Session 4

Equations of Higher Degree, Rational Algebraic Inequalities, Roots of Equation with the Help of Graphs,

Equations of Higher Degree

The equation $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0,$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants but $a_0 \neq 0$, is a polynomial equation of degree n. It has n and only n roots. Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n$ be n roots, then

- $\Sigma\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1} + \alpha_n = (-1)^1 \dfrac{a_1}{a_0}$ **[sum of all roots]**
- $\Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \ldots + \alpha_2\alpha_n + \ldots + \alpha_{n-1}\alpha_n$

  $= (-1)^2 \dfrac{a_2}{a_0}$ [sum of products taken two at a time]
- $\Sigma\alpha_1\alpha_2\alpha_3 = (-1)^3 \dfrac{a_3}{a_0}$ **[sum of products taken three at a time]**
- $\alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \dfrac{a_n}{a_0}$ **[product of all roots]**

In general, $\Sigma\alpha_1\alpha_2\alpha_3 \ldots \alpha_p = (-1)^p \dfrac{a_p}{a_0}$

Remark

1. A polynomial equation of degree n has n roots (real or imaginary).
2. If all the coefficients, i.e., $a_0, a_1, a_2, \ldots, a_n$ are real, then the imaginary roots occur in pairs, i.e. number of imaginary roots is always even.
3. If the degree of a polynomial equation is odd, then atleast one of the roots will be real.
4. $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$
   $= x^n + (-1)^1\Sigma\alpha_1 \cdot x^{n-1} + (-1)^2\Sigma\alpha_1\alpha_2 \cdot x^{n-2} + \ldots + (-1)^n \alpha_1\alpha_2\alpha_3 \ldots \alpha_n$

In Particular

(i) For $n = 3$, if α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, where a, b, c, d are constants and $a \neq 0$, then $\Sigma\alpha = \alpha + \beta + \gamma = (-1)^1 \dfrac{b}{a} = -\dfrac{b}{a}$,

$$\Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

and $\alpha\beta\gamma = (-1)^3 \dfrac{d}{a} = -\dfrac{d}{a}$

or $ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$

$= a(x^3 - \Sigma\alpha \cdot x^2 + \Sigma\alpha\beta \cdot x - \alpha\beta\gamma)]$

(ii) For $n = 4$, if $\alpha, \beta, \gamma, \delta$ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, where a, b, c, d, e are constants and $a \neq 0$, then

$$\Sigma\alpha = \alpha + \beta + \gamma + \delta = (-1)^1 \frac{b}{a} = -\frac{b}{a},$$

$$\Sigma\alpha\beta = (\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = (-1)^2 \frac{c}{a} = \frac{c}{a},$$

$$\Sigma\alpha\beta\gamma = \alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

and $\alpha\beta\gamma\delta = (-1)^4 \dfrac{e}{a} = \dfrac{e}{a}$

or $ax^4 + bx^3 + cx^2 + dx + e = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= a(x^4 - \Sigma\alpha \cdot x^3 + \Sigma\alpha\beta \cdot x^2 - \Sigma\alpha\beta\gamma \cdot x + \alpha\beta\gamma\delta)$

| Example 42. Find the conditions, if roots of the equation $x^3 - px^2 + qx - r = 0$ are in

(i) AP (ii) GP

(iii) HP

Sol. (i) Let roots of the given equation are $A - D, A, A + D$, then

$$A - D + A + A + D = p \Rightarrow A = \frac{p}{3}$$

Now, A is the roots of the given equation, then it must be satisfy

$$A^3 - pA^2 + qA - r = 0$$

$$\Rightarrow \left(\frac{p}{3}\right)^3 - p\left(\frac{p}{3}\right)^2 + q\left(\frac{p}{3}\right) - r = 0$$

$$\Rightarrow p^3 - 3p^3 + 9qp - 27r = 0$$

or $$2p^3 - 9pq + 27r = 0,$$

which is the required condition.

(ii) Let roots of the given equation are $\frac{A}{R}$, A, AR, then

$$\frac{A}{R}\cdot A\cdot AR = (-1)^3\cdot\left(-\frac{r}{1}\right) = r$$

$$\Rightarrow \quad A^3 = r$$

$$\Rightarrow \quad A = r^{\frac{1}{3}}$$

Now, A is the roots of the given equation, then

$$A^3 - pA^2 + qA - r = 0$$

$$\Rightarrow \quad r - p(r)^{2/3} = q(r)^{1/3} - r = 0$$

or $$p(r)^{2/3} = q(r)^{1/3}$$

or $$p^3r^2 = q^3r$$

or $$p^3r = q^3$$

which is the required condition.

(iii) Given equation is

$$x^3 - px^2 + qx - r = 0 \qquad \text{...(i)}$$

On replacing x by $\frac{1}{x}$ in Eq. (i), then

$$\left(\frac{1}{x}\right)^3 - p\left(\frac{1}{x}\right)^2 + q\left(\frac{1}{x}\right) - r = 0$$

$$\Rightarrow \quad rx^3 - qx^2 + px - 1 = 0 \qquad \text{...(ii)}$$

Now, roots of Eq. (ii) are in AP.

Let roots of Eq. (ii) are $A - P, A, A + P$, then

$$A - P + A + A + P = \frac{q}{r} \quad \text{or} \quad A = \frac{q}{3r}$$

$\because A$ is a root of Eq. (ii), then

$$rA^3 - qA^2 + pA - 1 = 0$$

$$\Rightarrow \quad r\left(\frac{q}{3r}\right)^3 - q\left(\frac{q}{3r}\right)^2 + p\left(\frac{q}{3r}\right) - 1 = 0$$

$$\Rightarrow \quad q^3 - 3q^3 + 9pqr - 27r^2 = 0$$

$$\Rightarrow \quad 2q^3 - 9pqr + 27r^2 = 0,$$

which is the required condition.

| Example 43. Solve $6x^3 - 11x^2 + 6x - 1 = 0$, if roots of the equation are in HP.

Sol. Put $x = \frac{1}{y}$ in the given equation, then

$$\frac{6}{y^3} - \frac{11}{y^2} + \frac{6}{y} - 1 = 0$$

$$\Rightarrow \quad y^3 - 6y^2 + 11y - 6 = 0 \qquad \text{...(i)}$$

Now, roots of Eq. (i) are in AP.

Let the roots be $\alpha - \beta, \alpha, \alpha + \beta$.

Then, sum of roots $= \alpha - \beta + \alpha + \alpha + \beta = 6$

$$\Rightarrow \quad 3\alpha = 6$$

$$\therefore \quad \alpha = 2$$

Product of roots $= (\alpha - \beta)\cdot\alpha\cdot(\alpha + \beta) = 6$

$$\Rightarrow \quad (2 - \beta)2(2 + \beta) = 6 \Rightarrow 4 - \beta^2 = 3$$

$$\beta = \pm 1$$

$\therefore$ Roots of Eqs. (i) are 1, 2, 3 or 3, 2, 1.

Hence, roots of the given equation are $1, \frac{1}{2}, \frac{1}{3}$ or $\frac{1}{3}, \frac{1}{2}, 1$.

| Example 44. If α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find

~~(i) $\Sigma\alpha^2$~~. (ii) $\boxed{\Sigma\alpha^2\beta}$. (iii) $\Sigma\alpha^3$.

Sol. Since, α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$.

$$\therefore \quad \Sigma\alpha = p, \Sigma\alpha\beta = q \text{ and } \alpha\beta\gamma = r$$

(i) $\because \Sigma\alpha\cdot\Sigma\alpha = p\cdot p$

$$\Rightarrow \quad (\alpha + \beta + \gamma)(\alpha + \beta + \gamma) = p^2$$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) = p^2$$

or $$\Sigma\alpha^2 + 2\Sigma\alpha\beta = p^2$$

or $$\Sigma\alpha^2 = p^2 - 2q$$

(ii) $\because \Sigma\alpha\cdot\Sigma\alpha\beta = p\cdot q$

$$\Rightarrow \quad (\alpha + \beta + \gamma)\cdot(\alpha\beta + \beta\gamma + \gamma\alpha) = pq$$

$$\Rightarrow \alpha^2\beta + \alpha\beta\gamma + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \alpha\beta\gamma + \gamma^2\beta + \gamma^2\alpha = pq$$

$$\Rightarrow \quad (\alpha^2\beta + \alpha^2\alpha + \beta^2\gamma + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta) + 3\alpha\beta\gamma = pq$$

or $$\Sigma\alpha^2\beta + 3r = pq$$

or $$\Sigma\alpha^2\beta = pq - 3r$$

(iii) $\because \Sigma\alpha^2\cdot\Sigma\alpha = (p^2 - 2q)\cdot p$ [from result (i)]

$$\Rightarrow \quad (\alpha^2 + \beta^2 + \gamma^2)(\alpha + \beta + \gamma) = p^3 - 2pq$$

$$\Rightarrow \quad \alpha^3 + \beta^3 + \gamma^3 + (\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta) = p^3 - 2pq$$

$$\Rightarrow \quad \Sigma\alpha^3 + \Sigma\alpha^2\beta = p^3 - 2pq$$

$$\Rightarrow \quad \Sigma\alpha^3 + pq - 3r = p^3 - 2pq \qquad \text{[from result (ii)]}$$

or $$\Sigma\alpha^3 = p^3 - 3pq + 3r$$

| Example 45. If α, β, γ are the roots of the cubic equation $x^3 + qx + r = 0$, then find the equation whose roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$.

Sol. $\because \alpha, \beta, \gamma$ are the roots of the cubic equation

$$x^3 + qx + r = 0 \qquad \text{...(i)}$$

Then, $$\Sigma\alpha = 0, \Sigma\alpha\beta = q, \alpha\beta\gamma = -r \qquad \text{...(ii)}$$

If y is a root of the required equation, then

$$y = (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$= (\alpha + \beta + \gamma - \gamma)^2 - \frac{4\alpha\beta\gamma}{\gamma}$$

$$= (0 - \gamma)^2 + \frac{4r}{\gamma} \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad y = \gamma^2 + \frac{4r}{\gamma}$$

[replacing γ by x which is a root of Eq. (i)]

$$\therefore \quad y = x^2 + \frac{4r}{x}$$

or $$x^3 - yx + 4r = 0 \quad \text{...(iii)}$$

The required equation is obtained by eliminating x between Eqs. (i) and (iii).

Now, subtracting Eq. (iii) from Eq. (i), we get

$$(q + y)x - 3r = 0$$

or $$x = \frac{3r}{q + y}$$

On substituting the value of x in Eq. (i), we get

$$\left(\frac{3r}{q + y}\right)^3 + q\left(\frac{3r}{q + y}\right) + r = 0$$

Thus, $$y^3 + 6qy^2 + 9q^2y + (4q^3 + 27r^2) = 0$$

which is the required equation.

Remark

$\Sigma(\alpha - \beta)^2 = -6q$, $\Pi(\alpha - \beta)^2 = -(4q^3 + 27r^2)$

Some Results on Roots of a Polynomial Equation

1. **Remainder Theorem** If a polynomial $f(x)$ is divided by a linear function $x - \lambda$, then the remainder is $f(\lambda)$,

   i.e. Dividend = Divisor × Quotient + Remainder

   Let $Q(x)$ be the quotient and R be the remainder, thus

   $$f(x) = (x - \lambda)Q(x) + R$$

   $$\Rightarrow \quad f(\lambda) = (\lambda - \lambda)Q(\lambda) + R = 0 + R = R$$

| Example 46. If the expression $2x^3 + 3px^2 - 4x + p$ has a remainder of 5 when divided by $x + 2$, find the value of p.

Sol. Let $$f(x) = 2x^3 + 3px^2 - 4x + p$$

$$\because \quad f(x) = (x + 2)Q(x) + 5$$

$$\Rightarrow \quad f(-2) = 5$$

$$\Rightarrow \quad 2(-2)^3 + 3p(-2)^2 - 4(-2) + p = 5 \text{ or } 13p = 13$$

$$\therefore \quad p = 1$$

2. **Factor Theorem** Factor theorem is a special case of Remainder theorem.

   Let $f(x) = (x - \lambda)Q(x) + R = (x - \lambda)Q(x) + f(\lambda)$

   If $f(\lambda) = 0$, $f(x) = (x - \lambda)Q(x)$, therefore $f(x)$ is exactly divisible by $x - \lambda$.

   or

   If λ is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x - \lambda)$ and conversely, if $f(x)$ is exactly divisible by $(x - \lambda)$, then λ is a root of the equation $f(x) = 0$ and the remainder obtained is $f(\lambda)$.

| Example 47. If $x^2 + ax + 1$ is a factor of $ax^3 + bx + c$, find the conditions.

Sol. $\because ax^3 + bx + c = (x^2 + ax + 1)Q(x)$

Let $$Q(x) = Ax + B,$$

then $$ax^3 + bx + c = (x^2 + ax + 1)(Ax + B)$$

On comparing coefficients of x^3, x^2, x and constant on both sides, we get

$$a = A, \quad \text{...(i)}$$

$$0 = B + aA, \quad \text{...(ii)}$$

$$b = aB + A, \quad \text{...(iii)}$$

and $$c = B \quad \text{...(iv)}$$

From Eqs. (i) and (iv), we get

$$A = a \text{ and } B = c$$

From Eqs. (ii) and (iii), $a^2 + c = 0$ and $b = ac + a$ are the required conditions.

| Example 48. A certain polynomial $f(x), x \in R$, when divided by $x - a$, $x - b$ and $x - c$ leaves remainders a, b and c, respectively. Then, find the remainder when $f(x)$ is divided by $(x - a)(x - b)(x - c)$, where a, b, c are distinct.

Sol. By Remainder theorem $f(a) = a$, $f(b) = b$ and $f(c) = c$

Let the quotient be $Q(x)$ and remainder is $R(x)$.

$$\therefore \quad f(x) = (x - a)(x - b)(x - c)Q(x) + R(x)$$

$$\therefore \quad f(a) = 0 + R(a) \Rightarrow R(a) = a$$

$$f(b) = 0 + R(b) \Rightarrow R(b) = b \text{ and } f(c) = 0 + R(c)$$

$$\Rightarrow \quad R(c) = c$$

So, the equation $R(x) - x = 0$ has three roots a, b and c. But its degree is atmost two. So, $R(x) - x$ must be zero polynomial (or identity).

Hence, $R(x) = x$.

3. Every equation of an odd degree has atleast one real root, whose sign is opposite to that of its last term, provided that the coefficient of the first term is positive.

4. Every equation of an even degree has atleast two real roots, one positive and one negative, whose last term is negative, provided that the coefficient of the first term is positive.

5. If an equation has no odd powers of x, then all roots of the equation are complex provided all the coefficients of the equation have positive sign.

6. If $x = \alpha$ is root repeated m times in $f(x) = 0$
($f(x) = 0$ is an nth degree equation in x), then
$f(x) = (x - \alpha)^m \, g(x)$
where, $g(x)$ is a polynomial of degree $(n - m)$ and the root $x = \alpha$ is repeated $(m - 1)$ time in $f'(x) = 0$, $(m - 2)$ times in $f''(x) = 0, \ldots, (m - (m - 1))$ times in $f^{m-1}(x) = 0$.

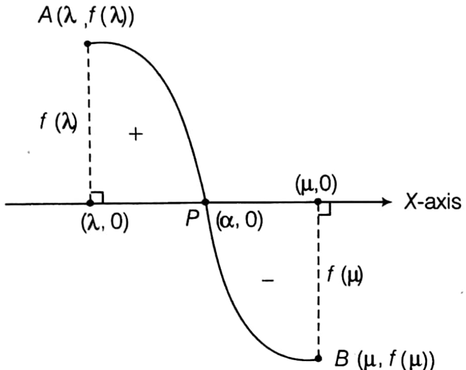
7. Let $f(x) = 0$ be a polynomial equation and λ, μ are two real numbers.

Then, $f(x) = 0$ will have atleast one real root or an odd number of roots between λ and μ, if $f(\lambda)$ and $f(\mu)$ are of opposite signs.

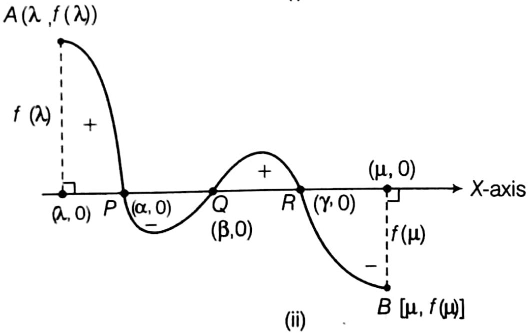
But if $f(\lambda)$ and $f(\mu)$ are of same signs, then either $f(x) = 0$ has no real roots or an even number of roots between λ and μ.

Illustration by Graphs

Since, $f(x)$ be a polynomial in x, then graph of $y = f(x)$ will be continuous in every interval.



(i)



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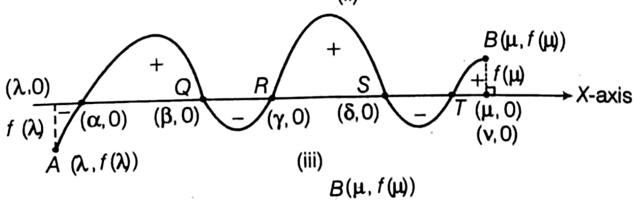

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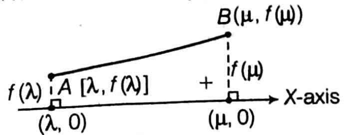

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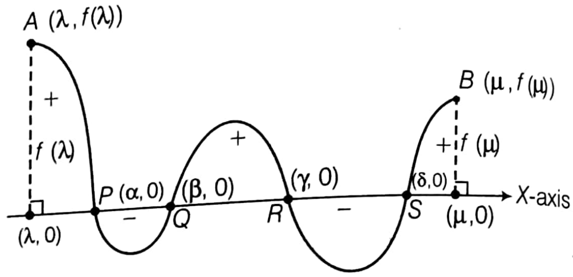

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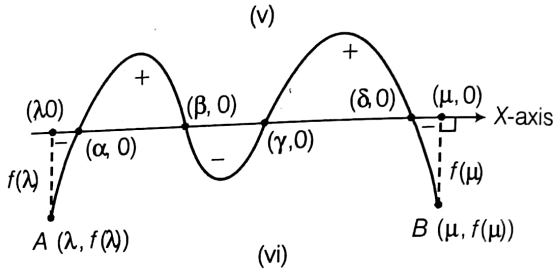


(vi)

(a) In figure (i), (ii) and (iii), $f(\lambda)$ and $f(\mu)$ have opposite signs and equation $f(x) = 0$, has one, three, five roots between λ and μ, respectively.

(b) In figure (iv), (v) and (vi), $f(\lambda)$ and $f(\mu)$ have same signs and equation $f(x) = 0$, has no, four and four roots between λ and μ, respectively.

Example 49. If a, b, c are real numbers, $a \neq 0$. If α is root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, show that the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies $\alpha < \gamma < \beta$.

Sol. Since, α is a root of $a^2x^2 + bx + c = 0$.

Then, $$a^2\alpha^2 + b\alpha + c = 0 \qquad \text{...(i)}$$

and β is a root of $a^2x^2 - bx - c = 0$,

then $$a^2\beta^2 - b\beta - c = 0 \qquad \text{...(ii)}$$

Let $$f(x) = a^2x^2 + 2bx + 2c$$

$\therefore$ $$f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = a^2\alpha^2 - 2a^2\alpha^2 \qquad \text{[from Eq. (i)]}$$

$$= -a^2\alpha^2$$

$\Rightarrow$ $f(\alpha) < 0$ and $f(\beta) = a^2\beta^2 + 2b\beta + 2c$

$$= a^2\beta^2 + 2a^2\beta^2 \qquad \text{[from Eq. (ii)]}$$

$$= 3a^2\beta^2$$

$\Rightarrow$ $$f(\beta) > 0$$

Since, $f(\alpha)$ and $f(\beta)$ are of opposite signs, then it is clear that a root γ of the equation $f(x) = 0$ lies between α and β.

Hence, $$\alpha < \gamma < \beta \qquad [\because \alpha < \beta]$$

Example 50. If $a < b < c < d$, then show that $(x - a)(x - c) + 3(x - b)(x - d) = 0$ has real and distinct roots.

Sol. Let $f(x) = (x - a)(x - c) + 3(x - b)(x - d)$

Then, $f(a) = 0 + 3(a - b)(a - d) > 0$ $[\because a - b < 0, a - d < 0]$
and $f(b) = (b - a)(b - c) + 0 < 0$ $[\because b - a > 0, b - c < 0]$
Thus, one root will lie between a and b.
and $f(c) = 0 + 3(c - b)(c - d) < 0$ $[\because c - b > 0, c - d < 0]$
and $f(d) = (d - a)(d - c) + 0 > 0$ $[\because d - a > 0, d - c > 0]$
Thus, one root will lie between c and d. Hence, roots of equation are real and distinct.

8. Let $f(x) = 0$ be a polynomial equation then
   (a) the number of positive roots of a polynomial equation $f(x) = 0$ (arranged in decreasing order of the degree) cannot exceed the number of changes of signs in $f(x) = 0$ as we move from left to right.
   For example, Consider the equation
   $$2x^3 - x^2 - x + 1 = 0.$$
   The number of changes of signs from left to right is 2 (+ to –, then – to +). Then, number of positive roots cannot exceed 2.
   (b) The number of negative roots of a polynomial equation $f(x) = 0$ cannot exceed the number of changes of signs in $f(-x)$.
   For example, Consider the equation
   $$5x^4 + 3x^3 - 2x^2 + 5x - 8 = 0$$
   Let $f(x) = 4x^4 + 3x^3 - 2x^2 + 5x - 8$
   $\therefore$ $f(-x) = 5x^4 - 3x^3 - 2x^2 - 5x - 8$
   The number of changes of signs from left to right is (+ to –). Then number of negative roots cannot exceed 1.
   (c) If equation $f(x) = 0$ have atmost r positive roots and atmost t negative roots, then equation $f(x) = 0$ will have atmost $(r + t)$ real roots, i.e. it will have atleast $n - (r + t)$ imaginary roots, where n is the degree of polynomial.
   For example, Consider the equation
   $$5x^6 - 8x^3 + 3x^5 + 5x^2 + 8 = 0$$
   The given equation can be written as
   $$5x^6 + 3x^5 - 8x^3 + 5x^2 + 8 = 0$$
   Let $f(x) = 5x^6 + 3x^5 - 8x^3 + 5x^2 + 8$
   Here, $f(x)$ has two changes in signs.
   So, $f(x)$ has atmost two positive real roots
   and $f(-x) = 5x^6 - 3x^5 + 8x^3 + 5x^2 + 8$
   Here, $f(-x)$ has two changes in signs.
   So, $f(x)$ has atmost two negative real roots.
   and $x = 0$ cannot be root of $f(x) = 0$.
   Hence, $f(x) = 0$ has atmost four real roots, therefore atleast two imaginary roots.

9. **Rolle's Theorem** If $f(x)$ is continuous function in the interval $[a, b]$ and differentiable in interval (a, b) and $f(a) = f(b)$, then equation $f'(x) = 0$ will have atleast one root between a and b. Since, every polynomial $f(x)$ is always continuous and differentiable in every interval. Therefore, Rolle's theorem is always applicable to polynomial function in every interval $[a, b]$ if $f(a) = f(b)$.

Example 51. If $2a + 3b + 6c = 0$; $a, b, c \in R$, then show that the equation $ax^2 + bx + c = 0$ has atleast one root between 0 and 1.

Sol. Given, $2a + 3b + 6c = 0$

$$\Rightarrow \quad \frac{a}{3} + \frac{b}{2} + c = 0 \quad \text{...(i)}$$

Let $f'(x) = ax^2 + bx + c$,

Then, $$f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$$

Now, $f(0) = d$ and $f(1) = \frac{a}{3} + \frac{b}{2} + c + d$
$= 0 + d$ [from Eq. (i)]

Since, $f(x)$ is a polynomial of three degree, then $f(x)$ is continuous and differentiable everywhere and $f(0) = f(1)$, then by Rolle's theorem $f'(x) = 0$ i.e., $ax^2 + bx + c = 0$ has atleast one real root between 0 and 1.

Reciprocal Equation of the Standard Form can be Reduced to an Equation of Half Its Dimensions

Let the equation be
$$ax^{2m} + bx^{2m-1} + cx^{2m-2} + \dots + kx^m + \dots + cx^2 + bx + a = 0$$
On dividing by x^m, then
$$ax^m + bx^{m-1} + cx^{m-2} + \dots + k + \dots + \frac{c}{x^{m-2}} + \frac{b}{x^{m-1}} + \frac{a}{x^m} = 0$$
On rearranging the terms, we have
$$a\left(x^m + \frac{1}{x^m}\right) + b\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + c\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \dots + k = 0$$
Now, $$x^{p+1} + \frac{1}{x^{p+1}} = \left(x^p + \frac{1}{x^p}\right)\left(x + \frac{1}{x}\right) - \left(x^{p-1} + \frac{1}{x^{p-1}}\right)$$

Hence, writing z for $x+\frac{1}{x}$ and given to p succession the values 1, 2, 3, ..., we obtain

$$x^2+\frac{1}{x^2}=z^2-2$$

$$x^3+\frac{1}{x^3}=z(z^2-2)-z=z^3-3z$$

$$x^4+\frac{1}{x^4}=z(z^3-3z)-(z^2-2)=z^4-4z^2+2$$

and so on and **generally $x^m+\frac{1}{x^m}$ is of m dimensions in z and therefore the equation in z is of m dimensions.**

| Example 52. Solve the equation $2x^4+x^3-11x^2+x+2=0$.

Sol. Since, $x=0$ is not a solution of the given equation.
On dividing by x^2 in both sides of the given equation, we get

$$2\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-11=0 \qquad \text{...(i)}$$

Put $x+\frac{1}{x}=y$ in Eq. (i), then Eq. (i) reduce in the form

$$2(y^2-2)+y-11=0$$

$$\Rightarrow \quad 2y^2+y-15=0$$

$$\therefore \quad y_1=-3 \text{ and } y_2=\frac{5}{2}$$

Consequently, the original equation is equivalent to the collection of equations

$$\left[\begin{array}{l} x+\frac{1}{x}=-3 \\ x+\frac{1}{x}=\frac{5}{2} \end{array}\right.,$$

we find that, $x_1=\frac{-3-\sqrt{5}}{2}, x_2=\frac{-3+\sqrt{5}}{2}, x_3=\frac{1}{2}, x_4=2$

Equations which can be Reduced to Linear, Quadratic and Biquadratic Equations

***Type* I An equation of the form**

$$(x-a)(x-b)(x-c)(x-d)=A$$

where, $a<b<c<d$, $b-a=d-c$, can be solved by a change of variable.

i.e. $$y=\frac{(x-a)+(x-b)+(x-c)+(x-d)}{4}$$

$$y=x-\frac{(a+b+c+d)}{4}$$

| Example 53. Solve the equation $(12x-1)(6x-1)(4x-1)(3x-1)=5$.

Sol. The given equation can be written as

$$\left(x-\frac{1}{12}\right)\left(x-\frac{1}{6}\right)\left(x-\frac{1}{4}\right)\left(x-\frac{1}{3}\right)=\frac{5}{12\cdot6\cdot4\cdot3} \qquad \text{...(i)}$$

Since, $\frac{1}{12}<\frac{1}{6}<\frac{1}{4}<\frac{1}{3}$ and $\frac{1}{6}-\frac{1}{12}=\frac{1}{3}-\frac{1}{4}$

We can introduced a new variable,

$$y=\frac{1}{4}\left[\left(x-\frac{1}{12}\right)+\left(x-\frac{1}{6}\right)+\left(x-\frac{1}{4}\right)+\left(x-\frac{1}{3}\right)\right]$$

$$y=x-\frac{5}{24}$$

On substituting $x=y+\frac{5}{24}$ in Eq. (i), we get

$$\left(y+\frac{3}{24}\right)\left(y+\frac{1}{24}\right)\left(y-\frac{1}{24}\right)\left(y-\frac{3}{24}\right)=\frac{5}{12\cdot6\cdot4\cdot3}$$

$$\Rightarrow \quad \left[y^2-\left(\frac{1}{24}\right)^2\right]\left[y^2-\left(\frac{3}{24}\right)^2\right]=\frac{5}{12\cdot6\cdot4\cdot3}$$

Hence, we find that

$$y^2=\frac{49}{24^2}$$

i.e. $$y_1=\frac{7}{24} \text{ and } y_2=-\frac{7}{24}$$

Hence, the corresponding roots of the original equation are $-\frac{1}{12}$ and $\frac{1}{2}$.

***Type* II** An equation of the form

$$(x-a)(x-b)(x-c)(x-d)=Ax^2$$

where, $ab=cd$ can be reduced to a collection of two quadratic equations by a change of variable $y=x+\frac{ab}{x}$.

| Example 54. Solve the equation $(x+2)(x+3)(x+8)(x+12)=4x^2$.

Sol. Since, $(-2)(-12)=(-3)(-8)$, so we can write given equation as

$$(x+2)(x+12)(x+3)(x+8)=4x^2$$

$$\Rightarrow \quad (x^2+14x+24)(x^2+11x+24)=4x^2 \qquad \text{...(i)}$$

Now, $x=0$ is not a root of given equation.
On dividing by x^2 in both sides of Eq. (i), we get

$$\left(x+\frac{24}{x}+14\right)\left(x+\frac{24}{x}+11\right)=4 \qquad \text{...(ii)}$$

Put $x+\frac{24}{x}=y$, then Eq. (ii) can be reduced in the form

$$(y+14)(y+11)=4 \quad \text{or} \quad y^2+25y+150=0$$

$$\therefore \quad y_1=-15 \text{ and } y_2=-10$$

Thus, the original equation is equivalent to the collection of equations

$$\left[\begin{aligned} x + \frac{24}{x} &= -15, \\ x + \frac{24}{x} &= -10, \end{aligned}\right.$$

i.e.
$$\left[\begin{aligned} x^2 + 15x + 24 &= 0 \\ x^2 + 10x + 24 &= 0 \end{aligned}\right.$$

On solving these collection, we get

$$x_1 = \frac{-15 - \sqrt{129}}{2},\ x_2 = \frac{-15 + \sqrt{129}}{2},\ x_3 = -6,\ x_4 = -4$$

Type **III An equation of the form $(x-a)^4 + (x-b)^4 = A$** can also be solved by a change of variable, i.e. making a substitution $y = \dfrac{(x-a)+(x-b)}{2}$.

| Example 55. Solve the equation $(6-x)^4 + (8-x)^4 = 16$.

Sol. After a change of variable,

$$y = \frac{(6-x)+(8-x)}{2}$$

$\therefore \quad y = 7 - x$ or $x = 7 - y$

Now, put $x = 7 - y$ in given equation, we get

$$(y-1)^4 + (y+1)^4 = 16$$

$$\Rightarrow \quad y^4 + 6y^2 - 7 = 0$$

$$\Rightarrow \quad (y^2+7)(y^2-1) = 0$$

$$y^2 + 7 \neq 0$$

[y gives imaginary values]

$$\therefore \quad y^2 - 1 = 0$$

Then, $y_1 = -1$ and $y_2 = 1$

Thus, $x_1 = 8$ and $x_2 = 6$ are the roots of the given equation.

Rational Algebraic Inequalities

Consider the following types of rational algebraic inequalities

$$\frac{P(x)}{Q(x)} > 0,\ \frac{P(x)}{Q(x)} < 0,$$

$$\frac{P(x)}{Q(x)} \geq 0,\ \frac{P(x)}{Q(x)} \leq 0$$

If $P(x)$ and $Q(x)$ can be resolved in linear factors, then use *Wavy curve method*, otherwise we use the following statements for solving inequalities of this kind.

(1) $\dfrac{P(x)}{Q(x)} > 0 \Rightarrow \{P(x)\,Q(x) > 0 \Rightarrow \begin{cases} P(x) > 0, Q(x) > 0 \\ \text{or} \\ P(x) < 0, Q(x) < 0 \end{cases}$

(2) $\dfrac{P(x)}{Q(x)} < 0 \Rightarrow \{P(x)\,Q(x) < 0 \Rightarrow \begin{cases} P(x) > 0, Q(x) < 0 \\ \text{or} \\ P(x) < 0, Q(x) > 0 \end{cases}$

(3) $\dfrac{P(x)}{Q(x)} \geq 0 \Rightarrow \begin{cases} P(x)\,Q(x) \geq 0 \\ Q(x) \neq 0 \end{cases} \Rightarrow \begin{cases} P(x) \geq 0, Q(x) > 0 \\ \text{or} \\ P(x) \leq 0, Q(x) < 0 \end{cases}$

(4) $\dfrac{P(x)}{Q(x)} \leq 0 \Rightarrow \begin{cases} P(x)\,Q(x) \leq 0 \\ Q(x) \neq 0 \end{cases} \Rightarrow \begin{cases} P(x) \geq 0, Q(x) < 0 \\ \text{or} \\ P(x) \leq 0, Q(x) > 0 \end{cases}$

| Example 56. Find all values of a for which the set of all solutions of the system

$$\begin{cases} \dfrac{x^2 + ax - 2}{x^2 - x + 1} < 2 \\ \dfrac{x^2 + ax - 2}{x^2 - x + 1} > -3 \end{cases}$$

is the entire number line.

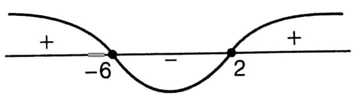
Sol. The system is equivalent to

$$\begin{cases} \dfrac{x^2 - (a+2)x + 4}{x^2 - x + 1} > 0 \\ \dfrac{4x^2 + (a-3)x + 1}{x^2 - x + 1} > 0 \end{cases}$$

Since, $x^2 - x + 1 = \left(x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$, this system is equivalent to $\begin{cases} x^2 - (a+2)x + 4 > 0 \\ 4x^2 + (a-3)x + 1 > 0 \end{cases}$

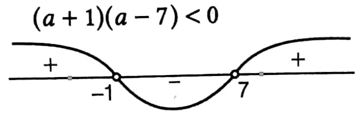
Hence, the discriminants of the both equations of this system are negative.

i.e., $\begin{cases} (a+2)^2 - 16 < 0 \\ (a-3)^2 - 16 < 0 \end{cases} \Rightarrow (a+6)(a-2) < 0$



i.e., $\quad x \in (-6, 2)$ ...(i)

$\Rightarrow \quad (a+1)(a-7) < 0$



i.e. $\quad x \in (-1, 7)$ ...(ii)

Hence, from Eqs. (i) and (ii), we get

$$x \in (-1, 2)$$

Equations Containing Absolute Values

By definition, $|x| = x$, if $x \geq 0$ $|x| = -x$, if $x < 0$

Example 57. Solve the equation $x^2 - 5|x| + 6 = 0$.

Sol. The given equation is equivalent to the collection of systems

$$\begin{cases} x^2 - 5x + 6 = 0, \text{ if } x \geq 0 \\ x^2 + 5x + 6 = 0, \text{ if } x < 0 \end{cases} \Rightarrow \begin{cases} (x-2)(x-3) = 0, \text{ if } x \geq 0 \\ (x+2)(x+3) = 0, \text{ if } x < 0 \end{cases}$$

Hence, the solutions of the given equation are

$$x_1 = 2, x_2 = 3, x_3 = -2, x_4 = -3$$

Example 58. Solve the equation

$$\left|\frac{x^2 - 8x + 12}{x^2 - 10x + 21}\right| = -\frac{x^2 - 8x + 12}{x^2 - 10x + 21}.$$

Sol. This equation has the form $|f(x)| = -f(x)$

when, $f(x) = \dfrac{x^2 - 8x + 12}{x^2 - 10x + 21}$

such an equation is equivalent to the collection of systems

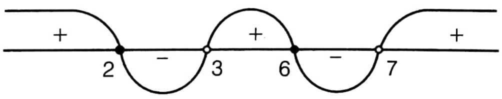
$$\begin{cases} f(x) = -f(x), \text{ if } f(x) \geq 0 \\ f(x) = f(x), \text{ if } f(x) < 0 \end{cases}$$

The first system is equivalent to $f(x) = 0$ and the second system is equivalent to $f(x) < 0$ the combining both systems, we get

$$\boxed{f(x) \leq 0}$$

$$\therefore \quad \frac{x^2 - 8x + 12}{x^2 - 10x + 21} \leq 0$$

$$\Rightarrow \quad \frac{(x-2)(x-6)}{(x-3)(x-7)} \leq 0$$



Hence, by *Wavy curve method,*

$$x \in [2, 3) \cup [6, 7)$$

Example 59. Solve the equation $|x - |4 - x|| - 2x = 4$.

Sol. This equation is equivalent to the collection of systems

$$\begin{cases} |x - (4 - x)| - 2x = 4, \text{ if } 4 - x \geq 0 \\ |x + (4 - x)| - 2x = 4, \text{ if } 4 - x < 0 \end{cases}$$

$$\Rightarrow \begin{cases} |2x - 4| - 2x = 4, & \text{if } x \leq 4 \\ 4 - 2x = 4, & \text{if } x > 4 \end{cases} \quad \text{...(i)}$$

The second system of this collection

gives $\quad x = 0$

but $\quad x > 4$

Hence, second system has no solution.

The first system of collection Eq. (i) is equivalent to the system of collection

$$\begin{cases} 2x - 4 - 2x = 4, \text{ if } 2x \geq 4 \\ -2x + 4 - 2x = 4, \text{ if } 2x < 4 \end{cases}$$

$$\Rightarrow \begin{cases} -4 = 4, \text{ if } x \geq 2 \\ -4x = 0, \text{ if } x < 2 \end{cases}$$

The first system is failed and second system gives $x = 0$.
Hence, $x = 0$ is unique solution of the given equation.

Important Forms Containing Absolute Values

Form 1 **The equation of the form**

$$|f(x) + g(x)| = |f(x)| + |g(x)|$$

is equivalent of the system

$$f(x)\,g(x) \geq 0.$$

Example 60. Solve the equation

$$\left|\frac{x}{x-1}\right| + |x| = \frac{x^2}{|x-1|}.$$

Sol. Let $f(x) = \dfrac{x}{x-1}$ and $g(x) = x$,

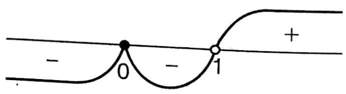
Then, $\quad f(x) + g(x) = \dfrac{x}{x-1} + x = \dfrac{x^2}{x-1}$

$\therefore$ The given equation can be reduced in the form

$$|f(x)| + |g(x)| = |f(x) + g(x)|$$

Hence, $\quad f(x) \cdot g(x) \geq 0$

$$\Rightarrow \quad \frac{x^2}{x-1} \geq 0$$



From *Wavy curve method,* $x \in (1, \infty) \cup \{0\}$.

Form 2 **The equation of the form**

$$|f_1(x)| + |f_2(x)| + \ldots + |f_n(x)| = g(x) \quad \text{...(i)}$$

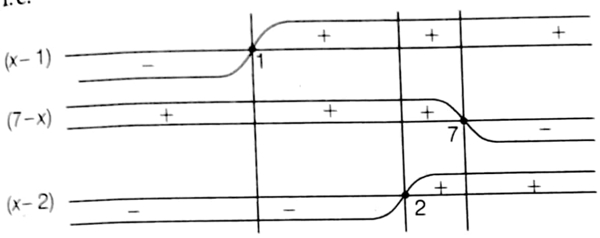
where, $f_1(x), f_2(x), \ldots, f_n(x), g(x)$ are functions of x and $g(x)$ may be constant.

Equations of this form solved by the **method of intervals**. We first find all critical points of $f_1(x), f_2(x), \ldots, f_n(x)$, if coefficient of x is positive, then graph start with positive sign (+) and if coefficient of x is negative, then graph start with negative sign (–). Then, using the definition of the absolute value, we pass from Eq. (i) to a collection of systems which do not contain the absolute value symbols.

| Example 61. Solve the equation
$|x-1|+|7-x|+2|x-2|=4.$

Sol. Here, critical points are 1, 2, 7 using the method of intervals, we find intervals when the expressions $x-1, 7-x$ and $x-2$ are of constant signs.

i.e. $\quad x<1, 1<x<2, 2<x<7, x>7$



Thus, the given equation is equivalent to the collection of four systems,

$$\left[\begin{array}{l}\begin{cases}x<1\\-(x-1)+(7-x)-2(x-2)=4\end{cases}\\ \begin{cases}1\le x<2\\(x-1)+(7-x)-2(x-2)=4\end{cases}\\ \begin{cases}2\le x<7\\(x-1)+(7-x)+2(x-2)=4\end{cases}\\ \begin{cases}x\ge 7\\(x-1)-(7-x)+2(x-2)=4\end{cases}\end{array}\right. \Rightarrow \left[\begin{array}{l}\begin{cases}x<1\\x=2\end{cases}\\ \begin{cases}1\le x<2\\x=3\end{cases}\\ \begin{cases}2\le x<7\\x=1\end{cases}\\ \begin{cases}x\ge 7\\x=4\end{cases}\end{array}\right.$$

From the collection of four systems, the given equation has no solution.

Inequations Containing Absolute Values

By definition, $\quad |x|<a \Rightarrow -a<x<a\,(a>0)$

$|x|\le a \Rightarrow -a\le x\le a$

$|x|>a \Rightarrow x<-a$ and $x>a$

and $\quad |x|\ge a \Rightarrow x\le -a$ and $x\ge a$.

| Example 62. Solve the inequation $\left|1-\dfrac{|x|}{1+|x|}\right|\ge\dfrac{1}{2}$.

Sol. The given inequation is equivalent to the collection of systems

$$\left[\begin{array}{l}\left|1-\dfrac{x}{1+x}\right|\ge\dfrac12, \text{ if } x\ge 0\\ \left|1+\dfrac{x}{1-x}\right|\ge\dfrac12, \text{ if } x<0\end{array}\right\} \Rightarrow \left[\begin{array}{l}\dfrac{1}{|1+x|}\ge\dfrac12, \text{ if } x\ge 0\\ \dfrac{1}{|1-x|}\ge\dfrac12, \text{ if } x<0\end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l}\dfrac{1}{1+x}\ge\dfrac12, \text{ if } x\ge 0\\ \dfrac{1}{1-x}\ge\dfrac12, \text{ if } x<0\end{array}\right. \Rightarrow \left\{\begin{array}{l}\dfrac{1-x}{1+x}\ge 0, \text{ if } x\ge 0\\ \dfrac{1+x}{1-x}\ge 0, \text{ if } x<0\end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l}\dfrac{x-1}{x+1}\le 0, \text{ if } x\ge 0\\ \dfrac{x+1}{x-1}\le 0, \text{ if } x<0\end{array}\right.$$

For $\quad \dfrac{x-1}{x+1}\le 0$, if $x\ge 0$

$\therefore \quad 0\le x\le 1 \quad$ ...(i)

For $\quad \dfrac{x+1}{x-1}\le 0$, if $x<0$

$\therefore \quad -1\le x<0 \quad$ ...(ii)

Hence, from Eqs. (i) and (ii), the solution of the given equation is $x\in[-1,1]$.

Aliter

$$\left|1-\frac{|x|}{1+|x|}\right|\ge\frac12 \Rightarrow \left|\frac{1}{1+|x|}\right|\ge\frac12$$

$$\Rightarrow \quad \frac{1}{1+|x|}\ge\frac12 \Rightarrow 1+|x|\le 2 \text{ or } |x|\le 1$$

$$\therefore \quad -1\le x\le 1 \text{ or } x\Rightarrow[-1,1]$$

Equations Involving Greatest Integer, Least Integer and Fractional Part

1. Greatest Integer

$[x]$ denotes the greatest integer less than or equal to x i.e., $[x]\le x$. It is also known as **floor** of x.

Thus, $\quad [3.5779]=3, [0.89]=0, [3]=3$

$[-8.7285]=-9$

$[-0.6]=-1$

$[-7]=-7$

In general, if n is an integer and x is any real number between n and $n+1$.

i.e. $\quad n\le x<n+1$, then $[x]=n$

Properties of Greatest Integer

(i) $[x\pm n]=[x]\pm n, n\in I$

(ii) $[-x]=-[x], x\in I$

(iii) $[-x]=-1-[x], x\notin I$

(iv) $[x]-[-x]=2n$, if $x=n, n\in I$

(v) $[x]-[-x]=2n+1$, if $x=n+\{x\}, n\in I$ and $0<\{x\}<1$

(vi) $[x]\ge n \Rightarrow x\ge n, n\in I$

(vii) $[x]>n \Rightarrow x\ge n+1, n\in I$

(viii) $[x]\le n \Rightarrow x<n+1, n\in I$

(ix) $[x]<n \Rightarrow x<n, n\in I$

(x) $n_2\le[x]\le n_1 \Rightarrow n_2\le x<n_1+1, n_1, n_2\in I$

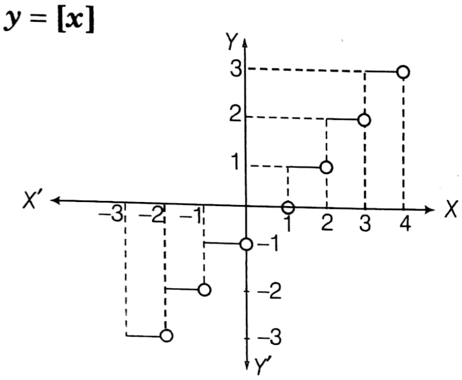
(xi) $[x+y]\ge[x]+[y]$

(xii) $\left[\dfrac{[x]}{n}\right]=\left[\dfrac{x}{n}\right], n\in N$

(xiii) $\left[\dfrac{n+1}{2}\right]+\left[\dfrac{n+2}{4}\right]+\left[\dfrac{n+4}{8}\right]+\left[\dfrac{n+8}{16}\right]+\ldots=n, n\in N$

(xiv) $[x]+\left[x+\dfrac{1}{n}\right]+\left[x+\dfrac{2}{n}\right]+\ldots+\left[x+\dfrac{n-1}{n}\right]=[nx],$
$n\in N$

Graph of $y=[x]$



Remark

Domain and Range of $[x]$ are R and I, respectively.

Example 63. If $[x]$ denotes the integral part of x for real x, then find the value of

$$\left[\frac{1}{4}\right]+\left[\frac{1}{4}+\frac{1}{200}\right]+\left[\frac{1}{4}+\frac{2}{200}\right]+\left[\frac{1}{4}+\frac{3}{200}\right]+\ldots+\left[\frac{1}{4}+\frac{199}{200}\right].$$

Sol. The given expression can be written as

$$\left[\frac{1}{4}\right]+\left[\frac{1}{4}+\frac{1}{200}\right]+\left[\frac{1}{4}+\frac{2}{200}\right]+\left[\frac{1}{4}+\frac{3}{200}\right]+\ldots+\left[\frac{1}{4}+\frac{199}{200}\right]$$

$$=\left[200\cdot\frac{1}{4}\right]=[50]=50 \qquad \text{[from property (xiv)]}$$

Example 64. Let $[a]$ denotes the larger integer not exceeding the real number a. If x and y satisfy the equations $y=2[x]+3$ and $y=3[x-2]$ simulaneously, determine $[x+y]$.

Sol. We have, $\quad y=2[x]+3=3[x-2] \qquad$ ...(i)

$\Rightarrow \quad 2[x]+3=3([x]-2) \qquad$ [from property (i)]

$\Rightarrow \quad 2[x]+3=3[x]-6$

$\Rightarrow \quad [x]=9$

From Eq. (i), $y=2\times 9+3=21$

$\therefore \quad [x+y]=[x+21]=[x]+21=9+21=30$

Hence, the value of $[x+y]$ is 30.

2. Least Integer

(x) or $\lceil x\rceil$ denotes the least integer greater than or equal to x i.e., $(x)\geq x$ or $\lceil x\rceil \geq x$. It is also known as **ceilling** of x.

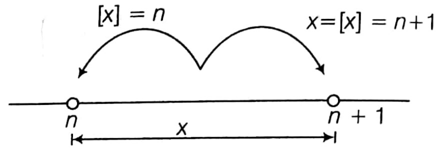
Thus, $\quad (3.578)=4, (0.87)=1,$

$(4)=4$

$\lceil -8.239\rceil=-8, \lceil -0.7\rceil=0$

In general, if n is an integer and x is any real number between n and $n+1$

i.e., $\quad n<x\leq n+1$, then $(x)=n+1$



Relation between Greatest Integer and Least Integer

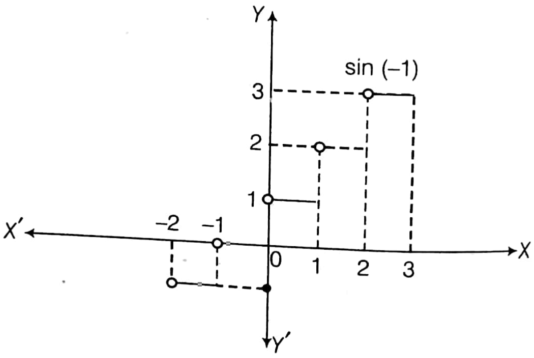
$$(x)=\begin{cases}[x], & x\in I\\ [x]+1, & x\notin I\end{cases}$$

i.e. If $x\in I$, then $x=[x]=(x)$. [remember]

Remark

If $(x)=n$, then $(n-1)<x\leq n$

Graph of $y=(x)=\lceil x\rceil$



Remark

Domain and Range of (x) are R and $[x]+1$, respectively.

Example 65. If $[x]$ and (x) are the integral part of x and nearest integer to x, then solve $(x)[x]=1$.

Sol. ***Case I*** If $x\in I$, then $x=[x]=(x)$

$\therefore$ Given equation convert in $x^2=1$.

$\therefore \quad x=(\pm 1)$

Case II If $x\notin I$, then $(x)=[x]+1$

$\therefore$ Given equation convert in

$$([x]+1)[x]=1 \Rightarrow [x]^2+[x]-1=0$$

or $$[x]=\frac{-1\pm\sqrt{5}}{2} \quad \text{[impossible]}$$

Then, final answer is $x=\pm 1$.

| Example 66. Find the solution set of $(x)^2+(x+1)^2=25$, where (x) is the least integer greater than or equal to x.

Sol. *Case I* If $x \in I$, then $x=(x)=[x]$

Then, $(x)^2+(x+1)^2=25$ reduces to

$$x^2+\overline{x+1}^2=25 \Rightarrow 2x^2+2x-24=0$$

$$\Rightarrow \quad x^2+x-12=0 \Rightarrow (x+4)(x-3)=0$$

$$\therefore \quad x=-4, 3 \quad \text{...(i)}$$

Case II If $x \notin I$, then $(x)=[x]+1$

Then, $(x)^2+(x+1)^2=25$ reduces to

$$\{[x]+1\}^2+\{[x+1]+1\}^2=25$$

$$\Rightarrow \quad \{[x]+1\}^2+\{[x]+2\}^2=25$$

$$\Rightarrow \quad 2[x]^2+6[x]-20=0$$

$$\Rightarrow \quad [x]^2+3[x]-10=0$$

$$\Rightarrow \quad \{[x]+5\}\{[x]-2\}=0$$

$$\therefore \quad [x]=-5 \text{ and } [x]=2$$

$$\Rightarrow \quad x\in[-5,-4)\cup[2,3)$$

$$\because \quad x\notin I,$$

$$\therefore \quad x\in(-5,-4)\cup(2,3) \quad \text{...(ii)}$$

On combining Eqs. (i) and (ii), we get

$$x\in(-5,-4]\cup(2,3]$$

3. Fractional Part

$\{x\}$ denotes the fractional part of x, i.e. $0\le\{x\}<1$.

Thus, $\{2\cdot 7\}=0.7$, $\{5\}=0$, $\{-3.72\}=0.28$

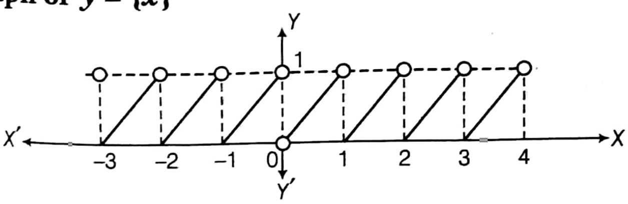
If x is a real number, then $x=[x]+\{x\}$

i.e., $x=n+f$, where $n\in I$ and $0\le f<1$

Properties of Fractional Part of x

(i) $\{x\pm n\}=\{x\}, n\in I$ (ii) If $0\le x<1$, then $\{x\}=x$

Graph of $y=\{x\}$



Remark

1. For proper fraction $0<\{x\}<1$.
2. Domain and range of $\{x\}$ are R and $[0, 1)$, respectively.
3. $\{-5.238\}=\{-5-0.238\}=\{-5-1+1-0.238\}$
$=\{-6+0.762\}=\{\bar{6}.762\}=0.762$

| Example 67. If $\{x\}$ and $[x]$ represent fractional and integral part of x respectively, find the value of $[x]+\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}$.

Sol. $$[x]+\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}=[x]+\sum_{r=1}^{2000}\frac{\{x\}}{2000} \quad \text{[from property (i)]}$$

$$=[x]+\frac{\{x\}}{2000}\sum_{r=1}^{2000}1=[x]+\frac{\{x\}}{2000}\times 2000=[x]+\{x\}=x$$

| Example 68. If $\{x\}$ and $[x]$ represent fractional and integral part of x respectively, then solve the equation $x-1=(x-[x])(x-\{x\})$.

Sol. $\because \quad x=[x]+\{x\}, 0\le\{x\}<1$

Thus, given equation reduces to

$$[x]+\{x\}-1=\{x\}[x]$$

$$\Rightarrow \quad \{x\}[x]-[x]-\{x\}+1=0$$

$$\Rightarrow \quad ([x]-1)(\{x\}-1)=0$$

Now, $$\{x\}-1\ne 0 \quad [\because 0\le\{x\}<1]$$

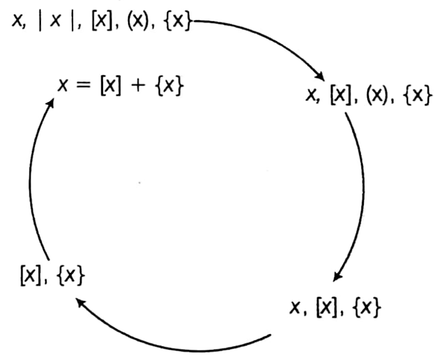
$$\therefore \quad [x]-1=0$$

$$\Rightarrow \quad [x]=1$$

$$\therefore \quad x\in[1,2)$$

Problem Solving Cycle

If a problem has $x, |x|, [x], (x), \{x\}$, then first solve $|x|$, then problem convert in $x, [x], (x), \{x\}$.



Secondly, solve $(x)=\begin{cases}[x], & x\in I\\ [x]+1, & x\notin I\end{cases}$

Then, problem convert in $x, [x], \{x\}$.

Now, put $x=[x]+\{x\}$

Then, problem convert in $[x]$ and $\{x\}$. ...(i)

Since, $0\le\{x\}<1$, then we get $[x]$

From Eq. (i), we get $\{x\}$

Hence, final solution is $x=[x]+\{x\}$.

Example 69. Let $\{x\}$ and $[x]$ denotes the fractional and integral parts of a real number x, respectively. Solve $4\{x\} = x + [x]$.

Sol. $\because \quad x = [x] + \{x\}$ ...(i)

Then, given equation reduces to

$$4\{x\} = [x] + \{x\} + [x]$$

$\Rightarrow \quad \{x\} = \frac{2}{3}[x]$ ...(ii)

$\because \quad 0 \le \{x\} < 1 \Rightarrow 0 \le \frac{2}{3}[x] < 1$ or $0 \le [x] < \frac{3}{2}$

$\therefore \quad [x] = 0, 1$

From Eq. (ii), $\{x\} = 0, \frac{2}{3}$

From Eq. (i), $x = 0, 1 + \frac{2}{3}$ i.e., $x = 0, \frac{5}{3}$

Example 70. Let $\{x\}$ and $[x]$ denotes the fractional and integral part of a real number (x), respectively. Solve $|2x - 1| = 3[x] + 2\{x\}$.

Sol. *Case* **I** $2x - 1 \ge 0$ or $x \ge \frac{1}{2}$

Then, given equation convert to

$2x - 1 = 3[x] + 2\{x\}$ ...(i)

$\because \quad x = [x] + \{x\}$ ...(ii)

From Eqs. (i) and (ii), we get

$$2([x] + \{x\}) - 1 = 3[x] + 2\{x\}$$

$\therefore \quad [x] = -1$

$\therefore \quad -1 \le x < 0$

No solution $\left[\because x \ge \frac{1}{2}\right]$

Case **II** $2x - 1 < 0$ or $x < \frac{1}{2}$

Then, given equation reduces to

$1 - 2x = 3[x] + 2\{x\}$ ...(iii)

$\because \quad x = [x] + \{x\}$ ...(iv)

From Eqs. (iii) and (iv), we get

$$1 - 2([x] + \{x\}) = 3[x] + 2\{x\}$$

$\Rightarrow \quad 1 - 5[x] = 4\{x\}$

$\therefore \quad \{x\} = \frac{1 - 5[x]}{4}$ ...(v)

Now, $\quad 0 \le \{x\} < 1$

$\Rightarrow \quad 0 \le \frac{1 - 5[x]}{4} < 1$

$\Rightarrow \quad 0 \le 1 - 5[x] < 4$

$\Rightarrow \quad 0 \ge -1 + 5[x] > -4$

$\Rightarrow \quad 1 \ge 5[x] > -3$ or $-\frac{3}{5} < [x] \le \frac{1}{5}$

$\therefore \quad [x] = 0$

From Eq. (v), $\{x\} = \frac{1}{4}$

$\therefore \quad x = 0 + \frac{1}{4} = \frac{1}{4}$

Example 71. Solve the equation

$$(x)^2 = [x]^2 + 2x$$

where, $[x]$ and (x) are integers just less than or equal to x and just greater than or equal to x, respectively.

Sol. *Case* **I** If $x \in I$ then

$$x = [x] = (x)$$

The given equation reduces to

$$x^2 = x^2 + 2x$$

$\Rightarrow \quad 2x = 0$ or $x = 0$ ...(i)

Case **II** If $x \notin I$, then $(x) = [x] + 1$

The given equation reduces to

$$([x] + 1)^2 = [x]^2 + 2x$$

$\Rightarrow \quad 1 = 2(x - [x])$ or $\{x\} = \frac{1}{2}$

$\therefore \quad x = [x] + \frac{1}{2} = n + \frac{1}{2}, n \in I$ ...(ii)

Hence, the solution of the original equation is $x = 0, n + \frac{1}{2}$, $n \in I$.

Example 72. Solve the system of equations in x, y and z satisfying the following equations:

must

$$x + [y] + \{z\} = 3 \cdot 1$$
$$\{x\} + y + [z] = 4 \cdot 3$$
$$[x] + \{y\} + z = 5 \cdot 4$$

where, $[\cdot]$ and $\{\cdot\}$ denotes the greatest integer and fractional parts, respectively.

Sol. $\because [x] + \{x\} = x$, $[y] + \{y\} = y$ and $[z] + \{z\} = z$,

On adding all the three equations, we get

$$2(x + y + z) = 12.8$$

$\Rightarrow \quad x + y + z = 6.4$ ...(i)

Now, adding first two equations, we get

$$x + y + z + [y] + \{x\} = 7.4$$

$\Rightarrow \quad 6.4 + [y] + \{x\} = 7.4$ [from Eq. (i)]

$\Rightarrow \quad \boxed{[y] + \{x\} = 1}$

$\therefore \quad [y] = 1$ and $\{x\} = 0$ ...(ii)

On adding last two equations, we get

$$x + y + z + \{y\} + [z] = 9.7$$

$$\{y\} + [z] = 3.3$$ [from Eq. (ii)]

$\therefore \quad [z] = 3$ and $\{y\} = 0.3$ ...(iii)

On adding first and last equations, we get

$$x + y + z + [x] + \{z\} = 8.5$$

$\Rightarrow \quad [x] + \{z\} = 2.1$ [from Eq. (i)]

$\therefore \quad [x] = 2, \{z\} = 0.1$ ...(iv)

From Eqs. (i), (ii) and (iii), we get

$$x = [x] + \{x\} = 2 + 0 = 2$$
$$y = [y] + \{y\} = 1 + 0.3 = 1.3$$

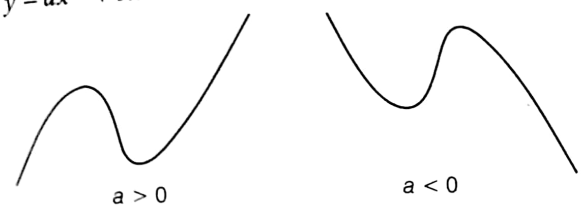
and $\quad z = [z] + \{z\} = 3 + 0.1 = 3.1$

Roots of Equation with the Help of Graphs

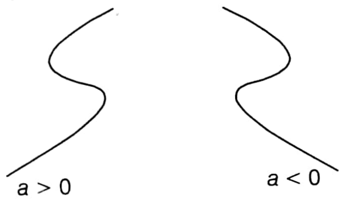
Here, we will discuss some examples to find the roots of equations with the help of graphs.

Important Graphs

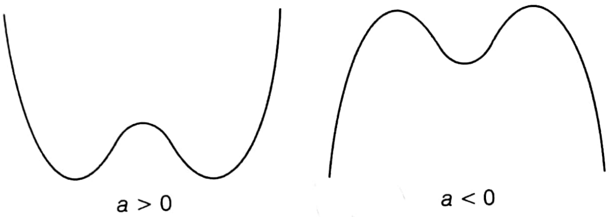
1. $y = ax^3 + bx^2 + cx + d$



2. $x = ay^3 + by^2 + cy + d$



3. $y = ax^4 + bx^3 + cx^2 + dx + e$



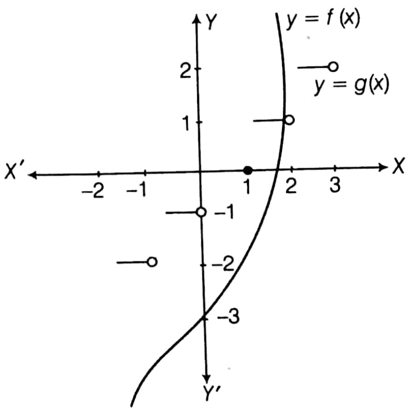
| Example 73. Solve the equation $x^3 - [x] = 3$, where $[x]$ denotes the greatest integer less than or equal to x.

Sol. We have, $x^3 - [x] = 3$

$\Rightarrow \quad x^3 - 3 = [x]$

Let $\quad f(x) = x^3 - 3$ and $g(x) = [x]$.

It is clear from the graphs, the point of intersection of two curves $y = f(x)$ and $y = g(x)$ lies between $(1, 0)$ and $(2, 0)$.



$\therefore \quad 1 < x < 2$

We have, $f(x) = x^3 - 3$ and $g(x) = 1$

or $\quad x^3 - 3 = 1 \Rightarrow x^3 = 4$

$\therefore \quad x = (4)^{1/3}$

Hence, $x = 4^{1/3}$ is the solution of the equation $x^3 - [x] = 3$.

Aliter

$\because \quad x = [x] + f, 0 \le f < 1,$

Then, given equation reduces to

$$x^3 - (x - f) = 3 \Rightarrow x^3 - x = 3 - f$$

Hence, it follows that

$$2 < x^3 - x \le 3$$

$$\Rightarrow \quad 2 < x(x + 1)(x - 1) \le 3$$

Further for $x \ge 2$, we have $x(x + 1)(x - 1) \ge 6 > 3$

For $x < -1$, we have $x(x + 1)(x - 1) < 0 < 2$

For $x = -1$, we have $x(x + 1)(x - 1) = 0 < 2$

For $-1 < x \le 0$, we have $x(x + 1)(x - 1) \le -x < 1$

and for $0 < x \le 1$, we have $x(x + 1)(x - 1) < x < x^3 \le 1$

Therefore, x must be $1 < x < 2$

$\therefore \quad [x] = 1$

Now, the original equation can be written as

$$x^3 - 1 = 3 \Rightarrow x^3 = 4$$

Hence, $x = 4^{1/3}$ is the solution of the given equation.

must

| Example 74. Solve the equation $x^3 - 3x - a = 0$ for different values of a.

Sol. We have, $x^3 - 3x - a = 0 \Rightarrow x^3 - 3x = a$

Let $f(x) = x^3 - 3x$ and $g(x) = a$

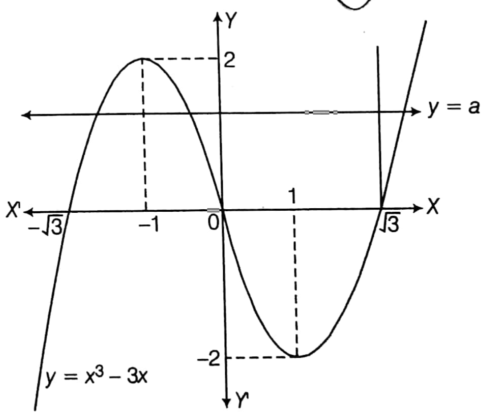
$\because \quad f'(x) = 0$

$\Rightarrow \quad 3x^2 - 3 = 0$

$\Rightarrow \quad x = -1, 1$

$\quad f''(x) = 6x$

this method



$\therefore \quad f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$

$\therefore f(x)$ local maximum at $x = (-1)$ and local minimum at $x = 1$ and $f(-1) = 2$ and $f(1) = -2$ and $y = g(x) = a$ is a straight line parallel to X-axis.

Following cases arise

Case I When $a > 2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at only one point, so $x^3 - 3x - a = 0$ has only one real root.

Case II When $a = 2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at two points, so $x^3 - 3x - a = 0$ has three real roots, two are equal and one different.

Case III When $-2 < a < 2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at three points, so $x^3 - 3x - a = 0$ has three distinct real roots.

Case IV When $a = -2$,

In this case $y = f(x)$ and $y = g(x)$ touch at one point and intersect at other point, so $x^3 - 3x - a = 0$ has three real roots, two are equal and one different.

Case V When $a < -2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at only one point, so $x^3 - 3x - a = 0$ has only one real root.

| Example 75. Show that the equation $x^3 + 2x^2 + x + 5 = 0$ has only one real root, such that $[\alpha] = -3$, where $[x]$ denotes the integral point of x.

Sol. We have, $x^3 + 2x^2 + x + 5 = 0$

$$\Rightarrow \quad x^3 + 2x^2 + x = -5$$

Let $\quad f(x) = x^3 + 2x^2 + x$ and $g(x) = -5$

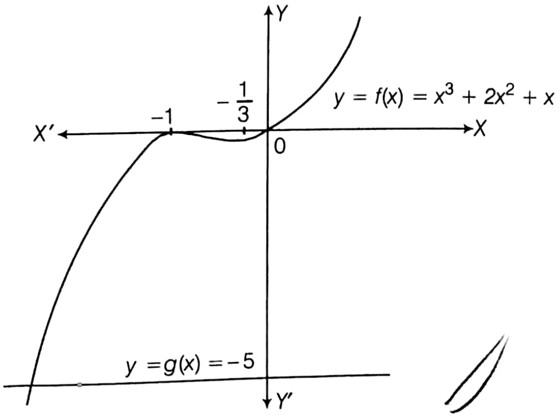
$\because \quad f'(x) = 0 \Rightarrow 3x^2 + 4x + 1 = 0$

$\Rightarrow \quad x = -1, -\dfrac{1}{3}$ and $f''(x) = 6x + 4$

$\therefore \quad f''(-1) = -2 < 0$ and $f''\left(-\dfrac{1}{3}\right) = -2 + 4 = 2 > 0$

$\therefore f(x)$ local maximum at $x = -1$ and local minimum at $x = -\dfrac{1}{3}$

and $\quad f(-1) = 0, f\left(-\dfrac{1}{3}\right) = -\dfrac{4}{27}$



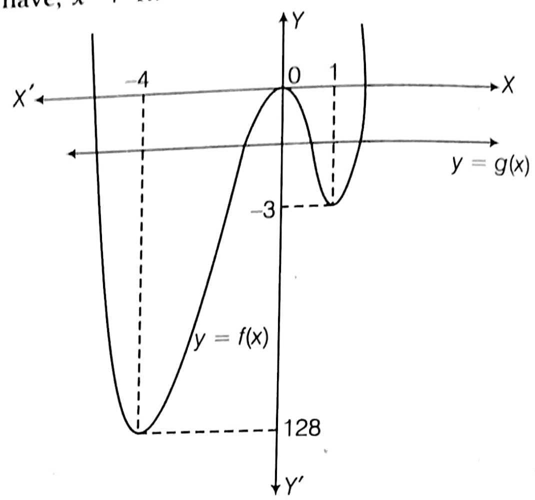
and $\quad f(-2) = -2$ and $f(-3) = -12$

Therefore, x must lie between (-3) and (-2).

i.e. $\quad -3 < \alpha < -2 \Rightarrow [\alpha] = -3$

| Example 76. Find all values of the parameter k for which all the roots of the equation $x^4 + 4x^3 - 8x^2 + k = 0$ are real.

Sol. We have, $x^4 + 4x^3 - 8x^2 + k = 0$



$$\Rightarrow \quad x^4 + 4x^3 - 8x^2 = -k$$

Let $\quad f(x) = x^4 + 4x^3 - 8x^2$ and $g(x) = -k$

$\because \quad f'(x) = 0$

$\Rightarrow \quad 4x^3 + 12x^2 - 16x = 0 \Rightarrow x = -4, 0, 1$

and $\quad f''(x) = 12x^2 + 24x - 16$

$\therefore \quad f''(-4) = 80, f''(0) = -16, f''(1) = 20$

$\therefore f(x)$ has local minimum at $x = -4$ and $x = 1$ and local maximum at $x = 0$

and $\quad f(-4) = -128, f(0) = 0, f(1) = -3.$

Following cases arise

Case I When $-k > 0$ i.e., $k < 0$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = (-k)$ intersect at two points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two real roots.

Case II When $-k = 0$ and $-k = -3$, i.e. $k = 0, 3$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at four points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two distinct real roots and two equal roots.

Case III When $-3 < -k < 0$, i.e. $0 < k < 3$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at four distinct points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has four distinct real roots.

Case IV When $-128 < -k < -3$, i.e. $3 < k < 128$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at two distinct points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two distinct real roots.

Case V When $-k = -128$ i.e., $k = 128$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ touch at one point, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two real and equal roots.

Case VI When $-k < -128$, i.e. $k > 128$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ do not intersect, so there is no real root.

Example 77. Let $-1 \le p \le 1$, show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it.

Sol. We have, $4x^3 - 3x - p = 0$

$\Rightarrow \quad 4x^3 - 3x = p$

Let $\quad f(x) = 4x^3 - 3x$ and $g(x) = p$

$\therefore \quad f'(x) = 0$

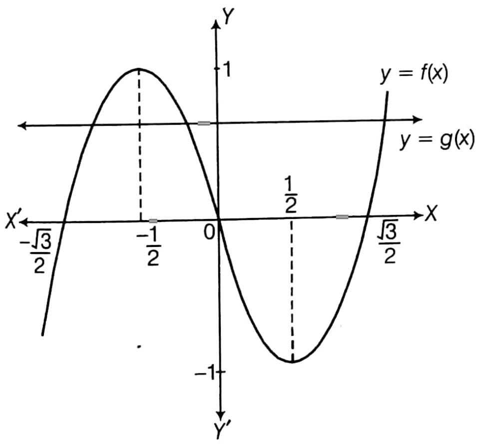
$\Rightarrow \quad 12x^2 - 3 = 0$

$\Rightarrow \quad x = -\frac{1}{2}, \frac{1}{2}$ and $f''(x) = 24x$

$\therefore \quad f''\left(-\frac{1}{2}\right) = -12 < 0$ and $f''\left(\frac{1}{2}\right) = 12 > 0$

$\therefore f(x)$ has local maximum at $\left(x = -\frac{1}{2}\right)$ and local minimum at $\left(x = \frac{1}{2}\right)$.

Also, $f\left(-\frac{1}{2}\right) = -\frac{4}{8} + \frac{3}{2} = 1$ and $f\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{3}{2} = -1$



We observe that, the line $y = g(x) = p$, where $-1 \le p \le 1$ intersect the curve $y = f(x)$ exactly at point $\alpha \in \left[\frac{1}{2}, 1\right]$.

Hence, $4x^3 - 3x - p = 0$ has exactly one root in the interval $\left[\frac{1}{2}, 1\right]$.

Now, we have to find the value of root α.

Let $\alpha = \cos\theta$, then $4\cos^3\theta - 3\cos\theta - p = 0$

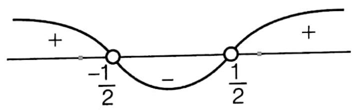
$\Rightarrow \quad \cos 3\theta = p \Rightarrow 3\theta = \cos^{-1}(p)$ or $\theta = \frac{1}{3}\cos^{-1}(p)$

$\therefore \quad \alpha = \cos\theta = \cos\left\{\frac{1}{3}\cos^{-1}(p)\right\}$

Aliter

Let $\quad \phi(x) = 4x^3 - 3x - p$

$\therefore \quad \phi'(x) = 12x^2 - 3 = 12\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$



Clearly, $\phi'(x) > 0$ for $x \in \left[\frac{1}{2}, 1\right]$.

Hence, $\phi(x)$ can have atmost one root in $\left[\frac{1}{2}, 1\right]$.

Also, $\phi\left(\frac{1}{2}\right) = -1 - p$ and $\phi(1) = 1 - p$

$\therefore \quad \phi\left(\frac{1}{2}\right)\phi(1) = -(1 - p^2) = (p^2 - 1) \le 0 \qquad [\because -1 \le p \le 1]$

Since, $\phi(x)$ being a polynomial, continuous on $\left[\frac{1}{2}, 1\right]$ and $\phi\left(\frac{1}{2}\right)\phi(1) \le 0$. Therefore, by intermediate value theorem $\phi(x)$ has atleast one root in $\left[\frac{1}{2}, 1\right]$. ?

Hence, $\phi(x)$ has exactly one root in $\left[\frac{1}{2}, 1\right]$.

c, c, c, d, a, d, d, c, b, d

Exercise for Session 4

1. If α, β, γ are the roots of $x^3 - x^2 - 1 = 0$, the value of $\sum\left(\frac{1+\alpha}{1-\alpha}\right)$, is equal to

(a) -7 (b) -6
(c) -5 (d) -4

2. If r, s, t are the roots of the equation $8x^3 + 1001x + 2008 = 0$. The value of $(r+s)^3 + (s+t)^3 + (t+r)^3$ is

(a) 751 (b) 752
(c) 753 (d) 754

3. If $\alpha, \beta, \gamma, \delta$ are the roots of equation $x^4 + 4x^3 - 6x^2 + 7x - 9 = 0$, the value of $\Pi(1+\alpha^2)$ is

(a) 9 (b) 11
(c) 13 (d) 15

4. If a, b, c, d are four consecutive terms of an increasing AP, the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are

(a) non-real complex (b) real and equal
(c) integers (d) real and distinct

5. If $x^2 + px + 1$ is a factor of the expression $ax^3 + bx + c$ then

(a) $a^2 - c^2 = ab$ (b) $a^2 + c^2 = -ab$
(c) $a^2 - c^2 = -ab$ (d) None of these

6. The number of real roots of the equation $x^2 - 3|x| + 2 = 0$ is

(a) 1 (b) 2
(c) 3 (d) 4

7. Let $a \neq 0$ and $p(x)$ be a polynomial of degree greater than 2, if $p(x)$ leaves remainder a and $(-a)$ when divided respectively by $x + a$ and $x - a$, the remainder when $p(x)$ is divided by $x^2 - a^2$, is

(a) $2x$ (b) $-2x$
(c) x (d) $-x$

8. The product of all the solutions of the equation $(x-2)^2 - 3|x-2| + 2 = 0$ is

(a) 2 (b) -4
(c) 0 (d) None of these

9. If $0 < x < 1000$ and $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x$, where $[x]$ is the greatest integer less than or equal to x, the number of possible values of x is

(a) 32 (b) 33
(c) 34 (d) None of these

10. If $[x]$ is the greatest integer less than or equal to x and (x) be the least integer greater than or equal to x and $[x]^2 + (x)^2 > 25$ then x belongs to

(a) $[3, 4]$ (b) $(-\infty, -4]$
(c) $[4, \infty)$ (d) $(-\infty, -4] \cup [4, \infty)$

Session 5

Irrational Equations, Irrational Inequations, Exponential Equations, Exponential Inequations, Logarithmic Equations, Logarithmic Inequations

Irrational Equations

Here, we consider equations of the type which contain the unknown under the radical sign and the value under the radical sign is known as radicand.

- If roots are all even (i.e. $\sqrt{x}$, $\sqrt[4]{x}, \sqrt[6]{x}, \ldots$, etc) of an equation are arithmetic. In other words, if the radicand is negative (i.e. $x < 0$), then the root is imaginary, if the radicand is zero, then the root is also zero and if the radicand is positive, then the value of the root is also positive.
- If roots are all odd (i.e. $\sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x}, \ldots$ etc) of an equation, then it is defined for all real values of the radicand. If the radicand is negative, then the root is negative, if the radicand is zero, then the root is zero and if the radicand is positive, then the root is positive.

Some Standard Formulae to Solve Irrational Equations

If f and g be functions of $x, k \in N$. Then,

1. $\sqrt[2k]{f}\,\sqrt[2k]{g} = \sqrt[2k]{fg}, f \geq 0, g \geq 0$
2. $\sqrt[2k]{f}\,/\,\sqrt[2k]{g} = \sqrt[2k]{(f/g)}, f \geq 0, g > 0$
3. $|f|\sqrt[2k]{g} = \sqrt[2k]{(f^{2k} g)}, g \geq 0$
4. $\sqrt[2k]{(f/g)} = \sqrt[2k]{|f|}\,/\,\sqrt[2k]{|g|}, fg \geq 0, g \neq 0$
5. $\sqrt[2k]{fg} = \sqrt[2k]{|f|}\,\sqrt[2k]{g}, fg \geq 0$

| Example 78. Prove that the following equations has no solutions.

(i) $\sqrt{(2x+7)} + \sqrt{(x+4)} = 0$ (ii) $\sqrt{(x-4)} = -5$

(iii) $\sqrt{(6-x)} - \sqrt{(x-8)} = 2$ (iv) $\sqrt{-2-x} = \sqrt[5]{(x-7)}$

(v) $\sqrt{x} + \sqrt{(x+16)} = 3$ (vi) $7\sqrt{x} + 8\sqrt{-x} + \dfrac{15}{x^3} = 98$

(vii) $\sqrt{(x-3)} - \sqrt{x+9} = \sqrt{(x-1)}$

Sol. (i) We have, $\sqrt{(2x+7)} + \sqrt{(x+4)} = 0$

This equation is defined for $2x + 7 \geq 0$

and $\quad x + 4 \geq 0 \Rightarrow \begin{cases} x \geq -\dfrac{7}{2} \\ x \geq -4 \end{cases}$

$\therefore \quad x \geq -\dfrac{7}{2}$

For $x \geq -\dfrac{7}{2}$, the left hand side of the original equation is positive, but right hand side is zero. Therefore, the equation has no roots.

(ii) We have, $\sqrt{(x-4)} = -5$

The equation is defined for $x - 4 \geq 0$

$\therefore \quad x \geq 4$

For $x \geq 4$, the left hand side of the original equation is positive, but right hand side is negative.

Therefore, the equation has no roots.

(iii) We have, $\sqrt{(6-x)} - \sqrt{x-8} = 2$

The equation is defined for

$$6 - x \geq 0 \text{ and } x - 8 \geq 0$$

$\therefore \quad \begin{cases} x \leq 6 \\ x \geq 8 \end{cases}$

Consequently, there is no x for which both expressions would have sense. Therefore, the equation has no roots.

(iv) We have, $\sqrt{(-2-x)} = \sqrt[5]{(x-7)}$

This equation is defined for

$-2 - x \geq 0 \quad \Rightarrow \quad x \leq -2$

For $x \leq -2$ the left hand side is positive, but right hand side is negative.

Therefore, the equation has no roots.

(v) We have, $\sqrt{x} + \sqrt{(x+16)} = 3$

The equation is defined for

$$x \geq 0 \text{ and } x + 16 \geq 0 \Rightarrow \begin{cases} x \geq 0 \\ x \geq -16 \end{cases}$$

Hence, $\quad x \geq 0$

For $x \geq 0$ the left hand side ≥ 4, but right hand side is 3. Therefore, the equation has no roots.

(vi) We have, $7\sqrt{x} + 8\sqrt{-x} + \dfrac{15}{x^3} = 98$

For $x < 0$, the expression $7\sqrt{x}$ is meaningless,

For $x > 0$, the expression $8\sqrt{-x}$ is meaningless

and for $x = 0$, the expression $\frac{15}{x^3}$ is meaningless.

Consequently, the left hand side of the original equation is meaningless for any $x \in R$. Therefore, the equation has no roots.

(vii) We have, $\sqrt{(x-3)} - \sqrt{(x+9)} = \sqrt{x-1}$

This equation is defined for

$$\begin{cases} x - 3 \ge 0 \\ x + 9 \ge 0 \\ x - 1 \ge 0 \end{cases} \Rightarrow \begin{cases} x \ge 3 \\ x \ge -9 \\ x \ge 1 \end{cases}$$

Hence, $x \ge 3$

For $x \ge 3$, $\sqrt{x-3} < \sqrt{x+9}$ i.e. $\sqrt{(x-3)} - \sqrt{(x+9)} < 0$

Hence, for $x \ge 3$, the left hand side of the original equation is negative and right hand side is positive. Therefore, the equation has no roots.

Some Standard Forms to Solve Irrational Equations

Form 1 An equation of the form

$f^{2n}(x) = g^{2n}(x), n \in N$ is equivalent to $f(x) = g(x)$.

Then, find the roots of this equation. If root of this equation satisfies the original equation, then its root of the original equation, otherwise, we say that this root is its **extraneous root**.

Remark

Squaring an Equation May Give Extraneous Roots

Squaring should be avoided as for as possible. If squaring is necessary, then the roots found after squaring must be checked whether they satisfy the original equation or not. If some values of x which do not satisfy the original equation. These values of x are called extraneous roots and are rejected.

Example 79. Solve the equation $\sqrt{x} = x - 2$.

Sol. We have, $\sqrt{x} = x - 2$

On squaring both sides, we obtain

$$x = (x-2)^2$$

$\Rightarrow \quad x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) = 0$

$\therefore \quad x_1 = 1$ and $x_2 = 4$

Hence, $x_1 = 4$ satisfies the original equation, but $x_2 = 1$ does not satisfy the original equation.

$\therefore \quad x_2 = 1$ is the extraneous root.

Example 80. Solve the equation

$$3\sqrt{(x+3)} - \sqrt{(x-2)} = 7.$$

Sol. We have, $3\sqrt{(x+3)} - \sqrt{x-2} = 7$

$\Rightarrow \quad 3\sqrt{(x+3)} = 7 + \sqrt{(x-2)}$

On squaring both sides of the equation, we obtain

$$9x + 27 = 49 + x - 2 + 14\sqrt{x-2}$$

$\Rightarrow \quad 8x - 20 = 14\sqrt{(x-2)}$

$$(4x - 10) = 7\sqrt{x-2}$$

Again, squaring both sides, we obtain

$$16x^2 + 100 - 80x = 49x - 98$$

$\Rightarrow \quad 16x^2 - 129x + 198 = 0$

$\Rightarrow \quad (x-6)\left(x - \frac{33}{16}\right) = 0$

$$x_1 = 6 \text{ and } x_2 = \frac{33}{16}$$

Hence, $x_1 = 6$ satisfies the original equation, but $x_2 = \frac{33}{16}$ does not satisfy the original equation.

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Form 2 An equation in the form
$\sqrt[2n]{f(x)} = g(x), n \in N$

is equivalent to the system $\boxed{\begin{cases} g(x) \ge 0 \\ f(x) = g^{2n}(x) \end{cases}}$

Example 81. Solve the equation

$$\sqrt{(6 - 4x - x^2)} = x + 4.$$

do not remember do by traditional method

Sol. We have, $\sqrt{(6 - 4x - x^2)} = x + 4$

This equation is equivalent to the system

$$\begin{cases} x + 4 \ge 0 \\ 6 - 4x - x^2 = (x+4)^2 \end{cases}$$

$$\Rightarrow \begin{cases} x \ge -4 \\ x^2 + 6x + 5 = 0 \end{cases}$$

On solving the equation $x^2 + 6x + 5 = 0$

We find that, $x_1 = (-1)$ and $x_2 = (-5)$ only $x_1 = (-1)$ satisfies the condition $x \ge -4$.

Consequently, the number -1 is the only solution of the given equation.

Form 3 An equation in the form

$$\sqrt[3]{f(x)} + \sqrt[3]{g(x)} = h(x) \quad \text{...(i)}$$

where $f(x)$, $g(x)$ are the functions of x, but $h(x)$ is a function of x or constant, can be solved as follows cubing both sides of the equation, we obtain

$$f(x) + g(x) + 3\sqrt[3]{f(x)\,g(x)}\,(\sqrt[3]{f(x)} + \sqrt[3]{g(x)}) = h^3(x)$$

$$\Rightarrow \quad f(x) + g(x) + 3\sqrt[3]{f(x)\,g(x)}\,(h(x)) = h^3(x)$$

[from Eq. (i)]

For $x < 0$, the expression $7\sqrt{x}$ is meaningless,

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$$\Rightarrow \quad (x-6)\left(x - \frac{33}{16}\right) = 0$$

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Hence, $x_1 = 6$ satisfies the original equation, but $x_2 = \frac{33}{16}$ does not satisfy the original equation.

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where $f(x), g(x)$ are the functions of x, but $h(x)$ is a function of x or constant, can be solved as follows cubing both sides of the equation, we obtain

$$f(x) + g(x) + 3\sqrt[3]{f(x)\,g(x)}\,(\sqrt[3]{f(x)} + \sqrt[3]{g(x)}) = h^3(x)$$

$$\Rightarrow \quad f(x) + g(x) + 3\sqrt[3]{f(x)\,g(x)}\,(h(x)) = h^3(x)$$

[from Eq. (i)]

We find its roots and then substituting, then into the original equation, we choose those which are the roots of the original equation.

| Example 82. Solve the equation

$$\sqrt[3]{(2x-1)}+\sqrt[3]{(x-1)}=1.$$

Sol. We have, $\sqrt[3]{(2x-1)}+\sqrt[3]{(x-1)}=1$ ...(i)

Cubing both sides of Eq. (i), we obtain

$$2x-1+x-1+3\cdot\sqrt[3]{(2x-1)(x-1)}$$
$$(\sqrt[3]{(2x-1)}+\sqrt[3]{(x-1)})=1$$

$\Rightarrow$ $3x-2+3\cdot\sqrt[3]{(2x^2-3x+1)}\,(1)=1$ [from Eq. (i)]

$\Rightarrow$ $3\cdot\sqrt[3]{(2x^2-3x+1)}=3-3x$

$\Rightarrow$ $\sqrt[3]{(2x^2-3x+1)}=(1-x)$

Again cubing both sides, we obtain

$$2x^2-3x+1=(1-x)^3$$

$\Rightarrow$ $(2x-1)(x-1)=(1-x)^3$

$\Rightarrow$ $(2x-1)(x-1)=-(x-1)^3$

$\Rightarrow$ $(x-1)\{2x-1+(x-1)^2\}=0$

$\Rightarrow$ $(x-1)(x^2)=0$

$\therefore$ $x_1=0$ and $x_2=1$

$\because x_1=0$ is not satisfies the Eq. (i), then $x_1=0$ is an extraneous root of the Eq. (i), thus $x_2=1$ is the only root of the original equation.

Form 4 **An equation of the form**

$$\sqrt[n]{a-f(x)}+\sqrt[n]{b+f(x)}=g(x).$$

Let $u=\sqrt[n]{a-f(x)},\ v=\sqrt[n]{b+f(x)}$

Then, the given equation reduces to the solution of the system of algebraic equations.

$$\begin{cases} u+v=g(x) \\ u^n+v^n=a+b \end{cases}$$

| Example 83. Solve the equation

$$\sqrt{(2x^2+5x-2)}-\sqrt{2x^2+5x-9}=1.$$

Sol. Let $u=\sqrt{(2x^2+5x-2)}$

and $v=\sqrt{(2x^2+5x-9)}$

$\therefore$ $u^2=2x^2+5x-2$

and $v^2=2x^2+5x-9$

Then, the given equation reduces to the solution of the system of algebraic equations.

$$u-v=1$$
$$u^2-v^2=7$$

$\Rightarrow$ $(u+v)(u-v)=7$

$\Rightarrow$ $u+v=7$ [$\because u-v=1$]

We get, $u=4,\ v=3$

$\therefore$ $\sqrt{2x^2+5x-2}=4$

$\therefore$ $2x^2+5x-18=0$

$\therefore$ $x_1=2$ and $x_2=-9/2$

Both roots satisfies the original equation.

Hence, $x_1=2$ and $x_2=-9/2$ are the roots of the original equation.

Irrational Inequations

We consider, here inequations which contain the unknown under the radical sign.

Some Standard Forms to Solve Irrational Inequations

Form 1 An inequation of the form

$$\sqrt[2n]{f(x)}<\sqrt[2n]{g(x)},\ n\in N$$

is equivalent to the system $\begin{cases} f(x)\ge 0 \\ g(x)>f(x) \end{cases}$

and inequation of the form $\sqrt[2n+1]{f(x)}<\sqrt[2n+1]{g(x)},\ n\in N$ is equivalent to the inequation $f(x)<g(x)$.

| Example 84. Solve the inequation

$$\sqrt[5]{\left[\frac{3}{x+1}+\frac{7}{x+2}\right]}<\sqrt[5]{\frac{6}{x-1}}.$$

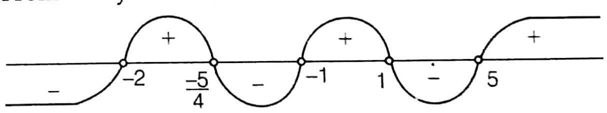
Sol. The given inequation is equivalent to

$$\frac{3}{x+1}+\frac{7}{x+2}<\frac{6}{x-1}$$

$\Rightarrow$ $\dfrac{4x^2-15x-25}{(x+1)(x+2)(x-1)}<0$

$\Rightarrow$ $\dfrac{(x+5/4)(x-5)}{(x+1)(x+2)(x-1)}<0$

From Wavy Curve Method :



$$x\in(-\infty,-2)\cup\left(-\frac{5}{4},1\right)\cup(1,5)$$

Form 2 **An inequation of the form**

$$\sqrt[2n]{f(x)}<g(x),\ n\in N.$$

is equivalent to the system $\begin{cases} f(x)\ge 0 \\ g(x)>0 \\ f(x)<g^{2n}(x), \end{cases}$

and inequation of the form $\sqrt[2n+1]{f(x)} < g(x), n \in N$ is equivalent to the inequation $f(x) < g^{2n+1}(x)$.

Example 85. Solve the inequation $\sqrt{(x+14)} < (x+2)$.

Sol. We have, $\sqrt{(x+14)} < (x+2)$

This inequation is equivalent to the system

$$\begin{cases} x+14 \geq 0 \\ x+2 > 0 \\ x+14 < (x+2)^2 \end{cases} \Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ x^2+3x-10 > 0 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ (x+5)(x-2) > 0 \end{cases} \Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ x < -5 \text{ and } x > 2 \end{cases}$$

On combining all three inequation of the system, we get

$$x > 2, \text{ i.e. } x \in (2, \infty)$$

***Form* 3** An inequation of the form

$$\sqrt[2n]{f(x)} > g(x), n \in N$$

is equivalent to the collection of two systems of inequations

i.e. $\begin{cases} g(x) \geq 0 \\ f(x) > g^{2n}(x) \end{cases}$ and $\begin{cases} g(x) < 0 \\ f(x) \geq 0 \end{cases}$

and inequation of the form $\sqrt[2n+1]{f(x)} > g(x), n \in N$ is equivalent to the inequation $f(x) > g^{2n+1}(x)$.

Example 86. Solve the inequation $\sqrt{(-x^2+4x-3)} > 6-2x$.

Sol. We have, $\sqrt{(-x^2+4x-3)} > 6-2x$

This inequation is equivalent to the collection of two systems, of inequations

i.e. $\begin{cases} 6-2x \geq 0 \\ -x^2+4x-3 > (6-2x)^2 \end{cases}$ and $\begin{cases} 6-2x < 0 \\ -x^2+4x-3 \geq 0 \end{cases}$

$\Rightarrow \begin{cases} x \leq 3 \\ (x-3)(5x-13) < 0 \end{cases}$ and $\begin{cases} x > 3 \\ (x-1)(x-3) \leq 0 \end{cases}$

$\Rightarrow \begin{cases} x \leq 3 \\ \frac{13}{5} < x < 3 \end{cases}$ and $\begin{cases} x > 3 \\ 1 \leq x < 3 \end{cases}$

The second system has no solution and the first system has solution in the interval $\left(\frac{13}{5} < x < 3\right)$.

Hence, $x \in \left(\frac{13}{5}, 3\right)$ is the set of solution of the original inequation.

Exponential Equations

If we have an equation of the form $a^x = b (a > 0)$, then

(i) $x \in \phi$, if $b \leq 0$
(ii) $x = \log_a b$, if $b > 0, a \neq 1$
(iii) $x \in \phi$, if $a = 1, b \neq 1$
(iv) $x \in R$, if $a = 1, b = 1$ (since, $1^x = 1 \Rightarrow 1 = 1, x \in R$)

Example 87. Solve the equation $\sqrt{(6-x)}\,(3^{x^2-7.2x+3.9} - 9\sqrt{3}) = 0$.

Sol. We have,

$$\sqrt{(6-x)}\,(3^{x^2-7.2x+3.9} - 9\sqrt{3}) = 0$$

This equation is defined for

$$6 - x \geq 0 \text{ i.e., } x \leq 6 \quad \text{...(i)}$$

This equation is equivalent to the collection of equations

$$\sqrt{6-x} = 0 \text{ and } 3^{x^2-7.2x+3.9} - 9\sqrt{3} = 0$$

$\therefore$ $x_1 = 6$ and $3^{x^2-7.2x+3.9} = 3^{2.5}$

then $x^2 - 7.2x + 3.9 = 2.5$

$$x^2 - 7.2x + 1.4 = 0$$

We find that, $x_2 = \frac{1}{5}$ and $x_3 = 7$

Hence, solution of the original equation are [which satisfies Eq. (i)]

$$x_1 = 6, x_2 = \frac{1}{5}.$$

Some Standard Forms to Solve Exponential Equations

***Form* 1 An equation in the form $a^{f(x)} = 1, a > 0, a \neq 1$** is equivalent to the equation $f(x) = 0$

Example 88. Solve the equation $5^{x^2+3x+2} = 1$.

Sol. This equation is equivalent to

$$x^2 + 3x + 2 = 0$$

$$\Rightarrow (x+1)(x+2) = 0$$

$\therefore x_1 = -1, x_2 = -2$ consequently, this equation has two roots $x_1 = -1$ and $x_2 = -2$.

***Form* 2** An equation in the form

$$f(a^x) = 0$$

is equivalent to the equation $f(t) = 0$, where $t = a^x$.

If $t_1, t_2, t_3, \ldots, t_k$ are the roots of $f(t) = 0$, then

$$a^x = t_1, a^x = t_2, a^x = t_3, \ldots, a^x = t_k$$

| Example 89. Solve the equation $5^{2x} - 24 \cdot 5^x - 25 = 0$.

Sol. Let $5^x = t$, then the given equation can reduce in the form

$$t^2 - 24t - 25 = 0$$

$$\Rightarrow \quad (t - 25)(t + 1) = 0 \Rightarrow t \neq -1,$$

$$\therefore \quad t = 25,$$

then $$5^x = 25 = 5^2, \text{ then } x = 2$$

Hence, $x_1 = 2$ is only one root of the original equation.

Form **3** An equation of the form

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} = 0,$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \neq 0$ and the bases satisfy the condition $b^2 = ac$ is equivalent to the equation

$$\alpha t^2 + \beta t + \gamma = 0, \text{ where } t = (a / b)^{f(x)}$$

If roots of this equation are t_1 and t_2, then

$$(a / b)^{f(x)} = t_1 \text{ and } (a / b)^{f(x)} = t_2$$

| Example 90. Solve the equation $64 \cdot 9^x - 84 \cdot 12^x + 27 \cdot 16^x = 0$.

Sol. Here, $9 \times 16 = (12)^2$.

Then, we divide its both sides by 12^x and obtain

$$\Rightarrow \quad 64 \cdot \left(\frac{3}{4}\right)^x - 84 + 27 \cdot \left(\frac{4}{3}\right)^x = 0 \quad \text{...(i)}$$

Let $\left(\frac{3}{4}\right)^x = t$, then Eq. (i) reduce in the form

$$64t^2 - 84t + 27 = 0$$

$$\therefore \quad t_1 = \frac{3}{4} \text{ and } t_2 = \frac{9}{16}$$

then, $$\left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^1 \text{ and } \left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^2$$

$$\therefore \quad x_1 = 1 \text{ and } x_2 = 2$$

Hence, roots of the original equation are $x_1 = 1$ and $x_2 = 2$.

Form **4 An equation in the form**

$$\alpha \cdot a^{f(x)} + \beta \cdot b^{f(x)} + c = 0,$$

where $\alpha, \beta, c \in R$ and $\alpha, \beta, c \neq 0$ and $ab = 1$ (a and b are inverse positive numbers) is equivalent to the equation

$$\alpha t^2 + ct + \beta = 0, \text{ where } t = a^{f(x)}.$$

If roots of this equation are t_1 and t_2, then $a^{f(x)} = t_1$ and $a^{f(x)} = t_2$.

| Example 91. Solve the equation $15 \cdot 2^{x+1} + 15 \cdot 2^{2-x} = 135$.

Sol. This equation rewrite in the form

$$30.2^x + \frac{60}{2^x} = 135$$

Let $$t = 2^x,$$

Then, $$30t^2 - 135t + 60 = 0$$

$$\Rightarrow \quad 6t^2 - 27t + 12 = 0$$

$$\Rightarrow \quad 6t^2 - 24t - 3t + 12 = 0$$

$$\Rightarrow \quad (t - 4)(6t - 3) = 0$$

Then, $$t_1 = 4 \text{ and } t_2 = \frac{1}{2}$$

Thus, given equation is equivalent to

$$2^x = 4 \text{ and } 2^x = \frac{1}{2}$$

Then, $$x_1 = 2 \text{ and } x_2 = -1$$

Hence, roots of the original equation are $x_1 = 2$ and $x_2 = -1$.

Form **5 An equation of the form $a^{f(x)} + b^{f(x)} = c$,**

where $a, b, c \in R$ and a, b, c satisfies the condition $a^2 + b^2 = c$, then solution of this equation is $f(x) = 2$ and no other solution of this equation.

| Example 92. Solve the equation $3^{x-4} + 5^{x-4} = 34$.

Sol. Here, $3^2 + 5^2 = 34$, then given equation has a solution $x - 4 = 2$.

$\therefore$ $x_1 = 6$ is a root of the original equation.

Form **6 An equation of the form $\{f(x)\}^{g(x)}$ is equivalent to the equation**

$$\{f(x)\}^{g(x)} = 10^{g(x) \log f(x)},$$

where $f(x) > 0$.

| Example 93. Solve the equation $5^x \sqrt[x]{8^{x-1}} = 500$.

Sol. We have, $$5^x \sqrt[x]{8^{x-1}} = 5^3 \cdot 2^2$$

$$\Rightarrow \quad 5^x \cdot 8^{\left(\frac{x-1}{x}\right)} = 5^3 \cdot 2^2$$

$$\Rightarrow \quad 5^x \cdot 2^{\frac{3x-3}{x}} = 5^3 \cdot 2^2$$

$$\Rightarrow \quad 5^{x-3} \cdot 2^{\left(\frac{x-3}{x}\right)} = 1$$

$$\Rightarrow \quad (5.2^{1/x})^{(x-3)} = 1$$

is equivalent to the equation

$$10^{(x-3)\log(5 \cdot 2^{1/x})} = 1$$

$$\Rightarrow \quad (x - 3)\log(5 \cdot 2^{1/x}) = 0$$

Thus, original equation is equivalent to the collection of equations

$$x - 3 = 0, \ \log(5 \cdot 2^{1/x}) = 0$$

$$\therefore \quad x_1 = 3, 5 \cdot 2^{1/x} = 1 \Rightarrow 2^{1/x} = \left(\frac{1}{5}\right)$$

$$\therefore \quad x_2 = -\log_5 2$$

Hence, roots of the original equation are $x_1 = 3$ and $x_2 = -\log_5 2$.

Exponential Inequations

When we solve exponential inequation $a^{f(x)} > b\ (a > 0)$, we have

(i) $x \in D_f$, if $b \le 0$

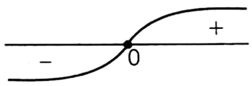
(ii) If $b > 0$, then we have $f(x) > \log_a b$, if $a > 1$ and $f(x) < \log_a b$, if $0 < a < 1$ for $a = 1$, then $b < 1$.

Remark

The inequation $a^{f(x)} \le b$ has no solution for $b \le 0$, $a > 0$, $a \ne 1$.

| Example 94. Solve the inequation $3^{x+2} > \left(\frac{1}{9}\right)^{1/x}$.

Sol. We have, $\quad 3^{x+2} > (3^{-2})^{1/x} \Rightarrow 3^{x+2} > 3^{-2/x}$



Here, base $3 > 1$

$$\Rightarrow \quad x + 2 > -\frac{2}{x} \Rightarrow \frac{x^2 + 2x + 2}{x} > 0$$

$$\Rightarrow \quad \frac{(x+1)^2 + 1}{x} > 0 \Rightarrow x > 0$$

$$\therefore \quad x \in (0, \infty)$$

Some Standard Forms to Solve Exponential Inequations

Form 1 An inequation of the form

$$f(a^x) \ge 0 \text{ or } f(a^x) \le 0$$

is equivalent to the system of collection

$$\begin{cases} t > 0, & \text{where } t = a^x \\ f(t) \ge 0 \text{ or } & f(t) \le 0 \end{cases}$$

| Example 95. Solve the inequation $4^{x+1} - 16^x < 2\log_4 8$.

Sol. Let $4^x = t$, then given inequation reduce in the form

$$4t - t^2 > 2 \cdot \frac{3}{2}$$

$$\Rightarrow \quad t^2 - 4t + 3 < 0 \Rightarrow (t-1)(t-3) < 0$$

$$\Rightarrow \quad 1 < t < 3 \qquad [\because t > 0]$$

$$\Rightarrow \quad 1 < 4^x < 3$$

$$\Rightarrow \quad 0 < x < \log_4 3$$

$$\therefore \quad x \in (0, \log_4 3)$$

Form 2 An inequation of the form

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} \ge 0$$

or

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} \le 0$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \ne 0$ and the bases satisfy the condition $b^2 = ac$ is equivalent to the inequation

$$\alpha t^2 + \beta t + \gamma \ge 0 \text{ or } \alpha t^2 + \beta t + \gamma \le 0,$$

where $t = (a/b)^{f(x)}$.

Form 3 An inequation of the form

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma \ge 0$$

or

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma \le 0$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \ne 0$ and $ab = 1$(a and b are inverse (+ve) numbers) is equivalent to the inequation

$$\alpha t^2 + \beta t + \gamma \ge 0 \text{ or } \alpha t^2 + \beta t + \gamma \le 0$$

where $\quad t = a^{f(x)}$

Form 4 If an inequation of the exponential form reduces to the solution of homogeneous algebraic inequation, i.e.

$$a_0 f^n(x) + a_1 f^{n-1}(x)\, g(x) + a_2 f^{n-2}(x) g^2(x) + \dots + a_{n-1} f(x)\, g^{n-1}(x) + a_n\, g^n(x) \ge 0,$$

where $a_0, a_1, a_2, \dots, a_n$ are constants ($a_0 \ne 0$) and $f(x)$ and $g(x)$ are functions of x.

| Example 96. Solve the inequation $2^{2x^2 - 10x + 3} + 6^{x^2 - 5x + 1} \ge 3^{2x^2 - 10x + 3}$.

Sol. The given inequation is equivalent to

$$8 \cdot 2^{2(x^2 - 5x)} + 6 \cdot 2^{x^2 - 5x} \cdot 3^{x^2 - 5x} - 27 \cdot 3^{2(x^2 - 5x)} \ge 0$$

Let $\quad 2^{x^2 - 5x} = f(x)$ and $3^{x^2 - 5x} = g(x)$,

then $\quad 8 \cdot f^2(x) + 6 f(x) \cdot g(x) - 27 g^2(x) \ge 0$

On dividing in each by $g^2(x)$ $\qquad [\because g(x) > 0]$

Then, $\quad 8\left(\frac{f(x)}{g(x)}\right)^2 + 6\left(\frac{f(x)}{g(x)}\right) - 27 \ge 0$

and let $\quad \frac{f(x)}{g(x)} = t \qquad [\because t > 0]$

then $\quad 8t^2 + 6t - 27 \ge 0$

$$\Rightarrow \quad \left(t - \frac{3}{2}\right)(t + 9/4) \ge 0$$

$$\Rightarrow \quad t \ge 3/2 \text{ and } t \le -9/4$$

The second inequation has no root. $\qquad [\because t > 0]$

From the first inequation, $t > 3/2$

$$\left(\frac{2}{3}\right)^{x^2 - 5x} \ge \left(\frac{2}{3}\right)^{-1} \qquad \left[\because \frac{2}{3} < 1\right]$$

$$\Rightarrow \quad x^2 - 5x \le -1 \Rightarrow x^2 - 5x + 1 \le 0$$

$$\therefore \quad \frac{5 - \sqrt{21}}{2} \le x \le \frac{5 + \sqrt{21}}{2}$$

Hence, $\quad x \in \left[\frac{5 - \sqrt{21}}{2}, \frac{5 + \sqrt{21}}{2}\right]$.

Logarithmic Equations

If we have an equation of the form

$$\log_a f(x) = b, (a > 0), a \neq 1$$

is equivalent to the equation

$$f(x) = a^b \quad (f(x) > 0).$$

| Example 97. Solve the equation $\log_3(5 + 4\log_3(x - 1)) = 2$.

Sol. We have, $\log_3(5 + 4\log_3(x - 1)) = 2$

is equivalent to the equation (here, base $\neq 1, > 0$).

$\therefore \quad 5 + 4\log_3(x - 1) = 3^2$

$\Rightarrow \quad \log_3(x - 1) = 1 \Rightarrow x - 1 = 3^1$

$\therefore \quad x = 4$

Hence, $x_1 = 4$ is the solution of the original equation.

Some Standard Formulae to Solve Logarithmic Equations

f and g are some functions and $a > 0, a \neq 1$, then, if $f > 0, g > 0$, we have

(i) $\log_a(fg) = \log_a f + \log_a g$

(ii) $\log_a(f / g) = \log_a f - \log_a g$

(iii) $\log_a f^{2\alpha} = 2\alpha \log_a |f|$ (iv) $\log_{a^\beta} f^\alpha = \dfrac{\alpha}{\beta} \log_a f$

(v) $\boxed{f^{\log_a g} = g^{\log_a f}}$ (vi) $a^{\log_a f} = f$

| Example 98. Solve the equation $2x^{\log_4 3} + 3^{\log_4 x} = 27$.

Sol. The domain of the admissible values of the equation is $x > 0$. The given equation is equivalent to

$2.3^{\log_4 x} + 3^{\log_4 x} = 27$ [from above result (v)]

$\Rightarrow \quad 3.3^{\log_4 x} = 27$

$\Rightarrow \quad 3^{\log_4 x} = 9$

$\Rightarrow \quad 3^{\log_4 x} = 3^2$

$\Rightarrow \quad \log_4 x = 2$

$\Rightarrow \quad x_1 = 4^2 = 16$ is its only root.

Some Standard Forms to Solve Logarithmic Equations

Form 1 An equation of the form $\log_x a = b, a > 0$ has

(i) Only root $x = a^{1/b}$, if $a \neq 1$ and $b = 0$.

(ii) Any positive root different from unity, if $a = 1$ and $b = 0$.

(iii) No roots, if $a = 1, b \neq 0$.

(iv) No roots, if $a \neq 1, b = 0$.

| Example 99. Solve the equation $\log_{(\log_5 x)} 5 = 2$.

Sol. We have, $\log_{(\log_5 x)} 5 = 2$

Base of logarithm > 0 and $\neq 1$.

$\therefore \quad \log_5 x > 0$ and $\log_5 x \neq 1$

$\Rightarrow \quad x > 1$ and $x \neq 5$

$\therefore$ The original equation is equivalent to

$$\log_5 x = 5^{1/2} = \sqrt{5}$$

$\therefore \quad x_1 = 5^{\sqrt{5}}$

Hence, $5^{\sqrt{5}}$ is the only root of the original equation.

Form 2 Equations of the form

(i) $f(\log_a x) = 0, a > 0, a \neq 1$ and

(ii) $g(\log_x A) = 0, A > 0$

Then, Eq. (i) is equivalent to

$f(t) = 0$, where $t = \log_a x$

If $t_1, t_2, t_3, \ldots, t_k$ are the roots of $f(t) = 0$, then

$\log_a x = t_1, \log_a x = t_2, \ldots, \log_a x = t_k$

and Eq. (ii) is equivalent to $f(y) = 0$, where $y = \log_x A$.

If $y_1, y_2, y_3, \ldots, y_k$ are the roots of $f(y) = 0$, then

$\log_x A = y_1, \log_x A = y_2, \ldots, \log_x A = y_k$

| Example 100. Solve the equation

$$\frac{1 - 2(\log x^2)^2}{\log x - 2(\log x)^2} = 1.$$

Sol. The given equation can rewrite in the form

$$\frac{1 - 2(2\log x)^2}{\log x - 2(\log x)^2} = 1$$

$$\Rightarrow \quad \frac{1 - 8(\log x)^2}{\log x - 2(\log x)^2} - 1 = 0$$

Let $\quad \log x = t,$

then $\quad \dfrac{1 - 8t^2}{t - 2t^2} - 1 = 0 \Rightarrow \dfrac{1 - 8t^2 - t + 2t^2}{t - 2t^2} = 0$

$$\Rightarrow \quad \frac{1 - t - 6t^2}{(t - 2t^2)} = 0 \Rightarrow \frac{(1 + 2t)(1 - 3t)}{t(1 - 2t)} = 0$$

$$\Rightarrow \quad \left\{\begin{array}{l} t = -\dfrac{1}{2} \\ t = \dfrac{1}{3} \end{array}\right. \Rightarrow \left\{\begin{array}{l} \log x = -\dfrac{1}{2} \\ \log x = \dfrac{1}{3} \end{array}\right. \Rightarrow \left\{\begin{array}{l} x_1 = 10^{-1/2} \\ x_2 = 10^{1/3} \end{array}\right.$$

Hence, $x_1 = \dfrac{1}{\sqrt{10}}$ and $x_2 = \sqrt[3]{10}$ are the roots of the original equation.

Example 101. Solve the equation
$\log_x^3 10 - 6\log_x^2 10 + 11\log_x 10 - 6 = 0$.

Sol. Put $\log_x 10 = t$ in the given equation, we get

$$t^3 - 6t^2 + 11t - 6 = 0 \Rightarrow (t-1)(t-2)(t-3) = 0,$$

then $\begin{cases} t = 1 \\ t = 2 \\ t = 3 \end{cases}$

It follows that

$$\begin{cases} \log_x 10 = 1 \\ \log_x 10 = 2 \\ \log_x 10 = 3 \end{cases} \Rightarrow \begin{cases} x = 10 \\ x^2 = 10 \\ x^3 = 10 \end{cases} \Rightarrow \begin{cases} x = 10 \\ x = \sqrt{10} \\ x = \sqrt[3]{10} \end{cases} \quad [\because x > 0 \text{ and} \neq 1]$$

$[\because x > 0 \text{ and} \neq 1]$

$\therefore x_1 = 10$, $x_2 = \sqrt{10}$ and $x_3 = \sqrt[3]{10}$ are the roots of the original equation.

***Form* 3** Equations of the form

(i) $\log_a f(x) = \log_a g(x)$, $a > 0$, $a \neq 1$ is equivalent to two ways.

Method I $\begin{cases} g(x) > 0 \\ f(x) = g(x) \end{cases}$

Method II $\begin{cases} f(x) > 0 \\ f(x) = g(x) \end{cases}$

(ii) $\log_{f(x)} A = \log_{g(x)} A$, $A > 0$ is equivalent to two ways.

Method I $\begin{cases} g(x) > 0 \\ g(x) \neq 1 \\ f(x) = g(x) \end{cases}$

Method II $\begin{cases} f(x) > 0 \\ f(x) \neq 1 \\ f(x) = g(x) \end{cases}$

Example 102. Solve the equation
$\log_{1/3}\left[2\left(\frac{1}{2}\right)^x - 1\right] = \log_{1/3}\left[\left(\frac{1}{4}\right)^x - 4\right]$.

Sol. The given equation is equivalent to

$$\begin{cases} 2\left(\frac{1}{2}\right)^x - 1 > 0 \\ 2\left(\frac{1}{2}\right)^x - 1 = \left(\frac{1}{4}\right)^x - 4 \end{cases}$$

$$\Rightarrow \begin{cases} \left(\frac{1}{2}\right)^x > \frac{1}{2} \\ \left(\frac{1}{2}\right)^{2x} - 2\left(\frac{1}{2}\right)^x - 3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x < 1 \\ \left[\left(\frac{1}{2}\right)^x - 3\right]\left[\left(\frac{1}{2}\right)^x + 1\right] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x < 1 \\ \left(\frac{1}{2}\right)^x = 3, \left(\frac{1}{2}\right)^x + 1 \neq 0 \end{cases} \Rightarrow \begin{cases} x < 1 \\ x = (-\log_2 3) \end{cases}$$

Hence, $x_1 = -\log_2 3$ is the root of the original equation.

Example 103. Solve the equation $\log_{\left(\frac{2+x}{10}\right)} 7 = \log_{\left(\frac{2}{x+1}\right)} 7$.

Sol. The given equation is equivalent to

$$\begin{cases} \frac{2}{x+1} > 0 \\ \frac{2}{x+1} \neq 1 \\ \frac{2+x}{10} = \frac{2}{x+1} \end{cases} \Rightarrow \begin{cases} x + 1 > 0 \\ x \neq 1 \\ x = -6, 3 \end{cases}$$

$\therefore x_1 = 3$ is root of the original equation.

***Form* 4** Equations of the form

(i) $\log_{f(x)} g(x) = \log_{f(x)} h(x)$ is equivalent to two ways.

Method I $\begin{cases} g(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases}$ Method II $\begin{cases} h(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases}$

(ii) $\log_{g(x)} f(x) = \log_{h(x)} f(x)$ is equivalent to two ways.

Method I $\begin{cases} f(x) > 0 \\ g(x) > 0 \\ g(x) \neq 1 \\ g(x) = h(x) \end{cases}$

Method II $\begin{cases} f(x) > 0 \\ h(x) > 0 \\ h(x) \neq 1 \\ g(x) = h(x) \end{cases}$

Example 104. Solve the equation
$\log_{(x^2-1)}(x^3 + 6) = \log_{(x^2-1)}(2x^2 + 5x)$.

Sol. This equation is equivalent to the system

$$\begin{cases} 2x^2 + 5x > 0 \\ x^2 - 1 > 0 \\ x^2 - 1 \neq 1 \\ x^3 + 6 = 2x^2 + 5x \end{cases} \Rightarrow \begin{cases} x < -\frac{5}{2} \text{ and } x > 0 \\ x < -1 \text{ and } x > 1 \\ x \neq \pm\sqrt{2} \\ x = -2, 1, 3 \end{cases}$$

Hence, $x_1 = 3$ is only root of the original equation.

Example 101. Solve the equation
$\log_x^3 10 - 6\log_x^2 10 + 11\log_x 10 - 6 = 0$.

Sol. Put $\log_x 10 = t$ in the given equation, we get

$$t^3 - 6t^2 + 11t - 6 = 0 \Rightarrow (t-1)(t-2)(t-3) = 0,$$

then
$$\begin{cases} t = 1 \\ t = 2 \\ t = 3 \end{cases}$$

It follows that

$$\begin{cases} \log_x 10 = 1 \\ \log_x 10 = 2 \\ \log_x 10 = 3 \end{cases} \Rightarrow \begin{cases} x = 10 \\ x^2 = 10 \\ x^3 = 10 \end{cases} \Rightarrow \begin{cases} x = 10 \\ x = \sqrt{10} \\ x = \sqrt[3]{10} \end{cases} \quad [\because x > 0 \text{ and} \neq 1]$$

$[\because x > 0 \text{ and } \neq 1]$

$\therefore x_1 = 10$, $x_2 = \sqrt{10}$ and $x_3 = \sqrt[3]{10}$ are the roots of the original equation.

***Form* 3** Equations of the form

(i) $\boldsymbol{\log_a f(x) = \log_a g(x), a > 0, a \neq 1}$ is equivalent to two ways.

$$\text{Method I} \begin{cases} g(x) > 0 \\ f(x) = g(x) \end{cases}$$

$$\text{Method II} \begin{cases} f(x) > 0 \\ f(x) = g(x) \end{cases}$$

(ii) $\boldsymbol{\log_{f(x)} A = \log_{g(x)} A, A > 0}$ is equivalent to two ways.

$$\text{Method I} \begin{cases} g(x) > 0 \\ g(x) \neq 1 \\ f(x) = g(x) \end{cases}$$

$$\text{Method II} \begin{cases} f(x) > 0 \\ f(x) \neq 1 \\ f(x) = g(x) \end{cases}$$

Example 102. Solve the equation
$\log_{1/3}\left[2\left(\frac{1}{2}\right)^x - 1\right] = \log_{1/3}\left[\left(\frac{1}{4}\right)^x - 4\right]$.

Sol. The given equation is equivalent to

$$\begin{cases} 2\left(\frac{1}{2}\right)^x - 1 > 0 \\ 2\left(\frac{1}{2}\right)^x - 1 = \left(\frac{1}{4}\right)^x - 4 \end{cases}$$

$$\Rightarrow \begin{cases} \left(\frac{1}{2}\right)^x > \frac{1}{2} \\ \left(\frac{1}{2}\right)^{2x} - 2\left(\frac{1}{2}\right)^x - 3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x < 1 \\ \left[\left(\frac{1}{2}\right)^x - 3\right]\left[\left(\frac{1}{2}\right)^x + 1\right] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x < 1 \\ \left(\frac{1}{2}\right)^x = 3, \left(\frac{1}{2}\right)^x + 1 \neq 0 \end{cases} \Rightarrow \begin{cases} x < 1 \\ x = (-\log_2 3) \end{cases}$$

Hence, $x_1 = -\log_2 3$ is the root of the original equation.

Example 103. Solve the equation $\log_{\left(\frac{2+x}{10}\right)} 7 = \log_{\left(\frac{2}{x+1}\right)} 7$.

Sol. The given equation is equivalent to

$$\begin{cases} \dfrac{2}{x+1} > 0 \\ \dfrac{2}{x+1} \neq 1 \\ \dfrac{2+x}{10} = \dfrac{2}{x+1} \end{cases} \Rightarrow \begin{cases} x + 1 > 0 \\ x \neq 1 \\ x = -6, 3 \end{cases}$$

$\therefore x_1 = 3$ is root of the original equation.

***Form* 4** Equations of the form

(i) $\boldsymbol{\log_{f(x)} g(x) = \log_{f(x)} h(x)}$ is equivalent to two ways.

$$\text{Method I} \begin{cases} g(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases} \quad \text{Method II} \begin{cases} h(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases}$$

(ii) $\boldsymbol{\log_{g(x)} f(x) = \log_{h(x)} f(x)}$ is equivalent to two ways.

$$\text{Method I} \begin{cases} f(x) > 0 \\ g(x) > 0 \\ g(x) \neq 1 \\ g(x) = h(x) \end{cases}$$

$$\text{Method II} \begin{cases} f(x) > 0 \\ h(x) > 0 \\ h(x) \neq 1 \\ g(x) = h(x) \end{cases}$$

Example 104. Solve the equation
$\log_{(x^2-1)}(x^3 + 6) = \log_{(x^2-1)}(2x^2 + 5x)$.

Sol. This equation is equivalent to the system

$$\begin{cases} 2x^2 + 5x > 0 \\ x^2 - 1 > 0 \\ x^2 - 1 \neq 1 \\ x^3 + 6 = 2x^2 + 5x \end{cases} \Rightarrow \begin{cases} x < -\frac{5}{2} \text{ and } x > 0 \\ x < -1 \text{ and } x > 1 \\ x \neq \pm\sqrt{2} \\ x = -2, 1, 3 \end{cases}$$

Hence, $x_1 = 3$ is only root of the original equation.

| Example 105. Solve the equation $\log_{(x^3+6)}(x^2-1)=\log_{(2x^2+5x)}(x^2-1)$.

Sol. This equation is equivalent to

$$\begin{cases} x^2-1>0 \\ 2x^2+5x>0 \\ 2x^2+5x\neq 1 \\ x^3+6=2x^2+5x \end{cases}$$

$$\Rightarrow \begin{cases} x<-1 \text{ and } x>1 \\ x<-\dfrac{5}{2} \text{ and } x>0 \\ x\neq \dfrac{-5\pm\sqrt{33}}{4} \\ x=-2,1,3 \end{cases}$$

Hence, $x_1=3$ is only root of the original equation.

Form 5 **An equation of the form**
$\log_{h(x)}(\log_{g(x)} f(x))=0$ is equivalent to the system

$$\begin{cases} h(x)>0 \\ h(x)\neq 1 \\ g(x)>0 \\ g(x)\neq 1 \\ f(x)=g(x) \end{cases}$$

| Example 106. Solve the equation $\log_{x^2-6x+8}[\log_{2x^2-2x+8}(x^2+5x)]=0$.

Sol. This equation is equivalent to the system

$$\begin{cases} x^2-6x+8>0 \\ x^2-6x+8\neq 1 \\ 2x^2-2x-8>0 \\ 2x^2-2x-8\neq 1 \\ x^2+5x=2x^2-2x-8 \end{cases}$$

Solve the equations of this system

$$\Rightarrow \begin{cases} x<2 \text{ and } x>4 \\ x\neq 3\pm\sqrt{2} \\ x<\dfrac{1-\sqrt{17}}{2} \text{ and } x>\dfrac{1+\sqrt{17}}{2} \\ x\neq\dfrac{1\pm\sqrt{19}}{2} \\ x=-1,8 \end{cases}$$

$x=-1$, does not satisfy the third relation of this system.
Hence, $x_1=8$ is only root of the original equation.

Form 6 **An equation of the form**
$2m\log_a f(x)=\log_a g(x), a>0, a\neq 1, m\in N$ is equivalent to the system

$$\begin{cases} f(x)>0 \\ f^{2m}(x)=g(x) \end{cases}$$

| Example 107. Solve the equation
$2\log 2x=\log(7x-2-2x^2)$.

Sol. This equation is equivalent to the system

$$\begin{cases} 2x>0 \\ (2x)^2=7x-2-2x^2 \end{cases}$$

$$\Rightarrow \begin{cases} x>0 \\ 6x^2-7x+2=0 \end{cases}$$

$$\Rightarrow \begin{cases} x>0 \\ (x-1/2)(x-2/3)=0 \end{cases}$$

$$\Rightarrow \begin{cases} x=1/2 \\ x=2/3 \end{cases}$$

Hence, $x_1=1/2$ and $x_2=2/3$ are the roots of the original equation.

Form 7 **An equation of the form**

$$(2m+1)\log_a f(x)=\log_a g(x), a>0, a\neq 1, m\in N$$

is equivalent to the system $\begin{cases} g(x)>0 \\ f^{2m+1}(x)=g(x) \end{cases}$.

| Example 108. Solve the equation
$\log(3x^2+x-2)=3\log(3x-2)$.

Sol. This equation is equivalent to the system

$$\begin{cases} 3x^2+x-2>0 \\ 3x^2+x-2=(3x-2)^3 \end{cases}$$

$$\Rightarrow \begin{cases} (x-2/3)(x-2)>0 \\ (x-2/3)(9x^2-13x+3)=0 \end{cases}$$

$$\Rightarrow \begin{cases} x<2/3 \text{ and } x>2 \\ x=\dfrac{2}{3}, x=\dfrac{13\pm\sqrt{61}}{18} \end{cases}$$

Original equation has the only root $x_1=\dfrac{13-\sqrt{61}}{18}$.

Form 8 **An equation of the form**

$$\log_a f(x)+\log_a g(x)=\log_a m(x), a>0, a\neq 1$$

is equivalent to the system

$$\begin{cases} f(x)>0 \\ g(x)>0 \\ f(x)\,g(x)=m(x) \end{cases}$$

| Example 109. Solve the equation
$2\log_3 x+\log_3(x^2-3)=\log_3 0.5+5^{\log_5(\log_3 8)}$

Sol. This equation can be written as

$$\log_3 x^2+\log_3(x^2-3)=\log_3 0.5+\log_3 8$$

$$\log_3 x^2+\log_3(x^2-3)=\log_3(4)$$

This is equivalent to the system

$$\begin{cases} x^2 > 0 \\ x^2 - 3 > 0 \\ x^2(x^2 - 3) = 4 \end{cases} \Rightarrow \begin{cases} x < 0 \text{ and } x > 0 \\ x < -\sqrt{3} \text{ and } x > \sqrt{3} \\ (x^2 - 4)(x^2 + 1) = 0 \end{cases}$$

$$\Rightarrow \quad x^2 - 4 = 0 \;\therefore\; x = \pm 2, \text{but } x > 0$$

Consequently, $x_1 = 2$ is only root of the original equation.

***Form* 9 An equation of the form**

$\log_a f(x) - \log_a g(x) = \log_a h(x) - \log_a t(x), a > 0, a \neq 1$

is equivalent to the equation

$\log_a f(x) + \log_a t(x) = \log_a g(x) + \log_a h(x)$,

which is equivalent to the system

$$\begin{cases} f(x) > 0 \\ t(x) > 0 \\ g(x) > 0 \\ h(x) > 0 \\ f(x) \cdot t(x) = g(x) \cdot h(x) \end{cases}$$

| Example 110. Solve the equation

$$\log_2(3 - x) - \log_2\left(\frac{\sin\frac{3\pi}{4}}{5 - x}\right) = \frac{1}{2} + \log_2(x + 7).$$

Sol. This equation is equivalent to

$$\log_2(3 - x) = \log_2\left(\frac{\sin\frac{3\pi}{4}}{5 - x}\right) + \frac{1}{2}\log_2 2 + \log_2(x + 7)$$

$$\Rightarrow \log_2(3 - x) = \log_2\left(\frac{1}{\sqrt{2}(5 - x)}\right) + \log_2\sqrt{2} + \log_2(x + 7)$$

which is equivalent to the system

$$\begin{cases} 3 - x > 0 \\ \dfrac{1}{\sqrt{2}(5 - x)} > 0 \\ x + 7 > 0 \\ (3 - x) = \dfrac{\sqrt{2}(x + 7)}{\sqrt{2}(5 - x)} \end{cases}$$

$$\Rightarrow \begin{cases} x < 3 \\ x < 5 \\ x > -7 \\ (x - 1)(x - 8) = 0 \end{cases}$$

Hence, $x_1 = 1$ is only root of the original equation.

Logarithmic Inequations

When we solve logarithmic inequations

(i) $\begin{cases} \log_a f(x) > \log_a g(x) \\ a > 1 \end{cases}$

$$\Rightarrow \begin{cases} g(x) > 0 \\ a > 1 \\ f(x) > g(x) \end{cases}$$

(ii) $\begin{cases} \log_a f(x) > \log_a g(x) \\ 0 < a < 1 \end{cases}$

$$\Rightarrow \begin{cases} f(x) > 0 \\ 0 < a < 1 \\ f(x) < g(x) \end{cases}$$

| Example 111. Solve the inequation

$$\log_{2x+3} x^2 < \log_{2x+3}(2x + 3).$$

Sol. This inequation is equivalent to the collection of the systems

$$\left[\begin{array}{l} \begin{cases} 2x + 3 > 1 \\ x^2 < 2x + 3 \end{cases} \\ \begin{cases} 0 < 2x + 3 < 1 \\ x^2 > 2x + 3 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x > -1 \\ (x - 3)(x + 1) < 0 \end{cases} \\ \begin{cases} -\frac{3}{2} < x < -1 \\ (x - 3)(x + 1) > 0 \end{cases} \end{array}\right.$$

$$\Rightarrow \left[\begin{array}{l} \begin{cases} x > -1 \\ -1 < x < 3 \end{cases} \\ \begin{cases} -\frac{3}{2} < x < -1 \\ x < -1 \text{ and } x > 3 \end{cases} \end{array}\right. \begin{array}{l} \Rightarrow -1 < x < 3 \\ \\ \Rightarrow -\frac{3}{2} < x < -1 \end{array}$$

Hence, the solution of the original inequation is

$$x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 3).$$

Canonical Logarithmic Inequalities

1. $\begin{cases} \log_a x > 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} x > 1 \\ a > 1 \end{cases}$

2. $\begin{cases} \log_a x > 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ 0 < a < 1 \end{cases}$

3. $\begin{cases} \log_a x < 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ a > 1 \end{cases}$

4. $\begin{cases} \log_a x < 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} x > 1 \\ 0 < a < 1 \end{cases}$

Some Standard Forms to Solve Logarithmic Inequations

Form 1 **Inequations of the form**

Forms		Collection of systems
(a) $\log_{g(x)} f(x) > 0$	$\Leftrightarrow$	$\begin{cases} f(x) > 1, \\ g(x) > 1, \end{cases} \begin{cases} 0 < f(x) < 1 \\ 0 < g(x) < 1 \end{cases}$
(b) $\log_{g(x)} f(x) \geq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1, \\ g(x) > 1, \end{cases} \begin{cases} 0 < f(x) \leq 1 \\ 0 < g(x) < 1 \end{cases}$
(c) $\log_{g(x)} f(x) < 0$	$\Leftrightarrow$	$\begin{cases} f(x) > 1, \\ 0 < g(x) < 1, \end{cases} \begin{cases} 0 < f(x) < 1 \\ g(x) > 1 \end{cases}$
(d) $\log_{g(x)} f(x) \leq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1, \\ 0 < g(x) < 1, \end{cases} \begin{cases} 0 < f(x) \leq 1 \\ g(x) > 1 \end{cases}$

| Example 112. Solve the inequation

$$\log_{\left(\frac{x^2-12x+30}{10}\right)}\left(\log_2 \frac{2x}{5}\right) > 0.$$

Sol. This inequation is equivalent to the collection of two systems

$$\begin{cases} \dfrac{x^2 - 12x + 30}{10} > 1, \\ \log_2\left(\dfrac{2x}{5}\right) > 1, \end{cases}$$

$$\begin{cases} 0 < \dfrac{x^2 - 12x + 30}{10} < 1 \\ 0 < \log_2\left(\dfrac{2x}{5}\right) < 1 \end{cases}$$

On solving the first system, we have

$$\Rightarrow \begin{cases} x^2 - 12x + 20 > 0 \\ \dfrac{2x}{5} > 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x - 10)(x - 2) > 0 \\ x > 5 \end{cases}$$

$$\Leftrightarrow \begin{cases} x < 2 \text{ and } x > 10 \\ x > 5 \end{cases}$$

Therefore, the system has solution $x > 10$.

On solving the second system, we have

$$\Rightarrow \begin{cases} 0 < x^2 - 12x + 30 < 10 \\ 1 < \dfrac{2x}{5} < 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2 - 12x + 30 > 0 \text{ and } x^2 - 12x + 20 < 0 \\ 5/2 < x < 5 \end{cases}$$

$$\Leftrightarrow \begin{cases} x < 6 - \sqrt{6} \text{ and } x > 6 + \sqrt{6} \text{ and } 2 < x < 10 \\ 0 < x < 5 \end{cases}$$

Therefore, the system has solution $2 < x < 6 - \sqrt{6}$ combining both systems, then solution of the original inequations is
$x \in (2, 6 - \sqrt{6}) \cup (10, \infty)$.

Form 2 Inequations of the form

Forms		Collection of systems
(a) $\log_{\phi(x)} f(x) > \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) > g(x), \\ g(x) > 0, \\ \phi(x) > 1, \end{cases}$ $\begin{cases} f(x) < g(x) \\ f(x) > 0 \\ 0 < \phi(x) < 1 \end{cases}$
(b) $\log_{\phi(x)} f(x) \geq \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) \geq g(x), \\ g(x) > 0, \\ \phi(x) > 1, \end{cases}$ $\begin{cases} f(x) \leq g(x) \\ f(x) > 0 \\ 0 < \phi(x) < 1 \end{cases}$
(c) $\log_{\phi(x)} f(x) < \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) < g(x), \\ f(x) > 0, \\ \phi(x) > 1, \end{cases}$ $\begin{cases} f(x) > g(x) \\ g(x) > 0 \\ 0 < \phi(x) < 1 \end{cases}$
(d) $\log_{\phi(x)} f(x) \leq \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) \leq g(x), \\ f(x) > 0, \\ \phi(x) > 1, \end{cases}$ $\begin{cases} f(x) \geq g(x) \\ g(x) > 0 \\ 0 < \phi(x) < 1 \end{cases}$

| Example 113. Solve the inequation

$$\log_{(x-3)}(2(x^2 - 10x + 24)) \geq \log_{(x-3)}(x^2 - 9).$$

Sol. This inequation is equivalent to the collection of systems

$$\begin{cases} 2(x^2 - 10x + 24) \geq x^2 - 9, \\ x^2 - 9 > 0, \\ x - 3 > 1, \end{cases}$$

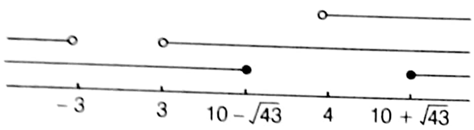
$$\begin{cases} 2(x^2 - 10x + 24) \le x^2 - 9 \\ 2(x^2 - 10x + 24) > 0 \\ 0 < x - 3 < 1 \end{cases}$$

On solving the first system, we have

$$\begin{cases} x^2 - 20x + 57 \ge 0, \\ (x + 3)(x - 3) > 0, \\ x > 4, \end{cases}$$

$$\Leftrightarrow \begin{cases} x \in (-\infty, 10 - \sqrt{43}] \cup [10 + \sqrt{43}, \infty) \\ x \in (-\infty, -3) \cup (3, \infty) \\ x \in (4, \infty) \end{cases}$$

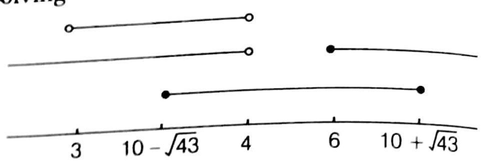
Therefore, the system has solution



$$x \ge 10 + \sqrt{43}$$

i.e. $\quad x \in [10 + \sqrt{43}, \infty)$

On solving the second system, we have



$$\begin{cases} x^2 - 20x + 57 \le 0, \\ (x - 6)(x - 4) > 0, \\ 3 < x < 4, \end{cases}$$

$$\Leftrightarrow \begin{cases} x \in [10 - \sqrt{43}, 10 + \sqrt{43}] \\ x \in (-\infty, 4) \cup (6, \infty) \\ x \in (3, 4) \end{cases}$$

Therefore, the system has solution

$$10 - \sqrt{43} \le x < 4,$$

i.e., $\quad x \in [10 - \sqrt{43}, 4)$

On combining the both systems, the solution of the original inequation is

$$x \in [10 - \sqrt{43}, 4) \cup [10 + \sqrt{43}, \infty).$$

a, a, b, c, c, b, a, b, b, d

Exercise for Session 5

1. The equation $\sqrt{(x+1)} - \sqrt{(x-1)} = \sqrt{(4x-1)}$ has
   (a) no solution (b) one solution (c) two solutions (d) more than two solutions
2. The number of real solutions of $\sqrt{(x^2 - 4x + 3)} + \sqrt{(x^2 - 9)} = \sqrt{(4x^2 - 14x + 6)}$ is
   (a) one (b) two (c) three (d) None of these
3. The number of real solutions of $\sqrt{(3x^2 - 7x - 30)} - \sqrt{(2x^2 - 7x - 5)} = x - 5$ is
   (a) one (b) two (c) three (d) None of these
4. o The number of integral values of x satisfying $\sqrt{(-x^2 + 10x - 16)} < x - 2$ is
   (a) 0 (b) 1 (c) 2 (d) 3
5. The number of real solutions of the equation $\left(\frac{9}{10}\right)^x = -3 + x - x^2$ is
   (a) 2 (b) 1 (c) 0 (d) None of these
6. The set of all x satisfying $3^{2x} - 3^x - 6 > 0$ is given by
   (a) $0 < x < 1$ (b) $x > 1$ (c) $x > 3^{-2}$ (d) None of these
7. The number of real solutions of the equation $2^{x/2} + (\sqrt{2} + 1)^x = (3 + 2\sqrt{2})^{x/2}$ is
   (a) one (b) two (c) four (d) infinite
8. even function The sum of the values of x satisfying the equation $(31 + 8\sqrt{15})^{x^2 - 3} + 1 = (32 + 8\sqrt{15})^{x^2 - 3}$ is
   (a) 3 (b) 0 (c) 2 (d) None of these
9. The number of real solutions of the equation $\log_{0.5} x = |x|$ is
   (a) 0 (b) 1 (c) 2 (d) None of these
10. The inequality $(x - 1)\ln(2 - x) < 0$ holds, if x satisfies
    (a) $1 < x < 2$ (b) $x > 0$ (c) $0 < x < 1$ (d) None of these

Shortcuts and Important Results to Remember

1 '0' is neither positive nor negative even integer, '2' is the only even prime number and all other prime numbers are odd, '1' (i.e. unity) is neither a composite nor a prime number and 1, −1 are two units in the set of integers.

2 (i) If $a > 0, b > 0$ and $a < b \Rightarrow a^2 < b^2$

(ii) If $a < 0, b < 0$ and $a < b \Rightarrow a^2 > b^2$

(iii) If $a_1, a_2, a_3, \ldots, a_n \in R$ and $a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = 0$

$\Rightarrow a_1 = a_2 = a_3 = \ldots = a_n = 0$

3 (i) $\text{Max}(a, b) = \frac{1}{2}(|a+b| + |a-b|)$

(ii) $\text{Min}(a, b) = \frac{1}{2}(|a+b| - |a-b|)$

4 If the equation $f(x) = 0$ has two real roots α and β, then $f'(x) = 0$ will have a real root lying between α and β.

5 If two quadratic equations $P(x) = 0$ and $Q(x) = 0$ have an irrational common root, both roots will be common.

6 In the equation $ax^2 + bx + c = 0\ [a, b, c \in R]$, if $a + b + c = 0$, the roots are $1, \frac{c}{a}$ and if $a - b + c = 0$, the roots are -1 and $\frac{c}{a}$.

7 The condition that the roots of $ax^2 + bx + c = 0$ may be in the ratio $p : q$, is

$pq\, b^2 = ac\,(p+q)^2$ (here, $\alpha : \beta = p : q$)

i.e., $$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} = \pm\sqrt{\frac{b^2}{ac}}$$

(i) If one root of $ax^2 + bx + c = 0$ is n times that of the other, then $nb^2 = ac\,(n+1)^2$, here $\alpha : \beta = n : 1$.

(ii) If one root of $ax^2 + bx + c = 0$ is double of the other here $n = 2$, then $2b^2 = 9ac$.

8 If one root of $ax^2 + bx + c = 0$ is nth power of the other, then $(a^n c)^{\frac{1}{n+1}} + (ac^n)^{\frac{1}{n+1}} = -b$.

9 If one root of $ax^2 + bx + c = 0$ is square of the other, then $a^2c + ac^2 + b^3 = 3abc$.

10 If the ratio of the roots of the equation $ax^2 + bx + c = 0$ is equal to the ratio of the roots of $Ax^2 + Bx + C = 0$ and $a \neq 0, A \neq 0$, then $\boxed{\frac{b^2}{ac} = \frac{B^2}{AC}}$.

11 If sum of the roots is equal to sum of their squares then $2ac = ab + b^2$.

12 If sum of roots of $ax^2 + bx + c = 0$ is equal to the sum of their reciprocals, then

$2a^2c = ab^2 + bc^2$, i.e. ab^2, bc^2, ca^2 are in AP

or $\frac{2a}{b} = \frac{b}{c} + \frac{c}{a}$ i.e. $\frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in AP.

13 Given, $y = ax^2 + bx + c$

(i) If $a > 0$, $y_{\min} = \frac{4ac - b^2}{4a}$

(ii) If $a < 0$, $y_{\max} = \frac{4ac - b^2}{4a}$

14 If α, β are the roots of $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$, then $aS_{n+1} + bS_n + cS_{n-1} = 0$.

15 If D_1 and D_2 are discriminants of two quadratics $P(x) = 0$ and $Q(x) = 0$, then

(i) If $D_1D_2 < 0$, then the equation $P(x)\cdot Q(x) = 0$ will have two real roots.

(ii) If $D_1D_2 > 0$, then the equation $P(x)\cdot Q(x) = 0$ has either four real roots or no real root.

(iii) If $D_1D_2 = 0$, then the equation $P(x)\cdot Q(x) = 0$ will have

(a) two equal roots and two distinct roots such that $D_1 > 0$ and $D_2 = 0$ or $D_1 = 0$ and $D_2 > 0$.

(b) only one real solution such that $D_1 < 0$ and $D_2 = 0$ or $D_1 = 0$ and $D_2 < 0$.

16 If $a > 0$ and $x = \sqrt{a + \sqrt{a + \sqrt{a + \ldots + \infty}}}$, then $x = \frac{1 + \sqrt{(4a+1)}}{2}$.

17 If $a_1, a_2, a_3, \ldots, a_n$ are positive real numbers, then least value of $(a_1 + a_2 + a_3 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)$ is n^2.

(i) Least value of $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3^2 = 9$

(ii) Least value of
$$(a+b+c+d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) = 4^2 = 16$$

18 **Law of Proportions** If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$, then each of these ratios is also equal to

(i) $\frac{a + c + e + \ldots}{b + d + f + \ldots}$

(ii) $\left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{1/n}$ (where, $p, q, r, \ldots, n \in R$)

(iii) $\frac{\sqrt{ac}}{\sqrt{bd}} = \frac{\sqrt[n]{(ace\ldots)}}{\sqrt[n]{(bdf\ldots)}}$

19 **Lagrange's Mean Value Theorem** Let $f(x)$ be a function defined on $[a, b]$ such that

contd...

(i) $f(x)$ is continuous on $[a, b]$ and

(ii) $f(x)$ is derivable on (a, b), then $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

20 Lagrange's Identity If $a_1, a_2, a_3, b_1, b_2, b_3 \in R$, then

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$
$$= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2$$

or $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2$$

Remark

If $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \le (a_1b_1 + a_2b_2 + a_3b_3)^2$,

then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

21 Horner's Method of Synthetic, Division When, we divide a polynomial of degree ≥ 1 by a linear monic polynomial, the quotient and remainder can be found by this method. Consider

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$

where $a_0 \ne 0$ and $a_0, a_1, a_2, \ldots, a_n \in R$.

Let $g(x) = (x - \alpha)$ be a linear monic polynomial $\alpha \in R$.

When $g(x) \mid f(x)$; we can find quotient and remainder as follows :

α	a_0	a_1	a_2	$\ldots$	a_n
	0	αa_0	$b_1\alpha$		αb_{n-1}
	$a_0 = b_0$	$a_1 + \alpha a_0 = b_1$	$a_2 + b_1\alpha = b_2$		$a_n + \alpha b_{n-1} = 0$

$\therefore \; f(x) = (x - \alpha)(b_0 x^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + \ldots + b_{n-1})$

e.g. Find all roots of $x^3 - 6x^2 + 11x - 6 = 0$.

$\because (x - 1)$ is a factor of $x^3 - 6x^2 + 11x - 6$, then

$x = 1$	1	−6	11	−6
	0	1	−5	6
	1	−5	6	0

$$\therefore \; x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$$
$$= (x - 1)(x - 2)(x - 3)$$

Hence, roots of $x^3 - 6x^2 + 11x - 6 = 0$ are 1, 2 and 3.

JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1 *If α and $\beta (\alpha < \beta)$, are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then*

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

Sol. (*b*) $\because \quad \alpha + \beta = -b, \ \alpha\beta = c \quad$ ...(i)

$\because \quad c < 0 \Rightarrow \alpha\beta < 0$

Let $\quad \alpha < 0, \beta > 0$

$\therefore \quad |\alpha| = -\alpha$ and $\alpha < 0 < \beta \quad [\because \alpha < \beta]$ ...(ii)

From Eq. (i), we get $-|\alpha| + \beta < 0$

$\Rightarrow \quad \beta < |\alpha| \quad$ ...(iii)

From Eqs. (ii) and (iii), we get

$$\alpha < 0 < \beta < |\alpha|$$

• Ex. 2 *Let α, β be the roots of the equation $x^2 - x + p = 0$ and γ, δ be the roots of the equation $x^2 - 4x + q = 0$. If α, β, γ and δ are in GP, the integral values of p and q respectively, are*

(a) $-2, -32$ (b) $-2, 3$
(c) $-6, 3$ (d) $-6, -32$

Sol. (*a*) Let r be the common ratio of the GP, then

$$\beta = \alpha r, \gamma = \alpha r^2 \text{ and } \delta = \alpha r^3$$

$\therefore \quad \alpha + \beta = 1 \Rightarrow \alpha + \alpha r = 1$

or $\quad \alpha(1 + r) = 1 \quad$ ...(i)

and $\quad \alpha\beta = p \Rightarrow \alpha(\alpha r) = p$

or $\quad \alpha^2 r = p \quad$ ...(ii)

and $\quad \gamma + \delta = 4 \Rightarrow \alpha r^2 + \alpha r^3 = 4$

or $\quad \alpha r^2(1 + r) = 4 \quad$ ...(iii)

and $\quad \gamma\delta = q$

$\Rightarrow \quad (\alpha r^2)(\alpha r^3) = q$

or $\quad \alpha^2 r^5 = q \quad$ ...(iv)

On dividing Eq. (iii) by Eq. (i), we get

$$r^2 = 4 \Rightarrow r = -2, 2$$

If we take $r = 2$, then α is not integer, so we take $r = -2$.

On substituting $r = -2$ in Eq. (i), we get $\alpha = -1$

Now, from Eqs. (ii) and (iv), we get

$$p = \alpha^2 r = (-1)^2(-2) = -2$$

and $\quad q = \alpha^2 r^5 = (-1)^2(-2)^5 = -32$

Hence, $\quad (p, q) = (-2, -32)$

• Ex. 3 *Let $f(x) = \int_1^x \sqrt{(2 - t^2)}\, dt$, the real roots of the equation $x^2 - f'(x) = 0$ are*

(a) ± 1 (b) $\pm \frac{1}{\sqrt{2}}$
(c) $\pm \frac{1}{2}$ (d) 0 and 1

Sol. (*a*) We have, $f(x) = \int_1^x \sqrt{(2 - t^2)}\, dt$

$\Rightarrow \quad f'(x) = \sqrt{(2 - x^2)}$

$\therefore \quad x^2 - f'(x) = 0$

$\Rightarrow \quad x^2 - \sqrt{(2 - x^2)} = 0 \Rightarrow x^4 + x^2 - 2 = 0$

$\Rightarrow \quad x^2 = 1, -2$

$\Rightarrow \quad x = \pm 1 \quad$ [only for real value of x]

• Ex. 4 *If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have a common root and $a, b, c \in N$, the minimum value of $a + b + c$ is*

(a) 3 (b) 9
(c) 6 (d) 12

Sol. (*b*) $\because$ Roots of the equation $x^2 + 3x + 5 = 0$ are non-real.

Thus, given equations will have two common roots.

$\Rightarrow \quad \frac{a}{1} = \frac{b}{3} = \frac{c}{5} = \lambda \quad$ [say]

$\therefore \quad a + b + c = 9\lambda$

Thus, minimum value of $a + b + c = 9 \quad [\because a, b, c \in N]$

• Ex. 5 *If $x_1, x_2, x_3, \ldots, x_n$ are the roots of the equation $x^n + ax + b = 0$, the value of $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \ldots (x_1 - x_n)$ is*

(a) $nx_1 + b$
(b) $n(x_1)^{n-1}$
(c) $n(x_1)^{n-1} + a$
(d) $n(x_1)^{n-1} + b$

Sol. (*c*) $\because x^n + ax + b = (x - x_1)(x - x_2)(x - x_3) \ldots (x - x_n)$

$$\Rightarrow (x - x_2)(x - x_3) \ldots (x - x_n) = \frac{x^n + ax + b}{x - x_1}$$

On taking $\lim_{x \to x_1}$ both sides, we get

$$(x_1 - x_2)(x_1 - x_3) \ldots (x_1 - x_n) = \lim_{x \to x_1} \frac{x^n + ax + b}{x - x_1} \left[\frac{0}{0} \text{form}\right]$$

$$= \lim_{x \to x_1} \frac{nx^{n-1} + a}{1} = n(x_1)^{n-1} + a$$

• **Ex. 6** *If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $A_n = \alpha^n + \beta^n$, then $aA_{n+2} + bA_{n+1} + cA_n$ is equal to*

(a) 0 (b) 1 (c) $a + b + c$ (d) abc

Sol. (a) $\because \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$$\therefore \quad A_{n+2} = \alpha^{n+2} + \beta^{n+2}$$
$$= (\alpha + \beta)(\alpha^{n+1} + \beta^{n+1}) - \alpha\beta^{n+1} - \beta\alpha^{n+1}$$
$$= (\alpha + \beta)(\alpha^{n+1} + \beta^{n+1}) - \alpha\beta(\alpha^n + \beta^n)$$
$$= -\frac{b}{a}A_{n+1} - \frac{c}{a}A_n$$
$$\Rightarrow \quad aA_{n+2} + bA_{n+1} + cA_n = 0$$

• **Ex. 7** *If x and y are positive integers such that $xy + x + y = 71$, $x^2y + xy^2 = 880$, then $x^2 + y^2$ is equal to*

(a) 125 (b) 137 (c) 146 (d) 152

Sol. (c) $\because \; xy + x + y = 71 \Rightarrow xy + (x + y) = 71$

and $\quad x^2y + xy^2 = 880 \Rightarrow \quad xy(x + y) = 880$

$\Rightarrow xy$ and $(x + y)$ are the roots of the quadratic equation.

$$t^2 - 71t + 880 = 0$$
$$\Rightarrow \quad (t - 55)(t - 16) = 0$$
$$\Rightarrow \quad t = 55, 16$$
$$\therefore \quad x + y = 16 \text{ and } xy = 55$$

So, $\quad x^2 + y^2 = (x + y)^2 - 2xy = (16)^2 - 110 = 146$

• **Ex. 8** *If α, β are the roots of the equation $x^2 - 3x + 5 = 0$ and γ, δ are the roots of the equation $x^2 + 5x - 3 = 0$, then the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$, is*

(a) $x^2 - 15x - 158 = 0$ (b) $x^2 + 15x - 158 = 0$

(c) $x^2 - 15x + 158 = 0$ (d) $x^2 + 15x + 158 = 0$

Sol. (d) $\because \; \alpha + \beta = 3, \alpha\beta = 5, \gamma + \delta = (-5), \gamma\delta = (-3)$

Sum of roots $= (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma)$

$= (\alpha + \beta)(\gamma + \delta) = 3 \times (-5) = (-15)$

Product of roots $= (\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma)$

$$= \alpha^2\gamma\delta + \alpha\beta\gamma^2 + \beta\alpha\delta^2 + \beta^2\gamma\delta$$
$$= \gamma\delta(\alpha^2 + \beta^2) + \alpha\beta(\gamma^2 + \delta^2)$$
$$= -3(\alpha^2 + \beta^2) + 5(\gamma^2 + \delta^2)$$
$$= -3[(\alpha + \beta)^2 - 2\alpha\beta] + 5[(\gamma + \delta)^2 - 2\gamma\delta]$$
$$= -3[9 - 10] + 5[25 + 6] = 158$$

$\therefore$ Required equation is $x^2 + 15x + 158 = 0$.

• **Ex. 9** *The number of roots of the equation $\frac{1}{x} + \frac{1}{\sqrt{(1 - x^2)}} = \frac{35}{12}$ is*

(a) 0 (b) 1

(c) 2 (d) 3

Sol. (d) Let $\frac{1}{x} = u$ and $\frac{1}{\sqrt{(1 - x^2)}} = v$, then

$u + v = \frac{35}{12}$ and $u^2 + v^2 = u^2v^2$

$$\Rightarrow \quad (u + v)^2 = \left(\frac{35}{12}\right)^2$$
$$\Rightarrow \quad u^2 + v^2 + 2uv = \left(\frac{35}{12}\right)^2$$
$$\Rightarrow \quad u^2v^2 + 2uv = \left(\frac{35}{12}\right)^2 \quad [\because u^2 + v^2 = u^2v^2]$$
$$\Rightarrow \quad u^2v^2 + 2uv - \left(\frac{35}{12}\right)^2 = 0$$
$$\Rightarrow \quad \left(uv + \frac{49}{12}\right)\left(uv - \frac{25}{12}\right) = 0$$
$$\Rightarrow \quad uv = -\frac{49}{12}, uv = \frac{25}{12}$$

Case I If $uv = -\frac{49}{12}$, then

$$\frac{1}{x} \cdot \frac{1}{\sqrt{(1 - x^2)}} = -\frac{49}{12} \quad [\text{here } x < 0]$$
$$\Rightarrow \quad x^4 - x^2 + \frac{(12)^2}{(49)^2} = 0$$
$$\Rightarrow \quad x = -\frac{(5 + \sqrt{73})}{14}$$

Case II If $uv = \frac{25}{12}$, then

$$\frac{1}{x} \cdot \frac{1}{\sqrt{(1 - x^2)}} = \frac{25}{12} \quad [\text{here } x > 0]$$
$$\Rightarrow \quad x^4 - x^2 + \frac{(12)^2}{(25)^2} = 0$$
$$\Rightarrow \quad \left(x^2 - \frac{9}{25}\right)\left(x^2 - \frac{16}{25}\right) = 0 \Rightarrow x = \frac{3}{5}, \frac{4}{5}$$

On combining both cases,

$$x = -\frac{(5 + \sqrt{73})}{14}, \frac{3}{5}, \frac{4}{5}$$

Hence, number of roots = 3

• **Ex. 10** *The sum of the roots of the equation $2^{33x - 2} + 2^{11x + 2} = 2^{22x + 1} + 1$ is*

(a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{3}{11}$ (d) $\frac{4}{11}$

Sol. (b) Let $2^{11x} = t$, given equation reduces to

$$\frac{t^3}{4} + 4t = 2t^2 + 1$$
$$\Rightarrow \quad t^3 - 8t^2 + 16t - 4 = 0 \Rightarrow t_1 \cdot t_2 \cdot t_3 = 4$$
$$\Rightarrow \quad 2^{11x_1} \cdot 2^{11x_2} \cdot 2^{11x_3} = 4 \Rightarrow 2^{11(x_1 + x_2 + x_3)} = 2^2$$
$$\Rightarrow \quad 11(x_1 + x_2 + x_3) = 2$$
$$\therefore \quad x_1 + x_2 + x_3 = \frac{2}{11}$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples**. Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

• Ex. 11 *For the equation $2x^2 - 6\sqrt{2}x - 1 = 0$*

(a) roots are rational
(b) roots are irrational
(c) if one root is $(p + \sqrt{q})$, the other is $(-p + \sqrt{q})$
(d) if one root is $(p + \sqrt{q})$, the other is $(p - \sqrt{q})$

Sol. (*b,c*) As the coefficients are not rational, irrational roots need not appear in conjugate pair.

Here, $\alpha + \beta = 3\sqrt{2}$ and $\alpha\beta = -\dfrac{1}{2}$

Let $\alpha = p + \sqrt{q}$, then prove that other root $\beta = -p + \sqrt{q}$

• Ex. 12 *Given that α, γ are roots of the equation $Ax^2 - 4x + 1 = 0$ and β, δ the roots of the equation $Bx^2 - 6x + 1 = 0$, such that α, β, γ and δ are in HP then*

(a) $A = 3$ (b) $A = 4$ (c) $B = 2$ (d) $B = 8$

Sol. (*a,d*) Since, α, β, γ and δ are in HP, hence $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ and $\dfrac{1}{\delta}$ are in AP and they may be taken as $a - 3d$, $a - d$, $a + d$ and $a + 3d$. Replacing x by $\dfrac{1}{x}$, we get the equation whose roots are $a - 3d$, $a + d$ is $x^2 - 4x + A = 0$ and equation whose roots are $a - d$, $a + 3d$ is $x^2 - 6x + B = 0$, then

$$(a - 3d) + (a + d) = 4 \Rightarrow 2(a - d) = 4$$

and $$(a - d) + (a + 3d) = 6 \Rightarrow 2(a + d) = 6$$

$$\therefore \quad a = \frac{5}{2} \text{ and } d = \frac{1}{2}$$

Now, $A = (a - 3d)(a + d) = \left(\dfrac{5}{2} - \dfrac{3}{2}\right)\left(\dfrac{5}{2} + \dfrac{1}{2}\right) = 3$

and $B = (a - d)(a + 3d) = \left(\dfrac{5}{2} - \dfrac{1}{2}\right)\left(\dfrac{5}{2} + \dfrac{3}{2}\right) = 8$

• Ex. 13 *If $|ax^2 + bx + c| \le 1$ for all x in [0, 1], then*

(a) $|a| \le 8$ (b) $|b| > 8$
(c) $|c| \le 1$ (d) $|a| + |b| + |c| \le 17$

Sol. (*a,c,d*) On putting $x = 0, 1$ and $\dfrac{1}{2}$, we get

$$|c| \le 1 \quad \text{...(i)}$$

$$|a + b + c| \le 1 \quad \text{...(ii)}$$

and $$|a + 2b + 4c| \le 4 \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$|b| \le 8 \text{ and } |a| \le 8$$

$$\Rightarrow \quad |a| + |b| + |c| \le 17$$

• Ex. 14 *If $\cos^4\theta + p$, $\sin^4\theta + p$ are the roots of the equation $x^2 + a(2x + 1) = 0$ and $\cos^2\theta + q$, $\sin^2\theta + q$ are the roots of the equation $x^2 + 4x + 2 = 0$ then a is equal to*

(a) -2 (b) -1 (c) 1 (d) 2

Sol. (*b,d*)

$$\because \quad \cos^4\theta - \sin^4\theta = \cos 2\theta$$

$$\Rightarrow \quad \cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$$

$$\Rightarrow \quad (\cos^4\theta + p) - (\sin^4\theta + p) = (\cos^2\theta + q) - (\sin^2\theta + q)$$

$$\Rightarrow \quad \frac{\sqrt{4a^2 - 4a}}{1} = \frac{\sqrt{16 - 8}}{1} \quad \left[\because \alpha - \beta = \frac{\sqrt{D}}{a}\right]$$

$$\Rightarrow \quad 4a^2 - 4a = 8 \text{ or } a^2 - a - 2 = 0$$

or $$(a - 2)(a + 1) = 0 \text{ or } a = 2, -1$$

• Ex. 15 *If α, β, γ are the roots of $x^3 - x^2 + ax + b = 0$ and β, γ, δ are the roots of $x^3 - 4x^2 + mx + n = 0$. If α, β, γ and δ are in AP with common difference d then*

(a) $a = m$ (b) $a = m - 5$
(c) $n = b - a - 2$ (d) $b = m + n - 3$

Sol. (*b,c,d*)

$\because$ $\alpha, \beta, \gamma, \delta$ are in AP with common difference d, then

$$\beta = \alpha + d, \gamma = \alpha + 2d \text{ and } \delta = \alpha + 3d \quad \text{...(i)}$$

Given, α, β, γ are the roots of $x^3 - x^2 + ax + b = 0$, then

$$\alpha + \beta + \gamma = 1 \quad \text{...(ii)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = a \quad \text{...(iii)}$$

$$\alpha\beta\gamma = -b \quad \text{...(iv)}$$

Also, β, γ, δ are the roots of $x^3 - 4x^2 + mx + n = 0$, then

$$\beta + \gamma + \delta = 4 \quad \text{...(v)}$$

$$\beta\gamma + \gamma\delta + \delta\beta = m \quad \text{...(vi)}$$

$$\beta\gamma\delta = -n \quad \text{...(vii)}$$

From Eqs. (i) and (ii), we get

$$3\alpha + 3d = 1 \quad \text{...(viii)}$$

and from Eqs. (i) and (v), we get

$$3\alpha + 6d = 4 \quad \text{...(ix)}$$

From Eqs. (viii) and (ix), we get

$$d = 1, \alpha = -\frac{2}{3}$$

Now, from Eq. (i), we get

$$\beta = \frac{1}{3}, \gamma = \frac{4}{3} \text{ and } \delta = \frac{7}{3}$$

From Eqs. (iii), (iv), (vi) and (vii), we get

$$a = -\frac{2}{3}, b = \frac{8}{27}, m = \frac{13}{3}, n = -\frac{28}{27}$$

$$\therefore \quad a = m - 5, n = b - a - 2 \text{ and } b = m + n - 3$$

JEE Type Solved Examples : Passage Based Questions

- This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

If G and L are the greatest and least values of the expression $\dfrac{x^2-x+1}{x^2+x+1}$, $x \in R$ *respectively, then*

16. The least value of $G^5 + L^5$ is

(a) 0 (b) 2 (c) 16 (d) 32

Sol. (b) Let $y = \dfrac{x^2-x+1}{x^2+x+1}$

$\Rightarrow \quad x^2y + xy + y = x^2 - x + 1$

$\Rightarrow \quad (y-1)x^2 + (y+1)x + y - 1 = 0 \quad [\because x \in R]$

$\therefore \quad (y+1)^2 - 4\cdot(y-1)(y-1) \geq 0 \quad [\because b^2 - 4ac \geq 0]$

$\Rightarrow \quad (y+1)^2 - (2y-2)^2 \geq 0$

$\Rightarrow \quad (3y-1)(y-3) \leq 0$

$\therefore \quad \dfrac{1}{3} \leq y \leq 3 \Rightarrow G = 3$ and $L = \dfrac{1}{3} \quad \therefore GL = 1$

$\dfrac{G^5 + L^5}{2} \geq (GL)^{1/5} = (1)^{1/5} = 1$

$\Rightarrow \quad \dfrac{G^5 + L^5}{2} \geq 1$ or $G^5 + L^5 \geq 2$

$\therefore$ Minimum value of $G^5 + L^5$ is 2.

17. G and L are the roots of the equation

(a) $3x^2 - 10x + 3 = 0$ (b) $4x^2 - 17x + 4 = 0$
(c) $x^2 - 7x + 10 = 0$ (d) $x^2 - 5x + 6 = 0$

Sol. (a) Equation whose roots are G and L, is

$$x^2 - (G+L)x + GL = 0$$

$\Rightarrow \quad x^2 - \dfrac{10}{3}x + 1 = 0$ or $3x^2 - 10x + 3 = 0$

18. If $L < \lambda < G$ and $\lambda \in N$, the sum of all values of λ is

(a) 2 (b) 3 (c) 4 (d) 5

Sol. (b) $\because L < \lambda < G \Rightarrow \dfrac{1}{3} < \lambda < 3 \quad \therefore \lambda = 1, 2$

Sum of values of $\lambda = 1 + 2 = 3$

Passage II

(Ex. Nos. 19 to 21)

Let a, b, c and d are real numbers in GP. Suppose u, v, w satisfy the system of equations $u + 2v + 3w = 6$, $4u + 5v + 6w = 12$ *and* $6u + 9v = 4$. *Further, consider the expressions*

$f(x) = \left(\dfrac{1}{u} + \dfrac{1}{v} + \dfrac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]$
$x + u + v + w = 0$ *and* $g(x) = 20x^2 + 10(a-d)^2x - 9 = 0$

19. $(b-c)^2 + (c-a)^2 + (d-b)^2$ is equal to

(a) $a - d$ (b) $(a-d)^2$ (c) $a^2 - d^2$ (d) $(a+d)^2$

Sol. (b) Let $b = ar, c = ar^2$ and $d = ar^3$

Now, $(b-c)^2 + (c-a)^2 + (d-b)^2$

$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$

$= a^2r^2(1-r)^2 + a^2(r^2-1)^2 + a^2r^2(r^2-1)^2$

$= a^2(1-r)^2\{r^2 + (r+1)^2 + r^2(r+1)^2\}$

$= a^2(1-r)^2(r^4 + 2r^3 + 3r^2 + 2r + 1)$

$= a^2(1-r)^2(1 + r + r^2)^2 = a^2(1-r^3)^2$

$= (a - ar^3)^2 = (a-d)^2$

20. $(u + v + w)$ is equal to

(a) 2 (b) $\dfrac{1}{2}$ (c) 20 (d) $\dfrac{1}{20}$

Sol. (a) Now, $u + 2v + 3w = 6 \quad$ ...(i)

$4u + 5v + 6w = 12 \quad$ ...(ii)

and $6u + 9v = 4 \quad$ ...(iii)

From Eqs. (i) and (ii), we get

$2u + v = 0 \quad$ ...(iv)

Solving Eqs. (iii) and (iv), we get

$$u = -\frac{1}{3}, v = \frac{2}{3}$$

Now, from Eq. (i), we get $w = \dfrac{5}{3}$

$\therefore \quad v + u + w = -\dfrac{1}{3} + \dfrac{2}{3} + \dfrac{5}{3} = 2$

21. If roots of $f(x) = 0$ be α, β, the roots of $g(x) = 0$ will be

(a) α, β (b) $-\alpha, -\beta$ (c) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ (d) $-\dfrac{1}{\alpha}, -\dfrac{1}{\beta}$

Sol. (c) Now, $f(x) = \left(\dfrac{1}{u} + \dfrac{1}{v} + \dfrac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]x + u + v + w = 0$

$\Rightarrow \quad f(x) = -\dfrac{9}{10}x^2 + (a-d)^2x + 2 = 0$

$\Rightarrow \quad f(x) = -9x^2 + 10(a-d)^2x + 20 = 0 \quad$ ...(v)

Given, roots of $f(x) = 0$ are α and β.

Now, replace x by $\dfrac{1}{x}$ in Eq. (v), then

$$\frac{-9}{x^2} + \frac{10(a-d)^2}{x} + 20 = 0$$

$\Rightarrow \quad 20x^2 + 10(a-d)^2x - 9 = 0$

$g(x) = 0$

$\therefore \quad$ Roots of $g(x) = 0$ are $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$.

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

• Ex. 22 *If the roots of the equation* $10x^3 - cx^2 - 54x - 27 = 0$ *are in harmonic progression, the value of c is*

Sol. (9) Given, roots of the equation

$$10x^3 - cx^2 - 54x - 27 = 0 \text{ are in HP.} \quad ...(i)$$

Now, replacing x by $\frac{1}{x}$ in Eq. (i), we get

$$27x^3 + 54x^2 + cx - 10 = 0 \quad ...(ii)$$

Hence, the roots of Eq. (ii) are in AP.

Let $a - d$, a and $a + d$ are the roots of Eq. (ii).

Then, $a - d + a + a + d = -\frac{54}{27}$

$$\Rightarrow \quad a = -\frac{2}{3} \quad ...(iii)$$

Since, a is a root of Eq. (ii), then

$$27a^3 + 54a^2 + ca - 10 = 0$$

$$\Rightarrow 27\left(-\frac{8}{27}\right) + 54\left(\frac{4}{9}\right) + c\left(-\frac{2}{3}\right) - 10 = 0 \quad \text{[from Eq. (iii)]}$$

$$\Rightarrow \quad 6 - \frac{2c}{3} = 0 \text{ or } c = 9$$

• Ex. 23 *If a root of the equation* $n^2 \sin^2 x - 2\sin x - (2n+1) = 0$ *lies in* $[0, \pi/2]$, *the minimum positive integer value of n is*

Sol. (3) $\because \; n^2 \sin^2 x - 2\sin x - (2n+1) = 0$

$$\Rightarrow \quad \sin x = \frac{2 \pm \sqrt{4 + 4n^2(2n+1)}}{2n^2}$$

[by Shridharacharya method]

$$= \frac{1 \pm \sqrt{(2n^3 + n^2 + 1)}}{n^2}$$

$$\because \quad 0 \le \sin x \le 1 \quad [\because x \in [0, \pi/2]]$$

$$\Rightarrow \quad 0 \le \frac{1 + \sqrt{(2n^3 + n^2 + 1)}}{n^2} \le 1$$

$$\Rightarrow \quad 0 \le 1 + \sqrt{(2n^3 + n^2 + 1)} \le n^2$$

$$\Rightarrow \sqrt{(2n^3 + n^2 + 1)} \le (n^2 - 1) \quad [\because n > 1]$$

On squaring both sides, we get

$$2n^3 + n^2 + 1 \le n^4 - 2n^2 + 1$$

$$\Rightarrow \quad n^4 - 2n^3 - 3n^2 \ge 0$$

$$\Rightarrow \quad n^2 - 2n - 3 \ge 0 \Rightarrow (n-3)(n+1) \ge 0$$

$$\Rightarrow \quad n \ge 3$$

$$\therefore \quad n = 3, 4, 5, \ldots$$

Hence, the minimum positive integer value of n is 3.

JEE Type Solved Examples :
Matching Type Questions

- This section contains **2 examples.** Examples 24 and 25 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

• Ex. 24 *Column I contains rational algebraic expressions and Column II contains possible integers which lie in their range. Match the entries of Column I with one or more entries of the elements of Column II.*

	Column I		Column II
(A)	$y = \frac{x^2 - 2x + 9}{x^2 + 2x + 9}, x \in R$	(p)	1
(B)	$y = \frac{x^2 - 3x - 2}{2x - 3}, x \in R$	(q)	3
(C)	$y = \frac{2x^2 - 2x + 4}{x^2 - 4x + 3}, x \in R$	(r)	−4
		(s)	−9

Sol. (A) → (p); (B) → (p, q, r, s); (C) → (p, q, s)

(A) $y = \frac{x^2 - 2x + 9}{x^2 + 2x + 9} \Rightarrow x^2y + 2xy + 9y = x^2 - 2x + 9$

$$\Rightarrow \quad (y-1)x^2 + 2x(y+1) + 9(y-1) = 0$$

$$\because \quad x \in R$$

$$\therefore \quad 4(y+1)^2 - 4 \cdot 9 \cdot (y-1)^2 \ge 0$$

$$\Rightarrow \quad (y+1)^2 - (3y-3)^2 \ge 0$$

$$\Rightarrow \quad (4y-2)(-2y+4) \ge 0$$

$$\Rightarrow \quad (2y-1)(y-2) \le 0$$

$$\therefore \quad \frac{1}{2} \le y \le 2 \Rightarrow y = 1, 2 \text{ (p)}$$

(B) $\because \quad y = \frac{x^2 - 3x - 2}{2x - 3} \Rightarrow 2xy - 3y = x^2 - 3x - 2$

$$\Rightarrow \quad x^2 - x(3 + 2y) + 3y - 2 = 0 \quad \because x \in R$$

$$\therefore \quad (3 + 2y)^2 - 4 \cdot 1 \cdot (3y - 2) \ge 0$$

$$\Rightarrow \quad 4y^2 + 17 \ge 0$$

$$\therefore \quad y \in R \text{ (p, q, r, s)}$$

(C) $\because$ $y = \dfrac{2x^2 - 2x + 4}{x^2 - 4x + 3}$

$\Rightarrow$ $x^2y - 4xy + 3y = 2x^2 - 2x + 4$

$\Rightarrow$ $x^2(y-2) + 2x(1-2y) + 3y - 4 = 0$

$\because$ $x \in R$

$\therefore$ $4(1-2y)^2 - 4(y-2)(3y-4) \geq 0$

$\Rightarrow$ $(4y^2 - 4y + 1) - (3y^2 - 10y + 8) \geq 0$

$\Rightarrow$ $y^2 + 6y - 7 \geq 0$

$\Rightarrow$ $(y+7)(y-1) \geq 0$

$\therefore$ $y \leq -7$ or $y \geq 1$(p,q,s)

• ***Ex. 25*** *Entries of Column I are to be matched with one or more entries of Column II.*

	Column I		Column II
(A)	If $a + b + 2c = 0$ but $c \neq 0$, then $ax^2 + bx + c = 0$ has	(p)	atleast one root in $(-2, 0)$
(B)	If $a, b, c \in R$ such that $2a - 3b + 6c = 0$, then equation has	(q)	atleast one root in $(-1, 0)$
(C)	Let a, b, c be non-zero real numbers such that	(r)	atleast one root in $(-1, 1)$
	$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)dx$, the equation $ax^2 + bx + c = 0$ has	(s)	atleast one root in $(0, 1)$
		(t)	atleast one root in $(0, 2)$

Sol. (A) $\rightarrow$ (r, s, t); (B) $\rightarrow$ (p, q, r); (C) $\rightarrow$ (r, s, t)

(A) Let $f(x) = ax^2 + bx + c$

Then, $f(1) = a + b + c = -c$ $[\because a + b + 2c = 0]$

and $f(0) = c$

$\therefore$ $f(0)\, f(1) = -c^2 < 0$ $[\because c \neq 0]$

$\therefore$ Equation $f(x) = 0$ has a root in $(0, 1)$.

$\therefore$ $f(x)$ has a root in $(0, 2)$ as well as in $(-1, 1)$ (r)

(B) Let $f'(x) = ax^2 + bx + c$

$\therefore$ $f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx + d$

$\therefore$ $f(0) = d$

and $f(-1) = -\dfrac{a}{3} + \dfrac{b}{2} + c + d = -\left(\dfrac{2a - 3b + 6c}{6}\right) + d$

$= 0 + d = d$ $[\because 2a - 3b + 6c = 0]$

Hence, $f(0) = f(-1)$

Hence, $f'(x) = 0$ has atleast one root in $(-1, 0)$ (q)

$\therefore$ $f(x) = 0$ has a root in $(-2, 0)$ (p) as well as $(-1, 1)$ (r)

(C) Let $f(x) = \int (1 + \cos^8 x)(ax^2 + bx + c)dx$

Given, $f(1) - f(0) = f(2) - f(0)$

$\Rightarrow$ $f(1) = f(2)$

$\Rightarrow$ $f'(x) = 0$ has atleast one root in $(0, 1)$.

$\Rightarrow$ $(1 + \cos^8 x)(ax^2 + bx + c) = 0$ has atleast one root in $(0, 1)$.

$\Rightarrow$ $ax^2 + bx + c = 0$ has atleast one root in $(0, 1)$ (s)

$\therefore$ $ax^2 + bx + c = 0$ has a root in $(0, 2)$ (t) as well as in $(-1, 1)$(r)

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** Example numbers 26 and 27 are Assertion-Reason type examples. Each of these examples contains two statements:
  Statement-1 (Assertion) and **Statement-2** (Reason)
  Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
  (c) Statement-1 is true, Statement-2 is false
  (d) Statement-1 is false, Statement-2 is true

• ***Ex. 26*** **Statement 1** *Roots of $x^2 - 2\sqrt{3}x - 46 = 0$ are rational.*

Statement 2 *Discriminant of $x^2 - 2\sqrt{3}x - 46 = 0$ is a perfect square.*

Sol. (*d*) In $ax^2 + bx + c = 0$, $a, b, c \in Q$

[here Q is the set of rational number]

If $D > 0$ and is a perfect square, then roots are real, distinct and rational.

But, here $2\sqrt{3} \notin Q$

$\therefore$ Roots are not rational.

Here, roots are $\dfrac{2\sqrt{3} \pm \sqrt{(12 + 184)}}{2}$

i.e. $\sqrt{3} \pm 7$. [irrational]

But $D = 12 + 184 = 196 = (14)^2$

$\therefore$ Statement-1 is false and Statement-2 is true.

• ***Ex. 27*** **Statement 1** *The equation $a^x + b^x + c^x - d^x = 0$ has only one real root, if $a > b > c > d$.*

Statement 2 *If $f(x)$ is either strictly increasing or decreasing function, then $f(x) = 0$ has only one real root.*

Sol. (c) $\because$ $a^x + b^x + c^x - d^x = 0$

$\Rightarrow$ $a^x + b^x + c^x = d^x$

Let $f(x) = \left(\frac{a}{d}\right)^x + \left(\frac{b}{d}\right)^x + \left(\frac{c}{d}\right)^x - 1$

$\therefore \quad f'(x) = \left(\frac{a}{d}\right)^x \ln\left(\frac{a}{d}\right) + \left(\frac{b}{d}\right)^x \ln\left(\frac{b}{d}\right) + \left(\frac{c}{d}\right)^x \ln\left(\frac{c}{d}\right) > 0$

and $f(0) = 2$

$\therefore$ $f(x)$ is increasing function and $\lim_{x \to -\infty} f(x) = -1$

$\Rightarrow$ $f(x)$ has only one real root.

But Statement-2 is false.

For example, $f(x) = e^x$ is increasing but $f(x) = 0$ has no solution.

Theory of Equations Exercise 8 :
Questions Asked in Previous 13 Years' Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

112. If α, β are the roots of $ax^2 + bx + c = 0$ and $\alpha + \beta$, $\alpha^2 + \beta^2, \alpha^3 + \beta^3$ are in GP, where $\Delta = b^2 - 4ac$, then **[IIT-JEE 2005, 3M]**

(a) $\Delta \neq 0$ (b) $b\Delta = 0$ (c) $cb \neq 0$ (d) $c\Delta = 0$

113. If S is a set of $P(x)$ is polynomial of degree ≤ 2 such that $P(0) = 0$, $P(1) = 1$, $P'(x) > 0, \forall x \in (0, 1)$, then **[IIT-JEE 2005, 3M]**

(a) $S = 0$

(b) $S = ax + (1 - a)x^2, \forall a \in (0, \infty)$

(c) $S = ax + (1 - a)x^2, \forall a \in R$

(d) $S = ax + (1 - a)x^2, \forall a \in (0, 2)$

114. If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is **[AIEEE 2005, 3M]**

(a) 1 (b) 2
(c) 3 (d) 4

115. If the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x = 0$, $a_1 \neq 0$, $n \geq 2$, has a positive root $x = \alpha$, then the equation $na_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots + a_1 = 0$ has a positive root, which is **[AIEEE 2005, 3M]**

(a) greater than or equal to α
(b) equal to α
(c) greater than α
(d) smaller than α

116. If both the roots of the quadratic equation

$$x^2 - 2kx + k^2 + k - 5 = 0$$

are less than 5, k lies in the interval **[AIEEE 2005, 3M]**

(a) $(-\infty, 4)$ (b) $[4, 5]$
(c) $(5, 6)$ (d) $(6, \infty)$

117. Let a and b be the roots of equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c and d, the value of $a + b + c + d$, when $a \neq b \neq c \neq d$, is **IIT-JEE 2006, 6M]**

118. Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a+b+c)x + 3\lambda(ab+bc+ca) = 0$ are real, then **[IIT-JEE 2006, 3M]**

(a) $\lambda < \frac{4}{3}$ (b) $\lambda < \frac{5}{3}$
(c) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (d) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

119. All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4, lie in the interval **[AIEEE 2006, 3M]**

(a) $-2 < m < 0$ (b) $m > 3$
(c) $-1 < m < 3$ (d) $1 < m < 4$

120. If the roots of the quadratic equation $x^2 + px + q = 0$ are $\tan 30°$ and $\tan 15°$, respectively, the value of $2 + q - p$ is **[AIEEE 2006, 3M]**

(a) 2 (b) 3
(c) 0 (d) 1

121. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. The value of r is **[IIT-JEE 2007, 3M]**

(a) $\frac{2}{9}(p-q)(2q-p)$ (b) $\frac{2}{9}(q-p)(2p-q)$
(c) $\frac{2}{9}(q-2p)(2q-p)$ (d) $\frac{2}{9}(2p-q)(2q-p)$

122. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, the set of possible values of a, is **[AIEEE 2007, 3M]**

(a) $(-3, 3)$ (b) $(-3, \infty)$
(c) $(3, \infty)$ (d) $(-\infty, -3)$

123. Let a, b, c, p, q be real numbers. Suppose α, β are roots of the equation $x^2 + 2px + q = 0$ and $\alpha, \frac{1}{\beta}$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$.

Statement-1 $(p^2 - q)(b^2 - ac) \geq 0$ and

Statement-2 $b \neq pa$ or $c \neq qa$ **[IIT-JEE 2008, 3M]**

(a) Statement-1 is true, Statement-2, is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

124. The quadratic equation $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3. The common root is **[AIEEE 2008, 3M]**

(a) 4 (b) 3 (c) 2 (d) 1

125. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? **[AIEEE 2008, 3M]**

(a) 1 (b) 3 (c) 5 (d) 7

126. Suppose the cubic $x^3 - px + q = 0$ has three distinct real roots, where $p > 0$ and $q < 0$. Which one of the following holds? **[AIEEE 2008, 3M]**

(a) The cubic has minima at $\left(-\sqrt{\frac{p}{3}}\right)$ and maxima at $\sqrt{\frac{p}{3}}$
(b) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $\left(-\sqrt{\frac{p}{3}}\right)$
(c) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $\left(-\sqrt{\frac{p}{3}}\right)$
(d) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $\left(-\sqrt{\frac{p}{3}}\right)$

127. The smallest value of k, for which both roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have value at least 4, is **[IIT-JEE 2009, 4M]**

(a) 6 (b) 4 (c) 2 (d) 0

128. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$, is **[AIEEE 2009, 4M]**

(a) less than $(-4ab)$ (b) greater than $4ab$
(c) less than $4ab$ (d) greater than $(-4ab)$

129. Let p and q be real numbers such that $p \neq 0, p^3 \neq -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots, is **[IIT-JEE 2010, 3M]**

(a) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(b) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(c) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(d) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

130. Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$, real number s lies in the interval **[IIT-JEE 2010, 3M]**

(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, \frac{3}{4}\right)$ (c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

131. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[IIT-JEE 2011, 3 and JEE Main 2015,4M]**

(a) 1 (b) 2 (c) 3 (d) 4

132. A value of b for which the equations

$$x^2 + bx - 1 = 0 \quad x^2 + x + b = 0$$

have one root in common, is **[IIT-JEE 2011, 3M]**

(a) $-\sqrt{2}$ (b) $-i\sqrt{3}, i = \sqrt{-1}$
(c) $i\sqrt{5}, i = \sqrt{-1}$ (d) $\sqrt{2}$

133. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[IIT-JEE 2011, 4M]**

134. Let for $a \neq a_1 \neq 0$, $f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = (-1)$ and $p(-2) = 2$, the value of $p(2)$ is **[AIEEE 2011, 4M]**

(a) 18 (b) 3 (c) 9 (d) 6

135. Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are **[AIEEE 2011, 4M]**

(a) $-4, -3$ (b) 6, 1 (c) 4, 3 (d) $-6, -1$

136. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation

$$(\sqrt[3]{(1+a)} - 1)x^2 + (\sqrt{(1+a)} - 1)x + (\sqrt[6]{(1+a)} - 1) = 0,$$

where $a > -1$, then $\lim_{a \to 0^+} \alpha(a)$ and $\lim_{a \to 0^+} \beta(a)$, are **[IIT-JEE 2012, 3M]**

(a) $\left(-\frac{5}{2}\right)$ and 1 (b) $\left(-\frac{1}{2}\right)$ and (-1)
(c) $\left(-\frac{7}{2}\right)$ and 2 (d) $\left(-\frac{9}{2}\right)$ and 3

137. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has **[AIEEE 2012, 4M]**

(a) exactly one real root
(b) exactly four real roots
(c) infinite number of real roots
(d) no real roots

138. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$ have a common root, then $a : b : c$ is **[JEE Main 2013, 4M]**

(a) 3 : 2 : 1 (b) 1 : 3 : 2 (c) 3 : 1 : 2 (d) 1 : 2 : 3

139. If $a \in R$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where [·] denotes the greatest integer function) has no integral solution, then all possible values of a lie in the interval **[JEE Main 2014, 4M]**

(a) $(-2, -1)$
(b) $(-\infty, -2) \cup (2, \infty)$
(c) $(-1, 0) \cup (0, 1)$
(d) $(1, 2)$

140. Let α, β be the roots of the equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q, r are in AP and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, the value of $\boxed{|\alpha - \beta|}$ is **[JEE Main 2014, 4M]**

(a) $\frac{\sqrt{34}}{9}$ (b) $\frac{2\sqrt{13}}{9}$
(c) $\frac{\sqrt{61}}{9}$ (d) $\frac{2\sqrt{17}}{9}$

141. Let $a \in R$ and let $f : R \to R$ be given by $f(x) = x^5 - 5x + a$. Then, **[JEE Advanced 2014, 3M]**

(a) $f(x)$ has three real roots, if $a > 4$
(b) $f(x)$ has only one real root, if $a > 4$
(c) $f(x)$ has three real roots, if $a < -4$
(d) $f(x)$ has three real roots, if $-4 < a < 4$

142. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then, $p(p(x)) = 0$ has **[JEE Advanced 2014, 3M]**

(a) only purely imaginary roots
(b) all real roots
(c) two real and two purely imaginary roots
(d) neither real nor purely imaginary roots

143. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$.

Which of the following intervals is (are) a subset(s) of S? **[JEE Advanced 2015, 4M]**

(a) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (b) $\left(-\frac{1}{\sqrt{5}}, 0\right)$
(c) $\left(0, \frac{1}{\sqrt{5}}\right)$ (d) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

must

144. The sum of all real values of x satisfying the equation $(x^2 - 5x + 5)^{x^2 + 4x - 60} = 1$ is **[JEE Main 2016, 4M]**

(a) 6 (b) 5 (c) 3 (d) −4

145. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals **[JEE Advanced 2016, 3M]**

(a) $2(\sec\theta - \tan\theta)$ (b) $2\sec\theta$
(c) $-2\tan\theta$ (d) 0

146. If for a positive integer n, the quadratic equation $x(x+1) + (x+1)(x+2) \ldots + (x + \overline{n-1})(x+n) = 10n$ has two consecutive integral solutions, then n is equal to **[JEE Main 2017, 4M]**

(a) 11 (b) 12
(c) 9 (d) 10

Answers

Exercise for Session 1

1.(b)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (a)
7.(c)	**8.** (b)	**9.** (c)	**10.** (d)	**11.** (b)	

Exercise for Session 2

1.(a)	**2.** (c)	**3.** (b)	**4.** (a)	**5.**(d)	**6.** (c)
7. (c)	**8.** (b)	**9.** (c)	**10.** (a)		

Exercise for Session 3

1.(a)	**2.** (b)	**3.** (c)	**4.** (c)	**5.**(d)	**6.** (c)
7.(c)	**8.** (a)	**9.** (a)	**10.** (d)		

Exercise for Session 4

1.(c)	**2.** (c)	**3.** (c)	**4.** (d)	**5.**(a)	**6.** (d)
7.(d)	**8.** (c)	**9.** (b)	**10.** (d)		

Exercise for Session 5

1.(a)	**2.** (a)	**3.** (b)	**4.** (c)	**5.**(c)	**6.** (b)
7.(a)	**8.** (b)	**9.** (b)	**10.** (d)		

Chapter Exercises

1.(b)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (a)
7.(c)	**8.** (b)	**9.** (c)	**10.** (a)	**11.** (b)	**12.** (c)
13.(b)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (b)	**18.** (b)
19.(c)	**20.** (d)	**21.** (b)	**22.** (a)	**23.** (c)	**24.** (b)
25.(a)	**26.** (a)	**27.** (b)	**28.** (c)	**29.** (a)	**30.** (c)
31.(a,b)	**32.**(b,c)	**33.**(a,d)	**34.**(a,b,c,d)	**35.** (b,d)	**36.** (a,b,c,d)
37.(a, b,c,d)	**38.** (a,c)	**39.** (a,b)	**40.** (a,b,c,d)	**41.** (a,c)	**42.** (b,c)
43.(c,d)	**44.**(a,c,d)	**45.**(a,c,d)			
46.(d)	**47.** (b)	**48.** (d)	**49.**(d)	**50.**(d)	**51.** (c)
52.(b)	**53.** (b)	**54.** (c)	**55.** (b)	**56.** (d)	**57.** (a)
58.(c)	**59.** (a)	**60.** (b)	**61.** (c)	**62.** (b)	**63.** (c)
64.(d)	**65.** (a)	**66.** (b)	**67.** (4)	**68.** (4)	**69.** (9)
70. (6)	**71.** (3)	**72.** (4)	**73.** (2)	**74.** (5)	**75.**(3)

76.(7) **77.** (A) → (r,s), (B) → (p,q,r,s,t), (C) → (p,q,t)

78. (A) → (q,r,s), (B) → (p), (C) → (q)

79. (A) → (q,r,s,t), (B) → (q,r), (C) → (p,q)

80.(A) → (p,q,r,s), (B) → (p,q), (C) → (s) **81.**(d) **82.** (a)

83. (a) **84.** (a) **85.** (a) **86.** (d) **87.**(a)

88. (i) $m \in (0, 3)$ (ii) $m = 0, 3$
(iii) $m \in (-\infty, 0) \cup (3, \infty)$ (iv) $m \in (-\infty, -1) \cup [3, \infty)$
(v) $m \in \phi$ (vi) $m \in (-1, -1/8)$
(vii) $m = -1/3$ (viii) $m \in (-\infty, -1) \cup (-1, -1/8) \cup [3, \infty)$
(ix) $m \in (-1, -1/8)$ (x) $m = \dfrac{81 \pm \sqrt{6625}}{32}$

89. (i) $m \in \left(-\infty, \frac{7-\sqrt{33}}{2}\right)$ (ii) $m \in \left(\frac{7+\sqrt{33}}{2}, \infty\right)$ (iii) $m \in \phi$
(iv) $m \in \left(\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2}\right)$
(v) $m \in (0, 3)$ (vi) $m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right)$
(vii) $m \in \left(\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2}\right)$
(viii) $m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \infty\right)$
(ix) $m \in \left(-\infty, \frac{7-\sqrt{33}}{2}\right) \cup \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right)$
(x) $m \in \left(\frac{11-\sqrt{73}}{2}, \frac{7+\sqrt{33}}{2}\right)$

93. $a^2l^2x^2 - ablmx + (b^2 - 2ac)ln + (m^2 - 2ln)ac = 0$

97. $x \in \phi$

98. $x_1 = 1 + \sqrt{1 + \log_{2+\sqrt{3}} 10}, x_2 = 1 - \sqrt{1 + \log_{2+\sqrt{3}} 10}$

99. $x_1 = 2, x_2 = -1 + \sqrt{3}$ and $x_3 = -1 - \sqrt{3}$

100. $x_1 = 2$

101. $a \in (-\infty, -1) \cup \left(\frac{\sqrt{3}}{2}, \infty\right)$

102. $x \in (-1 - \sqrt{5}, -3) \cup (\sqrt{5} - 1, 5)$

103. The pairs (0, 1), (1, 0), $\left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ are solutions of the original system of equations.

104. (i) $ry^3 - q(r+1)y^2 + p(r+1)^2 y - (r+1)^3 = 0$
(ii) $y^3 - py^2 + (4q - p^2)y + (8r - 4pq + p^3) = 0$ and $4pq - p^3 - 8r$

106. $a \in \left(\frac{3}{4}, \infty\right)$ **107.** $x_1 = -1, x_2 = -1/2$ **109.** Four

110. $a \in \left(-\frac{1}{4}, 1\right)$ **111.** 80 **112.** (d) **113.** (d) **114.** (a)

115. (d)	**116.** (a)	**117.** 1210	**118.**(a)	**119.** (c)	**120.** (b)
121. (d)	**122.** (a)	**123.** (b)	**124.** (c)	**125.** (a)	**126.** (d)
127. (c)	**128.** (d)	**129.** (b)	**130.** (c)	**131.** (c)	**132.** (b)
133.(2)	**134.** (a)	**135.** (b)	**136.** (b)	**137.** (d)	**138.** (d)
139. (c)	**140.** (b)	**141.**(b,d)	**142.** (d)	**143.** (a, d)	
144. (c)	**145.** (c)	**146.** (a)			