CHAPTER

06

Ellipse

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Session 1

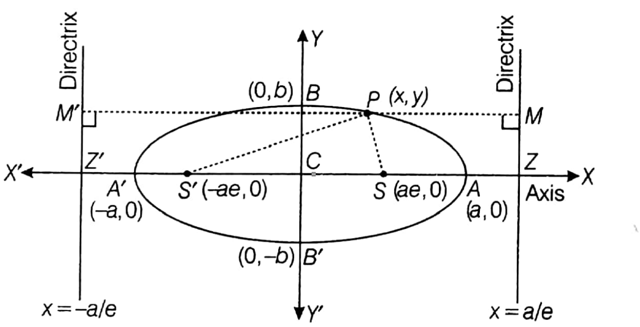
Ellipse Definition, Standard Equation of Ellipse, Tracing of the Ellipse,Focal Distances of a Point, The Shape of the Ellipse $x^2/a^2 + y^2/b^2 = 1$, When $b > a$, Mechanical Construction of an Ellipse,

Ellipse Definition

An ellipse is the locus of a point which moves in a plane such that its distance from a fixed point (i.e. focus) is a constant ratio from a fixed line (i.e. directrix).
This ratio is called eccentricity and is denoted by e.
For an ellipse $\boxed{0 < e < 1.}$

Standard Equation of Ellipse

Let S be the focus and ZM the directrix of the ellipse. Drow $SZ \perp ZM$. Divide SZ internally and externally in the ratio $e:1 (0 < e < 1)$ and let A and A' be these internal and external point of division



Then, $\quad SA = eAZ \quad$ ...(i)

and $\quad SA' = eA'Z \quad$ ...(ii)

Clearly A and A' will lie on the ellipse

Let $AA' = 2a$ and take C the mid point of AA' as origin

$\therefore \quad CA = CA' = a \quad$ ...(iii)

Let $P(x, y)$ by any point on the ellipse referred to CA and CB as co-ordinate axes

Then, adding Eqs. (i) and (ii),

$$SA + SA' = e(AZ + A'Z)$$

$$\Rightarrow \quad AA' = e(CZ - CA + CA' + CZ) \quad \text{(from figure)}$$

$$\Rightarrow \quad AA' = e(2CZ) \quad (\because CA = CA')$$

$$\Rightarrow \quad 2a = 2eCZ$$

$$\because \quad \boxed{CZ = a/e}$$

$\therefore$ The directrix MZ is $x = CZ = a/e$

or $$\boxed{\frac{a}{e} - x = 0} \quad \left(\because e < 1, \therefore \frac{a}{e} > 1\right)$$

and subtracting Eqs. (i) from (ii), then

$$SA' - SA = e(A'Z - AZ)$$

$$\Rightarrow \quad (CA' + CS) - (CA - CS) = e(AA')$$

$$\Rightarrow \quad 2CS = e(AA') \quad (\because CA = CA')$$

$$\Rightarrow \quad 2CS = e(2a)$$

$$\therefore \quad \boxed{CS = ae}$$

$\therefore$ The focus S is $(CS, 0)$ i.e. $(ae, 0)$

Now draw $PM \perp MZ$

$$\therefore \quad \frac{SP}{PM} = e \text{ or } (SP)^2 = e^2 (PM)^2$$

$$(x - ae)^2 + (y - 0)^2 = e^2 \left(\frac{a}{e} - x\right)^2$$

$$\Rightarrow \quad (x - ae)^2 + y^2 = (a - ex)^2$$

$$\Rightarrow \quad x^2 + a^2e^2 - 2aex + y^2 = a^2 - 2aex + e^2x^2$$

$$\Rightarrow \quad x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$$

$$\Rightarrow \quad \boxed{\frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or

$$\boxed{b^2=a^2(1-e^2)}$$

where,

This is the standard equation of an ellipse, AA' and BB' are called the major and minor axes of the ellipse.

(Here $b<a$) and A and A' are the vertices of the ellipse.

Remark

Two ellipse are said to be similar if they have the same value of eccentricity.

Generally,

The equation to the ellipse, whose focus is the point (h,k) and directrix is $lx+my+n=0$ and whose eccentricity is e, is

$$(x-h)^2+(y-k)^2=e^2\cdot\frac{(lx+my+n)^2}{(l^2+m^2)}$$

The Foci and Two Directrices of an Ellipse

On the negative side of origin take a point S' which is such that

$$CS=CS'=ae$$

and another point Z', then

$$CZ=CZ'=\frac{a}{e}$$

$\therefore$ Coordinates of S' are $(-ae,0)$ and equation of second directrix (i.e., $Z'M'$) is $\boxed{x=-\frac{a}{e}}$

Let $P(x,y)$ be any point on the ellipse, then

$$S'P=ePM'$$

or

$$(S'P)^2=e^2(PM')^2$$

or

$$(x+ae)^2+(y-0)^2=e^2\left(x+\frac{a}{e}\right)^2$$

or

$$(x+ae)^2+y^2=(ex+a)^2$$

or

$$x^2(1-e^2)+y^2=a^2(1-e^2)$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where,

$$b^2=a^2(1-e^2)$$

The equation being the same as that of the ellipse when $S(ae,0)$ is focus and MZ i.e. $x=a/e$ is directrix.

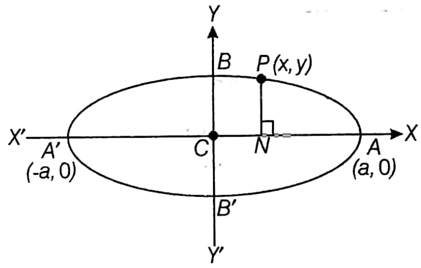
Hence, coordinates of foci are $\boxed{(\pm ae,0)}$ and equations of directrices are $\boxed{x=\pm a/e.}$

Remarks

1. Distance between foci $SS'=2ae$ and distance between directrices $ZZ'=2a/e$
2. If $e=0$
   then $\quad b^2=a^2(1-0)$
   $\therefore \quad b^2=a^2$
   then equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ changes in circle
   i.e. $\quad x^2+y^2=a^2$
3. Since, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\Rightarrow \quad \frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$$

$$\boxed{\frac{y^2}{b^2}=\frac{(a+x)(a-x)}{a^2}}$$



or $$\frac{(PN)^2}{b^2}=\frac{A'N\cdot AN}{a^2}$$

or $$\frac{(PN)^2}{AN\cdot A'N}=\frac{b^2}{a^2}=\frac{(BC)^2}{(AC)^2}$$

i.e. $$(PN)^2:AN\cdot A'N::(BC)^2:(AC)^2$$

4. The distance of every focus from the extremity of minor axis is equal to a.
   i.e. $\quad \sqrt{(a^2e^2+b^2)}=\sqrt{(a^2-b^2+b^2)}=a$

Tracing of the Ellipse

Equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

1. The ellipse (i) cuts X-axis at $A(a,0)$ and $A'(-a,0)$ and cuts Y-axis at $B(0,b)$ and $B'(0,-b)$.
2. The Eq. (i) does not change when y is replaced by $-y$ and x is replaced by $-x$, Hence ellipse (i) is symmetrical about both axes

Method of Checking Symmetricity

3. The equation (i), may be written in either of the form

$$y = \pm b\sqrt{\left(1 - \frac{x^2}{a^2}\right)} \qquad \text{...(ii)}$$

or $$x = \pm a\sqrt{\left(1 - \frac{y^2}{b^2}\right)} \qquad \text{...(iii)}$$

From Eq. (ii), it follows that (y is real)

if $$1 - \frac{x^2}{a^2} \geq 0 \text{ or } a^2 - x^2 \geq 0$$

or $$x^2 \leq a^2 \text{ or } \boxed{-a \leq x \leq a}$$

Also from Eq. (iii), it follows that (x is real)

if $$1 - \frac{y^2}{b^2} \geq 0 \text{ or } b^2 - y^2 \geq 0 \text{ or } y^2 \leq b^2$$

$\therefore$ $$\boxed{-b \leq y \leq b}$$

Ellipse (i) is a closed curve lies entirely between the lines $x = a$ and $x = -a$ and the lines $y = b$ and $y = -b$.

Since, when x increases, then y decreases from Eq. (ii) and when y decreases, then x increases from Eq. (iii).

Remark

Area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\boxed{\pi ab}$.

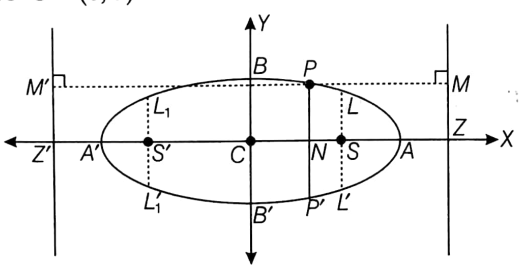
Some Terms Related to an Ellipse

Let the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$(a > b)$

1. **Centre :** All chords passing through C is bisected at C

   Here $C \equiv (0, 0)$



2. **Foci :** S and S' are two foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively.
3. **Directrices :** ZM and $Z'M'$ are two directrices of the ellipse and their equation are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.
4. **Axes :** The lines AA' and BB' are called the **major** and **minor** axes of the ellipse

$$\because \quad 0 < e < 1$$
$$\text{or} \quad 0 < e^2 < 1 \qquad (\therefore 0 > -e^2 > -1)$$
$$\text{or} \quad 0 < 1 - e^2 < 1 \qquad (\text{or } 1 > 1 - e^2 > 1 - 1)$$
$$\text{or} \quad a^2(1 - e^2) < a^2 \qquad (\text{or } 0 < 1 - e^2 < 1)$$
$$\text{or} \quad b^2 < a^2$$
$$\text{i.e.} \quad b < a$$

Remark

The major and minor axis together are called **principal axes** of the ellipse.

5. **Double ordinates :** If P be a point on the ellipse draw PN perpendicular to the axis of the ellipse and produced to meet the curve again at P'. Then PP' is called a double ordinate.

   If abscissa of P is h, then ordinate of P,

$$\frac{y^2}{b^2} = 1 - \frac{h^2}{a^2}$$

$$y = \frac{b}{a}\sqrt{(a^2 - h^2)} \qquad \text{(for first quadrant)}$$

   and ordinate of P' is

$$y = -\frac{b}{a}\sqrt{(a^2 - h^2)} \qquad \text{(for fourth quadrant)}$$

   Hence, coordinates of P and P' are

$$\left(h, \frac{b}{a}\sqrt{(a^2 - h^2)}\right) \text{ and } \left(h, -\frac{b}{a}\sqrt{(a^2 - h^2)}\right)$$

   respectively.

6. **Latusrectum** The double ordinates LL' and L_1L_1' are latusrectums of the ellipse. These line are perpendicular to major axis $A'A$ and through the foci S and S', respectively.

   Length of the latusrectum

   Now let $\quad LL' = 2k$

   then $\quad LS = L'S = k$

   Coordinates of L and L' are (ae, k) and $(ae, -k)$ lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \quad \frac{a^2e^2}{a^2} + \frac{k^2}{b^2} = 1 \text{ or } k^2 = b^2(1 - e^2)$$

$$= b^2\left(\frac{b^2}{a^2}\right) \qquad [\because b^2 = a^2(1 - e^2)]$$

$$\therefore \quad k = \frac{b^2}{a} \quad (\because k > 0)$$

$$\therefore \quad \boxed{2k = \frac{2b^2}{a} = LL'}$$

$\therefore$ Length of latusrectum $LL' = L_1 L_1' = \dfrac{2b^2}{a}$ and end of points of latusrectum are

$$L \equiv \left(ae, \frac{b^2}{a}\right); \; L' \equiv \left(ae, -\frac{b^2}{a}\right)$$

$$L_1 \equiv \left(-ae, \frac{b^2}{a}\right); \; L_1' \equiv \left(-ae, -\frac{b^2}{a}\right)$$

respectively.

Remark

Latusrectum

$$= LL' = \frac{2b^2}{a} = \frac{(2b)^2}{(2a)} = \frac{(\text{Minor axis})^2}{(\text{Major axis})}$$

$$= 2a(1-e^2) = \boxed{2e\left(\frac{a}{e} - ae\right)}$$

= 2e (distance from focus to the corresponding directrix)

7. **Focal chord :** A chord of the ellipse passing through its focus is called a focal chord.

$\dfrac{2(SP)(SQ)}{SP+SQ} = \dfrac{b^2}{a}$

Remark

Semi latusrectum is the harmonic mean of the segments of focal chord or $\dfrac{1}{SP} + \dfrac{1}{SQ} = \dfrac{2a}{b^2}$, $(a > b)$, where PQ is the focal chord and S is the focus.

8. **Vertices :** The vertices of the ellipse are the points where the ellipse meets its major axis.

Hence, A and A' are the vertices

$$\therefore \quad A \equiv (a, 0) \quad \text{and} \quad A' \equiv (-a, 0)$$

| Example 1 If PSQ is a focal chord of the ellipse $16x^2 + 25y^2 = 400$, such that $SP = 8$, then find the length of SQ.

Sol. The given ellipse is $16x^2 + 25y^2 = 400$ or $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

$$\because \quad \frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$$

$$\Rightarrow \quad \frac{1}{8} + \frac{1}{SQ} = \frac{2(5)}{16} = \frac{5}{8} \text{ or } \frac{1}{SQ} = \frac{1}{2}$$

$$\therefore \quad SQ = 2$$

| Example 2 If the latusrectum of an ellipse is equal to half of its minor-axis, then find its eccentricity.

Sol. $\because$ Latusrectum $= \dfrac{1}{2}$(minor axis)

$$\Rightarrow \quad \frac{2b^2}{a} = \frac{1}{2}(2b)$$

$$\text{or} \quad 2b = a$$

$$\text{or} \quad 4b^2 = a^2$$

$$\Rightarrow \quad 4a^2(1-e^2) = a^2$$

$$\text{or} \quad 4e^2 = 3$$

$$\therefore \quad e = \frac{\sqrt{3}}{2}$$

| Example 3 If the distance between the directrices is thrice the distance between the foci, then find eccentricity of the ellipse.

Sol. Given, $\dfrac{2a}{e} = 3(2ae) \Rightarrow e^2 = \dfrac{1}{3}$

$$\therefore \quad e = \frac{1}{\sqrt{3}}$$

| Example 4 If $P(x, y)$ be any point on the ellipse $16x^2 + 25y^2 = 400$ and $F_1 \equiv (3, 0)$, $F_2 \equiv (-3, 0)$, then find the value of $PF_1 + PF_2$. always 2a

Sol. We have, $16x^2 + 25y^2 = 400$

$$\Rightarrow \quad \frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$$

$\because$ Coordinates of foci an $(\pm ae, 0)$

or $\left(\pm\sqrt{(a^2 - b^2)}, 0\right)$

i.e. $(\pm\sqrt{(25-16)}, 0)$ or $(\pm 3, 0)$

Thus, F_1 and F_2 are foci of the ellipse.

$$\therefore \quad PF_1 + PF_2 = \text{Length of major axis}$$
$$= 2a = 2 \times 5 = 10$$

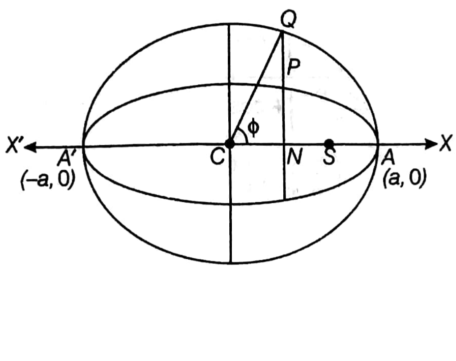
9. **Parametric equation of the ellipse :** The circle described on the major-axis of an ellipse as diameter is called the **auxiliary circle** of the ellipse.

Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\therefore$ Equation of its auxiliary circle is

$x^2 + y^2 = a^2$ ($\because AA'$ is diameter of the circle)



Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the X-axis.

Then P and Q are the **corresponding points** on the ellipse and the auxiliary circle respectively.

Let $\angle QCA = \phi \qquad (0 \le \phi < 2\pi)$

i.e. the eccentric angle of P on an ellipse is the angle which the radius (or **radius vector)** through the corresponding point on the auxiliary circle makes with the major axis.

$\therefore \quad Q \equiv (a\cos\phi, a\sin\phi)$

$\therefore$ Now x-coordinate of P is $a\cos\phi$

Let y-coordinate of P is y, then $(a\cos\phi, y)$ lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{a^2\cos^2\phi}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad y^2 = b^2\sin^2\phi$$

$$\therefore \quad \boxed{y = \pm b\sin\phi}$$

$\because$ P is in I quadrant

$$\therefore \quad y = b\sin\phi$$

Coordinates of P are $(a\cos\phi, b\sin\phi)$. We have $x = a\cos\phi$, $y = b\sin\phi$ are called parametric equations of the ellipse.

This point $\boxed{(a\cos\phi, b\sin\phi)}$ is also called the point 'ϕ'.

Remark

The equation of the chord joining the points

$$P \equiv (a\cos\phi_1, b\sin\phi_1)$$

and $\quad Q \equiv (a\cos\phi_2, b\sin\phi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\phi_1+\phi_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right) = \cos\left(\frac{\phi_1-\phi_2}{2}\right)$$

If its focal chord, then it pass through $(ae, 0)$ or $(-ae, 0)$, then

$$\pm e\cos\left(\frac{\phi_1+\phi_2}{2}\right) + 0 = \cos\left(\frac{\phi_1-\phi_2}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)} = \pm\frac{e}{1}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right) - \cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1-\phi_2}{2}\right) + \cos\left(\frac{\phi_1+\phi_2}{2}\right)} = \left(\frac{\pm e-1}{\pm e+1}\right)$$

$$\Rightarrow \quad \tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \left(\frac{\pm e-1}{\pm e+1}\right)$$

if focal chord pass through $(ae, 0)$, then

$$\boxed{\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \left(\frac{e-1}{e+1}\right)}$$

and if focal chord pass through $(-ae, 0)$, then

$$\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \left(\frac{e+1}{e-1}\right)$$

or $$\boxed{\cot\left(\frac{\phi_1}{2}\right)\cot\left(\frac{\phi_2}{2}\right) = \left(\frac{e-1}{e+1}\right)}$$

Remark

Circle described on focal length as diameter always touches the auxiliary circle. Remember

Proof

Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

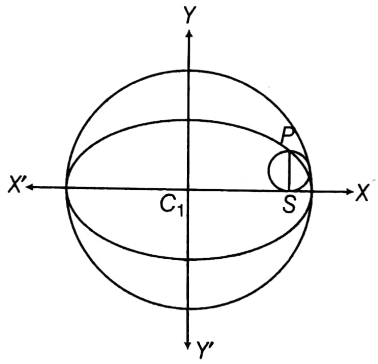
Let P be $(a\cos\theta, b\sin\theta)$

and S be $(ae, 0)$

$$\therefore \quad SP = ePM = e\left(\frac{a}{e} - a\cos\theta\right)$$

$$= (a - ae\cos\theta)$$

The auxiliary circle $x^2 + y^2 = a^2$ having center $C_1\,(0,0)$ and radius $r_1 = a$



The circle having SP as the diameter has center

$$C_2\left(\frac{ae + a\cos\theta}{2}, \frac{b\sin\theta}{2}\right)$$

and $\quad$ radius $r_2 = \frac{SP}{2} = \frac{a(1 - e\cos\theta)}{2}$

Now, $C_1C_2 = \sqrt{\left(\frac{ae + a\cos\theta}{2}\right)^2 + \left(\frac{b\sin\theta}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{a^2(e + \cos\theta)^2 + a^2(1 - e^2)\sin^2\theta}$$

$$= \frac{a}{2}\sqrt{(e^2 + \cos^2\theta + 2e\cos\theta + \sin^2\theta - e^2\sin^2\theta)}$$

$$= \frac{a}{2}\sqrt{(e^2\cos^2\theta + 2e\cos\theta + 1)}$$

$$= \frac{a}{2}(1 + e\cos\theta)$$

$$= r_1 - r_2$$

Hence, the circle on SP as diameter touches the auxiliary circle internally.

Example 5 Find the eccentric angle of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$. Whose distance from the center of the ellipse is $\sqrt{5}$.

Sol. Any point on the ellipse is $P(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$

Where, $0 \le \theta < 2\pi$ and $C(0,0)$ is center, given $CP=\sqrt{5}$

$$(CP)^2 = 5$$

$$\Rightarrow \quad 6\cos^2\theta + 2\sin^2\theta = 5$$

$$\Rightarrow \quad 6(1-\sin^2\theta) + 2\sin^2\theta = 5$$

$$\Rightarrow \quad \sin^2\theta = \frac{1}{4}$$

or $$\sin\theta = \pm\frac{1}{2} = \pm\sin\frac{\pi}{6}$$

or $$\therefore \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Focal Distances of a Point

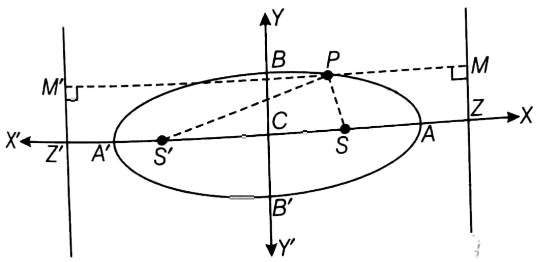
The sum of focal distances of any point on the ellipse is equal to the major axis. The ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{...(i)}$$

The foci S and S' are $(ae, 0)$ and $(-ae, 0)$.

The equations of its directrices MZ and $M'Z'$ are $x = a/e$ and $x=-a/e$

Let $P(x_1, y_1)$ be any point on Eq. (i)



Now $$SP = ePM = e\left(\frac{a}{e} - x_1\right)$$
$$\boxed{= a - ex_1}$$

and $$S'P = ePM' = e\left(\frac{a}{e} + x_1\right)$$
$$\boxed{= a + ex_1}$$

$$SP + S'P = (a - ex_1) + (a + ex_1)$$
$$= 2a = AA' = \text{major axis}$$

Hence, the sum of the focal distances of a point on the ellipse is constant and is equal to the length of the major axis of the ellipse.

Another definition of ellipse : An ellipse is the locus of a point which moves in a plane such that the sum of its distances from two fixed points in the same plane is always constant.

Remark

SP and *S'P* are also called focal radii of the ellipse

$\therefore \quad SP = a - ex_1$ and $S'P = a + ex_1$

Example 6 An ellipse having foci at (3, 3) and (−4, 4) and passing through the origin, then find eccentricity of the ellipse.

Sol. The ellipse is passing through $O(0,0)$ and has foci $P(3,3)$ and $Q(-4,4)$, then

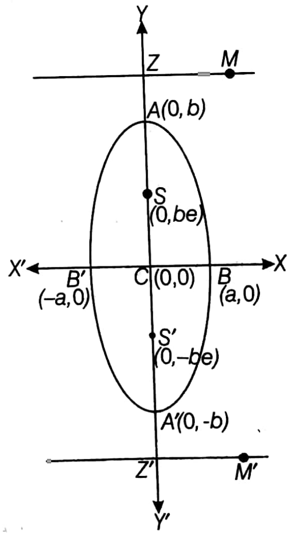
$$OP + OQ = 2a \text{ and } PQ = 2ae$$

$$\therefore \quad e = \frac{PQ}{OP+OQ} = \frac{\sqrt{50}}{3\sqrt{2}+4\sqrt{2}} = \frac{5}{7}$$

The Shape of the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, when $b>a$

In this case major and minor-axis of the ellipse along Y-axis and X-axis respectively.

then $\quad AA' = 2b \quad$ and $\quad BB' = 2a$



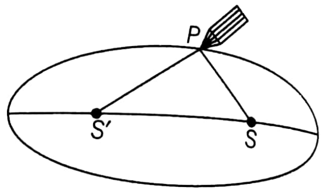
The foci S and S' are $(0, be)$ and $(0, -be)$ respectively

The directrices are MZ and $M'Z'$ are

$$y = \frac{b}{e} \quad \text{and} \quad y = -\frac{b}{e}$$

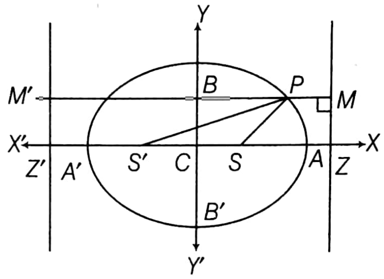
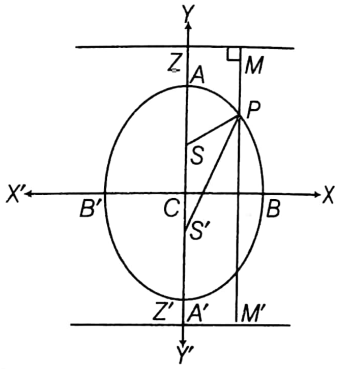
are respectively,

Mechanical Construction of an Ellipse



Let S and S' be two drawing pins and let an inextensible string whose ends at S and S' and length is equal to sum of SP and $S'P$ i.e. $2a$, where P is point of pencil. The point of pencil move on paper and the fixed ends always tight. So as to satisfy these conditions it will trace out the curve on the paper. This curve is an ellipse. Hence the locus of the point of pencil is an ellipse.

Smart Table : Difference between both (Horizontal and Vertical) Ellipses will be clear from the following table

Equation and graph of the ellipse → / Basic fundamentals ↓	**Horizontal ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; a>b$ 	**Vertical ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; a<b$ 
Centre	$(0, 0)$	$(0, 0)$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$ or $(\pm\sqrt{(a^2-b^2)}, 0)$	$(0, \pm be)$ or $(0, \pm\sqrt{(b^2-a^2)})$
Distance between foci	$2ae$ or $2\sqrt{(a^2-b^2)}$	$2be$ or $2\sqrt{(b^2-a^2)}$
Equation of directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Distance between directrices	$\frac{2a}{e}$	$\frac{2b}{e}$
Relation between a, b and e	$b^2=a^2(1-e^2)$	$a^2=b^2(1-e^2)$
Length of latusrectum	$\frac{2b^2}{a}$ or $2a(1-e^2)$	$\frac{2a^2}{b}$ or $2b(1-e^2)$
End points of latusrectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Focal radii	$SP=a-ex_1$, $S'P=a+ex_1$ and $SP+S'P=2a$	$SP=b-ey_1$, $S'P=b+ey_1$ and $SP+S'P=2b$
Parametric Coordinates	$(a\cos\theta, b\sin\theta)$, $0\le\theta<2\pi$	$(a\cos\theta, b\sin\theta)$, $0\le\theta<2\pi$
Tangents at the vertices	$x=\pm a$	$y=\pm b$

| Example 7 Find the lengths of major and minor axes, the coordinates of foci, vertices and the eccentricity of the ellipse $3x^2 + 2y^2 = 6$. Also, find the equation of the directrices.

Sol. The equation of ellipse is

$$3x^2 + 2y^2 = 6$$

$$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{3} = 1$$

$$\Rightarrow \quad \frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{3})^2} = 1$$

Comparing this equation with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \quad a = \sqrt{2}, b = \sqrt{3} \qquad \text{(Here } b > a\text{)}$$

Length of major axis $= 2b = 2\sqrt{3}$

and Length of minor axis $= 2a = 2\sqrt{2}$

If e be the eccentricity, then $a^2 = b^2(1 - e^2)$

$$\Rightarrow \quad 2 = 3(1 - e^2) \Rightarrow e^2 = \frac{1}{3}$$

$$\therefore \quad e = \frac{1}{\sqrt{3}}$$

Vertices $= (0, \pm b) = (0, \pm\sqrt{3})$

and foci are $(0, \pm be) = (0, \pm 1)$

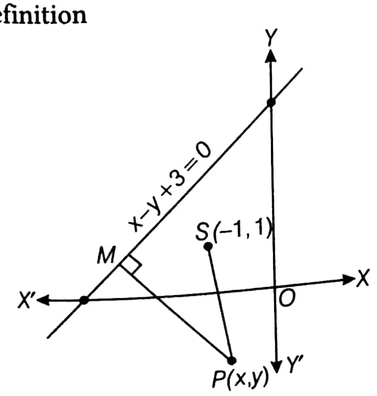
and equation of the directrices are $y = \pm b/e$

$$y = \pm \frac{\sqrt{3}}{(1/\sqrt{3})}$$

$$\therefore \quad y = \pm 3$$

| Example 8 Find the equation of an ellipse whose focus is $(-1, 1)$, eccentricity is $\frac{1}{2}$ and the directrix is $x - y + 3 = 0$.

Sol. Let $P(x, y)$ be any point on the ellipse whose focus is $S(-1, 1)$ and the directrix is $x - y + 3 = 0$. Draw PM perpendicular from $P(x, y)$ on the directrix $x - y + 3 = 0$. Then by definition



$$SP = ePM$$

$$\Rightarrow \quad (SP)^2 = e^2(PM)^2$$

$$\Rightarrow \quad (x+1)^2 + (y-1)^2 = \frac{1}{4}\left\{\frac{x - y + 3}{\sqrt{2}}\right\}^2$$

$$\Rightarrow \quad 8(x^2 + y^2 + 2x - 2y + 2)$$
$$= x^2 + y^2 + 9 - 2xy + 6x - 6y$$

$$\Rightarrow \quad 7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$$

which is the required equation of the ellipse.

| Example 9 Show that the line $lx + my + n = 0$ will cut the ellipse $x^2/a^2 + y^2/b^2 = 1$ in points whose eccentric angles differ by $(\pi/2)$, if $a^2l^2 + b^2m^2 = 2n^2$.

Sol. Let eccentric angles are θ and ϕ, then

$$\theta - \phi = \frac{\pi}{2} \qquad \text{(given)}$$

$$\therefore \quad \theta = \frac{\pi}{2} + \phi$$

The line joining the point 'θ' and 'ϕ' is

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

or $$\frac{x}{a}\cos\left(\frac{\pi}{4}+\phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{4}+\phi\right) = \cos\frac{\pi}{4} \qquad \left(\because \theta = \frac{\pi}{2} + \phi\right)$$

or $$\frac{x}{a}\cos\left(\frac{\pi}{4}+\phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{4}+\phi\right) = \frac{1}{\sqrt{2}} \qquad \text{...(i)}$$

and the given line is $lx + my + n = 0$

or $$lx + my = -n \qquad \text{...(ii)}$$

Now, Eqs. (i) and (ii) represent the same line, so comparing them, we get

$$\frac{\cos\left(\frac{\pi}{4}+\phi\right)}{la} = \frac{\sin\left(\frac{\pi}{4}+\phi\right)}{mb} = -\frac{1}{n\sqrt{2}}$$

$$\therefore \quad \cos\left(\frac{\pi}{4}+\phi\right) = -\frac{la}{n\sqrt{2}} \qquad \text{...(iii)}$$

and $$\sin\left(\frac{\pi}{4}+\phi\right) = -\frac{mb}{n\sqrt{2}} \qquad \text{...(iv)}$$

Squaring and adding Eqs. (iii) and (iv), then

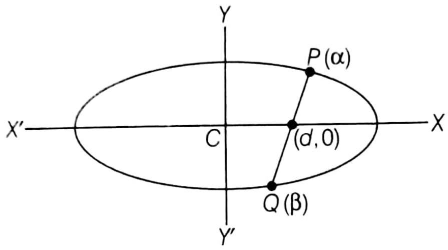
$$\frac{l^2a^2}{2n^2} + \frac{m^2b^2}{2n^2} = 1$$

$$l^2a^2 + m^2b^2 = 2n^2$$

| Example 10 If a chord joining two points whose eccentric angles are α, β cut the major axis of an ellipse at a distance d from the centre. Show that $\tan(\alpha/2)\tan(\beta/2) = (d - a)/(d + a)$, where $2a$ is the length of major axis.

Sol. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$



The equation of the line joining 'α' and 'β' on the ellipse Eq. (i) is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \text{...(ii)}$$

Since, it cuts the major axis of the ellipse at a distance d from the centre i.e. passes through the point $(d, 0)$, then

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+0=\cos\left(\frac{\alpha-\beta}{2}\right)$$

or
$$\frac{d}{a}=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

or
$$\frac{d-a}{d+a}=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)}$$

(By componendo and dividendo)

$$=\frac{2\sin(\alpha/2)\sin(\beta/2)}{2\cos(\alpha/2)\cos(\beta/2)}$$

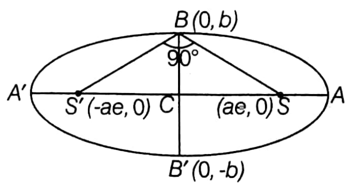
$$=\tan(\alpha/2)\tan(\beta/2)$$

$$\therefore \quad \tan(\alpha/2)\tan(\beta/2)=\frac{d-a}{d+a}$$

Example 11 If the angle between the straight lines joining foci and the ends of the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is 90°. Find its eccentricity.

Sol. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let $a > b$



$\therefore$ The ends of minor axis are $B(0, b)$ and $B'(0, -b)$. If the eccentricity of the ellipse is e, then the foci are $S(ae, 0)$ and $S'(-ae, 0)$

$$\therefore \quad \text{Slope of } BS \text{ is } m_1=\frac{b-0}{0-ae}=-\frac{b}{ae}$$

$$\text{and slope of } BS' \text{ is } \quad m_2=\frac{b-0}{0+ae}=\frac{b}{ae}$$

$\because$ The angle between BS and BS' is 90°,

$$\therefore \quad m_1m_2=-1$$

$$\Rightarrow \quad -\frac{b}{ae}\times\frac{b}{ae}=-1$$

$$\Rightarrow \quad b^2=a^2e^2$$

$$a^2(1-e^2)=a^2e^2$$

$$\Rightarrow \quad 1-e^2=e^2$$

$$\Rightarrow \quad 2e^2=1$$

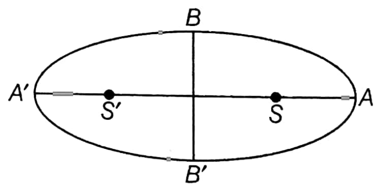
$$\therefore \quad e=\frac{1}{\sqrt{2}}$$

Example 12 Find the equation of the ellipse referred to its centre whose minor axis is equal to the distance between the foci and whose latusrectum is 10.

Sol. Let the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (\text{let } a > b)$$

Then, the foci are $S(ae, 0)$ and $S'(-ae, 0)$, length of minor axis $BB' = 2b$ and length of latusrectum $=\frac{2b^2}{a}$



$\therefore$ According to the question

$$BB' = SS' \Rightarrow 2b = 2ae$$

$$\Rightarrow \quad b = ae \qquad \text{...(i)}$$

$$\text{and} \quad \frac{2b^2}{a}=10$$

$$\Rightarrow \quad b^2=5a \qquad \text{...(ii)}$$

$$\text{also we have } \; b^2=a^2(1-e^2) \qquad \text{...(iii)}$$

Putting the value of b from Eq. (i) in Eq. (iii), we have

$$a^2e^2=a^2(1-e^2)$$

$$\Rightarrow \quad e^2=1-e^2$$

$$\Rightarrow \quad 2e^2=1$$

$$\therefore \quad e=\frac{1}{\sqrt{2}}$$

From Eq. (i), we have

$$b=\frac{a}{\sqrt{2}}$$

$$\therefore \quad b^2=\frac{a^2}{2}$$

$$\Rightarrow \quad 5a = \frac{a^2}{2} \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad a = 10$$

From Eq. (ii)

$$b^2 = 5 \times 10 = 50$$

Putting the values of a and b in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of required ellipse is

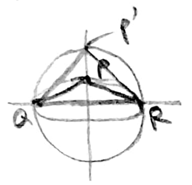
$$\frac{x^2}{100} + \frac{y^2}{50} = 1$$

$$x^2 + 2y^2 = 100$$

| Example 13 Prove that the ratio of area of any triangle PQR inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and that of triangle formed by the corresponding points on the auxiliary circle is $\frac{b}{a}$.

Sol. Let the three points on the ellipse be $P(a\cos\alpha, b\sin\alpha)$, $Q(a\cos\beta, b\sin\beta)$ and $R(a\cos\gamma, b\sin\gamma)$.

Then, the corresponding points on the auxiliary circle are $A(a\cos\alpha, a\sin\alpha)$, $B(a\cos\beta, a\sin\beta)$ and $C(a\cos\gamma, a\sin\gamma)$, then



$$\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix}}{\frac{1}{2}\begin{vmatrix} a\cos\alpha & a\sin\alpha & 1 \\ a\cos\beta & a\sin\beta & 1 \\ a\cos\gamma & a\sin\gamma & 1 \end{vmatrix}}$$

$$\frac{\frac{1}{2}\times\cancel{\Delta}\times ab}{\frac{1}{2}\times\cancel{\Delta}\times a} = \frac{b}{a}$$

$$= \frac{ab\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}}{a^2\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}}$$

$$= \frac{b}{a}$$

| Example 14 The extremities of a line segment of length l move in two fixed perpendicular straight lines. Find the locus of that point which divides this line segment in ratio 1 : 2.

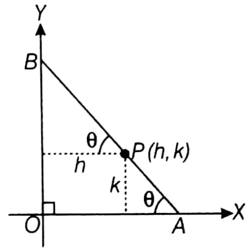
Sol. Let $PA : PB = 1 : 2$

$$PA = \frac{l}{3} \quad \text{and} \quad PB = \frac{2l}{3}$$

$$k = \frac{l}{3}\sin\theta$$

or $\quad 3k = l\sin\theta \qquad \ldots\text{(i)}$

and $\quad h = \frac{2l}{3}\cos\theta$



or $\quad \frac{3h}{2} = l\cos\theta \qquad \ldots\text{(ii)}$

Squaring and adding Eqs. (i) and (ii), then

$$9k^2 + \frac{9h^2}{4} = l^2$$

or $\quad 9h^2 + 36k^2 = 4l^2$

$\therefore$ Locus of $P(h, k)$ is

$$9x^2 + 36y^2 = 4l^2$$

Aliter :

Let $A \equiv (a, 0)$ and $B \equiv (0, b)$

$\because \quad P(h, k)$ divide AB in the ratio 1 : 2 (internally), then

$$h = \frac{1.0 + 2.a}{1 + 2} \Rightarrow a = \frac{3h}{2}$$

and $\quad k = \frac{1.b + 2.0}{1 + 2} \Rightarrow b = 3k$

$\because \quad a^2 + b^2 = l^2$

$\Rightarrow \quad \frac{9h^2}{4} + 9k^2 = l^2$

or $\quad 9h^2 + 36k^2 = 4l^2$

$\therefore$ Locus of $P(h, k)$ is

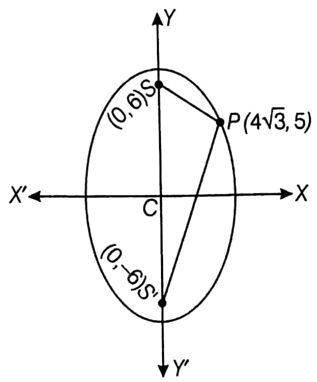
$$9x^2 + 36y^2 = 4l^2$$

| Example 15 Find the lengths and equations of the focal radii drawn from the point $(4\sqrt{3}, 5)$ on the ellipse $25x^2 + 16y^2 = 1600$.

Sol. The equation of the ellipse is

$$25x^2 + 16y^2 = 1600$$

or $\quad \frac{x^2}{64} + \frac{y^2}{100} = 1$



Here, $b > a$

$$a^2 = 64,\ b^2 = 100$$

$$a^2 = b^2(1 - e^2)$$

$\therefore$ $$64 = 100(1 - e^2)$$

$\Rightarrow$ $$e = 3/5$$

Let $P(x_1, y_1) \equiv (4\sqrt{3}, 5)$

be a point on the ellipse, then SP and $S'P$ are the focal radii

$\therefore$ $SP = b - ey_1$ and $S'P = b + ey_1$

$\therefore$ $SP = 10 - \frac{3}{5} \times 5$ and $S'P = 10 + \frac{3}{5} \times 5$

$\Rightarrow$ $SP = 7$ and $S'P = 13$

Also, S is $(0, be)$

i.e. $\left(0, 10 \times \frac{3}{5}\right)$ i.e. $(0, 6)$

and S' is $(0, -be)$

i.e. $\left(0, -10 \times \frac{3}{5}\right)$

i.e. $(0, -6)$

$\therefore$ Equation of SP is

$$y - 5 = \frac{6 - 5}{0 - 4\sqrt{3}}(x - 4\sqrt{3})$$

$$-4\sqrt{3}y + 20\sqrt{3} = x - 4\sqrt{3}$$

or $$x + 4\sqrt{3}y - 24\sqrt{3} = 0$$

and equation of $S'P$ is

$\therefore$ $$y - 5 = \frac{-6 - 5}{0 - 4\sqrt{3}}(x - 4\sqrt{3})$$

$\Rightarrow$ $$-4\sqrt{3}y + 20\sqrt{3} = -11x + 44\sqrt{3}$$

or $$11x - 4\sqrt{3}y - 24\sqrt{3} = 0$$

Exercise for Session 1

1. The length of the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is three times the length of minor axis, it eccentricity is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{2\sqrt{2}}{3}$ (d) $\frac{2\sqrt{2}}{5}$

2. The equation $\frac{x^2}{10 - a} + \frac{y^2}{4 - a} = 1$, represents an ellipse, if

(a) $a < 4$ (b) $a > 4$ (c) $4 < a < 10$ (d) $a > 10$

3. The eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose latusrectum is half of its major axis, is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{\left(\frac{2}{3}\right)}$

4. If the eccentricity of an ellipse is $\frac{1}{\sqrt{2}}$, then its latusrectum is equal to its

(a) minor axis (b) semi minor axis (c) major axis (d) semi major axis

5. If the distance between the foci of an ellipse is equal to its minor axis, then its eccentricity is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{3}$ (d) $\frac{1}{\sqrt{3}}$

6. The eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$, whose distance from the centre of ellipse is 2, is

(a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{2}$ (d) $\frac{5\pi}{3}$

7. If $\tan\alpha \tan\beta = -\frac{a^2}{b^2}$, then the chord joining two points α and β on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, will subtend a right angle at

(a) focus (b) centre (c) end of major axis (d) end of minor axis

8. If the eccentricity of the two ellipses $\frac{x^2}{169}+\frac{y^2}{25}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are equal, then the value of a/b is

(a) $\frac{5}{13}$
(b) $\frac{6}{13}$
(c) $\frac{13}{5}$
(d) $\frac{13}{6}$

9. The ratio of the areas of triangle inscribed in ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, to that of triangle formed by the corresponding points on the auxiliary circle is $\frac{1}{2}$, then the eccentricity of the ellipse is

(a) $\frac{1}{2}$
(b) $\frac{\sqrt{3}}{2}$
(c) $\frac{1}{\sqrt{2}}$
(d) $\frac{1}{\sqrt{3}}$

10. If PSQ is a focal chord of the ellipse $16x^2+25y^2=400$ such that $SP=16$, then the length SQ is

(a) $\frac{2}{9}$
(b) $\frac{4}{9}$
(c) $\frac{8}{9}$
(d) $\frac{16}{9}$

11. Let P be a variable point on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ with foci at S and S'. If A be the area of triangle PSS', then the maximum value of A is

(a) 12 sq units
(b) 24 sq units
(c) 36 sq units
(d) 48 sq units

12. If S and S' are the foci of an ellipse of major axis of length 10 units and P is any point on the ellipse such that the perimeter of triangle PSS' is 15 units, then the eccentricity of the ellipse is

(a) $\frac{1}{2}$
(b) $\frac{1}{4}$
(c) $\frac{7}{25}$
(d) $\frac{3}{4}$

13. Find the latusrectum, eccentricity, coordinates of the foci, coordinates of the vertices, the length of the axes and the centre of the ellipse

$4x^2+9y^2-8x-36y+4=0$

14. The distance between the foci of an ellipse is 10 and its latusrectum is 15 ; find its equation referred to its axes as axes of coordinates.

15. Find the equation of the ellipse whose axes are parallel to the coordinate axes having its centre at the point $(2,-3)$ one focus at $(3,-3)$ and one vertex at $(4,-3)$.

16. Find the equation of the ellipse whose foci are $(2,3)$, $(-2,3)$ and whose semi-minor axes is $\sqrt{5}$.

17. Show that the equation $(10x-5)^2+(10y-5)^2=(3x+4y-1)^2$ represents an ellipse, find the eccentricity of the ellipse.

18. Find the locus of the extremities of the latusrectum of the family of ellipses $b^2x^2+y^2=a^2b^2$ having a given major axes.

Session 2

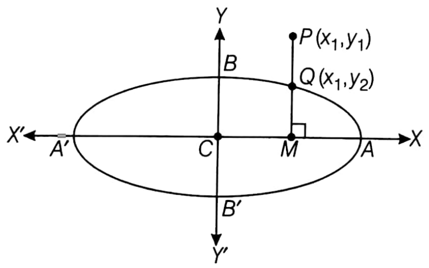
Position of a Point with Respect to an Ellipse, Intersection of a Line and an Ellipse, Equation of Tangent in Different Forms, Equations of Normals in Different Forms, Properties of Eccentric Angles of the Co-normal Points, Co-normal Points Lie on a Fixed Curve

Position of a Point with Respect to an Ellipse

Theorem : Prove that the point $P(x_1, y_1)$ lies outside, on, or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>,=\text{or},<0$$

Proof : From point $P(x_1, y_1)$ draw perpendicular PM on AA' to meet the ellipse at $Q(x_1, y_2)$.



Since, $Q(x_1, y_2)$ lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

then, $$\frac{x_1^2}{a^2}+\frac{y_2^2}{b^2}=1$$

$$\Rightarrow \quad \frac{y_2^2}{b^2}=1-\frac{x_1^2}{a^2} \qquad \text{...(i)}$$

Now, point P lies outside, on or inside the ellipse according as

$$PM>,=\text{or},<QM$$

$$\Rightarrow \quad y_1>,=\text{or},<y_2$$

$$\Rightarrow \quad \frac{y_1^2}{b^2}>,=\text{or},<\frac{y_2^2}{b^2}$$

$$\Rightarrow \quad \frac{y_1^2}{b^2}>,=\text{or},<1-\frac{x_1^2}{a^2} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}>,=\text{or},<1,$$

or $$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>,=\text{or},<0$$

Hence, the point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0,=\text{or},<0$$

Remark

Let $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$, then $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

The point (x_1, y_1) lies outside, on or inside the ellipse $S=0$ according as $S_1>,=\text{or}<0$.

I Example 16 Find the position of the point $(4,-3)$ relative to the ellipse $5x^2+7y^2=140$.

Sol. The given ellipse can be written as $\frac{x^2}{28}+\frac{y^2}{20}-1=0$

Let $$S=\frac{x^2}{28}+\frac{y^2}{20}-1$$

$$\therefore \quad S_1=\frac{(4)^2}{28}+\frac{(-3)^2}{20}-1=\frac{3}{140}>0$$

So, the point $(4, -3)$ lies outside the ellipse $5x^2 + 7y^2 = 140$.

Example 17 Find the integral value of α for which the point $\left(7 - \frac{5\alpha}{4}, -\alpha\right)$ lies inside the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Sol. Since, the point $\left(7 - \frac{5\alpha}{4}, -\alpha\right)$ lies inside the ellipse

$\frac{x^2}{25} + \frac{y^2}{16} = 1$, then $\frac{1}{25}\left(7 - \frac{5\alpha}{4}\right)^2 + \frac{1}{16}(-\alpha)^2 - 1 < 0$

$$\Rightarrow \quad (28 - 5\alpha)^2 + 25\alpha^2 - 400 < 0$$

$$\Rightarrow \quad 50\alpha^2 - 280\alpha + 384 < 0$$

$$\Rightarrow \quad 25\alpha^2 - 140\alpha + 192 < 0$$

$$\Rightarrow \quad (5\alpha - 12)(5\alpha - 16) < 0$$

$$\therefore \quad \frac{12}{5} < \alpha < \frac{16}{5}$$

Hence, integral value of α is 3

Intersection of a Line and an Ellipse

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

and the given line be $y = mx + c$ ...(ii)

Eliminating y from Eqs. (i) and (ii), then

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow \quad (a^2m^2 + b^2)x^2 + 2mca^2x + c^2a^2 - a^2b^2 = 0 \quad \text{...(iii)}$$

Above equation being a quadratic in x gives two values of x. Shows that every straight line will cut the ellipse in two points may be real, coincident or imaginary according as

Discriminant of Eq. (iii) $>, =, < 0$

i.e. $\quad 4m^2c^2a^4 - 4(a^2m^2 + b^2)(c^2a^2 - a^2b^2) >, =, < 0$

or $\quad -a^2b^2c^2 + a^4b^2m^2 + a^2b^4 >, =, < 0$

or $\quad a^2m^2 + b^2 >, =, < c^2$ ...(iv)

Condition of Tangency : If the line Eq. (ii) touches the ellipse Eq. (i), then Eq. (iii) has equal roots.

$\therefore$ Discriminant of Eq. (iii) $= 0$

$$\Rightarrow \quad c^2 = a^2m^2 + b^2 \quad \text{or} \quad c = \pm\sqrt{(a^2m^2 + b^2)} \quad \text{...(v)}$$

So, the line $y = mx + c$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{if} \quad \boxed{c^2 = a^2m^2 + b^2}$$

(which is condition of tangency)

Substituting the value of c from Eq. (v) in Eq. (ii), then

$$y = mx \pm \sqrt{(a^2m^2 + b^2)}$$

Hence, the lines $y = mx \pm \sqrt{(a^2m^2 + b^2)}$ will always tangents to the ellipse.

Point of contact : Substituting $c = \pm\sqrt{(a^2m^2 + b^2)}$ in Eq. (iii), then

$$(a^2m^2 + b^2)x^2 \pm 2ma^2x\sqrt{(a^2m^2 + b^2)} + (a^2m^2 + b^2)a^2 - a^2b^2 = 0$$

or $\quad (a^2m^2 + b^2)x^2 \pm 2ma^2x\sqrt{(a^2m^2 + b^2)} + a^4m^2 = 0$

or $\quad (x\sqrt{(a^2m^2 + b^2)} \pm a^2m)^2 = 0$

$$\therefore \quad x = \pm\frac{a^2m}{\sqrt{(a^2m^2 + b^2)}} = \pm\frac{a^2m}{c}$$

From Eq. (i), $\quad \frac{a^4m^2}{c^2}\cdot\frac{1}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - \frac{a^2m^2}{c^2} = \frac{c^2 - a^2m^2}{c^2} = \frac{b^2}{c^2}$$

$$y = \pm\frac{b^2}{c}$$

Hence, the point of contact is $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$ this known as m-point on the ellipse.

Remark

If $m = 0$, then Eq. (iii) gives $b^2x^2 + c^2a^2 - a^2b^2 = 0$

or $\quad b^2x^2 + (a^2m^2 + b^2)a^2 - a^2b^2 = 0$

$$\therefore \quad \boxed{x = \pm\frac{a^2m}{b}}$$

which gives two values of x.

Example 18 Prove that the straight line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $a^2l^2 + b^2m^2 = n^2$.

Sol. The given line is

$$lx + my + n = 0$$

or $\quad y = -\frac{l}{m}x - \frac{n}{m}$ ...(i)

Comparing this line with $y = Mx + c$

$$\therefore \quad M = -\frac{l}{m} \quad \text{and} \quad c = -\frac{n}{m}$$

The line Eq. (i) will touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$c^2 = a^2M^2 + b^2$$
$$\frac{n^2}{m^2} = \frac{a^2l^2}{m^2} + b^2$$
$$a^2l^2 + b^2m^2 = n^2$$

Example 19 Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$ and that point of contact is $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$.

Sol. The given line is $x\cos\alpha + y\sin\alpha = p$

$$y = -x\cot\alpha + p\operatorname{cosec}\alpha$$

Comparing this line with $y = mx + c$

$\therefore$ $m = -\cot\alpha$ and $c = p\operatorname{cosec}\alpha$

Hence, the given line touches the ellipse, then

$$c^2 = a^2m^2 + b^2$$
$$\Rightarrow \quad p^2\operatorname{cosec}^2\alpha = a^2\cot^2\alpha + b^2$$
$$\Rightarrow \quad p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

and point of contact is $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

i.e. $\left(-\frac{a^2(-\cot\alpha)}{p\operatorname{cosec}\alpha}, \frac{b^2}{p\operatorname{cosec}\alpha}\right)$

i.e. $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$

Example 20 For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Sol. $\because$ Equation of ellipse is

$$9x^2 + 16y^2 = 144 \quad \text{or} \quad \frac{x^2}{16}+\frac{y^2}{9}=1$$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

then, we get $a^2 = 16$ and $b^2 = 9$

and comparing the line $y = x + \lambda$ with $y = mx + c$

$\therefore$ $m = 1$

and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse

$$9x^2 + 16y^2 = 144$$

then $c^2 = a^2m^2 + b^2$

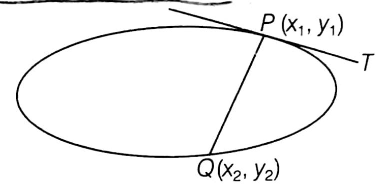
$$\Rightarrow \quad \lambda^2 = 16 \times 1^2 + 9$$
$$\Rightarrow \quad \lambda^2 = 25$$
$$\therefore \quad \lambda = \pm 5$$

Equation of Tangent in Different Forms

1. Point Form

Theorem : The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Proof : (By first Principal Method)



$\because$ Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be any two point on Eq. (i), then

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \quad \text{...(ii)}$$

and

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1 \quad \text{...(iii)}$$

Subtracting Eqs. (ii) from (iii), then

$$\frac{1}{a^2}(x_2^2 - x_1^2) + \frac{1}{b^2}(y_2^2 - y_1^2) = 0$$
$$\Rightarrow \quad \frac{(x_2+x_1)(x_2-x_1)}{a^2} + \frac{(y_2+y_1)(y_2-y_1)}{b^2} = 0$$
$$\Rightarrow \quad \frac{y_2-y_1}{x_2-x_1} = -\frac{b^2(x_1+x_2)}{a^2(y_1+y_2)} \quad \text{...(iv)}$$

Equation of PQ is

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1) \quad \text{...(v)}$$

From Eqs. (iv) and (v), then

$$y - y_1 = -\frac{b^2(x_1+x_2)}{a^2(y_1+y_2)}(x - x_1) \quad \text{...(vi)}$$

Now, for tangent at $P, Q \to P$ i.e., $x_2 \to x_1$ and $y_2 \to y_1$, then Eq. (vi) becomes

$$y - y_1 = -\frac{b^2(2x_1)}{a^2(2y_1)}(x - x_1)$$

or

$$\frac{yy_1 - y_1^2}{b^2} = -\left(\frac{xx_1 - x_1^2}{a^2}\right)$$

or

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \text{ or } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \text{[from (ii)]}$$

which is required equation of tangent at (x_1, y_1).

Remark

The equation of tangent at (x_1, y_1) can be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$.

This method is applicable only when the equation of ellipse is a polynomial of second degree in x and y.

2. Parametric form

Theorem : The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

Proof : The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ (by point form)

Replacing x_1 by $a\cos\phi$ and y_1 by $b\sin\phi$, then we get

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

Remark

Point of intersection of tangent at 'θ' and 'ϕ' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$

Remembering method : $\because$ Equation of chord joining $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\Rightarrow \frac{x}{a}\left\{\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}+\frac{y}{b}\left\{\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}=1$$

or $$\frac{x}{a^2}\left\{\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}+\frac{y}{b^2}\left\{\frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}=1$$

3. Slope form

Theorem : The equations of tangents of slope m to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $y=mx\pm\sqrt{(a^2m^2+b^2)}$ and the coordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{(a^2m^2+b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2+b^2)}}\right)$$

Proof : Let $y=mx+c$ be a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then the equation $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

$$\Rightarrow \quad x^2(a^2m^2+b^2)+2a^2mcx+a^2(c^2-b^2)=0 \quad \text{...(i)}$$

must have equal roots

$$4a^4m^2c^2-4(a^2m^2+b^2)a^2(c^2-b^2)=0$$

$$\{\because B^2-4AC=0\}$$

$$\Rightarrow \quad a^2m^2c^2-(a^2m^2+b^2)(c^2-b^2)=0$$

$$\Rightarrow \quad a^2m^2c^2-a^2m^2c^2+a^2b^2m^2-b^2c^2+b^4=0$$

$$\Rightarrow \quad a^2b^2m^2-b^2c^2+b^4=0$$

$$\Rightarrow \quad c^2=a^2m^2+b^2$$

$$\therefore \quad c=\pm\sqrt{(a^2m^2+b^2)}$$

Substituting this value of c in $y=mx+c$, we get

$$y=mx\pm\sqrt{(a^2m^2+b^2)}$$

as the required equations of tangent of ellipse in terms of slope, putting $c=\pm\sqrt{(a^2m^2+b^2)}$ in (i), we get

$$x^2(a^2m^2+b^2)\pm2a^2m\sqrt{(a^2m^2+b^2)}\,x+a^4m^2=0$$

$$\Rightarrow \quad (\sqrt{(a^2m^2+b^2)}\,x\pm a^2m)^2=0$$

$$\Rightarrow \quad x=\mp\frac{a^2m}{\sqrt{(a^2m^2+b^2)}}$$

Substituting this value of x in

$$y=mx\pm\sqrt{a^2m^2+b^2}$$

we obtained $$y=\pm\frac{b^2}{\sqrt{(a^2m^2+b^2)}}$$

Thus, the coordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{(a^2m^2+b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2+b^2)}}\right)$$

| Example 21 If the line $3x+4y=\sqrt{7}$ touches the ellipse $3x^2+4y^2=1$, then find the point of contact.

Sol. Let the given line touches the ellipse at point $P(x_1, y_1)$.

The equation of tangent at P is

$$3xx_1+4yy_1=1 \quad \text{...(i)}$$

Comparing Eq. (i) with the given equation of line $3x + 4y = \sqrt{7}$, we get

$$\frac{3x_1}{3} = \frac{4y_1}{4} = \frac{1}{\sqrt{7}}$$

$$\therefore \quad x_1 = y_1 = \frac{1}{\sqrt{7}}$$

Hence, point of contact (x_1, y_1) is $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$.

| Example 22 Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$.

$$\therefore \quad m \times -2 = -1$$

$$\Rightarrow \quad m = \frac{1}{2}$$

Note :- If find point, then use parametric form, Because in this only one variable i.e θ

Since $\quad 3x^2 + 4y^2 = 12$

or $\quad \dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

Comparing this with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\therefore \quad a^2 = 4$$

and $\quad b^2 = 3$

So the equations of the tangents are

$$y = \frac{1}{2}x \pm \sqrt{4 \times \frac{1}{4} + 3}$$

$$\Rightarrow \quad y = \frac{1}{2}x \pm 2$$

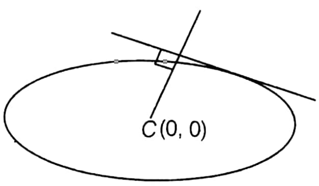
or $\quad x - 2y \pm 4 = 0$

| Example 23 Find the locus of the foot of the perpendicular drawn from centre upon any tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Sol. Any tangent of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$y = mx + \sqrt{(a^2m^2 + b^2)} \quad \text{...(i)}$$

Equation of the line perpendicular to Eq. (i) and passing through (0, 0) is



$$y = -\frac{1}{m}x \text{ or } m = -\frac{x}{y} \quad \text{...(ii)}$$

Substituting the value of m from Eq. (ii) in Eq. (i), then

$$y = -\frac{x^2}{y} + \sqrt{\left(a^2\frac{x^2}{y^2} + b^2\right)}$$

$$\Rightarrow \quad (x^2 + y^2)^2 = a^2x^2 + b^2y^2$$

or changing to polars by putting $x = r\cos\theta, y = r\sin\theta$ it becomes

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

| Example 24 Find the point on the ellipse $16x^2 + 11y^2 = 256$, where the common tangent to it and the circle $x^2 + y^2 - 2x = 15$ touch.

Sol. The given ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{(256/11)} = 1$

Equation of tangent to it at point $\left(4\cos\theta, \dfrac{16}{\sqrt{11}}\sin\theta\right)$ is

$$\frac{x}{4}\cos\theta + y\frac{\sqrt{11}}{16}\sin\theta = 1$$

It also touch the circle $(x-1)^2 + (y-0)^2 = 4^2$

Therefore,

$$\frac{\left|\frac{1}{4}\cos\theta - 1\right|}{\sqrt{\left(\frac{\cos^2\theta}{16} + \frac{11}{256}\sin^2\theta\right)}} = 4$$

$$\Rightarrow \quad |\cos\theta - 4| = \sqrt{(16\cos^2\theta + 11\sin^2\theta)}$$

or $\quad 4\cos^2\theta + 8\cos\theta - 5 = 0$

or $\quad (2\cos\theta - 1)(2\cos\theta + 5) = 0$

or $\quad \cos\theta = \dfrac{1}{2} \quad \left(\because \cos\theta \neq -\dfrac{5}{2}\right)$

$$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Therefore, points are $\left(2, \pm\dfrac{8\sqrt{33}}{11}\right)$.

Must

| Example ~~25~~ Find the maximum area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ which touches the line $y = 3x + 2$.

Sol. $\because$ Line $y = 3x + 2$ touches ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Here, $\quad m = 3$ and $c = 2$

Substituting in $c^2 = a^2m^2 + b^2$

or $\quad 4 = 9a^2 + b^2 \quad$ ...(i)

Note:- If θ given any, then choose simplest to find

So, $(a-ae)(a+ae)=b^2$.

Now, AM $\geq$ GM

$$\frac{9a^2+b^2}{2} \geq \sqrt{(9a^2)b^2} \Rightarrow \frac{9a^2+b^2}{2} \geq 3ab$$

$\Rightarrow$ $2 \geq 3ab$ [from Eq. (i)]

$\Rightarrow$ $\dfrac{2\pi}{3} \geq \pi ab$

or $\dfrac{2\pi}{3} \geq$ Area of ellipse

or

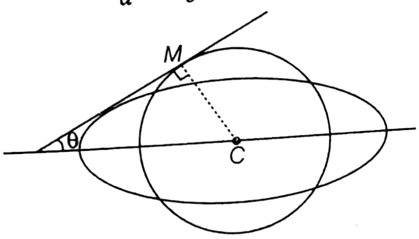
Therefore, the maximum area of the ellipse is $\dfrac{2\pi}{3}$.

| Example 26 A circle of radius r is concentric with the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Prove that the common tangent is inclined to the major axis at an angle $\tan^{-1}\sqrt{\left(\dfrac{r^2-b^2}{a^2-r^2}\right)}$.

Sol. Equation of the circle of radius r and concentric with ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

$$x^2+y^2=r^2 \quad \text{...(i)}$$

any tangent to ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is



$$y = mx+\sqrt{(a^2m^2+b^2)} \quad \text{(where } m=\tan\theta)$$

If it is a tangent to circle, then perpendicular from (0, 0) is equal to radius r,

$$\therefore \quad \frac{\sqrt{(a^2m^2+b^2)}}{\sqrt{(m^2+1)}}=|r| \text{ or } a^2m^2+b^2=m^2r^2+r^2$$

$$(a^2-r^2)m^2=r^2-b^2$$

$$\therefore \quad m=\sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)}$$

$$\tan\theta=\sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)}$$

$$\theta=\tan^{-1}\sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)}$$

| Example 27 Show that the product of the perpendiculars from the foci of any tangent to an ellipse is equal to the square of the semi minor axis, and the feet of a these perpendiculars lie on the auxiliary circle.

Sol. Let equation of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{... (i)}$$

Equation of any tangent in term of slope (m) of (i) is

$$y=mx+\sqrt{(a^2m^2+b^2)}$$

or

$$y-mx=\sqrt{a^2m^2+b^2} \quad \text{...(ii)}$$

Equation of a line perpendicular to Eq. (ii) and passing through $S(ae, 0)$ is

$$y-0=-\frac{1}{m}(x-ae)$$

or

$$x+my=ae \quad \text{...(iii)}$$

The lines Eq. (ii) and Eq. (iii) will meet at the foot of perpendicular whose locus is obtained by eliminating the variable m between Eq. (ii) and Eq. (iii), then squaring and adding Eq. (ii) and Eq. (iii), we get

$$(y-mx)^2+(x+my)^2=a^2m^2+b^2+a^2e^2$$

$$\Rightarrow \quad (1+m^2)(x^2+y^2)=a^2m^2+b^2+a^2-b^2$$

$$\Rightarrow \quad (1+m^2)(x^2+y^2)=a^2(1+m^2)$$

or

$$x^2+y^2=a^2$$

which is auxiliary circle of ellipse, similarly we can show that the other foot drawn from second focus also lies on $x^2+y^2=a^2$.

Again if p_1 and p_2 be perpendiculars from foci $S(ae, 0)$ and $S'(-ae, 0)$ on (ii), then

$$p_1=\frac{|\sqrt{(a^2m^2+b^2)}+mae|}{\sqrt{(1+m^2)}}$$

and

$$p_2=\frac{|\sqrt{(a^2m^2+b^2)}-mae|}{\sqrt{(1+m^2)}}$$

$$\therefore \quad p_1p_2=\frac{|a^2m^2+b^2-a^2e^2m^2|}{(1+m^2)}$$

$$=\frac{|a^2m^2+b^2-(a^2-b^2)m^2|}{(1+m^2)}$$

$$=\frac{b^2(1+m^2)}{(1+m^2)}$$

$$=b^2=(\text{semi minor axis})^2$$

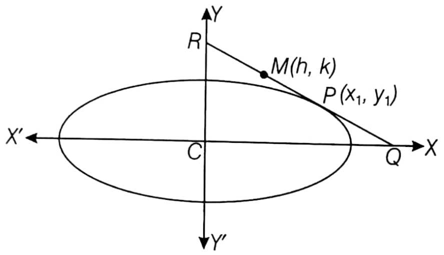
Must

Example 28 Prove that the locus of mid-points of the portion of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepted between the axes is $a^2y^2+b^2x^2=4x^2y^2$.

Sol. Let $P(x_1, y_1)$ be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

$\therefore$ Equation of tangent at (x_1, y_1) to (i) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$



This meet the coordinate axes at

$$Q\left(\frac{a^2}{x_1}, 0\right) \text{ and } R\left(0, \frac{b^2}{y_1}\right)$$

Let $M(h, k)$ be the mid-point of QR then,

$$h=\frac{\frac{a^2}{x_1}+0}{2}, k=\frac{0+\frac{b^2}{y_1}}{2}$$

$$\Rightarrow \quad x_1=\frac{a^2}{2h}, y_1=\frac{b^2}{2k}$$

Since, (x_1, y_1) lies on Eq. (i)

$$\therefore \quad \frac{\left(\frac{a^2}{2h}\right)^2}{a^2}+\frac{\left(\frac{b^2}{2k}\right)^2}{b^2}=1$$

$$\Rightarrow \quad \frac{a^2}{4h^2}+\frac{b^2}{4k^2}=1$$

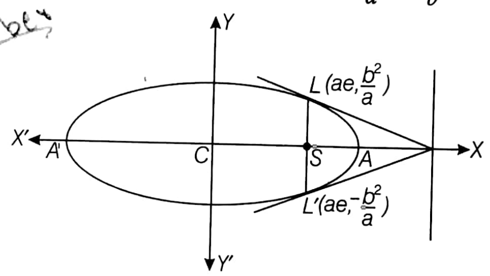
$$\Rightarrow \quad a^2k^2+b^2h^2=4h^2k^2$$

Hence, the locus of $M(h, k)$ is $a^2y^2+b^2x^2=4x^2y^2$.

Example 29 Prove that the tangents at the extremities of latusrectum of an ellipse intersect on the corresponding directrix.

Sol. Let LSL' be a latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Remember



$\therefore$ The coordinates of L and L' are

$$\left(ae, \frac{b^2}{a}\right) \text{ and } \left(ae, -\frac{b^2}{a}\right) \text{ respectively}$$

$\therefore$ Equation of tangent at $L\left(ae, \frac{b^2}{a}\right)$ is

$$\Rightarrow \quad \frac{x(ae)}{a^2}+\frac{y\left(\frac{b^2}{a}\right)}{b^2}=1$$

$$\Rightarrow \quad xe+y=a \qquad \text{...(i)}$$

The equation of the tangent at $L'\left(ae, -\frac{b^2}{a}\right)$ is

$$\frac{x(ae)}{a^2}+\frac{y\left(-\frac{b^2}{a}\right)}{b^2}=1$$

$$\Rightarrow \quad ex-y=a \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$x=\frac{a}{e} \text{ and } y=0$$

Thus, the tangents at L and L' intersect at $(a/e, 0)$ which is a point lying on the corresponding directrix i.e. $x=\frac{a}{e}$.

Equations of Normals in Different Forms

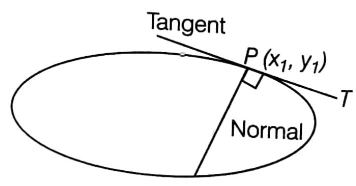
1. Point form

Theorem : The equation of normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

Proof : Since the equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$



The slope of the tangent at $(x_1, y_1)=-\frac{b^2x_1}{a^2y_1}$

$\therefore$ Slope of Normals at $(x_1, y_1) = \dfrac{a^2 y_1}{b^2 x_1}$

Hence, the equation of normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

or $$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

Remark

The equation of normal at (x_1, y_1) can also be obtained by this method

$$\frac{x - x_1}{a'x_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + b'y_1 + f} \quad \text{...(i)}$$

a', b', g, f, h are obtained by comparing the given ellipse with

$$a'x^2 + 2hxy + b'y^2 + 2gx + 2fy + c = 0 \quad \text{...(ii)}$$

The denominors of (i) can easily remembered by the first two rows of this determinant

i.e. $$\begin{vmatrix} a' & h & g \\ h & b' & f \\ g & f & c \end{vmatrix}$$

Since, first row, $a'(x_1) + h(y_1) + g(1)$

and second row, $h(x_1) + b'(y_1) + f(1)$

Here ellipse $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad \text{... (iii)}$$

Comparing Eqs. (ii) and (iii), then we get

$$a' = \frac{1}{a^2}, b' = \frac{1}{b^2}, g = 0, f = 0, h = 0$$

From, Eq. (i), equation of normal of Eq. (iii) at (x_1, y_1) is

$$\frac{x - x_1}{\frac{1}{a^2}x_1 + 0 + 0} = \frac{y - y_1}{0 + \frac{1}{b^2}y_1 + 0}$$

or $$\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}$$

or $$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

2. Parametric form

Theorem : The equation of normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (a\cos\phi, b\sin\phi) \text{ is}$$

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$$

Proof : Since, the equation of normal of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2 \quad \text{...(i)}$$

Replacing x_1 by $a\cos\phi$ and y_1 by $b\sin\phi$, then Eq.(i) becomes

$$\frac{a^2 x}{a\cos\phi} - \frac{b^2 y}{b\sin\phi} = a^2 - b^2$$

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$$

is the equation of normal at $(a\cos\phi, b\sin\phi)$

3. Slope form

Theorem : The equations of the normals of slope m to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are given by

$$y = mx \mp \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2 m^2)}}$$

at the points $\left(\pm \dfrac{a^2}{\sqrt{(a^2 + b^2 m^2)}}, \pm \dfrac{mb^2}{\sqrt{(a^2 + b^2 m^2)}} \right)$.

Proof : The equation of normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2 \quad \text{...(i)}$$

Since, 'm' is the slope of the normal, then

$$m = \frac{a^2 y_1}{b^2 x_1}$$

$$y_1 = \frac{b^2 x_1 m}{a^2} \quad \text{...(ii)}$$

Since, (x_1, y_1) lies on $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\therefore$ $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

or $$\frac{x_1^2}{a^2} + \frac{b^4 x_1^2 m^2}{a^4 b^2} = 1$$

or $$\frac{x_1^2}{a^2} + \frac{b^2 x_1^2 m^2}{a^4} = 1 \text{ or } x_1^2 = \frac{a^4}{(a^2 + b^2 m^2)}$$

$\therefore$ $$x_1 = \pm \frac{a^2}{\sqrt{a^2 + b^2 m^2}}$$

From Eq. (ii),

$$y_1 = \pm \frac{mb^2}{\sqrt{(a^2 + b^2m^2)}}$$

$\therefore$ Equation of normal in terms of slope is

$$y - \left(\pm \frac{mb^2}{\sqrt{a^2 + b^2m^2}} \right) = m \left(x - \left(\pm \frac{a^2}{\sqrt{a^2 + b^2m^2}} \right) \right)$$

$$\Rightarrow \quad y = mx \mp \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}} \quad \text{...(iii)}$$

Thus $y = mx \pm \dfrac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}}$ is a normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where m is the slope of the normal.

The coordinates of the point of contact are

$$\left(\pm \frac{a^2}{\sqrt{a^2 + b^2m^2}}, \pm \frac{mb^2}{\sqrt{a^2 + b^2m^2}} \right)$$

Comparing Eq. (iii) with,

$$y = mx + c$$

$$\therefore \quad c = \mp \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}}$$

$$\text{or} \quad c^2 = \frac{m^2(a^2 - b^2)^2}{(a^2 + b^2m^2)}$$

which is condition of normality, when $y = mx + c$ is the normal of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

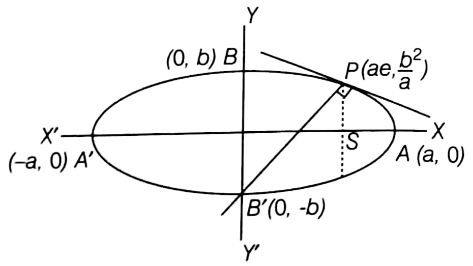
I Example 30 If the normal at an end of a latusrectum of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by

$$e^4 + e^2 - 1 = 0 \text{ or } e^2 = \frac{\sqrt{5} - 1}{2}$$

Sol. The coordinates of an end of the latusrectum are $(ae, b^2/a)$. The equation of normal at $P(ae, b^2/a)$ is

$$\frac{a^2x}{ae} - \frac{b^2(y)}{b^2/a} = a^2 - b^2$$

$$\text{or} \quad \frac{ax}{e} - ay = a^2 - b^2$$



It passes through one extremity of the minor axis whose coordinates are $(0, -b)$

$$\therefore \quad 0 + ab = a^2 - b^2$$

$$\text{or} \quad (a^2b^2) = (a^2 - b^2)^2$$

$$\text{or} \quad a^2a^2(1 - e^2) = (a^2e^2)^2$$

$$\text{or} \quad 1 - e^2 = e^4$$

$$\text{or} \quad e^4 + e^2 - 1 = 0$$

$$\text{or} \quad (e^2)^2 + e^2 - 1 = 0$$

$$\therefore \quad e^2 = \frac{-1 \pm \sqrt{1 + 4}}{2}$$

$$\Rightarrow \quad e^2 = \frac{\sqrt{5} - 1}{2} \quad \text{(taking +ve sign)}$$

I Example 31 Prove that the straight line $lx + my + n = 0$ is a normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ if $\dfrac{a^2}{l^2} + \dfrac{b^2}{m^2} = \dfrac{(a^2 - b^2)^2}{n^2}$.

Sol. The equation of any normal to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2 \quad \text{... (i)}$$

The straight line $lx + my + n = 0$ will be a normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Therefore, Eq. (i) and $lx + my + n = 0$ represent the same line

$$\frac{a \sec \phi}{l} = \frac{-b \operatorname{cosec} \phi}{m} = \frac{a^2 - b^2}{-n}$$

$$\cos \phi = \frac{-na}{l(a^2 - b^2)}$$

$$\text{and} \quad \sin \phi = \frac{nb}{m(a^2 - b^2)}$$

$$\because \quad \sin^2 \phi + \cos^2 \phi = 1$$

$$\therefore \quad \frac{n^2b^2}{m^2(a^2 - b^2)^2} + \frac{n^2a^2}{l^2(a^2 - b^2)^2} = 1$$

$$\Rightarrow \quad \frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$$

Example 32 A normal inclined at an angle of 45° to x-axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is drawn. It meets the major and minor axes in P and Q respectively. If C is the centre of the ellipse, prove that area of ΔCPQ is $\frac{(a^2-b^2)^2}{2(a^2+b^2)}$ sq units.

Sol. Let $R(a\cos\phi, b\sin\phi)$ be any point on the ellipse, then equation of normal at R is

$$ax\sec\phi - by\,\text{cosec}\,\phi = a^2-b^2$$

or
$$\frac{x}{\frac{\cos\phi(a^2-b^2)}{a}}+\frac{y}{\frac{-\sin\phi(a^2-b^2)}{b}}=1$$

If meets the major and minor axes at $P\left(\frac{(a^2-b^2)}{a}\cos\phi, 0\right)$ and $Q\left(0, -\frac{(a^2-b^2)}{b}\sin\phi\right)$ are respectively

$m=1$

So, $y = x \pm \frac{a^2-b^2}{\sqrt{a^2+b^2}}$

put $x=0$ then $y=0$

$\frac{1}{2}(x)(y)$

$$\therefore \quad CP=\left(\frac{a^2-b^2}{a}\right)|\cos\phi|$$

and
$$CQ=\left(\frac{a^2-b^2}{b}\right)|\sin\phi|$$

$$\therefore \text{ Area of } \Delta CPQ=\frac{1}{2}\times CP\times CQ$$
$$=\frac{(a^2-b^2)^2\,|\sin\phi\cos\phi|}{2ab} \qquad \text{... (i)}$$

But slope of normal $=\frac{a}{b}\tan\phi=\tan 45°$ (given)

$$\frac{a}{b}\tan\phi=1$$
$$\tan\phi=\frac{b}{a}$$
$$\therefore \quad \sin 2\phi=\frac{2\tan\phi}{1+\tan^2\phi}=\frac{2ab}{a^2+b^2}$$

$$\therefore \text{ From Eq. (i), Area of } \Delta CPQ=\frac{(a^2-b^2)^2\left|\frac{\sin 2\phi}{2}\right|}{2ab}$$
$$=\frac{(a^2-b^2)^2\frac{ab}{(a^2+b^2)}}{2ab}$$
$$=\frac{(a^2-b^2)^2}{2(a^2+b^2)}\text{ sq units.}$$

Example 33 Any ordinate MP of an ellipse meets the auxiliary circle in Q. Prove that the locus of the point of intersection of the normals at P and Q is the circle $x^2+y^2=(a^2+b^2)^2$.

Sol. Let equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ its auxiliary circle is

$$x^2+y^2=a^2$$

Coordinates of P and Q are $(a\cos\phi, b\sin\phi)$ and $(a\cos\phi, a\sin\phi)$ respectively. Equation of normal at P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$ax\sec\phi - by\,\text{cosec}\,\phi=a^2-b^2 \qquad \text{...(i)}$$

and equation of normal at Q to the circle $x^2+y^2=a^2$ is

$$y=x\tan\phi \qquad \text{...(ii)}$$

From Eq. (ii), $\tan\phi=\frac{y}{x}$

$$\therefore \quad \sin\phi=\frac{y}{\sqrt{(x^2+y^2)}} \quad\text{and}\quad \cos\phi=\frac{x}{\sqrt{(x^2+y^2)}}$$

or
$$\text{cosec}\,\phi=\frac{\sqrt{(x^2+y^2)}}{y}$$

and
$$\sec\phi=\frac{\sqrt{(x^2+y^2)}}{x} \qquad \text{...(iii)}$$

Substituting the values of $\sec\phi$ and $\text{cosec}\,\phi$ from Eq. (iii) in Eq. (i)

$$\therefore \quad ax\times\frac{\sqrt{(x^2+y^2)}}{x}-by\times\frac{\sqrt{(x^2+y^2)}}{y}=a^2-b^2$$

or
$$(a-b)\sqrt{(x^2+y^2)}=(a+b)(a-b)$$

or
$$\sqrt{x^2+y^2}=a+b$$

$$\therefore \quad x^2+y^2=(a+b)^2$$

which is required locus.

Properties of Eccentric Angles of the Co-normal Points

1. **In general, four normals can be drawn to an ellipse from any point and if $\alpha, \beta, \gamma, \delta$ the eccentric angles of these four co-normal points, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.**

   Let $Q(h, k)$ be any given point and let $P(a\cos\phi, b\sin\phi)$ be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Equation of normal at $P(a\cos\phi, b\sin\phi)$ is

$$ax\sec\phi - by\,\text{cosec}\,\phi=a^2-b^2$$

it passes through $Q(h, k)$

$\therefore \quad ah \sec\phi - bk \operatorname{cosec}\phi = a^2 - b^2$

or $$\frac{ah}{\cos\phi} - \frac{bk}{\sin\phi} = a^2 - b^2 \quad \ldots\text{(i)}$$

or $$\frac{ah}{\left(\dfrac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)}\right)} - \frac{bk}{\left(\dfrac{2\tan(\phi/2)}{1+\tan^2(\phi/2)}\right)} = a^2 - b^2 \quad \ldots\text{(ii)}$$

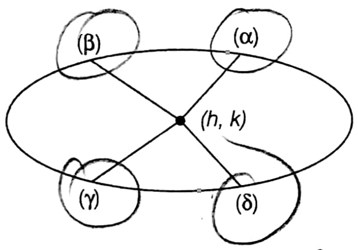
Let $\quad \tan\phi/2 = t$

then, Eq. (ii) reduced to

$$bkt^4 + 2\{ah + (a^2 - b^2)\}t^3 + 2\{ah - (a^2 - b^2)\}t - bk = 0 \quad \ldots\text{(iii)}$$

Which is a fourth degree equation in t, hence four normals can be drawn to an ellipse from any point. Consequently, it has four values of ϕ say $\alpha, \beta, \gamma, \delta$ ($\because t = \tan\phi/2$).

Now, $$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4}$$



$$= \frac{\dfrac{-2\{ah + (a^2 - b^2)\}}{bk} + \dfrac{2\{(ah - (a^2 - b^2)\}}{bk}}{1 - 0 - 1}$$

$= \infty \quad$ (From trigonometry) $\quad (\because a \neq b)$

or $$\cot\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = 0$$

or $$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2} = \text{an odd multiple of } \pi/2$$

$\Rightarrow \quad \alpha + \beta + \gamma + \delta = $ an odd multiple of π

Aliter :

Let $\quad z = e^{i\phi} = \cos\phi + i\sin\phi$

$\therefore \quad \dfrac{1}{z} = e^{-i\phi} = \cos\phi - i\sin\phi$

$\therefore \quad \cos\phi = \dfrac{z + \frac{1}{z}}{2} = \dfrac{z^2 + 1}{2z}$

and $\quad \sin\phi = \dfrac{z - \frac{1}{z}}{2i} = \dfrac{z^2 - 1}{2iz}$

Now, Eq. (i), reduces to

$$\frac{ah}{\left(\dfrac{z^2+1}{2z}\right)} - \frac{bk}{\left(\dfrac{z^2-1}{2iz}\right)} = a^2 - b^2$$

$$\Rightarrow (a^2 - b^2)z^4 - 2(ah - ibk)z^3 + 2(ah + ibk)z - (a^2 - b^2) = 0 \quad \ldots\text{(iv}$$

Consequently $z = e^{i\phi}$

gives four values of ϕ, say $\alpha, \beta, \gamma, \delta$ (Here, sum of four angles)

$\therefore \quad z_1 . z_2 . z_3 . z_4 = -1$

$\Rightarrow \quad e^{i\alpha} \cdot e^{i\beta} \cdot e^{i\gamma} \cdot e^{i\delta} = -1$

$\Rightarrow \quad e^{i(\alpha + \beta + \gamma + \delta)} = -1$

$\cos(\alpha + \beta + \gamma + \delta) + i\sin(\alpha + \beta + \gamma + \delta) = -1$

or $\quad \cos(\alpha + \beta + \gamma + \delta) = -1$

and $\quad \sin(\alpha + \beta + \gamma + \delta) = 0$

$\quad \alpha + \beta + \gamma + \delta = (2n + 1)\pi$

and $\quad \alpha + \beta + \gamma + \delta = r\pi$

where, $n\, r, \in I$

Hence, $\alpha + \beta + \gamma + \delta$ = odd multiple of π

2. If α, β, γ are the eccentric angles of three points on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the normals at which are concurrent, then

$$\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$$

Here, in each term sum of two eccentric angles

$\therefore$ From Eq. (iv),

$$\Sigma z_1 z_2 = 0$$

or $\quad z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 = 0$

$\Rightarrow \quad e^{i(\alpha+\beta)} + e^{i(\alpha+\gamma)} + e^{i(\alpha+\delta)} + e^{i(\beta+\gamma)} + e^{i(\beta+\delta)} + e^{i(\gamma+\delta)} = 0$

$\Rightarrow \quad [\cos(\alpha + \beta) + \cos(\alpha + \gamma) + \cos(\alpha + \delta) + \cos(\beta + \gamma) + \cos(\beta + \delta) + \cos(\gamma + \delta)] + i[(\sin(\alpha + \beta) + \sin(\alpha + \gamma) + \sin(\alpha + \delta) + \sin(\beta + \gamma) + \sin(\beta + \delta) + \sin(\gamma + \delta)] = 0$

Comparing the imaginary part, then

$$\sin(\alpha + \beta) + \sin(\alpha + \gamma) + \sin(\alpha + \delta) + \sin(\beta + \gamma) + \sin(\beta + \delta) + \sin(\gamma + \delta) = 0 \quad \ldots\text{(v)}$$

Since, from property Eq. (i)

$\alpha + \beta + \gamma + \delta$ = odd multiple of π

$(\alpha + \delta)$ = odd multiple of $\pi - (\beta + \gamma)$

$(\beta + \delta)$ = odd multiple of $\pi - (\alpha + \gamma)$

$$(\gamma+\delta) = \text{odd multiple of } \pi - (\alpha+\beta)$$

$$\left.\begin{aligned}\sin(\alpha+\delta) &= \sin(\beta+\gamma)\\ \sin(\beta+\delta) &= \sin(\alpha+\gamma)\\ \sin(\gamma+\delta) &= \sin(\alpha+\beta)\end{aligned}\right\}$$

$\{\because \sin(n\pi-\alpha) = \sin\alpha$, if n is integer$)\}$ ...(vi)

From Eqs. (v) and (vi), we get

$$2\sin(\alpha+\beta) + 2\sin(\beta+\gamma) + 2\sin(\gamma+\alpha) = 0$$

Hence, $\quad \sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$

Aliter :

From Eq. (iii), $\quad \Sigma t_1 t_2 = 0$ ...(vii)

and $\quad t_1 t_2 t_3 t_4 = -1$ ...(viii)

Now, $\quad \Sigma t_1 t_2 = 0$

$$\Rightarrow \quad t_1t_2 + t_2t_3 + t_3t_1 = -t_4(t_1+t_2+t_3)$$

$$\Rightarrow \quad t_1t_2 + t_2t_3 + t_3t_1 = \frac{t_1+t_2+t_3}{t_1t_2t_3} \qquad \{\text{from (viii)}\}$$

$$\Rightarrow \quad t_1t_2 + t_2t_3 + t_3t_1 = \frac{1}{t_2t_3} + \frac{1}{t_3t_1} + \frac{1}{t_1t_2}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2}$$

$$= \cot\frac{\beta}{2}\cot\frac{\gamma}{2} + \cot\frac{\gamma}{2}\cot\frac{\alpha}{2} + \cot\frac{\alpha}{2}\cot\frac{\beta}{2}$$

$$\Rightarrow \sum\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2} - \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\right) = 0$$

$$\Rightarrow \sum\left(\frac{\sin^2(\alpha/2)\sin^2(\beta/2) - \cos^2(\alpha/2)\cos^2(\beta/2)}{\sin(\alpha/2)\sin(\beta/2)\cos(\alpha/2)\cos(\beta/2)}\right) = 0$$

$$\Rightarrow \sum -4\left(\frac{\{\cos(\alpha/2)\cos(\beta/2)+\sin(\alpha/2)\sin(\beta/2)\}\{\cos(\alpha/2)\cos(\beta/2)-\sin(\alpha/2)\sin(\beta/2)\}}{\sin\alpha\sin\beta}\right) = 0$$

$$\Rightarrow \quad \sum -4\left(\frac{\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\alpha\sin\beta}\right) = 0$$

$$\Rightarrow \quad \sum -2\frac{[\cos\alpha+\cos\beta]}{\sin\alpha\sin\beta} = 0$$

$$\Rightarrow \quad \sum \frac{\sin\gamma(\cos\alpha+\cos\beta)}{\sin\alpha\sin\beta\sin\gamma} = 0$$

$$\Rightarrow \quad \sum \sin\gamma(\cos\alpha+\cos\beta) = 0$$

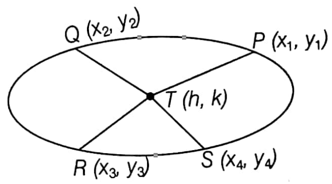
$$\Rightarrow \quad \sin\gamma(\cos\alpha+\cos\beta) + \sin\alpha(\cos\beta+\cos\gamma) + \sin\beta(\cos\gamma+\cos\alpha) = 0$$

$$\Rightarrow \quad \sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$$

Co-normal Points Lie on a Fixed Curve

Let $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$ and $T(x_4, y_4)$ be conormal points so that normal drawn from them meet in $T(h, k)$.

Then, equation of normal at $P(x_1, y_1)$ is



$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

or $\quad (a^2-b^2)x_1y_1 + b^2yx_1 - a^2xy_1 = 0$

but the point $T(h, k)$ lies on it

$\therefore \quad (a^2-b^2)x_1y_1 + b^2kx_1 - a^2hy_1 = 0$

Similarly, for points Q, R and S are

$$(a^2-b^2)x_2y_2 + b^2kx_2 - a^2hy_2 = 0$$

$$(a^2-b^2)x_3y_3 + b^2kx_3 - a^2hy_3 = 0$$

and $\quad (a^2-b^2)x_4y_4 + b^2kx_4 - a^2hy_4 = 0$

Hence, P, Q, R, S all lie on the curve

$$(a^2-b^2)xy + b^2kx - a^2hy = 0$$

This curve is called **Apollonian rectangular hyperbola.**

Remark

The feet of the normals from any fixed point to the ellipse lie at the intersections of the **Apollonian rectangular hyperbola** with the ellipse.

Exercise for Session 2

1. The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$, is

(a) 0 (b) 1 (c) 2 (d) infinite

2. If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts off intercepts of length h and k on the axes, then $\frac{a^2}{h^2} + \frac{b^2}{k^2}$ is equal to

(a) -1 (b) 0 (c) 1 (d) None of these

3. The equations of the tangents to the ellipse $3x^2 + y^2 = 3$ making equal intercepts on the axes are

(a) $y = \pm x \pm 2$ (b) $y = \pm x \pm 4$ (c) $y = \pm x \pm \sqrt{30}$ (d) $y = \pm x \pm \sqrt{35}$

4. If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then its, eccentric angle θ is equal to

(a) 0 (b) 45° (c) 60° (d) 90°

5. The number of values of $\theta \in [0, 2\pi]$ for which the line $2x\cos\theta + 3y\sin\theta = 6$ touches the ellipse $4x^2 + 9y^2 = 36$ is

(a) 1 (b) 2 (c) 4 (d) infinite

6. The common tangent of $x^2 + y^2 = 4$ and $2x^2 + y^2 = 2$ is

(a) $x + y + 4 = 0$ (b) $x - y + 7 = 0$ (c) $2x + 3y + 8 = 0$ (d) None of these

7. If the normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axes in G and g respectively, then $PG \cdot Pg =$

(a) $a : b$ (b) $a^2 : b^2$ (c) $b : a$ (d) $b^2 : a^2$

8. Number of distinct normal lines that can be drawn to the ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$, from the point $(0, 6)$ is

(a) one (b) two (c) three (d) four

9. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is

(a) 4 (b) 2 (c) 1 (d) None of these

10. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$, intersect it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

11. The line $5x - 3y = 8\sqrt{2}$ is a normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. If '$\theta$' be eccentric angle of the foot of this normal, then 'θ' is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

12. If the tangent drawn at point $(\lambda^2, 2\lambda)$ on the parabola $y^2 = 4x$ is same as the normal drawn at a point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2 + 5y^2 = 20$. Find the values of λ and θ.

13. If the normal at any point P of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, meets the major and minor axes in G and H respectively and C in the centre of the ellipse, then prove that

$$a^2(CG)^2 + b^2(CH)^2 = (a^2 - b^2)^2$$

14. If the normal at the point $P(\theta)$ to the ellipse $5x^2 + 14y^2 = 70$ intersects it again at the point $Q(2\theta)$, show that $\cos\theta = -\frac{2}{3}$.

15. The tangent and normal at any point P of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cut its major axis in point Q and R respectively. If $QR = a$ prove that the eccentric angle of the point P is given by

$$e^2\cos^2\phi + \cos\phi - 1 = 0$$

Session 3

Pair of Tangents, Chord of Contact, Chord Bisected at a Given Point, Diameter, Conjugate Diameters, Equi-Conjugate Diameters, Director Circle, Sub-Tangent and Sub-Normal, Concyclic Points, Some Standard Properties of the Ellipse, Reflection Property of an Ellipse, Equation of an Ellipse Referred to Two Perpendicular Lines

Pair of Tangents

Theorem : The combined equation of the pair of tangents drawn from a point (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

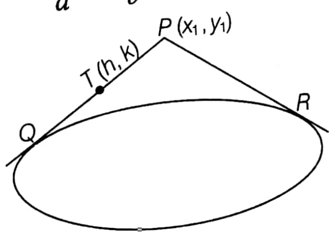
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

or $$\boxed{SS_1=T^2}$$

where $$S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1;\ S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

and $$T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

Proof : Let $T(h, k)$ be any point on the pair of tangents PQ or PR drawn from any external point $P(x_1, y_1)$ to the ellipse $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$\therefore$ Equation of PT is

$$y-y_1=\frac{k-y_1}{h-x_1}(x-x_1)$$

or $$y=\left(\frac{k-y_1}{h-x_1}\right)x+\left(\frac{hy_1-kx_1}{h-x_1}\right)$$

which is the tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\therefore$ $$c^2=a^2m^2+b^2$$

or $$\left(\frac{hy_1-kx_1}{h-x_1}\right)^2=a^2\left(\frac{k-y_1}{h-x_1}\right)^2+b^2$$

$\Rightarrow$ $$(hy_1-kx_1)^2=a^2(k-y_1)^2+b^2(h-x_1)^2$$

Hence, locus of (h, k) is

$$(xy_1-x_1y)^2=a^2(y-y_1)^2+b^2(x-x_1)^2$$

or $$(xy_1-x_1y)^2=(b^2x^2+a^2y^2)+(b^2x_1^2+a^2y_1^2)-2(b^2xx_1+a^2yy_1)$$

or $$\left(\frac{xy_1-x_1y}{ab}\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)-2\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)$$

or $$\left(\frac{xy_1-x_1y}{ab}\right)^2-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)+\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)^2$$

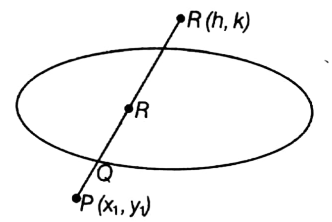
$$=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)^2+1-2\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)$$

or $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

or $$SS_1=T^2$$

Aliter : Let the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ...(i)

Let $P(x_1, y_1)$ be any point outside the ellipse let a chord of the ellipse through the point $P(x_1, y_1)$ cut the ellipse at Q and let $R(h, k)$ be any arbitrary point on the line PQ (R inside or outside). Let Q divides PR in the ratio $\lambda : 1$ then coordinates of Q is



$$\left(\frac{\lambda h+x_1}{\lambda+1}, \frac{\lambda k+y_1}{\lambda+1}\right) \quad (\because PQ:QR=\lambda:1)$$

since Q lies on ellipse Eq. (i), then

$$\frac{1}{a^2}\left(\frac{\lambda h+x_1}{\lambda+1}\right)^2+\frac{1}{b^2}\left(\frac{\lambda k+y_1}{\lambda+1}\right)^2=1$$

$$\Rightarrow \quad b^2(\lambda h+x_1)^2+a^2(\lambda k+y_1)^2=a^2b^2(\lambda+1)^2$$

$$\Rightarrow \quad (a^2k^2+b^2h^2-a^2b^2)\lambda^2+2$$

$$(hx_1b^2+ky_1a^2-a^2b^2)\lambda+(b^2x_1^2+a^2y_1^2-a^2b^2)=0 \quad \text{...(ii)}$$

Line PR will become tangent to ellipse Eq. (i) then roots of Eq. (ii) are equal

$$\therefore \quad 4(hx_1b^2+ky_1a^2-a^2b^2)^2$$
$$-4(a^2k^2+b^2h^2-a^2b^2)(b^2x_1^2+a^2y_1^2-a^2b^2)=0$$

Dividing by $4a^4b^4$

$$\therefore \quad \left(\frac{hx_1}{a^2}+\frac{ky_1}{b^2}-1\right)^2=\left(\frac{k^2}{b^2}+\frac{h^2}{a^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$$

Hence, locus of $R(h, k)$ i.e. equation of pair of tangents from $P(x_1, y_1)$ is

$$\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$$

i.e. $$T^2=SS_1 \quad \text{or} \quad SS_1=T^2$$

Remark

$S=0$ is the equation of the curve, S_1 is obtained from S by replacing x by x_1 and y by y_1 and $T=0$ is the equation of tangent at (x_1, y_1) to $S=0$

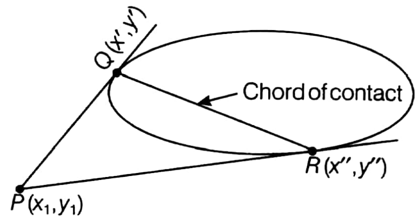
Chord of Contact

Theorem : The equation of chord of contact of tangents drawn from a point (x_1, y_1) to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

T = 0

Proof : Let PQ and PR be the tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ such that



$$Q\equiv(x', y') \quad \text{and} \quad R\equiv(x'', y'')$$

are the points of contacts of these tangents the chord QR is called chord of contact of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Equations of tangents at $Q(x', y')$ and $R(x'', y'')$ are

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$$

and $$\frac{xx''}{a^2}+\frac{yy''}{b^2}=1, \text{ respectively}$$

These tangents pass through $P(x_1, y_1)$ therefore,

$$\frac{x'x_1}{a^2}+\frac{y'y_1}{b^2}=1 \quad \text{and} \quad \frac{x''x_1}{a^2}+\frac{y''y_1}{b^2}=1$$

$\Rightarrow$ (x', y') and (x'', y'') lie on $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$

Hence, the equation QR is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$

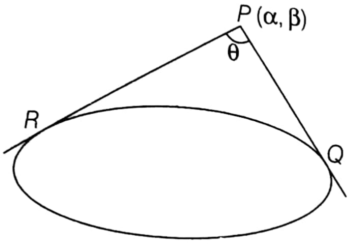
which is same as the equation of tangent but position of point differ.

| Example 34 Find the locus of the points of the intersection of tangents to ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ which make an angle θ.

Method

Sol. Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)

Equation of any tangent to ellipse Eq. (i) in terms of slope (m) is $y = mx + \sqrt{(a^2m^2+b^2)}$



Since, its passes through $P(\alpha,\beta)$ then

$$\beta = m\alpha + \sqrt{(a^2m^2+b^2)}$$

$$\Rightarrow (\beta - m\alpha) = \sqrt{(a^2m^2+b^2)}$$

$$\Rightarrow (\beta - m\alpha)^2 = a^2m^2+b^2$$

$$\Rightarrow m^2(a^2-\alpha^2)+2\alpha\beta m+(b^2-\beta^2)=0 \quad ...(ii)$$

Eq. (ii) being a quadratic equation in m.

Let roots of Eq. (ii) are m_1 and m_2, then

$$\therefore m_1+m_2 = -\frac{2\alpha\beta}{(a^2-\alpha^2)},\ m_1m_2=\frac{b^2-\beta^2}{a^2-\alpha^2}$$

$$\therefore (m_1-m_2)=\sqrt{(m_1+m_2)^2-4m_1m_2}$$

$$=\sqrt{\frac{4\alpha^2\beta^2}{(a^2-\alpha^2)^2}-\frac{4(b^2-\beta^2)}{(a^2-\alpha^2)}}$$

$$=\sqrt{\frac{4\alpha^2\beta^2-4(b^2-\beta^2)(a^2-\alpha^2)}{(a^2-\alpha^2)^2}}$$

$$=\sqrt{\frac{4\{\alpha^2\beta^2-a^2b^2+b^2\alpha^2+a^2\beta^2-\alpha^2\beta^2\}}{(a^2-\alpha^2)^2}}$$

$$=\frac{2}{|(a^2-\alpha^2)|}\sqrt{(a^2\beta^2+b^2\alpha^2-a^2b^2)}$$

$\because$ θ be the angle between these two tangents, then

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$=\left|\frac{\frac{2}{(a^2-\alpha^2)}\sqrt{(a^2\beta^2+b^2\alpha^2-a^2b^2)}}{\left(1+\frac{b^2-\beta^2}{a^2-\alpha^2}\right)}\right|$$

$$\tan\theta = \left|\frac{2\sqrt{(a^2\beta^2+b^2\alpha^2-a^2b^2)}}{a^2+b^2-\alpha^2-\beta^2}\right|$$

or $(a^2+b^2-\alpha^2-\beta^2)^2\tan^2\theta = 4\ (a^2\beta^2+b^2\alpha^2-a^2b^2)$

$\Rightarrow (\alpha^2+\beta^2-a^2-b^2)^2\tan^2\theta = 4\ (b^2\alpha^2+a^2\beta^2-a^2b^2)$

$\therefore$ Locus of $P(\alpha,\beta)$ is

$$(x^2+y^2-a^2-b^2)^2\tan^2\theta = 4\ (b^2x^2+a^2y^2-a^2b^2)$$

Example 35 Prove that the chord of contact of tangents drawn from the point (h,k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at the centre, if $\frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$. Also, find the locus of (h,k).

Sol. The equation of chord of contact of tangents drawn from $P(h,k)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{hx}{a^2}+\frac{ky}{b^2}=1$ ...(i)

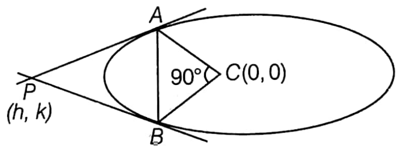
The equation of the straight lines CA and CB is obtained by making homogeneous ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the help of Eq. (i)

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{hx}{a^2}+\frac{ky}{b^2}\right)^2$$

$$\Rightarrow \left(\frac{k^2}{a^4}-\frac{1}{a^2}\right)x^2+\left(\frac{k^2}{b^4}-\frac{1}{b^2}\right)y^2+\frac{2hk}{a^2b^2}xy=0 \quad ...(ii)$$

But given $\angle ACB = 90°$

$\therefore$ Coefficient of x^2 + Coefficient of $y^2=0$



$$\Rightarrow \frac{h^2}{a^4}-\frac{1}{a^2}+\frac{k^2}{b^4}-\frac{1}{b^2}=0$$

$$\Rightarrow \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

Hence, locus of (h,k) is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

Chord Bisected at a Given Point

Theorem : The equation of a chord of the ellipse

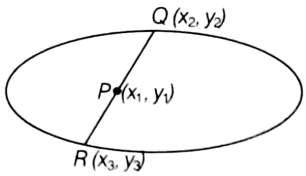
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

bisected at the point (x_1,y_1) is given by

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

or $\boxed{T=S_1}$

Proof : Let $Q\equiv(x_2,y_2)$ and $R\equiv(x_3,y_3)$ be the end points of a chord QR of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let $P\equiv(x_1,y_1)$ be its mid point.



Now, $Q \equiv (x_2, y_2)$ and $R \equiv (x_3, y_3)$ lie on $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1 \quad \text{...(i)}$$

and
$$\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} = 1 \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i),

$$\therefore \quad \frac{1}{a^2}(x_2^2 - x_3^2) + \frac{1}{b^2}(y_2^2 - y_3^2) = 0$$

$$\Rightarrow \quad \frac{(x_2 + x_3)(x_2 - x_3)}{a^2} + \frac{(y_2 + y_3)(y_2 - y_3)}{b^2} = 0$$

$$\Rightarrow \quad \frac{y_2 - y_3}{x_2 - x_3} = -\frac{b^2 (x_2 + x_3)}{a^2 (y_2 + y_3)} = -\frac{b^2}{a^2} \cdot \frac{2x_1}{2y_1}$$

$$\left(\because x_1 = \frac{x_2 + x_3}{2} \text{ and } y_1 = \frac{y_2 + y_3}{2} \right)$$

$$= -\frac{b^2 x_1}{a^2 y_1} \quad \text{...(iii)}$$

$\therefore$ Equation of QR is

$$y - y_1 = \frac{y_2 - y_3}{x_2 - x_3}(x - x_1)$$

$$\Rightarrow \quad y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1) \quad \text{[from Eq. (iii)]}$$

$$\Rightarrow \quad \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{xx_1}{a^2} + \frac{x_1^2}{a^2}$$

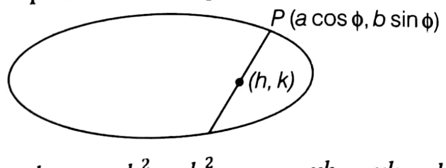
$$\Rightarrow \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \text{ or } T = S_1$$

where,
$$T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$

and
$$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

| Example 36 Prove that the locus of the middle points of normal chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is the curve $\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)^2 \left(\dfrac{a^6}{x^2} + \dfrac{b^6}{y^2}\right) = (a^2 - b^2)^2$.

Sol. Let (h, k) be the middle point of any chord of an ellipse, then its equation is $T = S_1$



$$\text{or} \quad \frac{xh}{a^2} + \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 \text{ or } \frac{xh}{a^2} + \frac{yk}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \quad \text{...(i)}$$

If Eq. (i) is a normal chords, then it must be of the form

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2 \quad \text{...(ii)}$$

Thus, the Eqs. (i) and (ii) represents the same normal chord of the ellipse with its middle point (h, k).

Hence, they are identical and comparing their co-efficients,

we get
$$\frac{h/a^2}{a \sec \phi} = \frac{k/b^2}{-b \operatorname{cosec} \phi} = \frac{\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}}{(a^2 - b^2)}$$

$$\Rightarrow \quad \cos \phi = \frac{a^3}{h} \frac{\left(\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}\right)}{(a^2 - b^2)} \quad \text{...(iii)}$$

and
$$\sin \phi = -\frac{b^3}{k} \frac{\left(\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}\right)}{(a^2 - b^2)} \quad \text{...(iv)}$$

Squaring and adding Eqs. (iii) and (iv), then

$$\cos^2 \phi + \sin^2 \phi = \frac{\left(\dfrac{a^6}{h^2} + \dfrac{b^6}{k^2}\right)\left(\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}\right)^2}{(a^2 - b^2)^2}$$

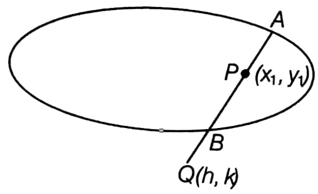
$$\Rightarrow \quad 1 = \frac{\left(\dfrac{a^6}{h^2} + \dfrac{b^6}{k^2}\right)\left(\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}\right)^2}{(a^2 - b^2)^2}$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 \left(\frac{a^6}{h^2} + \frac{b^6}{k^2}\right) = (a^2 - b^2)^2$$

Hence, locus of (h, k) is $\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)^2 \left(\dfrac{a^6}{x^2} + \dfrac{b^6}{y^2}\right) = (a^2 - b^2)^2$

| Example 37 Show that the locus of the middle points of chords of an ellipse which pass through a fixed point, is another ellipse

Sol. Let $P(x_1, y_1)$ be the middle point of any chord AB of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then equation of chord AB is



$$T = S_1$$

$$\Rightarrow \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

$$\Rightarrow \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \quad \text{... (i)}$$

But it passes through a fixed point $Q(h, k)$ its coordinates must satisfy Eq. (i),

$$\therefore \quad \frac{hx_1}{a^2} + \frac{ky_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

This can be re-written as

$$\frac{\left(x_1 - \frac{h}{2}\right)^2}{a^2} + \frac{\left(y_1 - \frac{k}{2}\right)^2}{b^2} = \frac{1}{4}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)$$

Hence, locus of $P(x_1, y_1)$ is

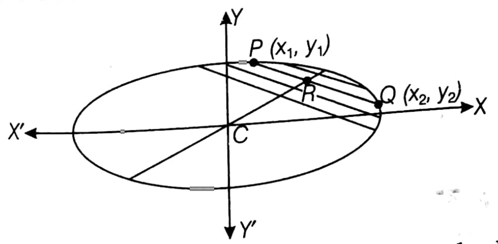
$$\frac{\left(x - \frac{h}{2}\right)^2}{a^2} + \frac{\left(y - \frac{k}{2}\right)^2}{b^2} = \frac{1}{4}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)$$

Its obviously an ellipse with centre at $\left(\frac{h}{2}, \frac{k}{2}\right)$ and axes parallel to coordinates axes.

Diameter

The locus of the middle points of a system of parallel chords of an ellipse is called a diameter and the point where the diameter intersects the ellipse is called the vertex of the diameter.

Let $y = mx + c$ be system of parallel chords to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for different chords c varies, m remains constant.



Let the extremities of any chord PQ of the set be $P(x_1, y_1)$ and $Q(x_2, y_2)$ and let its middle point be $R(h, k)$, then solving equations.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } y = mx + c$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow \quad (a^2m^2 + b^2)x^2 + 2mca^2x + a^2(c^2 - b^2) = 0$$

Since, x_1 and x_2 be the roots of this equation, then

$$x_1 + x_2 = -\frac{2mca^2}{a^2m^2 + b^2} \quad \text{...(i)}$$

Since, (h, k) be the middle point of QR, then

$$h = \frac{x_1 + x_2}{2}$$

then, from Eq. (i),

$$2h = -\frac{2mca^2}{a^2m^2 + b^2}$$

$$\Rightarrow \quad h = -\frac{mca^2}{a^2m^2 + b^2} \quad \text{...(ii)}$$

but (h, k) lies on $y = mx + c$

$$\therefore \quad k = mh + c, \; c = k - mh \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), then, $h = -\frac{ma^2(k - mh)}{a^2m^2 + b^2}$

$$\Rightarrow \quad a^2m^2h + b^2h = -mka^2 + m^2a^2h$$

$$\Rightarrow \quad b^2h = -mka^2 \text{ or } k = -\frac{b^2h}{a^2m}$$

Hence, locus of $R(h, k)$ is $y = -\frac{b^2x}{a^2m}$

which is diameter of the ellipse passing through $(0, 0)$.

Aliter : Let (h, k) be the middle point of the chord $y = mx + c$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

$$T = S_1$$

$$\Rightarrow \quad \frac{xh}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

$$\therefore \quad \text{Slope} = -\frac{b^2h}{a^2k} = m$$

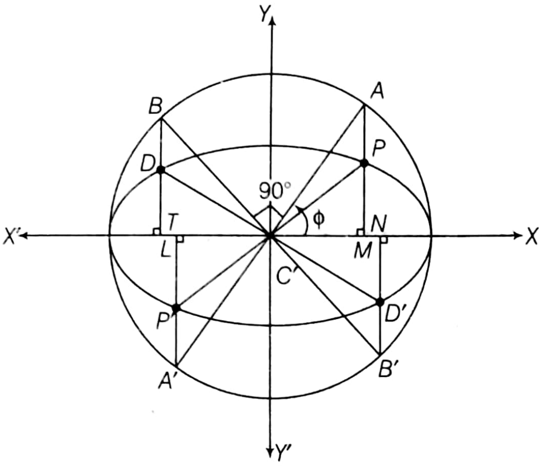
$$\Rightarrow \quad k = -\frac{b^2h}{a^2m}$$

Hence, locus of the mid-point is

$$y = -\frac{b^2x}{a^2m}$$

Conjugate Diameters

Two diameters are said to be conjugate when each bisects all chords parallel to the other. If $y = mx$ and $y = m_1x$ be two conjugate diameters of an ellipse, then $mm_1 = \frac{b^2}{a^2}$.



Conjugate diameters of circle i.e. AA' and BB' are perpendicular to each other. Hence, conjugate diameters of ellipse are PP' and DD'.

Hence, angle between conjugate diameters of ellipse $\boxed{\neq 90°}$.

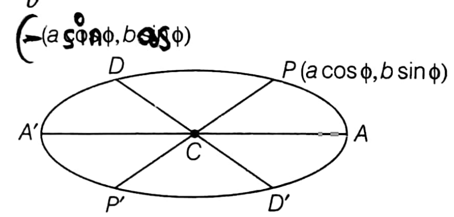
Now the co-ordinates of the four extremities of two conjugate diameters are

$$\left.\begin{array}{l} P\,(a\cos\phi,\, b\sin\phi),\ P'\,(-a\cos\phi, -b\sin\phi), \\ D\,(-a\sin\phi,\, b\cos\phi),\ D'\,(a\sin\phi, -b\cos\phi) \end{array}\right\}$$

Properties of Conjugate Diameters

Prop. 1 : *The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.*

Let PCP' and DCD' be two conjugate diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and let the eccentric angles of the



extremities P and D be ϕ and ϕ' respectively. Then, the co-ordinates of P and D are $(a\cos\phi, b\sin\phi)$ and $(a\cos\phi', b\sin\phi')$ respectively.

Now $m_1 =$ slope of $CP = \dfrac{b}{a}\tan\phi$

and $m_2 =$ slope of $CD = \dfrac{b}{a}\tan\phi'$

since, the diameters PCP' and DCD' are conjugate diameters.

$$\therefore \qquad m_1 m_2 = -\frac{b^2}{a^2}$$

$$\Rightarrow \qquad \frac{b^2}{a^2}\tan\phi\tan\phi' = -\frac{b^2}{a^2}$$

$$\Rightarrow \qquad \tan\phi\tan\phi' = -1$$

$$\Rightarrow \qquad \tan\phi = -\cot\phi' = \tan\left(\frac{\pi}{2}+\phi'\right)$$

$$\Rightarrow \qquad \boxed{\phi = \frac{\pi}{2}+\phi'} \Rightarrow \phi-\phi' = \frac{\pi}{2}$$

Prop. 2 : *The sum of the squares of any two conjugate semi diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse* i.e.

$$\boxed{CP^2 + CD^2 = a^2 + b^2}$$

Let CP and CD be two conjugate semi-diameters of an ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and let eccentric angle of P is ϕ. The eccentric angle of D is $\dfrac{\pi}{2}+\phi$. So the coordinates of P and D are

$$(a\cos\phi, b\sin\phi) \quad \text{and} \quad \left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right)$$

i.e. $(-a\sin\phi, b\cos\phi)$ respectively

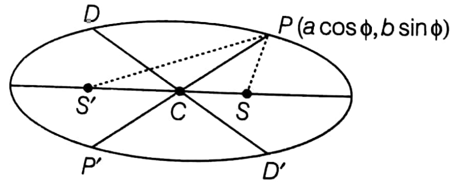
$$\therefore \qquad CP^2 + CD^2 = (a^2\cos^2\phi + b^2\sin^2\phi) + (a^2\sin^2\phi + b^2\cos^2\phi)$$

$$= a^2 + b^2$$

Prop. 3 : *The product of the focal distances of a point on an ellipse is equal to the square of the semi diameter which is conjugate to the diameter through the point.*

Let PCP' and DCD' be the conjugate diameters of an ellipse and let the eccentric angle of P is ϕ then coordinates of P is $(a\cos\phi, b\sin\phi)$

$\therefore$ Coordinates of D is $(-a\sin\phi, b\cos\phi)$



Let S and S' be two foci of the ellipse. Then

$$SP \cdot S'P = (a - ae\cos\phi)\cdot(a + ae\cos\phi)$$

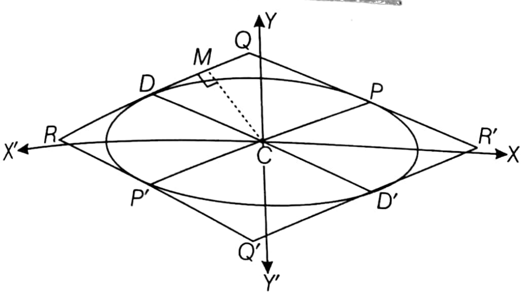
$$= a^2 - a^2e^2\cos^2\phi$$

$$= a^2 - (a^2 - b^2)\cos^2\phi \quad \{\because b^2 = a^2(1-e^2)\}$$

$$= a^2\sin^2\phi + b^2\cos^2\phi = CD^2$$

$$\{\therefore a^2 - b^2 = a^2e^2\}$$

Prop. 4 : *The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to product of the axes.*



Let PCP' and DCD' be a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let the eccentric angle of P be ϕ. Then the eccentric angle of D is $\left(\frac{\pi}{2}+\phi\right)$ so the coordinates of P and D are

$$(a\cos\phi, b\sin\phi) \text{ and } \left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right)$$

i.e. $(-a\sin\phi, b\cos\phi)$

Similarly the coordinates of P' and D' are

$(-a\cos\phi, -b\sin\phi)$ and $(a\sin\phi, -b\cos\phi)$ respectively.

Equation of tangents at P, D, P' and D' are respectively.

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1, -\frac{x}{a}\sin\phi+\frac{y}{b}\cos\phi=1,$$

$$-\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi=1$$

and $\frac{x}{a}\sin\phi-\frac{y}{b}\cos\phi=1$

Clearly the tangents at P and P' are parallel. Also, the tangents at D and D' are parallel. Hence, the tangents at P, D, P', D' form a parallelogram.

Area of parallelogram $QRQ'R' = 4$ (the area of parallelogram $QDCP$)

$$= 4\cdot|QD|\cdot\{\perp \text{ from } C \text{ on } QD\}$$

$$= 4\cdot|CP|\cdot\{\perp \text{ from } C \text{ on } QD\} \quad \text{...(i)}$$

Now $\quad |CP| = \sqrt{(a^2\cos^2\phi+b^2\sin^2\phi)}$

$\because$ tangent at D is $-\frac{x}{a}\sin\phi+\frac{y}{b}\cos\phi=1$

$\therefore$ $\perp$ from C on

$$QD = \frac{1}{\sqrt{\frac{\sin^2\phi}{a^2}+\frac{\cos^2\phi}{b^2}}} = \frac{ab}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}$$

Now from Eq. (i),

Area of parallelogram $QRQ'R'$

$$= 4\times\sqrt{(a^2\cos^2\phi+b^2\sin^2\phi)}\times\frac{ab}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi)}}$$

$$= 4ab\,(=\text{constant}) = \boxed{(2a)(2b)}$$

= Area of rectangle contained under major and minor axes.

Prop. 5 : *The polar of any point with respect to ellipse is parallel to the diameter to the one on which the point lies. Hence obtain the equation of the chord whose mid-point is (h, k).*

Let (h, k) be the point on the diameter $y = m_1x$

$\therefore \quad m_1 = k/h$

any diameter conjugate to it is $y = m_2x$

but $\quad m_1m_2 = -\frac{b^2}{a^2} \Rightarrow \frac{k}{h}m_2 = -\frac{b^2}{a^2}$

$\therefore \quad m_2 = -\frac{b^2h}{a^2k}$

Polar of (h, k) is $\frac{hx}{a^2}+\frac{ky}{b^2}=1 \quad$ ...(i)

Its slope is $-\frac{b^2h}{a^2k}=m_2$ and hence parallel.

Now, equation of chord parallel to the Eq. (i) is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\lambda \quad \text{...(ii)}$$

It is passes through points (h, k)

$$\therefore \quad \lambda = \frac{h^2}{a^2}+\frac{k^2}{b^2} \quad \text{... (iii)}$$

From Eqs. (ii) and (iii), $\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$

i.e. $\boxed{T = S_1}$

which is the equation of chord of the ellipse, if mid-points is (h, k).

Equi-Conjugate Diameters

Two conjugate diameters are called *equi-conjugate* if their lengths are equal. In such cases therefore.

$$(CP)^2 = (CD)^2$$

$\therefore \quad a^2\cos^2\phi+b^2\sin^2\phi = a^2\sin^2\phi+b^2\cos^2\phi$

$\Rightarrow \quad (a^2\cos^2\phi-\sin^2\phi)-b^2(\cos^2\phi-\sin^2\phi)=0$

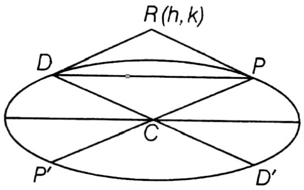
$$\Rightarrow \qquad (a^2 - b^2)\cos 2\phi = 0$$
$$\because \qquad (a^2 - b^2) \neq 0$$
$$\therefore \qquad \cos 2\phi = 0$$
$$\therefore \qquad \phi = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$
$$\therefore \qquad \boxed{(CP) = (CD) = \sqrt{\frac{(a^2 + b^2)}{2}}}$$

Example 38 Show that the tangents at the ends of conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$ intersect on the ellipse $x^2/a^2 + y^2/b^2 = 2$.

Or

Prove that the locus of the poles of the line joining the extremities of two conjugate diameters is the ellipse $x^2/a^2 + y^2/b^2 = 2$.

Sol. Let CP and CD be two semi-conjugate diameters, so that if eccentric angle of P is ϕ then eccentric angle of D is $\boxed{\frac{\pi}{2} + \phi}$



$\therefore$ Coordinates of P and D are

$(a\cos\phi, b\sin\phi)$ and $\left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right)$

respectively

$\therefore$ Equation of (PD) is

$$\frac{x}{a}\cos\left(\frac{\phi + \frac{\pi}{2} + \phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\phi + \frac{\pi}{2} + \phi}{2}\right) = \cos\left(\frac{\frac{\pi}{2} + \phi - \phi}{2}\right)$$

$$\Rightarrow \quad \frac{x}{a}\cos\left(\frac{\pi}{4}+\phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{4}+\phi\right) = \frac{1}{\sqrt{2}} \qquad \text{...(i)}$$

If its pole or point of intersection of tangents at its extremities be (h, k), then its equation is the same as that of the polar or the chord of contact of (h, k).

i.e.
$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \qquad \text{...(ii)}$$

Since, Eqs. (i) and (ii) are identical, comparing

$$\frac{h}{a\cos\left(\frac{\pi}{4}+\phi\right)} = \frac{k}{b\sin\left(\frac{\pi}{4}+\phi\right)} = \sqrt{2}$$

or
$$\sqrt{2}\cos\left(\frac{\pi}{4}+\phi\right) = \frac{h}{a} \qquad \text{...(iii)}$$

or
$$\sqrt{2}\sin\left(\frac{\pi}{4}+\phi\right) = \frac{k}{b} \qquad \text{...(iv)}$$

Squaring and adding Eqs. (iii) and (iv), then

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = 2\left(\cos^2\left(\frac{\pi}{4}+\phi\right) + \sin^2\left(\frac{\pi}{4}+\phi\right)\right)$$

$$\therefore \quad \frac{h^2}{a^2} + \frac{k^2}{b^2} = 2$$

Hence, locus of (h, k) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

Aliter : Equation of tangents at P and D are

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1 \qquad \text{...(i)}$$

and
$$\frac{x}{a}\cos\left(\frac{\pi}{2}+\phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{2}+\phi\right) = 1$$

i.e.
$$-\frac{x}{a}\sin\phi + \frac{y}{b}\cos\phi = 1 \qquad \text{...(ii)}$$

Squaring and adding Eqs. (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

which is required locus.

Example 39 If $x\cos\alpha + y\sin\alpha = p$ is a chord joining the ends P and D of conjugate semi-diameters, of the ellipse then prove that $a^2\cos^2\alpha + b^2\sin^2 a = 2p^2$ and hence or otherwise deduce that the line PD always touches a similar ellipse.

Sol. Let equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, eccentric angle of P is ϕ, then eccentric angle of D is $\frac{\pi}{2} + \phi$

$\therefore$ Coordinates of P and D are

$(a\cos\phi, b\sin\phi)$ and $\left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right)$

$\therefore$ Equation of PD is

$$\frac{x}{a}\cos\left(\frac{\phi + \frac{\pi}{2} + \phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\phi + \frac{\pi}{2} + \phi}{2}\right) = \cos\left(\frac{\frac{\pi}{2} + \phi - \phi}{2}\right)$$

$$\Rightarrow \quad \frac{x}{a}\cos\left(\frac{\pi}{4}+\phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{4}+\phi\right) = \frac{1}{\sqrt{2}} \qquad \text{...(i)}$$

If it is same as $x\cos\alpha + y\sin\alpha = p$ ...(ii)

then on comparing, we get

$$\frac{\cos\left(\frac{\pi}{4}+\phi\right)}{a\cos\alpha} = \frac{\sin\left(\frac{\pi}{4}+\phi\right)}{b\sin\alpha} = \frac{1}{p\sqrt{2}}$$

or $$a\cos\alpha = p\sqrt{2}\cos\left(\frac{\pi}{4}+\phi\right) \quad \text{...(iii)}$$

and $$b\sin\alpha = p\sqrt{2}\sin\left(\frac{\pi}{4}+\phi\right) \quad \text{...(iv)}$$

Squaring and adding Eq. (iii) and (iv), we get

$$a^2\cos^2\alpha + b^2\sin^2\alpha = (p\sqrt{2})^2$$
$$\left\{\cos^2\left(\frac{\pi}{4}+\phi\right)+\sin^2\left(\frac{\pi}{4}+\phi\right)\right\}$$

$\Rightarrow$ $$a^2\cos^2\alpha + b^2\sin^2\alpha = 2p^2$$

Again, line Eq. (i) can be written as

$$\frac{x}{a/\sqrt{2}}\cos\theta + \frac{y}{b/\sqrt{2}}\sin\theta = 1,$$

where $$\theta = \frac{\pi}{4}+\phi$$

which is clearly a tangent to the ellipse

$$\frac{x^2}{(a/\sqrt{2})^2}+\frac{y^2}{(b\sqrt{2})^2}=1 \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2} \quad \text{...(v)}$$

If e' be its eccentricity, then

$$(b/\sqrt{2})^2 = (a/\sqrt{2})^2(1-e'^2)$$

$\Rightarrow$ $$b^2 = a^2(1-e'^2)$$

but $$b^2 = a^2(1-e^2)$$

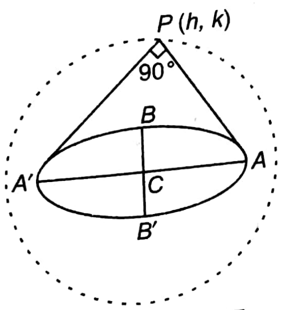
$\therefore$ $$e = e'$$

Hence, ellipse (v) is a similar ellipse.

Director Circle

The locus of the point of intersection of the tangents to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which are perpendicular to each other is called director circle.

Let any tangent in terms of slope of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is



$$y = mx + \sqrt{(a^2m^2+b^2)}$$

It is passes through (h, k)

$\therefore$ $$k = mh + \sqrt{(a^2m^2+b^2)}$$

or $$(k-mh)^2 = a^2m^2+b^2$$

$\Rightarrow$ $$k^2+m^2h^2-2mhk = a^2m^2+b^2$$

$\Rightarrow$ $$m^2(h^2-a^2)-2hkm+k^2-b^2=0$$

It is quadratic equation in m let slope of two tangents are m_1 and m_2

$\therefore$ $$m_1m_2 = \frac{k^2-b^2}{h^2-a^2}$$
$$-1 = \frac{k^2-b^2}{h^2-a^2}$$

($\because$ tangents are perpendicular)

$\Rightarrow$ $$-h^2+a^2 = k^2-b^2$$

or $$h^2+k^2 = a^2+b^2$$

Hence, locus of $P(h, k)$ is

$$\boxed{x^2+y^2=a^2+b^2}$$

Aliter :

If any tangent $y = mx+\sqrt{(a^2m^2+b^2)}$ ...(i)

and $$y = -\frac{x}{m}+\sqrt{\left\{a^2\left(-\frac{1}{m}\right)^2+b^2\right\}} \quad \text{...(ii)}$$

touch the ellipse and intersect at right angles.

From Eq. (i), $$y-mx = \sqrt{(a^2m^2+b^2)} \quad \text{...(iii)}$$

Eq. (ii) can be re-written as

$$x+my = \sqrt{(a^2+b^2m^2)} \quad \text{...(iv)}$$

Squaring and adding Eqs. (iii) and (iv), then

$$(y-mx)^2+(x+my)^2 = a^2m^2+b^2+a^2+b^2m^2$$

$\Rightarrow$ $$(1+m^2)(x^2+y^2) = (1+m^2)(a^2+b^2)$$

Hence, $x^2+y^2=a^2+b^2$ is the **director circle** of the ellipse.

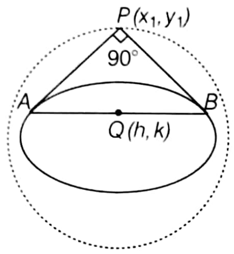
| Example 40 Tangents at right angles are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show that the locus of the middle points of the chord of contact is the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 = \frac{x^2+y^2}{a^2+b^2}$$

Sol. Let $Q(h, k)$ be the middle point of the chord of contact

$\therefore$ Equation of chord AB whose mid point $Q(h, k)$ is

$T = S_1$



$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \quad \text{...(i)}$$

and equation of chord of contact AB with respect to $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{...(ii)}$$

the Eqs. (i) and (ii) are identical, hence comparing their coefficient, we get

$$\frac{x_1}{h} = \frac{y_1}{k} = \frac{1}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)}$$

$$\therefore \quad x_1 = \frac{h}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)} \quad \text{...(iii)}$$

and
$$y_1 = \frac{k}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)} \quad \text{...(iv)}$$

Since tangents are at right angles, then the point (x_1, y_1) must lie on the director circle $x^2 + y^2 = a^2 + b^2$, of the ellipse

$$x_1^2 + y_1^2 = a^2 + b^2$$

$$\frac{h^2}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2} + \frac{k^2}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2} = a^2 + b^2$$

[from Eqs. (iii) and (iv)]

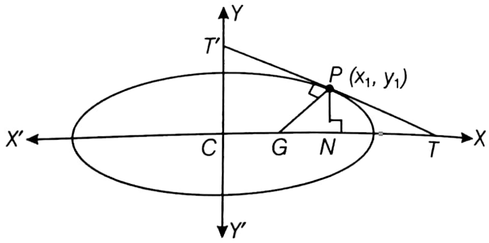
$$\therefore \quad \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 = \frac{h^2 + k^2}{a^2 + b^2}$$

Hence, locus of mid-point $Q(h, k)$ is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2 + y^2}{a^2 + b^2}$

Sub-Tangent and Sub-Normal

Let the tangent and normal at $P(x_1, y_1)$ meet the axes at T and G respectively.

Equation of tangent at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is



$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{...(i)}$$

$\because$ T lies on X-axis.

Put $y = 0$ in Eq. (i) $\Rightarrow x = CT$

$$\therefore \quad CT = \frac{a^2}{x_1} \text{ and } CN = x_1$$

and **length of sub-tangent** $NT = CT - CN = \frac{a^2}{x_1} - x_1$

Equation of normal at $P(x_1, y_1)$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is}$$

$$\frac{x - x_1}{x_1 / a^2} = \frac{y - y_1}{y_1 / b^2} \quad \text{...(ii)}$$

$\because$ G lies on X-axis. Put $y = 0$ in Eq. (ii)

$$\Rightarrow \quad x = CG$$

$$\therefore \quad CG = x_1 - \frac{b^2}{a^2} x_1$$

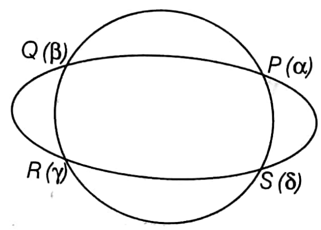
$\therefore$ **Length of sub-normal**

$$GN = CN - CG = x_1 - \left(x_1 - \frac{b^2}{a^2} x_1\right)$$

$$= \frac{b^2}{a^2} x_1 = (1 - e^2)\, x_1$$

Concyclic Points

Any circle intersects an ellipse in two or four real points. They are called concyclic points and the sum of their eccentric angles is an even multiple of π. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four concyclic points on an ellipse, then prove that $\alpha + \beta + \gamma + \delta = 2n\pi$ where n is any integer.



Let the given circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{...(i)}$$

and the given ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{...(ii)}$$

Let $(a\cos\phi, b\sin\phi)$ be a point of intersection of Eqs. (i) and (ii).

As it lies on the circle Eq. (i).

$$\therefore \quad a^2\cos^2\phi+b^2\sin^2\phi+2ga\cos\phi$$
$$+2fb\sin\phi+c=0 \quad \text{...(iii)}$$

or $$a^2\left(\frac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)}\right)^2+b^2\left(\frac{2\tan(\phi/2)}{1+\tan^2(\phi/2)}\right)^2$$
$$+2ga\left(\frac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)}\right)+2fb\left(\frac{2\tan(\phi/2)}{1+\tan^2(\phi/2)}\right)+c=0 \quad \text{...(iv)}$$

Put $\tan(\phi/2)=t$

$\therefore$ Eq. (iv) reduces to

$$a^2\left(\frac{1-t^2}{1+t^2}\right)^2+b^2\left(\frac{2t}{1+t^2}\right)^2+2ga\left(\frac{1-t^2}{1+t^2}\right)+2fb\left(\frac{2t}{1+t^2}\right)+c=0$$

or $$(a^2-2ga+c)t^4+4bf\,t^3+(4b^2-2a^2+2c)\,t^2$$
$$+4bft+(a^2+2ga+c)=0 \quad \text{...(v)}$$

which is biquadratic equation in t.

i.e. it has four values of t

$$t=\tan(\phi/2)$$

Since, four values of eccentric angles are $\alpha,\beta,\gamma,\delta$

$$\therefore \quad \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\frac{S_1-S_3}{1-S_2+S_4}$$
$$=\frac{\Sigma t_1-\Sigma t_1t_2t_3}{1-\Sigma t_1t_2+t_1t_2t_3t_4}=0$$

$$\Rightarrow \quad \frac{1}{2}(\alpha+\beta+\gamma+\delta)=n\pi$$

$\therefore$ $\alpha+\beta+\gamma+\delta=2n\pi$, when n is any integer.

Aliter : Let P, Q, R, S be four concyclic points on an ellipse, whose eccentric angles $\alpha,\beta,\gamma,\delta$ respectively.

Then equation of the chords PQ and RS are (Take any two chords)

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)=0$$

and $$\frac{x}{a}\cos\left(\frac{\gamma+\delta}{2}\right)+\frac{y}{b}\sin\left(\frac{\gamma+\delta}{2}\right)-\cos\left(\frac{\gamma-\delta}{2}\right)=0$$

Now, the equation of any curve passing through P, Q, R and S is given by

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\lambda$$
$$\left(\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)\right)$$
$$\times\left(\frac{x}{a}\cos\left(\frac{\gamma+\delta}{2}\right)+\frac{y}{b}\sin\left(\frac{\gamma+\delta}{2}\right)-\cos\left(\frac{\gamma-\delta}{2}\right)\right)=0$$

But the given points are concyclic. Hence this equation will represent a circle, if co-efficient of x^2 = co-efficient y^2. and co-efficient of $xy=0$

Now equation of the co-efficient of $xy=0$

$$\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\gamma+\delta}{2}\right)+\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\gamma+\delta}{2}\right)=0$$

or $$\sin\left(\frac{\alpha+\beta+\gamma+\delta}{2}\right)=0=\sin n\pi$$

$$\therefore \quad \frac{1}{2}(\alpha+\beta+\gamma+\delta)=n\pi \quad \text{or} \quad \alpha+\beta+\gamma+\delta=2n\pi$$

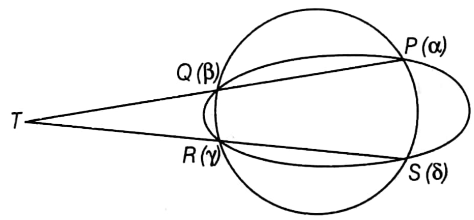
where, n is any integer.

Hence, the sum of eccentric angles of four concylic points on an ellipse is always an even multiple of π

Corollary 1 : *Prove that the common chords of a circle and an ellipse are equally inclined to the axes of the ellipse.*

If the point of intersection of chords PQ and RS is T, then equation of chord PQ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$



$$\therefore \text{ Slope of } PQ=-\frac{b}{a}\cot\left(\frac{\alpha+\beta}{2}\right)$$
$$=-\frac{b}{a}\cot\left(n\pi-\frac{\gamma+\delta}{2}\right) \quad (\because \alpha+\beta+\gamma+\delta=2n\pi)$$
$$=\frac{b}{a}\cot\left(\frac{\gamma+\delta}{2}\right)=-(\text{slope of } RS)$$

Hence, PQ and RS are equally inclined to the axis of x.

Corollary 2 : *Find the centre of the circle passing through the three points on an ellipse whose eccentric angles are α, β, γ.*

Let the point of intersection of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

and circle $\quad x^2 + y^2 + 2gx + 2fy + c = 0 \quad$ ...(ii)

be $\alpha, \beta, \gamma, \delta$

$\therefore \quad \alpha + \beta + \gamma + \delta = 2n\pi \quad$ (where n is an integer)

Let ϕ be any point on Eq. (i)

$\therefore \quad x = a\cos\phi, y = b\sin\phi$

This point also lie on Eq. (ii)

$\therefore \quad a^2\cos^2\phi + b^2\sin^2\phi + 2ga\cos\phi + 2fb\sin\phi + c = 0$...(iii)

$$\Rightarrow \{(a^2 - b^2)\cos^2\phi + 2ga\cos\phi + (b^2 + c)\}^2 = 4f^2b^2(1 - \cos^2\phi)$$

$$\Rightarrow (a^2 - b^2)^2\cos^4\phi + 4ga(a^2 - b)\cos^3\phi + \{2(a^2 - b^2)(b^2 + c) + 4g^2a^2 + 4f^2b^2\}\cos^2\phi + 4ga(b^2 + c)\cos\phi + \{b^2 + c^2 - 4f^2b^2\} = 0$$

This is a fourth degree equation in $\cos\phi$.

It has four roots (i.e. $\cos\alpha, \cos\beta, \cos\gamma, \cos\delta$)

$$\therefore \quad \cos\alpha + \cos\beta + \cos\gamma + \cos\delta = -\frac{4ga}{(a^2 - b^2)} \quad \text{...(iv)}$$

Similarly changing Eq. (iii) in $\sin\phi$, we get

$$\sin\alpha + \sin\beta + \sin\gamma + \sin\delta = -\frac{4fb}{b^2 - a^2} \quad \text{...(v)}$$

$\because \quad \alpha + \beta + \gamma + \delta = 2n\pi$

$\therefore \quad \delta = 2n\pi - (\alpha + \beta + \gamma)$

$\therefore \quad \sin\delta = -\{\sin(\alpha + \beta + \gamma)\}$ and $\cos\delta = \cos(\alpha + \beta + \gamma)$

then, from Eqs. (iv) and (v), we get

$$-g = \left(\frac{a^2 - b^2}{4a}\right)\{\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma)\}$$

and $$-f = \left(\frac{b^2 - a^2}{4b}\right)\{\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma)\}$$

which give co-ordinate of centre of circle through α, β and γ.

Corollary 3 : *$P'CP$ and $D'CD$ are conjugate diameters of an ellipse and α is the eccentric angle of P. Prove that the eccentric angle of the point where the circle through P, P', D again cuts the ellipse is $\frac{\pi}{2} - 3\alpha$.*

The eccentric angles of P, P' and D are $\boxed{\alpha, \pi + \alpha, \frac{\pi}{2} + \alpha}$ respectively. Let β be the eccentric angle of the fourth point. As above

$$\alpha + (\pi + \alpha) + \left(\frac{\pi}{2} + \alpha\right) + \beta = 2n\pi$$

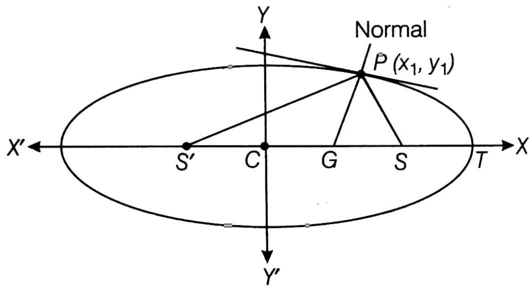
$$\therefore \quad \beta = 2n\pi - \left(\frac{3\pi}{2} + 3\alpha\right) = \frac{\pi}{2} - 3\alpha \quad (\text{for } n = 1)$$

Note

Any other values of n gives the same point on the ellipse.

Some Standard Properties of the Ellipse

(i) If S be the focus and G be the point where the normal at P meets the axis of an ellipse, then $SG = e \cdot SP$ and the tangent and normal at P bisects the external and internal angles between the focal distances of P.



Let P be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\therefore$ Equation of normals PG is

$$(x - x_1)\frac{a^2}{x_1} = (y - y_1)\frac{b^2}{y_1}$$

Putting $y = 0$. For the point G, we have

$$(x - x_1)\frac{a^2}{x_1} = -b^2$$

$$\therefore x = CG = \left(\frac{a^2 - b^2}{a^2}\right)x_1 = \frac{a^2e^2}{a^2}x_1 = e^2x_1$$

$$\therefore \quad SG = CS - CG = ae - e^2x_1 = e(a - ex_1)$$

$$= eSP$$

Similarly $\quad S'G = eS'P$

$$\therefore \quad \frac{SG}{S'G} = \frac{eSP}{eS'P} = \frac{SP}{S'P}$$

$\therefore$ The normal PG bisects the internal $\angle SPS'$ between the focal distances but tangent and normal are at right angles, the tangent PT bisects the external angle SPL between them.

(ii) The locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle.

The equation of any tangent in terms slope (m) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$y = mx + \sqrt{(a^2m^2+b^2)}$$

or $$y - mx = \sqrt{(a^2m^2+b^2)} \qquad \text{...(i)}$$

Equation of perpendicular line of Eq. (i) and passes through $(\pm ae, 0)$ is

$$my + x = \pm ae \qquad \text{...(ii)}$$

The locus of the point of intersection of the line given by Eqs. (i) and (ii) can be obtained by eliminating m between them, squaring and adding Eqs. (i) and (ii), we get

$$y^2(1+m^2)+x^2(1+m^2) = a^2m^2+b^2+a^2e^2$$

$$\Rightarrow \quad (1+m^2)(x^2+y^2) = a^2m^2+a^2$$

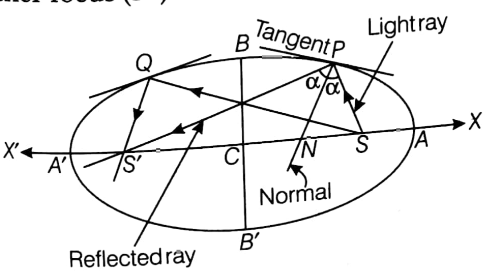
$$= a^2(1+m^2)$$

or $$x^2+y^2=a^2$$

which is the equation of the auxiliary circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Reflection Property of an Ellipse

If an incoming light ray passes through on focus (S) strike the concave side of the ellipse, then it will get reflected towards other focus (S').



and $\angle SPS' = \angle SQS'$

I Example 41 A ray emanating from the point $(-3, 0)$ is incident on the ellipse $16x^2+25y^2=400$ at the point P with ordinate 4. Find the equation of the reflected ray after first reflection.

Sol. For point P, y-coordinate $= 4$

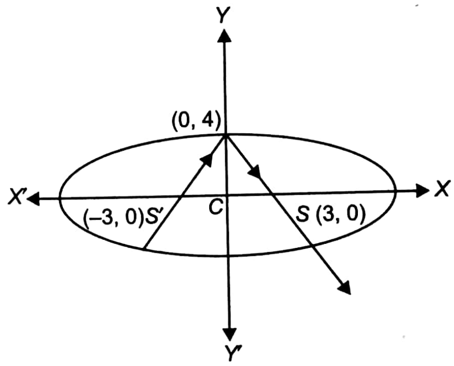
$\because$ Given ellipse is $16x^2+25y^2=400$

$$16x^2+25(4)^2=400$$

Coordinate of P is $(0, 4)$

$$e^2 = 1-\frac{16}{25}=\frac{9}{25}$$

$$e=\frac{3}{5}$$



foci $(\pm ae, 0)$ i.e. $(\pm 3, 0)$

Equation of reflected ray (i.e. PS) is

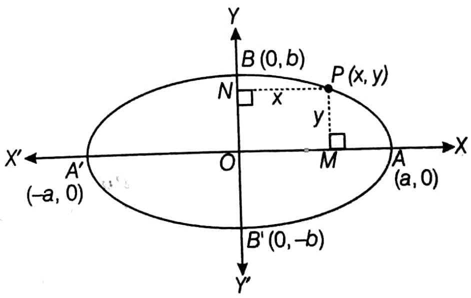
$$\frac{x}{3}+\frac{y}{4}=1 \text{ or } 4x+3y=12.$$

Equation of an Ellipse Referred to Two Perpendicular Lines

Let $P(x, y)$ be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

then, $$y = PM, x = PN$$

$$\therefore \quad \frac{(PN)^2}{a^2}+\frac{(PM)^2}{b^2}=1$$

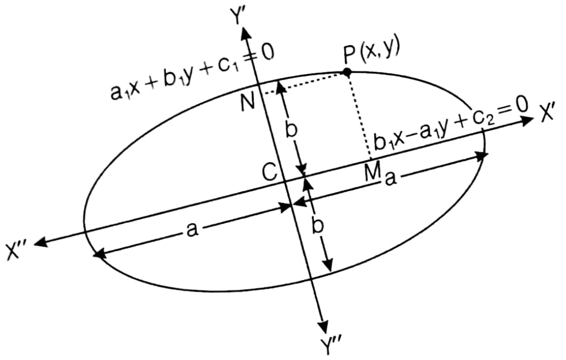


suppose if axes along the lines

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad b_1x - a_1y + c_2 = 0$$

then
$$PN = \frac{|a_1x + b_1y + c_1|}{\sqrt{(a_1^2 + b_1^2)}}$$

$$PM = \frac{|b_1x - a_1y + c_2|}{\sqrt{(b_1^2 + a_1^2)}}$$



then, equation of ellipse is

$$\frac{\left\{\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}}\right\}^2}{a^2} + \frac{\left\{\frac{b_1x - a_1y + c_2}{\sqrt{(b_1^2 + a_1^2)}}\right\}^2}{b^2} = 1$$

Centre : Is the point of intersection of $a_1x + b_1y + c_1 = 0$ and $b_1x - a_1y + c_2 = 0$

Equations of Major and Minor Axes :

(i) If $a > b$, then major axis lies along $b_1x - a_1y + c_2 = 0$ and minor axis lies along $a_1x + b_1y + c_1 = 0$.

(ii) If $a < b$, then major axis lies along $a_1x + b_1y + c_1 = 0$ and minor axis lies along $b_1x - a_1y + c_2 = 0$

Eccentricity :

(i) If $a > b$, $b^2 = a^2(1 - e^2)$ (ii) If $a < b$, $a^2 = b^2(1 - e^2)$

Foci :

(i) If $a > b$

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm ae, \frac{b_1x - a_1y + c_2}{\sqrt{(b_1^2 + a_1^2}} = 0$$

we get after solving (x, y)

(ii) If $a < b$

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = 0, \frac{b_1x - a_1y + c_2}{\sqrt{b_1^2 + a_1^2}} = \pm be$$

we get after solving (x, y)

Directrices :

(i) If $a > b$,

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{a}{e}$$

(ii) If $a < b$,

$$\frac{b_1x - a_1y + c_2}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{b}{e}$$

Example 42 Determine the equations of major and minor axes of the ellipse

$$4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 25$$

Also, find its centre, length of the latusrectum and eccentricity.

Sol. The equation of the ellipse can be written as

$$4 \times 5\left(\frac{x - 2y + 1}{\sqrt{5}}\right)^2 + 9 \times 5\left(\frac{2x + y + 2}{\sqrt{5}}\right)^2 = 25$$

or
$$\frac{\left(\frac{x - 2y + 1}{\sqrt{5}}\right)^2}{(5/4)} + \frac{\left(\frac{2x + y + 2}{\sqrt{5}}\right)^2}{(5/9)} = 1$$

or
$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

Here $a > b$

$\therefore$ **Equation of major axis is** $Y = 0$

i.e. $2x + y + 2 = 0$

and **Equation of minor axis is** $X = 0$

i.e. $x - 2y + 1 = 0$

Centre : $X = 0, Y = 0$

$\Rightarrow \quad x - 2y + 1 = 0, 2x + y + 2 = 0$

we get $\quad x = -1, y = 0$

$\therefore$ Centre is $(-1, 0)$

Length of latusrectum $= \dfrac{2b^2}{a} = \dfrac{2 \times 5/9}{5/4} = \dfrac{8}{9}$

Eccentricity : $b^2 = a^2(1 - e^2)$

$$\Rightarrow \quad \frac{5}{9} = \frac{5}{4}(1 - e^2)$$

$$\Rightarrow \quad \frac{4}{9} = 1 - e^2$$

$$\Rightarrow \quad e^2 = \frac{5}{9}$$

$$\therefore \quad e = \frac{\sqrt{5}}{3}$$

Exercise for Session 3

1. The angle between the pair of tangents drawn from the point (1, 2) to the ellipse $3x^2 + 2y^2 = 5$ is

   (a) $\tan^{-1}\left(\frac{12}{5}\right)$ (b) $\tan^{-1}\left(\frac{6}{\sqrt{5}}\right)$ (c) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$ (d) $\tan^{-1}(12\sqrt{5})$

2. If chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are at right angles, then $\frac{x_1x_2}{y_1y_2}$ is equal to

   (a) $\frac{a^2}{b^2}$ (b) $-\frac{b^2}{a^2}$ (c) $-\frac{a^4}{b^4}$ (d) $-\frac{b^4}{a^4}$

3. From the point $(\lambda, 3)$ tangents are drawn to $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and are perpendicular to each other than λ is

   (a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

4. The eccentric angle of one end of a diameter of $x^2 + 3y^2 = 3$ is $\pi/6$, then the eccentric angle of the other end will be

   (a) $\frac{5\pi}{6}$ (b) $-\frac{5\pi}{6}$ (c) $-\frac{2\pi}{3}$ (d) $\frac{2\pi}{3}$

5. The locus of the mid-points of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

   (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{ex}{a}$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{ex}{a}$ (c) $x^2 + y^2 = a^2 + b^2$ (d) $x^2 - y^2 = a^2 + b^2$

6. The centre of the ellipse $\frac{(x + y - 2)^2}{9} + \frac{(x - y)^2}{16} = 1$ is

   (a) (0, 0) (b) (1, 0) (c) (0, 1) (d) (1, 1)

7. The locus of the point of intersection of two perpendicular tangents of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is

   (a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 9$ (c) $x^2 + y^2 = 13$ (d) $x^2 + y^2 = 5$

8. The area of the parallelogram inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose diagonals are the conjugate diameters of the ellipse is given by

   (a) $2ab$ (b) $3ab$ (c) $4ab$ (d) $5ab$

9. Find the locus of the vertices of equilateral triangle circumscribing the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

10. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

11. Find the locus of the mid-point of chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$ passing through the point $(2a, 0)$.

12. Find the locus of the point the chord of contact of tangents from which to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, touches the circle $x^2 + y^2 = c^2$.

13. Find the centre and eccentricity of the ellipse

$$4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 5$$

14. A ray emanating from the point $(0, -\sqrt{5})$ is incident on the ellipse $9x^2 + 4y^2 = 36$ at the point P with abscissa 2. Find the equation of the reflected ray after first reflection.

Shortcuts and Important Results to Remember

1 If S and S' are foci and P be a point, then

(a) If $|SP| + |S'P| > |SS'|$, then the locus of P is an ellipse.

(b) If $|SP| + |S'P| = |SS'|$, then the locus of P is a straight line.

(c) If $|SP| + |S'P| < |SS'|$, then the locus of P is an empty set.

2 If the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ and $\frac{x^2}{\theta^2} + \frac{y^2}{\phi^2} = 1$ have a common tangent, then $\begin{vmatrix} a^2 & b^2 & 1 \\ \alpha^2 & \beta^2 & 1 \\ \theta^2 & \phi^2 & 1 \end{vmatrix} = 0$

3 Area of the quadrilateral formed by the common tangents of the circle $x^2 + y^2 = c^2$ and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $c \in (a, b)$ is $\frac{2c^2|a^2 - b^2|}{\sqrt{(a^2 - c^2)(c^2 - b^2)}}$

4 The product of the two perpendicular distances from the foci on any tangent of an ellipse is b^2.

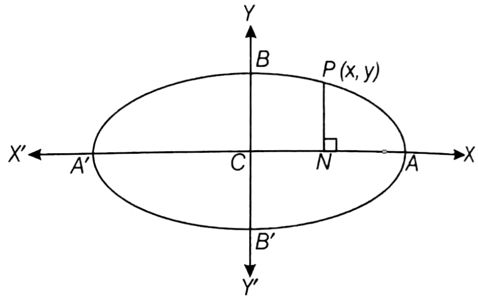
5 If the normals at the point $P(x_1, y_1)$; $Q(x_2, y_2)$ and $R(x_3, y_3)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are concurrent, then

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0$$

and if points $P(\alpha)$, $Q(\beta)$ and $R(\gamma)$, then

$$\begin{vmatrix} \sec\alpha & \operatorname{cosec}\alpha & 1 \\ \sec\beta & \operatorname{cosec}\beta & 1 \\ \sec\gamma & \operatorname{cosec}\gamma & 1 \end{vmatrix} = 0$$

6 If ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

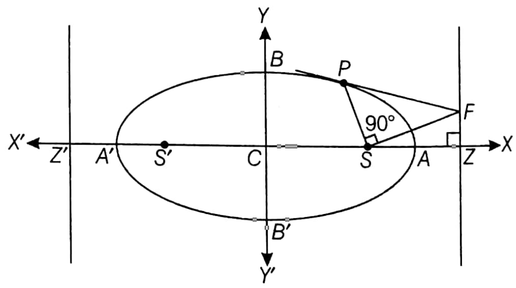


$$(PN)^2 : AN \cdot A'N = (BC)^2 : (AC)^2$$

7 If α and β are the eccentric angles of extremittes of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\tan(\alpha/2).\tan(\beta/2) = \frac{e-1}{e+1}$ or $\frac{e+1}{e-1}$ according as focus $(ae, 0)$ or $(-ae, 0)$.

8 If the tangent at P on an ellipse meets the directrix in F; then the PF will subtend a right angle at the corresponding focus.

i.e. $\angle PSF = \pi/2$



JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Must

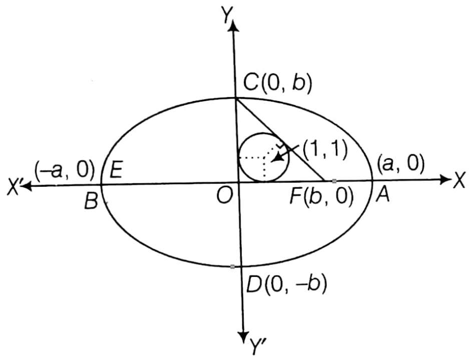
Ex. 1 *Point 'O' is the centre of the ellipse with major axis AB and minor axis CD. Point F is one focus of the ellipse. If $OF = 6$ and the diameter of the inscribed circle of triangle OCF is 2, then the product $(AB)(CD)$ is equal to*

(a) 52 (b) 56 (c) 78 (d) None of these

Sol. (b) $\because$ Diameter of the inscribed circle of triangle OCF is 2.

$\therefore$ Radius $= 1$

Centre of the circle is (1, 1) and equation of CF is $\frac{x}{6} + \frac{y}{b} = 1$



Now, length of perpendicular from (1, 1) on $CF = 1$ (radius)

$$\Rightarrow \quad \frac{\left|\frac{1}{6} + \frac{1}{b} - 1\right|}{\sqrt{\left(\frac{1}{36} + \frac{1}{b^2}\right)}} = 1 \text{ or } -\left(\frac{1}{b} - \frac{5}{6}\right) = \sqrt{\left(\frac{1}{36} + \frac{1}{b^2}\right)}$$

$$\Rightarrow \quad \left(\frac{5}{6} - \frac{1}{b}\right) = \sqrt{\left(\frac{1}{36} + \frac{1}{b^2}\right)}$$

On squaring both sides, then $\left(\frac{5}{6} - \frac{1}{b}\right)^2 = \frac{1}{36} + \frac{1}{b^2}$

$$\Rightarrow \quad \frac{25}{36} + \frac{1}{b^2} - \frac{5}{3b} = \frac{1}{36} + \frac{1}{b^2}$$

$$\Rightarrow \quad \frac{5}{3b} = \frac{24}{36} = \frac{2}{3}$$

$$\therefore \quad b = \frac{5}{2}$$

Also, $b^2 = a^2(1 - e^2)$

$b^2 = a^2 - (ae)^2$

$$\Rightarrow \quad a^2 = b^2 + (ae)^2 = \frac{25}{4} + 36 \qquad [\because ae = 6]$$

$$\Rightarrow \quad a = \frac{13}{2}$$

Hence, $(AB)(CD) = (2a)(2b) = 13 \times 5 = 65$

Ex. 2 *Let P_i and P_i' be the feet of the perpendiculars drawn from the foci S and S' on a tangent T_i to an ellipse whose length of semi-major axis is 20. If $\sum_{i=1}^{10}(SP_i)(S'P_i') = 2560$, then the value of eccentricity is*

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

Sol. (c) $\therefore$ Product of length of perpendiculars from foci on a tangent to an ellipse $= b^2$

$$\therefore \quad (SP_1)(S'P_1) = (SP_2)(S'P_2) = \ldots = (SP_{10})(S'P_{10}') = b^2$$

Given $\sum_{i=1}^{10}(SP_i)(S'P_i') = 2560$

$\Rightarrow \quad 10b^2 = 2560$ or $b^2 = 256$

or $\quad b = 16$

and $\quad b^2 = a^2(1 - e^2)$

$\Rightarrow \quad (16)^2 = (20)^2(1 - e^2)$

or $\quad 1 - e^2 = \frac{16}{25}$

or $\quad e^2 = \frac{9}{25}$ or $e = \frac{3}{5}$

Ex. 3 *Coordinates of the vertices B and C of a ΔABC are (2, 0) and (8, 0) respectively. The vertex A is varing in such a way that $4\tan\left(\frac{B}{2}\right) \cdot \tan\left(\frac{C}{2}\right) = 1$. Then, the locus of A is*

(a) $\frac{(x-5)^2}{25} + \frac{y^2}{16} = 1$ (b) $\frac{(x-5)^2}{16} + \frac{y^2}{9} = 1$

(c) $\frac{(x-5)^2}{25} + \frac{y^2}{9} = 1$ (d) $\frac{(x-5)^2}{16} + \frac{y^2}{25} = 1$

Sol. (a) $\because \; 4\tan\left(\frac{B}{2}\right) \cdot \tan\left(\frac{C}{2}\right) = 1$

$$\Rightarrow \quad 4 \times \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \times \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = 1$$

$$\Rightarrow \quad \frac{s-a}{s} = \frac{1}{4}$$

$$\Rightarrow \quad 3s = 4a$$

$$\Rightarrow \quad 3\left(\frac{a+b+c}{2}\right) = 4a$$

or $\quad b + c = \frac{5a}{3} = 10 = 2a_1$ $\qquad [\because a = BC = 6]$

Since, the sum of distances of A from two given fixed points B and C is always 10.

Here, B and C are foci.

$\therefore$ Centre (5, 0) and distance between foci $=6$

$$\Rightarrow \quad 2a_1 e = 6$$

$$\Rightarrow \quad e = \frac{6}{10} = \frac{3}{5} \qquad [\because a_1 = 5]$$

and $$b^2 = a_1^2(1-e^2) = 25\left(1-\frac{9}{25}\right) = 16 \text{ or } b = 4$$

$\because$ A lies on the ellipse

$\therefore$ Locus of A is

$$\frac{(x-5)^2}{(5)^2} + \frac{(y-0)^2}{(4)^2} = 1$$

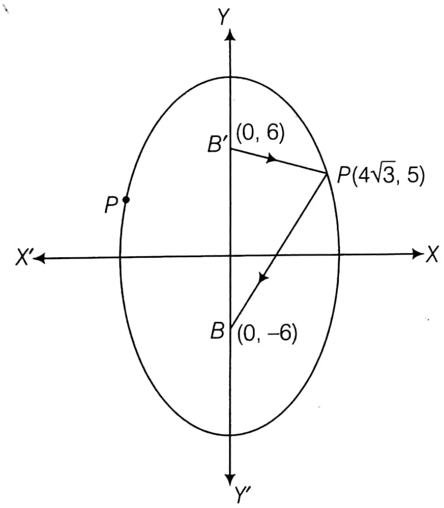
or $$\frac{(x-5)^2}{25} + \frac{y^2}{16} = 1$$

Ex. 4 *A ray emanating from the point (0, 6) is incident on the ellipse $25x^2 + 16y^2 = 1600$ at the point P with ordinate 5. After reflection, ray cuts the Y-axis at B. The length of PB is*

(a) 5 (b) 7 (c) 12 (d) 13

Sol. (d) $\because$ Ellipse is $25x^2 + 16y^2 = 1600$

or $$\frac{x^2}{8^2} + \frac{y^2}{10^2} = 1$$



Coordinating foci an $\left(0, \pm\sqrt{(10^2-8^2)}\right)$

i.e. $(0, \pm 6)$

Let coordinates of P are $(\lambda, 5)$

$$\therefore \quad \frac{\lambda^2}{8^2} + \frac{5^2}{10^2} = 1$$

$$\Rightarrow \quad \lambda = \pm 4\sqrt{3}$$

$\therefore$ $P \equiv (\pm 4\sqrt{3}, 5)$ (lie in I or II quadrants)

According to reflection property, a ray passing through focus $B'(0,6)$ will passing through $B(0,-6)$ (other focus). If P lies in I quadrant, then

$$PB = \sqrt{48 + (5+6)^2} = \sqrt{169} = 13$$

done by me
Just look at diagram.

Ex. 5 *If the ellipse $\frac{x^2}{4} + y^2 = 1$ meets the ellipse $x^2 + \frac{y^2}{a^2} = 1$ in four distinct points and $a = b^2 - 5b + 7$, then b belongs to*

(a) (1, 4) (b) $(-\infty, 2) \cup (3, \infty)$

(c) (2, 3) (d) None of these

Sol. (b) The ellipse $\frac{x^2}{1^2} + \frac{y^2}{a^2} = 1$ will intersect in four distinct points with ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$, where $a^2 > 1$

$$\Rightarrow \quad a > 1 \qquad [\because \text{a always positive}]$$

Now, $$a = b^2 - 5b + 7$$

$$\Rightarrow \quad b^2 - 5b + 7 > 1$$

or $$b^2 - 5b + 6 > 0$$

or $$(b-2)(b-3) > 0$$

or $$b \in (-\infty, 2) \cup (3, \infty)$$

Must

Ex. 6 *The normal at a variable point P on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of eccentricity e meets the axes of the ellipse in Q and R, then the locus of the mid-point of QR is a conic with eccentricity e′ such that*

(a) e' is independent of e (b) $e' = 1$

(c) $e' = e$ (d) $e' = \frac{1}{e}$

Sol. (c) Normal at $P(a\cos\theta, b\sin\theta)$ is

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2 \qquad (\text{let } a > b)$$

It meets the axes at

$$Q\left(\frac{(a^2-b^2)}{a}\cos\theta,\ 0\right) \text{ and } R\left(0,\ -\frac{(a^2-b^2)}{b}\sin\theta\right)$$

Let mid-point of QR is $T(x, y)$, then

$$2x = \frac{(a^2-b^2)}{a}\cos\theta$$

or $$2ax = (a^2-b^2)\cos\theta \qquad \text{...(i)}$$

and $$2y = -\frac{(a^2-b^2)}{b}\sin\theta$$

or $$2by = -(a^2-b^2)\sin\theta \qquad \text{...(ii)}$$

On squaring and adding Eq. (i) and Eq.(ii), we get

$$4a^2x^2 + 4b^2y^2 = (a^2-b^2)^2$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{a^2-b^2}{2a}\right)^2} + \frac{y^2}{\left(\frac{a^2-b^2}{2b}\right)^2} = 1$$

which is an ellipse, having eccentricity e', then

Let $$A = \frac{a^2-b^2}{2a},\ B = \frac{a^2-b^2}{2b}$$

$$A^2 = B^2(1-e'^2)$$

$\Rightarrow$ $$e'^2 = 1 - \frac{A^2}{B^2} = 1 - \frac{b^2}{a^2} = e^2$$ $[\because B > A]$

$\therefore$ $$e' = e$$

by diagram.

Ex. 7 *If the curves $\frac{x^2}{4} + y^2 = 1$ and $\frac{x^2}{a^2} + y^2 = 1$ for a suitable value of a cut on four concyclic points, the equation of the circle passing through these four points is*

(a) $x^2 + y^2 = 8$ (b) $x^2 + y^2 = 4$

(c) $x^2 + y^2 = 2$ (d) $x^2 + y^2 = 1$

Sol. (d) The equation of conic through the point of intersection of given two ellipses is

$$\left(\frac{x^2}{4} + y^2 - 1\right) + \lambda\left(\frac{x^2}{a^2} + y^2 - 1\right) = 0$$

or $$x^2\left(\frac{1}{4} + \frac{\lambda}{a^2}\right) + y^2(1+\lambda) = (1+\lambda)$$

or $$x^2\left(\frac{a^2 + 4\lambda}{4a^2(1+\lambda)}\right) + y^2 = 1$$

for circle $$\frac{a^2 + 4\lambda}{4a^2(1+\lambda)} = 1$$

$\Rightarrow$ $$\lambda = -\frac{3a^2}{4(a^2-1)}$$

Therefore, the circle is $x^2 + y^2 = 1$

Must

Ex. 8 *If p is the length of perpendicular drawn from the origin to any normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, then the maximum value of p is*

(a) 5 (b) 4

(c) 2 (d) 1

Sol. (d) The equation of any normal at $(5\cos\theta, 4\sin\theta)$ to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is

$$5x\sec\theta - 4y\,\text{cosec}\,\theta = 9$$

$\therefore$ $$p = \frac{|0-0-9|}{\sqrt{(25\sec^2\theta + 16\,\text{cosec}^2\theta)}} = \frac{9}{\sqrt{(25\tan^2\theta + 16\cot^2\theta + 41)}}$$

Now, AM $\geq$ GM

$\Rightarrow$ $$\frac{25\tan^2\theta + 16\cot^2\theta}{2} \geq \sqrt{25 \times 16} = 20$$

$\therefore$ $$25\tan^2\theta + 16\cot^2\theta \geq 40$$

$\Rightarrow$ $$25\tan^2\theta + 16\cot^2\theta + 41 \geq 81$$

or $$\frac{9}{\sqrt{(25\tan^2\theta + 16\cot^2\theta + 41)}} \leq 1$$

$\Rightarrow$ $$p \leq 1$$

Thus, maximum value of p is 1.

Ex. 9 *If $f(x)$ is a decreasing function, then the set of values of 'k', for which the major axis of the ellipse $\frac{x^2}{f(k^2+2k+5)} + \frac{y^2}{f(k+11)} = 1$ is the X-axis, is*

(a) $k \in (-2, 3)$

(b) $k \in (-3, 2)$

(c) $k \in (-\infty, -3) \cup (2, \infty)$

(d) $k \in (-\infty, -2) \cup (3, \infty)$

Sol. (b) $\because$ $f(x)$ is a decreasing function and for major axis to be X-axis.

$\therefore$ $$f(k^2+2k+5) > (f(k+11))$$ Must

$\Rightarrow$ $$k^2 + 2k + 5 < k + 11$$

or $$k^2 + k - 6 < 0$$

or $$(k+3)(k-2) < 0$$

or $$k \in (-3, 2)$$

Ex. 10 *If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is*

(a) 1 (b) 2

(c) 4 (d) 8

Sol. (c) Equation of tangent is $y = 2x + \sqrt{(4a^2 + b^2)}$

This is normal to the circle

$$x^2 + y^2 + 4x + 1 = 0$$

This tangent passes through $(-2, 0)$, then

$$0 = -4 + \sqrt{(4a^2 + b^2)}$$

$\Rightarrow$ $$4a^2 + b^2 = 16$$

$\because$ $$AM \geq GM$$

$\therefore$ $$\frac{4a^2 + b^2}{2} \sqrt{(4a^2)(b^2)}$$

$\Rightarrow$ $$\frac{16}{2} \geq 2ab$$

or $$ab \leq 4$$

Hence, maximum value of ab is 4.

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

• ***Ex. 11*** *Extremities of the latusrectum of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$ *having a given major axis* $2a$ *lies on*

(a) $x^2=a(a-y)$ (b) $x^2=a(a+y)$
(c) $y^2=a(a+x)$ (d) $y^2=a(a-x)$

Sol. (a, b) $\because$ Extremities of the latusrectum are $\left(ae, \pm\frac{b^2}{a}\right)$

Let $x=\pm ae$ and $y=\pm\frac{b^2}{a}$

or $x^2=a^2e^2$ and $b^2=\pm ay$

or $x^2=a^2-b^2$ and $b^2=\pm ay$

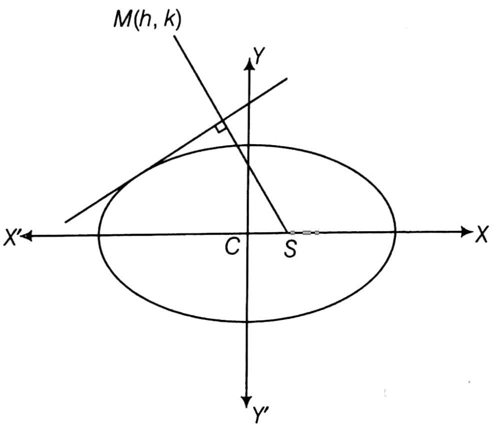
$\therefore$ $x^2=a^2\pm ay$ or $x^2=a(a\pm y)$

• ***Ex. 12*** *The locus of the image of the focus of the ellipse* $\frac{x^2}{25}+\frac{y^2}{9}=1$, *with respect to any of the tangent to the ellipse is*

(a) $(x+4)^2+y^2=100$ (b) $(x+2)^2+y^2=50$
(c) $(x-4)^2+y^2=100$ (d) $(x-2)^2+y^2=50$

Sol. (a, c) Let $M(h,k)$ be the image SM cuts a tangent at a point which lies on the auxiliary circle of the ellipse, therefore

$$\left(\frac{h\pm 4}{2}\right)^2+\frac{k^2}{4}=(5)^2 \quad [\because \text{foci of the given ellipse are } (\pm 4, 0)]$$



or $(h\pm 4)^2+k^2=100$

Hence, the locus is $(x\pm 4)^2+y^2=100$

• ***Ex. 13*** *A tangent to the ellipse* $4x^2+9y^2=36$ *is cut by the tangent at the extremities of the major axis at* T *and* T'. *The circle* TT' *as diameter passes through the point*

(a) $(-\sqrt{5}, 0)$ (b) $(\sqrt{5}, 0)$
(c) $(\sqrt{3}, 0)$ (d) $(-\sqrt{3}, 0)$

Sol. (a, b) Given equation of the ellipse is $4x^2+9y^2=36$

i.e., $$\frac{x^2}{9}+\frac{y^2}{4}=1 \qquad \text{...(i)}$$

The equation of tangent at $(3\cos\theta, 2\sin\theta)$ is

$$\frac{x}{3}\cos\theta+\frac{y}{2}\sin\theta=1$$

which meets the tangent at $x=3$ and $x=-3$ at the extremities of major axis

$$T\equiv\left(3, \frac{2(1-\cos\theta)}{\sin\theta}\right)$$

and $$T'\equiv\left(-3, \frac{2(1+\cos\theta)}{\sin\theta}\right)$$

$\therefore$ Equation of circle on TT' as diameter is

$$(x-3)(x+3)+\left(y-\frac{2(1-\cos\theta)}{\sin\theta}\right)\left(y-\frac{2(1+\cos\theta)}{\sin\theta}\right)=0$$

$\Rightarrow$ $$x^2+y^2-\frac{4}{\sin\theta}\cdot y-5=0$$

or $$(x^2+y^2-5)-\frac{4}{\sin\theta}y=0 \qquad \text{...(ii)}$$

Clearly Eq. (ii), passes through point of intersection of $x^2+y^2-5=0$ and $y=0$ i.e., $(\pm\sqrt{5}, 0)$

• ***Ex. 14*** *Consider the ellipse* $\frac{x^2}{\tan^2\alpha}+\frac{y^2}{\sec^2\alpha}=1$, *where* $\alpha\in(0,\pi/2)$. *Which of the following quantities would vary as* α *varies?*

(a) degree of flatness (b) ordinate of the vertex
(c) coordinate of the foci (d) length of latusrectum

Sol. (a, b, d) In $\alpha\in(0,\pi/2)$

$$\sec^2\alpha>\tan^2\alpha$$

$\therefore$ Coordinates of foci $(0, \pm\sqrt{(\sec^2\alpha-\tan^2\alpha)})$ i.e., $(0,\pm 1)$ which is independent of α.

Vertices are $(0, \pm\sec\alpha)$ and latusrectum

$$=\frac{2a^2}{b}=\frac{2\tan^2\alpha}{\sec\alpha}$$

$\because$ $\alpha\in(0,\pi/2)$

$\Rightarrow$ $\tan\alpha\in(0,\infty)$ and $\sec\alpha\in(1,\infty)$

Hence, $\alpha\propto$ degree of flatness.

Ex. 15 *Let $A(\theta)$ and $B(\phi)$ be the extremities of a chord of an ellipse. If the slope of AB is equal to the slope of the tangent at a point $C(\alpha)$ on the ellipse, then the value of α is*

(a) $\frac{\theta+\phi}{2}$ (b) $\frac{\theta-\phi}{2}$

(c) $\frac{\theta+\phi}{2}+\pi$ (d) $\frac{\theta+\phi}{2}-\pi$

Sol. (a, c) $\because$ Slope of AB = Slope of tangent at C

$$\Rightarrow \left(\frac{b\sin\phi - b\sin\theta}{a\cos\phi - a\cos\theta}\right) = -\frac{b}{a}\frac{\cos\alpha}{\sin\alpha}$$

$$\Rightarrow \frac{b.2\cos\left(\frac{\theta+\phi}{2}\right).\sin\left(\frac{\phi-\theta}{2}\right)}{-a.2\sin\left(\frac{\theta+\phi}{2}\right).\sin\left(\frac{\phi-\theta}{2}\right)} = -\frac{b}{a}\cot\alpha$$

$$\therefore \tan\alpha = \tan\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \alpha = n\pi + \left(\frac{\theta+\phi}{2}\right), n \in I$$

JEE Type Solved Examples : Paragraph Based Questions

- This section contains **2 solved Paragraphs** based upon each of the Paragraph **3 multiple choice** questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to 18)

A sequence of ellipses $E_1, E_2, E_3, \ldots, E_n$ is constructed as follows : Ellipse E_n is drawn so as to touch ellipse E_{n-1} as the extremities of the major axis of E_{n-1} and to have its foci at the extremities of the mirror axis of E_{n-1}.

16. If E_n is independent of n, then the eccentricity of ellipse E_{n-2} is

(a) $\left(\frac{3-\sqrt{5}}{2}\right)$ (b) $\left(\frac{\sqrt{5}-1}{2}\right)$

(c) $\left(\frac{2-\sqrt{3}}{2}\right)$ (d) $\left(\frac{\sqrt{3}-1}{2}\right)$

17. If eccentricity of ellipse E_n is e_n, then the locus of (e_n^2, e_{n-1}^2) is

(a) a parabola (b) an ellipse

(c) a hyperbola (d) a rectangular hyperbola

18. If equation of ellipse E_1 is $\frac{x^2}{9}+\frac{y^2}{16}=1$, then the equation of ellipse E_3 is

(a) $\frac{x^2}{9}+\frac{y^2}{16}=1$ (b) $\frac{x^2}{25}+\frac{y^2}{49}=1$

(c) $\frac{x^2}{25}+\frac{y^2}{41}=1$ (d) $\frac{x^2}{16}+\frac{y^2}{25}=1$

Sol.

16. (b) If $E_n : \frac{x^2}{a_n^2}+\frac{y^2}{b_n^2}=1$ and eccentricity of E_n is e_n

If $a_n > b_n$

Then, $b_n^2 = a_n^2(1-e_n^2)$ ...(i)

According to the question, $b_n = b_{n-1}$ ...(ii)

and $a_{n-1} = a_n e_n$ ...(iii)

For ellipse E_{n-1}, $a^2_{n-1} = b_{n-1}^2(1-e_{n-1}^2)$ ...(iv)

From Eqs. (i) and (ii), we get $b_{n-1}^2 = a_n^2(1-e_n^2)$ ...(v)

Substituting the values of a_{n-1} and b_{n-1}^2 from Eqs. (iii) and (v) in Eq. (iv), then

$$a_n^2e_n^2 = a_n^2(1-e_n^2)(1-e_{n-1}^2)$$

$$\Rightarrow e_n^2 = (1-e_n^2)(1-e_{n-1}^2) \quad \text{...(vi)}$$

$\because$ E_n is independent of n

$\therefore$ $e_n = e_{n-1} = e$ [say]

From Eq. (vi), we get

$$e^2 = (1-e^2)^2$$

$$\Rightarrow e^4 - 3e^2 + 1 = 0$$

$$\therefore e^2 = \frac{3\pm\sqrt{5}}{2} = \frac{6\pm2\sqrt{5}}{4} = \left(\frac{\sqrt{5}\pm1}{2}\right)^2$$

$$\therefore e = \frac{\sqrt{5}-1}{2} \quad [\because 0<e<1]$$

17. (d) From Eq. (vi), $e_n^2 = (1-e_n^2)(1-e_{n-1}^2)$

Locus of (e_n^2, e_{n-1}^2) is

$$x = (1-x)(1-y)$$

$$\Rightarrow xy - 2x - y + 1 = 0$$

Here, $a=0, b=0, c=1,$

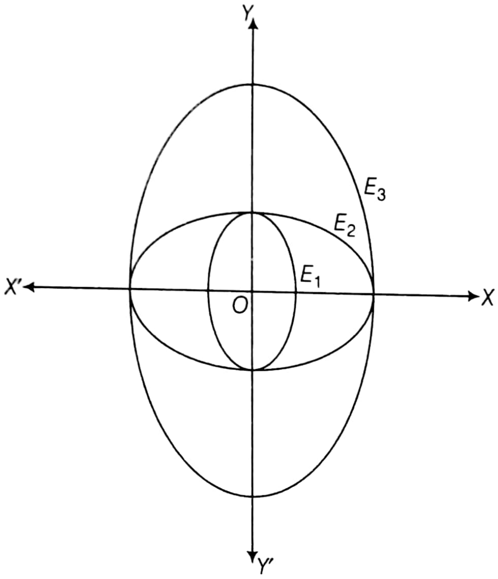
$$f=-\frac{1}{2}, g=-1, h=\frac{1}{2}$$

$$\therefore \Delta = 0+2\times-\frac{1}{2}\times-1\times\frac{1}{2}-0-0-1\times\frac{1}{4} = \frac{1}{4} \neq 0$$

and $h^2 > ab, a+b=0$

$\Rightarrow$ rectangular hyperbola.

18. (c) From Eq. (vi), $e_n = \sqrt{\left(\dfrac{1-e_{n-1}^2}{2-e_{n-1}^2}\right)}$ ...(vii)



$\because \quad E_1 \equiv \dfrac{x^2}{9}+\dfrac{y^2}{16}=1$

$\therefore \quad a_1 = 3, b_1 = 4$ and $9 = 16(1-e_1^2)$

$\therefore \quad e_1 = \dfrac{\sqrt{7}}{4}$

From Eq. (vii), $e_2 = \sqrt{\left(\dfrac{1-e_1^2}{2-e_1^2}\right)} = \sqrt{\left(\dfrac{1-\dfrac{7}{16}}{2-\dfrac{7}{16}}\right)} = \dfrac{3}{5}$

and then, $e_3 = \sqrt{\left(\dfrac{1-\dfrac{9}{25}}{2-\dfrac{9}{25}}\right)} = \dfrac{4}{\sqrt{41}}$

Also, $a_1 = a_2 e_2$ and $b_1 = b_2$

$\therefore \quad b_2 = 4 = b_3 e_3 \Rightarrow b_3 = \sqrt{41}$

and $\quad a_3^2 = b_3^2(1-e_3^2) = 41\left(1-\dfrac{16}{41}\right) = 25$

$\therefore$ Ellipse E_3 is $\dfrac{x^2}{25}+\dfrac{y^2}{41}=1$

Paragraph II

(Q. Nos. 19 to 21)

Consider an ellipse $E: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, centred at point 'O' and having AB and CD as its major and minor axes respectively if S_1 be one of the focus of the ellipse, radius of incircle of ΔOCS_1 be 1 unit and $OS_1 = 6$ units.

19. If area of ellipse (E) is Δ sq unit, then the value of 4Δ is

(a) 63π (b) 64π (c) 65π (d) 66π

20. If perimeter of ΔOCS_1 is p units, then the value of p is

(a) 10 (b) 15 (c) 20 (d) 25

21. The equation of the director circle of (E) is

(a) $x^2+y^2 = 48.5$ (b) $x^2+y^2 = 97$

(c) $x^2+y^2 = \sqrt{48.5}$ (d) $x^2+y^2 = \sqrt{97}$

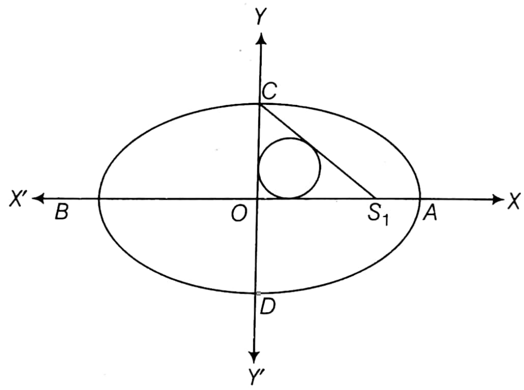
Sol.

$\because \quad OS_1 = ae = 6, \; OC = b$

$\therefore \quad CS_1 = \sqrt{(OS_1)^2+(OC)^2} = \sqrt{(a^2e^2+b^2)} = a$

Area of $\Delta OCS_1 = \dfrac{1}{2}\times(OS_1)\times(OC) = \dfrac{1}{2}\times 6\times b = 3b$

and semi-perimeter of $\Delta OCS_1 = \dfrac{1}{2}(OS_1+OC+CS_1) = \dfrac{1}{2}(6+a+b)$



$\because$ In radius of $\Delta OCS_1 = 1$

$\therefore \quad \dfrac{3b}{\dfrac{1}{2}(6+a+b)} = 1$

$\Rightarrow \quad b = \dfrac{1}{5}(6+a)$ ...(i)

Also, $\quad b^2 = a^2(1-e^2) = a^2 - 36$ ...(ii)

From Eqs. (i) and (ii), we get

$\dfrac{1}{25}(6+a)^2 = a^2 - 36$

$\Rightarrow \quad 2a^2 - a - 78 = 0$

or $\quad a = \dfrac{13}{2}, -6$

19. (c) $\therefore \quad a = \dfrac{13}{2}$ and $b = \dfrac{5}{2}$ [from Eq. (i)]

$\because \quad \Delta = \pi ab = \pi \times \dfrac{13}{2}\times\dfrac{5}{2}$

$\therefore \quad 4\Delta = 65\pi$

20. (b) $p = (OS_1 + OC + CS_1) = 6 + b + a$

$= 6 + \dfrac{5}{2} + \dfrac{13}{2} = 15$

21. (a) Equation of director circle of E is

$x^2+y^2 = a^2+b^2$

$= \dfrac{169+25}{4} = 48.5$

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer**, ranging from 0 to 9 (both inclusive).

O✰

- *Ex. 22 If the normals at the four points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then the value of $\left(\sum_{i=1}^{4} x_i\right)\left(\sum_{i=1}^{4}\frac{1}{x_i}\right)$ is*

Sol. (4) Let point of concurrent is (h, k).

Equation of normal at (x', y') is

$$\frac{x-x'}{x'a^2}=\frac{y-y'}{y'b^2}$$

It is passes through (h, k), then

$$y'^2\{a^2(h-x')+b^2x'\}^2=b^4k^2x'^2 \quad \text{...(i)}$$

But $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$ or $y'^2=\frac{b^2}{a^2}(a^2-x'^2)$ ...(ii)

Value of y'^2 from Eq. (ii), putting in Eq. (i), we get

$$\frac{b^2}{a^2}(a^2-x'^2)\{a^2h+(b^2-a^2)x'\}^2=b^4k^2x'^2$$

$$\Rightarrow \frac{b^2}{a^2}(a^2-x'^2)\{a^4h^2+(b^2-a^2)^2x'^2+2a^2hx'(b^2-a^2)\}$$

$$=b^4k^2x'^2$$

Arranging above as a fourth degree equation in x', we get

$$\Rightarrow -(a^2-b^2)^2x'^4+2ha^2(a^2-b^2)x'^3+x'^2(\ldots)$$

$$-2a^4h(a^2-b^2)x'+a^6h^2=0$$

above equation being of fourth degree in x', therefore roots of the above equation are x_1, x_2, x_3, x_4, then

$$(x_1+x_2+x_3+x_4)=-\frac{2ha^2(a^2-b^2)}{-(a^2-b^2)^2}=\frac{2ha^2}{(a^2-b^2)} \quad \text{...(iii)}$$

$$\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=\frac{\Sigma x_1x_2x_3}{x_1\cdot x_2\cdot x_3\cdot x_4}$$

$$=\frac{\frac{2a^4h(a^2-b^2)}{-(a^2-b^2)^2}}{\frac{a^6h^2}{-(a^2-b^2)^2}}=\frac{2(a^2-b^2)}{a^2h} \quad \text{...(iv)}$$

Multiplying Eqs. (iii) and (iv), we get

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

or $\left(\sum_{i=1}^{4} x_i\right)\left(\sum_{i=1}^{4}\frac{1}{x_i}\right)=4$

✰

- *Ex. 23 If $x, y \in R$, satisfies the equation $\frac{(x-4)^2}{4}+\frac{y^2}{9}=1$, then the difference between the largest and the smallest value of the expression $\frac{x^2}{4}+\frac{y^2}{9}$ is*

Sol. (8) Parametric coordinates on $\frac{(x-4)^2}{4}+\frac{y^2}{9}=1$ are $(4+2\cos\theta, 3\sin\theta)$

Now, let $E=\frac{x^2}{4}+\frac{y^2}{9}$

$$=\frac{(4+2\cos\theta)^2}{4}+\frac{(3\sin\theta)^2}{9}$$

$$=(2+\cos\theta)^2+\sin^2\theta$$

$$=4+4\cos\theta+\cos^2\theta+\sin^2\theta$$

$$=5+4\cos\theta$$

$\therefore$ $E_{\max}=5+4(1)=9$ $(\because -1\le\cos\theta\le 1)$

and $E_{\min}=5+4(-1)=1$

Hence, $E_{\max}-E_{\min}=9-1=8$

JEE Type Solved Examples : Matching Type Questions

- This section contains **only one example**. This example has four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

Ex. 24 *Match the following*

Column I	Column II
A. Let $f(x) = \begin{cases} a^x, x < 2 \\ 8, x = 2 \\ \dfrac{b(x^2 - b^2)}{(x-2)}, x > 2 \end{cases}$ If f is continuous at $x = 2$, then the locus of the pair of perpendicular tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $x^2 + y^2 = r^2$, then r^2 is divisible by	(p) 3
B. If the ellipse $\dfrac{(x-h)^2}{M} + \dfrac{(y-k)^2}{N} = 1$ has major axis on the line $y = 2$, minor-axis on the line $x = -1$, major axis has length 10 and minor axis has length 4. Then, $h + k + M + N$ is divisible by	(q) 4
C. If PQ is a focal chord of ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$, which passes through $S(3, 0)$ and $PS = 2$, then length of PQ is divisible by	(r) 5
D. A tangent to the ellipse $\dfrac{x^2}{27} + \dfrac{y^2}{48} = 1$ having slope $\left(-\dfrac{4}{3}\right)$ cuts the x and y-axis at the points A and B respectively. If O is the origin, then area of ΔOAB is divisible by	(s) 6

Sol. (A) → (p, q, s); (B) → (p, r, s); (C) → (r); (D) → (p, q, s)

(A) $\underset{x \to 2^-}{\text{Lt}} f(x) = \underset{x \to 2^+}{\text{Lt}} f(x) = f(2)$

$$\Rightarrow \underset{h \to 0}{\text{Lt}}\, a^{2-h} = \underset{h \to 0}{\text{Lt}}\, \frac{b((2+h)^2 - b^2)}{2+h-2} = 8$$

$$\Rightarrow \quad a^2 = b \underset{h \to 0}{\text{Lt}}\, \frac{(2+h)^2 - b^2}{h} = 8$$

at $h \to 0, (2+h)^2 - b^2 \to 0$

$\therefore \quad b^2 = 4$ and $a^2 = 8$

$\therefore$ Locus of the pair of perpendicular tangents to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\therefore$ Required locus is

$$x^2 + y^2 = a^2 + b^2 = 8 + 4 = 12$$

$$\Rightarrow \quad r^2 = 12$$

(B) $\because$ Major axis on the line $y = 2$ and minor axis on the line $x = -1$

$\therefore$ Centre of ellipse is $(-1, 2)$

$$\Rightarrow \quad h = -1, k = 2$$

Also, $2a = 10$ and $2b = 4$

$$\therefore \quad M = a^2 = 25$$

and $\quad N = b^2 = 4$

Now, $h + k + M + N = -1 + 2 + 25 + 4 = 30$

(C) Here, $a = 5$, $b = 4$

$$\because \quad b^2 = a^2(1 - e^2) \Rightarrow 16 = 25(1 - e^2)$$

$$\therefore \quad e = \frac{3}{5}$$

Foci $(\pm 3, 0)$

Here, $SA = 2$ [A and A' are vertices]

Also gives $PS = 2$

$\therefore$ P coincides with A and Q coincides with A'

$\therefore$ $PQ = 2a = 10$

(D) Let $(\sqrt{27}\cos\theta, \sqrt{48}\sin\theta)$ be a point on the ellipse $\dfrac{x^2}{27} + \dfrac{y^2}{48} = 1$

$\therefore$ Equation of tangent at $(\sqrt{27}\cos\theta, \sqrt{48}\sin\theta)$ is

$$\frac{x\cos\theta}{\sqrt{27}} + \frac{y\sin\theta}{\sqrt{48}} = 1$$

$$\therefore \quad \text{slope} = -\frac{\sqrt{48}}{\sqrt{27}} \cdot \frac{\cos\theta}{\sin\theta} = -\frac{4}{3} \quad \text{[given]}$$

$$\therefore \quad \tan\theta = 1$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

$\therefore$ Equation of tangent is $\dfrac{x}{\sqrt{54}} + \dfrac{y}{\sqrt{96}} = 1$

$\therefore$ Area of triangle $= \dfrac{1}{2} \times 3\sqrt{6} \times 4\sqrt{6} = 36$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** (Ex. Nos. 25 and 26) are Assertion. Reason type examples. Each of these examples contains two statements.

Statement I (Assertion) and **Statement II** (Reason)

Each of these example also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for statement I

(b) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is flase

(d) Statement I is false, Statement II is true

• **Ex. 25 Statement I** *Feet of perpendiculars drawn from foci of an ellipse* $4x^2 + y^2 = 16$ *on the line* $2\sqrt{3}x + y = 8$ *lie on the circle* $x^2 + y^2 = 16$.

Statement II *If perpendiculars are drawn from foci of an ellipse to its any tangent, the feet of these perpendiculars lie on director circle of the ellipse.*

Sol. (c) Simultaneously solving the equations of ellipse and the given line, we get

$$4x^2 + (8 - 2\sqrt{3}x)^2 = 16$$

$$\Rightarrow \quad x^2 + (4 - \sqrt{3}x)^2 = 4$$

$$\Rightarrow \quad 4x^2 - 8\sqrt{3}\,x + 12 = 0$$

or $$x^2 - 2\sqrt{3}x + 3 = 0$$

or $$(x - \sqrt{3})^2 = 0$$

$\therefore$ $2\sqrt{3}x + y = 8$ is a tangent to the ellipse, the auxiliary circle is $x^2 + y^2 = 16$.

Hence, Statement I is true and Statement II is false.

• **Ex. 26 Statement II** *The condition on a and b for which two distinct chords of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$ *passing through* $(a, -b)$ *are bisected by the line* $x + y = b$ *is* $a^2 + 6ab - 7b^2 > 0$.

Statement II *Equation of chord of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ *whose mid-point* (x_1, y_1) *is* $T = S_1$

Sol. (a) Let $(\lambda, b - \lambda)$ is a point on the line $x + y = b$, then equation of chord whose mid-point $(\lambda, b - \lambda)$ is $T = S_1$

or $$\frac{\lambda x}{2a^2} + \frac{(b-\lambda)y}{2b^2} - 1 = \frac{\lambda^2}{2a^2} + \frac{(b-\lambda)^2}{2b^2} - 1 \quad \text{...(i)}$$

$(a, -b)$ lies on Eq. (i), then

$$\frac{\lambda a}{2a^2} - \frac{b(b-\lambda)}{2b^2} = \frac{\lambda^2}{2a^2} + \frac{(b-\lambda)^2}{2b^2}$$

$$\Rightarrow \quad \lambda^2(a^2 + b^2) - ab\lambda(3a + b) + 2a^2b^2 = 0$$

For two distinct chords $D > 0$

$$\Rightarrow \quad a^2b^2(3a + b)^2 - 8a^2b^2(a^2 + b^2) > 0$$

$$\Rightarrow \quad a^2 + 6ab - 7b^2 > 0$$

Hence, both Statements are true and Statement II is correct explanation for Statement I.

Subjective Type Questions

- In this section, there are **12 subjective solved examples.**

• **Ex. 27** *Find the locus of the centroid of an equilateral triangle inscribed in the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Sol. Let the vertices of the equilateral triangle P, Q and R and whose eccentric angles are α, β and γ.

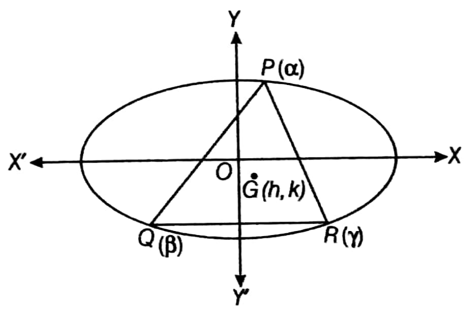
Let the centroid of ΔPQR be (h, k) then

$$h = \frac{a}{3}(\cos\alpha + \cos\beta + \cos\gamma) \quad \text{...(i)}$$

and $$k = \frac{b}{3}(\sin\alpha + \sin\beta + \sin\gamma) \quad \text{...(ii)}$$

$\because$ ΔPQR is equilateral.

$\therefore$ Centroid of the ΔPQR is same as the circumcentre.



$\because$ Circumcentre of ΔPQR be

Ellipse Exercise 8 :
Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

94. The minimum area of triangle formed by the tangent to the $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and coordinate axes is **[IIT-JEE 2003, 3M]**

(a) ab sq units (b) $\frac{a^2+b^2}{2}$ sq units

(c) $\frac{(a+b)^2}{2}$ sq units (d) $\frac{a^2+ab+b^2}{3}$ sq units

95. Find the equation of the common tangent in 1st quadrant to the circle $x^2+y^2=16$ and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also find the length of the intercept of the tangent between the coordinate axes. **[IIT-JEE 2005, 4M]**

96. An ellipse has OB as semi minor axis, F and F' its focii and the angle FBF' is a right angle. Then, the eccentricity of the ellipse is **[AIEEE 2005, 3M]**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{\sqrt{3}}$

97. In an ellipse, the distance between its foci is 6 and minor axis is 8. Then, its eccentricity is **[AIEEE 2006, 4.5M]**

(a) $\frac{3}{5}$ (b) $\frac{1}{2}$

(c) $\frac{4}{5}$ (d) $\frac{1}{\sqrt{5}}$

98. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1<0, y_2<0$ be the end points of the latusrectum of the, ellipse $x^2+4y^2=4$. The equations of parabolas with latusrectum PQ are **[IIT-JEE 2008, 4M]**

(a) $x^2+2\sqrt{3}y=3+\sqrt{3}$ (b) $x^2-2\sqrt{3}y=3+\sqrt{3}$

(c) $x^2+2\sqrt{3}y=3-\sqrt{3}$ (d) $x^2-2\sqrt{3}y=3-\sqrt{3}$

99. A focus of an ellipse is tat the origin. The directrix is the line $x=4$ and the eccentricity is $\frac{1}{2}$. Then, the length of the semi-major axis is **[AIEEE 2008, 3M]**

(a) $\frac{8}{3}$ (b) $\frac{2}{3}$

(c) $\frac{4}{3}$ (d) $\frac{5}{3}$

100. The line passing through the extermity A of the major axis and extremity B of the minor axis of the ellipse $x^2+9y^2=9$ meets its auxiliary circle at the point M. Then, the area of the triangle with vertices at A, M and the origin O is **[IIT-JEE 2009, 3M]**

(a) $\frac{31}{10}$ (b) $\frac{29}{10}$

(c) $\frac{21}{10}$ (d) $\frac{27}{10}$

101. The normal at a point P on the ellipse $x^2+4y^2=16$ meets the X-axis at Q. If M is the mid-point of the line segment PQ, then the locus of M intersects the latusrectum of the given ellipse at the points **[IIT-JEE 2009, 3M]**

(a) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (b) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\sqrt{\frac{19}{4}}\right)$

(c) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (d) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

102. In a triangle ABC with fixed based BC, the vertex A moves such that

$$\cos B+\cos C=4\sin^2\frac{A}{2}$$

If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then **[IIT-JEE 2009, 4M]**

(a) $b+c=4a$
(b) $b+c=2a$
(c) locus of point A is an ellipse
(d) locus of point A is a pair of straight lines

103. The conic having parametric representation $x=\sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y=\frac{2t}{1+t^2}$ is **[IIT-JEE 2009, 2M]**

(a) a circle (b) a parabola
(c) an ellipse (d) a hyperbola

104. The ellipse $x^2+4y^2=4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then, the equation of the ellipse is **[AIEEE 2009, 4M]**

(a) $x^2+12y^2=16$ (b) $4x^2+48y^2=48$
(c) $4x^2+64y^2=48$ (d) $x^2+16y^2=16$

Paragraph

(Q. Nos. 105 to 107)

Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

105. The coordinates of A and B are

(a) $(3, 0)$ and $(0, 2)$

(b) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

(c) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $(0, 2)$

(d) $(3, 0)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

106. The orthocenter of the triangle PAB is

(a) $\left(5, \frac{8}{7}\right)$ (b) $\left(\frac{7}{5}, \frac{25}{8}\right)$

(c) $\left(\frac{11}{5}, \frac{8}{5}\right)$ (d) $\left(\frac{8}{25}, \frac{7}{5}\right)$

107. The equation of the locus of the point whose distances from the point P and the line AB are equal, is **[IIT-JEE 2010, (3+3+3)M]**

(a) $9x^2+y^2-6xy-54x-62y+241=0$

(b) $x^2+9y^2+6xy-54x+62y-241=0$

(c) $9x^2+9y^2-6xy-54x-62y-241=0$

(d) $x^2+y^2-2xy+27x+31y-120=0$

108. Equation of the ellipse whose axes are the axes of coordinates an which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$ is **[AIEEE 2011, 4M]**

(a) $5x^2+3y^2-48=0$ (b) $3x^2+5y^2-15=0$

(c) $5x^2+3y^2-32=0$ (d) $3x^2+5y^2-32=0$

109. The ellipse $E_1:\frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle R. the eccentricity of the ellipse E_2 is **[IIT-JEE 2012, 4M]**

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

110. Statement I : An equation of a common tangent to the parabola $y^2=16\sqrt{3}x$ and the ellipse $2x^2+y^2=4$ is $y=2x+2\sqrt{3}$

Statement II : If the line $y=mx+\frac{4\sqrt{3}}{m}, (m\neq 0)$ is a common tangent to the parabola $y^2=16\sqrt{3}x$ and the ellipse $2x^2+y^2=4$, then m satisfies $m^4+2m^2=24$ **[AIEEE 2012, 4M]**

(a) Statement I is false, statement-II is true

(b) Statement I is true, statementII is true; statement-II is a correct explanation for statement-I

(c) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(d) Statement I is true, statement II is false

111. An ellipse is drawn by taking a diameter of the circle $(x-1)^2+y^2=1$ as its semi-minor axis and diameter of the circle $x^2+(y-2)^2=4$ is semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is **[AIEEE 2012, 4M]**

(a) $4x^2+y^2=4$ (b) $x^2+4y^2=8$

(c) $4x^2+y^2=8$ (d) $x^2+4y^2=16$

112. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, and having centre at $(0, 3)$ is **[JEE Main 2013, 4M]**

(a) $x^2+y^2-6y-7=0$ (b) $x^2+y^2-6y+7=0$

(c) $x^2+y^2-6y-5=0$ (d) $x^2+y^2-6y+5=0$

113. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \frac{1}{2}\leq h\leq \max \Delta(h)$ and $\Delta_2 = \frac{1}{2}\leq h\leq \min \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ **[JEE Advanced 2013, 3M]**

114. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2+3y^2=6$ on any tangent to it is **[JEE Main 2014, 4M]**

(a) $(x^2+y^2)^2=6x^2+2y^2$ (b) $(x^2+y^2)^2=6x^2-2y^2$

(c) $(x^2-y^2)^2=6x^2+2y^2$ (d) $(x^2-y^2)^2=6x^2-2y^2$

115. The area (in sq units) of the quadrilateral formed by the tangents at the end points of the latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is **[JEE Main 2015, 4M]**

(a) $\frac{27}{2}$ (b) 27

(c) $\frac{27}{4}$ (d) 18

Must

116. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the X-axis and the Y-axis, respectively. Let S be the circle $x^2+(y-1)^2=2$. The straight line $x+y=3$ touches the curves, S, E_1 and E_2 at P, Q and R respectively. Suppose that $PQ=PR=\frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are) **[JEE Advanced, 2015 4M]**

(a) $e_1^2+e_2^2=\frac{43}{40}$ (b) $e_1e_2=\frac{\sqrt{7}}{2\sqrt{10}}$

(c) $|e_1^2-e_2^2|=\frac{5}{8}$ (d) $e_1e_2=\frac{\sqrt{3}}{4}$

117. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1,0)$ and $(f_2\ 0)$ where $f_1>0$ and $f_2>0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1,0)$ and $(2f_2,0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2,0)$ and T_2 be a tangent to P_2 which passes through $(f_1,0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the9 value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is **[JEE Advanced 2015, 4M]**

Paragraph

(Q. Nos. 118 and 119)

Let $F_1(x_1,0)$ and $F_2(x_2,0)$ for $x_1<0$ and $x_2>0$, be the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{8}=1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

118. The orthocentre of the triangle F_1MN is

(a) $\left(-\frac{9}{10},0\right)$ (b) $\left(\frac{2}{3},0\right)$ (c) $\left(\frac{9}{10},0\right)$ (d) $\left(\frac{2}{3},\sqrt{6}\right)$

119. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the X-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is **[JEE Advanced 2016, (3+3)M]**

(a) 3 : 4 (b) 4 : 5 (c) 5 : 8 (d) 2 : 3

120. The eccentricity of an ellipse whose centre is at the origin is 1/2. If one of its directices is $x=-4$, then the equation of the normal to it at $(1, 3/2)$ is **[JEE Main 2017, 4M]**

(a) $x+2y=4$ (b) $2y-x=2$

(c) $4x-2y=1$ (d) $4x+2y=7$

Answers

Answers

Exercise for Session 1

1. (c) 2. (a) 3. (b) 4. (d) 5. (b) 6. (a,b) 7. (b) 8. (c) 9. (b) 10. (d) 11. (a) 12. (a)

13. $\frac{8}{3}, \frac{\sqrt{5}}{3}$; $(1\pm\sqrt{5}, 2)$; $(-2, 2)$ and $(4, 2)$; 6 and 4; $(1, 2)$

14. $3x^2+4y^2=300$ 15. $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{3}=1$

16. $5x^2+9y^2-54y+36=0$ 17. $\frac{1}{2}$

18. $x^2\pm ay=a^2$

Exercise for Session 2

1. (c) 2. (c) 3. (a) 4. (b) 5. (d) 6. (d) 7. (d) 8. (c) 9. (a) 10. (b)

11. (b) 12. $\lambda=-\frac{1}{\sqrt{5}}$; $\theta=\cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$

Exercise for Session 3

1. (c) 2. (c) 3. (b) 4. (b) 5. (a) 6. (d) 7. (b) 8. (a)

9. $3(x^2+y^2-a^2-b^2)^2=4(b^2x^2+a^2y^2-a^2b^2)$

11. $\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1$ 12. $b^4c^2x^2+a^2c^2y^2=a^4b^4$

13. $(-1, 0)$; $5\sqrt{3}$ 14. $x\sqrt{5}+2y=2\sqrt{5}$

Chapter Exercises

1. (c) 2. (d) 3. (a) 4. (b) 5. (c) 6. (d) 7. (a) 8. (c) 9. (a) 10. (c) 11. (b) 12. (b) 13. (d) 14. (c) 15. (c) 16. (a) 17. (c) 18. (c) 19. (d) 20. (a) 21. (b) 22. (a) 23. (a) 24. (a) 25. (a) 26. (a) 27. (d) 28. (d) 29. (a) 30. (c) 31. (a,c) 32. (a,b,c,d) 33. (b,d) 34. (a,c) 35. (a,d) 36. (a, c) 37. (a,b) 38. (a,b,c) 39. (a,b) 40. (a,c) 41. (a,b) 42. (a,c) 43. (a,b,c) 44. (a,b,c) 45. (a,c) 46. (d) 47. (b) 48. (a) 49. (a) 50. (c) 51. (b) 52. (d) 53. (d) 54. (b) 55. (c) 56. (a) 57. (b) 58. (c) 59. (b) 60. (c) 61. (6) 62. (9) 63. (4) 64. (5) 65. (2) 66. (8) 67. (1) 68. (5) 69. (9) 70. (6)

71. (A) $\rightarrow$ (p,q); (B) $\rightarrow$ (p,q,s); (C) $\rightarrow$ (p,r); (D) $\rightarrow$ (p,r)

72. (A) $\rightarrow$ (q); (B) $\rightarrow$ (p); (C) $\rightarrow$ (p,r,s); (D) $\rightarrow$ (q)

73. (A) $\rightarrow$ (p,s); (B) $\rightarrow$ (q); (C) $\rightarrow$ (q,s); (D) $\rightarrow$ (p,r)

74. (a) 75. (b) 76. (a) 77. (a) 78. (b) 79. (c)

80. (c) 81. (a) 86. $y=\pm\sqrt{\left(\frac{r^2-4}{(25-r^2)}\right)}x$ 90. (2b)

94. (a) 95. $\left(\frac{14}{\sqrt{3}}\right)$ 96. (a) 97. (a) 98. (b,c) 99. (a)

100. (d) 101. (c) 102. (b,c) 103. (c) 104. (a) 105. (d) 106. (c) 107. (a) 108. (d) 109. (c) 110. (b) 111. (d) 112. (a) 113. (9) 114. (a) 115. (b) 116. (a,b) 117. (4) 118. (a) 119. (c) 120. (c)