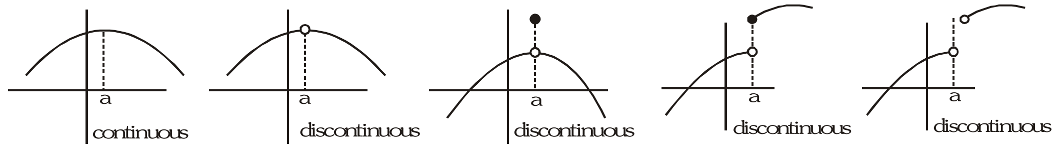
CONTINUITY

1. CONTINUOUS FUNCTIONS :

A function for which a small change in the independent variable causes only a small change and not a sudden jump in the dependent variable are called continuous functions. Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function around that point without lifting the pen from the plane of the paper.



Continuity of a function at a point :

A function f(x) is said to be continuous at x = a, if $\lim\limits_{x\to a} f(x) = f(a)$. Symbolically f is continuous at x = a if $\lim\limits_{h\to 0} f(a-h) = \lim\limits_{h\to 0} f(a+h) = f(a)$, h > 0

i.e. ($LHL_{x=a} = RHL_{x=a}$) equals value of 'f' at x = a. It should be noted that continuity of a function at x = a can be discussed only if the **function is defined in the immediate neighbourhood of x = a, not necessarily at x = a.**

Ex. Continuity at x = 0 for the curve can not be discussed.

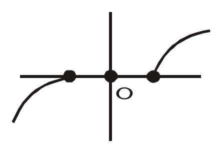


Illustration 1 : If $f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, & x < 1 \\ [x] & x \ge 1 \end{cases}$ then find whether f(x) is continuous or not at x = 1, where [ ] denotes greatest integer function.

Solution :

$$f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, & x < 1 \\ [x] & , x \ge 1 \end{cases}$$

For continuity at x = 1, we determine, f(1), $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$.

Now, f(1) = [1] = 1

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin\frac{\pi x}{2} = \sin\frac{\pi}{2} = 1 \text{ and } \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} [x] = 1$$

so $\quad f(1) = \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

$\therefore \quad$ f(x) is continuous at x = 1

Illustration 2 : Consider $f(x) = \begin{cases} \dfrac{8^x - 4^x - 2^x + 1}{x^2}, & x > 0 \\ e^x \sin x + \pi x + k\ell n4, & x < 0 \end{cases}$ Define the function at x = 0 if possible, so that f(x) becomes continuous at x = 0.

Solution :

$$f(0^+) = \lim_{h\to 0} \frac{8^h - 4^h - 2^h + 1}{h^2} = \lim_{h\to 0} \frac{4^h(2^h - 1) - (2^h - 1)}{h^2}$$

$$= \lim_{h\to 0} \frac{(4^h - 1)}{h}\frac{(2^h - 1)}{h} = \ell n4 \,.\, \ell n2$$

$$f(0^-) = \lim_{x\to 0^-} \left(e^x \sin x + \pi x + k\ell n4\right) = k\ell n4$$

f(x) is continuous at x = 0,

$\Rightarrow \quad f(0^+) = f(0^-) = f(0) \;\Rightarrow\; \ell n4 . \ell n2 = k\ell n4 \;\Rightarrow\; k = \ell n2 \;\Rightarrow f(0) = (\ell n4)(\ell n2)$

Illustration 3 : Let $f(x) = \begin{cases} \dfrac{a(1 - x\sin x) + b\cos x + 5}{x^2} & x < 0 \\ 3 & x = 0 \\ \left(1 + \left(\dfrac{cx + dx^3}{x^2}\right)\right)^{\frac{1}{x}} & x > 0 \end{cases}$

If f is continuous at x = 0, then find out the values of a, b, c and d.

Solution : Since f(x) is continuous at x = 0, so at x = 0, both left and right limits must exist and both must be equal to 3.

Now

$$\lim_{x\to 0^-} \frac{a(1 - x\sin x) + b\cos x + 5}{x^2} = \lim_{x\to 0^-} \frac{(a+b+5) + \left(-a - \frac{b}{2}\right)x^2 + \ldots}{x^2} = 3 \text{ (By the expansions of sinx and cosx)}$$

If $\lim\limits_{x\to 0^-} f(x)$ exists then a + b + 5 = 0 and $-a - \dfrac{b}{2} = 3 \Rightarrow a = -1$ and $b = -4$

since $\lim\limits_{x\to 0^+} \left(1 + \left(\dfrac{cx + dx^3}{x^2}\right)\right)^{\frac{1}{x}}$ exists $\Rightarrow \lim\limits_{x\to 0^+} \dfrac{cx + dx^3}{x^2} = 0 \Rightarrow c = 0$

Now $\lim\limits_{x\to 0^+} (1 + dx)^{\frac{1}{x}} = \lim\limits_{x\to 0^+} \left[(1 + dx)^{\frac{1}{dx}}\right]^d = e^d$

So $e^d = 3 \Rightarrow d = \ell n\ 3$,

Hence a = – 1, b = – 4, c = 0 and d = ℓn 3.

Do yourself -1 :

(i) If $f(x) = \begin{cases} \cos x; x \ge 0 \\ x + k; x < 0 \end{cases}$ find the value of k if f (x) is continuous at x = 0.

(ii) If $f(x) = \begin{cases} \dfrac{|x+2|}{\tan^{-1}(x+2)} & ; \ x \ne -2 \\ 2 & ; \ x = -2 \end{cases}$ then discuss the continuity of f (x) at x=–2

2. CONTINUITY OF THE FUNCTION IN AN INTERVAL :

(a) A function is said to be continuous in (a,b) if f is continuous at each & every point belonging to (a, b).

(b) A function is said to be continuous in a closed interval [a,b] if :

(i) f is continuous in the open interval (a,b)

(ii) f is right continuous at 'a' i.e. $\lim\limits_{x\to a^+} f(x) = f(a) =$ a finite quantity

(iii) f is left continuous at 'b' i.e. $\lim\limits_{x\to b^-} f(x) = f(b) =$ a finite quantity

Note :

(i) Obseve that $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to b^+} f(x)$ do not make sense. As a consequence of this definition, if $f(x)$ is defined only at one point, it is continuous there, i.e., if the domain of $f(x)$ is a singleton, $f(x)$ is a continuous function.

Example : Consider $f(x) = \sqrt{a - x} + \sqrt{x - a}$.
$f(x)$ is a singleton function defined only at x = a. Hence $f(x)$ is a continuous function.

(ii) All polynomials, trigonometrical functions, exponential & logarithmic functions are continuous in their domains.

(iii) If f (x) & g(x) are two functions that are continuous at x = c then the function defined by :

$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, where K is any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x = c.

Further, if g(c) is not zero, then $F_4(x) = \dfrac{f(x)}{g(x)}$ is also continuous at x = c.

(iii) Some continuous functions :

Function f(x)	Interval in which f(x) is continuous
Constant function	$(-\infty, \infty)$
x^n, n is an integer ≥ 0	$(-\infty, \infty)$
x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
$\lvert x - a \rvert$	$(-\infty, \infty)$
$p(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$	$(-\infty, \infty)$
$\frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomial in x	$(-\infty, \infty) - \{x : q(x) = 0\}$
sinx, cosx, e^x	$(-\infty, \infty)$
tanx, secx	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$
cotx, cosecx	$(-\infty, \infty) - \{n\pi : n \in I\}$
ℓnx	$(0, \infty)$

(iv) Some Discontinuous Functions :

Functions	Points of discontinuity
[x], {x}	Every Integer
tanx, secx	$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$
cotx, cosecx	$x = 0, \pm\pi, \pm 2\pi, ...$
$\sin\frac{1}{x}$, $\cos\frac{1}{x}$, $\frac{1}{x}$, $e^{1/x}$	$x = 0$

Illustration 4 : Discuss the continuity of $f(x) = \begin{cases} |x+1| & , \quad x < -2 \\ 2x+3 & , \quad -2 \leq x < 0 \\ x^2+3 & , \quad 0 \leq x < 3 \\ x^3-15 & , \quad x \geq 3 \end{cases}$

Solution : We write f(x) as $f(x) = \begin{cases} -x-1 & , \quad x < -2 \\ 2x+3 & , \quad -2 \leq x < 0 \\ x^2+3 & , \quad 0 \leq x < 3 \\ x^3-15 & , \quad x \geq 3 \end{cases}$

As we can see, f(x) is defined as a polynomial function in each of intervals $(-\infty, -2)$, $(-2,0)$, $(0,3)$ and $(3,\infty)$. Therefore, it is continuous in each of these four open intervals. Thus we check the continuity at $x = -2, 0, 3$.

At the point $x = -2$

$$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} (-x - 1) = +2 - 1 = 1$$

$$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} (2x + 3) = 2.(-2) + 3 = -1$$

Therefore, $\lim\limits_{x\to -2} f(x)$ does not exist and hence f(x) is discontinuous at x = –2.

At the point x = 0

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x + 3) = 3$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2 + 3) = 3$$

$f(0) = 0^2 + 3 = 3$

Therefore f(x) is continuous at x = 0.

At the point x = 3

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 + 3) = 3^2 + 3 = 12$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^3 - 15) = 3^3 - 15 = 12$$

$f(3) = 3^3 - 15 = 12$

Therefore, f(x) is continuous at x = 3.

We find that f(x) is continuous at all points in R except at x = – 2

Do yourself -2 :

(i) If $f(x) = \begin{cases} \dfrac{x^2}{a} & ; \quad 0 \le x < 1 \\ -1 & ; \quad 1 \le x < \sqrt{2} \\ \dfrac{2b^2 - 4b}{x^2} & ; \quad \sqrt{2} \le x < \infty \end{cases}$ then find the value of a & b if f (x) is continuous in $[0,\infty)$

(ii) Discuss the continuity of f (x) = $\begin{cases} |x-3| & ; \quad 0 \le x < 1 \\ \sin x & ; \quad 1 \le x \le \dfrac{\pi}{2} \\ \log_{\frac{\pi}{2}} x & ; \quad \dfrac{\pi}{2} < x < 3 \end{cases}$ in [0,3)

3. REASONS OF DISCONTINUITY :

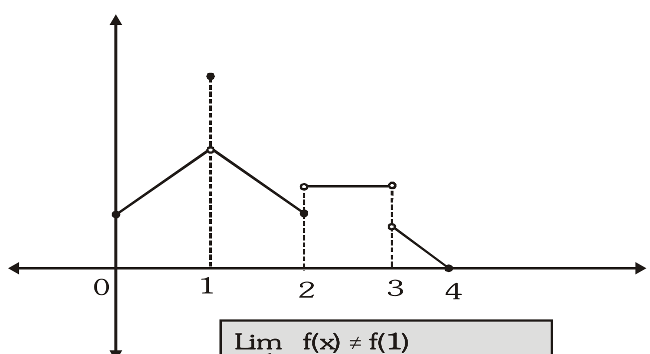
(a) Limit does not exist

i.e. $\lim\limits_{x\to a^-} f(x) \ne \lim\limits_{x\to a^+} f(x)$

(b) f(x) is not defined at x = a

(c) $\lim\limits_{x\to a} f(x) \ne f(a)$

Geometrically, the graph of the function will exhibit a break at x = a, if the function is discontinuous at x = a. The graph as shown is discontinuous at x = 1 , 2 and 3.



$\lim\limits_{x\to 1} f(x) \ne f(1)$

$\lim\limits_{x\to 2} f(x)$ does not exist

f(x) is not defined at x = 3

4. TYPES OF DISCONTINUITIES :

Type-1 : (Removable type of discontinuities) : - In case $\lim\limits_{x\to a} f(x)$ exists but is not equal to f(a) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim\limits_{x\to a} f(x) = f(a)$ & make it continuous at x = a. Removable type of discontinuity can be further classified as:

(a) Missing point discontinuity :

Where $\lim\limits_{x\to a} f(x)$ exists but f(a) is not defined.

(b) Isolated point discontinuity :

Where $\lim\limits_{x\to a} f(x)$ exists & f(a) also exists but; $\lim\limits_{x\to a} f(x) \neq f(a)$.

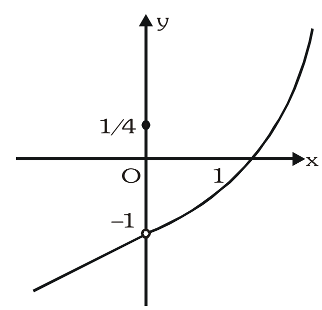
Illustration 5 : Examine the function , $f(x) = \begin{cases} x-1 & , \ x<0 \\ 1/4 & , \ x=0 \\ x^2-1 & , \ x>0 \end{cases}$. Discuss the continuity, and if discontinuous remove the discontinuity by redefining the function (if possible).

Solution : Graph of f(x) is shown, from graph it is seen that

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = -1$, but f(0) = 1/4

Thus, f(x) has removable discontinuity and f(x) could be made continuous by taking f(0) = – 1

$$\Rightarrow f(x) = \begin{cases} x-1 & , \ x<0 \\ -1 & , \ x=0 \\ x^2-1 & , \ x>0 \end{cases}$$



y = f(x) before redefining

Do yourself -3 :

(i) If $f(x) = \begin{cases} \dfrac{1}{x-1} & ; \ 0 \le x < 2 \\ x^2-3 & ; \ 2 \le x < 4 \\ 5 & ; \ x = 4 \\ 14 - \dfrac{x^{1/2}}{2} & ; \ x > 4 \end{cases}$, then discuss the types of discontinuity for the function.

Type-2 : (Non-Removable type of discontinuities) :-

In case $\lim\limits_{x\to a} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

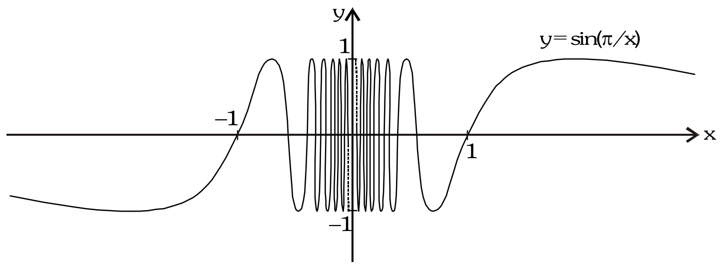
(i) **Finite type discontinuity** : In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

(ii) **Infinite type discontinuity** : In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

(iii) Oscillatory type discontinuity :

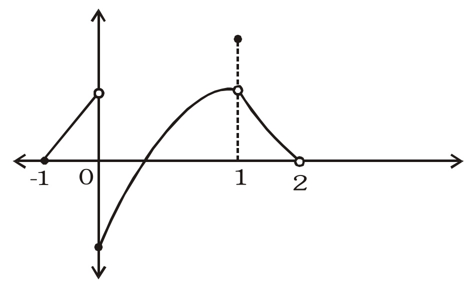
e.g. $f(x) = \sin\frac{\pi}{x}$ at x = 0

$$f(x) = \sin\frac{\pi}{x}$$



f(x) has non removable oscillatory type discontinuity at x = 0

Example : From the adjacent graph note that



(i) f is continuous at x = –1

(ii) f has isolated discontinuity at x = 1

(iii) f has missing point discontinuity at x = 2

(iv) f has non removable (finite type) discontinuity at the origin.

Note : In case of non-removable (finite type) discontinuity the non-negative difference between the value of the RHL at x = a & LHL at x = a is called **the jump of discontinuity**. A function having a finite number of jumps in a given interval I is called a **piece wise continuous or sectionally continuous** function in this interval.

Illustration 6 : Show that the function, $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & ;\ \text{when } x \neq 0 \\ 0, & ;\ \text{when } x = 0 \end{cases}$ has non-removable discontinuity at x = 0.

Solution : We have, $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & ;\ \text{when } x \neq 0 \\ 0, & ;\ \text{when } x = 0 \end{cases}$

$\Rightarrow \lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} \dfrac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1} = \lim\limits_{h\to 0} \dfrac{1-\dfrac{1}{e^{1/h}}}{1+\dfrac{1}{e^{1/h}}} = 1$ $\quad [\because\ e^{1/h} \to \infty]$

$\Rightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} \dfrac{e^{-1/h}-1}{e^{-1/h}+1} = \dfrac{0-1}{0+1} = -1$ $\quad [\because\ h \to 0\ ;\ e^{-1/h} \to 0]$

$\lim\limits_{x\to 0^-} f(x) = -1$

$\Rightarrow \lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$. Thus f(x) has non-removable discontinuity.

Illustration 7 : $f(x) = \begin{cases} \cos^{-1}\{\cot x\} & x < \dfrac{\pi}{2} \\ \pi[x]-1 & x \geq \dfrac{\pi}{2} \end{cases}$; find jump of discontinuity, where [ ] denotes greatest integer & { } denotes fractional part function.

Solution : $f(x) = \begin{cases} \cos^{-1}\{\cot x\} & x < \dfrac{\pi}{2} \\ \pi[x] - 1 & x \ge \dfrac{\pi}{2} \end{cases}$

$$\lim_{x\to\frac{\pi^-}{2}} f(x) = \lim_{x\to\frac{\pi^-}{2}} \cos^{-1}\{\cot x\} = \lim_{h\to 0}\cos^{-1}\left\{\cot\left(\frac{\pi}{2}-h\right)\right\} = \lim_{h\to 0}\cos^{-1}\{\tan h\} = \frac{\pi}{2}$$

$$= \lim_{x\to\frac{\pi^+}{2}} f(x) = \lim_{x\to\frac{\pi^+}{2}} \pi[x]-1 = \lim_{h\to 0}\pi\left[\frac{\pi}{2}+h\right]-1 = \pi - 1$$

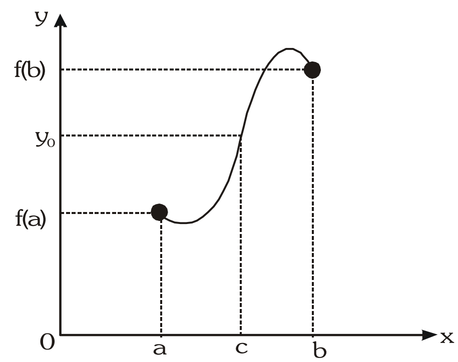
$\therefore$ jump of discontinuity $= \pi - 1 - \dfrac{\pi}{2} = \dfrac{\pi}{2} - 1$

Do yourself -4 :

(i) Discuss the type of discontinuity for $f(x) = \begin{cases} -1 & ; \quad x \le -1 \\ |x| & ; \quad -1 < x < 1 \\ (x+1) & ; \quad x \ge 1 \end{cases}$

5. THE INTERMEDIATE VALUE THEOREM :

Suppose f(x) is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b), there exists a number c between a and b such that $f(c) = y_0$



The function f, being continuous on [a,b] takes on every value between f(a) and f(b)

Note that a function f which is continuous in [a,b] possesses the following properties :

(i) If f(a) & f(b) posses opposite signs, then there exists atleast one root of the equation $f(x) = 0$ in the open interval (a,b).

(ii) If K is any real number between f(a) & f(b), then there exists atleast one root of the equation $f(x) = K$ in the open interval (a,b).

Note : In above cases the number of roots is always odd.

Illustration 8 : Show that the function, $f(x) = (x - a)^2(x - b)^2 + x$, takes the value $\dfrac{a+b}{2}$ for some $x_0 \in (a, b)$

Solution : $f(x) = (x - a)^2(x - b)^2 + x$

$f(a) = a$

$f(b) = b$

& $\dfrac{a+b}{2} \in (f(a), f(b))$

$\therefore$ By intermediate value theorem, there is atleast one $x_0 \in (a, b)$ such that $f(x_0) = \dfrac{a+b}{2}$.

Illustration 9 : Let $f : [0, 1] \xrightarrow{\text{onto}} [0, 1]$ be a continuous function, then prove that $f(x) = x$ for atleast one $x \in [0, 1]$

Solution : Consider $g(x) = f(x) - x$

$g(0) = f(0) - 0 = f(0) \geq 0$ $\quad \{\because \; 0 \leq f(x) \leq 1\}$

$g(1) = f(1) - 1 \leq 0$

$\Rightarrow$ $g(0) \, . \, g(1) \leq 0$

$\Rightarrow$ $g(x) = 0$ has atleast one root in $[0, 1]$

$\Rightarrow$ $f(x) = x$ for atleast one $x \in [0, 1]$

Do yourself -5 :

(i) If $f(x)$ is continuous in $[a,b]$ such that $f(c) = \dfrac{2f(a)+3f(b)}{5}$, then prove that $c \in (a,b)$

6. SOME IMPORTANT POINTS :

(a) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ will **not necessarily be discontinuous at x = a,** *e.g.*

$$f(x) = x \;\&\; g(x) = \begin{cases} \sin\dfrac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f(x) is continuous at x = 0 & g(x) is discontinuous at x = 0, but f(x).g(x) is continuous at x = 0.

(b) If f (x) and g (x) both are discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ **is not necessarily be discontinuous at x = a** , *e.g.*

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

f(x) & g(x) both are discontinuous at x = 0 but the product function f.g(x) is still continuous at x = 0

(c) If f (x) and g (x) both are discontinuous at x = a then f(x) ± g(x) is not necessarily be discontinuous at x = a

(d) A continuous function whose domain is closed must have a range also in closed interval.

(e) If f is continuous at x = a & g is continuous at x = f (a) then the composite g[f(x)] is continuous at x = a. *eg.*

$f(x) = \dfrac{x\sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x =0, hence the composite $(gof)(x) = \left|\dfrac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0

Illustration 10 : If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{1}{x-2}$, then discuss the continuity of f(x), g(x) and fog (x) in R.

Solution : $f(x) = \dfrac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain and f is not defined at x = 1.

$\therefore$ f is discontinuous at x = 1

$g(x) = \dfrac{1}{x-2}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 2.

$\therefore$ g is discontinuous at x = 2

Now fog(x) will be discontinuous at x = 2 (point of discontinuity of g(x))

Consider g(x) = 1 (when g(x) = point of discontinuity of f(x))

$\dfrac{1}{x-2} = 1 \quad \Rightarrow \quad x = 3$

$\therefore$ fog(x) is discontinuous at x = 2 & x = 3.

Do yourself -6 :

(i) Let $f(x) = [x]$ & $g(x) = \text{sgn}(x)$ (where [.] denotes greatest integer function) , then discuss the continuity of $f(x) \pm g(x)$, $f(x).g(x)$ & $\frac{f(x)}{g(x)}$ at x= 0.

(ii) If $f(x) = \sin|x|$ & $g(x) = \tan|x|$ then discuss the continuity of $f(x) \pm g(x)$; $\frac{f(x)}{g(x)}$ & $f(x)\, g(x)$

7. SINGLE POINT CONTINUITY :

Functions which are continuous only at one point are said to exhibit single point continuity

Illustration 11: If $f(x) = \begin{cases} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{cases}$, find the points where $f(x)$ is continuous

Solution : Let x = a be the point at which $f(x)$ is continuous.

$\Rightarrow \lim\limits_{\substack{x \to a \\ \text{through rational}}} f(x) = \lim\limits_{\substack{x \to a \\ \text{through irrational}}} f(x)$

$\Rightarrow$ $a = -a$

$\Rightarrow$ $a = 0 \Rightarrow$ function is continuous at x = 0.

Do yourself -7 :

(i) If $g(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$, then find the points where function is continuous.

(ii) If $f(x) = \begin{cases} x^2 & ; x \in Q \\ 1 - x^2 & ; x \notin Q \end{cases}$, then find the points where function is continuous.

ANSWERS FOR DO YOURSELF

1. (i) 1 (ii) discontinuous at x = –2
2. (i) a = –1 & b = 1 (ii) Discontinuous at x = 1 & continuous at $x = \frac{\pi}{2}$
3. (i) Missing point removable discontinuity at x = 1, isolated point removable discontinuity at x = 4 .
4. (i) Finite type non-removable discontinuity at x = –1, 1
6. (i) All are discontinuous at x = 0.

   (ii) $f(x)\, g(x)$ & $f(x) \pm g(x)$ are discontinuous at $x = (2n+1)\frac{\pi}{2}$; $n \in I$

   $\frac{f(x)}{g(x)}$ is discontinuous at $x = \frac{n\pi}{2}$; $n \in I$
7. (i) x = 0 (ii) $x = \pm\frac{1}{\sqrt{2}}$

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. If $f(x)=\begin{cases} x+2 & , \text{ when } x<1 \\ 4x-1 & , \text{ when } 1\le x\le 3 \\ x^2+5 & , \text{ when } x>3 \end{cases}$, then correct statement is -

(A) $\lim\limits_{x\to1} f(x)=\lim\limits_{x\to3} f(x)$
(B) f(x) is continuous at x = 3
(C) f(x) is continuous at x = 1
(D) f(x) is continuous at x = 1 and 3

2. If $f(x)=\begin{cases} \dfrac{1}{e^{1/x}+1} & , x\ne 0 \\ 0 & , x=0 \end{cases}$, then -

(A) $\lim\limits_{x\to0+} f(x)=1$
(B) $\lim\limits_{x\to0-} f(x)=0$
(C) f(x) is discontinuous at x = 0
(D) f(x) is continuous

3. If function f(x) = $\dfrac{\sqrt{1+x}-\sqrt[3]{1+x}}{x}$, is continuous function, then f(0) is equal to -

(A) 2 (B) 1/4 (C) 1/6 (D) 1/3

4. If $f(x)=\begin{cases} \dfrac{x^2-(a+2)x+2a}{x-2} & , x\ne 2 \\ 2 & , x=2 \end{cases}$ is continuous at x = 2, then a is equal to -

(A) 0 (B) 1 (C) –1 (D) 2

5. If f(x) = $\begin{cases} \dfrac{\log(1+2ax)-\log(1-bx)}{x} & , x\ne 0 \\ k & , x=0 \end{cases}$, is continuous at x = 0 , then k is equal to -

(A) 2a + b (B) 2a – b (C) b – 2a (D) a + b

6. If $f(x)=\begin{cases} [x]+[-x], & x\ne 2 \\ \lambda, & x=2 \end{cases}$, f is continuous at x = 2 then λ is (where [.] denotes greatest integer) -

(A) –1 (B) 0 (C) 1 (D) 2

7. If f(x) = $\begin{cases} \dfrac{1-\cos 4x}{x^2} & , x<0 \\ a & , x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & , x>0 \end{cases}$, then correct statement is -

(A) f(x) is discontinuous at x = 0 for any value of a
(B) f(x) is continuous at x = 0 when a = 8
(C) f(x) is continuous at x = 0 when a = 0
(D) none of these

8. Function $f(x) = \dfrac{1}{\log|x|}$ is discontinuous at -

(A) one point (B) two points (C) three points (D) infinite number of points

9. Which of the following functions has finite number of points of discontinuity in R (where [.] denotes greatest integer)

(A) $\tan x$ (B) $|x| / x$ (C) $x + [x]$ (D) $\sin [\pi x]$

10. If $f(x) = \dfrac{1-\tan x}{4x-\pi}, x \neq \dfrac{\pi}{4},\ x \in \left[0, \dfrac{\pi}{2}\right)$ is a continuous functions, then $f(\pi/4)$ is equal to -

(A) $-1/2$ (B) $1/2$ (C) 1 (D) -1

11. The value of f(0), so that function, $f(x) = \dfrac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$ becomes continuous for all x, is given by -

(A) $a\sqrt{a}$ (B) $-\sqrt{a}$ (C) $\sqrt{a}$ (D) $-a\sqrt{a}$

12. If $f(x) = \dfrac{x-e^x+\cos 2x}{x^2}$, $x \neq 0$ is continuous at x = 0, then -

(A) $f(0) = \dfrac{5}{2}$ (B) $[f(0)] = -2$ (C) $\{f(0)\} = -0.5$ (D) $[f(0)].\{f(0)\} = -1.5$

where [x] and {x} denotes greatest integer and fractional part function.

13. Let $f(x) = \dfrac{x(1+a\cos x)-b\sin x}{x^3}$, $x \neq 0$ and $f(0) = 1$. The value of a and b so that f is a continuous function are -

(A) 5/2, 3/2 (B) 5/2, −3/2 (C) −5/2, −3/2 (D) none of these

14. 'f' is a continuous function on the real line. Given that $x^2+(f(x)-2)x-\sqrt{3}.f(x)+2\sqrt{3}-3=0$. Then the value of $f(\sqrt{3})$ is -

(A) $\dfrac{2(\sqrt{3}-2)}{\sqrt{3}}$ (B) $2(1-\sqrt{3})$ (C) zero (D) cannot be determined

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

15. The value(s) of x for which $f(x) = \dfrac{e^{\sin x}}{4-\sqrt{x^2-9}}$ is continuous, is (are) -

(A) 3 (B) −3 (C) 5 (D) all $x \in (-\infty, -3] \cup [3, \infty)$

16. Which of the following function(s) not defined at x = 0 has/have removable discontinuity at the origin ?

(A) $f(x) = \dfrac{1}{1+2^{\cot x}}$ (B) $f(x) = \cos\left(\dfrac{|\sin x|}{x}\right)$

(C) $f(x) = x\sin\dfrac{\pi}{x}$ (D) $f(x) = \dfrac{1}{\ell n|x|}$

17. Function whose jump (non-negative difference of LHL & RHL) of discontinuity is greater than or equal to one, is/are -

(A) $f(x)=\begin{cases}\dfrac{(e^{1/x}+1)}{(e^{1/x}-1)} & ; \quad x<0\\ \dfrac{(1-\cos x)}{x} & ; \quad x>0\end{cases}$

(B) $g(x)=\begin{cases}\dfrac{x^{1/3}-1}{x^{1/2}-1} & ; \quad x>1\\ \dfrac{\ell nx}{(x-1)} & ; \quad \dfrac{1}{2}<x<1\end{cases}$

(C) $u(x)=\begin{cases}\dfrac{\sin^{-1}2x}{\tan^{-1}3x} & ; \quad x\in\left(0,\dfrac{1}{2}\right]\\ \dfrac{|\sin x|}{x} & ; \quad x<0\end{cases}$

(D) $v(x)=\begin{cases}\log_3(x+2) & ; \quad x>2\\ \log_{1/2}(x^2+5) & ; \quad x<2\end{cases}$

18. If $f(x)=\dfrac{1}{x^2-17x+66}$, then $f\left(\dfrac{2}{x-2}\right)$ is discontinuous at x =

(A) 2
(B) $\dfrac{7}{3}$
(C) $\dfrac{24}{11}$
(D) 6,11

19. Let f (x) = [x] & $g(x)=\begin{cases}0; & x\in Z\\ x^2; & x\in R-Z\end{cases}$, then (where [.] denotes greatest integer function) -

(A) $\lim\limits_{x\to 1} g(x)$ exists, but g(x) is not continuous at x= 1.

(B) $\lim\limits_{x\to 1} f(x)$ does not exist and f (x) is not continuous at x=1.

(C) gof is continuous for all x.

(D) fog is continuous for all x.

CHECK YOUR GRASP — ANSWER KEY — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C	C	A	A	A	B	C	B	A
Que.	11	12	13	14	15	16	18	18	19	
Ans.	B	D	C	B	A,B	B,C,D	A,C,D	A,B,C	A,B,C	

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. Consider the piecewise defined function $f(x)=\begin{cases}\sqrt{-x} & \text{if } x<0\\ 0 & \text{if } 0\le x\le 4\\ x-4 & \text{if } x>4\end{cases}$ choose the answer which best describes the continuity of this function -

(A) the function is unbounded and therefore cannot be continuous

(B) the function is right continuous at x = 0

(C) the function has a removable discontinuity at 0 and 4, but is continuous on the rest of the real line

(D) the function is continuous on the entire real line

2. f(x) is continuous at x=0, then which of the following are always true ?

(A) $\lim\limits_{x\to 0} f(x)=0$ (B) f(x) is non continuous at x=1

(C) $g(x) = x^2 f(x)$ is continuous at x = 0 (D) $\lim\limits_{x\to 0^+}(f(x)-f(0))=0$

3. Indicate all correct alternatives if, $f(x)=\frac{x}{2}-1$, then on the interval $[0,\pi]$

(A) $\tan(f(x))$ & $\frac{1}{f(x)}$ are both continuous (B) $\tan(f(x))$ & $\frac{1}{f(x)}$ are both discontinuous

(C) $\tan(f(x))$ & $f^{-1}(x)$ are both continuous (D) $\tan(f(x))$ is continuous but $\frac{1}{f(x)}$ is not

4. If f(x) = sgn(cos2x − 2 sinx + 3), where sgn ( ) is the signum function, then f(x) -

(A) is continuous over its domain (B) has a missing point discontinuity

(C) has isolated point discontinuity (D) has irremovable discontinuity.

5. $f(x)=\frac{2\cos x-\sin 2x}{(\pi-2x)^2}$; $g(x)=\frac{e^{-\cos x}-1}{8x-4\pi}$

h(x) = f(x) for $x<\pi/2$

= g(x) for $x>\pi/2$

then which of the followings does not holds ?

(A) h is continuous at $x = \pi/2$ (B) h has an irremovable discontinuity at $x=\pi/2$

(C) h has a removable discontinuity at $x = \pi/2$ (D) $f\left(\frac{\pi^+}{2}\right)=g\left(\frac{\pi^-}{2}\right)$

6. The number of points where f(x) = [sinx + cosx] (where [ ] denotes the greatest integer function), $x \in (0, 2\pi)$ is not continuous is -

(A) 3 (B) 4 (C) 5 (D) 6

7. On the interval I = [−2, 2], the function $f(x)=\begin{cases}(x+1)e^{-\left[\frac{1}{|x|}+\frac{1}{x}\right]} & (x\ne 0)\\ 0 & (x=0)\end{cases}$

then which one of the following hold good ?

(A) is continuous for all values of $x \in I$ (B) is continuous for $x \in I-(0)$

(C) assumes all intermediate values from f(−2) & f(2) (D) has a maximum value equal to 3/e

8. If $f(x)=\cos\left[\frac{\pi}{x}\right]\cos\left(\frac{\pi}{2}(x-1)\right)$; where [x] is the greatest integer function of x, then f(x) is continuous at -

(A) x = 0 (B) x = 1 (C) x = 2 (D) none of these

9. Given $f(x)=\begin{cases}3-\left[\cot^{-1}\left(\frac{2x^3-3}{x^2}\right)\right] & \text{for } x>0\\ \left\{x^2\right\}\cos\left(e^{1/x}\right) & \text{for } x<0\end{cases}$ where { } & [ ] denotes the fractional part and the integral part functions respectively, then which of the following statement does not hold good -

(A) $f(0^-)=0$
(B) $f(0^+)=3$
(C) $f(0)=0 \Rightarrow$ continuity of f at x = 0
(D) irremovable discontinuity of f at x = 0

10. Let 'f' be a continuous function on R. If $f(1/4^n)=(\sin e^n)e^{-n^2}+\frac{n^2}{n^2+1}$ then f(0) is -

(A) not unique
(B) 1
(C) data sufficient to find f(0)
(D) data insufficient to find f(0)

11. Given $f(x) = b([x]^2+[x])+1$ for $x \ge -1$

$= \sin(\pi(x+a))$ for $x < -1$

where [x] denotes the integral part of x, then for what values of a, b the function is continuous at x = – 1 ?

(A) $a=2n+(3/2);\ b\in R\ ;\ n\in I$
(B) $a=4n+2\ ;\ b\in R\ ;\ n\in I$
(C) $a=4n+(3/2)\ ;\ b\in R^+\ ;\ n\in I$
(D) $a=4n+1\ ;\ b\in R^+\ ;\ n\in I$

12. Consider $f(x)=\begin{cases}x[x]^2\log_{(1+x)}2 & \text{for } -1<x<0\\ \dfrac{\ln\left(e^{x^2}+2\sqrt{\{x\}}\right)}{\tan\sqrt{x}} & \text{for } 0<x<1\end{cases}$ where [*] & {*} are the greatest integer function & fractional part function respectively, then -

(A) $f(0)=\ln 2 \Rightarrow$ f is continuous at x = 0
(B) $f(0)=2 \Rightarrow$ f is continuous at x = 0
(C) $f(0)=e^2 \Rightarrow$ f is continuous at x = 0
(D) f has an irremovable discontinuity at x = 0

13. Let $f(x)=\begin{cases}a\sin^{2n}x & \text{for } x\ge 0 \text{ and } n\to\infty\\ b\cos^{2m}x-1 & \text{for } x<0 \text{ and } m\to\infty\end{cases}$ then -

(A) $f(0^-)\ne f(0^+)$
(B) $f(0^+)\ne f(0)$
(C) $f(0^-)=f(0)$
(D) f is continuous at x = 0

14. Consider $f(x)=\lim\limits_{n\to\infty}\frac{x^n-\sin x^n}{x^n+\sin x^n}$ for $x>0, x\ne 1$ f(1)=0 then -

(A) f is continuous at x = 1
(B) f has a finite discontinuity at x = 1
(C) f has an infinite or oscillatory discontinuity at x = 1
(D) f has a removable type of discontinuity at x=1

BRAIN TEASERS **ANSWER KEY** EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	C,D	C,D	C	A,C,D	C	B,C,D	B,C	B,D	B,C
Que.	11	12	13	14						
Ans.	A,C	D	A	B						

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

TRUE / FALSE

1. $\dfrac{1}{x+[x]}$ is discontinuous at infinite points. ([ ] denotes greatest integer function)
2. $\sin|x| + |\sin x|$ is not continuous for all x.
3. If f is continuous and g is discontinuous at x = a, then f(x).g(x) is discontinuous at x = a.
4. There exists a continuous onto function $f : [0, 1] \longrightarrow [0, 10]$, but there exists no continuous onto function $g : [0, 1] \longrightarrow (0, 10)$
5. If $f(x) = \dfrac{\tan(\pi/4 - x)}{\cos 2x}$ for $x \neq \dfrac{\pi}{4}$, then the value which can be given to f(x) at $x = \dfrac{\pi}{4}$ so that the function becomes continuous every where in $(0, \pi/2)$ is 1/4.
6. The function f, defined by $f(x) = \dfrac{1}{1+2^{\tan x}}$ is continuous for real x.
7. $f(x) = \lim\limits_{n\to\infty} \dfrac{1}{1+n\sin^2 \pi x}$ is continuous at x = 1.
8. If f(x) is continuous in [0, 1] and f(x) = 1 for all rational numbers in [0, 1] then $f\left(\dfrac{1}{\sqrt{2}}\right) = 1$.

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

1.

	Column-I		Column-II
(A)	If $f(x) = \begin{cases} \sin\{x\}; & x < 1 \\ \cos x + a; & x \geq 1 \end{cases}$ where {.} denotes the fractional part function, such that f(x) is continuous at x = 1. If $\lvert k \rvert = \dfrac{a}{\sqrt{2}\sin\dfrac{(4-\pi)}{4}}$ then k is	(p)	1
(B)	If the function $f(x) = \dfrac{(1-\cos(\sin x))}{x^2}$ is continuous at x = 0, then f(0) is	(q)	0
(C)	$f(x) = \begin{cases} x & , x \in Q \\ 1-x & , x \notin Q \end{cases}$, then the values of x at which f(x) is continuous	(r)	–1
(D)	If f(x) = x + {–x} + [x], where [x] and {x} represents integral and fractional part of x, then the values of x at which f(x) is discontinuous	(s)	$\dfrac{1}{2}$

2.

	Column-I		Column-II
(A)	If f(x) = 1/(1–x), then the points at which the function fofof(x) is discontinuous	(p)	$\frac{1}{2}$
(B)	$f(u) = \frac{1}{u^2+u-2}$, where $u = \frac{1}{x-1}$. The values of x at which 'f' is discontinuous	(q)	0
(C)	$f(x) = u^2$, where $u = \begin{cases} x-1, x \geq 0 \\ x+1, x < 0 \end{cases}$ The number of values of x at which 'f' is discontinuous	(r)	2
(D)	The number of value of x at which the function $f(x) = \frac{2x^5-8x^2+11}{x^4+4x^3+8x^2+8x+4}$ is discontinuous	(s)	1

ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : f(x) = sinx + [x] is discontinuous at x = 0

Because

Statement-II : If g(x) is continuous & h(x) is discontinuous at x = a, then g(x) + h(x) will necessarily be discontinuous at x = a

(A) A (B) B (C) C (D) D

2. Consider $f(x) = \begin{cases} 2\sin(a\cos^{-1} x) & \text{if } x \in (0,1) \\ \sqrt{3} & \text{if } x = 0 \\ ax+b & \text{if } x < 0 \end{cases}$

Statement-I : If $b = \sqrt{3}$ and $a = \frac{2}{3}$ then $f(x)$ is continuous in $(-\infty, 1)$

Because

Statement-II : If a function is defined on an interval I and limit exist at every point of interval I then function is continuous in I.

(A) A (B) B (C) C (D) D

3. Let $f(x) = \begin{cases} \frac{\cos x - e^{-x^2/2}}{x^3} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$ then

Statement-I : $f(x)$ is continuous at x = 0.

Because

Statement-II : $\lim_{x \to 0} \frac{\cos x - e^{-x^2/2}}{x^4} = \frac{-1}{12}$.

(A) A (B) B (C) C (D) D

4. **Statement-I** : The equation $\frac{x^3}{4} - \sin \pi x + 3 = 2\frac{1}{3}$ has atleast one solution in [–2, 2]

Because

Statement-II : If f:[a, b] → R be a function & let 'c' be a number such that f(a) < c < f(b), then there is atleast one number n ∈ (a, b) such that f(n) = c.

(A) A (B) B (C) C (D) D

5. **Statement-I** : Range of $f(x) = x\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right) + x^2 + x^4$ is not R.

Because

Statement-II : Range of a continuous even function can not be R.

(A) A (B) B (C) C (D) D

6. Let $f(x) = \begin{cases} Ax - B & x \le -1 \\ 2x^2 + 3Ax + B & x \in (-1, 1] \\ 4 & x > 1 \end{cases}$

Statement-I : $f(x)$ is continuous at all x if $A = \frac{3}{4}$, $B = -\frac{1}{4}$.

Because

Statement-II : Polynomial function is always continuous.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

If $S_n(x) = \frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} + \dots\dots + \frac{x^{2^n}}{(x+1)(x^2+1)\dots\dots(x^{2^n}+1)}$ and x > 1

$\lim_{n \to \infty} S_n(x) = \ell$

$g(x) = \begin{cases} \frac{\sqrt{ax+b} - 1}{x} & , \ x \ne 0 \\ 1 & , \ x = 0 \end{cases}$

h : R → R $h(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 7$

On the basis of above information, answer the following questions :

1. If g(x) is continuous at x = 0 then a + b is equal to -

(A) 0 (B) 1 (C) 2 (D) 3

2. If g(x) is continuous at x = 0 then g'(0) is equal to -

(A) ℓ (B) $\frac{h(6)}{2}$ (C) a – 2b (D) does not exist

3. Identify the incorrect option -

(A) h(x) is surjective (B) domain of g(x) is [–1/2, ∞)

(C) h(x) is bounded (D) ℓ = 1

Comprehension # 2

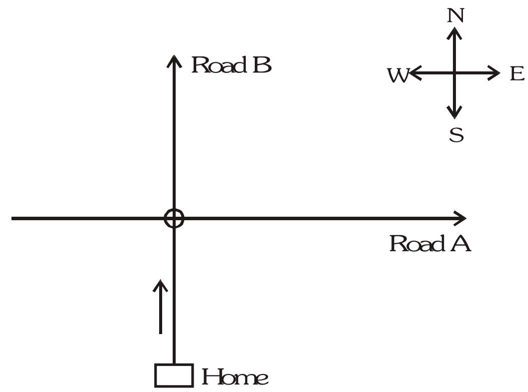
A man leaves his home early in the morning to have a walk. He arrives at a junction of road A & road B as shown in figure. He takes the following steps in later journey :

(a) 1 km in north direction

(b) changes direction & moves in north-east direction for $2\sqrt{2}$ kms.

(c) changes direction & moves southwards for distance of 2 km.

(d) finally he changes the direction & moves in south-east direction to reach road A again.



Visible/Invisible path :- The path traced by the man in the direction parallel to road A & road B is called invisible path, the remaining path traced is visible.

Visible points :- The points about which the man changes direction are called visible points except the point from where he changes direction last time

Now if road A & road B are taken as x-axis & y-axis then visible path & visible point represents the graph of y = f(x).

On the basis of above information, answer the following questions :

1. The value of x at which the function is discontinuous -

(A) 2 (B) 0 (C) 1 (D) 3

2. The value of x at which fof(x) is discontinuous -

(A) 0 (B) 1 (C) 2 (D) 3

3. If f(x) is periodic with period 3, then f(19) is -

(A) 2 (B) 3 (C) 19 (D) none of these

MISCELLANEOUS TYPE QUESTION **ANSWER KEY** EXERCISE -3

- ***True / False***

1. T **2.** F **3.** F **4.** T **5.** F **6.** F **7.** F **8.** T

- ***Match the Column***

1. (A) → (p, r); (B) → (s); (C) → (s); (D) → (p, q, r) **2.** (A) → (q, s); (B) → (p, r, s); (C) → (q); (D) → (q)

- ***Assertion & Reason***

1. A **2.** C **3.** A **4.** C **5.** A **6.** B

- ***Comprehension Based Questions***

Comprehension # 1 : **1.** D **2.** B **3.** C *Comprehension # 2* : **1.** A **2.** B,C **3.** A

EXERCISE - 4 [A]

CONCEPTUAL SUBJECTIVE EXERCISE

1. If $f(x)=\begin{cases}-x^2, \text{ when } & x\le 0\\ 5x-4, \text{ when } & 0<x\le 1\\ 4x^2-3x, \text{ when } & 1<x<2\\ 3x+4, \text{ when } & x\ge 2\end{cases}$, discuss the continuity of f(x) in R.

2. Let $f(x)=\begin{cases}-2\sin x & \text{for } -\pi\le x\le -\frac{\pi}{2}\\ a\sin x+b & \text{for } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \cos x & \text{for } \frac{\pi}{2}\le x\le \pi\end{cases}$. If f is continuous on $[-\pi,\pi]$ then find the values of a & b.

3. Determine the values of a,b & c for which the function $f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x} & \text{for } x<0\\ c & \text{for } x=0\\ \dfrac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} & \text{for } x>0\end{cases}$

is continuous at x = 0

4. Determine the kind of discontinuity of the function $y=-\dfrac{2^{1/x}-1}{2^{1/x}+1}$ at the point x = 0

5. Suppose that $f(x)=x^3-3x^2-4x+12$ and $h(x)=\begin{cases}\dfrac{f(x)}{x-3} & , x\ne 3\\ K & x=3\end{cases}$ then

(a) find all zeros of 'f'
(b) find the value of K that makes 'h' continuous at x =3
(c) using the value of K found in (b) determine whether 'h' is an even function.

6. Draw the graph of the function $f(x)=x-|x-x^2|,\ -1\le x\le 1$ & discuss the continuity or discontinuity of f in the interval $-1\le x\le 1$.

7. If $f(x)=\dfrac{\sin 3x+A\sin 2x+B\sin x}{x^5}(x\ne 0)$ is continuous at x = 0, then find A & B. Also find f(0).

8. (a) Let f(x + y) = f(x) + f(y) for all x, y & if the function f(x) is continuous at x = 0, then show that f(x) is continuous at all x.
(b) If f(x . y) = f(x) . f(y) for all x, y and f(x) is continuous at x = 1. Prove that f(x) is continuous for all x except at x = 0. Given f(1) ≠ 0.

9. Examine the continuity at x= 0 of the sum function of the infinite series :

$$\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\ldots\ldots\ldots\infty$$

10. Show that : (a) a polynomial of an odd degree has at least one real root
(b) a polynomial of an even degree has at least two real roots if it attains at least one value opposite in sign to the coefficient of its highest-degree term.

CONCEPTUAL SUBJECTIVE EXERCISE	ANSWER KEY		EXERCISE-4(A)
1.	continuous every where except at x = 0	2.	a = –1 b = 1
3.	a = –3/2, $b\ne 0$, c = 1/2	4.	non-removable - finite type
5.	(a) –2, 2, 3 (b) K= 5 (c) even	6.	f is continuous in $-1\le x\le 1$
7.	A = –4, B = 5, f(0) = 1	9.	discontinuous at x = 0

EXERCISE - 4 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. Given $f(x)=\sum_{r=1}^{n}\tan\left(\frac{x}{2^r}\right)\sec\left(\frac{x}{2^{r-1}}\right)$; $r,n\in N$

$$g(x)=\begin{cases}\lim\limits_{n\to\infty}\dfrac{\ln\left(f(x)+\tan\frac{x}{2^n}\right)-\left(f(x)+\tan\frac{x}{2^n}\right)^n\cdot\left[\sin\left(\tan\frac{x}{2}\right)\right]}{1+\left(f(x)+\tan\frac{x}{2^n}\right)^n} & ; \quad x\neq\pi/4\\ K & ; \quad x=\pi/4\end{cases}$$

where [ ] denotes the greatest integer function and the domain of g(x) is $\left(0,\frac{\pi}{2}\right)$. Find the value of k, if possible, so that g(x) is continuous at $x=\pi/4$. Also state the points of discontinuity of g(x) in $(0,\pi/4)$, if any.

2. Let $f(x)=\begin{cases}1+x^3, & x<0\\ x^2-1, & x\geq 0\end{cases}$; $g(x)=\begin{cases}(x-1)^{1/3}, & x<0\\ (x+1)^{1/2}, & x\geq 0\end{cases}$ Discuss the continuity of g(f(x)).

3. Discuss the continuity of 'f' in [0,2] where $f(x)=\begin{cases}|4x-5|[x] & \text{for } x>1\\ [\cos\pi x] & \text{for } x\leq 1\end{cases}$; where [x] is the greatest integer not greater than x. Also draw the graph

4. Discuss the continuity of the function $f(x)=\lim\limits_{n\to\infty}\frac{\ln(2+x)-x^{2n}\sin x}{1+x^{2n}}$ at x = 1

5. Consider the function $g(x)=\begin{cases}\dfrac{1-a^x+xa^x\ln a}{a^x x^2} & \text{for } x<0\\ \dfrac{2^x a^x-x\ln 2-x\ln a-1}{x^2} & \text{for } x>0\end{cases}$ where a > 0.

Find the value of 'a' & 'g(0)' so that the function g(x) is continuous at x = 0.

6. Let $f(x)=\begin{cases}\dfrac{\left(\frac{\pi}{2}-\sin^{-1}\left(1-\{x\}^2\right)\right).\sin^{-1}\left(1-\{x\}\right)}{\sqrt{2}\left(\{x\}-\{x\}^3\right)} & \text{for } x\neq 0\\ \dfrac{\pi}{2} & \text{for } x=0\end{cases}$ where {x} is the fractional part of x.

Consider another function g(x); such that

$$g(x)=\begin{cases}f(x) & \text{for } x\geq 0\\ 2\sqrt{2}\, f(x) & \text{for } x<0\end{cases}$$

Discuss the continuity of the functions f(x) & g(x) at x = 0.

7. $f(x)=\dfrac{a^{\sin x}-a^{\tan x}}{\tan x-\sin x}$ for $x>0$

$=\dfrac{\ell n(1+x+x^2)+\ell n(1-x+x^2)}{\sec x-\cos x}$ for x < 0, if 'f' is continuous at x = 0, find 'a'

now if $g(x)=\ell n\left(2-\frac{x}{a}\right).\cot(x-a)$ for x ≠ a, a ≠ 0, a > 0. If 'g' is continuous at x = a then show that $g(e^{-1})=-e$

8. Let [x] denote the greatest integer function & f(x) be defined in a neighbourhood of 2 by

$$f(x)=\begin{cases}\dfrac{\left(\exp\left\{(x+2)\ell n4\right\}\right)^{\frac{[x+1]}{4}}-16}{4^x-16}, & x<2\\[2ex] A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)}, & x>2\end{cases}$$

Find the value of A & f(2) in order that f(x) may be continuous at x = 2.

9. If g : [a, b] onto [a, b] is continuous show that there is some c ∈ [a, b] such that g(c) = c.

10. Let $y_n(x)=x^2+\dfrac{x^2}{1+x^2}+\dfrac{x^2}{\left(1+x^2\right)^2}+.........\dfrac{x^2}{\left(1+x^2\right)^{n-1}}$ and $y(x)=\lim\limits_{n\to\infty}y_n(x)$. Discuss the continuity of $y_n(x)\,(n=1,2,3.....n)$ and y (x) at x = 0

BRAIN STORMING SUBJECTIVE EXERCISE	ANSWER KEY	EXERCISE-4(B)

1. $k=0\ ;\ g(x)=\begin{cases} \ln(\tan x) & \text{if } 0<x<\frac{\pi}{4}\\ 0 & \text{if } \frac{\pi}{4}\le x<\frac{\pi}{2}\end{cases}$. Hence g(x) is continuous everywhere.

2. gof is discontinuous at x = 0, 1 and –1

3. the function 'f' is continuous everywhere in [0,2] except for x = 0, $\frac{1}{2}$, 1 & 2

4. discontinuous at x = 1

5. a = $\frac{1}{\sqrt{2}}$, g(0)= $\frac{(\ell n2)^2}{8}$

6. $f(0^+)=\frac{\pi}{2}$; $f(0^-)=\frac{\pi}{4\sqrt{2}}$ ⇒ 'f' is dicontinuous at x = 0 ; $g(0^+)=g(0^-)=g(0)=\frac{\pi}{2}$ ⇒ 'g' is continuous at x = 0

7. a = e^{-1}

8. A = 1 ; f(2) = 1/2

10. y_n (x) is continuous at x = 0 for all n and y (x) is discontinuous at x = 0

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. If $f(x) = \begin{cases} x & x \in Q \\ -x & x \notin Q \end{cases}$, then f is continuous at- [AIEEE 2002]

(1) Only at zero (2) only at 0, 1 (3) all real numbers (4) all rational numbers

2. If $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0 & , x=0 \end{cases}$ then f(x) is- [AIEEE 2003]

(1) discontinuous everywhere
(2) continuous as well as differentiable for all x
(3) continuous for all x but not differentiable at x=0
(4) neither differentiable nor continuous at x = 0

3. Let $f(x) = \frac{1-\tan x}{4x-\pi}$, $x \neq \frac{\pi}{4}$, $x \in \left[0, \frac{\pi}{2}\right]$, If f(x) is continuous in $\left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is- [AIEEE 2004]

(1) 1 (2) 1/2 (3) –1/2 (4) –1

4. The function $f : R/\{0\} \to R$ given by $f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$ can be made continuous at x = 0 by defining f(0) as- [AIEEE 2007]

(1) 2 (2) –1 (3) 0 (4) 1

5. The values of p and q for which the function $f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x} & , x < 0 \\ q & , x = 0 \\ \frac{\sqrt{x+x^2}-\sqrt{x}}{x^{\frac{3}{2}}} & , x > 0 \end{cases}$ is continuous for all x in R,

are:- [AIEEE 2011]

(1) $p = -\frac{3}{2}, q = \frac{1}{2}$ (2) $p = \frac{1}{2}, q = \frac{3}{2}$ (3) $p = \frac{1}{2}, q = -\frac{3}{2}$ (4) $p = \frac{5}{2}, q = \frac{1}{2}$

6. Define F(x) as the product of two real functions $f_1(x) = x$, $x \in IR$, and $f_2(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ as follows:

$F(x) = \begin{cases} f_1(x).f_2(x) & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ [AIEEE 2011]

Statement-1 : F(x) is continuous on IR.
Statement-2 : $f_1(x)$ and $f_2(x)$ are continuous on IR.
(1) Statemen-1 is false, statement-2 is true.
(2) Statemen-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.
(3) Statement-1 is true, statement-2 is true, statement-2 is not a correct explanation for statement-1
(4) Statement-1 is true, statement-2 is false

7. Consider the function, $f(x) = |x - 2| + |x - 5|$, $x \in R$.
Statement–1 : f'(4) = 0.
Statement–2 : f is continuous in [2, 5], differentiable in (2, 5) and f(2) = f(5). [AIEEE 2012]
(1) Statement–1 is true, Statement–2 is false.
(2) Statement–1 is false, Statement–2 is true.
(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.
(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

PREVIOUS YEARS QUESTIONS — **ANSWER KEY** — EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7
Ans	1	3	3	4	1	4	4

EXERCISE - 05 [B]

JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. Discuss the continuity of the function $f(x)=\begin{cases}\dfrac{e^{1/(x-1)}-2}{e^{1/(x-1)}+2}, & x\neq 1\\ 1, & x=1\end{cases}$ at x = 1.

[REE 2001 (Mains), 3]

2. For every integer n, let a_n and b_n be real numbers. Let function $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x)=\begin{cases}a_n+\sin\pi x, & \text{for } x\in[2n,2n+1]\\ b_n+\cos\pi x, & \text{for } x\in(2n-1,2n)\end{cases}, \text{ for all integers n.}$$

If f is continuous, then which of the following holds(s) for all n ? [JEE 2012, 4]

(A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$ (C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

PREVIOUS YEARS QUESTIONS	ANSWER KEY	EXERCISE-5 [B]
1. Discontinuous at x = 1 ; $f(1^+) = 1$ and $f(1^-) = -1$	2. B,D	