CHAPTER

08

Matrices

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J. J. Sylvester was the first to use the word "Matrix" in 1850 and later on in 1858 Arthur Cayley developed the theory of matrices in a systematic way. 'Matrices' is a powerful tool in mathematics and its study is becoming important day by day due to its wide applications in almost every branch of science. This mathematical tool is not only used in certain branches of sciences but also in genetics, economics, sociology, modern psychology and industrial management.

Session 1

Definition, Types of Matrices, Difference Between a Matrix and a Determinant, Equal Matrices, Operations of Matrices, Various Kinds of Matrices

Definition

A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix (which is read as m by n matrix).

An $m \times n$ matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m_1} & a_{m_2} & a_{m_3} & \cdots & a_{mn} \end{bmatrix}$$

In a compact form the above matrix is represented by $[a_{ij}]$, $i = 1, 2, 3, \ldots, m$, $j = 1, 2, 3, \ldots, n$ or simply by $[a_{ij}]_{m \times n}$, where the symbols a_{ij} represent any numbers (a_{ij} lies in the ith row (from top) and jth column (from left)).

Notations A matrix is denoted by capital letter such as $A, B, C, \ldots, X, Y, Z$.

Note

1. A matrix may be represented by the symbols $[a_{ij}]$, (a_{ij}), $\| a_{ij} \|$ or by a single capital letter A (say)
   $A = [a_{ij}]_{m \times n}$ or $(a_{ij})_{m \times n}$ or $\| a_{ij} \|$
   Generally, the first system is adopted.
2. The numbers $a_{11}, a_{12}, \ldots$, etc., of rectangular array are called the elements or entries of the matrix.
3. A matrix is essentially an arrangement of elements and has no value.
4. The plural of 'matrix' is 'matrices'.

Example 1. If a matrix has 12 elements, what are the possible orders it can have? What will be the possible orders if it has 7 elements?

Sol. We know that, if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 12 elements, we will find all ordered pairs of natural numbers, whose product is 12.

Thus, all possible ordered pairs are (1, 12), (12, 1), (2, 6), (6, 2), (3, 4), (4, 3).

Hence, possible orders are 1×12, 12×1, 2×6, 6×2, 3×4 and 4×3.

If the matrix has 7 elements, then the possible orders will be 1×7 and 7×1.

Example 2. Construct a 2×3 matrix $A = [a_{ij}]$, whose elements are given by

(i) $a_{ij} = \dfrac{(i + 2j)^2}{2}$. (ii) $a_{ij} = \dfrac{1}{2}|2i - 3j|$.

(iii) $a_{ij} = \begin{cases} i - j, & i \ge j \\ i + j, & i < j \end{cases}$

(iv) $a_{ij} = \left[\dfrac{i}{j}\right]$,

where [.] denotes the greatest integer function.

(v) $a_{ij} = \left\{\dfrac{2i}{3j}\right\}$,

where {.} denotes the fractional part function.

(vi) $a_{ij} = \left(\dfrac{3i + 4j}{2}\right)$

where (.) denotes the least integer function.

Sol. We have, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3}$

(i) Since, $a_{ij} = \dfrac{(i + 2j)^2}{2}$, therefore

$a_{11} = \dfrac{(1+2)^2}{2} = \dfrac{9}{2}$, $a_{12} = \dfrac{(1+4)^2}{2} = \dfrac{25}{2}$,

$a_{13} = \dfrac{(1+6)^2}{2} = \dfrac{49}{2}$, $a_{21} = \dfrac{(2+2)^2}{2} = 8$,

$a_{22} = \dfrac{(2+4)^2}{2} = 18$ and $a_{23} = \dfrac{(2+6)^2}{2} = 32$

Hence, the required matrix is $A = \begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32 \end{bmatrix}$

(ii) Since, $a_{ij} = \dfrac{1}{2}|2i - 3j|$, therefore

$a_{11} = \dfrac{1}{2}|2 - 3| = \dfrac{1}{2}|-1| = \dfrac{1}{2}$,

$a_{12} = \dfrac{1}{2}|2 - 6| = \dfrac{1}{2}|-4| = \dfrac{4}{2} = 2$,

$a_{13} = \dfrac{1}{2}|2 - 9| = \dfrac{1}{2}|-7| = \dfrac{7}{2}$,

$a_{21} = \dfrac{1}{2}|2 - 3| = \dfrac{1}{2}|-1| = \dfrac{1}{2}$,

$a_{22} = \dfrac{1}{2}|2 - 6| = \dfrac{1}{2}|-4| = \dfrac{4}{2} = 2$

and $a_{23} = \dfrac{1}{2}|4 - 9| = \dfrac{1}{2}|-5| = \dfrac{5}{2}$

Hence, the required matrix is $A = \begin{bmatrix} \frac{1}{2} & 2 & \frac{7}{2} \\ \frac{1}{2} & 2 & \frac{5}{2} \end{bmatrix}$.

(iii) Since, $a_{ij} = \begin{cases} i - j, & i \ge j \\ i + j, & i < j \end{cases}$, therefore

$a_{11} = 1 - 1 = 0$, $a_{12} = 1 + 2 = 3$, $a_{13} = 1 + 3 = 4$,
$a_{21} = 2 - 1 = 1$, $a_{22} = 2 - 2 = 0$ and $a_{23} = 2 + 3 = 5$

Hence, the required matrix is

$$A = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$$

(iv) Since, $a_{ij} = \left[\dfrac{i}{j}\right]$, therefore $[\because [x] \le x]$

$a_{11} = \left[\dfrac{1}{1}\right] = [1] = 1$, $a_{12} = \left[\dfrac{1}{2}\right] = [0.5] = 0$,

$a_{13} = \left[\dfrac{1}{3}\right] = [0.33] = 0$, $a_{21} = \left[\dfrac{2}{1}\right] = [2] = 2$

and $a_{22} = \left[\dfrac{2}{2}\right] = [1] = 1$ and $a_{23} = \left[\dfrac{2}{3}\right] = [0.67] = 0$

Hence, the required matrix is $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

(v) Since, $a_{ij} = \left\{\dfrac{2i}{3j}\right\}$, therefore $[\because 0 \le \{x\} < 1]$

$a_{11} = \left\{\dfrac{2}{3}\right\} = \dfrac{2}{3}$, $a_{12} = \left\{\dfrac{2}{6}\right\} = \left\{\dfrac{1}{3}\right\} = \dfrac{1}{3}$,

$a_{13} = \left\{\dfrac{2}{9}\right\} = \dfrac{2}{9}$, $a_{21} = \left\{\dfrac{4}{3}\right\} = \left\{1 + \dfrac{1}{3}\right\} = \dfrac{1}{3}$,

$a_{22} = \left\{\dfrac{4}{6}\right\} = \left\{\dfrac{2}{3}\right\} = \dfrac{2}{3}$ and $a_{23} = \left\{\dfrac{4}{9}\right\} = \dfrac{4}{9}$

Hence, the required matrix is $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{9} \end{bmatrix}$.

(vi) Since, $a_{ij} = \left(\dfrac{3i + 4j}{2}\right)$, therefore $[\because (x) \ge x]$

$a_{11} = \left(\dfrac{3+4}{2}\right) = \left(\dfrac{7}{2}\right) = (3.5) = 4$,

$a_{12} = \left(\dfrac{3+8}{2}\right) = \left(\dfrac{11}{2}\right) = (5.5) = 6$,

$a_{13} = \left(\dfrac{3+12}{2}\right) = \left(\dfrac{15}{2}\right) = (7.5) = 8$,

$a_{21} = \left(\dfrac{6+4}{2}\right) = \left(\dfrac{10}{2}\right) = (5) = 5$,

$a_{22} = \left(\dfrac{6+8}{2}\right) = \left(\dfrac{14}{2}\right) = (7) = 7$

and $a_{23} = \left(\dfrac{6+12}{2}\right) = \left(\dfrac{18}{2}\right) = (9) = 9$

Hence, the required matrix is

$$A = \begin{bmatrix} 4 & 6 & 8 \\ 5 & 7 & 9 \end{bmatrix}$$

Types of Matrices

1. Row Matrix or Row Vector

A matrix is said to be row matrix or row vector, if it contains only one row, i.e. a matrix $A = [a_{ij}]_{m \times n}$ is said to be row matrix, if $m = 1$.

For example,

(i) $A = [a_{11} \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n}]_{1 \times n}$

(ii) $B = [3 \quad 5 \quad -7 \quad 9]_{1 \times 4}$

are called row matrices.

2. Column Matrix or Column Vector

A matrix is said to be column matrix or column vector, if it contains only one column, i.e., a matrix $A = [a_{ij}]_{m \times n}$ is said to be column matrix, if $n = 1$. *For example,*

(i) $A=\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{mn} \end{bmatrix}_{m\times 1}$ (ii) $B=\begin{bmatrix} 7 \\ 0 \\ -8 \\ 2 \\ 1 \end{bmatrix}_{5\times 1}$

are called column matrices.

3. Rectangular Matrix

A matrix is said to be rectangular matrix, if the number of rows and the number of columns are not equal i.e., a matrix $A=[a_{ij}]_{m\times n}$ is called a rectangular matrix, iff $\boxed{m\neq n}$. For example,

(i) $A=\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 0 & -3 & 8 \\ 7 & 4 & 2 & 5 \end{bmatrix}_{3\times 4}$ (ii) $B=\begin{bmatrix} 2 & -3 \\ 3 & 0 \\ 4 & 8 \end{bmatrix}_{3\times 2}$

are called rectangular matrices.

4. Square Matrix

A matrix is said to be a square matrix, if the number of rows and the number of columns are equal i.e., a matrix $A=[a_{ij}]_{m\times n}$ is called a square matrix, iff $\boxed{m=n}$.

For example,

(i) $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$ (ii) $B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

are called square matrices.

Remark

If $A=[a_{ij}]$ is a square matrix of order n, then elements (entries) $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are said to constitute the *diagonal* of the matrix A. The line along which the diagonal elements lie is called principal or leading diagonal. Thus, if $A=\begin{bmatrix} 1 & 4 & 0 \\ 8 & 3 & -2 \\ 9 & 2 & 5 \end{bmatrix}$, then the elements of the diagonal of A are 1, 3, 5.

5. Diagonal Matrix

A square matrix is said to be a diagonal matrix, if all its non-diagonal elements are zero. Thus, $A=[a_{ij}]_{n\times n}$ is called a diagonal matrix, if $a_{ij}=0$, when $i\neq j$.

For example,

(i) $A=[2]$ (ii) $B=\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ (iii) $C=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

are diagonal matrices of order 1, 2 and 3, respectively. A diagonal matrix of order n having $d_1, d_2, d_3, \ldots, d_n$ as diagonal elements may be denoted by $\text{diag}(d_1, d_2, d_3, \ldots, d_n)$.

Thus, $A=\text{diag}\,(2)$, $B=\text{diag}\,(-1, 2)$ and $C=\text{diag}\,(3, 5, 7)$.

Remark

(i) No element of principal diagonal in a diagonal matrix is zero.

(ii) Minimum number of zero in a diagonal matrix is given by $\boxed{n(n-1)}$, where n is order of matrix.

6. Scalar Matrix

A diagonal matrix is said to be a scalar matrix, if its diagonal elements are equal. Thus, $A=[a_{ij}]_{n\times n}$ is called scalar matrix, if

$$a_{ij}=\begin{cases} 0, \text{ if } i\neq j \\ k, \text{ if } i=j \end{cases}, \text{ where } k \text{ is scalar.}$$

For example,

(i) $[7]$ (ii) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

are scalar matrices of order 1, 2 and 3, respectively. They can be written as diag (7), diag (2, 2) and diag (5, 5, 5), respectively.

7. Unit or Identity Matrix

A diagonal matrix is said to be an identity matrix, if its diagonal elements are equal to 1.

Thus, $A=[a_{ij}]_{n\times n}$ is called unit or identity matrix, if

$$a_{ij}=\begin{cases} 0, \text{ if } i\neq j \\ 1, \text{ if } i=j \end{cases}$$

A unit matrix of order n is denoted by I_n or I. *For example,*

(i) $I_1=[1]$ (ii) $I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iii) $I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are identity matrices of order 1, 2 and 3, respectively.

8. Singleton Matrix

A matrix is said to be singleton matrix, if it has only one element i.e. a matrix $A=[a_{ij}]_{m\times n}$ is said to be singleton matrix, iff $\boxed{m=n=1}$.

For example, $[3], [k], [-2]$ are singleton matrices.

9. Triangular Matrix

A square matrix is called a triangular matrix, if its each element above or below the principal diagonal is zero. It is of two types:

(a) **Upper Triangular Matrix** A square matrix in which all elements below the principal diagonal are zero is called an upper triangular matrix i.e., a matrix $A=[a_{ij}]_{n\times n}$ is said to be an upper triangular matrix, if $a_{ij}=0$, when $i>j$.

For example,

(i) $\begin{bmatrix} 3 & -2 & 4 & 1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 8 \end{bmatrix}_{4\times 4}$

(ii) $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}_{5\times 5}$

are upper triangular matrices.

(b) **Lower Triangular Matrix** A square matrix in which all elements above the principal diagonal are zero is called a lower triangular matrix i.e., a matrix $A=[a_{ij}]_{n\times n}$ is said to be a lower triangular matrix, if $a_{ij}=0$, when $i<j$. *For example,*

(i) $\begin{bmatrix} 7 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}_{3\times 3}$ (ii) $\begin{bmatrix} 10 & 0 & 0 & 0 \\ 8 & 9 & 0 & 0 \\ 5 & 6 & 7 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}_{4\times 4}$

are lower triangular matrices.

Note

Minimum number of zeroes in a triangular matrix is given by $\frac{n(n-1)}{2}$, where n is order of matrix.

10. Horizontal Matrix

A matrix is said to be horizontal matrix, if the number of rows is less than the number of columns i.e., a matrix $A=[a_{ij}]_{m\times n}$ is said to horizontal matrix, iff $\boxed{m<n}$.

For example, $A=\begin{bmatrix} 2 & 3 & 4 & 5 \\ 8 & 9 & 7 & -2 \\ 2 & -2 & -3 & 4 \end{bmatrix}_{3\times 4}$ is a horizontal matrix. [$\because$ number of rows (3) < number of columns (4)]

11. Vertical Matrix

A matrix is said to be vertical matrix, if the number of rows is greater than the number of columns i.e., a matrix $A=[a_{ij}]_{m\times n}$ is said to vertical matrix, iff $m>n$.

For example, $A=\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 7 \\ 3 & 5 & 4 \\ 2 & 7 & 9 \\ -1 & 2 & -5 \end{bmatrix}_{5\times 3}$ is a vertical matrix.

[$\because$ number of rows (5) > number of columns (3)]

12. Null Matrix or Zero Matrix

A matrix is said to be null matrix or zero matrix, if all elements are zero i.e., a matrix $A=[a_{ij}]_{m\times n}$ is said to be a zero or null matrix, iff $a_{ij}=0, \forall\, i, j$. It is denoted by O.

For example,

(i) $O_{2\times 3}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $O_{3\times 3}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are called the null matrices.

13. Sub-Matrix

A matrix which is obtained from a given matrix by deleting any number of rows and number of columns is called a sub-matrix of the given matrix.

For example, $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$ is a sub-matrix of $\begin{bmatrix} 8 & 9 & 5 \\ 2 & 3 & 4 \\ 3 & -2 & 5 \end{bmatrix}$.

14. Trace of a Matrix

The sum of all diagonal elements of a square matrix $A=[a_{ij}]_{n\times n}$ (say) is called the trace of a matrix A and is denoted by Tr (A).

Thus, $$\text{Tr}(A)=\sum_{i=1}^{n} a_{ii}$$

For example, If $A=\begin{bmatrix} 2 & -7 & 9 \\ 0 & 3 & 2 \\ 8 & 9 & 4 \end{bmatrix}$, then

$$\text{Tr}(A)=2+3+4=9$$

Properties of Trace of a Matrix

Let $A=[a_{ij}]_{n\times n}$, $B=[b_{ij}]_{n\times n}$ and k is a scalar, then

(i) $\text{Tr}(kA)=k\cdot\text{Tr}(A)$

(ii) $\text{Tr}(A\pm B)=\text{Tr}(A)\pm\text{Tr}(B)$

(iii) $\text{Tr}(AB)=\text{Tr}(BA)$

(iv) $\text{Tr}(A)=\text{Tr}(A')$

(v) $\text{Tr}(I_n)=n$

(vi) $\text{Tr}(AB)\neq\text{Tr}(A)\,\text{Tr}(B)$

(vii) $\text{Tr}(A)=\text{Tr}(CAC^{-1})$, where C is a non-singular square matrix of order n.

15. Determinant of Square Matrix

Let $A=[a_{ij}]_{n\times n}$ be a matrix. The determinant formed by the elements of A is said to be the determinant of matrix A. This is denoted by $|A|$.

For example,

If $A=\begin{bmatrix}3&4&5\\6&7&8\\2&-3&5\end{bmatrix}$, then $|A|=\begin{vmatrix}3&4&5\\6&7&8\\2&-3&5\end{vmatrix}=-39.$

Remark

1. If $A_1, A_2, A_3, \ldots, A_n$ are square matrices of the same order, then $|A_1A_2A_3\ldots A_n|=|A_1||A_2||A_3|\ldots|A_n|$.
2. If k is a scalar and A is a square matrix of order n, then $|kA|=k^n|A|$

16. Comparable Matrices

Two matrices $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{p\times q}$ are said to be comparable, if $\boxed{m=p \text{ and } n=q}$.

For example,

The matrices $\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}$ and $\begin{bmatrix}p&q&r\\s&t&u\end{bmatrix}$ are comparable

but the matrices $\begin{bmatrix}1&2\\4&8\end{bmatrix}$ and $\begin{bmatrix}2&4&6\\5&3&1\end{bmatrix}$ are not comparable.

Difference Between a Matrix and a Determinant

(i) A matrix cannot be reduced to a number but determinant can be reduced to a number.

(ii) The number of rows may or may not be equal to the number of columns in matrices but in determinant the number of rows is equal to the number of columns.

(iii) On interchanging the rows and columns, a different matrix is formed but in determinant it does not change the value.

(iv) A square matrix A such that $|A|\neq 0$, is called a non-singular matrix. If $|A|=0$, then the matrix A is called a singular matrix. Imp

(v) Matrices represented by [ ], ( ), ‖ ‖ but determinant is represented by | |.

Equal Matrices

Two matrices are said to be equal, if

(i) they are of the same order i.e., if they have same number of rows and columns.

(ii) the elements in the corresponding positions of the two matrices are equal.

Thus, if $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{p\times q}$, then $A=B$, iff

(i) $m=p, n=q$ (ii) $a_{ij}=b_{ij}, \forall\, i, j$

For example, If $A=\begin{bmatrix}-1&2&4\\3&0&5\end{bmatrix}_{2\times 3}$ and

$B=\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}_{2\times 3}$ are equal matrices, then

$a=-1, b=2, c=4, d=3, e=0, f=5$

I Example 3. If $\begin{bmatrix}x+3 & 2y+x\\ z-1 & 4w-8\end{bmatrix}=\begin{bmatrix}-x-1 & 0\\ 3 & 2w\end{bmatrix}$, then find the value of $|x+y|+|z+w|$.

Sol. As the given matrices are equal so their corresponding elements are equal.

$$x+3=-x-1 \Rightarrow 2x=-4$$

$\therefore$ $x=-2$ ...(i)

$2y+x=0$

$\Rightarrow$ $2y-2=0$ [from Eq. (i)]

$\Rightarrow$ $y=1$ ...(ii)

$z-1=3$

$\Rightarrow$ $z=4$ ...(iii)

$4w-8=2w$

$\Rightarrow$ $2w=8$

$\therefore$ $w=4$ ...(iv)

Hence, $|x+y|+|z+w|=|-2+1|+|4+4|=1+8=9$

I Example 4. If $\begin{bmatrix}2\alpha+1 & 3\beta\\ 0 & \beta^2-5\beta\end{bmatrix}=\begin{bmatrix}\alpha+3 & \beta^2+2\\ 0 & -6\end{bmatrix}$ find the equation whose roots are α and β.

Sol. The given matrices will be equal, iff

$2\alpha+1=\alpha+3 \Rightarrow \alpha=2$

$3\beta=\beta^2+2 \Rightarrow \beta^2-3\beta+2=0$

$\therefore$ $\beta=1, 2$ and $\beta^2-5\beta=-6$ ...(i)

$\Rightarrow$ $\beta^2-5\beta+6=0$

$\therefore$ $\beta=2, 3$ ...(ii)

From Eqs. (i) and (ii), we get $\beta=2$

$\Rightarrow$ $\alpha=2, \beta=2$

$\therefore$ Required equation is $x^2-(2+2)x+2\cdot 2=0$

$\Rightarrow$ $x^2-4x+4=0$

Operations of Matrices

Addition of Matrices

Let A, B be two matrices, each of order $m\times n$. Then, their sum $A+B$ is a matrix of order $m\times n$ and is obtained by adding the corresponding elements of A and B.

Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$, then $A+B=[a_{ij}+b_{ij}]_{m\times n}, \forall\, i, j$

I Example 5. Given, $A=\begin{bmatrix}1&3&5\\-2&0&2\\0&4&-3\end{bmatrix}$, $B=\begin{bmatrix}0&3\\-2&0\\0&-4\end{bmatrix}$ and $C=\begin{bmatrix}4&1&-2\\3&2&1\\2&-1&7\end{bmatrix}$. Find (whichever defined)

(i) $A+B$. (ii) $A+C$.

Sol. (i) Given, A is a matrix of the type 3×3 and B is a matrix of the type 3×2.
Since, A and B are not of the same type.
$\therefore$ Sum $A+B$ is not defined.

(ii) As A and C are two matrices of the same type, therefore the sum $A+C$ is defined.

$$\therefore A+C=\begin{bmatrix}1&3&5\\-2&0&2\\0&4&-3\end{bmatrix}+\begin{bmatrix}4&1&-2\\3&2&1\\2&-1&7\end{bmatrix}$$

$$=\begin{bmatrix}1+4&3+1&5-2\\-2+3&0+2&2+1\\0+2&4-1&-3+7\end{bmatrix}=\begin{bmatrix}5&4&3\\1&2&3\\2&3&4\end{bmatrix}$$

I Example 6. If $a, b; b, c$ and c, a are the roots of $x^2-4x+3=0$, $x^2-8x+15=0$ and $x^2-6x+5=0$, respectively. Compute $\begin{bmatrix}a^2+c^2 & a^2+b^2\\ b^2+c^2 & a^2+c^2\end{bmatrix}+\begin{bmatrix}2ac & -2ab\\ -2bc & -2ac\end{bmatrix}$

Sol. $\because$ $x^2-4x+3=0$

$\Rightarrow$ $(x-1)(x-3)=0$ $\therefore x=1, 3$

$x^2-8x+15=0$

$\Rightarrow$ $(x-3)(x-5)=0$ $\therefore x=3, 5$

and $x^2-6x+5=0$

$\Rightarrow$ $(x-5)(x-1)=0$ $\therefore x=5, 1$

It is clear that $a=1, b=3$ and $c=5$

Now, $\begin{bmatrix}a^2+c^2 & a^2+b^2\\ b^2+c^2 & a^2+c^2\end{bmatrix}+\begin{bmatrix}2ac & -2ab\\ -2bc & -2ac\end{bmatrix}$

$$=\begin{bmatrix}a^2+c^2+2ac & a^2+b^2-2ab\\ b^2+c^2-2bc & a^2+c^2-2ac\end{bmatrix}=\begin{bmatrix}(a+c)^2 & (a-b)^2\\ (b-c)^2 & (a-c)^2\end{bmatrix}$$

$$=\begin{bmatrix}(1+5)^2 & (1-3)^2\\ (3-5)^2 & (1-5)^2\end{bmatrix}=\begin{bmatrix}36 & 4\\ 4 & 16\end{bmatrix}$$

Properties of Matrix Addition

Property 1 Addition of matrices is commutative,
i.e. $A+B=B+A$
where A and B are any two $m\times n$ matrices, i.e. matrices of the same order.

Property 2 Addition of matrices is associative
i.e. $(A+B)+C=A+(B+C)$
where A, B and C are any three matrices of the same order $m\times n$ (say).

Property 3 Existence of additive identity
i.e. $A+O=A=O+A$
where A be any $m\times n$ matrix and O be the $m\times n$ null matrix. The null matrix O is the identity element for matrix addition.

Property 4 Existence of additive inverse
If A be any $m\times n$ matrix, then there exists another $m\times n$ matrix B, such that $A+B=O=B+A$
where O is the $m\times n$ null matrix.
Here, the matrix B is called the additive inverse of the matrix A or the negative of A.

Property 5 Cancellation laws
If A, B and C are matrices of the same order $m\times n$ (say),
then $A+B=A+C \Rightarrow B=C$ [left cancellation law]
and $B+A=C+A \Rightarrow B=C$ [right cancellation law]

Scalar Multiplication

Let $A=[a_{ij}]_{m\times n}$ be a matrix and k be any number called a scalar. Then, the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA.

Thus, $kA=[ka_{ij}]_{m\times n}$

Properties of Scalar Multiplication

If $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$ are two matrices and k, l are scalars, then

(i) $k(A+B)=kA+kB$ (ii) $(k+l)A=kA+lA$

(iii) $(kl)A=k(lA)=l(kA)$

(iv) $(-k)A=-(kA)=k(-A)$

(v) $1A=A, (-1)A=-A$

I Example 7. Determine the matrix A,

when $A=4\begin{bmatrix}1&2&3\\-1&-2&-3\\4&2&6\end{bmatrix}+2\begin{bmatrix}5&4&1\\3&2&4\\3&8&2\end{bmatrix}$.

Sol. $A = \begin{bmatrix} 4 & 8 & 12 \\ -4 & -8 & -12 \\ 16 & 8 & 24 \end{bmatrix} + \begin{bmatrix} 10 & 8 & 2 \\ 6 & 4 & 8 \\ 6 & 16 & 4 \end{bmatrix}$

$= \begin{bmatrix} 4+10 & 8+8 & 12+2 \\ -4+6 & -8+4 & -12+8 \\ 16+6 & 8+16 & 24+4 \end{bmatrix} = \begin{bmatrix} 14 & 16 & 14 \\ 2 & -4 & -4 \\ 22 & 24 & 28 \end{bmatrix}$

| Example 8. If $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ and $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$, then find the value of $b - a - k$.

Sol. We have, $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix} \Rightarrow kA = \begin{bmatrix} 0 & 2k \\ 3k & -4k \end{bmatrix}$

But $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$

$\therefore \begin{bmatrix} 0 & 2k \\ 3k & -4k \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$

$\Rightarrow \quad 2k = 3a, 3k = 2b, -4k = 24$

$\Rightarrow \quad k = -6, a = -4, b = -9$

Hence, $b - a - k = -9 - (-4) - (-6) = -9 + 4 + 6 = 1$

Subtraction of Matrices

Let A, B be two matrices, each of order $m \times n$. Then, their subtraction $A - B$ is a matrix of order $m \times n$ and is obtained by subtracting the corresponding elements of A and B. Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A - B = [a_{ij} - b_{ij}]_{m \times n}, \forall i, j$

For example, If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$,

then $A - B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 2-a & 3-b \\ 4-c & 5-d \\ 6-e & 7-f \end{bmatrix}$

| Example 9. Given, $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$. Find the matrix C such that $A + 2C = B$.

Sol. Given, $A + 2C = B$

$2C = B - A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3-1 & -1-2 & 2+3 \\ 4-5 & 2-0 & 5-2 \\ 2-1 & 0+1 & 3-1 \end{bmatrix}$

$\therefore \quad 2C = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow C = \frac{1}{2}\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & -3/2 & 5/2 \\ -1/2 & 1 & 3/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$

| Example 10. Solve the following equations for X and Y. $2X - Y = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$, $2Y + X = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$

Sol. Given, $2X - Y = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$

On multiplying both sides by 2, we get

$4X - 2Y = 2\begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$; $4X - 2Y = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix}$ ...(i)

also given $X + 2Y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$ ...(ii)

Adding Eqs. (i) and (ii), we get

$5X = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$

$= \begin{bmatrix} 6+4 & -6+1 & 0+5 \\ 6-1 & 6+4 & 4-4 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix}$

$\therefore \quad X = \frac{1}{5}\begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

Putting the value of X in Eq. (ii), we get

$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} + 2Y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$

$\Rightarrow \quad 2Y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 4-2 & 1+1 & 5-1 \\ -1-1 & 4-2 & -4-0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ -2 & 2 & -4 \end{bmatrix}$

$\therefore \quad Y = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$

Hence, $X = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$

Remark

If two matrices A and B are of the same order, then only their addition and subtraction is possible and these matrices are said to be conformable for addition or subtraction. On the other hand, if the matrices A and B are of different orders, then their addition and subtraction is not possible and these matrices are called non-conformable for addition and subtraction.

Multiplication

Conformable for Multiplication

If A and B be two matrices which are said to be conformable for the product AB. If the number of columns in A (called the pre-factor) is equal to the number of rows in B (called the post-factor) otherwise non-conformable for multiplication. Thus,

(i) AB is defined, if number of columns in A = number of rows in B.

(ii) BA is defined, if number of columns in B = number of rows in A.

Multiplication of Matrices

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ be two matrices, then the product AB is defined as the matrix $C = [C_{ij}]_{m \times p}$,

where $C_{ij} = \sum_{j=1}^{n} a_{ij}\, b_{jk}, 1 \le i \le m, 1 \le k \le p$

$= a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + a_{i3}\, b_{3k} + \ldots + a_{in}\, b_{nk}$

i.e., (i, k) th entry of the product AB is the sum of the product of the corresponding elements of the ith row of A (pre-factor) and kth column of B (post-factor).

Note

In the product AB, $\begin{cases} A = \text{Pre-factor} \\ B = \text{Post-factor} \end{cases}$

| Example 11. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$, obtain the product AB and explain why BA is not defined?

Sol. Here, the number of columns in $A = 3$ = the number of rows in B. Therefore, the product AB is defined.

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \times \begin{array}{cc} C_1 & C_2 \\ \end{array} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$$

R_1, R_2, R_3 are rows of A and C_1, C_2 are columns of B.

$\therefore \quad AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \\ R_3C_1 & R_3C_2 \end{bmatrix}_{3 \times 2}$

$$= \begin{bmatrix} \boxed{0\ 1\ 2}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} & \boxed{0\ 1\ 2}\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \\ \boxed{1\ 2\ 3}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} & \boxed{1\ 2\ 3}\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \\ \boxed{2\ 3\ 4}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} & \boxed{2\ 3\ 4}\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \end{bmatrix}_{3 \times 2}$$

For convenience of multiplication we write columns in horizontal rectangles.

$$\begin{bmatrix} \boxed{0\ 1\ 2} \; \boxed{1\ -1\ 2} & \boxed{0\ 1\ 2} \; \boxed{-2\ 0\ -1} \\ \boxed{1\ 2\ 3} \; \boxed{1\ -1\ 2} & \boxed{1\ 2\ 3} \; \boxed{-2\ 0\ -1} \\ \boxed{2\ 3\ 4} \; \boxed{1\ -1\ 2} & \boxed{2\ 3\ 4} \; \boxed{-2\ 0\ -1} \end{bmatrix}$$

$= \begin{bmatrix} 0 \times 1 + 1 \times (-1) + 2 \times 2 & 0 \times (-2) + 1 \times 0 + 2 \times (-1) \\ 1 \times 1 + 2 \times (-1) + 3 \times 2 & 1 \times (-2) + 2 \times 0 + 3 \times (-1) \\ 2 \times 1 + 3 \times (-1) + 4 \times 2 & 2 \times (-2) + 3 \times 0 + 4 \times (-1) \end{bmatrix}_{3 \times 2}$

$= \begin{bmatrix} 0-1+4 & 0+0-2 \\ 1-2+6 & -2+0-3 \\ 2-3+8 & -4+0-4 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}_{3 \times 2}$

Since, the number of columns of B is 2 and the number of rows of A is 3, BA is not defined $(\because 2 \ne 3)$.

Remark

Verification for the product to be correct.

From above example

$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$

Sum 3 6 9   15 −15

Now, $\boxed{3\ 6\ 9}$ $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ $= 3 \times 1 + 6 \times (-1) + 9 \times 2$
$= 3 - 6 + 18$
$= 15$

and $\boxed{3\ 6\ 9}$ $\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$ $= 3(-2) + 6 \times 0 + 9 \times (-1)$
$= -6 + 0 - 9$
$= -15$

| Example 12. If $A = \begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$ and I is a 2×2 unit matrix, prove that

$$I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Sol. Since, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and given $A = \begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$

$\therefore \quad I + A = \begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix}$ ...(i)

$\text{RHS} = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} 1 & \tan(\alpha/2) \\ -\tan(\alpha/2) & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$=\begin{bmatrix}1 & \tan(\alpha/2)\\ -\tan(\alpha/2) & 1\end{bmatrix}\begin{bmatrix}\dfrac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} & \dfrac{-2\tan(\alpha/2)}{1+\tan^2(\alpha/2)}\\ \dfrac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} & \dfrac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}\end{bmatrix}$$

Let $\tan(\alpha/2)=\lambda$, then

$$\text{RHS}=\begin{bmatrix}1 & \lambda\\ -\lambda & 1\end{bmatrix}\begin{bmatrix}\dfrac{1-\lambda^2}{1+\lambda^2} & \dfrac{-2\lambda}{1+\lambda^2}\\ \dfrac{2\lambda}{1+\lambda^2} & \dfrac{1-\lambda^2}{1+\lambda^2}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{1-\lambda^2+2\lambda^2}{1+\lambda^2} & \dfrac{-2\lambda+\lambda(1-\lambda^2)}{1+\lambda^2}\\ \dfrac{-\lambda(1-\lambda^2)+2\lambda}{1+\lambda^2} & \dfrac{2\lambda^2+1-\lambda^2}{1+\lambda^2}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{1+\lambda^2}{1+\lambda^2} & \dfrac{-\lambda(1+\lambda^2)}{1+\lambda^2}\\ \dfrac{\lambda(1+\lambda^2)}{1+\lambda^2} & \dfrac{1+\lambda^2}{1+\lambda^2}\end{bmatrix}=\begin{bmatrix}1 & -\lambda\\ \lambda & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & -\tan(\alpha/2)\\ \tan(\alpha/2) & 1\end{bmatrix}\quad[\because \lambda=\tan(\alpha/2)]$$

$= I + A$ [from Eq. (i)]

$=$ LHS

Pre-multiplication and Post-multiplication of Matrices

The matrix AB is the matrix B pre-multiplied by A and the matrix BA is the matrix B post-multiplied by A.

Properties of Multiplication of Matrices

Property 1 Multiplication of matrices is not commutative

i.e. $\qquad AB \neq BA$

Note

1. If $AB=-BA$, then A and B are said to anti-commute.
2. If $A=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ and $B=\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}$, then $AB=BA=\begin{bmatrix}3 & 0\\ 0 & 8\end{bmatrix}$

Observe that multiplication of diagonal matrices of same order will be commutative.

Property 2 Matrix multiplication associative if conformability assumed.

i.e. $\qquad A(BC)=(AB)C$

Property 3 Matrix multiplication is distributive with respect to addition. i.e. $A(B+C)=AB+AC$, whenever both sides of equality are defined.

Property 4 If A is an $m\times n$ matrix, then $I_m A = A = A I_n$.

Property 5 If product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

For example,

(i) $$\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}\times\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}1\cdot(-1)+1.1 & 1\cdot1+1\cdot(-1)\\ 2\cdot(-1)+2.1 & 2\cdot1+2\cdot(-1)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$$

(ii) $$\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\times\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0\cdot0+0\cdot0 & 0\cdot0+0\cdot1\\ 1\cdot0+0\cdot0 & 1\cdot0+0\cdot1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$$

None of the matrices on the LHS is a null matrix whereas their product is a null matrix.

Note If A and B are two non-zero matrices such that $AB=O$, then A and B are called the divisors of zero. Also, if

$AB=0 \Rightarrow |AB|=0 \Rightarrow |A||B|=0$

$\Rightarrow\quad |A|=0$ or $|B|=0$ but not the converse.

Property 6 Multiplication of a matrix A by a null matrix conformable with A for multiplication.

For example, If $A=\begin{bmatrix}3 & 4\\ 5 & 6\\ 7 & 8\end{bmatrix}_{3\times2}$ and $O=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}_{2\times3}$,

then $AO=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}_{3\times3}$, which is a 3×3 null matrix.

Property 7 Multiplication of a matrix by itself

The product of $A\,A\,A \ldots m$ times $=A^m$ and $(A^m)^n=A^{mn}$

Note

1. If I be unit matrix, then $I^2=I^3=\ldots=I^m=I\ (m\in I_+)$
2. If A and B are two matrices of the same order, then
   (i) $(A+B)^2=A^2+AB+BA+B^2$
   (ii) $(A-B)^2=A^2-AB-BA+B^2$
   (iii) $(A-B)(A+B)=A^2+AB-BA+B^2$
   (iv) $(A+B)(A-B)=A^2-AB+BA-B^2$
   (v) $A(-B)=(-A)(B)=-AB$

| Example 13. If $A=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}$ and $C=\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}$, verify that $(AB)C=A(BC)$ and $A(B+C)=AB+AC$.

Sol. We have, $AB=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\times\begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}$

$$=\begin{bmatrix}1\cdot2+2\cdot2 & 1\cdot1+2\cdot3\\ (-2)\cdot2+3\cdot2 & (-2)\cdot1+3\cdot3\end{bmatrix}=\begin{bmatrix}6 & 7\\ 2 & 7\end{bmatrix}$$

$$BC=\begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}\times\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}2\cdot(-3)+1\cdot2 & 2\cdot1+1\cdot0\\ 2\cdot(-3)+3\cdot2 & 2\cdot1+3\cdot0\end{bmatrix}$$

$$=\begin{bmatrix}-6+2 & 2+0\\ -6+6 & 2+0\end{bmatrix}=\begin{bmatrix}-4 & 2\\ 0 & 2\end{bmatrix}$$

$$AC=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\times\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}1\cdot(-3)+2\cdot2 & 1\cdot1+2\cdot0\\ (-2)\cdot(-3)+3\cdot2 & (-2)\cdot1+3\cdot0\end{bmatrix}$$

$$=\begin{bmatrix}-3+4 & 1+0\\ 6+6 & -2+0\end{bmatrix}=\begin{bmatrix}1 & 1\\ 12 & -2\end{bmatrix}$$

$$B+C=\begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}+\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}2-3 & 1+1\\ 2+2 & 3+0\end{bmatrix}=\begin{bmatrix}-1 & 2\\ 4 & 3\end{bmatrix}$$

Now, $(AB)C=\begin{bmatrix}6 & 7\\ 2 & 7\end{bmatrix}\times\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}-18+14 & 6+0\\ -6+14 & 2+0\end{bmatrix}$

$$=\begin{bmatrix}-4 & 6\\ 8 & 2\end{bmatrix}\quad\ldots\text{(i)}$$

$$A(BC)=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\times\begin{bmatrix}-4 & 2\\ 0 & 2\end{bmatrix}$$

$$=\begin{bmatrix}-4+0 & 2+4\\ 8+0 & -4+6\end{bmatrix}=\begin{bmatrix}-4 & 6\\ 8 & 2\end{bmatrix}\quad\ldots\text{(ii)}$$

Thus, from Eqs. (i) and (ii), we get, $(AB)C=A(BC)$

Now, $A(B+C)=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}\times\begin{bmatrix}-1 & 2\\ 4 & 3\end{bmatrix}=\begin{bmatrix}-1+8 & 2+6\\ 2+12 & -4+9\end{bmatrix}$

$$=\begin{bmatrix}7 & 8\\ 14 & 5\end{bmatrix}\quad\ldots\text{(iii)}$$

and $\quad AB+AC=\begin{bmatrix}6 & 7\\ 2 & 7\end{bmatrix}+\begin{bmatrix}1 & 1\\ 12 & -2\end{bmatrix}=\begin{bmatrix}6+1 & 7+1\\ 2+12 & 7-2\end{bmatrix}$

$$=\begin{bmatrix}7 & 8\\ 14 & 5\end{bmatrix}\quad\ldots\text{(iv)}$$

Thus, from Eqs. (iii) and (iv), we get

$A(B+C)=AB+AC$

| Example 14. If $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ p & q & r\end{bmatrix}$, show that $A^3=pI+qA+rA^2$.

Sol. We have, $A^2=A\cdot A$

$$=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ p & q & r\end{bmatrix}\times\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ p & q & r\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 1\\ p & q & r\\ pr & p+qr & q+r^2\end{bmatrix}$$

$$\therefore\quad A^3=A^2\cdot A=\begin{bmatrix}0 & 0 & 1\\ p & q & r\\ pr & p+qr & q+r^2\end{bmatrix}\times\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ p & q & r\end{bmatrix}$$

$$=\begin{bmatrix}p & q & r\\ pr & p+qr & q+r^2\\ pq+r^2p & pr+q^2+qr^2 & p+2qr+r^3\end{bmatrix}\quad\ldots\text{(i)}$$

and $pI+qA+rA^2=p\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+q\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ p & q & r\end{bmatrix}+r\begin{bmatrix}0 & 0 & 1\\ p & q & r\\ pr & p+qr & q+r^2\end{bmatrix}$

$$=\begin{bmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{bmatrix}+\begin{bmatrix}0 & q & 0\\ 0 & 0 & q\\ pq & q^2 & qr\end{bmatrix}+\begin{bmatrix}0 & 0 & r\\ pr & qr & r^2\\ pr^2 & pr+qr^2 & qr+r^3\end{bmatrix}$$

$$=\begin{bmatrix}p+0+0 & 0+q+0 & 0+0+r\\ 0+0+pr & p+0+qr & 0+q+r^2\\ 0+pq+pr^2 & 0+q^2+pr+qr^2 & p+qr+qr+r^3\end{bmatrix}$$

$$=\begin{bmatrix}p & q & r\\ pr & p+qr & q+r^2\\ pq+pr^2 & q^2+pr+qr^2 & p+2qr+r^3\end{bmatrix}\quad\ldots\text{(ii)}$$

Thus, from Eqs. (i) and (ii), we get $A^3=pI+qA+rA^2$

| Example 15. Find x, so that $[1\ \ x\ \ 1]\begin{bmatrix}1 & 3 & 2\\ 0 & 5 & 1\\ 0 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\\ x\end{bmatrix}=O$.

Sol. We have, $[1\ \ x\ \ 1]\begin{bmatrix}1 & 3 & 2\\ 0 & 5 & 1\\ 0 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\\ x\end{bmatrix}=O$

$\Rightarrow\quad [1\ \ 5x+6\ \ x+4]\begin{bmatrix}1\\ 1\\ x\end{bmatrix}=O$

$\Rightarrow\quad [1+5x+6+x^2+4x]=O$

or $\quad x^2+9x+7=0$

$\therefore\quad x=\dfrac{-9\pm\sqrt{(81-28)}}{2}\Rightarrow x=\dfrac{-9\pm\sqrt{53}}{2}$

Various Kinds of Matrices

Idempotent Matrix

A square matrix A is called idempotent provided it satisfies the relation $\boxed{A^2 = A.}$

Note

$\boxed{A^n = A \ \forall\ n \geq 2,\ n \in N}$

Example 16. Show that the matrix

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \text{ is idempotent.}$$

Sol. $A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot 2+(-2)\cdot(-1)+(-4)\cdot 1 & 2\cdot(-2)+(-2)\cdot 3+(-4)\cdot(-2) & 2\cdot(-4)+(-2)\cdot 4+(-4)\cdot(-3) \\ (-1)\cdot 2+3\cdot(-1)+4\cdot 1 & (-1)\cdot(-2)+3\cdot 3+4\cdot(-2) & (-1)\cdot(-4)+3\cdot 4+4\cdot(-3) \\ 1\cdot 2+(-2)\cdot(-1)+(-3)\cdot 1 & 1\cdot(-2)+(-2)\cdot 3+(-3)\cdot(-2) & 1\cdot(-4)+(-2)\cdot 4+(-3)\cdot(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, the matrix A is idempotent.

Periodic Matrix Must

A square matrix A is called periodic, if $\boxed{A^{k+1} = A,}$ where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then k is said to be **period** of A. For $k = 1$, we get $\boxed{A^2 = A}$ and we called it to be **idempotent matrix.**

Note

Period of an idempotent matrix is 1.

Nilpotent Matrix

A square matrix A is called nilpotent matrix of order m provided it satisfies the relation $A^k = O$ and $A^{k-1} \neq O$, where k is positive integer and O is null matrix and k is the order of the nilpotent matrix A.

Example 17. Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

Sol. Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$\therefore A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore \quad A^3 = O$ i.e., $A^k = O$

Here, $k = 3$

Hence, the matrix A is nilpotent of order 3.

Involutory Matrix

A square matrix A is called involutory provided it satisfies the relation $A^2 = I$, where I is identity matrix.

Note $A = A^{-1}$ for an involutory matrix.

Example 18. Show that the matrix

$$A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \text{ is involutory.}$$

Sol. $A^2 = A \cdot A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, the given matrix A is involutory.

b, b, d, b, a, b, b, b, c

Exercise for Session 1

1. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, α is equal to
   (a) ± 2 (b) ± 3
   (c) ± 5 (d) 0

2. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$, the value of $a + b$ is
   (a) 4 (b) 5
   (c) 6 (d) 7

3. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^2 - \lambda A - I_2 = O$, then λ is equal to
   (a) -4 (b) -2
   (c) 2 (d) 4

4. Let $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A + I)^{50} - 50A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the value of $a + b + c + d$, is
   (a) 1 (b) 2
   (c) 4 (d) None of these

5. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^2 = I$ is true for
   (a) $\theta = 0$ (b) $\theta = \frac{\pi}{4}$
   (c) $\theta = \frac{\pi}{2}$ (d) None of these

6. If $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is to be the square root of two rowed unit matrix, then α, β and γ should satisfy the relation
   (a) $1 - \alpha^2 + \beta\gamma = 0$ (b) $\alpha^2 + \beta\gamma - 1 = 0$
   (c) $1 + \alpha^2 + \beta\gamma = 0$ (d) $1 - \alpha^2 - \beta\gamma = 0$

7. If $A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$, then A^{100} is equal to
   (a) $\begin{bmatrix} 1 & 0 \\ 25 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 0 \\ (1/2)^{100} & 1 \end{bmatrix}$ (d) None of these

8. If the product of n matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is equal to the matrix $\begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$, the value of n is equal to
   (a) 26 (b) 27
   (c) 377 (d) 378

9. If A and B are two matrices such that $AB = B$ and $BA = A$, then $A^2 + B^2$ is equal to
   (a) $2AB$ (b) $2BA$
   (c) $A + B$ (d) AB

Session 2

Transpose of a Matrix, Symmetric Matrix, Orthogonal Matrix, Complex Conjugate (or Conjugate) of a Matrix, Hermitian Matrix, Unitary Matrix, Determinant of a Matrix, Singular and Non-Singular Matrices,

Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$ be any given matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A' or A^T or A^t. In other words, if $A = [a_{ij}]_{m \times n}$, then $A' = [a_{ji}]_{n \times m}$.

For example,

If
$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ -2 & -1 & 4 & 8 \\ 7 & 5 & 3 & 1 \end{bmatrix}_{3 \times 4},$$
then
$$A' = \begin{bmatrix} 2 & -2 & 7 \\ 3 & -1 & 5 \\ 4 & 4 & 3 \\ 5 & 8 & 1 \end{bmatrix}_{4 \times 3}$$

Properties of Transpose Matrices

If A' and B' denote the transpose of A and B respectively, then

(i) $(A')' = A$

(ii) $(A \pm B)' = A' \pm B'$; A and B are conformable for matrix addition.

(iii) $(kA)' = kA'$; k is a scalar.

(iv) $(AB)' = B'A'$; A and B are conformable for matrix product AB.

In general, $(A_1 A_2 A_3 \ldots A_{n-1} A_n)' = A'_n A'_{n-1} \ldots A'_3 A'_2 A'_1$ (reversal law for transpose).

Remark

$I' = I$, where I is an identity matrix.

| Example 19. If $A = \begin{bmatrix} \theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, find the values of θ satisfying the equation $A^T + A = I_2$.

Sol. We have, $A^T + A = I_2$

$$\Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \cos\theta = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in$$

Symmetric Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be symmetric, if $A' = A$ i.e., $a_{ij} = a_{ji}$, $\forall\, i, j$.

For example,

If $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, then $A' = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Here, A is symmetric matrix as $A' = A$.

Note

1. Maximum number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.
2. For any square matrix A with real number entries, then $A + A'$ is a symmetric matrix.

Proof $(A + A')' = A' + (A')' = A' + A = A + A'$

Skew-Symmetric Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be skew-symmetric matrix, if $A' = -A$, i.e. $a_{ij} = -a_{ji}, \forall\, i, j$. (the pair of conjugate elements are additive inverse of each other)

Now, if we put $i = j$, we have $a_{ii} = -a_{ii}$.

Therefore, $2a_{ii} = 0$ or $a_{ii} = 0$, $\forall\, i$'s.

This means that all the diagonal elements of a skew-symmetric matrix are zero, but not the converse.

For example,

If $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$, then

$$A' = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix} = -\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix} = -A$$

Here, A is skew-symmetric matrix as $\boxed{A' = -A}$.

Note

1. Trace of a skew-symmetric matrix is always 0.
2. For any square matrix A with real number entries, then $A - A'$ is a skew-symmetric matrix.

Proof $(A - A')' = A' - (A')' = A' - A = -(A - A')$

3. Every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

i.e. If A is a square matrix, then we can write
$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

| Example 20. The square matrix $A = [a_{ij}]_{m \times m}$ given by $a_{ij} = (i - j)^n$, show that A is symmetric and skew-symmetric matrices according as n is even or odd, respectively.

Sol. $\because a_{ij} = (i - j)^n = (-1)^n (j - i)^n$

$$= (-1)^n a_{ji} = \begin{cases} a_{ji}, & n \text{ is even integer} \\ -a_{ji}, & n \text{ is odd integer} \end{cases}$$

Hence, A is symmetric if n is even and skew-symmetric if n is odd integer.

| Example 21. Express A as the sum of a symmetric and a skew-symmetric matrix, where $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$

Sol. We have,

$$A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$$

Let $P = \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = P'$

Thus, $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

Also, let $Q = \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

Then, $Q' = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} = -Q$

Thus, $Q = \frac{1}{2}(A - A')$ is a skew-symmetric matrix.

Now, $P + Q = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} = A$

Hence, A is represented as the sum of a symmetric and a skew-symmetric matrix.

Properties of Symmetric and Skew-Symmetric Matrices

(i) If A be a square matrix, then AA' and $A'A$ are symmetric matrices.

(ii) All positive integral powers of a symmetric matrix are symmetric, because
$$(A^n)' = (A')^n$$

(iii) All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric, because $(A^n)' = (A')^n$

(iv) If A be a symmetric matrix and B be a square matrix of order that of A, then $-A$, kA, A', A^{-1}, A^n and $B'AB$ are also symmetric matrices, where $n \in N$ and k is a scalar.

(v) If A be a skew-symmetric matrix, then

(a) A^{2n} is a symmetric matrix for $n \in N$.

(b) A^{2n+1} is a skew-symmetric matrix for $n \in N$.

(c) kA is a skew-symmetric matrix, where k is scalar.

(d) $B'AB$ is also skew-symmetric matrix, where B is a square matrix of order that of A.

(vi) If A and B are two symmetric matrices, then

(a) $A \pm B$, $AB + BA$ are symmetric matrices.

(b) $AB - BA$ is a skew-symmetric matrix.

(c) AB is a symmetric matrix, iff $AB = BA$ (where A and B are square matrices of same order)

(vii) If A and B are two skew-symmetric matrices, then

(a) $A \pm B$, $AB - BA$ are skew-symmetric matrices.

(b) $AB + BA$ is a symmetric matrix.

(where A and B are square matrices of same order)

(viii) If A be a skew-symmetric matrix and C is a column matrix, then $C'AC$ is a zero matrix, where $C'AC$ is conformable.

Orthogonal Matrix

A square matrix A is said to be orthogonal matrix, iff $\boxed{AA' = I}$ where I is an identity matrix.

Note

1. If $AA' = I$, then $\boxed{A^{-1} = A'}$
2. If A and B are orthogonal, then AB is also orthogonal.
3. If A is orthogonal, then A^{-1} and A' are also orthogonal.

I Example 22. If $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal, then find the value of $2\alpha^2 + 6\beta^2 + 3\gamma^2$.

Sol. Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$, then $A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

Since, A is orthogonal.

$\therefore \quad AA' = I$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we get

$$4\beta^2 + \gamma^2 = 1 \quad \text{...(i)}$$
$$2\beta^2 - \gamma^2 = 0 \quad \text{...(ii)}$$
and $$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{...(iii)}$$

From Eqs. (i) and (ii), we get

$$\beta^2 = \frac{1}{6} \text{ and } \gamma^2 = \frac{1}{3}$$

From Eq. (iii),

$$\alpha^2 = 1 - \beta^2 - \gamma^2 = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$$

Hence, $2\alpha^2 + 6\beta^2 + 3\gamma^2 = 2 \times \frac{1}{2} + 6 \times \frac{1}{6} + 3 \times \frac{1}{3} = 3$

Aliter

The rows of matrix A are unit orthogonal vectors

$$\vec{R_1} \cdot \vec{R_2} = 0 \Rightarrow 2\beta^2 - \gamma^2 = 0 \Rightarrow 2\beta^2 = \gamma^2 \quad \text{...(i)}$$
$$\vec{R_2} \cdot \vec{R_3} = 0 \Rightarrow \alpha^2 - \beta^2 - \gamma^2 = 0 \Rightarrow \beta^2 + \gamma^2 = \alpha^2 \quad \text{...(ii)}$$
and $$\vec{R_3} \cdot \vec{R_3} = 1 \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$\alpha^2 = \frac{1}{2}, \beta^2 = \frac{1}{6} \text{ and } \gamma^2 = \frac{1}{3}$$

$\therefore \quad 2\alpha^2 + 6\beta^2 + 3\gamma^2 = 3$

I Example 23. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying $AA' = 9I_3$, find the value of $|a| + |b|$.

Sol. Since, $AA' = 9I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a-2b+2 \\ a+2b+4 & 2a-2b+2 & a^2+b^2+4 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Equating the corresponding elements, we get

$$a + 2b + 4 = 0 \quad \text{...(i)}$$
$$2a - 2b + 2 = 0 \quad \text{...(ii)}$$
and $$a^2 + b^2 + 4 = 9 \quad \text{...(iii)}$$

From Eqs. (i) and (ii), we get

$$a = -2 \text{ and } b = -1$$

Hence, $|a| + |b| = |-2| + |-1| = 2 + 1 = 3$

Complex Conjugate (or Conjugate) of a Matrix

If a matrix A is having complex numbers as its elements, the matrix obtained from A by replacing each element of A by its conjugate $(\overline{a \pm ib} = a \mp ib$, where $i = \sqrt{-1}$) is called the conjugate of matrix A and is denoted by $\overline{A}$.

For example, If $A = \begin{bmatrix} 2+5i & 3-i & 7 \\ -2i & 6+i & 7-5i \\ 1-i & 3 & 6i \end{bmatrix}$, where $i = \sqrt{-1}$,

then $\overline{A} = \begin{bmatrix} 2-5i & 3+i & 7 \\ 2i & 6-i & 7+5i \\ 1+i & 3 & -6i \end{bmatrix}$

Note

If all elements of A are real, then $\boxed{\overline{A} = A}$.

Properties of Complex Conjugate of a Matrix

If A and B are two matrices of same order, then

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$, where A and B being conformable for addition.

(iii) $\overline{(kA)} = \overline{k}\,\overline{A}$, where k is real.

(iv) $\overline{(AB)} = \overline{A}\,\overline{B}$, where A and B being conformable for multiplication.

Conjugate Transpose of a Matrix

The conjugate of the transpose of a matrix A is called the conjugate transpose of A and is denoted by A^θ i.e.

A^θ = Conjugate of $A' = (\overline{A'})$

For example,

If $A = \begin{bmatrix} 2+4i & 3 & 5-9i \\ 4 & 5+2i & 3i \\ -2 & -5 & 4-i \end{bmatrix}$,

where $i = \sqrt{-1}$,

then $A^\theta = (\overline{A'}) = \begin{bmatrix} 2-4i & 4 & 2 \\ 3 & 5-2i & -5 \\ 5+9i & -3i & 4+i \end{bmatrix}$

Properties of Transpose Conjugate Matrix

If A and B are two matrices of same order, then

(i) $(\overline{A})' = (\overline{A'})$ (ii) $(A^\theta)^\theta = A$

(iii) $(A+B)^\theta = A^\theta + B^\theta$, where A and B being conformable for addition.

(iv) $(kA)^\theta = \overline{k}\,A^\theta$, where k is real.

(v) $(AB)^\theta = B^\theta A^\theta$, where A and B being conformable for multiplication

Hermitian Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be hermitian, if $\boxed{A^\theta = A}$ i.e., $a_{ij} = \overline{a}_{ji}$, $\forall\, i, j$. If we put $j = i$, we have $a_{ii} = \overline{a}_{ii}$

$\Rightarrow a_{ii}$ is purely real for all i's.

This means that all the diagonal elements of a hermitian matrix must be purely real.

For example,

If $A = \begin{bmatrix} \alpha & \lambda + i\mu & \theta + i\phi \\ \lambda - i\mu & \beta & x + iy \\ \theta - i\phi & x - iy & \gamma \end{bmatrix}$

where $\alpha, \beta, \gamma, \lambda, \mu, \theta, \phi, x, y \in R$ and $i = \sqrt{-1}$, then

$$A' = \begin{bmatrix} \alpha & \lambda - i\mu & \theta - i\phi \\ \lambda + i\mu & \beta & x - iy \\ \theta + i\phi & x + iy & \gamma \end{bmatrix}$$

$$\therefore \quad A^\theta = (\overline{A'}) = \begin{bmatrix} \alpha & \lambda + i\mu & \theta + i\phi \\ \lambda - i\mu & \beta & x + iy \\ \theta - i\phi & x - iy & \gamma \end{bmatrix} = A$$

Here, A is hermitian matrix as $A^\theta = A$.

Note

For any square matrix A with complex number entries, then $A + A^\theta$ is a Hermitian matrix.

Proof $(A + A^\theta)^\theta = A^\theta + (A^\theta)^\theta = A^\theta + A = A + A^\theta$

Skew-Hermitian Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be skew-hermitian matrix. If $A^\theta = -A$, i.e. $a_{ij} = -\overline{a}_{ji}$, $\forall\, i, j$. If we put $j = i$, we have $a_{ii} = -\overline{a}_{ii} \Rightarrow a_{ii} + \overline{a}_{ii} = 0 \Rightarrow a_{ii}$ is purely imaginary for all i's. This means that all the diagonal elements of a skew-hermitian matrix must be purely imaginary or zero.

For example,

If $A = \begin{bmatrix} 2i & -2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$,

then $A' = \begin{bmatrix} 2i & 2-3i & 2+i \\ -2-3i & -i & 3i \\ -2+i & 3i & 0 \end{bmatrix}$

$$\therefore \quad A^\theta = (\overline{A'}) = \begin{bmatrix} -2i & 2+3i & 2-i \\ -2+3i & i & -3i \\ -2-i & -3i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 2i & -2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix} = -A$$

Hence, A is skew-hermitian matrix.

Note

1. For any square matrix A with complex number entries, then $A - A^\theta$ is a skew-hermitian matrix.
   Proof $(A - A^\theta)^\theta = (A^\theta) - (A^\theta)^\theta = A^\theta - A = -(A - A^\theta)$
2. Every square matrix (with complex elements) can be uniquely expressed as the sum of a hermitian and a skew-hermitian matrix i.e.
   If A is a square matrix, then we can write
   $A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$

I Example 24. Express A as the sum of a hermitian and a skew-hermitian matrix, where

$A = \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix}$, $i = \sqrt{-1}$.

Sol. We have, $A = \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix}$, then $A^\theta = (\overline{A'}) = \begin{bmatrix} 2-3i & 1+i \\ 7 & -2i \end{bmatrix}$

Let $P = \frac{1}{2}(A + A^\theta) = \frac{1}{2}\begin{bmatrix} 4 & 8+i \\ 8-i & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4+\frac{i}{2} \\ 4-\frac{i}{2} & 0 \end{bmatrix} = P^\theta$

Thus, $P = \frac{1}{2}(A + A^\theta)$ is a hermitian matrix.

Also, let $Q = \frac{1}{2}(A - A^\theta) = \frac{1}{2}\begin{bmatrix} 6i & 6-i \\ -6-i & 4i \end{bmatrix}$

$$= \begin{bmatrix} 3i & 3-\frac{i}{2} \\ -3-\frac{i}{2} & 2i \end{bmatrix} = -\begin{bmatrix} -3i & -3+\frac{i}{2} \\ 3+\frac{i}{2} & -2i \end{bmatrix} = -Q^\theta$$

Thus, $Q = \frac{1}{2}(A - A^\theta)$ is a skew-hermitian matrix.

Now, $P + Q = \begin{bmatrix} 2 & 4+\frac{i}{2} \\ 4-\frac{i}{2} & 0 \end{bmatrix} + \begin{bmatrix} 3i & 3-\frac{i}{2} \\ -3-\frac{i}{2} & 2i \end{bmatrix}$

$$= \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix} = A$$

Hence, A is represented as the sum of a hermitian and a skew-hermitian matrix.

Properties of Hermitian and Skew-Hermitian Matrices

(i) If A be a square matrix, then AA^θ and $A^\theta A$ are hermitian matrices.

(ii) If A is a hermitian matrix, then
  (a) iA is skew-hermitian matrix, where $i = \sqrt{-1}$.
  (b) iff $\overline{A}$ is hermitian matrix.
  (c) kA is hermitian matrix, where $k \in R$.

(iii) If A is a skew-hermitian matrix, then
  (a) iA is hermitian matrix, where $i = \sqrt{-1}$.
  (b) iff $\overline{A}$ is skew-hermitian matrix.
  (c) kA is skew-hermitian matrix, where $k \in R$.

(iv) If A and B are hermitian matrices of same order, then
  (a) $k_1 A + k_2 B$ is also hermitian, where $k_1, k_2 \in R$.
  (b) AB is also hermitian, if $AB = BA$.
  (c) $AB + BA$ is a hermitian matrix.
  (d) $AB - BA$ is a skew-hermitian matrix.

(v) If A and B are skew-hermitian matrices of same order, then $k_1 A + k_2 B$ is also skew-hermitian matrix.

Unitary Matrix

A square matrix A is said to be unitary matrix iff $\boxed{AA^\theta = I}$, where I is an identity matrix.

Note

1. If $AA^\theta = I$, then $A^{-1} = A^\theta$
2. If A and B are unitary, then AB is also unitary.
3. If A is unitary, then A^{-1} and A' are also unitary.

I Example 25. Verify that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary, where $i = \sqrt{-1}$.

Sol. Let $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$, then $A^\theta = (\overline{A}') = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$\therefore \quad AA^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is unitary matrix.

Determinant of a Matrix

Let A be a square matrix, then the determinant formed by the elements of A without changing their respective positions is called the determinant of A and is denoted by det A or $|A|$.

i.e., If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, then $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Properties of the Determinant of a Matrix

If A and B are square matrices of same order, then

(i) $|A|$ exists $\Leftrightarrow A$ is a square matrix.

(ii) $|A'| = |A|$

(iii) $|AB| = |A||B|$ and $|AB| = |BA|$

(iv) If A is orthogonal matrix, then $\boxed{|A| = \pm 1}$

(v) If A is skew-symmetric matrix of odd order, then $|A| = 0$

(vi) If A is skew-symmetric matrix of even order, then $|A|$ is a perfect square.

(vii) $|kA| = k^n |A|$, where n is order of A and k is scalar.

(viii) $|A^n| = |A|^n$, where $n \in N$

(ix) If $A = \text{diag}(a_1, a_2, a_3, \ldots, a_n)$, then $|A| = a_1 \cdot a_2 \cdot a_3 \ldots a_n$

I Example 26. If A, B and C are square matrices of order n and $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 5$, then find the value of $10\det(A^3 B^2 C^{-1})$.

Sol. Given, $|A| = 2$, $|B| = 3$ and $|C| = 5$.

Now, $10\det(A^3 B^2 C^{-1}) = 10 \times |A^3 B^2 C^{-1}|$

$$= 10 \times |A^3| \times |B^2| \times |C^{-1}| = 10 \times |A|^3 \times |B|^2 \times |C|^{-1}$$

$$= \frac{10 \times |A|^3 \times |B^2|}{|C|} = \frac{10 \times 2^3 \times 3^2}{5} = 144$$

I Example 27. If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, $abc = 1$, $A^T A = I$, then find the value of $a^3 + b^3 + c^3$.

Sol. $\because \quad A^T A = I$

$\Rightarrow \quad |A^T A| = |I| \Rightarrow |A^T||A| = 1$

$\Rightarrow \quad |A||A| = 1 \qquad [\because |A^T| = |A|]$

$\Rightarrow \quad |A| = \pm 1$

$\Rightarrow \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \pm 1$

$\Rightarrow \quad 3abc - (a^3 + b^3 + c^3) = \pm 1$

or $\quad 3 - (a^3 + b^3 + c^3) = \pm 1 \qquad [\because abc = 1]$

or $\quad a^3 + b^3 + c^3 = 3 \pm 1 = 2$ or 4

Singular and Non-Singular Matrices

A square matrix A is said to be a singular, if $|A| = 0$ and a square matrix A is said to be non-singular, if $|A| \neq 0$.

For example,

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 4 & 6 \end{bmatrix}$ is a singular matrix.

Since, $|A| = 0$.

(ii) $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ is a non-singular matrix.

Since, $|A| = 10 - 12 = -2 \neq 0$

I Example 28. If $\omega \neq 1$ is a complex cube root of unity, then prove that

$$\begin{bmatrix} 1+2\omega^{2017}+\omega^{2018} & \omega^{2018} & 1 \\ 1 & 1+\omega^{2018}+2\omega^{2017} & \omega^{2017} \\ \omega^{2017} & \omega^{2018} & 2+\omega^{2017}+2\omega^{2018} \end{bmatrix}$$ is singular.

Sol. Let $A = \begin{bmatrix} 1+2\omega^{2017}+\omega^{2018} & \omega^{2018} & 1 \\ 1 & 1+\omega^{2018}+2\omega^{2017} & \omega^{2017} \\ \omega^{2017} & \omega^{2018} & 2+\omega^{2017}+2\omega^{2018} \end{bmatrix}$

$\because \quad \omega^3 = 1 \Rightarrow \omega^{2017} = \omega$

and $\quad \omega^{2018} = \omega^2$, then

$$A = \begin{bmatrix} 1+2\omega+\omega^2 & \omega^2 & 1 \\ 1 & 1+\omega^2+2\omega & \omega \\ \omega & \omega^2 & 2+\omega+2\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} \omega & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & \omega^2 & -\omega \end{bmatrix} \qquad [\because 1+\omega+\omega^2 = 0]$$

Now, $|A| = \begin{vmatrix} \omega & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & \omega^2 & -\omega \end{vmatrix} = \omega \begin{vmatrix} \omega & \omega & 1 \\ 1 & 1 & \omega \\ \omega & \omega & -\omega \end{vmatrix} = 0 \qquad [\because C_1 = C_2]$

Thus, $|A| = 0$.

Hence, A is singular matrix.

I Example 29. Find the real values of x for which the matrix $\begin{bmatrix} x+1 & 3 & 5 \\ 1 & x+3 & 5 \\ 1 & 3 & x+5 \end{bmatrix}$ is non-singular.

Sol. Let $A = \begin{bmatrix} x+1 & 3 & 5 \\ 1 & x+3 & 5 \\ 1 & 3 & x+5 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} x+1 & 3 & 5 \\ 1 & x+3 & 5 \\ 1 & 3 & x+5 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$|A| = \begin{vmatrix} x+9 & 3 & 5 \\ x+9 & x+3 & 5 \\ x+9 & 3 & x+5 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$|A| = \begin{vmatrix} x+9 & 3 & 5 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = x^2(x+9)$$

$\because \quad A$ is non-singular.

$\therefore \quad |A| \neq 0 \Rightarrow x^2(x+9) \neq 0$

$\therefore \quad x \neq 0, -9$

Hence, $x \in R - \{0, -9\}$.

d, a, c, a, b, c, d, b, d, c, b, d, b, a, b, b, c, b, b

Exercise for Session 2

1 If $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is symmetric, then x is equal to

(a) 2 (b) 3
(c) 4 (d) 5

2 If A and B are symmetric matrices, then ABA is

(a) symmetric matrix (b) skew-symmetric matrix
(c) diagonal matrix (d) scalar matrix

3 If A and B are symmetric matrices of the same order and $P = AB + BA$ and $Q = AB - BA$, then $(PQ)'$ is equal to

(a) PQ (b) QP
(c) $-QP$ (d) None of these

4 If A is a skew-symmetric matrix and n is odd positive integer, then A^n is

(a) a skew-symmetric matrix (b) a symmetric matrix
(c) a diagonal matrix (d) None of these

5 If A is symmetric as well as skew-symmetric matrix, then A is

(a) diagonal matrix (b) null matrix
(c) triangular matrix (d) None of these

6 If A is square matrix order 3, then $|(A - A')^{2015}|$ is

(a) $|A|$ (b) $|A'|$
(c) 0 (d) None of these

7 The maximum number of different elements required to form a symmetric matrix of order 6 is

(a) 15 (b) 17
(c) 19 (d) 21

8 If A and B are square matrices of order 3×3 such that A is an orthogonal matrix and B is a skew-symmetric matrix, then which of the following statement is true?

(a) $|AB| = 1$ (b) $|AB| = 0$
(c) $|AB| = -1$ (d) None of these

9 The matrix $A = \begin{bmatrix} i & 1-2i \\ -1-2i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$, is

(a) symmetric (b) skew-symmetric
(c) hermitian (d) skew-hermitian

10 If A and B are square matrices of same order such that $A^* = A$ and $B^* = B$, where A^* denotes the conjugate transpose of A, then $(AB - BA)^*$ is equal to

(a) null matrix (b) $AB - BA$
(c) $BA - AB$ (d) None of these

11 If matrix $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & a \end{bmatrix}, i = \sqrt{-1}$ is unitary matrix, a is equal to

(a) 2 (b) -1
(c) 0 (d) 1

12 If A is a 3×3 matrix and $\det(3A) = k\{\det(A)\}$, k is equal to

(a) 9 (b) 6
(c) 1 (d) 27

13 If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then $|3AB|$ is equal to

(a) -9 (b) -81
(c) -27 (d) 81

14 If A is a square matrix such that $A^2 = A$, then $\det(A)$ is equal to

(a) 0 or 1 (b) -2 or 2
(c) -3 or 3 (d) None of these

15 If I is a unit matrix of order 10, the determinant of I is equal to

(a) 10 (b) 1
(c) $\frac{1}{10}$ (d) 9

16 If $A_i = \begin{bmatrix} 2^{-i} & 3^{-i} \\ 3^{-i} & 2^{-i} \end{bmatrix}$, then $\sum_{i=1}^{\infty} \det(A_i)$ is equal to

(a) $\frac{3}{4}$ (b) $\frac{5}{24}$
(c) $\frac{5}{4}$ (d) $\frac{7}{144}$

17 The number of values of x for which the matrix $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular, is

(a) 0 (b) 1
(c) 2 (d) 3

18 The number of values of x in the closed interval $[-4, -1]$, the matrix $\begin{bmatrix} 3 & -1+x & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{bmatrix}$ is singular, is

(a) 0 (b) 1
(c) 2 (d) 3

19 The values of x for which the given matrix $\begin{bmatrix} -x & x & 2 \\ 2 & x & -x \\ x & -2 & -x \end{bmatrix}$ will be non-singular are

(a) $-2 \le x \le 2$ (b) for all x other than 2 and -2
(c) $x \ge 2$ (d) $x \le -2$

Session 3

Adjoint of a Matrix, Inverse of a Matrix (Reciprocal Matrix), Elementary Row Operations (Transformations), Equivalent Matrices, Matrix Polynomial, Use of Mathematical Induction,

Adjoint of a Matrix

Let $A=[a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then, the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj (A).

Thus, $\text{adj}(A)=[C_{ij}]'$

$\Rightarrow \quad (\text{adj } A)_{ij}=C_{ji}=$ Cofactor of a_{ji} in A

i.e. if $\quad A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$, then

$$\text{adj } A=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33}\end{bmatrix}'=\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33}\end{bmatrix}$$

where C_{ij} denotes the cofactor of a_{ij} in A.

Here, $\quad C_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix}=a_{22}a_{33}-a_{23}\,a_{32}$,

$C_{12}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix}=a_{31}a_{23}-a_{33}a_{21}$,

$C_{13}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}=a_{21}a_{32}-a_{31}a_{22}$,

$C_{21}=-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33}\end{vmatrix}=a_{13}a_{32}-a_{12}a_{33}$,

$C_{22}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{vmatrix}=a_{11}a_{33}-a_{31}a_{13}$,

$C_{23}=-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32}\end{vmatrix}=a_{12}a_{31}-a_{11}a_{32}$,

$C_{31}=\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{vmatrix}=a_{12}a_{23}-a_{22}a_{13}$,

$C_{32}=-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23}\end{vmatrix}=a_{13}a_{21}-a_{11}a_{23}$

and $\quad C_{33}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$

Rule to Write Cofactors of an Element a_{ij}

Cross the row and column intersection at the element a_{ij} and the determinant which is left be denoted by D, then

$$\text{Cofactors of } a_{ij}=\begin{cases} D, & \text{if } i+j=\text{even integer} \\ -D, & \text{if } i+j=\text{odd integer}\end{cases}$$

I Example 30. Find the cofactor of a_{23} in $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & -1 \\ 1 & -3 & 5\end{bmatrix}$

Sol. Let $\quad A=\begin{bmatrix} 3 & 1 & 4 \\ 0 & \cdots 2 & \cdots -1 \\ 1 & -3 & 5\end{bmatrix}$

$\therefore$ Cofactor of $a_{23}=-D \qquad [\because 2+3=\text{odd}]$

where $D=\begin{vmatrix} 3 & 1 \\ 1 & -3\end{vmatrix}$

[after crossing the 2nd row and 3rd column]

$=-9-1=-10$

Hence, cofactor of $a_{23}=-(-10)=10$

Note

The adjoint of a square matrix of order 2 is obtained by interchanging the diagonal elements and changing signs of off-diagonal elements.

If $\quad A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$, then

$$(\text{adj } A)=\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$$

I Example 31. Find the adjoint of the matrix

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3\end{bmatrix}.$$

Sol. If C be the matrix of cofactors of the element in $|A|$, then

$$C=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33}\end{bmatrix}$$

$$=\begin{bmatrix} \begin{vmatrix} 5 & 0 \\ 4 & 3\end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 2 & 3\end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 2 & 4\end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 4 & 3\end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 3\end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 4\end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 0\end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 0\end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 5\end{vmatrix}\end{bmatrix}=\begin{bmatrix} 12 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5\end{bmatrix}$$

$$\Rightarrow \text{adj}A=C'=\begin{bmatrix} 12 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5\end{bmatrix}$$

Maha Shortcut for Adjoint (Goyal's Method)

This method applied only for third order square matrix.

Method : Let $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3\end{bmatrix}$

Step I Write down the three rows of A and rewrite first two rows.

i.e.

$$\begin{matrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \\ 1 & 2 & 3 \\ 0 & 5 & 0\end{matrix}$$

Step II After Step I, rewrite first two columns.

i.e.

$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 5 & 0 & 0 & 5 \\ 2 & 4 & 3 & 2 & 4 \\ 1 & 2 & 3 & 1 & 2 \\ 0 & 5 & 0 & 0 & 5\end{matrix}$$

Step III After Step II, deleting first row and first column, then we get all elements of adj A i.e.,

$$\begin{array}{ccccccccc} 1 & \cdots & 2 & \cdots & 3 & \cdots & 1 & \cdots & 2 \\ \vdots \\ 0 & & 5 & & 0 & & 0 & & 5 \\ \vdots & & & \times & & \times & & \times & \Rightarrow 15 \;\; 0 \;\; -10 \text{ first column of adj } A \\ 2 & & 4 & & 3 & & 2 & & 4 \\ \vdots & & & \times & & \times & & \times & \Rightarrow 6 \;\; -3 \;\; 0 \text{ second column of adj } A \\ 1 & & 2 & & 3 & & 1 & & 2 \\ \vdots & & & \times & & \times & & \times & \Rightarrow -15 \;\; 0 \;\; 5 \text{ third column of adj } A \\ 0 & & 5 & & 0 & & 0 & & 5\end{array}$$

or $\quad \text{adj } A=\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5\end{bmatrix}$

Properties of Adjoint Matrix

Property 1 If A be a square matrix order n, then

$$A(\text{adj } A)=(\text{adj } A)A=|A|I_n$$

i.e., the product of a matrix and its adjoint is commutative.

Deductions of Property 1

Deduction 1 If A be a square singular matrix of order n, then $\quad A(\text{adj } A)=(\text{adj } A)\,A=O \qquad$ [null matrix]

Since, for singular matrix, $|A|=0$.

Deduction 2 If A be a square non-singular matrix of order n, then $\quad |\text{adj } A|=|A|^{n-1}$

Since, for non-singular matrix, $|A|\neq 0$.

Proof $\because \quad A(\text{adj } A)=|A|I_n$

Taking determinant on both sides, then

$$|A(\text{adj } A)|=||A|\,I_n|$$

$\Rightarrow \quad |A||\text{adj } A|=|A|^n|I_n|=|A|^n \qquad [\because |I_n|=1]$

$\Rightarrow \quad |\text{adj } A|=|A|^{n-1} \qquad [\because |A|\neq 0]$

Note

In general $\underbrace{|\text{adj}\,(\text{adj}\,(\text{adj}\ldots(\text{adj } A)))|}_{\text{adj repeat } m \text{ times}}=|A|^{(n-1)^m}$

Property 2 If A and B are square matrices of order n, then

$$\text{adj}\,(AB)=(\text{adj } B)(\text{adj } A)$$

Property 3 If A is a square matrix of order n, then

$$(\text{adj } A)'=\text{adj } A'$$

Property 4 If A be a square non-singular matrix of order n, then $\quad \boxed{\text{adj}\,(\text{adj} A)=|A|^{n-2}\,A}$

Proof $\because \quad A(\text{adj } A)=|A|I_n \qquad \ldots(i)$

Replace A by adj A, then

$(\text{adj } A)(\text{adj}\,(\text{adj } A))=|\text{adj } A|\,I_n$

$=|A|^{n-1}\,I_n \qquad [\because |\text{adj } A|=|A|^{n-1}]$

$=I_n|A|^{n-1}$

Pre-multiplying both sides by matrix A, then

$A(\text{adj } A)(\text{adj}\,(\text{adj } A))=AI_n|A|^{n-1}=A|A|^{n-1}$

$\Rightarrow \quad |A|\,I_n(\text{adj}(\,\text{adj}A))=A|A|^{n-1}$

$\Rightarrow \quad (\text{adj}\,(\text{adj } A))=A|A|^{n-2}$

or $\quad \text{adj}\,(\text{adj} A)=|A|^{n-2}\,A$

Property 5 If A be a square non-singular matrix of order n, then

$$|\text{adj}\,(\text{adj } A)|=|A|^{(n-1)^2}$$

Proof $\because \quad \text{adj}\;\;\text{adj } A\;=|A|^{n-2}\,A$

Taking determinant on both sides, then

$$|\text{adj}\,(\text{adj } A)|=||A|^{n-2}\,A|$$

$$= |A|^{n(n-2)}\,|A| \qquad [\because |kA| = k^n |A|]$$
$$= |A|^{n^2-2n+1} = |A|^{(n-1)^2}$$

Note

In general, $|\underbrace{\text{adj}\,(\text{adj}\,(\text{adj}\ldots(\text{adj}\,A)))}_{\text{adj } m \text{ times}}| = |A|^{(n-1)^m}$

Property 6 If A be a square matrix of order n and k is a scalar, then

$$\text{adj}\,(kA) = k^{n-1}\cdot(\text{adj}\,A)$$

Proof $\because$ $\quad A(\text{adj}\,A) = |A|\,I_n \qquad \ldots\text{(i)}$

Replace A by kA, then

$$kA(\text{adj}\,(kA)) = |kA|\,I_n = k^n|A|\,I_n$$
$$\Rightarrow \quad A(\text{adj}\,(kA)) = k^{n-1}|A|\,I_n$$
$$= k^{n-1}A(\text{adj}\,A) \qquad [\text{from Eq. (i)}]$$

Hence, $\quad \text{adj}\,(kA) = k^{n-1}(\text{adj}\,A)$

Property 7 If A be a square matrix of order n and $m \in N$, then $\quad (\text{adj}\,A^m) = (\text{adj}\,A)^m$

Property 8 If A and B be two square matrices of order n such that B is the adjoint of A and k is a scalar, then

$$|AB + kI_n| = (|A| + k)^n$$

Proof $\because$ $\quad B = \text{adj}\,A$

$\therefore \quad AB = A(\text{adj}\,A) = |A|\,I_n$

$$\text{LHS} = |AB + kI_n| = ||A|\,I_n + kI_n| = |(|A| + k)I_n|$$
$$= (|A| + k)^n|I_n| = (|A| + k)^n = \text{RHS}$$

Property 9 Adjoint of a diagonal matrix is a diagonal matrix.

i.e. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $\text{adj}\,A = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$

Note

$\boxed{\text{adj}\,(I_n) = I_n.}$

| Example 32. If $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$, find the values of

(i) $|A|\,|\text{adj}\,A|$ $\qquad$ (ii) $|\text{adj}\,(\text{adj}\,(\text{adj}\,A))|$

(iii) $|\text{adj}\,(3A)|$ $\qquad$ (iv) $\text{adj}\,(\text{adj}\,A)$

Sol. $\because$ $\quad A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$\therefore |A| = (-1)(1-1) - (1)(-1-1) + (1)(1+1) = 4 \neq 0$

$\Rightarrow$ **A is non-singular.**

(i) $|A|\,|\text{adj}\,A| = |A|\,|A|^2$
$$= |A|^3 = 4^3 = 64 \qquad [\because n = 3]$$

(ii) $|\text{adj}\,(\text{adj}\,(\text{adj}\,A))| = |A|^{(3-1)^3} = |A|^8 = 4^8 = 2^{16}$

(iii) $|\text{adj}\,(3A)| = |3^2\,\text{adj}\,A| = (3^2)^3|\text{adj}\,A|$
$$= 3^6|A|^2 = 729 \times 4^2 = 11664$$

(iv) $\text{adj}(\,\text{adj}\,A) = |A|^2 A = 16A$ (4A)

| Example 33. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and B is the adjoint of A, find the value of $|AB + 2I|$, where I is the identity matrix of order 3.

Sol. $\because$ $\quad A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$
$$= 3(-3+4) + 3(2-0) + 4(-2-0) = 1 \neq 0$$

$\therefore \ |AB + 2I| = (|A| + 2)^3 \qquad [\text{by property 8}]$
$$= (1+2)^3 = 3^3 = 27$$

Inverse of a Matrix
(Reciprocal Matrix)

A square matrix A (non-singular) of order n is said to be invertible, if there exists a square matrix B of the same order such that $AB = I_n = BA$,

then B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus, $\quad A^{-1} = B \Leftrightarrow AB = I_n = BA$

We have, $\quad A(\text{adj}\,A) = |A|\,I_n$

$\Rightarrow \quad A^{-1}A(\text{adj}\,A) = A^{-1}I_n|A|$

$\Rightarrow \quad I_n(\text{adj}\,A) = A^{-1}|A|\,I_n$

$\therefore \quad A^{-1} = \dfrac{\text{adj}\,A}{|A|}$; provided $|A| \neq 0$

Note The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Properties of Inverse of a Matrix

Property 1 (Uniqueness of inverse) Every invertible matrix possesses a unique inverse.

Proof Let A be an invertible matrix of order $n \times n$. Let B and C be two inverses of A. Then,

$$AB = BA = I_n \qquad \ldots\text{(i)}$$

and $\quad AC = CA = I_n \qquad \ldots\text{(ii)}$

Now, $\quad AB = I_n$

$\Rightarrow \quad C(AB) = CI_n \qquad [\text{pre-multiplying by } C]$

$\Rightarrow \quad (CA)B = CI_n \qquad [\text{by associativity}]$

$\Rightarrow \quad I_nB = CI_n \qquad [\because CA = I_n \text{ by Eq. (ii)}]$

$\Rightarrow \quad B = C$

Hence, an invertible matrix possesses a unique inverse.

Property 2 (Reversal law) If A and B are invertible matrices of order $n \times n$, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof It is given that A and B are invertible matrices.

$$|A| \neq 0 \text{ and } |B| \neq 0 \Rightarrow |A|\,|B| \neq 0$$
$$\Rightarrow \quad |AB| \neq 0$$

Hence, AB is an invertible matrix.

Now, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} \qquad [\text{by associativity}]$
$$= (AI_n)A^{-1} \qquad [\because BB^{-1} = I_n]$$
$$= AA^{-1} \qquad [\because AI_n = A]$$
$$= I_n \qquad [\because AA^{-1} = I_n]$$

Also, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B \qquad [\text{by associativity}]$
$$= B^{-1}(I_nB) \qquad [\because A^{-1}A = I_n]$$
$$= B^{-1}B \qquad [\because I_nB = B]$$
$$= I_n \qquad [\because B^{-1}B = I_n]$$

Thus, $(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$

Note

If $A, B, C, \ldots, Y, Z$ are invertible matrices, then $(ABC\ldots YZ)^{-1} = Z^{-1}Y^{-1}\ldots C^{-1}B^{-1}A^{-1}$ $\qquad$ [reversal law]

Property 3 Let A be an invertible matrix of order n, then A' is also invertible and $(A')^{-1} = (A^{-1})'$.

Proof $\because$ A is invertible matrix

$\therefore \quad |A| \neq 0 \Rightarrow |A'| \neq 0 \qquad [\because |A| = |A'|]$

Hence, A^{-1} is also invertible.

Now, $\quad AA^{-1} = I_n = A^{-1}A$

$\Rightarrow \quad (AA^{-1})' = (I_n)' = (A^{-1}A)'$

$\Rightarrow \quad (A^{-1})'A' = I_n = A'(A^{-1})'$

$\qquad\qquad [\text{by reversal law for transpose}]$

$\Rightarrow \quad (A')^{-1} = (A^{-1})' \qquad [\text{by definition of inverse}]$

Property 4 Let A be an invertible matrix of order n and $k \in N$, then

$$(A^k)^{-1} = (A^{-1})^k = A^{-k}$$

Proof We have,

$$(A^k)^{-1} = (\underbrace{A \times A \times A \times \ldots \times A}_{\text{repeat } k \text{ times}})^{-1}$$
$$= \underbrace{A^{-1} \times A^{-1} \times A^{-1} \times \ldots \times A^{-1}}_{\text{repeat } k \text{ times}}$$
$$[\text{by reversal law for inverse}]$$
$$= (A^{-1})^k = A^{-k}$$

Property 5 Let A be an invertible matrix of order n, then $(A^{-1})^{-1} = A$.

Proof We have, $\quad A^{-1}A = I_n$

$\therefore$ Inverse of $\quad A^{-1} = A \Rightarrow (A^{-1})^{-1} = A$

Note

$I_n^{-1} = I_n$ as $I_n^{-1}\,I_n = I_n$

Property 6 Let A be an invertible matrix of order n, then $|A^{-1}| = \dfrac{1}{|A|}$.

Proof $\because$ A is invertible, then $|A| \neq 0$.

Now, $\quad AA^{-1} = I_n = A^{-1}A$

$\Rightarrow \quad |AA^{-1}| = |I_n|$

$\Rightarrow \quad |A|\,|A^{-1}| = 1$

$\qquad\qquad [\because |AB| = |A|\,|B| \text{ and } |I_n| = 1]$

$\Rightarrow \quad |A^{-1}| = \dfrac{1}{|A|} \qquad [\because |A| \neq 0]$

Property 7 Inverse of a non-singular diagonal matrix is a diagonal matrix.

i.e. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $|A| \neq 0$, then

$$A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$$

Note

The inverse of a non-singular square matrix A of order 2 is obtained by interchanging the diagonal elements and changing signs of off-diagonal elements and dividing by $|A|$.

For example,

If $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| = (ad - bc) \neq 0$, then

$$A^{-1} = \frac{1}{(ad - bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 34. Compute the inverse of the matrix

$$A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}.$$

Sol. We have, $A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$

Then, $|A|=\begin{vmatrix}0&1&2\\1&2&3\\3&1&1\end{vmatrix}=0\cdot(2-3)-1(1-9)+2(1-6)$

$=-2\neq 0$

$\therefore$ A^{-1} exists.

Now, cofactors along $R_1=-1,8,-5$

cofactors along $R_2=1,-6,3$

cofactors along $R_3=-1,2,-1$

Let C is a matrix of cofactors of the elements in $|A|$

$$\therefore\quad C=\begin{bmatrix}-1&8&-5\\1&-6&3\\-1&2&-1\end{bmatrix}$$

$$\therefore\quad \text{adj } A=C'=\begin{bmatrix}-1&1&-1\\8&-6&2\\-5&3&-1\end{bmatrix}$$

Hence, $A^{-1}=\dfrac{\text{adj } A}{|A|}=-\dfrac{1}{2}\begin{bmatrix}-1&1&-1\\8&-6&2\\-5&3&-1\end{bmatrix}$

$$=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&-\frac{3}{2}&\frac{1}{2}\end{bmatrix}$$

Example 35. If A and B are symmetric non-singular matrices of same order, $AB=BA$ and $A^{-1}B^{-1}$ exist, prove that $A^{-1}B^{-1}$ is symmetric.

Sol. $\because$ $A'=A$, $B'=B$ and $|A|\neq 0, |B|'\neq 0$

$\therefore$ $(A^{-1}B^{-1})'=(B^{-1})'(A^{-1})'$ [by reversal law of transpose]

$=(B')^{-1}(A')^{-1}$ [by property 3]

$=B^{-1}A^{-1}$ [$\because A'=A$ and $B'=B$]

$=(AB)^{-1}$ [by reversal law of inverse]

$=(BA)^{-1}$ [$\because AB=BA$]

$=A^{-1}B^{-1}$ [by reversal law of inverse]

Hence, $A^{-1}B^{-1}$ is symmetric.

Must

Example 36. Matrices A and B satisfy $AB=B^{-1}$, where $B=\begin{bmatrix}2&-2\\-1&0\end{bmatrix}$, find the value of λ for which $\lambda A-2B^{-1}+I=O$, without finding B^{-1}.

Sol. $\because$ $AB=B^{-1}$ or $AB^2=I$

Now, $\lambda A-2B^{-1}+I=O$

$\Rightarrow$ $\lambda AB-2B^{-1}B+IB=O$ [post-multiplying by B]

$\Rightarrow$ $\lambda AB-2I+B=O$

$\Rightarrow$ $\lambda AB^2-2IB+B^2=O$ [again post-multiplying by B]

$\Rightarrow$ $\lambda AB^2-2B+B^2=O$

$\Rightarrow$ $\lambda I-2B+B^2=O$ [$\because AB^2=I$]

$$\Rightarrow \lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}-2\begin{bmatrix}2&-2\\-1&0\end{bmatrix}+\begin{bmatrix}2&-2\\-1&0\end{bmatrix}\begin{bmatrix}2&-2\\-1&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}-\begin{bmatrix}4&-4\\-2&0\end{bmatrix}+\begin{bmatrix}6&-4\\-2&2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\lambda+2&0\\0&\lambda+2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$\Rightarrow$ $\lambda+2=0$

$\therefore$ $\lambda=-2$

Example 37. If A, B and C are three non-singular square matrices of order 3 satisfying the equation $A^2=A^{-1}$ and let $B=A^8$ and $C=A^2$, find the value of $\det(B-C)$.

Sol. $\because$ $B=A^8=(A^2)^4=(A^{-1})^4$ [$\because A^{-1}=A^2$]

$=(A^4)^{-1}=(A^{2\cdot 2})^{-1}$

$=((A^2)^2)^{-1}=((A^2)^{-1})^2$

$=((A^{-1})^{-1})^2=A^2=C$

So, $B=C\Rightarrow B-C=0$

$\therefore$ $\det(B-C)=0$

Elementary Row Operations (Transformations)

The following three types of operations (transformations) on the rows of a given matrix are known as elementary row operation (transformations).

(i) The interchange of ith and jth rows is denoted by $R_i\leftrightarrow R_j$ or R_{ij}.

(ii) The multiplication of the ith row by a constant $k(k\neq 0)$ is denoted by $R_i\to kR_i$ or $R_i(k)$

(iii) The addition of the ith row to the elements of the jth row multiplied by constant $k(k\neq 0)$ is denoted by $R_i\to R_i+kR_j$ or $R_{ij}(k)$.

Note

Similarly, we can define the three column operations, $C_{ij}(C_i\leftrightarrow C_j)$, $C_i(k)(C_i\to kC_i)$ and $C_{ij}(k)(C_i\to C_i+kC_j)$.

Equivalent Matrices

Two matrices are said to be equivalent if one is obtained from the other by elementary operations (transformations). The symbol ~ is used for equivalence.

Properties of Equivalent Matrices

(i) If A and B are equivalent matrices, there exist non-singular matrices P and Q such that $B=PAQ$

(ii) If A and B are equivalent matrices such that $B=PAQ$, then $\boxed{P^{-1}BQ^{-1}=A}$

(iii) Every non-singular square matrix can be expressed as the product of elementary matrices.

Example 38. Transform $\begin{bmatrix}1&3&3\\2&4&10\\3&8&4\end{bmatrix}$ into a unit matrix.

Note:- If using Row ...

Sol. Let $A=\begin{bmatrix}1&3&3\\2&4&10\\3&8&4\end{bmatrix}$

Applying $R_2\to R_2-2R_1$ and $R_3\to R_3-3R_1$, we get

$$A\sim\begin{bmatrix}1&3&3\\0&-2&4\\0&-1&-5\end{bmatrix}$$

Applying $R_2\to\left(-\frac{1}{2}\right)R_2$ and $R_2\to(-1)R_2$, we get

$$A\sim\begin{bmatrix}1&3&3\\0&1&-2\\0&1&5\end{bmatrix}$$

Applying $R_1\to R_1-3R_2$ and $R_3\to R_3-R_2$, we get

$$A\sim\begin{bmatrix}1&0&9\\0&1&-2\\0&0&7\end{bmatrix}$$

Applying $R_3\to\left(\frac{1}{7}\right)R_3$, we get

$$A\sim\begin{bmatrix}1&0&9\\0&1&-2\\0&0&1\end{bmatrix}$$

Applying $R_1\to R_1-9R_3$ and $R_2\to R_2+2R_3$, we get

$$A\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Hence, $A\sim I$

Example 39. Given $A=\begin{bmatrix}1&1&1\\2&4&1\\2&3&1\end{bmatrix}$, $B=\begin{bmatrix}2&3\\3&4\end{bmatrix}$. Find P such that $BPA=\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}$

Sol. Given, $BPA=\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}$

$\therefore$ $P=B^{-1}\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}A^{-1}$ ...(i) [by property]

$\because$ $B=\begin{bmatrix}2&3\\3&4\end{bmatrix}\Rightarrow B^{-1}=\dfrac{1}{(-1)}\begin{bmatrix}4&-3\\-3&2\end{bmatrix}=\begin{bmatrix}-4&3\\3&-2\end{bmatrix}$

$\therefore$ $B^{-1}=\begin{bmatrix}-4&3\\3&-2\end{bmatrix}$ ...(ii)

and $A=\begin{bmatrix}1&1&1\\2&4&1\\2&3&1\end{bmatrix}$

$\therefore$ $|A|=1(4-3)-1(2-2)+1(6-8)=-1\neq 0$

$\Rightarrow$ A^{-1} exists.

Now, adj $A=\begin{bmatrix}1&2&-3\\0&-1&1\\-2&-1&2\end{bmatrix}$ [by shortcut method]

$\therefore$ $A^{-1}=\dfrac{\text{adj } A}{|A|}=\begin{bmatrix}-1&-2&3\\0&1&-1\\2&1&-2\end{bmatrix}$ ...(iii)

Substituting the values of A^{-1} and B^{-1} from Eqs. (ii) and (iii) in Eq. (i), then

$$P=\begin{bmatrix}-4&3\\3&-2\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}-1&-2&3\\0&1&-1\\2&1&-2\end{bmatrix}$$

$$=\begin{bmatrix}-4&3\\3&-2\end{bmatrix}\begin{bmatrix}1&-1&1\\0&1&-1\end{bmatrix}=\begin{bmatrix}-4&7&-7\\3&-5&5\end{bmatrix}$$

To Compute the Inverse of a Non-Singular Matrix by Elementary Operations (Gauss-Jordan Method)

If A be a non-singular matrix of order n, then write $A = I_n A$.

If A is reduced to I_n by elementary operations (LHS), then suppose I_n is reduced to P(RHS) and not change A in RHS, then after elementary operations, we get

$$I_n = PA,$$

then P is the inverse of A

$$\therefore \quad P = A^{-1}$$

I Example 40. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, using elementary row operations.

Sol. Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1(3-1) - 2(2+1) + 5(2+3) = 21 \neq 0$$

$\therefore$ A^{-1} exists.

We write $A = IA$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$, we get

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to (-1)R_2$ and $R_3 \to \left(\frac{1}{3}\right)R_3$, we get

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 9 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 9 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ -\frac{5}{3} & 1 & \frac{1}{3} \end{bmatrix} A$$

Applying $R_3 \to \left(-\frac{1}{7}\right)R_3$, we get

$$\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix} A$$

Applying $R_2 \to R_2 - 9R_3$ and $R_1 \to R_1 + 13R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix} A$$

Hence, $$A^{-1} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix}$$

Matrix Polynomial

Let $f(x) = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_{m-1}x + a_m$ be a polynomial in x and let $A = [a_{ij}]_{n \times n}$, then expression of the form

$f(A) = a_0 A^m + a_1 A^{m-1} + a_2 A^{m-2} + \ldots + a_{m-1} A + a_m I_n$ is called a matrix polynomial.

Thus, to obtain $f(A)$ replace x by A in $f(x)$ and the constant term is multiplied by the identity matrix of order equal to that of A.

For example, If $f(x) = x^2 - 7x + 32$ is a polynomial in x and A is a square matrix of order 3, then $f(A) = A^2 - 7A + 32 I_3$ is a matrix polynomial.

Note

1. The polynomial equation $f(x) = 0$ is satisfied by the matrix $A = [a_{ij}]_{n \times n}$, then $f(A) = 0$.
2. Let $A = [a_{ij}]_{n \times n}$ satisfies the equation $a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r = 0$, then A is invertible of $a_0 \neq 0$, $|A| = 0$ and its inverse is given by $$A^{-1} = \frac{1}{a_0}(a_1 I_n + a_2 A + \ldots + a_r A^{r-1}).$$

I Example 41. If $A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$ and $kn \neq lm$, show that $A^2 - (k+n)A + (kn - lm)I = O$. Hence, find A^{-1}.

Sol. We have, $A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$, then $|A| = \begin{vmatrix} k & l \\ m & n \end{vmatrix}$

$$= kn - ml \neq 0 \qquad \text{[given]}$$

$\therefore$ A^{-1} exists.

Now, $A^2 = A \cdot A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}\begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} k^2 + lm & kl + ln \\ mk + nm & ml + n^2 \end{bmatrix}$

$\therefore$ $A^2 - (k+n)A + (kn - lm)I$

$$= \begin{bmatrix} k^2 + lm & kl + ln \\ mk + nm & ml + n^2 \end{bmatrix} - (k+n)\begin{bmatrix} k & l \\ m & n \end{bmatrix} + (kn - lm)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k^2 + lm & kl + ln \\ mk + nm & ml + n^2 \end{bmatrix} - \begin{bmatrix} k^2 + nk & kl + nl \\ km + nm & kn + n^2 \end{bmatrix} + \begin{bmatrix} kn - lm & 0 \\ 0 & kn - lm \end{bmatrix}$$

$$= \begin{bmatrix} k^2 + lm - k^2 - nk + kn - lm & kl + ln - kl - ln \\ mk + nm - km - nm & ml + n^2 - kn - n^2 + kn - lm \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

As $A^2 - (k+n)A + (kn - lm)I = O$

$$\Rightarrow \quad (kn - lm)I = (k+n)A - A^2$$

$$\Rightarrow \quad (kn - lm)IA^{-1} = ((k+n)A - A^2)A^{-1}$$

$$\Rightarrow \quad (kn - lm)A^{-1} = (k+n)AA^{-1} - A(AA^{-1})$$

$$= (k+n)I - AI \qquad [\because AA^{-1} = I]$$

$$= (k+n)I - A$$

$$= (k+n)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} k & l \\ m & n \end{bmatrix}$$

$$= \begin{bmatrix} k+n & 0 \\ 0 & k+n \end{bmatrix} - \begin{bmatrix} k & l \\ m & n \end{bmatrix}$$

$$\Rightarrow \quad (kn - lm)A^{-1} = \begin{bmatrix} n & -l \\ -m & k \end{bmatrix}$$

Hence, $$A^{-1} = \frac{1}{(kn - lm)}\begin{bmatrix} n & -l \\ -m & k \end{bmatrix}$$

I Example 42. If $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, find the value of $|a| + |b|$ such that $A^2 + aA + bI = O$. Hence, find A^{-1}.

Sol. We have, $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, then $|A| = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 3 - 2 = 1 \neq 0$

$\therefore$ A^{-1} exists.

Now, $A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix}$

Since, $A^2 + aA + bI = O$

$$\Rightarrow \quad \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 11 + 3a + b & 4 + a \\ 8 + 2a & 3 + a + b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the corresponding elements, we get

$$11 + 3a + b = 0 \qquad \ldots(\text{i})$$
$$4 + a = 0 \qquad \ldots(\text{ii})$$
$$8 + 2a = 0 \qquad \ldots(\text{iii})$$
$$3 + a + b = 0 \qquad \ldots(\text{iv})$$

From Eqs. (ii) and (iv), we get $a = -4$ and $b = 1$

$$\therefore \quad |a| + |b| = |-4| + |1| = 4 + 1 = 5$$

As $\quad A^2 + aA + bI = O$

$$\Rightarrow \quad A^2 - 4A + I = O \Rightarrow I = 4A - A^2$$

$$\Rightarrow \quad IA^{-1} = (4A - A^2)A^{-1}$$

$$\Rightarrow \quad A^{-1} = 4(AA^{-1}) - A(AA^{-1})$$

$$= 4I - AI = 4I - A$$

$$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Use of Mathematical Induction

I Example 43. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bA)^n = a^n I + na^{n-1} bA$, $\forall\, n \in N$.

Sol. Let $P(n): (aI + bA)^n = a^n I + na^{n-1} bA$

Step I For $n = 1$,

LHS $= (aI + bA)^1 = aI + bA$

and RHS $= a^1 I + 1 \cdot a^0 bA = aI + bA$

LHS = RHS

Therefore, $P(1)$ is true.

Step II Assume that $P(k)$ is true, then

$$P(k): (aI + bA)^k = a^k I + ka^{k-1} bA$$

Step III For $n = k + 1$, we have to prove that

$$P(k+1): (aI + bA)^{k+1} = a^{k+1} I + (k+1) a^k bA$$

$$\text{LHS} = (aI + bA)^{k+1} = (aI + bA)^k (aI + bA)$$

$$= (a^k I + ka^{k-1} bA)(aI + bA) \qquad \text{[from step II]}$$

$$= a^{k+1} I^2 + a^k b(IA) + ka^k b(AI) + k a^{k-1} b^2 A^2$$

$$= a^{k+1} I + (k+1)a^k b A + 0$$

$$[\because AI = A,\ A^2 = 0 \text{ and } I^2 = I]$$

$$= a^{k+1} I + (k+1)a^k bA = \text{RHS}$$

Therefore, $P(k+1)$ is true.

Hence, by the principal of mathematical2 induction $P(n)$ is true for all $n \in N$.

| Example 44. If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, use mathematical induction to show that $A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$, $\forall\, n\in N$.

Sol. Let $P(n): A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$

Step I For $n=1$,

$$\text{LHS}=A^1=A$$

and $\text{RHS}=\begin{bmatrix}1+2 & -4\\ 1 & 1-2\end{bmatrix}=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=A$

$\Rightarrow$ LHS = RHS

Therefore, $P(1)$ is true.

Step II Assume that $P(k)$ is true, then

$$P(k): A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$$

Step III For $n=k+1$, we have to prove that

$$P(k+1): A^{k+1}=\begin{bmatrix}3+2k & -4(k+1)\\ k+1 & -1-2k\end{bmatrix}$$

$$\text{LHS}=A^{k+1}=A^k\cdot A$$

$$=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}\quad\text{[from step II]}$$

$$=\begin{bmatrix}3(1+2k)-4k & -4(1+2k)+4k\\ 3k+1(1-2k) & -4k-1(1-2k)\end{bmatrix}$$

$$=\begin{bmatrix}3+2k & -4(k+1)\\ k+1 & -1-2k\end{bmatrix}=\text{RHS}$$

Therefore, $P(k+1)$ is true.

Hence, by the principal of mathematical induction $P(n)$ is true for all $n\in N$.

Exercise for Session 3

1. If $A=\begin{bmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$, then adj A equals to
(a) A (b) A^T (c) $3A$ (d) $3A^T$

2. If A is a 3×3 matrix and B is its adjoint such that $|B|=64$, then $|A|$ is equal to
(a) 64 (b) $\pm$ 64 (c) $\pm$ 8 (d) 18

3. For any 2×2 matrix A, if $A(\text{adj }A)=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}$, then $|A|$ is equal to
(a) 0 (b) 10 (c) 20 (d) 100

4. If A is a singular matrix, then adj A is
(a) singular (b) non-singular (c) symmetric (d) not defined

5. If $A=\begin{bmatrix}1 & 2 & -1\\ -1 & 1 & 2\\ 2 & -1 & 1\end{bmatrix}$, then det (adj (adj A)) is
(a) 14^4 (b) 14^3 (c) 14^2 (d) 14

6. If $k\in R_0$, then det adj $(k\,I_n)$ is equal to
(a) k^{n-1} (b) $k^{n(n-1)}$ (c) k^n (d) k

7. With $1,\omega,\omega^2$ as cube roots of unity, inverse of which of the following matrices exists?
(a) $\begin{bmatrix}1 & \omega\\ \omega & \omega^2\end{bmatrix}$ (b) $\begin{bmatrix}\omega^2 & 1\\ 1 & \omega\end{bmatrix}$
(c) $\begin{bmatrix}\omega & \omega^2\\ \omega^2 & 1\end{bmatrix}$ (d) None of these

8. If the matrix A is such that $A\begin{bmatrix}-1 & 2\\ 3 & 1\end{bmatrix}=\begin{bmatrix}-4 & 1\\ 7 & 7\end{bmatrix}$, then A is equal to
(a) $\begin{bmatrix}1 & 1\\ 2 & -3\end{bmatrix}$ (b) $\begin{bmatrix}1 & 1\\ -2 & 3\end{bmatrix}$ (c) $\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$ (d) $\begin{bmatrix}-1 & 1\\ 2 & 3\end{bmatrix}$

9. If $A=\begin{bmatrix}\cos x & \sin x & 0\\ -\sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}=f(x)$, then A^{-1} is equal to
(a) $f(-x)$ (b) $f(x)$
(c) $-f(x)$ (d) $-f(-x)$

10. The element in the first row and third column of the inverse of the matrix $\begin{bmatrix}1 & 2 & -3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$, is
(a) -2 (b) 0 (c) 1 (d) None of these

11. If $A=\begin{bmatrix}0 & 1 & -1\\ 2 & 1 & 3\\ 3 & 2 & 1\end{bmatrix}$, then $(A(\text{adj }A)A^{-1})A$ is equal to
(a) $\begin{bmatrix}-6 & 0 & 0\\ 0 & -6 & 0\\ 0 & 0 & -6\end{bmatrix}$ (b) $\begin{bmatrix}0 & \frac{1}{6} & -\frac{1}{6}\\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6}\end{bmatrix}$ (c) $2\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$ (d) $2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

12. A is an involutory matrix given by $A=\begin{bmatrix}0 & 1 & -1\\ 4 & -3 & 4\\ 3 & -3 & 4\end{bmatrix}$, then the inverse of $\frac{A}{2}$ will be
(a) $2A$ (b) $\frac{A^{-1}}{2}$
(c) $\frac{A}{2}$ (d) A^2

13. If A satisfies the equation $x^3-5x^2+4x+\lambda=0$, then A^{-1} exists, if
(a) $\lambda\neq1$ (b) $\lambda\neq2$
(c) $\lambda\neq-1$ (d) $\lambda\neq0$

14. A square non-singular matrix A satisfies the equation $x^2-x+2=0$, then A^{-1} is equal to
(a) $I-A$ (b) $(I-A)/2$
(c) $I+A$ (d) $(I+A)/2$

15. Matrix A is such that $A^2=2A-I$, where I is the identity matrix, then for $n\geq2$, A^n is equal to $\to A^3=2A^2-AI$
$A^3=2(2A-I)-A$
(a) $nA-(n-1)I$ (b) $nA-I$
(c) $2^{n-1}A-(n-1)I$ (d) $2^{n-1}A-I$

16. If $X=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, the value of X^n is
(a) $\begin{bmatrix}3n & -4n\\ n & -n\end{bmatrix}$ (b) $\begin{bmatrix}2n+n & 5-n\\ n & -n\end{bmatrix}$
(c) $\begin{bmatrix}3^n & (-4)^n\\ 1^n & (-1)^n\end{bmatrix}$ (d) None of these

Session 4

Solutions of Linear Simultaneous Equations Using Matrix Method

Solutions of Linear Simultaneous Equations Using Matrix Method

Let us consider a system of n linear equations in n unknowns say $x_1, x_2, x_3, \dots, x_n$ given as below

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3 \\ \dots \quad \dots \\ \dots \quad \dots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n \end{array}\right\} \quad \dots(i)$$

If $b_1 = b_2 = b_3 = \dots = b_n = 0$, then the system of Eq.(i) is called a system of homogeneous linear equations and if atleast one of $b_1, b_2, b_3, \dots, b_n$ is non-zero, then it is called a system of non-homogeneous linear equation. We write the above system of Eq. (i) in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ \dots \\ b_n \end{bmatrix}$$

$$\Rightarrow \quad AX = B \quad \dots(ii)$$

where
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ \dots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ \dots \\ b_n \end{bmatrix}$$

Pre-multiplying Eq. (ii) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow \quad IX = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}B = \frac{(\text{adj } A)B}{|A|}$$

Types of Equations

(1) **When system of equations is non-homogeneous**

(i) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$ and $(\text{adj}A)\cdot B \neq 0$, then the system of equations is inconsistent and has no solution.

(iii) If $|A| = 0$ and $(\text{adj}A)\cdot B = O$, then the system of equations is consistent and has an infinite number of solutions.

(2) **When system of equations is homogeneous**

(i) If $|A| \neq 0$, then the system of equations has only trivial solution and it has one solution.

(ii) If $|A| = 0$, then the system of equations has non-trivial solution and it has infinite solutions.

(iii) If number of equations < number of unknowns, then it has non-trivial solution.

Note

Non-homogeneous linear equations can also be solved by Cramer's rule, this method has been discussed in the chapter on determinants.

Example 45. Solve the system of equations $x + 2y + 3z = 1$, $2x + 3y + 2z = 2$ and $3x + 3y + 4z = 1$ with the help of matrix inversion.

Sol. We have,

$$x + 2y + 3z = 1,\ 2x + 3y + 2z = 2 \text{ and } 3x + 3y + 4z = 1$$

The given system of equations in the matrix form are written as below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \quad \dots(i)$$

where
$$\Rightarrow \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$|A| = 1(12-6) - 2(8-6) + 3(6-9) = 6 - 4 - 9 = -7 \neq 0$$

$\therefore$ A^{-1} exists and has unique solution.

Let C be the matrix of cofactors of elements in $|A|$.

Now, cofactors along $R_1 = 6, -2, -3$

cofactors along $R_2 = 1, -5, 3$

and cofactors along $R_3 = -5, 4, -1$

$$\therefore \quad C = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = C^T$$

$$\Rightarrow \quad \text{adj } A = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{7}\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$$

From Eq. (i), $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ \frac{8}{7} \\ -\frac{2}{7} \end{bmatrix}$$

Hence, $x = -\frac{3}{7}, y = \frac{8}{7}$ and $z = -\frac{2}{7}$ is the required solution.

Example 46. Solve the system of equations $x + y + z = 6$, $x + 2y + 3z = 14$ and $x + 4y + 7z = 30$ with the help of matrix method.

Sol. We have,
$$x + y + z = 6,$$
$$x + 2y + 3z = 14$$
and
$$x + 4y + 7z = 30$$

The given system of equations in the matrix form are written as below :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$$AX = B \quad \dots(i)$$

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

$$|A| = 1(14-12) - 1(7-3) + 1(4-2) = 2 - 4 + 2 = 0$$

$\therefore$ The equation either has no solution or an infinite number of solutions. To decide about this, we proceed to find (adj A) B.

Let C be the matrix of cofactors of elements in $|A|$.

Now, cofactors along $R_1 = 2, -4, 2$

cofactors along $R_2 = -3, 6, -3$

and cofactors along $R_3 = 1, -2, 1$

$$\therefore \quad C = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 6 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \text{adj} = C^T = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix},$$

then
$$(\text{adj } A)\,B = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$

Hence, both conditions $|A| = 0$ and (adj A) $B = O$ are satisfied, then the system of equations is consistent and has an infinite number of solutions.

Proceed as follows :

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 4 & 16 \end{array}\right]$$

Applying $R_3 \to R_3 - 2R_2$, then

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then, Eq. (i) reduces to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x + y + z \\ y + 2z \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

On comparing $x+y+z=6$ and $y+2z=8$

Taking $z=k\in R$, then $y=8-2k$ and $x=k-2$.

Since, k is arbitrary, hence the number of solutions is infinite.

| Example 47. Solve the system of equations $x+3y-2z=0$, $2x-y+4z=0$ and $x-11y+14z=0$.

Sol. We have,
$$x+3y-2z=0$$
$$2x-y+4z=0$$
$$x-11y+14z=0$$

The given system of equations in the matrix form are written as below.

$$\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$AX=O \quad \text{...(i)}$$

where $A=\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $O=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore \quad |A|=1(-14+44)-3(28-4)-2(-22+1)$
$$=30-72+42=0$$

and therefore the system has a non-trivial solution. Now, we may write first two of the given equations

$$x+3y=2z \text{ and } 2x-y=-4z$$

Solving these equations in terms of z, we get

$$x=-\frac{10}{7}z \text{ and } y=\frac{8}{7}z$$

Putting $x=-\frac{10}{7}z$ and $y=\frac{8}{7}z$ in third equation of the given system,

we get, LHS $=-\frac{10}{7}z-\frac{88}{7}z+14z=0=$ RHS

Now, if $z=7k$, then $x=-10k$ and $y=8k$.

Hence, $x=-10k$, $y=8k$ and $z=7k$ (where k is arbitrary) are the required solutions.

| Example 48. Solve the system of equations
$$2x+3y-3z=0,$$
$$3x-3y+z=0$$
and
$$3x-2y-3z=0$$

Sol. We have,
$$2x+3y-3z=0$$
$$3x-3y+z=0$$
$$3x-2y-3z=0$$

The given system of equations in the matrix form are written as below.

$$\begin{bmatrix}2 & 3 & -3\\ 3 & -3 & 1\\ 3 & -2 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$AX=O$$

where $A=\begin{bmatrix}2 & 3 & -3\\ 3 & -3 & 1\\ 3 & -2 & -3\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $O=\begin{bmatrix}0\\0\\0\end{bmatrix}$ ...(i)

$\therefore \quad |A|=2(9+2)-3(-9-3)-3(-6+9)$
$$=22+36-9=49\neq 0$$

Hence, the equations have a unique trivial solution $x=0$, $y=0$ and $z=0$ only.

Echelon Form of a Matrix

A matrix A is said to be in echelon form, if

(i) The first non-zero element in each row is 1.

(ii) Every non-zero row in A preceds every zero-row.

(iii) The number of zeroes before the first non-zero element in 1st, 2nd, 3rd, ... rows should be in increasing order.

For example,

(i) $\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 4 & 5\\ 0 & 0 & 1 & 9\\ 0 & 0 & 0 & 1\end{bmatrix}$

(iii) $\begin{bmatrix}1 & 2 & 3 & 4 & 5\\ 0 & 1 & 2 & 4 & 3\\ 0 & 0 & 1 & 2 & 5\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

Rank of Matrix

The rank of a matrix is said to be r, if

(i) It has atleast minors of order r is different from zero.

(ii) All minors of A of order higher than r are zero.

The rank of A is denoted by $\rho(A)$.

Note

1. The rank of a zero matrix is zero and the rank of an identity matrix of order n is n.
2. The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.
3. The rank of a non-singular matrix ($|A|\neq 0$) of order n is n.

Properties of Rank of Matrices

(i) If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$, then
$$\rho(A+B)\le\rho(A)+\rho(B)$$

(ii) If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$, then
$$\rho(AB)\le\rho(A) \text{ and } \rho(AB)\le\rho(B)$$

(iii) If $A=[a_{ij}]_{n\times n}$, then $\rho(A)=\rho(A')$

| Example 49. Find the rank of $\begin{bmatrix}3 & -1 & 2\\ -3 & 1 & 2\\ -6 & 2 & 4\end{bmatrix}$.

Sol. We have,

Let $A=\begin{bmatrix}3 & -1 & 2\\ -3 & 1 & 2\\ -6 & 2 & 4\end{bmatrix}$

Applying $R_2\to R_2+R_1$ and $R_3\to R_3+2R_1$, we get
$$A=\begin{bmatrix}3 & -1 & 2\\ 0 & 0 & 4\\ 0 & 0 & 8\end{bmatrix}$$

Applying $R_3\to R_3-2R_2$, we get
$$A=\begin{bmatrix}3 & -1 & 2\\ 0 & 0 & 4\\ 0 & 0 & 0\end{bmatrix}$$

Applying $R_1\to\left(\frac{1}{3}\right)R_1$ and $R_2\to\left(\frac{1}{4}\right)R_2$, then
$$A=\begin{bmatrix}1 & -\frac{1}{3} & \frac{2}{3}\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

This is Echelon form of matrix A.

$\therefore$ Rank = Number of non-zero rows $\Rightarrow \rho(A)=2$

Aliter $|A|=\begin{vmatrix}3 & -1 & 2\\ -3 & 1 & 2\\ -6 & 2 & 4\end{vmatrix}$
$$=3(4-4)+1(-12+12)+2(-6+6)=0$$

$\therefore$ Rank of $A\neq 3$ but less than 3.

There will be ${}^3C_2\times{}^3C_2=9$ square minors of order 2. Now, we consider of there minors.

(i) $\begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix}=0$ (ii) $\begin{vmatrix}3 & 2\\ -6 & 4\end{vmatrix}=24\neq 0$

Hence, all minors are not zero.

Hence, rank of A is 2. $\Rightarrow \rho(A)=2$

Solutions of Linear Simultaneous Equations Using Rank Method

Let us consider a system of n linear equations in n unknowns say $x_1, x_2, x_3, \ldots, x_n$ given as below.

$$\left.\begin{array}{l}a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots+a_{1n}x_n=b_1\\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots+a_{2n}x_n=b_2\\ a_{31}x_1+a_{32}x_2+a_{33}x_3+\ldots+a_{3n}x_n=b_3\\ \ldots\quad\ldots\quad\ldots\quad\ldots\\ \ldots\quad\ldots\quad\ldots\quad\ldots\\ a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\ldots+a_{mn}x_n=b_m\end{array}\right\} \quad \text{...(i)}$$

We write the above system of Eq. (i) in the matrix form as

$$\begin{bmatrix}a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n}\\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ \cdots\\ \cdots\\ x_n\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\\ \cdots\\ \cdots\\ b_m\end{bmatrix}$$

$$\Rightarrow \quad AX=B \quad \text{...(ii)}$$

where $A=\begin{bmatrix}a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n}\\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn}\end{bmatrix}$,

$X=\begin{bmatrix}x_1\\ x_2\\ x_3\\ \cdots\\ \cdots\\ x_n\end{bmatrix}$ and $B=\begin{bmatrix}b_1\\ b_2\\ b_3\\ \cdots\\ \cdots\\ b_m\end{bmatrix}$

The matrix A is called the coefficient matrix and the matrix

$$C=[A:B]=\begin{bmatrix}a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} & \vdots & b_1\\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} & \vdots & b_2\\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} & \vdots & b_3\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots\\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} & \vdots & b_m\end{bmatrix}$$

is called the augmented matrix of the given system of equations.

Types of Equations

1. **Consistent Equation** If $\rho(A)=\rho(C)$
   - (i) **Unique Solution** If $\rho(A)=\rho(C)=n$, where $n=$ number of knowns.
   - (ii) **Infinite Solution** If $\rho(A)=\rho(C)=r$, where $r<n$.
2. **Inconsistent Equation** If $\rho(A)\neq\rho(C)$, then no solution.

| Example 50. Determine for what values of λ and μ the following system of equations
$$x+y+z=6,$$
$$x+2y+3z=10$$

and $x + 2y + \lambda z = \mu$
have (i) no solution? (ii) a unique solution?
(iii) an infinite number of solutions?

Sol. We can write the above system of equations in the matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$\Rightarrow \quad AX = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

$\therefore$ The augmented matrix

$$C = [A : B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$C = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right],$$

Applying $R_3 \to R_3 - R_2$, we get

$$C = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

(i) No solution $\rho(A) \neq \rho(C)$
i.e. $\lambda - 3 = 0$ and $\mu - 10 \neq 0$
$\therefore \quad \lambda = 3$ and $\mu \neq 10$

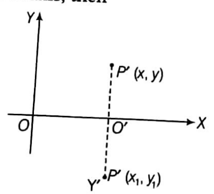
(ii) A unique solution $\rho(A) = \rho(C) = 3$
i.e., $\lambda - 3 \neq 0$ and $\mu \in R$
$\therefore \quad \lambda \neq 3$ and $\mu \in R$

(iii) Infinite number of solutions
$\rho(A) = \rho(C) (\angle 3)$
i.e. $\lambda - 3 = 0$ and $\mu - 10 = 0$
$\therefore \quad \lambda = 3$ and $\mu = 10$

Reflection Matrix

(i) Reflection in the X-axis

Let $P(x, y)$ be any point and $P'(x_1, y_1)$ be its image after reflection in the X-axis, then



$$\begin{cases} x_1 = x \\ y_1 = -y \end{cases} \quad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be rewritten as

$$\begin{cases} x_1 = 1 \cdot x + 0 \cdot y \\ y_1 = 0 \cdot x + (-1) \cdot y \end{cases}$$

These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of a point $P(x, y)$ in the X-axis.

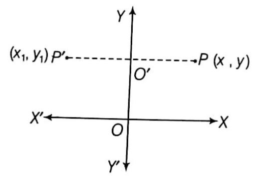
(ii) Reflection in the Y-axis

Let $P(x, y)$ be any point and $P'(x_1, y_1)$ be its image after reflection in the Y-axis, then

$$\begin{cases} x_1 = -x \\ y_1 = y \end{cases} \quad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be written as

$$\begin{cases} x_1 = (-1) \cdot x + 0 \cdot y \\ y_1 = 0 \cdot x + 1 \cdot y \end{cases}$$



These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

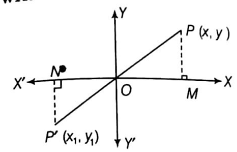
Thus, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ describes the reflection of a point $P(x, y)$ in the Y-axis.

(iii) Reflection through the origin

Let $P(x, y)$ be any point and $P'(x_1, y_1)$ be its image after reflection through the origin, then

$$\begin{cases} x_1 = -x \\ y_1 = -y \end{cases} \quad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be written as



$$\begin{cases} x_1 = (-1)\, x + 0 \cdot y \\ y_1 = 0 \cdot x + (-1) \cdot y \end{cases}$$

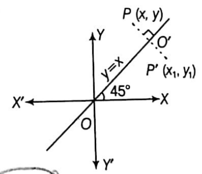
These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of a point $P(x, y)$ through the origin.

(iv) Reflection in the line y = x

Let $P(x, y)$ be any point and $P'(x_1, y_1)$ be its image after reflection in the line $y = x$, then



$$\begin{cases} x_1 = y \\ y_1 = x \end{cases} \quad [O' \text{ is the mid-point of } P \text{ and } P']$$

remember.

These may be written as

$$\begin{cases} x_1 = 0 \cdot x + 1 \cdot y \\ y_1 = 1 \cdot x + 0 \cdot y \end{cases}$$

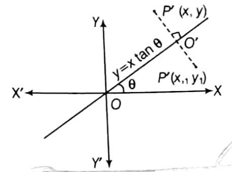
These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ describes the reflection of a point $P(x, y)$ in the line $y = x$.

(v) Reflection in the line y = x tan θ

Let $P(x, y)$ be any point and $P'(x_1, y_1)$ be its image after reflection in the line $y = x \tan\theta$, then



Remember

$$\begin{cases} x_1 = x\cos 2\theta + y \sin 2\theta \\ y_1 = x \sin 2\theta - y\cos 2\theta \end{cases}$$

$[O' \text{ is the mid-point of } P \text{ and } P']$

These may be written as

$$\begin{cases} x_1 = x \cdot \cos 2\theta + y \cdot \sin 2\theta \\ y_1 = x \cdot \sin 2\theta + y \cdot (-\cos 2\theta) \end{cases}$$

These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ describes the reflection of a point $P(x, y)$ in the line $y = x\tan\theta$.

Note

By putting $\theta = 0, \frac{\pi}{2}, \frac{\pi}{4}$, we can get the reflection matrices in the X-axis, Y-axis and the line $y = x$, respectively.

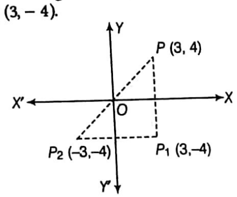
| Example 51. The point $P(3, 4)$ undergoes a reflection in the X-axis followed by a reflection in the Y-axis. Show that their combined effect is the same as the single reflection of $P(3, 4)$ in the origin.

Sol. Let $P_1(x_1, y_1)$ be the image of $P(3, 4)$ after reflection in the X-axis. Then,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$[x][y]$ multiply both matrix

Therefore, the image of $P(3, 4)$ after reflection in the X-axis is $P_1(3, -4)$.



Now, let $P_2(x_2, y_2)$ be the image of $P_1(3, -4)$ after reflection in the Y-axis, then

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

Therefore, the image of $P_1(3, -4)$ after reflection in the Y-axis is $P_2(-3, -4)$.

Further, let $P_3(x_3, y_3)$ be the image of $P(3, 4)$ in the origin O. Then,

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

Therefore, the image of $P(3, 4)$ after reflection in the origin is $P_3(-3, -4)$. It is clear that $P_2 = P_3$.

Hence, the image of P_2 of P often successive reflections in their X-axis and Y-axis is the same as P_3, which is single reflection of P in the origin.

| Example 52. Find the image of the point $(-2, -7)$ under the transformations $(x, y) \to (x - 2y, -3x + y)$.

Sol. Let (x_1, y_1) be the image of the point (x, y) under the given transormations, then

$$\begin{cases} x_1 = x - 2y = 1 \cdot x + (-2) \cdot y \\ y_1 = -3x + y = (-3) \cdot x + 1 \cdot y \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -7 \end{bmatrix} = \begin{bmatrix} -2+14 \\ 6-7 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \end{bmatrix}$$

Therefore, the required image is $(12, -1)$.

| Example 53. The image of the point $A(2, 3)$ by the line mirror $y = x$ is the point B and the image of B by the line mirror $y = 0$ is the point (α, β). Find α and β.

Sol. Let $B(x_1, y_1)$ be the image of the point $A(2,3)$ about the line $y = x$, then

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Therefore, the image of $A(2, 3)$ by the line mirror $y = x$ is $B(3, 2)$.

Given, image of B by the line mirror $y = 0$ (X-axis) is (α, β), then

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

On comparing, we get $\alpha = 3$ and $\beta = -2$

| Example 54. Find the image of the point $(-\sqrt{2}, \sqrt{2})$ by the line mirror $y = x \tan\left(\frac{\pi}{8}\right)$

Sol. Let (x_1, y_1) be the image of $(-\sqrt{2}, \sqrt{2})$ about the line $y = x \tan\left(\frac{\pi}{8}\right)$.

On comparing $y = x \tan\left(\frac{\pi}{8}\right)$ by $y = x \tan\theta$

$$\therefore \quad \theta = \frac{\pi}{8}$$

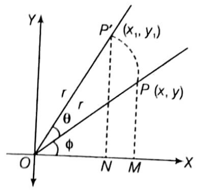
Now, $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

On comparing $x_1 = 0$ and $y_1 = -2$.

Therefore, the required image is $(0, -2)$.

Rotation Through an Angle θ



Let $P(x, y)$ be any point such that $OP = r$ and $\angle POX = \phi$. Let OP rotate through an angle θ in the anti-clockwise direction such that $P'(x_1, y_1)$ is the new position.

$$\therefore \quad OP' = r, \qquad [\because OP = OP']$$

then $$\begin{cases} x_1 = x\cos\theta - y\sin\theta \\ y_1 = x\sin\theta + y\cos\theta \end{cases}$$

These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ describes a rotation of a line segment through an angle θ.

Remember Use of complex number

$$OP' = OP\, e^{i\theta}, i = \sqrt{-1}$$

$$(x_1 + iy_1) = (x + iy)(\cos\theta + i\sin\theta)$$

$$= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$$

$$\therefore \quad x_1 = x\cos\theta - y\sin\theta$$

and $$y_1 = x\sin\theta + y\cos\theta$$

| Example 55. Find the matrices of transformation T_1T_2 and T_2T_1 when T_1 is rotation through an angle $60°$ and T_2 is the reflection in the Y-axis. Also, verify that $T_1T_2 \neq T_2T_1$.

Sol. $$T_1 = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

and $$T_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad T_1T_2 = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1+0 & 0-\sqrt{3} \\ -\sqrt{3}+0 & 0+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad \text{...(i)}$$

and $$T_2T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1+0 & \sqrt{3}+0 \\ 0+\sqrt{3} & 0+1 \end{bmatrix}$$

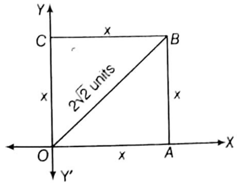
$$= \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad \text{...(ii)}$$

It is clear from Eqs.(i) and (ii), then

$$T_1T_2 \neq T_2T_1$$

| Example 56. Write down 2×2 matrix A which corresponds to a counterclockwise rotation of $60°$ about the origin. In the diagram the square $OABC$ has its diagonal OB of $2\sqrt{2}$ units in length. The square is rotated counterclockwise about O through $60°$. Find the coordiates of the vertices of the square after rotating.

Sol. The matrix describes a rotation through an angle $60°$ in counterclockwise direction is



$$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

Since, each side of the square be x,

then $$x^2 + x^2 = (2\sqrt{2})^2$$

$$\Rightarrow \quad 2x^2 = 8 \Rightarrow x^2 = 4$$

$$\therefore \quad x = 2 \text{ units}$$

Therefore, the coordinates of the vertices O, A, B and C are $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$, respectively. Let after rotation A map into A', B map into B', C map into C' but the O map into itself.

If coordinates of A', B' and C' are (x', y'), (x'', y'') and (x''', y'''), respectively.

$$\therefore \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

$$\therefore \quad x' = 1, y' = \sqrt{3} \Rightarrow A(2, 0) \to A'(1, \sqrt{3})$$

and $$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2-2\sqrt{3} \\ 2\sqrt{3}+2 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{3} \\ \sqrt{3}+1 \end{bmatrix}$$

$$\therefore \quad x'' = 1 - \sqrt{3}, y'' = \sqrt{3} + 1$$

$$\Rightarrow \quad B(2, 2) \to B'(1 - \sqrt{3}, \sqrt{3} + 1)$$

$$\begin{bmatrix} x''' \\ y''' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2\sqrt{3} \\ 2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

$$\therefore \quad x''' = -\sqrt{3}, y''' = 1$$

$$\Rightarrow \quad C(0, 2) \to C'(-\sqrt{3}, 1)$$

Eigen Values or Characteristic roots and Characteristic Vectors of a square matrix

Let X be any non-zero vector satisfying

$$AX = \lambda X \qquad \text{...(i)}$$

where λ is any scalar, then λ is said to be eigen value or characteristic root of square matrix A and the vector X is called eigen vector or characteristic vector of matrix A.

Now, from Eq. (i), we have

$$(A - \lambda I)X = O$$

Since, $X \neq O$, we deduce that the matrix $(A - \lambda I)$ is singular, so that its determinant is 0

i.e. $$\boxed{|A - \lambda I| = 0} \qquad \text{...(ii)}$$

is called characteristic equation of matrix A.

If A be $n \times n$ matrix, then equation $|A - \lambda I| = 0$ reduces to polynomial equation of nth from degree in λ, which given n values of λ i.e., matrix A will have n characteristic roots or eigen values.

Important Properties of Eigen Values

(i) Any square matrix A and its transpose A^T have the same eigen values.

(ii) The sum of the eigen values of a matrix is equal to the trace of the matrix.

(iii) The product the eigen values of a matrix A is equal to the determinant of A.

(iv) If $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n$ are the eigen values of A, then the eigen values of

(a) kA are $k\lambda_1, k\lambda_2, k\lambda_3, k\lambda_4, \ldots, k\lambda_n$.

(b) A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \lambda_4^m, \ldots, \lambda_n^m$.

(c) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_4}, \ldots, \frac{1}{\lambda_n}$.

Remark

1. All the eigen values of a real symmetric matrix are real and the eigen vectors corresponding to two distinct eigen values are orthogonal.
2. All the eigen values of a real skew-symmetric matrix are purely imaginary or zero. An odd order skew-symmetric matrix is singular and hence has zero as an eigen value.

| Example 57. Let matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$, find the non-zero column vector X such that $AX = \lambda X$ for some scalar λ.

Sol. The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

or $(\lambda + 1)(\lambda - 1)(\lambda - 4) = 0$

The eigen values are $\lambda = -1, 1, 4$

If $\lambda = -1$, we get $5x + 6y + 6z = 0, x + 4y + 2z = 0$

and $-x - 4y - 2z = 0$

Giving $\frac{x}{6} = \frac{y}{2} = \frac{z}{-7}, X = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$

If $\lambda = 1$, we get $3x + 6y + 6z = 0, x + 2y + 2z = 0$

and $-x - 4y - 4z = 0$

Giving, $\frac{x}{0} = \frac{y}{1} = \frac{z}{-1}, X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

If $\lambda = 4$, we get $0 \cdot x + 6y + 6z = 0, x - y + 2z = 0$

and $-x - 4y - 7z = 0$

Giving, $\frac{x}{3} = \frac{y}{1} = \frac{3}{-1}, x = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

Hence, vector are $X = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$.

| Example 58. If A and P are the square matrices of the same order and if P be invertible, show that the matrices A and $P^{-1}AP$ have the same characteristic roots.

Sol. Let $P^{-1}AP = B$

$\therefore$
$$|B - \lambda I| = |P^{-1}AP - \lambda I|$$
$$= |P^{-1}AP - P^{-1}\lambda P| \quad [\because P^{-1}P = I]$$
$$= |P^{-1}(A - \lambda I)P|$$
$$= |P^{-1}||A - \lambda I||P|$$
$$= \frac{1}{|P|}|A - \lambda I||P| = |A - \lambda I|$$

| Example 59. Show that the characteristic roots of an idempotent matrix are either zero or unity.

Sol. Let A be an idempotent matrix, then

$$A^2 = A \quad \ldots(i)$$

If λ be an eigen value of the matrix A corresponding to eigen vector X, so that

$$AX = \lambda X \quad \ldots(ii)$$

where $X \neq 0$

From Eq. (ii), $A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(\lambda X)$ [from Eq. (ii)]

$\Rightarrow AX = \lambda^2 X$ [from Eq. (i)]

$\Rightarrow \lambda X = \lambda^2 X$ [from Eq. (ii)]

$\Rightarrow (\lambda - \lambda^2)X = 0$

$\Rightarrow \lambda - \lambda^2 = 0$ $[\because X \neq 0]$

$\therefore \lambda = 0$

or $\lambda = 1$

| Example 60. If $3, -2$ are the eigen values of a non-singular matrix A and $|A| = 4$, find the eigen values of adj (A).

Sol. $\because A^{-1} = \frac{\text{adj } A}{|A|}$, if λ is eigen value of A, then λ^{-1} is eigen value of A^{-1}.

Thus, for adj $(A)X = (A^{-1}X)|A| = |A|\lambda^{-1}I$

Thus, eigen value corresponding to $\lambda = 3$ is $\frac{4}{3}$ and corresponding to $\lambda = -2$ is $\frac{4}{-2} = -2$

Cayley-Hamilton Theorem

Every square matrix A satisfies its characteristic equation $|A - \lambda I| = 0$

i.e., $a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0$

$\therefore$ By Cayley-Hamilton theorem

$$a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_nI = O$$

$$\Rightarrow A^{-1} = -\left\{\frac{a_0}{a_n}A^{n-1} + \frac{a_1}{a_n}A^{n-2} + \frac{a_2}{a_n}A^{n-3} + \ldots + \frac{a_{n-1}}{a_n}I\right\}$$

| Example 61. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and hence find its inverse using Cayley-hamilton theorem.

Sol. Characteristic equation is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2 - \lambda)^2 - 3 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 1 = 0$

$\therefore$ By Cayley-hamilton theorem,

$A^2 - 4A + I = O$ or $I = 4A - A^2$

Multiplying by A^{-1}, we get

$$A^{-1} = 4A^{-1}A - A^{-1}AA$$
$$= 4I - IA = 4I - A$$
$$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

a, c, b, d, d, d, b, d,

Exercise for Session 4

1 If the system of equations $ax + y = 1, x + 2y = 3, 2x + 3y = 5$ are consistent, then a is given by
(a) 0 (b) 1 (c) 2 (d) None of these

2 The system of equations $x + y + z = 2, 2x + y - z = 3, 3x + 2y + \lambda z = 4$ has unique solution if
(a) $\lambda \neq 0$ (b) $-1 < \lambda < 1$ (c) $\lambda = 0$ (d) $-2 < \lambda < 2$

3 The value of a for which the following system of equations $a^3x + (a+1)^3y + (a+2)^3z = 0$, $ax + (a+1)y + (a+2)z = 0, x + y + z = 0$ has a non-trivial solution is equal to
(a) 2 (b) 1 (c) 0 (d) -1

4 The number of solutions of the set of equations
$\frac{2x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, -\frac{x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 0, -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2z^2}{c^2} = 0$ is
(a) 6 (b) 7 (c) 8 (d) 9

5 The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the matrix reflection in the line
(a) $x = 1$ (b) $x + y = 1$ (c) $y = 1$ (d) $x = y$

6 The matrix S is rotation through an angle 45° and G is the reflection about the line $y = 2x$, then $(SG)^2$ is equal to
(a) $7I$ (b) $5I$ (c) $3I$ (d) I

7 If $A = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$, then A^3 is equal to
(a) $2A$ (b) A (c) $2I$. (d) I

8 If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and the sum of eigen values of A is m and product of eigen values of A is n, then $m + n$ is equal to
(a) 10 (b) 12 (c) 14 (d) 16

9 If $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ and θ be the angle between the two non-zero column vectors X such that $AX = \lambda X$ for some scalar λ, then $9\sec^2\theta$ is equal to
(a) 13 (b) 12 (c) 11 (d) 10

Shortcuts And Important Results To Remember

1 $|A|$ exists $\Leftrightarrow$ A is square matrix.

2 No element of principal diagonal in a diagonal matrix is zero.

3 If A is a diagonal matrix of order n, then

(a) Number of zeroes in A is $n(n-1)$

(b) If $d_1, d_2, d_3, \ldots, d_n$ are diagonal elements, then

$$A = \text{diag}\{d_1, d_2, d_3, \ldots, d_n\}$$

and $\quad |A| = d_1 d_2 d_3 \ldots d_n$

$$A^{-1} = \text{diag}(d_1^{-1}, d_2^{-1}, d_3^{-1}, \ldots, d_n^{-1})$$

(c) Diagonal matrix is both upper and lower triangular.

(d) $\text{diag}\{a_1, a_2, a_3, \ldots, a_n\} \times \text{diag}\{b_1, b_2, b_3, \ldots, b_n\}$
$= \text{diag}\{a_1b_1, a_2b_2, a_3b_3, \ldots, a_nb_n\}$

4 If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $B^k = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, $\forall k \in N$.

5 If A and B are square matrices of order n, then

(a) $|kA| = k^n |A|$, k is scalar

(b) $|AB| = |A||B|$

(c) $|kAB| = k^n |A||B|$, k is scalar

(d) $|AB| = |BA|$

(e) $|A^T| = |A| = |A^\theta|$, where A^θ is conjugate transpose matrix of A

(f) $|A|^m = |A^m|$, $m \in N$

6 Minimum number of zeroes in a triangular matrix is given by $\frac{n(n-1)}{2}$, where n is order of matrix.

7 If A is a skew-symmetric matrix of odd order, then $|A| = 0$ and of even order is a non-zero perfect square.

8 If A is involutory matrix, then

(a) $|A| = \pm 1$

(b) $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I + A) \cdot \frac{1}{2}(I - A) = 0$

9 If A is orthogonal matrix, then $|A| = \pm 1$

10 To obtain an orthogonal matrix B from a skew-symmetric matrix A, then

$$B = (I - A)^{-1}(I + A) \text{ or } B = (I - A)(I + A)^{-1}$$

11 The sum of two orthogonal matrices is not orthogonal while the sum of two symmetric (skew-symmetric) matrices is symmetric (skew-symmetric)

12 The product of two orthogonal matrices is orthogonal while the product of two symmetric (skew-symmetric) matrices need not be symmetric (skew-symmetric)

13 The adjoint of a square matrix of order 2 can be easily obtained by interchanging the principal diagonal elements and changing the sign of the other diagonal.

i.e., If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

14 If $|A| \neq 0$, then $|A^{-1}| = \frac{1}{|A|}$.

15 If A and B are invertible matrices such that $AB = C$, then $|B| = \frac{|C|}{|A|}$.

16 Commutative law does not necessarily hold for matrices.

17 If $AB = -BA$, then matrices A and B are called anti-commutative matrices.

18 If $AB = O$, it is not necessary that atleast one of the matrix should be zero matrix.

For example, If $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, then

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ while neither A nor B is the null matrix.

19 If A, B and C are invertible matrices, then

(a) $(AB)^{-1} = B^{-1}A^{-1}$

(b) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

20 If B is a non-singular matrix and A is any square matrix, then $\det(B^{-1}AB) = \det(A)$

21 If A is a non-singular square matrix of order n, then $\text{adj}(\text{adj } A) = |A|^{n-2} A$

22 If A is a non-singular square matrix of order n, then

$$|\underbrace{\text{adj}(\text{adj}(\text{adj} \ldots (\text{adj}(\text{adj } A))))}_{m \text{ times}}| = |A|^{(n-1)^m}$$

23 If $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$, $A^m = O, \forall m \geq 2$

$\therefore \quad (A + I)^n = I + nA$

24 If A and B are two symmetric matrices, then $A \pm B$, $AB + BA$ are symmetric matrices and $AB - BA$ is a skew-symmetric matrix.

25 If A and B are two square matrices of order n and λ be a scalar, then

(i) $\text{Tr}(\lambda A) = \lambda \text{Tr}(A)$

(ii) $\text{Tr}(A \pm B) = \text{Tr}(A) \pm \text{Tr}(B)$

(iii) $\text{Tr}(AB) = \text{Tr}(BA)$

(iv) $\text{Tr}(A) = \text{Tr}(A')$

(v) $\text{Tr}(I_n) = n$

(vi) $\text{Tr}(O) = 0$

(vii) $\text{Tr}(AB) \neq \text{Tr}(A) \cdot \text{tr}(B)$

contd...

26 If rank of a matrix A is denoted by $\rho(A)$, then

(i) $\rho(A) = 0$, if A is zero matrix.

(ii) $\rho(A) = 1$, if every element of A is same.

(iii) If A and B are square matrices of order n each and $\rho(A) = \rho(B) = n$, then $\rho(AB) = n$

(iv) If A is a square matrix of order n and $\rho(A) = n - 1$, then $\rho(\text{adj } A) = 1$ and if $\rho(A) < n - 1$, then $\rho(\text{adj } A) = 0$

27 System of planes

$$a_{11}x + a_{12}y + a_{13}z = b_1,$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
and $\quad a_{31}x + a_{32}y + a_{33}z = b_3$

Augmented matrix $C = [A : B]$ and if Rank of $A = r$ and Rank of $C = s$, then

(i) If $r = s = 1$, then planes are coincident

(ii) If $r = 1$, $s = 2$, then planes are parallel

(iii) If $r = s = 2$, then planes intersect along a single straight line

(iv) If $r = 2$, $s = 3$, then planes form a triangular prism

(v) If $r = s = 3$, then planes meet at a single point

28 If P is an orthogonal matrix, then $\det(P) = \pm 1$

(i) P represents a reflection about a line, then $\det(P) = -1$.

(ii) P represents a rotation about a point, then $\det(P) = 1$.

29 **Cayley-Hamilton Theorem** : Every matrix satisfies its characteristic equation.

For Example, Let A be a square matrix, then $|A - \lambda I| = 0$ is the characteristic equation for A.

If $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ is the characteristic equation for A, then $A^3 - 6A^2 + 11A - 6I = O$. Roots of characteristic equation for A are called eigen values of A or characteristic roots of A or latent roots of A. If λ is a characteristic root of A, then λ^{-1} is characteristic root of A^{-1}.

JEE Type Solved Examples :
Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1 *If A is a square matrix of order 2 such that* $A\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ *and* $A^2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$. *The sum of elements and product of elements of A are S and P, then S + P is*

(a) −1 (b) 2 (c) 4 (d) 5

Sol. (d) Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

From first part, $A\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ ...(i)

$$\Rightarrow \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$$

or $a-b=-1$ ...(ii)

and $c-d=2$ ...(iii)

From second part,

$$A^2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\Rightarrow A\left(A\begin{bmatrix}1\\-1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$

From Eq. (i), we get

$$A\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\Rightarrow\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

or $-a+2b=1$ ...(iv)

and $-c+2d=0$ ...(v)

From Eqs. (ii) and (iv), we get

$a=-1,\ b=0$

and from Eqs. (iii) and (v), we get

$c=4,\ d=2$

$\therefore$ $S=a+b+c+d=5$

and $P=abcd=0$

Hence, $S+P=5$

• Ex. 2 *If P is an orthogonal matrix and* $Q=PAP^T$ *and* $B=P^TQ^{1000}P$, *then* B^{-1} *is, where A is involutory matrix*

(a) A (b) A^{1000} (c) I (d) None of these

Sol. (c) Given, P is orthogonal

$\therefore$ $P^TP=I$ ...(i)

and $Q=PAP^T$ ...(ii)

Now, $B=P^TQ^{1000}P=P^T(PAP^T)^{1000}P$ [from Eq. (ii)]

$=P^TPAP^T(PAP^T)^{999}P$

$=IAP^T\cdot PAP^T(PAP^T)^{998}P$

$=AIAP^T(PAP^T)^{998}P$

$=A^2P^TPAP^T(PAP^T)^{997}P$

$=A^2IAP^T(PAP^T)^{997}P$

$=A^3P^T(PAP^T)^{997}P$

... ...

$=A^{1000}P^TP=A^{1000}=I$ [$\because$ A is involutory]

Hence, $B^{-1}=I^{-1}=I$

• Ex. 3 *If A is a diagonal matrix of order 3 × 3 is commutative with every square matrix of order 3 × 3 under multiplication and trace (A) = 12, then*

(a) $|A|=64$ (b) $|A|=16$
(c) $|A|=12$ (d) $|A|=4$

Sol. (a) A diagonal matrix is commutative with every square matrix, if it is scalar matrix so every diagonal element is 4.

$$\therefore\quad A=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$$

$\Rightarrow$ $|A|=4\cdot4\cdot4=64$

• Ex. 4 *If* $A=[a_{ij}]_{4\times4}$, *such that* $a_{ij}=\begin{cases}2, & \text{when } i=j\\0, & \text{when } i\neq j\end{cases}$, *then* $\left\{\dfrac{\det(\text{adj}(\text{adj } A))}{7}\right\}$ *is [when {·} represents fractional part function]*

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) $\frac{4}{7}$

Sol. (a) $\because$

$$A=\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{bmatrix}$$

$$\therefore\quad |A|=\begin{vmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{vmatrix}=2^4=16$$

$\because$ $\det(\text{adj}(\text{adj } A))=|\text{adj}(\text{adj } A)|=|A|^{3^2}=|A|^9$

$=(2^4)^9=2^{36}=(2^3)^{12}=(1+7)^{12}$

$=1+{}^{12}C_1(7)+{}^{12}C_2(7)^2+\dots$

$$\frac{\det(\text{adj}(\text{adj } A))}{7}=\frac{1}{7}+\text{Positive integer}$$

$$\therefore\quad\left\{\frac{\det(\text{adj}(\text{adj } A))}{7}\right\}=\frac{1}{7}$$

• Ex. 5 *If* $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ *and* $\det(A^n-I)=1-\lambda^n,\ n\in N$, *then the value of* λ, *is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. (b) $\because$ $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$

$$\therefore\quad A^2=\begin{bmatrix}2&2\\2&2\end{bmatrix}=2\begin{bmatrix}1&1\\1&1\end{bmatrix}=2A$$

$\Rightarrow$ $A^3=A^2\cdot A=2A^2=2^2A$

Similarly, $A^n=2^{n-1}A$

$$\therefore\quad A^n-I=\begin{bmatrix}2^{n-1}&2^{n-1}\\2^{n-1}&2^{n-1}\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}2^{n-1}-1&2^{n-1}\\2^{n-1}&2^{n-1}-1\end{bmatrix}$$

$\Rightarrow$ $\det(A^n-I)=(2^{n-1}-1)^2-(2^{n-1})^2$

$=1-2^n=1-\lambda^n$ [given]

$\therefore$ $\lambda=2$

• Ex. 6 *If* $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$ *and* $f(x)=\dfrac{1+x}{1-x}$, *then f(A) is*

(a) $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ (b) $\begin{bmatrix}2&2\\2&2\end{bmatrix}$

(c) $\begin{bmatrix}-1&-1\\-1&-1\end{bmatrix}$ (d) None of these

Sol. (c) $\because$ $f(x)=\dfrac{1+x}{1-x}$

$\Rightarrow$ $(1-x)f(x)=1+x$

$\Rightarrow$ $(I-A)f(A)=(I+A)$

$\Rightarrow$ $\boxed{f(A)=(I-A)^{-1}(I+A)}$

$$=\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}1&2\\2&1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}1&2\\2&1\end{bmatrix}\right)$$

$$=\begin{bmatrix}0&-2\\-2&0\end{bmatrix}^{-1}\begin{bmatrix}2&2\\2&2\end{bmatrix}=-\frac{1}{4}\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}2&2\\2&2\end{bmatrix}$$

$$=-\frac{1}{4}\begin{bmatrix}4&4\\4&4\end{bmatrix}=\begin{bmatrix}-1&-1\\-1&-1\end{bmatrix}$$

• Ex. 7 *The number of solutions of the matrix equation* $X^2=\begin{bmatrix}1&1\\2&3\end{bmatrix}$ *is*

(a) more than 2 (b) 2
(c) 0 (d) 1

Sol. (a) Let $X=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$$\therefore\quad X^2=\begin{bmatrix}a^2+bc&b(a+d)\\c(a+d)&bc+d^2\end{bmatrix}=\begin{bmatrix}1&1\\2&3\end{bmatrix}\quad\text{[given]}$$

$\Rightarrow$ $a^2+bc=1,\ b(a+d)=1,$

$c(a+d)=2$ and $bc+d^2=3$

$\Rightarrow$ $d^2-a^2=2$

$\Rightarrow$ $d-a=\dfrac{2}{d+a}=2b$ and $d+a=\dfrac{1}{b}$

$\therefore$ $2d=2b+\dfrac{1}{b}$ and $2a=\dfrac{1}{b}-2b$

Also, $c=2b$

Now, from $bc+d^2=3$

$\Rightarrow$ $2b^2+\left(b+\dfrac{1}{2b}\right)^2=3\Rightarrow 3b^2+\dfrac{1}{4b^2}-2=0$

$\Rightarrow$ $12b^4-8b^2+1=0$

or $(6b^2-1)(2b^2-1)=0$

$\Rightarrow$ $b=\pm\dfrac{1}{\sqrt6}$ or $b=\pm\dfrac{1}{\sqrt2}$

Therefore, matrices are

$$\begin{bmatrix}0&\frac{1}{\sqrt2}\\\sqrt2&\sqrt2\end{bmatrix},\begin{bmatrix}0&-\frac{1}{\sqrt2}\\-\sqrt2&-\sqrt2\end{bmatrix},\begin{bmatrix}\frac{2}{\sqrt6}&\frac{1}{\sqrt6}\\\frac{2}{\sqrt6}&\frac{4}{\sqrt6}\end{bmatrix}\text{ and }\begin{bmatrix}-\frac{2}{\sqrt6}&-\frac{1}{\sqrt6}\\-\frac{2}{\sqrt6}&-\frac{4}{\sqrt6}\end{bmatrix}.$$

• Ex. 8 *For a matrix* $A=\begin{bmatrix}1&2r-1\\0&1\end{bmatrix}$, *the value of* $\prod_{r=1}^{50}\begin{bmatrix}1&2r-1\\0&1\end{bmatrix}$ *is equal to*

Use this method in multiplicat

(a) $\begin{bmatrix}1&100\\0&1\end{bmatrix}$ (b) $\begin{bmatrix}1&4950\\0&1\end{bmatrix}$ (c) $\begin{bmatrix}1&5050\\0&1\end{bmatrix}$ (d) $\begin{bmatrix}1&2500\\0&1\end{bmatrix}$

Sol. (d) $\prod_{r=1}^{50}\begin{bmatrix}1&2r-1\\0&1\end{bmatrix}=\begin{bmatrix}1&1+3+5+\dots+99\\0&1\end{bmatrix}$

$$=\begin{bmatrix}1&(50)^2\\0&1\end{bmatrix}=\begin{bmatrix}1&2500\\0&1\end{bmatrix}$$

• Ex. 9 *If* $A_1, A_2, A_3, \dots, A_{2n-1}$ *are n skew-symmetric matrices of same order, then* $B=\sum_{r=1}^{n}(2r-1)(A_{2r-1})^{2r-1}$ *will be*

Pg no. 619
By Prop. of Skew n sym. matrix.

(a) symmetric
(b) skew-symmetric
(c) neither symmetric nor skew-symmetric
(d) data not adequate

Sol. (b) $\because\ B = A_1 + 3A_3^3 + 5A_5^5 + \ldots + (2n-1)(A_{2n-1})^{2n-1}$

$\therefore\ B^T = (A_1 + 3A_3^3 + 5A_5^5 + \ldots + (2n-1)(A_{2n-1})^{2n-1})^T$

$= A_1^T + 3(A_3^T)^3 + 5(A_5^T)^5 + \ldots + (2n-1)(A_{2n-1}^T)^{2n-1}$

$= -A_1 + 3(-A_3)^3 + 5(-A_5)^5 + \ldots + (2n-1)(-A_{2n-1})^{2n-1}$

$= -(A_1 + 3A_3^3 + 5A_5^5 + \ldots + (2n-1)A_{2n-1}^{2n-1})$

$= -B$

Hence, B is skew-symmetric.

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

• **Ex. 11** *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *(where* $bc \neq 0$*) satisfies the equations* $x^2 + k = 0$*, then*

(a) $a + d = 0$ (b) $k = -|A|$
(c) $k = |A|$ (d) None of these

Sol. (a, c) We have, $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

As A satisfies $x^2 + k = 0$, therefore

$$A^2 + kI = 0$$

$\Rightarrow \begin{bmatrix} a^2+bc+k & (a+d)b \\ (a+d)c & bc+d^2+k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow\ a^2 + bc + k = 0, (a+d)b = 0,$

$(a+d)c = 0$ and $bc + d^2 + k = 0$

As $\quad bc \neq 0 \Rightarrow b \neq 0, c \neq 0$

So, $\quad a + d = 0 \Rightarrow a = -d$

Also, $k = -(a^2 + bc) = -(-ad + bc) = (ad - bc) = |A|$

• **Ex. 12** *If* $A = [a_{ij}]_{n\times n}$ *and f is a function, we define* $f(A) = [f(a_{ij})]_{n\times n}$*. Let* $A = \begin{bmatrix} \frac{\pi}{2}-\theta & \theta \\ -\theta & \frac{\pi}{2}-\theta \end{bmatrix}$*, then*

(a) $\sin A$ is invertible (b) $\sin A = \cos A$
(c) $\sin A$ is orthogonal (d) $\sin 2A = 2\sin A\cos A$

Sol. (a, c) $\sin A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $\cos\theta = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$\therefore\ |\sin A| = \cos^2\theta + \sin^2\theta = 1 \neq 0$

Hence, $\sin A$ is invertible.

Also, $(\sin A)(\sin A)^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Hence, $\sin A$ is orthogonal.

Also, $2\sin A\cos A = 2\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$= 2\begin{bmatrix} \sin 2\theta & 1 \\ \cos 2\theta & 0 \end{bmatrix} \neq \sin 2A$

• **Ex. 13** *Let A and B are two square idempotent matrices such that* $AB \pm BA$ *is a null matrix, the value of* $\det(A - B)$ *can be equal*

(a) -1 (b) 0
(c) 1 (d) 2

Sol. (a, b, c)

$\because (A - B)^2 = A^2 - AB - BA + B^2$

$= A + B \quad [\because AB + BA = 0 \text{ and } A^2 = A, B^2 = B]$

$\therefore\ |A - B|^2 = |A + B| \quad \ldots(i)$

and $(A + B)^2 = A^2 + AB + BA + B^2$

$= A + B \quad [\because AB + BA = 0 \text{ and } A^2 = A, B^2 = B]$

$\Rightarrow\ |A + B|^2 = |A + B|$

$\Rightarrow\ |A + B|(|A + B| - 1) = 0$

$\therefore\ |A + B| = 0, 1$

• **Ex. 10** *Elements of a matrix A of order* 10×10 *are defined as* $a_{ij} = \omega^{i+j}$ *(where* ω *is cube root of unity), then trace(A) of the matrix is*

(a) 0 (b) 1 (c) 3 (d) None of these

Sol. (d) $\text{tr}(A) = \sum_{i=j=1}^{10} a_{ij} = \sum_{i=j=1}^{10} \omega^{i+j} = \sum_{i=1}^{10} \omega^{2i}$

$= \omega^2 + \omega^4 + \omega^6 + \omega^8 + \ldots + \omega^{20}$

$= (\omega^2 + \omega + 1) + (\omega^2 + \omega + 1) + (\omega^2 + \omega + 1) + \omega^{20}$

$= 0 + 0 + 0 + \omega^2 = \omega^2$

From Eq. (i),

$|A - B|^2 = 0, 1 \Rightarrow |A - B| = 0, \pm 1$

or $\quad \det(A - B) = 0, -1, 1$

• **Ex. 14** *If* $AB = A$ *and* $BA = B$*, then*

(a) $A^2B = A^2$ (b) $B^2A = B^2$
(c) $ABA = A$ (d) $BAB = B$

Sol. (a, b, c, d)

We have, $A^2B = A(AB) = A \cdot A = A^2$,

$B^2A = B(BA) = BB = B^2$,

$ABA = A(BA) = AB = A$, $BAB = B(AB) = BA = B$

• **Ex. 15** *If A is a square matrix of order 3 and I is an Identity matrix of order 3 such that* $A^3 - 2A^2 - A + 2I = 0$*, then A is equal to*

(a) I (b) $2I$ (c) $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Sol. (a, b, d) It is clear that $A = I$ and $A = 2I$ satisfy the given equaion $A^3 - 2A^2 - A + 2I = 0$ and the characteristic equation of the matrix in (c) is

$$\begin{vmatrix} 2-\lambda & -1 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow\ \lambda^3 - 2\lambda^2 + \lambda - 2 = 0,$

giving $\ A^3 - 2A^2 + A - 2I = 0$

$\neq A^3 - 2A^2 - A + 2I = 0$

and the characteristic equation of the matrix in (d) is

$$\begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow\ \lambda^3 - 2\lambda^2 - \lambda + 2 = 0,$

giving $\ A^3 - 2A^2 - A + 2I = 0$

JEE Type Solved Examples : Passage Based Questions

- This section contains **2 solved passages**. Base upon each of the passage **3 multiple choice question** have to be answered. Each of these question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 16 to 18)

If $A_0 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ and $B_0 = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ and $B_n = \text{adj}(B_{n-1}), n \in N$ and I is an identity matrix of order 3.

16. $\det(A_0 + A_0^2 B_0^2 + A_0^3 + A_0^4 B_0^4 + \ldots$ upto 12 terms) is equal to

(a) 1200 (b) -960
(c) 0 (d) -9600

17. $B_2 + B_3 + B_4 + \ldots + B_{50}$ is equal to

(a) B_0 (b) $7B_0$
(c) $49B_0$ (d) $49I$

18. For a variable matrix X, the equation $A_0X = B_0$ will have

(a) unique solution
(b) infinite solution
(c) finitely many solution
(d) no solution

Sol. (Ex. Nos 16 to 18)

$\because\ A_0 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \Rightarrow |A_0| = 0$

and $\ \text{adj}\,B_0 = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = B_0$

$\because\ B_n = \text{adj}(B_{n-1}), n \in N$

$\therefore\ B_1 = \text{adj}(B_0) = B_0$

$\Rightarrow\ B_2 = \text{adj}(B_1) = \text{adj}(B_0) = B_0,$

Similarly $B_3 = B_0, B_4 = B_0, \ldots$

$\therefore\ B_n = B_0\ \forall\ n \in N$

16. (c) $\det(A_0 + A_0^2B_0^2 + A_0^3 + A_0^4B_0^4 + \ldots$ upto 12 terms) $= \det\{A_0(I + A_0B_0^2 + A_0^2 + A_0^3B_0^4 + \ldots$ upto 12 terms)\}

$= |A_0|(I + A_0B_0^2 + A_0^2 + A_0^3B_0^4 + \ldots$ upto 12 terms)

$= 0 \quad [\because |A_0| = 0]$

17. (c) $B_2 + B_3 + B_4 + \ldots + B_{50} = B_0 + B_0 + B_0 + \ldots + B_0 = 49B_0$

18. (d) $\because |A_0| = 0$

$\Rightarrow A_0^{-1}$ is not possible.

Hence, system of equation $A_0X = B_0$ has no Sol.

Passage II
(Ex. Nos. 19 to 21)

Let $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$ satisfies $A^n=A^{n-2}+A^2-I$ for $n\ge 3$ and consider a matrix $\underset{3\times 3}{U}$ with its columns as U_1, U_2, U_3, such that

$$A^{50}U_1=\begin{bmatrix}1\\25\\25\end{bmatrix}, A^{50}U_2=\begin{bmatrix}0\\1\\0\end{bmatrix} \text{ and } A^{50}U_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

19. The value of $|A^{50}|$ equals

(a) −1 (b) 0 (c) 1 (d) 25

20. Trace of A^{50} equals

(a) 0 (b) 1 (c) 2 (d) 3

21. The value of $|U|$ equals

(a) −1 (b) 0 (c) 1 (d) 2

Sol. (Ex. Nos. 19 to 21)

$\because \quad A^n=A^{n-2}+A^2-I \Rightarrow A^{50}=A^{48}+A^2-I$

Further, $A^{48}=A^{46}+A^2-I$

$A^{46}=A^{44}+A^2-I$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$A^4=A^2+A^2-I^4$

On adding all, we get

$$A^{50}=25A^2-24I \quad \text{...(i)}$$

19. (c) $|A^{50}|=|A|^{50}=\begin{vmatrix}1&0&0\\1&0&1\\0&1&0\end{vmatrix}^{50}=(-1)^{50}=1$

20. (d) $\because A^2=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}$

$\therefore A^{50}=25A^2-24I=\begin{bmatrix}25&0&0\\25&25&0\\25&0&25\end{bmatrix}-\begin{bmatrix}24&0&0\\0&24&0\\0&0&24\end{bmatrix}$ [from Eq. (i)]

$=\begin{bmatrix}1&0&0\\25&1&0\\25&0&1\end{bmatrix}$ ...(ii)

Hence, trace of $A^{50}=1+1+1=3$

21. (c) Let $U_1=\begin{bmatrix}x\\y\\z\end{bmatrix}$

Given, $A^{50}U_1=\begin{bmatrix}1\\25\\25\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0\\25&1&0\\25&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\25\\25\end{bmatrix}$ [from Eq. (ii)]

$\Rightarrow \begin{bmatrix}x\\25x+y\\25x+z\end{bmatrix}=\begin{bmatrix}1\\25\\25\end{bmatrix}$, we get $x=1, y=20$ and $z=0$

$\therefore U_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, similarly $U_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$

and $U_3=\begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow U=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

$\therefore |U|=1$

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

• Ex. 22 *Let A be a 3×3 diagonal matrix which commutes with every 3×3 matrix. If $\det(A)=8$, then tr A is*

Sol. (6) Let $A=\begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{bmatrix}$

$\therefore \begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{bmatrix}\begin{bmatrix}a&h&g\\h&b&f\\g&f&c\end{bmatrix}=\begin{bmatrix}a&h&g\\h&b&f\\g&f&c\end{bmatrix}\begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{bmatrix}$

$\Rightarrow \alpha=\beta=\gamma$

$\therefore A=\begin{bmatrix}\alpha&0&0\\0&\alpha&0\\0&0&\alpha\end{bmatrix}$

$\Rightarrow \det(A)=\alpha^3=8$ [given]

$\therefore \alpha=2$

$\therefore A=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$

$\Rightarrow$ tr $A=2+2+2=6$

• Ex. 23 *Let A and B be two non-singular matrices such that $A\ne I$, $B^3=I$ and $AB=BA^2$, where I is the identity matrix, the least value of k such that $A^k=I$ is*

Sol. (7) Given, $AB=BA^2 \Rightarrow B=A^{-1}BA^2 \Rightarrow B^3=I$

$\Rightarrow (A^{-1}BAA)(A^{-1}BAA)(A^{-1}BAA)=I$

$\Rightarrow (A^{-1}BA)(BA)(BAA)=I$ $[\because A^{-1}A=I]$

$\Rightarrow A^{-1}B(AB)(AB)AA=I$

$\Rightarrow A^{-1}B(BA^2)(BA^2)AA=I$ $[\because AB=BA^2]$

$\Rightarrow A^{-1}BBA(AB)A^4=I$

$\Rightarrow A^{-1}BBA(BA^2)A^4=I$ $[\because AB=BA^2]$

$\Rightarrow A^{-1}BB(AB)A^6=I$

$\Rightarrow A^{-1}BB(BA^2)A^6=I$ $[\because AB=BA^2]$

$\Rightarrow A^{-1}B^3A^8=I$

$\Rightarrow (A^{-1}I)A^8=I$ $[\because B^3=I]$

$\Rightarrow A^{-1}A^8=I$

$\Rightarrow A^7=I=A^k$ $[\because A^k=I]$

$\Rightarrow A^k=A^7$

$\therefore$ Least value of k is 7.

JEE Type Solved Examples : Matching Type Questions

- This section contains **2 examples.** Example 24 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II** and example 25 have three statements (A, B and C) given in **Column I** and five statements (p, q, r, s and t). In **Column II** any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

• Ex. 24

	Column I		Column II
(A)	If A is a square matrix of order 3 and $\det(A)=3$, then $\det(6A^{-1})$ is divisible by	(p)	3
(B)	If A is a square matrix of order 3 and $\det(A)=\frac{1}{4}$, then det [adj (adj (2A))] is divisible by	(q)	4
(C)	If A and B are square matrices of odd order and $(A+B)^2=A^2+B^2$, if det $(A)=2$, then det(B) is divisible by	(r)	5
		(s)	6

Sol. (A) → (p, q, s); (B) →(q); (C) →(p, q, r, s)

(A) $\det(6A^{-1})=6^3\det(A^{-1})=\frac{216}{\det(A)}=\frac{216}{3}=72$

(B) $\det[\text{adj}(\text{adj}(2A))]=[\det(2A)]^4=[2^3\det(A)]^4$

$=2^{12}[\det(A)]^4$

$=2^{12}\left(\frac{1}{4}\right)^4=2^4=16$

(C) $\because \quad (A+B)^2=A^2+AB+BA+B^2$

$\Rightarrow A^2+B^2=A^2+AB+BA+B^2$ $[\because (A+B)^2=A^2+B^2]$

$\Rightarrow AB+BA=O \Rightarrow AB=-BA$

$\therefore \det(AB)=\det(-BA)=-\det(BA)$

$\Rightarrow \det(A)\cdot\det(B)=-\det B\cdot\det(A)$

$\Rightarrow 2\det(A)\cdot\det(B)=0 \Rightarrow 4\det(B)=0$ $[\because \det(A)=2]$

$\therefore \det(B)=0$

• Ex. 25

	Column I		Column II
(A)	If $\begin{bmatrix}1&2&a\\0&1&4\\0&0&1\end{bmatrix}^n=\begin{bmatrix}1&18&2007\\0&1&36\\0&0&1\end{bmatrix}$, then $(n+a)$ is divisible by	(p)	4
(B)	If A is a square matrix of order 3 such that $\|A\|=a$, $B=\text{adj}(A)$ and $\|B\|=b$, then $(ab^2+a^2b+1)\lambda$ is divisible by, where $\frac{1}{2}\lambda=\frac{a}{b}+\frac{a^2}{b^3}+\frac{a^3}{b^5}+\ldots$ upto ∞ and $a=3$	(q)	6
(C)	Let $A=\begin{bmatrix}a&b&c\\p&q&r\\1&1&1\end{bmatrix}$ and $B=A^2$. If $(a-b)^2+(p-q)^2=25$, $(b-c)^2+(q-r)^2=36$ and $(c-a)^2+(r-p)^2=49$, then det $\left(\frac{B}{2}\right)$ is divisible by	(r)	10
		(s)	12
		(t)	15

Sol. (A) → (p, r); (B) →(t); (C) → (q, s)

(A) $\because \quad A=\begin{bmatrix}1&2&a\\0&1&4\\0&0&1\end{bmatrix}$

$\therefore A^2=\begin{bmatrix}1&2&a\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&a\\0&1&4\\0&0&1\end{bmatrix}=\begin{bmatrix}1&4&2a+8\\0&1&8\\0&0&1\end{bmatrix}$

$$\Rightarrow \quad A^3 = \begin{bmatrix} 1 & 4 & 2a+8 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 3a+24 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, we get

$$A^n = \begin{bmatrix} 1 & 2n & na + 8\sum_{r=0}^{n-1} r \\ 0 & 1 & 4n \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 18 & 2007 \\ 0 & 1 & 36 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{[given]}$$

$$\Rightarrow \quad 2n = 18 \Rightarrow n = 9$$

$$\therefore \quad na + 8\sum_{r=0}^{n-1} r = 2007 \Rightarrow 9a + 8\sum_{r=0}^{8} r = 2007$$

$$\Rightarrow \quad 9a + 8\cdot\left(\frac{8 \times 9}{2}\right) = 2007 \Rightarrow 9a = 2007 - 288 = 1719$$

$$\therefore \quad a = 191$$

Hence, $n + a = 9 + 191 = 200$

(B) $B = \text{adj } A$

$$\Rightarrow \quad b = |B| = |\text{adj } A| = |A|^2 = a^2 = 9 \Rightarrow a = 3, b = 9$$

and $\quad \frac{1}{2}\lambda = \frac{3}{9} + \frac{3^2}{9^3} + \frac{3^3}{9^5} + \ldots + \infty$

$$= \frac{1}{3} + \frac{1}{81} + \frac{1}{27 \times 81} + \ldots + \infty = \frac{\frac{1}{3}}{1 - \frac{1}{27}} = \frac{9}{26}$$

$$\Rightarrow \quad \lambda = \frac{9}{13}$$

Now, $(ab^2 + a^2b + 1)\lambda = (3 \times 81 + 9 \times 9 + 1) \times \frac{9}{13} = 225$

(C) $\det(A) = \begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix} = 2 \times \frac{1}{2}\begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix}$

$= 2 \times$ Area of the triangle with vertices $(a, p), (b, q)$ and (c, r) with sides 5, 6, 7

$$= 2 \times \sqrt{s(s-a)(s-b)(s-c)} = 2 \times 6\sqrt{6} = 12\sqrt{6}$$

Hence, $\det\left(\frac{B}{2}\right) = \left(\frac{1}{2}\right)^3 \det(B) = \frac{1}{8}\det(A^2)$

$$= \frac{1}{8}(\det A)^2 = \frac{1}{8}(12\sqrt{6})^2 = 108$$

JEE Type Solved Examples : Statement I and II Type Questions

- **Direction** example numbers 26 and 27 are Assertion-Reason type examples. Each of these examples contains two statements:

Statement-1 (Assertion) and **Statement-2** (Reason)

Each of these examples also has four alternative choices, **ONLY ONE** of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

Ex. 26 Statement-1 *A is singular matrix of order $n \times n$, then adj A is singular.*

Statement-2 $|\text{adj } A| = |A|^{n-1}$

Sol. (*d*) If A is non-singular matrix of order $n \times n$, then

$$|\text{adj } A| = |A|^{n-1}$$

Hence, Statement-1 is false and Statement-2 is true.

Ex. 27 Statement-1 *If A and B are two matrices such that $AB = B$, $BA = A$, then $A^2 + B^2 = A + B$.*

Statement-2 *A and B are idempotent matrices, then $A^2 = A$, $B^2 = B$.*

Sol. (*b*) $\because AB = B$

$$\Rightarrow \quad B(AB) = B \cdot B$$

$$\Rightarrow \quad (BA)B = B^2 \quad \text{[by associative law]}$$

$$\Rightarrow \quad AB = B^2 \quad [\because BA = A]$$

$$\Rightarrow \quad B = B^2 \quad [\because AB = B]$$

and $\quad BA = A$

$$\Rightarrow \quad A(BA) = A \cdot A$$

$$\Rightarrow \quad (AB)A = A^2 \quad \text{[by associative law]}$$

$$\Rightarrow \quad BA = A^2 \quad [\because AB = B]$$

$$\Rightarrow \quad A = A^2 \quad [\because BA = A]$$

Hence, $\therefore \quad A^2 + B^2 = A + B$

Here, both statements are true and Statement-2 is not a correct explanation for Statement-1.

Matrices Exercise 8 :
Questions Asked in Previous 13 Year's Exam

This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year 2005 to year **2017.**

98. $A=\begin{bmatrix}1&0&0\\0&1&1\\0&-2&4\end{bmatrix}$; $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ and

$A^{-1}=\frac{1}{6}[A^2+cA+dI]$ where $c, d\in R$, the pair of values (c, d) **[IIT- JEE 2005, 3M]**

(a) $(6, 11)$ (b) $(6, -11)$ (c) $(-6, 11)$ (d) $(-6, -11)$

99. If $P=\begin{bmatrix}\frac{\sqrt{3}}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{\sqrt{3}}{2}\end{bmatrix}$, $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ and $Q=PAP^T$, the $P^T(Q^{2005})P$ equal to **[IIT- JEE 2005, 3M]**

(a) $\begin{bmatrix}1&2005\\0&1\end{bmatrix}$ (b) $\begin{bmatrix}\sqrt{3}/1&2005\\1&0\end{bmatrix}$

(c) $\begin{bmatrix}1&2005\\\sqrt{3}/2&1\end{bmatrix}$ (d) $\begin{bmatrix}1&\sqrt{3}/2\\0&2005\end{bmatrix}$

100. If $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, which one of the following holds for all $n\geq 1$, (by the principal of mathematical induction) **[AIEEE 2005, 3M]**

(a) $A^n=nA+(n-1)I$ (b) $A^n=2^{n-1}A+(n-1)I$

(c) $A^n=nA-(n-1)I$ (d) $A^n=2^{n-1}A-(n-1)I$

101. If $A^2-A+I=0$, then A^{-1} is equal to **[AIEEE 2005, 3M]**

(a) A^{-2} (b) $A+I$ (c) $I-A$ (d) $A-I$

102. If $A=\begin{pmatrix}1&0&0\\2&1&0\\3&2&1\end{pmatrix}$, U_1, U_2 and U_3 are column matrices satisfying $AU_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $AU_2=\begin{pmatrix}2\\3\\0\end{pmatrix}$ and $AU_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$ and U is 3×3 matrix when columns are U_1, U_2, U_3, then answer the following questions

(i) The value of $|U|$ is

(a) 3 (b) -3 (c) 3/2 (d) 2

(ii) The sum of the elements of U^{-1} is

(a) -1 (b) 0 (c) 1 (d) 3

(iii) The value of $(3\ 2\ 0)\ U\begin{pmatrix}3\\2\\0\end{pmatrix}$ is **[IIT- JEE 2006, 5+5+5M]**

(a) 5 (b) 5/2 (c) 4 (d) 3/2

103. Let $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $B=\begin{pmatrix}a&0\\0&b\end{pmatrix}$, $a, b\in N$. Then, **[AIEEE 2006, 4½M]**

(a) there cannot exist any B such that $AB=BA$

(b) there exist more than one but finite number of B 's such that $AB=BA$

(c) there exists exactly one B such that $AB=BA$

(d) there exist infinitely among B 's such that $AB=BA$

104. If A and B are square matrices of size $n\times n$ such that $A^2-B^2=(A-B)(A+B)$, which of the following will be always true? **[AIEEE 2006, 3M]**

(a) $A=B$ (b) $AB=BA$

(c) Either of A or B is a zero matrix

(d) Either of A or B is identity matrix

105. Let $A=\begin{bmatrix}5&5\alpha&\alpha\\0&\alpha&5\alpha\\0&0&5\end{bmatrix}$. If $|A^2|=25$, then $|\alpha|$ equals to **[AIEEE 2007, 3M]**

(a) 5^2 (b) 1 (c) 1/5 (d) 5

106. Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric and $(A+B)(A-B)=(A-B)(A+B)$. If $(AB)^t=(-1)^k\ AB$, where $(AB)^t$ is the transpose of matrix AB, the value of k is **[IIT- JEE 2008, 1½M]**

(a) 0 (b) 1 (c) 2 (d) 3

107. Let A be a square matrix all of whose entries are integers. Which one of the following is true? **[AIEEE 2008, 3M]**

(a) If det $A\neq\pm 1$, then A^{-1} exists and all its entries are non-integers

(b) If det $A=\pm 1$, then A^{-1} exists and all its entries are integers

(c) If det $A=\pm 1$, then A^{-1} need not exist

(d) If det $A=\pm 1$, then A^{-1} exists but all its entries are not necessarily integers

108. Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denote by tr(A), the sum of diagonal entries of A. Assume that $A^2 = I$. **[AIEEE 2008, 3M]**

Statement-1 If $A \neq I$ and $A \neq -I$, then det $A = -1$.

Statement-2 If $A \neq I$ and $A \neq -1$, then tr$(A) \neq 0$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

109. Let A be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0. **[IIT- JEE 2009, 4+4+4M]**

(i) The number of matrices in A is
(a) 12 (b) 6
(c) 9 (d) 3

(ii) The number of matrices A for which the system of linear equations $A\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ has a unique solution, is
(a) less than 4 (b) atleast 4 but less than 7
(c) atleast 7 but less than 10
(d) atleast 10

(iii) The number of matrices A in which the system of linear equations $A\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ is inconsistent is
(a) 0 (b) more than 2
(c) 2 (d) 1

110. Let A be a 2×2 matrix

Statement-1 adj (adj A) = A

Statement-2 | adj A | = | A | **[AIEEE 2009, 4M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

111. The number of 3×3 matrices A whose are either 0 or 1 and for which the system $A\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ has exactly two distinct solutions, is **[IIT- JEE 2010, 3M]**
(a) 0 (b) 2^9-1
(c) 168 (d) 2

112. Let p be an odd prime number and T_p be the following set of 2×2 matrices.

$$T_p=\left\{A=\begin{bmatrix}a & b\\ c & a\end{bmatrix};\ a,b,c\in\{0,1,2,\dots,p-1\}\right\}$$

[IIT- JEE 2010, 3+3+3M]

(i) The number of A in T_p such that A is either symmetric or skew-symmetric or both and det (A) divisible by p, is
(a) $(p-1)^2$ (b) $2(p-1)$
(c) $(p-1)^2+1$ (d) $2p-1$

(ii) The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p, is
[**Note** The trace of a matrix is the sum of its diagonal entries]
(a) $(p-1)(p^2-p+1)$ (b) $p^3-(p-1)^2$
(c) $(p-1)^2$ (d) $(p-1)(p^2-2)$

(iii) The number of A in T_p such that det (A) is not divisible by p, is
(a) $2p^2$ (b) p^3-5p (c) p^3-3p (d) p^3-p^2

113. Let k be a positive real number and let

$$A=\begin{bmatrix}2k-1 & 2\sqrt{k} & 2\sqrt{k}\\ 2\sqrt{k} & 1 & -2k\\ -2\sqrt{k} & 2k & -1\end{bmatrix} \text{ and } \begin{bmatrix}0 & 2k-1 & \sqrt{k}\\ 1-2k & 0 & 2\sqrt{k}\\ -\sqrt{k} & -2\sqrt{k} & 0\end{bmatrix}.$$

If det (adj A) + det (adj B) = 10^6, then [k] is equal to **[IIT- JEE 2010, 3M]**

Note adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k}.

114. The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is **[AIEEE 2010, 8M]**
(a) 5 (b) 6
(c) atleast 7 (d) less than 4

115. Let A be a 2×2 matrix with non-zero entries and let $A^2 = I$, where I is 2×2 identity matrix. Define Tr(A) = sum of diagonal elements of A and | A | = determinant of matrix A.

Statement-1 Tr$(A) = 0$

Statement-2 | A | = 1 **[AIEEE 2010, 4M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

116. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to **[IIT- JEE 2011, 4M]**
(a) M^2 (b) $-N^2$ (c) $-M^2$ (d) MN

117. Let a, b and c be three real numbers satisfying

$$[a\ b\ c]\begin{bmatrix}1 & 9 & 7\\ 8 & 2 & 7\\ 7 & 3 & 7\end{bmatrix}=[0\ 0\ 0] \qquad \dots(E)$$

(i) If the point $P(a,b,c)$, with reference to (E), lies on the plane $2x+y+z=1$, then the value of $7a+b+c$ is
(a) 0 (b) 12 (c) 7 (d) 6

(ii) Let ω be a solution of $x^3-1=0$ with Im$(\omega)>0$. If $a=2$ with b and c satisfying (E), the value of $\dfrac{3}{\omega^a}+\dfrac{1}{\omega^b}+\dfrac{3}{\omega^c}$ is equal to
(a) -2 (b) 2 (c) 3 (d) -3

(iii) Let $b=6$ with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2+bx+c=0$, then $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is **[IIT- JEE 2011, 3+3+3M]**
(a) 6 (b) 7
(c) $\frac{6}{7}$ (d) ∞

118. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix}1 & a & b\\ \omega & 1 & c\\ \omega^2 & \omega & 1\end{bmatrix}$, where each of a, b and c is either ω or ω^2. The number of distinct matrices in the set S is **[IIT- JEE 2011, 3M]**
(a) 2 (b) 6
(c) 4 (d) 8

119. Let M be a 3×3 matrix satisfying $M\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$,

$M\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$ and $M\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\12\end{bmatrix}$.

The sum of the diagonal entries of M is **[IIT- JEE 2011, 4M]**

120. Let A and B are symmetric matrices of order 3.

Statement-1 $A(BA)$ and $(AB)A$ are symmetric matrices.

Statement-2 AB is symmetric matrix, if matrix multiplication of A with B is commutative.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is false
(c) Statement-1 is false, Statement-2 is true
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1 **[AIEEE 2011, 4M]**

121. Let $P=[a_{ij}]$ be a 3×3 matrix and $Q=[b_{ij}]$, where $b_{ij}=2^{i+j}a_{ij}$ for $1\le i,j\le3$. If the determinant of P is 2, the determinant of the matrix Q is **[IIT- JEE 2012, 3M]**
(a) 2^{11} (b) 2^{12}
(c) 2^{13} (d) 2^{10}

122. If P is a 3×3 matrix such that $P^T=2P+I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X=\begin{bmatrix}x\\y\\z\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\end{bmatrix}$ such that **[IIT- JEE 2012, 3M]**
(a) $PX=\begin{bmatrix}0\\0\\0\end{bmatrix}$ (b) $PX=X$ (c) $PX=2X$ (d) $PX=-X$

123. If the adjoint of a 3×3 matrix P is $\begin{bmatrix}1 & 4 & 4\\ 2 & 1 & 7\\ 1 & 1 & 3\end{bmatrix}$, then the possible value(s) of the determinant of P is (are) **[IIT- JEE 2012, 4M]**
(a) -2 (b) -1 (c) 1 (d) 2

124. If $A=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{pmatrix}$, u_1 and u_2 are the column matrices such that $Au_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $Au_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$, then u_1+u_2 is equal to **[AIEEE 2012, 4M]**
(a) $\begin{pmatrix}-1\\-1\\0\end{pmatrix}$ (b) $\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ (c) $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ (d) $\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

125. Let P and Q be 3×3 matrices with $P\neq Q$. If $P^3=Q^3$ and $P^2Q=Q^2P$, the determinant of (P^2+Q^2) is equal to **[AIEEE 2012, 4M]**
(a) 0 (b) -1 (c) -2 (d) 1

126. If $P=\begin{bmatrix}1 & \alpha & 3\\ 1 & 3 & 3\\ 2 & 4 & 4\end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A|=4$, then α is equal to **[JEE Main 2013, 4M]**
(a) 11 (b) 5 (c) 0 (d) 4

127. For 3×3 matrices M and N, which of the following statement(s) is (are) **not** correct?
(a) N^TMN is symmetric or skew-symmetric, according as M is symmetric or skew-symmetric
(b) $MN-NM$ is skew-symmetric for all symmetric matrices M and N
(c) MN is symmetric for all symmetric matrices M and N
(d) (adj M) (adj N) = adj (MN) for all invertible matrices M and N **[JEE Advanced 2013, 4M]**

128. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then, $P^2 \neq 0$, when n is equal to **[JEE Advanced 2013, 3M]**

(a) 55 (b) 56 (c) 57 (d) 58

129. If A is a 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' equals to **[JEE Main 2014, 4M]**

(a) B^{-1} (b) $(B^{-1})'$ (c) $I + B$ (d) I

130. Let M be a 2×2 symmetric matrix with integer entries. Then, M is invertible, if

(a) the first column of M is the transpose of the second row of M

(b) the second row of M is the transpose of the first column of M

(c) m is a diagonal matrix with non-zero entries in the main diagonal

(d) the product of entries in the main diagonal of M is not the square of an integer **[JEE Advanced 2014, 3M]**

131. Let M and N be two 3×3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(a) determinant of $(M^2 + MN^2)$ is 0

(b) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix

(c) determinant of $(M^2 + MN^2) \geq 1$

(d) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix, then U is the zero matrix

[JEE Advanced 2014, 3M]

132. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to **[JEE Main 2015, 4M]**

(a) $(2, 1)$ (b) $(-2, -1)$ (c) $(2, -1)$ (d) $(-2, 1)$

133. Let X and Y be two arbitrary 3×3 non-zero, skew-symmetric matrices and Z be an arbitrary 3×3 non-zero, symmetric matrix. Then, which of the following matrices is (are) skew-symmetric?

[JEE Advanced 2015, 4M]

(a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$

(c) $X^4Z^3 - Z^3X^4$ (d) $X^{23} + Y^{23}$

134. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \operatorname{adj} A = AA^T$, then $5a + b$ is equal to **[JEE Main 2016, 4M]**

(a) 5 (b) 13

(c) 4 (d) -1

135. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in R, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then **[JEE Advanced 2016, 4M]**

(a) $\alpha = 0, k = 8$ (b) $4\alpha - k + 8 = 0$

(c) $\det(P \operatorname{adj}(Q)) = 2^9$ (d) $\det(Q \operatorname{adj}(P)) = 2^{13}$

136. Let $z = \frac{-1 + \sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s = \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of oreder 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is **[JEE Advanced 2016, 3M]**

(a) $\frac{1}{2}|\mathbf{a} - \mathbf{b}|$ (b) $\frac{1}{2}|\mathbf{a} + \mathbf{b}|$

(c) $|\mathbf{a} - \mathbf{b}|$ (d) $|\mathbf{a} + \mathbf{b}|$

137. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals **[JEE Advanced 2016, 3M]**

(a) 52 (b) 103

(c) 201 (d) 205

138. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then $\operatorname{adj}(3A^2 + 12A)$ is equal to **[JEE Main 2017, 4M]**

(a) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ (b) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$

(c) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (d) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

Answers

Exercise for Session 1

1. (b) 2. (b) 3. (d) 4. (b) 5. (a) 6. (b) 7. (b) 8. (b) 9. (c)

Exercise for Session 2

1. (d) 2. (a) 3. (c) 4. (a) 5. (b) 6. (c) 7. (d) 8. (b) 9. (d) 10. (c) 11. (b) 12. (d) 13. (b) 14. (a) 15. (b) 16. (b) 17. (c) 18. (b) 19. (b)

Exercise for Session 3

1. (d) 2. (c) 3. (b) 4. (d) 5. (a) 6. (b) 7. (d) 8. (c) 9. (a) 10. (d) 11. (c) 12. (a) 13. (d) 14. (b) 15. (a) 16. (d)

Exercise for Session 4

1. (a) 2. (a) 3. (d) 4. (d) 5. (d) 6. (d) 7. (d) 8. (b) 9. (d)

Chapter Exercises

1. (d) 2. (c) 3. (b) 4. (d) 5. (c) 6. (b) 7. (a) 8. (a) 9. (a) 10. (b) 11. (b) 12. (b) 13. (b) 14. (d) 15. (d) 16. (b) 17. (b) 18. (d) 19. (d) 20. (d) 21. (c) 22. (d) 23. (c) 24. (d) 25. (b) 26. (a) 27. (d) 28. (b) 29. (a) 30. (c) 31. (a, d) 32. (a, b, d) 33. (a, b, d) 34. (b, c) 35. (b, d) 36. (a, b, c) 37. (a, c, d) 38. (a, d) 39. (a,c,d) 40. (a, c) 41. (a, c, d) 42. (a,b,c,d) 43. (c, d) 44. (a, b, c) 45. (a, c) 46. (b) 47. (b) 48. (c) 49. (b) 50. (d) 51. (d) 52. (d) 53. (c) 54. (d) 55. (a) 56. (c) 57. (a) 58. (b) 59. (c) 60. (a) 61. (a) 62. (3) 63. (2) 64. (1) 65. (9) 66. (2) 67. (7)

68. (9) 69. (1) 70. (2) 71. (6)

72. (A) $\to$ (p, r); (B) $\to$ (s); (C) $\to$ (q); (D) $\to$ (s)

73. (A) $\to$ (r, t); (B) $\to$ (s); (C) $\to$ (p); (D) $\to$ (q)

74. (A) $\to$ (q, s); (B) $\to$ (p, t); (C) $\to$ (p, q, r, s); (D) $\to$ (q, s)

75. (A) $\to$ (q, t); (B) $\to$ (p, s); (C) $\to$ (p, r, s); (D) $\to$ (q, r, t)

76. (d) 77. (c) 78. (d) 79. (c) 80. (a) 81. (d) 82. (d) 83. (d) 84. (d) 85. (a)

88. $\alpha = \frac{2\pi}{3}$ 89. $\begin{bmatrix} \cos\theta\cos\phi\cos(\theta - \phi) & \cos\theta\sin\phi\cos(\theta - \phi) \\ \sin\theta\cos\phi\cos(\theta - \phi) & \sin\theta\sin\phi\cos(\theta - \phi) \end{bmatrix}$

91. (i) Number of posts in all the offices taken together are 5 office superintendents; 235 head clerks; 235 cashiers; 275 clerks; 5 typists and 270 peons.
(ii) Total basic monthly salary bill of each division or district and taluka offices an ₹1675, ₹875 and ₹625, respectively.
(iii) Total basic monthly salary bill of all the offices taken together is ₹ 159625.

92. ₹53000; ₹44500; ₹34000, respectively

94. $x = 1, u = -1, y = 3, v = 2, z = 5, w = 1$

95. $x_1 = z_1 - 2z_2 + 9z_3, x_2 = 9z_1 + 10z_2 + 11z_3, x_3 = 7z_1 + z_2 - 2z_3$

96. (i) $k \neq 7$ (ii) $k = 7$

98. (c) 99. (a) 100. (c) 101. (c) 102. (i) (a), (ii) (b), (iii) (a) 103. (b) 104. (b) 105. (c) 106. (b,d) 107. (d) 108. (c)

109. (i) (a), (ii) (b), (iii) (b) 110. (b) 111. (a)

112. (i) (d), (ii) (c), (iii) (d) 113. (4) 114. (c) 115. (b) 116. (c)

117. (i) (d), (ii) (a), (iii) (b)

118. (a) 119. (9) 120. (a) 121. (c) 122. (d) 123. (a, d) 124. (b) 125. (a) 126. (a) 127. (c,d) 128. (a,b,d) 129. (d) 130. (c, d) 131. (a, b) 132. (b) 133. (c, d) 134. (a) 135. (b,c) 136. (1) 137. (b) 138. (c)