CHAPTER

01

Coordinate System and Coordinates

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Session 1

Introduction, Coordinate Axes, Rectangular Cartesian Coordinates of a Point, Polar Coordinates of a Point, Relation between the Polar and Cartesian Coordinates

Introduction

The great philospher and mathematician of France **Rane Descartes** (1596-1665) published a book 'La Geometric' in 1637.

Descartes gave a new idea i.e. each point in a plane is expressed by an ordered pair of algebraic real numbers like $(x, y), (r, \theta)$ etc., called coordinates of the point.

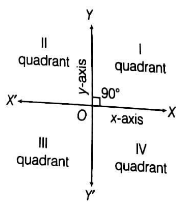
The point (x, y) is called cartesian coordinates and (r, θ) is called polar coordinates of the point. Then represents different forms of equations which are developed for all types of straight lines and curves.

Thus the Coordinate Geometry (or Analytical Geometry) is that branch of mathematics in which geometrical problems are solved with the help of Algebra.

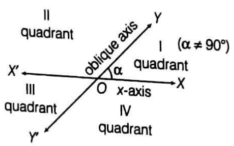
Coordinate Axes

The position of a point in a plane is determined with reference to two intersecting straight lines called the **coordinate axes** and their point of intersection is called the **origin** of coordinates.

If these two axes of reference (generally we call them x and y axes) cut each other at right angle, they are called **rectangular axes** otherwise they are called **oblique axes**. The axes divide the coordinate plane in four quadrants.



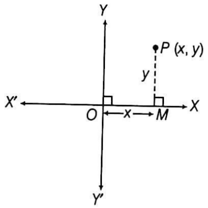
(a) Rectangular axes



(b) Oblique axes

Rectangular Cartesian Coordinates of a Point

Let $X'OX$ and $Y'OY$ be two perpendicular axes in the plane of paper intersecting at O. Let P be any point in the plane of the paper. Draw PM perpendicular to OX. Then the lengths OM and PM are called the rectangular cartesian coordinates or briefly the coordinates of P.



Let $\qquad OM = x$ and $MP = y$

Then, the position of the point P in the plane with respect to the coordinate axes is represented by the ordered pair (x, y). The ordered pair (x, y) is called the coordinates of point P.

i.e. $\qquad OM = x$-coordinate or abscissa of the point P

and $\qquad MP = y$-coordinate or ordinate of the point P.

Remarks

1. The ordinate of every point on X-axis is 0.
2. The abscissa of every point on Y-axis is 0.
3. The abscissa and ordinate of the origin $O(0, 0)$ are both zero.
4. The abscissa and ordinate of a point are at perpendicular distance from Y-axis and X-axis respectively.
5. Table for conversion sign of coordinates :

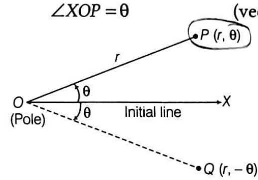
Quadrants	XOY (I)	$X'OY$ (II)	$X'OY'$ (III)	XOY' (IV)
Sign of x coordinates	+	−	−	+
Sign of y coordinates	+	+	−	−
Sign of (x, y)	$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$

6. Equation of X-axis, $y = 0$ and equation of Y-axis, $x = 0$.

Polar Coordinates of a Point

If $\qquad OP = r \qquad$ (radius vector)

and $\qquad \angle XOP = \theta \qquad$ (vectorial angle)



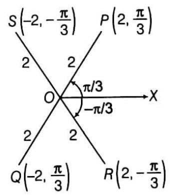
Then, the ordered pair of real numbers (r, θ) called the polar coordinates of the point P.

Remarks

1. r may be positive or negative according as θ is measured in anticlockwise or clockwise direction. θ lies between $-\pi$ to π i.e. $-\pi < \theta \le \pi$. If it is greater than π, then we subtract 2π from it and if it is less than $-\pi$, then we add 2π, to it. It is also known as principal value of P.
2. Always taken θ in radian.

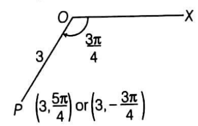
Example 1. Draw the polar coordinates $\left(2, \frac{\pi}{3}\right), \left(-2, \frac{\pi}{3}\right), \left(-2, -\frac{\pi}{3}\right)$ and $\left(2, -\frac{\pi}{3}\right)$ on the plane.

Sol.



Example 2. Draw the polar coordinate $\left(3, \frac{5\pi}{4}\right)$ on the plane.

Sol. Here, $\qquad \theta = \frac{5\pi}{4} > \pi$



then, $\qquad \theta - 2\pi = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$

$\boxed{\therefore \left(3, \frac{5\pi}{4}\right) \text{ is same } \left(3, -\frac{3\pi}{4}\right)}$

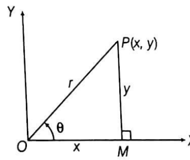
Relation between the Polar and Cartesian Coordinates

Let $P(x, y)$ be the cartesian coordinates with respect to axes OX and OY and (r, θ) be its polar coordinates with respect to pole O and initial line OX.

It is clear from figure

$$OM = x = r\cos\theta \qquad \dots(i)$$

and
$$MP = y = r\sin\theta \qquad \dots(ii)$$



Squaring and adding Eqs. (i) and (ii), we get

$$x^2 + y^2 = r^2 \quad \text{or} \quad r = \sqrt{(x^2 + y^2)}$$

Dividing Eqs. (ii) by (i), then

$$\tan\theta = \left(\frac{y}{x}\right) \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

i.e. $\qquad (r\cos\theta, r\sin\theta) \Rightarrow (x, y) \qquad \dots(iii)$

and $\qquad \left(\sqrt{(x^2 + y^2)}, \tan^{-1}\left(\frac{y}{x}\right)\right) \Rightarrow (r, \theta) \qquad \dots(iv)$

If r and θ are known then we can find (x, y) from Eq. (iii) and if x and y are known then we can find (r, θ) from Eq. (iv).

$\because \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

If $\qquad \alpha = \tan^{-1}\left|\frac{y}{x}\right|$

Then, values of θ in four quadrants Remember

Quadrant	I	II	III	IV
θ	α	$\pi - \alpha$	$-\pi + \alpha$	$-\alpha$

Example 3. Find the cartesian coordinates of the points whose polar coordinates are

(i) $\left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$ $\qquad$ (ii) $\left(5\sqrt{2}, \frac{\pi}{4}\right)$

Sol. (i) Given, $r = 5, \theta = \pi - \tan^{-1}\left(\frac{4}{3}\right)$

Now, $x = r\cos\theta = 5\cos\left(\pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$

$= -5\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

$= -5\cos\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = -5 \times \frac{3}{5} = -3$

and $y = r\sin\theta = 5\sin\left(\pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$

$= 5\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

$= 5\sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right) = 5 \times \frac{4}{5} = 4$

Hence, cartesian coordinates of the given point will be $(-3, 4)$.

(ii) Given, $r = 5\sqrt{2}, \theta = \frac{\pi}{4}$

Now, $x = r\cos\theta = 5\sqrt{2}\cos\left(\frac{\pi}{4}\right) = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$

and $y = r\sin\theta = 5\sqrt{2}\sin\left(\frac{\pi}{4}\right) = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$

Hence, cartesian coordinates of the given point, will be $(5, 5)$.

| Example 4. Find the polar coordinates of the points whose cartesian coordinates are

(i) $(-2, -2)$ (ii) $(-3, 4)$

Sol. (i) Given, $x = -2, y = -2$

$\therefore$ $r = \sqrt{(x^2 + y^2)} = \sqrt{(4 + 4)} = 2\sqrt{2}$

and $\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-2}{-2}\right| = \tan^{-1} 1 = \frac{\pi}{4}$

Since, point $(-2, -2)$ lies in III quadrant.

$\therefore$ $\theta = -\pi + \alpha = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

Hence, polar coordinates of the given points will be $\left(2\sqrt{2}, -\frac{3\pi}{4}\right)$.

Remark : If we find θ, from the equation,

$\tan\theta = \frac{y}{x} = \frac{-2}{-2} = 1$

then, $\theta = \frac{\pi}{4}$ and then

$(x, y) = (r\cos\theta, r\sin\theta) = \left(2\sqrt{2} \times \frac{1}{\sqrt{2}}, 2\sqrt{2} \times \frac{1}{\sqrt{2}}\right)$

$= (2, 2) \neq (-2, -2)$

(ii) Given, $x = -3, y = 4$

$\therefore$ $r = \sqrt{(x^2 + y^2)} = \sqrt{(9 + 16)} = 5$

and $\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{4}{-3}\right| = \tan^{-1}\left(\frac{4}{3}\right)$

Since, point $(-3, 4)$ lies in II quadrant

$\therefore$ $\theta = \pi - \alpha = \pi - \tan^{-1}\left(\frac{4}{3}\right)$

Hence, polar coordinates of the given points, will be $\left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$.

| Example 5. Transform the equation $r^2 = a^2\cos 2\theta$ into cartesian form.

Sol. $\because$ $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

or $r^2 = (x^2 + y^2)$ and $\tan\theta = \frac{y}{x}$

Given, $r^2 = a^2\cos 2\theta = a^2\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)$

or $(x^2 + y^2) = a^2\left(\dfrac{1 - \frac{y^2}{x^2}}{1 + \frac{y^2}{x^2}}\right)$

or $(x^2 + y^2)^2 = a^2(x^2 - y^2)$

This is the required equation in cartesian form.

Aliter :

$r^2 = a^2\cos 2\theta$

or $r^2 = a^2(\cos^2\theta - \sin^2\theta)$

$\because$ $x = r\cos\theta$ and $y = r\sin\theta$

and $r^2 = x^2 + y^2$

then $r^2 = a^2\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$

or $r^4 = a^2(x^2 - y^2)$

or $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

| Example 6. Transform the equation $x^2 + y^2 = ax$ into polar form.

Sol. $\because$ $x = r\cos\theta, y = r\sin\theta$

Given, $x^2 + y^2 = ax$

$\Rightarrow$ $r^2 = a(r\cos\theta)$

or $r = a\cos\theta$

This is the required equation in polar form.

Exercise for Session 1

1. The polar coordinates of the point whose cartesian coordinates are $(-1, -1)$ is

(a) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ (b) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$ (c) $\left(\sqrt{2}, -\frac{\pi}{4}\right)$ (d) $\left(\sqrt{2}, -\frac{3\pi}{4}\right)$

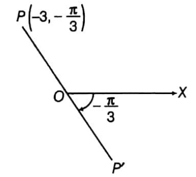
2. The cartesian coordinates of the point whose polar coordinates are $\left(13, \pi - \tan^{-1}\left(\frac{5}{12}\right)\right)$ is

(a) $(12, 5)$ (b) $(-12, 5)$ (c) $(-12, -5)$ (d) $(12, -5)$

3. The transform equation of $r^2\cos^2\theta = a^2\cos 2\theta$ to cartesian form is $(x^2 + y^2)x^2 = a^2\lambda$, then value of λ is

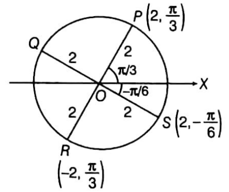
(a) $y^2 - x^2$ (b) $x^2 - y^2$ (c) xy (d) x^2y^2

4. The coordinates of P' in the figure is



(a) $\left(3, \frac{\pi}{3}\right)$ (b) $\left(3, -\frac{\pi}{3}\right)$ (c) $\left(-3, -\frac{\pi}{3}\right)$ (d) $\left(-3, \frac{\pi}{3}\right)$

5. The cartesian coordinates of the point Q in the figure is



(a) $(\sqrt{3}, 1)$ (b) $(-\sqrt{3}, 1)$ (c) $(-\sqrt{3}, -1)$ (d) $(\sqrt{3}, -1)$

6. A point lies on X-axis at a distance 5 units from Y-axis. What are its coordinates ?

7. A point lies on Y-axis at a distance 4 units from X-axis. What are its coordinates ?

8. A point lies on negative direction of X-axis at a distance 6 units from Y-axis. What are its coordinates ?

9. Transform the equation $y = x\tan\alpha$ to polar form.

10. Transform the equation $r = 2a\cos\theta$ to cartesian form.

Session 2

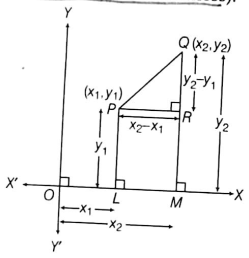
Distance between Two Points, Choice of Axes, Distance between Two Points in Polar Coordinates

Distance Between Two Points

Theorem : The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points in the plane. Let us assume that the points P and Q are both in 1st quadrant (for the sake of exactness).



From P and Q draw PL and QM perpendiculars to X-axis. From P draw PR perpendicular to QM and join PQ. Then

$OL = x_1$, $OM = x_2$, $PL = y_1$, $QM = y_2$

$\therefore$ $PR = LM = OM - OL = x_2 - x_1$

and $QR = QM - RM = QM - PL = y_2 - y_1$

Since, PRQ is a right angled triangle, therefore by pythagoras theorem.

$$(PQ)^2 = (PR)^2 + (QR)^2$$

$\therefore$ $|PQ| = \sqrt{(PR)^2 + (QR)^2}$ ($\because$ PQ is always positive)

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$\therefore$ The distance PQ between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

or $\sqrt{\text{(difference in x coordinates)}^2 + \text{(difference in y coordinates)}^2}$

or $\sqrt{\text{(difference of abscissaes)}^2 + \text{(difference of ordinates)}^2}$

Notations : We shall denote the distance between two points P and Q of the coordinate plane, either by $|PQ|$ or by $\overline{PQ}$.

***Corollary* 1 :** The above formula is true for all positions of the points (i.e. either point or both points are not in the 1st quadrant) keeping in mind, the proper signs of their coordinates.

***Corollary* 2 :** The distance of the point $P(x, y)$ from the origin $O(0, 0)$ is given by

$$|OP| = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{(x^2 + y^2)}$$

***Corollary* 3 :** The above formula can also be used as

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

***Corollary* 4 :** (i) If PQ is parallel to X-axis, then $y_1 = y_2$ and so

$$|PQ| = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$$

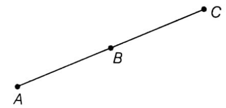
(ii) If PQ is parallel to Y-axis, then $x_1 = x_2$ and so

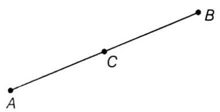
$$|PQ| = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|$$

***Corollary* 5 :** If distance between two points is given, then use $\pm$ sign.

Remarks

1. If three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, then
$$|AB| \pm |BC| = |AC|$$





2. When three points are given and it is required to :
   (i) an **Isosceles triangle**, show that two of its sides (or two angles) are equal.

   (ii) an **Equilateral triangle**, show that its all sides are equal or each angle is of 60°.
   (iii) a **Right angle triangle**, show that the sum of the squares of two sides is equal to the square of the third side.
   (iv) an **Isosceles right angled triangle**, show that two of its sides are equal and the sum of the squares of two equal sides is equal to the square of the third side.
   (v) a **Scalene triangle**, show that its all sides are unequal.
3. When four points are given and it is required to
   (i) a **Square**, show that the four sides are equal and the diagonals are also equal.
   (ii) a **Rhombus, (or equilateral trapezium)** show that the four sides are equal and the diagonals are not equal.
   (iii) a **Rectangle**, show that the opposite sides are equal and the diagonals are also equal.
   (iv) a **Parallelogram**, show that the opposite sides are equal and the diagonals are not equal.
   (v) a **Trapezium**, show that the two sides are parallel and the other two sides are not parallel.
   (vi) An Isosceles Trapezium, show that the two sides are parallel and the other two sides are not parallel but equal.
4. If A, B, C be the vertices of a triangle and we have to find the coordinates of the circumcentre then, let the circumcentre be $P(x, y)$ and use $PA^2 = PB^2$ and $PA^2 = PC^2$ this will give two equations in x and y then solve these two equations and (x, y).

Important Remarks for Objective Questions

(i) If (x_1, y_1) and (x_2, y_2) are the ends of the hypotenuse of a right angled isosceles triangle, then the third vertex is given by

$$\left(\frac{(x_1 + x_2) \pm (y_1 - y_2)}{2}, \frac{(y_1 + y_2) \mp (x_1 - x_2)}{2}\right)$$

(ii) If two vertices of an equilateral triangle are (x_1, y_1) and (x_2, y_2), then coordinates of the third vertex are

$$\left(\frac{x_1 + x_2 \mp \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right)$$

| Example 7. Prove that the distance of the point $(a\cos\alpha, a\sin\alpha)$ from the origin is independent of α.

Sol. Let $P \equiv (a\cos\alpha, a\sin\alpha)$ and $O \equiv (0, 0)$

then $|OP| = \sqrt{(a\cos\alpha - 0)^2 + (a\sin\alpha - 0)^2}$

$= \sqrt{(a^2\cos^2\alpha + a^2\sin^2\alpha)}$

$= \sqrt{a^2(\cos^2\alpha + \sin^2\alpha)} = \sqrt{a^2}$

$= |a|$, which is independent of α.

| Example 8. Find the distance between the points $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$, where $a > 0$.

Sol. Let $P \equiv (a\cos\alpha, a\sin\alpha)$ and $Q \equiv (a\cos\beta, a\sin\beta)$

then $|PQ| = \sqrt{(a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2}$

$= \sqrt{a^2\{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2\}}$

$= \sqrt{a^2\{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\}}$

$= \sqrt{a^2\{1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\}}$

$= \sqrt{a^2(2 - 2\cos(\alpha - \beta))}$

$= \sqrt{2a^2(1 - \cos(\alpha - \beta))}$

$= \sqrt{\left(2a^2 \cdot 2\sin^2\left(\frac{\alpha - \beta}{2}\right)\right)}$

$= \sqrt{4a^2\sin^2\left(\frac{\alpha - \beta}{2}\right)}$

$= \left|2a\sin\left(\frac{\alpha - \beta}{2}\right)\right|$

$= 2a\left|\sin\left(\frac{\alpha - \beta}{2}\right)\right|$ $(\because a > 0)$

| Example 9. If the point (x, y) be equidistant from the points $(6, -1)$ and $(2, 3)$, prove that $x - y = 3$.

Sol. Let $P \equiv (x, y)$, $A \equiv (6, -1)$ and $B \equiv (2, 3)$

By the given condition, $PA = PB$

$\Rightarrow$ $\sqrt{(x-6)^2 + (y+1)^2} = \sqrt{(x-2)^2 + (y-3)^2}$

or $(x-6)^2 + (y+1)^2 = (x-2)^2 + (y-3)^2$

or $x^2 - 12x + 36 + y^2 + 2y + 1$
$= x^2 - 4x + 4 + y^2 - 6y + 9$

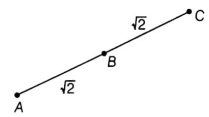
or $8x - 8y = 24$

or $x - y = 3$

| Example 10. Using distance formula, show that the points $(1, 5)$, $(2, 4)$ and $(3, 3)$ are collinear.

Sol. Let $A \equiv (1, 5)$, $B \equiv (2, 4)$ and $C \equiv (3, 3)$ be the given points, then

$$|AB| = \sqrt{(1-2)^2 + (5-4)^2} = \sqrt{2}$$



$|BC| = \sqrt{(2-3)^2 + (4-3)^2} = \sqrt{2}$

and $|AC| = \sqrt{(1-3)^2 + (5-3)^2} = 2\sqrt{2}$

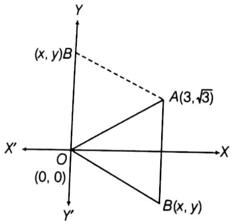
Clearly, $|AB| + |BC| = \sqrt{2} + \sqrt{2} = 2\sqrt{2} = |AC|$.

Hence, A, B, C are collinear.

Example 11. An equilateral triangle has one vertex at the point (0, 0) and another at $(3, \sqrt{3})$. Find the coordinates of the third vertex.

Sol. Let $O \equiv (0, 0)$ and $A \equiv (3, \sqrt{3})$ be the given points and let $B \equiv (x, y)$ be the required point. Then

$$OA = OB = AB$$



$$\Rightarrow \quad (OA)^2 = (OB)^2 = (AB)^2$$

$$\Rightarrow \quad (3-0)^2 + (\sqrt{3}-0)^2 = (x-0)^2 + (y-0)^2$$

$$= (x-3)^2 + (y-\sqrt{3})^2$$

$$\Rightarrow \quad 12 = x^2 + y^2 = x^2 + y^2 - 6x - 2\sqrt{3}y + 12$$

Taking first two members then

$$x^2 + y^2 = 12 \qquad \ldots\text{(i)}$$

and taking last two members, then

$$6x + 2\sqrt{3}y = 12 \text{ or } y = \sqrt{3}(2-x) \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$x^2 + 3(2-x)^2 = 12$$

or $\quad 4x^2 - 12x = 0$

$\Rightarrow \quad x = 0, 3$

Putting $x = 0, 3$ in Eq. (ii), we get $y = 2\sqrt{3}, -\sqrt{3}$

Hence, the coordinates of the third vertex B are $(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$.

Short Cut Method : According to important note :

$$\left(\frac{x_1 + x_2 \mp \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right)$$

i.e. $\quad \left(\dfrac{0 + 3 \mp \sqrt{3}(\sqrt{3} - 0)}{2}, \dfrac{0 + \sqrt{3} \pm \sqrt{3}(3 - 0)}{2}\right)$

or $\quad \left(\dfrac{3 \mp 3}{2}, \dfrac{\sqrt{3} \pm 3\sqrt{3}}{2}\right)$

$\Rightarrow \quad (0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

Example 12. Show that four points (1, −2), (3, 6), (5, 10) and (3, 2) are the vertices of a parallelogram.

Sol. Let $A \equiv (1, -2)$, $B \equiv (3, 6)$, $C \equiv (5, 10)$ and $D \equiv (3, 2)$ be the given points. Then

$$|AB| = \sqrt{(1-3)^2 + (-2-6)^2} = \sqrt{4 + 64} = 2\sqrt{17}$$

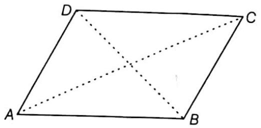
$$|BC| = \sqrt{(3-5)^2 + (6-10)^2} = \sqrt{4 + 16} = 2\sqrt{5}$$

$$|CD| = \sqrt{(5-3)^2 + (10-2)^2} = \sqrt{4 + 64} = 2\sqrt{17}$$

$$|AD| = \sqrt{(1-3)^2 + (-2-2)^2} = \sqrt{4 + 16} = 2\sqrt{5}$$

$$|AC| = \sqrt{(1-5)^2 + (-2-10)^2} = \sqrt{16 + 144} = 4\sqrt{10}$$

and $\quad |BD| = \sqrt{(3-3)^2 + (6-2)^2} = 4$



Clearly, $|AB| = |CD|$, $|BC| = |AD|$ and $|AC| \neq |BD|$

Hence, $ABCD$ is a parallelogram.

Example 13. Let the opposite angular points of a square be (3, 4) and (1, −1). Find the coordinates of the remaining angular points.

Sol. Let $A(3, 4)$ and $C(1, -1)$ be the given angular points of a square $ABCD$ and let $B(x, y)$ be the unknown vertex. Then

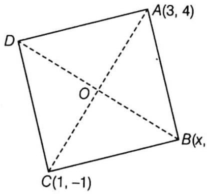
$$AB = BC$$

$$\Rightarrow \quad (AB)^2 = (BC)^2$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-1)^2 + (y+1)^2$$

$$\Rightarrow \quad 4x + 10y - 23 = 0$$

$$\Rightarrow \quad x = \left(\frac{23 - 10y}{4}\right) \qquad \ldots\text{(i)}$$



Also, in ΔABC,

$$(AB)^2 + (BC)^2 = (AC)^2$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 + (x-1)^2 + (y+1)^2$$

$$= (3-1)^2 + (4+1)^2$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 3y - 1 = 0 \qquad \ldots\text{(ii)}$$

Substituting the value of x from Eqs. (i) into (ii), we get

$$\left(\frac{23 - 10y}{4}\right)^2 + y^2 - 4\left(\frac{23 - 10y}{4}\right) - 3y - 1 = 0$$

$$\Rightarrow \quad 4y^2 - 12y + 5 = 0$$

or $\quad (2y - 1)(2y - 5) = 0$

$\therefore \quad y = \dfrac{1}{2}$ or $\dfrac{5}{2}$

Putting $y = \dfrac{1}{2}$ in Eq. (i), we get $x = \dfrac{9}{2}$,

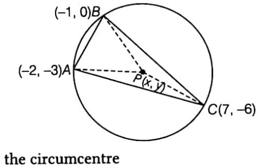
and putting $y = \dfrac{5}{2}$ in Eq. (i), we get $x = -\dfrac{1}{2}$

Hence, the required vertices of the square are $\left(\dfrac{9}{2}, \dfrac{1}{2}\right)$ and $\left(-\dfrac{1}{2}, \dfrac{5}{2}\right)$.

Example 14. Find the circumcentre of the triangle whose vertices are (−2, −3), (−1, 0) and (7, −6). Also find the radius of the circumcircle.

Sol. Let $A \equiv (-2, -3)$, $B \equiv (-1, 0)$ and $C \equiv (7, -6)$.

Let $P \equiv (x, y)$ be the circumcentre of ΔABC.



Since, P is the circumcentre

$$\therefore \quad |PA| = |PB| = |PC| \Rightarrow (PA)^2 = (PB)^2 = (PC)^2$$

$$(x+2)^2 + (y+3)^2 = (x+1)^2 + (y-0)^2$$

$$= (x-7)^2 + (y+6)^2$$

$$\Rightarrow \quad x^2 + y^2 + 4x + 6y + 13 = x^2 + y^2 + 2x + 1$$

$$= x^2 + y^2 - 14x + 12y + 85$$

Taking first two members, we get

$$x + 3y + 6 = 0 \qquad \ldots\text{(i)}$$

and taking 1st and last member then, we get

$$3x - y - 12 = 0 \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$x = 3, y = -3$$

Hence, circumcentre is (3, −3).

Radius of the circumcircle

$$= PB = \sqrt{(3+1)^2 + (-3-0)^2}$$

$$= \sqrt{16 + 9} = 5 \text{ units}$$

Example 15. If the line segment joining the points $A(a, b)$ and $B(c, d)$ subtends an angle θ at the origin O, prove that

$$\cos\theta = \frac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$$

or $\quad OA . OB \cos\theta = ac + bd$

Sol. Let $OA = r_1$ and $OB = r_2$

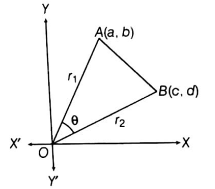
Now, $\quad r_1 = |OA| = \sqrt{(a^2 + b^2)} \qquad \ldots\text{(i)}$

and $\quad r_2 = |OB| = \sqrt{(c^2 + d^2)} \qquad \ldots\text{(ii)}$

Also, $\quad |AB| = \sqrt{(a-c)^2 + (b-d)^2}$

$$= \sqrt{a^2 + b^2 + c^2 + d^2 - 2ac - 2bd}$$

$$= \sqrt{r_1^2 + r_2^2 - 2(ac + bd)} \text{ [from Eqs. (i) and (ii)]}$$



By using **Cosine formula** in ΔAOB

$$\cos\theta = \frac{(OA)^2 + (OB)^2 - (AB)^2}{2 \cdot OA \,.\, OB}$$

$$= \frac{r_1^2 + r_2^2 - (r_1^2 + r_2^2 - 2(ac + bd))}{2r_1 r_2}$$

$$= \frac{2(ac + bd)}{2r_1 r_2} = \frac{(ac + bd)}{r_1 r_2} \qquad \ldots\text{(iii)}$$

$$= \frac{(ac + bd)}{\sqrt{(a^2 + b^2)}\sqrt{(c^2 + d^2)}} \text{ [from Eqs. (i) and (ii)]}$$

$$= \frac{(ac + bd)}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$$

Also from Eq. (iii),

$$r_1 r_2 \cos\theta = ac + bd \quad \text{or} \quad OA . OB \cos\theta = ac + bd$$

Choice of Axes

For simplification we carefully choose the axes or the origin. Some situations are given below :

(i) If two lines are perpendicular then point of intersection is taken as origin and these lines must be taken as the coordinate axes.

(ii) If two fixed points A and B are given then we take $|AB| = 2a$ and the mid-point of AB as origin 'O', line AOB as X-axis and the line perpendicular to AB through O is taken as Y-axis then the coordinates of the fixed points are $(\pm a, 0)$. Similarly if AOB as Y-axis and the line perpendicular to AB through O is taken as X-axis then the coordinates of the fixed points are $(0, \pm a)$.

(iii) If there is a symmetry of any kind then take the coordinates of the points in a general way i.e. (x_i, y_i), $i = 1, 2, 3, \ldots$ etc.

Example 16. Show that the triangle, the coordinates of whose vertices are given by integers, can never be an equilateral triangle.

Sol. Let $A \equiv (0, 0)$, $B \equiv (a, 0)$ and $C \equiv (b, c)$ be the vertices of equilateral triangle ABC where a, b, c are integers then,

$$|AB| = |BC| = |CA|$$

$$\Rightarrow \quad (AB)^2 = (BC)^2 = (CA)^2$$

$$\Rightarrow \quad a^2 = (a-b)^2 + c^2 = b^2 + c^2$$

From first two members, we get

$$b^2 + c^2 = 2ab \quad \text{...(i)}$$

and taking first and third members, then

$$b^2 + c^2 = a^2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii) we get

$$a = 2b \quad (\because a \neq 0)$$

From Eq. (ii), $b^2 + c^2 = (2b)^2$

or $\quad c^2 = 3b^2$

or $\quad c = \pm b\sqrt{3}$

which is impossible, since b and c are integers.

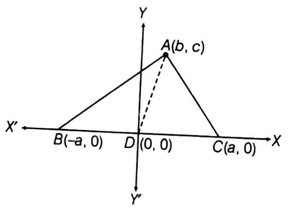
Example 17. In any triangle ABC, show that

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

where, D is the middle point of BC.

Sol. Let D as the origin and DC and DY as the X and Y-axes respectively. Let $BC = 2a$, then

$$B \equiv (-a, 0),\ C \equiv (a, 0) \text{ and let } A \equiv (b, c)$$



Now, $\quad LHS = AB^2 + AC^2$

$$= (b+a)^2 + (c-0)^2 + (b-a)^2 + (c-0)^2$$

$$= 2(a^2 + b^2 + c^2) \quad \text{...(i)}$$

and $\quad RHS = 2(AD^2 + BD^2)$

$$= 2\{(b-0)^2 + (c-0)^2 + a^2\}$$

$$= 2(a^2 + b^2 + c^2) \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

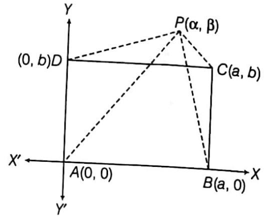
$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

Example 18. Let $ABCD$ be a rectangle and P be any point in its plane. Show that $PA^2 + PC^2 = PB^2 + PD^2$

Sol. Let A as the origin and AB and AD as the X and Y-axes respectively. Let $AB = a$ and $AD = b$, then

$$B \equiv (a, 0),\ D \equiv (0, b) \text{ and } C \equiv (a, b)$$

Let $\quad P \equiv (\alpha, \beta)$



Now, $LHS = PA^2 + PC^2$

$$= (\alpha - 0)^2 + (\beta - 0)^2 + (\alpha - a)^2 + (\beta - b)^2$$

$$= 2\alpha^2 + 2\beta^2 - 2a\alpha - 2b\beta + a^2 + b^2 \quad \text{...(i)}$$

and $\quad RHS = PB^2 + PD^2$

$$= (\alpha - a)^2 + (\beta - 0)^2 + (\alpha - 0)^2 + (\beta - b)^2$$

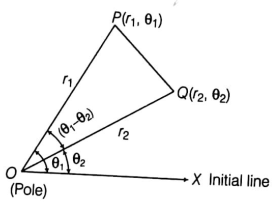
$$= 2\alpha^2 + 2\beta^2 - 2a\alpha - 2b\beta + a^2 + b^2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$PA^2 + PC^2 = PB^2 + PD^2$$

Distance between Two Points in Polar Coordinates

Let O be the pole and OX be the initial line. Let P and Q be two given points whose polar coordinates are (r_1, θ_1) and (r_2, θ_2) respectively.



Then, $\quad OP = r_1,\ OQ = r_2$

and $\quad \angle POX = \theta_1,\ \angle QOX = \theta_2$

$\therefore \quad \angle POQ = \theta_1 - \theta_2$

By using **Cosine formula** in ΔPOQ,

$$\cos(\angle POQ) = \frac{(OP)^2 + (OQ)^2 - (PQ)^2}{2(OP)(OQ)}$$

or $\quad \cos(\theta_1 - \theta_2) = \dfrac{r_1^2 + r_2^2 - (PQ)^2}{2 r_1 r_2}$

$$\therefore \quad |PQ| = \sqrt{(r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_1 - \theta_2))}$$

Remark

Always taking θ_1 and θ_2 in radians.

Example 19. Prove that the points $(0, 0)$, $\left(3, \frac{\pi}{2}\right)$ and $\left(3, \frac{\pi}{6}\right)$ are the vertices of an equilateral triangle.

Sol. Let $A \equiv (0, 0)$, $B \equiv \left(3, \frac{\pi}{2}\right)$ and $C \equiv \left(3, \frac{\pi}{6}\right)$

Here, given coordinates are in polar form

$$\therefore \quad |AB| = \sqrt{\left(0^2 + 3^2 - 2 \cdot 0 \cdot 3 \cos\left(\frac{\pi}{2} - 0\right)\right)} = 3 \text{ units}$$

$$|BC| = \sqrt{\left(3^2 + 3^2 - 2 \cdot 3 \cdot 3 \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\right)}$$

$$= \sqrt{\left(18 - 18 \sin\frac{\pi}{6}\right)} = \sqrt{(18 - 9)} = 3 \text{ units}$$

and $\quad |CA| = \sqrt{\left(3^2 + 0^2 - 2 \cdot 3 \cdot 0 \cos\left(\frac{\pi}{6} - 0\right)\right)} = 3$ units

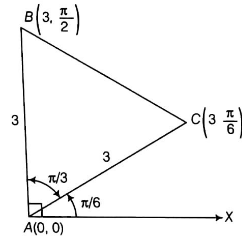
$$\therefore \quad |AB| = |BC| = |CA|$$

Hence, points A, B, C are the vertices of an equilateral triangle.

Aliter :

$\because \quad \angle BAX = \dfrac{\pi}{2}$

and $\quad \angle CAX = \dfrac{\pi}{6}$



$\therefore \quad \angle BAC = \dfrac{\pi}{2} - \dfrac{\pi}{6} = \dfrac{\pi}{3}$

$\because$ In ΔABC

$$AB = AC$$

$\therefore \quad \angle ACB = \angle ABC = \alpha \quad$ (say)

$\therefore \quad \alpha + \alpha + \dfrac{\pi}{3} = \pi$

or $\quad \alpha = \dfrac{\pi}{3}$

Hence, $\quad |AB| = |BC| = |CA|$.

Exercise for Session 2

1. If the distance between the points $(a, 2)$ and $(3, 4)$ be 8, then a equals to
(a) $2+3\sqrt{15}$ (b) $2-3\sqrt{15}$ (c) $2\pm 3\sqrt{15}$ (d) $3\pm 2\sqrt{15}$

2. The three points $(-2, 2)$, $(8, -2)$ and $(-4, -3)$ are the vertices of
(a) an isosceles triangle (b) an equilateral triangle (c) a right angled triangle (d) None of these

3. The distance between the points $\left(3, \frac{\pi}{4}\right)$ and $\left(7, \frac{5\pi}{4}\right)$ is
(a) 8 (b) 10 (c) 12 (d) 14

4. Let $A(6, -1)$, $B(1, 3)$ and $C(x, 8)$ be three points such that $AB = BC$, then the value of x are
(a) 3, 5 (b) -3, 5 (c) 3, -5 (d) -3, -5

5. The points $(a+1, 1)$, $(2a+1, 3)$ and $(2a+2, 2a)$ are collinear, if
(a) $a=-1, 2$ (b) $a=\frac{1}{2}, 2$ (c) $a \ne 2, 1$ (d) $a=-\frac{1}{2}, 2$

6. If $A \equiv (3, 4)$ and B is a variable point on the lines $|x| = 6$. If $AB \le 4$ then the number of positions of B with integral coordinates is
(a) 5 (b) 6 (c) 10 (d) 12

7. The number of points on X-axis which are at a distance c units $(c < 3)$ from $(2, 3)$ is
(a) 1 (b) 2 (c) 0 (d) 3

8. The point on the axis of y which its equidistant from $(-1, 2)$ and $(3, 4)$, is
(a) $(0, 3)$ (b) $(0, 4)$ (c) $(0, 5)$ (d) $(0, -6)$

9. Find the distance between the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, where t_1 and t_2 are the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$ and $a > 0$.

10. If $P(at^2, 2at)$, $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ and $S(a, 0)$ be any three points, show that $\frac{1}{SP} + \frac{1}{SQ}$ is independent of t.

11. Prove that the points $(3, 4)$, $(8, -6)$ and $(13, 9)$ are the vertices of a right angled triangle.

12. Show that the points $(0, -1)$, $(6, 7)$, $(-2, 3)$ and $(8, 3)$ are the vertices of a rectangle.

13. Find the circumcentre and circumradius of the triangle whose vertices are $(-2, 3)$, $(2, -1)$ and $(4, 0)$.

14. The vertices of a triangle are $A(1, 1)$, $B(4, 5)$ and $C(6, 13)$. Find $\cos A$.

15. Two opposite vertices of a square are $(2, 6)$ and $(0, -2)$. Find the coordinates of the other vertices.

16. If the point (x, y) is equidistant from the points $(a+b, b-a)$ and $(a-b, a+b)$, prove that $bx = ay$.

17. If a and b are real numbers between 0 and 1 such that the points $(a, 1)$, $(1, b)$ and $(0, 0)$ form an equilateral triangle, find a and b.

18. An equilateral triangle has one vertex at $(3, 4)$ and another at $(-2, 3)$. Find the coordinates of the third vertex.

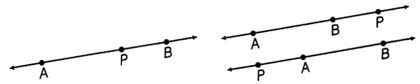
19. If P be any point in the plane of square $ABCD$, prove that
$$PA^2 + PC^2 = PB^2 + PD^2$$

Session 3

Section Formula, Centroid of a Triangle, Incentre, Some Standard Results, Area of Triangle

Section Formula

Definition : If P be any point on the line AB between A and B then we say that P divides segment AB *internally* in the ratio $AP : PB$.



Also, if P be any point on the line AB but not between A and B (P may be to the right or the left of the points A, B) then P divides AB *externally* in the ratio $AP : PB$

Note

$$\frac{AP}{PB} = \begin{cases} \text{Positive, in internally division} \\ \text{Negative, in externally division} \end{cases}$$

(i) Formula for Internal Division

Theorem : If the point $P(x, y)$ divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$, then prove that

$$x = \frac{mx_2 + nx_1}{m+n}$$

$$y = \frac{my_2 + ny_1}{m+n}$$

Proof : The given points are $A(x_1, y_1)$ and $B(x_2, y_2)$. Let us assume that the points A and B are both in 1st quadrant (for the sake of exactness). Since $P(x, y)$ divides AB internally in the ratio $m : n$ i.e. $AP : PB = m : n$. From A, B and P draw AL, BM and PN perpendiculars to X-axis. From A and P draw AH and PJ perpendiculars to PN and BM respectively, then

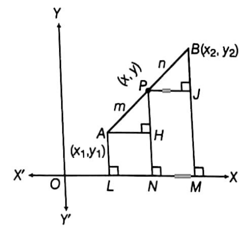
$OL = x_1$, $ON = x$, $OM = x_2$, $AL = y_1$, $PN = y$ and $BM = y_2$

$\therefore$ $AH = LN = ON - OL = x - x_1$

$PJ = NM = OM - ON = x_2 - x$

$PH = PN - HN = PN - AL = y - y_1$

and $BJ = BM - JM = BM - PN = y_2 - y$



Clearly, the Δs AHP and PJB are similar and therefore, their sides are proportional

$$\therefore \quad \frac{AH}{PJ} = \frac{PH}{BJ} = \frac{AP}{PB}$$

or

$$\underset{\text{(i)}}{\frac{x - x_1}{x_2 - x}} = \underset{\text{(ii)}}{\frac{y - y_1}{y_2 - y}} = \underset{\text{(iii)}}{\frac{m}{n}}$$

From Eqs. (i) and (iii), we have

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}$$

$\Rightarrow$ $nx - nx_1 = mx_2 - mx$

$\Rightarrow$ $(m+n)x = mx_2 + nx_1$

$$\therefore \quad x = \frac{mx_2 + nx_1}{m+n}$$

and from Eqs. (ii) and (iii), we have

$$\frac{y - y_1}{y_2 - y} = \frac{m}{n}$$

$\Rightarrow$ $ny - ny_1 = my_2 - my$

$\Rightarrow$ $(m+n)y = my_2 + ny_1$

$$\therefore \quad y = \frac{my_2 + ny_1}{m+n}$$

Thus, the coordinates of P are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

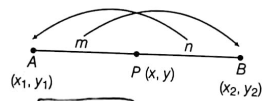
***Corollary* 1 :** The above section formula is true for all positions of the points (i.e. either point or both points are not in the 1st quadrant), keeping in mind, the proper signs of their coordinates.

***Corollary* 2 :** If P is the mid-point of AB then $m = n$, the coordinates of the middle-point of AB are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

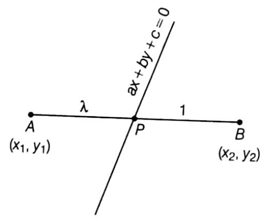
Remarks

1. If $P(\alpha, \beta)$ be the mid-point of AB and if coordinates of A are (λ, μ) then the coordinates of B are $(2\alpha - \lambda, 2\beta - \mu)$, i.e. (Double the x-co-ordinate of mid point – x-coordinate of given point. Double the y-co-ordinate of mid point – y-coordinate of given point).
2. The following diagram will help to remember the section formula.



3. For finding ratio, use ratio $\lambda : 1$, then coordinates of P are $\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right)$. If λ is positive then divides internally and if λ is negative, then divides externally.
4. The straight line $ax + by + c = 0$ divides the joint of points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio

$$\frac{AP}{PB} = \frac{\lambda}{1} = -\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$$



If ratio is positive, then divides internally and if ratio is negative then divides externally.

Proof : Coordinates of P are $\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right)$

$\because P$ lies on the line $ax + by + c = 0$, then

$$a\left(\frac{x_1 + \lambda x_2}{1 + \lambda}\right) + b\left(\frac{y_1 + \lambda y_2}{1 + \lambda}\right) + c = 0$$

or $\quad (ax_1 + by_1 + c) + \lambda(ax_2 + by_2 + c) = 0$

or $\quad \frac{\lambda}{1} = -\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$

5. The line joining the points (x_1, y_1) and (x_2, y_2) is divided by the X-axis in the ratio $-\frac{y_1}{y_2}$ and by Y-axis in the ratio $-\frac{x_1}{x_2}$.
6. In square, rhombus, rectangle and parallelogram diagonals bisect to each other.

| Example 20. Find the coordinates of the point which divides the line segment joining the points $(5, -2)$ and $(9, 6)$ in the ratio $3 : 1$.

Sol. Let the required point be (x, y), then

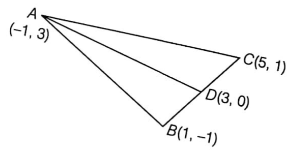
$$x = \left(\frac{3 \times 9 + 1 \times 5}{3 + 1}\right) = 8$$

and $$y = \left(\frac{3 \times 6 + 1 \times (-2)}{3 + 1}\right) = 4$$

Thus, the coordinates of the required point are $(8, 4)$.

| Example 21. Find the length of median through A of a triangle whose vertices are $A(-1, 3)$, $B(1, -1)$ and $C(5, 1)$.

Sol. Let D be the mid-point of BC, then coordinates of D are $\left(\frac{1+5}{2}, \frac{-1+1}{2}\right)$ i.e. $(3, 0)$



$\therefore$ $$\text{Median } AD = \sqrt{(3+1)^2 + (0-3)^2}$$
$$= \sqrt{16 + 9} = \sqrt{25}$$
$$= 5 \text{ units}$$

| Example 22. Determine the ratio in which $y - x + 2 = 0$ divides the line joining $(3, -1)$ and $(8, 9)$.

Sol. Suppose the line $y - x + 2 = 0$ divides the line segment joining $A(3, -1)$ and $B(8, 9)$ in the ratio $\lambda : 1$ at point P, then the coordinates of the point P are $\left(\frac{8\lambda + 3}{\lambda + 1}, \frac{9\lambda - 1}{\lambda + 1}\right)$.

But P lies on $y - x + 2 = 0$ therefore

$$\left(\frac{9\lambda - 1}{\lambda + 1}\right) - \left(\frac{8\lambda + 3}{\lambda + 1}\right) + 2 = 0$$

$\Rightarrow \quad 9\lambda - 1 - 8\lambda - 3 + 2\lambda + 2 = 0$

$\Rightarrow \quad 3\lambda - 2 = 0 \quad$ or $\quad \lambda = \frac{2}{3}$

So, the required ratio is $\frac{2}{3} : 1$, i.e. $2 : 3$ (internally) since here λ is positive.

Shortcut method

According to Remark 4 :

$$\lambda = -\left(\frac{-1 - 3 + 2}{9 - 8 + 2}\right) = \frac{2}{3}$$

or $\quad \lambda : 1 = 2 : 3$

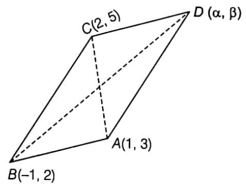
Must

| Example 23. The coordinates of three consecutive vertices of a parallelogram are $(1, 3)$, $(-1, 2)$ and $(2, 5)$. Then find the coordinates of the fourth vertex.

Sol. Let the fourth vertex be $D(\alpha, \beta)$. Since $ABCD$ is a parallelogram, the diagonals bisect to each other.
i.e. mid-point of BD = mid-point of AC

$\therefore \quad \left(\frac{\alpha - 1}{2}, \frac{\beta + 2}{2}\right) = \left(\frac{2 + 1}{2}, \frac{5 + 3}{2}\right)$

or $\quad \left(\frac{\alpha - 1}{2}, \frac{\beta + 2}{2}\right) = \left(\frac{3}{2}, 4\right)$



On equating abscissaes and ordinates, we get

$\frac{\alpha - 1}{2} = \frac{3}{2} \quad$ or $\quad \alpha - 1 = 3 \quad$ or $\quad \alpha = 4$

and $\quad \frac{\beta + 2}{2} = 4 \quad$ or $\quad \beta + 2 = 8 \quad$ or $\quad \beta = 6$

Hence, the coordinates of the fourth vertex $D(\alpha, \beta)$ is $(4, 6)$.

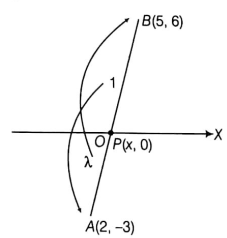
| Example 24. In what ratio does X-axis divide the line segment joining $(2, -3)$ and $(5, 6)$?

Sol. Let the given points be $A(2, -3)$ and $B(5, 6)$. Let AB be divided by the X-axis at $P(x, 0)$ in the ratio $\lambda : 1$ internally. Considering the ordinate of P, then

$$0 = \frac{\lambda \times 6 + 1 \times (-3)}{\lambda + 1}$$

or $\quad \lambda = \frac{1}{2}$

$\therefore$ The ratio is $\frac{1}{2} : 1$ i.e. $1 : 2$ (Internally)



Shortcut Method

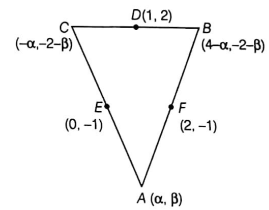
According to Remark 5 :

$$\frac{\lambda}{1} = -\frac{y_1}{y_2} = \frac{-(-3)}{6} = \frac{1}{2}$$

$\therefore$ The ratio is $\frac{1}{2} : 1$ i.e. $1 : 2$ (internally)

| Example 25. The mid-points of the sides of a triangle are $(1, 2)$, $(0, -1)$ and $(2, -1)$. Find the coordinates of the vertices of a triangle with the help of two unknowns.

Sol. Let $D(1, 2)$, $E(0, -1)$ and $F(2, -1)$ be the mid-points of BC, CA and AB respectively.
Let the coordinates of A be (α, β) then coordinates of B and C are $(4 - \alpha, -2 - \beta)$ and $(-\alpha, -2 - \beta)$ respectively (see note 1)
$\because$ D is the mid-point of B and C



then $\quad 1 = \frac{4 - \alpha - \alpha}{2}$

$\Rightarrow \quad 1 = 2 - \alpha \quad$ or $\quad \alpha = 1$

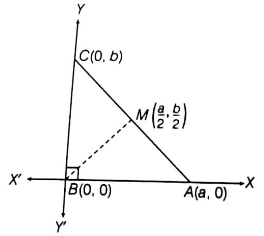
and $\quad 2 = \frac{-2 - \beta - 2 - \beta}{2}$

$\Rightarrow \quad 2 = -2 - \beta \quad$ or $\quad \beta = -4$

Hence, coordinates of A, B and C are $(1, -4)$, $(3, 2)$ and $(-1, 2)$ respectively.

| Example 26. Prove that in a right angled triangle the mid-point of the hypotenuse is equidistant from its vertices.

Sol. Let the given right angled triangle be ABC, with right angled at B. We take B as the origin and BA and BC as the X and Y-axes respectively.
Let $BA = a$ and $BC = b$
then $A \equiv (a, 0)$ and $C \equiv (0, b)$
Let M to be the mid-point of the hypotenuse AC, then coordinates of M are $\left(\frac{a}{2}, \frac{b}{2}\right)$



$$\therefore \quad |AM| = \sqrt{\left(a-\frac{a}{2}\right)^2 + \left(0-\frac{b}{2}\right)^2} = \frac{\sqrt{(a^2+b^2)}}{2} \quad \ldots (i)$$

$$|BM| = \sqrt{\left(0-\frac{a}{2}\right)^2 + \left(0-\frac{b}{2}\right)^2} = \frac{\sqrt{(a^2+b^2)}}{2} \quad \ldots (ii)$$

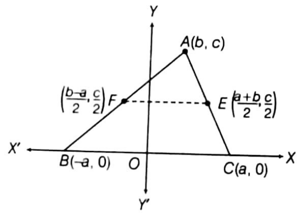
$$\text{and} \quad |CM| = \sqrt{\left(0-\frac{a}{2}\right)^2 + \left(b-\frac{b}{2}\right)^2} = \frac{\sqrt{(a^2+b^2)}}{2} \quad \ldots (iii)$$

From Eqs. (i), (ii) and (iii), we get

$$|AM| = |BM| = |CM|$$

| Example 27 Show that the line joining the mid-points of any two sides of a triangle is half the third side.

Sol. We take O as the origin and OC and OY as the X and Y-axes respectively.



Let $BC = 2a$, then $B \equiv (-a, 0)$, $C \equiv (a, 0)$

Let $A \equiv (b, c)$, if E and F are the mid-points of sides AC and AB respectively.

Then, $\quad E \equiv \left(\frac{a+b}{2}, \frac{c}{2}\right)$ and $F \equiv \left(\frac{b-a}{2}, \frac{c}{2}\right)$

Now, $$FE = \sqrt{\left(\frac{a+b}{2} - \frac{b-a}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2} = a$$

$$= \frac{1}{2}(2a) = \frac{1}{2}(BG)$$

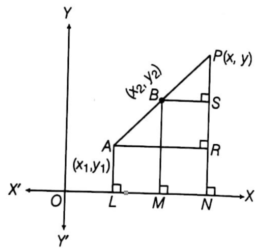
Hence, the line joining the mid-points of any two sides of a triangle is half the third side.

(ii) Formula for External Division

Theorem : If the point $P(x, y)$ divides the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m:n$ then prove that

$$x = \frac{mx_2 - nx_1}{m-n}, \; y = \frac{my_2 - ny_1}{m-n}$$

Proof : The given points are $A(x_1, y_1)$ and $B(x_2, y_2)$. Let us assume that the points A and B are both in the 1st quadrant (for the sake of exactness). Let $P(x, y)$ be the point which divides AB externally in the ratio $m:n$, so that $\frac{AP}{BP} = \frac{m}{n}$.



From A, B and P draw AL, BM and PN perpendiculars on X-axis. Also, from A and B draw AR and BS perpendiculars on PN,

then
$$AR = LN = ON - OL = x - x_1$$
$$BS = MN = ON - OM = x - x_2$$
$$PR = PN - RN = PN - AL = y - y_1$$
and
$$PS = PN - SN = PN - BM = y - y_2$$

Clearly, the Δs APR and BPS are similar and therefore their sides are proportional.

$$\therefore \quad \frac{AP}{PB} = \frac{AR}{BS} = \frac{PR}{PS}$$

or
$$\frac{m}{n} = \frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2}$$
$$\text{(i)} \qquad \text{(ii)} \qquad \text{(iii)}$$

From Eqs. (i) and (ii), we have

$$\frac{m}{n} = \frac{x - x_1}{x - x_2}$$

$\Rightarrow$ $$mx - mx_2 = nx - nx_1$$

$\Rightarrow$ $$(m-n)x = mx_2 - nx_1$$

or $$x = \frac{mx_2 - nx_1}{m-n}$$

Also, from Eqs. (i) and (iii), we have

$$\frac{m}{n} = \frac{y - y_1}{y - y_2}$$

$\Rightarrow$ $$my - my_2 = ny - ny_1$$

$\Rightarrow$ $$(m-n)y = my_2 - ny_1$$

or $$y = \frac{my_2 - ny_1}{m-n}$$

Thus, the coordinates of P are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$.

(Here, $m \neq n$)

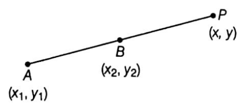
***Corollary* 1 :** The above formula is true for all positions of the points, keeping in mind, the proper signs of their coordinates.

***Corollary* 2 :** The above coordinates can also be expressed as

$$\left(\frac{mx_2 + (-n)x_1}{m + (-n)}, \frac{my_2 + (-n)y_1}{m + (-n)}\right)$$

and this can be thought of as the coordinates of the point dividing AB internally in the ratio $m:-n$

***Corollary* 3 :** $\because \quad \frac{AP}{PB} = \frac{m}{n}$



or $$\frac{AP}{PB} - 1 = \frac{m}{n} - 1$$

or $$\frac{AP - PB}{PB} = \frac{m-n}{n}$$

or $$\frac{AB}{PB} = \frac{m-n}{n}$$

Now, we can say that B divides AP in the ratio $\boxed{m-n:n}$ internally.

i.e. $\quad x_2 = \frac{(m-n)x + nx_1}{(m-n)+n} \Rightarrow x = \frac{mx_2 - nx_1}{m-n}$

and $\quad y_2 = \frac{(m-n)y + ny_1}{(m-n)+n} \Rightarrow y = \frac{my_2 - ny_1}{m-n}$

***Corollary* 4 :** (for proving A, B and C are collinear)

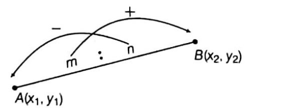
If A, B, C three points are collinear then let C divides AB in the ratio $\lambda : 1$ internally.

If $\lambda = +$ve rational, then divide internally

and if $\lambda = -$ve rational, then divide externally.

Remarks

1. The following diagram will help to remember the section formula



2. Let $\frac{m}{n} = \lambda$, then $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$

or $\left(\frac{\frac{m}{n}x_2 - x_1}{\frac{m}{n} - 1}, \frac{\frac{m}{n}y_2 - y_1}{\frac{m}{n} - 1}\right)$ or $\left(\frac{\lambda x_2 - x_1}{\lambda - 1}, \frac{\lambda y_2 - y_1}{\lambda - 1}\right)$

| Example 28. Find the coordinates of a point which divides externally the line joining $(1, -3)$ and $(-3, 9)$ in the ratio $1:3$.

Sol. Let the coordinates of the required point be $P(x, y)$.

Then, $$x = \left(\frac{1 \times (-3) - 3 \times 1}{1-3}\right)$$

and $$y = \left(\frac{1 \times 9 - 3 \times (-3)}{1-3}\right)$$

i.e. $\quad x = 3$ and $y = -9$

Hence, the required point is $(3, -9)$.

| Example 29. The line segment joining $A(6, 3)$ to $B(-1, -4)$ is doubled in length by having its length added to each end. Find the coordinates of the new ends.

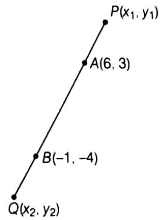
Sol. Let P and Q be the required new ends

Let the coordinates of P be (x_1, y_1)

Given, $\quad AB = 2AP$

$\Rightarrow$ $$\frac{AB}{AP} = \frac{2}{1}$$

i.e. A divides BP internally in the ratio $2:1$.



Then, $$6 = \frac{2 \times x_1 + 1 \times (-1)}{2+1}$$

$\Rightarrow \quad 19 = 2x_1$ or $x_1 = \dfrac{19}{2}$

and $\quad 3 = \dfrac{2 \times y_1 + 1 \times (-4)}{2+1}$

$\Rightarrow \quad 13 = 2y_1$ or $y_1 = \dfrac{13}{2}$

$\therefore$ Coordinates of P are $\left(\dfrac{19}{2}, \dfrac{13}{2}\right)$.

Also, let coordinates of Q be (x_2, y_2)

Given, $\quad AB = 2BQ \Rightarrow \dfrac{AB}{BQ} = \dfrac{2}{1}$

i.e. B divides AQ internally in the ratio $2:1$

Then $\quad -1 = \dfrac{2 \times x_2 + 1 \times 6}{2+1}$

$\Rightarrow \quad -9 = 2x_2$ or $x_2 = -\dfrac{9}{2}$

and $\quad -4 = \dfrac{2 \times y_2 + 1 \times 3}{2+1}$

$\Rightarrow \quad -15 = 2y_2$ or $y_2 = -\dfrac{15}{2}$

$\therefore$ Coordinates of Q are $\left(-\dfrac{9}{2}, -\dfrac{15}{2}\right)$

Aliter : $\because \quad AB = 2AP$

$\Rightarrow \quad \dfrac{AB}{AP} = \dfrac{2}{1} \Rightarrow \dfrac{AB}{AP} + 1 = \dfrac{2}{1} + 1$

$\Rightarrow \quad \dfrac{AB + AP}{AP} = \dfrac{3}{1} \Rightarrow \dfrac{BP}{AP} = \dfrac{3}{1}$

$\therefore$ P divides AB externally in the ratio $1:3$

Then, $\quad x_1 = \dfrac{1 \times (-1) - 3 \times 6}{1-3} = \dfrac{19}{2}$

and $\quad y_1 = \dfrac{1 \times (-4) - 3 \times 3}{1-3} = \dfrac{13}{2}$

$\therefore$ Coordinates of P are $\left(\dfrac{19}{2}, \dfrac{13}{2}\right)$

Also, $\quad AB = 2BQ$

$\Rightarrow \quad \dfrac{AB}{BQ} = \dfrac{2}{1} \Rightarrow \dfrac{AB}{BQ} + 1 = \dfrac{2}{1} + 1$

$\Rightarrow \quad \dfrac{AB + BQ}{BQ} = \dfrac{3}{1} \Rightarrow \dfrac{AQ}{BQ} = \dfrac{3}{1}$

$\therefore$ Q divides AB externally in the ratio $3:1$

then, $x_2 = \dfrac{3 \times (-1) - 1 \times 6}{3-1} = -\dfrac{9}{2}$

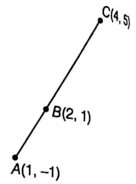
and $\quad y_2 = \dfrac{3 \times (-4) - 1 \times 3}{3-1} = -\dfrac{15}{2}$

$\therefore$ Coordinates of Q are $\left(-\dfrac{9}{2}, -\dfrac{15}{2}\right)$.

| Example 30. Using section formula show that the points $(1, -1), (2, 1)$ and $(4, 5)$ are collinear.

Sol. Let $A \equiv (1, -1)$, $B \equiv (2, 1)$ and $C \equiv (4, 5)$

Suppose C divides AB in the ratio $\lambda : 1$ internally, then



$$4 = \frac{\lambda \times 2 + 1 \times 1}{\lambda + 1}$$

$\Rightarrow \quad 4\lambda + 4 = 2\lambda + 1$

or $\quad \lambda = -\dfrac{3}{2}$

i.e. C divides AB in the ratio $3:2$ (externally).

Hence, A, B, C are collinear.

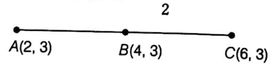
| Example 31. Find the ratio in which the point $(2, y)$ divides the line segment joining $(4, 3)$ and $(6, 3)$ and hence find the value of y.

Sol. Let $A \equiv (4, 3)$, $B \equiv (6, 3)$ and $P \equiv (2, y)$

Let P divides AB internally in the ratio $\lambda : 1$

then, $\quad 2 = \dfrac{6\lambda + 4}{\lambda + 1} \Rightarrow 2\lambda + 2 = 6\lambda + 4$

$\Rightarrow \quad -4\lambda = 2$ or $\lambda = -\dfrac{1}{2}$



$\therefore$ P divides AB externally in the ratio $1:2$ ($\because \lambda$ is negative)

Now, $y = \dfrac{1 \times 3 - 2 \times 3}{1 - 2} = 3$

(iii) Harmonic Conjugates

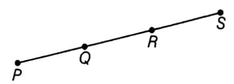
If four points in a line, then the system is said to form a range. Let four points say P, Q, R, S.

If the range (PQ, RS) has a cross ratio equal to -1, then it is called harmonic.

i.e. $\quad \dfrac{PR}{RQ} \cdot \dfrac{SQ}{SP} = -1 \Rightarrow \dfrac{PR}{RQ} = -\dfrac{SP}{SQ} = \lambda \quad$ (say)

$\therefore \quad \dfrac{PR}{RQ} = \dfrac{\lambda}{1} \Rightarrow PR : RQ = \lambda : 1 \quad$ (internally)

and $\quad \dfrac{SP}{SQ} = -\dfrac{\lambda}{1} \Rightarrow PS : SQ = \lambda : 1 \quad$ (externally)

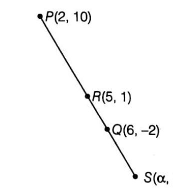


Hence, R and S are called the **harmonic conjugates** to each other with respect to the points P and Q.

| Example 32 Find the harmonic conjugates of the point $R(5, 1)$ with respect to the points $P(2, 10)$ and $Q(6, -2)$.

Sol. Let $S(\alpha, \beta)$ (be the harmonic conjugates of the point $R(5, 1)$).

Suppose R divides PQ in the ratio $\lambda : 1$ internally, then S divides PQ in the ratio $\lambda : 1$ externally, then



$$5 = \frac{6\lambda + 2}{\lambda + 1} \Rightarrow 5\lambda + 5 = 6\lambda + 2$$

$\therefore \quad \lambda = 3$

Also, $\quad 1 = \dfrac{-2\lambda + 10}{\lambda + 1}$

$\lambda + 1 = -2\lambda + 10 \Rightarrow 3\lambda = 9$

$\therefore \quad \lambda = 3$

Now, $\quad \alpha = \dfrac{3 \times 6 - 1 \times 2}{3 - 1} = 8$

and $\quad \beta = \dfrac{3 \times (-2) - 1 \times 10}{3 - 1}$

$= -8$

Hence, harmonic conjugates of $R(5, 1)$ is $S(8, -8)$.

Centroid of a Triangle

Definition : The point of intersection of the medians of a triangle is called the centroid of the triangle and it divides the median internally in the ratio $2:1$.

Theorem : Prove that the coordinates of the centroid of the triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

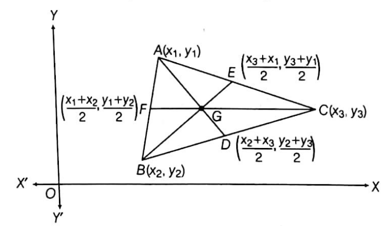
Also, deduce that the medians of a triangle are concurrent.

Proof : Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ be the vertices of the triangle ABC. Let us assume that the points A, B and C are in the 1st quadrant (for the sake of exactness) whose medians are AD, BE and CF respectively so D, E and F are respectively the mid-points of BC, CA and AB then the coordinates of D, E, F are

$$D \equiv \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

$$E \equiv \left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$$

and $$F \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



The coordinates of a point dividing AD in the ratio $2:1$ are

$$\left(\frac{2 \cdot \left(\frac{x_2 + x_3}{2}\right) + 1 \cdot x_1}{2+1}, \frac{2 \cdot \left(\frac{y_2 + y_3}{2}\right) + 1 \cdot y_1}{2+1}\right)$$

or $$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

and the coordinates of a point dividing BE in the ratio $2:1$ are

$$\left(\frac{2 \cdot \left(\frac{x_3 + x_1}{2}\right) + 1 \cdot x_2}{2+1}, \frac{2 \cdot \left(\frac{y_3 + y_1}{2}\right) + 1 \cdot y_2}{2+1}\right)$$

or $$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Similarly the coordinates of a point dividing CF in the ratio $2:1$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

$\therefore$ The common point which divides AD, BE and CF in the ratio $2:1$ is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Hence, medians of a triangle are concurrent and the coordinates of the centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Important Theorem

Must

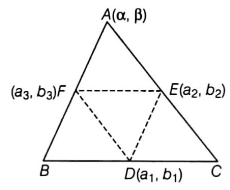
Centroid of the triangle obtained by joining the middle points of the sides of a triangle is the same as the centroid of the original triangle.

Or

If $(a_1, b_1), (a_2, b_2)$ and (a_3, b_3) are the mid-points of the sides of a triangle, then its centroid is given by

$$\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right)$$

Proof : Let D, E, F are the mid-points of BC, CA and AB respectively now let coordinates of A are (α, β) then coordinates of B and C are $(2a_3 - \alpha, 2b_3 - \beta)$ and $(2a_2 - \alpha, 2b_2 - \beta)$ are respectively.



$\because D(a_1, b_1)$ is the mid-point of B and C, then

$$2a_1 = 2a_3 - \alpha + 2a_2 - \alpha \Rightarrow \alpha = a_2 + a_3 - a_1$$

and $$2b_1 = 2b_3 - \beta + 2b_2 - \beta \Rightarrow \beta = b_2 + b_3 - b_1$$

Now, coordinates of B are $(2a_3 - \alpha, 2b_3 - \beta)$

or $$(a_3 + a_1 - a_2, b_3 + b_1 - b_2)$$

and coordinates of C are $(2a_2 - \alpha, 2b_2 - \beta)$

or $$(a_2 + a_1 - a_3, b_2 + b_1 - b_3)$$

Hence, coordinates of A, B and C are

$$A \equiv (a_2 + a_3 - a_1, b_2 + b_3 - b_1),$$
$$B \equiv (a_3 + a_1 - a_2, b_3 + b_1 - b_2)$$
and $$C \equiv (a_2 + a_1 - a_3, b_2 + b_1 - b_3)$$

$\therefore$ Coordinates of centroid of triangle ABC are

$$\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right)$$

which is same as the centroid of triangle DEF.

***Corollary* 1 (Finger Rule) :** If mid-points of the sides of a triangle are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), then coordinates of the original triangle are

$$(x_2 + x_3 - x_1, y_2 + y_3 - y_1),$$
$$(x_3 + x_1 - x_2, y_3 + y_1 - y_2)$$
and $$(x_1 + x_2 - x_3, y_1 + y_2 - y_3).$$

***Corollary* 2 :** If two vertices of a triangle are (x_1, y_1) and (x_2, y_2) and the coordinates of centroid are (α, β), then coordinates of the third vertex are

$$(3\alpha - x_1 - x_2, 3\beta - y_1 - y_2)$$

***Corollary* 3 :** According to important theorem Δs ABC and DEF are similar

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{(BC)^2}{(EF)^2}$$

$$= \frac{4\{(a_2 - a_3)^2 + (b_2 - b_3)^2\}}{\{(a_2 - a_3)^2 + (b_2 - b_3)^2\}} = 4$$

$$\therefore \quad \boxed{\text{Area of } \Delta ABC = 4 \times \text{Area of } \Delta DEF}$$

i.e. Area of a triangle is four times the area of the triangle formed by joining the mid-points of its sides.

| Example 33. Two vertices of a triangle are $(-1, 4)$ and $(5, 2)$. If its centroid is $(0, -3)$, find the third vertex.

Sol. Let the third vertex be (x, y) then the coordinates of the centroid of triangle are

$$\left(\frac{-1 + 5 + x}{3}, \frac{4 + 2 + y}{3}\right) \text{ i.e. } \left(\frac{4 + x}{3}, \frac{6 + y}{3}\right)$$

Now, $$\left(\frac{4 + x}{3}, \frac{6 + y}{3}\right) = (0, -3)$$

$$\Rightarrow \quad \frac{4 + x}{3} = 0 \quad \text{and} \quad \frac{6 + y}{3} = -3$$

$$\Rightarrow \quad 4 + x = 0 \quad \text{and} \quad y + 6 = -9$$

or $$x = -4 \quad \text{and} \quad y = -15$$

Hence, the third vertex is $(-4, -15)$.

Shortcut Method

According to corollary 2

$$(x, y) = (3 \times 0 - (-1) - 5, 3 \times (-3) - 4 - 2)$$
$$= (-4, -15)$$

| Example 34. The vertices of a triangle are $(1, 2), (h, -3)$ and $(-4, k)$. Find the value of $\sqrt{\{(h + k)^2 + (h + 3k)^2\}}$. If the centroid of the triangle be at the point $(5, -1)$.

Sol. Here, $$\frac{1 + h - 4}{3} = 5 \quad \text{and} \quad \frac{2 - 3 + k}{3} = -1$$

then, we get $h = 18, k = -2$

$$\therefore \quad = \sqrt{(h + k)^2 + (h + 3k)^2}$$
$$= \sqrt{(18 - 2)^2 + (18 - 6)^2}$$
$$= \sqrt{(16^2 + 12^2)} = 20$$

| Example 35. If $D(-2, 3), E(4, -3)$ and $F(4, 5)$ are the mid-points of the sides BC, CA and AB of triangle ABC, then find $\sqrt{(|AG|^2 + |BG|^2 - |CG|^2)}$ where, G is the centroid of ΔABC.

Sol. Let the coordinates of A be (α, β)

then coordinates of B are $(8 - \alpha, 10 - \beta)$

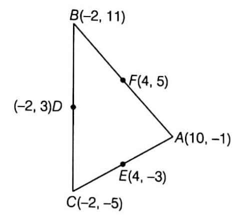
and coordinates of C are $(8 - \alpha, -6 - \beta)$

$\because D$ is the mid-point of BC, then

$$\frac{8 - \alpha + 8 - \alpha}{2} = -2$$

and $$\frac{10 - \beta - 6 - \beta}{2} = 3$$

i.e. $$\alpha = 10 \quad \text{and} \quad \beta = -1$$



$\therefore$ Coordinates of A, B, C are $(10, -1), (-2, 11)$ and $(-2, -5)$ respectively.

Now, coordinates of centroid

$$G \equiv \left(\frac{10 - 2 - 2}{3}, \frac{-1 + 11 - 5}{3}\right)$$

i.e. $$G \equiv \left(2, \frac{5}{3}\right)$$

$$\therefore \quad AG = \sqrt{(10 - 2)^2 + \left(-1 - \frac{5}{3}\right)^2}$$
$$= \sqrt{\left(64 + \frac{64}{9}\right)} = \frac{8}{3}\sqrt{(10)}$$
$$BG = \sqrt{(-2 - 2)^2 + \left(11 - \frac{5}{3}\right)^2}$$
$$= \sqrt{16 + \frac{(28)^2}{9}} = \frac{4}{3}\sqrt{(58)}$$
and $$CG = \sqrt{(-2 - 2)^2 + \left(-5 - \frac{5}{3}\right)^2}$$
$$= \sqrt{\left(16 + \frac{400}{9}\right)} = \frac{4}{3}\sqrt{(34)}$$

Hence, $$\sqrt{(|AG|^2 + |BG|^2 - |CG|^2)}$$
$$= \sqrt{\left(\frac{64}{9} \times 10 + \frac{16}{9} \times 58 - \frac{16}{9} \times 34\right)}$$
$$= \sqrt{\frac{32}{9}(20 + 29 - 17)}$$
$$= \sqrt{\left(\frac{32}{9} \times 32\right)} = \frac{32}{3}$$

| Example 36. If G be the centroid of the ΔABC and O be any other point in the plane of the triangle ABC, then show that
$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$.

Sol. Let G be the origin and GO be X-axis.

$$O \equiv (a, 0), A \equiv (x_1, y_1), B \equiv (x_2, y_2)$$
and $$C \equiv (x_3, y_3)$$

Now, LHS $= OA^2 + OB^2 + OC^2$

$$= (x_1 - a)^2 + y_1^2 + (x_2 - a)^2 + y_2^2 + (x_3 - a)^2 + y_3^2$$
$$= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) - 2a(x_1 + x_2 + x_3) + 3a^2$$
$$= \sum x_1^2 + \sum y_1^2 - 0 + 3a^2 \quad \left(\because \frac{x_1 + x_2 + x_3}{3} = 0 \text{ i.e., } x_1 + x_2 + x_3 = 0\right)$$
$$= \sum x_1^2 + \sum y_1^2 + 3a^2 \quad \text{...(i)}$$

and RHS $= GA^2 + GB^2 + GC^2 + 3GO^2$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 + 3\{(a - 0)^2\}$$
$$= \sum x_1^2 + \sum y_1^2 + 3a^2 \quad \text{... (ii)}$$

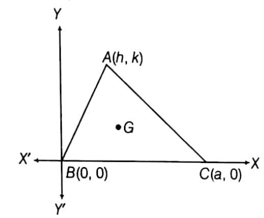
Hence, from Eqs. (i) and (ii), we get

$$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2.$$

| Example 37. If G be the centroid of ΔABC, show that

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

Sol. We take B as the origin and BC and BY as the X and Y-axes respectively.



Let $BC = a$, then $B \equiv (0, 0)$ and $C \equiv (a, 0)$
and let $A \equiv (h, k)$
then, coordinates of G will be

$$\left(\frac{h+0+a}{3}, \frac{k+0+0}{3}\right), \text{ i.e. } \left(\frac{h+a}{3}, \frac{k}{3}\right)$$

Take ΔABC as in 1st quadrant (for the sake of exactness).

Now, $\text{LHS} = (AB)^2 + (BC)^2 + (CA)^2$

$$= (h-0)^2 + (k-0)^2 + a^2 + (h-a)^2 + (k-0)^2$$
$$= 2h^2 + 2k^2 - 2ah + 2a^2 \quad \text{... (i)}$$

$$\text{RHS} = 3((GA)^2 + (GB)^2 + (GC)^2)$$
$$= 3\left\{\left(\frac{a+h}{3} - h\right)^2 + \left(\frac{k}{3} - k\right)^2 + \left(\frac{a+h}{3} - 0\right)^2 + \left(\frac{k}{3} - 0\right)^2 + \left(\frac{a+h}{3} - a\right)^2 + \left(\frac{k}{3} - 0\right)^2\right\}$$
$$= \frac{3}{9}\{(a-2h)^2 + (-2k)^2 + (a+h)^2 + k^2 + (h-2a)^2 + k^2\}$$
$$= \frac{1}{3}\{6a^2 + 6h^2 + 6k^2 - 6ah\}$$
$$= 2h^2 + 2k^2 - 2ah + 2a^2 \quad \text{... (ii)}$$

Hence, from Eqs. (i) and (ii), we get

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$$

Example 38. The vertices of a triangle are $(1, a)$, $(2, b)$ and $(c^2, -3)$

(i) Prove that its centroid can not lie on the Y-axis.

(ii) Find the condition that the centroid may lie on the X-axis.

Sol. Centroid of the triangle is

$$G \equiv \left(\frac{1+2+c^2}{3}, \frac{a+b-3}{3}\right) \text{ i.e. } \left(\frac{3+c^2}{3}, \frac{a+b-3}{3}\right)$$

(i) $\because$ G will lie on Y-axis, then

$$\frac{3+c^2}{3} = 0$$
$$\Rightarrow \quad c^2 = -3$$
$$\text{or} \quad c = \pm i\sqrt{3}$$

$\because$ Both values of c are imaginary.

Hence, G can not lie on Y-axis.

(ii) $\because$ G will lie on X-axis, then $\frac{a+b-3}{3} = 0$

$$\Rightarrow \quad a + b - 3 = 0$$
$$\text{or} \quad a + b = 3$$

Incentre

Definition : The point of intersection of internal angle bisectors of triangle is called the incentre of the triangle.

Theorem : Prove that the coordinates of the incentre of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are

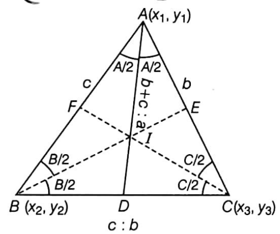
$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

where, a, b, c are the lengths of sides BC, CA and AB respectively.

Also, prove that the internal bisectors of the angles of a triangle are concurrent.

Proof : Given $A \equiv (x_1, y_1)$, $B(x_2, y_2)$, $C \equiv (x_3, y_3)$ be the vertices of ΔABC and $BC = a$, $CA = b$ and $AB = c$. Let AD be the bisector of A. We know that the bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the triangle.

$$\therefore \quad \boxed{\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}} \quad \text{... (i)}$$



Hence, D divides BC in the ratio $c : b$

$$\therefore \text{ Coordinates of } D \text{ are } \left(\frac{cx_3 + bx_2}{c+b}, \frac{cy_3 + by_2}{c+b}\right)$$

From Eq. (i), $\frac{DC}{BD} = \frac{b}{c}$ or $\frac{DC}{BD} + 1 = \frac{b}{c} + 1$

or $\frac{DC + BD}{BD} = \left(\frac{b+c}{c}\right)$ or $\frac{a}{BD} = \left(\frac{b+c}{c}\right)$

$$\therefore \quad BD = \frac{ac}{(b+c)}$$

Also, in ΔABD, BI is the bisector of B.

Then, $$\frac{AI}{ID} = \frac{AB}{BD} = \frac{c}{\left(\frac{ac}{b+c}\right)} = \frac{b+c}{a}$$

$\therefore$ I divides AD in the ratio $b + c : a$

$\therefore$ Coordinates of I are

$$\left(\frac{(b+c)\cdot\frac{cx_3 + bx_2}{c+b} + a\cdot x_1}{b+c+a}, \frac{(b+c)\cdot\frac{cy_3 + by_2}{c+b} + b\cdot y_1}{b+c+a}\right)$$

i.e. $$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

Similarly we can show that the coordinates of the point which divides BE internally in the ratio $c + a : b$ and the coordinates of the point which divides CF internally in the ratio $a + b : c$ will be each

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

and $$CE = \frac{ab}{(c+a)}, AE = \frac{bc}{(c+a)},$$
$$AF = \frac{bc}{a+b}, BF = \frac{ac}{a+b}$$

Thus, the three internal bisectors of the angles of a triangle meet in a point I.

$$I \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

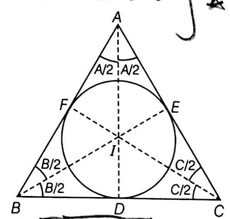
***Corollary* 1 :** If ΔABC is equilateral, then $a = b = c$

$$\text{incentre} = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \text{centroid}$$

i.e. incentre and centroid coincide in equilateral, triangle.

***Corollary* 2 :**

$$AE = AF = s - a$$
$$BD = BF = s - b$$
$$CD = CE = s - c$$



where, $$\boxed{s = \frac{a+b+c}{2}}$$

and $$|BC| = a, |CA| = b, |AB| = c$$

Proof : Let $AE = AF = \alpha$

($\because$ Lengths of tangents are equal from a point to a circle)

$$BD = BF = \beta$$
$$CD = CE = \gamma$$

Also, $$a = BC = BD + DC = \beta + \gamma \quad \text{....(i)}$$
$$b = CA = CE + AE = \gamma + \alpha \quad \text{...(ii)}$$
and $$c = AB = AF + BF = \alpha + \beta \quad \text{...(iii)}$$

Adding all, we get

$$a + b + c = 2(\alpha + \beta + \gamma)$$
$$\text{or} \quad 2s = 2(\alpha + \beta + \gamma)$$
$$\therefore \quad s = \alpha + \beta + \gamma$$

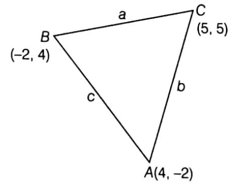
From Eqs. (i), (ii) and (iii), we get

$$\alpha = s - a, \beta = s - b, \gamma = s - c$$

Example 39. Find the coordinates of incentre of the triangle whose vertices are $(4, -2)$, $(-2, 4)$ and $(5, 5)$.

Sol. Let $A(4, -2)$, $B(-2, 4)$ and $C(5, 5)$ be the vertices of the given triangle. Then

$$a = BC = \sqrt{(-2-5)^2 + (4-5)^2} = \sqrt{50} = 5\sqrt{2}$$
$$b = CA = \sqrt{(5-4)^2 + (5+2)^2} = \sqrt{50} = 5\sqrt{2}$$
and $$c = AB = \sqrt{(4+2)^2 + (-2-4)^2} = \sqrt{72} = 6\sqrt{2}$$



Let (x, y) be the coordinates of incentre of ΔABC. Then

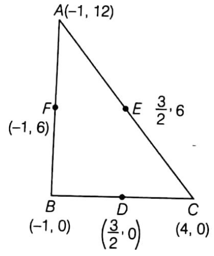
$$x = \frac{ax_1 + bx_2 + cx_3}{a+b+c}$$
$$= \frac{5\sqrt{2} \times 4 + 5\sqrt{2} \times (-2) + 6\sqrt{2} \times 5}{5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}}$$
$$= \frac{20\sqrt{2} - 10\sqrt{2} + 30\sqrt{2}}{16\sqrt{2}}$$
$$= \frac{40}{16} = \frac{5}{2}$$

and $$y = \frac{ay_1 + by_2 + cy_3}{a+b+c}$$
$$= \frac{5\sqrt{2} \times (-2) + 5\sqrt{2} \times 4 + 6\sqrt{2} \times 5}{5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}}$$
$$= \frac{40}{16} = \frac{5}{2}$$

$\therefore$ The coordinates of the incentre are $\left(\frac{5}{2}, \frac{5}{2}\right)$

Example 40. If $\left(\frac{3}{2}, 0\right), \left(\frac{3}{2}, 6\right)$ and $(-1, 6)$ are mid-points of the sides of a triangle, then find
(i) Centroid of the triangle
(ii) Incentre of the triangle

Sol. Let $A \equiv (\alpha, \beta)$, then coordinates of $B \equiv (-2-\alpha, 12-\beta)$ and coordinates of $C \equiv (3-\alpha, 12-\beta)$. But mid-point of BC is $\left(\frac{3}{2}, 0\right)$



then $\quad 3 = -2 - \alpha + 3 - \alpha$

$\Rightarrow \quad \alpha = -1$

and $\quad 0 = 12 - \beta + 12 - \beta$

$\Rightarrow \quad \beta = 12$

$\therefore$ Coordinates of vertices are

$A \equiv (-1, 12), B \equiv (-1, 0)$ and $C \equiv (4, 0)$

(i) Centroid : The centroid of ΔABC is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

or $\left(\frac{-1-1+4}{3}, \frac{12+0+0}{3}\right)$ i.e. $\left(\frac{2}{3}, 4\right)$

(ii) Incentre : We have

$$a = BC = \sqrt{(-1-4)^2 + (0-0)^2} = 5$$

$$b = CA = \sqrt{(4+1)^2 + (0-12)^2} = 13$$

and $\quad c = AB = \sqrt{(-1+1)^2 + (12-0)^2} = 12$

$\therefore$ The incentre of ΔABC is

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

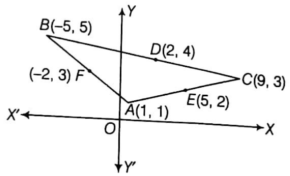
or $\left(\frac{5 \times (-1) + 13 \times (-1) + 12 \times 4}{5+13+12}, \frac{5 \times 12 + 13 \times 0 + 12 \times 0}{5+13+12}\right)$

i.e. $\quad (1, 2)$

Example 41. If a vertex of a triangle be (1, 1) and the middle points of two sides through it be (–2, 3) and (5, 2), then find the centroid and the incentre of the triangle.

Sol. Let coordinate of A be (1, 1) and mid-points of AB and AC are F and E.

$\therefore \quad F \equiv (-2, 3)$ and $E \equiv (5, 2)$



Hence, coordinates of B and C are $(2 \times (-2) - 1, (2 \times 3 - 1)$ and $(2 \times 2 - (-5), 2 \times 4 - 5)$ respectively.

i.e. $\quad B \equiv (-5, 5)$ and $C \equiv (9, 3)$

Then, centroid is $\left(\frac{1-5+9}{3}, \frac{1+5+3}{3}\right)$ i.e., $\left(\frac{5}{3}, 3\right)$

Also, $\quad a = |BC| = \sqrt{(-5-9)^2 + (5-3)^2} = \sqrt{200} = 10\sqrt{2}$

$$b = |CA| = \sqrt{(9-1)^2 + (3-1)^2} = \sqrt{68} = 2\sqrt{17}$$

and $\quad c = |AB| = \sqrt{(1+5)^2 + (1-5)^2} = \sqrt{52} = 2\sqrt{13}$

Then, incentre is

$$\left(\frac{10\sqrt{2} \times 1 + 2\sqrt{17} \times (-5) + 2\sqrt{13} \times 9}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}}, \frac{10\sqrt{2} \times 1 + 2\sqrt{17} \times 5 + 2\sqrt{13} \times 3}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}}\right)$$

i.e. $\quad \left(\frac{5\sqrt{2} - 5\sqrt{17} + 9\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}, \frac{5\sqrt{2} + 5\sqrt{17} + 3\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}\right)$

Example 42. If G be the centroid and I be the incentre of the triangle with vertices $A(-36, 7), B(20, 7)$ and $C(0, -8)$ and $GI = \frac{25}{3}\sqrt{(205)}\, \lambda$, then find the value of λ.

Sol. Coordinates of centroid are

$$G \equiv \left(-\frac{16}{3}, 2\right)$$

and $\quad a = |BC| = \sqrt{(20-0)^2 + (7+8)^2}$

$$= \sqrt{625} = 25$$

$$b = |CA| = \sqrt{(0+36)^2 + (-8-7)^2}$$

$$= \sqrt{1521} = 39$$

$$c = |AB| = \sqrt{(-36-20)^2 + (7-7)^2}$$

$$= \sqrt{(56)^2} = 56$$

Therefore, the coordinates of incentre are

$$I \equiv \left(\frac{25 \times (-36) + 39 \times 20 + 56 \times 0}{25 + 39 + 56}, \frac{25 \times 7 + 39 \times 7 + 56 \times (-8)}{25 + 39 + 56}\right)$$

i.e., $\quad I \equiv (-1, 0)$

$\therefore \quad GI = \sqrt{\left(-\frac{16}{3} + 1\right)^2 + (2-0)^2} = \frac{\sqrt{(205)}}{3}$

but given $\quad GI = \frac{25}{3}\sqrt{(205)}\, \lambda$

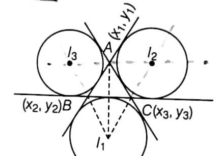
$\therefore \quad \frac{1}{3}\sqrt{(205)} = \frac{25}{3}\sqrt{(205)}\, \lambda$

$\Rightarrow \quad \lambda = \frac{1}{25}$

Some Standard Results

1. Excentres of a Triangle

This is the point of intersection of the external bisectors of the angles of a triangle.



The circle opposite to the vertex A is called the escribed circle opposite A or the circle escribed to the side BC. If I_1 is the point of intersection of internal bisector of $\angle BAC$ and external bisector of $\angle ABC$ and $\angle ACB$, then

$$I_1 \equiv \left(\frac{ax_1 - bx_2 - cx_3}{a-b-c}, \frac{ay_1 - by_2 - cy_3}{a-b-c}\right)$$

or $\quad I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$

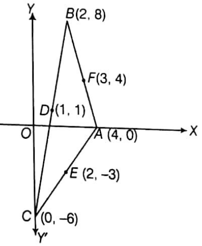
Similarly, $\quad I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$

and $\quad I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$

where, $|BC| = a, |CA| = b$ and $|AB| = c$

Example 43. If the coordinates of the mid-points of sides BC, CA and AB of triangle ABC are (1, 1), (2, –3) and (3, 4), then find the excentre opposite to the vertex A.

Sol. Let $D(1, 1)$, $E(2, -3)$ and $F(3, 4)$ are the mid-points of the sides of the triangle BC, CA and AB respectively. Let $A \equiv (\alpha, \beta)$



then $\quad B \equiv (6 - \alpha, 8 - \beta)$

and $\quad C \equiv (4 - \alpha, -6 - \beta)$

Also, D is the mid-point of B and C, then

$$1 = \frac{6 - \alpha + 4 - \alpha}{2} \Rightarrow \alpha = 4$$

and $\quad 1 = \frac{8 - \beta - 6 - \beta}{2} \Rightarrow \beta = 0$

$\therefore \quad A \equiv (4, 0), B \equiv (2, 8)$ and $C \equiv (0, -6)$, then

$$a = |BC| = \sqrt{(2-0)^2 + (8+6)^2} = \sqrt{200} = 10\sqrt{2}$$

$$b = |CA| = \sqrt{(0-4)^2 + (-6-0)^2} = \sqrt{52} = 2\sqrt{13}$$

and $\quad c = |AB| = \sqrt{(4-2)^2 + (0-8)^2} = \sqrt{68} = 2\sqrt{17}$

Hence, the coordinates of the excentre opposite to A are

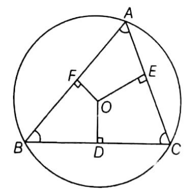
$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

i.e., $\quad \left(\frac{-10\sqrt{2} \times 4 + 2\sqrt{13} \times 2 + 2\sqrt{17} \times 0}{-10\sqrt{2} + 2\sqrt{13} + 2\sqrt{17}}, \frac{-10\sqrt{2} \times 0 + 2\sqrt{13} \times 8 + 2\sqrt{17} \times (-6)}{-10\sqrt{2} + 2\sqrt{13} + 2\sqrt{17}}\right)$

or $\quad \left(\frac{-20\sqrt{2} + 2\sqrt{13}}{-5\sqrt{2} + \sqrt{13} + \sqrt{17}}, \frac{8\sqrt{13} - 6\sqrt{17}}{-5\sqrt{2} + \sqrt{13} + \sqrt{17}}\right)$

2. Circumcentre of a Triangle

The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle (i.e., the lines through the mid-point of a side and perpendicular to it). Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and if angles of ΔABC are given, then coordinates of circumcentre



are $\left(\dfrac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \dfrac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$

Or

$$\left(\frac{ax_1 \cos A + bx_2 \cos B + cx_3 \cos C}{a\cos A + b\cos B + c\cos C}, \frac{ay_1 \cos A + by_2 \cos B + cy_3 \cos C}{a\cos A + b\cos B + c\cos C}\right)$$

where, $|BC| = a, |CA| = b$ and $|AB| = c$

| Example 44. In a ΔABC with vertices $A(1,2)$, $B(2,3)$ and $C(3,1)$ and $\angle A = \angle B = \cos^{-1}\left(\dfrac{1}{\sqrt{10}}\right)$, $\angle C = \cos^{-1}\left(\dfrac{4}{5}\right)$, then find the circumcentre of ΔABC.

Sol. Since, $\angle A = \angle B = \cos^{-1}\left(\dfrac{1}{\sqrt{10}}\right)$

$\Rightarrow \quad \cos A = \cos B = \dfrac{1}{\sqrt{10}}$

then, $\sin A = \sin B = \dfrac{3}{\sqrt{10}}$

$\therefore \quad \sin 2A = \sin 2B = 2 \times \dfrac{3}{\sqrt{10}} \times \dfrac{1}{\sqrt{10}} = \dfrac{3}{5}$

and $\quad \angle C = \cos^{-1}\left(\dfrac{4}{5}\right)$

$\Rightarrow \quad \cos C = \dfrac{4}{5}$ then, $\sin C = \dfrac{3}{5}$

$\therefore \quad \sin 2C = 2 \times \dfrac{3}{5} \times \dfrac{4}{5} = \dfrac{24}{25}$

Let the circumcenter be (x, y), then

$$x = \frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} = \frac{1 \times \frac{3}{5} + 2 \times \frac{3}{5} + 3 \times \frac{24}{25}}{\frac{3}{5} + \frac{3}{5} + \frac{24}{25}} = \frac{13}{6}$$

and
$$y = \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} = \frac{2 \times \frac{3}{5} + 3 \times \frac{3}{5} + 1 \times \frac{24}{25}}{\frac{3}{5} + \frac{3}{5} + \frac{24}{25}} = \frac{11}{6}$$

Hence, coordinates of circumcenter are $\left(\dfrac{13}{6}, \dfrac{11}{6}\right)$.

Two Important Tricks for Circumcentre

(a) If angles of triangle ABC are not given and the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are given, then the circumcentre of the ΔABC is given by

$$\left(\frac{(x_1 + x_2) + \lambda(y_1 - y_2)}{2}, \frac{(y_1 + y_2) - \lambda(x_1 - x_2)}{2}\right)$$

Here, we observe that

$$P = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$

$$\therefore \quad \lambda = \frac{\vec{R}_1 \cdot \vec{R}_2}{|P|} \rightarrow R_i = \text{column}$$

(b) If the angle C is given instead of coordinates of the vertex C and the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ of ΔABC are given, then the circumcentre of ΔABC is given by

$$\left(\frac{(x_1 + x_2) \pm \cot C(y_1 - y_2)}{2}, \frac{(y_1 + y_2) \pm \cot C(x_1 - x_2)}{2}\right)$$

Remark

Circumcentre of the right angled triangle ABC, right angled at A is $\dfrac{B + C}{2}$.

| Example 45. Find the circumcentre of the triangle whose vertices are (2, 2), (4, 2) and (0, 4).

Sol. Let the given points are (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.

for the matrix

$$P = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix}$$

$$\therefore \quad \lambda = \frac{\vec{R}_1 \cdot \vec{R}_2}{|P|} = \frac{2 \times 4 + (-2) \times (-2)}{2 \times (-2) - 4 \times (-2)} = \frac{12}{4} = 3$$

$\therefore$ Circumcentre of the triangle

$$\equiv \left(\frac{(x_1 + x_2) + \lambda(y_1 - y_2)}{2}, \frac{(y_1 + y_2) - \lambda(x_1 - x_2)}{2}\right)$$
$$\equiv \left(\frac{2 + 4 + 3(2 - 2)}{2}, \frac{2 + 2 - 3(2 - 4)}{2}\right)$$
$$\equiv (3, 5)$$

| Example 46. Find the circumcentre of triangle ABC if $A \equiv (7, 4)$, $B \equiv (3, -2)$ and $\angle C = \dfrac{\pi}{3}$.

Sol. Here, $x_1 = 7, y_1 = 4, x_2 = 3, y_2 = -2$ and $\angle C = \dfrac{\pi}{3}$

$\therefore$ The circumcentre of ΔABC

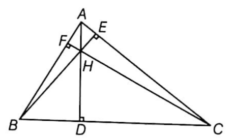
$$\equiv \left(\frac{(x_1 + x_2) \pm \cot C(y_1 - y_2)}{2}, \frac{(y_1 + y_2) \mp \cot C(x_1 - x_2)}{2}\right)$$
$$\equiv \left(\frac{(7 + 3) \pm \frac{1}{\sqrt{3}}(4 + 2)}{2}, \frac{(4 - 2) \mp \frac{1}{\sqrt{3}}(7 - 3)}{2}\right)$$
$$\equiv \left(5 + \sqrt{3}, 1 - \frac{2}{\sqrt{3}}\right) \text{ or } \left(5 - \sqrt{3}, 1 + \frac{2}{\sqrt{3}}\right)$$

3. Orthocentre of a Triangle

The orthocentre of a triangle is the point of intersection of altitudes

(i.e., the lines through the vertices and perpendicular to opposite sides).

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and if angles of ΔABC are given, then coordinates of orthocentre are



$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

or

$$\left(\frac{ax_1 \sec A + bx_2 \sec B + cx_3 \sec C}{a\sec A + b\sec B + c\sec C}, \frac{ay_1 \sec A + by_2 \sec B + cy_3 \sec C}{a\sec A + b\sec B + c\sec C}\right)$$

where, $|BC| = a, |CA| = b$ and $|AB| = c$

Important trick for orthocentre :

orthocentre of the triangle whose vertices are $(0, 0)$, (x_1, y_1) is given by

$$(\lambda(y_2 - y_1), -\lambda(x_2 - x_1))$$

where, $\quad \lambda = \dfrac{x_1 x_2 + y_1 y_2}{x_1 y_2 - x_2 y_1}$

Remarks

1. The orthocentre of a triangle having vertices (α, β), (β, α) and (α, α) is (α, α).
2. The orthocentre of a triangle having vertices is $\left(-\dfrac{1}{\alpha\beta\gamma}, -\alpha\beta\gamma\right)$
3. The orthocentre of right angled triangle ABC, right angled at A is A.

Ex.3 Alter in JEE Type

| Example 47. Find the orthocentre of ΔABC if $A \equiv (0, 0)$, $B \equiv (3, 5)$ and $C \equiv (4, 7)$.

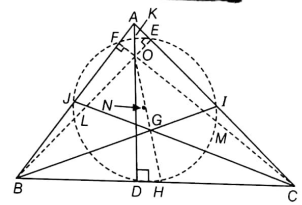
Sol. Here, $x_1 = 3, y_1 = 5, x_2 = 4, y_2 = 7$

$$\therefore \quad \lambda = \frac{3 \times 4 + 5 \times 7}{3 \times 7 - 4 \times 5} = 47$$

$\Rightarrow$ Orthocentre of $\Delta ABC \equiv (47(7 - 5), -47(4 - 3))$
$\equiv (94, -47)$

4. Nine Point Centre of a Triangle

If a circle passing through the feet of perpendiculars (i.e., D, E, F) mid-points of sides BC, CA, AB respectively (i.e., H, I, J) and the n x mid-points of the line joining the orthocentre O to the angular points A, B, C (i.e., K, L, M) thus the nine points $D, E, F, H, I, J, K, L, M$ all lie on a circle.



This circle is known as nine point circle and its centre is called the nine point centre. The nine-point centre of a triangle is collinear with the circumcentre and the orthocentre and bisects the segments joining them and radius of nine point circle of a triangle is half the radius of the circumcircle.

Corollary 1 : The orthocentre, the nine point centre, the centroid and the circumcentre therefore all lie on a straight line.

Corollary 2 : If O is orthocentre, N is nine point centre, G is centroid and C is circumcentre, then to remember it see ONGC (i.e. Oil Natural Gas Corporation) in left of G are 2 and in right is 1, therefore G divides O and C in the ratio 2 : 1 (internally).

Corollary 3 : N is the mid-point of O and C

Corollary 4 : Radius of nine point circle $= \dfrac{1}{2} \times$ Radius of circumcircle

Remarks

1. The distance between the orthocentre and circumcentre in an equilateral triangle is zero.
2. If the circumcentre and centroid of a triangle are respectively (α, β) (γ, δ) then orthocentre will be $(3\gamma - 2\alpha, 3\delta - 2\beta)$.

Example 48. If a triangle has its orthocentre at (1,1) and circumcentre at $\left(\frac{3}{2}, \frac{3}{4}\right)$, then find the centroid and nine point centre.

Sol. Since, centroid divides the orthocentre and circumcentre in the ratio 2 : 1 (internally) and if centroid $G(x, y)$, then

$$O(1,1) \overset{2}{\text{———}} G(x,y) \overset{1}{\text{———}} C\left(\frac{3}{2}, \frac{3}{4}\right)$$

$$x = \frac{2 \times \frac{3}{2} + 1 \times 1}{2+1} = \frac{4}{3}$$

and $$y = \frac{2 \times \frac{3}{4} + 1 \times 1}{2+1} = \frac{5}{6}$$

$\therefore$ Centroid is $\left(\frac{4}{3}, \frac{5}{6}\right)$ and nine point centre is the mid-point of orthocentre and circumcentre.

$\therefore$ Nine point centre is $\left(\frac{1+\frac{3}{2}}{2}, \frac{1+\frac{3}{4}}{2}\right)$, i.e. $\left(\frac{5}{4}, \frac{7}{8}\right)$.

Must

Example 49. The vertices of a triangle are $A(a, a\tan\alpha)$, $B(b, b\tan\beta)$ and $C(c, c\tan\gamma)$. If the circumcentre of ΔABC coincides with the origin and $H(\bar{x}, \bar{y})$ is the orthocentre, then show that

$$\frac{\bar{y}}{\bar{x}} = \left(\frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma}\right).$$

Sol. If R be the circumradius and O be the circumcentre

$\therefore$ $OA = OB = OC = R$

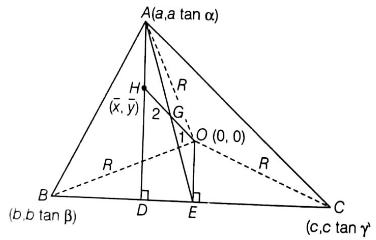
or $\sqrt{(a^2 + a^2\tan^2\alpha)} = \sqrt{(b^2 + b^2\tan^2\beta)}$

$= \sqrt{(c^2 + c^2\tan^2\gamma)} = R$

or $a\sec\alpha = b\sec\beta = c\sec\gamma = R$

or $a = R\cos\alpha$, $b = R\cos\beta$, $c = R\cos\gamma$

then, $a\tan\alpha = R\cos\alpha \cdot \frac{\sin\alpha}{\cos\alpha} = R\sin\alpha$



Similarly,

$b\tan\beta = R\sin\beta$ and $c\tan\gamma = R\sin\gamma$

$\therefore$ Coordinates of the vertices of a triangle are :

$A(R\cos\alpha, R\sin\alpha)$, $B(R\cos\beta, R\sin\beta)$

and $C(R\cos\gamma, R\sin\gamma)$

$\therefore$ Centroid

$$(G) \equiv \left(\frac{R(\cos\alpha + \cos\beta + \cos\gamma)}{3}, \frac{R(\sin\alpha + \sin\beta + \sin\gamma)}{3}\right)$$

Since, G divides H and O in the ratio 2 : 1 (internally), then

$$\frac{R}{3}(\cos\alpha + \cos\beta + \cos\gamma) = \frac{2\cdot 0 + 1\cdot\bar{x}}{2+1}$$

or $$\frac{R}{3}(\cos\alpha + \cos\beta + \cos\gamma) = \frac{\bar{x}}{3} \quad \text{...(i)}$$

and $$\frac{R}{3}(\sin\alpha + \sin\beta + \sin\gamma) = \frac{2\cdot 0 + 1\cdot\bar{y}}{2+1}$$

or $$\frac{R}{3}(\sin\alpha + \sin\beta + \sin\gamma) = \frac{\bar{y}}{3} \quad \text{...(ii)}$$

Dividing Eqs. (ii) by (i), then we get

$$\frac{\bar{y}}{\bar{x}} = \left(\frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma}\right)$$

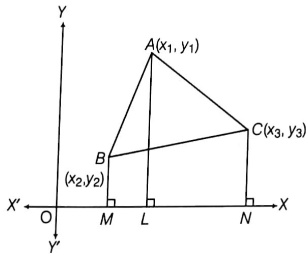
Area of a Triangle

Theorem : The area of a triangle, the coordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

or $$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

Proof : Let ABC be a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let us assume that the points A, B and C are in 1st quadrant (for the sake of exactness). Draw AL, BM and CN perpendicular on X-axis. Let Δ be the required area of the triangle ABC, then



Δ = Area of triangle ABC

= [Area of trapezium $ABML$ + Area of trapezium $ALNC$ − Area of trapezium $BMNC$]

[$\because$ Area of trapezium $= \frac{1}{2}$ (Sum of parallel sides) × (Distance between them)]

$$\therefore \Delta = \left[\frac{1}{2}(BM + AL)(ML) + \frac{1}{2}(AL + CN)(LN) - \frac{1}{2}(BM + CN)(MN)\right]$$

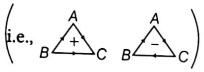
$$= \left[\frac{1}{2}(BM + AL)(OL - OM) + \frac{1}{2}(AL + CN)(ON - (OL) - \frac{1}{2}(BM + CN)(ON - OM)\right]$$

$$= \left[\frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)\right]$$

$$= \frac{1}{2}[x_1(y_2 + y_1 - y_1 - y_3) + x_2(-y_2 - y_1 + y_2 + y_3) + x_3(y_1 + y_3 - y_2 - y_3)]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

The area of triangle ABC will come out to be a positive quantity only when the vertices A, B, C are taken in anticlockwise direction and if points A, B, C are taken in clockwise direction then the area will be negative and if the points A, B, C are taken arbitrary then the area will be positive or negative, the numerical value being the same in all cases.

Thus in general (i.e., )

$$\text{Area of } \Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

This expression can be written in determinant form as follows

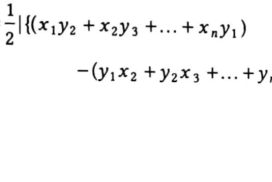
$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

Corollary 1 : Area of triangle can also be found by easy method

$$\Delta = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right|$$

and **Area of quadrilateral $ABCD$:** The area of a quadrilateral can be found out by dividing the quadrilateral into two triangles.

$\therefore$ Area of quadrilateral $ABCD$



= Area of ΔABC + Area of ΔDAC

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}$$

$$= \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}\right| + \frac{1}{2}\begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}$$

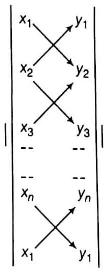
$$\left(\because \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} = -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}\right)$$

$\therefore$ **Area of polygon whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n) is**

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \ldots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right|$$

Or

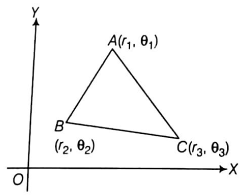
Stair Method Repeat first coordinates one time in last for down arrow use positive sign and for up arrow use negative sign.

$\therefore$ Area of polygon $= \frac{1}{2}$ 

$$= \frac{1}{2}|\{(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (y_1x_2 + y_2x_3 + \ldots + y_nx_1)\}|$$

Corollary 2 : If the coordinates of the vertices of the triangle are given in polar form i.e.,

$A(r_1, \theta_1), B(r_2, \theta_2), C(r_3, \theta_3)$.



Then, area of triangle

$$= \frac{1}{2} | r_1 r_2 \sin(\theta_1 - \theta_2) + r_2 r_3 \sin(\theta_2 - \theta_3) + r_3 r_1 \sin(\theta_3 - \theta_1) |$$

$$= \frac{1}{2} | \sum r_1 r_2 \sin(\theta_1 - \theta_2) |$$

Corollary 3 : If $a_1 x + b_1 y + c_1 = 0$, $a_2 x + b_2 y + c_2 = 0$ and $a_3 x + b_3 y + c_3 = 0$ are the sides of a triangle, then the area of the triangle is given by (without solving the vertices)

$$\Delta = \frac{1}{2|C_1 C_2 C_3|} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where, C_1, C_2, C_3 are the cofactors of c_1, c_2, c_3 in the determinant

Here, $C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)$

$C_2 = \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = (a_3 b_1 - a_1 b_3)$

and $C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (a_1 b_2 - a_2 b_1)$

and $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = c_1 C_1 + c_2 C_2 + c_3 C_3$

Or

Area of triangle $= \dfrac{\Delta^2}{2|\Delta_1 \Delta_2 \Delta_3|}$

where, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

and $\Delta_3 = \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}$

Corollary 4 : Area of the triangle formed by the lines of the form $y = m_1 x + c_1$, $y = m_2 x + c_2$ and $y = m_3 x + c_3$ is

$$\Delta = \frac{1}{2} \left| \frac{(c_2 - c_3)^2}{(m_2 - m_3)} + \frac{(c_3 - c_1)^2}{(m_3 - m_1)} + \frac{(c_1 - c_2)^2}{(m_1 - m_2)} \right|$$

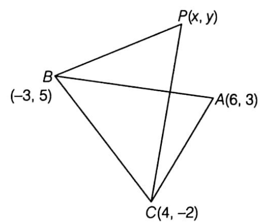
Remarks

1. If area of a triangle is given then, use ± sign.
2. The points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, then area of $(\Delta ABC) = 0$.
3. Four given points will be collinear, then area of the quadrilateral is zero.
4. Area of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\Delta = \frac{1}{2} \left\| \begin{matrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{matrix} \right\|$
5. If one vertex (x_3, y_3) is at (0, 0) then, $\Delta = \frac{1}{2} | x_1 y_2 - x_2 y_1 |$

| Example 50. The coordinates of A, B, C are $(6, 3), (-3, 5)$ and $(4, -2)$ respectively and P is any points (x, y). Show that the ratio of the areas of the triangles PBC and ABC is $\dfrac{|x + y - 2|}{7}$.

Sol. We have

$$\frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2} |\{x(5 + 2) - 3(-2 - y) + 4(y - 5)\}|}{\frac{1}{2} |\{6(5 + 2) - 3(-2 - 3) + 4(3 - 5)\}|}$$



$$= \frac{|7x + 7y - 14|}{|49|} = \frac{7|x + y - 2|}{49} = \frac{|x + y - 2|}{7}$$

| Example 51. Find the area of the pentagon whose vertices are $A(1, 1)$, $B(7, 21)$, $C(7 - 3)$, $D(12, 2)$ and $E(0, -3)$.

Sol. The required area

$$= \frac{1}{2} \left| \begin{vmatrix} 1 & 1 \\ 7 & 21 \end{vmatrix} + \begin{vmatrix} 7 & 21 \\ 7 & -3 \end{vmatrix} + \begin{vmatrix} 7 & -3 \\ 12 & 2 \end{vmatrix} + \begin{vmatrix} 12 & 2 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} |(21 - 7) + (-21 - 147) + (14 + 36) + (-36 - 0) + (0 + 3)|$$

$$= \frac{1}{2} | -137 | = \frac{137}{2} \text{ sq units}$$

| Example 52. Show that the points $(a, 0)$, $(0, b)$ and $(1, 1)$ are collinear, if $\dfrac{1}{a} + \dfrac{1}{b} = 1$

Sol. Let $A \equiv (a, 0)$, $B \equiv (0, b)$ and $C \equiv (1, 1)$

Now, points A, B, C will be collinear, if area of $\Delta ABC = 0$

or $\quad \frac{1}{2} \left| \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} + \begin{vmatrix} 0 & b \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ a & 0 \end{vmatrix} \right| = 0$

$\Rightarrow \quad |(ab - 0) + (0 - b) + (0 - a)| = 0$

or $\quad ab - a - b = 0$

or $\quad a + b = ab$ or $\dfrac{1}{a} + \dfrac{1}{b} = 1$

| Example 53. Prove that the coordinates of the vertices of an equilateral triangle can not all be rational.

Sol. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible let $x_1, y_1, x_2, y_2, x_3, y_3$ be all rational.

Now, area of $\Delta ABC = \frac{1}{2} | x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) |$

$= \text{Rational} \quad \ldots(i)$

Since, ΔABC is equilateral

$\therefore$ Area of $\Delta ABC = \dfrac{\sqrt{3}}{4} (\text{side})^2$

$= \dfrac{\sqrt{3}}{4} (AB)^2$

$= \dfrac{\sqrt{3}}{4} \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$

$= \text{Irrational} \quad \ldots(ii)$

From Eqs. (i) and (ii),

Rational = Irrational

which is contradiction.

Hence, $x_1, y_1, x_2, y_2, x_3, y_3$ cannot all be rational.

| Example 54. The coordinates of two points A and B are (3, 4) and (5, –2) respectively. Find the coordinates of any point P if $PA = PB$ and area of ΔAPB is 10.

Sol. Let coordinates of P be (h, k).

$\because \quad PA = PB \Rightarrow (PA)^2 = (PB)^2$

$\Rightarrow \quad (h - 3)^2 + (k - 4)^2 = (h - 5)^2 + (k + 2)^2$

$\Rightarrow \quad (h - 3)^2 - (h - 5)^2 + (k - 4)^2 - (k + 2)^2 = 0$

$\Rightarrow \quad (2h - 8)(2) + (2k - 2)(-6) = 0$

$\Rightarrow \quad (h - 4) - 3(k - 1) = 0$

$\Rightarrow \quad h - 3k - 1 = 0 \quad \ldots(i)$

Now, Area of $\Delta PAB = \frac{1}{2} \left\| \begin{matrix} h & k & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{matrix} \right\| = 10$

or $\quad 6h + 2k - 26 = \pm 20$

$\Rightarrow \quad 6h + 2k - 46 = 0$ or $6h + 2k - 6 = 0$

$\Rightarrow \quad 3h + k - 23 = 0$ or $3h + k - 3 = 0$

Solving $\quad h - 3k - 1 = 0$ and $3h + k - 23 = 0$,

we get $\quad h = 7, k = 2$

Solving $\quad h - 3k - 1 = 0$ and $3h + k - 3 = 0$,

we get $\quad h = 1, k = 0$

Hence, the coordinates of P are (7, 2) or (1, 0).

| Example 55. Find the area of the triangle formed by the straight lines $7x - 2y + 10 = 0$, $7x + 2y - 10 = 0$ and $9x + y + 2 = 0$ (without solving the vertices of the triangle).

Sol. The given lines are :

$7x - 2y + 10 = 0$

$7x + 2y - 10 = 0$

$9x + y + 2 = 0$

$\therefore$ Area of triangle $\Delta = \dfrac{1}{2|C_1 C_2 C_3|} \begin{vmatrix} 7 & -2 & 10 \\ 7 & 2 & -10 \\ 9 & 1 & 2 \end{vmatrix}^2 \quad \ldots(i)$

where, $\quad C_1 = \begin{vmatrix} 7 & 2 \\ 9 & 1 \end{vmatrix} = 7 - 18 = -11,$

$C_2 = \begin{vmatrix} 9 & 1 \\ 7 & -2 \end{vmatrix} = -18 - 7 = -25$

and $\quad C_3 = \begin{vmatrix} 7 & -2 \\ 7 & 2 \end{vmatrix} = 14 + 14 = 28,$

and $\quad \begin{vmatrix} 7 & -2 & 10 \\ 7 & 2 & -10 \\ 9 & 1 & 2 \end{vmatrix} = 10C_1 - 10C_2 + 2C_3$

$= 10 \times (-11) - 10 \times (-25) + 2 \times 28 = 196$

$\therefore$ From Eq. (i), $\Delta = \dfrac{1}{2|-11 \times (-25) \times 28|} \times (196)^2$

$= \dfrac{196 \times 196}{2 \times 11 \times 25 \times 28} = \dfrac{686}{275}$ sq units

| Example 56. If Δ_1 is the area of the triangle with vertices $(0, 0)$, $(a \tan\alpha, b \cot\alpha)$, $(a \sin\alpha, b \cos\alpha)$; Δ_2 is the area of the triangle with vertices (a, b), $(a \sec^2\alpha, b \operatorname{cosec}^2\alpha)$, $(a + a \sin^2\alpha, b + b \cos^2\alpha)$ and Δ_3 is the area of the triangle with vertices $(0, 0)$, $(a \tan\alpha, -b \cot\alpha)$, $(a \sin\alpha, b \cos\alpha)$. Show that there is no value of α for which Δ_1, Δ_2 and Δ_3 are in GP.

Sol. We have, $\Delta_1 = \frac{1}{2}|(a\tan\alpha)(b\cos\alpha) - (a\sin\alpha)(b\cot\alpha)|$ ($\because$ one vertex is (0,0))

$= \frac{1}{2}|ab||\sin\alpha - \cos\alpha|$ ... (i)

and $\Delta_2 = \frac{1}{2}\left|\begin{vmatrix} a-(a+a\sin^2\alpha) & a\sec^2\alpha-(a+a\sin^2\alpha) \\ b-(b+b\cos^2\alpha) & b\,\mathrm{cosec}^2\alpha-(b+b\cos^2\alpha) \end{vmatrix}\right|$ (See remark 4)

$= \frac{1}{2}\left|\begin{vmatrix} -a\sin^2\alpha & a(\tan^2\alpha-\sin^2\alpha) \\ -b\cos^2\alpha & b(\cot^2\alpha-\cos^2\alpha) \end{vmatrix}\right|$

$= \frac{1}{2}|ab|\times\left|\begin{vmatrix} -\sin^2\alpha & \sin^2\alpha(\sec^2\alpha-1) \\ -\cos^2\alpha & \cos^2\alpha(\mathrm{cosec}^2\alpha-1) \end{vmatrix}\right|$

$= \frac{1}{2}|ab|\times\left|\begin{vmatrix} -\sin^2\alpha & \sin^2\alpha\tan^2\alpha \\ -\cos^2\alpha & \cos^2\alpha\cot^2\alpha \end{vmatrix}\right|$

$= \frac{1}{2}|ab|\times|-\sin^2\alpha\cos^2\alpha\cot^2\alpha + \sin^2\alpha\cos^2\alpha\tan^2\alpha|$

$= \frac{1}{2}|ab|\times|-\cos^4\alpha+\sin^4\alpha|$

$= \frac{1}{2}|ab|\times|\sin^2\alpha+\cos^2\alpha|\times|\sin^2\alpha-\cos^2\alpha|$

$= \frac{1}{2}|ab|\times|1|\times|-\cos 2\alpha|$

$= \frac{1}{2}|ab|\times|\cos 2\alpha|$ ... (ii)

and $\Delta_3 = \frac{1}{2}|(a\tan\alpha)(b\cos\alpha) - (-b\cot\alpha)(a\sin\alpha)|$

$= \frac{1}{2}|ab||\sin\alpha+\cos\alpha|$ ... (iii)

Since, $\Delta_1, \Delta_2, \Delta_3$ are in GP, then $\Delta_1\Delta_3 = \Delta_2^2$

$\Rightarrow \frac{1}{2}|ab||\sin\alpha-\cos\alpha|\times\frac{1}{2}|ab||\sin\alpha+\cos\alpha|$

$= \frac{1}{4}|ab|^2|\cos 2\alpha|^2$ [from Eqs. (i), (ii) and (iii)]

$\Rightarrow |\sin^2\alpha-\cos^2\alpha| = |\cos 2\alpha|^2$

$\Rightarrow |-\cos 2\alpha| = |\cos 2\alpha|^2$

$\Rightarrow |\cos 2\alpha| = |\cos 2\alpha|^2$

$\Rightarrow |\cos 2\alpha|(1-|\cos 2\alpha|) = 0$

$\therefore 1-|\cos 2\alpha| = 0$ ($\because |\cos 2\alpha| \neq 0$)

$\Rightarrow |\cos 2\alpha| = 1$

or $\cos 2\alpha = \pm 1$ or $\cos 2\alpha = 1$

and $\cos 2\alpha = -1$

or $2\alpha = 2n\pi, 2\alpha = (2p+1)\pi$

or $\alpha = n\pi, \alpha = p\pi + \frac{\pi}{2}; n, p \in I$

For these values of α the vertices of the given triangles are not defined. Hence Δ_1, Δ_2 and Δ_3 cannot be in GP for any value of α.

Exercise for Session 3

1. The coordinates of the middle points of the sides of a triangle are (4, 2), (3, 3) and (2, 2), then coordinates of centroid are

(a) (3, 7/3) (b) (3, 3) (c) (4, 3) (d) (3, 4)

2. The incentre of the triangle whose vertices are (−36, 7), (20, 7) and (0, −8) is

(a) (0, −1) (b) (−1, 0) (c) (1, 1) (d) $\left(\frac{1}{2}, 1\right)$

3. If the orthocentre and centroid of a triangle are (−3, 5) and (3, 3) then its circumcentre is

(a) (6, 2) (b) (3, −1) (c) (−3, 5) (d) (−3, 1)

4. An equilateral triangle has each side equal to a. If the coordinates of its vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) then the square of the determinant $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ equals

(a) $3a^4$ (b) $\frac{3a^4}{2}$ (c) $\frac{3}{4}a^4$ (d) $\frac{3}{8}a^4$

5. The vertices of a triangle are $A(0,0)$, $B(0,2)$ and $C(2,0)$. The distance between circumcentre and orthocentre is

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 2 (d) $\frac{1}{2}$

6. $A(a, b)$, $B(x_1, y_1)$ and $C(x_2, y_2)$ are the vertices of a triangle. If a, x_1, x_2 are in GP with common ratio r and b, y_1, y_2 are in GP with common ratio s, then area of ΔABC is

(a) $ab(r-1)(s-1)(s-r)$ (b) $\frac{1}{2}ab(r+1)(s+1)(s-r)$

(c) $\frac{1}{2}ab(r-1)(s-1)(s-r)$ (d) $ab(r+1)(s+1)(r-s)$

7. The points $(x+1, 2)$, $(1, x+2)$, $\left(\frac{1}{x+1}, \frac{2}{x+1}\right)$ are collinear, then x is equal to

(a) −4 (b) −8 (c) 4 (d) 8

8. The vertices of a triangle are (6, 0), (0, 6) and (6, 6). Then distance between its circumcentre and centroid, is

(a) $2\sqrt{2}$ (b) 2 (c) $\sqrt{2}$ (d) 1

9. The nine point centre of the triangle with vertices $(1, \sqrt{3})$, (0, 0) and (2, 0) is

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$ (c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

10. The vertices of a triangle are (0, 0), (1, 0) and (0, 1). Then excentre opposite to (0, 0) is

(a) $\left(1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$ (b) $\left(1+\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$ (c) $\left(1+\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$ (d) $\left(1-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$

11. If α, β, γ are the real roots of the equation $x^3 - 3px^2 + 3qx - 1 = 0$, then find the centroid of the triangle whose vertices are $\left(\alpha, \frac{1}{\alpha}\right)$, $\left(\beta, \frac{1}{\beta}\right)$ and $\left(\gamma, \frac{1}{\gamma}\right)$.

12. If centroid of a triangle be (1, 4) and the coordinates of its any two vertices are (4, −8) and (−9, 7), find the area of the triangle.

13. Find the centroid and incentre of the triangle whose vertices are (1, 2), (2, 3) and (3, 4).

14. Show that the area of the triangle with vertices $(\lambda, \lambda-2)$, $(\lambda+3, \lambda)$ and $(\lambda+2, \lambda+2)$ is independent of λ.

15. Prove that the points $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ are collinear.

16. Prove that the points (a, b), (c, d) and $(a-c, b-d)$ are collinear, if $ad = bc$.

17. If the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, show that $\sum\left(\frac{y_1-y_2}{x_1x_2}\right) = 0$, i.e.

$$\frac{y_1-y_2}{x_1x_2} + \frac{y_2-y_3}{x_2x_3} + \frac{y_3-y_1}{x_3x_1} = 0$$

18. The coordinates of points A, B, C and D are (−3, 5), (4, −2), $(x, 3x)$ and (6, 3) respectively and $\frac{\Delta ABC}{\Delta BCD} = \frac{2}{3}$, find x.

19. Find the area of the hexagon whose vertices taken in order are (5, 0), (4, 2), (1, 3), (−2, 2), (−3, −1) and (0, −4).

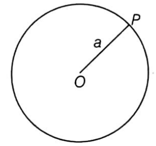
Session 4

Locus and Its Equation, Change of Axes the Transformation of Axes, Removal of the Term xy from $F(x, y) = ax^2 + 2hxy + by^2$ without Changing the Origin, Position of a Point which lies Inside a Triangle

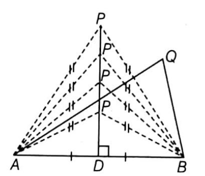
Locus and Its Equation

Locus : The locus of a moving point is the path traced out by that point under one or more given conditions.

For example 1. If a point P moves in a plane such that whose distance from a fixed point O (say) in the plane is always constant distance a. Thus the locus of the moving point P is clearly a circle with centre O and radius a.



For example 2. If a point P moves in a plane such that whose distance from two fixed points A and B (say) are always equal i.e. $PA = PB$



(The point P cannot be at Q because $AQ \neq BQ$)

Obviously all the positions of the moving point P lies on the right bisector of AB. Thus the locus of the moving point P is the right bisector of AB.

Equation of a Locus

A relation $f(x, y) = 0$ between x and y which is satisfied by each point on the locus and such that each point satisfying the equation is on the locus is called the ***equation of the locus.***

How to Find the Locus of a Point

Let (x_1, y_1) be the coordinates of the moving point say P. Now, apply the geometrical conditions on x_1, y_1. This gives a relation between x_1 and y_1. Now replace x_1 by x and y_1 by y in the eliminant and resulting equation would be the equation of the locus.

***Corollary* 1 :** If x and y are not there in the question, the coordinates of P may also be taken as (x, y).

***Corollary* 2 :** If coordinates and equation are not given in the question, suitable choice of origin and axes may be made.

***Corollary* 3 :** To find the locus of the point of intersection of two straight lines, eliminate the parameter or parameters from the given lines. If more than one parameter, then additional condition or conditions will also be given.

Note

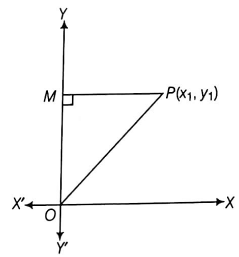
Simplify the equation by squaring both sides if square roots are there and taking LCM to remove the denominators.

| Example 57. Find the locus of a point which moves such that its distance from the point (0, 0) is twice its distance from the Y-axis.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required.

By hypothesis $|OP| = 2|PM|$ $\quad$ ($\because$ P lies in any quadrant)

$$\Rightarrow \sqrt{(x_1^2 + y_1^2)} = 2|x_1|$$



Squaring both sides, then

$$x_1^2 + y_1^2 = 4x_1^2$$

$$\Rightarrow \quad 3x_1^2 - y_1^2 = 0$$

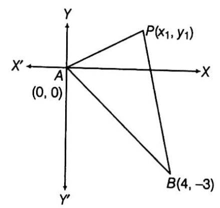
Changing (x_1, y_1) to (x, y), then

$$3x^2 - y^2 = 0$$

which is the required locus of P.

| Example 58. Find the locus of the moving point P such that $2PA = 3PB$, where A is (0, 0) and B is $(4, -3)$.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required.



By hypothesis,

$$2PA = 3PB \text{ or } 4(PA)^2 = 9(PB)^2$$

$$\Rightarrow \quad 4\{x_1^2 + y_1^2\} = 9\{(x_1 - 4)^2 + (y_1 + 3)^2\}$$

$$\Rightarrow \quad 4(x_1^2 + y_1^2) = 9(x_1^2 + y_1^2 - 8x_1 + 6y_1 + 25)$$

$$\text{or} \quad 5x_1^2 + 5y_1^2 - 72x_1 + 54y_1 + 225 = 0$$

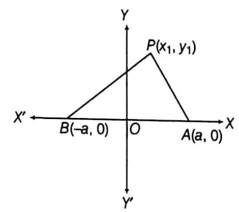
Changing (x_1, y_1) to (x, y), then

$$5x^2 + 5y^2 - 72x + 54y + 225 = 0$$

which is the required locus of P.

| Example 59. A point moves so that the sum of the squares of its distances from two fixed points $A(a, 0)$ and $B(-a, 0)$ is constant and equal to $2c^2$, find the locus of the point.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required.



By hypothesis, $\quad (PA)^2 + (PB)^2 = 2c^2$

$$\Rightarrow \quad (x_1 - a)^2 + (y_1 - 0)^2 + (x_1 + a)^2 + (y_1 - 0)^2 = 2c^2$$

$$\Rightarrow \quad 2x_1^2 + 2y_1^2 + 2a^2 = 2c^2$$

$$\text{or} \quad x_1^2 + y_1^2 = c^2 - a^2$$

Changing (x_1, y_1) to (x, y), then

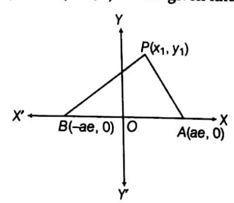
$$x^2 + y^2 = c^2 - a^2$$

which is the required locus of P.

| Example 60. A point moves such that the sum of its distances from two fixed points $(ae, 0)$ and $(-ae, 0)$ is always $2a$. Prove that the equation of the locus is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } b^2 = a^2(1 - e^2)$$

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required and $A(ae, 0)$ and $B(-ae, 0)$ be the given fixed points.



By hypothesis, $\quad |PA| + |PB| = 2a$

or $\sqrt{(x_1 - ae)^2 + (y_1 - 0)^2} + \sqrt{(x_1 + ae)^2 + (y_1 - 0)^2} = 2a$

or $\sqrt{(x_1^2 + y_1^2 - 2aex_1 + a^2e^2)} + \sqrt{(x_1^2 + y_1^2 + 2aex_1 + a^2e^2)}$
$= 2a$ $\quad$ ...(i)

Let $\quad l = x_1^2 + y_1^2 - 2aex_1 + a^2e^2$

and $\quad m = x_1^2 + y_1^2 + 2aex_1 + a^2e^2$ $\quad$ ($l - m$ method)

then, Eq. (i) can be written as

$$\sqrt{l} + \sqrt{m} = 2a \quad \text{...(ii)}$$

and $\quad l - m = -4aex_1$

or $\quad (\sqrt{l} + \sqrt{m})(\sqrt{l} - \sqrt{m}) = -4aex_1$

or $\quad 2a(\sqrt{l} - \sqrt{m}) = -4aex_1$ $\quad$ [from Eq. (ii)]

or $\quad \sqrt{l} - \sqrt{m} = -2ex_1$ $\quad$ ...(iii)

Adding Eqs. (ii) and (iii), then

$$2\sqrt{l} = 2a - 2ex_1 \text{ or } \sqrt{l} = a - ex_1$$

Squaring both sides,

$$l = a^2 - 2aex_1 + e^2x_1^2$$

$$\Rightarrow \quad x_1^2 + y_1^2 - 2aex_1 + a^2e^2 = a^2 - 2aex_1 + e^2x_1^2$$

$$\Rightarrow \quad (1 - e^2)x_1^2 + y_1^2 = a^2(1 - e^2)$$

$$\text{or} \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2(1 - e^2)} = 1$$

$$\text{or} \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad [\because b^2 = a^2(1 - e^2)]$$

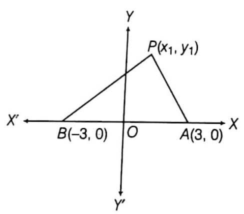
Changing (x_1, y_1) to (x, y), then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the required locus of P.

Example 61. Find the equation of the locus of a point which moves so that the difference of its distances from the points $(3, 0)$ and $(-3, 0)$ is 4 units.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required and $A(3, 0)$ and $B(-3, 0)$ be the given fixed points.



By hypothesis

$$|PB| - |PA| = 4 \text{ (assume } |PB| > |PA|)$$

$\Rightarrow$ $\sqrt{(x_1+3)^2+(y_1-0)^2} - \sqrt{(x_1-3)^2+(y_1-0)^2} = 4$

$\Rightarrow$ $\sqrt{(x_1^2+y_1^2+6x_1+9)} = 4 + \sqrt{(x_1^2+y_1^2-6x_1+9)}$

Squaring both sides then,

$$x_1^2+y_1^2+6x_1+9 = 16 + x_1^2+y_1^2-6x_1+9+8\sqrt{(x_1^2+y_1^2-6x_1+9)}$$

or $(12x_1 - 16) = 8\sqrt{(x_1^2+y_1^2-6x_1+9)}$

or $(3x_1 - 4) = 2\sqrt{(x_1^2+y_1^2-6x_1+9)}$

Again, squaring both sides, then

$$9x_1^2 - 24x_1 + 16 = 4x_1^2 + 4y_1^2 - 24x_1 + 36$$

or $5x_1^2 - 4y_1^2 = 20$

$\Rightarrow$ $\dfrac{x_1^2}{4} - \dfrac{y_1^2}{5} = 1$

Changing (x_1, y_1) by (x, y), then

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

which is the required locus of P.

Aliter ($l - m$ method):

Since, $|PB| - |PA| = 4$

$\Rightarrow$ $\sqrt{(x_1+3)^2+(y_1-0)^2} - \sqrt{(x_1-3)^2+(y_1-0)^2} = 4$

$\Rightarrow$ $\sqrt{(x_1^2+y_1^2+6x_1+9)} - \sqrt{(x_1^2+y_1^2-6x_1+9)} = 4$ ...(i)

Let $l = x_1^2+y_1^2+6x_1+9$

and $m = x_1^2+y_1^2-6x_1+9$

then, Eq. (i) can be written as

$$\sqrt{l} - \sqrt{m} = 4 \qquad \text{... (ii)}$$

and $l - m = 12x_1$

$\Rightarrow$ $(\sqrt{l}+\sqrt{m})(\sqrt{l}-\sqrt{m}) = 12x_1$

$\Rightarrow$ $(\sqrt{l}+\sqrt{m})(4) = 12x_1$ [from Eq. (ii)]

$\Rightarrow$ $\sqrt{l}+\sqrt{m} = 3x_1$ ...(iii)

Adding Eqs. (ii) and (iii),

$$2\sqrt{l} = (3x_1 + 4)$$

Squaring both sides,

$$4l = 9x_1^2 + 24x_1 + 16$$

$\Rightarrow$ $4(x_1^2+y_1^2+6x_1+9) = 9x_1^2+24x_1+16$

$\Rightarrow$ $5x_1^2 - 4y_1^2 = 20$ or $\dfrac{x_1^2}{4} - \dfrac{y_1^2}{5} = 1$

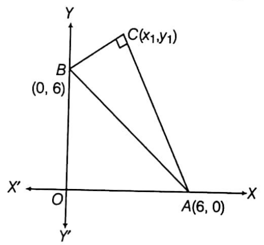
Changing (x_1, y_1) by (x, y), then

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

which is the required locus of P.

Example 62. The ends of the hypotenuse of a right angled triangle are $(6, 0)$ and $(0, 6)$. Find the locus of the third vertex.

Sol. Let $C(x_1, y_1)$ be the moving point (third vertex) whose locus is required and $A(6, 0)$ and $B(0, 6)$ be the given vertices.



By hypothesis

$$(AC)^2 + (BC)^2 = (AB)^2 \qquad (\because \angle ACB = 90°)$$

$\Rightarrow$ $(x_1-6)^2+(y_1-0)^2+(x_1-0)^2+(y_1-6)^2 = 6^2+6^2$

$\Rightarrow$ $2x_1^2 + 2y_1^2 - 12x_1 - 12y_1 = 0$

or $x_1^2 + y_1^2 - 6x_1 - 6y_1 = 0$

Changing (x_1, y_1) by (x, y), then

$$x^2 + y^2 - 6x - 6y = 0$$

which is the required locus of third vertex C.

Aliter 1. Slope of $AC \times$ slope of $BC = -1$

$\Rightarrow$ $\left(\dfrac{y_1 - 0}{x_1 - c}\right) \times \left(\dfrac{y_1 - r}{x_1 - 0}\right) = -1$

or $x_1^2 + y_1^2 - 6x_1 - 6y_1 = 0$

Aliter 2. Mid-point of AB is $M \equiv (3, 3)$

$\because$ $MA = MB = MC \Rightarrow (MA)^2 \equiv (MC)^2$

$$(3-0)^2 + (3-6)^2 = (x_1-3)^2(y_1-3)^2$$

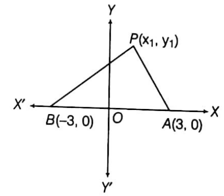
or $x_1^2 + y_1^2 - 6x_1 - 6y_1 = 0$

$\therefore$ Required locus is

$$x^2 + y^2 - 6x - 6y = 0$$

Example 63. Find the equation of the locus of a point which moves so that the sum of their distances from $(3, 0)$ and $(-3, 0)$ is less than 9.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required and $A(3, 0)$ and $B(-3, 0)$ are the given points.



By hypothesis,

$$|PA| + |PB| < 9$$

$\Rightarrow$ $\sqrt{\{(x_1-3)^2+(y_1-0)^2\}} + \sqrt{\{(x_1+3)^2+(y_1-0)^2\}} < 9$

$\Rightarrow$ $\sqrt{(x_1^2+y_1^2-6x_1+9)} < 9 - \sqrt{(x_1^2+y_1^2+6x_1+9)}$

On squaring, we get

$$(x_1^2+y_1^2-6x_1+9) < (81 + x_1^2+y_1^2+6x_1+9) - 18\sqrt{(x_1^2+y_1^2+6x_1+9)}$$

$(\because a < b \Rightarrow a^2 < b^2$ provided $a > 0)$

$\Rightarrow$ $-12x_1 - 81 < -18\sqrt{(x_1^2+y_1^2+6x_1+9)}$

$\Rightarrow$ $(4x_1 + 27) > 6\sqrt{(x_1^2+y_1^2+6x_1+9)}$

$(\because$ If $a > b$, then $-a < -b)$

On squaring, we get

$$16x_1^2 + 729 + 216x > 36x_1^2 + 36y_1^2 + 216x_1 + 324$$

$\Rightarrow$ $20x_1^2 + 36y_1^2 < 405$

Changing (x_1, y_1) by (x, y), then

$$20x^2 + 36y^2 < 405$$

which is the required locus of P.

Example 64. Find the locus of a point whose coordinate are given by $x = t + t^2$, $y = 2t + 1$, where t is variable.

Sol. Given, $x = t + t^2$ ...(i)

and $y = 2t + 1$ ...(ii)

From Eq. (ii), $t = \left(\dfrac{y-1}{2}\right)$ ...(iii)

On eliminating t from Eqs. (i) and (iii), we get required locus as

$$x = \left(\frac{y-1}{2}\right) + \left(\frac{y-1}{2}\right)^2$$

$\Rightarrow$ $x = \left(\dfrac{y-1}{2}\right)\left(1 + \dfrac{y-1}{2}\right)$

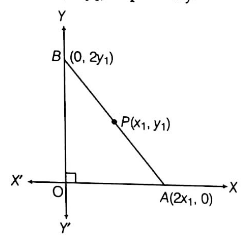
$\Rightarrow$ $x = \left(\dfrac{y-1}{2}\right)\left(\dfrac{y+1}{2}\right)$

or $4x = y^2 - 1$

or $y^2 = 4x + 1$

Example 65. A stick of length l rests against the floor and a wall of a room. If the stick begins to slide on the floor, find the locus of its middle point.

Sol. Let the cross section of the floor and wall be taken as the coordinate axes and AB be one of the position of the stick. Let the mid-point of AB be $P(x_1, y_1)$, then coordinates of A and B are $(2x_1, 0)$ and $(0, 2y_1)$ respectively.



But given, $|AB| = l$

$\Rightarrow$ $(AB)^2 = l^2$

$\Rightarrow$ $(2x_1 - 0)^2 + (0 - 2y_1)^2 = l^2$

$\Rightarrow$ $4x_1^2 + 4y_1^2 = l^2$

Changing (x_1, y_1) by (x, y), then

$$4(x^2 + y^2) = l^2$$

which is the required locus of P.

Aliter :

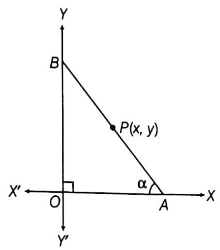
Since, $|AB| = l$

Let $\angle OAB = \alpha$

$\therefore$ $OA = l\cos\alpha$ and $OB = l\sin\alpha$

then, $A \equiv (l\cos\alpha, 0)$

and $B \equiv (0, l\sin\alpha)$



Let $P(x, y)$ be the mid-point of AB, then

$$2x = l\cos\alpha \quad \text{...(i)}$$
$$2y = l\sin\alpha \quad \text{...(ii)}$$

Squaring and adding Eqs. (i) and (ii), then

$$4(x^2 + y^2) = l^2$$

which is the required locus of P.

Example 66. Find the locus of the point of intersection of the lines $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$, where α is variable.

Sol. Given equations are

$$x\cos\alpha + y\sin\alpha = a \quad \text{...(i)}$$
and
$$x\sin\alpha - y\cos\alpha = b \quad \text{...(ii)}$$

Here, α is a variable, on eliminating α. Squaring and adding Eqs. (i) and (ii), we get required locus as

$$(x\cos\alpha + y\sin\alpha)^2 + (x\sin\alpha - y\cos\alpha)^2 = a^2 + b^2$$

or
$$(x^2\cos^2\alpha + y^2\sin^2\alpha + 2xy\cos\alpha\sin\alpha) + (x^2\sin^2\alpha + y^2\cos^2\alpha - 2xy\sin\alpha\cos\alpha) = a^2 + b^2$$

$$\Rightarrow \quad x^2(\cos^2\alpha + \sin^2\alpha) + y^2(\sin^2\alpha + \cos^2\alpha) = a^2 + b^2$$

$$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$$

Example 67. A variable line cuts X-axis at A, Y-axis at B, where $OA = a$, $OB = b$ (O as origin) such that $a^2 + b^2 = 1$.

Find the locus of

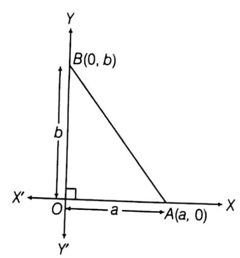
(i) centroid of ΔOAB

(ii) circumcentre of ΔOAB

Sol. (i) Coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively. If centroid of ΔOAB be $G(x, y)$

then
$$x = \frac{0 + a + 0}{3} \Rightarrow a = 3x$$

and
$$y = \frac{0 + 0 + b}{3} \Rightarrow b = 3y$$



On substituting in $a^2 + b^2 = 1$, we get

$$(3x)^2 + (3y)^2 = 1$$

or
$$x^2 + y^2 = \frac{1}{9}$$

which is the required locus of G.

(ii) If the circumcentre be $C(x, y)$. Since in semicircle angle $90°$, then AB is the diameter of the circumcircle OAB.

$\therefore$ Circumcentre is the mid-point of AB.

Then
$$x = \frac{a + 0}{2} \Rightarrow a = 2x$$

and
$$y = \frac{0 + b}{2} \Rightarrow b = 2y$$

On substituting in $a^2 + b^2 = 1$, we get

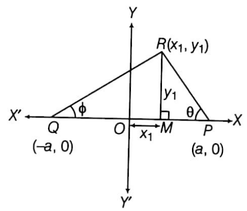
$$(2x)^2 + (2y)^2 = 1$$

or
$$x^2 + y^2 = \frac{1}{4}$$

which is the required locus of C.

Example 68. Two points P and Q are given, R is a variable point on one side of the line PQ such that $\angle RPQ - \angle RQP$ is a positive constant 2α. Find the locus of the point R.

Sol. Let the X-axis along QP and the middle point of PQ is origin and let coordinates of moving point R be (x_1, y_1).



Let
$$OP = OQ = a$$

then coordinates of P and Q are $(a, 0)$ and $(-a, 0)$ respectively.

Draw RM perpendicular on QP

$\therefore \quad OM = x_1$ and $MR = y_1$

and let $\angle RPM = \theta$ and $\angle RQM = \phi$

Now, in ΔRMP,

$$\tan\theta = \frac{RM}{MP} = \frac{RM}{OP - OM} = \frac{y_1}{a - x_1} \quad \text{...(i)}$$

and in ΔRMQ,

$$\tan\phi = \frac{RM}{QM} = \frac{RM}{OQ + OM} = \frac{y_1}{a + x_1} \quad \text{...(ii)}$$

But given

$$\angle RPQ - \angle RQP = 2\alpha \quad \text{(constant)}$$

$$\Rightarrow \quad \theta - \phi = 2\alpha$$

$$\therefore \quad \tan(\theta - \phi) = \tan 2\alpha$$

$$\Rightarrow \quad \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \tan 2\alpha$$

$$\Rightarrow \quad \frac{\dfrac{y_1}{a - x_1} - \dfrac{y_1}{a + x_1}}{1 + \left(\dfrac{y_1}{a - x_1}\right)\cdot\left(\dfrac{y_1}{a + x_1}\right)} = \tan 2\alpha \quad \text{[from Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad \frac{2x_1y_1}{a^2 - x_1^2 + y_1^2} = \tan 2\alpha$$

or
$$a^2 - x_1^2 + y_1^2 = 2x_1y_1\cot 2\alpha$$

or
$$x_1^2 - y_1^2 + 2x_1y_1\cot 2\alpha = a^2$$

Hence, locus of the point $R(x_1, y_1)$ is

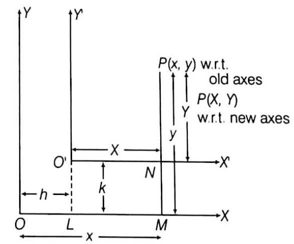
$$x^2 - y^2 + 2xy\cot 2\alpha = a^2$$

Change of Axes OR the Transformations of Axes

In coordinate geometry we have discussed the coordinates of a point or the equation of a curve are always considered on taking a fixed point O as the origin and two perpendicular straight lines through O as the coordinates axes. For convenient the coordinates of the point or the equation of the curve changes when either the origin is changed or the direction of axes or both are suitably. These processes in coordinate geometry are known as the *transformations or change of axes.* This process of transformation of coordinates will be of great advantage to solve most of the problems very easily.

(i) Change of origin OR Shifting of origin (Translation of Axes)

To change the origin of coordinates to another point (h, k) whereas the directions of axes remain unaltered.



Let O be the origin of coordinates and OX, OY be the original coordinate axes. Let O' be the new origin and (h, k) its coordinates referred to the original axes. Draw two lines $O'X'$ and $O'Y'$ through O' and parallel to OX and OY respectively. Let $P(x, y)$ be any point referred to the original axes OX, OY. Again suppose that the coordinates of the same point P referred to the new axes $O'X', O'Y'$ are (X, Y).

From O' draw $O'L$ perpendicular to OX. from P draw PM perpendicular to OX to meet $O'X'$ in N. Then

$$OL = h, O'L = k, OM = x, PM = y, O'N = X$$

and $\quad PN = Y$

we have $\quad x = OM = OL + LM = OL + O'N = h + X$

i.e. $\quad \boxed{x = X + h} \quad \text{...(i)}$

and $\quad y = PM = PN + NM = PN + O'L = Y + k$

i.e. $\quad \boxed{y = Y + k} \quad \text{...(ii)}$

from Eqs. (i) and (ii),

$$X = x - h \quad \text{and} \quad Y = y - k$$

Thus, if origin is shifted to point (h, k) without rotation of axes, then new equation of curve can be obtained by putting $x + h$ in place of x and $y + k$ in place of y.

Remarks

1. In this case axes are shifted parallel to themselves, then it is also called **Transformation by parallel axes.**
2. **Inverse translation or shifting the origin back :** Some times it is required to shift the new origin back. Then putting $x - h$ in place of x and $y - k$ in place of y in any equation of curve referred to the new origin to get the corresponding equation referred to the old origin.
3. The above transformation is true whether the axes be rectangular or oblique.

Example 69. Find the equation of the curve $2x^2 + y^2 - 3x + 5y - 8 = 0$ when the origin is transferred to the point $(-1, 2)$ without changing the direction of axes.

Sol. Here, we want to shift the origin to the point $(-1, 2)$ without changing the direction of axes. Then we replace x by $x - 1$ and y by $y + 2$ in the equation of given curve, then the transformed equation is

$$2(x - 1)^2 + (y + 2)^2 - 3(x - 1) + 5(y + 2) - 8 = 0$$

$$\Rightarrow \quad 2x^2 + y^2 - 7x + 9y + 11 = 0$$

Example 70. The equation of a curve referred to the new axes, axes retaining their direction and origin is $(4, 5)$ is $x^2 + y^2 = 36$. Find the equation referred to the original axes.

Sol. Here we want to shift the $(4, 5)$ to the origin without changing the direction of axes. Then we replace x by $x - 4$ and y by $y - 5$ in the equation of given curve then the required equation is

$$(x - 4)^2 + (y - 5)^2 = 36$$

$$\Rightarrow \quad x^2 + y^2 - 8x - 10y + 5 = 0$$

Must

Example 71. Shift the origin to a suitable point so that the equation $y^2+4y+8x-2=0$ will not contain term in y and the constant.

Sol. Let the origin be shifted to the point (h, k) without changing the direction of axes. Then we replace x by $x+h$ and y by $y+k$ in the equation of the given curve then the transformed equation is

$$(y+k)^2+4(y+k)+8(x+h)-2=0$$

$$\Rightarrow \quad y^2+(2k+4)y+8x+(k^2+4k+8h-2)=0$$

Since, this equation is required to be free from the term containing y and the constant, we have

$$2k+4=0 \quad \text{and} \quad k^2+4k+8h-2=0$$

$$\therefore \quad k=-2 \quad \text{and} \quad h=\frac{3}{4}$$

Hence, the point to which the origin be shifted is $\left(\frac{3}{4}, -2\right)$.

Example 72. At what point the origin be shifted, if the coordinates of a point $(-1, 8)$ become $(-7, 3)$?

Sol. Let the origin be shifted to the point (h, k) without changing the direction of axes. Then we replace x by $x+h$ and y by $y+k$ and we get new co-ordinates. Here, given old coordinates and new coordinates are $(-1, 8)$ and $(-7, 3)$ respectively.

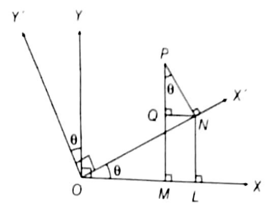
We have $\quad -1+h=-7 \quad \text{and} \quad 8+k=3$

$\Rightarrow \quad h=-6 \quad \text{and} \quad k=-5$

Hence, the origin must be shifted to $(-6, -5)$.

(ii) Rotation of Axes (Change of Directions of Axes)

To find the change in the coordinates of a point when the directions of axes are rotated through an angle θ the origin being fixed.



Let OX and OY be the original system of coordinate axes. Let OX' and OY' be the new axes obtained by rotating the original axes through an angle θ. Let P be a point in the plane whose coordinates are (x, y) and (X, Y) referred to old and new axes respectively. Draw PM and PN perpendiculars to OX and OX' and also NL and NQ perpendiculars to OX and PM. We have

$$OM=x,\ PM=y,\ ON=X,\ PN=Y$$

Now, $\quad x=OM=OL-ML$

$\left(\because \text{Angle between any two lines = Angle between their perpendiculars i.e. } \angle XOX' = \angle NPM = \theta\right)$

$$=OL-QN=ON\cos\theta-PN\sin\theta$$

$$=X\cos\theta-Y\sin\theta$$

i.e. $\quad \boxed{x=X\cos\theta-Y\sin\theta}$ ...(i)

and $\quad y=PM=PQ+QM=PQ+NL$

$$=PN\cos\theta+ON\sin\theta$$

$$=Y\cos\theta+X\sin\theta$$

i.e. $\quad \boxed{y=X\sin\theta+Y\cos\theta}$ ...(ii)

Now, multiplying Eqs. (i) by $\cos\theta$ and Eq. (ii) by $\sin\theta$ and adding we get

$\boxed{X=x\cos\theta+y\sin\theta}$ ...(iii)

Also, subtracting the product of Eq. (i) by $\sin\theta$ from the product of Eq. (ii) by $\cos\theta$, we get

$Y=-x\sin\theta+y\cos\theta$ ...(iv)

also $x^2+y^2=X^2+Y^2=OP^2$ are unchanged i.e. the distance of the point P from the origin O remains unaffected by the rotation of axes.

Rule : When the axes are rotated through θ, replace (x, y) by $(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta)$.

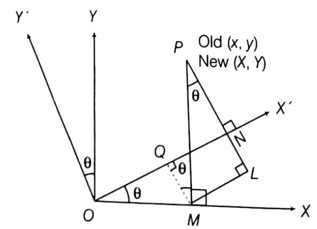
Shifting the coordinate axes back : Some times it is required to shift the new coordinates axes back. Then replace (x, y) by

$$(x\cos\theta+y\sin\theta, -x\sin\theta+y\cos\theta).$$

Independent Proof (Aliter)

The relations $X=x\cos\theta+y\sin\theta$ and $Y=-x\sin\theta+y\cos\theta$ can be obtained independently.

Proof : Draw PM and PN perpendiculars to OX and OX' and also ML and MQ perpendiculars to PN and OX' respectively.



We have, $\quad OM=x,\ PM=y,\ ON=X$

and $\quad PN=Y$

Also, angle between any two lines

= Angle between their perpendiculars lines

i.e. $\quad \angle XOX' = \angle MPN = \angle PMQ = \theta$

$\therefore \quad X=ON=OQ+QN$

$$=OQ+ML$$

$$=OM\cos\theta+PM\sin\theta$$

$$(\because OQ=OM\cos\theta \text{ and } ML=PM\sin\theta)$$

$$=x\cos\theta+y\sin\theta$$

i.e. $\quad \mathbf{X=x\cos\theta+y\sin\theta}$

and $\quad Y=PN=PL-NL=PL-QM$

$$=PM\cos\theta-OM\sin\theta$$

$$(\because PL=PM\cos\theta \text{ and } QM=OM\sin\theta)$$

$$=y\cos\theta-x\sin\theta$$

i.e. $\quad \mathbf{Y=-x\sin\theta+y\cos\theta}$

Remark :

The results Eqs. (i), (ii), (iii) and (iv) can be conveniently remembered by the following methods.

(i) Light heavy method : Let x, y be light and X, Y be heavy then heavy X, Y down and x, y up then

	x	y
X	$\cos\theta$	$\sin\theta$
Y	$-\sin\theta$	$\cos\theta$

third row is obtained by differentiating second row with respect to θ.

For remembrance $\quad$ **C** $\quad$ **S** $\quad$ (Civil Services)

$$\begin{matrix} \downarrow & \downarrow \\ \cos\theta & \sin\theta \\ \downarrow & \downarrow \\ -\sin\theta & \cos\theta \end{matrix} \quad \left(\text{i.e. } \begin{matrix} C\to\cos\theta \\ S\to\sin\theta \end{matrix}\right) \quad \left(\begin{matrix} (\cos\theta)'=-\sin\theta \\ (\sin\theta)'=\cos\theta \end{matrix}\right)$$

(a) Finding x and y in terms of X and Y

x = Sum of the products of the elements in the left most column with the corresponding elements of the first column

i.e. $\quad x=X\cos\theta-Y\sin\theta$

and y = Sum of the products of the elements in the left most column with the corresponding elements of the second column.

i.e. $\quad y=X\sin\theta+Y\cos\theta$

Hence, $\quad \left.\begin{matrix} x=X\cos\theta-Y\sin\theta \\ y=X\sin\theta+Y\cos\theta \end{matrix}\right\}$

(b) Finding X and Y in terms of x and y

X = Sum of the products of the elements of top-row with the corresponding elements of first row.

i.e. $\quad X=x\cos\theta+y\sin\theta$

and Y = Sum of the products of the elements of top-row with the corresponding elements of second row.

i.e. $\quad Y=-x\sin\theta+y\cos\theta$

Hence, $\quad \left.\begin{matrix} X=x\cos\theta+y\sin\theta \\ Y=-x\sin\theta+y\cos\theta \end{matrix}\right\}$

(ii) Matrix method :

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \text{(say)}$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = A'\begin{bmatrix} X \\ Y \end{bmatrix}$$

where, A' is the transpose matrix of A.

(iii) Complex number method :

Let $\quad z=x+iy$

and $\quad Z=X+iY$, where $i=\sqrt{-1}$

then $\quad z=Ze^{i\theta}$ ...(i)

i.e. $\quad (x+iy)=(X+iY)(\cos\theta+i\sin\theta)$

On comparing real and imaginary parts, we get

$$\left.\begin{matrix} x=X\cos\theta-Y\sin\theta \\ y=X\sin\theta+Y\cos\theta \end{matrix}\right\}$$

Again from Eq. (i), $\quad Z=ze^{-i\theta}$

i.e. $\quad (X+iY)=(x+iy)(\cos\theta-i\sin\theta)$

On comparing real and imaginary parts, we get

$$\left.\begin{matrix} X=x\cos\theta+y\sin\theta \\ Y=-x\sin\theta+y\cos\theta \end{matrix}\right\}$$

Example 73. If the axes are turned through 45°, find the transformed form of the equation $3x^2+3y^2+2xy=2$.

Sol. Here, $\theta=45^\circ$ so $\sin\theta=\cos\theta=\frac{1}{\sqrt{2}}$

Replacing (x, y) by $(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta)$

i.e. $\quad \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$

Then, $3x^2+3y^2+2xy=2$ becomes

$$3\left(\frac{x-y}{\sqrt{2}}\right)^2+3\left(\frac{x+y}{\sqrt{2}}\right)^2+2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right)=2$$

$\Rightarrow \quad 3(2x^2+2y^2)+2(x^2-y^2)=4$

$\Rightarrow \quad 8x^2+4y^2=4$

or $\quad 2x^2+y^2=1$

which is free from the term containing xy.

I Example 74. Prove that if the axes be turned through $\frac{\pi}{4}$ the equation $x^2-y^2=a^2$ is transformed to the form $xy=\lambda$. Find the value of λ.

Sol. Here, $\theta=\frac{\pi}{4}$ so $\sin\theta=\cos\theta=\frac{1}{\sqrt{2}}$

Replacing (x, y) by $(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta)$

i.e. $\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$

then, $x^2-y^2=a^2$ becomes

$$\left(\frac{x-y}{\sqrt{2}}\right)^2-\left(\frac{x+y}{\sqrt{2}}\right)^2=a^2$$

$$\Rightarrow \left(\frac{x-y}{\sqrt{2}}+\frac{x+y}{\sqrt{2}}\right)\left(\frac{x-y}{\sqrt{2}}-\frac{x+y}{\sqrt{2}}\right)=a^2$$

$$\Rightarrow \left(\frac{2x}{\sqrt{2}}\right)\left(\frac{-2y}{\sqrt{2}}\right)=a^2$$

or $\quad xy=-\frac{a^2}{2}$

Comparing it with $xy=\lambda$, then we get $\lambda=-\frac{a^2}{2}$.

I Example 75. Through what angle should the axes be rotated so that the equation $9x^2-2\sqrt{3}\,xy+7y^2=10$ may be changed to $3x^2+5y^2=5$?

Sol. Let angle be θ then replacing (x, y) by $(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta)$

then, $\quad 9x^2-2\sqrt{3}xy+7y^2=10$ becomes

$9(x\cos\theta-y\sin\theta)^2-2\sqrt{3}$
$(x\cos\theta-y\sin\theta)(x\sin\theta+y\cos\theta)$
$+7(x\sin\theta+y\cos\theta)^2=10$

$\Rightarrow \quad x^2(9\cos^2\theta-2\sqrt{3}\sin\theta\cos\theta+7\sin^2\theta)$
$+2xy(-9\sin\theta\cos\theta-\sqrt{3}\cos 2\theta+7\sin\theta\cos\theta)$
$+y^2(9\cos^2\theta+2\sqrt{3}\sin\theta\cos\theta+7\cos^2\theta)=10$

On comparing with $3x^2+5y^2=5$ (coefficient of xy)

we get $-9\sin\theta\cos\theta-\sqrt{3}\cos 2\theta+7\sin\theta\cos\theta=0$

or $\quad \sin 2\theta=-\sqrt{3}\cos 2\theta$

or $\quad \tan 2\theta=-\sqrt{3}=\tan(180^\circ-60^\circ)$

or $\quad 2\theta=120^\circ$

$\therefore \quad \theta=60^\circ$

I Example 76. If (x, y) and (X, Y) be the coordinates of the same point referred to two sets of rectangular axes with the same origin and if $ux+vy$, when u and v are independent of X and Y become $VX+UY$, show that $u^2+v^2=U^2+V^2$

Sol. Let the axes rotate an angle θ and if (x, y) be the point with respect to old axes and (X, Y) be the co-ordinates with respect to new axes, then

$$x+iy=(X+iY)e^{i\theta}=(X+iY)(\cos\theta+i\sin\theta)$$

On comparing real and imaginary parts, we get

$$\begin{cases} x=X\cos\theta-Y\sin\theta \\ y=X\sin\theta+Y\cos\theta \end{cases}$$

Then, $ux+vy=u(X\cos\theta-Y\sin\theta)+v(X\sin\theta+Y\cos\theta)$
$=(u\cos\theta+v\sin\theta)X+(-u\sin\theta+v\cos\theta)Y$

but given new curve $VX+UY$

then, $VX+UY=(u\cos\theta+v\sin\theta)$
$X+(-u\sin\theta+v\cos\theta)Y$

On comparing the coefficients of X and Y, we get

$V=u\cos\theta+v\sin\theta$ ...(i)

and $\quad U=-u\sin\theta+v\cos\theta$ ...(ii)

Squaring and adding Eqs. (i) and (ii), we get

$$V^2+U^2=(u\cos\theta+v\sin\theta)^2+(-u\sin\theta+v\cos\theta)^2$$
$$=u^2+v^2$$

Hence, $\quad u^2+v^2=U^2+V^2$

Aliter 1 (By matrix method) :

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}=\begin{bmatrix} X\cos\theta-Y\sin\theta \\ X\sin\theta+Y\cos\theta \end{bmatrix} \quad ...(i)$$

$$\because \quad ux+vy=[u \;\; v]\begin{bmatrix} x \\ y \end{bmatrix}=[u \; v]\begin{bmatrix} X\cos\theta+Y\sin\theta \\ X\sin\theta+Y\cos\theta \end{bmatrix} \quad \text{[from (i)]}$$

$=uX\cos\theta-uY\sin\theta+vX\sin\theta+vY\cos\theta$
$=X(u\cos\theta+v\sin\theta)+Y(-u\sin\theta+v\cos\theta)$

but given new curve $VX+UY$

Then, $VX+UY=X(u\cos\theta+v\sin\theta)+Y$
$(-u\sin\theta+v\cos\theta)$

On comparing the coefficients of X and Y, we get

$V=u\cos\theta+v\sin\theta$ ...(ii)

$U=-u\sin\theta+v\cos\theta$ ...(iii)

Squaring and adding Eqs. (ii) and (iii), we get

$$u^2+v^2=U^2+V^2$$

Aliter 2 (Best approach) :

$ux+vy=R_e((u-iv)(x+iy))$
$=R_e((u-iv)(X+iY)e^{i\theta})$ ...(i)

and $\quad VX+UY=R_e((V-iU)(X+iY))$ ...(ii)

From Eqs. (i) and (ii), we get

$$V-iU=(u-iv)e^{i\theta}$$

Taking modulus both sides, then

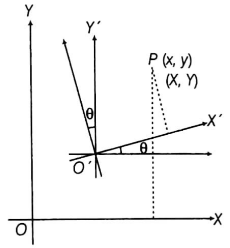
$$|V-iU|=|u-iv||e^{i\theta}|$$

$\Rightarrow \quad \sqrt{(V^2+U^2)}=\sqrt{u^2+v^2}\cdot 1$

or $\quad u^2+v^2=U^2+V^2$

(iii) Double Transformation (Origin Shifted and Axes Rotated)

If origin is shifted to the point. $O'(h, k)$ and at the same time the directions of axes are rotated through an angle θ in the anticlockwise sense such that new coordinates of $P(x, y)$ become (X, Y).



Then, we get $\quad \boxed{x=h+X\cos\theta-Y\sin\theta}$ ...(i)

and $\quad \boxed{y=k+X\sin\theta+Y\cos\theta}$ ...(ii)

In practice we have to replace x by $h+x\cos\theta-y\sin\theta$ and y by $k+x\sin\theta+y\cos\theta$.

Again, if we want to shift the coordinate axes back to their original positions, then we obtained X and Y by solving Eqs. (i) and (ii), then

$$\boxed{X=(x-h)\cos\theta+(y-k)\sin\theta}$$

and $\quad \boxed{Y=-(x-h)\sin\theta+(y-k)\cos\theta}$

I Example 77. What does the equation $2x^2+4xy-5y^2+20x-22y-14=0$ becomes when referred to rectangular axes through the point $(-2, -3)$, the new axes being inclined at an angle of 45° with the old ?

Sol. Let $O'(-2, -3)$ be the new origin and axes are rotated about O' through an angle 45° in anticlockwise direction then replacing x and y by

$-2+x\cos 45^\circ-y\sin 45^\circ$

and $\quad -3+x\sin 45^\circ+y\cos 45^\circ$

i.e. $-2+\left(\frac{x-y}{\sqrt{2}}\right)$ and $-3+\left(\frac{x+y}{\sqrt{2}}\right)$ respectively in the given curve, then the new equation of curve will be

$$2\left\{-2+\left(\frac{x-y}{\sqrt{2}}\right)\right\}^2+4\left\{-2+\left(\frac{x-y}{\sqrt{2}}\right)\right\}\left\{-3+\left(\frac{x+y}{\sqrt{2}}\right)\right\}$$

$$-5\left\{-3+\left(\frac{x+y}{\sqrt{2}}\right)\right\}^2+20\left\{-2+\left(\frac{x-y}{\sqrt{2}}\right)\right\}$$
$$-22\left\{-3+\left(\frac{x+y}{\sqrt{2}}\right)\right\}-14=0$$

$\Rightarrow \quad x^2-14xy-7y^2-2=0$

Removal of the Term xy from $f(x, y)=ax^2+\boxed{2hxy}+by^2$ without Changing the Origin

Clearly, $h\neq 0$

Rotating the axes through an angle θ, we have

$x=X\cos\theta-Y\sin\theta$

and $\quad y=X\sin\theta+Y\cos\theta$

$\because \quad f(x, y)=ax^2+2hxy+by^2$

After rotation, new equation is

$$F(X, Y)=(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)X^2$$
$$+2\{(b-a)\cos\theta\sin\theta+h(\cos^2\theta-\sin^2\theta)\}XY$$
$$+(a\sin^2\theta-2h\cos\theta\sin\theta+b\cos^2\theta)Y^2$$

Now, coefficient of $XY=0$

Then, we get $\quad \cot 2\theta=\frac{a-b}{2h}$

Remember

Remark

Usually, we use the formula $\boxed{\tan 2\theta=\frac{2h}{a-b}}$ for finding the angle of rotation, θ.

However, if $\boxed{a=b}$, we use $\boxed{\cot 2\theta=\frac{a-b}{2h}}$ as in this case $\tan 2\theta$ is not defined.

I Example 78. Given the equation $4x^2+2\sqrt{3}xy+2y^2=1$, through what angle should the axes be rotated so that the term in xy be wanting from the transformed equation.

Sol. Comparing the given equation with $ax^2+2hxy+by^2$, we get $a=4, h=\sqrt{3}\; b=2$. If axes are to be rotated at θ, then

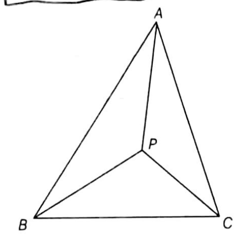
$$\tan 2\theta=\frac{2h}{a-b}=\frac{2\sqrt{3}}{2}=\sqrt{3}=\tan\frac{\pi}{3}$$

$$2\theta=\frac{\pi}{3}, \pi+\frac{\pi}{3}\Rightarrow 2\theta=\frac{\pi}{3}, \frac{4\pi}{3}$$

$$\therefore \quad \theta=\frac{\pi}{6}, \frac{2\pi}{3}$$

Position of a Point which Lies Inside a Triangle

If any point say (P) lies within the triangle ABC, then $\boxed{\Delta_1 + \Delta_2 + \Delta_3 = \Delta}$



where, $\Delta =$ Area of triangle ABC,

$\Delta_1 =$ Area of ΔPBC,

$\Delta_2 =$ Area of ΔPCA,

$\Delta_3 =$ Area of ΔPAB

Also, $\Delta_1 \neq 0, \Delta_2 \neq 0, \Delta_3 \neq 0, \Delta \neq 0$

(Each individual area must be non-zero)

Example 79. Find λ if $(\lambda, \lambda+1)$ is an interior point of ΔABC where, $A \equiv (0,3); B \equiv (-2,0)$ and $C \equiv (6,1)$.

Sol. The point $P(\lambda, \lambda+1)$ will be inside the triangle ABC, then

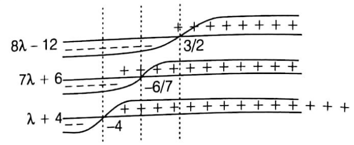
Area of ΔPBC + Area of ΔPCA + Area of ΔPAB = Area of ΔABC

$$\Rightarrow \frac{1}{2}\left|\begin{vmatrix} \lambda & \lambda+1 & 1 \\ -2 & 0 & 1 \\ 6 & 1 & 1 \end{vmatrix}\right| + \frac{1}{2}\left|\begin{vmatrix} \lambda & \lambda+1 & 1 \\ 6 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix}\right| + \frac{1}{2}\left|\begin{vmatrix} \lambda & \lambda+1 & 1 \\ 0 & 3 & 1 \\ -2 & 0 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|\begin{vmatrix} 0 & 3 & 1 \\ -2 & 0 & 1 \\ 6 & 1 & 1 \end{vmatrix}\right|$$

$$\Rightarrow \boxed{|7\lambda + 6| + |8\lambda - 12| + |\lambda + 4| = 22}$$

For $\lambda < -4$:



Then, $-(7\lambda + 6) - (8\lambda - 12) - (\lambda + 4) = 22$

$\Rightarrow \quad -16\lambda = 20 \quad \therefore \lambda = -\frac{5}{4}$

which is impossible.

For $-4 \leq \lambda < -\frac{6}{7}$:

Then, $-(7\lambda + 6) - (8\lambda - 12) + (\lambda + 4) = 22$

$\Rightarrow \quad -14\lambda = 12 \quad \therefore = -\frac{6}{7}$

which is impossible.

For $-\frac{6}{7} \leq \lambda < \frac{3}{2}$:

Then, $(7\lambda + 6) - (8\lambda - 12) + \lambda + 4 = 22 \Rightarrow 22 = 22$

$\because$ at $\lambda = -\frac{6}{7}$, area of $\Delta PBC = 0$

$\therefore \quad \lambda \neq \frac{-6}{7}$

$\therefore \quad \lambda \in \left(-\frac{6}{7}, \frac{3}{2}\right)$

For $\lambda \geq \frac{3}{2}$:

Then, $7\lambda + 6 + 8\lambda - 12 + \lambda + 4 = 22 \Rightarrow 16\lambda = 24$

$\therefore \quad \lambda = \frac{3}{2}$

$\because$ at $\lambda = \frac{3}{2}$, area of $\Delta PCA = 0$

$\therefore \quad \lambda \neq \frac{3}{2}$

Hence, value of $\lambda \in \left(-\frac{6}{7}, \frac{3}{2}\right)$.

Exercise for Session 4

1. The equation of the locus of points equidistant from $(-1, -1)$ and $(4, 2)$ is
(a) $3x - 5y - 7 = 0$ (b) $5x + 3y - 9 = 0$ (c) $4x + 3y + 2 = 0$ (d) $x - 3y + 5 = 0$

2. The equation of the locus of a point which moves so that its distance from the point $(ak, 0)$ is k times its distance from the point $\left(\frac{a}{k}, 0\right)$, $(k \neq 1)$ is
(a) $x^2 - y^2 = a^2$ (b) $2x^2 - y^2 = 2a^2$ (c) $xy = a^2$ (d) $x^2 + y^2 = a^2$

3. If the coordinates of a variable point P be $\left(t + \frac{1}{t}, t - \frac{1}{t}\right)$, where t is the variable quantity, then the locus of P is
(a) $xy = 8$ (b) $2x^2 - y^2 = 8$ (c) $x^2 - y^2 = 4$ (d) $2x^2 + 3y^2 = 5$

4. If the coordinates of a variable point P be $(\cos\theta + \sin\theta, \sin\theta - \cos\theta)$, where θ is the parameter, then the locus of P is
(a) $x^2 - y^2 = 4$ (b) $x^2 + y^2 = 2$ (c) $xy = 3$ (d) $x^2 + 2y^2 = 3$

5. If a point moves such that twice its distance from the axis of x exceeds its distance from the axis of y by 2, then its locus is
(a) $x - 2y = 2$ (b) $x + 2y = 2$ (c) $2y - x = 2$ (d) $2y - 3x = 5$

6. The equation $4xy - 3x^2 = a^2$ become when the axes are turned through an angle $\tan^{-1} 2$ is
(a) $x^2 + 4y^2 = a^2$ (b) $x^2 - 4y^2 = a^2$ (c) $4x^2 + y^2 = a^2$ (d) $4x^2 - y^2 = a^2$

7. Transform the equation $x^2 - 3xy + 11x - 12y + 36 = 0$ to parallel axes through the point $(-4, 1)$ becomes $ax^2 + bxy + 1 = 0$ then $b^2 - a =$
(a) $\frac{1}{4}$ (b) $\frac{1}{16}$ (c) $\frac{1}{64}$ (d) $\frac{1}{256}$

8. Find the equation of the locus of all points equidistant from the point $(2, 4)$ and the Y-axis.

9. Find the equation of the locus of the points twice as far from $(-a, 0)$ as from $(a, 0)$.

10. OA and OB are two perpendicular straight lines. A straight line AB is drawn in such a manner that $OA + OB = 8$. Find the locus of the mid point of AB.

11. The ends of a rod of length l move on two mutually perpendicular lines. Find the locus of the point on the rod which divides it in the ratio $1 : 2$.

12. The coordinates of three points O, A, B are $(0, 0)$, $(0, 4)$ and $(6, 0)$ respectively. A point P moves so that the area of ΔPOA is always twice the area of ΔPOB. Find the equation to both parts of the locus of P.

13. What does the equation $(a - b)(x^2 + y^2) - 2abx = 0$ become, if the origin be moved to the point $\left(\frac{ab}{a - b}, 0\right)$?

14. The equation $x^2 + 2xy + 4 = 0$ is transformed to the parallel axes through the point $(6, \lambda)$. For what value of λ its new form passes through the new origin ?

15. Show that if the axes be turned through $7\frac{1}{2}°$; the equation $\sqrt{3}x^2 + (\sqrt{3} - 1)xy - y^2 = 0$ become free of xy in its new form.

16. Find the angle through which the axes may be turned so that the equation $Ax + By + C = 0$ may reduce to the form $x =$ constant, and determine the value of this constant.

17. Transform $12x^2 + 7xy - 12y^2 - 17x - 31y - 7 = 0$ to rectangular axes through the point $(1, -1)$ inclined at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ to the original axes.

Shortcuts and Important Results to Remember

1 If D, E, F are the mid-points of the sides BC, CA, AB of ΔABC, the $A = E + F - D$
$B = F + D - E$
and $C = D + E - F$

2 Orthocentre, nine point centre, centroid, circumcentre of a triangle are collinear. Centroid divides the line joining orthocentre and circumcentre in the ration 2 :1 (Internally) and nine point centre is the mid-point of orthocentre and circumcentre.

3 The circumcentre of a right angled triangle is the mid-point of the hypotenuse.

4 In an equilateral triangle orthocentre, nine point centre, centroid, circumcentre, incentre coincide.

5 The distance between the orthocentre and circumcentre in an equilateral triangle is zero.

6 The orthocentre of a triangle having vertices (α, β), (β, α) and (α, α) is (α, α).

7 Orthocentre of the triangle formed by the points $\left(\alpha, \frac{1}{\alpha}\right); \left(\beta, \frac{1}{\beta}\right); \left(\gamma, \frac{1}{\gamma}\right)$ is $\left(-\frac{1}{\alpha\beta\gamma}, -\alpha\beta\gamma\right)$

i.e. all points and orthocentre lie on $xy = 1$.

8 **Points in a triangle :** Centroid (G), Incentre (I), Excentres (I_1, I_2, I_3), Orthocentre (O), Circumcentre (C) are given by
$\left(\frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}, \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}\right)$, where

	m_1	m_2	m_3
G	1	1	1
I	$\sin A$	$\sin B$	$\sin C$
I_1	$-\sin A$	$\sin B$	$\sin C$
I_2	$\sin A$	$-\sin B$	$\sin C$
I_3	$\sin A$	$\sin B$	$-\sin C$
O	$\tan A$	$\tan B$	$\tan C$
C	$\sin 2A$	$\sin 2B$	$\sin 2B$

and vertices

$A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, $C \equiv (x_3, y_3)$ and A, B, C are the angles of ΔABC.

9 If the circumcentre and centroid of a triangle are respectively (α, β), (γ, δ), then orthocentre will be $(3\gamma - 2\alpha, 3\delta - 2\beta)$.

10 If $ABCD$ is a parallelogram, then $D = A - B + C$.

11 If D, E, F are the mid-points of the sides BC, CA, AB of ΔABC, then the centroid of ΔABC = centroid of ΔDEF. If area of $\Delta ABC = \Delta$, then area of ΔAFE = area of ΔBDF = area of ΔCED = area of $\Delta DEF = \frac{\Delta}{4}$ and area of parallelogram $CEFD$ = area of parallelogram $BDEF$ = area of parallelogram $AEDF = \frac{\Delta}{2}$

12 Orthocentre of the right angle triangled ABC, right angled at A is A.

13 Circumcentre of the right angled triangle ABC, right angled at A is $\frac{B + C}{2}$.

14 X-axis divides the line segment joining (x_1, y_1), (x_2, y_2) in the ratio $-y_1 : y_2$ and Y-axis divides the same line segment in the ratio $-x_1 : x_2$.

15 Area of the triangle formed by (x_1, y_1), (x_2, y_2), (x_3, y_3) is $\frac{1}{2}\left|\begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix}\right|$.

16 The area of the triangle formed by $y = m_1x + c_1$, $y = m_2x + c_2$, $y = m_3x + c_3$ is $\frac{1}{2}\Sigma\frac{(c_1 - c_2)^2}{|m_1 - m_2|}$.

17 Area of the quadrilateral formed by $(x_1, y_1)(x_2, y_2)(x_3, y_3)$, (x_4, y_4) is
$$\frac{1}{2}\left|\begin{vmatrix} x_1 - x_3 & x_2 - x_4 \\ y_1 - y_3 & y_2 - y_4 \end{vmatrix}\right|$$

18 If (x_1, y_1), (x_2, y_2) are the ends of the hypotenuse of a right angled isosceles triangle, then the third vertex is given by $\left(\frac{x_1 + x_2 \pm (y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp (x_1 - x_2)}{2}\right)$

19 Given the two vertices (x_1, y_1) and (x_2, y_2) of an equilateral triangle, then its third vertex is given by $\left(\frac{x_1 + x_2 \pm \sqrt{3}(y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp \sqrt{3}(x_1 - x_2)}{2}\right)$

20 Circumcentre of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is same as that of triangle formed by the points $(0, 0)$, $(x_2 - x_1, y_2 - y_1)$, $(x_3 - x_1, y_3 - y_1)$.

JEE Type Solved Examples : Single Option Correct Type Questions

This section contains **5 multiple choice examples**. Each example has four choice (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1. *Locus of centroid of the triangle whose vertices are $(a\cos t, a\sin t)$, $(b\sin t, -b\cos t)$ and $(1, 0)$, where t is a parameter, is*

(a) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$
(b) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
(c) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$
(d) $(3x + 1)^2 + 3y^2 = a^2 - b^2$

Sol. (b) Let $A \equiv (a\cos t, a\sin t)$, $B \equiv (b\sin t, -b\cos t)$ and $C \equiv (1, 0)$

$\therefore$ Centroid $\equiv \left(\frac{a\cos t + b\sin t + 1}{3}, \frac{a\sin t - b\cos t}{3}\right)$

Let $x = \frac{a\cos t + b\sin t + 1}{3} \Rightarrow 3x - 1 = a\cos t + b\sin t$ ...(i)

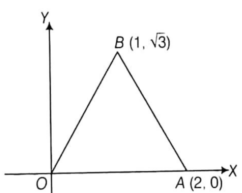
and $y = \frac{a\sin t - b\cos t}{3} \Rightarrow 3y = a\sin t - b\cos t$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get
$$(3x - 1)^2 + (3y)^2 = a^2 + b^2$$
which is locus of centroid.

• Ex. 2. *The incentre of triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$ is*

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$ (c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

Sol. (d)



Let $O \equiv (0, 0)$, $A \equiv (2, 0)$
and $B \equiv (1, \sqrt{3})$
$\because$ $OA = 2 = OB = AB$
$\Rightarrow \Delta OAB$ is an equilateral.
$\therefore$ $\boxed{\text{Incentre = Centroid}}$
$$\equiv \left(\frac{0 + 2 + 1}{3}, \frac{0 + 0 + \sqrt{3}}{3}\right)$$
$\therefore$ Incentre $\equiv \left(1, \frac{1}{\sqrt{3}}\right)$

• Ex. 3. *Orthocentre of triangle with vertices $(0, 0)$, $(3, 4)$ and $(4, 0)$ is*

(a) $\left(3, \frac{5}{4}\right)$ (b) $(3, 12)$
(c) $\left(3, \frac{3}{4}\right)$ (d) $(3, 9)$

Sol. (c) Denote the points are (x_1, y_1), (x_2, y_2) and (x_3, y_3) from the matrix
$$P = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ -1 & 4 \end{bmatrix}$$

$\therefore$ $\lambda = \frac{\vec{R}_1 \cdot \vec{R}_2}{|P|} = \frac{(-4)(-1) + 0}{-16} = -\frac{1}{4}$

$\therefore$ Circumcentre of the triangle
$$\equiv \left(\frac{x_1 + x_2 + \lambda(y_1 - y_2)}{2}, \frac{y_1 + y_2 - \lambda(x_1 - x_2)}{2}\right)$$
$$\equiv \left(\frac{3 - 4\lambda}{2}, \frac{4 + 3\lambda}{2}\right) \equiv \left(2, \frac{13}{8}\right)$$

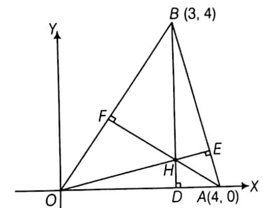
i.e. $C \equiv \left(2, \frac{13}{8}\right)$

and centroid $G \equiv \left(\frac{7}{3}, \frac{4}{3}\right)$

$\therefore$ Orthocentre
$$H \equiv \left(3 \times \frac{7}{3} - 2 \times 2, 3 \times \frac{4}{3} - 2 \times \frac{13}{8}\right) \equiv \left(3, \frac{3}{4}\right)$$

Aliter : Let orthocentre $H \equiv (\alpha, \beta)$

Must



$\therefore$ Slope of $BH \times$ slope of $OA = -1$

$\Rightarrow$ $\left(\frac{\beta - 4}{\alpha - 3}\right) \times \left(\frac{0}{4}\right) = -1$

$$\alpha - 3 = 0$$

$\therefore \quad \alpha = 3 \quad$ ...(i)

$\Rightarrow$ and slope of $AH \times$ slope of $OB = -1$

$\Rightarrow \quad \left(\frac{\beta - 0}{\alpha - 4}\right) \times \left(\frac{4}{3}\right) = -1$

From Eq. (i), $\quad \beta = \frac{3}{4}$

Hence, orthocentre is $\left(3, \frac{3}{4}\right)$.

• **Ex. 4.** *If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in GP, with the same common ratio, then the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3)*

(a) lie on a straight line
(b) lie on an ellipse
(c) lie on a circle
(d) are vertices of a triangle

Sol. (a) Let common ratio of GP is r, then $x_2 = x_1 r$, $x_3 = x_1 r^2$, $y_2 = y_1 r$ and $y_3 = y_1 r^2$.

Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$

$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$

$= \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 r & y_1 r & 1 \\ x_1 r^2 & y_1 r^2 & 1 \end{vmatrix} \right|$

$= \frac{1}{2} |x_1 y_1| \left| \begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix} \right| = 0 \quad$ ($\because C_1$ and C_2 are identical)

$\Rightarrow$ Points A, B, C are collinear.

Must

• **Ex. 5.** *Let A be the image of $(2, -1)$ with respect to Y-axis. Without transforming the origin, coordinate axis are turned at an angle 45° in the clockwise direction. Then, the coordiates of A in the new system are*

(a) $\left(-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$ (b) $\left(-\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
(c) $\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ (d) $\left(\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Sol. (a) Since, the image of (h, k) w.r.t. Y-axis is $(-h, k)$.

$\therefore$ Coordinate of A are $(-2, -1)$.

If (X, Y) are the coordinates of A w.r.t. the new coordinate axes obtained by turning the axes through an angle 45° in the clockwise direction, then

$$X = -2\cos(-45°) - \sin(-45°)$$
$$= -\frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

and $\quad Y = 2\sin(-45°) - \cos(-45°)$

$$= -\frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{3}{\sqrt{2}}$$

$\therefore \quad$ Required coordinates are $\left(-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$.

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **3 multiple choice examples**. Each example has four choices (a), (b), (c) and (d). Out of which **MORE THAN ONE** may be correct.

• **Ex. 6.** *Let $S_1, S_2, \ldots$, be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side S_1 is 10 cm, then for which of the following value of n is the area of S_n less than 1 sq cm?*

(a) 7 (b) 8 (c) 9 (d) 10

Sol. (b,c,d)

If a be the side of the square, then diagonal $d = a\sqrt{2}$ by hypothesis

$$a_n = \sqrt{2}\, a_{n+1}$$

$\Rightarrow \quad a_{n+1} = \frac{a_n}{\sqrt{2}} = \frac{a_{n-1}}{(\sqrt{2})^2} = \frac{a_{n-2}}{(\sqrt{2})^3} = \ldots = \frac{a_1}{(\sqrt{2})^n}$

$\Rightarrow \quad a_n = \frac{a_1}{(\sqrt{2})^{n-1}} = \frac{10}{(2)^{(n-1)/2}}$

$\because \quad$ Area of $S_n < 1 \Rightarrow a_n^2 < 1$

$\Rightarrow \quad \frac{100}{2^{n-1}} < 1$

$\Rightarrow \quad 2^{n-1} > 100 > 2^6$

$\Rightarrow \quad n - 1 > 6$

$\Rightarrow \quad n > 7$

$\therefore \quad n = 8, 9, 10, \ldots$

• **Ex 7.** *If each of the vertices of a triangle has integral coordinates, then the triangles may be*

(a) right angled (b) equilateral
(c) isosceles (d) scalene

Sol. (a,c,d) Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ be the vertices of triangle ABC. Given $x_1, y_1, x_2, y_2, x_3, y_3$ be all integers.

Now, area of $\Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

$= \text{Rational} \quad$ ...(i)

If ΔABC is equilateral then,

Area of $\Delta ABC = \frac{\sqrt{3}}{4}(\text{side})^2$

$= \frac{\sqrt{3}}{4}\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$

$= \text{Irrational} \quad$ ...(ii)

It is clear from Eqs. (i) and (ii), ΔABC can not be equilateral.

• **Ex. 8.** *ABC is an isosceles triangle. If the coordinates of the base are $B(1, 3)$ and $C(-2, 7)$. The coordinates of vertex A can be*

(a) $(1, 6)$ (b) $\left(-\frac{1}{2}, 5\right)$ (c) $\left(\frac{5}{6}, 6\right)$ (d) $\left(-7, \frac{1}{8}\right)$

Sol. (b,c,d) Let vertex of the ΔABC be $A(x, y)$

$\therefore \quad AB = AC$

$\Rightarrow \quad (AB)^2 = (AC)^2$

$\Rightarrow \quad (x - 1)^2 + (y - 3)^2 = (x + 2)^2 + (y - 7)^2$

$\Rightarrow \quad 6x - 8y + 43 = 0 \quad$ ...(i)

Here, use observe that the coordinates $\left(-\frac{1}{2}, 5\right), \left(\frac{5}{6}, 6\right)$ and $\left(-7, \frac{1}{8}\right)$ satisfy the Eq. (i).

JEE Type Solved Examples : Paragraph Based Questions

- This section contains **one solved paragraph** based 3 **multiple choice questions**. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph

(Q. Nos. 9 to 11)

If $A\left(\alpha, \frac{1}{\alpha}\right)$, $B\left(\beta, \frac{1}{\beta}\right)$, $C\left(\gamma, \frac{1}{\gamma}\right)$ be the vertices of a ΔABC, where α, β are the roots of $x^2 - 6ax + 2 = 0$; β, γ are the roots of $x^2 - 6bx + 3 = 0$ and γ, α are the roots of $x^2 - 6cx + 6 = 0$; a, b, c being positive.

9. The value of $a + b + c$ is

(a) 1 (b) 2
(c) 3 (d) 5

10. The coordinates of centroid of ΔABC is

(a) $\left(1, \frac{11}{9}\right)$ (b) $\left(\frac{1}{3}, \frac{11}{18}\right)$
(c) $\left(2, \frac{11}{18}\right)$ (d) $\left(\frac{2}{3}, \frac{11}{19}\right)$

11. The coordinates of orthocentre of ΔABC is

(a) $\left(-\frac{1}{2}, -2\right)$ (b) $\left(-\frac{1}{3}, -3\right)$
(c) $\left(-\frac{1}{5}, -5\right)$ (d) $\left(-\frac{1}{6}, -6\right)$

Sol. $\because \alpha, \beta$ are the roots of $x^2 - 6ax + 2 = 0$

$\therefore \quad \alpha + \beta = 6a \quad$ ...(i)

and $\quad \alpha\beta = 2 \quad$ ...(ii)

Again, β, γ are the roots of $x^2 - 6bx + 3 = 0$

$\therefore \quad \beta + \gamma = 6b \quad$ ...(iii)

and $\quad \beta\gamma = 3 \quad$ ...(iv)

Again, γ, α are the roots of $x^2 - 6cx + 6 = 0$

$\therefore \quad \gamma + \alpha = 6c \quad$ ...(v)

and $\quad \gamma\alpha = 6 \quad$ ...(vi)

from Eqs. (ii), (iv) and (vi), we get

$$\alpha\beta \cdot \beta\gamma \cdot \gamma\alpha = 2 \cdot 3 \cdot 6$$

$\Rightarrow \quad \alpha\beta\gamma = 6$

$\therefore \quad \alpha = 2, \beta = 1, \gamma = 3$

9. (b) Adding Eqs. (i), (iii) and (v), we get

$$2(\alpha + \beta + \gamma) = 6(a + b + c)$$

or $\quad a + b + c = \frac{1}{3}(2 + 1 + 3) = 2$

10. (c) Centroid of $\Delta ABC \equiv \left(\frac{\Sigma\alpha}{3}, \frac{\Sigma\alpha\beta}{3\alpha\beta\gamma}\right)$

$\equiv \left(\frac{2 + 1 + 3}{3}, \frac{2 \cdot 1 + 1 \cdot 3 + 3 \cdot 2}{3 \cdot 2 \cdot 1 \cdot 3}\right)$

$\equiv \left(2, \frac{11}{18}\right)$

11. (d) Orthocentre of $\Delta ABC \equiv \left(-\frac{1}{\alpha\beta\gamma}, -\alpha\beta\gamma\right)$

$\equiv \left(-\frac{1}{6}, -6\right)$

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains **one solved example**. The answer to this example is **a single digit integer** ranging from 0 to 9 (both inclusive).

Ex. 12. *If the points $(-2, 0)$, $\left(-1, \frac{1}{\sqrt{3}}\right)$ and $(\cos\theta, \sin\theta)$ are collinear, then the number of values of $\theta \in [0, 2\pi]$ is*

Sol. (1) Since, the given points are collinear, then

$$\begin{vmatrix} -2 & 0 & 1 \\ -1 & \frac{1}{\sqrt{3}} & 1 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad -2\left(\frac{1}{\sqrt{3}} - \sin\theta\right) - 0 + 1\left(-\sin\theta - \frac{\cos\theta}{\sqrt{3}}\right) = 0$$

$$\Rightarrow \quad \sqrt{3}\sin\theta - \cos\theta = 2$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = 1$$

$$\Rightarrow \quad \sin\left(\theta - \frac{\pi}{6}\right) = 1$$

or $$\theta - \frac{\pi}{6} = 2n\pi + \frac{\pi}{2}$$

or for $n = 0$, $$\theta = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3} \in [0, 2\pi]$$

Number of values θ is 1.

JEE Type Solved Examples : Matching Type Questions

- This section contains **one solved example**. Which has four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statements in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

Ex. 13. *Match the following*

Column I	Column II
A. The points $(\lambda + 1, 1)$, $(2\lambda + 1, 3)$ and $(2\lambda + 2, 2\lambda)$ are collinear then number of values of λ is	(p) a prime number
B. Area of ΔABC is 20 sq units, where A, B and C are $(4, 6)$, $(10, 14)$ and (x, y) respectively. AC is perpendicular to BC, then number of positions of C is	(q) an odd number
C. In a ΔABC coordinates of orthocentre, centroid and vertex A are respectively $(2, 2)$, $(2, 1)$ and $(0, 2)$. The x-coordinate of vertex B is	(r) a composite number
D. A man starts from $P(-3, 4)$ and reaches the point $Q(0, 1)$ touching the X-axis at $R(\lambda, 0)$ such that $PR + RQ$ is minimum, then $10\|\lambda\|$ is	(s) a perfect number

Sol. (A) → (p); (B) → (r); (C) → (p, q); (D) → (r, s)

(A) $\because$ Points are collinear

$$\therefore \quad \begin{vmatrix} \lambda+1 & 1 & 1 \\ 2\lambda+1 & 3 & 1 \\ 2\lambda+2 & 2\lambda & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda+1)(3-2\lambda) - 1(2\lambda+1-2\lambda-2) + 1(4\lambda^2 + 2\lambda - 6\lambda - 6) = 0$$

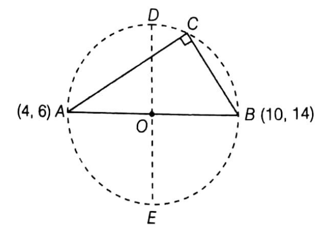
or $$2\lambda^2 - 3\lambda - 2 = 0$$

or $$(2\lambda+1)(\lambda-2) = 0$$

$$\therefore \quad \lambda = 2, -\frac{1}{2}$$

Number of values of λ is 2.

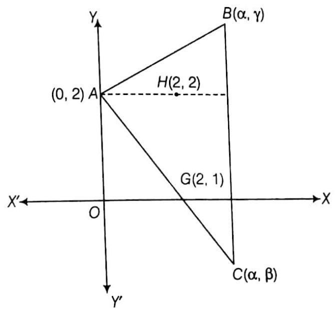
(B) $\because$ Area of triangles ABC is 20 sq units.



$\because$ C can not be at D and E

$\therefore$ Four positions are possible two above AB and two below AB.

(C)

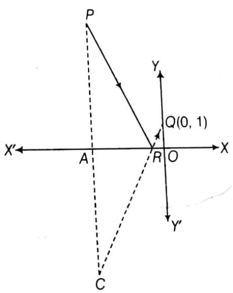


$$\because \quad \frac{\alpha + \alpha + 0}{3} = 2$$

$$\therefore \quad \alpha = 3$$

$\Rightarrow$ x-coordinates of vertex $B = 3$

(D) For $PR + RQ$ to be minimum, it should be the path of light



$$\because \quad \angle PRA = \angle QRO$$

From similar triangles PAR and QOR

$$\frac{AR}{RO} = \frac{PA}{QO} \quad \text{or} \quad \frac{\lambda+3}{0-\lambda} = \frac{4}{1}$$

or $$\lambda = -\frac{3}{5} \quad \therefore \quad 10|\lambda| = 6.$$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** (Ex. Nos. 14 and 15) are Assertion-Reason type examples. Each of these examples contains two statements.

Statement I (Assertion) and **Statement II** (Reason)

Each of these examples also has four alternative choices, (a), (b), (c) and (d) only one out of which is the correct answer. You have to select the correct choice as given below :

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

Ex. 14. Statement I *The area of the triangle formed by the points $A(100, 102)$, $B(102, 105)$, $C(104, 107)$ is same as the area formed by $A'(0, 0)$, $B'(2, 3)$, $C'(4, 5)$.*

Statement II *The area of the triangle is constant with respect to translation :*

Sol. (a) Area of triangle is unaltered by shifting origin to any point. If origin is shifted to $(100, 102)$, then A, B, C becomes $A'(0, 0)$, $B'(2, 3)$, $C'(4, 5)$ respectively. Hence, both statements are true and Statement II is correct explanation for Statement I.

Ex. 15. Statement I *If centroid and circumcentre of a triangle are known its orthocentre can be found*

Statement II *Centroid, orthocentre and circumcentre of a triangle are collinear.*

Sol. (b) $\because$ Centroid divides orthocentre and circumcentre in the ratio 2 : 1 (internally).

$\therefore$ We can find easily orthocentre.

$\Rightarrow$ Statement I is true, and centroid, orthocentre and circumcentre are collinear Statement II is true but Statement II is not correct explanation for Statement I.

Coordinate System and Coordinates Exercise 8 :
Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

47. If a vertex of a triangle is (1, 1) and the mid-points of two side through this vertex are (– 1, 2) and (3, 2), then the centroid of the triangle is **[AIEEE 2005, 3M]**

(a) $\left(\frac{1}{3}, \frac{7}{3}\right)$ (b) $\left(1, \frac{7}{3}\right)$

(c) $\left(-\frac{1}{3}, \frac{7}{3}\right)$ (d) $\left(-1, \frac{7}{3}\right)$

48. Let $O(0, 0)$, $P(3, 4)$, $Q(6, 0)$ be the vertices of the triangle OPQ. The point R inside the triangle OPQ is such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are **[IIT-JEE 2007, 3M]**

(a) $\left(\frac{4}{3}, 3\right)$ (b) $\left(3, \frac{2}{3}\right)$

(c) $\left(3, \frac{4}{3}\right)$ (d) $\left(\frac{4}{3}, \frac{2}{3}\right)$

49. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of values which 'k' can take is given by **[AIEEE 2007, 3M]**

(a) {1, 3} (b) {0, 2}

(c) (– 1, 3} (d) {–3, – 2}

50. Three distinct points A, B and C are given in the 2-dimensional coordinates plane such that the ratio of the distance of any one of them from the point (1, 0) to the distance from the point (–1, 0) is equal to $\frac{1}{3}$. Then, the circumcentre of the triangle ABC is at the point **[AIEEE 2009, 4M]**

Must

(a) $\left(\frac{5}{4}, 0\right)$ (b) $\left(\frac{5}{2}, 0\right)$

(c) $\left(\frac{5}{3}, 0\right)$ (d) (0, 0)

51. The x-coordinate of the incentre of the triangle that has the coordinates of mid-points of its sides are (0, 1), (1, 1) and (1, 0) is **[JEE Main 2013, 4M]**

(a) $2+\sqrt{2}$ (b) $2-\sqrt{2}$

(c) $1+\sqrt{2}$ (d) $1-\sqrt{2}$

52. The number of points, having both coordinates are integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is **[JEE Main 2015, 4M]**

By A.P

(a) 820 (b) 780

(c) 901 (d) 861

39+38+37 ... 1

53. Let k be an integer such that the triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq units. Then, the orthocentre of this triangle is at the point **[JEE Main 2017, 4M]**

(a) $\left(2, \frac{1}{2}\right)$ (b) $\left(2, -\frac{1}{2}\right)$

(c) $\left(1, \frac{3}{4}\right)$ (d) $\left(1, -\frac{3}{4}\right)$