CHAPTER

09

Probability

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Session 1

Some Basic Definitions, Mathematical or Priori or Classical Definition of Probability, Odds in Favour and Odds Against the Event

Some Basic Definitions

1. Random Experiment

An experiment whose outcome cannot be predicted with certainty, is called a random experiment.

Or

If in each trial of an experiment, which when repeated under identical conditions, the outcome is not unique but the outcome in a trial is one of the several possible outcomes, then such an experiment is known as a random experiment.

For example,

(i) "Throwing an unbiased die" is a random experiment because when a die is thrown, we cannot say with certainty which one of the numbers 1, 2, 3, 4, 5 and 6 will come up.

(ii) "Tossing of a fair coin" is a random experiment because when a coin is tossed, we cannot say with certainty whether either a head or a tail will come up.

(iii) "Drawing a card from a well-shuffled pack of cards" is a random experiment.

Remark

1. A die is a solid cube which has six faces and numbers 1, 2, 3, 4, 5 and 6 marked on the faces, respectively. In throwing or rolling a die, then any one number can be on the uppermost face.
2. (i) A pack of cards consists of 52 cards in 4 suits, i.e (a) Spades (♠) (b) Clubs (♣) (c) Hearts (♥) (d) Diamonds (♦). Each suit consists of 13 cards. Out of these, spades and clubs are black faced cards, while hearts and diamonds are red faced cards. The King, Queen, Jack (or Knave) are called face cards or honour cards.
   (ii) **Game of bridge** It is played by 4 players, each player is given 13 cards.
   (iii) **Game of whist** It is played by two pairs of persons.

2. Sample Space

The set of all possible results of a random experiment is called the sample space of that experiment and it is generally denoted by S.

Each element of a sample space is called a **sample point.**

For example,

(i) If we toss a coin, there are two possible results, namely a head (H) or a tail (T).
So, the sample space in this experiment is given by
$$S = \{H, T\}.$$

(ii) When two coins are tossed, the sample space
$$S = \{HH, HT, TH, TT\}$$
where, HH denotes the head on the first coin and head on the second coin. Similarly, HT denotes the head on the first coin and tail on the second coin.

(iii) When we throw a die, then any one of the numbers 1, 2, 3, 4, 5 and 6 will come up. So, the sample space
$$S = \{1, 2, 3, 4, 5, 6\}.$$

3. Elementary Event

An event having only a single sample point is called an elementary or simple event.

For example, When two coins are tossed, the sample space, $S = \{HH, HT, TH, TT\}$, then the event, $E_1 = \{HH\}$ of getting both the heads is a simple event.

4. Mixed Event or Compound Event or Composite Event

An event other than elementary or simple event is called mixed event.

For example,

(i) When two coins are tossed, the sample space
$$S = \{HH, HT, TH, TT\}$$
Then, the event $E = \{HH, HT, TH\}$ of getting atleast one head, is a mixed event.

(ii) When a die is thrown, the sample space
$$S = \{1, 2, 3, 4, 5, 6\}$$
Let $A = \{2, 4, 6\}$ = the event of occurrence of an even number
and $B = \{3, 6\}$ = the event of occurrence of a number divisible by 3.
Here, A and B are mixed events.

5. Equally likely Events

The given events are said to be equally likely, if none of them is expected to occur in preference to the other.

For example,

(i) When an unbiased coin is tossed, then occurrence of head or tail are equally likely cases and there is no reason to expect a 'head' or a 'tail' in preference to the other.

(ii) When an unbiased die is thrown, all the six faces 1, 2, 3, 4, 5 and 6 are equally likely to come up. There is no reason to expect 1 or 2 or 3 or 4 or 5 or 6 in preference to the other.

6. Independent Events

Two events are said to be independent, if the occurrence of one does not depend on the occurrence of the other.

For example, When an unbiased die is thrown, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$

Let $E_1 = \{1, 3, 5\}$ = the event of occurrence of an odd number and $E_2 = \{2, 4, 6\}$ = the event of occurrence of an even number. Clearly, the occurrence of odd number does not depend on the occurrence of even number. So, E_1 and E_2 are independent events.

7. Complementary Event

Let E be an event and S be the sample space for a random experiment, then complement of E is denoted by E' or E^c or $\bar{E}$. Clearly, E' means E does not occur.

Thus, E' occurs $\Leftrightarrow E$ does not occur.

For example, When an unbiased die is thrown, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

If $\quad E = \{1, 4, 6\}$, then $E' = \{2, 3, 5\}$

8. Mutually Exclusive Events

A set of events is said to be mutually exclusive, if occurrence of one of them precludes the occurrence of any of the remaining events. If a set of events $E_1, E_2, \ldots, E_n$ for mutually exclusive events.

Then, $\quad E_1 \cap E_2 \cap \ldots \cap E_n = \phi$

For example, If we thrown an unbiased die, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$ in which $E_1 = \{1, 2, 3\}$ = the event of occurrence of a number less than 4 and $E_2 = \{5, 6\}$ = the event of occurrence of a number greater than 4. Clearly, $E_1 \cap E_2 = \phi$

So, E_1 and E_2 are mutually exclusive.

9. Exhaustive Events

A set of events is said to be exhaustive, if the performance of the experiment results in the occurrence of atleast one of them. If a set of events $E_1, E_2, \ldots, E_n$ for exhaustive events.

Then, $\quad E_1 \cup E_2 \cup \ldots \cup E_n = S$

For example, If we thrown an unbiased die, then sample space $S = \{1, 2, 3, 4, 5, 6\}$ in which

$E_1 = \{1, 2, 3, 4\}$ = the event of occurrence of a number less than 5 and $E_2 = \{3, 4, 5, 6\}$ = the event of occurrence of a number greater than 2.

Then, $E_1 \cup E_2 = \{1, 2, 3, 4, 5, 6\}$ and $E_1 \cap E_2 = \{3, 4\}$

So, $\quad E_1 \cup E_2 = S$ and $E_1 \cap E_2 \neq \phi$

Hence, E_1 and E_2 are exhaustive events.

10. Mutually Exclusive and Exhaustive Events

A set of events is said to be mutually exclusive and exhaustive, if above two conditions are satisfied. If a set of events $E_1, E_2, \ldots, E_n$ for mutually exclusive and exhaustive events.

Then, $E_1 \cup E_2 \cup \ldots \cup E_n = S$ and $E_1 \cap E_2 \cap \ldots \cap E_n = \phi$

For example, If we thrown an unbiased die, then sample space

$S = \{1, 2, 3, 4, 5, 6\}$ in which

$E_1 = \{1, 3, 5\}$ = the event of occurrence of an odd number and $E_2 = \{2, 4, 6\}$ = the event of occurrence of an even number.

Then, $E_1 \cup E_2 = \{1, 2, 3, 4, 5, 6\}$ and $E_1 \cap E_2 = \phi$

So, $E_1 \cup E_2 = S$ and $E_1 \cap E_2 = \phi$.

Hence, E_1 and E_2 are mutually exclusive and exhaustive events.

Mathematical or Priori or Classical Definition of Probability

The probability of an event E to occur is the ratio of the number of cases in its favour to the total number of cases (equally likely).

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{\text{Number of cases favourable to event E}}{\text{Total number of cases}}$$

Range of Value of $P(E)$

Probability of occurrence of an event is a number lying between 0 and 1.

Proof Let S be the sample space and E be an event. Then,

$$E \subseteq S \quad \ldots \text{(i)}$$

Also, $$\phi \subseteq S \quad \ldots \text{(ii)}$$

where ϕ is a null set. From Eqs. (i) and (ii), we get

$$\phi \subseteq S \supseteq E \Rightarrow n(\phi) \le n(E) \le n(S)$$

$$\Rightarrow \quad 0 \le \frac{n(E)}{n(S)} \le 1 \quad [\because n(\phi) = 0]$$

$$\Rightarrow \quad 0 \le P(E) \le 1$$

Remark

1. For impossible event ϕ: $P(\phi) = 0$
2. For sure event S, $P(S) = 1$

Relationship between $P(E)$ and $P(E')$

If E is any event and E' be the complement of event E, then

$$P(E) + P(E') = 1$$

Proof Let S be the sample space, then

$$E' = S - E$$

$\Rightarrow \quad n(E') = n(S) - n(E)$

$\Rightarrow \quad \dfrac{n(E')}{n(S)} = 1 - \dfrac{n(E)}{n(S)}$

$\Rightarrow \quad P(E') = 1 - P(E)$

i.e. $\quad P(E) + P(E') = 1$

Odds in Favour and Odds Against the Event

Let S be the sample space. If a is the number of cases favourable to the event E, b is the number of cases favourable to the event E', the **odds in favour** of E are defined by $a : b$ and **odds against** of E are $b : a$.

i.e. odds in favour of event E is

$$\frac{a}{b} = \frac{n(E)}{n(E')} = \frac{\frac{n(E)}{n(S)}}{\frac{n(E')}{n(S)}} = \frac{P(E)}{P(E')} \Rightarrow \frac{P(E')}{P(E)} = \frac{b}{a}$$

$\Rightarrow \quad \dfrac{P(E') + P(E)}{P(E)} = \dfrac{b+a}{a}$

$\Rightarrow \quad \dfrac{1}{P(E)} = \dfrac{b+a}{a}$

$\Rightarrow \quad P(E) = \dfrac{a}{a+b}$ and $P(E') = \dfrac{b}{a+b}$

Remark

We use the sign '+' for the operation 'or' and '×' for the operation 'and' in order to solve the problems on definition of probability.

| Example 1. If three coins are tossed, represent the sample space and the event of getting atleast two heads, then find the number of elements in them.

Sol. Let S be the sample space and E be the event of occurrence of atleast two heads and let H denote the occurrence of head and T denote the occurrence of tail, when one coin is tossed.

Then, $S = \{H, T\} \times \{H, T\} \times \{H, T\}$

$S = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (T, T, H), (T, H, T), (H, T, T), (T, T, T)\}$

and $\quad E = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H)\}$

Also, $n(S) = 8$ and $n(E) = 4$

| Example 2. One ticket is drawn at random from a bag containing 24 tickets numbered 1 to 24. Represent the sample space and the event of drawing a ticket containing number which is a prime. Also, find the number of elements in them.

Sol. Let S be the sample space and E be the event of occurrence a prime number.

Then, $\quad S = \{1, 2, 3, 4, 5, \ldots, 24\}$

and $\quad E = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$

Also, $\quad n(S) = 24$ and $n(E) = 9$

| Example 3. Two dice are thrown simultaneously. What is the probability obtaining a total score less than 11?

Sol. Let S be the sample space and E be the event of obtaining a total less than 11.

Then, $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6 \times 6 = 36$

Let E' be the event of obtaining a total score greater than or equal to 11.

Also, $\quad E' = \{(5, 6), (6, 5), (6, 6)\}$; $\therefore n(E') = 3$

Then, probability of obtaining a total score greater than or equal to 11,

$$P(E') = \frac{n(E')}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

$\therefore \quad P(E) = 1 - P(E') = 1 - \dfrac{1}{12} = \dfrac{11}{12}$

Hence, required probability is $\dfrac{11}{12}$.

| Example 4. If a leap-leap year is selected at random, then what is the chance it will contain 53 Sunday?

Sol. A leap-leap year has 367 days i.e., 52 complete week and three days more. These three days will be three consecutive days of a week. A leap-leap year will have 53 Sundays, if out of the three consecutive days of a week selected at random one is a Sunday.

Let be the sample space and E be the event that out of the three consecutive days of a week one is Sunday, then

$S =$ {(Sunday, Monday, Tuesday), (Monday, Tuesday, Wednesday), (Tuesday, Wednesday, Thursday), (Wednesday, Thursday, Friday), (Thursday, Friday, Saturday), (Friday, Saturday, Sunday), (Saturday, Sunday, Monday)}; $n(S) = 7$

and $\quad E =$ {(Sunday, Monday, Tuesday), (Friday, Saturday, Sunday), (Saturday, Sunday, Monday)}

$\therefore \quad n(E) = 3$

Now, required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{3}{7}$

| Example 5. From a pack of 52 playing cards, three cards are drawn at random. Find the probability of drawing a King, a Queen and a Knave.

Sol. Let S be the sample space and E be the event that out of the three cards drawn one is a King, one is a Queen and one is a Knave.

$\therefore n(S) =$ Total number of selecting 3 cards out of 52 cards $= {}^{52}C_3$

and $n(E) =$ Number of selecting 3 cards out of one is King, one is Queen and one is Knave $= {}^4C_1 \cdot {}^4C_1 \cdot {}^4C_1 = 64$

$\therefore$ Required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{64}{{}^{52}C_3} = \dfrac{64}{\frac{52 \cdot 51 \cdot 50}{1 \cdot 2 \cdot 3}} = \dfrac{16}{5525}$

| Example 6. A bag contains 8 red and 5 white balls. Three balls are drawn at random. Find the probability that

(i) all the three balls are white.
(ii) all the three balls are red.
(iii) one ball is red and two balls are white.

Sol. Let S be the sample space, E_1 be the event of getting 3 white balls, E_2 be the event of getting 3 red balls and E_3 be the event of getting one red ball and two white balls.

$\therefore n(S) =$ Number of ways of selecting 3 balls out of

$$13(8+5) = {}^{13}C_3 = \frac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3} = 286$$

(i) $n(E_1) =$ Number of ways of selecting 3 white balls out of 5

$$= {}^5C_3 = {}^5C_2 = \frac{5 \cdot 4}{1 \cdot 2} = 10$$

$\therefore P$ (getting 3 white balls) $= \dfrac{n(E_1)}{n(S)} = \dfrac{10}{286} = \dfrac{5}{143}$

(ii) $n(E_2) =$ Number of ways of selecting 3 red balls out of 8

$$= {}^8C_3 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

$\therefore P$ (getting 3 red balls) $= \dfrac{n(E_2)}{n(S)} = \dfrac{56}{286} = \dfrac{28}{143}$

(iii) $n(E_3) =$ Number of ways of selecting 1 red ball out of 8 and 2 black balls out of $5 = {}^8C_1 \cdot {}^5C_2 = 8 \cdot 10 = 80$

$\therefore P$ (getting 1 red and 2 black balls)

$$= \frac{n(E_3)}{n(S)} = \frac{80}{286} = \frac{40}{143}$$

Exercise for Session 1

1. A problem in mathematics is given to three students and their respective probabilities of solving the problem are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. The probability that the problem is solved, is

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

2. A dice is thrown 3 times and the sum of the 3 numbers thrown is 15. The probability that the first throw was a four, is

(a) $\frac{1}{5}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{2}{5}$

3. Three faces of a fair dice are yellow, two faces red and one blue. The dice is tossed three times. The probability that the colours yellow, red and blue appear in the first, second and third toss respectively, is

(a) $\frac{1}{6}$ (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{36}$

4. A speaks truth in 75% of cases and B in 80% of cases. The percentage of cases they are likely to contradict each other in stating the same fact, is

(a) 30% (b) 35% (c) 45% (d) 25%

5. An unbiased dice with faces marked 1, 2, 3, 4, 5, 6 is rolled four times. Out of four face values obtained the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5, is

(a) $\frac{16}{81}$ (b) $\frac{1}{81}$ (c) $\frac{80}{81}$ (d) $\frac{65}{81}$

6. Three numbers are chosen at random without replacement from {1, 2, 3, ..., 10}. The probability that the minimum of the chosen number is 3 or their maximum is 7, is

(a) $\frac{11}{20}$ (b) $\frac{7}{20}$ (c) $\frac{11}{40}$ (d) $\frac{7}{40}$

7. Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently, is

(a) $\frac{1}{2}$ (b) $\frac{7}{15}$ (c) $\frac{2}{15}$ (d) $\frac{1}{3}$

8. Two numbers are selected randomly from the set $S = \{1, 2, 3, 4, 5, 6\}$ without replacement. The probability that minimum of the two numbers is less than 4, is

(a) $\frac{1}{15}$ (b) $\frac{14}{15}$ (c) $\frac{1}{5}$ (d) $\frac{4}{5}$

9. If $\frac{1+3p}{3}$, $\frac{1-p}{4}$ and $\frac{1-2p}{2}$ are the probabilities of the three mutually exclusive events, then $p \in$

(a) $[0, 1]$ (b) $\left[0, \frac{1}{2}\right]$ (c) $\left[\frac{1}{3}, 1\right]$ (d) $\left[\frac{1}{3}, \frac{1}{2}\right]$

10. Three identical dice are rolled once. The probability that the same number will appear on each of them, is

(a) $\frac{1}{6}$ (b) $\frac{1}{36}$ (c) $\frac{1}{18}$ (d) $\frac{3}{28}$

11. If the letters of the word ASSASSIN are written down in a row, the probability that no two S's occur together, is

(a) $\frac{1}{35}$ (b) $\frac{1}{21}$ (c) $\frac{1}{14}$ (d) $\frac{1}{28}$

12. A box contains 2 black, 4 white and 3 red balls. One ball is drawn at random from the box and kept aside. From the remaining balls in the box another ball is drawn and kept beside the first. This process is repeated till all the balls are drawn from the box. The probability that the balls drawn from the box are in the sequence 2 black, 4 white and 3 red, is

(a) $\frac{1}{126}$ (b) $\frac{1}{630}$ (c) $\frac{1}{1260}$ (d) $\frac{1}{2520}$

13. If three distinct numbers are chosen randomly from the first 100 natural numbers, then the probability that all three of them are divisible by both 2 and 3, is

(a) $\frac{4}{55}$ (b) $\frac{4}{35}$ (c) $\frac{4}{33}$ (d) $\frac{4}{1155}$

14. There are 2 vans each having numbered seats, 3 in the front and 4 at the back. There are 3 girls and 9 boys to be seated in the vans. The probability of 3 girls sitting together in a back row on adjacent seats, is

(a) $\frac{1}{13}$ (b) $\frac{1}{39}$ (c) $\frac{1}{65}$ (d) $\frac{1}{91}$

15. A and B stand in a ring along with 10 other persons. If the arrangement is at random, then the probability that there are exactly 3 persons between A and B, is

(a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{3}{11}$ (d) $\frac{4}{11}$

16. The first 12 letters of English alphabet are written down at random in a row. The probability that there are exactly 4 letters between A and B, is

(a) $\frac{7}{33}$ (b) $\frac{7}{66}$ (c) $\frac{7}{99}$ (d) $\frac{5}{33}$

17. Six boys and six girls sit in a row randomly. The probability that the six girls sit together or the boys and girls sit alternately, is

(a) $\frac{3}{308}$ (b) $\frac{1}{100}$ (c) $\frac{2}{205}$ (d) $\frac{4}{407}$

18. If from each of three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn, the probability of drawing 2 white and 1 black ball, is

(a) $\frac{13}{32}$ (b) $\frac{1}{4}$ (c) $\frac{1}{32}$ (d) $\frac{3}{16}$

19. The probability that a year chosen at random has 53 Sundays, is

(a) $\frac{5}{7}$ (b) $\frac{3}{7}$ (c) $\frac{5}{28}$ (d) $\frac{3}{28}$

20. If the letters of the word MATHEMATICS are arranged arbitrarily, the probability that C comes before E, E before H, H before I and I before S, is

(a) $\frac{3}{10}$ (b) $\frac{1}{20}$ (c) $\frac{1}{120}$ (d) $\frac{1}{720}$

Session 2

Some Important Symbols, Conditional Probability

Some Important Symbols

If A, B and C are any three events, then

(i) $A \cap B$ or AB denotes the event of simultaneous occurrence of both the events A and B.

(ii) $A \cup B$ or $A + B$ denotes the event of occurrence of atleast one of the events A or B.

(iii) $A - B$ denotes the occurrence of event A but not B.

(iv) $\overline{A}$ denotes the not occurrence of event A.

(v) $A \cap \overline{B}$ denotes the occurrence of event A but not B.

(vi) $\overline{A} \cap \overline{B} = (\overline{A \cup B})$ denotes the occurrence of neither A nor B.

(vii) $A \cup B \cup C$ denotes the occurrence of atleast one event A, B or C.

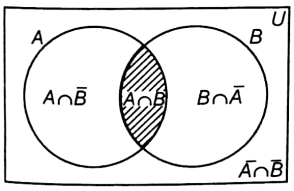
(viii) $(A \cap \overline{B}) \cup (\overline{A} \cap B)$ denotes the occurrence of exactly one of A and B.

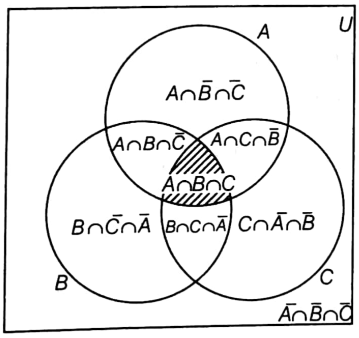
(ix) $A \cap B \cap C$ denotes the occurrence of all three A, B and C.

(x) $(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$ denotes the occurrence of exactly two of A, B and C.

Remark

Remember with the help of figures





Important Results

1. If A and B are arbitrary events, then

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof Let S be the sample space. Since, we know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad \frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Remark

If A and B are mutually exclusive events, then $A \cap B = \phi$. Hence, $P(A \cap B) = 0$.

$\therefore P(A \cup B) = P(A) + P(B)$

(b) P (exactly one of A, B occurs)

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$$
$$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$$
$$= P(A) + P(B) - 2P(A \cap B)$$
$$= P(A \cup B) - P(A \cap B)$$

(c) P (neither A nor B)

$$= P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$$

Remark

$P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B)$

2. If A, B and C are three events, then

(a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Remark

If A, B and C are mutually exclusive events, then

$$A \cap B = \phi, B \cap C = \phi, C \cap A = \phi, A \cap B \cap C = \phi$$

$\Rightarrow P(A \cap B) = 0, P(B \cap C) = 0, P(C \cap A) = 0, P(A \cap B \cap C) = 0$

$\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C)$

General form of Addition Theorem of Probability

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Remark

If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$\sum_{i<j} P(A_i \cap A_j) = 0, \quad \sum_{i<j<k} P(A_i \cap A_j \cap A_k) = 0$

and $\quad P(A_1 \cap A_2 \cap \ldots \cap A_n) = 0$

$\therefore \quad P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i)$

(b) P (atleast two of A, B, C occur)
$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$

(c) P (exactly two of A, B, C occur)
$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$

(d) P (exactly one of A, B, C occur)
$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(C \cap A) + 3P(A \cap B \cap C)$

3. (a) If $A_1, A_2, \ldots, A_n$ are independent events, then
$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\,P(A_2)\ldots P(A_n)$$

(b) If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then
$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

(c) If $A_1, A_2, \ldots, A_n$ are exhaustive events, then
$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1$$

(d) If $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive events, then
$$P(A_1 \cup A_2 \cap \ldots \cap A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) = 1$$

4. If $A_1, A_2, \ldots, A_n$ are n events, then

(a) $P(A_1 \cup A_2 \cup \ldots \cup A_n) \le P(A_1) + P(A_2) + \ldots + P(A_n)$

(b) $P(A_1 \cap A_2 \cap \ldots \cap A_n) \ge 1 - P(\overline{A}_1) - P(\overline{A}_2) - \ldots - P(\overline{A}_n)$

Important Result

If E_1 and E_2 are independent events, then

(a) E_1 and $\overline{E}_2$ are independent events.

(b) $\overline{E}_1$ and E_2 are independent events.

(c) $\overline{E}_1$ and $\overline{E}_2$ are independent events.

Proof Given, E_1 and E_2 are independent events, then
$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

(a) $P(E_1 \cap \overline{E}_2) = P(E_1) - P(E_1 \cap E_2)$
$= P(E_1) - P(E_1) \cdot P(E_2)$
$= P(E_1)[1 - P(E_2)] = P(E_1) \cdot P(\overline{E}_2)$

So, E_1 and $\overline{E}_2$ are independent events.

(b) Same as in part (i).

(c) $P(\overline{E}_1 \cap \overline{E}_2) = P(\overline{E_1 \cup E_2})$
$= 1 - P(E_1 \cup E_2) = 1 - [P(E_1) + P(E_2) - P(E_1 \cap E_2)]$
$= 1 - P(E_1) - P(E_2) + P(E_1) \cdot P(E_2)$
$= P(\overline{E}_1) - P(E_2)[1 - P(E_1)]$
$= P(\overline{E}_1) - P(E_2) \cdot P(\overline{E}_1) = P(\overline{E}_1)[1 - P(E_2)]$
$= P(\overline{E}_1)\,P(\overline{E}_2)$

Remark

If $E_1, E_2, \ldots, E_n$ are independent events, then $P(E_1 \cup E_2 \cup \ldots \cup E_n)$
$= 1 - P(E_1 \cup E_2 \cup \ldots \cup E_n)' = 1 - P(E_1' \cap E_2' \cap \ldots \cap E_n')$
$= 1 - P(E_1') \cdot P(E_2') \ldots P(E_n')$

Example 7. For a post, three persons A, B and C appear in the interiew. The probability of A being selected is twice that of B and the probability of B being selected is thrice that of C. What are the individual probabilities of A, B and C being selected?

Sol. Let E_1, E_2 and E_3 be the events of selection of A, B and C respectively.

Let $P(E_3) = x$.

Then, $P(E_2) = 3P(E_3) = 3x$ and $P(E_1) = 2P(E_2) = 6x$

Since, E_1, E_2 and E_3 are mutually exclusive and exhaustive events.

$\therefore \quad P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) = 1$

$\therefore \quad P(E_1) + P(E_2) + P(E_3) = 1$

$\Rightarrow \quad 6x + 3x + x = 1$

$\therefore \quad x = \frac{1}{10}$

Hence, $P(E_1) = 6x = \frac{6}{10} = \frac{3}{5}$

$P(E_2) = 3x = \frac{3}{10}$ and $P(E_3) = x = \frac{1}{10}$

Example 8. If A and B are independent events, the probability that both A and B occur is $\frac{1}{8}$ and the probability that none of them occurs is $\frac{3}{8}$. Find the probability of the occurrence of A.

Sol. We have,
$$P(A \cap B) = \frac{1}{8} \Rightarrow P(A)\,P(B) = \frac{1}{8} \quad \ldots\text{(i)}$$
[$\because$ A and B are independent]

and $P(\overline{A} \cap \overline{B}) = \frac{3}{8} \Rightarrow P(\overline{A})\,P(\overline{B}) = \frac{3}{8}$

$\Rightarrow \quad (1 - P(A))(1 - P(B)) = \frac{3}{8}$

$\Rightarrow \quad 1 - P(A) - P(B) + \frac{1}{8} = \frac{3}{8}$ [from Eq. (i)]

$\Rightarrow \quad P(A) + P(B) = \frac{3}{4}$ ...(ii)

The quadratic equation whose roots are $P(A)$ and $P(B)$ is
$x^2 - [P(A) + P(B)]x + P(A) \cdot P(B) = 0$

$\Rightarrow \quad x^2 - \frac{3}{4}x + \frac{1}{8} = 0$ [from Eqs. (i) and (ii)] or $8x^2 - 6x + 1 = 0$ or $x = \frac{1}{2}, \frac{1}{4}$

Hence, $P(A) = \frac{1}{2}$ or $\frac{1}{4}$

| Example 9. A and B are two candidates seeking admission in IIT. The probability that A is selected is 0.5 and the probability that both A and B are selected is atmost 0.3. Is it possible that the probability of B getting selected is 0.9?

Sol. Let E_1 and E_2 are the events of A and B selected, respectively.

Given, $P(E_1 \cap E_2) \le 0.3$ and $P(E_1) = 0.5$

Since, $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\because \quad P(E_1 \cup E_2) \le 1$

$\therefore \quad P(E_1) + P(E_2) - P(E_1 \cap E_2) \le 1$

$\Rightarrow \quad P(E_1) + P(E_2) \le 1 + P(E_1 \cap E_2)$

$\Rightarrow \quad 0.5 + P(E_2) \le 1 + 0.3 \Rightarrow P(E_2) \le 0.8$

Hence, $P(E_2) \ne 0.9$

| Example 10. Let A, B and C be three events. If the probability of occurring exactly one event out of A and B is $1-a$, out of B and C is $1-2a$, out of C and A is $1-a$ and that of occurring three events simultaneously is a^2, then prove that the probability that atleast one out of A, B and C will occur is greater than $\frac{1}{2}$.

Sol. Given,

$$P(A) + P(B) - 2P(A \cap B) = 1 - a \quad \text{...(i)}$$

$$P(B) + P(C) - 2P(B \cap C) = 1 - 2a \quad \text{...(ii)}$$

and $$P(C) + P(A) - 2P(C \cap A) = 1 - a \quad \text{...(iii)}$$

$\therefore$ $$P(A \cap B \cap C) = a^2 \quad \text{...(iv)}$$

$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= \frac{1}{2}\{P(A) + P(B) - 2P(A \cap B) + P(B) + P(C) - 2P(B \cap C) + P(C) + P(A) - 2P(C \cap A)\} + P(A \cap B \cap C)$

$= \frac{1}{2}\{1 - a + 1 - 2a + 1 - a\} + a^2$ [from Eqs. (i), (ii), (iii) and (iv)]

$= \frac{3}{2} - 2a + a^2 = (a-1)^2 + \frac{1}{2} > \frac{1}{2}$ $\quad [\because a \ne 1]$

| Example 11. If A, B and C are three events, such that $P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(AB) = 0.08$, $P(AC) = 0.28$, $P(ABC) = 0.09$. If $P(A \cup B \cup C) \ge 0.75$, then show that $P(BC)$ lies in the interval $0.23 \le x \le 0.48$.

Sol. Let $P(BC) = x$

Since, $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$

$\therefore \quad = 0.3 + 0.4 + 0.8 - 0.08 - x - 0.28 + 0.09 = 1.23 - x$

But given that, $P(A \cup B \cup C) \ge 0.75$ and $P(A \cup B \cup C) \le 1$

$\therefore \quad 0.75 \le 1.23 - x \le 1 \Rightarrow -0.75 \ge -1.23 + x \ge -1$

or $\quad 1.23 - 0.75 \ge x \ge 1.23 - 1$ or $0.23 \le x \le 0.48$

Conditional Probability

The probability of occurrence of an event E_1, given that E_2 has already occurred is called the conditional probability of occurrence of E_1 on the condition that E_2 has already occurred, it is denoted by $P\left(\frac{E_1}{E_2}\right)$.

Thus, $$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}, E_2 \ne \phi = \frac{\frac{n(E_1 \cap E_2)}{n(S)}}{\frac{n(E_2)}{n(S)}}$$

$$\Rightarrow \quad = \frac{n(E_1 \cap E_2)}{n(E_2)}$$

Remark

1. If E_1 and E_2 are independent events, then $P\left(\frac{E_2}{E_1}\right) = P(E_2)$
2. If E_1 and E_2 are two events such that $E_2 \ne \phi$, then $P\left(\frac{E_1}{E_2}\right) + P\left(\frac{\overline{E}_1}{E_2}\right) = 1$
3. If $E_1, E_2, E_3, \ldots, E_4$ are independent events, then $P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n) = 1 - P(\overline{E}_1) \cdot P(\overline{E}_2) \cdot P(\overline{E}_3) \ldots P(\overline{E}_n)$
4. If E_1, E_2 and E_3 are three events such that $E_1 \ne \phi$, $E_1E_2 \ne \phi$, then $P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) \cdot P\left(\frac{E_3}{E_1E_2}\right)$

Generalised form

If $E_1, E_2, E_3, \ldots, E_n$ are n events such that $E_1 \ne \phi$, $E_1E_2 \ne \phi$, $E_1E_2E_3 \ne \phi$, ..., $E_1E_2E_3 \ldots E_{n-1} \ne \phi$, then $P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n)$

$$= P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) \cdot P\left(\frac{E_3}{E_1E_2}\right) \cdot P\left(\frac{E_4}{E_1E_2E_3}\right) \ldots P\left(\frac{E_n}{E_1E_2E_3 \ldots E_{n-1}}\right).$$

| Example 12. Two dice are thrown. Find the probability that the sum of the numbers coming up on them is 9, if it is known that the number 5 always occurs on the first dice.

Sol. Let S be the sample space

$\therefore \quad S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

$\therefore \quad n(S) = 36$

and let $E_1 \equiv$ The event that the sum of the numbers coming up is 9.

and $\quad E_2 \equiv$ The event of occurrence of 5 on the first dice.

$\therefore \quad E_1 \equiv \{(3, 6), (6, 3), (4, 5), (5, 4)\}$

$\therefore \quad n(E_1) = 4$

and $E_2 = \{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$

$\therefore \quad n(E_2) = 6$

$$E_1 \cap E_2 = \{(5,4)\}$$

$\therefore \quad n(E_1 \cap E_2) = 1$

Now, $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{1}{36}$

and $P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$

$\therefore$ Required probability,

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

Aliter $P\left(\dfrac{E_1}{E_2}\right) = \dfrac{n(E_1 \cap E_2)}{n(E_2)} = \dfrac{1}{6}$

| Example 13. In a class, 30% students fail in English; 20% students fail in Hindi and 10% students fail in English and Hindi both. A student is chosen at random, then what is the probability that he will fail in English, if he has failed in Hindi?

Sol. Let S be the sample space.

If $n(S) = 100$, then

$E_1 \equiv$ The event that the student chosen fail in English

$\therefore n(E_1) = 30$

and $E_2 \equiv$ The event that the student chosen fail in Hindi

$\therefore \quad n(E_2) = 20$ and $n(E_1 \cap E_2) = 10$

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{20}{100} = \frac{1}{5}$$

and $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{10}{100} = \dfrac{1}{10}$

$\therefore$ Required probability, $P\left(\dfrac{E_1}{E_2}\right) = \dfrac{P(E_1 \cap E_2)}{P(E_2)} = \dfrac{\frac{1}{10}}{\frac{1}{5}} = \dfrac{1}{2}$

Aliter $P\left(\dfrac{E_1}{E_2}\right) = \dfrac{n(E_1 \cap E_2)}{n(E_2)} = \dfrac{10}{20} = \dfrac{1}{2}$

Exercise for Session 2

1 If $P(A) = 0{\cdot}8$, $P(B) = 0{\cdot}5$, then $P(A \cap B)$ lies in the interval

(a) $[0{\cdot}2, 0{\cdot}5]$ (b) $[0{\cdot}2, 0{\cdot}3]$ (c) $[0{\cdot}3, 0{\cdot}5]$ (d) $[0{\cdot}1, 0{\cdot}5]$

2 If $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{13}$ and $P(A \cap B) = \frac{1}{52}$, then the value of $P(\bar{A} \cap \bar{B})$, is

(a) $\frac{3}{13}$ (b) $\frac{5}{13}$ (c) $\frac{7}{13}$ (d) $\frac{9}{13}$

3 If A and B are independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(A \cap \bar{B}) = \frac{1}{6}$, then $P(B)$ is

(a) $\frac{1}{5}$ (b) $\frac{1}{6}$ (c) $\frac{4}{5}$ (d) $\frac{5}{6}$

4 If A and B are two events such that $P(A \cup B) = \frac{5}{6}$, $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{4}$, then A and B are

(a) mutually exclusive (b) dependent (c) independent (d) None of these

5 If A, B and C are mutually exclusive and exhaustive events associated with a random experiment. If $P(B) = \frac{3}{2}P(A)$ and $P(C) = \frac{1}{2}P(B)$, then $P(A)$ is equal to

(a) $\frac{2}{13}$ (b) $\frac{4}{13}$ (c) $\frac{6}{13}$ (d) $\frac{8}{13}$

6 If A and B are two events, then $P(A) + P(B) = 2P(A \cap B)$ if and only if

(a) $P(A) + P(B) = 1$ (b) $P(A) = P(B)$ (c) $P(A) + P(B) > 1$ (d) None of these

7 If A and B are two events such that $P(A \cap B) = \frac{1}{4}$, $P(\bar{A} \cap \bar{B}) = \frac{1}{5}$ and $P(A) = P(B) = p$, then p is equal to

(a) $\frac{17}{40}$ (b) $\frac{19}{40}$ (c) $\frac{21}{40}$ (d) $\frac{23}{40}$

8 If A and B are two events such that $P(A \cup B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$, $P(\bar{A}) = \frac{2}{3}$. Then $(\bar{A} \cap B)$ is equal to

(a) $\frac{5}{12}$ (b) $\frac{3}{8}$ (c) $\frac{5}{8}$ (d) $\frac{1}{4}$

9 If $P(B) = \frac{3}{4}$, $P(A \cap B \cap \bar{C}) = \frac{1}{3}$ and $P(\bar{A} \cap B \cap \bar{C}) = \frac{1}{3}$, then $P(B \cap C)$ is equal to

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{1}{15}$ (d) $\frac{1}{9}$

10 If A and B are two events such that $P(A) > 0$ and $P(B) \neq 1$, then $P\left(\frac{\bar{A}}{\bar{B}}\right)$ is equal to

(a) $1 - P\left(\frac{A}{B}\right)$ (b) $1 - P\left(\frac{A}{\bar{B}}\right)$ (c) $\frac{1 - P(A \cup B)}{P(\bar{B})}$ (d) $\frac{P(\bar{A})}{P(\bar{B})}$

11 If $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, then $P\left(\frac{\bar{A}}{\bar{B}}\right)$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

12 If two events A and B are such that $P(\bar{A}) = 0 \cdot 3$, $P(B) = 0 \cdot 4$ and $P(A \cap \bar{B}) = 0 \cdot 5$, then $P\left(\frac{B}{A \cup \bar{B}}\right)$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

13 Two dice are thrown. The probability that the number appeared have a sum of 8. If it is known that the second die always exhibits 4, is

(a) $\frac{5}{6}$ (b) $\frac{1}{6}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

14 A is targetting to B, B and C are targetting to A. The probability of hitting the target by A, B and C are $\frac{2}{3}, \frac{1}{2}, \frac{1}{3}$ respectively. If A is hit, the probability that B hits the target and C does not, is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

15 If A and B are two events such that $A \cap B \neq \phi$, $P\left(\frac{A}{B}\right) = P\left(\frac{B}{A}\right)$. Then,

(a) $A = B$ (b) $P(A) = P(B)$
(c) A and B are independent (d) All of these

Session 3

Total Probability Theorem, Baye's Theorem or Inverse Probability

Total Probability Theorem

Let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events i.e.,$E_i \cap E_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{n} E_i = S$.

Suppose that, $P(E_i) > 0, \forall\ 1 \leq i \leq n$

Then for any event E

$$P(E) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)$$

Proof Since, $E_1, E_2, \ldots, E_n$ are disjoint

$\therefore E \cap E_1, E \cap E_2, \ldots, E \cap E_n$ are also disjoint.

Now, $$E = E \cap S = E \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (E \cap E_i)$$

$$\therefore \quad P(E) = \sum_{i=1}^{n} P(E \cap E_i) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)$$

| Example 14. The probability that certain electronic component fails, when first used is 0.10. If it does not fail immediately, then the probability that it lasts for one year is 0.99. What is the probability that a new component will last for one year?

Sol. Given probability of electronic component fails, when first used = 0.10

i.e., $P(F) = 0.10$

$\therefore \quad P(\overline{F}) = 1 - P(F) = 0.90$

and let $P(Y)$ = Probability of new component to last for one year

$\because \quad P(F) + P(\overline{F}) = 1$

Obviously, the two events are mutually exclusive and exhaustive

$$\therefore \quad P\left(\frac{Y}{F}\right) = 0 \text{ and } P\left(\frac{Y}{\overline{F}}\right) = 0.99$$

$$\therefore \quad P(Y) = P(F) \cdot P\left(\frac{Y}{F}\right) + P(\overline{F}) \cdot P\left(\frac{Y}{\overline{F}}\right)$$

$$= 0.10 \times 0 + 0.90 \times 0.99$$
$$= 0 + (0.9)(0.99) = 0.891$$

| Example 15. Three groups A, B and C are contesting for positions on the Board of Directors of a company. The probabilities of their winning are 0.5, 0.3 and 0.2, respectively. If the group A wins, then the probability of introducing a new product is 0.7 and the corresponding probabilities for groups B and C are 0.6 and 0.5, respectively. Find the probability that the new product will be introduced.

Sol. Given, $P(A) = 0.5$, $P(B) = 0.3$ and $P(C) = 0.2$

$\therefore \quad P(A) + P(B) + P(C) = 1$

Then, events A, B, C are exhaustive.

If $P(E)$ = Probability of introducing a new product, then as given

$$P\left(\frac{E}{A}\right) = 0.7,\ P\left(\frac{E}{B}\right) = 0.6 \text{ and } P\left(\frac{E}{C}\right) = 0.5$$

$$P(E) = P(A) \cdot P\left(\frac{E}{A}\right) + P(B) \cdot P\left(\frac{E}{B}\right) + P(C) \cdot P\left(\frac{E}{C}\right)$$

$$= 0.5 \times 0.7 + 0.3 \times 0.6 + 0.2 \times 0.5$$
$$= 0.35 + 0.18 + 0.10 = 0.63$$

| Example 16. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replace into urn, otherwise it is replaced along with another ball of the same colour. The process is repeated, find the probability that the third ball drawn is black.

Sol. For the first two draw, the balls taken out may be

Let E_1 = White and White; E_2 = White and Black

E_3 = Black and White; E_4 = Black and Black

$$\therefore \quad P(E_1) = P(W) \cdot P\left(\frac{W}{W}\right) = \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{6}$$

$$P(E_2) = P(W) \cdot P\left(\frac{B}{W}\right) = \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{3}$$

$$P(E_3) = P(B) \cdot P\left(\frac{W}{B}\right) = \frac{2}{4} \cdot \frac{2}{5} = \frac{1}{5}$$

$$\text{and } P(E_4) = P(B) \cdot P\left(\frac{B}{B}\right) = \frac{2}{4} \cdot \frac{3}{5} = \frac{3}{10}$$

$$\therefore P(E_1)+P(E_2)+P(E_3)+P(E_4)=\frac{1}{6}+\frac{1}{3}+\frac{1}{5}+\frac{3}{10}$$
$$=\frac{10+20+12+18}{60}=1$$

Then, events E_1, E_2, E_3 and E_4 are exhaustive. Obviously, these events are mutually exclusive, then

$$P\left(\frac{B}{E_1}\right)=\frac{2}{2}=1;\ P\left(\frac{B}{E_2}\right)=\frac{3}{4}$$
$$P\left(\frac{B}{E_3}\right)=\frac{3}{4} \text{ and } P\left(\frac{B}{E_4}\right)=\frac{4}{6}=\frac{2}{3}$$

$\therefore$ Required probability,

$$P(B)=P(E_1)\cdot P\left(\frac{B}{E_1}\right)+P(E_2)\cdot P\left(\frac{B}{E_2}\right)+P(E_3)\cdot P\left(\frac{B}{E_3}\right)+P(E_4)\cdot P\left(\frac{B}{E_4}\right)$$
$$=\frac{1}{6}\times 1+\frac{1}{3}\times\frac{3}{4}+\frac{1}{5}\times\frac{3}{4}+\frac{3}{10}\times\frac{2}{3}$$
$$=\frac{1}{6}+\frac{1}{4}+\frac{3}{20}+\frac{1}{5}$$
$$=\frac{10+15+9+12}{60}=\frac{46}{60}=\frac{23}{30}$$

Baye's Theorem or Inverse Probability

If an event E can occur only with one of the n mutually exclusive and exhaustive events $E_1, E_2, E_3, \dots, E_n$ and the probabilities $P(E/E_1), P(E/E_2), \dots, P(E/E_n)$ are known, then

$$P\left(\frac{E_k}{E}\right)=\frac{P(E_k)\cdot P\left(\frac{E}{E_k}\right)}{\sum_{k=1}^{n}P(E_k)\cdot P\left(\frac{E}{E_k}\right)}$$

Proof The event E occurs with one of the n mutually exclusive and exhaustive events $E_1, E_2, E_3, \dots, E_n$, then

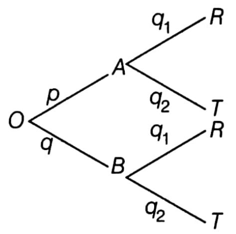
$$E=EE_1+EE_2+EE_3+\dots+EE_n$$
$$\Rightarrow P(E)=P(EE_1)+P(EE_2)+P(EE_3)+\dots+P(EE_n)$$
$$=\sum_{k=1}^{n}P(EE_k)=\sum_{k=1}^{n}P(E_k)\cdot P\left(\frac{E}{E_k}\right)$$
$$\therefore P\left(\frac{E_k}{E}\right)=\frac{P(E_k\,E)}{P(E)}=\frac{P(E_k)\cdot P\left(\frac{E}{E_k}\right)}{\sum_{k=1}^{n}P(E_k)\cdot P\left(\frac{E}{E_k}\right)}$$

Remark

The probabilities $P(E_k)$ and $P\left(\frac{E_k}{E}\right)$ are known as **priori** and **posteriori** probabilities, respectively.

Remarks We can visualise a tree structure here

$$P(A)=p,\ P(B)=q$$
$$P\left(\frac{R}{A}\right)=p_1,\ P\left(\frac{T}{A}\right)=q_1$$
$$P\left(\frac{R}{B}\right)=p_2,\ P\left(\frac{T}{B}\right)=q_2$$



If we are to find $P\left(\frac{A}{R}\right)$, we go

$$\therefore P\left(\frac{A}{R}\right)=\frac{P(A)\cdot P\left(\frac{R}{A}\right)}{P(A)\cdot P\left(\frac{R}{A}\right)+P(B)\cdot P\left(\frac{R}{B}\right)}$$

| Example 17. A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Then, find the probability that it was drawn from the bag B.

Sol. Let $E_1 \equiv$ The event of ball being drawn from bag A.

$E_2 \equiv$ The event of ball being drawn from bag B.

and $E \equiv$ The event of ball being red.

Since, both the bag are equally likely to be selected, therefore

$$P(E_1)=P(E_2)=\frac{1}{2} \text{ and } P\left(\frac{E}{E_1}\right)=\frac{3}{5} \text{ and } P\left(\frac{E}{E_2}\right)=\frac{5}{9}$$

$\therefore$ Required probability,

$$P\left(\frac{E_2}{E}\right)=\frac{P(E_2)\cdot P\left(\frac{E}{E_2}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)}$$
$$=\frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}}=\frac{\frac{5}{9}}{\frac{3}{5}+\frac{5}{9}}=\frac{25}{52}$$

| Example 18. A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Sol. Let E_1 be the event that the man reports that it is a six and E be the event that a six occurs.

Then, $P(E)=\frac{1}{6}$

$\therefore \quad P(\overline{E})=1-P(E)=1-\frac{1}{6}=\frac{5}{6}$

$\therefore \quad P\left(\frac{E_1}{E}\right)=P$ (man speaking the truth) $=\frac{3}{4}$

and $P\left(\frac{E_1}{\overline{E}}\right)=P$ (man not speaking the truth) $=1-\frac{3}{4}=\frac{1}{4}$

Clearly, $\left(\frac{E}{E_1}\right)$ is the event that it is actually a six, when it is known that the man reports a six.

$$P\left(\frac{E}{E_1}\right)=\frac{P(E)\cdot P\left(\frac{E_1}{E}\right)}{P(E)\cdot P\left(\frac{E_1}{E}\right)+P(\overline{E})\cdot P\left(\frac{E_1}{\overline{E}}\right)}$$

$$=\frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}}=\frac{3}{8}$$

| Example 19. In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is $\frac{1}{3}$ and the probability that he copies the answer is $\frac{1}{6}$. The probability that his answer is correct given that he copied it is $\frac{1}{8}$. Find the probability that he knew the answer to the question given that he correctly answered it.

Sol. Lest E_1 be the event that the answer is guessed, E_2 be the event that the answer is copied, E_3 be the event that the examinee knows the answer and E be the event that the examinee answers correctly.

Given, $P(E_1)=\frac{1}{3}$, $P(E_2)=\frac{1}{6}$

Assume that events E_1, E_2 and E_3 are exhaustive

$\therefore \quad P(E_1)+P(E_2)+P(E_3)=1$

$\therefore \quad P(E_3)=1-P(E_1)-P(E_2)=1-\frac{1}{3}-\frac{1}{6}=\frac{1}{2}$

Now, $P\left(\frac{E}{E_1}\right)$

$\equiv$ Probability of getting correct answer by guessing

$=\frac{1}{4}$ [since 4 alternatives]

$P\left(\frac{E}{E_2}\right)\equiv$ Probability of answering correctly by copying $=\frac{1}{8}$

and $P\left(\frac{E}{E_3}\right)\equiv$ Probability of answering correctly by knowing $=1$

Clearly, $\left(\frac{E_3}{E}\right)$ is the event he knew the answer to the question, given that he correctly answered it.

$$\therefore \quad P\left(\frac{E_3}{E}\right)=\frac{P(E_3)\cdot P\left(\frac{E}{E_3}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)+P(E_2)\cdot P\left(\frac{E}{E_3}\right)}$$

$$=\frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1}=\frac{24}{29}$$

| Example 20. A and B are two independent witnesses (i.e., there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statements. Show that the probability that the statements is true, is $\frac{xy}{1-x-y+2xy}$.

Sol. Let E_1 be the event that both A and B speak the truth, E_2 be the event that both A and B tell a lie and E be the event that A and B agree in a certain statements.

And also, let C be the event that A speak the truth and D be the event that B speaks the truth.

$\therefore \quad E_1=C\cap D$

[$\because C$ and D are independent events]

and $\quad E_2=\overline{C}\cap\overline{D}$

then, $\quad P(E_1)=(C\cap D)=P(C)\cdot P(D)=xy$

and $\quad P(E_2)=P(\overline{C}\cap\overline{D})=P(\overline{C})\,P(\overline{D})$

$=\{1-P(C)\}\{1-P(D)\}=(1-x)(1-y)$

$=1-x-y-xy$

Now, $P\left(\frac{E}{E_1}\right)\equiv$ Probability that A and B will agree, when both of them speak the truth $=1$

and $P\left(\frac{E}{E_2}\right)=$ Probability that A and B will agree, when both of them tell a lie $=1$

Clearly, $\left(\frac{E_1}{E}\right)$ be the event that the statement is true

$$\therefore \quad P\left(\frac{E_1}{E}\right)=\frac{P(E_1)\cdot P\left(\frac{E}{E_1}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)}$$

$$=\frac{xy\cdot 1}{xy\cdot 1+(1-x-y+xy)\cdot 1}=\frac{xy}{1-x-y+2xy}$$

Exercise for Session 3

1. A bag A contains 3 white and 2 black balls and another bag B contains 2 white and 4 black balls. A bag and a ball out of it are picked at random. The probability that the ball is white, is

(a) $\frac{2}{7}$ (b) $\frac{7}{9}$ (c) $\frac{4}{15}$ (d) $\frac{7}{15}$

2. There are two bags, one of which contains 3 black and 4 white balls, while the other contains 4 black and 3 white balls. A die is cast. If the face 1 or 3 turns up a ball is taken out from the first bag and if any other face turns up, a ball is taken from the second bag. The probability of choosing a black ball, is

(a) $\frac{7}{15}$ (b) $\frac{8}{15}$ (c) $\frac{10}{21}$ (d) $\frac{11}{21}$

3. There are two groups of subjects, one of which consists of 5 Science subjects and 3 Engineering subjects and the other consists of 3 Science and 5 Engineering subjects. An unbiased die is cast. If number 3 or 5 turns up, a subject from group I is selected, otherwise a subject is selected from group II. The probability that an Engineering subject is selected ultimately, is

(a) $\frac{7}{13}$ (b) $\frac{9}{17}$ (c) $\frac{13}{24}$ (d) $\frac{11}{20}$

4. Urn A contains 6 red and 4 white balls and urn B contains 4 red and 6 white balls. One ball is drawn at random from urn A and placed in urn B. Then a ball is drawn from urn B and placed in urn A. Now, if one ball is drawn from urn A, the probability that it is red, is

(a) $\frac{6}{11}$ (b) $\frac{17}{50}$ (c) $\frac{16}{55}$ (d) $\frac{32}{55}$

5. A box contains N coins, of which m are fair and the rest are biased. The probability of getting head when a fair coin is tossed is $\frac{1}{2}$, while it $\frac{2}{3}$ when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. The probability that the coin drawn is fair, is

(a) $\frac{5m}{m+8N}$ (b) $\frac{3m}{m+8N}$ (c) $\frac{7m}{m+8N}$ (d) $\frac{9m}{m+8N}$

6. A pack of playing cards was found to contain only 51 cards. If the first 13 cards which are examined are all red, then the probability that the missing card is black, is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{15}{26}$ (d) $\frac{16}{39}$

7. A purse contains n coins of unknown values. A coin is drawn from it at random and is found to be a rupee. Then the chance that it is the only rupee coin in the purse, is

(a) $\frac{1}{n}$ (b) $\frac{2}{n+1}$ (c) $\frac{1}{n(n+1)}$ (d) $\frac{2}{n(n+1)}$

8. A card is lost from a pack of 52 playing cards. From the remainder of the pack, one card is drawn and is found to be a spade. The probability that the missing card is a spade, is

(a) $\frac{2}{17}$ (b) $\frac{3}{17}$ (c) $\frac{4}{17}$ (d) $\frac{5}{17}$

9. A person is known to speak the truth 4 times out of 5. He throws a die and reports that it is an ace. The probability that it is actually an ace, is

(a) $\frac{1}{3}$ (b) $\frac{2}{9}$ (c) $\frac{4}{9}$ (d) $\frac{5}{9}$

10. Each of the n urns contains 4 white and 6 black balls, the $(n+1)$ th urn contains 5 white and 5 black balls. Out of $(n+1)$ urns an urn is chosen at random and two balls are drawn from it without replacement. Both the balls are found to be black. If the probability that the $(n+1)$th urn was chosen to drawn the balls is $\frac{1}{16}$, the value of n, is

(a) 10 (b) 11 (c) 12 (d) 13

Session 4

Binomial Theorem on Probability, Poisson Distribution, Expectation, Multinomial Theorem, Uncountable Uniform Spaces (Geometrical Problems)

Binomial Theorem on Probability

Suppose, a binomial experiment has probability of success p and that of failure q (i.e., $p+q=1$). If E be an event and let X = number of successes i.e., number of times event E occurs in n trials. Then, the probability of occurrence of event E exactly r times in n trials is denoted by $P(X=r)$ or $P(r)$ and is given by $P(X=r)$

or $P(r) = {}^nC_r\, p^r q^{n-r}$

$=(r+1)$ th terms in the expansion of $(q+p)^n$

where, $r = 0, 1, 2, 3, \ldots, n$.

Remark

1. The probability of getting atleast k success is
$$P(r \geq k) = \sum_{r=k}^{n} {}^nC_r p^r q^{n-r}.$$
2. The probability of getting atmost k success is
$$P(0 \leq r \leq k) = \sum_{r=0}^{k} {}^nC_r p^r q^{n-r}.$$
3. The probability distribution of the random variable X is as given below

X	0	1	2	...	r	...	n
P(X)	q^n	${}^nC_1\, pq^{n-1}$	${}^nC_2\, p^2 q^{n-2}$	...	${}^nC_r\, p^r q^{n-r}$		p^n

4. The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.
5. **Mode of binomial distribution** Mode of Binomial distribution is the value of r when $P(X=r)$ is maximum.
$$(n+1)\,p - 1 \leq r \leq (n+1)p$$

| Example 21. If on an average, out of 10 ships, one is drowned, then what is the probability that out of 5 ships, atleast 4 reach safely?

Sol. Let p be the probability that a ship reaches safely.

$\therefore \quad p = \dfrac{9}{10}$

$\therefore \quad q$ = Probability that a ship is drowned $= 1 - p = 1 - \dfrac{9}{10}$

$\therefore \quad q = \dfrac{1}{10}$

Let X be the random variable, showing the number of ships reaching safely.

Then, P (atleast 4 reaching safely) $= P(X=4 \text{ or } X=5)$

$= P(X=4) + P(X=5)$

$$= {}^5C_4\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right)^{5-4} + {}^5C_5\left(\frac{9}{10}\right)^5\left(\frac{1}{10}\right)^{5-5}$$

$$= \frac{5\times 9^4}{10^5} + \frac{9^5}{10^5} = \frac{9^4\times 14}{10^5}$$

| Example 22. Numbers are selected at random one at a time, from the numbers 00, 01, 02, ..., 99 with replacement. An event E occurs, if and only if the product of the two digits of a selected number is 18. If four numbers are selected, then find the probability that E occurs atleast 3 times.

Sol. Out of the numbers 00, 01, 02, ..., 99, those numbers the product of whose digits is 18 are 29, 36, 63, 92 i.e., only 4.

$$p = P(E) = \frac{4}{100} = \frac{1}{25},\ q = P(\bar{E}) = 1 - \frac{1}{25} = \frac{24}{25}$$

Let X be the random variable, showing the number of times E occurs in 4 selections.

Then, $P(E$ occurs atleast 3 times$) = P(X=3 \text{ or } X=4)$

$$= P(X=3) + P(X=4) = {}^4C_3\, p^3 q^1 + {}^4C_4\, p^4 q^0$$

$$= 4p^3 q + p^4 = 4\times\left(\frac{1}{25}\right)^3 \times \frac{24}{25} + \left(\frac{1}{25}\right)^4$$

$$= \frac{97}{390625}$$

| Example 23. A man takes a step forward with probability 0.4 and backward with probability 0.6. Then, find the probability that at the end of eleven steps he is one step away from the starting point.

Sol. Since, the man is one step away from starting point mean that either

(i) man has taken 6 steps forward and 5 steps backward.

(ii) man has taken 5 steps forward and 6 steps backward.

Taking, movement 1 step forward as success and 1 step backward as failure.

$\therefore\quad p =$ Probability of success $= 0.4$

and $q =$ Probability of failure $= 0.6$

$\therefore$ Required probability $= P(X = 6 \text{ or } X = 5)$

$$= P(X = 6) + P(X = 5) = {}^{11}C_6\, p^6 q^5 + {}^{11}C_5\, p^5 q^6$$

$$= {}^{11}C_5 (p^6 q^5 + p^5 q^6)$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\{(0 \cdot 4)^6 (0 \cdot 6)^5 + (0 \cdot 4)^5 (0 \cdot 6)^6\}$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(0 \cdot 24)^5 = 0 \cdot 37$$

Hence, the required probability is $0 \cdot 37$.

| Example 24. Find the minimum number of tosses of a pair of dice, so that the probability of getting the sum of the digits on the dice equal to 7 on atleast one toss, is greater than 0.95. (Given, $\log_{10} 2 = 0 \cdot 3010$, $\log_{10} 3 = 0 \cdot 4771$)

Sol. The sample space,

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

$\therefore\quad n(S) = 36$ and let E be the event getting the sum of digits on the dice equal to 7, then

$$E = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$$

$\therefore\quad n(E) = 6$

$p =$ Probability of getting the sum 7

$$p = \frac{6}{36} = \frac{1}{6} \quad \therefore \quad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

$\because$ Probability of not throwing the sum 7 in first m trials $= q^m$

$\therefore P$ (atleast one 7 in m throws) $= 1 - q^m = 1 - \left(\frac{5}{6}\right)^m$

According to the question, $1 - \left(\frac{5}{6}\right)^m > 0 \cdot 95$

$$\Rightarrow \left(\frac{5}{6}\right)^m < 1 - 0 \cdot 95 \Rightarrow \left(\frac{5}{6}\right)^m < 0.05$$

$$\Rightarrow \left(\frac{5}{6}\right)^m < \frac{1}{20}$$

Taking logarithm,

$$\Rightarrow m\{\log_{10} 5 - \log_{10} 6\} < \log_{10} 1 - \log_{10} 20$$

$$\Rightarrow m\{1 - \log_{10} 2 - \log_{10} 2 - \log_{10} 3\} < 0 - \log_{10} 2 - \log_{10} 10$$

$$\Rightarrow m\{1 - 2\log_{10} 2 - \log_{10} 3\} < -\log_{10} 2 - 1$$

$$\Rightarrow m\{1 - 0.6020 - 0.4771\} < -0.3010 - 1$$

$$\Rightarrow -0.079\, m < -1.3010$$

$$\Rightarrow m > \frac{1.3010}{0.079} = 16.44$$

$$\therefore \quad m > 16.44$$

Hence, the least number of trials is 17.

| Example 25. Write probability distribution, when three coins are tossed.

Sol. Let X be a random variable denoting the number of heads occurred, then $P(X = 0) =$ Probability of occurrence of zero head

$$= P(TTT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$P(X = 1) =$ Probability of occurrence of one head

$$= P(HTT) + P(THT) + P(TTH)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$$

$P(X = 2) =$ Probability of occurrence of two heads

$$= P(HHT) + P(HTH) + P(THH)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$$

$P(X = 3) =$ Probability of occurrence of three heads

$$= P(HHH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Thus, the probability distribution when three coins are tossed is as given below

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

another form,

$$\begin{pmatrix} X: & 0 & 1 & 2 & 3 \\ P(X): & \frac{1}{3} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}$$

| Example 26. The mean and variance of a binomial variable X are 2 and 1, respectively. Find the probability that X takes values greater than 1.

Sol. Given, mean, $np = 2$ ...(i)

and variance, $npq = 1$ ...(ii)

On dividing Eq. (ii) by Eq. (i), we get $q = \frac{1}{2}$

$$\therefore \quad p = 1 - q = \frac{1}{2}$$

From Eq. (i), $n \times \frac{1}{2} = 2 \;\therefore\; n = 4$

The binomial distribution is $\left(\frac{1}{2} + \frac{1}{2}\right)^4$

Now, $P(X > 1) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_4\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4$$

$$= \frac{6 + 4 + 1}{16} = \frac{11}{16}$$

Aliter $P(X > 1) = 1 - \{P(X = 0) + P(X = 1)\}$

$$= 1 - \left\{{}^4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3\right\} = 1 - \left(\frac{1 + 4}{16}\right) = \frac{11}{16}$$

Poisson Distribution

It is the limiting case of binomial distribution under the following conditions :

(i) Number of trails are very large i.e. $n \to \infty$

(ii) $p \to 0$

(iii) $nq \to \lambda$, a finite quantity (λ is called parameter)

(a) Probability of r success for poisson distribution is given by $P(X = r) = \dfrac{e^{-\lambda}\lambda^r}{r!}, r = 0, 1, 2, \ldots$

(b) Recurrence formula for poisson distribution is given by $P(r+1) = \dfrac{\lambda}{(r+1)} P(r)$

Remark

1. For poisson distribution, mean = variance = $\lambda = np$
2. If X and Y are independent poisson variates with parameters λ_1 and λ_2, then $X + Y$ has poisson distribution with parameter $\lambda_1 + \lambda_2$.

Expectation

If p be the probability of success of a person in any venture and m be the sum of money which he will receive in case of success, the sum of money denoted by pm is called his expectation.

| Example 27. A random variable X has Poisson's distribution with mean 3. Then find the value of $P(X > 2.5)$

Sol. $P(X > 2.5) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$

$$\because \quad P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$\therefore \quad P(X > 2.5) = 1 - \frac{e^{-\lambda}}{0!} - \frac{e^{-\lambda} \cdot \lambda^1}{1!} - \frac{e^{-\lambda} \cdot \lambda^2}{2!}$$

$$= 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

$$= 1 - e^{-3}\left(1 + 3 + \frac{9}{2}\right) \quad (\because \lambda = np = 3)$$

$$= 1 - \frac{17}{2e^3}$$

| Example 28. A and B throw with one die for a stake of ₹ 11 which is to be won by the player who first throw 6. If A has the first throw, then what are their respective expectations?

Sol. Since, A can win the game at the 1st, 3rd, 5th,..., trials.

If p be the probability of success and q be the probability of fail, then

$$p = \frac{1}{6} \text{ and } q = \frac{5}{6}$$

$$P(A \text{ wins at the first trial}) = \frac{1}{6}$$

$$P(A \text{ wins at the 3rd trials}) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$$

$$P(A \text{ wins at the 5th trials}) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \text{ and so on.}$$

$$\text{Therefore, } P(A \text{ wins}) = \frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \ldots \infty$$

$$= \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{6}{11}$$

$$\text{Similarly, } P(B \text{ wins}) = \frac{5}{6} \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 \frac{1}{6} + \left(\frac{5}{6}\right)^5 \frac{1}{6} + \ldots \infty$$

$$= \frac{\frac{5}{6} \cdot \frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{5}{11}$$

Hence, expectations of A and B are ₹ $\frac{6}{11} \times 11$ and ₹ $\frac{5}{11} \times 11$, respectively. i.e. Expectations of A and B are ₹ 6 and ₹ 5, respectively.

Multinomial Theorem

If a dice has m faces marked 1, 2, 3,..., m and if such n dice are thrown, then the probability that the sum of the numbers of the upper faces is equal to r is given by the coefficient of x^r in $\dfrac{(x + x^2 + \ldots + x^m)^n}{m^n}$.

| Example 29. A person throws two dice, one the common cube and the other a regular tetrahedron, the number on the lowest face being taken in the case of the tetrahedron, then find the probability that the sum of the numbers appearing on the dice is 6.

Sol. Let S be the sample space, then

$$S = \{1, 2, 3, 4\} \times \{1, 2, 3, 4, 5, 6\}$$

$$\therefore \quad n(S) = 24$$

If E be the event that the sum of the numbers on dice is 6.

Then, $n(E)$ = Coefficient of x^6 in

$$(x^1 + x^2 + x^3 + x^4) \times (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$$

$$= 1 + 1 + 1 + 1 = 4$$

$$\therefore \text{Required probability, } P(E) = \frac{n(E)}{n(S)} = \frac{4}{24} = \frac{1}{6}$$

| Example 30. Five ordinary dice are rolled at random and the sum of the numbers shown on them is 16. What is the probability that the numbers shown on each is any one from 2, 3, 4 or 5?

Sol. If the integers x_1, x_2, x_3, x_4 and x_5 are shown on the dice, then $x_1 + x_2 + x_3 + x_4 + x_5 = 16$

where, $1 \le x_i \le 6$ $(i = 1, 2, 3, 4, 5)$

The number of total solutions of this equation.

$= $ Coefficient of x^{16} in $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^5$

$= $ Coefficient of x^{16} in $x^5(1 + x + x^2 + x^3 + x^4 + x^5)^5$

$= $ Coefficient of x^{11} in $(1 + x + x^2 + x^3 + x^4 + x^5)^5$

$= $ Coefficient of x^{11} in $\left\{\left(\frac{1-x^6}{1-x}\right)^5\right\}$

$= $ Coefficient of x^{11} in $(1 - x^6)^5(1 - x)^{-5}$

$= $ Coefficient of x^{11} in

$(1 - 5x^6 + \dots)(1 + {}^5C_1x + {}^6C_2x^2 + \dots + {}^9C_5x^5 + \dots + {}^{15}C_{11}x^{11} + \dots)$

$= {}^{15}C_{11} - 5 \cdot {}^9C_5$

$= {}^{15}C_4 - 5 \cdot {}^9C_4 = \frac{15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4} - 5 \cdot \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 735$

If S be the sample space

$\therefore \quad n(S) = 735$

Let E be the occurrence event, then

$n(E) = $ The number of integral solutions of

$x_1 + x_2 + x_3 + x_4 + x_5 = 16,$

where $2 \le x_i \le 5$ $(i = 1, 2, 3, 4, 5)$

$= $ Coefficient of x^{16} in $(x^2 + x^3 + x^4 + x^5)^5$

$= $ Coefficient of x^{16} in $x^{10}(1 + x + x^2 + x^3)^5$

$= $ Coefficient of x^6 in $(1 + x + x^2 + x^3)^5$

$= $ Coefficient of x^6 in $\left\{\left(\frac{1-x^4}{1-x}\right)^5\right\}$

$= $ Coefficient of x^6 in $(1 - x^4)^5(1 - x)^{-5}$

$= $ Coefficient of x^6 in

$(1 - 5x^4 + \dots)(1 + {}^5C_1x + {}^6C_2x^2 + \dots + {}^{10}C_6x^6 + \dots)$

$= {}^{10}C_6 - 5 \cdot {}^6C_2 = {}^{10}C_4 - 5 \cdot {}^6C_2$

$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} - 5 \cdot \frac{6 \cdot 5}{1 \cdot 2} = 210 - 75 = 135$

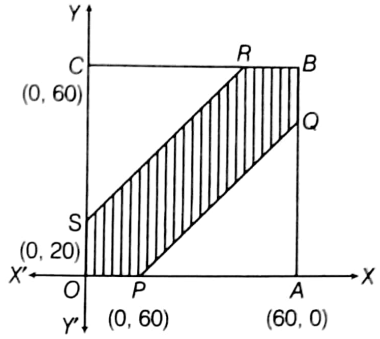
$\therefore$ The required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{135}{735} = \frac{9}{49}$

Uncountable Uniform Spaces

(Geometrical Problems)

| Example 31. Two persons A and B agree to meet at a place between 11 to 12 noon. The first one to arrive waits for 20 min and then leave. If the time of their arrival be independent and at random, then what is the probability that A and B meet?

Sol. Let A and B arrive at the place of their meeting x minutes and y minutes after 11 noon.



The given condition $\Rightarrow$ their meeting is possible only if

$|x - y| \le 20$ ...(i)

$OABC$ is a square, where $A \equiv (60, 0)$ and $C \equiv (0, 60)$

Considering the equality part of Eq. (i)

i.e., $|x - y| = 20$

$\therefore$ The area representing the favourable cases

$= $ Area $OPQBRSO$

$= $ Area of square $OABC$ $-$ Area of ΔPAQ $-$ Area of ΔSRC

$= (60)(60) - \frac{1}{2}(40)(40) - \frac{1}{2}(40)(40)$

$= 3600 - 1600 = 2000$ sq units

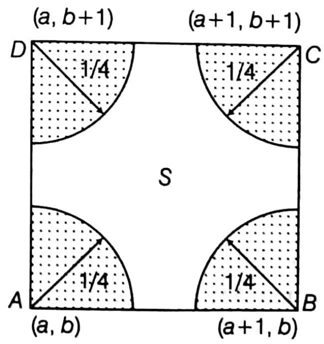
Total way $=$ Area of square $OABC = (60)(60) = 3600$ sq units

Required probability $= \frac{2000}{3600} = \frac{5}{9}$

| Example 32. Consider the cartesian plane R^2 and let X denote the subset of points for which both coordinates are integers. A coin of diameter $\frac{1}{2}$ is tossed randomly onto the plane. Find the probability p that the coin covers a point of X.

Sol. Let S denote the set of points inside a square with corners

$(a, b), (a, b + 1), (a + 1, b), (a + 1, b + 1) \in X$

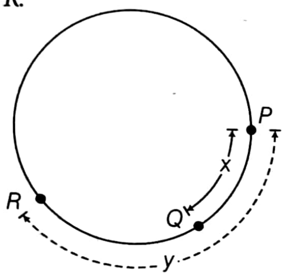


Let P denotes the set of points in S with distance less than $\frac{1}{4}$ from any corner point. (observe that the area of P is equal to the area inside a circle of radius $\frac{1}{4}$). Thus a coin, whose centre falls in S, will cover a point of X if and only if its centre falls in a point of P.

$$\text{Hence, } p = \frac{\text{area of } P}{\text{area of } S} = \frac{\pi\left(\frac{1}{4}\right)^2}{1} = \frac{\pi}{15} \approx 0.2$$

| Example 33. Three points P, Q and R are selected at random from the circumference of a circle. Find the probability p that the points lie on a semi-circle.

Sol. Let the length of the circumference is $2s$. Let x denote the clockwise arc length of PQ and let y denote the clockwise arc length of PR.



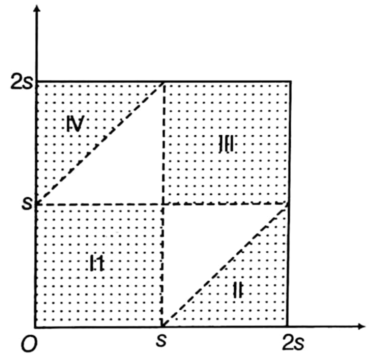
Thus, $0 < x < 2s$ and $0 < y < 2s$

Let A denotes the subset of S for which any of the following conditions holds:

(i) $x, y < s$ (ii) $x < s$ and $y - x > s$

(iii) $x, y > s$ (iv) $y < s$ and $x - y > s$

Then, A consists of those points for which P, Q and R lie on a semi-circle. Thus,



$$p = \frac{\text{area of } A}{\text{area of } S} = \frac{3s^2}{4s^2} = \frac{3}{4}$$

| Example 34. A wire of length l is cut into three pieces. Find the probability that the three pieces form a triangle.

Sol. Let the lengths of three parts of the wire be x, y and $l - (x + y)$. Then, $x > 0$, $y > 0$

and $\quad l - (x + y) > 0$

i.e., $\quad x + y < l$ or $y < l - x$

Since, in a triangle, the sum of any two sides is greater than third side, so

$$x + y > l - (x + y) \Rightarrow y > \frac{l}{2} - x$$

and $$x + l - (x + y) > y \Rightarrow y < \frac{l}{2}$$

and $$y + l - (x + y) > x \Rightarrow x < \frac{l}{2}$$

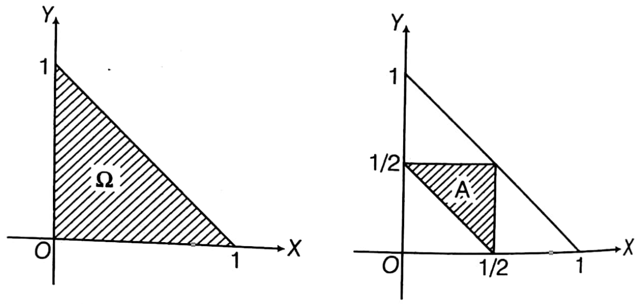
$$\Rightarrow \quad \frac{l}{2} - x < y < \frac{l}{2} \text{ and } 0 < x < \frac{l}{2}$$

So, required probability $= \dfrac{\int_0^{l/2}\int_{l/2-x}^{l/2} dy\,dx}{\int_0^l\int_0^{l-x} dy\,dx}$

$$= \frac{\int_0^{l/2}\left\{\frac{l}{2} - \left(\frac{l}{2} - x\right)\right\}dx}{\int_0^l (l - x)dx} = \frac{\int_0^{l/2} x\,dx}{\int_0^l (l - x)\,dx} = \frac{l^2/8}{l^2/2} = \frac{1}{4}$$

Aliter

The elementary event w is characterised by two parameters x and y [since $z = l - (x + y)$]. We depict the event by a point on x, y plane. The conditions $x > 0, y > 0, x + y < l$ are imposed on the quantities x and y, the sample space is the interior of a right angled triangle with unit legs i.e. $S_\Omega = \frac{1}{2}$.



The condition A requiring that a triangle could be formed from the segments $x, y, l - (x + y)$ reduces to the following two conditions: (1) The sum of any two sides is larger than the third side, (2) The difference between any two sides is smaller than the third side. This condition is associated with the triangular domain A with area.

$$S_A = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8} \therefore P(A) = \frac{S_A}{S_\Omega} = \frac{\left(\frac{1}{8}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{4}$$

Exercise for Session 4

1 A coin is tossed three times. The probability of getting exactly 2 heads is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{3}{8}$ (d) $\frac{5}{8}$

2 A coin is tossed 4 times. The probability that atleast one head turns up is

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{7}{8}$ (d) $\frac{15}{16}$

3 The following is the probability distribution of a random variable X.

X	1	2	3	4	5
$P(X)$	0.1	0.2	k	0.3	$2k$

The value of k is

(a) $\frac{4}{15}$ (b) $\frac{1}{15}$ (c) $\frac{1}{5}$ (d) $\frac{2}{15}$

4 A random variable X has the distribution

X	2	3	4
$P(X = x)$	0.3	0.4	0.3

Then, variance of the distribution, is

(a) 0.6 (b) 0.7 (c) 0.77 (d) 1.55

5. In a box containing 100 bulbs, 10 bulbs are defective. Probability that out of a sample of 5 bulbs, none is defective, is

(a) 10^{-5} (b) 2^{-5} (c) $(0.9)^5$ (d) 0.9

6. A pair of dice is rolled together till a sum of either 5 or 7 is obtained. The probability that 5 comes before 7, is

(a) $\frac{2}{5}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) None of these

7. If X follows the binomial distribution with parameters $n = 6$ and p and $9P(X = 4) = P(X = 2)$, then p is

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

8. If probability of a defective bolt is 0.1, then mean and standard deviation of distribution of bolts in a total of 400, are

(a) 30, 3 (b) 40, 5 (c) 30, 4 (d) 40, 6

9. The mean and variance of a binomial distribution are $\frac{5}{4}$ and $\frac{15}{16}$ respectively, then value of p, is

(a) $\frac{1}{2}$ (b) $\frac{15}{16}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

10. The mean and variance of a binomial distribution are 6 and 4, then n is

(a) 9 (b) 12 (c) 18 (d) 10

11. A die is thrown 100 times. Getting an even number is considered a success. Variance of number of successes, is

(a) 10 (b) 20 (c) 25 (d) 50

12. 10% of tools produced by a certain manufacturing process turn out to be defective. Assuming binomial distribution, the probability of 2 defective in sample of 10 tools chosen at random, is

(a) 0.368 (b) 0.194 (c) 0.271 (d) None of these

13. If X follows a binomial distribution with parameters $n = 100$ and $p = \frac{1}{3}$, then $P(X = r)$ is maximum, when r equals

(a) 16 (b) 32 (c) 33 (d) None of these

14. The expected value of the number of points, obtained in a single throw of die, is

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{9}{2}$

15. Two points P and Q are taken at random on a line segment OA of length a. The probability that $PQ > b$, where $0 < b < a$, is

(a) $\frac{b}{a}$ (b) $\frac{b^2}{a^2}$ (c) $\left(\frac{a-b}{a}\right)^2$ (d) $\left(\frac{a-2b}{a-b}\right)^2$

Shortcuts and Important Results to Remember

1 If n letters corresponding to n envelopes are placed in the envelopes at random, then

(i) probability that all letters are in right envelopes $= \frac{1}{n!}$.

(ii) probability that all letters are not in right envelopes $= 1 - \frac{1}{n!}$.

(iii) probability that no letter is in right envelopes $= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}$.

(iv) probability that exactly r letters are in right envelopes $= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right]$.

2 When two dice are thrown, the probability of getting a total r (sum of numbers on upper faces), is

(i) $\frac{(r-1)}{36}$, if $2 \le r \le 7$ (ii) $\frac{(13-r)}{36}$, if $8 \le r \le 12$

3 When three dice are thrown, the probability of getting a total r (sum of numbers on upper faces), is

(i) $\frac{{}^{r-1}C_2}{216}$, if $3 \le r \le 8$ (ii) $\frac{25}{216}$, if $r = 9$

(iii) $\frac{27}{216}$, if $r = 10, 11$ (iv) $\frac{25}{216}$, if $r = 12$

(v) $\frac{{}^{(20-r)}C_2}{216}$, if $13 \le r \le 18$

4 If A and B are two finite sets (Let $n(A) = n$ and $n(B) = m$) and if a mapping is selected at random from the set of all mappings from A to B, the probability that the mapping is

(i) a one-one function is $\frac{{}^mP_n}{m^n}$.

(ii) a one-one onto function is $\frac{n!}{m^n}$.

(iii) a many one function is $1 - \frac{{}^mP_n}{m^n}$.

5 If r squares are selected from a chess board of size 8×8, then the probability that they lie on a diagonal line, is $\frac{4({}^7C_r + {}^6C_r + {}^5C_r + \ldots + {}^rC_r) + 2({}^8C_r)}{{}^{64}C_r}$ for $1 \le r \le 7$.

6 If n objects are distributed among n persons, then the probability that atleast one of them will not get anything, is $\frac{n^n - n!}{n^n}$.

7 Points about coin, dice and playing cards:

(a) **Coin** If 'one' coin is tossed n times 'n' coins are tossed once, then number of simple events (or simple points) in the space of the experiment is 2^n. All these events are equally likely.

(b) **Dice** If 'one' die is thrown 'n' times or 'n' dice are thrown once, then number of simple events (or simple points) in the space of the experiment is 6^n (here dice is cubical). All events are equally likely.

(c) **Playing Cards** A pack of playing cards has 52 cards. There are four suits Spade (♠ black face), Heart (♥ red face), Diamond (♦ red face) and Club (♣ black face) each having 13 cards. In 13 cards of each suit, there are 3 face (or court) cards namely King, Queen and Jack (or knave), so there are in all 12 face cards 4 King, 4 Queen and 4 Jacks (or knaves). 4 of each suit namely Ace (or Ekka), King, Queen and Jack (or knave).

(i) **Game of bridge** It is played by 4 players, each player is given 13 cards.

(ii) **Game of whist** It is played by two pairs of persons.

(iii) If two cards (one after the other) can be drawn out of a well-shuffled pack of 52 cards, then number of ways; ($\times$) With replacement is $52 \times 52 = (52)^2 = 2704$ (β) Without replacement is $52 \times 51 = 2652$.

(iv) Two cards (simultaneously) can be drawn out of a well-shuffled pack of 52 cards, then number of ways is ${}^{52}C_2 = \frac{52 \times 51}{2} = 1326$.

8 Out of $(2n+1)$ tickets consecutively numbered, three are drawn at random, then the probability that the numbers on them are in AP, is $\frac{3n}{4n^2 - 1}$.

9 Out of $3n$ consecutive integers, three are selected at random, then the probability that their sum is divided by 3, is $\frac{(3n^2 - 3n + 2)}{(3n-1)(3n-2)}$.

10 Two numbers a and b are chosen at random from the set $\{1, 2, 3, \ldots, 5n\}$, the probability that $a^4 - b^4$ is divisible by 5, is $\frac{17n - 5}{5(5n-1)}$.

11 Two numbers a and b are chosen at random from the set $\{1, 2, 3, \ldots, 3n\}$ the probability that $a^2 - b^2$ is divisible by 3, is $\frac{(5n-3)}{3(3n-1)}$.

12 Two numbers a and b are chosen at random from the set $\{1, 2, 3, \ldots, 3n\}$, the probability that $a^3 + b^3$ is divisible by 3, is $\frac{1}{3}$.

13 There are n stations between two cities A and B. A train is to stop at three of these n stations. The probability that no two of these three stations are consecutive, is $\frac{(n-3)(n-4)}{n(n-1)}$.

JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1 *The probability that in a year of 22nd century chosen at random, there will be 53 Sundays, is*

(a) $\frac{3}{28}$ (b) $\frac{2}{28}$ (c) $\frac{7}{28}$ (d) $\frac{5}{28}$

Sol. (d) In the 22nd century, there are 25 leap years viz. 2100, 2104, 2108,..., 2196 and 75 non-leap years.

Consider the following events:

E_1 = Selecting a leap year from 22nd century

E_2 = Selecting a non-leap year from 22nd century

E = There are 53 Sundays in a year of 22nd century

We have,

$$P(E_1) = \frac{25}{100},\ P(E_2) = \frac{75}{100},\ P\left(\frac{E}{E_1}\right) = \frac{2}{7} \text{ and } P\left(\frac{E}{E_2}\right) = \frac{1}{7}$$

$$\text{Required probability } = P(E) = P((E \cap E_1) \cup (E \cap E_2))$$
$$= P(E \cap E_1) + P(E \cap E_2)$$
$$= P(E_1).P\left(\frac{E}{E_1}\right) + P(E_2).P\left(\frac{E}{E_2}\right)$$
$$= \frac{25}{100} \times \frac{2}{7} + \frac{75}{100} \times \frac{1}{7} = \frac{5}{28}$$

• Ex. 2 *In a convex hexagon two diagonals are drawn at random. The probability that the diagonals intersect at an interior point of the hexagon, is*

(a) $\frac{5}{12}$ (b) $\frac{7}{12}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

Sol. (a) We have,

Number of diagonals of a hexagon $= {}^6C_2 - 6 = 9$

$\therefore$ $n(s)$ = Total number of selections of two diagonals $= {}^9C_2 = 36$

and $n(E)$ = The number of selections of two diagonals which intersect at an interior point

= The number of selections of four vertices $= {}^6C_4 = 15$

$$\text{Hence, required probability} = \frac{n(E)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

• Ex. 3 *If three integers are chosen at random from the set of first 20 natural numbers, the chance that their product is a multiple of 3, is*

(a) $\frac{1}{57}$ (b) $\frac{13}{19}$

(c) $\frac{2}{19}$ (d) $\frac{194}{285}$

Sol. (d) $n(S)$ = Total number of ways of selecting 3 integers from 20 natural numbers $= {}^{20}C_3 = 1140.$

Their product is multiple of 3 means atleast one number is divisible by 3. The number which are divisible by 3 are 3, 6, 9, 12, 15 and 16.

$\therefore$ $n(E)$ = The number of ways of selecting atleast one of them multiple of 3

$$= {}^6C_1 \times {}^{14}C_2 + {}^6C_2 \times {}^{14}C_1 \times {}^6C_3 = 776$$

$$\therefore \text{ Required probability} = \frac{n(E)}{n(S)}$$
$$= \frac{776}{1140} = \frac{194}{285}$$

• Ex. 4 *If three numbers are selected from the set of the first 20 natural numbers, the probability that they are in GP, is*

(a) $\frac{1}{285}$ (b) $\frac{4}{285}$

(c) $\frac{11}{1140}$ (d) $\frac{1}{71}$

Sol. (c) $n(S)$ = Total number of ways of selecting 3 numbers from first 20 natural numbers $= {}^{20}C_3 = 1140$

Three numbers are in GP, the favourable cases are 1, 2, 4; 1, 3, 9; 1, 4, 16; 2, 4, 8; 2, 6, 18; 3, 6, 12; 4, 8, 16; 5, 10, 20; 4, 6, 9; 8, 12, 18; 9, 12, 16

$\therefore$ $n(E)$ = The number of favourable cases = 11

$$\therefore \text{ Required probability} = \frac{n(E)}{n(S)} = \frac{11}{1140}$$

• Ex. 5 *Two numbers b and c are chosen at random with replacement from the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9. The probability that $x^2 + bx + c > 0$ for all $x \in R$, is*

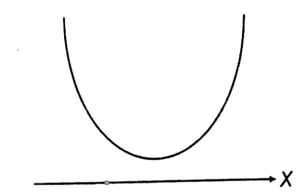
(a) $\frac{17}{123}$ (b) $\frac{32}{81}$

(c) $\frac{82}{125}$ (d) $\frac{45}{143}$

Sol. (b) Here, $x^2 + bx + c > 0, \forall\ x \in R$

$\therefore$ $D < 0$

$\Rightarrow$ $b^2 < 4c$



Value of b	Possible values of c		
1	$1 < 4c$	$\Rightarrow c > \frac{1}{4} \Rightarrow$	{1, 2, 3, 4, 5, 6, 7, 8, 9}
2	$4 < 4c$	$\Rightarrow c > 1 \Rightarrow$	{2, 3, 4, 5, 6, 7, 8, 9}
3	$9 < 4c$	$\Rightarrow c > \frac{9}{4} \Rightarrow$	{3, 4, 5, 6, 7, 8, 9}
4	$16 < 4c$	$\Rightarrow c > 4 \Rightarrow$	{5, 6, 7, 8, 9}
5	$25 < 4c$	$\Rightarrow c > 6.25 \Rightarrow$	{7, 8, 9}
6	$36 < 4c$	$\Rightarrow c > 9 \Rightarrow$	Impossible
7	Impossible		
8	Impossible		
9	Impossible		

$n(E)$ = Number of favourable cases $= 9 + 8 + 7 + 5 + 3 = 32$

$n(S)$ = Total ways $= 9 \times 9 = 81$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{32}{81}$

• **Ex. 6** *Three dice are thrown. The probability of getting a sum which is a perfect square, is*

(a) $\frac{2}{5}$ (b) $\frac{9}{20}$

(c) $\frac{1}{4}$ (d) None of these

Sol. (d) $n(S)$ = Total number of ways $= 6 \times 6 \times 6 = 216$

The sum of the numbers on three dice varies from 3 to 18 and among these 4, 9 and 16 are perfect squares.

$\therefore n(E)$ = Number of favourable ways

= Coefficient of x^4 in $(x + x^2 + \ldots + x^6)^3$ + Coefficient of x^9 in $(x + x^2 + \ldots + x^6)^3$ + Coefficient of x^{16} in $(x + x^2 + \ldots + x^6)^3$

= Coefficient of x in $(1 + x + \ldots + x^5)^3$ + Coefficient of x^6 in $(1 + x + x^2 + \ldots + x^5)^3$ + Coefficient of x^{13} in $(1 + x + x^2 + \ldots + x^5)^3$

= Coefficient of x in $(1 - x^6)^3(1 - x)^{-3}$ + Coefficient of x^6 in $(1 - x^6)^3(1 - x)^{-3}$ + Coefficient of x^{13} in $(1 - x^6)^3(1 - x)^{-3}$

= Coefficient of x in $(1)(1 + {}^3C_1x + \ldots)$ + Coefficient of x^6 in $(1 - 3x^6)(1 + {}^3C_1x + \ldots)$ + Coefficient of x^{13} in $(1 - 3x^6 + 3x^{12} + \ldots); (1 + {}^3C_1x + \ldots)$

$= {}^3C_1 + ({}^8C_6 - 3) + ({}^{15}C_{13} - 3 \times {}^9C_7 + 9)$

$= {}^3C_1 + ({}^8C_2 - 3) + ({}^{15}C_2 - 3 \times {}^9C_2 + 9)$

$= 3 + 25 + 6$

$= 34$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{34}{216} = \frac{17}{108}$

• **Ex. 7** *A quadratic equation is chosen from the set of all quadratic equations which are unchanged by squaring their roots. The chance that the chosen equation has equal roots, is*

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) None of these

Sol. (a) Let α and β be the roots of the quadratic equation.

According to question,

$\alpha + \beta = \alpha^2 + \beta^2$ and $\alpha\beta = \alpha^2\beta^2 \Rightarrow \alpha\beta(\alpha\beta - 1) = 0$

$\Rightarrow \alpha\beta = 1$ or $\alpha\beta = 0$

$\Rightarrow \alpha = 1, \beta = 1; \alpha = \omega, \beta = \omega^2$ [cube roots and unity]

$\alpha = 1, \beta = 0; \alpha = 0, \beta = 0$

$\therefore n(S)$ = Number of quadratic equations which are unchanged by squaring their roots = 4

and $n(E)$ = Number of quadratic equations have equal roots $= 2$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

• **Ex. 8** *Three-digit numbers are formed using the digits 0, 1, 2, 3, 4, 5 without repetition of digits. If a number is chosen at random, then the probability that the digits either increase or decrease, is*

(a) $\frac{1}{10}$ (b) $\frac{2}{11}$ (c) $\frac{3}{10}$ (d) $\frac{4}{11}$

Sol. (c) $n(S)$ = Total number of three digit numbers

$= {}^6P_3 - {}^5P_2 = 120 - 20 = 100$

$n(E)$ = Number of numbers with digits either increase or decrease

= Number of numbers with increasing digits + Number of numbers with decreasing digits

$= {}^5C_3 + {}^6C_3 = 10 + 20 = 30$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{30}{100} = \frac{3}{10}$

• **Ex. 9** *If X follows a binomial distribution with parameters $n = 8$ and $p = \frac{1}{2}$, then $p(|x - 4| \le 2)$ is equal to*

(a) $\frac{121}{128}$ (b) $\frac{119}{128}$ (c) $\frac{117}{128}$ (d) $\frac{115}{128}$

Sol. (b) Here, $p = \frac{1}{2}, n = 8$

$\therefore \quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore$ The binomial distribution is $\left(\frac{1}{2} + \frac{1}{2}\right)^8$

Also, $|x - 4| \le 2$

$\Rightarrow \quad -2 \le x - 4 \le 2 \Rightarrow 2 \le x \le 6$

$\therefore p(|x-4| \le 2) = p(x=2) + p(x=3) + p(x=4)$
$+ p(x=5) + p(x=6)$

$$= {}^8C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6 + {}^8C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5 + {}^8C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4 + {}^8C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3 + {}^8C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2$$

$$= \frac{{}^8C_2 + {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6}{2^8}$$

$$= \frac{238}{256} = \frac{119}{128}$$

• ***Ex. 10*** *A doctor is called to see a sick child. The doctor knows (prior to the visit) that 90% of the sick children in that neighbourhood are sick with the flue, denoted by F, while 10% are sick with the measles, denoted by M. A well-known symptom of measles is a rash, denoted by R. The probability of having a rash for a child sick with the measles is 0.95. However, occasionally children with the flue also develop a rash with conditional probability 0.08. Upon examination the child, the doctor finds a rash, then the probability that the child has the measles, is*

(a) $\frac{89}{167}$ (b) $\frac{91}{167}$ (c) $\frac{93}{167}$ (d) $\frac{95}{167}$

Sol. (d) $\because P(F) = 0\cdot90,\ P(M) = 0\cdot10,$

$$P\left(\frac{R}{F}\right) = 0\cdot08,\ P\left(\frac{R}{M}\right) = 0\cdot95$$

$$\therefore P\left(\frac{M}{R}\right) = \frac{P(M)\cdot P\left(\frac{R}{M}\right)}{P(M)\cdot P\left(\frac{R}{M}\right) + P(F)\cdot P\left(\frac{R}{F}\right)}$$

$$= \frac{0\cdot10\times0\cdot95}{0\cdot10\times0\cdot95 + 0\cdot90\times0\cdot08} = \frac{0\cdot095}{0\cdot167} = \frac{95}{167}$$

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

• ***Ex. 11*** *Let p_n denote the probability of getting n heads, when a fair coin is tossed m times. If p_4, p_5, p_6 are in AP, then values of m can be*

(a) 5 (b) 7 (c) 10 (d) 14

Sol. (b, d) $\because p_4 = {}^mC_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{m-4} = {}^mC_4\left(\frac{1}{2}\right)^m$

$$p_5 = {}^mC_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{m-5} = {}^mC_5\left(\frac{1}{2}\right)^m$$

and $p_6 = {}^mC_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{m-6} = {}^mC_6\left(\frac{1}{2}\right)^m$

According to the question, p_4, p_5, p_6 are in AP

$\therefore \quad 2p_5 = p_4 + p_6$

$\Rightarrow \quad 2\times{}^mC_5\left(\frac{1}{2}\right)^m = {}^mC_4\left(\frac{1}{2}\right)^m + {}^mC_6\left(\frac{1}{2}\right)^m$

or $\quad 2\times{}^mC_5 = {}^mC_4 + {}^mC_6$

or $\quad 2 = \frac{{}^mC_4}{{}^mC_5} + \frac{{}^mC_6}{{}^mC_5} \Rightarrow 2 = \frac{5}{m-5+1} + \frac{m-6+1}{6}$

$\Rightarrow \quad 2 = \frac{5}{m-4} + \frac{m-5}{6} \Rightarrow (m^2-2)\,m + 98 = 0$

$\Rightarrow \quad (m-14)(m-7) = 0$

$\Rightarrow \quad m = 7$ or 14

• ***Ex. 12*** *A random variable X follows binomial distribution with mean a and variance b. Then,*

(a) $a > b > 0$ (b) $\frac{a}{b} > 1$

(c) $\frac{a^2}{a-b}$ is an integer (d) $\frac{a^2}{a+b}$ is an integer

Sol. (a, b, c) Suppose, $X \sim B(n, p)$ i.e. $(q+p)^n$

Here, $np = a$ and $npq = b$

$\therefore \quad q = \frac{b}{a}$, then $p = 1 - q = 1 - \frac{b}{a}$

Now, $0 < q < 1 \Rightarrow 0 < \frac{b}{a} < 1 \Rightarrow a > b > 0$ [alternate (a)]

and $\frac{a}{b} > 1$ [alternate (b)]

Also, $\frac{a^2}{a-b} = \frac{(np)^2}{np - npq} = \frac{np}{1-q} = \frac{np}{p} = n =$ Integer [alternate (c)]

• ***Ex. 13*** *If $A_1, A_2, \ldots, A_n$ are n independent events, such that $P(A_i) = \frac{1}{i+1}$, $i = 1, 2, \ldots, n$, then the probability that none of $A_1, A_2, A_3, \ldots, A_n$ occur, is*

(a) $\frac{n}{n+1}$ (b) $\frac{1}{n+1}$

(c) less than $\frac{1}{n}$ (d) greater than $\frac{1}{n+2}$

Sol. (b, c, d) $\because A_1, A_2, A_3, \ldots, A_n$ are n independent, then

Required probability $= P(A_1' \cap A_2' \cap A_3' \cap \ldots \cap A_n')$

$= P(A_1').P(A_2').P(A_3') \ldots P(A_n')$

$= (1 - P(A_1))(1 - P(A_2))(1 - P(A_3)) \ldots (1 - P(A_n))$

$= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \ldots \left(1 - \frac{1}{n+1}\right)$

$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \ldots \times \frac{n}{n+1} = \frac{1}{n+1}$

$\because n + 2 > n + 1 > n; \ \therefore \ \frac{1}{n+2} < \frac{1}{n+1} < \frac{1}{n}$

• **Ex. 14** *A and B are two events, such that* $P(A \cup B) \geq \frac{3}{4}$ *and* $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$, *then*

(a) $P(A) + P(B) \leq \frac{11}{8}$ (b) $P(A) \cdot P(B) \leq \frac{3}{8}$

(c) $P(A) + P(B) \geq \frac{7}{8}$ (d) None of these

Sol. (a, c) $\because \ \frac{3}{4} \leq P(A \cup B) \leq 1$

$\Rightarrow \ \frac{3}{4} \leq P(A) + P(B) - P(A \cap B) \leq 1$

As the minimum value of $P(A \cap B) = \frac{1}{8}$, we get

$P(A) + P(B) - \frac{1}{8} \geq \frac{3}{4} \Rightarrow P(A) + P(B) \geq \frac{7}{8}$ [alternate (c)]

As the maximum value of $P(A \cap B) = \frac{3}{8}$, we get

$P(A) + P(B) - \frac{3}{8} \leq 1 \Rightarrow P(A) + P(B) \leq \frac{11}{8}$ [alternate (a)]

• **Ex. 15** *A, B, C and D cut a pack of 52 cards successively in the order given. If the person who cuts a spade first receives ₹ 350, then the expectations of*

(a) B is ₹ 96 (b) D is ₹ 54

(c) $(A + C)$ is ₹ 200 (d) $(B - D)$ is ₹ 56

Sol. (a, b, c) Let E be the event of any one cutting a spade in one cut and let S be the sample space, then

$n(E) = {}^{13}C_1 = 13$ and $n(S) = {}^{52}C_1 = 52$

$\therefore \quad p = P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$ and $q = p(\bar{E}) = 1 - p = \frac{3}{4}$

The probability of A winning (when A starts the game)

$= p + q^4 p + q^8 p + \ldots \infty = \frac{p}{1 - q^4} = \frac{\frac{1}{4}}{1 - \left(\frac{3}{4}\right)^4} = \frac{64}{175}$

$\therefore \quad E(A) = ₹\, 350 \times \frac{64}{175} = ₹\, 128$

$E(B) = ₹128 \times q = ₹128 \times \frac{3}{4} = ₹\, 96$

$E(C) = ₹96 \times q = ₹96 \times \frac{3}{4} = ₹\, 72$

and $E(D) = ₹72 \times q = ₹72 \times \frac{3}{4} = ₹\, 54$

$\therefore E(A + C) = ₹\, 200$ and $E(B - D) = ₹\, 42$.

JEE Type Solved Examples : Passage Based Questions

- This section contains **3 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

Each coefficient in the equation $ax^2 + bx + c = 0$ *is determined by throwing an ordinary die.*

16. *The probability that roots of quadratic are real and distinct, is*

(a) $\frac{5}{216}$ (b) $\frac{19}{108}$ (c) $\frac{173}{216}$ (d) $\frac{17}{108}$

Sol. (b) For roots of $ax^2 + bx + c = 0$ to be real and distinct,

$$b^2 - 4ac > 0$$

Value of b	Possible values of a and c
1, 2	No values of a and c
3	(1, 1), (1, 2), (2, 1)
4	(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)
5	(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2), (1, 4), (4, 1), (1, 5), (5, 1), (2, 3), (3, 2), (1, 6), (6, 1)
6	(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2), (1, 4), (4, 1), (1, 5), (5, 1), (2, 3), (3, 2), (1, 6), (6, 1), (2, 4), (4, 2)

If E be the event of favourable cases, then $n(E) = 38$

Total ways, $n(S) = 6 \times 6 \times 6 = 216$

Hence, the required probability, $p_1 = \frac{n(E)}{n(S)} = \frac{38}{216} = \frac{19}{108}$

17. *The probability that roots of quadratic are equal, is*

(a) $\frac{5}{216}$ (b) $\frac{7}{216}$ (c) $\frac{11}{216}$ (d) $\frac{17}{216}$

Sol. (a) For roots of $ax^2 + bx + c = 0$ to be equal $b^2 = 4ac$

i.e. b^2 must be even.

Value of b	Possible values of a and c
2	(1, 1)
4	(2, 2), (1, 4), (4, 1)
6	(3, 3)

If E be the event of favourable cases, then $n(E) = 5$

Total ways, $n(S) = 6 \times 6 \times 6 = 216$

Hence, the required probability, $p_2 = \dfrac{n(E)}{n(S)} = \dfrac{5}{216}$

18. *The probability that roots of quadratic are imaginary, is*

(a) $\dfrac{103}{216}$ (b) $\dfrac{133}{216}$ (c) $\dfrac{157}{216}$ (d) $\dfrac{173}{216}$

Sol. (d) Let p_3 = Probability that roots of $ax^2 + bx + c = 0$ are imaginary

$= 1 -$ (Probability that roots of $ax^2 + bx + c = 0$ are real)

$= 1 - (p_1 + p_2)$ [from above]

$= 1 - \dfrac{43}{216} = \dfrac{173}{216}$

Passage II

(Ex. Nos. 19 to 21)

A box contains n coins. Let $P(E_i)$ be the probability that exactly i out of n coins are biased. If $P(E_i)$ is directly proportional to $i(i+1); 1 \le i \le n$.

19. *Proportionality constant k is equal to*

(a) $\dfrac{3}{n(n^2+1)}$ (b) $\dfrac{1}{(n^2+1)(n+2)}$

(c) $\dfrac{3}{n(n+1)(n+2)}$ (d) $\dfrac{1}{(n+1)(n+2)(n+3)}$

Sol. (c) $\because P(E_i) \propto i(i+1)$

$\Rightarrow P(E_i) = k\,i(i+1)$, where k is proportionality constant.

We have, $P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_n) = 1$

($\because E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events)

$$\Rightarrow \sum_{i=1}^{n} P(E_i) = 1$$

$$\Rightarrow k\sum_{i=1}^{n}(i^2 + i) = 1$$

$$\Rightarrow k\left[\sum n^2 + \sum n\right] = 1$$

$$\Rightarrow k\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = 1$$

$$\therefore \quad k = \frac{3}{n(n+1)(n+2)} \quad \ldots\text{(i)}$$

20. *If P be the probability that a coin selected at random is biased, then $\lim_{x \to \infty} P$ is*

(a) $\dfrac{1}{4}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{3}{5}$ (d) $\dfrac{7}{8}$

Sol. (b) $\because P = P(E) = \sum_{i=1}^{n} \cdot P(E_i)\, P\left(\dfrac{E}{E_i}\right)$ ...(ii)

$$= \sum_{i=1}^{n} k\, i(i+1) \cdot \frac{i}{n}$$

$$= \frac{k}{n}\sum_{i=1}^{n}(i^3 + i^2) = \frac{k}{n}\left[\sum n^3 + \sum n^2\right]$$

$$= \frac{k}{n}\left[\left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{k(n+1)(n+2)(3n+1)}{12}$$

$$= \frac{3}{n(n+1)(n+2)} \cdot \frac{(n+1)(n+2)(3n+1)}{12} \quad \text{[from Eq. (i)]}$$

$$= \frac{3n+1}{4n} = \frac{3}{4} + \frac{1}{4n}$$

$$\therefore \lim_{n \to \infty} P = \lim_{n \to \infty}\left[\frac{3}{4} + \frac{1}{3n}\right] = \frac{3}{4} + 0 = \frac{3}{4}$$

21. *If a coin is selected at random is found to be biased, the probability that it is the only biased coin the box, is*

(a) $\dfrac{1}{(n+1)(n+2)(n+3)(n+4)}$ (b) $\dfrac{12}{n(n+1)(n+2)(3n+1)}$

(c) $\dfrac{24}{n(n+1)(n+2)(2n+1)}$ (d) $\dfrac{24}{n(n+1)(n+2)(3n+1)}$

Sol. (d) $P\left(\dfrac{E_1}{E}\right) = \dfrac{P(E_1) \cdot P\left(\dfrac{E}{E_1}\right)}{\sum_{i=1}^{n} P(E_i) \cdot P\left(\dfrac{E}{E_i}\right)} = \dfrac{2k \times \dfrac{1}{n}}{P(E)}$ [from Eq. (ii)]

$$= \frac{\dfrac{2k}{n}}{\left(\dfrac{3n+1}{4n}\right)} = \frac{8k}{(3n+1)}$$

$$= \frac{24}{n(n+1)(n+2)(3n+1)} \quad \text{[from Eq. (i)]}$$

Passage III

(Ex. Nos. 22 to 24)

Let S be the set of the first 21 natural numbers, then the probability of

22. *Choosing $\{x, y\} \subseteq S$, such that $x^3 + y^3$ is divisible by 3, is*

(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{5}$ (c) $\dfrac{1}{4}$ (d) $\dfrac{1}{3}$

Sol. (d) $\because S = \{1, 2, 3, 4, 5, \ldots, 21\}$

Total number of ways choosing x and y is

$${}^{21}C_2 = \frac{21 \cdot 20}{1 \cdot 2} = 210$$

Now, arrange the given numbers as below:

1	4	7	10	13	16	19
2	5	8	11	14	17	20
3	6	9	12	15	18	21

We see that, $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$ will be divisible by 3 in the following cases:

One of two numbers belongs to the first row and one of the two numbers belongs to the second row or both numbers occurs in third row.

$\therefore$ Number of favourable cases $= ({}^7C_1)({}^7C_1) + {}^7C_2 = 70$

$\therefore$ Required probability $= \frac{70}{210} = \frac{1}{3}$

23. *Choosing $\{x, y, z\} \subseteq S$, such that x, y, z are in AP, is*

(a) $\frac{5}{133}$ (b) $\frac{10}{133}$ (c) $\frac{3}{133}$ (d) $\frac{2}{133}$

Sol. (b) Given, x, y, z are in AP

$\therefore$ $2y = x + z$

It is clear that sum of x and z is even.

$\therefore$ x and z both are even or odd out of set S.

i.e., 11 numbers (1, 3, 5,..., 21) are odd and 10 numbers (2, 4, 6,..., 20) are even.

$\therefore$ Number of favourable cases

$$= {}^{21}C_2 + {}^{10}C_2 = \frac{11 \cdot 10}{1 \cdot 2} + \frac{10 \cdot 9}{1 \cdot 2} = 100$$

and total number of ways choosing x, y and z is

$${}^{21}C_3 = \frac{21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3} = 1330$$

$\therefore$ Required probability $= \frac{100}{1330} = \frac{10}{133}$

24. *Choosing $\{x, y, z\} \subseteq S$, such that x, y, z are not consecutive, is*

(a) $\frac{17}{70}$ (b) $\frac{34}{70}$ (c) $\frac{51}{70}$ (d) $\frac{34}{35}$

Sol. (c) Given, x, y and z are not consecutive.

$\therefore$ Number of favourable ways $= {}^{21-3+1}C_3$

$$= {}^{19}C_3 = \frac{19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3} = 969$$

and total number of ways $= {}^{21}C_3 = \frac{21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3} = 1330$

$\therefore$ Required probability $= \frac{969}{1330} = \frac{51}{70}$

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● **Ex. 25** *The altitude through A of ΔABC meets BC at D and the circumscribed circle at E. If $D \equiv (2, 3)$, $E \equiv (5, 5)$, the ordinate of the orthocentre being a natural number. If the probability that the orthocentre lies on the lines $y = 1$; $y = 2$; $y = 3 \ldots\ldots y = 10$ is $\frac{m}{n}$, where m and n are relative primes, the value of $m + n$ is*

Sol. (8) Let the orthocentre be $O(x, y)$.

It is clear from the OE is perpendicular bisector of line BC.

$\therefore$ $OD = DE$

$\Rightarrow \sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(5-2)^2 + (5-3)^2}$

$\Rightarrow (x-2)^2 + (y-3)^2 = (5-2)^2 + (5-3)^2$

$\Rightarrow x^2 + y^2 - 4x - 6y = 0 \Rightarrow x = 2 \pm \sqrt{13 - (y-3)^2}$

$\Rightarrow y$ can take the values as 1, 2, 3, 4, 5, 6

$\therefore$ Required probability $= \frac{6}{10} = \frac{3}{5} = \frac{m}{n}$ [given]

$\Rightarrow$ $m = 3$ and $n = 5$

$\therefore$ $m + n = 8$

● **Ex. 26** *The digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are written in random order to form a nine digit number. Let p be the probability that this number is divisible by 36, the value of 9p is*

Sol. (2) $\because$ $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, a number consisting all these digits will be divisible by 9. Thus, the number will be divisible by 36, if and only if it is divisible by 4. The number formed by its last two digits must be divisible by 4. The possible values of the last pair to the following:

12, 16, 24, 28, 32, 36, 48, 52, 56, 64, 68, 72, 76, 84, 92, 96.

i.e., There are 16 ways of choosing last two digits.

The remaining digits can be arranged in ${}^7P_7 = 7!$ ways.

Therefore, number of favourable ways $= 16 \times 7!$

and number of total ways $= 9!$

$\therefore$ Required probability, $p = \frac{16 \times 7!}{9!} = \frac{16}{9 \times 8} = \frac{2}{9}$

$\Rightarrow$ $9p = 2$

JEE Type Solved Examples : Matching Type Questions

- This section contains **2 examples.** Examples 27 and 28 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

- ***Ex. 27*** *If n positive integers taken at random are multiplied together.*

	Column I		Column II
(A)	The probability that the last digit is 1, 3, 7 or 9 is $P(n)$, then 100 $P(2)$ is divisible by	(p)	3
(B)	The probability that the last digit is 2, 4, 6 or 8 is $Q(n)$, then 100 $Q(2)$ is divisible by	(q)	4
(C)	The probability that the last digit is 5 is $R(n)$, then 100 $R(2)$ is divisible by	(r)	6
(D)	The probability that the last digit is zero is $S(n)$, then 100 $S(2)$ is divisible by	(s)	9

Sol. A $\rightarrow$ (q); B $\rightarrow$ (p, q, r); C $\rightarrow$ (p, s); D $\rightarrow$ (p, s)

Let n positive integers be $x_1, x_2, x_3, \ldots, x_n$

Let $$a = x_1 . x_2 . x_3 \ldots x_n$$

Since, the last digit in each of the numbers $x_1, x_2, \ldots, x_n$ can be any one of the digits

0, 1, 2,..., 9 (total 10)

$$\therefore \quad n(S) = 10^n$$

Let E_1, E_2, E_3 and E_4 are the events given in A, B, C and D, respectively.

(A) $n(E_1) = 4^n \Rightarrow P(E_1) = \left(\frac{4}{10}\right)^n = P(n)$ [given]

$$\therefore \quad 100\,P(2) = 16$$

(B) $n(E_2) = n$ (last digit is 1 or 2 or 3 or 4 or 6 or 7 or 8 or 9) $- n(E_1) = 8^n - 4^n$

$$\Rightarrow \quad P(E_2) = \frac{8^n - 4^n}{10^n} = Q(n) \quad \text{[given]}$$

$$\therefore \quad 100\,Q(2) = 64 - 16 = 48$$

(C) $n(E_3) = n$ (last digit is 1 or 3 or 5 or 7 or 9) $- n(E_1)$

$$= 5^n - 4^n$$

$$\Rightarrow \quad P(E_3) = \frac{5^n - 4^n}{10^n} = R(n) \quad \text{[given]}$$

$$\therefore \quad 100\,R(2) = 25 - 16 = 9$$

(D) $n(E_4) = n(S) - n$ (last digit is 1 or 2 or 3 or 4 or 6 or 7 or 8 or 9)$- n(E_3) = 10^n - 8^n - (5^n - 4^n)$

$$\therefore \quad P(E_4) = \frac{10^n - 8^n - 5^n + 4^n}{10^n} = S(n) \quad \text{[given]}$$

$$\therefore \quad 100\,S(2) = 27$$

- ***Ex. 28*** *If A and B are two independent events, such that* $P(A) = \frac{1}{3}$ *and* $P(B) = \frac{1}{4}$.

	Column I		Column II
(A)	If $P\left(\frac{A}{B}\right) = \lambda_1$, then $12\lambda_1$ is	(p)	a prime number
(B)	If $P\left(\frac{A}{A \cup B}\right) = \lambda_2$, then $9\lambda_2$ is	(q)	a composite number
(C)	If $P[(A \cap \overline{B}) \cup (\overline{A} \cap B)] = \lambda_3$, then $12\lambda_3$ is	(r)	a natural number
(D)	If $P(\overline{A} \cup B) = \lambda_4$, then $12\lambda_4$ is	(s)	a perfect number

Sol. A $\rightarrow$ (q, r); B (q, r, s); C $\rightarrow$ (p, r); D $\rightarrow$ (q, r)

$\because$ A and B are independent events.

$$\therefore \quad P(A \cap B) = P(A) \cdot P(B) = \frac{1}{12},$$

$$P(A \cap \overline{B}) = P(A) \cdot P(\overline{B}) = \frac{1}{3} \times \left(1 - \frac{1}{4}\right) = \frac{1}{4},$$

$$P(\overline{A} \cap B) = P(\overline{A}) \cdot P(B) = \left(1 - \frac{1}{3}\right) \cdot \frac{1}{4} = \frac{1}{6}$$

(A) $$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3} = \lambda_1 \quad \text{[given]}$$

$\therefore$ $12\lambda_1 = 4$ [natural number and composite number]

(B) $$P\left(\frac{A}{A \cup B}\right) = \frac{P(A \cap (A \cup B))}{P(A \cup B)}$$

$$= \frac{P(A)}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A \cap B)}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{4} - \frac{1}{12}} = \frac{2}{3} = \lambda_2 \quad \text{[given]}$$

$\therefore$ $9\lambda_2 = 6$

[natural number, composite number and perfect number]

(C) $$P(A \cap \overline{B}) \cup (\overline{A} \cap B)) = P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

$$= \frac{1}{4} + \frac{1}{6} = \frac{5}{12} = \lambda_3 \quad \text{[given]}$$

$\therefore$ $12\lambda_3 = 5$ [prime number and natural number]

(D) $$P(\overline{A} \cup B) = P(\overline{A}) + P(B) - P(\overline{A} \cap B)$$

$$= \left(1 - \frac{1}{3}\right) + \frac{1}{4} - \frac{1}{6} = \frac{3}{4} = \lambda_4 \quad \text{[given]}$$

$\therefore$ $12\lambda_4 = 9$ [natural number and composite number]

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** Example numbers 29 and 30 are Assertion-Reason type examples. Each of these examples contains two statements:

Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

- **Ex. 29.** *A man P speaks truth with probability p and another man Q speaks truth with probability 2p.*

Statement-1 *If P and Q contradict each other with probability* $\frac{1}{2}$, *then there are two values of p.*

Statement-2 *A quadratic equation with real coefficients has two real roots.*

Sol. (c) Let E_1 be the event that P speaks the truth, then $P(E_1) = p$ and let E_2 be the event that Q speaks the truth, then $P(E_2) = 2p$.

Statement-1 If P and Q contradict each other with probability $\frac{1}{2}$, then $P(E_1) \cdot P(E_2') + P(E_1') \cdot P(E_2) = \frac{1}{2}$

$$\Rightarrow p \cdot (1-2p) + (1-p) \cdot 2p = \frac{1}{2} \Rightarrow 8p^2 - 6p + 1 = 0$$

$$\Rightarrow \quad (2p-1)(4p-1) = 0 \Rightarrow p = \frac{1}{2} \text{ and } p = \frac{1}{4}$$

$\therefore$ Statement-1 is true.

Statement-2 Let quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$

If $\qquad b^2 - 4ac < 0$

then, roots are imaginary.

$\therefore$ Statement-2 is false.

- **Ex. 30** *A fair die thrown twice. Let* (a, b) *denote the outcome in which the first throw shows a and the second shows b. Let A and B be the following two events:*

$A = \{(a, b) | a \text{ is even}\}$, $B = \{(a, b) | b \text{ is even}\}$

Statement-1 If $C = \{(a, b) | a + b \text{ is odd}\}$, then $P(A \cap B \cap C) = \frac{1}{8}$.

Statement-2 If $D = \{(a, b) | a + b \text{ is even}\}$, then

$$P[(A \cap B \cap D) | (A \cup B)] = \frac{1}{3}$$

Sol. (c) If a and b are both even, then $a + b$ is even, therefore $P(A \cap B \cap C) = 0$

$\therefore$ Statement-1 is false.

Also, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$

$$\therefore P[(A \cap B \cap D) | (A \cup B)] = \frac{P((A \cap B \cap D) \cap (A \cup B))}{P(A \cup B)}$$

$$= \frac{P(A \cap B)}{P(A \cup B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \quad [\because A \cap B \subseteq D]$$

$\therefore$ Statement-2 is true.

Probability Exercise 8 : Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

116. A person goes to office either by car, scooter, bus or train. The probability of which being $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}$ and $\frac{1}{7}$, respectively. Probability that he reaches office late, if he takes car, scooter, bus or train is $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}$ and $\frac{1}{9}$, respectively. Given that he reached office in time, then what is the probability that he travelled by a car. **[IIT-JEE 2005, 2M]**

117. A six faced fair die is thrown until 1 comes. Then, the probability that 1 comes in even number of trials, is **[IIT-JEE 2005, 3M]**

(a) $\frac{5}{11}$ (b) $\frac{5}{6}$ (c) $\frac{6}{11}$ (d) $\frac{1}{6}$

118. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for complement of event A. Then, events A and B are **[IIT-JEE 2005, 3M]**

(a) independent but not equally likely
(b) mutually exclusive and independent
(c) equally likely and mutually exclusive
(d) equally likely but not independent

119. Three houses are available in a locality. Three persons apply for the houses. Each applies for one house without consulting others. The probability that all the three apply for the same house, is **[AIEEE 2005, 3M]**

(a) $\frac{8}{9}$ (b) $\frac{7}{9}$ (c) $\frac{2}{9}$ (d) $\frac{1}{9}$

120. A random variable X has Poisson's distribution with mean 2. Then, $P(X > 1.5)$ is equal to **[AIEEE 2005, 3M]**

(a) $1 - \frac{3}{e^2}$ (b) $\frac{3}{e^2}$ (c) $\frac{2}{e^2}$ (d) 0

121. There are n urns each containing $(n + 1)$ balls such that the ith urn contains i white balls and $(n + 1 - i)$ red balls. Let u_i be the event of selecting ith urn, $i = 1, 2, 3, \ldots, n$ and w denotes the event of getting a white ball. **[IIT-JEE 2006, 5+5+5M]**

(i) If $P(u_i) \propto i$, where $i = 1, 2, 3, \ldots, n$, then $\lim_{n \to \infty} P(w)$, is

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{1}{4}$

(ii) If $P(u_i) = c$, where c is a constant, then $P\left(\frac{u_n}{w}\right)$, is

(a) $\frac{2}{n+1}$ (b) $\frac{1}{n+1}$ (c) $\frac{n}{n+1}$ (d) $\frac{1}{2}$

(iii) If n is even and E denotes the event of choosing even numbered urn $\left(P(u_i) = \frac{1}{n}\right)$, then the value of $P\left(\frac{w}{E}\right)$, is

(a) $\frac{n+2}{2n+1}$ (b) $\frac{n+2}{2(n+1)}$ (c) $\frac{n}{n+1}$ (d) $\frac{1}{n+1}$

122. At a telephone enquiry system, the number of phone calls regarding relevant enquiry follow Poisson's distribution with an average of 5 phone calls during 10 min time interval. The probability that there is atmost one phone call during a 10 min time period, is **[AIEEE 2006, 4, 5M]**

(a) $\frac{6}{5^e}$ (b) $\frac{5}{6}$ (c) $\frac{6}{55}$ (d) $\frac{6}{e^5}$

123. One Indian and four American men and their wives are to be seated randomly around a circular table. Then, the conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife, is **[IIT-JEE 2007, 3M]**

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

124. Let $H_1, H_2, \ldots, H_n$ be mutually exclusive events with $P(H_i) > 0, i = 1, 2, \ldots, n$. Let E be any other event with $0 < P(E) < 1$.

Statement-1 $P(H_i / E) > P(E / H_i)\, P(H_i)$, for $i = 1, 2, \ldots, n$.

Statement-2 $\sum_{i=1}^{n} P(H_i) = 1$ **[IIT-JEE 2007, 3M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

125. Let E^c denote the complement of an event E. Let E, F and G be pairwise independent events with $P(G) > 0$ and $P(E \cap F \cap G) = 0$, then $P(E^c \cap F^c / G)$, is **[IIT-JEE 2007, 3M]**

(a) $P(E^c) + P(F^c)$ (b) $P(E^c) - P(F^c)$
(c) $P(E^c) - P(F)$ (d) $P(E) - P(F^c)$

126. A pair of fair dice is thrown independently three times. Then, the probability of getting a score of exactly 9 twice, is **[AIEEE 2007, 3M]**

(a) $\frac{1}{729}$ (b) $\frac{8}{9}$ (c) $\frac{8}{729}$ (d) $\frac{8}{243}$

127. Two aeroplanes I and II bomb a target in successions. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2, respectively. The second plane will bomb only if the first misses the target. The probability that the target is hit by the second plane, is **[AIEEE 2007, 3M]**

(a) 0.06 (b) 0.14 (c) 0.2 (d) 0.7

128. An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, then the number of outcomes that B must have, so that A and B are independent, is **[IIT-JEE 2008, 3M]**

(a) 2, 4 or 8 (b) 3, 6 or 9 (c) 4 or 8 (d) 5 or 10

129. Consider the system of equations $ax + by = 0$ and $cx + dy = 0$, where $a, b, c, d \in \{0, 1\}$.

Statement-1 The probability that the system of equations has a unique solution is 3/8 and **[IIT-JEE 2008, 3M]**

Statement-2 The probability that the system of equations has a solution is 1.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

130. A die is thrown. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is **[AIEEE 2008, 3M]**

(a) 0 (b) 1 (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

131. It is given that the events A and B are such that $P(A) = \frac{1}{4}$, $P\left(\frac{A}{B}\right) = \frac{1}{2}$ and $P\left(\frac{B}{A}\right) = \frac{2}{3}$. Then $P(B)$ is **[AIEEE 2008, 3M]**

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{6}$

▪ **Passage for Question Nos. 132 to 134**

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

132. The probability that $X = 3$ is

(a) $\frac{25}{216}$ (b) $\frac{25}{36}$ (c) $\frac{5}{36}$ (d) $\frac{125}{216}$

133. The probability that $X \geq 3$ is

(a) $\frac{125}{216}$ (b) $\frac{25}{36}$ (c) $\frac{5}{36}$ (d) $\frac{25}{216}$

134. The conditional probability that $X \geq 6$ given $X > 3$, is

(a) $\frac{125}{216}$ (b) $\frac{25}{216}$ (c) $\frac{5}{36}$ (d) $\frac{25}{36}$

[IIT-JEE 2009, 4+4+4M]

135. In a binomial distribution $B\left(n, p = \frac{1}{4}\right)$, if the probability of atleast one success is greater than or equal to $\frac{9}{10}$, then n is greater than **[AIEEE 2009, 4M]**

(a) $\frac{4}{\log_{10} 4 - \log_{10} 3}$ (b) $\frac{1}{\log_{10} 4 - \log_{10} 3}$

(c) $\frac{1}{\log_{10} 4 + \log_{10} 3}$ (d) $\frac{9}{\log_{10} 4 - \log_{10} 3}$

136. One ticket is selected at random from 50 tickets numbered 00, 01, 02, ..., 49. Then, the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, is **[AIEEE 2009, 4M]**

(a) $\frac{1}{50}$ (b) $\frac{1}{14}$

(c) $\frac{1}{7}$ (d) $\frac{5}{14}$

137. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$, is **[IIT-JEE 2010, 3M]**

(a) $\frac{1}{18}$ (b) $\frac{1}{9}$ (c) $\frac{2}{9}$ (d) $\frac{1}{36}$

138. A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green, is **[IIT-JEE 2010, 5M]**

(a) $\frac{3}{5}$ (b) $\frac{6}{7}$ (c) $\frac{20}{23}$ (d) $\frac{9}{20}$

139. Four numbers are chosen at random (without replacement) from the set $\{1, 2, 3, \ldots, 20\}$.

Statement-1 The probability that the chosen numbers, when arranged in some order will form an AP is $\frac{1}{85}$.

Statement-2 If the four chosen number form an AP, then the set of all possible values of common difference is $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$. **[AIEEE 2010, 8M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is false
(c) Statement-1 is false, Statement-2 is true
(d) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

140. An urn contains nine balls of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that the three balls have different colour, is **[AIEEE 2010, 4M]**

(a) $\frac{2}{7}$ (b) $\frac{1}{21}$ (c) $\frac{2}{23}$ (d) $\frac{1}{3}$

Passage for Question Nos. 141 and 142

Let U_1 and U_2 be two urns such that U_1 contains 3 white balls and 2 red balls and U_2 contains only 1 white ball. A fair coin is tossed. If head appears, then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears, then 2 balls are drawn at random from U_1 and put into U_2. Now, 1 ball is drawn at random from U_2.

141. The probability of the drawn ball from U_2 being white, is

(a) $\frac{13}{30}$ (b) $\frac{23}{30}$ (c) $\frac{19}{30}$ (d) $\frac{11}{30}$

142. Given that the drawn ball from U_2 is white, then the probability that head appeared on the coin, is **[IIT-JEE 2011, 3+3M]**

(a) $\frac{17}{23}$ (b) $\frac{11}{23}$ (c) $\frac{15}{23}$ (d) $\frac{12}{23}$

143. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If $P(T)$ denotes the probability of occurrence of the event T, then **[IIT-JEE 2011, 4M]**

(a) $P(E)=\frac{4}{5}, P(F)=\frac{3}{5}$ (b) $P(E)=\frac{1}{5}, P(F)=\frac{2}{5}$

(c) $P(E)=\frac{2}{5}, P(F)=\frac{1}{5}$ (d) $P(E)=\frac{3}{5}, P(F)=\frac{4}{5}$

144. Consider 5 independent Bernoulli's trials each with probability of success P. If the probability of atleast one failure is greater than or equal to $\frac{31}{32}$, then P lies in the interval **[AIEEE 2011, 4M]**

(a) $\left(\frac{3}{4}, \frac{11}{12}\right]$ (b) $\left[0, \frac{1}{2}\right]$ (c) $\left(\frac{11}{12}, 1\right]$ (d) $\left(\frac{1}{2}, \frac{3}{4}\right]$

145. If C and D are two events, such that $C \subset D$ and $P(D) \neq 0$, then the correct statement among the following, is **[AIEEE 2011, 4M]**

(a) $P\left(\frac{C}{D}\right) \geq P(C)$ (b) $P\left(\frac{C}{D}\right) < P(C)$

(c) $P\left(\frac{C}{D}\right) = \frac{P(D)}{P(C)}$ (d) $P\left(\frac{C}{D}\right) = P(C)$

146. Let A, B and C are pairwise independent events with $P(C) > 0$ and $P(A \cap B \cap C) = 0$. Then, $P\left(\frac{(A^c \cap B^c)}{C}\right)$ is

(a) $P(A^c) - P(B)$ (b) $P(A) - P(B^c)$

(c) $P(A^c) + P(B^c)$ (d) $P(A^c) - P(B^c)$

147. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$, respectively. For the ship to be operational atleast two of its engines must function. Let X denote the event that the ship is operational and let X_1, X_2 and X_3 denote respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is (are) true? **[IIT-JEE 2012, 4M]**

(a) $P[X_1^c / X] = \frac{3}{16}$

(b) P [exactly two engines of the ship are functioning $/X] = \frac{7}{8}$

(c) $P[X / X_2] = \frac{5}{16}$

(d) $P[X / X_1] = \frac{7}{16}$

148. Four fair dice D_1, D_2, D_3 and D_4 each having six faces numbered 1, 2, 3, 4, 5 and 6 are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3, is **[IIT-JEE 2012, 3M]**

(a) $\frac{91}{216}$ (b) $\frac{108}{216}$ (c) $\frac{25}{216}$ (d) $\frac{127}{216}$

149. Let X and Y be two events, such that $P(X/Y) = \frac{1}{2}$, $P(Y/X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is (are) correct? **[IIT-JEE 2012, 4M]**

(a) $P(X \cup Y) = \frac{2}{3}$ (b) X and Y are independent

(c) X and Y are not independent

(d) $P(X^c \cap Y) = \frac{1}{3}$

150. Three numbers are chosen at random without replacement from $\{1, 2, 3, \ldots, 8\}$. The probability that their minimum is 3, given that their maximum is 6, is **[AIEEE 2012, 4M]**

(a) $\frac{1}{4}$ (b) $\frac{2}{5}$ (c) $\frac{3}{8}$ (d) $\frac{1}{5}$

151. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing, is **[JEE Main 2013, 4M]**

(a) $\frac{13}{3^5}$ (b) $\frac{11}{3^5}$ (c) $\frac{10}{3^5}$ (d) $\frac{17}{3^5}$

152. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}$ and $\frac{1}{8}$. Then, the probability that the problem is solved correctly by atleast one of them, is **[JEE Advanced 2013, 2M]**

(a) $\frac{235}{256}$ (b) $\frac{21}{256}$ (c) $\frac{3}{256}$ (d) $\frac{253}{256}$

153. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equation $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval $(0, 1)$.

Then, $\dfrac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3}$ is

[JEE Advanced 2013, 4M]

▪ Passage for Question Nos. 154 and 155

A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

154. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, then the probability that these 2 balls are drawn from box B_2, is

(a) $\frac{116}{181}$ (b) $\frac{126}{181}$ (c) $\frac{65}{181}$ (d) $\frac{55}{181}$

155. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3, then the probability that all 3 drawn balls are of the same colour, is **[JEE Advanced 2013, 3+3M]**

(a) $\frac{82}{648}$ (b) $\frac{90}{648}$ (c) $\frac{558}{648}$ (d) $\frac{566}{648}$

156. Let A and B be two events, such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for the complement of the event A. Then, the events A and B are

(a) independent but not equally likely **[JEE Main 2014, 4M]**
(b) independent and equally likely
(c) mutually exclusive and independent
(d) equally likely but not independent

157. Three boys and two girls stand in a queue. The probability that the number of boys ahead of every girl is atleast one more than the number of girls ahead of her, is **[JEE Advanced 2014, 3M]**

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

▪ Passage for Question Nos. 158 and 159

Box 1 contains three cards bearing numbers 1, 2, 3, box 2 contains five cards bearing numbers 1, 2, 3, 4, 5 and box 3 contains seven cards bearing numbers 1, 2, 3, 4, 5, 6, 7. A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the ith box, $i = 1, 2, 3$.

158. The probability that $x_1 + x_2 + x_3$ is odd, is

(a) $\frac{29}{105}$ (b) $\frac{53}{105}$ (c) $\frac{57}{105}$ (d) $\frac{1}{2}$

159. The probability that x_1, x_2 and x_3 are in arithmetic progression, is **[JEE Advanced 2014, 3+3M]**

(a) $\frac{9}{105}$ (b) $\frac{10}{105}$ (c) $\frac{11}{105}$ (d) $\frac{7}{105}$

160. If 12 identical balls are to be placed in 3 identical boxes, then the probability that one of the boxes contains exactly 3 balls, is **[JEE Main 2015, 4M]**

(a) $220\left(\frac{1}{3}\right)^{12}$ (b) $22\left(\frac{1}{3}\right)^{11}$ (c) $\frac{55}{3}\left(\frac{2}{3}\right)^{11}$ (d) $55\left(\frac{2}{3}\right)^{10}$

161. The minimum number of times a fair coin needs to be tossed, so that the probability of getting atleast two heads is atleast 0.96, is **[JEE Advanced 2015, 4M]**

▪ Passage for Question Nos. 162 and 163

Let n_1 and n_2 be the number of red and black balls respectively, in box I. Let n_3 and n_4 be the number of red and black balls respectively, in box II.

162. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is 1 / 3, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is (are)

(a) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$ (b) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$
(c) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$ (d) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

163. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer is $\frac{1}{3}$, then correct option(s) with possible values of n_1 and n_2 is (are) **[JEE Advanced 2015, 4+4M]**

(a) $n_1 = 4$ and $n_2 = 6$ (b) $n_1 = 2$ and $n_2 = 3$
(c) $n_1 = 10$ and $n_2 = 20$ (d) $n_1 = 3$ and $n_2 = 6$

164. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is NOT true? **[JEE Main 2016, 4M]**

(a) E_2 and E_3 are independent (b) E_1 and E_3 are independent
(c) E_1, E_2 and E_3 are independent (d) E_1 and E_2 are independent

165. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced 7% of computers produced in the factory turn out to be defective. It is known that P (computer terms out to be defective given that it is produced in plant T_1) = $10P$ (computer terms out to be defective given that it is produced in plant T_2), when $P(E)$ denotes the probability of an event E. A computer produced in the factory is randomly selected and it does not turn out to be defective. Then, the probability that it is produced in plant T_2 is

[JEE Advanced 2016, 3M]

(a) $\frac{36}{73}$ (b) $\frac{47}{79}$ (c) $\frac{78}{93}$ (d) $\frac{75}{83}$

Passage for Question Nos. 166 and 167

Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$ respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2 respectively, after two games. **[JEE Advanced 2016, 3+3M]**

(a) $\frac{1}{4}$ (b) $\frac{5}{12}$ (c) $\frac{1}{2}$ (d) $\frac{7}{12}$

167. $P(X = Y)$ is

(a) $\frac{11}{36}$ (b) $\frac{1}{3}$ (c) $\frac{13}{36}$ (d) $\frac{1}{2}$

168. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn, one-by-one with replacement, then the variance of the number of green balls drawn is **[JEE Main 2017, 4M]**

(a) $\frac{6}{25}$ (b) $\frac{12}{5}$ (c) 6 (d) 4

169. If two different numbers are taken from the set $\{0, 1, 2, 3, \ldots, 10\}$, then the probability that their sum as well as absolute difference are both multiple of 4, is **[JEE Main 2017, 4M]**

(a) $\frac{7}{55}$ (b) $\frac{6}{55}$ (c) $\frac{12}{55}$ (d) $\frac{14}{45}$

170. For three events A, B and C.

P(Exactly one of A or B or C occurs)

$= P$(Exactly one of B or C occurs)

$= P$(Exactly one of C or A occurs) $= \frac{1}{4}$

and P (All the three events occur simultaneously) $= \frac{1}{16}$,

Then the probability that atleast one of the events occurs, is **[JEE Main 2017, 4M]**

(a) $\frac{3}{16}$ (b) $\frac{7}{32}$

(c) $\frac{7}{16}$ (d) $\frac{7}{64}$

Answers

Exercise for Session 1

1. (a)	2. (a)	3. (d)	4. (b)	5. (a)	6. (c)
7. (b)	8. (d)	9. (d)	10. (d)	11. (c)	12. (c)
13. (d)	14. (d)	15. (b)	16. (b)	17. (a)	18. (a)
19. (c)	20. (c)				

Exercise for Session 2

1. (c)	2. (d)	3. (b)	4. (c)	5. (b)	6. (b)
7. (c)	8. (a)	9. (a)	10. (b)	11. (c)	12. (a)
13. (b)	14. (b)	15. (b)			

Exercise for Session 3

1. (d)	2. (d)	3. (c)	4. (d)	5. (d)	6. (b)
7. (d)	8. (c)	9. (c)	10. (a)		

Exercise for Session 4

1. (c)	2. (d)	3. (d)	4. (a)	5. (c)	6. (a)
7. (a)	8. (d)	9. (c)	10. (c)	11. (c)	12. (b)
13. (c)	14. (c)	15. (a)			

Chapter Exercise

1. (d)	2. (d)	3. (c)	4. (d)	5. (b)	6. (b)
7. (d)	8. (b)	9. (d)	10. (c)	11. (c)	12. (b)
13. (d)	14. (b)	15. (a)	16. (c)	17. (a)	18. (b)
19. (d)	20. (a)	21. (b)	22. (a)	23. (c)	24. (d)
25. (a)	26. (d)	27. (b)	28. (a)	29. (b)	30. (c)
31. (a,b,c)	32. (b,c,d)	33. (a,c)	34. (a,d)	35. (a,b)	36. (c, d)
37. (a,b,c,d)	38. (a,b,c,d)		39. (a,b,c)		40. (a,c)
41. (a,b,c)	42. (b,c)	43. (a,b,c,d)		44. (b,c,d)	
45. (a,c)	46. (c)	47. (d)	48. (b)	49. (b)	50. (d)
51. (c)	52. (b)	53. (c)	54. (c)	55. (d)	56. (c)
57. (a)	58. (c)	59. (c)	60. (d)	61. (c)	62. (a)
63. (c)	64. (c)	65. (b)	66. (a)	67. (d)	68. (d)
69. (c)	70. (a)	71. (b)	72. (c)	73. (5)	74. (3)
75. (2)	76. (5)	77. (4)			
78. (1)	79. (1)	80. (0)	81. (2)	82. (2)	

83. (A) $\rightarrow$ (q,r); (B) $\rightarrow$ (p,r); (C) $\rightarrow$ (p,r); (D) $\rightarrow$ (q,r,s)

84. (A) $\rightarrow$ (s); (B) $\rightarrow$ (p); (C) $\rightarrow$ (r); (D) $\rightarrow$ (q)

85. (A) $\rightarrow$ (p,r,s); (B) $\rightarrow$ (p,r,s); (C) $\rightarrow$ (s); (D) $\rightarrow$ (q,r)

86. (A) $\rightarrow$ (q); (B) $\rightarrow$ (r); (C) $\rightarrow$ (p); (D) $\rightarrow$ (s)

87. (A) $\rightarrow$ (r); (B) $\rightarrow$ (s); (C) $\rightarrow$ (p); (D) $\rightarrow$ (q)

88. (A) $\rightarrow$ (q); (B) $\rightarrow$ (s); (C) $\rightarrow$ (p); (D) $\rightarrow$ (r)

89. (a)	90. (a)	91. (a)	92. (c)	93. (c)	
94. (a)	95. (d)	96. (c)	97. (d)	98. (c)	
99. (a)	100. (a)				
101. $\left(\frac{1}{5}\right)$	102. $\left(\frac{5}{54}\right)$	103. $\left(\frac{(N-r)(N-r-1)}{(N-1)(N-2)}\right)$			
105. (12)	106. $\left(\frac{213}{1001}\right)$	107. $\left(\frac{a}{a+b}\right)$			
108. $\left(\frac{377}{4096}\right)$	109. $\left(\frac{1}{2}\right)$	112. $\left(\frac{{}^nC_r 3^r}{4^n}\right)$			
116. $\frac{1}{7}$	117. (a)	118. (a)	119. (d)	120. (a)	
121. (i) (b), (ii) (a), (iii) (b)			122. (d)	123. (c)	124. (d)
125. (c)	126. (d)	127. (b)	128. (d)	129. (b)	130. (b)
131. (a)	132. (a)	133. (b)	134. (d)	135. (b)	136. (b)
137. (c)	138. (c)	139. (b)	140. (a)	141. (b)	142. (d)
143. (a, d)	144. (b)	145. (a)	146. (a)	147. (b,d)	148. (a)
149. (a,b)	150. (d)	151. (b)	152. (a)	153. (b)	154. (d)
155. (a)	156. (a)	157. (a)	158. (b)	159. (b)	160. (c)
161. (8)	162. (a,b)	163. (c,d)	164. (c)	165. (c)	166. (b)
167. (c)	168. (b)	169. (c)	170. (c)		