Session 4

Harmonic Sequence or Harmonic Progression (HP)

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A Harmonic Progression (HP) is a sequence, if the reciprocals of its terms are in Arithmetic Progression (AP)

i.e., $t_1, t_2, t_3, \ldots$ is HP if and only if $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, \ldots$ is an AP.

For example, The sequence

(i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ (ii) $2, \frac{5}{2}, \frac{10}{3}, \ldots$

(iii) $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$ are HP's.

Remark

1. No term of HP can be zero.
2. The most general or standard HP is
$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots$$

Example 52. If a, b, c are in HP, then show that $\frac{a-b}{b-c} = \frac{a}{c}$.

Sol. Since, a, b, c are in HP, therefore

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

i.e. $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$

or $\frac{a-b}{ab} = \frac{b-c}{bc}$ or $\frac{a-b}{b-c} = \frac{a}{c}$

Remark

A HP may also be defined as a series in which every three consecutive terms (say I, II, III) satisfy $\frac{\text{I}-\text{II}}{\text{II}-\text{III}} = \frac{\text{I}}{\text{III}}$ this relation.

Example 53. Find the first term of a HP whose second term is $\frac{5}{4}$ and the third term is $\frac{1}{2}$.

Sol. Let a be the first term. Then, $a, \frac{5}{4}, \frac{1}{2}$ are in HP.

Then, $$\frac{a - \frac{5}{4}}{\frac{5}{4} - \frac{1}{2}} = \frac{a}{\frac{1}{2}}$$ [from above note]

$$\Rightarrow \frac{4a-5}{5-2} = 2a$$

$$\Rightarrow 4a - 5 = 6a \text{ or } 2a = -5$$

$$\therefore a = -\frac{5}{2}$$

(i) nth Term of HP from Beginning

Let a be the first term, d be the common difference of an AP. Then, nth term of an AP from beginning $= a + (n-1)d$

Hence, the nth term of HP from beginning

$$= \frac{1}{a+(n-1)d}, \forall n \in N$$

(ii) nth Term of HP from End

Let l be the last term, d be the common difference of an AP. Then,

nth term of an AP from end $= l - (n-1)d$

Hence, the nth term of HP from end $= \frac{1}{l-(n-1)d}, \forall n \in N$

Remark

1. $\frac{1}{n\text{th term of HP from beginning}} + \frac{1}{n\text{th term of HP from end}}$
$= a + l = \frac{1}{\text{first term of HP}} + \frac{1}{\text{last term of HP}}$
2. There is no general formula for the sum of any number of quantities in HP are generally solved by inverting the terms and making use of the corresponding AP.

Example 54. If $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$, then prove that a, b, c are in HP, unless $b = a + c$.

Sol. We have, $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$

$$\Rightarrow \left(\frac{1}{a} + \frac{1}{c-b}\right) + \left(\frac{1}{c} + \frac{1}{a-b}\right) = 0$$

$$\Rightarrow \frac{(c-b+a)}{a(c-b)} + \frac{(a-b+c)}{c(a-b)} = 0$$

$$\Rightarrow (a+c-b)\left[\frac{1}{a(c-b)} + \frac{1}{c(a-b)}\right] = 0$$

$$\Rightarrow (a+c-b)[2ac - b(a+c)] = 0$$

If $a + c - b \neq 0$, then $2ac - b(a + c) = 0$

or $$b = \frac{2ac}{a + c}$$

Therefore, a, b, c are in HP and if $2ac - b(a + c) \neq 0$, then $a + c - b = 0$ i.e., $b = a + c$.

| Example 55. If $a_1, a_2, a_3, \ldots, a_n$ are in HP, then prove that $a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}a_n = (n - 1)a_1a_n$

Sol. Given, $a_1, a_2, a_3, \ldots, a_n$ are in HP.

$\therefore \quad \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in AP.

Let D be the common difference of the AP, then

$$\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \frac{1}{a_4} - \frac{1}{a_3} = \ldots = \frac{1}{a_n} - \frac{1}{a_{n-1}} = D$$

$$\Rightarrow \quad \frac{a_1 - a_2}{a_1a_2} = \frac{a_2 - a_3}{a_2a_3} = \frac{a_3 - a_4}{a_3a_4} = \ldots = \frac{a_{n-1} - a_n}{a_{n-1}a_n} = D$$

$$\Rightarrow \quad a_1a_2 = \frac{a_1 - a_2}{D}, a_2a_3 = \frac{a_2 - a_3}{D}, a_3a_4 = \frac{a_3 - a_4}{D},$$

$$\ldots, a_{n-1}a_n = \frac{a_{n-1} - a_n}{D}$$

On adding all such expressions, we get

$$a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}a_n = \frac{a_1 - a_n}{D} = \frac{a_1a_n}{D}\left(\frac{1}{a_n} - \frac{1}{a_1}\right)$$

$$= \frac{a_1a_n}{D}\left[\frac{1}{a_1} + (n - 1)D - \frac{1}{a_1}\right] = (n - 1)a_1a_n$$

Hence, $a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}a_n = (n - 1)a_1a_n$

Remark

In particular case,

1. when $n = 4$ $a_1a_2 + a_2a_3 + a_3a_4 = 3a_1a_4$
2. when $n = 6$ $a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 = 5a_1a_6$

| Example 56. The sum of three numbers in HP is 37 and the sum of their reciprocals is $\frac{1}{4}$. Find the numbers.

Sol. Three numbers in HP can be taken as $\frac{1}{a - d}, \frac{1}{a}, \frac{1}{a + d}$.

Then, $$\frac{1}{a - d} + \frac{1}{a} + \frac{1}{a + d} = 37 \quad \ldots(i)$$

and $$a - d + a + a + d = \frac{1}{4}$$

$$\therefore \quad a = \frac{1}{12}$$

From Eq. (i), $$\frac{12}{1 - 12d} + 12 + \frac{12}{1 + 12d} = 37$$

$$\Rightarrow \quad \frac{12}{1 - 12d} + \frac{12}{1 + 12d} = 25 \Rightarrow \frac{24}{1 - 144d^2} = 25$$

$$\Rightarrow \quad 1 - 144d^2 = \frac{24}{25} \quad \text{or} \quad d^2 = \frac{1}{25 \times 144}$$

$$\therefore \quad d = \pm\frac{1}{60}$$

$\therefore$ $a - d, a, a + d$ are $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$ or $\frac{1}{10}, \frac{1}{12}, \frac{1}{15}$.

Hence, three numbers in HP are 15, 12, 10 or 10, 12, 15.

| Example 57. If pth, qth and rth terms of a HP be respectively a, b and c, then prove that $(q - r)bc + (r - p)ca + (p - q)ab = 0$.

Sol. Let A and D be the first term and common difference of the corresponding AP. Now, a, b, c are respectively the p th, q th and r th terms of HP.

$\therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ will be respectively the p th, q th and r th terms of the corresponding AP.

$$\Rightarrow \quad \frac{1}{a} = A + (p - 1)D \quad \ldots(i)$$

$$\frac{1}{b} = A + (q - 1)D \quad \ldots(ii)$$

$$\frac{1}{c} = A + (r - 1)D \quad \ldots(iii)$$

On subtracting Eq. (iii) from Eq. (ii), we get

$$\frac{1}{b} - \frac{1}{c} = (q - r)D \Rightarrow bc(q - r) = \frac{(c - b)}{D} = -\frac{(b - c)}{D}$$

So, LHS $= (q - r)bc + (r - p)ca + (p - q)ab$

$$= -\frac{1}{D}\{b - c + c - a + a - b\} = 0 = \text{RHS}$$

Theorem Relating to the Three Series

If a, b, c are three consecutive terms of a series, then

if $\frac{a - b}{b - c} = \frac{a}{a}$, then a, b, c are in AP.

if $\frac{a - b}{b - c} = \frac{a}{b}$, then a, b, c are in GP and if $\frac{a - b}{b - c} = \frac{a}{c}$, then a, b, c are in HP.

Mixed Examples on AP, GP and HP

| Example 58. If a, b, c are in AP and a^2, b^2, c^2 be in HP. Then, prove that $-\frac{a}{2}, b, c$ are in GP or else $a = b = c$.

Sol. Given, a, b, c are in AP.

$$\therefore \quad b = \frac{a + c}{2} \quad \ldots(i)$$

and a^2, b^2, c^2 are in HP.

$$\therefore \quad b^2 = \frac{2a^2c^2}{a^2 + c^2} \quad \ldots(ii)$$

From Eq. (ii) $b^2\{(a + c)^2 - 2ac\} = 2a^2c^2$

$$b^2\{(2b)^2 - 2ac\} = 2a^2c^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 2b^4 - acb^2 - a^2c^2 = 0$$

$$\Rightarrow \quad (2b^2 + ac)(b^2 - ac) = 0$$

$$\Rightarrow \quad 2b^2 + ac = 0 \text{ or } b^2 - ac = 0$$

If $2b^2 + ac = 0$, then $b^2 = -\frac{1}{2}ac$ or $-\frac{a}{2}, b, c$ are in GP

and if $b^2 - ac = 0 \Rightarrow a, b, c$ are in GP.

But given, a, b, c are in AP.

Which is possible only when $a = b = c$

| Example 59. If a, b, c are in HP, b, c, d are in GP and c, d, e are in AP, then show that $e = \frac{ab^2}{(2a - b)^2}$.

Sol. Given, a, b, c are in HP.

$$\therefore \quad b = \frac{2ac}{a + c} \quad \text{or} \quad c = \frac{ab}{2a - b} \quad \ldots(i)$$

Given, b, c, d are in GP.

$$\therefore \quad c^2 = bd \quad \ldots(ii)$$

and given, c, d, e are in AP.

$$\therefore \quad d = \frac{c + e}{2}$$

$$\Rightarrow \quad e = 2d - c$$

$$e = \left(\frac{2c^2}{b} - c\right) \quad \text{[from Eq. (ii)]} \ldots(iii)$$

From Eqs. (i) and (iii), $$e = \frac{2}{b}\left(\frac{ab}{2a - b}\right)^2 - \left(\frac{ab}{2a - b}\right)$$

$$= \frac{ab}{(2a - b)^2}\{2a - (2a - b)\}$$

$$= \frac{ab^2}{(2a - b)^2}$$

| Example 60. If a, b, c, d and e be five real numbers such that a, b, c are in AP; b, c, d are in GP; c, d, e are in HP. If $a = 2$ and $e = 18$, then find all possible values of b, c and d.

Sol. Given, a, b, c are in AP,

$$\therefore \quad b = \frac{a + c}{2} \quad \ldots(i)$$

b, c, d are in GP,

$$\therefore \quad c^2 = bd \quad \ldots(ii)$$

and c, d, e are in HP.

$$\therefore \quad d = \frac{2ce}{c + e} \quad \ldots(iii)$$

Now, substituting the values of b and d in Eq. (ii), then

$$c^2 = \left(\frac{a + c}{2}\right)\left(\frac{2ce}{c + e}\right)$$

$$\Rightarrow \quad c(c + e) = e(a + c)$$

$$\Rightarrow \quad c^2 = ae \quad \ldots(iv)$$

Given, $\quad a = 2, e = 18$

From Eq. (iv), $\quad c^2 = (2)(18) = 36$

$$\therefore \quad c = \pm 6$$

From Eq. (i), $\quad b = \frac{2 \pm 6}{2} = 4, -2$

and from Eq. (ii), $\quad d = \frac{c^2}{b} = \frac{36}{b} = \frac{36}{4}$ or $\frac{36}{-2}$

$$\therefore \quad d = 9 \text{ or } -18$$

Hence, $\quad c = 6, b = 4, d = 9$ or $c = -6, b = -2, d = -18$

| Example 61. If three positive numbers a, b and c are in AP, GP and HP as well, then find their values.

Sol. Since a, b, c are in AP, GP and HP as well

$$\therefore \quad b = \frac{a + c}{2} \quad \ldots(i)$$

$$b^2 = ac \quad \ldots(ii)$$

and $$b = \frac{2ac}{a + c} \quad \ldots(iii)$$

From Eqs. (i) and (ii), we have

$$\left(\frac{a + c}{2}\right)^2 = ac$$

or $\quad (a + c)^2 = 4ac$

or $\quad (a + c)^2 - 4ac = 0$

or $\quad (a - c)^2 = 0$

$$\therefore \quad a = c \quad \ldots(iv)$$

On putting $c = a$ in Eq. (i), we get $b = \frac{a + a}{2} = a \quad \ldots(v)$

From Eqs. (iv) and (v), $a = b = c$, thus the three numbers will be equal.

Remark

1. If three positive numbers are in any two of AP, GP and HP, then it will be also in third.
2. Thus, if three positive numbers are in any two of AP, GP and HP, then they will be in the third progression and the numbers will be equal.

Exercise for Session 4

1. If a, b, c are in AP and b, c, d be in HP, then
(a) $ab = cd$ (b) $ad = bc$ (c) $ac = bd$ (d) $abcd = 1$

2. If a, b, c are in AP, then $\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ are in
(a) AP (b) GP (c) HP (d) None of these

3. If a, b, c are in AP and a, b, d are in GP, then $a, a-b, d-c$ will be in
(a) AP (b) GP (c) HP (d) None of these

4. If $x, 1, z$ are in AP and $x, 2, z$ are in GP, then $x, 4, z$ will be in
(a) AP (b) GP (c) HP (d) None of these

5. If a, b, c are in GP, $a-b, c-a, b-c$ are in HP, then $a+4b+c$ is equal to
(a) 0 (b) 1 (c) -1 (d) None of these

6. If $(m+1)$th, $(n+1)$th and $(r+1)$th terms of an AP are in GP and m, n, r are in HP, then the value of the ratio of the common difference to the first term of the AP is
(a) $-\frac{2}{n}$ (b) $\frac{2}{n}$ (c) $-\frac{n}{2}$ (d) $\frac{n}{2}$

7. If a, b, c are in AP and a^2, b^2, c^2 are in HP, then
(a) $a = b = c$ (b) $2b = 3a + c$ (c) $b^2 = \sqrt{\left(\frac{ac}{8}\right)}$ (d) None of these

8. If a, b, c are in HP, then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in
(a) AP (b) GP (c) HP (d) None of these

9. If $\frac{x+y}{2}, y, \frac{y+z}{2}$ are in HP, then x, y, z are in
(a) AP (b) GP (c) HP (d) None of these

10. If $\frac{a+b}{1-ab}, b, \frac{b+c}{1-bc}$ are in AP, then $a, \frac{1}{b}, c$ are in
(a) AP (b) GP (c) HP (d) None of these

Session 5

Mean

Mean

Arithmetic Mean

If three terms are in AP, then the middle term is called the Arithmetic Mean (or shortly written as AM) between the other two, so if a, b, c are in AP, then b is the AM of a and c.

(i) Single AM of n Positive Numbers

Let n positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and A be the AM of these numbers, then

$$A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

In particular Let a and b be two given numbers and A be the AM between them, then a, A, b are in AP.

$$\therefore \quad A = \frac{a+b}{2}$$

Remark

1. AM of $2a, 3b, 5c$ is $\frac{2a+3b+5c}{3}$.
2. AM of $a_1, a_2, a_3, \ldots, a_{n-1}, 2a_n$ is $\frac{a_1 + a_2 + a_3 + \ldots + a_{n-1} + 2a_n}{n}$.

(ii) Insert n-Arithmetic Mean Between Two Numbers

Let a and b be two given numbers and $A_1, A_2, A_3, \ldots, A_n$ are AM's between them.

Then, $a, A_1, A_2, A_3, \ldots, A_n, b$ will be in AP.

Now, $b = (n+2)$ th term $= a + (n+2-1)\, d$

$$\therefore \quad \boxed{d = \left(\frac{b-a}{n+1}\right)}$$

[Remember] [where, d = common difference] ...(i)

$\therefore \quad A_1 = a + d, A_2 = a + 2d, \ldots, A_n = a + nd$

$\Rightarrow \quad A_1 = a + \left(\frac{b-a}{n+1}\right), A_2 = a + 2\left(\frac{b-a}{n+1}\right), \ldots, A_n$

$= a + n\left(\frac{b-a}{n+1}\right)$

Corollary I The sum of n AM's between two given quantities is equal to n times the AM between them.

Let two numbers be a and b and $A_1, A_2, A_3, \ldots, A_n$ are n AM's between them.

Then, $a, A_1, A_2, A_3, \ldots, A_n, b$ will be in AP.

$\therefore$ Sum of n AM's between a and b

$= A_1 + A_2 + A_3 + \ldots + A_n$

$= \frac{n}{2}(A_1 + A_n)$ [$\because A_1, A_2, A_3, \ldots, A_n$ are in AP]

$= \frac{n}{2}(a + d + a + nd) = \frac{n}{2}[2a + (n+1)\, d]$

$= \frac{n}{2}(2a + b - a)$ [from Eq. (i)]

$= n\left(\frac{a+b}{2}\right) = n$ [AM between a and b]

[Remember]

Aliter $A_1 + A_2 + A_3 + \ldots + A_n$

$= (a + A_1 + A_2 + A_3 + \ldots + A_n + b) - (a+b)$

$= \frac{(n+2)}{2}(a+b) - (a+b) = n\left(\frac{a+b}{2}\right)$

$= n$ [AM of a and b]

Aliter

[This method is applicable only when n is even]

$A_1 + A_2 + A_3 + \ldots + A_{n-2} + A_{n-1} + A_n$

$= (A_1 + A_n) + (A_2 + A_{n-1}) + (A_3 + A_{n-2}) + \ldots$ upto $\frac{n}{2}$ terms

$= (a+b) + (a+b) + (a+b) + \ldots$ upto $\frac{n}{2}$ times [$\because T_n + T'_n = a + l$]

$= \frac{n}{2}(a+b) = n\left(\frac{a+b}{2}\right) = n$ [AM of a and b]

Corollary II The sum of m AM's between any two numbers is to the sum of n AM's between them as $m : n$.

Let two numbers be a and b.

$\therefore$ Sum of m AM's between a and $b = m$ [AM of a and b] ...(i)

Similarly, sum of n AM's between a and $b = n$ [AM of a and b] ...(ii)

$$\therefore \quad \frac{\text{Sum of } m \text{ AM's}}{\text{Sum of } n \text{ AM's}} = \frac{m(\text{AM of } a \text{ and } b)}{n(\text{AM of } a \text{ and } b)} = \frac{m}{n}$$

| Example 62. If a, b, c are in AP and p is the AM between a and b and q is the AM between b and c, then show that b is the AM between p and q.

Sol. $\because$ a, b, c are in AP.

$\therefore \quad 2b = a + c \quad$ ...(i)

$\because$ p is the AM between a and b.

$\therefore \quad p = \frac{a+b}{2} \quad$ ...(ii)

$\because$ q is the AM between b and c.

$\therefore \quad q = \frac{b+c}{2} \quad$ ...(iii)

On adding Eqs. (ii) and (iii), then

$$p + q = \frac{a+b}{2} + \frac{b+c}{2} = \frac{a+c+2b}{2} = \frac{2b+2b}{2} \quad \text{[using Eq. (i)]}$$

$\therefore \quad p + q = 2b$ or $b = \frac{p+q}{2}$

Hence, b is the AM between p and q.

| Example 63. Find n, so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ $(a \ne b)$ be the AM between a and b.

Sol. $\because \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$

$$\Rightarrow \quad \frac{b^{n+1}\left[\left(\frac{a}{b}\right)^{n+1}+1\right]}{b^n\left[\left(\frac{a}{b}\right)^n+1\right]} = \frac{b}{2}\left[\left(\frac{a}{b}\right)+1\right]$$

$$\Rightarrow \quad 2\left[\left(\frac{a}{b}\right)^{n+1}+1\right] = \left[\left(\frac{a}{b}\right)^n+1\right]\left(\frac{a}{b}+1\right)$$

Let $\quad \frac{a}{b} = \lambda$

$\therefore \quad 2\lambda^{n+1} + 2 = (\lambda^n + 1)(\lambda + 1)$

$\Rightarrow \quad 2\lambda^{n+1} + 2 = \lambda^{n+1} + \lambda^n + \lambda + 1$

$\Rightarrow \quad \lambda^{n+1} - \lambda^n - \lambda + 1 = 0 \Rightarrow (\lambda^n - 1)(\lambda - 1) = 0$

$\lambda - 1 \ne 0 \quad [\because a \ne b]$

$\therefore \quad \lambda^n - 1 = 0 \Rightarrow \lambda^n = 1 = \lambda^0$

$\Rightarrow \quad \boxed{n = 0}$ remember.

| Example 64. There are n AM's between 3 and 54 such that 8th mean is to $(n-2)$th mean as 3 to 5. Find n.

Sol. Let $A_1, A_2, A_3, \ldots, A_n$ be n AM's between 3 and 54.

If d be the common difference, then

$$d = \frac{54-3}{n+1} = \frac{51}{n+1} \quad \text{...(i)}$$

According to the example,

$$\frac{A_8}{A_{n-2}} = \frac{3}{5}$$

$\Rightarrow \quad 5(3+8d) = 3[3+(n-2)d] \Rightarrow 6 = d(3n-46)$

$\Rightarrow \quad 6 = (3n-46)\frac{51}{(n+1)} \quad$ [from Eq. (i)]

$\Rightarrow \quad 6n + 6 = 153n - 2346 \Rightarrow 147n = 2352$

$\therefore \quad n = 16$

| Example 65. If 11 AM's are inserted between 28 and 10, then find the three middle terms in the series.

Sol. Let $A_1, A_2, A_3, \ldots, A_{11}$ be 11 AM's between 28 and 10.

If d be the common difference, then

$$d = \frac{10-28}{12} = -\frac{3}{2}$$

Total means = 11 (odd)

$\therefore \quad$ Middle mean $= \left(\frac{11+1}{2}\right)$th $=$ 6th $= A_6$

Then, three middle terms are A_5, A_6 and A_7.

$\therefore \quad A_5 = 28 + 5d = 28 - \frac{15}{2} = \frac{41}{2}$

$A_6 = 28 + 6d = 28 - 9 = 19$

and $\quad A_7 = 28 + 7d = 28 - \frac{21}{2} = \frac{35}{2}$

| Example 66. If a, b, c are in AP, then show that $a^2(b+c) + b^2(c+a) + c^2(a+b) = \frac{2}{9}(a+b+c)^3$.

Sol. $\because$ a, b, c are in AP.

$\therefore \quad b = \frac{a+c}{2}$ i.e., $2b = a + c \quad$ ...(i)

$$\begin{aligned} \text{LHS} &= a^2(b+c) + b^2(c+a) + c^2(a+b) \\ &= (a^2b + a^2c) + b^2(2b) + (c^2a + c^2b) \\ &= b(a^2+c^2) + ac(a+c) + 2b^3 \\ &= b[(a+c)^2 - 2ac] + ac(2b) + 2b^3 \\ &= b(a+c)^2 + 2b^3 = b(2b)^2 + 2b^3 = 6b^3 \end{aligned}$$

$$\text{RHS} = \frac{2}{9}(a+b+c)^3 = \frac{2}{9}(2b+b)^3 = \frac{2}{9} \times 27b^3 = 6b^3$$

Hence, LHS = RHS

Geometric Mean

If three terms are in GP, then the middle term is called the **Geometric Mean** (or shortly written as GM) between the other two, so if a, b, c are in GP, then b is the GM of a and c.

(i) Single GM of n Positive Numbers

Let n positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and G be the GM of these numbers, then $G = (a_1a_2a_3 \ldots a_n)^{1/n}$

In particular Let a and b be two numbers and G be the GM between them, then a, G, b are in GP.

Hence, $G = \sqrt{ab}$; $a > 0, b > 0$

Remark

1. If $a < 0, b < 0$, then $G = -\sqrt{ab}$
2. If $a < 0, b > 0$ or $a > 0, b < 0$, then GM between a and b does not exist.

Example

(i) The GM between 4 and 9 is given by

$$G = \sqrt{4 \times 9} = 6$$

(ii) The GM between -4 and -9 is given by

$$G = \sqrt{-4 \times -9} = -6$$

(iii) The GM between -4 and 9 or 4 and -9 does not exist.

i.e. $\sqrt{(-4) \times 9} = \sqrt{-1}\sqrt{36} = 6i$

and $\sqrt{4 \times (-9)} = \sqrt{-1}\sqrt{36} = 6i$

(ii) Insert n-Geometric Mean Between Two Numbers

Let a and b be two given numbers and $G_1, G_2, G_3, \ldots, G_n$ are n GM's between them.

Then, $a, G_1, G_2, G_3, \ldots, G_n, b$ will be in GP.

Now, $b = (n+2)$ th term $= ar^{n+2-1}$

$$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \quad \text{[where } r = \text{common ratio] [Remember]} \quad \text{...(i)}$$

$\therefore \quad G_1 = ar, G_2 = ar^2, \ldots, G_n = ar^n$

$$\Rightarrow \quad G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots, G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Corollary The product of n geometric means between a and b is equal to the nth power of the geometric mean between a and b.

Let two numbers be a and b and $G_1, G_2, G_3, \ldots, G_n$ are n GM's between them.

Then, $a, G_1, G_2, G_3, \ldots, G_n, b$ will be in GP.

$\therefore$ Product of n GM's between a and b

$$\begin{aligned} &= G_1G_2G_3 \ldots G_n = (ar)(ar^2)(ar^3)\ldots(ar^n) \\ &= a^{1+1+1+\ldots+1} \cdot r^{1+2+3+\ldots+n} \\ &= a^n \cdot r^{\frac{n(n+1)}{2}} = a^n \cdot \left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}} \quad \text{[from Eq. (i)]} \\ &= a^n\left(\frac{b}{a}\right)^{\frac{n}{2}} = a^{n/2}b^{n/2} = (\sqrt{ab})^n \\ &= [\text{GM of } a \text{ and } b]^n \quad \text{[Remember]} \end{aligned}$$

Aliter [This method is applicable only when n is even]

$$G_1G_2G_3 \ldots G_{n-2}G_{n-1}G_n = (G_1G_n)(G_2G_{n-1})(G_3G_{n-2})\ldots\frac{n}{2}\text{ factors}$$

$$= (ab)(ab)(ab)\ldots\frac{n}{2}\text{ factors} \quad [\because T_n \times T_n' = a \times l]$$

$$= (ab)^{n/2} = (\sqrt{ab})^n = [\text{GM of } a \text{ and } b]^n$$

| Example 67. If a be one AM and G_1 and G_2 be two geometric means between b and c, then prove that $G_1^3 + G_2^3 = 2abc$.

Sol. Given, a = AM between b and c

$\Rightarrow \quad a = \frac{b+c}{2} \Rightarrow 2a = b + c \quad$ ...(i)

Again, b, G_1, G_2, c are in GP.

$\therefore \quad \frac{G_1}{b} = \frac{G_2}{G_1} = \frac{c}{G_2} \Rightarrow b = \frac{G_1^2}{G_2}, c = \frac{G_2^2}{G_1}$

and $\quad G_1G_2 = bc \quad$ ...(ii)

From Eqs. (i) and (ii),

$$2a = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = \frac{G_1^3 + G_2^3}{G_1G_2} = \frac{G_1^3 + G_2^3}{bc} \quad [\because G_1G_2 = bc]$$

$\Rightarrow \quad G_1^3 + G_2^3 = 2abc$

| Example 68. If one geometric mean G and two arithmetic means p and q be inserted between two quantities, then show that $G^2 = (2p - q)(2q - p)$.

Sol. Let the two quantities be a and b, then

$$G^2 = ab \quad \text{...(i)}$$

Again, a, p, q, b are in AP.

$\therefore \quad p - a = q - p = b - q$

$\Rightarrow \quad a = 2p - q$

$b = 2q - p \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$G^2 = (2p - q)(2q - p)$$

| Example 69. Find n, so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ $(a \ne b)$ be the GM between a and b.

Sol. $\because \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \sqrt{ab}$

$$\Rightarrow \frac{b^{n+1}\left\{\left(\frac{a}{b}\right)^{n+1}+1\right\}}{b^n\left\{\left(\frac{a}{b}\right)^{n}+1\right\}} = b\sqrt{\frac{a}{b}} \Rightarrow \frac{\left(\frac{a}{b}\right)^{n+1}+1}{\left(\frac{a}{b}\right)^{n}+1} = \left(\frac{a}{b}\right)^{\frac{1}{2}}$$

Let $\frac{a}{b} = \lambda$

$$\Rightarrow \frac{\lambda^{n+1}+1}{\lambda^n+1} = \lambda^{\frac{1}{2}} \Rightarrow \lambda^{n+1}+1 = \lambda^{n+\frac{1}{2}} + \lambda^{\frac{1}{2}}$$

$$\Rightarrow \lambda^{n+\frac{1}{2}}(\lambda^{\frac{1}{2}}-1) - (\lambda^{\frac{1}{2}}-1) = 0$$

$$\Rightarrow (\lambda^{\frac{1}{2}}-1)(\lambda^{n+\frac{1}{2}}-1) = 0$$

$$\Rightarrow \lambda^{\frac{1}{2}} - 1 \neq 0 \quad [\because a \neq b]$$

$$\therefore \lambda^{n+\frac{1}{2}} - 1 = 0$$

$$\Rightarrow \lambda^{n+\frac{1}{2}} = 1 = \lambda^0$$ remember

$$\Rightarrow n + \frac{1}{2} = 0 \text{ or } \boxed{n = -\frac{1}{2}}$$

| Example 70. Insert five geometric means between $\frac{1}{3}$ and 9 and verify that their product is the fifth power of the geometric mean between $\frac{1}{3}$ and 9.

Sol. Let G_1, G_2, G_3, G_4, G_5 be 5 GM's between $\frac{1}{3}$ and 9.

Then, $\frac{1}{3}, G_1, G_2, G_3, G_4, G_5, 9$ are in GP.

Here, r = common ratio $= \left(\frac{9}{\frac{1}{3}}\right)^{1/6} = 3^{\frac{1}{2}} = \sqrt{3}$

$$\therefore G_1 = ar = \frac{1}{3}\cdot\sqrt{3} = \frac{1}{\sqrt{3}}$$

$$G_2 = ar^2 = \frac{1}{3}\cdot 3 = 1$$

$$G_3 = ar^3 = \frac{1}{3}\cdot 3\sqrt{3} = \sqrt{3}$$

$$G_4 = ar^4 = \frac{1}{3}\cdot 9 = 3$$

$$G_5 = ar^5 = \frac{1}{3}\cdot 9\sqrt{3} = 3\sqrt{3}$$

Now, Product $= G_1 \times G_2 \times G_3 \times G_4 \times G_5$

$$= \frac{1}{\sqrt{3}} \times 1 \times \sqrt{3} \times 3 \times 3\sqrt{3} = 9\sqrt{3} = (3)^{\frac{5}{2}} = \left(\sqrt{\frac{1}{3}\times 9}\right)^5$$

$$= \left[\text{GM of } \frac{1}{3} \text{ and } 9\right]^5$$

An Important Theorem

Let a and b be two real, positive and unequal numbers and A, G are arithmetic and geometric means between them, then

(i) a and b are the roots of the equation

$$x^2 - 2Ax + G^2 = 0 \quad \text{[Remember]}$$

(ii) a and b are given by $A \pm \sqrt{(A+G)(A-G)}$ [Remember]

(iii) $\boxed{A > G}$ [Remember]

Proof $\because A$ is the AM between a and b, then

$$A = \frac{a+b}{2} \Rightarrow a + b = 2A$$

and G is the GM between a and b, then

$$G = \sqrt{ab} \Rightarrow ab = G^2$$

$\therefore a$ and b are the roots of the equation, then

x^2 − (sum of roots) x + product of roots = 0

$$\Rightarrow x^2 - (a+b)x + ab = 0$$

i.e. $x^2 - 2Ax + G^2 = 0$ is the required equation.

$$\Rightarrow x = \frac{2A \pm \sqrt{(-2A)^2 - 4\cdot 1\cdot G^2}}{2\cdot 1} = A \pm \sqrt{(A^2 - G^2)}$$

$$\therefore x = A \pm \sqrt{(A+G)(A-G)}$$

Now, for real, positive and unequal numbers of a and b,

$$(A+G)(A-G) > 0 \Rightarrow (A-G) > 0$$

$$\therefore A > G$$

Remark

1. If a and b are real and positive numbers, then $A \geq G$
2. If $a_1, a_2, a_3, \ldots, a_n$ are n positive numbers, then AM ≥ GM i.e.

$$\boxed{\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{1/n}}$$

3. (i) If $a > 0, b > 0$ or $a < 0, b < 0$ and $\lambda_1 > 0, \lambda_2 > 0$, then $\lambda_1\frac{a}{b} + \lambda_2\frac{b}{a} \geq 2\sqrt{\lambda_1\lambda_2}$

if $\frac{a}{b} = x > 0$ and $\lambda_1 = \lambda_2 = 1$, then $\boxed{x + \frac{1}{x} \geq 2}$

(ii) If $a > 0, b < 0$ or $a < 0, b > 0$ and $\lambda_1 > 0, \lambda_2 > 0$, then $\lambda_1\frac{a}{b} + \lambda_2\frac{b}{a} \leq -2\sqrt{\lambda_1\lambda_2}$

if $\frac{a}{b} = x < 0$ and $\lambda_1 > 0, \lambda_2 > 0$ then $\boxed{x + \frac{1}{x} \leq -2}$

| Example 71. AM between two numbers whose sum is 100 is to the GM as 5 : 4, find the numbers.

Sol. Let the numbers be a and b.

Then, $a + b = 100$

or $2A = 100$

$$\Rightarrow A = 50 \quad \ldots\text{(i)}$$

and given, $\frac{A}{G} = \frac{5}{4} \Rightarrow \frac{50}{G} = \frac{5}{4}$ [from Eq. (i)]

$$\therefore G = 40 \quad \ldots\text{(ii)}$$

From important theorem $a, b = A \pm \sqrt{(A+G)(A-G)}$

$$= 50 \pm \sqrt{(50+40)(50-40)}$$

$$= 50 \pm 30 = 80, 20$$

$$\therefore a = 80, b = 20$$

or $a = 20, b = 80$

| Example 72. If $a_1, a_2, \ldots, a_n$ are positive real numbers whose product is a fixed number c, then find the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 3a_n$.

must

Sol. $\because$ AM ≥ GM

$$\therefore \frac{a_1 + a_2 + \ldots + a_{n-1} + 3a_n}{n} \geq (a_1 a_2 \ldots a_{n-1} 3a_n)^{1/n} = (3c)^{1/n}$$

$$\Rightarrow a_1 + a_2 + \ldots + a_{n-1} + 3a_n \geq n(3c)^{1/n}$$

Hence, the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 3a_n$ is $n(3c)^{1/n}$.

Harmonic Mean

If three terms are in HP, then the middle term is called the **Harmonic Mean** (or shortly written as HM) between the other two, so if a, b, c are in HP, then b is the HM of a and c.

[i] Single HM of n Positive Numbers

Let n positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and H be the HM of these numbers, then

$$H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)}$$

In particular Let a and b be two given numbers and H be the HM between them a, H, b are in HP.

Hence, $H = \frac{2}{\frac{1}{a} + \frac{1}{b}}$ i.e., $\boxed{H = \frac{2ab}{(a+b)}}$

Remark

HM of a, b, c is $\frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$ or $\frac{3abc}{ab + bc + ca}$

Caution The AM between two numbers a and b is $\frac{a+b}{2}$. It does not follow that HM between the same numbers is $\frac{2}{a+b}$. The HM is the reciprocals of $\frac{\frac{1}{a} + \frac{1}{b}}{2}$ i.e., $\frac{2ab}{(a+b)}$.

[ii] Insert n-Harmonic Mean Between Two Numbers

Let a and b be two given numbers and $H_1, H_2, H_3, \ldots, H_n$ are n HM's between them.

Then, $a, H_1, H_2, H_3, \ldots, H_n, b$ will be in HP. If D be the common difference of the corresponding AP

$\therefore b = (n+2)$ th term of HP.

$$= \frac{1}{(n+2)\text{th term of corresponding AP}}$$

$$= \frac{1}{\frac{1}{a} + (n+2-1)D}$$

$$\Rightarrow D = \frac{\frac{1}{b} - \frac{1}{a}}{(n+1)} \quad \text{[Remember]}$$

$$\therefore \frac{1}{H_1} = \frac{1}{a} + D, \frac{1}{H_2} = \frac{1}{a} + 2D, \ldots, \frac{1}{H_n} = \frac{1}{a} + nD$$

$$\Rightarrow \frac{1}{H_1} = \frac{1}{a} + \frac{(a-b)}{ab(n+1)}, \frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{ab(n+1)}, \ldots, \frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{ab(n+1)}$$

Corollary **The sum of reciprocals of n harmonic means between two given numbers is n times the reciprocal of single HM between them.**

Let two numbers be a and b and $H_1, H_2, H_3, \ldots, H_n$ are n HM's between them. Then, $a, H_1, H_2, H_3, \ldots, H_n, b$ will be in HP.

$$\therefore \frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} + \ldots + \frac{1}{H_n} = \frac{n}{2}\left(\frac{1}{H_1} + \frac{1}{H_n}\right) \quad \left[\because S_n = \frac{n}{2}(a+l)\right]$$

$$= \frac{n}{2}\left(\frac{1}{a} + D + \frac{1}{b} - D\right) = \frac{n}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$= \frac{n}{\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right)} = \frac{n}{[\text{HM of } a \text{ and } b]}$$

Aliter [This method is applicable only when n is even]

$$\frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} + \ldots + \frac{1}{H_{n-2}} + \frac{1}{H_{n-1}} + \frac{1}{H_n}$$

$$= \left(\frac{1}{H_1} + \frac{1}{H_n}\right) + \left(\frac{1}{H_2} + \frac{1}{H_{n-1}}\right) + \left(\frac{1}{H_3} + \frac{1}{H_{n-2}}\right) + \ldots \text{ upto } \frac{n}{2} \text{ terms}$$

$$=\left(\frac{1}{a}+D+\frac{1}{b}-D\right)+\left(\frac{1}{a}+2D+\frac{1}{b}-2D\right)+\left(\frac{1}{a}+3D+\frac{1}{b}-3D\right)+\ldots\text{upto }\frac{n}{2}\text{ terms}$$

$$=\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{1}{a}+\frac{1}{b}\right)+\ldots\text{ upto }\frac{n}{2}\text{ terms}$$

$$=\frac{n}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{n}{\left(\dfrac{2}{\frac{1}{a}+\frac{1}{b}}\right)}=\frac{n}{(\text{HM of } a \text{ and } b)}$$

| Example 73. If H be the harmonic mean between x and y, then show that $\dfrac{H+x}{H-x}+\dfrac{H+y}{H-y}=2$

Sol. We have, $H=\dfrac{2xy}{x+y}$

$\therefore \quad \dfrac{H}{x}=\dfrac{2y}{x+y}$ and $\dfrac{H}{y}=\dfrac{2x}{x+y}$

By componendo and dividendo, we have

$$\frac{H+x}{H-x}=\frac{2y+x+y}{2y-x-y}=\frac{x+3y}{y-x}$$

and $\quad \dfrac{H+y}{H-y}=\dfrac{2x+x+y}{2x-x-y}=\dfrac{3x+y}{x-y}$

must remember this method

$$\therefore \quad \frac{H+x}{H-x}+\frac{H+y}{H-y}=\frac{x+3y}{y-x}+\frac{3x+y}{x-y}$$

$$=\frac{x+3y-3x-y}{y-x}=\frac{2(y-x)}{(y-x)}=2$$

Aliter $\dfrac{H+x}{H-x}+\dfrac{H+y}{H-y}=2$

$$\Rightarrow \left(\frac{H+x}{H-x}-1\right)=\left(1-\frac{H+y}{H-y}\right) \Rightarrow \frac{2x}{H-x}=\frac{-2y}{H-y}$$

i.e. $\quad Hx-xy=-Hy+xy \Rightarrow H(x+y)=2xy$

i.e. $\quad \boxed{H=\dfrac{2xy}{(x+y)}}$

which is true as, x, H, y are in HP. Hence, the required result.

| Example 74. If $a_1, a_2, a_3, \ldots, a_{10}$ be in AP and $h_1, h_2, h_3, \ldots, h_{10}$ be in HP. If $a_1=h_1=2$ and $a_{10}=h_{10}=3$, then find the value of $a_4 h_7$.

Sol. $\because a_1, a_2, a_3, \ldots, a_{10}$ are in AP.

If d be the common difference, then

$$d=\frac{a_{10}-a_1}{9}=\frac{3-2}{9}=\frac{1}{9}$$

$\therefore \quad a_4=a_1+3d=2+\dfrac{3}{9}=2+\dfrac{1}{3}=\dfrac{7}{3}$ ...(i)

and given $h_1, h_2, h_3, \ldots, h_{10}$ are in HP.

If D be common difference of corresponding AP.

Then, $\quad D=\dfrac{\frac{1}{h_{10}}-\frac{1}{h_1}}{9}=\dfrac{\frac{1}{3}-\frac{1}{2}}{9}=-\dfrac{1}{54}$

$\therefore \quad \dfrac{1}{h_7}=\dfrac{1}{h_1}+6D=\dfrac{1}{2}-\dfrac{6}{54}=\dfrac{1}{2}-\dfrac{1}{9}=\dfrac{7}{18} \Rightarrow h_7=\dfrac{18}{7}$

Hence, $\quad a_4 \cdot h_7=\dfrac{7}{3}\times\dfrac{18}{7}=6$

| Example 75. Find n, so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ $(a \neq b)$ be HM between a and b.

Sol. $\because \quad \dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}=\dfrac{2ab}{a+b}$

$$\Rightarrow \quad \frac{b^{n+1}\left[\left(\frac{a}{b}\right)^{n+1}+1\right]}{b^n\left[\left(\frac{a}{b}\right)^n+1\right]}=\frac{b^2\left[2\left(\frac{a}{b}\right)\right]}{b\left(\frac{a}{b}+1\right)}$$

$$\Rightarrow \quad \frac{\left(\frac{a}{b}\right)^{n+1}+1}{\left(\frac{a}{b}\right)^n+1}=\frac{2\left(\frac{a}{b}\right)}{\left(\frac{a}{b}\right)+1}$$

Let $\quad \dfrac{a}{b}=\lambda$

Then, $\quad \dfrac{\lambda^{n+1}+1}{\lambda^n+1}=\dfrac{2\lambda}{\lambda+1}$

$\Rightarrow \quad (\lambda+1)(\lambda^{n+1}+1)=2\lambda(\lambda^n+1)$

$\Rightarrow \quad \lambda^{n+2}+\lambda+\lambda^{n+1}+1=2\lambda^{n+1}+2\lambda$

$\Rightarrow \quad \lambda^{n+2}-\lambda^{n+1}-\lambda+1=0$

$\Rightarrow \quad \lambda^{n+1}(\lambda-1)-1(\lambda-1)=0$

$\Rightarrow \quad (\lambda-1)(\lambda^{n+1}-1)=0$

$\Rightarrow \quad \lambda-1 \neq 0$ $[\because a \neq b]$

$\therefore \quad \lambda^{n+1}-1=0$

$\Rightarrow \quad \lambda^{n+1}=1=\lambda^0$

$\Rightarrow \quad n+1=0$ or $\boxed{n=-1}$

| Example 76. Insert 6 harmonic means between 3 and $\dfrac{6}{23}$.

Sol. Let $H_1, H_2, H_3, H_4, H_5, H_6$ be 6 HM's between 3 and $\dfrac{6}{23}$.

Then, $3, H_1, H_2, H_3, H_4, H_5, H_6, \dfrac{6}{23}$ are in HP.

$\Rightarrow \quad \dfrac{1}{3}, \dfrac{1}{H_1}, \dfrac{1}{H_2}, \dfrac{1}{H_3}, \dfrac{1}{H_4}, \dfrac{1}{H_5}, \dfrac{1}{H_6}, \dfrac{23}{6}$ are in AP.

Let common difference of this AP be D.

$\therefore \quad D=\dfrac{\frac{23}{6}-\frac{1}{3}}{7}=\dfrac{(23-2)}{7\times 6}=\dfrac{21}{7\times 6}=\dfrac{1}{2}$

$\therefore \quad \dfrac{1}{H_1}=\dfrac{1}{3}+D=\dfrac{1}{3}+\dfrac{1}{2}=\dfrac{5}{6}$

$\Rightarrow \quad H_1=\dfrac{6}{5}=1\dfrac{1}{5}$

$\dfrac{1}{H_2}=\dfrac{1}{3}+2D=\dfrac{1}{3}+1=\dfrac{4}{3} \Rightarrow H_2=\dfrac{3}{4}$

$\dfrac{1}{H_3}=\dfrac{1}{3}+3D=\dfrac{1}{3}+\dfrac{3}{2}=\dfrac{11}{6} \Rightarrow H_3=\dfrac{6}{11}$

$\dfrac{1}{H_4}=\dfrac{1}{3}+4D=\dfrac{1}{3}+2=\dfrac{7}{3} \Rightarrow H_4=\dfrac{3}{7}$

$\dfrac{1}{H_5}=\dfrac{1}{3}+5D=\dfrac{1}{3}+\dfrac{5}{2}=\dfrac{17}{6} \Rightarrow H_5=\dfrac{6}{17}$

and $\quad \dfrac{1}{H_6}=\dfrac{1}{3}+6D=\dfrac{1}{3}+3=\dfrac{10}{3} \Rightarrow H_6=\dfrac{3}{10}$

$\therefore$ HM's are $1\dfrac{1}{5}, \dfrac{3}{4}, \dfrac{6}{11}, \dfrac{3}{7}, \dfrac{6}{17}, \dfrac{3}{10}$.

Important Theorem 1

Let a and b be two real, positive and unequal numbers and A, G and H are arithmetic, geometric and harmonic means respectively between them, then

(i) A, G, H form a GP i.e. $\boxed{G^2=AH}$ [Remember]

(ii) $\boxed{A>G>H}$ [Remember]

Proof

(i) $\because \quad A=\dfrac{a+b}{2}, G=\sqrt{ab}$ and $H=\dfrac{2ab}{a+b}$

Now, $\quad AH=\left(\dfrac{a+b}{2}\right)\left(\dfrac{2ab}{a+b}\right)=ab=G^2$

Therefore, $G^2=AH$ i.e. A, G, H are in GP.

Remark

The result $AH=G^2$ will be true for n numbers, if they are in GP.

(ii) $\because \quad A>G$ [from important theorem of GM] ...(i)

or $\quad \dfrac{A}{G}>1$

$\Rightarrow \quad \dfrac{G}{H}>1 \quad \left[\because \dfrac{A}{G}=\dfrac{G}{H} \Rightarrow G^2=AH\right]$

$\Rightarrow \quad G>H$ ...(ii)

From Eqs. (i) and (ii), we get

$$A>G>H$$

Remark

If $a_1, a_2, a_3, \ldots, a_n$ are n positive numbers, then AM $\geq$ GM $\geq$ HM i.e.,

$$\frac{a_1+a_2+\ldots+a_n}{n} \geq (a_1 a_2 \ldots a_n)^{1/n} \geq \frac{n}{\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)}$$

Sign of equality (AH = GM = HM) holds when numbers are equal i.e., $a_1=a_2=\ldots=a_n$.

Important Theorem 2

If A, G, H are arithmetic, geometric and harmonic means of three given numbers a, b and c, then the equation having a, b, c as its roots is

$$x^3-3Ax^2+\frac{3G^3}{H}x-G^3=0 \quad \text{[Remember]}$$

Proof $\because A=$ AM of $a, b, c=\dfrac{a+b+c}{3}$

i.e., $\quad a+b+c=3A$ ...(i)

$G=$ GM of $a, b, c=(abc)^{1/3}$

i.e. $\quad abc=G^3$ ...(ii)

and $\quad H=$ HM of a, b, c

$$=\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{3abc}{ab+bc+ca}=\frac{3G^3}{ab+bc+ca}$$

[from Eq. (ii)]

i.e. $\quad ab+bc+ca=\dfrac{3G^3}{H}$ ...(iii)

$\therefore a, b, c$ are the roots of the equation

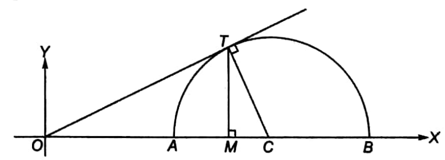
$$x^3-(a+b+c)x^2+(ab+bc+ca)x-abc=0$$

i.e., $\quad x^3-3Ax^2+\dfrac{3G^3}{H}x-G^3=0$

[from Eqs. (i), (ii) and (iii)]

Geometrical Proof of $A>G>H$

Let $OA=a$ unit and $OB=b$ unit and AB be a diameter of semi-circle. Draw tangent OT to the circle and TM perpendicular to AB.



Let C be the centre of the semi-circle.

$\because \quad \dfrac{OA+OB}{2}=\dfrac{(OC-AC)+(OC+CB)}{2}$

$=\dfrac{2OC}{2}=OC \quad [\because AC=CB=\text{radius of circle}]$

$\therefore \quad OC = \dfrac{a+b}{2}$ [i.e. OC = arithmetic mean]

$\Rightarrow \quad A = \dfrac{a+b}{2}$

Now, from geometry

$(OT)^2 = OA \times OB = ab = G^2$

$\therefore \quad OT = G$, the geometric mean

Now, from similar ΔOCT and ΔOMT, we have

$$\frac{OM}{OT} = \frac{OT}{OC} \text{ or } OM = \frac{(OT)^2}{OC} = \frac{ab}{\frac{a+b}{2}} = \frac{2ab}{a+b}$$

$\therefore \quad OM = H$, the harmonic mean

Also, it is clear from the figure, that

$OC > OT > OM$ i.e. $A > G > H$

| Example 77. If $A^x = G^y = H^z$, where A, G, H are AM, GM and HM between two given quantities, then prove that x, y, z are in HP.

Sol. Let $A^x = G^y = H^z = k$

Then, $\quad A = k^{1/x}, G = k^{1/y}, H = k^{1/z}$

$\because \quad G^2 = AH \Rightarrow (k^{1/y})^2 = k^{1/x} \cdot k^{1/z}$

$\Rightarrow \quad k^{2/y} = k^{1/x + 1/z} \Rightarrow \dfrac{2}{y} = \dfrac{1}{x} + \dfrac{1}{z} \Rightarrow \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ are in AP.

Hence, x, y, z are in HP.

| Example 78. The harmonic mean of two numbers is 4, their arithmetic mean A and geometric mean G satisfy the relation $2A + G^2 = 27$. Find the numbers.

Sol. Let the numbers be a and b.

Given, $\quad H = 4$

$\because \quad G^2 = AH = 4A$ ...(i)

and given $\quad 2A + G^2 = 27$

$\Rightarrow \quad 2A + 4A = 27$ [from Eq. (i)]

$\therefore \quad A = \dfrac{9}{2}$

From Eq. (i), $G^2 = 4 \times \dfrac{9}{2} = 18$

Now, from important theorem of GM

$$a, b = A \pm \sqrt{(A^2 - G^2)} = \frac{9}{2} \pm \sqrt{\left(\frac{81}{4} - 18\right)}$$

$$= \frac{9}{2} \pm \frac{3}{2} = 6, 3 \text{ or } 3, 6$$

| Example 79. If the geometric mean is $\dfrac{1}{n}$ times the harmonic mean between two numbers, then show that the ratio of the two numbers is

$$1 + \sqrt{(1-n^2)} : 1 - \sqrt{(1-n^2)}.$$

Sol. Let the two numbers be a and b.

Given, $\quad G = \dfrac{1}{n} H$ ...(i)

Now, $\quad G^2 = AH$

$\Rightarrow \quad \dfrac{H^2}{n^2} = AH$ [from Eq. (i)]

$\therefore \quad A = \dfrac{H}{n^2}$ ...(ii)

Now, from important theorem of GM

$$a, b = A \pm \sqrt{(A^2 - G^2)} = \frac{H}{n^2} \pm \sqrt{\left(\frac{H^2}{n^4} - \frac{H^2}{n^2}\right)}$$

$$= \frac{H}{n^2}[1 \pm \sqrt{(1-n^2)}]$$

$$\therefore \quad \frac{a}{b} = \frac{\frac{H}{n^2}[1 + \sqrt{(1-n^2)}]}{\frac{H}{n^2}[1 - \sqrt{(1-n^2)}]}$$

$\therefore \quad a : b = 1 + \sqrt{(1-n^2)} : 1 - \sqrt{(1-n^2)}$

| Example 80. If three positive unequal quantities a, b, c be in HP, then prove that $a^n + c^n > 2b^n, n \in N$

Sol. $\because \quad G > H$

$\therefore \quad \sqrt{ac} > b$

$\Rightarrow \quad (ac)^{\frac{n}{2}} > b^n$ or $a^{\frac{n}{2}} c^{\frac{n}{2}} > b^n$ ...(i)

Also, $\quad (a^{\frac{n}{2}} - c^{\frac{n}{2}})^2 > 0 \Rightarrow a^n + c^n - 2a^{\frac{n}{2}} c^{\frac{n}{2}} > 0$

$\Rightarrow \quad a^n + c^n > 2a^{\frac{n}{2}} c^{\frac{n}{2}} > 2b^n$ [from Eq. (i)]

$\therefore \quad a^n + c^n > 2b^n$

| Example 81.

(i) If a, b, c, d be four distinct positive quantities in AP, then

(a) $bc > ad$

(b) $c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(ii) If a, b, c, d be four distinct positive quantities in GP, then

(a) $a + d > b + c$

(b) $c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(iii) If a, b, c, d be four distinct positive quantities in HP, then

(a) $a + d > b + c$ (b) $ad > bc$

Sol. (i) $\because a, b, c, d$ are in AP.

(a) Applying AM > GM

For first three members, $b > \sqrt{ac}$

$\Rightarrow \quad b^2 > ac$ ...(i)

and for last three members, $c > \sqrt{bd}$

$\Rightarrow \quad c^2 > bd$ ...(ii)

From Eqs. (i) and (ii), we get

$b^2c^2 > (ac)(bd)$

Hence, $\quad bc > ad$

(b) Applying AM > HM

For first three members,

$b > \dfrac{2ac}{a+c}$

$\Rightarrow \quad ab + bc > 2ac$ ...(iii)

For last three members, $c > \dfrac{2bd}{b+d}$

$bc + cd > 2bd$ ...(iv)

From Eqs. (iii) and (iv), we get

$ab + bc + bc + cd > 2ac + 2bd$

or $\quad ab + cd > 2(ac + bd - bc)$

Dividing in each term by $abcd$, we get

$c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(ii) $\because a, b, c, d$ are in GP.

(a) Applying AM > GM

For first three members,

$\dfrac{a+c}{2} > b$

$\Rightarrow \quad a + c > 2b$ ...(v)

For last three members, $\dfrac{b+d}{2} > c$

$\Rightarrow \quad b + d > 2c$ ...(vi)

From Eqs. (v) and (vi), we get

$a + c + b + d > 2b + 2c$ or $a + d > b + c$

(b) Applying GM > HM

For first three members, $b > \dfrac{2ac}{a+c}$

$\Rightarrow \quad ab + bc > 2ac$ ...(vii)

For last three members, $c > \dfrac{2bd}{b+d}$

$\Rightarrow \quad bc + cd > 2bd$ ...(viii)

From Eqs. (vii) and (viii), we get

$ab + bc + bc + cd > 2ac + 2bd$

or $\quad ab + cd > 2(ac + bd - bc)$

Dividing in each term by $abcd$, we get

$c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(iii) $\because a, b, c, d$ are in HP.

(a) Applying AM > HM

For first three members,

$\dfrac{a+c}{2} > b$

$\Rightarrow \quad a + c > 2b$ ...(ix)

For last three members, $\dfrac{b+d}{2} > c$

$\Rightarrow \quad b + d > 2c$ ...(x)

From Eqs. (ix) and (x), we get

$a + c + b + d > 2b + 2c$

or $\quad a + d > b + c$

(b) Applying GM > HM

For first three members, $\sqrt{ac} > b$

$\Rightarrow \quad ac > b^2$ ...(xi)

For last three members,

$\sqrt{bd} > c$

$\Rightarrow \quad bd > c^2$ ...(xii)

From Eqs. (xi) and (xii), we get

$(ac)(bd) > b^2c^2$

or $\quad ad > bc$

Exercise for Session 5

1. If the AM of two positive numbers a and b $(a > b)$ is twice of their GM, then $a : b$ is

(a) $2+\sqrt{3} : 2-\sqrt{3}$ (b) $7+4\sqrt{3} : 7-4\sqrt{3}$
(c) $2 : 7+4\sqrt{3}$ (d) $2 : \sqrt{3}$

2. If $A_1, A_2; G_1, G_2$ and H_1, H_2 are two arithmetic, geometric and harmonic means, respectively between two quantities a and b, then which of the following is not the value of ab is?

(a) A_1H_2 (b) A_2H_1
(c) G_1G_2 (d) None of these

3. The GM between -9 and -16, is

(a) 12 (b) -12
(c) -13 (d) None of these

4. Let $n \in N, n > 25$. If A, G and H denote the arithmetic mean, geometric mean and harmonic mean of 25 and n. Then, the least value of n for which $A, G, H \in \{25, 26, \ldots, n\}$, is

(a) 49 (b) 81
(c) 169 (d) 225

5. If 9 harmonic means be inserted between 2 and 3, then the value of $A + \frac{6}{H} + 5$ (where A is any of the AM's and H is the corresponding HM), is

(a) 8 (b) 9
(c) 10 (d) None of these

6. If $H_1, H_2, \ldots, H_n$ be n harmonic means between a and b, then $\frac{H_1+a}{H_1-a} + \frac{H_n+b}{H_n-b}$ is

(a) n (b) $n+1$
(c) $2n$ (d) $2n-2$

7. The AM of two given positive numbers is 2. If the larger number is increased by 1, the GM of the numbers becomes equal to the AM to the given numbers. Then, the HM of the given numbers is

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$
(c) $\frac{1}{2}$ (d) 2

8. If $a, a_1, a_2, a_3, \ldots, a_{2n}, b$ are in AP and $a, b_1, b_2, b_3, \ldots, b_{2n}, b$ are in GP and h is the HM of a and b, then $\frac{a_1+a_{2n}}{b_1b_{2n}} + \frac{a_2+a_{2n-1}}{b_2b_{2n-1}} + \ldots + \frac{a_n+a_{n+1}}{b_nb_{n+1}}$ is equal to

(a) $\frac{2n}{h}$ (b) $2nh$
(c) nh (d) $\frac{n}{h}$

Session 6

Arithmetico-Geometric Series (AGS), Sigma (Σ) Notation, Natural Numbers

Arithmetico-Geometric Series (AGS)

Definition

A series formed by multiplying the corresponding terms of an AP and a GP is called **Arithmetico - Geometric Series** (or shortly written as AGS)

For example, $1+4+7+10+\ldots$ is an AP and $1+x+x^2+x^3+\ldots$ is a GP.

Multiplying together the corresponding terms of these series, we get

$1+4x+7x^2+10x^3+\ldots$ which is an Arithmetico-Geometric Series.

Again, $a+(a+d)+(a+2d)+\ldots+[a+(n-1)d]$ is a typical AP

and $1+r+r^2+\ldots+r^{n-1}$ is a typical GP.

Multiplying together the corresponding terms of these series, we get

$$a+(a+d)r+(a+2d)r^2+\ldots+[a+(n-1)d]r^{n-1}$$

which is called a standard Arithmetico-Geometric series.

Sum of n Terms of an Arithmetico-Geometric Series

Let the series be $a+(a+d)r+(a+2d)r^2+\ldots+[a+(n-1)d]r^{n-1}$

Let S_n denotes the sum to n terms, then

$$S_n = a+(a+d)r+(a+2d)r^2+\ldots+[a+(n-2)d]r^{n-2} + [a+(n-1)d]r^{n-1} \quad \ldots(i)$$

Multiplying both sides of Eq. (i) by r, we get

$$rS_n = ar+(a+d)r^2+(a+2d)r^3+\ldots + [a+(n-2)d]r^{n-1}+[a+(n-1)d]r^n \quad \ldots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1-r)S_n = a+(dr+dr^2+\ldots+dr^{n-1})-[a+(n-1)d]r^n$$

$$= a+\frac{dr(1-r^{n-1})}{(1-r)}-[a+(n-1)d]r^n$$

$$\therefore \quad S_n = \frac{a}{(1-r)}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{[a+(n-1)d]r^n}{(1-r)} \quad \ldots(iii)$$

Remark

The above result (iii) is not used as standard formula in any examination. You should follow all steps as shown above.

To Deduce the Sum upto Infinity from the Sum upto n Terms of an Arithmetico - Geometric Series, when $|r|<1$

From Eq. (iii), we have

$$S_n = \frac{a}{1-r}+\frac{dr}{(1-r)^2}-\frac{dr^n}{(1-r)^2}-\frac{[a+(n-1)d]r^n}{(1-r)}$$

If $|r|<1$, when $n \to \infty, r^n \to 0$

and $\frac{dr^n}{(1-r)^2}$ and $\frac{[a+(n-1)d]r^n}{(1-r)}$ both $\to 0$

$$\therefore \quad S_\infty = \frac{a}{(1-r)}+\frac{dr}{(1-r)^2}$$

Independent method Let S_∞ denotes the sum to infinity, then

$$S_\infty = a+(a+d)r+(a+2d)r^2+(a+3d)r^3 + \ldots \text{ upto } \infty \quad \ldots(iv)$$

Multiplying both sides of Eq. (iv) by r, we get

$$rS_\infty = ar+(a+d)r^2+(a+2d)r^3+\ldots \text{ upto } \infty \quad \ldots(v)$$

Subtracting Eq. (v) from Eq. (iv), we get

$$(1-r)S_\infty = a+(dr+dr^2+dr^3+\ldots \text{ upto } \infty)$$

$$= a+\frac{dr}{(1-r)}$$

$$\therefore \quad S_\infty = \frac{a}{(1-r)}+\frac{dr}{(1-r)^2}$$

Example 82. Find the sum of the series
$1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\dots$

(i) to n terms. (ii) to infinity.

Sol. The given series can be written as

$$1+4\left(\frac{1}{5}\right)+7\left(\frac{1}{5}\right)^2+10\left(\frac{1}{5}\right)^3+\dots$$

The series is an Arithmetico-Geometric series, since each term is formed by multiplying corresponding terms of the series 1, 4, 7, ... which are in AP and $1, \frac{1}{5}, \frac{1}{5^2}, \dots$ which are in GP.

$$\therefore \quad T_n = [n\text{ th term of } 1, 4, 7, \dots]\left[n\text{ th term of } 1, \frac{1}{5}, \left(\frac{1}{5}\right)^2, \dots\right]$$

$$= [1+(n-1)3]\times 1\cdot\left(\frac{1}{5}\right)^{n-1} = (3n-2)\left(\frac{1}{5}\right)^{n-1}$$

$$\therefore T_{n-1} = (3n-5)\left(\frac{1}{5}\right)^{n-2}$$

(i) Let sum of n terms of the series is denoted by S_n.

Then, $S_n = 1+4\left(\frac{1}{5}\right)+7\left(\frac{1}{5}\right)^2+\dots$

$$+(3n-5)\left(\frac{1}{5}\right)^{n-2}+(3n-2)\left(\frac{1}{5}\right)^{n-1} \quad \dots(i)$$

Multiplying both the sides of Eq. (i) by $\frac{1}{5}$, we get

$$\therefore \quad \frac{1}{5}S_n = \frac{1}{5}+4\left(\frac{1}{5}\right)^2+7\left(\frac{1}{5}\right)^3+\dots+(3n-5)\left(\frac{1}{5}\right)^{n-1}$$
$$+(3n-2)\left(\frac{1}{5}\right)^n \quad \dots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$\left(1-\frac{1}{5}\right)S_n = 1+3\left[\frac{1}{5}+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\dots+\left(\frac{1}{5}\right)^{n-1}\right]$$
$$-(3n-2)\left(\frac{1}{5}\right)^n$$

or $$\frac{4}{5}S_n = 1+3\left[\left(\frac{1}{5}\right)+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\dots+(n-1)\text{ terms}\right]$$
$$-(3n-2)\left(\frac{1}{5}\right)^n$$

$$= 1+3\left\{\frac{\frac{1}{5}\left[1-\left(\frac{1}{5}\right)^{n-1}\right]}{1-\frac{1}{5}}\right\}-(3n-2)\left(\frac{1}{5}\right)^n$$

$$= 1+\frac{3}{4}\left\{1-\left(\frac{1}{5}\right)^{n-1}\right\}-(3n-2)\left(\frac{1}{5}\right)^n$$

$$\therefore \quad S_n = \frac{5}{4}+\frac{15}{16}\left[1-\left(\frac{1}{5}\right)^{n-1}\right]-\frac{(3n-2)}{4}\left(\frac{1}{5}\right)^{n-1}$$

$$= \frac{35}{16}-\frac{(12n+7)}{16}\left(\frac{1}{5}\right)^{n-1}$$

(ii) $$S_\infty = 1+4\left(\frac{1}{5}\right)+7\left(\frac{1}{5}\right)^2+10\left(\frac{1}{5}\right)^3+\dots\text{ upto }\infty \quad \dots(iii)$$

Multiplying both sides of Eq. (i) by $\frac{1}{5}$, we get

$$\frac{1}{5}S_\infty = \left(\frac{1}{5}\right)+4\left(\frac{1}{5}\right)^2+7\left(\frac{1}{5}\right)^3+\dots\text{ upto }\infty \quad \dots(iv)$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$\left(1-\frac{1}{5}\right)S_\infty = 1+3\left[\left(\frac{1}{5}\right)+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\dots\text{upto }\infty\right]$$

$$= 1+3\left[\frac{\frac{1}{5}}{1-\frac{1}{5}}\right] = 1+\frac{3}{4}$$

$$\Rightarrow \quad \frac{4}{5}S_\infty = \frac{7}{4}$$

$$\therefore \quad S_\infty = \frac{35}{16}$$

Example 83. If the sum to infinity of the series $1+4x+7x^2+10x^3+\dots$ is $\frac{35}{16}$, find x.

Sol. Let $S_\infty = 1+4x+7x^2+10x^3+\dots$ upto ∞ ...(i)

Multiplying both sides of Eq. (i) by x we get

$$x\,S_\infty = x+4x^2+7x^3+10x^4+\dots\text{ upto }\infty \quad \dots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1-x)S_\infty = 1+3x+3x^2+3x^3+\dots\text{ upto }\infty$$

$$= 1+3(x+x^2+x^3+\dots\text{upto }\infty) = 1+3\left(\frac{x}{1-x}\right) = \frac{(1+2x)}{(1-x)}$$

$$\therefore \quad S_\infty = \frac{(1+2x)}{(1-x)^2} = \frac{35}{16} \quad \left[\because S_\infty = \frac{35}{16}\right]$$

$$\Rightarrow \quad 16+32x = 35-70x+35x^2$$

$$\Rightarrow \quad 35x^2-102x+19=0$$

$$\Rightarrow \quad (7x-19)(5x-1)=0$$

$$x \neq \frac{19}{7}$$

[$\because$ for infinity series common ratio $-1<x<1$]

Hence, $x = \frac{1}{5}$

Example 84. Find the sum of the series $1+2^2x+3^2x^2+4^2x^3+\dots$ up to ∞, $|x|<1$.

Sol. Here, the numbers $1^2, 2^2, 3^2, 4^2, \dots$ i.e. 1, 4, 9, 16, ... are not in AP but $1, 4-1=3, 9-4=5, 16-9=7, \dots$ are in AP.

Let $$S_\infty = 1+2^2x+3^2x^2+4^2x^3+\dots\text{ upto }\infty$$
$$= 1+4x+9x^2+16x^3+\dots\text{ upto }\infty \quad \dots(i)$$

Multiplying both sides of Eq. (i) by x, we get

$$xS_\infty = x+4x^2+9x^3+16x^4+\dots\text{ upto }\infty \quad \dots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1-x)S_\infty = 1+3x+5x^2+7x^3+\dots\text{ upto }\infty \quad \dots(iii)$$

Again, multiplying both sides of Eq. (iii) by x, we get

$$x(1-x)S_\infty = x+3x^2+5x^3+7x^4+\dots\text{ upto }\infty \quad \dots(iv)$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$(1-x)(1-x)S_\infty = 1+2x+2x^2+2x^3+\dots\text{ upto }\infty$$
$$= 1+2(x+x^2+x^3+\dots\text{ upto }\infty)$$
$$= 1+2\left(\frac{x}{1-x}\right) = \frac{(1+x)}{(1-x)}$$

$$\therefore \quad S_\infty = \frac{(1+x)}{(1-x)^3}$$

Sigma (Σ) Notation

Σ is a letter of greek alphabets and it is called 'sigma'. The symbol sigma (Σ) represents the sum of similar terms. Usually sum of n terms of any series is represented by placing Σ the nth term of the series. But if we have to find the sum of k terms of a series whose nth term is u_n, this will be represented by $\sum_{n=1}^{k} u_n$.

For example, $\sum_{n=1}^{n=9} n$, i.e. $\sum_{1}^{9} n$ only means the sum of n similar terms when n varies from 1 to 9.

Thus, $\sum_{1}^{9} n = 1+2+3+4+5+6+7+8+9$

Remark

Shortly Σ is written in place of $\sum_{1}^{n}$.

Properties of Sigma Notation

1. $\sum_{r=1}^{n} T_r = T_1+T_2+T_3+\dots+T_n$, when T_n is the general term of the series.

2. $\sum_{r=1}^{n}(T_r \pm T_r') = \sum_{r=1}^{n} T_r \pm \sum_{r=1}^{n} T_r'$

[sigma operator is distributive over addition and subtraction]

3. $\sum_{r=1}^{n} T_r T_r' \neq \left(\sum_{r=1}^{n} T_r\right)\left(\sum_{r=1}^{n} T_r'\right)$

[sigma operator is not distributive over multiplication]

4. $\sum_{r=1}^{n}\left(\frac{T_r}{T_r'}\right) \neq \frac{\left(\sum_{r=1}^{n} T_r\right)}{\left(\sum_{r=1}^{n} T_r'\right)}$

[sigma operator is not distributive over division]

5. $\sum_{r=1}^{n} aT_r = a\sum_{r=1}^{n} T_r$ [where a is constant]

6. $\sum_{j=1}^{n}\sum_{i=1}^{n} T_i T_j = \left(\sum_{i=1}^{n} T_i\right)\left(\sum_{j=1}^{n} T_j\right)$

[where i and j are independent]

Examples on Sigma Notation

(i) $\sum_{i=1}^{m} a = a+a+a+\dots$ upto m times $= am$

(ii) $\sum a = a+a+a+\dots$ upto n times $= an$

i.e. $\sum 5 = 5n$, $\sum 3 = 3n$

(iii) $\sum_{i=1}^{5}(i^2-3i) = \sum_{i=1}^{5} i^2 - 3\sum_{i=1}^{5} i$

$= (1^2+2^2+3^2+4^2+5^2)-3(1+2+3+4+5)$

$= 55-45 = 10$

(iv) $\sum_{r=1}^{3}\left(\frac{r+1}{2r+4}\right) = \left(\frac{1+1}{2\cdot 1+4}\right)+\left(\frac{2+1}{2\cdot 2+4}\right)+\left(\frac{3+1}{2\cdot 3+4}\right)$

$= \frac{2}{6}+\frac{3}{8}+\frac{4}{10} = \frac{40+45+48}{120} = \frac{133}{120} = 1\frac{13}{120}$

Important Theorems on Σ (Sigma) Operator

Theorem 1 $\sum_{r=1}^{n} f(r+1)-f(r) = f(n+1)-f(1)$

Theorem 2

$$\sum_{r=1}^{n} f(r+2)-f(r) = f(n+2)+f(n+1)-f(2)-f(1)$$

Proof (Theorem 1) $\sum_{r=1}^{n} f(r+1) - f(r)$

$$= [f(2) - f(1)] + [f(3) - f(2)] + [f(4) - f(3)] + \ldots + [f(n+1) - f(n)]$$
$$= f(n+1) - f(1)$$

Proof (Theorem 2)

$$\sum_{r=1}^{n} f(r+2) - f(r) = \sum_{r=1}^{n} [f(r+2) - f(r+1)] + [f(r+1) - f(r)]$$
$$= \sum_{r=1}^{n} f(r+2) - f(r+1) + \sum_{r=1}^{n} f(r+1) - f(r)$$
$$= [f(n+2) - f(2)] + [f(n+1) - f(1)] \quad \text{[from Theorem 1]}$$
$$= f(n+2) + f(n+1) - f(2) - f(1)$$

Remark

1. $\sum_{r=1}^{n} f(r+k) - f(r) = \sum_{m=1}^{k} f(n+m) - \sum_{m=1}^{k} f(m), \forall\, k \in N$
2. $\sum_{r=1}^{n} f(2r+1) - f(2r-1) = f(2n+1) - f(1)$
3. $\sum_{r=1}^{n} f(2r) - f(2r-2) = f(2n) - f(0)$

Natural Numbers

The positive integers 1, 2, 3, ... are called natural numbers. These form an AP with first term and common difference, each equal to unity.

(i) Sum of the First n Natural Numbers

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = \Sigma n$$

$$\Rightarrow \quad \Sigma n = \frac{n(n+1)}{2} \quad \text{[Remember]}$$

(ii) Sum of the First n Odd Natural Numbers

$$1 + 3 + 5 + \ldots \text{ upto } n \text{ terms} = \frac{n}{2}[2 \cdot 1 + (n-1) \cdot 2] = n^2$$

$$\Rightarrow \quad \sum (2n-1) = n^2 \quad \text{[Remember]}$$

(iii) Sum of the Squares of the First n Natural Numbers

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof We know that, $r^3 - (r-1)^3 = 3r^2 - 3r + 1$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^3 - (r-1)^3 = 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad n^3 - 0^3 = 3\sum n^2 - 3\sum n + n \quad \text{...(i)}$$

[from important Theorem 1]

Substituting the value of $\sum n$ in Eq. (i), we get

$$\Rightarrow \quad n^3 = 3\sum n^2 - \frac{3 \cdot n(n+1)}{2} + n$$

$$\Rightarrow \quad 3\sum n^2 = n^3 + \frac{3n(n+1)}{2} - n = \frac{n}{2}(2n^2 + 3n + 1)$$
$$= \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \quad \sum n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{[Remember]}$$

Independent Proof We know that,

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\boxed{\sum_{r=1}^{n} (2r+1)^3 - (2r-1)^3} = \sum_{r=1}^{n} (24r^2 + 2)$$

$$\Rightarrow \quad (2n+1)^3 - 1^3 = 24\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} 1$$

[from points to consider-2]

$$\Rightarrow \quad (2n+1)^3 - 1 = 24\sum n^2 + 2n$$
$$\Rightarrow \quad (2n+1)^3 - (2n+1) = 24\sum n^2$$
$$\Rightarrow \quad (2n+1)[(2n+1)^2 - 1] = 24\sum n^2$$
$$\Rightarrow \quad (2n+1)(2n+1+1)(2n+1-1) = 24\sum n^2$$
$$\Rightarrow \quad \Sigma n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{[Remember]}$$

(iv) Sum of the Cubes of the First n Natural Numbers

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \Sigma n^3 = (\Sigma n)^2 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Proof We know that,

$$r^4 - (r-1)^4 = 4r^3 - 6r^2 + 4r - 1$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^4 - (r-1)^4 = 4\sum_{r=1}^{n} r^3 - 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad n^4 - 0^4 = 4\sum n^3 - 6\sum n^2 + 4\sum n - n \quad \text{...(i)}$$

[from important theorem 1]

Substituting the values of $\sum n^2$ and $\sum n$ in Eq. (i), we get

$$\Rightarrow \quad n^4 = 4\sum n^3 - \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} - n$$

$$\Rightarrow 4\sum n^3 = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$
$$= n[n^3 + (n+1)(2n+1) - 2(n+1) + 1]$$
$$= n(n^3 + 2n^2 + n)$$
$$= n^2(n+1)^2$$

$$\therefore \quad \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = (\Sigma n)^2 \quad \text{[Remember]}$$

Independent Proof We know that,

$$r^2(r+1)^2 - r^2(r-1)^2 = 4r^3$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^2(r+1)^2 - r^2(r-1)^2 = 4\sum_{r=1}^{n} r^3$$

$$\Rightarrow \quad n^2(n+1)^2 - 1^2 \cdot 0^2 = 4\Sigma n^3$$

[from important Theorem 1]

$$\Rightarrow \quad \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = (\Sigma n)^2 \quad \text{[Remember]}$$

Corollary $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$

(v) Sum of the Powers Four of the First n Natural Numbers

$$1^4 + 2^4 + 3^4 + \ldots + n^4 = \Sigma n^4$$
$$= \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

Proof We know that,

$$r^5 - (r-1)^5 = 5r^4 - 10r^3 + 10r^2 - 5r + 1$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^5 - (r-1)^5 = 5\sum_{r=1}^{n} r^4 - 10\sum_{r=1}^{n} r^3 + 10\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad n^5 - 0^5 = 5\Sigma n^4 - 10\Sigma n^3 + 10\Sigma n^2 - 5\Sigma n + n \quad \text{...(i)}$$

[from important Theorem 1]

Substituting the values of $\Sigma n, \Sigma n^2, \Sigma n^3$ in Eq. (i), we get

$$\Rightarrow \quad n^5 = 5\Sigma n^4 - \frac{10n^2(n+1)^2}{4} + \frac{10n(n+1)(2n+1)}{6} - \frac{5n(n+1)}{2} + n$$

$$\therefore \quad 5\Sigma n^4 = n\left\{n^4 + \frac{5n(n+1)^2}{2} - \frac{5(n+1)(2n+1)}{3} + \frac{5(n+1)}{2} - 1\right\}$$

$$= \frac{n}{6}\{6n^4 + 15n(n^2 + 2n + 1) - 10(2n^2 + 3n + 1) + 15n + 15 - 6\}$$

$$\Rightarrow \quad \Sigma n^4 = \frac{n}{30}(6n^4 + 15n^3 + 10n^2 - 1)$$
$$= \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

Remark

If nth term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constants.

Then, sum of n terms, $S_n = \Sigma T_n = a\Sigma n^3 + b\,\Sigma n^2 + c\,\Sigma n + d\Sigma 1$

This can be evaluated using the above results.

| Example 85. Find the sum to n terms of the series $1^2 + 3^2 + 5^2 + \ldots$ upto n terms.

Sol. Let T_n be the nth term of this series, then

$$T_n = [1 + (n-1)2]^2 = (2n-1)^2 = 4n^2 - 4n + 1$$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = 4\Sigma n^2 - 4\Sigma n + \Sigma 1$

$$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$
$$= \frac{n}{3}(4n^2 + 6n + 2 - 6n - 6 + 3)$$
$$= \frac{n(4n^2 - 1)}{3}$$

| Example 86. Find the sum to n terms of the series $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \ldots$.

Sol. Let T_n be the nth term of this series, then

T_n = (nth term of 1, 2, 3, ...) (nth term of $2^2, 3^2, 4^2, \ldots$)

$$= n(n+1)^2 = n^3 + 2n^2 + n$$

$\therefore$ Sum of n terms $S_n = \Sigma T_n$

$$2 = \Sigma n^3 + 2\Sigma n^2 + \Sigma n$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right\}$$
$$= \frac{n(n+1)}{12}(3n^2 + 3n + 8n + 4 + 6)$$
$$= \frac{n(n+1)(3n^2+11n+10)}{12} = \frac{n(n+1)(n+2)(3n+5)}{12}$$

| Example 87. Find the sum of n terms of the series whose nth terms is (i) $n(n-1)(n+1)$ (ii) $n^2 + 3^n$.

Sol. (i) We have, $T_n = n(n-1)(n+1) = n^3 - n$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \Sigma n^3 - \Sigma n$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 - \left\{\frac{n(n+1)}{2}\right\}$$
$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} - 1\right\}$$
$$= \frac{n(n+1)(n-1)(n+2)}{4}$$

(ii) We have, $T_n = n^2 + 3^n$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \Sigma n^2 + \Sigma 3^n$

$$= \Sigma n^2 + (3^1 + 3^2 + 3^3 + \ldots + 3^n)$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{3(3^n-1)}{(3-1)}$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{3}{2}(3^n - 1)$$

| Example 88. Find the sum of the series
$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots \text{ upto } n \text{ terms.}$$

Sol. Let T_n be the nth term of the given series. Then,

$$T_n = \frac{(1^3+2^3+3^3+\ldots+n^3)}{(1+3+5+\ldots+(2n-1))} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}(1+2n-1)}$$
$$= \frac{(n+1)^2}{4} = \frac{1}{4}(n^2+2n+1)$$

Let S_n denotes the sum of n terms of the given series. Then,

$$S_n = \Sigma T_n = \frac{1}{4}\Sigma(n^2+2n+1)$$
$$= \frac{1}{4}(\Sigma n^2 + 2\Sigma n + \Sigma 1)$$
$$= \frac{1}{4}\left\{\frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} + n\right\}$$
$$= \frac{n}{24}\{2n^2 + 3n + 1 + 6n + 6 + 6\}$$

Hence, $S_n = \dfrac{n(2n^2+9n+13)}{24}$

| Example 89. Show that
$$\frac{1\cdot 2^2 + 2\cdot 3^2 + \ldots + n\cdot(n+1)^2}{1^2\cdot 2 + 2^2\cdot 3 + \ldots + n^2\cdot(n+1)} = \frac{3n+5}{3n+1}.$$

Sol. Let T_n and T_n' be the nth terms of the series in numerator and denominator of LHS. Then,

$\therefore \quad T_n = n(n+1)^2$ and $T_n' = n^2(n+1)$

$$\therefore \quad \text{LHS} = \frac{\Sigma T_n}{\Sigma T_n'} = \frac{\Sigma n(n+1)^2}{\Sigma n^2(n+1)} = \frac{\Sigma(n^3+2n^2+n)}{\Sigma(n^3+n^2)}$$
$$= \boxed{\frac{\Sigma n^3 + 2\Sigma n^2 + \Sigma n}{\Sigma n^3 + \Sigma n^2}}$$
$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}}{\left\{\frac{n(n+1)}{2}\right\}^2 + \left\{\frac{n(n+1)(2n+1)}{6}\right\}}$$
$$= \frac{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right\}}{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{(2n+1)}{3}\right\}}$$
$$= \frac{\frac{1}{6}(3n^2+3n+8n+4+6)}{\frac{1}{6}(3n^2+3n+4n+2)}$$
$$= \frac{(3n^2+11n+10)}{(3n^2+7n+2)} = \frac{(3n+5)(n+2)}{(3n+1)(n+2)} = \frac{(3n+5)}{(3n+1)} = \text{RHS}$$

| Example 90. Find the sum of the series $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots$ upto n terms.

Sol. Here, $T_n = \{n\text{th term of } 1, 2, 3, \ldots\}$
$\times \{n\text{th term of } 2, 3, 4, \ldots\} \times \{n\text{th term of } 3, 4, 5, \ldots\}$

$\therefore \quad T_n = n(n+1)(n+2) = n^3 + 3n^2 + 2n$

$\therefore \quad S_n$ = Sum of n terms of the series

$$= \Sigma T_n = \Sigma n^3 + 3\Sigma n^2 + 2\Sigma n$$
$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 3\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 2\left\{\frac{n(n+1)}{2}\right\}$$
$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$
$$= \frac{n(n+1)}{4}(n^2 + n + 4n + 2 + 4)$$
$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

| Example 91. Find sum to n terms of the series
$$1 + (2+3) + (4+5+6) + \ldots.$$

Sol. Now, number of terms in first bracket is 1, in the second bracket is 2, in the third bracket is 3, etc. Therefore, the number of terms in the nth bracket will be n.

Let the sum of the given series of n terms $= S$

$\therefore$ Number of terms in $S = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

Also, the first term of S is 1 and common difference is also 1.

$$\therefore \quad S = \frac{\left\{\frac{n(n+1)}{2}\right\}}{2}\left[2\cdot1 + \left(\frac{n(n+1)}{2} - 1\right)\cdot 1\right]$$
$$= \frac{n(n+1)}{8}(4 + n^2 + n - 2)$$
$$= \frac{n(n+1)(n^2+n+2)}{8}$$

| Example 92. Find the sum of the series
$1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + 4\cdot(n-3) + \ldots + (n-1)\cdot 2 + n\cdot 1$
also, find the coefficient of x^{n-1} in the expansion of $(1 + 2x + 3x^2 + \ldots + nx^{n-1})^2$.

Sol. The rth term of the given series is

$T_r = r\cdot(n - r + 1) = (n+1)r - r^2$

$\therefore$ Sum of the series

$$S_n = \sum_{r=1}^{n} T_r = (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 = (n+1)\Sigma n - \Sigma n^2$$
$$= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{n(n+1)}{6}(3n + 3 - 2n - 1) = \frac{n(n+1)(n+2)}{6}$$

Now,
$$(1 + 2x + 3x^2 + \ldots + nx^{n-1})^2 = (1 + 2x + 3x^2 + \ldots + nx^{n-1}) \times (1 + 2x + 3x^2 + \ldots + nx^{n-1})$$

$\therefore$ Coefficient of x^{n-1} in $(1 + 2x + 3x^2 + \ldots + nx^{n-1})^2$

$$= 1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \ldots + n\cdot 1$$
$$= S_n = \frac{n(n+1)(n+2)}{6}$$

Method of Differences

If the differences of the successive terms of a series are in AP or GP, we can find the nth term of the series by the following steps.

Step I Denote the nth term and the sum of the series upto n terms of the series by T_n and S_n, respectively.

Step II Rewrite the given series with each term shifted by one place to the right.

Step III Then, subtract the second expression of S_n from the first expression to obtain T_n.

| Example 93. Find the nth term and sum of n terms of the series, $1 + 5 + 12 + 22 + 35 + \ldots$.

Sol. The sequence of differences between successive terms is 4, 7, 10, 13,... . Clearly, it is an AP with common difference 3. So, let the nth term of the given series be T_n and sum of n terms be S_n.

Then, $S_n = 1 + 5 + 12 + 22 + 35 + \ldots + T_{n-1} + T_n$ ...(i)

$S_n = 1 + 5 + 12 + 22 + \ldots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$0 = 1 + 4 + 7 + 10 + 13 + \ldots + (T_n - T_{n-1}) - T_n$

$\Rightarrow \quad T_n = 1 + 4 + 7 + 10 + 13 + \ldots n$ terms

$$= \frac{n}{2}\{2\cdot1 + (n-1)3\} = \frac{1}{2}(3n^2 - n)$$

Hence, $T_n = \dfrac{3}{2}n^2 - \dfrac{1}{2}n$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \dfrac{3}{2}\Sigma n^2 - \dfrac{1}{2}\Sigma n$

$$= \frac{3}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{1}{2}\left(\frac{n(n+1)}{2}\right)$$
$$= \frac{n(n+1)}{4}(2n + 1 - 1)$$
$$= \frac{1}{2}n^2(n+1) = \frac{1}{2}(n^3 + n^2)$$

| Example 94. Find the nth term and sum of n terms of the series, $1 + 3 + 7 + 15 + 31 + \ldots$.

Sol. The sequence of differences between successive terms is 2, 4, 8, 16, ... Clearly, it is a GP with common ratio 2. So, let the nth term and sum of the series upto n terms of the series be T_n and S_n, respectively. Then,

$S_n = 1 + 3 + 7 + 15 + 31 + \ldots + T_{n-1} + T_n$ ...(i)

$S_n = 1 + 3 + 7 + 15 + \ldots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$0 = 1 + 2 + 4 + 8 + 16 + \ldots + (T_n - T_{n-1}) - T_n$

$\Rightarrow \quad T_n = 1 + 2 + 4 + 8 + 16 + \ldots$ upto n terms

$$= \frac{1\cdot(2^n - 1)}{2 - 1}$$

Hence, $\quad T_n = (2^n - 1)$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \Sigma(2^n - 1) = \Sigma 2^n - \Sigma 1$

$$= (2 + 2^2 + 2^3 + \ldots + 2^n) - n$$
$$= \frac{2\cdot(2^n - 1)}{(2-1)} - n = 2^{n+1} - 2 - n$$

must

| Example 95. Find the nth term of the series $1 + 4 + 10 + 20 + 35 + \ldots$

Sol. The sequence of first consecutive differences is 3, 6, 10, 15, ... and second consecutive differences is 3, 4, 5, ... Clearly, it is an AP with common difference 1. So, let the nth term and sum of the series upto n terms of the series be T_n and S_n, respectively.

Then,

$S_n = 1 + 4 + 10 + 20 + 35 + \ldots + T_{n-1} + T_n$ ...(i)

$S_n = \quad 1 + 4 + 10 + 20 + \ldots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 3 + 6 + 10 + 15 + \ldots + (T_n - T_{n-1}) - T_n$$

$\Rightarrow \quad T_n = 1 + 3 + 6 + 10 + 15 + \ldots$ upto n terms

or $\quad T_n = 1 + 3 + 6 + 10 + 15 + \ldots + t_{n-1} + t_n$ ...(iii)

$T_n = 1 + 3 + 6 + 10 + \ldots + t_{n-1} + t_n$ ...(iv)

Now, subtracting Eq. (iv) from Eq. (iii), we get

$$0 = 1 + 2 + 3 + 4 + 5 + \ldots + (t_n - t_{n-1}) - t_n$$

or $\quad t_n = 1 + 2 + 3 + 4 + 5 + \ldots$ upto n terms

$$= \Sigma n = \frac{n(n+1)}{2}$$

$$\therefore \quad T_n = \Sigma t_n = \frac{1}{2}(\Sigma n^2 + \Sigma n)$$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{n(n+1)}{6}(2n+1+3) = \frac{1}{6}n(n+1)(n+2)$$

I Example 96. Find the nth term of the series $1 + 5 + 18 + 58 + 179 + \ldots$

Sol. The sequence of first consecutive differences is 4, 13, 40, 121, ... and second consecutive differences is 9, 27, 81, ... Clearly, it is a GP with common ratio 3. So, let the nth term and sum of the series upto n terms of the series be T_n and S_n, respectively. Then,

$S_n = 1 + 5 + 18 + 58 + 179 + \ldots + T_{n-1} + T_n$ ...(i)

$S_n = 1 + 5 + 18 + 58 + \ldots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 4 + 13 + 40 + 121 + \ldots + (T_n - T_{n-1}) - T_n$$

$\Rightarrow \quad T_n = 1 + 4 + 13 + 40 + 121 + \ldots$ upto n terms

or $\quad T_n = 1 + 4 + 13 + 40 + 121 + \ldots + t_{n-1} + t_n$ ...(iii)

$T_n = \quad 1 + 4 + 13 + 40 + \ldots + t_{n-1} + t_n$ ...(iv)

Now, subtracting Eq. (iv) from Eq. (iii), we get

$$0 = 1 + 3 + 9 + 27 + 81 + \ldots + (t_n - t_{n-1}) - t_n$$

or $\quad t_n = 1 + 3 + 9 + 27 + 81 + \ldots$ upto n terms

$$= \frac{1 \cdot (3^n - 1)}{(3-1)} = \frac{1}{2}(3^n - 1)$$

$$\therefore \quad T_n = \Sigma t_n = \frac{1}{2}(\Sigma 3^n - \Sigma 1)$$

$$= \frac{1}{2}\{(3 + 3^2 + 3^3 + \ldots + 3^n) - n\}$$

$$= \frac{1}{2}\left\{\frac{3(3^n - 1)}{(3-1)} - n\right\}$$

$$= \frac{3}{4}(3^n - 1) - \frac{1}{2}n$$

Use this when exact series are not making

Method of Differences (Shortcut) to find nth term of a Series

The nth term of the series can be written directly on the basis of successively differences, we use the following steps to find the nth term T_n of the given sequence.

Step I If the first consecutive differences of the given sequence are in AP, then take

$T_n = a(n-1)(n-2) + b(n-1) + c$, where a, b, c are constants. Determine a, b, c by putting $n = 1, 2, 3$ and putting the values of T_1, T_2, T_3.

Step II If the first consecutive differences of the given sequence are in GP, then take

$T_n = ar^{n-1} + bn + c$, where a, b, c are constants and r is the common ratio of GP. Determine a, b, c by putting $n = 1, 2, 3$ and putting the values of T_1, T_2, T_3.

Step III If the differences of the differences computed in Step I are in AP, then take

$T_n = a(n-1)(n-2)(n-3) + b(n-1)(n-2) + c(n-1) + d$, where a, b, c, d are constants. Determine by putting $n = 1, 2, 3, 4$ and putting the values of T_1, T_2, T_3, T_4.

Step IV If the differences of the differences computed in Step I are in GP with common ratio r, then take

$T_n = ar^{n-1} + bn^2 + cn + d$, where a, b, c, d are constants. Determine by putting $n = 1, 2, 3, 4$ and putting the values of T_1, T_2, T_3, T_4.

I Example 97. Find the nth term and sum of n terms of the series $2 + 4 + 7 + 11 + 16 + \ldots$.

Sol. The sequence of first consecutive differences is 2, 3, 4, 5, ... Clearly, it is an AP.

Then, nth term of the given series be

$T_n = a(n-1)(n-2) + b(n-1) + c$ ...(i)

Putting $n = 1, 2, 3$, we get

$$2 = c \Rightarrow 4 = b + c \Rightarrow 7 = 2a + 2b + c$$

After solving, we get $a = \frac{1}{2}, b = 2, c = 2$

Putting the values of a, b, c in Eq. (i), we get

$$T_n = \frac{1}{2}(n-1)(n-2) + 2(n-1) + 2 = \frac{1}{2}(n^2 + n + 2)$$

Hence, sum of series $S_n = \Sigma T_n = \frac{1}{2}(\Sigma n^2 + \Sigma n + 2\Sigma 1)$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + 2n\right)$$

$$= \frac{1}{6}n(n^2 + 3n + 8)$$

I Example 98. Find the nth term and sum of n terms of the series $5 + 7 + 13 + 31 + 85 + \ldots$.

Sol. The sequence of first consecutive differences is 2, 6, 18, 54, ... Clearly, it is a GP with common ratio 3. Then, nth term of the given series be

$T_n = a(3)^{n-1} + bn + c$ ...(i)

Putting $n = 1, 2, 3$, we get

$5 = a + b + c$ ...(ii)

$7 = 3a + 2b + c$ ...(iii)

$13 = 9a + 3b + c$ ...(iv)

Solving these equations, we get

$$a = 1, b = 0, c = 4$$

Putting the values of a, b, c in Eq. (i), we get

$$T_n = 3^{n-1} + 4$$

Hence, sum of the series

$$S_n = \Sigma T_n = \Sigma(3^{n-1} + 4) = \Sigma(3^{n-1}) + 4\Sigma 1$$

$$= (1 + 3 + 3^2 + \ldots + 3^{n-1}) + 4n$$

$$= 1 \cdot \frac{(3^n - 1)}{(3-1)} + 4n = \frac{1}{2}(3^n + 8n - 1)$$

I Example 99. Find the nth term of the series $1 + 2 + 5 + 12 + 25 + 46 + \ldots$.

Sol. The sequence of first consecutive differences is 1, 3, 7, 13, 21, ... The sequence of the second consecutive differences is 2, 4, 6, 8, ... Clearly, it is an AP. Then, nth term of the given series be

$T_n = a(n-1)(n-2)(n-3) + b(n-1)(n-2) + c(n-1) + d$ ...(i)

Putting $n = 1, 2, 3, 4$, we get

$1 = d$ ...(ii)

$2 = c + d$ ...(iii)

$5 = 2b + 2c + d$ ...(iv)

$12 = 6a + 6b + 3c + d$ ...(v)

After, solving these equations, we get

$$a = \frac{1}{3}, b = 1, c = 1, d = 1$$

Putting the values of a, b, c, d in Eq. (i), we get

$$T_n = \frac{1}{3}(n^3 - 6n^2 + 11n - 6) + (n^2 - 3n + 2) + (n-1) + 1$$

$$= \frac{1}{3}(n^3 - 3n^2 + 5n) = \frac{n}{3}(n^2 - 3n + 5)$$

I Example 100. Find the nth term of the series $2 + 5 + 12 + 31 + 86 + \ldots$.

Sol. The sequence of first consecutive differences is 3, 7, 19, 55, ... The sequence of the second consecutive differences is 4, 12, 36, ... Clearly, it is a GP with common ratio 3. Then, nth term of the given series be

$T_n = a(3)^{n-1} + bn^2 + cn + d$ ...(i)

Putting $n = 1, 2, 3, 4$, we get

$2 = a + b + c + d$ ...(ii)

$5 = 3a + 4b + 2c + d$ ...(iii)

$12 = 9a + 9b + 3c + d$ ...(iv)

$31 = 27a + 16b + 4c + d$ ...(v)

After, solving these equations, we get

$$a = 1, b = 0, c = 1, d = 0$$

Putting the values of a, b, c, d in Eq. (i), we get

$$T_n = 3^{n-1} + n$$

Method of Differences (Maha Shortcut)

To find $t_1 + t_2 + t_3 + \ldots + t_{n-1} + t_n$

Let $S_n = t_1 + t_2 + t_3 + \ldots + t_{n-1} + t_n$

Then, $\quad \Delta t_1, \Delta t_2, \Delta t_3, \ldots, \Delta t_{n-1}$ [1st order differences]

$\Delta^2 t_1, \Delta^2 t_2, \Delta^2 t_3, \ldots, \Delta^2 t_{n-1}$ [2nd order differences]

$\vdots \quad \vdots \quad \vdots$

$$\therefore \quad t_n = {}^{n-1}C_0\, t_1 + {}^{n-1}C_1\, \Delta t_1 + {}^{n-1}C_2\, \Delta^2 t_1 + \ldots + {}^{n-1}C_{r-1}\, \Delta^{r-1} t_1$$

and $\quad S_n = {}^nC_1\, t_1 + {}^nC_2\, \Delta t_1 + {}^nC_3\, \Delta^2 t_1 + \ldots + {}^nC_r\, \Delta^r t_1$

where, $\quad \Delta t_1 = t_2 - t_1, \Delta t_2 = t_3 - t_2$, etc.

$\Delta^2 t_1 = \Delta t_2 - \Delta t_1, \Delta^3 t_1 = \Delta^2 t_2 - \Delta^2 t_1$, etc.

I Example 101. Find the nth term and sum to n terms of the series $12 + 40 + 90 + 168 + 280 + 432 + \ldots$.

Sol. Let $S_n = 12 + 40 + 90 + 168 + 280 + 432 + \ldots$, then

1st order differences are 28, 50, 78, 112, 152, ... (i.e. $\Delta t_1, \Delta t_2, \Delta t_3, \ldots$)

and 2nd order differences are 22, 28, 34, 40, ... (i.e. $\Delta^2 t_1, \Delta^2 t_2, \Delta^2 t_3, \ldots$)

and 3rd order differences are 6, 6, 6, 6,... (i.e. $\Delta^3 t_1, \Delta^3 t_2, \Delta^3 t_3, \ldots$)

and 4th order differences are 0, 0, 0, 0, ... (i.e. $\Delta^4 t_1, \Delta^4 t_2, \Delta^4 t_3, \ldots$)

$$\therefore \quad t_n = 12 \cdot {}^{n-1}C_0 + 28 \cdot {}^{n-1}C_1 + 22 \cdot {}^{n-1}C_2 + 6 \cdot {}^{n-1}C_3$$

$$= 12 + 28(n-1) + \frac{22(n-1)(n-2)}{2} + \frac{6(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3}$$

$$= n^3 + 5n^2 + 6n$$

and $S_n = 12 \cdot {}^nC_1 + 28 \cdot {}^nC_2 + 22 \cdot {}^nC_3 + 6 \cdot {}^nC_4$

$$= 12n + \frac{28n(n-1)}{2} + \frac{22n(n-1)(n-2)}{1\cdot2\cdot3} + \frac{6\cdot n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}$$

$$= \frac{n}{12}(n+1)(3n^2+23n+46)$$

V_n Method

To find the sum of the series of the forms

I. $a_1a_2\dots a_r + a_2a_3\dots a_{r+1} + \dots + a_na_{n+1}\dots a_{n+r-1}$

II. $\dfrac{1}{a_1a_2\dots a_r} + \dfrac{1}{a_2a_3\dots a_{r+1}} + \dots + \dfrac{1}{a_na_{n+1}\dots a_{n+r-1}}$

where, $a_1, a_2, a_3, \dots, a_n, \dots$ are in AP.

***Solution of form* I** Let S_n be the sum and T_n be the nth term of the series, then

$$S_n = a_1a_2\dots a_r + a_2a_3\dots a_{r+1} + \dots + a_na_{n+1} + \dots + a_{n+r-1}$$

$\therefore$ $T_n = a_na_{n+1}a_{n+2}\dots a_{n+r-2}\,a_{n+r-1}$ ...(i)

Let $V_n = a_na_{n+1}a_{n+2}\dots a_{n+r-2}\,a_{n+r-1}\,a_{n+r}$

[taking one extra factor in T_n for V_n]

$\therefore$ $V_{n-1} = a_{n-1}a_n\,a_{n+1}\dots a_{n+r-3}\,a_{n+r-2}\,a_{n+r-1}$

$\Rightarrow$ $V_n - V_{n-1} = a_na_{n+1}a_{n+2}\dots a_{n+r-1}(a_{n+r} - a_{n-1})$

$= T_n(a_{n+r} - a_{n-1})$ [from Eq. (i)] ...(ii)

Let d be the common difference of AP, then

$$a_n = a_1 + (n-1)d$$

Then, from Eq. (ii)

$$V_n - V_{n-1} = T_n[\{a_1 + (n+r-1)d\} - \{a_1 + (n-2)d\}] = (r+1)d\,T_n$$

$\Rightarrow$ $T_n = \dfrac{1}{(r+1)d}(V_n - V_{n-1})$

$\therefore$ $S_n = \Sigma T_n = \displaystyle\sum_{n=1}^{n} T_n = \frac{1}{(r+1)d}\sum_{n=1}^{n}(V_n - V_{n-1})$

$= \dfrac{1}{(r+1)d}(V_n - V_0)$

[from important Theorem 1 of Σ]

$= \dfrac{1}{(r+1)(a_2-a_1)}(a_na_{n+1}\dots a_{n+r} - a_0a_1\dots a_r)$

Corollary I If $a_1, a_2, a_3, \dots, a_n, \dots$ are in AP, then

(i) For $r = 2$, $a_1a_2 + a_2a_3 + \dots + a_na_{n+1} = \dfrac{1}{3(a_2-a_1)}(a_na_{n+1}a_{n+2} - a_0a_1a_2)$

(ii) For $r = 3$,

$a_1a_2a_3 + a_2a_3a_4 + \dots + a_na_{n+1}a_{n+2} = \dfrac{1}{4(a_2-a_1)}(a_na_{n+1}a_{n+2}a_{n+3} - a_0a_1a_2a_3)$

Corollary II

(i) $1\cdot2 + 2\cdot3 + \dots + n(n+1) = \dfrac{1}{3}\{n(n+1)(n+2) - 0\cdot1\cdot2\} = \dfrac{n(n+1)(n+2)}{3}$

(ii) $1\cdot3\cdot5\cdot7 + 3\cdot5\cdot7\cdot9 + \dots + (2n-1)\cdot(2n+1)(2n+3)(2n+5)$

$= \dfrac{1}{10}\{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7) - (-1)\cdot1\cdot3\cdot5\cdot7\}$

$= \dfrac{1}{10}\{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7) + 105\}$

***Solution of form* II** Let d be the common difference of AP, then $a_n = a_1 + (n-1)d$

Let sum of the series and nth term are denoted by S_n and T_n, respectively. Then,

$$S_n = \frac{1}{a_1a_2\dots a_r} + \frac{1}{a_2a_3\dots a_{r+1}} + \dots + \frac{1}{a_na_{n+1}\dots a_{n+r-1}}$$

$\therefore$ $T_n = \dfrac{1}{a_na_{n+1}a_{n+2}\dots a_{n+r-2}\,a_{n+r-1}}$ ...(i)

Let $V_n = \dfrac{1}{a_{n+1}\,a_{n+2}\dots a_{n+r-2}\,a_{n+r-1}}$ ...(ii)

[leaving first factor from denominator of T_n]

So, $V_{n-1} = \dfrac{1}{a_na_{n+1}\dots a_{n+r-3}\,a_{n+r-2}}$

$\Rightarrow$ $V_n - V_{n-1} = \dfrac{1}{a_{n+1}\,a_{n+2}\dots a_{n+r-2}\,a_{n+r-1}} - \dfrac{1}{a_na_{n+1}\dots a_{n+r-3}\,a_{n+r-2}}$

$= \dfrac{a_n - a_{n+r-1}}{a_na_{n+1}\,a_{n+2}\dots a_{n+r-2}\,a_{n+r-1}}$

$= T_n(a_n - a_{n+r-1})$ [from Eq. (i)]

$= T_n[\{a_1 + (n-1)d\} - \{a_1 + (n+r-2)d\}]$

$= d(1-r)T_n$

$\therefore$ $T_n = \dfrac{(V_n - V_{n-1})}{d(1-r)}$

$\therefore$ $S_n = \Sigma T_n = \displaystyle\sum_{n=1}^{n}\frac{(V_n - V_{n-1})}{d(1-r)} = \frac{1}{d(1-r)}(V_n - V_0)$

[from important Theorem 1 of Σ]

$= \dfrac{1}{(a_2-a_1)(1-r)}\left\{\dfrac{1}{a_{n+1}\,a_{n+2}\dots a_{n+r-2}\,a_{n+r-1}} - \dfrac{1}{a_1\,a_2\dots a_{r-2}\,a_{r-1}}\right\}$

Hence, the sum of n terms is $S_n = \dfrac{1}{(r-1)(a_2-a_1)}\left\{\dfrac{1}{a_1a_2\dots a_{r-1}} - \dfrac{1}{a_{n+1}\,a_{n+2}\dots a_{n+r-1}}\right\}$

Corollary I If $a_1, a_2, a_3, \dots, a_n, \dots$ are in AP, then

(i) For $r = 2$,

$\dfrac{1}{a_1a_2} + \dfrac{1}{a_2a_3} + \dots + \dfrac{1}{a_na_{n+1}} = \dfrac{1}{(a_2-a_1)}\left\{\dfrac{1}{a_1} - \dfrac{1}{a_{n+1}}\right\} = \dfrac{1}{d}\left(\dfrac{a_{n+1}-a_1}{a_1\,a_{n+1}}\right)$

$= \dfrac{1}{d}\left(\dfrac{a_1 + nd - a_1}{a_1a_{n+1}}\right) = \dfrac{n}{a_1\,a_{n+1}}$

(ii) For $r = 3$, $\dfrac{1}{a_1a_2a_3} + \dfrac{1}{a_2a_3a_4} + \dots + \dfrac{1}{a_n\,a_{n+1}\,a_{n+2}}$

$= \dfrac{1}{2(a_2-a_1)}\left\{\dfrac{1}{a_1a_2} - \dfrac{1}{a_{n+1}\,a_{n+2}}\right\}$

(iii) For $r = 4$,

$\dfrac{1}{a_1a_2a_3a_4} + \dfrac{1}{a_2a_3a_4a_5} + \dots + \dfrac{1}{a_na_{n+1}a_{n+2}a_{n+3}}$

$= \dfrac{1}{3(a_2-a_1)}\left\{\dfrac{1}{a_1a_2a_3} - \dfrac{1}{a_{n+1}\,a_{n+2}\,a_{n+3}}\right\}$

Corollary II

(i) $\dfrac{1}{1\cdot2} + \dfrac{1}{2\cdot3} + \dfrac{1}{3\cdot4} + \dots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

(ii) $\dfrac{1}{1\cdot2\cdot3} + \dfrac{1}{2\cdot3\cdot4} + \dfrac{1}{3\cdot4\cdot5} + \dots + \dfrac{1}{n(n+1)(n+2)}$

$= \dfrac{1}{2}\left\{\dfrac{1}{1\cdot2} - \dfrac{1}{(n+1)(n+2)}\right\} = \dfrac{1}{4} - \dfrac{1}{2(n+1)(n+2)}$

(iii) $\dfrac{1}{1\cdot3\cdot5\cdot7} + \dfrac{1}{3\cdot5\cdot7\cdot9} + \dots + \dfrac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$

$= \dfrac{1}{6}\left\{\dfrac{1}{1\cdot3\cdot5} - \dfrac{1}{(2n+1)(2n+3)(2n+5)}\right\}$

$= \dfrac{1}{90} - \dfrac{1}{6(2n+1)(2n+3)(2n+5)}$

| Example 102. Find the sum upto n terms of the series $1\cdot4\cdot7\cdot10 + 4\cdot7\cdot10\cdot13 + 7\cdot10\cdot13\cdot16 + \dots$

Sol. Let T_n be the nth term of the given series.

$\therefore$ T_n = (nth term of 1, 4, 7, ...)(nth term of 4, 7, 10, ...)(nth term of 7, 10, 13, ...) (nth term of 10, 13, 16, ...)

$T_n = (3n-2)(3n+1)(3n+4)(3n+7)$ ...(i)

$\therefore V_n = (3n-2)(3n+1)(3n+4)(3n+7)(3n+10)$

$V_{n-1} = (3n-5)(3n-2)(3n+1)(3n+4)(3n+7)$

$\Rightarrow$ $V_n = (3n+10)\,T_n$ [from Eq. (i)]

and $V_{n-1} = (3n-5)\,T_n$

$\therefore$ $V_n - V_{n-1} = 15\,T_n$

$\therefore$ $T_n = \dfrac{1}{15}(V_n - V_{n-1})$

$\therefore$ $S_n = \Sigma T_n = \displaystyle\sum_{n=1}^{n}\frac{1}{15}(V_n - V_{n-1})$

$= \dfrac{1}{15}(V_n - V_0)$

[from important Theorem 1 of Σ]

$= \dfrac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) - (-2)(1)(4)(7)(10)\}$

$= \dfrac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) + 560\}$

Shortcut Method

$S_n = \dfrac{1}{(\text{last factor of III term} - \text{first factor of I term})}$

(Taking one extra factor in T_n in last
– Taking one extra factor in I term in start)

$= \dfrac{1}{(16-1)}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) - (-2)\cdot1\cdot4\cdot7\cdot10\}$

$= \dfrac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) + 560\}$

| Example 103. Find the sum to n terms of the series $\dfrac{1}{1\cdot3\cdot5\cdot7\cdot9} + \dfrac{1}{3\cdot5\cdot7\cdot9\cdot11} + \dfrac{1}{5\cdot7\cdot9\cdot11\cdot13} + \dots$. Also, find the sum to infinity terms.

Sol. Let T_n be the nth term of the given series.

Then, $T_n = \dfrac{1}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}$ ...(i)

$\therefore$ $V_n = \dfrac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}$

[leaving first factor from denominator of T_n]

$V_{n-1} = \dfrac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$

$\Rightarrow$ $V_n - V_{n-1} = \dfrac{1}{(2n+1)(2n+3)(2n+5)(2n+7)} - \dfrac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$

$= \dfrac{(2n-1)-(2n+7)}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}$

$= -8\,T_n$ [from Eq. (i)]

$\therefore \quad T_n = -\frac{1}{8}(V_n - V_{n-1})$

$\therefore \quad S_n = \Sigma T_n = \sum_{n=1}^{n} T_n = -\frac{1}{8}\sum_{n=1}^{n}(V_n - V_{n-1}) = -\frac{1}{8}(V_n - V_0)$

[from Important Theorem 1 of Σ]

$= \frac{1}{8}(V_0 - V_n)$

$= \frac{1}{8}\left\{\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$

$= \frac{1}{840} - \frac{1}{8(2n+1)(2n+3)(2n+5)(2n+7)}$

and $S_\infty = \frac{1}{840} - \frac{1}{\infty} = \frac{1}{840} - 0 = \frac{1}{840}$

Shortcut Method

$$\frac{1}{1\cdot3\cdot5\cdot7\cdot9} + \frac{1}{3\cdot5\cdot7\cdot9\cdot11} + \frac{1}{5\cdot7\cdot9\cdot11\cdot13} + \dots + \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)} \quad \dots(i)$$

Now, in each term in denominator

$9 - 1 = 11 - 3 = 13 - 5 = \dots = (2n+7) - (2n-1) = 8$

Then, Eq. (i) can be written as

$$= \frac{1}{8}\left\{\frac{9-1}{1\cdot3\cdot5\cdot7\cdot9} + \frac{11-3}{3\cdot5\cdot7\cdot9\cdot11} + \frac{13-5}{5\cdot7\cdot9\cdot11\cdot13} + \dots + \frac{(2n+7)-(2n-1)}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{3\cdot5\cdot7\cdot9} + \frac{1}{3\cdot5\cdot7\cdot9} - \frac{1}{5\cdot7\cdot9\cdot11} + \frac{1}{5\cdot7\cdot9\cdot11} - \frac{1}{7\cdot9\cdot11\cdot13} + \dots + \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

[middle terms are cancelled out]

$= \frac{1}{840} - \frac{1}{8(2n+1)(2n+3)(2n+5)(2n+7)} = S_n$ [say]

$\therefore$ Sum to infinity terms $= S_\infty = \frac{1}{840} - 0 = \frac{1}{840}$

Maha Shortcut Method

Taking $\frac{1}{8}$ outside the bracket

$\left(\text{i.e. } \frac{1}{9-1} = \frac{1}{11-3} = \frac{1}{13-5} = \dots\right)$ and in bracket leaving last factor of denominator of first term – leaving first factor of denominator of last term

i.e., $S_n = \frac{1}{8}\left(\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right)$

$\therefore \quad S_\infty = \frac{1}{8}\left(\frac{1}{1\cdot3\cdot5\cdot7} - 0\right) = \frac{1}{840}$

| Example 104. If $\sum_{r=1}^{n} T_r = \frac{n(n+1)(n+2)(n+3)}{12}$, where T_r denotes the rth term of the series. Find $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{T_r}$.

Sol. We have, $T_n = \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r$

$= \frac{n(n+1)(n+2)(n+3)}{12} - \frac{(n-1)n(n+1)(n+2)}{12}$

$= \frac{n(n+1)(n+2)}{12}[(n+3)-(n-1)]$

$= \frac{n(n+1)(n+2)}{3} \qquad \frac{1}{T_n} = \frac{3}{n(n+1)(n+2)}$

$\therefore \quad \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{T_r} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{3}{r(r+1)(r+2)}$

$= 3 \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$

$= 3 \lim_{n\to\infty}\left(\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \dots + \frac{1}{n(n+1)(n+2)}\right)$

Maha Shortcut Method

$= 3 \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{1\cdot2} - \frac{1}{(n+1)(n+2)}\right)$

$= \frac{3}{2}\left(\frac{1}{2} - 0\right) = \frac{3}{4}$

Exercise for Session 6

1. The sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \dots$ is
(a) $2^n - n - 1$ (b) $1 - 2^{-n}$ (c) $n + 2^{-n} - 1$ (d) $2^n - 1$

2. $2^{1/4} \cdot 4^{1/8} \cdot 8^{1/16} \cdot 16^{1/32} \dots$ is equal to
(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) $\frac{5}{2}$

3. $1 + 3 + 7 + 15 + 31 + \dots$ upto n terms equals
(a) $2^{n+1} - n$ (b) $2^{n+1} - n - 2$ (c) $2^n - n - 2$ (d) None of these

4. 99th term of the series $2 + 7 + 14 + 23 + 34 + \dots$ is
(a) 9998 (b) 9999 (c) 10000 (d) 100000

5. The sum of the series $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \dots$ upto n terms is
(a) $n(n+1)(n+2)$ (b) $(n+1)(n+2)(n+3)$
(c) $\frac{1}{4}n(n+1)(n+2)(n+3)$ (d) $\frac{1}{4}(n+1)(n+2)(n+3)$

6. $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots + \frac{1}{n(n+1)}$ equals
(a) $\frac{1}{n(n+1)}$ (b) $\frac{n}{n+1}$
(c) $\frac{2n}{n+1}$ (d) $\frac{2}{n(n+1)}$

7. Sum of the n terms of the series $\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^3} + \dots$ is
(a) $\frac{2n}{n+1}$ (b) $\frac{4n}{n+1}$
(c) $\frac{6n}{n+1}$ (d) $\frac{9n}{n+1}$

8. If $t_n = \frac{1}{4}(n+2)(n+3)$ for $n = 1, 2, 3, \dots$, then $\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \dots + \frac{1}{t_{2003}}$ equals
(a) $\frac{4006}{3006}$ (b) $\frac{4003}{3007}$
(c) $\frac{4006}{3008}$ (d) $\frac{4006}{3009}$

9. The value of $\frac{1}{(1+a)(2+a)} + \frac{1}{(2+a)(3+a)} + \frac{1}{(3+a)(4+a)} + \dots$ upto ∞ is
(where, a is constant)
(a) $\frac{1}{1+a}$ (b) $\frac{2}{1+a}$
(c) ∞ (d) None of these

10. If $f(x)$ is a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$. Then, the value of n is
(a) 4 (b) 5 (c) 6 (d) None of these

Session 7

Application to Problems of Maxima and Minima (Without Calculus)

Application to Problems of Maxima and Minima (Without Calculus)

Suppose that $a_1, a_2, a_3, \ldots, a_n$ are n positive variables and k is constant, then

(i) If $a_1 + a_2 + a_3 + \ldots + a_n = k$ (constant), the value of $a_1 a_2 a_3 \ldots a_n$ is greatest when $a_1 = a_2 = a_3 = \ldots = a_n$, so that the greatest value of $a_1 a_2 a_3 \ldots a_n$ is $\boxed{\left(\frac{k}{n}\right)^n}$.

Proof $\because$ AM $\geq$ GM

$$\therefore \quad \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{1/n}$$

$$\Rightarrow \quad \frac{k}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{1/n}$$

$$\text{or} \quad (a_1 a_2 a_3 \ldots a_n) \leq \left(\frac{k}{n}\right)^n$$

Here, $a_1 = a_2 = a_3 = \ldots = a_n$

$\therefore$ Greatest value of $a_1 a_2 a_3 \ldots a_n$ is $\left(\frac{k}{n}\right)^n$.

| Example 105. Find the greatest value of xyz for positive values of x, y, z subject to the condition $yz + zx + xy = 12$.

Sol. Given, $yz + zx + xy = 12$ (constant), the value of $(yz)(zx)(xy)$ is greatest when $yz = zx = xy$

Here, $n = 3$ and $k = 12$

Hence, greatest value of $(yz)(zx)(xy)$ is $\left(\frac{12}{3}\right)^3$ i.e. 64.

$\therefore$ Greatest value of $x^2y^2z^2$ is 64.

Thus, greatest value of xyz is 8.

Aliter

Given $yz + zx + xy = 12$, the greatest value of $(yz)(zx)(xy)$ is greatest when

$$yz = zx = xy = c \qquad \text{[say]}$$

Since, $\quad yz + zx + xy = 12$

$\therefore \quad c + c + c = 12$

$\Rightarrow \quad 3c = 12$ or $c = 4$

$\therefore \quad yz = zx = xy = 4$

Hence, greatest value of $(yz)(zx)(xy)$ is $4 \cdot 4 \cdot 4$

i.e., greatest value of $x^2y^2z^2$ is 64.

Hence, greatest value of xyz is 8.

| Example 106. Find the greatest value of x^3y^4, if $2x + 3y = 7$ and $x \geq 0, y \geq 0$.

Sol. To find the greatest value of x^3y^4 or $(x)(x)(x)(y)(y)(y)(y)$

Here, x repeats 3 times and y repeats 4 times.

Given, $\quad 2x + 3y = 7$,

then multiplying and dividing coefficients of x and y by 3 and 4, respectively.

$$\text{Rewrite} \quad 3\left(\frac{2x}{3}\right) + 4\left(\frac{3y}{4}\right) = 7$$

$$\text{or} \quad \underset{1}{\left(\frac{2x}{3}\right)} + \underset{2}{\left(\frac{2x}{3}\right)} + \underset{3}{\left(\frac{2x}{3}\right)} + \underset{4}{\left(\frac{3y}{4}\right)} + \underset{5}{\left(\frac{3y}{4}\right)} + \underset{6}{\left(\frac{3y}{4}\right)} + \underset{7}{\left(\frac{3y}{4}\right)} = 7$$

Here, $k = 7$ and $n = 7$

Hence, greatest value of

$$\left(\frac{2x}{3}\right)\left(\frac{2x}{3}\right)\left(\frac{2x}{3}\right)\left(\frac{3y}{4}\right)\left(\frac{3y}{4}\right)\left(\frac{3y}{4}\right)\left(\frac{3y}{4}\right) \text{ is } \left(\frac{7}{7}\right)^7.$$

or greatest value of $\frac{2^3 \cdot 3^4}{3^3 \cdot 4^4} x^3y^4$ is 1.

Thus, greatest value of x^3y^4 is $\frac{32}{3}$.

(ii) If $a_1 a_2 a_3 \ldots a_n = k$ (constant), the value of $a_1 + a_2 + a_3 + \ldots + a_n$ is least when $a_1 = a_2 = a_3 = \ldots = a_n$, so that the least of $a_1 + a_2 + a_3 + \ldots + a_n$ is $n(k)^{1/n}$.

Proof $\because$ AM $\geq$ GM

$$\therefore \quad \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{1/n} = (k)^{1/n}$$

$$\Rightarrow \quad \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (k)^{1/n}$$

$$\text{or} \quad a_1 + a_2 + a_3 + \ldots + a_n \geq n(k)^{1/n}$$

Here, $a_1 = a_2 = a_3 = \ldots = a_n$

$\therefore$ Least value of $a_1 + a_2 + a_3 + \ldots + a_n$ is $n(k)^{1/n}$

| Example 107. Find the least value of $3x + 4y$ for positive values of x and y, subject to the condition $x^2y^3 = 6$.

Sol. Given, $x^2y^3 = 6$

or $\quad (x)(x)(y)(y)(y) = 6$

Here, x repeats 2 times and y repeats 3 times

$$\therefore \quad 3x + 4y = 2\left(\frac{3x}{2}\right) + 3\left(\frac{4y}{3}\right)$$

$$= \underset{1}{\left(\frac{3x}{2}\right)} + \underset{2}{\left(\frac{3x}{2}\right)} + \underset{3}{\left(\frac{4y}{3}\right)} + \underset{4}{\left(\frac{4y}{3}\right)} + \underset{5}{\left(\frac{4y}{3}\right)}$$

multiplying and dividing coefficient of x and y by 2 and 3 respectively and write $x^2y^3 = 6$

$$\Rightarrow \left(\frac{3x}{2}\right)\left(\frac{3x}{2}\right)\left(\frac{4y}{3}\right)\left(\frac{4y}{3}\right)\left(\frac{4y}{3}\right) = \frac{3^2}{2^2} \times \frac{4^3}{3^3} \times 6 = 32$$

Here, $n = 5$ and $k = 32$

Hence, least value of $\frac{3x}{2} + \frac{3x}{2} + \frac{4y}{3} + \frac{4y}{3} + \frac{4y}{3}$

$$= 5(32)^{1/5} = 10$$

i.e. least value of $3x + 4y = 10$

| Example 108. Find the minimum value of $bcx + cay + abz$, when $xyz = abc$.

Sol. To find the minimum value of

$$bcx + cay + abz,$$

write, $\quad xyz = abc$

or $\quad (bcx)(cay)(abz) = a^3b^3c^3 = k \qquad$ [constant]

Here, $\quad n = 3$

Hence, minimum value of $bcx + cay + abz = n(k)^{1/n}$

$$= 3(a^3b^3c^3)^{1/3} = 3abc$$

An Important Result

If $a_i > 0, i = 1, 2, 3, \ldots, n$ which are not identical, then

(i) $\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$; If $m < 0$ or $m > 1$

(ii) $\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$; If $0 < m < 1$

Remark

If $a_1 = a_2 = \ldots = a_n$, then use equal sign in inequalities.

| Example 109. If a, b, c be positive real numbers, prove that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$.

Sol. Arithmetic mean of (-1) th powers $\geq (-1)$ th power of arithmetic mean

$$\frac{\left(\frac{b+c}{a+b+c}\right)^{-1} + \left(\frac{c+a}{a+b+c}\right)^{-1} + \left(\frac{a+b}{a+b+c}\right)^{-1}}{3} \geq \left(\frac{\frac{b+c}{a+b+c} + \frac{c+a}{a+b+c} + \frac{a+b}{a+b+c}}{3}\right)^{-1}$$

$$\Rightarrow \quad \frac{\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b}}{3} \geq \left(\frac{2}{3}\right)^{-1}$$

$$\Rightarrow \quad \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b} + 1 \geq \frac{9}{2}$$

$$\Rightarrow \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{9}{2} - 3$$

$$\text{or} \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

| Example 110. If a and b are positive and $a + b = 1$, show that $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 > \frac{25}{2}$.

Sol. Since, AM of 2nd powers > 2nd power of AM

$$\therefore \quad \frac{\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2}{2} > \left(\frac{a + \frac{1}{a} + b + \frac{1}{b}}{2}\right)^2$$

$$= \frac{1}{4}(a + b + a^{-1} + b^{-1})^2 = \frac{1}{4}(1 + a^{-1} + b^{-1})^2 \qquad [\because a + b = 1]$$

$$\therefore \quad \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 > \frac{1}{2}(1 + a^{-1} + b^{-1})^2 \qquad \ldots(i)$$

$$\text{Again,} \quad \frac{a^{-1} + b^{-1}}{2} > \left(\frac{a+b}{2}\right)^{-1} = \left(\frac{1}{2}\right)^{-1} = 2$$

$$\text{or} \quad \frac{a^{-1} + b^{-1}}{2} > 2$$

$$\Rightarrow \quad a^{-1} + b^{-1} > 4$$

$$\therefore \quad (1 + a^{-1} + b^{-1}) > 5 \text{ or } (1 + a^{-1} + b^{-1})^2 > 25$$

$$\Rightarrow \quad \frac{1}{2}(1 + a^{-1} + b^{-1})^2 > \frac{25}{2} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 > \frac{25}{2}$$

Exercise for Session 7

1. The minimum value of $4^x + 4^{2-x}$, $x \in R$ is

(a) 0 (b) 2
(c) 4 (d) 8

2. If $0 < \theta < \pi$, then the minimum value of $\sin^3 \theta + \operatorname{cosec}^3 \theta + 2$, is

(a) 0 (b) 2
(c) 4 (d) 8

3. If a, b, c and d are four real numbers of the same sign, then the value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$ lies in the interval

(a) $[2, \infty)$ (b) $[3, \infty)$
(c) $(4, \infty)$ (d) $[4, \infty)$

4. If $0 < x < \frac{\pi}{2}$, then the minimum value of $2(\sin x + \cos x + \operatorname{cosec} 2x)^3$ is

(a) 27 (b) 13.5
(c) 6.75 (d) 0

5. If $a + b + c = 3$ and $a > 0, b > 0, c > 0$, then the greatest value of $a^2 b^3 c^2$ is

(a) $\frac{3^4 \cdot 2^{10}}{7^7}$ (b) $\frac{3^{10} \cdot 2^4}{7^7}$
(c) $\frac{3^2 \cdot 2^{12}}{7^7}$ (d) $\frac{3^{12} \cdot 2^2}{7^7}$

6. If $x + y + z = a$ and the minimum value of $\frac{a}{x} + \frac{a}{y} + \frac{a}{z}$ is 81^{λ}, then the value of λ is

(a) $\frac{1}{2}$ (b) 1
(c) $\frac{1}{4}$ (d) 2

7. a, b, c are three positive numbers and abc^2 has the greatest value $\frac{1}{64}$, then

(a) $a = b = \frac{1}{2}, c = \frac{1}{4}$ (b) $a = b = c = \frac{1}{3}$
(c) $a = b = \frac{1}{4}, c = \frac{1}{2}$ (d) $a = b = c = \frac{1}{4}$

Shortcuts and Important Results to Remember

1 If $T_n = An + B$, i.e. nth term of an AP is a linear expression in n, where A, B are constants, then coefficient of n i.e., A is the common difference.

2 If $S_n = Cn^2 + Dn$ is the sum of n terms of an AP, where C and D are constants, then common difference of AP is $2C$ i.e., 2 times the coefficient of n^2.

3 (i) $d = T_n - T_{n-1}$ $[n \geq 2]$ (ii) $T_n = S_n - S_{n-1}$ $[n \geq 2]$
(iii) $d = S_n - 2S_{n-1} + S_{n-2}$ $[n \geq 3]$

4 If for two different AP's

$$\frac{S_n}{S_n'} = \frac{An^2 + Bn}{Cn^2 + Cn} \text{ or } \frac{An + B}{Cn + D}$$

Then, $\frac{T_n}{T_n'} = \frac{A(2n-1) + B}{C(2n-1) + D}$

5 If for two different AP's

$$\frac{T_n}{T_n'} = \frac{An + B}{Cn + D}, \text{ then } \frac{S_n}{S_n'} = \frac{A\left(\frac{n+1}{2}\right) + B}{C\left(\frac{n+1}{2}\right) + D}$$

6 If $T_p = q$ and $T_q = p$, then $T_{p+q} = 0$, $T_r = p + q - r$

7 If $pT_p = qT_q$ of an AP, then $T_{p+q} = 0$

8 If $S_p = S_q$ for an AP, then $S_{p+q} = 0$

9 If $S_p = q$ and $S_q = p$ of an AP, then $S_{p+q} = -(p+q)$

10 If $T_p = P$ and $T_q = Q$ for a GP, then $T_n = \left[\frac{P^{n-q}}{Q^{n-p}}\right]^{1/(p-q)}$

11 If $T_{m+n} = p$, $T_{m-n} = q$ for a GP, then

$$T_m = \sqrt{pq}, \quad T_n = p\left(\frac{q}{p}\right)^{m/2n}$$

12 If $T_m = n$, $T_n = m$ for a HP, then

$$T_{m+n} = \frac{mn}{(m+n)}, T_{mn} = 1, T_p = \frac{mn}{p}$$

13 If $T_p = qr$, $T_q = pr$ for a HP, then $T_r = pq$

14 No term of HP can be zero and there is no formula to find S_n for HP.

15 a, b, c are in AP, GP or HP as $\frac{a-b}{b-c} = \frac{a}{a}$ or $\frac{a}{b}$ or $\frac{a}{c}$.

16 If A, G, H be AM, GM and HM between a and b, then

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \begin{cases} A, & \text{when } n = 0 \\ G, & \text{when } n = -\frac{1}{2} \\ H, & \text{when } n = -1 \end{cases}$$

17 If A and G are the AM and GM between two numbers a, b, then a, b are given by $A \pm \sqrt{(A+G)(A-G)}$

18 If a, b, c are in GP, then $a + b, 2b, b + c$ are in HP.

19 If a, b, c are in AP, then $\lambda^a, \lambda^b, \lambda^c$ are in GP, where $\lambda > 0, \lambda \neq 1$.

20 If $-1 < r < 1$, then GP is said to be convergent, if $r < -1$ or $r > 1$, then GP is said to be divergent and if $r = -1$, then series is oscillating.

21 If a, b, c, d are in GP, then
$(a \pm b)^n, (b \pm c)^n, (c \pm d)^n$ are in GP, $\forall\, n \in I$

22 If a, b, c are in AP as well as in GP, then $a = b = c$.

23 The equations $a_1 x + a_2 y = a_3$, $a_4 x + a_5 y = a_6$ has a unique solution, if $a_1, a_2, a_3, a_4, a_5, a_6$ are in AP and common difference $\neq 0$.

24 For n positive quantities $a_1, a_2, a_3, \ldots, a_n$

$$\text{AM} \geq \text{GM} \geq \text{HM}$$

sign of equality (AM = GM = HM) holds when quantities are equal

i.e. $a_1 = a_2 = a_3 = \ldots = a_n$.

25 For two positive numbers a and b (AM) (HM) $= (\text{GM})^2$, the result will be true for n numbers, if they are in GP.

26 If odd numbers of (say $2n + 1$) AM's, GM's and HM's be inserted between two numbers, then their middle means [i.e., $(n+1)$th mean] are in GP.

27 If a^2, b^2, c^2 are in AP.

$$\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in AP.}$$

28 Coefficient of x^{n-1} and x^{n-2} in
$(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$
are $-(a_1 + a_2 + a_3 + \ldots + a_n)$ and $\Sigma a_1 a_2$, respectively
where, $\Sigma a_1 a_2 = \frac{(\Sigma a_1)^2 - \Sigma a_1^2}{2}$.

29 $1 + 3 + 5 + \ldots$ upto n terms $= n^2$

30 $2 + 6 + 12 + 20 + \ldots$ upto n terms $= \frac{n(n+1)(n+2)}{3}$

31 $1 + 3 + 7 + 13 + \ldots$ upto n terms $= \frac{n(n^2+2)}{3}$

32 $1 + 5 + 14 + 30 + \ldots$ upto n terms $= \frac{n(n+1)^2(n+2)}{12}$

33 If $a_1, a_2, a_3, \ldots, a_n$ are the non-zero terms of a non-constant AP, then

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \ldots + \frac{1}{a_{n-1} a_n} = \frac{(n-1)}{a_1 a_n}$$