RELATIONS

INTRODUCTION :

Let A and B be two sets. Then a relation R from A to B is a subset of A B.

thus, R is a relation from A to B $\Leftrightarrow$ R $\subseteq$ A B.

Ex. If A = {1, 2, 3} and B = {a, b, c}, then R = {(1, b), (2, c), (1, a), (3, a)} being a subset of A B, is a relation from A to B. Here (1, b), (2, c), (1, a) and (3, a) $\in$ R, so we write 1 Rb, 2Rc, 1Ra and 3Ra. But (2, b) $\notin$ R, so we write $2 \not R\ b$

Total Number of Realtions : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then A B consists of mn ordered pairs. So, total number of subsets of A B is 2^{mn}.

Domain and Range of a relation : Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom (R) = {a : (a, b) $\in$ R}

and, Range (R) = {b : (a, b) $\in$ R}

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

e.g. Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} be two sets and let R be a relation from A to B defined by the phrase "(x, y) $\in$ R $\Leftrightarrow$ x > y". Under this relation R, we have

3R2, 5R2, 5R4, 7R2, 7R4 and 7R6

i.e. R = {(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)}

$\therefore$ Dom (R) = {3, 5, 7} and Range (R) = {2, 4, 6}

Inverse Relation : Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Dom(R) = Range(R^{-1}) and Range (R) = Dom (R^{-1})

Illustration 1 :

Let A be the set of first ten natural numbers and let R be a relation on A defined by (x, y) $\in$ R $\Leftrightarrow$ x + 2y = 10, i.e. R = {(x, y) : x $\in$ A, y $\in$ A and x + 2y = 10}. Express R and R^{-1} as sets of ordered pairs. Determine also (i) domain of R and R^{-1} (ii) range of R and R^{-1}

Solution :

We have $(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \dfrac{10-x}{2}$, x, y $\in$ A

where A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Now, $x = 1 \Rightarrow y = \dfrac{10-1}{2} = \dfrac{9}{2} \notin A$.

This shows that 1 is not related to any element in A. Similarly we can observe. that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation

Further we find that :

For $x = 2,\ y = \dfrac{10-2}{2} = 4 \in A$ $\quad\therefore$ (2, 4) $\in$ R

For $x = 4,\ y = \dfrac{10-4}{2} = 3 \in A$ $\quad\therefore$ (4, 3) $\in$ R

For $x = 6,\ y = \dfrac{10-6}{2} = 2 \in A$ $\quad\therefore$ (6, 2) $\in$ R

For $x = 8,\ y = \dfrac{10-8}{2} = 1 \in A$ $\quad\therefore\ (8, 1) \in R$

Thus, $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow \quad R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, $Dom(R) = \{2, 4, 6, 8\} = Range(R^{-1})$

and, $Range\ (R) = \{4, 3, 2, 1\} = Dom(R^{-1})$

Do yourself - 1 :

(i) If $A = \{2, 4, 6, 9\}$ and $B = \{4, 6, 18, 27, 54\}$, $a \in A$, $b \in B$, find the set of ordered pairs such that 'a' is a factor of 'b' and $a < b$.

(ii) Find the domain and range of the relation R given by $R = \{(x, y) : y = x + \dfrac{6}{x}$, where $x, y \in N$ and $x < 6\}$

TYPES OF RELATIONS :

In this section we intend to define various types of relations on a given set A.

Void Relation : Let A be a set. Then $\phi \subseteq A \quad A$ and so it is a relation on A. This relation is called the void or empty relation on A.

Universal Relation : Let A be a set. Then $A \quad A \subseteq A \quad A$ and so it is a relation on A. This relation is called the universal relation on A.

Identity Relation : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

e.g. The relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set $A = \{1, 2, 3\}$. But relations $R_1 = \{(1, 1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in R_2 element 1 is related to elements 1 and 3.

Reflexive Relation : A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R on a set A is not reflexive if there exists an element $A \in A$ such that $(a\ ,\ a) \notin R$.

e.g. Let $A = \{1, 2, 3\}$ be a set. Then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A. But $R_1 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.

Note : Every Identity relation is reflexive but every reflexive ralation is not identity.

Symmetric Relation : A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $a\,R\,b \Rightarrow bRa$ for all $a, b, \in A$.

e.g. Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R \Leftrightarrow x$ is perpendicular to y. Then R is a symmetric relation on L, because $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$

i.e. $(L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$.

e.g. Let $A = \{1, 2, 3, 4\}$ and Let R_1 and R_2 be realtion on A given by $R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$. Clearly, R_1 is a symmetric relation on A. However, R_2 is not so, because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$

Transitive Relation : Let A be any set. A relation R on A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. $a\,R\,b$ and $b\,R\,c \Rightarrow a\,R\,c$ for all $a, b, c \in A$

e,g. On the set N of natural numbers, the relation R defined by $x\ R\ y \Rightarrow x$ is less than y is transitive, because for any $x, y, z \in N$

$x < y$ and $y < z \Rightarrow x < z \Rightarrow x\ R\ y$ and $y\ R\ z \Rightarrow x\ R\ z$

e.g. Let L be the set of all straight lines in a plane. Then the realtion 'is parallel to' on L is a transitive relation, because from any $\ell_1, \ell_2, \ell_3 \in L$.

$$\ell_1 \parallel \ell_2 \text{ and } \ell_2 \parallel \ell_3 \Rightarrow \ell_1 \parallel \ell_3$$

Antisymmetric Relation : Let A be any set. A relation R on set A is said to be an antisymmetric relation iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

e.g. Let R be a relation on the set N of natural numbers defined by

$$x\ R\ y \Leftrightarrow \text{'x divides y' for all } x, y \in N$$

This relation is an antisymmetric relation on N. Since for any two numbers $a, b \in N$

$$a|b \text{ and } b|a \Rightarrow a = b \quad \text{i.e. } a\ R\ b \text{ and } b\ R\ a \Rightarrow a = b$$

Equivalence Relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

e.g. Let R be a relation on the set of all lines in a plane defined by $(\ell_1, \ell_2) \in R \Leftrightarrow$ line ℓ_1 is parallel to line ℓ_2. R is an equivalence relation.

Note : It is not neccessary that every relation which is symmetric and transitive is also reflexive.

PARTIAL ORDER RELATION :

A relation R on set A is said to be an partial order relation on A if

(i) R is reflexive i.e. $(a, a) \in R, \forall a \in A$

(ii) R is antisymmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ only Possible When $a = b \ \forall a, b \in A$

(iii) R is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R \ \forall a, b, c \in R$

e.g. R be a relation on the set N of natural numbers defined by

$x\ R\ y \Rightarrow$ 'x divides y' $\forall x, y \in N$ then R is a partial order Relation.

Illustration 2 :

Three relation R_1, R_2 and R_3 are defined on set $A = \{a, b, c\}$ as follows :

(i) R_1 $\{(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ (ii) R_2 $\{(a, b), (b, a), (a, c), (c, a)\}$

(iii) $R_3\{(a, b), (b, c), (c, a)\}$

Find whether each of R_1, R_2 and R_3 is reflexive, symmetric and transitive.

Solution :

(i) Reflexive : Clearly, $(a, a), (b, b), (c, c) \in R_1$. So, R_1 is reflexive on A.

Symmetric : We observe that $(a, b) \in R_1$ but $(b, a) \notin R_1$. So, R_1 is not symmetric on A.

Transitive : We find that $(b, c) \in R_1$ and $(c, a) \in R_1$ but $(b, a) \notin R_1$. So, R is not transitive on A.

(ii) Reflexive : Since (a, a), (b, b) and (c, c) are not in R_2. So, it is not a reflexive realtion on A.

Symmetric : We find that the ordered pairs obtained by interchanging the components of ordered pairs in R_2 are also in R_2. So, R_2 is a symmetric relation on A.

Transitive : Clearly $(c, a) \in R_2$ and $(a, b) \in R_2$ but $(c, b) \notin R_2$. So, it is not a transitive relation on R_2.

(iii) Reflexive : Since non of (a, a), (b, b) and (c, c) is an element of R_3. So, R_3 is not reflexive on A.

Symmetric : Clearly, $(b, c) \in R_3$ but $(c, b) \notin R_3$. so, is not symmetric on A.

Transitive : Clearly, $(b, c) \in R_3$ and $(c, a) \in R_3$ but $(b, a) \notin R_3$. So, R_3 is not transitive on A.

Illustration 3 :

Prove that the relation R on the set Z of all integers defined by

$$(x, y) \in R \Leftrightarrow x - y \text{ is divisible by } n$$

is an equivalence relation on Z.

Solution :

We observe the following properties

Reflexivity : For any $a \in N$, we have

$$a - a = 0 = 0 \quad n \Rightarrow a - a \text{ is divisible by } n \Rightarrow (a, a) \in R$$

Thus, $(a, a) \in R$ for all $a \in Z$

So, R is reflexive on Z

symmetry : Let $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow (a - b)$ is divisible by n

$\Rightarrow$ $a - b = np$ for some $p \in Z$

$\Rightarrow$ $b - a = n(-p)$

$\Rightarrow$ $b - a$ is divisible by n $\quad [\because p \in Z \Rightarrow -p \in Z]$

$\Rightarrow$ $(b, a) \in R$

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all a, b, $\in Z$

So, R is symmetric on Z.

Transitivity : Let $a, b, c \in Z$ such that $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R \Rightarrow (a - b)$ is divisible by n

$\Rightarrow$ $a - b = np$ for some $p \in Z$

$(b, c) \in R \Rightarrow (b - c)$ is divisible by n

$\Rightarrow$ $b - c = nq$ for some $q \in Z$

$\therefore$ $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a - b = np$ and $b - c - nq$

$\Rightarrow$ $(a - b) + (b - c) = np + nq$

$\Rightarrow$ $a - c = n(p + q)$

$\Rightarrow$ $a - c$ is divisible by n $\quad [\because p, q \in Z \Rightarrow p + q = Z]$

$\Rightarrow$ $(a, c) \in R$

thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in Z$. so, R is transitive realtion in Z.

Illustration 4 :

Show that the relation is congruent to' on the set of all triangles in a plane is an equivalence relation.

Solution :

Let S be the set of all triangles in a plane and let R be the relation on S defined by $(\Delta_1, \Delta_2) \in R \Leftrightarrow$ triangle Δ_1 is congruent to triangle Δ_2. We observe the following properties.

Reflexivity : For each triangle $\Delta \in S$, we have

$\Delta \cong \Delta \Rightarrow (\Delta, \Delta) \in R$ for all $\Delta \in S \Rightarrow R$ is reflexive on S

Symmetry : Let $\Delta_1, \Delta_2 \in S$ such that $(\Delta_1, \Delta_2) \in R$. Then, $(\Delta_1, \Delta_2) \in R \Rightarrow \Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1 \Rightarrow (\Delta_2, \Delta_1) \in R$

So, R is symmetric on S

Transitivity : Let $\Delta_1, \Delta_2, \Delta_3 \in S$ such that $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R$. Then,

$(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow \Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3 \Rightarrow \Delta_1 \cong \Delta_3 \Rightarrow (\Delta_1, \Delta_3) \in R$

So, R is transitive on S.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on S.

Do yourself - 2 :

(i) Show that the relation R defined on the set N of natural number by $xRy \Leftrightarrow 2x^2 - 3xy + y^2 = 0$, i.e. by $R = \{(x, y);\ x, y \in N \text{ and } 2x^2 - 3xy + y^2 = 0\}$ is not symmetric but it is reflexive.

ANSWERS FOR DO YOURSELF

1. **(i)** $\{(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54), (9, 18), (9, 27), (9, 54)\}$

(ii) Domain of $R = \{1, 2, 3\}$, Range of $R = \{7, 5\}$