

SOLUTIONS OF TRIANGLE

The process of calculating the sides and angles of triangle using given information is called solution of triangle.

In a $\triangle ABC$, the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.

1. SINE FORMULAE :

In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and Δ is area of triangle.

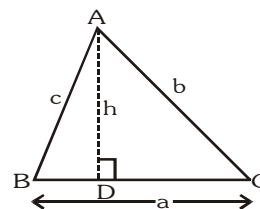


Illustration 1 : Angles of a triangle are in 4 : 1 : 1 ratio. The ratio between its greatest side and perimeter is

- (A) $\frac{3}{2+\sqrt{3}}$ (B) $\frac{\sqrt{3}}{2+\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2-\sqrt{3}}$ (D) $\frac{1}{2+\sqrt{3}}$

Solution : Angles are in ratio 4 : 1 : 1.

\Rightarrow angles are 120, 30, 30.

If sides opposite to these angles are a, b, c respectively, then a will be the greatest side. Now from

$$\text{Sine formula } \frac{a}{\sin 120^\circ} = \frac{b}{\sin 30^\circ} = \frac{c}{\sin 30^\circ}$$

$$\Rightarrow \frac{a}{\sqrt{3}/2} = \frac{b}{1/2} = \frac{c}{1/2}$$

$$\Rightarrow \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1} = k \text{ (say)}$$

$$\text{then } a = \sqrt{3}k, \text{ perimeter} = (2 + \sqrt{3})k$$

$$\therefore \text{ required ratio} = \frac{\sqrt{3}k}{(2 + \sqrt{3})k} = \frac{\sqrt{3}}{2 + \sqrt{3}}$$

Ans. (B)

Illustration 2 : In triangle ABC, if $b = 3$, $c = 4$ and $\angle B = \pi/3$, then number of such triangles is -

- (A) 1 (B) 2 (C) 0 (D) infinite

Solution : Using sine formulae $\frac{\sin B}{b} = \frac{\sin C}{c}$

$$\Rightarrow \frac{\sin \pi/3}{3} = \frac{\sin C}{4} \Rightarrow \frac{\sqrt{3}}{6} = \frac{\sin C}{4} \Rightarrow \sin C = \frac{2}{\sqrt{3}} > 1 \text{ which is not possible.}$$

Hence there exist no triangle with given elements.

Ans. (C)

Illustration 3 : The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Solution : Let the sides be n , $n + 1$, $n + 2$ cms.

$$\text{i.e. } AC = n, AB = n + 1, BC = n + 2$$

Smallest angle is B and largest one is A.

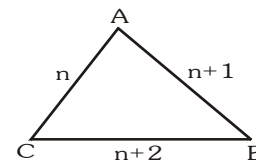
$$\text{Here, } \angle A = 2\angle B$$

$$\text{Also, } \angle A + \angle B + \angle C = 180$$

$$\Rightarrow 3\angle B + \angle C = 180 \Rightarrow \angle C = 180 - 3\angle B$$

We have, sine law as,

$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \Rightarrow \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180-3B)}{n+1}$$



$$\Rightarrow \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin 3B}{n+1}$$

(i) (ii) (iii)

from (i) and (ii);

$$\frac{2 \sin B \cos B}{n+2} = \frac{\sin B}{n} \Rightarrow \cos B = \frac{n+2}{2n} \quad \dots\dots\dots (iv)$$

and from (ii) and (iii);

$$\frac{\sin B}{n} = \frac{3 \sin B - 4 \sin^3 B}{n+1} \Rightarrow \frac{\sin B}{n} = \frac{\sin B(3 - 4 \sin^2 B)}{n+1}$$

$$\Rightarrow \frac{n+1}{n} = 3 - 4(1 - \cos^2 B) \quad \dots\dots\dots (v)$$

from (iv) and (v), we get

$$\frac{n+1}{n} = -1 + 4 \left(\frac{n+2}{2n} \right)^2 \Rightarrow \frac{n+1}{n} + 1 = \left(\frac{n^2 + 4n + 4}{n^2} \right)$$

$$\Rightarrow \frac{2n+1}{n} = \frac{n^2 + 4n + 4}{n^2} \Rightarrow 2n^2 + n = n^2 + 4n + 4$$

$$\Rightarrow n^2 - 3n - 4 = 0 \Rightarrow (n - 4)(n + 1) = 0$$

$$n = 4 \text{ or } -1$$

where $n \neq -1$

$\therefore n = 4$. Hence the sides are 4, 5, 6

Ans.

Do yourself - 1 :

(i) If in a $\triangle ABC$, $\angle A = \frac{\pi}{6}$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.

(ii) Show that, in any $\triangle ABC$: $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0$.

(iii) If in a $\triangle ABC$, $\frac{\sin A}{\sin C} = \frac{\sin(A - B)}{\sin(B - C)}$, show that a^2, b^2, c^2 are in A.P.

(iv) If in a $\triangle ABC$, $\angle A = 3\angle B$, then prove that $\sin B = \frac{1}{2} \sqrt{\frac{3b-a}{b}}$.

2. COSINE FORMULAE :

(a) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
 or $a^2 = b^2 + c^2 - 2bc \cos A$

(b) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$

(c) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Illustration 4 : In a triangle ABC, if $B = 30^\circ$ and $c = \sqrt{3}b$, then A can be equal to -

(A) 45°

(B) 60°

(C) 90°

(D) 120°

Solution :

$$\text{We have } \cos B = \frac{c^2 + a^2 - b^2}{2ca} \Rightarrow \frac{\sqrt{3}}{2} = \frac{3b^2 + a^2 - b^2}{2 \times \sqrt{3}b \times a}$$

$$\Rightarrow a^2 - 3ab + 2b^2 = 0 \Rightarrow (a - 2b)(a - b) = 0$$

$$\Rightarrow \text{Either } a = b \Rightarrow A = 30^\circ$$

$$\text{or } a = 2b \Rightarrow a^2 = 4b^2 = b^2 + c^2 \Rightarrow A = 90^\circ$$

Ans. (C)

Illustration 5 : In a triangle ABC, $(a^2 - b^2 - c^2) \tan A + (a^2 - b^2 + c^2) \tan B$ is equal to -

- (A) $(a^2 + b^2 - c^2) \tan C$ (B) $(a^2 + b^2 + c^2) \tan C$
 (C) $(b^2 + c^2 - a^2) \tan C$ (D) none of these

Solution : Using cosine law :

The given expression is equal to $-2bc \cos A \tan A + 2ac \cos B \tan B$

$$= 2abc \left(-\frac{\sin A}{a} + \frac{\sin B}{b} \right) = 0$$

Ans. (D)

Illustration 6 : If in a triangle ABC, $\frac{2 \cos A}{a} + \frac{\cos B}{b} + \frac{2 \cos C}{c} = \frac{a}{bc} + \frac{b}{ac}$, find the $\angle A =$

- (A) 90 (B) 60 (C) 30 (D) none of these

Solution : We have $\frac{2 \cos A}{a} + \frac{\cos B}{b} + \frac{2 \cos C}{c} = \frac{a}{bc} + \frac{b}{ac}$

Multiplying both sides by abc , we get

$$\Rightarrow 2bc \cos A + ac \cos B + 2ab \cos C = a^2 + b^2$$

$$\Rightarrow (b^2 + c^2 - a^2) + \frac{(a^2 + c^2 - b^2)}{2} + (a^2 + b^2 - c^2) = a^2 + b^2$$

$$\Rightarrow c^2 + a^2 - b^2 = 2a^2 - 2b^2 \quad \Rightarrow b^2 + c^2 = a^2$$

$$\therefore \Delta ABC \text{ is right angled at A.} \quad \Rightarrow \angle A = 90$$

Ans. (A)

Illustration 7 : A cyclic quadrilateral ABCD of area $\frac{3\sqrt{3}}{4}$ is inscribed in unit circle. If one of its side $AB = 1$,

and the diagonal $BD = \sqrt{3}$, find lengths of the other sides.

Solution : $AB = 1$, $BD = \sqrt{3}$, $OA = OB = OD = 1$

The given circle of radius 1 is also circumcircle of ΔABD

$\Rightarrow R = 1$ for ΔABD

$$\Rightarrow \frac{a}{\sin A} = 2R \Rightarrow A = 60$$

and hence $C = 120$

$$\text{Also by cosine rule on } \Delta ABD, (\sqrt{3})^2 = 1^2 + x^2 - 2x \cos 60^\circ$$

$$\Rightarrow x = 2$$

Now, area ABCD = $\Delta ABD + \Delta BCD$

$$\Rightarrow \frac{3\sqrt{3}}{4} = \frac{1}{2}(1 \cdot 2 \cdot \sin 60^\circ) + \frac{1}{2}(c \cdot d \cdot \sin 120^\circ)$$

$$\Rightarrow cd = 1, \text{ or } c^2 d^2 = 1$$

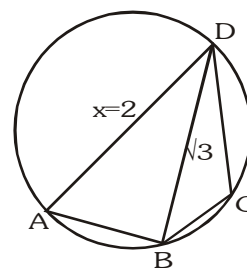
Also by cosine rule on triangle BCD we have

$$(\sqrt{3})^2 = c^2 + d^2 - 2cd \cos 120^\circ = c^2 + d^2 + cd$$

$$\Rightarrow c^2 + d^2 = 2 \text{ or } cd = 1$$

$$\Rightarrow c^2 \text{ and } d^2 \text{ are the roots of } t^2 - 2t + 1 = 0$$

$$\therefore c^2 = d^2 = 1 \therefore BC = 1 = CD \text{ and } AD = x = 2.$$



Do yourself - 2 :

(i) If $a : b : c = 4 : 5 : 6$, then show that $\angle C = 2\angle A$.(ii) In any $\triangle ABC$, prove that

$$(a) \quad \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

$$(b) \quad \frac{b^2}{a} \cos A + \frac{c^2}{b} \cos B + \frac{a^2}{c} \cos C = \frac{a^4 + b^4 + c^4}{2abc}$$

3. PROJECTION FORMULAE :

(a) $b \cos C + c \cos B = a$

(b) $c \cos A + a \cos C = b$

(c) $a \cos B + b \cos A = c$

Illustration 8 : In a $\triangle ABC$, $c \cos^2 \frac{A}{2} + a \cos^2 \frac{C}{2} = \frac{3b}{2}$, then show a, b, c are in A.P.

Solution : Here, $\frac{c}{2}(1 + \cos A) + \frac{a}{2}(1 + \cos C) = \frac{3b}{2}$

$$\Rightarrow a + c + (c \cos A + a \cos C) = 3b$$

$$\Rightarrow a + c + b = 3b \quad \{\text{using projection formula}\}$$

$$\Rightarrow a + c = 2b$$

which shows a, b, c are in A.P.

Do yourself - 3 :

(i) In a $\triangle ABC$, if $\angle A = \frac{\pi}{4}$, $\angle B = \frac{5\pi}{12}$, show that $a + c\sqrt{2} = 2b$.

(ii) In a $\triangle ABC$, prove that : (a) $b(a \cos C - c \cos A) = a^2 - c^2$ (b) $2\left(b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2}\right) = a + b + c$

4. NAPIER'S ANALOGY (TANGENT RULE) :

$$(a) \quad \tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c} \cot \frac{A}{2}$$

$$(b) \quad \tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a} \cot \frac{B}{2}$$

$$(c) \quad \tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot \frac{C}{2}$$

Illustration 9 : In a $\triangle ABC$, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Solution : Here, $\tan\left(\frac{A-B}{2}\right) = \frac{1}{3} \tan\left(\frac{A+B}{2}\right)$ (i)

using Napier's analogy, $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot\left(\frac{C}{2}\right)$ (ii)

from (i) & (ii) ;

$$\frac{1}{3} \tan\left(\frac{A+B}{2}\right) = \frac{a-b}{a+b} \cot\left(\frac{C}{2}\right) \Rightarrow \frac{1}{3} \cot\left(\frac{C}{2}\right) = \frac{a-b}{a+b} \cot\left(\frac{C}{2}\right)$$

$$\{\text{as } A + B + C = \pi \therefore \tan\left(\frac{B+C}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot \frac{C}{2}\}$$

$$\Rightarrow \frac{a-b}{a+b} = \frac{1}{3} \quad \text{or} \quad 3a - 3b = a + b$$

$$2a = 4b \quad \text{or} \quad \frac{a}{b} = \frac{2}{1} \Rightarrow \frac{b}{a} = \frac{1}{2}$$

Thus the ratio of the sides opposite to the angles is $b : a = 1 : 2$.

Ans.

Do yourself - 4 :

(i) In any $\triangle ABC$, prove that $\frac{b-c}{b+c} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$

(ii) If $\triangle ABC$ is right angled at C, prove that : (a) $\tan \frac{A}{2} = \sqrt{\frac{c-b}{c+b}}$ (b) $\sin(A-B) = \frac{a^2 - b^2}{a^2 + b^2}$

(iii) If in a $\triangle ABC$, two sides are $a = 3$, $b = 5$ and $\cos(A-B) = \frac{7}{25}$, find $\tan \frac{C}{2}$.

5. HALF ANGLE FORMULAE :

$$s = \frac{a+b+c}{2} = \text{semi-perimeter of triangle.}$$

(a) (i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$	(ii) $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$	(iii) $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$
(b) (i) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$	(ii) $\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$	(iii) $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$
(c) (i) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$	(ii) $\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$	(iii) $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$
$= \frac{\Delta}{s(s-a)}$	$= \frac{\Delta}{s(s-b)}$	$= \frac{\Delta}{s(s-c)}$

(d) Area of Triangle

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3, \text{ where } p_1, p_2, p_3 \text{ are altitudes from vertices A, B, C respectively.}$$

Illustration 10 : If in a triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to -

(A) $\frac{a+b}{2ab} \cos \frac{C}{2}$ (B) $\frac{2ab}{a+b} \sin \frac{C}{2}$ (C) $\frac{2ab}{a+b} \cos \frac{C}{2}$ (D) $\frac{b \sin \angle DAC}{\sin(B+C/2)}$

Solution :

$$\triangle CAB = \triangle CAD + \triangle CDB$$

$$\Rightarrow \frac{1}{2} ab \sin C = \frac{1}{2} b \cdot CD \cdot \sin\left(\frac{C}{2}\right) + \frac{1}{2} a \cdot CD \cdot \sin\left(\frac{C}{2}\right)$$

$$\Rightarrow CD(a+b) \sin\left(\frac{C}{2}\right) = ab \left(2 \sin\left(\frac{C}{2}\right) \cos\left(\frac{C}{2}\right)\right)$$

$$\text{So } CD = \frac{2ab \cos(C/2)}{(a+b)}$$

$$\text{and in } \triangle CAD, \frac{CD}{\sin \angle DAC} = \frac{b}{\sin \angle CDA} \quad (\text{by sine rule})$$

$$\Rightarrow CD = \frac{b \sin \angle DAC}{\sin(B+C/2)}$$

Ans. (C,D)

Illustration 11 : If Δ is the area and $2s$ the sum of the sides of a triangle, then show $\Delta \leq \frac{s^2}{3\sqrt{3}}$.

Solution : We have, $2s = a + b + c$, $\Delta^2 = s(s-a)(s-b)(s-c)$

Now, A.M. \geq G.M.

$$\frac{(s-a) + (s-b) + (s-c)}{3} \geq \{(s-a)(s-b)(s-c)\}^{1/3}$$

$$\text{or } \frac{3s-2s}{3} \geq \left(\frac{\Delta^2}{s}\right)^{1/3}$$

$$\text{or } \frac{s}{3} \geq \left(\frac{\Delta^2}{s}\right)^{1/3}$$

$$\text{or } \frac{\Delta^2}{s} \leq \frac{s^3}{27} \Rightarrow \Delta \leq \frac{s^2}{3\sqrt{3}}$$

Ans.

Do yourself - 5 :

(i) Given $a = 6$, $b = 8$, $c = 10$. Find

(a) $\sin A$ (b) $\tan A$ (c) $\sin \frac{A}{2}$ (d) $\cos \frac{A}{2}$ (e) $\tan \frac{A}{2}$ (f) Δ

(ii) Prove that in any ΔABC , $(abc) \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} = \Delta^2$.

(iii) Show that if $\left(\tan \frac{A}{2} + \tan \frac{C}{2}\right) = \frac{2}{3} \cot \frac{B}{2}$, then a, b, c are in A.P.

6. m-n THEOREM :

$$(m+n) \cot \theta = m \cot \alpha - n \cot \beta$$

$$(m+n) \cot \theta = n \cot B - m \cot C.$$

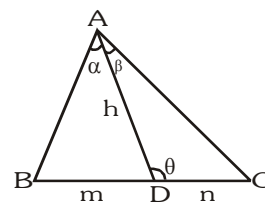


Illustration 12 : The base of a Δ is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that :

$$\left(\frac{1}{t_1} + \frac{1}{t_2}\right) \left(\frac{1}{t_2} + \frac{1}{t_3}\right) = 4 \left(1 + \frac{1}{t_2^2}\right)$$

Solution : Let the points P and Q divide the side BC in three equal parts :

Such that $BP = PQ = QC = x$

Also let,

$$\angle BAP = \alpha, \angle PAQ = \beta, \angle QAC = \gamma$$

$$\text{and } \angle AQC = \theta$$

From question, $\tan \alpha = t_1$, $\tan \beta = t_2$, $\tan \gamma = t_3$.

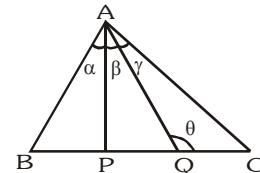
Applying

$m : n$ rule in triangle ABC we get,

$$(2x + x) \cot \theta = 2x \cot(\alpha + \beta) - x \cot \gamma \quad \dots\dots (i)$$

from ΔAPC , we get

$$(x + x) \cot \theta = x \cot \beta - x \cot \gamma \quad \dots\dots (ii)$$



dividing (i) and (ii), we get

$$\frac{3}{2} = \frac{2 \cot(\alpha + \beta) - \cot \gamma}{\cot \beta - \cot \gamma}$$

$$\text{or } 3 \cot \beta - \cot \gamma = \frac{4(\cot \alpha \cdot \cot \beta - 1)}{\cot \beta + \cot \alpha}$$

$$\text{or } 3 \cot^2 \beta - \cot \beta \cot \gamma + 3 \cot \alpha \cdot \cot \beta - \cot \alpha \cdot \cot \gamma = 4 \cot \alpha \cdot \cot \beta - 4$$

$$\text{or } 4 + 4 \cot^2 \beta = \cot^2 \beta + \cot \alpha \cdot \cot \beta + \cot \beta \cdot \cot \gamma + \cot \gamma \cdot \cot \alpha$$

$$\text{or } 4(1 + \cot^2 \beta) = (\cot \beta + \cot \alpha)(\cot \beta + \cot \gamma)$$

$$\text{or } 4 \left(1 + \frac{1}{t_2^2} \right) = \left(\frac{1}{t_1} + \frac{1}{t_2} \right) \left(\frac{1}{t_2} + \frac{1}{t_3} \right)$$

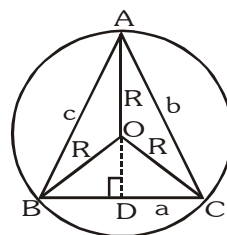
Do yourself - 6 :

(i) The median AD of a ΔABC is perpendicular to AB, prove that $\tan A + 2 \tan B = 0$

7. RADIUS OF THE CIRCUMCIRCLE 'R' :

Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4 \Delta}$$



8. RADIUS OF THE INCIRCLE 'r' :

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.

$$r = \frac{\Delta}{s} = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2} = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= a \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} = b \frac{\sin \frac{A}{2} \sin \frac{C}{2}}{\cos \frac{B}{2}} = c \frac{\sin \frac{B}{2} \sin \frac{A}{2}}{\cos \frac{C}{2}}$$

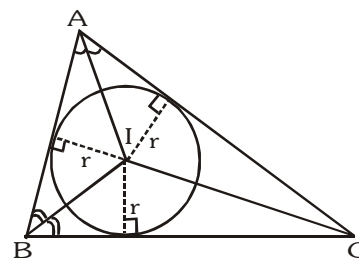


Illustration 13 : In a triangle ABC, if $a : b : c = 4 : 5 : 6$, then ratio between its circumradius and inradius is-

(A) $\frac{16}{7}$

(B) $\frac{16}{9}$

(C) $\frac{7}{16}$

(D) $\frac{11}{7}$

Solution : $\frac{R}{r} = \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = \frac{(abc)s}{4\Delta^2} \Rightarrow \frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)} \dots (i)$

$$\therefore a : b : c = 4 : 5 : 6 \Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k \text{ (say)}$$

$$\Rightarrow a = 4k, b = 5k, c = 6k$$

$$\therefore s = \frac{a+b+c}{2} = \frac{15k}{2}, s-a = \frac{7k}{2}, s-b = \frac{5k}{2}, s-c = \frac{3k}{2}$$

$$\text{using (i) in these values } \frac{R}{r} = \frac{(4k)(5k)(6k)}{4 \left(\frac{7k}{2} \right) \left(\frac{5k}{2} \right) \left(\frac{3k}{2} \right)} = \frac{16}{7}$$

Ans. (A)

Illustration 14 : If A, B, C are the angles of a triangle, prove that : $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$.

Solution :

$$\begin{aligned}\cos A + \cos B + \cos C &= 2 \cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right) + \cos C \\&= 2 \sin \frac{C}{2} \cdot \cos\left(\frac{A-B}{2}\right) + 1 - 2 \sin^2 \frac{C}{2} = 1 + 2 \sin \frac{C}{2} \left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right) \right] \\&= 1 + 2 \sin \frac{C}{2} \left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right) \right] \quad \left\{ \because \frac{C}{2} = 90^\circ - \left(\frac{A+B}{2}\right) \right\} \\&= 1 + 2 \sin \frac{C}{2} \cdot 2 \sin \frac{A}{2} \cdot \sin \frac{B}{2} = 1 + 4 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} \\&= 1 + \frac{r}{R} \quad \left\{ \text{as, } r = 4R \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} \right\} \\ \Rightarrow \quad \cos A + \cos B + \cos C &= 1 + \frac{r}{R}. \text{ Hence proved.}\end{aligned}$$

Do yourself - 7 :

(i) If in $\triangle ABC$, $a = 3$, $b = 4$ and $c = 5$, find

(a) Δ (b) R (c) r

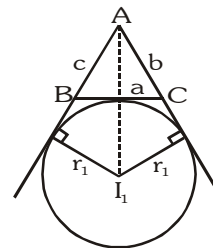
(ii) In a $\triangle ABC$, show that :

(a) $\frac{a^2 - b^2}{c} = 2R \sin(A - B)$ (b) $r \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{\Delta}{4R}$ (c) $a + b + c = \frac{abc}{2Rr}$

(iii) Let Δ & Δ' denote the areas of a Δ and that of its incircle. Prove that $\Delta : \Delta' = \left(\cot \frac{A}{2} \cdot \cot \frac{B}{2} \cdot \cot \frac{C}{2} \right) : \pi$

9. RADII OF THE EX-CIRCLES :

Point of intersection of two external angles and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to $\angle A$ of $\triangle ABC$ and so on, then -



(a) $r_1 = \frac{\Delta}{s-a} = s \tan \frac{A}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$

(b) $r_2 = \frac{\Delta}{s-b} = s \tan \frac{B}{2} = 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} = \frac{b \cos \frac{A}{2} \cos \frac{C}{2}}{\cos \frac{B}{2}}$

(c) $r_3 = \frac{\Delta}{s-c} = s \tan \frac{C}{2} = 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} = \frac{c \cos \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{C}{2}}$

I_1, I_2 and I_3 are taken as ex-centre opposite to vertex A, B, C respectively.

Illustration 15 : Value of the expression $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$ is equal to -

- (A) 1 (B) 2 (C) 3 (D) 0

Solution :

$$\frac{(b-c)}{r_1} + \frac{(c-a)}{r_2} + \frac{(a-b)}{r_3}$$

$$\Rightarrow (b-c)\left(\frac{s-a}{\Delta}\right) + (c-a)\left(\frac{s-b}{\Delta}\right) + (a-b)\left(\frac{s-c}{\Delta}\right)$$

$$\Rightarrow \frac{(s-a)(b-c) + (s-b)(c-a) + (s-c)(a-b)}{\Delta}$$

$$= \frac{s(b-c+c-a+a-b) - [ab-ac+bc-ba+ac-bc]}{\Delta} = \frac{0}{\Delta} = 0$$

Thus, $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$

Ans. (D)

Illustration 16 : If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Solution : We have, $r_1 - r = r_2 + r_3$

$$\Rightarrow \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c} \Rightarrow \frac{s-s+a}{s(s-a)} = \frac{s-c+s-b}{(s-b)(s-c)}$$

$$\Rightarrow \frac{a}{s(s-a)} = \frac{2s-(b+c)}{(s-b)(s-c)} \quad \{as, 2s = a+b+c\}$$

$$\Rightarrow \frac{a}{s(s-a)} = \frac{a}{(s-b)(s-c)} \Rightarrow s^2 - (b+c)s + bc = s^2 - as$$

$$\Rightarrow s(-a+b+c) = bc \Rightarrow \frac{(b+c-a)(a+b+c)}{2} = bc$$

$$\Rightarrow (b+c)^2 - (a)^2 = 2bc \Rightarrow b^2 + c^2 + 2bc - a^2 = 2bc$$

$$\Rightarrow b^2 + c^2 = a^2$$

$$\therefore \angle A = 90^\circ$$

Ans.

Do yourself - 8 :

(i) In an equilateral $\triangle ABC$, $R = 2$, find

(a) r (b) r_1 (c) a

(ii) In a $\triangle ABC$, show that

(a) $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$

(b) $\frac{1}{4} r^2 s^2 \left(\frac{1}{r} - \frac{1}{r_1} \right) \left(\frac{1}{r} - \frac{1}{r_2} \right) \left(\frac{1}{r} - \frac{1}{r_3} \right) = \frac{r+r_1+r_2-r_3}{4 \cos C} = R$

(c) $\sqrt{rr_1 r_2 r_3} = \Delta$

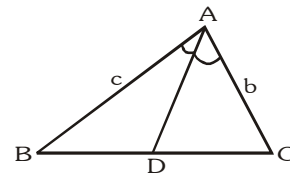
10. ANGLE BISECTORS & MEDIANS :

An angle bisector divides the base in the ratio of corresponding sides.

$$\frac{BD}{CD} = \frac{c}{b} \Rightarrow BD = \frac{ac}{b+c} \quad \& \quad CD = \frac{ab}{b+c}$$

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

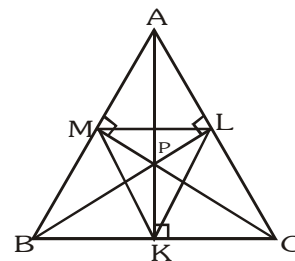
$$m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2} \quad \text{and} \quad \beta_a = \frac{2bc \cos \frac{A}{2}}{b+c}$$



Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

11. ORTHOCENTRE :

- Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.
- The distances of the orthocentre from the angular points of the ΔABC are $2R \cos A$, $2R \cos B$, & $2R \cos C$.
- The distance of P from sides are $2R \cos B \cos C$, $2R \cos C \cos A$ and $2R \cos A \cos B$.



Do yourself - 9 :

- If x, y, z are the distance of the vertices of ΔABC respectively from the orthocentre, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$
- If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that
 - $p_1 p_2 p_3 = \frac{a^2 b^2 c^2}{8R^3}$
 - $\Delta = \sqrt{\frac{1}{2} R p_1 p_2 p_3}$
- In a ΔABC , AD is altitude and H is the orthocentre prove that $AH : DH = (\tan B + \tan C) : \tan A$
- In a ΔABC , the lengths of the bisectors of the angle A, B and C are x, y, z respectively. Show that $\frac{1}{x} \cos \frac{A}{2} + \frac{1}{y} \cos \frac{B}{2} + \frac{1}{z} \cos \frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Also show that $\frac{a}{b+c} = \sqrt{1 - \frac{x^2}{bc}}$

12. THE DISTANCES BETWEEN THE SPECIAL POINTS :

- The distance between circumcentre and orthocentre is $= R\sqrt{1 - 8 \cos A \cos B \cos C}$
- The distance between circumcentre and incentre is $= \sqrt{R^2 - 2Rr}$
- The distance between incentre and orthocentre is $= \sqrt{2r^2 - 4R^2 \cos A \cos B \cos C}$
- The distances between circumcentre & excentres are

$$OI_1 = R\sqrt{1 + 8 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \sqrt{R^2 + 2Rr_1} \text{ \& so on.}$$

Illustration 17 : Prove that the distance between the circumcentre and the orthocentre of a triangle ABC is

$$R\sqrt{1 - 8 \cos A \cos B \cos C}.$$

Solution : Let O and P be the circumcentre and the orthocentre respectively. If OF is the perpendicular to AB, we have $\angle OAF = 90^\circ - \angle AOF = 90^\circ - C$. Also $\angle PAL = 90^\circ - C$.

$$\text{Hence, } \angle OAP = A - \angle OAF - \angle PAL = A - 2(90^\circ - C) = A + 2C - 180^\circ$$

$$= A + 2C - (A + B + C) = C - B.$$

$$\text{Also } OA = R \text{ and } PA = 2R \cos A.$$

Now in $\triangle AOP$,

$$OP^2 = OA^2 + PA^2 - 2OA \cdot PA \cos OAP$$

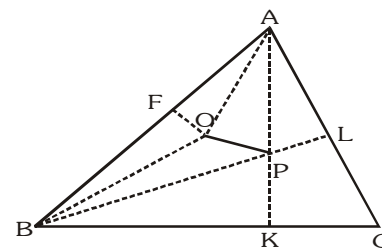
$$= R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos(C - B)$$

$$= R^2 + 4R^2 \cos A [\cos A - \cos(C - B)]$$

$$= R^2 - 4R^2 \cos A [\cos(B + C) + \cos(C - B)] = R^2 - 8R^2 \cos A \cos B \cos C.$$

$$\text{Hence } OP = R \sqrt{1 - 8 \cos A \cos B \cos C}.$$

Ans.



Do yourself - 10 :

- (i) Show that in an equilateral triangle, circumcentre, orthocentre and incentre overlap each other.
- (ii) If the incentre and circumcentre of a triangle are equidistant from the side BC, show that $\cos B + \cos C = 1$.

13. SOLUTION OF TRIANGLES :

The three sides a, b, c and the three angles A, B, C are called the elements of the triangle ABC . When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

* If the three sides a, b, c are given, angle A is obtained from $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$. B and C can be obtained in the similar way.

* If two sides b and c and the included angle A are given, then $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$ gives $\frac{B-C}{2}$. Also

$$\frac{B+C}{2} = 90^\circ - \frac{A}{2}, \text{ so that } B \text{ and } C \text{ can be evaluated. The third side is given by } a = b \frac{\sin A}{\sin B}$$

$$\text{or } a^2 = b^2 + c^2 - 2bc \cos A.$$

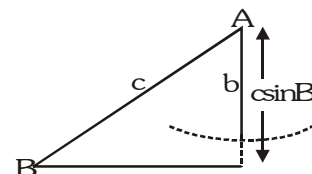
* If two sides b and c and an angle opposite the one of them (say B) are given then

$$\sin C = \frac{c}{b} \sin B, \quad A = 180^\circ - (B + C) \quad \text{and} \quad a = \frac{b \sin A}{\sin B} \text{ given the remaining elements.}$$

Case I :

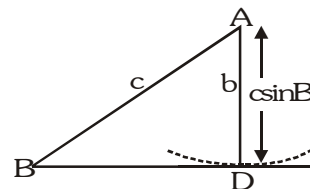
$$b < c \sin B.$$

We draw the side c and angle B . Now it is obvious from the figure that there is no triangle possible.



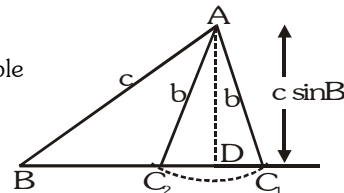
Case II :

$b = c \sin B$ and B is an acute angle, there is only one triangle possible. and it is right-angled at C .



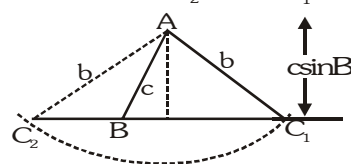
Case III :

$b > c \sin B$, $b < c$ and B is an acute angle, then there are two triangles possible for two values of angle C .



Case IV :

$b > c \sin B$, $c < b$ and B is an acute angle, then there is only one triangle.



Case V :

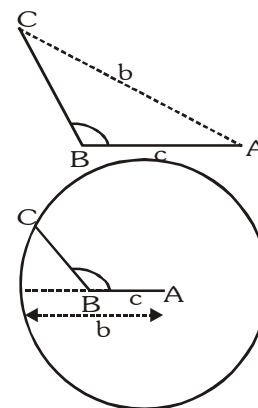
$b > c \sin B$, $c > b$ and B is an obtuse angle.

For any choice of point C , b will be greater than c which is a contradiction as $c > b$ (given). So there is no triangle possible.

Case VI :

$b > c \sin B$, $c < b$ and B is an obtuse angle.

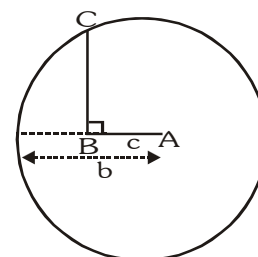
We can see that the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



Case VII :

$b > c$ and $B = 90^\circ$.

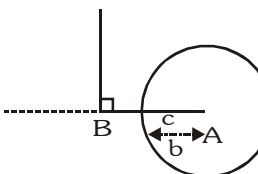
Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



Case VIII :

$b \leq c$ and $B = 90^\circ$.

The circle with A as centre and b as radius will not cut the line in any point. So no triangle is possible.



This is, sometimes, called an ambiguous case.

Alternative Method :

By applying cosine rule, we have $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$$\Rightarrow a^2 - (2c \cos B)a + (c^2 - b^2) = 0 \Rightarrow a = c \cos B \pm \sqrt{(c \cos B)^2 - (c^2 - b^2)}$$

$$\Rightarrow a = c \cos B \pm \sqrt{b^2 - (c \sin B)^2}$$

This equation leads to following cases :

Case-I : If $b < c \sin B$, no such triangle is possible.

Case-II: Let $b = c \sin B$. There are further following case :

(a) B is an obtuse angle $\Rightarrow \cos B$ is negative. There exists no such triangle.

(b) B is an acute angle $\Rightarrow \cos B$ is positive. There exists only one such triangle.

Case-III: Let $b > c \sin B$. There are further following cases :

(a) B is an acute angle $\Rightarrow \cos B$ is positive. In this case triangle will exist if and only if $c \cos B >$

$\sqrt{b^2 - (c \sin B)^2}$ or $c > b \Rightarrow$ Two such triangle is possible. If $c < b$, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow \cos B$ is negative. In this case triangle will exist if and only if

$\sqrt{b^2 - (c \sin B)^2} > |c \cos B| \Rightarrow b > c$. So in this case only one such triangle is possible. If $b < c$ there exists no such triangle.

This is called an ambiguous case.

* If one side a and angles B and C are given, then $A = 180 - (B + C)$, and $b = \frac{a \sin B}{\sin A}$, $c = \frac{a \sin C}{\sin A}$.

* If the three angles A, B, C are given, we can only find the ratios of the sides a, b, c by using sine rule (since there are infinite similar triangles possible).

Illustration 18 : In the ambiguous case of the solution of triangles, prove that the circumcircles of the two triangles are of same size.

Solution : Let us say b, c and angle B are given in the ambiguous case. Both the triangles will have b and its opposite angle as B . so $\frac{b}{\sin B} = 2R$ will be given for both the triangles. So their circumradii and therefore their sizes will be same.

Illustration 19 : If a, b and A are given in a triangle and c_1, c_2 are the possible values of the third side, prove that $c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = 4a^2 \cos^2 A$.

Solution :

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow c^2 - 2bc \cos A + b^2 - a^2 = 0.$$

$$c_1 + c_2 = 2b \cos A \text{ and } c_1 c_2 = b^2 - a^2.$$

$$\Rightarrow c_1^2 + c_2^2 - 2c_1 c_2 \cos 2A = (c_1 + c_2)^2 - 2c_1 c_2 (1 + \cos 2A)$$

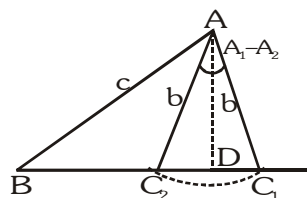
$$= 4b^2 \cos^2 A - 2(b^2 - a^2) 2 \cos^2 A = 4a^2 \cos^2 A.$$

Illustration 20 : If b, c, B are given and $b < c$, prove that $\cos\left(\frac{A_1 - A_2}{2}\right) = \frac{c \sin B}{b}$.

Solution : $\angle C_2 A C_1$ is bisected by AD .

$$\Rightarrow \text{In } \triangle A C_2 D, \cos\left(\frac{A_1 - A_2}{2}\right) = \frac{AD}{AC_2} = \frac{c \sin B}{b}$$

Hence proved.



Do yourself - 11 :

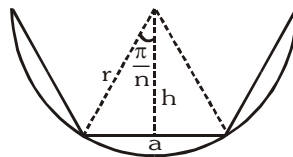
- (i) If b, c, B are given and $b < c$, prove that $\sin\left(\frac{A_1 - A_2}{2}\right) = \frac{a_1 - a_2}{2b}$
- (ii) In a $\triangle ABC$, b, c, B ($c > b$) are given. If the third side has two values a_1 and a_2 such that $a_1 = 3a_2$, show that $\sin B = \sqrt{\frac{4b^2 - c^2}{3c^2}}$.

14. REGULAR POLYGON :

A regular polygon has all its sides equal. It may be inscribed or circumscribed.

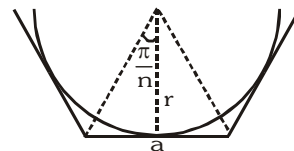
(a) Inscribed in circle of radius r :

- (i) $a = 2h \tan \frac{\pi}{n} = 2r \sin \frac{\pi}{n}$
- (ii) Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr \sin \frac{\pi}{n}$ and $A = \frac{1}{2} nr^2 \sin \frac{2\pi}{n}$



(b) Circumscribed about a circle of radius r :

- (i) $a = 2r \tan \frac{\pi}{n}$
- (ii) Perimeter (P) and area (A) of a regular polygon of n sides



circumscribed about a given circle of radius r is given by $P = 2nr \tan \frac{\pi}{n}$ and $A = nr^2 \tan \frac{\pi}{n}$

Do yourself - 12 :

- (i) If the perimeter of a circle and a regular polygon of n sides are equal, then

prove that $\frac{\text{area of the circle}}{\text{area of polygon}} = \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}}$.

- (ii) The ratio of the area of n -sided regular polygon, circumscribed about a circle, to the area of the regular polygon of equal number of sides inscribed in the circle is $4 : 3$. Find the value of n .

15. IMPORTANT POINTS :

- (a) (i) If $a \cos B = b \cos A$, then the triangle is isosceles.
(ii) If $a \cos A = b \cos B$, then the triangle is isosceles or right angled.
- (b) In right angle triangle
(i) $a^2 + b^2 + c^2 = 8R^2$ (ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$
- (c) In equilateral triangle
(i) $R = 2r$ (ii) $r_1 = r_2 = r_3 = \frac{3R}{2}$
(iii) $r : R : r_1 = 1 : 2 : 3$ (iv) $\text{area} = \frac{\sqrt{3}a^2}{4}$ (v) $R = \frac{a}{\sqrt{3}}$
- (d) (i) The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.
(ii) The orthocentre of right angled triangle is the vertex at the right angle.
(iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio $2 : 1$ except in case of equilateral triangle. In equilateral triangle, all these centres coincide
- (e) Area of a cyclic quadrilateral $= \sqrt{s(s-a)(s-b)(s-c)(s-d)}$
where a, b, c, d are lengths of the sides of quadrilateral and $s = \frac{a+b+c+d}{2}$.

Illustration 21 : For a ΔABC , it is given that $\cos A + \cos B + \cos C = 3/2$. Prove that the triangle is equilateral.

Solution : If a, b, c are the sides of the ΔABC , then $\cos A + \cos B + \cos C = 3/2$

$$\Rightarrow \frac{b^2 + c^2 - a^2}{2bc} + \frac{a^2 + c^2 - b^2}{2ac} + \frac{a^2 + b^2 - c^2}{2ab} = \frac{3}{2}$$

$$\Rightarrow ab^2 + ac^2 - a^3 + bc^2 + ba^2 - b^3 + ca^2 + cb^2 - c^3 = 3abc$$

$$\Rightarrow ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - 6abc = a^3 + b^3 + c^3 - 3abc$$

$$\Rightarrow a(b-c)^2 + b(c-a)^2 + c(a-b)^2 = \frac{(a+b+c)}{2} \left\{ (a-b)^2 + (b-c)^2 + (c-a)^2 \right\}$$

$$\Rightarrow (a+b-c)(a-b)^2 + (b+c-a)(b-c)^2 + (c+a-b)(c-a)^2 = 0 \quad \dots\dots\dots (i)$$

as we know $a+b > c, b+c > a, c+a > b$

\therefore each term on the left side of equation (i) has positive coefficient multiplied by perfect square, each must be separately zero.

$$\Rightarrow a = b = c.$$

Hence Δ is equilateral.

Ans.

Illustration 22 : In a triangle ABC , if $\cos A + 2 \cos B + \cos C = 2$. Prove that the sides of the triangle are in A.P.

Solution : $\cos A + 2 \cos B + \cos C = 2$ or $\cos A + \cos C = 2(1 - \cos B)$

$$\Rightarrow 2 \cos \left(\frac{A+C}{2} \right) \cdot \cos \left(\frac{A-C}{2} \right) = 4 \sin^2 B / 2$$

$$\Rightarrow \cos \left(\frac{A-C}{2} \right) = 2 \sin \frac{B}{2} \quad \left\{ \text{as } \cos \left(\frac{A+C}{2} \right) = \cos \left(\frac{\pi}{2} - \frac{B}{2} \right) = \sin \frac{B}{2} \right\}$$

$$\Rightarrow \cos \left(\frac{A-C}{2} \right) = 2 \cos \left(\frac{A+C}{2} \right)$$

$$\Rightarrow \cos \frac{A}{2} \cdot \cos \frac{C}{2} + \sin \frac{A}{2} \cdot \sin \frac{C}{2} = 2 \cos \frac{A}{2} \cdot \cos \frac{C}{2} - 2 \sin \frac{A}{2} \cdot \sin \frac{C}{2}$$

$$\Rightarrow \cot \frac{A}{2} \cdot \cot \frac{C}{2} = 3 \Rightarrow \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 3$$

$$\Rightarrow \frac{s}{(s-b)} = 3 \Rightarrow s = 3s - 3b \Rightarrow 2s = 3b$$

$$\Rightarrow a + c = 2b, \quad \therefore a, b, c \text{ are in A.P.}$$

Ans.

ANSWERS FOR DO YOURSELF

1 : (i) 90

4 : (iii) $\frac{1}{3}$

5 : (i) (a) $\frac{3}{5}$ (b) $\frac{3}{4}$ (c) $\frac{1}{\sqrt{10}}$ (d) $\frac{3}{\sqrt{10}}$ (e) $\frac{1}{3}$ (f) 24

7 : (i) (a) 6 (b) $\frac{5}{2}$ (c) 1

8 : (i) (a) 1 (b) 3 (c) $2\sqrt{3}$

12 : (ii) 6