Session 3

Point of Intersection of Two Lines, Concurrent Lines Family of Lines, How to Find Circumcentre and Orthocentre by Slopes

Points of Intersection of Two Lines

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be two non-parallel lines. If (x_1, y_1) be the coordinates of their point of intersection,

then $a_1x_1 + b_1y_1 + c_1 = 0$ and $a_2x_1 + b_2y_1 + c_2 = 0$

Solving these two by cross multiplication, then

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

we get $$(x_1, y_1) \equiv \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

$$\equiv \left(\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}, \frac{\begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}\right)$$

Remarks

1. Here lines are not parallel, they have unequal slopes, then, $a_1b_2 - a_2b_1 \neq 0$
2. In solving numerical questions, we should not be remember the coordinates (x_1, y_1) given above, but we solve the equations directly.

Concurrent Lines

The three given lines are concurrent, if they meet in a point. Hence to prove that three given lines are concurrent, we proceed as follows :

I Method : Find the point of intersection of any two lines by solving them simultaneously. If this point satisfies the third equation also, then the given lines are concurrent.

II Method : The three lines $a_ix + b_iy + c_i = 0, i = 1, 2, 3$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

III Method : The condition for the lines $P = 0, Q = 0$ and $R = 0$ to be concurrent is that three constants l, m, n (not all zeros at the same time) can be obtained such that

$$\boxed{lP + mQ + nR = 0}$$

Remarks

1. The reader is advised to follow method I in numerical problems.
2. For finding unknown quantity applying method II.

I Example 65. Show that the lines $2x + 3y - 8 = 0$, $x - 5y + 9 = 0$ and $3x + 4y - 11 = 0$ *are concurrent.*

Sol. **I Method :** Solving the first two equations, we see that their point of intersection is (1, 2) which also satisfies the third equation

$$3 \times 1 + 4 \times 2 - 11 = 0$$

Hence the given lines are concurrent.

II Method : We have $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 8 \\ 1 & -5 & 9 \\ 3 & 4 & -11 \end{vmatrix}$

Applying $C_3 \to C_3 + C_1 + 2C_2$

$$= \begin{vmatrix} 2 & 3 & 0 \\ 1 & -5 & 0 \\ 3 & 4 & 0 \end{vmatrix} = 0$$

Hence the given lines are concurrent.

III Method : Suppose

$$l(2x + 3y - 8) + m(x - 5y + 9) + n(3x + 4y - 11) = 0$$
$$\Rightarrow x(2l + m + 3n) + y(3l - 5m + 4n) + (-8l + 9m - 11n) = 0$$
$$= 0 \cdot x + 0 \cdot y + 0$$

On comparing,

$$2l + m + 3n = 0, 3l - 5m + 4n = 0, -8l + 9m - 11n = 0$$

After solving, we get $l = 19, m = 1, n = -13$

Hence, $19(2x + 3y - 8) + (x - 5y + 9) - 13(3x + 4y - 11) = 0$

Hence the given lines are concurrent.

Must

| Example 66. If the lines $ax+y+1=0$, $x+by+1=0$ and $x+y+c=0$ (a, b and c being distinct and different from 1) are concurrent, then find the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$.

Sol. The given lines are concurrent, then

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}=0 \Rightarrow \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix}=0$$

(applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

Expanding along first row

$$\Rightarrow \quad a(b-1)(c-1)-(1-a)(c-1)-(1-a)(b-1)=0$$

$$\Rightarrow \quad a(1-b)(1-c)+(1-a)(1-c)+(1-a)(1-b)=0$$

Dividing by $(1-a)(1-b)(1-c)$, then

$$\frac{a}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=0$$

$$\Rightarrow \quad -1+\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=0$$

Hence, $$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$$

| Example 67. Show that the three straight lines $2x-3y+5=0$, $3x+4y-7=0$ and $9x-5y+8=0$ meet in a point.

Sol. If we multiply these three equations by 3, 1 and -1, we have

$$3(2x-3y+5)+(3x+4y-7)-(9x-5y+8)=0$$

which is an identity.

Hence, three lines meet in a point.

Family of Lines

Theorem : Any line through the point of intersection of the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ can be represented by the equation

$$(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0$$

where λ is a parameter which depends on the other property of line.

Proof : The equations of the lines are

$$a_1x+b_1y+c_1=0 \qquad \text{...(i)}$$

and $$a_2x+b_2y+c_2=0 \qquad \text{...(ii)}$$

Multiplying μ and ν in Eqs. (i) and (ii) and adding, we get

$$\mu(a_1x+b_1y+c_1)+\nu(a_2x+b_2y+c_2)=0$$

where μ, ν are any constants not both zero.

Dividing both sides by μ, then

$$(a_1x+b_1y+c_1)+\frac{\nu}{\mu}(a_2x+b_2y+c_2)=0$$

$$\Rightarrow (a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0 \left(\text{where, } \lambda=\frac{\nu}{\mu}\right)$$

It is a first degree equation in x and y. So it represents family of lines through the point of intersection of Eqs. (i) and (ii).

Thus, the family of straight lines through the intersection of lines

$$L_1 \equiv a_1x+b_1y+c_1=0$$

and $$L_2 \equiv a_2x+b_2y+c_2=0 \text{ is}$$

$$(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0$$

i.e. $$L_1+\lambda L_2=0$$

Corollaries :

1. The equation $L_1+\lambda L_2=0$ or $\mu L_1+\nu L_2=0$ represent a line passing through the intersection of the lines $L_1=0$ and $L_2=0$ which is a fixed point, where λ, μ, ν are constants.
2. For finding fixed point, the number of constants in family of lines are one or two. If number of constants more than two, then convert in two or one constant form.

| Example 68. Find the equation of the straight line passing through the point (2, 1) and through the point of intersection of the lines $x+2y=3$ and $2x-3y=4$.

Sol. Equation of any straight line passing through the intersection of the lines $x+2y=3$ and $2x-3y=4$ is

$$\lambda(x+2y-3)+(2x-3y-4)=0 \qquad \text{...(i)}$$

Since, it passes through the point (2, 1)

$$\therefore \quad \lambda(2+2-3)+(4-3-4)=0$$

$$\Rightarrow \quad \lambda-3=0$$

$$\therefore \quad \lambda=3$$

Now, substituting this value of λ in (i), we get

$$3(x+2y-3)+(2x-3y-4)=0$$

i.e. $$5x+3y-13=0$$

which is the equation of required line.

| Example 69. The family of lines $x(a+2b)+y(a+3b)=a+b$ passes through the point for all values of a and b. Find the point.

Sol. The given equation can be written as

$$a(x+y-1)+b(2x+3y-1)=0$$

which is equation of a line passing through the point of intersection of the lines $x+y-1=0$ and $2x+3y-1=0$. The point of intersection of these lines is $(2, -1)$. Hence the given family of lines passes through the point $(2, -1)$ for all values of a and b.

Example 70. If $3a+2b+6c=0$ the family of straight lines $ax+by+c=0$ passes through a fixed point. Find the coordinates of fixed point.

Sol. Given, $3a+2b+6c=0$

or $$\frac{a}{2}+\frac{b}{3}+c=0 \quad \text{...(i)}$$

and family of straight lines is

$$ax+by+c=0 \quad \text{...(ii)}$$

Subtracting Eqs. (i) from (ii), then

$$a\left(x-\frac{1}{2}\right)+b\left(y-\frac{1}{3}\right)=0$$

which is equation of a line passing through the point of intersection of the lines

$$x-\frac{1}{2}=0 \quad \text{and} \quad y-\frac{1}{3}=0$$

$\therefore$ The coordinates of fixed point are $\left(\frac{1}{2},\frac{1}{3}\right)$.

Example 71. If $4a^2+9b^2-c^2+12ab=0$, then the family of straight lines $ax+by+c=0$ is either concurrent at ... or at ....

Sol. Given, $4a^2+9b^2-c^2+12ab=0$

or $$(2a+3b)^2-c^2=0$$

or $$c=\pm(2a+3b) \quad \text{...(i)}$$

and family of straight lines is

$$ax+by+c=0 \quad \text{...(ii)}$$

Substituting the value of c from Eqs. (i) in (ii), then

$$ax+by\pm(2a+3b)=0$$

$$\Rightarrow \quad a(x\pm2)+b(y\pm3)=0$$

Taking '+' sign : $a(x+2)+b(y+3)=0$

which is equation of a line passing through the point of intersection of the lines $x+2=0$ and $y+3=0$

$\therefore$ coordinates of fixed point are $(-2,-3)$.

Taking '–' sign : $a(x-2)+b(y-3)=0$

which is equation of a line passing through the point of intersection of the lines

$$x-2=0 \quad \text{and} \quad y-3=0$$

$\therefore$ coordinates of fixed point are $(2,3)$

Hence, the family of straight lines $ax+by+c=0$ is either concurrent at $(-2,-3)$ or at $(2,3)$.

Example 72. Find the equation of the line passing through the point of intersection of the lines $x+5y+7=0$, $3x+2y-5=0$ and (a) parallel to the line $7x+2y-5=0$

(b) perpendicular to the line $7x+2y-5=0$

Sol. Any line passing through the point of intersection of the given lines is

$$(x+5y+7)+\lambda(3x+2y-5)=0$$

$$\Rightarrow \quad x(1+3\lambda)+y(5+2\lambda)+(7-5\lambda)=0 \quad \text{...(i)}$$

Its slope $=-\dfrac{(1+3\lambda)}{(5+2\lambda)}$

(a) Line Eq. (i) is to be parallel to $7x+2y-5=0$

then $$-\frac{(1+3\lambda)}{(5+2\lambda)}=-\frac{7}{2}$$

$$\Rightarrow \quad 2+6\lambda=35+14\lambda$$

$$\Rightarrow \quad 8\lambda=-33$$

$$\Rightarrow \quad \lambda=-\frac{33}{8}$$

Substituting this value of λ in Eq. (i), we get the required equation as $7x+2y-17=0$

(b) Line, (i) is to be perpendicular to $7x+2y-5=0$

$$\therefore \quad -\frac{(1+3\lambda)}{(5+2\lambda)}\times-\left(\frac{7}{2}\right)=-1$$

or $$7+21\lambda=-10-4\lambda$$

$$\therefore \quad \lambda=-\frac{17}{25}$$

Substituting this value of λ is Eq. (i), we get the required equation as

$$2x-7y-20=0.$$

Aliter :

The point of intersection of the given lines

$x+5y-7=0$ and $3x+2y-5=0$ is $(3,-2)$.

$\therefore$ Equation of line through $(3,-2)$ is

$$y+2=m(x-3) \quad \text{...(ii)}$$

(a) Line (ii) is parallel to $7x+2y-5=0$

$$\therefore \quad m=-\frac{7}{2}$$

Hence, the equation of the required line is

$$y+2=-\frac{7}{2}(x-3)$$

or $$7x+2y-17=0$$

(b) Line (ii) is perpendicular to $7x+2y-5=0$

then $$m\times\left(-\frac{7}{2}\right)=-1$$

or $$m=\frac{2}{7}$$

Hence, the equation of the required line is

$$y+2=\frac{2}{7}(x-3)$$

or $$2x-7y-20=0$$

Example 73. Find the equation of straight line which passes through the intersection of the straight lines

$$3x - 4y + 1 = 0 \quad and \quad 5x + y - 1 = 0$$

and cuts off equal intercepts from the axes.

Solution : Equation of any line passing through the intersection of the given lines is

$$(3x - 4y + 1) + \lambda (5x + y - 1) = 0$$

$$\Rightarrow \quad x(3 + 5\lambda) + y(-4 + \lambda) + (1 - \lambda) = 0 \qquad \text{...(i)}$$

$$\Rightarrow \quad \frac{x}{\left(\frac{\lambda - 1}{3 + 5\lambda}\right)} + \frac{y}{\left(\frac{\lambda - 1}{\lambda - 4}\right)} = 1$$

but given x-intercept $= y$-intercept

i.e. $$\left(\frac{\lambda - 1}{3 + 5\lambda}\right) = \left(\frac{\lambda - 1}{\lambda - 4}\right)$$

$$\Rightarrow \quad \frac{1}{3 + 5\lambda} = \frac{1}{\lambda - 4}$$

($\lambda \neq 1 \because$ if $\lambda = 1$ then line (i) pass through origin)

$$\therefore \quad \lambda - 4 = 3 + 5\lambda$$

or $$4\lambda = -7$$

$$\therefore \quad \lambda = -\frac{7}{4}$$

Substituting the value of λ in Eq. (i), we get required equation is $23x + 23y = 11$.

Example 74. If t_1 and t_2 are roots of the equation $t^2 + \lambda t + 1 = 0$, where λ is an arbitrary constant. Then prove that the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ always passes through a fixed point. Also find that point.

Sol. $\because t_1$ and t_2 are the roots of the equation $t^2 + \lambda t + 1 = 0$

$$\therefore \quad t_1 + t_2 = -\lambda \quad \text{and} \quad t_1 t_2 = 1 \qquad \text{...(i)}$$

Equation of the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

$$\Rightarrow \quad y - 2at_1 = \frac{2}{(t_2 + t_1)}(x - at_1^2)$$

$$\Rightarrow \quad y(t_1 + t_2) - 2at_1t_2 - 2at_1^2 = 2x - 2at_1^2$$

$$\Rightarrow \quad y(t_1 + t_2) - 2at_1t_2 = 2x$$

$$\Rightarrow \quad y(-\lambda) - 2a = 2x \qquad \text{[from Eq. (i)]}$$

or $$(x + a) + \lambda\left(\frac{y}{2}\right) = 0$$

which is equation of a line passing through the point of intersection of the lines $x + a = 0$ and $\frac{y}{2} = 0$.

$\therefore$ coordinates of fixed point are $(-a, 0)$.

Example 75. A variable straight line through the point of intersection of the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ meets the coordinate axes in A and B. Show that the locus of the mid-point of AB is the curve $2xy(a + b) = ab(x + y)$.

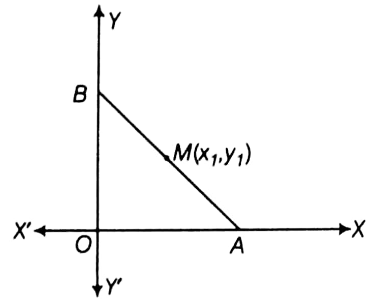
Sol. Any line through the point of intersection of given lines is

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right) + \lambda\left(\frac{x}{b} + \frac{y}{a} - 1\right) = 0$$

$$x\left(\frac{1}{a} + \frac{\lambda}{b}\right) + y\left(\frac{1}{b} + \frac{\lambda}{a}\right) = (1 + \lambda)$$

$$\Rightarrow \quad x\left(\frac{b + a\lambda}{ab}\right) + y\left(\frac{a + b\lambda}{ab}\right) = (1 + \lambda)$$

$$\Rightarrow \quad \frac{x}{\left\{\frac{ab(1 + \lambda)}{b + a\lambda}\right\}} + \frac{y}{\left\{\frac{ab(1 + \lambda)}{a + b\lambda}\right\}} = 1$$



This meets the X-axis at

$$A \equiv \left(\frac{ab(1 + \lambda)}{b + a\lambda}, 0\right)$$

and meets the Y-axis at

$$B \equiv \left(0, \frac{ab(1 + \lambda)}{a + b\lambda}\right)$$

Let the mid-point of AB is $M(x_1, y_1)$, then

$$x_1 = \frac{ab(1 + \lambda)}{2(b + a\lambda)} \quad \text{and} \quad y_1 = \frac{ab(1 + \lambda)}{2(a + b\lambda)}$$

$$\therefore \quad \frac{1}{x_1} + \frac{1}{y_1} = \frac{2(b + a\lambda)}{ab(1 + \lambda)} + \frac{2(a + b\lambda)}{ab(1 + \lambda)}$$

$$= \frac{2}{ab(1 + \lambda)}(b + a\lambda + a + b\lambda)$$

$$= \frac{2}{ab(1 + \lambda)}(a + b)(1 + \lambda)$$

$$\Rightarrow \quad \frac{(x_1 + y_1)}{x_1 y_1} = \frac{2(a + b)}{ab}$$

$$\Rightarrow \quad 2x_1y_1(a + b) = ab(x_1 + y_1)$$

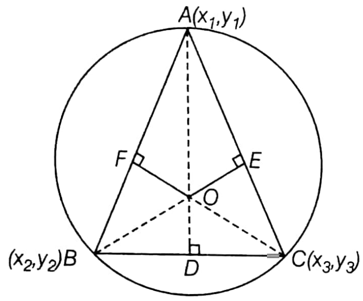
Hence, the locus of mid point of AB is

$$2xy(a + b) = ab(x + y).$$

How to Find Circumcentre and Orthocentre by Slopes

(i) Circumcentre from mid point

The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle which passes through the vertices of the triangle and so its distance from the vertices of the triangle is the same and this distance is known as the **circumradius** of the triangle.



Let $O(x, y)$ be the circumcentre.

If D, E and F are the mid points of BC, CA and AB respectively and $OD \perp BC, OE \perp CA$ and $OF \perp AB$

$\therefore$ slope of $OD \times$ slope of $BC = -1$

and slope of $OE \times$ slope of $CA = -1$

and slope of $OF \times$ slope of $AB = -1$

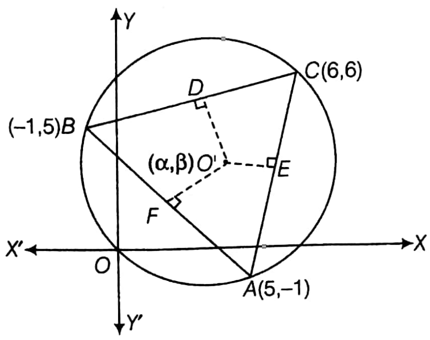
Solving any two, we get (x, y).

Example 76. Find the coordinates of the circumcentre of the triangle whose vertices are $A(5, -1), B(-1, 5)$ and $C(6, 6)$. Find its radius also.

Sol. Let circumcentre be $O'(\alpha, \beta)$ and mid points of sides BC, CA and AB are $D\left(\frac{5}{2}, \frac{11}{2}\right)$, $E\left(\frac{11}{2}, \frac{5}{2}\right)$ and $F(2, 2)$ respectively. Since $O'D \perp BC$.

$\therefore$ Slope of $O'D \times$ slope of $BC = -1$

$$\Rightarrow \quad \frac{\beta - \frac{11}{2}}{\alpha - \frac{5}{2}} \times \frac{6-5}{6+1} = -1$$



$$\Rightarrow \quad \frac{2\beta - 11}{7(2\alpha - 5)} = -1$$

$$\Rightarrow \quad 2\beta - 11 = -14\alpha + 35$$

$$\Rightarrow \quad 14\alpha + 2\beta = 46$$

$$\therefore \quad 7\alpha + \beta = 23 \quad \text{...(i)}$$

and $\quad O'E \perp CA$

$\therefore$ Slope of $O'E \times$ Slope of $CA = -1$

$$\Rightarrow \quad \frac{\beta - \frac{5}{2}}{\alpha - \frac{11}{2}} \times \frac{-1-6}{5-6} = -1$$

$$\Rightarrow \quad \frac{2\beta - 5}{2\alpha - 11} \times \frac{7}{1} = -1$$

$$\Rightarrow \quad 14\beta - 35 = -2\alpha + 11$$

$$\therefore \quad 2\alpha + 14\beta = 46$$

$$\therefore \quad \alpha + 7\beta = 23 \quad \text{... (ii)}$$

Solving Eqs. (i) and (ii), we get

$$\alpha = \beta = \frac{23}{8}$$

$$\therefore \text{ Circumcentre} = \left(\frac{23}{8}, \frac{23}{8}\right)$$

$$\therefore \text{ Circumradius} = O'A = O'B = O'C$$

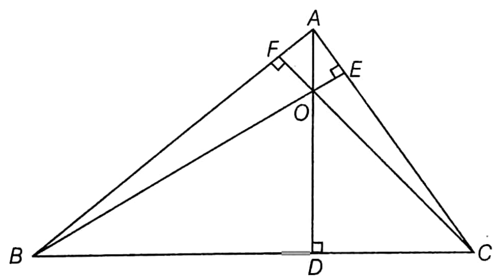
$$= O'C = \sqrt{(\alpha - 6)^2 + (\beta - 6)^2}$$

$$= \sqrt{\left(\frac{23}{8} - 6\right)^2 + \left(\frac{23}{8} - 6\right)^2}$$

$$= \sqrt{\left(\frac{25}{8}\right)^2 + \left(\frac{25}{8}\right)^2} = \frac{25\sqrt{2}}{8} \text{ units.}$$

(ii) Orthocentre from vertex.

The orthocentre of a triangle is the point of intersection of altitudes.



Here O is the orthocentre since $AD \perp BC, BE \perp CA$ and $CF \perp AB$, then $OA \perp BC, OB \perp CA$, and $OC \perp AB$

Solving any two we can get coordinates of O.

Remarks

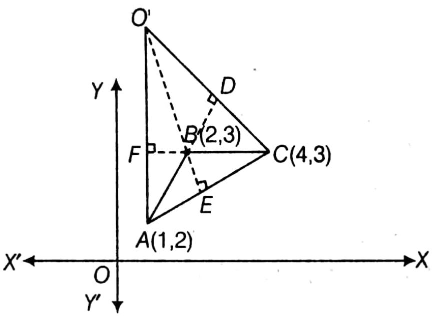
1. If any two lines out of three lines i.e. AB, BC and CA are perpendicular, then orthocentre is the point of intersection of two perpendicular lines.
2. Firstly find the slope of lines BC, CA and AB.

Example 77. Find the orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$.

Sol. Three sides of the triangle are $x = 0$, $y = 0$ and $x + y = 1$. The coordinates of the vertices are $O(0, 0)$, $A(1, 0)$ and $B(0, 1)$. The triangle OAB is a right angled triangle having right angle at O. Therefore $O(0, 0)$ is the orthocentre. Since we know that the point of intersection of two perpendicular lines is the orthocentre of the triangle OAB.

Example 78. Find the orthocentre of the triangle ABC whose angular points are $A(1, 2)$, $B(2, 3)$ and $C(4, 3)$..

Sol. Now, Slope of $BC = \dfrac{3-3}{4-2} = 0$



$$\text{Slope of } CA = \frac{2-3}{1-4} = \frac{1}{3}$$

and $\quad$ Slope of $AB = \dfrac{3-2}{2-1} = 1$

Let orthocentre be $O'(\alpha, \beta)$ then

Slope of $O'A \times$ slope of $BC = -1$

$$\frac{2-\beta}{1-\alpha} \times 0 = -1$$

$$\Rightarrow \quad \frac{0}{1-\alpha} = -1$$

$$\Rightarrow \quad 1 - \alpha = 0$$

$$\therefore \quad \alpha = 1$$

and $\quad$ Slope of $OB \times$ slope of $CA = -1$

$$\Rightarrow \quad \frac{3-\beta}{2-\alpha} \times \frac{1}{3} = -1$$

$$\Rightarrow \quad 3 - \beta = 3\alpha - 6$$

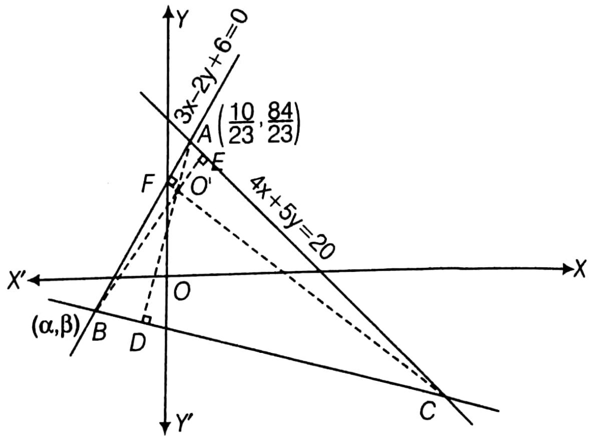
$$\Rightarrow \quad 3\alpha + \beta = 9$$

$$\therefore \quad \beta = 6 \qquad (\because \alpha = 1)$$

Hence, orthocentre of the given triangle is $(1, 6)$.

Example 79. The equations of two sides of a triangle are $3x - 2y + 6 = 0$ and $4x + 5y = 20$ and the orthocentre is $(1, 1)$. Find the equation of the third side.

Sol. Let $3x - 2y + 6 = 0$ and $4x + 5y = 20$ are the equations of the sides AB and AC. The point of intersection of AB and AC is $\left(\frac{10}{23}, \frac{84}{23}\right)$. Let slope of BC is m. Since $O'A \perp BC$



$\therefore$ Slope of $O'A \times$ Slope of $BC = -1$

$$\Rightarrow \quad \frac{\frac{84}{23}-1}{\frac{10}{23}-1} \times m = -1$$

$$\Rightarrow \quad \frac{61}{-13} m = -1$$

$$\therefore \quad m = \frac{13}{61}$$

Let the vertex B is (α, β).

(α, β) lies on $3x - 2y + 6 = 0$

$$\therefore \quad 3\alpha - 2\beta + 6 = 0 \qquad \text{...(i)}$$

and $\quad O'B \perp AC$

$\therefore$ Slope of $O'B \times$ slope of $AC = -1$

$$\frac{\beta - 1}{\alpha - 1} \times \left(-\frac{4}{5}\right) = -1$$

$$\Rightarrow \quad 4\beta - 4 = 5\alpha - 5$$

$$\Rightarrow \quad 5\alpha - 4\beta - 1 = 0 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\alpha = -13 \quad \text{and} \quad \beta = -\frac{33}{2}$$

Since, third side passes through $\left(-13, -\frac{33}{2}\right)$ with slope $\frac{13}{61}$, therefore its equation is

$$y + \frac{33}{2} = \frac{13}{61}(x + 13)$$

$$\Rightarrow \quad 122y + 33 \times 61 = 26x + 2 \times 169$$

$$\Rightarrow \quad 26x - 122y - 1675 = 0$$

Aliter : The equation of line through A. i.e. point of intersection of AB and AC is

$$(3x - 2y + 6) + \lambda(4x + 5y - 20) = 0 \qquad \text{...(i)}$$

it passes through $(1, 1)$, then

$$(3 - 2 + 6) + \lambda(4 + 5 - 20) = 0$$

$$\Rightarrow \quad 7 - 11\lambda = 0$$

$$\therefore \quad \lambda = \frac{7}{11}$$

From Eq. (i), $(3x - 2y + 6) + \frac{7}{11}(4x + 5y - 20) = 0$

$\Rightarrow \quad 61x + 13y - 74 = 0$

$\therefore \quad$ Slope of $AD = -\frac{61}{13}$

$\Rightarrow \quad$ Slope of $BC = \frac{13}{61}$

If coordinates of $B(\alpha, \beta)$, B lies on AB

$\therefore \quad 3\alpha - 2\beta + 6 = 0 \quad$ ...(ii)

and $\quad O'B \perp CA$

then $\quad \frac{\beta - 1}{\alpha - 1} \times \left(-\frac{4}{5}\right) = -1$

$\Rightarrow \quad 5\alpha - 4\beta - 1 = 0 \quad$ ...(iii)

Solving Eqs. (ii) and (iii), we get

$$\alpha = -13 \quad \text{and} \quad \beta = -\frac{33}{2}$$

$\therefore$ Equation of third side i.e. BC is

$$y + \frac{33}{2} = \frac{13}{61}(x + 13)$$

$\therefore \quad 26x - 122y - 1675 = 0$

Example 80. If the orthocentre of the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 1 = 0$, $ax + by - 1 = 0$ is at origin, then find (a, b).

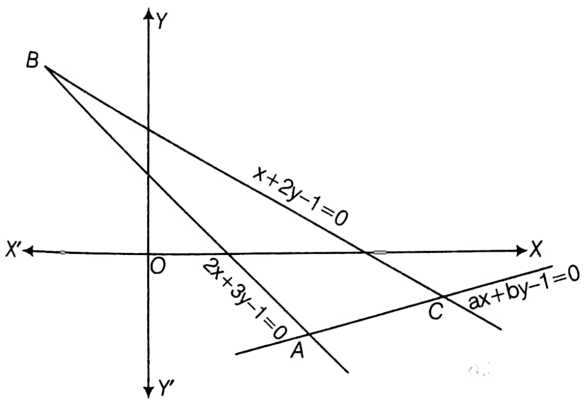
Sol. The equation of a line through A i.e. the point of intersection of AB and AC, is

$$(2x + 3y - 1) + \lambda(ax + by - 1) = 0 \quad \text{...(i)}$$

It passes through $O(0, 0)$, then

$$-1 - \lambda = 0$$

$\therefore \quad \lambda = -1$



From Eq. (i),

$$2x + 3y - 1 - ax - by + 1 = 0$$

$\Rightarrow \quad (2 - a)x + (3 - b)y = 0$

Since, $AD \perp BC$

$\therefore \quad -\frac{(2 - a)}{(3 - b)} \times \left(-\frac{1}{2}\right) = -1$

$\Rightarrow \quad 2 - a = -6 + 2b$

$\Rightarrow \quad a + 2b = 8 \quad$ ...(ii)

Similarly, $BE \perp AC$, we get

$$a + b = 0 \quad \text{...(iii)}$$

Solving Eqs. (ii) and (iii), we get

$$b = 8 \quad \text{and} \quad a = -8$$

$\therefore \quad (a, b)$ is $(-8, 8)$.

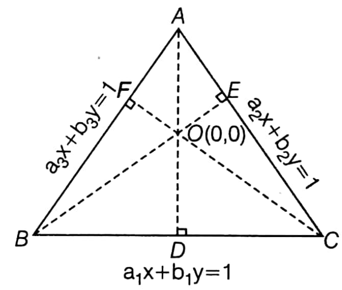
Example 81. If the equations of the sides of a triangle are $a_r x + b_r y = 1$; $r = 1, 2, 3$ and the orthocentre is the origin, then prove that

$$a_1 a_2 + b_1 b_2 = a_2 a_3 + b_2 b_3 = a_3 a_1 + b_3 b_1.$$

Sol. The equation of the line through A, i.e. the point of intersection of AB and AC is

$$(a_2 x + b_2 y - 1) + \lambda(a_3 x + b_3 y - 1) = 0 \quad \text{...(i)}$$

It passes through $(0, 0)$, then



$$-1 - \lambda = 0$$

$\therefore \quad \lambda = -1$

From Eq. (i), $a_2 x + b_2 y - 1 - a_3 x - b_3 y + 1 = 0$

$\therefore \quad (a_2 - a_3)x + (b_2 - b_3)y = 0$

Since, $AD \perp BC$

$\therefore$ Slope of $AD \times$ slope of $BC = -1$

$$-\frac{(a_2 - a_3)}{(b_2 - b_3)} \times \left(-\frac{a_1}{b_1}\right) = -1$$

$\Rightarrow \quad a_1 a_2 - a_3 a_1 = -b_1 b_2 + b_1 b_3$

$\Rightarrow \quad a_1 a_2 + b_1 b_2 = a_3 a_1 + b_3 b_1 \quad$ ...(ii)

Similarly, $BE \perp CA$, then we get

$$a_1 a_2 + b_1 b_2 = a_2 a_3 + b_2 b_3 \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$a_1 a_2 + b_1 b_2 = a_2 a_3 + b_2 b_3 = a_3 a_1 + b_3 b_1$$

Exercise for Session 3

1. The locus of the point of intersection of lines $x \cos\alpha + y \sin\alpha = a$ and $x \sin\alpha - y \cos\alpha = b$ (α is a parameter) is

(a) $2(x^2 + y^2) = a^2 + b^2$
(b) $x^2 - y^2 = a^2 - b^2$
(c) $x^2 + y^2 = a^2 + b^2$
(d) $x^2 - y^2 = a^2 + b^2$

2. If a, b, c are in AP then $ax + by + c = 0$ represents

(a) a straight line
(b) a family of concurrent lines
(c) a family of parallel lines
(d) None of these

3. If the lines $x + 2ay + a = 0$, $x + 3by + b = 0$ and $x + 4cy + c = 0$ are concurrent, then a, b, c are in

(a) AP
(b) GP
(c) HP
(d) AGP

4. The set of lines $ax + by + c = 0$, where $3a + 2b + 4c = 0$ is concurrent at the point

(a) $\left(\frac{3}{4}, \frac{1}{2}\right)$
(b) $\left(\frac{1}{2}, \frac{3}{4}\right)$
(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$
(d) $\left(-\frac{1}{2}, -\frac{3}{4}\right)$

5. If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$, (a, b and c being distinct and different from 1) are concurrent, then the value of $\frac{a}{a-1} + \frac{b}{b-1} + \frac{c}{c-1}$ is

(a) -2
(b) -1
(c) 1
(d) 2

6. If $u \equiv a_1x + b_1y + c_1 = 0$ and $v \equiv a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then $u + kv = 0$ represents

(a) $u = 0$
(b) a family of concurrent lines
(c) a family of parallel lines
(d) None of these

7. The straight lines $x + 2y - 9 = 0$, $3x + 5y - 5 = 0$ and $ax + by - 1 = 0$ are concurrent, if the straight line $35x - 22y + 1 = 0$ passes through the point

(a) (a, b)
(b) (b, a)
(c) $(a, -b)$
(d) $(-a, b)$

8. If the straight lines $x + y - 2 = 0$, $2x - y + 1 = 0$ and $ax + by - c = 0$ are concurrent, then the family of lines $2ax + 3by + c = 0$ (a, b, c are non-zero) is concurrent at

(a) $(2, 3)$
(b) $\left(\frac{1}{2}, \frac{1}{3}\right)$
(c) $\left(-\frac{1}{6}, -\frac{5}{9}\right)$
(d) $\left(\frac{2}{3}, -\frac{7}{5}\right)$

9. The straight line through the point of intersection of $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel to Y-axis has the equation

(a) $x(ab' - a'b) + (cb' - c'b) = 0$
(b) $x(ab' + a'b) + (cb' + c'b) = 0$
(c) $y(ab' - a'b) + (c'a - ca') = 0$
(d) $y(ab' + a'b) + (c'a + ca') = 0$

10. If the equations of three sides of a triangle are $x + y = 1$, $3x + 5y = 2$ and $x - y = 0$, then the orthocentre of the triangle lies on the line/lines

(a) $5x - 3y = 1$
(b) $5y - 3x = 1$
(c) $2x - 3y = 1$
(d) $5x - 3y = 2$

11. Find the equation of the line through the intersection of $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and perpendicular to $6x - 7y + c = 0$

(a) $199y + 120x = 125$
(b) $199y - 120x = 125$
(c) $119x + 102y = 125$
(d) $119x - 102y = 125$

12. The locus of the point of intersection of the lines $\frac{x}{a} - \frac{y}{b} = m, \frac{x}{a} + \frac{y}{b} = \frac{1}{m}$ is

(a) a circle
(b) an ellipse
(c) a hyperbola
(d) a parabola

13. Find the condition on a and b, such that the portion of the line $ax + by - 1 = 0$, intercepted between the lines $ax + y + 1 = 0$ and $x + by = 0$ subtends a right angled at the origin.

14. If the lines $(a - b - c)x + 2ay + 2a = 0$, $2bx + (b - c - a)y + 2b = 0$ and $(2c + 1)x + 2cy + c - a - b = 0$ are concurrent, then prove that either $a + b + c = 0$ or $(a + b + c)^2 + 2a = 0$.

15. Prove that the lines $ax + by + c = 0, bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent if $a + b + c = 0$ or $a + b\omega + c\omega^2 = 0$ or $a + b\omega^2 + c\omega = 0$, where ω is a complex cube root of unity.

16. Find the equation of the straight line which passes through the intersection of the lines $x - y - 1 = 0$ and $2x - 3y + 1 = 0$ and is parallel to (i) X-axis (ii) Y-axis (iii) $3x + 4y = 14$.

17. Let a, b, c be parameters. Then, the equation $ax + by + c = 0$ will represent a family of straight lines passing through a fixed-point, if there exists a linear relation between a, b and c.

18. Prove that the family of lines represented by $x(1 + \lambda) + y(2 - \lambda) + 5 = 0, \lambda$ being arbitrary, pass through a fixed point. Also find the fixed point.

19. Prove that $\left(-a, -\frac{a}{2}\right)$ is the orthocentre of the triangle formed by the lines $y = m_i x + \frac{a}{m_i}, i = 1, 2, 3; m_1, m_2, m_3$ being the roots of the equation $x^3 - 3x^2 + 2 = 0$.

Session 4

Equations of Straight Lines Passing Through a Given Point and Making a Given Angle with a Given Line, A Line Equally Inclined With Two Lines, Equation of the Bisectors, Bisector of the Angle Containing the Origin, Equation of that Bisector of the Angle Between Two Lines which Contains a Given Point, How to Distinguish the Acute (Internal) and Obtuse (External) Angle Bisectors

Equations of Straight Lines Passing Through a Given Point and Making a Given Angle with a Given Line

Theorem : Prove that the equations of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are

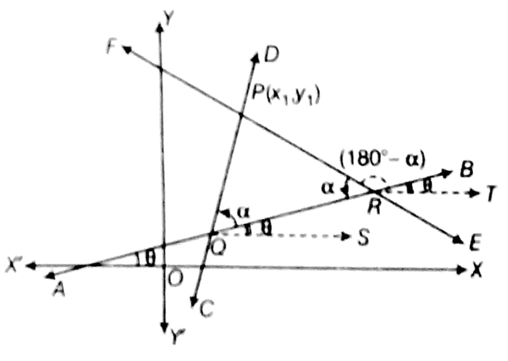
$$y - y_1 = \tan(\theta \pm \alpha)(x - x_1)$$

where, $\quad m = \tan\theta.$

Proof : Let AB be the given line which makes an angle θ with X-axis.

$\therefore \quad m = \tan\theta$

Let CD and EF are two required lines which make angle α with the given line. Let these lines meet the given line AB at Q and R respectively



$\therefore \quad \angle DQS = \angle PQR + \angle RQS = (\alpha + \theta)$

$\therefore$ Equation of line CD is

$$y - y_1 = \tan(\theta + \alpha)(x - x_1) \quad \text{...(i)}$$

and $\quad \angle FRT = \angle FRB + \angle BRT$

$$= 180^\circ - \alpha + \theta$$

$$= 180^\circ + (\theta - \alpha)$$

$\therefore$ Equation of line EF is

$$y - y_1 = \tan(180^\circ + \theta - \alpha)(x - x_1)$$

or $\quad y - y_1 = \tan(\theta - \alpha)(x - x_1) \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$y - y_1 = \tan(\theta \pm \alpha)(x - x_1)$$

These are the equations of the two required lines.

| Example 82. Find the equations of the straight lines passing through the point (2, 3) and inclined at $\pi/4$ radians to the line $2x + 3y = 5$.

Sol. Let the line $2x + 3y = 5$ make an angle θ with positive X-axis.

Then $\quad \tan\theta = -\frac{2}{3}$

Now $\quad \tan\theta \cdot \tan\frac{\pi}{4} = -\frac{2}{3} \times 1$

$$= -\frac{2}{3} \neq \pm 1$$

Slopes of required lines are

$$\tan\left(\theta + \frac{\pi}{4}\right) \text{ and } \tan\left(\theta - \frac{\pi}{4}\right)$$

$$\therefore \quad \tan\left(\theta+\frac{\pi}{4}\right)=\frac{\tan\theta+\tan\left(\frac{\pi}{4}\right)}{1-\tan\theta\tan\left(\frac{\pi}{4}\right)}=\frac{\left(-\frac{2}{3}\right)+1}{1-\left(-\frac{2}{3}\right)(1)}=\frac{1}{5}$$

$$\text{and} \quad \tan\left(\theta-\frac{\pi}{4}\right)=\frac{\tan\theta-\tan\left(\frac{\pi}{4}\right)}{1+\tan\theta\tan\left(\frac{\pi}{4}\right)}$$

$$=\frac{\left(-\frac{2}{3}\right)-1}{1+\left(-\frac{2}{3}\right)(1)}=-5$$

$\therefore$ Equations of required lines are

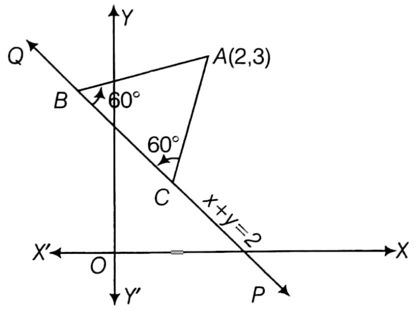
$$y-3=\frac{1}{5}(x-2) \quad \text{and} \quad y-3=-5(x-2)$$

i.e. $\quad x-5y+13=0 \quad$ and $\quad 5x+y-13=0$

| Example 83. A vertex of an equilateral triangle is $(2,3)$ and the opposite side is $x+y=2$. Find the equations of the other sides.

Sol. Let $A(2,3)$ be one vertex and $x+y=2$ be the opposite side of an equilateral triangle. Clearly remaining two sides pass through the point $A(2,3)$ and make an angle 60° with $x+y=2$

$\because$ Slope of $x+y=2$ is -1



Let $\quad \tan\theta=-1$

$\therefore \quad \theta=135°$

$\therefore$ Equations of the other two sides are

$$y-3=\tan(135°\pm60°)(x-2)$$

i.e. sides are

$y-3=\tan(195°)(x-2)$ (taking '+' sign)

$\Rightarrow \quad y-3=\tan(180°+15°)(x-2)$

$\Rightarrow \quad y-3=\tan 15°(x-2)$

$\Rightarrow \quad y-3=(2-\sqrt{3})(x-2)$

$\Rightarrow \quad (2-\sqrt{3})x-y=1-2\sqrt{3}$

and $\quad y-3=\tan(75°)(x-2)$

(taking '–' sign)

$\Rightarrow \quad y-3=\cot 15°(x-2)$

$\Rightarrow \quad y-3=(2+\sqrt{3})(x-2)$

$\Rightarrow \quad (2+\sqrt{3})x-y=1+2\sqrt{3}$

Hence, equations of other sides are

$$(2-\sqrt{3})x-y=1-2\sqrt{3}$$

and $\quad (2+\sqrt{3})x-y=1+2\sqrt{3}$

| Example 84. The straight lines $3x+4y=5$ and $4x-3y=15$ intersect at a point A. On these lines, the points B and C are chosen so that $AB=AC$. Find the possible equations of the line BC passing through the point $(1,2)$.

By pg. 120 method.

Sol. Clearly $\angle BAC=90°$

$\because \quad AB=AC$

$\therefore \quad \angle ABC=\angle BCA=45°$

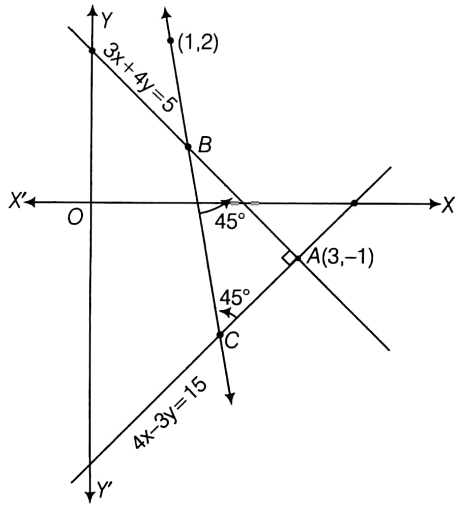
$\alpha=45°$

$\because$ Slope of $3x+4y=5$ is $-\frac{3}{4}$

Let $\tan\theta=-\frac{3}{4}$

So, possible equations of BC are given by

$$y-2=\tan(\theta\pm\alpha)(x-1)$$



$$\Rightarrow \quad y-2=\left(\frac{\tan\theta\pm\tan\alpha}{1\mp\tan\theta\tan\alpha}\right)(x-1)$$

$$\Rightarrow \quad y-2=\left(\frac{-\frac{3}{4}\pm1}{1\mp\left(-\frac{3}{4}\right)(1)}\right)(x-1)$$

$$\Rightarrow \quad y-2=\left(\frac{-3\pm4}{4\mp(-3)}\right)(x-1)$$

$$\Rightarrow \quad y-2=\frac{1}{4-(-3)}(x-1) \qquad \text{(taking upper sign)}$$

or $\quad x-7y+13=0$

and $\quad y - 2 = \dfrac{(-3-4)}{(4+(-3))}(x-1)$ (taking below sign)

or $\quad 7x + y - 9 = 0$

Hence, possible equation of the line BC are $x - 7y + 13 = 0$ and $7x + y - 9 = 0$

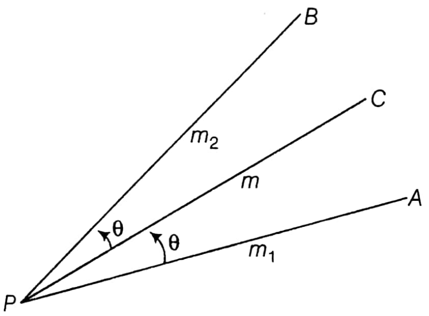
A Line Equally Inclined with Two Lines

Theorem : If two lines with slopes m_1 and m_2 be equally inclined to a line with slope m, then

$$\left(\frac{m_1 - m}{1 + mm_1}\right) = -\left(\frac{m_2 - m}{1 + mm_2}\right)$$

Proof : Let be two lines of slopes m_1 and m_2 intersecting at a point P.

Let $\angle CPA = \angle BPC = \theta$



$\therefore \quad \tan(\angle CPA) = \left(\dfrac{m - m_1}{1 + mm_1}\right)$

or $\quad \tan\theta = \left(\dfrac{m - m_1}{1 + mm_1}\right) \quad (\because m > m_1)$ ...(i)

and $\quad \tan(\angle BPC) = \left(\dfrac{m_2 - m}{1 + m_2 m}\right)$

or $\quad \tan\theta = \left(\dfrac{m_2 - m}{1 + m_2 m}\right) \quad (\because m_2 > m)$ ... (ii)

From Eqs. (i) and (ii), we get

$$\left(\frac{m - m_1}{1 + mm_1}\right) = \left(\frac{m_2 - m}{1 + m_2 m}\right) \text{ or } \left(\frac{m_1 - m}{1 + mm_1}\right) = -\left(\frac{m_2 - m}{1 + mm_2}\right)$$

Remarks

1. The above equation gives two values of m which are the slopes of the lines parallel to the bisectors of the angles between the two given lines.
2. Sign of m in both brackets is same.

Example 85. Find the equations to the straight lines passing through the point (2, 3) and equally inclined to the lines $3x - 4y - 7 = 0$ and $12x - 5y + 6 = 0$.

Sol. Let m be the slope of the required line. Then its equation is

$$y - 3 = m(x - 2) \quad \text{...(i)}$$

It is given that line (i) is equally inclined to the lines $3x - 4y - 7 = 0$ and $12x - 5y + 6 = 0$ then

$$\Rightarrow \left(\frac{\frac{3}{4} - m}{1 + \frac{3}{4}m}\right) = -\left(\frac{\frac{12}{5} - m}{1 + \frac{12}{5}m}\right)$$

$\left(\begin{array}{l}\text{slope of } 3x - 4y - 7 = 0 \text{ is } \frac{3}{4} \\ \text{and slope of } 12x - 5y + 6 = 0 \text{ is } \frac{12}{5}\end{array}\right)$

$$\Rightarrow \left(\frac{3 - 4m}{4 + 3m}\right) = -\left(\frac{12 - 5m}{5 + 12m}\right)$$

$$\Rightarrow (3 - 4m)(5 + 12m) + (4 + 3m)(12 - 5m) = 0$$

$$\Rightarrow 63m^2 - 32m - 63 = 0$$

$$\Rightarrow (7m - 9)(9m + 7) = 0$$

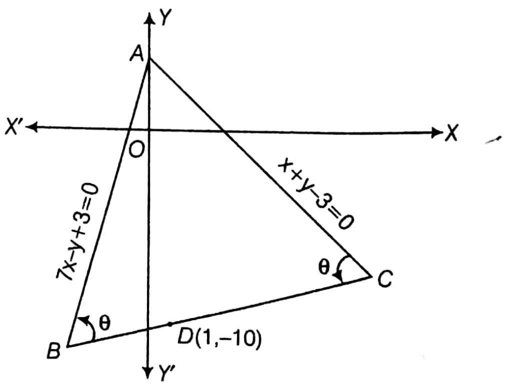
$$\therefore m = \frac{9}{7}, -\frac{7}{9}$$

Putting these values of m in Eq. (i) we obtain the equations of required lines as $9x - 7y + 3 = 0$ and $x + 9y - 41 = 0$.

Example 86. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point $(1, -10)$. Determine the equation of the third side.

Sol. Let m be the slope of BC. Since $AB = AC$.

Therefore BC makes equal angles with AB and AC.



Then $\left(\dfrac{7 - m}{1 + 7m}\right) = -\left(\dfrac{-1 - m}{1 + (-1)m}\right)$

$$\Rightarrow (7 - m)(1 - m) - (1 + 7m)(1 + m) = 0$$

$$\Rightarrow 6m^2 + 16m - 6 = 0$$

$$\Rightarrow 3m^2 + 8m - 3 = 0$$

$$\Rightarrow \quad (3m-1)(m+3)=0$$

$$\Rightarrow \quad m=\frac{1}{3},-3$$

Equation of third side BC is $y+10=m(x-1)$

i.e. $y+10=\frac{1}{3}(x-1)$ and $y+10=-3(x-1)$

or $x-3y-31=0$ and $3x+y+7=0$

Equation of the Bisectors

Theorem : Prove that the equation of the bisectors of the angles between the lines

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0$$

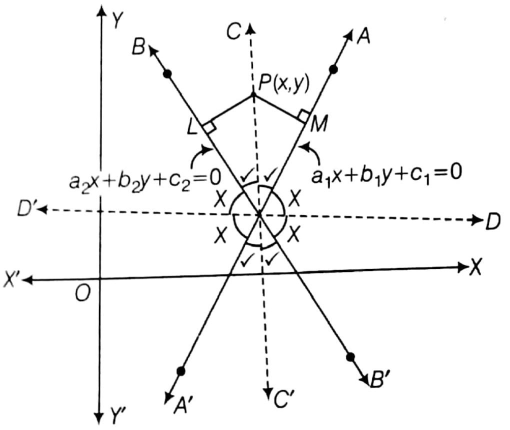
are given by $$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

Proof : Let the given lines be AA' and BB' whose equations are

$$a_1x+b_1y+c_1=0 \quad \text{...(i)}$$

and $$a_2x+b_2y+c_2=0 \quad \text{...(ii)}$$

Since bisectors of the angles between the two lines are the locus of a point which moves in a plane such that whose distance from two lines are equal.



Let CC' and DD' be the two bisectors of the angle between the lines AA' and BB'. Let $P(x,y)$ be any point on CC', then

Length of the perpendicular from P on AA'

= length of the perpendicular from P on BB'

$$\therefore \quad \frac{|a_1x+b_1y+c_1|}{\sqrt{(a_1^2+b_1^2)}}=\frac{|a_2x+b_2y+c_2|}{\sqrt{(a_2^2+b_2^2)}}$$

or $$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

These are the required equations of the bisectors.

Note

The two bisectors are perpendicular to each other.

| Example 87. Find the equations of the bisectors of the angles between the straight lines $3x-4y+7=0$ and $12x+5y-2=0$.

Sol. The equations of the bisectors of the angles between $3x-4y+7=0$ and $12x+5y-2=0$ are

$$\frac{(3x-4y+7)}{\sqrt{(3)^2+(-4)^2}}=\pm\frac{(12x+5y-2)}{\sqrt{(12)^2+(5)^2}}$$

or $$\frac{(3x-4y+7)}{5}=\pm\frac{(12x+5y-2)}{13}$$

or $$(39x-52y+91)=\pm(60x+25y-10)$$

Taking the positive sign, we get

$$21x+77y-101=0$$

as one bisector.

Taking the negative sign, we get $99x-77y+81=0$ as the second bisector.

Bisector of the Angle Containing the Origin

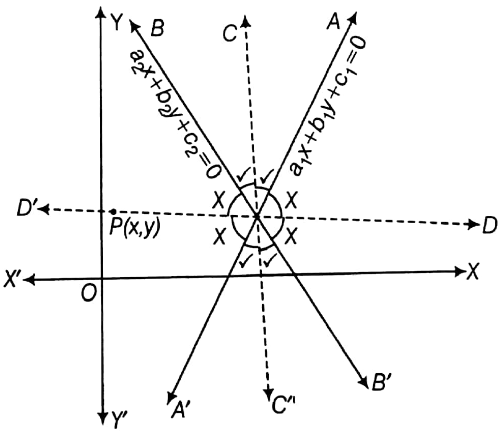
Let equations of lines be

$$a_1x+b_1y+c_1=0 \quad \text{...(i)}$$

$$a_2x+b_2y+c_2=0 \quad \text{...(ii)}$$

where c_1 and c_2 are positive.

Let $P(x,y)$ be taken on the bisector of the angle which contains the origin.



(i) Let $P(x,y)$ lies on DD' ,then either $O(0,0)$ and $P(x,y)$ will lie on the same side of the two lines Eqs. (i) and (ii), then

$$\frac{a_1x+b_1y+c_1}{0+0+c_1}>0$$

and $$\frac{a_2x+b_2y+c_2}{0+0+c_2}>0$$

or $$a_1x+b_1y+c_1>0$$

and $$a_2x+b_2y+c_2>0 \quad (\because c_1,c_2>0)$$

If the origin $O(0,0)$ and $P(x,y)$ lie on the opposite side of the two lines Eqs. (i) and (ii), then

$$\frac{a_1x+b_1y+c_1}{0+0+c_1}<0 \text{ and } \frac{a_2x+b_2y+c_2}{0+0+c_2}<0$$

or $\quad a_1x+b_1y+c_1<0$ and $a_2x+b_2y+c_2<0$ $\quad (\because c_1, c_2>0)$

Then equation of bisectors will be

$$\frac{|a_1x+b_1y+c_1|}{\sqrt{(a_1^2+b_1^2)}}=\frac{|a_2x+b_2y+c_2|}{\sqrt{(a_2^2+b_2^2)}}$$

Case I : If $a_1x+b_1y+c_1>0$ and $a_2x+b_2y+c_2>0$

then $\quad \dfrac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\dfrac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$

Case II : If $a_1x+b_1y+c_1<0$ and $a_2x+b_2y+c_2<0$

then $\quad -\dfrac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=-\dfrac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$

i.e. $\quad \dfrac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\dfrac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$

Thus is both cases equation of the bisector containing the origin, when c_1 and c_2 are positive is

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

and equation of the bisector of the angle between the lines

$$a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0$$

which does not contain the origin when c_1 and c_2 are positive is

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=-\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

Working Rule :

(i) First re-write the equations of the two lines so that their constant terms are positive.

(ii) The bisector of the angle containing the origin and does not containing the origin, then taking +ve and − ve sign in

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}} \text{ respectively.}$$

Example 88. Find the equations of angular bisector bisecting the angle containing the origin and not containing the origin of the lines $4x+3y-6=0$ and $5x+12y+9=0$.

Sol. Firstly make the constant terms (c_1, c_2) positive, then

$$-4x-3y+6=0 \text{ and } 5x+12y+9=0$$

$\therefore$ The equation of the bisector bisecting the angle containing origin is

$$\frac{(-4x-3y+6)}{\sqrt{(-4^2)+(-3)^2}}=\frac{(5x+12y+9)}{\sqrt{(5)^2+(12)^2}}$$

$\Rightarrow \quad \left(\dfrac{-4x-3y+6}{5}\right)=\left(\dfrac{5x+12y+9}{13}\right)$

$\Rightarrow \quad -52x-39y+78=25x+60y+45$

$\Rightarrow \quad 77x+99y-33=0$ or $7x+9y-3=0$

and the equation of the bisector bisecting the angle not containing origin is

$$\frac{(-4x-3y+6)}{\sqrt{(-4)^2+(-3)^2)}}=-\frac{(5x+12y+9)}{\sqrt{(5^2)+(12)^2}}$$

$\Rightarrow \quad \left(\dfrac{-4x-3y+6}{5}\right)=-\left(\dfrac{5x+12y+9}{13}\right)$

$\Rightarrow \quad -52x-39y+78=-25x-60y-45$

$\Rightarrow \quad 27x-21y-123=0$ or $9x-7y-41=0$

Equation of that Bisector of the Angle between Two Lines which Contains a Given Point

Let the equations of the two lines be

$$a_1x+b_1y+c_1=0 \quad \text{...(i)}$$

and $$a_2x+b_2y+c_2=0 \quad \text{...(ii)}$$

The equation of the bisector of the angle between the two lines containing the points (h, k) will be

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

or $$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=-\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

according as $a_1h+b_1k+c_1$ and $a_2h+b_2k+c_2$ are of the same sign or opposite sign.

Example 89. Find the bisector of the angle between the lines $2x+y-6=0$ and $2x-4y+7=0$ which contains the point (1, 2).

Sol. Value of $2x+y-6$ at (1, 2) is − 2 (negative)

and value of $2x-4y+7$ at (1, 2) is 1 (positive)

i.e. opposite sign.

$\therefore$ Equation of bisector containing the point (1, 2) is

$$\frac{(2x+y-6)}{\sqrt{(2^2+1^2)}}=-\frac{(2x-4y+7)}{\sqrt{(2)^2+(-4)^2}}$$

$\Rightarrow \quad 2(2x+y-6)+(2x-4y+7)=0$

or $\quad 6x-2y-5=0$

How to Distinguish the Acute (Internal) and Obtuse (External) Angle Bisectors?

Let the equations of the two lines be

$$a_1x+b_1y+c_1=0 \quad \text{...(i)}$$

and

$$a_2x+b_2y+c_2=0 \quad \text{...(ii)}$$

where, $c_1>0, c_2>0$.

$\because$ Equations of bisectors are

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}} \quad \text{...(iii)}$$

when Eq. (iii) be simplified, let the bisectors be

$$p_1x+q_1y+r_1=0 \quad \text{...(iv)}$$

and

$$p_2x+q_2y+r_2=0 \quad \text{...(v)}$$

Since the two bisectors are at right angles, the angle α between the acute (internal) bisector and any one of the given lines must lie between 0 and 45° i.e. $0<\alpha<45°$.

$\therefore \quad 0<\tan\alpha<1$

If m_1 and m_2 are the slopes of Eqs. (i) and (iii) respectively.

Then, $\quad m_1=-\dfrac{a_1}{b_1}$ and $m_2=-\dfrac{p_1}{q_1}$

$$\therefore \quad \tan\alpha=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$=\left|\frac{\left(-\frac{a_1}{b_1}\right)-\left(-\frac{p_1}{q_1}\right)}{1+\left(-\frac{a_1}{b_1}\right)\left(-\frac{p_1}{q_1}\right)}\right|=\left|\frac{a_1q_1-b_1p_1}{b_1q_1+a_1p_1}\right|$$

Hence, if $\boxed{0<\tan\alpha<1}$, $p_1x+q_1y+r_1=0$ is the acute (internal) bisector and if $\boxed{\tan\alpha>1}$, $p_2x+q_2y+r_2=0$ is the obtuse (external) bisector.

Shortcut Method for finding Acute (Internal) and Obtuse (External) Angle Bisectors

Let the equations of the two lines be

$$a_1x+b_1y+c_1=0$$

$$a_2x+b_2y+c_2=0$$

Taking $\quad c_1>0, c_2>0$ and $a_1b_2\neq a_2b_1$

Then equations of the bisectors are

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

Conditions	Acute angle bisector	Obtuse angle bisector
$a_1a_2+b_1b_2>0$	–	+
$a_1a_2+b_1b_2<0$	+	–

Remarks

1. Bisectors are perpendiculars to each other.
2. '+' sign gives the bisector of the angle containing origin.
3. If $a_1a_2+b_1b_2>0$ then the origin lies in obtuse angle and if $a_1a_2+b_1b_2<0$, then the origin lies in acute angle.

Explanation : Equations of given lines in normal form will be respectively

$$-\frac{a_1x}{\sqrt{(a_1^2+b_1^2)}}-\frac{b_1y}{\sqrt{(a_1^2+b_1^2)}}=\frac{c_1}{\sqrt{(a_1^2+b_1^2)}} \quad (\because c_1>0)$$

and

$$-\frac{a_2x}{\sqrt{(a_2^2+b_2^2)}}-\frac{b_2y}{\sqrt{(a_2^2+b_2^2)}}=\frac{c_2}{\sqrt{(a_2^2+b_2^2)}} \quad (\because c_2>0)$$

If $\quad \cos\alpha=-\dfrac{a_1}{\sqrt{(a_1^2+b_1^2)}}$ then $\sin\alpha=-\dfrac{b_1}{\sqrt{(a_1^2+b_1^2)}}$

and $\quad \cos\beta=-\dfrac{a_2}{\sqrt{(a_2^2+b_2^2)}}$ then $\sin\beta=-\dfrac{b_2}{\sqrt{(a_2^2+b_2^2)}}$

Now, $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$$=\frac{(a_1a_2+b_1b_2)}{\sqrt{(a_1^2+b_1^2)}\sqrt{(a_2^2+b_2^2)}}$$

$$\cos(\alpha-\beta)>0 \quad \text{or} \quad <0$$

according as $(\alpha-\beta)$ is acute or obtuse.

i.e. $\quad a_1a_2+b_1b_2>0$ or <0

Hence, bisector of the angle between the lines will be the bisector of the acute or obtuse angle according as origin lies in the acute or obtuse angle according as $a_1a_2+b_1b_2<0$ or >0.

| Example 90. Find the equation of the bisector of the obtuse angle between the lines $3x-4y+7=0$ and $12x+5y-2=0$.

Sol. Firstly make the constant terms (c_1, c_2) positive

$$3x-4y+7=0 \quad \text{and} \quad -12x-5y+2=0$$

$\because \quad a_1a_2+b_1b_2=(3)(-12)+(-4)(-5)=-36+20=-16$

$\therefore \quad a_1a_2+b_1b_2<0$

Hence "–" sign gives the obtuse bisector.

$\therefore$ Obtuse bisector is

$$\frac{(3x-4y+7)}{\sqrt{(3)^2+(-4)^2}}=-\frac{(-12x-5y+2)}{\sqrt{(-12)^2+(-5)^2}}$$

$\Rightarrow \quad 13(3x-4y+7)=-5(-12x-5y+2)$

$\Rightarrow \quad 21x+77y-101=0$ is the obtuse angle bisector.

Example 91. Find the bisector of acute angle between the lines $x+y-3=0$ and $7x-y+5=0$.

Sol. Firstly, make the constant terms (c_1, c_2) positive then

$-x-y+3=0$ and $7x-y+5=0$

$\because \quad a_1a_2+b_1b_2=(-1)(7)+(-1)(-1)=-7+1=-6$

i.e. $\quad a_1a_2+b_1b_2<0$

Hence "+" sign gives the acute bisector.

$\therefore$ Acute bisector is $\dfrac{-x-y+3}{\sqrt{(-1)^2+(-1)^2}}=+\dfrac{7x-y+5}{\sqrt{(7)^2+(-1)^2}}$

$$\Rightarrow \quad \frac{-x-y+3}{\sqrt{2}}=\frac{7x-y+5}{5\sqrt{2}}$$

$$\Rightarrow \quad -5x-5y+15=7x-y+5$$

$\therefore \quad 12x+4y-10=0$ or $6x+2y-5=0$

is the acute angle bisector.

Example 92. Find the coordinates of incentre of the triangle. The equation of whose sides are

$AB: x+y-1=0, BC: 7x-y-15=0$

and $CA: x-y-1=0$.

Sol. Firstly, make the constant terms $(c_1, c_2,$ and $c_3)$ positive

i.e. $\quad AB: -x-y+1=0 \quad$ ...(i)

$BC: -7x+y+15=0 \quad$ ...(ii)

$CA: -x+y+1=0 \quad$ ...(iii)

$\because$ The incentre of triangle is the point of intersection of internal or acute angle bisectors.

Internal bisector of AB and BC:

$$-x-y+1=0$$
$$-7x+y+15=0$$

$\because \quad a_1a_2+b_1b_2=(-1)(-7)+(-1)(1)=6>0$

$\therefore$ Acute or internal bisector is

$$\frac{(-x-y+1)}{\sqrt{(-1)^2+(-1)^2}}=-\frac{(-7x+y+15)}{\sqrt{(-7)^2+(1)^2}}$$

$$\Rightarrow \quad \frac{(-x-y+1)}{\sqrt{2}}=-\frac{(-7x+y+15)}{5\sqrt{2}}$$

$$\Rightarrow \quad -5x-5y+5=7x-y-15$$

or $\quad 12x+4y-20=0$

or $\quad 3x+y-5=0 \quad$ ...(iv)

Internal bisector of BC and CA:

$$-7x+y+15=0$$
$$-x+y+1=0$$

$\because \quad a_1a_2+b_1b_2=(-7)(-1)+(1)(1)=8>0$

$\because$ Acute or internal bisector is

$$\frac{(-7x+y+15)}{\sqrt{(-7)^2+(1)^2}}=-\frac{(-x+y+1)}{\sqrt{(-1)^2+(1)^2}}$$

$$\Rightarrow \quad \frac{-7x+y+15}{5\sqrt{2}}=\frac{(x-y-1)}{\sqrt{2}}$$

$$\Rightarrow \quad -7x+y+15=5x-5y-5$$

or $\quad 12x-6y-20=0$

or $\quad 6x-3y-10=0 \quad$ ...(v)

Finally, solve Eqs. (iv) and (v), we get

$$x=\frac{5}{3} \text{ and } y=0$$

Hence coordinates of incentre are $\left(\dfrac{5}{3}, 0\right)$.

Exercise for Session 4

1. The straight lines $2x+11y-5=0$, $24x+7y-20=0$ and $4x-3y-2=0$
   (a) form a triangle
   (b) are only concurrent
   (c) are concurrent with one line bisecting the angle between the other two
   (d) None of the above

2. The line $x+3y-2=0$ bisects the angle between a pair of straight lines of which one has the equation $x-7y+5=0$. The equation of other line is
   (a) $3x+3y-1=0$ (b) $x-3y+2=0$ (c) $5x+5y+3=0$ (d) $5x+5y-3=0$

3. P is a point on either of the two lines $y-\sqrt{3}|x|=2$ at a distance of 5 units from their point of intersection. The coordinates of the foot of the perpendicular from P on the bisector of the angle between them are
   (a) $\left(0, \dfrac{4+5\sqrt{3}}{2}\right)$ or $\left(0, \dfrac{4-5\sqrt{3}}{2}\right)$ depending on which the point P is taken
   (b) $\left(0, \dfrac{4+5\sqrt{3}}{2}\right)$ (c) $\left(0, \dfrac{4-5\sqrt{3}}{2}\right)$ (d) $\left(\dfrac{5}{2}, \dfrac{5\sqrt{3}}{2}\right)$

4. In a triangle ABC, the bisectors of angles B and C lie along the lines $x = y$ and $y = 0$. If A is (1, 2), then the equation of line BC is

(a) $2x + y = 1$ (b) $3x - y = 5$ (c) $x - 2y = 3$ (d) $x + 3y = 1$

5. In ΔABC, the coordinates of the vertex A are (4, –1) and lines $x - y - 1 = 0$ and $2x - y = 3$ are the internal bisecters of angles B and C. Then, the radius of the incircle of triangle ABC is

(a) $\frac{5}{\sqrt{5}}$ (b) $\frac{3}{\sqrt{5}}$ (c) $\frac{6}{\sqrt{5}}$ (d) $\frac{7}{\sqrt{5}}$

6. The equation of the straight line which bisects the intercepts made by the axes on the lines $x + y = 2$ and $2x + 3y = 6$ is

(a) $2x = 3$ (b) $y = 1$ (c) $2y = 3$ (d) $x = 1$

7. The equation of the bisector of the acute angle between the lines $2x - y + 4 = 0$ and $x - 2y = 1$ is

(a) $x + y + 5 = 0$ (b) $x - y + 1 = 0$ (c) $x - y = 5$ (d) $x - y + 5 = 0$

8. The equation of the bisector of that angle between the lines $x + y = 3$ and $2x - y = 2$ which contains the point (1, 1) is

(a) $(\sqrt{5} - 2\sqrt{2})x + (\sqrt{5} + \sqrt{2})y = 3\sqrt{5} - 2\sqrt{2}$ (b) $(\sqrt{5} + 2\sqrt{2})x + (\sqrt{5} - \sqrt{2})y = 3\sqrt{5} + 2\sqrt{2}$

(c) $3x = 10$ (d) $3x - 5y + 2 = 0$

9. Find the equations of the two straight lines through (7, 9) and making an angle of 60° with the line $x - \sqrt{3}y - 2\sqrt{3} = 0$.

10. Equation of the base of an equilateral triangle is $3x + 4y = 9$ and its vertex is at the point (1, 2). Find the equations of the other sides and the length of each side of the triangle.

11. Find the coordinates of those points on the line $3x + 2y = 5$ which are equidistant from the lines $4x + 3y - 7 = 0$ and $2y - 5 = 0$.

12. Two sides of rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonal of the rhombus intersect at the point (1, 2) and the vertex A lies on Y-axis, find the possible coordinates of A.

13. The bisector of two lines L_1 and L_2 are given by $3x^2 - 8xy - 3y^2 + 10x + 20y - 25 = 0$. If the line L_1 passes through origin, find the equation of line L_2.

14. Find the equation of the bisector of the angle between the lines $x + 2y - 11 = 0$ and $3x - 6y - 5 = 0$ which contains the point (1, –3).

15. Find the equation of the bisector of the angle between the lines $2x - 3y - 5 = 0$ and $6x - 4y + 7 = 0$ which is the supplement of the angle containing the point (2, –1).

Session 5

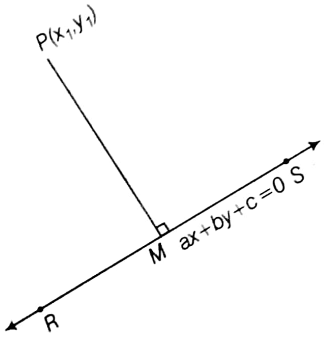
The Foot of Perpendicular Drawn from the Point (x_1, y_1) to the Line $ax + by + c = 0$, Image or Reflection of a Point (x_1, y_1) about a Line Mirror, Image or Reflection of a Point In Different Cases, Use of Image or Reflection

The Foot of Perpendicular Drawn from the Point (x_1, y_1) to the Line $ax + by + c = 0$

Let $P \equiv (x_1, y_1)$ and let M be the foot of perpendicular drawn from P on $ax + by + c = 0$.

In order to find the coordinates of M, find the equation of the line PM which is perpendicular to RS and passes through $P(x_1, y_1)$, i.e. $bx - ay = bx_1 - ay_1$

or $b(x - x_1) - a(y - y_1) = 0$ and solving it with $ax + by + c = 0$, then we get coordinates of M.



Aliter I : Let the coordinates of M are (x_2, y_2) then $M(x_2, y_2)$ lies on $ax + by + c = 0$

$$\Rightarrow \quad ax_2 + by_2 + c = 0 \quad \text{...(i)}$$

and $\because PM \perp RS$

then (Slope of PM) (Slope of RS) $= -1$

$$\Rightarrow \quad \left(\frac{y_2 - y_1}{x_2 - x_1}\right) \times \left(-\frac{a}{b}\right) = -1$$

$$\text{or} \quad bx_2 - ay_2 = bx_1 - ay_1 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get (x_2, y_2).

Aliter II : Let the coordinates of M are (x_2, y_2)

$$\because \quad PM \perp RS$$

and M lies on $\quad ax + by + c = 0$

$$\text{i.e.} \quad ax_2 + by_2 + c = 0 \quad \text{...(iii)}$$

$$\text{and} \quad \left(\frac{y_2 - y_1}{x_2 - x_1}\right) \times \left(-\frac{a}{b}\right) = -1 \quad (\because PM \perp RS)$$

$$\text{or} \quad \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b}$$

$$\text{or} \quad \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{a(x_2 - x_1) + b(y_2 - y_1)}{(a^2 + b^2)}$$

(By ratio proportion method)

$$= \frac{(ax_2 + by_2) - (ax_1 + by_1)}{a^2 + b^2}$$

$$= \frac{-c - (ax_1 + by_1)}{a^2 + b^2} \quad \text{[from Eq. (iii)]}$$

$$\text{or} \quad \boxed{\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{(a^2 + b^2)}}$$

| Example 93. Find the coordinates of the foot of the perpendicular drawn from the point (2, 3) to the line $y = 3x + 4$.

Sol. Given line is

$$3x - y + 4 = 0 \quad \text{...(i)}$$

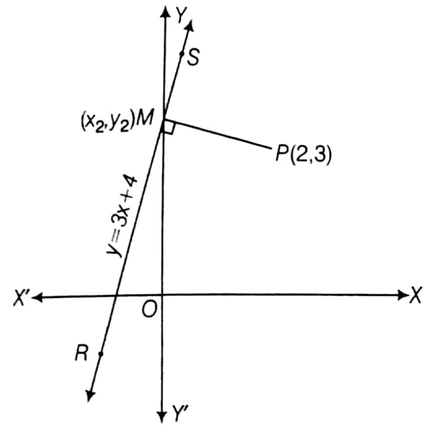
Let Eq. $\quad P \equiv (2, 3)$.

Let, M be the foot of perpendicular drawn from P on RS.

Then equation of PM passes through $P(2, 3)$ and perpendicular to RS is

$$x + 3y - (2 + 3 \times 3) = 0$$

$$\text{i.e.} \quad x + 3y - 11 = 0 \quad \text{...(ii)}$$



Solving Eqs. (i) and (ii), we get

$$x = -\frac{1}{10}, y = \frac{37}{10}$$

$\therefore$ $$M \equiv \left(-\frac{1}{10}, \frac{37}{10}\right)$$

Aliter I : Let the coordinates of M be (x_2, y_2) then $M(x_2, y_2)$ lies on $3x - y + 4 = 0$

$\Rightarrow$ $$3x_2 - y_2 + 4 = 0 \qquad \text{...(iii)}$$

and $\because PM \perp RS$

$\therefore$ (Slope of PM) $\times$ (Slope of RS) = – 1

$\Rightarrow$ $$\left(\frac{y_2 - 3}{x_2 - 2}\right) \times (3) = -1$$

or $$x_2 + 3y_2 - 11 = 0 \qquad \text{...(iv)}$$

Solving Eqs. (iii) and (iv), we get

$$x_2 = -\frac{1}{10}, y_2 = \frac{37}{10}$$

$\therefore$ $$M \equiv \left(-\frac{1}{10}, \frac{37}{10}\right)$$

Aliter II : By Ratio Proportion Method :

$$\frac{x_2 - 2}{3} = \frac{y_2 - 3}{-1} = \frac{-(3 \times 2 - 3 + 4)}{3^2 + (-1)^2}$$

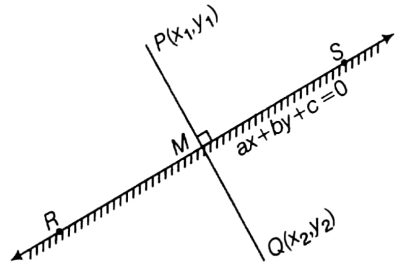
$\Rightarrow$ $$\frac{x_2 - 2}{3} = \frac{y_2 - 3}{-1} = -\frac{7}{10}$$

$$x_2 = -\frac{1}{10} \text{ and } y_2 = \frac{37}{10}$$

$\therefore$ $$M \equiv \left(-\frac{1}{10}, \frac{37}{10}\right)$$

Image or Reflection of a Point (x_1, y_1) About a Line Mirror

Let $Q \equiv (x_2, y_2)$ be the image of $P \equiv (x_1, y_1)$ then find coordinates of the foot of perpendicular M drawn from the point $P(x_1, y_1)$ on RS and use fact that M is the mid- point P and Q.



i.e. $$M \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Aliter I : Ratio Proportion Method :

$\because$ $$PQ \perp RS$$

$\therefore$ (Slope of PQ) $\times$ (Slope of RS) = – 1

or $$\left(\frac{y_2 - y_1}{x_2 - x_1}\right) \times \left(-\frac{a}{b}\right) = -1$$

or $$\frac{(x_2 - x_1)}{a} = \frac{(y_2 - y_1)}{b}$$

or $$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{a(x_2 - x_1) + b(y_2 - y_1)}{a^2 + b^2}$$

$$= \frac{a(2x - x_1 - x_1) + b(2y - y_1 - y_1)}{a^2 + b^2}$$

$$\left(\begin{array}{l}\because M(x, y) \text{ is mid-point of } P \text{ and } Q \\ \therefore x_2 = 2x - x_1 \text{ and } y_2 = 2y - y_1\end{array}\right)$$

$$= \frac{-2ax_1 - 2by_1 + 2(ax + by)}{a^2 + b^2}$$

$$= \frac{-2ax_1 - 2by_1 + 2(-c)}{a^2 + b^2} \quad (\because ax + by = -c)$$

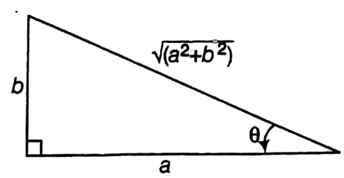
$$= \frac{-2(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

i.e. $$\boxed{\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{(a^2 + b^2)}}$$

Aliter II : By Distance form or Symmetric form or parametric form :

$\because$ $$\text{Slope of } RS = -\frac{a}{b}$$

$\therefore$ $$\text{Slope of } PQ = \frac{b}{a}$$



Let $\tan\theta = \dfrac{b}{a}$

$\therefore$ $\sin\theta = \dfrac{b}{\sqrt{(a^2+b^2)}}$

and $\cos\theta = \dfrac{a}{\sqrt{(a^2+b^2)}}$

Put the equation of the mirror line such that the coefficient of y becomes negative.

Suppose if $b > 0$

then $ax + by + c = 0$

becomes $-ax - by - c = 0$

and $p = PM =$ Directed distance from $P(x_1, y_1)$ on $-ax - by - c = 0$ (i.e. p +ve or −ve)

$$= \left(\frac{-ax_1 - by_1 - c}{\sqrt{(a^2+b^2)}}\right)$$

$\therefore$ $PQ = 2PM = 2p = 2\left(\dfrac{-ax_1 - by_1 - c}{\sqrt{(a^2+b^2)}}\right) = r$

$\Rightarrow$ Required image has the coordinates $(x_1 + r\cos\theta, y_1 + r\sin\theta)$.

Example 94. Find the image of the point $(4, -13)$ with respect to the line mirror $5x + y + 6 = 0$.

Sol. Let, $P \equiv (4, -13)$ and Let, $Q \equiv (x_2, y_2)$ be mirror image P with respect to line mirror $5x + y + 6 = 0$.

Let, $M(\alpha, \beta)$ be the foot of perpendicular from $P(4, -13)$ on the line mirror $5x + y + 6 = 0$, then

$$\frac{\alpha - 4}{5} = \frac{\beta + 13}{1} = \frac{-(5 \times 4 - 13 + 6)}{5^2 + 1^2}$$

or $\dfrac{\alpha - 4}{5} = \dfrac{\beta + 13}{1} = -\dfrac{1}{2}$

$\therefore$ $M \equiv \left(\dfrac{3}{2}, -\dfrac{27}{2}\right)$

$\because$ M is the mid-point of P and Q, then

$$Q \equiv (x_2, y_2) \equiv \left(2 \times \frac{3}{2} - 4, 2 \times \left(-\frac{27}{2}\right) + 13\right)$$

i.e. $Q \equiv (-1, -14)$

Aliter I :

By Ratio Proportion Method : Let, $Q(x_2, y_2)$ the image of $P(4, -13)$ with respect to line mirror $5x + y + 6 = 0$, then

$$\frac{x_2 - 4}{5} = \frac{y_2 + 13}{1} = -\frac{2(5 \times 4 - 13 + 6)}{5^2 + 1^2} = -1$$

or $x_2 - 4 = -5$ and $y_2 + 13 = -1$

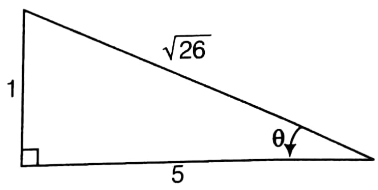
$\therefore$ $x_2 = -1$ and $y_2 = -14$

Hence $Q \equiv (-1, -14)$.

Aliter II :

By distance form or Symmetric form or Parametric form : Let, $P \equiv (4, -13)$ and Q be the image of P with respect to line mirror (RS) $5x + y + 6 = 0$

$\because$ Slope of $RS = -5$



$\therefore$ Slope of $PQ = \dfrac{1}{5} = \tan\theta$

$\therefore$ $\sin\theta = \dfrac{1}{\sqrt{26}}$ and $\cos\theta = \dfrac{5}{\sqrt{26}}$

Now, put the equation of the mirror line such that the coefficient of y becomes negative.

Then, $5x + y + 6 = 0$ becomes $-5x - y - 6 = 0$ and $p = \perp$ Directed distance from $P(4, -13)$ on $(-5x - y - 6 = 0)$

$$= \frac{-5 \times 4 + 13 - 6}{\sqrt{(-5)^2 + (-1)^2}} = -\frac{13}{\sqrt{26}}$$

$\therefore$ $PQ = r = 2p = -\dfrac{26}{\sqrt{26}} = -\sqrt{26}$

Hence, required image has the coordinates

$$Q \equiv (4 - \sqrt{26}\cos\theta, -13 - \sqrt{26}\sin\theta)$$

i.e. $\left(4 - \sqrt{26} \times \dfrac{5}{\sqrt{26}}, -13 - \sqrt{26} \times \dfrac{1}{\sqrt{26}}\right)$

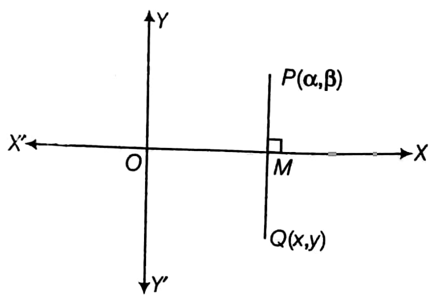
i.e. $(4 - 5, -13 - 1)$

Hence, $Q \equiv (-1, -14)$

Image or Reflection of a Point in Different Cases

(i) The image or reflection of a point with respect to X-axis

Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about X-axis, then (M is the mid-point of P and Q)



$$x = \alpha \text{ and } y = -\beta$$

$$\therefore \quad Q \equiv (\alpha, -\beta)$$

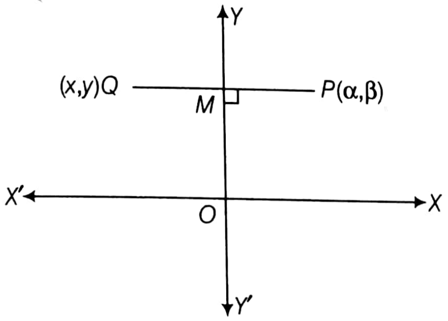
i.e. sign change of ordinate.

Remark

The image of the line $ax + by + c = 0$ about X-axis is $ax - by + c = 0$

(ii) The image or reflection of a point with respect to Y-axis

Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about Y-axis, then (M is the mid-point of PQ)



$$x = -\alpha \quad \text{and} \quad y = \beta$$

$$\therefore \quad Q \equiv (-\alpha, \beta)$$

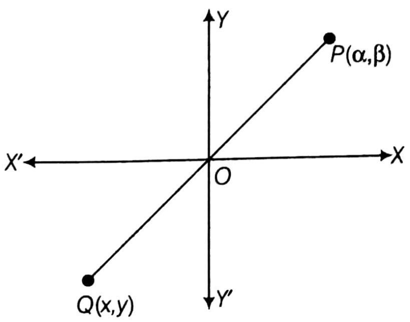
i.e. sign change of abscissae.

Remark

The image of the line $ax + by + c = 0$ about Y-axis is $-ax + by + c = 0$

(iii) The image or reflection of a point with respect to origin

Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about the origin (O is the mid point of PQ), then



$$x = -\alpha \quad \text{and} \quad y = -\beta$$

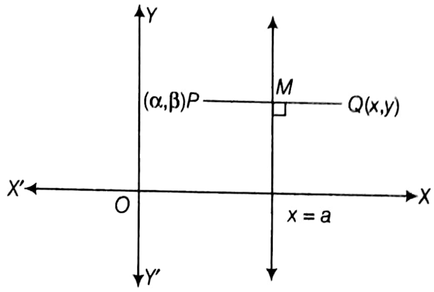
$$\therefore \quad Q \equiv (-\alpha, -\beta)$$

i.e. sign change of abscissae and ordinate.

Remark

The image of the line $ax + by + c = 0$ about origin is $-ax - by + c = 0$.

(iv) The image or reflection of a point with respect to the line $x = a$



Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about the line $x = a$, then $y = \beta$

$\therefore$ Coordinates of M are (a, β)

$\because$ M is the mid-point of PQ

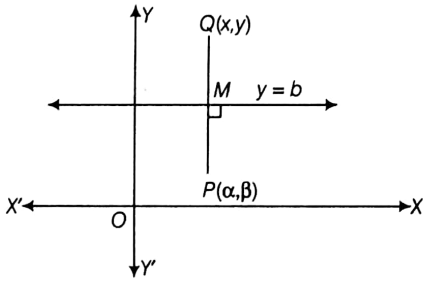
$\therefore$ $Q \equiv (2a - \alpha, \beta)$

Remark

The image of the line $ax + by + c = 0$ about the line $x = \lambda$ is $a(2\lambda - x) + by + c = 0$

(v) The image or reflection of a point with respect to the line $y = b$

Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about the line y =b, then $x = \alpha$



$\therefore$ Co-ordinates of M are (α, b)

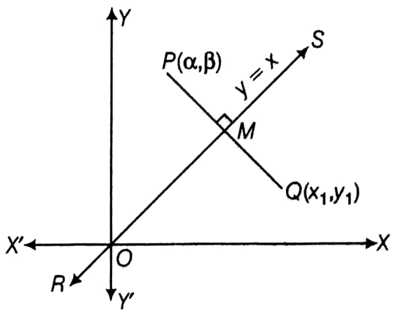
$\because$ M is the mid-point of PQ $\therefore$ $Q \equiv (\alpha, 2b - \beta)$

Remark

The image of the line $ax + by + c = 0$ about the line $y = \mu$ is $ax + b(2\mu - y) + c = 0$.

(vi) The image or reflection of a point with respect to the line $y = x$

Let $P(\alpha, \beta)$ be any point and $Q(x_1, y_1)$ be its image about the line $y = x$ (RS), then $PQ \perp RS$



$\therefore$ (Slope of PQ) $\times$ (Slope of RS) $=-1$

or $$\frac{y_1-\beta}{x_1-\alpha}\times 1=-1$$

or $$x_1-\alpha=\beta-y_1 \qquad \text{...(i)}$$

and mid--point of PQ lie on $y=x$

i.e. $$\left(\frac{y_1+\beta}{2}\right)=\left(\frac{x_1+\alpha}{2}\right)$$

or $$x_1+\alpha=\beta+y_1 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get $x_1=\beta$ and $y_1=\alpha$

$\therefore$ $Q\equiv(\beta,\alpha)$ i.e. interchange of x and y.

Remark

The image of the line $ax+by+c=0$ about the line $y=x$ is $ay+bx+c=0$.

(vii) The image or reflection of a point with respect to the line $y=x\tan\theta$

Let $P(\alpha,\beta)$ be any point and $Q(x_1,y_1)$ be its image about the line $y=x\tan\theta$ (RS), then $PQ\perp RS$

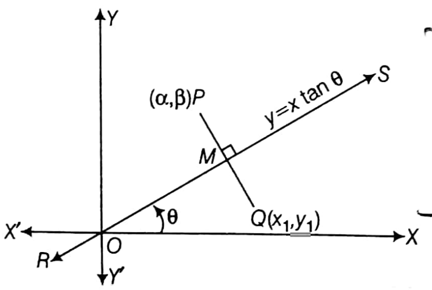
$\therefore$ (Slope of PQ) $\times$ (Slope of RS) $=-1$

or $$\frac{y_1-\beta}{x_1-\alpha}\times\tan\theta=-1$$

$\Rightarrow$ $$y_1-\beta=(\alpha-x_1)\cot\theta \qquad \text{...(i)}$$

and mid-point of PQ lie on $y=x\tan\theta$

i.e. $$\left(\frac{y_1+\beta}{2}\right)=\left(\frac{x_1+\alpha}{2}\right)\tan\theta$$



or $$y_1+\beta=(x_1+\alpha)\tan\theta \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$x_1=\alpha\cos 2\theta+\beta\sin 2\theta$$

and $$y_1=\alpha\sin 2\theta-\beta\cos 2\theta$$

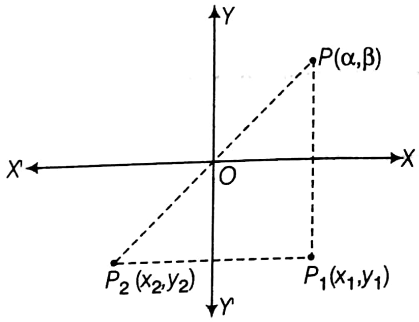
$\therefore$ $$Q\equiv(\alpha\cos 2\theta+\beta\sin 2\theta,\alpha\sin 2\theta-\beta\cos 2\theta)$$

Example 95. The point $P(\alpha,\beta)$ undergoes a reflection in the X-axis followed by a reflection in the Y-axis. Show that their combined effect is the same as the single reflection of $P(\alpha,\beta)$ in the origin when $\alpha,\beta>0$.

Sol. Let $P_1(x_1,y_1)$ be the image of (α,β) after reflection in the X-axis. Then

$$x_1=\alpha \quad \text{and} \quad y_1=-\beta \qquad \text{...(i)}$$

Now, let $P_2(x_2,y_2)$ be the image of $P_1(x_1,y_1)$ in the Y-axis. Then



$$x_2=-x_1,\ y_2=y_1$$

$\Rightarrow$ $$x_2=-\alpha,\ y_2=-\beta \qquad \text{[from Eq. (i)] ...(ii)}$$

further let $P_3(x_3,y_3)$ be the image of $P(\alpha,\beta)$ in the origin O. Then

$$x_3=-\alpha,\ y_3=-\beta \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii) ,we get

$$x_3=x_2 \text{ and } y_3=y_2.$$

Hence the image of P_2 of P after successive reflection in their X-axis and Y-axis is the same as the single reflection of P in the origin.

Example 96. Find the image of the point $(-2,-7)$ under the transformations $(x,y)\rightarrow(x-2y,-3x+y)$.

Sol. Let (x_1,y_1) be the image of the point (x,y) under the given transformation. Then

$$x_1=x-2y=(-2)-2(-7)=12$$

$\therefore$ $x_1=12$ and $y_1=-3x+y=-3(-2)-7=-1$

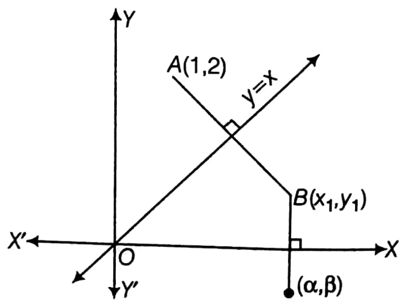
$\therefore$ $y_1=-1$

Hence, the image is $(12,-1)$.

Example 97. The image of the point $A(1,2)$ by the line mirror $y=x$ is the point B and the image of B by the line mirror $y=0$ is the point (α,β). Find α and β.

Sol. Let (x_1,y_1) be the image of the point $(1,2)$ about the line $y=x$.

Then $$x_1=2,\ y_1=1 \qquad \text{...(i)}$$



Also given image of $B(x_1,y_1)$ by the line mirror $y=0$ is (α,β). Then $\alpha=x_1=2$

and $$\beta=-y_1=-1 \qquad \text{[from Eq. (i)]}$$

Hence, $$\alpha=2 \quad \text{and} \quad \beta=-1$$

Example 98. The point (4, 1) undergoes the following three transformations successively :

(i) Reflection about the line $y = x$.

(ii) Translation through a distance 2 units along the positive direction of X-axis.

(iii) Rotation through an angle $\pi/4$ about the origin in the anticlockwise direction.

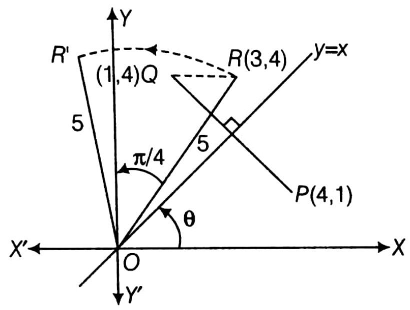
Then, find the coordinates of the final position.

Sol. Let $Q(x_1, y_1)$ be the reflection of P about the line $y = x$. Then

$$\left.\begin{aligned} x_1 &= 1 \\ y_1 &= 4 \end{aligned}\right\}$$

$\therefore$ Coordinates of Q is (1, 4).

Given that Q move 2 units along the positive direction of X-axis.



$\therefore$ Coordinates of R is $(x_1 + 2, y_1)$

or $\quad R(3, 4)$

If OR makes an angle θ , then

$$\tan\theta = \frac{4}{3}$$

$$\therefore \quad \sin\theta = \frac{4}{5} \quad \text{and} \quad \cos\theta = \frac{3}{5}$$

After rotation of $\frac{\pi}{4}$ let new position of R is R' and

$$OR = OR' = \sqrt{3^2 + 4^2} = 5$$

$\therefore \quad OR'$ makes an angle $(\pi/4 + \theta)$ with X-axis.

Coordinates of $R'\left(OR' \cos\left(\frac{\pi}{4} + \theta\right), OR' \sin\left(\frac{\pi}{4} + \theta\right)\right)$

i.e. $$R'\left(OR'\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right), OR'\sin\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\right)$$

$$\Rightarrow \quad R'\left(5\left(\frac{3}{5\sqrt{2}} - \frac{4}{5\sqrt{2}}\right), 5\left(\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}}\right)\right)$$

$$\Rightarrow \quad R'\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$$

Aliter (Use of complex number) :

Let Q be the reflection of $P(4, 1)$ about the line $y = x$, then $Q \equiv (1, 4)$

$\because Q$ move 2 units along the +ve direction of X-axis, if new point is ,R then $R \equiv (3, 4)$.

If $\quad R(3, 4) = R(z_1)$

when $\quad z_1 = (3 + 4i)$

then $\quad R'(x, y) = R'(z_2)$

$$\therefore \quad z_2 = z_1 e^{i\pi/4} \qquad \left(\because \angle ROR' = \frac{\pi}{4}\right)$$

$$= (3 + 4i)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

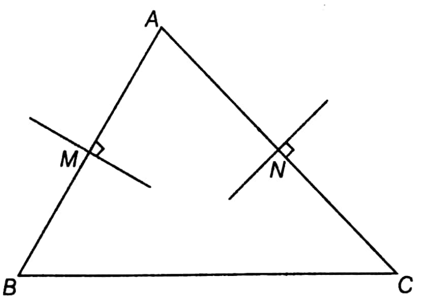
$$= (3 + 4i)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}} + \frac{7i}{\sqrt{2}}\right)$$

Hence, new coordinates are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$.

Use of Image or Reflection

To make problems simpler and easier use Image or reflection.

Types of problems : (i) If vertex of a $\Delta\, ABC$ and equations of perpendicular bisectors of AB and AC are given, then B and C are the images or reflections of A about the perpendicular bisectors of AB and AC (where M and N are the mid-points of AB and AC).



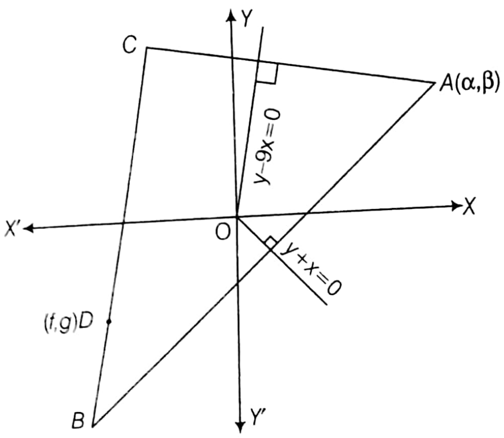
Example 99. The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles by the lines $y + x = 0$ and $y - 9x = 0$. Determine the locus of its vertex.

Sol. Let $A \equiv (\alpha, \beta)$ the image of $A(\alpha, \beta)$ about $y + x = 0$ is B, then $B \equiv (-\beta, -\alpha)$ and if image of $A(\alpha, \beta)$ about $y - 9x = 0$ is $C(x_2, y_2)$, then

$$\frac{x_2 - \alpha}{-9} = \frac{y_2 - \beta}{1} = \frac{-2(\beta - 9\alpha)}{1 + 81}$$

$$\therefore \quad x_2 = \frac{9\beta - 40\alpha}{41} \text{ and } y_2 = \frac{40\beta + 9\alpha}{41}$$

$$\therefore \quad C \equiv \left(\frac{9\beta - 40\alpha}{41}, \frac{40\beta + 9\alpha}{41}\right)$$



Hence, B, D, C are collinear, then

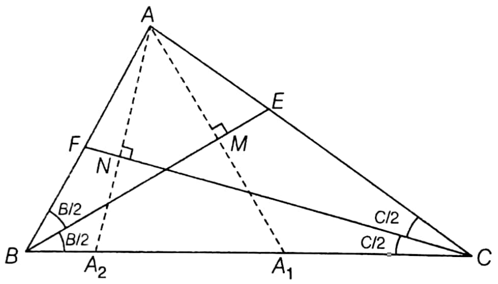
$$\frac{1}{2}\begin{vmatrix} -\beta & -\alpha & 1 \\ f & g & 1 \\ \frac{9\beta-40\alpha}{41} & \frac{40\beta+9\alpha}{41} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 4(\alpha^2+\beta^2)+(4g+5f)\alpha+(4f-5g)\beta=0$$

Hence, locus of vertex is

$$4(x^2+y^2)+(4g+5f)x+(4f-5g)y=0$$

(ii) The images or reflections of vertex A of a $\Delta\, ABC$ about the angular bisectors of angles B and C lie on the side BC. (By congruence) A_1 and A_2 are the images of A about the angle bisectors BE and CF respectively, where M and N are the mid-points of AA_1 and AA_2.



Example 100. Find the equations of the sides of the triangle having $(3,-1)$ as a vertex, $x-4y+10=0$ and $6x+10y-59=0$ being the equations of an angle bisector and a median respectively drawn from different vertices.

Sol. Let BE be the angle bisector and CF be the median. Given equations of BE and CF are $x-4y+10=0$ and $6x+10y-59=0$ respectively.

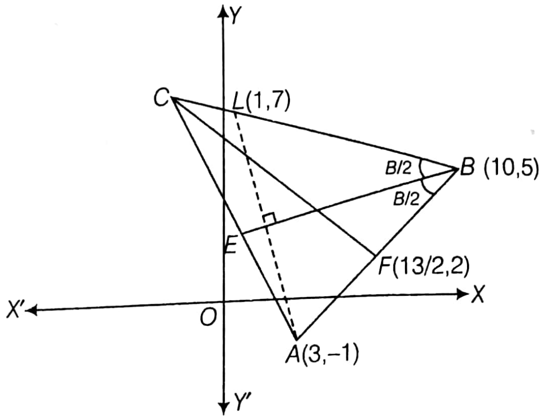
Since, image of A with respect to BE lie on BC. If image of A is $L(h,k)$.

then $$\frac{h-3}{1}=\frac{k+1}{-4}=\frac{-2(3+4+10)}{1^2+(-4)^2}=-2$$

i.e. $\quad h=1, k=7$

$\therefore \quad L\equiv(1,7)$

$\because$ F be the mid-point of AB.



Let $\quad F\equiv(\alpha,\beta)$

then $\quad B\equiv(2\alpha-3, 2\beta+1)$

$\because$ B lie on BE, then

$$(2\alpha-3)-4(2\beta+1)+10=0$$

i.e. $\quad 2\alpha-8\beta+3=0 \quad$ ...(i)

and $\quad F$ lie on CF, then

$$6\alpha+10\beta-59=0 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\alpha=\frac{13}{2}, \beta=2$$

then $\quad F\equiv\left(\frac{13}{2},2\right)$

and $\quad B=(10,5)$

Equation of AB is

$$y+1=\frac{2+1}{\frac{13}{2}-3}(x-3)$$

$$\Rightarrow \quad y+1=\frac{6}{7}(x-3)$$

or $\quad 6x-7y-25=0$

Equation of BC is

$$y-5=\frac{7-5}{1-10}(x-10)$$

$$\Rightarrow \quad y-5=-\frac{2}{9}(x-10)$$

or $\quad 2x+9y-65=0$

$\because$ CA is the family of lines of CB and CF

then $\quad (2x+9y-65)+\lambda(6x+10y-59)=0 \quad$ ...(iii)

it pass through $A(3,-1)$

then $\quad (6-9-65)+\lambda(18-10-59)=0$

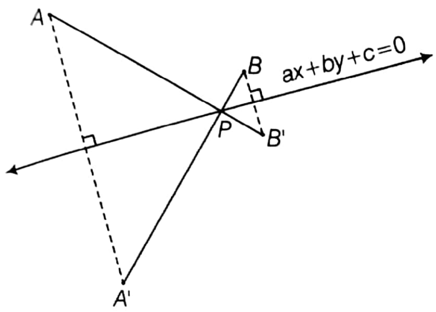
$$\therefore \quad \lambda=-\frac{4}{3}$$

From Eq. (iii), we get equation of AC is

$$18x+13y-41=0$$

(iii) Optimization (Minimization or Maximization)

(a) **Minimization :** Let A and B are two given points on the same side of $ax+by+c=0$. Suppose we want to determine a point P on $ax+by+c=0$ such that $PA+PB$ is minimum. Then find the image of A or B about the line $ax+by+c=0$ (say A' or B') then join B' with A or A' with B wherever it intersects $ax+by+c=0$ is the required point.



$$\therefore \quad PA+PB = PA+PB'$$

or $$PA+PB = PA'+PB$$

Remark

By triangle in equality

Sum of two sides of a triangle > Third side

i.e. $|PA+PB| = |PA+PB'| = |AB'|$ (minimum value).

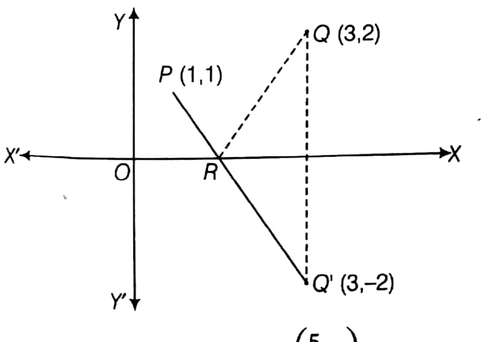
| Example 101. Find a point R on the X-axis such that $PR+RQ$ is the minimum, when $P\equiv(1,1)$ and $Q\equiv(3,2)$.

Sol. Since P and Q lie on the same side of X-axis.

The image of Q (3, 2) about X-axis is Q' (3, – 2) then the equation of line PQ' is

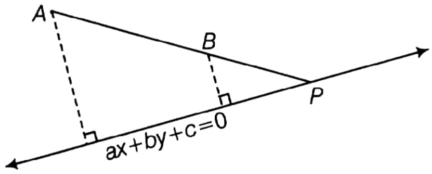
$$y-1 = \frac{-2-1}{3-1}(x-1)$$

$$\Rightarrow \quad 3x+2y-5=0$$



This line meets X-axis at $R\left(\frac{5}{3}, 0\right)$ which is the required point.

(b) **Maximization :** Let A and B are two given points on the same side of $ax+by+c=0$. Suppose we want to determine a point P on $ax+by+c=0$ such that $|PA-PB|$ is maximum, then find the equation of line AB wherever it intersects $ax+by+c=0$ is the required point.



Remark

By triangle inequality

Difference of two sides of a triangle < Third side

i.e. $\boxed{|PA-PB| = |AB| \text{ (maximum value)}}$

| Example 102. Find a point P on the line $3x+2y+10=0$ such that $|PA-PB|$ is maximum where A is (4, 2) and B is (2, 4).

Sol. Let, $\quad L(x, y) = 3x+2y+10$

$\therefore \quad L(4, 2) = 12+4+10 = 26$

and $\quad L(2, 4) = 6+8+10 = 24$

$\therefore$ A and B lie on the same side of the line

$$3x+2y+10=0 \qquad \text{...(i)}$$

Equation of line AB is

$$y-2 = \frac{4-2}{2-4}(x-4)$$

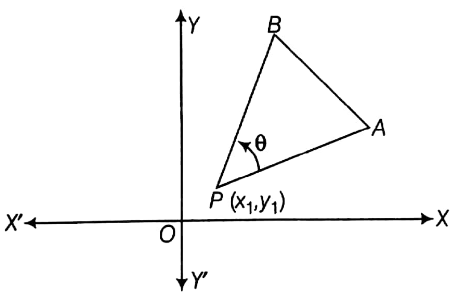
or $$x+y-6=0 \qquad \text{...(ii)}$$

This line Eq. (ii) meets Eq. (i) at $P\equiv(-22, 28)$ which is the required point.

Aliter

Let $\quad$ P be (x_1, y_1) and $\angle APB = \theta$

then $$\cos\theta = \frac{(PA)^2+(PB)^2-(AB)^2}{2PA\cdot PB}$$



since $\cos\theta \le 1$

$$\Rightarrow \quad \frac{(PA)^2 + (PB)^2 - (AB)^2}{2PA \cdot PB} \le 1$$

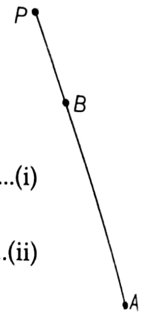
$$\Rightarrow \quad (PA - PB)^2 \le (AB)^2$$

$$\Rightarrow \quad |PA - PB| \le |AB|$$

$$\Rightarrow \quad |PA - PB| \le 2\sqrt{2}$$

Maximum value of $|PA - PB|$ is $2\sqrt{2}$ when, $\theta = 0$.

i.e. P lies on the line AB as well as on the given line.



$\therefore$ Equation of AB is

$$y - 2 = \frac{4-2}{2-4}(x - 4)$$

$$\Rightarrow \quad y - 2 = -x + 4$$

$$\Rightarrow \quad x + y = 6 \qquad \text{...(i)}$$

and given line

$$3x + 2y + 10 = 0 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get $P(-22, 28)$.

Exercise for Session 5

1. The coordinates of the foot of the perpendicular from (2, 3) to the line $3x + 4y - 6 = 0$ are

(a) $\left(-\frac{14}{25}, -\frac{27}{25}\right)$ (b) $\left(\frac{14}{25}, -\frac{17}{25}\right)$
(c) $\left(-\frac{14}{25}, \frac{17}{25}\right)$ (d) $\left(\frac{14}{25}, \frac{27}{25}\right)$

2. If the foot of the perpendicular from the origin to a straight line is at the point (3, −4). Then the equation of the line is

(a) $3x - 4y = 25$ (b) $3x - 4y + 25 = 0$
(c) $4x + 3y - 25 = 0$ (d) $4x - 3y + 25 = 0$

3. The coordinates of the foot of the perpendicular from (a, 0) on the line $y = mx + \frac{a}{m}$ are

(a) $\left(0, -\frac{1}{a}\right)$ (b) $\left(0, \frac{a}{m}\right)$ (c) $\left(0, -\frac{a}{m}\right)$ (d) $\left(0, \frac{1}{a}\right)$

4. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of c is

(a) $a_1^2 - a_2^2 + b_1^2 - b_2^2$ (b) $\sqrt{(a_1^2 + b_1^2 - a_2^2 - b_2^2}$
(c) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ (d) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$

5. The image of the point (3, 8) in the line $x + 3y = 7$ is

(a) (1, 4) (b) (4, 1) (c) (−1 − 4) (d) (−4, − 1)

6. The image of the point (4, −3) with respect to the line $y = x$ is

(a) (−4, − 3) (b) (3, 4) (c) (−4, 3) (d) (−3, 4)

7. The coordinates of the image of the origin O with respect to the straight line $x + y + 1 = 0$ are

(a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) (−2, − 2)
(c) (1, 1) (d) (−1, − 1)

8. If (−2, 6) is the image of the point (4, 2) with respect to the line $L = 0$, then $L \equiv$

(a) $6x - 4y - 7 = 0$ (b) $2x - 3y - 5 = 0$
(c) $3x - 2y + 5 = 0$ (d) $3x - 2y + 10 = 0$

9. The image of $P(a, b)$ on the line $y = -x$ is Q and the image of Q on the line $y = x$ is R. Then the mid-point of PR is

(a) $(a + b, a + b)$
(b) $\left(\frac{a+b}{2}, \frac{b+2}{2}\right)$
(c) $(a - b, b - a)$
(d) $(0, 0)$

10. The nearest point on the line $3x - 4y = 25$ from the origin is

(a) $(3, 4)$
(b) $(3, -4)$
(c) $(3, 5)$
(d) $(-3, 5)$

11. Consider the points $A(0, 1)$ and $B(2, 0)$, P be a point on the line $4x + 3y + 9 = 0$. The coordinates of P such that $|PA - PB|$ is maximum are

(a) $\left(-\frac{12}{5}, \frac{17}{5}\right)$
(b) $\left(-\frac{84}{5}, \frac{13}{5}\right)$
(c) $\left(-\frac{6}{5}, \frac{17}{5}\right)$
(d) $(0, -3)$

12. Consider the points $A(3, 4)$ and $B(7, 13)$. If P is a point on the line $y = x$ such that $PA + PB$ is minimum, then the coordinates of P are

(a) $\left(\frac{12}{7}, \frac{12}{7}\right)$
(b) $\left(\frac{13}{7}, \frac{13}{7}\right)$
(c) $\left(\frac{31}{7}, \frac{31}{7}\right)$
(d) $(0, 0)$

13. The image of a point $P(2, 3)$ in the line mirror $y = x$ is the point Q and the image of Q in the line mirror $y = 0$ is the point $R(x, y)$. Find the coordinates of R.

14. The equations of perpendicular bisector of the sides AB and AC of a ΔABC are $x - y + 5 = 0$ and $x + 2y = 0$ respectively. If the point A is $(1, -2)$, find the equation of line BC.

15. In a ΔABC, the equation of the perpendicular bisector of AC is $3x - 2y + 8 = 0$. If the coordinates of the point A and B are $(1, -1)$ and $(3, 1)$ respectively, find the equation of the line BC and the centre of the circumcircle of ΔABC.

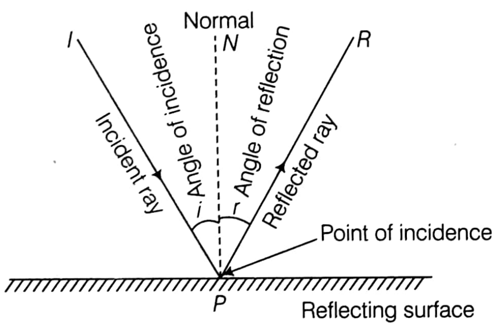
16. Is there a real value of λ for which the image of the point $(\lambda, \lambda - 1)$ by the line mirror $3x + y = 6\lambda$ is the point $(\lambda^2 + 1, \lambda)$? If so find λ.

Session 6

Reflection of Light, Refraction of Light, Conditions of Collinearity if Three Given Points be in Cyclic Order

Reflection of Light

When a ray of light falls on a smooth polished surface (Mirror) separating two media, a part of it is reflected back into the first medium.



IP is the incident ray and PR is the reflected ray. A perpendicular drawn to the surface, at the point of incidence P is called the normal. Hence PN is the normal. The angle between the incident ray and the normal ($\angle IPN$) is called the **angle of incidence** which is represented by $\angle i$

i.e. $\angle IPN = \angle i =$ Angle of incidence and the angle between the reflected ray and the normal ($\angle IPR$) is called the **angle of reflection** which is represented by $\angle r$.

i.e. $\angle IPR = \angle r =$ Angle of reflection.

Laws of Reflection :

(i) The incident ray, normal and the reflected ray to a surface at the point of incidence all lie in the same plane.

(ii) The angle of incidence = angle of reflection

i.e. $\qquad \angle i = \angle r$

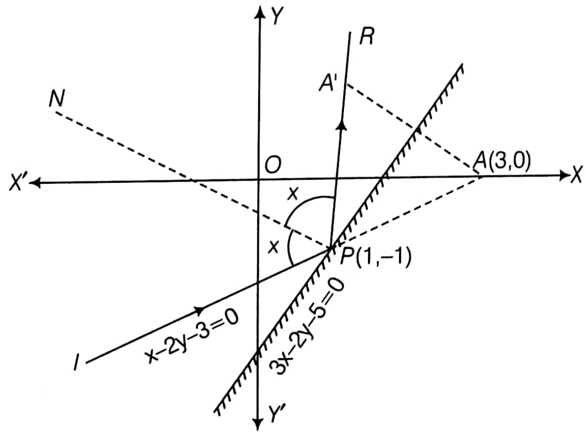
| Example 103. A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Sol. To get coordinates of point P, we solve the given equation of lines together as

$$x - 2y - 3 = 0$$
$$3x - 2y - 5 = 0$$

$\therefore \qquad x = 1, y = -1$

$\therefore$ Coordinates of P are $(1, -1)$.



Let slope of reflected ray be m.

Since, slope of line mirror is 3/2.

$\therefore \qquad$ Slope of $PN = -2/3$ and

slope of $IP = 1/2$, line PN is equally inclined to IP and PR, then

$$\left(\frac{m - \left(-\frac{2}{3}\right)}{1 + m\left(-\frac{2}{3}\right)}\right) = -\left(\frac{\frac{1}{2} - \left(-\frac{2}{3}\right)}{1 + \frac{1}{2}\left(-\frac{2}{3}\right)}\right)$$

$$\Rightarrow \qquad \frac{3m + 2}{3 - 2m} = -\frac{7}{4}$$

$$\Rightarrow \qquad 12m + 8 = -21 + 14m$$

$$\therefore \qquad 2m = 29$$

$$\Rightarrow \qquad m = \frac{29}{2}$$

$\therefore$ Equation of reflected ray $y + 1 = \frac{29}{2}(x - 1)$

$$\Rightarrow \qquad 2y + 2 = 29x - 29$$

$$\Rightarrow \qquad 29x - 2y - 31 = 0.$$

Aliter (Image method) : Take A (3, 0) be any point on IP and if $A'(\alpha, \beta)$ be the image of A about the mirror line $3x - 2y - 5 = 0$, then

$$\frac{\alpha - 3}{3} = \frac{\beta - 0}{-2} = \frac{-2(9 - 0 - 5)}{9 + 4}$$

$\therefore \quad \alpha = \frac{15}{13}$ and $\beta = \frac{16}{13}$

$\therefore \quad A' \equiv \left(\frac{15}{13}, \frac{16}{13}\right)$

$\therefore$ Equation of $A'P$ is the equation of the reflected ray then its equation is,

$$y + 1 = \frac{\left(\frac{16}{13} + 1\right)}{\left(\frac{15}{13} - 1\right)}(x - 1)$$

$$\Rightarrow \quad y + 1 = \frac{29}{2}(x - 1) \text{ or } 29x - 2y - 31 = 0$$

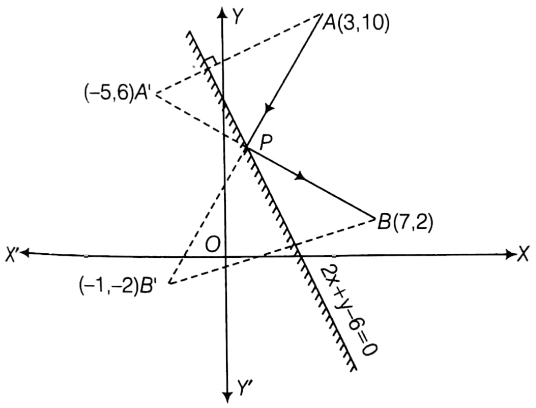
| Example 104. A light beam, emanating from the point (3, 10) reflects from the straight line $2x + y - 6 = 0$ and, then passes through the point (7, 2). Find the equations of the incident and reflected beams.

Sol. Let images of A and B about the line $2x + y - 6 = 0$ are $A'(\alpha, \beta)$ and $B'(\gamma, \delta)$ respectively.

Then, $$\frac{\alpha - 3}{2} = \frac{\beta - 10}{1} = \frac{-2(6 + 10 - 6)}{2^2 + 1^2} = -4$$

$\therefore \quad \alpha = -5, \beta = 6$

i.e. $\quad A' \equiv (-5, 6)$



and $$\frac{\gamma - 7}{2} = \frac{\delta - 2}{1} = \frac{-2(14 + 2 - 6)}{2^2 + 1^2} = -4$$

$\therefore \quad \gamma = -1, \delta = -2$

i.e. $\quad B' \equiv (-1, -2)$.

$\therefore$ Equation of incident ray AB' is

$$y + 2 = \frac{10 + 2}{3 + 1}(x + 1) \text{ or } 3x - y + 1 = 0$$

and equation of reflected ray $A'B$ is

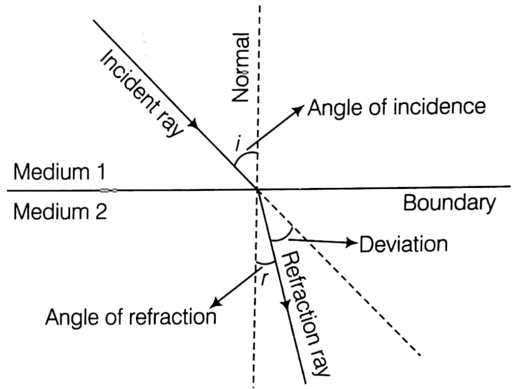
$$y - 6 = \frac{2 - 6}{7 + 5}(x + 5)$$

$$\Rightarrow \quad y - 6 = -\frac{1}{3}(x + 5)$$

or $\quad x + 3y - 13 = 0$

Refraction of Light

When a ray of light falls on the boundary separating the two transparent media, there is a change in direction of ray. This phenomenon is called refraction.



Laws of Refraction

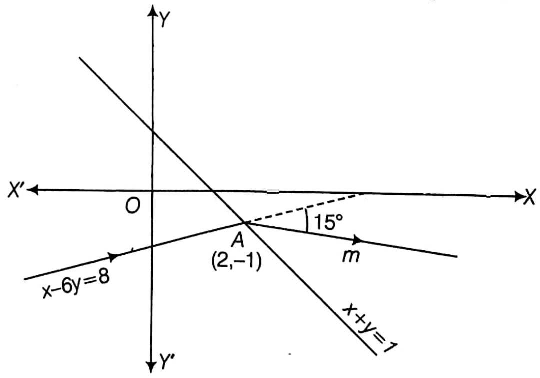
(i) The incident ray, normal and the refracted ray to the surface separating the two transparent media all lie in the same plane.

(ii) The ratio of sine of angle of incidence to the sine of the angle of refraction is constant for the two given media. The constant is called the refractive index of medium 2 with respect to medium 1.

i.e. $$_1\mu_2 = \frac{\sin i}{\sin r}$$

| Example 105. A ray of light is sent along the line $x - 6y = 8$. After refracting across the line $x + y = 1$ it enters the opposite side after turning by 15° away from the line $x + y = 1$. Find the equation of the line along which the refracted ray travels.

Sol. The point of intersection of $x - 6y = 8$ and $x + y = 1$ is $A \equiv (2, -1)$. Let the required ray have the slope $= m$, then



$$\tan 15° = \left|\frac{m - \frac{1}{6}}{1 + \frac{m}{6}}\right|$$

$$\Rightarrow \quad 2-\sqrt{3}=\pm\left(\frac{6m-1}{m+6}\right)$$

then, $$\frac{6m-1}{m+6}=2-\sqrt{3} \text{ or } \sqrt{3}-2$$

$$\Rightarrow \quad m=\frac{70-37\sqrt{3}}{13}$$

or $$m=\frac{37\sqrt{3}-70}{61}$$

Let the angle between $x+y=1$ and the line through $A(2,-1)$ with the slope $\frac{70-37\sqrt{3}}{13}$ be α, then

$$\tan\alpha=\left|\frac{\frac{70-37\sqrt{3}}{13}-(-1)}{1-\left(\frac{70-37\sqrt{3}}{13}\right)}\right|=\left|\frac{83-37\sqrt{3}}{37\sqrt{3}-57}\right|$$

$$=\frac{83-37\sqrt{3}}{37\sqrt{3}-57}$$

and if angle between $x+y=1$ and the line through $A(2,-1)$ with the slope $\frac{37\sqrt{3}-70}{61}$ be β, then

$$\tan\beta=\left|\frac{\frac{37\sqrt{3}-70}{61}-(-1)}{1-\left(\frac{37\sqrt{3}-70}{61}\right)}\right|=\left|\frac{37\sqrt{3}-9}{131-37\sqrt{3}}\right|$$

$$=\frac{37\sqrt{3}-9}{131-37\sqrt{3}}$$

Here $\tan\alpha>\tan\beta, \therefore \alpha>\beta$

therefore the slope of the refracted ray $=\frac{70-37\sqrt{3}}{13}$

$\therefore$ The equation of the refracted ray is

$$y+1=\frac{(70-37\sqrt{3})}{13}(x-2)$$

$$\Rightarrow \quad 13y+13=(70-37\sqrt{3})x-140+74\sqrt{3}$$

or $$(70-37\sqrt{3})x-13y-153+74\sqrt{3}=0$$

Conditions of Collinearity if Three given points are in Cyclic Order

Let the three given points

$$A\equiv(f(a),g(a)),\ B\equiv(f(b),g(b))$$

and $C\equiv(f(c),g(c))$ lie on the line

$lx+my+n=0$, where l, m and n are constants.

Then $$lf(t)+m\,g(t)+n=0 \qquad \text{...(i)}$$

where, $t=a,b,c$

i.e. a,b,c are the roots of the Eq. (i).

In this case Eq. (i) must be cubic in t.

$$At^3+Bt^2+Ct+D=0 \qquad \text{(say)}$$

then $$a+b+c=-\frac{B}{A},\ ab+bc+ca=\frac{C}{A}$$

and $$abc=-\frac{D}{A}$$

which are the required conditions.

Example 106. If the points $\left(\frac{a^3}{a-1},\frac{a^2-3}{a-1}\right),\left(\frac{b^3}{b-1},\frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1},\frac{c^2-3}{c-1}\right)$ are collinear for three distinct values a,b,c and different from 1, then show that $abc-(bc+ca+ab)+3(a+b+c)=0$.

Sol. Let the three given points lie on the line

$lx+my+n=0$, where l, m and n are constants.

Then, $$l\left(\frac{t^3}{t-1}\right)+m\left(\frac{t^2-3}{t-1}\right)+n=0$$

$$\Rightarrow \quad lt^3+mt^2+nt-(3m+n)=0$$

for $t=a,b,c$

i.e. a,b,c are the roots of

$$lt^3+mt^2+nt-3m-n=0$$

then $$a+b+c=-\frac{m}{l},\ ab+bc+ca=\frac{n}{l}$$

and $$abc=\left(\frac{3m+n}{l}\right)$$

Now, $abc-(bc+ca+ab)+3(a+b+c)$

$$=\left(\frac{3m+n}{l}\right)-\frac{n}{l}-\frac{3m}{l}=0$$

Hence, $abc-(bc+ca+ab)+3(a+b+c)=0$

Example 107. If t_1, t_2 and t_3 are distinct, the points $(t_1, 2at_1+at_1^3), (t_2, 2at_2+at_2^3)$ and $(t_3, 2at_3+at_3^3)$ are collinear, then prove that $t_1+t_2+t_3=0$.

Sol. Let the three given points lie on the line $lx+my+n=0$, where l, m and n are constants. Then,

$$l(t)+m(2at+at^3)+n=0$$

$$\Rightarrow \quad (am)t^3+(2am+l)t+n=0 \qquad \text{...(i)}$$

for $t=t_1,t_2,t_3$

i.e., t_1,t_2,t_3 are the roots of Eq. (i), then

$$t_1+t_2+t_3=0$$

Exercise for Session 6

1. A ray of light passing through the point (1, 2) is reflected on the X-axis at a point P and passes through the point (5, 3). The abscissae of the point P is

(a) 3 (b) $\frac{13}{3}$ (c) $\frac{13}{5}$ (d) $\frac{13}{4}$

2. The equation of the line segment AB is $y = x$. If A and B lie on the same side of the line mirror $2x - y = 1$, then the image of AB has the equation

(a) $x + y = 2$ (b) $8x + y = 9$ (c) $7x - y = 6$ (d) None of these

3. A ray of light travelling along the line $x + y = 1$ is incident on the X-axis and after refraction it enters the other side of the X-axis by turning $\pi/6$ away from the X-axis. The equation of the line along which the refracted ray travels is

(a) $x + (2 - \sqrt{3})\, y = 1$ (b) $x\,(2 + \sqrt{3}) + y = 2 + \sqrt{3}$

(c) $(2 - \sqrt{3})x + y = 1$ (d) $x + (2 + \sqrt{3})y = (2 + \sqrt{3})$

4. All the points lying inside the triangle formed by the points $(0, 4)$, $(2, 5)$ and $(6, 2)$ satisfy

(a) $3x + 2y + 8 \geq 0$ (b) $2x + y - 10 \geq 0$

(c) $2x - 3y - 11 \geq 0$ (d) $-2x + y - 3 \geq 0$

5. Let O be the origin and let $A(1, 0)$, $B(0, 1)$ be two points. If $P(x, y)$ is a point such that $xy > 0$ and $x + y < 1$ then

(a) P lies either inside in ΔOAB or in third quadrant (b) P cannot be inside in ΔOAB

(c) P lies inside the ΔOAB (d) None of these

6. A ray of light coming along the line $3x + 4y - 5 = 0$ gets reflected from the line $ax + by - 1 = 0$ and goes along the line $5x - 12y - 10 = 0$ then

(a) $a = \frac{64}{115}, b = \frac{112}{15}$ (b) $a = -\frac{64}{115}, b = \frac{8}{115}$

(c) $a = \frac{64}{115}, b = \frac{8}{115}$ (d) $a = -\frac{64}{115}, b = \frac{-8}{115}$

7. Two sides of a triangle have the joint equation $x^2 - 2xy - 3y^2 + 8y - 4 = 0$. The third side, which is variable, always passes through the point $(-5, -1)$. Find the range of values of the slope of the third side, so that the origin is an interior point of the triangle.

8. Determine the range of values of $\theta \in [0, 2\pi]$ for which $(\cos\theta, \sin\theta)$ lies inside the triangle formed by the lines $x + y - 2 = 0$, $x - y - 1 = 0$ and $6x + 2y - \sqrt{10} = 0$.

9. Let $P(\sin\theta, \cos\theta)$, where $0 \leq \theta \leq 2\pi$ be a point and let OAB be a triangle with vertices (0, 0), $\left(\frac{\sqrt{3}}{2}, 0\right)$ and $\left(0, \frac{\sqrt{3}}{2}\right)$.

Find θ if P lies inside the $\Delta\, OAB$.

10. Find all values of θ for which the point $(\sin^2\theta, \sin\theta)$ lies inside the square formed by the line $xy = 0$ and $4xy - 2x - 2y + 1 = 0$.

11. Determine whether the point $(-3, 2)$ lies inside or outside the triangle whose sides are given by the equations $x + y - 4 = 0$, $3x - 7y + 8 = 0$, $4x - y - 31 = 0$.

12. A ray of light is sent along the line $x - 2y + 5 = 0$, upon reaching the line $3x - 2y + 7 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Shortcuts and Important Results to Remember

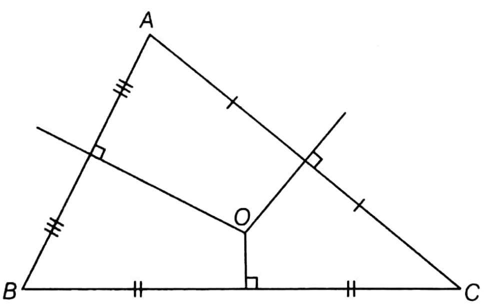
1. Area of parallelogram formed by the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + d_1 = 0$, $a_1x + b_1y + c_2 = 0$ and $a_2x + b_2y + d_2 = 0$ is
$$\frac{|c_1 - c_2||d_1 - d_2|}{\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|}$$

2. Area of parallelogram formed by the lines $y = m_1x + c_1$, $y = m_2x + d_1$, $y = m_1x + c_2$ and $y = m_2x + d_2$ is
$$\frac{|c_1 - c_2||d_1 - d_2|}{|m_1 - m_2|}$$

3. If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ are the vertices of a $\Delta\, ABC$, then angle A is acute or obtuse according as
$$(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3) > 0 \quad \text{or} \quad < 0$$
Similarly, for $\angle B$
$$(x_2 - x_3)(x_2 - x_1) + (y_2 - y_3)(y_2 - y_1) > 0 \quad \text{or} \quad < 0$$
and for $\angle C$
$$(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) > 0 \quad \text{or} \quad < 0$$

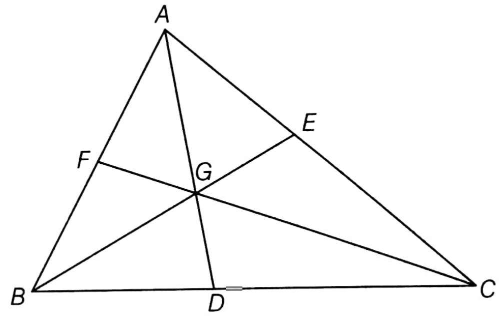
4. If the origin lies in the acute angle or obtuse angle between the lines
$$a_1x + b_1y + c_1 = 0$$
and $$a_2x + b_2y + c_2 = 0$$
according as $(a_1a_2 + b_1b_2)c_1c_2 < 0$ or > 0.

5. If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ are the vertices of a $\Delta\, ABC$ then the equations of the right bisectors (perpendicular bisectors) of the sides BC, CA and AB are
$$y(y_2 - y_3) + x(x_2 - x_3) = \left(\frac{x_2^2 - x_3^2}{2}\right) + \left(\frac{y_2^2 - y_3^2}{2}\right);$$



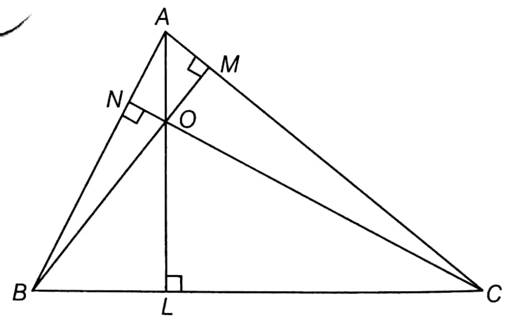
$$y(y_3 - y_1) + x(x_3 - x_1) = \left(\frac{x_3^2 - x_1^2}{2}\right) + \left(\frac{y_3^2 - y_1^2}{2}\right)$$
and $$y(y_1 - y_2) + x(x_1 - x_2) = \left(\frac{x_1^2 - x_2^2}{2}\right) + \left(\frac{y_1^2 - y_2^2}{2}\right)$$
respectively. Where O is the circumcentre of $\Delta\, ABC$.

6. If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ are the vertices of a $\Delta\, ABC$ then the equations of medians AD, BE and CF are
$$y(x_2 + x_3 - 2x_1) - x(y_2 + y_3 - 2y_1) = y_1(x_2 + x_3) - x_1(y_2 + y_3);$$
$$y(x_3 + x_1 - 2x_2) - x(y_3 + y_1 - 2y_2) = y_2(x_3 + x_1) - x_2(y_3 + y_1)$$
and $y(x_1 + x_2 - 2x_3) - x(y_1 + y_2 - 2y_3) = y_3(x_1 + x_2) - x_3(y_1 + y_2)$ respectively.



Where, G is the centroid of ΔABC.

7. If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ are the vertices of a $\Delta\, ABC$ then the equations of the altitudes AL, BM and CN are
$$y(y_2 - y_3) + x(x_2 - x_3) = y_1(y_2 - y_3) + x_1(x_2 - x_3);$$
$$y(y_3 - y_1) + x(x_3 - x_1) = y_2(y_3 - y_1) + x_2(x_3 - x_1)$$



and $y(y_1 - y_2) + x(x_1 - x_2) = y_3(x_1 - x_2) + x_3(x_1 - x_2)$ respectively. Where O is the orthocentre of $\Delta\, ABC$.

8. If sides of a triangle ABC are represented by
$$BC : a_1x + b_1y + c_1 = 0,$$
$$CA : a_2x + b_2y + c_2 = 0$$
and $$AB : a_3x + b_3y + c_3 = 0$$
then $|BC| : |CA| : |AB|$
$$= \sqrt{(a_1^2 + b_1^2)}\left|\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right| : \sqrt{(a_2^2 + b_2^2)}\left|\begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}\right| : \sqrt{(a_3^2 + b_3^2)}\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|$$

JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1 *A rectangle ABCD has its side AB parallel to $y = x$ and vertices A, B and D lie on $y = 1$, $x = 2$ and $x = -2$ respectively, then locus of vertex C is*

(a) $x = 5$ (b) $x - y = 5$

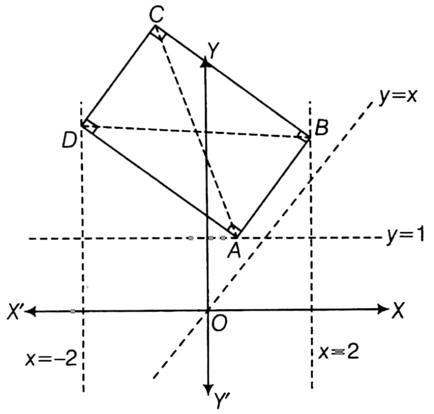
(c) $y = 5$ (d) $x + y = 5$

Sol. (c) Since AB is parallel to $y = x$.

$\therefore$ Equation of AB is $y = x + a$ (say)

$\because$ A lies on $y = 1$

$\therefore$ $A \equiv (1 - a, 1)$



Again, B lies on $x = 2$

$\therefore$ $B = (2, 2 + a)$

$\Rightarrow$ Equation of AD is

$$y - 1 = -[x - (1 - a)] \text{ or } y = 2 - x - a$$

$\because$ D lies on $x = -2$

$\therefore$ $D \equiv (-2, 4 - a)$

Let $C \equiv (h, k)$

$\because$ Diagonals of rectangle bisects to each other

$\therefore$ $h + 1 - a = 2 - 2 \Rightarrow a = 1 + h$

and $k + 1 = 2 + a + 4 - a$

$\Rightarrow$ $k = 5$

$\therefore$ Locus of C is $y = 5$

• Ex. 2 *The line $(\lambda + 1)^2 x + \lambda y - 2\lambda^2 - 2 = 0$ passes through a point regardless of the value λ. Which of the following is the line with slope 2 passing through the point?*

(a) $y = 2x - 8$ (b) $y = 2x - 5$

(c) $y = 2x - 4$ (d) $y = 2x + 8$

Sol. (a)

$\because$ $(\lambda + 1)^2 x + \lambda y - 2\lambda^2 - 2 = 0$

or $(\lambda^2 + 2\lambda + 1)x + \lambda y - 2\lambda^2 - 2 = 0$

or $(\lambda^2 + 1)(x - 2) + \lambda(2x + y) = 0$

$\therefore$ For fixed point

$x - 2 = 0$ and $2x + y = 0$

$\therefore$ Fixed point is $(2, -4)$

$\therefore$ Equation of required line is $y + 4 = 2(x - 2)$

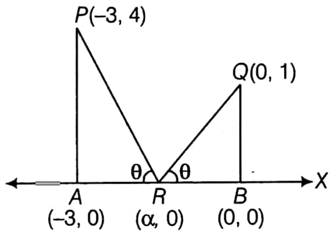
or $y = 2x - 8$

• Ex. 3 *A man starts from the point $P(-3, 4)$ and reaches point $Q(0, 1)$ touching X-axis at R such that $PR + RQ$ is minimum, then the point R is*

(a) $\left(\frac{3}{5}, 0\right)$ (b) $\left(-\frac{3}{5}, 0\right)$ (c) $\left(-\frac{2}{5}, 0\right)$ (d) $(-2, 0)$

Sol. (b) Let $R = (\alpha, 0)$

For $PR + PQ$ to be minimum it should be the path of light and thus we have



$\Delta APR \sim \Delta BQR$

$\Rightarrow$ $\frac{AR}{RB} = \frac{PA}{QB} \Rightarrow \frac{\alpha + 3}{0 - \alpha} = \frac{4}{1} \Rightarrow \alpha = -\frac{3}{5}$

Hence, $R \equiv \left(-\frac{3}{5}, 0\right)$

• Ex. 4 *If the point $P(a, a^2)$ lies inside the triangle formed by the lines $x = 0$, $y = 0$ and $x + y = 2$, then exhaustive range of 'a' is*

(a) $(0, 1)$ (b) $(1, \sqrt{2})$

(c) $(\sqrt{2} - 1, 1)$ (d) $(\sqrt{2} - 1, 2)$

Sol. (a) Since the point $P(a, a^2)$ lies on $y = x^2$

Solving, $y = x^2$

and $x + y = 2$, we get

$x^2 + x - 2 = 0$

$\Rightarrow$ $(x + 2)(x - 1) = 0$

$\Rightarrow$ $x = -2, 1$

It is clear from figure,

$A \equiv (1, 1)$

also $a > 0$ for I quadrant.

$\therefore$ $a \in (0, 1)$

• **Ex. 5** *If $5a + 4b + 20c = t$, then the value of t for which the line $ax + by + c - 1 = 0$ always passes through a fixed point is*

(a) 0 (b) 20

(c) 30 (d) None of these

Sol. (b) Equation of line $\frac{ax}{c-1} + \frac{by}{c-1} + 1 = 0$ has two independent parameters. It can pass through a fixed point if it contains only one independent parameter. Now there must be one relation between $\frac{a}{c-1}$ and $\frac{b}{c-1}$ independent of a, b and c so that $\frac{a}{c-1}$ can be expressed in terms of $\frac{b}{c-1}$ and straight line contains only one independent parameter. Now that given relation can be expressed as $\frac{5a}{c-1} + \frac{4b}{c-1} = \frac{t-20c}{c-1}$.

RHS is independent of c if $t = 20$.

• **Ex. 6** *If the straight lines, $ax + amy + 1 = 0$, $bx + (m+1)by + 1 = 0$ and $cx + (m+2)cy + 1 = 0$, $m \neq 0$ are concurrent, then a, b, c are in*

(a) AP only for $m = 1$ (b) AP for all m

(c) GP for all m (d) HP for all m

Sol. (d) The three lines are concurrent if

$$\begin{vmatrix} a & am & 1 \\ b & (m+1)b & 1 \\ c & (m+2)c & 1 \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - mC_1$, then

$$\begin{vmatrix} a & 0 & 1 \\ b & b & 1 \\ c & 2c & 1 \end{vmatrix} = 0$$

or $a(b - 2c) - 0 + 1(2bc - bc) = 0$

or $b = \frac{2ac}{a+c}$, which is independent of m.

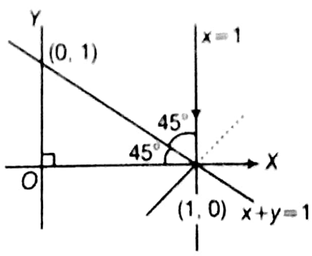
$\therefore$ a, b, c are in HP for all m.

• **Ex. 7** *If a ray travelling the line $x = 1$ gets reflected the line $x + y = 1$, then the equation of the line along which the reflected ray travels is*

(a) $y = 0$ (b) $x - y = 1$

(c) $x = 0$ (d) None of these

Sol. (a) Reflected ray is X-axis.



$\therefore$ Equation $y = 0$

• **Ex. 8** *Through the point $P(\alpha, \beta)$, when $\alpha\beta > 0$, the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is drawn so as to form with coordinate axes a triangle of area Δ. If $ab > 0$, then the least value of Δ is*

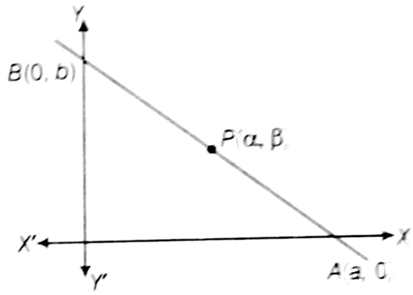
(a) $\alpha\beta$ (b) $2\alpha\beta$

(c) $4\alpha\beta$ (d) $8\alpha\beta$

Sol. (b) Given line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$\therefore$ $A \equiv (a, 0)$, $B \equiv (0, b)$



$\because$ Area of $(\Delta AOB) = \Delta$

$\therefore$ $\frac{1}{2}|ab| = \Delta \Rightarrow ab = 2\Delta$ $(\because ab > 0)$

Since, the line (i) passes through the point $P(\alpha, \beta)$.

$\therefore$ $\frac{\alpha}{a} + \frac{\beta}{b} = 1 \Rightarrow \frac{\alpha}{a} + \frac{a\beta}{2\Delta} = 1$ $\left(\because b = \frac{2\Delta}{a}\right)$

$\Rightarrow$ $a^2\beta - 2a\Delta + 2\Delta\alpha = 0$

$\because a$ is real

$\therefore$ $D \geq 0$

$\Rightarrow$ $4\Delta^2 - 4\beta(2\Delta\alpha) \geq 0$ or $\Delta \geq 2\alpha\beta$

$\therefore$ Least value of Δ is $2\alpha\beta$.

• **Ex. 9** *The coordinates of the point P on the line $2x + 3y + 1 = 0$, such that $|PA - PB|$ is maximum, where A is $(2, 0)$ and B is $(0, 2)$ is*

(a) $(5, -3)$ (b) $(7, -5)$

(c) $(9, -7)$ (d) $(11, -9)$

Sol. (b) $|PA - PB| \leq |AB|$

Maximum value of $|PA - PB|$ is $|AB|$, which is possible only when P, A, B are collinear

if $P(x, y)$, then equation AB is

$$\frac{x}{2} + \frac{y}{2} = 1$$

$\Rightarrow$ $x + y = 2$ ...(i)

Now solving Eq. (i)

and $2x + 3y + 1 = 0$ ...(ii)

Then, we get,

$$x = 7, y = -5$$

$\therefore$ $P \equiv (7, -5)$

• **Ex. 10** *Equation of the straight line which belongs to the system of straight lines $a(2x+y-3)+b(3x+2y-5)=0$ and is farthest from the point $(4,-3)$ is*

(a) $4x+11y-15=0$ (b) $3x-4y+1=0$

(c) $7x+y-8=0$ (d) None of these

Sol. (b) The system of straight lines $a(2x+y-3)+b(3x+2y-5)=0$) passes through the point of intersection of the lines $2x+y-3=0$ and $3x+2y-5=0$ i.e. $(1, 1)$

The line of this family which is farthest from $(4,-3)$ is the line through $(1, 1)$ and perpendicular to the line joining $(1, 1)$ and $(4,-3)$

$\therefore$ The required line is

$$y-1=\frac{3}{4}(x-1)$$

or $$3x-4y+1=0$$

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **5 multiple choice examples**. Each example has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

• **Ex. 11** *The vertices of a square inscribed in the triangle with vertices $A(0,0)$, $B(2,1)$ and $C(3,0)$, given that two of its vertices are on the side AC, are*

(a) $\left(\frac{3}{2},0\right)$ (b) $\left(\frac{3}{2},\frac{3}{4}\right)$ (c) $\left(\frac{9}{4},\frac{3}{4}\right)$ (d) $\left(\frac{9}{4},0\right)$

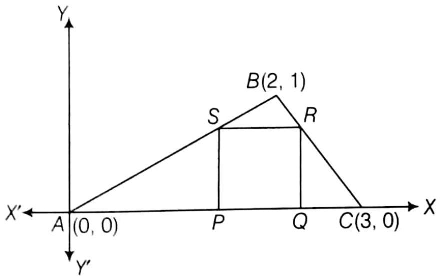
Sol. (a, b, c, d)

Let $PQRS$ be a square inscribed in ΔABC and

$$PQ=QR=RS=SP=\lambda \quad \text{(say)}$$

Let $P\equiv(a,0)$,

$\therefore$ $Q\equiv(a+\lambda,\lambda)$, $R\equiv(a+\lambda,\lambda)$ and $S\equiv(a,\lambda)$



Now equation of AB is

$$x-2y=0 \quad \text{...(i)}$$

and equation of BC is

$$x+y-3=0 \quad \text{...(ii)}$$

$\because$ S lies on AB, then

$$a-2\lambda=0 \quad \text{...(iii)}$$

and R lies on BC, then

$$a+\lambda+\lambda-3=0 \text{ or } a+2\lambda-3=0 \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get $a=\frac{3}{2}$, $\lambda=\frac{3}{4}$

Hence, $P\equiv\left(\frac{3}{2},0\right)$, $Q\equiv\left(\frac{9}{4},0\right)$,

$R\equiv\left(\frac{9}{4},\frac{3}{2}\right)$, $S=\left(\frac{3}{2},\frac{3}{4}\right)$

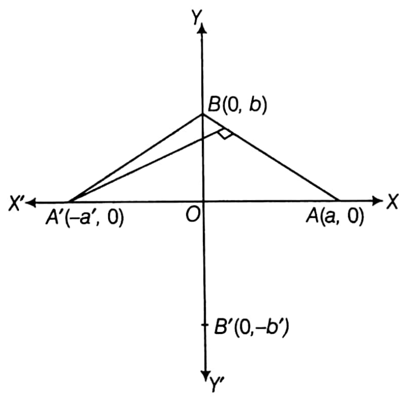
• **Ex. 12** *Line $\frac{x}{a}+\frac{y}{b}=1$ cuts the coordinate axes at $A(a,0)$ and $B(0,b)$ and the line $\frac{x}{a'}+\frac{y}{b'}=-1$ at $A'(-a',0)$ and $B'(0,-b')$. If the points A, B, A', B' are concyclic, then the orthocentre of the triangle ABA' is*

(a) $(0, 0)$ (b) $(0, b')$

(c) $\left(0,\frac{-aa'}{b}\right)$ (d) $\left(0,\frac{bb'}{a}\right)$

Sol. (b, c)

$\because$ A, B, A', B' are concyclic then,



$$OA\cdot OA'=OB\cdot OB'$$

or $$(a)\cdot(-a')=(b)\cdot(-b')$$

or $$aa'=bb' \quad \text{...(i)}$$

The equation of altitude through A' is

$$y-0=\frac{a}{b}(x+a')$$

It intersects the altitude

$$x=0 \text{ at } y=\frac{aa'}{b}$$

$\therefore$ Orthocentre is $\left(0,\frac{aa'}{b}\right)$ or $(0,b')$ [from Eq. (i)]

• **Ex. 13** *Two straight lines $u=0$ and $v=0$ passes through the origin and angle between them is $\tan^{-1}\left(\frac{7}{9}\right)$. If the ratio of the slope of $v=0$ and $u=0$ is $\frac{9}{2}$, then their equations are*

(a) $y = 3x$ and $3y = 2x$
(b) $2y = 3x$ and $3y = x$
(c) $y + 3x = 0$ and $3y + 2x = 0$
(d) $2y + 3x = 0$ and $3y + x = 0$

Sol. (a, b, c, d)

Let the slope of $u = 0$ be m, then the slope of $v = 0$ is $\frac{9m}{2}$.

Therefore, $$\left|\frac{m-\frac{9m}{2}}{1+m\times\frac{9m}{2}}\right| = \frac{7}{9}$$

or $$\left|\frac{-7m}{2+9m^2}\right| = \frac{7}{9}$$

$\Rightarrow$ $$9m^2 + 9m + 2 = 0 \text{ or } 9m^2 - 9m + 2 = 0$$

$\Rightarrow$ $$m = -\frac{2}{3}, -\frac{1}{3} \text{ or } m = \frac{2}{3}, \frac{1}{3}$$

Therefore, the equation of lines are
(i) $2x + 3y = 0$ and $3x + y = 0$
(ii) $x + 3y = 0$ and $3x + 2y = 0$
(iii) $2x = 3y$ and $3x = y$
(iv) $x = 3y$ and $3x = 2y$

• **Ex. 14** *Two sides of a rhombus OABC (lying entirely in the first or third quadrant) of are equal to 2 sq units are $y = \frac{x}{\sqrt{3}}$, $y = \sqrt{3}x$. Then the possible coordinates of B is/are (O being the origin)*

(a) $(1+\sqrt{3}, 1+\sqrt{3})$
(b) $(-1-\sqrt{3}, -1-\sqrt{3})$
(c) $(3+\sqrt{3}, 3+\sqrt{3})$
(d) $(\sqrt{3}-1, \sqrt{3}-1)$

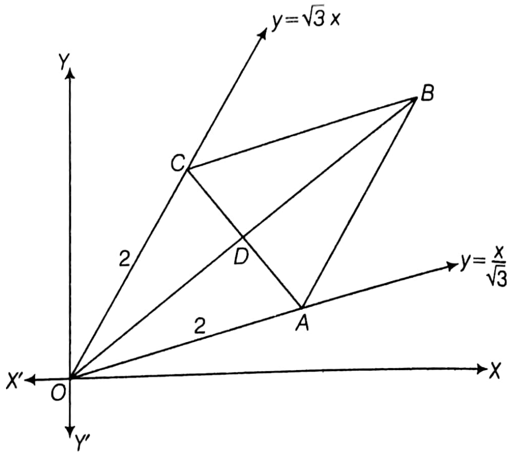
Sol. (a, b)

Here, $\angle COA = 30°$
Let $OA = AB = BC = CO = x$
$\because$ Area of rhombus $OABC$

$$= 2 \times \frac{1}{2} \times x \times x \sin 30°$$

$$= \frac{x^2}{2} = 2 \qquad \text{[given]}$$

$\therefore$ $x = 2$



Coordinates of A and C are $(\sqrt{3}, 1)$ and $(1, \sqrt{3})$ in I quadrant and in III quadrant are $(-\sqrt{3}, -1)$ and $(-1, -\sqrt{3})$

Hence, coordinates of B are $(\sqrt{3}+1, \sqrt{3}+1)$ and $(-\sqrt{3}-1, -\sqrt{3}-1)$

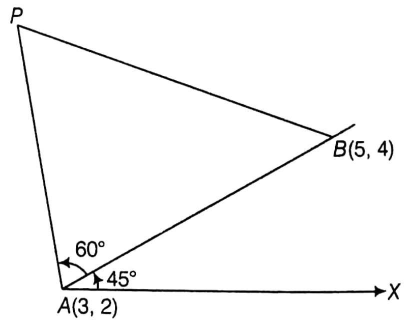
• **Ex. 15** *A and B are two fixed points whose coordinates are (3, 2) and (5, 4) respectively. The coordinates of a point P, if ABP is an equilateral triangle are*

(a) $(4-\sqrt{3}, 3+\sqrt{3})$
(b) $(4+\sqrt{3}, 3-\sqrt{3})$
(c) $(3-\sqrt{3}, 4+\sqrt{3})$
(d) $(3+\sqrt{3}, 4-\sqrt{3})$

Sol. (a, b)

$\because$ $$AB = AP = BP = 2\sqrt{2}$$

$\therefore$ Coordinates of P are $(3 + 2\sqrt{2}\cos 105°,\ 2 + 2\sqrt{2}\sin 105°)$



or $$(3-(\sqrt{3}-1), 2+\sqrt{3}+1)$$

or $$(4-\sqrt{3}, 3+\sqrt{3})$$

If P below AB, then coordinates of P are

$$(3 + 2\sqrt{2}\cos 15°, 2 - 2\sqrt{2}\sin 15°)$$

or $$[(3+\sqrt{3}+1, 2-(\sqrt{3}-1)]$$

or $$(4+\sqrt{3}, 3-\sqrt{3})$$

JEE Type Solved Examples : Paragraph Based Questions

- This section contains **2 solved paragraphs** based upon each of the paragraph **3 multiple choice questions** have to be answered. Each of these question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to18)

Let $d(P, OA) \le \min \cdot \{d(P, AB), d(P, BC), d(P, OC)\}$, where d denotes the distance from the point to the corresponding the line and R be the region consisting of all those points P inside the rectangle OABC such that $O \equiv (0,0)$, $A \equiv (3,0)$, $B \equiv (3,2)$ and $C \equiv (0,2)$. Let M be the peak of region R.

16. Length of the perpendicular from M to OA is

(a) 4 (b) 3 (c) 2 (d) 1

17. If λ be the perimeter of region R, then λ is

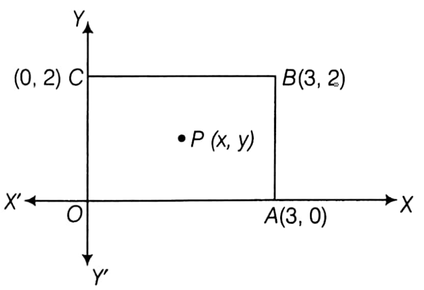
(a) $4-\sqrt{2}$ (b) $4+\sqrt{2}$ (c) $4+2\sqrt{2}$ (d) 10

18. If Δ be the area of region R, then Δ is

(a) 2 (b) 4 (c) 6 (d) 8

Sol. Let $P \equiv (x, y)$

$\because \quad d(P, OA) \le \min \cdot \{d(P, AB), d(P, BC), d(P, OC)\}$



$\Rightarrow \quad |y| \le \min \{|3-x|, |2-y|, |x|\}$

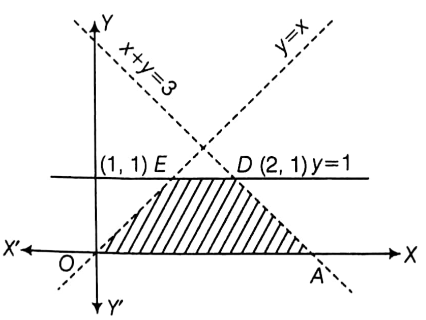
As the rectangle $OABC$ lies in I quadrant,

$\therefore \quad y \le \min \cdot [3-x, 2-y.x]$

We draw the graph of

$y = 3-x, y = 2-y, y = x$

or $\quad x+y=3, y=1, y=x$



$\therefore \quad E \equiv (1, 1)$ and $D \equiv (2, 1)$

16. (d) $\because$ M lie on $y=1$

$\therefore$ Length of $\perp$ from M to OA is 1.

17. (c) λ = Perimeter of region R

$$= OA + AD + DE + EO$$
$$= 3 + \sqrt{2} + 1 + \sqrt{2}$$
$$= 4 + 2\sqrt{2}$$

18. (a) Δ = Area of region R

$$= \frac{1}{2}(OA + ED) \times 1$$
$$= \frac{1}{2}(3+1) = 2$$

Paragraph II

(Q. Nos. 19 to 21)

A variable straight line 'L' is drawn through O(0, 0) to meet this lines $L_1 : y - x - 10 = 0$ and $L_2 : y - x - 20 = 0$ at the points A and B respectively.

19. A point P is taken on 'L' such that $\frac{2}{OP} = \frac{1}{OA} + \frac{1}{OB}$, then the locus of P is

(a) $3x+3y-40=0$ (b) $3x+3y+40=0$

(c) $3x-3y-40=0$ (d) $3x-3y+40=0$

20. A point P is taken on 'L' such that $(OP)^2 = OA \cdot OB$, then the locus of P is

(a) $(y-x)^2 = 25$ (b) $(y-x)^2 = 50$

(c) $(y-x)^2 = 100$ (d) $(y-x)^2 = 200$

21. A point P is taken on 'L' such that $\frac{1}{(OP)^2} = \frac{1}{(OA)^2} + \frac{1}{(OB)^2}$, then locus of P is

(a) $(y-x)^2 = 32$ (b) $(y-x)^2 = 64$

(c) $(y-x)^2 = 80$ (d) $(y-x)^2 = 100$

Sol. Let the equation of line 'L' through origin is

$$\frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta} = r$$

$\therefore \quad P \equiv (r\cos\theta, r\sin\theta)$

Let $\quad OA = r_1$ and $OB = r_2$

$\therefore \quad A \equiv (r_1\cos\theta, r_1\sin\theta)$

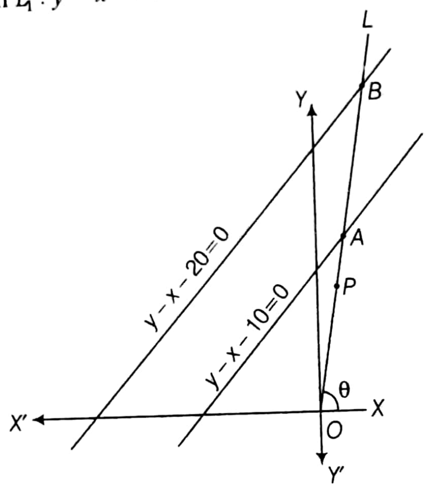
and $\quad B \equiv (r_2\cos\theta, r_2\sin\theta)$

A lies on $\quad L_1 : y - x - 10 = 0$

$\therefore \quad r_1\sin\theta - r_1\cos\theta - 10 = 0$

$\Rightarrow \quad r_1 = \frac{10}{\sin\theta - \cos\theta}$ ...(i)

$\Rightarrow$ B lies on $L_1 : y - x - 20 = 0$



$\therefore \quad r_2 \sin\theta - r_2 \cos\theta - 20 = 0$

$\Rightarrow \quad r_2 = \dfrac{20}{\sin\theta - \cos\theta}$ ...(ii)

19. (d) $\because \quad \dfrac{2}{OP} = \dfrac{1}{OA} + \dfrac{1}{OB} \Rightarrow \dfrac{2}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$

$\Rightarrow \quad \dfrac{2}{r} = \dfrac{\sin\theta - \cos\theta}{10} + \dfrac{\sin\theta - \cos\theta}{20}$ [from Eqs. (i) and (ii)]

or $\quad 2 = \dfrac{r\sin\theta - r\cos\theta}{10} + \dfrac{r\sin\theta - r\cos\theta}{20}$

or $\quad 2 = \dfrac{y - x}{10} + \dfrac{y - x}{20}$ $[\because P \equiv (r\cos, r\sin\theta)]$

$\therefore$ Locus of P is $3x - 3y + 40 = 0$

20. (d) $\because (OP)^2 = OA \cdot OB$

$\Rightarrow \quad r^2 = r_1 \cdot r_2$

$\Rightarrow \quad r^2 = \dfrac{10}{(\sin\theta - \cos\theta)} \cdot \dfrac{20}{(\sin\theta - \cos\theta)}$ [from Eqs. (i) and (ii)]

or $\quad (r\sin\theta - r\cos\theta)^2 = 200$

$\therefore$ Locus of P is $(y - x)^2 = 200$

21. (c) $\because \quad \dfrac{1}{(OP)^2} = \dfrac{1}{(OA)^2} + \dfrac{1}{(OB)^2}$

$\Rightarrow \quad \dfrac{1}{r^2} = \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2}$

$\Rightarrow \quad \dfrac{1}{r^2} = \dfrac{(\sin\theta - \cos\theta)^2}{100} + \dfrac{(\sin\theta - \cos\theta)^2}{400}$

or $\quad 400 = 4(r\sin\theta - r\cos\theta)^2 + (r\sin\theta - r\cos\theta)^2$

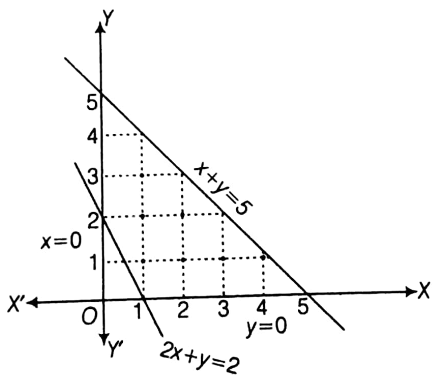
$\therefore$ Locus of P is $(y - x)^2 = 80$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples**. The answer to each example is **a single digit integer,** ranging from 0 to 9 (both inclusive).

• **Ex. 22** *$P(x, y)$ is called a natural point if $x, y \in N$. The total number of points lying inside the quadrilateral formed by the lines $2x + y = 2$, $x = 0$, $y = 0$ and $x + y = 5$ is*

Sol. (6) First, we construct the graph of the given quadrilateral.



It is clear from the graph that there are six points lying inside the quadrilateral.

• **Ex. 23** *The distance of the point (x, y) from the origin is defined as $d = \max \cdot \{|x|, |y|\}$. Then the distance of the common point for the family of lines $x(1 + \lambda) + \lambda y + 2 + \lambda = 0$ (λ being parameter) from the origin is*

Sol. (2) Given family of lines is

$$x(1 + \lambda) + \lambda y + 2 + \lambda = 0$$

$\Rightarrow \quad (x + 2) + \lambda(x + y + 1) = 0$

for common point or fixed point

$$x + 2 = 0$$

and $\quad x + y + 1 = 0$

or $\quad x = -2, y = 1$

$\therefore$ Common point is $(-2, 1)$

or $\quad d = \max\{|-2|, |1|\}$

$\quad = \max\{2, 1\} = 2$

JEE Type Solved Examples : Matching Type Questions

- This section contains **2 examples**. Examples 24 and 25 has four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

• ***Ex. 24*** *Consider the following linear equations x and y*

$$ax + by + c = 0$$
$$bx + cy + a = 0$$
$$cx + ay + b = 0$$

	Column I		Column II
(A)	$a+b+c \neq 0$ and $a^2+b^2+c^2 = ab+bc+ca$	(p)	Lines are sides of a triangle
(B)	$a+b+c = 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	(q)	Lines are different and concurrent
(C)	$a+b+c \neq 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	(r)	Number of pair (x, y) satisfying the equations are infinite
(D)	$a+b+c = 0$ and $a^2+b^2+c^2 = ab+bc+ca$	(s)	Lines are identical

Sol. (A) → (r, s); (B) → (q); (C) → (p); (D) → (r)

(A) if $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$

or $\frac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a)^2\} = 0$

or $a - b = 0, b - c = 0, c - a = 0$

or $a = b = c \Rightarrow$ All the lines an identical

and number of pair (x, y) are infinite.

(B) If $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$\Rightarrow a + b + c = 0$, but a, b, c are not simultaneously equal.

Hence, lines are different and concurrent.

(C) If $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$\Rightarrow \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \neq 0$ and a, b, c are not all simultaneously equal.

∴ Lines are sides of a triangle.

(D) If $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$

$\Rightarrow \Delta = 0$ and $a = b = c$

∴ Equations are satisfied for any (x, y).

• ***Ex. 25*** *The equation of the sides of a triangle are $x + 2y + 1 = 0$, $2x + y + 2 = 0$ and $px + qy + 1 = 0$ and area of triangle is Δ.*

	Column I		Column II
(A)	$p = 2, q = 3$, then 8Δ is divisible by	(p)	3
(B)	$p = 3, q = 2$, then 8Δ is divisible by	(q)	4
(C)	$p = 3, q = 4$, then 10Δ is divisible by	(r)	6
(D)	$p = 4, q = 3$, then 20Δ is divisible by	(s)	9

Sol. (A) → (p); (B) → (p, q, r); (C) → (p, r); (D) → (p, s)

$\because \quad \Delta = \frac{D^2}{2|C_1 C_2 C_3|}$, where

$$D = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ p & q & 1 \end{vmatrix} = 3(p-1)$$

and C_1, C_2, C_3 are co-factors of third column, then

$$C_1 = 2q - p, C_2 = 2p - q, C_3 = -3$$

$\therefore \quad \Delta = \frac{3(p-1)^2}{2|2q-p||2p-q|}$

(A) for $p = 2, \ q = 3$

$\Rightarrow \quad \Delta = \frac{3}{8}$

$\therefore \quad 8\Delta = 3$

(B) for $p = 3, \ q = 2$

$\Rightarrow \quad \Delta = \frac{3}{2}$

$\therefore \quad 8\Delta = 12$

(C) for $p = 3, \ q = 4$

$\Rightarrow \quad \Delta = \frac{6}{10}$

$\therefore \quad 10\Delta = 6$

(D) for $p = 4, \ q = 3$

$\Rightarrow \quad \Delta = \frac{27}{20}$

$\therefore \quad 20\Delta = 27$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** (Ex. Nos. 26 and 27) are Assertion-Reason type examples. Each of these examples contains two statements.

Statement I (Assertion) and **Statement II** (Reason)

Each of these examples also has four alternative choices. Only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement I is true, statement II is true; statement II is a correct explanation for statement I.
(b) Statement I is true, statement II is true; statement II is not a correct explanation for statement I.
(c) Statement I is true, statement II is false.
(d) Statement I is false, statement II is true.

- ***Ex. 26*** *Consider the lines,* $L_1 : \frac{x}{3} + \frac{y}{4} = 1$; $L_2 : \frac{x}{4} + \frac{y}{3} = 1$; $L_3 : \frac{x}{3} + \frac{y}{4} = 2$ *and* $L_4 : \frac{x}{4} + \frac{y}{3} = 2$

Statement I : The quadrilateral formed by these four lines is a rhombus.

Statement II : If diagonals of a quadrilateral formed by any four lines are unequal and intersect at right angle, then it is a rhombus.

Sol. (c) $\because$ L_1, L_3 are parallel.

$\therefore$ Distance between L_1 and $L_3 = \dfrac{1}{\sqrt{\left(\frac{1}{9} + \frac{1}{16}\right)}} = \dfrac{12}{5}$ and

L_2, L_4 are parallel.

$\therefore$ Distance between L_2 and $L_4 = \dfrac{1}{\sqrt{\left(\frac{1}{16} + \frac{1}{9}\right)}} = \dfrac{12}{5}$

$\therefore$ Distance between L_1 and L_3 = Distance between L_2 and L_4.
$\therefore$ Quadrilateral formed by L_1, L_2, L_3, L_4 is a rhombus.
Hence, statement I is true and statement II is false.

- ***Ex. 27***

Statement I : Incentre of the triangle formed by the lines whose sides are $3x + 4y = 0$; $5x - 12y = 0$ and $y - 15 = 0$ is the point P whose coordinates are (1, 8).

Statement II : Point P equidistant from the three lines forming the triangle.

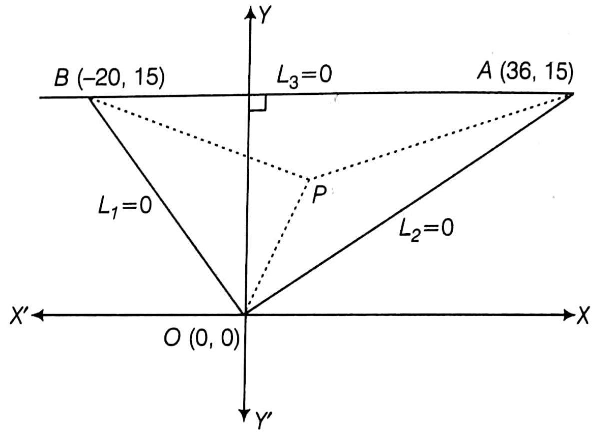
Sol. (b) Let $L_1 \equiv 3x + 4y = 0$,
$L_2 \equiv 5x - 12y = 0$ and $L_3 \equiv y - 15 = 0$

Length of $\perp$ from P to $L_1 = \dfrac{3 + 32}{5} = 7$

Length of $\perp$ from P to $L_2 = \dfrac{|5 - 96|}{13} = 7$

and Length of $\perp$ from P to $L_3 = \dfrac{|8 - 15|}{1} = 7$

$\therefore$ Statement II is true



Also, Area of $\Delta OPA = \frac{1}{2} \times OA \times 7$
$= \frac{1}{2} \times 39 \times 7 = \Delta_1$

Area of $\Delta OPB = \frac{1}{2} \times OB \times 7$
$= \frac{1}{2} \times 25 \times 7 = \Delta_2$

and Area of $\Delta APB = \frac{1}{2} \times AB \times 7$
$= \frac{1}{2} \times 56 \times 7 = \Delta_3$

$\therefore$ $\Delta_1 + \Delta_2 + \Delta_3 = \frac{7}{2}(39 + 25 + 56)$
$= \frac{7 \times 120}{2} = \frac{1}{2} \times 56 \times 15 =$ Area of ΔAOB

$\Rightarrow P$ inside the triangle.

Hence, both statements are true and statement II is not correct explanation of statement I.

The Straight Lines Exercise 8 :
Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

99. The line parallel to the X-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$ is **[AIEEE 2005, 3M]**

(a) below the X-axis at a distance of $\frac{3}{2}$ from it

(b) below the X-axis at a distance of $\frac{2}{3}$ from it

(c) above the X-axis at a distance of $\frac{3}{2}$ from it

(d) above the X-axis at a distance of $\frac{2}{3}$ from it

100. A straight line through the point $A(3, 4)$ is such that its intercept between the axes is bisected at A. Its equation is **[AIEEE 2006, 4.5M]**

(a) $x + y = 7$ (b) $3x - 4y + 7 = 0$
(c) $4x + 3y = 24$ (d) $3x + 4y = 25$

101. If (a, a^2) falls inside the angle made by the lines $y = \frac{x}{2}$, $x > 0$ and $y = 3x$, $x > 0$, then a belong to **[AIEEE 2006, 6M]**

(a) $\left(0, \frac{1}{2}\right)$ (b) $(3, \infty)$
(c) $\left(\frac{1}{2}, 3\right)$ (d) $\left(-3, -\frac{1}{2}\right)$

102. Lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R. **[IIT-JEE 2007, 3M]**

Statement I The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$ because

Statement II In any triangle, bisector of an angle divides the triangle into two similar triangles.

(a) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(b) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(c) Statement I is true, statement II is false

(d) Statement I is false, statement II is true

103. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three point. The equation of the bisector of the angle PQR is **[AIEEE 2007, 3M]**

(a) $\frac{\sqrt{3}}{2}x + y = 0$ (b) $x + \sqrt{3}y = 0$
(c) $\sqrt{3}x + y = 0$ (d) $x + \frac{\sqrt{3}}{2}y = 0$

104. Consider the lines given by

$$L_1 : x + 3y - 5 = 0$$
$$L_2 : 3x - ky - 1 = 0$$
$$L_3 : 5x + 2y - 12 = 0$$

Match the statements/Expressions in **Column I** with the statements/Expressions in **Column II**

	Column I		Column II
(A)	L_1, L_2, L_3 are concurrent, if	(p)	$k = -9$
(B)	one of L_1, L_2, L_3 is parallel to at least one of the other two, if	(q)	$k = -\frac{6}{5}$
(C)	L_1, L_2, L_3 form a triangle, if	(r)	$k = \frac{5}{6}$
(D)	L_1, L_2, L_3 do not form a triangle, if	(s)	$k = 5$

[IIT-JEE 2008, 6M]

105. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept -4. Then a possible value of k is **[AIEEE 2008, 3M]**

(a) 1 (b) 2 (c) -2 (d) -4

106. The lines $p(p^2+1)x-y+q=0$ and $(p^2+1)^2x+(p^2+1)y+2q=0$ are perpendicular to a common line for **[AIEEE 2009, 4M]**
(a) exactly one values of p
(b) exactly two values of p
(c) more than two values of p
(d) no value of p

107. The line L given by $\frac{x}{5}+\frac{y}{b}=1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c}+\frac{y}{3}=1$. Then the distance between L and K is **[AIEEE 2010, 4M]**
(a) $\sqrt{17}$ (b) $\frac{17}{\sqrt{15}}$
(c) $\frac{23}{\sqrt{17}}$ (d) $\frac{23}{\sqrt{15}}$

108. A straight line L through the point $(3,-2)$ is inclined at an angle $60°$ to the line $\sqrt{3}x+y=1$. If L also intersects the X-axis, then the equation of L is **[IIT-JEE 2011, 3M]**
(a) $y+\sqrt{3}x+2-3\sqrt{3}=0$ (b) $y-\sqrt{3}x+2+3\sqrt{3}=0$
(c) $\sqrt{3}y-x+3+2\sqrt{3}=0$ (d) $\sqrt{3}y+x-3+2\sqrt{3}=0$

109. The lines $L_1: y-x=0$ and $L_2: 2x+y=0$ intersect the line $L_3: y+2=0$ at P and Q respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R. **[AIEEE 2011, 4M]**

Statement I : The ratio $PR:RQ$ equals $2\sqrt{2}:\sqrt{5}$

Statement II : In any triangle, bisector of an angle divides the triangle into two similar triangles.

(a) Statement I is true, statement II is true; statement II is not a correct explanation for statement I.
(b) Statement I is true, statement II is false.
(c) Statement I is false, statement II is true.
(d) Statement I is true, statement II is true; statement II is a correct explanation for statement I

110. If the line $2x+y=k$ passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k equals **[AIEEE 2012, 4M]**
(a) $\frac{29}{5}$ (b) 5
(c) 6 (d) $\frac{11}{5}$

111. A ray of light along $x+\sqrt{3}y=\sqrt{3}$ gets reflected upon reaching X-axis, the equation of the reflected ray is **[JEE Main 2013, 4M]**
(a) $y=x+\sqrt{3}$ (b) $\sqrt{3}y=x-\sqrt{3}$
(c) $y=\sqrt{3}x-\sqrt{3}$ (d) $\sqrt{3}y=x-1$

112. For $a>b>c>0$, the distance between (1, 1) and the point of intersection of the lines $ax+by+c=0$ and $bx+ay+c=0$ is less than $2\sqrt{2}$. Then **[JEE Advanced 2013, 3M]**
(a) $a+b-c>0$ (b) $a-b+c<0$
(c) $a-b+c>0$ (d) $a+b-c<0$

113. Let PS be the median of the triangle with vertices $P(2,2)$, $Q(6,-1)$ and $R(7,3)$. The equation of the line passing through $(1,-1)$ and parallel to PS is **[JEE Main 2014, 4M]**
(a) $4x+7y+3=0$ (b) $2x-9y-11=0$
(c) $4x-7y-11=0$ (d) $2x+9y+7=0$

114. Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax+2ay+c=0$ and $5bx+2by+d=0$ lies in the fourth quadrant and is equidistant from the two axes, then **[JEE Main 2014, 4M]**
(a) $3bc-2ad=0$ (b) $3bc+2ad=0$
(c) $2bc-3ad=0$ (d) $2bc+3ad=0$

115. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distance of the point P from the lines $x-y=0$ and $x+y=0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2\le d_1(P)+d_2(P)\le 4$, is **[JEE Advanced 2014, 3M]**

116. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is **[JEE Advanced 2015, 4M]**
(a) 820 (b) 780 (c) 901 (d) 861

117. Two sides of a rhombus are along the lines, $x-y+1=0$ and $7x-y-5=0$. If its diagonals intersect at $(-1,-2)$, then which one of the following is a vertex of this rhombus? **[JEE Main 2016, 4M]**
(a) $\left(\frac{1}{3},-\frac{8}{3}\right)$ (b) $\left(-\frac{10}{3},-\frac{7}{3}\right)$
(c) $(-3,-9)$ (d) $(-3,-8)$

Answers

Exercise for Session 1

1. (c) **2.** (b) **3.** (b) **4.** (d) **5.** (b)
6. (d) **7.** (c) **8.** (d) **9.** (a,d) **10.** (c)
11. (d) **12.** $y = 9$ **13.** $(2 + 2\sqrt{3}, 4)$ and $(2 - 2\sqrt{3}, 0)$
14. $PQ = \dfrac{132}{12\sqrt{3} + 5}$ **15.** $4\sqrt{2}$ units
16. $83x - 35y + 92 = 0$ **17.** $x + y - 11 = 0$

Exercise for Session 2

1. (b) **2.** (c) **3.** (a) **4.** (c) **5.** (b) **6.** (d)
7. (c,d) **8.** (d) **9.** (d) **10.** (d) **11.** (b) **12.** (d)
13. The two points are on the opposite side of the given line.
15. $3x - 4y = 0$ and $3x - 4y - 10 = 0$
17. $7x + y - 31 = 0$ **18.** $2x + 2y + \sqrt{2} = 0$

Exercise for Session 3

1. (c) **2.** (b) **3.** (c) **4.** (a) **5.** (d) **6.** (a)
7. (a) **8.** (c) **9.** (a) **10.** (a,b) **11.** (c) **12.** (c)
13. $2a + b^2 + b = 0$ **16.** (i) $y = 3$ (ii) $x = 4$, (iii) $3x + 4y = 24$
18. $\left(\dfrac{-5}{3}, \dfrac{-5}{3}\right)$

Exercise for Session 4

1. (c) **2.** (d) **3.** (b) **4.** (b) **5.** (c) **6.** (b)
7. (c) **8.** (a) **9.** $x = 7$ and $x + \sqrt{3}y = 7 + 9\sqrt{3}$
10. $x(4\sqrt{3} + 3) + y(4 - 3\sqrt{3}) = 11 - 2\sqrt{3}$ and
$y(4 + 3\sqrt{3}) - x(4\sqrt{3} - 3) = 11 + 2\sqrt{3}, \dfrac{4\sqrt{3}}{15}$
11. $\left(-\dfrac{1}{14}, \dfrac{73}{28}\right)$ and $\left(\dfrac{1}{16}, \dfrac{77}{32}\right)$
12. $\left(0, \dfrac{5}{2}\right)$ and $(0, 0)$ **13.** $x + 2y - 6 = 0$
14. $3x = 19$ **15.** $10x - 10y - 3 = 0$

Exercise for Session 5

1. (d) **2.** (a) **3.** (b) **4.** (d) **5.** (c)
6. (d) **7.** (d) **8.** (c) **9.** (d) **10.** (b)
11. (b) **12.** (c) **13.** $(3, -2)$ **14.** $14x + 23y - 40 = 0$
15. $4x - y + 6 = 0, \left(-\dfrac{4}{5}, \dfrac{14}{5}\right)$ **16.** (2)

Exercise for Session 6

1. (c) **2.** (c) **3.** (a,b) **4.** (a) **5.** (a)
6. (c) **7.** $m \in \left(-1, \dfrac{1}{5}\right)$
8. $\theta \in \left(0, \dfrac{5\pi}{6} - \tan^{-1} 3\right)$ **9.** $\theta \in \left(0, \dfrac{\pi}{12}\right) \cup \left(\dfrac{5\pi}{12}, \dfrac{\pi}{2}\right)$
10. $\theta \in \left\{\bigcup_{n=z}\left(2n\pi, 2n\pi + \dfrac{\pi}{6}\right)\right\} \cup \left\{\bigcup_{m=z}\left(2m\pi + \dfrac{5\pi}{6}, 2m\pi\right)\right\}$
11. Outside **12.** $29x - 2y + 33 = 0$

Chapter Exercises

1. (b) **2.** (a) **3.** (c) **4.** (d) **5.** (c) **6.** (b)
7. (b) **8.** (b) **9.** (b) **10.** (c) **11.** (c) **12.** (b)
13. (c) **14.** (d) **15.** (a) **16.** (b) **17.** (b) **18.** (c)
19. (b) **20.** (b) **21.** (b) **22.** (b) **23.** (c) **24.** (a)
25. (a) **26.** (b) **27.** (a) **28.** (a) **29.** (b) **30.** (a)
31. (a,b,c,d) **32.** (a,b,c,d) **33.** (a,c) **34.** (a,d)
35. (b,d) **36.** (a,b,c,d) **37.** (a,c,d) **38.** (a,d) **39.** (a,b) **40.** (a,b)
41. (a,b,c,d) **42.** (b,d) **43.** (a,b,c) **44.** (a,b) **45.** (a,b,c)
46. (d) **47.** (d) **48.** (a) **49.** (a) **50.** (d) **51.** (a)
52. (b) **53.** (c) **54.** (a) **55.** (c) **56.** (d) **57.** (b)
58. (a) **59.** (a) **60.** (b) **61.** (3) **62.** (6) **63.** (5)
64. (8) **65.** (3) **66.** (9) **67.** (8) **68.** (2) **69.** (2)
70. (4) **71.** (A) $\to$ (p); (B) $\to$ (p,q); (C) $\to$ (p,r) (D) $\to$ (p,r,s)
72. (A) $\to$ (p,r); (B) $\to$ (q); (C) $\to$ (q,s) (D) $\to$ (p)
73. (A) $\to$ (p,q); (B) $\to$ (p, q,r,s); (C) $\to$ (p,q,r,s); (D) $\to$ (p,q,r,s)
74. (A) $\to$ (t); (B) $\to$ (p,q,r); (C) $\Rightarrow$ (s)
75. (A) $\to$ (q,s); (B) $\to$ (p,t); (C) $\to$ (r) **76.** (a) **77.** (a)
78. (d) **79.** (d) **80.** (c) **81.** (b) **82.** (b) **83.** (d)
85. $\left(\dfrac{6}{5}, \dfrac{-1}{10}\right), \left(-\dfrac{2}{5}, \dfrac{-13}{10}\right), \left(0, \dfrac{3}{2}\right), \left(\dfrac{-8}{5}, \dfrac{3}{10}\right)$
87. $\left(\dfrac{\pm\sum_{p=1}^{n}\dfrac{1}{\sqrt{(1+p^2)}}}{c}, \dfrac{\pm\sum_{p=1}^{n}\dfrac{p}{\sqrt{(1+p^2)}}}{c}\right)$ **94.** $2x + 3y + 22 = 0$
95. $\left(\dfrac{h + mk}{1 + m^2}, \dfrac{mh - k}{1 + m^2}\right)$ **96.** $3x + 4y - 18 = 0$ and $x - 2 = 0$
97. $(2 - \sqrt{3})$ sq units. **99.** (a) **100.** (c) **101.** (c) **102.** (c) **103.** (c)
103. (c) **104.** (A) $\to$ (s); (B) $\to$ (p,q); (C) $\to$ (r); (D) $\to$ (p,q,s)
105. (a) **106.** (a) **107.** (c) **108.** (b) **109.** (b) **110.** (c) **111.** (b)
112. (a) **113.** (d) **114.** (a) **115.** (6) **116.** (b) **117.** (a)