CHAPTER

06

Binomial Theorem

Learning Part

Session 1
- Binomial Theorem for Positive Integral Index
- Pascal's Triangle

Session 2
- General Term
- Middle Terms
- Greatest Term
- Trinomial Expansion

Session 3
- Two Important Theorems
- Divisibility Problems

Session 4
- Use of Complex Numbers in Binomial Theorem
- Multinomial Theorem
- Use of Differentiation
- Use of Integration
- When Each Term is Summation Contains the Product of Two Binomial Coefficients or Square of Binomial Coefficients
- Binomial Inside Binomial
- Sum of the Series

Practice Part
- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !
Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

Session 1

Binomial Theorem for Positive Integral Index, Pascal's Triangle

An algebraic expression consisting of two dissimilar terms with positive or negative sign between them is called a binomial expressions.

For example, $x + a, x^2 a - \frac{a}{x}, \frac{p}{x^2} - \frac{q}{x^4}, 5 - x,$

$(x^2 + 1)^{1/3} - \frac{1}{\sqrt{(x^3 + 1)}}$, etc., are called binomial expressions.

Remarks

1. An algebraic expression consisting of three dissimilar terms is called a trinomial. e.g. $a + 2b + c, x - 2y + 3z, 2\alpha - \frac{3}{\beta} + \gamma$, etc. are called the trinomials.
2. In general, expressions consisting more than two dissimilar terms are known as multinomial expressions.

Binomial Theorem for Positive Integral Index

If $x, a \in C$ and $n \in N$, then

$$(x + a)^n = {}^nC_0\, x^{n-0}\, a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2}\, a^2 + \dots + {}^nC_r\, x^{n-r}\, a^r + \dots + {}^nC_{n-1}\, x^1\, a^{n-1} + {}^nC_n\, x^0\, a^n \quad \text{...(i)}$$

or $$(x + a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r$$

Hence, ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ are called binomial coefficients.

Remark

1. In each term, the degree is n and the coefficient of $x^{n-r} a^r$ is equal to the number of ways $\underbrace{x, x, x, \dots, x}_{(n-r)\text{ times}}, \underbrace{a, a, a, \dots, a}_{r\text{ times}}$ can be arranged, which is given by $\boxed{\frac{n!}{(n-r)!\,r!} = {}^nC_r}$

For example, $(x + a)^5 = \frac{5!}{5!\,0!} x^5 a^0 + \frac{5!}{4!\,1!} x^4 a + \frac{5!}{3!\,2!} x^3 a^2 + \frac{5!}{2!\,3!} x^2 a^3 + \frac{5!}{1!\,4!} x a^4 + \frac{5!}{0!\,5!} x^0 a^5$

$= {}^5C_0\, x^5 + {}^5C_1\, x^4\, a + {}^5C_2\, x^3\, a^2 + {}^5C_3\, x^2\, a^3 + {}^5C_4\, x\, a^4 + {}^5C_5\, a^5$

2. Let $S = (x + a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r$

Replacing r by $n - r$, we have

$S = (x + a)^n = \sum_{r=0}^{n} {}^nC_{n-r}\, x^{n-(n-r)}\, a^{n-r} = \sum_{r=0}^{n} {}^nC_{n-r}\, x^r\, a^{n-r}$

$= {}^nC_n\, a^n + {}^nC_{n-1}\, a^{n-1}\, x + {}^nC_{n-2}\, a^{n-2}\, x^2 + \dots + {}^nC_0\, x^n$

Thus, replacing r by $n - r$, we are infact writing the binomial expansion in reverse order.

Some Important Points

1. Replacing a by $(-a)$ in Eq. (i), we get

$$(x - a)^n = {}^nC_0\, x^{n-0}\, a^0 - {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 - \dots + \dots + (-1)^r\, {}^nC_r\, x^{n-r}\, a^r + \dots + (-1)^n\, {}^nC_n\, x^0\, a^n \quad \text{...(ii)}$$

or $$(x - a)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r}\, a^r$$

2. On adding Eqs. (i) and (ii), we get

$$(x + a)^n + (x - a)^n = 2\{{}^nC_0\, x^{n-0}\, a^0 + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \dots\}$$
$$= 2\ \{\text{Sum of terms at odd places}\}$$

The last term is ${}^nC_n\, a^n$ or ${}^nC_{n-1}\, x\, a^{n-1}$, according as n is even or odd, respectively.

3. On subtracting Eq. (ii) from Eq. (i), we get

$$(x + a)^n - (x - a)^n = 2\{{}^nC_1\, x^{n-1}\, a^1 + {}^nC_3\, x^{n-3}\, a^3 + {}^nC_5\, x^{n-5}\, a^5 + \dots\}$$
$$= 2\ \{\text{Sum of terms at even places}\}$$

The last term is ${}^nC_{n-1}\, x\, a^{n-1}$ or ${}^nC_n\, a^n$, according as n is even or odd, respectively.

4. Replacing x by 1 and a by x in Eq. (i), we get

$$(1 + x)^n = {}^nC_0\, x^0 + {}^nC_1\, x^1 + {}^nC_2\, x^2 + \dots + {}^nC_r\, x^r + \dots + {}^nC_{n-1}\, x^{n-1} + {}^nC_n\, x^n \quad \text{...(iii)}$$

or $$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

5. Replacing x by $(-x)$ in Eq. (iii), we get

$$(1-x)^n = {}^nC_0\, x^0 - {}^nC_1\, x^1 + {}^nC_2\, x^2 - \ldots + (-1)^r\, {}^nC_r\, x^r + \ldots + {}^nC_n (-1)^n x^n$$

or $(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$

| Example 1. Expand $\left(2a - \frac{3}{b}\right)^5$ by binomial theorem.

Sol. Using binomial theorem, we get

$$\left(2a-\frac{3}{b}\right)^5 = {}^5C_0\,(2a)^{5-0}\left(-\frac{3}{b}\right)^0 + {}^5C_1(2a)^{5-1}\left(-\frac{3}{b}\right)^1$$
$$+\,{}^5C_2\,(2a)^{5-2}\left(-\frac{3}{b}\right)^2 + {}^5C_3\,(2a)^{5-3}\left(-\frac{3}{b}\right)^3$$
$$+\,{}^5C_4\,(2a)^{5-4}\left(-\frac{3}{b}\right)^4 + {}^5C_5\,(2a)^{5-5}\left(-\frac{3}{b}\right)^5$$
$$= {}^5C_0\,(2a)^5 - {}^5C_1\,(2a)^4\left(\frac{3}{b}\right) + {}^5C_2\,(2a)^3\left(\frac{3}{b}\right)^2$$
$$-\,{}^5C_3\,(2a)^2\left(\frac{3}{b}\right)^3 + {}^5C_4\,(2a)^1\left(\frac{3}{b}\right)^4 - {}^5C_5\left(\frac{3}{b}\right)^5$$
$$= 32a^5 - \frac{240\,a^4}{b} + \frac{720\,a^3}{b^2} - \frac{1080\,a^2}{b^3} + \frac{810\,a}{b^4} - \frac{243}{b^5}$$

| Example 2. Simplify

$$(x+\sqrt{(x^2-1)})^6 + (x-\sqrt{(x^2-1})^6.$$

Sol. Let $\sqrt{(x^2-1)} = a$

Then, $(x+a)^6 + (x-a)^6 = 2\{{}^6C_0\, x^{6-0}\, a^0 + {}^6C_2\, x^{6-2}\, a^2 + {}^6C_4\, x^{6-4}\, a^4 + {}^6C_6\, x^{6-6}\, a^6\}$

$= 2\{x^6 + 15x^4a^2 + 15x^2a^4 + a^6\}$ [from point (2)]

$= 2\{x^6 + 15x^4(x^2-1) + 15x^2(x^2-1)^2 + (x^2-1)^3\}$ $[\because a = \sqrt{x^2-1}]$

$= 2(32x^6 - 48x^4 + 18x^2 - 1)$

| Example 3. In the expansion of $(x+a)^n$, if sum of odd terms is P and sum of even terms is Q, prove that

(i) $P^2 - Q^2 = (x^2-a^2)^n$

(ii) $4PQ = (x+a)^{2n} - (x-a)^{2n}$

Sol. $\because (x+a)^n = {}^nC_0\, x^{n-0}a^0 + {}^nC_1\, x^{n-1}a^1 + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_3\, x^{n-3}\, a^3 + \ldots + \ldots + {}^nC_n\, x^{n-n}\, a^n$

$= ({}^nC_0\, x^n + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \ldots) + ({}^nC_1\, x^{n-1}\, a^1 + {}^nC_3\, x^{n-3}\, a^3 + {}^nC_5\, x^{n-5}\, a^5 + \ldots)$

$= P + Q$ (given) ...(i)

and $(x-a)^n = {}^nC_0\, x^{n-0}\, a^0 - {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2}\, a^2 - {}^nC_3\, x^{n-3}\, a^3 + \ldots + {}^nC_n\, x^{n-n}\, a^n$

$= ({}^nC_0\, x^n + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \ldots) - ({}^nC_1\, x^{n-1}\, a + {}^nC_3\, x^{n-3}\, a^3 + {}^nC_5\, x^{n-5}\, a^5 + \ldots)$

$= P - Q$ (given) ...(ii)

(i) $P^2 - Q^2 = (P+Q)(P-Q)$
$= (x+a)^n \cdot (x-a)^n$
$= (x^2-a^2)^n$ [from Eqs. (i) and (ii)]

(ii) $(x+a)^{2n} - (x-a)^{2n} = [(x+a)^n]^2 - [(x-a)^n]^2$
$= (P+Q)^2 - (P-Q)^2$
$= 4PQ$ [from Eqs. (i) and (ii)]

| Example 4. Show that $(101)^{50} > (100)^{50} + (99)^{50}$.

Sol. Since, $(101)^{50} - (99)^{50} = (100+1)^{50} - (100-1)^{50}$

$= 2\{{}^{50}C_1\,(100)^{49} + {}^{50}C_3\,(100)^{47} + {}^{50}C_5\,(100)^{45} + \ldots\}$

$= 2 \times {}^{50}C_1\,(100)^{49} + 2\{{}^{50}C_3\,(100)^{47} + {}^{50}C_5\,(100)^{45} + \ldots\}$

$= (100)^{50} + (\text{a positive number}) > (100)^{50}$

Hence, $(101)^{50} - (99)^{50} > (100)^{50}$

$\Rightarrow$ $(101)^{50} > (100)^{50} + (99)^{50}$

A must

| Example 5. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, find the value of $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$.

Sol. Let $P = \sum_{r=0}^{n} \frac{r}{{}^nC_r}$ ...(i)

Replacing r by $(n-r)$ in Eq. (i), we get

$$P = \sum_{r=0}^{n} \frac{(n-r)}{{}^nC_{n-r}} = \sum_{r=0}^{n} \frac{(n-r)}{{}^nC_r} \quad [\because {}^nC_r = {}^nC_{n-r}] \ldots\text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2P = \sum_{r=0}^{n} \frac{n}{{}^nC_r} = n \sum_{r=0}^{n} \frac{1}{{}^nC_r} = na_n \quad \text{[given]}$$

$\therefore$ $P = \frac{n}{2}\, a_n$

Hence, $\sum_{r=0}^{n} \frac{r}{{}^nC_r} = \frac{n}{2}\, a_n$

Properties of Binomial Expansion $(x + a)^n$

(i) This expansion has $(n + 1)$ terms.

(ii) Since, ${}^nC_r = {}^nC_{n-r}$, we have

$${}^nC_0 = {}^nC_n = 1$$

$${}^nC_1 = {}^nC_{n-1} = n$$

$${}^nC_2 = {}^nC_{n-2} = \frac{n(n-1)}{2!} \text{ and so on.}$$

(iii) In any term, the suffix of C is equal to the index of a and the index of $x = n -$ (suffix of C).

(iv) In each term, sum of the indices of x and a is equal to n.

Properties of Binomial Coefficient

(i) ${}^nC_r$ can also be represented by $C(n, r)$ or $\binom{n}{r}$.

(ii) ${}^nC_x = {}^nC_y$, then either $x = y$ or $n = x + y$.

So, ${}^nC_r = {}^nC_{n-r} = \dfrac{n!}{r!(n-r)!}$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(v) ${}^nC_r = \dfrac{n}{r} \cdot {}^{n-1}C_{r-1}$

Pascal's Triangle

Coefficients of binomial expansion can also be easily determined by Pascal's triangle.

$(x + a)^0$ 1

$(x + a)^1$ 1 1

$(x + a)^2$ 1 2 1

$(x + a)^3$ 1 3 3 1

$(x + a)^4$ 1 4 6 4 1

$(x + a)^5$ 1 5 10 10 5 1

Pascal triangle gives the direct binomial coefficients.

For example,

$$(x + a)^4 = 1 \cdot x^4 + 4 \cdot x^3 \cdot a + 6 \cdot x^2a^2 + 4 \cdot x a^3 + 1 \cdot a^4$$

$$= x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$$

How to Construct a Pascal's Triangle

Binomial coefficients in the expansion of $(x + a)^3$ are

1 3 3 1

1 3 3 1

1 $(1+3)$ $(3+3)$ $(3+1)$ 1

Then, 1 4 6 4 1

are the binomial coefficients in the expansion of $(x + a)^4$.

| Example 6. Find the number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{33}$.

Sol. $(1 - 3x + 3x^2 - x^3)^{33} = [(1 - x)^3]^{33} = (1 - x)^{99}$

Therefore, number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^3$ is 100.

| Example 7. Find the value of $\displaystyle\sum_{r=1}^{n} \frac{r \cdot {}^nC_r}{{}^nC_{r-1}}$.

Sol. $\because \quad \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

$\therefore \quad \dfrac{r \cdot {}^nC_r}{{}^nC_{r-1}} = (n - r + 1)$

$$\therefore \quad \sum_{r=1}^{n} \frac{r \cdot {}^nC_r}{{}^nC_{r-1}} = \sum_{r=1}^{n}(n - r + 1) = \sum_{r=1}^{n}(n + 1) - \sum_{r=1}^{n} r$$

$$= (n + 1)\sum_{r=1}^{n} 1 - (1 + 2 + 3 + \dots + n)$$

$$= (n + 1) \cdot n - \frac{n(n + 1)}{2} = \frac{n(n + 1)}{2}$$

| Example 8. Let C_r stands for ${}^nC_r$, prove that $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)\dots(C_{n-1} + C_n)$

$$= \frac{(n+1)^n}{n!}(C_0C_1C_2 \dots C_{n-1}).$$

Sol. LHS $= (C_0 + C_1)(C_1 + C_2)(C_2 + C_3)\dots(C_{n-1} + C_n)$

$$= \prod_{r=1}^{n}(C_{r-1} + C_r) = \prod_{r=1}^{n}({}^{n+1}C_r) \quad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$

$$= \prod_{r=1}^{n}\left(\frac{n+1}{r}\right){}^nC_{r-1} \quad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$= \prod_{r=1}^{n}(n + 1) \cdot \prod_{r=1}^{n}\frac{1}{r} \cdot \prod_{r=1}^{n} C_{r-1}$$

$$= (n + 1)^n \cdot \frac{1}{n!} \cdot (C_0C_1C_2 \dots C_{n-1})$$

$$= \frac{(n + 1)^n}{n!}(C_0\, C_1\, C_2 \dots C_{n-1}) = \text{RHS}$$

★ must

Example 9. Find the sum of the series

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left\{\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\dots\text{ upto } m \text{ terms}\right\}.$$

Sol. $\because (1-x)^n = \sum_{r=0}^{n}(-1)^r\,{}^nC_r\,x^r$ ...(i)

Let $P = \sum_{r=0}^{n}(-1)^r\,{}^nC_r\left\{\left(\frac{1}{2}\right)^r+\left(\frac{3}{4}\right)^r+\left(\frac{7}{8}\right)^r+\left(\frac{15}{16}\right)^r+\dots\text{ upto } m \text{ terms}\right\}$

$$= \sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{1}{2}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{3}{4}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{7}{8}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{15}{16}\right)^r+\dots\text{ upto } m \text{ terms}$$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\left(1-\frac{15}{16}\right)^n+\dots\text{ upto } m \text{ terms}$$

$$=\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^{2n}+\left(\frac{1}{2}\right)^{3n}+\left(\frac{1}{2}\right)^{4n}+\dots\text{ upto } m \text{ terms}$$

$$=\frac{\left(\frac{1}{2}\right)^n\left[1-\left\{\left(\frac{1}{2}\right)^n\right\}^m\right]}{1-\left(\frac{1}{2}\right)^n}$$

$$=\frac{(2^{mn}-1)}{2^{mn}(2^n-1)}$$

c, a, c, c, b, b, c, d.

Exercise for Session 1

1. The value of $\sum_{r=0}^{10} r\cdot{}^{10}C_r\cdot 3^r\cdot(-2)^{10-r}$ is

(a) 10 (b) 20 (c) 30 (d) 300

2. The number of dissimilar terms in the expansion of $\left(x+\frac{1}{x}+x^2+\frac{1}{x^2}\right)^{15}$ are

(a) 61 (b) 121 (c) 255 (d) 16

3. The expansion $\{x+(x^3-1)^{1/2}\}^5+\{x-(x^3-1)^{1/2}\}^5$ is a polynomial of degree

(a) 5 (b) 6 (c) 7 (d) 8

4. $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$ is equal to

(a) 101 (b) $70\sqrt{2}$ (c) $140\sqrt{2}$ (d) $120\sqrt{2}$

5. The total number of dissimilar terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplification will be

(a) 202 (b) 51
(c) 50 (d) 101

6. The number of non-zero terms in the expansion of $(1+3\sqrt{2}x)^9+(1-3\sqrt{2}x)^9$, is

(a) 0 (b) 5
(c) 9 (d) 10

7. If $(1+x)^n=\sum_{r=0}^{n}C_r\,x^r$, $\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)\dots\left(1+\frac{C_n}{C_{n-1}}\right)$ is equal to

(a) $\frac{n^{n-1}}{(n-1)!}$ (b) $\frac{(n+1)^{n-1}}{(n-1)!}$

(c) $\frac{(n+1)^n}{n!}$ (d) $\frac{(n+1)^{n+1}}{n!}$

8. If ${}^{n+1}C_{r+1}:{}^nC_r:{}^{n-1}C_{r-1}=11:6:3$, nr is equal to

(a) 20 (b) 30
(c) 0 (d) 50

Session 2

General Term, Middle Terms, Greatest Term, Trinomial Expansion

General Term

The term ${}^nC_r\, x^{n-r}\, a^r$ is the $(r+1)$ th term from beginning in the expansion of $(x+a)^n$. It is usually called the general term and it is denoted by T_{r+1}.
i.e., $T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

| Example 10. Find the 7th term in the expansion of $\left(4x - \dfrac{1}{2\sqrt{x}}\right)^{13}$.

Sol. Seventh term, $T_7 = T_{6+1} = {}^{13}C_6\,(4x)^{13-6}\left(-\dfrac{1}{2\sqrt{x}}\right)^6$

$$= {}^{13}C_6 \cdot 4^7 \cdot x^7 \cdot \frac{1}{2^6 \cdot x^3}$$

$$= {}^{13}C_6 \cdot 2^8 \cdot x^4$$

| Example 11. Find the coefficient of x^8 in the expansion of $\left(x^2 - \dfrac{1}{x}\right)^{10}$.

Sol. Here, $T_{r+1} = {}^{10}C_r\,(x^2)^{10-r}\left(-\dfrac{1}{x}\right)^r$

$$= {}^{10}C_r\, x^{20-2r} \cdot (-1)^r \cdot \frac{1}{x^r}$$

$$= {}^{10}C_r\,(-1)^r \cdot x^{20-3r} \qquad \text{...(i)}$$

Now, in order to find out the coefficient of x^8, $20-3r$ must be 8.

i.e. $\quad 20 - 3r = 8$

$\therefore \quad r = 4$

Hence, putting $r = 4$ in Eq. (i), we get

Required coefficient $= (-1)^4 \cdot {}^{10}C_4 = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210$

| Example 12. Find

(i) the coefficient of x^7 in the expansion of $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$.

(ii) the coefficient of x^{-7} in the expansion of $\left(ax - \dfrac{1}{bx^2}\right)^{11}$.

Also, find the relation between a and b, so that these coefficients are equal.

Sol. (i) Here, $T_{r+1} = {}^{11}C_r\,(ax^2)^{11-r}\left(\dfrac{1}{bx}\right)^r$

$$= {}^{11}C_r \cdot \frac{a^{11-r}}{b^r} \cdot x^{22-3r} \qquad \text{...(i)}$$

Now, in order to find out the coefficient of x^7, $22-3r$ must be 7,

i.e. $\quad 22 - 3r = 7$

$\therefore \quad r = 5$

Hence, putting $r = 5$ in Eq. (i), we get

Required coefficient $= {}^{11}C_5 \cdot \dfrac{a^6}{b^5}$

(ii) Here, $T_{R+1} = {}^{11}C_R\,(ax)^{11-R}\left(-\dfrac{1}{bx^2}\right)^R$

$$= {}^{11}C_R\,(a)^{11-R}\left(-\frac{1}{b}\right)^R \cdot x^{11-3R}$$

$$= (-1)^R \cdot {}^{11}C_R \cdot \frac{a^{11-R}}{b^R} \cdot x^{11-3R} \qquad \text{...(ii)}$$

Now, in order to find out the coefficient of x^{-7}, $11-3R$ must be -7.

i.e., $11 - 3R = -7 \Rightarrow R = 6$. Hence, putting $R = 6$ in Eq. (ii), we get

Required coefficient

$$= (-1)^6 \cdot {}^{11}C_6 \cdot \frac{a^5}{b^6} = {}^{11}C_5 \cdot \frac{a^5}{b^6} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

Also given, coefficient of x^7 in $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax - \dfrac{1}{bx^2}\right)^{11}$

$$\Rightarrow {}^{11}C_5 \cdot \frac{a^6}{b^5} = {}^{11}C_5 \cdot \frac{a^5}{b^6} \Rightarrow ab = 1$$

which is the required relation between a and b.

Example 13. Find the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

Sol. Here, $T_{r+1} = {}^9C_r \left(\frac{3}{2}x^2\right)^{9-r} \left(-\frac{1}{3x}\right)^r$

$$= (-1)^r \cdot {}^9C_r \cdot \left(\frac{3}{2}\right)^{9-r} \cdot \left(\frac{1}{3}\right)^r \cdot x^{18-3r} \quad \ldots\text{(i)}$$

If this term is independent of x, then the index of x must be zero, i.e., $18 - 3r = 0 \Rightarrow r = 6$

Therefore, $(r+1)$ th term, i.e., 7th term is independent of x and its value by putting $r = 6$ in Eq. (i)

$$= (-1)^6 \cdot {}^9C_6 \cdot \left(\frac{3}{2}\right)^3 \cdot \left(\frac{1}{3}\right)^6 = {}^9C_3 \cdot \frac{1}{2^3 \cdot 3^3}$$

$$= \frac{9 \cdot 8 \cdot 7}{(1 \cdot 2 \cdot 3)\, 2^3 \cdot 3^3} = \frac{7}{18}$$

$(p+1)$th Term From End in the Expansion of $(x+a)^n$

$(p+1)$ th term from end in the expansion of $(x+a)^n$

$= (p+1)$ th term from beginning in the expansion of $(a+x)^n$

$= {}^nC_p\, a^{n-p}\, x^p$

Example 14. Find the 4th term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^7$.

Sol. 4th term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^7$

= 4th term from beginning in the expansion of $\left(-\frac{2}{x^2} + \frac{x^3}{2}\right)^7$

$$= {}^7C_3 \left(-\frac{2}{x^2}\right)^{7-3} \left(\frac{x^3}{2}\right)^3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{2^4}{x^8} \cdot \frac{x^9}{2^3} = 70x$$

Example 15. Find the $(n+1)$th term from the end in the expansion of $\left(2x - \frac{1}{x}\right)^{3n}$.

Sol. $(n+1)$th term from the end in the expansion of $\left(2x - \frac{1}{x}\right)^{3n}$

$= (n+1)$ th term from beginning in the expansion of $\left(-\frac{1}{x} + 2x\right)^{3n}$

$$= T_{n+1} = {}^{3n}C_n \left(-\frac{1}{x}\right)^{3n-n} (2x)^n = {}^{3n}C_n \cdot 2^n \cdot x^{-n}$$

How to Find Free from Radical Terms or Rational Terms in the Expansion of $(a^{1/p} + b^{1/q})^N, \forall\, a, b \in$ Prime Numbers

First, find $T_{r+1} = {}^NC_r\, (a^{1/p})^{N-r}\, (b^{1/q})^r$

$\therefore \quad T_{r+1} = {}^NC_r \cdot a^{(N-r)/p} \cdot b^{r/q}$

By inspection, putting the values of $0 \le r \le N$, when indices of a and b are integers.

Remark

1. If indices of a and b are positive integers.
   Then, free from radical terms = Terms which are integers
   $\therefore$ Number of non-integral terms = Total terms − Number of integral terms
2. If indices of a and b both are not positive integers.
   Then, free from radical terms = Rational terms − Integral terms
3. Number of irrational terms = Total terms − Number of rational terms

Example 16. Find the number of terms in the expansion of $(\sqrt[4]{9} + \sqrt[6]{8})^{500}$ which are integers.

Sol. Since, $(\sqrt[4]{9} + \sqrt[6]{8})^{500} = (9^{1/4} + 8^{1/6})^{500} = (3^{1/2} + 2^{1/2})^{500}$

[$\because a, b \in$ prime numbers]

$\therefore$ General term, $T_{r+1} = {}^{500}C_r\, (3^{1/2})^{500-r} \cdot (2^{1/2})^r$

$$= {}^{500}C_r \cdot 3^{\frac{500-r}{2}} \cdot 2^{r/2}$$

$$= {}^{500}C_r \cdot 3^{250 - r/2} \cdot 2^{r/2}$$

Now, $\quad 0 \le r \le 500$

For $r = 0, 2, 4, 6, 8, \ldots, 500$, indices of 3 and 2 are positive integers.

Hence, number of terms which are integers $= 250 + 1 = 251$

Example 17. Find the sum of all rational terms in the expansion of $(3^{1/5} + 2^{1/3})^{15}$.

Sol. The general term in the expansion of $(3^{1/5} + 2^{1/3})^{15}$ is

$$T_{r+1} = {}^{15}C_r\, (3^{1/5})^{15-r} \cdot (2^{1/3})^r$$

$$= {}^{15}C_r \cdot 3^{3 - \frac{r}{5}} \cdot 2^{\frac{r}{3}}$$

Now, $\quad 0 \le r \le 15$

For $r = 0, 15$

Rational terms are T_{0+1} and T_{15+1}.

Then, $\quad T_{0+1} = {}^{15}C_0 \cdot 3^3 \cdot 2^0 = 27$

and $\quad T_{15+1} = {}^{15}C_{15} \cdot 3^0 \cdot 2^5 = 32$

$\therefore$ Sum of all rational terms $= 27 + 32 = 59$

| Example 18. Find the number of irrational terms in the expansion of $(\sqrt[8]{5}+\sqrt[6]{2})^{100}$.

Sol. Since, $(\sqrt[8]{5}+\sqrt[6]{2})^{100} = (5^{1/8}+2^{1/6})^{100}$

$\therefore$ General term, $T_{r+1} = {}^{100}C_r\,(5^{1/8})^{100-r}\,(2^{1/6})^r$

$= {}^{100}C_r\,(5)^{(100-r)/8}\cdot(2)^{r/6}$

As, 2 and 5 are coprime.

$\therefore$ T_{r+1} will be rational, if $(100-r)$ is a multiple of 8 and r is a multiple of 6.

Also, $0 \le r \le 100$

$\therefore$ $r = 0, 6, 12, 18, \ldots, 96$

Now, $100 - r = 4, 10, 16, \ldots, 100$ ...(i)

and $100 - r = 0, 8, 16, 24, \ldots, 100$ ...(ii)

The common terms in Eqs. (i) and (ii) are 16, 40, 64 and 88.

$\therefore$ $r = 84, 60, 36, 12$ gives rational terms.

$\therefore$ The number of irrational terms $= 101 - 4 = 97$

Problems Regarding Three/Four Consecutive Terms or Coefficients

(i) If consecutive coefficients are given

In this case, divide consecutive coefficients pairwise, we get equations and then solve them.

| Example 19. Let n be a positive integer. If the coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of $(1+x)^n$ are in AP, then find the relation between n and r.

Sol. $\because$ $T_r = T_{(r-1)+1} = {}^nC_{r-1}\,x^{r-1}$

$T_{r+1} = {}^nC_r\,x^r$ and $T_{r+2} = T_{(r+1)+1} = {}^nC_{r+1}\,x^{r+1}$

$\therefore$ Coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of

$(1+x)^n$ are ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$.

$\because$ Given, ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$ are in AP

and $n \ge r+1$

$\therefore$ $\dfrac{{}^nC_{r-1}}{{}^nC_r}, 1, \dfrac{{}^nC_{r+1}}{{}^nC_r}$ are also in AP.

dividing by ${}^nC_r$.

$\Rightarrow$ $\dfrac{r}{n-r+1}, 1, \dfrac{n-r}{r+1}$ are in AP.

$\Rightarrow$ $1 - \dfrac{r}{n-r+1} = \dfrac{n-r}{r+1} - 1 \Rightarrow \dfrac{n-2r+1}{n-r+1} = \dfrac{n-2r-1}{r+1}$

$\Rightarrow$ $nr - 2r^2 + r + n - 2r + 1$

$= n^2 - 2nr - n - nr + 2r^2 + r + n - 2r - 1$

$\Rightarrow$ $n^2 - 4nr + 4r^2 = n + 2 \Rightarrow (n-2r)^2 = n+2$

***Corollary* I** For $r = 2, n = 7$ $[\because n \ge 3]$

***Corollary* II** For $r = 5, n = 7, 14$ $[\because n \ge 6]$

| Example 20. If a, b, c and d are any four consecutive coefficients in the expansion of $(1+x)^n$, then prove that:

(i) $\dfrac{a}{a+b} + \dfrac{c}{c+d} = \dfrac{2b}{b+c}$.

(ii) $\left(\dfrac{b}{b+c}\right)^2 > \dfrac{ac}{(a+b)(c+d)}$, if $x > 0$.

Sol. Let a, b, c and d be the coefficients of the r th, $(r+1)$th, $(r+2)$th and $(r+3)$th terms respectively, in the expansion of $(1+x)^n$. Then,

$T_r = T_{r-1+1} = {}^nC_{r-1}\,x^{r-1}$

$\therefore$ $a = {}^nC_{r-1}$ ...(i)

$\because$ $T_{r+1} = {}^nC_r\,x^r$

$\therefore$ $b = {}^nC_r$ ...(ii)

$\because$ $T_{r+2} = T_{(r+1)+1} = {}^nC_{r+1}\,x^{r+1}$

$\therefore$ $c = {}^nC_{r+1}$ ...(iii)

and $T_{r+3} = T_{(r+2)+1} = {}^nC_{r+2}\,x^{r+2}$

$\therefore$ $d = {}^nC_{r+2}$ ...(iv)

From Eqs. (i) and (ii), we get

$a + b = {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

$= \dfrac{n+1}{r}\cdot{}^nC_{r-1} = \left(\dfrac{n+1}{r}\right)a$

$\therefore$ $\dfrac{a}{a+b} = \dfrac{r}{n+1}$ ...(v)

From Eqs. (ii) and (iii), we get

$b + c = {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

$= \left(\dfrac{n+1}{r+1}\right){}^nC_r = \left(\dfrac{n+1}{r+1}\right)b$

$\therefore$ $\dfrac{b}{b+c} = \dfrac{r+1}{n+1}$ ...(vi)

From Eqs. (iii) and (iv), we get

$c + d = {}^nC_{r+1} + {}^nC_{r+2} = {}^{n+1}C_{r+2}$

$= \left(\dfrac{n+1}{r+2}\right){}^nC_{r+1} = \left(\dfrac{n+1}{r+2}\right)c$

$\therefore$ $\dfrac{c}{c+d} = \dfrac{r+2}{n+1}$ ...(vii)

From Eqs. (v), (vi) and (vii), we get

$\dfrac{a}{a+b}, \dfrac{b}{b+c}$ and $\dfrac{c}{c+d}$ are in AP.

(i) $\dfrac{a}{a+b} + \dfrac{c}{c+d} = 2\left(\dfrac{b}{b+c}\right)$

or $\dfrac{a}{a+b} + \dfrac{c}{c+d} = \dfrac{2b}{b+c}$

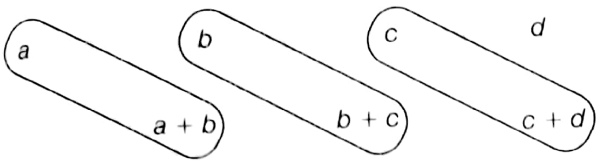
(ii) AM > GM

$$\therefore \quad \left(\frac{b}{b+c}\right) > \sqrt{\left(\frac{a}{a+b}\right)\left(\frac{c}{c+d}\right)}$$

$$\Rightarrow \quad \left(\frac{b}{b+c}\right)^2 > \frac{ac}{(a+b)(c+d)}$$

Remembering Method

$\because$



$$\therefore \quad \frac{a}{a+b}, \frac{b}{b+c} \text{ and } \frac{c}{c+d} \text{ are in AP.}$$

(ii) If consecutive terms are given

In this case, divide consecutive terms pairwise. i.e., If four consecutive terms are $T_r, T_{r+1}, T_{r+2}, T_{r+3}$. Then, find $\frac{T_{r+1}}{T_r}, \frac{T_{r+2}}{T_{r+1}}, \frac{T_{r+3}}{T_{r+2}} \Rightarrow \lambda_1, \lambda_2, \lambda_3$ (say). Then, divide λ_2 by λ_1 and λ_3 by λ_2 and solve.

| Example 21. If the 2nd, 3rd and 4th terms in the expansion of $(x+y)^n$ are 240, 720 and 1080 respectively, find x, y and n.

Sol. Given, $T_2 = T_{1+1} = {}^nC_1 \cdot x^{n-1} \cdot y = 240$ ...(i)

$T_3 = T_{2+1} = {}^nC_2 \cdot x^{n-2} \cdot y^2 = 720$ ...(ii)

and $T_4 = T_{3+1} = {}^nC_3 \cdot x^{n-3} \cdot y^3 = 1080$ ...(iii)

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{{}^nC_2 \cdot x^{n-2} \cdot y^2}{{}^nC_1 \cdot x^{n-1} \cdot y} = \frac{720}{240}$$

$$\Rightarrow \quad \left(\frac{n-2+1}{2}\right) \cdot \frac{y}{x} = 3 \Rightarrow \frac{y}{x} = \frac{6}{n-1} \quad \text{...(iv)}$$

Also, dividing Eq. (iii) by Eq. (ii), we get

$$\frac{{}^nC_3 \cdot x^{n-3} \cdot y^3}{{}^nC_2 \cdot x^{n-2} \cdot y^2} = \frac{1080}{720}$$

$$\Rightarrow \quad \left(\frac{n-3+1}{3}\right) \cdot \frac{y}{x} = \frac{3}{2} \Rightarrow \frac{y}{x} = \frac{9}{2(n-2)} \quad \text{...(v)}$$

From Eqs. (iv) and (v), we get

$$\frac{6}{n-1} = \frac{9}{2(n-2)}$$

$$\Rightarrow \quad 12n - 24 = 9n - 9$$

$$\Rightarrow \quad 3n = 15$$

$$\therefore \quad n = 5$$

From Eq. (iv), we get $y = \frac{3}{2}x$ ...(vi)

From Eqs. (i) and (vi), we get

$$ {}^5C_1 \cdot x^4 \cdot y = 240 \quad \Rightarrow 5 \cdot x^4 \cdot \frac{3}{2}x = 240$$

$$\therefore \quad x^5 = 32 = 2^5 \Rightarrow x = 2$$

From Eq. (vi), we get $y = 3$

Hence, $x = 2, y = 3$ and $n = 5$

Middle Terms

The middle term depends upon the value of n.

(i) **When n is even** The total number of terms in the expansion of $(x+a)^n$ is $n+1$ (odd). So, there is only one middle term, i.e., $\left(\frac{n}{2}+1\right)$th term is the middle term. It is given by $T_{n/2+1} = {}^nC_{n/2}\, x^{n/2}\, a^{n/2}$

(ii) **When n is odd** The total number of terms in the expansion of $(x+a)^n$ is $n+1$ (even). So, there are two middle terms, i.e., $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th are two middle terms. They are given by

$$T_{\frac{n+1}{2}} = T_{\left(\frac{n-1}{2}\right)+1} = {}^nC_{\frac{n-1}{2}} \cdot x^{\frac{n+1}{2}} \cdot a^{\frac{n-1}{2}}$$

$$\text{and} \quad T_{\frac{n+3}{2}} = T_{\left(\frac{n+1}{2}\right)+1} = {}^nC_{\frac{n+1}{2}} \cdot x^{\frac{n-1}{2}} \cdot a^{\frac{n+1}{2}}$$

| Example 22. Find the middle term in the expansion of $\left(\frac{a}{x}+bx\right)^{12}$.

Sol. The number of terms in the expansion of $\left(\frac{a}{x}+bx\right)^{12}$ is 13 (odd), its middle term is $\left(\frac{12}{2}+1\right)$ th, i.e., 7th term.

$$\therefore \text{ Required term, } T_7 = T_{6+1} = {}^{12}C_6 \left(\frac{a}{x}\right)^6 (bx)^6$$

$$= {}^{12}C_6\, a^6\, b^6 = 924\, a^6\, b^6$$

| Example 23. Find the middle term in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$.

Sol. The number of terms in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$ is 10 (even). So, there are two middle terms, i.e. $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+3}{2}\right)$th terms. They are given by T_5 and T_6.

$\therefore \quad T_5 = T_{4+1} = {}^9C_4 (3x)^5 \left(-\frac{x^3}{6}\right)^4$

$= \frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4}\cdot 3^5\, x^5 \cdot \frac{x^{12}}{6^4} = \frac{189}{8} x^{17}$

and $\quad T_6 = T_{5+1} = {}^9C_5 (3x)^4 \left(-\frac{x^3}{6}\right)^5$

$= -\, {}^9C_4 \cdot 3^4 \cdot x^4 \cdot \frac{x^{15}}{6^5}$

$= -\frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4}\cdot 3^4 \cdot \frac{x^{19}}{6^5} = -\frac{21}{16} x^{19}$

Example 24. Show that the middle term in the expansion of $(1+x)^{2n}$ is

$\frac{1\cdot 3\cdot 5 \ldots (2n-1)}{n!}\cdot 2^n\, x^n$, n being a positive integer.

Sol. The number of terms in the expansion of $(1+x)^{2n}$ is $2n+1$ (odd), its middle term is $(n+1)$th term.

$\therefore$ Required term $= T_{n+1}$

$= {}^{2n}C_n\, x^n = \frac{2n!}{n!\,n!} x^n = \frac{(1\cdot 2\cdot 3\cdot 4\ldots(2n-1)\cdot 2n)}{n!\,n!} x^n$

$= \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\{2\cdot 4\cdot 6\ldots 2n\}}{n!\,n!} x^n$

$= \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\, 2^n\, (1\cdot 2\cdot 3\ldots n)}{n!\,n!} x^n$

$= \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\, 2^n\, n!}{n!\,n!} x^n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!} 2^n\, x^n$

Greatest Term

If T_r and T_{r+1} are the rth and $(r+1)$th terms in the expansion of $(x+a)^n$, then

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r \cdot x^{n-r}\cdot a^r}{{}^nC_{r-1}\cdot x^{n-r+1}\cdot a^{r-1}} = \left(\frac{n-r+1}{r}\right)\cdot\frac{a}{x}$$

Let numerically, T_{r+1} be the greatest term in the above expansion. Then,

$$T_{r+1} \ge T_r \text{ or } \frac{T_{r+1}}{T_r} \ge 1 \Rightarrow \left(\frac{n-r+1}{r}\right)\left|\frac{a}{x}\right| \ge 1$$

[$\because$ a may be + ve or – ve]

or $$r \le \frac{(n+1)}{\left(1+\left|\frac{x}{a}\right|\right)} \quad \ldots(i)$$

Now, on substituting values of n, x and a in Eq. (i), we get

$r \le m + f$ or $r \le m$

where, $\quad m \in N$ and $0 < f < 1$

In the first case, T_{m+1} is the greatest term, while in the second case, T_m and T_{m+1} are the greatest terms and both are equal (numerically).

Shortcut Method

To find the greatest term (numerically) in the expansion of $(x+a)^n$.

Now, $\quad (x+a)^n = a^n\left(1+\frac{x}{a}\right)^n$

Calculate $\quad m = \frac{\left|\frac{x}{a}\right|(n+1)}{\left(\left|\frac{x}{a}\right|+1\right)}$

Case I If $m \in$ Integer, then T_m and T_{m+1} are the greatest terms and both are equal (numerically).

Case II If $m \notin$ Integer, then $T_{[m]+1}$ is the greatest term, where $[\cdot]$ denotes the greatest integer function.

Example 25. Find numerically the greatest term in the expansion of $(2+3x)^9$, when $x = 3/2$.

Sol. Let T_{r+1} be the greatest term in the expansion of $(2+3x)^9$, we have

$$\frac{T_{r+1}}{T_r} = \left(\frac{9-r+1}{r}\right)\left|\frac{3x}{2}\right| = \left(\frac{10-r}{r}\right)\left|\frac{3}{2}\times\frac{3}{2}\right| = \frac{90-9r}{4r}$$

[$\because x = 3/2$]

$\therefore \quad \frac{T_{r+1}}{T_r} \ge 1$

$\Rightarrow \quad \frac{90-9r}{4r} \ge 1 \Rightarrow 90 \ge 13r$

$\therefore \quad r \le \frac{90}{13} = 6\frac{12}{13}$

or $\quad r \le 6\frac{12}{13}$

$\therefore$ Maximum value of r is 6.

So, greatest term $= T_{6+1} = {}^9C_6\, (2)^{9-6}\, (3x)^6$

$= {}^9C_3 \cdot 2^3 \cdot \left(3\times\frac{3}{2}\right)^6$

$= \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{2^3\cdot 3^{12}}{2^6} = \frac{7\times 3^{13}}{2}$

Aliter Since, $(2+3x)^9 = 2^9\left(1+\frac{3x}{2}\right)^9$

Now, $m = \dfrac{(9+1)\left|\dfrac{3x}{2}\right|}{\left|\dfrac{3x}{2}\right|+1} = \dfrac{10 \times \dfrac{9}{4}}{\dfrac{9}{4}+1}$ $[\because x = 3/2]$

$= \dfrac{90}{13} = 6\dfrac{12}{13} \neq$ Integer

$\therefore$ The greatest term in the expansion is

$T_{[m]+1} = T_{6+1}$ in $(2+3x)^9$

$= {}^9C_6 (2)^{9-6} (3x)^6 = {}^9C_3 \cdot 2^3 \cdot \left(\dfrac{3^2}{2}\right)^6$ $[\because x = 3/2]$

$= \dfrac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} \cdot \dfrac{3^{12}}{2^3} = \dfrac{7 \times 3^{13}}{2}$

| Example 26. Find numerically the greatest term in the expansion of $(3-5x)^{11}$, when $x = \dfrac{1}{5}$.

Sol. Let T_{r+1} be the greatest term in the expansion of $(3-5x)^{11}$, we have

$$\frac{T_{r+1}}{T_r} = \left(\frac{11-r+1}{r}\right)\left|-\frac{5x}{3}\right|$$

$$= \left(\frac{12-r}{r}\right)\left|-\frac{1}{3}\right| = \frac{12-r}{3r} \quad [\because x = 1/5]$$

$$\therefore \quad \frac{T_{r+1}}{T_r} \geq 1 \Rightarrow \frac{12-r}{3r} \geq 1 \Rightarrow 12 \geq 4r$$

$$\therefore \quad r \leq 3 \Rightarrow r = 2, 3$$

So, the greatest terms are T_{2+1} and T_{3+1}.

$\therefore$ Greatest term (when $r = 2$) $= T_{2+1} = {}^{11}C_2 (3)^9 (-5x)^2$

$$= \frac{11\cdot 10}{1\cdot 2} \cdot 3^9 \cdot (1)^2 = 55 \times 3^9 \quad [\because x = 1/5]$$

and greatest term (when $r = 3$) $= T_{3+1}$

$$= \left| {}^{11}C_3 (3)^8 (-5x)^3 \right| = \left| {}^{11}C_3 (3)^8 (-1)^3 \right| \quad [\because x = 1/5]$$

$$= {}^{11}C_3 \cdot 3^8 = \frac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3} \cdot 3^8 = 55 \times 3^9$$

From above, we say that the values of both greatest terms are equal.

Aliter

Since, $(3-5x)^{11} = 3^{11}\left(1 - \dfrac{5x}{3}\right)^{11}$

Now, $m = \dfrac{(11+1)\left|-\dfrac{5x}{3}\right|}{\left|-\dfrac{5x}{3}\right|+1} = \dfrac{12 \times \left|-\dfrac{1}{3}\right|}{\left|-\dfrac{1}{3}\right|+1}$ $\left[\because x = \dfrac{1}{5}\right]$

$$= \frac{4}{\dfrac{1}{3}+1} = 3$$

Since, the greatest terms in the expansion are T_3 and T_4.

$\therefore$ Greatest term (when $r = 2$) $= {}^{11}C_2 (3)^9 (-5x)^2$

$$= {}^{11}C_2 (3)^9 (-1)^2 \quad \left[\because x = \frac{1}{5}\right]$$

$$= \frac{11\cdot 10}{1\cdot 2} \cdot 3^9 = 55 \times 3^9$$

and greatest term (when $r = 3$) $= \left| {}^{11}C_3 (3)^8 (-5x)^3 \right|$

$$= \left| {}^{11}C_3 (3)^8 (-1)^3 \right| \quad \left[\because x = \frac{1}{5}\right]$$

$$= \frac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3} \cdot 3^8 = 55 \times 3^9$$

Greatest Coefficient

(i) If n is even, then greatest coefficient is ${}^nC_{n/2}$.

(ii) If n is odd, then greatest coefficients are ${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$.

| Example 27. Show that, if the greatest term in the expansion of $(1+x)^{2n}$ has also the greatest coefficient, then x lies between $\dfrac{n}{n+1}$ and $\dfrac{n+1}{n}$.

Sol. In the expansion of $(1+x)^{2n}$, the middle term is $\left(\dfrac{2n}{2}+1\right)$th

i.e., $(n+1)$th term, we know that from binomial expansion, middle term has greatest coefficient.

[$\because$ Terms $T_1, T_2, T_3, \ldots, T_n, T_{n+1}, T_{n+2}, \ldots$]

$$\therefore \quad T_n < T_{n+1} > T_{n+2}$$

$$\Rightarrow \quad \frac{T_{n+1}}{T_n} = \frac{{}^{2n}C_n \cdot x^n}{{}^{2n}C_{n-1} \cdot x^{n-1}} = \frac{2n-n+1}{n} \cdot x$$

$$\Rightarrow \quad \frac{T_{n+1}}{T_n} > 1 \text{ or } \frac{n+1}{n} \cdot x > 1$$

$$\text{or} \quad x > \frac{n}{n+1} \quad \ldots\text{(i)}$$

$$\text{and} \quad \frac{T_{n+2}}{T_{n+1}} = \frac{{}^{2n}C_{n+1}\, x^{n+1}}{{}^{2n}C_n\, x^n} = \frac{2n-(n+1)+1}{n+1} \cdot x$$

$$= \frac{n}{n+1} \cdot x$$

$$\Rightarrow \quad \frac{T_{n+2}}{T_{n+1}} < 1 \Rightarrow \frac{n}{n+1} \cdot x < 1 \text{ or } x < \frac{n+1}{n} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{n}{n+1} < x < \frac{n+1}{n}$$

Corollary For $n = 5$

$$\frac{5}{6} < x < \frac{6}{5}$$

Important Properties of the Binomial Coefficients

In the binomial expansion of $(1+x)^n$. Let us denote the coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_r, \ldots, {}^nC_n$ by $C_0, C_1, C_2, \ldots, C_r, \ldots, C_n$, respectively.

(i) The coefficients of the terms equidistant from the beginning and the end are equal

The $(r+1)$th term from the beginning in the expansion of $(1+x)^n$ is ${}^nC_r\, x^r$.

$\therefore$ The coefficient of the $(r+1)$th term from the beginning is ${}^nC_r$ and the $(r+1)$th term from the end in the expansion of $(1+x)^n = (r+1)$ th term from the beginning in the expansion of $(x+1)^n = {}^nC_r\, x^{n-r}$

$\therefore$ The coefficient of the $(r+1)$th term from the end is ${}^nC_r$.

Hence, the coefficients of $(r+1)$th term from the beginning and the end are equal.

(ii) The sum of the binomial coefficients in the expansion of $(1+x)^n$

$$\because \quad (1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots + {}^nC_n\, x^n$$

Putting $x = 1$, we get

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$$

or $\boxed{C_0 + C_1 + C_2 + \ldots + C_n = 2^n}$

$\therefore$ Sum of binomial coefficients $= 2^n$

(iii) The sum of the coefficients of the odd terms = The sum of the coefficients of the even terms

$$\because \quad (1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots + {}^nC_n\, x^n$$

Putting $x = -1$, we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + \ldots$$

or $\quad {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots$

Since, the sum of all the coefficients is 2^n, therefore each side is equal to $\dfrac{2^n}{2}$ i.e. 2^{n-1}.

Hence, $C_1 + C_3 + C_5 + \ldots = C_0 + C_2 + C_4 + \ldots = 2^{n-1}$

Remark

1. In the expansion of $(x - 2y + 3z)^n$, putting $x = y = z = 1$, then we get the sum of coefficients $= (1 - 2 + 3)^n = 2^n$.
2. In the expansion of $(1 + x + x^2)^n$, putting $x = 1$, we get the sum of coefficients $= (1 + 1 + 1)^n = 3^n$.

Trinomial Expansion

For $n \in N, (1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$

$$= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots + a_{2n} x^{2n} \quad \ldots \text{(i)}$$

There are $(2n+1)$ terms. The middle coefficient is a_n which is also the greatest.

$$a_0 = a_{2n}, a_1 = a_{2n-1}, \ldots, a_r = a_{2n-r}$$

The coefficients of $(1 + x + x^2)^n$ for $n = 0, 1, 2, \ldots$ can be arranged in a triangle.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & 1 & 1 & & & & \\
 & & & 1 & 2 & 3 & 2 & 1 & & & \\
 & & 1 & 3 & 6 & 7 & 6 & 3 & 1 & & \\
 & \boxed{1 \;\; 4 \;\; 10} & & & 16 & 19 & \boxed{16 \;\; 10 \;\; 4} & & & 1 & \\
 & \downarrow & & & & & \downarrow & & & & \\
1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

i.e., The rows contains the coefficients for $n = 0, 1, 2, 3, \ldots$.

Each entry other than two entries at the ends is the sum of three entries above it.

$$15 = 1 + 4 + 10, 30 = 16 + 10 + 4, \text{ etc.}$$

Putting $x = 1$ and $x = -1$ in Eq. (i), we get

$$a_0 + a_1 + a_2 + a_3 + \ldots + a_{2n} = 3^n$$

[sum of all coefficients] ...(ii)

and $\quad a_0 - a_1 + a_2 - a_3 + \ldots + a_{2n} = 1 \quad$ ...(iii)

On adding Eqs. (ii) and (iii), we get

$$a_0 + a_2 + a_4 + \ldots + a_{2n} = \frac{3^n + 1}{2}$$

[sum of coefficients of even powers of x]

On subtracting Eq. (ii) from Eq. (i), we get

$$a_1 + a_3 + a_5 + \ldots + a_{2n-1} = \frac{3^n - 1}{2}$$

[sum of coefficients of odd powers of x]

Putting $x = i\,(\sqrt{-1})$ in Eq. (i), we get

$$a_0 + a_1\, i + a_2\, i^2 + a_3\, i^3 + a_4\, i^4 + a_5\, i^5 + \ldots + a_{2n}\, i^{2n} = i^n$$

$$\Rightarrow \quad (a_0 - a_2 + a_4 - \ldots) + i(a_1 - a_3 + a_5 - \ldots) = i^n$$

$$(a_0 - a_2 + a_4 - \ldots) + i(a_1 - a_3 + a_5 - \ldots)$$

$$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n = \cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)$$

On comparing real and imaginary parts, we get

$$a_0 - a_2 + a_4 - \ldots = \cos\left(\frac{n\pi}{2}\right)$$

and $a_1 - a_3 + a_5 - \ldots = \sin\left(\frac{n\pi}{2}\right)$

Putting $x = \omega$ and ω^2 (cube roots of unity) in Eq. (i), we get

$$a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + \ldots = 0 \quad \ldots(iv)$$

and $$a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + \ldots = 0 \quad \ldots(v)$$

On adding Eqs. (ii), (iv) and (v) and then dividing by 3, we get

$$a_0 + a_3 + a_6 + \ldots = 3^{n-1}$$

Note

(i) $a_1 + a_4 + a_7 + \ldots = a_2 + a_5 + a_8 + \ldots = 3^{n-1}$

(ii) $a_0 + a_4 + a_8 + \ldots = \frac{1}{4}\left\{3^n + 1 + 2\cos\left(\frac{n\pi}{2}\right)\right\}$

(iii) $a_1 + a_5 + a_9 + \ldots = \frac{1}{4}\left\{3^n - 1 + 2\sin\left(\frac{n\pi}{2}\right)\right\}$

(iv) $a_0 + a_6 + a_{12} + \ldots = \frac{1}{6}\left\{3^n + 1 + 2^{n+1}\cos\left(\frac{n\pi}{3}\right)\right\}$

(v) $\sum_{r=1}^{2n} r \cdot a_r = n \cdot 3^n$ (vi) $\sum_{r=1}^{2n} (-1)^{r-1} \cdot r \cdot a_r = -n$

| Example 28. Find the sum of coefficients in the expansion of the binomial $(5p - 4q)^n$, where n is a positive integer.

Sol. Putting $p = q = 1$ in $(5p - 4q)^n$, the required sum of coefficients $= (5-4)^n = 1^n = 1$

| Example 29. must In the expansion of $(3^{-x/4} + 3^{5x/4})^n$, if the sum of binomial coefficients is 64 and the term with the greatest binomial coefficient exceeds the third by $(n-1)$, find the value of x.

Sol. Given sum of the binomial coefficients in the expansion of $(3^{-x/4} + 3^{5x/4})^n = 64$

Then, putting $3^{-x/4} = 3^{5x/4} = 1$

$\therefore$ $(1+1)^n = 64 \Rightarrow 2^n = 2^6$

$\therefore$ $n = 6$

We know that, middle term has the greatest binomial coefficients. Here, $n = 6$

$\therefore$ Middle term $= \left(\frac{n}{2} + 1\right)$th term $= 4$th term $= T_4$

and given that $\boxed{T_4 = (n-1) + T_3}$

$\Rightarrow$ $T_{3+1} = (6-1) + T_{2+1}$

$\Rightarrow$ ${}^6C_3 (3^{-x/4})^3 (3^{5x/4})^3 = 5 + {}^6C_2 (3^{-x/4})^4 (3^{5x/4})^2$

$\Rightarrow$ $20 \cdot 3^{3x} = 5 + 15 \cdot 3^{3x/2}$

Let $3^{3x/2} = t$

$\therefore$ $20t^2 = 5 + 15t$

$\Rightarrow$ $4t^2 - 3t - 1 = 0$

$\Rightarrow$ $(4t+1)(t-1) = 0$

$\therefore$ $t = 1,\ t \neq -\frac{1}{4} \Rightarrow 3^{3x/2} = 1 = 3^0$

$\therefore$ $\frac{3x}{2} = 0$ or $x = 0$

| Example 30. Find the values of

(i) $\frac{1}{(n-1)!} + \frac{1}{(n-3)!\,3!} + \frac{1}{(n-5)!\,5!} + \ldots$

(ii) $\frac{1}{12!} + \frac{1}{10!\,2!} + \frac{1}{8!\,4!} + \ldots + \frac{1}{12!}$

Sol. (i) $\because 1! = 1$

$\therefore$ The given series can be written as

see this always

$$\frac{1}{(n-1)!\,1!} + \frac{1}{(n-3)!\,3!} + \frac{1}{(n-5)!\,5!} + \ldots \quad \ldots(i)$$

$\because$ Sum of values of each terms in factorial are equal.

i.e. $(n-1) + 1 = (n-3) + 3 = (n-5) + 5 = \ldots = n$

From Eq. (i),

$$\frac{1}{n!}\left[\frac{n!}{(n-1)!\,1!} + \frac{n!}{(n-3)!\,3!} + \frac{n!}{(n-5)!\,5!} + \ldots\right]$$

$$= \frac{1}{n!}({}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots) = \frac{2^{n-1}}{n!}$$

(ii) $\because 0! = 1$

$\therefore$ The given series can be written as

$$\frac{1}{12!\,0!} + \frac{1}{10!\,2!} + \frac{1}{8!\,4!} + \ldots + \frac{1}{0!\,12!} \quad \ldots(ii)$$

$\because$ Sum of values of each terms in factorial are equal

i.e., $12 + 0 = 10 + 2 = 8 + 4 = \ldots = 12$

From Eq. (ii), $\frac{1}{12!}\left[\frac{12!}{12!\,0!} + \frac{12!}{10!\,2!} + \frac{12!}{8!\,4!} + \ldots + \frac{12!}{0!\,12!}\right]$

$$= \frac{1}{12!}({}^{12}C_0 + {}^{12}C_2 + {}^{12}C_4 + \ldots + {}^{12}C_{12}) = \frac{2^{12-1}}{12!} = \frac{2^{11}}{12!}$$

| Example 31. Prove that the sum of the coefficients in the expansion of $(1 + x - 3x^2)^{2163}$ is -1.

Sol. Putting $x = 1$ in $(1 + x - 3x^2)^{2163}$, the required sum of coefficients $= (1 + 1 - 3)^{2163} = (-1)^{2163} = -1$

| Example 32. If the sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, find the value of α.

Sol. Given, sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$

$= $ Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$

Putting $x = y = 1$, we get

$$(\alpha - 1)^{35} = (1 - \alpha)^{35}$$
$$\Rightarrow \quad (\alpha - 1)^{35} = -(\alpha - 1)^{35}$$
$$\Rightarrow \quad 2(\alpha - 1)^{35} = 0$$
$$\Rightarrow \quad \alpha - 1 = 0$$
$$\therefore \quad \alpha = 1$$

Example 33. If $(1 + x - 2x^2)^{20} = \sum_{r=0}^{40} a_r x^r$, then find the value of $a_1 + a_3 + a_5 + \ldots + a_{39}$.

Sol. $\because \quad (1 + x - 2x^2)^{20} = \sum_{r=0}^{40} a_r x^r \quad \ldots(i)$

Putting $x = 1$, we get $0 = \sum_{r=0}^{40} a_r$

or $\quad a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_{39} + a_{40} = 0 \quad \ldots(ii)$

Putting $x = -1$ in Eq. (i), we get

$$(-2)^{20} = \sum_{r=0}^{40} (-1)^r a_r$$

or $a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \ldots - a_{39} + a_{40} = 2^{20} \quad \ldots(iii)$

On subtracting Eq. (iii) from Eq. (ii), we get

$$2[a_1 + a_3 + a_5 + \ldots + a_{39}] = -2^{20}$$

or $\quad a_1 + a_3 + a_5 + \ldots + a_{39} = -2^{19}$

Corollary On adding Eqs. (ii) and (iii) and then dividing by 2, we get $a_0 + a_2 + a_4 + \ldots + a_{40} = 2^{19}$

b, c, d, b, c, c, c, b, a, d

Exercise for Session 2

1. If the rth term in the expansion of $(1 + x)^{20}$ has its coefficient equal to that of the $(r + 4)$th term, then r is
   (a) 7 (b) 9 (c) 11 (d) 13
2. If the fourth term in the expansion of $\left(px + \frac{1}{x}\right)^n$ is $\frac{5}{2}$, then $n + p$ is equal to
   (a) $\frac{9}{2}$ (b) $\frac{11}{2}$ (c) $\frac{13}{2}$ (d) $\frac{15}{2}$
3. If in the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of 7th term from the beginning to the 7th term from the end is $\frac{1}{6}$, then n is
   (a) 3 (b) 5 (c) 7 (d) 9
4. The number of integral terms in the expansion of $(5^{1/2} + 7^{1/8})^{1024}$ is
   (a) 128 (b) 129 (c) 130 (d) 131
5. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radicals is
   (a) 715 (b) 725 (c) 730 (d) 750
6. If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are 165, 330 and 462 respectively, the value of n is
   (a) 7 (b) 9 (c) 11 (d) 13
7. If the coefficients of 5th, 6th and 7th terms in the expansion of $(1 + x)^n$ are in AP, then n is equal to
   (a) 7 only (b) 14 only (c) 7 or 14 (d) None of these
8. If the middle term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ is $924x^6$, the value of n is
   (a) 8 (b) 12 (c) 16 (d) 20
9. If the sum of the binomial coefficients in the expansion of $\left(x^2 + \frac{2}{x^3}\right)^n$ is 243, the term independent of x is equal to
   (a) 40 (b) 30 (c) 20 (d) 10
10. In the expansion of $(1 + x)(1 + x + x^2)\ldots(1 + x + x^2 + \ldots + x^{2n})$, the sum of the coefficients is
    (a) 1 (b) $2n!$ (c) $2n! + 1$ (d) $(2n + 1)!$

Session 3

Two Important Theorems, Divisibility Problems

Two Important Theorems

Theorem 1 If $(\sqrt{P} + Q)^n = I + f$, where I and n are positive integers, n being odd and $0 \le f < 1$, then show that $(I + f)f = k^n$, where $\boxed{P - Q^2 = k > 0}$ and $\boxed{\sqrt{P} - Q < 1}$.

Proof Given, $\sqrt{P} - Q < 1 \quad \therefore \boxed{0 < (\sqrt{P} - Q)^n < 1}$?

Now, let $(\sqrt{P} - Q)^n = f'$, where $0 < f' < 1$

Also
$$I + f = (\sqrt{P} + Q)^n \quad \ldots(i)$$
$$0 \le f < 1 \quad \ldots(ii)$$
$$f' = (\sqrt{P} - Q)^n \quad \ldots(iii)$$
and
$$0 < f' < 1 \quad \ldots(iv)$$

On subtracting Eq. (iii) from Eq. (i), we get

$$I + f - f' = (\sqrt{P} + Q)^n - (\sqrt{P} - Q)^n$$
$$= 2[{}^nC_1 (\sqrt{P})^{n-1} \cdot Q + {}^nC_3 (\sqrt{P})^{n-3} \cdot Q^3 + \ldots]$$
$$= 2\,(\text{integer}) = \text{Even integer} \quad \ldots(v)$$

[Since, n is odd, RHS contains even powers of $\sqrt{P}$, so RHS is an even integer]

$\therefore$ LHS is also an integer.

$\because$ I is an integer.

$\therefore$ $(f - f')$ is also an integer.

$\Rightarrow \quad f - f' = 0 \quad [\because -1 < (f - f') < 1]$

or $\quad f = f'$

From Eq. (v), I is an even integer and

$$(I + f)f = (I + f)f' = (\sqrt{P} + Q)^n (\sqrt{P} - Q)^n$$
$$= (P - Q^2)^n = k^n$$

Remark

If n is even integer, then $(\sqrt{P} + Q)^n + (\sqrt{P} - Q)^n = I + f + f'$

Since, LHS and I are integers.

$\therefore \quad (f + f')$ is also an integer.

$\Rightarrow \quad f + f' = 1 \quad [\because 0 < (f + f') < 2]$

$\therefore \quad f' = 1 - f$

Hence, $(I + f)(1 - f) = (I + f)f' = (\sqrt{P} + Q)^n (\sqrt{P} - Q)^n$
$= (P - Q^2)^n = k^n$

Theorem 2 If $(P + \sqrt{Q})^n = I + f$, where I and n are positive integers and $0 \le f < 1$, show that $(I + f)(1 - f) = k^n$, where $P^2 - Q = k > 0$ and $P - \sqrt{Q} < 1$.

Proof Given, $\quad P - \sqrt{Q} < 1$

$\therefore \quad 0 < (P - \sqrt{Q})^n < 1$

Now, let $(P - \sqrt{Q})^n = f'$, where $0 < f' < 1$

Also,
$$I + f = (P + \sqrt{Q})^n \quad \ldots(i)$$
$$0 \le f < 1 \quad \ldots(ii)$$
$$f' = (P - \sqrt{Q})^n \quad \ldots(iii)$$
and
$$0 < f' < 1 \quad \ldots(iv)$$

On adding Eqs. (i) and (iii), we get

$$I + f + f' = (P + \sqrt{Q})^n + (P - \sqrt{Q})^n$$
$$= 2[{}^nC_0 P^n + {}^nC_2 P^{n-2} (\sqrt{Q})^2 + {}^nC_4 P^{n-4} (\sqrt{Q})^4 + \ldots]$$
$$= 2\,(\text{integer}) = \text{Even integer} \quad \ldots(v)$$

[Since, RHS contains even power of $\sqrt{Q}$, so RHS is an even integer]

$\therefore$ LHS is also an integer.

$\because$ I is an integer.

$\Rightarrow$ $f + f'$ is also an integer.

$\therefore \quad f + f' = 1 \quad [\because 0 < (f + f') < 2]$

or $\quad f' = 1 - f$

From Eq. (v), I = even integer $- 1$ = odd integer and

$$(I + f)(1 - f) = (I + f)f'$$
$$= (P + \sqrt{Q})^n (P - \sqrt{Q})^n = (P^2 - Q)^n = k^n$$

Example 34. Show that the integral part of $(5 + 2\sqrt{6})^n$ is odd, where n is natural number.

Sol. $(5 + 2\sqrt{6})^n$ can be written as $(5 + \sqrt{24})^n$

Now, let
$$I + f = (5 + \sqrt{24})^n \quad \ldots(i)$$
$$0 \le f < 1 \quad \ldots(ii)$$
and let
$$f' = (5 - \sqrt{24})^n \quad \ldots(iii)$$
$$0 < f' < 1 \quad \ldots(iv)$$

On adding Eqs. (i) and (iii), we get

$$I + f + f' = (5 + \sqrt{24})^n + (5 - \sqrt{24})^n$$
$$I + 1 = 2p,$$
$$\forall\, p \in N = \text{Even integer} \quad \text{[from theorem 2]}$$
$$\therefore \quad I = 2p - 1 = \text{Odd integer}$$

Example 35. Show that the integral part of $(5\sqrt{5} + 11)^{2n+1}$ is even, where $n \in N$.

Sol. $(5\sqrt{5} + 11)^{2n+1}$ can be written as $(\sqrt{125} + 11)^{2n+1}$

Now, let
$$I + f = (\sqrt{125} + 11)^{2n+1} \quad \ldots(i)$$
$$0 \le f < 1 \quad \ldots(ii)$$
and let
$$f' = (\sqrt{125} - 11)^{2n+1} \quad \ldots(iii)$$
$$0 < f' < 1 \quad \ldots(iv)$$

On subtracting Eq. (iii) from Eq. (i), we get

$$I + f - f' = (\sqrt{125} + 11)^{2n+1} - (\sqrt{125} - 11)^{2n+1}$$

$$I + 0 = 2p, \forall\ p \in N = \text{Even integer} \quad \text{[from theorem 1]}$$

$$I = 2p = \text{Even integer}$$

Example 36. Let $R = (6\sqrt{6} + 14)^{2n+1}$ and $f = R - [R]$, where $[\cdot]$ denotes the greatest integer function. Find the value of $Rf, n \in N$.

Sol. $(6\sqrt{6} + 14)^{2n+1}$ can be written as $(\sqrt{216} + 14)^{2n+1}$ and given that $f = R - [R]$

and $\quad R = (6\sqrt{6} + 14)^{2n+1} = (\sqrt{216} + 14)^{2n+1}$

$\therefore \quad [R] + f = (\sqrt{216} + 14)^{2n+1} \quad$ ...(i)

$0 \le f < 1 \quad$ ...(ii)

Let $\quad f' = (\sqrt{216} - 14)^{2n+1} \quad$ ...(iii)

$0 < f' < 1 \quad$ ...(iv)

On subtracting Eq. (iii) from Eq. (i), we get

$$[R] + f - f' = (\sqrt{216} + 14)^{2n+1} - (\sqrt{216} - 14)^{2n+1}$$

$$[R] + 0 = 2p, \forall\ p \in N = \text{Even integer [from theorem 1]}$$

$\therefore \quad f - f' = 0$ or $f = f'$

Now, $\quad Rf = Rf' = (\sqrt{216} + 14)^{2n+1}(\sqrt{216} - 14)^{2n+1}$

$$= (216 - 196)^{2n+1} = (20)^{2n+1}$$

Example 37. If $(7 + 4\sqrt{3})^n = s + t$, where n and s are positive integers and t is a proper fraction, show that $(1 - t)(s + t) = 1$.

Sol. $(7 + 4\sqrt{3})^n$ can be written as $(7 + \sqrt{48})^n$

$\therefore \quad s + t = (7 + \sqrt{48})^n \quad$ ...(i)

$0 < t < 1 \quad$ ...(ii)

Now, let $\quad t' = (7 - \sqrt{48})^n \quad$ ...(iii)

$0 < t' < 1 \quad$ ...(iv)

On adding Eqs. (i) and (iii), we get

$$s + t + t' = (7 + \sqrt{48})^n + (7 - \sqrt{48})^n$$

$$s + 1 = 2p, \forall\ p \in N = \text{Even integer [from theorem 2]}$$

$\therefore \quad t + t' = 1$ or $1 - t = t'$

Then, $(1 - t)(s + t) = t'(s + t) = (7 - \sqrt{48})^n (7 + \sqrt{48})^n$ [from Eqs. (i) and (iii)]

$$= (49 - 48)^n = (1)^n = 1$$

Example 38. If $x = (8 + 3\sqrt{7})^n$, where n is a natural number, prove that the integral part of x is an odd integer and also show that $x - x^2 + x[x] = 1$, where $[\cdot]$ denotes the greatest integer function.

Sol. $(8 + 3\sqrt{7})^n$ can be written as $(8 + \sqrt{63})^n$

$\therefore \quad x = [x] + f$

or $\quad [x] + f = (8 + \sqrt{63})^n \quad$ ...(i)

$0 \le f < 1 \quad$ ...(ii)

Now, let $\quad f' = (8 - \sqrt{63})^n$

$0 < f' < 1 \quad$ ...(iii) ...(iv)

On adding Eqs. (i) and (iii), we get

$$[x] + f + f' = (8 + \sqrt{63})^n + (8 - \sqrt{63})^n$$

$$[x] + 1 = 2p, \forall\ p \in N = \text{Even integer} \quad \text{[from theorem 2]}$$

$\therefore \quad [x] = 2p - 1 = $ Odd integer

i.e., Integral part of x = Odd integer

$\because \quad f + f' = 1 \Rightarrow 1 - f = f' \quad$ ...(v)

LHS $= x - x^2 + x[x] = x - x(x - [x]) = x - xf \quad [\because x = [x] + f]$

$= x(1 - f) = xf' \quad$ [from Eq. (v)]

$= (8 + \sqrt{63})^n (8 - \sqrt{63})^n$ [from Eqs. (i) and (iii)]

$= (64 - 63)^n = (1)^n = 1 =$ RHS

Remark

Sometimes, students find it difficult to decide whether a problem is on addition or subtraction. Now, if $x = [x] + f$ and $0 < f' < 1$ and if $[x] + f + f' =$ Integer. Then, addition and if $[x] + f - f' =$ Integer, the subtraction and values of $(f + f')$ and $(f - f')$ are 1 and 0, respectively.

Divisibility Problems

Type I

(i) $(x^n - a^n)$ is divisible by $(x - a)$, $\forall\ n \in N$.

(ii) $(x^n + a^n)$ is divisible by $(x + a)$, $\forall\ n \in$ Only odd natural numbers.

Must

Example 39. Show that $1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$ is divisible by 1998.

Sol. Here, $n = 1998$ (Even)

$\therefore$ Only result (i) applicable.

Let $\quad P = 1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$

$$= \underbrace{(1992^{1998} - 1955^{1998})}_{\substack{\text{divisible by } (1992-1955) \\ \text{i.e. } 37}} - \underbrace{(1938^{1998} - 1901^{1998})}_{\substack{\text{divisible by } (1938-1901) \\ \text{i.e. } 37}}$$

$\therefore$ P is divisible by 37.

Also, $$P = \underbrace{(1992^{1998} - 1938^{1998})}_{\substack{\text{divisible by } (1992-1938) \\ \text{i.e., } 54}} - \underbrace{(1955^{1998} - 1901^{1998})}_{\substack{\text{divisible by } (1955-1901) \\ \text{i.e., } 54}}$$

$\therefore$ P is also divisible by 54.

Hence, P is divisible by 37×54, i.e., 1998.

Must

Example 40. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Sol. We have, $2222^{5555} + 5555^{2222}$

$$= (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - (4^{5555} - 4^{2222}) \quad \text{...(i)}$$

The number $(2222^{5555} + 4^{5555})$ is divisible by $2222 + 4 = 2226 = 7 \times 318$, which is divisible by 7 and the number $(5555^{2222} - 4^{2222})$ is divisible by

$5555 - 4 = 5551 = 7 \times 793$, which is divisible by 7 and the number

$(4^{5555} - 4^{2222}) = 4^{2222}(4^{3333} - 1) = 4^{2222}(64^{1111} - 1^{1111})$ is divisible by $64 - 1 = 63 = 7 \times 9$, which is divisible by 7.

Therefore, each brackets of Eq. (i) are divisible by 7. Hence, $2222^{5555} + 5555^{2222}$ is divisible by 7.

Type II To show that an Expression is Divisible by An Integer

Solution Process

(i) If a, p, n and r are positive integers, first of all write $a^{pn+r} = a^{pn} \cdot a^r = (a^p)^n \cdot a^r$

(ii) If we will show that the given expression is divisible by c. Then, expression $a^p = \{1 + (a^p - 1)\}$, if some power of $(a^p - 1)$ has c as a factor.

or $a^p = \{2 + (a^p - 2)\}$, if some power of $(a^p - 2)$ has c as a factor.

or $a^p = \{3 + (a^p - 3)\}$, if some power of $(a^p - 3)$ has c as a factor.

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

or $a^p = \{k + (a^p - k)\}$, if some power of $(a^p - k)$ has c as a factor.

Example 41. If n is any positive integer, show that $2^{3n+3} - 7n - 8$ is divisible by 49.

Sol. Given expression

$$= 2^{3n+3} - 7n - 8 = 2^{3n} \cdot 2^3 - 7n - 8$$
$$= 8^n \cdot 8 - 7n - 8 = 8(1 + 7)^n - 7n - 8$$
$$= 8(1 + {}^nC_1 \cdot 7 + {}^nC_2 \cdot 7^2 + \dots + {}^nC_n \cdot 7^n) - 7n - 8$$
$$= 8 + 56n + 8({}^nC_2 \cdot 7^2 + \dots + {}^nC_n \cdot 7^n) - 7n - 8$$
$$= 49n + 8({}^nC_2 \cdot 7^2 + \dots + {}^nC_n \cdot 7^n)$$
$$= 49\{n + 8({}^nC_2 + \dots + {}^nC_n \cdot 7^{n-2})\}$$

Hence, $2^{3n+3} - 7n - 8$ is divisible by 49.

Example 42. If 10^n divides the number $101^{100} - 1$, find the greatest value of n.

Sol. We have, $101^{100} - 1 = (1 + 100)^{100} - 1$

$$= 1 + {}^{100}C_1 \cdot 100 + {}^{100}C_2 \cdot 100^2 + \dots + {}^{100}C_{100}\, 100^{100} - 1$$
$$= {}^{100}C_1 \cdot 100 + {}^{100}C_2 \cdot 100^2 + \dots + {}^{100}C_{100} \cdot 10^{100}$$
$$= (100)(100) + {}^{100}C_2 \cdot 100^2 + \dots + {}^{100}C_{100} \cdot 100^{100}$$
$$= (100)^2[1 + {}^{100}C_2 + \dots + 100^{98}]$$
$$= 100^2 k, \text{ where } k \text{ is a positive integer}$$

Therefore, $101^{100} - 1$ is divisible by 100^2 i.e., 10^4.

$\therefore \quad n = 4$

How to Find Remainder by Using Binomial Theorem

If a, p, n and r are positive integers, then to find the remainder when a^{pn+r} is divided by b, we adjust power of a to a^{pn+r} which is very close to b, say with difference 1 i.e., $b \pm 1$. Also, the remainder is always positive. When number of the type $5n - 2$ is divided by 5, then we have

$$5)\,5n - 2\,(n$$
$$\underline{-\,5n}$$
$$\underline{-2}$$

We can write $-2 = -2 - 3 + 3 = -5 + 3$

or $$\frac{5n - 2}{5} = \frac{5n - 5 + 3}{5} = n - 1 + \frac{3}{5}$$

Hence, the remainder is 3.

Must

Example 43. If 7^{103} is divided by 25, find the remainder.

Soln. We have, $7^{103} = 7 \cdot 7^{102} = 7 \cdot (7^2)^{51} = 7(49)^{51} = 7(50 - 1)^{51}$

$$= 7[(50)^{51} - {}^{51}C_1(50)^{50} + {}^{51}C_2(50)^{49} - \dots - 1]$$
$$= 7[(50)^{51} - {}^{51}C_1(50)^{50} + {}^{51}C_2(50)^{49} - \dots + {}^{51}C_{50}(50)] - 7 - 18 + 18$$
$$= 7[50((50)^{50} - {}^{51}C_1(50)^{49} + {}^{51}C_2(50)^{48} - \dots + {}^{51}C_{50})] - 25 + 18$$
$$= 7[50k] - 25 + 18, \text{ where } k \text{ is an integer}$$
$$= 25[14k - 1] + 18 = 25p + 18 \quad \text{[where } p \text{ is an integer]}$$

Now, $\dfrac{7^{103}}{25} = p + \dfrac{18}{25}$ Hence, the remainder is 18.

Example 44. Find the remainder, when $5^{5^{5^{\cdot^{\cdot^{\cdot}}}}}$ (24 times 5) is divided by 24.

Sol. Here, $5^{5^{5^{\cdot^{\cdot^{\cdot}}}}}$ (23 times 5) is an odd natural number.

Let $5^{5^{5^{\cdot^{\cdot^{\cdot}}}}}$ (23 times 5) $= 2m + 1$

Now, let $x = 5^{5^{5^{\cdot^{\cdot^{\cdot}}}}}$ (24 times 5) $= 5^{2m+1} = 5 \cdot 5^{2m}$, where m is a natural number.

$\therefore \quad x = 5 \cdot (5^2)^m = 5(24 + 1)^m$

$$= 5[{}^mC_0(24)^m + {}^mC_1(24)^{m-1} + \dots + {}^mC_{m-1}(24) + 1]$$
$$= 5(24k + 1) = 24(5k) + 5$$

$\therefore \quad \dfrac{x}{24} = 5k + \dfrac{5}{24}$

Hence, the remainder is 5.

Example 45. If 7 divides $32^{32^{32}}$, then find the remainder

Solution. We have, $32 = 2^5$

$\therefore \quad 32^{32} = (2^5)^{32} = 2^{160} = (3 - 1)^{160}$

$$= {}^{160}C_0(3)^{160} - {}^{160}C_1(3)^{159} + \dots - {}^{160}C_{159}(3) + 1$$
$$= 3(3^{159} - {}^{160}C_1(3)^{158} + \dots - {}^{160}C_{159}) + 1$$
$$= 3m + 1, \; m \in I^+$$

Now, $32^{32^{32}} = 32^{3m+1} = 2^{5(3m+1)} = 2^{15m+5}$

$= 2^2 \cdot 2^{3(5m+1)} = 4(8)^{5m+1} = 4(7+1)^{5m+1}$

$= 4[\,^{5m+1}C_0 (7)^{5m+1} + \,^{5m+1}C_1 (7)^{5m} + \,^{5m+1}C_2 (7)^{5m-1} + \dots + \,^{5m+1}C_{5m} (7) + 1]$

$= 4[7(\,^{5m+1}C_0 (7)^{5m} + \,^{5m+1}C_1 (7)^{5m-1} + \dots + \,^{5m+1}C_{5m}) + 1]$

$= 4[7k+1]$, where k is positive integer $= 28k + 4$

$\therefore \quad \dfrac{32^{32^{32}}}{7} = 4k + \dfrac{4}{7}$

Hence, the remainder is 4.

How to Find Last Digit, Last Two Digits, Last Three Digits, ... and so on.

If a, p, n and r are positive integers, then a^{pn+r} is adjust of the form $(10k \pm 1)^m$, where k and m are positive integers. For last digit, take 10 common. For last two digits, take 100 common, for last three digits, take 1000 common, ... and so on.

i.e. $(10k \pm 1)^m = (10k)^m + \,^mC_1 (10k)^{m-1} (\pm 1) + \,^mC_2 (10k)^{m-2} (\pm 1)^2 + \dots + \,^mC_{m-2} (10k)^2 (\pm 1)^{m-2} + \,^mC_{m-1} (10k)(\pm 1)^{m-1} + (\pm 1)^m$

For last digit $= 10\lambda + (\pm 1)^m$

For last two digits $= 100\mu + \,^mC_{m-1} (10k)(\pm 1)^{m-1} + (\pm 1)^m$

For last three digits $= 1000\nu + \,^mC_{m-2} (10k)^2 (\pm 1)^{m-2} + \,^mC_{m-1} (10k)(\pm 1)^{m-1} + (\pm 1)^m$ and so on where $\lambda, \mu, \nu \in I$.

must

| Example 46. Find the last two digits of 3^{400}.

Sol. We have, $3^{400} = (3^2)^{200} = (9)^{200} = (10-1)^{200}$

$= (10)^{200} - \,^{200}C_1 (10)^{199} + \,^{200}C_2 (10)^{198} - \,^{200}C_3 (10)^{197} + \dots + \,^{200}C_{198} (10)^2 - \,^{200}C_{199} (10) + 1$

$= 100\mu - \,^{200}C_{199} (10) + 1$, where $\mu \in I$

$= 100\mu - \,^{200}C_1 (10) + 1 = 100\mu - 2000 + 1$

$= 100(\mu - 20) + 1 = 100p + 1$, where p is an integer.

Hence, the last two digits of 3^{400} is $00 + 1 = \boxed{01}$.

| Example 47. If the number is 17^{256}, find the
(i) last digit. (ii) last two digits.
(iii) last three digits of 17^{256}.

Sol. Since, $17^{256} = (17^2)^{128} = (289)^{128} = (290-1)^{128}$

$\therefore 17^{256} = \,^{128}C_0 (290)^{128} - \,^{128}C_1 (290)^{127} + \,^{128}C_2 (290)^{126} - \,^{128}C_3 (290)^{125} + \dots - \,^{128}C_{125} (290)^3 + \,^{128}C_{126} (290)^2 - \,^{128}C_{127} (290) + 1$

(i) **For last digit**

$17^{256} = 290[\,^{128}C_0 (290)^{127} - \,^{128}C_1 (290)^{126} + \,^{128}C_2 (290)^{125} - \dots - \,^{128}C_{127} (1)] + 1$

$= 290(k) + 1$, where k is an integer.

$\therefore$ Last digit $= 0 + 1 = 1$

(ii) **For last two digits,**

$17^{256} = (290)^2 [\,^{128}C_0 (290)^{126} - \,^{128}C_1 (290)^{125} + \,^{128}C_2 (290)^{124} - \dots + \,^{128}C_{126} (1)] - \,^{128}C_{127} (290) + 1$

$= 100m - \,^{128}C_{127} (290) + 1$, where m is an integer.

$= 100m - \,^{128}C_1 (290) + 1 = 100m - 128 \times 290 + 1$

$= 100m - 128 \times (300 - 10) + 1$

$= 100(m - 384) + 1281$

$= 100n + 1281$, where n is an integer.

$\therefore$ Last two digits $= 00 + 81 = 81$

(iii) **For last three digits,**

$17^{256} = (290)^3 [\,^{128}C_0 (290)^{125} - \,^{128}C_1 (290)^{124} + \,^{128}C_2 (290)^{123} - \dots - \,^{128}C_{125} (1)] + \,^{128}C_{126} (290)^2 - \,^{128}C_{127} (290) + 1$

$= 1000m + \,^{128}C_{126} (290)^2 - \,^{128}C_{127} (290) + 1$

where, m is an integer

$= 1000m + \,^{128}C_2 (290)^2 - \,^{128}C_1 (290) + 1$

$= 1000m + \dfrac{(128)(127)}{2} (290)^2 - 128 \times 290 + 1$

$= 1000m + (128)(127)(290)(145) - (128)(290) + 1$

$= 1000m + (128)(290)(127 \times 145 - 1) + 1$

$= 1000m + (128)(290)(18414) + 1$

$= 1000m + 683527680 + 1$

$= 1000m + 683527000 + 680 + 1$

$= 1000(m + 683527) + 681$

$\therefore$ Last three digits $= 000 + 681 = 681$

Two Important Results

(i) $2 \le \left(1 + \dfrac{1}{n}\right)^n < 3, n \ge 1, n \in N$

(ii) If $n > 6$, then $\left(\dfrac{n}{3}\right)^n < n! < \left(\dfrac{n}{2}\right)^n$

| Example 48. Find the positive integer just greater than $(1 + 0.0001)^{10000}$.

Sol. $(1 + 0.0001)^{10000} = \left(1 + \dfrac{1}{10000}\right)^{10000}$

We know that, $2 \le \left(1 + \dfrac{1}{n}\right)^n < 3, n \ge 1, n \in N$ [Result (i)]

Hence, positive integer just greater than $(1 + 0.0001)^{10000}$ is 3.

| Example 49. Find the greater number is 100^{100} and $(300)!$.

Sol. Using Result (ii), We know that, $\left(\dfrac{n}{3}\right)^n < n!$

Putting $n = 300$, we get

$(100)^{300} < (300)!$ ...(i)

But $(100)^{100} < (100)^{300}$ ...(ii)

From Eqs. (i) and (ii), we get

$(100)^{100} < (100)^{300} < (300)!$

$\Rightarrow (100)^{100} < (300)!$

Hence, the greater number is $(300)!$.

| Example 50. Find the greater number in $300!$ and $\sqrt{300^{300}}$.

Sol. Since, $(100)^{150} > 3^{150}$

$\Rightarrow (100)^{150} \cdot (100)^{150} > 3^{150} \cdot (100)^{150}$

$\Rightarrow (100)^{300} > (300)^{150}$

or $(100)^{300} > \sqrt{300^{300}}$ ...(i)

Using result (ii), $\left(\dfrac{n}{3}\right)^n < n!$

Putting $n = 300$, we get $(100)^{300} < 300!$ ...(ii)

From Eqs. (i) and (ii), we get

$\sqrt{300^{300}} < (100)^{300} < 300!$

$\Rightarrow \sqrt{300^{300}} < 300!$

Hence, the greater number is $300!$.

Exercise for Session 3

1. If $x = (7 + 4\sqrt{3})^{2n} = [x] + f$, where $n \in N$ and $0 \le f < 1$, then $x(1-f)$ is equal to
(a) 1 (b) 0 (c) -1 (d) even integer

2. If $(5 + 2\sqrt{6})^n = I + f$; $n, I \in N$ and $0 \le f < 1$, then I equals
(a) $\dfrac{1}{f} - f$ (b) $\dfrac{1}{1+f} - f$ (c) $\dfrac{1}{1-f} - f$ (d) $\dfrac{1}{1+f} + f$

3. If $n > 0$ is an odd integer and $x = (\sqrt{2} + 1)^n$, $f = x - [x]$, then $\dfrac{1-f^2}{f}$ is
(a) an irrational number (b) a non-integer rational number (c) an odd number (d) an even number

4. Integral part of $(\sqrt{2} + 1)^6$ is
(a) 196 (b) 197 (c) 198 (d) 199

5. $(103)^{86} - (86)^{103}$ is divisible by
(a) 7 (b) 13 (c) 17 (d) 23

6. Fractional part of $\dfrac{2^{78}}{31}$ is
(a) $\dfrac{2}{31}$ (b) $\dfrac{4}{31}$ (c) $\dfrac{8}{31}$ (d) $\dfrac{16}{31}$

7. The unit digit of $17^{1983} + 11^{1983} - 7^{1983}$ is
(a) 1 (b) 2 (c) 3 (d) 0

8. The last two digits of the number $(23)^{14}$ are
(a) 01 (b) 03 (c) 09 (d) 27

9. The last four digits of the number 3^{100} are
(a) 2001 (b) 3211 (c) 1231 (d) 0001

10. The remainder when 23^{23} is divided by 53 is
(a) 17 (b) 21 (c) 30 (d) 47

Session 4

Use of Complex Numbers in Binomial Theorem, Multinomial Theorem, Use of Differentiation, Use of Integration, Binomial Inside Binomial, Sum of the Series

Use of Complex Numbers in Binomial Theorem

If $\theta \in R, n \in N$ and $i = \sqrt{-1}$, then

$$(\cos\theta + i\sin\theta)^n = {}^nC_0 (\cos\theta)^{n-0} (i\sin\theta)^0 + {}^nC_1 (\cos\theta)^{n-1} (i\sin\theta)^1 + {}^nC_2 (\cos\theta)^{n-2} (i\sin\theta)^2 + {}^nC_3 (\cos\theta)^{n-3} (i\sin\theta)^3 + \dots$$

or $\cos n\theta + i\sin n\theta = \cos^n\theta + i \cdot {}^nC_1 (\cos\theta)^{n-1} \sin\theta - {}^nC_2 (\cos\theta)^{n-2} \sin^2\theta - i \cdot {}^nC_3 (\cos\theta)^{n-3} \sin^3\theta + \dots$

On comparing real and imaginary parts, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2 (\cos\theta)^{n-2} \sin^2\theta - {}^nC_4 (\cos\theta)^{n-4} \sin^4\theta - \dots$$

and $\sin n\theta = {}^nC_1 (\cos\theta)^{n-1} \sin\theta - {}^nC_3 (\cos\theta)^{n-3} \sin^3\theta + {}^nC_5 (\cos\theta)^{n-5} \sin^5\theta - \dots$

| Example 51. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \dots$, find the values of

(i) $C_0 - C_2 + C_4 - C_6 + \dots$

(ii) $C_1 - C_3 + C_5 - C_7 + \dots$

(iii) $C_0 + C_3 + C_6 + \dots$

Sol. $\because (1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \dots$

Putting $x = i$, where $i = \sqrt{-1}$, then

$$(1+i)^n = C_0 + C_1 i + C_2 i^2 + C_3 i^3 + C_4 i^4 + C_5 i^5 + \dots$$
$$= (C_0 - C_2 + C_4 - \dots) + i(C_1 - C_3 + C_5 - \dots) \quad \dots\text{(i)}$$

Also, $(1+i)^n = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]^n$

$$= 2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$
$$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \quad \dots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$(C_0 - C_2 + C_4 - \dots) + i(C_1 - C_3 + C_5 - \dots) = 2^{n/2}\cos\left(\frac{n\pi}{4}\right) + i \cdot 2^{n/2}\sin\left(\frac{n\pi}{4}\right)$$

On comparing real and imaginary parts, we get

$$C_0 - C_2 + C_4 - \dots = 2^{n/2}\cos\left(\frac{n\pi}{4}\right) \quad \text{[part (i)]}$$

$$C_1 - C_3 + C_5 - \dots = 2^{n/2}\sin\left(\frac{n\pi}{4}\right) \quad \text{[part (ii)]}$$

We have, $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + \dots$

Putting $x = 1, \omega, \omega^2$ (cube roots of unity) and adding, we get

$$3(C_0 + C_3 + C_6 + \dots) = 2^n + (1+\omega)^n + (1+\omega^2)^n$$
$$= 2^n + (-\omega^2)^n + (-\omega)^n = 2^n + (-1)^n(\omega^{2n} + \omega^n)$$
$$= 2^n + (-1)^n\left\{e^{\frac{4\pi in}{3}} + e^{\frac{2\pi in}{3}}\right\}$$
$$= 2^n + (-1)^n \cdot e^{n\pi i} \cdot 2\cos\left(\frac{n\pi}{3}\right)$$
$$= 2^n + (-1)^n \cdot (-1)^n \cdot 2\cos\left(\frac{n\pi}{3}\right)$$
$$= 2^n + (-1)^{2n} \cdot 2\cos\left(\frac{n\pi}{3}\right) = 2^n + 2\cos\left(\frac{n\pi}{3}\right)$$

$$\therefore C_0 + C_3 + C_6 + \dots = \frac{1}{3}\left\{2^n + 2\cos\left(\frac{n\pi}{3}\right)\right\}$$

| Example 52. Find the value of ${}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \dots + {}^{4n}C_{4n}$.

Sol. $\because 4 - 0 = 8 - 4 = \dots = 4$

$\therefore$ Four roots of unity $(1)^{1/4}$ are $1, -1, i, -i$, we have

$$(1+x)^{4n} = {}^{4n}C_0 + {}^{4n}C_1x + {}^{4n}C_2x^2 + {}^{4n}C_3x^3 + \dots$$

Putting $x = 1, -1, i, -i$ and then adding, we get,

$$4({}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \dots) = 2^{4n} + 0 + (1+i)^{4n} + (1-i)^{4n}$$
$$= 2^{4n} + (2i)^{2n} + (-2i)^{2n}$$

$$= 2^{4n} + 2^{2n}(-1)^n + 2^{2n}(-1)^n$$
$$= 2^{4n} + (-1)^n \cdot 2^{2n+1}$$

$$\therefore {}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \dots = 2^{4n-2} + (-1)^n \cdot 2^{2n-1}$$

Remark

If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$, then

(i) $C_0 + C_4 + C_8 + C_{12} + \dots = \frac{1}{2}\left\{2^{n-1} + 2^{n/2}\cos\left(\frac{n\pi}{4}\right)\right\}$

(ii) $C_1 + C_5 + C_9 + C_{13} + \dots = \frac{1}{2}\left\{2^{n-1} + 2^{n/2}\sin\left(\frac{n\pi}{4}\right)\right\}$

(iii) $C_0 + C_6 + C_{12} + \dots = \frac{1}{3}\left\{2^{n-1}\cos\left(\frac{n\pi}{4}\right) + 3^{n/2}\cos\left(\frac{n\pi}{6}\right)\right\}$

Multinomial Theorem

If n is a positive integer and $x_1, x_2, x_3, \dots, x_k \in C$, then

$$(x_1 + x_2 + x_3 + \dots + x_k)^n = \sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\dots(\alpha_k!)} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots x_k^{\alpha_k}$$

where, $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k$ are all non-negative integers such that $\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_k = n$.

Remark

The coefficient of $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot x_3^{\alpha_3} \dots x_k^{\alpha_k}$ in the expansion of $(x_1 + x_2 + x_3 + \dots + x_k)^n$ is $\sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\dots(\alpha_k!)}$.

In Particular

(i) $(a+b+c)^n = \sum \frac{n!}{(\alpha!)(\beta!)(\gamma!)} a^\alpha b^\beta c^\gamma$ such that $\alpha + \beta + \gamma = n$

(ii) $(a+b+c+d)^n = \sum \frac{n!}{(\alpha!)(\beta!)(\gamma!)(\delta!)} a^\alpha b^\beta c^\gamma d^\delta$ such that $\alpha + \beta + \gamma + \delta = n$

| Example 53. Find the coefficient of $a^4b^3c^2d$ in the expansion of $(a - b + c - d)^{10}$.

Sol. The coefficient of $a^4b^3c^2d$ in the expansion of $(a-b+c-d)^{10}$ is $(-1)^4 \frac{10!}{4!3!2!1!} = 12600$

[powers of b and d are 3 and 1 $\therefore (-1)^3(-1)$]

| Example 54. Find the coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$.

Sol. In this case, write $a^3b^4c^5 = (ab)^x(bc)^y(ca)^z$ say

$\therefore a^3b^4c^5 = a^{z+x} \cdot b^{x+y} \cdot c^{y+z}$

$\Rightarrow z + x = 3, x + y = 4$

$y + z = 5$

On adding all, we get $2(x+y+z) = 12$

$\therefore x + y + z = 6$

Then, $x = 1, y = 3, z = 2$

Therefore, the coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$ or the coefficient of $(ab)^1(bc)^3(ca)^2$ in the expansion of $(bc + ca + ab)^6$ is $\frac{6!}{1!3!2!}$, i.e. 60.

Aliter

Coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$

= Coefficient of $a^3b^4c^5$ in the expansion of $(abc)^6\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^6$

= Coefficient of $\left(\frac{1}{a}\right)^3\left(\frac{1}{b}\right)^2\left(\frac{1}{c}\right)^1$ in the expansion of $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^6$ is $\frac{6!}{3!2!1!} = 60$

Number of Distinct or Dissimilar Terms in the Multinomial Expansion

Statement The number of distinct or dissimilar terms in the multinomial expansion of $(x_1 + x_2 + x_3 + \dots + x_k)^n$ is ${}^{n+k-1}C_{k-1}$.

Proof We have, $(x_1 + x_2 + x_3 + \dots + x_k)^n$

$$= \sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\dots(\alpha_k!)} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots x_k^{\alpha_k}$$

where, $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k$ are non-negative integers such that

$$\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_k = n \quad \dots\text{(i)}$$

Here, the number of terms in the expansion of $(x_1 + x_2 + x_3 + \dots + x_k)^n$

= The number of non-negative integral solutions of the Eq. (i)

$= {}^{n+k-1}C_{k-1}$

| Example 55. Find the total number of distinct or dissimilar terms in the expansion of $(x + y + z + w)^n$, $n \in N$.

Sol. The total number of distinct or dissimilar terms in the expansion of $(x + y + z + w)^n$ is

$$= {}^{n+4-1}C_{4-1} = {}^{n+3}C_3 = \frac{(n+3)(n+2)(n+1)}{1 \cdot 2 \cdot 3}$$
$$= \frac{(n+1)(n+2)(n+3)}{6}$$

I. Aliter

We know that, $(x+y+z+w)^n = \{(x+y)+(z+w)\}^n$

$$= (x+y)^n + {}^nC_1 (x+y)^{n-1}(z+w) + {}^nC_2 (x+y)^{n-2}(z+w)^2 + \dots + {}^nC_n (z+w)^n$$

$\therefore$ Number of terms in RHS

$$= (n+1) + n\cdot 2 + (n-1)\cdot 3 + \dots + 1\cdot(n+1)$$

$$= \sum_{r=0}^{n} (n-r+1)(r+1)$$

$$= \sum_{r=0}^{n} (n+1) + nr - r^2 = (n+1)\sum_{r=0}^{n} 1 + n\sum_{r=0}^{n} r - \sum_{r=0}^{n} r^2$$

$$= (n+1)\cdot(n+1) + n\cdot\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

II. Aliter

$$(x+y+z+w)^n = \sum \frac{n!}{n_1!\,n_2!\,n_3!\,n_4!} x^{n_1} y^{n_2} z^{n_3} w^{n_4}$$

where, n_1, n_2, n_3, n_4 are non-negative integers subject to the condition $n_1 + n_2 + n_3 + n_4 = n$

Hence, number of the distinct terms

$$= \text{Coefficient of } x^n \text{ in } (x^0 + x^1 + x^2 + \dots + x^n)^4$$

$$= \text{Coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^4$$

$$= \text{Coefficient of } x^n \text{ in } (1-x^{n+1})^4 (1-x)^{-4}$$

$$= \text{Coefficient of } x^n \text{ in } (1-x)^{-4} \qquad [\because x^{n+1} > x^n]$$

$$= {}^{n+3}C_n = {}^{n+3}C_3 = \frac{(n+3)(n+2)(n+1)}{6}$$

Greatest Coefficient in Multinomial Expansion

The greatest coefficient in the expansion of $(x_1 + x_2 + x_3 + \dots + x_k)^n$ is $\dfrac{n!}{(q!)^{k-r}\,((q+1)!)^r}$, where q is the quotient and r is the remainder when n is divided by k i.e.

$$k)\, n\, (q$$
$$\underline{\quad}$$
$$r$$

must

| Example 56. Find the greatest coefficient in the expansion of $(a+b+c+d)^{15}$.

Sol. Here, $n = 15$ and $k = 4$ $\qquad$ [$\because a, b, c, d$ are four terms]

$$4)\,15\,(3$$
$$\underline{12}$$
$$\underline{3}$$

$\therefore \qquad q = 3$ and $r = 3$

Hence, greatest coefficient $= \dfrac{15!}{(3!)^1 (4!)^3}$

Coefficient of x^r in Multinomial Expansion

If n is a positive integer and $a_1, a_2, a_3, \dots, a_k \in C$, then coefficient of x^r in the expansion of $(a_1 + a_2 x + a_3 x^2 + \dots + a_k x^{k-1})^n$, is

$$\sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\dots(\alpha_k!)} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \dots a_k^{\alpha_k}$$

where, $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k$ are non-negative integers such that $\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_k = n$

and $\qquad \alpha_2 + 2\alpha_3 + 3\alpha_4 + \dots + (k-1)\alpha_k = r$

| Example 57. Find the coefficient of x^7 in the expansion of $(1+3x-2x^3)^{10}$.

Sol. Coefficient of x^7 in the expansion of $(1+3x-2x^3)^{10}$ is

$$= \sum \frac{10!}{\alpha!\,\beta!\,\gamma!} (1)^\alpha (3)^\beta (-2)^\gamma$$

where, $\alpha + \beta + \gamma = 10$ and $\beta + 3\gamma = 7$

The possible values of α, β and γ are given below

α	β	γ
3	7	0
5	4	1
7	1	2

$\therefore$ Coefficient of x^7

$$= \frac{10!}{3!\,7!\,0!}(1)^3(3)^7(-2)^0 + \frac{10!}{5!\,4!\,1!}(1)^5(3)^4(-2)^1 + \frac{10!}{7!\,1!\,2!}(1)^7(3)^1(-2)^2$$

$$= 262440 - 204120 + 4320 = 62640$$

Use of Differentiation

This method applied only when the numericals occur as the product of the binomial coefficients, if

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$$

Solution Process

(i) If last term of the series leaving the plus or minus sign is m, then divide m by n. If q is the quotient and r is the remainder.

i.e. $\quad m = nq + r \quad$ or $\quad n)\, m\, (q$

$$\underline{nq}$$
$$\underline{r}$$

Then, replace x by x^q in the given series and multiplying both sides of the expression by x^r.

(ii) After this, differentiate both sides w.r.t. x and put $x = 1$ or -1 or i ($i = \sqrt{-1}$), etc. According to the given series.

(iii) If product of two numericals (or square of numericals) or three numericals (or cube of numericals), then differentiate twice or thrice.

| Example 58. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$, prove that $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n\cdot 2^{n-1}$.

Sol. Here, last term of $C_1 + 2C_2 + 3C_3 + \dots + nC_n$ is nC_n i.e., n and last term with positive sign.

Then, $n = n\cdot 1 + 0 \quad$ or $\quad n)\, n\, (1$

$$\underline{n}$$
$$\underline{0}$$

Here, $q = 1$ and $r = 0$

Then, the given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2 x + 3C_3 x^2 + \dots + nC_n x^{n-1}$$

Putting $x = 1$, we get

$$n\cdot 2^{n-1} = C_1 + 2C_2 + 3C_3 + \dots + nC_n$$

or $\qquad C_1 + 2C_2 + 3C_3 + \dots + nC_n = n\cdot 2^{n-1}$

I. Aliter

$$C_1 + 2C_2 + 3C_3 + \dots + nC_n$$

$$= n + 2\cdot\frac{n(n-1)}{1\cdot 2} + 3\cdot\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \dots + n\cdot 1$$

$$= n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1\cdot 2} + \dots + 1\right\}$$

Let $n - 1 = N$, then

$$\text{LHS} = (1+N)\left\{1 + N + \frac{N(N-1)}{1\cdot 2} + \dots + 1\right\}$$

$$= (1+N)\{1 + {}^NC_1 + {}^NC_2 + \dots + {}^NC_N\}$$

$$= (1+N)\,2^N = n\cdot 2^{n-1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_1 + 2C_2 + 3C_3 + \dots + nC_n = \sum_{r=1}^{n} r\cdot {}^nC_r$$

$$= \sum_{r=1}^{n} r\cdot\frac{n}{r}\cdot {}^{n-1}C_{r-1} \qquad \left[\because {}^nC_r = \frac{n}{r}\cdot {}^{n-1}C_{r-1}\right]$$

$$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$$= n({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \dots + {}^{n-1}C_{n-1})$$

$$= n\cdot 2^{n-1} = \text{RHS}$$

| Example 59. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$, prove that $C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n = (n+2)2^{n-1}$.

Sol. Here, last term of $C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n$ is $(n+1)C_n$ i.e., $(n+1)$ and last term with positive sign.

and $\qquad n + 1 = n\cdot 1 + 1$

or $\qquad n)\, n+1\, (1$

$$\underline{-n}$$
$$\underline{1}$$

Here, $q = 1$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$$

Now, replacing x by x^1 and multiplying both sides by x, we get

$$x(1+x)^n = C_0 x + C_1 x^2 + C_2 x^3 + \dots + C_n x^{n+1}$$

Differentiating both sides w.r.t. x, we get

$$x\cdot n(1+x)^{n-1} + (1+x)^n\cdot 1 = C_0 + 2C_1 x + 3C_2 x^2 + \dots + (n+1)C_n x^n$$

Putting $x = 1$, we get

$$n(2)^{n-1} + 2^n = C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n$$

or $\qquad C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n = (n+2)2^{n-1}$

I. Aliter

$$\text{LHS} = C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n$$

$$= C_0 + (1+1)C_1 + (1+2)C_2 + \dots + (1+n)C_n$$

$$= (C_0 + C_1 + C_2 + \dots + C_n) + (C_1 + 2C_2 + \dots + nC_n)$$

[use example 58]

$$= 2^n + n\cdot 2^{n-1} = (n+2)2^{n-1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n$$

$$= \sum_{r=1}^{n+1} r\cdot {}^nC_{r-1} = \sum_{r=1}^{n+1} (r-1+1)\cdot {}^nC_{r-1}$$

$$= \sum_{r=1}^{n+1} (r-1)\cdot {}^nC_{r-1} + {}^nC_{r-1}$$

$$= \sum_{r=1}^{n+1} n\cdot {}^{n-1}C_{r-2} + \sum_{r=1}^{n+1} {}^nC_{r-1} \qquad \left[\because {}^nC_{r-1} = \frac{n}{r-1}\cdot {}^{n-1}C_{r-2}\right]$$

$$= n(0 + {}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \dots + {}^{n-1}C_{n-1}) + ({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n)$$

$$= n\cdot 2^{n-1} + 2^n = (n+2)\cdot 2^{n-1} = \text{RHS}$$

Example 60. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n = (n+1)2^n$.

Sol. Here, last term of $C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n$ is $(2n+1)C_n$ i.e., $(2n+1)$ and last term with positive sign.

Then, $\quad 2n+1 = n \cdot 2 + 1$

or $\quad n)\,2n+1\,(2$
$\quad -2n$
$\quad \overline{\quad 1 \quad}$

Here, $q = 2$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$$

Now, replacing x by x^2, we get

$$(1+x^2)^n = C_0 + C_1x^2 + C_2x^4 + \ldots + C_nx^{2n}$$

On multiplying both sides by x^1, we get

$$x(1+x^2)^n = C_0x + C_1x^3 + C_2x^5 + \ldots + C_nx^{2n+1}$$

On differentiating both sides w.r.t. x, we get

$$x \cdot n(1+x^2)^{n-1} \cdot 2x + (1+x^2)^n \cdot 1 = C_0 + 3C_1x^2 + 5C_2x^4 + \ldots + (2n+1)C_nx^{2n}$$

Putting $x = 1$, we get

$$n \cdot 2^{n-1} \cdot 2 + 2^n = C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n$$

or $\quad C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n = (n+1)2^n$

I. Aliter

$$\begin{aligned} \text{LHS} &= C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n \\ &= C_0 + (1+2)C_1 + (1+4)C_2 + \ldots + (1+2n)C_n \\ &= (C_0 + C_1 + C_2 + \ldots + C_n) + 2(C_1 + 2C_2 + \ldots + nC_n) \\ &= 2^n + 2 \cdot n \cdot 2^{n-1} = 2^n + n \cdot 2^n \quad \text{[from Illusration 58]} \\ &= (n+1)2^n = \text{RHS} \end{aligned}$$

II. Aliter

$$\begin{aligned} \text{LHS} &= C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n \\ &= \sum_{r=0}^{n} (2r+1)\,{}^nC_r = \sum_{r=0}^{n} 2r \cdot {}^nC_r + \sum_{r=0}^{n} {}^nC_r \\ &= 2\sum_{r=0}^{n} r \cdot {}^nC_r + \sum_{r=0}^{n} {}^nC_r \\ &= 2\sum_{r=0}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r \quad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right] \\ &= 2n \sum_{r=0}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r \\ &= 2n\,(0 + {}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}) \\ &\quad + ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n) \\ &= 2n \cdot 2^{n-1} + 2^n = (n+1) \cdot 2^n = \text{RHS} \end{aligned}$$

Example 61. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n = n(n+1) \cdot 2^{n-2}$.

Sol. Here, last term of $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n$ is $n^2 \cdot C_n$ i.e., n^2. Linear factors of n^2 are n and n; [start always with greater factor] and last term with positive sign.

and $\quad n = n \cdot 1 + 0 \quad$ or $\quad n)\,n\,(1$
$\quad -n$
$\quad \overline{\quad 0 \quad}$

Here, $q = 1$ and $r = 0$

Then, the given series is

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$$

On differentiating both sides w.r.t. x, we get

$$nx(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1} \quad \ldots(\text{i})$$

and in last term, numerical is $n\,C_n$ i.e., n and power of $(1+x)$ is $n-1$.

Then, $n = (n-1) \cdot 1 + 1 \quad$ or $\quad n-1)\,n\,(1$
$\quad n-1$
$\quad -\;+$
$\quad \overline{\quad 1 \quad}$

Here, $q = 1$ and $r = 1$

Now, multiplying both sides by x in Eq. (i), then

$$nx(1+x)^{n-1} = C_1x + 2C_2x^2 + 3C_3x^3 + \ldots + nC_nx^n$$

Differentiating on both sides w.r.t. x, we get

$$n\{x \cdot (n-1)(1+x)^{n-2} + (1+x)^{n-1} \cdot 1\} = C_1 \cdot 1 + 2^2C_2x + 3^2C_3x^2 + \ldots + n^2C_nx^{n-1}$$

Putting $x = 1$, we get

$$n\{1 \cdot (n-1) \cdot 2^{n-2} + 2^{n-1}\} = 1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n$$

or $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n = n(n+1)2^{n-2}$

Aliter

$$\begin{aligned} \text{LHS} &= 1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n \\ &= \sum_{r=1}^{n} r^2 \cdot {}^nC_r = \sum_{r=1}^{n} r^2 \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \quad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right] \\ &= n\sum_{r=1}^{n} r \cdot {}^{n-1}C_{r-1} = n\sum_{r=1}^{n} \{(r-1)+1\} \cdot {}^{n-1}C_{r-1} \\ &= n\sum_{r=1}^{n} (r-1) \cdot {}^{n-1}C_{r-1} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1} \\ &= n\sum_{r=1}^{n} (n-1) \cdot {}^{n-2}C_{r-2} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1} \\ &= n(n-1)\sum_{r=1}^{n} {}^{n-2}C_{r-2} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1} \\ &= n(n-1)(0 + {}^{n-2}C_0 + {}^{n-2}C_1 + {}^{n-2}C_2 + \ldots + {}^{n-2}C_{n-2}) + n({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}) \\ &= n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1} = n(n+1)2^{n-2} = \text{RHS} \end{aligned}$$

Example 62. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $(1 \cdot 2)C_2 + (2 \cdot 3)C_3 + \ldots + \{(n-1) \cdot n\}C_n = n(n-1)2^{n-2}$.

Sol. Here, last term of $(1 \cdot 2)C_2 + (2 \cdot 3)C_3 + \ldots + \{(n-1) \cdot n\}C_n$ is $(n-1)nC_n$ i.e. $(n-1)n$

[start with greater factor here greater factor is n] and last term with positive sign, then $n = n \cdot 1 + 0$

or $\quad n)\,n\,(1$
$\quad -n$
$\quad \overline{\quad 0 \quad}$

Here, $q = 1$ and $r = 0$

The given series is

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$$

Differentiating on both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}$$

Again, differentiating on both sides w.r.t. x, we get

$$n(n-1)(1+x)^{n-2} = 0 + 0 + (1 \cdot 2)C_2 + (2 \cdot 3)C_3x + \ldots + \{(n-1) \cdot n\}C_nx^{n-2}$$

Putting $x = 1$, we get

$$n(n-1)(1+1)^{n-2} = (1 \cdot 2)C_2 + (2 \cdot 3)C_3 + \ldots + \{(n-1)n\} \cdot C_n$$

or $(1 \cdot 2)C_2 + (2 \cdot 3)C_3 + \ldots + \{(n-1)n\} \cdot C_n = n(n-1)2^{n-2}$

I. Aliter

$$\begin{aligned} \text{LHS} &= (1 \cdot 2)C_2 + (2 \cdot 3)C_3 + (3 \cdot 4)C_4 + \ldots + \{(n-1)n\} \cdot C_n \\ &= (1 \cdot 2)\frac{n(n-1)}{1 \cdot 2} + (2 \cdot 3)\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + (3 \cdot 4)\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots + (n-1)n \cdot 1 \\ &= n(n-1)\left\{1 + \frac{(n-2)}{1} + \frac{(n-2)(n-3)}{1 \cdot 2} + \ldots + 1\right\} \end{aligned}$$

Now, in bracket, let $n - 2 = N$, then

$$= n(n-1)\left\{1 + \frac{N}{1} + \frac{N(N-1)}{2!} + \ldots + 1\right\}$$
$$= n(n-1)\{{}^NC_0 + {}^NC_1 + \ldots + {}^NC_N\}$$
$$= n(n-1)2^N = n(n-1)2^{n-2} = \text{RHS}$$

II. Aliter

$$\begin{aligned} \text{LHS} &= (1 \cdot 2)C_2 + (2 \cdot 3)C_3 + \ldots + \{(n-1) \cdot n\}C_n \\ &= \sum_{r=2}^{n} (r-1) \cdot r \cdot {}^nC_r \\ &= \sum_{r=2}^{n} (r-1) \cdot r \cdot \frac{n}{r} \cdot \frac{n-1}{(r-1)} \cdot {}^{n-2}C_{r-2} \\ &= (n-1)n\sum_{r=2}^{n} {}^{n-2}C_{r-2} \\ &= (n-1)n({}^{n-2}C_0 + {}^{n-2}C_1 + {}^{n-2}C_2 + \ldots + {}^{n-2}C_{n-2}) \\ &= (n-1)n \cdot 2^{n-2} = \text{RHS} \end{aligned}$$

Example 63. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$, prove that $C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n(n+1)C_n = 0$.

Sol. Numerical value of last term of $C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n(n+1)C_n$ is $(n+1)C_n$ i.e., $(n+1)$, then

$n + 1 = n \cdot 1 + 1 \quad$ or $\quad n)\,n+1\,(1$
$\quad -n$
$\quad \overline{\quad 1 \quad}$

Here, $q = 1$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$$

On multiplying both sides by x, we get

$$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + C_3x^4 + \ldots + C_nx^{n+1}$$

On differentiating both sides w.r.t. x, we get

$$x \cdot n(1+x)^{n-1} + (1+x)^n \cdot 1 = C_0 + 2C_1x + 3C_2x^2 + 4C_3x^3 + \ldots + (n+1)C_nx^n$$

Putting $x = -1$, we get

$$0 = C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n(n+1)C_n$$

or $\quad C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n(n+1)C_n = 0$

I. Aliter

$$\begin{aligned} \text{LHS} &= C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n(n+1)C_n \\ &= C_0 - (C_1 + C_1) + (C_2 + 2C_2) - (C_3 + 3C_3) + \ldots + (-1)^n\{C_n + nC_n\} \\ &= \{C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^nC_n\} + \{-C_1 + 2C_2 - 3C_3 + \ldots + (-1)^n nC_n\} \\ &= (1-1)^n + \left\{-n + 2 \cdot \frac{n(n-1)}{1 \cdot 2} - 3\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \ldots + (-1)^n \cdot n\right\} \end{aligned}$$

$$= 0 + n\left\{-1 + (n-1) - \frac{(n-1)(n-2)}{1\cdot 2} + \dots + (-1)^n\right\}$$

$$= 0 - n\left\{1 - (n-1) + \frac{(n-1)(n-2)}{1\cdot 2} - \dots + (-1)^{n-1}\right\}$$

Let in bracket, put $n - 1 = N$, we get

$$\text{LHS} = 0 - n\left\{1 - N + \frac{N(N-1)}{1\cdot 2} - \dots + (-1)^N\right\}$$

$$= 0 - n\{{}^N C_0 - {}^N C_1 + {}^N C_2 - \dots + (-1)^N\, {}^N C_N\}$$

$$= 0 - n(1-1)^N = 0 - 0 = 0 = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 - 2C_1 + 3C_2 - 4C_3 + \dots + (-1)^n (n+1) C_n$$

$$= \sum_{r=0}^{n} (-1)^r (r+1)\, {}^nC_r = \sum_{r=0}^{n} (-1)^r [r\cdot {}^nC_r + {}^nC_r]$$

$$= \sum_{r=0}^{n} (-1)^r [n\cdot {}^{n-1}C_{r-1} + {}^nC_r] \quad \left[\because {}^nC_r = \frac{n}{r}\, {}^{n-1}C_{r-1}\right]$$

$$= n\sum_{r=0}^{n} (-1)^r \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r$$

$$= -n\sum_{r=0}^{n} (-1)^{r-1}\cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r$$

$$= -n(1-1)^{n-1} + (1-1)^n = 0 + 0 = 0 = \text{RHS}$$

Example 64. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$, prove that

$$C_1 - 2C_2 + 3C_3 - \dots + (-1)^{n-1} nC_n = 0.$$

Sol. Numerical value of last term of

$C_1 - 2C_2 + 3C_3 - \dots + (-1)^{n-1} n\, C_n$ is nC_n i.e., n, then

and $\quad n = n\cdot 1 + 0 \;$ or $\; n)\,n\,(1$

$$\begin{array}{r} -n \\ \hline 1 \end{array}$$

Here, $q = 1$ and $r = 0$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$$

On differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2 x + 3C_3 x^2 + \dots + nC_n x^{n-1}$$

Putting $\quad x = -1$, we get

$$0 = C_1 - 2C_2 + 3C_3 - \dots + (-1)^{n-1} n\, C_n$$

or $\quad C_1 - 2C_2 + 3C_3 - \dots + (-1)^{n-1} n\, C_n = 0$

I. Aliter

$$\text{LHS} = C_1 - 2C_2 + 3C_3 - \dots + (-1)^{n-1} n\cdot C_n$$

$$= n - 2\cdot\frac{n(n-1)}{1\cdot 2} + 3\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - \dots + (-1)^{n-1}\cdot n\cdot 1$$

$$= n\left\{1 - \frac{(n-1)}{1} + \frac{(n-1)(n-2)}{1\cdot 2} - \dots + (-1)^{n-1}\right\}$$

In bracket, put $n - 1 = N$, then

$$\text{LHS} = n\left\{1 - \frac{N}{1} + \frac{N(N-1)}{1\cdot 2} - \dots + (-1)^N\right\}$$

$$= n\{{}^NC_0 - {}^NC_1 + {}^NC_2 - \dots + (-1)^N\, {}^NC_N\}$$

$$= n(1-1)^N = 0 = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_1 - 2C_2 + 3C_3 - \dots + (-1)^{n-1}\cdot n\, C_n$$

$$= \sum_{r=1}^{n} (-1)^{r-1}\cdot r\cdot {}^nC_r$$

$$= \sum_{r=1}^{n} (-1)^{r-1}\cdot n\cdot {}^{n-1}C_{r-1} \quad \left[\because {}^nC_r = \frac{n}{r}\cdot {}^{n-1}C_{r-1}\right]$$

$$= n\sum_{r=1}^{n} (-1)^{r-1}\cdot {}^{n-1}C_{r-1}$$

$$= n(1-1)^{n-1} = 0 = \text{RHS}$$

Example 65. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$, prove that

$$C_0 - 3C_1 + 5C_2 - \dots + (-1)^n (2n+1) C_n = 0.$$

Sol. The numerical value of last term of

$C_0 - 3C_1 + 5C_2 - \dots + (-1)^n (2n+1) C_n$ is $(2n+1)C_n$

i.e. $\quad (2n+1)$

and $\quad 2n + 1 = 2\cdot n + 1 \;$ or $\; n)\,2n+1\,(2$

$$\begin{array}{r} -2n \\ \hline 1 \end{array}$$

Here, $q = 2$ and $r = 1$

The given series is

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$ now, replacing x by x^2, then we get

$$(1+x^2)^n = C_0 + C_1 x^2 + C_2 x^4 + \dots + C_n x^{2n}$$

On multiplying both sides by x, we get

$$x(1+x^2)^n = C_0 x + C_1 x^3 + C_2 x^5 + \dots + C_n x^{2n+1}$$

On differentiating both sides w.r.t. x, we get

$$x\cdot n(1+x^2)^{n-1} 2x + (1+x^2)^n\cdot 1 = C_0 + 3C_1 x^2 + 5C_2 x^4 + \dots + (2n+1) C_n x^{2n}$$

Putting $x = i$ in both sides, we get

$$0 + 0 = C_0 - 3C_1 + 5C_2 - \dots + (2n+1)(-1)^n C_n$$

or $\quad C_0 - 3C_1 + 5C_2 - \dots + (-1)^n (2n+1) C_n = 0$

I. Aliter

$$\text{LHS} = C_0 - 3C_1 + 5C_2 - \dots + (-1)^n (2n+1) C_n$$

$$= C_0 - (1+2) C_1 + (1+4) C_2 - \dots + (-1)^n (1+2n) C_n$$

$$= (C_0 - C_1 + C_2 - \dots + (-1)^n C_n) - 2(C_1 - 2C_2 + \dots + (-1)^{n-1} n\cdot C_n)$$

$$= (1-1)^n - 2\cdot 0 \qquad \text{[from Example 64]}$$

$$= 0 = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 - 3C_1 + 5C_2 - \dots + (-1)^n (2n+1) C_n$$

$$= \sum_{r=1}^{n} (-1)^r (2r+1)\, {}^nC_r = \sum_{r=1}^{n} (-1)^r [2r\cdot {}^nC_r + {}^nC_r]$$

$$= 2\sum_{r=1}^{n} n\cdot {}^{n-1}C_{r-1} + \sum_{r=1}^{n} (-1)^r\cdot {}^nC_r$$

$$= 2n(1-1)^{n-1} + (1-1)^n = 0 + 0 = 0 = \text{RHS}$$

Use of Integration

This method is applied only when the numericals occur as the denominator of the binomial coefficient.

Solution Process

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$, then integrate both sides between the suitable limits which gives the required series.

1. If the sum contains $C_0, C_1, C_2, \dots, C_n$ are all positive signs, then integrate between limits 0 to 1.
2. If the sum contains alternate signs (i.e., +, −), then integrate between limits −1 to 0.
3. If the sum contains odd coefficients (i.e., $C_0, C_2, C_4, \dots$), then integrate between −1 to +1.
4. If the sum contains even coefficients (i.e., $C_1, C_3, C_5, \dots$), then subtracting (2) from (1) and then dividing by 2.
5. If in denominator of binomial coefficient product of two numericals, then integrate two times first times taken limits between 0 to x and second times take suitable limits.

Example 66. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$, prove that

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}.$$

Sol. $\because (1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n \quad \dots(\text{i})$

Integrating both sides of Eq. (i) within limits 0 to 1, then we get

$$\int_0^1 (1+x)^n\, dx = \int_0^1 (C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n)\, dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \dots + \frac{C_n x^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \frac{2^{n+1} - 1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$

or $\quad C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \dots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1} - 1}{n+1}$

I. Aliter

$$\text{LHS} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$

$$= 1 + \frac{n}{1\cdot 2} + \frac{n(n-1)}{1\cdot 2\cdot 3} + \dots + \frac{1}{n+1}$$

$$= \frac{1}{n+1}\left[(n+1) + \frac{(n+1)n}{1\cdot 2} + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} + \dots + 1\right]$$

Put $n + 1 = N$, then

$$\text{LHS} = \frac{1}{N}\left[N + \frac{N(N-1)}{2!} + \frac{N(N-1)(N-2)}{3!} + \dots + 1\right]$$

$$= \frac{1}{N}[{}^NC_1 + {}^NC_2 + {}^NC_3 + \dots + {}^NC_N]$$

$$= \frac{1}{N}[(1+1)^N - 1] = \frac{2^N - 1}{N} = \frac{2^{n+1} - 1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \sum_{r=0}^{n} \frac{C_r}{r+1}$$

$$= \sum_{r=0}^{n} \frac{{}^nC_r}{(r+1)} = \sum_{r=0}^{n} \frac{{}^{n+1}C_{r+1}}{(n+1)} \quad \left[\because \frac{{}^{n+1}C_{r+1}}{n+1} = \frac{{}^nC_r}{r+1}\right]$$

$$= \frac{1}{(n+1)}\sum_{r=0}^{n} {}^{n+1}C_{r+1}$$

$$= \frac{1}{(n+1)}({}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \dots + {}^{n+1}C_{n+1})$$

$$= \frac{1}{n+1}(2^{n+1} - 1) = \frac{2^{n+1} - 1}{n+1} = \text{RHS}$$

Example 67. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$, prove that

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}.$$

Sol. $\because (1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n \quad \dots(\text{i})$

Integrating on both sides of Eq. (i) within limits −1 to 0, then we get

$$\int_{-1}^{0} (1+x)^n\, dx = \int_{-1}^{0} (C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n)\, dx$$

$$\Rightarrow \left[\frac{(1+x)^n}{n+1}\right]_{-1}^{0} = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \ldots + \frac{C_n x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\Rightarrow \frac{1-0}{n+1} = 0 - \left(-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots + (-1)^{n+1}\frac{C_n}{n+1}\right)$$

$$\Rightarrow \frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^{n+2}\frac{C_n}{n+1}$$

$$\Rightarrow \frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \frac{C_n}{n+1}$$

$$[\because (-1)^{n+2} = (-1)^n(-1)^2 = (-1)^n]$$

Hence, $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \ldots + (-1)^n \dfrac{C_n}{n+1} = \dfrac{1}{n+1}$

I. Aliter

$$\text{LHS} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n\frac{C_n}{n+1}$$

$$= 1 - \frac{n}{2} + \frac{n(n-1)}{1\cdot 2\cdot 3} - \ldots + (-1)^n\frac{1}{n+1} = \frac{1}{(n+1)}$$

$$\left[(n+1) - \frac{(n+1)n}{1\cdot 2} + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} - \ldots + (-1)^n\right]$$

Put $n+1 = N$, we get

$$= \frac{1}{N}\left[N - \frac{N(N-1)}{1\cdot 2} + \frac{N(N-1)(N-2)}{1\cdot 2\cdot 3} - \ldots + (-1)^{N-1}\right]$$

$$= \frac{1}{N}[{}^NC_1 - {}^NC_2 + {}^NC_3 - \ldots + (-1)^{N-1}]$$

$$= -\frac{1}{N}[-{}^NC_1 + {}^NC_2 - {}^NC_3 + \ldots + (-1)^N {}^NC_N]$$

$$= -\frac{1}{N}[{}^NC_0 - {}^NC_1 + {}^NC_2 - {}^NC_3 + \ldots + (-1)^N {}^NC_N - {}^NC_0]$$

$$= -\frac{1}{N}[(1-1)^N - {}^NC_0] = -\frac{1}{N}[0-1] = \frac{1}{N}$$

$$= \frac{1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n\frac{C_n}{n+1} = \sum_{r=0}^{n}\frac{(-1)^r \cdot C_r}{r+1}$$

$$= \sum_{r=0}^{n}\frac{(-1)^r \cdot {}^nC_r}{r+1}$$

$$= \sum_{r=0}^{n}(-1)^r \cdot \frac{{}^{n+1}C_{r+1}}{(n+1)} \qquad \left[\because \frac{{}^{n+1}C_{r+1}}{n+1} = \frac{{}^nC_r}{r+1}\right]$$

$$= \frac{1}{(n+1)}\sum_{r=0}^{n}(-1)^r \cdot {}^{n+1}C_{r+1}$$

$$= \frac{1}{(n+1)}({}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \ldots + (-1)^n \cdot {}^{n+1}C_{n+1})$$

$$= \frac{1}{(n+1)}\{{}^{n+1}C_0 - ({}^{n+1}C_0 - {}^{n+1}C_1 + {}^{n+1}C_2 - {}^{n+1}C_3 + \ldots + (-1)^{n+1}\,{}^{n+1}C_{n+1})\}$$

$$= \frac{1}{(n+1)}[1 - (1-1)^{n+1}] = \frac{1}{(n+1)}[1-0] = \frac{1}{n+1} = \text{RHS}$$

Example 68. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$, prove that

$$\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \ldots = \frac{2^n}{n+1}.$$

Sol. $\because (1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots + C_nx^n \quad \ldots(i)$

Integrating on both sides of Eq. (i) within limits -1 to 1, then we get

$$\int_{-1}^{1}(1+x)^n\,dx = \int_{-1}^{1}(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots + C_nx^n)\,dx$$

$$= \int_{-1}^{1}(C_0 + C_2x^2 + C_4x^4 + \ldots)\,dx + \int_{-1}^{1}(C_1x + C_3x^3 + \ldots)\,dx$$

$$= 2\int_0^1(C_0 + C_2x^2 + C_4x^4 + \ldots)\,dx + 0$$

[by property of definite integral]
[since, second integral contains odd function]

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{1} = 2\left[\left(C_0x + \frac{C_2x^3}{3} + \frac{C_4x^5}{5} + \ldots\right)\right]_0^1$$

$$\Rightarrow \frac{2^{n+1}}{n+1} = 2\left(C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots\right)$$

or $C_0 + \dfrac{C_2}{3} + \dfrac{C_4}{5} + \ldots = \dfrac{2^n}{n+1}$

I. Aliter

$$\text{LHS} = C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots$$

$$= 1 + \frac{n(n-1)}{1\cdot 2\cdot 3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \ldots$$

$$= \frac{1}{(n+1)}\left\{\frac{n+1}{1} + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} + \frac{(n+1)n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \ldots\right\}$$

$$= \frac{1}{n+1}\{{}^{n+1}C_1 + {}^{n+1}C_3 + {}^{n+1}C_5 + \ldots\}$$

$$= \frac{1}{(n+1)}[\text{sum of even binomial coefficients of } (1+x)^{n+1}]$$

$$= \frac{2^{n+1-1}}{n+1} = \frac{2^n}{n+1} = \text{RHS}$$

II. Aliter $\text{LHS} = \dfrac{C_0}{1} + \dfrac{C_2}{3} + \dfrac{C_4}{5} + \ldots$

Case I If n is odd say $n = 2m+1$, $\forall\ m \in W$, then

$$\text{LHS} = \sum_{r=0}^{m}\frac{{}^{2m+1}C_{2r}}{2r+1} = \sum_{r=0}^{m}\frac{{}^{2m+2}C_{2r+1}}{(2m+1)} \qquad \left[\because \frac{{}^{2m+1}C_{2r}}{2r+1} = \frac{{}^{2m+2}C_{2r+1}}{2m+1}\right]$$

$$= \frac{1}{(2m+1)}\cdot 2^{2m+2-1} = \frac{2^n}{n+1} = \text{RHS} \qquad [\because n = 2m+1]$$

Case II If n is even say $n = 2m$, $\forall\ m \in N$, then

$$\text{LHS} = \sum_{r=0}^{m}\frac{{}^{2m}C_{2r}}{2r+1} = \sum_{r=0}^{m}\frac{{}^{2m+1}C_{2r+1}}{(2m+1)} \qquad \left[\because \frac{{}^{2m+1}C_{2r+1}}{2m+1} = \frac{{}^{2m}C_{2r}}{2r+1}\right]$$

$$= \frac{2^{2m+1-1}}{2m+1} = \frac{2^n}{n+1} = \text{RHS} \qquad [\because n = 2m]$$

Example 69. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$, prove that $\dfrac{C_1}{2} + \dfrac{C_3}{4} + \dfrac{C_5}{6} + \ldots = \dfrac{2^n - 1}{n+1}$.

Sol. We know that, from Examples (66) and (67)

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} + \frac{C_5}{6} + \ldots = \frac{2^{n+1}-1}{n+1} \quad \ldots(i)$$

and $$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \frac{C_4}{5} - \frac{C_5}{6} + \ldots = \frac{1}{n+1} \quad \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2\left(\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots\right) = \frac{2^{n+1}-2}{n+1}$$

On dividing each sides by 2, we get

$$\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots = \frac{2^n - 1}{n+1}$$

I. Aliter $$\text{LHS} = \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots$$

$$= \frac{n}{1\cdot 2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots$$

$$= \frac{1}{n+1}\left[\frac{(n+1)n}{1\cdot 2} + \frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots\right]$$

Put $n+1 = N$, then

$$\text{LHS} = \frac{1}{N}\left[\frac{N(N-1)}{2!} + \frac{N(N-1)(N-2)(N-3)}{4!} + \frac{N(N-1)(N-2)(N-3)(N-4)(N-5)}{6!} + \ldots\right]$$

$$= \frac{1}{N}[{}^NC_2 + {}^NC_4 + {}^NC_6 + \ldots]$$

$$= \frac{1}{N}[({}^NC_0 + {}^NC_2 + {}^NC_4 + {}^NC_6 + \ldots) - {}^NC_0]$$

$$= \frac{1}{N}[2^{N-1} - 1] = \frac{2^n - 1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots$$

Case I If n is odd say $n = 2m+1$, $\forall\ m \in W$, then

$$\text{LHS} = \sum_{r=0}^{m}\frac{{}^{2m+1}C_{2r+1}}{2r+2} = \sum_{r=0}^{m}\frac{{}^{2m+2}C_{2r+2}}{(2m+2)} \qquad \left[\because \frac{{}^{2m+2}C_{2r+2}}{2m+2} = \frac{{}^{2m+1}C_{2r+1}}{2r+2}\right]$$

$$= \frac{1}{(2m+2)}({}^{2m+2}C_2 + {}^{2m+2}C_4 + \ldots + {}^{2m+2}C_{2m+2})$$

$$= \frac{1}{(2m+2)}\cdot(2^{2m+2-1} - {}^{2m+1}C_0) = \frac{2^n - 1}{n+1} \qquad [\because 2m+1 = n]$$

$= \text{RHS}$

Case II If n is even say $n = 2m$, $\forall\ m \in N$, then

$$\text{LHS} = \sum_{r=0}^{m-1}\frac{{}^{2m}C_{2r+1}}{(2r+2)} = \sum_{r=0}^{m-1}\frac{{}^{2m+1}C_{2r+2}}{(2m+1)} \qquad \left[\because \frac{{}^{2m+1}C_{2r+2}}{2m+1} = \frac{{}^{2m}C_{2r+1}}{2r+2}\right]$$

$$= \frac{1}{(2m+1)}\sum_{r=0}^{m-1}{}^{2m+1}C_{2r+2}$$

$$= \frac{1}{(2m+1)}({}^{2m+1}C_2 + {}^{2m+1}C_4 + {}^{2m+1}C_6 + \ldots + {}^{2m+1}C_{2m})$$

$$= \frac{1}{(2m+1)}\cdot(2^{2m+1-1} - {}^{2m+1}C_0)$$

$$= \frac{2^n - 1}{n+1} = \text{RHS} \qquad [\because n = 2m]$$

Example 70. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that

$$3C_0 + 3^2\frac{C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \ldots + \frac{3^{n+1}C_n}{n+1} = \frac{4^{n+1}-1}{n+1}.$$

Sol. $\because \quad (1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n \quad \dots(i)$

Integrating on both sides of Eq. (i) within limits 0 to 3, we get

$$\int_0^3 (1+x)^n dx = \int_0^3 (C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n)dx$$

$$\Rightarrow \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^3 = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \frac{C_3x^4}{4} + \dots + \frac{C_nx^{n+1}}{n+1}\right]_0^3$$

$$\Rightarrow \frac{4^{n+1}-1}{n+1} = 3C_0 + \frac{3^2C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \dots + \frac{3^{n+1}C_n}{n+1}$$

Hence,

$$3C_0 + \frac{3^2C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \dots + \frac{3^{n+1}C_n}{n+1} = \frac{4^{n+1}-1}{n+1}$$

I. Aliter

$$\text{LHS} = 3C_0 + \frac{3^2C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \dots + \frac{3^{n+1}C_n}{n+1}$$

$$= 3\cdot 1 + \frac{3^2\cdot n}{2} + \frac{3^3\cdot n(n-1)}{1\cdot 2\cdot 3} + \frac{3^4\cdot n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \dots + \frac{3^{n+1}}{n+1}$$

$$= \frac{1}{(n+1)}\left[3\cdot(n+1) + \frac{3^2(n+1)n}{1\cdot 2} + \frac{3^3(n+1)n(n-1)}{1\cdot 2\cdot 3} + \frac{3^4(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \dots + 3^{n+1}\right]$$

Put $n+1 = N$, then

$$\text{LHS} = \frac{1}{N}\left[3N + \frac{3^2N(N-1)}{2!} + \frac{3^3N(N-1)(N-2)}{3!} + \frac{3^4N(N-1)(N-2)(N-3)}{4!} + \dots + 3^N\right]$$

$$= \frac{1}{N}[{}^NC_1(3) + {}^NC_2(3)^2 + {}^NC_3(3)^3 + \dots + {}^NC_N(3)^N]$$

$$= \frac{1}{N}[{}^NC_0 + {}^NC_1(3) + {}^NC_2(3)^2 + {}^NC_3(3)^3 + \dots + {}^NC_N(3)^N - {}^NC_0]$$

$$= \frac{1}{N}\{(1+3)^N - 1\} = \frac{4^N-1}{N} = \frac{4^{n+1}-1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = 3C_0 + 3^2\frac{C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \dots + \frac{3^{n+1}C_n}{n+1}$$

$$= \sum_{r=0}^{n} \frac{3^{r+1}\cdot {}^nC_r}{(r+1)} = \sum_{r=0}^{n} \frac{3^{r+1}\cdot {}^{n+1}C_{r+1}}{(n+1)} \quad \left[\because \frac{{}^{n+1}C_{r+1}}{n+1} = \frac{{}^nC_r}{r+1}\right]$$

$$= \frac{1}{(n+1)}\sum_{r=0}^{n} {}^{n+1}C_{r+1}\cdot 3^{r+1}$$

$$= \frac{1}{(n+1)}({}^{n+1}C_1\cdot 3 + {}^{n+1}C_2\cdot 3^2 + {}^{n+1}C_3\cdot 3^3 + \dots + {}^{n+1}C_{n+1}\cdot 3^{n+1})$$

$$= \frac{1}{(n+1)}[(1+3)^{n+1} - {}^{n+1}C_0]$$

$$= \frac{4^{n+1}-1}{n+1} = \text{RHS}$$

Example 71. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, show that $\frac{2^2}{1\cdot 2}C_0 + \frac{2^3}{2\cdot 3}C_1 + \frac{2^4}{3\cdot 4}C_2 + \dots + \frac{2^{n+2}C_n}{(n+1)(n+2)} = \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$.

Sol. Given,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n \quad \dots(i)$$

Integrating both sides of Eq. (i) within limits 0 to x, we get

$$\int_0^x (1+x)^n dx = \int_0^x (C_0 + C_1x + C_2x^2 + \dots + C_nx^n)dx$$

$$\Rightarrow \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \frac{(1+x)^{n+1}-1}{(n+1)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1} \quad \dots(ii)$$

Again, integrating both sides of Eq. (ii) within limits 0 to 2, we get

$$\int_0^2 \frac{(1+x)^{n+1}-1}{(n+1)}dx = \int_0^2\left(C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}\right)dx$$

$$\Rightarrow \frac{1}{(n+1)}\left[\left(\frac{(1+x)^{n+2}}{n+2} - x\right)\right]_0^2 = \left[\frac{C_0x^2}{1\cdot 2} + \frac{C_1x^3}{2\cdot 3} + \frac{C_2x^4}{3\cdot 4} + \dots + \frac{C_nx^{n+2}}{(n+1)(n+2)}\right]_0^2$$

$$\Rightarrow \frac{1}{(n+1)}\left\{\frac{3^{n+2}}{n+2} - 2 - \frac{1}{n+2}\right\} = \frac{2^2}{1\cdot 2}C_0 + \frac{2^3}{2\cdot 3}C_1 + \frac{2^4}{3\cdot 4}C_2 + \dots + \frac{2^{n+2}C_n}{(n+1)(n+2)}$$

Hence, $\frac{2^2}{1\cdot 2}C_0 + \frac{2^3}{2\cdot 3}C_1 + \frac{2^4}{3\cdot 4}C_2 + \dots + \frac{2^{n+2}C_n}{(n+1)(n+2)}$

$$= \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

I. Aliter

$$\text{LHS} = \frac{2^2}{1\cdot 2}C_0 + \frac{2^3}{2\cdot 3}C_1 + \frac{2^4}{3\cdot 4}C_2 + \dots + \frac{2^{n+2}C_n}{(n+1)(n+2)}$$

$$= \frac{2^2}{1\cdot 2}(1) + \frac{2^3}{2\cdot 3}\cdot n + \frac{2^4}{3\cdot 4}\cdot\frac{n(n-1)}{1\cdot 2} + \dots + \frac{2^{n+2}\cdot 1}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}\left\{\frac{(n+2)(n+1)}{1\cdot 2}2^2 + \frac{(n+2)(n+1)n}{1\cdot 2\cdot 3}2^3 + \frac{(n+2)(n+1)n(n-1)}{1\cdot 2\cdot 3\cdot 4}2^4 + \dots + 2^{n+2}\right\}$$

Put $n+2 = N$, then we get

$$= \frac{1}{N(N-1)}\left\{\frac{N(N-1)}{1\cdot 2}2^2 + \frac{N(N-1)(N-2)}{1\cdot 2\cdot 3}2^3 + \frac{N(N-1)(N-2)(N-3)}{1\cdot 2\cdot 3\cdot 4}2^4 + \dots + 2^N\right\}$$

$$= \frac{1}{N(N-1)}\{{}^NC_2(2)^2 + {}^NC_3(2)^3 + {}^NC_4(2)^4 + \dots + {}^NC_N(2)^N\}$$

$$= \frac{1}{N(N-1)}\{{}^NC_0 + {}^NC_1(2) + {}^NC_2(2)^2 + {}^NC_3(2)^3 + {}^NC_4(2)^4 + \dots + {}^NC_N(2)^N - {}^NC_0 - {}^NC_1(2)\}$$

$$= \frac{1}{N(N-1)}\{(1+2)^N - 1 - 2N\}$$

$$= \frac{3^{n+2}-1-2(n+2)}{(n+2)(n+1)} = \frac{3^{n+2}-2n-5}{(n+1)(n+2)} = \text{RHS}$$

II. Aliter

$$\text{LHS} = \frac{2^2}{1\cdot 2}\cdot C_0 + \frac{2^3}{2\cdot 3}\cdot C_1 + \frac{2^4}{3\cdot 4}\cdot C_2 + \dots + \frac{2^{n+2}\cdot C_n}{(n+1)(n+2)}$$

$$= \sum_{r=1}^{n+1} \frac{2^{r+1}}{r(r+1)}\cdot {}^nC_{r-1}$$

$$= \sum_{r=1}^{n+1} \frac{2^{r+1}\cdot {}^{n+2}C_{r+1}}{(n+1)(n+2)} \quad \left[\because \frac{{}^{n+2}C_{r+1}}{(n+1)(n+2)} = \frac{{}^nC_{r-1}}{r(r+1)}\right]$$

$$= \frac{1}{(n+1)(n+2)}\sum_{r=1}^{n+1} {}^{n+2}C_{r+1}\cdot 2^{r+1}$$

$$= \frac{1}{(n+1)(n+2)}[{}^{n+2}C_2\cdot 2^2 + {}^{n+2}C_3\cdot 2^3 + \dots + {}^{n+2}C_{n+2}\cdot 2^{n+2}]$$

$$= \frac{1}{(n+1)(n+2)}[(1+2)^{n+2} - {}^{n+2}C_0 - {}^{n+2}C_1\cdot 2^1]$$

$$= \frac{(3^{n+2}-2n-5)}{(n+1)(n+2)} = \text{RHS}$$

When Each Term in Summation Contains the Product of Two Binomial Coefficients or Square of Binomial Coefficients

Solution Process

1. **If difference of the lower suffixes of binomial coefficients in each term is same.**

i.e. ${}^nC_0\,{}^nC_2 + {}^nC_1\cdot{}^nC_3 + {}^nC_2\cdot{}^nC_4 + \dots$

Here, $2 - 0 = 3 - 1 = 4 - 2 = \dots = 2$

Case I **If each term of series is positive, then**

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n \quad \dots(i)$$

Interchanging 1 and x, we get

$$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n \quad \dots(ii)$$

Then, multiplying Eqs. (i) and (ii) and equate the coefficients of suitable power of x on both sides.

Replacing x by $\frac{1}{x}$ in Eq. (i), then we get

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots + \frac{C_n}{x^n} \quad \dots(iii)$$

Then, multiplying Eqs. (i) and (iii) and equate the coefficients of suitable power of x on both sides.

Mult

Example 72. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$, prove that

$$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \dots + C_{n-r}C_n = \frac{2n!}{(n-r)!(n+r)!}.$$

Sol. Here, differences of lower suffixes of binomial coefficients in each term is r.

i.e., $r - 0 = r + 1 - 1 = r + 2 - 2 = \dots = n - (n-r) = r$

Given,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_{n-r}x^{n-r} + \dots + C_nx^n \quad \dots(i)$$

Now,

$$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_rx^{n-r} + C_{r+1}x^{n-r-1} + C_{r+2}x^{n-r-2} + \dots + C_n \quad \dots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \dots + C_{n-r}x^{n-r} + \dots + C_nx^n) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_rx^{n-r} + C_{r+1}x^{n-r-1} + C_{r+2}x^{n-r-2} + \dots + C_n) \quad \dots(iii)$$

Now, coefficient of x^{n-r} on LHS of Eq. (iii) $= {}^{2n}C_{n-r}$

$$= \frac{2n!}{(n-r)!(n+r)!}$$

and coefficient of x^{n-r} on RHS of Eq. (iii)

$$= C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n$$

But Eq. (iii) is an identity, therefore coefficient of x^{n-r} in RHS = coefficient of x^{n-r} in LHS.

$$\Rightarrow C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = \frac{2n!}{(n-r)!(n+r)!}$$

Aliter

Given,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_r x^r + C_{r+1}x^{r+1} + C_{r+2}x^{r+2} + \ldots + C_{n-r}x^r + \ldots + C_n x^n \quad \ldots \text{(i)}$$

$$\text{Now, } \left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + \frac{C_{r+2}}{x^{r+2}} + \ldots + \frac{C_{n-r}}{x^{n-r}} + \ldots + \frac{C_n}{x^n} \quad \ldots \text{(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + \ldots + C_r x^r + C_{r+1}x^{r+1} + C_{r+2}x^{r+2} + \ldots + C_{n-r}x^{n-r} + \ldots + C_n x^n)$$
$$\times \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + \frac{C_{r+2}}{x^{r+2}} + \ldots + \frac{C_{n-r}}{x^{n-r}} + \ldots + \frac{C_n}{x^n}\right) \quad \ldots \text{(iii)}$$

Now, coefficient of $\frac{1}{x^r}$ in RHS

$$= (C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n)$$

$\therefore$ Coefficient of $\frac{1}{x^r}$ in LHS = Coefficient of x^{n-r} in

$$(1+x)^{2n} = {}^{2n}C_{n-r} = \frac{2n!}{(n-r)!(n+r)!}$$

But Eq. (iii) is an identity, therefore coefficient of $\frac{1}{x^r}$ in RHS = coefficient of $\frac{1}{x^r}$ in LHS.

$$\Rightarrow \quad C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = \frac{2n!}{(n-r)!(n+r)!}$$

Corollary I For $r = 0$,

$$C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{2n!}{(n!)^2}$$

Corollary II For $r = 1$,

$$C_0C_1 + C_1C_2 + C_2C_3 + \ldots + C_{n-1}C_n = \frac{2n!}{(n-1)!(n+1)!}$$

Corollary III For $r = 2$,

$$C_0C_2 + C_1C_3 + C_2C_4 + \ldots + C_{n-2}C_n = \frac{2n!}{(n-2)!(n+2)!}$$

| Example 73. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_n x^n$, prove that

$$C_0^2 + C_1^2 + C_2^2 = \frac{2n!}{n!n!} = \frac{1\cdot3\cdot5\ldots(2n-1)}{n!}\cdot 2^n.$$

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_n x^n \quad \ldots$(i)

Now, $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n \quad \ldots$(ii)

On multiplying Eqs. (i) and (ii), we get

$$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \ldots + C_n x^n) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n) \quad \ldots \text{(iii)}$$

Now, coefficient of x^n in RHS

$$= C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$$

And coefficient of x^n in LHS $= {}^{2n}C_n = \frac{2n!}{n!n!}$

$$= \frac{1\cdot2\cdot3\cdot4\cdot5\ldots(2n-1)2n}{n!n!} = \frac{1\cdot3\cdot5\ldots(2n-1)2^n n!}{n!n!}$$

But Eq. (iii) is an identity, therefore coefficient of x^n in RHS = coefficient of x^n in LHS.

$$\Rightarrow C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{2n!}{n!n!} = \frac{1\cdot3\cdot5\ldots(2n-1)}{n!}\cdot 2^n$$

Aliter

Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_n x^n \quad \ldots$(i)

$$\text{Now, } \left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n} \quad \ldots \text{(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + \ldots + C_n x^n) \times \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n}\right) \quad \ldots \text{(iii)}$$

Now, constant term in RHS $= C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

Constant term in LHS = Constant term in $\frac{(1+x)^{2n}}{x^n}$

$$= \text{Coefficient of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n = \frac{2n!}{n!n!}$$

$$= \frac{n!\,2^n\,[1\cdot3\cdot5\ldots(2n-1)]}{n!n!} = \frac{2^n\,[1\cdot3\cdot5\ldots(2n-1)]}{n!}$$

But Eq. (iii) is an identity, therefore the constant term in RHS = constant term in LHS.

$$\Rightarrow C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{2n!}{n!n!} = \frac{\{1\cdot3\cdot5\ldots(2n-1)\}}{n!}2^n$$

Case II If terms of the series alternately positive and negative, then

$$(1-x)^n = C_0 - C_1x + C_2x^2 - \ldots + (-1)^n C_n x^n \quad \ldots \text{(i)}$$

$$\text{and } (x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n \quad \ldots \text{(ii)}$$

Then, multiplying Eqs. (i) and (ii) and equate the coefficient of suitable power of x on both sides.

Or

Replacing x by $\frac{1}{x}$ in Eq. (i), we get

$$\left(1-\frac{1}{x}\right)^n = C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \ldots + (-1)^n \frac{C_n}{x^n} \quad \ldots \text{(iii)}$$

Then, multiplying Eqs. (i) and (iii) and equate the coefficient of suitable power of x on both sides.

| Example 74. Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n$.

Sol. Since, $(1-x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - \ldots + (-1)^{2n}\cdot{}^{2n}C_{2n}x^{2n}$

or $(1-x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - \ldots + {}^{2n}C_{2n}x^{2n} \quad \ldots$(i)

and $(x+1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + \ldots + {}^{2n}C_{2n} \quad \ldots$(ii)

On multiplying Eqs. (i) and (ii), we get

$$(x^2-1)^{2n} = ({}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - \ldots + {}^{2n}C_{2n}x^{2n}) \times ({}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + \ldots + {}^{2n}C_{2n}) \quad \ldots \text{(iii)}$$

Now, coefficient of x^{2n} in RHS

$$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2$$

Now, LHS can also be written as $(1-x^2)^{2n}$.

$\therefore$ General term in LHS, $T_{r+1} = {}^{2n}C_r(-x^2)^r$

Putting $r = n$, we get $T_{n+1} = (-1)^n\cdot{}^{2n}C_n x^{2n}$

$\Rightarrow$ Coefficient of x^{2n} in LHS $= (-1)^n\cdot{}^{2n}C_n$

But Eq. (iii) is an identity, therefore coefficient of x^{2n} in RHS = coefficient of x^{2n} in LHS

$$\Rightarrow ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n\cdot{}^{2n}C_n$$

Aliter

Since, $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + \ldots + {}^{2n}C_{2n}x^{2n} \quad \ldots$(i)

$$\text{and } \left(1-\frac{1}{x}\right)^{2n} = {}^{2n}C_0 - \frac{{}^{2n}C_1}{x} + \frac{{}^{2n}C_2}{x^2} - \ldots + \frac{{}^{2n}C_{2n}}{x^{2n}} \quad \ldots \text{(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$\frac{(x^2-1)^{2n}}{x^{2n}} = ({}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + \ldots + {}^{2n}C_{2n}x^{2n}) \times \left({}^{2n}C_0 - \frac{{}^{2n}C_1}{x} + \frac{{}^{2n}C_2}{x^2} - \ldots + \frac{{}^{2n}C_{2n}}{x^{2n}}\right) \quad \ldots \text{(iii)}$$

Now, constant term in RHS

$$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2$$

Constant term in LHS = Constant term in $\frac{(x^2-1)^{2n}}{x^{2n}}$

$$= \text{Coefficient of } x^{2n} \text{ in } (x^2-1)^{2n}$$
$$= \text{Coefficient of } x^{2n} \text{ in } (1-x^2)^{2n}$$
$$= {}^{2n}C_n(-1)^n = (-1)^n\cdot{}^{2n}C_n$$

But Eq. (iii) is an identity, therefore the constant term in RHS = constant term in LHS.

$$\Rightarrow ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n\cdot{}^{2n}C_n$$

| Example 75. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_n x^n$, prove that $C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n\cdot C_n^2 = 0$ or $(-1)^{n/2}\cdot\frac{n!}{(n/2)!(n/2)!}$, according as n is odd or even. Also, evaluate $C_0^2 + C_1^2 + C_2^2 - \ldots + (-1)^n\cdot C_n^2$ for $n = 10$ and $n = 11$

Sol. Since, $(1-x)^n = C_0 - C_1x + C_2x^2 - \ldots + (-1)^n C_n x^n \quad \ldots$(i)

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n \quad \ldots$(ii)

On multiplying Eqs. (i) and (ii), we get

$$(1-x^2)^n = \{C_0 - C_1x + C_2x^2 - \ldots + (-1)^n C_n x^n\} \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n) \quad \ldots \text{(iii)}$$

Now, coefficient of x^n in RHS

$$= C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2$$

General term in LHS $= T_{r+1} = {}^nC_r(-x^2)^r = {}^nC_r(-1)^r x^{2r}$

Putting $2r = n$, we get $r = n/2$

$\therefore \quad T_{(n/2)+1} = {}^nC_{n/2}(-1)^{n/2}x^n$

$\therefore$ Coefficient of x^n in LHS $= {}^nC_{n/2}(-1)^{n/2}$

$$= (-1)^{n/2}\cdot\frac{n!}{(n/2)!(n/2)!}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}\dfrac{n!}{(n/2)!(n/2)!}, & \text{if } n \text{ is even} \end{cases} \quad \left[\because \left(\frac{\text{odd}}{2}\right)! = \infty\right]$$

But Eq. (iii) is an identity, therefore coefficient of x^n in RHS = coefficient of x^n in LHS.

$$\Rightarrow C_0^2 - C_1^2 + C_2^2 - \dots + (-1)^n C_n^2$$

$$= \begin{cases} 0 & \text{, if } n \text{ is odd} \\ (-1)^{n/2} \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

Now, for $n = 10$,

$$C_0^2 - C_1^2 + C_2^2 - \dots + C_{10}^2 = (-1)^{10/2} \frac{10!}{5!\,5!} = -252 \qquad [\because 10 \text{ is even}]$$

and from $n = 11$,

$$C_0^2 - C_1^2 + C_2^2 - \dots - C_{11}^2 = 0 \qquad [\because 11 \text{ is odd}]$$

Aliter

Since, $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ ...(i)

Replacing x by $-\dfrac{1}{x}$, then we get

$$\left(1 - \frac{1}{x}\right)^n = C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \dots + (-1)^n \frac{C_n}{x^n} \quad \text{...(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$\frac{(x^2-1)^n}{x^n} = (C_0 + C_1x + C_2x^2 + \dots + C_nx^n) \times \left(C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \dots + (-1)^n \frac{C_n}{x^n}\right) \quad \text{...(iii)}$$

Now, constant term in RHS

$$= C_0^2 - C_1^2 + C_2^2 - \dots + (-1)^n C_n^2$$

$\therefore$ Constant term in LHS

$$= \text{Constant term in } \frac{(x^2-1)^n}{x^n}$$

$$= \text{Coefficient of } x^n \text{ in } (x^2-1)^n$$

$$= \text{Coefficient of } x^n \text{ in } {}^nC_{n/2} (x^2)^{n-(n/2)} (-1)^{n/2}$$

$$= (-1)^{n/2} \cdot {}^nC_{n/2}$$

$$= (-1)^{n/2} \cdot \frac{n!}{(n/2)!\,(n/2)!}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

But Eq. (iii) is an identity, therefore the constant term in RHS = constant term in LHS.

$$\Rightarrow C_0^2 - C_1^2 + C_2^2 - \dots + (-1)^n C_n^2$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

2. If sum of the lower suffixes of binomial coefficients in each term is same.

i.e., $C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \dots + C_nC_0$

Here, $0 + n = 1 + (n-1) = 2 + (n-2) = \dots = n + 0 = n$

***Case* I** If each term of series is positive, then

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n \quad \text{...(i)}$$

and $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ ...(ii)

Then, multiplying Eqs. (i) and (ii) and equate the coefficient of suitable power of x on both sides.

must

| Example 76. Prove that

$${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \dots + {}^nC_r$$

if $r < m$, $r < n$ and m, n, r are positive integers.

Sol. Here, sum of lower suffixes of binomial coefficients in each term is r.

i.e. $r = r - 1 + 1 = r - 2 + 2 = \dots = r = r$

Since,

$$(1+x)^m = {}^mC_0 + {}^mC_1x + \dots + {}^mC_{r-2}x^{r-2} + {}^mC_{r-1}x^{r-1} + {}^mC_rx^r + \dots + {}^mC_mx^m \quad \text{...(i)}$$

and $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_rx^r + \dots + {}^nC_nx^n$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(1+x)^{m+n} = ({}^mC_0 + {}^mC_1x + \dots + {}^mC_{r-2}x^{r-2} + {}^mC_{r-1}x^{r-1} + {}^mC_rx^r + \dots + {}^mC_mx^m) \times ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_rx^r + \dots + {}^nC_nx^n) \quad \text{...(iii)}$$

Now, coefficient of x^r in RHS

$$= {}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \dots + {}^mC_0 \cdot {}^nC_r$$

$$= {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \dots + {}^nC_r$$

Coefficient of $\boxed{x^r}$ in LHS $= {}^{m+n}C_r$

But Eq. (iii) is an identity, therefore coefficient of x^r in LHS = coefficient of x^r in RHS.

$$\Rightarrow {}^{m+n}C_r = {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \dots + {}^nC_r$$

***Case* II** If terms of the series alternately positive and negative, then

$$(1-x)^n = C_0 - C_1x + C_2x^2 - \dots + (-1)^n C_nx^n \quad \text{...(i)}$$

and $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ ...(ii)

Then, multiplying Eqs. (i) and (ii) and equate the coefficient of suitable power of x on both sides.

| Example 77. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, prove that

$$C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \dots + (-1)^n C_nC_0 = 0 \text{ or } (-1)^{n/2} \frac{n!}{(n/2)!\,(n/2)!},$$

according as n is odd or even.

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_{n-2}x^{n-2} + C_{n-1}x^{n-1} + C_nx^n$ ...(i)

and $(1-x)^n = C_0 - C_1x + C_2x^2 - \dots + (-1)^n C_nx^n$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(1-x^2)^n = (C_0 + C_1x + C_2x^2 + \dots + C_{n-2}x^{n-2} + C_{n-1}x^{n-1} + C_nx^n) \times (C_0 - C_1x + C_2x^2 - \dots + (-1)^n C_nx^n) \quad \text{...(iii)}$$

Now, coefficient of x^n in RHS

$$= C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \dots + (-1)^n C_nC_0$$

Now, general term in LHS,

$$T_{r+1} = {}^nC_r(-x^2)^r = (-1)^r \cdot {}^nC_r\, x^{2r}$$

Putting $2r = n$, we get

$$r = n/2$$

Now, $T_{n/2+1} = (-1)^{n/2} \cdot {}^nC_{n/2}\, x^n$

$\therefore$ Coefficient of x^n in LHS $= (-1)^{n/2} \cdot {}^nC_{n/2}$

$$= (-1)^{n/2} \cdot \frac{n!}{(n/2)!\,(n/2)!}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

But Eq. (iii) is an identity, therefore the coefficient of x^n in RHS = coefficient of x^n in LHS.

$$\Rightarrow C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \dots + (-1)^n C_nC_0$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

3. If each term is the product of two binomial coefficient divided or multiplied by an integer, then integrating or differentiating by preceeding method. Then, multiplying two series and equate the coefficient of suitable power of x on both sides.

| Example 78. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$, prove that

$$C_1^2 + 2C_2^2 + 3C_3^2 + \dots + nC_n^2 = \frac{(2n-1)!}{((n-1)!)^2}.$$

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \dots + nC_nx^{n-1}$$

$$\Rightarrow n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \dots + nC_nx^{n-1} \quad \text{...(i)}$$

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \dots + C_n$ ...(ii)

On multiplying Eqs. (i) and (ii), then we get

$$n(1+x)^{2n-1} = (C_1 + 2C_2x + 3C_3x^2 + \dots + nC_nx^{n-1}) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \dots + C_n) \quad \text{...(iii)}$$

Now, coefficient of x^{n-1} on RHS

$$= C_1^2 + 2C_2^2 + 3C_3^2 + \dots + nC_n^2$$

and coefficient of x^{n-1} on LHS

$$= n \cdot {}^{2n-1}C_{n-1} = n \cdot \frac{(2n-1)!}{(n-1)!\,n!}$$

$$= \frac{(2n-1)!}{(n-1)!\,(n-1)!} = \frac{(2n-1)!}{\{(n-1)!)^2\}}$$

But Eq. (iii) is an identity, therefore the coefficient of x^{n-1} in RHS = coefficient of x^{n-1} in LHS.

$$\Rightarrow C_1^2 + 2C_2^2 + 3C_3^2 + \dots + nC_n^2 = \frac{(2n-1)!}{\{(n-1)!\}}$$

| Example 79. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, prove that

$$C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \dots + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{\{(n+1)!\}^2}.$$

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$

Integrating both sides w.r.t. x within limits 0 to x, then we get

$$\int_0^x (1+x)^n\,dx = \int_0^x (C_0 + C_1x + C_2x^2 + \dots + C_nx^n)\,dx$$

$$\frac{(1+x)^{n+1} - 1}{(1+n)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1} \quad \text{...(i)}$$

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n$ ...(ii)

Multiplying Eqs. (i) and (ii), we get

$$\frac{1}{(n+1)}\{(1+x)^{2n+1} - (1+x)^n\} = \left(C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}\right) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n) \quad \text{...(iii)}$$

Now, coefficient of x^{n+1} in RHS of Eq. (iii)

$$= C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \dots + \frac{C_n^2}{n+1}$$

and coefficient of x^{n+1} in LHS of Eq. (iii)

$$= \frac{1}{(n+1)}\{{}^{2n+1}C_{n+1} - 0\}$$

$$= \frac{1}{(n+1)} \cdot \frac{(2n+1)!}{(n+1)!\,n!}$$

$$= \frac{(2n+1)!}{(n+1)!\,(n+1)!} = \frac{(2n+1)!}{\{(n+1)!\}^2}$$

But Eq. (iii) is an identity, therefore coefficient of x^{n+1} in RHS of Eq. (iii) = coefficient of x^{n+1} in LHS of Eq. (iii).

$$\Rightarrow C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \ldots + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{\{(n+1)!\}^2}$$

Binomial Inside Binomial

The upper suffices of binomial coefficients are different but lower suffices are same.

| Example 80. Evaluate $\sum_{r=0}^{n} {}^{n+r}C_n$.

Sol. $\sum_{r=0}^{n} {}^{n+r}C_n = {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n}C_n$

= Coefficient of x^n in $[(1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \ldots + \ldots + (1+x)^{2n}]$

= Coefficient of x^n in $\left[\frac{(1+x)^n[(1+x)^{n+1}-1]}{(1+x)-1}\right]$

= Coefficient of x^{n+1} in $[(1+x)^{2n+1} - (1+x)^n]$

$= {}^{2n+1}C_{n+1} - 0 = {}^{2n+1}C_n$

| Example 81. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that

$C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-2}C_n + C_2 \cdot {}^{2n-4}C_n - \ldots = 2^n$

Sol. LHS $= C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-2}C_n + C_2 \cdot {}^{2n-4}C_n - \ldots$

= Coefficient of x^n in

$[C_0(1+x)^{2n} - C_1(1+x)^{2n-2} + C_2(1+x)^{2n-4} - \ldots]$

= Coefficient of x^n in

$[C_0((1+x)^2)^n - C_1[(1+x)^2]^{n-1} + C_2[(1+x)^2]^{n-2} - \ldots]$

= Coefficient of x^n in $[[(1+x)^2 - 1]^n]$

= Coefficient of x^n in $(2x + x^2)^n$

= Constant term in $(2+x)^n = 2^n$ = RHS

| Example 82. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$, prove that

$C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-1}C_n + C_2 \cdot {}^{2n-2}C_n - C_3 \cdot {}^{2n-3}C_n + \ldots + (-1)^n C_n \cdot {}^nC_n = 1$

Sol. LHS $= C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-1}C_n + C_2 \cdot {}^{2n-2}C_n - C_3 \cdot {}^{2n-3}C_n + \ldots + (-1)^n C_n \cdot {}^nC_n$

= Coefficient of x^n in

$[C_0(1+x)^{2n} - C_1(1+x)^{2n-1} + C_2(1+x)^{2n-2} - C_3(1+x)^{2n-3} + \ldots + (-1)^n C_n(1+x)^n]$

= Coefficient of x^n in

$(1+x)^n[C_0(1+x)^n - C_1(1+x)^{n-1} + C_2(1+x)^{n-2} - C_3(1+x)^{n-3} + \ldots + (-1)^n C_n]$

= Coefficient of x^n in $(1+x)^n[((1+x)-1)^n]$

= Coefficient of x^n in $(1+x)^n \cdot x^n$d

= Constant term in $(1+x)^n = 1$ = RHS

Sum of the Series

***Case I* When *i* and *j* are independent.**

In this summation, three types of terms occur, when $i<j$, $i=j$ and $i>j$,

i.e., $\sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j = \sum_{i=0}^{n}\left\{a_i\left(\sum_{j=0}^{n} a_j\right)\right\}$

$= \sum_{i=0}^{n} a_i \sum_{j=0}^{n} a_j = \left(\sum_{i=0}^{n} a_i\right)^2$ or $\left(\sum_{j=0}^{n} a_j\right)^2$

Corollary I $\sum_{i=0}^{n}\sum_{j=0}^{n} {}^nC_i\,{}^nC_j = \left(\sum_{i=0}^{n} {}^nC_i\right)^2 = (2^n)^2 = 2^{2n}$

| Example 83. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, find the values of the following

(i) $\sum_{i=0}^{n}\sum_{j=0}^{n} (C_i + C_j)$

(ii) $\sum_{i=0}^{n}\sum_{j=0}^{n} (i+j)C_iC_j$

Sol. (i) $\sum_{i=0}^{n}\sum_{j=0}^{n}(C_i + C_j) = \sum_{i=0}^{n}\sum_{j=0}^{n} C_i + \sum_{i=0}^{n}\sum_{j=0}^{n} C_j$

$= \sum_{j=0}^{n}\left(\sum_{i=0}^{n} C_i\right) + \sum_{i=0}^{n}\left(\sum_{j=0}^{n} C_j\right)$

$= \sum_{j=0}^{n}(2^n) + \sum_{i=0}^{n}(2^n) = (n+1)\cdot 2^n + (n+1)\cdot 2^n$

$= 2(n+1)2^n = (n+1)2^{n+1}$

(ii) $\sum_{i=0}^{n}\sum_{j=0}^{n}(i+j)C_iC_j = \sum_{i=0}^{n}\sum_{j=0}^{n} iC_iC_j + \sum_{i=0}^{n}\sum_{j=0}^{n} jC_iC_j$

$= \sum_{i=0}^{n} iC_i\left(\sum_{j=0}^{n} C_j\right) + \sum_{j=0}^{n} jC_j\left(\sum_{i=0}^{n} C_i\right)$

$= \sum_{i=0}^{n} iC_i(2^n) + \sum_{j=0}^{n} jC_j(2^n)$

$= 2^n \sum_{i=0}^{n} i\,{}^nC_i + 2^n \sum_{j=0}^{n} j\,{}^nC_j$

$= 2^n \sum_{i=0}^{n} i \cdot \frac{n}{i} \cdot {}^{n-1}C_{i-1} + 2^n \sum_{j=0}^{n} j \cdot \frac{n}{j} \cdot {}^{n-1}C_{j-1}$

$= n \cdot 2^n \sum_{i=0}^{n} {}^{n-1}C_{i-1} + n \cdot 2^n \sum_{j=0}^{n} {}^{n-1}C_{j-1}$

$= n \cdot 2^n \cdot 2^{n-1} + n \cdot 2^n \cdot 2^{n-1}$

$= n \cdot 2 \cdot 2^{2n-1} = n \cdot 2^{2n}$

***Case II* When *i* and *j* are dependent.**

In this summation, when $i<j$ is equal to the sum of the terms when $i>j$, if a_i and a_j are symmetrical. So, in this case

$\sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j = \sum\sum_{0 \le i<j \le n} a_i a_j + \sum\sum_{i=j} a_i a_j + \sum\sum_{0 \le j<i \le n} a_i a_j$

$= 2\sum\sum_{0 \le i<j \le n} a_i a_j + \sum\sum_{i=j} a_i a_j$

$$\Rightarrow \sum\sum_{0 \le i<j \le n} a_i a_j = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j - \sum\sum_{i=j} a_i a_j}{2}$$

When a_i and a_j are not symmetrical, we find the sum by listing all the terms.

Corollary I

$$\sum\sum_{0 \le i<j \le n} {}^nC_i\,{}^nC_j = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} {}^nC_i\,{}^nC_j - \sum\sum_{i=j} {}^nC_i \cdot {}^nC_j}{2}$$

$$= \frac{(2^n)^2 - \sum_{i=0}^{n}({}^nC_i)^2}{2} = \frac{2^{2n} - {}^{2n}C_n}{2} = 2^{2n-1} - \frac{2n!}{2(n!)^2}$$

| Example 84. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, find the values of the following.

(i) $\sum\sum_{0 \le i<j \le n} C_i$ (ii) $\sum\sum_{0 \le i<j \le n} jC_i$

(iii) $\sum\sum_{i \ne j} C_iC_j$ (iv) $\sum\sum_{0 \le i \le j \le n} C_iC_j$

(v) $\sum\sum_{0 \le i<j \le n} (C_i \pm C_j)^2$

(vi) $\sum\sum_{0 \le i<j \le n} (i+j)C_iC_j$

(vii) $\sum\sum_{0 \le i<j \le n} (i \cdot j)C_iC_j$

Sol. (i) $\sum\sum_{0 \le i<j \le n} C_i = \frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n} C_i\right) - \sum\sum_{i=j} C_i}{2}$

$= \frac{(n+1)\sum_{i=0}^{n} C_i - \sum_{i=0}^{n} C_i}{2} = n \cdot 2^{n-1}$

(ii) $\sum\sum_{0 \le i<j \le n} jC_i = \sum_{r=0}^{n-1} {}^nC_r$ $\{(r+1) + (r+2) + (r+3) + \ldots + n\}$

$= \sum_{r=0}^{n-1} {}^nC_r \cdot \frac{(n-r)(n+r+1)}{2}$

$= \frac{1}{2}\sum_{r=0}^{n-1} {}^nC_r(n^2 - r^2 + n - r)$

$= \frac{1}{2}(n^2+n)\sum_{r=0}^{n-1} {}^nC_r - \frac{1}{2}\sum_{r=0}^{n-1} r \cdot {}^nC_r - \frac{1}{2}\sum_{r=0}^{n-1} r^2 \cdot {}^nC_r$

$= \frac{1}{2}(n^2+n)(2^n - 1) - \frac{1}{2} \cdot n \cdot (2^{n-1} - 1)$

$- \frac{1}{2} \cdot n[(n-1)(2^{n-2} - 1) + 2^{n-1} - 1]$

$= n(3n+1) \cdot 2^{n-3}$

Remark

Here, j and C_i are not symmetrical.

(iii) Here, $i \ne j$ i.e., $i>j$ or $i<j$

But C_i and C_j are symmetrical.

$\therefore \sum\sum_{i \ne j} C_iC_j = 2\sum\sum_{0 \le i<j \le n} C_iC_j$

$= 2\left(\frac{2^{2n} - {}^{2n}C_n}{2}\right)$ [from corollary I]

$= 2^{2n} - {}^{2n}C_n$

(iv) $\sum\sum_{0 \le i \le j \le n} C_iC_j = \sum\sum_{0 \le i<j \le n} C_iC_j + \sum\sum_{i=j} C_iC_j$

$= \frac{1}{2}(2^{2n} - {}^{2n}C_n) + {}^{2n}C_n$ [from corollary I]

$= \frac{1}{2}(2^{2n} + {}^{2n}C_n)$

(v) $\sum\sum_{0 \le i<j \le n}(C_i \pm C_j)^2 = \sum\sum_{0 \le i<j \le n}(C_i^2 + C_j^2 \pm 2C_iC_j)$

$= \sum\sum_{0 \le i<j \le n}(C_i^2 + C_j^2) \pm 2\sum\sum_{0 \le i<j \le n} C_iC_j$

$\because \sum\sum_{0 \le i<j \le n}(C_i^2 + C_j^2)$

$$= \frac{\sum_{i=0}^{n}\sum_{j=0}^{n}(C_i^2 + C_j^2) - 2\sum_{i=0}^{n} C_i^2}{2}$$

$$= \frac{\sum_{i=0}^{n}\left(\sum_{j=0}^{n} C_i^2 + \sum_{j=0}^{n} C_j^2\right) - 2\cdot {}^{2n}C_n}{2}$$

$$= \frac{\sum_{i=0}^{n}((n+1)C_i^2 + {}^{2n}C_n) - 2\cdot {}^{2n}C_n}{2}$$

$$= \frac{(n+1)\sum_{i=0}^{n} C_i^2 + {}^{2n}C_n \sum_{i=0}^{n} 1 - 2\cdot {}^{2n}C_n}{2}$$

$$= \frac{(n+1)\cdot {}^{2n}C_n + {}^{2n}C_n\cdot(n+1) - 2\cdot {}^{2n}C_n}{2}$$

$$= n\cdot {}^{2n}C_n$$

$$\therefore \sum\sum_{0\le i<j\le n}(C_i \pm C_j)^2 = n\cdot {}^{2n}C_n \pm (2^{2n} - {}^{2n}C_n)$$

[from corollary 1]

$$= (n \mp 1)\, {}^{2n}C_n \pm 2^{2n}; \quad \sum\sum_{0\le i<j\le n}(i+j)C_iC_j$$

Remark

$$\sum\sum_{0\le i<j\le n}(C_i + C_j) = n\cdot 2^n$$

(vi) $\sum\sum_{0\le i<j\le n}(i+j)C_iC_j$

Let $P = \sum\sum_{0\le i<j\le n}(i+j)C_iC_j$ ...(i)

Replacing i by $n-i$ and j by $n-j$ in Eq. (i), then we get

$$P = \sum\sum_{0\le i<j\le n}(n-i+n-j)C_{n-i}C_{n-j}$$

[$\because$ sum of binomial expansion does not change if we replace r by $n-r$]

$$P = \sum\sum_{0\le i<j\le n}(2n-i-j)C_iC_j$$

[$\because {}^nC_r = {}^nC_{n-r}$] ...(ii)

On adding Eqs. (i) and (ii), we get

$$2P = 2n\sum\sum_{0\le i<j\le n} C_iC_j$$

or $$P = n\sum\sum_{0\le i<j\le n} C_iC_j = \frac{n}{2}(2^{2n} - {}^{2n}C_n)$$

[from corollary I]

(vii) $$\sum\sum_{0\le i<j\le n}(i\cdot j)C_iC_j = \sum\sum_{0\le i<j\le n}(i\,{}^nC_i)(j\cdot{}^nC_j)$$

$$= n^2\sum\sum_{0\le i<j\le n} {}^{n-1}C_{i-1}\,{}^{n-1}C_{j-1}$$

$$= n^2\left[\frac{2^{2(n-1)} - {}^{2n-2}C_{n-1}}{2}\right] \text{ [from corollary I]}$$

$$= n^2\left(2^{2n-3} - \frac{1}{2}\cdot {}^{2n-2}C_{n-1}\right)$$

Exercise for Session 4

1. The coefficient of $a^4b^8c^9d^9$ in the expansion of $(abc + abd + acd + bcd)^{10}$ is
(a) $10!$ (b) $\frac{10!}{4!8!9!9!}$ (c) 2520 (d) None of these

2. If $(1+2x+3x^2)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$, then a_1 equals
(a) 210 (b) 20 (c) 10 (d) None of these

3. If $(1+x+x^2+x^3)^5 = a_0 + a_1x + a_2x^2 + \ldots + a_{15}x^{15}$, then a_{10} equals
(a) 99 (b) 100 (c) 101 (d) 110

4. Coefficient of x^{15} in $(1+x+x^3+x^4)^n$ is
(a) $\sum_{r=0}^{5} {}^nC_{5-r}\cdot {}^nC_{3r}$ (b) $\sum_{r=0}^{5} {}^nC_{5r}$ (c) $\sum_{r=0}^{5} {}^nC_{2r}$ (d) $\sum_{r=0}^{3} {}^nC_{3-r}\cdot {}^nC_{5r}$

5. The number of terms in the expansion of $\left(x^2 + 1 + \frac{1}{x^2}\right)^n, n \in N$ is
(a) ${}^{n+2}C_2$ (b) ${}^{n+3}C_2$ (c) ${}^{2n+1}C_{2n}$ (d) ${}^{3n+1}C_{3n}$

6. If $(1+x)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{10}x^{10}$, then $(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is equal to
(a) 2^9 (b) 3^9 (c) 2^{10} (d) 3^{10}

7. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$, n being even the value of $C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \ldots + (C_0 + C_1 + C_2 + \ldots + C_{n-1})$ is equal to
(a) $n\cdot 2^n$ (b) $n\cdot 2^{n-1}$ (c) $n\cdot 2^{n-2}$ (d) $n\cdot 2^{n-3}$

8. The value of $\frac{C_0}{1\cdot 3} - \frac{C_1}{2\cdot 3} + \frac{C_2}{3\cdot 3} - \frac{C_3}{4\cdot 3} + \ldots + (-1)^n\frac{C_n}{(n+1)\cdot 3}$ is
(a) $\frac{3}{n+1}$ (b) $\frac{n+1}{3}$ (c) $\frac{1}{3(n+1)}$ (d) None of these

9. The value of $\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + \ldots + \binom{50}{49}\binom{50}{50}$, where ${}^nC_r = \binom{n}{r}$, is
(a) $\binom{100}{50}$ (b) $\binom{100}{51}$ (c) $\binom{50}{25}$ (d) $\binom{50}{25}^2$

10. If C_r stands for ${}^4C_r$, then $C_0C_4 - C_1C_3 + C_2C_2 - C_3C_1 + C_4C_0$ is equal to
(a) C_1 (b) C_2 (c) C_3 (d) C_4

11. The sum $\sum_{r=0}^{n}(r+1)({}^nC_r)^2$ is equal to
(a) $\frac{(n+2)(2n-1)!}{n!(n-1)!}$ (b) $\frac{(n+2)(2n+1)!}{n!(n-1)!}$ (c) $\frac{(n+2)(2n+1)!}{n!(n+1)!}$ (d) $\frac{(n+2)(2n-1)!}{n!(n+1)!}$

12. $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r\, {}^rC_p\, 2^p\right)$ is equal to
(a) $4^n - 3^n + 1$ (b) $4^n - 3^n - 1$ (c) $4^n - 3^n + 2$ (d) $4^n - 3^n$

13. $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{m=0}^{10}(-1)^m\frac{{}^{10}C_m}{2^m}\right)$ is equal to
(a) 1 (b) 2^5 (c) 2^{10} (d) 2^{20}

14. The value of $\sum\sum\sum\sum_{0\le i<j<k<l\le n} 2$ is equal to
(a) $2(n+1)^3$ (b) $2\cdot {}^{n+1}C_4$ (c) $2(n+1)^4$ (d) $2\cdot {}^{n+2}C_3$

Shortcuts and Important Results to Remember

1. $(r+1)$th term from end in the expansion of $(x+y)^n = (r+1)$th term from beginning in the expansion of $(y+x)^n$.

2. If ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$ are in AP, then $(n-2r)^2 = n+2$ or $r = \frac{1}{2}(n \pm \sqrt{(n+2)})$ for $r=2$, $n=7$ and for $r=5, n=7, 14$.

3. Four consecutive binomial coefficients can never be in AP.

4. Three consecutive binomial coefficients can never be in GP or HP.

5. If a, b, c, d are four consecutive coefficients in the expansion of $(1+x)^n$, then $\frac{a}{a+b}, \frac{b}{b+c}, \frac{c}{c+d}$ are in AP.

   (i) $\frac{a}{a+b} + \frac{c}{c+d} = 2\left(\frac{b}{b+c}\right)$

   (ii) $\left(\frac{b}{b+c}\right)^2 > \frac{ac}{(a+b)(c+d)}$

6. If greatest term in $(1+x)^{2n}$ has the greatest coefficient, then $\frac{n}{n+1} < x < \frac{n+1}{n}$.

7. (a) The coefficient of x^{n-1} in the expansion of
   $$(x-1)(x-2)(x-3)\ldots(x-n) = -(1+2+3+\ldots+n) = -\frac{n(n+1)}{2} = -{}^{n+1}C_2$$
   (b) The coefficient of x^{n-1} in the expansion of
   $$(x+1)(x+2)(x+3)\ldots(x+n) = (1+2+3+\ldots+n) = \frac{n(n+1)}{2} = {}^{n+1}C_2$$

8. The number of terms in the expansion of
   $$(x+a)^n + (x-a)^n = \begin{cases} \frac{n+2}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

9. The number of terms in the expansion of
   $$(x+a)^n - (x-a)^n = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

10. The number of terms in the expansion of multinomial $(x_1 + x_2 + x_3 + \ldots + x_m)^n$, when $x_1, x_2, x_3, \ldots, x_m \in C$ and $n \in N$, is $\boxed{{}^{n+m-1}C_{m-1}}$.

11. The number of terms in the expansion of $\left(ax^p + \frac{b}{x^p} + c\right)^n$, where $n, p \in N$ and a, b, c are constants, is $2n+1$.

12. If the coefficients of pth and qth terms in the expansion of $(1+x)^n$ are equal, then $p+q = n+2$, where $p, q, n \in N$.

13. If the coefficients of x^r, x^{r+1} in the expansion of $\left(a + \frac{x}{b}\right)^n$ are equal, then $n = (r+1)(ab+1) - 1$, where $n, r \in N$ and a, b are constants.

14. Coefficient of x^m in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ = Coefficient of T_{r+1}, where $r = \frac{np-m}{p+q}$, where $p, q, n \in N$ and a, b are constants.

15. The term independent of x in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ is T_{r+1}, where $r = \frac{np}{p+q}$, where $n, p, q \in N$ and a, b are constants.

16. Sum of the coefficients in the expansion of $(ax+by)^n$ is $(a+b)^n$, where $n \in N$ and a, b are constants.

17. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ and $p+q=1$, then

    (i) $\sum_{r=0}^{n} r \cdot C_r \cdot p^r \cdot q^{n-r} = np$

    (ii) $\sum_{r=0}^{n} r^2 \cdot C_r \cdot p^r \cdot q^{n-r} = n^2p^2 + npq$

18. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then

    (i) $\sum_{r=0}^{n} r \cdot C_r = n \cdot 2^{n-1}$ (ii) $\sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{2^{n+1}-1}{n+1}$

    (iii) $\sum_{r=0}^{n} r^2 \cdot C_r = n(n+1)2^{n-2}$ (iv) $\sum_{r=0}^{n} (-1)^r \cdot r \cdot C_r = 0$

    (v) $\sum_{r=0}^{n} \frac{(-1)^r C_r}{r+1} = \frac{1}{n+1}$

    (vi) $\sum_{r=0}^{n} (-1)^r \frac{C_r}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

    (vii) $\sum_{r=0}^{n} (-1)^r \cdot r^2 \cdot C_r = 0$

    (viii) $\sum_{r=0}^{n} (-1)^r \cdot (a-r)(b-r)C_r = 0, \forall n > 3$

    (ix) $\sum_{r=0}^{n} (-1)^r (a-r)(b-r)(c-r)C_r = 0, \forall n > 3$

    (x) $\sum_{r=0}^{n} (-1)^r (a-r)^3 C_r = 0, \forall n > 3$

    (xi) $\sum_{r=0}^{n} r(r-1)(r-2)\ldots(r-k+1)C_r x^{r-k} = \frac{d^k}{dx^k}(1+x)^n$

    for $k=2$, $\sum_{r=0}^{n} r(r-1)C_r = \frac{d^2}{dx^2}[(1+x)^n]_{x=1} = n(n-1)2^{n-2}$

    and for $k=3$; $\sum_{r=0}^{n} r(r-1)(r-2)(-1)^{r-3}C_r$
    $$= \frac{d^3}{dx^3}[(1+x)^n]_{x=-1} = 0$$