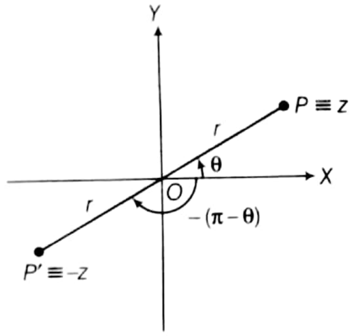
Session 3

amp(z) − amp (−z) = ± π; According as amp (z) is Positive or Negative, Square Root of a Complex Number, Solution of Complex Equations, De-Moivre's Theorem, Cube Roots of Unity

$\text{amp}(z) - \text{amp}(-z) = \pm\pi$, According as amp (z) is Positive or Negative

Case I **amp (z) is positive.**

If $\text{amp}(z) = \theta$, we have



$$\text{amp}(-z) = -(\angle P'OX) = -(\pi - \theta)$$

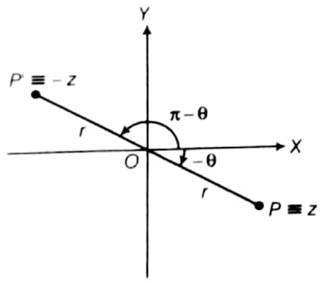
$\therefore$ $\quad \textbf{amp}(z) - \textbf{amp}(-z) = \pi$ [here, $OP = OP'$]

Case II **amp (z) is negative.**

If $\quad \text{amp}(z) = -\theta$

We have, $\quad \text{amp}(-z) = \angle P'OX = \pi - \theta$

$\therefore$ $\textbf{amp}(z) - \textbf{amp}(-z) = -\pi$ [here, $OP = OP'$]



| Example 47. If $|z_1| = |z_2|$ and $\arg(z_1/z_2) = \pi$, then find the value of $z_1 + z_2$.

Sol. $\because \arg\left(\frac{z_1}{z_2}\right) = \pi$

$\Rightarrow \quad \arg(z_1) - \arg(z_2) = \pi$ ...(i)

$\because \quad z_1 = |z_1|(\cos(\arg z_1) + i\sin(\arg z_1))$ ...(ii)

and $\quad z_2 = |z_2|(\cos(\arg z_2) + i\sin(\arg z_2))$ ...(iii)

From Eq. (ii), we get

$$z_1 = |z_2|(\cos(\pi + \arg(z_2)) + i\sin(\pi + \arg(z_2)))$$

[from Eq. (i) and $|z_1| = |z_2|$]

$$= |z_2|(-\cos(\arg z_2) - i\sin(\arg z_2)) = -z_2$$

[from Eq. (iii)]

$\therefore z_1 + z_2 = 0$

| Example 48. Let z and w be two non-zero complex numbers, such that $|z| = |w|$ and $\text{amp}(z) + \text{amp}(w) = \pi$, then find the relation between z and w.

Sol. Given, $\text{amp}(z) + \text{amp}(w) = \pi$

$\Rightarrow \quad \text{amp}(z) - \text{amp}(\overline{w}) = \pi$

Here, $\quad |z| = |\overline{w}| = |w|$ [given $|z| = |w|$]

and $\quad \text{amp}(z) > 0$

Then, $\quad z + \overline{w} = 0$

Square Root of a Complex Number

Let $\quad z = x + iy,$

where $x, y \in R$ and $i = \sqrt{-1}$.

Suppose $\sqrt{(x + iy)} = a + ib$ ...(i)

On squaring both sides, we get

$$(x + iy) = (a^2 - b^2) + 2iab$$

On comparing the real and imaginary parts, we get

$$a^2 - b^2 = x \quad \text{...(ii)}$$

and $\quad 2ab = y$ ...(iii)

$\therefore \quad a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = \sqrt{(x^2 + y^2)}$

$$a^2 + b^2 = |z| \quad \text{...(iv)}$$

From Eqs. (ii) and (iv), we get

$$a = \pm\sqrt{\left(\frac{|z| + x}{2}\right)}, \; b = \pm\sqrt{\left(\frac{|z| - x}{2}\right)}$$

or $\quad a = \pm\sqrt{\left(\frac{|z| + \text{Re}(z)}{2}\right)}, \; b = \pm\sqrt{\left(\frac{|z| - \text{Re}(z)}{2}\right)}$

Now, from Eq. (i), the required square roots,

i.e. $\sqrt{z} = \begin{cases} \pm\left(\sqrt{\dfrac{|z| + \text{Re}(z)}{2}} + i\sqrt{\dfrac{|z| - \text{Re}(z)}{2}}\right), \text{ if Im}(z) > 0 \\ \pm\left(\sqrt{\dfrac{|z| + \text{Re}(z)}{2}} - i\sqrt{\dfrac{|z| - \text{Re}(z)}{2}}\right), \text{ if Im}(z) < 0 \end{cases}$

Aliter

If $\sqrt{(x+iy)}$, where $x, y \in R$ and $i = \sqrt{-1}$, then

(i) If y is not even, then multiply and divide in y by 2, then $\sqrt{(x+iy)}$ convert in

$$\sqrt{x + y\sqrt{-1}} = \sqrt{\left(x + 2\sqrt{-\frac{y^2}{4}}\right)}.$$

(ii) Factorise: $-\dfrac{y^2}{4}$ say $\alpha, \beta\ (\alpha < \beta)$.

Take that possible factor which satisfy

$x = (\alpha i)^2 + \beta^2$, if $x > 0$ or $x = \alpha^2 + (i\beta)^2$, if $x < 0$

(iii) Finally, write $x + iy = (\alpha i)^2 + \beta^2 + 2i\alpha\beta$

or $\alpha^2 + (i\beta)^2 + 2i\alpha\beta$

and take their square root.

(iv) $\sqrt{(x+iy)} = \begin{cases} \pm(\alpha i + \beta) \\ \text{or} \pm(\alpha + i\beta) \end{cases}$ and $\sqrt{(x-iy)} = \begin{cases} \pm(\beta - i\alpha) \\ \text{or} \pm(\alpha - i\beta) \end{cases}$

Remark

1. The square root of i is $\pm\left(\dfrac{1+i}{\sqrt{2}}\right)$, where $i = \sqrt{-1}$.
2. The square root of $(-i)$ is $\left(\dfrac{1-i}{\sqrt{2}}\right)$.

| Example 49. Find the square roots of the following

(i) $4 + 3i$ (ii) $-5 + 12i$

(iii) $-8 - 15i$ (iv) $7 - 24i$ (where, $i = \sqrt{-1}$)

Sol. (i) Let $z = 4 + 3i$

$\therefore\ |z| = 5, \text{Re}(z) = 4, \text{Im}(z) = 3 > 0$

$\because\ \sqrt{z} = \pm\left(\sqrt{\dfrac{|z| + \text{Re}(z)}{2}} + i\sqrt{\dfrac{|z| - \text{Re}(z)}{2}}\right)$

$\therefore\ \sqrt{(4+3i)} = \pm\left(\sqrt{\left(\dfrac{5+4}{2}\right)} + i\sqrt{\left(\dfrac{5-4}{2}\right)}\right) = \pm\left(\dfrac{3+i}{\sqrt{2}}\right)$

Aliter

$$\sqrt{(4+3i)} = \sqrt{4 + 3\sqrt{-1}} = \sqrt{4 + 2\sqrt{\left(-\frac{9}{4}\right)}}$$

$$= \sqrt{\frac{9}{2} - \frac{1}{2} + 2\sqrt{\left(-\frac{9}{4}\right)}}$$

$$= \sqrt{\left\{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{i}{\sqrt{2}}\right)^2 + 2\cdot\frac{3}{\sqrt{2}}\cdot\frac{i}{\sqrt{2}}\right\}}$$

$$= \sqrt{\left(\frac{3+i}{\sqrt{2}}\right)^2} = \pm\left(\frac{3+i}{\sqrt{2}}\right)$$

(ii) Let $z = -5 + 12i$

$\therefore\ |z| = 13, \text{Re}(z) = -5, \text{Im}(z) = 12 > 0$

$\because\ \sqrt{z} = \pm\left(\sqrt{\dfrac{|z| + \text{Re}(z)}{2}} + i\sqrt{\dfrac{|z| - \text{Re}(z)}{2}}\right)$

$\therefore\ \sqrt{(-5+12i)} = \pm\left(\sqrt{\left(\dfrac{13-5}{2}\right)} + i\sqrt{\left(\dfrac{13+5}{2}\right)}\right)$

$= \pm(2 + 3i)$

Aliter

$$\sqrt{(-5+12i)} = \sqrt{(-5 + 12\sqrt{-1})}$$
$$= \sqrt{(-5 + 2\sqrt{(-36)})}$$
$$= \sqrt{(-5 + 2\sqrt{(-9\times 4)})}$$
$$= \sqrt{(-9 + 4 + 2\sqrt{(-9\times 4)})}$$
$$= \sqrt{(3i)^2 + 2^2 + 2\cdot 3i\cdot 2}$$
$$= \sqrt{(2+3i)^2} = \pm(2+3i)$$

(iii) Let $z = -8 - 15i$

$\therefore\ |z| = 17, \text{Re}(z) = -8, \text{Im}(z) = -15 < 0$

$\therefore\ \sqrt{(-8-15i)} = \pm\left(\sqrt{\left(\dfrac{17-8}{2}\right)} - i\sqrt{\left(\dfrac{17+8}{2}\right)}\right)$

$= \pm\left(\dfrac{3-5i}{\sqrt{2}}\right)$

Aliter $\sqrt{(-8-15i)} = \sqrt{(-8 - 15\sqrt{-1})}$

$$= \sqrt{\left(-8 - 2\sqrt{\left(-\frac{225}{4}\right)}\right)} = \sqrt{\left(-8 - 2\sqrt{\left(-\frac{25}{2}\times\frac{9}{2}\right)}\right)}$$

$$= \sqrt{\left(\frac{9}{2} - \frac{25}{2} - 2\sqrt{\left(-\frac{25}{2}\times\frac{9}{2}\right)}\right)}$$

$$= \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{5i}{\sqrt{2}}\right)^2 - 2\cdot\frac{3}{\sqrt{2}}\cdot\frac{5i}{\sqrt{2}}}$$

$$= \sqrt{\left(\frac{3-5i}{\sqrt{2}}\right)^2} = \pm\left(\frac{3-5i}{\sqrt{2}}\right)$$

(iv) Let $z = 7 - 24i$

$\therefore |z| = 25, \text{Re}(z) = 7, \text{Im}(z) = -24 < 0$

$$\because \sqrt{z} = \pm\left(\sqrt{\frac{|z| + \text{Re}(z)}{2}} - i\sqrt{\frac{|z| - \text{Re}(z)}{2}}\right)$$

$$\therefore \sqrt{(7 - 24i)} = \pm\left(\sqrt{\left(\frac{25 + 7}{2}\right)} - i\sqrt{\left(\frac{25 - 7}{2}\right)}\right)$$

$$= \pm(4 - 3i)$$

Aliter

$$\sqrt{(7 - 24i)} = \sqrt{(7 - 24\sqrt{-1})} = \sqrt{7 - 2\sqrt{(-144)}}$$
$$= \sqrt{7 - 2\sqrt{(16 \times -9)}}$$
$$= \sqrt{16 - 9 - 2\sqrt{(16 \times -9)}}$$
$$= \sqrt{(4)^2 + (3i)^2 - 2 \cdot 4 \cdot 3i}$$
$$= \sqrt{(4 - 3i)^2} = \pm(4 - 3i)$$

| Example 50. Find the square root of $x + \sqrt{(-x^4 - x^2 - 1)}$.

Sol. Let $z = x + \sqrt{(-x^4 - x^2 - 1)}$

$$= x + i\sqrt{(x^4 + x^2 + 1)} \qquad [\because \sqrt{-1} = i]$$

$$\therefore \quad |z| = \sqrt{x^2 + (x^4 + x^2 + 1)}$$
$$= \sqrt{(x^4 + 2x^2 + 1)} = \sqrt{(x^2 + 1)^2}$$

$$\therefore \quad |z| = (x^2 + 1)$$

$\text{Re}(z) = x$

$\text{Im}(z) = \sqrt{(x^4 + x^2 + 1)} > 0$

$$\because \quad \sqrt{z} = \pm\left(\sqrt{\frac{|z| + \text{Re}(z)}{2}} + i\sqrt{\frac{|z| - \text{Re}(z)}{2}}\right)$$

$$\therefore \quad \sqrt{x + \sqrt{(-x^4 - x^2 - 1)}}$$
$$= \pm\left(\sqrt{\left(\frac{x^2 + 1 + x}{2}\right)} + i\sqrt{\left(\frac{x^2 + 1 - x}{2}\right)}\right)$$

Aliter

$$\sqrt{x + \sqrt{(-x^4 - x^2 - 1)}} = \sqrt{x + 2\sqrt{\left(\frac{-x^4 - x^2 - 1}{4}\right)}}$$
$$= \sqrt{x + 2\sqrt{\left(\frac{-(x^2 + x + 1)(x^2 - x + 1)}{4}\right)}}$$
$$= \sqrt{x + 2\sqrt{\left[\left(\frac{x^2 + x + 1}{2}\right) \times -\left(\frac{x^2 - x + 1}{2}\right)\right]}}$$
$$= \sqrt{\left(\frac{x^2 + x + 1}{2}\right) - \left(\frac{x^2 - x + 1}{2}\right) + 2\sqrt{\left[\left(\frac{x^2 + x + 1}{2}\right) \times -\left(\frac{x^2 - x + 1}{2}\right)\right]}}$$
$$= \sqrt{\left[\left(\sqrt{\left(\frac{x^2 + x + 1}{2}\right)}\right)^2 + \left(i\sqrt{\left(\frac{x^2 - x + 1}{2}\right)}\right)^2 + 2\sqrt{\left(\frac{x^2 + x + 1}{2}\right)} \cdot i\sqrt{\left(\frac{x^2 - x + 1}{2}\right)}\right]}$$
$$= \sqrt{\left(\sqrt{\left(\frac{x^2 + x + 1}{2}\right)} + i\sqrt{\left(\frac{x^2 - x + 1}{2}\right)}\right)^2}$$
$$= \pm\left(\sqrt{\left(\frac{x^2 + x + 1}{2}\right)} + i\sqrt{\left(\frac{x^2 - x + 1}{2}\right)}\right)$$

Solution of Complex Equations

Putting $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$ in the given equation and equating the real and imaginary parts, we get x and y, then required solution is $z = x + iy$.

| Example 51. Solve the equation $z^2 + |z| = 0$.

Sol. Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$ ...(i)

$\Rightarrow \quad z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$

and $|z| = \sqrt{(x^2 + y^2)}$

Then, given equation reduces to

$$x^2 - y^2 + 2ixy + \sqrt{(x^2 + y^2)} = 0$$

On comparing the real and imaginary parts, we get

$$x^2 - y^2 + \sqrt{(x^2 + y^2)} = 0 \qquad \text{...(ii)}$$

and $$2xy = 0 \qquad \text{...(iii)}$$

From Eq. (iii), let $x = 0$ and from Eq. (ii),

$$-y^2 + \sqrt{y^2} = 0$$

$\Rightarrow \quad -|y|^2 + |y| = 0$

$\therefore \quad |y| = 0, 1$

$\Rightarrow \quad y = 0, \pm 1$

From Eq. (iii), let $y = 0$ and from Eq. (ii),

$$x^2 + \sqrt{x^2} = 0$$

$\Rightarrow \quad x^2 + |x| = 0$

$\Rightarrow \quad |x|^2 + |x| = 0 \Rightarrow x = 0$

$\therefore \quad x + iy$ are $0 + 0 \cdot i,\ 0 + i,\ 0 - i$

i.e. $z = 0, i, -i$ are the solutions of the given equation.

Example 52. Find the number of solutions of the equation $z^2 + |z|^2 = 0$.

Sol. $\because$ $z^2 + |z|^2 = 0$ or $z^2 + z\bar{z} = 0$

$\Rightarrow$ $z(z + \bar{z}) = 0$

$\therefore$ $z = 0$ ...(i)

and $z + \bar{z} = 0 \Rightarrow 2\text{Re}(z) = 0$

$\therefore$ $\text{Re}(z) = 0$

If $z = x + iy$ $[\because x = \text{Re}(z)]$

$= 0 + iy, y \in R$

and $i = \sqrt{-1}$ ...(ii)

On combining Eqs. (i) and (ii), then we can say that the given equation has infinite solutions.

Example 53. Find all complex numbers satisfying the equation $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$, where $i = \sqrt{-1}$.

Sol. Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$

$\Rightarrow$ $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$

and $|z| = \sqrt{(x^2 + y^2)}$

Then, given equation reduces to

$$2(x^2 + y^2) + x^2 - y^2 + 2ixy - 5 + i\sqrt{3} = 0$$

$\Rightarrow$ $(3x^2 + y^2 - 5) + i(2xy + \sqrt{3}) = 0 = 0 + i \cdot 0$

On comparing the real and imaginary parts, we get

$3x^2 + y^2 - 5 = 0$ ...(i)

and $2xy + \sqrt{3} = 0$ ...(ii)

On substituting the value of x from Eq. (ii) in Eq. (i), we get

$$3\left(-\frac{\sqrt{3}}{2y}\right)^2 + y^2 - 5 = 0$$

$\Rightarrow$ $\frac{9}{4y^2} + y^2 = 5$

or $4y^4 - 20y^2 + 9 = 0$

$\Rightarrow$ $(2y^2 - 9)(2y^2 - 1) = 0$

$\therefore$ $y^2 = \frac{9}{2}, y^2 = \frac{1}{2}$ or $y = \pm\frac{3}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{2}}$

or $y = -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

From Eq. (ii), we get

$$x = \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$$

$\therefore$ $z = x + iy$

$$= \frac{1}{\sqrt{6}} - \frac{3i}{\sqrt{2}}, -\frac{1}{\sqrt{6}} + \frac{3i}{\sqrt{2}}, \sqrt{\frac{3}{2}} - \frac{i}{\sqrt{2}}, -\sqrt{\frac{3}{2}} + \frac{i}{\sqrt{2}}$$

are the solutions of the given equation.

De-Moivre's Theorem

Statements

(i) If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n \in R$ and $i = \sqrt{-1}$, then

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$$

(ii) If $\theta \in R, n \in I$ (set of integers) and $i = \sqrt{-1}$, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(iii) If $\theta \in R$, $n \in Q$ (set of rational numbers) and $i = \sqrt{-1}$, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Proof

(i) By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\text{LHS} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= e^{i\theta_1} \cdot e^{i\theta_2} \cdot e^{i\theta_3} \ldots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)}$$
$$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) = \text{RHS}$$

(ii) If $\theta_1 = \theta_2 = \theta_3 = \ldots = \theta_n = \theta$, then from the above result (i), $(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$ $(\cos\theta + i\sin\theta)\ldots$ upto n factors

$$= \cos(\theta + \theta + \theta + \ldots \text{ upto } n \text{ times}) + i\sin(\theta + \theta + \theta + \ldots \text{ upto } n \text{ times})$$

i.e., $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(iii) Let $n = \frac{p}{q}$, where $p, q \in I$ and $q \neq 0$, from above result (ii),

we have $\left(\cos\left(\frac{p}{q}\theta\right) + i\sin\left(\frac{p}{q}\theta\right)\right)^q$

$$= \cos\left(\left(\frac{p}{q}\theta\right)q\right) + i\sin\left(\left(\frac{p}{q}\theta\right)q\right) = \cos p\theta + i\sin p\theta$$

$\Rightarrow$ $\cos\left(\frac{p\theta}{q}\right) + i\sin\left(\frac{p\theta}{q}\right)$ is one of the values of $(\cos p\theta + i\sin p\theta)^{1/q}$

$\Rightarrow$ $\cos\left(\frac{p\theta}{q}\right) + i\sin\left(\frac{p\theta}{q}\right)$ is one of the values of $[(\cos\theta + i\sin\theta)^p]^{1/q}$

$\Rightarrow \cos\left(\frac{p\theta}{q}\right) + i\sin\left(\frac{p\theta}{q}\right)$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$

Other Forms of De-Moivre's Theorem

1. $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta, \forall n \in I$

Proof $(\cos\theta - i\sin\theta)^n = (\cos(-\theta) + i\sin(-\theta))^n$
$= \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

2. $(\sin\theta + i\cos\theta)^n = (i)^n (\cos n\theta - i\sin n\theta), \forall n \in I$

Proof $(\sin\theta + i\cos\theta)^n = (i(\cos\theta - i\sin\theta))^n$
$= i^n (\cos\theta - i\sin\theta)^n = (i)^n (\cos n\theta - i\sin n\theta)$ [from remark (1)]

3. $(\sin\theta - i\cos\theta)^n = (-i)^n (\cos n\theta + i\sin n\theta), \forall n \in I$

Proof $(\sin\theta - i\cos\theta)^n = (-i(\cos\theta + i\sin\theta))^n$
$= (-i)^n (\cos\theta + i\sin\theta)^n$
$= (-i)^n (\cos n\theta + i\sin n\theta)$

4. $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi, \forall n \in I$

[here, $\theta \neq \phi \therefore$ De-Moivre's theorem is not applicable]

5. $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1}$
$= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

| Example 54. If $z_r = \cos\left(\frac{\pi}{3^r}\right) + i\sin\left(\frac{\pi}{3^r}\right)$, where $i = \sqrt{-1}$, prove that $z_1 z_2 z_3 \ldots$ upto infinity $= i$.

Sol. We have, $z_r = \cos\left(\frac{\pi}{3^r}\right) + i\sin\left(\frac{\pi}{3^r}\right)$

$$\therefore z_1 z_2 z_3 \ldots \infty = \cos\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots + \infty\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots + \infty\right)$$

$$= \cos\left(\frac{\frac{\pi}{3}}{1 - \frac{1}{3}}\right) + i\sin\left(\frac{\frac{\pi}{3}}{1 - \frac{1}{3}}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$

$= 0 + i \cdot 1 = i$

| Example 55. Express $\dfrac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5}$ in $a + ib$ form, where $i = \sqrt{-1}$.

Sol. $\because (\sin\theta + i\cos\theta)^5 = (i)^5 (\cos\theta - i\sin\theta)^5$
$= i(\cos\theta + i\sin\theta)^{-5}$

$$\therefore \frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \frac{(\cos\theta + i\sin\theta)^4}{i(\cos\theta + i\sin\theta)^{-5}}$$
$$= \frac{(\cos\theta + i\sin\theta)^9}{i}$$
$$= \frac{\cos 9\theta + i\sin 9\theta}{i} = -i\cos 9\theta + \sin 9\theta$$
$$= \sin 9\theta - i\cos 9\theta$$

To Find the Roots of $(a+ib)^{p/q}$, where $a, b \in R$; $p, q \in I, q \neq 0$ and $i = \sqrt{-1}$

Let $a + ib = r(\cos\theta + i\sin\theta)$ [polar form]

$$\therefore (a+ib)^{p/q} = \{r(\cos(2n\pi + \theta) + i\sin(2n\pi + \theta))\}^{p/q}, n \in I$$
$$= r^{p/q} (\cos(2n\pi + \theta) + i\sin(2n\pi + \theta))^{p/q}$$
$$= r^{p/q}\left(\cos\left(\frac{p}{q}(2n\pi + \theta)\right) + i\sin\left(\frac{p}{q}(2n\pi + \theta)\right)\right),$$

where, $n = 0, 1, 2, 3, \ldots, q - 1$

| Example 56. Find all roots of $x^5 - 1 = 0$.

Sol. $\because x^5 - 1 = 0 \Rightarrow x^5 = 1$

$\therefore x = (1)^{1/5} = (\cos 0 + i\sin 0)^{1/5}$,

where $i = \sqrt{-1}$

$$= [\cos(2n\pi + 0) + i\sin(2n\pi + 0)]^{1/5}$$
$$= \cos\left(\frac{2n\pi}{5}\right) + i\sin\left(\frac{2n\pi}{5}\right),$$

where, $n = 0, 1, 2, 3, 4$

$\therefore$ Roots are

$1, \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right), \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right),$
$\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right), \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$

Now, $\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)$
$$= \cos\left(2\pi - \frac{4\pi}{5}\right) + i\sin\left(2\pi - \frac{4\pi}{5}\right)$$
$$= \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)$$

and $\cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$
$$= \cos\left(2\pi - \frac{2\pi}{5}\right) + i\sin\left(2\pi - \frac{2\pi}{5}\right)$$
$$= \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)$$

Hence, roots are 1, $\cos\left(\frac{2\pi}{5}\right) \pm i\sin\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right) \pm i\sin\left(\frac{4\pi}{5}\right)$.

Remark

Five roots are 1, $z_1, z_2, \bar{z}_1, \bar{z}_2$ (one real, two complex and two conjugate of complex roots).

| Example 57. Find all roots of the equation $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$. G.P

Sol. $\because \quad 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 = 0$

$$\Rightarrow \quad 1 \cdot \frac{[1-(-x)^7]}{1-(-x)} = 0, 1 + x \neq 0$$

or $\quad 1 + x^7 = 0, \ x \neq -1$ or $x^7 = -1$

$$\therefore \quad x = (-1)^{1/7} = (\cos \pi + i \sin \pi)^{1/7}, i = \sqrt{-1}$$

$$= [\cos (2n+1)\pi + i \sin (2n+1)\pi]^{1/7}$$

$$= \cos\left(\frac{(2n+1)\pi}{7}\right) + i \sin\left(\frac{(2n+1)\pi}{7}\right)$$

for $n = 0, 1, 2, 4, 5, 6$.

Remark

$\because$ For $n = 3$, $x = -1$ but here $x \neq -1$

$\therefore \quad n \neq 3$

Cube Roots of Unity

Let $z = (1)^{1/3} \Rightarrow z^3 = 1 \Rightarrow z^3 - 1 = 0$

$\Rightarrow (z-1)(z^2 + z + 1) = 0 \Rightarrow \quad z - 1 = 0$ or $z^2 + z + 1 = 0$

$$\therefore \quad z = 1 \text{ or } z = \frac{-1 \pm \sqrt{(1-4)}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Therefore, $z = 1, \dfrac{-1 + i\sqrt{3}}{2}, \dfrac{-1 - i\sqrt{3}}{2}$, where $i = \sqrt{-1}$.

If second root is represented by ω (omega), third root will be ω^2.

$\therefore$ Cube roots of unity are $1, \omega, \omega^2$ and ω, ω^2 are called non-real complex cube roots of unity.

Remark

1. $\bar{\omega} = \omega^2, (\bar{\omega})^2 = \omega$
2. $\sqrt{\omega} = \pm \omega^2, \sqrt{\omega^2} = \pm \omega$
3. $|\omega| = |\omega^2| = 1$

Aliter

Let $\quad z = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3}, i = \sqrt{-1}$

$$= [\cos (2n\pi + 0) + i \sin (2n\pi + 0)]^{1/3}$$

$$= \cos\left(\frac{2n\pi}{3}\right) + i \sin\left(\frac{2n\pi}{3}\right), \text{ where, } n = 0, 1, 2$$

Therefore, roots are

$$1, \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right), \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)$$

or $\quad 1, e^{2\pi i/3}, e^{4\pi i/3}$

If second root is represented by ω, then third root will be ω^2 or if third root is represented by ω, then second root will be ω^2.

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

(ii) **To find the value of $\omega^n (n > 3)$.**

First divide n by 3. Let q be the quotient and r be the remainder.

$$\begin{array}{r} 3\,)\,n\,(q \\ -3q \\ \hline r \end{array}$$

i.e. $\quad n = 3q + r$, where $0 \leq r \leq 2$

$$\therefore \quad \omega^n = \omega^{3q+r} = (\omega^3)^q \cdot \omega^r = \omega^r$$

In general, $\omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2$

(iii) $1 + \omega^r + \omega^{2r} = \begin{cases} 3, \text{ when } n \text{ is a multiple of } 3 \\ 0, \text{ when } n \text{ is not a multiple of } 3 \end{cases}$

(iv) Cube roots of -1 are $-1, -\omega$ and $-\omega^2$.

(v) $a + b\omega + c\omega^2 = 0 \Rightarrow a = b = c$, if $a, b, c \in R$.

(vi) If a, b, c are non-zero numbers such that $a + b + c = 0 = a^2 + b^2 + c^2$, then $a : b : c = 1 : \omega : \omega^2$.

(vii) A complex number $a + ib$ (where $i = \sqrt{-1}$), for which $|a : b| = 1 : \sqrt{3}$ or $\sqrt{3} : 1$ can always be expressed in terms of ω or ω^2.

For example,

(a) $1 + i\sqrt{3} = -(-1 - i\sqrt{3}) \qquad [\because |1 : \sqrt{3}| = 1 : \sqrt{3}]$

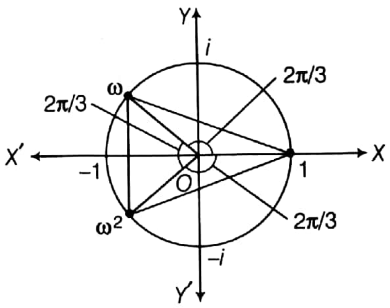
$$= -2\left(\frac{-1 - i\sqrt{3}}{2}\right) = -2\omega^2$$

(b) $\sqrt{3} + i = \dfrac{i(\sqrt{3} + i)}{i} = \dfrac{(-1 + i\sqrt{3})}{i}$

$$= \left(\frac{-1 + i\sqrt{3}}{2}\right)\left(\frac{2}{i}\right) \qquad [\because |\sqrt{3} : 1| = \sqrt{3} : 1]$$

$$= \frac{2\omega}{i} = -2i\omega$$

(viii) The cube roots of unity when represented on complex plane lie on vertices of an equilateral triangle inscribed in a unit circle, having centre at origin. One vertex being on positive real axis.



Important Relations in Terms of Cube Root of Unity

(i) $a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2)$

(ii) $a^2 - ab + b^2 = (a + b\omega)(a + b\omega^2)$

(iii) $a^3 + b^3 = (a + b)(a + b\omega)(a + b\omega^2)$

(iv) $a^3 - b^3 = (a - b)(a - b\omega)(a - b\omega^2)$

(v) $a^2 + b^2 + c^2 - ab - bc - ca$
$$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

(vi) $a^3 + b^3 + c^3 - 3abc$
$$= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

| Example 58. If ω is a non-real complex cube root of unity, find the values of the following.

(i) ω^{1999}

(ii) ω^{-998}

(iii) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^{3n+2}$, $n \in N$ and $i = \sqrt{-1}$

(iv) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots$ upto $2n$ factors

(v) $\left(\dfrac{\alpha+\beta\omega+\gamma\omega^2+\delta\omega^2}{\beta+\alpha\omega^2+\gamma\omega+\delta\omega}\right)$, where $\alpha, \beta, \gamma, \delta \in R$

(vi) $1\cdot(2-\omega)(2-\omega^2)+2\cdot(3-\omega)(3-\omega^2)+3\cdot(4-\omega)(4-\omega^2)+\ldots+\ldots+(n-1)\cdot(n-\omega)(n-\omega^2)$

Sol. (i) $\omega^{1999} = \omega^{3\times 666+1} = \omega$

(ii) $\omega^{-998} = \dfrac{1}{\omega^{998}} = \dfrac{\omega}{\omega^{999}} = \omega$

(iii) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^{3n+2} = \omega^{3n+2} = \omega^{3n}\cdot\omega^2 = (\omega^3)^n\cdot\omega^2$
$$= (1)^n \cdot \omega^2 = \omega^2$$

(iv) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots$ upto $2n$ factors
$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\ldots$ upto $2n$ factors
$= (-\omega^2)(-\omega)(-\omega^2)(-\omega)\ldots$ upto $2n$ factors
$= (\omega^3)(\omega^3)\ldots$ upto n factors $= 1\cdot1\cdot1\cdot\ldots$upto n factors
$= (1)^n = 1$

(v) $\left(\dfrac{\alpha+\beta\omega+\gamma\omega^2+\delta\omega^2}{\beta+\alpha\omega^2+\gamma\omega+\delta\omega}\right) = \dfrac{\omega(\alpha+\beta\omega+\gamma\omega^2+\delta\omega^2)}{(\beta\omega+\alpha\omega^3+\gamma\omega^2+\delta\omega^2)}$
$$= \frac{\omega(\alpha+\beta\omega+\gamma\omega^2+\delta\omega^2)}{(\beta\omega+\alpha+\gamma\omega^2+\delta\omega^2)} = \omega$$

(vi) $\Sigma(n-1)(n-\omega)(n-\omega^2) = \Sigma(n^3-1) = \Sigma n^3 - \Sigma 1$
$$= \left\{\frac{n(n+1)}{2}\right\}^2 - n$$

| Example 59. If α, β and γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$, find the value of $\dfrac{\alpha-1}{\beta-1} + \dfrac{\beta-1}{\gamma-1} + \dfrac{\gamma-1}{\alpha-1}$.

Sol. We have, $x^3 - 3x^2 + 3x + 7 = 0$

$\Rightarrow (x-1)^3 + 8 = 0$

$\Rightarrow (x-1)^3 + 2^3 = 0$

$\Rightarrow (x-1+2)(x-1+2\omega)(x-1+2\omega^2) = 0$

$\Rightarrow (x+1)(x-1+2\omega)(x-1+2\omega^2) = 0$

$\therefore x = -1, 1-2\omega, 1-2\omega^2$

$\Rightarrow \alpha = -1, \beta = 1-2\omega, \gamma = 1-2\omega^2$

Then, $\dfrac{\alpha-1}{\beta-1} + \dfrac{\beta-1}{\gamma-1} + \dfrac{\gamma-1}{\alpha-1} = \dfrac{-2}{-2\omega} + \dfrac{-2\omega}{-2\omega^2} + \dfrac{-2\omega^2}{-2}$
$$= \frac{1}{\omega} + \frac{1}{\omega} + \omega^2 = \omega^2 + \omega^2 + \omega^2 = 3\omega^2$$

| Example 60. If $z = \dfrac{\sqrt{3}+i}{2}$, where $i = \sqrt{-1}$, find the value of $(z^{101} + i^{103})^{105}$.

Sol. $\because$ $z = \dfrac{\sqrt{3}+i}{2} = \dfrac{1}{i}\left(\dfrac{i\sqrt{3}+i^2}{2}\right)$ $[\because i^2 = -1]$
$$= -i\left(\frac{-1+i\sqrt{3}}{2}\right) = -i\omega$$

$\therefore z^{101} = (-i\omega)^{101} = -i^{101}\cdot\omega^{101} = -i\omega^2$ and $i^{103} = i^3 = -i$

Then, $z^{101} + i^{103} = -i\omega^2 - i = -i(\omega^2+1)$
$$= -i(-\omega) = i\omega$$

Hence, $(z^{101} + i^{103})^{105} = (i\omega)^{105} = i^{105}\cdot\omega^{105} = i\cdot 1 = i$

| Example 61. If $\left(\dfrac{3}{2} + \dfrac{i\sqrt{3}}{2}\right)^{50} = 3^{25}(x - iy)$, where $x, y \in R$ and $i = \sqrt{-1}$, find the ordered pair of (x, y).

Sol. $\because$ $\dfrac{3}{2} + \dfrac{i\sqrt{3}}{2} = \sqrt{3}\left(\dfrac{\sqrt{3}+i}{2}\right) = \dfrac{\sqrt{3}}{i}\left(\dfrac{i\sqrt{3}+i^2}{2}\right)$
$$= -i\sqrt{3}\left(\frac{-1+i\sqrt{3}}{2}\right) = -i\sqrt{3}\,\omega$$

$\therefore \left(\dfrac{3}{2} + \dfrac{i\sqrt{3}}{2}\right)^{50} = (-i\sqrt{3}\,\omega)^{50} = i^{50}\cdot 3^{25}\cdot\omega^{50}$
$$= -1\cdot 3^{25}\cdot\omega^2 = -3^{25}\cdot\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$= 3^{25}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = 3^{25}(x - iy) \quad \text{[given]}$$

$$\therefore \quad x = \frac{1}{2}, y = -\frac{\sqrt{3}}{2}$$

$\Rightarrow$ Ordered pair is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

| Example 62. If the polynomial $7x^3 + ax + b$ is divisible by $x^2 - x + 1$, find the value of $2a + b$.

Sol. Let $f(x) = 7x^3 + ax + b$

and $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

$\because f(x)$ is divisible by $x^2 - x + 1$

Then, $f(-\omega) = 0$ and $f(-\omega^2) = 0$

$\Rightarrow \quad -7\omega^3 - a\omega + b = 0$ and $-7\omega^6 - a\omega^2 + b = 0$

or $\quad -7 - a\omega + b = 0$

and $\quad -7 - a\omega^2 + b = 0$

On adding, we get

$$-14 - a(\omega + \omega^2) + 2b = 0$$

or $\quad -14 + a + 2b = 0$ or $a + 2b = 14$ ...(i)

and on subtracting, we get

$$-a(\omega - \omega^2) = 0$$

$\Rightarrow \quad a = 0 \quad [\because \omega - \omega^2 \neq 0]$

From Eq. (i), we get $b = 7$

$\therefore \quad 2a + b = 7$

Exercise for Session 3

1. The real part of $(1 - i)^{-i}$, where $i = \sqrt{-1}$ is
   (a) $e^{-\pi/4} \cos\left(\frac{1}{2}\log_e 2\right)$ (b) $-e^{-\pi/4} \sin\left(\frac{1}{2}\log_e 2\right)$
   (c) $e^{\pi/4} \cos\left(\frac{1}{2}\log_e 2\right)$ (d) $e^{-\pi/4} \sin\left(\frac{1}{2}\log_e 2\right)$

2. The amplitude of $e^{e^{-i\theta}}$, where $\theta \in R$ and $i = \sqrt{-1}$ is
   (a) $\sin\theta$ (b) $-\sin\theta$
   (c) $e^{\cos\theta}$ (d) $e^{\sin\theta}$

3. If $z = i \log_e (2 - \sqrt{3})$, where $i = \sqrt{-1}$, then the $\cos z$ is equal to
   (a) i (b) $2i$ (c) 1 (d) 2

4. If $z = i^{i^i}$, where $i = \sqrt{-1}$, then $|z|$ is equal to
   (a) 1 (b) $e^{-\pi/2}$ (c) $e^{-\pi}$ (d) e^{π}

5. $\sqrt{(-8 - 6i)}$ is equal to (where, $i = \sqrt{-1}$)
   (a) $1 \pm 3i$ (b) $\pm(1 - 3i)$ (c) $\pm(1 + 3i)$ (d) $\pm(3 - i)$

6. $\dfrac{\sqrt{(5 + 12i)} + \sqrt{(5 - 12i)}}{\sqrt{(5 + 12i)} - \sqrt{(5 - 12i)}}$ is equal to (where, $i = \sqrt{-1}$)
   (a) $-\frac{3}{2}i$ (b) $\frac{3}{4}i$ (c) $-\frac{3}{4}i$ (d) $-\frac{3}{2}$

7. If $0 < \text{amp}(z) < \pi$, then $\text{amp}(z) - \text{amp}(-z)$ is equal to
   (a) 0 (b) $2\,\text{amp}(z)$ (c) π (d) $-\pi$

8. If $|z_1| = |z_2|$ and $\text{amp}(z_1) + \text{amp}(z_2) = 0$, then
   (a) $z_1 = z_2$ (b) $\bar{z}_1 = z_2$ (c) $z_1 + z_2 = 0$ (d) $\bar{z}_1 = \bar{z}_2$

9. The solution of the equation $|z| - z = 1 + 2i$, where $i = \sqrt{-1}$, is
   (a) $2 - \frac{3}{2}i$ (b) $\frac{3}{2} + 2i$ (c) $\frac{3}{2} - 2i$ (d) $-2 + \frac{3}{2}i$

10 The number of solutions of the equation $z^2 + \bar{z} = 0$, is

(a) 1
(b) 2
(c) 3
(d) 4

11 If $z_r = \cos\left(\frac{r\alpha}{n^2}\right) + i \sin\left(\frac{r\alpha}{n^2}\right)$, where $r = 1, 2, 3, \dots, n$ and $i = \sqrt{-1}$, then $\lim_{n \to \infty} z_1 z_2 z_3 \dots z_n$ is equal to

(a) $e^{i\alpha}$
(b) $e^{-i\alpha/2}$
(c) $e^{i\alpha/2}$
(d) $\sqrt[3]{e^{i\alpha}}$

12 If $\theta \in R$ and $i = \sqrt{-1}$, then $\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n$ is equal to

(a) $\cos\left(\frac{n\pi}{2} - n\theta\right) + i \sin\left(\frac{n\pi}{2} - n\theta\right)$
(b) $\cos\left(\frac{n\pi}{2} + n\theta\right) + i \sin\left(\frac{n\pi}{2} + n\theta\right)$
(c) $\sin\left(\frac{n\pi}{2} - n\theta\right) + i \cos\left(\frac{n\pi}{2} - n\theta\right)$
(d) $\cos\left(n\left(\frac{\pi}{2} + 2\theta\right)\right) + i \sin\left(n\left(\frac{\pi}{2} + 2\theta\right)\right)$

13 If $i z^4 + 1 = 0$, where $i = \sqrt{-1}$, then z can take the value

(a) $\frac{1+i}{\sqrt{2}}$
(b) $\cos\left(\frac{\pi}{8}\right) + i \sin\left(\frac{\pi}{8}\right)$
(c) $\frac{1}{4i}$
(d) i

14 If $\omega (\neq 1)$ is a cube root of unity, then $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \dots$ upto $2n$ factors, is

(a) 2^n
(b) 2^{2n}
(c) 0
(d) 1

15 If α, β and γ are the cube roots of p $(p < 0)$, then for any x, y and z, $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}$ is equal to

(a) $\frac{1}{2}(-1 - i\sqrt{3}), i = \sqrt{-1}$
(b) $\frac{1}{2}(1 + i\sqrt{3}), i = \sqrt{-1}$
(c) $\frac{1}{2}(1 - i\sqrt{3}), i = \sqrt{-1}$
(d) None of these

Session 4

nth Root of Unity, Vector Representation of Complex Numbers, Geometrical Representation of Algebraic Operation on Complex Numbers, Rotation Theorem (Coni Method), Shifting the Origin in Case of Complex Numbers, Inverse Points, Dot and Cross Product, Use of Complex Numbers in Coordinate Geometry

nth Root of Unity

Let x be the nth root of unity, then

$$x = (1)^{1/n} = (\cos 0 + i \sin 0)^{1/n}$$
$$= (\cos (2k\pi + 0) + i \sin (2k\pi + 0))^{1/n}$$
[where k is an integer]
$$= (\cos 2k\pi + i \sin 2k\pi)^{1/n}$$
$$\therefore \quad x = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$$

where, $k = 0, 1, 2, 3, \dots, n-1$

Let $\alpha = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, the n, nth roots of unity are α^k $(k = 0, 1, 2, 3, \dots, n-1)$ i.e, the n, nth roots of unity are $1, \alpha, \alpha^2, \alpha^3, \dots, \alpha^{n-1}$ which are in GP with common ratio $= e^{2\pi i/n}$.

(a) **Sum of n, nth roots of unity**

$$1 + \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{n-1} = \frac{1 \cdot (1 - \alpha^n)}{(1 - \alpha)}$$
$$= \frac{1 - (\cos 2\pi + i \sin 2\pi)}{1 - \alpha}$$
$$= \frac{1 - (1 + 0)}{1 - \alpha} = 0$$

Remark

$1 + \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{n-1} = 0$ is the basic concept to be understood.

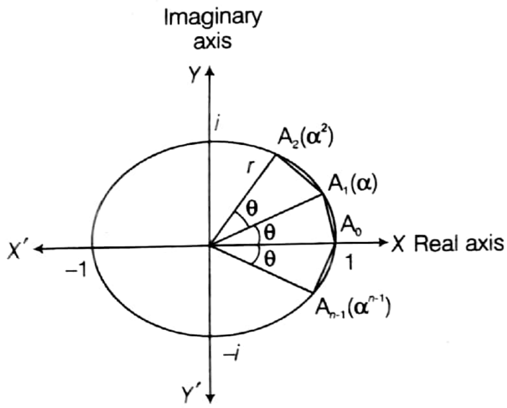
(b) **Product of n, nth roots of unity**

$$1 \times \alpha \times \alpha^2 \times \alpha^3 \times \dots \times \alpha^{n-1} = \alpha^{1+2+3+\dots+(n-1)}$$
$$= \alpha^{\frac{(n-1)n}{2}} = \left(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}\right)^{\frac{(n-1)n}{2}}$$
$$= \cos (n-1)\pi + i \sin (n-1)\pi$$
$$= (\cos \pi + i \sin \pi)^{n-1} = (-1)^{n-1}$$

Remark

$1 \cdot \alpha \cdot \alpha^2 \cdot \alpha^3 \dots \alpha^{n-1} = (-1)^{n-1}$ is the basic concept to be understood.

(c) If α is an imaginary nth root of unity, then other roots are given by $\alpha^2, \alpha^3, \alpha^4, \dots, \alpha^n$.



(d) $\because \quad 1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} = 0$

$$\Rightarrow \quad \sum_{k=0}^{n-1} \alpha^k = 0$$

or $$\sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) + i \sum_{k=0}^{n-1} \sin\left(\frac{2\pi k}{n}\right) = 0$$

$$\Rightarrow \quad \sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) = 0$$

and $$\sum_{k=0}^{n-1} \sin\left(\frac{2\pi k}{n}\right) = 0$$

These roots are located at the vertices of a regular plane polygon of n sides inscribed in a unit circle having centre at origin, one vertex being on positive real axis.

(e) $x^n - 1 = (x - 1)(x - \alpha)(x - \alpha^2) \dots (x - \alpha^{n-1})$.

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Important Benefits

1. If $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are the n, nth root of unity, then
$$(1)^p + (\alpha_1)^p + (\alpha_2)^p + \ldots + (\alpha_{n-1})^p = \begin{cases} 0, & \text{if } p \text{ is not an integral multiple of } n \\ n, & \text{if } p \text{ is an integral multiple of } n \end{cases}$$
2. $(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_{n-1}) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$
3. $(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1}) = n$
4. $z^n - 1 = (z-1)(z+1)\prod_{r=1}^{(n-2)/2}\left(z^2 - 2z\cos\frac{2r\pi}{n} + 1\right)$, if '$n$' is even.
5. $z^n + 1 = \prod_{r=0}^{(n-2)/2}\left(z^2 - 2z\cos\left(\frac{(2r+1)\pi}{n}\right) + 1\right)$, if n is even.
6. $z^n + 1 = (z+1)\prod_{r=0}^{(n-3)/2}\left(z^2 - 2z\cos\left(\frac{(2r+1)\pi}{n}\right) + 1\right)$, if '$n$' is odd.

The Sum of the Following Series Should be Remembered

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$
$$= \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \cos\left[\left(\frac{n+1}{2}\right)\theta\right]$$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$
$$= \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \sin\left[\left(\frac{n+1}{2}\right)\theta\right]$$

Proof

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$
$$= \text{Re}\{e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{ni\theta}\}, \text{ where } i = \sqrt{-1}$$
$$= \text{Re}\left\{\frac{e^{i\theta}\{(e^{i\theta})^n - 1\}}{e^{i\theta} - 1}\right\} = \text{Re}\left\{\frac{e^{i\theta} \cdot e^{ni\theta/2} \cdot 2i\sin\left(\frac{n\theta}{2}\right)}{e^{i\theta/2} \cdot 2i\sin(\theta/2)}\right\}$$
$$= \text{Re}\left\{\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot e^{\left(\frac{n+1}{2}\right)i\theta}\right\} = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \cos\left[\left(\frac{n+1}{2}\right)\theta\right]$$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$
$$= \text{Im}\{e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{ni\theta}\}, \text{ where } i = \sqrt{-1}$$
$$= \text{Im}\left\{\frac{e^{i\theta}\{(e^{i\theta})^n - 1\}}{e^{i\theta} - 1}\right\} = \text{Im}\left\{\frac{e^{i\theta} \cdot e^{\frac{ni\theta}{2}} \cdot 2i\sin\left(\frac{n\theta}{2}\right)}{e^{i\theta/2} \cdot 2i\sin\left(\frac{\theta}{2}\right)}\right\}$$
$$= \text{Im}\left\{\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot e^{\left(\frac{n+1}{2}\right)i\theta}\right\} = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \sin\left[\left(\frac{n+1}{2}\right)\theta\right]$$

Remark

For $\theta = \frac{2\pi}{n}$, we get

1. $1 + \cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cos\left(\frac{6\pi}{n}\right) + \ldots + \cos\left(\frac{(2n-2)\pi}{n}\right) = 0$

2. $\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \sin\left(\frac{6\pi}{n}\right) + \ldots + \sin\left(\frac{(2n-2)\pi}{n}\right) = 0$

| Example 63. If $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are n, nth roots of unity, find the value of $(9-\omega)(9-\omega^2)\ldots(9-\omega^{n-1})$.

Sol. Let $x = (1)^{1/n} \Rightarrow x^n - 1 = 0$

has n roots $1, \omega, \omega^2, \ldots, \omega^{n-1}$

$\therefore \quad x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$

On putting $x = 9$ in both sides, we get
$$\frac{9^n - 1}{9-1} = (9-\omega)(9-\omega^2)(9-\omega^3)\ldots(9-\omega^{n-1})$$
or $\quad (9-\omega)(9-\omega^2)\ldots(9-\omega^{n-1}) = \frac{9^n - 1}{8}$

Remark
$$\frac{x^n - 1}{x-1} = (x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$$
$$\therefore \lim_{x\to 1}\frac{x^n - 1}{x-1} = \lim_{x\to 1}(x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$$
$$\Rightarrow \quad n = (1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1})$$

| Example 64. If $a = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$, where $i = \sqrt{-1}$, find the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$.

Sol. $\because \quad a = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$

$\therefore \quad a^7 = \cos 2\pi + i\sin 2\pi = 1 + 0 = 1$

or $\quad a = (1)^{1/7}$

$\therefore \quad 1, a, a^2, a^3, a^4, a^5, a^6$ are 7, 7 th roots of unity.

$\therefore \quad 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 = 0 \quad$ ...(i)

$\Rightarrow (a + a^2 + a^4) + (a^3 + a^5 + a^6) = -1$ or $\alpha + \beta = -1$

and $\alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6)$

$$= a^4 + a^6 + a^7 + a^5 + a^7 + a^8 + a^7 + a^9 + a^{10}$$

$$= a^4 + a^6 + 1 + a^5 + 1 + a + 1 + a^2 + a^3 \quad [\because a^7 = 1]$$

$$= (1 + a + a^2 + a^3 + a^4 + a^5 + a^6) + 2$$

$$= 0 + 2 \quad \text{[from Eq. (i)]}$$

$$= 2$$

Therefore, the required equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \text{ or } x^2 + x + 2 = 0$$

| Example 65. Find the value of $\sum_{k=1}^{10}\left[\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right]$, where $i = \sqrt{-1}$.

Sol. $\sum_{k=1}^{10}\left[\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right]$

$$= -i\sum_{k=1}^{10}\left[\cos\left(\frac{2\pi k}{11}\right) + i\sin\left(\frac{2\pi k}{11}\right)\right]$$

$$= -i\left\{\sum_{k=0}^{10}\left[\cos\left(\frac{2\pi k}{11}\right) + i\sin\left(\frac{2\pi k}{11}\right)\right] - 1\right\}$$

$$= -i(0 - 1) \quad \text{[sum of 11, 11th roots of unity]}$$

$$= i$$

| Example 66. If $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the n, nth roots of the unity, then find the value of $\sum_{i=0}^{n-1}\frac{\alpha_i}{2 - \alpha_i}$.

Sol. Let $x = (1)^{1/n} \Rightarrow x^n = 1 \quad \therefore \quad x^n - 1 = 0$

or $\quad x^n - 1 = (x - \alpha_0)(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_{n-1})$

$$= \prod_{i=0}^{n-1}(x - \alpha_i)$$

On taking logarithm both sides, we get

$$\log_e(x^n - 1) = \sum_{i=0}^{n-1}\log_e(x - \alpha_i)$$

On differentiating both sides w.r.t. x, we get

$$\frac{nx^{n-1}}{x^n - 1} = \sum_{i=0}^{n-1}\left(\frac{1}{x - \alpha_i}\right)$$

On putting $x = 2$, we get

$$\frac{n(2)^{n-1}}{2^n - 1} = \sum_{i=0}^{n-1}\frac{1}{(2 - \alpha_i)} \quad \ldots\text{(i)}$$

Now, $\sum_{i=0}^{n-1}\frac{\alpha_i}{(2 - \alpha_i)} = \sum_{i=0}^{n-1}\left(-1 + \frac{2}{2 - \alpha_i}\right)$

$$= -\sum_{i=0}^{n-1}1 + 2\sum_{i=0}^{n-1}\frac{1}{(2 - \alpha_i)} = -(n) + \frac{2\cdot n\cdot 2^{n-1}}{2^n - 1} \quad \text{[from Eq. (i)]}$$

$$= -n + \frac{n\cdot 2^n}{2^n - 1} = \frac{n}{2^n - 1}$$

| Example 67. If $n \geq 3$ and $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are the n, nth roots of unity, then find the value of $\sum\sum_{1 \leq i < j \leq n-1}\alpha_i\alpha_j$.

Sol. Let $\quad x = (1)^{1/n}$

$\therefore \quad x^n = 1$ or $x^n - 1 = 0$

$\therefore \quad 1 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1} = 0$

or $\quad \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1} = -1$

On squaring both sides, we get

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \ldots + \alpha^2_{n-1} + 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_1\alpha_{n-1} + \alpha_2\alpha_3 + \ldots + \alpha_2\alpha_{n-1} + \ldots + \alpha_{n-2}\alpha_{n-1}) = 1$$

or $1^2 + (\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 + \ldots + (\alpha_{n-1})^2 + 2\sum\sum_{1 \leq i < j \leq n-1}\alpha_i\alpha_j = 1 + 1^2$

$$0 + 2\sum\sum_{1 \leq i < j \leq n-1}\alpha_i\alpha_j = 2$$

[here, p is not a multiple of n]

$$\therefore \quad \sum\sum_{1 \leq i < j \leq n-1}\alpha_i\alpha_j = 1$$

Aliter

$\because \quad x^n - 1 = (x - 1)(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_{n-1})$

On comparing the coefficient of x^{n-2} both sides, we get

$$0 = \sum\sum_{0 \leq i < j \leq n-1}\alpha_i\alpha_j + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}$$

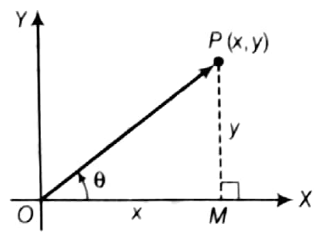
$$0 = \sum\sum_{1 \leq i < j \leq n-1}\alpha_i\alpha_j - 1$$

$$[\because 1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} = 0]$$

$$\therefore \quad \sum\sum_{1 \leq i < j \leq n-1}\alpha_i\alpha_j = 1$$

Vector Representation of Complex Numbers

If P is the point (x, y) on the argand plane corresponding to the complex number $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$.



Then, $\quad \overrightarrow{\mathbf{OP}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \Rightarrow \left|\overrightarrow{\mathbf{OP}}\right| = \sqrt{(x^2 + y^2)} = |z|$

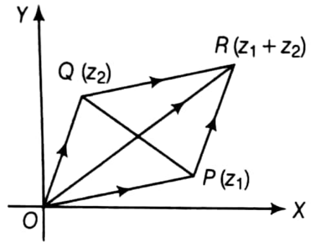
and $\arg(z) =$ direction of the vector $\overrightarrow{\mathbf{OP}} = \tan^{-1}(y/x) = \theta$

Therefore, complex number z can also be represented by $\overrightarrow{\mathbf{OP}}$.

Geometrical Representation of Algebraic Operation on Complex Numbers

(a) Sum

Let the complex numbers $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$ be represented by the points P and Q on the argand plane.



Complete the parallelogram $OPRQ$. Then, the mid-points of PQ and OR are the same. The mid-point of

$$PQ = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Hence, $R = (x_1 + x_2, y_1 + y_2)$

Therefore, complex number z can also be represented by

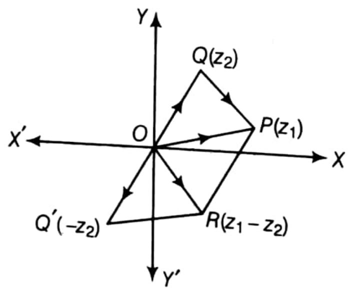
$$\overrightarrow{\mathbf{OR}} = (x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$$

In vector notation, we have

$$z_1 + z_2 = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PR}} = \overrightarrow{\mathbf{OR}}$$

(b) Difference

We first represent $-z_2$ by Q', so that QQ' is bisected at O. Complete the parallelogram $OPRQ'$. Then, the point R represents the difference $z_1 - z_2$.



We see that $ORPQ$ is a parallelogram, so that $\overrightarrow{\mathbf{OR}} = \overrightarrow{\mathbf{QP}}$

We have in vectorial notation,

$$z_1 - z_2 = \overrightarrow{\mathbf{OP}} - \overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{QO}}$$
$$= \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PR}} = \overrightarrow{\mathbf{OR}} = \overrightarrow{\mathbf{QP}}$$

(c) Product

Let $\quad z_1 = r_1 (\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$

$\therefore \quad |z_1| = r_1$ and $\arg(z_1) = \theta_1$

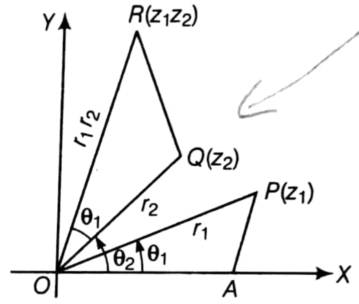
and $\quad z_2 = r_2 (\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$\therefore \quad |z_2| = r_2$ and $\arg(z_2) = \theta_2$

Then, $\quad z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$\quad = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\therefore \quad |z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$



Let P and Q represent the complex numbers z_1 and z_2, respectively.

$$\therefore \quad OP = r_1, OQ = r_2$$
$$\angle POX = \theta_1 \quad \text{and} \quad \angle QOX = \theta_2$$

Take a point A on the real axis OX, such that $OA = 1$ unit. Complete the $\angle OPA$

Now, taking OQ as the base, construct a ΔOQR similar to ΔOPA, so that $\dfrac{OR}{OQ} = \dfrac{OP}{OA}$

i.e. $\quad OR = OP \cdot OQ = r_1 r_2 \quad$ [since, $OA = 1$ unit]

and $\quad \angle ROX = \angle ROQ + \angle QOX = \theta_1 + \theta_2$

Hence, R is the point representing product of complex numbers z_1 and z_2.

Remark

1. Multiplication by i

Since, $z = r(\cos\theta + i\sin\theta)$ and $i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$$\therefore \quad iz = r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right]$$

Hence, multiplication of z with i, then vector for z rotates a right angle in the positive sense.

2. Thus, to multiply a vector by (-1) is to turn it through two right angles.
3. Thus, to multiply a vector by $(\cos\theta + i\sin\theta)$ is to turn it through the angle θ in the positive sense.

(d) Division

Let $\quad z_1 = r_1 (\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$

$\therefore \quad |z_1| = r_1$ and $\arg(z_1) = \theta_1$

and $\quad z_2 = r_2 (\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$\therefore \quad |z_2| = r_2$ and $\arg(z_2) = \theta_2$

Then, $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \cdot \dfrac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}$ $\quad [z_2 \neq 0, r_2 \neq 0]$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

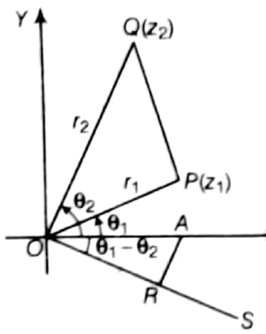
$$\therefore \quad \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}, \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

Let P and Q represent the complex numbers z_1 and z_2, respectively.

$\therefore \quad OP = r_1, OQ = r_2, \angle POX = \theta_1$ and $\angle QOX = \theta_2$

Let OS be new position of OP, take a point A on the real axis OX, such that $OA = 1$ unit and through A draw a line making with OA an angle equal to the $\angle OQP$ and meeting OS in R.

Then, R represented by (z_1/z_2).



Now, in similar ΔOPQ and ΔOAR.

$$\frac{OR}{OA} = \frac{OP}{OQ} \Rightarrow OR = \frac{r_1}{r_2}$$

since $OA = 1$ and $\angle AOR = \angle POR - \angle POX = \theta_2 - \theta_1$

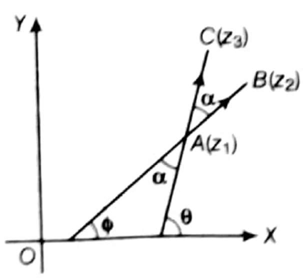
Hence, the vectorial angle of R is $-(\theta_2 - \theta_1)$ i.e., $\theta_1 - \theta_2$.

Remark

If θ_1 and θ_2 are the principal values of z_1 and z_2 (then $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ are not necessarily the principal value of $\arg(z_1 z_2)$ and $\arg(z_1/z_2)$.)

Rotation Theorem (Coni Method)

Let z_1, z_2 and z_3 be the affixes of three points A, B and C respectively taken on argand plane.



Then, we have $\overrightarrow{\mathbf{AC}} = z_3 - z_1$ and $\overrightarrow{\mathbf{AB}} = z_2 - z_1$

and let $\quad \arg \overrightarrow{\mathbf{AC}} = \arg(z_3 - z_1) = \theta$

and $\quad \arg \overrightarrow{\mathbf{AB}} = \arg(z_2 - z_1) = \phi$

Let $\quad \angle CAB = \alpha$

$$\angle CAB = \alpha = \theta - \phi = \arg \overrightarrow{\mathbf{AC}} - \arg \overrightarrow{\mathbf{AB}}$$

$$= \arg(z_3 - z_1) - \arg(z_2 - z_1)$$

$$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

or angle between AC and AB

$$= \arg\left(\frac{\text{affix of } C - \text{affix of } A}{\text{affix of } B - \text{affix of } A}\right)$$

For any complex number z, we have

$$z = |z| e^{i(\arg z)}$$

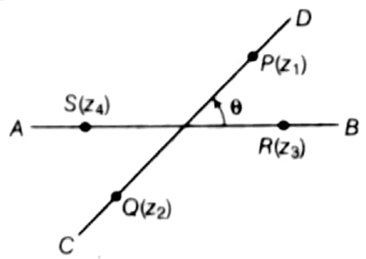
Similarly, $\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = \left|\dfrac{z_3 - z_1}{z_2 - z_1}\right| e^{i\left[\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)\right]}$

or $\quad \dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|} e^{i(\angle CAB)} = \dfrac{AC}{AB} e^{i\alpha}$

Remark

1. Here, only principal values of the arguments are considered.
2. $\arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right) = \theta$, if AB coincides with CD, then

   $\arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right) = 0$ or π, so that $\dfrac{z_1 - z_2}{z_3 - z_4}$ is real. It follows that if $\dfrac{z_1 - z_2}{z_3 - z_4}$ is real, then the points A, B, C, D are collinear.



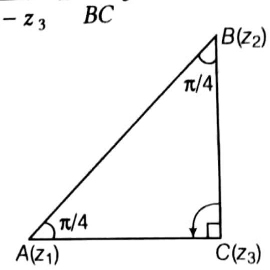
3. If AB is perpendicular to CD, then

   $\arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right) = \pm \dfrac{\pi}{2}$, so $\dfrac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary.
4. It follows that, if $z_1 - z_2 = \pm k(z_3 - z_4)$, where k is purely imaginary number, then AB and CD are perpendicular to each other.

Example 68. Complex numbers z_1, z_2 and z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Sol. Since, $\angle ACB = 90^\circ$ and $AC = BC$, then by Coni method

$$\frac{z_1 - z_3}{z_2 - z_3} = \frac{AC}{BC} e^{i\pi/2} = i$$



$$\Rightarrow \quad z_1 - z_3 = i(z_2 - z_3)$$

On squaring both sides, we get

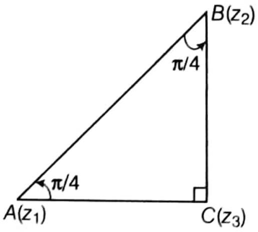
$$(z_1 - z_3)^2 = -(z_2 - z_3)^2$$

$$\Rightarrow \quad z_1^2 + z_3^2 - 2z_1z_3 = -(z_2^2 + z_3^2 - 2z_2z_3)$$

$$\Rightarrow \quad z_1^2 + z_2^2 - 2z_1z_2 = 2(z_1z_3 - z_1z_2 - z_3^2 + z_2z_3)$$

Therefore, $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

Aliter $\qquad CA = CB = \frac{1}{\sqrt{2}} BA$



$\because \quad \angle BAC = (\pi/4)$

$\therefore \quad \frac{z_2 - z_1}{z_3 - z_1} = \frac{BA}{CA} e^{(i\pi/4)}$

or $\quad \frac{z_1 - z_2}{z_1 - z_3} = \sqrt{2}\, e^{(i\pi/4)}$ ...(i)

and $\quad \angle CBA = (\pi/4)$

$\therefore \quad \frac{z_3 - z_2}{z_1 - z_2} = \frac{CB}{AB} e^{(i\pi/4)}$ or $\frac{z_3 - z_2}{z_1 - z_2} = \frac{1}{\sqrt{2}} e^{(i\pi/4)}$ ...(ii)

On dividing Eq. (i) by Eq. (ii), we get

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

Example 69. Complex numbers z_1, z_2, z_3 are the vertices of A, B, C respectively of an equilateral triangle. Show that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

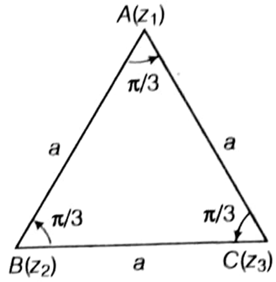
Sol. Let $\quad AB = BC = CA = a$

$\because \quad \angle ABC = \frac{\pi}{3}$

From Coni method, $\frac{z_1 - z_2}{z_3 - z_2} = \frac{a}{a} e^{i\pi/3}$ ...(i)

and $\quad \angle BAC = \frac{\pi}{3}$

From Coni method, $\frac{z_3 - z_1}{z_2 - z_1} = \frac{a}{a} e^{i\pi/3}$ ...(ii)



From Eqs. (i) and (ii), we get $\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_3 - z_1}{z_2 - z_1}$

$$\Rightarrow \quad (z_1 - z_2)(z_2 - z_1) = (z_3 - z_1)(z_3 - z_2)$$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Remark

Triangle with vertices z_1, z_2, z_3, then

(i) $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

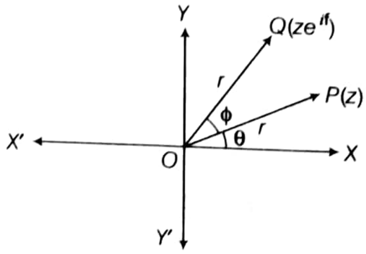
(ii) $(z_1 - z_2)^2 = (z_2 - z_3)(z_3 - z_1)$

(iii) $\sum (z_1 - z_2)(z_2 - z_3) = 0$ $\qquad$ (iv) $\sum \frac{1}{(z_1 - z_2)} = 0$

Complex Number as a Rotating Arrow in the Argand Plane

Let $\qquad z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ ...(i)

be a complex number representing a point P in the argand plane.



Then, $OP = |z| = r$ and $\angle POX = \theta$

Now, consider complex number $z_1 = ze^{i\phi}$

or $\qquad z_1 = re^{i\theta} \cdot e^{i\phi} = re^{i(\theta + \phi)}$ [from Eq. (i)]

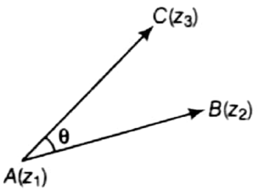
Clearly, the complex number z_1 represents a point Q in the argand plane, when $OQ = r$ and $\angle QOX = \theta + \phi$

Clearly, multiplication of z with $e^{i\phi}$ rotates the vector $\overrightarrow{\mathbf{OP}}$ through angle ϕ in anti-clockwise sense. Similarly, multiplication of z with $e^{-i\phi}$ will rotate the vector $\overrightarrow{\mathbf{OP}}$ in clockwise sense.

Remark

1. If z_1, z_2 and z_3 are the affixes of the three points A, B and C, such that $AC = AB$ and $\angle CAB = \theta$. Therefore,

$$\overrightarrow{\mathbf{AB}} = z_2 - z_1, \ \overrightarrow{\mathbf{AC}} = z_3 - z_1.$$



Then, $\overrightarrow{\mathbf{AC}}$ will be obtained by rotating $\overrightarrow{\mathbf{AB}}$ through an angle θ in anti-clockwise sense and therefore,

$$\boxed{\overrightarrow{\mathbf{AC}} = \overrightarrow{\mathbf{AB}}\, e^{i\theta}}$$

or $\quad (z_3 - z_1) = (z_2 - z_1)\, e^{i\theta}$ or $\dfrac{z_3 - z_1}{z_2 - z_1} = e^{i\theta}$ Proof & Coni method

2. If A, B and C are three points in argand plane, such that $AC = AB$ and $\angle CAB = \theta$, then use the rotation about A to find $e^{i\theta}$, but if $AC \neq AB$, then use Coni method.

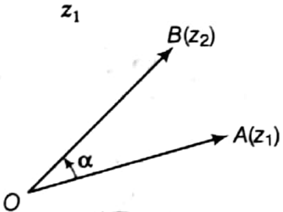
I Example 70. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2(\alpha/2)$.

Sol. Clearly, $\overrightarrow{\mathbf{OB}}$ is obtained by rotating $\overrightarrow{\mathbf{OA}}$ through angle α.

$$\therefore \quad \overrightarrow{\mathbf{OB}} = \overrightarrow{\mathbf{OA}}\, e^{i\alpha}$$

$$\Rightarrow \quad z_2 = z_1 e^{i\alpha}$$

$$\Rightarrow \quad \frac{z_2}{z_1} = e^{i\alpha} \qquad \text{...(i)}$$



$$\text{or} \quad \frac{z_2}{z_1} + 1 = (e^{i\alpha} + 1)$$

$$\Rightarrow \quad \frac{(z_1 + z_2)}{z_1} = e^{i\alpha/2} \cdot 2\cos(\alpha/2)$$

On squaring both sides, we get

$$\frac{(z_1 + z_2)^2}{z_1^2} = e^{i\alpha} \cdot (4\cos^2 \alpha/2)$$

$$\Rightarrow \quad \frac{(z_1 + z_2)^2}{z_1^2} = \frac{z_2}{z_1} \cdot (4\cos^2 \alpha/2) \qquad \text{[from Eq. (i)]}$$

$$(z_1 + z_2)^2 = 4 z_1 z_2 \cos^2(\alpha/2)$$

$$(-p)^2 = 4q\cos^2(\alpha/2)$$

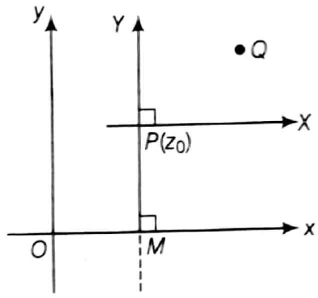
$$\left[\begin{array}{l} \because z_1 \text{ and } z_2 \text{ are the roots of } z^2 + pz + q = 0 \\ \therefore z_1 + z_2 = -p \text{ and } z_1 z_2 = q \end{array}\right]$$

$$\text{or} \quad p^2 = 4q\cos^2(\alpha/2)$$

Shifting the Origin in Case of Complex Numbers

Let O be the origin and P be a point with affix z_0. Let a point Q has affix z with respect to the coordinate system passing through O. When origin is shifted to the point P (z_0), then the new affix Z of the point Q with respect to new origin P is given by $Z = z - z_0$.

i.e., to shift the origin at z_0, we should replace z by $Z + z_0$.



I Example 71. If z_1, z_2 and z_3 are the vertices of an equilateral triangle with z_0 as its circumcentre, then changing origin to z_0, show that $Z_1^2 + Z_2^2 + Z_3^2 = 0$, where Z_1, Z_2, Z_3 are new complex numbers of the vertices.

Sol. In an equilateral triangle, the circumcentre and the centroid are the same point.

$$\text{So,} \quad z_0 = \frac{z_1 + z_2 + z_3}{3}$$

$$\therefore \quad z_1 + z_2 + z_3 = 3z_0 \qquad \text{...(i)}$$

To shift the origin at z_0, we have to replace z_1, z_2, z_3 and z_0 by $Z_1 + z_0, Z_2 + z_0, Z_3 + z_0$ and $0 + z_0$.

Then, Eq. (i) becomes

$$(Z_1 + z_0) + (Z_2 + z_0) + (Z_3 + z_0) = 3(0 + z_0)$$

$$\Rightarrow \quad Z_1 + Z_2 + Z_3 = 0$$

On squaring, we get

$$Z_1^2 + Z_2^2 + Z_3^2 + 2(Z_1Z_2 + Z_2Z_3 + Z_3Z_1) = 0 \qquad \text{...(ii)}$$

But triangle with vertices Z_1, Z_2 and Z_3 is equilateral, then

$$Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1 \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$3(Z_1^2 + Z_2^2 + Z_3^2) = 0$$

$$\text{Therefore,} \quad Z_1^2 + Z_2^2 + Z_3^2 = 0$$

Inverse Points

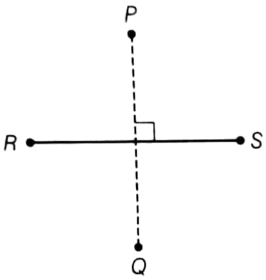
(a) **Inverse points with respect to a line** Two points P and Q are said to be the inverse points with respect to the line RS. If Q is the image of P in RS, i.e., if the line RS is the right bisector of PQ.

| Example 72. Show that z_1, z_2 are the inverse points with respect to the line $z\bar{a}+a\bar{z}=b$, if $z_1\bar{a}+a\bar{z}_2=b$.

Sol. Let RS be the line represented by the equation,

$$z\bar{a}+a\bar{z}=b \quad \text{...(i)}$$

Let P and Q are the inverse points with respect to the line RS. The point Q is the reflection (inverse) of the point P in the line RS, if the line RS is the right bisector of PQ. Take any point z in the line RS, then lines joining z to P and z to Q are equal.

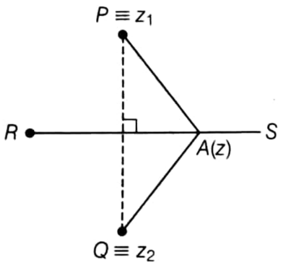


i.e., $|z-z_1|=|z-z_2|$ or $|z-z_1|^2=|z-z_2|^2$

i.e., $(z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$

$\Rightarrow z(\bar{z}_2-\bar{z}_1)+\bar{z}(z_2-z_1)+(z_1\bar{z}_1-z_2\bar{z}_2)=0$ ...(ii)

Hence, Eqs. (i) and (ii) are identical, therefore, comparing coefficients, we get



$$\frac{\bar{a}}{\bar{z}_2-\bar{z}_1}=\frac{a}{z_2-z_1}=\frac{-b}{z_1\bar{z}_1-z_2\bar{z}_2}$$

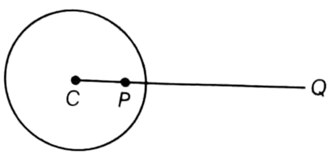
So that, $\dfrac{z_1\bar{a}}{z_1(\bar{z}_2-\bar{z}_1)}=\dfrac{a\bar{z}_2}{\bar{z}_2(z_2-z_1)}$

$$=\frac{-b}{z_1\bar{z}_1-z_2\bar{z}_2}=\frac{z_1\bar{a}+a\bar{z}_2-b}{0}$$

[by ratio and proportion rule]

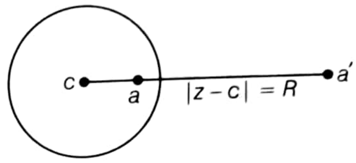
$$z_1\bar{a}+a\bar{z}_2-b=0 \text{ or } z_1\bar{a}+a\bar{z}_2=b$$

(b) **Inverse points with respect to a circle** If C is the centre of the circle and P, Q are the inverse points with respect to the circle, then three points C, P, Q are collinear and also $CP \cdot CQ = r^2$, where r is the radius of the circle.



| Example 73. Show that inverse of a point a with respect to the circle $|z-c|=R$ (a and c are complex numbers, centre c and radius R) is the point $c+\dfrac{R^2}{\bar{a}-\bar{c}}$.

Sol. Let a' be the inverse point of a with respect to the circle $|z-c|=R$, then by definition,



The points c, a, a' are collinear.

We have, $\arg(a'-c)=\arg(a-c)$

$=-\arg(\bar{a}-\bar{c})$ $\quad [\because \arg\bar{z}=-\arg z]$

$\Rightarrow \arg(a'-c)+\arg(\bar{a}-\bar{c})=0$

$\Rightarrow \arg\{(a'-c)(\bar{a}-\bar{c})\}=0$

$\therefore (a'-c)(\bar{a}-\bar{c})$ is purely real and positive.

By definition, $|a'-c||a-c|=R^2$ $\quad [\because CP \cdot CQ = r^2]$

$\Rightarrow |a'-c||\bar{a}-\bar{c}|=R^2$ $\quad [\because |z|=|\bar{z}|]$

$\Rightarrow |(a'-c)(\bar{a}-\bar{c})|=R^2$

$\Rightarrow (a'-c)(\bar{a}-\bar{c})=R^2$

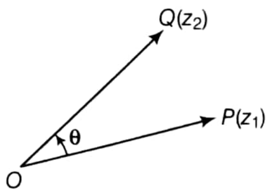
$[\because (a'-c)(\bar{a}-\bar{c})$ is purely real and positive$]$

$\Rightarrow a'-c=\dfrac{R^2}{\bar{a}-\bar{c}} \Rightarrow a'=c+\dfrac{R^2}{\bar{a}-\bar{c}}$

Dot and Cross Product

Let $z_1=x_1+iy_1\equiv(x_1,y_1)$ and $z_2=x_2+iy_2\equiv(x_2,y_2)$, where $x_1,y_1,x_2,y_2\in R$ and $i=\sqrt{-1}$, be two complex numbers.

If $\angle POQ=\theta$, then from Coni method,



$$\frac{z_2-0}{z_1-0}=\frac{|z_2|}{|z_1|}e^{i\theta}$$

$$\Rightarrow \frac{z_2\bar{z}_1}{z_1\bar{z}_1}=\frac{|z_2|}{|z_1|}e^{i\theta}$$

$$\Rightarrow \frac{z_2\bar{z}_1}{|z_1|^2}=\frac{|z_2|}{|z_1|}e^{i\theta}$$

$$z_2\bar{z}_1=|z_1||z_2|e^{i\theta}$$

$$z_2\bar{z}_1=|z_1||z_2|(\cos\theta+i\sin\theta)$$

$\Rightarrow$ $\text{Re}(z_2\bar{z}_1)=|z_1||z_2|\cos\theta$ ...(i)

and $\text{Im}(z_2\bar{z}_1)=|z_1||z_2|\sin\theta$ ...(ii)

The dot product z_1 and z_2 is defined by,

$$z_1 \cdot z_2=|z_1||z_2|\cos\theta$$

$$=\text{Re}(\bar{z}_1 z_2)=x_1x_2+y_1y_2 \quad \text{[from Eq. (i)]}$$

and cross product of z_1 and z_2 is defined by

$$z_1 \times z_2 = |z_1||z_2|\sin\theta$$
$$= \text{Im}(\bar{z}_1 z_2) = x_1y_2 - x_2y_1 \quad \text{[from Eq. (ii)]}$$

Hence, $z_1 \cdot z_2 = x_1x_2 + y_1y_2 = \boxed{\text{Re}(\bar{z}_1 z_2)}$

and $z_1 \times z_2 = x_1y_2 - x_2y_1 = \boxed{\text{Im}(\bar{z}_1 z_2)}$

Results for Dot and Cross Products of Complex Number

1. If z_1 and z_2 are perpendicular, then $z_1 \cdot z_2 = 0$
2. If z_1 and z_2 are parallel, then $z_1 \times z_2 = 0$
3. Projection of z_1 on $z_2 = (z_1 \cdot z_2)/|z_2|$
4. Projection of z_2 on $z_1 = (z_1 \cdot z_2)/|z_1|$
5. Area of triangle, if two sides represented by z_1 and z_2 is $\frac{1}{2}|z_1 \times z_2|$.
6. Area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$.
7. Area of parallelogram, if diagonals represented by z_1 and z_2 is $\frac{1}{2}|z_1 \times z_2|$.

Example 74. If $z_1 = 2 + 5i, z_2 = 3 - i$, where $i = \sqrt{-1}$, find

(i) $z_1 \cdot z_2$ (ii) $z_1 \times z_2$

(iii) $z_2 \cdot z_1$ (iv) $z_2 \times z_1$

(v) acute angle between z_1 and z_2.

(vi) projection of z_1 on z_2.

Sol. (i) $z_1 \cdot z_2 = x_1x_2 + y_1y_2 = (2)(3) + (5)(-1) = 1$

(ii) $z_1 \times z_2 = x_1y_2 - x_2y_1 = (2)(-1) - (3)(5) = -17$

(iii) $z_2 \cdot z_1 = x_1x_2 + y_1y_2 = (2)(3) + (5)(-1) = 1$

(iv) $z_2 \times z_1 = x_2y_1 - x_1y_2 = (3)(5) - (2)(-1) = 17$

(v) Let angle between z_1 and z_2 be θ, then

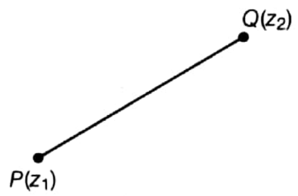
$$z_1 \cdot z_2 = |z_1||z_2|\cos\theta$$
$$\Rightarrow \quad 1 = \sqrt{(4+25)}\sqrt{(9+1)}\cos\theta$$
$$\therefore \quad \cos\theta = \frac{1}{\sqrt{290}} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{290}}\right)$$

(vi) Projection of z_1 on $z_2 = \dfrac{z_1 \cdot z_2}{|z_2|} = \dfrac{1}{\sqrt{(9+1)}} = \dfrac{1}{\sqrt{10}}$

Use of Complex Numbers in Coordinate Geometry

(a) Distance Formula

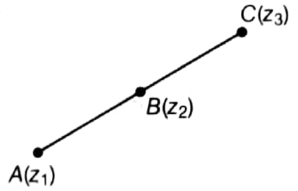
The distance between two points $P(z_1)$ and $Q(z_2)$ is given by



$$PQ = |z_2 - z_1| = |\text{affix of } Q - \text{affix of } P|$$

Remark

1. The distance of a point z from origin, $|z - 0| = |z|$
2. Three points $A(z_1)$, $B(z_2)$ and $C(z_3)$ are collinear, then $AB + BC = AC$



i.e. $|z_1 - z_2| + |z_2 - z_3| = |z_1 - z_3|$.

Example 75. Show that the points representing the complex numbers $(3 + 2i), (2 - i)$ and $-7i$, where $i = \sqrt{-1}$, are collinear.

Sol. Let $z_1 = 3 + 2i, z_2 = 2 - i$ and $z_3 = -7i$.

Then, $|z_1 - z_2| = |1 + 3i| = \sqrt{10}, |z_2 - z_3| = |2 + 6i|$
$$= \sqrt{40} = 2\sqrt{10}$$

and $|z_1 - z_3| = |3 + 9i| = \sqrt{90} = 3\sqrt{10}$

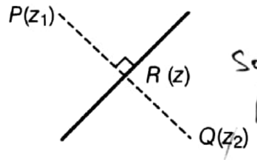
$\therefore \quad |z_1 - z_2| + |z_2 - z_3| = |z_1 - z_3|$

Hence, the points $(3 + 2i), (2 - i)$ and $-7i$ are collinear.

(b) Equation of the Perpendicular Bisector

If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is moving point, such that it is always at equal distance from $P(z_1)$ and $Q(z_2)$.

Let, $z_1 - z_2 = a$
$|z_1|^2 - |z_2|^2 = b$
So $\boxed{z\bar{a} + \bar{z}a = b}$



i.e. $PR = QR$

or $|z - z_1| = |z - z_2|$ square both sides & solve!

or $z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = z_1\bar{z}_1 - z_2\bar{z}_2$

or $z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$

Hence, z lies on the perpendicular bisectors of z_1 and z_2.

Example 76. Find the perpendicular bisector of $3 + 4i$ and $-5 + 6i$, where $i = \sqrt{-1}$.

Sol. Let $z_1 = 3 + 4i$ and $z_2 = -5 + 6i$

If z is moving point, such that it is always equal distance from z_1 and z_2.

i.e. $|z - z_1| = |z - z_2|$

or $z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$

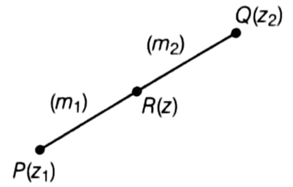
$\Rightarrow z((3 - 4i) - (-5 - 6i)) + \bar{z}((3 + 4i) - (-5 + 6i)) = 25 - 61$

Hence, $(8 + 2i)z + (8 - 2i)\bar{z} + 36 = 0$

which is required perpendicular bisector.

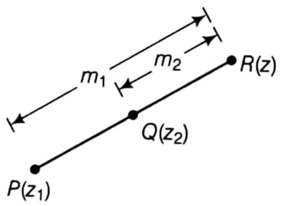
(c) Section Formula

If $R(z)$ divides the joining of $P(z_1)$ and $Q(z_2)$ in the ratio $m_1 : m_2$ $(m_1, m_2 > 0)$.



(i) If $R(z)$ divides the segment PQ internally in the ratio of $m_1 : m_2$, then $z = \dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$

(ii) If $R(z)$ divides the segment PQ externally in the ratio of $m_1 : m_2$, then $z = \dfrac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$

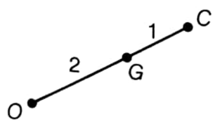


Remark

1. If $R(z)$ is the mid-point of PQ, then affix of R is $\dfrac{z_1 + z_2}{2}$.
2. If z_1, z_2 and z_3 are affixes of the vertices of a triangle, then affix of its centroid is $\dfrac{z_1 + z_2 + z_3}{3}$.
3. In acute angle triangle, orthocentre (O), nine point centre (N), centroid (G) and circumcentre (C) are collinear and $\dfrac{OG}{GC} = \dfrac{2}{1}$, $\dfrac{ON}{NG} = \dfrac{1}{1}$.
4. If z_1, z_2, z_3 and z_4 are the affixes of the vertices of a parallelogram taken in order, then $z_1 + z_3 = z_2 + z_4$.

I Example 77. If z_1, z_2 and z_3 are the affixes of the vertices of a triangle having its circumcentre at the origin. If z is the affix of its orthocentre, prove that $z_1 + z_2 + z_3 - z = 0$.

Sol. We know that orthocentre O, centroid G and circumcentre C of a triangle are collinear, such that G divides OC in the ratio $2:1$. Since, affix of G is $\dfrac{z_1 + z_2 + z_3}{3}$ and C is the origin. Therefore, by section formula, we get



$$\Rightarrow \quad \frac{z_1 + z_2 + z_3}{3} = \frac{2 \cdot 0 + 1 \cdot z}{2 + 1}$$

$$\Rightarrow \quad z_1 + z_2 + z_3 = z$$

Therefore, $z_1 + z_2 + z_3 - z = 0$

I Example 78. Let z_1, z_2 and z_3 be three complex numbers and $a, b, c \in R$, such that $a + b + c = 0$ and $az_1 + bz_2 + cz_3 = 0$, then show that z_1, z_2 and z_3 are collinear.

Sol. Given, $\quad a + b + c = 0 \quad$ ...(i)

and $\quad az_1 + bz_2 + cz_3 = 0 \quad$ ...(ii)

$\Rightarrow \quad az_1 + bz_2 - (a + b) z_3 = 0 \quad$ [from Eq. (i)]

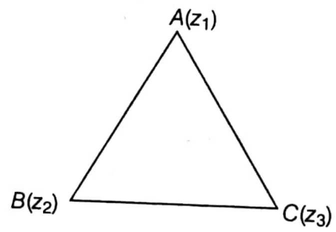
or $\quad z_3 = \dfrac{az_1 + bz_2}{a + b}$

It follows that z_3 divides the line segment joining z_1 and z_2 internally in the ratio $b : a$. (If a, b are of same sign and opposite sign, then externally.)

Hence, z_1, z_2 and z_3 are collinear.

(d) Area of Triangle

If z_1, z_2 and z_3 are the affixes of the vertices of a triangle, then its area $= \dfrac{1}{4} \left| \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} \right|$



Remark

The area of the triangle with vertices z, ωz and $z + \omega z$ is $\dfrac{\sqrt{3}}{4} |z|^2$, where ω is the cube root of unity.

I Example 79. Show that the area of the triangle on the argand plane formed by the complex numbers z, iz and $z + iz$ is $\dfrac{1}{2}|z|^2$, where $i = \sqrt{-1}$.

Sol. Required area $= \dfrac{1}{4} \left| \begin{vmatrix} z & \bar{z} & 1 \\ iz & \overline{iz} & 1 \\ z + iz & \overline{z + iz} & 1 \end{vmatrix} \right|$

$$= \frac{1}{4} \left| \begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ z + iz & \bar{z} - i\bar{z} & 1 \end{vmatrix} \right|$$

$$= \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ z+iz & \bar{z}-i\bar{z} & 1 \end{vmatrix}\right|$$

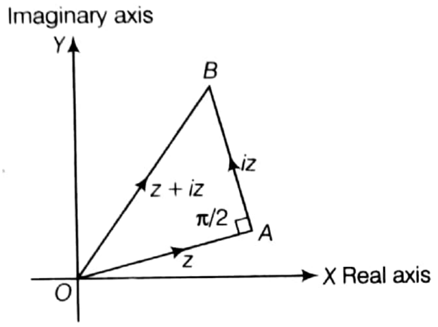
On applying $R_3 \to R_3 - (R_1 + R_2)$, we get

$$\text{Area} = \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ 0 & 0 & -1 \end{vmatrix}\right| = \frac{1}{4}\left|(-1)(-iz\bar{z} - iz\bar{z})\right|$$

$$= \frac{1}{4}\left|2iz\bar{z}\right| = \frac{1}{2}|i||z\bar{z}| = \frac{1}{2}|z|^2$$

Aliter

We have, $iz = z(\cos(\pi/2) + i\sin(\pi/2)) = ze^{(i\pi/2)}$ iz is the vector obtained by rotating vector z in anti-clockwise direction through $(\pi/2)$. Therefore, $OA \perp AB$,



Now, area of $\Delta OAB = \frac{1}{2} OA \times AB = \frac{1}{2}|z||iz|$

$$= \frac{1}{2}|z||i||z| = \frac{1}{2}|z|^2$$

(e) Equation of a Straight Line

(i) Parametric form

Equation of the straight line joining the points having affixes z_1 and z_2 is

$$z = tz_1 + (1-t)z_2, \text{ where } t \in R \sim \{0\}$$

Proof

$$\because \quad z = tz_1 + (1-t)z_2 = \frac{tz_1 + (1-t)z_2}{t + (1-t)}$$

Hence, z divides the line joining z_1 and z_2 in the ratio $1-t : t$. Thus, the points z_1, z_2, z are collinear.

(ii) Non-parametric form

Equation of the straight line joining the points having affixes z_1 and z_2 is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$

Proof Equation of the straight line joining points having affixes z_1 and z_2 is

$$z = tz_1 + (1-t)z_2, \text{ where } t \in R \sim \{0\}$$

$$\Rightarrow \quad z - z_2 = t(z_1 - z_2) \quad \text{...(i)}$$

and $\quad \overline{z - z_2} = t\,\overline{(z_1 - z_2)}$

or $\quad \bar{z} - \bar{z}_2 = t(\bar{z}_1 - \bar{z}_2) \quad \text{.. (ii)}$

From Eqs. (i) and (ii), we get

$$\frac{z - z_2}{\bar{z} - \bar{z}_2} = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \Rightarrow \frac{z - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

$$\Rightarrow \begin{vmatrix} z - z_2 & \bar{z} - \bar{z}_2 & 0 \\ z_1 - z_2 & \bar{z}_1 - \bar{z}_2 & 0 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

Now, applying $R_1 \to R_1 + R_3$ and $R_2 \to R_2 + R_3$, we get

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$

Aliter

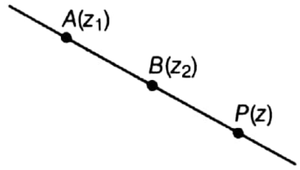
Let $P(z)$ be an arbitrary point on the line, which pass through $A(z_1)$ and $B(z_2)$.

$\therefore \quad \angle BAP = 0$ or π

$\therefore \quad \arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$ or π [by rotation theorem]

$\Rightarrow \frac{z - z_1}{z_2 - z_1}$ is purely real.

$$\therefore \quad \left(\frac{z - z_1}{z_2 - z_1}\right) = \left(\overline{\frac{z - z_1}{z_2 - z_1}}\right) \Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$



$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

or
$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

Remark

If z_1, z_2 and z_3 are collinear, $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

or $\quad \sum \bar{z}_1(z_2 - z_3) = 0.$

(iii) General form The general equation of a straight line is of the form $\bar{a} z + a \bar{z} + b = 0$, where a is a complex number and b is a real number.

Sol. The equation of a straight line passing through points having affixes z_1 and z_2 is given by

$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0 \quad \text{...(i)}$$

On multiplying Eq. (i) by i (where, $i = \sqrt{-1}$), we get

$$zi(\bar{z}_1 - \bar{z}_2) - \bar{z}\, i(z_1 - z_2) + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

$$\Rightarrow \bar{z}\{-i(z_1 - z_2)\} + z\{i(\bar{z}_1 - \bar{z}_2)\} + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

$$\Rightarrow \bar{z}\{-i(z_1 - z_2)\} + z\{\overline{-i(z_1 - z_2)}\} + \{i(2i\,\mathrm{Im}(z_1\bar{z}_2))\} = 0$$

$$\Rightarrow \bar{z}\{-i(z_1 - z_2)\} + z\{\overline{-i(z_1 - z_2)}\} + \{(-2\,\mathrm{Im}(z_1\bar{z}_2)\} = 0$$

$$\Rightarrow \quad \bar{z} a + z \bar{a} + b = 0,$$

where, $a = -i(z_1 - z_2), b = -2\,\mathrm{Im}(z_1\bar{z}_2)$

Hence, the general equation of a straight line is of the form $\bar{a} z + a \bar{z} + b = 0$,

where a is complex number and b is a real number.

(iv) Slope of the line $\bar{a} z + a \bar{z} + b = 0$

Let $A(z_1)$ and $B(z_2)$ be two points on the line $\bar{a} z + a \bar{z} + b = 0$, then

$Z = y$
$\bar{Z} = x$

$$\bar{a} z_1 + a \bar{z}_1 + b = 0$$

and
$$\bar{a} z_2 + a \bar{z}_2 + b = 0$$

$$\therefore \quad \bar{a}(z_1 - z_2) + a(\bar{z}_1 - \bar{z}_2) = 0$$

$$\Rightarrow \quad \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = -\frac{a}{\bar{a}} \quad \text{[Remember]}$$

$$\text{Complex slope of } AB = -\frac{a}{\bar{a}} = -\frac{\text{coefficient of } \bar{z}}{\text{coefficient of } z}$$

Thus, the complex slope of the line $\bar{a} z + a \bar{z} + b = 0$ is $-\frac{a}{\bar{a}}$.

Remark

The real slope of the line $\bar{a} z + a\bar{z} + b = 0$ is $-\frac{\mathrm{Re}(a)}{\mathrm{Im}(a)}$, i.e. $-\frac{\mathrm{Re}(\text{coefficient of } \bar{z})}{\mathrm{Im}(\text{coefficient of } \bar{z})}$.

Important Theorem

If α_1 and α_2 are the complex slopes of two lines on the argand plane, then prove that the lines are

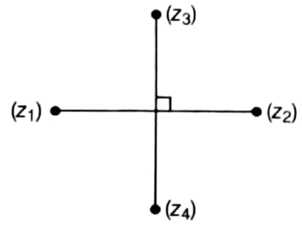
(i) perpendicular, if $\alpha_1 + \alpha_2 = 0$.

(ii) parallel, if $\alpha_1 = \alpha_2$.

Proof Let z_1 and z_2 be the affixes of two points on one line with complex slope α_1 and z_3 and z_4 be the affixes of two points another line with complex slope α_2. Then,

$$\alpha_1 = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \text{ and } \alpha_2 = \frac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4} \quad \text{...(i)}$$

(i) If the lines are perpendicular, then



$$\frac{(z_1 - z_2)}{|z_1 - z_2|} = \frac{(z_3 - z_4)}{|z_3 - z_4|} e^{i\pi/2}$$

$$\Rightarrow \frac{(z_1 - z_2)^2}{|z_1 - z_2|^2} = \frac{(z_3 - z_4)^2}{|z_3 - z_4|^2} e^{i\pi}$$

$$\Rightarrow \frac{(z_1 - z_2)^2}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} = \frac{(z_3 - z_4)^2}{(z_3 - z_4)(\bar{z}_3 - \bar{z}_4)} e^{i\pi}$$

$$\Rightarrow \frac{(z_1 - z_2)}{(\bar{z}_1 - \bar{z}_2)} = \frac{(z_3 - z_4)}{(\bar{z}_3 - \bar{z}_4)}(-1)$$

$$\Rightarrow \alpha_1 = -\alpha_2 \quad \text{[from Eq. (i)]}$$

$$\therefore \alpha_1 + \alpha_2 = 0$$

(ii) If the lines are parallel, then

$$\frac{z_1 - z_2}{|z_1 - z_2|} = \frac{z_3 - z_4}{|z_3 - z_4|} e^0$$

$$\Rightarrow \frac{(z_1 - z_2)^2}{|z_1 - z_2|^2} = \frac{(z_3 - z_4)^2}{|z_3 - z_4|^2}$$

$$\Rightarrow \frac{(z_1 - z_2)^2}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} = \frac{(z_3 - z_4)^2}{(z_3 - z_4)(\bar{z}_3 - \bar{z}_4)}$$

$$\Rightarrow \frac{(z_1 - z_2)}{(\bar{z}_1 - \bar{z}_2)} = \frac{(z_3 - z_4)}{(\bar{z}_3 - \bar{z}_4)}$$

$$\Rightarrow \alpha_1 = \alpha_2$$

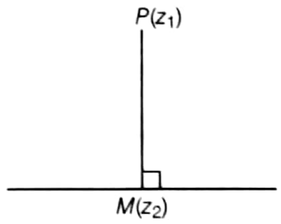
Remark

1. The equation of a line parallel to the line $\bar{a} z + a\bar{z} + b = 0$ is $\bar{a} z + a\bar{z} + \lambda = 0$, where $\lambda \in R$.
2. The equation of a line perpendicular to the line $\bar{a} z + a\bar{z} + b = 0$ is $\bar{a} z - a\bar{z} + i\lambda = 0$ where, $\lambda \in R$ and $i = \sqrt{-1}$

(v) Length of perpendicular from a given point on a given line

The length of perpendicular from a point $P(z_1)$ to the line $\bar{a} z + a \bar{z} + b = 0$ is given by $\frac{|\bar{a} z_1 + a \bar{z}_1 + b|}{2|a|}$.

Proof Let PM be perpendicular from P on the line $\bar{a}z + a\bar{z} + b = 0$ and let the affix of M be z_2, then



$$PM = |z_1 - z_2|$$

$$\bar{a}z + a\bar{z} + b = 0$$

and $M(z_2)$ lies on $\bar{a}z + a\bar{z} + b = 0$, then

$$\bar{a}z_2 + a\bar{z}_2 + b = 0 \quad \text{...(i)}$$

Since, PM perpendicular to the line $(\bar{a}z + a\bar{z} + b = 0)$.

Therefore, $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} + \left(-\dfrac{a}{\bar{a}}\right) = 0$

$$\Rightarrow \quad \bar{a}z_1 - \bar{a}z_2 - a\bar{z}_1 + a\bar{z}_2 = 0$$

$$\Rightarrow \quad \bar{a}z_1 + a\bar{z}_1 + b = 2a\bar{z}_1 + \bar{a}z_2 - a\bar{z}_2 + b$$

$$= 2a\bar{z}_1 - a\bar{z}_2 + (\bar{a}z_2 + b)$$

$$= 2a\bar{z}_1 - a\bar{z}_2 - a\bar{z}_2 \quad [\because \bar{a}z_2 + b = -a\bar{z}_2]$$

$$= 2a(\bar{z}_1 - \bar{z}_2)$$

or $\quad |\bar{a}z_1 + a\bar{z}_1 + b| = 2|a||\bar{z}_1 - \bar{z}_2|$

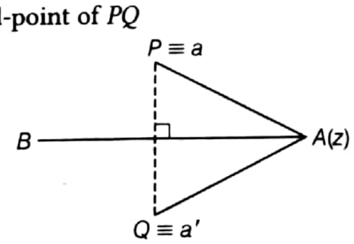
$$= 2|a||z_1 - z_2| \quad [\because |\bar{z}| = |z|]$$

$$= 2|a|PM$$

$$\therefore \quad PM = \frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$$

| Example 80. Show that the point a' is the reflection of the point a in the line $z\bar{b} + \bar{z}b + c = 0$, if $a'\bar{b} + \bar{a}b + c = 0$.

Sol. Since, a' is the reflection of point a through the line. So, the mid-point of PQ



i.e., $\dfrac{a + a'}{2}$ lies on $z\bar{b} + \bar{z}b + c = 0$

or $\quad \bar{b}\left(\dfrac{a + a'}{2}\right) + b\left(\dfrac{\overline{a + a'}}{2}\right) + c = 0$

$$\Rightarrow \quad \bar{b}(a + a') + b(\bar{a} + \bar{a}') + 2c = 0 \quad \text{...(i)}$$

Since, $PQ \perp AB$. Therefore,

Complex slope of PQ + Complex slope of $AB = 0$

$$\Rightarrow \quad \frac{a - a'}{\bar{a} - \bar{a}'} + \left(-\frac{b}{\bar{b}}\right) = 0$$

$$\Rightarrow \quad \bar{b}(a - a') - b(\bar{a} - \bar{a}') = 0 \quad \text{...(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$a'\bar{b} + \bar{a}b + c = 0$$

Aliter

Equation of perpendicular bisector of PQ is

$$z(\bar{a}' - \bar{a}) + \bar{z}(a' - a) - a'\bar{a}' + a\bar{a} = 0 \quad \text{...(i)}$$

and given line $\quad z\bar{b} + \bar{z}b + c = 0 \quad$ ...(ii)

Since, Eqs. (i) and (ii) are identical, we have

$$\frac{\bar{a}' - \bar{a}}{\bar{b}} = \frac{a' - a}{b} = \frac{a\bar{a} - a'\bar{a}'}{c} = k \quad \text{[say]}$$

$\because \quad a' - \bar{a} = \bar{b}k, a' - a = bk$

and $\quad a\bar{a} - a'\bar{a}' = ck$

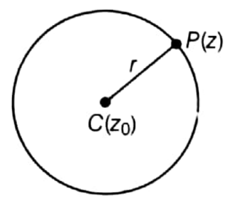
Now, $\quad a'\bar{b} + \bar{a}b = \left\{a'\left(\dfrac{\bar{a}' - \bar{a}}{k}\right) + \bar{a}\left(\dfrac{a' - a}{k}\right)\right\}$

$$= \frac{1}{k}\{a'\bar{a}' - a\bar{a}\} = \frac{1}{k}(-ck) = -c$$

Hence, $a'\bar{b} + \bar{a}b + c = 0$

(f) Circle

The equation of a circle whose centre is at point affix z_0 and radius r, is $|z - z_0| = r$.



Remark

1. If the centre of the circle is at origin and radius r, then its equation is $|z| = r$.
2. $|z - z_0| < r$ represents interior of a circle $|z - z_0| = r$ and $|z - z_0| > r$ represent exterior of the circle $|z - z_0| = r$.
3. $r < |z - z_0| < R$, this region is known as **annulus**.

(i) General Equation of a Circle

The general equation of the circle is

$$z\bar{z} + \bar{a}z + a\bar{z} + b = 0,$$

where a is a complex number and $b \in R$, having centre at $(-a)$ and radius $= \sqrt{|a|^2 - b}$.

Proof The equation of circle having centre at z_0 and radius r is

$$|z - z_0| = r$$

$$\Rightarrow \quad |z - z_0|^2 = r^2$$

$$\Rightarrow \quad (z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

$$\Rightarrow \quad z\bar{z} - z\bar{z}_0 - z_0\bar{z} + z_0\bar{z}_0 = r^2$$

$$\Rightarrow \quad z\bar{z} + (-\bar{z}_0)z + (-z_0)\bar{z} + |z_0|^2 - r^2 = 0$$

$$\Rightarrow \quad z\bar{z} + \bar{a}z + a\bar{z} + b = 0$$

where, $\quad a = -z_0$ and $b = |z_0|^2 - r^2$

$$\Rightarrow \quad z\bar{z} + \bar{a}z + a\bar{z} + b = 0$$

where, $b \in R$ represents a circle having centre at $(-a)$ and radius $= \sqrt{|z_0|^2 - b} = \sqrt{|a|^2 - b}$.

Remark

Rule to find the centre and radius of a circle whose equation is given

1. Make the coefficient of $z\bar{z}$ equal to 1 and right hand side equal to zero.
2. The centre of circle will be $= (-a) = (-$ coefficient of $\bar{z})$.
3. Radius $= \sqrt{(|a|^2 - \text{constant term})}$

| Example 81. Find the centre and radius of the circle $2z\bar{z} + (3 - i)z + (3 + i)\bar{z} - 7 = 0$, where $i = \sqrt{-1}$.

Sol. The given equation can be written as

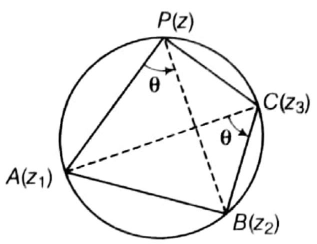
$$z\bar{z} + \left(\frac{3+i}{2}\right)z + \left(\frac{3+i}{2}\right)\bar{z} - \frac{7}{2} = 0$$

So, it represent a circle with centre at $-\left(\frac{3+i}{2}\right)$ and radius

$$= \sqrt{\left(\left|-\left(\frac{3+i}{2}\right)\right|^2 + \frac{7}{2}\right)} = \sqrt{\left(\frac{9}{4} + \frac{1}{4} + \frac{7}{2}\right)} = \sqrt{6}$$

(ii) Equation of Circle Through Three Non-Collinear Points

Let $A(z_1), B(z_2), C(z_3)$ be three points on the circle and $P(z)$ be any point on the circle, then



$$\angle ACB = \angle APB$$

Using Coni method,

in ΔACB, $$\frac{z_2 - z_3}{z_1 - z_3} = \frac{BC}{CA}e^{i\theta} \quad \text{...(i)}$$

in ΔAPB, $$\frac{z_2 - z}{z_1 - z} = \frac{BP}{AP}e^{i\theta} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)} = \text{Real} \quad \text{...(iii)}$$

Remark

If four points z_1, z_2, z_3, z_4 are concyclic, then $\frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_2)(z_1 - z_3)} =$ real [replacing z by z_4 in Eq. (iii)]

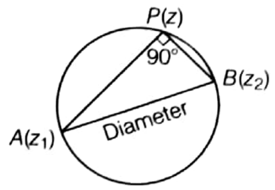
or $$\arg\left[\frac{(z_2 - z_3)(z_4 - z_1)}{(z_1 - z_3)(z_4 - z_2)}\right] = \pi, 0.$$

(iii) Equation of Circle in Diametric Form

If end points of diameter represented by $A(z_1)$ and $B(z_2)$ and $P(z)$ is any point on circle.

$$\therefore \quad \angle APB = 90^\circ$$

$\therefore$ Complex slope of PA + Complex slope of $PB = 0$



$$\Rightarrow \quad \left(\frac{z - z_1}{\bar{z} - \bar{z}_1}\right) + \left(\frac{z - z_2}{\bar{z} - \bar{z}_2}\right) = 0$$

Hence, $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$ which is required equation of circle in diametric form.

(iv) Other Forms of Circle

(a) Equation of all circles which are orthogonal to

$$|z - z_1| = r_1 \text{ and } |z - z_2| = r_2.$$

Let the circle be $|z - \alpha| = r$ cut given circles orthogonally.

$$\therefore \quad r^2 + r_1^2 = |\alpha - z_1|^2 \quad \text{...(i)}$$

and $$r^2 + r_2^2 = |\alpha - z_2|^2 \quad \text{...(ii)}$$

On solving,

$$r_2^2 - r_1^2 = \alpha(\bar{z}_1 - \bar{z}_2) + \bar{\alpha}(z_1 - z_2) + |z_2|^2 - |z_1|^2$$

and let $\alpha = a + ib$, $i = \sqrt{-1}$, $a, b \in R$

(b) **Apollonius circle** $\left|\frac{z - z_1}{z - z_2}\right| = k \neq 1$

It is the circle with join of z_3 and z_4 as a diameter, where $z_3 = \frac{z_1 + kz_2}{1 + k}$, $z_4 = \frac{z_1 - kz_2}{1 - k}$

for $k = 1$, the circle reduces to the straight line which is perpendicular bisector of the line segment from z_1 to z_2.

(c) **Circular arc** $\arg\left(\frac{z-z_1}{z-z_2}\right)=\alpha$

This is an arc of a circle in which the chord joining z_1 and z_2 subtends angle α at any point on the arc.

If $\alpha=\pm\frac{\pi}{2}$, then locus of z is a circle with the join of z_1 and z_2 as diameter. If $\alpha=0$ or π, then locus is a straight line through the points z_1 and z_2.

(d) The equation $|z-z_1|^2+|z-z_2|^2=k$, will represent a circle, if $k\geq\frac{1}{2}|z_1-z_2|^2$.

| Example 82. Find all circles which are orthogonal to $|z|=1$ and $|z-1|=4$.

Sol. Let $\quad |z-\alpha|=k \quad$ ...(i)

(where, $\alpha=a+ib$ and $a,b,k\in R$ and $i=\sqrt{-1}$) be a circle which cuts the circles

$$|z|=1 \quad \text{...(ii)}$$

and $\quad |z-1|=4 \quad$ ...(iii)

Orthogonally, then using the property that the sum of squares of their radii is equal to square of distance between centres. Thus, the circle (i) will cut the circles (ii) and (iii) orthogonally, if

$$k^2+1=|\alpha-0|^2=\alpha\bar{\alpha}$$

and
$$k^2+16=|\alpha-1|^2=(\alpha-1)(\bar{\alpha}-1)$$
$$=\alpha\bar{\alpha}-(\alpha+\bar{\alpha})+1$$

$\therefore \quad 1-(\alpha+\bar{\alpha})-15=0 \Rightarrow \alpha+\bar{\alpha}=-14$

$\therefore \quad 2a=-14 \Rightarrow a=-7$

$\Rightarrow \quad \alpha=a+ib=-7+ib$

Also, $\quad k^2=|\alpha|^2-1=(-7)^2+b^2-1=b^2+48$

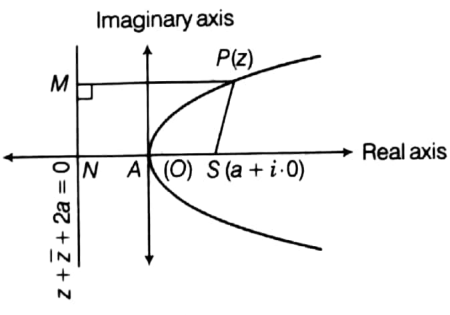
$\Rightarrow \quad k=\sqrt{(b^2+48)}$

Therefore, required family of circles is given by

$$|z+7-ib|=\sqrt{(48+b^2)}.$$

(g) Equation of Parabola

Now, for parabola



$$SP=PM$$

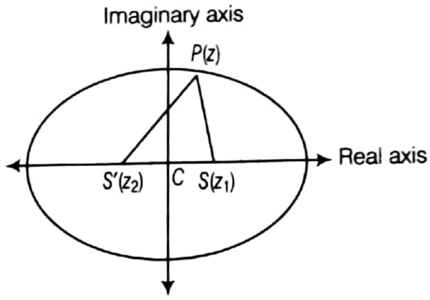
$$|z-a|=\frac{|z+\bar{z}+2a|}{2}$$

or $\quad z\bar{z}-4a(z+\bar{z})=\frac{1}{2}\{z^2+(\bar{z})^2\}$

where, $a\in R$ (focus), directrix is $z+\bar{z}+2a=0$.

(h) Equation of Ellipse

For ellipse



$$SP+S'P=2a$$

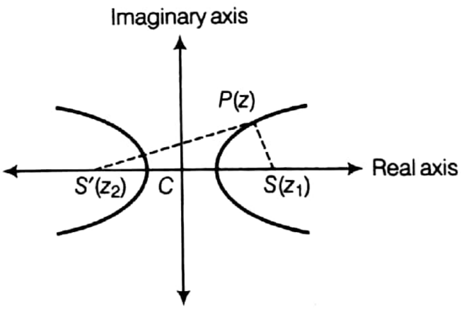
$\Rightarrow \quad |z-z_1|+|z-z_2|=2a$

where, $2a>|z_1-z_2|$ [since, eccentricity < 1]

Then, point z describes an ellipse having foci at z_1 and z_2 and $a\in R^+$.

(i) Equation of Hyperbola

For hyperbola



$$SP-S'P=2a \Rightarrow |z-z_1|-|z-z_2|=2a$$

where, $\quad 2a<|z_1-z_2|$ [since, eccentricity > 1]

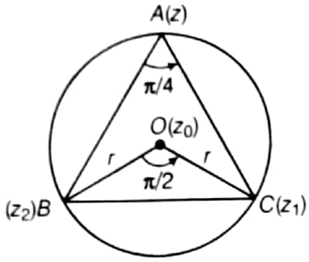
Then, point z describes a hyperbola having foci at z_1 and z_2 and $a\in R^+$.

Examples on Geometry

| Example 83. Let $z_1=10+6i, z_2=4+6i$, where $i=\sqrt{-1}$. If z is a complex number, such that the argument of $(z-z_1)/(z-z_2)$ is $\pi/4$, then prove that $|z-7-9i|=3\sqrt{2}$.

Sol. $\because \quad \arg\left(\frac{z-z_1}{z-z_2}\right)=\frac{\pi}{4}$

It is clear that z, z_1, z_2 are non-collinear points. Always a circle passes through z, z_1 and z_2. Let z_0 be the centre of the circle.



On applying rotation theorem in ΔBOC,

$$\frac{z_1 - z_0}{z_2 - z_0} = \frac{OC}{OB} e^{(\pi/2)i} = i \qquad [\because OC = OB]$$

$$\Rightarrow \quad (z_1 - z_0) = i(z_2 - z_0)$$

$$\Rightarrow \quad 10 + 6i - z_0 = i(4 + 6i - z_0)$$

$$\Rightarrow \quad 16 + 2i = (1 - i) z_0$$

or
$$z_0 = \frac{(16 + 2i)}{(1 - i)} \cdot \frac{(1 + i)}{(1 + i)}$$

$$= \frac{16 + 16i + 2i + 2i^2}{2}$$

$$= \frac{14 + 18i}{2} = 7 + 9i$$

and radius, $r = OC = |z_0 - z_1| = |7 + 9i - 10 - 6i|$

$$= |-3 + 3i|$$

$$= \sqrt{(9 + 9)} = 3\sqrt{2}$$

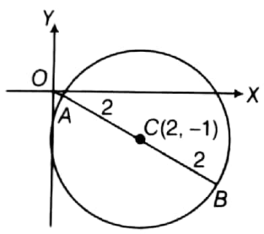
Hence, required equation is

$$|z - z_0| = r$$

$$\Rightarrow \quad |z - 7 - 9i| = 3\sqrt{2}$$

| Example 84. If $|z - 2 + i| \le 2$, where $i = \sqrt{-1}$, then find the greatest and least value of $|z|$.

Sol. $\because$ Radius = 2 units



i.e., $AC = CB = 2$ units

$\therefore$ Least value of $|z| = OA = OC - AC = \sqrt{5} - 2$

and greatest value of $|z| = OB = OC + CB = \sqrt{5} + 2$

Hence, greatest value of $|z|$ is $\sqrt{5} + 2$ and least value of $|z|$ is $\sqrt{5} - 2$.

By Similar Δ in graph.

| Example 85. In the argand plane, the vector $z = 4 - 3i$, where $i = \sqrt{-1}$, is turned in the clockwise sense through $180°$ and stretched three times. Then, find the complex number represented by the new vector.

Sol. $\because \quad z = 4 - 3i \Rightarrow |z| = \sqrt{(4)^2 + (-3)^2} = 5$

Let z_1 be the new vector obtained by rotating z in the clockwise sense through 180°, therefore

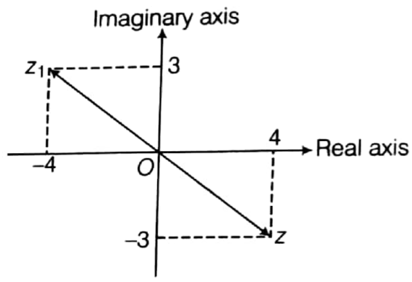
$$z_1 = z e^{-i\pi} = -z = -(4 - 3i) = -4 + 3i.$$

The unit vector in the direction of z_1 is $-\frac{4}{5} + \frac{3}{5} i$.

Therefore, required vector $= 3|z|\left(-\frac{4}{5} + \frac{3}{5} i\right)$

$$= 15\left(-\frac{4}{5} + \frac{3}{5} i\right) = -12 + 9i$$

Aliter



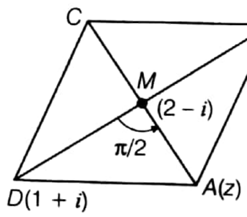
Here, $z_1 = -4 + 3i$

Hence, $3z_1 = -12 + 9i$

| Example 86. *ABCD* is a rhombus. Its diagonals *AC* and *BD* intersect at the point *M* and satisfy $BD = 2AC$. If the points *D* and *M* represents the complex numbers $1 + i$ and $2 - i$, where $i = \sqrt{-1}$, respectively, find *A*.

Sol. Let $A \equiv z$

$\because \quad BD = 2AC$ or $DM = 2AM$



Now, in ΔDMA,

Applying Coni method, we have

$$\frac{z - (2 - i)}{(1 + i) - (2 - i)} = \frac{AM}{DM} e^{i\pi/2} = \frac{1}{2} i$$

$$\Rightarrow \quad z - 2 + i = \frac{i}{2}(-1 + 2i) = -\frac{i}{2} - 1 \text{ or } z = 1 - \frac{3}{2} i$$

$$\therefore \quad A \equiv 1 - \frac{3}{2} i \text{ or } 3 - \frac{i}{2}$$

[if positions of *A* and *C* interchange]

If $\left|z \pm \frac{b}{z}\right| = a$, then the greatest and least values of $|z|$ are $\frac{a+\sqrt{(a^2+4|b|)}}{2}$ and $\frac{-a+\sqrt{(a^2+4|b|)}}{2}$, respectively.

Proof $$\left|z \pm \frac{b}{z}\right| \geq \left||z| - \left|\frac{b}{z}\right|\right|$$

$\Rightarrow$ $$a \geq \left||z| - \frac{|b|}{|z|}\right|$$

or $$-a \leq |z| - \frac{|b|}{|z|} \leq a$$

Now, $$|z| - \frac{|b|}{|z|} \leq a$$

$\Rightarrow$ $$|z|^2 - a|z| - |b| \leq 0$$

$\therefore$ $$\frac{a-\sqrt{(a^2+4|b|)}}{2} \leq |z| \leq \frac{a+\sqrt{(a^2+4|b|)}}{2}$$

or $$0 \leq |z| \leq \frac{a+\sqrt{(a^2+4|b|)}}{2} \quad \text{...(i)}$$

and $$|z| - \frac{|b|}{|z|} \geq -a \Rightarrow |z|^2 + a|z| - |b| \geq 0$$

$\therefore$ $$|z| \geq \frac{-a+\sqrt{(a^2+4|b|)}}{2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{-a+\sqrt{(a^2+4|b|)}}{2} \leq |z| \leq \frac{a+\sqrt{(a^2+4|b|)}}{2}$$

Hence, the greatest value of $|z|$ is $\frac{a+\sqrt{(a^2+4|b|)}}{2}$

and the least value of $|z|$ is $\frac{-a+\sqrt{(a^2+4|b|)}}{2}$.

Corollary For $b = 1$, $\left|z \pm \frac{1}{z}\right| = a$

Then, $$\frac{-a+\sqrt{(a^2+4)}}{2} \leq |z| \leq \frac{a+\sqrt{(a^2+4)}}{2}$$

| Example 87. Find the maximum and minimum values of $|z|$ satisfying $\left|z + \frac{1}{z}\right| = 2$.

Sol. Here, $b = 1$ and $a = 2$

$\therefore$ Maximum and minimum values of $|z|$ are $\frac{2+\sqrt{(4+4)}}{2}$ and $\frac{-2+\sqrt{(4+4)}}{2}$ i.e., $1+\sqrt{2}$ and $-1+\sqrt{2}$, respectively.

| Example 88. If $\left|z + \frac{4}{z}\right| = 2$, find the maximum and minimum values of $|z|$.

Sol. Here, $b = 4$ and $a = 2$.

$\therefore$ Maximum and minimum values of $|z|$ are $\frac{2+\sqrt{(4+16)}}{2}$ and $\frac{-2+\sqrt{(4+16)}}{2}$

i.e. $1+\sqrt{5}$ and $-1+\sqrt{5}$, respectively.

| Example 89. If $|z| \geq 3$, then determine the least value of $\left|z + \frac{1}{z}\right|$.

Sol. $\because$ $$\left|z + \frac{1}{z}\right| \geq \left||z| - \left|\frac{1}{z}\right|\right| = \left||z| - \frac{1}{|z|}\right| \quad \text{...(i)}$$

$\because$ $$|z| \geq 3 \Rightarrow \frac{1}{|z|} \leq \frac{1}{3} \text{ or } -\frac{1}{|z|} \geq -\frac{1}{3}$$

$\therefore$ $$|z| - \frac{1}{|z|} \geq 3 - \frac{1}{3} = \frac{8}{3} \Rightarrow |z| - \frac{1}{|z|} \geq \frac{8}{3}$$

or $$\left||z| - \frac{1}{|z|}\right| \geq \frac{8}{3} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\left|z + \frac{1}{z}\right| \geq \frac{8}{3}$$

$\therefore$ Least value of $\left|z + \frac{1}{z}\right|$ is $\frac{8}{3}$.

Exercise for Session 4

even function
So summation of odd powers of roots = 0

1 If z_1, z_2, z_3 and z_4 are the roots of the equation $z^4 = 1$, the value of $\sum_{i=1}^{4} z_i^3$ is

(a) 0 (b) 1 (c) $i, i = \sqrt{-1}$ (d) $1 + i, i = \sqrt{-1}$

2 If $z_1, z_2, z_3, \ldots, z_n$ are n, nth roots of unity, then for $k = 1, 2, 3, \ldots, n$

(a) $|z_k| = k|z_{k+1}|$ (b) $|z_{k+1}| = k|z_k|$
(c) $|z_{k+1}| = |z_k| + |z_{k-1}|$ (d) $|z_k| = |z_{k+1}|$

$\lim_{x \to 1} \frac{x^n - 1}{x - 1}$

3 If $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are n, nth roots of unity, then $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_{n-1})$ equals to

(a) 0 (b) 1 (c) n (d) n^2

4 The value of $\sum_{k=1}^{6}\left(\sin\left(\frac{2\pi k}{7}\right) - i\cos\left(\frac{2\pi k}{7}\right)\right)$, where $i = \sqrt{-1}$, is

(a) -1 (b) 0 (c) $-i$ (d) i

5 If $\alpha \neq 1$ is any nth root of unity, then $S = 1 + 3\alpha + 5\alpha^2 + \ldots$ upto n terms is equal to

(a) $\frac{2n}{1-\alpha}$ (b) $-\frac{2n}{1-\alpha}$ (c) $\frac{n}{1-\alpha}$ (d) $-\frac{n}{1-\alpha}$

6 If a and b are real numbers between 0 and 1, such that the points $z_1 = a + i, z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then

(a) $a = b = 2 + \sqrt{3}$ (b) $a = b = 2 - \sqrt{3}$
(c) $a = 2 - \sqrt{3}, b = 2 + \sqrt{3}$ (d) None of these

0 **7** If $|z| = 2$, the points representing the complex numbers $-1 + 5z$ will lie on

(a) a circle (b) a straight line (c) a parabola (d) an ellipse

8 If $|z - 2| / |z - 3| = 2$ represents a circle, then its radius is equal to

appolonius circle

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) $\frac{2}{3}$

9 If centre of a regular hexagon is at origin and one of the vertex on argand diagram is $1 + 2i$, where $i = \sqrt{-1}$, its perimeter is

(a) $2\sqrt{5}$ (b) $6\sqrt{2}$ (c) $4\sqrt{5}$ (d) $6\sqrt{5}$

10 If z is a complex number in the argand plane, the equation $|z - 2| + |z + 2| = 8$ represents

(a) a parabola (b) an ellipse (c) a hyperbola (d) a circle

11 If $|z - 2 - 3i| + |z + 2 - 6i| = 4$, where $i = \sqrt{-1}$, then locus of $P(z)$ is

(a) an ellipse (b) ϕ
(c) line segment of points $2 + 3i$ and $-2 + 6i$ (d) None of these

12 Locus of the point z satisfying the equation $|iz - 1| + |z - 1| = 2$, is (where, $i = \sqrt{-1}$)

(a) a straight line (b) a circle (c) an ellipse (d) a pair of straight lines

13 If z, iz and $z + iz$ are the vertices of a triangle whose area is 2 units, the value of $|z|$ is

(a) 1 (b) 2 (c) 4 (d) 8

14 If $\left|z - \frac{4}{z}\right| = 2$, the greatest value of $|z|$ is

(a) $\sqrt{5} - 1$ (b) $\sqrt{3} + 1$ (c) $\sqrt{5} + 1$ (d) $\sqrt{3} - 1$

Shortcuts and Important Results to Remember

1 $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

Thus, $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$.

2 If $\left|z \pm \dfrac{b}{z}\right| = a$, then the greatest and least values of $|z|$ are

$\dfrac{a + \sqrt{(a^2 + 4|b|)}}{2}$ and $\dfrac{-a + \sqrt{(a^2 + 4|b|)}}{2}$, respectively.

3 $\left|z_1 + \sqrt{(z_1^2 - z_2^2)}\right| + \left|z_2 - \sqrt{(z_1^2 - z_2^2)}\right|$

$$= |z_1 + z_2| + |z_1 - z_2|$$

4 $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$

i.e. z_1 and z_2 are parallel.

5 $|z_1 + z_2| = |z_1| - |z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi$

6 $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pm\pi/2$

7 If $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = 0$, then z_1 and z_2 are conjugate complex numbers of each other.

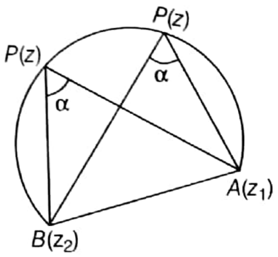
8 The equation $|z - z_1|^2 + |z - z_2|^2 = k, k \in R$ will represent a circle with centre at $\frac{1}{2}(z_1 + z_2)$ and radius is $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$ provided $k \ge \frac{1}{2}|z_1 - z_2|^2$.

9 Area of triangle whose vertices are z, iz and $z + iz$, where $i = \sqrt{-1}$, is $\frac{1}{2}|z|^2$.

10 Area of triangle whose vertices are z, ωz and $z + \omega z$ is $\frac{\sqrt{3}}{4}|z|^2$, where ω is cube root of unity.

11 $\arg(z) - \arg(-z) = \pi$ or $-\pi$ according as $\arg(z)$ is positive or negative.

12 If $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$ (fixed), then the locus of z is a segment of circle.



13 If $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \pm\pi/2$, the locus of z is a circle with z_1 and z_2 as the vertices of diameter.

14 If $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = 0$ or π, the locus of z is a straight line passing through z_1 and z_2.

15 If three complex numbers are in AP, they lie on a straight line in the complex plane.

16 If three points z_1, z_2, z_3 connected by relation $a z_1 + b z_2 + c z_3 = 0$, where $a + b + c = 0$, the three points are collinear.

17 If z_1, z_2, z_3 are vertices of a triangle, its centroid

$$z_0 = \frac{z_1 + z_2 + z_3}{3}, \text{ circumcentre } z_1 = \frac{\sum |z_1|^2 (z_2 - z_3)}{\sum \bar{z}_1(z_2 - z_3)},$$

$$\text{orthocentre } z = \frac{\sum \bar{z}_1(\bar{z}_2 - \bar{z}_3) + \sum |z_1|^2 (z_2 - z_3)}{\sum (z_1\bar{z}_2 - \bar{z}_1 z_2)}$$

$$\text{and its area} = \frac{1}{4}\left|\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|.$$

18 If $|z_1| = n_1, |z_2| = n_2, |z_3| = n_3, \ldots, |z_m| = n_m$,

then $\left|\dfrac{n_1^2}{z_1} + \dfrac{n_2^2}{z_2} + \dfrac{n_3^2}{z_3} + \ldots + \dfrac{n_m^2}{z_m}\right| = |z_1 + z_2 + z_3 + \ldots + z_m|$.