PERMUTATION & COMBINATION

1. FUNDAMENTAL PRINCIPLE OF COUNTING (counting without actual counting):

If an event A can occur in 'm' different ways and another event B can occur in 'n' different ways, then the total number of different ways of-

(a) simultaneous occurrence of both events in a definite order is m n. This can be extended to any number of events (known as multiplication principle).

(b) happening exactly one of the events is m + n (known as addition principle).

Example : There are 15 IITs in India and let each IIT has 10 branches, then the IITJEE topper can select the IIT and branch in 15 10 = 150 number of ways.

Example : There are 15 IITs & 20 NITs in India, then a student who cleared both IITJEE & AIEEE exams can select an institute in (15 + 20) = 35 number of ways.

Illustration 1 : A college offers 6 courses in the morning and 4 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is-

(A) 24 (B) 2 (C) 12 (D) 10

Solution : The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways 6 4 = 24. **Ans.(A)**

Illustration 2 : A college offers 6 courses in the morning and 4 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening-

(A) 6 (B) 4 (C) 10 (D) 24

Solution : The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways 6 + 4 = 10. **Ans. (C)**

Do yourself - 1 :

(i) There are 3 ways to go from A to B, 2 ways to go from B to C and 1 way to go from A to C. In how many ways can a person travel from A to C ?

(ii) There are 2 red balls and 3 green balls. All balls are identical except colour. In how many ways can a person select two balls ?

2. PERMUTATION & COMBINATION :

(a) Factorial : A Useful Notation : $n! = n.(n-1).(n-2)..............3.\ 2.\ 1;$

$n! = n.\ (n-1)!$ where $n \in N$

Note :

(i) $0! = 1! = 1$

(ii) Factorials of negative integers are not defined.

(iii) $n!$ is also denoted by $\underline{|n}$

(iv) $(2n)! = 2^n.n!\ [1.\ 3.\ 5.\ 7........(2n-1)]$

(v) Prime factorisation of n! : Let p be a prime number and n be a positive integer, then exponent of p in n! is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... + \left[\frac{n}{p^k}\right]$$

where $p^k \le n < p^{k+1}$ and [x] denotes the integral part of x.

If we isolate the power of each prime contained in any number n, then n can be written as

$n = 2^{\alpha_1}\ 3^{\alpha_2}\ 5^{\alpha_3}\ 7^{\alpha_4}\ ....$ where α_i are whole numbers.

(b) **Permutation** : Each of the arrangements in a definite order which can be made by taking some or all of the things at a time is called a PERMUTATION. In permutation, order of appearance of things is taken into account; when the order is changed, a different permutation is obtained.

Generally, it involves the problems of arrangements (standing in a line, seated in a row), problems on digit, problems on letters from a word etc.

${}^nP_r$ denotes the number of permutations of n **different** things, taken r at a time ($n \in N$, $r \in W$, $r \le n$)

$${}^nP_r = n(n-1)(n-2)............(n-r+1) = \frac{n!}{(n-r)!}$$

Note :

(i) ${}^nP_n = n!$, ${}^nP_0 = 1$, ${}^nP_1 = n$

(ii) Number of arrangements of n **distinct** things taken all at a time = n!

(iii) ${}^nP_r$ is also denoted by A_r^n or P(n,r).

(c) **Combination :**

Each of the groups or selections which can be made by taking some or all of the things without considering the order of the things in each group is called a COMBINATION.

Generally, involves the problem of selections, choosing, distributed groups formation, committee formation, geometrical problems etc.

${}^nC_r$ denotes the number of combinations of n different things taken r at a time ($n \in N$, $r \in W$, $r \le n$)

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Note :

(i) ${}^nC_r$ is also denoted by $\binom{n}{r}$ or C (n, r).

(ii) ${}^nP_r = {}^nC_r.\ r!$

Illustration 3: Find the exponent of 6 in 50!

Solution:

$$E_2(50!) = \left[\frac{50}{2}\right] + \left[\frac{50}{4}\right] + \left[\frac{50}{8}\right] + \left[\frac{50}{16}\right] + \left[\frac{50}{32}\right] + \left[\frac{50}{64}\right]$$ (where [ ] denotes integral part)

$E_2(50!) = 25 + 12 + 6 + 3 + 1 + 0 = 47$

$$E_3(50!) = \left[\frac{50}{3}\right] + \left[\frac{50}{9}\right] + \left[\frac{50}{27}\right] + \left[\frac{50}{81}\right]$$

$E_3(50!) = 16 + 5 + 1 + 0 = 22$

$\Rightarrow$ 50! can be written as $50! = 2^{47}.\ 3^{22}.........$

Therefore exponent of 6 in 50! = 22 **Ans.**

Illustration 4 : If *a* denotes the number of permutations of (x + 2) things taken all at a time, *b* the number of permutations of x things taken 11 at a time and *c* the number of permutations of (x − 11) things taken all at a time such that *a = 182 bc,* then the value of x is

(A) 15 (B) 12 (C) 10 (D) 18

Solution :

$$^{x+2}P_{x+2} = a \Rightarrow a = (x+2)!$$

$$^{x}P_{11} = b \Rightarrow b = \frac{x!}{(x-11)!}$$

and $^{x-11}P_{x-11} = c \Rightarrow c = (x-11)!$

$$\because a = 182bc$$

$$(x+2)! = 182\frac{x!}{(x-11)!}(x-11)! \Rightarrow (x+2)(x+1) = 182 = 14 \times 13$$

$\therefore x+1 = 13 \Rightarrow x = 12$ **Ans. (B)**

Illustration 5 : A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be drawn so that there are atleast two balls of each colour ?

Solution : The selections of 6 balls, consisting of atleast two balls of each colour from 5 red and 6 white balls, can be made in the following ways

Red balls (5)	White balls (6)	Number of ways
2	4	$^5C_2 \times {}^6C_4 = 150$
3	3	$^5C_3 \times {}^6C_3 = 200$
4	2	$^5C_4 \times {}^6C_2 = 75$

Therefore total number of ways = 425 **Ans.**

Illustration 6 : How many 4 letter words can be formed from the letters of the word 'ANSWER' ? How many of these words start with a vowel ?

Solution : Number of ways of arranging 4 different letters from 6 different letters are $^6C_4 4! = \frac{6!}{2!} = 360$.

There are two vowels (A & E) in the word 'ANSWER'.

Total number of 4 letter words starting with A : A _ _ _ = $^5C_3 3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with E : E _ _ _ = $^5C_3 3! = \frac{5!}{2!} = 60$

$\therefore$ Total number of 4 letter words starting with a vowel = 60 + 60 = 120. **Ans.**

Illustration 7 : If all the letters of the word 'RAPID' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RAPID'.

Solution : First of all, arrange all letters of given word alphabetically : 'ADIPR'

Total number of words starting with A _ _ _ _ = 4! = 24

Total number of words starting with D _ _ _ _ = 4! = 24

Total number of words starting with I _ _ _ _ = 4! = 24

Total number of words starting with P _ _ _ _ = 4! = 24

Total number of words starting with RAD _ _ = 2! = 2

Total number of words starting with RAI _ _ = 2! = 2

Total number of words starting with RAPD _ = 1

Total number of words starting with RAPI _ = 1

$\therefore$ Rank of the word RAPID = 24 + 24 + 24 + 24 + 2 + 2 + 1 + 1 = 102 **Ans.**

Do yourself -2 :

(i) Find the exponent of 10 in ${}^{75}C_{25}$.

(ii) If ${}^{10}P_r = 5040$, then find the value of r.

(iii) Find the number of ways of selecting 4 even numbers from the set of first 100 natural numbers.

(iv) If all letters of the word 'RANK' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RANK'.

(v) How many words can be formed using all letters of the word 'LEARN' ? In how many of these words vowels are together ?

3. PROPERTIES OF ${}^nP_r$ and ${}^nC_r$:

(a) The number of permutation of n different objects taken r at a time, when p particular objects are always to be included is $r!.{}^{n-p}C_{r-p}$ $(p \le r \le n)$

(b) The number of permutations of n different objects taken r at a time, when repetition is allowed any number of times is n^r.

(c) Following properties of ${}^nC_r$ should be remembered :

(i) ${}^nC_r = {}^nC_{n-r}$; ${}^nC_0 = {}^nC_n = 1$

(ii) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n$

(v) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1}$

(vi) ${}^nC_r$ is maximum when $r = \dfrac{n}{2}$ if n is even & $r = \dfrac{n-1}{2}$ or $r = \dfrac{n+1}{2}$ if n is odd.

(d) The number of combinations of n different things taking r at a time,

(i) when p particular things are always to be included = ${}^{n-p}C_{r-p}$

(ii) when p particular things are always to be excluded = ${}^{n-p}C_r$

(iii) when p particular things are always to be included and q particular things are to be excluded = ${}^{n-p-q}C_{r-p}$

Illustration 8 : There are 6 pockets in the coat of a person. In how many ways can he put 4 pens in these pockets?

(A) 360 (B) 1296 (C) 4096 (D) none of these

Solution : First pen can be put in 6 ways.

Similarly each of second, third and fourth pen can be put in 6 ways.

Hence total number of ways = 6 6 6 6 = 1296 **Ans.(B)**

Illustration 9 : A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected, if-

(a) all the students are equally willing ?

(b) two particular students have to be included in the delegation ?

(c) two particular students do not wish to be together in the delegation ?

(d) two particular students wish to be included together only ?

(e) two particular students refuse to be together and two other particular students wish to be together only in the delegation ?

Solution : (a) Formation of delegation means selection of 4 out of 12.

Hence the number of ways = ${}^{12}C_4 = 495$.

(b) If two particular students are already selected. Here we need to select only 2 out of the remaining 10. Hence the number of ways = ${}^{10}C_2 = 45$.

(c) The number of ways in which both are selected = 45. Hence the number of ways in which the two are not included together = 495 – 45 = 450

(d) There are two possible cases

(i) Either both are selected. In this case, the number of ways in which the selection can be made = 45.

(ii) Or both are not selected. In this case all the four students are selected from the remaining ten students. This can be done in ${}^{10}C_4 = 210$ ways.

Hence the total number of ways of selection = 45 + 210 = 255

(e) We assume that students A and B wish to be selected together and students C and D do not wish to be together. Now there are following 6 cases.

(i) (A, B, C) selected, (D) not selected

(ii) (A, B, D) selected, (C) not selected

(iii) (A, B) selected, (C, D) not selected

(iv) (C) selected, (A, B, D) not selected

(v) (D) selected, (A, B, C) not selected

(vi) A, B, C, D not selected

For (i) the number of ways of selection = ${}^8C_1 = 8$

For (ii) the number of ways of selection = ${}^8C_1 = 8$

For (iii) the number of ways of selection = ${}^8C_2 = 28$

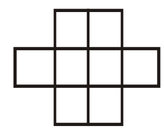
For (iv) the number of ways of selection = ${}^8C_3 = 56$

For (v) the number of ways of selection = ${}^8C_3 = 56$

For (vi) the number of ways of selection = ${}^8C_4 = 70$

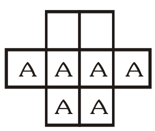
Hence total number of ways = 8 + 8 + 28 + 56 + 56 + 70 = 226. **Ans.**

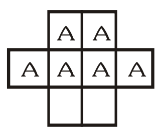
Illustration 10: In the given figure of squares, 6 A's should be written in such a manner that every row contains at least one 'A'. In how many number of ways is it possible ?



(A) 24 (B) 25 (C) 26 (D) 27

Solution : There are 8 squares and 6 'A' in given figure. First we can put 6 'A' in these 8 squares by ${}^8C_6$ number of ways.

(I) 

(II) 

According to question, atleast one 'A' should be included in each row. So after subtracting these two cases, number of ways are = $({}^8C_6 - 2) = 28 - 2 = 26$. **Ans. (C)**

Illustration 11: There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices at these points is :

(A) $3p^2 (p - 1) + 1$ (B) $3p^2 (p - 1)$ (C) $p^2 (4p - 3)$ (D) none of these

Solution : The number of triangles with vertices on different lines = ${}^pC_1 \quad {}^pC_1 \quad {}^pC_1 = p^3$

The number of triangles with two vertices on one line and the third vertex on any one of the other two lines = ${}^3C_1 \{{}^pC_2 \quad {}^{2p}C_1\} = 6p.\dfrac{p(p-1)}{2}$

So, the required number of triangles = $p^3 + 3p^2 (p - 1) = p^2 (4p - 3)$ **Ans. (C)**

Illustration 12: There are 10 points in a row. In how many ways can 4 points be selected such that no two of them are consecutive ?

Solution : Total number of remaining non-selected points = 6

. . . . . .

Total number of gaps made by these 6 points = 6 + 1 = 7

If we select 4 gaps out of these 7 gaps and put 4 points in selected gaps then the new points will represent 4 points such that no two of them are consecutive.

x . . x . x . . x .

Total number of ways of selecting 4 gaps out of 7 gaps = ${}^7C_4$ **Ans.**

In general, total number of ways of selection of r points out of n points in a row such that no two of them are consecutive : ${}^{n-r+1}C_r$

Do yourself-3 :

(i) Find the number of ways of selecting 5 members from a committee of 5 men & 2 women such that all women are always included.

(ii) Out of first 20 natural numbers, 3 numbers are selected such that there is exactly one even number. How many different selections can be made ?

(iii) How many four letter words can be made from the letters of the word 'PROBLEM'. How many of these start as well as end with a vowel ?

4. FORMATION OF GROUPS :

(a) (i) The number of ways in which (m + n) different things can be divided into two groups such that one of them contains m things and other has n things, is $\frac{(m+n)!}{m!\ n!}$ $(m \neq n)$.

(ii) If m = n, it means the groups are equal & in this case the number of divisions is $\frac{(2n)!}{n!\ n!\ 2!}$. As in any one way it is possible to interchange the two groups without obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways : $\frac{(2n)!}{n!\ n!\ (2!)} \times 2!$.

(b) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is : $\frac{(m+n+p)!}{m!\ n!\ p!}$, $m \neq n \neq p$.

(ii) If m = n = p then the number of groups = $\frac{(3n)!}{n!\ n!\ n!\ 3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

(c) In general, the number of ways of dividing n distinct objects into ℓ groups containing p objects each and m groups containing q objects each is equal to $\frac{n!(\ell+m)!}{(p!)^{\ell}(q!)^{m}\ell!m!}$

Here $\ell p + mq = n$

Illustration 13 : In how many ways can 15 students be divided into 3 groups of 5 students each such that 2 particular students are always together ? Also find the number of ways if these groups are to be sent to three different colleges.

Solution : Assuming two particular students as one student (as they are always together), we have to make groups of 5 + 5 + 4 students out of 14 students.

Therefore total number of ways = $\frac{14!}{5!5!4!2!}$

Now if these groups are to be sent to three different colleges, the total number of ways

$= \frac{14!}{5!5!4!2!} \times 3!$ **Ans.**

Illustration 14 : Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

Solution : Total number of ways of dividing 48 cards (Excluding 4Aces) in 4 groups $= \frac{48!}{(12!)^4 4!}$

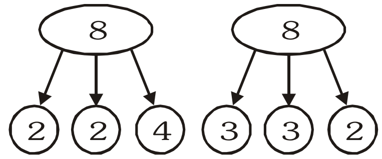
Now, distribute exactly one Ace to each group of 12 cards. Total number of ways = $\frac{48!}{(12!)^4 4!} \times 4!$

Now, distribute these groups of cards among four players

$= \frac{48!}{(12!)^4 4!} \times 4!4! = \frac{48!}{(12!)^4} \times 4!$ **Ans.**

Illustration 15 : In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books ?

Solution : If each receives at least two books, then the division trees would be as shown below :



(i) (ii)

The number of ways of division for tree in figure (i) is $\left[\dfrac{8!}{(2!)^2 4!2!}\right]$.

The number of ways of division for tree in figure (ii) is $\left[\dfrac{8!}{(3!)^2 2!2!}\right]$.

The total number of ways of distribution of these groups among 3 students

is $\left[\dfrac{8!}{(2!)^2 4!2!}+\dfrac{8!}{(3!)^2 2!2!}\right]\times 3!$. **Ans.**

Do yourself-4 :

(i) Find the number of ways in which 16 constables can be assigned to patrol 8 villages, 2 for each.

(ii) In how many ways can 6 different books be distributed among 3 students such that none gets equal number of books ?

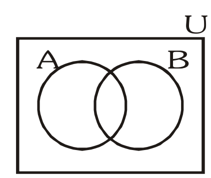
(iii) n different toys are to be distributed among n children. Find the number of ways in which these toys can be distributed so that exactly one child gets no toy.

5. PRINCIPLE OF INCLUSION AND EXCLUSION :

In the Venn's diagram (i), we get

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A' \cap B') = n(U) - n(A \cup B)$

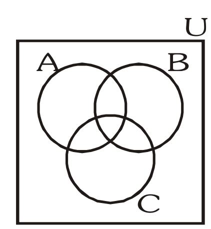


(i)

In the Venn's diagram (ii), we get

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$

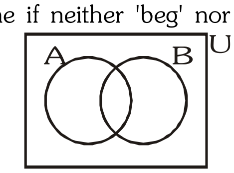


(ii)

In general, we have $n(A_1 \cup A_2 \cup \ldots\ldots\ldots \cup A_n)$

$$= \sum n(A_i) - \sum_{i \neq j} n(A_i \cap A_j) + \sum_{i \neq j \neq k} n(A_i \cap A_j \cap A_k) + \ldots.. + (-1)^n \sum n(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Illustration 16 : Find the number of permutations of letters a,b,c,d,e,f,g taken all at a time if neither 'beg' nor 'cad' pattern appear.



Solution : The total number of permutations without any restrictions; n(U) = 7!

beg acdf

Let A be the set of all possible permutations in which 'beg' pattern always appears : n(A) = 5!

cad befg

Let B be the set of all possible permutations in which 'cad' pattern always appears : n(B) = 5!

cad beg f

$n(A \cap B)$: Number of all possible permutations when both 'beg' and 'cad' patterns appear.

$n(A \cap B) = 3!$.

Therefore, the total number of permutations in which 'beg' and 'cad' patterns do not appear

$n(A' \cap B') = n(U) - n(A \cap B) = n(U) - n(A) - n(B) + n(A \cap B)$

$= 7! - 5! - 5! + 3!$. **Ans.**

> **Do yourself-5 :**
>
> (i) Find the number of n digit numbers formed using first 5 natural numbers, which contain the digits 2 & 4 essentially.

6. PERMUTATIONS OF ALIKE OBJECTS :

Case-I : Taken all at a time -

The number of permutations of n things taken all at a time : when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n - (p + q + r)$ are all different is : $\dfrac{n!}{p!\ q!\ r!}$.

Illustration 17 : In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative position of vowels & consonants.

Solution : The consonants in their positions can be arranged in $\dfrac{4!}{2!} = 12$ ways.

The vowels in their positions can be arranged in $\dfrac{3!}{2!} = 3$ ways

$\therefore$ Total number of arrangements = 12 3 = 36 **Ans.**

Illustration 18 : How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

(A) 17 (B) 18 (C) 19 (D) 20

Solution : There are 4 odd digits (1, 1, 3, 3) and 4 odd places (first, third, fifth and seventh). At these places the odd digits can be arranged in $\dfrac{4!}{2!2!} = 6$ ways

Then at the remaining 3 places, the remaining three digits (2, 2, 4) can be arranged in $\dfrac{3!}{2!} = 3$ ways

$\therefore$ The required number of numbers = 6 3 = 18. **Ans. (B)**

Illustration 19 :

(a) How many permutations can be made by using all the letters of the word HINDUSTAN ?

(b) How many of these permutations begin and end with a vowel ?

(c) In how many of these permutations, all the vowels come together ?

(d) In how many of these permutations, none of the vowels come together ?

(e) In how many of these permutations, do the vowels and the consonants occupy the same relative positions as in HINDUSTAN ?

Solution :

(a) The total number of permutations = Arrangements of nine letters taken all at a time $= \dfrac{9!}{2!} = 181440$.

(b) We have 3 vowels and 6 consonants, in which 2 consonants are alike. The first place can be filled in 3 ways and the last in 2 ways. The rest of the places can be filled in $\dfrac{7!}{2!}$ ways.

Hence the total number of permutations = 3 2 $\dfrac{7!}{2!}$ = 15120.

(c) Assume the vowels (I, U, A) as a single letter. The letters (IUA), H, D, S, T, N, N can be arranged in $\frac{7!}{2!}$ ways. Also IUA can be arranged among themselves in 3! = 6 ways.

Hence the total number of permutations = $\frac{7!}{2!}$ 6 = 15120.

(d) Let us divide the task into two parts. In the first, we arrange the 6 consonants as shown below in $\frac{6!}{2!}$ ways.

C C C C C C (Here C stands for a consonant and stands for a gap between two consonants)

Now 3 vowels can be placed in 7 places (gaps between the consonants) in $^7C_3.3!$ = 210 ways.

Hence the total number of permutations = $\frac{6!}{2!}$ 210 = 75600.

(e) In this case, the vowels can be arranged among themselves in 3! = 6 ways.

Also, the consonants can be arranged among themselves in $\frac{6!}{2!}$ ways.

Hence the total number of permutations = $\frac{6!}{2!}$ 6 = 2160. **Ans.**

Illustration 20 : If all the letters of the word 'PROPER' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'PROPER' .

Solution : First of all, arrange all letters of given word alphabetically : EOPPRR

Total number of words starting with-

E _ _ _ _ _ = $\frac{5!}{2!2!} = 30$

O _ _ _ _ _ = $\frac{5!}{2!2!} = 30$

PE _ _ _ _ = $\frac{4!}{2!} = 12$

PO _ _ _ _ = $\frac{4!}{2!} = 12$

PP _ _ _ _ = $\frac{4!}{2!} = 12$

PRE _ _ _ = 3! = 6

PROE _ _ = 2! = 2

PROPER = 1= 1

Rank of the word PROPER = 105 **Ans.**

Case-II : Taken some at a time

Illustration 21 : Find the total number of 4 letter words formed using four letters from the word "PARALLELOPIPED".

Solution : Given letters are PPP, LLL, AA, EE, R, O, I, D.

Cases	No. of ways of selection	No. of ways of arrangements	Total
All distinct	$^8C_4$	$^8C_4 \times 4!$	1680
2 alike, 2 distinct	$^4C_1 \times {}^7C_2$	$^4C_1 \times {}^7C_2 \times \frac{4!}{2!}$	1008
2 alike, 2 other alike	$^4C_2$	$^4C_2 \times \frac{4!}{2!2!}$	36
3 alike, 1 distinct	$^2C_1 \times {}^7C_1$	$^2C_1 \times {}^7C_1 \times \frac{4!}{3!}$	56
		Total	2780

Ans.

Illustration 22 : Find the number of all 6 digit numbers such that all the digits of each number are selected from the set {1,2,3,4,5} and any digit that appears in the number appears at least twice.

Solution :

Cases	No. of ways of selection	No. of ways of arrangements	Total
All alike	${}^5C_1$	${}^5C_1 \times 1$	5
4 alike + 2 other alike	${}^5C_2 \times 2!$	${}^5C_2 \times 2 \times \frac{6!}{2!4!}$	300
3 alike + 3 other alike	${}^5C_2$	${}^5C_2 \times \frac{6!}{3!3!}$	200
2 alike + 2 other alike +2 other alike	${}^5C_3$	${}^5C_3 \times \frac{6!}{2!2!2!}$	900
		Total	1405

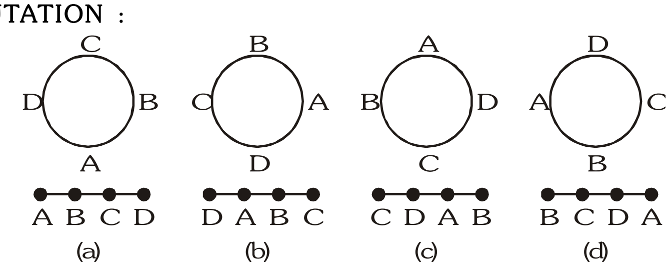
Ans.

Do yourself-6 :

(i) In how many ways can the letters of the word 'ALLEN' be arranged ? Also find its rank if all these words are arranged as they are in dictionary.

(ii) How many numbers greater than 1000 can be formed from the digits 1, 1, 2, 2, 3 ?

7. CIRCULAR PERMUTATION :



Let us consider that persons A,B,C,D are sitting around a round table. If all of them (A,B,C,D) are shifted by one place in anticlockwise order, then we will get Fig.(b) from Fig.(a). Now, if we shift A,B,C,D in anticlockwise order, we will get Fig.(c). Again, if we shift them, we will get Fig.(d) and in the next time, Fig.(a).

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements, thus obtained will be the same, because anticlockwise order of A,B,C,D does not change.

But if A,B,C,D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is n   (number of circular arrangements of n different things). Hence, the number of circular arrangements of n different things is -

1/n   (number of linear arrangements of n different things) = $\frac{n!}{n}$ = (n–1)!

Therefore note that :

(i) The number of circular permutations of n different things taken all at a time is : (n – 1)!.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is : $\frac{(n-1)!}{2}$.

(ii) The number of circular permutations of n different things taking r at a time distinguishing clockwise & anticlockwise arrangements is : $\frac{{}^nP_r}{r}$

Illustration 23 : In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls are together?

(A) 5!   5! (B) 5!   4! (C) $\frac{1}{2}(5!)^2$ (D) $\frac{1}{2}(5!\quad 4!)$

Solution : Leaving one seat vacant between two boys, 5 boys may be seated in 4! ways. Then at remaining 5 seats, 5 girls sit in 5! ways. Hence the required number of ways = 4!   5! **Ans. (B)**

Illustration 24 : The number of ways in which 7 girls can stand in a circle so that they do not have same neighbours in any two arrangements ?

(A) 720 (B) 380 (C) 360 (D) none of these

Solution : Seven girls can stand in a circle by $\frac{(7-1)!}{2!}$ number of ways, because there is no difference in anticlockwise and clockwise order of their standing in a circle.

$\therefore \quad \frac{(7-1)!}{2!} = 360$ **Ans. (C)**

Illustration 25 : The number of ways in which 20 different pearls of two colours can be set alternately on a necklace, there being 10 pearls of each colour, is

(A) 9! 10! (B) $5(9!)^2$ (C) $(9!)^2$ (D) none of these

Solution : Ten pearls of one colour can be arranged in $\frac{1}{2}.(10-1)!$ ways. The number of arrangements of 10 pearls of the other colour in 10 places between the pearls of the first colour = 10!

$\therefore$ The required number of ways $= \frac{1}{2} \times 9! \times 10! = 5\,(9!)^2$ **Ans. (B)**

Illustration 26 : A person invites a group of 10 friends at dinner. They sit

(i) 5 on one round table and 5 on other round table,

(ii) 4 on one round table and 6 on other round table.

Find the number of ways in each case in which he can arrange the guests.

Solution : (i) The number of ways of selection of 5 friends for first table is $^{10}C_5$. Remaining 5 friends are left for second table.

The total number of permutations of 5 guests at a round table is 4!. Hence, the total number of arrangements is $^{10}C_5 \quad 4! \quad 4! = \frac{10!4!4!}{5!5!} = \frac{10!}{25}$

(ii) The number of ways of selection of 6 guests is $^{10}C_6$.

The number of ways of permutations of 6 guests on round table is 5!. The number of permutations of 4 guests on round table is 3!

Therefore, total number of arrangements is : $^{10}C_6 5! \times 3! = \frac{(10)!}{6!4!} 5!3! = \frac{(10)!}{24}$ **Ans. (B)**

Do yourself-7 :

(i) In how many ways can 3 men and 3 women be seated around a round table such that all men are always together ?

(ii) Find the number of ways in which 10 different diamonds can be arranged to make a necklace.

(iii) Find the number of ways in which 6 persons out of 5 men & 5 women can be seated at a round table such that 2 men are never together.

(iv) In how many ways can 8 persons be seated on two round tables of capacity 5 & 3.

8. TOTAL NUMBER OF COMBINATIONS :

(a) Given n different objects , the number of ways of selecting atleast one of them is, $^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots.. + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(b) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r +......things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : (p + 1) (q + 1) (r + 1).........–1.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is given by : $(p + 1)\,(q + 1)\,(r + 1)\,2^n - 1$

Illustration 27 : A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q = \phi$ is :-

(A) $2^{2n} - {}^{2n}C_n$ (B) 2^n (C) $2^n - 1$ (D) 3^n

Solution : Let $A = \{a_1, a_2, a_3, ... a_n\}$. For $a_i \in A$, we have the following choices :

(i) $a_i \in P$ and $a_i \in Q$ (ii) $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \notin P$ and $a_i \in Q$ (iv) $a_i \notin P$ and $a_i \notin Q$

Out of these only (ii), (iii) and (iv) imply $a_i \notin P \cap Q$. Therefore, the number of ways in which none of $a_1, a_2, ...a_n$ belong to $P \cap Q$ is 3^n. **Ans. (D)**

Illustration 28 : A student is allowed to select at most n books from a collection of (2n + 1) books. If the total number of ways in which he can select books is 63, find the value of n.

Solution : Given student selects at most n books from a collection of (2n + 1) books. It means that he selects one book or two books or three books or .... or n books. Hence, by the given condition-

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n = 63 \quad ...(i)$$

But we know that

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_{2n+1} = 2^{2n+1} \quad ...(ii)$$

Since ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$, equation (ii) can also be written as

$$2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n) +$$
$$({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + {}^{2n+1}C_{n+3} + ... + {}^{2n+1}C_{2n-1} + {}^{2n+1}C_{2n}) = 2^{2n+1}$$

$$\Rightarrow 2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n)$$
$$+ ({}^{2n+1}C_n + {}^{2n+1}C_{n-1} + ... + {}^{2n+1}C_2 + {}^{2n+1}C_1) = 2^{2n+1}$$

$(\because {}^{2n+1}C_r = {}^{2n+1}C_{2n+1-r})$

$$\Rightarrow 2 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n) = 2^{2n+1} \quad \text{[from (i)]}$$

$\Rightarrow 2 + 2.63 = 2^{2n+1}$ $\Rightarrow 1 + 63 = 2^{2n}$

$\Rightarrow 64 = 2^{2n} \Rightarrow 2^6 = 2^{2n}$ $\therefore 2n = 6$

Hence, n = 3. **Ans.**

Illustration 29 : There are 3 books of mathematics, 4 of science and 5 of english. How many different collections can be made such that each collection consists of-

(i) one book of each subject ?

(ii) at least one book of each subject ?

(iii) at least one book of english ?

Solution : (i) ${}^3C_1 \ {}^4C_1 \ {}^5C_1 = 60$

(ii) $(2^3-1)(2^4-1)(2^5-1) = 7 \ 15 \ 31 = 3255$

(iii) $(2^5-1)(2^3)(2^4) = 31 \ 128 = 3968$ **Ans.**

Illustration 30 : Find the number of groups that can be made from 5 red balls, 3 green balls and 4 black balls, if at least one ball of all colours is always to be included. Given that all balls are identical except colours.

Solution : After selecting one ball of each colour, we have to find total number of combinations that can be made from 4 red. 2 green and 3 black balls. These will be (4 + 1) (2 + 1) (3 + 1) = 60 **Ans.**

Do yourself-8 :

(i) There are p copies each of n different books. Find the number of ways in which atleast one book can be selected ?

(ii) There are 10 questions in an examination. In how many ways can a candidate answer the questions, if he attempts atleast one question.

9. **DIVISORS :**

Let $N = p^a. q^b. r^c$ .... where p, q, r........ are distinct primes & a, b, c....... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is = (a + 1) (b + 1) (c + 1).......

(b) The sum of these divisors is = $(p^0 + p^1 + p^2 + ...+ p^a)(q^0 + q^1 + q^2 + ...+ q^b)(r^0 + r^1 + r^2 + ...+ r^c)$...

(c) Number of ways in which N can be resolved as a product of two factor is =

$\frac{1}{2}(a+1)(b+1)(c+1)......$ if N is not a perfect square

$\frac{1}{2}\left[(a+1)(b+1)(c+1)......+1\right]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

Note :

(i) Every natural number except 1 has atleast 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begin with 2. All primes except 2 are odd.

(ii) A number having more than 2 divisors is called composite. 2 is the only even number which is not composite.

(iii) Two natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively prime, it is not necessary that one or both should be prime. It is possible that they both are composite but still coprime, eg. 4 and 25.

(iv) 1 is neither prime nor composite however it is co-prime with every other natural number.

(v) Two prime numbers are said to be twin prime numbers if their non-negative difference is 2 (e.g.5 & 7, 19 & 17 etc).

(vi) All divisors except 1 and the number itself are called proper divisors.

Illustration 31: Find the number of proper divisors of the number 38808. Also find the sum of these divisors.

Solution :

(i) The number $38808 = 2^3 . 3^2 . 7^2 . 11$

Hence the total number of divisors (excluding 1 and itself i.e.38808)

$= (3 + 1)(2 + 1)(2 + 1)(1 + 1) - 2 = 70$

(ii) The sum of these divisors

$=(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)(7^0 + 7^1 + 7^2)(11^0 + 11^1) - 1 - 38808$

$= (15)(13)(57)(12) - 1 - 38808 = 133380 - 1 - 38808 = 94571.$ **Ans.**

Illustration 32: In how many ways the number 18900 can be split in two factors which are relative prime (or coprime) ?

Solution: Here $N = 18900 = 2^2 . 3^3 . 5^2 . 7^1$

Number of different prime factors in 18900 = n = 4

Hence number of ways in which 18900 can be resolved into two factors which are relative prime (or coprime) $= 2^{4-1} = 2^3 = 8.$ **Ans.**

Illustration 33: Find the total number of proper factors of the number 35700. Also find

(i) sum of all these factors,

(ii) sum of the odd proper divisors,

(iii) the number of proper divisors divisible by 10 and the sum of these divisors.

Solution: $35700 = 5^2 \quad 2^2 \quad 3^1 \quad 7^1 \quad 17^1$

The total number of factors is equal to the total number of selections from (5,5), (2,2), (3), (7) and (17), which is given by 3 3 2 2 2 = 72.

These include 1 and 35700. Therefore, the number of proper divisors (excluding 1 and 35700) is 72 – 2 = 70

(i) Sum of all these factors (proper) is :

$(5 \ + 5^1 + 5^2)\ (2 \ + 2^1 + 2^2)\ (3 \ + 3^1)\ (7 \ + 7^1)\ (17 \ + 17^1) - 1 - 35700$

$= 31 \quad 7 \quad 4 \quad 8 \quad 18 - 1 - 35700 = 89291$

(ii) The sum of odd proper divisors is :

$(5 \ + 5^1 + 5^2)\ (3 \ + 3^1)\ (7 \ + 7^1)\ (17 \ + 17^1) - 1$

$= 31 \quad 4 \quad 8 \quad 18 - 1 = 17856 - 1 = 17855$

(iii) The number of proper divisors divisible by 10 is equal to number of selections from (5,5), (2,2), (3), (7), (17) consisting of at least one 5 and at least one 2 and 35700 is to be excluded and is given by 2 2 2 2 2 – 1= 31.

Sum of these divisors is :

$(5^1 + 5^2)\ (2^1 + 2^2)\ (3 \ + 3^1)\ (7 \ + 7^1)\ (17 \ + 17^1) - 35700$

$= 30 \quad 6 \quad 4 \quad 8 \quad 18 - 35700 = 67980$ **Ans.**

Do yourself-9 :

(i) Find the number of ways in which the number 94864 can be resolved as a product of two factors.

(ii) Find the number of different sets of solution of xy = 1440.

10. TOTAL DISTRIBUTION :

(a) Distribution of distinct objects : Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is given by : p^n

(b) Distribution of alike objects : Number of ways to distribute n alike things among p persons so that each may get none, one or more thing(s) is given by ${}^{n+p-1}C_{p-1}$.

Illustration 34: In how many ways can 5 different mangoes, 4 different oranges & 3 different apples be distributed among 3 children such that each gets alteast one mango ?

Solution : 5 different mangoes can be distributed by following ways among 3 children such that each gets atleast 1 :

3 1 1

2 2 1

Total number of ways : $\left(\dfrac{5!}{3!1!1!2!}+\dfrac{5!}{2!2!2!}\right)\times 3!$

Now, the number of ways of distributing remaining fruits (i.e. 4 oranges + 3 apples) among 3 children = 3^7 (as each fruit has 3 options).

$\therefore$ Total number of ways $=\left(\dfrac{5!}{3!2!}+\dfrac{5!}{(2!)^3}\right)\times 3!\times 3^7$ **Ans.**

Illustration 35: In how many ways can 12 identical apples be distributed among four children if each gets atleast 1 apple and not more than 4 apples.

Solution : Let x,y,z & w be the number of apples given to the children.

$\Rightarrow x + y + z + w = 12$

Giving one-one apple to each

Now, $x + y + z + w = 8$ ....(i)

Here, $0 \le x \le 3,\ 0 \le y \le 3,\ 0 \le z \le 3,\ 0 \le w \le 3$

$x = 3 - t_1,\ y = 3 - t_2,\ z = 3 - t_3,\ w = 3 - t_4.$

Putting value of x, y, z, w in equation (i)

Put $12 - 8 = t_1 + t_2 + t_3 + t_4$

$\Rightarrow t_1 + t_2 + t_3 + t_4 = 4$

(Here max. value that t_1, t_2, t_3 & t_4 can attain is 3, so we have to remove those cases when any of t_i getting value 4)

$= {}^7C_3$ – (all cases when atleast one is 4)

$= {}^7C_3 - 4 = 35 - 4 = 31$ **Ans.**

Illustration 36: Find the number of non negative integral solutions of the inequation $x + y + z \le 20$.

Solution : Let w be any number ($0 \le w \le 20$), then we can write the equation as :

$x + y + z + w = 20$ (here $x, y, z, w \ge 0$)

Total ways $= {}^{23}C_3$ **Ans.**

Illustration 37: Find the number of integral solutions of $x + y + z + w < 25$, where $x > -2$, $y > 1$, $z \ge 2$, $w \ge 0$.

Solution : Given $x + y + z + w < 25$

$x + y + z + w + v = 25$ ....(i)

Let $x = -1 + t_1$, $y = 2 + t_2$, $z = 2 + t_3$, $w = t_4$, $v = 1 + t_5$ where $(t_1, t_2, t_3, t_4 \ge 0)$

Putting value of x, y, z, w, v in equation (i)

$\Rightarrow t_1 + t_2 + t_3 + t_4 + t_5 = 21.$

Number of solutions $= {}^{25}C_4$ **Ans.**

Illustration 38: Find the number of positive integral solutions of the inequation $x + y + z \ge 150$, where $0 < x \le 60$, $0 < y \le 60$, $0 < z \le 60$.

Solution : Let $x = 60 - t_1$, $y = 60 - t_2$, $z = 60 - t_3$ (where $0 \le t_1 \le 59$, $0 \le t_2 \le 59$, $0 \le t_3 \le 59$)

Given $x + y + z \ge 150$

or $x + y + z - w = 150$ (where $0 \le w \le 147$) ....(i)

Putting values of x, y, z in equation (i)

$60 - t_1 + 60 - t_2 + 60 - t_3 - w = 150$

$30 = t_1 + t_2 + t_3 + w$

Total solutions $= {}^{33}C_3$ **Ans.**

Illustration 39: Find the number of positive integral solutions of $xy = 12$

Solution : $xy = 12$

$xy = 2^2 \quad 3^1$

(i) 3 has 2 ways either 3 can go to x or y

(ii) 2^2 can be distributed between x & y as distributing 2 identical things between 2 persons (where each person can get 0, 1 or 2 things). Let two person be ℓ_1 & ℓ_2

$\Rightarrow \quad \ell_1 + \ell_2 = 2$

$\Rightarrow \quad {}^{2+1}C_1 = {}^3C_1 = 3$

So total ways $= 2 \quad 3 = 6$.

Alternatively :

$xy = 12 = 2^2 \quad 3^1$

$x = 2^{a_1} 3^{a_2}$ $\quad 0 \le a_1 \le 2$

$0 \le a_2 \le 1$

$y = 2^{b_1} 3^{b_2}$ $\quad 0 \le b_1 \le 2$

$0 \le b_2 \le 1$

$2^{a_1+b_1} 3^{a_2+b_2} = 2^2 3^1$

$\Rightarrow a_1 + b_1 = 2 \quad \rightarrow \quad {}^3C_1$ ways

$a_2 + b_2 = 1 \quad \rightarrow \quad {}^2C_1$ ways

Number of solutions $= {}^3C_1 \quad {}^2C_1 = 3 \quad 2 = 6$ **Ans.**

Illustration 40 : Find the number of solutions of the equation xyz = 360 when **(i)** $x,y,z \in N$ **(ii)** $x,y,z \in I$

Solution : (i) $xyz = 360 = 2^3 \quad 3^2 \quad 5$ $(x,y,z \in N)$

$x = 2^{a_1}3^{a_2}5^{a_3}$ (where $0 \le a_1 \le 3$, $0 \le a_2 \le 2$, $0 \le a_3 \le 1$)

$y = 2^{b_1}3^{b_2}5^{b_3}$ (where $0 \le b_1 \le 3$, $0 \le b_2 \le 2$, $0 \le b_3 \le 1$)

$z = 2^{c_1}3^{c_2}5^{c_3}$ (where $0 \le c_1 \le 3$, $0 \le c_2 \le 2$, $0 \le c_3 \le 1$)

$\Rightarrow \quad 2^{a_1}3^{a_2}5^{a_3}.2^{b_1}3^{b_2}5^{b_3}.2^{c_1}3^{c_2}5^{c_3} = 2^3 \times 3^2 \times 5^1$

$\Rightarrow \quad 2^{a_1+b_1+c_1}.3^{a_2+b_2+c_2}.5^{a_3+b_3+c_3} = 2^3 \times 3^3 \times 5^1$

$\Rightarrow \quad a_1 + b_1 + c_1 = 3 \rightarrow {}^5C_2 = 10$

$a_2 + b_2 + c_2 = 2 \rightarrow {}^4C_2 = 6$

$a_3 + b_3 + c_3 = 1 \rightarrow {}^3C_2 = 3$

Total solutions = 10 6 3 = 180.

(ii) If $x,y,z \in I$ then, (a) all positive (b) 1 positive and 2 negative.

Total number of ways = $180 + {}^3C_2 \quad 180 = 720$ **Ans.**

Do yourself -10 :

(i) In how many ways can 12 identical apples be distributed among 4 boys. (a) If each boy receives any number of apples. (b) If each boy receives atleast 2 apples.

(ii) Find the number of non-negative integral solutions of the equation $x + y + z = 10$.

(iii) Find the number of integral solutions of $x + y + z = 20$, if $x \ge -4$, $y \ge 1$, $z \ge 2$

11. DEARRANGEMENT :

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots.. + \frac{(-1)^n}{n!}\right]$$

Proof : n letters are denoted by 1,2,3,........,n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelope) so that the i^{th} letter is placed in the corresponding envelope. Then,

$n(A_i) = 1 \quad (n-1)!$ [since the remaining n–1 letters can be placed in n –1 envelops in (n–1)! ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then,

$n(A_i \cap A_j) = 1 \quad 1 \quad (n-2)!$

Also $n(A_i \cap A_j \cap A_k) = 1 \quad 1 \quad 1 \quad (n-3)!$

Hence, the required number is

$n(A_1' \cup A_2' \cup \ldots.. \cup A_n') = n! - n(A_1 \cup A_2 \cup \ldots\ldots\ldots \cup A_n)$

$$= n! - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + \ldots\ldots. + (-1)^n \sum n(A_i \cap A_2 \ldots.. \cap A_n)\right]$$

$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + \ldots\ldots.+ (-1)^{n-1} \quad {}^nC_n 1]$

$$= n! - \left[\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + \ldots\ldots. + (-1)^{n-1}\right] = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots.. + \frac{(-1)^n}{n!}\right]$$

Illustration 41: A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) all the letters are in the wrong envelopes. (ii) at least two of them are in the wrong envelopes.

Solution : (i) The number of ways is which all letters be placed in wrong envelopes

$$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right) = 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\right)$$

$= 360 - 120 + 30 - 6 + 1 = 265.$

(i) The number of ways in which at least two of them in the wrong envelopes

$$= {}^6C_4 \,.\, 2!\left(1-\frac{1}{1!}+\frac{1}{2!}\right) + {}^6C_3 \,.\, 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right) + {}^6C_2 \,.\, 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)$$

$$+ {}^6C_1 .\, 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) + {}^6C_0\, 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

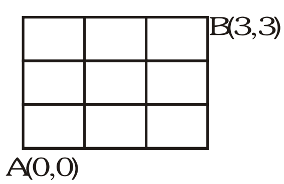
$= 15 + 40 + 135 + 264 + 265 = 719.$ **Ans.**

Do yourself - 11 :

(i) There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one in each box, could be placed in such a way that a ball does not go to box of its own colour.

Miscellaneous Illustrations :

Illustration 42: In how many ways can a person go from point A to point B if he can travel only to the right or upward along the lines (Grid Problem) ?



Solution : To reach the point B from point A, a person has to travel along 3 horizontal and 3 vertical strips.

Therefore, we have to arrange 3H and 3V in a row. Total number of ways $= \frac{6!}{3!3!} = 20$ ways **Ans.**

Illustration 43: Find sum of all numbers formed using the digits 2,4,6,8 taken all at a time and no digit being repeated.

Solution : All possible numbers = 4! = 24

If 2 occupies the unit's place then total numbers = 6

Hence, 2 comes at unit's place 6 times.

Sum of all the digits occuring at unit's place

$= 6 \quad (2 + 4 + 6 + 8)$

Same summation will occur for ten's, hundred's & thousand's place. Hence required sum

$= 6 \quad (2 + 4 + 6 + 8) \quad (1 + 10 + 100 + 1000) = 133320$ **Ans.**

Illustration 44: Find the sum of all the numbers greater than 1000 using the digits 0,1,2,2.

Solution : (i) When 1 is at thousand's place, total numbers formed will be = $\dfrac{3!}{2!} = 3$

(ii) When 2 is at thousand's place, total numbers formed will be = 3! = 6

(iii) When 1 is at hundred's, ten's or unit's place then total numbers formed will be-

Thousand's place is fixed i.e. only the digit 2 will come here, remaining two places can be filled in 2! ways.

So total numbers = 2!

(iv) When 2 is at hundred's, ten's or unit's place then total numbers formed will be-

Thousand's place has 2 options and other two places can be filled in 2 ways.

So total numbers = $2 \quad 2 = 4$

Sum = $10^3 (1 \quad 3 + 2 \quad 6) + 10^2 (1 \quad 2 + 2 \quad 4) + 10^1(1 \quad 2 + 2 \quad 4) + (1 \quad 2 + 2 \quad 4)$

$= 15 \quad 10^3 + 10^3 + 10^2 + 10$

$= 16110$ **Ans.**

Illustration 45 : Find the number of positive integral solutions of $x + y + z = 20$, if $x \neq y \neq z$.

Solution : $x \geq 1$

$y = x + t_1$ $\qquad t_1 \geq 1$

$z = y + t_2$ $\qquad t_2 \geq 1$

$x + x + t_1 + x + t_1 + t_2 = 20$

$3x + 2t_1 + t_2 = 20$

(i) $x = 1$ $\qquad 2t_1 + t_2 = 17$

$t_1 = 1,2 \,\ldots\ldots\, 8 \Rightarrow 8$ ways

(ii) $x = 2$ $\qquad 2t_1 + t_2 = 14$

$t_1 = 1,2 \,\ldots\ldots\, 6 \Rightarrow 6$ ways

(iii) $x = 3$ $\qquad 2t_1 + t_2 = 11$

$t_1 = 1,2 \,\ldots\ldots\, 5 \Rightarrow 5$ ways

(vi) $x = 4$ $\qquad 2t_1 + t_2 = 8$

$t_1 = 1,2,3 \Rightarrow 3$ ways

(v) $x = 5$ $\qquad 2t_1 + t_2 = 5$

$t_1 = 1, 2 \Rightarrow 2$ ways

Total = 8 + 6 + 5 + 3 + 2 = 24

But each solution can be arranged by 3! ways.

So total solutions = $24 \quad 3! = 144$. **Ans.**

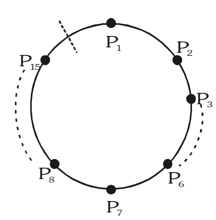
Illustration 46: A regular polygon of 15 sides is constructed. In how many ways can a triangle be formed using the vertices of the polygon such that no side of triangle is same as that of polygon ?

Solution : Select one point out of 15 point, therefore total number of ways = ${}^{15}C_1$

Suppose we select point P_1. Now we have to choose 2 more point which are not consecutive. since we can not select P_2 & P_{15}.

Total points left are 12.

Now we have to select 2 points out of 12 points which are not consecutive



Total ways = ${}^{12-2+1}C_2 = {}^{11}C_2$

Every select triangle will be repeated 3 times.

So total number of ways $= \dfrac{{}^{15}C_1 \times {}^{11}C_2}{3} = 275$

Alternative :

First of all let us cut the polygon between points P_1 & P_{15}. Now there are 15 points on a straight line and we have to select 3 points out of these, such that the selected points are not consecutive.

x O y O z O w

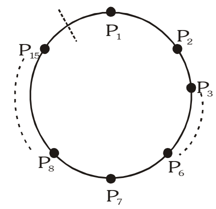
Here bubbles represents the selected points,

x represents the number of points before first selected point,

y represents the number of points between Ist & IInd selected point,

z represents the number of points between IInd & IIIrd selected point

and w represents the number of points after IIIrd selected point.



$x + y + z + w = 15 - 3 = 12$

here $x \geq 0,\ y \geq 1,\ z \geq 1,\ w \geq 0$

Put $y = 1 + y'$ & $z = 1 + z'$ $(y' \geq 0,\ z' \geq 0)$

$\Rightarrow x + y' + z' + w = 10$

Total number of ways $= {}^{13}C_3$

These selections include the cases when both the points P_1 & P_{15} are selected. We have to remove those cases. Here a represents number of points between P_1 & 3^{rd} selected point & b represents number of points between 3^{rd} selected point and P_{15}

$\Rightarrow a + b = 15 - 3 = 12 \quad (a \geq 1, b \geq 1)$

put $a = 1 + t_1$ & $b = 1 + t_2$

$t_1 + t_2 = 10$

Total number of ways $= {}^{11}C_1 = 11$

Therefore required number of ways $= {}^{13}C_3 - {}^{11}C_1 = 286 - 11 = 275$ **Ans.**

Illustration 47: Find the number of ways in which three numbers can be selected from the set $\{5^1, 5^2, 5^3, ...5^{11}\}$ so that they form a G.P.

Solution : Any three selected numbers which are in G.P. have their powers in A.P.

Set of powers is $= \{1, 2, ...6, 7, ...11\}$

By selecting any two numbers from $\{1,3,5,7,9,11\}$, the middle number is automatically fixed. Total number of ways $= {}^6C_2$

Now select any two numbers from $\{2,4,6,8,10\}$ and again middle number is automatically fixed. Total number of ways $= {}^5C_2$

$\therefore$ Total number of ways are $= {}^6C_2 + {}^5C_2 = 15 + 10 = 25$ **Ans.**

ANSWERS FOR DO YOURSELF

1 :	(i)	7	(ii)	3						
2 :	(i)	0	(ii)	r=4	(iii)	${}^{50}C_4$	(iv)	20	(v)	120, 48
3 :	(i)	10	(ii)	450	(iii)	840, 40				
4 :	(i)	$\frac{16!}{(2!)^8 8!} \times 8!$	(ii)	360	(iii)	${}^nC_2.n!$				
5 :	(i)	$5^n - 4^n - 4^n + 3^n$								
6 :	(i)	60, 6^{th}	(ii)	60						
7 :	(i)	36	(ii)	$\frac{9!}{2} = 181440$	(iii)	5400	(iv)	2688		
8 :	(i)	$(p + 1)^n - 1$	(ii)	$2^{10} - 1$						
9 :	(i)	23	(ii)	36						
10 :	(i)	(a) ${}^{15}C_3$ (b) ${}^7C_3$	(ii)	${}^{12}C_2$	(iii)	${}^{23}C_2$				
11 :	(i)	9								

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. The total number of words which can be formed using all the letters of the word "AKSHI" if each word begins with vowel or terminates with vowel -

(A) 84 (B) 12 (C) 48 (D) 60

2. The number of different seven digit numbers that can be written using only three digits 1, 2 & 3 under the condition that the digit 2 occurs exactly twice in each number is -

(A) 672 (B) 640 (C) 512 (D) none of these

3. Out of seven consonants and four vowels, the number of words of six letters, formed by taking four consonants and two vowels is (Assume that each ordered group of letter is a word) -

(A) 210 (B) 462 (C) 151200 (D) 332640

4. A 5 digit number divisible by 3 is to be formed using the numerals 0, 1, 2,3,4 & 5 without repetition. The total number of ways this can be done is -

(A) 3125 (B) 600 (C) 240 (D) 216

5. The number of ways in which 5 different books can be distributed among 10 people if each person can get at most one book is -

(A) 252 (B) 10^5 (C) 5^{10} (D) ${}^{10}C_5 . 5!$

6. Number of ways in which 9 different prizes can be given to 5 students, if one particular student receives 4 prizes and the rest of the students can get any numbers of prizes is -

(A) ${}^{9}C_4 . 2^{10}$ (B) ${}^{9}C_5 . 5^4$ (C) $4 . 4^5$ (D) none of these

7. Boxes numbered 1, 2, 3, 4 and 5 are kept in a row and they are necessarily to be filled with either a red or a blue ball such that no two adjacent boxes can be filled with blue balls. How many different arrangements are possible, given that the balls of a given colour are exactly identical in all respects ?

(A) 8 (B) 10 (C) 13 (D) 22

8. Ten different letters of alphabet are given. Words with four letters are formed from these letters, then the number of words which have at least one letter repeated is -

(A) 10^4 (B) ${}^{10}P_4$ (C) ${}^{10}C_4$ (D) 4960

9. If all the letters of the word "QUEUE" are arranged in all possible manner as they are in a dictionary, then the rank of the word QUEUE is -

(A) 15^{th} (B) 16^{th} (C) 17^{th} (D) 18^{th}

10. Number of ways in which 9 different toys can be distributed among 4 children belonging to different age groups in such a way that distribution among the 3 elder children is even and the youngest one is to receive one toy more is -

(A) $\frac{(5!)^2}{8}$ (B) $\frac{9!}{2}$ (C) $\frac{9!}{3!(2!)^3}$ (D) none of these

11. The number of ways of arranging the letters AAAAA, BBB, CCC, D, EE & F in a row if no two 'C's are together :

(A) ${}^{13}C_3 . \frac{12!}{5!3!2!}$ (B) $\frac{13!}{5!3!3!2!}$ (C) $\frac{14!}{5!3!2!}$ (D) $11.\frac{13!}{6!}$

12. Number of numbers greater than a million and divisible by 5 which can be formed by using only the digits 1, 2, 1, 2, 0, 5 & 2 is -

(A) 120 (B) 110 (C) 90 (D) none of these

13. A set contains (2n + 1) elements. The number of subset of the set which contain at most n elements is : -

(A) 2^n (B) 2^{n+1} (C) 2^{n-1} (D) 2^{2n}

14. The maximum number of different permutations of 4 letters of the word "EARTHQUAKE" is -

(A) 2910 (B) 2550 (C) 2190 (D) 2091

15. The number of ways in which we can arrange n ladies & n gentlemen at a round table so that 2 ladies or 2 gentlemen may not sit next to one another is -

(A) (n – 1)! (n – 2)! (B) (n)! (n – !)! (C) (n + 1)! (n)! (D) none of these

16. The number of proper divisors of $a^p b^q c^r d^s$ where a, b, c, d are primes & p, q, r, s $\in$ N is -

(A) pqrs (B) (p + 1) (q + 1) (r + 1) (s + 1) – 4

(C) pqrs – 2 (D) (p + 1) (q + 1) (r + 1) (s + 1) – 2

17. The sum of all numbers greater than 1000 formed by using the digits 1, 3, 5, 7 such that no digit is being repeated in any number is -

(A) 72215 (B) 83911 (C) 106656 (D) 114712

18. The number of way in which 10 identical apples can be distributed among 6 children so that each child receives atleast one apple is -

(A) 126 (B) 252 (C) 378 (D) none of these

19. Number of ways in which 25 identical pens can be distributed among Keshav, Madhav, Mukund and Radhika such that at least 1, 2, 3 and 4 pens are given to Keshav, Madhav, Mukund and Radhika respectively, is -

(A) ${}^{18}C_4$ (B) ${}^{28}C_3$ (C) ${}^{24}C_3$ (D) ${}^{18}C_3$

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

20. There are (p + q) different books on different topics in Mathematics. (p $\neq$ q)

If L = the number of ways in which these books are distributed between two students X and Y such that X get p books and Y gets q books.

M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books and another gets q books.

N = The number of ways in which these books are divided into two groups of p books and q books then -

(A) L = N (B) L = 2M = 2N (C) 2L = M (D) L = M

21. Number of dissimilar terms in the expansion of $(x_1 + x_2 + \ldots\ldots + x_n)^3$ is -

(A) $\frac{n^2(n+1)^2}{4}$ (B) $\frac{n(n+1)(n+2)}{6}$ (C) ${}^{n+1}C_2 + {}^{n+1}C_3$ (D) $\frac{n^3+3n^2}{4}$

22. A persons wants to invite one or more of his friend for a dinner party. In how many ways can he do so if he has eight friends : -

(A) 2^8 (B) $2^8 - 1$ (C) 8^2 (D) ${}^8C_1 + {}^8C_2 + \ldots\ldots + {}^8C_8$

23. If P(n, n) denotes the number of permutations of n different things taken all at a time then P(n, n) is also identical to :-

(A) n.P(n − 1, n − 1) (B) P(n, n − 1) (C) r! . P(n, n − r) (D) (n − r) . P(n, r)

where $0 \le r \le n$

24. Which of the following statement(s) is/are true :-

(A) $^{100}C_{50}$ is not divisible by 10

(B) n(n − 1)(n − 2) ....(n − r + 1) is always divisible by r! ($n \in N$ and $0 \le r \le n$)

(C) Morse telegraph has 5 arms and each arm moves on 6 different positions including the position of rest. Number of different signals that can be transmitted is $5^6 - 1$.

(D) There are 5 different books each having 5 copies. Number of different selections is $6^5 - 1$.

BRAIN TEASERS **ANSWER KEY** EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A	A	C	D	D	A	C	D	C	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	A	B	D	C	B	D	C	A	D	A,C
Que.	21	22	23	24						
Ans.	B,C	B,D	A,B,C	A,B,D						

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. 5 Indian & 5 American couples meet at a party & shake hands. If no wife shakes hands with her own husband & no Indian wife shakes hands with a male, then the number of hand shakes that takes place in the party is -

(A) 95 (B) 110 (C) 135 (D) 150

2. The number of ways in which a mixed double tennis game can be arranged from amongst 9 married couple if no husband & wife plays in the same game is -

(A) 756 (B) 3024 (C) 1512 (D) 6048

3. There are n identical red balls & m identical green balls. The number of different linear arrangements consisting of "n red balls but not necessarily all the green balls" is ${}^{x}C_{y}$ then -

(A) $x = m + n$, $y = m$ (B) $x = m + n + 1$, $y = m$

(C) $x = m + n + 1$, $y = m + 1$ (D) $x = m + n$, $y = n$

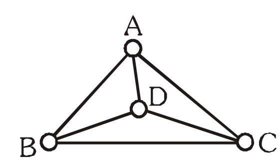
4. Number of different words that can be formed using all the letters of the word "DEEPMALA" if two vowels are together and the other two are also together but separated from the first two is -

(A) 960 (B) 1200 (C) 2160 (D) 1440

5. In a unique hockey series between India & Pakistan, they decide to play on till a team wins 5 matches. The number of ways in which the series can be won by India, if no match ends in a draw is -

(A) 126 (B) 252 (C) 225 (D) none of these

6. A road network as shown in the figure connect four cities. In how many ways can you start from any city (say A) and come back to it without travelling on the same road more than once ?



(A) 8 (B) 12

(C) 9 (D) 16

7. The number of ways of choosing a committee of 2 women & 3 men from 5 women & 6 men, if Mr. A refuses to serve on the committee if Mr. B is a member & Mr. B can only serve, if Miss C is the member of the committee, is -

(A) 60 (B) 84 (C) 124 (D) none of these

8. Six persons A, B, C, D, E and F are to be seated at a circular table. The number of ways this can be done if A must have either B or C on his right and B must have either C or D on his right is -

(A) 36 (B) 12 (C) 24 (D) 18

9. Sum of all the numbers that can be formed using all the digits 2, 3, 3, 4, 4, 4 is -

(A) 22222200 (B) 11111100 (C) 55555500 (D) 20333280

10. $N = 2^2 . 3^3 . 5^4 . 7$, then -

(A) Number of proper divisors of N(excluding 1 & N) is 118

(B) Number of proper divisors of N(excluding 1 & N) is 120

(C) Number of positive integral solutions of xy = N is 60

(D) Number of positive integral solutions of xy = N is 120

11. Sameer has to make a telephone call to his friend Harish, Unfortunately he does not remember the 7 digit phone number. But he remembers that the first three digits are 635 or 674, the number is odd and there is exactly one 9 in the number. The maximum number of trials that Sameer has to make to be successful is -

(A) 10,000 (B) 3402 (C) 3200 (D) 5000

12. Let P_n denotes the number of ways in which three people can be selected out of 'n' people sitting in a row, if no two of them are consecutive. If $P_{n+1} - P_n = 15$ then the value of 'n' is -

(A) 7 (B) 8 (C) 9 (D) 10

13. The number of solutions of $x_1 + x_2 + x_3 = 51$ (x_1, x_2, x_3 being odd natural numbers) is : -

(A) 300 (B) 325 (C) 330 (D) 350

14. The number of positive integral solutions of the equation $x_1x_2x_3 = 60$ is : -

(A) 54 (B) 27 (C) 81 (D) none of these

15. Total number of even divisors of 2079000 which are divisible by 15 are -

(A) 54 (B) 128 (C) 108 (D) 72

16. The number of five digit numbers that can be formed using all the digits 0, 1, 3, 6, 8 which are -

(A) divisible by 4 is 30

(B) greater than 30,000 and divisible by 11 is 12

(C) smaller than 60,000 when digit 8 always appears at ten's place is 6

(D) between 30,000 and 60,000 and divisible by 6 is 18.

17. All the 7 digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once and not divisible by 5 are arranged in the increasing order. Then -

(A) 1800^{th} number in the list is 3124567 (B) 1897^{th} number in the list is 4213567

(C) 1994^{th} number in the list is 4312567 (D) 2001^{th} number in the list is 4315726

BRAIN TEASERS — ANSWER KEY — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C	B	D	A	B	C	D	A	A,D
Que.	11	12	13	14	15	16	17			
Ans.	B	B	B	A	C	A,B,D	B,D			

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

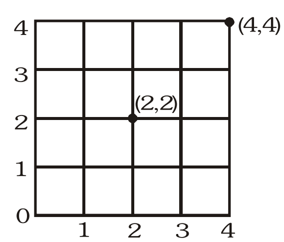
1. 5 balls are to be placed in 3 boxes. Each box can hold all the 5 balls. Number of ways in which the balls can be placed so that no box remains empty, if :

	Column-I		Column-II
(A)	balls are identical but boxes are different	(p)	2
(B)	balls are different but boxes are identical	(q)	25
(C)	balls as well as boxes are identical	(r)	50
(D)	balls as well as boxes are identical but boxes are kept in a row	(s)	6

2. Consider all the different words that can be formed using the letters of the word HAVANA, taken 4 at a time.

	Column-I		Column-II
(A)	Number of such words in which all the 4 letters are different	(p)	36
(B)	Number of such words in which there are 2 alike letters & 2 different letters.	(q)	42
(C)	Number of such words in which A's never appear together	(r)	37
(D)	If all such 4 letters words are written, by the rule of dictionary then the rank of the word HANA	(s)	24

3.

	Column-I		Column-II
(A)	${}^{24}C_2 + {}^{23}C_2 + {}^{22}C_2 + {}^{21}C_2 + {}^{20}C_2 + {}^{20}C_3$ is equal to	(p)	102
(B)	In the adjoining figure number of progressive ways to reach from (0,0) to (4, 4) passing through point (2, 2) are (particle can move on horizontal or vertical line) 	(q)	2300
(C)	The number of 4 digit numbers that can be made with the digits 1, 2, 3, 4, 3, 2	(r)	82
(D)	If $\left\{\frac{500!}{14^k}\right\} = 0$, then the maximum natural value of k is equal to (where {.} is fractional part function)	(s)	36

ASSERTION & REASON

These questions contain, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I :** If a polygon has 45 diagonals, then its number of sides is 10.

Because

Statement-II : Number of ways of selecting 2 points from n non collinear points is ${}^nC_2$.

(A) A (B) B (C) C (D) D

2. **Statement-I** : The expression $n!(20 - n)!$ is minimum where $n = 10$

Because

Statement-II : ${}^{2p}C_r$ is maximum where $r = p$, where p is a constant.

(A) A (B) B (C) C (D) D

3. **Statement-I** : The number of non negative integral solutions of $x_1 + x_2 + x_3 + \ldots\ldots + x_n = r$ is ${}^{r+n-1}C_r$.

Because

Statement-II : The number of ways in which n identical things can be distributed among r students is ${}^{n+r-1}C_n$.

(A) A (B) B (C) C (D) D

4. **Statement-I** : If a, b, c are positive integers such that $a + b + c \le 8$, then the number of possible values of the ordered triplets (a, b, c) is 56.

Because

Statement-II : The number of ways in which n distinct things can be distributed among r girls such that each get at least one is ${}^{n-1}C_{r-1}$.

(A) A (B) B (C) C (D) D

5. **Statement-I** : Number of terms in the expansion of $(x_1 + x_2 + x_3 + \ldots. + x_{11})^6 = {}^{16}C_6$.

Because

Statement-II : Number of ways of distributing n identical things among r persons when each person get zero or more things = ${}^{n+r-1}C_n$

(A) A (B) B (C) C (D) D

6. **Statement-I** : Number of ways in which 400 different things can be distributed between Ramu & Shamu so that each receives 200 things > Number of ways in which 400 different things can be distributed between Sita & Geeta. So that Sita receives 238 things & Geeta receives 162 things.

Because

Statement-II : Number of ways in which (m + n) different things can be distributed between two receivers such that one receives m and other receives n is equal to ${}^{m+n}C_m$, for any two non-negative integers m and n.

(A) A (B) B (C) C (D) D

7. **Statement-I** : The number of positive integral solutions of the equation $xyzw = 770$ is 2^8.

Because

Statement-II : The number of ways of selection of atleast one thing from n things of which 'p' are alike of one kind, q are alike of 2nd kind and rest of the things are different is $(p + 1)(q + 1)\, 2^{n-(p+q)} - 1$.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

S = {0, 2, 4, 6, 8}. A natural number is said to be divisible by 2 if the digit at the unit place is an even number. The number is divisible by 5, if the number at the unit place is 0 or 5. If four numbers are selected from S and a four digit number ABCD is formed.

On the basis of above information, answer the following questions :

1. The number of such numbers which are even (all digits are different) is

(A) 60 (B) 96 (C) 120 (D) 204

2. The number of such numbers which are even (all digits are not different) is

(A) 404 (B) 500 (C) 380 (D) none of these

3. The number of such numbers which are divisible by two and five (all digits are not different) is

(A) 125 (B) 76 (C) 65 (D) 100

Comprehension # 2

Let p be a prime number and n be a positive integer, then exponent of p is n! is denoted by E_p (n!) and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots.. + \left[\frac{n}{p^k}\right]$$

where $p^k < n < p^{k+1}$

and [x] denotes the integral part of x.

If we isolate the power of each prime contained in any number N, then N can be written as

$$N = 2^{\alpha_1}\ 3^{\alpha_2}\ 5^{\alpha_3}\ 7^{\alpha_4}\ \ldots.$$

where α_i are whole numbers.

On the basis of above information, answer the following questions :

1. The exponent of 7 in ${}^{100}C_{50}$ is -

(A) 0 (B) 1 (C) 2 (D) 3

2. The number of zeros at the end of 108! is -

(A) 10 (B) 13 (C) 25 (D) 26

3. The exponent of 12 in 100! is -

(A) 32 (B) 48 (C) 97 (D) none of these

Comprehension # 3

We have to choose 11 players for cricket team from 8 batsmen. 6 bowlers, 4 allrounders and 2 wicketkeeper, in the following conditions.

On the basis of above information, answer the following questions :

1. The number of selections when at most 1 allrounder and 1 wicketkeeper will play -

(A) ${}^4C_1 . {}^{14}C_{10} + {}^2C_1 . {}^{14}C_{10} + {}^4C_1 . {}^2C_1 . {}^{14}C_9 + {}^{14}C_{11}$ (B) ${}^4C_1 . {}^{15}C_{11} + {}^{15}C_{11}$

(C) ${}^4C_1 . {}^{15}C_{10} + {}^{15}C_{11}$ (D) none of these

2. Number of selections when 2 particular batsmen don't want to play, if a particular bowler will play -

(A) ${}^{17}C_{10} + {}^{19}C_{11}$ (B) ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$ (C) ${}^{17}C_{10} + {}^{20}C_{11}$ (D) ${}^{19}C_{10} + {}^{19}C_{11}$

3. Number of selections when a particular batsman and a particular wicketkeeper don't want to play together -

(A) $2\ {}^{18}C_{10}$ (B) ${}^{19}C_{11} + {}^{18}C_{10}$ (C) ${}^{19}C_{10} + {}^{19}C_{11}$ (D) none of these

MISCELLANEOUS TYPE QUESTION	ANSWER KEY	EXERCISE -3

- ***Match the Column***

1. (A) → (s), (B) → (q), (C) → (p), (D) → (s) **2.** (A) → (s), (B) → (p), (C) → (q), (D) → (r)

3. (A)→(q), (B)→(s), (C)→(p), (D)→(r)

- ***Assertion & Reason***

1. D **2.** A **3.** A **4.** C **5.** A **6.** C **7.** B

- ***Comprehension Based Questions***

Comprehension # 1 : 1. B 2. A 3. B

Comprehension # 2 : 1. A 2. C 3. B

Comprehension # 3 : 1. A 2. A 3. B

EXERCISE - 04 [A]

CONCEPTUAL SUBJECTIVE EXERCISE

1. (a) Prove that : ${}^{n}P_{r} = {}^{n-1}P_{r} + r.\ {}^{n-1}P_{r-1}$

(b) If ${}^{20}C_{r+2} = {}^{20}C_{2r-3}$ find ${}^{12}C_{r}$

(c) Find r if ${}^{15}C_{3r} = {}^{15}C_{r+3}$

(d) Find the ratio ${}^{20}C_{r}$ to ${}^{25}C_{r}$ when each of them has the greatest value possible.

2. In how many ways can a team of 6 horses be selected out of a stud of 16, so that there shall always be 3 out of ABC A'B'C', but never AA', BB' or CC' together ?

3. How many 4 digit numbers are there which contain not more than 2 different digits ?

4. An examination paper consists of 12 questions divided into parts A & B. Part-A contains 7 questions & Part -B contains 5 questions. A candidate is required to attempt 8 questions selecting atleast 3 from each part. In how many maximum ways can the candidate select the questions ?

5. The straight lines ℓ_1, ℓ_2, and ℓ_3 are parallel & lie in the same plane. A total of m points are taken on the line ℓ_1, n points on ℓ_2 and k points on ℓ_3. How many maximum number of triangles are there whose vertices are at these points ?

6. Prove that if each of m points in one straight line be joined to each of n in another by straight lines terminated by the points, then excluding the given points, the lines will intersect $\frac{1}{4}mn(m-1)(n-1)$ times.

7. A man has 7 relatives, 4 of them are ladies & 3 gentlemen; his wife has also 7 relatives, 3 of them are ladies & 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies & 3 gentlemen so that there are 3 of the man's relatives & 3 of the wife's relatives ?

8. 5 boys & 4 girls sit in a straight line. Find the number of ways in which they can be seated if 2 girls are together & the other 2 are also together but separated from the first 2.

9. In how many ways 8 persons can be seated on a round table

(a) If two of them (say A and B) must not sit in adjacent seats ?

(b) If 4 of the persons are men and 4 ladies and if no two men are to be in adjacent seats?

(c) If 8 persons constitute 4 married couples and if no husband and wife, as well as no two men are to be in adjacent seats ?

10. There are 2 women participating in a chess tournament. Every participant played 2 games with the other participants. The number of games that the men played between themselves exceeded by 66 as compared to the number of games that the men played with the women. Find the number of participants & the total number of games played in the tournament.

11. In how many ways can you divide a pack of 52 cards equally among 4 players. In how many ways the cards can be divided in 4 sets, 3 of them having 17 cards each & the 4^{th} with 1 card ?

12. (a) How many divisors are there of the number 21600 ? Also find the sum of these divisors.

(b) In how many ways the number 7056 can be resolved as a product of 2 factors.

(c) Find the number of ways in which the number 300300 can be split into 2 factors which are relatively prime.

13. How many different ways can 15 candy bars be distributed between Ram, Shyam, Ghanshyam and Balram, if Ram can not have more than 5 candy bars and Shyam must have at least two ? Assume all candy bars to be alike.

14. Find the sum of all numbers greater than 10000 formed by using the digits 0, 1, 2, 4, 5 & no digit being repeated in any number.

15. Find the number of ways in which the letters of the word 'MUNMUN' can be arranged so that no two alike letters are together.

16. A shop sells 6 different flavours of ice-creams. In how many ways can a customer choose 4 ice-cream cones if

(a) they are all of different flavours ?

(b) they are non necessarily of different flavours ?

(c) they contain only 3 different flavours ?

(d) they contain only 2 or 3 different flavours ?

17. Find the number of ways in which a selection of 100 balls, can be made out of 100 identical red balls, 100 identical blue balls & 100 identical white balls.

18. There are 5 balls of different colours & 5 boxes of colours same as those of the balls. The number of ways in which the balls, one in each box could be placed such that exactly one ball goes to the box of its own colour.

CONCEPTUAL SUBJECTIVE EXERCISE	ANSWER KEY	EXERCISE-4(A)

1. (b) 792, (c) r = 3 (d) $\frac{143}{4025}$ **2.** 960 **3.** 576 **4.** 420

5. ${}^{m+n+k}C_3 - ({}^{m}C_3 + {}^{n}C_3 + {}^{k}C_3)$ **7.** 485 **8.** 43200 **9.** (a) 5.(6!) (b) 3!4! (c) 12

10. 13, 156 **11.** $\frac{52!}{(13!)^4}; \frac{52!}{3!(17!)^3}$ **12.** (a) 72 ; 78120 (b) 23 (c) 32 **13.** 440

14. 3119976 **15.** 30 **16.** (a) 15; (b) 126 ; (c) 60 ; (d) 105 **17.** 5151 **18.** 45

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. There are counters available in 7 different colours. Counters are all alike except colour and they are atleast ten of each colour. Find the number of ways in which an arrangement of 10 counters can be made. How many of these will have counters of each colour ?

2. How many integral solutions are there for the equation; $x + y + z + w = 29$ when $x > 0$, $y > 1$, $z > 2$ & $w \ge 0$?

3. A party of 10 consists of 2 Americans, 2 Britishmen, 2 Chinese & 4 men of other nationalities (all different). Find the number of ways in which they can stand in a row so that no two men of the same nationality are next to one another. Find also the number of ways in which they can sit at a round table.

4. Find the number of words each consisting of 3 consonants & 3 vowels that can be formed from the letters of the word "CIRCUMFERENCE". In how many of these 'C's will be together.

5. Find the number of ways in which the number 30 can be partitioned into three unequal parts, each part being a natural number. What this number would be if equal parts are also included.

6. Prove by combinatorial argument that :

(a) $^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

(b) $^{n+m}C_r = {}^nC_0 \cdot {}^mC_r + {}^nC_1 \cdot {}^mC_{r-1} + {}^nC_2 \cdot {}^mC_{r-2} + \ldots.. + {}^nC_r \cdot {}^mC_0$

7. How many 6 digits odd numbers greater than 60,0000 can be formed from the digits 5,6,7,8,9.0 if

(a) repetitions are not allowed ? (b) repetitions are allowed ?

8. All the 7 digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once and not divisible by 5 are arranged in the increasing order. Find the $(2004)^{th}$ number in this list.

9. A firm of Chartered Accountants in Bombay has to send 10 clerks to 5 different companies, two clerks in each. Two of the companies are in Bombay and the others are outside. Two of the clerks prefer to work in Bombay while three others prefer to work outside. In how many ways can the assignment be made if the preferences are to be satisfied ?

10. There are 5 white, 4 yellow, 3 green, 2 blue & 1 red ball. The balls are all identical except colour. These are to be arranged in a line at 5 places. Find the number of distinct arrangements.

11. A crew of an eight oar boat has to be chosen out of 11 men five of whom can row on stroke side only, four on the bow side only and the remaining two on either side. How many different selections can be made?

12. There are n straight lines in a plane, no 2 of which are parallel & no 3 pass through the same point. Their points of intersection are joined. Show that the number of fresh lines introduced is $\dfrac{n(n-1)(n-2)(n-3)}{8}$.

13. There are 20 books on Algebra & Calculus in our library. Prove that the greatest number of selections each of which consists of 5 books on each topic is possible only when there are 10 books on each topic in the library.

14. Find the number of ways to invite one of the three friends for dinner on 6 successive nights such that no friend is invited more than 3 times.

BRAIN STORMING SUBJECTIVE EXERCISE	ANSWER KEY			EXERCISE-4(B)
1. $7^{10}; \left(\dfrac{49}{6}\right)10!$	2. 2600	3. (47)8! ; (244)6!	4. 22100, 52	5. 61, 75
7. (a) 240 (b) 15552	8. 4316527	9. 5400	10. 2111 11. 145	14. 510

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. Numbers greater than 1000 but not greater than 4000 which can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is allowed), are **[AIEEE 2002]**
(1) 350 (2) 375 (3) 450 (4) 576

2. A five digit number divisible by 3 has to formed using the numerals 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is **[AIEEE 2002]**
(1) 216 (2) 240 (3) 600 (4) 3125

3. Total number of four digit odd numbers that can be formed using 0, 1, 2, 3, 5, 7 are **[AIEEE 2002]**
(1) 192 (2) 375 (3) 400 (4) 720

4. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by **[AIEEE 2003]**
(1) 6! 5! (2) 30 (3) 5! 4! (4) 7! 5!

5. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five question. The number of choices available to him is **[AIEEE 2003]**
(1) 140 (2) 196 (3) 280 (4) 346

6. If ${}^nC_r$ denots the number of combinations of n things taken r at a time, then the expression ${}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$ equals **[AIEEE 2003]**
(1) ${}^{n+2}C_r$ (2) ${}^{n+2}C_{r+1}$ (3) ${}^{n+1}C_r$ (4) ${}^{n+1}C_{r+1}$

7. How many ways are there to arrange the letters in the word 'GARDEN' with the vowels in alphabetical order? **[AIEEE 2004]**
(1) 120 (2) 240 (3) 360 (4) 480

8. The number of ways of distributing 8 identical balls in 3 distinct boxes so that none of the boxes is empty is **[AIEEE 2004]**
(1) 5 (2) 21 (3) 3^8 (4) ${}^8C_3$

9. If the letters of the word 'SACHIN' are arranged in all possible ways and these words are written out as in dictionary, then the word 'SACHIN' appears at serial number **[AIEEE 2005]**
(1) 602 (2) 603 (3) 600 (4) 601

10. The value of ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is **[AIEEE 2005]**
(1) ${}^{56}C_4$ (2) ${}^{56}C_3$ (3) ${}^{55}C_3$ (4) ${}^{55}C_4$

11. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidated and 4 are to be elected. If a voter votes for at least one candidates, then the number of ways in which he can vote is **[AIEEE-2006]**
(1) 385 (2) 1110 (3) 5040 (4) 6210

12. The set S = {1, 2, 3,.....,12} is to partitioned into three sets A, B, C of equal size. Thus,$A \cup B \cup C = S$, $A \cap B = B \cap C = C \cap A = \phi$, then number of ways to partition S are- **[AIEEE-2007]**
(1) $\frac{12!}{3!(3!)^4}$ (2) $\frac{12!}{(4!)^3}$ (3) $\frac{12!}{(3!)^4}$ (4) $\frac{12!}{3!(4!)^3}$

13. In a shop there are five types of ice–creams available. A child buys six ice–creams. **[AIEEE 2008]**
Statement –1 : The number of different ways the child can buy the six ice–creams is ${}^{10}C_5$.
Statement –2 : The number of different ways the child can buy the six ice–creams is equal to the number of different ways of arranging 6 A's and 4 B's in a row.
(1) Statement –1 is false, Statement –2 is true
(2) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1
(3) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for Statement–1
(4) Statement–1 is true, Statement–2 is false

14. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangements is :- **[AIEEE 2009]**

(1) At least 750 but less than 1000 (2) At least 1000

(3) Less than 500 (4) At least 500 but less than 750

15. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is :- **[AIEEE-2010]**

(1) 3 (2) 36 (3) 66 (4) 108

16. **Statement - 1 :** The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is $^9C_3$. **[AIEEE-2011]**

Statement - 2 : The number of ways of choosing any 3 places from 9 different places is $^9C_3$.

(1) Statement-1 is true, Statement-2 is false.

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

17. There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then : **[AIEEE-2011]**

(1) $N > 190$ (2) $N \leq 100$ (3) $100 < N \leq 140$ (4) $140 < N \leq 190$

18. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is :- **[AIEEE-2012]**

(1) 879 (2) 880 (3) 629 (4) 630

19. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of A B having 3 or more elements is **[JEE (Main)-2013]**

(1) 256 (2) 220 (3) 219 (4) 211

20. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is : **[JEE (Main)-2013]**

(1) 7 (2) 5 (3) 10 (4) 8

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans	2	1	1	1	2	2	3	2	4	1	1	2	1	2	4
Que.	16	17	18	19	20										
Ans	3	2	1	3	2										

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. How many different nine digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions ? [JEE 2000, Screening 1M]
(A) 16 (B) 36 (C) 60 (D) 180

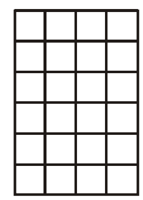
2. (a) Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of 'n' sides. If $T_{n+1} - T_n = 21$, then 'n' equals - [JEE 2001, Screening 1+1M]
(A) 5 (B) 7 (C) 6 (D) 4
(b) Let E = {1, 2, 3, 4} and F = {1, 2}. Then the number of onto functions from E to F is -
(A) 14 (B) 16 (C) 12 (D) 8

3. The number of arrangements of the letters of the word BANANA in which two 'N's do not appear adjacently is - [JEE 2002, Screening 3M]
(A) 40 (B) 60 (C) 80 (D) 100

4. Number of points with integral co-ordinates that lie inside a triangle whose co-ordinates are (0, 0), (0, 21) and (21, 0) [JEE 2003, Screening 3M]
(A) 210 (B) 190 (C) 220 (D) none of these

5. Using permutation or otherwise prove that $\dfrac{(n^2)!}{(n!)^n}$ is an integer, where n is a positive integer.
[JEE 2004, (Mains) 2M out of 60]

6. A rectangle with sides 2m − 1 and 2n − 1 is divided into squares of unit length by drawing parallel lines as shown then number of rectangles possible with odd side lengths is - [JEE 2005, Screening 3M]



(A) $(m + n + 1)^2$ (B) 4^{m+n-1}
(C) $m^2 n^2$ (D) mn (m + 1) (n + 1)

7. If r, s, t are the prime numbers and p, q are the positive integers such that the LCM of p & q is $r^2t^4s^2$, then the number of ordered pair (p, q) is : [JEE 2006, (3M, −1M) out of 184]
(A) 252 (B) 254 (C) 225 (D) 224

8. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word COCHIN is - [JEE 2007, 3M]
(A) 360 (B) 192 (C) 96 (D) 48

9. Consider all possible permutations of the letters of the word ENDEANOEL. [JEE 2008, 6M]
Match the Statements / Expressions in Column I with the Statements / Expressions in Column II.

	Column I		Column II
(A)	The number of permutations containing the word ENDEA is	(p)	5!
(B)	The number of permutations in which the letter E occurs in the first and the last positions is	(q)	2 5!
(C)	The number of permutations in which none of the letters D, L, N occurs in the last five positions is	(r)	7 5!
(D)	The number of permutations in which the letters A, E, O occur only in odd positions is	(s)	21 5!

10. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only is - [JEE 2009, 3M, −1M]
(A) 55 (B) 66 (C) 77 (D) 88

11. Let S = {1,2,3,4}. The total number of unordered pairs of disjoint subsets of S is equal to -
[JEE 10, 5M, −2M]
(A) 25 (B) 34 (C) 42 (D) 41

12. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets at least one ball is - [JEE 2012, 3M, −1M]
(A) 75 (B) 150 (C) 210 (D) 243

Paragraph for Question 13 and 14 :

Let a_n denotes the number of all n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

13. The value of b_6 is **[JEE 2012, 3M, –1M]**

(A) 7 (B) 8 (C) 9 (D) 11

14. Which of the following is correct ? **[JEE 2012, 3M, –1M]**

(A) $a_{17} = a_{16} + a_{15}$ (B) $c_{17} \neq c_{16} + c_{15}$ (C) $b_{17} \neq b_{16} + c_{16}$ (D) $a_{17} = c_{17} + b_{16}$

PREVIOUS YEARS QUESTIONS		ANSWER KEY			EXERCISE-5 [B]	
1. C	**2.** (a) B; (b) 14	**3.** A	**4.** B	**6.** C	**7.** C	**8.** C
9. (A)→(p), (B)→(s), (C)→(q), (D)→(q)		**10.** C	**11.** D	**12.** B	**13.** B	**14.** A