POINT & STRAIGHT LINE

BASIC THEOREMS & RESULTS OF TRIANGLES :

(a) Two polygons are similar if (i) their corresponding angles are equal, (ii) the length of their corresponding sides are proportional. (Both condition are independent & necessary)

In case of a triangle, any one of the condition is sufficient, other satisfies automatically.

(b) **Thales Theorem (Basic Proportionality Theorem) :** In a triangle, a line drawn parallel to one side, to intersect the other sides in distinct points, divides the two sides in the same ratio.

Converse : If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side.

(c) **Similarity Theorem :**

(i) **AAA similarity** : If in two triangles, corresponding angles are equal i.e. two triangles are equiangular, then the triangles are similar.

(ii) **SSS similarity** : If the corresponding sides of two triangles are proportional, then they are similar.

(iii) **SAS similarity :** If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

(iv) If two triangles are similar then

(1) They are equiangular.

(2) The ratio of the corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes are same (converse also true)

(3) The ratio of the areas is equal to the ratio of the squares of corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes (converse also true)

(d) **Congruency theorem :**

Congruent triangles : Two triangles are conguent, iff one of them can be made to superpose on the other, so as to cover it exactly.

Sufficient-conditions (criteria) for congruence of triangles :

(i) **Side-Angle-Side (SAS) :** Two triangles are congruent, if two sides and the included angle of one triangle are equal to the corresponding sides and the included angle of other triangle.

(ii) **Angle-Side-Angle (ASA) :** Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

(iii) **Angle-Angle-Side(AAS) :** If any two angles and a non-included side of one triangle are equal to the corresponding to angles & the non-included side of the other triangle then the two triangles are congruent.

(iv) **Side-Side-Side (SSS) :** Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

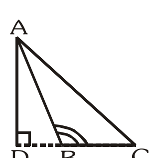
(v) **Right angle-Hypotenuse-Side(RHS) :** Two right-triangles are congruent, if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of other triangle.

(d) **Pythagoras theorem :**

(i) In a right triangle the square of hypotenuse is equal to the sum of squares of the other two sides.

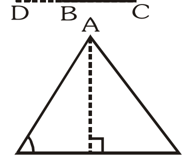
Converse : In a triangle if square of one side is equal to sum of the squares of the other two side. then the angle opposite to the side is a right angle.

(ii) In obtuse Δ :



$AC^2 = AB^2 + BC^2 + 2BC \,.\, BD$ & $AC^2 > AB^2 + BC^2$

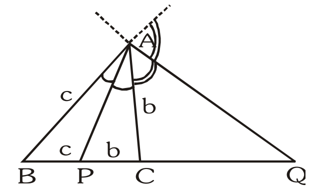
(iii) In Acute Δ :



$AC^2 = AB^2 + BC^2 - 2BC \,.\, BD$ & $AC^2 < AB^2 + BC^2$

(e) In any triangle ABC, $AB^2 + AC^2 = 2(AD^2 + DC^2)$, where D is the mid point of BC

(f) The internal/external bisector of an angle of a triangle divides the opposite side internally/externally in the ratio of sides containing the angle (converse is also true). $\frac{BP}{PC} = \frac{BQ}{CQ} = \frac{AB}{AC}$



(g) The line joining the mid points of two sides of a Δ is parallel & half of the third side. (It's converse also true)

(h) (i) The diagonals of a trapezium divide each other proportionally. (converse is also true)

(ii) Any line parallel to the parallel sides of a trapezium divides the non parallel sides proportionally.

(iii) If three or more parallel lines are intersected by two transversals, then intercepts made by them on the transversals are proportional.

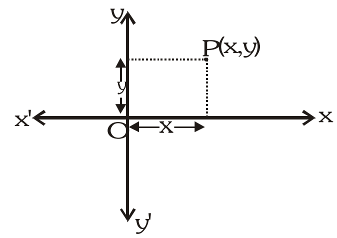
(i) In any triangle three times the sum of squares of the sides of a triangle is equal to four times the sum of the squares of its medians.

(j) The altitudes, medians and angle bisectors of a triangle are concurrent among themselves.

1. INTRODUCTION OF COORDINATE GEOMETRY :

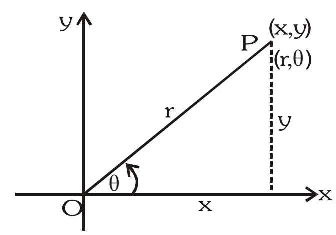
Coordinate geometry is the combination of algebra and geometry. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes. The resulting combination of analysis and geometry is referred as ***analytical geometry***.

2. CARTESIAN CO-ORDINATES SYSTEM :



In two dimensional coordinate system, two lines are used; the lines are at right angles, forming a rectangular coordinate system. The horizontal axis is the x-axis and the vertical axis is y-axis. The point of intersection O is the origin of the coordinate system. Distances along the x-axis to the right of the origin are taken as positive, distances to the left as negative. Distances along the y-axis above the origin are positive; distances below are negative. The position of a point anywhere in the plane can be specified by two numbers, the coordinates of the point, written as (x, y). The x-coordinate (or abscissa) is the distance of the point from the y-axis in a direction parallel to the x-axis (i.e. horizontally). The y-coordinate (or ordinate) is the distance from the x-axis in a direction parallel to the y-axis (vertically). The origin O is the point (0, 0).

3. POLAR CO-ORDINATES SYSTEM :



A coordinate system in which the position of a point is determined by the length of a line segment from a fixed origin together with the angle that the line segment makes with a fixed line. The origin is called the pole and the line segment is the radius vector (r).

The angle θ between the polar axis and the radius vector is called the vectorial angle. By convention, positive values of θ are measured in an anticlockwise sense, negative values in clockwise sense. The coordinates of the point are then specified as (r, θ).

If (x,y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

4. DISTANCE FORMULA AND ITS APPLICATIONS :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note :

(i) Three given points A,B and C are collinear, when sum of any two distances out of AB,BC, CA is equal to the remaining third otherwise the points will be the vertices of a triangle.

(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if AB = BC = CD = DA & AC = BD ; $AC \perp BD$

(b) Rhombus if AB = BC = CD = DA and AC $\neq$ BD ; $AC \perp BD$

(c) Parallelogram if AB = DC, BC = AD; AC $\neq$ BD ; $AC \not\perp BD$

(d) Rectangle if AB = CD, BC = DA, AC = BD ; $AC \not\perp BD$

Illustration 1 : The number of points on x-axis which are at a distance c(c < 3) from the point (2, 3) is

(A) 2 (B) 1 (C) infinite (D) no point

Solution : Let a point on x-axis is $(x_1, 0)$ then its distance from the point (2, 3)

$$= \sqrt{(x_1-2)^2+9} = c \quad \text{or} \quad (x_1-2)^2 = c^2-9$$

$$\therefore\ x_1 - 2 = \pm\sqrt{c^2-9} \quad \text{since } c<3 \Rightarrow c^2-9<0$$

$\therefore$ x_1 will be imaginary. **Ans. (D)**

Illustration 2 : The distance between the point $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$ is -

(A) $4a\sin\dfrac{\alpha-\beta}{2}$ (B) $2a\sin\dfrac{\alpha+\beta}{2}$ (C) $2a\sin\dfrac{\alpha-\beta}{2}$ (D) $2a\cos\dfrac{\alpha-\beta}{2}$

Solution : $d^2 = (a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2 = a^2(\cos\alpha-\cos\beta)^2 + a^2(\sin\alpha-\sin\beta)^2$

$$= a^2\left\{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}\right\}^2 + a^2\left\{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right\}^2$$

$$= 4a^2\sin^2\frac{\alpha-\beta}{2}\left\{\sin^2\frac{\alpha+\beta}{2}+\cos^2\frac{\alpha+\beta}{2}\right\} = 4a^2\sin^2\frac{\alpha-\beta}{2} \Rightarrow d = 2a\sin\frac{\alpha-\beta}{2}$$ **Ans. (C)**

Do yourself - 1 :

(i) Find the distance between the points P(–3, 2) and Q(2, –1).

(ii) If the distance between the points P(–3, 5) and Q(–x, – 2) is $\sqrt{58}$, then find the value(s) of x.

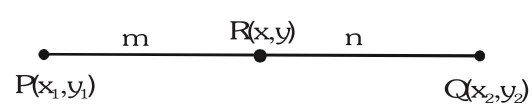
(iii) A line segment is of the length 15 units and one end is at the point (3, 2), if the abscissa of the other end is 15, then find possible ordinates.

5. SECTION FORMULA :

The co-ordinates of a point dividing a line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in the ratio m:n is given by :

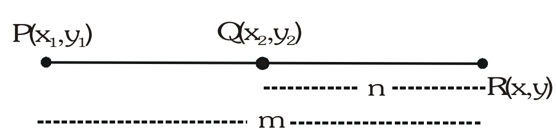
(a) For internal division : P - R - Q $\Rightarrow$ R divides line segment PQ, internally.

$$(x, y) \equiv \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$



(b) For external division : R - P - Q or P - Q - R $\Rightarrow$ R divides line segment PQ, externally.

$$(x, y) \equiv \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$



$\dfrac{(PR)}{(QR)} < 1 \Rightarrow$ R lies on the left of P & $\dfrac{(PR)}{(QR)} > 1 \Rightarrow$ R lies on the right of Q

(c) Harmonic conjugate : If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically ; $\dfrac{2}{AB} = \dfrac{1}{AP} + \dfrac{1}{AQ}$ i.e. AP, AB & AQ are in **H.P.**

Illustration 3 : Determine the ratio in which y − x + 2 = 0 divides the line joining (3, −1) and (8, 9).

Solution : Suppose the line y − x + 2 = 0 divides the line segment joining A(3, −1) and B(8, 9) in the ratio λ : 1 at a point P, then the co-ordinates of the point P are $\left(\frac{8\lambda+3}{\lambda+1}, \frac{9\lambda-1}{\lambda+1}\right)$

But P lies on y − x + 2 = 0 therefore $\left(\frac{9\lambda-1}{\lambda+1}\right)-\left(\frac{8\lambda+3}{\lambda+1}\right)+2=0$

⇒ 9λ − 1 − 8λ − 3 + 2λ + 2 = 0

⇒ 3λ − 2 = 0 or $\lambda = \frac{2}{3}$

So, the required ratio is $\frac{2}{3}$: 1, i.e., 2 : 3 (internally) since here λ is positive.

Do yourself - 2 :

(i) Find the co-ordinates of the point dividing the join of A(1, − 2) and B(4, 7) :

(a) Internally in the ratio 1 : 2 (b) Externally in the ratio of 2 : 1

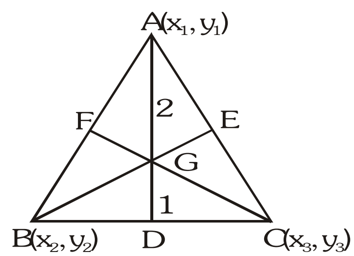
(ii) In what ratio is the line joining A(8, 9) and B(− 7, 4) is divided by

(a) the point (2, 7) (b) the x-axis (c) the y-axis.

6. CO-ORDINATES OF SOME PARTICULAR POINTS :

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle ABC, then

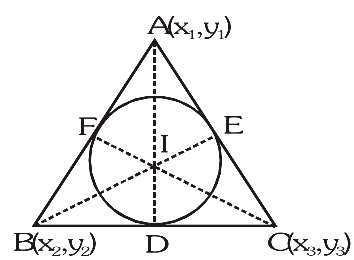
(a) Centroid :



The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices). Centroid divides each median in the ratio of 2 : 1.

Co-ordinates of centroid $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

(b) Incenter :



The incenter is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of the circle touching all the sides of a triangle.

Co-ordinates of incenter $I\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

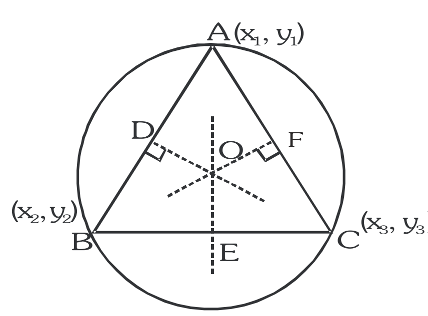
where a, b, c are the sides of triangle ABC.

Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g. $\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b+c):a, (c+a):b, (a+b):c$.

(c) Circumcenter :



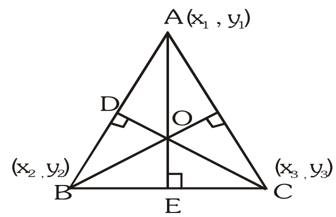
It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then $OA^2 = OB^2 = OC^2$. Also it is a centre of a circle touching all the vertices of a triangle.

Note :

(i) If the triangle is right angled, then its circumcenter is the mid point of hypotenuse.

(ii) Co-ordinates of circumcenter $\left(\dfrac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \dfrac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$

(d) Orthocenter :



It is the point of intersection of perpendiculars drawn from vertices on the opposite sides of a triangle and it can be obtained by solving the equation of any two altitudes.

Note :

(i) If a triangle is right angled, then orthocenter is the point where right angle is formed.

(ii) Co-ordinates of circumcenter $\left(\dfrac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \dfrac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$

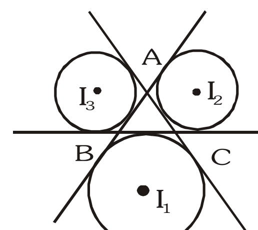
Remarks :

(i) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincide.

(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1

(iii) In an isosceles triangle centroid, orthocentre, incentre & circumcentre lie on the same line.

(e) Ex-centers :



The centre of a circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are

$$I_1\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and

$$I_2\left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right), \quad I_3\left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

Illustration 4 : If $\left(\dfrac{5}{3}, 3\right)$ is the centroid of a triangle and its two vertices are (0, 1) and (2, 3), then find its third vertex, circumcentre, circumradius & orthocentre.

Solution : Let the third vertex of triangle be (x, y), then

$\dfrac{5}{3} = \dfrac{x+0+2}{3} \Rightarrow x = 3$ and $3 = \dfrac{y+1+3}{3} \Rightarrow y = 5$. So third vertex is (3, 5).

Now three vertices are A(0, 1), B(2, 3) and C(3, 5)

Let circumcentre be P(h, k),

then AP = BP = CP = R (circumradius) $\Rightarrow$ $AP^2 = BP^2 = CP^2 = R^2$

$h^2 + (k-1)^2 = (h-2)^2 + (k-3)^2 = (h-3)^2 + (k-5)^2 = R^2$ .... (i)

from the first two equations, we have

$h + k = 3$ .... (ii)

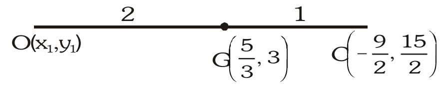
from the first and third equation, we obtain

$6h + 6k = 33$ .... (iii)

On solving, (ii) & (iii), we get

$h = -\dfrac{9}{2}$, $k = \dfrac{15}{2}$

Substituting these values in (i), we have



$$R = \frac{5}{2}\sqrt{10}$$

Let $O(x_1, y_1)$ be the orthocentre, then $\dfrac{x_1 + 2\left(-\dfrac{9}{2}\right)}{3} = \dfrac{5}{3} \Rightarrow x_1 = 14$, $\dfrac{y_1 + 2\left(\dfrac{15}{2}\right)}{3} = 3$

$\Rightarrow$ $y_1 = -6$. Hence orthocentre of the triangle is (14, –6).

Illustration 5 : The vertices of a triangle are A(0, –6), B(–6, 0) and C(1,1) respectively, then coordinates of the ex-centre opposite to vertex A is :

(A) $\left(\dfrac{-3}{2}, \dfrac{-3}{2}\right)$ (B) $\left(-4, \dfrac{3}{2}\right)$ (C) $\left(\dfrac{-3}{2}, \dfrac{3}{2}\right)$ (D) (–4, 6)

Solution :

$$a = BC = \sqrt{(-6-1)^2 + (0-1)^2} = \sqrt{50} = 5\sqrt{2}$$

$$b = CA = \sqrt{(1-0)^2 + (1+6)^2} = \sqrt{50} = 5\sqrt{2}$$

$$c = AB = \sqrt{(0+6)^2 + (-6-0)^2} = \sqrt{72} = 6\sqrt{2}$$

coordinates of ex-centre opposite to vertex A will be :

$$x = \frac{-ax_1 + bx_2 + cx_3}{-a+b+c} = \frac{-5\sqrt{2}.0 + 5\sqrt{2}(-6) + 6\sqrt{2}(1)}{-5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}} = \frac{-24\sqrt{2}}{6\sqrt{2}} = -4$$

$$y = \frac{-ay_1 + by_2 + cy_3}{-a+b+c} = \frac{-5\sqrt{2}(-6) + 5\sqrt{2}.0 + 6\sqrt{2}(1)}{-5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}} = \frac{36\sqrt{2}}{6\sqrt{2}} = 6$$

Hence coordinates of ex-centre is (–4, 6) **Ans. (D)**

Do yourself - 3 :

(i) The coordinates of the vertices of a triangle are (0, 1), (2, 3) and (3, 5) :

(a) Find centroid of the triangle.

(b) Find circumcentre & the circumradius.

(c) Find Orthocentre of the triangle.

7. AREA OF TRIANGLE :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left| \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right| = \frac{1}{2}\,|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$$

To remember the above formula, take the help of the following method :

$$= \frac{1}{2}\begin{bmatrix} x_1 & & x_2 & & x_3 & & x_1 \\ & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_1 \end{bmatrix} = \left| \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)] \right|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) **Area of Equilateral triangle :** If altitude of any equilateral triangle is P, then its area = $\dfrac{P^2}{\sqrt{3}}$. If 'a' be the side of equilateral triangle, then its area $= \left(\dfrac{a^2\sqrt{3}}{4}\right)$.

(iii) Area of quadrilateral with given vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$

Area of quad. ABCD $= \dfrac{1}{2}\left|\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{bmatrix}\right|$

Note : Area of a polygon can be obtained by dividing the polygon into disjoined triangles and then adding their areas.

Illustration 6 : If the vertices of a triangle are (1, 2), (4, –6) and (3, 5) then its area is

(A) $\dfrac{25}{2}$ sq. units (B) 12 sq. units (C) 5 sq. units (D) 25 sq. units

Solution : $\Delta = \dfrac{1}{2}\left[1(-6-5)+4(5-2)+3(2+6)\right] = \dfrac{1}{2}\left[-11+12+24\right] = \dfrac{25}{2}$ square units **Ans. (A)**

Illustration 7 : The point A divides the join of the points (–5, 1) and (3, 5) in the ratio k:1 and coordinates of points B and C are (1, 5) and (7, –2) respectively. If the area of ΔABC be 2 units, then k equals -

(A) 7, 9 (B) 6, 7 (C) 7, $\dfrac{31}{9}$ (D) 9, $\dfrac{31}{9}$

Solution : $A \equiv \left(\dfrac{3k-5}{k+1}, \dfrac{5k+1}{k+1}\right)$

Area of $\Delta ABC = 2$ units $\Rightarrow \dfrac{1}{2}\left[\dfrac{3k-5}{k+1}(5+2)+1\left(-2-\dfrac{5k+1}{k+1}\right)+7\left(\dfrac{5k+1}{k+1}-5\right)\right] = \pm 2$

$\Rightarrow 14k - 66 = \pm 4(k+1)$ $\Rightarrow k = 7$ or $\dfrac{31}{9}$ **Ans. (C)**

Illustration 8 : Prove that the co-ordinates of the vertices of an equilateral triangle can not all be rational.

Solution : Let $A(x_1\ y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible let x_1, y_1, x_2, y_2, x_3, y_3 be all rational.

Now area of $\Delta ABC = \dfrac{1}{2}\,|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$ = Rational .... (i)

Since ΔABC is equilateral

$\therefore$ Area of $\Delta ABC = \dfrac{\sqrt{3}}{4}(\text{side})^2 = \dfrac{\sqrt{3}}{4}(AB)^2 = \dfrac{\sqrt{3}}{4}\{(x_1-x_2)^2+(y_1-y_2)^2\}$ = Irrational .... (ii)

From (i) and (ii),

Rational = Irrational

which is contradiction.

Hence x_1, y_1, x_2, y_2, x_3, y_3 cannot all be rational.

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8. CONDITIONS FOR COLLINEARITY OF THREE GIVEN POINTS :

Three given points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear if any one of the following conditions are satisfied.

(a) Area of triangle ABC is zero i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(b) Slope of AB = slope of BC = slope of AC. i.e. $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_3 - y_2}{x_3 - x_2} = \dfrac{y_3 - y_1}{x_3 - x_1}$

(c) Find the equation of line passing through 2 given points, if the third point satisfies the given equation of the line, then three points are collinear.

Do yourself - 4 :

(i) Find the area of the triangle whose vertices are A(1,1), B(7, – 3) and C(12, 2)

(ii) Find the area of the quadrilateral whose vertices are A(1,1) B(7, – 3), C(12,2) and D(7, 21)

(iii) Prove that the points A(a, b + c), B(b, c + a) and C(c, a + b) are collinear (By determinant method)

(iv) Prove that the points (–1, –1), (2, 3) and (8, 11) are collinear.

(v) Find the value of x so that the points (x, –1), (2, 1) and (4, 5) are collinear.

9. LOCUS :

The locus of a moving point is the path traced out by that point under one or more geometrical conditions.

(a) Equation of Locus :

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

(b) Procedure for finding the equation of the locus of a point :

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given condition as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

(iii) Eliminate the parameters, so that the eliminant contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the eliminant. The resulting equation would be the equation of the locus of P.

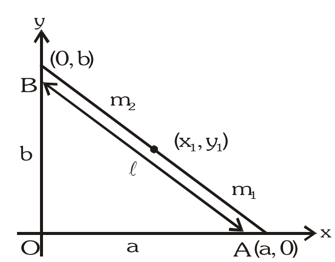
Illustration 9 : The ends of the rod of length ℓ moves on two mutually perpendicular lines, find the locus of the point on the rod which divides it in the ratio $m_1 : m_2$

(A) $m_1^2x^2 + m_2^2y^2 = \dfrac{\ell^2}{(m_1+m_2)^2}$ (B) $(m_2x)^2 + (m_1y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$

(C) $(m_1x)^2 + (m_2y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$ (D) none of these

Solution : Let (x_1, y_1) be the point that divide the rod $AB = \ell$, in the ratio $m_1 : m_2$, and OA = a, OB = b say

$\therefore\ a^2 + b^2 = \ell^2$ ..... (i)



Now $x_1 = \left(\dfrac{m_2a}{m_1+m_2}\right) \Rightarrow a = \left(\dfrac{m_1+m_2}{m_2}\right)x_1$

$y_1 = \left(\dfrac{m_1b}{m_1+m_2}\right) \Rightarrow b = \left(\dfrac{m_1+m_2}{m_1}\right)y_1$

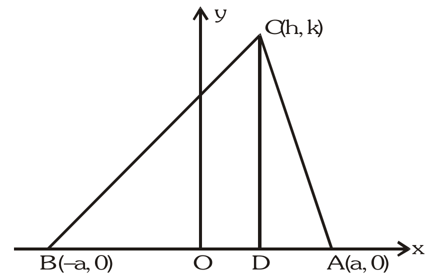
putting these values in (i) $\dfrac{(m_1+m_2)^2}{m_2^2}x_1^2 + \dfrac{(m_1+m_2)^2}{m_1^2}y_1^2 = \ell^2$

$\therefore$ Locus of (x_1, y_1) is $m_1^2x^2 + m_2^2y^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$ **Ans. (C)**

Illustration 10 : A(a, 0) and B(–a, 0) are two fixed points of ΔABC. If its vertex C moves in such a way that cotA + cotB = λ, where λ is a constant, then the locus of the point C is -

(A) yλ = 2a (B) y = λa (C) ya =2λ (D) none of these

Solution : Given that coordinates of two fixed points A and B are (a, 0) and (–a, 0) respectively. Let variable point C is (h, k). From the adjoining figure



$\cot A = \dfrac{DA}{CD} = \dfrac{a-h}{k}$

$\cot B = \dfrac{BD}{CD} = \dfrac{a+h}{k}$

But cotA + cotB = λ, so we have

$\dfrac{a-h}{k} + \dfrac{a+h}{k} = \lambda \Rightarrow \dfrac{2a}{k} = \lambda$

Hence locus of C is $y\lambda = 2a$ **Ans. (A)**

Do yourself - 5 :

(i) Find the locus of a variable point which is at a distance of 2 units from the y-axis.

(ii) Find the locus of a point which is equidistant from both the axes.

(iii) Find the locus of a point whose co-ordinates are given by $x = at^2$, $y = 2at$, where 't' is a parameter.

10. STRAIGHT LINE :

Introduction : A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, remember that every one degree equation in variable x and y always represents a straight line i.e. **ax + by + c = 0 ; a & b ≠ 0** simultaneously.

(a) Equation of a line parallel to x-axis at a distance 'a' is **y = a** or **y = – a**

(b) Equation of x-axis is **y = 0**

(c) Equation of a line parallel to y-axis at a distance 'b' is **x = b** or **x = – b**

(d) Equation of y-axis is **x = 0**

Illustration 11 : Prove that every first degree equation in x, y represents a straight line.

Solution : Let ax + by + c = 0 be a first degree equation in x, y

where a, b, c are constants.

Let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be any two points on the curve represented by ax + by + c = 0. Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$

Let R be any point on the line segment joining P & Q

Suppose R divides PQ in the ratio λ : 1. Then, the coordinates of R are $\left(\dfrac{\lambda x_2 + x_1}{\lambda+1}, \dfrac{\lambda y_2 + y_1}{\lambda+1}\right)$

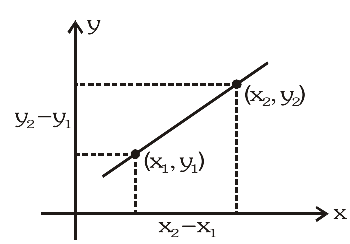
We have $a\left(\dfrac{\lambda x_2 + x_1}{\lambda+1}\right) + b\left(\dfrac{\lambda y_2 + y_1}{\lambda+1}\right) + c = \lambda\ 0 + 0 = 0$

∴ $R\left(\dfrac{\lambda x_2 + x_1}{\lambda+1}, \dfrac{\lambda y_2 + y_1}{\lambda+1}\right)$ lies on the curve represented by ax + by + c = 0. Thus every point on the line segment joining P & Q lies on ax + by + c = 0.

Hence ax + by + c = 0 represents a straight line.

11. SLOPE OF LINE :

If a given line makes an angle $\theta(0 \le \theta < 180, \theta \ne 90)$ with the positive direction of x-axis, then slope of this line will be tanθ and is usually denoted by the letter **m** i.e. **m = tanθ**. If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \ne x_2$ then slope of line AB $= \dfrac{y_2 - y_1}{x_2 - x_1}$



Remark :

(i) If $\theta = 90$, **m does not exist** and line is parallel to **y-axis.**

(ii) If $\theta = 0$, **m = 0** and the line is parallel to **x-axis.**

(iii) Let m_1 and m_2 be slopes of two given lines (none of them is parallel to y-axis)

(a) If lines are parallel, $\mathbf{m_1 = m_2}$ and vice-versa.

(b) If lines are perpendicular, $\mathbf{m_1 m_2 = -1}$ and vice-versa

12. STANDARD FORMS OF EQUATIONS OF A STRAIGHT LINE :

(a) **Slope Intercept form :** Let m be the slope of a line and c its intercept on y-axis. Then the equation of this straight line is written as : $y = mx + c$

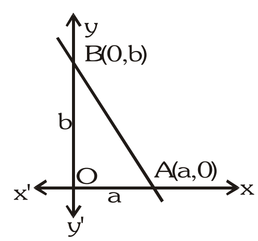
If the line passes through origin, its equation is written as $y = mx$

(b) **Point Slope form :** If m be the slope of a line and it passes through a point (x_1, y_1), then its equation is written as : $y - y_1 = m(x - x_1)$

(c) **Two point form :** Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(d) **Intercept form :** If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\dfrac{x}{a} + \dfrac{y}{b} = 1$



(i) Length of intercept of line between the coordinate axes $= \sqrt{a^2 + b^2}$

(ii) Area of triangle AOB $= \dfrac{1}{2}OA.OB = \left|\dfrac{1}{2}ab\right|$

Illustration 12 : The equation of the lines which passes through the point (3, 4) and the sum of its intercepts on the axes is 14 is -

(A) $4x - 3y = 24$, $x - y = 7$ (B) $4x + 3y = 24$, $x + y = 7$

(C) $4x + 3y + 24 = 0$, $x + y + 7 = 0$ (D) $4x - 3y + 24 = 0$, $x - y + 7 = 0$

Solution : Let the equation of the line be $\dfrac{x}{a} + \dfrac{y}{b} = 1$ .....(i)

This passes through (3 , 4) , therefore $\dfrac{3}{a} + \dfrac{4}{b} = 1$ .....(ii)

It is given that $a + b = 14 \Rightarrow b = 14 - a$. Putting $b = 14 - a$ in (ii) , we get

$$\frac{3}{a} + \frac{4}{14 - a} = 1 \Rightarrow a^2 - 13a + 42 = 0 \quad \Rightarrow (a - 7)(a - 6) = 0 \Rightarrow a = 7 , 6$$

For $a = 7$, $b = 14 - 7 = 7$ and for $a = 6$, $b = 14 - 6 = 8$

Putting the values of a and b in (i) , we get the equations of the lines

$\dfrac{x}{7} + \dfrac{y}{7} = 1$ and $\dfrac{x}{6} + \dfrac{y}{8} = 1$ or $x + y = 7$ and $4x + 3y = 24$ **Ans. (B)**

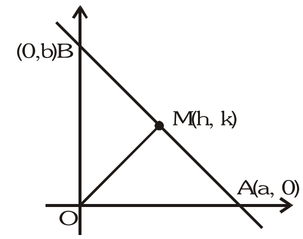
Illustration 13 : Two points A and B move on the positive direction of x-axis and y-axis respectively, such that OA + OB = K. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x + y)(x^2 + y^2) = Kxy$.

Solution : Let the equation of AB be $\frac{x}{a}+\frac{y}{b}=1$ .... (i)

given, a + b = K .... (ii)

now, m_{AB} m_{OM} = –1 ⇒ ah = bk .... (iii)

from (ii) and (iii),



$a = \frac{kK}{h+k}$ and $b = \frac{hK}{h+k}$

∴ from (i) $\frac{x(h+k)}{k.K}+\frac{y(h+k)}{h.K}=1$

as it passes through (h, k)

$\frac{h(h+k)}{k.K}+\frac{k(h+k)}{h.K}=1$ ⇒ $(h + k)(h^2 + k^2) = Khk$

∴ locus of (h, k) is $(x + y)(x^2 + y^2) = Kxy$.

(e) Normal form : If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as : **xcosα + ysinα = p** (p is always positive) where **$0 \leq \alpha < 2\pi$.**

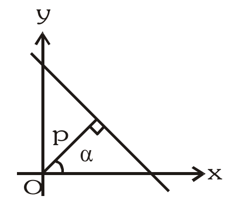
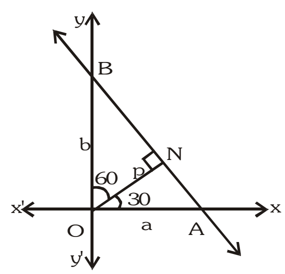


Illustration 14 : Find the equation of the straight line on which the perpendicular from origin makes an angle 30 with positive x-axis and which forms a triangle of area $\left(\frac{50}{\sqrt{3}}\right)$ sq. units with the co-ordinates axes.

Solution : ∠NOA = 30

Let ON = p > 0, OA = a, OB = b



In ΔONA, cos30 = $\frac{ON}{OA}=\frac{p}{a}$ ⇒ $\frac{\sqrt{3}}{2}=\frac{p}{a}$

or $a = \frac{2p}{\sqrt{3}}$

and in ΔONB, cos60 = $\frac{ON}{OB}=\frac{p}{b}$ ⇒ $\frac{1}{2}=\frac{p}{b}$

or b = 2p

∵ Area of ΔOAB = $\frac{1}{2}$ ab = $\frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)(2p) = \frac{2p^2}{\sqrt{3}}$

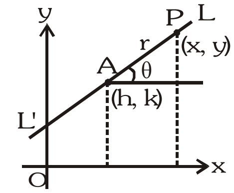
∴ $\frac{2p^2}{\sqrt{3}}=\frac{50}{\sqrt{3}}$ ⇒ $p^2 = 25$

or p = 5

∴ Using xcosα + ysinα = p, the equation of the line AB is xcos30 + y sin30 = 5

or $x\sqrt{3} + y = 10$

(f) Parametric form : To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'.



Let AP = r, then **x – h = r cosθ, y – k = r sinθ** & $\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}$ **= r** is the equation of the straight line LAL'.

Any point P on the line will be of the form (h + r cosθ, k + r sinθ), where |r| gives the distance of the point P from the fixed point (h, k).

Illustration 15 : Equation of a line which passes through point A(2, 3) and makes an angle of 45 with x axis. If this line meet the line x + y + 1 = 0 at point P then distance AP is -

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{5}$

Solution : Here $x_1 = 2$, $y_1 = 3$ and $\theta = 45$ hence $\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r$

from first two parts $\Rightarrow$ $x - 2 = y - 3 \Rightarrow x - y + 1 = 0$

Co-ordinate of point P on this line is $\left(2+\frac{r}{\sqrt{2}}, 3+\frac{r}{\sqrt{2}}\right)$.

If this point is on line x + y + 1 = 0 then

$\left(2+\frac{r}{\sqrt{2}}\right) + \left(3+\frac{r}{\sqrt{2}}\right) + 1 = 0 \Rightarrow r = -3\sqrt{2}$; $|r| = 3\sqrt{2}$ **Ans. (B)**

Illustration 16 : A variable line is drawn through O, to cut two fixed straight lines L_1 and L_2 in A_1 and A_2, respectively.

A point A is taken on the variable line such that $\frac{m+n}{OA} = \frac{m}{OA_1} + \frac{n}{OA_2}$.

Show that the locus of A is a straight line passing through the point of intersection of L_1 and L_2 where O is being the origin.

Solution : Let the variable line passing through the origin is $\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = r_i$ ...... (i)

Let the equation of the line L_1 is $p_1x + q_1y = 1$ ...... (ii)

Equation of the line L_2 is $p_2x + q_2y = 1$ ...... (iii)

the variable line intersects the line (ii) at A_1 and (iii) at A_2.

Let $OA_1 = r_1$.

Then $A_1 = (r_1\cos\theta, r_1\sin\theta) \Rightarrow A_1$ lies on L_1

$\Rightarrow \quad r_1 = OA_1 = \frac{1}{p_1\cos\theta + q_1\sin\theta}$

Similarly, $r_2 = OA_2 = \frac{1}{p_2\cos\theta + q_2\sin\theta}$

Let OA = r

Let co-ordinate of A are (h, k) $\Rightarrow$ (h, k) $\equiv$ (rcosθ, rsinθ)

Now $\frac{m+n}{r} = \frac{m}{OA_1} + \frac{n}{OA_2} \Rightarrow \frac{m+n}{r} = \frac{m}{r_1} + \frac{n}{r_2}$

$\Rightarrow \quad m + n = m(p_1r\cos\theta + q_1r\sin\theta) + n(p_2r\cos\theta + q_2r\sin\theta)$

$\Rightarrow \quad (p_1h + q_1k - 1) + \frac{n}{m}(p_2h + q_2k - 1) = 0$

Therefore, locus of A is $(p_1x+q_1y - 1) + \frac{n}{m}(p_2x + q_2y - 1) = 0$

$\Rightarrow \quad L_1 + \lambda L_2 = 0$ where $\lambda = \frac{n}{m}$.

This is the equation of the line passing through the intersection of L_1 and L_2.

Illustration 17 : A straight line through P(–2, –3) cuts the pair of straight lines $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ in Q and R. Find the equation of the line if PQ. PR = 20.

Solution : Let line be $\frac{x+2}{\cos\theta} = \frac{y+3}{\sin\theta} = r$

$\Rightarrow \quad x = r\cos\theta - 2,\ y = r\sin\theta - 3$ .... (i)

Now, $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ .... (ii)

Taking intersection of (i) with (ii) and considering terms of r^2 and constant (as we need PQ . PR = $r_1 . r_2$ = product of the roots)

$r^2(\cos^2\theta + 3\sin^2\theta + 4\sin\theta\cos\theta) + (\text{some terms})r + 80 = 0$

$\therefore \quad r_1.r_2 = PQ.\ PR = \dfrac{80}{\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta}$

$\therefore \quad \cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta = 4 \qquad (\because\ PQ\ .\ PR = 20)$

$\therefore \quad \sin^2\theta - 4\sin\theta\cos\theta + 3\cos^2\theta = 0$

$\Rightarrow \quad (\sin\theta - \cos\theta)(\sin\theta - 3\cos\theta) = 0$

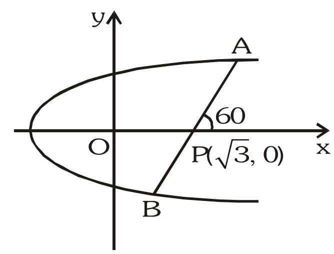
$\therefore \quad \tan\theta = 1,\ \tan\theta = 3$

hence equation of the line is $y + 3 = 1(x + 2) \Rightarrow x - y = 1$

and $y + 3 = 3(x + 2) \Rightarrow 3x - y + 3 = 0$.

Illustration 18 : If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then find the value of PA.PB {where $P \equiv (\sqrt{3}, 0)$}

Solution : Slope of line $y - \sqrt{3}x + 3 = 0$ is $\sqrt{3}$



If line makes an angle θ with x-axis, then $\tan\theta = \sqrt{3}$

$\therefore \quad \theta = 60°$

$\dfrac{x-\sqrt{3}}{\cos 60°} = \dfrac{y-0}{\sin 60°} = r \Rightarrow \left(\sqrt{3} + \dfrac{r}{2}, \dfrac{r\sqrt{3}}{2}\right)$

be a point on the parabola $y^2 = x + 2$

then $\dfrac{3}{4}r^2 = \sqrt{3} + \dfrac{r}{2} + 2 \Rightarrow 3r^2 - 2r - 4(2 + \sqrt{3}) = 0$

$\therefore \quad PA.PB = r_1r_2 = \left|\dfrac{-4\left(2+\sqrt{3}\right)}{3}\right| = \dfrac{4\left(2+\sqrt{3}\right)}{3}$

Do yourself - 6 :

(i) Reduce the line $2x - 3y + 5 = 0$,

(a) In slope- intercept form and hence find slope & Y-intercept

(b) In intercept form and hence find intercepts on the axes.

(c) In normal form and hence find perpendicular distance from the origin and angle made by the perpendicular with the positive x-axis.

(ii) Find distance of point A (2, 3) measured parallel to the line $x - y = 5$ from the line $2x + y + 6 = 0$.

(g) **General form :** We know that a first degree equation in x and y, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \dfrac{-a}{b} = -\dfrac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $= -\dfrac{c}{a}$ and intercept by this line on y-axis $= -\dfrac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2 + b^2}$.

13. ANGLE BETWEEN TWO LINES :

(a) If θ be the angle between two lines : $y = m_1x + c$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\dfrac{m_1 - m_2}{1 + m_1m_2}\right)$

Note :

(i) There are two angles formed between two lines but usually the acute angle is taken as the angle between the lines. So we shall find θ from the above formula only by taking positive value of $\tan\theta$.

(ii) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the Δ ABC found by these formulas are given by,

$$\tan A = \frac{m_1 - m_2}{1 + m_1 m_2} \;;\; \tan B = \frac{m_2 - m_3}{1 + m_2 m_3} \quad \& \quad \tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$$

(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these line are -

(i) Parallel $\Leftrightarrow$ $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow$ $a_1a_2+b_1b_2=0$

(iii) Coincident $\Leftrightarrow$ $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Illustration 19 : If $x + 4y - 5 = 0$ and $4x + ky + 7 = 0$ are two perpendicular lines then k is -

(A) 3 (B) 4 (C) –1 (D) –4

Solution : $m_1 = -\dfrac{1}{4}$ $m_2 = -\dfrac{4}{k}$

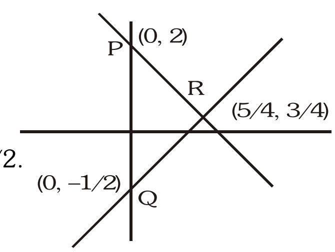
Two lines are perpendicular if $m_1 m_2 = -1$

$\Rightarrow \left(-\dfrac{1}{4}\right) \times \left(-\dfrac{4}{k}\right) = -1 \Rightarrow k = -1$ **Ans. (C)**

Illustration 20 : A line L passes through the points (1, 1) and (0, 2) and another line M which is perpendicular to L passes through the point (0, –1/2).The area of the triangle formed by these lines with y-axis is -

(A) 25/8 (B) 25/16 (C) 25/4 (D) 25/32

Solution : Equation of the line L is $y - 1 = \dfrac{-1}{1}(x - 1) \Rightarrow y = -x + 2$



Equation of the line M is $y = x - 1/2$.

If these lines meet y-axis at P (0, –1/2) and Q (0, 2) then PQ = 5/2.

Also x-coordinate of their point of intersection R = 5/4

$\therefore$ area of the $\Delta PQR = \dfrac{1}{2}\left(\dfrac{5}{2} \times \dfrac{5}{4}\right) = 25/16.$ **Ans. (B)**

Illustration 21 : If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y-axis, then the value of k is -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : A straight line is parallel to y-axis, if its y - coefficient is zero, i.e. $4 - k = 0$ i.e. $k = 4$ **Ans. (D)**

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14. EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE :

(a) Equation of line parallel to line $ax + by + c = 0$

$ax + by + \lambda = 0$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$bx - ay + k = 0$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

15. STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE :

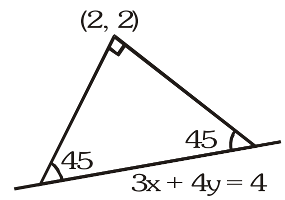
Equations of lines passing through a point (x_1,y_1) and making an angle α, with the line $y=mx+c$ is written as:

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Illustration 22 : Find the equation to the sides of an isosceles right-angled triangle, the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2).

Solution : The problem can be restated as :

Find the equations of the straight lines passing through the given point (2, 2) and making equal angles of 45 with the given straight line $3x + 4y - 4 = 0$. Slope of the line $3x + 4y - 4 = 0$ is $m_1 = -3/4$.



$\Rightarrow \quad \tan 45 = \pm\dfrac{m - m_1}{1 + m_1 m}$, i.e., $1 = \pm\dfrac{m + 3/4}{1 - \frac{3}{4}m}$

$m_A = \dfrac{1}{7}$, and $m_B = -7$

Hence the required equations of the two lines are

$y - 2 = m_A(x - 2)$ and $y - 2 = m_B(x - 2)$

$\Rightarrow \quad 7y - x - 12 = 0$ and $7x + y = 16$ **Ans.**

Do yourself - 7 :

(i) Find the angle between the lines $3x + y - 7 = 0$ and $x + 2y - 9 = 0$.

(ii) Find the line passing through the point (2, 3) and perpendicular to the straight line $4x - 3y = 10$.

(iii) Find the equation of the line which has positive y-intercept 4 units and is parallel to the line $2x - 3y - 7 = 0$. Also find the point where it cuts the x-axis.

(iv) Classify the following pairs of lines as coincident, parallel or intersecting :

(a) $x + 2y - 3 = 0$ & $-3x - 6y + 9 = 0$

(b) $x + 2y + 1 = 0$ & $2x + 4y + 3 = 0$

(c) $3x - 2y + 5 = 0$ & $2x + y - 5 = 0$

16. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

Length of perpendicular from a point (x_1,y_1) on the line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

In particular, the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \dfrac{|c|}{\sqrt{a^2 + b^2}}$

Illustration 23 : If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Solution : Let n given points be (x_i, y_i) where $i = 1, 2.... n$ and the variable straight line is $ax + by + c = 0$.

Given that $\displaystyle\sum_{i=1}^{n}\left(\frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}}\right) = 0 \quad \Rightarrow \quad a\Sigma x_i + b\Sigma y_i + cn = 0 \Rightarrow a\frac{\Sigma x_i}{n} + b\frac{\Sigma y_i}{n} + c = 0$.

Hence the variable straight line always passes through the fixed point $\left(\dfrac{\Sigma x_i}{n}, \dfrac{\Sigma y_i}{n}\right)$. **Ans.**

Illustration 24 : Prove that no line can be drawn through the point (4, –5) so that its distance from (–2, 3) will be equal to 12.

Solution : Suppose, if possible.

Equation of line through (4, –5) with slope of m is

$y + 5 = m(x - 4)$

$\Rightarrow \quad mx - y - 4m - 5 = 0$

Then $\dfrac{|m(-2) - 3 - 4m - 5|}{\sqrt{m^2+1}} = 12$

$\Rightarrow \quad |-6m - 8| = 12\sqrt{(m^2+1)}$

On squaring, $(6m + 8)^2 = 144(m^2 + 1)$

$\Rightarrow \quad 4(3m + 4)^2 = 144(m^2 + 1)$

$\Rightarrow \quad (3m + 4)^2 = 36(m^2 + 1)$

$\Rightarrow \quad 27m^2 - 24m + 20 = 0$ .... (i)

Since the discriminant of (i) is $(-24)^2 - 4.27.20 = -1584$ which is negative, there is no real value of m. Hence no such line is possible.

---

17. DISTANCE BETWEEN TWO PARALLEL LINES :

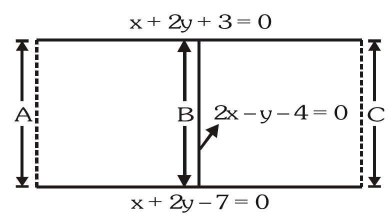
(a) The distance between two parallel lines $ax + by + c_1 = 0$ and $ax+by+c_2=0$ is $= \dfrac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$

(**Note** : The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram $= \dfrac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides . Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\dfrac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

Illustration 25 : Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

Solution : Distance between the two parallel lines is $\dfrac{|7+3|}{\sqrt{5}} = 2\sqrt{5}$.



The equations of sides A and C are of the form $2x - y + k = 0$.

Since distance between sides A and B = distance between sides B and C $\dfrac{|k-(-4)|}{\sqrt{5}} = 2\sqrt{5} \Rightarrow \dfrac{k+4}{\sqrt{5}} = \pm 2\sqrt{5} \Rightarrow k = 6, -14$.

Hence the fourth sides of the two squares are (i) $2x - y + 6 = 0$ (ii) $2x - y - 14 = 0$. **Ans.**

Do yourself - 8 :

(i) Find the distances between the following pair of parallel lines :

(a) $3x + 4y = 13,\ 3x + 4y = 3$

(b) $3x - 4y + 9 = 0,\ 6x - 8y - 15 = 0$

(ii) Find the points on the x-axis such that their perpendicular distance from the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ is 'a', a, b > 0.

(iii) Show that the area of the parallelogram formed by the lines

$2x - 3y + a = 0$, $3x - 2y - a = 0$, $2x - 3y + 3a = 0$ and $3x - 2y - 2a = 0$ is $\dfrac{2a^2}{5}$ square units.

18. POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE :

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

Illustration 26 : Let P $(\sin\theta, \cos\theta)$ $(0 \le \theta \le 2\pi)$ be a point and let OAB be a triangle with vertices (0, 0), $\left(\sqrt{\frac{3}{2}}, 0\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$. Find θ if P lies inside the ΔOAB.

Solution : Equations of lines along OA, OB and AB are $y = 0$, $x = 0$ and $x + y = \sqrt{\frac{3}{2}}$ respectively. Now P and B will lie on the same side of $y = 0$ if $\cos\theta > 0$. Similarly P and A will lie on the same side of $x = 0$ if $\sin\theta > 0$ and P and O will lie on the same side of $x + y = \sqrt{\frac{3}{2}}$ if $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$.

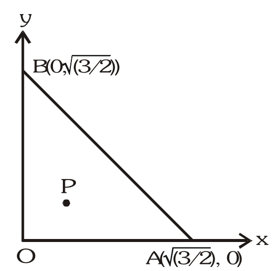
Hence P will lie inside the ΔABC, if $\sin\theta > 0$, $\cos\theta > 0$ and $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$.

Now $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$

$\Rightarrow \quad \sin\left(\theta + \frac{\pi}{4}\right) < \frac{\sqrt{3}}{2}$

i.e. $0 < \theta + \frac{\pi}{4} < \pi/3$

or $\frac{2\pi}{3} < \theta + \frac{\pi}{4} < \pi$



Since $\sin\theta > 0$ and $\cos\theta > 0$, so $0 < \theta < \frac{\pi}{12}$ or $\frac{5\pi}{12} < \theta < \frac{3\pi}{4}$.

19. CONCURRENCY OF LINES :

(a) Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix} = 0$

(b) To test the concurrency of three lines, first find out the point of intersection of any two of the three lines. If this point lies on the remaining line (i.e. coordinates of the point satisfy the equation of the line) then the three lines are concurrent otherwise not concurrent.

Illustration 27 : If the lines $ax + by + p = 0$, $x\cos\alpha + y\sin\alpha - p = 0$ $(p \ne 0)$ and $x\sin\alpha - y\cos\alpha = 0$ are concurrent and the first two lines include an angle $\frac{\pi}{4}$, then $a^2 + b^2$ is equal to -

(A) 1 (B) 2 (C) 4 (D) p^2

Solution : Since the given lines are concurrent,

$$\begin{vmatrix} a & b & p \\ \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad a\cos\alpha + b\sin\alpha + 1 = 0 \qquad \ldots\ldots\ldots \text{(i)}$

As $ax + by + p = 0$ and $x\cos\alpha + y\sin\alpha - p = 0$ include an angle $\frac{\pi}{4}$.

$$\pm\tan\frac{\pi}{4} = \frac{-\frac{a}{b} + \frac{\cos\alpha}{\sin\alpha}}{1 + \frac{a}{b}\frac{\cos\alpha}{\sin\alpha}}$$

$\Rightarrow$ $-a\sin\alpha + b\cos\alpha = \pm(b\sin\alpha + a\cos\alpha)$

$\Rightarrow$ $-a\sin\alpha + b\cos\alpha = \pm 1$ [from (i)] ……… (ii)

Squaring and adding (i) & (ii), we get

$a^2 + b^2 = 2$. **Ans. (B)**

Do yourself - 9 :

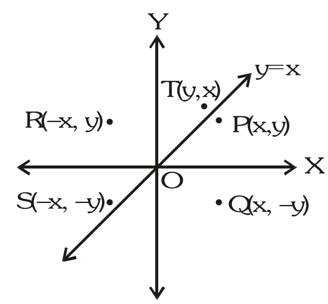
(i) Examine the positions of the points (3, 4) and (2, – 6) w.r.t. $3x - 4y = 8$

(ii) If (2, 9), (– 2, 1) and (1, –3) are the vertices of a triangle, then prove that the origin lies inside the triangle.

(iii) Find the equation of the line joining the point (2, – 9) and the point of intersection of lines $2x + 5y - 8 = 0$ and $3x - 4y - 35 = 0$.

(iv) Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33 = 0$ and $2x - 3y + \lambda = 0$ are concurrent.

20. REFLECTION OF A POINT :



Let P(x, y) be any point, then its image with respect to

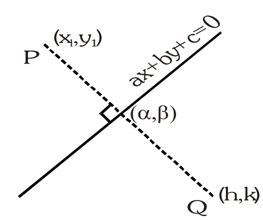
(a) x-axis is Q(x, –y)

(b) y-axis is R(–x, y)

(c) origin is S(–x,–y)

(d) line y = x is T(y, x)

(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1, y_1)$ about the line $ax + by + c = 0$ is given by following formula.



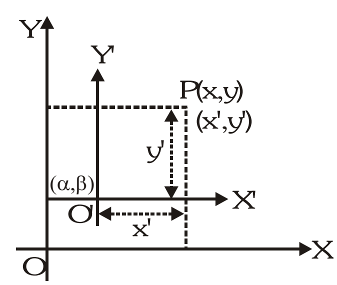
$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line $ax + by + c = 0$ is given by following formula.

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

21. TRANSFORMATION OF AXES

(a) Shifting of origin without rotation of axes :



Let P (x, y) with respect to axes OX and OY.

Let O' (α, β) is new origin with respect to axes OX and OY and let P (x', y') with respect to axes O'X' and O'Y' , where OX and O'X' are parallel and OY and O'Y' are parallel.

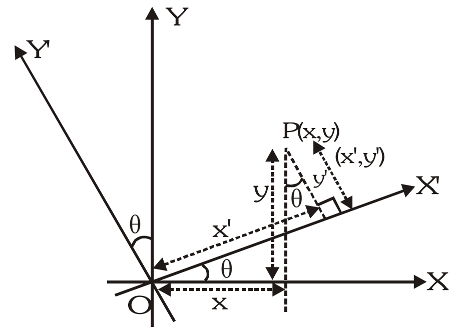
Then $x = x' + \alpha$, $y = y' + \beta$

or $x' = x - \alpha$, $y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

(b) **Rotation of axes without shifting the origin :**

Let O be the origin. Let P (x, y) with respect to axes OX and OY and let P (x', y') with respect to axes OX' and OY' where $\angle X'OX = \angle YOY' = \theta$, where θ is measured in anticlockwise direction.



then $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	$x\downarrow$	$y\downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

Illustration 28 : Through what angle should the axes be rotated so that the equation $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$?

Solution : Let angle be θ then replacing (x, y) by $(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$

then $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ becomes

$$9(x\cos\theta - y\sin\theta)^2 - 2\sqrt{3}(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 7(x\sin\theta + y\cos\theta)^2 = 10$$

$$\Rightarrow \quad x^2(9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta) + 2xy(-9\sin\theta\cos\theta - \sqrt{3}\cos 2\theta + 7\sin\theta\cos\theta)$$

$$+ y^2(9\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta) = 10$$

On comparing with $3x^2 + 5y^2 = 5$ (coefficient of xy = 0)

We get $\quad -9\sin\theta\cos\theta - \sqrt{3}\cos 2\theta + 7\sin\theta\cos\theta = 0$

or $\quad \sin 2\theta = -\sqrt{3}\cos 2\theta \quad$ or $\quad \tan 2\theta = -\sqrt{3} = \tan(180 - 60)$

or $\quad 2\theta = 120 \quad \therefore \quad \theta = 60$

Do yourself - 10 :

(i) The point (4, 1) undergoes the following transformations, then the match the correct alternatives :

Column-I	Column-II
(A) Reflection about x-axis is	(p) (4, –1)
(B) Reflection about y-axis is	(q) (–4, –1)
(C) Reflection about origin is	(r) $\left(-\frac{12}{25}, -\frac{59}{25}\right)$
(D) Reflection about the line y = x is	(s) (–4, 1)
(E) Reflection about the line $4x + 3y - 5 = 0$ is	(t) (1, 4)

(ii) On what point must the origin be shifted, if the coordinates of a point (4, 5) become (– 3, 9).

(iii) If the axes be turned through an angle $\tan^{-1}2$ (in anticlockwise direction), what does the equation $4xy - 3x^2 = a^2$ become ?

22. EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES :

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad \text{..........(i)}$$

(a) Equation of bisector of angle containing origin :

If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (i)

(b) Equation of bisector of acute/obtuse angles :

To find the equation of the bisector of the acute or obtuse angle :

(i) let ϕ be the angle between one of the two bisectors and one of two given lines. Then if $\tan\phi < 1$ i.e. $\phi < 45$ i.e. $2\phi < 90$, the angle bisector will be bisector of acute angle.

(ii) See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by -1 to make the constant terms positive.

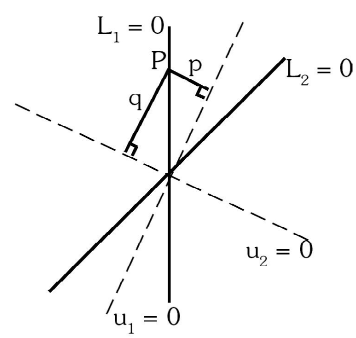
Determine the sign of $a_1a_2 + b_1b_2$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use − sign in eq. (1)
−	use − sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

(iii) Another way of identifying an acute and obtuse angle bisector is as follows :



Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown . If,

$|p| < |q| \Rightarrow u_1$ is the acute angle bisector .

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector .

$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular .

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

Illustration 29 : For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them.

(ii) bisector of the acute angle between them.

(iii) bisector of the angle which contains origin.

(iv) bisector of the angle which contains (1, 2).

Solution : Equations of bisectors of the angles between the given lines are

$$\frac{4x + 3y - 6}{\sqrt{4^2 + 3^2}} = \pm\frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}} \Rightarrow 9x - 7y - 41 = 0 \text{ and } 7x + 9y - 3 = 0$$

If θ is the acute angle between the line $4x + 3y - 6 = 0$ and the bisector

$9x - 7y - 41 = 0$, then $\tan\theta = \left|\dfrac{-\frac{4}{3} - \frac{9}{7}}{1 + \left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \dfrac{11}{3} > 1$

Hence

(i) bisector of the obtuse angle is $9x - 7y - 41 = 0$

(ii) bisector of the acute angle is $7x + 9y - 3 = 0$

(iii) bisector of the angle which contains origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

(iv) $L_1(1,\ 2) = 4 \quad 1 + 3 \quad 2 - 6 = 4 > 0$
$L_2(1,\ 2) = 5 \quad 1 + 12 \quad 2 + 9 = 38 > 0$

+ve sign will give the required bisector, $\dfrac{4x+3y-6}{5} = +\dfrac{5x+12y+9}{13}$

$\Rightarrow \quad 9x - 7y - 41 = 0.$

Alternative :

Making c_1 and c_2 positive in the given equation, we get $-4x - 3y + 6 = 0$ and $5x + 12y + 9 = 0$

Since $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$, so the origin will lie in the acute angle.

Hence bisector of the acute angle is given by

$$\frac{-4x-3y+6}{\sqrt{4^2+3^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \quad \Rightarrow \quad 7x+9y-3=0$$

Similarly bisector of obtuse angle is $9x - 7y - 41 = 0$.

Illustration 30 : A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line mirror $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Solution : Let Q be the point of intersection of the incident ray and the line mirror, then

$x_1 - 2y_1 - 3 = 0 \ \ \& \ \ 3x_1 - 2y_1 - 5 = 0$

on solving these equations, we get

$x_1 = 1 \ \ \& \ \ y_1 = -1$

Since P(−1, −2) be a point lies on the incident ray, so we can find the image of the point P on the reflected ray about the line mirror (by property of reflection).

Let P'(h, k) be the image of point P about line mirror, then

$$\frac{h+1}{3} = \frac{k+2}{-2} = \frac{-2(-3+4-5)}{13} \quad \Rightarrow \quad h = \frac{11}{13} \text{ and } k = \frac{-42}{13}.$$

So $\quad P'\left(\dfrac{11}{13}, \dfrac{-42}{13}\right)$

Then equation of reflected ray will be

$$(y+1) = \frac{\left(\frac{-42}{13}+1\right)(x-1)}{\left(\frac{11}{13}-1\right)}$$

$\Rightarrow \quad 2y - 29x + 31 = 0$ is the required equation of reflected ray.

23. FAMILY OF LINES :

If equation of two lines be $P \equiv a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

Illustration 31 : Prove that each member of the family of straight lines $(3\sin\theta + 4\cos\theta)x + (2\sin\theta - 7\cos\theta)y + (\sin\theta + 2\cos\theta) = 0$ (θ is a parameter) passes through a fixed point.

Solution : The given family of straight lines can be rewritten as

$(3x + 2y + 1)\sin\theta + (4x - 7y + 2)\cos\theta = 0$

or, $(4x - 7y + 2) + \tan\theta(3x + 2y + 1) = 0$ which is of the form $L_1 + \lambda L_2 = 0$

Hence each member of it will pass through a fixed point which is the intersection of

$4x - 7y + 2 = 0$ and $3x + 2y + 1 = 0$ i.e. $\left(\frac{-11}{29}, \frac{2}{29}\right)$.

Do yourself - 11 :

(i) Find the equations of bisectors of the angle between the lines $4x + 3y = 7$ and $24x + 7y - 31 = 0$. Also find which of them is (a) the bisector of the angle containing origin (b) the bisector of the acute angle.

(ii) Find the equations of the line which pass through the point of intersection of the lines $4x - 3y = 1$ and $2x - 5y + 3 = 0$ and is equally inclined to the axis.

(iii) Find the equation of the line through the point of intersection of the lines $3x - 4y + 1 = 0$ & $5x + y - 1 = 0$ and perpendicular to the line $2x - 3y = 10$.

24. PAIR OF STRAIGHT LINES :

(a) Homogeneous equation of second degree :

Let us consider the homogeneous equation of 2nd degree as

$ax^2 + 2hxy + by^2 = 0$ ....(i)

which represents pair of straight lines passing through the origin.

Now, we divide by x^2, we get

$$a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2 = 0$$

$$\frac{y}{x} = m \quad \text{(say)}$$

then $a + 2hm + bm^2 = 0$ ....(ii)

if m_1 & m_2 are the roots of equation (ii), then $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$

and also, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right|$

$$= \left|\frac{\sqrt{\frac{4h^2}{b} - \frac{4a}{b}}}{1 + \frac{a}{b}}\right| = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

These line will be :

(i) **Real and different**, if $h^2 - ab > 0$

(ii) **Real and coincident**, if $h^2 - ab = 0$

(iii) **Imaginary**, if $h^2 - ab < 0$

(iv) The condition that these lines are :

(1) At **right angles** to each other is **a + b = 0.** i.e. coefficient of **x** + coefficient of **y** = 0.

(2) **Coincident** is **h = ab.**

(3) **Equally inclined** to the axes of x is **h = 0.** i.e. coefficient of **xy = 0.**

Homogeneous equation of 2nd degree $ax^2 + 2hxy + by^2 = 0$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x \equiv y = m_1x \;\&\; y = m_2x \quad \text{and} \quad m_1 + m_2 = -\frac{2h}{b} \; ; \; m_1m_2 = \frac{a}{b}$$

These straight lines passes through the origin.

Note : A homogeneous equation of degree n represents n straight lines passing through **origin**.

(b) **The combined equation of angle bisectors :**

The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$, $a \neq b$, $h \neq 0$.

Note :

(i) If $a = b$, the bisectors are $x^2 - y^2 = 0$ i.e. $x - y = 0$, $x + y = 0$

(ii) If $h = 0$, the bisectors are $xy = 0$ i.e. $x = 0$, $y = 0$.

(iii) The two bisectors are always at **right angles**, since we have coefficient of x^2 + coefficient of $y^2 = 0$

(c) **General Equation and Homogeneous Equation of Second Degree :**

(i) The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) If θ be the angle between the lines, then $\tan\theta = \pm\dfrac{2\sqrt{h^2 - ab}}{a + b}$

Obviously these lines are

(1) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(2) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(iii) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(iv) The product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is $\left|\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a - b)^2 + 4h^2}}\right|$

(v) The product of the perpendiculars drawn from the origin to the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left|\dfrac{c}{\sqrt{(a - b)^2 + 4h^2}}\right|$

Illustration 32 : If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then λ is equal to -

(A) 4 (B) 3 (C) 2 (D) 1

Solution : Here $a = \lambda$, $b = 12$, $c = -3$, $f = -8$, $g = 5/2$, $h = -5$

Using condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, we have

$\lambda(12)(-3) + 2(-8)(5/2)(-5) - \lambda(64) - 12(25/4) + 3(25) = 0$

$\Rightarrow \quad -36\lambda + 200 - 64\lambda - 75 + 75 = 0 \quad \Rightarrow \quad 100\lambda = 200$

$\therefore \quad \lambda = 2$ **Ans. (C)**

Illustration 33 : Show that the two straight lines $x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$ represented by the equation are such that the difference of their slopes is 2.

Solution : The given equation is $x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$ .... (i)

and general equation of second degree $ax^2 + 2hxy + by^2 = 0$ .... (ii)

Comparing (i) and (ii), we get $a = \tan^2\theta + \cos^2\theta$

$h = -\tan\theta$

and $b = \sin^2\theta$

Let separate lines of (ii) are $y = m_1x$ and $y = m_2x$

where $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

therefore, $m_1+m_2=-\dfrac{2h}{b}=\dfrac{2\tan\theta}{\sin^2\theta}$

and $m_1.m_2=\dfrac{\tan^2\theta+\cos^2\theta}{\sin^2\theta}$

$\therefore \quad m_1-m_2=\sqrt{(m_1+m_2)^2-4m_1m_2}$

$\Rightarrow \quad \tan\theta_1-\tan\theta_2=\sqrt{\dfrac{4\tan^2\theta}{\sin^4\theta}-\dfrac{4(\tan^2\theta+\cos^2\theta)}{\sin^2\theta}}=\dfrac{2}{\sin^2\theta}\sqrt{\tan^2\theta-\sin^2\theta(\tan^2\theta+\cos^2\theta)}$

$=\dfrac{2\sin\theta}{\sin^2\theta}\sqrt{(\sec^2\theta-\tan^2\theta-\cos^2\theta)}=\dfrac{2\sin\theta}{\sin^2\theta}\sqrt{(1-\cos^2\theta)}=\dfrac{2}{\sin\theta}\sin\theta=2$ **Ans.**

Illustration 34 : If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $pq = -1$.

Solution : According to the question, the equation of the bisectors of the angle between the lines

$x^2 - 2pxy - y^2 = 0$ .... (i)

is $x^2 - 2qxy - y^2 = 0$ .... (ii)

$\therefore$ The equation of bisectors of the angle between the lines (i) is $\dfrac{x^2-y^2}{1-(-1)}=\dfrac{xy}{-p}$

$\Rightarrow \quad -px^2 - 2xy + py^2 = 0$

Since (ii) and (iii) are identical, comparing (ii) and (iii), we get $\dfrac{1}{-p}=\dfrac{-2q}{-2}=\dfrac{-1}{p} \Rightarrow pq = -1$

Do yourself - 12 :

(i) Prove that the equation $x^2 - 5xy + 4y^2 = 0$ represents two lines passing through the origin. Also find their equations.

(ii) If the equation $3x^2 + kxy - 10y^2 + 7x + 19y = 6$ represents a pair of lines, find the value of k.

(iii) If the equation $6x^2 - 11xy - 10y^2 - 19y + c = 0$ represents a pair of lines, find their equations. Also find the angle between the two lines.

(iv) Find the point of intersection and the angle between the lines given by the equation : $2x^2 - 3xy - 2y^2 + 10x + 5y + 12 = 0$.

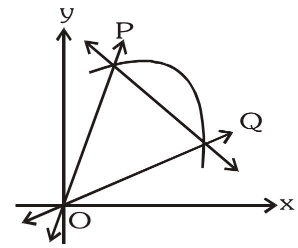
25. EQUATIONS OF LINES JOINING THE POINTS OF INTERSECTION OF A LINE AND A CURVE TO THE ORIGIN :

(a) Let the equation of curve be :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

and straight line be

$lx + my + n = 0$ ...(ii)



Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation (i) homogenous with the help of equation of the line. Thus required equation is given by

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{\ell x+my}{-n}\right)+c\left(\frac{\ell x+my}{-n}\right)^2=0$$

$\Rightarrow$ $(an^2 + 2gln + cl^2)x^2 + 2(hn^2 + gmn + fln + clm)xy + (bn^2 + 2fmn + cm^2)y^2 = 0$ ......(iii)

All points which satisfy (i) and (ii) simultaneously, will satisfy (iii)

(b) Any second degree curve through the four points of intersection of $f(x, y) = 0$ & $xy = 0$ is given by $f(x, y) + \lambda xy = 0$ where $f(x, y) = 0$ is also a second degree curve.

Illustration 35 : The chord $\sqrt{6}y=\sqrt{8}px+\sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at origin then find the value of p.

Solution : $\sqrt{3}y-2px=1$ is the given chord. Homogenizing the equation of the curve, we get,

$py^2 - 4x(\sqrt{3}\,y - 2px)+(\sqrt{3}\,y - 2px)^2 = 0$

$\Rightarrow$ $(4p^2 + 8p)x^2 + (p + 3)y^2 - 4\sqrt{3}\,xy - 4\sqrt{3}\,pxy = 0$

Now, angle at origin is 90

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore$ $4p^2 + 8p + p + 3 = 0$ $\Rightarrow$ $4p^2 + 9p + 3 = 0$

$\therefore$ $p=\dfrac{-9\pm\sqrt{81-48}}{8}=\dfrac{-9\pm\sqrt{33}}{8}$.

Do yourself - 13 :

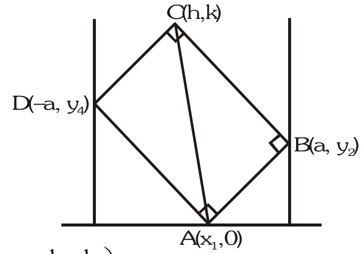
(i) Find the angle subtended at the origin by the intercept made on the curve $x^2 - y^2 - xy + 3x - 6y + 18 = 0$ by the line $2x - y = 3$.

(ii) Find the equation of the lines joining the origin to the points of intersection of the curve $2x^2 + 3xy - 4x + 1 = 0$ and the line $3x + y = 1$.

Miscellaneous Illustration :

Illustration 36 : ABCD is a variable rectangle having its sides parallel to fixed directions (say m). The vertices B and D lie on x = a and x = –a and A lies on the line y = 0. Find the locus of C.

Solution : Let A be $(x_1, 0)$, $B(a, y_2)$ and D be $(-a, y_4)$. We are given AB and AD have fixed directions and hence their slopes are constants. i.e. m & m_1 (say)



$\therefore \quad \dfrac{y_2}{a-x_1} = m$ and $\dfrac{y_4}{-a-x_1} = m_1$.

Further, $mm_1 = -1$. Since ABCD is a rectangle.

$\dfrac{y_2}{a-x_1} = m$ and $\dfrac{y_4}{-a-x_1} = -\dfrac{1}{m}$

The mid point of BD is $\left(0, \dfrac{y_2+y_4}{2}\right)$ and mid point of AC is $\left(\dfrac{x_1+h}{2}, \dfrac{k}{2}\right)$, where C is taken to be (h, k). This gives $h = -x_1$ and $k = y_2 + y_4$. So C is $(-x_1, y_2 + y_4)$.

Also, $\dfrac{y_2}{a-x_1} = m$ and $\dfrac{y_4}{a+x_1} = \dfrac{1}{m}$ gives $m(k - y_2) = a + x_1 = m(k - m(a - x_1)) = a + x_1$

$\Rightarrow \quad mk - m^2(a - x_1) = a + x_1 \quad \Rightarrow \quad m^2(a + h) - mk + a - h = 0$

$\Rightarrow \quad (m^2 - 1)h - mk = -(m^2 + 1)a \quad \Rightarrow \quad (1 - m^2)h + mk = (m^2 + 1)a$

$\Rightarrow \quad (1 - m^2)x + my = (m^2 + 1)a$

The locus of C is $(1 - m^2)x + my = (m^2 + 1)a$.

ANSWERS FOR DO YOURSELF

1 : (i) $PQ = \sqrt{34}$; (ii) x = 6 or x = 0 (iii) 11, –7

2 : (i) (a) (2,1) ; (b) (7,16) ; (ii) (a) 2 : 3 (internally) ; (b) 9 : 4 (externally) ; (c) 8 : 7 (internally)

3 : (i) (a) $\left(\dfrac{5}{3}, 3\right)$; (b) $\left(-\dfrac{9}{2}, \dfrac{15}{2}\right)$, $\dfrac{5\sqrt{10}}{2}$, (c) (14, – 6)

4 : (i) 25 square units ; (ii) 132 square units ; (v) 1

5 : (i) $x = \pm 2$; (ii) $y = \pm x$; (iii) $y^2 = 4ax$

6 : (i) (a) $y = \dfrac{2x}{3} + \dfrac{5}{3}$, $\dfrac{2}{3}$, $\dfrac{5}{3}$; (b) $\dfrac{x}{(-5/2)} + \dfrac{y}{(5/3)} = 1$, $-\dfrac{5}{2}$, $\dfrac{5}{3}$;

(c) $-\dfrac{2x}{\sqrt{13}} + \dfrac{3y}{\sqrt{13}} = \dfrac{5}{\sqrt{13}}$, $\dfrac{5}{\sqrt{13}}$, $\alpha = -\tan^{-1}\left(\dfrac{3}{2}\right)$; (ii) $13\sqrt{2}/3$ units

7 : (i) θ = 135 or 45 ; (ii) 3x + 4y = 18 ; (iii) 2x – 3y + 12 = 0, (–6, 0)

(iv) (a) Coincident, (b) Parallel, (c) Intersecting

8 : (i) (a) 2 (b) 33/10 ; (ii) $\left(\dfrac{a}{b}\left(b \pm \sqrt{a^2+b^2}\right), 0\right)$

9 : (i) opposite sides of the line; (iii) – y + x = 11; (iv) λ = – 7

10 : (i) (A) → (p), (B) → (s), (C) → (q), (D) → (t), (E) → (r), ; (ii) (7, – 4) ; (iv) $x^2 - 4y^2 = a^2$

11 : (i) x – 2y + 1 = 0 & 2x + y – 3 = 0; (a) x – 2y + 1 = 0 ; (b) 2x + y – 3 = 0

(ii) x + y = 2, x = y; (iii) 69x + 46y – 25 = 0

12 : (i) x – y = 0 & x – 4y = 0; (ii) k = 1, or $\dfrac{127}{6}$;

(iii) 2x – 5y – 2 = 0 & 3x + 2y + 3 = 0 ; $\pm\tan^{-1}\left(\dfrac{19}{4}\right)$; (iv) $\left(\dfrac{11}{5}, \dfrac{2}{5}\right)$, 90° ;

13 : (i) $\theta = \pm\tan^{-1}\dfrac{4}{7}$; (ii) $x^2 - y^2 - 5xy = 0$;

EXERCISE-01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. If (3, –4) and (–6, 5) are the extremities of a diagonal of a parallelogram and (2, 1) is its third vertex, then its fourth vertex is -
(A) (–1, 0) (B) (–1, 1) (C) (0, –1) (D) (–5, 0)

2. The ratio in which the line joining the points (3, –4) and (–5, 6) is divided by x-axis -
(A) 2 : 3 (B) 6 : 4 (C) 3 : 2 (D) none of these

3. The circumcentre of the triangle with vertices (0, 0), (3, 0) and (0, 4) is -
(A) (1, 1) (B) (2, 3/2) (C) (3/2, 2) (D) none of these

4. The mid points of the sides of a triangle are (5, 0), (5, 12) and (0, 12), then orthocentre of this triangle is -
(A) (0, 0) (B) (0, 24) (C) (10, 0) (D) $\left(\frac{13}{3}, 8\right)$

5. Area of a triangle whose vertices are $(a\cos\theta, b\sin\theta)$, $(-a\sin\theta, b\cos\theta)$ and $(-a\cos\theta, -b\sin\theta)$ is -
(A) $ab\sin\theta\cos\theta$ (B) $a\cos\theta\sin\theta$ (C) $\frac{1}{2}ab$ (D) ab

6. The point A divides the join of the points (–5,1) and (3,5) in the ratio k : 1 and coordinates of points B and C are (1,5) and (7,–2) respectively. If the area of ΔABC be 2 units, then k equals -
(A) 7,9 (B) 6,7 (C) 7,31/9 (D) 9,31/9

7. If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, C (1,2) are the vertices of a ΔABC, then as α varies, the locus of its centroid is -
(A) $x^2 + y^2 - 2x - 4y + 3 = 0$ (B) $x^2 + y^2 - 2x - 4y + 1 = 0$
(C) $3(x^2 + y^2) - 2x - 4y + 1 = 0$ (D) none of these

8. The points with the co-ordinates (2a, 3a), (3b, 2b) & (c, c) are collinear-
(A) for no value of a, b, c (B) for all values of a, b, c
(C) if a, $\frac{c}{5}$, b are in H.P. (D) if a, $\frac{2}{5}$c, b are in H.P.

9. A stick of length 10 units rests against the floor and a wall of a room. If the stick begins to slide on the floor then the locus of its middle point is -
(A) $x^2 + y^2 = 2.5$ (B) $x^2 + y^2 = 25$ (C) $x^2 + y^2 = 100$ (D) none

10. The equation of the line cutting an intercept of 3 units on negative y-axis and inclined at an angle $\tan^{-1}\frac{3}{5}$ to the x-axis is -
(A) $5y - 3x + 15 = 0$ (B) $5y - 3x = 15$ (C) $3y - 5x + 15 = 0$ (D) none of these

11. The equation of a straight line which passes through the point (–3, 5) such that the portion of it between the axes is divided by the point in the ratio 5 : 3, internally (reckoning from x-axis) will be -
(A) $x + y - 2 = 0$ (B) $2x + y + 1 = 0$ (C) $x + 2y - 7 = 0$ (D) $x - y + 8 = 0$

12. The points $\left(0, \frac{8}{3}\right)$, (1, 3) and (82, 30) are vertices of- [IIT-JEE 1986]
(A) an obtuse angled triangle (B) an acute angled triangle
(C) a right angled triangle (D) an isosceles triangle

13. The straight lines $x + y = 0$, $3x + y - 4 = 0$, $x + 3y - 4 = 0$ form a triangle which is- [IIT-JEE 1983]
(A) isosceles (B) equilateral (C) right angled (D) none of these

14. The co-ordinates of the vertices P, Q, R & S of square PQRS inscribed in the triangle ABC with vertices $A \equiv (0, 0)$, B (3, 0) & $C \equiv (2, 1)$ given that two of its vertices P, Q are on the side AB are respectively :
(A) $\left(\frac{1}{4}, 0\right), \left(\frac{3}{8}, 0\right), \left(\frac{3}{8}, \frac{1}{8}\right)$ & $\left(\frac{1}{4}, \frac{1}{8}\right)$ (B) $\left(\frac{1}{2}, 0\right), \left(\frac{3}{4}, 0\right), \left(\frac{3}{4}, \frac{1}{4}\right)$ & $\left(\frac{1}{2}, \frac{1}{4}\right)$
(C) $(1, 0), \left(\frac{3}{2}, 0\right), \left(\frac{3}{2}, \frac{1}{2}\right)$ & $\left(1, \frac{1}{2}\right)$ (D) $\left(\frac{3}{2}, 0\right), \left(\frac{9}{4}, 0\right), \left(\frac{9}{4}, \frac{3}{4}\right)$ & $\left(\frac{3}{2}, \frac{3}{4}\right)$

15. The equation of perpendicular bisector of the line segment joining the points (1, 2) and (–2, 0) is -

(A) $5x + 2y = 1$ (B) $4x + 6y = 1$ (C) $6x + 4y = 1$ (D) none of these

16. The number of possible straight lines, passing through (2, 3) and forming a triangle with coordinate axes, whose area is 12 sq. units, is -

(A) one (B) two (C) three (D) four

17. Points A & B are in the first quadrant ; point 'O' is the origin. If the slope of OA is 1, slope of OB is 7 and OA = OB, then the slope of AB is -

(A) $-1/5$ (B) $-1/4$ (C) $-1/3$ (D) $-1/2$

18. A line is perpendicular to $3x + y = 3$ and passes through a point (2, 2). Its y intercept is -

(A) 2/3 (B) 1/3 (C) 1 (D) 4/3

19. The equation of the line passing through the point (c, d) and parallel to the line $ax + by + c = 0$ is -

(A) $a(x + c) + b(y + d) = 0$ (B) $a(x + c) - b(y + d) = 0$ (C) $a(x - c) + b(y - d) = 0$ (D) none of these

20. The position of the point (8,–9) with respect to the lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$ is -

(A) point lies on the same side of the lines
(B) point lies on one of the lines
(C) point lies on the different sides of the line
(D) point lies between the lines

21. If origin and (3, 2) are contained in the same angle of the lines $2x + y - a = 0$, $x - 3y + a = 0$, then 'a' must lie in the interval -

(A) $(-\infty, 0) \cup (8, \infty)$ (B) $(-\infty, 0) \cup (3, \infty)$ (C) (0, 3) (D) (3, 8)

22. The line $3x + 2y = 6$ will divide the quadrilateral formed by the lines $x + y = 5$, $y - 2x = 8$, $3y + 2x = 0$ & $4y - x = 0$ in -

(A) two quadrilaterals
(B) one pentagon and one triangle
(C) two triangles
(D) one triangle and one quadrilateral

23. If the point (a, 2) lies between the lines $x - y - 1 = 0$ and $2(x - y) - 5 = 0$, then the set of values of a is -

(A) $(-\infty, 3) \cup (9/2, \infty)$ (B) (3, 9/2) (C) $(-\infty, 3)$ (D) $(9/2, \infty)$

24. $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are three non-collinear points in cartesian plane. Number of parallelograms that can be drawn with these three points as vertices is -

(A) one (B) two (C) three (D) four

25. If P = (1,0) ; Q = (–1,0) & R = (2,0) are three given points, then the locus of the points S satisfying the relation, $SQ^2 + SR^2 = 2\ SP^2$ is -

(A) A straight line parallel to x-axis
(B) A circle passing through the origin
(C) A circle with the centre at the origin
(D) A straight line parallel to y-axis

26. The area of triangle formed by the lines $x + y - 3 = 0$, $x - 3y + 9 = 0$ and $3x - 2y + 1 = 0$ is -

(A) $\frac{16}{7}$ sq. units (B) $\frac{10}{7}$ sq. units (C) 4 sq. units (D) 9 sq. units

27. The co-ordinates of foot of the perpendicular drawn on line $3x - 4y - 5 = 0$ from the point (0, 5) is -

(A) (1, 3) (B) (2, 3) (C) (3, 2) (D) (3, 1)

28. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is- **[JEE 1992]**

(A) square (B) circle (C) straight line (D) two intersecting lines

29. Distance of the point (2, 5) from the line $3x + y + 4 = 0$ measured parallel to the line $3x - 4y + 8 = 0$ is -

(A) 15/2 (B) 9/2 (C) 5 (D) none

30. Three vertices of triangle ABC are A(–1, 11), B(–9, –8) and C(15, –2). The equation of angle bisector of angle A is -

(A) $4x - y = 7$ (B) $4x + y = 7$ (C) $x + 4y = 7$ (D) $x - 4y = 7$

31. Given the four lines with the equations

$$x + 2y - 3 = 0,\ 3x + 4y - 7 = 0$$
$$2x + 3y - 4 = 0,\ 4x + 5y - 6 = 0$$

then **[JEE 1980]**

(A) they are all concurrent (B) they are the sides of a quadrilateral

(C) only three lines are concurrent (D) none of the above

32. The co-ordinates of the point of reflection of the origin (0, 0) in the line $4x - 2y - 5 = 0$ is -

(A) (1, –2) (B) (2, –1) (C) $\left(\frac{4}{5},-\frac{2}{5}\right)$ (D) (2, 5)

33. If the axes are rotated through an angle of 30 in the anti-clockwise direction, the coordinates of point $(4,-2\sqrt{3})$ with respect to new axes are-

(A) $(2,\sqrt{3})$ (B) $(\sqrt{3},-5)$ (C) (2, 3) (D) $(\sqrt{3},2)$

34. If one diagonal of a square is along the line x = 2y and one of its vertex is (3, 0), then its sides through this vertex are given by the equations -

(A) $y - 3x + 9 = 0,\ x - 3y - 3 = 0$ (B) $y - 3x + 9 = 0,\ x - 3y - 3 = 0$

(C) $y + 3x - 9 = 0,\ x + 3y - 3 = 0$ (D) $y - 3x + 9 = 0,\ x + 3y - 3 = 0$

35. The line $(p + 2q)x + (p - 3q)y = p - q$ for different values of p and q passes through a fixed point whose co-ordinates are -

(A) $\left(\frac{3}{2},\frac{5}{2}\right)$ (B) $\left(\frac{2}{5},\frac{2}{5}\right)$ (C) $\left(\frac{3}{5},\frac{3}{5}\right)$ (D) $\left(\frac{2}{5},\frac{3}{5}\right)$

36. The equation $2x^2 + 4xy - py^2 + 4x + qy + 1 = 0$ will represent two mutually perpendicular straight lines, if -

(A) p=1 and q = 2 or 6 (B) p = – 2 and q = – 2 or 8

(C) p = 2 and q = 0 or 8 (D) p = 2 and q = 0 or 6

37. Equation of the pair of straight lines through origin and perpendicular to the pair of straight lines $5x^2 - 7xy - 3y^2 = 0$ is -

(A) $3x^2 - 7xy - 5y^2 = 0$ (B) $3x^2 + 7xy + 5y^2 = 0$ (C) $3x^2 - 7xy + 5y^2 = 0$ (D) $3x^2 + 7xy - 5y^2 = 0$

38. If the straight lines joining the origin and the points of intersection of the curve $5x^2 + 12xy - 6y^2 + 4x - 2y + 3 = 0$ and $x + ky - 1 = 0$ are equally inclined to the co-ordinate axis, then the value of k -

(A) is equal to 1 (B) is equal to –1 (C) is equal to 2 (D) does not exist in the set of real numbers

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

39. Coordinates of a point which is at 3 units distance from the point (1, –3) on the line $2x + 3y + 7 = 0$ is/are -

(A) $\left(1+\frac{9}{\sqrt{13}},3-\frac{6}{\sqrt{13}}\right)$ (B) $\left(1-\frac{9}{\sqrt{13}},-3+\frac{6}{\sqrt{13}}\right)$ (C) $\left(1+\frac{9}{\sqrt{13}},-3-\frac{6}{\sqrt{13}}\right)$ (D) $\left(1-\frac{9}{\sqrt{13}},3-\frac{6}{\sqrt{13}}\right)$

40. The angle between the lines $y - x + 5 = 0$ and $\sqrt{3}x - y + 7 = 0$ is/are -

(A) 15 (B) 60 (C) 165 (D) 75

41. If line $y - x + 2 = 0$ is shifted parallel to itself towards the x-axis by a perpendicular distance of $3\sqrt{2}$ units, then the equation of the new line is may be -

(A) $y = x + 4$ (B) $y = x + 1$ (C) $y = x - (2 + 3\sqrt{2})$ (D) $y = x - 8$

42. Three lines $px + qy + r = 0$, $qx + ry + p = 0$ and $rx + py + q = 0$ are concurrent if - **[JEE 1985]**

(A) $p + q + r = 0$ (B) $p^2 + q^2 + r^2 = pr + qr + pq$

(C) $p^3 + q^3 + r^3 = 3pqr$ (D) none of these

43. All points lying inside the triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy - **[JEE 1986]**

(A) $3x + 2y \geq 0$ (B) $2x + y - 13 \geq 0$ (C) $2x - 3y - 12 \leq 0$ (D) $-2x + y \geq 0$

44. The diagonals of a square are along the pair of lines whose equation is $2x^2 - 3xy - 2y^2 = 0$. If (2, 1) is a vertex of the square, then the vertex of the square adjacent to it may be -

(A) (1, 4) (B) (–1, –4) (C) (–1, 2) (D) (1, –2)

45. Equation of two equal sides of a triangle are the lines $7x + 3y - 20 = 0$ and $3x + 7y - 20 = 0$ and the third side passes through the point (–3, 3), then the equation of the third side can be -

(A) $x + y = 0$ (B) $x - y + 6 = 0$ (C) $x + 3 = 0$ (D) $y = 3$

CHECK YOUR GRASP				ANSWER KEY					EXERCISE-1
Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	A	C	A	D	C	C	D	B	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	D	A	D	C	C	D	D	C	A
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	A	A	B	C	D	B	D	A	C	B
Que.	31	32	33	34	35	36	37	38	39	40
Ans.	C	B	B	D	D	C	A	B	B,C	A,C
Que.	41	42	43	44	45					
Ans.	A,D	A,B,C	A,C	C,D	A,B					

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. The co-ordinates of a point P on the line $2x - y + 5 = 0$ such that $|PA - PB|$ is maximum where A is (4, – 2) and B is (2, – 4) will be -

(A) (11, 27) (B) (–11, – 17) (C) (–11, 17) (D) (0, 5)

2. The line $x + y = p$ meets the axis of x and y at A and B respectively. A triangle APQ is inscribed in the triangle OAB, O being the origin, with right angle at Q. P and Q lie respectively on OB and AB. If the area of the triangle APQ is $3/8^{th}$ of the area of the triangle OAB, then $\frac{AQ}{BQ}$ is equal to -

(A) 2 (B) 2/3 (C) 1/3 (D) 3

3. Lines, $L_1 : x + \sqrt{3}y = 2$, and $L_2 : ax + by = 1$, meet at P and enclose an angle of 45 between them. Line L_3 : $y = \sqrt{3}x$, also passes through P then -

(A) $a^2 + b^2 = 1$ (B) $a^2 + b^2 = 2$ (C) $a^2 + b^2 = 3$ (D) $a^2 + b^2 = 4$

4. A triangle is formed by the lines $2x - 3y - 6 = 0$; $3x - y + 3 = 0$ and $3x + 4y - 12 = 0$. If the points $P(\alpha,0)$ and $Q(0,\beta)$ always lie on or inside the ΔABC, then range of α & β -

(A) $\alpha \in [-1, 2]$ & $\beta \in [-2, 3]$ (B) $\alpha \in [-1, 3]$ & $\beta \in [-2, 4]$

(C) $\alpha \in [-2, 4]$ & $\beta \in [-3, 4]$ (D) $\alpha \in [-1, 3]$ & $\beta \in [-2, 3]$

5. The line $x + 3y - 2 = 0$ bisects the angle between a pair of straight lines of which one has equation $x - 7y + 5 = 0$. The equation of the other line is -

(A) $3x + 3y - 1 = 0$ (B) $x - 3y + 2 = 0$ (C) $5x + 5y - 3 = 0$ (D) none

6. A ray of light passing through the point A (1, 2) is reflected at a point B on the x-axis line mirror and then passes through (5, 3). Then the equation of AB is -

(A) $5x + 4y = 13$ (B) $5x - 4y = -3$ (C) $4x + 5y = 14$ (D) $4x - 5y = -6$

7. Let the algebraic sum of the perpendicular distances from the points (3, 0), (0, 3) & (2, 2) to a variable straight line be zero, then the line passes through a fixed point whose co-ordinates are-

(A) (3, 2) (B) (2, 3) (C) $\left(\frac{3}{5}, \frac{3}{5}\right)$ (D) $\left(\frac{5}{3}, \frac{5}{3}\right)$

8. The image of the pair of lines respresented by $ax^2 + 2h\,xy + by^2 = 0$ by the line mirror $y = 0$ is :

(A) $ax^2 - 2hxy + by^2 = 0$ (B) $bx^2 - 2h\,xy + ay^2 = 0$

(C) $bx^2 + 2h\,xy + ay^2 = 0$ (D) $ax^2 - 2h\,xy - by^2 = 0$

9. The pair of straight lines $x^2 - 4xy + y^2 = 0$ together with the line $x + y + 4\sqrt{6} = 0$ form a triangle which is :

(A) right angled but not isosscles (B) right isosceles

(C) scalene (D) equilateral

10. Let A ≡ (3, 2) and B ≡ (5, 1). ABP is an equilateral triangle is constructed on the side of AB remote from the origin then the orthocentre of triangle ABP is -

(A) $\left(4 - \frac{1}{2}\sqrt{3}, \frac{3}{2} - \sqrt{3}\right)$ (B) $\left(4 + \frac{1}{2}\sqrt{3}, \frac{3}{2} + \sqrt{3}\right)$ (C) $\left(4 - \frac{1}{6}\sqrt{3}, \frac{3}{2} - \frac{1}{3}\sqrt{3}\right)$ (D) $\left(4 + \frac{1}{6}\sqrt{3}, \frac{3}{2} + \frac{1}{3}\sqrt{3}\right)$

11. The line PQ whose equation is $x - y = 2$ cuts the x axis at P and Q is (4,2). The line PQ is rotated about P through 45 in the anticlockwise direction. The equation of the line PQ in the new position is -

(A) $y = -\sqrt{2}$ (B) $y = 2$ (C) $x = 2$ (D) $x = -2$

12. Distance between two lines respresented by the line pair, $x^2 - 4xy + 4y^2 + x - 2y - 6 = 0$ is -

(A) $\frac{1}{\sqrt{5}}$ (B) $\sqrt{5}$ (C) $2\sqrt{5}$ (D) none

13. The circumcentre of the triangle formed by the lines, $xy + 2x + 2y + 4 = 0$ and $x + y + 2 = 0$ is -

(A) (–1, –1) (B) (–2, –2) (C) (0, 0) (D) (–1, –2)

14. Area of the rhombus bounded by the four lines, $ax \pm by \pm c = 0$ is -

(A) $\frac{c^2}{2ab}$ (B) $\frac{2c^2}{ab}$ (C) $\frac{4c^2}{ab}$ (D) $\frac{ab}{4c^2}$

15. If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ & $x + y + c = 0$ where a, b & c are distinct real numbers different from 1 are concurrent, then the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=$

(A) 4 (B) 3 (C) 2 (D) 1

16. If one vertex of an equilateral triangle of side 'a' lies at the origin and the other lies on the line $x-\sqrt{3}y=0$, then the co-ordinates of the third vertex are -

(A) (0, a) (B) $\left(\frac{\sqrt{3}}{2}a, -\frac{a}{2}\right)$ (C) (0, –a) (D) $\left(-\frac{\sqrt{3}}{2}a, \frac{a}{2}\right)$

17. The area enclosed by $2|x| + 3|y| \le 6$ is -

(A) 3 sq. units (B) 4 sq. units (C) 12 sq. units (D) 24 sq. units

18. The point (4, 1) undergoes the following three transformations successively -

(i) Reflection about the line $y = x$

(ii) Translation through a distance 2 units along the positive directions of x-axis.

(iii) Rotation through an angle $\pi/4$ about the origin.

The final position of the point is given by the coordinates :

(A) $\left(\frac{7}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ (B) $\left(\frac{7}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (C) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (D) none of these

19. If the equation $ax^2 - 6xy + y^2 + bx + cx + d = 0$ represents a pair of lines whose slopes are m and m^2, then value(s) of a is/are -

(A) a = –8 (B) a = 8 (C) a = 27 (D) a = –27

20. Given the family of lines, $a(3x + 4y + 6) + b(x + y + 2) = 0$. The line of the family situated at the greatest distance from the point P (2,3) has equation -

(A) $4x + 3y + 8 = 0$ (B) $5x + 3y + 10 = 0$ (C) $15x + 8y + 30 = 0$ (D) none

21. If the vertices P, Q, R of a triangle PQR are rational points, which of the following points of the triangle PQR is/are always rational point (s) ? **[JEE 98]**

(A) centriod (B) incentre (C) circumcentre (D) orthocentre

22. Let PQR be a right angled isosceles triangle, right angled at P (2, 1). If the equation of the line QR is $2x + y = 3$, then the equation representing the pair of lines PQ and PR is - **[JEE 99]**

(A) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$ (B) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$

(C) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$ (D) $3x^2 - 3y^2 - 8xy - 10x - 15y - 20 = 0$

BRAIN TEASERS — ANSWER KEY — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	D	B	D	C	A	D	A	D	D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	C	B	A	B	D	A,B,C,D	C	C	B,D	A
Que.	21	22								
Ans.	A,C,D	B								

EXERCISE-03
MISCELLANEOUS TYPE QUESTIONS

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1.

	Column-I		Column-II
(A)	If $3a - 2b + 5c = 0$, then family of straight lines $ax + by + c = 0$ are always concurrent at a point whose co-ordinates is (a, b), then the values of $a - 5b$	(p)	$3\sqrt{2}$
(B)	Number of integral values of b for which the origin and the point (1, 1) lie on the same side of the straight line $a^2x + aby + 1 = 0$ for all $a \in R - \{0\}$ is	(q)	5
(C)	Vetices of a right angled triangle lie on a circle and extrimites of whose hypotenuse are (6, 0) and (0, 6), then radius of circle is	(r)	12
(D)	If the slope of one of the lines represented by $ax^2 - 6xy + y^2 = 0$ is square of the other, then a is	(s)	3
		(t)	8

2.

	Column-I		Column-II
(A)	Two adjacent sides of a parallogram are $4x + 5y = 0$ and $7x + 2y = 0$ and one diagonal is $ax + by + c = 0$, then $a + b + c$ is equal to	(p)	1
(B)	If line $2x - by + 1 = 0$ intersects the curve $2x^2 - by^2 + (2b - 1)xy - x - by = 0$ at points A & B and AB subtends a right angle at origin, then value of $b + b^2$ is equal to	(q)	0
(C)	A line passes through point (3, 4) and the point of intersection of the lines $4x + 3y = 12$ and $3x + 4y = 12$ and length of intercepts on the co-ordinate axes are a and b, then ab is equal to	(r)	5
(D)	A light ray emerging from the point source placed at P(2, 3) is reflected at a point 'Q' on the y-axis and then passes through the point R(5, 10). If co-ordinates of Q are (a, b), then a + b is	(s)	4

ASSERTION & REASON

These questions contain, Statement-I (assertion) and Statement-II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for Statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. Let $L_1 : a_1x + b_1y + c_1 = 0$, $L_2 : a_2x + b_2y + c_2 = 0$ and $L_3 : a_3x + b_3y + c_3 = 0$.

Statement-I : If L_1, L_2 and L_3 are three concurrent lines, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Because

Statement-II : If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then the lines L_1, L_2 and L_3 must be concurrent.

(A) A (B) B (C) C (D) D

2. **Statement-I** : The diagonals of the parallelogram whose sides are $\ell x + my + n = 0$, $\ell x + my + n' = 0$, $mx + \ell y + n = 0$, $mx + \ell y + n' = 0$ are perpendicular.

Because

Statement-II : If the perpedicular distances between parallel sides of a parallelogram are equal, then it is a rhombus.

(A) A (B) B (C) C (D) D

3. **Statement-I** : The equation $2x^2 + 3xy - 2y^2 + 5x - 5y + 3 = 0$ represents a pair of perpendicular straight lines.

Because

Statement-II : A pair of lines given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular, if $a + b = 0$

(A) A (B) B (C) C (D) D

4. **Statement-I** : The joint equation of lines $2y = x+1$ and $2y = -(x+1)$ is $4y^2 = -(x + 1)^2$.

Because

Statement-II : The joint equation of two lines satisfy every point lying on any one of the line.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1 :

A locus is the curve traced out by a point which moves under certain geomatrical conditions:

To find the locus of a point first we assume the co-ordinates of the moving point as (h,k) and then try to find a relation between h and k with the help of the given conditions of the problem. If there is any variable involved in the process then we eliminate them. At last we replace h by x and k by y and get the locus of the point which will be an equation in x and y.

On the basis of above information, answer the following questions :

1. Locus of centroid of the triangle whose vertices are $(a\cos t, a\sin t)$, $(b\sin t, -b\cos t)$ and $(1, 0)$ where t is a parameter is -

(A) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$ (B) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$

(C) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$ (D) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$

2. A variable line cuts x-axis at A, y-axis at B where OA = a, OB = b (O as origin) such that $a^2 + b^2 = 1$ then the locus of circumcentre of Δ OAB is -

(A) $x^2 + y^2 = 4$ (B) $x^2 + y^2 = 1/4$ (C) $x^2 - y^2 = 4$ (D) $x^2 - y^2 = 1/4$

3. The locus of the point of intersection of the lines $x \cos\alpha + y \sin\alpha = a$ and $x \sin \alpha - y \cos \alpha = b$ where α is variable is -

(A) $x^2 + y^2 = a^2 + b^2$ (B) $x^2 + y^2 = a^2 - b^2$ (C) $x^2 - y^2 = a^2 - b^2$ (D) $x^2 - y^2 = a^2 + b^2$

Comprehension # 2 :

For points $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ of the coordinate plane, a new distance d(P, Q) is defined by

$$d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$$

Let $O \equiv (0, 0)$, $A \equiv (1, 2)$, $B \equiv (2, 3)$ and $C \equiv (4, 3)$ are four fixed points on $x - y$ plane.

On the basis of above information, answer the following questions :

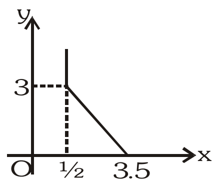
1. Let R(x, y), such that R is equidistant from the points O and A with respect to new distance and if $0 \le x < 1$ and $0 \le y < 2$, then R lies on a line segment whose equation is -

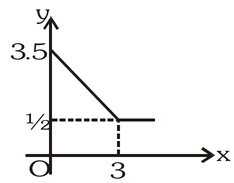
(A) $x + y = 3$ (B) $x + 2y = 3$ (C) $2x + y = 3$ (D) $2x + 2y = 3$

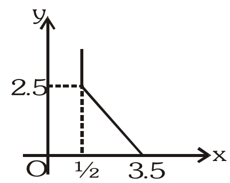
2. Let S(x, y), such that S is equidistant from points O and B with respect to new distance and if $x \geq 2$ and $0 \leq y < 3$, then locus of S is -

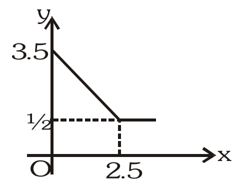
(A) a line segment
(B) a line
(C) a vertical ray
(D) a horizontal ray

3. Let T(x, y), such that T is equidistant from point O and C with respect to new distance and if T lies in first quadrant, then T consists of the union of a line segment of finite length and an infinite ray whose labelled diagram is -

(A) 

(B) 

(C) 

(D) 

MISCELLANEOUS TYPE QUESTION — **ANSWER KEY** — EXERCISE-3

- ***Match the Column***

1. (A)→(q); (B)→(s); (C)→(p); (D)→(t)
2. (A)→(q); (B)→(s); (C)→(p); (D)→(r)

- ***Assertion & Reason***

1. C 2. A 3. D 4. D

- ***Comprehension Based Questions***

Comprehension # 1 : 1. B 2. B 3. A

Comprehension # 2 : 1. D 2. D 3. A

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

1. Determie the ratio in which the point P(3, 5) divides the join of A(1, 3) & B(7, 9). Find the harmonic conjugate of P w.r.t. A & B.

2. The area of a triangle is 5. Two of its vertices are (2, 1) & (3, − 2). The third vertex lies on y = x + 3. Find the third vertex.

3. Two vertices of a triangle are (4, − 3) & (– 2, 5). If the orthocentre of the triangle is at (1, 2), find the coordinates of the third vertex.

4. The line 3x + 2y = 24 meets the y-axis at A & the x-axis at B. The perpendicular bisector of AB meets the line through (0,–1) parallel to x-axis at C. Find the area of the triangle ABC.

5. A line is such that its segment between the straight lines 5x − y − 4 = 0 and 3x + 4y − 4 = 0 is bisected at the point (1, 5). Obtain the equation.

6. A straight line L is perpendicular to the line 5x − y = 1. The area of the triangle formed by the line L & the coordinate axes is 5. Find the equation of the line.

7. A line cuts the x-axis at A(7, 0) and the y-axis at B(0, –5). A variable line PQ is drawn perpendicular to AB cutting the x-axis in P and the y-axis in Q. If AQ and BP intersect at R, find the locus of R.

[IIT-JEE 1990]

8. The vertices of a triangle OBC are O(0, 0), B (–3, –1), C (–1, –3). Find the equation of the line parallel to BC & intersecting the sides OB & OC, whose perpendicular distance from the point (0, 0) is half.

9. If the straight line drawn through the point $P\left(\sqrt{3},2\right)$ & making an angle $\frac{\pi}{6}$ with the x-axis, meets the line $\sqrt{3}x-4y+8=0$ at Q. Find the length PQ.

10. The points (1, 3) & (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line y = 2x + c. Find c & the remaining vertices.

11. If a, b, c are all different and the points $\left(\frac{r^3}{r-1},\frac{r^2-3}{r-1}\right)$ where r = a, b, c are collinear, then prove that 3(a + b + c) = ab + bc + ca − abc.

12. Two sides of a rhombus ABCD are parallel to the lines y = x + 2 and y = 7x + 3. If the diagonals of the rhombus intersect at the point (1, 2) and the vertex A is on the y-axis. find possible co-ordinates of A. **[IIT-JEE 1985]**

13. Find the direction in which a straight line may be drawn through the point (2, 1) so that its point of intersection with the line 4y − 4x + 4 + $3\sqrt{2}$ + $3\sqrt{10}$ = 0 is at a distance of 3 unit from (2, 1).

14. Straight lines 3x + 4y = 5 and 4x − 3y = 15 intersect at the point A. Points B and C are chosen on these two lines such that AB = AC. Determine the possible equations of the line BC passing through the point (1, 2) **[IIT-JEE 1990]**

15. Find the equation of the line which bisects the obtuse angle between the lines x − 2y + 4 = 0 and 4x − 3y + 2 = 0. **[IIT-JEE 1978]**

16. A line through A (–5, –4) meets the line x + 3y + 2 = 0, 2x + y + 4 = 0 and x − y − 5 = 0 at the points B, C & D respectively, if $\left(\frac{15}{AB}\right)^2+\left(\frac{10}{AC}\right)^2=\left(\frac{6}{AD}\right)^2$. Find the equation of the line.

17. A variable line, drawn through the point of intersection of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ & $\frac{x}{b}+\frac{y}{a}=1$, meets the coordinate axes in A & B. Show that the locus of the mid point of AB is the curve 2xy (a + b) = ab (x + y).

18. In a triangle ABC, D is a point on BC such that $\frac{BD}{DC}=\frac{AB}{AC}$. The equation of the line AD is 2x + 3y + 4 = 0 & the equation of the line AB is 3x + 2y + 1 = 0. Find the equation of the line AC.

19. Show that all the chords of the curve $3x^2 + 3y^2 - 2x + 4y = 0$ which subtend a right angle at the origin are concurrent. Also find the point of concurrency.

ANSWER KEY

CONCEPTUAL SUBJECTIVE EXERCISE — EXERCISE-4(A)

1. 1 : 2 ; Q (–5, –3) **2.** $\left(\frac{7}{2}, \frac{13}{2}\right)$ or $\left(-\frac{3}{2}, \frac{3}{2}\right)$ **3.** (33, 26) **4.** 91 sq. units

5. 83x – 35y + 92 = 0 **6.** $x + 5y + 5\sqrt{2} = 0$ or $x + 5y - 5\sqrt{2} = 0$

7. $x^2 + y^2 - 7x + 5y = 0$ **8.** $2x + 2y + \sqrt{2} = 0$ **9.** 6 units

10. C = – 4 ; B (2, 0) ; D (4, 4) **12.** $\left(0,\frac{5}{2}\right),(0,0)$ **13.** 171 , 99 **14.** x– 7y + 13 = 0 and 7x + y – 9 = 0

15. $\left(4+\sqrt{5}\right)x-\left(2\sqrt{5}+3\right)y+\left(4\sqrt{5}+2\right)=0$ **16.** 2x + 3y + 22 = 0 **18.** 9x + 46y + 83 = 0

19. $\left(\frac{1}{3}, -\frac{2}{3}\right)$

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. Find the equation of the straight lines passing through $(-2, -7)$ & having an intercept of length 3 between the straight lines $4x + 3y = 12$, $4x + 3y = 3$.

2. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines

$2x + 3y - 1 = 0$; $x + 2y - 3 = 0$; $5x - 6y - 1 = 0$.

3. Find the co-ordinates of the orthocentre of the triangle, the equations of whose sides are $x + y = 1$, $2x + 3y = 6$, $4x - y + 4 = 0$, without finding the co-ordinates of its vertices.

4. Let ABC be a triangle with AB = AC. If D is the midpoint of BC, E is the foot of the perpendicular drawn from D to AC and F the mid-point of DE, prove that AF is perpendicular to BE. **[IIT-JEE 1989]**

5. Find the condition that the diagonals of the parallelogram formed by the lines

$ax + by + c = 0$; $ax + by + c' = 0$; $a'x + b'y + c = 0$ & $a'x + b'y + c' = 0$ are at right angles. Also find the equation to the diagonals of the parallelogram.

6. Find the co-ordinates of the incentre of the triangle formed by the line $x + y + 1 = 0$; $x - y + 3 = 0$ & $7x - y + 3 = 0$. Also find the centre of the circle escribed to $7x - y + 3 = 0$.

7. Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv \ell x + my + n = 0$ intersect at the point P and makes an angle θ with each other. Find the equation of a line L different from L_2 which passes through P and makes the same angle θ with L_1 **[IIT-JEE 1988]**

8. A triangle is formed by the lines whose equations are AB : $x + y - 5 = 0$, BC : $x + 7y - 7 = 0$ and CA : $7x + y - 14 = 0$. Find the bisector of the interior angle at B and the exterior angle at C. Determine the nature of the interior angle at A and find the equation of the bisector.

9. The distance of a point (x_1, y_1) from each of two straight lines which passes through the origin of co-ordinates is δ ; find the combined equation of these straight lines.

10. Equation of a line is given by $y + 2at = t(x - at^2)$, t being the parameter. Find the locus of the point intersection of the lines which are at right angles.

11. A rectangle PQRS has its side PQ parallel to the line $y = mx$ and vertices P,Q and S on the lines $y = a$, $x = b$ and $x = -b$, respectively. Find the locus of the vertex R. **[IIT-JEE 1996]**

12. A variable straight line of slope 4 intersects the hyperbola $xy=1$ at two points. Find the locus of the point which divides the line segment between these two points in the ratio 1 : 2. **[IIT-JEE 1997]**

13. The vertices of a triangle are $A(x_1, x_1 \tan\theta_1)$, $B(x_2, x_2 \tan\theta_2)$ & $C(x_3, x_3 \tan\theta_3)$. If the circumcentre O of the triangle ABC is at the origin & $H(\bar{x}, \bar{y})$ be its orthocentre, then show that $\dfrac{\bar{x}}{\bar{y}} = \dfrac{\cos\theta_1 + \cos\theta_2 + \cos\theta_3}{\sin\theta_1 + \sin\theta_2 + \sin\theta_3}$.

14. The ends A, B of a straight line segment of constant length 'c' slide upon the fixed rectangular axes OX & OY respectively. If the rectangle OAPB be completed then show that the locus of the foot of the perpendicular drawn from P to AB is $x^{2/3} + y^{2/3} = c^{2/3}$.

BRAIN STORMING SUBJECTIVE EXERCISE **ANSWER KEY** EXERCISE-4(B)

1. $7x + 24y + 182 = 0$ or $x = -2$ **2.** $-\dfrac{3}{2} < \alpha < -1 \cup \dfrac{1}{2} < \alpha < 1$ **3.** $\left(\dfrac{3}{7}, \dfrac{22}{7}\right)$

5. $a^2 + b^2 = a'^2 + b'^2$; $(a + a')x + (b + b')y + (c + c') = 0$; $(a - a')x + (b - b')y = 0$ **6.** $(-1, 1)$; $(4, 1)$

7. $2(a\ell + bm)(ax + by + c) - (a^2 + b^2)(\ell x + my + n) = 0$

8. $3x + 6y - 16 = 0$; $8x + 8y - 21 = 0$; acute angle bisector, $12x + 6y - 39 = 0$

9. $(y_1^2 - \delta^2)x^2 - 2x_1y_1xy + (x_1^2 - \delta^2)y^2 = 0$ **10.** $y^2 = a(x - 3a)$

11. $(m^2 - 1)x - my + b(m^2 + 1) + am = 0$ **12.** $16x^2 + y^2 + 10xy = 2$

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. The angle between the straight lines $x^2 + 4xy + y^2 = 0$ is- [AIEEE 2002]
(1) 30 (2) 45 (3) 60 (4) 90

2. The distance between a pair of parallel lines $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ [AIEEE 2002]
(1) 5 (2) 8 (3) 8/5 (4) 5/8

3. A square of sides a lies above the x-axis and has one vertex at the origin. The side passing through the origin makes an angle α $(0 < \alpha < \pi/4)$ with the positive direction of x-axis.The equation of its diagonal not passing through the origin is- [AIEEE 2003]
(1) $y(\cos\alpha + \sin\alpha) + x(\cos\alpha - \sin\alpha) = a$
(2) $y(\cos\alpha - \sin\alpha) - x(\sin\alpha - \cos\alpha) = a$
(3) $y(\cos\alpha + \sin\alpha) + x(\sin\alpha - \cos\alpha) = a$
(4) $y(\cos\alpha + \sin\alpha) + x(\sin\alpha + \cos\alpha) = a$

4. If the pair of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then- [AIEEE 2003]
(1) $pq = -1$
(2) $p = q$
(3) $p = -q$
(4) $pq = 1$

5. Locus of centroid of the triangle whose vertices are (a cos t, a sin t), (b sin t, – b cos t) and (1,0), where t is a parameter, is- [AIEEE 2003]
(1) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$
(2) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$
(3) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
(4) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$

6. If the equation of the locus of a point equidistant from the points (a_1,b_1) and (a_2,b_2) is $(a_1-a_2)x + (b_1-b_2)y + c = 0$, then the value of 'c' is- [AIEEE 2003]
(1) $\sqrt{a_1^2+b_1^2-a_2^2-b_2^2}$
(2) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$
(3) $a_1^2 - a_2^2 + b_1^2 - b_2^2$
(4) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_1^2)$

7. The equation of the straight line passing through the point (4,3) and making intercepts on the coordinate axes whose sum is –1 is- [AIEEE 2004]
(1) $\frac{x}{2}+\frac{y}{3}=-1$ and $\frac{x}{-2}+\frac{y}{1}=-1$
(2) $\frac{x}{2}-\frac{y}{3}=-1$ and $\frac{x}{-2}+\frac{y}{1}=-1$
(3) $\frac{x}{2}+\frac{y}{3}=1$ and $\frac{x}{2}+\frac{y}{1}=1$
(4) $\frac{x}{2}-\frac{y}{3}=1$ and $\frac{x}{-2}+\frac{y}{1}=1$

8. If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then c has the value- [AIEEE 2004]
(1) 1 (2) –1 (3) 2 (4) –2

9. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then c equals- [AIEEE 2004]
(1) 1 (2) –1 (3) 3 (4) –3

10. The line parallel to the x-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ is, (where $(a,b) \neq (0,0)$) [AIEEE 2005]
(1) below the x-axis at a distance of $\frac{3}{2}$ from it
(2) below the x-axis at a distance of $\frac{2}{3}$ from it
(3) above the x-axis at a distance of $\frac{3}{2}$ from it
(4) above the x-axis at a distance of $\frac{2}{3}$ from it

11. If non-zero numbers a,b,c are in H.P., then the straight line $\frac{x}{a}+\frac{y}{b}+\frac{1}{c}=0$ always passes through a fixed point that point is- **[AIEEE 2005]**

(1) (–1,2) (2) (–1,–2) (3) (1,–2) (4) $\left(1,-\frac{1}{2}\right)$

12. A straight line passing through the point A(3,4) is such that its intercept between the axes is bisected at A. Then its equation is- **[AIEEE 2006]**

(1) $3x - 4y + 7 = 0$ (2) $4x + 3y = 24$ (3) $3x + 4y = 25$ (4) $x + y = 7$

13. If (a,a^2) falls inside the angle made by the lines $y=\frac{x}{2}$, $x > 0$ and $y = 3x$, $x > 0$, then a belongs to- **[AIEEE 2006]**

(1) $(3,\infty)$ (2) $\left(\frac{1}{2},3\right)$ (3) $\left(-3,-\frac{1}{2}\right)$ (4) $\left(0,\frac{1}{2}\right)$

14. Let P(–1,0) Q(0,0) and R(3, $3\sqrt{3}$) be three points. The equation of the bisector of the angle PQR is- **[AIEEE 2007], [IIT Scr. 2002]**

(1) $\sqrt{3}x + y = 0$ (2) $x+\frac{\sqrt{3}}{2}y=0$ (3) $\frac{\sqrt{3}}{2}x+y=0$ (4) $x + \sqrt{3}y = 0$

15. If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines xy=0, then m is- **[AIEEE 2007]**

(1) $-\frac{1}{2}$ (2) –2 (3) 1 (4) 2

16. The perpendicular bisector of the line segment joining P (1, 4) and Q (k, 3) has y–intercept –4. Then a possible value of k is- **[AIEEE 2008]**

(1) 1 (2) 2 (3) –2 (4) –4

17. The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2x + (p^2 + 1)y + 2q = 0$ are **[AIEEE 2009]**

Perpendicular to a common line for :

(1) Exactly two values of p (2) More than two values of p

(3) No value of p (4) Exactly one value of p

18. The line L given by $\frac{x}{5}+\frac{y}{b}=1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c}+\frac{y}{3}=1$. Then the distance between L and K is : **[AIEEE-2010]**

(1) $\frac{23}{\sqrt{15}}$ (2) $\sqrt{17}$ (3) $\frac{17}{\sqrt{15}}$ (4) $\frac{23}{\sqrt{17}}$

19. The lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.

Statement - 1 : The ratio PR : RQ equals $2\sqrt{2}:\sqrt{5}$

Statement - 2 : In any triangle, bisector of an angle divides the triangle into two similar triangles. **[AIEEE 2011]**

(1) Statement-1 is true, Statement-2 is false.

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

20. The lines x + y = |a| and ax − y = 1 intersect each other in the first quadrant. Then the set of all possible values of a is the interval : **[AIEEE 2011]**

(1) (–1, 1] (2) (0, ∞) (3) [1, ∞) (4) (–1, ∞)

21. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a triangle OPQ, where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is : **[AIEEE 2012]**

(1) $-\frac{1}{2}$ (2) $-\frac{1}{4}$ (3) –4 (4) –2

22. If the line 2x + y = k passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k equals : **[AIEEE 2012]**

(1) $\frac{11}{5}$ (2) $\frac{29}{5}$ (3) 5 (4) 6

23. A ray of light along $x+\sqrt{3}y=\sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is : **[JEE(Main)-2013]**

(1) $y=x+\sqrt{3}$ (2) $\sqrt{3}y=x-\sqrt{3}$ (3) $y=\sqrt{3}x-\sqrt{3}$ (4) $\sqrt{3}y=x-1$

24. The x-coordinate of the incentre of the triangle that has the coordinates of mid points of its sides as (0, 1)(1, 1) and (1, 0) is : **[JEE(Main)-2013]**

(1) $2+\sqrt{2}$ (2) $2-\sqrt{2}$ (3) $1+\sqrt{2}$ (4) $1-\sqrt{2}$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans	3	3	1	1	3	2	4	3	4	1	3	2	2	1	3
Que.	16	17	18	19	20	21	22	23	24						
Ans	4	4	4	1	3	4	4	2	2						

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. (a) Area of the parallelogram formed by the lines y = mx, y = mx + 1, y = nx and y = nx + 1 equals

(A) $\frac{|m+n|}{(m-n)^2}$ (B) $\frac{2}{|m+n|}$ (C) $\frac{1}{|m+n|}$ (D) $\frac{1}{|m-n|}$

(b) The number of integer values of m, for which the x co-ordinate of the point of intersection of the lines 3x + 4y = 9 and y = mx + 1 is also an integer, is

(A) 2 (B) 0 (C) 4 (D) 1

[JEE 2001 (Screening)]

2. (a) Let P = (–1, 0), Q = (0, 0) and R = (3, $3\sqrt{3}$) be three points. Then the equation of the bisector of the angle PQR is

(A) $\frac{\sqrt{3}}{2}x + y = 0$ (B) $x + \sqrt{3}y = 0$ (C) $\sqrt{3}x + y = 0$ (D) $x + \frac{\sqrt{3}}{2}y = 0$

(b) A straight line through the origin O meets the parallel lines 4x + 2y = 9 and 2x + y + 6 = 0 at points P and Q respectively. Then the point O divides the segment PQ in the ratio

(A) 1 : 2 (B) 3 : 4 (C) 2 : 1 (D) 4 : 3

(c) The area bounded by the curves y = |x| – 1 and y = –|x| + 1 is

(A) 1 (B) 2 (C) $2\sqrt{2}$ (D) 4

[JEE 2002 (Screening)]

(d) A straight line L through the origin meets the line x+ y = 1 and x + y = 3 at P and Q respectively. Through P and Q two straight lines L_1 and L_2 are drawn, parallel to 2x – y = 5 and 3x + y = 5 respectively. Lines L_1 and L_2 intersect at R. Show that the locus of R, as L varies, is a straight line. [JEE 2002 (Mains)]

(e) A straight line L with negative slope passes through the point (8,2) and cuts the positive coordinates axes at points P and Q. Find the absolute minimum value of OP + OQ, as L varies, where O is the origin. [JEE 2002 Mains, 5]

3. The area bounded by the angle bisectors of the lines $x^2 - y^2 + 2y = 1$ and the line x + y = 3, is

(A) 2 (B) 3 (C) 4 (D) 6

[JEE 2004 (Screening)]

4. The area of the triangle formed by the intersection of a line parallel to x-axis and passing through P(h,k) with the lines y = x and x + y = 2 is $4h^2$. Find the locus of the point P. [JEE 2005, Mains, 2]

5. (a) Let O(0, 0), P (3, 4), Q(6, 0) be the vertices of the triangle OPQ. The point R inside the triangle OPQ is such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are

(A) $(4/3,\ 3)$ (B) $(3,\ 2/3)$ (C) $(3,\ 4/3)$ (D) $(4/3,\ 2/3)$

(b) Lines L_1 : y – x = 0 and L_2 : 2x + y = 0 intersect the line L_3 : y + 2 = 0 at P and Q, respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.

Statement-1 : The ratio PR : RQ equals $2\sqrt{2} : \sqrt{5}$

because

Statement-2 : In any triangle, bisector of an angle divides the triangle into two similar triangles.

(A) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true. [JEE 2007, 3+3]

6. Consider the lines given by

$L_1 = x + 3y - 5 = 0$

$L_2 = 3x - ky - 1 = 0$

$L_3 = 5x + 2y - 12 = 0$

Match the statements / Expression in **Column-I** with the statements / Expressions in **Column-II** and indicate your answer by darkening the appropriate bubbles in the 4 4 matrix given in OMR.

	Column-I		Column-II
(A)	L_1, L_2, L_3 are concurrent, if	(P)	$k = -9$
(B)	One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(Q)	$k = -\frac{6}{5}$
(C)	L_1, L_2, L_3 form a triangle, if	(R)	$k = \frac{5}{6}$
(D)	L_1, L_2, L_3 do not form a triangle, if	(S)	$k = 5$

[JEE 2008, 6]

7. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i}-\hat{j}, 4\hat{i}, 3\hat{i}+3\hat{j}$ and $-3\hat{i}+2\hat{j}$ respectively.

The quadrilateral PQRS must be a

(A) parallelogram, which is neither a rhombus nor a rectangle

(B) square

(C) rectangle, but not a square

(D) rhombus, but not a square **[JEE 2010, 3]**

8. A straight line L through the point (3, –2) is inclined at an angle 60 to the line $\sqrt{3}x + y = 1$. If L also intersect the x-axis, then the equation of L is **[JEE 2011, 3 (–1)]**

(A) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$ (B) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$

(C) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$ (D) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

9. For $a > b > c > 0$, the distance between (1, 1) and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then **[JEE(Advanced) 2013, 2M]**

(A) $a + b - c > 0$ (B) $a - b + c < 0$

(C) $a - b + c > 0$ (D) $a + b - c < 0$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [B]

1. (a) D; (b) A 2. (a) C; (b) B; (c) B; (d) $x - 3y + 5 = 0$; (e) 18

3. A 4. $y = 2x + 1$, $y = -2x + 1$ 5. (a) C; (b) C

6. (A) S; (B) P,Q; (C) R ; (D) P,Q,S 7. A 8. B 9. A or C or A,C