CHAPTER

07

Hyperbola

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- The Foci and Two Directrices of a Hyperbola
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Session 1

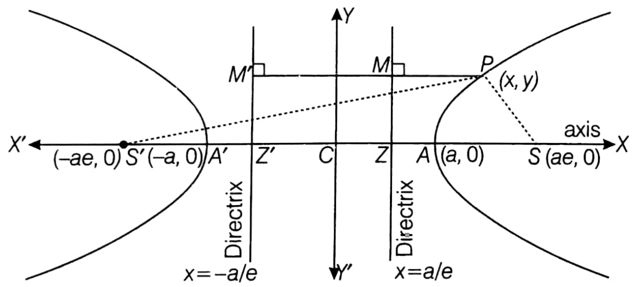
Hyperbola : Definition, Standard Equation of Hyperbola, The Foci and Two Directrices of a Hyperbola, Tracing of the Hyperbola, Focal Distances of a Point, Conjugate Hyperbola, Position of a Point with Respect to a Hyperbola, Intersection of a Line and a Hyperbola

Hyperbola : Definition

The locus of a point which moves in a plane such that its distance from a fixed point (i.e. focus) is e times its distance from a fixed line (i.e. directrix) is known as hyperbola. For hyperbola $e > 1$.

Standard Equation of Hyperbola

Let S be the focus and ZM the directrix of the hyperbola. Draw $SZ \perp ZM$. Divide SZ internally and externally in the ratio $e:1 (e > 1)$ and let A and A' be their internal and external points of division.



then $\quad SA = e\,AZ \quad$ ...(i)

and $\quad SA' = e\,A'Z \quad$ ...(ii)

Clearly A and A' will lie on the hyperbola. Let $AA' = 2a$ and take C the mid-point of AA' as origin

$\therefore \quad CA = CA' = a$

Let $P(x, y)$ be any point on the hyperbola and CA as X-axis, the line through C perpendicular to CA as Y-axis.

Then adding Eqs. (i) and (ii)

$\therefore \quad SA + SA' = e(AZ + A'Z)$

$\Rightarrow \quad CS - CA + CS + CA' = e(AA')$

$\Rightarrow \quad 2CS = e(2a) \quad (\because CA = CA')$

$\therefore \quad \boxed{CS = ae}$

$\therefore$ The focus S is $(CS, 0)$ i.e. $(ae, 0)$ and subtracting Eq. (i) from Eq. (ii), then

$$SA' - SA = e(A'Z - AZ)$$

$$AA' = e[(CA' + CZ) - (CA - CZ)]$$

$\Rightarrow \quad AA' = e(2CZ) \quad (\because CA = CA')$

$\Rightarrow \quad 2a = e(2CZ)$

$\therefore \quad CZ = a/e$

$\therefore$ The directrix MZ is $x = CZ = a/e$

or $\quad x - a/e = 0 \quad \left(\because e > 1, \therefore \frac{a}{e} < 1\right)$

Now, draw $PM \perp MZ$,

$\therefore \quad \frac{SP}{PM} = e$ or $(SP)^2 = e^2 (PM)^2$

or $\quad (x - ae)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2$

or $\quad (x - ae)^2 + y^2 = (ex - a)^2$

$\Rightarrow \quad x^2 + a^2e^2 - 2aex + y^2 = e^2x^2 - 2aex + a^2$

$\Rightarrow \quad x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$

$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$

or $\quad \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ where, $\boxed{b^2 = a^2(e^2 - 1)}$

This is the standard equation of the hyperbola.

Generally : The equation of the hyperbola whose focus is the point (h, k) and directrix is $lx + my + n = 0$ and whose eccentricity is e, is

$$(x - h)^2 + (y - k)^2 = e^2 \frac{(lx + my + n)^2}{(l^2 + m^2)}$$

The Foci and Two Directrices of a Hyperbola

On the negative side of origin take a point S' which is such that

$$CS = CS' = ae$$

and another point Z', then $CZ = CZ' = \frac{a}{e}$

$\therefore$ Coordinates of S' are $(-ae, 0)$ and equation of second directrix (i.e. $Z'M'$) is

$$x = -\frac{a}{e}$$

Let $P(x, y)$ be any point on the hyperbola, then

$$S'P = ePM' \text{ or } (S'P)^2 = e^2 (PM')^2$$

or $$(x + ae)^2 + (y - 0)^2 = e^2 \left(x + \frac{a}{e}\right)^2$$

or $$(x + ae)^2 + y^2 = (ex + a)^2$$

or $$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2$$

or $$x^2 (e^2 - 1) - y^2 = a^2(e^2 - 1)$$

or $$\frac{x^2}{a^2} - \frac{y^2}{a^2 (e^2 - 1)} = 1$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where, } b^2 = a^2 (e^2 - 1)$$

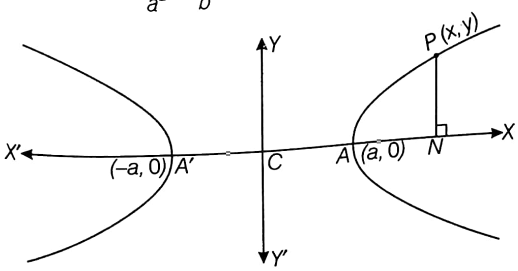
The equation being the same as that of hyperbola when $S(ae, 0)$ is focus and MZ i.e. $x = \frac{a}{e}$ is directrix.

Hence, coordinates of foci are $(\pm ae, 0)$ and equations of directrices are

$$x = \pm a / e$$

Remarks

1. Distance between foci $SS' = 2ae$ and distance between directrices $ZZ' = 2a/e$
2. Two hyperbolas are said to be similar if they have the same value of eccentricity.
3. Since, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$\Rightarrow$ $$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\frac{y^2}{b^2} = \frac{(x - a)(x + a)}{a^2} \quad \text{...(i)}$$

$\because$ $AN = CN - CA = (x - a)$

and $A'N = CN + CA' = (x + a)$

and $PN = y$

$\therefore$ From Eq. (i), $\frac{(PN)^2}{AN \cdot A'N} = \frac{b^2}{a^2}$

Tracing of the Hyperbola

Equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

(i) Since only even powers of x and y occur in this equation so it (hyperbola) is symmetrical about the both axes.

(ii) The hyperbola (i) does not cut Y-axis in real points where as it cuts X-axis at $(a, 0)$ and $(-a, 0)$.

(iii) The Eq. (i) may be written as

$$y = \pm \frac{b}{a} \sqrt{(x^2 - a^2)}$$

If follows that $x^2 - a^2 \geq 0$

$\therefore$ $x^2 \geq a^2$

$\Rightarrow$ $x \leq -a$ or $x \geq a$

Hence, $x \notin (-a, a)$

The curve does not exist in the region

$$x = -a \text{ to } x = a.$$

(iv) As x increases, y also increases i.e. the curve extends to infinity.

Some Terms Related to Hyperbola

Let the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(1) Centre : All chords passing through C and bisected at C.

Here $C \equiv (0, 0)$

(2) Eccentricity : For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we have

$$b^2 = a^2 (e^2 - 1)$$

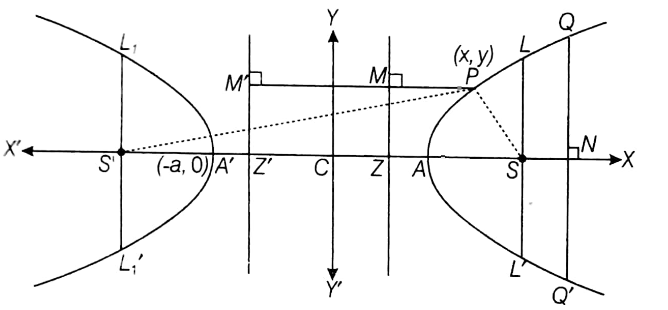
$\Rightarrow$ $$e^2 = \frac{a^2 + b^2}{a^2}$$

$$\Rightarrow \qquad e=\sqrt{1+\left(\frac{b^2}{a^2}\right)} \Rightarrow e=\sqrt{\left\{1+\frac{(2b)^2}{(2a)^2}\right\}}$$

$$\Rightarrow \qquad e=\sqrt{\left\{1+\frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}\right\}}$$

(3) **Foci and directrices :** S and S' are the foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively and ZM and $Z'M'$ are two directrices of the hyperbola and their equations are

$$x=\frac{a}{e} \text{ and } x=-\frac{a}{e} \text{ respectively.}$$



(4) **Axes :** The points $A(a, 0)$ and $A'(-a, 0)$ are called the vertices of the hyperbola and line AA' is called **transverse axis** and the line perpendicular to its through the centre $(0, 0)$ of the hyperbola is called **conjugate axis.**

The length of transverse and conjugate axes are taken as $2a$ and $2b$ respectively.

(5) **Double ordinates :** If Q be a point on the hyperbola draw QN perpendicular to the axis of the hyperbola and produced to meet the curve again at Q'. Then QQ' is called a double ordinate of Q.

If abscissa of Q is h, then ordinates of Q are

$$\frac{y^2}{b^2}=\frac{h^2}{a^2}-1$$

$$\therefore \qquad y=\pm\frac{b}{a}\sqrt{(h^2-a^2)}$$

$$\therefore \qquad y=\frac{b}{a}\sqrt{(h^2-a^2)} \qquad \text{(for I quadrant)}$$

and ordinate of Q' is

$$y=-\frac{b}{a}\sqrt{(h^2-a^2)} \qquad \text{(for IV quadrant)}$$

Hence, coordinates of Q and Q' are

$$\left(h, \frac{b}{a}\sqrt{(h^2-a^2)}\right) \text{ and } \left(h, -\frac{b}{a}\sqrt{(h^2-a^2)}\right)$$

respectively.

(6) **Latusrectum :** The double ordinates LL' and L_1L_1' are the latusrectums of the hyperbola. These lines are perpendicular to transverse axis AA' and through the foci S and S' respectively.

Length of latusrectum

Now, let $LL'=2k$, then $LS=L'S=k$

Coordinates of L and L' are (ae, k) and $(ae, -k)$

lie on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$\therefore \qquad \frac{a^2e^2}{a^2}-\frac{k^2}{b^2}=1$$

$$\text{or} \qquad k^2=b^2(e^2-1)=b^2\left(\frac{b^2}{a^2}\right) \qquad [\because b^2=a^2(e^2-1)]$$

$$\therefore \qquad k=\frac{b^2}{a} \qquad (\because k>0)$$

$$\therefore \qquad \boxed{2k=\frac{2b^2}{a}=LL'}$$

$\therefore$ Length of latusrectum $LL'=L_1L'_1=\frac{2b^2}{a}$ and end points of latusrectums are

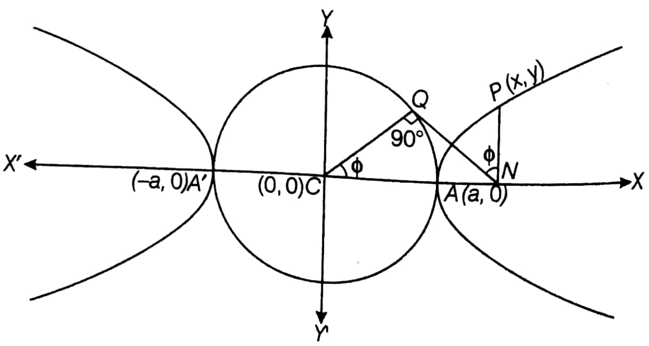
$$L\equiv\left(ae, \frac{b^2}{a}\right); L'\equiv\left(ae, -\frac{b^2}{a}\right);$$

$$L_1\equiv\left(-ae, \frac{b^2}{a}\right); L_1'\equiv\left(-ae, -\frac{b^2}{a}\right) \text{ respectively.}$$

(7) **Focal chord :** A chord of hyperbola passing through its focus is called a focal chord.

(8) **Parametric equations of the hyperbola :** Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the hyperbola with centre C and transverse axis $A'A$. Therefore, circle drawn with centre C and segment $A'A$ as a diameter is called auxiliary circle of the hyperbola.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$



$\therefore$ Equation of the auxiliary circle is $x^2 + y^2 = a^2$

Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw PN perpendicular to X-axis.

Let NQ be a tangent to the auxiliary circle $x^2 + y^2 = a^2$. Join CQ

Let $\angle QCN = \phi$

Here, P and Q are the **corresponding points** of the hyperbola and the auxiliary circle. ϕ is the eccentric angle of $P \cdot (0 \le \phi < 2\pi)$

Since, $Q \equiv (a\cos\phi, a\sin\phi)$

Now, $x = CN = \frac{CN}{CQ} \cdot CQ = \sec\phi \cdot a$

$\therefore$ $\boxed{x = CN = a\sec\phi}$

$\therefore$ $P(x, y) \equiv (a\sec\phi, y)$

$\because$ P lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore$ $\frac{a^2\sec^2\phi}{a^2} - \frac{y^2}{b^2} = 1$

or $\frac{y^2}{b^2} = \sec^2\phi - 1 = \tan^2\phi$

$\therefore$ $y = \pm b\tan\phi$

$\therefore$ $\boxed{y = b\tan\phi}$ (P lies in I quadrant)

The equations of $x = a\sec\phi$ and $y = b\tan\phi$ are known as the parametric equations of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Position of points Q an auxiliary circle and the corresponding point P which describes the hyperbola and $0 \le \phi < 2\pi$.

ϕ varies from	$Q(a\cos\phi, a\sin\phi)$	$P(a\sec\phi, b\tan\phi)$
0 to $\frac{\pi}{2}$	I	I
$\frac{\pi}{2}$ to π	II	III
π to $\frac{3\pi}{2}$	III	II
$\frac{3\pi}{2}$ to 2π	IV	IV

Remark

Equation of the chord joining the points $P \equiv (a\sec\phi_1, b\tan\phi_1)$ and $Q \equiv (a\sec\phi_2, b\tan\phi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\phi_1 - \phi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\phi_1 + \phi_2}{2}\right) = \cos\left(\frac{\phi_1 + \phi_2}{2}\right)$$

If it is focal chord, then pass through $(ae, 0)$ or $(-ae, 0)$. Suppose it pass through $(ae, 0)$, then

$$e\cos\left(\frac{\phi_1 - \phi_2}{2}\right) - 0 = \cos\left(\frac{\phi_1 + \phi_2}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1 - \phi_2}{2}\right)}{\cos\left(\frac{\phi_1 + \phi_2}{2}\right)} = \frac{1}{e}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1 - \phi_2}{2}\right) - \cos\left(\frac{\phi_1 + \phi_2}{2}\right)}{\cos\left(\frac{\phi_1 - \phi_2}{2}\right) + \cos\left(\frac{\phi_1 + \phi_2}{2}\right)} = \frac{1 - e}{1 + e}$$

or $$\boxed{\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \frac{1 - e}{1 + e}}$$

Hence, if ϕ_1 and ϕ_2 are the eccentric angles of extremities of a focal chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then

$$\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \frac{1 - e}{1 + e} \quad \text{or} \quad \frac{1 + e}{1 - e}$$

according as focus $(ae, 0)$ or $(-ae, 0)$.

Focal Distances of a Point

The difference of the focal distances of any point on the hyperbola is constant and equal to length of the transverse axis of the hyperbola.

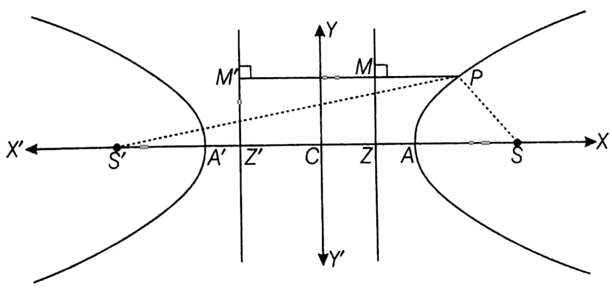
The hyperbola is $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

The foci S and S' are $(ae, 0)$ and $(-ae, 0)$.

The equation of its directrices MZ and $M'Z'$ are

$$x = \frac{a}{e} \quad \text{and} \quad x = -\frac{a}{e}$$

respectively.



Let $P(x_1, y_1)$ be any point on Eq. (i).

Now $SP = ePM = e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$

and $S'P' = ePM' = e\left(x_1 + \frac{a}{e}\right) = ex_1 + a$

$\therefore$ $S'P - SP = (ex_1 + a) - (ex_1 - a) = \boxed{2a} = AA'$

$= \text{Transverse axis}$

A hyperbola is the locus of a point which moves in a plane such that the difference of its distances from two fixed point (foci) is always constant.

Example 1 To find the equation of the hyperbola from the definition that hyperbola is the locus of a point which moves such that the difference of its distances from two fixed points is constant with the fixed point as foci.

Sol. Let two fixed point be $S(ae, 0)$ and $S'(-ae, 0)$. Let $P(x, y)$ be a moving point such that

$$S'P - SP = \text{constant} = 2a \text{ (say)}$$

i.e. $$\sqrt{(x+ae)^2+(y-0)^2} - \sqrt{(x-ae)^2+(y-0)^2} = 2a$$

or $$\sqrt{(x^2+y^2+2aex+a^2e^2)} - \sqrt{(x^2+y^2-2aex+a^2e^2)} = 2a \quad \text{...(i)}$$

Let $$l = x^2+y^2+2aex+a^2e^2 \quad \text{...(ii)}$$

and $$m = x^2+y^2-2aex+a^2e^2 \quad \text{...(iii)}$$

Use this method

Eq. (i) can be re-written as

$$\sqrt{l} - \sqrt{m} = 2a \quad \text{...(iv)}$$

From Eqs. (ii) and (iii),

$$l - m = 4aex$$

$$\Rightarrow (\sqrt{l}+\sqrt{m})(\sqrt{l}-\sqrt{m}) = 4aex$$

$$\Rightarrow 2a(\sqrt{l}+\sqrt{m}) = 4aex \quad \text{[from Eq. (iv)]}$$

or $$\sqrt{l}+\sqrt{m} = 2ex \quad \text{...(v)}$$

Adding Eqs. (iv) and (v), then

$$2\sqrt{l} = 2a + 2ex$$

$$\Rightarrow \sqrt{l} = a + ex \Rightarrow l = (a+ex)^2$$

$$x^2+y^2+2aex+a^2e^2 = a^2+2aex+e^2x^2$$

or $$x^2(e^2-1) - y^2 = a^2(e^2-1)$$

or $$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(e^2-1).$$

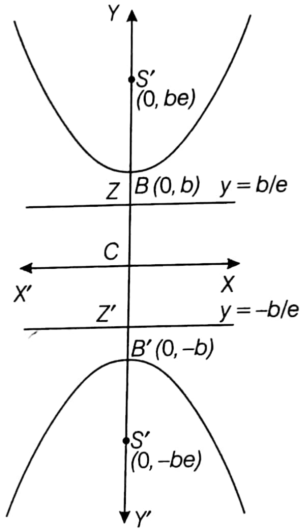
Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \boxed{-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

its shape is shown alongside.



Various results related to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

are given in the following table

Basic fundamentals \ Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1}$
Centre	$(0, 0)$	$(0, 0)$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e = \sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e = \sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of latusrectum	$\frac{2b^2}{a}$	$\boxed{\frac{2a^2}{b}}$
Parametric coordinates	$(a\sec\phi, b\tan\phi)$, $0 \le \phi < 2\pi$	$(a\tan\phi, b\sec\phi)$, $0 \le \phi < 2\pi$
Focal radii	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii $(S'P - SP)$	$2a$	$2b$
Tangents at the vertices	$x = -a, x = a$	$y = -b, y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

Remarks

1. If the centre of hyperbola is (h,k) and axes are parallel to the coordinates axes, then its equation is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1.$$

By shifting the origin at (h, k) without rotating the coordinate axes, the above equation reduces to

$$\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$$

where, $X = X + h, y = Y + k$

2. If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, respectively, then $\boxed{e_1^2+e_2^2=1.}$ Remember

Proof For hyperbola $b^2 = a^2(e_1^2 - 1)$

or $$e_1^2 = 1+\frac{b^2}{a^2}=\frac{a^2+b^2}{a^2} \quad \dots \text{(i)}$$

and for conjugate hyperbola $a^2 = b^2(e_2^2 - 1)$

or $$e_2^2 = 1+\frac{a^2}{b^2}=\frac{a^2+b^2}{b^2} \quad \dots \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=1 \text{ or } e_1^{-2}+e_2^{-2}=1$$

3. The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

4. A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree two should be remembered as :

Let $\phi(x, y) = 0$ represents a hyperbola.

Find $\frac{\partial\phi}{\partial x}$ (differentiate w.r.t x keeping y as constant) and $\frac{\partial\phi}{\partial y}$ (differentiate w.r.t. y keeping x as constant).

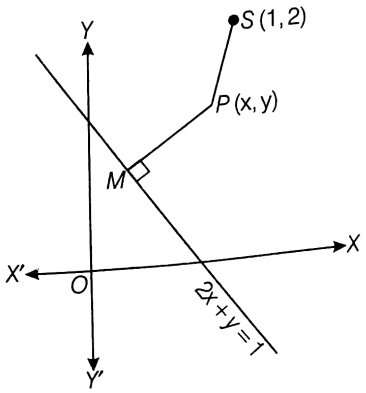
Then the point of intersection of $\frac{\partial\phi}{\partial x}=0$ and $\frac{\partial\phi}{\partial y}=0$ gives the centre of the hyperbola.

| Example 2 Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Sol. Let $P(x, y)$ be any point on the hyperbola. Draw PM perpendicular from P on the directrix.

Then, by definition $\quad SP = e\ PM$

$\Rightarrow \quad (SP)^2 = e^2\,(PM)^2$



$$\Rightarrow (x-1)^2+(y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2$$

$$\Rightarrow 5\{x^2+y^2-2x-4y+5\}$$
$$=3(4x^2+y^2+1+4xy-2y-4x)$$

$$\Rightarrow 7x^2-2y^2+12xy-2x+14y-22=0$$

which is the required hyperbola.

| Example 3 Find the lengths of transverse axis and conjugate axis, eccentricity, the coordinates of foci, vertices, lengths of the latusrectum and equations of the directrices of the following hyperbolas

(i) $9x^2 - y^2 = 1$ (ii) $16x^2 - 9y^2 = -144$

Sol. (i) The equation $9x^2 - y^2 = 1$ can be written as

$$\frac{x^2}{(1/9)}-\frac{y^2}{1}=1.$$

This is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore \quad a^2 = \frac{1}{9}, b^2 = 1$

$\Rightarrow \quad a = \frac{1}{3}, b = 1$

Length of transverse axis : The length of transverse, axis $= 2a = \frac{2}{3}$

Length of conjugate axis : The length of conjugate axis $= 2b = 2$

Eccentricity : $e = \sqrt{\left(1+\frac{b^2}{a^2}\right)}=\sqrt{1+\frac{1}{(1/9)}}=\sqrt{10}$

Foci : The coordinates of the foci are $(\pm ae, 0)$ i.e. $\left(\pm\frac{\sqrt{10}}{3}, 0\right)$.

Vertices : The coordinates of the vertices are $(\pm a, 0)$ i.e. $\left(\pm\frac{1}{3}, 0\right)$.

Length of latusrectum : The length of latusrectum

$$=\frac{2b^2}{a}=\frac{2(1)^2}{1/3}=6$$

Equation of the directrices : The equations of the directrices are

$$x = \pm\frac{a}{e}$$

i.e. $$x = \pm\frac{1/3}{\sqrt{10}}$$

or $$x = \pm\frac{1}{3\sqrt{10}}$$

(ii) The equation $16x^2 - 9y^2 = -144$ can be written as

$$\frac{x^2}{9} - \frac{y^2}{16} = -1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore \quad a^2 = 9, b^2 = 16$

$\Rightarrow \quad a = 3, b = 4$

Length of transverse axis : The length of transverse axis $= 2b = 8$.

Length of conjugate axis : The length of conjugate axis $= 2a = 6$.

Eccentricity : $e = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$

Foci : The coordinates of the foci are $(0, \pm be)$ i.e. $(0, \pm 5)$

Vertices : The coordinates of the vertices are $(0, \pm b)$ i.e. $(0, \pm 4)$.

Length of latusrectum : The length of latusrectum $= \frac{2a^2}{b}$

$$= \frac{2(3)^2}{4} = \frac{9}{2}$$

Equation of directrices : The equation of directrices are

$$y = \pm \frac{b}{e}$$

$$y = \pm \frac{4}{(5/4)}$$

$$\Rightarrow \quad y = \pm \frac{16}{5}$$

Example 4 Find the eccentricity of the hyperbola whose latusrectum is half of its transverse axis.

Sol. Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Then, transverse axis $= 2a$ and latusrectum $= \frac{2b^2}{a}$

According to question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow \quad 2b^2 = a^2 \quad (\because b^2 = a^2(e^2 - 1))$

$\Rightarrow \quad 2a^2(e^2 - 1) = a^2$

$\Rightarrow \quad 2e^2 - 2 = 1$

$\Rightarrow \quad e^2 = \frac{3}{2}$

$\therefore \quad e = \sqrt{\frac{3}{2}}$

Hence, the required eccentricity is $\sqrt{\frac{3}{2}}$.

Example 5 Prove that the point $\left\{\frac{a}{2}\left(t + \frac{1}{t}\right), \frac{b}{2}\left(t - \frac{1}{t}\right)\right\}$ lies on the hyperbola for all values of $t \ (t \neq 0)$.

Sol. Let $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$

or $\quad \frac{2x}{a} = t + \frac{1}{t}$

or $\quad \left(\frac{2x}{a}\right)^2 = t^2 + \frac{1}{t^2} + 2 \quad$ ...(i)

and let $\quad y = \frac{b}{2}\left(t - \frac{1}{t}\right)$

or $\quad \frac{2y}{b} = t - \frac{1}{t}$

or $\quad \left(\frac{2y}{b}\right)^2 = t^2 + \frac{1}{t^2} - 2 \quad$ ...(ii)

Subtracting Eqs. (ii) from (i),

$$\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4$$

or $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

which is hyperbola.

Example 6 Show that the equation $7y^2 - 9x^2 + 54x - 28y - 116 = 0$ represent a hyperbola. Find the coordinates of the centre, lengths of transverse and conjugate axes, eccentricity, latusrectum, coordinate of foci, vertices and equations of the directrices of the hyperbola.

Sol. We have, $7y^2 - 9x^2 + 54x - 28y - 116 = 0$

or $\quad 7(y^2 - 4y) - 9(x^2 - 6x) - 116 = 0$

or $\quad 7(y^2 - 4y + 4) - 9(x^2 - 6x + 9) = 116 + 28 - 81$

or $\quad 7(y-2)^2 - 9(x-3)^2 = 63$

or $\quad \frac{(y-2)^2}{9} - \frac{(x-3)^2}{7} = 1$

or $\quad \frac{Y^2}{9} - \frac{X^2}{7} = 1$ solve by partial diff.

where $X = x - 3$ and $Y = y - 2$

This equation represents conjugate hyperbola. Comparing it with

$$\frac{Y^2}{b^2} - \frac{X^2}{a^2} = 1$$

we get, $\quad b^2 = 9$ and $a^2 = 7$

$\therefore \quad b = 3$ and $a = \sqrt{7}$

Centre : $\quad X = 0, Y = 0$

i.e. $\quad x - 3 = 0, y - 2 = 0$

$\therefore$ Centre is $(3, 2)$.

Length of transverse axis : Length of transverse axis $= 2b = 6.$

Length of conjugate axis : Length of conjugate axis $= 2a = 2\sqrt{7}.$

Eccentricity : The eccentricity e is given by

$$e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{7}{9}} = \frac{4}{3}$$

Length of latusrectum : The length of latusrectum

$$= \frac{2a^2}{b} = \frac{2(7)}{3} = \frac{14}{3}$$

Foci : The coordinates of foci are $(0, \pm be)$

$\therefore$ $X = 0, Y = \pm be$

$\Rightarrow$ $x - 3 = 0, y - 2 = \pm 3 \times \frac{4}{3}$

or $(3, 2 \pm 4)$

i.e. $(3, -2)$ and $(3, 6)$

Vertices : The coordinates of vertices are $(0, \pm b)$

or $X = 0, Y = \pm b$

or $x - 3 = 0, y - 2 = \pm 3$ or $(3, 2 \pm 3)$

or vertices are $(3, -1)$ and $(3, 5)$

Equation of directrices : The equation of directrices are

$$Y = \pm \frac{b}{e}$$

$\Rightarrow$ $y - 2 = \pm \frac{3}{4/3}$

$\Rightarrow$ $y = \left(2 \pm \frac{9}{4}\right)$

i.e. $y = \frac{17}{4}$ and $y = -\frac{1}{4}$

| Example 7 Find the equation of the hyperbola whose foci are $(6, 4)$ and $(-4, 4)$ and eccentricity is 2.

Sol. The centre of the hyperbola is the mid-point of the line joining the two foci. So, the coordinates of the centre are

$$\left(\frac{6-4}{2}, \frac{4+4}{2}\right) \text{ i.e. } (1, 4)$$

Let $2a$ and $2b$ be the lengths of transverse and conjugate axes and let e be the eccentricity. Then, equation of hyperbola is

$$\frac{(x-1)^2}{a^2} - \frac{(y-4)^2}{b^2} = 1$$

$\because$ Distance between the foci $= 2ae$

$$\sqrt{(6+4)^2 + (4-4)^2} = 2a \times 2$$

$\Rightarrow$ $10 = 4a$

$\therefore$ $a = 5/2$

$\therefore$ $b^2 = a^2(e^2 - 1) = \frac{25}{4}(4 - 1) = \frac{75}{4}$

Thus, the equation of the hyperbola is

$$\frac{(x-1)^2}{\left(\frac{25}{4}\right)} - \frac{(y-4)^2}{\left(\frac{75}{4}\right)} = 1$$

or $12(x-1)^2 - 4(y-4)^2 = 75$

or $12(x^2 - 2x + 1) - 4(y^2 - 8y + 16) = 75$

or $12x^2 - 4y^2 - 24x + 32y - 127 = 0$

| Example 8 Obtain the equation of a hyperbola with coordinate axes as principal axes given that the distances of one of its vertices from the foci are 9 and 1 units.

Sol. Let equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

If vertices are $A(a, 0)$ and $A'(-a, 0)$ and foci are $S(ae, 0)$ and $S'(-ae, 0)$.

Given $l(S'A) = 9$ and $l(SA) = 1$

$\Rightarrow$ $a + ae = 9$ and $ae - a = 1$

or $a(1 + e) = 9$ and $a(e - 1) = 1$

$\therefore$ $\frac{a(1+e)}{a(e-1)} = \frac{9}{1}$

$\Rightarrow$ $1 + e = 9e - 9 \Rightarrow e = \frac{5}{4}$

$\because$ $a(1 + e) = 9$

$\therefore$ $a\left(1 + \frac{5}{4}\right) = 9 \Rightarrow a = 4$

$b^2 = a^2(e^2 - 1) = 16\left(\frac{25}{16} - 1\right)$

$\therefore$ $b^2 = 9$

From Eq. (i) equation of hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

| Example 9 The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the equation of the hyperbola if its eccentricity is 2.

Sol. The given ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore$ $a^2 = 25$ and $b^2 = 9$

then eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

$\therefore$ Foci of ellipse are $(\pm ae, 0)$ i.e. $(\pm 4, 0)$

So, the coordinates of foci of the hyperbola are $(\pm 4, 0)$.

Let e' be the eccentricity of the required hyperbola and its equation be

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 \quad \text{...(i)}$$

the coordinates of foci are $(\pm Ae', 0)$

$\therefore \quad Ae' = 4 \Rightarrow A \times 2 = 4 \Rightarrow A = 2$

Also $\quad B^2 = A^2(e'^2 - 1) \quad (\because \text{given } e' = 2)$

$= 4(4-1) = 12$

Substituting the values of A and B in Eq. (i), we get

$$\frac{x^2}{4} - \frac{y^2}{12} = 1 \text{ or } 3x^2 - y^2 - 12 = 0$$

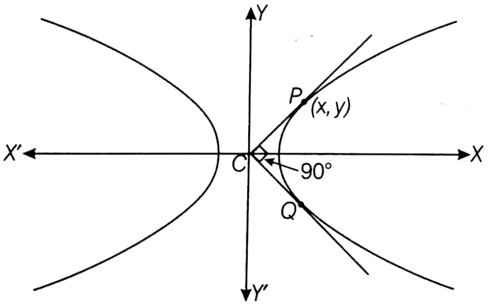
which is required hyperbola.

Example 10 If two points P and Q on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose centre is C, are such that CP is perpendicular to CQ, $a < b$, then prove that

$$\frac{1}{(CP)^2} + \frac{1}{(CQ)^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

Sol. Let $P(x, y)$ be any point on the given hyperbola. Let slope of CP is m, then equation of CP is $y = mx$.

Solving, $y = mx$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$$\frac{x^2}{a^2} - \frac{m^2x^2}{b^2} = 1$$

$$\therefore \quad x^2 = \frac{a^2b^2}{(b^2 - a^2m^2)}$$

and $$y^2 = m^2x^2 = \frac{a^2m^2b^2}{b^2 - a^2m^2}$$

$$\therefore \quad x^2 + y^2 = \frac{a^2b^2}{(b^2 - a^2m^2)}(1 + m^2)$$

$$\therefore \quad (CP)^2 = x^2 + y^2 = \frac{a^2b^2(1+m^2)}{(b^2 - a^2m^2)}$$

$$\Rightarrow \quad \frac{1}{(CP)^2} = \frac{b^2 - a^2m^2}{a^2b^2(1+m^2)} \quad \text{...(i)}$$

and equation of CQ is $\quad y = -\frac{1}{m}x$

replacing m by $-\frac{1}{m}$ in Eq. (i)

then $$\frac{1}{(CQ)^2} = \frac{b^2 - a^2\left(-\frac{1}{m}\right)^2}{a^2b^2\left(1 + \left(-\frac{1}{m}\right)^2\right)} = \frac{b^2m^2 - a^2}{a^2b^2(1+m^2)} \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), then

$$\frac{1}{(CP)^2} + \frac{1}{(CQ)^2} = \frac{b^2(1+m^2) - a^2(1+m^2)}{a^2b^2(1+m^2)}$$

$$= \frac{b^2 - a^2}{a^2b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Position of a Point with Respect to a Hyperbola

Theorem : The point (x_1, y_1) lies outside, on or inside the hyperbola

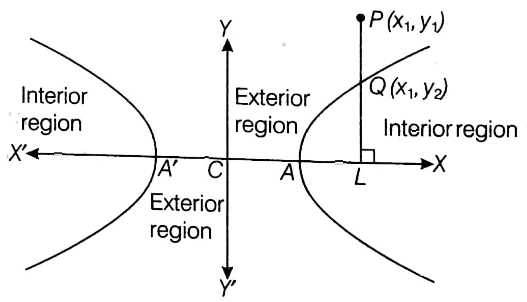
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Remember

according as $\quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 <, = \text{ or } > 0$

Proof : Let $P \equiv (x_1, y_1)$ then $Q \equiv (x_1, y_2)$

Draw PL perpendicular to X-axis



Clearly, $PL > QL$

$$\Rightarrow \quad y_1 > y_2$$

$$\Rightarrow \quad \frac{y_1^2}{b^2} > \frac{y_2^2}{b^2}$$

$$\Rightarrow \quad -\frac{y_1^2}{b^2} < -\frac{y_2^2}{b^2}$$

$$\Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2}$$

$$\Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1 \quad \left(\because Q(x_1, y_2) \text{ lies on } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$$

$$\Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$$

Thus, the point $P(x_1, y_1)$ lies outside the hyperbola.
Hence the point (x_1, y_1) lies outside, on or inside the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 <, = \text{ or } > 0.$

Must

| Example 11 Find the position of the point $(5, -4)$ relative to the hyperbola $9x^2 - y^2 = 1$.

Sol. Since, $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$,

So, the point $(5, -4)$ inside the hyperbola $9x^2 - y^2 = 1$.

Intersection of a Line and a Hyperbola

Let the hyperbola be $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

and the given line be $$y = mx + c \quad \text{...(ii)}$$

Eliminating y from Eqs. (i) and (ii),

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1 \Rightarrow b^2x^2 - a^2(mx+c)^2 = a^2b^2$$

$$\Rightarrow \quad (a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2 + c^2) = 0 \quad \text{...(iii)}$$

Above equation being a quadratic in x gives two values of x. It shows that every straight line will cut the hyperbola in two points, may be real, coincident or imaginary, according as discriminant of Eq. (iii) $>, =, < 0$

i.e. $$4m^2c^2a^4 - 4(a^2m^2 - b^2)a^2(b^2 + c^2) >, =, < 0$$

or $$-a^2m^2 + b^2 + c^2 >, =, < 0$$

or $$c^2 >, =, < a^2m^2 - b^2 \quad \text{...(iv)}$$

Condition of Tangency : If the line (ii) touches the hyperbola (i), then Eq. (iii) has equal roots.

$\therefore$ Discriminant of Eq. (iii) $= 0$

$$\Rightarrow \quad \boxed{c^2 = a^2m^2 - b^2}$$

or $$c = \pm\sqrt{(a^2m^2 - b^2)} \quad \text{...(v)}$$

So, the line $y = mx + c$ touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{if } c^2 = a^2m^2 - b^2$$

(which is condition of tangency)

Substituting the value of c from Eq. (v) in Eq. (ii)

$$\boldsymbol{y = mx \pm \sqrt{(a^2m^2 - b^2)}}$$

Hence, the line $y = mx \pm \sqrt{(a^2m^2 - b^2)}$ will always be tangent to the hyperbola.

Point of contact : Substituting $c = \pm\sqrt{(a^2m^2 - b^2)}$ in Eq. (iii)

$$(a^2m^2 - b^2)x^2 \pm 2ma^2\sqrt{(a^2m^2 - b^2)}\,x + a^4m^2 = 0$$

or $$(x\sqrt{(a^2m^2 - b^2)} \pm a^2m)^2 = 0$$

$$\therefore \quad x = \pm\frac{a^2m}{\sqrt{(a^2m^2 - b^2)}} = \pm\frac{a^2m}{c}$$

From Eq. (i), $$\frac{a^4m^2}{c^2} \cdot \frac{1}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{y^2}{b^2} = \frac{a^2m^2}{c^2} - 1 = -\frac{(c^2 - a^2m^2)}{c^2} = \frac{b^2}{c^2}$$

$$\therefore \quad y = \pm\frac{b^2}{c}$$

Hence, the point of contact is $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$.

This is known as m-point on the hyperbola.

Remark

If $m = 0$, then Eq. (iii) gives $-b^2x^2 + a^2(b^2 + c^2) = 0$

$$x^2 = \frac{a^2(b^2 + c^2)}{b^2}$$

$$\therefore \quad x = \pm\frac{a}{b}\sqrt{(b^2 + c^2)}$$

which gives two values of x.

| Example 12 Prove that the straight line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2l^2 - b^2m^2 = n^2$.

Sol. The given line is $lx + my + n = 0$ or $y = -\frac{l}{m}x - \frac{n}{m}$

Comparing this line with

$$y = Mx + c \quad \text{...(i)}$$

$$\therefore \quad M = -\frac{l}{m} \text{ and } c = -\frac{n}{m}$$

This line (i) will touch the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ if } c^2 = a^2M^2 - b^2$$

$$\Rightarrow \quad \frac{n^2}{m^2} = \frac{a^2l^2}{m^2} - b^2 \text{ or } a^2l^2 - b^2m^2 = n^2$$

Example 13 Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \textit{if} \quad a^2\cos^2\alpha - b^2\sin^2\alpha = p^2.$$

Sol. The given line is

$$x\cos\alpha + y\sin\alpha = p$$

$$\Rightarrow \quad y\sin\alpha = -x\cos\alpha + p$$

$$\Rightarrow \quad y = -x\cot x + p\operatorname{cosec}\alpha$$

Comparing this line with $y = mx + c$

$$\Rightarrow \quad m = -\cot\alpha, c = p\operatorname{cosec}\alpha$$

Since, the given line touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ then } c^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad p^2\operatorname{cosec}^2\alpha = a^2\cot^2\alpha - b^2$$

or $$p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$$

Example 14 For what value of λ does the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$?

Sol. Equation of hyperbola is

$$16x^2 - 9y^2 = 144$$

or $$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get $a^2 = 9, b^2 = 16$

and comparing this line $y = 2x + \lambda$ with $y = mx + c$

$$\therefore \quad m = 2 \text{ and } c = \lambda$$

If the line $y = 2x + \lambda$ touches the hyperbola

$$16x^2 - 9y^2 = 144$$

then $$c^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad \lambda^2 = 9(2)^2 - 16 = 36 - 16 = 20$$

$$\therefore \quad \lambda = \pm 2\sqrt{5}$$

Example 15 If it possible to draw the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having slope 2, then find its range of eccentricity.

Must

Sol. Tangent having slope m is $y = mx \pm \sqrt{(a^2m^2 - b^2)}$

The tangent having slope 2 is $y = 2x \pm \sqrt{(4a^2 - b^2)}$, which is real if

$$4a^2 - b^2 \geq 0$$

or $$\frac{b^2}{a^2} \leq 4 \text{ or } e^2 - 1 \leq 4 \text{ or } e^2 \leq 5$$

or $$1 < e \leq \sqrt{5} \qquad \text{(for hyperbola } e > 1)$$

Exercise for Session 1

1. The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is
(a) 1 (b) $\frac{1}{2}$ (c) -1 (d) $\sqrt{2}$

2. If e_1 and e_2 are the eccentricities of the conic sections $16x^2 + 9y^2 = 144$ and $9x^2 - 16y^2 = 144$, then
(a) $e_1^2 - e_2^2 = 1$ (b) $e_1^2 + e_2^2 < 3$ (c) $e_1^2 + e_2^2 = 3$ (d) $e_1^2 + e_2^2 > 3$

3. The transverse axis of a hyperbola is of length $2a$ and a vertex divides the segment of the axis between the centre and the corresponding focus in the ratio 2 : 1, then the equation of the hyperbola is
(a) $4x^2 - 5y^2 = 4a^2$ (b) $4x^2 - 5y^2 = 5a^2$ (c) $5x^2 - 4y^2 = 4a^2$ (d) $5x^2 - 4y^2 = 5a^2$

4. The eccentricity of the hyperbola whose latusrectum is 8 and conjugate axis is equal to half of the distance between the foci, is
(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{3}{\sqrt{3}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $\frac{5}{\sqrt{3}}$

5. The straight line $x + y = \sqrt{2}p$ will touch the hyperbola $4x^2 - 9y^2 = 36$, if
(a) $p^2 = 2$ (b) $p^2 = 5$ (c) $5p^2 = 2$ (d) $2p^2 = 5$

6. The equation of the tangent, parallel to $y - x + 5 = 0$ drawn to $\frac{x^2}{3} - \frac{y^2}{2} = 1$ is
(a) $x - y - 1 = 0$ (b) $x - y + 2 = 0$ (c) $x + y - 1 = 0$ (d) $x + y + 2 = 0$

7. If e and e' are the eccentricities of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$, then the point $\left(\frac{1}{e},\frac{1}{e'}\right)$ lies on the circle

(a) $x^2+y^2=1$ (b) $x^2+y^2=2$ (c) $x^2+y^2=3$ (d) $x^2+y^2=4$

8. If e and e' are the eccentricities of the ellipse $5x^2+9y^2=45$ and the hyperbola $5x^2-4y^2=45$ respectively, then $ee'=$

(a) -1 (b) 1 (c) -4 (d) 9

9. The equation $\frac{x^2}{10-\lambda}+\frac{y^2}{6-\lambda}=1$ represents

(a) a hyperbola if $\lambda<6$ (b) an ellipse if $\lambda>6$ (c) a hyperbola if $6<\lambda<10$ (d) an ellipse if $0<\lambda<6$

10. The eccentricity of the hyperbola conjugate to $x^2-3y^2=2x+8$ is

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$ (c) 2 (d) $\sqrt{2}$

11. For hyperbola $\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$ which of the following remains constant with change in 'α'

(a) Abscissae of vertices (b) Abscissae of foci (c) Eccentricity (d) Directrix

12. If the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then the value of b^2 is

(a) 1 (b) 5 (c) 7 (d) 9

13. Find the equation of the hyperbola whose foci are $(0,\pm\sqrt{10})$ and which passes through the point $(2,3)$.

14. Find the equation of the hyperbola whose foci are $(10,5)$ and $(-2,5)$ and eccentricity 3.

15. Prove that the straight lines $\frac{x}{a}-\frac{y}{b}=m$ and $\frac{x}{a}+\frac{y}{b}=\frac{1}{m}$ always meet on a hyperbola.

16. Find the centre, eccentricity and length of axes of the hyperbola $3x^2-5y^2-6x+20y-32=0$.

17. Find the eccentricity of the hyperbola conjugate to the hyperbola $x^2-3y^2=1$.

18. For what value of λ, does the line $y=3x+\lambda$ touch the hyperbola $9x^2-5y^2=45$?

19. Find the equation of the tangent to the hyperbola $4x^2-9y^2=1$ which is parallel to the line $4y=5x+7$. Also find the point of contact.

Session 2

Equations of Tangents in Different Forms, Equations of Normals in Different Forms, Pair of Tangents, Chord of Contact, Equation of the Chord Bisected at a Given Point

Equations of Tangents in Different Forms

1. Point form (first principal method) :

Theorem : The equation of the tangent to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ at } (x_1, y_1) \text{ is}$$

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

Proof : Equation of hyperbola is

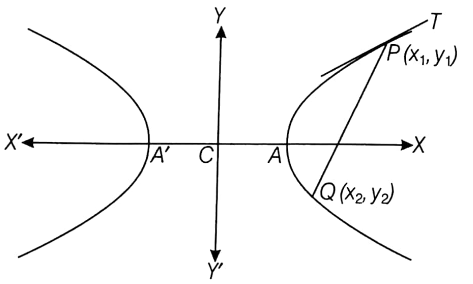
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{...(i)}$$

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be any two points in Eq. (i), then

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \quad \text{...(ii)}$$

and

$$\frac{x_2^2}{a^2}-\frac{y_2^2}{b^2}=1 \quad \text{...(iii)}$$



Subtracting Eq. (ii) from Eq. (iii), then

$$\frac{1}{a^2}(x_2^2-x_1^2)-\frac{1}{b^2}(y_2^2-y_1^2)=0$$

$$\Rightarrow \quad \frac{(x_2-x_1)(x_2+x_1)}{a^2}-\frac{(y_2-y_1)(y_2+y_1)}{b^2}=0$$

$$\Rightarrow \quad \frac{y_2-y_1}{x_2-x_1}=\frac{b^2}{a^2}\frac{(x_1+x_2)}{(y_1+y_2)} \quad \text{...(iv)}$$

Equation of PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \text{...(v)}$$

From Eqs. (iv) and (v),

$$y-y_1=\frac{b^2}{a^2}\cdot\frac{(x_1+x_2)}{(y_1+y_2)}(x-x_1) \quad \text{...(vi)}$$

Now, tangent at $P, Q \to P$

i.e. $x_2 \to x_1$ and $y_2 \to y_1$, then Eq. (vi) becomes

$$y-y_1=\frac{b^2}{a^2}\cdot\frac{(2x_1)}{(2y_1)}(x-x_1)$$

or $\quad \dfrac{yy_1-y_1^2}{b^2}=\dfrac{xx_1-x_1^2}{a^2}$ or $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=\dfrac{x_1^2}{a^2}-\dfrac{y_1^2}{b^2}$

or $\quad \dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=1$ [from Eq. (ii)]

which is the required equation of tangent at (x_1, y_1).

Remark

The equation of tangent at (x_1, y_1) can be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$. This method is applied only when the equation of conic is a polynomial of second degree in x and y.

2. Parametric form :

Theorem : The equation of tangent to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ at } (a\sec\phi, b\tan\phi) \text{ is}$$

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1$$

Proof : Since, equation of tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Now, replacing x_1 by $a \sec\phi$ and y_1 by $b \tan\phi$, we get

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1$$

Remark

The point of intersection of tangents at 'θ' and 'ϕ' on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\left(\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right)$$

Remembering method :

∵ Equations of chord joining 'θ' and 'ϕ' is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

or
$$\frac{x}{a}\left\{\frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\} - \frac{y}{b}\left\{\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\} = 1$$

or
$$\frac{x}{a^2}\left\{\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\} - \frac{y}{b^2}\left\{\frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\} = 1$$

i.e.
$$\left\{\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\}$$

3. Slope form :

Theorem : The equations of tangents of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by

$$y = mx \pm \sqrt{(a^2m^2 - b^2)}$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{(a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{(a^2m^2 - b^2)}}\right)$$

Proof : Let $y = mx + c$ be a tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Substituting the value of $y = mx + c$ in Eq. (i), then

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

or $(a^2m^2 - b^2)x^2 + 2mca^2x + a^2(c^2 + b^2) = 0$

must have equal roots

$$\therefore \quad 4a^4m^2c^2 - 4(a^2m^2 - b^2)a^2(c^2 + b^2) = 0$$

$$[\because B^2 - 4AC = 0]$$

$$\Rightarrow \quad a^2m^2c^2 - (a^2m^2 - b^2)(b^2 + c^2) = 0$$

$$\Rightarrow \quad a^2m^2c^2 - a^2b^2m^2 - a^2m^2c^2 + b^4 + b^2c^2 = 0$$

$$\Rightarrow \quad -a^2b^2m^2 + b^4 + b^2c^2 = 0 \Rightarrow c^2 = a^2m^2 - b^2$$

$$\therefore \quad c = \pm\sqrt{(a^2m^2 - b^2)}$$

Substituting this value of c in $y = mx + c$, we get

$$\boldsymbol{y = mx \pm \sqrt{(a^2m^2 - b^2)}} \quad \text{...(ii)}$$

as the required equation of tangents of hyperbola in terms of slope.

∵ Tangent at (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \text{...(iii)}$$

On comparing Eqs. (ii) and (iii) we get

$$\frac{x_1/a^2}{m} = \frac{y_1/b^2}{1} = \pm\frac{1}{\sqrt{a^2m^2 - b^2}}$$

Thus, the coordinates of the points of contact are

$$\left(\frac{\pm a^2m}{\sqrt{(a^2m - b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m - b^2)}}\right)$$

Remark

The equations of tangents of slope m to the hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given by $y = mx \pm \sqrt{(b^2 - a^2m^2)}$ and the coordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{(b^2 - a^2m^2)}}, \mp\frac{b^2}{\sqrt{(b^2 - a^2m^2)}}\right)$$

| Example 16 Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Sol. Let m be the slope of the tangent. Since, the tangent is perpendicular to the line $x - y + 4 = 0$

$$\therefore \quad m \times 1 = -1$$

$$\Rightarrow \quad m = -1$$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36} - \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore$ $a^2 = 36$ and $b^2 = 9$

So the equation of tangents are

$$y = (-1)\,x \pm \sqrt{36 \times (-1)^2 - 9}$$

$\Rightarrow$ $y = -x \pm \sqrt{27}$

or $x + y \pm 3\sqrt{3} = 0$

Must

| Example 17 Find the equation and the length of the common tangents to hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad and \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Sol. Tangent at $(a \sec \phi, b \tan \phi)$ on the 1st hyperbola is

$$\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1 \qquad \text{...(i)}$$

Similarly tangent at any point $(b \tan\theta, a \sec \theta)$ on 2nd hyperbola is

$$\frac{y}{a} \sec \theta - \frac{x}{b} \tan\theta = 1 \qquad \text{...(ii)}$$

If Eqs. (i) and (ii) are common tangents then they should be identical. Comparing the coefficients of x and y

$\Rightarrow$ $\frac{\sec \theta}{a} = -\frac{\tan \phi}{b}$

or $\sec \theta = -\frac{a}{b} \tan \phi$ ...(iii)

and $-\frac{\tan\theta}{b} = \frac{\sec \phi}{a}$

or $\tan\theta = -\frac{b}{a} \sec \phi$ ...(iv)

$\because$ $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow$ $\frac{a^2}{b^2} \tan^2 \phi - \frac{b^2}{a^2} \sec^2 \phi = 1$ [from Eqs. (iii) and (iv)]

or $\frac{a^2}{b^2} \tan^2 \phi - \frac{b^2}{a^2} (1 + \tan^2 \phi) = 1$

or $\left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right) \tan^2 \phi = 1 + \frac{b^2}{a^2}$

$\therefore$ $\tan^2 \phi = \frac{b^2}{a^2 - b^2}$

and $\sec^2\phi = 1 + \tan^2 \phi = \frac{a^2}{a^2 - b^2}$

Hence, the points of contact are

$$\left\{\pm \frac{a^2}{\sqrt{(a^2 - b^2)}}, \pm \frac{b^2}{\sqrt{(a^2 - b^2)}}\right\}$$

and $\left\{\mp \frac{b^2}{\sqrt{(a^2 - b^2)}}, \mp \frac{a^2}{\sqrt{(a^2 - b^2)}}\right\}$ [from Eqs. (iii) and (iv)]

Length of common tangent i.e. the distance between the above points is $\sqrt{2}\,\frac{(a^2 + b^2)}{\sqrt{(a^2 - b^2)}}$ and equation of common tangent on putting the values of $\sec \phi$ and $\tan \phi$ in Eqs. (i) is

$$\pm \frac{x}{\sqrt{(a^2 - b^2)}} \mp \frac{y}{\sqrt{(a^2 - b^2)}} = 1$$

or $x \mp y = \pm \sqrt{(a^2 - b^2)}$

Aliter : The given two hyperbolas are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

and $\frac{x^2}{(-b^2)} - \frac{y^2}{(-a^2)} = 1$ ...(ii)

we know that

$$y = mx \pm \sqrt{(a^2m^2 - b^2)} \qquad \text{...(iii)}$$

is tangent to Eq. (i) for all m.

Similarly $y = m_1x \pm \sqrt{(-b^2)\, m_1^2 - (-a^2)}$

$$y = m_1x \pm \sqrt{(a^2 - b^2m_1^2)} \qquad \text{...(iv)}$$

will be tangent to Eq. (ii)

For common tangents to Eqs. (i) and (ii), the lines (iii) and (iv) must be identical

i.e. $m = m_1$ and $a^2m^2 - b^2 = a^2 - b^2m_1^2$

i.e. $(a^2 + b^2)(m^2 - 1) = 0$

$\Rightarrow$ $m^2 = 1 \Rightarrow m = \pm 1$

$\therefore$ The equation of common tangent lines are

$$\boldsymbol{y = \pm x \pm \sqrt{(a^2 - b^2)}} \qquad \text{[from Eq. (iii)]}$$

or $y \mp x = \pm\sqrt{(a^2 - b^2)}$ ...(v)

Equation of tangent to Eq. (i) at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad \text{...(vi)}$$

On comparing Eqs.(v) and (vi), then

$$\frac{x_1 / a^2}{\mp 1} = \frac{-y_1 / b^2}{1} = \frac{1}{\pm \sqrt{(a^2 - b^2)}}$$

i.e. $\left(\mp \frac{a^2}{\sqrt{(a^2 - b^2)}}, \mp \frac{b^2}{\sqrt{(a^2 - b^2)}}\right)$

and equation of tangent to Eq. (ii) at (x_2, y_2) is

$$\frac{xx_2}{(-b^2)} - \frac{yy_2}{(-a^2)} = 1 \qquad \text{...(vii)}$$

On comparing Eqs. (v) and (vii), then

$$\frac{x_2 / (-b^2)}{\mp 1} = \frac{y_2}{1} = \frac{1}{\pm\sqrt{(a^2 - b^2)}}$$

i.e. $\left(\pm\dfrac{b^2}{\sqrt{(a^2-b^2)}}, \pm\dfrac{a^2}{\sqrt{(a^2-b^2)}}\right)$

The points of contact are

$$\left(\pm\frac{a^2}{\sqrt{(a^2-b^2)}}, \pm\frac{b^2}{\sqrt{(a^2-b^2)}}\right)$$

and $\left(\mp\dfrac{b^2}{\sqrt{(a^2-b^2)}}, \mp\dfrac{a^2}{\sqrt{(a^2-b^2)}}\right)$

Hence, the length of common tangent is

$$\sqrt{2}\cdot\frac{(a^2+b^2)}{\sqrt{(a^2-b^2)}}$$

| Example 18 PQ is the chord joining the points ϕ_1 and ϕ_2 on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. If $\phi_1-\phi_2=2\alpha$, where α is constant, prove that PQ touches the hyperbola

$$\frac{x^2}{a^2}\cos^2\alpha-\frac{y^2}{b^2}=1.$$

Sol. Given hyperbola is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ ...(i)

Equation of the chord PQ to the hyperbola (i) is

$$\frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

$$\Rightarrow \quad \frac{x}{a}\cos\alpha-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

(Given $\phi_1-\phi_2=2\alpha$)

i.e. $$y=\frac{b}{a}\frac{\cos\alpha}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}x+\frac{b\cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)} \quad \text{...(ii)}$$

Comparing this line with $y=mx+c$

$$\therefore \quad m=\frac{b}{a}\cdot\frac{\cos\alpha}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)} \text{ and } c=\frac{b\cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}$$

For line $y=mx+c$ to be a tangent on $\dfrac{x^2}{a^2}\cos^2\alpha-\dfrac{y^2}{b^2}=1$, we have

$$c^2=\frac{a^2}{\cos^2\alpha}m^2-b^2$$

$$\therefore \quad \text{LHS}=c^2=\frac{b^2\cos^2\left(\frac{\phi_1+\phi_2}{2}\right)}{\sin^2\left(\frac{\phi_1+\phi_2}{2}\right)}$$

and
$$\text{RHS}=\frac{a^2}{\cos^2\alpha}m^2-b^2$$
$$=\frac{a^2}{\cos^2\alpha}\times\frac{b^2\cos^2\alpha}{a^2\sin^2\left(\frac{\phi_1+\phi_2}{2}\right)}-b^2$$
$$=\frac{b^2}{\sin^2\left(\frac{\phi_1+\phi_2}{2}\right)}-b^2$$
$$=\frac{b^2\cos^2\left(\frac{\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}$$

Hence proved.

| Example 19 If the line $y=mx+\sqrt{(a^2m^2-b^2)}$ touches the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$, show that $\theta=\sin^{-1}\left(\dfrac{b}{am}\right)$.

Sol. Since $(a\sec\theta, b\tan\theta)$ lies on

$$y=mx+\sqrt{(a^2m^2-b^2)}$$

$$\therefore \quad b\tan\theta=am\sec\theta+\sqrt{(a^2m^2-b^2)}$$
$$\Rightarrow \quad (b\tan\theta-am\sec\theta)^2=a^2m^2-b^2$$
$$\Rightarrow \quad b^2\tan^2\theta+a^2m^2\sec^2\theta-2abm\tan\theta\sec\theta=a^2m^2-b^2$$
$$\Rightarrow \quad a^2m^2\tan^2\theta-2\,abm\tan\theta\;\;\theta+b^2\sec^2\theta=0$$
or $$a^2m^2\sin^2\theta-2abm\sin\theta+b^2=0 \qquad (\because \cos\theta\neq 0)$$
$$\therefore \quad \sin\theta=\frac{2abm\pm\sqrt{4a^2b^2m^2-4a^2b^2m^2}}{2a^2m^2}=\left(\frac{b}{am}\right)$$
$$\therefore \quad \theta=\sin^{-1}\left(\frac{b}{am}\right)$$

| Example 20 If SY and $S'Y'$ be drawn perpendiculars from foci to any tangent to a hyperbola. Prove that Y and Y' lie on the auxiliary circle and that product of these perpendiculars is constant.

Sol. Let hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{...(i)}$$

Tangent at $P(a\sec\phi, b\tan\phi)$ on Eq. (i) is

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1 \quad \text{...(ii)}$$

Its slope is $\dfrac{(b\sec\phi)}{(a\tan\phi)}$

Equation of SY which is perpendicular to Eq. (i) and passes through focus S i.e. $(ae, 0)$ is

$$y - 0 = -\frac{a \tan\phi}{b \sec\phi}(x - ae)$$

or $$\frac{x}{b}\tan\phi + \frac{y}{a}\sec\phi = \frac{ae}{b}\tan\phi \quad \text{...(iii)}$$

$\because$ Lines (ii) and (iii) intersect at Y and in order to find its locus we have to eliminate ϕ between Eqs. (ii) and (iii), for which squaring and adding Eqs. (ii) and (iii) then, we get

$$(x^2 + y^2)\left(\frac{\sec^2\phi}{a^2} + \frac{\tan^2\phi}{b^2}\right) = 1 + \frac{a^2e^2}{b^2}\tan^2\phi$$

$$= 1 + \frac{a^2 + b^2}{b^2}\tan^2\phi = (1 + \tan^2\phi) + \frac{a^2}{b^2}\tan^2\phi$$

$$= \left(\sec^2\phi + \frac{a^2}{b^2}\tan^2\phi\right) = a^2\left(\frac{\sec^2\phi}{a^2} + \frac{\tan^2\phi}{b^2}\right)$$

$\therefore$ $x^2 + y^2 = a^2$ is the required locus.

Similarly the point Y' also lies on it. Again, if p_1 and p_2 be the length of perpendiculars from $S(ae, 0)$ and $S'(-ae, 0)$ on the tangent (ii), then

$$p_1p_2 = \frac{(e\sec\phi - 1)\cdot(e\sec\phi + 1)}{\sqrt{\left(\frac{\sec^2\phi}{a^2} + \frac{\tan^2\phi}{b^2}\right)}\sqrt{\left(\frac{\sec^2\phi}{a^2} + \frac{\tan^2\phi}{b^2}\right)}}$$

$$= \frac{a^2b^2(e^2\sec^2\phi - 1)}{b^2\sec^2\phi + a^2\tan^2\phi}$$

$$= \frac{a^2b^2(e^2\sec^2\phi - 1)}{a^2(e^2 - 1)\sec^2\phi + a^2\tan^2\phi}$$

$$= \frac{b^2(e^2\sec^2\phi - 1)}{(e^2\sec^2\phi - 1)} = b^2 \quad [(\because b^2 = a^2(e^2 - 1)]$$

Equations of Normals in Different Forms

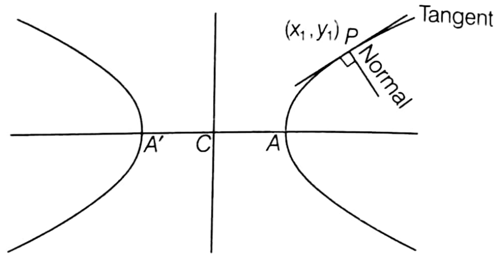
1. Point form :

Theorem : The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

Proof : Since the equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

The slope of tangent at $(x_1, y_1) = \frac{b^2x_1}{a^2y_1}$

$\therefore$ Slope of normal at $(x_1, y_1) = -\frac{a^2y_1}{b^2x_1}$



Hence, the equation of normal at (x_1, y_1)

$$y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$$

or $$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

Remark

The equation of normal at (x_1, y_1) can also be obtained by this method

$$\frac{x - x_1}{a'x_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + b'y_1 + f} \quad \text{...(i)}$$

a', b', g, f, h are obtained by comparing the given hyperbola with

$$a'x^2 + 2hxy + b'y^2 + 2gx + 2fy + c = 0 \quad \text{...(ii)}$$

The denominator of Eq. (i) can easily be remembered by the first two rows of this determinant

i.e. $$\begin{vmatrix} a' & h & g \\ h & b' & f \\ g & f & c \end{vmatrix}$$

Since first row is $a'(x_1) + h(y_1) + g(1)$

and second row i $h(x_1) + b'(y_1) + f(1)$

Here, hyperbola $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \quad \text{...(iii)}$$

On comparing Eqs. (ii) and (iii), we get

$$a' = \frac{1}{a^2}, b' = -\frac{1}{b^2}, g = 0, f = 0, h = 0$$

From Eq. (i), equation of normal of (iii) at (x_1, y_1) is

$$\frac{x - x_1}{\frac{1}{a^2}x_1 + 0 + 0} = \frac{y - y_1}{0 - \frac{y_1}{b^2} + 0}$$

or $$\frac{a^2x}{x_1} - a^2 = -\frac{b^2y}{y_1} + b^2$$

or $$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

2. Parametric form :

Theorem : The equation of normal at $(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $ax\cos\phi + by\cot\phi = a^2 + b^2$

Proof : Since the equation of normals of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2 \quad \text{...(i)}$$

Replacing x_1 by $a\sec\phi$ and y_1 by $b\tan\phi$, then Eq. (i) becomes

$$\frac{a^2x}{a\sec\phi}+\frac{b^2y}{b\tan\phi}=a^2+b^2$$

or $$ax\cos\phi+by\cot\phi=a^2+b^2$$

is equation of normal at $(a\sec\phi, b\tan\phi)$.

3. Slope form :

Theorem : The equations of normals of slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by

$$y=mx\mp\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$$

at the points $\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$

Proof : The equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2 \quad \text{...(i)}$$

Since 'm' is the slope of the normal, then

$$m=-\frac{a^2y_1}{b^2x_1}$$

$$y_1=-\frac{b^2x_1m}{a^2} \quad \text{...(ii)}$$

Since (x_1, y_1) lies on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore$ $$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$$

or $$\frac{x_1^2}{a^2}-\frac{b^4x_1^2m^2}{a^4b^2}=1$$

or $$x_1^2=\frac{a^4}{a^2-b^2m^2}$$

$\therefore$ $$x_1=\pm\frac{a^2}{\sqrt{(a^2-b^2m^2)}}$$

From Eq. (ii), $$y_1=\mp\frac{mb^2}{\sqrt{(a^2-b^2m^2)}}$$

$\therefore$ Equation of normal in terms of slope is

$$y-\left(\mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)=m\left(x-\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}\right)\right)$$

$\Rightarrow$ $$\boldsymbol{y=mx\mp\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}} \quad \text{...(iii)}$$

Thus Eq. (iii) is a normal to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

where m is the slope of the normal.

The coordinates of the point of contact are

$$\left(\pm\frac{a^2}{\sqrt{(a^2-b^2m^2)}}, \mp\frac{mb^2}{\sqrt{(a^2-b^2m^2)}}\right)$$

Comparing Eq. (iii) with $y=mx+c$

$\therefore$ $$c=\mp\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$$

or $$c^2=\frac{m^2(a^2+b^2)^2}{(a^2-m^2b^2)}$$

which is condition of normality, where $y=mx+c$ is the normal of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Remark

Normal other than transverse axis, never passes through the focus.

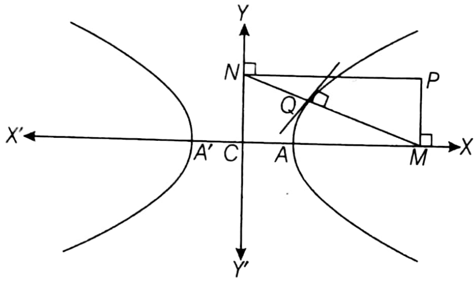
Example 21 A normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N and lines MP and NP are drawn perpendiculars to the axes meeting at P. Prove that the locus of P is the hyperbola

$$a^2x^2-b^2y^2=(a^2+b^2)^2.$$

Sol. The equation of normal at the point $(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$ax\cos\phi + by\cot\phi = a^2 + b^2 \quad \text{...(i)}$$

The normal (i) meets the X-axis in $M\left(\frac{a^2+b^2}{a}\sec\phi, 0\right)$ and Y-axis in $N\left(0, \frac{a^2+b^2}{b}\tan\phi\right)$



$\therefore$ Equation of MP, the line through M and perpendicular to X axis, is

$$x = \left(\frac{a^2+b^2}{a}\right)\sec\phi$$

or
$$\sec\phi = \frac{ax}{(a^2+b^2)} \quad \text{...(ii)}$$

and the equation of NP, the line through N and perpendicular to the Y-axis, is

$$y = \left(\frac{a^2+b^2}{b}\right)\tan\phi$$

or
$$\tan\phi = \frac{by}{(a^2+b^2)} \quad \text{...(iii)}$$

The locus of the point of intersection of MP and NP will be obtained by elminating ϕ from Eqs. (ii) and (iii), we have

$$\sec^2\phi - \tan^2\phi = 1$$

$$\Rightarrow \quad \frac{a^2x^2}{(a^2+b^2)^2} - \frac{b^2y^2}{(a^2+b^2)^2} = 1$$

or
$$a^2x^2 - b^2y^2 = (a^2+b^2)^2$$

is the required locus of P.

Example 22 Prove that the line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.

Sol. The equation of any normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$ax\cos\phi + by\cot\phi = a^2 + b^2$$

or
$$ax\cos\phi + by\cot\phi - (a^2+b^2) = 0 \quad \text{...(i)}$$

The straight line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then Eq. (i) and $lx + my - n = 0$ represent the same line

$$\frac{a\cos\phi}{l} = \frac{b\cot\phi}{m} = \frac{(a^2+b^2)}{n}$$

or
$$\sec\phi = \frac{na}{l(a^2+b^2)} \text{ and } \tan\phi = \frac{nb}{m(a^2+b^2)}$$

$$\because \quad \sec^2\phi - \tan^2\phi = 1$$

$$\therefore \quad \frac{n^2a^2}{l^2(a^2+b^2)^2} - \frac{n^2b^2}{m^2(a^2+b^2)^2} = 1$$

$$\Rightarrow \quad \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$$

Example 23 If the normal at 'ϕ' on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet transverse axis at G, prove that

$$AG \cdot A'G = a^2(e^4\sec^2\phi - 1).$$

where A and A′ are the vertices of the hyperbola.

Sol. The equation of normal at $(a\sec\phi, b\tan\phi)$ to the given hyperbola is $ax\cos\phi + by\cot\phi = (a^2+b^2)$

This meets the transverse axis i.e. X-axis at G. So, the coordinates of G are $\left(\left(\frac{a^2+b^2}{a}\right)\sec\phi, 0\right)$ and the coordinates of the vertices A and A' are $A(a, 0)$ and $A'(-a, 0)$ respectively.

$$\therefore AG \cdot A'G = \left(-a + \left(\frac{a^2+b^2}{a}\right)\sec\phi\right)\left(a + \left(\frac{a^2+b^2}{a}\right)\sec\phi\right)$$

$$= \left(\frac{a^2+b^2}{a}\right)^2\sec^2\phi - a^2$$

$$= (ae^2)^2\sec^2\phi - a^2 = a^2(e^4\sec^2\phi - 1)$$

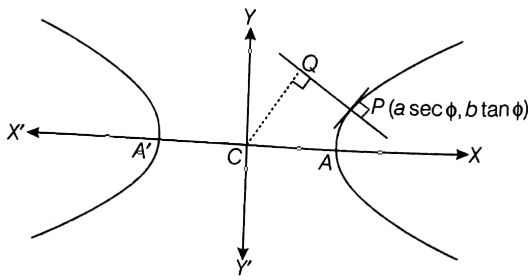
Example 24 Find the locus of the foot of perpendicular from the centre upon any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Sol. Normal at $P(a\sec\phi, b\tan\phi)$ is

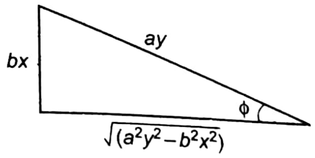
$$ax\cos\phi + by\cot\phi = a^2 + b^2 \quad \text{...(i)}$$

and equation of line perpendicular to Eq. (i) and passing through origin is

$$bx - ay\sin\phi = 0 \quad \text{...(ii)}$$



Eliminating ϕ from Eqs. (i) and (ii), we will get the equation of locus of Q, as from Eq. (ii),



$$\sin\phi = \frac{bx}{ay}$$

$$\therefore \quad \cos\phi = \frac{\sqrt{(a^2y^2 - b^2x^2)}}{ay}$$

$$\text{and} \quad \cot\phi = \frac{\sqrt{(a^2y^2 - b^2x^2)}}{bx}$$

From Eq. (i),

$$ax \times \frac{\sqrt{a^2y^2 - b^2x^2}}{ay} + by \times \frac{\sqrt{(a^2y^2 - b^2x^2)}}{bx} = a^2 + b^2$$

$$\Rightarrow \quad (x^2 + y^2)\sqrt{(a^2y^2 - b^2x^2)} = (a^2 + b^2)xy$$

$$\text{or} \quad (x^2 + y^2)^2(a^2y^2 - b^2x^2) = (a^2 + b^2)^2x^2y^2$$

which is required locus.

Pair of Tangents

Theorem : The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$, lying outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

or

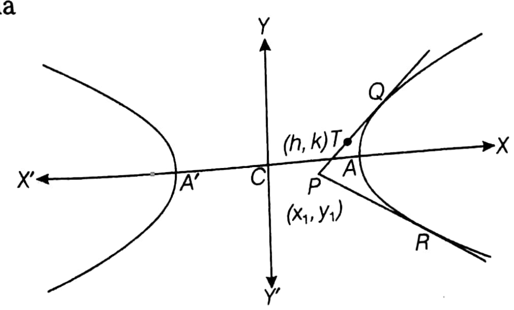
$$\boxed{SS_1 = T^2}$$

where

$$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1;$$

$$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 \text{ and } T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

Proof : Let $T(h, k)$ be any point on the pair of tangents PQ or PR drawn from any external point $P(x_1, y_1)$ to the hyperbola



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \text{ Equation of } PT \text{ is } \quad y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1)$$

$$\text{or} \quad y = \left(\frac{k - y_1}{h - x_1}\right)x + \left(\frac{hy_1 - kx_1}{h - x_1}\right)$$

which is the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \quad c^2 = a^2m^2 - b^2$$

$$\text{or} \quad \left(\frac{hy_1 - kx_1}{h - x_1}\right)^2 = a^2\left(\frac{k - y_1}{h - x_1}\right)^2 - b^2$$

$$\Rightarrow \quad (hy_1 - kx_1)^2 = a^2(k - y_1)^2 - b^2(h - x_1)^2$$

Hence, locus of (h, k) is

$$(xy_1 - x_1y)^2 = a^2(y - y_1)^2 - b^2(x - x_1)^2$$

$$\text{or} \quad (xy_1 - x_1y)^2 = -(b^2x^2 - a^2y^2) - (b^2x_1^2 - a^2y_1^2) - 2(a^2yy_1 - b^2xx_1)$$

$$\text{or} \quad \left(\frac{xy_1 - x_1y}{ab}\right)^2 = -\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) + 2\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)$$

$$\text{or} \quad -\left(\frac{xy_1 - x_1y}{ab}\right)^2 - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) + 1 + \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)^2$$

$$= \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)^2 - 2\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right) + 1$$

$$\text{or} \quad \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

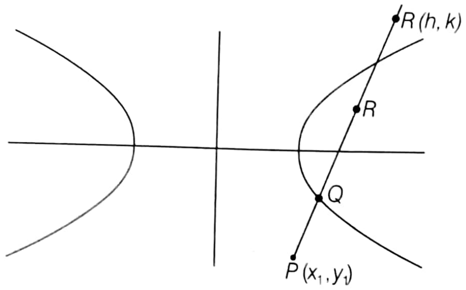
$$\text{or} \quad SS_1 = T^2$$

Aliter :

Let the hyperbola be $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad$ ...(i)

Let $P(x_1, y_1)$ be any point outside the hyperbola.

Let a chord of the hyperbola through the point $P(x_1, y_1)$ cut the hyperbola at Q and R. Let $R(h, k)$ be any arbitrary point on the line PQ (R inside or outside).



Let Q divides PR in the ratio $\lambda : 1$, then coordinates of Q are

$$\left(\frac{\lambda h + x_1}{\lambda + 1}, \frac{\lambda k + y_1}{\lambda + 1}\right) \qquad (\because PQ:QR = \lambda : 1)$$

Since, Q lies on hyperbola (i), then

$$\frac{1}{a^2}\left(\frac{\lambda h + x_1}{\lambda + 1}\right)^2 - \frac{1}{b^2}\left(\frac{\lambda k + y_1}{\lambda + 1}\right)^2 = 1$$

$$\Rightarrow \quad b^2(\lambda h + x_1)^2 - a^2(\lambda k + y_1)^2 = a^2b^2(\lambda + 1)^2$$

$$\Rightarrow \quad (b^2h^2 - a^2k^2 - a^2b^2)\lambda^2 + 2(b^2hx_1 - a^2ky_1 - a^2b^2)\lambda + (b^2x_1^2 - a^2y_1^2 - a^2b^2) = 0 \quad \text{...(ii)}$$

Let PR will become tangent to the hyperbola (i), then roots of Eq. (ii) are equal

$$4(b^2hx_1 - a^2ky_1 - a^2b^2)^2 - 4(b^2h^2 - a^2k^2 - a^2b^2) \times (b^2x_1^2 - a^2y_1^2 - a^2b^2) = 0$$

Dividing by $4a^4b^4$

$$\therefore \quad \left(\frac{hx_1}{a^2} - \frac{ky_1}{b^2} - 1\right)^2 = \left(\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)$$

Hence, locus of $R(h, k)$, i.e. equation of pair of tangents from $P(x_1, y_1)$ is

$$\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)$$

i.e. $$T^2 = SS_1 \quad \text{or} \quad SS_1 = T^2$$

Remark

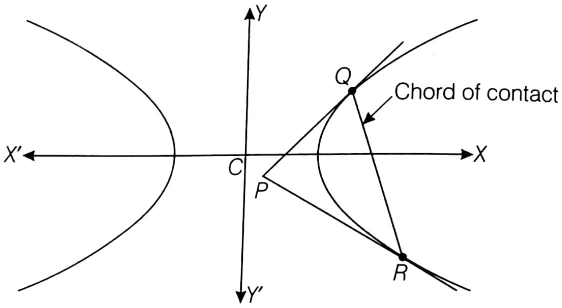
$S = 0$ is the equation of the curve, S_1 is obtained from S by replacing x by x_1 and y by y_1 and $T = 0$ is the equation of the tangent at (x_1, y_1) to $S = 0$.

Chord of Contact

Theorem : If the tangents from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ touch the hyperbola at Q and R, then the equation of the chord of contact QR is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Proof : Let $Q \equiv (x', y')$ and $R \equiv (x'', y'')$



Now, equation of tangents PQ and PR are

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1 \quad \text{...(i)}$$

and $$\frac{xx''}{a^2} - \frac{yy''}{b^2} = 1 \quad \text{...(ii)}$$

Since, Eqs. (i) and (ii) pass through $P(x_1, y_1)$, then

$$\frac{x'x_1}{a^2} - \frac{y'y_1}{b^2} = 1 \quad \text{...(iii)}$$

and $$\frac{x''x_1}{a^2} - \frac{y''y_1}{b^2} = 1 \quad \text{...(iv)}$$

Hence, it is clear that $Q(x', y')$ and $R(x'', y'')$ lie on

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \text{or} \quad \boxed{T = 0}$$

which is **chord of contact** QR.

Equation of the Chord Bisected at a Given Point

Theorem : The equation of the chord of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

bisected at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

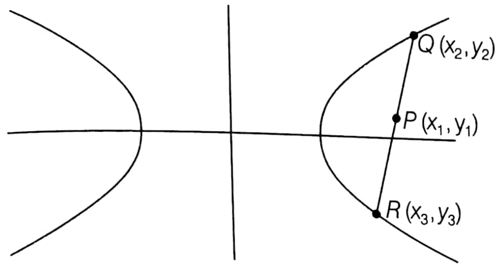
or $\boxed{T = S_1}$ where $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

and $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1.$

Proof : Since equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Let QR be the chord of the hyperbola whose mid-point is $P(x_1, y_1)$. Since Q and R lie on the hyperbola (i).



$$\therefore \quad \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1 \quad \text{...(ii)}$$

and
$$\frac{x_3^2}{a^2} - \frac{y_3^2}{b^2} = 1 \quad \text{...(iii)}$$

Subtracting Eq. (iii) from Eq. (ii),

$$\frac{1}{a^2}(x_2^2 - x_3^2) - \frac{1}{b^2}(y_2^2 - y_3^2) = 0$$

$$\Rightarrow \quad \frac{(x_2 + x_3)(x_2 - x_3)}{a^2} - \frac{(y_2 + y_3)(y_2 - y_3)}{b^2} = 0$$

$$\Rightarrow \quad \frac{y_2 - y_3}{x_2 - x_3} = \frac{b^2(x_2 + x_3)}{a^2(y_2 + y_3)}$$

$$= \frac{b^2\,(2x_1)}{a^2\,(2y_1)} \quad [P \text{ is the mid-point of } QR]$$

$$= \frac{b^2 x_1}{a^2 y_1} \quad \text{...(iv)}$$

$\therefore$ Equation of QR is

$$y - y_1 = \frac{y_2 - y_3}{x_2 - x_3}(x - x_1)$$

$$\Rightarrow \quad y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1) \quad [\text{from Eq. (iv)}]$$

$$\Rightarrow \quad \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$$

$$\Rightarrow \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

$$\Rightarrow \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

$$\Rightarrow \quad T = S_1$$

| Example 25 Find the locus of the mid-points of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtend a right angle at the origin.

Sol. Let (h, k) be the mid-point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2} - \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$$

or
$$\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \quad \text{...(i)}$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (i) is obtained by making homogeneous hyperbola with the help of Eq. (i)

$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\left(\frac{hx}{a^2} - \frac{ky}{b^2}\right)^2}{\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 x^2 - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 y^2$$

$$= \frac{h^2}{a^4}x^2 + \frac{k^2}{b^4}y^2 - \frac{2hk}{a^2b^2}xy \quad \text{...(ii)}$$

The lines represented by Eq. (ii) will be at right angle if

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{h^2}{a^4} - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{k^2}{b^4} = 0$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{h^2}{a^4} + \frac{k^2}{b^4}$$

Hence, the locus of (h, k) is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{x^2}{a^4} + \frac{y^2}{b^4}$$

| Example 26 From the points on the circle $x^2 + y^2 = a^2$, tangents are drawn to the hyperbola $x^2 - y^2 = a^2$; prove that the locus of the middle-points of the chords of contact is the curve $(x^2 - y^2)^2 = a^2(x^2 + y^2)$.

Sol. Since any point on the circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$ chord of contact of this point w.r.t. hyperbola $x^2 - y^2 = a^2$ is

$$x\,(a\cos\theta) - y\,(a\sin\theta) = a^2$$

or
$$x\cos\theta - y\sin\theta = a \quad \text{...(i)}$$

If its mid-point be (h, k), then it is same as

$$T = S_1$$

i.e.
$$hx - ky - a^2 = h^2 - k^2 - a^2$$

or
$$hx - ky = h^2 - k^2 \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$\frac{\cos\theta}{h} = \frac{\sin\theta}{k} = \frac{a}{(h^2 - k^2)}$$

or $\qquad (h^2 - k^2)\cos\theta = ah \qquad$ ...(iii)

and $\qquad (h^2 - k^2)\sin\theta = ak \qquad$ ...(iv)

Squaring and adding Eqs. (iii) and (iv), we get

$$(h^2 - k^2)^2 = a^2h^2 + a^2k^2$$

$$\Rightarrow \quad (h^2 - k^2)^2 = a^2(h^2 + k^2)$$

Hence, the required locus is

$$(x^2 - y^2)^2 = a^2(x^2 + y^2).$$

Example 27 Prove that the locus of the middle-points of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which pass through a fixed point (α, β) is a hyperbola whose centre is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

Sol. Let the mid-point of the chord be (h, k). The equation of the chord whose mid-point is (h, k) is

$$\frac{hx}{a^2} - \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1 \text{ or } \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

It passes through (α, β), then

$$\frac{\alpha h}{a^2} - \frac{\beta k}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

$\therefore$ locus of (h, k) is

$$\frac{x\alpha}{a^2} - \frac{y\beta}{b^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$\Rightarrow \quad \frac{1}{a^2}(x^2 - \alpha x) - \frac{1}{b^2}(y^2 - \beta y) = 0$$

$$\Rightarrow \quad \frac{1}{a^2}\left\{\left(x - \frac{\alpha}{2}\right)^2 - \frac{\alpha^2}{4}\right\} - \frac{1}{b^2}\left\{\left(y - \frac{\beta}{2}\right)^2 - \frac{\beta^2}{4}\right\} = 0$$

or $$\frac{\left(x - \frac{\alpha}{2}\right)^2}{a^2} - \frac{\left(y - \frac{\beta}{2}\right)^2}{b^2} = \frac{1}{4}\left\{\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2}\right\} = \lambda \text{ (say)}$$

$$\therefore \quad \frac{\left(x - \frac{\alpha}{2}\right)^2}{a^2\lambda} - \frac{\left(y - \frac{\beta}{2}\right)^2}{b^2\lambda} = 1$$

The centre of this hyperbola is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

Exercise for Session 2

1. The tangents from $(1, 2\sqrt{2})$ to the hyperbola $16x^2 - 25y^2 = 400$ include between them an angle equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

2. If $4x^2 + \lambda y^2 = 45$ and $x^2 - 4y^2 = 5$ cut orthogonally, then the value of λ is

(a) $\frac{1}{9}$ (b) $\frac{1}{3}$ (c) 9 (d) 18

3. If the tangent at the point $(2\sec\phi, 3\tan\phi)$ of the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is parallel to $3x - y + 4 = 0$, then the value of ϕ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{12}$

4. If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is

(a) $(-2, \sqrt{6})$ (b) $(-5, 2\sqrt{6})$ (c) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (d) $(4, -\sqrt{6})$

5. The equation of the chord of hyperbola $25x^2 - 16y^2 = 400$, whose mid-point is (5, 3), is

(a) $115x - 47y = 434$ (b) $125x - 48y = 481$
(c) $127x - 49y = 488$ (d) $155x - 67y = 574$

6. The value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$ is

(a) $\sqrt{\left(\frac{17}{20}\right)}$ (b) $-\sqrt{\left(\frac{17}{21}\right)}$ (c) $\sqrt{\left(\frac{20}{17}\right)}$ (d) $-\sqrt{\left(\frac{21}{17}\right)}$

7. P is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then $OT \cdot ON$ is equal to

(a) a^2 (b) b^2 (c) e^2 (d) b^2/a

8. If $x=9$ is the chord of contact of the hyperbola $x^2-y^2=9$, then the equation of the corresponding pair of tangents, is

(a) $9x^2-8y^2+18x-9=0$ (b) $9x^2-8y^2-18x+9=0$

(c) $9x^2-8y^2-18x-9=0$ (d) $9x^2-8y^2+18x+9=0$

9. Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, when $\theta+\phi=\frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h,k) is the point of intersection of the normals at P and Q, then k is equal to

(a) $\left(\frac{a^2+b^2}{a}\right)$ (b) $-\left(\frac{a^2+b^2}{a}\right)$ (c) $\left(\frac{a^2+b^2}{b}\right)$ (d) $-\left(\frac{a^2+b^2}{b}\right)$

10. The tangent at a point P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ passes through the point $(0,-b)$ and the normal at P passes through $(2a\sqrt{2},0)$; then eccentricity of the hyperbola is

(a) $\frac{5}{4}$ (b) $\frac{3}{2}$ (c) $\sqrt{2}$ (d) $2\sqrt{2}$

11. A tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points P and Q. Find the locus of the mid-point of PQ.

12. A line through the origin meets the circle $x^2+y^2=a^2$ at P and the hyperbola $x^2-y^2=a^2$ at Q. Prove that the locus of the point of intersection of tangent at P to the circle with the tangent at Q to the hyperbola is the curve.

13. Normals are drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the points $P(a\sec\theta_1, b\tan\theta_1)$ and $Q(a\sec\theta_2, b\tan\theta_2)$ meeting the conjugate axis at G_1 and G_2 respectively. If $\theta_1+\theta_2=\pi/2$, prove that

$$CG_1 \cdot CG_2 = \frac{a^2e^4}{(e^2-1)}$$

where C is the centre of the hyperbola and e is its eccentricity.

14. Chords of the hyperbola $x^2-y^2=a^2$ touch the parabola $y^2=4ax$. Prove that the locus of their middle-points is the curve $y^2(x-a)=x^3$.

Session 3

Diameter, Conjugate Diameters, Properties of Hyperbola, Intersection of Conjugate Diameters and Hyperbola, Director Circle, Asymptotes, Rectangular Hyperbola, The Rectangular Hyperbola $xy=c^2$, Reflection Property of a Hyperbola, Equation of a Hyperbola Referred to Two Perpendicular Lines

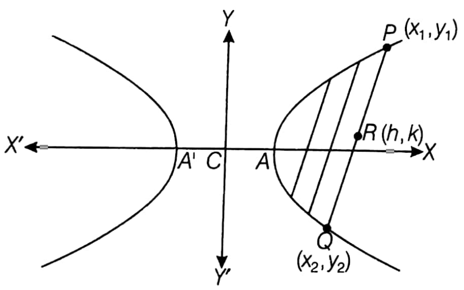
Diameter

The locus of the middle points of a system of parallel chords of a hyperbola is called a diameter and the point where the diameter intersects the hyperbola is called the vertex of the diameter.

Theorem : The equation of a diameter bisecting a system of parallel chords of slope m of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$y=\frac{b^2}{a^2m}x$$

Proof : Let $y=mx+c$ be a system of a parallel chords to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for different chords. As c varies, m remain constant.



Let the extremities of any chord PQ of the set be $P(x_1,y_1)$ and $Q(x_2,y_2)$ and let its middle point be $R(h,k)$. Then solving equations

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ and } y=mx+c$$

we get
$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

$\Rightarrow$
$$(a^2m^2-b^2)x^2+2mca^2x+a^2(c^2+b^2)=0$$

Since, x_1 and x_2 be the roots of this equation, then

$$x_1+x_2=\frac{-2mca^2}{a^2m^2-b^2} \quad \text{...(i)}$$

Since, (h,k) be the middle point of QR, then

$$h=\frac{x_1+x_2}{2}$$

then from Eq. (i), $h=-\frac{mca^2}{a^2m^2-b^2}$

but (h,k) lies on $y=mx+c$

$\therefore$ $k=mh+c$

or $c=k-mh$ ...(ii)

From Eqs. (i) and (ii),

$$h=-\frac{ma^2(k-mh)}{a^2m^2-b^2}$$

$$\Rightarrow \quad a^2m^2h-b^2h=-ma^2k+m^2a^2h$$

$$\Rightarrow \quad -b^2h=-ma^2k \text{ or } k=\frac{b^2h}{a^2m}$$

Hence, locus of $R(h,k)$ is $y=\frac{b^2x}{a^2m}$

which is the diameter of the hyperbola passing through (0, 0).

Aliter :

Let (h,k) be the middle-point of the chord $y=mx+c$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then

$$T = S_1$$

$$\Rightarrow \quad \frac{xh}{a^2} - \frac{yk}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

$$\therefore \quad \text{Slope} = \frac{hb^2}{a^2k} = m$$

$$\Rightarrow \quad k = \frac{b^2h}{a^2m}$$

Hence, the locus of mid-point is $y = \frac{b^2x}{a^2m}$.

Conjugate Diameters

Two diameters are said to be conjugate when each bisects all chords parallel to the others.

If $y = mx$, $y = m_1x$ be conjugate diameters, then $mm_1 = \frac{b^2}{a^2}$.

Let $y = m_1x + c$ be a set of chords parallel to $y = m_1x$, then the diameter $y = \frac{b^2}{a^2m}x$ bisects them all. But being the conjugate diameter $y = mx$ also bisects them.

Hence, these two lines must be identical

$$\therefore \quad m = \frac{b^2}{a^2m_1} \Rightarrow \boxed{mm_1 = \frac{b^2}{a^2}}$$

Properties of Hyperbola

Prop. 1. *If a pair of diameters be conjugate with respect to a hyperbola, they are conjugate with respect to its conjugate hyperbola also.*

Let $y = mx$ and $y = m_1x$ be two conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then

$$mm_1 = \frac{b^2}{a^2} \qquad \text{...(i)}$$

Now conjugate hyperbola of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\frac{x^2}{-a^2} - \frac{y^2}{-b^2} = 1 \qquad \text{...(ii)}$$

If $y = m_2x$ and $y = m_3x$ are the conjugate diameters of Eq. (ii), then

$$m_2m_3 = \frac{(-b^2)}{(-a^2)}$$

$$= \frac{b^2}{a^2} = mm_1 \qquad \text{[from Eq. (i)]}$$

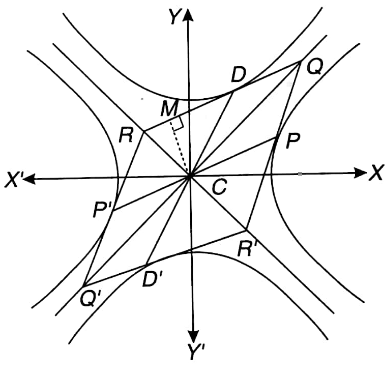
This proves the proposition.

Prop. 2. *The parallelogram formed by the tangents at the extremities of conjugate diameters of a hyperbola has its vertices lying on the asymptotes and is of constant area.*

Since, P and D lie on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and its conjugate diameter $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

then coordinates of P and D are $(a \sec \phi, b \tan \phi)$ and $(a \tan \phi, b \sec \phi)$ respectively.



Then $D' \equiv (-a \tan \phi, -b \sec \phi)$

and $P' \equiv (-a \sec \phi, -b \tan \phi)$

Since, equations of asymptotes CQ and CR are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ respectively and the equations of tangents at P, P', D and D' are

$$\frac{x}{a}\sec \phi - \frac{y}{b}\tan \phi = 1 \qquad \text{...(i)}$$

$$-\frac{x}{a}\sec \phi + \frac{y}{b}\tan \phi = 1 \qquad \text{...(ii)}$$

$$-\frac{x}{a}\tan \phi + \frac{y}{b}\sec \phi = 1 \qquad \text{...(iii)}$$

and

$$\frac{x}{a}\tan \phi - \frac{y}{b}\sec \phi = 1 \qquad \text{...(iv)}$$

respectively.

Now, the lines (i) and (ii) are parallel and so are Eqs. (iii) and (iv).

Hence these tangents form a parallelogram. Solving Eqs. (i) and (iii), we get the coordinate of Q as $[a(\sec\phi+\tan\phi), b(\sec\phi+\tan\phi)]$ which clearly lies on the asymptote $y=\frac{b}{a}x$ similarly the other points of intersection lies on the asymptotes.

The equations of PCP' and DCD' are

$$y=\frac{b}{a}\frac{\tan\phi}{\sec\phi}x \text{ or } bx\tan\phi-ay\sec\phi=0 \quad \text{...(v)}$$

and $$y=\frac{b}{a}\frac{\sec\phi}{\tan\phi}x \text{ or } bx\sec\phi-ay\tan\phi=0 \quad \text{...(vi)}$$

Hence by symmetry.

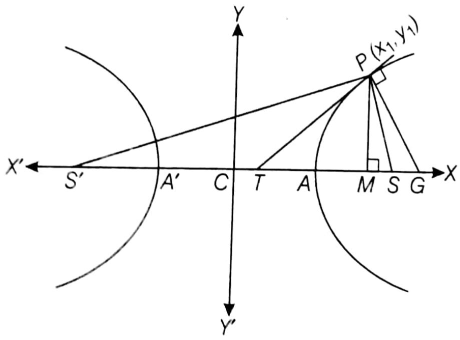
Area of parallelogram $QRQ'R'=4$ (Area of parallelogram $QDCP$)

$$=4\cdot CP\times(\text{Perpendicular length from } C \text{ on } QD)$$

$$=4\cdot\sqrt{(a^2\sec^2\phi+b^2\tan^2\phi)}\times\frac{1}{\sqrt{\frac{\tan^2\phi}{a^2}+\frac{\sec^2\phi}{b^2}}}$$

$$\boxed{=4ab=\text{ Constant.}}$$ Remember.

Prop. 3. *If the normal at P meets the transverse axis in G, then SG = e · SP. Prove also that the tangent and normal bisect the angle between the focal distances of P.*



Let the coordinates of P be (x_1, y_1). The equation of normal at P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2 \quad \text{... (i)}$$

The normal (i) meets the X-axis i.e. $y=0$ in Eq. (i), then coordinates of G are

$$\left(\frac{(a^2+b^2)}{a^2}x_1, 0\right) \text{ or } (e^2x_1, 0)$$

$\therefore$ $$CG=e^2x_1$$

Now $$SG=CG-CS$$

$$=e^2x_1-ae=e(ex_1-a)=e\cdot SP$$

Similarly, $$S'G=e\cdot S'P$$

$\therefore$ $$\frac{SG}{S'G}=\frac{SP}{S'P}$$

This relation shows that the normal PG is the external bisector of the angle SPS'. The tangent PT being perpendicular to PG is therefore the internal bisector of the angle SPS'.

Prop. 4. *If a pair of conjugate diameters meet the hyperbola in P, P' and its conjugate in D, D', then the asymptotes bisect PD, PD', P'D and P'D'.*

The coordinates of four points P, D, P', D' are $(a\sec\phi, b\tan\phi)$; $(a\tan\phi, b\sec\phi)$; $(-a\sec\phi, -b\tan\phi)$; $(-a\tan\phi, -b\sec\phi)$ respectively.

If (h, k) be the middle point of PD, then

$$h=\frac{a}{2}(\sec\phi+\tan\phi)$$

and $$k=\frac{b}{2}(\tan\phi+\sec\phi)$$

$\therefore$ $$\frac{h}{k}=\frac{a}{b}$$

$\therefore$ Locus of mid-point (h, k) is $y=\frac{b}{a}x$ which is equation of asymptote of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Similarly other mid-points lie on the other asymptotes.

Intersection of Conjugate Diameters and Hyperbola

To prove that of a pair of conjugate diameters of a hyperbola, only one meets the curve in real points

Let $$y=mx \quad \text{...(i)}$$

and $$y=m_1x \quad \text{...(ii)}$$

be a pair of conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

$$mm_1=\frac{b^2}{a^2} \quad \text{...(iii)}$$

On solving Eq. (i) and the equation of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$$\frac{x^2}{a^2}-\frac{m^2x^2}{b^2}=1$$

or $$x^2=\frac{a^2b^2}{b^2-a^2m^2} \quad \text{...(iv)}$$

Similarly Eq. (ii) meets the hyperbola at points whose abscissa are given by

$$x^2=\frac{a^2b^2}{b^2-a^2m_1^2} \quad \text{...(v)}$$

The two values of x given by Eq. (iv) will be real if $b^2-a^2m^2>0$ i.e. $m<\left(\frac{b}{a}\right)$

i.e. $$\frac{b^2}{a^2m_1}<\left(\frac{b}{a}\right) \qquad \left\{\because mm_1=\frac{b^2}{a^2}\right\}$$

$\Rightarrow$ $$\frac{b}{a}<m_1 \text{ i.e. if } b^2<a^2m_1^2$$

i.e. if $b^2-a^2m_1^2<0$

Then from Eq. (v) the values of x are imaginary.

Hence, if Eq. (i) meets the hyperbola in real points then Eq. (ii) meets it in imaginary points and *vice-versa.*

Remark

If CD is the conjugate diameter of a diameter CP of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where P is $(a\sec\phi, b\tan\phi)$, then equation of CP, where C is $(0,0)$ is

$$y-0=\frac{b\tan\phi-0}{a\sec\phi-0}(x-0) \Rightarrow y=\frac{b}{a}\sin\phi.x$$

Comparing this with $y=mx$

$$m=\frac{b}{a}\sin\phi$$

$\because$ $$mm'=\frac{b^2}{a^2} \Rightarrow \frac{b}{a}\sin\phi\times m'=\frac{b^2}{a^2}$$

$\therefore$ $$m'=\left(\frac{b}{a}\right)\text{cosec }\phi$$

$\therefore$ Equation of conjugate diameter is $y=m'x=(b/a)\text{cosec }\phi.x$ on solving

$$y=(b/a)\text{cosec }\phi\cdot x \quad \text{and} \quad -\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then we get coordinates of D

i.e. $\quad D\equiv(a\tan\phi, b\sec\phi)$

Example 28 If a pair of conjugate diameters meets the hyperbola and its conjugate in P and D respectively, then prove that $CP^2-CD^2=a^2-b^2$.

Sol. $\because$ Coordinates of P and D are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.

Then $(CP)^2-(CD)^2=a^2\sec^2\phi$

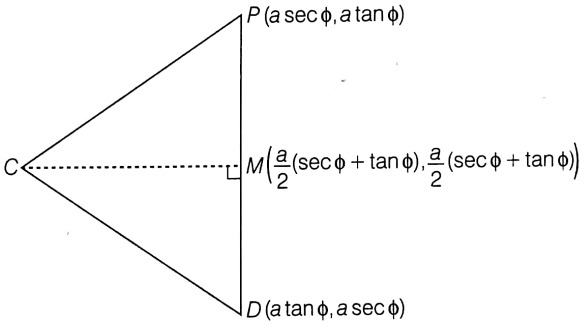
$$+b^2\tan^2\phi-a^2\tan^2\phi-b^2\sec^2\phi$$

$$=a^2(\sec^2\phi-\tan^2\phi)-b^2(\sec^2\phi-\tan^2\phi)$$

$$=a^2(1)-b^2(1)=a^2-b^2$$

Example 29 For the hyperbola $x^2-y^2=a^2$, prove that the triangle CPD is isosceles and has constant area, where CP and CD are a pair of its conjugate diameters.

Sol. Since, CP and CD are the conjugate diameters of $x^2-y^2=a^2$, hence coordinates of P and D are $P\equiv(a\sec\phi, a\tan\phi)$, $D\equiv(a\tan\phi, a\sec\phi)$ respectively.



$\therefore$ $$CP=\sqrt{a^2(\sec^2\phi+\tan^2\phi)}=CD \qquad [\because C=0(0,0)]$$

Hence, ΔCPD is isosceles triangle.

Draw $\perp$ from C on PD, where M is the mid-point of PD

$\therefore$ $$M\equiv\left(\frac{a}{2}(\sec\phi+\tan\phi), \frac{a}{2}(\sec\phi+\tan\phi)\right)$$

$\therefore$ $$CM=\frac{|(\sec\phi+\tan\phi)|}{2}\sqrt{(a^2+a^2)}$$

and $$PD=\sqrt{[a(\sec\phi-\tan\phi)]^2+[a(\sec\phi-\tan\phi)]^2}$$

$$=|\sec\phi-\tan\phi|\sqrt{(a^2+a^2)}$$

$\therefore$ Area of $\Delta PCD=\frac{1}{2}\cdot PD\cdot CM$

$$=\frac{1}{4}(a^2+a^2)=\boxed{\frac{a^2}{2}=\text{constant}}$$

Example 30 Find the condition for the lines $Ax^2+2Hxy+By^2=0$ to be conjugate diameters of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Sol. Let the lines represented by

$$Ax^2+2Hxy+By^2=0$$

are $$y=mx \quad \text{and} \quad y=m_1x$$

$\therefore$ $$mm_1=\frac{A}{B} \quad \text{...(i)}$$

But $y=mx$ and $y=m_1x$ are the conjugate diameters of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

then $\qquad mm_1=\dfrac{b^2}{a^2}$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), $\dfrac{A}{B}=\dfrac{b^2}{a^2}$

or $\qquad a^2A=b^2B$

which is the required condition.

| Example 31 If the lines $lx+my+n=0$ passes through the extremities of a pair of conjugate diameters of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, show that $a^2l^2-b^2m^2=0$.

Sol. The extremities of a pair of conjugate diameters of $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.

According to the question, since extremities of a pair of conjugate diameters lie on $lx+my+n=0$

$\therefore \qquad l(a\sec\phi)+m(b\tan\phi)+n=0$ ...(i)

$\Rightarrow \qquad l(a\tan\phi)+m(b\sec\phi)+n=0$ ...(ii)

then from Eq. (i), $al\sec\phi+bm\tan\phi=-n$

or $a^2l^2\sec^2\phi+b^2m^2\tan^2\phi+2ablm\sec\phi\tan\phi=n^2$ ...(iii)

and from Eq. (ii), $al\tan\phi+bm\sec\phi=-n$

or $a^2l^2\tan^2\phi+b^2m^2\sec^2\phi+2ablm\sec\phi\tan\phi=n^2$ ...(iv)

then subtracting Eq. (iv) from Eq. (iii),

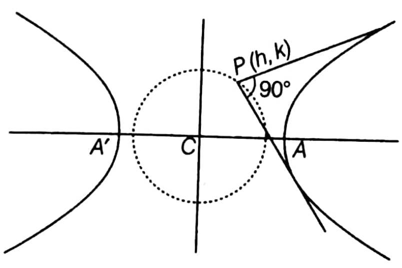
$$a^2l^2(\sec^2\phi-\tan^2\phi)+b^2m^2(\tan^2\phi-\sec^2\phi)=0$$

or $\qquad a^2l^2-b^2m^2=0$

Director Circle

The locus of the point of intersection of the tangents to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, which are perpendicular to each other is called director circle.

Let any tangent in terms of slope of hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is



$$y=mx+\sqrt{(a^2m^2-b^2)}$$

It passes through (h,k)

$\therefore \qquad k=mh+\sqrt{(a^2m^2-b^2)}$

or $\qquad (k-mh)^2=a^2m^2-b^2$

$\Rightarrow \qquad k^2+m^2h^2-2mhk=a^2m^2-b^2$

$\Rightarrow \qquad m^2(h^2-a^2)-2hkm+k^2+b^2=0$

It is quadratic equation in m. Let slopes of two tangents are m_1 and m_2

$\therefore \qquad m_1m_2=\dfrac{k^2+b^2}{h^2-a^2}$

$\qquad -1=\dfrac{k^2+b^2}{h^2-a^2}$ ($\because$ tangents are perpendicular)

$\Rightarrow \qquad -h^2+a^2=k^2+b^2$

or $\qquad h^2+k^2=a^2-b^2$

Hence, locus of $P(h,k)$ is

$$x^2+y^2=a^2-b^2 \qquad (a>b)$$

Aliter :

If $\quad$ tangents $y=mx+\sqrt{(a^2m^2-b^2)}$ ...(i)

and $\qquad y=-\dfrac{x}{m}+\sqrt{\left\{a^2\left(-\dfrac{1}{m}\right)^2-b^2\right\}}$ ...(ii)

touch the hyperbola and intersects at right angles

$\therefore$ From Eq. (i),

$$y-mx=\sqrt{(a^2m^2-b^2)} \qquad \text{...(iii)}$$

Eq. (ii) can be rewritten as

$$x+my=\sqrt{(a^2-b^2m^2)} \qquad \text{...(iv)}$$

Squaring and adding Eqs. (iii) and (iv), then

$$(y-mx)^2+(x+my)^2=a^2m^2-b^2+a^2-b^2m^2$$

$$(1+m^2)(x^2+y^2)=(1+m^2)(a^2-b^2)$$

Hence, $x^2+y^2=a^2-b^2$ is the **director circle** of the hyperbola.

Remarks

1. For director circle of $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, a must be greater than b. If $a<b$, then director circle $x^2+y^2=a^2-b^2$ does not exist.
2. The equation of director circle of $-\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $x^2+y^2=b^2-a^2$ $(b>a)$. If $b<a$, then director circle does not exist.

Example 32 If any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre C, meets its director circle in P and Q, show that CP and CQ are conjugate semi-diameters of the hyperbola.

Sol. Since, equation of hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

Equation of tangent at $(a \sec \phi, b \tan \phi)$ on Eq. (i) is

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1 \qquad \text{...(ii)}$$

Equation of director circle of Eq. (i) is

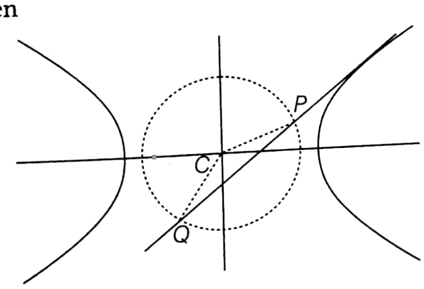
$$x^2+y^2=a^2-b^2 \qquad \text{...(iii)}$$

Equation of lines joining the point of intersection of Eqs. (i) and (ii) to the origin is obtained by making Eq. (iii) homogeneous with the help of Eq. (ii).

$$\therefore \quad x^2+y^2=(a^2-b^2)\left(\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi\right)^2$$

or
$$x^2\left(1-\frac{(a^2-b^2)\sec^2\phi}{a^2}\right)+y^2\left(1-\frac{(a^2-b^2)\tan^2\phi}{b^2}\right)+\frac{2(a^2-b^2)\sec\phi\tan\phi}{ab}xy=0 \qquad \text{...(iv)}$$

Let m and m' represent the slopes of the lines given by Eq. (iv), then



$$mm'=\frac{1-\dfrac{(a^2-b^2)\sec^2\phi}{a^2}}{1-\dfrac{(a^2-b^2)\tan^2\phi}{b^2}}=\frac{b^2}{a^2}\cdot\frac{a^2-(a^2-b^2)\sec^2\phi}{b^2-(a^2-b^2)\tan^2\phi}$$

$$=\frac{b^2}{a^2}\cdot\frac{b^2\sec^2\phi-a^2\tan^2\phi}{b^2\sec^2\phi-a^2\tan^2\phi}=\frac{b^2}{a^2}$$

Hence, the lines CP and CQ are conjugate semi-diameters of the hyperbola.

Asymptotes

An asymptotes of any hyperbola or a curve is a straight line which touches in it two points at infinity.

Asymptotes of hyperbola : *The equations of two asymptotes of the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *are* $y=\pm\frac{b}{a}x$ *or* $\boxed{\frac{x}{a}\pm\frac{y}{b}=0.}$

Proof : Let $y=mx+c$ be an asymptote of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

Substituting the value of y in Eq. (i),

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

or
$$(a^2m^2-b^2)x^2+2a^2mcx+a^2(b^2+c^2)=0 \qquad \text{...(ii)}$$

If the line $y=mx+c$ is an asymptote to the given hyperbola, then it touches the hyperbola at infinity. So both roots of Eq. (ii) must be infinite.

$$\therefore \quad a^2m^2-b^2=0$$

and
$$-2a^2mc=0$$

then
$$m=\pm\frac{b}{a} \quad \text{and} \quad c=0$$

Substituting the value of m and c in $y=mx+c$, we get

$$y=\pm\frac{b}{a}x \Rightarrow \frac{x}{a}\pm\frac{y}{b}=0$$

Aliter :

(The difference between the second degree curve and pair of asymptotes is constant.)

$\because$ Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore$ Pair of asymptotes is $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\lambda=0 \qquad \text{... (i)}$

Eq. (i) represents a pair of lines, then $\Delta=0$

$$\therefore \quad \frac{1}{a^2}\cdot\left(-\frac{1}{b^2}\right)\cdot\lambda+0-0-0-\lambda\cdot 0=0$$

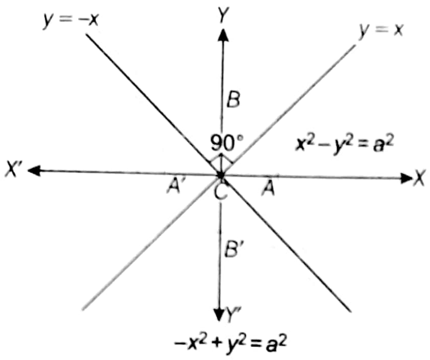
$$\therefore \quad \lambda=0$$

From Eq. (i), pair of asymptotes is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

or
$$y=\pm\frac{b}{a}x \text{ or } \frac{x}{a}\pm\frac{y}{b}=0$$

Remarks

1. If $b=a$, then $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ reduces to $x^2-y^2=a^2$. The asymptotes of rectangular hyperbola $x^2-y^2=a^2$ are $y=\pm x$ which are at right angles.
2. A hyperbola and its conjugate hyperbola have the same asymptotes.
3. The angle between the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\boxed{2\tan^{-1}\left(\frac{b}{a}\right)}$



4. If the angle between the asymptotes of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\boxed{2\theta, \text{ then } e=\sec\theta.}$

5. The asymptotes pass through the centre of the hyperbola.

6. The bisectors of the angles between the asymptotes are the coordinate axes.

7. Let $H \equiv \frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$

$$A \equiv \frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \quad \text{and} \quad C \equiv \frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$$

be the equation of the hyperbola, asymptotes and the conjugate hyperbola respectively, then clearly $\boxed{C+H=2A}$

| Example 33 Find the asymptotes of the hyperbola $xy-3y-2x=0$.

Sol. Since equation of a hyperbola and its asymptotes differ in constant terms only,

$\therefore$ Pair of asymptotes is given by

$$xy-3y-2x+\lambda=0 \qquad \text{...(i)}$$

where, λ is any constant such that it represents two straight lines.

$$\therefore \quad abc+2fgh-af^2-bg^2-ch^2=0$$

$$\Rightarrow \quad 0+2\times-\frac{3}{2}\times-1\times\frac{1}{2}-0-0-\lambda\left(\frac{1}{2}\right)^2=0$$

$$\therefore \quad \lambda=6$$

From Eq. (i), the asymptotes of given hyperbola are given by

$$xy-3y-2x+6=0$$

or
$$(y-2)(x-3)=0.$$

$\therefore$ Asymptotes are $x-3=0$ and $y-2=0$.

| Example 34 The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x+3y=0$ and $3x+2y=0$. If the hyperbola passes through the point (5, 3), show that its equation is

$$(2x+3y-8)(3x+2y+7)=154.$$

Sol. Let the asymptotes be $2x+3y+\lambda=0$ and $3x+2y+\mu=0$.

Since asymptotes passes through (1, 2), then

$$\lambda=-8 \text{ and } \mu=-7$$

Thus, the equation of asymptotes are

$$2x+3y-8=0 \quad \text{and} \quad 3x+2y-7=0$$

Let the equation of hyperbola be

$$(2x+3y-8)(3x+2y-7)+\nu=0 \qquad \text{... (i)}$$

It passes through (5,3), then

$$(10+9-8)(15+6-7)+\nu=0$$

$$\Rightarrow \quad 11\times14+\nu=0$$

$$\therefore \quad \gamma=-154$$

Putting the value of ν in Eq. (i), we obtain

$$(2x+3y-8)(3x+2y-7)-154=0$$

which is the equation of required hyperbola.

| Example 35 Show that the tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes and that the portion of it intercepted between the asymptotes is bisected at the point of contact.

Sol. Let $P(a\sec\phi, b\tan\phi)$ be any point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{... (i)}$$

Asymptotes of Eq. (i) are

$$y=\pm\frac{b}{a}x$$

Equation of tangent of Eq. (i) at $P(a\sec\phi, b\tan\phi)$ is

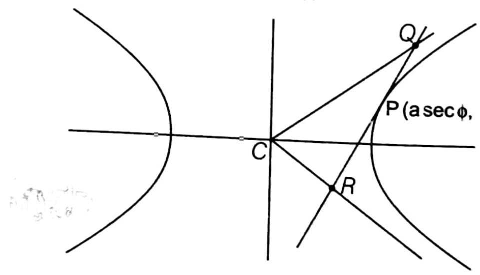
$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1 \qquad \text{... (ii)}$$

Solving Eq. (ii) and $y=\frac{b}{a}x$

We get $\frac{x}{a}(\sec\phi-\tan\phi)=(\sec^2\phi-\tan^2\phi)$

$$\therefore \quad x=a(\sec\phi+\tan\phi)$$

and
$$y=b(\sec\phi+\tan\phi)$$



Let $Q\equiv[a(\sec\phi+\tan\phi), b(\sec\phi+\tan\phi)]$

Now solving Eq. (ii) and $y=-\frac{b}{a}x$

We get, $\frac{x}{a}(\sec\phi+\tan\phi)=(\sec^2\phi-\tan^2\phi)$

or
$$x=a(\sec\phi-\tan\phi)$$

and
$$y=-b(\sec\phi-\tan\phi)$$

Let $R\equiv[a(\sec\phi-\tan\phi), -b(\sec\phi-\tan\phi)]$

Mid-point of QR is $(a\sec\phi, b\tan\phi)$ which is coordinate of P.

Area of $\Delta CQR=\frac{1}{2}|(x_1y_2-x_2y_1)|$

$$=\frac{1}{2}|-ab-ab|=ab=\text{constant.}$$

Rectangular Hyperbola

A hyperbola whose asymptotes include a right angle is said to be rectangular hyperbola.

OR

If the lengths of transverse and conjugate axes of any hyperbola be equal, it is called **rectangular or equilateral hyperbola**.

According to the first definition

$$2\tan^{-1}\left(\frac{b}{a}\right)=\frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{b}{a}\right)=\frac{\pi}{4} \Rightarrow \frac{b}{a}=1$$

$$\boxed{a=b}$$

then, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ becomes $x^2-y^2=a^2$

According to the second definition

When $a=b$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ becomes

$$x^2-y^2=a^2$$

$$\text{Eccentricity, } e=\sqrt{\left(1+\frac{b^2}{a^2}\right)}=\boxed{\sqrt{2}}$$

Hence, $x^2-y^2=a^2$ is the general form of the equation of the rectangular hyperbola.

Remark

All the results of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are applicable to the hyperbola $x^2-y^2=a^2$ after changing b by a.

The Rectangular Hyperbola $xy=c^2$

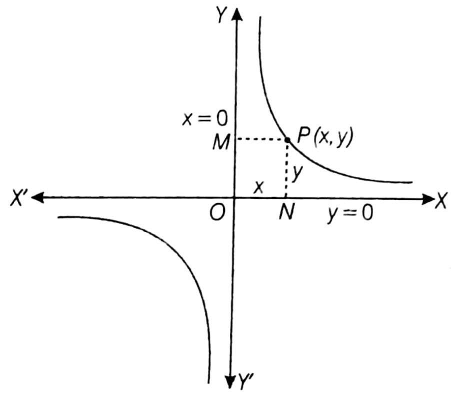
When the centre of any rectangular hyperbola be at the origin and its asymptotes coincide with the coordinate axes its, equation is $\boxed{xy=c^2.}$

Since, asymptotes coincides the coordinate axes.

Hence, asymptotes are $y=0$ and $x=0$.

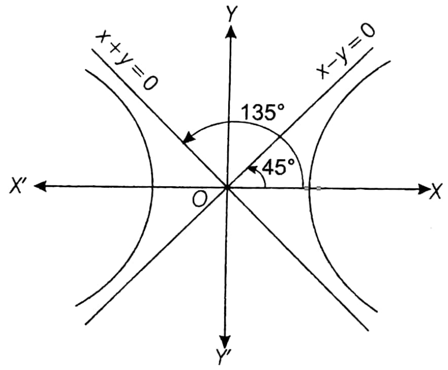
$\therefore$ Combined equation of asymptotes is $\boxed{xy=0}$ Now equation of any hyperbola and its asymptotes differ in respect of constant terms only.

$\therefore$ Equation of rectangular hyperbola is $xy=c^2$ where c is any constant.



Second method (By rotation of axes) :

The equation of rectangular hyperbola is $x^2-y^2=a^2$ and its asymptotes are $x-y=0$ and $x+y=0$. Since, asymptotes are inclined at 45° and 135° to the X-axis respectively.



If we rotate the axes through $\theta=-45°$ without changing the origin. Thus, when we replace (x, y) by

$$[x\cos(-45°)-y\sin(-45°),\ x\sin(-45°)+y\cos(-45°)]$$

i.e. $\left(\frac{x+y}{\sqrt{2}}, \frac{-x+y}{\sqrt{2}}\right)$

then equation $x^2-y^2=a^2$ reduces to

$$\left(\frac{x+y}{\sqrt{2}}\right)^2-\left(\frac{-x+y}{\sqrt{2}}\right)^2=a^2$$

or $\quad \frac{1}{2}\{(x+y)^2-(-x+y)^2\}=a^2$

$\Rightarrow \quad \frac{1}{2}(2y)(2x)=a^2$

or $\quad \boxed{xy=\frac{a^2}{2}}=\left(\frac{a}{\sqrt{2}}\right)^2=c^2 \quad$ (say)

or $\quad xy=c^2$

Study of Hyperbola $xy = c^2$

(i) Vertices : $A(c,c)$ and $A'(-c,-c)$.

(ii) Transverse axis : $y = x$.

(iii) Conjugate axis : $y = -x$.

(iv) Foci : $S(c\sqrt{2}, c\sqrt{2})$ and $S'(-c\sqrt{2}, -c\sqrt{2})$.

(v) Length of latusrectum = Length of $AA' = 2\sqrt{2}\,c$.

(vi) Equation of auxiliary circle : $x^2 + y^2 = 2c^2$.

(vii) Equation of director circle $x^2 + y^2 = 0$.

(viii) Asymptotes : $x = 0, y = 0$.

Remarks

1. The equations of the asymptotes and the conjugate hyperbola of the rectangular hyperbola $xy = c^2$, where the axes are the asymptotes, are $xy = 0$ and $xy = -c^2$ respectively.
2. The equation of a rectangular hyperbola having coordinate axes as its asymptotes is $xy = c^2$. If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$ then its equation is $(x-\alpha)(y-\beta) = c^2$ or $xy - \alpha y - \beta x + \lambda = 0$.
3. Since $x = ct$, $y = \frac{c}{t}$ satisfies $xy = c^2$

   $\therefore (x, y) = \left(ct, \frac{c}{t}\right) (t \neq 0)$ is called a 't' point on the rectangular hyperbola. The set $\left\{x = ct, y = \frac{c}{t}\right\}$ represents its parametric equations with parameter 't'.

Properties of Rectangular Hyperbola $xy = c^2$

(i) Equation of the chord joining 't_1' and 't_2' is
$x + yt_1t_2 - c(t_1 + t_2) = 0$.

(ii) Equation of tangent at (x_1, y_1) is $xy_1 + x_1y = 2c^2$.

(iii) Equation of tangent at 't' is $\frac{x}{t} + yt = 2c$.

(iv) Point of intersection of tangents at 't_1' and 't_2' is
$\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.

(v) Equation of normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(vi) Equation of normal at 't' is $xt^3 - yt - ct^4 + c = 0$.

(vii) Point of intersection of normals at 't_1' and 't_2' is.

$$\left(\frac{c\{t_1t_2(t_1^2 + t_1t_2 + t_2^2) - 1\}}{t_1t_2(t_1 + t_2)}, \frac{c\{t_1^3t_2^3 + (t_1^2 + t_1t_2 + t_2^2)\}}{t_1t_2(t_1 + t_2)}\right).$$

Example 36 If the normal at the point 't_1' to the rectangular hyperbola $xy = c^2$ meets it again at the point 't_2', prove that $t_1^3t_2 = -1$.

Sol. Since, the equation of normal at $\left(ct_1, \frac{c}{t_1}\right)$ to the hyperbola $xy = c^2$ is

$$xt_1^3 - yt_1 - ct_1^4 + c = 0$$

but this passes through $\left(ct_2, \frac{c}{t_2}\right)$, then

$$ct_2t_1^3 - \frac{c}{t_2}t_1 - ct_1^4 + c = 0$$

$$\Rightarrow \quad t_2^2t_1^3 - t_1 - t_1^4t_2 + t_2 = 0$$

$$\Rightarrow \quad t_2t_1^3(t_2 - t_1) + (t_2 - t_1) = 0$$

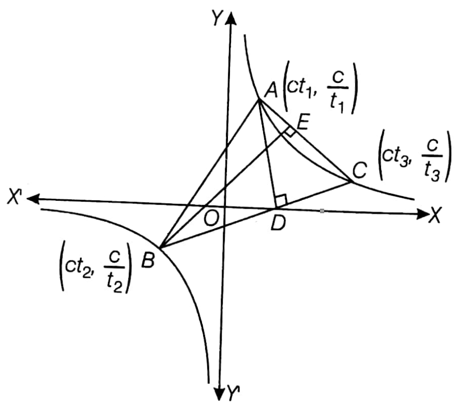
$$\Rightarrow \quad (t_1^3t_2 + 1)(t_2 - t_1) = 0$$

$$\Rightarrow \quad t_1^3t_2 + 1 = 0 \qquad [\because t_2 \neq t_1]$$

$$\therefore \quad t_1^3t_2 = -1$$

Example 37 A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola. Remember.

Sol. Let "t_1", "t_2" and "t_3" are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.



$\therefore$ Coordinates of A, B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively.

Now, slope of BC is $\dfrac{\frac{c}{t_3} - \frac{c}{t_2}}{ct_3 - ct_2} = -\dfrac{1}{t_2t_3}$

$\therefore$ Slope of AD is t_2t_3

Equation of altitude AD is

$$y - \frac{c}{t_1} = t_2t_3(x - ct_1)$$

or $$t_1y - c = x\, t_1t_2t_3 - ct_1^2t_2t_3 \quad \text{...(i)}$$

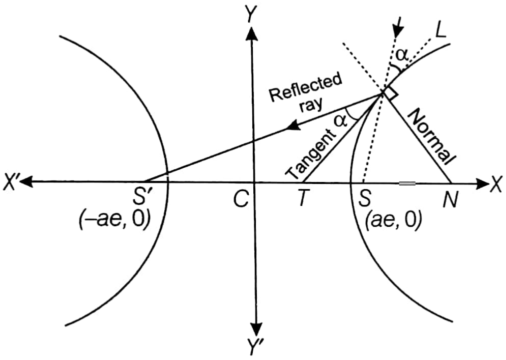
Similarly equation of altitude BE is

$$t_2y - c = x\, t_1t_2t_3 - c\, t_1t_2^2t_3 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get the orthocentre $\left(-\dfrac{c}{t_1t_2t_3}, -c\, t_1t_2t_3\right)$ which lies on $xy = c^2$.

Reflection Property of a Hyperbola

If an incoming light ray passing through one focus S strike convex side of the hyperbola then it will get reflected towards other focus S'.

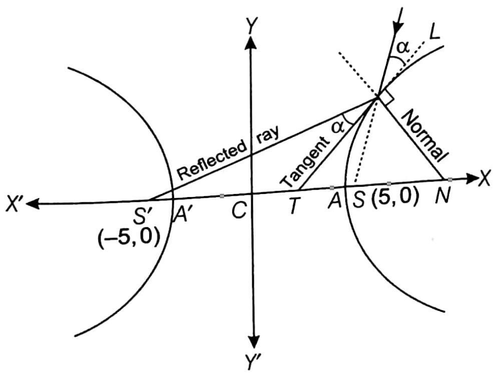


Example 38 A ray emanating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lies in first quadrant.

Sol. Given, hyperbola is $9x^2 - 16y^2 = 144$. This equation can be rewritten as

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \quad \text{...(i)}$$

Coordinates of foci are $(0, \pm\sqrt{16+9})$ i.e., $(0, \pm 5)$



Since x-coordinate of P is 8. Let y-coordinate of P is α.

$\therefore$ $(8, \alpha)$ lies on Eq. (i)

$$\therefore \quad \frac{64}{16} - \frac{\alpha^2}{9} = 1$$

$$\Rightarrow \quad \alpha^2 = 27$$

$$\Rightarrow \quad \alpha = 3\sqrt{3} \quad (\because P \text{ lies in first quadrant})$$

Hence coordinate of point P is $(8, 3\sqrt{3})$.

$\because$ Equation of reflected ray passing through $P(8, 3\sqrt{3})$ and $S'(-5, 0)$.

$\therefore$ Its equation is $y - 3\sqrt{3} = \dfrac{0 - 3\sqrt{3}}{-5-8}(x - 8)$

or $$13y - 39\sqrt{3} = 3\sqrt{3}x - 24\sqrt{3}$$

or $$3\sqrt{3}x - 13y + 15\sqrt{3} = 0.$$

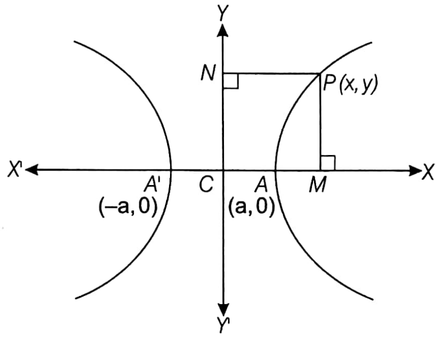
Equation of a Hyperbola Referred to Two Perpendicular Lines

Let $P(x, y)$ be any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, then

$$PM = y \text{ and } PN = x$$

$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{(PN)^2}{a^2} - \frac{(PM)^2}{b^2} = 1$$

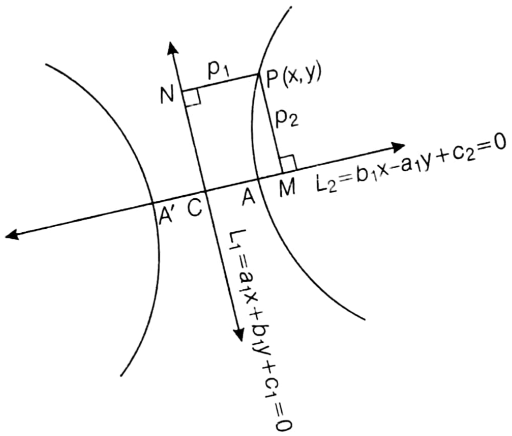


If perpendicular lines represented by

$$L_1 \equiv a_1x + b_1y + c_1 = 0$$

and $$L_2 \equiv b_1x - a_1y + c_2 = 0$$

then $$PN = p_1 = \left(\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}}\right)$$



and $\quad PM = p_2 = \left(\dfrac{b_1x - a_1y + c_2}{\sqrt{(b_1^2 + a_1^2)}}\right)$

Now equation of hyperbola becomes

$$\frac{\left(\dfrac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}}\right)^2}{a^2} - \frac{\left(\dfrac{b_1x - a_1y + c_2}{\sqrt{(b_1^2 + a_1^2)}}\right)^2}{b^2} = 1$$

Then the point P describes a hyperbola in the plane of the given lines such that :

(i) **Centre :** The centre of the hyperbola is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$.

(ii) **Transverse axis :** The transverse axis lies along $L_2 = 0$.

Conjugate axis : The conjugate axis lies along $L_1 = 0$.

(iii) **Length of transverse and conjugate axes :** The length of transverse and conjugate axes are $2a$ and $2b$ respectively.

(iv) **Foci :** The foci of the hyperbola is the point of intersection of the lines

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm ae \text{ and } L_2 = 0.$$

(v) **Directrix :** The directrices of the hyperbola are

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{a}{e}$$

(vi) **Latusrectum :** The latusrectum of the hyperbola is $\dfrac{2b^2}{a}$.

Example 39 The equations of the transverse and conjugate axes of a hyperbola are respectively $3x + 4y - 7 = 0$, $4x - 3y + 8 = 0$ and their respective lengths are 4 and 6. Find the equation of the hyperbola.

Sol. The equation of the required hyperbola is

$$\frac{\left(\dfrac{3x + 4y - 7}{\sqrt{(3^2 + 4^2)}}\right)^2}{\left(\dfrac{4}{2}\right)^2} - \frac{\left(\dfrac{4x - 3y + 8}{\sqrt{4^2 + (-3)^2}}\right)^2}{\left(\dfrac{6}{2}\right)^2} = 1$$

$$\Rightarrow \frac{1}{100}(3x + 4y - 7)^2 - \frac{1}{225}(4x - 3y + 8)^2 = 1$$

$$\Rightarrow 9(3x + 4y - 7)^2 - 4(4x - 3y + 8)^2 = 900$$

$$\Rightarrow 9(9x^2 + 16y^2 + 49 + 24xy - 42x - 56y) - 4(16x^2 + 9y^2 + 64 - 24xy + 64x - 48y) = 900$$

$$\Rightarrow 17x^2 + 312xy + 108y^2 - 634x - 312y - 715 = 0.$$

Exercise for Session 3

1. The diameter of $16x^2 - 9y^2 = 144$ which is conjugate to $x = 2y$ is

(a) $y = \dfrac{16}{9}x$ (b) $y = \dfrac{32}{9}x$ (c) $x = \dfrac{16}{9}y$ (d) $x = \dfrac{32}{9}y$

2. Tangents drawn from a point on the circle $x^2 + y^2 = 9$ to the hyperbola $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$, then tangents are at angle

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{2}$

3. If $H \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$, $C \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + 1 = 0$ and $A \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$ then H, A and C are in

(a) AP (b) GP (c) HP (d) AGP

4. The angle between the asymptotes of $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is equal to

(a) $\tan^{-1}\left(\frac{2}{3}\right)$ (b) $\tan^{-1}\left(\frac{3}{2}\right)$ (c) $2\tan^{-1}\left(\frac{2}{3}\right)$ (d) $2\tan^{-1}\left(\frac{3}{2}\right)$

5. If e and e_1 are the eccentricities of the hyperbolas $xy = c^2$ and $x^2 - y^2 = a^2$, then $(e + e_1)^2$ is equal to

(a) 2 (b) 4 (c) 6 (d) 8

6. The product of the lengths of perpendiculars drawn from any point on the hyperbola $\frac{x^2}{2} - y^2 = 1$ to its asymptotes is

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

7. The number of points on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 3$ from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$ is/are

(a) 0 (b) 2 (c) 3 (d) 4

8. If the sum of the slopes of the normal from a point P to the hyperbola $xy = c^2$ is equal to $\lambda(\lambda \in R^+)$, then the locus of point P is

(a) $x^2 = \lambda c^2$ (b) $y^2 = \lambda c^2$ (c) $xy = \lambda c^2$ (d) None of these

9. If $S \equiv x^2 + 4xy + 3y^2 - 4x + 2y + 1 = 0$, then the value of λ for which $S + \lambda = 0$ represents its asymptotes is

(a) 20 (b) 18 (c) -16 (d) -22

10. A ray emanating from the point $(-\sqrt{41}, 0)$ is incident on the hyperbola $16x^2 - 25y^2 = 400$ at the point P with abscissa 10. Then the equation of the reflected ray after first reflection and point P lies in second quadrant is

(a) $4\sqrt{3}x - (10 - \sqrt{41})\,y + 4\sqrt{123} = 0$ (b) $4\sqrt{3}x + (10 - \sqrt{41})\,y - 4\sqrt{123} = 0$

(c) $4\sqrt{3}x + (10 - \sqrt{41})\,y + 4\sqrt{123} = 0$ (d) $4\sqrt{3}x - (10 - \sqrt{41})\,y - 4\sqrt{123} = 0$

11. A ray of light incident along the line $3x + (5 - 4\sqrt{2})y = 15$ gets reflected from the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, then its reflected ray goes along the line

(a) $x\sqrt{2} - y + 5 = 0$ (b) $y\sqrt{2} - x + 5 = 0$ (c) $y\sqrt{2} - x - 5 = 0$ (d) None of these

12. The equations of the transverse and conjugate axes of a hyperbola are $x + 2y - 3 = 0$ and $2x - y + 4 = 0$ respectively and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$. The equation of the hyperbola is

(a) $2(x + 2y - 3)^2 - 3(2x - y + 4)^2 = 5$ (b) $2(2x - y + 4)^2 - 3(x + 2y - 3)^2 = 5$

(c) $2(x + 2y - 3)^2 - 3(2x - y + 4)^2 = 1$ (d) $2(2x - y + 4)^2 - 3(x + 2y - 3)^2 = 1$

13. Find the equation of that diameter which bisects the chord $7x + y - 2 = 0$ of the hyperbola $\frac{x^2}{3} - \frac{y^2}{7} = 1$.

14. Find the equation of the hyperbola which has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ for its asymptotes and which passes through the origin.

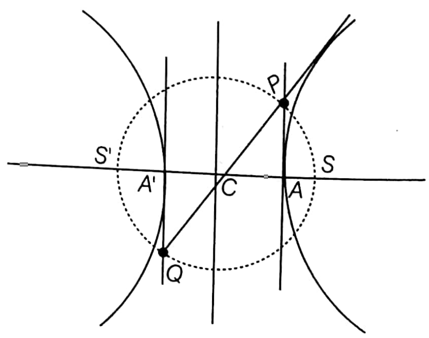
15. The asymptotes of a hyperbola are parallel to lines $2x + 3y = 0$ and $3x + 2y = 0$. The hyperbola has its centre at $(1, 2)$ and it passes through $(5, 3)$, find its equation.

16. If the pair of straight lines $Ax^2 + 2Hxy + By^2 = 0$ be conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then prove that $Aa^2 = Bb^2$.

17. A circle cuts the rectangular hyperbola $xy = 1$ in points (x_r, y_r), $r = 1, 2, 3, 4$ then prove that $x_1x_2x_3x_4 = y_1y_2y_3y_4 = 1$.

Shortcuts and Important Results to Remember

1. If P be any point and F_1 and F_2 are any other two points then :
   (a) If $|PF_1 - PF_2| < |F_1F_2|$, then the locus of P is a hyperbola.
   (b) If $|PF_1 - PF_2| = |F_1F_2|$, then the locus of P is a straight line.
   (c) If $|PF_1 - PF_2| > |F_1F_2|$, then the locus of P is an empty set.
2. The orthocentre of triangle inscribed in the hyperbola $xy = c^2$ lies on it.
3. Length of the chord of the rectangular hyperbola $xy = c^2$ whose middle-point is (h, k) is $2\sqrt{\left(\frac{(h^2+k^2)(hk-c^2)}{hk}\right)}$.
4. The product of length of perpendicular drawn from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is $\frac{a^2b^2}{(a^2+b^2)}$.
5. Asymptotes are the tangents from the centre of a hyperbola.
6. If the angle between the asymptotes is 2α, then eccentricity of the hyperbola is $\sec\alpha$.
7. If the tangent and normal to a rectangular hyperbola $xy = c^2$ at a point cuts off intercepts a_1 and a_2 on one axis and b_1, b_2 on the other axis, then $a_1a_2 + b_1b_2 = 0$.
8. The equation of common tangents to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $y = \pm x \pm \sqrt{(a^2 - b^2)}$.
9. The director circle of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ will be imaginary if $a < b$ and will become a circle, if $a > b$ (for $a = b$, point circle).
10. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2-k^2} - \frac{y^2}{k^2-b^2} = 1$ $(a > k > b)$ are confocal and therefore orthogonal.
11. If four normals can be drawn to a hyperbola from any point and if $\alpha, \beta, \gamma, \delta$ be eccentric angles of these four co-normal points, then $\alpha + \beta + \gamma + \delta$ = odd multiple of π.
12. If α, β, γ are the eccentric angles of three points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the normals at which are concurrent, then $\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$.
13. The locus of the foot of the perpendiculars drawn from the focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ and product of the perpendiculars is b^2.
14. The portion of tangent between the point of the contact and the directrix subtends a right angle at the corresponding focus.
15. The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only.
16. The asymptotes pass through the centre of the hyperbola and the bisectors of the angles between the asymptotes are the axes of the hyperbola.
17. The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.
18. Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix and the common points of intersection lie on the auxiliary circle.
19. If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve is always equal to the square of the semi conjugate axis.
20. The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a ΔCQR of constant area equal to ab from the asymptotes and the portion of the tangent intercepted between the asymptotes is bisected at the point of the contact. This implies that locus of the centre of the circle circumscribing the ΔCQR in case of rectangular hyperbola is the hyperbola itself and for a standard hyperbola the locus would be the curve $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.
21. If a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intersects a rectangular hyperbola $xy = \lambda^2$ or $x^2 - y^2 = a^2$ at four points then the Arithmetic mean of the points of intersection lies on the middle of the line joining the centres of the circle and hyperbola.
22. The points (two) in which any tangent meets the tangents at the vertices and the foci of the hyperbola are concyclic i.e. S, P, S' and Q are lie on circle.



JEE Type Solved Examples :
Single Option Correct Type Questions

- This section contains 10 **multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1 *If the eccentricity of the hyperbola $x^2 - y^2\sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\alpha + y^2 = 25$, then a value of α is*

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Sol. (b) For the hyperbola $\frac{x^2}{5} - \frac{y^2}{5\cos^2\alpha} = 1$

we have, $$e_1^2 = 1 + \frac{5\cos^2\alpha}{5} = 1 + \cos^2\alpha \quad \text{... (i)}$$

For the ellipse $\frac{x^2}{25\cos^2\alpha} + \frac{y^2}{25} = 1$

we have, $$e_2^2 = 1 - \frac{25\cos^2\alpha}{25} = 1 - \cos^2\alpha = \sin^2\alpha \quad \text{... (ii)}$$

Given that, $e_1 = \sqrt{3}\, e_2$ or $e_1^2 = 3e_2^2$

$\Rightarrow \quad 1 + \cos^2\alpha = 3\sin^2\alpha$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad 2 = 4\sin^2\alpha$

or $\quad \sin\alpha = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \alpha = \frac{\pi}{4}$

• Ex. 2 *The asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form with any tangent to the hyperbola a triangle whose area is $a^2\tan\lambda$ in magnitude, then its eccentricity is*

(a) $\sec\lambda$ (b) $\operatorname{cosec}\lambda$

(c) $\sec^2\lambda$ (d) $\operatorname{cosec}^2\lambda$

Sol. (a) Any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ forms a triangle with the asymptotes which has constant area ab. Remember

Given, $\quad ab = a^2\tan\lambda$

or $\quad b = a\tan\lambda$

or $\quad b^2 = a^2\tan^2\lambda$

or $\quad a^2(e^2 - 1) = a^2\tan^2\lambda$

or $\quad e^2 = 1 + \tan^2\lambda = \sec^2\lambda$

$\therefore \quad e = \sec\lambda$

• Ex. 3 *The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is*

(a) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$ (b) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$

(c) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$ (d) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

Sol. (a) The mid-point of the chord is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$\therefore$ The equation of chord whose mid-point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is $\boxed{T = S_1}$

or $$\frac{1}{2}\left(x\left(\frac{y_1 + y_2}{2}\right) + y\left(\frac{x_1 + x_2}{2}\right)\right) - c^2 = \left(\frac{x_1 + x_2}{2}\right)\left(\frac{y_1 + y_2}{2}\right) - c^2$$

$\Rightarrow \quad x(y_1 + y_2) + y(x_1 + x_2) = (x_1 + x_2)(y_1 + y_2)$

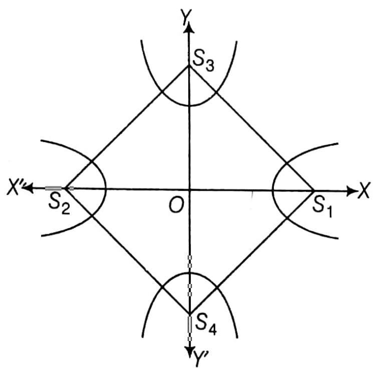
or $\quad \frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$

• Ex. 4 *Area of quadrilateral formed with the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is*

(a) $4(a^2 + b^2)$ (b) $2(a^2 + b^2)$

(c) $(a^2 + b^2)$ (d) $\frac{1}{2}(a^2 + b^2)$

Sol. (b) Required area $= 4 \times$ Area of ΔS_1OS_3



$$= 4 \times \frac{1}{2} ae \times be_1 = (2ab)(ee_1) \quad \text{...(i)}$$

$\because \quad b^2 = a^2(e^2 - 1)$

$\therefore \quad e = \frac{\sqrt{(a^2 + b^2)}}{a} \quad \text{...(ii)}$

and $\quad a^2 = b^2(e_1^2 - 1)$

$$\therefore \qquad e_1 = \frac{\sqrt{(a^2+b^2)}}{b} \qquad \text{...(iii)}$$

Substituting the values of e and e_1 from Eqs. (ii) and (iii) in Eq. (i), then

$$\text{Required area} = 2ab \times \frac{\sqrt{(a^2+b^2)}}{a} \times \frac{\sqrt{(a^2+b^2)}}{b}$$

$$= 2(a^2+b^2)$$

• Ex. 5 *Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta+\phi=\frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h,k) is the point of intersection of normals at P and Q, then k is equal to*

(a) $\left(\frac{a^2+b^2}{a}\right)$ (b) $-\left(\frac{a^2+b^2}{a}\right)$

(c) $\left(\frac{a^2+b^2}{b}\right)$ (d) $-\left(\frac{a^2+b^2}{b}\right)$

Sol. (d) Equations of the normals at $P(\theta)$ and $Q(\phi)$ are

$$ax\cos\theta + by\cot\theta = a^2+b^2 \qquad \text{... (i)}$$

and $$ax\cos\phi + by\cot\phi = a^2+b^2 \qquad \text{... (ii)}$$

Now, dividing by $\cos\theta$ and $\cos\phi$ in Eqs. (i) and (ii) respectively, then

$$ax + by\operatorname{cosec}\theta = (a^2+b^2)\sec\theta \qquad \text{... (iii)}$$

and $$ax + by\operatorname{cosec}\phi = (a^2+b^2)\sec\phi \qquad \text{... (iv)}$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$y = \left(\frac{a^2+b^2}{b}\right)\cdot\left(\frac{\sec\theta-\sec\phi}{\operatorname{cosec}\theta-\operatorname{cosec}\phi}\right)$$

$$\therefore \quad k = y$$

$$= \left(\frac{a^2+b^2}{b}\right)\cdot\left(\frac{\sec\theta-\sec\left(\frac{\pi}{2}-\theta\right)}{\operatorname{cosec}\theta-\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)}\right) \qquad \left(\because \theta+\phi=\frac{\pi}{2}\right)$$

$$= \left(\frac{a^2+b^2}{b}\right)\cdot\left(\frac{\sec\theta-\operatorname{cosec}\theta}{\operatorname{cosec}\theta-\sec\theta}\right) = -\left(\frac{a^2+b^2}{b}\right)$$

• Ex. 6 *Let the major axis of a standard ellipse equals the transverse axis of a standard hyperbola and their director circles have radius equal to 2R and R respectively. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola, then the correct relation is*

(a) $4e_1^2 - e_2^2 = 6$ (b) $e_1^2 - 4e_2^2 = 2$

(c) $4e_2^2 - e_1^2 = 6$ (d) $e_2^2 - 4e_1^2 = 2$

Sol. (c) Let equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$$

$\therefore$ Director circle is $x^2+y^2=a^2+b^2$

and $$b^2 = a^2(1-e_1^2)$$

given $$a^2+b^2=(2R)^2$$

$$\Rightarrow \quad a^2+a^2(1-e_1^2)=4R^2$$

$$\Rightarrow \quad 2-e_1^2=\frac{4R^2}{a^2} \qquad \text{... (i)}$$

and equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b_1^2}=1$

$\therefore$ Director circle is $x^2+y^2=a^2-b_1^2$

and $$b_1^2=a^2(e_2^2-1)$$

Given $$a^2-b_1^2=R^2$$

$$\Rightarrow \quad a^2-a^2(e_2^2-1)=R^2$$

$$\Rightarrow \quad 2-e_2^2=\frac{R^2}{a^2} \qquad \text{... (ii)}$$

Dividing Eq. (i) by Eq. (ii), then

$$\frac{2-e_1^2}{2-e_2^2}=4 \Rightarrow 4e_2^2-e_1^2=6$$

• Ex. 7 *The tangent to the hyperbola $xy=c^2$ at point $P(t)$ intersects the X-axis at T and the Y-axis. at T′. The normal to the hyperbola at $P(t)$ intersects the X-axis at N and the Y-axis at N′. The areas of the triangles PNT and PN′T′ are Δ and Δ' respectively, then $\frac{1}{\Delta}+\frac{1}{\Delta'}$ is*

(a) equal to 1 (b) depends on t

(c) depends on c (d) equal to 2

Sol. (c) Equation of tangent at $P\left(ct, \frac{c}{t}\right)$ is $\frac{x}{t}+ty=2c$

or $$\frac{x}{2ct}+\frac{y}{(2c/t)}=1$$

$$\therefore \quad T\equiv(2ct, 0), T'\equiv\left(0, \frac{2c}{t}\right)$$

and equation of normal at $P\left(ct, \frac{c}{t}\right)$ is

$$xt^3-yt-ct^4+c=0$$

or $$\frac{x}{\left(\frac{c(t^4-1)}{t^3}\right)}+\frac{y}{\left(\frac{c(t^4-1)}{-t}\right)}=1$$

$$\therefore \quad N\equiv\left(\frac{c(t^4-1)}{t^3}, 0\right), N'\equiv\left(0, -\frac{c(t^4-1)}{t}\right)$$

$\because$ Area of triangle $PNT=\Delta$

$$\therefore \quad \Delta=\frac{1}{2}\left|\begin{vmatrix} ct-2ct & \frac{c}{t}-0 \\ \frac{c(t^4-1)}{t^3}-2ct & 0-0 \end{vmatrix}\right|$$

$$=\frac{1}{2}\left|-\frac{c}{t}\left(\frac{c(t^4-1)}{t^3}-2ct\right)\right|=\frac{c^2}{2}\left(\frac{t^4+1}{t^4}\right)$$

and Area of $\Delta PN'T' = \Delta'$

$$\therefore \quad \Delta' = \frac{1}{2}\begin{Vmatrix} ct-0 & \frac{c}{t}-\frac{2c}{t} \\ 0-0 & -\frac{c(t^4-1)}{t}-\frac{2c}{t} \end{Vmatrix}$$

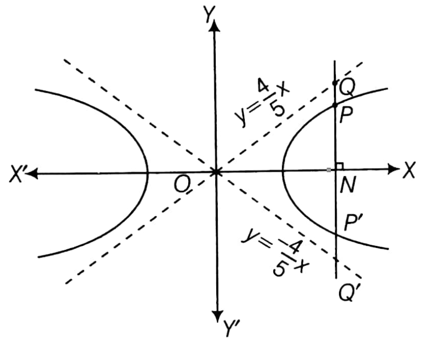
$$=\frac{1}{2}\left|ct\left(-\frac{c(t^4-1)}{t}-\frac{2c}{t}\right)\right|=\frac{c^2}{2}(t^4+1)$$

$$\therefore \quad \frac{1}{\Delta}+\frac{1}{\Delta'}=\frac{2t^4}{c^2(t^4+1)}+\frac{2}{c^2(t^4+1)}=\frac{2}{c^2}$$

• Ex. 8 *Let any double ordinate PNP' of the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ be produced both sides to meet the asymptotes is Q and Q', then $(PQ)(P'Q)$ is equal to*

(a) 9 (b) 16 (c) 25 (d) 41

Sol. (b) Let $Q \equiv (x_1, y_1)$



We have $\quad NP = \frac{4}{5}\sqrt{(x_1^2-25)}$

and Q is on $\quad y = \frac{4}{5}x$

$\therefore \quad NQ = \frac{4}{5}x_1$

Now, $\quad PQ = NQ - NP$

$$=\frac{4}{5}(x_1-\sqrt{(x_1^2-25)})$$

and $\quad P'Q = NP' + NQ = NP + NQ \quad (\because NP' = NP)$

$$=\frac{4}{5}(x_1+\sqrt{(x_1^2-25)})$$

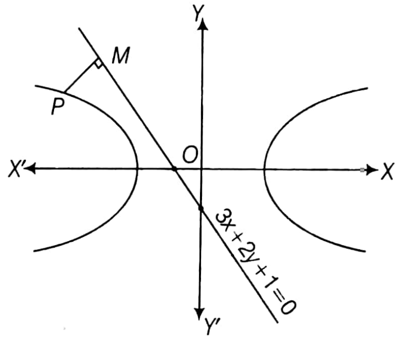
$$\therefore \quad (PQ)(P'Q) = \frac{16}{25}(x_1^2-(x_1^2-25))$$

$$=\frac{16}{25}\times 25 = 16$$

• Ex. 9 *The coordinates of a point on the hyperbola $\frac{x^2}{24}-\frac{y^2}{18}-1$. Which is nearest to the line $3x+2y+1=0$ are*

(a) (6, 3) (b) (−6, −3)
(c) (6, −3) (d) (−6, 3)

Sol. (d)



Let point P is nearest to the given line if the tangent at P is parallel to the given line.

Now, equation of tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{24}-\frac{yy_1}{18}=1$$

$\therefore$ Slope of tangent at $P(x_1, y_1)$ is

$$\frac{3}{4}\frac{x_1}{y_1}$$

which must be equal to $-\frac{3}{2}$

Therefore,

$$\frac{3}{4}\cdot\frac{x_1}{y_1}=-\frac{3}{2} \text{ or } x_1=-2y_1 \quad \text{...(i)}$$

Also, $P(x_1, y_1)$ lies on the curve

Hence, $$\frac{x_1^2}{24}-\frac{y_1^2}{18}=1 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get two points $(6,-3)$ and $(-6,3)$ of which $(-6,3)$ is the nearest.

Aliter :

The equation of normal at $P(2\sqrt{6}\sec\theta, 3\sqrt{2}\tan\theta)$ is

$$(2\sqrt{6}\cos\theta)x+(3\sqrt{2}\cot\theta)y=42$$

Now, this line will be perpendicular to

$$3x+2y+1=0$$

then, $\quad (2\sqrt{6}\cos\theta)\times 3+(3\sqrt{2}\cot\theta)\times 2=0$

or $\quad \sin\theta = -\frac{1}{\sqrt{3}}$

$\therefore \quad \tan\theta = \frac{1}{\sqrt{2}}$ and $\sec\theta = -\frac{\sqrt{3}}{\sqrt{2}}$

Hence, $\quad P \equiv (-6,3)$

• Ex. 10 *For each positive integer n, consider the point P with abscissa n on the curve $y^2-x^2=1$. If d_n represents the shortest distance from the point P to the line $y=x$, then $\lim_{n\to\infty}(n.d_n)$ has the value equal to*

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2}$
(c) $\frac{1}{\sqrt{2}}$ (d) 0

Sol. (a) Let $P \equiv (n, \sqrt{(n^2+1)})$

$\therefore \quad d_n = \left(\dfrac{\sqrt{(n^2+1)} - n}{\sqrt{2}}\right)$

or $\quad n \cdot d_n = \dfrac{n}{\sqrt{2}}(\sqrt{(n^2+1)} - n)$

or $\quad \lim_{n\to\infty} (n \cdot d_n) = \dfrac{1}{\sqrt{2}} \lim_{n\to\infty} n(\sqrt{(n^2+1)} - n)$

$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{n(\sqrt{(n^2+1)} - n)(\sqrt{(n^2+1)} + n)}{(\sqrt{(n^2+1)} + n)}$$

$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{n(1)}{n\left(\sqrt{\left(1+\frac{1}{n^2}\right)} + 1\right)}$$

$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \left(\frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)} + 1}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{1+1}\right) = \frac{1}{2\sqrt{2}}$$

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains 5 **multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

• Ex. 11 *If two tangents can be drawn to the different branches of hyperbola $\dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$ from the point (α, α^2), then*

(a) $\alpha \in (-\infty, -2)$ (b) $\alpha \in (-2, 0)$

(c) $\alpha \in (0, 2)$ (d) $\alpha \in (2, \infty)$

Sol. (a, d) $\because (\alpha, \alpha^2)$ lie on the parabola $y = x^2$

$\therefore (\alpha, \alpha^2)$ must lie between the asymptotes of hyperbola

$\dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$ in I and II quadrants.

$\because$ Asymptotes of $\dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$ are $y = \pm 2x$

then $\quad 2\alpha < \alpha^2$ and $-2\alpha < \alpha^2$

$\Rightarrow \quad \alpha(\alpha - 2) > 0$ and $\alpha(\alpha + 2) > 0$

$\therefore \quad \alpha < 0$ or $\alpha > 2$ and $\alpha < -2$ or $\alpha > 0$

$\therefore \quad \alpha \in (-\infty, -2) \cup (2, \infty)$

• Ex. 12 *If the ellipse $x^2 + \lambda^2 y^2 = \lambda^2 a^2; \lambda^2 > 1$ is confocal with the hyperbola $x^2 - y^2 = a^2$, then*

(a) ratio of eccentricities of ellipse and hyperbola is $1:\sqrt{3}$

(b) ratio of major axis of ellipse and transverse axis of hyperbola is $\sqrt{3}:1$

(c) The ellipse and hyperbola cuts each other orthogonally

(d) ratio of length of latusrectum of ellipse and hyperbola is 1 : 3

Sol. (a, b, c) Given ellipse is $\dfrac{x^2}{\lambda^2 a^2} + \dfrac{y^2}{a^2} = 1$; $\lambda^2 a^2 > a^2$ and let e_1 and e_2 be the eccentricities of ellipse and hyperbola, then

$$a^2 = \lambda^2 a^2 (1 - e_1^2)$$

or $\quad e_1 = \sqrt{\left(1 - \dfrac{1}{\lambda^2}\right)}$

and $\quad e_2 = \sqrt{2}$

Now, $\quad \lambda a e_1 = a e_2$

$\Rightarrow \quad \lambda\sqrt{\left(1 - \dfrac{1}{\lambda^2}\right)} = \sqrt{2}$

$\Rightarrow \quad \lambda^2 - 1 = 2$ or $\lambda = \pm\sqrt{3}$

$\therefore \quad \lambda = \sqrt{3} \quad (\because \lambda > 0)$

Alternate (a) : $\dfrac{e_1}{e_2} = \dfrac{1}{\lambda} = \dfrac{1}{\sqrt{3}}$

Alternate (b) : $= \dfrac{\text{Major axis of ellipse}}{\text{Transverse axis of hyperbola}}$

$$= \frac{2\lambda a}{2a} = \lambda = \sqrt{3}$$

Alternate (c) : Equations of tangents of ellipse and hyperbola at (x_1, y_1) are

$$\frac{xx_1}{\lambda^2 a^2} + \frac{yy_1}{a^2} = 1$$

and $\quad xx_1 - yy_1 = a^2$

i.e., $\quad$ slopes are $= -\dfrac{x_1}{\lambda^2 y_1}$ and $\dfrac{x_1}{y_1}$ $\quad$ (say m_1 and m_2)

According to alternate,

$$m_1 m_2 = -\frac{x_1}{\lambda^2 y_1} \times \frac{x_1}{y_1} = -\frac{x_1^2}{\lambda^2 y_1^2}$$

$$= -\frac{\frac{2a^2}{\lambda^2+1}}{\frac{a^2(\lambda^2-1)}{(\lambda^2+1)}} = -\frac{2}{\lambda^2 - 1} \quad \left(\because \begin{matrix} x_1^2 = \dfrac{2a^2\lambda^2}{\lambda^2+1} \\ \text{and } y_1^2 = \dfrac{a^2(\lambda^2-1)}{(\lambda^2+1)} \end{matrix}\right)$$

$$= -\frac{2}{3-1} = -1 \qquad (\because \lambda = \sqrt{3})$$

Alternate (d) : $= \dfrac{\text{Latus reaction of ellipse}}{\text{Latus reaction of hyperbola}}$

$$= \frac{\dfrac{2a^2}{\lambda a}}{\dfrac{2a^2}{a}} = \frac{1}{\lambda} = \frac{1}{\sqrt{3}}$$

• ***Ex. 13*** *If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$, then*

(a) $\Sigma x_1 = 0$ (b) $\Sigma y_1 = 0$

(c) $\Pi x_1 = 0$ (d) $\Pi y_1 = 0$

Sol. (a, b, c, d) Solving $x^2 + y^2 = a^2$ and $xy = c^2$,

we have, $$x^2 + \frac{c^4}{x^2} = a^2$$

or $$x^4 - a^2x^2 + c^4 = 0$$

or $$\Sigma x_1 = 0 \text{ and } \Pi x_1 = c^4$$

Similarly, if we eliminate x, then
$$y^4 - a^2y^2 + c^4 = 0$$

or $$\Sigma y_1 = 0 \text{ and } \Pi y_1 = c^4$$

• ***Ex. 14*** *A straight line touches the rectangular hyperbola $9x^2 - 9y^2 = 8$ and the parabola $y^2 = 32x$, the equation of the line is*

(a) $9x + 3y - 8 = \infty$ (b) $9x - 3y + 8 = 0$

(c) $9x + 3y + 8 = 0$ (d) $9x - 3y - 8 = 0$

Sol. (b, c) Equation of tangent to the parabola $y^2 = 32x$ is

$$y = mx + \frac{8}{m} \qquad \text{... (i)}$$

Which is also touches the hyperbola $9x^2 - 9y^2 = 8$

i.e. $$x^2 - y^2 = \frac{8}{9}$$

So that, $$\left(\frac{8}{m}\right)^2 = \frac{8}{9} \times m^2 - \frac{8}{9}$$

$\Rightarrow$ $$\frac{8}{m^2} = \frac{m^2}{9} - \frac{1}{9}$$

$\Rightarrow$ $$m^4 - m^2 - 72 = 0$$

$\Rightarrow$ $$(m^2 - 9)(m^2 + 8) = 0$$

$\therefore$ $$m^2 - 9 = 0$$

but $$m^2 + 8 \neq 0$$

$\therefore$ $$m = \pm 3$$

Hence, from Eq. (i), the equation of tangents are

$$y = 3x + \frac{8}{3}$$

and $$y = -3x - \frac{8}{3}$$

or $$9x - 3y + 8 = 0$$

and $$9x + 3y + 8 = 0$$

• ***Ex. 15*** *The differential equation $\dfrac{dx}{dy} = \dfrac{3y}{2x}$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity*

(a) $\sqrt{\frac{7}{3}}$ (b) $\sqrt{\frac{5}{3}}$

(c) $\sqrt{\frac{3}{2}}$ (d) $\sqrt{\frac{5}{2}}$

Sol. (b, d) $\because \dfrac{dx}{dy} = \dfrac{3y}{2x}$

or $$\int 2x\,dx = \int 3y\,dy$$

or $$x^2 = \frac{3y^2}{2} + c$$

or $$\frac{x^2}{3} - \frac{y^2}{2} = \frac{c}{3}$$

or $$\frac{x^2}{3} - \frac{y^2}{2} = c_1$$

Case I If $c_1 > 0$, then $e = \sqrt{\left(1 + \frac{2}{3}\right)} = \sqrt{\frac{5}{3}}$

Case II If $c_1 < 0$, then $e = \sqrt{\left(1 + \frac{3}{2}\right)} = \sqrt{\frac{5}{2}}$

JEE Type Solved Examples : Paragraph Based Questions

- This section contains 2 **solved paragraphs** based upon each of the paragraph 3 **multiple choice** questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to 18)

A conic C satisfies the differential equation. $(1+y^2)dx - xy\,dy = 0$ and passes through the point (1, 0). An ellipse E which is confocal with C having its eccentricity equal to $\sqrt{\left(\frac{2}{3}\right)}$.

16. Length of latusrectum of the conic C is

(a) 1 (b) 2

(c) 3 (d) 4

17. Equations of the ellipse E is

(a) $\frac{x^2}{3}+\frac{y^2}{1}=1$ (b) $\frac{x^2}{1}+\frac{y^2}{3}=1$

(c) $\frac{x^2}{4}+\frac{y^2}{9}=1$ (d) $\frac{x^2}{9}+\frac{y^2}{4}=1$

18. Locus of the point of intersection of the perpendicular tangents to the ellipse E, is

(a) $x^2+y^2=4$ (b) $x^2+y^2=8$

(c) $x^2+y^2=10$ (d) $x^2+y^2=13$

Sol. Given differential equation is $(1+y^2)\,dx - xy\,dy = 0$

$$\Rightarrow \quad \frac{dx}{x}-\frac{y\,dy}{(1+y^2)}=0 \text{ or } \frac{2dx}{x}-\frac{2y\,dy}{1+y^2}=0$$

On integrating, we have $2\ln x - \ln(1+y^2)=\ln c$

or $\ln x^2 = \ln(c(1+y^2))$ or $x^2 = c(1+y^2)$

or $\frac{x^2}{c}-\frac{y^2}{1}=1$ (Hyperbola)

It passes through (1, 0), then $\frac{1}{c}-0=1$

$\therefore$ $c=1$

$\therefore C$ is rectangular hyperbola $x^2-y^2=1$, having its eccentricity $=\sqrt{2}$ and its foci are $(\pm\sqrt{2}, 0)$.

If a is the length of semi major axis of ellipse E, then $ae=\sqrt{2}$

or $a\sqrt{\frac{2}{3}}=\sqrt{2}$ or $a=\sqrt{3}$

Now, $b^2=a^2(1-e^2)=3\left(1-\frac{2}{3}\right)=1$

16. (b) Length of latusrectum of the conic C is $\frac{2(1)^2}{1}=2$

17. (a) Equation of the ellipse E is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ i.e, } \frac{x^2}{3}+\frac{y^2}{1}=1$$

18. (a) Required locus is the director circle of E which is $x^2+y^2=3+1$ or $x^2+y^2=4$

Paragraph II

(Q. Nos. 19 to 21)

For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the normal at point P meets the transverse axis AA' in G and the conjugate axis BB' in g and CF be perpendicular to the normal from the centre.

19. The value of $\frac{PF.PG}{(CB)^2}$ is equal to

(a) 4 (b) 3

(c) 2 (d) 1

20. The value of $PF.Pg$ is equal to

(a) $(CA)^2$ (b) $(CF)^2$

(c) $(CB)^2$ (d) $CA.CB$

21. Locus of middle-point of G and g is a hyperbola of eccentricity

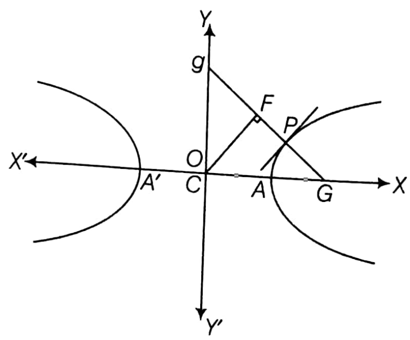
(a) $\frac{1}{\sqrt{(e^2-1)}}$ (b) $\frac{e}{\sqrt{(e^2-1)}}$

(c) $2\sqrt{(e^2-1)}$ (d) $\frac{e}{2}$

Sol. $C\equiv(0,0)$

$AA'=2a$, $BB'=2b$

$\therefore$ $CA=CA'=a$ and $CB=CB'=b$



Let $P\equiv(a\sec\theta, b\tan\theta)$

Equation of normals at P is

$$ax\cos\theta + by\cot\theta = a^2+b^2 \quad \text{... (i)}$$

$$\therefore \quad G\equiv\left(\frac{(a^2+b^2)}{a}\sec\theta, 0\right), g\equiv\left(0, \frac{(a^2+b^2)}{b}\tan\theta\right)$$

Equation of CF which is perpendicular to Eq. (i) and through origin is $bx\cot\theta - ay\cos\theta = 0$

$$\therefore \quad PF = \frac{ab}{\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}}$$

and
$$PG = \sqrt{\left(a\sec\theta - \frac{a^2+b^2}{a}\sec\theta\right)^2 + (b\tan\theta - 0)^2}$$
$$= \frac{b}{a}\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}$$

19. (d) $\because PF.PG = \dfrac{ab}{\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}} \cdot \dfrac{b}{a}\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}$

$$= b^2 = (CB)^2$$
$$\therefore \quad \frac{PF.PG}{(CB)^2} = 1$$

20. (a) $\because Pg = \sqrt{(a\sec\theta - 0)^2 + \left(b\tan\theta - \dfrac{(a^2+b^2)}{b}\tan\theta\right)^2}$

$$= \sqrt{\left(a^2\sec^2\theta + \frac{a^4}{b^2}\tan^2\theta\right)}$$
$$= \frac{a}{b}\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}$$

$$\therefore \quad PF.Pg = \frac{ab}{\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}} \times \frac{a}{b}\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}$$
$$= a^2 = (CA)^2$$

21. (b) $\because$ Mid-point of G and g is

$$\left(\frac{(a^2+b^2)}{2a}\sec\theta, \frac{(a^2+b^2)}{2b}\tan\theta\right)$$

$$\therefore \quad x = \left(\frac{a^2+b^2}{2a}\right)\sec\theta, \; y = \left(\frac{a^2+b^2}{2a}\right)\tan\theta$$

$$\therefore \quad (2ax)^2 - (2by)^2 = (a^2+b^2)^2$$

or
$$\frac{x^2}{\left(\dfrac{a^2+b^2}{2a}\right)^2} - \frac{y^2}{\left(\dfrac{a^2+b^2}{2b}\right)^2} = 1$$

$$\therefore \quad e_1 = \sqrt{\frac{\left(\dfrac{a^2+b^2}{2a}\right)^2 + \left(\dfrac{a^2+b^2}{2b}\right)^2}{\left(\dfrac{a^2+b^2}{2a}\right)^2}} = \sqrt{\frac{\dfrac{1}{a^2}+\dfrac{1}{b^2}}{\dfrac{1}{a^2}}}$$

$$= \sqrt{\frac{a^2+b^2}{2b^2}} = \sqrt{\frac{a^2+a^2(e^2-1)}{a^2(e^2-1)}} = \frac{e}{\sqrt{(e^2-1)}}$$

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains 2 **examples.** The answer to each example in **a single digit integer,** ranging from 0 to 9 (both inclusive).

• Ex. 22 *The equation of transverse axis of hyperbola (passing through origin) having asymptotes $3x - 4y - 1 = 0$ and $4x - 3y - 6 = 0$ is $ax + by - c = 0$, $a, b, c \in N$ and g.c.d $(a, b, c) = 1$, then the value of $a + b + c$ is*

Sol. (7) Since the equation of asymptotes are

$$3x - 4y - 1 = 0 \text{ and } 4x - 3y - 6 = 0$$

$\therefore$ Equation of transverse axis is given by

$$\frac{|(3x-4y-1)|}{\sqrt{(3)^2+(-4)^2}} = \frac{|4x-3y-6|}{\sqrt{(4)^2+(-3)^2}}$$

or $\quad |3x-4y-1| = |4x-3y-6|$

or $\quad (3x-4y-1) = \pm(4x-3y-6) \Rightarrow x+y-5=0$

and $\quad x - y - 7 = 0$

$\therefore$ Transverse axis is given by $ax + by - c = 0$; $a, b, c \in N$

$\Rightarrow \quad ax + by - c \equiv x + y - 5 = 0$

$\Rightarrow \quad a = 1, b = 1, c = 5$

Hence, $\quad a + b + c = 1 + 1 + 5 = 7$

• Ex. 23 *If a variable line has its intercepts on the coordinate axes are e and e′, where $\frac{e}{2}$ and $\frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where r is*

Sol. (2) Since $\frac{e}{2}$ and $\frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate, we have

$$\left(\frac{e}{2}\right)^{-2} + \left(\frac{e'}{2}\right)^{-2} = 1 \text{ or } \frac{4}{e^2} + \frac{4}{(e')^2} = 1$$

or
$$4 = \frac{e^2e'^2}{e^2+e'^2} \qquad \text{... (i)}$$

Equations of variable line is $\dfrac{x}{e} + \dfrac{y}{e'} = 1$

i.e. $\quad xe' + ye - ee' = 0.$

It is tangent to the circle $x^2 + y^2 = r^2$

$$\therefore \quad \frac{|0 - ee'|}{\sqrt{e^2+e'^2}} = r$$

$$\therefore \quad r = \frac{ee'}{\sqrt{(e^2+e'^2)}} = 2 \qquad \text{[from Eq. (i)]}$$

JEE Type Solved Examples : Matching Type Questions

- This section contain **only one example.** This example has three statements (A, B and C) given in **Column I** and four statements (p, q, r, and s) in **Column II**. Any given statment in **Column I** can have correct matching with one or more statements (s) given in **Column II.**

• Ex. 24 *Match the following.*

	Column I		Column II
(A)	The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}t = 0$ and $\sqrt{3}\,tx + ty - 4\sqrt{3} = 0$ (where t is a parameter) is a hyperbola whose eccentricity is	(p)	a natural number
(B)	If the product of the perpendicular distances from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of eccentricity $e = \sqrt{3}$ from its asymptotes is equal to 6, then the length of the transverse axis of the hyperbola is	(q)	a prime number
(C)	The area of the triangle that a tangent at a point of the hyperbola $\frac{x^2}{16} - \frac{y^2}{a} = 1$ makes with its asymptotes is	(r)	a composite number
		(s)	a perfect number

Sol. (A) → (p, q); (B) → (p, r, s); (C)→ (p, r)

(A) The given lines are

$$\sqrt{3}\,x - y - 4\sqrt{3}\,t = 0 \qquad \text{... (i)}$$

and $$\sqrt{3}\,tx + ty - 4\sqrt{3} = 0 \qquad \text{... (ii)}$$

Eliminate 't' from Eqs. (i) and (ii), then

$$\frac{\sqrt{3}x - y}{4\sqrt{3}} = \frac{4\sqrt{3}}{\sqrt{3}x + y}$$

$\Rightarrow$ $$3x^2 - y^2 = 48$$

or $$\frac{x^2}{16} - \frac{y^2}{48} = 1$$

or $$48 = 16(e^2 - 1)$$

$\Rightarrow$ $$e^2 = 4$$

$\therefore$ $$e = 2$$

(B) Here, $e = \sqrt{3}$

$\therefore$ $$b^2 = a^2(3-1) = 2a^2 \qquad \text{... (i)}$$

Now, hyperbola convert in the form

$$\frac{x^2}{a^2} - \frac{y^2}{2a^2} = 1$$

Let $P(a\sec\theta, a\sqrt{2}\tan\theta)$ be any point on the hyperbola.

$\because$ Asymptotes of hyperbola are

$$\frac{x^2}{a^2} - \frac{y^2}{2a^2} = 0$$

or $$2x^2 - y^2 = 0$$

or $$x\sqrt{2} + y = 0$$

and $$x\sqrt{2} - y = 0$$

$\because$ Product of the perpendiculars from P on asymptotes $= 6$

$\therefore$ $$\frac{|a\sqrt{2}\sec\theta + a\sqrt{2}\tan\theta|}{\sqrt{(2+1)}} \cdot \frac{|a\sqrt{2}\sec\theta - a\sqrt{2}\tan\theta|}{\sqrt{2+1}} = 6$$

or $$\frac{2a^2}{3} = 6$$

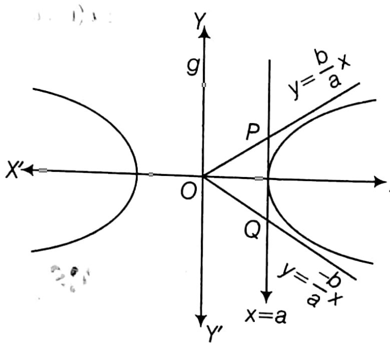
$\therefore$ $$a = 3$$

Hence, length of transverse axis $= 2a = 6$

(C) Equation of tangent at $(a, 0)$ is $x = a$

Equation of asymptotes are $y = \pm \frac{b}{a}x$

$\therefore$ $$P \equiv (a, b), Q \equiv (a, -b)$$



$\therefore$ Required area $= \frac{1}{2} \times a \times 2b$

$$= ab$$

$$= (4)(3) = 12 \qquad \text{(Here } a = 4, b = 3)$$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** (Ex. Nos. 25 and 26) are Assertion-Reason type examples. Each of these examples contains two statements. **Statement I** (Assertion) and **Statement II** (Reason) Each of these also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below :

(a) Statement I is true, statement II is true; statement II is a correct explanation for statement I

(b) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(c) Statement I is true, statement II is false

(d) Statement I is false, statement II is true

- **Ex. 25 Statement I** *Director circle of hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$ *is defined only when* $b \geq a$

Statement II *Director circle of hyperbola* $\frac{x^2}{25}-\frac{y^2}{9}=1$ *is* $x^2+y^2=16$.

Sol. (b) Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$ can be re-written as

$$\frac{x^2}{(-a)^2}-\frac{y^2}{(-b^2)}=1$$

So, the director circle will be

$$x^2+y^2=(-a^2)-(-b^2)=b^2-a^2;$$

Which will be defined only where $b \geq a$ (i.e. $b^2-a^2 \geq 0$)

$\therefore$ Statement I is true and director circle of hyperbola $\frac{x^2}{25}-\frac{y^2}{9}=1$

is $\qquad x^2+y^2=25-9=16$

$\therefore$ Statement II is true.

Hence, both statements are true but statement II is not a correct explanation of statement I.

- **Ex. 26 Statement I** *If a circle* $S \equiv 0$ *intersect a hyperbola* $xy=4$ *at four points, three of them being (2, 2), (4, 1) and (6, 2/3), then the coordinates of the fourth point are* $\left(\frac{1}{4}, 16\right)$.

Statement II *If a circle* $S \equiv 0$ *intersects a hyperbola* $xy=c^2$ *at* t_1, t_2, t_3 *and* t_4, *then* $t_1 t_2 t_3 t_4=1$.

Sol. (d) Let circle $S \equiv x^2+y^2-a^2=0$

and given hyperbola $\quad xy=c^2$

Then, $\qquad x^2+\frac{c^4}{x^2}=a^2$ or $x^4-a^2x^2+c^4=0$

If four intersecting points are

$\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right), \left(ct_3, \frac{c}{t_3}\right)$ and $\left(ct_4, \frac{c}{t_4}\right)$, then

$$(ct_1)(ct_2)(ct_3)(ct_4)=c^4$$

$\therefore \qquad t_1 t_2 t_3 t_4=1$

$\therefore$ Statement II is true

For the point (2, 2); $t_1=1$

For the point (4, 1); $t_2=2$

For the point (6, 2/3); $t_3=3$

For the point $\left(\frac{1}{4}, 16\right)$; $t_4=\frac{1}{8}$

Now, $\qquad t_1 t_2 t_3 t_4=\frac{3}{4} \neq 1$

$\therefore$ Statement I is false.

Hyperbola Exercise 8 :
Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

94. The locus of a point $P(\alpha,\beta)$ moving under the condition that the line $y=\alpha x+\beta$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is **[AIEEE 2005, 3M]**

(a) an ellipse
(b) a circle
(c) a parabola
(d) a hyperbola

95. Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$. The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse, also the product of eccentricities of given ellipse and hyperbola is 1, then **[IIT- JEE 2006, 5M]**

(a) the equation of hyperbola is $\frac{x^2}{9}-\frac{y^2}{16}=1$
(b) the equations of hyperbola is $\frac{x^2}{9}-\frac{y^2}{25}=1$
(c) focus of hyperbola is (5, 0)
(d) vertex of hyperbola is $(5\sqrt{3},0)$

96. A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2+4y^2=12$. Then, its equation is **[IIT- JEE 2007, 3M]**

(a) $x^2\text{cosec}^2\theta-y^2\sec^2\theta=1$
(b) $x^2\sec^2\theta-y^2\text{cosec}^2\theta=1$
(c) $x^2\sin^2\theta-y^2\cos^2\theta=1$
(d) $x^2\cos^2\theta-y^2\sin^2\theta=1$

97. Two branches of a hyperbola **[IIT- JEE 2007, 1.5M]**

(a) have a common tangent
(b) have a common normal
(c) do not have a common tangent
(d) do not have a common normal

98. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant when α varies **[AIEEE 2007, 3M]**

(a) abscissae of vertices (b) abscissae of foci
(c) eccentricity (d) directrix

99. Consider a branch of the hyperbola
$x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$
with vertex at the point A. Let B be one of the end points of its latusrectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is **[IIT- JEE 2008, 3M]**

(a) $1 - \sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{3}{2}} - 1$ (c) $1 + \sqrt{\frac{2}{3}}$ (d) $\sqrt{\frac{3}{2}} + 1$

100. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then **[IIT- JEE 2009, 4M]**

(a) equation of ellipse is $x^2 + 2y^2 = 2$
(b) the foci of ellipse are $(\pm 1, 0)$
(c) equation of ellipse is $x^2 + 2y^2 = 4$
(d) the foci of ellipse are $(\pm\sqrt{2}, 0)$

Paragraph

(Q. Nos. 100 and 102)

The circle $x^2 + y^2 - 8x = 0$ *and hyperbola* $\frac{x^2}{9} - \frac{y^2}{4} = 1$ *intersect at the points A and B.*

101. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

(a) $2x - \sqrt{5}y - 20 = 0$ (b) $2x - \sqrt{5}y + 4 = 0$
(c) $3x - 4y + 8 = 0$ (d) $4x - 3y + 4 = 0$

102. Equation of the circle with AB as its diameter is

(a) $x^2 + y^2 - 12x + 24 = 0$ (b) $x^2 + y^2 + 12x + 24 = 0$
(c) $x^2 + y^2 + 24x - 12 = 0$ (d) $x^2 + y^2 - 24x - 12 = 0$

[IIT- JEE 2010, 3 + 3M]

103. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the X-axis, then the eccentricity of the hyperbola is **[IIT- JEE 2010, 3M]**

104. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the X-axis at $(9, 0)$, then the eccentricity of the hyperbola is **[IIT- JEE 2011, 3M]**

(a) $\sqrt{\frac{5}{2}}$ (b) $\sqrt{\frac{3}{2}}$
(c) $\sqrt{2}$ (d) $\sqrt{3}$

105. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then **[IIT- JEE 2011, 4M]**

(a) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
(b) a focus of the hyperbola is $(2, 0)$
(c) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
(d) the equation of the hyperbola is $x^2 - 3y^2 = 3$

106. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are **[IIT- JEE 2012, 4M]**

(a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
(c) $(3\sqrt{3}, -2\sqrt{2})$ (d) $(-3\sqrt{3}, 2\sqrt{2})$

107. Consider the hyperbola $H: x^2 - y^2 = 1$ and a circle S with centre $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the X-axis at point M. If (l, m) is the centroid of the triangle PMN, then the correct expression(s) is(are) **[JEE (Advanced) 2015, 4M]**

(a) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$ (b) $\frac{dm}{dx_1} = \frac{x_1}{3(\sqrt{(x_1^2 - 1)})}$ for $x_1 > 1$
(c) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$ (d) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

108. The eccentricity of the hyperbola whose length of the latusrectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is **[JEE (Main) 2016, 4M]**

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$
(c) $\frac{4}{3}$ (d) $\frac{4}{\sqrt{3}}$

109. A hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at $(\pm 2, 0)$. Then the ~~taught~~ tangent to this hyperbola at ~~Pab~~ passes through the point **[JEE (Main) 2017, 4M]**

(a) $(-\sqrt{2}, -\sqrt{3})$ (b) $(3\sqrt{2}, 2\sqrt{3})$
(c) $(2\sqrt{2}, 3\sqrt{3})$ (d) $(\sqrt{3}, \sqrt{2})$

110. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following cannot be sides of a right angled triangle? **[JEE Advanced 2017, 4M]**

(a) $2a, 8, 1$ (b) $a, 4, 1$
(c) $a, 4, 2$ (d) $2a, 4, 1$

■ **Direction** (Q. No. 111 to 113) Matching the information given in the three columns of the following table. Columns 1, 2 and 3 contain conics, equations of tangents to the conics and points of contact, respectively.

[JEE Advanced 2017, (3 + 3 + 3) M]

Column 1	Column 2	Column 3
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2 - 1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

111. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only correct combination?

(a) (IV) (iii) (S)
(b) (II) (iv) (R)
(c) (IV) (iv) (S)
(d) (II) (iii) (R)

112. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact $(-1, 1)$, then which of the following options is the only correct combination for obtaining its equation?

(a) (III) (i) (P) (b) (I) (i) (P)
(c) (II) (ii) (Q) (d) (I) (ii) (Q)

113. If a tangent of a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is (8, 16), then which of the following options is the only correct combination?

(a) (III) (i) (P) (b) (III) (ii) (Q)
(c) (II) (iv) (R) (d) (I) (ii) (Q)

Answers

Exercise for Session 1

1. (d) **2.** (b) **3.** (d) **4.** (a) **5.** (d)
6. (a) **7.** (a) **8.** (b) **9.** (c,d) **10.** (c)
11. (b) **12.** (c) **13.** $y^2 - x^2 = 5$
14. $8x^2 - y^2 - 64x + 10y + 71 = 0$
16. (1, 2); $2\sqrt{\left(\frac{2}{5}\right)}$; $2\sqrt{5}$; $2\sqrt{3}$ **17.** 2 **18.** $\lambda = \pm 6$
19. $30x - 24y \pm \sqrt{(161)} = 0$; $\left(\pm\frac{15}{2\sqrt{161}}, \pm\frac{8}{3\sqrt{161}}\right)$

Exercise for Session 2

1. (d) **2.** (c) **3.** (a) **4.** (d) **5.** (b)
6. (a) **7.** (a) **8.** (b) **9.** (d) **10.** (c)
11. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Exercise for Session 3

1. (b) **2.** (d) **3.** (a) **4.** (d) **5.** (d)
6. (c) **7.** (a) **8.** (a) **9.** (d) **10.** (b)
11. (d) **12.** (b) **13.** $x + 3y = 0$
14. $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$
15. $6x^2 + 13xy + 6y^2 - 38x - 37y - 98 = 0$

Chapter Exercises

1. (a) **2.** (d) **3.** (b) **4.** (c) **5.** (a) **6.** (a)
7. (b) **8.** (a) **9.** (b) **10.** (a) **11.** (b) **12.** (c)
13. (d) **14.** (b) **15.** (c) **16.** (b) **17.** (b) **18.** (b)
19. (d) **20.** (a) **21.** (a) **22.** (c) **23.** (b) **24.** (c)
25. (b) **26.** (c) **27.** (a) **28.** (b) **29.** (b) **30.** (c)
31. (a,c) **32.** (c,d) **33.** (b,c) **34.** (a,b,d) **35.** (a,b,c) **36.** (b,c)
37. (b,c) **38.** (a,b,c) **39.** (a,b) **40.** (a,d) **41.** (a,b,c) **42.** (a,c,d)
43. (a,b,c,d) **44.** (a,b,c,d) **45.** (a,b,d)
46. (c) **47.** (c) **48.** (b) **49.** (b) **50.** (c) **51.** (d)
52. (c) **53.** (d) **54.** (b) **55.** (d) **56.** (a) **57.** (b)
58. (b) **59.** (c) **60.** (b) **61.** (2) **62.** (8) **63.** (4)
64. (3) **65.** (7) **66.** (3) **67.** (2) **68.** (5) **69.** (4)
70. (7)
71. (A) → (p,r,s); (B) → (p,q,r); (C) → (p,r,s); (D) → (p,r)
72. (A) → (p,q); (B) → (p,q); (C) → (p,r); (D) → (p,r,s)
73. (A) → (q,r,s); (B) → (p); (C) → (p); (D) → (q,r,s)
74. (b) **75.** (a) **76.** (d) **77.** (a) **78.** (b) **79.** (a)
80. (c) **81.** (d)
83. (i) $\frac{c^2}{2t_1t_2t_3}|(t_1 - t_2)((t_2 - t_3)(t_3 - t_1)|$

(ii) $2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

89. $x = 0$ and $x + y = 0$
94. (d) **95.** (a,c) **96.** (a) **97.** (b,c) **98.** (b) **99.** (b)
100. (a,b) **101.** (b) **102.** (a) **103.** (2) **104.** (b) **105.** (b,d)
106. (a,b) **107.** (a,b,d) **108.** (a) **109.** (c) **110.** (a,b,c)
111. (b) **112.** (d) **113.** (a)