CHAPTER

05

Permutations and Combinations

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- Factorial Notation

Session 2

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- Principle of Inclusion and Exclusion
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In everyday life, we need to know about the number of ways of doing certain work from given number of available options. *For example*, Three persons A, B and C are applying for a job in which only one post is vacant. Clearly, vacant post can be filled either by A or B or C i.e., total number of ways doing this work is three.

Again, let two persons A and B are to be seated in a row, then only two possible ways of arrangement is AB or BA. In two arrangements, persons are same but their order is different. Thus, in arranging things, order of things is important.

Session 1

Fundamental Principle of Counting, Factorial Notation

Fundamental Principle of Counting

(i) Multiplication Principle

If an operation can be performed in 'm' different ways, following which a second operation can be performed in 'n' different ways, then the two operations in succession can be performed in $m \times n$ ways. This can be extended to any finite number of operations.

Note For AND → '×' (multiply)

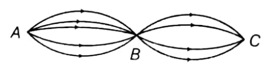
Example 1. A hall has 12 gates. In how many ways can a man enter the hall through one gate and come out through a different gate?

Sol. Since, there are 12 ways of entering into the hall. After entering into the hall, the man come out through a different gate in 11 ways.

Hence, by the fundamental principle of multiplication, total number of ways is $12 \times 11 = 132$ ways.

Example 2. There are three stations A, B and C, five routes for going from station A to station B and four routes for going from station B to station C. Find the number of different ways through which a person can go from A to C via B.

Sol. Since, there are five routes for going from A to B. So, there are four routes for going from B to C.



Hence, by the fundamental principle of multiplication, total number of different ways

$= 5 \times 4$ [i.e., A to B and then B to C]

$= 20$ ways

(ii) Addition Principle

If an operation can be performed in 'm' different ways and another operation, which is independent of the first operation, can be performed in 'n' different ways. Then, either of the two operations can be performed in $(m+n)$ ways. This can be extended to any finite number of mutually exclusive operation.

Note For OR → '+' (Addition)

Example 3. There are 25 students in a class in which 15 boys and 10 girls. The class teacher select either a boy or a girl for monitor of the class. In how many ways the class teacher can make this selection?

Sol. Since, there are 15 ways to select a boy, so there are 10 ways to select a girl.

Hence, by the fundamental principle of addition, either a boy or a girl can be performed in $15 + 10 = 25$ ways.

Example 4. There are 4 students for Physics, 6 students for Chemistry and 7 students for Mathematics gold medal. In how many ways one of these gold medals be awarded?

Sol. The Physics, Chemistry and Mathematics student's gold medal can be awarded in 4, 6 and 7 ways, respectively.

Hence, by the fundamental principle of addition, number ways of awarding one of the three gold medals.

$= 4 + 6 + 7 = 17$ ways.

Factorial Notation

Let n be a positive integer. Then, the continued product of first 'n' natural numbers is called factorial n, to be denoted by $n!$ or $\underline{n}$ i.e., $n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$

Note When n is negative or a fraction, $n!$ is not defined.

Some Important Properties

(i) $n! = n(n-1)! = n(n-1)(n-2)!$

(ii) $0! = 1! = 1$

(iii) $(2n)! = 2^n n![1\cdot 3\cdot 5\ldots(2n-1)]$

(iv) $\dfrac{n!}{r!} = n(n-1)(n-2)\ldots(r+1)$ $[r < n]$

(v) $\dfrac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-r+1)$ $[r < n]$

(vi) $\dfrac{1}{n!} + \dfrac{1}{(n+1)!} = \dfrac{\lambda}{(n+2)!}$, then $\lambda = (n+2)^2$

(vii) If $x! = y! \Rightarrow x = y$ or $x = 0, y = 1$ or $x = 1, y = 0$

Example 5. Find n, if $(n+2)! = 60 \times (n-1)!$.

Sol. $\because (n+2)! = (n+2)(n+1)n(n-1)!$

$\Rightarrow \dfrac{(n+2)!}{(n-1)!} = (n+2)(n+1)n$

$\Rightarrow 60 = (n+2)(n+1)n$ [given]

$\Rightarrow 5 \times 4 \times 3 = (n+2) \times (n+1) \times n$

$\therefore n = 3$

Example 6. Evaluate $\sum_{r=1}^{n} r \times r!$.

Sol. We have, $\sum_{r=1}^{n} r \times r! = \sum_{r=1}^{n}\{(r+1)-1\}r! = \sum_{r=1}^{n}(r+1)! - r!$

$= (n+1)! - 1!$ [put $r = n$ in $(r+1)!$ and $r = 1$ is $r!$]

$= (n+1)! - 1$

Example 7. Find the remainder when $\sum_{r=1}^{n} r!$ is divided by 15, if $n \geq 5$.

Sol. Let $N = \sum_{r=1}^{n} r! = 1! + 2! + 3! + 4! + 5! + 6! + 7! + \ldots + n!$

$= (1! + 2! + 3! + 4!) + (5! + 6! + 7! + \ldots + n!)$

$= 33 + (5! + 6! + 7! + \ldots + n!)$

$\Rightarrow \dfrac{N}{15} = \dfrac{33}{15} + \dfrac{(5! + 6! + 7! + \ldots + n!)}{15}$

$= 2 + \dfrac{3}{15} + \text{Integer}$ [as $5!, 6!, \ldots$ are divisible by 15]

$= \dfrac{3}{15} + \text{Integer}$

Hence, remainder is 3.

Exponent of prime p in $n!$

Exponent of prime p in $n!$ is denoted by $E_p(n!)$, where p is prime number and n is a natural number. The last integer amongst $1, 2, 3, \ldots, (n-1), n$ which is divisible by p is $\left[\dfrac{n}{p}\right]p$,

where $[\cdot]$ denotes the greatest integer function.

$\therefore E_p(n!) = E_p(1\cdot 2\cdot 3\ldots(n-1)\cdot n)$

$= E_p\left(p\cdot 2p\cdot 3p\ldots(n-1)p\cdot\left[\dfrac{n}{p}\right]p\right)$

[because the remaining natural numbers from 1 to n are not divisible by p]

$= \left[\dfrac{n}{p}\right] + E_p\left(1\cdot 2\cdot 3\ldots\left[\dfrac{n}{p}\right]\right)$ ...(i)

Now, the last integer amongs $1, 2, 3, \ldots, \left[\dfrac{n}{p}\right]$ which is divisible by p is $\left[\dfrac{\left[\frac{n}{p}\right]}{p}\right] = \left[\dfrac{n}{p^2}\right]$. Now, from Eq. (i), we get

$E_p(n!) = \left[\dfrac{n}{p}\right] + E_p\left(p, 2p, 3p, \ldots, \left[\dfrac{n}{p^2}\right]p\right)$

[because the remaining natural numbers from 1 to $\left[\dfrac{n}{p}\right]$ are not divisible by p]

$\therefore E_p(n!) = \left[\dfrac{n}{p}\right] + \left[\dfrac{n}{p^2}\right] + E_p\left(1\cdot 2\cdot 3\ldots\left[\dfrac{n}{p^2}\right]\right)$

Similarly, we get

$E_p(n!) = \left[\dfrac{n}{p}\right] + \left[\dfrac{n}{p^2}\right] + \left[\dfrac{n}{p^3}\right] + \ldots + \left[\dfrac{n}{p^s}\right]$

where, s is the largest natural number such that

$p^s \leq n < p^{s+1}$

Note Number of zeroes at the end of $n! = E_5(n!)$.

Example 8. Find the exponent of 3 in 100!.

Sol. In terms of prime factors 100! can be written as $2^a\cdot 3^b\cdot 5^c\cdot 7^d\ldots$

Now, $b = E_3(100!)$

$= \left[\dfrac{100}{3}\right] + \left[\dfrac{100}{3^2}\right] + \left[\dfrac{100}{3^3}\right] + \left[\dfrac{100}{3^4}\right] + \ldots$

$= 33 + 11 + 3 + 1 + 0 + \ldots = 48$

Hence, exponent of 3 is 48.

Aliter

$\because \quad 100! = 1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times 98 \times 99 \times 100$

$$= (1 \times 2 \times 4 \times 5 \times 7 \times \ldots \times 98 \times 100)(3 \times 6 \times 9 \times \ldots \times 96 \times 99)$$

$$= k \times 3^{33}(1 \times 2 \times 3 \times \ldots \times 32 \times 33)$$

$$= [k(1 \times 2 \times 4 \times 5 \times \ldots \times 31 \times 32)](3 \times 6 \times 9 \times \ldots \times 30 \times 33)$$

$$= 3^{33} k_1 \times 3^{11}(1 \times 2 \times 3 \times \ldots \times 10 \times 11)$$

$$= 3^{44}\,[k_1 (1 \times 2 \times 4 \times 5 \times \ldots \times 10 \times 11)](3 \times 6 \times 9)$$

$$= k_2 \times 3^{44} \times 3^4 \times 2 = k_3 \times 3^{48}$$

Hence, exponent of 3 in 100! is 48.

| Example 9. Prove that 33! is divisible by 2^{19} and what is the largest integer n such that 33! is divisible by 2^n?

Sol. In terms of prime factors, 33! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d \ldots$

Now, $E_2(33!) = \left[\frac{33}{2}\right] + \left[\frac{33}{2^2}\right] + \left[\frac{33}{2^3}\right] + \left[\frac{33}{2^4}\right] + \left[\frac{33}{2^5}\right] + \ldots$

$$= 16 + 8 + 4 + 2 + 1 + 0 + \ldots$$

$$= 31$$

Hence, the exponent of 2 in 33! is 31. Now, 33! is divisible by 2^{31} which is also divisible by 2^{19}.

$\therefore$ Largest value of n is 31.

| Example 10. Find the number of zeroes at the end of 100!.

Sol. In terms of prime factors, 100! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d \ldots$

Now, $E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$

$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and $E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right]$

$$= 20 + 4 = 24$$

$\therefore \quad 100! = 2^{97} \cdot 3^b \cdot 5^{24} \cdot 7^d \ldots = 2^{73} \cdot 3^b \cdot (2 \times 5)^{24} \cdot 7^d \ldots$

$$= 2^{73} \cdot 3^b \cdot (10)^{24} \cdot 7^d \ldots$$

Hence, number of zeroes at the end of 100! is 24.

or Exponent of 10 in 100! = min (97, 24) = 24.

Aliter

Number of zeroes at the end of 100!

$$= E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \ldots$$

$$= 20 + 4 + 0 + \ldots = 24$$

| Example 11. For how many positive integral values of n does $n!$ end with precisely 25 zeroes?

Sol. $\because$ Number of zeroes at the end of $n! = 25$ [given]

$\Rightarrow \quad E_5(x!) = 25$

$\Rightarrow \quad \left[\frac{n}{5}\right] + \left[\frac{n}{25}\right] + \left[\frac{n}{125}\right] + \ldots = 25$

It's easy to see that $n = 105$ is the smallest satisfactory value of n. The next four values of n will also work (i.e., n = 106, 107, 108, 109). Hence, the answer is 5.

| Example 12. Find the exponent of 80 in 180!.

Sol. $\because \quad 80 = 2^4 \times 5$

$\therefore \quad E_2(180!) = \left[\frac{180}{2}\right] + \left[\frac{180}{2^2}\right] + \left[\frac{180}{2^3}\right] + \left[\frac{180}{2^4}\right] + \left[\frac{180}{2^5}\right] + \left[\frac{180}{2^6}\right] + \left[\frac{180}{2^7}\right] + \ldots$

$$= 90 + 45 + 22 + 11 + 5 + 2 + 1 = 176$$

and $E_5(180!) = \left[\frac{180}{5}\right] + \left[\frac{180}{5^2}\right] + \left[\frac{180}{5^3}\right] + \ldots$

$$= 36 + 7 + 1 + 0 + \ldots$$

$$= 44$$

Now, exponent of 16 in 180! is $\left[\frac{176}{4}\right] = 44$, where $[\cdot]$ denotes the greatest integer function. Hence, the exponent of 80 in 180! is 44.

Exercise for Session 1

1. There are three routes: air, rail and road for going from Chennai to Hyderabad. But from Hyderabad to Vikarabad, there are two routes, rail and road. The number of routes from Chennai to Vikarabad via Hyderabad is

(a) 4 (b) 5 (c) 6 (d) 7

2. There are 6 books on Mathematics, 4 books on Physics and 5 books on Chemistry in a book shop. The number of ways can a student purchase either a book on Mathematics or a book on Chemistry, is

(a) 10 (b) 11 (c) 9 (d) 15

3. If a, b and c are three consecutive positive integers such that $a < b < c$ and $\frac{1}{a!} + \frac{1}{b!} = \frac{\lambda}{c!}$, the value of $\sqrt{\lambda}$ is

(a) a (b) b (c) c (d) $a + b + c$

4. If $n!$, $3 \times n!$ and $(n+1)!$ are in GP, then $n!$, $5 \times n!$ and $(n+1)!$ are in

(a) AP (b) GP (c) HP (d) AGP

5. Sum of the series $\sum_{r=1}^{n} (r^2 + 1)\, r!$ is

(a) $(n+1)!$ (b) $(n+2)! - 1$ (c) $n \cdot (n+1)!$ (d) $n \cdot (n+2)!$

6. If $15! = 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma} \cdot 7^{\delta} \cdot 11^{\theta} \cdot 13^{\phi}$, the value of $\alpha - \beta + \gamma - \delta + \theta - \phi$ is

(a) 4 (b) 6 (c) 8 (d) 10

7. The number of naughts standing at the end of 125! is

(a) 29 (b) 30 (c) 31 (d) 32

8. The exponent of 12 in 100! is

(a) 24 (b) 25 (c) 47 (d) 48

9. The number 24! is divisible by

(a) 6^{24} (b) 24^6 (c) 12^{12} (d) 48^5

10. The last non-zero digit in 20! is

(a) 2 (b) 4 (c) 6 (d) 8

11. The number of prime numbers among the numbers $105! + 2, 105! + 3, 105! + 4, \ldots, 105! + 104$ and $105! + 105$ is

(a) 31 (b) 32 (c) 33 (d) None of these

Session 2

Divisibility Test, Principle of Inclusion and Exclusion, Permutation

Divisibility Test

In decimal system all numbers are formed by the digits 0, 1, 2, 3, ..., 9. If $a\,b\,c\,d\,e$ is a five-digit number in decimal system, then we can write that.

$$\boxed{a\,b\,c\,d\,e = 10^4 \cdot a + 10^3 \cdot b + 10^2 \cdot c + 10 \cdot d + e.}$$

Number $a\,b\,c\,d\,e$ will be divisible

(1) **by 2,** if e is divisible by 2.

(2) **by 4,** if $2d + e$ is divisible by 4.

(3) **by 8,** if $4c + 2d + e$ is divisible by 8.

(4) **by 2^t**, if number formed by last t digits is divisible by 2^t.

For example, Number 820101280 is divisible by 2^5 because 01280 is divisible by 2^5.

(5) **by 5,** if $e = 0$ or 5.

(6) **by 5^t**, if number formed by last t digits is divisible by 5^t.

For example, Number 1128375 is divisible by 5^3 because 375 is divisible by 5^3.

(7) **by 3,** if $a + b + c + d + e$(sum of digits) is divisible by 3.

(8) **by 9,** if $a + b + c + d + e$ is divisible by 9.

(9) **by 6,** if $e =$ even and $a + b + c + d + e$ is divisible by 3.

(10) **by 18,** if $e =$ even and $a + b + c + d + e$ is divisible by 9.

(11) **by 7,** if $abcd - 2e$ is divisible by 7.

For example, Number 6552 is divisible by 7 because $655 - 2 \times 2 = 651 = 93 \times 7$ is divisible by 7.

(12) **by 11,** if $\underbrace{a + c + e}_{\text{Sum of digits at odd places}} - \underbrace{b + d}_{\text{Sum of digits at even places}}$ is divisible by 11.

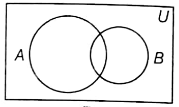
For example, Number 15222163 is divisible by 11 because
$(1 + 2 + 2 + 6) - (5 + 2 + 1 + 3) = 0$ is divisible by 11.

(13) **by 13,** if $abcd + 4e$ is divisible by 13.

For example, Number 1638 is divisible by 13 because $163 + 4 \times 8 = 195 = 15 \times 13$ is divisible by 13.

Principle of Inclusion and Exclusion

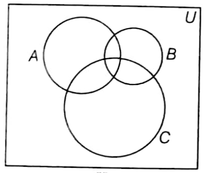
1. If A and B are finite sets, from the Venn diagram (i), it is clear that



(i)

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
and $n(A' \cap B') = n(U) - n(A \cup B)$

2. If A, B and C are three finite sets, then from the Venn diagram (ii), it is clear that



(ii)

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$
and $n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$

Note If $A_1, A_2, A_3, \ldots, A_n$ are finite sets, then
$n(A_1 \cup A_2 \cup \ldots \cup A_n) = \Sigma n(A_i) - \Sigma n(A_i \cap A_j) + \Sigma n(A_i \cap A_j \cap A_k) - \ldots + (-1)^n \Sigma n(A_1 \cap A_2 \cap \ldots \cap A_n)$
and $n(A'_1 \cap A'_2 \cap \ldots \cap A'_n) = n(U) - n(A_1 \cup A_2 \cup \ldots \cup A_n)$.

| Example 13. Find the number of positive integers from 1 to 1000, which are divisible by atleast 2, 3 or 5.

Sol. Let A_k be the set of positive integers from 1 to 1000, which is divisible by k. Obviously, we have to find $n(A_2 \cup A_3 \cup A_5)$. If $[\cdot]$ denotes the greatest integer function, then

$$n(A_2) = \left[\frac{1000}{2}\right] = [500] = 500$$
$$n(A_3) = \left[\frac{1000}{3}\right] = [333.33] = 333$$
$$n(A_5) = \left[\frac{1000}{5}\right] = [200] = 200$$
$$n(A_2 \cap A_3) = \left[\frac{1000}{6}\right] = [166.67] = 166$$
$$n(A_3 \cap A_5) = \left[\frac{1000}{15}\right] = [66.67] = 66$$
$$n(A_2 \cap A_5) = \left[\frac{1000}{10}\right] = [100] = 100$$
and $n(A_2 \cap A_3 \cap A_5) = \left[\frac{1000}{30}\right] = [33.33] = 33$

From Principle of Inclusion and Exclusion
$$n(A_2 \cup A_3 \cup A_5) = n(A_2) + n(A_3) + n(A_5) - n(A_2 \cap A_3) - n(A_3 \cap A_5) - n(A_2 \cap A_5) + n(A_2 \cap A_3 \cap A_5)$$
$$= 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734$$

Hence, the number of positive integers from 1 to 1000, which are divisible by atleast 2, 3 or 5 is 734.

Note
The number of positive integers from 1 to 1000, which are not divisible by 2, 3 or 5 is $n(A'_2 \cap A'_3 \cap A'_5)$.
$\therefore n(A'_2 \cap A'_3 \cap A'_5) = n(U) - n(A_2 \cup A_3 \cup A_5)$ [here, $n(U) = 1000$]
$= 1000 - 734 = 266$

Permutation

Each of the different arrangements which can be made by taking some or all of a number of things is called a permutation. In permutation, order of the arrangement is important.

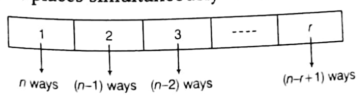
Important Results

1. **The number of permutations of n different things, taking r at a time is denoted by ^nP_r or $P(n, r)$ or $A(n, r)$, then**

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

where, $n \in N$, $r \in W$ and $0 \le r \le n$.

Proof LHS $= {}^nP_r =$ Number of ways of filling up r vacant places simultaneously from n different things



$$= n(n-1)(n-2)\ldots(n-r+1)$$
$$= \frac{n(n-1)(n-2)\ldots(n-r+1) \times (n-r)!}{(n-r)!}$$
$$= \frac{n!}{(n-r)!} = \text{RHS}$$

Note

(i) The number of permutations of n different things taken all at a time $= {}^nP_n = n!$

(ii) $^nP_0 = 1$, $^nP_1 = n$ and $\boxed{^nP_{n-1} = {}^nP_n = n!}$

(iii) $^nP_r = n({}^{n-1}P_{r-1}) = n(n-1)({}^{n-2}P_{r-2})$
$= n(n-1)(n-2)({}^{n-3}P_{r-3}) = \ldots$

(iv) $^{n-1}P_r = (n-r)\,{}^{n-1}P_{r-1}$

(v) $\dfrac{^nP_r}{^nP_{r-1}} = (n-r+1)$

| Example 14. If $^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, find $^rP_2$.

Sol. We have, $\dfrac{^{56}P_{r+6}}{^{54}P_{r+3}} = \dfrac{30800}{1}$

$\Rightarrow (56)(55)\dfrac{^{54}P_{r+4}}{^{54}P_{r+3}} = \dfrac{30800}{1}$ [from note (iii)]

$\Rightarrow \dfrac{^{54}P_{r+4}}{^{54}P_{r+3}} = 10$

$\Rightarrow 54 - (r + 4) + 1 = 10$ [from note (v)]

$r = 41$

$\therefore {}^rP_2 = {}^{41}P_2 = 41 \cdot 40 = 1640$

| Example 15. If $^{n+5}P_{n+1} = \dfrac{11(n-1)}{2} \cdot {}^{n+3}P_n$, find n.

Sol. We have, $\dfrac{^{n+5}P_{n+1}}{^{n+3}P_n} = \dfrac{11(n-1)}{2}$

$\Rightarrow \dfrac{(n+5)(n+4) \cdot {}^{n+3}P_{n-1}}{^{n+3}P_n} = \dfrac{11(n-1)}{2}$ [from note (iii)]

$\dfrac{(n+5)(n+4)}{(n+3-n+1)} = \dfrac{11(n-1)}{2}$ [from note (v)]

$\Rightarrow (n+5)(n+4) = 22(n-1)$

$\Rightarrow n^2 - 13n + 42 = 0$

$\Rightarrow (n-6)(n-7) = 0$

$\therefore n = 6, 7$

| Example 16. If $^{m+n}P_2 = 90$ and $^{m-n}P_2 = 30$, find the values of m and n.

Sol. $\because {}^{m+n}P_2 = 90 = 10 \times 9 = {}^{10}P_2$

$\therefore m + n = 10$ ...(i)

and $^{m-n}P_2 = 30 = 6 \times 5 = {}^6P_2$

$\therefore m - n = 6$ ...(ii)

From Eqs. (i) and (ii), we get

$m = 8$ and $n = 2$.

Example 17. Find the value of r, if

(i) ${}^{11}P_r = 990$

(ii) ${}^{8}P_5 + 5 \cdot {}^{8}P_4 = {}^{9}P_r$

(iii) ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$

Sol. (i) $\because {}^{11}P_r = 990 = 11 \times 10 \times 9 = {}^{11}P_3$

$\therefore \quad r = 3$

(ii) $\because \quad {}^{8}P_5 + 5 \cdot {}^{8}P_4 = {}^{9}P_r$

$\Rightarrow \quad {}^{8}P_4\left(\dfrac{{}^{8}P_5}{{}^{8}P_4} + 5\right) = {}^{9}P_r$

$\Rightarrow \quad {}^{8}P_4 (8 - 5 + 1 + 5) = {}^{9}P_r$ [from note (v)]

$\Rightarrow \quad 9 \cdot {}^{8}P_4 = {}^{9}P_r$

$\Rightarrow \quad {}^{9}P_5 = {}^{9}P_r$ [from note (iii)]

$\therefore \quad r = 5$

(iii) $\because \quad {}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$

$\Rightarrow \quad \dfrac{{}^{22}P_{r+1}}{{}^{20}P_{r+2}} = \dfrac{11}{52}$

$\Rightarrow \quad \dfrac{22 \cdot 21 \cdot {}^{20}P_{r-1}}{(19-r)\cdot(20-r)\cdot(21-r)\cdot {}^{20}P_{r-1}} = \dfrac{11}{52}$

[from note (iii) and (iv)]

$\Rightarrow \quad (21-r)\cdot(20-r)\cdot(19-r) = 52 \times 2 \times 21$

$= 14 \times 13 \times 12$

$\therefore \quad r = 7$

Example 18. Prove that ${}^{1}P_1 + 2 \cdot {}^{2}P_2 + 3 \cdot {}^{3}P_3 + \ldots + n \cdot {}^{n}P_n = {}^{n+1}P_{n+1} - 1$.

Sol. LHS $= {}^{1}P_1 + 2 \cdot {}^{2}P_2 + 3 \cdot {}^{3}P_3 + \ldots + n \cdot {}^{n}P_n$

$= \sum_{r=1}^{n} r \cdot {}^{r}P_r = \sum_{r=1}^{n} \{(r+1) - 1\} \cdot {}^{r}P_r$

$= \sum_{r=1}^{n} \{(r+1) \cdot {}^{r}P_r - {}^{r}P_r\}$

$= \sum_{r=1}^{n} ({}^{r+1}P_{r+1} - {}^{r}P_r)$ [from note (iii)]

$= {}^{n+1}P_{n+1} - {}^{1}P_1 = {}^{n+1}P_{n+1} - 1$

$=$ RHS

Example 19. Determine the number of permutations of the letters of the word 'SIMPLETON' taken all at a time.

Sol. There are 9 letters in the word 'SIMPLETON' and all the 9 letters are different. Hence, the number of permutations taking all the letters at a time

$= {}^{9}P_9 = 9! = 362880$

Note Total number of letters in English alphabet = 26

(i) Number of vowels = 5
i.e., A, E, I, O, U [strong vowels]

(ii) W and Y an half vowels. [weak vowels]

(iii) Number of consonants = 21 [except vowels]
i.e., B, C, D, F, G, ..., Y, Z

(iv) Words which contains all vowels are
EDUCATION, EQUATION, ...

(iv) Words which do not contains any vowels are
SKY, FLY, TRY, ...

Example 20. How many different signals can be given using any number of flags from 4 flags of different colours?

Sol. The signals can be made by using one or more flags at a time. Hence, by the fundamental principle of addition, the total number of signals

$= {}^{4}P_1 + {}^{4}P_2 + {}^{4}P_3 + {}^{4}P_4$

$= 4 + (4 \times 3) + (4 \times 3 \times 2) + (4 \times 3 \times 2 \times 1)$

$= 4 + 12 + 24 + 24 = 64$

Example 21. Find the total number of 9-digit numbers which have all different digits.

Sol. Number of digits are 10 (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

Total number of 9-digit numbers $= {}^{10}P_9$

Out of these, the number of numbers having zero at the first place $= {}^{9}P_8$

Hence, required number of numbers $= {}^{10}P_9 - {}^{9}P_8$

$= 10 \times {}^{9}P_8 - {}^{9}P_8 = 9 \times {}^{9}P_8$

$= 9 \times \dfrac{9!}{1!} = 9 \times 9!$

Note Total number of n digit numbers $(1 \le n \le 10)$, which have all different digits $= {}^{10}P_n - {}^{9}P_{n-1}$

Example 22. A 5-digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5 without repetition. Find total number of ways in which this can be done.

Sol. A number will be divisible by 3, if sum of the digits in number be divisible by 3.

Here, $0 + 1 + 2 + 3 + 4 + 5 = 15$, which is divisible by 3. Therefore, the digit that can be left out, while the sum still is multiple of 3, is either 0 or 3.

If 0 left out

Then, possible numbers $= {}^{5}P_5 = 5! = 120$

If 3 left out

Then, possible numbers $= {}^{5}P_5 - {}^{4}P_4 = 5! - 4! = 120 - 24 = 96$

Hence, required total numbers $= 120 + 96 = 216$

Example 23. A 5-digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. Find the number of the numbers, thus formed divisible by 4.

Sol. Let a 5-digit number be $abcde$.

It will be divisible by 4, if $2d + e$ is divisible by 4.

$\Rightarrow \quad \underbrace{2d + e}_{\text{Even}}$ is divisible by 4 $\quad \therefore e$ must be even.

$\Rightarrow \quad \underbrace{2\left(d + \dfrac{e}{2}\right)}_{\text{Should be even}}$ is divisible by 4

Then, $e = 2, d = 1, 3, 5$ and $e = 4, d = 2$ } Total four cases

$\therefore$ Required number of ways $= 4 \times \underbrace{3!}_{\text{Number of ways}} = 24$

filling abc after filling de.

Aliter A number will be divisible by 4, if the last two digits of the number is divisible by 4, then for divisible by 4, last two digits 12 or 24 or 32 or 52

3! ways — 4 ways

Hence, the number formed is divisible by $4 = 3! \times 4 = 24$.

Example 24. Find the number of permutations of letters $a\,b\,c\,d\,e\,f\,g$ taken all together if neither 'beg' nor 'cad' pattern appear.

Sol. The total number of permutations without any restriction is 7!

$n(U) = 7! = 5040$

Let $n(A)$ be the number of permutations in which 'beg' pattern always appears

$b\,e\,g\,a\,c\,d\,f$

i.e., $\quad n(A) = 5! = 120$.

and let $n(B)$ be the number of permutations in which 'cad' pattern always appears

$c\,a\,d\,b\,e\,f\,g$

i.e., $\quad n(B) = 5! = 120$

Now, $n(A \cap B)$ = Number of permutations in which 'beg' and 'cad' pattern appear

$b\,e\,g\,c\,a\,d\,f$

i.e., $\quad n(A \cap B) = 3! = 6$

Hence, the number of permutations in which 'beg' and 'cad' patterns do not appear is $n(A' \cap B')$.

or $\quad n(A' \cap B') = n(U) - n(A \cup B)$

$= n(U) - [n(A) + n(B) - n(A \cap B)]$

$= 5040 - 120 - 120 + 6 = 4806$

2. The number of permutations of n things taken all at a time, p are alike of one kind, q are alike of second kind and r are alike of a third kind and the rest $n - (p + q + r)$ are all different is $\dfrac{n!}{p!\,q!\,r!}$

Proof Let the required number of permutations be x. Since, p different things can be arranged among themselves in $p!$ ways, therefore if we replace p identical things by p different things, which are also different from the rest of things, the number of permutations will become $x \times p!$

Again, if we replace q identical things by q different things, the number of permutations will become $(x \times p!) \times q!$

Again, if we replace r identical things by r different things, the number of permutations will become $(x \times p! \times q!) \times r!$. Now, all the n things are different and therefore, number of permutations should be $n!$.

Thus, $x \times p! \times q! \times r! = n!$

$\therefore \quad x = \dfrac{n!}{p!\,q!\,r!}$

Remark

The above theorem can be extended further i.e. if there are n things taken all at a time, p_1 are alike of one kind, p_2 are alike of second kind, p_3 are alike of third kind, ..., p_r are alike of rth kind such that $p_1 + p_2 + p_3 + \ldots + p_r = n$, the number of permutations of these n things is $\dfrac{n!}{p_1!\,p_2!\,p_3! \ldots p_r!}$.

Example 25. How many words can be formed with the letters of the word 'ARIHANT' by rearranging them?

Sol. Here, total letters 7, in which 2A's but the rest are different. Hence, the number of words formed $= \dfrac{7!}{2!} = 2520$

Example 26. Find the number of permutations of the letters of the words 'DADDY DID A DEADLY DEED'.

Sol. Here, total letters 19, in which 9D's, 3A's, 2Y's, 3E's and rest occur only once.

$\therefore$ Required number of permutations $= \dfrac{19!}{9! \times 3! \times 2! \times 3!}$

Example 27. How many words can be formed with the letters of the words

(i) HIGH SCHOOL and

(ii) INTERMEDIATE by rearranging them?

Sol. (i) Here, total letters are 10, in which 3H's and 2O's, but the rest are different. Hence, the number of words formed $= \dfrac{10!}{3!\,2!}$

(ii) Here, total letters are 12, in which 2I's, 2T's and 3E's but the rest are different. Hence, the number of words formed $= \dfrac{12!}{2!\,2!\,3!}$

Note [For Remember]

High School = 10 th class = Total number of letters are 10
Intermediate = 12 th class = Total number of letters are 12

3. The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r.

Proof Since, the number of permutations of n different things taken r at a time = Number of ways in which r blank places can be filled by n different things.

Clearly, the first place can be filled in n ways. Since, each thing may be repeated, the second place can be filled in n ways. Similarly, each of the 3rd, 4th, …, rth place can be filled in n ways.

By multiplication principle, the number of permutations of n different things taken r at a time when each thing may be repeated any number of times

$$= n \times n \times n \times \ldots \times r \text{ factors}$$
$$= n^r$$

Corollary When $r = n$

i.e., the number of permutations of n different things, taken all at a time, when each thing may be repeated any number of times in each arrangements is n^n.

Example 28. A child has four pockets and three marbles. In how many ways can the child put the marbles in its pockets?

Sol. The first marble can be put into the pocket in 4 ways, so can the second. Thus, the number of ways in which the child can put the marbles $= 4 \times 4 \times 4 = 4^3 = 64$ ways

Example 29. There are m men and n monkeys $(n > m)$. If a man have any number of monkeys. In how many ways may every monkey have a master?

Sol. The first monkey can select his master by m ways and after that the second monkey can select his master again by m ways, so can the third and so on.

All monkeys can select master $= m \times m \times m \ldots$ upto n factors $= (m)^n$ ways

Example 30. How many four digit numbers can be formed by using the digits 1, 2, 3, 4, 5, 6, 7, if atleast one digit is repeated ?

Sol. The numbers that can be formed when repetition of digits is allowed are $7^4 = 2401$.

The numbers that can be formed when all the digits are distinct when repetition is not allowed are ${}^7P_4 = 840$.

Therefore, the numbers that can be formed when atleast one digit is repeated $= 7^4 - {}^7P_4$

$$= 2401 - 840 = 1561$$

Example 31. In how many ways can 4 prizes be distributed among 5 students, if no student gets all the prizes?

Sol. The number of ways in which the 4 prizes can be given away to the 5 students, if a student can get any number of prizes $= 5^4 = 625$.

Again, the number of ways in which a student gets all the 4 prizes = 5, since there are 5 students and any one of them may get all the four prizes.

Therefore, the required number of ways in which a student does not get all the prizes $= 625 - 5 = 620$.

Example 32. Find the number of n-digit numbers, which contain the digits 2 and 7, but not the digits 0, 1, 8, 9.

Sol. The total number without any restrictions containing digits 2, 3, 4, 5, 6, 7 is $n(U) = 6^n$.

The total number of numbers that contain 3, 4, 5, 6, 7 is $n(A) = 5^n$.

The total number of numbers that contain 2, 3, 4, 5, 6 is $n(B) = 5^n$.

The total number of numbers that contain 3, 4, 5, 6 is $n(A \cap B) = 4^n$.

The total number of numbers that do not contain digits 2 and 7 is $n(A \cup B)$

i.e., $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$$= 5^n + 5^n - 4^n = 2(5^n) - 4^n$$

Hence, the total number of numbers that contain 2 and 7 is $n(A' \cap B')$

$\therefore \quad n(A' \cap B') = n(U) - n(A \cup B) = 6^n - 2 \cdot (5^n) + 4^n$

Example 33. Show that the total number of permutations of n different things taken not more than r at a time, when each thing may be repeated any number of times is $\dfrac{n(n^r - 1)}{(n-1)}$.

Sol. Given, total different things $= n$

The number of permutations of n things taken one at a time $= {}^nP_1 = n$, now if we taken two at a time (repetition is allowed), then first place can be filled by n ways and second place can again be filled in n ways.

$\therefore$ The number of permutations of n things taken two at a time

$$= {}^nP_1 \times {}^nP_1 = n \times n = n^2$$

Similarly, the number of permutations of n things taken three at a time $= n^3$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

The number of permutations of n things taken r at a time $= n^r$. Hence, the total number of permutations

$$= n + n^2 + n^3 + \ldots + n^r$$
$$= \frac{n(n^r - 1)}{(n-1)} \qquad \text{[sum of } r \text{ terms of a GP]}$$

Exercise for Session 2

1. If ${}^nP_5 = 20 \cdot {}^nP_3$, then n equals
   (a) 4 (b) 8 (c) 6 (d) 7
2. If ${}^9P_5 + 5 \cdot {}^9P_4 = {}^nP_r$, then $n + r$ equals
   (a) 13 (b) 14 (c) 15 (d) 16
3. If ${}^{m+n}P_2 = 56$ and ${}^{m-n}P_3 = 24$, then $\dfrac{{}^mP_3}{{}^nP_2}$ equals
   (a) 20 (b) 40 (c) 60 (d) 80
4. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 7 : 10$, then ${}^nP_3$ equals
   (a) 60 (b) 24 (c) 120 (d) 6
5. In a train, five seats are vacant, the number of ways three passengers can sit, is
   (a) 10 (b) 20 (c) 30 (d) 60
6. If a denotes the number of permutations of $x + 2$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $(x - 11)$ things taken all at a time such that $a = 182\,bc$, the value of x is
   (a) 10 (b) 12 (c) 15 (d) 18
7. The number of nine non-zero digits such that all the digits in the first four places are less than the digit in the middle and all the digits in the last four places are greater than that in the middle, is
   (a) 48 (b) 7560 (c) 10080 (d) 576
8. Total number of words that can be formed using all letters of the word 'DIPESH' that neither begins with 'I' nor ends with 'D' is equal to
   (a) 504 (b) 480 (c) 624 (d) 696
9. The number of all five digit numbers which are divisible by 4 that can be formed from the digits 0, 1, 2, 3, 4 (without repetition), is
   (a) 36 (b) 30 (c) 34 (d) None of these
10. The number of words can be formed with the letters of the word 'MATHEMATICS' by rearranging them, is
    (a) $\dfrac{11!}{2!\,2!}$ (b) $\dfrac{11!}{2!}$ (c) $\dfrac{11!}{2!\,2!\,2!}$ (d) 11!
11. Six identical coins are arranged in a row. The number of ways in which the number of tails is equal to the number of heads, is
    (a) 9 (b) 20 (c) 40 (d) 120
12. A train time table must be compiled for various days of the week so that two trains twice a day depart for three days, one train daily for two days and three trains once a day for two days. How many different time tables can be compiled?
    (a) 140 (b) 210 (c) 133 (d) 72
13. Five persons entered the lift cabin on the ground floor of an 8 floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. The total number of ways in which each of the five persons can leave the cabin at any one of the floor, is
    (a) 5^7 (b) 7^5 (c) 35 (d) 2520
14. Four die are rolled. The number of ways in which atleast one die shows 3, is
    (a) 625 (b) 671 (c) 1256 (d) 1296
15. The number of 4-digit numbers that can be made with the digits 1, 2, 3, 4 and 5 in which atleast two digits are identical, is
    (a) $4^5 - 5!$ (b) 505 (c) 600 (d) 120
16. There are unlimited number of identical balls of three different colours. How many arrangements of atmost 7 balls in a row can be made by using them?
    (a) 2187 (b) 343 (c) 399 (d) 3279

Session 3

Number of Permutations Under Certain Conditions, Circular Permutations, Restricted Circular Permutations

Number of Permutations Under Certain Conditions

(i) Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$

Corollary Number of permutations of n different things, taken r at a time, when p particular things is to be always included in each arrangement, is

$$p!(r-(p-1))^{n-p}P_{r-p}.$$

(ii) Number of permutations of n different things, taken r at a time, when a particular thing is never taken in each arrangement, is

$${}^{n-1}P_r.$$

(iii) Number of permutations of n different things, taken all at a time, when m specified things always come together, is

$$m! \times (n-m+1)!$$

(iv) Number of permutations of n different things, taken all at a time, when m specified things never come together, is

$$n! - m! \times (n-m+1)!$$

| Example 34. How many permutations can be made out of the letters of the word 'TRIANGLE'? How many of these will begin with T and end with E?

Sol. The word 'TRIANGLE' has eight different letters, which can be arranged themselves in 8! ways.

$\therefore$ Total number of permutations $= 8! = 40320$

Again, when T is fixed at the first place and E at the last place, the remaining six can be arranged themselves in 6! ways.

$\therefore$ The number of permutations which begin with T and end with

$$E = 6! = 720.$$

| Example 35. In how many ways can the letters of the word 'INSURANCE' be arranged, so that the vowels are never separate?

Sol. The word 'INSURANCE' has nine different letters, combine the vowels into one bracket as (IUAE) and treating them as one letter we have six letters viz.

(IUAE) N S R N C and these can be arranged among themselves in $\frac{6!}{2!}$ ways and four vowels within the bracket can be arranged themselves in 4! ways.

$\therefore$ Required number of words $= \frac{6!}{2!} \times 4! = 8640$

| Example 36. How many words can be formed with the letters of the word 'PATALIPUTRA' without changing the relative positions of vowels and consonants?

Sol. The word 'PATALIPUTRA' has eleven letters, in which 2P's, 3A's, 2T's, 1L, 1U, 1R and 1I. Vowels are AAIUA

These vowels can be arranged themselves in $\frac{5!}{3!} = 20$ ways.

The consonants are PTLPTR these consonants can be arranged themselves in $\frac{6!}{2!\,2!} = 180$ ways

$\therefore$ Required number of words

$$= 20 \times 180 = 3600 \text{ ways.}$$

| Example 37. Find the number of permutations that can be had from the letters of the word 'OMEGA'

(i) O and A occuping end places.

(ii) E being always in the middle.

(iii) Vowels occuping odd places.

(iv) Vowels being never together.

Sol. There are five letters in the word 'OMEGA'.

(i) When O and A occuping end places

i.e., M E G (OA)

the first three letters (M, E, G) can be arranged themselves by $3! = 6$ ways and last two letters (O, A) can be arranged themselves by $2! = 2$ ways.

$\therefore$ Total number of such words

$$= 6 \times 2 = 12 \text{ ways.}$$

(ii) When E is the fixed in the middle, then there are four places left to be filled by four remaining letters O, M, G and A and this can be done in 4! ways.

$\therefore$ Total number of such words $= 4! = 24$ ways.

(iii) Three vowels (O, E, A) can be arranged in the odd places in 3! ways (1st, 3rd and 5th) and the two consonants (M, G) can be arranged in the even places in 2! ways (2nd and 4th)

$\therefore$ Total number of such words

$$= 3! \times 2! = 12 \text{ ways.}$$

(iv) Total number of words $= 5! = 120$

Combine the vowels into one bracket as (OEA) and treating them as one letter, we have

(OEA), M, G and these can be arranged themselves in 3! ways and three vowels with in the bracket can be arranged themselves in 3! ways.

$\therefore$ Number of ways when vowels come together

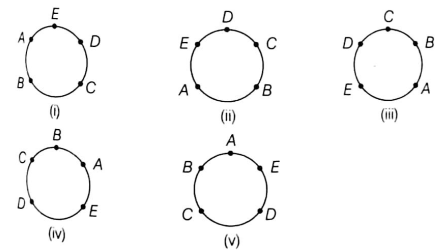
$$= 3! \times 3! = 36 \text{ ways.}$$

Hence, number of ways when vowels being never together $= 120 - 36 = 84$ ways.

Circular Permutations

(i) Arrangements round a circular table

Consider five persons A, B, C, D and E on the circumference of a circular table in order which has no head now, shifting A, B, C, D and E one position in anti-clockwise direction we will get arragements as follows



We see that, if 5 persons are sitting at a round table, they can be shifted five times and five different arrangements. Thus, obtained will be the same, because anti-clockwise order of A, B, C, D and E does not change.

But if A, B, C, D and E are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the five arrangements will be different. Thus, if there are 5 things, then for each circular arrangement number of linear arrangements is 5.

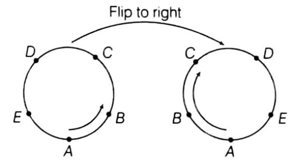
Similarly, if n different things are arranged along a circle for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things $= n \times$ number of circular arrangements of n different things

(ii) Arrangements of beads or flowers (all different) around a circular necklace or garland

Consider five beads A, B, C, D and E in a necklace or five flowers A, B, C, D and E in a garland, etc. If the necklace or garland on the left is turned over, we obtain the arrangement on the right i.e. anti-clockwise and clockwise order of arrangement is not different we will get arrangements as follows:

We see that arrangements in figures are not different.



Then, the number of circular permutations of n different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anti-clockwise orders are taken as not different.

| Example 38. Find the number of ways in which 12 different beads can be arranged to form a necklace.

Sol. 12 different beads can be arranged among themselves in a circular order in $(12-1)! = 11!$ ways. Now, in the case of necklace, there is no distinction between clockwise and anti-clockwise arrangements. So, the required number of arrangements $= \frac{1}{2}(11!)$.

| Example 39. Consider 21 different pearls on a necklace. How many ways can the pearls be placed in on this necklace such that 3 specific pearls always remain together?

Sol. After fixing the places of three pearls, treating 3 specific pearls = 1 unit. So, we have now

18 pearls + 1 unit = 19 and the number of arrangement will be $(19-1)! = 18!$

Also, the number of ways of 3 pearls can be arranged between themselves is $3! = 6$.

Since, there is no distinction between the clockwise and anti-clockwise arrangements.

So, the required number of arrangements $= \frac{1}{2} 18! \cdot 6 = 3(18!)$.

Restricted Circular Permutations

Case I If clockwise and anti-clockwise orders are taken as different, then the number of circular permutations of n different things taken r at a time.

$$\boxed{= \frac{{}^nP_r}{r} = \frac{1}{r} \cdot \frac{n!}{(n-r)!}}$$

Note For checking correctness of formula, put $r = n$, then we get $(n-1)!$ [result (5) (i)]

| Example 40. In how many ways can 24 persons be seated round a table, if there are 13 sets ?

Sol. In case of circular table, the clockwise and anti-clockwise orders are different, the required number of circular

permutations $= \frac{{}^{24}P_{13}}{13} = \frac{24!}{13 \times 11!}$.

$\Rightarrow n! = n \times$ number of circular arrangements of n different things

$\Rightarrow$ Number of circular arrangements of n different things

$$= \frac{n!}{n} = (n-1)!$$

Hence, the number of circular permutations of n different things taken all at a time is $(n-1)!$, if clockwise and anti-clockwise orders are taken as different.

| Example 41. Find the number of ways in which three Americans, two British, one Chinese, one Dutch and one Egyptian can sit on a round table so that persons of the same nationality are separated.

Sol. The total number of persons without any restrictions is

$$n(U) = (8-1)!$$
$$= 7! = 5040$$

When, three Americans (A_1, A_2, A_3) are sit together,

$$n(A) = 5! \times 3!$$
$$= 720$$

When, two British (B_1, B_2) are sit together

$$n(B) = 6! \times 2!$$
$$= 1440$$

When, three Americans (A_1, A_2, A_3) and two British (B_1, B_2) are sit together $n(A \cap B) = 4! \times 3! \times 2! = 288$

$\therefore \quad n(A \cup B) = n(A) + nB) - n(A \cap B)$

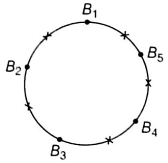
$$= 720 + 1440 - 288 = 1872$$

Hence, $\quad n(A \cap B') = n(U) - n(A \cup B)$

$$= 5040 - 1872$$
$$= 3168$$

| Example 42. In how many different ways can five boys and five girls form a circle such that the boys and girls alternate?

Sol. After fixing up one boy on the table, the remaining can be arranged in 4 ! ways but boys and girls are to alternate. There will be 5 places, one place each between two boys these five places can be filled by 5 girls in 5 ! ways.



Hence, by the principle of multiplication, the required number of ways $= 4! \times 5! = 2880$.

| Example 43. 20 persons were invited to a party. In how many ways can they and the host be seated at a circular table ? In how many of these ways will two particular persons be seated on either side of the host?

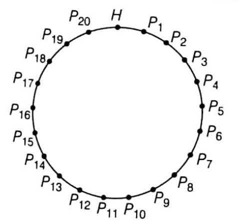
Sol. I Part Total persons on the circular table

$$= 20 \text{ guest} + 1 \text{ host} = 21$$

They can be seated in $(21-1)! = 20!$ ways.

II Part After fixing the places of three persons (1 host + 2 persons).

Treating (1 host + 2 persons) = 1 unit, so we have now {(remaining 18 persons + 1 unit) = 19} and the number of arrangement will be $(19-1)! = 18!$ also these two particular persons can be seated on either side of the host in $2!$ ways.



Hence, the number of ways of seating 21 persons on the circular table such that two particular persons be seated on either side of the host $= 18! \times 2! = 2 \times 18!$

Case II If clockwise and anti-clockwise orders are taken as not different, then the number of circular permutations of n different things taken r at a time $= \frac{{}^nP_r}{2r} = \frac{1}{2r} \cdot \frac{n!}{(n-r)!}$

Note

For checking correctness of formula put $r = n$, then we get $\frac{(n-1)!}{2}$ [result (5) (ii)]

| Example 44. How many necklace of 12 beads each can be made from 18 beads of various colours?

Sol. In the case of necklace, there is no distinction between the clockwise and anti-clockwise arrangements, the required number of circular permutations.

$$= \frac{{}^{18}P_{12}}{2 \times 12} = \frac{18!}{6! \times 24} = \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13!}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 24} = \frac{119 \times 13!}{2}$$

Exercise for Session 3

1. How many words can be formed from the letters of the word 'COURTESY' whose first letter is C and the last letter is Y ?
(a) 6! (b) 8! (c) 2(6)! (d) 2(7)!

2. The number of words that can be made by writing down the letters of the word 'CALCULATE' such that each word starts and ends with a consonant, is
(a) $\frac{3}{2}(7)!$ (b) 2(7)! (c) $\frac{5}{2}(7)!$ (d) 3(7)!

3. The number of words can be formed from the letters of the word 'MAXIMUM', if two consonants cannot occur together, is
(a) 4! (b) 3!× 4! (c) 3! (d) $\frac{4!}{3!}$

4. All the letters of the word 'EAMCET' are arranged in all possible ways. The number of such arrangements in which two vowels are not adjacent to each other, is
(a) 54 (b) 72 (c) 114 (d) 360

5. How many words can be made from the letters of the word 'DELHI', if L comes in the middle in every word?
(a) 6 (b) 12 (c) 24 (d) 60

6. In how many ways can 5 boys and 3 girls sit in a row so that no two girls are sit together?
(a) 5!× 3! (b) ${}^4P_3 \times 5!$ (c) ${}^6P_3 \times 5!$ (d) ${}^5P_3 \times 3!$

7. There are n numbered seats around a round table. Total number of ways in which n_1 $(n_1 < n)$ persons can sit around the round table, is equal to
(a) ${}^nC_{n_1}$ (b) ${}^nP_{n_1} \times \frac{1}{n_1}$ (c) ${}^nC_{n_1-1}$ (d) ${}^nP_{n_1-1}$

8. In how many ways can 7 men and 7 women can be seated around a round table such that no two women can sit together?
(a) 7! (b) 7!× 6! (c) $(6!)^2$ (d) $(7!)^2$

9. The number of ways that 8 beads of different colours be string as a necklace, is
(a) 2520 (b) 2880 (c) 4320 (d) 5040

10. If 11 members of a committee sit at a round table so that the President and secretary always sit together, then the number of arrangements, is
(a) 9! × 2 (b) 10! (c) 10!× 2 (d) 11!

11. In how many ways can 15 members of a council sit along a circular table, when the secretary is to sit on one side of the Chairman and the deputy secretary on the other side?
(a) 12!× 2 (b) 24 (c) 15!× 2 (d) 30

Session 4

Combination, Restricted Combinations

Combination

Each of the different groups or selections which can be made by some or all of a number of given things without reference to the order of the things in each group is called a combination.

Important Result

(1) The number of combinations of n different things taken r at a time is denoted by ^nC_r or $C(n, r)$ or $\binom{n}{r}$

Then,

$$^nC_r = \frac{n!}{r!(n-r)!} \quad [0 \le r \le n]$$

$$= \frac{^nP_r}{r!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots 2\cdot 1}, n \in N \text{ and } r \in W$$

Proof Let the number of combinations of n different things taken r at a time be ^nC_r.

Now, each combination consists of r different things and these r things can be arranged among themselves in $r!$ ways.

Thus, for one combination of r different things, the number of arrangements is $r!$.

Hence, for ^nC_r combinations, number of arrangements is

$$r! \times {^nC_r} \quad \ldots(i)$$

But number of permutations of n different things taken r at a time is ^nP_r. ...(ii)

From Eqs. (i) and (ii), we get

$$r! \times {^nC_r} = {^nP_r} = \frac{n!}{(n-r)!}$$

$$\therefore \quad ^nC_r = \frac{n!}{r!(n-r)!}, r \in W \text{ and } n \in N$$

Note the following facts:

(i) ^nC_r is a natural number

(ii) $^nC_r = 0$, if $r > n$

(iii) $^nC_0 = {^nC_n} = 1$, $^nC_1 = n$

(iv) $^nP_r = {^nC_r}$, if $r = 0$ or 1

(v) $^nC_r = {^nC_{n-r}}$, if $r > \frac{n}{2}$

(vi) If $^nC_x = {^nC_y} \Rightarrow x = y$ or $x + y = n$

(vii) $\boxed{^nC_r + {^nC_{r-1}} = {^{n+1}C_r}}$ [Pascal's rule]

(viii) $^nC_r = \frac{n}{r} \cdot {^{n-1}C_{r-1}}$

(ix) $n \cdot {^{n-1}C_{r-1}} = (n - r + 1) \cdot {^nC_{r-1}}$

(x) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(xi) (a) If n is even, ^nC_r is greatest for $\boxed{r = \frac{n}{2}}$

(b) If n is odd, ^nC_r is greatest for $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$

(xii) $^nC_0 + {^nC_1} + {^nC_2} + \ldots + {^nC_n} = 2^n$

(xiii) $^nC_0 + {^nC_2} + {^nC_4} + \ldots$
$$= {^nC_1} + {^nC_3} + {^nC_5} + \ldots = 2^{n-1}$$

(xiv) $^{2n+1}C_0 + {^{2n+1}C_1} + {^{2n+1}C_2} + \ldots + {^{2n+1}C_n} = 2^{2n}$

(xv) $^nC_n + {^{n+1}C_n} + {^{n+2}C_2} + {^{n+3}C_n} + \ldots$
$$+ {^{2n-1}C_n} = {^{2n}C_{n+1}}$$

| Example 45. If $^{15}C_{3r} = {^{15}C_{r+3}}$, find $^rC_2$.

Sol. We have, $^{15}C_{3r} = {^{15}C_{r+3}}$

$\Rightarrow \quad 3r = r + 3$

or $\quad 3r + r + 3 = 15$

$\Rightarrow \quad 2r = 3$ or $4r = 12$

$\Rightarrow \quad r = \frac{3}{2}$ or $r = 3$

but $r \in W$, so that $r \ne \frac{3}{2}$

$\therefore \quad r = 3$

Then, $\quad ^rC_2 = {^3C_2} = {^3C_1} = 3$

| Example 46. If $^nC_9 = {^nC_7}$, find n.

Sol. We have, $\quad ^nC_9 = {^nC_7} \Rightarrow n = 9 + 7 \quad [\because 9 \ne 7]$

$\therefore \quad n = 16$

| Example 47. Prove that

$$\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} = \binom{n+2}{r}$$

Sol. $\because \binom{n}{r} = {^nC_r}$

$$\therefore \text{LHS} = \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$$
$$= {^nC_r} + 2\,{^nC_{r-1}} + {^nC_{r-2}}$$
$$= ({^nC_r} + {^nC_{r-1}}) + ({^nC_{r-1}} + {^nC_{r-2}})$$
$$= {^{n+1}C_r} + {^{n+1}C_{r-1}} = {^{n+2}C_r}$$
$$= \binom{n+2}{r} = \text{RHS}$$

| Example 48. If $^{2n}C_3 : {^nC_3} = 11 : 1$, find the value of n.

Sol. We have,

$$^{2n}C_3 : {^nC_3} = 11 : 1$$

$$\Rightarrow \quad \frac{^{2n}C_3}{^nC_3} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{\frac{2n(2n-1)(2n-2)}{1\cdot 2\cdot 3}}{\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}} = 11 \Rightarrow \frac{4(2n-1)}{(n-2)} = 11$$

$$\Rightarrow \quad 8n - 4 = 11n - 22 \quad \Rightarrow 3n = 18$$

$$\therefore \quad n = 6$$

| Example 49. If $^{n+1}C_{r+1} : {^nC_r} : {^{n-1}C_{r-1}} = 11 : 6 : 3$, find the values of n and r.

Sol. Here, $\quad \frac{^{n+1}C_{r+1}}{^nC_r} = \frac{11}{6}$

$$\Rightarrow \quad \frac{n+1}{r+1} \cdot \frac{^nC_r}{^nC_r} = \frac{11}{6} \quad \left[\because {^nC_r} = \frac{n}{r} \cdot {^{n-1}C_{r-1}}\right]$$

$$\Rightarrow \quad \frac{n+1}{r+1} = \frac{11}{6}$$

$$\Rightarrow \quad 6n + 6 = 11r + 11$$

$$\Rightarrow \quad 6n - 11r = 5 \quad \ldots(i)$$

and $\quad \frac{^nC_r}{^{n-1}C_{r-1}} = \frac{6}{3}$

$$\Rightarrow \quad \frac{n}{r} \cdot \frac{^{n-1}C_{r-1}}{^{n-1}C_{r-1}} = \frac{6}{3} \quad \left[\because {^nC_r} = \frac{n}{r} \cdot {^{n-1}C_{r-1}}\right]$$

$$\Rightarrow \quad \frac{n}{r} = 2$$

$$\Rightarrow \quad n = 2r \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get $n = 10$ and $r = 5$.

| Example 50. If $^nC_{r-1} = 36$, $^nC_r = 84$ and $^nC_{r+1} = 126$, find r.

Sol. Here, $\quad \frac{^nC_r}{^nC_{r-1}} = \frac{84}{36}$

$$\Rightarrow \quad \frac{n-r+1}{r} = \frac{7}{3} \quad \left[\because \frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 3n - 3r + 3 = 7r$$

$$\Rightarrow \quad 10r - 3n = 3 \quad \ldots(i)$$

and $\quad \frac{^nC_{r+1}}{^nC_r} = \frac{n-(r+1)+1}{(r+1)} = \frac{126}{84} \quad \left[\because \frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}\right]$

$$\Rightarrow \quad \frac{n-r}{r+1} = \frac{3}{2}$$

$$\Rightarrow \quad 2n - 2r = 3r + 3$$

$$\Rightarrow \quad 5r - 2n = -3$$

or $\quad 10r - 4n = -6 \quad \ldots(ii)$

On subtracting Eq. (ii) from Eq. (i), we get

$$n = 9$$

From Eq. (i), we get

$$10r - 27 = 3 \Rightarrow 10r = 30$$

$$\therefore \quad r = 3$$

| Example 51. Prove that product of r consecutive positive integers is divisible by $r!$.

Sol. Let r consecutive positive integers be (m), $(m+1), (m+2), \ldots, (m+r-1)$, where $m \in N$.

$\therefore$ Product $= m(m+1)(m+2)\ldots(m+r-1)$

$$= \frac{(m-1)!\, m(m+1)(m+2)\ldots(m+r-1)}{(m-1)!}$$

$$= \frac{(m+r-1)!}{(m-1)!} = \frac{r! \cdot (m+r-1)!}{r!(m-1)!}$$

[$\because {^{m+r-1}C_r}$ is natural number]

$$= r! \cdot {^{m+r-1}C_r},$$

which is divisible by $r!$.

| Example 52. Evaluate $^{47}C_4 + \sum_{j=0}^{3} {^{50-j}C_3} + \sum_{k=0}^{5} {^{56-k}C_{53-k}}$.

Sol. We have, $^{47}C_4 + \sum_{j=0}^{3} {^{50-j}C_3} + \sum_{k=0}^{5} {^{56-k}C_{53-k}}$

$$= {}^{47}C_4 + \sum_{j=0}^{3} {}^{50-j}C_3 + \sum_{k=0}^{5} {}^{56-k}C_3 \; [\because {}^nC_r = {}^nC_{n-r}]$$

$$= {}^{47}C_4 + ({}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3)$$
$$+ ({}^{56}C_3 + {}^{55}C_3 + {}^{54}C_3 + {}^{53}C_3 + {}^{52}C_3 + {}^{51}C_3)$$

$$= {}^{47}C_4 + {}^{47}C_3 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$
$$+ {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3 + {}^{56}C_3$$

$$= ({}^{47}C_4 + {}^{47}C_3) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$
$$+ {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3 + {}^{56}C_3$$

$$= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 + \ldots + {}^{56}C_3$$

$$= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + \ldots + {}^{56}C_3$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$= {}^{56}C_4 + {}^{56}C_3 = {}^{57}C_4$$

| Example 53. Prove that the greatest value of ${}^{2n}C_r \; (0 \le r \le 2n)$ is ${}^{2n}C_n$ (for $1 \le r \le n$).

Sol. We have, $\dfrac{{}^{2n}C_r}{{}^{2n}C_{r-1}} = \dfrac{2n - r + 1}{r} \quad \left[\because \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n - r + 1}{r}\right]$

$$= \frac{2(n-r)+(r+1)}{r} = \frac{1 + 2(n-r)+1}{r} > 1$$

$$\Rightarrow \quad \frac{{}^{2n}C_r}{{}^{2n}C_{r-1}} > 1 \qquad [\text{for } 1 \le r \le n]$$

$$\therefore \quad {}^{2n}C_{r-1} < {}^{2n}C_r$$

On putting $r = 1, 2, 3, \ldots, n$,

then ${}^{2n}C_0 < {}^{2n}C_1,\; {}^{2n}C_1 < {}^{2n}C_2, \ldots, {}^{2n}C_{n-1} < {}^{2n}C_n$

On combining all inequalities, we get

$$\Rightarrow \quad {}^{2n}C_0 < {}^{2n}C_1 < {}^{2n}C_2 < {}^{2n}C_3 < \ldots < {}^{2n}C_{n-1} < {}^{2n}C_n$$

but ${}^{2n}C_r = {}^{2n}C_{2n-r}$, it follows that

$${}^{2n}C_{2n} < {}^{2n}C_{2n-1} < {}^{2n}C_{2n-2} < {}^{2n}C_{2n-3} < \ldots < {}^{2n}C_{n+1} < {}^{2n}C_n$$

Hence, the greatest value of ${}^{2n}C_r$ is ${}^{2n}C_n$.

| Example 54. Thirty six games were played in a football tournament with each team playing once against each other. How many teams were there?

Sol. Let the number of teams be n.

Then number of matches to be played is ${}^nC_2 = 36$

$$\Rightarrow \quad {}^nC_2 = \frac{9 \times 8}{1 \times 2} = {}^9C_2$$

$$\Rightarrow \quad n = 9$$

Restricted Combinations

(i) The number of selections (combinations) of r objects out of n different objects, when

(a) k particular things are always included $= {}^{n-k}C_{r-k}$.

(b) k particular things are never included $= {}^{n-k}C_r$.

(ii) The number of combinations of r things out of n different things, such that k particular things are not together in any selection $= {}^nC_r - {}^{n-k}C_{r-k}$

(iii) The number of combinations of n different objects taking r at a time when, p particular objects are always included and q particular objects are always excluded $= {}^{n-p-q}C_{r-p}$

Note

(i) The number of selections of r consecutive things out of n things in a row $= n - r + 1$.

(ii) The number of selections of r consecutive things out of n things along a circle $= \begin{cases} n, & \text{if } r < n \\ 1, & \text{if } r = n \end{cases}$

| Example 55. In how many ways can a cricket, eleven players by chosen out of a batch 15 players, if

(i) a particular is always chosen.

(ii) a particular player is never chosen?

Sol. (i) Since, particular player is always chosen. It means that $11 - 1 = 10$ players are selected out of the remaining $15 - 1 = 14$ players.

$\therefore$ Required number of ways $= {}^{14}C_{10} = {}^{14}C_4$

$$= \frac{14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4} = 1001$$

(ii) Since, particular player is never chosen. It means that 11 players are selected out of the remaining $15 - 1 = 14$ players.

$\therefore$ Required number of ways $= {}^{14}C_{11} = {}^{14}C_3$

$$= \frac{14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3} = 364$$

| Example 56. How many different selections of 6 books can be made from 11 different books, if

(i) two particular books are always selected.

(ii) two particular books are never selected?

Sol. (i) Since, two particular books are always selected. It means that $6 - 2 = 4$ books are selected out of the remaining $11 - 2 = 9$ books.

$\therefore$ Required number of ways $= {}^9C_4 = \dfrac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126$

(ii) Since, two particular books are never selected. It means that 6 books are selected out of the remaining $11 - 2 = 9$ books.

$\therefore$ Required number of ways $= {}^9C_6$

$$= {}^9C_3 = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84.$$

| Example 57. A person tries to form as many different parties as he can, out of his 20 friends. Each party should consist of the same number. How many friends should be invited at a time? In how many of these parties would the same friends be found?

Sol. Let the person invite r number of friends at a time. Then, the number of parties are ${}^{20}C_r$, which is maximum, when $r = 10$.

If a particular friend will be found in p parties, then p is the number of combinations out of 20 in which this particular friend must be included. Therefore, we have to select 9 more from 19 remaining friends.

Hence, $p = {}^{19}C_9$

(2) **The number of ways (or combinations) of n different things selecting atleast one of them is $2^n - 1$. This can also be stated as the total number of combinations of n different things.**

Proof For each things, there are two possibilities, whether it is selected or not selected.

Hence, the total number of ways is given by total possibilities of all the things which is equal to

$2 \times 2 \times 2 \times \ldots \times n$ factors $= 2^n$

But, this includes one case in which nothing is selected.

Hence, the total number of ways of selecting one or more of n different things $= 2^n - 1$

Aliter Number of ways of selecting one, two, three, ..., n things from n different things are

$${}^nC_1, {}^nC_2, {}^nC_3, \ldots, {}^nC_n, \text{ respectively.}$$

Hence, the total number of ways or selecting atleast one thing is

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$$
$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n) - {}^nC_0 = 2^n - 1$$

| Example 58. Mohan has 8 friends, in how many ways he invite one or more of them to dinner?

Sol. Mohan select one or more than one of his 8 friends. So, required number of ways

$$= {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_8$$
$$= 2^8 - 1 = 255.$$

| Example 59. A question paper consists of two sections having respectively, 3 and 5 questions. The following note is given on the paper "It is not necessary to attempt all the questions one question from each section is compulsory". In how many ways can a candidate select the questions?

Sol. Here, we have two sections A and B (say), the section A has 3 questions and section B has 5 questions and one question from each section is compulsory, according to the given direction.

$\therefore$ Number of ways selecting one or more than one question from section A is $2^3 - 1 = 7$

and number of ways selecting one or more than one question from section B is $2^5 - 1 = 31$

Hence, by the principle of multiplication, the required number of ways in which a candidate can select the questions

$$= 7 \times 31 = 217.$$

| Example 60. A student is allowed to select atleast one and atmost n books from a collection of $(2n+1)$ books. If the total number of ways in which he can select books is 63, find the value of n.

Sol. Given, student select atleast one and atmost n books from a collection of $(2n + 1)$ books. It means that he select one book or two books or three books or ... or n books. Hence, by the given hypothesis.

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots + {}^{2n+1}C_n = 63 \quad \ldots\text{(i)}$$

Also, the sum of binomial coefficients, is

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \ldots + {}^{2n+1}C_{2n+1}$$
$$= (1+1)^{2n+1} = 2^{2n+1}$$

$$\Rightarrow \quad {}^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1} \; [\because {}^nC_r = {}^nC_{n-r}]$$

$$\Rightarrow \quad 1 + 2 \times 63 + 1 = 2^{2n+1} \Rightarrow 128 = 2^{2n+1}$$

$$\Rightarrow \quad 2^7 = 2^{2n+1} \Rightarrow 7 = 2n + 1$$

$$\therefore \quad n = 3$$

| Example 61. There are three books of Physics, four of Chemistry and five of Mathematics. How many different collections can be made such that each collection consists of

(i) one book of each subject,

(ii) atleast one book of each subject,

(iii) atleast one book of Mathematics.

Sol. (i) ${}^3C_1 \times {}^4C_1 \times {}^5C_1 = 3 \times 4 \times 5 = 60$

(ii) $(2^3 - 1) \times (2^4 - 1) \times (2^5 - 1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5 - 1) \times 2^7 = 31 \times 128 = 3968$

Exercise for Session 4

1. If ${}^{43}C_{r-6} = {}^{43}C_{3r+1}$, the value of r is
(a) 6 (b) 8 (c) 10 (d) 12

2. If ${}^{18}C_{15} + 2({}^{18}C_{16}) + {}^{17}C_{16} + 1 = {}^nC_3$, the value of n is
(a) 18 (b) 20 (c) 22 (d) 24

3. If ${}^{20}C_{n+2} = {}^nC_{16}$, the value of n is
(a) 7 (b) 10 (c) 13 (d) None of these

4. If ${}^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3$ is equal to
(a) ${}^{47}C_6$ (b) ${}^{52}C_5$ (c) ${}^{52}C_4$ (d) None of these

5. If ${}^nC_3 + {}^nC_4 > {}^{n+1}C_3$ then
(a) $n > 6$ (b) $n < 6$ (c) $n > 7$ (d) $n < 7$

6. The Solution set of ${}^{10}C_{x-1} > 2 \cdot {}^{10}C_x$ is
(a) {1, 2, 3} (b) {4, 5, 6} (c) {8, 9, 10} (d) {9, 10, 11}

7. If ${}^{2n}C_2 : {}^nC_2 = 9:2$ and ${}^nC_r = 10$, then r is equal to
(a) 2 (b) 3 (c) 4 (d) 5

8. If ${}^{2n}C_3 : {}^nC_2 = 44:3$, for which of the following value of r, the value of ${}^nC_r$ will be 15.
(a) $r = 3$ (b) $r = 4$ (c) $r = 5$ (d) $r = 6$

9. If ${}^nC_r = {}^nC_{r-1}$ and ${}^nP_r = {}^nP_{r+1}$, the value of n is
(a) 2 (b) 3 (c) 4 (d) 5

10. If ${}^nP_r = 840$, ${}^nC_r = 35$, the value of n is
(a) 1 (b) 3 (c) 5 (d) 7

11. If ${}^nP_3 + {}^nC_{n-2} = 14n$, the value of n is
(a) 5 (b) 6 (c) 8 (d) 10

12. There are 12 volleyball players in all in a college, out of which a team of 9 players is to be formed. If the captain always remains the same, in how many ways can the team be formed ?
(a) 36 (b) 99 (c) 108 (d) 165

13. In how many ways a team of 11 players can be formed out of 25 players, if 6 out of them are always to be included and 5 are always to be excluded
(a) 2002 (b) 2008 (c) 2020 (d) 8002

14. A man has 10 friends. In how many ways he can invite one or more of them to a party?
(a) 10! (b) 2^{10} (c) $10! - 1$ (d) $2^{10} - 1$

15. In an examination, there are three multiple choice questions and each question has four choices. Number of ways in which a student can fail to get all answers correct, is
(a) 11 (b) 12 (c) 27 (d) 63

16. In an election, the number of candidates is 1 greater than the persons to be elected . If a voter can vote in 254 ways, the number of candidates is
(a) 6 (b) 7 (c) 8 (d) 10

17. The number of groups that can be made from 5 different green balls, 4 different blue balls and 3 different red balls, if atleast one green and one blue ball is to be included
(a) 3700 (b) 3720 (c) 4340 (d) None of these

18. A person is permitted to select atleast one and atmost n coins from a collection of $(2n + 1)$ distinct coins. If the total number of ways in which he can select coins is 255, then n equals
(a) 4 (b) 8 (c) 16 (d) 32

Session 5

Combinations from Identical Objects

Combinations from Identical Objects

(i) The number of combinations of n identical objects taking r objects $(r \le n)$ at a time = 1.

(ii) The number of combinations of zero or more objects from n identical objects $= n + 1$.

(iii) The total number of combinations of atleast one out of $a_1 + a_2 + a_3 + \ldots + a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, a_3 are alike of third kind, ..., a_n are alike of nth kind

$$= (a_1 + 1)(a_2 + 1)(a_3 + 1) \ldots (a_n + 1) - 1$$

| Example 62. How many selections of atleast one red ball from a bag containing 4 red balls and 5 black balls, balls of the same colour being identical?

Sol. Number of selections of atleast one red ball from 4 identical red balls = 4

Number of selections of any number of black balls from 5 identical black balls

$$= 5 + 1 = 6$$

∴ Required number of selections of balls

$$= 4 \times 6 = 24$$

| Example 63. There are p copies each of n different books. Find the number of ways in which a non-empty selection can be made from them.

Sol. Since, copies of the same book are identical.

∴ Number of selections of any number of copies of a book is $p + 1$. Similarly, in the case for each book.

Therefore, total number of selections is $(p + 1)^n$.

But this includes a selection, which is empty i.e., zero copy of each book. Excluding this, the required number of non-empty selections is $(p + 1)^n - 1$.

| Example 64. There are 4 oranges, 5 apples and 6 mangoes in a fruit basket and all fruits of the same kind are identical. In how many ways can a person make a selection of fruits from among the fruits in the basket?

Sol. Zero or more oranges can be selected out of 4 identical oranges = 4 + 1 = 5 ways.

Similarly, for apples number of selection = 5 + 1 = 6 ways and mangoes can be selected in 6 + 1 = 7 ways.

∴ The total number of selections when all the three kinds of fruits are selected $= 5 \times 6 \times 7 = 210$

But, in one of these selection number of each kind of fruit is zero and hence this selection must be excluded.

∴ Required number = 210 − 1 = 209

Combinations when both Identical and Distinct Objects are Present

The number of combinations (selections) of one or more objects out of $a_1 + a_2 + a_3 + \ldots + a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, a_3 are alike of third kind, ..., a_n are alike of nth kind and k are distinct.

$$= \{(a_1 + 1)(a_2 + 1)(a_3 + 1) \cdots (a_n + 1)\}$$
$$({}^kC_0 + {}^kC_1 + {}^kC_2 + \ldots + {}^kC_k) - 1$$
$$= (a_1 + 1)(a_2 + 1)(a_3 + 1) + \ldots + (a_n + 1)2^k - 1$$

| Example 65. Find the number of ways in which one or more letters can be selected from the letters

AAAAA BBBB CCC DD EFG.

Sol. Here, 5A's are alike, 4 B's are alike, 3 C's are alike, 2D's are alike and E, F, G are different.

∴ Total number of combinations

$$= (5 + 1)(4 + 1)(3 + 1)(2 + 1)2^3 - 1$$
$$= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 8 - 1$$
$$= 2879$$

[excluding the case, when no letter is selected]

Explanation Selection from (AAAAA) can be made by 6 ways such include no A, include one A, include two A, include three A, include four A, include five A. Similarly, selections from (BBBB) can be made in 5 ways, selections from (CCC) can be made in 4 ways, selections from (DD) can be made in 3 ways and from E, F, G can be made in $2 \times 2 \times 2$ ways.

Number of Divisors of N

Every natural number N can always be put in the form $N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \ldots p_k^{\alpha_k}$, where $p_1, p_2, p_3, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k \in W$.

(i) The total number of divisors of N including 1 and N
$= (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\dots(\alpha_k+1)$

(ii) The total number of divisors of N excluding 1 and N
$= (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\dots(\alpha_k+1)-2$

(iii) The total number of divisors of N excluding either 1 or $N = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\dots(\alpha_k+1)-1$

(iv) Sum of all divisors $= (p_1^0+p_1^1+p_1^2+p_1^3+\dots+p_1^{\alpha_1})$
$(p_2^0+p_2^1+p_2^2+p_2^3+\dots+p_2^{\alpha_2})\dots$
$(p_k^0+p_k^1+p_k^2+p_k^3+\dots+p_k^{\alpha_k})$

$$= \left(\frac{1-p_1^{\alpha_1+1}}{1-p_1}\right)\cdot\left(\frac{1-p_2^{\alpha_2+1}}{1-p_2}\right)\dots\left(\frac{1-p_k^{\alpha_k+1}}{1-p_k}\right)$$

(v) Sum of proper divisors (excluding 1 and the expression itself)
$=$ Sum of all divisors $-(N+1)$

(vi) The number of even divisors of N are possible only if $p_1 = 2$, otherwise there is no even divisor.
$\therefore$ Required number of **even** divisors
$= \alpha_1(\alpha_2+1)(\alpha_3+1)+\dots+(\alpha_k+1)$

(vii) The number of **odd** divisors of N

***Case* I** If $p_1 = 2$, the number of odd divisors
$= (\alpha_2+1)(\alpha_3+1)\dots(\alpha_k+1)$

***Case* II** If $p_1 \neq 2$, the number of odd divisors
$= (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\dots(\alpha_k+1)$

(viii) The number of ways in which N can be resolved as a product of two factors

$$= \begin{cases} \dfrac{1}{2}(\alpha_1+1)(\alpha_2+1)\dots(\alpha_k+1), & \text{if } N \text{ is not a perfect square} \\ \dfrac{1}{2}\{(\alpha_1+1)(\alpha_2+1)\dots(\alpha_k+1)+1\}, & \text{if } N \text{ is a perfect square} \end{cases}$$

(ix) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1}, where n is the number of different factors (or different primes) in N.

I Example 66. Find the number of proper factors of the number 38808. Also, find sum of all these divisors.

Sol. The number $38808 = 2^3 \cdot 3^2 \cdot 7^2 \cdot 11$

Hence, the total number of proper factors (excluding 1 and itself i.e., 38808)
$= (3+1)(2+1)(2+1)(1+1)-2$
$= 72-2 = 70$

And sum of all these divisors (proper)
$= (2^0+2^1+2^2+2^3)(3^0+3^1+3^2)$
$(7^0+7^1+7^2)(11^0+11^1)-1-38808$
$= (15)(13)(57)(12)-38809$
$= 133380-38809$
$= 94571$

I Example 67. Find the number of even proper divisors of the number 1008.

Sol. $\because 1008 = 2^4 \times 3^2 \times 7^1$

$\therefore$ Required number of even proper divisors
$=$ Total number of selections of atleast one 2 and any number of 3's or 7's.
$= 4\times(2+1)\times(1+1)-1 = 23$

I Example 68. Find the number of odd proper divisors of the number 35700. Also, find sum of the odd proper divisors.

Sol. $\because 35700 = 2^2 \times 3^1 \times 5^2 \times 7^1 \times 17^1$

$\therefore$ Required number of odd proper divisors
$=$ Total number of selections of zero 2 and any number of 3's or 5's or 7's or 17's
$= (1+1)(2+1)(1+1)(1+1)-1 = 23$

$\therefore$ The sum of odd proper divisors
$= (3^0+3^1)(5^0+5^1+5^2)(7^0+7^1)(17^0+7^1)-1$
$= 4\times 31\times 8\times 18-1$
$= 17856-1 = 17855$

I Example 69. If $N = 10800$, find the
(i) the number of divisors of the form $4m+2, \forall\, m \in W$.
(ii) the number of divisors which are multiple of 10.
(iii) the number of divisors which are multiple of 15.

Sol. We have, $N = 10800 = 2^4 \times 3^3 \times 5^2$

(i) $\because (4m+2) = 2(2m+1)$, in any divisor of the form $4m+2$, 2 should be exactly 1.
So, the number of divisors of the form
$(4m+2) = 1\times(3+1)\times(2+1) = 1\times 4\times 3 = 12$

(ii) $\therefore$ The required number of proper divisors
$=$ Total number of selections of atleast one 2 and one 5 from 2, 2, 2, 2, 3, 3, 3, 5, 5
$= 4\times(3+1)\times 2 = 32$

(iii) $\therefore$ The required number of proper divisors
$=$ Total number of selections of atleast one 3 and one 5 from 2, 2, 2, 2, 3, 3, 3, 5, 5
$= (4+1)\times 3\times 2 = 30$

I Example 70. Find the number of divisors of the number $N = 2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9 \cdot 9^{11}$, which are perfect square.

Sol. $\because N = 2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9 \cdot 9^{11}$
$= 2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9 \cdot 3^{22}$
$= 2^3 \cdot 3^{27} \cdot 5^7 \cdot 7^9$

For perfect square of N, each prime factor must occur even number of times.
2 can be taken in 2 ways (i.e., 2^0 or 2^2)
3 can be taken in 14 ways (i.e., 3^0 or 3^2 or 3^4 or 3^6 or 3^8 or 3^{10} or 3^{12} or 3^{14} or 3^{16} or 3^{18} or 3^{20} or 3^{22} or 3^{24} or 3^{26})
5 can taken in 4 ways (i.e., 5^0 or 5^2 or 5^4 or 5^6)
and 7 can taken in 5 ways (i.e., 7^0 or 7^2 or 7^4 or 7^6 or 7^8)
Hence, total divisors which are perfect squares
$= 2\times 14\times 4\times 5 = 560$

I Example 71. In how many ways the number 10800 can be resolved as a product of two factors?

Sol. Let $N = 10800 = 2^4 \times 3^3 \times 5^2$

Here, N is not a perfect square [$\because$ power of 3 is odd]

Hence, the number of ways $= \frac{1}{2}(4+1)(3+1)(2+1) = 30$

I Example 72. In how many ways the number 18900 can be split in two factors which are relatively prime (or coprime)?

Sol. Let $N = 18900 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7^1$

Relatively prime or coprime Two numbers not necessarily prime are said to be relatively prime or coprime, if their HCF (highest common factor) is one as 2, 3, 5, 7 are relatively prime numbers.

$\therefore$ $n = 4$ [number of different primes in N]

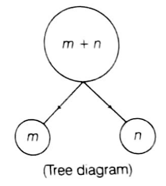
Hence, number of ways in which a composite number N can be resolved into two factors which are relatively prime or coprime $= 2^{4-1} = 2^3 = 8$

Division of Objects Into Groups

(a) Division of Objects Into Groups of Unequal Size

Theorem Number of ways in which $(m+n)$ distinct objects can be divided into two unequal groups containing m and n objects is $\dfrac{(m+n)!}{m!\,n!}$.

Proof The number of ways in which $(m+n)$ distinct objects are divided into two groups of the size m and n
$=$ The number of ways m objects are selected out of $(m+n)$ objects to form one of the groups, which can be done in ${}^{m+n}C_m$ ways. The other group of n objects is formed by the remaining n objects.

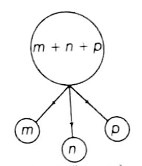


(Tree diagram)

$$= {}^{m+n}C_m \cdot {}^{n}C_n = \frac{(m+n)}{m!\,n!}$$

***Corollary* I** The number of ways to distribute $(m+n)$ distinct objects among 2 persons in the groups containing m and n objects
$=$ (Number of ways to divide) $\times$ (Number of groups)
$= \dfrac{(m+n)}{m!\,n!} \times 2!$

***Corollary* II** The number of ways in which $(m+n+p)$ distinct objects can be divided into three unequal groups containing m, n and p objects, is

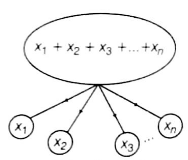


(Tree diagram)

$${}^{m+n+p}C_m \cdot {}^{n+p}C_n \cdot {}^{p}C_p = \frac{(m+n+p)!}{m!\,n!\,p!}$$

***Corollary* III** The number of ways to distribute $(m+n+p)$ distinct objects among 3 persons in the groups containing m, n and p objects
$=$ (Number of ways to divide) $\times$ (Number of groups)
$= \dfrac{(m+n+p)!}{m!\,n!\,p!} \times 3!$

***Corollary* IV** The number of ways in which $(x_1+x_2+x_3+\dots+x_n)$ distinct objects can be divided into n unequal groups containing $x_1, x_2, x_3, \dots, x_n$ objects, is



(Tree diagram)

$$\frac{(x_1 + x_2 + x_3 + \cdots + x_n)!}{x_1!\, x_2!\, x_3! \cdots x_n!}.$$

***Corollary* V** The number of ways to distribute $(x_1 + x_2 + x_3 + \ldots + x_n)$ distinct objects among n persons in the groups containing $x_1, x_2, \ldots, x_n$ objects

= (Number of ways to divide) × (Number of groups)

$$= \frac{(x_1 + x_2 + x_3 + \ldots + x_n)!}{x_1!\, x_2!\, x_3! \ldots x_n!} \times n!$$

(b) Division of Objects Into Groups of Equal Size

The number of ways in which mn distinct objects can be divided equally into m groups, each containing n objects and

(i) If order of groups is not important is.

$$= \left(\frac{(mn)!}{(n!)^m}\right) \times \frac{1}{m!}.$$

(ii) If order of groups is important is.

$$\left(\frac{(mn)!}{(n!)^m} \times \frac{1}{m!}\right) \times m! = \frac{(mn)!}{(n!)^m}.$$

Note Division of $14n$ objects into 6 groups of $2n, 2n, 2n, 2n, 3n, 3n$, size is

$$\frac{\left(\frac{(14n)!}{(2n)!\,(2n)!\,(2n)!\,(2n)!\,(3n)!\,(3n)!}\right)}{4!\,2!} = \frac{(14n)}{((2n)!)^4\,((3n)!)^2} \times \frac{1}{4!\,2!}$$

Now, the distribution ways of these 6 groups among 6 persons is

$$\frac{(14n)!}{[(2n)!]^4\,[(3n)!]^2} \times \frac{1}{4!\,2!} \times 6! = \frac{(14n)!}{[(2n)!]^4\,[(3n)!]^2} \times 15$$

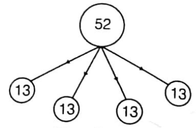
I Example 73. In how many ways can a pack of 52 cards be

(i) distributed equally among four players in order?

(ii) divided into four groups of 13 cards each?

(iii) divided into four sets, three of them having 17 cards each and fourth just one card?

Sol. (i) Here, order of group is important, then the numbers of ways in which 52 different cards can be divided equally into 4 players is



(Tree diagram)

$$\frac{52!}{4!\,(13!)^4} \times 4! = \frac{52!}{(13!)^4}$$

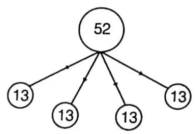
Aliter Each player will get 13 cards. Now, first player can be given 13 cards out of 52 cards in $^{52}C_{13}$ ways. Second player can be given 13 cards out of remaining 39 cards (i.e., $52 - 13 = 39$) in $^{39}C_{13}$ ways. Third player can be given 13 cards out of remaining 26 cards (i.e., $39 - 13 = 26$) in $^{26}C_{13}$ ways and fourth player can be given 13 cards out of remaining 13 cards (i.e., $26 - 13 = 13$) in $^{13}C_{13}$ ways.

Hence, required number of ways

$$= {^{52}C_{13}} \times {^{39}C_{13}} \times {^{26}C_{13}} \times {^{13}C_{13}}$$

$$= \frac{52!}{13!\,39!} \times \frac{39!}{13!\,26!} \times \frac{26!}{13!\,13!} \times 1 = \frac{52!}{(13!)^4}$$

(ii) Here, order of group is not important, then the number of ways in which 52 different cards can be divided equally into 4 groups is



(Tree diagram)

$$\frac{52!}{4!\,(13!)^4}$$

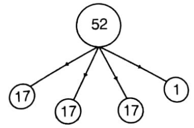
Aliter Each group will get 13 cards. Now, first group can be given 13 cards out of 52 cards in $^{52}C_{13}$ ways. Second group can be given 13 cards out of remaining 39 cards (i.e., $52 - 13 = 39$) in $^{39}C_{13}$ ways. Third group can be given 13 cards out of remaining 26 cards (i.e., $39 - 13 = 26$) in $^{26}C_{13}$ ways and fourth group can be given 13 cards out of remaining 13 cards (i.e., $26 - 13 = 13$) in $^{13}C_{13}$ ways. But the all (four) groups can be interchanged in 4! ways. Hence, the required number of ways

$$= {^{52}C_{13}} \times {^{39}C_{13}} \times {^{26}C_{13}} \times {^{13}C_{13}} \times \frac{1}{4!}$$

$$= \frac{52!}{13!\,39!} \times \frac{39!}{13!\,26!} \times \frac{26!}{13!\,13!} \times 1 \times \frac{1}{4!} = \frac{52!}{(13!)^4\,4!}$$

(iii) First, we divide 52 cards into two sets which contains 1 and 51 cards respectively, is

$$\frac{52!}{1!\,51!}$$



Now, 51 cards can be divided equally in three sets each contains 17 cards (here order of sets is not important) in $\frac{51!}{3!\,(17!)^3}$ ways.

Hence, the required number of ways

$$= \frac{52!}{1!\,51!} \times \frac{51!}{3!\,(17!)^3}$$

$$= \frac{52!}{1!\,3!\,(17)^3} = \frac{52!}{(17!)^3\,3!}$$

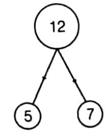
Aliter First set can be given 17 cards out of 52 cards in $^{52}C_{17}$. Second set can be given 17 cards out of remaining 35 cards (i.e., $52 - 17 = 35$) in $^{35}C_{17}$. Third set can be given 17 cards out of remaining 18 cards (i.e., $35 - 17 = 18$) in $^{18}C_{17}$ and fourth set can be given 1 card out of 1 card in $^1C_1$. But the first three sets can interchanged in 3! ways. Hence, the total number of ways for the required distribution

$$= {^{52}C_{17}} \times {^{35}C_{17}} \times {^{18}C_{17}} \times {^1C_1} \times \frac{1}{3}!$$

$$= \frac{52!}{17!\,35!} \times \frac{35!}{17!\,1!} \times \frac{18!}{17!\,18!} \times 1 \times \frac{1}{3!} = \frac{(52)!}{(17!)^3\,3!}$$

I Example 74. In how many ways can 12 different balls be divided between 2 boys, one receiving 5 and the other 7 balls? Also, in how many ways can these 12 balls be divided into groups of 5, 4 and 3 balls, respectively?

Sol. **I Part** Here, order is important, then the number of ways in which 12 different balls can be divided between two boys which contains 5 and 7 balls respectively, is



(Tree diagram)

$$= \frac{12!}{5!\,7!} \times 2! = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)7!} \cdot 2 = 1584$$

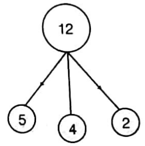
Aliter First boy can be given 5 balls out of 12 balls in $^{12}C_5$. Second boy can be given 7 balls out of 7 balls (i.e., $12 - 5 = 7$) but there order is important boys interchange by (2 types), then required number of ways

$$= {^{12}C_5} \times {^7C_7} \times 2! = \frac{12!}{5!\,7!} \times 1 \times 2!$$

$$= \frac{12! \times 2}{5! \times 7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7! \cdot 2}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 7!} = 1584.$$

II Part Here, order is not important, then the number of ways in which 12 different balls can be divided into three groups of 5,4 and 3 balls respectively, is

$$= \frac{12!}{5!\,4!\,3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 27720$$



(Tree diagram)

Aliter First group can be given 5 balls out of 12 balls in $^{12}C_5$ ways. Second group can be given 4 balls out of remaining 7 balls ($12 - 5 = 7$) in $^7C_4$ and 3 balls can be given out of remaining 3 balls in $^3C_3$.

Hence, the required number of ways (here order of groups are not important)

$$= {^{12}C_5} \times {^7C_4} \times {^3C_3}$$

$$= \frac{12!}{5!\,7!} \times \frac{7!}{4!\,3!} \times 1$$

$$= \frac{12!}{5!\,4!\,3!}$$

must

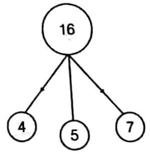
I Example 75. In how many ways can 16 different books be distributed among three students A, B, C so that B gets 1 more than A and C gets 2 more than B?

Sol. Let A gets n books, then B gets $n + 1$ and C gets $n + 3$.

Now, $n + (n + 1) + (n + 3) = 16$

$\Rightarrow \quad 3n = 12$

$\therefore \quad n = 4$



(Tree diagram)

This is condition so, we do not multiply by 3!

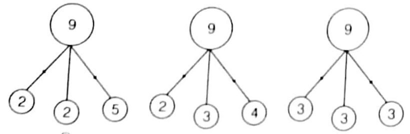
$\Rightarrow$ A, B, C gets 4, 5 and 7 books, respectively.

Hence, the total number of ways for the required distribution

$$= \frac{16!}{4!\,5!\,7!}$$

Example 76. In how many ways can 9 different books be distributed among three students if each receives atleast 2 books?

Sol. If each receives atleast 2 books, then the division as shown by tree diagrams



(i) (ii) (iii)

The number of division ways for tree diagrams (i), (ii) and (iii) are

$$\frac{9!}{(2!)^2\,(5!)} \times \frac{1}{2!}, \frac{9!}{2!\,3!\,4!} \text{ and } \frac{9!}{(3!)^3} \times \frac{1}{3!}, \text{ respectively.}$$

Hence, the total number of ways of distribution of these groups among 3 students is

$$\left[\frac{9!}{(2!)^2\,(5!)} \times \frac{1}{2!} + \frac{9!}{2!\,3!\,4!} + \frac{9!}{(3!)^3} \times \frac{1}{3!}\right] \times 3!$$

$$= [378 + 1260 + 280] \times 6$$

$$= 11508$$

c, c, d, a, b, b, d, a, c, c, d, a, b, c, d, c, b, c

Exercise for Session 5

1. There are 3 oranges, 5 apples and 6 mangoes in a fruit basket (all fruits of same kind are identical). Number of ways in which fruits can be selected from the basket, is
(a) 124 (b) 125 (c) 167 (d) 168

2. In a city no two persons have identical set of teeth and there is no person without a tooth. Also, no person has more than 32 teeth. If we disguard the shape and size of tooth and consider only the positioning of the teeth, the maximum population of the city is
(a) 2^{32} (b) $(32)^2 - 1$ (c) $2^{32} - 1$ (d) 2^{31}

3. If $a_1, a_2, a_3, \ldots, a_{n+1}$ be $(n+1)$ different prime numbers, then the number of different factors (other than 1) of $a_1^m \cdot a_2 \cdot a_3, \ldots, a_{n+1}$, is
(a) $m+1$ (b) $(m+1)2^n$ (c) $m \cdot 2^n + 1$ (d) None of these

4. Number of proper factors of 2400 is equal to
(a) 34 (b) 35
(c) 36 (d) 37

5. The sum of the divisors of $2^5 \cdot 3^4 \cdot 5^2$ is
(a) $3^2 \cdot 7^1 \cdot 11^2$ (b) $3^2 \cdot 7^1 \cdot 11^2 \cdot 31$
(c) $3 \cdot 7 \cdot 11 \cdot 31$ (d) None of these

6. The number of proper divisors of $2^p \cdot 6^q \cdot 21^r$, $\forall\, p, q, r \in N$, is
(a) $(p+q+1)(q+r+1)(r+1)$ (b) $(p+q+1)(q+r+1)(r+1)-2$
(c) $(p+q)(q+r)r-2$ (d) $(p+q)(q+r)r$

7. The number of odd proper divisors of $3^p \cdot 6^q \cdot 15^r$, $\forall\, p, q, r \in N$, is
(a) $(p+1)(q+1)(r+1)-2$ (b) $(p+1)(q+1)(r+1)-1$
(c) $(p+q+r+1)(r+1)-2$ (d) $(p+q+r+1)(r+1)-1$

8. The number of proper divisors of 1800, which are also divisible by 10, is
(a) 18 (b) 27 (c) 34 (d) 43

9. Total number of divisors of 480 that are of the form $4n+2$, $n \ge 0$, is equal to
(a) 2 (b) 3 (c) 4 (d) 5

10. Total number of divisors of $N = 2^5 \cdot 3^4 \cdot 5^{10} \cdot 7^6$ that are of the form $4n+2$, $n \ge 1$, is equal to
(a) 54 (b) 55 (c) 384 (d) 385

11. Total number of divisors of $N = 3^5 \cdot 5^7 \cdot 7^9$ that are of the form $4n+1$, $n \ge 0$ is equal to
(a) 15 (b) 30 (c) 120 (d) 240

12. Number of ways in which 12 different books can be distributed equally among 3 persons, is
(a) $\frac{12!}{(4!)^3}$ (b) $\frac{12!}{(3!)^4}$ (c) $\frac{12!}{(4!)^4}$ (d) $\frac{12!}{(3!)^3}$

13. Number of ways in which 12 different things can be distributed in 3 groups, is
(a) $\frac{12!}{(4!)^3}$ (b) $\frac{12!}{3!\,(4!)^3}$ (c) $\frac{12!}{4!\,(3!)^3}$ (d) $\frac{12!}{(3!)^4}$

14. Number of ways in which 12 different things can be distributed in 5 sets of 2, 2, 2, 3, 3, things is
(a) $\frac{12!}{(3!)^2\,(2!)^3}$ (b) $\frac{12!\,5!}{(3!)^2\,(2!)^3}$ (c) $\frac{12!}{(3!)^3\,(2!)^4}$ (d) $\frac{12!\,5!}{(3!)^2\,(2!)^4}$

15. Number of ways in which 12 different things can be divided among five persons so that they can get 2, 2, 2, 3, 3 things respectively, is
(a) $\frac{12!}{(3!)^2\,(2!)^3}$ (b) $\frac{12!\,5!}{(3!)^2\,(2!)^3}$ (c) $\frac{12!}{(3!)^2\,(2!)^4}$ (d) $\frac{12!\,5!}{(3!)^2\,(2!)^4}$

16. The total number of ways in which $2n$ persons can be divided into n couples, is
(a) $\frac{2n!}{(n!)^2}$ (b) $\frac{2n!}{(2n!)^n}$ (c) $\frac{2n!}{n!\,(2n!)^2}$ (d) None of these

17. n different toys have to be distributed among n children. Total number of ways in which these toys can be distributed so that exactly one child gets no toy, is equal to
(a) $n!$ (b) $n!\,{}^nC_2$ (c) $(n-1)!\,{}^nC_2$ (d) $n!\,{}^{n-1}C_2$

18. In how many ways can 8 different books be distributed among 3 students if each receives atleast 2 books?
(a) 490 (b) 980 (c) 2940 (d) 5880

Session 6

Arrangement in Groups, Multinomial Theorem, Multiplying Synthetically

Arrangement in Groups

(a) The number of ways in which n different things can be arranged into r different groups is $r(r+1)(r+2)\dots(r+n-1)$ or $n!\,{}^{n-1}C_{r-1}$ according as blank groups are or are not admissible.

Proof

(i) Let n letters $a_1, a_2, a_3, \dots, a_n$ be written in a row in any order. All the arrangements of the letters in r, groups, blank groups being admissible, can be obtained thus, place among the letters $(r-1)$ marks of partition and arrange the $(n+r-1)$ things (consisting of letters and marks) in all possible orders. Since, $(r-1)$ of the things are alike, the number of different arrangements is

$$\frac{(n+r-1)!}{(r-1)!} = r(r+1)(r-2)\dots(r+n-1).$$

(ii) All the arrangements of the letters in r groups, none of the groups being blank, can be obtained as follows:

(I) Arrange the letters in all possible orders. This can be done in $n!$ ways.

(II) In every such arrangement, place $(r-1)$ marks of partition in $(r-1)$ out of the $(n-1)$ spaces between the letters. This can be done in ${}^{n-1}C_{r-1}$ ways.

Hence, the required number is $n!\cdot{}^{n-1}C_{r-1}$.

| Example 77. In how many ways 5 different balls can be arranged into 3 different boxes so that no box remains empty?

Sol. The required number of ways $= 5!\cdot{}^{5-1}C_{3-1} = 5!\cdot{}^4C_2$

$$= (120)\cdot\left(\frac{4\cdot 3}{1\cdot 2}\right) = 720$$

Aliter

Each box must contain atleast one ball, since no box remains empty. Boxes can have balls in the following systems

Box	I	II	III
Number of balls	1	1	3

Or

Box	I	II	III
Number of balls	1	2	2

All 5 balls can be arranged by 5! ways and boxes can be arranged in each system by $\frac{3!}{2!}$.

Hence, required number of ways $= 5! \times \frac{3!}{2!} + 5! \times \frac{3!}{2!}$

$$= 120 \times 3 + 120 \times 3 = 720$$

(b) The number of ways in which n different things can be distributed into r different groups is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \dots + (-1)^{r-1}\cdot{}^rC_{r-1}$$

Or

$$\sum_{p=0}^{r} (-1)^p \cdot {}^rC_p \cdot (r-p)^n$$

Or

Coefficient of x^n in $n!(e^x-1)^r$.

Here, blank groups are not allowed.

Proof In any distribution, denote the groups by $g_1, g_2, g_3, \dots, g_r$ and consider the distributions in which blanks are allowed.

The total number of these is r^n.

The number in which g_1 is blank, is $(r-1)^n$.

Therefore, the number in which g_1 is not blank, is

$$r^n - (r-1)^n$$

of these last, the number in which g_2 is blank, is

$$(r-1)^n - (r-2)^n$$

Therefore, the number in which g_1, g_2 are not blank, is

$$r^n - 2(r-1)^n + (r-2)^n$$

of these last, the number in which g_3 is blank, is

$$(r-1)^n - 2(r-2)^n + (r-3)^n$$

Therefore, the number in which g_1, g_2, g_3 are not blank, is

$$r^n - 3(r-1)^n + 3(r-2)^n - (r-3)^n$$

This process can be continued as far as we like and it is obvious that the coefficients are formed as in a binomial expansion.

Hence, the number of distributions in which no one of x assigned groups is blank, is

$$r^n - {}^xC_1(r-1)^n + {}^xC_2(r-2)^n - \dots + (-1)^x(r-x)^n$$

when $x = r$, then

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \dots + (-1)^{r-1}\cdot{}^rC_{r-1}(r-(r-1))^n + (-1)^r\cdot{}^rC_r(r-r)^n$$

Or

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \dots + (-1)^{r-1}\cdot{}^rC_{r-1}$$

Aliter

By Principle of Inclusion and Exclusion

Let A_i denotes the set of distribution of things, if ith group gets nothing. Then, $n(A_i) = (r-1)^n$

[as n things can be distributed among $(r-1)$ groups in $(r-1)^n$ ways]

Then, $n(A_i \cap A_j)$ represents number of distribution ways in which groups i and j get no object. Then,

$$n(A_i \cap A_j) = (r-2)^n$$

Also, $n(A_i \cap A_j \cap A_k) = (r-3)^n$

This process can be continued, then the required number is

$$n(A_1' A_2' \cap \dots \cap A_r')$$
$$= n(U) - n(A_1 \cup A_2 \cup \dots \cup A_r)$$
$$= r^n - \left\{\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) \dots + (-1)^n \sum n(A_1 \cap A_2 \cap \dots \cap A_r)\right\}$$
$$= r^n - \{{}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \dots + (-1)^r\cdot{}^rC_{r-1}\}$$
$$= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \dots + (-1)^{r-1}\cdot{}^rC_{r-1}\cdot 1$$

Note Coefficient of x^r in $e^{px} = \frac{p^r}{r!}$.

| Example 78. In how many ways 5 different balls can be distributed into 3 boxes so that no box remains empty?

Sol. The required number of ways

$$= 3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5 - {}^3C_3(3-3)^5$$
$$= 243 - 96 + 3 - 0 = 150$$

Or

Coefficient of x^5 in $5!(e^x-1)^3$

$=$ Coefficient of x^5 in $5!(e^{3x} - 3e^{2x} + 3e^x - 1)$

$$= 5!\left(\frac{3^5}{5!} - 3 \times \frac{2^5}{5!} + 3 \times \frac{1}{5!}\right) = 3^5 - 3\cdot 2^5 + 3 = 243 - 96 + 3 = 150$$

Aliter

Each box must contain atleast one ball, since number box remains empty. Boxes can have balls in the following systems

Box	I	II	III
Number of balls	1	1	3

Or

Box	I	II	III
Number of balls	1	2	2

The number of ways to distribute the balls in I system

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_3$$

$\therefore$ The total number of ways to distribute 1, 1, 3 balls to the boxes

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!} = 5 \times 4 \times 1 \times 3 = 60$$

and the number of ways to distribute the balls in II system

$$= {}^5C_1 \times {}^4C_2 \times {}^2C_2$$

$\therefore$ The total number of ways to distribute 1, 2, 2 balls to the boxes

$$= {}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!}$$
$$= 5 \times 6 \times 1 \times 3 = 90$$

$\therefore$ The required number of ways $= 60 + 90 = 150$

| Example 79. In how many ways can 5 different books be tied up in three bundles?

Sol. The required number of ways $= \frac{1}{3!}(3^5 - {}^3C_1\cdot 2^5 + {}^3C_2\cdot 1^5)$

$$= \frac{150}{6} = 25$$

must

| Example 80. If $n(A) = 5$ and $n(B) = 3$, find number of onto mappings from A to B.

Sol. We know that in onto mapping, each image must be assigned atleast one pre-image.

This is equivalent to number of ways in which 5 different balls (pre-images) can be distributed in 3 different boxes (images), if no box remains empty. The total number of onto mappings from A to B

$$= 3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5$$
$$= 243 - 96 + 3 = 150$$

(c) The number of ways in which n identical things can be distributed into r different groups is

$${}^{n+r-1}C_{r-1} \text{ or } {}^{n-1}C_{r-1}$$

According, as blank groups are or are not admissible.

Proof

If blank groups are not allowed Any such distribution can be effected as follows: place the n things in a row and put marks of partition in a selection of $(r-1)$ out of the $(n-1)$ spaces between them. This can be done in ${}^{n-1}C_{r-1}$.

If blank groups are allowed The number of distribution is the same as that of $(n+r)$ things of the same sort into r groups with no blank groups. For such a distribution can be effected thus, put one of the

$(n+r)$ things into each of the r groups and distribute the remaining n things into r groups, blank lots being allowed. Hence, the required number is ${}^{n+r-1}C_{r-1}$.

Aliter The number of distribution of n identical things into r different groups is the coefficient of x^n in $(1+x+x^2+\ldots+\infty)^r$ or in $(x+x^2+x^3+\ldots+\infty)^r$ according as blank groups are or are not allowed.

These expressions are respectively equal to $(1-x)^{-r}$ and $x^r(1-x)^{-r}$

Hence, coefficient of x^n in two expressions are ${}^{n+r-1}C_{r-1}$ and ${}^{n-1}C_{r-1}$, respectively.

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| Example 81. In how many ways 5 identical balls can be distributed into 3 different boxes so that no box remains empty?

Sol. The required number of ways $= {}^{5-1}C_{3-1} = {}^4C_2 = \dfrac{4\cdot 3}{1\cdot 2} = 6$

Aliter Each box must contain atleast one ball, since no box remains empty. Boxes can have balls in the following systems.

Box	I	II	III
Number of balls	1	1	3

Or

Box	I	II	III
Number of balls	1	2	2

Here, balls are identical but boxes are different the number of combinations will be 1 in each systems.

$\therefore$ Required number of ways $= 1\times\dfrac{3!}{2!} + 1\times\dfrac{3!}{2!} = 3+3 = 6$

| Example 82. Four boys picked up 30 mangoes. In how many ways can they divide them, if all mangoes be identical?

Sol. Clearly, 30 mangoes can be distributed among 4 boys such that each boy can receive any number of mangoes.

Hence, total number of ways $= {}^{30+4-1}C_{4-1}$

$$= {}^{33}C_3 = \frac{33\cdot 32\cdot 31}{1\cdot 2\cdot 3} = 5456$$

| Example 83. Find the positive number of solutions of $x+y+z+w=20$ under the following conditions

(i) Zero value of x, y, z and w are included.

(ii) Zero values are excluded.

Sol. (i) Since, $x+y+z+w=20$

Here, $x\ge 0, y\ge 0, z\ge 0, w\ge 0$

The number of Sols of the given equation in this case is same as the number of ways of distributing 20 things among 4 different groups.

Hence, total number of Sols $= {}^{20+4-1}C_{4-1}$

$$= {}^{23}C_3 = \frac{23\cdot 22\cdot 21}{1\cdot 2\cdot 3} = 1771$$

(ii) Since, $x+y+z+w=20$ ...(i)

Here, $x\ge 1, y\ge 1, z\ge 1, w\ge 1$

or $x-1\ge 0, y-1\ge 0, z-1\ge 0, w-1\ge 0$

Let $x_1 = x-1 \Rightarrow x = x_1+1$

$y_1 = y-1 \Rightarrow y = y_1+1$

$z_1 = z-1 \Rightarrow z = z_1+1$

$w_1 = w-1 \Rightarrow w = w_1+1$

Then, from Eq. (i), we get

$$x_1+1+y_1+1+z_1+1+w_1+1 = 20$$

$\Rightarrow$ $x_1+y_1+z_1+w_1 = 16$

and $x_1\ge 0, y_1\ge 0, z_1\ge 0, w_1\ge 0$

Hence, total number of Solutions $= {}^{16+4-1}C_{4-1}$

$$= {}^{19}C_3 = \frac{19\cdot 18\cdot 17}{1\cdot 2\cdot 3} = 57\cdot 17 = 969$$

Aliter

Part (ii) $\because$ $x+y+z+w=20$

$x\ge 1, y\ge 1, z\ge 1, w\ge 1$

Hence, total number of solutions

$$= {}^{20-1}C_{4-1} = {}^{19}C_3 = 969$$

| Example 84. How many integral solutions are there to $x+y+z+t=29$, when $x\ge 1$, $y>1$, $z\ge 3$ and $t\ge 0$?

Sol. Since, $x+y+z+t=29$ ...(i)

and x, y, z, t are integers

$\therefore$ $x\ge 1, y\ge 2, z\ge 3, t\ge 0$

$\Rightarrow$ $x-1\ge 0, y-2\ge 0, z-3\ge 0, t\ge 0$

Let $x_1 = x-1, x_2 = y-2, x_3 = z-3$

or $x = x_1+1, y = x_2+2, z = x_3+3$ and then $x_1\ge 0, x_2\ge 0, x_3\ge 0, t\ge 0$

From Eq. (i), we get

$$x_1+1+x_2+2+x_3+3+t = 29$$

$\Rightarrow$ $x_1+x_2+x_3+t = 23$

Hence, total number of solutions $= {}^{23+4-1}C_{4-1}$

$$= {}^{26}C_3 = \frac{26\cdot 25\cdot 24}{1\cdot 2\cdot 3} = 2600$$

Aliter

$\because$ $x+y+z+t=29$ ...(i)

and $x\ge 1, y-1\ge 1, z-2\ge 1, t+1\ge 1$

Let $x_1 = x, y_1 = y-1, z_1 = z-2, t_1 = t+1$

or $x = x_1, y = y_1+1, z = z_1+2, t = t_1-1$

and then $x_1\ge 1, y_1\ge 1, z_1\ge 1, t_1\ge 1$

From Eq. (i), $x_1+y_1+1+z_1+2+t_1-1 = 29$

$\Rightarrow$ $x_1+y_1+z_1+t_1 = 27$

Hence, total number of solutions $= {}^{27-1}C_{4-1} = {}^{26}C_3$

$$= \frac{26\cdot 25\cdot 24}{1\cdot 2\cdot 3} = 2600$$

must

| Example 85. How many integral Solutions are there to the system of equations $x_1+x_2+x_3+x_4+x_5=20$ and $x_1+x_2=15$, when $x_k\ge 0$?

Sol. We have, $x_1+x_2+x_3+x_4+x_5 = 20$ ...(i)

and $x_1+x_2 = 15$ ...(ii)

Then, from Eqs. (i) and (ii), we get two equations

$x_3+x_4+x_5 = 5$ ...(iii)

$x_1+x_2 = 15$ ...(iv)

and given $x_1\ge 0, x_2\ge 0, x_3\ge 0, x_4\ge 0$ and $x_5\ge 0$

Then, number of solutions of Eq. (iii)

$$= {}^{5+3-1}C_{3-1} = {}^7C_2$$

$$= \frac{7\cdot 6}{1\cdot 2} = 21$$

and number of solutions of Eq. (iv)

$$= {}^{15+2-1}C_{2-1} = {}^{16}C_1 = 16$$

Hence, total number of solutions of the given system of equations

$$= 21\times 16 = 336$$

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| Example 86. Find the number of non-negative integral solutions of $3x+y+z=24$.

Sol. We have,

$3x+y+z=24$ and given $x\ge 0, y\ge 0, z\ge 0$

Let $x = k$

$\therefore$ $y+z = 24-3k$ ...(i)

Here, $24\ge 24-3k\ge 0$ [$\because x\ge 0$]

Hence, $0\le k\le 8$

The total number of integral solutions of Eq. (i) is

$${}^{24-3k+2-1}C_{2-1} = {}^{25-3k}C_1 = 25-3k$$

Hence, the total number of Sols of the original equation

$$= \sum_{k=0}^{8}(25-3k) = 25\sum_{k=0}^{8}1 - 3\sum_{k=0}^{8}k$$

$$= 25\cdot 9 - 3\cdot\frac{8\cdot 9}{2} = 225-108 = 117$$

(d) The number of ways in which n identical things can be distributed into r groups so that no group contains less than l things and more than m things ($l<m$) is coefficient of x^{n-lr} in the expansion of $(1-x^{m-l+1})^r(1-x)^{-r}$.

Proof Required number of ways

= Coefficient of x^n in the expansion of

$$(x^l + x^{l+1} + x^{l+2} + \ldots + x^m)^r$$

[$\because$ no group contains less than l things and more than m things, here r groups]

= Coefficient of x^n in the expansion of $x^{lr}(1+x+x^2+\ldots+x^{m-l})^r$

= Coefficient of x^{n-lr} in the expansion of $(1+x+x^2+\ldots+x^{m-l})^r$

= Coefficient of x^{n-lr} in the expansion of

$$\left(\frac{1\cdot(1-x^{m-l+1})}{(1-x)}\right)^r$$

[sum of $m-l+1$ terms of GP]

= Coefficient of x^{n-lr} in the expansion of

$$(1-x^{m-l+1})^r(1-x)^{-r}$$

| Example 87. In how many ways can three persons, each throwing a single dice once, make a sum of 15?

Sol. Number on the faces of the dice are 1, 2, 3, 4, 5, 6 (least number 1, greatest number 6)

Here, $l=1, m=6, r=3$ and $n=15$

$\therefore$ Required number of ways = Coefficient of $x^{15-1\times 3}$ in the expansion of $(1-x^6)^3(1-x)^{-3}$

= Coefficient of x^{12} in the expansion of

$$(1-3x^6+3x^{12})(1+{}^3C_1x+{}^4C_2x^2+\ldots+{}^8C_6x^6+\ldots+{}^{14}C_{12}x^{12}+\ldots)$$

$$= {}^{14}C_{12} - 3\times{}^8C_6 + 3 = {}^{14}C_2 - 3\times{}^8C_2 + 3$$

$$= 91-84+3 = 10$$

| Example 88. In how many ways in which an examiner can assign 30 marks to 8 questions, giving not less than 2 marks to any question.

Sol. If examiner given marks any seven question 2 (each) marks, then marks on remaining questions given by examiner $= -7\times 2+30 = 16$

If x_i are the marks assigned to ith question, then $x_1+x_2+x_3+\ldots+x_8 = 30$ and $2\le x_i\le 16$ for $i = 1, 2, 3, \ldots, 8$.

Here, $l=2, m=16, r=8$ and $n=30$

$\therefore$ Required number of ways

= Coefficient of $x^{30-2\times 8}$ in the expansion of

$$(1-x^{16-2+1})^8(1-x)^{-8}$$

= Coefficient of x^{14} in the expansion of

$$(1-x^{15})^8(1+{}^8C_1x+{}^9C_2x^2+\ldots+{}^{21}C_{14}x^{14}+\ldots)$$

= Coefficient of x^{14} in the expansion of

$$(1+{}^8C_1x+{}^9C_2x^2+\ldots+{}^{21}C_{14}x^{14}+\ldots)$$

$$= {}^{21}C_{14} = {}^{21}C_7$$

Note Coefficient of x^r in the expansion of $(1-x)^{-n}$ is ${}^{n+r-1}C_r$

(e) If a group has n things in which p are identical, then the number of ways of selecting r things from a group is

$$\sum_{r=0}^{r} {}^{n-p}C_r \text{ or } \sum_{r=r-p}^{r} {}^{n-p}C_r, \text{according as } r \le p \text{ or } r \ge p.$$

| Example 89. A bag has contains 23 balls in which 7 are identical. Then, find the number of ways of selecting 12 balls from bag.

Sol. Here, $n = 23$, $p = 7$, $r = 12$ $(r > p)$

$\therefore$ Required number of selections $= \sum_{r=5}^{12} {}^{16}C_r$

$= {}^{16}C_5 + {}^{16}C_6 + {}^{16}C_7 + {}^{16}C_8 + {}^{16}C_9 + {}^{16}C_{10} + {}^{16}C_{11} + {}^{16}C_{12}$

$= ({}^{16}C_5 + {}^{16}C_6) + ({}^{16}C_7 + {}^{16}C_8) + ({}^{16}C_9 + {}^{16}C_{10}) + ({}^{16}C_{11} + {}^{16}C_{12})$

$= {}^{17}C_6 + {}^{17}C_8 + {}^{17}C_{10} + {}^{17}C_{12}$ $\quad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$= {}^{17}C_{11} + {}^{17}C_9 + {}^{17}C_{10} + {}^{17}C_{12}$ $\quad [\because {}^nC_r = {}^nC_{n-r}]$

$= ({}^{17}C_{11} + {}^{17}C_{12}) + ({}^{17}C_9 + {}^{17}C_{10})$

$= {}^{18}C_{12} + {}^{18}C_{10} = {}^{18}C_6 + {}^{18}C_8$

Derangements Any change in the order of the things in a group is called a derangement.

Or

When 'n' things are to be placed at 'n' specific places but none of them is placed on its specified position, then we say that the 'n' things are deranged.

Or

Assume $a_1, a_2, a_3, \ldots, a_n$ be n distinct things such that their positions are fixed in a row. If we now rearrange $a_1, a_2, a_3, \ldots, a_n$ in such a way that no one occupy its original position, then such an arrangement is called a derangement.

Consider 'n' letters and 'n' corresponding envelops. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$

Proof n letters are denoted by 1, 2, 3, …, n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelop) so that the i th letter is placed in the corresponding envelope, then

$$n(A_i) = 1 \times (n-1)!$$

[$\because$ the remaining $(n-1)$ letters can be placed in $(n-1)$ envelopes is $(n-1)!$]

and $n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$ [$\because$ i and j can be placed in their corresponding envelopes and remaining $(n-2)$ letters can be placed in $(n-2)$ envelopes in $(n-2)!$ way]

Also, $n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$

Hence, the required number is

$n(A_1' \cap A_2' \cap A_3' \cap \ldots \cap A_n')$

$= n(U) - n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)$

$= n! - \{\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \ldots + (-1)^n \sum n(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)$

$= n! - \{{}^nC_1 \times (n-1)! - {}^nC_2 \times (n-2)! + {}^nC_3 \times (n-3)! - \ldots + (-1)^{n-1} \times {}^nC_n \times 1!\}$

$= n! - \left\{\frac{n \times (n-1)!}{1!} - \frac{n(n-1)}{2!} \times (n-2)! + \frac{n(n-1)(n-2)}{3!} \times (n-3)! - \ldots + (-1)^{n-1} \times 1\right\}$

$= n! - \left\{\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \ldots + (-1)^n \cdot 1\right\}$

$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right]$

Maha Short Cut Method

If $\quad D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right]$

Then, $\boldsymbol{D_{n+1} = (n+1)D_n + (-1)^{n+1}, \forall x \in N}$

and $\quad \boldsymbol{D_{n+1} = n(D_n + D_{n-1}), \forall x \in N - \{1\}}$

where $\quad D_1 = 0$

For $n = 1$, from result I

$D_2 = 2D_1 + (-1)^2 = 0 + 1 = 1$

For $n = 2$, from result I

$D_3 = 3D_2 + (-1)^3 = 3 \times 1 - 1 = 2$

For $n = 3$, from result I

$D_4 = 4D_3 + (-1)^4 = 4 \times 2 + 1 = 9$

For $n = 4$, from result I

$D_5 = 5D_4 + (-1)^5 = 5 \times 9 - 1 = 44$

For $n = 5$, from result I

$D_6 = 6D_5 + (-1)^6 = 6 \times 44 + 1 = 265$

Note $D_1 = 0, D_2 = 1, D_3 = 2, D_4 = 9, D_5 = 44, D_6 = 265$ [Remember]

Remark

If r things goes to wrong place out of n things, then $(n-r)$ things goes to original place (here $r < n$).

If D_n = Number of ways, if all n things goes to wrong places.

and D_r = Number of ways, if r things goes to wrong places.

If r goes to wrong places out of n, then $(n-r)$ goes to correct places.

Then, $\quad \boxed{D_n = {}^nC_{n-r}\, D_r}$

where, $\quad D_r = r!\left(1 - \frac{1}{1!} + \frac{1}{2!}! - \frac{1}{3!} + \ldots + (-1)^r \frac{1}{r!}\right)$

If atleast p things goes to wrong places, then $D_n = \sum_{r=p}^{n} {}^nC_{n-r} \cdot D_r$

must

| Example 90. A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) atleast two of them are in the wrong envelopes. (ii) all the letters are in the wrong envelopes.

Sol. (i) The number of ways in which atleast two of them in the wrong envelopes

$= \sum_{r=2}^{6} {}^6C_{6-r} \cdot D_r$

$= {}^6C_4 \times D_2 + {}^6C_3 \times D_3 + {}^6C_2 \times D_4 + {}^6C_1 \times D_5 + {}^6C_0 \times D_6$

$= 15D_2 + 20D_3 + 15D_4 + 6D_5 + D_6$ $\quad$ [from note]

$= 15 \times 1 + 20 \times 2 + 15 \times 9 + 6 \times 44 + 265$

$= 719$

(ii) The number of ways in which all letters be placed in wrong envelopes $= D_6 = 265$ $\quad$ [from note]

Aliter

(i) The number of all the possible ways of putting 6 letters into 6 envelopes is $6!$.

Number of ways to place all letters correctly into corresponding envelopes = 1

and number of ways to place one letter in the wrong envelope and other 5 letters in the write envelope = 0

[$\because$ It is not possible that only one letter goes in the wrong envelope, when if 5 letters goes in the right envelope, then remaining one letter also goes in the write envelope]

Hence, number of ways to place atleast two letters goes in the wrong envelopes

$= 6! - 0 - 1 = 6! - 1 = 720 - 1 = 719$

(ii) The number of ways 1 letter in 1 address envelope, so that one letter is in wrong envelope = 0 $\quad$ …(i)

[because it is not possible that only one letter goes in the wrong envelope]

The number of ways to put 2 letters in 2 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope

$= 2! - 1 - 0 = 2 - 1$

$= 1$ $\quad$ …(ii) [from Eq. (i)]

The number of ways to put 3 letters in 3 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope – The number of ways in which 2 letter are in correct envelope

$= 3! - 1 - {}^3C_1 \times 1 - 0$ [from Eqs. (i) and (ii)]

$= 2$

[${}^3C_1$ means that select one envelope to put the letter correctly]

The number of ways to put 4 letters in 4 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope – The number of ways in which 2 letters are in correct envelopes – The number of ways in which 3 letters are in correct envelopes

$= 4! - 1 - {}^4C_1 \times 2 - {}^4C_2 \times 1 - {}^4C_3 \times 0$

[from Eqs. (i), (ii) and (iii)]

$= 24 - 1 - 8 - 6 - 0 = 9$ $\quad$ …(iv)

The number of ways to put 5 letters in 5 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelopes – The number of ways in which 2 letters are in correct envelopes – The number of ways in which 3 letters are in correct envelopes – The number of ways in which 4 letters are in correct envelopes

$= 5! - 1 - {}^5C_1 \times 9 - {}^5C_2 \times 2 - {}^5C_3 \times 1 - {}^5C_4 \times 0$

[from Eqs. (i), (ii), (iii) and (iv)]

$= 120 - 1 - 45 - 20 - 10 - 0 = 44$

The number of ways to put 6 letters in 6 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope – The number of ways in which 2 letters are in correct envelopes – The number of ways in which 3 letters are in correct envelopes – The number of ways in which 4 letters are in correct envelopes – The number of ways in which 5 letters are in correct envelopes.

$$= 6! - 1 - {}^6C_1 \times 44 - {}^6C_2 \times 9 - {}^6C_3 \times 2 - {}^6C_4 \times 1 - {}^6C_5 \times 0$$

[from Eqs. (i), (ii), (iii), (iv) and (v)]

$$= 720 - 1 - 264 - 135 - 40 - 15 = 720 - 455 = 265$$

Multinomial Theorem

(i) If there are l objects of one kind, m objects of second kind, n objects of third kind and so on, then the number of ways of choosing r objects out of these objects (i.e., $l+m+n+\dots$) is the coefficient of x^r in the expansion of

$$(1+x+x^2+x^3+\dots+x^l)(1+x+x^2+\dots+x^m)(1+x+x^2+\dots+x^n)$$

Further, if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., $l+m+n+\dots$) is the coefficient of x^r in the expansion of

$$(x+x^2+x^3+\dots+x^l)(x+x^2+x^3+\dots+x^m)(x+x^2+x^3+\dots+x^n)\dots$$

(ii) If there are l objects of one kind, m objects of second kind, n objects of third kind and so on, then the number of possible arrangements/permutations of r objects out of these objects (i.e., $l+m+n+\dots$) is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^l}{l!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^m}{m!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}\right)\dots$$

Different Cases of Multinomial Theorem

***Case I* If upper limit of a variable is more than or equal to the sum required, then the upper limit of that variable can be taken as infinite.**

Example 91. In how many ways the sum of upper faces of four distinct die can be five?

Sol. Here, the number of required ways will be equal to the number of solutions of $x_1+x_2+x_3+x_4=5$ i.e., $1 \le x_i \le 6$ for $i = 1, 2, 3, 4$.

Since, upper limit is 6, which is greater than required sum, so upper limit taken as infiite. So, number of Sols is equal to coefficient of α^5 in the expansion of $(1+\alpha+\alpha^2+\dots+\infty)^4$

= Coefficient of α^5 in the expansion of $(1-\alpha)^{-4}$

= Coefficient of α^5 in the expansion of $(1+{}^4C_1\alpha+{}^5C_2\alpha^2+\dots)$

$$= {}^8C_5 = {}^8C_3 = \frac{8\cdot7\cdot6}{1\cdot2\cdot3} = 56$$

***Case II* If the upper limit of a variable is less than the sum required and the lower limit of all variables is non-negative, then the upper limit of that variable is that given in the problem.**

Example 92. In an examination, the maximum marks each of three papers is 50 and the maximum mark for the fourth paper is 100. Find the number of ways in which the candidate can score 60% marks in aggregate.

Sol. Aggregate of marks $= 50 \times 3 + 100 = 250$

$\therefore$ 60% of the aggregate $= \frac{60}{100} \times 250 = 150$

Let the marks scored by the candidate in four papers be x_1, x_2, x_3 and x_4. Here, the number of required ways will be equal to the number of Sols of $x_1+x_2+x_3+x_4=150$ i.e., $0 \le x_1, x_2, x_3 \le 50$ and $0 \le x_4 \le 100$.

Since, the upper limit is 100 < required sum (150).

The number of solutions of the equation is equal to coefficient of α^{150} in the expansion of

$$(\alpha^0+\alpha^1+\alpha^2+\dots+\alpha^{50})^3(\alpha^0+\alpha^1+\alpha^2+\dots+\alpha^{100})$$

= Coefficient of α^{150} in the expansion of $(1-\alpha^{51})^3(1-\alpha^{101})(1-\alpha)^{-4}$

= Coefficient of α^{150} in the expansion of $(1-3\alpha^{51}+3\alpha^{102})(1-\alpha^{101})(1+{}^4C_1\alpha+{}^5C_2\alpha^2+\dots+\infty)$

= Coefficient of α^{150} in the expansion of $(1-3\alpha^{51}-\alpha^{101}+3\alpha^{102})(1+{}^4C_1\alpha+{}^5C_2\alpha^2+\dots+\infty)$

$$= {}^{153}C_{150} - 3\times{}^{102}C_{99} - {}^{52}C_{49} + 3\times{}^{51}C_{48}$$

$$= {}^{153}C_3 - 3\times{}^{102}C_3 - {}^{52}C_3 + 3\times{}^{51}C_3$$

$$= 110556$$

Very Important Trick

On multiplying $p_0+p_1\alpha+p_2\alpha^2+p_3\alpha^3+\dots+p_n\alpha^n$ by $(1+\alpha)$, we get

$$p_0+(p_0+p_1)\alpha+(p_1+p_2)\alpha^2+(p_2+p_3)\alpha^3+\dots+(p_{n-2}+p_{n-1})\alpha^{n-1}+(p_{n-1}+p_n)\alpha^n+p_n\alpha^{n+1}$$

i.e., we just add coefficient of α^r with coefficient of α^{r-1} (i.e., previous term) to get coefficient α^r in product.

Now, coefficient of $\alpha^r = p_{r-1}+p_r$

On multiplying $p_0+p_1\alpha+p_2\alpha^2+p_3\alpha^3+\dots+p_n\alpha^n$ by $(1+\alpha+\alpha^2)$

we get, $p_0+(p_0+p_1)\alpha+(p_0+p_1+p_2)\alpha^2+(p_1+p_2+p_3)\alpha^3+(p_2+p_3+p_4)\alpha^4+\dots$

i.e., to find coefficient of α^r in product and add this with 2 preceding coefficients.

Now, coefficient of $\alpha^r = p_{r-2}+p_{r-1}+p_r$

Similarly, in product of $p_0+p_1\alpha+p_2\alpha^2+\dots$ with $(1+\alpha+\alpha^2+\alpha^3)$, the coefficient of α^r in product will be

$$\underbrace{p_{r-3}+p_{r-2}+p_{r-1}}_{\text{3 preceding coefficients}}+p_r$$

and in product of $p_0+p_1\alpha+p_2\alpha^2+\dots$ with $(1+\alpha+\alpha^2+\alpha^3+\alpha^4)$, the coefficient of α^r in product will be $\underbrace{p_{r-4}+p_{r-3}+p_{r-2}+p_{r-1}}_{\text{4 preceding coefficients}}+p_r$

Finally, in product of $p_0+p_1\alpha+p_2\alpha^2+\dots$ with $(1+\alpha+\alpha^2+\alpha^3+\dots+\text{upto}\,\infty)$, the coefficient of α^r in product will be $\underbrace{p_0+p_1+p_2+\dots+p_{r-1}}_{\text{all preceding coefficients}}+p_r$

Example 93. Find the coefficient of α^6 in the product $(1+\alpha+\alpha^2)(1+\alpha+\alpha^2)(1+\alpha+\alpha^2+\alpha^3)(1+\alpha)(1+\alpha)(1+\alpha)$.

Sol. The given product can be written as

$$(1+\alpha+\alpha^2)(1+\alpha+\alpha^2)(1+\alpha+\alpha^2+\alpha^3)(1+\alpha)^3$$

or $(1+\alpha+\alpha^2)(1+\alpha+\alpha^2)(1+\alpha+\alpha^2+\alpha^3)(1+3\alpha+3\alpha^2+\alpha^3)$

Multiplying Synthetically

1	α	α^2	α^3	α^4	α^5	α^6	...
1	3	3	1	0	0	0	

... on multiplying by $1+\alpha+\alpha^2+\alpha^3 \to$ To each coefficient add 3 preceding coefficients

1	4	7	8	7	4	1	...

...on multiplying by $1+\alpha+\alpha^2 \to$ To each coefficient add 2 preceding coefficients.

1	5	12	19	22	19	12	...

...on multiplying by $1+\alpha+\alpha^2 \to$ To each coefficient add 2 preceding coefficients.

...	...	...	...	...	...	53	...

Hence, required coefficient is 53.

Example 94. Find the number of different selections of 5 letters which can be made from 5A's, 4B 's, 3C's, 2D's and 1E

Sol. All selections of 5 letters are given by 5th degree terms in

$$(1+A+A^2+A^3+A^4+A^5)(1+B+B^2+B^3+B^4)(1+C+C^2+C^3)(1+D+D^2)(1+E)$$

$\therefore$ Number of 5 letter selections

= Coefficient of α^5 in $(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5)(1+\alpha+\alpha^2+\alpha^3+\alpha^4)(1+\alpha+\alpha^2+\alpha^3)(1+\alpha+\alpha^2)(1+\alpha)$

Multiplying synthetically

1	α	α^2	α^3	α^4	α^5...
1	1	1	1	1	1
					$\times 1+\alpha+\alpha+\alpha^2+\alpha^3+\alpha^4$
1	2	3	4	5	5...
					$\times 1+\alpha+\alpha^2+\alpha^3$
1	3	6	10	14	17...
					$\times 1+\alpha+\alpha^2$
1	4	10	19	30	41...
					$\times 1+\alpha$
1	5	14	29	49	71

Hence, required coefficient is 71.

Example 95. must Find the number of combinations and permutations of 4 letters taken from the word EXAMINATION.

Sol. There are 11 letters

A, A, N, N, X, M, T, O, E, I, I

Then, number of combinations

= coefficient of x^4 in $(1+x+x^2)^3(1+x)^5$

[$\because$ 2A's, 2I 's, 2N's, 1E, 1X, 1M, 1T and 1O]

$= \text{Coefficient of } x^4 \text{ in } \{(1+x)^3 + x^6 + 3(1+x)^2x^2 + 3(1+x)x^4\}(1+x)^5$

$= \text{Coefficient of } x^4 \text{ in}$

$\{(1+x)^8 + x^6(1+x)^5 + 3x^2(1+x)^7 + 3x^4(1+x)^6\}$

$= {}^8C_4 + 0 + 3\cdot{}^7C_2 + 3 = \dfrac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4} + 3\cdot\dfrac{7\cdot6}{1\cdot2} + 3 = 70 + 63 + 3$

$= 136$

and number of permutations

$= \text{Coefficient of } x^4 \text{ in } 4!\left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!}\right)^3\left(1 + \dfrac{x}{1!}\right)^5$

$= \text{Coefficient of } x^4 \text{ in } 4!\left(1 + x + \dfrac{x^2}{2}\right)^3 (1+x)^5$

$= \text{Coefficient of } x^4 \text{ in}$

$4!\left\{(1+x)^3 + \dfrac{x^6}{8} + \dfrac{3}{2}(1+x)^2x^2 + \dfrac{3}{4}x^4(1+x)\right\}(1+x)^5$

$= \text{Coefficient of } x^4 \text{ in}$

$4!\left\{(1+x)^8 + \dfrac{x^6}{8}(1+x)^5 + \dfrac{3}{2}x^2(1+x)^7 + \dfrac{3}{4}x^4(1+x)^6\right\}$

$= 4!\left\{{}^8C_4 + 0 + \dfrac{3}{2}\cdot{}^7C_2 + \dfrac{3}{4}\right\} = 24\left\{\dfrac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4} + \dfrac{3}{2}\cdot\dfrac{7\cdot6}{1\cdot2} + \dfrac{3}{4}\right\}$

$= 8\cdot7\cdot6\cdot5 + 6(3\cdot7\cdot6) + 6\cdot3 = 1680 + 756 + 18 = 2454$

Aliter There are 11 letters:

A, A, I, I, N, N, E, X, M, T, O

The following cases arise:

***Case* I All letters different** The required number of choosing 4 different letters from 8 different (A, I, N, E, X, M, T, O) types of the letters

$= {}^8C_4 = \dfrac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4} = 70$

and number of permutations $= {}^8P_4 = 8\cdot7\cdot6\cdot5 = 1680$

***Case* II Two alike of one type and two alike of another type** This must be 2A's, 2I's or 2I's, 2N's, or 2N's, 2A's.

$\therefore$ Number of selections $= {}^3C_2 = 3$

For example, [for arrangements]

A	A	I	I

and number of permutations $= 3\cdot\dfrac{4!}{2!2!} = 18$

***Case* III Two alike and two different** This must be 2A's or 2I's or 2N's

and for each case 7 different letters.

For example, for 2A's, 7 differents's are I, N, E, X, M, T, O

For example, [for arrangements]

A	X	A	O

$\therefore$ Number of selections $= {}^3C_1 \times {}^7C_2 = 3 \times \dfrac{7\times6}{1\times2} = 63$

and number of permutations $= 63\cdot\dfrac{4!}{2!} = 756$

From Case I, II and III

The required number of combinations $= 70 + 3 + 63 = 136$

and number of permutations $= 1680 + 18 + 756 = 2454$

Note Number of combinations and permutations of 4 letters taken from the word MATHEMATICS are 136 and 2454 respectively, as like of EXAMINATION.

Number of Solutions with the Help of Multinomial Theorem

***Case* I** If the equation

$$\alpha + 2\beta + 3\gamma + \ldots + q\theta = n \quad \ldots(\text{i})$$

(a) If zero included, the number of solution of Eq. (i)

$= \text{Coefficient of } x^n \text{ in } (1 + x + x^2 + \ldots)$

$(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)\ldots$

$(1 + x^q + x^{2q} + \ldots)$

$= \text{Coefficient of } x^n \text{ in}$

$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^q)^{-1}$

(b) If zero excluded, then the number of solutions of Eq. (i)

$= \text{Coefficient of } x^n \text{ in } (x + x^2 + x^3 + \ldots)$

$(x^2 + x^4 + x^6 + \ldots)(x^3 + x^6 + x^9 + \ldots)$

$\ldots(x^q + x^{2q} + \ldots)$

$= \text{Coefficient of } x^n \text{ in } x^{1+2+3+\ldots+q}(1-x)^{-1}$

$(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^q)^{-1}$

$= \text{Coefficient of } x^{n - \frac{q(q+1)}{2}} \text{ in}$

$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^q)^{-1}$

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Example 96. Find the number of non-negative integral solutions of $x_1 + x_2 + x_3 + 4x_4 = 20$.

Sol. Number of non-negative integral solutions of the given equation

$= \text{Coefficient of } x^{20} \text{ in } (1-x)^{-1}(1-x)^{-1}(1-x)^{-1}(1-x^4)^{-1}$

$= \text{Coefficient of } x^{20} \text{ in } (1-x)^{-3}(1-x^4)^{-1}$

$= \text{Coefficient of } x^{20} \text{ in } (1 + {}^3C_1 + {}^4C_2x^2 + {}^5C_3x^3 + {}^6C_4x^4 + \ldots + {}^{10}C_8x^8 + \ldots + {}^{14}C_{12}x^{12} + \ldots + {}^{18}C_{16}x^{16} + \ldots + {}^{22}C_{20}x^{20} + \ldots)(1 + x^4 + x^8 + x^{12} + x^{16} + x^{20} + \ldots)$

$= 1 + {}^6C_4 + {}^{10}C_8 + {}^{14}C_{12} + {}^{18}C_{16} + {}^{22}C_{20}$

$= 1 + {}^6C_2 + {}^{10}C_2 + {}^{14}C_2 + {}^{18}C_2 + {}^{22}C_2$

$= 1 + \left(\dfrac{6\cdot5}{1\cdot2}\right) + \left(\dfrac{10\cdot9}{1\cdot2}\right) + \left(\dfrac{14\cdot13}{1\cdot2}\right) + \left(\dfrac{18\cdot17}{1\cdot2}\right) + \left(\dfrac{22\cdot21}{1\cdot2}\right)$

$= 1 + 15 + 45 + 91 + 153 + 231 = 536$

Example 97. Find the number of positive unequal integral solutions of the equation $x + y + z + w = 20$.

Sol. We have,

$$x + y + z + w = 20 \quad \ldots(\text{i})$$

Assume $x < y < z < w$. Here, $x, y, z, w \geq 1$

Now, let $x = x_1, y - x = x_2, z - y = x_3$ and $w - z = x_4$

$\therefore$ $x = x_1, y = x_1 + x_2, z = x_1 + x_2 + x_3$ and

$w = x_1 + x_2 + x_3 + x_4$

From Eq. (i), $4x_1 + 3x_2 + 2x_3 + x_4 = 20$

Then, $x_1, x_2, x_3, x_4 \geq 1$

$\because$ $4x_1 + 3x_2 + 2x_3 + x_4 = 20 \quad \ldots(\text{ii})$

$\therefore$ Number of positive integral solutions of Eq. (ii)

$= \text{Coefficient of } x^{20-10} \text{ in}$

$(1-x^4)^{-1}(1-x^3)^{-1}(1-x^2)^{-1}(1-x)^{-1}$

$= \text{Coefficient of } x^{10} \text{ in}$

$(1-x^4)^{-1}(1-x^3)^{-1}(1-x^2)^{-1}(1-x)^{-1}$

$= \text{Coefficient of } x^{10} \text{ in } (1 + x^4 + x^8 + x^{12} + \ldots)$

$\times (1 + x^3 + x^6 + x^9 + x^{12} + \ldots)\times$

$(1 + x^2 + x^4 + x^6 + x^8 + x^{10} + \ldots)\times(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + \ldots)$

$= \text{Coefficient of } x^{10} \text{ in}$

$(1 + x^3 + x^6 + x^9 + x^4 + x^7 + x^{10} + x^8)$

$\times (1 + x^2 + x^4 + x^6 + x^8 + x^{10})(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})$

[neglecting higher powers]

$= \text{Coefficient of } x^{10} \text{ in}$

$(1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^3 + x^5 + x^7 + x^9 + x^6 + x^8 + x^{10} + x^9 + x^4 + x^6 + x^8 + x^{10} + x^7 + x^9 + x^{10} + x^8 + x^{10})(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})$ [neglecting higher powers]

$= 1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1 = 23$

But x, y, z and w can be arranged in ${}^4P_4 = 4! = 24$

Hence, required number of Sols $= (23)(24) = 552$

Example 98. In how many ways can 15 identical blankets be distribted among six beggars such that everyone gets atleast one blanket and two particular beggars get equal blankets and another three particular beggars get equal blankets.

Sol. The number of ways of distributing blankets is equal to the number of solutions of the equation $3x + 2y + z = 15$, where $x \geq 1, y \geq 1, z \geq 1$ which is equal to coefficient of α^{15} in

$(\alpha^3 + \alpha^6 + \alpha^9 + \alpha^{12} + \alpha^{15} + \ldots)$

$\times(\alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 + \alpha^{10} + \alpha^{12} + \alpha^{14} + \ldots)$

$\times(\alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{15} + \ldots)$

$= \text{Coefficient of } \alpha^9 \text{ in } (1 + \alpha^3 + \alpha^6 + \alpha^9)$

$\times(1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8)$

$\times(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^9)$

[neglecting higher powers]

$= \text{Coefficient of } \alpha^9 \text{ in } (1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 + \alpha^3 + \alpha^5 + \alpha^7 + \alpha^9 + \alpha^6 + \alpha^8 + \alpha^9) \times (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^9)$

[neglecting higher powers]

$= 1+1+1+1+1+1+1+1+1+1+1+1 = 12$

***Case* II** If the inequation

$$x_1 + x_2 + x_3 + \ldots + x_m \leq n \quad \ldots(\text{i})$$

[when the required sum is not fixed]

In this case, we introduce a dummy variable x_{m+1}.

So that,

$$x_1 + x_2 + x_3 + \ldots + x_m + x_{m+1} = n,$$

$$x_{m+1} \geq 0 \quad \ldots(\text{ii})$$

Here, the number of Sols of Eqs. (i) and (ii) will be same.

Example 99. Find the number of positive integral solutions of the inequation $3x + y + z \leq 30$.

Sol. Let dummy variable w, then

$$3x + y + z + w = 30, w \geq 0 \quad \ldots(\text{i})$$

Now, let $a = x - 1, b = y - 1, c = z - 1, d = w$, then

$$3a + b + c + d = 25, \text{ where } a, b, c, d \geq 0 \quad \ldots(\text{ii})$$

$\therefore$ Number of positive integral solutions of Eq. (i)

$=$ Number of non-negative integral solutions of Eq. (ii)

$= \text{Coefficient of } \alpha^{25} \text{ in } (1 + \alpha^3 + \alpha^6 + \ldots)(1 + \alpha + \alpha^2 + \ldots)^3$

$= \text{Coefficient of } \alpha^{25} \text{ in } (1 + \alpha^3 + \alpha^6 + \ldots)(1-\alpha)^{-3}$

$= \text{Coefficient of } \alpha^{25} \text{ in}$

$(1 + \alpha^3 + \alpha^6 + \ldots)(1 + {}^3C_1\alpha + {}^4C_2\alpha^2 + \ldots)$

$= {}^{27}C_{25} + {}^{24}C_{22} + {}^{21}C_{19} + {}^{18}C_{16} + {}^{15}C_{13} + {}^{12}C_{10} + {}^9C_7 + {}^6C_4 + {}^3C_1$

$= {}^{27}C_2 + {}^{24}C_2 + {}^{21}C_2 + {}^{18}C_2 + {}^{15}C_2 + {}^{12}C_2 + {}^9C_2 + {}^6C_2 + {}^3C_1$

$= 351 + 276 + 210 + 153 + 105 + 66 + 36 + 15 + 3 = 1215$

Aliter

From Eq. (ii), $3a + b + c + d = 25$, where $a, b, c, d \ge 0$

Clearly, $0 \le a \le 8$, if $a = k$, then

$b + c + d = 25 - 3k$ ...(iii)

Hence, number of non-negative integral solutions of Eq. (iii) is

$${}^{25-3k+3-1}C_{3-1} = {}^{27-3k}C_2 = \frac{(27-3k)(26-3k)}{2}$$

$$= \frac{3}{2}(3k^2 - 53k + 234)$$

Therefore, required number is

$$\frac{3}{2}\sum_{k=0}^{8}(3k^2 - 53k + 234)$$

$$= \frac{3}{2}\left[3\cdot\left(\frac{8\times 9\times 17}{6}\right) - 53\cdot\left(\frac{8\times 9}{2}\right) + 234\times 9\right] = 1215$$

| Example 100. In how many ways can we get a sum of atmost 15 by throwing six distinct dice ?

Sol. Let x_1, x_2, x_3, x_4, x_5 and x_6 be the number that appears on the six dice.

The number of ways = Number of solutions of the inequation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 15$$

Introducing a dummy variable $x_7 (x_7 \ge 0)$, the inequation becomes an equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 15$$

Here, $1 \le x_i \le 6$ for $i = 1, 2, 3, 4, 5, 6$ and $x_7 \ge 0$.

Therefore, number of solutions

$$= \text{Coefficient of } x^{15} \text{ in } (x + x^2 + x^3 + x^4 + x^5 + x^6)^6 \times (1 + x + x^2 + \dots)$$

$$= \text{Coefficient of } x^9 \text{ in } (1 - x^6)^6 (1 - x)^{-7}$$

$$= \text{Coefficient of } x^9 \text{ in } (1 - 6x^6)(1 + {}^7C_1 x + {}^8C_2 x^2 + \dots)$$

[neglecting higher powers]

$$= {}^{15}C_9 - 6 \times {}^9C_3 = {}^{15}C_6 - 6 \times {}^9C_3$$

$$= 5005 - 504 = 4501$$

***Case* III** If the inequation

$$x_1 + x_2 + x_3 + \dots + x_n \ge n$$

[when the values of $x_1, x_2, \dots, x_n$ are restricted]

In this case first find the number of solutions of $x_1 + x_2 + x_3 + \dots + x_n \le n - 1$ and then subtract it from the total number of solutions.

| Example 101. In how many ways can we get a sum greater than 15 by throwing six distinct dice?

Sol. Let x_1, x_2, x_3, x_4, x_5 and x_6 be the number that appears on the six dice.

The number of ways = Number of solutions of the inequation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 > 15$$

Here, $1 \le x_i \le 6, i = 1, 2, 3, 4, 5, 6$

Total number of cases $= 6^6 = 2^6 \times 3^6 = 64 \times 729 = 46656$

and number of ways to get the sum less than or equal to 15 which is 4501 [from Example 100]

Hence, the number of ways to get a sum greater than 15 is $46656 - 4501 = 42155$

***Case* IV** If the equation

$$x_1 x_2 x_3 \dots x_n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \dots$$

where $\alpha_1, \alpha_2, \alpha_3, \dots$ are natural numbers.

In this case number of positive integral solutions $(x_1, x_2, x_3, \dots, x_n)$ are

$$({}^{\alpha_1+n-1}C_{n-1})({}^{\alpha_2+n-1}C_{n-1})({}^{\alpha_3+n-1}C_{n-1})\dots$$

| Example 102. Find the total number of positive integral solutions for (x, y, z) such that $xyz = 24$.

Sol. $\because xyz = 24 = 2^3 \times 3^1$

Hence, total number of positive integral solutions

$$= ({}^{3+3-1}C_{3-1})({}^{1+3-1}C_{3-1})$$

$$= {}^5C_2 \times {}^3C_2 = 30$$

Aliter

$\because \quad xyz = 24 = 2^3 \times 3^1$

Now, consider three boxes x, y, z.

3 can be put in any of the three boxes.

Also, 2, 2, 2 can be distributed in the three boxes in ${}^{3+3-1}C_{3-1} = {}^5C_2 = 10$ ways. Hence, the total number of positive integral solutions = the number of distributions which is given by $3 \times 10 = 30$.

Geometrical Problems

(a) If there are n points in a plane out of these points no three are in the same line except m points which are collinear, then

(i) Total number of different lines obtained by joining these n points is ${}^nC_2 - {}^mC_2 + 1$

(ii) Total number of different triangles formed by joining these n points is ${}^nC_3 - {}^mC_3$

(iii) Total number of different quadrilateral formed by joining these n points is

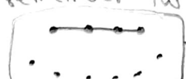
$${}^nC_4 - ({}^mC_3 \cdot {}^nC_1 + {}^mC_4 \cdot {}^nC_0)$$

| Example 103. There are 10 points in a plane out of these points no three are in the same straight line except 4 points which are collinear. How many

(i) straight lines (ii) trian-gles

(iii) quadrilateral, by joining them?

Dont remember these formula only remember this diagram → from these make shapes.



Sol. (i) Required number of straight lines

$$= {}^{10}C_2 - {}^4C_2 + 1 = \frac{10\cdot 9}{1\cdot 2} - \frac{4\cdot 3}{1\cdot 2} + 1 = 45 - 6 + 1 = 40$$

(ii) Required number of triangles

$$= {}^{10}C_3 - {}^4C_3 = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} - {}^4C_1 = 120 - 4 = 116$$

(iii) Required number of quadrilaterals

$$= {}^{10}C_4 - ({}^4C_3 \cdot {}^6C_1 + {}^4C_4 \cdot {}^6C_0)$$

$$= \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} - ({}^4C_1 \cdot {}^6C_1 + 1.1)$$

$$= 210 - (4 \times 6 + 1) = 210 - 25 = 185$$

(b) If there are n points in a plane out of these points no any three are collinear, then

(i) Total points of intersection of the lines joining these n points $= {}^pC_2$, where $p = {}^nC_2$

(ii) If n points are the vertices of a polygon, then total number of diagonals $= {}^nC_2 - n = \dfrac{n(n-3)}{2}$

| Example 104. How many number of points of intersection of n straight lines, if n satisfies

$${}^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{n+3}P_n \text{ ?}$$

Sol. We have, ${}^{n+5}P_{n+1} = \dfrac{11(n-1)}{2} \times {}^{n+3}P_n$

$$\Rightarrow \quad \frac{(n+5)!}{4!} = \frac{11(n-1)}{2} \times \frac{(n+3)!}{3!}$$

$$\Rightarrow \quad \frac{(n+5)(n+4)}{4} = \frac{11(n-1)}{2}$$

$$\Rightarrow \quad n^2 - 13n + 42 = 0 \Rightarrow (n-6)(n-7) = 0$$

$$\Rightarrow \quad n = 6 \text{ or } n = 7$$

The number of points of intersection of lines is ${}^6C_2$ or ${}^7C_2$

$$= 15 \text{ or } 21$$

| Example 105. The interior angles of a regular polygon measure 150° each. Then, find the number of diagonals of the polygon.

Sol. Each exterior angle = 30°

$$\therefore \text{Number of sides} = \frac{360°}{30°} = \frac{360 \times \frac{\pi}{180}}{30 \times \frac{\pi}{180}} = 12$$

$$\therefore \text{Number of diagonals} = {}^{12}C_2 - 12 = 66 - 12 = 54$$

Imp

| Example 106. In a polygon the number of diagonals is 77. Find the number of sides of the polygon.

Sol. Let number of sides of the polygon $= n$, then ${}^nC_2 - n = 77$

$$\Rightarrow \quad \frac{n(n-1)}{2} - n = 77 \Rightarrow \frac{n(n-3)}{2} = \frac{14 \times 11}{2}$$

we get, $\quad n = 14$

(c) n straight lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. Then, number of parts into which these lines divides the plane is equal to

$$1 + \sum_{k=1}^{n} k, \text{ i.e. } \frac{(n^2 + n + 2)}{2}$$

| Example 107. If n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent, such that these lines divide the plane in 67 parts, then find number of different points at which these lines will cut.

Sol. Given number of straight lines $= n$, then

$$1 + \sum_{k=1}^{n} k = 67 \Rightarrow \frac{n^2 + n + 2}{2} = 67$$

$$\Rightarrow \quad n^2 + n - 132 = 0 \Rightarrow (n + 12)(n - 11) = 0$$

$$\therefore \quad n = 11, n \ne -12$$

Hence, required number of points $= {}^nC_2 = {}^{11}C_2 = \dfrac{11\cdot 10}{2}$

$$= 55$$

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then, total number of parallelograms so formed

$$\boxed{= {}^mC_2 \cdot {}^nC_2} \text{ i.e., } \frac{mn(m-1)(n-1)}{4}$$

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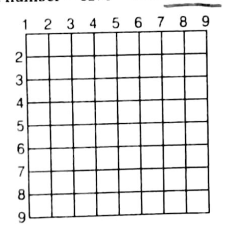
| Example 108. Find number of rectangles in a chess board, which are not a square.

Sol. Number of rectangles $= {}^9C_2 \times {}^9C_2 = (36)^2 = 1296$

Number of squares $= 8 \times 8 + 7 \times 7 + 6 \times 6 + \dots + 1 \times 1$

$$= 204$$

$\therefore$ Required number $= 1296 - 204 = 1092$



Square can be formed as follows :

To form the smallest square, select any two consecutive lines from the given (here 9) vertical and horizontal lines. This can be done in 8×8 ways (1-2, 2-3, 3-4, ..., 8-9)

Again, to form the square consists of four small squares, select the lines as follows (1-3, 2-4, 3-5,..., 7-9) from both vertical and horizontal lines, thus 7×7 squares are obtained. Proceed in the same way)

Note If n parallel lines are intersected by another n parallel lines, then number of rhombus $= \sum (n-1)^2 = \frac{(n-1)\,n\,(2n-1)}{6}$

(e) Number of Rectangles and Squares

(i) Number of rectangles of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

(ii) In a rectangle of $n \times p\ (n < p)$ number of rectangles of any size is $\frac{np}{4}(n+1)(p+1)$ and number of squares of any size is

$$\sum_{r=1}^{n} (n+1-r)(p+1-r).$$

Example 109. Find the number of rectangles excluding squares from a rectangle of size 9×6.

Sol. Here, $n = 6$ and $p = 9$

$\therefore$ Number of rectangles excluding square

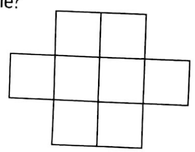
$$= \frac{6 \cdot 9}{4}(6+1)(9+1) - \sum_{r=1}^{6}(7-r)(10-r)$$

$$= 945 - \sum_{r=1}^{6}(70 - 17r + r^2) = 945 - 154 = 791$$

(f) If there are n rows, first row has α_1 squares, 2nd row has α_2 squares, 3rd row has α_3 squares, ... and nth row has α_n squares. If we have to filled up the squares with β X_s such that each row has atleast one X. The number of ways = Coefficient of x^{β} in

$$({}^{\alpha_1}C_1 x + {}^{\alpha_1}C_2 x^2 + \ldots + {}^{\alpha_1}C_{\alpha_1} x^{\alpha_1})$$
$$\times({}^{\alpha_2}C_1 x + {}^{\alpha_2}C_2 x^2 + \ldots + {}^{\alpha_2}C_{\alpha_2} x^{\alpha_2})$$
$$\times({}^{\alpha_3}C_1 x + {}^{\alpha_3}C_2 x^2 + \ldots + {}^{\alpha_3}C_{\alpha_3} x^{\alpha_3}) \times$$
$$\ldots \times({}^{\alpha_n}C_1 x + {}^{\alpha_n}C_2 x^2 + \ldots + {}^{\alpha_n}C_{\alpha_n} x^{\alpha_n})$$

Example 110. Six X's have to be placed in the squares of the figure below, such that each row contains atleast one X. In how many different ways can this be done?



Sol. The required number of ways

$= \text{Coefficient of } x^6 \text{ in } ({}^2C_1 x + {}^2C_2 x^2)({}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4 x^4)({}^2C_1 x + {}^2C_2 x^2)$

$= \text{Coefficient of } x^3 \text{ in } (2+x)^2 (4 + 6x + 4x^2 + x^3)$

$= \text{Coefficient of } x^3 \text{ in } (4 + 4x + x^2)(4 + 6x + 4x^2 + x^3)$

$= 4 + 16 + 6$

$= 26$

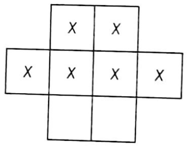
Aliter

In the given figure there are 8 squares and we have to place $6X$'s this can be done in

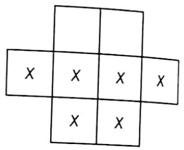
$${}^8C_6 = {}^8C_2 = \frac{8 \cdot 7}{1 \cdot 2} = 28 \text{ ways}$$

But these include the possibility that either headed row or lowest row may not have any X. These two possibilities are to be excluded.

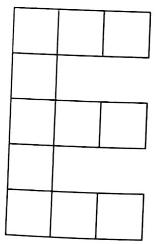
$\therefore$ Required number of ways $= 28 - 2 = 26$



OR



Example 111. In how many ways the letters of the word DIPESH can be placed in the squares of the adjoining figure so that no row remains empty?



Sol. If all letters are same, then number of ways

$= \text{Coefficient of } x^6 \text{ in } ({}^3C_1 x + {}^3C_2 x^2 + {}^3C_3 x^3)^3 ({}^1C_1 x)^2$

$= \text{Coefficient of } x \text{ in } (3 + 3x + x^2)^3$

$= \text{Coefficient of } x \text{ in } (3 + 3x)^3$

[neglecting higher degree term]

$= 27 \times {}^3C_1 = 81$

But in DIPESH all letters are different.

$\therefore$ Required number of ways $= 81 \times 6!$

Exercise for Session 6

1. If number of ways in which 7 different balls can be distributed into 4 different boxes, so that no box remains empty is 100 λ, the value of λ is
(a) 18 (b) 108 (c) 1008 (d) 10008

2. If number of ways in which 7 different balls can be distributed into 4 boxes, so that no box remains empty is 48 λ, the value of λ is
(a) 231 (b) 331 (c) 431 (d) 531

3. If number of ways in which 7 identical balls can be distributed into 4 boxes, so that no box remains empty is 4λ, the value of λ is
(a) 5 (b) 7 (c) 9 (d) 11

4. Number of non-negative integral solutions of the equation $a + b + c = 6$ is
(a) 28 (b) 32 (c) 36 (d) 56

5. Number of integral solutions of $a + b + c = 0$, $a \ge -5$, $b \ge -5$ and $c \ge -5$, is
(a) 272 (b) 136 (c) 240 (d) 120

6. If a, b and c are integers and $a \ge 1$, $b \ge 2$ and $c \ge 3$. If $a + b + c = 15$, the number of possible solutions of the equation is
(a) 55 (b) 66 (c) 45 (d) None of these

7. Number of integral solutions of $2x + y + z = 10\ (x \ge 0, y \ge 0, Z \ge 0)$ is
(a) 18 (b) 27 (c) 36 (d) 51

8. A person writes letters to six friends and addresses the corresponding envelopes. Let x be the number of ways so that atleast two of the letters are in wrong envelopes and y be the number of ways so that all the letters are in wrong envelopes. Then, $x - y$ is equal to
(a) 719 (b) 265 (c) 454 (d) None of these

9. A person goes for an examination in which there are four papers with a maximum of m marks from each paper. The number of ways in which one can get $2m$ marks, is
(a) ${}^{2m+3}C_3$
(b) $\left(\frac{1}{3}\right)(m+1)(2m^2 + 4m + 1)$
(c) $\left(\frac{1}{3}\right)(m+1)(2m^2 + 4m + 3)$
(d) None of these

10. The number of selections of four letters from the letters of the word ASSASSINATION, is
(a) 72 (b) 71 (c) 66 (d) 52

11. The number of positive integral solutions of $2x_1 + 3x_2 + 4x_3 + 5x_4 = 25$, is
(a) 20 (b) 22 (c) 23 (d) None of these

12. If $a, b,$ and c are positive integers such that $a + b + c \le 8$, the number of possible values of the ordered triplet (a, b, c) is
(a) 84 (b) 56 (c) 83 (d) None of these

13. The total number of positive integral solutions of $15 < x_1 + x_2 + x_3 \le 20$ is equal to
(a) 685 (b) 785 (c) 1125 (d) None of these

14. The total number of integral solutions for (x, y, z) such that $xyz = 24$, is
(a) 36 (b) 90 (c) 120 (d) None of these

15. There are 12 points in a plane in which 6 are collinear. Number of different straight lines that can be drawn by joining them, is
(a) 51 (b) 52 (c) 132 (d) 18

16. 4 points out of 11 points in a plane are collinear. Number of different triangles that can be drawn by joining them, is

(a) 165 (b) 161 (c) 152 (d) 159

17. The number of triangles that can be formed with 10 points as vertices, n of them being collinear, is 110. Then, n is

(a) 3 (b) 4 (c) 5 (d) 6

18. $ABCD$ is a convex quadrilateral. 3, 4, 5 and 6 points are marked on the sides AB, BC, CD and DA, respectively. The number of triangles with vertices on different sides, is

(a) 270 (b) 220 (c) 282 (d) None of these

19. There are 10 points in a plane of which no three points are collinear and 4 points are concyclic. The number of different circles that can be drawn through atleast 3 points of these points, is

(a) 116 (b) 120 (c) 117 (d) None of these

20. 4 points out of 8 points in a plane are collinear. Number of different quadrilateral that can be formed by joining them, is

(a) 56 (b) 60 (c) 76 (d) 53

21. There are $2n$ points in a plane in which m are collinear $(n > m > 4)$. Number of quadrilateral formed by joining these lines

(a) is equal to ${}^{2n}C_4 - {}^{m}C_4$ (b) is greater than ${}^{2n}C_4 - {}^{m}C_4$

(c) is less than ${}^{2n}C_4 - {}^{m}C_4$ (d) None of these

22. In a polygon the number of diagonals is 54. The number of sides of the polygon, is

(a) 10 (b) 12 (c) 9 (d) None of these

23. In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon be 70, then the number of diagonals of the polygon, is

(a) 20 (b) 28 (c) 8 (d) None of these

24. n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. The number of different points at which these lines will cut, is

(a) $\sum_{k=1}^{n-1} k$ (b) $n(n-1)$ (c) n^2 (d) None of these

25. Six straight lines are drawn in a plane such that no two lines are parallel and no three lines are concurrent. Then, the number of parts into which these lines divide the plane, is

(a) 15 (b) 22 (c) 29 (d) 36

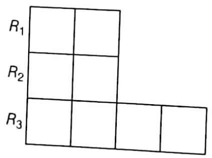
26. A parallelogram is cut by two sets of m lines parallel to its sides. The number of parallelogram thus formed, is

(a) $({}^{m}C_2)^2$ (b) $({}^{m+1}C_2)^2$ (c) $({}^{m+2}C_2)^2$ (d) None of these

27. The number of rectangles excluding squares from a rectangle of size 11×8 is 48λ, then the value of λ is

(a) 13 (b) 23 (c) 43 (d) 53

28. The number of ways the letters of the word PERSON can be placed in the squares of the figure shown so that no row remains empty, is



(a) $24 \times 6!$ (b) $26 \times 6!$ (c) $26 \times 7!$ (d) $27 \times 7!$

Session 7

Rank in a Dictionary, Gap Method

Rank in a Dictionary

The dictionary format means words are arranged in alphabetical order.

Following Examples will help you learn how to find the rank in the dictionary.

| Example 112. If the letters of the word are arranged as in dictionary, find the rank of the following words.

(i) RAJU (ii) UMANG (iii) AIRTEL

Sol. (i) In a dictionary, the letters in alphabetical order are A, J, R, U

$\therefore$ The first word is AJRU.

Number of words beginning with A = Number of ways arranging J, R, U $= 3! = 6$

Number of words beginning with J $= 3! = 6$

The next word begin with R and it is RAJU.

$\therefore$ Number of words before RAJU = 12

$\therefore$ Rank of word RAJU = 13

(ii) The letters in alphabetical order are A, G, M, N, U

$\therefore$ The first word is AGMNU

Number of words beginning with A $= 4! = 24$

Number of words beginning with G $= 4! = 24$

Number of words beginning with M $= 4! = 24$

Number of words beginning with N $= 4! = 24$

Number of words beginning with UA $= 3! = 6$

Number of words beginning with UG $= 3! = 6$

Number of words beginning with UMAG $= 1! = 1$

Number of words beginning with UMANG = 1

$\therefore$ Rank of the word

UMANG $= 24 + 24 + 24 + 24 + 6 + 6 + 1 + 1 = 110$

(iii) The letters in alphabetical order are A, E, I, L, R, T

$\therefore$ The first word is AEILRT

Number of words beginning with AE $= 4! = 24$

Number of words beginning with AIE $= 3! = 6$

Number of words beginning with AIL $= 3! = 6$

Number of words beginning with AIRE $= 2! = 2$

Number of words beginning with AIRL $= 2! = 2$

Number of words beginning with AIRTEL = 1

$\therefore$ Rank of the word AIRTEL $= 24 + 6 + 6 + 2 + 2 + 1 = 41$

| Example 113. If letters of the word are arranged as in dictionary, find the rank of the following words.

(i) INDIA (ii) SURITI (iii) DOCOMO

Sol. (i) The letters in alphabetical order are A, D, I, I, N

$\therefore$ The first word is ADIIN

Number of words beginning with A $= \frac{4!}{2!} = 12$

Number of words beginning with D $= \frac{4!}{2!} = 12$

Number of words beginning with IA $= 3! = 6$

Number of words beginning with ID $= 3! = 6$

Number of words beginning with II $= 3! = 6$

Number of words beginning with INA $= 2! = 2$

Number of words beginning with INDA $= 1! = 1$

Number of words beginning with INDIA = 1

$\therefore$ Rank of the word INDIA

$= 12 + 12 + 6 + 6 + 6 + 2 + 1 + 1 = 46$

(ii) The letters in alphabetical order are I, I, R, S, T, U

$\therefore$ The first word is IIRSTU

Number of words beginning with I $= 5! = 120$

Number of words beginning with R $= \frac{5!}{2!} = 60$

Number of words beginning with SI $= 4! = 24$

Number of words beginning with SR $= \frac{4!}{2!} = 12$

Number of words beginning with ST $= \frac{4!}{2!} = 12$

Number of words beginning with SUI $= 3! = 6$

Number of words beginning with SURII $= 1! = 1$

Number of words beginning with SURITI = 1

$\therefore$ Rank of the word SURITI

$= 120 + 60 + 24 + 12 + 12 + 6 + 1 + 1 = 236$

(iii) The letters in alphabetical order are C, D, M, O, O, O

$\therefore$ The first word is CDMOOO

Number of words beginning with C $= \frac{5!}{3!} = 20$

Number of words beginning with DC $= \frac{4!}{3!} = 4$

Number of words beginning with DM $= \frac{4!}{3!} = 4$

Number of words beginning with DOCM $= \frac{2!}{2!} = 1$

Number of words beginning with DOCOMO = 1

$\therefore$ Rank of the word DOCOMO $= 20 + 4 + 4 + 1 + 1 = 30$

Gap Method

[when particular objects are never together]

Example 114. There are 10 candidates for an examination out of which 4 are appearing in Mathematics and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two Mathematics candidates are together?

Sol. In this method first arrange the remaining candidates

Here, remaining candidates = 6

$\times 0 \times 0 \times 0 \times 0 \times 0 \times 0 \times$

$\times$: Places available for Mathematics candidates

0 : Places for others

Remaining candidates can be arranged in 6 ! ways. There are seven places available for Mathematics candidates so that no two Mathematics candidates are together. Now, four candidates can be placed in these seven places in ${}^7P_4$ ways.

${}^7C_4 \times 4!$

Hence, the total number of ways $= 6! \times {}^7P_4 = 720 \times 840$

$= 604800$

Example 115. In how many ways can 7 plus (+) and 5 minus (–) signs be arranged in a row so that no two minus (–) signs are together?

Sol. In this method, first arrange the plus (+) signs.

Here, minus (–) signs = 5

$0+0+0+0+0+0+0+0$

We can put minus (–) sign in any of the 8 places in the above arrangement i.e., we have to select 5 places out of 8 which can be done is ${}^8C_5$ ways $= {}^8C_3$ ways = 56 ways.

Example 116. Find the number of ways in which 5 girls and 5 boys can be arranged in a row, if no two boys are together.

Sol. In this example, there is no condition for arranging the girls. Now, 5 girls can be arranged in 5 ! ways.

$\times G \times G \times G \times G \times G \times$

When girls are arranged, six gaps are generated as shown above with '×'.

Now, boys must occupy the places with '×' marked, so that no two boys are together.

Therefore, five boys can be arranged in these six gaps in ${}^6P_5$ ways.

Hence, total number of arrangement is $5! \times {}^6P_5$.

Example 117. Find the number of ways in which 5 girls and 5 boys can be arranged in a row, if boys and girls are alternate.

Sol. First five girls can be arranged in 5 ! ways

i.e., $\times G \times G \times G \times G \times G$ or $G \times G \times G \times G \times G \times$

Now, if girls and boys are alternate, then boys can occupy places with '×' as shows above.

Hence, total number of arrangements is

$5! \times 5! + 5! \times 5! = 2 \times (5!)^2$

Use of Set Theory

A set is well defined collection of distinct objects.

Subset

If every element of a set A is also an element of a set B, then A is called the subset B, we write

$A \subset B \Leftrightarrow \{x \in A \Rightarrow x \in B\}$

Union

The union of two sets A and B is the set of all those elements which are either in A or in B or in both. This set is denoted by $A \cup B$ or $A + B$.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Intersection

The intersection of two sets A and B is the set of all elements which are common in A and B. This set is denoted by $A \cap B$ or AB.

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Example 118. A is a set containing n elements. A subset P_1 of A is chosen. The set A is reconstructed by replacing the elements of P_1. Next, a subset P_2 of A is chosen and again the set is reconstructed by replacing the elements of P_2. In this way m (> 1) subsets $P_1, P_2, \ldots, P_m$ of A are chosen. Find the number of ways of choosing $P_1, P_2, \ldots, P_m$, so that

(i) $P_1 \cap P_2 \cap P_3 \cap \ldots \cap P_m = \phi$

(ii) $P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_m = A$

Sol. Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$

(i) For each a_i $(1 \le i \le n)$, we have either $a \in P_j$ or $a_i \notin P_j$ $(1 \le j \le m)$. i.e., there are 2^m choices in which $a_i (1 \le i \le n)$ may belong to the P_j's.

Out of these, there is only one choice, in which $a_i \in P_j$ for all $j = 1, 2, \ldots, m$ which is not favourable for $P_1 \cap P_2 \cap P_3 \cap \ldots \cap P_m$ to be ϕ. Thus, $a_i \notin P_1 \cap P_2 \cap \ldots \cap P_m$ in $(2^m - 1)$ ways. Since, there are n elements in the set A, the total number of choices is $(2^m - 1)^n$.

(ii) There is exactly one choice, in which, $a_i \notin P_j$ for all $j = 1, 2, 3, \ldots, m$ which is not favourable for $P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_m$ to be equal to A. Thus, a_i can belong to $P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_m$ in $(2^m - 1)$ ways.

Since, there are n elements in the set A, the number of ways in which $P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_m$ can be equal to A is $(2^m - 1)^n$.

Example 119. A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset of A is again chosen. Find the number of ways of choosing P and Q, so that

(i) $P \cap Q$ contains exactly r elements.

(ii) $P \cap Q$ contains exactly 2 elements.

(iii) $P \cap Q = \phi$

Sol. Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$

(i) The r elements in P and Q such that $P \cap Q$ can be chosen out of n is ${}^nC_r$ ways a general element of A must satisfy one of the following possibilities [here, general element be $a_i (1 \le i \le n)$]

(i) $a_i \in P$ and $a_i \in Q$

(ii) $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \in P$ and $a_i \in Q$

(iv) $a_i \notin P$ and $a_i \notin Q$

Let $a_1, a_2, \ldots, a_r \in P \cap Q$

There is only one choice each of them (i.e., (i) choice) and three choices (ii), (iii) and (iv) for each of remaining $(n - r)$ elements.

Hence, number of ways of remaining elements $= 3^{n-r}$

Hence, number of ways in which $P \cap Q$ contains exactly r elements $= {}^nC_r \times 3^{n-r}$

(i) Put $r = 2$, then ${}^nC_2 \times 3^{n-2}$

(iii) Put $r = 0$, then ${}^nC_0 \times 3^n = 3^n$

Sum of digits

(i) The sum of the digits in the unit's place of all numbers formed with the help of $a_1, a_2, \ldots, a_n$ taken all at a time is $(n-1)!(a_1 + a_2 + \ldots + a_n)$ (repetition of digits not allowed)

Example 120. Find the sum of the digits in the unit's place of all numbers formed with the help of 3, 4, 5, 6 taken all at a time.

Sol. Sum of the digits in the unit's place

$= (4-1)!(3+4+5+6) = 6 \times 18 = 108$

(ii) The sum of all digit numbers that can be formed using the digits $a_1, a_2, \ldots, a_n$ (repetition of digits not allowed) is $= (n-1)!(a_1 + a_2 + \ldots + a_n)\dfrac{(10^n - 1)}{9}$

Example 121. Find the sum of all five digit numbers that can be formed using the digits 1, 2, 3, 4 and 5 (repetition of digits not allowed)

Sol. Required sum $= (5-1)!(1+2+3+4+5)\left(\dfrac{10^5 - 1}{9}\right)$

$= 24 \cdot 15 \cdot 11111 = 3999960$

Aliter

Since, one of the numbers formed with the 5 digits a, b, c, d and e is $10^4 a + 10^3 b + 10^2 c + 10d + e$;

Hence, $10^4 a$ will occur altogether in 4 ! ways similarly each of $10^4 b, 10^4 c, 10^4 d, 10^4 e$ will occur in 4 ! ways.

Hence, if all the numbers formed with the digits be written one below the other, thus

$10^4 \cdot a + 10^3 \cdot b + 10^2 \cdot c + 10 \cdot d + e$

$10^4 \cdot b + 10^3 \cdot c + 10^2 \cdot d + 10 \cdot e + a$

$10^4 \cdot c + 10^3 \cdot d + 10^2 \cdot e + 10 \cdot a + b$

$10^4 \cdot d + 10^3 \cdot e + 10^2 \cdot a + 10 \cdot b + c$

$10^4 \cdot e + 10^3 \cdot a + 10^2 \cdot b + 10 \cdot c + d$

Hence, the required sum

$= 4! \times (a + b + c + d + e) \times (10^4 + 10^3 + 10^2 + 10 + 1)$

$= 4! \times (1 + 2 + 3 + 4 + 5)(11111) = 3999960$

Difference between Permutation and Combination

Problems of permutations	Problems of combinations
1. Arrangements	Selections, choose
2. Standing in a line, seated in a row	Distributed group is formed
3. Problems on digits	Committee
4. Problems on letters from a word	Geometrical problems

Exercise for Session 7

1. The letters of the word "DELHI" are arranged in all possible ways as in a dictionary, the rank of the word "DELHI" is

(a) 4 (b) 5
(c) 6 (d) 7

2. The letters of the word "KANPUR" are arranged in all possible ways as in a dictionary, the rank of the word "KANPUR" from last is

(a) 121 (b) 122
(c) 598 (d) 599

3. The letters of the word "MUMBAI" are arranged in all possible ways as in a dictionary, the rank of the word "MUMBAI" is

(a) 297 (b) 295
(c) 299 (d) 301

4. The letters of the word "CHENNAI" are arranged in all possible ways as in a dictionary, then rank of the word "CHENNAI" from last is

(a) 2016 (b) 2017
(c) 2018 (d) 2019

5. If all permutations of the letters of the word "AGAIN" are arranged as in a dictionary, then 50th word is

(a) NAAGI (b) NAGAI
(c) NAAIG (d) NAIAG

Shortcuts and Important Results to Remember

1 When two dice are thrown, the number of ways of getting a total r (sum of numbers on upper faces), is

(i) $r-1$, if $2 \le r \le 7$
(ii) $13-r$, if $8 \le r \le 12$

2 When three dice are thrown, the number of ways of getting a total r (sum of numbers on upper faces), is

(i) ${}^{r-1}C_2$, if $3 \le r \le 8$
(ii) 25, if $r = 9$
(iii) 27, if $r = 10, 11$
(iv) 25, if $r = 12$
(v) ${}^{20-r}C_2$, if $13 \le r \le 18$

3 The product of k consecutive positive integers is divisible by $k!$.

4 Number of zeroes in $n! = E_5(n!)$

5 n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then, the number of parts into which these lines divides the plane is equal to $\dfrac{(n^2+n+2)}{2}$.

6 ${}^nC_r$ is divisible by n only, if n is a prime number $(1 \le r \le n-1)$.

7 The number of diagonals in n-gon (n sides closed polygon) is $\dfrac{n(n-3)}{2}$.

8 In n-gon no three diagonals are concurrent, then the total number of points of intersection of diagonals interior to the polygon is ${}^nC_4$.

9 Consider a polygon of n sides, then number of triangles in which no side is common with that of the polygon are $\dfrac{1}{6}n(n-4)(n-5)$.

10 If m parallel lines in a plane are intersected by a family of other n parallel lines. The total number of parallelograms so formed $= {}^mC_2 \cdot {}^nC_2 = \dfrac{mn(m-1)(n-1)}{4}$

11 Highest power of prime p in ${}^nC_r$, since

$$ {}^nC_r = \frac{n!}{r!(n-r)!} $$

If $H_p(n!) = \alpha$,
$H_p(r!) = \beta$
and $H_p\{(n-r)!\} = \gamma$
Then, $H_p({}^nC_r) = \alpha - (\beta + \gamma)$

12 Highest power of prime p in ${}^nP_r$, since

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

If $H_p(n!) = \lambda$, $H_p\{(n-r)!\} = \mu$. Then, $H_p({}^nP_r) = \lambda - \mu$

13 If there are n rows. Ist row has m_1 squares, IInd row has m_2 squares, IIIrd row has m_3 squares and so on. If we placed λ X's in the squares such that each row contains atleast one X. Then the number of ways = Coefficient of x^λ in

$({}^{m_1}C_1x + {}^{m_1}C_2x^2 + \ldots + {}^{m_1}C_{m_1}x^{m_1})$
$\times({}^{m_2}C_1x + {}^{m_2}C_2x^2 + {}^{m_2}C_3x^3 + \ldots + {}^{m_2}C_{m_2}x^{m_2}) \times$
$({}^{m_3}C_1x + {}^{m_3}C_2x^2 + \ldots + {}^{m_3}C_{m_3}x^{m_3}) \times \ldots$

14 If $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{n}$, $\forall\, x, y, n \in N$

$\Rightarrow \quad (x-n)(y-n) = n^2$

$\therefore \quad x = n + \lambda,$

$y = n + \dfrac{n^2}{\lambda},$

where λ is divisor of n^2.

Then, number of integral solutions (x, y) is equal to number of divisors of n^2.

If $n = 3$, $n^2 = 9 = 3^2$, the equation has 3 solutions.

$(x, y) = (4, 12), (6, 6), (12, 4)$

JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1 *Number of words of 4 letters that can be formed with the letters of the word IIT JEE, is*

(a) 42 (b) 82 (c) 102 (d) 142

Sol. (c) There are 6 letters I, I, E, E, T, J

The following cases arise:

Case I All letters are different

$$^4P_4 = 4! = 24$$

Case II Two alike and two different

$$^2C_1 \times {^3C_2} \times \frac{4!}{2!} = 72$$

Case III Two alike of one kind and two alike of another kind

$$^2C_2 \times \frac{4!}{2!\,2!} = 6$$

Hence, number of words $= 24 + 72 + 6 = 102$

Aliter

Number of words = Coefficient of x^4 in

$$4!\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)^2 (1 + x)^2$$

$= \text{Coefficient of } x^4 \text{ in } 6[(1 + x)^2 + 1]^2(1 + x)^2$

$= \text{Coefficient of } x^4 \text{ in } 6[(1 + x)^6 + 2(1 + x)^4 + (1 + x)^2]$

$= 6[{^6C_4} + 2\cdot{^4C_4} + 0] = 6(15 + 2) = 102$

• Ex. 2 *Let y be element of the set* $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ *and* x_1, x_2, x_3 *be integers such that* $x_1x_2x_3 = y$, *the number of positive integral solutions of* $x_1x_2x_3 = y$, *is*

(a) 27 (b) 64 (c) 81 (d) 256

Sol. (b) Number of solutions of the given equations is the same as the number of solutions of the equation

$$x_1\, x_2\, x_3\, x_4 = 30 = 2 \times 3 \times 5$$

Here, x_4 is infact a dummy variable.

If $x_1\, x_2\, x_3 = 15$, then $x_4 = 2$ and if $x_1x_2\, x_3 = 5$, then $x_4 = 6$, etc.

Thus, $\quad x_1\, x_2\, x_3\, x_4 = 2 \times 3 \times 5$

Each of 2, 3 and 5 will be factor of exactly one of x_1, x_2, x_3, x_4 in 4 ways.

$\therefore$ Required number $= 4^3 = 64$

• Ex. 3 *The number of positive integer solutions of* $a + b + c = 60$, *where a is a factor of b and c, is*

(a) 184 (b) 200 (c) 144 (d) 270

Sol. (c) $\because$ *a* is a factor of *b* and $c \Rightarrow a$ divides 60

$\therefore \quad a = 1, 2, 3, 4, 5, 6, 10, 12, 15, 30$

and $b = ma$, $c = na$, when $m, n \geq 1$

$\because \quad a + b + c = 60$

$\Rightarrow \quad a + ma + na = 60 \Rightarrow m + n = \left(\frac{60}{a} - 1\right)$ $\quad [\because a \neq 60]$

$\therefore$ Number of solutions $= {^{\frac{60}{a} - 1 - 1}C_{2-1}} = \left(\frac{60}{a} - 2\right)$

Hence, total number of solutions for all values of *a*

$$= 58 + 28 + 18 + 13 + 10 + 8 + 4 + 3 + 2 + 0 = 144$$

• Ex. 4 *The number of times the digit 3 will be written when listing the integers from 1 to 1000, is*

(a) 269 (b) 271 (c) 300 (d) 302

Sol. (c) Since, 3 does not occur in 1000. So, we have to count the number of times 3 occurs, when we list the integers from 1 to 999.

Any number between 1 and 999 is of the form *xyz*, where $0 \leq x, y, z \leq 9$.

Let us first count the number in which 3 occurs exactly once. Since, 3 can occur at one place in $^3C_1$ ways, there are $^3C_1 \times 9 \times 9 = 243$ such numbers. Next 3 can occur in exactly two places in $^3C_2 \times 9 = 27$ such numbers. Lastly, 3 can occur in all three digits in one number only. Hence, the number of times, 3 occurs is $1 \times 243 + 2 \times 27 + 3 \times 1 = 300$

• Ex. 5 *Number of points having position vector* $a\hat{i} + b\hat{j} + c\hat{k}$, *where* $a, b, c \in \{1, 2, 3, 4, 5\}$ *such that* $2^a + 3^b + 5^c$ *is divisible by 4, is*

(a) 70 (b) 140

(c) 210 (d) 280

Sol. (a) $\because 2^a + 3^b + 5^c = 2^a + (4 - 1)^b + (4 + 1)^c$

$$= 2^a + 4k + (-1)^b + (1)^c$$

$$= 2^a + 4k + (-1)^b + 1$$

I. $a = 1$, $b =$ even, $c =$ any number

II. $a \neq 1$, $b =$ odd, $c =$ any number

$\therefore$ Required number of ways $= 1 \times 2 \times 5 + 4 \times 3 \times 5 = 70$

[$\because$ even numbers = 2, 4; odd numbers = 1, 3, 5 and any numbers = 1, 2, 3, 4, 5]

• Ex. 6 *Number of positive unequal integral solutions of the equation* $x + y + z = 12$ *is*

(a) 21 (b) 42 (c) 63 (d) 84

Sol. (b) We have, $x + y + z = 12$ $\quad$ ...(i)

Assume $x < y < z$. Here, $x, y, z \geq 1$

$\therefore$ Solutions of Eq. (i) are

(1, 2, 9), (1, 3, 8), (1, 4, 7), (1, 5, 6), (2, 3, 7), (2, 4, 6) and (3, 4, 5).

Number of positive integral solutions of Eq. (i) = 7 but *x, y, z* can be arranged in $3! = 6$

Hence, required number of solutions $= 7 \times 6 = 42$

Aliter

Let $x = \alpha$, $y - x = \beta$, $z - y = \gamma$

$\therefore x = \alpha$, $y = \alpha + \beta$, $z = \alpha + \beta + \gamma$

From Eq. (i), $3\alpha + 2\beta + \gamma = 12$; $\alpha, \beta, \gamma \geq 1$

$\therefore$ Number of positive integral solutions of Eq. (i)

= Coefficient of λ^{12} in

$$(\lambda^3 + \lambda^6 + \lambda^9 + \lambda^{12} + \ldots)$$
$$(\lambda^2 + \lambda^4 + \lambda^6 + \lambda^8 + \lambda^{10} + \lambda^{12} + \ldots)$$
$$(\lambda + \lambda^2 + \lambda^3 + \ldots + \lambda^{12})$$

$= \text{Coefficient of } \lambda^6 \text{ in } (1 + \lambda^3 + \lambda^6)(1 + \lambda^2 + \lambda^4 + \lambda^6)$
$$(1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6)$$

$= \text{Coefficient of } \lambda^6 \text{ in } (1 + \lambda^2 + \lambda^4 + \lambda^6 + \lambda^3 + \lambda^5 + \lambda^6)$
$$\times (1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6)$$

$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$

but *x, y, z* can be arranged in $3! = 6$

Hence, required number of solutions $= 7 \times 6 = 42$

• Ex. 7 *12 boys and 2 girls are to be seated in a row such that there are atleast 3 boys between the 2 girls. The number of ways this can be done is* $\lambda \times 12!$, *the value of* λ *is*

(a) 55 (b) 110 (c) 20 (d) 45

Sol. (b) Let P = Number of ways, 12 boys and 2 girls are seated in a row

$$= 14! = 14 \times 13 \times 12! = 182 \times 12!$$

P_1 = Number of ways, the girls can sit together

$$= (14 - 2 + 1) \times 2! \times 12! = 26 \times 12!$$

P_2 = Number of ways, one boy sits between the girls

$$= (14 - 3 + 1) \times 2! \times 12! = 24 \times 12!$$

P_3 = Number of ways, two boys sit between the girls

$$= (14 - 4 + 1) \times 2! \times 12! = 22 \times 12!$$

$\therefore$ Required number of ways $= (182 - 26 - 24 - 22) \times 12!$

$= 110 \times 12! = \lambda \times 12!$ $\quad$ [given]

$\therefore \quad \lambda = 110$

• Ex. 8 *A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen, the number of ways of choosing so that* $(P \cup Q)$ *is a proper subset of A, is*

(a) 3^n (b) 4^n (c) $4^n - 2^n$ (d) $4^n - 3^n$

Sol. (d) Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$

a general element of *A* must satisfy one of the following possibilities.

[here, general element be $a_i (1 \leq i \leq n)$]

(i) $a_i \in P, a_i \in Q$ (ii) $a_i \in P, a_i \notin Q$

(iii) $a_i \notin P, a_i \in Q$ (iv) $a_i \notin P, a_i \notin Q$

Therefore, for one element a_i of *A*, we have four choices (i), (ii), (iii) and (iv).

$\therefore$ Total number of cases for all elements $= 4^n$

and for one element a_i of *A*, such that $a_i \in P \cup Q$, we have three choices (i), (ii) and (iii).

$\therefore$ Number of cases for all elements belong to $P \cup Q = 3^n$

Hence, number of ways in which atleast one element of *A* does not belong to

$$P \cup Q = 4^n - 3^n.$$

• Ex. 9 *Let N be a natural number. If its first digit (from the left) is deleted, it gets reduced to* $\frac{N}{29}$. *The sum of all the digits of N is*

(a) 14 (b) 17

(c) 23 (d) 29

Sol. (a) Let $N = a_n\, a_{n-1}\, a_{n-2} \ldots a_3\, a_2\, a_1\, a_0$

$$= a_0 + 10a_1 + 10^2a_2 + \ldots + 10^{n-1}a_{n-1} + 10^n a_n \quad \ldots\text{(i)}$$

Then, $\frac{N}{29} = a_{n-1}\, a_{n-2}\, a_{n-3} \ldots a_3\, a_2\, a_1\, a_0$

$$= a_0 + 10a_1 + 10^2a_2 + \ldots + 10^{n-2}a_{n-2} + 10^{n-1}a_{n-1}$$

or $\quad N = 29(a_0 + 10a_1 + 10^2a_2 + \ldots + 10^{n-2}a_{n-2} + 10^{n-1}a_{n-1}) \quad \ldots\text{(ii)}$

From Eqs. (i) and (ii), we get

$$10^n \cdot a_n = 28(a_0 + 10a_1 + 10^2a_2 + \ldots + 10^{n-1}a_{n-1})$$

$\Rightarrow$ 28 divides $10^n \cdot a_n \Rightarrow a_n = 7, n \geq 2 \Rightarrow 5^2 = a_0 + 10a_1$

The required *N* is 725 or 7250 or 72500, etc.

$\therefore$ The sum of the digits is 14.

• Ex. 10 *If the number of ways of selecting n cards out of unlimited number of cards bearing the number 0, 9, 3, so that they cannot be used to write the number 903 is 93, then n is equal to*

(a) 3 (b) 4

(c) 5 (d) 6

Sol. (c) We cannot write 903.

If in the selection of *n* cards, we get either (9 or 3), (9 or 0), (0 or 3), (only 0), (only 3) or (only 9).

For (9 or 3) can be selected $= 2 \times 2 \times 2 \times \ldots \times n$ factors $= 2^n$

Similarly, (9 or 0) or (0 or 3) can be selected $= 2^n$

In the above selection (only 0) or (only 3) or (only 9) is repeated twice.

$\therefore$ Total ways $= 2^n + 2^n + 2^n - 3 = 93$

$\Rightarrow \quad 3 \cdot 2^n = 96 \Rightarrow 2^n = 32 = 2^5$

$\therefore \quad n = 5$

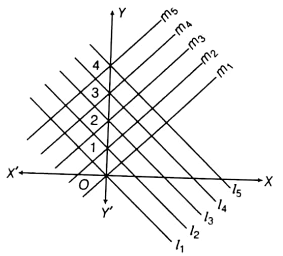
JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **more than one** may be correct.

● **Ex. 11** *In a plane, there are two families of lines $y = x + r, y = -x + r$, where $r \in \{0, 1, 2, 3, 4\}$. The number of squares of diagonals of the length 2 formed by the lines is*

(a) 9 (b) 16 (c) $\frac{3}{2}\cdot{}^4C_2$ (d) ${}^5C_2 + {}^3P_2$

Sol. (a, c) There are two sets of five parallel lines at equal distances. Clearly, lines like l_1, l_3, m_1 and m_3 form a square whose diagonal's length is 2.



$\therefore$ The number of required squares $= 3 \times 3 = 9 = \frac{3}{2}\cdot{}^4C_2$

[$\because$ choices are $(l_1, l_2), (l_2, l_4)$ and (l_3, l_5) for one set, etc.]

● **Ex. 12** *Number of ways in which three numbers in AP can be selected from $1, 2, 3, \ldots, n$, is*

(a) $\left(\frac{n-1}{2}\right)^2$, if n is even (b) $\frac{n(n-2)}{4}$, if n is even

(c) $\frac{(n-1)^2}{4}$, if n is odd (d) $\frac{n(n+1)}{2}$, if n is odd

Sol. (b, c) If a, b, c are in AP, then $a + c = 2b$

a and c both are odd or both are even.

***Case* I** If n is even

Let $n = 2m$ in which m are even and m are odd numbers.

$\therefore$ Number of ways $= {}^mC_2 + {}^mC_2 = 2\cdot{}^mC_2 = 2\cdot\frac{m(m-1)}{2}$

$= \frac{n}{2}\left(\frac{n}{2} - 1\right) = \frac{n(n-2)}{4}$ [$\because n = 2m$]

***Case* II** If n is odd

Let $n = 2m + 1$ in which m are even and $m + 1$ are odd numbers.

$\therefore$ Number of ways $= {}^mC_2 + {}^{m+1}C_2$

$= \frac{m(m-1)}{2} + \frac{(m+1)m}{2} = m^2 = \frac{(n-1)^2}{4}$ [$\because n = 2m + 1$]

● **Ex. 13** *If n objects are arranged in a row, then number of ways of selecting three of these objects so that no two of them are next to each other, is*

(a) ${}^{n-2}C_3$ (b) ${}^{n-3}C_3 + {}^{n-3}C_2$

(c) $\frac{(n-2)(n-3)(n-4)}{6}$ (d) ${}^nC_2$

Sol. (a, b, c) Let a_0 be the number of objects to the left of the first object chosen, a_1 be the number of objects between the first and the second, a_2 be the number of objects between the second and the third and a_3 be the number of objects to the right of the third object. Then,

$$a_0, a_3 \geq 0 \text{ and } a_1, a_2 \geq 1$$

—a_0—•—a_1—•—a_2—•—a_3—

also $a_0 + a_1 + a_2 + a_3 = n - 3$

Let $a = a_0 + 1, b = a_3 + 1$, then $a \geq 1, b \geq 1$ such that $a + a_1 + a_2 + b = n - 1$

The total number of positive integral solutions of this equation is ${}^{n-1-1}C_{4-1} = {}^{n-2}C_3 = {}^{n-3}C_3 + {}^{n-3}C_2$

$$= \frac{(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3}$$

● **Ex. 14** *Given that the divisors of $n = 3^p\cdot 5^q\cdot 7^r$ are of the form $4\lambda + 1, \lambda \geq 0$. Then,*

(a) $p + r$ is always even (b) $p + q + r$ is even or odd

(c) q can be any integer (d) if p is even, then r is odd

Sol. (a, b, c)

$\because$ $3^p = (4-1)^p = 4\lambda_1 + (-1)^p,$

$5^q = (4+1)^q = 4\lambda_2 + 1$

and $7^r = (8-1)^r = 8\lambda_3 + (-1)^r$

Hence, both p and r must be odd or both must be even. Thus, $p + r$ is always even. Also, $p + q + r$ can be odd or even.

● **Ex. 15** *Number of ways in which 15 identical coins can be put into 6 different bags*

(a) is coefficient of x^{15} in $x^6(1 + x + x^2 + \ldots \infty)^6$, if no bag remains empty

(b) is coefficient of x^{15} in $(1 - x)^{-6}$

(c) is same as number of the integral solutions of $a + b + c + d + e + f = 15$

(d) is same as number of non-negative integral solutions of $\sum_{i=1}^{6} x_i = 15$

Sol. (a, b, d) Let bags be x_1, x_2, x_3, x_4, x_5 and x_6, then $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$.

$\therefore$ For no bags remains empty, number of ways

$= \text{Coefficient of } x^{15} \text{ in } (x^1 + x^2 + x^3 + \ldots \infty)^6$

$= \text{Coefficient of } x^{15} \text{ in } x^6(1 + x + x^2 + \ldots \infty)^6$

$= \text{Coefficient of } x^9 \text{ in } (1 - x)^{-6}$

In option (c), it is not mentioned that solution is positive integral.

JEE Type Solved Examples : Passage Based Questions

- This section contains **3 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

All the letters of the word 'AGAIN' be arranged and the words thus formed are known as 'Simple Words'. Further two new types of words are defined as follows:

(i) ***Smart word:*** *All the letters of the word 'AGAIN' are being used, but vowels can be repeated as many times as we need.*

(ii) ***Dull word:*** *All the letters of the word 'AGAIN' are being used, but consonants can be repeated as many times as we need.*

16. If a vowel appears in between two similar letters, the number of simple words is

(a) 12 (b) 6 (c) 36 (d) 14

17. Number of 7 letter smart words is

(a) 1500 (b) 1050 (c) 1005 (d) 150

18. Number of 7 letter dull words in which no two vowels are together, is

(a) 402 (b) 420 (c) 840 (d) 42

Sol.

16. (b)

N	A-I-A	G

$\therefore$ Required number of simple words $= 3! = 6$

17. (b)

A	G	A	I	N	A	A
					I	I
					A	I

$\therefore$ Number of 7 letter smart words

$= \frac{7!}{4!} + \frac{7!}{2!\,3!} + \frac{7!}{3!\,2!} = 210 + 420 + 420 = 1050$

18. (b) Now, 3 vowels A, I, A are to be placed in the five available places.

× N× G× N× N×

OR

× N× G× G× G×

OR

× N× G× G× N×

Hence, required number of ways

$= {}^5C_3 \times \frac{3!}{2!} \times \left\{\frac{4!}{3!} + \frac{4!}{3!} + \frac{4!}{2!\,2!}\right\}$

$= 30(4 + 4 + 6) = 420$

Passage II (Ex. Nos. 19 to 21)

Consider a polygon of sides 'n' which satisfies the equation $3\cdot{}^nP_4 = {}^{n-1}P_5$.

19. Rajdhani express travelling from Delhi to Mumbai has n stations enroute. Number of ways in which a train can be stopped at 3 stations if no two of the stopping stations are consecutive, is

(a) 20 (b) 35 (c) 56 (d) 84

20. Number of quadrilaterals that can be formed using the vertices of a polygon of sides 'n' if exactly 1 side of the quadrilateral is common with side of the n-gon, is

(a) 96 (b) 100 (c) 150 (d) 156

21. Number of quadrilaterals that can be made using the vertices of the polygon of sides 'n' if exactly two adjacent sides of the quadrilateral are common to the sides of the n-gon, is

(a) 50 (b) 60 (c) 70 (d) 80

Sol. $\because$ $3\cdot{}^nP_4 = {}^{n-1}P_5$

It is clear that $n \geq 6$.

$\therefore$ $3\cdot n(n-1)(n-2)(n-3) = (n-1)(n-2)(n-3)(n-4)(n-5)$

$\Rightarrow$ $(n-1)(n-2)(n-3)(n^2 - 12n + 20) = 0$

$\Rightarrow$ $(n-1)(n-2)(n-3)(n-10)(n-2) = 0$

$\therefore$ $n = 10, n \neq 1, 2, 3$ [$\because n \geq 6$]

$\Rightarrow$ $n = 10$

19. (d) Let a_0 be the number of stations to the left of the station I chosen, a_1 be the number of stations between the station I and station II, a_2 be the number of stations between the station II and station III and a_3 be the number of stations to the right of the third station. Then,

$$a_0, a_3 \geq 0 \text{ and } a_1, a_2 \geq 1$$

Also, $a_0 + a_1 + a_2 + a_3 = n + 1 - 3$

Let $a = a_0 + 1, b = a_3 + 1$, then $a, b \geq 1$ such that $a + a_1 + a_2 + b = n$

$\therefore$ Required number of ways $= {}^{n-1}C_{4-1} = {}^9C_3$ [here, $n = 10$]

$= 84$

20. (c) Number of quadrilaterals of which exactly one side is the side of the n-gon

$= n \times {}^{n-4}C_2 = 10 \times {}^6C_2 = 150$ [$\because n = 10$]

21. (a) Number of quadrilaterals of which exactly two adjacent sides of the quadrilateral are common to the sides of the n-gon

$$= n \times {}^{n-5}C_1 = n(n-5) = 10 \times 5 \quad [\because n = 10]$$
$$= 50$$

Passage III
(Ex. Nos. 22 to 23)

Consider the number $N = 2016$.

22. Number of cyphers at the end of ${}^{N}C_{N/2}$ is

(a) 0 (b) 1
(c) 2 (d) 3

23. Sum of all even divisors of the number N is

(a) 6552 (b) 6448
(c) 6048 (d) 5733

Sol.

22. (c) $\because$ ${}^{N}C_{N/2} = {}^{2016}C_{1008} = \dfrac{(2016)!}{[(1008)!]^2}$

$$E_5(2016!) = \left[\frac{2016}{5}\right] + \left[\frac{2016}{5^2}\right] + \left[\frac{2016}{5^3}\right] + \left[\frac{2016}{5^4}\right]$$
$$= 403 + 80 + 16 + 3 = 502$$

and $E_5(1008!) = \left[\dfrac{1008}{5}\right] + \left[\dfrac{1008}{5^2}\right] + \left[\dfrac{1008}{5^3}\right] + \left[\dfrac{1008}{5^4}\right]$
$$= 201 + 40 + 8 + 1 = 250$$

Hence, the number of cyphers at the end of ${}^{2016}C_{1008}$
$$= 502 - 250 - 250 = 2$$

23. (b) $\because$ $N = 2016 = 2^5 \cdot 3^2 \cdot 7^1$

$\therefore$ Sum of all even divisors of the number N
$$= (2 + 2^2 + 2^3 + 2^4 + 2^5)(1 + 3 + 3^2)(1 + 7^1) = 6448$$

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0 to 9** (both inclusive).

● ***Ex. 24*** *If* $\dbinom{18}{r-2} + 2\dbinom{18}{r-1} + \dbinom{18}{r} \ge \dbinom{20}{13}$, *then the number of values of r are*

Sol. (7) We have, $\dbinom{18}{r-2} + 2\dbinom{18}{r-1} + \dbinom{18}{r} \ge \dbinom{20}{13}$

It means that ${}^{18}C_{r-2} + 2 \cdot {}^{18}C_{r-1} + {}^{18}C_r \ge {}^{20}C_{13}$

$\Rightarrow$ $({}^{18}C_{r-2} + {}^{18}C_{r-1}) + ({}^{18}C_{r-1} + {}^{18}C_r) \ge {}^{20}C_7$

$\Rightarrow$ ${}^{19}C_{r-1} + {}^{19}C_r \ge {}^{20}C_7$

$\Rightarrow$ ${}^{20}C_r \ge {}^{20}C_7$

or ${}^{20}C_r \ge {}^{20}C_{13}$

$\Rightarrow$ $7 \le r \le 13$

$\therefore$ $r = 7, 8, 9, 10, 11, 12, 13$

Hence, the number of values of r are 7.

● ***Ex. 25*** *If* λ *be the number of 3-digit numbers are of the form xyz with* $x < y$, $z < y$ *and* $x \ne 0$, *the value of* $\dfrac{\lambda}{30}$ *is*

Sol. (8) Since, $x \ge 1$, then $y \ge 2$ $[\because x < y]$

If $y = n$, then x takes values form 1 to $n - 1$ and z can take the values from 0 to $n - 1$ (i.e., n values).

Thus, for each values of $y (2 \le y \le 9)$, x and z take $n(n-1)$ values.

Hence, the 3-digit numbers are of the form xyz

$$= \sum_{n=2}^{9} n(n-1) = \sum_{n=1}^{9} n(n-1) \quad [\because \text{at } n = 1,\ n(n-1) = 0]$$
$$= \sum_{n=1}^{9} n^2 - \sum_{n=1}^{9} n$$
$$= \frac{9(9+1)(18+1)}{6} - \frac{9(9+1)}{2}$$
$$= 285 - 45$$
$$= 240 = \lambda \quad \text{[given]}$$

$\therefore$ $\dfrac{\lambda}{30} = 8$

JEE Type Solved Examples : Matching Type Questions

- This section contains **2 examples.** Examples 26 and 27 have four statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 26***

	Column I		Column II
(A)	The sum of the factors of 8! which are odd and are the form $3\lambda + 2$, $\lambda \in N$, is	(p)	384
(B)	The number of divisors of $n = 2^7 \cdot 3^5 \cdot 5^3$ which are the form $4\lambda + 1$, $\lambda \in N$, is	(q)	240
(C)	Total number of divisors of $n = 2^5 \cdot 3^4 \cdot 5^{10} \cdot 7^6$ which are the form $4\lambda + 2$, $\lambda \ge 1$, is	(r)	11
(D)	Total number of divisors of $n = 3^5 \cdot 5^7 \cdot 7^9$ which are the form $4\lambda + 1$, $\lambda \ge 0$, is	(s)	40

Sol. (A) $\to$ s; (B) $\to$ r; (C) $\to$ p; (D) $\to$ q

(A) Here, $8! = 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1$

So, the factors may be 1, 5, 7, 35 of which 5 and 35 are of the form $3\lambda + 2$.

$\therefore$ Sum is 40.

(B) Number of odd numbers $= (5 + 1)(3 + 1) = 24$

Required number = 12, but 1 is included.

$\therefore$ Required number of numbers $= 12 - 1 = 11$ of the form $4\lambda + 1$.

(C) Here, $4\lambda + 2 = 2(2\lambda + 1)$

$\therefore$ Total divisors $= 1 \cdot 5 \cdot 11 \cdot 7 - 1 = 384$

[$\because$ one is subtracted because there will be case when selected powers of 3, 5 and 7 are zero]

(D) Here, any positive integer power of 5 will be in the form of $4\lambda + 1$ when even powers of 3 and 7 will be in the form of $4\lambda + 1$ and odd powers of 3 and 7 will be in the form of $4\lambda - 1$.

$\therefore$ Required divisors $= 8(3 \cdot 5 + 3 \cdot 5) = 240$

● ***Ex. 27***

	Column I		Column II
(A)	Four dice (six faced) are rolled. The number of possible outcomes in which atleast one die shows 2, is	(p)	210
(B)	Let A be the set of 4-digit numbers $a_1a_2a_3a_4$, where $a_1 > a_2 > a_3 > a_4$. Then, $n(A)$ is equal to	(q)	480
(C)	The total number 3-digit numbers, the sum of whose digits is even, is equal to	(r)	671
(D)	The number of 4-digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, so that each number contains digit 1, is	(s)	450

Sol. (A) $\to$ r; (B) $\to$ p; (C) $\to$ s; (D) $\to$ q

(A) The number of possible outcomes with 2 on atleast one die
= The total number of outcomes with 2 on atleast one die
= (The total number of outcomes) – (The number of outcomes in which 2 does not appear on any dice)
$= 6^4 - 5^4 = 1296 - 625 = 671$

(B) Any selection of four digits from the 10 digits 0, 1, 2, 3,..., 9 gives one number. So, the required number of numbers is ${}^{10}C_4$ i.e., 210.

(C) Let the number be $n = pqr$. Since, $p + q + r$ is even, p can be filled in 9 ways and q can be filled in 10 ways.
r can be filled in number of ways depending upon what is the sum of p and q.
If $(p + q)$ is odd, then r can be filled with any one of five odd digits.
If $(p + q)$ is even, then r can be filled with any one of five even digits.
In any case, r can be filled in five ways.
Hence, total number of numbers is $9 \times 10 \times 5 = 450$

(D) After fixing 1 at one position out of 4 places, 3 places can be filled by ${}^7P_3$ ways. But for some numbers whose fourth digit is zero, such type of ways is ${}^6P_2$. Therefore, total number of ways is ${}^7P_3 - {}^6P_2 = 480$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** Example numbers 28 and 29 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

• ***Ex. 28*** **Statement-1** *Number of rectangles on a chess-board is* ${}^8C_2 \times {}^8C_2$.

Statement-2 *To form a rectangle, we have to select any two of the horizontal lines and any two of the vertical lines.*

Sol. (*d*) In a chessboard, there are 9 horizontal lines and 9 vertical lines.

$\therefore$ Number of rectangles of any size are ${}^9C_2 \times {}^9C_2$.

Hence, Statement-1 is false and Statement-2 is true.

• ***Ex. 29*** **Statement-1** *If* $f:\{a_1, a_2, a_3, a_4, a_5\} \to \{a_1, a_2, a_3, a_4, a_5\}$, *$f$ is onto and $f(x) \neq x$ for each $x \in \{a_1, a_2, a_3, a_4, a_5\}$, is equal to 44.*

• **Statement-2** *The number of derangement for n objects is* $n!\sum_{r=0}^{n} \frac{(-1)^r}{r!}$.

Sol. (*a*) $\because D_n = n!\sum_{r=0}^{n}\frac{(-1)^r}{r!} = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)$

$$\therefore \quad D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$
$$= 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$
$$= 6 - 20 + 5 - 1$$
$$= 65 - 21$$
$$= 44$$

Hence, Statement-1 is true, Statement-2 is true and Statement-2 is a correct explanation for Statement-1.

99. 6 balls marked as 1, 2, 3, 4, 5 and 6 are kept in a box. Two players A and B start to take out 1 ball at a time from the box one after another without replacing the ball till the game is over. The number marked on the ball is added each time to the previous sum to get the sum of numbers marked on the balls taken out.

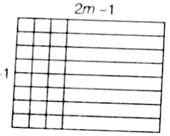
If this sum is even, then 1 point is given to the player. The first player to get 2 points is declared winner. At the start of the game, the sum is 0. If A starts to take out the ball, find the number of ways in which the game can be won.

Permutations and Combinations Exercise 8 :

Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

100. There is a rectangular sheet of dimension $(2m-1)\times(2n-1)$, (where $m>0, n>0$). It has been divided into square of unit area by drawing line perpendicular to the sides. Find the number of rectangles having sides of odd unit length. **[IIT-JEE 2005, 3M]**



(a) $(m+n+1)^2$ (b) $mn(m+1)(n+1)$
(c) 4^{m+n-2} (d) m^2n^2

101. If the letters of the word SACHIN arranged in all possible ways and these words are written out as in dictionary, then the word SACHIN appears at series number **[AIEEE 2005, 3M]**

(a) 603 (b) 602 (c) 601 (d) 600

102. If r, s, t are prime numbers and p, q are the positive integers such that LCM of p, q is $r^2t^4s^2$, then the number of ordered pair (p, q) is **[IIT-JEE 2006, 3M]**

(a) 252 (b) 254 (c) 225 (d) 224

103. At an election, a voter may vote for any number of candidates, not greater than number to be elected. There are 10 candidates and 4 are to be selected. If a voter votes for atleast one candidate, then number of ways in which he can vote, is **[AIEEE 2006, 4.5M]**

(a) 5040 (b) 6210
(c) 385 (d) 1110

104. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in English dictionary. The number of words that appear before the word COHIN is **[IIT-JEE 2007, 3M]**

(a) 360 (b) 192
(c) 96 (d) 48

105. The set $S=\{1, 2, 3, \ldots, 12\}$ to be partitioned into three sets A, B, C of equal size. Thus, $A\cup B\cup C=S$, $A\cap B=B\cap C=A\cap C=\phi$. The number of ways to partition S, is **[AIEEE 2007, 3M]**

(a) $\dfrac{12!}{3!(4!)^3}$ (b) $\dfrac{12!}{3!(3!)^4}$
(c) $\dfrac{12!}{(4!)^3}$ (d) $\dfrac{12!}{(3!)^4}$

106. Consider all possible permutations of the letters of the word ENDEANOEL. Match the statements/ expressions in **Column I** with the statements/ expressions in **Column II**. **[IIT-JEE 2008, 6M]**

	Column I		Column II
(A)	The number of permutations containing the word ENDEA is	(p)	5!
(B)	The number of permutations in which the letters E occurs in the first and the last positions, is	(q)	2×5!
(C)	The number of permutations in which none of the letters D, L, N occurs in the last five positions, is	(r)	7×5!
(D)	The number of permutations in which the letters A, E, O occur only in odd positions, is	(s)	21×5!

107. How many different words can be formed by Jumbling the letters in the word MISSISSIPPI in which no two S's are adjacent? **[AIEEE 2008, 3M]**

(a) $6\cdot 7\cdot {}^8C_4$ (b) $6\cdot 8\cdot {}^7C_4$
(c) $7\cdot {}^6C_4\cdot {}^8C_4$ (d) $8\cdot {}^6C_4\cdot {}^7C_4$

108. In a shop, there are five types of ice-creams available. A child buys six ice-creams. **[AIEEE 2008, 3M]**

Statement-1 The number of different ways the child can buy the six ice-creams is ${}^{10}C_4$.

Statement-2 The number of different ways the child can buy six ice-creams is equal to the number of different ways to arranging 6A's and 4B's in a row. **[AIEEE 2008, 3M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

109. The number of 7-digit integers, with sum of the digits equal to10 and formed by using the digits 1, 2 and 3 only, is **[IIT-JEE 2009, 3M]**

(a) 55 (b) 66 (c) 77 (d) 88

110. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then, the number of such arrangements is **[AIEEE 2009, 4M]**

(a) atleast 1000
(b) less than 500
(c) atleast 500 but less than 750
(d) atleast 750 but less than 1000

111. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn, two balls are taken out at random and then transferred to the other. The number of ways in which this can be done, is **[AIEEE 2010, 4M]**

(a) 36 (b) 66 (c) 108 (d) 3

112. **Statement-1** The number of ways distributing 10 identical balls in 4 distinct boxes such that no box is empty, is ${}^9C_3$.

Statement-2 The number of ways of choosing any 3 places from 9 different places is ${}^9C_3$ **[AIEEE 2011, 4M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

113. There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then **[AIEEE 2011, 4M]**

(a) $N>190$ (b) $N\le 100$
(c) $100<N\le 140$ (d) $140<N\le 190$

114. The total number of ways in which 5 balls of different colours can be distributed among 3 persons, so that each person gets atleast one ball is **[IIT-JEE 2012, 3M]**

(a) 75 (b) 150 (c) 210 (d) 243

- **Directions** (Q. Nos. 115 to 116) Let a_n denotes the number of all n-digits positive integer formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n be the number of such n-digit integers ending with digit 1 and c_n be the number of such n digits integers ending with digit 0.

115. The value of b_6, is **[IIT-JEE 2012, 3M]**

(a) 7 (b) 8 (c) 9 (d) 11

116. Which of the following is correct?

(a) $a_{17}=a_{16}+a_{15}$ (b) $c_{17}\ne c_{16}+c_{15}$
(c) $b_{17}\ne b_{16}+c_{16}$ (d) $a_{17}=c_{17}+b_{16}$

117. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls, is **[AIEEE 2012, 4M]**

(a) 630 (b) 879 (c) 880 (d) 629

118. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1}-T_n=10$, the value of n is **[JEE Main 2013, 4M]**

(a) 5 (b) 10 (c) 8 (d) 7

119. Consider the set of eight vectors $V=[a\hat{i}+b\hat{j}+c\hat{k}: a, b, c\in\{-1, 1\}]$. Three non-coplanar vectors can be chosen from V in 2^p ways, then p is **[JEE Advanced 2013, 4M]**

120. Let $n_1<n_2<n_3<n_4<n_5$ be positive integers such that $n_1+n_2+n_3+n_4+n_5=20$, the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is **[JEE Advanced 2014, 3M]**

121. For $n\ge 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, the value of n is **[JEE Advanced 2014, 3M]**

122. Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes, so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 is always placed in envelope numbered 2. Then the number of ways it can be done, is **[JEE Advanced 2014, 3M]**

(a) 264 (b) 265
(c) 53 (d) 67

123. The number of integers greater than 6000 that can be formed using the digits 3,5,6,7 and 8 without repetition, is **[JEE Main 2015, 4M]**

(a) 120 (b) 72
(c) 216 (d) 192

124. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such away that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue, the value of $\dfrac{m}{n}$ is **[JEE Advanced 2015, 3M]**

125. If all the words (with or without meaning having five letters, formed using the letters of the word SMALL and arranged as in a dictionary, then the position of the word SMALL is **[JEE Main 2016, 4M]**

(a) 59th (b) 52nd (c) 58th (d) 46th

126. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is **[JEE Advanced 2016, 3M]**

(a) 380 (b) 320
(c) 260 (d) 95

127. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is **[JEE Main 2017, 4M]**

(a) 484 (b) 485
(c) 468 (d) 469

Answers

Exercise for Session 1

1. (c)	2. (b)	3. (c)	4. (a)	5. (c)	6. (b)
7. (c)	8. (c)	9. (b)	10. (b)	11. (d)	

Exercise for Session 2

1. (b)	2. (c)	3. (c)	4. (d)	5. (d)	6. (b)
7. (d)	8. (a)	9. (b)	10. (c)	11. (b)	12. (b)
13. (b)	14. (b)	15. (b)	16. (d)		

Exercise for Session 3

1. (a)	2. (c)	3. (a)	4. (b)	5. (c)	6. (c)
7. (b)	8. (b)	9. (a)	10. (a)	11. (a)	

Exercise for Session 4

1. (d)	2. (b)	3. (d)	4. (c)	5. (a)	6. (c)
7. (a)	8. (b)	9. (b)	10. (d)	11. (a)	12. (d)
13. (a)	14. (d)	15. (d)	16. (c)	17. (b)	18. (a)

Exercise for Session 5

1. (c)	2. (c)	3. (d)	4. (a)	5. (b)	6. (b)
7. (d)	8. (a)	9. (c)	10. (c)	11. (d)	12. (a)
13. (b)	14. (c)	15. (d)	16. (c)	17. (b)	18. (c)

Exercise for Session 6

1. (c)	2. (c)	3. (a)	4. (a)	5. (b)	6. (a)
7. (c)	8. (c)	9. (c)	10. (a)	11. (d)	12. (b)
13. (a)	14. (c)	15. (b)	16. (b)	17. (c)	18. (d)
19. (c)	20. (d)	21. (c)	22. (b)	23. (a)	24. (a)
25. (b)	26. (c)	27. (c)	28. (b)		

Exercise for Session 7

1. (b)	2. (d)	3. (a)	4. (c)	5. (c)

Chapter Exercises

1. (c) 2. (a) 3. (c) 4. (b) 5. (c) 6. (c)
7. (a) 8. (b) 9. (b) 10. (d) 11. (a) 12. (d)
13. (b) 14. (b) 15. (c) 16. (a) 17. (d) 18. (c)
19. (d) 20. (b) 21. (c) 22. (d) 23. (d) 24. (d)
25. (c) 26. (a) 27. (d) 28. (a) 29. (c) 30. (b)
31. (a, c) 32. (a,b,c,d) 33. (a,b,c,d) 34. (a,d) 35. (a, c)
36. (b,d) 37. (a, c) 38. (b,c) 39. (a,b,c,d) 40. (b,c,d)
41. (a) 42. (c) 43. (c) 44. (c) 45. (b) 46. (a)
47. (c) 48. (c) 49. (c) 50. (c) 51. (c) 52. (b)
53. (b) 54. (c) 55. (d) 56. (1) 57. (0) 58. (8)
59. (3) 60. (7) 61. (8) 62. (6) 63. (7) 64. (8)
65. (8) 66. (A) $\to$ (q); (B) $\to$ (r); (C) $\to$ (s); (D) $\to$ (p)
67. (A) $\to$ (r); (B) $\to$ (s); (C) $\to$ (q); (D) $\to$ (p)
68. (A) $\to$ (r); (B) $\to$ (s); (C) $\to$ (p); (D) $\to$ (q)
69. (A) $\to$ (s); (B) $\to$ (r); (C) $\to$ (s); (D) $\to$ (p)
70. (A) $\to$ (s); (B) $\to$ (q); (C) $\to$ (p); (D) $\to$ (s)
71. (b) 72. (c) 73. (d) 74. (c) 75. (d) 76. (b)
77. (b) 78. (d) 79. (a) 80. (d) 81. (d) 82. (a)
83. $x = n + 3$ 84. $x = 3$ 85. 4 87. 60 88. 42
89. 11520 90. ${}^8P_4 \times {}^8P_2 \times 10!$ 92. ${}^{45}C_6$
93. $\dfrac{n(n-4)(n-5)}{6}$ 94. 1728 95. $\dfrac{(3n)!}{6(n!)^3}$ 96. $\dfrac{n^2(n+1)}{2}$
97. 4530 99. 96 100. (d) 101. (c) 102. (d)
103. (c) 104. (c) 105. (c)
106. (A) $\to$ (p); (B) $\to$ (s); (C) $\to$ (q); (D) $\to$ (q)
107. (c) 108. (a) 109. (c) 110. (a) 111. (c) 112. (a)
113. (b) 114. (b) 115. (b) 116. (a) 117. (b) 118. (a)
119. (5) 120. (7) 121. (5) 122. (c) 123. (d) 124. (5)
125. (c) 126. (a) 127. (b)