CHAPTER

02

Theory of Equations

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Session 1

Polynomial in One Variable, Identity, Linear Equation, Quadratic Equations, Standard Quadratic Equation

Polynomial in One Variable

An algebraic expression containing many terms of the form cx^n, n being a non-negative integer is called a polynomial,

i.e., $$f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \dots + a_{n-1} \cdot x + a_n,$$

where x is a variable, $a_0, a_1, a_2, \dots, a_n$ are constants and $a_0 \neq 0$.

1. Real Polynomial

Let $a_0, a_1, a_2, \dots, a_n$ be real numbers and x is a real variable.

Then,
$f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \dots + a_{n-1} \cdot x + a_n$

is called a real polynomial of real variable (x) with real coefficients.

For example, $5x^3 - 3x^2 + 7x - 4$, $x^2 - 3x + 1$, etc., are real polynomials.

2. Complex Polynomial

Let $a_0, a_1, a_2, \dots, a_n$ are complex numbers and x is a varying complex number.

Then $f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \dots + a_{n-1} \cdot x + a_n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.

For example, $x^3 - 7ix^2 + (3 - 2i)x + 13$, $3x^2 - (2 + 3i)x + 5i$, etc. (where $i = \sqrt{-1}$) are complex polynomials.

3. Rational Expression or Rational Function

An expression of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x, is called a rational expression. As a particular case when $Q(x)$ is a non-zero constant, $\frac{P(x)}{Q(x)}$ reduces to a polynomial.

Thus, every polynomial is a rational expression but a rational expression may or may not be a polynomial.

For example,

(i) $x^2 - 7x + 8$ (ii) $\frac{2}{x - 3}$

(iii) $\frac{x^3 - 6x^2 + 11x - 6}{(x - 4)}$ (iv) $x + \frac{3}{x}$ or $\frac{x^2 + 3}{x}$

4. Degree of Polynomial

The highest power of variable (x) present in the polynomial is called the degree of the polynomial.

For example, $f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \dots + a_{n-1} \cdot x + a_n$ is a polynomial in x of degree n.

Remark

A polynomial of degree one is generally called a linear polynomial. Polynomials of degree 2, 3, 4 and 5 are known as quadratic, cubic, biquadratic and pentic polynomials, respectively.

5. Polynomial Equation

If $f(x)$ is a polynomial, real or complex, then $f(x) = 0$ is called a polynomial equation.

(i) A polynomial equation has atleast one root.

(ii) A polynomial equation of degree n has n roots.

Remarks

1. A polynomial equation of degree one is called a **linear equation** i.e. $ax + b = 0$, where $a, b \in C$, set of all complex numbers and $a \neq 0$.
2. A polynomial equation of degree two is called a **quadratic equation** i.e., $ax^2 + bx + c$, where $a, b, c \in C$ and $a \neq 0$.
3. A polynomial equation of degree three is called a **cubic equation** i.e., $ax^3 + bx^2 + cx + d = 0$, where $a, b, c, d \in C$ and $a \neq 0$.
4. A polynomial equation of degree four is called a **biquadratic equation** i.e., $ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a, b, c, d, e \in C$ and $a \neq 0$.
5. A polynomial equation of degree five is called a **pentic equation** i.e., $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, where $a, b, c, d, e, f \in C$ and $a \neq 0$.

6. Roots of an Equation

The values of the variable for which an equation is satisfied are called the roots of the equation.

If $x=\alpha$ is a root of the equation $f(x)=0$, then $f(\alpha)=0$.

Remark

The real roots of an equation $f(x)=0$ are the values of x, where the curve $y=f(x)$ crosses X-axis.

7. Solution Set

The set of all roots of an equation in a given domain is called the solution set of the equation.

For example, The roots of the equation $x^3-2x^2-5x+6=0$ are $1,-2,3$, the solution set is $\{1,-2,3\}$.

Remark

Solve or solving an equation means finding its solution set or obtaining all its roots.

Identity

If two expressions are equal for all values of x, then the statement of equality between the two expressions is called an identity.

For example, $(x+1)^2 = x^2+2x+1$ is an identity in x.

or

If $f(x)=0$ is satisfied by every value of x in the domain of $f(x)$, then it is called an identity.

For example, $f(x)=(x+1)^2-(x^2+2x+1)=0$ is an identity in the domain C, as it is satisfied by every complex number.

or

If $f(x)=a_0\cdot x^n+a_1\cdot x^{n-1}+a_2\cdot x^{n-2}+\ldots+a_{n-1}\cdot x+a_n=0$ have more than n distinct roots, it is an identity, then

$$a_0=a_1=a_2=\ldots=a_{n-1}=a_n=0$$

For example, If $ax^2+bx+c=0$ is satisfied by more than two values of x, then $a=b=c=0$.

or

In an identity in x coefficients of similar powers of x on the two sides are equal.

For example, If $ax^4+bx^3+cx^2+dx+e=5x^4-3x^3+4x^2-7x-9$ be an identity in x, then

$$a=5, b=-3, c=4, d=-7, e=-9.$$

Thus, an identity in x satisfied by all values of x, where as an equation in x is satisfied by some particular values of x.

Example 1. If equation $(\lambda^2-5\lambda+6)x^2+(\lambda^2-3\lambda+2)x+(\lambda^2-4)=0$ is satisfied by more than two values of x, find the parameter λ.

Sol. If an equation of degree two is satisfied by more than two values of unknown, then it must be an identity. Then, we must have

$$\lambda^2-5\lambda+6=0,\ \lambda^2-3\lambda+2=0,\ \lambda^2-4=0$$

$$\Rightarrow \quad \lambda=2,3 \text{ and } \lambda=2,1 \text{ and } \lambda=2,-2$$

Common value of λ which satisfies each condition is $\lambda=2$.

Example 2. Show that

$$\frac{(x+b)(x+c)}{(b-a)(c-a)}+\frac{(x+c)(x+a)}{(c-b)(a-b)}+\frac{(x+a)(x+b)}{(a-c)(b-c)}=1$$

is an identity.

Sol. Given relation is

$$\frac{(x+b)(x+c)}{(b-a)(c-a)}+\frac{(x+c)(x+a)}{(c-b)(a-b)}+\frac{(x+a)(x+b)}{(a-c)(b-c)}=1 \quad \ldots(\text{i})$$

When $x=-a$, then LHS of Eq. (i) $=\dfrac{(b-a)(c-a)}{(b-a)(c-a)}=1$

$=$ RHS of Eq. (i)

When $x=-b$, then LHS of Eq. (i)

$=\dfrac{(c-b)(a-b)}{(c-b)(a-b)}=1=$ RHS of Eq. (i)

and when $x=-c$, then LHS of Eq. (i) $=\dfrac{(a-c)(b-c)}{(a-c)(b-c)}=1$

$=$ RHS of Eq. (i).

Thus, highest power of x occurring in relation of Eq. (i) is 2 and this relation is satisfied by three distinct values of $x\ (=-a,-b,-c)$. Therefore, it cannot be an equation and hence it is an identity.

Example 3. Show that $x^2-3|x|+2=0$ is an equation.

Sol. Put $x=0$ in $x^2-3|x|+2=0$

$$\Rightarrow \quad 0^2-3|0|+2=2\neq 0$$

Since, the relation $x^2-3|x|+2=0$ is not satisfied by $x=0$. Hence, it is an equation.

Linear Equation

An equation of the form

$$ax+b=0 \quad \ldots(\text{i})$$

where $a,b\in R$ and $a\neq 0$, is a linear equation.

Eq. (i) has an unique root equal to $-\dfrac{b}{a}$.

Example 4. Solve the equation $\frac{x}{2}+\frac{(3x-1)}{6}=1-\frac{x}{2}$

Sol. We have, $\frac{x}{2}+\frac{(3x-1)}{6}=1-\frac{x}{2}$

or $\frac{x}{2}+\frac{x}{2}+\frac{x}{2}=1+\frac{1}{6}$

or $\frac{3x}{2}=\frac{7}{6}$

or $x=\frac{7}{9}$

Example 5. Solve the equation $(a-3)x+5=a+2$.

Sol. *Case* **I** For $a \neq 3$, this equation is linear, then

$(a-3)\,x=(a-3)$

$\therefore \quad x=\frac{(a-3)}{(a-3)}=1$

Case **II** For $a=3$,

$$0 \cdot x+5=3+2$$

$\Rightarrow \quad 5=5$

Therefore, any real number is its solution.

Quadratic Equations

An equation in which the highest power of the unknown quantity is 2, is called a quadratic equation.

Quadratic equations are of two types :

1. Purely Quadratic Equation

A quadratic equation in which the term containing the first degree of the unknown quantity is absent, is called a purely quadratic equation.

i.e., $\quad ax^2+c=0$,

where $a, c \in C$ and $a \neq 0$.

2. Adfected Quadratic Equation

A quadratic equation in which it contains the terms of first as well as second degrees of the unknown quantity, is called an adfected (or complete) quadratic equation.

i.e., $\quad ax^2+bx+c=0$,

where $a, b, c \in C$ and $a \neq 0, b \neq 0$.

Standard Quadratic Equation

An equation of the form

$$ax^2+bx+c=0 \qquad \text{...(i)}$$

where $a, b, c \in C$ and $a \neq 0$, is called a standard quadratic equation.

The numbers a, b, c are called the coefficients of this equation.

A root of the quadratic Eq. (i) is a complex number α, such that $a\alpha^2+b\alpha+c=0$. Recall that $D=b^2-4ac$ is the discriminant of the Eq. (i) and its roots are given by the following formula.

$$x=\frac{-b \pm \sqrt{D}}{2a} \quad \text{[Shridharacharya method]}$$

Nature of Roots

1. If $a, b, c \in R$ and $a \neq 0$, then
   (i) If $D<0$, then Eq. (i) has non-real complex roots.
   (ii) If $D>0$, then Eq. (i) has real and distinct roots, namely
   $$x_1=\frac{-b+\sqrt{D}}{2a},\ x_2=\frac{-b-\sqrt{D}}{2a} \text{ and then}$$
   $$ax^2+bx+c=a(x-x_1)(x-x_2). \qquad \text{...(ii)}$$
   (iii) If $D=0$, then Eq. (i) has real and equal roots, then
   $x_1=x_2=-\frac{b}{2a}$ and then
   $$ax^2+bx+c=a(x-x_1)^2. \qquad \text{...(iii)}$$
   To represent the quadratic ax^2+bx+c in form Eqs. (ii) or (iii), is to expand it into linear factors.
   (iv) If $D \geq 0$, then Eq. (i) has real roots.
   (v) If D_1 and D_2 be the discriminants of two quadratic equations, then
   (a) If $D_1+D_2 \geq 0$, then
   - atleast one of D_1 and $D_2 \geq 0$.
   - if $D_1<0$, then $D_2>0$ and if $D_1>0$, then $D_2<0$.

   (b) If $D_1+D_2<0$, then
   - atleast one of D_1 and $D_2<0$.
   - If $D_1<0$, then $D_2>0$ and if $D_1>0$, then $D_2<0$.
2. If $a, b, c \in Q$ and D is a perfect square of a rational number, the roots are rational and in case it is not a perfect square, the roots are irrational.
3. If $a, b, c \in R$ and $p+iq$ is one root of Eq. (i) $(q \neq 0)$, then the other must be the conjugate $(p-iq)$ and *vice-versa* (where, $p, q \in R$ and $i=\sqrt{-1}$).
4. If $a, b, c \in Q$ and $p+\sqrt{q}$ is one root of Eq. (i), then the other must be the conjugate $p-\sqrt{q}$ and *vice-versa* (where, p is a rational and $\sqrt{q}$ is a surd).
5. If $a=1$ and $b, c \in I$ and the roots of Eq. (i) are rational numbers, these roots must be integers.

6. If $a+b+c=0$ and a, b, c are rational, 1 is a root of the Eq. (i) and roots of the Eq. (i) are rational.

7. $a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}$
$\{(a-b)^2+(b-c)^2+(c-a)^2\}$
$=-\{(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)\}$

| Example 6. Find all values of the parameter a for which the quadratic equation

$$(a+1)x^2+2(a+1)x+a-2=0$$

(i) has two distinct roots.
(ii) has no roots.
(iii) has two equal roots.

Sol. By the hypothesis, this equation is quadratic and therefore $a \neq -1$ and the discriminant of this equation,

$$D = 4(a+1)^2 - 4(a+1)(a-2)$$
$$= 4(a+1)(a+1-a+2)$$
$$= 12(a+1)$$

(i) For $a > (-1)$, then $D > 0$, this equation has two distinct roots.
(ii) For $a < (-1)$, then $D < 0$, this equation has no roots.
(iii) This equation cannot have two equal roots. Since, $D = 0$ only for $a = -1$ and this contradicts the hypothesis.

| Example 7. Solve for x,
$(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10.$

Sol. $\because \quad (5+2\sqrt{6})(5-2\sqrt{6})=1$

$\therefore \quad (5-2\sqrt{6})=\frac{1}{(5+2\sqrt{6})}$

$\therefore \quad (5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$

reduces to $(5+2\sqrt{6})^{x^2-3}+\left(\frac{1}{5+2\sqrt{6}}\right)^{x^2-3}=10$

Put $(5+2\sqrt{6})^{x^2-3}=t$, then $t+\frac{1}{t}=10$

$\Rightarrow \quad t^2-10t+1=0$

or $\quad t=\frac{10\pm\sqrt{(100-4)}}{2}=(5\pm 2\sqrt{6})$

$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3}=(5\pm 2\sqrt{6})=(5+2\sqrt{6})^{\pm 1}$

$\therefore \quad x^2-3=\pm 1$

$\Rightarrow \quad x^2-3=1$ or $x^2-3=-1$

$\Rightarrow \quad x^2=4$ or $x^2=2$

Hence, $\quad x=\pm 2, \pm\sqrt{2}$

| Example 8. Show that if p, q, r and s are real numbers and $pr=2(q+s)$, then atleast one of the equations $x^2+px+q=0$ and $x^2+rx+s=0$ has real roots.

Sol. Let D_1 and D_2 be the discriminants of the given equations $x^2+px+q=0$ and $x^2+rx+s=0$, respectively.

Now, $D_1+D_2=p^2-4q+r^2-4s=p^2+r^2-4(q+s)$
$=p^2+r^2-2pr$ [given, $pr=2(q+s)$]
$=(p-r)^2 \geq 0$ [$\because p$ and q are real]

or $\quad D_1+D_2 \geq 0$

Hence, atleast one of the equations $x^2+px+q=0$ and $x^2+rx+s=0$ has real roots.

| Example 9. If α, β are the roots of the equation $(x-a)(x-b)=c, c\neq 0$. Find the roots of the equation $(x-\alpha)(x-\beta)+c=0$.

Sol. Since, α, β are the roots of

$$(x-a)(x-b)=c$$

or $\quad (x-a)(x-b)-c=0,$

Then $\quad (x-a)(x-b)-c=(x-\alpha)(x-\beta)$

$\Rightarrow \quad (x-\alpha)(x-\beta)+c=(x-a)(x-b)$

Hence, roots of $(x-\alpha)(x-\beta)+c=0$ are a, b.

| Example 10. Find all roots of the equation $x^4+2x^3-16x^2-22x+7=0$, if one root is $2+\sqrt{3}$.

Sol. All coefficients are real, irrational roots will occur in conjugate pairs.

Hence, another root is $2-\sqrt{3}$.

$\therefore$ Product of these roots $=(x-2-\sqrt{3})(x-2+\sqrt{3})$
$=(x-2)^2-3=x^2-4x+1.$

On dividing $x^4+2x^3-16x^2-22x+7$ by x^2-4x+1, then the other quadratic factor is x^2+6x+7.

Then, the given equation reduce in the form

$$(x^2-4x+1)(x^2+6x+7)=0$$

$\therefore \quad x^2+6x+7=0$

Then, $\quad x=\frac{-6\pm\sqrt{36-28}}{2}=-3\pm\sqrt{2}$

Hence, the other roots are $2-\sqrt{3}, -3\pm\sqrt{2}$.

Relation between Roots and Coefficients

1. **Relation between roots and coefficients of quadratic equation** If roots of the equation $ax^2+bx+c=0\ (a\neq 0)$ be real and distinct and $\alpha<\beta$, then $\alpha=\frac{-b+\sqrt{D}}{2a}, \beta=\frac{-b-\sqrt{D}}{2a}$.

(i) Sum of roots

$$= S = \alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}.$$

(ii) Product of roots

$$= P = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

(iii) Difference of roots

$$= D' = \alpha - \beta = \frac{\sqrt{D}}{a} = \frac{\sqrt{\text{Discriminant}}}{\text{Coefficient of } x^2}.$$

2. **Formation of an equation with given roots** A quadratic equation whose roots are α and β, is given by $(x-\alpha)(x-\beta)=0$ or $x^2-(\alpha+\beta)x+\alpha\beta=0$ i.e. x^2- (Sum of roots) x + Product of roots $=0$

$$\therefore \qquad x^2 - Sx + P = 0.$$

3. **Symmetric function of roots** A function of α and β is said to be symmetric function, if it remains unchanged, when α and β are interchanged.

For example, $\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$ is a symmetric function of α and β, whereas $\alpha^3-\beta^3+5\alpha\beta$ is not a symmetric function of α and β. In order to find the value of a symmetric function in terms of $\alpha+\beta, \alpha\beta$ and $\alpha-\beta$ and also in terms of a, b and c.

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2-2ac}{a^2}.$$

(ii) $\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)$

$$= \left(-\frac{b}{a}\right)\left(\frac{\sqrt{D}}{a}\right) = -\frac{b\sqrt{D}}{a^2}.$$

(iii) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$= \left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right) = -\left(\frac{b^3-3abc}{a^3}\right).$$

(iv) $\alpha^3-\beta^3=(\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$

$$= \left(\frac{\sqrt{D}}{a}\right)^3 + 3\left(\frac{c}{a}\right)\left(\frac{\sqrt{D}}{a}\right) = \frac{\sqrt{D}(D+3ac)}{a^3}.$$

(v) $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$

$$= \left(\frac{b^2-2ac}{a^2}\right)^2 - 2\left(\frac{c}{a}\right)^2 = \frac{b^4+2a^2c^2-4acb^2}{a^4}.$$

(vi) $\alpha^4-\beta^4=(\alpha^2+\beta^2)(\alpha^2-\beta^2)$

$$= -\frac{b\sqrt{D}(b^2-2ac)}{a^4}.$$

(vii) $\alpha^5+\beta^5=(\alpha^2+\beta^2)(\alpha^3+\beta^3)-\alpha^2\beta^2(\alpha+\beta)$

$$= \left(\frac{b^2-2ac}{a^2}\right)\left(-\frac{(b^3-3abc)}{a^3}\right) - \frac{c^2}{a^2}\left(-\frac{b}{a}\right)$$

$$= \frac{-(b^5-5ab^3c+5a^2bc^2)}{a^5}.$$

(viii) $\alpha^5-\beta^5=(\alpha^2+\beta^2)(\alpha^3-\beta^3)+\alpha^2\beta^2(\alpha-\beta)$

$$= \left(\frac{b^2-2ac}{a^2}\right)\left(\frac{\sqrt{D}(D+3ac)}{a^3}\right) + \left(\frac{c}{a}\right)^2\left(\frac{\sqrt{D}}{a}\right)$$

$$= \frac{\sqrt{D}(b^4-3acb^2+3a^2c^2)}{a^5}.$$

| Example 11. If one root of the equation $x^2-ix-(1+i)=0, (i=\sqrt{-1})$ is $1+i$, find the other root.

Sol. All coefficients of the given equation are not real, then other root $\neq 1-i$.

Let other root be α, then sum of roots $= i$

i.e. $\qquad 1+i+\alpha = i \Rightarrow \alpha = (-1)$

Hence, the other root is (-1).

| Example 12. If one root of the equation $x^2-\sqrt{5}x-19=0$ is $\dfrac{9+\sqrt{5}}{2}$, then find the other root.

Sol. All coefficients of the given equation are not rational, then other root $\neq \dfrac{9-\sqrt{5}}{2}$.

Let other root be α, sum of roots $=\sqrt{5}$

$$\Rightarrow \quad \frac{9+\sqrt{5}}{2}+\alpha=\sqrt{5} \Rightarrow \alpha = \frac{-9+\sqrt{5}}{2}$$

Hence, other root is $\dfrac{-9+\sqrt{5}}{2}$.

| Example 13. If the difference between the corresponding roots of the equations $x^2+ax+b=0$ and $x^2+bx+a=0\ (a\neq b)$ is the same, find the value of $a+b$.

Sol. Let α, β be the roots of $x^2+ax+b=0$ and γ, δ be the roots of $x^2+bx+a=0$, then given

$$\alpha-\beta=\gamma-\delta$$

$$\Rightarrow \quad \frac{\sqrt{a^2-4b}}{1} = \frac{\sqrt{b^2-4a}}{1} \qquad \left[\because \alpha-\beta=\frac{\sqrt{D}}{a}\right]$$

$$\Rightarrow \quad a^2-4b=b^2-4a$$

$$\Rightarrow \quad (a^2-b^2)+4(a-b)=0 \Rightarrow (a-b)(a+b+4)=0$$

$$\because \quad a-b\neq 0$$

$$\therefore \quad a+b+4=0 \text{ or } a+b=-4.$$

| Example 14. If $a+b+c=0$ and a,b,c are rational. Prove that the roots of the equation

$$(b+c-a)x^2+(c+a-b)x+(a+b-c)=0$$

are rational.

Sol. Given equation is

$$(b+c-a)x^2+(c+a-b)x+(a+b-c)=0 \quad \text{...(i)}$$

$\because (b+c-a)+(c+a-b)+(a+b-c)=a+b+c=0$

$\therefore$ $x=1$ is a root of Eq. (i), let other root of Eq. (i) is α, then

Product of roots $=\dfrac{a+b-c}{b+c-a}$

$\Rightarrow \quad 1\times\alpha=\dfrac{-c-c}{-a-a} \quad [\because a+b+c=0]$

$\therefore \quad \alpha=\dfrac{c}{a} \quad$ [rational]

Hence, both roots of Eq. (i) are rational.

Aliter

Let $b+c-a=A, c+a-b=B, a+b-c=C$

Then, $A+B+C=0 \quad [\because a+b+c=0]$ ...(ii)

Now, Eq. (i) becomes

$$Ax^2+Bx+C=0 \quad \text{...(iii)}$$

Discriminant of Eq. (iii),

$$D=B^2-4AC$$
$$=(-C-A)^2-4AC \quad [\because A+B+C=0]$$
$$=(C+A)^2-4AC$$
$$=(C-A)^2=(2a-2c)^2$$
$$=4(a-c)^2=\text{A perfect square}$$

Hence, roots of Eq. (i) are rational.

| Example 15. If the roots of equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ be equal, prove that a,b,c are in HP.

Sol. Given equation is

$$a(b-c)x^2+b(c-a)x+c(a-b)=0 \quad \text{...(i)}$$

Here, coefficient of x^2 + coefficient of x + constant term = 0

i.e., $a(b-c)+b(c-a)+c(a-b)=0$ always check this.

Then, 1 is a root of Eq. (i).

Since, its roots are equal.

Therefore, its other root will be also equal to 1.

Then, product of roots $=1\times1=\dfrac{c(a-b)}{a(b-c)}$

$\Rightarrow \quad ab-ac=ca-bc$

$\therefore \quad b=\dfrac{2ac}{a+c}$

Hence, a, b and c are in HP.

| Example 16. If α is a root of $4x^2+2x-1=0$. Prove that $4\alpha^3-3\alpha$ is the other root.

Sol. Let other root is β,

then $\alpha+\beta=-\dfrac{2}{4}=-\dfrac{1}{2}$ or $\beta=-\dfrac{1}{2}-\alpha \quad$ ...(i)

and so $4\alpha^2+2\alpha-1=0$, because α is a root of $4x^2+2x-1=0$.

Now, $\beta=4\alpha^3-3\alpha=\alpha(4\alpha^2-3)$

$$=\alpha(1-2\alpha-3) \quad [\because 4\alpha^2+2\alpha-1=0]$$
$$=-2\alpha^2-2\alpha$$
$$=-\frac{1}{2}(4\alpha^2)-2\alpha$$
$$=-\frac{1}{2}(1-2\alpha)-2\alpha \quad [\because 4\alpha^2+2\alpha-1=0]$$
$$=-\frac{1}{2}-\alpha=\beta \quad \text{[from Eq. (i)]}$$

Hence, $4\alpha^3-3\alpha$ is the other root.

| Example 17. If α,β are the roots of the equation $\lambda(x^2-x)+x+5=0$. If λ_1 and λ_2 are two values of λ for which the roots α,β are related by $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{4}{5}$, find the value of $\dfrac{\lambda_1}{\lambda_2}+\dfrac{\lambda_2}{\lambda_1}$.

Sol. The given equation can be written as

$$\lambda x^2-(\lambda-1)x+5=0$$

$\because \alpha,\beta$ are the roots of this equation.

$\therefore \quad \alpha+\beta=\dfrac{\lambda-1}{\lambda}$ and $\alpha\beta=\dfrac{5}{\lambda}$

But, given $\quad \dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{4}{5}$

$\Rightarrow \quad \dfrac{\alpha^2+\beta^2}{\alpha\beta}=\dfrac{4}{5}$

$\Rightarrow \quad \dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\dfrac{4}{5} \Rightarrow \dfrac{\dfrac{(\lambda-1)^2}{\lambda^2}-\dfrac{10}{\lambda}}{\dfrac{5}{\lambda}}=\dfrac{4}{5}$

$\Rightarrow \quad \dfrac{(\lambda-1)^2-10\lambda}{5\lambda}=\dfrac{4}{5} \Rightarrow \lambda^2-12\lambda+1=4\lambda$

$\Rightarrow \quad \lambda^2-16\lambda+1=0$

It is a quadratic in λ, let roots be λ_1 and λ_2, then

$\lambda_1+\lambda_2=16$ and $\lambda_1\lambda_2=1$

$\therefore \quad \dfrac{\lambda_1}{\lambda_2}+\dfrac{\lambda_2}{\lambda_1}=\dfrac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}=\dfrac{(\lambda_1+\lambda_2)^2-2\lambda_1\lambda_2}{\lambda_1\lambda_2}$

$$=\frac{(16)^2-2(1)}{1}=254$$

| Example 18. If α, β are the roots of the equation $x^2 - px + q = 0$, find the quadratic equation the roots of which are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$ and $\alpha^3\beta^2 + \alpha^2\beta^3$.

Sol. Since, α, β are the roots of $x^2 - px + q = 0$.

$\therefore \quad \alpha + \beta = p, \alpha\beta = q$

$\Rightarrow \quad \alpha - \beta = \sqrt{(p^2 - 4q)}$

Now, $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$

$$= (\alpha + \beta)(\alpha - \beta)(\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$$

$$= (\alpha + \beta)(\alpha - \beta)^2 \{(\alpha + \beta)^2 - \alpha\beta\}$$

$$= p(p^2 - 4q)(p^2 - q)$$

and $\alpha^3\beta^2 + \alpha^2\beta^3 = \alpha^2\beta^2(\alpha + \beta) = pq^2$

$$S = \text{Sum of roots} = p(p^2 - 4q)(p^2 - q) + pq^2$$

$$= p(p^4 - 5p^2q + 5q^2)$$

$$P = \text{Product of roots} = p^2q^2(p^2 - 4q)(p^2 - q)$$

$\therefore$ Required equation is $x^2 - Sx + P = 0$

i.e. $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

Exercise for Session 1

1. If $(a^2 - 1)x^2 + (a - 1)x + a^2 - 4a + 3 = 0$ be an identity in x, then the value of a is/are
   (a) -1 (b) 1 (c) 3 (d) $-1, 1, 3$

2. The roots of the equation $x^2 + 2\sqrt{3}x + 3 = 0$ are
   (a) real and unequal (b) rational and equal
   (c) irrational and equal (d) irrational and unequal

3. If $a, b, c \in Q$, then roots of the equation $(b + c - 2a)x^2 + (c + a - 2b)x + (a + b - 2c) = 0$ are
   (a) rational (b) non-real (c) irrational (d) equal

4. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then $P(x)Q(x) = 0$ has atleast
   (a) four real roots (b) two real roots
   (c) four imaginary roots (d) None of these

5. If roots of the equation $(q - r)x^2 + (r - p)x + (p - q) = 0$ are equal, then p, q, r are in
   (a) AP (b) GP (c) HP (d) AGP

6. If one root of the quadratic equation $ix^2 + 2(i + 1)x + (2 - i) = 0, i = \sqrt{-1}$ is $2 - i$, the other root is
   (a) $-i$ (b) i (c) $2 + i$ (d $2 - i$

7. If the difference of the roots of $x^2 - \lambda x + 8 = 0$ be 2, the value of λ is
   (a) ± 2 (b) ± 4 (c) ± 6 (d) ± 8

8. If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$ where $p \neq q$, pq is equal to
   (a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

9. If α, β are the roots of the quadratic equation $x^2 + bx - c = 0$, the equation whose roots are b and c, is
   (a) $x^2 + \alpha x - \beta = 0$ (b) $x^2 - [(\alpha + \beta) + \alpha\beta]x - \alpha\beta(\alpha + \beta) = 0$
   (c) $x^2 + [(\alpha + \beta) + \alpha\beta]x + \alpha\beta(\alpha + \beta) = 0$ (d) $x^2 + [(\alpha + \beta) + \alpha\beta]x - \alpha\beta(\alpha + \beta) = 0$

10. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots, is
    (a) 15 (b) 9 (c) 8 (d) 7

11. If α and β are the roots of the equation $ax^2 + bx + c = 0$ ($a \neq 0, a, b, c$ being different), then $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$ is equal to
    (a) zero (b) positive (c) negative (d) None of these

Session 2

Transformation of Quadratic Equations, Condition for Common Roots

Transformation of Quadratic Equations

Let α, β be the roots of the equation $ax^2 + bx + c = 0$, then the equation

(i) whose roots are $\alpha + k, \beta + k$, is

$a(x-k)^2 + b(x-k) + c = 0$ [replace x by $(x-k)$]

(ii) whose roots are $\alpha - k, \beta - k$, is

$a(x+k)^2 + b(x+k) + c = 0$ [replace x by $(x+k)$]

(iii) whose roots are $\alpha k, \beta k$, is

$ax^2 + kbx + k^2 c = 0$ $\left[\text{replace } x \text{ by} \left(\frac{x}{k}\right)\right]$

(iv) whose roots are $\frac{\alpha}{k}, \frac{\beta}{k}$, is

$ak^2x^2 + bkx + c = 0$ [replace x by xk]

(v) whose roots are $-\alpha, -\beta$, is

$ax^2 - bx + c = 0$ [replace x by $(-x)$]

(vi) whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$, is

$cx^2 + bx + a = 0$ $\left[\text{replace } x \text{ by} \left(\frac{1}{x}\right)\right]$

(vii) whose roots are $-\frac{1}{\alpha}, -\frac{1}{\beta}$, is

$cx^2 - bx + a = 0$ $\left[\text{replace } x \text{ by} \left(-\frac{1}{x}\right)\right]$

(viii) whose roots are $\frac{k}{\alpha}, \frac{k}{\beta}$, is

$cx^2 + kbx + k^2 a = 0$ $\left[\text{replace } x \text{ by} \left(\frac{k}{x}\right)\right]$

(ix) whose roots are $p\alpha + q, p\beta + q$, is

$a\left(\frac{x-q}{p}\right)^2 + b\left(\frac{x-q}{p}\right) + c = 0$ $\left[\text{replace } x \text{ by} \left(\frac{x-q}{p}\right)\right]$

(x) whose roots are $\alpha^n, \beta^n, n \in N$, is

$a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ [replace x by $(x^{1/n})$]

(xi) whose roots are $\alpha^{1/n}, \beta^{1/n}, n \in N$ is

$a(x^n)^2 + b(x^n) + c = 0$ [replace x by (x^n)]

| Example 19. If α, β be the roots of the equation $x^2 - px + q = 0$, then find the equation whose roots are $\frac{q}{p-\alpha}$ and $\frac{q}{p-\beta}$.

Sol. Let $\frac{q}{p-\alpha} = x \Rightarrow \alpha = p - \frac{q}{x}$

So, we replacing x by $p - \frac{q}{x}$ in the given equation, we get

$$\left(p - \frac{q}{x}\right)^2 - p\left(p - \frac{q}{x}\right) + q = 0$$

$$\Rightarrow \quad p^2 + \frac{q^2}{x^2} - \frac{2pq}{x} - p^2 + \frac{pq}{x} + q = 0$$

$$\Rightarrow \quad q - \frac{pq}{x} + \frac{q^2}{x^2} = 0$$

or $qx^2 - pqx + q^2 = 0$ or $x^2 - px + q = 0$

is the required equation whose roots are $\frac{q}{p-\alpha}$ and $\frac{q}{p-\beta}$.

| Example 20. If α and β are the roots of $ax^2 + bx + c = 0$, then find the roots of the equation $ax^2 - bx(x-1) + c(x-1)^2 = 0$.

Sol. $\because \; ax^2 - bx(x-1) + c(x-1)^2 = 0$ ...(i)

$$\Rightarrow \quad a\left(\frac{x}{x-1}\right)^2 - b\left(\frac{x}{x-1}\right) + c = 0$$

or $$a\left(\frac{x}{1-x}\right)^2 + b\left(\frac{x}{1-x}\right) + c = 0$$

Now, α, β are the roots of $ax^2 + bx + c = 0$.

Then, $\alpha = \frac{x}{1-x}$ and $\beta = \frac{x}{1-x}$

$$\Rightarrow \quad x = \frac{\alpha}{\alpha+1} \text{ and } x = \frac{\beta}{\beta+1}$$

Hence, $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ are the roots of the Eq. (i).

| Example 21. If α, β be the roots of the equation $3x^2 + 2x + 1 = 0$, then find value of $\left(\frac{1-\alpha}{1+\alpha}\right)^3 + \left(\frac{1-\beta}{1+\beta}\right)^3$.

Sol. Let $\frac{1-\alpha}{1+\alpha} = x \Rightarrow \alpha = \frac{1-x}{1+x}$

So, replacing x by $\frac{1-x}{1+x}$ in the given equation, we get

$$3\left(\frac{1-x}{1+x}\right)^2+2\left(\frac{1-x}{1+x}\right)+1=0 \Rightarrow x^2-2x+3=0 \quad \text{...(i)}$$

It is clear that $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$ are the roots of Eq. (i).

$$\therefore \quad \left(\frac{1-\alpha}{1+\alpha}\right)+\left(\frac{1-\beta}{1+\beta}\right)=2 \quad \text{...(ii)}$$

and
$$\left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)=3 \quad \text{...(iii)}$$

$$\therefore \quad \left(\frac{1-\alpha}{1+\alpha}\right)^3+\left(\frac{1-\beta}{1+\beta}\right)^3=\left(\frac{1-\alpha}{1+\alpha}+\frac{1-\beta}{1+\beta}\right)^3-3$$

$$\left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)\left(\frac{1-\alpha}{1+\alpha}+\frac{1-\beta}{1+\beta}\right)=2^3-3\cdot3\cdot2=8-18=-10$$

Roots Under Special Cases

Consider the quadratic equation $\quad ax^2+bx+c=0 \quad$ ...(i)

where $a, b, c \in R$ and $a \neq 0$. Then, the following hold good :

(i) If roots of Eq. (i) are equal in magnitude but opposite in sign, then sum of roots is zero as well as $D>0$, i.e. $b=0$ and $D>0$.

(ii) If roots of Eq. (i) are reciprocal to each other, then product of roots is 1 as well as $D \geq 0$ i.e., $a=c$ and $D \geq 0$.

(iii) If roots of Eq. (i) are of opposite signs, then product of roots <0 as well as $D>0$ i.e., $a>0, c<0$ and $D>0$ or $a<0, c>0$ and $D>0$.

(iv) If both roots of Eq. (i) are positive, then sum and product of roots >0 as well as $D \geq 0$ i.e., $a>0, b<0, c>0$ and $D \geq 0$ or $a<0, b>0, c<0$ and $D \geq 0$.

(v) If both roots of Eq. (i) are negative, then sum of roots <0, product of roots >0 as well as $D \geq 0$ i.e., $a>0, b>0, c>0$ and $D \geq 0$ or $a<0, b<0, c<0$ and $D \geq 0$.

(vi) If atleast one root of Eq. (i) is positive, then either one root is positive or both roots are positive i.e., point (iii) $\cup$ (iv).

(vii) If atleast one root of Eq. (i) is negative, then either one root is negative or both roots are negative i.e., point (iii) $\cup$ (v).

(viii) If greater root in magnitude of Eq. (i) is positive, then sign of b = sign of $c \neq$ sign of a.

(ix) If greater root in magnitude of Eq. (i) is negative, then sign of a = sign of $b \neq$ sign of c.

(x) If both roots of Eq. (i) are zero, then $b=c=0$.

(xi) If roots of Eq. (i) are 0 and $\left(-\frac{b}{a}\right)$, then $c=0$.

(xii) If roots of Eq. (i) are 1 and $\frac{c}{a}$, then $a+b+c=0$.

Example 22. For what values of m, the equation $x^2+2(m-1)x+m+5=0$ has $(m \in R)$

(i) roots are equal in magnitude but opposite in sign?

(ii) roots are reciprocals to each other?

(iii) roots are opposite in sign?

(iv) both roots are positive?

(v) both roots are negative?

(vi) atleast one root is positive?

(vii) atleast one root is negative?

Sol. Here, $a=1, b=2(m-1)$ and $c=m+5$

$\therefore \quad D=b^2-4ac=4(m-1)^2-4(m+5)$
$=4(m^2-3m-4)$

$\therefore \quad D=4(m-4)(m+1)$ and here $a=1>0$

(i) $b=0$ and $D>0$

$\Rightarrow 2(m-1)=0$ and $4(m-4)(m+1)>0$

$\Rightarrow \quad m=1$ and $m \in(-\infty,-1) \cup(4, \infty)$

$\therefore \quad m \in \phi$ [null set]

(ii) $a=c$ and $D \geq 0$

$\Rightarrow \quad 1=m+5$ and $4(m-4)(m+1) \geq 0$

$\Rightarrow \quad m=-4$ and $m \in(-\infty,-1] \cup[4, \infty)$

$\therefore \quad m=-4$

(iii) $a>0, c<0$ and $D>0$

$\Rightarrow \quad 1>0, m+5<0$ and $4(m-4)(m+1)>0$

$\Rightarrow \quad m<-5$ and $m \in(-\infty,-1) \cup(4, \infty)$

$\therefore \quad m \in(-\infty,-5)$

(iv) $a>0, b<0, c>0$ and $D \geq 0$

$\Rightarrow 1>0, 2(m-1)<0, m+5>0$

and $\quad 4(m-4)(m+1) \geq 0$

$\Rightarrow \quad m<1, m>-5$ and $m \in(-\infty,-1] \cup[4, \infty)$

$\Rightarrow \quad m \in(-5,-1]$

(v) $a>0, b>0, c>0$ and $D \geq 0$

$\Rightarrow 1>0, 2(m-1)>0, m+5>0$

and $\quad 4(m-4)(m+1) \geq 0$

$\Rightarrow m>1, m>-5$ and $m \in(-\infty,-1] \cup[4, \infty)$

$\therefore \quad m \in[4, \infty)$

(vi) Either one root is positive or both roots are positive

i.e., $\quad$ (c) $\cup$ (d)

$\Rightarrow m \in(-\infty,-5) \cup(-5,-1]$

(vii) Either one root is negative or both roots are negative

i.e., $\quad$ (c) $\cup$ (e)

$\Rightarrow \quad m \in(-\infty,-5) \cup[4, \infty)$

Condition for Common Roots

1. Only One Root is Common

Consider two quadratic equations

$$ax^2 + bx + c = 0 \text{ and } a'x^2 + b'x + c' = 0$$

[where $a, a' \neq 0$ and $ab' - a'b \neq 0$]

Let α be a common root, then

$$a\alpha^2 + b\alpha + c = 0 \text{ and } a'\alpha^2 + b'\alpha + c' = 0.$$

On solving these two equations by cross-multiplication, we have

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}$$

From first two relations, we get

$$\alpha = \frac{bc' - b'c}{ca' - c'a} \quad \text{...(i)}$$

and from last two relations, we get

$$\alpha = \frac{ca' - c'a}{ab' - a'b} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{bc' - b'c}{ca' - c'a} = \frac{ca' - c'a}{ab' - a'b}$$

$$\Rightarrow \quad (ab' - a'b)(bc' - b'c) = (ca' - c'a)^2$$

or $$\boxed{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \times \begin{vmatrix} b & c \\ b' & c' \end{vmatrix} = \begin{vmatrix} c & a \\ c' & a' \end{vmatrix}^2}$$ [remember]

This is the required condition for one root of two quadratic equations to be common.

2. Both Roots are Common

Let α, β be the common roots of the equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$, then

$$\alpha + \beta = -\frac{b}{a} = -\frac{b'}{a'} \Rightarrow \frac{a}{a'} = \frac{b}{b'} \quad \text{...(iii)}$$

and $$\alpha\beta = \frac{c}{a} = \frac{c'}{a'} \Rightarrow \frac{a}{a'} = \frac{c}{c'} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get $\boxed{\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}}$

This is the required condition for both roots of two quadratic equations to be identical.

Remark

To find the common root between the two equations, make the same coefficient of x^2 in both equations and then subtract of the two equations.

| Example 23. Find the value of λ, so that the equations $x^2 - x - 12 = 0$ and $\lambda x^2 + 10x + 3 = 0$ may have one root in common. Also, find the common root.

Sol. $\because$ $$x^2 - x - 12 = 0$$

$$\Rightarrow \quad (x-4)(x+3) = 0$$

$$\therefore \quad x = 4, -3$$

If $x = 4$ is a common root, then

$$\lambda(4)^2 + 10(4) + 3 = 0$$

$$\therefore \quad \lambda = -\frac{43}{16}$$

and if $x = -3$ is a common root, then

$$\lambda(-3)^2 + 10(-3) + 3 = 0$$

$$\therefore \quad \lambda = 3$$

Hence, for $\lambda = -\frac{43}{16}$, common root is $x = 4$

and for $\lambda = 3$, common root is $x = -3$.

| Example 24. If equations $ax^2 + bx + c = 0$, (where $a, b, c \in R$ and $a \neq 0$) and $x^2 + 2x + 3 = 0$ have a common root, then show that $a : b : c = 1 : 2 : 3$. conjugate of roots

Sol. Given equations are

$$ax^2 + bx + c = 0 \quad \text{...(i)}$$

and $$x^2 + 2x + 3 = 0 \quad \text{...(ii)}$$

Clearly, roots of Eq. (ii) are imaginary, since Eqs. (i) and (ii) have a common root. Therefore, common root must be imaginary and hence both roots will be common.

Therefore, Eqs. (i) and (ii) are identical.

$$\therefore \quad \boxed{\frac{a}{1} = \frac{b}{2} = \frac{c}{3} \text{ or } a : b : c = 1 : 2 : 3}$$

| Example 25. If a, b, c are in GP, show that the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, if $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ are in HP.

Sol. Given equations are

$$ax^2 + 2bx + c = 0 \quad \text{...(i)}$$

and $$dx^2 + 2ex + f = 0 \quad \text{...(ii)}$$

Since, a, b, c are in GP.

$$\therefore \quad b^2 = ac \text{ or } b = \sqrt{ac}$$

From Eq. (i), $ax^2 + 2\sqrt{ac}\,x + c = 0$

or $$(\sqrt{a}\,x + \sqrt{c})^2 = 0 \text{ or } x = -\frac{\sqrt{c}}{\sqrt{a}}$$

$\because$ Given Eqs. (i) and (ii) have a common root.

Hence, $x = -\frac{\sqrt{c}}{\sqrt{a}}$ also satisfied Eq. (ii), then

$$d\left(\frac{c}{a}\right) - 2e\frac{\sqrt{c}}{\sqrt{a}} + f = 0$$

$$\Rightarrow \quad \frac{d}{a} - \frac{2e}{\sqrt{ac}} + \frac{f}{c} = 0$$

or $$\frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0 \qquad [\because b = \sqrt{ac}]$$

or $$\frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$$

$\therefore \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

Hence, $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ are in HP.

Exercise for Session 2

1. If α and β are the roots of the equation $2x^2 - 3x + 4 = 0$, then the equation whose roots are α^2 and β^2, is

(a) $4x^2 + 7x + 16 = 0$ (b) $4x^2 + 7x + 6 = 0$ (c) $4x^2 + 7x + 1 = 0$ (d) $4x^2 - 7x + 16 = 0$

2. If α, β are the roots of $x^2 - 3x + 1 = 0$, then the equation whose roots are $\left(\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}\right)$, is

(a) $x^2 + x - 1 = 0$ (b) $x^2 + x + 1 = 0$ (c) $x^2 - x - 1 = 0$ (d) None of these

$\alpha - 1 = x$

3. The equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$, then

(a) $a = -b$ (b) $b = -c$ (c) $c = -a$ (d) $b = a + c$

4. If the roots of equation $\frac{x^2 - bx}{ax - c} = \frac{m - 1}{m + 1}$ are equal but opposite in sign, then the value of m will be

very easy

(a) $\frac{a - b}{a + b}$ (b) $\frac{b - a}{a + b}$ (c) $\frac{a + b}{a - b}$ (d) $\frac{b + a}{b - a}$

5. If $x^2 + px + q = 0$ is the quadratic equation whose roots are $a - 2$ and $b - 2$, where a and b are the roots of $x^2 - 3x + 1 = 0$, then

believe in yourself!!

Ans — $p = 1, q =$

(a) $p = 1, q = 5$ (b) $p = 1, q = -5$ (c) $p = -1, q = 1$ (d) None of these

6. If both roots of the equation $x^2 - (m - 3)x + m = 0 \ (m \in R)$ are positive, then

behtareen

(a) $m \in (3, \infty)$ (b) $m \in (-\infty, 1]$ (c) $m \in [9, \infty)$ (d) $m \in (1, 3)$

7. If the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$, where $m \in R \sim \{-1\}$, has atleast one root is negative, then

what if $\alpha \geq 0$

(a) $m \in (-\infty, -1)$ (b) $m \in \left(-\frac{1}{8}, \infty\right)$ (c) $m \in \left(-1, -\frac{1}{8}\right)$ (d) $m \in R$

8. If both the roots of $\lambda(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6\lambda(2x^2 + 1) + px + 4x^2 - 2 = 0$ are common, then $2r - p$ is equal to

(a) -1 (b) 0 (c) 1 (d) 2

9. If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root $a \neq 0$, then $\frac{a^3 + b^3 + c^3}{abc}$ is equal to

(a) 1 (b) 2 (c) 3 (d) None of these

10. If $a(p + q)^2 + 2bpq + c = 0$ and $a(p + r)^2 + 2bpr + c = 0$, then qr is equal to

(a) $p^2 + \frac{c}{a}$ (b) $p^2 + \frac{a}{c}$ (c) $p^2 + \frac{a}{b}$ (d) $p^2 + \frac{b}{a}$

Session 3

Quadratic Expression, Wavy Curve Method, Condition for Resolution into Linear Factors, Location of Roots

Quadratic Expression

An expression of the form $ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$ is called a quadratic expression in x. So, in general quadratic expression is represented by

$$f(x) = ax^2 + bx + c \text{ or } y = ax^2 + bx + c.$$

Graph of a Quadratic Expression

We have, $\quad y = ax^2 + bx + c = f(x), \quad [a \neq 0]$

$$\Rightarrow \quad y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

or $\quad \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$

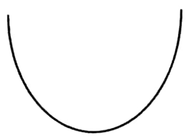
Now, let $y + \dfrac{D}{4a} = Y$ and $x + \dfrac{b}{2a} = X$

$$\therefore \quad Y = aX^2$$

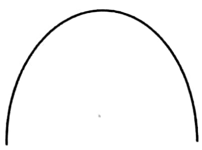
$$\Rightarrow \quad X^2 = \frac{Y}{a}$$

1. The shape of the curve $y = f(x)$ is parabolic.
2. The axis of parabola is $X = 0$ or $x + \dfrac{b}{2a} = 0$

   or $x = -\dfrac{b}{2a}$ i.e. parallel to Y-axis.
3. (i) If $\dfrac{1}{a} > 0 \Rightarrow a > 0$, the parabola open upwards.



   (ii) If $\dfrac{1}{a} < 0 \Rightarrow a < 0$, the parabola open downwards.



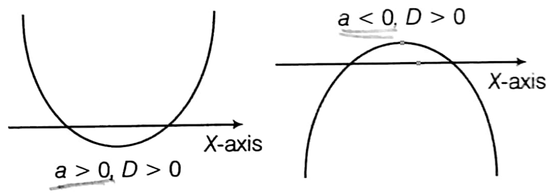
4. **Intersection with axes**

   (i) Intersection with X-axis

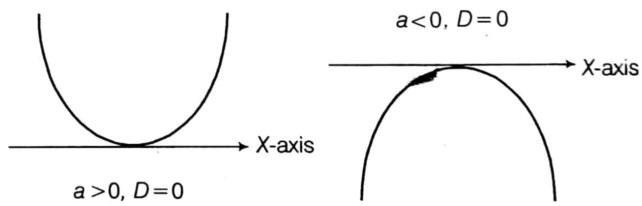
   For X-axis, $y = 0$.

$$\therefore \quad ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{D}}{2a}$$

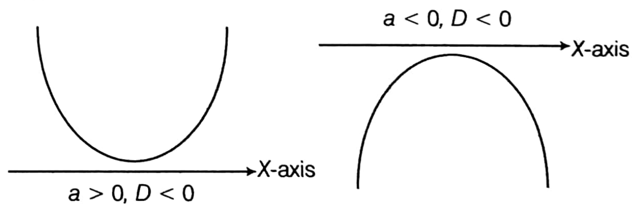
   For $D > 0$, parabola cuts X-axis in two real and distinct points



   For $D = 0$, parabola touches X-axis in one point i.e., $x = -\dfrac{b}{2a}$.



   For $D < 0$, parabola does not cut X-axis i.e., imaginary values of x.



   (ii) Intersection with Y-axis

   For Y-axis, $x = 0$.

$$\therefore \quad \boxed{y = c}$$

5. **Greatest and least values of $f(x)$**

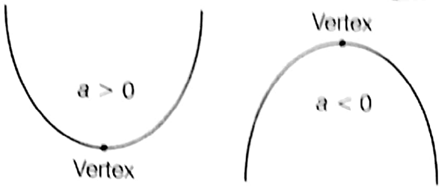
   Vertex of the parabola $X^2 = \dfrac{1}{a}Y$ is

$$X = 0, Y = 0$$

$$\Rightarrow \qquad x + \frac{b}{2a} = 0,\ y + \frac{D}{4a} = 0$$

or
$$x = -\frac{b}{2a},\ y = -\frac{D}{4a}$$

Hence, vertex of $y = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$.



For $a > 0$, $f(x)$ has least value at $x = -\frac{b}{2a}$.

This least value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$

or $\boxed{y_{\text{least}} = -\frac{D}{4a}}$.

$\therefore$ Range of $y = ax^2 + bx + c$ is $\left[-\frac{D}{4a}, \infty\right)$.

For $a < 0$, $f(x)$ has greatest value at $x = -\frac{b}{2a}$.

This greatest value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$

or $\quad y_{\text{greatest}} = -\frac{D}{4a}$

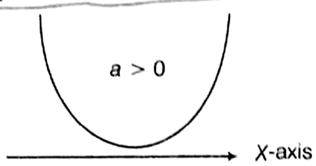
$\therefore$ Range of $y = ax^2 + bx + c$ is $\left(-\infty, -\frac{D}{4a}\right]$.

Sign of Quadratic Expression

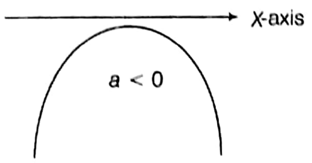
Let $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$,

where $a, b, c \in R$ and $a \neq 0$, for some values of x, $f(x)$ may be positive, negative or zero. This gives the following cases :

1. **$a > 0$ and $D < 0$.**
   So, $f(x) > 0$ for all $x \in R$,
   i.e. $f(x)$ is positive for all real values of x.

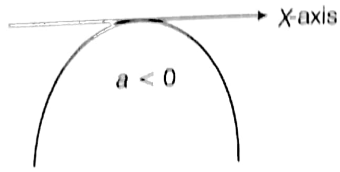


2. **$a < 0$ and $D < 0$.** So, $f(x) < 0$ for all $x \in R$,
   i.e. $f(x)$ is negative for all real values of x.

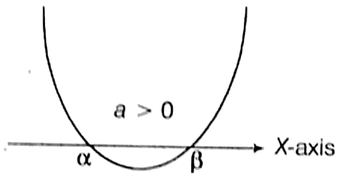


3. **$a > 0$ and $D = 0$.** So, $f(x) \geq 0$ for all $x \in R$,
   i.e. $f(x)$ is positive for all real values of x except at vertex, where $f(x) = 0$.

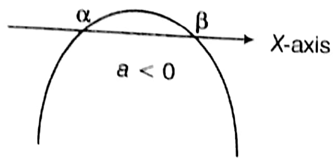
4. **$a < 0$ and $D = 0$.** So, $f(x) \leq 0$ for all $x \in R$,
   i.e. $f(x)$ is negative for all real values of x except at vertex, where $f(x) = 0$.



5. **$a > 0$ and $D > 0$.**
   Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$, then $f(x) > 0$ for all $x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0$ for all $x \in (\alpha, \beta)$.



6. **$a < 0$ and $D > 0$**
   Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$, then $f(x) < 0$ for all $x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0$ for all $x \in (\alpha, \beta)$.



Wavy Curve Method
(Generalised Method of Intervals)

Wave Curve Method is used for solving inequalities of the form

$$f(x) = \frac{(x - a_1)^{k_1}(x - a_2)^{k_2} \ldots (x - a_m)^{k_m}}{(x - b_1)^{p_1}(x - b_2)^{p_2} \ldots (x - b_n)^{p_n}} > 0$$

$(< 0, \geq 0 \text{ or } \leq 0)$,

where, $k_1, k_2, \ldots, k_m, p_1, p_2, \ldots, p_n$ are natural numbers and such that $a_i \neq b_j$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

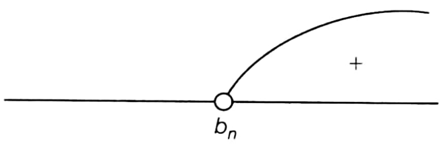
We use the following methods:

1. Solve $(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_m)^{k_m}=0$ and $(x-b_1)^{p_1}(x-a_2)^{p_2}\ldots(x-b_n)^{p_n}=0$, then we get $x=a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_n$ [critical points]
2. Assume $a_1<a_2<\ldots<a_m<b_1<b_2<\ldots<b_n$
   Plot them on the real line. Arrange inked (black) circles (●) and un-inked (white) circles (○), such that

	$a_1\ a_2\ \ldots\ a_m$	$b_1\ b_2\ \ldots\ b_n$
If $f(x)>0$	○ ○ ... ○	○ ○ ... ○
$f(x)<0$	○ ○ ... ○	○ ○ ... ○
$f(x)\geq 0$	● ● ... ●	○ ○ ... ○
$f(x)\leq 0$	● ● ... ●	○ ○ ... ○

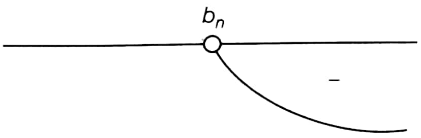
3. Obviously, b_n is the greatest root. If in all brackets before x positive sign and expression has also positive sign, then wave start from right to left, beginning above the number line, i.e.

$$+\frac{(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_m)^{k_m}}{(x-b_1)^{p_1}(x-b_2)^{p_2}\ldots(x-b_n)^{p_n}}, \text{ then}$$



and if in all brackets before x positive sign and expression has negative sign, then wave start from right to left, beginning below the number line, i.e.

$$-\frac{(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_m)^{k_m}}{(x-b_1)^{p_1}(x-b_2)^{p_2}\ldots(x-b_n)^{p_n}}, \text{ then}$$



4. If roots occur even times, then sign remain same from right to left side of the roots and if roots occur odd times, then sign will change from right to left through the roots of $x=a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_n$.
5. The solution of $f(x)>0$ or $f(x)\geq 0$ is the union of all intervals in which we have put the plus sign and the solution of $f(x)<0$ or $f(x)\leq 0$ is the union of all intervals in which we have put the minus sign.

Important Results

1. The point where denominator is zero or function approaches infinity, will never be included in the answer.
2. For $x^2<a^2$ or $|x|<a \Leftrightarrow -a<x<a$
   i.e., $x\in(-a,a)$
3. For $0<x^2<a^2$ or $0<|x|<a$
   $\Leftrightarrow$ $-a<x<a\sim\{0\}$
   i.e., $x\in(-a,a)\sim\{0\}$
4. For $x^2\geq a^2$ or $|x|\geq a \Leftrightarrow x\leq -a$ or $x\geq a$
   i.e., $x\in(-\infty,-a]\cup[a,\infty)$
5. For $x^2>a^2$ or $|x|>a \Leftrightarrow x<-a$ or $x>a$
   i.e., $x\in(-\infty,-a)\cup(a,\infty)$
6. For $a^2\leq x^2\leq b^2$ or $a\leq|x|\leq b$
   $\Leftrightarrow$ $a\leq x\leq b$ or $-b\leq x\leq -a$
   i.e., $x\in[-b,-a]\cup[a,b]$
7. For $a^2<x^2\leq b^2$ or $a<|x|\leq b$
   $\Leftrightarrow$ $a<x\leq b$ or $-b\leq x<-a$
   i.e., $x\in[-b,-a)\cup(a,b]$
8. For $a^2\leq x^2<b^2$ or $a\leq|x|<b$
   $\Leftrightarrow$ $a\leq x<b$ or $-b<x\leq -a$
   i.e., $x\in(-b,-a]\cup[a,b)$
9. For $a^2<x^2<b^2$ or $a<|x|<b$
   $\Leftrightarrow$ $a<x<b$ or $-b<x<-a$
   i.e., $x\in(-b,-a)\cup(a,b)$
10. For $(x-a)(x-b)<0$ and $a<b$, then $a<x<b$
    i.e., $x\in(a,b)$
11. If $(x-a)(x-b)\leq 0$ and $a<b$,
    then $a\leq x\leq b$, $x\in[a,b]$
12. If $(x-a)(x-b)>0$ and $a<b$, then $x<a$ or $x>b$
    i.e., $x\in(-\infty,a)\cup(b,\infty)$
13. If $(x-a)(x-b)\geq 0$ and $a<b$,
    then $x\leq a$ or $x\geq b$
    i.e., $x\in(-\infty,a]\cup[b,\infty)$

Example 26. Solve the inequality $(x+3)(3x-2)^5(7-x)^3(5x+8)^2\geq 0$.

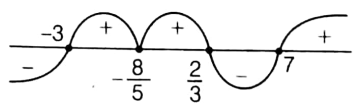
Sol. We have, $(x+3)(3x-2)^5(7-x)^3(5x+8)^2\geq 0$

$\Rightarrow -(x+3)(3x-2)^5(x-7)^3(5x+8)^2\geq 0$

$\Rightarrow (x+3)(3x-2)^5(x-7)^3(5x+8)^2\leq 0$

[take before x, + ve sign in all brackets]

So, at $-\frac{8}{5}=x$ $y=0$ so, we have to include $-\frac{8}{5}$



The critical points are $(-3),\left(-\frac{8}{5}\right),\frac{2}{3},7$.

Hence, $x\in(-\infty,-3]\cup\left[\frac{2}{3},7\right]\cup\left\{-\frac{8}{5}\right\}$.

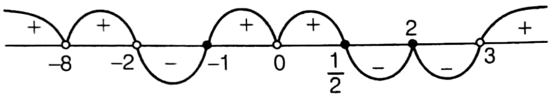
| Example 27. Solve the inequality

$$\frac{(x-2)^{10000}(x+1)^{253}\left(x-\frac{1}{2}\right)^{971}(x+8)^4}{x^{500}(x-3)^{75}(x+2)^{93}} \geq 0$$

Sol. We have, $\dfrac{(x-2)^{10000}(x+1)^{253}\left(x-\frac{1}{2}\right)^{971}(x+8)^4}{x^{500}(x-3)^{75}(x+2)^{93}} \geq 0$

The critical points are $(-8), (-2), (-1), 0, \frac{1}{2}, 2, 3.$

$[\because x \neq -2, 0, 3]$



Hence, $x \in (-\infty, -8] \cup [-8, -2) \cup [-1, 0) \cup \left(0, \frac{1}{2}\right] \cup (3, \infty)$

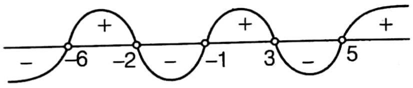
or $\quad x \in (-\infty, -2) \cup [-1, 0) \cup \left(0, \frac{1}{2}\right] \cup (3, \infty)$

| Example 28. Let $f(x) = \dfrac{(x-3)(x+2)(x+6)}{(x+1)(x-5)}$.

Find intervals, where $f(x)$ is positive or negative.

Sol. We have, $\quad f(x) = \dfrac{(x-3)(x+2)(x+6)}{(x+1)(x-5)}$

The critical points are $(-6), (-2), (-1), 3, 5$



For $f(x) > 0, \forall x \in (-6, -2) \cup (-1, 3) \cup (5, \infty)$

For $f(x) < 0, \forall x \in (-\infty, -6) \cup (-2, -1) \cup (3, 5)$

| Example 29. Find the set of all x for which

$$\frac{2x}{(2x^2+5x+2)} > \frac{1}{(x+1)}$$

Sol. We have, $\quad \dfrac{2x}{(2x^2+5x+2)} > \dfrac{1}{(x+1)}$

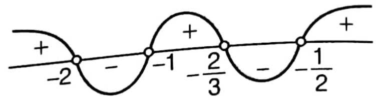
$\Rightarrow \quad \dfrac{2x}{(x+2)(2x+1)} - \dfrac{1}{(x+1)} > 0$

$\Rightarrow \quad \dfrac{(2x^2+2x)-(2x^2+5x+2)}{(x+2)(x+1)(2x+1)} > 0$

$\Rightarrow \quad -\dfrac{(3x+2)}{(x+2)(x+1)(2x+1)} > 0$

or $\quad \dfrac{(3x+2)}{(x+2)(x+1)(2x+1)} < 0$

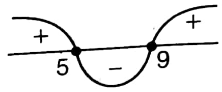
The critical points are $(-2), (-1), \left(-\frac{2}{3}\right), \left(-\frac{1}{2}\right)$.



Hence, $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$.

| Example 30. For $x \in R$, prove that the given expression $\dfrac{x^2+34x-71}{x^2+2x-7}$ cannot lie between 5 and 9.

Sol. Let $\dfrac{x^2+34x-71}{x^2+2x-7} = y$



$\Rightarrow \quad x^2(y-1) + (2y-34)x + 71 - 7y = 0$

For real values of x, discriminant ≥ 0

$\therefore \quad (2y-34)^2 - 4(y-1)(71-7y) \geq 0$

$\Rightarrow \quad 8y^2 - 112y + 360 \geq 0$

$\Rightarrow \quad y^2 - 14y + 45 \geq 0$

$\Rightarrow \quad (y-9)(y-5) \geq 0$

$\Rightarrow \quad y \in (-\infty, 5] \cup [9, \infty)$

Hence, y can never lie between 5 and 9.

| Example 31. For what values of the parameter k in the inequality $\left|\dfrac{x^2+kx+1}{x^2+x+1}\right| < 3$, satisfied for all real values of x?

Sol. We have, $\quad \left|\dfrac{x^2+kx+1}{x^2+x+1}\right| < 3$

$\Rightarrow \quad -3 < \dfrac{x^2+kx+1}{x^2+x+1} < 3$

Since, $\quad x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} > 0$

$\therefore \quad -3(x^2+x+1) < x^2+kx+1 < 3(x^2+x+1)$

$\therefore \quad 4x^2 + (k+3)x + 4 > 0 \quad$ ...(i)

and $\quad 2x^2 - (k-3)x + 2 > 0 \quad$ ...(ii)

$\because \quad 4 > 0$ and $2 > 0$

The inequality (i) will be valid, if

$(k+3)^2 - 4 \cdot 4 \cdot 4 < 0 \Rightarrow (k+3)^2 < 64$

or $\quad -8 < k+3 < 8$

or $\quad -11 < k < 5 \quad$ ...(iii)

and the inequality (ii) will be valid, if

$(k-3)^2 - 4 \cdot 2 \cdot 2 < 0$ or $(k-3)^2 < 16$

or $\quad -4 < k-3 < 4$

or $\quad -1 < k < 7 \quad$ ...(iv)

The conditions (iii) and (iv) will hold simultaneously, if

$-1 < k < 5$

Condition for Resolution into Linear Factors

The quadratic function

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

may be resolved into two linear factors, iff

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

i.e.,
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

| Example 32. Find the value of m for which the expression $12x^2 - 10xy + 2y^2 + 11x - 5y + m$ can be resolved into two rational linear factors.

Sol. Comparing the given expression with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, we have

$$a = 12, h = -5, b = 2, g = \frac{11}{2}, f = \left(-\frac{5}{2}\right), c = m$$

The given expression will have two linear factors, if and only if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

or $$(12)(2)(m) + 2\left(-\frac{5}{2}\right)\left(\frac{11}{2}\right)(-5) - (12)\left(-\frac{5}{2}\right)^2 - (2)\left(\frac{11}{2}\right)^2 - (m)(-5)^2 = 0$$

$$\Rightarrow \quad 24m + \frac{275}{2} - 75 - \frac{121}{2} - 25m = 0 \text{ or } m = 2$$

| Example 33. If the expression $ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy$ can be resolved into two rational factors, prove that

$$a^3 + b^3 + c^3 = 3abc.$$

Sol. Given expression is

$$ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy \quad \text{...(i)}$$

$$= z^2\left[a\left(\frac{x}{z}\right)^2 + b\left(\frac{y}{z}\right)^2 + c + 2a\left(\frac{y}{z}\right) + 2b\left(\frac{x}{z}\right) + 2c\left(\frac{x}{z}\right)\left(\frac{y}{z}\right)\right]$$

$$= z^2[aX^2 + bY^2 + c + 2aY + 2bX + 2cXY]$$

where, $\frac{x}{z} = X$ and $\frac{y}{z} = Y$

Expression (i) will have two rational linear factors in x, y and z, if expression $aX^2 + bY^2 + 2cXY + 2bX + 2aY + c$ will have two linear factors, if

$$abc + 2abc - aa^2 - bb^2 - cc^2 = 0$$

or $$a^3 + b^3 + c^3 = 3abc$$

| Example 34. Find the linear factors of $x^2 - 5xy + 4y^2 + x + 2y - 2$.

Sol. Given expression is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 \quad \text{...(i)}$$

Its corresponding equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

or $$x^2 - x(5y - 1) + 4y^2 + 2y - 2 = 0$$

$$\therefore \quad x = \frac{(5y-1) \pm \sqrt{(5y-1)^2 - 4 \cdot 1 \cdot (4y^2 + 2y - 2)}}{2}$$

$$= \frac{(5y-1) \pm \sqrt{(9y^2 - 18y + 9)}}{2}$$

$$= \frac{(5y-1) \pm \sqrt{(3y-3)^2}}{2}$$

$$= \frac{(5y-1) \pm (3y-3)}{2} = 4y - 2, y + 1$$

$\therefore$ The required linear factors are $(x - 4y + 2)$ and $(x - y - 1)$.

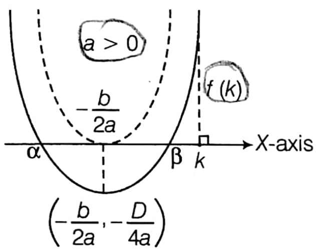
Location of Roots

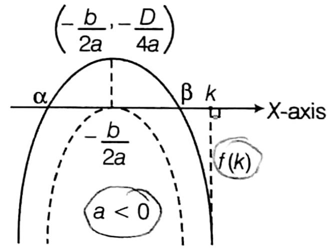
(Interval in which Roots Lie)

Let $f(x) = ax^2 + bx + c$, $a, b, c \in R$, $a \neq 0$ and α, β be the roots of $f(x) = 0$. Suppose $k, k_1, k_2 \in R$ and $k_1 < k_2$. Then, the following hold good :

1. Conditions for Number k

(If both the roots of $f(x) = 0$ are less than k)





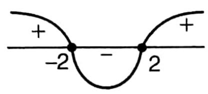
(i) $D \geq 0$ (roots may be equal)

(ii) $af(k) > 0$

(iii) $k > -\frac{b}{2a}$, where $\alpha \leq \beta$.

| Example 35. Find the values of m, for which both roots of equation $x^2 - mx + 1 = 0$ are less than unity.

Sol. Let $f(x) = x^2 - mx + 1$, as both roots of $f(x) = 0$ are less than 1, we can take $D \geq 0$, $af(1) > 0$ and $-\frac{b}{2a} < 1$.



(i) **Consider $D \geq 0$** $(-m)^2 - 4 \cdot 1 \cdot 1 \geq 0$

$$\Rightarrow \quad (m+2)(m-2) \geq 0$$

$$\Rightarrow \quad m \in (-\infty, -2] \cup [2, \infty) \qquad \text{...(i)}$$

(ii) **Consider $af(1) > 0$** $1(1 - m + 1) > 0$

$$\Rightarrow \quad m - 2 < 0 \Rightarrow m < 2$$

$$\Rightarrow \quad m \in (-\infty, 2) \qquad \text{...(ii)}$$

(iii) **Consider $\left(-\dfrac{b}{2a} < 1\right)$**

$$\frac{m}{2} < 1 \Rightarrow m < 2$$

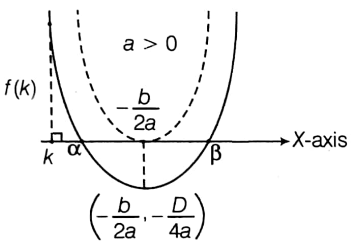
$$\Rightarrow \quad m \in (-\infty, 2) \qquad \text{...(iii)}$$

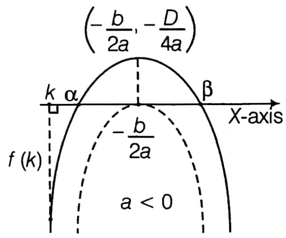
Take intersection

Hence, the values of m satisfying Eqs. (i), (ii) and (iii) at the same time are $m \in (-\infty, -2]$.

2. Conditions for a Number k

If both the roots of $f(x) = 0$ are greater than k





(i) $D \geq 0$ (roots may be equal)

(ii) $af(k) > 0$

(iii) $k < -\dfrac{b}{2a}$, where $\alpha \leq \beta$.

Example 36. For what values of $m \in R$, both roots of the equation $x^2 - 6mx + 9m^2 - 2m + 2 = 0$ exceed 3?

Sol. Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$

As both roots of $f(x) = 0$ are greater than 3, we can take $D \geq 0$, $af(3) > 0$ and $-\dfrac{b}{2a} > 3$.

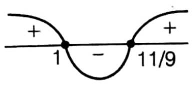
(i) **Consider $D \geq 0$**

$$(-6m)^2 - 4 \cdot 1(9m^2 - 2m + 2) \geq 0 \Rightarrow 8m - 8 \geq 0$$

$$\therefore \quad m \geq 1 \text{ or } m \in [1, \infty) \qquad \text{...(i)}$$

(ii) **Consider $af(3) \geq 0$**

$$1 \cdot (9 - 18m + 9m^2 - 2m + 2) > 0$$



$$\Rightarrow \quad 9m^2 - 20m + 11 > 0$$

$$\Rightarrow \quad (9m - 11)(m - 1) > 0$$

$$\Rightarrow \quad \left(m - \frac{11}{9}\right)(m - 1) > 0$$

$$\Rightarrow \quad m \in (-\infty, 1) \cup \left(\frac{11}{9}, \infty\right) \qquad \text{...(ii)}$$

(iii) **Consider $\left(-\dfrac{b}{2a} > 3\right)$**

$$\frac{6m}{2} > 3$$

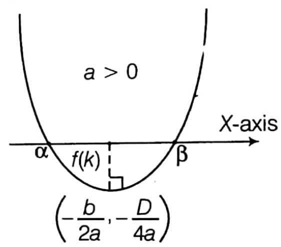
$$\Rightarrow \quad m > 1$$

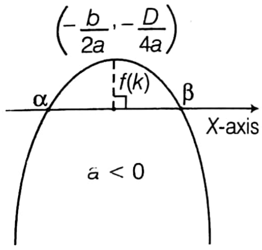
$$\Rightarrow \quad m \in (1, \infty) \qquad \text{...(iii)}$$

Hence, the values of m satisfying Eqs. (i), (ii) and (iii) at the same time are $m \in \left(\dfrac{11}{9}, \infty\right)$.

3. Conditions for a Number k

If k lies between the roots of $f(x) = 0$



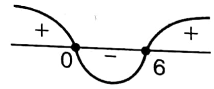


(i) $D > 0$ (ii) $af(k) < 0$, where $\alpha < \beta$

Example 37. Find all values of p, so that 6 lies between roots of the equation $x^2 + 2(p-3)x + 9 = 0$.

Sol. Let $f(x) = x^2 + 2(p-3)x + 9$, as 6 lies between the roots of $f(x) = 0$, we can take $D > 0$ and $af(6) < 0$

(i) **Consider $D > 0$**



$$\{2(p-3)\}^2 - 4 \cdot 1 \cdot 9 > 0$$

$$\Rightarrow \quad (p-3)^2 - 9 > 0$$

$$\Rightarrow \quad p(p-6) > 0$$

$$\Rightarrow \quad p \in (-\infty, 0) \cup (6, \infty) \qquad \text{...(i)}$$

(ii) **Consider $a\,f(6) < 0$**

$$1 \cdot \{36 + 12(p-3) + 9\} < 0$$

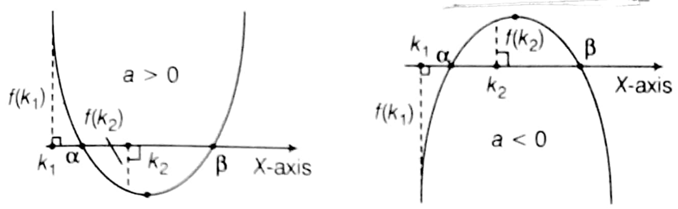
$$\Rightarrow \quad 12p + 9 < 0 \Rightarrow p + \frac{3}{4} < 0$$

$$\Rightarrow \quad p \in \left(-\infty, -\frac{3}{4}\right) \qquad \text{...(ii)}$$

Hence, the values of p satisfying Eqs. (i) and (ii) at the same time are $p \in \left(-\infty, -\dfrac{3}{4}\right)$.

4. Conditions for Numbers k_1 and k_2

If exactly one root of $f(x)=0$ lies in the interval (k_1, k_2)



(i) $D > 0$

(ii) $f(k_1) f(k_2) < 0$, where $\alpha < \beta$.

| Example 38. Find the values of m, for which exactly one root of the equation $x^2 - 2mx + m^2 - 1 = 0$ lies in the interval $(-2, 4)$.

Sol. Let $f(x) = x^2 - 2mx + m^2 - 1$, as exactly one root of $f(x) = 0$ lies in the interval $(-2, 4)$, we can take $D > 0$ and $f(-2) f(4) < 0$.

(i) **Consider $D > 0$**

$$(-2m)^2 - 4 \cdot 1(m^2 - 1) > 0 \Rightarrow 4 > 0$$

$$\therefore \quad m \in R \quad \ldots\text{(i)}$$

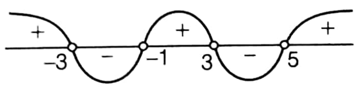
(ii) **Consider $f(-2) f(4) < 0$**

$$(4 + 4m + m^2 - 1)(16 - 8m + m^2 - 1) < 0$$

$$\Rightarrow \quad (m^2 + 4m + 3)(m^2 - 8m + 15) < 0$$

$$\Rightarrow \quad (m+1)(m+3)(m-3)(m-5) < 0$$

$$\Rightarrow \quad (m+3)(m+1)(m-3)(m-5) < 0$$

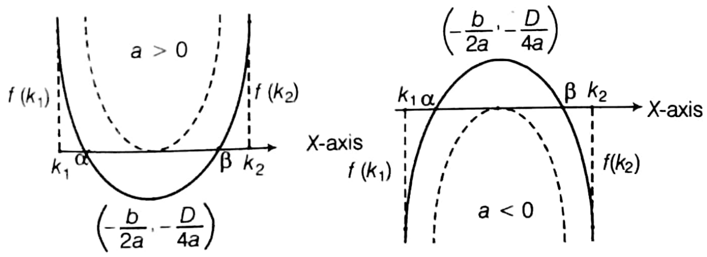


$$\therefore \quad m \in (-3, -1) \cup (3, 5) \quad \ldots\text{(ii)}$$

Hence, the values of m satisfying Eqs. (i) and (ii) at the same time are $m \in (-3, -1) \cup (3, 5)$.

5. Conditions for Numbers k_1 and k_2

(If both roots $f(x) = 0$ are confined between k_1 and k_2)



(i) $D \geq 0$ (roots may be equal)

(ii) $af(k_1) > 0$

(iii) $af(k_2) > 0$

(iv) $k_1 < -\dfrac{b}{2a} < k_2$, where $\alpha \leq \beta$ and $k_1 < k_2$.

| Example 39. Find all values of a for which the equation $4x^2 - 2x + a = 0$ has two roots lie in the interval $(-1, 1)$.

Sol. Let $f(x) = 4x^2 - 2x + a$ as both roots of the equation, $f(x) = 0$ are lie between $(-1, 1)$, we can take $D \geq 0$, $af(-1) > 0$, $af(1) > 0$ and $-1 < \dfrac{1}{4} < 1$.

(i) **Consider $D \geq 0$**

$$(-2)^2 - 4 \cdot 4 \cdot a \geq 0 \Rightarrow a \leq \frac{1}{4} \quad \ldots\text{(i)}$$

(ii) **Consider $a\,f(-1) > 0$**

$$4(4 + 2 + a) > 0$$

$$\Rightarrow \quad a > -6 \Rightarrow a \in (-6, \infty) \quad \ldots\text{(ii)}$$

(iii) **Consider $a\,f(1) > 0$**

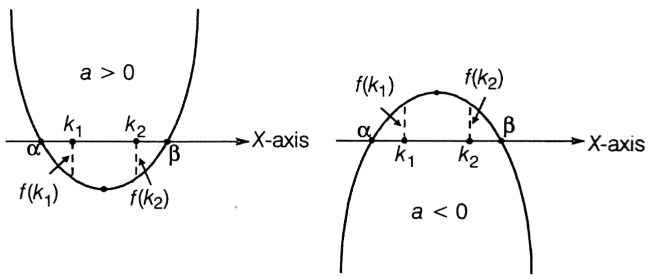
$$4(4 - 2 + a) > 0 \Rightarrow a > -2$$

$$\Rightarrow \quad a \in (-2, \infty) \quad \ldots\text{(iii)}$$

Hence, the values of a satisfying Eqs. (i), (ii) and (iii) at the same time are $\quad a \in \left(-2, \dfrac{1}{4}\right]$.

6. Conditions for Numbers k_1 and k_2

(If k_1 and k_2 lie between the roots of $f(x) = 0$)



(i) $D > 0$

(ii) $af(k_1) < 0$

(iii) $af(k_2) < 0$, where $\alpha < \beta$.

| Example 40. Find the values of a for which one root of equation $(a-5)x^2 - 2ax + a - 4 = 0$ is smaller than 1 and the other greater than 2.

Sol. The given equation can be written as

$$x^2 - \left(\frac{2a}{a-5}\right)x + \left(\frac{a-4}{a-5}\right) = 0,\ a \neq 5.$$

Now, let $f(x) = x^2 - \left(\dfrac{2a}{a-5}\right)x + \left(\dfrac{a-4}{a-5}\right)$

As 1 and 2 lie between the roots of $f(x) = 0$, we can take $D > 0$, $1 \cdot f(1) < 0$ and $1 \cdot f(2) < 0$.

(i) **Consider $D > 0$**

$$\left(-\left(\frac{2a}{a-5}\right)\right)^2 - 4\cdot 1\cdot\left(\frac{a-4}{a-5}\right) > 0$$

$$\Rightarrow \qquad \frac{36\left(a-\frac{20}{9}\right)}{(a-5)^2} > 0 \qquad [\because a \neq 5]$$

or $$a > \frac{20}{9} \qquad ...(i)$$

(ii) **Consider $1 \cdot f(1) < 0$**

$$1^2 - \left(\frac{2a}{a-5}\right) + \left(\frac{a-4}{a-5}\right) < 0 \Rightarrow \frac{9}{(a-5)} > 0 \text{ or } a > 5 \ ...(ii)$$

(iii) **Consider $1 \cdot f(2) < 0$**

$$4 - \frac{4a}{(a-5)} + \left(\frac{a-4}{a-5}\right) < 0$$

$$\Rightarrow \qquad \frac{(4a-20-4a+a-4)}{(a-5)} < 0 \Rightarrow \frac{(a-24)}{(a-5)} < 0$$

or $$5 < a < 24 \qquad ...(iii)$$

Hence, the values of a satisfying Eqs. (i), (ii) and (iii) at the same time are $a \in (5, 24)$.

Example 41. Let $x^2 - (m-3)x + m = 0$ $(m \in R)$ be a quadratic equation. Find the value of m for which

(i) both the roots are smaller than 2.

(ii) both the roots are greater than 2.

(iii) one root is smaller than 2 and the other root is greater than 2.

(iv) exactly one root lies in the interval (1, 2).

(v) both the roots lie in the interval (1, 2).

(vi) one root is greater than 2 and the other root is smaller than 1.

(vii) atleast one root lie in the interval (1, 2).

(viii) atleast one root is greater than 2.

Sol. Let $f(x) = x^2 - (m-3)x + m$

Here, $a = 1, b = -(m-3), c = m$

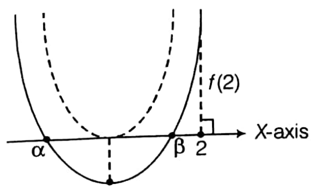
and $D = b^2 - 4ac = (m-3)^2 - 4m$

$= m^2 - 10m + 9 = (m-1)(m-9)$

and x-coordinate of vertex $= -\frac{b}{2a} = \frac{(m-3)}{2}$

(i) **Both the roots are smaller than 2**

$D \geq 0$



i.e., $(m-1)(m-9) \geq 0$

$\therefore \quad m \in (-\infty, 1] \cup [9, \infty) \qquad ...(i)$

$f(2) > 0$

i.e., $4 - 2(m-3) + m > 0$

$\Rightarrow \quad m < 10$

$\therefore \quad m \in (-\infty, 10) \qquad ...(ii)$

and x-coordinate of vertex < 2

i.e., $\frac{(m-3)}{2} < 2 \Rightarrow m < 7$

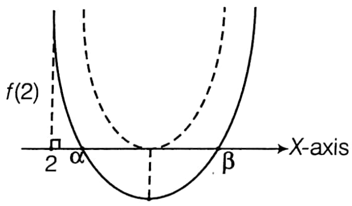
$\therefore \quad m \in (-\infty, 7) \qquad ...(iii)$

On combining Eqs. (i), (ii) and (iii), we get

$m \in (-\infty, 1]$

(ii) **Both the roots are greater than 2**

$D \geq 0$



i.e. $(m-1)(m-9) \geq 0$

$\therefore \quad m \in (-\infty, 1] \in [9, \infty) \qquad ...(i)$

$f(2) > 0$

i.e. $4 - 2(m-3) + m > 0$

$\Rightarrow \quad m < 10$

$\therefore \quad m \in (-\infty, 10) \qquad ...(ii)$

and x-coordinate of vertex > 2

i.e., $\frac{(m-3)}{2} > 2 \Rightarrow m > 7$

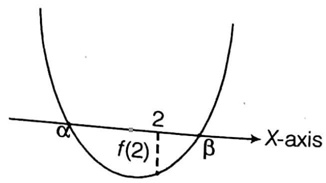
$\therefore \quad m \in (7, \infty) \qquad ...(iii)$

On combining Eqs. (i), (ii) and (iii), we get

$m \in [9, 10)$

(iii) **One root is smaller than 2 and the other root is greater than 2**

$D > 0$



i.e., $(m-1)(m-9) > 0$

$\therefore \quad m \in (-\infty, 1) \cup (9, \infty) \qquad ...(i)$

$f(2) < 0$

i.e. $4 - 2(m-3) + m < 0$

$\therefore \quad m > 10$

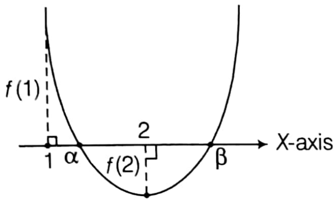
$\therefore \quad m \in (10, \infty)$ ...(ii)

On combining Eqs. (i) and (ii), we get

$m \in (10, \infty)$.

(iv) **Exactly one root lies in the interval (1, 2)**

$D > 0$



i.e., $(m-1)(m-9) > 0$

$\therefore \quad m \in (-\infty, 1) \cup (9, \infty)$ ...(i)

$f(1) f(2) < 0$

$(1-(m-3)+m)(4-2(m-3)+m) < 0$

$\Rightarrow \quad 4(-m+10) < 0$

$\Rightarrow \quad m - 10 > 0 \Rightarrow m > 10$

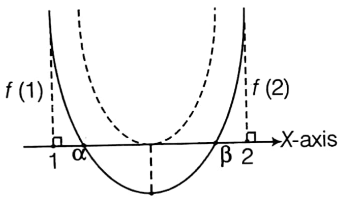
$\therefore \quad m \in (10, \infty)$ ...(ii)

On combining Eqs. (i) and (ii), we get

$m \in (10, \infty)$

(v) **Both the roots lie in the interval (1, 2)**

$D \geq 0$



i.e., $(m-1)(m-9) \geq 0$

$\therefore \quad m \in (-\infty, 1] \cup [9, \infty)$ ...(i)

$f(1) > 0$

i.e., $(1-(m-3)+m) > 0 \Rightarrow 4 > 0$

$\therefore \quad m \in R$ ...(ii)

$f(2) > 0$

i.e., $4 - 2(m-3) + m > 0 \Rightarrow m < 10$

$\therefore \quad m \in (-\infty, 10)$ ...(iii)

$1 < x$-coordinate of vertex < 2

i.e., $1 < \dfrac{(m-3)}{2} < 2$

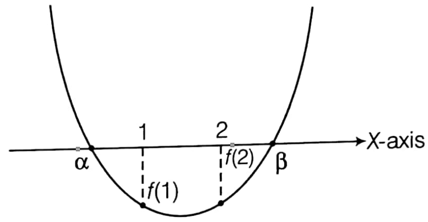
$\Rightarrow \quad 2 < m - 3 < 4 \quad \text{or} \quad 5 < m < 7$

$\therefore \quad m \in (5, 7)$ ...(iv)

On combining Eqs. (i), (ii), (iii) and (iv), we get

$m \in \phi$

(vi) **One root is greater than 2 and the other root is smaller than 1 $D > 0$**



i.e., $(m-1)(m-9) > 0$

$\therefore \quad m \in (-\infty, 1) \cup (9, \infty)$ ...(i)

$f(1) < 0$

i.e., $4 < 0$ which is not possible.

Thus, no such 'm' exists.

(vii) **At least one root lie in the interval (1, 2)**

***Case* I** Exactly one root lies in (1, 2)

$m \in (10, \infty)$ [from (iv) part]

***Case* II** Both roots lie in the interval (1, 2).

$m \in \phi$ [from (v) part]

Hence, at least one root lie in the interval (1, 2)

$m \in (10, \infty) \cup \phi$ or $m \in (10, \infty)$

(viii) **Atleast one root is greater than 2**

***Case* I** One root is smaller than 2 and the other root is greater than 2.

Then, $m \in (10, \infty)$ [from (iii) part]

***Case* II** Both the roots are greater than 2, then $m \in [9, 10)$.

Hence, atleast one root is greater than 2.

$\therefore \quad m \in (10, \infty) \cup [9, 10)$ or $m \in [9, 10) \cup (10, \infty)$

Exercise for Session 3

By putting values & by taking this expression $= y$ as eg:-30

1. If x is real, the maximum and minimum values of expression $\frac{x^2+14x+9}{x^2+2x+3}$ will be

   (a) 4, – 5 (b) 5, – 4 (c) – 4, 5 (d) – 4, – 5

2. If x is real, the expression $\frac{x+2}{(2x^2+3x+6)}$ takes all values in the interval

   (a) $\left(\frac{1}{13}, \frac{1}{3}\right)$ (b) $\left[-\frac{1}{13}, \frac{1}{3}\right]$ (c) $\left(-\frac{1}{3}, \frac{1}{13}\right)$ (d) None of these

3. If x be real, then the minimum value of $x^2-8x+17$, is

   (a) – 1 (b) 0 (c) 1 (d) 2

4. If the expression $\left(mx-1+\frac{1}{x}\right)$ is non-negative for all positive real x, the minimum value of m must be

   (a) $-\frac{1}{2}$ (b) 0
   (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

5. If the inequality $\frac{mx^2+3x+4}{x^2+2x+2}<5$ is satisfied for all $x \in R$ then

   (a) $1<m<5$ (b) $-1<m<5$
   (c) $1<m<6$ (d) $m<\frac{71}{24}$

6. The largest negative integer which satisfies $\frac{(x^2-1)}{(x-2)(x-3)}>0$, is

   (a) – 4 (b) – 3
   (c) – 2 (d) – 1

7. If the expression $2x^2+mxy+3y^2-5y-2$ can be resolved into two rational factors, the value of $|m|$ is

   (a) 3 (b) 5
   (c) 7 (d) 9

By checking options

8. If $c>0$ and $4a+c<2b$, then $ax^2-bx+c=0$ has a root in the interval

   (a) (0, 2) (b) (2, 4)
   (c) (0, 1) (d) (– 2, 0)

9. If the roots of the equation $x^2-2ax+a^2+a-3=0$ are less than 3 then

   (a) $a<2$ (b) $2\le a\le 3$
   (c) $3<a\le 4$ (d) $a>4$

10. The set of values of a for which the inequation $x^2+ax+a^2+6a<0$ is satisfied for all $x \in (1,2)$ lies in the interval

    (a) (1, 2) (b) [1, 2]
    (c) [– 7, 4] (d) None of these