CHAPTER

07

Determinants

Learning Part

Session 1

- Definition of Determinants
- Expansion of Determinant
- Sarrus Rule for Expansion
- Window Rule for Expansion

Session 2

- Minors and Cofactors
- Use of Determinants in Coordinate Geometry
- Properties of Determinants

Session 3

- Examples on Largest Value of a Third Order Determinant
- Multiplication of Two Determinants of the Same Order
- System of Linear Equations
- Cramer's Rule
- Nature of Solutions of System of Linear Equations
- System of Homogeneous Linear Equations

Session 4

- Differentiation of Determinant
- Integration of a Determinant
- Walli's Formula
- Use of Σ in Determinant

Practice Part

- JEE Type Examples
- Chapter Exercises

Session 1

Definition of Determinants, Expansion of Determinant, Sarrus Rule for Expansion, Window Rule for Expansion

Determinants were invented independently by **Gabriel Cramer**, whose now well-known rule for solving linear system was published in 1750, although not in present day notation. The now-standard "Vertical line notation", i.e. "| |" was given in 1841 by **Arthur Cayley**. The working knowledge of determinants is a basic necessity for a student. Determinants have wide applications in Engineering, Science, Economics, Social science, etc.

Definition of Determinants

Consider the system of two homogeneous linear equations

$$a_1 x + b_1 y = 0 \quad \text{...(i)}$$
$$a_2 x + b_2 y = 0 \quad \text{...(ii)}$$

in the two variables x and y. From these equations, we obtain

$$-\frac{a_1}{b_1} = \frac{y}{x} = -\frac{a_2}{b_2} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$$\Rightarrow \quad a_1 b_2 - a_2 b_1 = 0$$

The result $a_1 b_2 - a_2 b_1$ is represented by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

which is known as determinant of order two. The quantities a_1, b_1, a_2 and b_2 are called constituents or elements of the determinant and $a_1 b_2 - a_2 b_1$ is called its value.

The horizontal lines are called rows and vertical lines are called columns. Here, this determinant consists two rows and two columns.

For example, The value of the determinant

$$\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = 2 \times (-5) - 3 \times 4 = -10 - 12 = -22$$

Now, let us consider the system of three homogeneous linear equations

$$a_1 x + b_1 y + c_1 z = 0 \quad \text{...(i)}$$
$$a_2 x + b_2 y + c_2 z = 0 \quad \text{...(ii)}$$
$$a_3 x + b_3 y + c_3 z = 0 \quad \text{...(iii)}$$

On solving Eqs. (ii) and (iii) for x, y and z by cross-multiplication, we get

$$\frac{x}{b_2 c_3 - b_3 c_2} = \frac{y}{c_2 a_3 - c_3 a_2} = \frac{z}{a_2 b_3 - a_3 b_2} = k \quad \text{[say]}$$

$$\Rightarrow \quad x = k(b_2 c_3 - b_3 c_2),\ y = k(c_2 a_3 - c_3 a_2)$$

and $\quad z = k(a_2 b_3 - a_3 b_2)$

On putting these values of x, y and z in Eq. (i), we get

$$a_1(b_2 c_3 - b_3 c_2) + b_1(c_2 a_3 - c_3 a_2) + c_1(a_2 b_3 - a_3 b_2) = 0$$

or $$a_1(b_2 c_3 - b_3 c_2) - b_1(c_3 a_2 - c_2 a_3) + c_1(a_2 b_3 - a_3 b_2) = 0 \quad \text{...(iv)}$$

or $$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0 \quad \text{...(v)}$$

Usually this is written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Here, the expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of three rows and three columns, is called determinant of order three.

The quantities $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 are called constituents or elements of the determinant.

Remark

1. A determinant is generally denoted by D or Δ.
2. A determinant of the nth order consists of n rows and n columns and its expansion contains $n!$ terms.
3. A determinant of nth order consists of n rows and n columns.
   $\therefore$ Number of constituents in determinant $= n^2$
4. In a determinant the horizontal lines counting from top to bottom 1st, 2nd, 3rd, ... respectively, known as rows and denoted by $R_1, R_2, R_3, \ldots$ and vertical lines from left to right 1st, 2nd, 3rd, ... respectively, known as columns and denoted by $C_1, C_2, C_3, \ldots$.
5. Shape of every determinant is square.
6. Sign system for order 2, order 3, order 4, ... are given by

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}, \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}, \ldots$$

Expansion of Determinant

(i) Expansion of two order

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & \\ & b_2 \end{vmatrix} - \begin{vmatrix} & b_1 \\ a_2 & \end{vmatrix} = a_1 b_2 - b_1 a_2$$

For example, $\begin{vmatrix} 5 & -4 \\ -3 & 2 \end{vmatrix} = \begin{vmatrix} 5 & \\ & 2 \end{vmatrix} - \begin{vmatrix} & -4 \\ -3 & \end{vmatrix}$

$$= 10 - 12 = -2$$

(ii) Expansion of third order

(a) *With respect to first row.*

$$\begin{vmatrix} a_1 \cdots & b_1 \cdots & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$

(b) *With respect to second column.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ & \vdots & \\ a_2 & b_2 & c_2 \\ & \vdots & \\ a_3 & b_3 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= -b_1(a_2 c_3 - a_3 c_2) + b_2(a_1 c_3 - a_3 c_1) - b_3(a_1 c_2 - a_2 c_1)$$

Remark

A determinant can be expanded along any of its row or column. Value of the determinant remains same in any of the cases.

Example 1. Find the value of the determinant
$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 4 & 9 \\ 2 & -1 & 6 \end{vmatrix}$$

Sol. Expanding the determinant along the first row

$$\Delta_1 = 1 \begin{vmatrix} 4 & 9 \\ -1 & 6 \end{vmatrix} - 2 \begin{vmatrix} 3 & 9 \\ 2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}$$
$$= 1(24 + 9) - 2(18 - 18) + 4(-3 - 8)$$
$$= 33 - 0 - 44$$
$$= -11$$

and expanding the determinant along third column

$$\Delta_2 = 4 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} - 9 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 6 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$
$$= 4(-3 - 8) - 9(-1 - 4) + 6(4 - 6)$$
$$= -44 + 45 - 12$$
$$= 1 - 12$$
$$= -11$$

and expanding the determinant along second column

$$\Delta_3 = -2 \begin{vmatrix} 3 & 9 \\ 2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 4 \\ 3 & 9 \end{vmatrix}$$
$$= -2(18 - 18) + 4(6 - 8) + 1(9 - 12)$$
$$= 0 - 8 - 3$$
$$= -11$$

Hence, $\Delta_1 = \Delta_2 = \Delta_3$

Example 2. If $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$, prove that $2 \le \Delta \le 4$.

Sol. Given, $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = 1 \begin{vmatrix} 1 & \sin\theta \\ -\sin\theta & 1 \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & \sin\theta \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} -\sin\theta & 1 \\ -1 & -\sin\theta \end{vmatrix}$$

$$= (1 + \sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + (\sin^2\theta + 1)$$
$$= 2(1 + \sin^2\theta)$$

Again, $0 \le \sin^2\theta \le 1$

$\Rightarrow \quad 1 \le (1 + \sin^2\theta) \le 1 + 1$

$\Rightarrow \quad 2 \le 2(1 + \sin^2\theta) \le 4$

$\therefore \quad 2 \le \Delta \le 4$

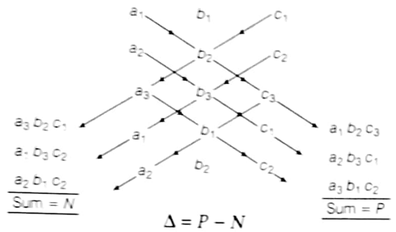
Sarrus Rule for Expansion

Sarrus gave a rule for a determinant of order 3.

Rule Write down the three rows of the Δ and rewrite the first two rows. The three diagonals sloping down to the right given the three terms and the three diagonals sloping down to the left also given the three terms.

If $\quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Rule

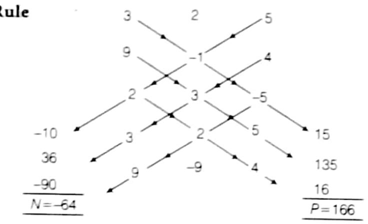


$\therefore \quad \Delta = P - N$

| Example 3. Expand $\begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$ by Sarrus rule.

Sol. Let $\Delta = \begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$

Rule

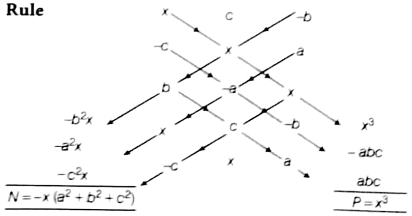


$\therefore \quad \Delta = P - N = 166 - (-64) = 230$

| Example 4. If $a, b, c \in R$, find the number of real roots of the equation $\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix} = 0$

Sol. Let $\Delta = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Rule



$\therefore \quad \Delta = P - N$

$= x^3 + x(a^2 + b^2 + c^2) = 0$ [given]

$\therefore \quad x = 0$ or $x^2 = -(a^2 + b^2 + c^2)$

$\Rightarrow \quad x = 0$ or $x = \pm i\sqrt{(a^2 + b^2 + c^2)}$, where $i = \sqrt{-1}$

Hence, number of real roots is one.

Window Rule for Expansion

Window rule valid only for third order determinant.

Let $\quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

In this method, rewrite first two elements of second row and third row, then

Rule
$$\begin{array}{ccccc} a_1 & b_1 & c_1 & & \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}$$

Now, taking positive sign with a_1, b_1 and c_1.

$$\Delta = a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$$

| Example 5. Expand $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 2 \\ 5 & 9 & 4 \end{vmatrix}$ by window rule.

Sol. Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 2 \\ 5 & 9 & 4 \end{vmatrix}$

Rule:
$$\begin{array}{ccccc} 1 & 2 & 3 & & \\ 4 & 6 & 2 & 4 & 6 \\ 5 & 9 & 4 & 5 & 9 \end{array}$$

$\therefore \quad \Delta = 1(24 - 18) + 2(10 - 16) + 3(36 - 30)$

$= 6 - 12 + 18 = 12$

| Example 6. Find the value of the determinant $\begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$.

Sol. Let $\Delta = \begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$ and let $2\sqrt{2} = \lambda$,

then $\quad \Delta = \begin{vmatrix} -1 & 2 & 1 \\ 3+\lambda & 2+\lambda & 1 \\ 3-\lambda & 2-\lambda & 1 \end{vmatrix}$

Rule
$$\begin{array}{ccccc} -1 & 2 & 1 & & \\ 3+\lambda & 2+\lambda & 1 & 3+\lambda & 2+\lambda \\ 3-\lambda & 2-\lambda & 1 & 3-\lambda & 2-\lambda \end{array}$$

Now, $\Delta = -1(2 + \lambda - 2 + \lambda) + 2(3 - \lambda - 3 - \lambda) + 1[(3 + \lambda)(2 - \lambda) - (3 - \lambda)(2 + \lambda)]$

$= -2\lambda - 4\lambda + (-2\lambda) = -8\lambda = -16\sqrt{2}$ $\quad [\because \lambda = 2\sqrt{2}]$

d, d, c, b, c, d, b

Exercise for Session 1

1 Sum of real roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is

(a) -2 (b) -1 (c) 0 (d) 1

2 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy, i = \sqrt{-1}$, then

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$
(c) $x = 0, y = 3$ (d) $x = 0, y = 0$

3 If $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda^2 + 1 & 2 - \lambda & \lambda - 3 \\ \lambda^2 - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$, then t is equal to

(a) 7 (b) 14
(c) 21 (d) 28

4 If one root of the equation $\begin{vmatrix} 7 & 6 & x^2 - 13 \\ 2 & x^2 - 13 & 2 \\ x^2 - 13 & 3 & 7 \end{vmatrix} = 0$ is $x = 2$, the sum of all other five roots is

even functn

(a) $2\sqrt{15}$ (b) -2
(c) $\sqrt{20} + \sqrt{15} - 2$ (d) None of these

5 If A, B and C are the angles of a non-right angled ΔABC, the value of $\begin{vmatrix} \tan A & 1 & 1 \\ 1 & \tan B & 1 \\ 1 & 1 & \tan C \end{vmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

d **6** If $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$, the maximum value of Δ is

(a) -10 (b) $-\sqrt{10}$ (c) $\sqrt{10}$ (d) 10

O **7** If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then $(a, b, c > 0)$

(a) $abc > 1$ (b) $abc > -8$ (c) $abc < -8$ (d) $abc > -2$

Session 2

Minors and Cofactors, Use of Determinants in Coordinate Geometry, Properties of Determinants

Minors and Cofactors

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$

be a determinant of order n, $n \geq 2$, then the determinant of order $n-1$ obtained from the determinant Δ after deleting the ith row and jth column is called the **minor of the element a_{ij}** and it is usually denoted by M_{ij}, where $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, n$.

If M_{ij} is the minor of the element a_{ij} in the determinant Δ, then $(-1)^{i+j} M_{ij}$ is called the **cofactor of the element a_{ij}**. It is usually denoted by C_{ij}.

Thus, $C_{ij} = (-1)^{i+j} M_{ij}$

$$= \begin{cases} M_{ij}, & \text{if } i+j \text{ is an even integer} \\ -M_{ij}, & \text{if } i+j \text{ is an odd integer} \end{cases}$$

(i) Let $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, then

$M_{11} = |a_{22}| = a_{22}$, $M_{12} = |a_{21}| = a_{21}$,

$M_{21} = |a_{12}| = a_{12}$, $M_{22} = |a_{11}| = a_{11}$ and

$C_{11} = (-1)^{1+1} M_{11} = a_{22}$,

$C_{12} = (-1)^{1+2} M_{12} = -a_{21}$,

$C_{21} = (-1)^{2+1} M_{21} = -a_{12}$

and $C_{22} = (-1)^{2+2} M_{22} = a_{11}$

(ii) Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Determinants of minors and cofactors are

$$\Delta^M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}, \ \Delta^c = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & a_{32} & a_{33} \end{vmatrix}$$

where

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $C_{11} = (-1)^{1+1} M_{11} = M_{11}$

$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$

$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$, $C_{13} = (-1)^{1+3} M_{13} = M_{13}$

$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$, $C_{21} = (-1)^{2+1} M_{21} = -M_{21}$

$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$, $C_{22} = (-1)^{2+2} M_{22} = M_{22}$

$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$, $C_{23} = (-1)^{2+3} M_{23} = -M_{23}$

$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$, $C_{31} = (-1)^{3+1} M_{31} = M_{31}$

$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$, $C_{32} = (-1)^{3+2} M_{32} = -M_{32}$

$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $C_{33} = (-1)^{3+3} M_{33} = M_{33}$

Important Results for Cofactors

1. The sum of products of the elements of any row or column with their corresponding cofactors is equal to the value of the determinant.

i.e., $\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$
$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$
$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

Now, value of n order determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots + a_{1n}C_{1n}$

(when expanded along first row)

2. The sum of the product of element of any row (or column) with corresponding cofactors of another row (or column) is equal to zero.

i.e., $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$,
$a_{11}C_{13} + a_{21}C_{23} + a_{31}C_{33} = 0$, etc.

3. If the value of a n order determinant is Δ, then the value of the determinant formed by the cofactors of corresponding elements of the given determinant is given by

$$\boxed{\Delta^c = \Delta^{n-1}}$$

i.e., in case of second order determinant

$$\Delta^c = \Delta$$

and third order determinant $\Delta^c = \Delta^2$.

| Example 7. Find the determinants of minors and cofactors of the determinant $\begin{vmatrix} 2 & 3 & 4 \\ 7 & 2 & -5 \\ 8 & -1 & 3 \end{vmatrix}$.

Sol. Here, $M_{11} = \begin{vmatrix} 2 & -5 \\ -1 & 3 \end{vmatrix} = 6 - 5 = 1$

[delete 1st row and 1st column]

$\therefore \quad C_{11} = (-1)^{1+1} M_{11} = M_{11} = 1$

$M_{12} = \begin{vmatrix} 7 & -5 \\ 8 & 3 \end{vmatrix} = 21 + 40 = 61$

[delete 1st row and 2nd column]

$\therefore \quad C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -61$

$M_{13} = \begin{vmatrix} 7 & 2 \\ 8 & -1 \end{vmatrix} = -7 - 16 = -23$

[delete 1st row and 3rd column]

$\therefore \quad C_{13} = (-1)^{1+3} M_{13} = M_{13} = -23$

$M_{21} = \begin{vmatrix} 3 & 4 \\ -1 & 3 \end{vmatrix}$ [delete 2nd row and 1st column]

$= 9 + 4 = 13$

$\therefore \quad C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -13$

$M_{22} = \begin{vmatrix} 2 & 4 \\ 8 & 3 \end{vmatrix}$ [delete 2nd row and 2nd column]

$= 6 - 32 = -26$

$\therefore \quad C_{22} = (-1)^{2+2} M_{22} = M_{22} = -26$

$M_{23} = \begin{vmatrix} 2 & 3 \\ 8 & -1 \end{vmatrix}$ [delete 2nd row and 3rd column]

$= -2 - 24 = -26$

$\therefore \quad C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 26$

$M_{31} = \begin{vmatrix} 3 & 4 \\ 2 & -5 \end{vmatrix}$ [delete 3rd row and 1st column]

$= -15 - 8 = -23$

$\therefore \quad C_{31} = (-1)^{3+1} M_{31} = M_{31} = -23$

$M_{32} = \begin{vmatrix} 2 & 4 \\ 7 & -5 \end{vmatrix}$ [delete 3rd row and 2nd column]

$= -10 - 28 = -38$

$\therefore \quad C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 38$

and $M_{33} = \begin{vmatrix} 2 & 3 \\ 7 & 2 \end{vmatrix} = 4 - 21 = -17$

[delete 3rd row and 3rd column]

$\therefore \quad C_{33} = (-1)^{3+3} M_{33} = M_{33} = -17$

Hence, determinants of minors and cofactors are

$$\begin{vmatrix} 1 & 61 & -23 \\ 13 & -26 & -26 \\ -23 & -38 & -17 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & -61 & -23 \\ -13 & -26 & 26 \\ -23 & 38 & -17 \end{vmatrix}, \text{ respectively.}$$

Goyal's Method for Cofactors

(Direct Method)

This method applied only for third order determinant.

Method If $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

***Step* I** Write down the three rows of the Δ and rewrite first two rows.

i.e.
$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}$$

***Step* II** After Step I, rewrite first two columns.

i.e.,
$$\begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{matrix}$$

***Step* III** After step II, deleting first row and first column, then we get all cofactors i.e.

$$\begin{matrix} b_2 & b_3 & b_1 & b_2 \\ c_2 & c_3 & c_1 & c_2 \\ a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{matrix}$$

or $\Delta^c = \begin{vmatrix} b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \\ c_2a_3 - c_3a_2 & c_3a_1 - c_1a_3 & c_1a_2 - c_2a_1 \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \end{vmatrix}$

Example 8. Find the determinant of cofactors of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ by Direct Method.

Sol. Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$

***Step* I** Write down the three rows of the Δ and rewrite first two rows.

i.e.,
$$\begin{matrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \\ 1 & 2 & 3 \\ -4 & 3 & 6 \end{matrix}$$

***Step* II** After step I, rewrite first two columns

i.e.,
$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ -4 & 3 & 6 & -4 & 3 \\ 2 & -7 & 9 & 2 & -7 \\ 1 & 2 & 3 & 1 & 2 \\ -4 & 3 & 6 & -4 & 3 \end{matrix}$$

***Step* III** After Step II, deleting first row and first column, then we get all cofactors i.e.,

$$\begin{matrix} 3 & 6 & -4 & 3 \\ -7 & 9 & 2 & -7 \\ 2 & 3 & 1 & 2 \\ 3 & 6 & -4 & 3 \end{matrix} \quad \text{or} \quad \Delta^c = \begin{vmatrix} 69 & 48 & 22 \\ -39 & 3 & 11 \\ 3 & -18 & 11 \end{vmatrix}$$

Example 9. If the value of a third order determinant is 11, find the value of the square of the determinant formed by the cofactors.

Sol. Here, $n = 3$ and $\Delta = 11$

$\therefore \quad (\Delta^c)^2 = (\Delta^2)^2 = \Delta^4 = 11^4 = 14641$

Use of Determinants in Coordinate Geometry

(i) Area of triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(ii) If points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) and collinear, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

(iii) If $a_r x + b_r y + c_r = 0; r = 1, 2, 3$ are the sides of a triangle, then the area of the triangle is given by

$$\Delta = \frac{1}{|2C_1C_2C_3|}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where C_1, C_2 and C_3 are the cofactors of the elements c_1, c_2 and c_3 respectively, in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(iv) Equation of straight line passing through two points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(v) If three lines $a_r x + b_r y + c_r = 0;\ r = 1, 2, 3$ are concurrent, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(vi) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(vii) Equation of circle through three non-collinear points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Some Useful Operations

(i) The interchange of ith row and jth row is denoted by $R_i \leftrightarrow R_j$. (In case of column $C_i \leftrightarrow C_j$)

(ii) The addition of m times the elements of jth row to the corresponding elements of ith row is denoted by $R_i \to R_i + mR_j$.
(In case of column $C_i \to C_i + mC_j$)

(iii) The addition of m times the elements of jth row and n times the elements of k th row to the corresponding elements of ith row is denoted by $R_i \to R_i + mR_j + nR_k$.
(In case of column $C_i \to C_i + mC_j + nC_k$)

Properties of Determinants

We shall establish certain properties of a determinant of the third order but reader should note that these are capable of application to a determinant of any order.

Property I The value of a determinant remains unaltered when rows are changed into corresponding columns and columns are changed into corresponding rows.

Proof Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row, then

$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \Delta'$, where Δ' be the value of the determinant when rows of determinant Δ are changed into corresponding columns.

Property II If any two rows (or two columns) of a determinant are interchanged, then the sign of determinant is changed and the numerical value remains unaltered.

Proof Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row, then

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= -a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1)$$

$$= -[a_2(b_1c_3 - b_3c_1) - b_2(a_1c_3 - a_3c_1) + c_2(a_1b_3 - a_3b_1)]$$

$$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad [\text{by } R_1 \leftrightarrow R_2]$$

Hence,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Remark

If any row (or column) of a determinant Δ be passed over m rows (or columns), then the resulting determinant $= (-1)^m \Delta$.

Property III If two rows (or columns) of a determinant are identical, then the value of the determinant is zero.

Proof Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\Delta$ $\quad$ [by $R_1 \leftrightarrow R_3$]

$\Rightarrow \quad 2\Delta = 0$

$\therefore \quad \Delta = 0$

Property IV If the elements of any row (or any column) of a determinant be each multiplied by the same factor k, then the value of the determinant is multiplied by k.

Proof Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1C_{11} + b_1C_{12} + c_1C_{13}$

Then, $\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = ka_1C_{11} + kb_1C_{12} + kc_1C_{13}$

(where C_{11}, C_{12} and C_{13} are the cofactors of a_1, b_1 and c_1 in Δ)

$$= k(a_1C_{11} + b_1C_{12} + c_1C_{13}) = \boxed{k\Delta}$$

Property V If every element of some column (or row) is the sum of two items, then the determinant is equal to the sum of two determinants; one containing one the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as in the given determinant i.e.,

$$\begin{vmatrix} a_1 + x & b_1 & c_1 \\ a_2 + y & b_2 & c_2 \\ a_3 + z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & b_1 & c_1 \\ y & b_2 & c_2 \\ z & b_3 & c_3 \end{vmatrix}$$

Proof Let
$$\Delta = \begin{vmatrix} a_1 + x & b_1 & c_1 \\ a_2 + y & b_2 & c_2 \\ a_3 + z & b_3 & c_3 \end{vmatrix}$$

Expanding the determinant along first column, then

$$\Delta=(a_1+x)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-(a_2+y)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+(a_3+z)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}+x\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-y\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+z\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} x & b_1 & c_1 \\ y & b_2 & c_2 \\ z & b_3 & c_3 \end{vmatrix}$$

Remark

1. $$\begin{vmatrix} a_1+b_1+c_1 & d_1+e_1 & f_1 \\ a_2+b_2+c_2 & d_2+e_2 & f_2 \\ a_3+b_3+c_3 & d_3+e_3 & f_3 \end{vmatrix}=\begin{vmatrix} a_1 & d_1 & f_1 \\ a_2 & d_2 & f_2 \\ a_3 & d_3 & f_3 \end{vmatrix}+\begin{vmatrix} a_1 & e_1 & f_1 \\ a_2 & e_2 & f_2 \\ a_3 & e_3 & f_3 \end{vmatrix}+\begin{vmatrix} b_1 & d_1 & f_1 \\ b_2 & d_2 & f_2 \\ b_3 & d_3 & f_3 \end{vmatrix}+\begin{vmatrix} b_1 & e_1 & f_1 \\ b_2 & e_2 & f_2 \\ b_3 & e_3 & f_3 \end{vmatrix}+\begin{vmatrix} c_1 & d_1 & f_1 \\ c_2 & d_2 & f_2 \\ c_3 & d_3 & f_3 \end{vmatrix}+\begin{vmatrix} c_1 & e_1 & f_1 \\ c_2 & e_2 & f_2 \\ c_3 & e_3 & f_3 \end{vmatrix}$$
2. If each element of first row of a determinant consists of algebraic sum of p elements, second row consists of algebraic sum of q elements, third row consists of algebraic sum of r elements and so on.
   Then, given determinant is equivalent to the sum of $p\times q\times r\times\ldots$ other determinants in each of which the elements consists of single term.

Property VI The value of the determinant does not change, if the elements of any row (or column) are increased or diminished by equimultiples of the corresponding elements of any other row (or column) of the determinant.

i.e., $$\begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Proof $$\begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+m\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}+n\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

$$=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+m\cdot 0+n\cdot 0=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property VII If each element on one side or other side or both side of the principal diagonal of determinant is zero, then the value of the determinant is the product of the diagonal element.

i.e., $$\begin{vmatrix} a & 0 & 0 \\ f & b & 0 \\ e & d & c \end{vmatrix}=\begin{vmatrix} a & i & h \\ 0 & b & g \\ 0 & 0 & c \end{vmatrix}=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}=abc$$

Proof Let $\Delta=\begin{vmatrix} a\cdots & 0\cdots & 0 \\ f & b & 0 \\ e & d & c \end{vmatrix}$

Expanding along R_1, we get $a\begin{vmatrix} b & 0 \\ d & c \end{vmatrix}=a(bc)=abc$

Property VIII If determinant Δ becomes zero on putting $x=\alpha$, then we say that $(x-\alpha)$ is a factor of Δ

i.e., if $$\Delta=\begin{vmatrix} x & 5 & 2 \\ x^2 & 9 & 4 \\ x^3 & 16 & 8 \end{vmatrix}$$

at $x=2$, $\Delta=0$ [because C_1 and C_3 are identical at $x=2$]

and at $x=0$, $\Delta=0$ [because all elements of C_1 are zero]

Hence, $(x-0)$ and $(x-2)$ are the factors of Δ.

Remark

1. It should be noted that while applying operations on determinant that atleast one row (or column) must remain unchanged.
2. Maximum number of operations at a time = $\boxed{\text{order} - 1}$
3. It should be noted that, if the row (or column) which is changed by multiplied a non-zero number, then the determinant will be divided by that number.

Examples on Properties

Example 10. Evaluate $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$.

Sol. Let $\Delta=\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$, then

$$\Delta=\begin{vmatrix} 13 & 16 & 19 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix}=2\begin{vmatrix} 13 & 16 & 19 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}=0$$

[$\because R_2$ and R_3 are identical]

Example 11. Prove that $\begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \psi \\ \lambda & \mu & \nu \end{vmatrix}=\begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix}$.

Sol. LHS $=\begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \psi \\ \lambda & \mu & \nu \end{vmatrix}=\begin{vmatrix} \alpha & \theta & \lambda \\ \beta & \phi & \mu \\ \gamma & \psi & \nu \end{vmatrix}$

[interchanging rows and columns]

$$=(-1)\begin{vmatrix} \alpha & \lambda & \theta \\ \beta & \mu & \phi \\ \gamma & \nu & \psi \end{vmatrix}\qquad [C_2\leftrightarrow C_3]$$

$$=(-1)^2\begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix}\qquad [R_1\leftrightarrow R_2]$$

$$=\begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix}=\text{RHS}$$

Example 12. Use the properties of determinant and without expanding, prove that

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}=2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}.$$

Sol. Let LHS $=\Delta=\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

Applying $R_1\to R_1+R_2+R_3$, then

$$\Delta=\begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}=2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$, then

$$\Delta=2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix}$$

Applying $R_1\to R_1+R_2+R_3$, then

$$\Delta=2\begin{vmatrix} a & p & x \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix}=2(-1)(-1)\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$=2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}=\text{RHS}$$

Example 13. Without expanding as far as possible, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}=(x-y)(y-z)(z-x)(x+y+z).$$

Sol. Let $$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$$

for $x=y$, $\Delta=0$ [$\because C_1$ and C_2 are identical]

Hence, $(x-y)$ is a factor of Δ. Similarly, $(y-z)$ and $(z-x)$ are factors of Δ. But Δ is a homogeneous expression of the 4 th degree in x, y and z.

There must be one more factor of the 1st degree in x, y and z say $k(x+y+z)$, where k is a constant.

Let $\Delta=k(x-y)(y-z)(z-x)(x+y+z)$

On putting $x=0$, $y=1$ and $z=2$, then

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 8 \end{vmatrix}=k(0-1)(1-2)(2-0)(0+1+2)$$

$\Rightarrow$ $1\cdot(8-2)=k(-1)(-1)(2)(3)$ $\therefore$ $k=1$

$\Rightarrow$ $\Delta=(x-y)(y-z)(z-x)(x+y+z)=\text{RHS}$

Example 14. Solve for x,

$$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix}=0.$$

Sol. Applying $C_2\to C_2-\frac{3}{2}C_1$ and $C_3\to C_3-2C_1$

Then, $$\begin{vmatrix} 4x & 2 & 1 \\ 6x+2 & 0 & -4 \\ 8x+1 & -(3/2) & 0 \end{vmatrix}=0$$

$\Rightarrow$ $4x(0-6)-(6x+2)\left(0+\frac{3}{2}\right)+(8x+1)(-8-0)=0$

$\Rightarrow$ $-97x-11=0$ $\Rightarrow$ $x=-\frac{11}{97}$

Example 15. Prove that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}=1+a^2+b^2+c^2.$$

Sol. Let LHS $=\Delta=\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

On taking common a, b and c from R_1, R_2 and R_3 respectively, then

$$\Delta = abc\begin{vmatrix} \frac{a^2+1}{a} & b & c \\ a & \frac{b^2+1}{b} & c \\ a & b & \frac{c^2+1}{c} \end{vmatrix}$$

Now, multiplying in C_1, C_2 and C_3 by a, b and c respectively, then

$$\Delta = \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta = \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\cdot 1\cdot 1\cdot 1 = (1+a^2+b^2+c^2) = \text{RHS}$$

| Example 16. If a, b and c are all different and if $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, prove that $abc = -1$.

Sol. Let $\Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad [\text{by } C_1 \leftrightarrow C_3]$$

$$= (-1)^2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad [\text{by } C_2 \leftrightarrow C_3]$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}(1+abc)$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}(1+abc)$$

$$= (b-a)(c-a)(1+abc)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, then

$$= (b-a)(c-a)(1+abc)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

$$= (b-a)(c-b)(c-a)(1+abc)$$
$$= (a-b)(b-c)(c-a)(1+abc)$$

But given that, $\Delta = 0$

$\therefore \quad (a-b)(b-c)(c-a)(1+abc) = 0$

$\Rightarrow \quad 1 + abc = 0$

[since a, b and c are different, so $a \neq b, b \neq c, c \neq a$]

Hence, $\quad abc = -1$

c,d,a,c,a,b,b,b,d,c,d

Exercise for Session 2

1 If λ and μ are the cofactors of 3 and -2 respectively, in the determinant $\begin{vmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{vmatrix}$, the value of $\lambda + \mu$ is

(a) 5 (b) 7 (c) 9 (d) 11

2 If a, b and c are distinct and $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then the square of the determinant of its cofactors is divisible by

(a) $(a^2+b^2+c^2)^2$ (b) $(ab+bc+ca)^2$ (c) $(a+b+c)^2$ (d) $(a+b+c)^4$

3 An equilateral triangle has each of its sides of length 4 cm. If $(x_r, y_r)(r = 1, 2, 3)$ are its vertices, the value of $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$ is

(a) 192 (b) 768 (c) 1024 (d) 128

4 If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b and c being distinct and different from 1) are concurrent, the value of $\frac{a}{a-1} + \frac{b}{b-1} + \frac{c}{c-1}$ is

(a) 0 (b) 1 (c) 2 (d) 3

5 If $p + q + r = 0 = a + b + c$, the value of the determinant $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$ is

(a) 0 (b) $pa + qb + rc$ (c) 1 (d) None of the above

6 If p, q and r are in AP, the value of determinant $\begin{vmatrix} a^2+2^{n+1}+2p & b^2+2^{n+2}+3q & c^2+p \\ 2^n+p & 2^{n+1}+q & 2q \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix}$ is

(a) 1 (b) 0 (c) $a^2b^2c^2 - 2^n$ (d) $(a^2+b^2+c^2) - 2^n q$

7 Let $\{D_1, D_2, D_3, \ldots, D_n\}$ be the set of third order determinants that can be made with the distinct non-zero real numbers $a_1, a_2, \ldots, a_9$. Then,

(a) $\sum_{i=1}^{n} D_i = 1$ (b) $\sum_{i=1}^{n} D_i = 0$ (c) $D_i = D_j, \forall i, j$ (d) None of these

8 If $\begin{vmatrix} x & 3 & 6 \\ 3 & 6 & x \\ 6 & x & 3 \end{vmatrix} = \begin{vmatrix} 2 & x & 7 \\ x & 7 & 2 \\ 7 & 2 & x \end{vmatrix} = \begin{vmatrix} 4 & 5 & x \\ 5 & x & 4 \\ x & 4 & 5 \end{vmatrix} = 0$, then x is equal to

(a) 0 (b) -9 (c) 3 (d) None of these

9 If $a + b + c = 0$, the one root of $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ is

(a) 1 (b) 2 (c) $a^2+b^2+c^2$ (d) 0

10 If $a^2+b^2+c^2 = -2$ and $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$, the $f(x)$ is a polynomial of degree

(a) 0 (b) 1 (c) 2 (d) 3

11 If a, b, c, d, e and f are in GP, the value of $\begin{vmatrix} a^2 & d^2 & x \\ b^2 & e^2 & y \\ c^2 & f^2 & z \end{vmatrix}$, is

$a, ar, ar^2 \ldots$

(a) depends on x and y (b) depends on x and z (c) depends on y and z (d) independent of x, y and z

Session 3

Examples on Largest Value of a Third Order Determinant, Multiplication of Two Determinants of the Same Order, System of Linear Equations, Cramer's Rule, Nature of Solutions of System of Linear Equations, System of Homogeneous Linear Equations

Examples on Largest Value of a Third Order Determinant

Example 17. Find the largest value of a third order determinant whose elements are 0 or 1.

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$
$$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (b_1c_3a_2 + b_2c_1a_3 + b_3c_2a_1)$$

Since, each element of Δ is either 0 or 1, therefore the value of the Δ cannot exceed 3. But to attain this value, each expression with a positive sign must equal 1, while those with a negative sign must be 0. However, if $a_1b_2c_3 = a_2b_3c_1 = a_3b_1c_2 = 1$, every element of the determinant must be 1, making its value zero. Thus, noting that

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$$

The largest value of Δ is 2.

Example 18. Find the largest value of a third order determinant, whose elements are 1 or -1.

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\therefore \Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$
$$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (b_1c_3a_2 + b_2c_1a_3 + b_3c_2a_1)$$

Since, each element of Δ is either 1 or -1, therefore the value of the Δ cannot exceed 6. But it can be 6 only if

$$a_1b_2c_3 = a_2b_3c_1 = a_3b_1c_2 = 1 \quad \text{...(i)}$$

and $$b_1c_3a_2 = b_2c_1a_3 = b_3c_2a_1 = -1 \quad \text{...(ii)}$$

In the first case, the product of the nine elements of the determinant equals 1, while it is -1 in the second case, so the two cannot occur simultaneously i.e., the determinant cannot equal 6. The following determinant satisfies the given conditions and equals the largest value

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1(1-1) - 1(-1-1) + 1(1+1) = 4$$

Example 19. Show that the value of a third order determinant whose all elements are 1 or -1 is an even number.

Must see the approach

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Applying $R_2 \to R_2 - \frac{a_2}{a_1}R_1$ and $R_3 \to R_3 - \frac{a_3}{a_1}R_1$, then

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 - \frac{a_2}{a_1}b_1 & c_2 - \frac{a_2}{a_1}c_1 \\ 0 & b_3 - \frac{a_3}{a_1}b_1 & c_3 - \frac{a_3}{a_1}c_1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = a_1\left\{\left(b_2 - \frac{a_2}{a_1}b_1\right)\left(c_3 - \frac{a_3}{a_1}c_1\right) - \left(b_3 - \frac{a_3}{a_1}b_1\right)\left(c_2 - \frac{a_2}{a_1}c_1\right)\right\} \quad \text{...(i)}$$

Since, $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are 1 or -1.

$\therefore \quad b_2, \frac{a_2}{a_1}b_1, c_3, \frac{a_3}{a_1}c_1, b_3, \frac{a_3}{a_1}b_1, c_2, \frac{a_2}{a_1}c_1$ are 1 or -1

$\Rightarrow b_2 - \frac{a_2}{a_1}b_1, c_3 - \frac{a_3}{a_1}c_1, b_3 - \frac{a_3}{a_1}b_1, c_2 - \frac{a_2}{a_1}c_1$ are 2, -2 or 0

$\therefore \quad \left(b_2 - \frac{a_2}{a_1}b_1\right)\left(c_3 - \frac{a_3}{a_1}c_1\right)$

and $\quad \left(b_3 - \frac{a_3}{a_1}b_1\right)\left(c_2 - \frac{a_2}{a_1}c_1\right)$ are 4, -4

or $\quad$ 0 = an even number

From Eq. (i), Δ = an even number ($a_1 = 1$ or -1)

Multiplication of Two Determinants of the Same Order

Let the two determinants of third order be

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Let Δ be their product.

Method of Multiplication (Row by Row)

Take the first row of Δ_1 and the first row of Δ_2 i.e., a_1, b_1, c_1 and $\alpha_1, \beta_1, \gamma_1$ multiplying the corresponding elements and add. The result is $a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1$ is the first element of first row of Δ.

Now, similar product first row of Δ_1 and second row of Δ_2 gives $a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2$ is the second element of first row of Δ and the product of first row of Δ_1 and third row of Δ_2 gives $a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3$ is the third element of first row of Δ. The second row and third row of Δ is obtained by multiplying second row and third row of Δ_1 with 1st, 2nd, 3rd row of Δ_2 in the above manner.

Hence, $\Delta = \Delta_1 \times \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Multiplication can also be performed row by column or column by row or column by column as required in the problem.

Example 20. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & 4 & -4 \end{vmatrix} \times \begin{vmatrix} -2 & 1 & 3 \\ 3 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix}$.

Using the concept of multiplication of determinants.

Sol. Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & 4 & -4 \end{vmatrix} \times \begin{vmatrix} -2 & 1 & 3 \\ 3 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix}$

On multiplying row by row, we get

$$\Delta = \begin{vmatrix} -2+2+9 & 3-4+3 & 2+2-6 \\ 4+3+6 & -6-6+2 & -4+3-4 \\ -6+4-12 & 9-8-4 & 6+4+8 \end{vmatrix}$$

$$= \begin{vmatrix} 9 & 2 & -2 \\ 13 & -10 & -5 \\ -14 & -3 & 18 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$ and $C_2 \to C_2 + C_3$, then

$$\Delta = \begin{vmatrix} 7 & 0 & -2 \\ 8 & -15 & -5 \\ 4 & 15 & 18 \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$, then

$$\Delta = \begin{vmatrix} 7 & 0 & -2 \\ 12 & 0 & 13 \\ 4 & 15 & 18 \end{vmatrix}$$

Expanding along C_2, we get

$$-15\begin{vmatrix} 7 & -2 \\ 12 & 13 \end{vmatrix} = -15(91 + 24) = -15 \times 115 = -1725$$

Example 21. If $ax_1^2 + by_1^2 + cz_1^2 = ax_2^2 + by_2^2 + cz_2^2 = ax_3^2 + by_3^2 + cz_3^2 = d$, $ax_2x_3 + by_2y_3 + cz_2z_3 = ax_3x_1 + by_3y_1 + cz_3z_1 = ax_1x_2 + by_1y_2 + cz_1z_2 = f$, then prove that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (d-f)\left\{\frac{(d+2f)}{abc}\right\}^{1/2}$$

Sol. Let LHS $= \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$\therefore \Delta^2 = \Delta \times \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} ax_1^2 + by_1^2 + cz_1^2 & ax_1x_2 + by_1y_2 + cz_1z_2 & ax_3x_1 + by_3y_1 + cz_3z_1 \\ ax_1x_2 + by_1y_2 + cz_1z_2 & ax_2^2 + by_2^2 + cz_2^2 & ax_2x_3 + by_2y_3 + cz_2z_3 \\ ax_3x_1 + by_3y_1 + cz_3z_1 & ax_2x_3 + by_2y_3 + cz_2z_3 & ax_3^2 + by_3^2 + cz_3^2 \end{vmatrix}$$ [multiplying row by row]

$$= \frac{1}{abc}\begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix}$$ [given]

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$= \frac{1}{abc}\begin{vmatrix} d+2f & f & f \\ d+2f & d & f \\ d+2f & f & d \end{vmatrix} = \frac{(d+2f)}{abc}\begin{vmatrix} 1 & f & f \\ 1 & d & f \\ 1 & f & d \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$= \frac{(d+2f)}{abc}\begin{vmatrix} 1 & f & f \\ 0 & d-f & 0 \\ 0 & 0 & d-f \end{vmatrix} = \frac{(d+2f)}{abc}(d-f)^2$$

$$\therefore \Delta = (d-f)\left\{\frac{d+2f}{abc}\right\}^{1/2} = \text{RHS}$$

An Important Property

If A_1, B_1 and $C_1, \ldots$ are respectively the cofactors of the elements a_1, b_1 and $c_1, \ldots$ of the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta \neq 0, \text{ then } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Proof Consider

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

[multiplying row by row]

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3 \qquad \left(\text{as } a_iA_j + b_iB_j + c_iC_j = \begin{cases} \Delta, & i = j \\ 0, & i \neq j \end{cases}\right)$$

$$\Rightarrow \Delta\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^3 \text{ or } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

$[\because \Delta \neq 0]$

Note Let $\Delta \neq 0$ and Δ^c denotes the determinant formed by the cofactors of Δ and n is order of determinant, then

$$\boxed{\Delta^c = \Delta^{n-1}}$$

This is known as **power cofactor formula.**

Example 22. Show that

$$\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2.$$

Sol. Let $\Delta = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Cofactors of 1st row of Δ are $x^2 + a^2, cx + ab, ac - bx$,
cofactors of 2nd row of Δ are $ab - cx, x^2 + b^2, ax + bc$ and
cofactors of 3rd row of Δ are $ac + bx, bc - ax, x^2 + c^2$.
Hence, the determinant of the cofactors of Δ is

$$\Delta^c = \begin{vmatrix} a^2+x^2 & ab+cx & ac-bx \\ ab-cx & b^2+x^2 & bc+ax \\ ac+bx & bc-ax & c^2+x^2 \end{vmatrix}$$

Interchanging rows into columns, we get

$$\Delta^c = \begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2 \quad [\because \Delta^c = \Delta^2]$$

Example 23. Prove the following by multiplication of determinants and power cofactor formula

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Sol. Let $\Delta = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$. Expanding along R_1, then

$$\Delta = 0 - c(0 - ab) + b(ac - 0) = 2abc$$

$$\therefore \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \Delta^2 = (2abc)^2 = 4a^2b^2c^2$$

Also,

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

[multiplying row by row]

and

$$\Delta^c = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = \Delta^{3-1} = \Delta^2$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 \quad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Express a Determinant Into Product of Two Determinants

Consider the determinant $\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$

Let $\Delta = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$

By the property of determinant, Δ can be written as

$$\Delta = \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 \\ a_2\alpha_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 \\ a_2\alpha_1 & b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 \\ b_2\beta_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & b_1\beta_2 \\ b_2\beta_1 & b_2\beta_2 \end{vmatrix}$$

$$= \alpha_1\alpha_2\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} + \alpha_1\beta_2\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \beta_1\alpha_2\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} + \beta_1\beta_2\begin{vmatrix} b_1 & b_1 \\ b_2 & b_2 \end{vmatrix}$$

$$= 0 + \alpha_1\beta_2\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} - \beta_1\alpha_2\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + 0$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}(\alpha_1\beta_2 - \alpha_2\beta_1)$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

$$\therefore \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

Example 24. Prove that

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix} \quad \text{[row by row]}$$

$= 0 \times 0 = 0 =$ RHS

Example 25. Prove that

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix} \quad \text{[row by row]}$$

$= 0 \times 0 = 0 =$ RHS

Example 26. Prove that

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} \quad \text{[row by row]}$$

$= 0 \times 0 = 0 =$ RHS

Example 27. If α, β and γ are real numbers, without expanding at any stage, prove that

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$

$$= \begin{vmatrix} \cos(\alpha-\alpha) & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & \cos(\beta-\beta) & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & \cos(\gamma-\gamma) \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$= 0 \times 0 = 0 = $ RHS

Example 28. If $a, b, c, x, y, z \in R$, prove that

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}.$$

Sol. LHS $= \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2cz+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & -a & 1 \\ b^2 & -b & 1 \\ c^2 & -c & 1 \end{vmatrix} \quad \text{[row by row]}$$

$$= \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \times (-1)(-1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

[$C_1 \leftrightarrow C_3$ and taking (-1) common from second determinant]

$$= \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+2ax+a^2x^2 & 1+2bx+b^2x^2 & 1+2cx+c^2x^2 \\ 1+2ay+a^2y^2 & 1+2by+b^2y^2 & 1+2cy+c^2y^2 \\ 1+2az+a^2z^2 & 1+2bz+b^2z^2 & 1+2cz+c^2z^2 \end{vmatrix}$$

[multiplying row by row]

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix} = \text{RHS}$$

System of Linear Equations

(i) Consistent equations *Definite and unique solution* [Intersecting lines]

A system of (linear) equations is said to be consistent, if it has atleast one solution.

For example, System of equations $\left.\begin{matrix} x+y=2 \\ x-y=6 \end{matrix}\right\}$ is consistent because it has a solution $x = 4, y = -2$. Here, two lines intersect at one point.

i.e., intersecting lines.

(ii) Inconsistent equations *No solution* [Parallel lines]

A system of (linear) equations is said to be inconsistent, if it has no solution.

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

$\Rightarrow$ Given equations are inconsistent.

For example, System of equations $\left.\begin{matrix} x+y=2 \\ 2x+2y=5 \end{matrix}\right\}$ is inconsistent because it has no solution i.e., there is no value of x and y which satisfy both the equations. Here, the two lines are parallel.

(iii) Dependent equations *Infinite solutions* [Identical lines]

A system of (linear) equations is said to be dependent, if it has infinite solutions.

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow$ Given equations are dependent.

For example, System of equations $\left.\begin{matrix} x+2y=3 \\ 2x+4y=6 \end{matrix}\right\}$ is dependent because it has infinite solutions i.e., there are infinite values of x and y satisfy both the equations. Here, the two lines are identical.

Cramer's Rule

System of linear equations in two variables

Let us consider a system of equations be

$$\left.\begin{matrix} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{matrix}\right\} \text{ where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

On solving by cross-multiplication, we get

$$\frac{x}{(b_1c_2-b_2c_1)} = \frac{y}{(c_1a_2-c_2a_1)} = \frac{1}{(a_1b_2-a_2b_1)}$$

or $\dfrac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \dfrac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \dfrac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

or $x = \dfrac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, y = \dfrac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

System of Linear Equations in Three Variables

Let us consider a system of linear equations be

$$a_1x + b_1y + c_1z = d_1 \quad \text{...(i)}$$
$$a_2x + b_2y + c_2z = d_2 \quad \text{...(ii)}$$
$$a_3x + b_3y + c_3z = d_3 \quad \text{...(iii)}$$

Here, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$

$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

If $\boxed{\Delta \neq 0}$ then

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix}$$

Applying $C_1 \to C_1 - yC_2 - zC_3$, then

$$\Delta_1 = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} = x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = x\Delta$$

$\therefore$ $x = \dfrac{\Delta_1}{\Delta}$, where $\Delta \neq 0$

Similarly, $\Delta_2 = y\Delta$ and $\Delta_3 = z\Delta$

$\therefore$ $y = \dfrac{\Delta_2}{\Delta}$ and $z = \dfrac{\Delta_3}{\Delta}$

Thus, $x = \dfrac{\Delta_1}{\Delta}, y = \dfrac{\Delta_2}{\Delta}, z = \dfrac{\Delta_3}{\Delta}$, where $\Delta \neq 0$ ...(iv)

The rule given in Eq. (iv) to find the values of x, y and z is called the **CRAMER'S RULE**.

Remark

1. Δ_i is obtained by replacing elements of ith columns by d_1, d_2, d_3, where $i = 1, 2, 3$
2. Cramer's rule can be used only when $\Delta \neq 0$.

Nature of Solution of System of Linear Equations

Let us consider a system of linear equations be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Now, there are two cases arise:

Case I If $\Delta \neq 0$

In this case, $x = \dfrac{\Delta_1}{\Delta}, y = \dfrac{\Delta_2}{\Delta}, z = \dfrac{\Delta_3}{\Delta}$

Then, system will have unique finite solutions and so equations are **consistent**.

Case II If $\Delta = 0$

(a) **When atleast one of $\Delta_1, \Delta_2, \Delta_3$ be non-zero**

(i) Let $\Delta_1 \neq 0$, then from $\Delta_1 = x\Delta$ will not be satisfied for any value of x because $\Delta = 0$ and $\Delta_1 \neq 0$ and hence no value of x is possible.

(ii) Let $\Delta_2 \neq 0$, then from $\Delta_2 = y\Delta$ will not be satisfied for any value of y because $\Delta = 0$ and $\Delta_2 \neq 0$ and hence no value of y is possible.

(iii) Let $\Delta_3 \neq 0$, then from $\Delta_3 = z\Delta$ will not be satisfied for any value of z because $\Delta = 0$ and $\Delta_3 \neq 0$ and hence no value of z is possible.

Thus, if $\Delta = 0$ and any of $\Delta_1, \Delta_2, \Delta_3$ is non-zero. Then, the system has no solution i.e., equations are **inconsistent**.

(b) **When $\Delta_1 = \Delta_2 = \Delta_3 = 0$**

In this case, $\left.\begin{matrix} \Delta_1 = x\Delta \\ \Delta_2 = y\Delta \\ \Delta_3 = z\Delta \end{matrix}\right\}$ will be true for all values of x, y and z.

But, since $a_1x + b_1y + c_1z = d_1$, therefore only two of x, y and z will be independent and third will be dependent on the other two.

Thus, the system will have infinite number of solutions i.e., equations are **consistent**.

Remark

1. If $\Delta \neq 0$, the system will have unique finite solution and so equations are consistent.
2. If $\Delta = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3$ be non-zero, then the system has no solution i.e., equations are inconsistent.
3. If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, the equations will have infinite number of solutions i.e. equations are consistent.

| Example 29. Solve the following system of equations by Cramer's rule.

$$x + y = 4 \text{ and } 3x - 2y = 9$$

Sol. Here, $\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2 - 3 = -5 \neq 0$

$$\Delta_1 = \begin{vmatrix} 4 & 1 \\ 9 & -2 \end{vmatrix} = -8 - 9 = -17$$

and $\Delta_2 = \begin{vmatrix} 1 & 4 \\ 3 & 9 \end{vmatrix} = 9 - 12 = -3$

Then, by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-17}{-5} = \frac{17}{5} \text{ and } y = \frac{\Delta_2}{\Delta} = \frac{-3}{-5} = \frac{3}{5}$$

$$\therefore \quad x = \frac{17}{5}, y = \frac{3}{5}$$

| Example 30. Solve the following system of equations by Cramer's rule.

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

Sol. Here, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$= \begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 2 & & 3 & & 5 \\ \vdots & & & & \\ 2 & & -1 & & -3 \end{vmatrix}$$

Expanding along R_1, then

$$\Delta = 1 \begin{vmatrix} 3 & 5 \\ -1 & -3 \end{vmatrix} = -9 + 5 = -4 \neq 0, \quad \Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}$$

Applying $C_2 \to C_2 + C_3$, then

$$\Delta_1 = \begin{vmatrix} 9 & 2 & 1 \\ & & \vdots \\ 52 & 12 & 7 \\ & & \vdots \\ 0 & \cdots\ 0 & \cdots\ -1 \end{vmatrix}$$

Expanding along R_3, then

$$\Delta_1 = (-1) \begin{vmatrix} 9 & 2 \\ 52 & 12 \end{vmatrix} = -(108 - 104) = -4$$

$$\Rightarrow \quad \Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + 2C_3$, then

$$\Delta_2 = \begin{vmatrix} 3 & 9 & 1 \\ & & \vdots \\ 16 & 52 & 7 \\ & & \vdots \\ 0 & \cdots\ 0 & \cdots\ -1 \end{vmatrix}$$

Expanding along R_3, then

$$\Delta_2 = (-1) \begin{vmatrix} 3 & 9 \\ 16 & 52 \end{vmatrix} = -(156 - 144) = -12 \text{ and } \Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$$

Applying $C_1 \to C_1 - 2C_2$, then

$$\Delta_3 = \begin{vmatrix} -1 & 1 & 9 \\ & \vdots & \\ -8 & 5 & 52 \\ & \vdots & \\ 0 & \cdots\ 1 & \cdots\ 0 \end{vmatrix}$$

Expanding along R_3, then

$$\Delta_3 = (-1) \begin{vmatrix} -1 & 9 \\ -8 & 52 \end{vmatrix}$$
$$= -(-52 + 72) = -20$$

Then, by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1, \; y = \frac{-12}{-4} = 3$$

and $z = \dfrac{\Delta_3}{\Delta} = \dfrac{-20}{-4} = 5$

$$\therefore \quad x = 1, y = 3, z = 5$$

| Example 31. For what values of p and q, the system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + pz = q \text{ has}$$

(i) unique solution?
(ii) an infinitely many solutions?
(iii) no solution?

Sol. Given equations are

$$x + y + z = 6 \quad \Rightarrow \quad x + 2y + 3z = 10$$
$$x + 2y + pz = q$$

$$\therefore \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & p \end{vmatrix} = (p - 3) \Rightarrow \Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ q & 2 & p \end{vmatrix}$$

$$= 6(2p - 6) - 1(10p - 3q) + (20 - 2q)$$
$$= 2p + q - 16$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & q & p \end{vmatrix}$$

$$= 1(10p - 3q) - 6(p - 3) + 1(q - 10) = 4p - 2q + 8$$

and $\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & q \end{vmatrix}$

$$= 1(2q - 20) - 1(q - 10) + 6(2 - 2) = q - 10$$

(i) For unique solution, $A \neq 0 \Rightarrow p \neq 3, q \in R$
(ii) For infinitely many solutions, $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$$\therefore \quad p = 3, q = 10$$

(iii) For no solution, $\Delta = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero is $p = 3$ and $q \neq 10$.

Condition for Consistency of Three Linear Equations in Two Unknowns

Let us consider a system of linear equations in x and y

$$a_1 x + b_1 y + c_1 = 0 \quad \text{...(i)}$$
$$a_2 x + b_2 y + c_2 = 0 \quad \text{...(ii)}$$
$$a_3 x + b_3 y + c_3 = 0 \quad \text{...(iii)}$$

will be consistent, the values of x and y obtained from any two equations satisfy the third equation.

On solving Eqs. (ii) and (iii) by Cramer's rule, we have

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$

These values of x and y will satisfy Eq. (i), then

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

$$\Rightarrow \quad a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

which is the required condition.

Remark

For consistency of three linear equations in two knowns, the number of solution is one.

| Example 32. Find the value of λ, if the following equations are consistent only two variables

$$x + y - 3 = 0$$
$$(1 + \lambda) x + (2 + \lambda) y - 8 = 0$$
$$x - (1 + \lambda) y + (2 + \lambda) = 0$$

Sol. The given equations in two unknowns are consistent, then

$$\begin{vmatrix} 1 & 1 & -3 \\ (1 + \lambda) & (2 + \lambda) & -8 \\ 1 & -(1 + \lambda) & (2 + \lambda) \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$, then

$$\begin{vmatrix} 1 & \cdots\ 0 & \cdots\ 0 \\ \vdots & & \\ (1 + \lambda) & 1 & (3\lambda - 5) \\ \vdots & & \\ 1 & -(2 + \lambda) & (5 + \lambda) \end{vmatrix} = 0$$

Expanding along R_1, then

$$1 \cdot \begin{vmatrix} 1 & 3\lambda - 5 \\ -(2 + \lambda) & (5 + \lambda) \end{vmatrix} = 0$$

$$\Rightarrow \quad (5 + \lambda) + (2 + \lambda)(3\lambda - 5) = 0$$
$$\Rightarrow \quad 3\lambda^2 + 2\lambda - 5 = 0 \text{ or } (3\lambda + 5)(\lambda - 1) = 0$$
$$\therefore \quad \lambda = 1, -5/3$$

System of Homogeneous Linear Equations

Let us consider a system of homogeneous linear equations in three unknown x, y and z be

$$a_1 x + b_1 y + c_1 z = 0 \quad \text{...(i)}$$
$$a_2 x + b_2 y + c_2 z = 0 \quad \text{...(ii)}$$
$$a_3 x + b_3 y + c_3 z = 0 \quad \text{...(iii)}$$

Here, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

***Case* I** If $\Delta \neq 0$, then $x = 0, y = 0, z = 0$ is the only solution of above system. This solution is called a **Trivial solution.**

***Case* II** If $\Delta = 0$, atleast one of x, y and z is non-zero. This solution is called a **Non-trivial solution.**

Explanation From Eqs. (ii) and (iii), we get

$$\frac{x}{(b_2 c_3 - b_3 c_2)} = \frac{y}{(c_2 a_3 - c_3 a_2)} = \frac{z}{(a_2 b_3 - a_3 b_2)}$$

or $\dfrac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \dfrac{y}{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = k \text{[say]} \; (\neq 0)$

$$\therefore \quad x = k \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, y = k \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} \text{ and } z = k \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

On putting these values of x, y and z in Eq. (i), we get

$$a_1\left\{k\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3\end{vmatrix}\right\}+b_1\left\{k\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3\end{vmatrix}\right\}+c_1\left\{k\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3\end{vmatrix}\right\}=0$$

$$\Rightarrow \quad a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3\end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3\end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3\end{vmatrix}=0 \quad [\because k \neq 0]$$

or
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3\end{vmatrix}=0 \text{ or } \Delta=0$$

This is the condition for system have **Non-trivial solution.**

Remark

1. If $\Delta \neq 0$, the given system of equations has **only zero** solution for all its variables, then the given equations are said to have **Trivial solution**.
2. If $\Delta = 0$, the given system of equations has **no solution** or **infinite solutions** for all its variables, then the given equations are said to have **Non-trivial solution**.

Example 33. Find all values of λ for which the equations

$$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$$
$$(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$$
$$2x+(3\lambda+1)y+3(\lambda-1)z=0$$

possess non-trivial solution and find the ratios $x:y:z$, where λ has the smallest of these values.

Sol. The given system of linear equations has non-trivial solution, then we must have

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1)\end{vmatrix}=0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & -\lambda+3 \\ 3-\lambda & 0 & \lambda-3\end{vmatrix}=0$$

Applying $C_3 \to C_3 + C_2$, then

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ & \vdots & \\ 0 \;\cdots & \lambda-3 & \cdots\; 0 \\ & \vdots & \\ 3-\lambda & 0 & \lambda-3\end{vmatrix}=0$$

Expanding along R_2, we get

$$(\lambda-3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 3-\lambda & \lambda-3\end{vmatrix}=0$$

$$\Rightarrow \quad (\lambda-3)[(\lambda-1)(\lambda-3)-(3-\lambda)(5\lambda+1)]=0$$

$$\Rightarrow \quad (\lambda-3)^2\cdot 6\lambda=0$$

$$\therefore \quad \lambda=0,3$$

Here, smallest value of λ is 0.

$\therefore$ The first two equations can be written as $x-y=0$ and $x+2y-3z=0$.

Using Cramer's rule, we get

$$\frac{x}{\begin{vmatrix} -1 & 0 \\ 2 & -3\end{vmatrix}}=\frac{y}{\begin{vmatrix} 0 & 1 \\ -3 & 1\end{vmatrix}}=\frac{z}{\begin{vmatrix} 1 & -1 \\ 1 & 2\end{vmatrix}}$$

$$\Rightarrow \quad \frac{x}{3}=\frac{y}{3}=\frac{z}{3} \Rightarrow \frac{x}{1}=\frac{y}{1}=\frac{z}{1}$$

$$\therefore \quad x:y:z=1:1:1$$

Example 34. Given, $x=cy+bz$, $y=az+cx$ and $z=bx+ay$, where x, y and z are not all zero, prove that $a^2+b^2+c^2+2abc=1$.

Sol. The given equation can be rewritten as

$$x-cy-bz=0$$
$$-cx+y-az=0$$
$$-bx-ay+z=0$$

Since, x, y and z are not all zero, the system will have non-trivial solution, if

$$\begin{vmatrix} 1 & -c & -b \\ -c & 1 & -a \\ -b & -a & 1\end{vmatrix}=0$$

Applying $C_2 \to C_2 + cC_1$ and $C_3 \to C_3 + bC_1$, then

$$\begin{vmatrix} 1 \;\cdots & 0 \;\cdots & 0 \\ \vdots & & \\ -c & 1-c^2 & -a-bc \\ \vdots & & \\ -b & -a-bc & 1-b^2\end{vmatrix}=0$$

Expanding along R_1, we get

$$1\begin{vmatrix} 1-c^2 & -a-bc \\ -a-bc & 1-b^2\end{vmatrix}=0$$

$$\Rightarrow \quad (1-c^2)(1-b^2)-(a+bc)^2=0$$

$$\Rightarrow \quad 1-b^2-c^2+b^2c^2-a^2-b^2c^2-2abc=0$$

$$\Rightarrow \quad a^2+b^2+c^2+2abc=1$$

Exercise for Session 3

b, c, c, b, b, d, d, a, b, c, c, a, a

1. Number of second order determinants which have maximum values whose each entry is either -1 or 1 is equal to
 (a) 2 (b) 4 (c) 6 (d) 8
2. Minimum value of a second order determinant whose each entry is either 1 or 2 is equal to
 (a) 0 (b) -1 (c) -2 (d) -3
3. If $l_i^2+m_i^2+n_i^2=1, (i=1,2,3)$ and $l_il_j+m_im_j+n_in_j=0, (i\neq j; i, j=1,2,3)$ and $\Delta=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3\end{vmatrix}$, then
 (a) $|\Delta|=3$ (b) $|\Delta|=2$ (c) $|\Delta|=1$ (d) $|\Delta|=0$
4. Let $\Delta_0=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}$ and Δ_1 denotes the determinant formed by the cofactors of elements of Δ_0 and Δ_2 denote the determinant formed by the cofactors of Δ_1 and so on. Δ_n denotes the determinant formed by the cofactors of Δ_{n-1}, the determinant value of Δ_n is
 (a) Δ_0^{2n} (b) $\Delta_0^{2^n}$ (c) $\Delta_0^{n^2}$ (d) Δ_0^2
5. If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x\end{vmatrix}=3$, then the value of $\begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0\end{vmatrix}$, is
 (a) 6 (b) 9 (c) 18 (d) 27
6. The value of the determinant $\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 & (a_1-b_4)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 & (a_2-b_4)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 & (a_3-b_4)^2 \\ (a_4-b_1)^2 & (a_4-b_2)^2 & (a_4-b_3)^2 & (a_4-b_4)^2\end{vmatrix}$, is
 (a) depends on $a_i, i=1,2,3,4$ (b) depends on $b_i, i=1,2,3,4$ (c) depends on $c_i, i=1,2,3,4$ (d) 0
7. Value of $\begin{vmatrix} 1+x_1 & 1+x_1x & 1+x_1x^2 \\ 1+x_2 & 1+x_2x & 1+x_2x^2 \\ 1+x_3 & 1+x_3x & 1+x_3x^2\end{vmatrix}$ depends upon =0
 (a) only x (b) only x_1 (c) only x_2 (d) None of these
8. If the system of linear equations $x+y+z=6$, $x+2y+3z=14$ and $2x+5y+\lambda z=\mu\ (\lambda, \mu \in R)$ has a unique solution, then
 (a) $\lambda \neq 8$ (b) $\lambda=8$ and $\mu \neq 36$ (c) $\lambda=8$ and $\mu=36$ (d) None of these
9. The system of equations $ax-y-z=a-1$, $x-ay-z=a-1$, $x-y-az=a-1$ has no solution, if a is
 (a) either -2 or 1 (b) -2 (c) 1 (d) not (-2)
10. The system of equations $x+2y-4z=3$, $2x-3y+2z=5$ and $x-12y+16z=1$ has
 (a) inconsistent solution (b) unique solution (c) infinitely many solutions (d) None of these
11. If $c<1$ and the system of equations $x+y-1=0$, $2x-y-c=0$ and $-bx+3by-c=0$ is consistent, then the possible real values of b are
 (a) $b\in\left(-3,\frac{3}{4}\right)$ (b) $b\in\left(-\frac{3}{2},4\right)$ (c) $b\in\left(-\frac{3}{4},3\right)$ (d) None of these
12. The equations $x+2y=3$, $y-2x=1$ and $7x-6y+a=0$ are consistent for
 (a) $a=7$ (b) $a=1$ (c) $a=11$ (d) None of these
13. Values of k for which the system of equations $x+ky+3z=0$, $kx+2y+2z=0$ and $2x+3y+4z=0$ possesses non-trivial solution
 (a) $\left\{2,\frac{5}{4}\right\}$ (b) $\left\{-2,\frac{5}{4}\right\}$ (c) $\left\{2,-\frac{5}{9}\right\}$ (d) $\left\{-2,-\frac{5}{4}\right\}$

Session 4

Differentiation of Determinant, Integration of a Determinant, Walli's Formula, Use of Σ in Determinant

Differentiation of Determinant

Let $\Delta(x)$ be a determinant of order n. If we write $\Delta(x) = [C_1\ C_2\ C_3 \dots C_n]$, where $C_1, C_2, C_3, \dots, C_n$ denotes 1st, 2nd, 3rd, ..., nth columns respectively, then

$$\Delta'(x) = [C_1'\ C_2\ C_3 \cdots C_n] + [C_1\ C_2'\ C_3 \cdots C_n] + [C_1\ C_2\ C_3' \cdots C_n] + \dots + [C_1\ C_2\ C_3 \cdots C_n']$$
$$= \Sigma [C_1'\ C_2\ C_3 \cdots C_n]$$

where C_i' denotes the column which contains the derivative of all the functions in the ith column C_i. Also, if

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}$$

where $R_1, R_2, R_3, \dots, R_n$ denote 1st, 2nd, 3rd, ..., nth rows respectively, then

$$\Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \\ \vdots \\ R_n \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \\ \vdots \\ R_n \end{bmatrix} + \dots + \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n' \end{bmatrix} = \Sigma \begin{bmatrix} R_1' \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}$$

where R_i' denotes the row which contains the derivative of all the functions in the ith row R_i.

***Corollary* I** For $n = 2$,

$$\Delta(x) = [C_1\ C_2], \text{ then } \Delta'(x) = [C_1'\ C_2] + [C_1\ C_2']$$

Also, if $\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$, then $\Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \end{bmatrix}$

For example, Let $\Delta(x) = \begin{vmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} a_1'(x) & b_1'(x) \\ a_2(x) & b_2(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & b_1(x) \\ a_2'(x) & b_2'(x) \end{vmatrix}$$

[derivative according to rowwise]

***Corollary* II For $n = 3$**, $\Delta(x) = [C_1\ C_2\ C_3]$, then

$$\Delta'(x) = [C_1'\ C_2\ C_3] + [C_1\ C_2'\ C_3] + [C_1\ C_2\ C_3']$$

Also, if $\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$, then $\Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \end{bmatrix}$

For example, Let $\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1(x) & c_2(x) & c_3(x) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} a_1'(x) & a_2'(x) & a_3'(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1(x) & c_2(x) & c_3(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1'(x) & b_2'(x) & b_3'(x) \\ c_1(x) & c_2(x) & c_3(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1'(x) & c_2'(x) & c_3'(x) \end{vmatrix}$$

[derivative according to rowwise]

Remark

1. In a third order determinant, if two rows (columns) consist functions of x and third row (column) is constant, let

$$\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ then}$$

$$\Delta'(x) = \begin{vmatrix} a_1'(x) & a_2'(x) & a_3'(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1'(x) & b_2'(x) & b_3'(x) \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. In a third order determinant, if only one row (column) consists functions of x and other rows (columns) are constant, let

$$\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ then } \Delta'(x) = \begin{vmatrix} a_1'(x) & a_2'(x) & a_3'(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and in general

$$\frac{d^n}{dx^n}\{\Delta(x)\} = \begin{vmatrix} \frac{d^n}{dx^n}\{a_1(x)\} & \frac{d^n}{dx^n}\{a_2(x)\} & \frac{d^n}{dx^n}\{a_3(x)\} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Important Derivatives (Committed to Memory)

If a and b are constants and $n \in N$, then

1. if $y = (ax + b)^n$, then $\boxed{\frac{d^n y}{dx^n} = n!\, a^n}$
2. if $y = \sin(ax + b)$, then $\frac{d^n y}{dx^n} = \sin\left(\frac{n\pi}{2} + ax + b\right) \cdot a^n$
3. if $y = \cos(ax + b)$, then $\frac{d^n y}{dx^n} = \cos\left(\frac{n\pi}{2} + ax + b\right) \cdot a^n$

| Example 35. If $f(x) = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$, find the value of $2^{f'(0)} + \{f'(1)\}^2$.

Sol. $f'(x) = \begin{vmatrix} \cos x & -\sin x & \cos x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \sin x & \cos x & \sin x \\ -\sin x & -\cos x & -\sin x \\ x & 1 & 1 \end{vmatrix}$

$+ \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ 1 & 0 & 0 \end{vmatrix}$ [derivative according to rowwise]

$$= 0 + 0 + 1\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$\therefore f'(x) = 1 \Rightarrow f'(0) = 1$ and $f'(1) = 1$

$\Rightarrow 2^{f'(0)} + \{f'(1)\}^2 = 2^1 + 1^2 = 3$

| Example 36. Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$, then find the value of $f'\left(\frac{\pi}{2}\right)$.

Sol. Given, $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$

$\therefore \quad f'(x) = \begin{vmatrix} -\sin x & \sin x & \cos x \\ -2\sin 2x & \sin 2x & 2\cos 2x \\ -3\sin 3x & \sin 3x & 3\cos 3x \end{vmatrix}$

$+ \begin{vmatrix} \cos x & \cos x & \cos x \\ \cos 2x & 2\cos 2x & 2\cos 2x \\ \cos 3x & 3\cos 3x & 3\cos 3x \end{vmatrix} + \begin{vmatrix} \cos x & \sin x & -\sin x \\ \cos 2x & \sin 2x & -4\sin 2x \\ \cos 3x & \sin 3x & -9\sin 3x \end{vmatrix}$

[derivative according to columnwise]

$$\Rightarrow f'\left(\frac{\pi}{2}\right) = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & -2 \\ 3 & -1 & 0 \end{vmatrix} + 0 + \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 9 \end{vmatrix}$$

[$\because C_2 = C_3$ in second determinant]

$= 2(1 - 3) + 1(9 - 1) = -4 + 8 = 4$

| Example 37. Let α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomials of degree 3, 4, and 5 respectively, show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where prime ($'$) denotes the derivatives.

Sol. Since, α is a repeated root of the quadratic equation $f(x) = 0$, then $f(x)$ can be written as $\boxed{f(x) = a(x - \alpha)^2}$ where a is some non-zero constant.

Let $\quad g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$g(x)$ is divisible by $f(x)$, if it is divisible by $(x - \alpha)^2$ i.e., $g(\alpha) = 0$ and $g'(\alpha) = 0$. As $A(x)$, $B(x)$ and $C(x)$ are polynomials of degree 3, 4 and 5, respectively.

$\therefore$ Degree of $g(x) \geq 2$

Now, $\quad g(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$

[$\because R_1$ and R_2 are identical]

Also, $\quad g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$\therefore \quad g'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$

[$\because R_1$ and R_3 are identical]

This implies that $f(x)$ divides $g(x)$.

| Example 38. Find the coefficient of x in the determinant

$$\begin{vmatrix} (1+x)^{a_1 b_1} & (1+x)^{a_1 b_2} & (1+x)^{a_1 b_3} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & (1+x)^{a_2 b_3} \\ (1+x)^{a_3 b_1} & (1+x)^{a_3 b_2} & (1+x)^{a_3 b_3} \end{vmatrix}$$

Aliky

Sol. We know that, if $f(x)$ be a polynomial, then coefficient of $\boxed{x^n \text{ in } f(x) = \frac{1}{n!} f^n(0).}$

Let $\quad f(x) = \begin{vmatrix} (1+x)^{a_1 b_1} & (1+x)^{a_1 b_2} & (1+x)^{a_1 b_3} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & (1+x)^{a_2 b_3} \\ (1+x)^{a_3 b_1} & (1+x)^{a_3 b_2} & (1+x)^{a_3 b_3} \end{vmatrix}$

$$\therefore f'(x) = \begin{vmatrix} a_1 b_1 (1+x)^{a_1 b_1 - 1} & a_1 b_2 (1+x)^{a_1 b_2 - 1} & a_1 b_3 (1+x)^{a_1 b_3 - 1} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & (1+x)^{a_2 b_3} \\ (1+x)^{a_3 b_1} & (1+x)^{a_3 b_2} & (1+x)^{a_3 b_3} \end{vmatrix}$$

$$+\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\ a_2b_1(1+x)^{a_2b_1-1} & a_2b_2(1+x)^{a_2b_2-1} & a_2b_3(1+x)^{a_2b_3-1}\\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix}$$

$$+\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\ a_3b_1(1+x)^{a_3b_1-1} & a_3b_2(1+x)^{a_3b_2-1} & a_3b_3(1+x)^{a_3b_3-1}\end{vmatrix}$$

$$\therefore\ f'(0)=\begin{vmatrix}a_1b_1 & a_1b_2 & a_1b_3\\ 1 & 1 & 1\\ 1 & 1 & 1\end{vmatrix}+\begin{vmatrix}1 & 1 & 1\\ a_2b_1 & a_2b_2 & a_2b_3\\ 1 & 1 & 1\end{vmatrix}+\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ a_3b_1 & a_3b_2 & a_3b_3\end{vmatrix}$$

$$=0+0+0=0$$

$\therefore$ Coefficient of x in $f(x)=\dfrac{f'(0)}{1!}=0$

Aliter

Let $\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix}=A+Bx+Cx^2+\dots$

On differentiating both sides w.r.t. x and then put $x=0$ in both sides, we get

$$B=\begin{vmatrix}a_1b_1 & a_1b_2 & a_1b_3\\ 1 & 1 & 1\\ 1 & 1 & 1\end{vmatrix}+\begin{vmatrix}1 & 1 & 1\\ a_2b_1 & a_2b_2 & a_2b_3\\ 1 & 1 & 1\end{vmatrix}+\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ a_3b_1 & a_3b_2 & a_3b_3\end{vmatrix}$$

$$=0+0+0=0$$

Hence, coefficient of x in given determinant is 0.

| Example 39. If $\Delta(x)=\begin{vmatrix}\alpha+x & \theta+x & \lambda+x\\ \beta+x & \phi+x & \mu+x\\ \gamma+x & \psi+x & \nu+x\end{vmatrix}$, show that $\Delta''(x)=0$ and $\Delta(x)=\Delta(0)+Sx$, where S denotes the sum of all the cofactors of all elements in $\Delta(0)$ and dash denotes the derivative.

Sol. We have, $\Delta'(x)=\begin{vmatrix}1 & \theta+x & \lambda+x\\ 1 & \phi+x & \mu+x\\ 1 & \psi+x & \nu+x\end{vmatrix}+\begin{vmatrix}\alpha+x & 1 & \lambda+x\\ \beta+x & 1 & \mu+x\\ \gamma+x & 1 & \nu+x\end{vmatrix}+\begin{vmatrix}\alpha+x & \theta+x & 1\\ \beta+x & \phi+x & 1\\ \gamma+x & \psi+x & 1\end{vmatrix}$

Applying $C_2\to C_2-xC_1$ and $C_3\to C_3-xC_1$ in first, $C_1\to C_1-xC_2$ and $C_3\to C_3-xC_2$ in second and $C_1\to C_1-xC_3$ and $C_2\to C_2-xC_3$ in third, then

$$\Delta'(x)=\begin{vmatrix}1 & \theta & \lambda\\ 1 & \phi & \mu\\ 1 & \psi & \nu\end{vmatrix}+\begin{vmatrix}\alpha & 1 & \lambda\\ \beta & 1 & \mu\\ \gamma & 1 & \nu\end{vmatrix}+\begin{vmatrix}\alpha & \theta & 1\\ \beta & \phi & 1\\ \gamma & \psi & 1\end{vmatrix}$$

= sum of all cofactors in $\Delta(0)$, where $\Delta(0)=\begin{vmatrix}\alpha & \theta & \lambda\\ \beta & \phi & \mu\\ \gamma & \psi & \nu\end{vmatrix}$

$\therefore\quad \Delta''(x)=0$ $\quad$ [$\because$ S is constant]

Since, $\quad \Delta'(x)=S$

On integrating $\Delta(x)=Sx+C$

$\therefore\quad \Delta(0)=0+C$

Hence, $\quad \Delta(x)=Sx+\Delta(0)$

| Example 40. If $f(x)=\begin{vmatrix}x^n & \sin x & \cos x\\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right)\\ \pi & \pi^2 & \pi^3\end{vmatrix}$, then find the value of $\dfrac{d^n}{dx^n}\{f(x)\}$ at $x=0$, $n\in I$.

Sol. $\dfrac{d^n}{dx^n}\{f(x)\}=\begin{vmatrix}\dfrac{d^n}{dx^n}(x^n) & \dfrac{d^n}{dx^n}(\sin x) & \dfrac{d^n}{dx^n}(\cos x)\\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right)\\ \pi & \pi^2 & \pi^3\end{vmatrix}$

$$=\begin{vmatrix}n! & \sin\left(\frac{n\pi}{2}+x\right) & \cos\left(\frac{n\pi}{2}+x\right)\\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right)\\ \pi & \pi^2 & \pi^3\end{vmatrix}$$

$$\therefore\ \frac{d^n}{dx^n}\{f(x)\}\text{at}(x=0)=\begin{vmatrix}n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right)\\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right)\\ \pi & \pi^2 & \pi^3\end{vmatrix}$$

$=0$ $\quad$ [$\because$ R_1 and R_2 are identical]

Integration of a Determinant

Let $\quad \Delta(x)=\begin{vmatrix}f(x) & g(x) & h(x)\\ p & q & r\\ l & m & n\end{vmatrix}$

where p,q,r,l,m and n are constants, then

$$\int_a^b \Delta(x)\,dx=\begin{vmatrix}\int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx\\ p & q & r\\ l & m & n\end{vmatrix}$$

Remark

If in a determinant, the elements of more than one columns or rows are functions of x, then the integration can be done only after evaluation or expansion of the determinant.

Important Integrals (Committed to Memory)

1. (i) $\displaystyle\int_0^{\pi/2}\frac{\sin^n x}{\sin^n x+\cos^n x}dx=\frac{\pi}{4}$

$\displaystyle=\int_0^{\pi/2}\frac{\cos^n x}{\sin^n x+\cos^n x}dx,\ \forall n\in R$

(ii) $\displaystyle\int_0^{\pi/2}\frac{\tan^n x}{1+\tan^n x}dx=\frac{\pi}{4}=\int_0^{\pi/2}\frac{dx}{1+\tan^n x},\ \forall n\in R$

(iii) $\displaystyle\int_0^{\pi/2}\frac{dx}{1+\cot^n x}=\frac{\pi}{4}=\int_0^{\pi/2}\frac{\cot^n x}{1+\cot^n x}dx,\ \forall n\in R$

2. (i) $\displaystyle\int_0^{\pi/2}\ln\sin x\,dx=\int_0^{\pi/2}\ln\cos x\,dx=-\frac{\pi}{2}\ln 2$

or $\dfrac{\pi}{2}\ln\left(\dfrac{1}{2}\right)$

(ii) $\displaystyle\int_0^{\pi/2}\ln\tan x\,dx=\int_0^{\pi/2}\ln\cot x\,dx=0$

(iii) $\displaystyle\int_0^{\pi/2}\ln\sec x\,dx=\int_0^{\pi/2}\ln\operatorname{cosec} x\,dx=\frac{\pi}{2}\ln 2$

Walli's Formula

(An easy way to evaluate $\int_0^{\pi/2}\sin^m x\cos^n x\,dx$, where $m,n\in W$) We have, $\int_0^{\pi/2}\sin^m x\cos^n x\,dx$

$$=\frac{\{(m-1)(m-3)\dots 2\text{ or }1\}\{(n-1)(n-3)\dots 2\text{ or }1\}}{\{(m+n)(m+n-2)(m+n-4)\dots 2\text{ or }1\}}$$

where, p is $\pi/2$, if m and n are both even, otherwise $p=1$. The last factor in each of three products is either 1 or 2. In case any of m or n is 1, we simply write 1 as the only factor to replace its product. If any of m or n is zero provided, we put 1 as the only factor in its product and we regard 0 as even.

For example,

1. $\displaystyle\int_0^{\pi/2}\sin^6 x\cos^4 x\,dx=\frac{[5\cdot3\cdot1][3\cdot1]}{[10\cdot8\cdot6\cdot4\cdot2]}\cdot\frac{\pi}{2}=\frac{3\pi}{512}$
2. $\displaystyle\int_0^{\pi/2}\sin^6 x\cos^3 x\,dx=\frac{[5\cdot3\cdot1][2]}{[9\cdot7\cdot5\cdot3\cdot1]}\cdot1=\frac{2}{63}$
3. $\displaystyle\int_0^{\pi/2}\sin^5 x\cos^7 x\,dx=\frac{[4\cdot2][6\cdot4\cdot2]}{[12\cdot10\cdot8\cdot6\cdot4\cdot2]}\cdot1=\frac{1}{120}$
4. $\displaystyle\int_0^{\pi/2}\sin^8 x\,dx=\frac{[7\cdot5\cdot3\cdot1]}{[8\cdot6\cdot4\cdot2]}\cdot\frac{\pi}{2}=\frac{35\pi}{256}$
5. $\displaystyle\int_0^{\pi/2}\cos^7 x\,dx=\frac{[6\cdot4\cdot2]}{[7\cdot5\cdot3\cdot1]}\cdot1=\frac{16}{35}$
6. $\displaystyle\int_0^{\pi/2}\sin^{10} x\cos x\,dx=\frac{[9\cdot7\cdot5\cdot3\cdot1][1]}{[11\cdot9\cdot7\cdot5\cdot3\cdot1]}\cdot1=\frac{1}{11}$

| Example 41. If $\Delta(x)=\begin{vmatrix}a & b & c\\ 6 & 4 & 3\\ x & x^2 & x^3\end{vmatrix}$, then find $\int_0^1\Delta(x)\,dx$.

Sol. $\displaystyle\int_0^1\Delta(x)\,dx=\begin{vmatrix}a & b & c\\ 6 & 4 & 3\\ \int_0^1 x\,dx & \int_0^1 x^2dx & \int_0^1 x^3dx\end{vmatrix}$

$$=\begin{vmatrix}a & b & c\\ 6 & 4 & 3\\ \left[\frac{x^2}{2}\right]_0^1 & \left[\frac{x^3}{3}\right]_0^1 & \left[\frac{x^4}{4}\right]_0^1\end{vmatrix}=\begin{vmatrix}a & b & c\\ 6 & 4 & 3\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\end{vmatrix}$$

Applying $R_2\to R_2-12R_3$, then $\int_0^1\Delta(x)dx=\begin{vmatrix}a & b & c\\ 0 & 0 & 0\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\end{vmatrix}=0$

| Example 42. If

$$f(x)=\begin{vmatrix}\sin^5 x & \ln\sin x & \dfrac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\\ n & \sum_{k=1}^{n}k & \prod_{k=1}^{n}k\\ \dfrac{8}{15} & \dfrac{\pi}{2}\ln\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4}\end{vmatrix},$$

then find the value of $\int_0^{\pi/2}f(x)dx$.

Sol. $\displaystyle\int_0^{\pi/2}f(x)\,dx$

$$=\begin{vmatrix}\int_0^{\pi/2}\sin^5 x\,dx & \int_0^{\pi/2}\ln\sin x\,dx & \int_0^{\pi/2}\dfrac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\\ n & \sum_{k=1}^{n}k & \prod_{k=1}^{n}k\\ \dfrac{8}{15} & \dfrac{\pi}{2}\ln\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4}\end{vmatrix},$$

$$=\begin{vmatrix}\frac{4}{5}\cdot\frac{2}{3} & -\frac{\pi}{2}\ln 2 & \frac{\pi}{4}\\ n & \sum_{k=1}^{n}k & \prod_{k=1}^{n}k\\ \frac{8}{15} & \frac{\pi}{2}\ln\left(\frac{1}{2}\right) & \frac{\pi}{4}\end{vmatrix}\quad\text{[by Walli's formula]}$$

$$=\begin{vmatrix}\frac{8}{15} & \frac{\pi}{2}\ln\left(\frac{1}{2}\right) & \frac{\pi}{4}\\ n & \sum_{k=1}^{n}k & \prod_{k=1}^{n}k\\ \frac{8}{15} & \frac{\pi}{2}\ln\left(\frac{1}{2}\right) & \frac{\pi}{4}\end{vmatrix}=0\quad\text{[since } R_1 \text{ and } R_3 \text{ are identical]}$$

Example 43. Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$, then find the value of $\int_0^{\pi/2} f(x)\,dx$.

Sol. Applying $C_2 \to C_2 - \cos^2 x C_1$, then

$$f(x) = \begin{vmatrix} \sec x & 0 & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x - \cos^4 x & \operatorname{cosec}^2 x \\ 1 & 0 & \cos^2 x \end{vmatrix}$$

[expanding along C_2]

$$= (\cos^2 x - \cos^4 x)\begin{vmatrix} \sec x & \sec^2 x + \cot x \operatorname{cosec} x \\ 1 & \cos^2 x \end{vmatrix}$$

$$= (\cos^2 x - \cos^4 x)(\cos x - \sec^2 x - \cot x \operatorname{cosec} x)$$

$$= \cos^2 x(1-\cos^2 x)\left(\cos x - \frac{1}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)$$

$$= \cos^2 x \sin^2 x\left(\cos x - \frac{1}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)$$

$$= \cos^3 x \sin^2 x - \sin^2 x - \cos^3 x$$

$$= -\cos^3 x(1-\sin^2 x) - \sin^2 x$$

$$f(x) = -\cos^5 x - \sin^2 x$$

$$\therefore \int_0^{\pi/2} f(x)\,dx = -\int_0^{\pi/2} \cos^5 x\,dx - \int_0^{\pi/2} \sin^2 x\,dx$$

$$= -\left(\frac{4}{5}\cdot\frac{2}{3}\cdot 1\right) - \left(\frac{1}{2}\cdot\frac{\pi}{2}\right) = -\left(\frac{8}{15} + \frac{\pi}{4}\right)$$

[by Walli's formula]

Use of Σ in Determinant

If $\Delta(r) = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

where a, b, c, a_1, b_1 and c_1 are constants, independent of r, then

$$\sum_{r=1}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Remark

If in a determinant, the elements of more than one columns or rows are function of r, then the Σ can be done only after evaluation or expansion of the determinant.

Important Summation (Committed to Memory)

1. $\sum_{r=1}^{n} r = \Sigma n = 1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$

2. $\sum_{r=1}^{n} r^2 = \Sigma n^2 = 1^2+2^2+3^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

3. $\sum_{r=1}^{n} r^3 = \Sigma n^3 = 1^3+2^3+3^3+\ldots+n^3$
$= \left[\dfrac{n(n+1)}{2}\right]^2 = (\Sigma n)^2$

4. $\sum_{r=1}^{n} a = \Sigma a = \underbrace{a+a+a+\ldots+a}_{n \text{ times}} = an$

5. $\sum_{r=1}^{n} (\lambda - 1)\lambda^{r-1} = \lambda^n - 1, \forall \lambda \neq 1$ and $\lambda > 1$

6. $\sum_{r=1}^{n} \sin[\alpha + (r-1)\beta] = \dfrac{\sin\left\{\alpha + \frac{(n-1)\beta}{2}\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Particular For $\alpha = \beta = \theta$.

$$\sum_{r=1}^{n} \sin r\theta = \frac{\sin\left\{\left(\frac{n+1}{2}\right)\theta\right\}\cdot\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

7. $\sum_{r=1}^{n} \cos\{\alpha + (r-1)\beta\} = \dfrac{\cos\left\{\alpha + \frac{(n-1)}{2}\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Particular For $\alpha = \beta = \theta$.

$$\sum_{r=1}^{n} \cos r\theta = \frac{\cos\left\{\left(\frac{n+1}{2}\right)\theta\right\}\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

8. $\sum_{r=1}^{n} \{f(r+1) - f(r)\} = f(n+1) - f(1)$

Particular $\sum_{r=1}^{n} \dfrac{1}{r(r+1)} = \sum_{r=1}^{n}\left(\dfrac{1}{r} - \dfrac{1}{r+1}\right)$
$= \dfrac{1}{1} - \dfrac{1}{n+1} = \dfrac{n}{n+1}$

9. $\sum_{r=1}^{n} {}^nC_r = 2^n$

Remark

Capital pie Π is not direct applicable in determint i.e.,

$$\prod_{r=1}^{n} \Delta(r) \neq \begin{vmatrix} \prod_{r=1}^{n} f(r) & \prod_{r=1}^{n} g(r) & \prod_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Explanation $\prod_{r=1}^{n} \Delta(r) = \Delta(1) \times \Delta(2) \times \ldots \times \Delta(n)$

$$= \begin{vmatrix} f(1) & g(1) & h(1) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} \times \begin{vmatrix} f(2) & g(2) & h(2) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} \times \ldots \times \begin{vmatrix} f(n) & g(n) & h(n) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$\neq \begin{vmatrix} \prod_{r=1}^{n} f(r) & \prod_{r=1}^{n} g(r) & \prod_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Example 44. Let n be a positive integer and $\Delta_r = \begin{vmatrix} 2r-1 & {}^nC_r & 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2 n^2 & \cos^2 n & \cos^2(n+1) \end{vmatrix}$, prove that $\sum_{r=0}^{n} \Delta_r = 0$.

Sol. We have, $\Delta_r = \begin{vmatrix} 2r-1 & {}^nC_r & 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$

$$\therefore \sum_{r=0}^{n} \Delta_r = \begin{vmatrix} \sum_{r=0}^{n}(2r-1) & \sum_{r=0}^{n} {}^nC_r & \sum_{r=0}^{n} 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$$

Now, $\sum_{r=0}^{n}(2r-1) = 2\sum_{r=0}^{n} r - \sum_{r=0}^{n} 1$

$= 2(0+1+2+3+\ldots+n) - \underbrace{(1+1+1+\ldots+1)}_{(n+1)\text{ times}}$

$= \dfrac{2n(n+1)}{2} - (n+1) = (n+1)(n-1) = n^2 - 1$

$$\sum_{r=0}^{n} \Delta_r = \begin{vmatrix} n^2-1 & 2^n & n+1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix} = 0$$

[since R_1 and R_2 are identical]

Example 45. Let n be a positive integer and $\Delta_r = \begin{vmatrix} r^2+r & r+1 & r-2 \\ 2r^2+3r-1 & 3r & 3r-3 \\ r^2+2r+3 & 2r-1 & 2r-1 \end{vmatrix}$ and $\sum_{r=1}^{n} \Delta_r = an^2 + bn + c$, find the value of $a+b+c$.

Sol. We have, $\Delta_r = \begin{vmatrix} r^2+1r & r+1 & r-2 \\ 2r^2+3r-1 & 3r & 3r-3 \\ r^2+2r+3 & 2r-1 & 2r-1 \end{vmatrix}$

Applying $R_2 \to R_2 - (R_1 + R_3)$, then

$$\Delta_r = \begin{vmatrix} r^2+r & r+1 & r-2 \\ \vdots & & \\ -4 & \ldots\ 0 & \ldots\ 0 \\ \vdots & & \\ r^2+2r+3 & 2r-1 & 2r-1 \end{vmatrix}$$

Expanding along R_2, we get

$$= 4\begin{vmatrix} r+1 & r-2 \\ 2r-1 & 2r-1 \end{vmatrix}$$

$= 4[(r+1)(2r-1) - (r-2)(2r-1)]$

$= 24r - 12$

Now, $\sum_{r=1}^{n} \Delta_r = 24\sum_{r=1}^{n} r - 12\sum_{r=1}^{n} 1$

$= 24\dfrac{n(n+1)}{2} - 12n = 12n(n+1-1)$

$= 12n^2 = an^2 + bn + c$ [given]

For $n = 1$, we have

$a + b + c = 12$

Exercise for Session 4

1. If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3 \\ 0 & 1 & x \end{vmatrix}$, $f'(1)$ is equal to

(a) -1 (b) 0 (c) 1 (d) 2

2. Let $f(x) = \begin{vmatrix} \sec x & x^2 & x \\ 2\sin x & x^3 & 2x^2 \\ \tan 3x & x^2 & x \end{vmatrix}$, $\lim_{x \to 0} \dfrac{f(x)}{x^4}$ is equal to

(a) 0 (b) -1 (c) 2 (d) 3

3. Let $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = Ax^4 + Bx^3 + Cx^2 + Dx + E$, the value of $5A + 4B + 3C + 2D + E$ is equal to

(a) -16 (b) -11 (c) 0 (d) 16

4. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is a constant. Then $\dfrac{d^3}{dx^3}\{f(x)\}$ at $x = 0$ is

(a) p (b) $p + p^2$ (c) $p + p^3$ (d) independent of p

5. If $y = \sin mx$, the value of the determinant $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$, where $y_n = \dfrac{d^n y}{dx^n}$, is

(a) m^2 (b) m^3 (c) m^9 (d) None of these

6. Let $f(x) = \begin{vmatrix} 2\cos^2 x & \sin 2x & -\sin x \\ \sin 2x & 2\sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$, the value of $\int_0^{\pi/2} \{f(x) + f'(x)\}\, dx$, is

(a) $\dfrac{\pi}{2}$ (b) π (c) $\dfrac{3\pi}{2}$ (d) 2π

7. If $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2\left(\dfrac{x}{2}\right) \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x + \tan x \end{vmatrix}$, the value of $\int_{-\pi/2}^{\pi/2} (x^2 + 1)[f(x) + f''(x)]\, dx$, is

(a) -1 (b) 0 (c) 1 (d) 2

8. If $f(x) = \begin{vmatrix} \sin^2 x + & \cos^4 x \ln \cos x & \dfrac{1}{1 + (\tan x)^{\sqrt{2}}} \\ \pi & \pi^2 & \pi^4 \\ \dfrac{7}{16} & -\dfrac{1}{2}\ln 2 & \dfrac{1}{4} \end{vmatrix}$, the value of $\int_0^{\pi/2} f(x)\, dx$ is

(a) 2 (b) -1 (c) 0 (d) None of these

9. If $\Delta_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2 + n + 1 & n^2 + n \\ 2k - 1 & n^2 & n^2 + n + 1 \end{vmatrix}$ and $\sum_{k=1}^{n} \Delta_k = 56$, then n is equal to

(a) 4 (b) 6 (c) 8 (d) None of these

10. The value of $\sum_{r=2}^{n} (-2)^r \begin{vmatrix} {}^{n-2}C_{r-2} & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ -3 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix}$ $(n > 2)$ is

(a) $2n - 1 + (-1)^n$ (b) $2n + 1 + (-1)^n$ (c) $2n - 3 + (-1)^n$ (d) None of these

Shortcuts and Important Results to Remember

1 **Symmetric Determinant** The elements situated at equal distance from the diagonal are equal both in magnitude and sign. i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

2 **Skew-symmetric Determinant** All the diagonal elements are zero and the elements situated at equal distance from the diagonal are equal in magnitude but opposite in sign. The value of skew-symmetric determinant of even order is always a perfect square and that of odd order is always zero i.e. $\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2$ and $\begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = 0$

3 **Circulant Determinant** The elements of the rows (or columns) are in cyclic order. i.e.,

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

(ii) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$
$= (a - b)(b - c)(c - a)(ab + bc + ca)$

(iii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a - b)(b - c)(c - a)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

(v) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

Remark

These results direct applicable in lengthy questions as behaviour of standard results.

4 (i) If $\Delta = 0$, then $\Delta^c = 0$, where Δ^c denotes the determinant of cofactors of elements of Δ.

(ii) If $\Delta \neq 0$, then $\Delta^c = \Delta^{n-1}$, where n is order of Δ.

(iii) Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The sum of products of the elements of any row or column with the corresponding cofactors is equal to the value of determinant, i.e.

$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$
$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = \Delta$

and sum of products of the elements of any row or column with the cofactors of the corresponding elements of any other row or column is zero, i.e.,

$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33}$
$= 0$

5 A homogeneous system of equations is never consistent.

6 **Conjugate of a Determinant** If a_i, b_i and $c_i \in C$ $(i = 1, 2, 3)$

and $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\bar{\Delta} = \begin{vmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 \end{vmatrix}$

(i) If Δ is purely real, then $\bar{\Delta} = \Delta$.

(ii) If Δ is purely imaginary, then $\boxed{\bar{\Delta} = -\Delta}$

(i) If $x_1, x_2, x_3, \ldots$ are in AP or $a^{x_1}, a^{x_2}, a^{x_3}, \ldots$ are in GP,

then $\begin{vmatrix} x_1 & x_2 & x_3 \\ x_{n+1} & x_{n+2} & x_{n+3} \\ x_{2n+1} & x_{2n+2} & x_{2n+3} \end{vmatrix} = 0$

(ii) If $a_1, a_2, a_3, \ldots$ are in GP and $a_i > 0, i = 1, 2, 3, \ldots$,

then $\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix} = 0$

JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1 *If* $(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$, $(x_2 - x_3)^2 + (y_2 - y_3)^2 = b^2$ *and*

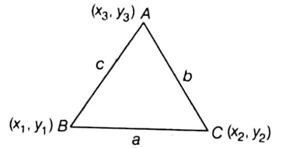
$$(x_3 - x_1)^2 + (y_3 - y_1)^2 = c^2 \text{ and } k\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

$= (a+b+c)(b+c-a)(c+a-b)(a+b-c)$, *the value of k is*

(a) 1 (b) 2 (c) 4 (d) 8

Sol. (c) Consider the triangle with vertices $B(x_1, y_1)$, $C(x_2, y_2)$ and $A(x_3, y_3)$ and

$$AB = c,\ BC = a \text{ and } CA = b$$



$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{...(i)}$$

Also, area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$,

where $2s = a + b + c$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \sqrt{s(s-a)(s-b)(s-c)}$$

On squaring and simplifying, we get

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Hence, the value of k is 4.

• Ex. 2 *If a, b and c are complex numbers, the determinant*

$$\Delta = \begin{vmatrix} 0 & -b & -c \\ \bar{b} & 0 & -a \\ \bar{c} & \bar{a} & 0 \end{vmatrix} \text{ is}$$

(a) a non-zero real number (b) purely imaginary
(c) 0 (d) None of these

Sol. (b) We observe that,

$$\bar{\Delta} = \begin{vmatrix} 0 & -\bar{b} & -\bar{c} \\ b & 0 & -\bar{a} \\ c & a & 0 \end{vmatrix}$$

$$\Rightarrow \quad \bar{\Delta} = \begin{vmatrix} 0 & b & c \\ -\bar{b} & 0 & a \\ -\bar{c} & -\bar{a} & 0 \end{vmatrix}$$

[interchanging rows and columns]

$$= -\begin{vmatrix} 0 & -b & -c \\ \bar{b} & 0 & -a \\ \bar{c} & \bar{a} & 0 \end{vmatrix}$$

[taking (– 1) common from each row]

$$\Rightarrow \quad \bar{\Delta} = -\Delta$$

Hence, Δ is purely imaginary.

• Ex. 3 *The equation* $\boxed{\text{Uses of } \Sigma}$

$$\begin{vmatrix} (1+x)^2 & (1-x)^2 & -(2+x^2) \\ 2x+1 & 3x & 1-5x \\ x+1 & 2x & 2-3x \end{vmatrix} + \begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ 1-2x & 3x-2 & 2x-3 \end{vmatrix} = 0 \text{ has}$$

(a) no real solution
(b) 4 real solutions
(c) two real and two non-real solutions
(d) infinite number of solutions, real or non-real

Sol. (d) Interchanging rows and columns in first determinant, then

$$\begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ -(2+x^2) & 1-5x & 2-3x \end{vmatrix} + \begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ 1-2x & 3x-2 & 2x-3 \end{vmatrix} = 0$$

Add w.r.t Row 3

$$\Rightarrow \quad \begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ -(1+x)^2 & -2x-1 & -x-1 \end{vmatrix} = 0$$

Applying $R_3 \to R_3 + R_1$, then

$$\Rightarrow \quad \begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad 0 = 0$$

which is true for all values of x.

Hence, given equation has infinite number of solutions, real or non-real.

• Ex. 4 *If X, Y and Z are positive numbers such that Y and Z have respectively 1 and 0 at their unit's place and*

$$\Delta = \begin{vmatrix} X & 4 & 1 \\ Y & 0 & 1 \\ Z & 1 & 0 \end{vmatrix}$$

If $(\Delta + 1)$ *is divisible by 10, then X has at its unit's place*

(a) 0 (b) 1
(c) 2 (d) 3

Sol. (c) Let $X = 10x + \lambda$, $Y = 10y + 1$ and $Z = 10z$, where $x, y, z \in N$, then

$$\Delta = \begin{vmatrix} X & 4 & 1 \\ Y & 0 & 1 \\ Z & 1 & 0 \end{vmatrix} = \begin{vmatrix} 10x+\lambda & 4 & 1 \\ 10y+1 & 0 & 1 \\ 10z & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 10x & 4 & 1 \\ 10y & 0 & 1 \\ 10z & 1 & 0 \end{vmatrix} + \begin{vmatrix} \lambda & 4 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 10\begin{vmatrix} x & 4 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix} + (1 - \lambda)$$

$\Rightarrow \Delta + 1 = 10k + (2 - \lambda)$,

$$\text{where } k = \begin{vmatrix} x & 4 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix}.$$

It is given that $(\Delta + 1)$ is divisible by 10. Therefore, $2 - \lambda = 0$ i.e., $\lambda = 2$

$$\therefore \quad X = 10x + 2$$

$\Rightarrow$ 2 is at unit's place of X.

• Ex. 5 *The number of distinct values of a 2 × 2 determinant whose entries are from the set* $\{-1, 0, 1\}$, *is*

(a) 3 (b) 4 (c) 5 (d) 6

Sol. (c) Possible values are $-2, -1, 0, 1, 2$

i.e., $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$, $\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0$, $\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$,

$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$, $\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$

• Ex. 6 *If* $f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$; *a, b being positive integers, then*

(a) constant term in $f(x)$ is 4
(b) coefficient of x in $f(x)$ is 0
(c) constant term in $f(x)$ is $(a - b)$
(d) constant term in $f(x)$ is $(a + b)$

Sol. (b) Let $\begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$

$$= A + Bx + Cx^2 + \dots$$

Put $x = 0$, then $A = 0$

On differentiating both sides w.r.t. x and then put $x = 0$

$$\begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix} = B$$

$$\therefore \quad 0 + 0 + 0 = B$$

$$\Rightarrow \quad B = 0$$

Hence, constant term in $f(x)$ is zero and coefficient of x in $f(x)$ is 0.

• Ex. 7 *If* $f_j = \sum_{i=0}^{2} a_{ij} x^i$, $j = 1, 2, 3$ *and* f_j' *and* f_j'' *are denoted by* $\frac{df_j}{dx}$ *and* $\frac{d^2 f_j}{dx^2}$ *respectively, then* $g(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$ *is*

(a) a constant (b) a linear in x
(c) a quadratic in x (d) a cubic in x

Sol. (a) $\because g'(x) = \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1'' & f_2'' & f_3'' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}$

$= 0 + 0 + 0$ [$\because f_j$ is a quadratic function]

$\therefore \quad g(x) = c = \text{constant}$

• Ex. 8 *Let* $\Delta_a = \begin{vmatrix} (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2 - 3n \end{vmatrix}$, *the value of* $\sum_{a=1}^{n} \Delta_a$ *is*

(a) 0 (b) $\frac{(n-1)n}{2}$
(c) $\frac{(n-1)n^2}{2}$ (d) $\frac{(n-1)n(2n-1)}{3}$

Sol. (a) $\sum_{a=1}^{n} \Delta_a = \begin{vmatrix} \sum_{a=1}^{n}(a-1) & n & 6 \\ \sum_{a=1}^{n}(a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n}(a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

$$= \begin{vmatrix} \frac{(n-1)n}{2} & n & 6 \\ \frac{(n-1)n(2n-1)}{6} & 2n^2 & 4n-2 \\ \frac{(n-1)^2 n^2}{4} & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{(n-1)n^2}{2} \begin{vmatrix} 1 & 1 & 6 \\ \frac{2n-1}{3} & 2n & 4n-2 \\ \frac{(n-1)n}{2} & 3n^2 & 3n^2-3n \end{vmatrix}$$

[taking $\frac{(n-1)n}{2}$ and n common from C_1 and C_2]

Applying $C_3 \to C_3 - 6C_1$, then

$$\sum_{a=1}^{n} \Delta_a = \frac{(n-1)n^2}{2} \begin{vmatrix} 1 & 1 & 0 \\ \frac{2n-1}{3} & 2n & 0 \\ \frac{(n-1)n}{2} & 3n^2 & 0 \end{vmatrix} = 0$$

• ~~**Ex. 9**~~ *If* $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1-\cos x \\ 1+\sin x & \cos x & 1+\sin x - \cos x \\ \sin x & \sin x & 1 \end{vmatrix}$, *then* $\int_0^{\pi/2} \Delta(x)\,dx$ *is equal to*

(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

Sol. (a) Applying $C_3 \to C_3 + C_2 - C_1$, then

$$\Delta(x) = \begin{vmatrix} 1 & \cos x & 0 \\ \vdots & & \vdots \\ 1+\sin x & \cos x & 0 \\ \vdots & & \vdots \\ \sin x & \dots \sin x & \dots 1 \end{vmatrix}$$

$$= 1(\cos x - \cos x - \cos x \sin x) = -\frac{1}{2}\sin 2x$$

$$\therefore \int_0^{\pi/2} \Delta(x)dx = -\frac{1}{2}\int_0^{\pi/2} \sin 2x\, dx$$

$$= \frac{1}{4}[\cos 2x]_0^{\pi/2} = \frac{1}{4}(-1-1) = -\frac{1}{2}$$

Must.

• ~~**Ex. 10**~~ *Number of values of a for which the system of equations* $a^2x + (2-a)y = 4 + a^2$ *and* $ax + (2a-1)y = a^5 - 2$ *possess no solution, is*

(a) 0 (b) 1
(c) 2 (d) infinite

Sol. (c) $\because \Delta = \begin{vmatrix} a^2 & 2-a \\ a & 2a-1 \end{vmatrix} = a^2(2a-1) - a(2-a)$

$$= 2a(a+1)(a-1)$$

For no solution, $\Delta = 0$

$\therefore$ $a = -1, 0, 1$

$\Rightarrow$ $\Delta_1 = \begin{vmatrix} 4+a^2 & 2-a \\ a^5-2 & 2a-1 \end{vmatrix}$

Values of Δ_1 at $a = -1, 0, 1$ are $-6, 0, 6$ respectively and

$$\Delta_2 = \begin{vmatrix} a^2 & 4+a^2 \\ a & a^5-2 \end{vmatrix}$$

Values of Δ_2 at $a = -1, 0, 1$ are $2, 0, -6$, respectively.

For no solution,

$\Delta = 0$ and atleast one of Δ_1, Δ_2 is non-zero.

$\therefore$ $a = -1, 1$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

• **Ex. 11** *The determinant* $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$ *is divisible by*

(a) $a+b+c$ (b) $(a+b)(b+c)(c+a)$
(c) $a^2+b^2+c^2$ (d) $(a-b)(b-c)(c-a)$

Sol. (a,c,d) Applying $C_2 \to C_2 - C_1 - 2C_3$, then

$$\begin{vmatrix} a^2 & -(b^2+c^2) & bc \\ b^2 & -(c^2+a^2) & ca \\ c^2 & -(a^2+b^2) & ab \end{vmatrix} = -\begin{vmatrix} a^2 & b^2+c^2 & bc \\ b^2 & c^2+a^2 & ca \\ c^2 & a^2+b^2 & ab \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, then

$$= -\begin{vmatrix} a^2 & a^2+b^2+c^2 & bc \\ b^2 & a^2+b^2+c^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$= -\begin{vmatrix} a^2 & \dots a^2+b^2+c^2 \dots & bc \\ & \vdots & \\ b^2-a^2 & 0 & c(a-b) \\ & \vdots & \\ c^2-a^2 & 0 & -b(c-a) \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} -(a+b)(a-b) & c(a-b) \\ (c+a)(c-a) & -b(c-a) \end{vmatrix}$$

$$= (a-b)(c-a)(a^2+b^2+c^2)\begin{vmatrix} -(a+b) & c \\ c+a & -b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, then

$$= (a-b)(c-a)(a^2+b^2+c^2)\begin{vmatrix} -(a+b+c) & c \\ (a+b+c) & -b \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

• **Ex. 12** *The value of* θ *lying between* $-\frac{\pi}{4}$ *and* $\frac{\pi}{2}$ *and* $0 \le A \le \frac{\pi}{2}$ *and satisfying the equation*

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} = 0, \text{ are}$$

(a) $A = \frac{\pi}{4}, \theta = -\frac{\pi}{8}$ (b) $A = \frac{3\pi}{8} = \theta$
(c) $A = \frac{\pi}{5}, \theta = -\frac{\pi}{8}$ (d) $A = \frac{\pi}{6}, \theta = \frac{3\pi}{8}$

Sol. (a, b, c, d)

$$\because \begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2$, then

$$\begin{vmatrix} 2 & \cos^2 A & 2\sin 4\theta \\ & \vdots & \\ 0 \dots & 1 & \dots 0 \\ & \vdots & \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $1(2 + 2\sin 4\theta) = 0$

$\therefore$ $\sin 4\theta = -1$

$\Rightarrow$ $4\theta = (2n-1)\frac{\pi}{2} \Rightarrow \theta = (2n-1)\frac{\pi}{8}$

For $n = 0, 2$ then $\theta = -\frac{\pi}{8}, \frac{3\pi}{8}$ and $A \in R$

• **Ex. 13** *The digits A, B, C are such that the three digit numbers A88, 6B8, 86C are divisible by 72, the determinant* $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ *is divisible by*

(a) 72 (b) 144 (c) 288 (d) 216

Sol. (a, b, c)

$\because$ A88, 6B8, 86C are divisible by 72.

$\therefore$ A88, 6B8, 86C are also divisible by 9.

$\Rightarrow$ $A + 8 + 8, 6 + B + 8, 8 + 6 + C$

are divisible by 9, then $A = 2, B = 4, C = 4$

Let $\Delta = \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix} = \begin{vmatrix} 2 & 6 & 8 \\ 8 & 4 & 6 \\ 8 & 8 & 4 \end{vmatrix}$

$$= 2(16-48) - 6(32-48) + 8(64-32) = 288$$

Hence, Δ is divisible by 72, 144 and 288.

• **Ex. 14** *If p, q, r and s are in AP and*

$$f(x) = \begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ q+\sin x & r+\sin x & -1+\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

such that $\int_0^1 f(x)\,dx = -2$, *the common difference of the AP can be*

(a) -1 (b) 1/2 (c) 1 (d) 2

Sol. (a, c)

$$\because f(x) = \frac{1}{2}\begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ 2q+2\sin x & 2r+2\sin x & -2+2\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - (R_1 + R_3)$, then

$$f(x) = \frac{1}{2}\begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ & & \vdots \\ 0 \dots & 0 & \dots -2 \\ & & \vdots \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

[$\because 2q = p + r, 2r = q + s$ and $p + s = q + r$]

$$= -\frac{(-2)}{2}\begin{vmatrix} p+\sin x & q+\sin x \\ r+\sin x & s+\sin x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, then

$$f(x) = \begin{vmatrix} p + \sin x & D \\ p + 2D + \sin x & D \end{vmatrix}$$

[where D = common difference]

$$= D[p + \sin x - p - 2D - \sin x] = -2D^2$$

and $\int_0^1 f(x)dx = -4$

$\Rightarrow \int_0^1 (-2D^2)dx = -4 \Rightarrow -2D^2 = -2$

$\therefore D^2 = 1 \Rightarrow D = \pm 1$

• **Ex. 15** *If the system of equations $a^2x - by = a^2 - b$ and $bx - b^2y = 2 + 4b$ possess an infinite number of solutions, the possible values of a and b are*

(a) $a = 1, b = -1$ (b) $a = 1, b = -2$
(c) $a = -1, b = -1$ (d) $a = -1, b = -2$

Sol. (a, b, c, d)

Here, $\Delta = \begin{vmatrix} a^2 & -b \\ b & -b^2 \end{vmatrix} = -a^2b^2 + b^2 = -(a^2 - 1)b^2$

If $\Delta = 0$, then $a^2 = 1, b = 0$

Now, $\Delta_1 = \begin{vmatrix} a^2 - b & -b \\ 2 + 4b & -b^2 \end{vmatrix}$

For $a^2 = 1, \Delta_1 = \begin{vmatrix} 1 - b & -b \\ 2 + 4b & -b^2 \end{vmatrix} = b(b+1)(b+2)$

and $\Delta_2 = \begin{vmatrix} a^2 & a^2 - b \\ b & 2 + 4b \end{vmatrix}$

For $a^2 = 1, \Delta_2 = \begin{vmatrix} 1 & 1 - b \\ b & 2 + 4b \end{vmatrix} = (b+1)(b+2)$

For infinite number of solutions, $\Delta = \Delta_1 = \Delta_2 = 0$

$\therefore a^2 = 1, b = -1, -2 \Rightarrow a = \pm 1, b = -1, b = -2$

JEE Type Solved Examples : Passage Based Questions

- This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

Consider the system of equations

$$2x + ay + 6z = 8;\quad x + 2y + bz = 5$$
$$x + y + 3z = 4$$

16. The system has unique solution, if

(a) $a = 2, b = 3$ (b) $a = 2, b \neq 3$
(c) $a \neq 2, b = 3$ (d) $a \neq 2, b \neq 3$

17. The system has infinite solutions, if

(a) $a = 2, b \in R$ (b) $a = 3, b \in R$
(c) $a \in R, b = 2$ (d) $a \in R, b = 3$

18. The system has no solution, if

(a) $a = 2, b = 3$ (b) $a = 2, b \neq 3$
(c) $a \neq 2, b = 3$ (d) $a \neq 2, b \neq 3$

Sol. $\Delta = \begin{vmatrix} 2 & a & 6 \\ 1 & 2 & b \\ 1 & 1 & 3 \end{vmatrix} = 2(6 - b) - a(3 - b) + 6(1 - 2)$

$= ab - 3a - 2b + 6 = (a - 2)(b - 3)$

$\Delta_1 = \begin{vmatrix} 8 & a & 6 \\ 5 & 2 & b \\ 4 & 1 & 3 \end{vmatrix} = 8(6 - b) - a(15 - 4b) + 6(5 - 8)$

$= 4ab - 15a - 8b + 30 = (a - 2)(4b - 15)$

$\Delta_2 = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & b \\ 1 & 4 & 3 \end{vmatrix} = 0$ $[\because R_1 = 2R_3]$

$\Delta_3 = \begin{vmatrix} 2 & a & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = 2(8 - 5) - a(4 - 5) + 8(1 - 2)$

$= 6 + a - 8 = a - 2$

16. (*d*) The system has unique solution, if

$\Delta \neq 0$

$\Rightarrow (a - 2)(b - 3) \neq 0$

$\Rightarrow a \neq 2, b \neq 3$

17. (*a*) The system has infinite solution, if

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow a - 2 = 0$

or $a = 2, b \in R$

18. (*c*) The system has no solution, if

$\Delta = 0$ and atleast one of Δ_1, Δ_2 and Δ_3 is non-zero.

$\Rightarrow a \neq 2, b = 3$

Passage II

(Ex. Nos. 19 to 20)

Let ${}^xC_i, {}^{x^2}C_i$ and ${}^{x^3}C_i (i = 1, 2, 3)$ be Binomial coefficients, where $x \in N$

and $f(x) = 12\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^{x^2}C_1 & {}^{x^2}C_2 & {}^{x^2}C_3 \\ {}^{x^3}C_1 & {}^{x^3}C_2 & {}^{x^3}C_3 \end{vmatrix}$, *then*

19. $f(x)$ is a polynomial of degree

(a) 6 (b) 10
(c) 14 (d) 18

20. If $f(x) = (x - 1)^m x^n (x + 1)^p$, where $m, n, p \in N$, then the value of $\Sigma\, mn$ is

(a) 32 (b) 43
(c) 44 (d) 56

Sol. $\because f(x) = 12\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^{x^2}C_1 & {}^{x^2}C_2 & {}^{x^2}C_3 \\ {}^{x^3}C_1 & {}^{x^3}C_2 & {}^{x^3}C_3 \end{vmatrix}$

$$= 12\begin{vmatrix} x & \dfrac{x(x-1)}{2} & \dfrac{x(x-1)(x-2)}{6} \\ x^2 & \dfrac{x^2(x^2-1)}{2} & \dfrac{x^2(x^2-1)(x^2-2)}{6} \\ x^3 & \dfrac{x^3(x^3-1)}{2} & \dfrac{x^3(x^3-1)(x^3-2)}{6} \end{vmatrix}$$

Taking x, x^2 and x^3 common from R_1, R_2 and R_3, then

$$f(x) = x \cdot x^2 \cdot x^3 \begin{vmatrix} 1 & (x-1) & (x-1)(x-2) \\ 1 & (x^2-1) & (x^2-1)(x^2-2) \\ 1 & (x^3-1) & (x^3-1)(x^3-2) \end{vmatrix}$$

$$= x^6(x-1)^2 \begin{vmatrix} 1 & 1 & x-2 \\ 1 & x+1 & (x+1)(x^2-2) \\ 1 & x^2+x+1 & (x^2+x+1)(x^3-2) \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$, then

$$f(x) = x^6(x-1)^2 \begin{vmatrix} 0 & -x & x(3 - x - x^2) \\ 1 & x+1 & (x+1)(x^2-2) \\ 0 & x^2 & x^2(x^2 + x^3 - 3) \end{vmatrix}$$

Expanding along C_1, then

$$= -x^6(x-1)^2 \begin{vmatrix} -x & x(3 - x - x^2) \\ x^2 & x^2(x^2 + x^3 - 3) \end{vmatrix}$$

$$= -x^9(x-1)^2 \begin{vmatrix} -1 & 3 - x - x^2 \\ 1 & x^2 + x^3 - 3 \end{vmatrix}$$

$$= -x^9(x-1)^2(-x^2 - x^3 + 3 - 3 + x + x^2)$$

$$= x^{10}(x-1)^2(x^2 - 1) = x^{10}(x-1)^3(x+1)$$

19. (c) $f(x)$ is a polynomial of degree 14.

20. (b) Here, $m = 3, n = 10$ and $p = 1$

$\therefore \Sigma\, mn = mn + np + pm = 30 + 10 + 3 = 43$

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

• **Ex. 21** *If $\Delta_r = \begin{vmatrix} r & r-1 \\ r-1 & r \end{vmatrix}$, where r is a natural number, the value of $\sqrt[10]{\sum_{r=1}^{1024} \Delta_r}$ is*

Sol. (4) $\because \Delta_r = r^2 - (r-1)^2$

$\therefore \sum_{r=1}^{1024} \Delta_r = (1024)^2 - (1-1)^2 = (1024)^2 = 2^{20}$

$\Rightarrow \sqrt[10]{\sum_{r=1}^{1024} \Delta_r} = 2^2 = 4$

• **Ex. 22** *If P, Q and R are the angles of a triangle, the value of $\begin{vmatrix} \tan P & 1 & 1 \\ 1 & \tan Q & 1 \\ 1 & 1 & \tan R \end{vmatrix}$ is*

Sol. (2) $\begin{vmatrix} \tan P & 1 & 1 \\ 1 & \tan Q & 1 \\ 1 & 1 & \tan R \end{vmatrix} = \tan P(\tan Q \tan R - 1)$

$- 1(\tan R - 1) + 1(1 - \tan Q)$

$= \tan P \tan Q \tan R - (\tan P + \tan Q + \tan R) + 2$

$= 0 + 2$

[$\because$ In ΔPQR, $\tan P + \tan Q + \tan R = \tan P \tan Q \tan R$]

$= 2$

JEE Type Solved Examples : Matching Type Questions

- This section contains 2 eamples. Example 23 have four statements (A, B, C and D) given in **Column I** and four statement (p, q, r and s) in **Column II** and example 24 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statements(s) given in **Column II**.

• **Ex. 23** *Let $f(x)$ denotes the determinant*

$$f(x)=\begin{vmatrix} x^2 & 2x & 1+x^2 \\ x^2+1 & x+1 & 1 \\ x & -1 & x-1 \end{vmatrix}.$$

On expansion $f(x)$ is seen to be a 4th degree polynomial given by $f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$.
Using differentiation of determinant or otherwise match the entries in Column I with one or more entries of the elements of Column II.

	Column I		Column II
(A)	$a_0^2+a_1$ is divisible by	(p)	2
(B)	$a_2^2+a_4$ is divisible by	(q)	3
(C)	$a_0^2+a_2$ is divisible by	(r)	4
(D)	$a_4^2+a_3^2+a_1^2$ is divisible by	(s)	5

Sol. (A) → (p, s); (B) → (p, r); (C) → (p, q); (D) → (q)

$$\because \quad f(x)=\begin{vmatrix} x^2 & 2x & 1+x^2 \\ x^2+1 & x+1 & 1 \\ x & -1 & x-1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$, then

$$f(x)=\begin{vmatrix} -1 & 2x & 1+x^2 \\ x^2 & x+1 & 1 \\ 1 & -1 & x-1 \end{vmatrix}$$

Expanding along R_1, then

$$f(x)=-(x^2-1+1)-2x(x^3-x^2-1)+(1+x^2)(-x^2-x-1)$$

$$=-3x^4+x^3-3x^2+x-1 \quad \text{...(i)}$$

According to the question, we get

$$f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$a_0=-3, a_1=1, a_2=-3, a_3=1, a_4=-1$$

(A) $a_0^2+a_1=(-3)^2+1=9+1=10=2\times5$

(B) $a_2^2+a_4=(-3)^2-1=9-1=8=2\times4$

(C) $a_0^2+a_2=(-3)^2-3=9-3=6=2\times3$

(D) $a_4^2+a_3^2+a_1^2=(-1)^2+(1)^2+(1)^2=1+1+1=3$

• **Ex. 24** *Suppose a, b and c are distinct and x, y and z are connected by the system of equations $x+ay+a^2z=a^3$, $x+by+b^2z=b^3$ and $x+cy+c^2z=c^3$.*

	Column I		Column II
(A)	For $x=1, y=2$ and $z=3, (a+b+c)^{-(ab+bc+ca)}$ is divisible by	(p)	3
(B)	For $x=4, y=3$ and $z=2, (ab+bc+ca)^{abc}$ is divisible by	(q)	6
(C)	For $x=6, y=4$ and $z=2, (abc)^{a+b+c}$ is divisible by	(r)	9
		(s)	12

Sol. (A) → (p, r) (B) → (p, r); (C) → (p, q, r, s)

$$a \ne b \ne c$$

$$\Delta=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=(a-b)(b-c)(c-a)$$

$$\Delta_1=\begin{vmatrix} a^3 & a & a^2 \\ b^3 & b & b^2 \\ c^3 & c & c^2 \end{vmatrix}=\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}=abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$=abc(a-b)(b-c)(c-a)$$

$$\Delta_2=\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}=-\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$=-(a-b)(b-c)(c-a)(ab+bc+ca)$$

and $$\Delta_3=\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$$

By Cramer's rule, we get

$$x=\frac{\Delta_1}{\Delta}=abc$$

$$y=\frac{\Delta^2}{\Delta}=-(ab+bc+ca),\ z=\frac{\Delta_3}{\Delta}=a+b+c$$

(A) $(a+b+c)^{-(ab+bc+ca)}=z^y=3^2=9$, which is divisible by 3 and 9.

(B) $(ab+bc+ca)^{abc}=(-y)^x=(-3)^4=81$, which is divisible by 3 and 9.

(C) $(abc)^{a+b+c}=x^3=6^2=36$, which is divisible by 3, 6, 9 and 12.

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** Example numbers 25 and 26 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

• **Ex. 25 Statement-1** *Let*

$$\Delta_r=\begin{vmatrix} (r-1) & n! & 6 \\ (r-1)^2 & (n!)^2 & 4n-2 \\ (r-1)^3 & (n!)^3 & 3n^2-2n \end{vmatrix},\ then \prod_{r=2}^{n+1}\Delta_r=0.$$

Statement-2 $\prod_{r=2}^{n+1}\Delta_r=\Delta_2\cdot\Delta_3\cdot\Delta_4\ldots\Delta_{n+1}$

Sol. (d) $\because \prod_{r=2}^{n+1}\Delta_r=\Delta_2\cdot\Delta_3\cdot\Delta_4\ldots\Delta_{n+1}$

$$=\begin{vmatrix} 1 & n! & 6 \\ 1 & (n!)^2 & 4n-2 \\ 1 & (n!)^3 & 3n^3-2n \end{vmatrix}\times\begin{vmatrix} 2 & n! & 6 \\ 4 & (n!)^2 & 4n-2 \\ 8 & (n!)^3 & 3n^2-2n \end{vmatrix}$$

$$\times\ldots\times\begin{vmatrix} n & n! & 6 \\ n^2 & (n!)^2 & 4n-2 \\ n^3 & (n!)^3 & (3n^2-2n) \end{vmatrix}\ne0$$

∴ Statement-1 is false and Statement-2 is true.

• **Ex. 26** *Consider the determinant*

$$f(x)=\begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix}.$$

Statement-1 *$f(x)=0$ has one root $x=0$.*
Statement-2 *The value of skew-symmetric determinant of odd order is always zero.*

Sol. (a) For $x=0$, the determinant reduces to the determinant of a skew-symmetric of odd order which is always zero. Hence, $x=0$ is the solution of given equation $f(x)=0$.

Subjective Type Examples

- In this section, there are **20 subjective** solved examples.

• **Ex. 27** *A determinant of second order is made with the elements 0 and 1. Find the number of determinants with non-negative values.*

Sol. The number of determinants that can be made with 0 and 1

$$=2\times2\times2\times2=16$$

and there are only three determinants of second order with negative values

i.e., $$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix},\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix},\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

Therefore, number of determinants with non-negative values $=16-3=13$

• **Ex. 28** *Prove that* $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$ $=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$, *hence find the value of the determinant, if a, b and c are the roots of the equation $px^3-qx^2+rx-s=0$.*

Sol. Let $$\Delta=\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

Since, the answer contain abc, then taking a, b and c common from R_1, R_2 and R_3 respectively, then

$$\Delta=abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

But answer also contains $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$,

then applying $R_1 \to R_1+R_2+R_3$

$$\therefore \quad \Delta=abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from R_1, then

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, then

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{b} & 1 & \frac{1}{b} \\ \frac{1}{c} & 0 & \frac{1}{c}+1 \end{vmatrix}$$

Expanding along C_2, then

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 \\ \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Hence, $\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

2nd Part $\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ca + ab$

$$= \frac{s}{p} + \frac{r}{p} = \left(\frac{s+r}{p}\right)$$

• **Ex. 29** *If a, b and c are positive and are the pth, qth and rth terms, respectively of a GP. Show without expanding that*

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$$

Sol. Let A be the first term and R be the common ratio of GP, then

$$a = p\text{th term} = AR^{p-1}$$
$$b = q\text{th term} = AR^{q-1}$$
$$c = r\text{th term} = AR^{r-1}$$

$\therefore$ $\log a = \log A + (p-1)\log R$,
$\log b = \log A + (q-1)\log R$ and
$\log c = \log A + (r-1)\log R$

$$\therefore \quad \text{LHS} = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - (\log A)C_3$, then

$$= \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix} = \log R \begin{vmatrix} (p-1) & p & 1 \\ (q-1) & q & 1 \\ (r-1) & r & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$, then

$$= \log R \begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} = 0 \text{ [since } C_1 \text{ and } C_2 \text{ are identical]}$$

$$= \text{RHS}$$

• **Ex. 30** *Prove that*

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b).$$

Sol. Let $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

On putting $a+b=0$, $b=-a$

Then, $\Delta = \begin{vmatrix} -2a & 0 & a+c \\ 0 & 2a & c-a \\ c+a & c-a & -2c \end{vmatrix}$

Expanding along R_1, then

$$\Delta = -2a\{-4ac-(c-a)^2\} - 0 + (a+c)\{0-2a(c+a)\}$$
$$= 2a(c+a)^2 - 2a(c+a)^2 = 0$$

Hence, $(a+b)$ is a factor of Δ, similarly $(b+c)$ and $(c+a)$ are the factors of Δ.

On expansion of determinant we can see that each term of the determinant is a homogeneous expression in a, b and c of degree 3 and also RHS is a homogeneous expression of degree 3.

Let $\Delta = k(a+b)(b+c)(c+a)$

or $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a+b)(b+c)(c+a)$

On putting $a=0$, $b=1$ and $c=2$, we get

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = k(0+1)(1+2)(2+0)$$

$\Rightarrow$ $0 - 1(-4-6) + 2(3+4) = 6k$
$\Rightarrow$ $24 = 6k$
$\therefore$ $k = 4$

Hence, $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$

• **Ex. 31** *If $bc+qr = ca+rp = ab+pq = -1$, show that* $\begin{vmatrix} ap & bp & cr \\ a & b & c \\ p & q & r \end{vmatrix} = 0.$

Sol. Given equations can be rewritten as

$$bc + qr + 1 = 0 \quad \ldots\text{(i)}$$
$$ca + rp + 1 = 0 \quad \ldots\text{(ii)}$$
$$ab + pq + 1 = 0 \quad \ldots\text{(iii)}$$

On multiplying Eqs. (i), (ii) and (iii) by ap, bq and cr respectively, we get

$$(abc)p + (pqr)a + ap = 0$$
$$(abc)q + (pqr)b + bq = 0$$
$$(abc)r + (pqr)c + cr = 0$$

These equations are consistent, given equations three but abc and pqr are two.

Hence, $\begin{vmatrix} p & a & ap \\ q & b & bq \\ r & c & cr \end{vmatrix} = 0$

$\Rightarrow$ $\begin{vmatrix} p & q & r \\ a & b & c \\ ap & bq & cr \end{vmatrix} = 0$

[interchanging rows into columns]

$\Rightarrow$ $(-1)\begin{vmatrix} ap & bq & cr \\ a & b & c \\ p & q & r \end{vmatrix} = 0$ $\quad [R_1 \leftrightarrow R_3]$

Hence, $\begin{vmatrix} ap & bq & cr \\ a & b & c \\ p & q & r \end{vmatrix} = 0$

• **Ex. 32** *If α and β are the roots of the equations $x^2-2x+4=0$, find the value of* $\begin{vmatrix} \Sigma\alpha & \Sigma\alpha^2 & \Sigma\alpha^3 \\ \Sigma\alpha^2 & \Sigma\alpha^3 & \Sigma\alpha^4 \\ \Sigma\alpha^3 & \Sigma\alpha^4 & \Sigma\alpha^5 \end{vmatrix}$.

Sol. Given, $x^2 - 2x + 4 = 0$

$\therefore$ $x = 1 \pm i\sqrt{3}$
$\therefore$ $\alpha = 1 + i\sqrt{3}$
and $\beta = 1 - i\sqrt{3}$

$\Rightarrow$ $\alpha = -2\left(\frac{-1-i\sqrt{3}}{2}\right)$ and $\beta = -2\left(\frac{-1+i\sqrt{3}}{2}\right)$

$\alpha = -2\omega^2$ and $\beta = -2\omega$, where ω is the cube root of unity.

$$\Sigma\alpha = \alpha+\beta = -2(\omega+\omega)^2 = -2(-1) = 2$$
$$\Sigma\alpha^2 = \alpha^2+\beta^2 = 4\omega^4 + 4\omega^2 = 4(\omega+\omega)^2 = 4(-1) = -4$$
$$\Sigma\alpha^3 = \alpha^3+\beta^3 = -8\omega^6 - 8\omega^3 = -8-8 = -16$$
$$\Sigma\alpha^4 = \alpha^4+\beta^4 = 16\omega^8 + 16\omega^4 = 16(\omega^2+\omega)$$
$$= 16(-1) = -16$$

and $\Sigma\alpha^5 = \alpha^5+\beta^5 = -32\omega^{10} - 32\omega^5$
$$= -32(\omega+\omega^2) = -32(-1) = 32$$

Let $\Delta = \begin{vmatrix} \Sigma\alpha & \Sigma\alpha^2 & \Sigma\alpha^3 \\ \Sigma\alpha^2 & \Sigma\alpha^3 & \Sigma\alpha^4 \\ \Sigma\alpha^3 & \Sigma\alpha^4 & \Sigma\alpha^5 \end{vmatrix} = \begin{vmatrix} 2 & -4 & -16 \\ -4 & -16 & -16 \\ -16 & -16 & 32 \end{vmatrix}$

$$= 2(-4)(-16)\begin{vmatrix} 1 & -2 & -8 \\ 1 & 4 & 4 \\ 1 & 1 & -2 \end{vmatrix} = 128\begin{vmatrix} 1 & -2 & -8 \\ 1 & 4 & 4 \\ 1 & 1 & -2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = 128\begin{vmatrix} 1 & -2 & -8 \\ 0 & 6 & 12 \\ 0 & 3 & 6 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = 128 \cdot 1 \cdot \begin{vmatrix} 6 & 12 \\ 3 & 6 \end{vmatrix} = 128(36-36) = 0$$

• **Ex. 33** *If $a^2+b^2+c^2=1$, prove that*

$$\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

is independent of a, b and c.

Sol. Let

$$\Delta = \begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

On multiplying C_1, C_2 and C_3 by a, b and c respectively and taking a, b and c common from R_1, R_2 and R_3 respectively, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta = \begin{vmatrix} a^2+b^2+c^2 & b^2 1-\cos\phi) & c^2(1-\cos\phi) \\ a^2+b^2+c^2 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ a^2+b^2+c^2 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Taking $a^2+b^2+c^2$ common from C_1, then

$$\Delta = (a^2+b^2+c^2)\begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 1 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ 1 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = 1\begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 0 & (a^2+b^2+c^2)\cos\phi & 0 \\ 0 & 0 & (a^2+b^2+c^2)\cos\phi \end{vmatrix}$$

$$= (a^2+b^2+c^2)^2\cos^2\phi$$

[by property, since all elements zero below leading diagonal]

$$= 1^2\cdot\cos\phi = \cos^2\phi \qquad [\because a^2+b^2+c^2 = 1]$$

which is independent of a, b and c.

• **Ex. 34** *If $a \neq 0$ and $a \neq 1$, show that*

$$\begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = a^3\left[1 + \frac{x(a^3-1)}{a^2(a-1)}\right].$$

Sol. Let

$$\text{LHS} = \Delta = \begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = \begin{vmatrix} x+1 & x & x \\ x+0 & x+a & x \\ x+0 & x & x+a^2 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} + \begin{vmatrix} 1 & x & x \\ 0 & x+a & x \\ 0 & x & x+a^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ in first determinant, then

$$\Delta = \begin{vmatrix} x & x & x \\ 0 & a & 0 \\ 0 & 0 & a^2 \end{vmatrix} + \begin{vmatrix} 1 & x & x \\ 0 & x+a & x \\ 0 & x & x+a^2 \end{vmatrix}$$

Expanding first determinant by property, since all elements below leading diagonal are zero and expanding second determinant along C_1, then

$$\Delta = x\cdot a\cdot a^2 + 1\cdot\begin{vmatrix} x+a & x \\ x & x+a^2 \end{vmatrix}$$

$$= xa^3 + \{(x+a)(x+a^2) - x^2\}$$

$$= xa^3 + (x^2 + a^2x + ax + a^3 - x^2)$$

$$= xa^3 + a^2x + ax + a^3 = a^3 + x(a^3 + a^2 + a)$$

$$= a^3 + \frac{x\cdot a(a^3-1)}{(a-1)} = a^3\left[1 + \frac{x(a^3-1)}{a^2(a-1)}\right] = \text{RHS}$$

• **Ex. 35** (i) *Prove that*

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix},$$

where $\alpha^2 = a^2+b^2+c^2$ and $\beta^2 = ab+bc+ca$.

(ii) *Prove that* $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ *is divisible by $(a+b+c)^2$. Find the quotient.*

(iii) *Prove that*

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}.$$

(iv) *Prove that*

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2.$$

Sol. (i) Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$\therefore$ Determinant of cofactors of Δ is

$$\Delta^c = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \Delta^{3-1} = \Delta^2$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{vmatrix} \quad \text{[row by row]}$$

$$= \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix}$$

Hence, $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix}$

(ii) From Eq. (i), we get

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= (a^3+b^3+c^3-3abc)^2$$

$$= (a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2$$

Therefore, $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is divisible by $(a+b+c)^2$.

Hence, the quotient is $(a^2+b^2+c^2-ab-bc-ca)^2$.

(iii) From Eq. (i), we get

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & -c & b \\ b & -a & c \\ c & -b & a \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix} \quad \text{[row by row]}$$

Hence, $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}.$$

(iv) $\text{LHS} = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} \quad \text{[row by row]}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= (a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2 \quad \text{[from Eq. (ii)]}$$

$$= [-(a^3+b^3+c^3-3abc)]^2$$

$$= (a^3+b^3+c^3-3abc)^2 = \text{RHS}$$

• **Ex. 36** *Let α and β be the roots of the equation $ax^2+bx+c=0$. Let $S_n = \alpha^n + \beta^n$ for $n \geq 1$. Evaluate the determinant* $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

Sol. Since, α and β are the roots of the equation $ax^2+bx+c=0$.

$$\therefore \alpha+\beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a} \text{ and } \alpha-\beta = \frac{\sqrt{D}}{a}$$

Let $\Delta = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$

$$= \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \Delta_1 \times \Delta_1 \quad \text{[say]}$$

$$\therefore \quad \Delta = \Delta_1^2 \quad \text{...(i)}$$

$$\therefore \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & \beta-1 \\ 1 & \alpha^2-1 & \beta^2-1 \end{vmatrix}$$

Expanding along R_1, then

$$\Delta_1 = \begin{vmatrix} \alpha-1 & \beta-1 \\ \alpha^2-1 & \beta^2-1 \end{vmatrix} = (\alpha-1)(\beta-1)\begin{vmatrix} 1 & 1 \\ \alpha+1 & \beta+1 \end{vmatrix}$$

$$= \{\alpha\beta-(\alpha+\beta)+1\}(\beta-\alpha)$$

$$\therefore \quad \Delta = \Delta_1^2 = [\alpha\beta-(\alpha+\beta)+1]^2(\beta-\alpha)^2$$

$$= \left(\frac{c}{a}+\frac{b}{a}+1\right)^2 \cdot \frac{D}{a^2} = \frac{(a+b+c)^2(b^2-4ac)}{a^4}$$

• Ex. 37 *If A, B and C are the angles of a triangle, show that*

(i) $\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0.$

(ii) $\begin{vmatrix} -1+\cos B & \cos C+\cos B & \cos B \\ \cos C+\cos A & -1+\cos A & \cos A \\ -1+\cos B & -1+\cos A & -1 \end{vmatrix} = 0.$

Sol. (i) LHS $= \begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix}$

$$= \begin{vmatrix} 2ka\cos A & kc & kb \\ kc & 2kb\cos B & ka \\ kb & ka & 2kc\cos C \end{vmatrix} \quad \text{[from sine rule]}$$

$$= k^3\begin{vmatrix} 2a\cos A & c & b \\ c & 2b\cos B & a \\ b & a & 2c\cos C \end{vmatrix}$$

$$= k^3\begin{vmatrix} a\cos A+a\cos A & a\cos B+b\cos A & a\cos C+c\cos A \\ a\cos B+b\cos A & b\cos B+b\cos B & b\cos C+c\cos B \\ a\cos C+c\cos A & c\cos B+b\cos C & c\cos C+c\cos C \end{vmatrix}$$

$$= k^3\begin{vmatrix} \cos A & a & 0 \\ \cos B & b & 0 \\ \cos C & c & 0 \end{vmatrix} \times \begin{vmatrix} a & \cos A & 0 \\ b & \cos B & 0 \\ c & \cos C & 0 \end{vmatrix} = 0\times 0 = 0 = \text{RHS}$$

(ii) LHS $= \begin{vmatrix} -1+\cos B & \cos C+\cos B & \cos B \\ \cos C+\cos A & -1+\cos A & \cos A \\ -1+\cos B & -1+\cos A & -1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, then

$$= \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} -a & \cos C & \cos B \\ a\cos C & -1 & \cos A \\ a\cos B & \cos A & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$, then

$$= \frac{1}{a}\begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0 = \text{RHS}$$

• Ex. 38 *Without expanding at any stage, evaluate the value of the determinant*

$$\begin{vmatrix} 2 & \tan A\cot B+\cot A\tan B & \tan A\cot C+\cot A\tan C \\ \tan B\cot A+\cot B\tan A & 2 & \tan B\cot C+\cot B\tan C \\ \tan C\cot A+\cot C\tan A & \tan C\cot B+\cot C\tan B & 2 \end{vmatrix}.$$

Sol. The given determinant can be written as the product of two determinants

$$\begin{vmatrix} \tan A & \cot A & 0 \\ \tan B & \cot B & 0 \\ \tan C & \cot C & 0 \end{vmatrix} \times \begin{vmatrix} \cot A & \tan A & 0 \\ \cot B & \tan B & 0 \\ \cot C & \tan C & 0 \end{vmatrix} = 0\times 0 = 0$$

• Ex. 39 *Suppose that digit numbers A28, 3B9 and 62C, where A, B and C are integers between 0 and 9 are divisible by a fixed integer k, prove that the determinant* $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ *is also divisible by k.*

Sol. Given, $A28$, $3B9$ and $62C$ are divisible by k, then

$$A28 = 100A + 20 + 8 = n_1 k \quad \text{...(i)}$$

$$3B9 = 300 + 10B + 9 = n_2 k \quad \text{...(ii)}$$

and $$62C = 600 + 20 + C = n_3 k \quad \text{...(iii)}$$

where $n_1, n_2, n_3 \in I$ (integers).

Let $$\Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$

Applying $R_2 \to R_2 + 10R_3 + 100R_1$, then

$$A = \begin{vmatrix} A & 3 & 6 \\ 100A+20+8 & 300+10B+9 & 600+20+C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ n_1k & n_2k & n_3k \\ 2 & B & 2 \end{vmatrix} \quad \text{[using Eqs. (i), (ii) and (iii)]}$$

$$= k\begin{vmatrix} A & 3 & 6 \\ n_1 & n_2 & n_3 \\ 2 & B & 2 \end{vmatrix}$$

Hence, Δ is divisible by k.

• Ex. 40 *If* $\Delta = \begin{vmatrix} \sin x & \sin(x+h) & \sin(x+2h) \\ \sin(x+2h) & \sin x & \sin(x+h) \\ \sin(x+h) & \sin(x+2h) & \sin x \end{vmatrix}$, *find* $\lim_{h\to 0}\left(\frac{\Delta}{h^2}\right)$.

Sol. Let $a = \sin x$, $b = \sin(x+h)$ and $c = \sin(x+2h)$

$$\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a^3+b^3+c^3-3abc)$$

$$= \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$

Now, $a - b = \sin x - \sin(x+h) = -2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$

$b - c = \sin(x+h) - \sin(x+2h) = -2\cos\left(x+\frac{3h}{2}\right)\sin\frac{h}{2}$

and $c - a = \sin(x+2h) - \sin x = 2\cos(x+h)\sin h$

$$\therefore \quad \frac{\Delta}{h^2} = \frac{1}{2}(a+b+c)\left[\left(\frac{a-b}{h}\right)^2+\left(\frac{b-c}{h}\right)^2+\left(\frac{c-a}{h}\right)^2\right]$$

$$= \frac{1}{2}[\sin x+\sin(x+h)+\sin(x+2h)]\times\left[\left(\frac{-2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}\right)^2+\left(\frac{-2\cos\left(x+\frac{3h}{2}\right)\sin\frac{h}{2}}{h}\right)^2+\left(\frac{2\cos(x+h)\sin h}{h}\right)^2\right]$$

$$\therefore \quad \lim_{h\to 0}\frac{\Delta}{h^2} = \frac{1}{2}(3\sin x)(\cos^2 x+\cos^2 x+4\cos^2 x)$$

$$\therefore \quad = 9\sin x\cos^2 x$$

• Ex. 41 *If* $f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$, *show that f(x) is linear in x. Hence, deduce that* $f(0) = \frac{bg(a)-ag(b)}{(b-a)}$, *where* $g(x) = (c_1-x)(c_2-x)(c_3-x)$.

Sol. Since, $f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, then

$$f(x) = \begin{vmatrix} x+c_1 & a-c_1 & 0 \\ x+b & c_2-b & a-c_2 \\ x+b & 0 & c_3-b \end{vmatrix}$$

$$f(x) = x\begin{vmatrix} 1 & a-c_1 & 0 \\ 1 & c_2-b & a-c_2 \\ 1 & 0 & c_3-b \end{vmatrix} + \begin{vmatrix} c_1 & a-c_1 & 0 \\ b & c_2-b & a-c_2 \\ b & 0 & c_3-b \end{vmatrix}$$

So, $f(x)$ is linear.

Let $\quad f(x) = Px + Q$

Then, $\quad f(-a) = -aP + Q,\ f(-b) = -bP + Q$

$$f(0) = 0\cdot P + Q = Q$$

$$= \frac{bf(-a)-af(-b)}{(b-a)} \quad \text{...(ii)}$$

From Eq. (i), we get

$$f(-a) = \begin{vmatrix} c_1-a & 0 & 0 \\ b-a & c_2-a & 0 \\ b-a & b-a & c_3-a \end{vmatrix}$$

$$= (c_1-a)(c_2-a)(c_3-a)$$

Similarly, $f(-b) = (c_1-b)(c_2-b)(c_3-b)$

and $\quad g(x) = (c_1-x)(c_2-x)(c_3-x)$

$$g(a) = f(-a)$$

$$\therefore \quad g(b) = f(-b)$$

Now, from Eq. (ii), we get

$$f(0) = \frac{bg(a)-ag(b)}{(b-a)}$$

• Ex. 42 *If f(x) is a polynomial of degree < 3, prove that*

$$\begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \frac{f(x)}{(x-a)(x-b)(x-c)}.$$

Sol. $$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)} \quad \text{[let]} \quad \text{...(i)}$$

On comparing the various powers of x, we get

$$\Rightarrow \begin{cases} A = \dfrac{f(a)}{(a-b)(a-c)} = -\dfrac{f(a)}{(a-b)(c-a)} \\ B = \dfrac{f(b)}{(b-a)(b-c)} = -\dfrac{f(b)}{(a-b)(b-c)} \\ C = \dfrac{f(c)}{(c-a)(c-b)} = -\dfrac{f(c)}{(b-c)(c-a)} \end{cases}$$

Now, from Eq. (i), we get

$$\frac{f(x)}{(x-a)(x-b)(x-c)}$$

$$= \frac{(c-b)\dfrac{f(a)}{(x-a)} - (c-a)\dfrac{f(b)}{(x-b)} + (b-a)\dfrac{f(c)}{(x-c)}}{(a-b)(b-c)(c-a)}$$

$$= \frac{\begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix}}{\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}}$$

• **Ex. 43** *If* $f(a,b) = \dfrac{f(b)-f(a)}{b-a}$ *and* $f(a,b,c) = \dfrac{f(b,c)-f(a,b)}{(c-a)}$, *prove that*

$$f(a,b,c) = \begin{vmatrix} f(a) & f(b) & f(c) \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

Sol. LHS $= f(a,b,c) = \dfrac{f(b,c)-f(a,b)}{(c-a)}$

$$= \frac{\dfrac{f(c)-f(b)}{(c-b)} - \dfrac{f(b)-f(a)}{(b-a)}}{(c-a)}$$

$$= \frac{(b-a)\{f(c)-f(b)\} - (c-b)\{f(b)-f(a)\}}{(b-a)(c-b)(c-a)}$$

$$= \frac{(f(a)\cdot(c-b) - f(b)\cdot(c-a) + f(c)\cdot(b-a))}{(a-b)(b-c)(c-a)}$$

$$= \frac{f(a)\begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} - f(b)\begin{vmatrix} 1 & 1 \\ a & c \end{vmatrix} + f(c)\begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} f(a) & f(b) & f(c) \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}}$$

$$= \begin{vmatrix} f(a) & f(b) & f(c) \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \text{RHS}$$

• **Ex. 44** *Let S be the sum of all possible determinants of order 2 having 0, 1, 2 and 3 as their elements. Find the common root* α *of the equations*

$$x^2 + ax + [m+1] = 0,$$
$$x^2 + bx + [m+4] = 0$$

and $x^2 - cx + [m+15] = 0$,

such that $\alpha > S$, *where* $a+b+c=0$ *and*

$$m = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} \frac{r}{\sqrt{(n^2+r^2)}}$$

and [.] denotes the greatest integer function.

Sol. Let α be a common root of the given equations, then

$$\alpha^2 + a\alpha + [m+1] = 0$$

$\Rightarrow$ $\quad \alpha^2 + a\alpha + [m] + 1 = 0 \quad$ ...(i)

$$\alpha^2 + b\alpha + [m+4] = 0$$

$\Rightarrow$ $\quad \alpha^2 + b\alpha + [m] + 4 = 0 \quad$ ...(ii)

and $\quad \alpha^2 - c\alpha + [m+15] = 0$

$\Rightarrow$ $\quad \alpha^2 - c\alpha + [m] + 15 = 0 \quad$ ...(iii)

On adding Eqs. (i) and (ii) and subtracting Eq. (iii), we get

$$a^2 + (a+b+c)\alpha + [m] - 10 = 0$$

$$\alpha^2 + 0 + [m] - 10 = 0 \quad [\because a+b+c=0]$$

$\Rightarrow$ $\quad \alpha^2 + [m] - 10 = 0 \quad$ ...(iv)

Also, $\quad m = \lim_{n\to\infty} \dfrac{1}{n}\sum_{r=1}^{2n} \dfrac{r}{\sqrt{n^2+r^2}}$

$$= \lim_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{n}\cdot\frac{r/n}{\sqrt{1+(r/n)^2}} = \int_0^2 \frac{x}{\sqrt{(1+x^2)}}dx$$

$$[\sqrt{(1+x^2)}]_0^2 = \sqrt{5}-1$$

Now, $\quad [m] = [\sqrt{5}-1] = 1$

From Eq. (iv), we get

$$\alpha^2 + 1 - 10 = 0 \Rightarrow \alpha^2 = 9$$

$\therefore$ $\quad \alpha = \pm 3$

Now, number of determinants of order 2 having

$$0, 1, 2, 3 = 4! = 24$$

Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}$ be one such determinant and their exists another determinant.

Let $\Delta_2 = \begin{vmatrix} a_3 & a_4 \\ a_1 & a_2 \end{vmatrix}$ [obtained on interchanging R_1 and R_2]

such that $\Delta_1 + \Delta_2 = 0$

$\because S =$ Sum of all the 24 determinants $= 0$

Since, $\quad \alpha > S \Rightarrow \alpha > 0$

$\therefore$ $\quad \alpha = 3$

• **Ex. 45** *If* a_1, a_2, a_3 *and* $b_1, b_2, b_3 \in R$ *and are such that* $a_ib_j \neq 1$ *for* $1 \le i, j \le 3$,

$$\begin{vmatrix} \dfrac{1-a_1^3b_1^3}{1-a_1b_1} & \dfrac{1-a_1^3b_2^3}{1-a_1b_2} & \dfrac{1-a_1^3b_3^3}{1-a_1b_3} \\ \dfrac{1-a_2^3b_1^3}{1-a_2b_1} & \dfrac{1-a_2^3b_2^3}{1-a_2b_2} & \dfrac{1-a_2^3b_3^3}{1-a_2b_3} \\ \dfrac{1-a_3^3b_1^3}{1-a_3b_1} & \dfrac{1-a_3^3b_2^3}{1-a_3b_2} & \dfrac{1-a_3^3b_3^3}{1-a_3b_3} \end{vmatrix} > 0 \text{ provided either}$$

$a_1 < a_2 < a_3$ *and* $b_1 < b_2 < b_3$ *or* $a_1 > a_2 > a_3$ *and* $b_1 > b_2 > b_3$.

Sol. Since, $\quad \dfrac{x^3-y^3}{x-y} = \dfrac{(x-y)(x^2+xy+y^2)}{(x-y)} = x^2+xy+y^2$

Hence, the given determinant becomes

$$\begin{vmatrix} 1+a_1b_1+a_1^2b_1^2 & 1+a_1b_2+a_1^2b_2^2 & 1+a_1b_3+a_1^2b_3^2 \\ 1+a_2b_1+a_2^2b_1^2 & 1+a_2b_2+a_2^2b_2^2 & 1+a_2b_3+a_2^2b_3^2 \\ 1+a_3b_1+a_3^2b_1^2 & 1+a_3b_2+a_3^2b_2^2 & 1+a_3b_3+a_3^2b_3^2 \end{vmatrix} > 0$$

$$\Rightarrow \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} \times \begin{vmatrix} 1 & b_1 & a_1^2 \\ 1 & b_2 & b_2^2 \\ 1 & b_3 & b_3^2 \end{vmatrix} > 0$$

$$\Rightarrow (a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1) > 0$$

$$\left\{\because \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)\right\}$$

Case I If $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$, then

$$(a_1-a_2) < 0, (a_2-a_3) < 0$$

and $\quad (b_1-b_2) < 0, (b_2-b_3) < 0$

$$(a_1-a_3) < 0$$

and $\quad (b_1-b_3) < 0$

$\therefore$ $\quad (a_3-a_1) > 0$

$\therefore$ $\quad (b_3-b_1) > 0$

Then, $\quad (a_1-a_2)(a_2-a_3)(a_3-a_1) > 0$

and $\quad (b_1-b_2)(b_2-b_3)(b_3-b_1) > 0$

$\therefore$ $\quad (a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1) > 0$

which is true.

Case II If $a_1 > a_2 > a_3$ and $b_1 > b_2 > b_3$

$\therefore$ $\quad a_1-a_2 > 0, a_2-a_3 > 0$

and $\quad b_1-b_2 > 0, b_2-b_3 > 0$

$$a_1-a_3 > 0 \Rightarrow a_3-a_1 < 0$$

and $\quad b_1-b_3 > 0 \Rightarrow b_3-b_1 < 0$

Hence, $\quad (a_1-a_2)(a_2-a_3)(a_3-a_1) < 0$

and $\quad (b_1-b_2)(b_2-b_3)(b_3-b_1) < 0$

$\therefore$ $\quad (a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1) > 0$

which is true.

• **Ex. 46** *Show that a six-digit number abcdef is divisible by 11, if and only if ab + cd + ef is divisible by 11. Hence or otherwise, find one set of values of two-digit numbers x, y and z, so that the value of the determinant* $\begin{vmatrix} x & 23 & 42 \\ 13 & 37 & y \\ 19 & z & 34 \end{vmatrix}$ *is divisible by 99 (without expanding the determinant).*

Sol. Since, $abcdef = ab0000 + cd00 + ef$

$$= (9999+1)\,ab + (99+1)\,cd + ef$$
$$= 9999\,ab + 99\,cd + ab + cd + ef$$

Given, $abcdef$ is divisible by 11, if and only if $ab + cd + ef$ is divisible by 11. Now, let $x = ab$, $y = cd$ and $z = ef$. [each being a two-digit number]

Again, let $\quad \Delta = \begin{vmatrix} x & 23 & 42 \\ 13 & 37 & y \\ 19 & z & 34 \end{vmatrix} = \begin{vmatrix} ab & 23 & 42 \\ 13 & 37 & cd \\ 19 & ef & 34 \end{vmatrix}$

Applying $R_1 \to R_1 + 100R_2 + 10000R_3$, we get

$$\Delta = \begin{vmatrix} 1913ab & ef3723 & 34cd42 \\ 13 & 37 & cd \\ 19 & ef & 34 \end{vmatrix}$$

Now, $1913ab$ is divisible by 11, if and only if $19 + 13 + ab = 32 + ab$ is divisible by $11 \Rightarrow ab = 01, 12, 23, \ldots$

Again, $1913ab$ is divisible by 9, if and only if $1+9+1+3+a+b = 14+a+b$ is divisibe by 9.

The above two conditions are satisfied for $a = 6$, $b = 7$. Thus, $x = 67$. Similarly, $y = 23$ and $z = 39$.

Determinants Exercise 1 :
Single Option Correct Type Questions

- This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$ $= k(1-\alpha)^2 (1-\beta)^2 (\alpha-\beta)^2$, $k^2 d$ is equal to

(a) 1 (b) -1
(c) $\alpha\beta$ (d) $\alpha\beta\gamma$

2. Let $\Delta(x) = \begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$ and $\int_0^2 \Delta(x)\,dx = -16$, where a, b, c and d are in AP, then the common difference of the AP is equal to

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

3. If $\Delta(x) = \begin{vmatrix} x & 1+x^2 & x^3 \\ \log_e(1+x^2) & e^x & \sin x \\ \cos x & \tan x & \sin^2 x \end{vmatrix}$, then

(a) $\Delta(x)$ is divisible by x (b) $\Delta(x) = 0$
(c) $\Delta'(x) = 0$ (d) None of these

4. If a, b and c are sides of a triangle and $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$, then

(a) ΔABC is an equilateral triangle
(b) ΔABC is a right angled isosceles triangle
(c) ΔABC is an isosceles triangle
(d) None of the above

5. If $\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - xf'(x)$, then $f(x)$ is equal to

(a) $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(b) $(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)$
(c) $2(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(d) None of the above

6. If $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = 0$, the line $ax + by + c = 0$ passes through the fixed point which is

(a) $(1, 2)$ (b) $(1, 1)$ (c) $(-2, 1)$ (d) $(1, 0)$

7. If $f(x) = a + bx + cx^2$ and α, β and γ are the roots of the equation $x^3 = 1$, then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is equal to

(a) $f(\alpha) + f(\beta) + f(\gamma)$
(b) $f(\alpha) f(\beta) + f(\beta) f(\gamma) + f(\gamma) f(\alpha)$
(c) $f(\alpha) f(\beta) f(\gamma)$
(d) $-f(\alpha) f(\beta) f(\gamma)$

8. When the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of $\sin x$, the constant term in that expression is

(a) 1 (b) 0 (c) -1 (d) 2

9. If [ ] denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0, 0 \le y < 1, 1 \le z < 2$, the value of the determinant $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is

(a) $[x]$ (b) $[y]$
(c) $[z]$ (d) None of these

10. The determinant $\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix}$ is equal to

(a) $\begin{vmatrix} bx+ay & cx+by \\ b'x+a'y & c'x+b'y \end{vmatrix}$ (b) $\begin{vmatrix} a'x+b'y & bx+cy \\ ax+by & b'x+c'y \end{vmatrix}$

(c) $\begin{vmatrix} bx+cy & ax+by \\ b'x+c'y & a'x+b'y \end{vmatrix}$ (d) $\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$

11. If A, B and C are angles of a triangle, the value of $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iBA} & e^{2iC} \end{vmatrix}$ is (where $i = \sqrt{-1}$)

(a) 1 (b) -1 (c) -2 (d) -4

12. If $\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^a & a \\ x^{n+5} & x^{a+6} & x^{2n+5} \end{vmatrix} = 0, \forall x \in R$, where $n \in N$, the value of a is

(a) n (b) $n-1$
(c) $n+1$ (d) None of these

13. If x, y and z are the integers in AP lying between 1 and 9 and $x51, y41$ and $z31$ are three digits number, the value of $\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ is

(a) $x + y + z$
(b) $x - y + z$
(c) 0
(d) None of the above

14. If $a_1 b_1 c_1, a_2 b_2 c_2$ and $a_3 b_3 c_3$ are three digit even natural numbers and $\Delta = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$, then Δ is

(a) divisible by 2 but not necessarily by 4
(b) divisible by 4 but not necessarily by 8
(c) divisible by 8
(d) None of the above

15. If a, b and c are sides of ΔABC such that
$\begin{vmatrix} c & b\cos B + c\beta & a\cos A + b\alpha + c\gamma \\ a & c\cos B + a\beta & b\cos A + c\alpha + a\gamma \\ b & a\cos B + b\beta & c\cos A + a\alpha + b\gamma \end{vmatrix} = 0$
$\left(\text{where } \alpha, \beta, \gamma \in R^+ \text{ and } \angle A, \angle B, \angle C \ne \frac{\pi}{2}\right)$ ΔABC is

(a) an isosceles (b) an equilateral
(c) can't say (d) None of these

16. If x_1, x_2 and y_1, y_2 are the roots of the equations $3x^2 - 18x + 9 = 0$ and $y^2 - 4y + 2 = 0$, the value of the determinant $\begin{vmatrix} x_1 x_2 & y_1 y_2 & 1 \\ x_1 + x_2 & y_1 + y_2 & 2 \\ \sin(\pi x_1 x_2) & \cos(\pi/2 y_1 y_2) & 1 \end{vmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) None of these

17. If the value of $\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix}$ is equal to zero, then m is equal to

(a) 6 (b) 4
(c) 5 (d) None of these

18. The value of the determinant $\begin{vmatrix} 1 & \sin(\alpha-\beta)\theta & \cos(\alpha-\beta)\theta \\ a & \sin\alpha\theta & \cos\alpha\theta \\ a^2 & \sin(\alpha-\beta)\theta & \cos(\alpha-\beta)\theta \end{vmatrix}$ is independent of

(a) α (b) β
(c) θ (d) a

19. If $f(x), g(x)$ and $h(x)$ are polynomials of degree 4 and $\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ p & q & r \end{vmatrix} = mx^4 + nx^3 + rx^2 + sx + t$ be an identity in x, then $\begin{vmatrix} f'''(0) - f''(0) & g'''(0) - g''(0) & h'''(0) - h''(0) \\ a & b & c \\ p & q & r \end{vmatrix}$ is equal to

(a) $2(3n + r)$ (b) $3(2n - r)$
(c) $3(2n + r)$ (d) $2(3n - r)$

20. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$, then $f(\theta) - 2f(\phi) + f(\psi)$ is equal to

(a) 0 (b) $\alpha - \beta$
(c) $\alpha + \beta + \gamma$ (d) $\alpha + \beta - \gamma$

21. If $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$, where a, b and c are all different, then the determinant $\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$ vanishes when

(a) $a + b + c = 0$ (b) $x = \frac{1}{3}(a + b + c)$
(c) $x = \frac{1}{2}(a + b + c)$ (d) $x = a + b + c$

22. Let $a, b, c \in R$ such that no two of them are equal and satisfy $\begin{vmatrix} 2a & b & c \\ b & c & 2a \\ c & 2a & b \end{vmatrix} = 0$, the equation $24ax^2 + 4bx + c = 0$ has

(a) atleast one root in $\left[0, \frac{1}{2}\right]$
(b) atleast one root in $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(c) atleast one root in $[-1, 0]$
(d) atleast two roots in $[0, 2]$

23. The number of positive integral solution of the equation $\begin{vmatrix} x^3 + 1 & x^2 y & x^2 z \\ xy^2 & y^3 + 1 & y^2 z \\ xz^2 & yz^2 & z^3 + 1 \end{vmatrix} = 11$ is

(a) 0 (b) 3 (c) 6 (d) 12

24. If $f(x)=ax^2+bx+c$, $a,b,c\in R$ and equation $f(x)-x=0$ has imaginary roots $\alpha,\beta\,\gamma$ and δ be the roots of $f(f(x))-x=0$, then $\begin{vmatrix}2 & \alpha & \delta\\ \beta & 0 & \alpha\\ \gamma & \beta & 1\end{vmatrix}$ is

(a) 0 (b) purely real
(c) purely imaginary (d) None of these

25. If the system of equations $2x-y+z=0$, $x-2y+z=0$, $tx-y+2z=0$ has infinitely many solutions and $f(x)$ be a continuous function, such that $f(5+x)+f(x)=2$, then $\int_0^{-2t} f(x)\,dx$ is equal to

(a) 0 (b) $-2t$ (c) 5 (d) t

26. If $(1+ax+bx^2)^4=a_0+a_1x+a_2x^2+\ldots+a_8x^8$, where $a,b,a_0,a_1,\ldots,a_8\in R$ such that $a_0+a_1+a_2\neq 0$ and $\begin{vmatrix}a_0 & a_1 & a_2\\ a_1 & a_2 & a_0\\ a_2 & a_0 & a_1\end{vmatrix}=0$, then

(a) $a=\dfrac{3}{4},b=\dfrac{5}{8}$ (b) $a=\dfrac{1}{4},b=\dfrac{5}{32}$
(c) $a=1,b=\dfrac{2}{3}$ (d) None of these

27. Given, $f(x)=\log_{10}x$ and $g(x)=e^{\pi ix}$.
If $\phi(x)=\begin{vmatrix}f(x)\cdot g(x) & (f(x))^{g(x)} & 1\\ f(x^2)\cdot g(x^2) & (f(x^2))^{g(x^2)} & 0\\ f(x^3)\cdot g(x^3) & (f(x^3))^{g(x^3)} & 1\end{vmatrix}$, the value of $\phi(10)$, is

(a) 1 (b) 2 (c) 0 (d) None of these

28. The value of the determinant
$\begin{vmatrix}1 & (\alpha^{2x}-\alpha^{-2x})^2 & (\alpha^{2x}+\alpha^{-2x})^2\\ 1 & (\beta^{2x}-\beta^{-2x})^2 & (\beta^{2x}+\beta^{-2x})^2\\ 1 & (\gamma^{2x}-\gamma^{-2x})^2 & (\gamma^{2x}+\gamma^{-2x})^2\end{vmatrix}$, is

(a) 0 (b) $(\alpha\beta\gamma)^{2x}$ (c) $(\alpha\beta\gamma)^{-2x}$ (d) None of these

29. If a,b and c are non-zero real numbers and if the equations $(a-1)x=y+z$, $(b-1)y=z+x$, $(c-1)z=x+y$ has a non-trivial solution, then $ab+bc+ca$ equals to

(a) $a+b+c$ (b) abc (c) 1 (d) None of these

30. The set of equations $\lambda x-y+(\cos\theta)z=0$, $3x+y+2z=0$, $(\cos\theta)x+y+2z=0$, $0\le\theta\le 2\pi$, has non-trivial solution(s)

(a) for no value of λ and θ
(b) for all value of λ and θ
(c) for all values of λ and only two values of θ
(d) for only one value of λ and all values of θ

Determinants Exercise 2 :
More than One Correct Option Type Questions

- This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. The determinant $\Delta=\begin{vmatrix}a^2+x^2 & ab & ac\\ ab & b^2+x^2 & bc\\ ac & bc & c^2+x^2\end{vmatrix}$ is divisible by

(a) x (b) x^2 (c) x^3 (d) x^4

32. The value of the determinant
$\begin{vmatrix}\sqrt{6} & 2i & 3+\sqrt{6}\\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i\\ \sqrt{18} & \sqrt{2}+\sqrt{12}i & \sqrt{27}+2i\end{vmatrix}$, is (where $i=\sqrt{-1}$)

(a) complex (b) real (c) irrational (d) rational

33. If $D_k=\begin{vmatrix}2^{k-1} & \dfrac{1}{k(k+1)} & \sin k\theta\\ x & y & z\\ 2^n-1 & \dfrac{n}{n+1} & \dfrac{\sin\left(\dfrac{n+1}{2}\right)\theta\sin\dfrac{n}{2}\theta}{\sin\dfrac{\theta}{2}}\end{vmatrix}$, then $\sum_{k=1}^{n} D_k$ is equal to

(a) 0 (b) independent of n
(c) independent of θ (d) independent of x, y and z

34. The determinant $\begin{vmatrix}a & b & a\alpha+b\\ b & c & b\alpha+c\\ a\alpha+b & b\alpha+c & 0\end{vmatrix}$ is equal to zero, if

(a) a, b and c are in AP
(b) a, b, c, are in GP
(c) a, b and c are in HP
(d) $(x-\alpha)$ is a factor of $ax^2+2bx+c$

35. Let $f(x)=\begin{vmatrix}2\cos x & 1 & 0\\ 1 & 2\cos x & 1\\ 0 & 1 & 2\cos x\end{vmatrix}$, then

(a) $f\left(\dfrac{\pi}{3}\right)=-1$
(b) $f'\left(\dfrac{\pi}{3}\right)=\sqrt{3}$
(c) $\int_0^{\pi} f(x)dx=0$
(d) $\int_{-\pi}^{\pi} f(x)\,dx=0$

36. If $\Delta(x)=\begin{vmatrix}x^2-5x+3 & 2x-5 & 3\\ 3x^2+x+4 & 6x+1 & 9\\ 7x^2-6x+9 & 14x-6 & 21\end{vmatrix}$ $=ax^3+bx^2+cx+d$, then

(a) $a=0$ (b) $b=0$ (c) $c=0$ (d) $d=47$

37. If a, b and c are the sides of a triangle and A, B and C are the angles opposite to a, b and c respectively, then
$\Delta=\begin{vmatrix}a^2 & b\sin A & C\sin A\\ b\sin A & 1 & \cos A\\ C\sin A & \cos A & 1\end{vmatrix}$ is independent of

(a) a (b) b (c) c (d) A, B, C

38. Let $f(a,b)=\begin{vmatrix}a & a^2 & 0\\ 1 & (2a+b) & (a+b)^2\\ 0 & 1 & (2a+3b)\end{vmatrix}$, then

(a) $(a+b)$ is a factor of $f(a,b)$
(b) $(a+2b)$ is a factor of $f(a,b)$
(c) $(2a+b)$ is a factor of $f(a,b)$
(d) a is a factor of $f(a,b)$

39. If $f(x)=\begin{vmatrix}\sec^2x & 1 & 1\\ \cos^2x & \cos^2x & \operatorname{cosec}^2x\\ 1 & \cos^2x & \cot^2x\end{vmatrix}$, then

(a) $\int_{-\pi/4}^{\pi/4} f(x)\,dx=\dfrac{1}{16}(3\pi+8)$
(b) $f'\left(\dfrac{\pi}{2}\right)=0$
(c) maximum value of $f(x)$ is 1
(d) minimum value of $f(x)$ is 0

40. If $\begin{vmatrix}a & a+x^2 & a+x^2+x^4\\ 2a & 3a+2x^2 & 4a+3x^2+2x^4\\ 3a & 6a+3x^2 & 10a+6x^2+3x^4\end{vmatrix}$ $=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$ and $f(x)=+a_0x^2+a_3x+a_6$, then

(a) $f(x)\ge 0$, $\forall\, x\in R$ if $a>0$
(b) $f(x)=0$, only if $a=0$
(c) $f(x)=0$, has two equal roots
(d) $f(x)=0$, has more than two root if $a=0$

41. If $\Delta(x)=\begin{vmatrix}4x-4 & (x-2)^2 & x^3\\ 8x-4\sqrt{2} & (x-2\sqrt{2})^2 & (x+1)^3\\ 12x-4\sqrt{3} & (x-2\sqrt{3})^2 & (x-1)^3\end{vmatrix}$, then

(a) term independent of x in $\Delta(x)=16(5-\sqrt{2}-\sqrt{3})$
(b) coefficient of x in $\Delta(x)=48(1+\sqrt{2}-\sqrt{3})$
(c) coefficient of x in $\Delta(x)=16(5+\sqrt{2}-\sqrt{3})$
(d) coefficient of x in $\Delta(x)$ is divisible by 16

42. If
$f(x)=\begin{vmatrix}3 & 3x & 3x^2+2a^2\\ 3x & 3x^2+2a^2 & 3x^3+6a^2x\\ 3x^2+2a^3 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^4\end{vmatrix}$,
then

(a) $f'(x)=0$
(b) $y=f(x)$ is a straight line parallel to X-axis
(c) $\int_0^2 f(x)\,dx=32a^4$
(d) None of the above

43. If $a>b>c$ and the system of equations $ax+by+cz=0$, $bx+cy+az=0$, $cx+ay+bz=0$ has a non-trivial solution, then both the roots of the quadratic equation at^2+bt+c, are

(a) real
(b) of opposite sign
(c) positive
(d) complex

44. The values of λ and b for which the equations $x+y+z=3$, $x+3y+2z=6$ and $x+\lambda y+3z=b$ have

(a) a unique solution, if $\lambda\neq 5$, $b\in R$
(b) no solution, if $\lambda\neq 5$, $b=9$
(c) infinite many solution, $\lambda=5$, $b=9$
(d) None of the above

45. Let λ and α be real. Let S denote the set of all values of λ for which the system of linear equations
$\lambda x+(\sin\alpha)y+(\cos\alpha)z=0$
$x+(\cos\alpha)y+(\sin\alpha)z=0$
$-x+(\sin\alpha)y-(\cos\alpha)z=0$
has a non-trivial solution, then S contains

(a) $(-1,1)$ (b) $[-\sqrt{2},-1]$
(c) $[1,\sqrt{2}]$ (d) $(-2,2)$

Determinants Exercise 3 : Passage Based Questions

- This section contains **7 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Q. Nos. 46 to 48)

Consider the system of equations

$$x+y+z=5;\ x+2y+3z=9;\ x+3y+\lambda z=\mu$$

The system is called smart, brilliant, good and lazy according as it has solution, unique solution, infinitely many solutions and no solution, respectively.

46. The system is smart, if
(a) $\lambda \neq 5$ or $\lambda = 5$ and $\mu = 13$ (b) $\lambda \neq 5$ and $\mu = 13$
(c) $\lambda \neq 5$ and $\mu \neq 13$ (d) $\lambda \neq 5$ or $\lambda = 5$ and $\mu \neq 13$

47. The system is good, if
(a) $\lambda \neq 5$ or $\lambda = 5$ and $\mu = 13$ (b) $\lambda = 5$ and $\mu = 13$
(c) $\lambda = 5$ and $\mu \neq 13$
(d) $\lambda \neq 5$, μ is any real number

48. The system is lazy, if
(a) $\lambda \neq 5$ or $\lambda = 5$ and $\mu = 13$ (b) $\lambda = 5$ and $\mu = 13$
(c) $\lambda = 5$ and $\mu \neq 13$ (d) $\lambda \neq 5$ or $\lambda = 5$ and $\mu \neq 13$

Passage II
(Q. Nos. 49 to 51)

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ *and* $C_{ij} = (-1)^{i+j} M_{ij}$, *where* M_{ij} *is a determinant obtained by deleting ith row and jth column, then* $\begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$.

49. If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 5$ and $\Delta = \begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$, then sum of digits of Δ^2, is
(a) 7 (b) 8 (c) 13 (d) 11

50. Suppose $a, b, c \in R$, $a+b+c>0$, $A = bc - a^2$, $B = ca - b^2$ and $C = ab - c^2$ and $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = 49$, then the value of $a^3+b^3+c^3-3abc$, is
(a) −7 (b) 7 (c) −2401 (d) 2401

51. If $a^3+b^3+c^3-3abc=-3$ and $A = bc-a^2$, $B = ca-b^2$ and $C = ab-c^2$, then the value of $aA+bB+cC$, is
(a) −3 (b) 3 (c) −9 (d) 9

Passage III
(Q. Nos. 52 to 54)

If α, β, γ *are the roots of* $x^3+2x^2-x-3=0$

52. The value of $\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix}$ is equal to
(a) 14 (b) −2 (c) 10 (d) 14

53. If the absolute value of the expression $\frac{\alpha-1}{\alpha+2}+\frac{\beta-1}{\beta+2}+\frac{\gamma-1}{\gamma+2}$ can be expressed as $\frac{m}{n}$, where m and n are co-prime, the value of $\begin{vmatrix} m & n^2 \\ m-n & m+n \end{vmatrix}$, is
(a) 17 (b) 27 (c) 37 (d) 47

54. If $a=\alpha^2+\beta^2+\gamma^2$, $b=\alpha\beta+\beta\gamma+\gamma\alpha$, the value of $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$, is
(a) 14 (b) 49 (c) 98 (d) 196

Passage IV
(Q. Nos. 55 to 57)

Suppose $f(x)$ *is a function satisfying the following conditions:*

(i) $f(0)=2$, $f(1)=1$

(ii) $f(x)$ *has a minimum value at* $x=\frac{5}{2}$

(iii) *For all* x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

55. The value of $f(2)+f(3)$ is
(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) $\frac{5}{2}$

56. The number of solutions of the equation $f(x)+1=0$ is
(a) 0 (b) 1 (c) 2 (d) infinite

57. Range of $f(x)$ is
(a) $\left(-\infty, \frac{7}{16}\right]$ (b) $\left[\frac{3}{4}, \infty\right)$ (c) $\left[\frac{7}{16}, \infty\right)$ (d) $\left(-\infty, \frac{3}{4}\right]$

Passage V
(Q. Nos. 58 to 60)

If $\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x-\ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} = a_0 + a_1(x-1) + a_2(x-1)^2 + \dots$

58. The value of $\cos^{-1}(a_1)$, is
(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

59. The value of $\lim_{x \to a_0} (\sin x)^x$ is
(a) 1 (b) e (c) $e-1$ (d) None of these

60. The equation whose roots are a_0 and a_1, is
(a) $x^2-x=0$ (b) $x^2-2x=0$
(c) $x^2-3x=0$ (d) None of these

Passage VI
(Q. Nos. 61 to 63)

Let $\Delta = \begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$ *and the equation* $x^3-px^2+qx-r=0$ *has roots* a, b, c, *where* $a, b, c \notin R^+$.

61. The value of Δ is
(a) $\leq 9r^3$ (b) $\geq 27r^2$ (c) $\leq 27r^2$ (d) $\geq 81r^3$

62. If a, b, c are in GP, then
(a) $r^3 = p^3q$ (b) $p^3 = r^3q$ (c) $p^3 = q^3r$ (d) $q^3 = p^3r$

63. If $\Delta = 27$ and $a^2+b^2+c^2=2$, then the value of $\sum a^2b$, is
(a) $3(2\sqrt{2}-p)$ (b) $3(2\sqrt{2}-r)$
(c) $3(2\sqrt{2}-q)$ (d) $3(2\sqrt{2}-p-q)$

Passage VII
(Q. Nos. 64 to 66)

If $\Delta n = \begin{vmatrix} a^2+n & ab & ac \\ ab & b^2+n & bc \\ ac & bc & c^2+n \end{vmatrix}$, $n \in N$ *and the equation* $x^3-\lambda x^2+11x-6=0$ *has roots* a, b, c *and* a, b, c *are in AP.*

64. The value of $\sum_{r=1}^{7} \Delta_n$ is
(a) $(12)^3$ (b) $(14)^3$ (c) $(26)^3$ (d) $(28)^3$

65. The value of $\frac{\Delta 2_n}{\Delta_n}$ is
(a) < 8 (b) = 8
(c) > 8 (d) None of these

66. The value of $\sum_{r=1}^{30} \left(\frac{27\Delta_r - \Delta 3_r}{27r^2}\right)$ is
(a) 130 (b) 190 (c) 280 (d) 340

Determinants Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

67. If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, the value of $\sqrt{2^k \sqrt{2^k \sqrt{2^k \dots \infty}}}$ is

68. Let α, β and γ are three distinct roots of $\begin{vmatrix} x-1 & -6 & 2 \\ -6 & x-2 & -4 \\ 2 & -4 & x-6 \end{vmatrix} = 0$, the value of $\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^{-1}$ is

69. If $\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x-\ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} = \sum_{r=0}^{n} a_r (x-1)^r$, the value of $(2^{a_0}+3^{a_1})^{a_1+1}$ is

70. If $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma$ is equal to

71. Let $f(a,b,c) = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$, the greatest integer $n \in N$ such that $(a+b+c)^n$ divides $f(a,b,c)$ is

72. If $0 \leq \theta \leq \pi$ and the system of equations

$$x = (\sin\theta)\,y + (\cos\theta)\,z$$
$$y = z + (\cos\theta)\,x$$
$$z = (\sin\theta)\,x + y$$

has a non-trivial solution, then $\frac{8\theta}{\pi}$ is equal to

73. The value of the determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$ is

74. If a, b, c and d are the roots of the equation $x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$, the value of the determinant $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$ is

75. If $a \neq 0, b \neq 0, c \neq 0$ and $\begin{vmatrix} 1+a & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix} = 0$, the value of $|a^{-1} + b^{-1} + c^{-1}|$ is equal to

76. If the system of equations

$$ax + hy + g = 0;\ hx + by + f = 0$$

and $ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0$ has a unique solution and

$$\frac{abc + 2fgh - af^2 - bg^2 - ch^2}{h^2 - ab} = 8,$$ the value of λ is

Determinants Exercise 5 : Matching Type Questions

- This section contains **5 questions**. Questions 77 to 81 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

77.

	Column I		Column II
(A)	If a,b,c are three complex numbers such that $a^2+b^2+c^2=0$ and $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \lambda a^2b^2c^2$, then λ is divisible by	(p)	2
(B)	If $a,b,c \in R$ and $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 5a+2b & 7a+5b+2c \\ 3a & 7a+3b & 9a+7b+3c \end{vmatrix} = -1024$, then a is divisible by	(q)	3
(C)	Let $\Delta(x) = \begin{vmatrix} x-1 & 2x^2-5 & x^3-1 \\ 2x^2+5 & 2x+2 & x^3+3 \\ x^3-1 & x+1 & 3x^2-2 \end{vmatrix}$ and $ax+b$ be the remainder, when $\Delta(x)$ is divided by x^2-1, then $4a+2b$ is divisible by	(r)	4
		(s)	5
		(t)	6

78.

	Column I		Column II
(A)	Let $f_1(x) = x + a_1$, $f_2(x) = x^2 + b_1x + b_2$, $x_1 = 2$, $x_2 = 3$ and $x_3 = 5$ and $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ f_1(x_1) & f_1(x_2) & f_1(x_3) \\ f_2(x_1) & f_2(x_2) & f_2(x_3) \end{vmatrix}$, then Δ is	(p)	Even number
(B)	If $\vert a_1 - b_1 \vert = 6$ and $f(x) = \begin{vmatrix} 1 & b_1 & a_1 \\ 1 & b_1 & 2a_1 - x \\ 1 & 2b_1 - x & a_1 \end{vmatrix}$, then the minimum value of $f(x)$ is	(q)	Prime number
(C)	If coefficient of x in $f(x) = \begin{vmatrix} x-2 & (x-1)^2 & x^3 \\ x-1 & x^2 & (x+1)^3 \\ x & (x+1)^2 & (x+2)^3 \end{vmatrix}$ is λ, then $\vert\lambda\vert$ is	(r)	Odd number
		(s)	Composite number
		(t)	Perfect number

79.

	Column I		Column II
(A)	If $\begin{vmatrix} x^2+3x & x-1 & x+3 \\ x^2+1 & 2+3x & x-3 \\ x^2-3 & x+4 & 3x \end{vmatrix} = ax^4+bx^3+cx^2+dx+e$, then $e+a$ is divisible by	(p)	2
(B)	If $\begin{vmatrix} x-1 & 5x & 7 \\ x^2-1 & x-1 & 8 \\ 2x & 3x & 0 \end{vmatrix} = ax^3+bx^2+cx+d$, then $(e+a-3)$ is divisible by	(q)	3
(C)	If $\begin{vmatrix} x^3+4x & x+3 & x-2 \\ x-2 & 5x & x-1 \\ x-3 & x+2 & 4x \end{vmatrix} = ax^5+bx^4+cx^3+dx^2+ex+f$, then $(f+e)$ is divisible by	(r)	5
		(s)	6
		(t)	7

80.

	Column I		Column II
(A)	If $a^2+b^2+c^2=1$ and $\Delta = \begin{vmatrix} a^2+(b^2+c^2)d & ab(1-d) & ca(1-d) \\ ab(1-d) & b^2+(c^2+a^2)d & bc(1-d) \\ ca(1-d) & bc(1-d) & c^2+(a+b^2)d \end{vmatrix}$, then Δ is	(p)	independent of a
(B)	If $\Delta = \begin{vmatrix} \frac{1}{c} & \frac{1}{c} & \frac{-(a+b)}{c^2} \\ \frac{-(b+c)}{a^2} & \frac{1}{a} & \frac{1}{a} \\ \frac{-bd(b+c)}{a^2c} & \frac{(ad+2bd+cd)}{ac} & \frac{-(a+b)bd}{ac^2} \end{vmatrix}$, then Δ is	(q)	independent of b
(C)	If $\Delta = \begin{vmatrix} \sin a & \cos a & \sin(a+d) \\ \sin b & \cos b & \sin(b+d) \\ \sin c & \cos c & \sin(c+d) \end{vmatrix}$, then Δ is	(r)	independent of c
		(s)	independent of d
		(t)	zero

81.

	Column I		Column II
(A)	If n be the number of distinct values of 2×2 determinant whose entries are from the set $\{-1, 0, 1\}$, then $(n-1)^2$ is divisible by	(p)	2
(B)	If n be the number of 2×2 determinants with non-negative values whose entries from the set $\{0, 1\}$, then $(n-1)$ is divisible by	(q)	3
(C)	If n be the number of 2×2 determinants with negative values whose entries from the set $\{-1, 1\}$, then $n(n+1)$ is divisible by	(r)	4
		(s)	5
		(t)	6

Determinants Exercise 6 : Statement I and II Type Questions

- **Directions** (Q. Nos. 82 to 87) are Assertion-Reason type questions. Each of these questions contains two statements:

Statement-1 (Assertion) and **Statement-2** (Reason) Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

82. **Statement-1** If $\Delta(r) = \begin{vmatrix} r & r+1 \\ r+3 & r+4 \end{vmatrix}$ then $\sum_{r=1}^{n} \Delta(r) = -3n$

Statement-2 If $\Delta(r) = \begin{vmatrix} f_1(r) & f_2(r) \\ f_3(r) & f_4(r) \end{vmatrix}$

then $\sum_{r=1}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) \\ \sum_{r=1}^{n} f_3(r) & \sum_{r=1}^{n} f_4(r) \end{vmatrix}$

83. Consider the determinant

$$\Delta = \begin{vmatrix} a_1 + b_1 x^2 & a_1 x^2 + b_1 & c_1 \\ a_2 + b_2 x^2 & a_2 x^2 + b_2 & c_2 \\ a_3 + b_3 x^2 & a_3 x^2 + b_3 & c_3 \end{vmatrix} = 0,$$

where $a_i, b_i, c_i \in R (i = 1, 2, 3)$ and $x \in R$.

Statement-1 The value of x satisfying $\Delta = 0$ are $x = 1, -1$

Statement-2 If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then $\Delta = 0$.

84. **Statement-1** The value of determinant

$$\begin{vmatrix} \sin \pi & \cos\left(x + \frac{\pi}{4}\right) & \tan\left(x - \frac{\pi}{4}\right) \\ \sin\left(x - \frac{\pi}{4}\right) & -\cos\left(\frac{\pi}{2}\right) & \ln\left(\frac{x}{y}\right) \\ \cot\left(\frac{\pi}{4} + x\right) & \ln\left(\frac{y}{x}\right) & \tan \pi \end{vmatrix}$$ is zero.

Statement-2 The value of skew-symmetric determinant of odd order equals zero.

85. **Statement-1** $f(x) = \begin{vmatrix} (1+x)^{11} & (1+x)^{12} & (1+x)^{13} \\ (1+x)^{21} & (1+x)^{22} & (1+x)^{23} \\ (1+x)^{31} & (1+x)^{32} & (1+x)^{33} \end{vmatrix}$,

the coefficient of x in $f(x) = 0$

Statement-2 If $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$, then $a_1 = P'(0)$, where dash denotes the differential coefficient.

86. **Statement-1** If system of equations $2x + 3y = a$ and $bx + 4y = 5$ has infinite solution,

then $a = \frac{15}{4}, b = \frac{8}{5}$

Statement-2 Straight lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are parallel,

if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

87. **Statement-1** The value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} \neq 0$

Statement-2 Neither of two rows or columns of $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}$ is identical.

88. **Statement-1** The digits A, B and C re such that the three digit numbers $A88$, $6B8$, $86C$ are divisible by 72, then the determinant $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ is divisible by 288.

Statement-2 $A = B = ?$

Determinants Exercise 8 :
Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

109. If $a^2+b^2+c^2=-2$ and

$$f(x)=\begin{vmatrix}1+a^2x & (1+b^2)x & (1+c^2)x\\ (1+a^2)x & 1+b^2x & (1+c^2)x\\ (1+a^2)x & (1+b^2)x & 1+c^2x\end{vmatrix}$$, then $f(x)$ is a polynomial of degree **[AIEEE 2005, 3M]**

(a) 3 (b) 2 (c) 1 (d) 0

110. The system of equations

$$\alpha x+y+z=\alpha-1,$$
$$x+\alpha y+z=\alpha-1$$
and $$x+y+\alpha z=\alpha-1$$

has no solution, if α is **[AIEEE 2005, 3M]**

(a) not -2 (b) 1
(c) -2 (d) Either -2 or 1

111. If $a_1, a_2, a_3, \dots, a_n, \dots$ are in GP, then the determinant

$$\Delta=\begin{vmatrix}\log a_n & \log a_{n+1} & \log a_{n+2}\\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5}\\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8}\end{vmatrix}$$ is equal to **[AIEEE 2005, 3M]**

(a) 1 (b) 0 (c) 4 (d) 2

112. If $D=\begin{vmatrix}1 & 1 & 1\\ 1 & 1+x & 1\\ 1 & 1 & 1+y\end{vmatrix}$ for $x\neq 0,\ y\neq 0$, then D is **[AIEEE 2007, 3M]**

(a) divisible by neither x nor y
(b) divisible by both x and y
(c) divisible by x but not y
(d) divisible by y but not x

113. Consider the system of equations

$$x-2y+3z=-1$$
$$-x+y-2z=k$$
$$x-3y+4z=1$$

Statement-1 The system of equations has no solutions for $k\neq 3$. **[IIT-JEE 2008, 3M]**

and

Statement-2 The determinant $\begin{vmatrix}1 & 3 & -1\\ -1 & -2 & k\\ 1 & 4 & 1\end{vmatrix}\neq 0$, for $k\neq 3$.

(a) Statement-1 is true, Statement-2 is true and Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true and Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

114. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that $x=cy+bz,\ y=az+cx$ and $z=bx+ay$. Then, $a^2+b^2+c^2+2abc$ is equal to **[AIEEE 2008, 3M]**

(a) -1 (b) 0 (c) 1 (d) 2

115. Let a, b, c be such that $b(a+c)\neq 0$. If

$$\begin{vmatrix}a & a+1 & a-1\\ -b & b+1 & b-1\\ c & c-1 & c+1\end{vmatrix}+\begin{vmatrix}a+1 & b+1 & c-1\\ a-1 & b-1 & c+1\\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c\end{vmatrix}=0,$$

then the value of n is **[AIEEE 2009, 4M]**

(a) any integer (b) zero
(c) an even integer (d) any odd integer

116. If $f(\theta)=\begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta):0\le\theta<\frac{\pi}{2}\right\}$ is [IIT-JEE 2011, 2M]

(a) $(-\infty,-1)\cup(1,\infty)$ (b) $[2,\infty)$
(c) $(-\infty,0]\cup[2,\infty)$ (d) $(-\infty,-1]\cup[1,\infty)$

117. The number of values of k for which the linear equations

$$4x+ky+2z=0$$
$$kx+4y+z=0$$
$$2x+2y+z=0$$

Possess a non-zero solution is [AIEEE 2011,4M]

(a) zero (b) 3 (c) 2 (d) 1

118. If the trivial solution is the only solution of the system of equations

$$x-ky+z=0$$
$$kx+3y-kz=0$$
$$3x+y-z=0$$

Then, the set of values of k is

(a) $\{2,-3\}$ (b) $R-\{2,-3\}$
(c) $R-\{2\}$ (d) $R-\{-3\}$ [AIEEE 2011, 4M]

119. The number of values of k for which the system of equations $(k+1)x+8y=4k;\ kx+(k+3)y=3k-1$ has no solution, is

(a) 1 (b) 2
(c) 3 (d) infinite [JEE Main 2013, 4M]

120. If $\alpha,\beta\ne0$ and $f(n)=\alpha^n+\beta^n$ and by putting values of α, β

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}=k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2,$$

then k is equal to [JEE Main 2014, 4M]

(a) 1 (b) -1
(c) $\alpha\beta$ (d) $1/\alpha\beta$

121. The set of all values of λ for which the system of linear equations

$$2x_1-2x_2+x_3=\lambda x_1$$
$$2x_1-3x_2+2x_3=\lambda x_2$$
$$-x_1+2x_2=\lambda x_3$$

has a non-trivial solution [JEE Main 2015, 4M]

(a) contains two elements
(b) contains more than two elements
(c) is an empty set
(d) is a singleton

122. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix}=-648\alpha?$$

[JEE Advanced 2015, 4M]

(a) -4 (b) 9
(c) -9 (d) 4

123. The system of linear equations

$$x+\lambda y-z=0$$
$$\lambda x-y-z=0$$
$$x+y-\lambda z=0$$

has a non-trivial solution for [JEE Main 2016, 4M]

(a) exactly one-value of λ
(b) exactly two values of λ
(c) exactly three values of λ
(d) infinitely many values of λ.

124. The total number of distinct $x\in R$ for which

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix}=10 \text{ is}$$

[JEE Advanced 2016, 3M]

125. Let $a,\lambda,\mu\in R$. Consider the system of linear equations

$$ax+2y=\lambda$$
$$3x-2y=\mu$$

Which of the following statement(s) is (are) correct? [JEE Advanced 2016, 4M]

(a) If $a=-3$, then the system has infinitely many solutions for all values of λ and μ
(b) If $a\ne-3$, then the system has a unique solution for all values of λ and μ
(c) If $\lambda+\mu=0$, then the system has infinitely many solutions for $a=-3$
(d) If $\lambda+\mu\ne0$, then the system has no solution for $a=-3$

126. If S is the set of distinct values of 'b' for which the following system of linear equations

$$x+y+z=1$$
$$x+ay+z=1$$
$$ax+by+z=0$$

has no solution, then S is [JEE Main 2017, 4M]

(a) an infinite set
(b) a finite set containing two or more elements
(c) a singleton
(d) an empty set

Answers

Exercise for Session 1

1. (d) **2.** (d) **3.** (c) **4.** (b) **5.** (c) **6.** (d)
7. (b)

Exercise for Session 2

1. (c) **2.** (d) **3.** (a) **4.** (c) **5.** (a) **6.** (b)
7. (b) **8.** (b) **9.** (d) **10.** (c) **11.** (d)

Exercise for Session 3

1. (b) **2.** (c) **3.** (c) **4.** (b) **5.** (b) **6.** (d)
7. (d) **8.** (a) **9.** (b) **10.** (c) **11.** (c) **12.** (a)
13. (a) **14.** (a)

Exercise for Session 4

1. (c) **2.** (b) **3.** (b) **4.** (d) **5.** (d) **6.** (b)
7. (b) **8.** (c) **9.** (d) **10.** (a)

Chapter Exercises

1. (a) **2.** (b) **3.** (a) **4.** (c) **5.** (a) **6.** (b)
7. (d) **8.** (c) **9.** (c) **10.** (d) **11.** (d) **12.** (c)
13. (c) **14.** (a) **15.** (b) **16.** (a) **17.** (c) **18.** (a)
19. (d) **20.** (a) **21.** (b) **22.** (a) **23.** (b) **24.** (b)
25. (b) **26.** (b) **27.** (c) **28.** (a) **29.** (b) **30.** (a)
31. (a, b, c, d) **32.** (b, d) **33.** (a, b, c, d) **34.**(b, d)
35. (a, c, d) **36.** (a, b, c) **37.** (a, b, c, d) **38.** (a, b, d)
39. (a, b, c, d) **40.** (a, c, d) **41.** (a, b) **42.** (a, b)
43. (a, b) **44.** (a, c) **45.** (a, b, c) **46.** (a)
47. (b) **48.** (c) **49.** (c) **50.** (b) **51.** (b)
52. (c) **53.** (c) **54.** (d) **55.** (a) **56.** (a) **57.** (c)
58. (c) **59.** (a) **60.** (d) **61.** (b) **62.** (d) **63.** (b)
64. (b) **65.** (a) **66.** (c) **67.** (2) **68.** (9) **69.** (2)
70. (1) **71.** (3) **72.** (6) **73.** (1) **74.** (8) **75.** (3)
76.(8) **77.** (A) $\to$ (p, r); (B) $\to$ (p, r); (C) $\to$ (p, q, s, t)
78. (A) $\to$ (p, s, t); (B) $\to$ (r, t); (C) $\to$ (p, q)
79. (A) $\to$ (r); (B) $\to$ (r, t); (C) $\to$ (p, q, s)
80. (A) $\to$ (p, q, r); (B) $\to$ (p, q, r, s, t); (C) $\to$ (p, q, r, s, t)
81. (A) $\to$ (p, r); (B) $\to$ (p, q, r, t); (C) $\to$ (p, r, s)
82. (c) **83.** (b) **84.** (a) **85.** (a) **86.** (b) **87.** (b)
88. (c) **91.** $15\sqrt{2} - 25\sqrt{3}$
92. 0 **99.** $\frac{1}{12}xyz(x-y)(y-z)(z-x)$
100. (i) 6 (ii) 0 **101.** 0
105. $-a^2(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)$
106. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$
107. (i) $\frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2} + 1 = 0$ (ii) $a^3 + b^3 + c^3 = 5abc$
109. (b) **110.** (c) **111.** (b) **112.** (b) **113.** (a) **114.** (c)
115. (d) **116.** (b) **117.** (c) **118.** (b) **119.** (a) **120.** (a)
121. (a) **122.** (b, c) **123.** (c) **124.** (2) **125.** (b,c,d)
126. (c)