CHAPTER

05

Parabola

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Session 1

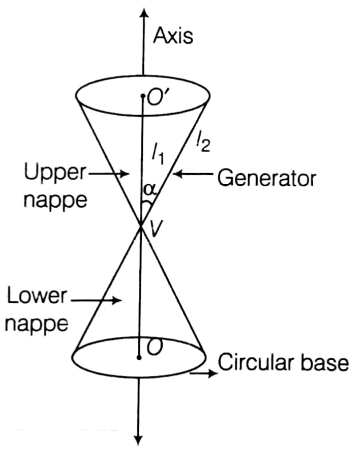
Introduction, Conic Section, Section of a Right Circular Cone by Different Planes, Conic Section : Definition, Equation of Conic Section, Recognisation of Conics, How to Find the Centre of Conics, Parabola : Definition, Other forms of Parabola with Latusrectum 4a, General Equation of a Parabola, The Generalised form $(y-k)^2 = 4a(x-h)$, Parabolic Curve

Introduction

The famous Greek mathematician Euclid, the father of creative Geometry, near about 300 BC considering various plane sections of a right circular cone found many curves, which are called conics or conic sections.

Conic Section

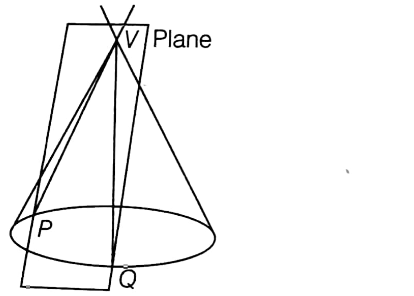
Let l_1 be a fixed vertical line and l_2 be another line intersecting it at a fixed point V and inclined to it at an angle α.



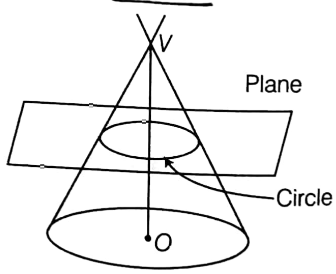
Suppose we rotate the line l_2 around the line l_1 in such a way the angle remains constant then, the surface generated is a double-napped right circular hollow cone. The point V is called the vertex, the line l_1 is the axis of the cone. The rotating line l_2 is called a generator of the cone. The vertex separates the cone into two parts called nappes. The constant angle α is called the semi-vertical angle of the cone.

Section of a Right Circular Cone by Different Planes

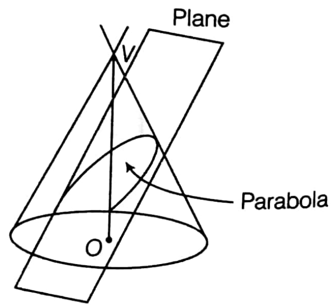
1. Section of a right circular cone by a plane which is passing through its vertex is a pair of straight lines. lines always passes through the vertex of the cone.



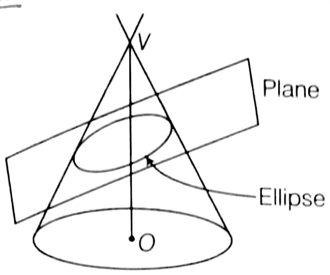
2. Section of a right circular cone by a plane which parallel to its base is a circle.



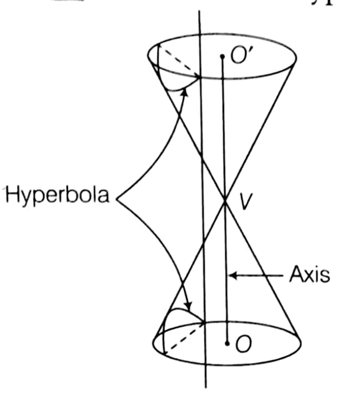
3. Section of a right circular cone by a plane which is parallel to a generator of the cone is a parabola.



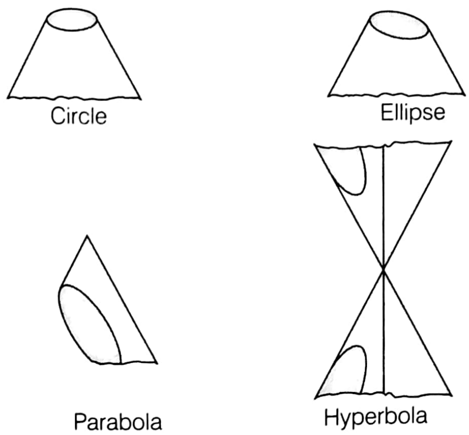
4. Section of a right circular cone by a plane which is not parallel to any generator and not parallel or perpendicular to the axis of the cone is an ellipse.



5. Section of a right circular cone by a plane which is parallel to the axis of the cone is a hyperbola.

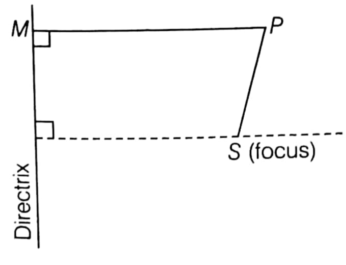


3D View



Conic Section : Definition

The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always constant, is known as a **conic section** or a **conic.**



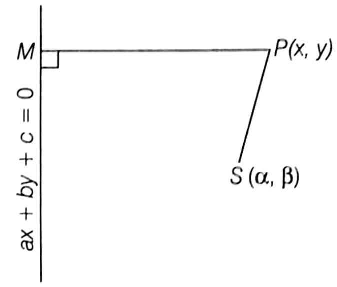
The fixed point is called the **focus** of the conic and this fixed line is called the **directrix** of the conic. Also, this constant ratio is called the **eccentricity** of the conic and is denoted by e.

In the figure, $\dfrac{SP}{PM} = \text{constant} = e$

$\Rightarrow \quad SP = e\,PM$

Equation of Conic Section

If the focus is (α, β) and the directrix is $ax + by + c = 0$, then the equation of the conic section whose eccentricity $= e$ is $SP = e\,PM$



$$\Rightarrow \quad \sqrt{(x-\alpha)^2 + (y-\beta)^2} = e \cdot \frac{|ax + by + c|}{\sqrt{(a^2 + b^2)}}$$

$$\Rightarrow \quad (x-\alpha)^2 + (y-\beta)^2 = e^2 \cdot \frac{(ax + by + c)^2}{(a^2 + b^2)}.$$

Important Terms

Axis The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

Vertex The points of intersection of the conic section and the axis is (are) called vertex (vertices) of the conic section.

Focal Chord Any chord passing through the focus is called focal chord of the conic section.

Double Ordinate A straight line drawn perpendicular to the axis and terminated at both end of the curve is a double ordinate of the conic section.

Latusrectum The double ordinate passing through the focus is called the latusrectum of the conic section.

Centre The point which bisects every chord of the conic passing through it, is called the centre of the conic section.

Remark

Parabola has no centre but circle, ellipse and hyperbola have centre.

Example 1 Find the locus of a point, which moves such that its distance from the point (0, −1) is twice its distance from the line $3x+4y+1=0$.

Sol. Let $P(x_1, y_1)$ be the point, whose locus is required.

Its distance from $(0, -1) = 2\times$ its distance from the line $3x+4y+1=0$.

$$\Rightarrow \quad \sqrt{(x_1-0)^2+(y_1+1)^2} = 2\times\frac{|3x_1+4y_1+1|}{\sqrt{(3^2+4^2)}}$$

$$\Rightarrow \quad 5\sqrt{x_1^2+(y_1+1)^2} = 2|3x_1+4y_1+1|$$

Squaring and simplifying, we have

$$25(x_1^2+y_1^2+2y_1+1) = 4(9x_1^2+16y_1^2+24x_1y_1+6x_1+8y_1+1)$$

or $11x_1^2+39y_1^2+96x_1y_1+24x_1-18y_1-21=0$

Hence, the locus of (x_1, y_1) is

$$11x^2+39y^2+96xy+24x-18y-21=0$$

must

Example 2 What conic does the equation $25(x^2+y^2-2x+1)=(4x-3y+1)^2$ represent ?

Sol. Given equation is

$$25(x^2+y^2-2x+1)=(4x-3y+1)^2 \quad \text{...(i)}$$

Write the right hand side of this equation, so that it appears in perpendicular distance form, then

$$(4x-3y+1)^2 = 25\left(\frac{4x-3y+1}{\sqrt{(4^2+3^2)}}\right)^2$$

then, Eq. (i) can be re-written as

$$25[(x-1)^2+(y-0)^2] = 25\left[\frac{4x-3y+1}{\sqrt{(4^2+3^2)}}\right]^2$$

or $$\sqrt{(x-1)^2+(y-0)^2} = \frac{|4x-3y+1|}{\sqrt{(4^2+3^2)}}$$

Here, $\boxed{e=1}$

Thus, the given equation represents a parabola. It may noted that (1,0) is the focus and $4x-3y+1=0$ is the directrix of the parabola.

Recognisation of Conics

The equation of conics represented by the general equation of second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{...(i)}$$

can be recognised easily by the condition given in the tabular form. For this, first we have to find discriminant of the equation. We know that the discriminant of above equation is represented by Δ, where

$$\Delta = abc+2fgh-af^2-bg^2-ch^2$$

The nature of the conic section depends upon the position of the focus S with respect to the directrix and also upon the value of the eccentricity e. Two different cases arise.

Case I (When the focus lies on the directrix)

In this case Eq. (i) represents the **Degenerate conic** whose nature is given in the following table :

Condition	Nature of Conic
$e>1$; $\Delta=0$, $h^2>ab$	The lines will be real and distinct intersecting at S.
$e=1$; $\Delta=0$, $h^2=ab$	The lines will coincident
$e<1$; $\Delta=0$, $h^2<ab$	The lines will be imaginary.

Case II (When the focus does not lie on the directrix)

In this case Eq. (i) represents the **Non-degenerate conic** whose nature is given in the following table :

Condition	Nature of Conic
$e=1$; $\Delta\neq0$, $h^2=ab$	a parabola
$0<e<1$; $\Delta\neq0$, $h^2<ab$	an ellipse
$e>1$; $\Delta\neq0$, $h^2>ab$	a hyperbola
$e>1$; $\Delta\neq0$, $h^2>ab$; $a+b=0$	rectangular hyperbola

Remark

1. If conic represents an empty set, then $\Delta\neq0$, $h^2<ab$.
2. If conic represents a single point, the $\Delta=0$, $h^2<ab$.

Example 3 What conic does $13x^2-18xy+37y^2+2x+14y-2=0$ represent?

Sol. Compare the given equation with

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

$\therefore \quad a=13, h=-9, b=37, g=1, f=7, c=-2,$

then, $\Delta = abc+2fgh-af^2-bg^2-ch^2$

$$= (13)(37)(-2)+2(7)(1)(-9) - 13(7)^2-37(1)^2+2(-9)^2$$

$$= -962-126-637-37+162 = -1600\neq0$$

and also $h^2=(-9)^2=81$ and $ab=13\times37=481$

Here, $h^2<ab$

So, we have $h^2<ab$ and $\Delta\neq0$.

Hence, the given equation represents an ellipse.

Example 4 What conic is represented by the equation $\sqrt{ax}+\sqrt{by}=1$?

Sol. Given conic is $\sqrt{ax}+\sqrt{by}=1$

On squaring both sides, we get

$$ax+by+2\sqrt{abxy}=1$$

$$\Rightarrow \quad ax+by-1=-2\sqrt{abxy}$$

Again, on squaring both sides, then

$$(ax+by-1)^2 = 4abxy$$

$$\Rightarrow \quad a^2x^2+b^2y^2+1+2abxy-2by-2ax = 4abxy$$

$$\Rightarrow \quad a^2x^2+b^2y^2-2abxy-2ax-2by+1=0$$

$$\Rightarrow \quad a^2x^2-2abxy+b^2y^2-2ax-2by+1=0 \quad \ldots(i)$$

Comparing the Eq. (i) with the equation

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

$$\therefore \quad A=a^2, H=-ab, B=b^2, G=-a, F=-b, C=1$$

then, $\Delta = ABC+2FGH-AF^2-BG^2-CH^2$

$$= a^2b^2-2a^2b^2-a^2b^2-a^2b^2-a^2b^2$$

$$= -4a^2b^2 \neq 0 \text{ and } H^2 = a^2b^2 = AB$$

So, we have $\Delta \neq 0$ and $H^2 = AB$.

Hence, the given equation represents a parabola.

| Example 5 If the equation $x^2-y^2-2x+2y+\lambda=0$ represents a degenerate conic, find the value of λ.

Sol. For degenerate conic $\Delta = 0$

Comparing the given equation of conic with

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

$$\therefore \quad a=1, b=-1, h=0, g=-1, f=1, c=\lambda$$

$$\therefore \quad \Delta = abc+2fgh-af^2-bg^2-ch^2=0$$

$$\Rightarrow (1)(-1)(\lambda)+0-1\times(1)^2+1\times(-1)^2-\lambda(0)^2=0$$

$$\Rightarrow \quad -\lambda-1+1=0 \Rightarrow \lambda=0$$

| Example 6 If the equation $x^2+y^2-2x-2y+c=0$ represents an empty set, then find the value of c.

Sol. For empty set $\Delta \neq 0$ and $h^2<ab$.

Now, comparing the given equation of conic with

$$ax^2+2hxy+by^2+2gx+2fy+c'=0$$

then $a=1, h=0, b=1, g=-1, f=-1, c'=c$

$$\because \quad h^2<ab$$

$$\therefore \quad 0<1 \text{ which is true}$$

and $\quad \Delta = abc'+2fgh-af^2-bg^2-c'h^2 \neq 0$

$$\Rightarrow \quad (1)(1)(c)+0-1\times(-1)^2-1\times(-1)^2-0\neq 0$$

$$\Rightarrow \quad c-2\neq 0$$

$$\therefore \quad c\neq 2$$

Hence, $\quad c\in R\text{~}(2)$

| Example 7 If the equation of conic $2x^2+xy+3y^2-3x+5y+\lambda=0$ represent a single point, then find the value of λ.

Sol. For single point,

$$h^2<ab \text{ and } \Delta=0$$

Comparing the given equation with

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

then, $a=2, h=\frac{1}{2}, b=3, g=-\frac{3}{2}, f=\frac{5}{2}, c=\lambda.$

$$\because \quad h^2=\frac{1}{4}, \; ab=6$$

$$\therefore \quad h^2<ab$$

and $\quad \Delta = abc+2fgh-af^2-bg^2-ch^2$

$$=(2)(3)(\lambda)+2\times\frac{5}{2}\times-\frac{3}{2}\times\frac{1}{2}-2\times\frac{25}{4}-3\times\frac{9}{4}-\lambda\times\frac{1}{4}$$

$$=6\lambda-\frac{15}{4}-\frac{25}{2}-\frac{27}{4}-\frac{\lambda}{4}$$

$$=\frac{23\lambda}{4}-23=0$$

$$\therefore \quad \lambda=4$$

| Example 8 For what value of λ the equation of conic $2xy+4x-6y+\lambda=0$ represents two intersecting straight lines, if $\lambda=17$, then this equation represents?

Sol. Comparing the given equation of conic with

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

$$\therefore \quad a=0, b=0, h=1, g=2, f=-3, c=\lambda$$

For two intersecting lines,

$$h^2>ab, \Delta=0$$

$$\because \quad ab=0, h=1$$

$$\therefore \quad h^2>ab$$

and $\quad \Delta=abc+2fgh-af^2-bg^2-ch^2$

$$=0+2\times-3\times2\times1-0-0-\lambda(1)^2$$

$$=-12-\lambda=0$$

$$\therefore \quad \lambda=-12$$

For $\lambda=17$, then the given equation of conic $2xy+4x-6y+17=0$ according to the first system but here $c=17$.

$$\therefore \quad a=0, b=0, h=1, g=2, f=-3, c=17,$$

$$\therefore \quad \Delta=abc+2f\,gh-af^2-bg^2-ch^2$$

$$=0+2\times-3\times2\times1-0-0-17\times(1)^2$$

$$=-12-17=-29\neq 0$$

$$\therefore \quad \Delta\neq 0 \text{ and } h^2>ab$$

So, we have $\Delta\neq 0$ and $h^2>ab$.

Hence, the given equation represents a hyperbola.

How to Find the Centre of Conics

If $S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$.

Partially differentiating w.r.t. x and y, we get

$$\frac{\partial S}{\partial x}=2ax+2hy+2g; \quad \frac{\partial S}{\partial y}=2hx+2by+2f$$

(Treating y as constant) (Treating x as constant)

For centre, $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$

$\therefore$ $2ax + 2hy + 2g = 0$ and $2hx + 2by + 2f = 0$

$\Rightarrow$ $ax + hy + g = 0$ and $hx + by + f = 0$

Solving these equations we get the centre

$$(x, y) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right).$$

Remembering Method

Since, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Write first two rows,

i.e. $\begin{matrix} a & h & g & a \\ h & b & f & h \end{matrix}$ (Repeat Ist member)

$\therefore$ $ab - h^2, hf - bg, gh - af$

or points $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$ or $\left(\frac{C_{13}}{C_{33}}, \frac{C_{23}}{C_{33}} \right)$.

OR

According to first two rows,

$$ax + hy + g = 0 \quad \text{and} \quad hx + by + f = 0.$$

After solving we get find the centre of conic.

Example 9 Find the centre of the conic $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$

Sol. Let $f(x, y) \equiv 14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$

Differentiating partially w.r.t. x and y, then

$\frac{\partial f}{\partial x} = 28x - 4y - 44$ and $\frac{\partial f}{\partial y} = -4x + 22y - 58$

For centre, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$,

$\therefore$ $28x - 4y - 44 = 0$

or $7x - y - 11 = 0$ ...(i)

and $-4x + 22y - 58 = 0$

or $-2x + 11y = 29$ ...(ii)

On solving Eqs. (i) and (ii) we get,

$x = 2$ and $y = 3$

$\therefore$ Centre is (2, 3).

Aliter : Comparing the given conic with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore$ $a = 14, h = -2, b = 11, g = -22, f = -29, c = 71$

$\therefore$ Centre $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$

or $= \left(\frac{(-2)(-29) - (11)(-22)}{(14)(11) - (-2)^2}, \frac{(-22)(-2) - (14)(-29)}{(14)(11) - (-2)^2} \right)$

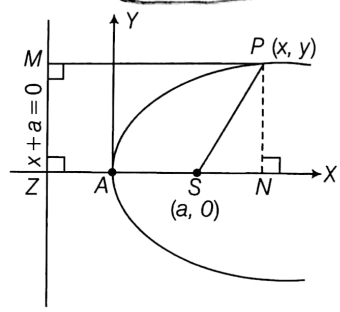
or $= (2, 3)$

Parabola : Definition

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (i.e. focus) is always equal to its distance from a fixed straight line (i.e., directrix).

Standard Equation of Parabola

Let S be the focus and ZM be the directrix of the parabola. Draw SZ perpendicular to ZM, let A be the mid point of SZ, then as $\boxed{AS = AZ}$



So, A lies on the parabola. Take A as the origin and a line AY through A perpendicular to AX as Y-axis.

Let $AS = AZ = a > 0$

then, coordinate of S is $(a, 0)$ and the equation of ZM is

$$x = -a \text{ or } x + a = 0$$

Now, take $P(x, y)$ be any point on the parabola. Join SP and from P draw PM perpendicular to the directrix ZM.

Then, $SP = \sqrt{(x - a)^2 + (y - 0)^2} = \sqrt{(x - a)^2 + y^2}$

and $PM = ZN = AZ + AN = a + x$

Now, for the parabola $SP = PM$

$\Rightarrow$ $(SP)^2 = (PM)^2 \Rightarrow (x - a)^2 + y^2 = (a + x)^2$

$\Rightarrow$ $y^2 = (a + x)^2 - (x - a)^2 = 4ax$

$\therefore$ $\boxed{y^2 = 4ax,}$

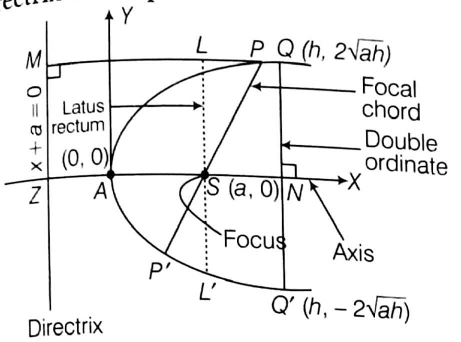
which is required equation of the parabola.

Remark

A parabola has two real foci situated on its axis one of which is the focus S and the other lies at infinity, the corresponding directrix is also at infinity.

Some Terms Related to Parabola

1. Axis The axis of the parabola is the straight line which is passes through focus and perpendicular to the directrix of the parabola.



For the parabola $y^2 = 4ax$, X-axis is the **axis.**

Here, all powers of y are even in $y^2 = 4ax$ then, parabola $y^2 = 4ax$ is symmetrical about its axis (i.e. X-axis).

or

If the point (x, y) lie on the parabola $y^2 = 4ax$, then the point $(x, -y)$ also lies on it. Hence, the parabola is symmetrical about X-axis (i.e. axis of parabola).

2. Vertex The point of intersection of the parabola and its axis is called the vertex of the parabola. For the parabola $y^2 = 4ax$.

$A(0,0)$ i.e. the origin is the vertex.

3. Double ordinate If Q be the point on the parabola, draw QN perpendicular to the axis of parabola and produced to meet the curve again at Q', then QQ' is called a double ordinate.

If abscissa of Q is h, then ordinate of Q,

$$y^2 = 4ah \quad \text{or } y = 2\sqrt{ah} \qquad \text{(for first quadrant)}$$

and ordinate of Q' is $y = -2\sqrt{ah}$ (for fourth quadrant)

Hence, coordinates of Q and Q' are $(h, 2\sqrt{ah})$ and $(h, -2\sqrt{ah})$, respectively.

4. Latusrectum The double ordinate LL' passes through the focus is called the latusrectum of the parabola.

Since focus $S(a, 0)$ the equation of the latusrectum of the parabola is $x = a$, then solving

$$x = a \quad \text{and} \quad y^2 = 4ax$$

then, we get $\boxed{y = \pm 2a}$

Hence, the coordinates of the extremities of the latusrectum are $L(a, 2a)$ and $L'(a, -2a)$, respectively.

Since,
$$LS = L'S = 2a$$

$\therefore$ Length of latusrectum $LL' = 2(LS) = 2(L'S) = \boxed{4a.}$

5. Focal chord A chord of a parabola which is passing through the focus is called a focal chord of the parabola. In the given figure, PP' and LL' are the focal chords.

Remarks

1. In objective questions use LL' as focal chord and in subjective questions use PP' as focal chord. Remember
2. Length of smallest focal chord of the parabola $4a$. Hence, the latusrectum of a parabola is the smallest focal chord.

6. Focal distance The focal distance of any point P on the parabola is its distance from the focus S i.e. SP

Also, $SP = PM =$ Distance of P from the directrix.

If $\quad P \equiv (x, y)$

then, $\quad SP = PM = \boxed{x + a}$

7. Parametric equations From the equation of the parabola $y^2 = 4ax$, we can write $\dfrac{y}{2a} = \dfrac{2x}{y} = t$

where 't' is a parameter.

Then, $\boxed{y = 2at \quad \text{and} \quad x = at^2}$

The equations $x = at^2$ and $y = 2at$ are called parametric equations. The point $(at^2, 2at)$ is also referred to as the point 't'.

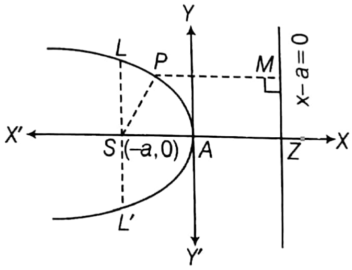
Remarks

1. Coordinates of any point on the parabola $y^2 = 4ax$, may be taken as $(at^2, 2at)$.
2. Equation of chord joining t_1 and t_2 is $2x - (t_1 + t_2)y + 2at_1t_2 = 0$.
3. If the chord joining t_1, t_2 and t_3, t_4 pass through a point $(c, 0)$ on the axis, then $\boxed{t_1t_2 = t_3t_4 = -\frac{c}{a}}$.

Other forms of Parabola with Latusrectum $4a$

(1) Parabola opening to left (i.e. $y^2 = -4ax$) : $(a > 0)$

(i) Vertex is $A(0,0)$.
(ii) Focus is $S(-a, 0)$.
(iii) Equation of the directrix MZ is $x - a = 0$.
(iv) Equation of the axis is $y = 0$ i.e. X-axis.
(v) Equation of the tangent at the vertex is $x = 0$ i.e. Y-axis.



(vi) Length of latusrectum $= LL' = 4a$.

(vii) Ends of latusrectum are $L(-a, 2a)$ and $L'(-a, -2a)$.

(viii) Equation of latusrectum is $x = -a$ i.e. $x + a = 0$.

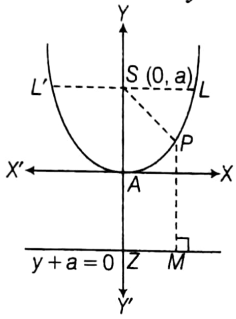
(ix) Parametric coordinates is $(-at^2, 2at)$.

(2) Parabola opening upwards (i.e. $x^2 = 4ay$) : $(a > 0)$

(i) Vertex is $A(0, 0)$.

(ii) Focus is $S(0, a)$.

(iii) Equation of the directrix MZ is $y + a = 0$.



(iv) Equations of the axis is $x = 0$ i.e. Y-axis.

(v) Equation of the tangent at the vertex is $y = 0$ i.e. X-axis.

(vi) Ends of latusrectum are $L(2a, a)$ and $L'(-2a, a)$.

(vii) Length of latusrectum $= LL' = 4a$.

(viii) Equation of latusrectum is $y = a$ i.e. $y - a = 0$.

(ix) Parametric coordinates is $(2at, at^2)$.

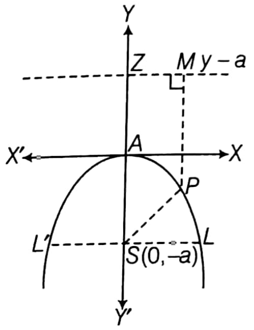
(3) Parabola opening downwards (i.e. $x^2 = -4ay$) : $(a > 0)$

(i) Vertex is $A(0, 0)$. (ii) Focus is $S(0, -a)$.

(iii) Equation of the directrix MZ is $y - a = 0$.

(iv) Equation of the axis is $x = 0$ i.e. Y-axis.

(v) Equation of the tangent at the vertex is $y = 0$ i.e. X-axis.



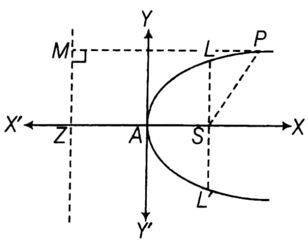
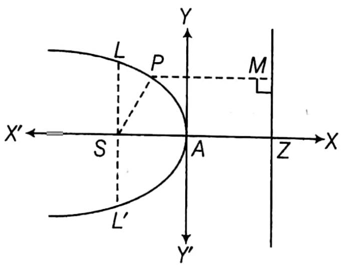
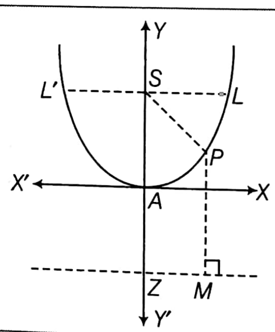
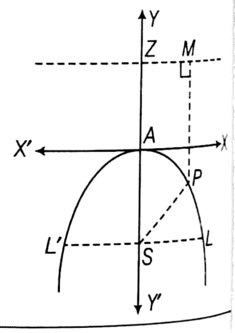
(vi) Length of latusrectum $= LL' = 4a$

(vii) Ends of latusrectum are $L(2a, -a)$ and $L'(-2a, -a)$

(viii) Equation of latusrectum is $y = -a$ i.e. $y + a = 0$.

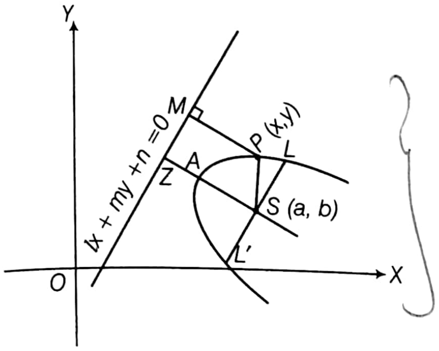
(ix) Parametric coordinates are $(2at, -at^2)$.

Smart Table : The Study of Standard Parabolas

Equation and Graph of the parabola	$y^2 = 4ax, a > 0$	$y^2 = -4ax, a > 0$	$x^2 = 4ay, a > 0$	$x^2 = -4ay, a > 0$
				
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
Length of latusrectum	$4a$	$4a$	$4a$	$4a$
Ends points of latusrectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Equation of latusrectum	$x - a = 0$	$x + a = 0$	$y - a = 0$	$y + a = 0$
Focal distance of a point $P(x, y)$	$x + a$	$a - x$	$y + a$	$a - y$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Eccentricity (e)	1	1	1	1

General Equation of a Parabola

Let $S(a,b)$ be the focus, and $lx+my+n=0$ is the equation of the directrix. Let $P(x,y)$ be any point on the parabola. Then by definition $SP = PM$



$\Rightarrow \sqrt{(x-a)^2+(y-b)^2} = \dfrac{|lx+my+n|}{\sqrt{(l^2+m^2)}}$

$\Rightarrow (x-a)^2+(y-b)^2 = \dfrac{(lx+my+n)^2}{(l^2+m^2)}$

$\Rightarrow m^2x^2+l^2y^2-2lmxy+x$ term$+\,y$ term$\,+$ constant $=0$

This is of the form $(mx-ly)^2+2gx+2fy+c=0$.

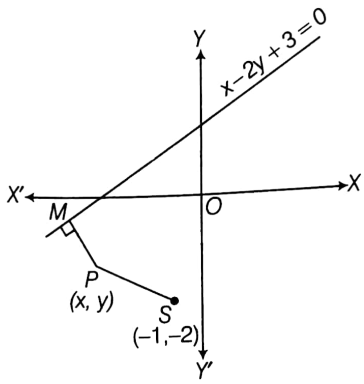
This equation is the general equation of parabola.

Remark

Second degree terms in the general equation of a parabola forms a perfect square.

Example 10 Find the equation of the parabola whose focus is at $(-1,-2)$ and the directrix is the straight line $x-2y+3=0$.

Sol. Let $P(x,y)$ be any point on the parabola whose focus is $S(-1,-2)$ and the directrix $x-2y+3=0$. Draw PM perpendicular from $P(x,y)$ on the directrix $x-2y+3=0$.



Then, by definition

$SP = PM$

$\Rightarrow (SP)^2 = (PM)^2$

$\Rightarrow (x+1)^2+(y+2)^2 = \left(\dfrac{|x-2y+3|}{\sqrt{(1)^2+(-2)^2}}\right)^2$

$\Rightarrow 5(x^2+y^2+2x+4y+5)$

$= (x^2+4y^2-4xy+6x-12y+9)$

$\therefore 4x^2+y^2+4xy+4x+32y+16=0$

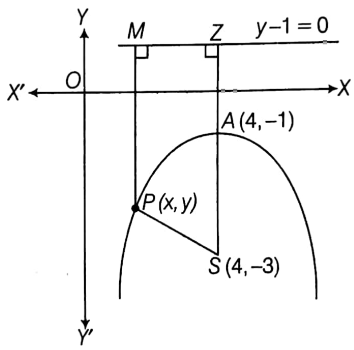
Example 11 Find the equation of the parabola whose focus is $(4,-3)$ and vertex is $(4,-1)$.

Sol. Let $A(4,-1)$ be the vertex and $S(4,-3)$ be the focus.

$\therefore$ Slope of $AS = \dfrac{-3+1}{4-4} = \infty$

which is parallel to Y-axis.

$\therefore$ Directrix parallel to X-axis.



Let $Z(x_1,y_1)$ be any point on the directrix, then A is the mid-point of SZ.

$\therefore 4 = \dfrac{x_1+4}{2} \Rightarrow x_1 = 4$

and $-1 = \dfrac{y_1-3}{2} \Rightarrow y_1 = 1$

$\therefore Z = (4,1)$

Also, directrix is parallel to X-axis and passes through $Z(4,1)$, so equation of directrix is

$y=1$ or $y-1=0$

Now, let $P(x,y)$ be any point on the parabola. Join SP and draw PM perpendicular to the directrix. Then, by definition

$SP = PM$

$\Rightarrow (SP)^2 = (PM)^2$

$\Rightarrow (x-4)^2+(y+3)^2 = \left(\dfrac{|y-1|}{\sqrt{1^2}}\right)^2$

$\Rightarrow (x-4)^2+(y+3)^2 = (y-1)^2$

$\therefore x^2-8x+8y+24=0$

Aliter :

Here $a = AS = 2$

$\therefore$ Length of latusrectum $= 4a = 8$

Equation of parabola with vertex $(0,0)$ and open downward is $x^2 = -8y$.

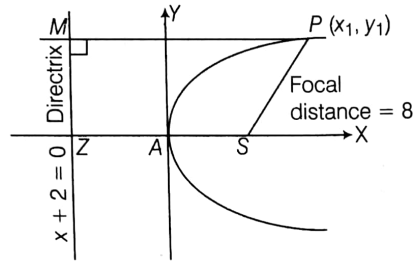
Shifting $(4,-1)$ on $(0,0)$, we get required parabola

$(x-4)^2 = -8(y+1)$

$\therefore x^2-8x+8y+24=0$

Example 12 The focal distance of a point on a parabola $y^2 = 8x$ is 8. Find it.

Sol. Comparing $y^2 = 8x$ with $y^2 = 4ax$



$\therefore \quad 4a = 8 \Rightarrow a = 2$

$\therefore$ Equation of directrix is $x + 2 = 0$.

Let $P(x_1, y_1)$ on the parabola

$$y^2 = 8x$$

$$\therefore \quad y_1^2 = 8x_1 \quad \ldots(i)$$

$$\because \quad SP = 8$$

$$\Rightarrow \quad PM = 8 \quad [\because SP = PM]$$

$$\Rightarrow \quad x_1 + 2 = 8$$

or $\quad x_1 = 6$

From Eq. (i), $\quad y_1^2 = 8 \times 6$

$$\therefore \quad y_1 = \pm 4\sqrt{3}$$

$\therefore$ The required points are $(6, 4\sqrt{3})$ and $(6, -4\sqrt{3})$.

Example 13 QQ' is a double ordinate of a parabola $y^2 = 4ax$. Find the locus of its point of trisection.

Sol. Let the double ordinate QQ' meet the axis of the parabola

$$y^2 = 4ax \quad \ldots(i)$$

Let coordinates of Q be (x_1, y_1), then coordinates of Q' be $(x_1 - y_1)$ since, Q and Q' lies on Eq. (i), then

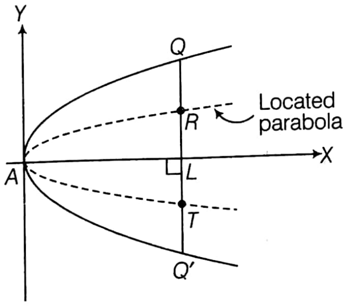
$$y_1^2 = 4ax_1 \quad \ldots(ii)$$

Let R and T be the points of trisection of QQ'. Then, the coordinates of R and T are

$$\left(\frac{1\cdot x_1 + 2\cdot x_1}{1+2}, \frac{1\cdot(-y_1) + 2\cdot y_1}{1+2}\right) \text{ or } \left(x_1, \frac{y_1}{3}\right)$$

and $$\left(\frac{2\cdot x_1 + 1\cdot x_1}{2+1}, \frac{2\cdot(-y_1) + 1\cdot y_1}{2+1}\right) \text{ or } \left(x_1, -\frac{y_1}{3}\right)$$

respectively.



Since, R divide QQ' in $1:2$ (internally)
and T divide QQ' in $2:1$ (internally).
For locus, let $R(h,k)$, then

$$x_1 = h \text{ and } \frac{y_1}{3} = k \text{ or } y_1 = 3k$$

On substituting the values of x_1 and y_1 in Eq. (ii), then

$$(3k)^2 = 4a(h) \text{ or } 9k^2 = 4ah$$

$\therefore$ The required locus is $9y^2 = 4ax$ similarly, let $T(h', k')$

then, $\quad x_1 = h'$ and $-\dfrac{y_1}{3} = k'$

or $\quad y_1 = -3k'$

On substituting the values of x_1 and y_1 in Eq. (ii), then

$$(-3k')^2 = 4a(h')$$

or $\quad 9k'^2 = 4ah'$

$\therefore$ The required locus is $9y^2 = 4ax$.

Hence, the locus of point of trisection is

$$9y^2 = 4ax.$$

Alter : Let R and T be the points of trisection of double ordinates QQ'. Let (h,k) be the coordinates of R,

then, $\quad AL = h$ and $RL = k$

$$RT = RL + LT = k + k = 2k.$$

Since, $\quad RQ = TR = Q'T = 2k$

$$\therefore \quad LQ = LR + RQ = k + 2k = 3k$$

Thus, the coordinates of Q are $(h, 3k)$.

Since, $(h, 3k)$ lies on $y^2 = 4ax$

$$\Rightarrow \quad 9k^2 = 4ah$$

Hence, the locus of (h, k) is $9y^2 = 4ax$.

Example 14 Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}(y_1 \sim y_2)(y_2 \sim y_3)(y_3 \sim y_1)$, where y_1, y_2, y_3 are the ordinates of the vertices.

Sol. Let the vertices of the triangle be (x_1, y_1), (x_2, y_2) and (x_3, y_3).

$\because (x_1, y_1)$ is a point on the parabola $y^2 = 4ax$.

$$\therefore \quad y_1^2 = 4ax_1$$

$$\therefore \quad x_1 = \frac{y_1^2}{4a}$$

Similarly, $\quad x_2 = \dfrac{y_2^2}{4a}$

and $\quad x_3 = \dfrac{y_3^2}{4a}$

Now, vertices of triangle are

$$\left(\frac{y_1^2}{4a}, y_1\right), \left(\frac{y_2^2}{4a}, y_2\right) \text{ and } \left(\frac{y_3^2}{4a}, y_3\right).$$

$\therefore$ Required area of the triangle $= \frac{1}{2}\left\| \begin{matrix} \frac{y_1^2}{4a} & y_1 & 1 \\ \frac{y_2^2}{4a} & y_2 & 1 \\ \frac{y_3^2}{4a} & y_3 & 1 \end{matrix} \right\|$

$$= \frac{1}{8a}\left\| \begin{matrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{matrix} \right\| = \frac{1}{8a}(y_1 \sim y_2)(y_2 \sim y_3)(y_3 \sim y_1)$$

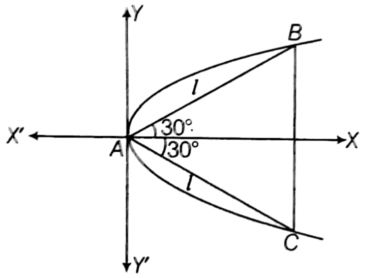
| Example 15 Find the length of the side of an equilateral triangle inscribed in the parabola $y^2 = 4ax$, so that one angular point is at the vertex.

Sol. Let ABC be the inscribed equilateral triangle, with one angular point at the vertex A of the parabola

$$y^2 = 4ax \qquad \text{...(i)}$$

Let the length of the side of equilateral triangle $= l$

$\therefore \quad AB = BC = CA = l$



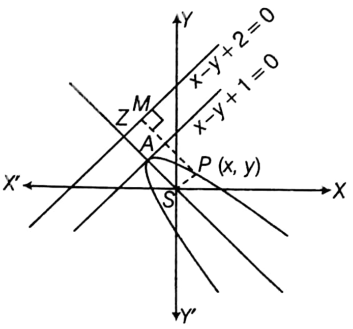
$\therefore$ The coordinates of B is $(l \cos 30°, l \sin 30°)$

i.e., $\left(\frac{l\sqrt{3}}{2}, \frac{l}{2}\right)$.

Since, B lies on Eq. (i), then $\left(\frac{l}{2}\right)^2 = 4a\left(\frac{l\sqrt{3}}{2}\right)$ or $l = 8a\sqrt{3}$

| Example 16 Prove that the equation of the parabola whose focus is (0,0) and tangent at the vertex is $x - y + 1 = 0$ is $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$.

Sol. Let focus is $S(0, 0)$ and A is the vertex of the parabola take any point Z such that $AS = AZ$ given tangent at vertex is $x - y + 1 = 0$, since directrix is parallel to the tangent at the vertex.



$\therefore$ Equation of directrix is $x - y + \lambda = 0$

where, λ is constant.

$\because A$ is the mid-point of SZ.

$$\therefore \quad SZ = 2SA$$

$$\Rightarrow \quad \frac{|0 - 0 + \lambda|}{\sqrt{(1^2 + (-1)^2)}} = 2 \times \frac{|0 - 0 + 1|}{\sqrt{(1^2 + (-1)^2)}}$$

$$\Rightarrow \quad \frac{|\lambda|}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$\therefore \quad \lambda = \pm 2$$

$$\Rightarrow \quad \lambda = 2$$

[$\because \lambda$ is positive since directrix in this case always lies in II quadrant]

$\therefore$ Equation of directrix is $x - y + 2 = 0$.

Now, take $P(x, y)$ be any point on the parabola, draw $PM \perp ZM$, then from definition,

$$SP = PM$$

$$\Rightarrow \quad (SP)^2 = (PM)^2$$

$$\Rightarrow \quad (x - 0)^2 + (y - 0)^2 = \left(\frac{|x - y + 2|}{\sqrt{2}}\right)^2$$

$$\Rightarrow \quad 2(x^2 + y^2) = (x - y + 2)^2$$

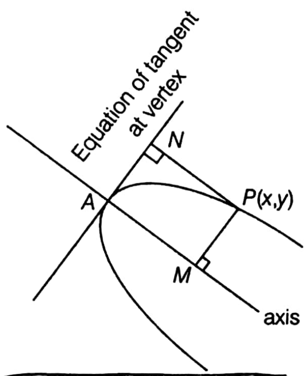
$$\Rightarrow \quad 2x^2 + 2y^2 = x^2 + y^2 - 2xy + 4x - 4y + 4$$

$$\therefore \quad x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$$

Equation of Parabola if Equation of axis, Tangent at Vertex and Latusrectum are given

Let equation of axis is $ax + by + c = 0$ and equation of tangent at vertex is $bx - ay + d = 0$.

Equation of parabola is

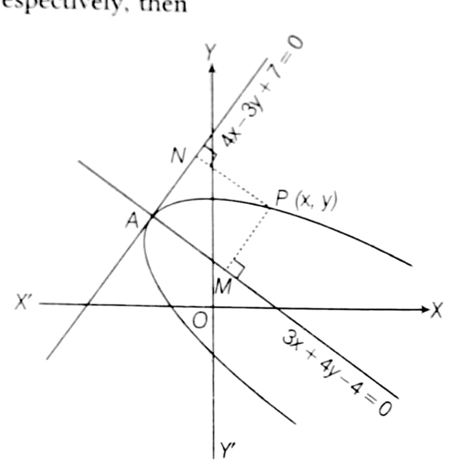


$$\boxed{(PM)^2 = (\text{Latusrectum})\ (PN)} \rightarrow y^2 = 4ax.$$

$$\Rightarrow \left(\frac{ax + by + c}{\sqrt{(a^2 + b^2)}}\right)^2 = (\text{Latusrectum})\left(\frac{bx - ay + d}{\sqrt{(b^2 + a^2)}}\right)$$

Example 17 Find the equation of the parabola whose latusrectum is 4 units, axis is the line $3x+4y-4=0$ and the tangent at the vertex is the line $4x-3y+7=0$.

Sol. Let $P(x, y)$ be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively, then



$$(PM)^2 = (\text{latusrectum})(PN)$$

$$\Rightarrow \quad \left(\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right)^2 = 4\left(\frac{4x-3y+7}{\sqrt{4^2+(-3)^2}}\right)$$

$$\therefore \quad (3x+4y-4)^2 = 20(4x-3y+7)$$

which is required parabola.

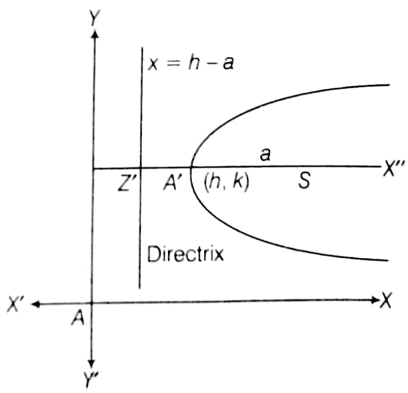
The Generalised form $(y-k)^2 = 4a(x-h)$

The parabola

$$y^2 = 4ax \qquad \text{...(i)}$$

can be written as $(y-0)^2 = 4a(x-0)$.

The vertex of this parabola is $A(0,0)$



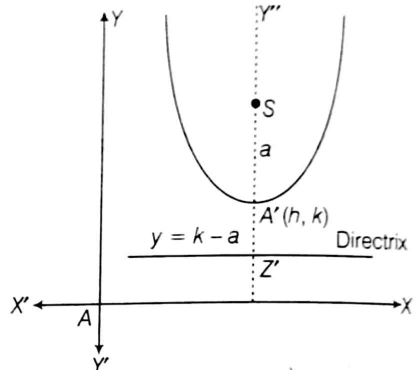
Now, when origin is shifted at $A'(h, k)$ without changing the direction of axes, its equation becomes

$$(y-k)^2 = 4a(x-h) \qquad \text{...(ii)}$$

This is called generalised form of the parabola Eq. (i) and axis $A'X'' \parallel AX$ with its vertex at $A'(h, k)$. Its focus is at $(a+h, k)$ and length of latusrectum $= 4a$, the equation of the directrix is add

$$x = h - a \Rightarrow x + a - h = 0$$

Another form is $(x-h)^2 = 4a(y-k)$ axis parallel to



add

Y-axis with its vertex (h,k) its focus is at $(h, a+k)$ and length of latusrectum $= 4a$, the equation of the directrix is

$$y = k - a \Rightarrow y + a - k = 0.$$

Remark

The parametric equation of $(y-k)^2 = 4a(x-h)$ are $x = h + at^2$ and $y = k + 2at$.

Parabolic Curve

The equations $y = Ax^2 + Bx + C$ and $x = Ay^2 + By + C$ are always represents parabolas generally called parabolic curve.

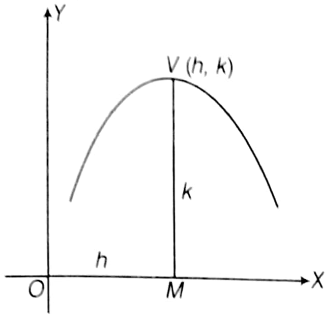
Now,
$$y = Ax^2 + Bx + C$$
$$= A\left\{x^2 + \frac{B}{A}x + \frac{C}{A}\right\}$$
$$= A\left\{\left(x + \frac{B}{2A}\right)^2 - \frac{B^2}{4A^2} + \frac{C}{A}\right\}$$
$$= A\left\{\left(x + \frac{B}{2A}\right)^2 - \frac{(B^2 - 4AC)}{4A^2}\right\}$$

or
$$\left(x + \frac{B}{2A}\right)^2 = \frac{1}{A}\left(y + \frac{B^2 - 4AC}{4A}\right)$$

Comparing it with $(x-h)^2 = 4a(y-k)$ it represent a parabola with vertex at $(h,k) = \left(-\frac{B}{2A}, -\frac{B^2-4AC}{4A}\right)$

and axis parallel to Y-axis and latusrectum $= \dfrac{1}{|A|}$

and the curve opening upwards and downwards depending upon the sign of A and B.



The optimum distance of its vertex V from OX is

$$\left|-\frac{B^2-4AC}{4A}\right|$$

and $\quad x = Ay^2 + By + C$

$$= A\left\{y^2 + \frac{B}{A}y + \frac{C}{A}\right\}$$

$$= A\left\{\left(y + \frac{B}{2A}\right)^2 - \frac{B^2}{4A^2} + \frac{C}{A}\right\}$$

$$= A\left\{\left(y + \frac{B}{2A}\right)^2 - \frac{B^2-4AC}{4A^2}\right\}$$

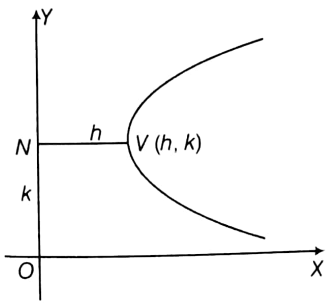
$$\Rightarrow \left(y + \frac{B}{2A}\right)^2 = \frac{1}{A}\left(x + \frac{B^2-4AC}{4A}\right)$$

Comparing it with $(y-k)^2 = 4a(x-h)$, it represent a parabola with vertex at

$$(h, k) = \left(-\frac{B^2-4AC}{4A}, -\frac{B}{2A}\right)$$

and axis parallel to X-axis and latusrectum $= \dfrac{1}{|A|}$

and the curve opening left and right depending upon the sign of A and B.



The optimum distance of its vertex V from OY is

$$\left|-\frac{B^2-4AC}{4A}\right|.$$

Remarks

1. The optimum distance of vertex from OX or OY can be easily obtained using calculus Method.
2. Equation of the parabola with axis parallel to the X-axis is of the form $x = Ay^2 + By + C$.
3. Equation of the parabola with axis parallel to the Y-axis is of the form $y = Ax^2 + Bx + C$.

Method to Make Perfect Square

If $\quad x = \alpha y^2 \pm \beta y + \gamma$

first make the coefficient of y^2 is unity

i.e., $\quad x = \alpha\left\{y^2 \pm \frac{\beta}{\alpha}y + \frac{\gamma}{\alpha}\right\}$

Now, in braces write y and put the sign after y which between y^2 and y i.e. $\pm$ and after this sign write the half the coefficient of y i.e. $\dfrac{\beta}{2\alpha}$.

Now, write in braces $\left(y \pm \dfrac{\beta}{2\alpha}\right)^2$

and $\boxed{\text{always subtract}}$ $\left(\dfrac{\beta}{2\alpha}\right)^2 = \dfrac{\beta^2}{4\alpha^2}$

$$\therefore \quad x = \alpha\left\{\left(y \pm \frac{\beta}{2\alpha}\right)^2 - \frac{\beta^2}{4\alpha^2} + \frac{\gamma}{\alpha}\right\}$$

$$= \alpha\left\{\left(y \pm \frac{\beta}{2\alpha}\right)^2 - \frac{(\beta^2 - 4\gamma\alpha)}{4\alpha^2}\right\}$$

Example 18 Find the vertex, focus, latusrectum, axis and the directrix of the parabola $x^2 + 8x + 12y + 4 = 0$.

Sol. The equation of parabola is

$$x^2 + 8x + 12y + 4 = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad (x+4)^2 - 16 + 12y + 4 = 0$$

$$\Rightarrow \quad (x+4)^2 - 12 + 12y = 0$$

$$\Rightarrow \quad (x+4)^2 = -12y + 12$$

$$\Rightarrow \quad (x+4)^2 = -12(y-1)$$

Let $\quad x + 4 = X, y - 1 = Y \quad$ ...(ii)

$\therefore \quad X^2 = -12Y \quad$ ...(iii)

Comparing it with $X^2 = -4aY$

$\therefore \quad a = 3$

$\therefore$ Vertex of Eq. (iii) is (0, 0)

i.e. $\quad X = 0, Y = 0$

From Eq. (ii),

$$x + 4 = 0, y - 1 = 0$$

$\therefore$ $$x = -4, y = 1$$

$\therefore$ Vertex of Eq. (i) is $(-4, 1)$.

Foucs of Eq. (iii) is $(0, -3)$

i.e. $$X = 0, Y = -3$$

From Eq. (ii),

$$x + 4 = 0, y - 1 = -3$$

$\therefore$ $$x = -4, y = -2$$

$\therefore$ Focus of Eq. (i) is $(-4, -2)$.

and latusrectum $= 4a = 12$.

Equation of axis of Eq. (iii) is $X = 0$

$\therefore$ Equation of axis of Eq. (i) is $x + 4 = 0$

Equation of directrix of Eq. (iii) is

$$Y = 3 \text{ or } y - 1 = 3$$

$\therefore$ $$y - 4 = 0$$

$\therefore$ Equation of directrix of Eq. (i) is

$$y - 4 = 0.$$

| Example 19 Prove that the equation $y^2 + 2ax + 2by + c = 0$ represents a parabola whose axis is parallel to the axis of x. Find its vertex.

Sol. The equation of parabola is

$$y^2 + 2ax + 2by + c = 0$$

$$(y + b)^2 - b^2 + 2ax + c = 0$$

$\Rightarrow$ $$(y + b)^2 = -2ax + b^2 - c$$

$\Rightarrow$ $$(y + b)^2 = -2a\left(x - \frac{b^2 - c}{2a}\right) \quad \text{...(i)}$$

Let $$y + b = Y, x - \frac{b^2 - c}{2a} = X$$

From Eq. (i),

$$Y^2 = -2aX \quad \text{...(ii)}$$

axis of its parabola is $Y = 0$

or $$y + b = 0,$$

which is parallel to X-axis

and vertex of Eq. (ii) is $X = 0, Y = 0$

$\Rightarrow$ $$x - \frac{b^2 - c}{2a} = 0, \quad y + b = 0$$

$\Rightarrow$ $$x = \frac{b^2 - c}{2a}, \quad y = -b$$

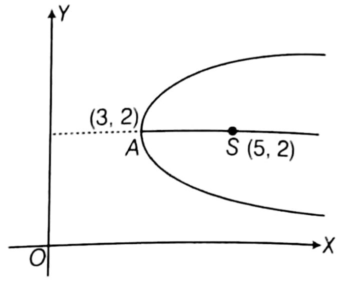
$\therefore$ Vertex of given parabola is $\left(\dfrac{b^2 - c}{2a}, -b\right)$.

| Example 20 Find the equation of the parabola with its vertex at (3, 2) and its focus at (5, 2).

Sol. Let Vertex $A(3,2)$ and focus is $S(5,2)$

Slope of $AS = \dfrac{2-2}{5-3} = 0$, which is parallel to X-axis.

Hence, axis of parabola parallel to X-axis.



The equation is of the form

$$(y - k)^2 = 4a(x - h)$$

or $$(y - 2)^2 = 4a(x - 3)$$

as (h,k) is the vertex (3,2)

$a =$ distance between the focus and the vertex

$$= \sqrt{(5-3)^2 + (2-2)^2} = 2$$

Hence, the required equation is

$$(y - 2)^2 = 8(x - 3)$$

or $$y^2 - 8x - 4y + 28 = 0.$$

| Example 21 Find the equation of the parabola with latusrectum joining the points (3, 6) and (3, –2).

Sol. Slope of (3,6) and (3, –2) is $\dfrac{-2-6}{3-3} = \infty$, since latusrectum is perpendicular to axis. Hence, axis parallel to X-axis. The equation of the two possible parabolas will be of the form

$$(y - k)^2 = \pm 4a(x - h) \quad \text{...(i)}$$

Since, latusrectum $= \sqrt{(3-3)^2 + (6+2)^2} = 8$

$\therefore$ $$4a = 8$$

$\Rightarrow$ $$a = 2$$

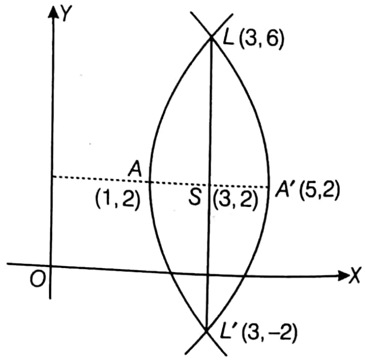
$\therefore$ From Eq. (i),

$$(y - k)^2 = \pm 8(x - h)$$

Since, (3,6) and (3, –2) lie on the parabola, then

$$(6 - k)^2 = \pm 8(3 - h) \quad \text{...(ii)}$$

and $$(-2 - k)^2 = \pm 8(3 - h) \quad \text{...(iii)}$$



On solving Eqs. (ii) and (iii), we get

$$k = 2$$

From Eq. (ii),

$$16 = \pm 8(3 - h),$$

$$\therefore \quad h = 3 \pm 2$$

$$\therefore \quad h = 5, 1$$

Hence, values of (h, k) are $(5,2)$ and $(1, 2)$.

The required parabolas are

$$(y-2)^2 = 8(x-5)$$

and $\quad (y-2)^2 = -8(x-1).$

Example 22 Find the equation to the parabola whose axis parallel to the Y-axis and which passes through the points (0, 4) (1, 9) and (4, 5) and determine its latusrectum .

Sol. The equation of parabola parallel to Y-axis is

$$\boxed{y = Ax^2 + Bx + C} \quad \text{...(i)}$$

The points (0,4) , (1,9) and (4,5) lie on Eq. (i), then

$$4 = 0 + 0 + C \Rightarrow C = 4 \quad \text{...(ii)}$$

$$\Rightarrow \quad 9 = A + B + C$$

$$\Rightarrow \quad 9 = A + B + 4 \quad [\because C = 4]$$

$$\therefore \quad A + B = 5 \quad \text{...(iii)}$$

and $\quad 5 = 16A + 4B + C \quad [\because C = 4]$

$$\Rightarrow \quad 5 = 16A + 4B + 4$$

$$\therefore \quad 16A + 4B = 1$$

$$\Rightarrow \quad 4A + B = \frac{1}{4} \quad \text{...(iv)}$$

On solving Eqs. (iii) and (iv), we get

$$A = -\frac{19}{12}, B = \frac{79}{12} \quad \text{...(v)}$$

On substituting the values of A, B and C from Eqs. (ii) and Eq. (v) in Eq. (i), then equation of parabola is

$$y = -\frac{19}{12}x^2 + \frac{79}{12}x + 4$$

Hence, length of latusrectum $= \dfrac{1}{\left|-\dfrac{19}{12}\right|} = \dfrac{12}{19}.$ $= \dfrac{1}{|A|}$

Exercise for Session 1

1. The vertex of the parabola $y^2 + 6x - 2y + 13 = 0$ is

(a) $(-2, 1)$ (b) $(2, -1)$
(c) $(1, 1)$ (d) $(1, -1)$

2. If the parabola $y^2 = 4ax$ passes through (3, 2), then the length of latusrectum is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) 1 (d) $\frac{4}{3}$

3. The value of p such that the vertex of $y = x^2 + 2px + 13$ is 4 units above the X-axis is

(a) ± 2 (b) 4
(c) ± 3 (d) 5

4. The length of the latusrectum of the parabola whose focus is (3,3) and directrix is $3x - 4y - 2 = 0$, is

(a) 1 (b) 2
(c) 4 (d) 8

5. If the vertex and focus of a parabola are (3,3) and (–3, 3) respectively , then its equation is

(a) $x^2 - 6x + 24y - 63 = 0$ (b) $x^2 - 6x + 24y + 81 = 0$
(c) $y^2 - 6y + 24x - 63 = 0$ (d) $y^2 - 6y - 24x + 81 = 0$

6. If the vertex of the parabola $y = x^2 - 8x + c$ lies on X-axis, then the value of c is

(a) 4 (b) -4
(c) 16 (d) -16

7. The parabola having its focus at (3,2) and directrix along the Y-axis has its vertex at

(a) $\left(\frac{3}{2}, 1\right)$ (b) $\left(\frac{3}{2}, 2\right)$
(c) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{3}{2}, -\frac{1}{2}\right)$

8. The directrix of the parabola $x^2 - 4x - 8y + 12 = 0$ is

(a) $y = 0$ (b) $x = 1$

(c) $y = -1$ (d) $x = -1$

9. The equation of the latusrectum of the parabola $x^2 + 4x + 2y = 0$ is

(a) $3y - 2 = 0$ (b) $3y + 2 = 0$

(c) $2y - 3 = 0$ (d) $2y + 3 = 0$

10. The focus of the parabola $x^2 - 8x + 2y + 7 = 0$ is

(a) $\left(0, -\frac{1}{2}\right)$ (b) $(4,4)$

(c) $\left(4, \frac{9}{2}\right)$ (d) $\left(-4, -\frac{9}{2}\right)$

11. The equation of the parabola with the focus $(3,0)$ and directrix $x + 3 = 0$ is

(a) $y^2 = 2x$ (b) $y^2 = 3x$

(c) $y^2 = 6x$ (d) $y^2 = 12x$

12. Equation of the parabola whose axis is parallel to Y-axis and which passes through the points $(1,0)$, $(0,0)$ and $(-2,4)$, is

(a) $2x^2 + 2x = 3y$ (b) $2x^2 - 2x = 3y$

(c) $2x^2 + 2x = y$ (d) $2x^2 - 2x = y$

13. Find the equation of the parabola whose focus is $(5,3)$ and directrix is the line $3x - 4y + 1 = 0$.

14. Find the equation of the parabola is focus is at $(-6, -6)$ and vertex is at $(-2, 2)$.

15. Find the vertex, focus, axis, directrix and latusrectum of the parabola $4y^2 + 12x - 20y + 67 = 0$.

16. Find the name of the conic represented by $\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(\frac{y}{b}\right)} = 1$.

17. Determine the name of the curve described parametrically by the equations
$x = t^2 + t + 1, y = t^2 - t + 1$.

18. Prove that the equation of the parabola whose vertex and focus are on the X-axis at a distance a and a' from the origin respectively is $y^2 = 4(a' - a)(x - a)$.

19. Find the equation of the parabola whose axis is parallel to X-axis and which passes through the points $(0, 4)$, $(1, 9)$ and $(-2, 6)$. Also, find its latusrectum.

20. The equation $ax^2 + 4xy + y^2 + ax + 3y + 2 = 0$ represents a parabola, then find the value of a.

Session 2

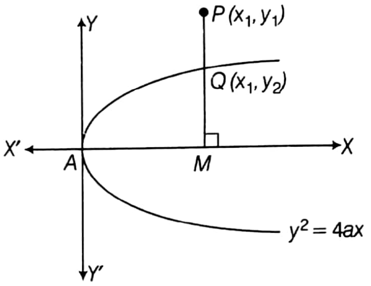
Position of a Point (x_1, y_1) with respect to a Parabola $y^2 = 4ax$, Parametric Relation between the Coordinates of the Ends of a Focal Chord of a Parabola, Intersection of a Line and a Parabola, Equation of Tangent in Different Forms, Point of Intersection of Tangents at any Two Points on the Parabola, Equation of Normals in Different Forms, Point of Intersection of Normals at any Two Points on the Parabola, Circle Through Co-normal Points

Position of a Point (x_1, y_1) with Respect to a Parabola $y^2 = 4ax$

Theorem The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as

$$y_1^2 - 4ax_1 >, =, \text{or} < 0.$$

Proof Let $P(x_1, y_1)$ be a point. From P draw $PM \perp AX$ (on the axis of parabola) meeting the parabola $y^2 = 4ax$ at Q let the coordinate of Q be (x_1, y_2).



Since, $Q(x_1, y_2)$ lies on the parabola

$$y_1^2 = 4ax$$

then, $$y_2^2 = 4ax_1 \quad \ldots\text{(i)}$$

Now, P will be outside, on or inside the parabola $y^2 = 4ax$ according as

$$PM >, =, \text{or} < QM$$

$\Rightarrow$ $$(PM)^2 >, =, \text{or} < (QM)^2$$

$\Rightarrow$ $$y_1^2 >, =, \text{or} < y_2^2$$

$\Rightarrow$ $$y_1^2 >, =, \text{or} < 4ax_1 \quad \text{[from Eq. (i)]}$$

Hence, $y_1^2 - 4ax_1 >, =, \text{or} < 0$

Remarks

1. The point (x_1, y_1) lies inside, on or outside $y^2 = -4ax$ according as $y_1^2 + 4ax_1 <, =, \text{or} > 0$
2. The point (x_1, y_1) lies inside, on or outside $x^2 = 4ay$ according as $x_1^2 - 4ay_1 <, =, \text{or} > 0$
3. The point (x_1, y_1) lies inside, on or outside $x^2 = -4ay$ according as $x_1^2 + 4ay_1 <, =, \text{or} > 0$

Example 23 Show that the point (2, 3) lies outside the parabola $y^2 = 3x$.

Sol. Let the point $(h, k) = (2, 3)$

We have, $k^2 - 3h = 3^2 - 3.2 = 9 - 6 = 3 > 0$

$\therefore$ $$k^2 - 3h > 0$$

This shows that (2,3) lies outside the parabola $y^2 = 3x$.

Example 24 Find the position of the point $(-2,2)$ with respect to the parabola $y^2 - 4y + 9x + 13 = 0$.

Sol. Let the point $(h,k) = (-2,2)$

We have, $k^2 - 4k + 9h + 13 = (2)^2 - 4(2) + 9(-2) + 13 = 4 - 8 - 18 + 13 = -9 < 0$

Hence, $$k^2 - 4k + 9h + 13 < 0$$

Therefore, the point $(-2,2)$ lies inside the parabola

$$y^2 - 4y + 9x + 13 = 0.$$

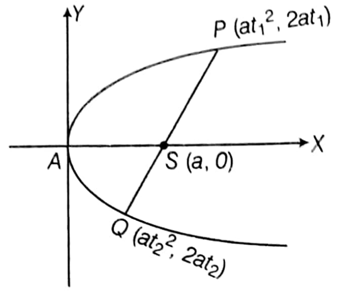
Parameteric Relation between the Coordinates of the Ends of a Focal Chord of a Parabola

Let $y^2 = 4ax$ be a parabola, if PQ be a focal chord.

Then, $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$

Since, PQ passes through the focus $S(a,0)$.

$\therefore$ Q, S, P are collinear.



$\therefore$ Slope of PS = Slope of QS

$$\Rightarrow \quad \frac{2at_1 - 0}{at_1^2 - a} = \frac{0 - 2at_2}{a - at_2^2} \Rightarrow \frac{2t_1}{t_1^2 - 1} = \frac{2t_2}{t_2^2 - 1}$$

$$\Rightarrow \quad t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$$

$$\Rightarrow \quad t_1 t_2(t_2 - t_1) + (t_2 - t_1) = 0$$

$$\Rightarrow \quad t_2 - t_1 \neq 0 \quad \text{or} \quad t_1 t_2 + 1 = 0$$

$$\Rightarrow \quad \boxed{t_1 t_2 = -1} \quad \text{or} \quad t_2 = -\frac{1}{t_1}, \qquad \ldots\text{(i)}$$

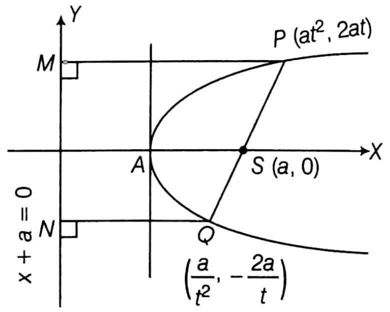
which is required relation.

Remark

If one extremity of a focal chord is $(at_1^2, 2at_1)$ then the other extremity $(at_2^2, 2at_2)$ becomes $\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$ by virtue of relation Eq. (i).

| Example 25 If the point $(at^2, 2at)$ be the extremity of a focal chord of parabola $y^2 = 4ax$ then show that the length of the focal chord is $a\left(t + \frac{1}{t}\right)^2$.

Sol. Since, one extremity of focal chord is $P(at^2, 2at)$, then the other extremity is $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ [Replacing t by $-1/t$]



$\therefore$ Length of focal chord $= PQ$

$$= \boxed{SP + SQ} \qquad [\because SP = PM \text{ and } SQ = QN]$$

$$= PM + QN$$

$$= at^2 + a + \frac{a}{t^2} + a$$

$$= a\left(t^2 + \frac{1}{t^2} + 2\right) = a\left(t + \frac{1}{t}\right)^2$$

Remark

$\because$ $\left|t + \frac{1}{t}\right| \geq 2$ for all $t \neq 0$ $\qquad [\because \text{AM} \geq \text{GM}]$

$\therefore$ $a\left(t + \frac{1}{t}\right)^2 \geq 4a$

$\Rightarrow$ Length of focal chord $\geq$ latusrectum i.e. The length of smallest focal chord of the parabola is $4a$. Hence, the latusrectum of a parabola is the smallest focal chord.

| Example 26 Prove that the semi-latusrectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.

Sol. Let parabola be $y^2 = 4ax$

If PQ be the focal chord, if

$$P \equiv (at^2, 2at), \text{ then } Q \equiv \left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$

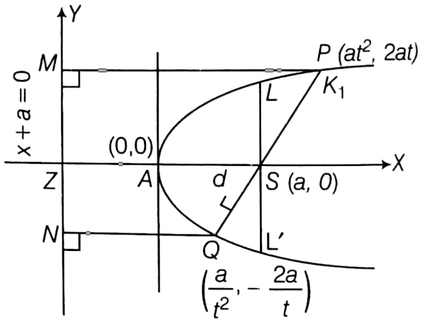
$\therefore$ Length of latusrectum $LL' = 4a$.

$\therefore$ Semi-latusrectum $= \frac{1}{2}(4a) = 2a$.

If sections of focal chord are k_1 and k_2,

then, $\quad k_1 = SP = PM = a + at^2 = a(1 + t^2)$

and $\quad k_2 = SQ = QN = a + \frac{a}{t^2} = \frac{a(1 + t^2)}{t^2}$



$\therefore$ Harmonic mean of k_1

and $\quad k_2 = \frac{2k_1 k_2}{k_1 + k_2}$

$$= \frac{2}{\frac{1}{k_2} + \frac{1}{k_1}} = \frac{2}{\frac{t^2}{a(1+t^2)} + \frac{1}{a(1+t^2)}}$$

$$= \frac{2}{\frac{1}{a}} = 2a = \text{Semi-latusrectum.}$$

Remarks

1. The length of focal chord having parameters t_1 and t_2 for its end points is $\boxed{a(t_2 - t_1)^2}$.
2. If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latusrectum is $\boxed{\frac{4l_1 l_2}{l_1 + l_2}}$.

| Example 27. Show that the focal chord of parabola $y^2 = 4ax$ makes an angle α with the X-axis is of length $4a\operatorname{cosec}^2\alpha$.

Sol. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the end points of a focal chord PQ which makes an angle α with the axis of the parabola. Then,

$$PQ = a(t_2 - t_1)^2$$
$$= a[(t_2 + t_1)^2 - 4t_1t_2] \quad \text{...(i)}$$
$$= a[(t_2 + t_1)^2 + 4] \quad [\because t_1t_2 = -1]$$

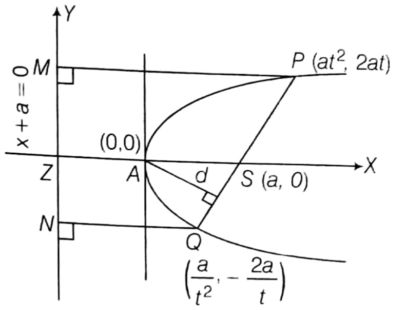
$$\therefore \quad \tan\alpha = \text{slope of } PQ = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}$$
$$\Rightarrow \quad \tan\alpha = \frac{2}{t_2 + t_1}$$
$$\Rightarrow \quad t_2 + t_1 = 2\cot\alpha \quad \text{...(ii)}$$

On substituting the value of $t_2 + t_1$ from Eq. (ii) in Eq. (i), then

$$PQ = a(4\cot^2\alpha + 4)$$
$$= 4a\operatorname{cosec}^2\alpha.$$

| Example 28 Prove that the length of a focal chord of a parabola varies inversly as the square of its distance from the vertex.

Sol. Let $P(at^2, 2at)$ be one end of a focal chord of the parabola $y^2 = 4ax$. The focus of its parabola is $S(a,0)$.



$\therefore$ Equation of focal chord is (i.e. equation of PS)

$$y - 0 = \frac{2at - 0}{at^2 - a}(x - a)$$
$$\Rightarrow \quad y = \frac{2t}{(t^2 - 1)}(x - a)$$
$$\Rightarrow \quad (t^2 - 1)y = 2tx - 2at$$
$$\Rightarrow \quad 2tx - (t^2 - 1)y - 2at = 0$$

If d be the distance of this focal chord from the vertex (0,0) of the parabola $y^2 = 4ax$, then

$$d = \frac{|0 - 0 - 2at|}{\sqrt{(2t)^2 + (t^2 - 1)^2}}$$
$$= \frac{|2at|}{(t^2 + 1)} = \frac{2a}{\left|t + \frac{1}{t}\right|}$$
$$\Rightarrow \quad d^2 = \frac{4a^2}{\left(t + \frac{1}{t}\right)^2} \quad \text{...(i)}$$

The other end of the focal chord is $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

If length of focal chord $= PQ = (l \text{ say})$

$$\therefore \quad l = PQ = PS + SQ = PM + QN$$
$$\therefore \quad l = at^2 + a + \frac{a}{t^2} + a$$
$$\Rightarrow \quad l = a\left(t^2 + \frac{1}{t^2} + 2\right) \Rightarrow l = a\left(t + \frac{1}{t}\right)^2$$
$$\Rightarrow \quad \frac{l}{a} = \left(t + \frac{1}{t}\right)^2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $d^2 = \dfrac{4a^2}{(l/a)} = \dfrac{4a^3}{l}$

$$\therefore \quad l = \frac{4a^3}{d^2} \Rightarrow l \propto \frac{1}{d^2}$$

i.e. the length of the focal chord varies inversely as the square of its distance from vertex.

Intersection of a Line and a Parabola

Let the parabola be $\quad y^2 = 4ax \quad$ ...(i)

and the given line be $\quad y = mx + c \quad$ ...(ii)

On eliminating x from Eqs. (i) and (ii), then

$$y^2 = 4a\left(\frac{y - c}{m}\right)$$
$$\Rightarrow \quad my^2 - 4ay + 4ac = 0 \quad \text{...(iii)}$$

This equation being quadratic in y, gives two values of y, shows that every straight line will cut the parabola in two points may be real, coincident or imaginary according as discriminant of Eq. (iii) $>, =, < 0$

i.e. $(-4a)^2 - 4 \cdot m \cdot 4ac >, =, < 0$ or $\boxed{a - mc >, =, < 0}$

or $\quad a >, =, < mc \quad$ ...(iv)

Condition of tangency

If the line Eq. (ii) touches the parabola Eq. (i), then Eq. (iii) has equal roots

$\therefore$ Discriminant of Eq. (iii) $= 0$

$$\Rightarrow \quad (-4a)^2 - 4m \cdot 4ac = 0$$

$$\Rightarrow \quad \boxed{c = \frac{a}{m}, m \neq 0} \quad \text{...(v)}$$

So, the line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ (which is condition of tangency).

Substituting the value of c from Eq. (v) in Eq. (ii), then

$$y = mx + \frac{a}{m}, m \neq 0$$

Hence, the line $y = mx + \frac{a}{m}$ will always be a tangent to the parabola $y^2 = 4ax$.

The point of contact Substituting $c = \frac{a}{m}$ in Eq. (iii), then

$$my^2 - 4ay + 4a\left(\frac{a}{m}\right) = 0$$

$$\Rightarrow \quad m^2y^2 - 4amy + 4a^2 = 0$$

$$\Rightarrow \quad (my - 2a)^2 = 0$$

$$\Rightarrow \quad my - 2a = 0$$

or

$$y = \frac{2a}{m}$$

Substituting this value of y in $y = mx + \frac{a}{m}$

$$\therefore \quad \frac{2a}{m} = mx + \frac{a}{m}$$

$$\Rightarrow \quad mx = \frac{a}{m} \text{ or } x = \frac{a}{m^2}$$

Hence, the point of contact is $\boxed{\left(\frac{a}{m^2}, \frac{2a}{m}\right)}$ $(m \neq 0)$ this known as ***m*-point** on the parabola.

Remark

If $m = 0$, then Eq. (iii) gives

$$0 - 4ay + 4ac = 0$$

$$\Rightarrow \quad y = c$$

which gives only one value of y and so every line parallel to X-axis cuts the parabola only in one real point.

| Example 29 Prove that the straight line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, if $ln = am^2$.

Sol. The given line is $lx + my + n = 0$

or $\quad y = -\frac{l}{m}x - \frac{n}{m} \quad$ ...(i)

Comparing this line with $y = Mx + c$

$$\therefore \quad M = -\frac{l}{m} \text{ and } c = -\frac{n}{m}$$

The line Eq. (i) will touch the parabola $y^2 = 4ax$, if

$$c = \frac{a}{M} \Rightarrow cM = a$$

$$\Rightarrow \quad \left(-\frac{n}{m}\right)\left(-\frac{l}{m}\right) = a$$

$$\therefore \quad ln = am^2$$

Aliter :

Given line $\quad lx + my + n = 0 \quad$ ...(i)

and the parabola $\quad y^2 = 4ax \quad$ ...(ii)

Substituting the value of x from Eq. (i) i.e. $x = -\frac{n + my}{l}$ in Eq. (ii), then

(we should not substituting the value of y from Eq. (i), in Eq. (ii) since y is quadratic, substituting the value of x since x is linear).

$$y^2 = 4a\left(-\frac{n + my}{l}\right)$$

$$\Rightarrow \quad ly^2 + 4amy + 4an = 0 \quad \text{...(iii)}$$

Since, Eq. (i) touches the parabola Eq. (ii), then roots of Eq. (iii) must be coincident and condition for the same is $B^2 = 4AC$,

i.e., $$(4am)^2 = 4 \cdot l4an$$

$$\Rightarrow \quad am^2 = ln$$

$$\therefore \quad ln = am^2$$

| Example 30 Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the parabola $y^2 = 4ax$, if $p\cos\alpha + a\sin^2\alpha = 0$ and that the point of contact is $(a\tan^2\alpha, -2a\tan\alpha)$.

Sol. The given line is

$$x\cos\alpha + y\sin\alpha = p$$

$$\Rightarrow \quad y = -x\cot\alpha + p\operatorname{cosec}\alpha$$

Comparing this line with $y = mx + c$.

$$\therefore \quad m = -\cot\alpha \text{ and } c = p\operatorname{cosec}\alpha$$

since, the given line touches the parabola

$$\therefore \quad c = \frac{a}{m} \text{ or } cm = a$$

$$\Rightarrow \quad (p\operatorname{cosec}\alpha)(-\cot\alpha) = a$$

$$\Rightarrow \quad a\sin^2\alpha + p\cos\alpha = 0$$

and point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

i.e. $$\left(\frac{a}{\cot^2\alpha}, -\frac{2a}{\cot\alpha}\right)$$

or $$(a\tan^2\alpha, -2a\tan\alpha)$$

Example 31 Prove that the line $\frac{x}{l}+\frac{y}{m}=1$ touches the parabola $y^2=4a(x+b)$, if $m^2(l+b)+al^2=0$.

Sol. The given parabola is

$$y^2 = 4a(x+b) \quad \text{...(i)}$$

Vertex of this parabola is $(-b, 0)$.

Now, shifting $(0, 0)$ at $(-b, 0)$,

then, $x = X + (-b)$ and $y = Y + 0$

or $\quad x + b = X \quad$ and $\quad y = Y \quad$ ...(ii)

From Eq. (i), $Y^2 = 4aX \quad$ ...(iii)

and the line $\frac{x}{l}+\frac{y}{m}=1$

reduces to $\frac{X-b}{l}+\frac{Y}{m}=1$

$$Y = m\left(1-\frac{X-b}{l}\right)$$

$$\Rightarrow \quad Y = \left(-\frac{m}{l}\right)X + m\left(1+\frac{b}{l}\right) \quad \text{...(iv)}$$

The line Eq. (iv) will touch the parabola Eq. (iii), if

$$m\left(1+\frac{b}{l}\right) = \frac{a}{\left(-\frac{m}{l}\right)}$$

$$\Rightarrow \quad \frac{m^2}{l}\left(1+\frac{b}{l}\right) = -a$$

$$\therefore \quad m^2(l+b)+al^2 = 0$$

Aliter :

The given line and parabola are

$$\frac{x}{l}+\frac{y}{m}=1 \quad \text{...(i)}$$

and

$$y^2 = 4a(x+b) \quad \text{...(ii)}$$

respectively substituting the value of x from Eq. (i)

$$x = l\left(1-\frac{y}{m}\right)$$

in Eq. (ii), then $\quad y^2 = 4a\left\{l\left(1-\frac{y}{m}\right)+b\right\}$

or $\quad y^2 + \frac{4al}{m}y - 4a(l+b) = 0 \quad$ ...(iii)

Since, the line Eq. (i) touches the parabola Eq. (ii), then the roots of Eq. (iii) are equal.

$$\therefore \left(\frac{4al}{m}\right)^2 - 4\cdot 1\{-4a(l+b)\} = 0$$

$$\Rightarrow \quad \frac{al^2}{m^2} + (l+b) = 0$$

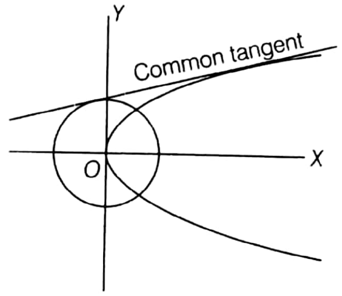
$$\Rightarrow \quad al^2 + m^2(l+b) = 0$$

$$\Rightarrow \quad m^2(l+b) + al^2 = 0$$

Must

Example 32 Find the equations of the straight lines touching both $x^2+y^2=2a^2$ and $y^2=8ax$.

Sol. The given curves are



$$x^2+y^2 = 2a^2 \quad \text{...(i)}$$

and

$$y^2 = 8ax \quad \text{...(ii)}$$

The parabola Eq. (ii) is $y^2 = 8ax$

or $\quad y^2 = 4(2a)x$

$\therefore$ Equation of tangent of Eq. (ii) is

$$y = mx + \frac{2a}{m}$$

First find tangent of parabola

or $\quad m^2x - my + 2a = 0 \quad$ ...(iii)

It is also tangent of Eq. (i), then the length of perpendicular from centre of Eq. (i) i.e. $(0, 0)$ to Eq. (iii) must be equal to the radius of Eq. (i) i.e. $a\sqrt{2}$.

$$\therefore \quad \frac{|0-0+2a|}{\sqrt{(m^2)^2+(-m)^2}} = a\sqrt{2} \Rightarrow \frac{4a^2}{m^4+m^2} = 2a^2$$

$$\Rightarrow \quad m^4+m^2-2=0$$

$$\Rightarrow \quad (m^2+2)(m^2-1)=0$$

$\because \quad m^2+2 \neq 0 \quad$ [gives the imaginary values]

$$\therefore \quad m^2-1=0$$

$$\Rightarrow \quad m = \pm 1$$

Hence, from Eq. (iii) the required tangents are

$$x \pm y + 2a = 0.$$

Equation of Tangent in Different Forms

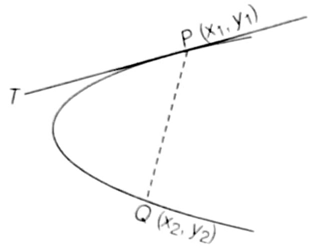
1. Point Form :

To find the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1).

(**First Principal Method**) Equation of parabola is

$$y^2 = 4ax \quad \text{...(i)}$$

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be any two points on parabola (i), then



$$y_1^2 = 4ax_1 \qquad \text{...(ii)}$$

and $$y_2^2 = 4ax_2 \qquad \text{...(iii)}$$

Subtracting Eq. (ii) from Eq. (iii), then

$$y_2^2 - y_1^2 = 4a(x_2 - x_1)$$

$$\Rightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1} \qquad \text{...(iv)}$$

Equation of PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{...(v)}$$

From Eqs. (iv) and (v), then

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1) \qquad \text{...(vi)}$$

Now, for tangent at $P, Q \to P$, i.e. $x_2 \to x_1$ and $y_2 \to y_1$, then Eq. (vi) becomes

$$y - y_1 = \frac{4a}{2y_1}(x - x_1) \Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$

$$\Rightarrow \quad yy_1 = 2ax + y_1^2 - 2ax_1$$

$$\Rightarrow \quad yy_1 = 2ax + 4ax_1 - 2ax_1 \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad yy_1 = 2ax + 2ax_1$$

$$\therefore \quad yy_1 = 2a(x + x_1),$$

which is the required equation of tangent at (x_1, y_1).

Remarks

1. The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + x_1y}{2}$ and without changing the constant (if any) in the equation of curve. This method is apply only when the equations of parabola is polynomial of second degree in x and y.
2. Equation of tangents of all standard parabolas at (x_1, y_1).

Equations of Parabolas	Tangent at(x_1, y_1)
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

2. Parametric Form :

To find the equation of tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ or 't'.

Since, the equation of tangent of the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$ ...(i)

replacing x_1 by at^2 and y_1 by $2at$, then Eq. (i) becomes

$$y(2at) = 2a(x + at^2) \Rightarrow ty = x + at^2$$

Remark

The equations of tangent of all standard parabolas at 't'.

Equations of Parabolas	Parametric coordinates 't'	Tangent at 't'
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

3. Slope Form :

To find the equation of tangent and point of contact in terms of m(slope) to the parabola $y^2 = 4ax$.

The equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$. ...(i)

Since, m is the slope of the tangent, then

$$m = \frac{2a}{y_1} \Rightarrow y_1 = \frac{2a}{m}$$

Since, (x_1, y_1) lies on $y^2 = 4ax$, therefore

$$y_1^2 = 4ax_1 \Rightarrow \frac{4a^2}{m^2} = 4ax_1$$

$$\therefore \quad x_1 = \frac{a}{m^2}$$

Substituting the values of x_1 and y_1 in Eq. (i), we get

$$y = mx + \frac{a}{m} \qquad \text{...(ii)}$$

Thus $\boxed{y = mx + \frac{a}{m}}$ is a tangent to the parabola $y^2 = 4ax$, where, m is the slope of the tangent.

The coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Comparing Eq. (ii) with $y = mx + c$,

$$c = \frac{a}{m}$$

which is condition of tangency.

when, $y = mx + c$ is the tangent of $y^2 = 4ax$.

Remark

The equation of tangent, condition of tangency and point of contact in terms of slope (m) of all standard parabolas.

Equation of parabolas	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of tangency
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$	$\boxed{c = -\frac{a}{m}}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$
$x^2 = -4ay$	$(2am, -am^2)$	$y = mx + am^2$	$c = am^2$
$(y-k)^2 = 4a(x-h)$	$\left(h + \frac{a}{m^2}, k + \frac{2a}{m}\right)$	$y = mx - mh + k + \frac{a}{m}$	$c + mh = k + \frac{a}{m}$
$(y-k)^2 = -4a(x-h)$	$\left(h - \frac{a}{m^2}, k + \frac{2a}{m}\right)$	$y = mx - mh + k - \frac{a}{m}$	$c + mh = k - \frac{a}{m}$
$(x-h)^2 = 4a(y-k)$	$(h + 2am, k + am^2)$	$y = mx - mh + k - am^2$	$c + mh = k - am^2$
$(x-h)^2 = -4a(y-k)$	$(h + 2am, k - am^2)$	$y = mx - mh + k + am^2$	$c + mh = k + am^2$

Point of Intersection of Tangents at any two Points on the Parabola

Let the parabola be $y^2 = 4ax$

let two points on the parabola are

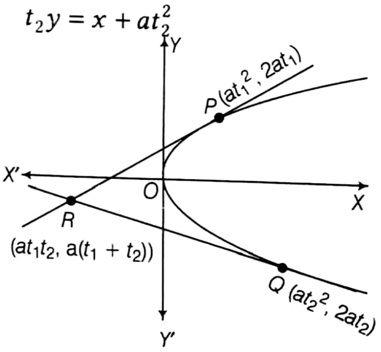
$$P \equiv (at_1^2, 2at_1) \quad \text{and} \quad Q \equiv (at_2^2, 2at_2).$$

Equation of tangents at $P\,(at_1^2, 2at_1)$

and $\quad Q\,(at_2^2, 2at_2)$

are $\quad t_1 y = x + at_1^2 \quad$ ...(i)

and $\quad t_2 y = x + at_2^2 \quad$ ...(ii)



On solving these equations, we get $x = at_1t_2, y = a(t_1 + t_2)$

Thus, the coordinates of the point of intersection of tangents at

$(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are $\boxed{(at_1t_2, a(t_1 + t_2)).}$

Remarks

1. The geometric mean of the x-coordinates of P and Q (i.e. $\sqrt{at_1^2 \times at_2^2} = at_1t_2$) is the x-coordinate of the point of intersection of tangents at P and Q on the parabola. If P and Q are the ends points of focal chord, then x-coordinate of point of intersection of tangents at P and Q is $\boxed{(-at_1t_2).}$
2. The arithmetic mean of the y-coordinates of P and Q $\left(\text{i.e. } \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.

Remembering Method :

G O A [GOA rule]

GM of at_1^2 and at_2^2 , AM of $2at_1$ and $2at_2$

i.e. at_1t_2 i.e. $\frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)$

Example 33 Find the equation of the common tangents to the parabola $y^2 = 4ax$ and $x^2 = 4by$.

Sol. The equation of any tangent in terms of slope (m) to the parabola $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \quad \text{...(i)}$$

If this line is also tangent to the parabola $x^2 = 4ay$, then Eq. (i) meets $x^2 = 4by$ in two coincident points.

Substituting the value of y from Eq. (i) in $x^2 = 4by$, we get

$$x^2 = 4b\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow \quad x^2 - 4bmx - \frac{4ab}{m} = 0$$

The roots of this quadratic are equal provided

$$B^2 = 4AC$$

i.e., $$(-4bm)^2 = 4 \cdot 1\left(\frac{-4ab}{m}\right)$$

$$\Rightarrow \quad 16b^2m^3 + 16ab = 0,\ m \neq 0$$

or $$m^3 = -a/b$$

$$\therefore \quad m = -a^{1/3}/b^{1/3}$$

Substituting the value of m in Eq. (i) the required equation is

$$y = -\frac{a^{1/3}}{b^{1/3}}x - \frac{ab^{1/3}}{a^{1/3}}$$

$$\Rightarrow \quad y = -\frac{a^{1/3}}{b^{1/3}}x - a^{2/3}b^{1/3}$$

$$\therefore \quad a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0$$

| Example 34 The tangents to the parabola $y^2 = 4ax$ make angle θ_1 and θ_2 with X-axis. Find the locus of their point of intersection, if $\cot\theta_1 + \cot\theta_2 = c$.

Sol. Let the equation of any tangent to the parabola $y^2 = 4ax$ is

$$y = mx + (a/m) \qquad \text{...(i)}$$

Let (x_1, y_1) be the point of intersection of the tangents to $y^2 = 4ax$, then Eq. (i) passes through (x_1, y_1).

$$\therefore \quad y_1 = mx_1 + (a/m)$$

or $$m^2x_1 - my_1 + a = 0$$

Let m_1 and m_2 be the roots of this quadratic equation, then

$$m_1 + m_2 = y_1/x_1 \quad \text{and} \quad m_1m_2 = a/x_1$$

or $$\tan\theta_1 + \tan\theta_2 = y_1/x_1$$

and $$\tan\theta_1\tan\theta_2 = a/x_1 \qquad \text{...(ii)}$$

Now, $$\cot\theta_1 + \cot\theta_2 = c \qquad \text{(given)}$$

$$\Rightarrow \quad \frac{1}{\tan\theta_1} + \frac{1}{\tan\theta_2} = c$$

$$\Rightarrow \quad \frac{\tan\theta_1 + \tan\theta_2}{\tan\theta_1\tan\theta_2} = c \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad \frac{y_1/x_1}{a/x_1} = c$$

$$\Rightarrow \quad \boxed{y_1 = ac}$$

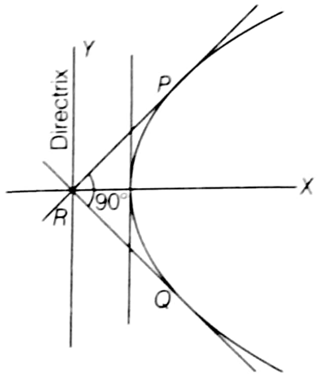
The required locus is $y = ac$, which is a line parallel to X-axis.

| Example 35. Show that the locus of the points of intersection of the mutually perpendicular tangents to a parabola is the directrix of the parabola.

Sol. Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ tangents at P and Q are

$$t_1y = x + at_1^2 \qquad \text{...(i)}$$

and $$t_2y = x + at_2^2 \qquad \text{...(ii)}$$



$\because$ Point of intersection of these tangents is $(at_1t_2, a(t_1 + t_2))$

Let this point is (h, k),

then, $$h = at_1t_2 \qquad \text{...(iii)}$$

and $$k = a(t_1 + t_2) \qquad \text{...(iv)}$$

Slope of tangents Eqs. (i) and (ii) are $\frac{1}{t_1}$ and $\frac{1}{t_2}$, respectively.

Since, tangents are perpendicular, then

$$\frac{1}{t_1} \times \frac{1}{t_2} = -1$$

or $$t_1t_2 = -1 \qquad \text{...(v)}$$

From Eqs. (iii) and (v), we get

$$h = -a \quad \text{or} \quad h + a = 0$$

$\therefore$ Locus of the point of intersection of tangents is

$$x + a = 0$$

which is directrix of $y^2 = 4ax$.

Aliter :

Let the equation of any tangent to the parabola $y^2 = 4ax$ is

$$y = mx + a/m \qquad \text{...(i)}$$

Let the point of intersection of the tangents to $y^2 = 4ax$ then, Eq (i) passes through (x_1, y_1).

$$\therefore \quad y_1 = mx_1 + a/m$$

or $$m^2x_1 - my_1 + a = 0$$

Let m_1, m_2 be the roots of this quadratic equation then

$$m_1m_2 = a/x_1 = -1$$

[since, tangents are perpendiculars]

$$\Rightarrow \quad a + x_1 = 0$$

$\therefore$ Locus of the point of intersection of tangents is $x + a = 0$ which is directrix of $y^2 = 4ax$.

Remark

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the director circle. Its equation is $x + a = 0$, which is parabola's own directrix.

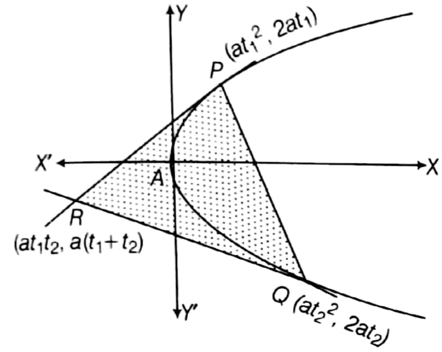
Example 36 The tangents to the parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at R. Prove that the area of the ΔPQR is $\frac{1}{2}a^2|(t_1 - t_2)|^3$.

Sol. Equations of tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$t_1 y = x + at_1^2 \quad \text{...(i)}$$

and

$$t_2 y = x + at_2^2 \quad \text{...(ii)}$$



Since, point of intersect of Eqs. (i) and (ii) is $R(at_1t_2, a(t_1 + t_2))$.

$$\therefore \text{ Area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_1t_2 & a(t_1+t_2) & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ a(t_2^2 - t_1^2) & 2a(t_2 - t_1) & 0 \\ at_1(t_2 - t_1) & a(t_2 - t_1) & 0 \end{vmatrix}$$

Expanding with respect to first row

$$= \frac{1}{2}\begin{vmatrix} a(t_2^2 - t_1^2) & 2a(t_2 - t_1) \\ at_1(t_2 - t_1) & a(t_2 - t_1) \end{vmatrix}$$

$$= \frac{1}{2}a^2(t_2 - t_1)^2\begin{vmatrix} t_2 + t_1 & 2 \\ t_1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}a^2(t_1 - t_2)^2|(t_2 - t_1)|$$

$$= \frac{1}{2}a^2(t_1 - t_2)^2|(t_1 - t_2)|$$

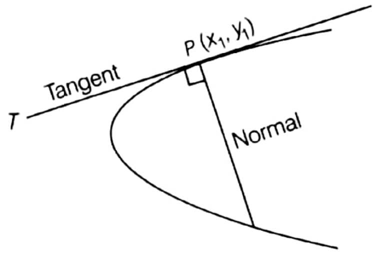
$$= \frac{1}{2}a^2|(t_1 - t_2)^3|.$$

Equations of Normals in Different Forms

1. Point Form: To find the equation of the normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1).

Since, the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1)$$



The slope of the tangent at $(x_1, y_1) = 2a/y_1$

Since, the normal at (x_1, y_1) is perpendicular to the tangent at (x_1, y_1).

$\therefore$ Slope of normal at $(x_1, y_1) = -y_1/2a$

Hence, the equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

Remarks

1. The equation of normal at (x_1, y_1) can also be obtained by this method

$$\frac{x - x_1}{a'x_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f} \quad \text{...(i)}$$

a', b, g, f, h are obtained by comparing the given parabola with

$$a'x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{...(ii)}$$

and denominators of Eq. (i) can easily remembered by the first two rows of this determinant

i.e. $\begin{vmatrix} a' & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Since, first row $a'(x_1) + h(y_1) + g(1)$

and second row, $h(x_1) + b(y_1) + f(1)$

Here, parabola $y^2 = 4ax$

or $y^2 - 4ax = 0 \quad \text{...(iii)}$

Comparing Eqs. (ii) and (iii), then we get

$a' = 0, b = 1, g = -2a, h = 0, f = 0$

From Eq. (i), equation of normal of Eq. (iii) is

$$\frac{x - x_1}{0 + 0 - 2a} = \frac{y - y_1}{0 + y_1 + 0}$$

or

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

2. Equations of normals of all standard parabolas at (x_1, y_1).

Equations of Parabola	Normal at (x_1, y_1)
$y^2 = 4ax$	$y - y_1 = -\frac{y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 = -\frac{2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$

2. Parametric form :

To find the equation of normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ or 't'.

Since, the equation of normal of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \text{...(i)}$$

Replacing x_1 by at^2 and y_1 by $2at$, then Eq. (i) becomes

$$y - 2at = -t(x - at^2)$$

or $\quad \boldsymbol{y + tx = 2at + at^3}$

Remark

The equations of normals of all standard parabolas at 't'

Equations of Parabolas	Parametric coordinates 't'	Normals at 't'
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

3. Slope form :

To find the Equation of normal, condition for normality and point of contact in terms of m (slope) to the parabola $y^2 = 4ax$

The equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \text{...(i)}$$

Since, m is the slope of the normal,

then, $\quad m = -\frac{y_1}{2a} \Rightarrow y_1 = -2am$

Since, (x_1, y_1) lies on $y^2 = 4ax$, therefore

$$y_1^2 = 4ax_1$$

$$\Rightarrow \quad 4a^2m^2 = 4ax_1$$

$$\therefore \quad x_1 = am^2$$

On substituting the values of x_1 and y_1 in Eq. (i) we get

$$y + 2am = m(x - am^2)$$

$$\therefore \quad y = mx - 2am - am^3 \quad \text{...(ii)}$$

Thus, $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$, where m is the slope of the normal. The coordinates of the point of contact are $(am^2, -2am)$

On comparing Eq. (ii) with

$$y = mx + c$$

$$\therefore \quad \boxed{c = -2am - am^3}$$

which is condition of normality when $y = mx + c$ is the normal of $y^2 = 4ax$.

Remark

The equations of normals, point of contact and condition of normality in terms of slope (m) of all standard parabolas.

Equation of parabolas	Point of contact in terms of slope (m)	Equation of normals in terms of slope (m)	Condition of normality
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$	$c = 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$
$(y-k)^2 = 4a(x-h)$	$(h + am^2, k - 2am)$	$y - k = m(x - h) - 2am - am^3$	$c = k - mh - 2am - am^3$
$(y-k)^2 = -4a(x-h)$	$(h - am^2, k + 2am)$	$y - k = m(x - h) + 2am + am^3$	$c = k - mh + 2am + am^3$
$(x-h)^2 = 4a(y-k)$	$\left(h - \frac{2a}{m}, k + \frac{a}{m^2}\right)$	$y - k = m(x - h) + 2a + \frac{a}{m^2}$	$c = k - mh + 2a + \frac{a}{m^2}$

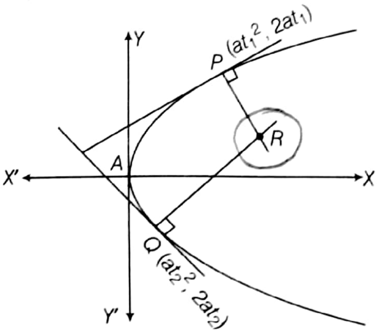
Notes- For tangent → $m \to \frac{1}{t}$.
For normal → $m \to t$

Point of Intersection of Normals at any Two Points on the Parabola

Let the parabola be $y^2 = 4ax$.

Let the points on the parabola are

$P \equiv (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$.



Equations of normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$y = -t_1x + 2at_1 + at_1^3 \quad \text{...(i)}$$

and

$$y = -t_2x + 2at_2 + at_2^3 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$x = 2a + a(t_1^2 + t_2^2 + t_1t_2) \text{ and } y = -at_1t_2(t_1 + t_2)$$

If R is the point of intersection, then

$$R \equiv [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$$

(Remember)

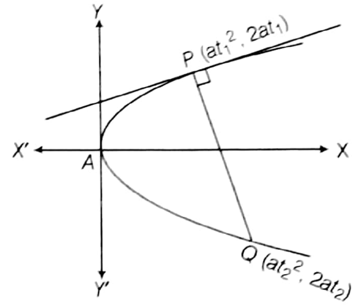
Point of intersection of normals at t_1 and t_2

Parabola	Equation of normal at any point 't'	Point of intersection of normals at t_1 and t_2
$y^2 = 4ax$	$y + tx = 2at + at^3$	$(2a + a(t_1^2 + t_1t_2 + t_2^2), -at_1t_2(t_1 + t_2))$
$y^2 = -4ax$	$y - tx = 2at + at^3$	$(-2a - a(t_1^2 + t_1t_2 + t_2^2), at_1t_2(t_1 + t_2))$
$x^2 = 4ay$	$x + ty = 2at + at^3$	$(-at_1t_2(t_1 + t_2), 2a + a(t_1^2 + t_1t_2 + t_2^2))$
$x^2 = -4ay$	$x - ty = 2at + at^3$	$(at_1t_2(t_1 + t_2), -2a - a(t_1^2 + t_1t_2 + t_2^2))$

Relation between 't_1' and 't_2' if Normal at 't_1' meets the Parabola Again at 't_2'

Let the parabola be $y^2 = 4ax$, equation of normal at $P(at_1^2, 2at_1)$ is

$$y = -t_1x + 2at_1 + at_1^3 \quad \text{...(i)}$$



Since, it meet the parabola again at $Q(at_2^2, 2at_2)$, then Eq. (i) passes through $Q(at_2^2, 2at_2)$.

$$\therefore \quad 2at_2 = -at_1t_2^2 + 2at_1 + at_1^3$$

$$\Rightarrow \quad 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$$

$$\Rightarrow \quad a(t_2 - t_1)[2 + t_1(t_2 + t_1)] = 0$$

$$\because \quad a(t_2 - t_1) \neq 0$$

[$\because t_1$ and t_2 are different]

$$\therefore \quad \boxed{2 + t_1(t_2 + t_1) = 0}$$

$$\therefore \quad t_2 = -t_1 - \frac{2}{t_1}$$

Remarks

1. If normals at 't_1' and 't_2' meets the parabola $y^2 = 4ax$ at same point, then $t_1t_2 = 2$.

Proof Suppose normals meet at 'T', then

$$T = -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$$\Rightarrow \quad (t_1 - t_2) = 2\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$$

or $\boxed{t_1t_2 = 2}$ [$\because t_1 \neq t_2$]

2. If the normals to the parabola $y^2 = 4ax$ at the points t_1 and t_2 intersect again on the parabola at the point t_3, then $t_3 = -(t_1 + t_2)$ and the line joining t_1 and t_2 passes through a fixed point $(-2a, 0)$.

Example 37 Show that normal to the parabola $y^2 = 8x$ at the point (2, 4) meets it again at (18, −12). Find also the length of the normal chord.

Sol. Comparing the given parabola (i.e. $y^2 = 8x$) with $y^2 = 4ax$.

$$\therefore \quad 4a = 8 \Rightarrow a = 2$$

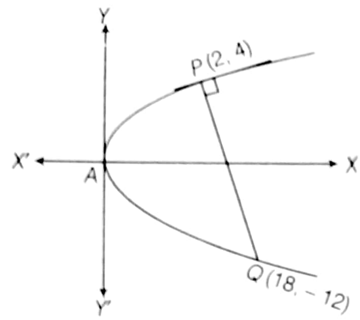
Since, normal at (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

Here, $x_1 = 2$ and $y_1 = 4$.

$\therefore$ Equation of normal is

$$y - 4 = -\frac{4}{4}(x - 2)$$



$$\Rightarrow \quad y - 4 = -x + 2$$

$$\Rightarrow \quad x + y - 6 = 0 \qquad \text{...(i)}$$

On solving Eq. (i) and $y^2 = 8x$,

then, $\quad y^2 = 8(6 - y)$

$$\Rightarrow \quad y^2 + 8y - 48 = 0$$

$$\Rightarrow \quad (y + 12)(y - 4) = 0$$

$$\therefore \quad y = -12 \quad \text{and} \quad y = 4$$

then, $x = 18$ and $x = 2$.

Hence, the point of intersection of normal and parabola are $(18, -12)$ and $(2, 4)$, therefore normal meets the parabola at $(18, -12)$ and length of normal chord is distance between their points

$$= PQ = \sqrt{(18-2)^2 + (-12-4)^2} = 16\sqrt{2}$$

| Example 38 Prove that the chord $y - x\sqrt{2} + 4a\sqrt{2} = 0$ is a normal chord of the parabola $y^2 = 4ax$. Also, find the point on the parabola when the given chord is normal to the parabola.

Sol. We have, $\quad y - x\sqrt{2} + 4a\sqrt{2} = 0$

i.e., $\quad y = x\sqrt{2} - 4a\sqrt{2} \qquad$ ...(i)

Comparing the Eq. (i) with the equation $y = mx + c$, then

$$m = \sqrt{2},\ c = -4a\sqrt{2}$$

Since, $\quad -2am - am^3 = -2a\sqrt{2} - a(\sqrt{2})^3$

$$= -2a\sqrt{2} - 2a\sqrt{2} = -4a\sqrt{2} = c$$

Hence, the given chord is normal to the parabola $y^2 = 4ax$.

The coordinates of the points are $(am^2, -2am)$ i.e. $(2a, -2\sqrt{2}a)$.

| Example 39 If the normal to a parabola $y^2 = 4ax$, makes an angle ϕ with the axis. Show that it will cut the curve again at an angle $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$.

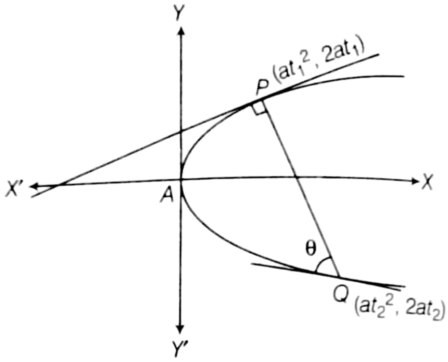
Sol. Let the normal at $P(at_1^2, 2at_1)$ be

$$y = -t_1 x + 2at_1 + at_1^3.$$

$\therefore \quad \tan\phi = -t_1 =$ slope of the normal, $\qquad$ ...(i)

it meet the curve again Q say $(at_2^2, 2at_2)$.

$$\therefore \quad t_2 = -t_1 - \frac{2}{t_1} \qquad \text{...(ii)}$$



Now, angle between the normal and parabola
= Angle between the normal and tangent at Q
(i.e. $t_2 y = x + at_2^2$)

If θ be the angle, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-t_1 - \frac{1}{t_2}}{1 + (-t_1)\left(\frac{1}{t_2}\right)} = -\frac{t_1 t_2 + 1}{t_2 - t_1}$$

$$= -\frac{t_1\left(-t_1 - \frac{2}{t_1}\right) + 1}{-t_1 - \frac{2}{t_1} - t_1} \qquad \text{[from Eq. (ii)]}$$

$$= -\frac{-t_1^2 - 1}{-2\left(\frac{1 + t_1^2}{t_1}\right)} = -\frac{t_1}{2}$$

$$= \frac{\tan\phi}{2} \qquad \text{[from Eq. (i)]}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{2}\tan\phi\right)$$

| Example 40 Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.

Sol. Let the normal at $P(at_1^2, 2at_1)$ meet the curve at $Q(at_2^2, 2at_2)$.

$\therefore \quad PQ$ is a normal chord

and $\qquad t_2 = -t_1 - \frac{2}{t_1}. \qquad$ ...(i)

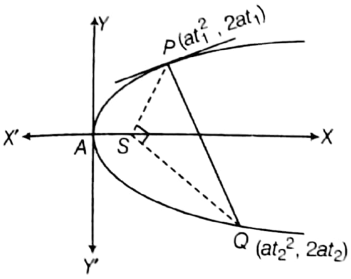
By given condition, $2at_1 = at_1^2$

$\therefore \quad t_1 = 2$ from Eq. (i), $t_2 = -3$

then, $P(4a, 4a)$ and $Q(9a, -6a)$

but focus $S(a, 0)$.

$$\therefore \quad \text{Slope of } SP = \frac{4a - 0}{4a - a} = \frac{4a}{3a} = \frac{4}{3}$$



and slope of $SQ = \frac{-6a-0}{9a-a} = -\frac{6a}{8a} = -\frac{3}{4}$

$\because$ Slope of $SP \times$ Slope of $SQ = \frac{4}{3} \times -\frac{3}{4} = -1$

$\therefore \quad \angle PSQ = \pi/2$

i.e. PQ subtends a right angle at the focus S.

| Example 41 If the normal to the parabola $y^2 = 4ax$ at point t_1 cuts the parabola again at point t_2, prove that $t_2^2 = 8$.

Sol. A normal at point t_1 cuts the parabola again at t_2. Then,

$$t_2 = -t_1 - \frac{2}{t_1} \Rightarrow t_1^2 + t_1 t_2 + 2 = 0$$

Since, t_1 is real, so $(t_2)^2 - 4 \cdot 1 \cdot 2 \geq 0$

$\Rightarrow \quad \boxed{t_2^2 \geq 8}$

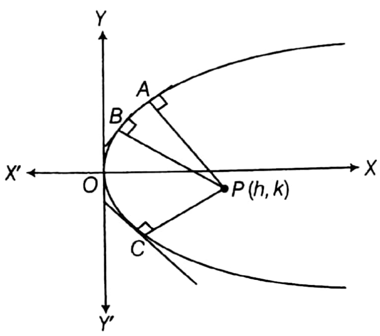
Co-normal Points

In general three normals can be drawn from a point to a parabola and their feet, points where they meet the parabola are called conormal points.

Let $P(h, k)$ be any given point and $y^2 = 4ax$ be a parabola.

The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$



If it passes through (h, k), then

$$k = mh - 2am - am^3$$

$$\Rightarrow am^3 + m(2a - h) + k = 0 \quad \text{...(i)}$$

This is a cubic equation in m, so it has three roots, say m_1, m_2 and m_3.

$$\therefore \quad \left.\begin{aligned} &\boxed{m_1 + m_2 + m_3 = 0,} \\ &m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{(2a-h)}{a}, \\ &m_1 m_2 m_3 = -\frac{k}{a} \end{aligned}\right\} \quad \text{...(ii)}$$

Hence, for any given point $P(h, k)$, Eq. (i) has three real or imaginary roots. Corresponding to each of these three roots, we have one normal passing through $P(h, k)$. Hence, in total, we have three normals PA, PB and PC drawn through P to the parabola.

Points A, B, C in which the three normals from $P(h, k)$ meet the parabola are **called co-normal points.**

Corollary 1 *The algebraic sum of the slopes of three concurrent normals is zero. This follows from Eq.* (ii).

Corollary 2 *The algebraic sum of ordinates of the feets of three normals drawn to a parabola from a given point is zero.*

Let the ordinates of A, B, C be y_1, y_2, y_3 respectively, then

$$y_1 = -2am_1, y_2 = -2am_2 \text{ and } y_3 = -2am_3$$

$\therefore$ Algebraic sum of these ordinates is

$$\begin{aligned} y_1 + y_2 + y_3 &= -2am_1 - 2am_2 - 2am_3 \\ &= -2a(m_1 + m_2 + m_3) \\ &= -2a \times 0 \quad \text{[from Eq. (ii)]} \\ &= 0 \end{aligned}$$

Corollary 3 *If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real then* $\boxed{h > 2a.}$

When normals are real, then all the three roots of Eq. (i) are real and in that case

$$m_1^2 + m_2^2 + m_3^2 > 0 \quad \text{(for any values of } m_1, m_2, m_3)$$

$$\Rightarrow (m_1 + m_2 + m_3)^2 - 2 \times (m_1 m_2 + m_2 m_3 + m_3 m_1) > 0$$

$$\Rightarrow (0)^2 - \frac{2(2a-h)}{a} > 0 \Rightarrow h - 2a > 0$$

$$\therefore \quad \boxed{h > 2a}$$

Remark

For $a = 1$ normals drawn to the parabola $y^2 = 4x$ from any point (h, k) are real, if $h > 2$.

Corollary 4 *If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real and distinct, then* $27ak^2 < 4(h - 2a)^3$

Let $\quad f(m) = am^3 + m(2a - h) + k$

$\therefore \quad f'(m) = 3am^2 + (2a - h)$

Two distinct roots of $f'(m) = 0$ are

$$\alpha = \sqrt{\left(\frac{h-2a}{3a}\right)} \quad \text{and} \quad \beta = -\sqrt{\left(\frac{h-2a}{3a}\right)},$$

Now, $\quad f(\alpha)\, f(\beta) < 0 \Rightarrow f(\alpha)\, f(-\alpha) < 0$

$\Rightarrow \quad (a\alpha^3 + \alpha(2a-h) + k)(-a\alpha^3 - \alpha(2a-h) + k) < 0$

$\Rightarrow \quad k^2 - (a\alpha^2 + (2a-h))^2 \alpha^2 < 0$

$$\Rightarrow \quad k^2 - \left(\frac{h-2a}{3} + (2a-h)\right)^2 \frac{(h-2a)}{3a} < 0$$

$$\Rightarrow \quad k^2 - \left(\frac{4a-2h}{3}\right)^2 \left(\frac{h-2a}{3a}\right) < 0$$

$$\Rightarrow \quad k^2 - \frac{4(h-2a)^3}{27a} < 0 \Rightarrow 27ak^2 - 4(h-2a)^3 < 0$$

$$\therefore \quad \boxed{27ak^2 < 4(h-2a)^3}$$

Corollary 5 *The centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola.*

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC, then its centroid is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, 0\right)$$

Since, $y_1 + y_2 + y_3 = 0$ (from corollary 2). Hence, the centroid lies on the X-axis OX, which is the axis of the parabola also.

$$\text{Now,} \quad \frac{x_1 + x_2 + x_3}{3} = \frac{1}{3}(am_1^2 + am_2^2 + am_3^2)$$

$$= \frac{a}{3}(m_1^2 + m_2^2 + m_3^2)$$

$$= \frac{a}{3}\{(m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)\}$$

$$= \frac{a}{3}\left[(0)^2 - 2\left\{\frac{2a-h}{a}\right\}\right] = \frac{2h-4a}{3}$$

$\therefore$ Centroid of ΔABC is $\boxed{\left(\frac{2h-4a}{3}, 0\right)}$.

Example 42 Show that the locus of points such that two of the three normals drawn from them to the parabola $y^2 = 4ax$ coincide is $27ay^2 = 4(x-2a)^3$.

Sol. Let (h, k) be the point of intersection of three normals to the parabola $y^2 = 4ax$. The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h, k), then

$$k = mh - 2am - am^3$$

$$\Rightarrow \quad am^3 + m(2a-h) + k = 0 \quad \text{...(i)}$$

Let the roots of Eq. (i) be m_1, m_2 and m_3.

Then, from Eq. (i), $\quad m_1 + m_2 + m_3 = 0 \quad$ ...(ii)

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a-h)}{a} \quad \text{...(iii)}$$

$$\text{and} \quad m_1m_2m_3 = -\frac{k}{a} \quad \text{...(iv)}$$

But here, two of the three normals are given to be coincident i.e. $m_1 = m_2$.

Putting $m_1 = m_2$ in Eqs. (ii) and (iv), we get

$$2m_1 + m_3 = 0 \quad \text{...(v)}$$

$$\text{and} \quad m_1^2 m_3 = -\frac{k}{a} \quad \text{...(vi)}$$

Putting $m_3 = -2m_1$ from Eq. (v) in Eq. (vi), we get

$$-2m_1^3 = -\frac{k}{a}$$

$$\Rightarrow \quad m_1^3 = \frac{k}{2a}$$

Since, m_1 is a root of Eq. (i).

$$\therefore \quad am_1^3 + m_1(2a-h) + k = 0$$

$$\Rightarrow \quad a\left(\frac{k}{2a}\right) + \left(\frac{k}{2a}\right)^{1/3}(2a-h) + k = 0$$

$$\left[\text{putting } m_1 = \left(\frac{k}{2a}\right)^{1/3}\right]$$

$$\Rightarrow \quad \left(\frac{k}{2a}\right)^{1/3}(2a-h) = -\frac{3k}{2}$$

$$\Rightarrow \quad \frac{k}{2a}(2a-h)^3 = -\frac{27k^3}{8}$$

$$\Rightarrow \quad 27ak^2 = 4(h-2a)^3$$

Hence, the locus of (h, k) is

$$27ay^2 = 4(x-2a)^3.$$

Example 43 Find the locus of the point through which pass three normals to the parabola $y^2 = 4ax$ such that two of them make angles α and β respectively with the axis such that $\tan\alpha \tan\beta = 2$.

Sol. Let (h, k) be the point of intersection of three normals to the parabola $y^2 = 4ax$.

The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h_1, k), then

$$k = mh - 2am - am^3$$

$$\Rightarrow \quad am^3 + m(2a-h) + k = 0 \quad \text{...(i)}$$

Let roots of Eq. (i) be m_1, m_2, m_3 then from Eq. (i)

$$m_1m_2m_3 = -\frac{k}{a} \quad \text{...(ii)}$$

Also $m_1 = \tan\alpha$, $m_2 = \tan\beta$ and $\tan\alpha \tan\beta = 2$

$$\therefore \quad m_1m_2 = 2 \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), $2m_3 = -\dfrac{k}{a}$

or $$m_3 = -\frac{k}{2a}$$

Which being a root of Eq. (i) must satisfy it

i.e. $$am_3^3 + m_3(2a-h) + k = 0$$

$$\Rightarrow \quad a\left(-\frac{k}{2a}\right)^3 - \frac{k}{2a}(2a-h) + k = 0$$

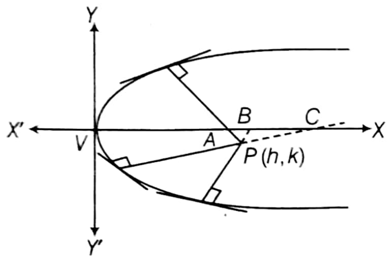
$$\Rightarrow \quad -\frac{k^3}{8a^2} - k + \frac{kh}{2a} + k = 0$$

$$\Rightarrow \quad k^2 - 4ah = 0$$

$\therefore$ Required locus of (h, k) is $y^2 - 4ax = 0$.

| Example 44 If the three normals from a point to the parabola $y^2 = 4ax$ cut the axis in points whose distance from the vertex are in AP, show that the point lies on the curve $27ay^2 = 2(x-2a)^3$.

Sol. Let (h, k) be the point of intersection of three normals to the parabola $y^2 = 4ax$. The equation of any normal to $y^2 = 4ax$ is $y = mx - 2am - am^3$ ...(i)



If it passes through (h, k) then

$$k = mh - 2am - am^3$$

$$\Rightarrow \quad am^3 + m(2a-h) + k = 0 \quad \text{...(ii)}$$

Let roots of Eq. (ii) be m_1, m_2, m_3 then from Eq. (ii)

$$m_1 + m_2 + m_3 = 0 \quad \text{...(iii)}$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a-h)}{a} \quad \text{...(iv)}$$

and $$m_1m_2m_3 = -\frac{k}{a} \quad \text{...(v)}$$

Since, Eq. (i) cuts the axis of parabola *viz.* $y = 0$ at $(2a + am^2, 0)$.

$\therefore$ The normal through (h, k) cut the axis at $A\,(2a + am_1^2, 0)$, $B\,(2a + am_2^2, 0)$ and $C\,(2a + am_3^2, 0)$ and let $V\,(0, 0)$ be the vertex of the parabola $y^2 = 4ax$.

Then, $VA = 2a + am_1^2$, $VB = 2a + am_2^2$ and $VC = 2a + am_3^2$.

Given, VA, VB and VC are in AP.

$$\therefore \quad 2VB = VA + VC$$

$$\Rightarrow \quad 4a + 2am_2^2 = 2a + am_1^2 + 2a + am_3^2$$

$$\Rightarrow \quad 2m_2^2 = m_1^2 + m_3^2$$

$$\Rightarrow \quad 2m_2^2 = (m_1 + m_3)^2 - 2m_1m_3$$

$$\Rightarrow \quad 2m_2^2 = (m_1 + m_2 + m_3 - m_2)^2 - \frac{2m_1m_2m_3}{m_2}$$

$$\Rightarrow \quad 2m_2^2 = (0 - m_2)^2 - \frac{2}{m_2}\left(-\frac{k}{a}\right)$$

[from Eqs. (iii) and (v)]

$$\Rightarrow \quad m_2^3 = \frac{2k}{a} \quad \text{...(vi)}$$

Now, from Eq. (iv), $m_2(m_1 + m_3) + m_3m_1 = \dfrac{(2a-h)}{a}$

$$\Rightarrow m_2(m_1 + m_2 + m_3 - m_2) + \frac{m_1m_2m_3}{m_2} = \frac{(2a-h)}{a}$$

$$\Rightarrow \quad m_2(0 - m_2) - \frac{k}{am_2} = \frac{(2a-h)}{a}$$

$$\Rightarrow \quad -am_2^3 - k = m_2(2a-h)$$

$$\Rightarrow \quad (-am_2^3 - k)^3 = m_2^3(2a-h)^3$$

$$\Rightarrow \quad (-2k - k)^3 = \frac{2k}{a}(2a-h)^3 \quad \text{[from Eq. (vi)]}$$

$$\Rightarrow \quad -27k^3 = -\frac{2k}{a}(h-2a)^3$$

$$\Rightarrow \quad 27ak^2 = 2(h-2a)^3$$

Hence, locus of (h, k) is

$$27ay^2 = 2(x-2a)^3.$$

| Example 45 The normals at P, Q, R on the parabola $y^2 = 4ax$ meet in a point on the line $y = k$. Prove that the sides of the ΔPQR touch the parabola $x^2 - 2ky = 0$.

Sol. Any normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3 \quad \text{...(i)}$$

Also, any point on the line $y = k$ is (x_1, k).

If Eq. (i) passes through (x_1, k) then $k = mx_1 - 2am - am^3$

or $$am^3 + m(2a - x_1) + k = 0$$

If the roots of this equation are m_1, m_2, m_3 then we get

$$m_1 + m_2 + m_3 = 0 \quad \text{...(ii)}$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a - x_1)}{a} \quad \text{...(iii)}$$

and $$m_1m_2m_3 = -\frac{k}{a} \quad \text{...(iv)}$$

Also, coordinates of three points P, Q and R are $(am_1^2, -2am_1)(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$, respectively.

$\therefore$ The equation of the line PQ is

$$y-(-2am_1)=\frac{(-2am_2)-(-2am_1)}{am_2^2-am_1^2}(x-am_1^2)$$

$$\Rightarrow \quad y+2am_1=-\frac{2}{(m_2+m_1)}(x-am_1^2)$$

$$\Rightarrow \quad y(m_1+m_2)+2am_1(m_1+m_2)=-2x+2am_1^2$$

$$\Rightarrow \quad y(m_1+m_2)+2am_1m_2=-2x$$

$$\Rightarrow \quad y(m_1+m_2+m_3-m_3)+\frac{2am_1m_2m_3}{m_3}=-2x$$

$$\Rightarrow \quad y(0-m_3)-\frac{2k}{m_3}=-2x \qquad \text{[from Eqs. (ii) and (iv)]}$$

$$\Rightarrow \quad -ym_3^2-2k=-2m_3x$$

$$\Rightarrow \quad ym_3^2-2m_3x+2k=0,$$

which is a quadratic in m_3.

Since, PQ will touch it, then

$$B^2-4AC=0$$

$$\Rightarrow \quad (-2x)^2-4\cdot y\cdot 2k=0$$

$$\therefore \quad x^2-2ky=0$$

Must

I Example 46 Find the point on the axis of the parabola $3y^2+4y-6x+8=0$ from when three distinct normals can be drawn.

Sol. Given, parabola is $3y^2+4y-6x+8=0$

$$\Rightarrow \quad 3\left(y^2+\frac{4}{3}y\right)=6x-8$$

$$\Rightarrow \quad 3\left\{\left(y+\frac{2}{3}\right)^2-\frac{4}{9}\right\}=6x-8$$

$$\Rightarrow \quad 3\left(y+\frac{2}{3}\right)^2=\left(6x-8+\frac{4}{3}\right)$$

$$\therefore \quad \left(y+\frac{2}{3}\right)^2=2\left(x-\frac{10}{9}\right)$$

Let $\quad y+\frac{2}{3}=Y,\ x-\frac{10}{9}=X$

Then, $\quad Y^2=2X$

Comparing with $\quad Y^2=4aX$

$$\therefore \quad a=\frac{1}{2}$$

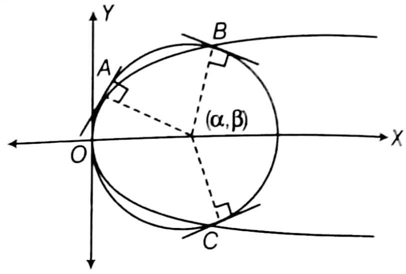
any point on the axis of parabola is $\left(x,-\frac{2}{3}\right)$

and $\quad X>2a \Rightarrow x-\frac{10}{9}>1$

$$\Rightarrow \quad x>\frac{19}{9}$$

Circle Through Co-normal Points

To find the equation of the circle passing through the three (conormal) points on the parabola, normals at which pass through a given point (α,β).



Let $A(am_1^2,-2am_1)$, $B(am_2^2,-2am_2)$ and $C(am_3^2,-2am_3)$ be the three points on the parabola

$$y^2=4ax$$

Since, point of intersection of normals is (α,β), then

$$am^3+(2a-\alpha)m+\beta=0 \qquad \text{...(E)}$$

$$\therefore \quad m_1+m_2+m_3=0 \qquad \text{...(i)}$$

$$m_1m_2+m_2m_3+m_3m_1=\frac{(2a-\alpha)}{a} \qquad \text{...(ii)}$$

and $$m_1m_2m_3=-\frac{\beta}{a} \qquad \text{...(iii)}$$

Let the equation of the circle through A, B, C be

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{...(iv)}$$

If the point $(am^2,-2am)$ lies on it, then

$$(am^2)^2+(-2am)^2+2g(am^2)+2f(-2am)+c=0$$

$$\Rightarrow \quad a^2m^4+(4a^2+2ag)m^2-4afm+c=0 \qquad \text{...(v)}$$

This is a $\boxed{\text{biquadratic equation}}$ in m. Hence, there are four values of m, say m_1, m_2, m_3 and m_4 such that the circle pass through the points.

$A(am_1^2,-2am_1)$, $B(am_2^2,-2am_2)$, $C(am_3^2,-2am_3)$ and $D(am_4^2,-2am_4)$.

$$\therefore \quad m_1+m_2+m_3+m_4=0 \qquad \text{...(F)}$$

$$\Rightarrow \quad 0+m_4=0 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad m_4=0$$

$$\therefore \quad (am_4^2,-2am_4)=(0,0)$$

Thus, the circle passes through the vertex of the parabola $y^2=4ax$ from Eq. (iv),

$$0+0+0+0+c=0$$

$$\therefore \quad c=0$$

From Eq. (v), $a^2m^4+(4a^2+2ag)m^2-4afm=0$

$$\Rightarrow \quad am^3+(4a+2g)m-4f=0 \qquad \text{...(vi)}$$

Now, Eqs. (E) and (vi) are identical.

$$\therefore \quad 1 = \frac{4a + 2g}{2a - \alpha} = -\frac{4f}{\beta}$$

$$\therefore \quad 2g = -(2a + \alpha), 2f = -\beta/2$$

$\therefore$ The equation of the required circle is

$$\boxed{x^2 + y^2 - (2a + \alpha)\,x - \frac{\beta}{2}y = 0}$$

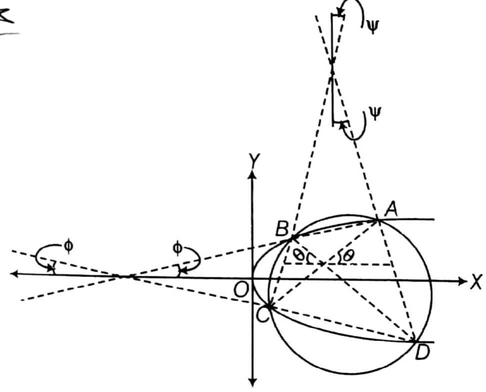
[from Eq. (iv)]

Corollary 1. *The algebraic sum of the ordinates of the four points of intersection of a circle and a parabola is zero.*

Sum of ordinates

$$= -2am_1 - 2am_2 - 2am_3 - 2am_4$$
$$= -2a(m_1 + m_2 + m_3 + m_4) \quad \text{[from Eq. (F)]}$$
$$= -2a \times 0 = 0$$

Corollary 2. *The common chords of a circle and a parabola are in pairs equally inclined to the axis of the parabola.*



Let A, B, C, D be the points of intersection of the circle and the parabola with $A(am_1^2, -2am_1)$, $B(am_2^2, -2am_2)$ $C(am_3^2, -2am_3)$ and $D(am_4^2, -2am_4)$ then equation of AC and BD are

$$y(m_1 + m_3) = -2x - 2am_1m_3$$

and

$$y(m_2 + m_4) = -2x - 2am_2m_4, \text{ respectively.}$$

$\therefore$ Slopes of the chords AC and BD are

$$-\frac{2}{m_1 + m_3} \text{ and } -\frac{2}{m_2 + m_4}, \text{ respectively.}$$

$$\therefore \quad \text{Slope of } AC = -\frac{2}{m_1 + m_3}$$
$$= \frac{2}{m_2 + m_4} \quad [\because m_1 + m_2 + m_3 + m_4 = 0]$$
$$= -\left(-\frac{2}{m_2 + m_4}\right) = -\text{ Slope of } BD$$

$\therefore$ Their slopes are equal in magnitude and opposite in sign.

$\therefore$ The chords of AC and BD are equally inclined to the axis.

Remark

This is likewise true for the pairs of chords AB, CD and AD, BC.

Corollary 3. *The circle through conormal point passes through the vertex $(0, 0)$ of the parabola.*

Corollary 4. *The centroid of four points; in which a circle intersects a parabola, lies on the axis of the parabola.*

$$\text{Centroid} = \left(\frac{\sum_{i=1}^{4} am_i^2}{4}, \frac{\sum_{i=1}^{4}(-2am_i)}{4}\right)$$

$$= \left(\frac{a}{4}\{(\Sigma m_1)^2 - 2\,\Sigma m_1m_2\}, -\frac{a}{2}(\Sigma m_1)\right)$$

$$= \left(\frac{a}{4}\left(0 - \frac{2(4a^2 + 2ag)}{a^2}\right), 0\right)$$

$$\boxed{= (-2a - g, 0)}$$

Here $y = 0$, which is axis of the parabola $y^2 = 4ax$.

| Example 47 A circle cuts the parabola $y^2 = 4ax$ at right angles and passes through the focus, show that its centre lies on the curve $y^2(a + 2x) = a(a + 3x)^2$.

Sol. Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

meet the parabola $y^2 = 4ax$ at any point $P(at^2, 2at)$ cutting it at right angles.

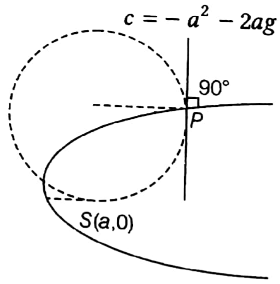
We have to find locus of centre of circle Eq. (i),

i.e. $(-g, -f)$

But given circle Eq. (i) passes through the focus $(a, 0)$, then

$$a^2 + 0 + 2ga + 0 + c = 0$$

$$\Rightarrow \quad c = -a^2 - 2ag \quad \text{...(ii)}$$



Now, the circle and parabola intersect at $P(at^2, 2at)$ at right angles.

Since, tangent at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$ty = x + at^2$$

Hence, this tangent must pass through the centre $(-g, -f)$ of the circle

$$\therefore \quad -ft = -g + at^2$$

$$\Rightarrow \quad ft = g - at^2 \quad \text{...(iii)}$$

Also the point $P(at^2, 2at)$ lies on the circle (i), then

$$a^2t^4 + 4a^2t^2 + 2agt^2 + 4aft + c = 0$$

Hence, from Eqs. (ii) and (iii) when we put values for c and ft.

$$\therefore \quad a^2t^4 + 4a^2t^2 + 2agt^2 + 4ag - 4a^2t^2 - a^2 - 2ag = 0$$

$$\Rightarrow \quad a^2t^4 + 2agt^2 + 2ag - a^2 = 0$$

$$\Rightarrow \quad at^4 + 2gt^2 + 2g - a = 0$$

$$\Rightarrow \quad a(t^4 - 1) + 2g(t^2 + 1) = 0$$

$$\Rightarrow \quad (t^2 + 1)[a(t^2 - 1) + 2g] = 0 \quad \text{But } t^2 + 1 \neq 0$$

$$\Rightarrow \quad a(t^2 - 1) + 2g = 0 \quad \Rightarrow t^2 = \frac{a - 2g}{a}$$

Hence, from Eq. (iii),

$$f\sqrt{\left(\frac{a-2g}{a}\right)} = g - a\left(\frac{a-2g}{a}\right)$$

$$\Rightarrow \quad f\sqrt{\left(\frac{a-2g}{a}\right)} = (3g - a)$$

$$\Rightarrow \quad f^2(a - 2g) = a(3g - a)^2$$

$$\Rightarrow \quad (-f)^2[a + 2(-g)] = a(a - 3g)^2$$

$$\Rightarrow \quad (-f)^2[a + 2(-g)] = a[a + 3(-g)]^2$$

Hence, locus of the centre $(-g, -f)$ is the curve

$$y^2(a + 2x) = a(a + 3x)^2.$$

Exercise for Session 2

1. If $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$, then the value of λ is

(a) -24 (b) -16 (c) -8 (d) 24

2. The slope of a chord of the parabola $y^2 = 4ax$ which is normal at one end and which subtends a right angle at the origin is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) $-\frac{1}{\sqrt{2}}$ (d) $-\sqrt{2}$

3. The common tangent to the parabola $y^2 = 4ax$ and $x^2 = 4ay$ is

(a) $x + y + a = 0$ (b) $x + y - a = 0$ (c) $x - y + a = 0$ (d) $x - y - a = 0$

4. The circle $x^2 + y^2 + 4\lambda x = 0$ which $\lambda \in R$ touches the parabola $y^2 = 8x$. The value of λ is given by

(a) $\lambda \in (0, \infty)$ (b) $\lambda \in (-\infty, 0)$ (c) $\lambda \in (1\ \infty)$ (d) $\lambda \in (-\infty, 1)$

5. If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of ordinates of P and Q is

(a) $4a^2$ (b) $2a^2$ (c) $-4a^2$ (d) $8a^2$

6. The normals at three points P, Q, R of the parabola $y^2 = 4ax$ meet in (h, k). The centroid of ΔPQR lies on

(a) $x = 0$ (b) $y = 0$ (c) $x = -a$ (d) $y = a$

7. The set of points on the axis of the parabola $y^2 - 4x - 2y + 5 = 0$ from which all the three normals to the parabola are real, is

(a) $(\lambda, 0)\,; \lambda > 1$ (b) $(\lambda, 1)\,; \lambda > 3$ (c) $(\lambda, 2)\,; \lambda > 6$ (d) $(\lambda, 3)\,; \lambda > 8$

8. Prove that any three tangents to a parabola whose slopes are in harmonic progression enclose a triangle of constant area.

9. A chord of parabola $y^2 = 4ax$ subtends a right angle at the vertex. Find the locus of the point of intersection of tangents at its extremities.

10. Find the equation of the normal to the parabola $y^2 = 4x$ which is

(a) parallel to the line $y = 2x - 5$. (b) perpendicular to the line $2x + 6y + 5 = 0$.

11. The ordinates of points P and Q on the parabola $y^2 = 12x$ are in the ratio 1 : 2. Find the locus of the point of intersection of the normals to the parabola at P and Q.

12. The normals at P, Q, R on the parabola $y^2 = 4ax$ meet in a point on the line $y = c$. Prove that the sides of the ΔPQR touch the parabola $x^2 = 2cy$.

13. The normals are drawn from $(2\lambda, 0)$ to the parabola $y^2 = 4x$. Show that λ must be greater than 1. One normal is always the X-axis. Find λ for which the other two normals are perpendicular to each other.

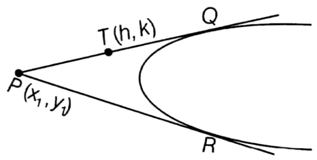
Session 3

Pair of Tangents $SS_1 = T^2$, Chord of Contact, Equation of the Chord Bisected at a Given Point, Diameter, Lengths of Tangent, Subtangent, Normal and Subnormal, Some Standard Properties of the Parabola, Reflection Property of a Parabola, Study of Parabola of the Form $(\alpha x^2 + \beta y)^2 + 2gx + 2fy + c = 0$

Pair of Tangents $SS_1 = T^2$

If $y_1^2 - 4ax_1 > 0$, then any point $P(x_1, y_1)$ lies outside the parabola and a pair of tangents PQ, PR can be drawn to it from P. We find their equation as follows.

Let $T(h, k)$ be any point on the pair of tangents PQ or PR drawn from any external point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$.



Equation of PT is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1)$$

$$\Rightarrow \quad y = \left(\frac{k - y_1}{h - x_1}\right)x + \left(\frac{hy_1 - kx_1}{h - x_1}\right)$$

which is tangent to the parabola

$$y^2 = 4ax$$

$$\therefore \quad c = \frac{a}{m}$$

$$\Rightarrow \quad cm = a \quad \Rightarrow \quad \left(\frac{hy_1 - kx_1}{h - x_1}\right)\left(\frac{k - y_1}{h - x_1}\right) = a$$

$$\Rightarrow \quad (k - y_1)(hy_1 - kx_1) = a(h - x_1)^2$$

$\therefore$ Locus of (h, k), equation of pair of tangents is

$$(y - y_1)(xy_1 - x_1 y) = a(x - x_1)^2$$

$$\Rightarrow (y^2 - 4ax)(y_1^2 - 4ax_1) = \{yy_1 - 2a(x + x_1)\}^2$$

$$\therefore \quad \boxed{SS_1 = T^2} \bigstar$$

where $S = y^2 - 4ax, S_1 = y_1^2 - 4ax_1$

and $T = yy_1 - 2a(x + x_1)$.

Aliter :

Let the parabola be $y^2 = 4ax$ ...(i)

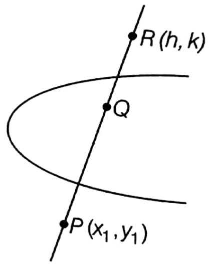
Let $P(x_1, y_1)$ be any point outside the parabola. Let a chord of the parabola through the point $P(x_1, y_1)$ cut the parabola at Q and let $R(h, k)$ be any arbitrary point on the line PQ (R inside or outside).

Let Q divide PR in the ratio $\lambda : 1$, then coordinates of Q are

$$\left(\frac{\lambda h + x_1}{\lambda + 1}, \frac{\lambda k + y_1}{\lambda + 1}\right) \qquad [\because PQ : QR = \lambda : 1]$$

Since, Q lies on parabola Eq. (i), then

$$\left(\frac{\lambda k + y_1}{\lambda + 1}\right)^2 = 4a\left(\frac{\lambda h + x_1}{\lambda + 1}\right)$$



$$\Rightarrow \quad (\lambda k + y_1)^2 - 4a(\lambda h + x_1)(\lambda + 1) = 0$$

$$\Rightarrow \quad (k^2 - 4ah)\lambda^2 + 2[ky_1 - 2a(h + x_1)]\lambda + (y_1^2 - 4ax_1) = 0 \quad \text{...(ii)}$$

Line PR will become tangent to parabola Eq. (i), then roots of Eq. (ii) are equal

$$\therefore \quad 4[ky_1 - 2a(h + x_1)]^2 - 4(k^2 - 4ah)(y_1^2 - 4ax_1) = 0$$

or $\{ky_1 - 2a(h + x_1)\}^2 = (k^2 - 4ah)(y_1^2 - 4ax_1)$

Hence, locus of $R(h, k)$ i.e. equation of pair of tangents from $P(x_1, y_1)$ is

$$\{yy_1 - 2a(x + x_1)\}^2 = (y^2 - 4ax)(y_1^2 - 4ax_1)$$

i.e. $T^2 = SS_1$ or $SS_1 = T^2$

Remark

$S = 0$ is the equation of the curve, S_1 is obtained from S by replacing x by x_1 and y by y_1 and $T = 0$ is the equation tangent at (x_1, y_1) to $S = 0$.

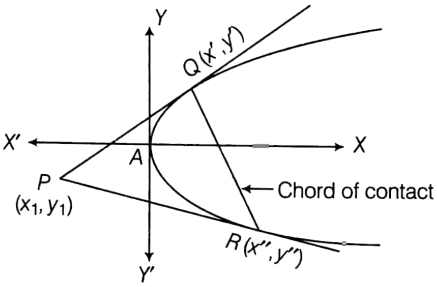
Chord of Contact

The chord joining the points of contact of two tangents drawn from an external point to a parabola is known as the chord of contact of tangents drawn from external point.

Theorem The chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$\boxed{yy_1 = 2a(x + x_1)}$$

Proof Let PQ and PR be tangents to the parabola $y^2 = 4ax$ drawn from any external point $P(x_1, y_1)$, then QR is called **chord of contact** of the parabola $y^2 = 4ax$.



Let $\qquad Q \equiv (x', y')$ and $R \equiv (x'', y'')$

Equation of tangent PQ is

$$yy' = 2a(x + x') \qquad \text{...(i)}$$

and equation of tangent PR is

$$yy'' = 2a(x + x'') \qquad \text{...(ii)}$$

Since, lines Eqs. (i) and (ii) pass through (x_1, y_1), then

$$y_1 y' = 2a(x_1 + x') \text{ and } y_1 y'' = 2a(x_1 + x'')$$

Hence, it is clear $Q(x', y')$ and $R(x'', y'')$ lie on

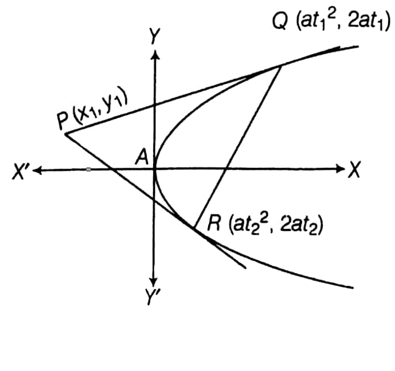
$$yy_1 = 2a(x + x_1)$$

which is **chord of contact** QR.

Example 48 Tangents are drawn from the point (x_1, y_1) to the parabola $y^2 = 4ax$, show that the length of their chord of contact is $\dfrac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$.

Sol. Given parabola is

$$y^2 = 4ax \qquad \text{...(i)}$$



Let $\qquad P \equiv (x_1, y_1)$

and the tangents from P touch the parabola at $Q(at_1^2, 2at_1)$ and $R(at_2^2, 2at_2)$ then P is the point of intersection of tangents.

$$\therefore \qquad x_1 = at_1t_2 \text{ and } y_1 = a(t_1 + t_2)$$

$$\Rightarrow \qquad t_1t_2 = \frac{x_1}{a} \text{ and } t_1 + t_2 = \frac{y_1}{a} \qquad \text{...(ii)}$$

Now,

$$QR = \sqrt{(at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2}$$

$$= \sqrt{a^2(t_1 - t_2)^2\,[(t_1 + t_2)^2 + 4]}$$

$$= |a||t_1 - t_2|\sqrt{\{(t_1 + t_2)^2 + 4\}}$$

$$= |a|\sqrt{\{(t_1 + t_2)^2 - 4t_1t_2\}}\sqrt{\{(t_1 + t_2)^2 + 4\}}$$

$$= |a|\sqrt{\left(\frac{y_1^2}{a^2} - \frac{4x_1}{a}\right)} \cdot \sqrt{\left(\frac{y_1^2}{a^2} + 4\right)}$$

[from Eq. (ii)]

$$= |a|\frac{\sqrt{(y_1^2 - 4ax_1)}}{|a|} \cdot \frac{\sqrt{(y_1^2 + 4a^2)}}{|a|}$$

$$= \frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$$

Aliter :

Equation of QR is $\quad yy_1 = 2a(x + x_1)$

$$\Rightarrow \qquad x = \frac{yy_1 - 2ax_1}{2a}$$

The ordinates of Q and R are the roots of the equation

$$y^2 = 4a\left(\frac{yy_1 - 2ax_1}{2a}\right)$$

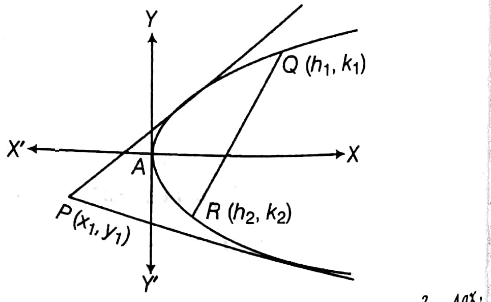
$$\Rightarrow \qquad y^2 = 2(yy_1 - 2ax_1) \qquad \text{...(i)}$$

$$\Rightarrow \qquad y^2 - 2yy_1 + 4ax_1 = 0$$

$$\therefore \qquad k_1 + k_2 = 2y_1 \text{ and } k_1k = 4ax_1$$

$$\therefore \qquad (k_2 - k_1) = \sqrt{(k_1 + k_2)^2 - 4k_1k_2}$$

$$= \sqrt{(4y_1^2 - 16ax_1)} = 2\sqrt{(y_1^2 - 4ax_1)} \qquad \text{...(ii)}$$



Since, $Q(h_1, k_1)$ and $R(h_2, k_2)$ lie on the parabola $y^2 = 4ax$, therefore

$$k_1^2 = 4ah_1 \text{ and } k_2^2 = 4ah_2$$

$$\Rightarrow \qquad k_2^2 - k_1^2 = 4a(h_2 - h_1)$$

$$(k_2 + k_1)(k_2 - k_1) = 4a(h_2 - h_1)$$

$$\Rightarrow \quad 2y_1(k_2 - k_1) = 4a(h_2 - h_1)$$

$$\Rightarrow \quad (h_2 - h_1) = \frac{y_1(k_2 - k_1)}{2a} \qquad \text{...(iii)}$$

Now, $QR = \sqrt{(k_2 - k_1)^2 + (h_2 - h_1)^2}$

$$= \sqrt{(k_2 - k_1)^2 + \frac{y_1^2 (k_2 - k_1)^2}{4a^2}} \qquad \text{[from Eq. (iii)]}$$

$$= \frac{(k_2 - k_1)}{2|a|}\sqrt{(y_1^2 + 4a^2)}$$

$$= \frac{2\sqrt{(y_1^2 - 4ax_1)}}{2|a|}\sqrt{(y_1^2 + 4a^2)} \qquad \text{[from Eq. (ii)]}$$

$$= \frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$$

| Example 49 Prove that the area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $(y_1^2 - 4ax_1)^{3/2}/2a$.

Sol. Equation of QR (chord of contact) is

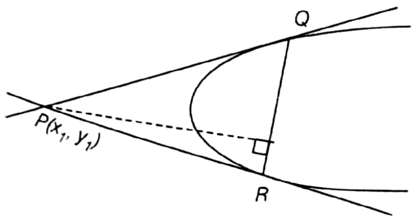
$$yy_1 = 2a(x + x_1)$$

$$\Rightarrow \quad yy_1 - 2a(x + x_1) = 0$$

$\because$ PM = Length of perpendicular from $P(x_1, y_1)$ on QR

$$= \frac{|y_1y_1 - 2a(x_1 + x_1)|}{\sqrt{(y_1^2 + 4a^2)}} = \frac{|(y_1^2 - 4ax_1)|}{\sqrt{(y_1^2 + 4a^2)}}$$

[Since, $P(x_1, y_1)$ lies outside the parabola. So, $y_1^2 - 4ax_1 > 0$]



Now, area of $\Delta PQR = \frac{1}{2} QR \cdot PM$

$$= \frac{1}{2}\frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}\,\frac{(y_1^2 - 4ax_1)}{\sqrt{(y_1^2 + 4a^2)}}$$

$$= (y_1^2 - 4ax_1)^{3/2}/2a, \text{ if } a > 0$$

Equation of the Chord Bisected at a Given Point

Theorem The equation of the chord of the parabola $y^2 = 4ax$ which is bisected at (x_1, y_1) is

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

or

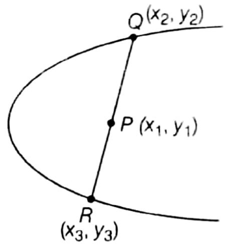
$$\boxed{T = S_1,}$$

where, $T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$.

Proof Since, equation of the parabola is

$$y^2 = 4ax \qquad \text{...(i)}$$

Let QR be the chord of the parabola whose mid-point is $P(x_1, y_1)$.



Since, Q and R lie on parabola (i),

$$y_2^2 = 4ax_2 \text{ and } y_3^2 = 4ax_3$$

$$\therefore \quad y_3^2 - y_2^2 = 4a(x_3 - x_2)$$

$$\Rightarrow \quad \frac{y_3 - y_2}{x_3 - x_2} = \frac{4a}{y_3 + y_2}$$

$$= \frac{4a}{2y_1} \quad [\because P(x_1, y_1) \text{ is mid-point of } QR]$$

$$\therefore \quad \frac{y_3 - y_2}{x_3 - x_2} = \frac{2a}{y_1} = \text{Slope of } QR$$

Equation of QR is $\quad y - y_1 = \frac{2a}{y_1}(x - x_1)$

$$\Rightarrow \quad yy_1 - y_1^2 = 2ax - 2ax_1$$

$$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

[subtracting $2ax_1$ from both sides]

$$\therefore \quad T = S_1,$$

where $\quad T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$.

| Example 50 Find the locus of the mid-points of the chords of the parabola $y^2 = 4ax$ which subtend a right angle at the vertex of the parabola.

Sol. Let $P(h, k)$ be the mid-point of a chord QR of the parabola $y^2 = 4ax$ then equation of chord QR is

$$T = S_1$$

or $\quad yk - 2a(x + h) = k^2 - 4ah$

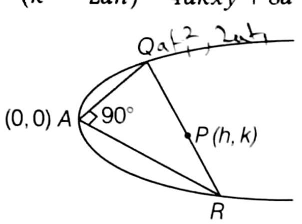
$$\Rightarrow \quad yk - 2ax = k^2 - 2ah \qquad \text{...(i)}$$

If A is the vertex of the parabola. For combined equation of AQ and AR making homogeneous of $y^2 = 4ax$ with the help of Eq. (i).

$$\therefore \quad y^2 = 4ax$$

$$\Rightarrow \quad y^2 = 4ax\left(\frac{yk - 2ax}{k^2 - 2ah}\right)$$

$\Rightarrow \quad y^2(k^2 - 2ah) - 4akxy + 8a^2x^2 = 0$



$\frac{at_1^2 + at_2^2}{2} = h$

& also

$\left(\frac{2at_1 - 0}{at_1^2 - 0}\right)\left(\;\right) = -1$

tan(90)

Since, $\angle QAR = 90°$

$\therefore$ Coefficient of x^2 + Coefficient of $y^2 = 0$

$$k^2 - 2ah + 8a^2 = 0$$

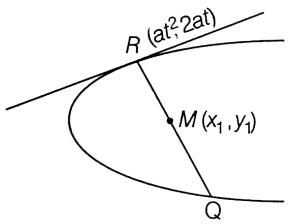
Hence, the locus of $P(h, k)$ is $y^2 - 2ax + 8a^2 = 0$.

| Example 51 Show that the locus of the middle points of normal chords of the parabola $y^2 = 4ax$ is

$$y^4 - 2a(x - 2a)y^2 + 8a^4 = 0.$$

Sol. Equation of the normal chord at any point $(at^2, 2at)$ of the parabola $y^2 = 4ax$ is

$$y + tx = 2at + at^3 \quad \text{...(i)}$$



$\left(t_2 = -t_1 - \frac{2}{t_1}\right)$

But if $M(x_1, y_1)$ be its middle point its equation must be also

$$T = S_1$$

$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

$\Rightarrow \quad yy_1 - 2ax = y_1^2 - 2ax_1 \quad \text{...(ii)}$

$\because$ From Eqs. (i) and (ii) are identical, comparing, them

$$\frac{1}{y_1} = \frac{t}{-2a} = \frac{2at + at^3}{y_1^2 - 2ax_1}$$

From first two relations, $t = -\frac{2a}{y_1}$ ...(iii)

From last two relations, $\frac{t}{-2a} = \frac{2at + at^3}{y_1^2 - 2ax_1}$

$\Rightarrow \quad \frac{y_1^2 - 2ax_1}{-2a} = 2a + at^2$

$\Rightarrow \quad \frac{y_1^2 - 2ax_1}{-2a} = 2a + a\left(\frac{-2a}{y_1}\right)^2$ [from Eq. (iii)]

$\Rightarrow \quad \frac{y_1^2 - 2ax_1}{-2a} = \frac{2ay_1^2 + 4a^3}{y_1^2}$

$\Rightarrow \quad y_1^4 - 2ax_1y_1^2 = -4a^2y_1^2 - 8a^4$

$\Rightarrow \quad y_1^4 - 2a(x_1 - 2a)y_1^2 + 8a^4 = 0$

Hence, the locus of middle point (x_1, y_1) is

$$y^4 - 2a(x - 2a)y^2 + 8a^4 = 0.$$

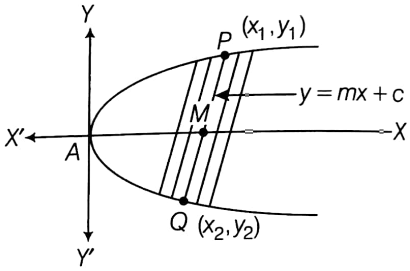
Diameter

The locus of the middle points of a system of parallel chords is called a diameter and in case of a parabola this diameter is shown to be a straight line which is parallel to the axis of the parabola.

Theorem The equation of the diameter bisecting chords of slope m of the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

Proof Let $y = mx + c$ be system of parallel chords to $y^2 = 4ax$ for different chords c varies, m remains constant.

Let the extremities of any chord PQ of the set be $P(x_1, y_1)$ and $Q(x_2, y_2)$ and let its middle point be $M(h, k)$.



On solving equations

$$y^2 = 4ax \quad \text{and} \quad y = mx + c.$$

$\therefore \quad y^2 = 4a\left(\frac{y - c}{m}\right)$

$\therefore \quad my^2 - 4ay + 4ac = 0$

$\therefore \quad y_1 + y_2 = \frac{4a}{m} \quad \text{or} \quad \frac{y_1 + y_2}{2} = \frac{2a}{m}$

[$\because (h, k)$ is the mid-point of PQ]

Hence, locus of $M(h, k)$ is $\boxed{y = \frac{2a}{m}}$.

Aliter :

Let (h, k) be the middle point of the chord $y = mx + c$ of the parabola $y^2 = 4ax$ then

$$T = S_1 \Rightarrow ky - 2a(x + h) = k^2 - 4ah$$

$$\text{slope} = \frac{2a}{k} = m \Rightarrow k = \frac{2a}{m}$$

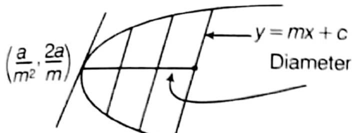
Hence, locus of the mid-point is $y = \frac{2a}{m}$.

Remarks

1. The point in which any diameter meets the curve is called the extremity of the diameter.
2. Any line which is parallel to the axis of the parabola drawn through any point on the parabola is called diameter and its equation is y-coordinate of that point.

   If point on diameter (x_1, y_1), then diameter is $y = y_1$.

Corollary 1. *The tangent at the extremity of a diameter of a parabola is parallel to the system of chords it bisects.*



Let $y = mx + c$ (c variable) represents the system of parallel chords, then the equation of diameter of $y^2 = 4ax$ is $y = \dfrac{2a}{m}$.

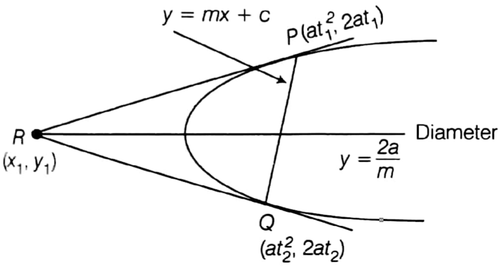
The diameter meets the parabola $y^2 = 4ax$ at $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

and tangent is $y = mx + \dfrac{a}{m}$ which is parallel to $y = mx + c$.

Corollary 2. *Tangents at the end of any chord meet on the diameter which bisects the chords.*

If extremities of the chord be $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ then its slope

$$m = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} \Rightarrow m = \frac{2}{(t_2 + t_1)}.$$



$\therefore$ Equation of diameter is

$$y = 2a/m \Rightarrow y = a(t_1 + t_2) \quad \text{...(i)}$$

Now, tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ meet at a point $[at_1t_2, a(t_1 + t_2)]$ which lies on Eq. (i).

Aliter

Let equation of any chord PQ be $y = mx + c$.

If tangents at P and Q meet at $R(x_1, y_1)$, then PQ is the chord of contact with respect to $R(x_1, y_1)$.

$\therefore$ Equation of PQ is

$$yy_1 = 2x(x + x_1) \quad \text{or} \quad y = \frac{2a}{y_1}x + \frac{2ax_1}{y_1}$$

which is identical to $y = mx + c$

$$m = \frac{2a}{y_1} \quad \text{or} \quad y_1 = \frac{2a}{m}.$$

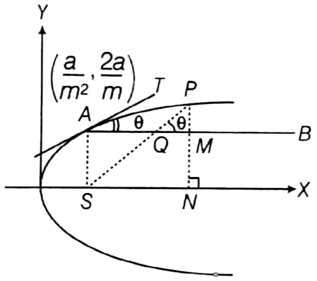
Hence, locus of $R(x_1, y_1)$ is $y = \dfrac{2a}{m}$, which bisects the chord PQ.

Corollary 3. *To find the equation of a parabola when the axes are any diameter and the tangent to the parabola at the point where this diameter meets the curve.*

Let the equation of the parabola be

$$y^2 = 4ax \quad \text{...(i)}$$

Let AB be the diameter of the parabola Eq. (i), then its equation is $y = \dfrac{2a}{m}$



Since, A is the extremity of the diameter

$\therefore$ Coordinates of A is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

where, $m = \tan\theta$

then, the equation of tangent AT at A is

$$y = mx + \frac{a}{m}.$$

Now, let P be any point on the parabola Eq. (i), whose coordinates referred to Vx and Vy are (x, y) and referred to diameter AB and tangent AT are (X, Y).

then $\quad X = AQ$ and $Y = QP$ $\quad$ [since, $PQ \| AT$]

Now, $\quad VN = VL + LN = VL + AM = VL + AQ + QM$

$$= \frac{a}{m^2} + X + QP\cos\theta$$

or
$$x = \frac{a}{m^2} + X + Y\cos\theta \quad \text{...(ii)}$$

and $\quad PN = PM + MN = PM + AL$

$$= QP\sin\theta + \frac{2a}{m}$$

$$\therefore \quad y = Y\sin\theta + \frac{2a}{m} \quad \text{... (iii)}$$

From Eqs. (ii) and (iii) coordinates of P are

$$\left(\frac{a}{m^2} + X + Y\cos\theta, Y\sin\theta + \frac{2a}{m}\right)$$

Now, P lies on Eq. (i).

$$\therefore \quad \left(Y\sin\theta + \frac{2a}{m}\right)^2 = 4a\left(\frac{a}{m^2} + X + Y\cos\theta\right)$$

$$\Rightarrow \quad Y^2 \sin^2\theta + \frac{4a^2}{m^2} + \frac{4a}{m} Y \sin\theta$$

$$= \frac{4a^2}{m^2} + 4aX + 4aY\cos\theta$$

$$\Rightarrow \quad Y^2 \sin^2\theta + 4a^2\cot^2\theta + 4a\cos\theta Y$$

$$= 4a^2\cot^2\theta + 4aX + 4aY\cos\theta \quad [\because m = \tan\theta]$$

$$\Rightarrow \quad \boxed{Y^2\sin^2\theta = 4aX}$$

$$\therefore \quad Y^2 = (4a\,\text{cosec}^2\theta)\,X,$$

which is the required parabola referred to diameter and tangent at the extremity of the diameter as axes.

Remark

The quantity $4a\text{cosec}^2\theta$ is called the parameter of the diameter AQ. It is equal to length of the chord which is parallel to AT and passes through the focus.

i.e. $\quad a\,\text{cosec}^2\theta = a(1+\cot^2\theta) = a + a\cot^2\theta$

$$= a + \frac{a}{m^2}$$

$$= a + VL = SP$$

But length of focal chord if $P(at^2, 2at)$ is $[\because S(a, 0)]$ $a\left(t + \frac{1}{t}\right)^2$.

$$\therefore \quad \tan\theta = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2-1} \text{ or } t - \frac{1}{t} = 2\cot\theta$$

$$\therefore \quad a\left(t+\frac{1}{t}\right)^2 = a\left\{\left(t-\frac{1}{t}\right)^2 + 4\right\}$$

$$= a\{(2\cot\theta)^2 + 4\}$$

$$= 4a\,\text{cosec}^2\theta = 4 \cdot SP$$

I Example 52 If the diameter through any point P of a parabola meets any chord in A and the tangent at the end of the chord meets the diameter in B and C, then prove that $PA^2 = PB \cdot PC$.

Sol. The equation of the parabola referred to the diameter through P and tangent at P as axes is

$$y^2 = 4\lambda x \quad \text{...(i)}$$

where, $\quad \lambda = a\,\text{cosec}^2\theta \quad$ [from previous corollary]

Let QR be any chord of the parabola Eq. (i). Let the extremities Q and R be $(\lambda t_1^2, 2\lambda t_1)$ and $(\lambda t_2^2, 2\lambda t_2)$.

Then, the equation of QR is

$$y(t_1+t_2) - 2x - 2\lambda t_1 t_2 = 0 \quad \text{... (ii)}$$

It meets the diameter through P i.e. X-axis or $y = 0$, then Eq. (ii) reduces

$$0 - 2x - 2\lambda\, t_1 t_2 = 0$$

$$\Rightarrow \quad x = -\lambda t_1 t_2 = PA$$

Now, tangent at Q is $\quad t_1 y = x + \lambda t_1^2 \quad$ ...(iii)

It meets the diameter through P i.e. X-axis or $y = 0$, then Eq. (iii) reduces

$$0 = x + \lambda t_1^2$$

$$\Rightarrow \quad x = -\lambda t_1^2 = PB \quad \text{...(iv)}$$

Similarly, $-\lambda t_2^2 = PC \quad$ ...(v)

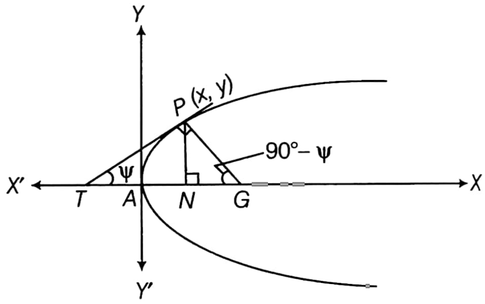
$$\therefore \quad (PA)^2 = (-\lambda t_1 t_2)^2 = \lambda^2 t_1^2 t_2^2$$

$$= (-\lambda t_1^2)(-\lambda t_2^2) \quad \text{[from Eqs. (iv) and (v)]}$$

$$= PB \cdot PC$$

Lengths of Tangent, Subtangent, Normal and Subnormal

Let the parabola $y^2 = 4ax$. Let the tangent and normal at $P(x, y)$ meet the axis of parabola at T and G respectively and let tangent at $P(x, y)$ makes angle ψ with the positive direction of X-axis.



Then, $\quad PT =$ Length of Tangent

$PG =$ Length of Normal

$TN =$ Length of Subtangent

and $\quad NG =$ Length of Subnormal

If $A(0, 0)$ is the vertex of the parabola.

$$\because \quad PN = y$$

$$\therefore \quad PT = PN\,\text{cosec}\,\psi = y\,\text{cosec}\,\psi$$

$$PG = PN\,\text{cosec}\,(90° - \psi) = y\sec\psi$$

$$TN = PN\cot\psi = y\cot\psi$$

and $\quad NG = PN\cot(90° - \psi) = y\tan\psi$

where $\boxed{\tan\psi = \frac{2a}{y} = m}$ $\quad$ [slope of tangent at $P(x, y)$]

$yy_1 = 2ax + 2ax_1$

I Example 53 Find the length of tangent, subtangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$.

Sol. $\because$ Equation of tangent of $(at^2, 2at)$ of parabola $y^2 = 4ax$ is

$$ty = x + at^2$$

Slope of this tangent $m = \frac{1}{t}$

Let tangent makes angle ψ with positive direction of X-axis

$$\tan\psi = \frac{1}{t}$$

then $\quad t = \cot\psi$

$\therefore$ Length of tangent at

$$(at^2, 2at) = 2at \operatorname{cosec}\psi$$

$$= 2at\sqrt{(1+\cot^2\psi)} = 2at\sqrt{(1+t^2)}$$

Length of normals at

$$(at^2, 2at) = 2at \sec\psi = 2at\sqrt{(1+\tan^2\psi)}$$

$$= 2a\sqrt{(t^2 + t^2\tan^2\psi)} = 2a\sqrt{(t^2+1)}$$

Length of subtangent at $(at^2, 2at) = 2at \cot\psi$

$$= 2at^2$$

Length of subnormal at $(at^2, 2at) = 2at \tan\psi$

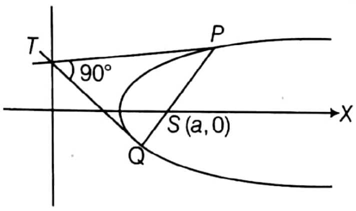
$$= 2a$$

Some Standard Properties of the Parabola

(1) The tangents at the extremities of a focal chord intersect at right angles on the directrix.

The extremities of a focal chord PQ are

$$P \equiv (at^2, 2at) \text{ and } Q \equiv \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$$



Let parabola is $y^2 = 4ax$

$\therefore$ Tangent at $P(at^2, 2at)$ is

$$ty = x + at^2 \qquad \text{...(i)}$$

Its slope $\frac{1}{t} = m_1$ (say)

and tangent at $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ is

$$-\frac{y}{t} = x + \frac{a}{t^2} \qquad \text{...(ii)}$$

$\Rightarrow$

$$-ty = xt^2 + a$$

Its slope $\quad -t = m_2$ (say)

$\because \quad m_1 m_2 = -1$

$\therefore \quad \angle PTQ = 90°$

and adding Eqs. (i) and (ii), we get

$$(x+a)(1+t^2) = 0 \Rightarrow 1 + t^2 \neq 0$$

$\therefore$ $x + a = 0$, which is directrix.

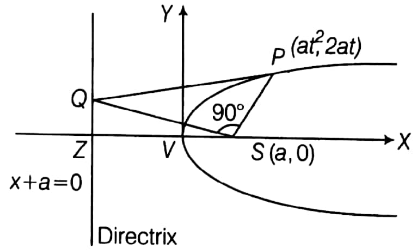
(2) The portion of a tangent to a parabola intercepted between the directrix and the curve subtends a right angle at the focus.

The equation of the tangent to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ is

$$ty = x + at^2 \qquad \text{...(i)}$$

Let Eq. (i) meet the directrix $x + a = 0$ at Q,

then coordinates of Q are $\left(-a, \frac{at^2 - a}{t}\right)$, also focus S is $(a, 0)$.



$\therefore \quad$ Slope of $SP = \frac{2at - 0}{at^2 - a}$

$$= \frac{2t}{t^2 - 1} = m_1 \qquad \text{(say)}$$

and $\quad$ slope of $SQ = \frac{\frac{at^2 - a}{t} - 0}{-a - a} = \frac{t^2 - 1}{-2t} = m_2 \qquad$ (say)

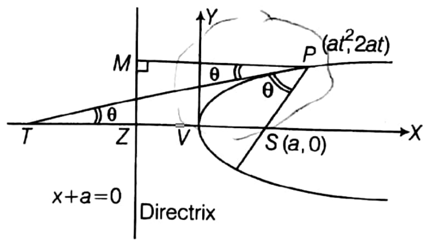
$\therefore \quad m_1 m_2 = -1$

i.e. SP is perpendicular to SQ i.e. $\angle PSQ = 90°$

(3) The tangent at any point P of a parabola bisects the angle between the focal chord through P and the perpendicular from P to the directrix.

Let the tangent at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meets the axis of the parabola i.e. X-axis or $y = 0$ at T. The equation of tangent to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ is

$$ty = x + at^2$$



For coordinates of T solve it with $y = 0$.

$\therefore \quad T(-at^2, 0)$

$\therefore \quad ST = SV + VT = a + at^2 = a(1 + t^2)$

Also, $\quad SP = PM = a + at^2 = a(1 + t^2)$

$\therefore \quad SP = ST$, i.e. $\angle STP = \angle SPT$

But $\quad \angle STP = \angle MPT \quad$ [alternate angles]

$\angle SPT = \angle MPT$

(4) The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.

Equation of tangent at $P(at^2, 2at)$

On the parabola $y^2 = 4ax$

is $\quad ty = x + at^2$

$\Rightarrow \quad x - ty + at^2 = 0 \quad$ ...(i)

Now, the equation of line through $S(a, 0)$ and perpendicular to Eq. (i) is

$$tx + y = \lambda$$

Since, it passes through $(a, 0)$.

$\therefore \quad ta + 0 = \lambda$

$\therefore \quad$ Equation $tx + y = ta$ or $t^2x + ty - at^2 = 0 \quad$ ...(ii)

By adding Eqs. (i) and (ii), we get

$$x(1 + t^2) = 0$$

$\Rightarrow \quad x = 0 \quad [\because 1 + t^2 \neq 0]$

Hence, the point of intersection of Eq. (i) and (ii) lies on $x = 0$ i.e. on Y-axis (which is tangent at the vertex of a parabola).

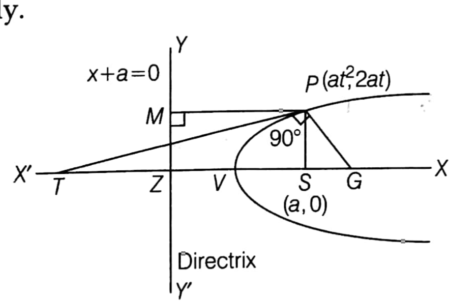
(5) If S be the focus of the parabola and tangent and normal at any point P meet its axis in T and G respectively then $\boxed{ST = SG = SP}$.

Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$, then equation of tangent and normal at $P(at^2, 2at)$ are

$ty = x + at^2$ and $y = -tx + 2at + at^2$, respectively.

Since, tangent and normal meet its axis in T and G.

$\therefore$ Coordinates of T' and G are $(-at^2, 0)$ and $(2a + at^2, 0)$ respectively.



$\therefore \quad SP = PM = a + at^2$

$SG = VG - VS = 2a + at^2 - a$

$= a + at^2$

and $\quad ST = VS + VT = a + at^2$

Hence, $SP = SG = ST$

(6) If S be the focus and SH be perpendicular to the tangent at P, then H lies on the tangent at the vertex and $\boxed{SH^2 = OS \cdot SP}$, where O is the vertex of the parabola.

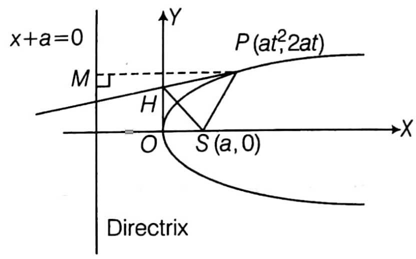
Let $P(at^2, 2at)$ be any point on the parabola

$$y^2 = 4ax \quad \text{...(i)}$$

then, tangent at $P(at^2, 2at)$ to the parabola Eq. (i) is

$$ty = x + at^2$$

It meets the tangent at the vertex i.e. $x = 0$.



$\therefore$ Coordinates of H is $(0, at)$

and $\quad SP = PM = a + at^2 \Rightarrow \quad OS = a$

and $\quad SH = \sqrt{(a - 0)^2 + (0 - at)^2} = \sqrt{a^2 + a^2t^2}$

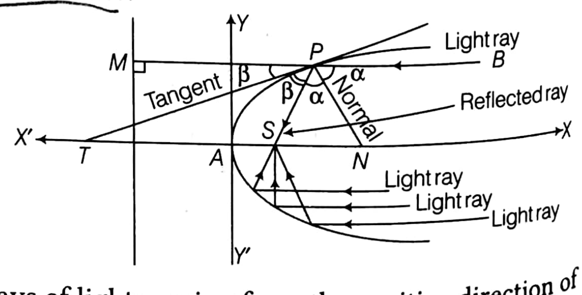
or $(SH)^2 = a\{a(1 + t^2)\} = OS \cdot SP$.

Reflection Property of Parabola

The tangent (PT) and normal (PN) of the parabola

$$y^2 = 4ax$$

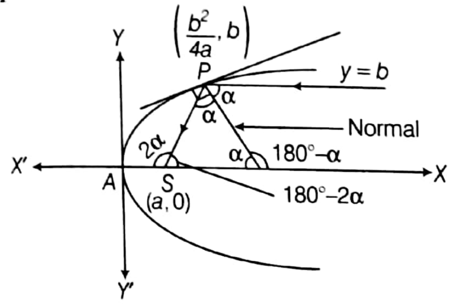
at P are the internal and external bisectors of $\angle SPM$ and BP is parallel to the axis of the parabola and $\angle BPN = \angle SPN$.



All rays of light coming from the positive direction of X-axis and parallel to the axis of the parabola after reflection pass through the focus of the parabola.

| Example 54 A ray of light is coming along the line $y = b$ from the positive direction of X-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $y^2 = 4ax$. Find the equation of the reflected ray and show that it passes through the focus of the parabola. Both a and b are positive.

Sol. Given parabola is $y^2 = 4ax$



Equation of tangent at $P\left(\frac{b^2}{4a}, b\right)$ is $yb = 2a\left(x + \frac{b^2}{4a}\right)$,

Slope of tangent is $\frac{2a}{b}$.

Hence, slope of normal $= -\frac{b}{2a} = \tan(180° - \alpha)$.

$$\therefore \quad \tan\alpha = \frac{b}{2a}$$

$\therefore$ Slope of reflected ray $= \tan(180° - 2\alpha)$
$= -\tan 2\alpha$

$$= -\left\{\frac{2\tan\alpha}{1 - \tan^2\alpha}\right\} = -\left\{\frac{2 \cdot \frac{b}{2a}}{1 - \frac{b^2}{4a^2}}\right\} = -\frac{4ab}{(4a^2 - b^2)}$$

Hence, equation of reflected ray is

$$y - b = -\frac{4ab}{(4a^2 - b^2)}\left(x - \frac{b^2}{4a}\right)$$

$$\Rightarrow \quad (y - b)(4a^2 - b^2) = -(4ax - b^2)$$

which obviously passes through the focus $S(a, 0)$.

Study of Parabola of the Form $(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$

Given equation can be written as

$$(\alpha x + \beta y)^2 = -2gx - 2fy - c$$

Now, add an arbitrary constant λ in the square root of the second degree terms. Then the equation will be of the form

i.e. $$(\alpha x + \beta y + \lambda)^2 = xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda) \quad \text{...(i)}$$

Now, choose λ such that the lines

$\alpha x + \beta y + \lambda = 0$ and $xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda) = 0$ are perpendicular

i.e. (slope of $\alpha x + \beta y + \lambda = 0$) $\times$ (slope of $xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda) = 0$) $= -1$

$$\Rightarrow \quad -\frac{\alpha}{\beta} \times -\frac{f_1(\lambda)}{f_2(\lambda)} = -1$$

$$\Rightarrow \quad \alpha f_1(\lambda) + \beta f_2(\lambda) = 0 \quad \text{...(ii)}$$

Now, substitute the value of λ in Eq. (i) from Eq. (ii).

Multiply and divide $(\alpha^2 + \beta^2)$ in LHS of Eq. (i)

i.e. $$(\alpha x + \beta y + \lambda)^2 = (\alpha^2 + \beta^2)\left(\frac{\alpha x + \beta y + \lambda}{\sqrt{\alpha^2 + \beta^2}}\right)^2$$

and RHS of Eq. (i) by $\sqrt{(\alpha^2 + \beta^2)}$

i.e. $$xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda)$$
$$= \sqrt{(\alpha^2 + \beta^2)}\left(\frac{xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda)}{\sqrt{(\alpha^2 + \beta^2)}}\right)$$

Then, Eq. (i) reduce in the form

$$\left(\frac{\alpha x + \beta y + \lambda}{\sqrt{\alpha^2 + \beta^2}}\right)^2 = 4\rho\left(\frac{\beta x - \alpha y + \mu}{\sqrt{(\alpha^2 + \beta^2)}}\right)$$

which is of the form $Y^2 = 4\rho X$

$$Y = \frac{\alpha x + \beta y + \lambda}{\sqrt{(\alpha^2 + \beta^2)}}, X = \frac{\beta x - \alpha y + \mu}{\sqrt{(\alpha^2 + \beta^2)}} \text{ and } 4\rho = \frac{1}{\sqrt{(\alpha^2 + \beta^2)}}.$$

Latusrectum is $4\rho = \frac{1}{\sqrt{(\alpha^2 + \beta^2)}}$.

Axis is $Y = 0$ or $\alpha x + \beta y + \lambda = 0$.

Equation of tangent at vertex is

$$X = 0 \quad \text{or} \quad \beta x - \alpha y + \mu = 0.$$

Vertex is the point of intersection of

$$X = 0 \quad \text{and} \quad Y = 0$$

i.e. $\beta x - \alpha y + \mu = 0$ and $\alpha x + \beta y + \lambda = 0$.

Equation of directrix is $\boxed{X + \rho = 0.}$

Equation of latusrectum is $X - \rho = 0$.

Focus Since, axis and latusrectum intersect at the focus S its coordinates are detained by solving

$$X - \rho = 0 \quad \text{and} \quad Y = 0.$$

| Example 55. Find the length of latusrectum of the parabola $(a^2 + b^2)(x^2 + y^2) = (bx + ay - ab)^2$.

Sol. Given equation may be written as

$$a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2 = b^2x^2 + a^2y^2 + a^2b^2 + 2abxy - 2a^2by - 2ab^2x$$

$$\Rightarrow \quad a^2x^2 - 2abxy + b^2y^2 = -2ab^2x - 2a^2by + a^2b^2$$

$$\Rightarrow \quad (ax - by)^2 = -2ab\left(bx + ay - \frac{ab}{2}\right)$$

Since, $ax - by = 0$ and $bx + ay - \dfrac{ab}{2} = 0$ are perpendicular.

$$\therefore (a^2 + b^2)\left(\frac{ax - by}{\sqrt{a^2 + b^2}}\right)^2 = -2ab\sqrt{(a^2 + b^2)}\left(\frac{bx + ay - \frac{ab}{2}}{\sqrt{(a^2 + b^2)}}\right)$$

$$\Rightarrow \quad \left(\frac{ax - by}{\sqrt{a^2 + b^2}}\right)^2 = \frac{-2ab}{\sqrt{(a^2 + b^2)}}\left(\frac{bx + ay - \frac{ab}{2}}{\sqrt{(a^2 + b^2)}}\right)$$

which is of the form $Y^2 = -4\rho X$.

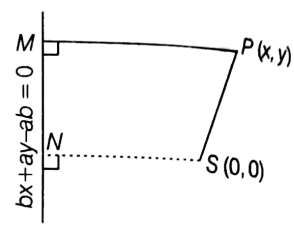
Therefore, the latusrectum $= 4\rho = \dfrac{2ab}{\sqrt{(a^2 + b^2)}}$.

Aliter :

Given equation may be written as

$$x^2 + y^2 = \frac{(bx + ay - ab)^2}{(a^2 + b^2)}$$

$$\Rightarrow \sqrt{(x^2 + y^2)} = \frac{|bx + ay - ab|}{\sqrt{(a^2 + b^2)}}$$



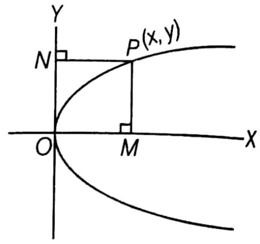
$$\Rightarrow \quad \sqrt{(x - 0)^2 + (y - 0)^2} = \frac{|bx + ay - ab|}{\sqrt{(a^2 + b^2)}}$$

which is of the form $SP = PM$.

Since, distance from focus S to ($bx + ay - ab = 0$) $= \dfrac{1}{2}(4\rho)$

$$\Rightarrow \quad \frac{1}{2}(4\rho) = \frac{ab}{\sqrt{a^2 + b^2}}$$

$$\therefore \quad 4\rho = \frac{2ab}{\sqrt{(a^2 + b^2)}}$$



Remark

Consider the equation of parabola is $y^2 = 4ax$.

i.e. $(MP)^2 =$ (Latusrectum) NP.

i.e. if P is any point on the given parabola, then (the distance of P from its axis)^2 = (Latusrectum) (The distance of P from the tangent at its vertex).

Exercise for Session 3

1. If m_1, m_2 are slopes of the two tangents that are drawn from (2, 3) to the parabola $y^2 = 4x$, then the value of $\dfrac{1}{m_1} + \dfrac{1}{m_2}$ is

(a) -3 (b) 3 (c) $\dfrac{2}{3}$ (d) $\dfrac{3}{2}$

2. The angle between the tangents drawn from the origin to the parabola $y^2 = 4a(x - a)$ is

(a) 90° (b) 30° (c) $\tan^{-1}\left(\dfrac{1}{2}\right)$ (d) 45°

3. If (a, b) is the mid-point of chord passing through the vertex of the parabola $y^2 = 4x$, then

(a) $a = 2b$ (b) $2a = b$ (c) $a^2 = 2b$ (d) $2a = b^2$

4. The diameter of the parabola $y^2 = 6x$ corresponding to the system of parallel chords $3x - y + c = 0$ is

(a) $y - 1 = 0$ (b) $y - 2 = 0$ (c) $y + 1 = 0$ (d) $y + 2 = 0$

5. From the point $(-1, 2)$ tangent lines are drawn to the parabola $y^2 = 4x$, the area of triangle formed by chord of contact and the tangents is given by

(a) 8 (b) $8\sqrt{3}$ (c) $8\sqrt{2}$ (d) None of these

6. For parabola $x^2 + y^2 + 2xy - 6x - 2y + 3 = 0$, the focus is

(a) $(1 - 1)$ (b) $(-1, 1)$ (c) $(3, 1)$ (d) None of these

7. The locus of the mid-point of that chord of parabola which subtends right angle on the vertex will be

(a) $y^2 - 2ax + 8a^2 = 0$ (b) $y^2 = a(x - 4a)$ (c) $y^2 = 4a(x - 4a)$ (d) $y^2 + 3ax + 4a^2 = 0$

8. A ray of light moving parallel to the X-axis gets reflected from a parabolic mirror whose equation is $(y-2)^2 = 4(x+1)$. After reflection, the ray must pass through the point

(a) $(-2, 0)$ (b) $(-1, 2)$ (c) $(0, 2)$ (d) $(2, 0)$

9. Prove that the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle α is $(x+a)^2 \tan^2 \alpha = y^2 - 4ax$.

10. Find the locus of the middle points of the chords of the parabola $y^2 = 4ax$ which pass through the focus.

11. From the point $P(-1, 2)$ tangents are drawn to the parabola $y^2 = 4x$. Find the equation of the chord of contact. Also, find the area of the triangle formed by the chord of contact and the tangents.

Shortcuts and Important Results to Remember

1 Second degree terms in the equation of a parabola should make perfect squares.

2 If l_1 and l_2 are the lengths of segments of a focal chord then the latusrectum of the parabola is $\dfrac{4l_1l_2}{l_1 + l_2}$.

3 If α be the inclination of a focal chord with axis of the parabola then its length is $4a\operatorname{cosec}^2\alpha$.

4 If tangents of $y^2 = 4ax$ at $P(t_1)$ and $Q(t_2)$ meets at R, then area of ΔPQR is $\dfrac{1}{2} a^2 (t_1 - t_2)^3$.

5 The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.

6 The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents, at these points.

7 The equation of the common tangent to the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $\boxed{x + y + a = 0.}$

8 If the chord joining t_1, t_2 and t_3, t_4 pass through a point $(c, 0)$ on the axis, then
$$t_1t_2 = t_3t_4 = -c/a.$$

9 If the normals to the parabola $y^2 = 4ax$ at the points t_1 and t_2 intersect again on the parabola at the point 't_3', then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 and t_2 passes through a fixed point $(-2a, 0)$.

10 If the length of a focal chord of $y^2 = 4ax$ at a distance b from the vertex is c then $b^2c = 4a^3$.

11 From an external point only one normal can be drawn.

12 If a normal to $y^2 = 4ax$ makes an angle θ with the axis of $y^2 = 4ax$ then it will cut the curve again at an angle of $\tan^{-1}\left(\dfrac{\tan\theta}{2}\right)$.

13 The orthocentre of any triangle formed by the three tangents to a parabola $y^2 = 4ax$ 't_1', 't_2' and 't_3' lies on the directrix and has the coordinates $(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$.

14 Normals at the end points of the latusrectum of a parabola $y^2 = 4ax$ intersect at right angle on the axis of the parabola and their point of intersection is $(3a, 0)$.

15 A line ray parallel to axis of the parabola after reflection passes through the focus.

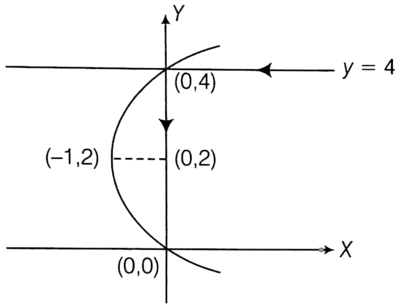
JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

- ***Ex. 1*** *A ray of light travels along a line $y = 4$ and strikes the surface of curves $y^2 = 4(x + y)$, then the equation of the line along which the reflected ray travels is*

  (a) $x = 0$ (b) $x = 2$
  (c) $x + y = 4$ (d) $2x + y = 4$

Sol. (a) The given curve is $(y-2)^2 = 4(x+1)$

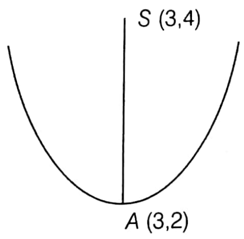


The focus is $(0, 2)$.
The point of intersection of the curve and $y = 4$ is $(0, 4)$. From the reflection property of parabola the reflected ray passes through the focus.
Therefore, $x = 0$ is the reflected ray.

- ***Ex. 2*** *A parabola is drawn with focus at $(3, 4)$ and vertex at the focus of the parabola $y^2 - 12x - 4y + 4 = 0$. The equation of the parabola is*

  (a) $x^2 - 6x - 8y + 25 = 0$ (b) $y^2 - 8x - 6y + 25 = 0$
  (c) $x^2 - 6x + 8y - 25 = 0$ (d) $x^2 + 6x - 8y - 25 = 0$

Sol. (a) $y^2 - 12x - 4y + 4 = 0 \Rightarrow (y-2)^2 = 12x$



Its vertex is $(0, 2)$ and $a = 3$,
its focus $= (3, 2)$.
Hence, for the required parabola; focus is $(3, 4)$, vertex $= (3, 2)$ and $a = 2$,
Hence, the equation of the parabola is

$$(x-3)^2 = 4(2)(y-2)$$

or $\quad x^2 - 6x - 8y + 25 = 0$

- ***Ex. 3*** *Two parabolas have the same focus. If their directrices are the X-axis and the Y-axis, respectively, then the slope of their common chord is*

  (a) ± 1 (b) $\frac{4}{3}$ (c) $\frac{3}{4}$ (d) None of these

Sol. (a) Let the focus be (a, b)

Then, equations are $(x-a)^2 + (y-b)^2 = y^2$ and
$$(x-a)^2 + (y-b)^2 = x^2$$

If $\quad S_1 \equiv (x-a)^2 + (y-b)^2 - y^2 = 0$

and $\quad S_2 \equiv (x-a)^2 + (y-b)^2 - x^2 = 0$

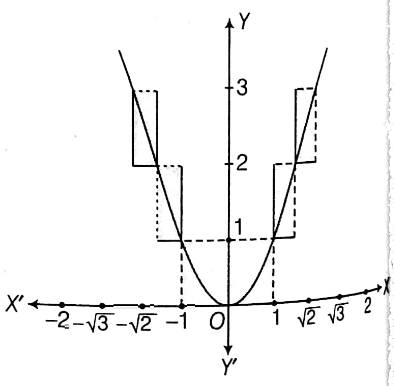
$\therefore$ Equation of common chord $S_1 - S_2 = 0$ gives
$$x^2 - y^2 = 0 \quad \text{or} \quad y = \pm x$$

Hence, slope of common chord is ± 1.

- ***Ex. 4*** *Let us define a region R in xy-plane as a set of points (x, y) satisfying $[x^2] = [y]$ (where $[x]$ denotes greatest integer $\leq x$), then the region R defines*

  (a) a parabola whose axis is horizontal
  (b) a parabola whose axis is vertical
  (c) integer point of the parabola $y = x^2$
  (d) None of the above

Sol. (d) $\because [x^2] = [y]$



If $\quad 0 \leq y < 1$,

then, $\quad [y] = 0$

$\therefore \quad [x^2] = 0$

$\quad 0 \leq x^2 < 1$

$\Rightarrow \quad x \in (-1, 1)$

for $\quad 1 \leq y < 2$,

then $\quad [y] = 1$

$\therefore \quad [x^2] = 1$

$\Rightarrow \quad 1 \leq x^2 < 2$

$\Rightarrow \quad x \in (-\sqrt{2}, -1] \cup [1, \sqrt{2})$

for $\quad 2 \leq y < 3$, then $[y] = 2$

then, $\quad [x^2] = 2 \Rightarrow 2 \leq x^2 < 3$

$\therefore \quad x \in (-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3})$

The graph of the region will not only contain of the parabola $y = x^2$ but $[x^2] = [y]$ contain points within the rectangles of side 1, 2, ; 1, $\sqrt{2} - 1$; 1, $\sqrt{3} - \sqrt{2}$ etc.

Hence, a, b, c are incorrects.

ellipse pg. no 463 last line.

Must.

• Ex. 5 *The minimum area of circle which touches the parabolas $y = x^2 + 1$ and $x = y^2 + 1$ is*

(a) $\frac{9\pi}{16}$ sq units (b) $\frac{9\pi}{32}$ sq units

(c) $\frac{9\pi}{8}$ sq units (d) $\frac{9\pi}{4}$ sq units

If we change y and x then other eq form

Sol. (b) The parabolas $y = x^2 + 1$ and $x = y^2 + 1$ are symmetrical about $y = x$.

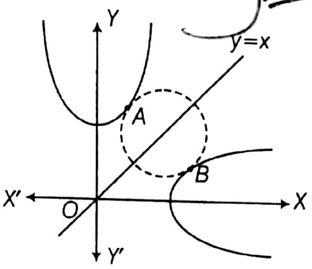
Therefore, the tangent at point A is parallel to $y = x$

then $\frac{dy}{dx} = 2x = 1$

or $x = \frac{1}{2}, y = \frac{5}{4}$

$\therefore$ $A \equiv \left(\frac{1}{2}, \frac{5}{4}\right)$

and $B \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$



Hence, Radius $(r) = \frac{1}{2} AB = \frac{1}{2}\sqrt{\left(\frac{1}{2} - \frac{5}{4}\right)^2 + \left(\frac{5}{4} - \frac{1}{2}\right)^2} = \frac{3\sqrt{2}}{8}$

$\therefore$ Area $= \pi r^2 = \frac{9\pi}{32}$ sq units

Must / Mast

• Ex. 6 *Let the line $lx + my = 1$ cut the parabola $y^2 = 4ax$ in the points A and B. Normals at A and B meet at point C. Normal from C other than these two meet the parabola at D, then the coordinate of D are*

(a) $(a, 2a)$ (b) $\left(\frac{4am}{l^2}, \frac{4a}{l}\right)$

(c) $\left(\frac{2am^2}{l^2}, \frac{2a}{l}\right)$ (d) $\left(\frac{4am^2}{l^2}, \frac{4am}{l}\right)$

Sol. (d) Let $A \equiv (am_1^2, -2am_1)$ and $B \equiv (am_2^2, -2am_2)$

Now, A and B lie on $lx + my = 1$

$\Rightarrow$ $l(am_1^2) + m(-2am_1) = 1$ ...(i)

and $l(am_2^2) + m(-2am_2) = 1$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), then

$$la(m_1^2 - m_2^2) - 2am(m_1 - m_2) = 0$$

$\Rightarrow$ $a(m_1 - m_2) \neq 0$

$\therefore$ $l(m_1 + m_2) - 2m = 0$

$\Rightarrow$ $m_1 + m_2 = \frac{2m}{l}$ ...(iii)

Let $D \equiv (am_3^2, -2am_3)$ and $C \equiv (h, k)$

$\therefore$ Equation of normal in terms of slope

$$y = Mx - 2aM - aM^3$$

then $aM^3 - (h - 2a)M + k = 0$

$\therefore$ $m_1 + m_2 + m_3 = 0$

$\Rightarrow$ $\frac{2m}{l} + m_3 = 0$

$\therefore$ $m_3 = -\frac{2m}{l}$

$\therefore$ $D \equiv \left(a\left(\frac{-2m}{l}\right)^2, -2a\left(\frac{-2m}{l}\right)\right)$

$\Rightarrow$ $D \equiv \left(\frac{4am^2}{l^2}, \frac{4am}{l}\right)$

• Ex. 7 *If d is the distance between the parallel tangents with positive slope to $y^2 = 4x$ and $x^2 + y^2 - 2x + 4y - 11 = 0$, then*

(a) $10 < d < 20$ (b) $4 < d < 6$

(c) $d < 4$ (d) None of these

Sol. (c) Tangent to the parabola $y^2 = 4x$ having slope m is

$$y = mx + \frac{1}{m}$$

and tangent to the circle $(x-1)^2 + (y+2)^2 = 4^2$ having slope m is

$$y + 2 = m(x-1) + 4\sqrt{(1+m^2)}$$

$\therefore$ Distance between tangents $(d) = \left|\frac{4\sqrt{(1+m^2)} - m - 2 - \frac{1}{m}}{\sqrt{(1+m^2)}}\right|$

$$= \left|4 - \frac{2}{\sqrt{(1+m^2)}} - \frac{\sqrt{(1+m^2)}}{m}\right|$$

As $m > 0$ (given)

we get $d < 4$

Must

• Ex. 8 *Two parabolas C and D intersect at two different points, where C is $y = x^2 - 3$ and D is $y = kx^2$. The intersection at which the x value is positive is designated point A, and $x = a$ at this intersection the tangent line l at A to the curve D intersects curve C at point B, other than A. If x-value of point B is 1, then a is equal to*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. (c) $C: y = x^2 - 3$ and $D: y = kx^2$

Solving C and D, then

$$kx^2 = x^2 - 3$$ ...(i)

or $x^2 = \frac{3}{1-k}$,

then, $y = \frac{3k}{1-k}$

$\therefore$ $A \equiv \left(\sqrt{\frac{3}{1-k}}, \frac{3k}{1-k}\right)$ (given x-value of A is positive)

and $a = \sqrt{\left(\frac{3}{1-k}\right)}$

then $A \equiv (a, ka^2) \equiv (a, a^2 - 3)$ [from Eq. (i)] tangent 'l' at A to the curve D is

$$\frac{y+a^2-3}{2}=kx\cdot a$$

$\Rightarrow$ $y+a^2-3=2ax\left(1-\frac{3}{a^2}\right)$ [from Eq. (i)] ...(ii)

$\therefore$ $B\equiv(1,-2)\ (a\neq 1)$

From Eq. (ii), $-2+a^2-3=2a\left(1-\frac{3}{a^2}\right)$

$\Rightarrow$ $a^3-5a=2a^2-6$

$\Rightarrow$ $a^3-2a^2-5a+6=0$

$\Rightarrow$ $(a-1)(a+2)(a-3)=0$

$\therefore$ $a=3$ $(\because a\neq 1, a\neq -2)$

• **Ex. 9** $\min[(x_1-x_2)^2+(3+\sqrt{(1-x_1^2)}-\sqrt{4x_2})^2], \forall$ $x_1, x_2\in R$, *is*

(a) $4\sqrt{5}+1$ (b) $3-2\sqrt{2}$ (c) $\sqrt{5}+1$ (d) $\sqrt{5}-1$

Sol. (b) Let $y_1=3+\sqrt{(1-x_1^2)}$ and $y_2=\sqrt{4x_2}$

or $x_1^2+(y_1-3)^2=1$ and $y_2^2=4x_2$

Thus, (x_1, y_1) lies on the circle $x^2+(y-3)^2=1$ and (x_2, y_2) lies on the parabola $y^2=4x$.

Thus, the given expression is the shortest distance between the curve $x^2+(y-3)^2=1$ and $y^2=4x$.

Now, the shortest distance always occurs along the common normal to the curves and normal to the circle passes through the centre of the circle.

Normal to parabola $y^2=4x$ is $y=mx-2m-m^3$. It passes through centre of circle $(0, 3)$.

Therefore, $3=-2m-m^3 \Rightarrow m^3+2m+3=0$ which has only one real value $m=-1$.

Hence, the corresponding point on the parabola is $(1, 2)$

$\therefore \sqrt{(x_1-x_2)^2+(3+\sqrt{(1-x_1^2)}-\sqrt{4x_2})^2}=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

= Distance between $(1, 2)$ and $(0, 3)$ – radius

$=\sqrt{(1+1)}-1=(\sqrt{2}-1)$

or $(x_1-x_2)^2+(3+\sqrt{(1-x_1^2)}-\sqrt{4x_2})^2=(\sqrt{2}-1)^2$

$\therefore$ $\min\cdot[(x_1-x_2)^2+(3+\sqrt{(1-x_1^2)}-\sqrt{4x_2})^2]=3-2\sqrt{2}$

• **Ex. 10** *The condition that the parabolas* $y^2=4c(x-d)$ *and* $y^2=4ax$ *have a common normal other than X-axis* $(a>0, c>0)$ *is*

(a) $2a<2c+d$ (b) $2c<2a+d$

(c) $2d<2a+c$ (d) $2d<2c+a$

Sol. (a) Normals of parabolas $y^2=4ax$ and $y^2=4c(x-d)$ in terms of slope are

$$y=mx-2am-am^3 \quad \text{...(i)}$$

and $$y=m(x-d)-2cm-cm^3 \quad \text{...(ii)}$$

Subtracting Eqs. (ii) from (i), then

$$md-2am-am^3+2cm+cm^3=0$$

$$m\neq 0$$

$\therefore$ $d-2a-am^2+2c+cm^2=0$

$\Rightarrow$ $(a-c)m^2=d-2a+2c$

$\Rightarrow$ $m^2=\frac{d-2a+2c}{(a-c)}$

$\Rightarrow$ $\frac{d}{a-c}-2>0$

$\Rightarrow$ $d>2a-2c$

$\Rightarrow$ $2a<2c+d$

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which MORE THAN ONE** may be correct.

• **Ex. 11** *The locus of the mid-point of the focal radii of a variable point moving on the parabola* $y^2=4ax$ *is a parabola whose*

(a) latusrectum is half the latusrectum of the original parabola

(b) vertex is $\left(\frac{a}{2}, 0\right)$

(c) directrix is Y-axis

(d) focus has the coordinates $(a, 0)$

Sol. (a, b, c, d) Let $P(at^2, 2at)$ be a point on the parabola $y^2=4ax$ with focus $S(a, 0)$

Now, mid-point of focal radii SP is $M\left(\frac{a}{2}(t^2+1), at\right)$.

Let $x=\frac{a}{2}(t^2+1)$ and $y=at$

or $x=\frac{a}{2}\left(\frac{y^2}{a^2}+1\right)$

$\Rightarrow$ $y^2=2ax-a^2$ or $y^2=2a\left(x-\frac{a}{2}\right)$

which is a parabola with vertex $\left(\frac{a}{2}, 0\right)$ and latusrectum is $2a$, directrix is $x-\frac{a}{2}=\frac{a}{2}$ i.e. $x=0$ (Y) and focus $\left(\frac{a}{2}+\frac{a}{2}, 0\right)$ i.e. $(a, 0)$

• **Ex. 12** *If P_1P_2 and Q_1Q_2, two focal chords of a parabola $y^2 = 4ax$ at right angles, then*

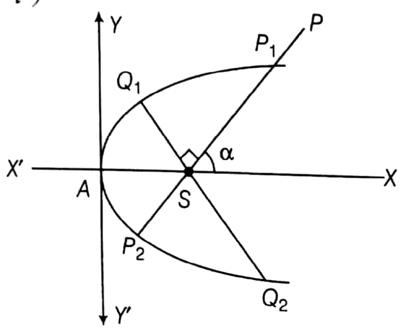
(a) area of the quadrilateral $P_1Q_1P_2Q_2$ is minimum when the chords are inclined at an angle $\pi/4$ to the axis of the parabola.

(b) minimum area is twice the area of the square on the latusrectum of the parabola.

(c) minimum area of quadrilateral $P_1Q_1P_2Q_2$ cannot be found.

(d) minimum area is thrice the area of the square on the latusrectum of the parabola.

Sol. (a, b) Let coordinates of P_1 are $(at^2, 2at)$, then coordinates of P_2 are $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.



Let focal chord P_1P_2 makes an angle α with X-axis, then

$$\tan\alpha = \frac{2at - \left(-\frac{2a}{t}\right)}{at^2 - \frac{a}{t^2}} = \frac{2}{t - \frac{1}{t}}$$

$$\Rightarrow \quad t - \frac{1}{t} = 2\cot\alpha \quad \text{... (i)}$$

Now, $\quad P_1P_2 = a\left(t + \frac{1}{t}\right)^2 = a\left\{\left(t - \frac{1}{t}\right)^2 + 4\right\}$

$= a\{4\cot^2\alpha + 4\} = 4a\,\mathrm{cosec}^2\alpha$ [from Eq. (i)]

Similarly, $\quad Q_1Q_2 = 4a\,\mathrm{cosec}^2(90 - \alpha) = 4a\sec^2\alpha$

$\therefore$ Area of quadrilateral $P_1Q_1P_2Q_2 = \frac{1}{2}(P_1P_2)(Q_1Q_2)$

$= 8a^2\sec^2\alpha\,\mathrm{cosec}^2\alpha = 32a^2\,\mathrm{cosec}^2 2\alpha$

$\therefore$ Minimum area $= 32a^2 = 2(\text{latusrectum})^2$ and is inclined at

$\alpha = \pi/4 \quad (\because \mathrm{cosec}\,2\alpha = 1)$

• **Ex. 13** *The equation of the line that touches the curves $y = x|x|$ and $x^2 + (y-2)^2 = 4$, where $x \neq 0$, is*

(a) $y = 4\sqrt{5}x + 20$ (b) $y = 4\sqrt{3}x - 12$

(c) $y = 0$ (d) $y = -4\sqrt{5}x - 20$

Sol. (a, b, c) $\because y = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

$\because$ Equation of tangent in terms of slope (m) of $x^2 = 4ay$ is

$$y = mx - am^2 \quad \text{... (i)}$$

Also, line (i) touches the circle $x^2 + (y-2)^2 = 4$, then

$$\frac{|2 + am^2|}{\sqrt{(1 + m^2)}} = 2$$

$$\Rightarrow \quad 4 + a^2m^4 + 4am^2 = 4 + 4m^2$$

$$\therefore \quad m^2 = \frac{4 - 4a}{a^2} \text{ and } m^2 = 0$$

Put $\quad 4a = 1 \quad$ for $y = x^2$, $x \geq 0$, then $m^2 = 48$

and put $4a = -1$ for $y = -x^2$, $x < 0$, then $m^2 = 80$

$\therefore$ Common tangents are

$$y = 0,\ y = 4\sqrt{3}x - 12 \text{ and } y = 4\sqrt{5}x + 20$$

• **Ex. 14** *Let V be the vertex and L be the latusrectum of the parabola $x^2 = 2y + 4x - 4$. Then, the equation of the parabola whose vertex is at V, latusrectum $L/2$ and axis is perpendicular to the axis of the given parabola.*

(a) $y^2 = x - 2$ (b) $y^2 = x - 4$

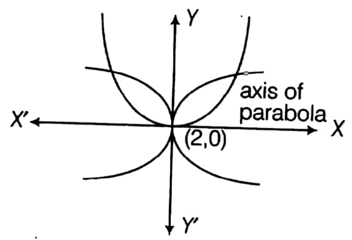
(c) $y^2 = 2 - x$ (d) $y^2 = 4 - x$

Sol. (a, c) Given parabola is $x^2 = 2y + 4x - 4$

$$\Rightarrow \quad (x-2)^2 = 2y$$

Vertex of the parabola is (2, 0) and length of latusrectum $= 2$

$\therefore \quad V(2, 0)$ and $L = 2$



Length of latusrectum of required parabola $= L/2 = 1$

$\therefore$ Equation of the required parabola is $(y - 0)^2 = \pm 1(x - 2)$

$$\Rightarrow \quad y^2 = x - 2 \text{ or } y^2 = -x + 2$$

• **Ex. 15** *Consider a circle with its centre lying on the focus of the parabola $y^2 = 2ax$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and the parabola is*

(a) $\left(\frac{a}{2}, a\right)$ (b) $\left(\frac{a}{2}, -a\right)$

(c) $\left(-\frac{a}{2}, a\right)$ (d) $\left(-\frac{a}{2}, -a\right)$

Sol. (a, b) Given parabola is $y^2 = 2ax$ ...(i)

$\therefore$ Focus and equation of directrix are $S\left(\frac{a}{2}, 0\right)$ and $x = -\frac{a}{2}$ respectively.

$\therefore$ Equation of circle is $\left(x - \frac{a}{2}\right)^2 + (y - 0)^2 = a^2$ ... (ii)

($\because$ radius = distance between focus and directrix)

From Eqs. (i) and (ii),

$$\left(x-\frac{a}{2}\right)^2+2ax=a^2 \Rightarrow x^2+ax-\frac{3a^2}{4}=0$$

or $\quad x=\frac{a}{2}, x\neq-\frac{3a}{2} \quad (\because \text{ for } x=-\frac{3a}{2}, y=\text{imaginary})$

From Eq. (i),

$$y^2=a^2$$

$\therefore \quad y=\pm a$

Hence, required point of intersection are $\left(\frac{a}{2},\pm a\right)$.

JEE Type Solved Examples : Paragraph Based Questions

- This section contains **2 Solved Paragraphs** based upon each of the paragraph **3 multiple choice question.** Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to 18)

Tangents are drawn to the parabola $y^2=4x$ at the point P which is the upper end of latusrectum.

16. Image of the parabola $y^2=4x$ in the tangent line at the point P is

(a) $(x+4)^2=16y$ (b) $(x+2)^2=8(y-2)$
(c) $(x+1)^2=4(y-1)$ (d) $(x-2)^2=2(y-2)$

17. Radius of the circle touching the parabola $y^2=4x$ at the point P and passing through its focus is

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

18. Area enclosed by the tangent line at P, X-axis and the parabola is

(a) $\frac{2}{3}$ sq units (b) $\frac{4}{3}$ sq units
(c) $\frac{14}{3}$ sq units (d) $\frac{16}{3}$ sq units

Sol. Upper end of latusrectum is $P(1,2)$

$\therefore$ The equation of tangent at $P(1,2)$ is

$$y\cdot 2=2(x+1)\Rightarrow x-y+1=0$$

16. (c) Any point on the given parabola is $(t^2,2t)$. The image of (h,k) of the point $(t^2,2t)$ on $x-y+1=0$ is given by

$$\frac{h-t^2}{1}=\frac{k-2t}{-1}=\frac{-2(t^2-2t+1)}{1+1}$$

$\Rightarrow \quad h=2t-1$ and $k=t^2+1$

or $\quad k=\left(\frac{h+1}{2}\right)^2+1 \Rightarrow (h+1)^2=4(k-1)$

The required equation of image is $(x+1)^2=4(y-1)$.

17. (c) $\because$ Focus is $S(1,0)$ and P is $(1,2)$.

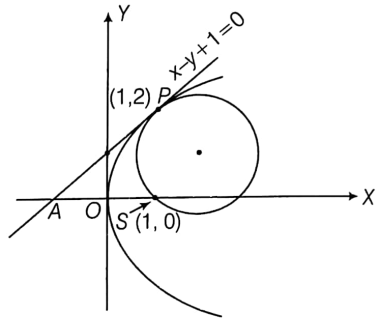
Equation of circle touching the parabola at (1, 2) is $(x-1)^2+(y-2)^2+\lambda(x-y+1)=0$ it passes through (1, 0).

Therefore, $\quad 4+2\lambda=0$ or $\lambda=-2$

Thus, the required circle is

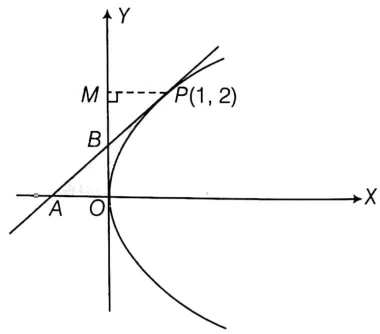
$(x-1)^2+(y-2)^2-2(x-y+1)=0$

or $\quad x^2+y^2-4x-2y+3=0$



Its radius is $\sqrt{(4+1-3)}=\sqrt{2}$.

18. (a) Area bounded by $AOPA$ = Area of ΔAOB + Area of $OPBO$



$$=\frac{1}{2}\times1\times1+\int_0^2\frac{y^2}{4}dy-\frac{1}{2}\times1\times1=\left[\frac{y^3}{12}\right]_0^2=\frac{8}{12}=\frac{2}{3}\text{ sq unit}$$

Paragraph II

(Q. Nos. 19 to 21)

Let C_1 and C_2 be respectively, the parabolas $x^2=y-1$ and $y^2=x-1$. Let P be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflections of P and Q, respectively with respect to the line $y=x$.

19. P_1 and Q_1 lie on

(a) C_1 and C_2 respectively (b) C_2 and C_1 respectively
(c) Cannot be determined (d) None of these

20. If the point $P(\lambda, \lambda^2+1)$ and $Q(\mu^2+1, \mu)$, then P_1 and Q_1 are

(a) (λ^2+1, λ) and (μ^2+1, μ) (b) (λ^2+1, λ) and (μ, μ^2+1)
(c) (λ, λ^2+1) and (μ, μ^2+1) (d) (λ, λ^2+1) and (μ^2+1, μ)

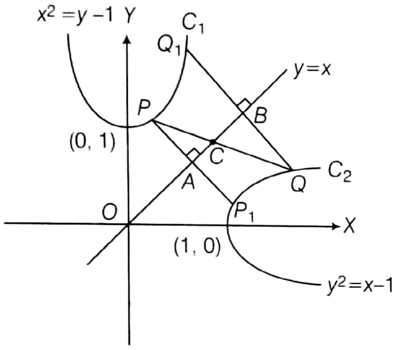
21. Arithmetic mean of PP_1 and QQ_1 is always less than

(a) PQ (b) $\frac{1}{2}PQ$ (c) $2PQ$ (d) $\frac{3}{2}PQ$

Sol. Since, the reflection of a point (p, q) with respect to line $y = x$ is (q, p).

Let $P(\lambda, \lambda^2+1)$ and $Q(\mu^2+1, \mu)$ be points on C_1 and C_2, respectively.

$\therefore$ Reflection of $P(\lambda, \lambda^2+1)$ with respect to line $y = x$ is $P_1(\lambda^2+1, \lambda)$ and reflection of $Q(\mu^2+1, \mu)$ with respect to line $y = x$ is $Q_1(\mu, \mu^2+1)$.



$\therefore$ P_1 and Q_1 are (λ^2+1, λ)

and (μ, μ^2+1) **(Ans. 20(b))**

Also, P_1 and Q_1 lie on $y^2 = x - 1$

and $x^2 = y - 1$.

Hence, P_1 and Q_1 lie on C_2 and C_1, respectively. **(Ans. 19(b))**

21. (a) $\because$ A is mid-point of PP_1 and B is mid-point of QQ_1.

$\therefore$ $PA = \frac{1}{2}PP_1$

and $QB = \frac{1}{2}QQ_1$ ...(i)

$\Rightarrow$ $PC \geq PA$ ...(ii)

and $QC \geq QB$ ...(iii)

On adding Eqs. (ii) and (iii), then

$$PC + QC \geq PA + QB$$

$$= \frac{1}{2}PP_1 + \frac{1}{2}QQ_1 \quad \text{[from Eq. (i)]}$$

$$= \left(\frac{PP_1 + QQ_1}{2}\right)$$

$\therefore$ $PC + QC \geq \left(\frac{PP_1 + QQ_1}{2}\right)$

$\Rightarrow$ $PQ \geq$ (AM of PP_1 and QQ_1)

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains **2 solved examples.** The answer to each example is **a single digit integer,** ranging from 0 to 9 (both inclusive).

• Ex. 22 *Points A, B, C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B, C taken in pairs intersect at points P, Q, R, then, the ratio of the areas of the ΔABC and ΔPQR is*

Sol. (2) Let $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$ be the points A, B and C respectively.

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2}\left|\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}\right| = a^2\left|\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix}\right|$$

$$= a^2 |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

Coordinates of P, Q and R are $(at_2t_3, a(t_2+t_3))$, $(at_3t_1, a(t_3+t_1))$, $(at_1t_2, a(t_1+t_2))$ respectively, then

$$\text{Area of } \Delta PQR = \frac{1}{2}\left|\begin{vmatrix} at_2t_3 & a(t_2+t_3) & 1 \\ at_3t_1 & a(t_3+t_1) & 1 \\ at_1t_2 & a(t_1+t_2) & 1 \end{vmatrix}\right| = \frac{a^2}{2}\left|\begin{vmatrix} t_2t_3 & t_2+t_3 & 1 \\ t_3t_1 & t_3+t_1 & 1 \\ t_1t_2 & t_1+t_2 & 1 \end{vmatrix}\right|$$

$$= \frac{a^2}{2}|\Sigma t_2t_3(t_3 - t_2)|$$

$$= \frac{a^2}{2}|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

$$= \frac{1}{2} \times \text{Area of } \Delta ABC$$

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = 2$$

• Ex. 23 *If the orthocentre of the triangle formed by the points t_1, t_2, t_3 on the parabola $y^2 = 4ax$ is the focus, the value of $|t_1t_2 + t_2t_3 + t_3t_1|$ is*

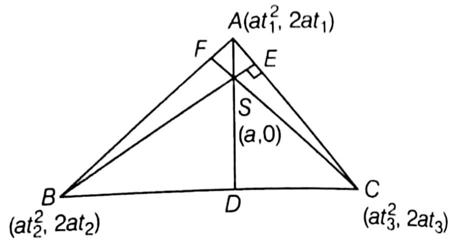
Sol. (5) $\because SA$ is perpendicular to BC[S is focus $(a, 0)$]

$$\Rightarrow \quad \left(\frac{2at_1 - 0}{at_1^2 - a}\right)\left(\frac{2at_3 - 2at_2}{at_3^2 - at_2^2}\right) = -1$$

$$\Rightarrow \quad \left(\frac{2t_1}{t_1^2 - 1}\right)\left(\frac{2}{t_3 + t_2}\right) = -1$$

$$\Rightarrow \quad 4t_1 + t_1^2(t_2 + t_3) = t_2 + t_3 \qquad \text{...(i)}$$

Similarly, $\quad 4t_2 + t_2^2(t_3 + t_1) = t_3 + t_1 \qquad$ ...(ii)



On subtracting Eq. (ii) from Eq. (i), then

$$4 + (t_1t_2 + t_2t_3 + t_3t_1) = -1 \Rightarrow t_1t_2 + t_2t_3 + t_3t_1 = -5$$

$$\therefore \quad |t_1t_2 + t_2t_3 + t_3t_1| = 5$$

JEE Type Solved Examples : Matching Type Questions

- This section contains **one solved example.** Example 24 has three statements (A, B and C) given in **Column I** and five statements (p, q, r, s and t) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

• Ex. 24 *Match the following.*

Column I	Column II
(A) If PQ is any focal chord of the parabola $y^2 = 32x$ and length of PQ can never be less than λ units, then λ is divisible by	(p) 2
(B) A tangent is drawn to the parabola $y^2 = 4x$ at the point 'P' whose abscissa lies in the interval $[1, 4]$. If maximum possible area of the triangle formed by the tangent at 'P', ordinate of the point 'P' and the X-axis is λ sq units, then λ is divisible by	(q) 3
(C) The normal at the ends of the latusrectum of the parabola $y^2 = 4x$ meet the parabola again at A and A'. If length $AA' = \lambda$ unit, then λ is divisible by	(r) 4
	(s) 6
	(t) 8

Sol. (A) $\rightarrow$ (p, r, t); (B) $\rightarrow$ (p, r, t); (C) $\rightarrow$ (p, q, r, s)

(A) Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$ and $S(a, 0)$ be the focus, then the other end of focal chord through P will be $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Then, length of focal chord $PQ = a\left(t + \frac{1}{t}\right)^2$

$$\because \quad t + \frac{1}{t} \geq 2 \Rightarrow a\left(t + \frac{1}{t}\right)^2 \geq 4a$$

or $\quad PQ \geq 32 \qquad [\because 4a = 32]$

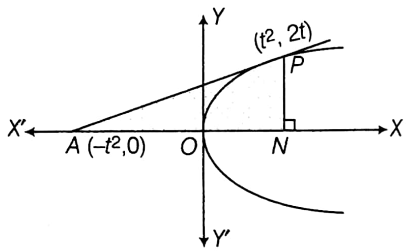
$\therefore \quad \lambda = 32$

(B) Equation of tangent to parabola $y^2 = 4x$ at $P(t^2, 2t)$ is

$$ty = x + t^2$$

Given $\quad t^2 \in [1, 4]$

$\therefore$ Area of $\Delta APN = \frac{1}{2}(AN)(PN)$



$$= \frac{1}{2}(2t^2)(2t)$$

$\Rightarrow \quad \Delta = 2t^3 = 2(t^2)^{\frac{3}{2}}$

$\because \quad t^2 \in [1, 4]$

$\Rightarrow \quad \Delta_{max}$ occurs, when $t^2 = 4$

$\Rightarrow \quad \Delta_{max} = 2(4)^{\frac{3}{2}} = 16$ sq units

$\therefore \quad \lambda = 16$

(C) Given parabola $y^2 = 4x$. Now, the ends of latusrectum are $P(1, 2)$ and $Q(1, -2)$ or $P(t^2, 2t)$ and $Q(t_1^2, 2t_1)$, where $t = 1, t_1 = -1$.

We know that the other end of normal is given by $t_2 = -t - \frac{2}{t}$

$\Rightarrow \quad A(t_2^2, 2t_2)$ and $A'(t_3^2, 2t_3)$,

where $\quad t_2 = -3, t_3 = 3$

or $\quad A(9, -6)$ and $A'(9, 6)$

$\therefore \quad AA' = 12$ units

$\therefore \quad \lambda = 12$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** (Ex. Nos. 25 and 26) are Assertion-Reason type questions. Each of these question contains two statements :
Statement I (Assertion) and
Statement II (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer.
You have to select the correct choice
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

- **Ex. 25 Statement I** *Through the point $(\lambda, \lambda+1)$, $\lambda < 2$, there cannot be more than one normal to the parabola $y^2 = 4x$.*

Statement II *The point $(\lambda, \lambda+1)$ cannot lie inside the parabola $y^2 = 4x$.*

Sol. (b) Let $S = y^2 - 4x$

$\therefore \quad S_1 = (\lambda+1)^2 - 4\lambda = (\lambda-1)^2 \geq 0$

$\therefore$ Point $(\lambda, \lambda+1)$ cannot lie inside the parabola $y^2 = 4x$.

$\therefore$ Statement II is true.

Now, equation of normal at $(t^2, 2t)$ is $y + tx = 2t + t^3$ passes through $(\lambda, \lambda+1)$.

$$\Rightarrow \quad \lambda + 1 + t\lambda = 2t + t^3$$

$$\Rightarrow \quad t^3 + (2-\lambda)t - (\lambda+1) = 0 \qquad \text{...(i)}$$

On differentiating w.r.t. t, we get

$$3t^2 + (2-\lambda) > 0, \text{ for } \lambda < 2$$

The cubic Eq. (i) has only one real root.

$\therefore$ Statement I is true.

Hence, both statements are true but Statement II is not correct explanation for Statement I.

- **Ex. 26 Statement I** *If there exist points on the circle $x^2 + y^2 = \lambda^2$ from which two perpendicular tangents can be drawn to the parabola $y^2 = 2x$, then $\lambda \geq \frac{1}{2}$.*

Statement II *Perpendicular tangents to the parabola meet at the directrix.*

Sol. (a) Statement II is true as it is property of parabola. Equation of directrix of parabola $y^2 = 2x$ is $x = -\frac{1}{2}$.

Any point on directrix is $\left(-\frac{1}{2}, y\right)$, now this point exists on the circle, then

$$\frac{1}{4} + y^2 = \lambda^2$$

$$\Rightarrow \quad y^2 = \lambda^2 - \frac{1}{4} \geq 0$$

$$\therefore \quad \lambda \geq \frac{1}{2}$$

Hence, both statements are true and Statement II is correct explanation for Statement I.

Parabola Exercise 8 :
Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

106. Tangent to the curve $y = x^2 + 6$ at a point (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are **[IIT-JEE 2005, 3M]**

(a) $(-6, -11)$ (b) $(-9, -13)$
(c) $(-10, -15)$ (d) $(-6, -7)$

107. Let P be a point (1, 0) and Q a point on the locus $y^2 = 8x$. The locus of mid-point of PQ is **[AIEEE 2005, 3M]**

(a) $x^2 - 4y + 2 = 0$ (b) $x^2 + 4y + 2 = 0$
(c) $y^2 + 4x + 2 = 0$ (d) $y^2 - 4x + 2 = 0$

108. The axis of a parabola is along the line $y = x$ and the distance of its vertex from origin is $\sqrt{2}$ and that from its focus is $2\sqrt{2}$. If vertex and focus both lie in the first quadrant, the equation of the parabola is **[IIT-JEE 2006, 3M]**

(a) $(x + y)^2 = (x - y - 2)$ (b) $(x - y)^2 = (x + y + 2)$
(c) $(x - y)^2 = 4(x + y - 2)$ (d) $(x - y)^2 = 8(x + y - 2)$

109. The equations of the common tangents to the parabolas $y = x^2$ and $y = -(x - 2)^2$ is/are **[IIT-JEE 2006, 5M]**

(a) $y = 4(x - 1)$ (b) $y = 0$
(c) $y = -4(x - 1)$ (d) $y = -30x - 50$

110. The locus of the vertices of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$, is **[AIEEE 2006, 4.5 M]**

(a) $xy = \frac{105}{64}$ (b) $xy = \frac{3}{4}$
(c) $xy = \frac{35}{16}$ (d) $xy = \frac{64}{105}$

111. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is **[AIEEE 2006, 4.5 M]**

(a) $\pi / 3$ (b) $\pi / 2$
(c) $\pi / 6$ (d) $\pi / 4$

112. Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and fourth quadrants, respectively. Tangents to the circle at P and Q intersect the X-axis at R and tangents to the parabola at P and Q intersect the X-axis at S.

(i) The ratio of the areas of the ΔPQS and ΔPQR is

(a) $1 : \sqrt{2}$ (b) $1 : 2$
(c) $1 : 4$ (d) $1 : 8$

(ii) The radius of the circumcircle of the ΔPRS is

(a) 5 (b) $3\sqrt{3}$
(c) $3\sqrt{2}$ (d) $2\sqrt{3}$

(iii) The radius of the incircle of the ΔPQR is **[IIT-JEE 2007, (4 + 4 + 4) M]**

(a) 4 (b) 3
(c) 8/3 (d) 2

113. **Statement I** The curve $y = -\frac{x^2}{2} + x + 1$ is symmetric with respect to the line $x = 1$ because

Statement II A parabola is symmetric about its axis. **[IIT-JEE 2007, 3M]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

114. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is **[AIEEE 2007, 3M]**

(a) $(-1, 1)$ (b) $(0, 2)$
(c) $(2, 4)$ (d) $(-2, 0)$

115. Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$, then **[IIT-JEE 2008, 3M]**

(a) C_1 and C_2 touch each other only at one point
(b) C_1 and C_2 touch each other exactly at two points
(c) C_1 and C_2 intersect (but do not touch) at exactly two points
(d) C_1 and C_2 neither intersect nor touch each other

116. A parabola has the origin as its focus and the line $x = 2$ as the directrix. The vertex of the parabola is at **[AIEEE 2008, 3M]**

(a) (0, 2) (b) (1, 0)
(c) (0, 1) (d) (2, 0)

117. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the ΔPTN is a parabola whose **[IIT-JEE 2009, 4M]**

(a) vertex is $\left(\frac{2a}{3}, 0\right)$ (b) directrix is at $x = 0$
(c) latusrectum is $\frac{2a}{3}$ (d) focus is $(a, 0)$

118. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, The slope of the line joining A and B can be **[IIT-JEE 2010, 3M]**

(a) $-\frac{1}{r}$ (b) $\frac{1}{r}$ (c) $\frac{2}{r}$ (d) $-\frac{2}{r}$

119. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, the locus of P is **[AIEEE 2010, 4M]**

(a) $2x + 1 = 0$ (b) $x = -1$
(c) $2x - 1 = 0$ (d) $x = 1$

120. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latusrectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola and Δ_2 be the area of the triangle formed by drawing tangent at P and at the end points of the latusrectum. Then, $\frac{\Delta_1}{\Delta_2}$ is **[IIT-JEE 2011, 4M]**

121. Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from $(0, 0)$ to (x, y) in the ratio $1 : 3$. Then, the locus of P is **[IIT-JEE 2011, 3M]**

(a) $x^2 = y$ (b) $y^2 = 2x$
(c) $y^2 = x$ (d) $x^2 = 2y$

122. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9, 6)$, then L is given by **[IIT-JEE 2011, 4M]**

(a) $y - x + 3 = 0$ (b) $y + 3x - 33 = 0$
(c) $y + x - 15 = 0$ (d) $y - 2x + 12 = 0$

123. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is **[AIEEE 2011, 4M]**

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{8}{3\sqrt{2}}$
(c) $\frac{4}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{4}$

124. Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the ΔPQS is **[IIT-JEE 2012, 4M]**

Paragraph

(Q. Nos. 125 and 126)

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangent to the parabola at P and Q meet at a point lying on the line $y = 2x + a, a > 0$. **[JEE Advanced 2013, 3+3 M]**

125. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, them $\tan\theta$ is equal to

(a) $\frac{2}{3}\sqrt{7}$ (b) $-\frac{2}{3}\sqrt{7}$ (c) $\frac{2}{3}\sqrt{5}$ (d) $-\frac{2}{3}\sqrt{5}$

126. Length of chord PQ is

(a) $7a$ (b) $5a$ (c) $2a$ (d) $3a$

127. The slope of the line touching the parabolas $y^2 = 4x$ and $x^2 = -32y$ is **[JEE Main 2014, 4M]**

(a) 1/8 (b) 2/3
(c) 1/2 (d) 3/2

128. The common tangent to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then, the area of the quadrilateral $PQRS$ is **[JEE Advanced 2014, 3 M]**

(a) 3 (b) 6 (c) 9 (d) 15

Paragraph

(Q. Nos. 129 and 130)

Let a, r, s and t be non-zero real numbers. Let $P(at^2\ 2at)$, $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$. **[JEE Advanced 2014, (3 + 3) M]**

129. The value of r is

(a) $-\frac{1}{t}$ (b) $\frac{t^2+1}{t}$ (c) $\frac{1}{t}$ (d) $\frac{t^2-1}{t}$

130. If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

(a) $\frac{(t^2+1)^2}{2t^3}$ (b) $\frac{a(t^2+1)^2}{2t^3}$
(c) $\frac{a(t^2+1)^2}{t^3}$ (d) $\frac{a(t^2+2)^2}{t^3}$

131. Let O be the vertex and Q be any point on the parabola $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is **[JEE Main 2015, 4M]**

(a) $x^2 = y$ (b) $y^2 = x$
(c) $y^2 = 2x$ (d) $x^2 = 2y$

132. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latusrectum are tangents to the circle $(x-3)^2 + (y+2)^2 = r^2$, then the value of r^2 is **[JEE Advanced 2015, 4M]**

133. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, the distance between A and B is **[JEE Advanced 2015, 4M]**

134. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P? **[JEE Advanced 2015, 4M]**

(a) $(4, 2\sqrt{2})$ (b) $(9, 3\sqrt{2})$
(c) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (d) $(1, \sqrt{2})$

135. Let P be the point on the parabola $y^2 = 8x$, which is at a minimum distance from the centre C of the circle $x^2 + (y+6)^2 = 1$, the equation of the circle passing through C and having its centre at P, is **[JEE Main 2016, 4M]**

(a) $x^2 + y^2 - 4x + 8y + 12 = 0$
(b) $x^2 + y^2 - x + 4y - 12 = 0$
(c) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$
(d) $x^2 + y^2 - 4x + 9y + 18 = 0$

136. The circle $C_1 : x^2 + y^2 = 3$ with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the Y-axis, then **[JEE Advanced 2016, 4M]**

(a) $Q_2Q_3 = 12$
(b) $R_2R_3 = 4\sqrt{6}$
(c) area of ΔOR_2R_3 is $6\sqrt{2}$
(d) area of ΔPQ_2Q_3 is $4\sqrt{2}$

137. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the centre S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then,

(a) $SP = 2\sqrt{5}$ **[JEE Advanced 2016, 4M]**

(b) $SQ : QP = (\sqrt{5} + 1) : 2$

(c) the x-intercept of the normal to the parabola at P is 6

(d) the slope of the tangent to the circle at Q is $\frac{1}{2}$

138. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is **[JEE Main 2017, 4M]**

(a) $4(\sqrt{2} + 1)$
(b) $2(\sqrt{2} + 1)$
(c) $2(\sqrt{2} - 1)$
(d) $4(\sqrt{2} - 1)$

139. If a chord, which is not a tangent of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and mid-point (h, k), then which of the following is (are) possible value(s) of p, h and k? **[JEE Advanced 2017, 4M]**

(a) $p = 2, h = 3, k = -4$
(b) $p = -1, h = 1, k = -3$
(c) $p = -2, h = 2, k = -4$
(d) $p = 5, h = 4, k = -3$

Answers

Exercise for Session 1

1. (a) **2.** (d) **3.** (c) **4.** (b) **5.** (c) **6.** (c)
7. (b) **8.** (c) **9.** (c) **10.** (b) **11.** (d) **12.** (b)
13. $16x^2 + 9y^2 + 24xy - 256x - 142y + 849 = 0$
14. $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$
15. $\left(\frac{-7}{2}, \frac{5}{2}\right); \left(\frac{-17}{2}, \frac{5}{2}\right); y = \frac{5}{2}; x + \frac{11}{4} = 0$; 3
16. Parabola **17.** Parabola
19. $x = \frac{2}{5}y^2 - 5y + \frac{68}{5}; \frac{5}{2}$ **20.** 4

Exercise for Session 2

1. (d) **2.** (b, d) **3.** (a) **4.** (a) **5.** (d) **6.** (d)
7. (b) **9.** $x + 4a = 0$ **10.** (a) $y = 2x - 12$ (b) $y = 3x - 33$
11. $y^2 = \frac{12}{343}(x-6)^2$ **13.** $\lambda = \frac{3}{2}$

Exercise for Session 3

1. (b) **2.** (a) **3.** (d) **4.** (a) **5.** (c) **6.** (c)
7. (a) **8.** (c) **10.** $y^2 = 2a(x - a)$ **11.** $y = x - 1, 8\sqrt{2}$ sq units

Chapter Exercise

1. (a) **2.** (b) **3.** (b) **4.** (d) **5.** (d) **6.** (a)
7. (a) **8.** (a) **9.** (d) **10.** (b) **11.** (a) **12.** (c)
13. (a) **14.** (d) **15.** (d) **16.** (a) **17.** (a) **18.** (c)
19. (d) **20.** (c) **21.** (d) **22.** (a) **23.** (c) **24.** (a)
25. (a) **26.** (c) **27.** (b) **28.** (b) **29.** (c) **30.** (d)
31. (b,c) **32.** (a,b) **33.** (b,d) **34.** (b,c,d) **35.** (a,b,c, d) **36.** (a, c)
37. (a,c) **38.** (b,d) **39.** (a,c) **40.** (a,c) **41.** (b,c,d) **42.** (a,b,c,d)
43. (a,c,d) **44.** (a,b,c) **45.** (a,b) **46.** (d) **47.** (d) **48.** (c)
49. (b) **50.** (c) **51.** (d) **52.** (d) **53.** (c) **54.** (d)
55. (b) **56.** (b) **57.** (d) **58.** (d) **59.** (b) **60.** (b)
61. (a) **62.** (b) **63.** (b) **64.** (c) **65.** (d) **66.** (a)
67. (c) **68.** (b) **69.** (d) **70.** (3) **71.** (6) **72.** (8)
73. (3) **74.** (6) **75.** (9) **76.** (8) **77.** (4) **78.** (5)
79. (0)
80. (A) $\rightarrow$ (q,r,s); (B) $\rightarrow$ (q,s); (C) $\rightarrow$ (p); (D) $\rightarrow$ (q)
81. (A) $\rightarrow$ (p,q); (B) $\rightarrow$ (p,r); (C) $\rightarrow$ (q,r,s); (D) $\rightarrow$ (s)
82. (A) $\rightarrow$ (p,r); (B) $\rightarrow$ (p,q); (C) $\rightarrow$ (r); (D) $\rightarrow$ (p,s)
83. (a) **84.** (a) **85.** (a) **88.** (c) **87.** (d) **88.** (c)
89. (c) **90.** (d) **92.** $x = 1$
94. $4(\sqrt{5} - 1)$ **95.** $\left(\frac{8}{9}, \frac{2}{9}\right)$ **98.** (2)
99. $2y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2$
104. $\frac{15a^2}{4}$ sq units **106.** (d) **107.** (d) **108.** (d) **109.** (a, b)
110. (a) **111.** (b) **112.** [i] (c) [ii] (b) [iii] (d) **113.** (a)
114. (d) **115.** (b) **116.** (b) **117.** (a,d) **118.** (c,d) **119.** (b)
120. (2) **121.** (c) **122.** (a,b,d) **123.** (a) **124.** 4 sq units
125. (d) **126.** (b) **127.** (c) **128.** (d) **129.** (d) **130.** (b)
131. (d) **132.** (2) **133.** (4) **134.** (a,d)
135. (a) **136.** (a,b,c) **137.** (a,c,d) **138.** (d) **139.** (a)