(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2}$$

$$= \sqrt{\frac{(c + d)^2}{(f^2 - d)^2} - \frac{4(g^2 - c)}{(f^2 - d)}}$$

$$= \frac{\sqrt{(c + d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2}}{(f^2 - d)} \quad \text{...(i)}$$

$$\therefore \quad \lambda_1 = \frac{(c + d) + \sqrt{(c - d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2}}{2(f^2 - d)} \quad \text{...(ii)}$$

and $\lambda_2 = \dfrac{(c + d) - \sqrt{\{(c - d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2\}}}{2(f^2 - d)}$ ...(iii)

Hence, limiting points are

$$\left(\frac{-g}{1 + \lambda_1}, \frac{-f\lambda_1}{1 + \lambda_1}\right) \text{ and } \left(\frac{-g}{1 + \lambda_2}, \frac{-f\lambda_2}{1 + \lambda_2}\right)$$

Substituting the values of λ_1 and λ_2 from Eqs. (ii) and (iii) square of the distance between limiting points

$$= \left(\frac{-g}{1 + \lambda_1} + \frac{g}{1 + \lambda_2}\right)^2 + \left(\frac{-f\lambda_1}{1 + \lambda_1} + \frac{f\lambda_2}{1 + \lambda_2}\right)^2$$

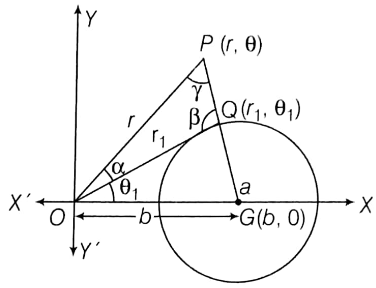
$$= \frac{(g^2 + f^2)(\lambda_1 - \lambda_2)^2}{[1 + (\lambda_1 + \lambda_2) + \lambda_1\lambda_2]^2}$$

$$= \frac{(g^2 + f^2)\dfrac{\{(c - d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2\}}{(f^2 - d)^2}}{\left(\dfrac{g^2 + f^2}{f^2 - d}\right)^2}$$

$$\Rightarrow \quad \frac{[(c - d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2]}{(g^2 + f^2)}$$

Ex. 43 *One vertex of a triangle of given species is fixed and another moves along circumference of a fixed circle. Prove that the locus of the remaining vertex is a circle and find its radius.*

Sol. Let OPQ be a triangle of given species. Then the angles α, β, γ will be fixed.



Let the polar coordinates of Q be (r_1, θ_1), we have to find the locus of $P(r, \theta)$. In $\angle OCQ$

$$\cos\theta_1 = \frac{r_1^2 + b^2 - a^2}{2r_1 b} \quad \text{...(i)}$$

$$\because \quad \theta = \alpha + \theta_1, \quad \therefore \theta_1 = \theta - \alpha \quad \text{...(ii)}$$

using sine rule in ΔOPQ

$$\frac{r}{\sin\beta} = \frac{r_1}{\sin\gamma}$$

$$\therefore \quad r_1 = \frac{r\sin\gamma}{\sin\beta} \quad \text{...(iii)}$$

Substituting the values of θ_1 and r_1 from Eqs. (ii)and (iii) in Eq. (i)

$$2b\frac{r\sin\gamma}{\sin\beta}\cos(\theta - \alpha) = \frac{r^2\sin^2\gamma}{\sin^2\beta} + b^2 - a^2$$

$$\Rightarrow \quad \frac{a^2\sin^2\beta}{\sin^2\gamma} = r^2 + \frac{b^2\sin^2\beta}{\sin^2\gamma} - 2b\frac{r\sin\beta}{\sin\gamma}\cos(\theta - \alpha)$$

This is an equation of circle in polar form with radius $\dfrac{\sin\beta}{\sin\gamma}$.