CHAPTER

02

The Straight Lines

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- Slope or Gradient of a Line
- Lines Parallel to Coordinate Axes
- Different Forms of the Equation of a Straight Line
- The Distance Form or Symmetric Form or Parametric Form of a Line
- Angle of Inclination of a Line
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Session 1

Definition, Angle of Inclination of a Line, Slope or Gradient of a Line, Angle Between Two Lines, Intercepts of a Lines Parallel to Coordinate Axes, Different Forms of the Equation of Line on Axes, Different Forms of the Equation of A Straight Line, Reduction of General Equation to Standard Form, The Distance Form or Symmetric Form or Parametric Form of a Line

Definition

A straight line defined as the curve which is such that the line segment joining any two points on it lies wholly on it.

Theorem : Show that the general equation of the first degree in x, y represents a straight line.

Proof : The general equation of the first degree is

$$ax + by + c = 0 \qquad \text{...(i)}$$

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the coordinates of any two points on the curve given by Eq. (i), then

$$ax_1 + by_1 + c = 0 \qquad \text{...(ii)}$$

$$ax_2 + by_2 + c = 0 \qquad \text{...(iii)}$$

Multiplying Eq. (iii) by λ and adding to Eq. (ii), we have

$$a(x_1 + \lambda x_2) + b(y_1 + \lambda y_2) + c(1 + \lambda) = 0$$

or
$$a\left(\frac{x_1 + \lambda x_2}{1+\lambda}\right) + b\left(\frac{y_1 + \lambda y_2}{1+\lambda}\right) + c = 0 \quad (\lambda \neq -1)$$

This relation shows that the point

$$\left(\frac{x_1 + \lambda x_2}{1+\lambda}, \frac{y_1 + \lambda y_2}{1+\lambda}\right) \text{ lies on Eq. (i).}$$

But from previous chapter we know that this point divides the join of $P(x_1, y_1)$ and $Q(x_2, y_2)$ is the ratio $\lambda : 1$.

Since, λ can have any value, so each point on the line PQ lies on Eq. (i) i.e. the line wholly lies on Eq. (i). Hence, by the definition of the straight line as given above we conclude that Eq. (i) represents a straight line.

Hence, the general equation of first degree in x, y viz $ax + by + c = 0$ represents a straight line.

Remarks

1. The number of **arbitrary constants** in the equation of a straight line is two (we observe three constants a, b and c in the equation $ax + by + c = 0$ of a straight line. The given equation of line can be rewritten as $\left(\frac{a}{c}\right)x + \left(\frac{b}{c}\right)y + 1 = 0$ or

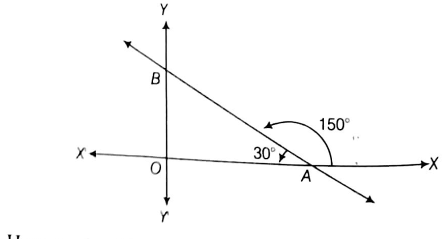
   $px + qy + 1 = 0$ where $p = \frac{a}{c}$ and $q = \frac{b}{c}$.

   Thus, we have only two arbitrary constants p and q in the equation of a straight line.

   Hence, to completely determine the equation of a straight line, we require two conditions to determine the two unknowns in general.
2. A straight line is briefly written as a 'line.'
3. The equation of a straight line is the relation between x and y, which is satisfied by the coordinates of each and every point on the line.

Angle of Inclination of a Line

The angle of inclination of a line is the measure of the angle between the X-axis and the line measured in the anticlockwise direction.



Here, angle of inclination of line $AB = 150°$.

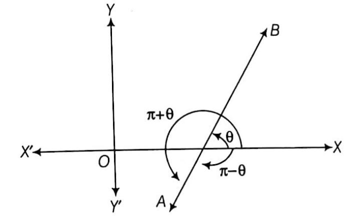
Remarks

1. When two lines are parallel, they have the same inclination.
2. The inclination of a line which is parallel to X-axis or coinciding with X-axis is 0°.
3. The angle of inclination of the line lies between 0° and 180° i.e. $0 < \theta \leq \pi$ and $\theta \neq \frac{\pi}{2}$.

Slope or Gradient of a Line

If inclination of a line is $(\theta \neq 90°)$, then $\tan\theta$ is called the slope or gradient of the line. It is usually denoted by m.

θ is positive or negative according as it is measured in anticlockwise or clockwise direction.



i.e. Slope of $AB = m$ of $AB = m(AB)$

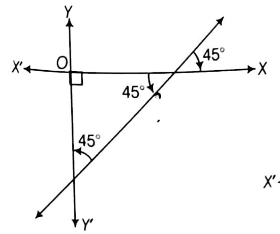
$$= \tan\theta \text{ or } \tan[-(\pi - \theta)]$$
$$= \tan(\pi + \theta)$$
$$= \text{slope of } BA = m \text{ of } BA$$
$$= m(BA)$$

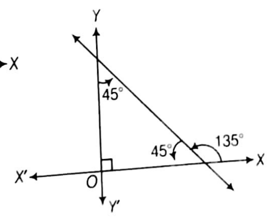
$\therefore \qquad m(AB) = m(BA)$

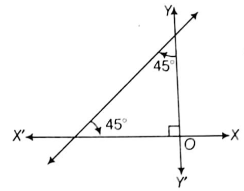
Hence, we do not take into consideration the direction of a line segment while talking of its slope.

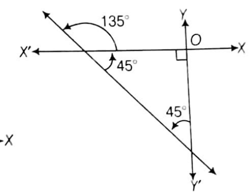
Remarks

1. Slope of a line is not the angle but is the tangent of the inclination of the line.
2. If a line is parallel to X-axis, then its slope = tan0° = 0.
3. Slope of a line parallel to Y-axis or perpendicular to X-axis is not defined. Whenever we say that the slope of a line is not defined.
4. If a line is equally inclined to the axes, then it will make an angle of 45° or 135° with the positive direction of X-axis. Slope in this case will be tan 45° or tan135°. i.e. ± 1





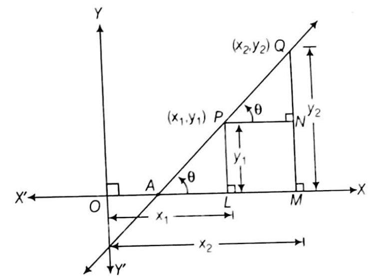




Theorem : If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line l, then the slope m of the line l is given by

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2}$$

If $x_1 = x_2$, then m is not defined. In that case the line is perpendicular to X-axis.



Proof : Given $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line l, let line l makes an angle θ with positive direction of X-axis. Draw PL, QM perpendiculars on X-axis and $PN \perp QM$

Then, $\qquad PN = LM = OM - OL = x_2 - x_1$

and $\qquad QN = QM - NM = QM - PL$

$$= y_2 - y_1$$

Also, $\qquad \angle QAM = \angle QPN = \theta$

Now, in ΔQPN

$$\tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissaes}}$$

or
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If $x_1 = x_2$, then $\tan\theta = \infty$ or $\theta = \frac{\pi}{2}$ i.e. m is not defined or the line is perpendicular to X-axis.

Remarks

1. When the two lines are parallel, then their slopes are equal i.e. $m_1 = m_2$.
2. If three points A, B, C are collinear, then slope of AB = slope of BC = slope of AC.

Example 1. Find the inclination of the line whose slope is $-\frac{1}{\sqrt{3}}$.

Sol. Let α be the inclination of a line then its slope $= \tan\alpha$

$$\therefore \quad \tan\alpha = -\frac{1}{\sqrt{3}} = -\tan 30^\circ$$
$$= \tan(180^\circ - 30^\circ) = \tan 150^\circ$$
$$\Rightarrow \quad \alpha = 150^\circ$$

Example 2. Find the slope of the line through the points $(4, -6), (-2, -5)$.

Sol. Slope of the line $m = \frac{-5-(-6)}{-2-(4)} = -\frac{1}{6}$

Example 3. Determine λ, so that 2 is the slope of the line through $(2, 5)$ and $(\lambda, 3)$.

Sol. Slope of the line joining $(2, 5)$ and $(\lambda, 3)$

$$= \frac{3-5}{\lambda-2} = \frac{-2}{\lambda-2} = 2 \quad \text{(given)}$$
$$\Rightarrow \quad -2 = 2\lambda - 4$$
$$\Rightarrow \quad 2\lambda = 2$$
$$\therefore \quad \lambda = 1$$

Example 4. Show that the line joining the points $(2, -3)$ and $(-5, 1)$ is parallel to the line joining $(7, -1)$ and $(0, 3)$.

Sol. Slope of the line joining the points $(2, -3)$ and $(-5, 1)$ is

$$m_1 = \frac{1-(-3)}{-5-2} = -\frac{4}{7}$$

and slope of the line joining the points $(7, -1)$ and $(0, 3)$ is

$$m_2 = \frac{3-(-1)}{0-7} = \frac{-4}{7}$$

Here, $m_1 = m_2$

Hence, lines are parallel.

Example 5. Find whether the points $(-a, -b), [-(s+1)a, -(s+1)b]$ and $[(t-1)a, (t-1)b]$ are collinear ?

Sol. Let $A \equiv (-a, -b)$, $B \equiv [-(s+1)a, -(s+1)b]$ and $C \equiv ((t-1)a, (t-1)b)$

Then, slope of $AB = \frac{-(s+1)b+b}{-(s+1)a+a} = \frac{b}{a}$

and slope of $BC = \frac{(t-1)b+(s+1)b}{(t-1)a+(s+1)a} = \frac{b}{a}$

Hence, given points are collinear.

Example 6. For what value of k the points $(k, 2-2k), (-k+1, 2k)$ and $(-4-k, 6-2k)$ are collinear ?

Sol. Let $A \equiv (k, 2-2k)$, $B \equiv (-k+1, 2k)$ and $C \equiv (-4-k, 6-2k)$ are collinear, then

Slope of $AB =$ Slope of AC

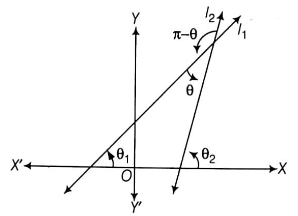
$$\Rightarrow \quad \frac{2k-(2-2k)}{-k+1-k} = \frac{6-2k-(2-2k)}{-4-k-k}$$
$$\Rightarrow \quad \frac{4k-2}{-2k+1} = \frac{4}{-4-2k} \quad \left(k \neq \frac{1}{2} (\because \text{Denominator} \neq 0)\right)$$
$$\Rightarrow \quad (4k-2)(-4-2k) = 4(-2k+1)$$
$$\Rightarrow \quad (2k-1)(-2-k) - (-2k+1) = 0$$
$$\Rightarrow \quad (2k-1)(-2-k+1) = 0$$
$$\therefore \quad k \neq \frac{1}{2}, \quad \therefore \quad k = -1$$

Angle Between Two Lines

Theorem : The acute angle θ between the lines having slopes m_1 and m_2 is given by

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Proof : Let l_1 and l_2 be two non-perpendicular lines, neither of which is parallel to the Y-axis.



Let m_1 and m_2 be the slopes of two given lines l_1 and l_2 respectively. Let θ_1 and θ_2 be the inclinations of these lines.

$$\therefore \quad m_1 = \tan\theta_1 \text{ and } m_2 = \tan\theta_2$$

Let θ and $\pi - \theta$ be the angles between the lines $\left(\theta \neq \frac{\pi}{2}\right)$.

Then, $\quad \theta_2 = \theta + \theta_1$ or $\theta = \theta_2 - \theta_1$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1)$$
$$\Rightarrow \quad \tan\theta = \left(\frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}\right) = \left(\frac{m_2 - m_1}{1 + m_1 m_2}\right) \quad \text{...(i)}$$

Also, $\quad \tan(\pi - \theta) = -\tan\theta = -\left(\frac{m_2 - m_1}{1 + m_2 m_1}\right) \quad \text{...(ii)}$

From Eqs. (i) and (ii) the angle between two lines of slopes m_1 and m_2 is given by

$$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$
$$\Rightarrow \quad \theta = \tan^{-1}\left[\pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)\right]$$

Hence, the acute angle between the lines l_1 and l_2 is given by

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

Corollary 1 **:** If two lines, whose slopes are m_1 and m_2 are parallel,

iff $\quad \theta = 0^\circ \text{ (or } \pi) \Leftrightarrow \tan\theta = 0$

$$\Leftrightarrow \boxed{m_1 = m_2}$$

Thus, when two lines are parallel, their slopes are equal.

Corollary 2 **:** If two lines, whose slopes are m_1 and m_2 are perpendicular,

iff $\quad \theta = \frac{\pi}{2}\left(\text{or } -\frac{\pi}{2}\right) \Leftrightarrow \cot\theta = 0$

$$\Leftrightarrow \boxed{m_1 m_2 = -1}$$

Thus, when two lines are perpendicular, the product of their slopes is -1. The slope of each is the negative reciprocal of the slope of other i.e. if m is the slope of a line, then the slope of a line perpendicular to it is $-\frac{1}{m}$.

Example 7. Find the angle between the lines joining the points $(0, 0), (2, 3)$ and $(2, -2), (3, 5)$.

Sol. Let the given points be $A \equiv (0, 0)$, $B \equiv (2, 3)$, $C \equiv (2, -2)$ and $D \equiv (3, 5)$. Let m_1 and m_2 be the slopes of the lines AB and CD respectively.

$$\therefore \quad m_1 = \frac{3-0}{2-0} = \frac{3}{2} \text{ and } m_2 = \frac{5-(-2)}{3-2} = 7$$

Let θ be the acute angle between the lines

$$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{3}{2} - 7}{1 + \left(\frac{3}{2}\right)\cdot 7}\right|$$
$$= \left|\frac{-11}{23}\right| = \frac{11}{23}$$
$$\therefore \quad \theta = \tan^{-1}\left(\frac{11}{23}\right)$$

Example 8. The angle between two lines is $\frac{\pi}{4}$ and the slope of one of them is $\frac{1}{2}$. Find the slope of the other line.

Sol. If θ be the acute angle between the lines with slopes m_1 and m_2, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Let $\quad \theta = \frac{\pi}{4}$ and $m_1 = \frac{1}{2}$

then $\quad \tan\frac{\pi}{4} = \left|\frac{\frac{1}{2} - m_2}{1 + \frac{1}{2}\cdot m_2}\right|$

$$\Rightarrow \quad 1 = \left|\frac{1 - 2m_2}{2 + m_2}\right|$$
$$\Rightarrow \quad \frac{1 - 2m_2}{2 + m_2} = \pm 1$$

Taking positive sign then,

$$1 - 2m_2 = 2 + m_2$$
$$\therefore \quad m_2 = -\frac{1}{3}$$

and taking negative sign then,

$$1 - 2m_2 = -2 - m_2$$
$$\therefore \quad m_2 = 3$$

Hence, the slope of the other line is either $-\frac{1}{3}$ or 3.

Example 9. Without using pythagoras theorem, show that the points $A(-1, 3), B(0, 5)$ and $C(3, 1)$ are the vertices of a right angled triangle.

Sol. In ΔABC, we have

Slope of side $AB = \frac{5-3}{0-(-1)} = 2 = m_1 \quad$ (say)

Slope of side $BC = \frac{1-5}{3-0} = -\frac{4}{3} = m_2 \quad$ (say)

and $\quad$ Slope of side $CA = \frac{3-1}{-1-3} = -\frac{1}{2} = m_3 \quad$ (say)

Clearly, $\quad m_1 m_3 = 2 \times \left(-\frac{1}{2}\right) = -1$

$\therefore$ AB and CA are perpendicular to each other i.e.

$$\angle BAC = 90^\circ$$

Hence, the given points are the vertices of a right angled triangle.

Example 10. A line passes through the points $A(2,-3)$ and $B(6,3)$. Find the slopes of the lines which are

(i) parallel to AB (ii) perpendicular to AB

Sol. Let m be the slope of AB. Then $m = \frac{3-(-3)}{6-2} = \frac{6}{4} = \frac{3}{2}$

(i) Let m_1 be the slope of a line parallel to AB, then

$$m_1 = m = \frac{3}{2}$$

(ii) The slope of a line perpendicular to AB is

$$-\frac{1}{m} = -\frac{1}{\frac{3}{2}} = -\frac{2}{3}$$

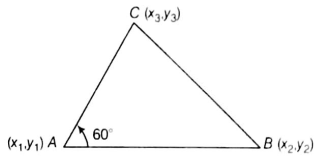
Example 11. Show that the triangle which has one of the angles as 60°, can not have all vertices with integral coordinates.

Sol. Let ABC be a triangle whose vertices are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$. Assume $x_1, x_2, x_3, y_1, y_2, y_3$ all are integers.

Let $\angle BAC = 60°$

Slope of $AC = \frac{y_3 - y_1}{x_3 - x_1} = m_1$ (say)

and Slope of $AB = \frac{y_2 - y_1}{x_2 - x_1} = m_2$ (say)



Here, m_1 and m_2 are rational numbers

($\because x_1, x_2, x_3, y_1, y_2, y_3$ are integers)

$\therefore$ $\tan(\angle BAC) = \frac{m_1 - m_2}{1 + m_1 m_2}$

= Rational ($\because m_1$ and m_2 are rational)

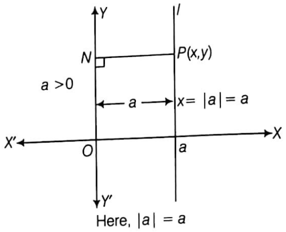
But $\tan(\angle BAC) = \tan 60° = \sqrt{3}$ = Irrational

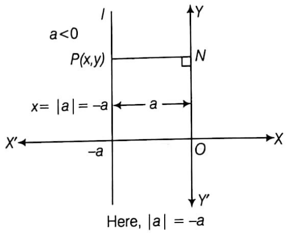
$\because$ Rational number $\neq$ Irrational number

Which is contradiction so our assumption that the vertices are integers is wrong. Hence, the triangle having one angle of 60° can not have all vertices with integral coordinates.

Lines Parallel to Coordinate Axes

(i) Equation of a line parallel to Y-axis : Let l be a straight line parallel to Y-axis and at a distance a from it, a being the directed distance of the line from the Y-axis. Therefore, the line lies on the right of Y-axis if $a > 0$ and if $a < 0$, then the line would lies on the left of Y-axis.



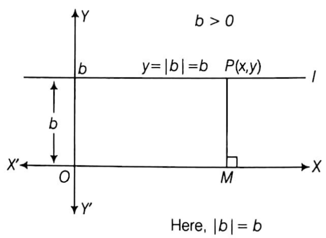


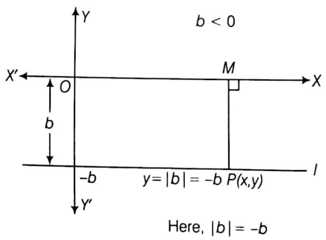
Let $P(x, y)$ be any point on the line l, then $x = a$ is the required equation. (Here, $|a| = a$)

Remarks

1. In particular equation of Y-axis is $x = 0$ ($\because a = 0$)
2. A line is parallel to Y-axis, at a distance from it and is on the negative side of Y-axis, then its equation is $x = -a$.

(ii) Equation of a line parallel to X-axis : Let l be a straight line parallel to X-axis and at a distance b from it, b being the directed distance of the line from the X-axis. Therefore, the line lies above the X-axis, if $b > 0$ and if $b < 0$, then the line would lie below the X-axis.





Let $P(x, y)$ be any point on the line l, then $y = b$ is the required equation (Here, $|b| = b$).

Remarks

1. In particular equation of X-axis is $y = 0$ ($\because b = 0$)
2. A line parallel to X-axis at a distance b from it and is on the negative side of X-axis, then its equation is $y = -b$.

Example 12. Find the equation of the straight line parallel to Y-axis and at a distance (i) 3 units to the right (ii) 2 units to the left.

Sol. (i) Equation of straight line parallel to Y-axis at a distance a units to the right is $x = a$.

$\therefore$ Required equation is $x = 3$

(ii) Equation of straight line parallel to Y-axis at a distance a units to the left is $x = -a$.

$\therefore$ Required equation is $x = -2$.

Example 13. Find the equation of the straight line parallel to X-axis and at a distance

(i) 5 units above the X-axis

(ii) 9 units below the X-axis.

Sol. (i) Equation of a straight line parallel to X-axis at a distance b units above the X-axis is $y = b$.

$\therefore$ Required equation is $y = 5$

(ii) Equation of a straight line parallel to X-axis at a distance b units below the X-axis is $y = -b$.

$\therefore$ Required equation is $y = -9$

Example 14. Find the equation of the straight line which passes through the point $(2,-3)$ and is

(i) parallel to the X-axis

(ii) perpendicular to the X-axis

Sol. (i) Let equation of any line parallel to X-axis is

$$y = b \quad \text{...(i)}$$

Since, it passes through the point $(2,-3)$.

Putting $y = -3$ in Eq. (i), then

$$b = -3$$

Hence, required equation of the line is $y = -3$.

Eq. (i) Let equation of any line perpendicular to X-axis = Equation of any line parallel to Y-axis is

$$x = a \quad \text{...(ii)}$$

Since, it passes through the point $(2,-3)$ putting $x = 2$ in Eq. (ii)

Then, $2 = a \Rightarrow a = 2$

Hence, required equation of the line $x = 2$.

Example 15. Find the equation of a line which is equidistant from the lines $x = -\frac{7}{2}$ and $x = \frac{15}{2}$.

Sol. Since, the given (both) lines are parallel to Y-axis and the required line is equidistant from these lines, so it is also parallel to Y-axis. Let equation of any line parallel to Y-axis is

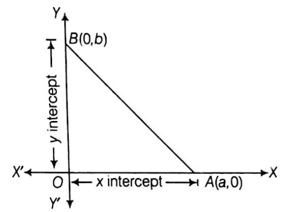
$$x = a$$

Here, $a = \frac{\left(-\frac{7}{2} + \frac{15}{2}\right)}{2} = \frac{8}{4} = 2$ units

Hence, its equation is $x = 2$.

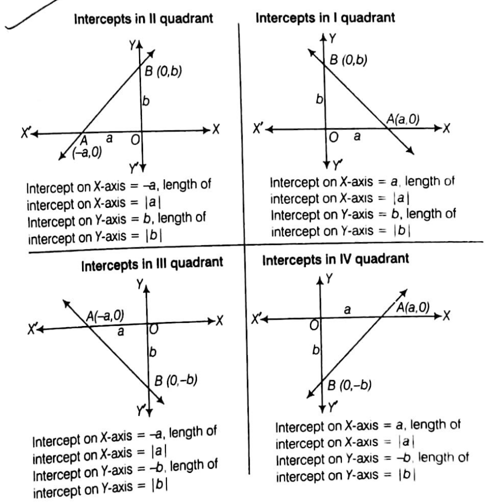
Intercepts of a Line on Axes

If a line cuts X-axis at $A(a, 0)$ and the Y-axis at $B(0, b)$ then OA and OB are known as the intercepts of the line on X-axis and Y-axis respectively. $|a|$ is called the length of intercept of the line on X-axis. Intercept of a line on X-axis may be positive or negative and $|b|$ is called the length of intercept of the line on Y-axis. Intercept of a line on Y-axis may be positive or negative.



Remark

If a line parallel to Y-axis, then its intercept on Y-axis is not defined and if a line parallel to X-axis, then its intercept on X-axis is not defined.

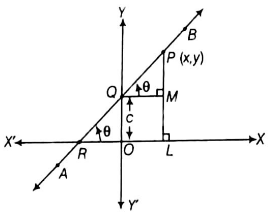


Different Forms of the Equation of a Straight Line

(i) Slope-Intercept Form

Theorem : The equation of the straight line whose slope is m and which cuts an intercept c on the Y-axis is

$$y = mx + c$$



Proof : Let AB be a line whose slope is m and which cuts an intercept c on Y-axis. Let $P(x, y)$ be any point on the line. Draw $PL \perp$ to X-axis and $QM \perp$ to PL.

Then, from figure,

$$\angle PRL = \angle PQM = \theta, OQ = c$$

$\therefore$ $PM = PL - ML = PL - OQ = y - c$

and $QM = OL = x$

Now in ΔPQM, $\tan\theta = \dfrac{PM}{QM}$

$\Rightarrow$ $m = \dfrac{y-c}{x} \Rightarrow \boxed{y = mx + c}$

which is the required equation of the line.

Remarks

1. If the line passes through the origin, then $c = 0$ ($\because 0 = m.0 + c \Rightarrow c = 0$) and hence equation of the line will become $\boxed{y = mx}$.
2. Equation of any line may be taken as $y = mx + c$.
3. If the line is parallel to X-axis, then $\theta = 0°$ i.e. $m = \tan 0° = 0$. Hence, equation of the line parallel to X-axis is $y = c$.

Example 16. If the straight line $y = mx + c$ passes through the points $(2, 4)$ and $(-3, 6)$, find the values of m and c.

Sol. Since, $(2, 4)$ lies on $y = mx + c$

$\therefore$ $4 = 2m + c$ ...(i)

Again, $(-3, 6)$ lies on $y = mx + c$

$\therefore$ $6 = -3m + c$ ...(ii)

On solving Eqs. (i) and (ii), we get $m = -\dfrac{2}{5}, c = \dfrac{24}{5}$

Example 17. What are the inclination to the X-axis and intercept on Y-axis of the line

$$3y = \sqrt{3}x + 6 ?$$

Sol. The given equation can be written as

$$y = \frac{x}{\sqrt{3}} + 2 \quad \text{...(i)}$$

Now, comparing Eq. (i) with $y = mx + c$, then we get

$$m = \frac{1}{\sqrt{3}}$$

Let θ be the inclination to the X-axis, then

$$\tan\theta = \tan 30°$$

$\therefore$ $\theta = 30°$ and $c = 2$.

Example 18. Find the equation of the straight line cutting off an intercept of 3 units on negative direction of Y-axis and inclined at an angle $\tan^{-1}\left(\dfrac{3}{5}\right)$ to the axis of x.

Sol. Here, $c = -3$ and $\theta = \tan^{-1}\left(\dfrac{3}{5}\right)$

or $\tan\theta = \dfrac{3}{5} = m$

Hence, the equation of the line

$$y = mx + c$$

i.e. $y = \dfrac{3}{5}x - 3$

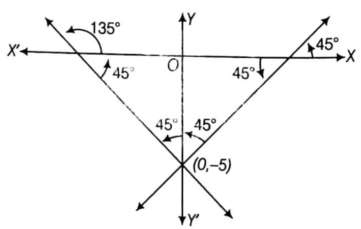
or $3x - 5y - 15 = 0$

Example 19. Find the equation to the straight line cutting off an intercept of 5 units on negative direction of Y-axis and being equally inclined to the axes.

Sol. Here, $c = -5$

$$m = \tan 45° \quad \text{or} \quad \tan 135°$$

i.e. $m = \pm 1$



Hence, required equation is

$$y = (\pm 1)x - 5$$

or $y = \pm x - 5$

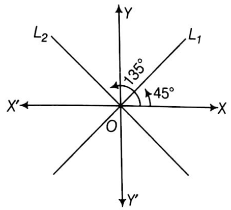
Example 20. Find the equations of the bisectors of the angle between the coordinate axes.

Sol. Let L_1 and L_2 be the straight lines bisecting the co-ordinate axes.

Both L_1 and L_2 pass through origin

$\therefore$ Equation of line through origin is $y = mx$

for L_1, $m = \tan 45° = 1$



$\therefore$ Equation of line L_1 is $y = x$

i.e. $x - y = 0$

For L_2, $m = \tan 135° = -1$

$\therefore$ Equation of line L_2 is $y = -x$

i.e. $x + y = 0$

Hence, equations of the bisectors of the angle between the coordinate axes are $x \pm y = 0$.

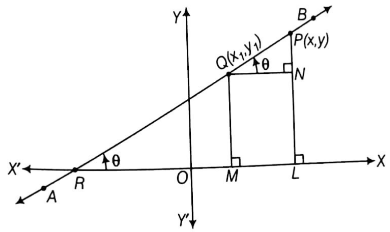
(ii) The Point-Slope Form of a Line

Theorem : The equation of the straight line which passes through the point (x_1, y_1) and has the slope 'm' is

$$y - y_1 = m(x - x_1)$$

Proof : Let AB be a straight line whose slope is m and which pass through the point $Q(x_1, y_1)$. Let the line AB cuts X-axis at R and $\angle BRX = \theta$, then

$$\tan\theta = m$$



Let $P(x, y)$ be any point on the line AB. Draws PL and QM perpendiculars from P and Q on X-axis respectively. Also draw QN perpendicular from Q on PL, then from figure

$$\angle PRL = \angle PQN = \theta, OL = x, OM = x_1, PL = y, QM = y_1$$

Then, $QN = ML = OL - OM = x - x_1$

and $PN = PL - NL = PL - QM = y - y_1$

Now, in triangle PQN,

$$\tan\theta = \frac{PN}{QN} = \frac{y - y_1}{x - x_1}$$

$\therefore$ $m = \dfrac{y - y_1}{x - x_1}$

or $y - y_1 = m(x - x_1)$

which is the required equation of the line.

Aliter : Let the equation of the required line be

$$y = mx + c \quad \text{...(i)}$$

where, m is the slope of the line.

Since line Eq. (i) passes through the point (x_1, y_1), therefore

$$y_1 = mx_1 + c \quad \text{...(ii)}$$

Subtracting (ii) form (i), we get

$$\mathbf{y - y_1 = m(x - x_1)}$$

which is the required equation of the line.

***Corollary*:** If the line passes through the origin, then putting $x_1 = 0$ and $y_1 = 0$ in $y - y_1 = m(x - x_1)$.

It becomes $y = mx$, which is the equation of the line passing through the origin and having slope m.

Remark

The equation $y - y_1 = m(x - x_1)$ is called **point-slope** form or **one point** form of the equation.

Example 21. Find the equation of a line which makes an angle of 135° with the positive direction of X-axis and passes through the point $(3, 5)$.

Sol. The slope of the line $= m = \tan 135° = -1$

Here $x_1 = 3, y_1 = 5$.

$\therefore$ The required equation of the line is

$$y - 5 = -1(x - 3)$$

or $x + y - 8 = 0$

Example 22. Find the equation of the straight line bisecting the segment joining the points $(5, 3)$ and $(4, 4)$ and making an angle of 45° with the positive direction of X-axis.

Sol. Here, m = slope of the line $= \tan 45° = 1$.

Let A be the mid-point of $(5, 3)$ and $(4, 4)$. Then, the coordinates of A are

$$\left(\frac{5+4}{2}, \frac{3+4}{2}\right) \text{ i.e. } \left(\frac{9}{2}, \frac{7}{2}\right)$$

Hence, the required equation of the line is

$$y - \frac{7}{2} = 1\left(x - \frac{9}{2}\right)$$

or $x - y - 1 = 0$

Example 23. Find the equation of the right bisector of the line joining (1, 1) and (3, 5).

Sol. Let m be the slope of the line joining (1, 1) and (3, 5).

Then, $m = \dfrac{5-1}{3-1} = \dfrac{4}{2} = 2$

$\therefore$ Slope (M) of right bisector of the join of (1, 1)

and (3, 5) $= -\dfrac{1}{m}$

$\therefore$ $M = -\dfrac{1}{2}$

Mid-point of the join of (1, 1) and (3, 5) is $\left(\dfrac{1+3}{2}, \dfrac{1+5}{2}\right)$

i.e. (2, 3).

Hence, equation of the right bisector passing through (2, 3) and having slope $M = -\dfrac{1}{2}$ is

$$y - 3 = -\frac{1}{2}(x - 2)$$

or $x + 2y - 8 = 0$

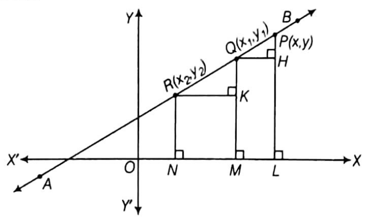
(iii) The Two-Point Form of a Line

Theorem : The equation of a line passing through two given points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Proof :

Let AB be a line which passes through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$. Let $P(x, y)$ be any point on the line AB.



Draws PL, QM and RN are perpendiculars from P, Q and R on X-axis respectively. Also draws QH and RK are perpendiculars on PL and QM respectively. Then from figure

$ON = x_2, OM = x_1, OL = x, RN = y_2,$

$QM = y_1$ and $PL = y$

then, $RK = NM = OM - ON = x_1 - x_2$

$QH = ML = OL - OM = x - x_1$

$QK = QM - KM = QM - RN = y_1 - y_2$

and $PH = PL - HL = PL - QM = y - y_1$

Now, triangles PHQ and QKR are similar, then

$$\frac{PH}{QK} = \frac{QH}{RK}$$

$\Rightarrow$ $$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

or $$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

which is the required equation of the line.

Aliter I : Let the equation of the required line be

$$y = mx + c \quad \text{...(i)}$$

where m is the slope of the line.

Since, line Eq. (i) passes through the points (x_1, y_1) and (x_2, y_2) therefore

$$y_1 = mx_1 + c \quad \text{...(ii)}$$

and $$y_2 = mx_2 + c \quad \text{...(iii)}$$

Now, subtracting Eqs. (ii) from (i), we get

$$y - y_1 = m(x - x_1) \quad \text{...(iv)}$$

and subtracting Eqs. (ii) from (iii), we get

$$y_2 - y_1 = m(x_2 - x_1) \quad \text{...(v)}$$

Dividing Eqs. (iv) by (v) then, we get

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

or $$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

which is the required equation of the line.

Aliter II : Since points, $P(x, y)$, $Q(x_1, y_1)$ and $R(x_2, y_2)$ are collinear then area of $\Delta PQR = 0$

i.e. $$\frac{1}{2}\left|\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}\right| = 0 \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

which is the required equation of the line.

Example 24. Find the equation to the straight line joining the points $\left(at_1, \dfrac{a}{t_1}\right)$ and $\left(at_2, \dfrac{a}{t_2}\right)$.

Sol. The equation of the line joining the points $\left(at_1, \dfrac{a}{t_1}\right)$ and $\left(at_2, \dfrac{a}{t_2}\right)$ is

$$y - \frac{a}{t_1} = \left(\frac{\dfrac{a}{t_2} - \dfrac{a}{t_1}}{at_2 - at_1}\right)(x - at_1)$$

$\Rightarrow$ $$y - \frac{a}{t_1} = -\frac{a(t_1 - t_2)}{at_1t_2(t_1 - t_2)}(x - at_1)$$

or $$y - \frac{a}{t_1} = -\frac{1}{t_1t_2}(x - at_1)$$

or $t_1t_2y - at_2 = -x + at_1$

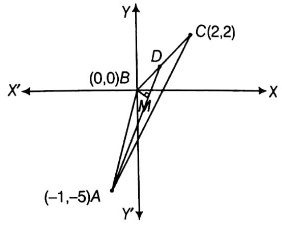
or $x + t_1t_2y = a(t_1 + t_2)$

which is the required equation of the line.

Example 25. Let ABC be a triangle with $A(-1, -5)$, $B(0, 0)$ and $C(2, 2)$ and let D be the middle point of BC. Find the equation of the perpendicular drawn from B to AD.

Sol. $\because$ D is the middle point of BC.

$\therefore$ Coordinates of D are $\left(\dfrac{0+2}{2}, \dfrac{0+2}{2}\right)$



i.e. $D(1, 1)$

Slope of median $AD = \dfrac{1+5}{1+1} = 3$

$\therefore$ Slope of BM which is perpendicular to $AD = -\dfrac{1}{3}$.

Hence, equation of the line BM is

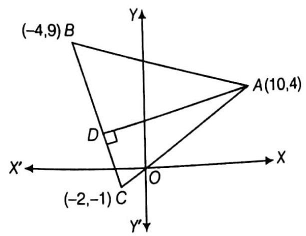
$$y - 0 = -\frac{1}{3}(x - 0) \Rightarrow x + 3y = 0$$

which is the required equation of the line.

Example 26. The vertices of a triangle are $A(10, 4)$, $B(-4, 9)$ and $C(-2, -1)$. Find the equation of the altitude through A.

Sol. $\because$ Slope of $BC = \dfrac{-1-9}{-2+4} = \dfrac{-10}{2} = -5$

$\therefore$ Slope of altitude $AD = -\dfrac{1}{-5} = \dfrac{1}{5}$



Hence, equation of altitude AD which passes through (0, 4) and having slope $\dfrac{1}{5}$ is

$$y - 4 = \frac{1}{5}(x - 10)$$

or $x - 5y + 10 = 0$

Example 27. Find the equations of the medians of a triangle, the coordinates of whose vertices are $(-1, 6)$, $(-3, -9)$ and $(5, -8)$.

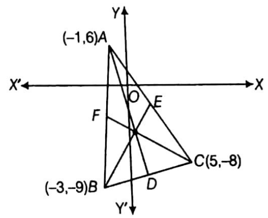
Sol. Let $A(-1, 6)$, $B(-3, -9)$ and $C(5, -8)$ be the vertices of ΔABC. Let D, E and F be the mid-points of the sides BC, CA and AB respectively.

Coordinates of $D \equiv \left(\dfrac{-3+5}{2}, \dfrac{-9-8}{2}\right)$

i.e. $\left(1, -\dfrac{17}{2}\right)$

Coordinates of $E \equiv \left(\dfrac{5-1}{2}, \dfrac{-8+6}{2}\right)$

i.e. $(2, -1)$



and coordinates of $F \equiv \left(\dfrac{-1-3}{2}, \dfrac{6-9}{2}\right)$

i.e. $(-2, -3/2)$

$\therefore$ Equation of the median AD = Equation of line through $(-1, 6)$ and $\left(1, -\dfrac{17}{2}\right)$ is

$$y - 6 = \frac{-\dfrac{17}{2} - 6}{1 + 1}(x + 1)$$

$\Rightarrow$ $y - 6 = -\dfrac{29}{4}(x + 1)$ or $29x + 4y + 5 = 0$

Equation of median BE is

$$y + 9 = \frac{-1+9}{2+3}(x + 3) \text{ or } 8x - 5y - 21 = 0$$

and equation of median CF is

$$y + 8 = \frac{-\dfrac{3}{2} + 8}{-2-5}(x - 5)$$

or $13x + 14y + 47 = 0$

Example 28. Find the ratio in which the line segment joining the points $(2, 3)$ and $(4, 5)$ is divided by the line joining $(6, 8)$ and $(-3, -2)$.

Sol. The equation of line passing through $(6, 8)$ and $(-3, -2)$ is

$$y - 8 = \frac{-2-8}{-3-6}(x-6)$$

$\Rightarrow$ $9y - 72 = 10x - 60$

or $10x - 9y + 12 = 0$ ...(i)

Let the required ratio be $\lambda : 1$.

Now, the coordinates of the point P which divide the line segment joining the points $(2, 3)$ and $(4, 5)$ in the ratio $\lambda : 1$ is

$$P\left(\frac{2+4\lambda}{1+\lambda}, \frac{3+5\lambda}{1+\lambda}\right)$$

Clearly P lies on Eq. (i), then

$$10\left(\frac{2+4\lambda}{1+\lambda}\right) - 9\left(\frac{3+5\lambda}{1+\lambda}\right) + 12 = 0$$

By trick on pg. no = 14

or $20 + 40\lambda - 27 - 45\lambda + 12 + 12\lambda = 0$

or $7\lambda + 5 = 0$ or $\lambda = -\frac{5}{7}$

$\therefore$ The required ratio $= \lambda : 1 = -\frac{5}{7} : 1 = -5 : 7$

Hence, the required ratio is $5 : 7$ (externally).

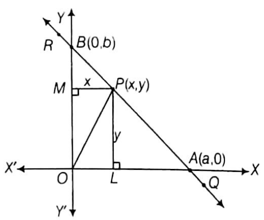
(iv) The Intercept Form of a Line :

Theorem : The equation of the straight line which cuts off intercepts of lengths of a and b on X-axis and Y-axis respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Proof : Let QR be a line which cuts off intercepts $OA = a$ and $OB = b$ on the X-axis and Y-axis respectively, where $a \neq 0$. The line is non vertical, because b is finite. Let $P(x, y)$ be a general point on the line.

Draws PL and PM perpendiculars on X-axis and Y-axis respectively. Then $PL = y$ and $OL = x$ also join OP. Clearly,



Area of ΔOAB = Area of ΔOAP + Area of ΔOPB

$\Rightarrow$ $\frac{1}{2}\cdot OA \cdot OB = \frac{1}{2}\cdot OA \cdot PL + \frac{1}{2}\cdot OB \cdot PM$

$\Rightarrow$ $\frac{1}{2}\cdot OA \cdot OB = \frac{1}{2}\cdot OA \cdot PL + \frac{1}{2}\cdot OB \cdot OL$

or $\frac{1}{2}ab = \frac{1}{2}ay + \frac{1}{2}bx$

$\Rightarrow$ $ab = ay + bx$

or $\frac{x}{a} + \frac{y}{b} = 1$

which is the required equation of the line.

Aliter I : Equation of the line through $A(a, 0)$ and $B(0, b)$ is

$$y - 0 = \frac{b-0}{0-a}(x-a)$$

or $-ay = bx - ab$

or $bx + ay = ab$

or $\frac{x}{a} + \frac{y}{b} = 1$

which is the required equation of the line.

Aliter II : Points $A(a, 0)$, $P(x, y)$ and $B(0, b)$ are collinear, we have

slope of AB = slope of AP

$\Rightarrow$ $\frac{b-0}{0-a} = \frac{y-0}{x-a} \Rightarrow bx - ab = -ay$

$\Rightarrow$ $bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$

or

$$\begin{vmatrix} a & 0 & 1 \\ x & y & 1 \\ 0 & b & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $a(y-b) - 0 + 1\cdot(bx) = 0$

or $bx + ay = ab$

or $\frac{x}{a} + \frac{y}{b} = 1$

which is the required equation of the line.

Example 29. Find the equation of the line through $(2, 3)$ so that the segment of the line intercepted between the axes is bisected at this point.

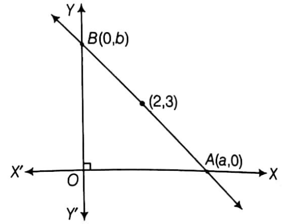
Sol. Let the required line segment be AB.

Let O be the origin and $OA = a$ and $OB = b$.

Then the coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.

$\therefore$ $\frac{a+0}{2} = 2 \Rightarrow a = 4$

and $\frac{0+b}{2} = 3 \Rightarrow b = 6$



Hence, the equation of the required line is

$$\frac{x}{4} + \frac{y}{6} = 1$$

i.e. $3x + 2y = 12$

Example 30. Find the equation to the straight line which passes through the points $(3, 4)$ and have intercepts on the axes :

(a) *equal in magnitude but opposite in sign*

(b) *such that their sum is 14*

Sol. (a) Let intercepts on the axes be a and $-a$ respectively.

$\therefore$ The equation of the line in intercept form is

$\frac{x}{a} + \frac{y}{-a} = 1$ or $x - y = a$ ...(i)

Since, Eq. (i) passes through $(3, 4)$, then

$3 - 4 = a$

$\therefore$ $a = -1$

From Eq. (i), $x - y + 1 = 0$

which is the required equation of the line.

(b) Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$

This passes through $(3, 4)$.

Therefore $\frac{3}{a} + \frac{4}{b} = 1$ ...(ii)

It is given that $a + b = 14$

$\therefore$ $b = 14 - a$

Putting $b = 14 - a$ in Eq. (ii), we get

$$\frac{3}{a} + \frac{4}{14-a} = 1$$

$\Rightarrow$ $42 - 3a + 4a = 14a - a^2$

$\Rightarrow$ $a^2 - 13a + 42 = 0$

$\Rightarrow$ $(a-7)(a-6) = 0$

$\therefore$ $a = 6, 7$

Then, $b = 8, 7$ $(\because b = 14 - a)$

Hence, the required equations are

$\frac{x}{6} + \frac{y}{8} = 1$ and $\frac{x}{7} + \frac{y}{7} = 1$

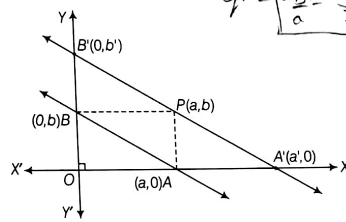
i.e. $4x + 3y = 24$ and $x + y = 7$

Example 31. Find the equation of the straight line through the point $P(a, b)$ parallel to the line $\frac{x}{a} + \frac{y}{b} = 1$. Also find the intercepts made by it on the axes.

Sol. Let the line $\frac{x}{a} + \frac{y}{b} = 1$ $y = -\frac{b}{a}x + b$

meets the axes in A and B respectively. So that So, $m = -\frac{b}{a}$

$OA = a$, $OB = b$ eq. = $\boxed{-\frac{b}{a} = \frac{y-b}{x-a}}$



Let the required parallel line meet in A' and B' respectively, so that

$OA' = a'$ (say)

and $OB' = b'$ (say)

$\therefore$ Equation of required line is

$$\frac{x}{a'} + \frac{y}{b'} = 1 \quad \text{...(i)}$$

Since, Δ's OAB and $OA'B'$ are similar, then

$$\frac{OA'}{OA} = \frac{OB'}{OB}$$

i.e. $\frac{a'}{a} = \frac{b'}{b} = \lambda$ (say)

$\Rightarrow$ $a' = a\lambda$, $b' = b\lambda$

Substituting these values in Eq. (i), then

$$\frac{x}{a\lambda} + \frac{y}{b\lambda} = 1 \quad \text{...(ii)}$$

It passes through (a, b), then

$$\frac{a}{a\lambda} + \frac{b}{b\lambda} = 1$$

$\Rightarrow$ $\frac{2}{\lambda} = 1$ or $\lambda = 2$

From Eq. (ii) required equation is

$$\frac{x}{2a} + \frac{y}{2b} = 1$$

Evidently intercepts on the axes are $2a$ and $2b$.

(v) The Normal Form or Perpendicular Form of a Line

Theorem : The equation of the straight line upon which the length of perpendicular from the origin is p and this normal makes an angle α with the positive direction of X-axis is

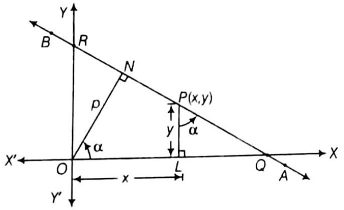
$$x\cos\alpha + y\sin\alpha = p.$$

Proof : Let AB be a line such that the length of perpendicular from O to the line be p

i.e. $ON = p$

and $\angle NOX = \alpha$

Let $P(x, y)$ be any point on the line. Draw PL perpendicular from P on X-axis.



Let line AB cuts X and Y-axes at Q and R respectively.

Now, $\angle NQO = 90° - \alpha$

$\therefore$ $\angle LPQ = 90° - (90° - \alpha) = \alpha$

In ΔPLQ, $\tan\alpha = \dfrac{LQ}{PL} = \dfrac{LQ}{y}$

$\therefore$ $LQ = y\tan\alpha$ ...(i)

Also, in ΔONQ, $\cos\alpha = \dfrac{ON}{OQ}$

$\Rightarrow$ $\cos\alpha = \dfrac{p}{OL + LQ}$

$\Rightarrow$ $OL\cos\alpha + LQ\cos\alpha = p$

$\Rightarrow$ $x\cos\alpha + y\tan\alpha\cos\alpha = p$

$(\because OL = x \text{ and } LQ = y\tan\alpha)$

$\therefore$ $\boldsymbol{x\cos\alpha + y\sin\alpha = p}$

which is the required equation of the line AB.

Aliter I : $\because \angle NOQ = \alpha$

then $\angle NOR = 90° - \alpha$

Now, in ΔONQ, $\sec\alpha = \dfrac{OQ}{ON} = \dfrac{OQ}{p}$

or $OQ = p\sec\alpha$

Also in ΔONR, $\sec(90° - \alpha) = \dfrac{OR}{ON}$

$\Rightarrow$ $\operatorname{cosec}\alpha = \dfrac{OR}{p}$

$OR = p\operatorname{cosec}\alpha$

or

Thus, AB makes intercepts $p\sec\alpha$ and $p\operatorname{cosec}\alpha$ on X-axis and Y-axis respectively.

$\therefore$ Equation of AB is $\dfrac{x}{p\sec\alpha} + \dfrac{y}{p\operatorname{cosec}\alpha} = 1$

$$\boldsymbol{x\cos\alpha + y\sin\alpha = p}$$

or

which is the required equation of the line AB.

Aliter II : The points $Q(p\sec\alpha, 0)$, $P(x, y)$ and $R(0, p\operatorname{cosec}\alpha)$ are collinear, then

$$\begin{vmatrix} p\sec\alpha & 0 & 1 \\ x & y & 1 \\ 0 & p\operatorname{cosec}\alpha & 1 \end{vmatrix} = 0$$

or $p\sec\alpha(y - p\operatorname{cosec}\alpha) - 0 + 1(px\operatorname{cosec}\alpha) = 0$

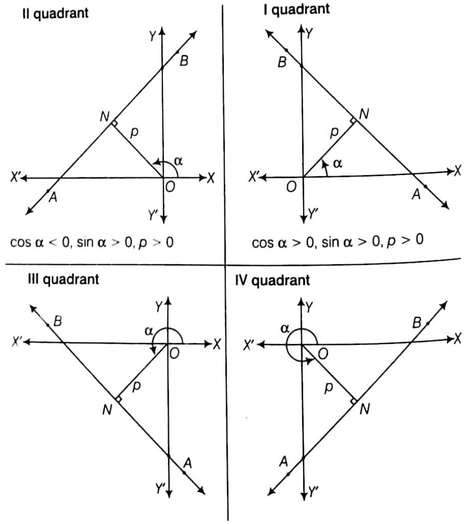
or $p(y\sin\alpha - p) + px\cos\alpha = 0$

or $\boldsymbol{x\cos\alpha + y\sin\alpha = p}$

which is the required equation of the line AB.

Remarks

1. Here, p is always taken as positive and α is measured from positive direction of X-axis in anticlockwise direction between 0 and 2π (i.e. $0 \le \alpha < 2\pi$).
2. (Coefficient of x)^2 + (Coefficient of y)$^2 = \cos^2\alpha + \sin^2\alpha = 1$
3. $\cos\alpha$ and $\cos(90° - \alpha)$ are the direction cosines of ON.
4.



$\cos\alpha < 0, \sin\alpha > 0, p > 0$ (II quadrant)

$\cos\alpha > 0, \sin\alpha > 0, p > 0$ (I quadrant)

$\cos\alpha < 0, \sin\alpha < 0, p > 0$ (III quadrant)

$\cos\alpha > 0, \sin\alpha < 0, p > 0$ (IV quadrant)

Corollary 1 : If $\alpha = 0°$, then equation $x\cos\alpha + y\sin\alpha = p$ becomes $x\cos 0° + y\sin 0° = p$

i.e. $x = p$ (Equation of line parallel to Y-axis)

Corollary 2 : If $\alpha = \dfrac{\pi}{2}$, then equation $x\cos\alpha + y\sin\alpha = p$

becomes $x\cos\left(\dfrac{\pi}{2}\right) + y\sin\left(\dfrac{\pi}{2}\right) = p$

i.e. $y = p$ (Equation of line parallel to X-axis).

Corollary 3 : If $\alpha = 0°$, $p = 0$ then equation $x\cos\alpha + y\sin\alpha = p$ becomes $x\cos 0° + y\sin 0° = 0$

i.e. $x = 0$ (Equation of Y-axis)

Corollary 4 : If $\alpha = \dfrac{\pi}{2}$, $p = 0$ then equation

$x\cos\alpha + y\sin\alpha = p$ becomes $x\cos\left(\dfrac{\pi}{2}\right) + y\sin\left(\dfrac{\pi}{2}\right) = 0$

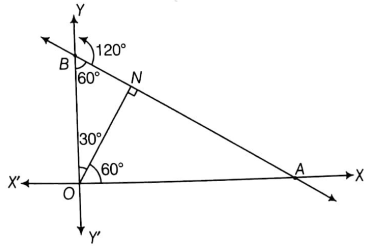
i.e. $y = 0$ (Eq. (i) of X-axis).

Example 32. The length of perpendicular from the origin to a line is 9 and the line makes an angle of $120°$ with the positive direction of Y-axis. Find the equation of the line.

Sol. Here, $\alpha = 60°$ and $p = 9$.

$\therefore$ Equation of the required line is

$$x\cos 60° + y\sin 60° = 9$$



$\Rightarrow$ $x\left(\dfrac{1}{2}\right) + y\left(\dfrac{\sqrt{3}}{2}\right) = 9$

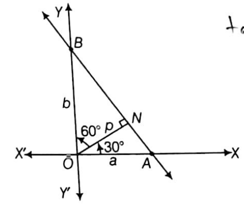
or $x + y\sqrt{3} = 18$

Example 33. Find the equation of the straight line on which the perpendicular from origin makes an angle of $30°$ with X-axis and which forms a triangle of area $\left(\dfrac{50}{\sqrt{3}}\right)$ sq units with the coordinates axes.

Sol. Let $\angle NOA = 30°$

Let $ON = p > 0$, $OA = a$, $OB = b$

In ΔONA, $\cos 30° = \dfrac{ON}{OA} = \dfrac{p}{a}$



$\tan 60° = \dfrac{b}{a}$

$ab = \dfrac{50}{\sqrt{3}}$

$\Rightarrow$ $\dfrac{\sqrt{3}}{2} = \dfrac{p}{a}$ or $a = \dfrac{2p}{\sqrt{3}}$

and in ΔONB, $\cos 60° = \dfrac{ON}{OB} = \dfrac{p}{b}$

$\Rightarrow$ $\dfrac{1}{2} = \dfrac{p}{b}$ or $b = 2p$

$\because$ Area of $\Delta OAB = \dfrac{1}{2}ab = \dfrac{1}{2}\left(\dfrac{2p}{\sqrt{3}}\right)(2p) = \dfrac{2p^2}{\sqrt{3}}$

$\therefore$ $\dfrac{2p^2}{\sqrt{3}} = \dfrac{50}{\sqrt{3}}$ (given)

$\Rightarrow$ $p^2 = 25$

or $p = 5$ $(\because p > 0)$

$\therefore$ Using $x\cos\alpha + y\sin\alpha = p$, the equation of the line AB is

$x\cos 30° + y\sin 30° = 5$

or $x\sqrt{3} + y = 10$

Reduction of General Equation to Standard Form

Let $Ax + By + C = 0$ be the general equation of a straight line where A and B are not both zero.

(i) Reduction of 'Slope-Intercept' Form

Given equation is $Ax + By + C = 0$

$\Rightarrow$ $By = -Ax - C$

$\Rightarrow$ $y = \left(-\dfrac{A}{B}\right)x + \left(-\dfrac{C}{B}\right)$ (Assuming $B \ne 0$)

Comparing it with $y = mx + c$

we get slope $(m) = -\dfrac{A}{B} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y}$

and y intercept $(c) = -\dfrac{C}{B} = -\dfrac{\text{constant term}}{\text{coefficient of } y}$

Corollary 1 : Find angle between the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$.

Slope of the line

$$A_1x + B_1y + C_1 = 0 \text{ is } -\frac{A_1}{B_1} = m_1 \quad \text{(say)}$$

and slope of the line

$$A_2x + B_2y + C_2 = 0 \text{ is } -\frac{A_2}{B_2} = m_2 \quad \text{(say)}$$

If θ is the angle between the two lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\left(-\frac{A_1}{B_1}\right) - \left(-\frac{A_2}{B_2}\right)}{1 + \left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right)}\right|$$

$$= \left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right|$$

$$\therefore \quad \theta = \tan^{-1}\left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right|$$

Corollary 2 : Find the condition of (i) parallelism (ii) perpendicularity of the lines

$$A_1x + B_1y + C_1 = 0$$

and $\quad A_2x + B_2y + C_2 = 0$

(i) If the two lines are parallel, $\theta = 0°$

$\therefore \quad \tan\theta = \tan 0° = 0$

$\Rightarrow \quad \left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right| = 0$

$\Rightarrow \quad A_1B_2 - A_2B_1 = 0$

or $\quad \boxed{\frac{A_1}{A_2} = \frac{B_1}{B_2}}$ **(Remember)**

which is required condition of parallelism.

(ii) If the two lines are perpendicular, $\theta = 90°$

$\therefore \quad \tan\theta = \tan 90° = \infty$

$\Rightarrow \quad \left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right| = \infty$

$\Rightarrow \quad \boxed{A_1A_2 + B_1B_2 = 0}$ **(Remember)**

which is required condition of perpendicularity.

Remark

If two lines are coincident, then

$$\boxed{\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}} \quad \textbf{(Remember)}$$

(ii) Reduction to 'Intercept' Form

Given equation is $Ax + By + C = 0$

$\Rightarrow \quad Ax + By = -C$

$\Rightarrow \quad \frac{A}{-C}x + \frac{B}{-C}y = 1 \quad$ (Assuming $C \neq 0$)

$\Rightarrow \quad \frac{x}{(-C/A)} + \frac{y}{(-C/B)} = 1$ (Assuming $A \neq 0, B \neq 0$)

Comparing with $\frac{x}{a} + \frac{y}{b} = 1$

we get, x-intercept $(a) = -\frac{C}{A} = -\frac{\text{constant term}}{\text{coefficient of } x}$

and $\quad y$-intercept $(b) = -\frac{C}{B} = -\frac{\text{constant term}}{\text{coefficient of } y}$

(iii) Reduction to 'Normal' Form

Given equation is $Ax + By + C = 0$. Let its **normal form** be $x\cos\alpha + y\sin\alpha = p$.

Clearly, equations $Ax + By + C = 0$ and $x\cos\alpha + y\sin\alpha = p$ represent the same line.

Therefore, $\quad \frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \frac{C}{-p}$

$\Rightarrow \quad \cos\alpha = -\frac{Ap}{C}$

and $\quad \sin\alpha = -\frac{Bp}{C} \quad$ ...(i)

$$\cos^2\alpha + \sin^2\alpha = 1$$

$$\left(-\frac{Ap}{C}\right)^2 + \left(-\frac{Bp}{C}\right)^2 = 1$$

or $\quad p^2 = \frac{C^2}{A^2 + B^2}$

$\Rightarrow \quad p = \frac{|C|}{\sqrt{(A^2 + B^2)}}$

From Eq. (i), $\cos\alpha = -\frac{|C|}{C} \cdot \frac{A}{\sqrt{(A^2 + B^2)}}$,

$$\sin\alpha = -\frac{|C|}{C} \cdot \frac{B}{\sqrt{(A^2 + B^2)}}.$$

Putting the values of $\cos\alpha$, $\sin\alpha$ and p in $x\cos\alpha + y\sin\alpha = p$, we get

$$x\left(\frac{-|C|}{C} \cdot \frac{A}{\sqrt{(A^2 + B^2)}}\right)x + y\left(\frac{-|C|}{C} \cdot \frac{B}{\sqrt{(A^2 + B^2)}}\right) = \frac{|C|}{\sqrt{(A^2 + B^2)}}$$

$$\Rightarrow \left(-\frac{A}{\sqrt{(A^2 + B^2)}}\right)x + \left(-\frac{B}{\sqrt{(A^2 + B^2)}}\right)y = \frac{C}{\sqrt{(A^2 + B^2)}}$$

This is the normal form of the line $Ax + By + C = 0$.

Rule : First shift the constant term on the RHS and make it positive, if it is not so by multiplying the whole equation by '-1' and then divide both sides by

$$\sqrt{(\text{coefficient of } x)^2 + (\text{coefficient of } y)^2}$$

Example 34. Reduce $x + \sqrt{3}y + 4 = 0$ to the :

(i) Slope-intercept form and find its slope and y-intercept

(ii) Intercept form and find its intercepts on the axes

(iii) Normal form and find the values of p and α

Sol. (i) Given equation is $x + \sqrt{3}\,y + 4 = 0$

$\Rightarrow \quad \sqrt{3}y = -x - 4$

$\Rightarrow \quad y = \left(-\frac{1}{\sqrt{3}}\right)x + \left(-\frac{4}{\sqrt{3}}\right)$

which is in the slope-intercept form $y = mx + c$

Where slope $(m) = -\frac{1}{\sqrt{3}}$ and y-intercept $(c) = -\frac{4}{\sqrt{3}}$

(ii) Given equation is

$$x + \sqrt{3}y + 4 = 0$$

$\Rightarrow \quad x + \sqrt{3}y = -4$

$\Rightarrow \quad \frac{x}{-4} + \frac{\sqrt{3}y}{-4} = 1$

$\Rightarrow \quad \frac{x}{-4} + \frac{y}{-4/\sqrt{3}} = 1$

which is in the intercept form $\frac{x}{a} + \frac{y}{b} = 1$

where x-intercept $(a) = -4$ and y-intercept $(b) = -\frac{4}{\sqrt{3}}$

(iii) Given equation is $x + \sqrt{3}y + 4 = 0$

$\Rightarrow \quad x + \sqrt{3}y = -4$

$\Rightarrow \quad -x - \sqrt{3}y = 4 \quad$ (RHS made positive)

Dividing both sides by $\sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$, we get

$$\left(-\frac{1}{2}\right)x + \left(-\frac{\sqrt{3}}{2}\right)y = 2$$

Which is the normal form $x\cos\alpha + y\sin\alpha = p$.

where, $\quad \cos\alpha = -\frac{1}{2} = -\cos 60° = \cos(180° - 60°)$

or $\quad \cos(180° + 60°)$

$\therefore \quad \alpha = 120°$ or $240°$

and $\quad \sin\alpha = -\frac{\sqrt{3}}{2} = -\sin 60° = \sin(180° + 60°)$

or $\quad \sin(360° - 60°)$

$\therefore \quad \alpha = 240°$ or $300°$

Hence, $\quad \alpha = 240°$, $p = 2$

$\therefore$ Required normal form is

$$x\cos 240° + y\sin 240° = 2$$

Example 35. Find the measure of the angle of intersection of the lines whose equations are $3x + 4y + 7 = 0$ and $4x - 3y + 5 = 0$.

Sol. Given lines are $3x + 4y + 7 = 0$, $4x - 3y + 5 = 0$. Comparing the given lines with $A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$ respectively, we get

$$A_1 = 3, B_1 = 4 \quad \text{and} \quad A_2 = 4, B_2 = -3$$

$\because \quad A_1A_2 + B_1B_2 = 3 \times 4 + 4(-3) = 0$

Hence, the given lines are perpendicular.

Example 36. Find the angle between the lines $(a^2 - ab)\,y = (ab + b^2)\,x + b^3$ and $(ab + a^2)\,y = (ab - b^2)\,x + a^3$ where $a > b > 0$.

Sol. The given equations of lines can be written as

$(ab + b^2)\,x - (a^2 - ab)\,y + b^3 = 0 \quad$ ...(i)

and $\quad (ab - b^2)\,x - (ab + a^2)\,y + a^3 = 0 \quad$ ...(ii)

Comparing the given lines (i) and (ii) with the lines

$A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$

respectively, we get

$$A_1 = ab + b^2, B_1 = -(a^2 - ab)$$

and $\quad A_2 = ab - b^2, B_2 = -(ab + a^2)$

Let θ be the acute angle between the lines, then

$$\tan\theta = \left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right|$$

$$\tan\theta = \left|\frac{(ab + b^2) \times (-(ab + a^2)) - (ab - b^2) \times (-(a^2 - ab))}{(ab + b^2)(ab - b^2) + (a^2 - ab)(ab + a^2)}\right|$$

$$= \left|\frac{-\{a^2b^2 + a^3b + ab^3 + a^2b^2 - a^3b + a^2b^2 + a^2b^2 - b^3a\}}{(a^2b^2 - b^4 + a^4 - a^2b^2)}\right|$$

$$= \left|\frac{-4a^2b^2}{a^4 - b^4}\right| = \frac{4a^2b^2}{a^4 - b^4}$$

$$\therefore \theta = \tan^{-1}\left(\frac{4a^2b^2}{a^4 - b^4}\right)$$

I Example 37. Two equal sides of an isosceles triangle are given by the equations $7x-y+3=0$ and $x+y-3=0$ and its third side passes through the point $(1,-10)$. Determine the equation of the third side.

Sol. Given equations

$$7x-y+3=0 \quad \text{...(i)}$$

and $$x+y-3=0 \quad \text{...(ii)}$$

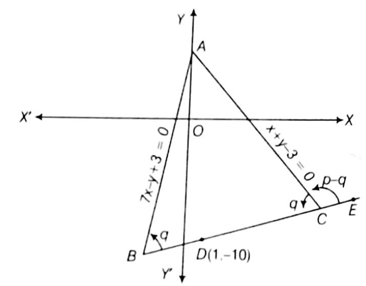
represents two equal sides AB and AC of an isosceles triangle ABC. Since its third side passes through $D(1,-10)$ then its equation is

$$y+10=m(x-1) \quad \text{...(iii)}$$

$\because$ $AB=AC$

Let $\angle ABC=\angle ACB=\theta$

then $\angle ACE=\pi-\theta$



From Eqs. (i) and (ii), slopes of AB and AC are $m_1=7$ and $m_2=-1$ respectively.

$\therefore$ $$\tan\theta=\frac{7-m}{1+7m}$$

and $$\tan(\pi-\theta)=\frac{-1-m}{1+(-1)m}=-\left(\frac{1+m}{1-m}\right)$$

$\Rightarrow$ $$-\tan\theta=-\left(\frac{1+m}{1-m}\right)\Rightarrow\tan\theta=\left(\frac{1+m}{1-m}\right)$$

$\therefore$ $$\frac{7-m}{1+7m}=\frac{1+m}{1-m}$$

$\Rightarrow$ $(7-m)(1-m)=(1+7m)(1+m)$

$\Rightarrow$ $6m^2+16m-6=0$

or $3m^2+8m-3=0$ or $(3m-1)(m+3)=0$

$\Rightarrow$ $$m=\frac{1}{3},-3$$

Hence from Eq. (iii), the third side BC has two equations

$$y+10=\frac{1}{3}(x-1) \text{ and } y+10=-3(x-1)$$

or $x-3y-31=0$ and $3x+y+7=0$

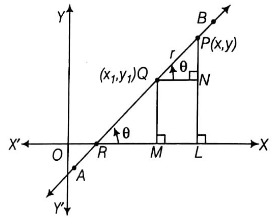
The Distance form or Symmetric form or Parametric form of a line

Theorem : The equation of the straight line passing through (x_1,y_1) and making an angle θ with the positive direction of X-axis is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

where, r is the directed distance between the points (x,y) and (x_1,y_1).

Proof : Let AB be a line which passes through the point $Q(x_1,y_1)$ and meet X-axis at R and makes an angle θ with the positive direction of X-axis.



Let $P(x,y)$ be any point on the line at a distance r from Q. Draws PL and QM are perpendiculars from P and Q on X-axis respectively and draw QN perpendicular on PL. Then,

$$QN=ML=OL-OM=x-x_1$$

and $$PN=PL-NL=PL-QM=y-y_1$$

from ΔPQN,

$$\cos\theta=\frac{QN}{PQ}=\frac{x-x_1}{r} \text{ or } \frac{x-x_1}{\cos\theta}=r \quad \text{...(i)}$$

and $$\sin\theta=\frac{PN}{PQ}=\frac{y-y_1}{r} \text{ or } \frac{y-y_1}{\sin\theta}=r \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

Corollary 1 : $\because$ $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r$, then

$$\left.\begin{aligned} x&=x_1+r\cos\theta\\ y&=y_1+r\sin\theta\end{aligned}\right\}$$

parametric equations of straight line AB.

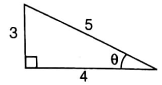
Corollary 2 : If P point above Q then r is positive then coordinates of P are $(x_1+r\cos\theta, y_1+r\sin\theta)$ and if P below Q then r is negative then coordinates of P are $(x_1-r\cos\theta, y_1-r\sin\theta)$.

I Example 38. The slope of a straight line through $A(3,2)$ is $\frac{3}{4}$. Find the coordinates of the points on the line that are 5 units away from A.

Sol. Let straight line makes an angle θ with positive direction of X-axis,

then $$\tan\theta=\frac{3}{4}$$

$\therefore$ $$\sin\theta=\frac{3}{5} \text{ and } \cos\theta=\frac{4}{5}$$



$\therefore$ Equation of the straight line through $A(3,2)$ in parametric form is

$$\frac{x-3}{\cos\theta}=\frac{y-2}{\sin\theta}=\pm 5$$

$\therefore$ $$x=3\pm5\cos\theta=3\pm5\times\frac{4}{5}=3\pm4=7 \text{ or } -1$$

and $$y=2\pm5\sin\theta=2\pm5\times\frac{3}{5}=2\pm3=5 \text{ or } -1$$

Hence, the coordinates of the points are $(7,5)$ and $(-1,-1)$.

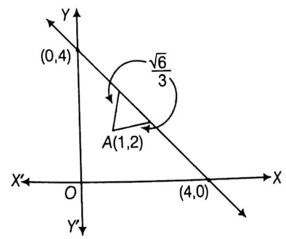
I Example 39. Find the direction in which a straight line must be drawn through the point $(1,2)$ so that its point of intersection with the line $x+y=4$ may be at a distance $\frac{1}{3}\sqrt{6}$ from this point.

Must

Sol. Let the straight line makes an angle θ with the positive direction of X-axis.

$\therefore$ Equation of the line through $(1,2)$ in parametric form is

$$\frac{x-1}{\cos\theta}=\frac{y-2}{\sin\theta}=\frac{1}{3}\sqrt{6}$$



or $$x=1+\frac{\sqrt6}{3}\cos\theta \text{ and } y=2+\frac{\sqrt6}{3}\sin\theta$$

Since, the point $\left(1+\frac{\sqrt6}{3}\cos\theta, 2+\frac{\sqrt6}{3}\sin\theta\right)$ lies on the line

$$x+y=4$$

$\therefore$ $$1+\frac{\sqrt6}{3}\cos\theta+2+\frac{\sqrt6}{3}\sin\theta=4$$

$\Rightarrow$ $$\frac{\sqrt6}{3}(\cos\theta+\sin\theta)=1$$

$\Rightarrow$ $$\cos\theta+\sin\theta=\frac{3}{\sqrt6}=\frac{\sqrt3}{\sqrt2}$$

$\Rightarrow$ $$\frac{1}{\sqrt2}\cos\theta+\frac{1}{\sqrt2}\sin\theta=\frac{\sqrt3}{2}$$

$\Rightarrow$ $$\cos\left(\theta-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{6}\right)$$

$$\theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{6};\ n\in I$$

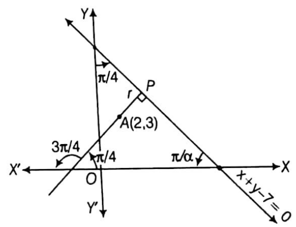
for $n=0$, $$\theta=\pm\frac{\pi}{6}+\frac{\pi}{4} \quad (\because 0\le\theta<\pi)$$

$$=15^\circ, 75^\circ$$

I Example 40. A line through $(2,3)$ makes an angle $\frac{3\pi}{4}$ with the negative direction of X-axis. Find the length of the line segment cut off between $(2,3)$ and the line $x+y-7=0$.

Sol. $\because$ Line makes an angle $\frac{3\pi}{4}$ with the negative direction of X-axis.

$\therefore$ Line makes an angle $\frac{\pi}{4}$ with the positive direction of X-axis.



$\therefore$ The equation of the line through $(1,2)$ in parametric form is

$$\frac{x-2}{\cos\left(\frac{\pi}{4}\right)}=\frac{y-3}{\sin\left(\frac{\pi}{4}\right)}=r$$

i.e. $$\frac{x-2}{\frac{1}{\sqrt2}}=\frac{y-3}{\frac{1}{\sqrt2}}=r \quad \text{...(i)}$$

$\therefore$ $$x=2+\frac{r}{\sqrt2} \text{ and } y=3+\frac{r}{\sqrt2}$$

Let the line (i) meet the line $x+y-7=0$ in P

$\therefore$ Coordinates of $P\left(2+\frac{r}{\sqrt2},3+\frac{r}{\sqrt2}\right)$ lies on $x+y-7=0$

then $$2+\frac{r}{\sqrt2}+3+\frac{r}{\sqrt2}-7=0$$

or $$\frac{2r}{\sqrt2}=2 \text{ or } r=\sqrt2$$

$\therefore$ $$AP=\sqrt2$$

Example 41. Find the distance of the point (2, 3) from the line $2x-3y+9=0$ measured along the line $2x-2y+5=0$.

Sol. Since, slope of the line $2x-2y+5=0$ is 1, its makes an angle $\frac{\pi}{4}$ with positive direction of X-axis.

The equation of the line through (2, 3) and making an angle $\frac{\pi}{4}$ in parametric form

$$\frac{x-2}{\cos\left(\frac{\pi}{4}\right)}=\frac{y-3}{\sin\left(\frac{\pi}{4}\right)}=r \text{ or } \frac{x-2}{\frac{1}{\sqrt{2}}}=\frac{y-3}{\frac{1}{\sqrt{2}}}=r$$

Coordinates of any point on this line are $\left(2+\frac{r}{\sqrt{2}},3+\frac{r}{\sqrt{2}}\right)$.

This point lies on the line $2x-3y+9=0$

$$\Rightarrow \quad 2\left(2+\frac{r}{\sqrt{2}}\right)-3\left(3+\frac{r}{\sqrt{2}}\right)+9=0$$

$$\Rightarrow \quad -\frac{r}{\sqrt{2}}+4=0$$

$$\therefore \quad r=4\sqrt{2}$$

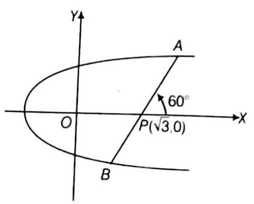
Example 42. If the line $y-\sqrt{3}x+3=0$ cuts the parabola $y^2=x+2$ at A and B, then find the value of $PA.PB$ {where $P\equiv(\sqrt{3},0)$}.

Sol. Slope of line $y-\sqrt{3}x+3=0$ is $\sqrt{3}$

If line makes an angle θ with X-axis, then $\tan\theta=\sqrt{3}$

$$\therefore \quad \theta=60^\circ$$

$$\frac{x-\sqrt{3}}{\cos 60^\circ}=\frac{y-0}{\sin 60^\circ}=r$$



$\Rightarrow \left(\sqrt{3}+\frac{r}{2},\frac{r\sqrt{3}}{2}\right)$ be a point on the parabola $y^2=x+2$

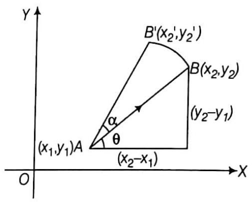
then, $\frac{3}{4}r^2=\sqrt{3}+\frac{r}{2}+2 \Rightarrow 3r^2-2r-4(2+\sqrt{3})=0$

$$\therefore \quad PA\cdot PB=r_1r_2=\left|\frac{-4(2+\sqrt{3})}{3}\right|=\frac{4(2+\sqrt{3})}{3}$$

Special Corollaries

(i) **Angle made by AB with positive X-axis (where A and B are given points)** :be two points and let AB makes an angle θ with the positive direction of X-axis and let d be the distance between A and B. Then Let $A(x_1,y_1)$ and $B(x_2,y_2)$

$$\cos\theta=\frac{x_2-x_1}{d},\sin\theta=\frac{y_2-y_1}{d}$$



where, $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=AB$

and $x_2=x_1+d\cos\theta, y_2=y_1+d\sin\theta$

If AB rotates an angle α about A, then new coordinates of B are

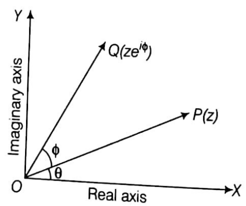
$$\boxed{\begin{aligned}x_2'&=x_1+d\cos(\theta+\alpha),\\ y_2'&=y_1+d\sin(\theta+\alpha)\end{aligned}}$$

and here, $AB=AB'=d$.

(ii) **Complex number as a rotating arrow in Argand plane** :

Let $z=r(\cos\theta+i\sin\theta)=re^{i\theta}$, where $i=\sqrt{-1}$ ... (i)

be a complex number representing a point P in the Argand plane.



Then, $OP=|z|=r$ and $\angle POX=\theta$

Now, consider complex number $z_1=ze^{i\phi}$

or $z_1=re^{i\theta}\cdot e^{i\phi}=r\cdot e^{i(\theta+\phi)}$ [from Eq. (i)]

Clearly the complex number z_1 represents a point Q in the Argand plane, when

$$OQ=r \text{ and } \angle QOX=\theta+\phi$$

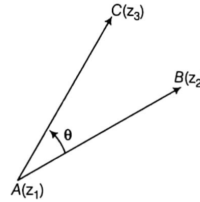
Clearly multiplication of z with $e^{i\phi}$ rotates the vector $\vec{OP}$ through angle ϕ in anti-clockwise sense. Similarly multiplication of z with $e^{-i\phi}$ will rotate the vector $\vec{OP}$ in clockwise sense.

Remark

If z_1, z_2 and z_3 are the affixes of the three points A, B and C such that $AC=AB$ and $\angle CAB=\theta$. Therefore

$$\vec{AB}=z_2-z_1, \vec{AC}=z_3-z_1$$

Then $\vec{AC}$ will be obtained by rotating $\vec{AB}$ through an angle θ in anticlockwise sense and therefore



$$\vec{AC}=\vec{AB}e^{i\theta}$$

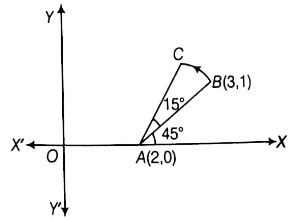
or $(z_3-z_1)=(z_2-z_1)e^{i\theta}$ or $\left(\frac{z_3-z_1}{z_2-z_1}\right)=e^{i\theta}$

Example 43. Must The line joining the points $A(2,0)$ and $B(3,1)$ is rotated about A in the anticlockwise direction through an angle of 15°. Find the equation of the line in the new position. If B goes to C in the new position, what will be the coordinates of C?

Sol. By special corollary (i)

Here $AB=\sqrt{(2-3)^2+(0-1)^2}=\sqrt{2}$

and slope of $AB=\frac{1-0}{3-2}=1=\tan 45^\circ$



$\therefore \quad \angle BAX=45^\circ$

Now line AB is rotated through an angle of 15°

$\Rightarrow \quad \angle CAX=45^\circ+15^\circ=60^\circ$

and $\quad AB=AC=\sqrt{2}$

Equation of line AC in parametric form is

$$\left.\begin{aligned}x&=2+r\cos 60^\circ\\ y&=0+r\sin 60^\circ\end{aligned}\right\} \quad ...(i)$$

Since, $AC=r=\sqrt{2}$

Put $r=\sqrt{2}$ in Eq. (i), then

$$x=2+\sqrt{2}\cdot\frac{1}{2}=\frac{4+\sqrt{2}}{2}$$

and $$y=\sqrt{2}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{6}}{2}$$

Equation of the line AC is

$$\frac{x-2}{y}=\cot 60^\circ=\frac{1}{\sqrt{3}}$$

or $$x\sqrt{3}-y-2\sqrt{3}=0$$

and coordinates of C are $\left(\frac{4+\sqrt{2}}{2},\frac{\sqrt{6}}{2}\right)$.

Aliter (By special corollary (ii))

$\because \quad A\equiv(2,0), B\equiv(3,1)$, let $C\equiv(x,y)$

$\therefore \quad z_A=2, z_B=3+i, z_C=x+iy$, where $i=\sqrt{-1}$

$$\frac{z_C-z_A}{z_B-z_A}=e^{i\frac{5\pi}{12}} \quad \left(\because 15^\circ=\frac{5\pi}{12}\right)$$

$\Rightarrow \quad z_C-2=(1+i)(\cos 15^\circ+i\sin 15^\circ)$

or $$z_C=2+(1+i)\left(\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)+i\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\right)$$

$$=\left(2+\frac{\sqrt{3}+1}{2\sqrt{2}}-\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+i\left(\frac{\sqrt{3}+1}{2\sqrt{2}}+\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$$

$$=\left(2+\frac{1}{\sqrt{2}}\right)+i\left(\frac{\sqrt{3}}{\sqrt{2}}\right)=\frac{4+\sqrt{2}}{2}+i\left(\frac{\sqrt{6}}{2}\right)$$

$$\therefore \quad C\equiv\left(\frac{4+\sqrt{2}}{2},\frac{\sqrt{6}}{2}\right)$$

and equation of AC

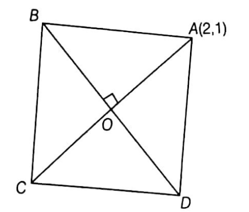
$$y-0=\tan 60^\circ(x-2)\Rightarrow x\sqrt{3}-y-2\sqrt{3}=0$$

Example 44. The centre of a square is at the origin and one vertex is $A(2,1)$. Find the coordinates of other vertices of the square. Must

Sol. [By special corollary (ii)]

$\because \quad A\equiv(2,1)$

$\therefore \quad z_A=2+i$, where $i=\sqrt{-1}$



Now, in triangle AOB,

$$OA = OB,\ \angle AOB = 90^\circ = \frac{\pi}{2}$$

$\therefore\quad z_B = z_A e^{i\frac{\pi}{2}} = iz_A = 2i - 1$

$\therefore\quad B \equiv (-1, 2)$

$\because$ O is the mid-point of AC and BD

$\therefore\quad C \equiv (-2, -1)$ and $D \equiv (1, -2)$.

Example 45. The extremities of the diagonal of a square are $(1, 1)$, $(-2, -1)$. Obtain the other two vertices and the equation of the other diagonal.

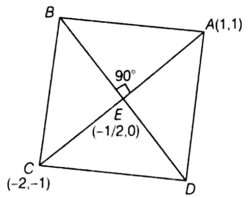
Sol. (By special corollary (ii))

$\because\quad A \equiv (1, 1)$

$\therefore\quad z_A = 1 + i$, where $i = \sqrt{-1}$

and $\quad C \equiv (-2, -1)$

$\therefore\quad z_C = -2 - i$



then centre of square $E \equiv \left(-\frac{1}{2}, 0\right)$

$\therefore\quad z_E = -\frac{1}{2}$

Now, in ΔAEB, $(EA = EB)$

$$\frac{z_B - z_E}{z_A - z_E} = e^{i\frac{\pi}{2}} = i$$

$\Rightarrow\quad z_B + \frac{1}{2} = i\left(1 + i + \frac{1}{2}\right)$

$\therefore\quad z_B = -\frac{3}{2} + \frac{3}{2}i$

$$B \equiv \left(-\frac{3}{2}, \frac{3}{2}\right)$$

then $\quad D \equiv \left(-1 + \frac{3}{2}, -\frac{3}{2}\right)$

or $\quad D \equiv \left(\frac{1}{2}, -\frac{3}{2}\right)$

Hence, equation of other diagonal BD is

$$y - 0 = \frac{\frac{3}{2} - 0}{-\frac{3}{2} + \frac{1}{2}}\left(x + \frac{1}{2}\right)$$

$\Rightarrow\quad 6x + 4y + 3 = 0$

Exercise for Session 1

1. The distance of the point (3, 5) from the line $2x + 3y - 14 = 0$ measured parallel to the line $x - 2y = 1$ is
(a) $\frac{7}{\sqrt{5}}$ (b) $\frac{7}{\sqrt{13}}$ (c) $\sqrt{5}$ (d) $\sqrt{13}$

2. The lines $x\cos\alpha + y\sin\alpha = p_1$ and $x\cos\beta + y\sin\beta = p_2$ will be perpendicular, if
(a) $\alpha = \beta$ (b) $|\alpha - \beta| = \pi/2$
(c) $\alpha = \pi/2$ (d) $\alpha \pm \beta = \pi/2$

3. If each of the points $(x_1, 4)$, $(-2, y_1)$ lies on the line joining the points $(2, -1)$, $(5, -3)$, then the point $P(x_1, y_1)$ lies on the line
(a) $6(x + y) - 25 = 0$ (d) $2x + 6y + 1 = 0$
(c) $2x + 3y - 6 = 0$ (d) $6(x + y) + 25 = 0$

4. The equation of the straight line passing through the point (4, 3) and making intercepts on the coordinate axes whose sum is –1 is
(a) $\frac{x}{2} + \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$ (b) $\frac{x}{2} - \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$
(c) $\frac{x}{2} + \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$ (d) $\frac{x}{2} - \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$

5. If the straight lines $ax + by + c = 0$ and $x\cos\alpha + y\sin\alpha = c$ enclose an angle $\pi/4$ between them and meet the straight line $x\sin\alpha - y\cos\alpha = 0$ in the same point, then
(a) $a^2 + b^2 = c^2$ (b) $a^2 + b^2 = 2$ (c) $a^2 + b^2 = 2c^2$ (d) $a^2 + b^2 = 4$

6. The angle between the lines $2x - y + 3 = 0$ and $x + 2y + 3 = 0$ is
(a) 30° (b) 45° (c) 60° (d) 90°

7. The inclination of the straight line passing through the point (–3, 6) and the mid-point of the line joining the points (4, –5) and (–2, 9) is
(a) $\pi/4$ (b) $\pi/2$ (c) $3\pi/4$ (d) π

8. A square of side a lies above the X-axis and has one vertex at the origin. The side passing through the origin makes an angle $\pi/6$ with the positive direction of X-axis. The equation of its diagonal not passing through the origin is
(a) $y(\sqrt{3} - 1) - x(1 - \sqrt{3}) = 2a$ (b) $y(\sqrt{3} + 1) + x(1 - \sqrt{3}) = 2a$
(c) $y(\sqrt{3} + 1) + x(1 + \sqrt{3}) = 2a$ (d) $y(\sqrt{3} + 1) + x(\sqrt{3} - 1) = 2a$

9. $A(1, 3)$ and $C(7, 5)$ are two opposite vertices of a square. The equation of side through A is
(a) $x + 2y - 7 = 0$ (b) $x - 2y + 5 = 0$
(c) $2x + y - 5 = 0$ (d) $2x - y + 1 = 0$

10. The equation of a straight line passing through the point $(-5, 4)$ and which cuts off an intercept of $\sqrt{2}$ units between the lines $x + y + 1 = 0$ and $x + y - 1 = 0$ is
(a) $x - 2y + 13 = 0$ (b) $2x - y + 14 = 0$
(c) $x - y + 9 = 0$ (d) $x - y + 10 = 0$

11. Equation to the straight line cutting off an intercept 2 from negative direction of the axis of y and inclined at 30° to the positive direction of axis of x is
(a) $y + x - \sqrt{3} = 0$ (b) $y - x + 2 = 0$
(c) $y - x\sqrt{3} - 2 = 0$ (d) $y\sqrt{3} - x + 2\sqrt{3} = 0$

12. What is the value of y so that the line through $(3, y)$ and $(2, 7)$ is parallel to the line through $(-1, 4)$ and $(0, 6)$?

13. A straight line is drawn through the point $P(2, 2)$ and is inclined at an angle of 30° with the X-axis. Find the coordinates of two points on it at a distance 4 from P on either side of P.

14. If the straight line through the point $P(3, 4)$ makes an angle $\frac{\pi}{6}$ with X-axis and meets the line $12x + 5y + 10 = 0$ at Q, find the length of PQ.

15. Find the distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along the line $x - y + 1 = 0$.

16. A line is such that its segment between the straight line $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point $(1, 5)$. Obtain the equation.

17. The side AB and AC of a ΔABC are respectively $2x + 3y = 29$ and $x + 2y = 16$. If the mid-point of BC is (5, 6), then find the equation of BC.

18. A straight line through $A(-15, -10)$ meets the lines $x - y - 1 = 0$, $x + 2y = 5$ and $x + 3y = 7$ respectively at A, B and C. If $\frac{12}{AB} + \frac{40}{AC} = \frac{52}{AD}$, prove that the line passes through the origin.

Session 2

Position of Two Points Relative to a Given Line, Position of a Point Which Lies Inside a Triangle, Equations of Lines Parallel and Perpendicular to a Given Line, Distance of a Point From a Line, Distance Between Two Parallel Lines, Area of Parallelogram,

Position of Two Points Relative to a Given Line

Theorem : The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same or opposite sides of the line $ax + by + c = 0$ according as

$$\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0 \text{ or } < 0.$$

Proof : Let the line PQ be divided by the line $ax + by + c = 0$ in the ratio $\lambda : 1$ (internally) at the point R.

$\therefore$ The coordinates of R are $\left(\frac{x_1 + \lambda x_2}{1+\lambda}, \frac{y_1 + \lambda y_2}{1+\lambda}\right)$

The point of R lies on the line $ax + by + c = 0$

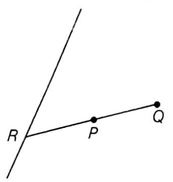
then $a\left(\frac{x_1 + \lambda x_2}{1+\lambda}\right) + b\left(\frac{y_1 + \lambda y_2}{1+\lambda}\right) + c = 0$

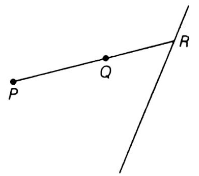
$\Rightarrow \quad \lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c) = 0$

$\Rightarrow \quad \lambda = -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) \qquad (\because ax_2 + by_2 + c \neq 0)$

Case I : Let P and Q are on same side of the line $ax + by + c = 0$.

R divides PQ externally.





$\therefore$ λ is negative

$\Rightarrow \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$

$\Rightarrow \quad \left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) > 0$

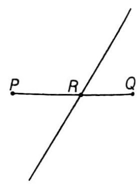
or $\quad \frac{f(x_1, y_1)}{f(x_2, y_2)} > 0$

where, $f(x, y) \equiv ax + by + c$.

Case II : Let P and Q are on opposite sides of the line $ax + by + c = 0$

$\therefore$ R divides PQ internally.

$\therefore$ λ is positive



$\Rightarrow \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) > 0$

$\Rightarrow \quad \left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$

or $\quad \frac{f(x_1, y_1)}{f(x_2, y_2)} < 0$

where, $f(x, y) = ax + by + c$

Remarks

1. The side of the line where origin lies is known as origin side.
2. A point (α, β) will lie on origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have same sign.
3. A point (α, β) will lie on non-origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have opposite sign.

| Example 46. Are the points $(2, 1)$ and $(-3, 5)$ on the same or opposite side of the line $3x - 2y + 1 = 0$?

Sol. Let $f(x, y) \equiv 3x - 2y + 1$

$\therefore \quad \frac{f(2,1)}{f(-3,5)} = \frac{3(2) - 2(1) + 1}{3(-3) - 2(5) + 1} = -\frac{5}{18} < 0$

Therefore, the two points are on the opposite sides of the given line.

| Example 47. Is the point $(2, -7)$ lies on origin side of the line $2x + y + 2 = 0$?

Sol. Let $f(x, y) \equiv 2x + y + 2$

$\therefore \quad f(2, -7) = 2(2) - 7 + 2 = -1$

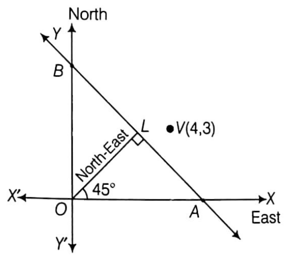
$f(2, -7) < 0$ and constant $2 > 0$

Hence, the point $(2, -7)$ lies on non-origin side.

| Example 48. A straight canal is at a distance of $4\frac{1}{2}$ km from a city and the nearest path from the city to the canal is in the north-east direction. Find whether a village which is at 3 km north and 4 km east from the city lies on the canal or not. If not, then on which side of the canal is the village situated ?

Sol. Let $O(0, 0)$ be the given city and AB be the straight canal.

Given, $OL = \frac{9}{2}$ km



$\therefore$ Equation of AB

i.e. Equation of canal is

$$x \cos 45° + y \sin 45° = \frac{9}{2}$$

or $\quad x + y = \frac{9}{\sqrt{2}} \quad$ ...(i)

Let V be the given village, then $V \equiv (4, 3)$

Putting $x = 4$ and $y = 3$ in Eq. (i),

then $4 + 3 = \frac{9}{\sqrt{2}}$, i.e. $7 = \frac{9}{\sqrt{2}}$ which is impossible.

Hence, the given village V does not lie on the canal.

Also if $f(x, y) \equiv x + y - \frac{9}{\sqrt{2}}$

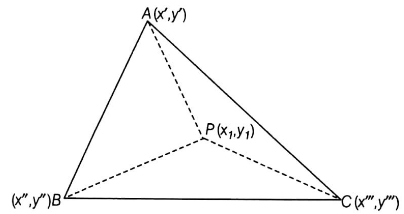
$\therefore \quad \frac{f(4,3)}{f(0,0)} = \left(\frac{4 + 3 - \frac{9}{\sqrt{2}}}{0 + 0 - \frac{9}{\sqrt{2}}}\right) = -\left(\frac{7\sqrt{2} - 9}{9}\right) < 0$

Hence, the village is on that side of the canal on which origin or the city lies.

Position of a Point Which Lies Inside a Triangle

Let $P(x_1, y_1)$ be the point and equations of the sides of a triangle are

$$BC : a_1x + b_1y + c_1 = 0$$
$$CA : a_2x + b_2y + c_2 = 0$$
and $$AB : a_3x + b_3y + c_3 = 0$$



First find the coordinates of A, B and C say,

$$A \equiv (x', y');\ B \equiv (x'', y'') \text{ and } C \equiv (x''', y''')$$

and if coordinates of A, B, C are given, then find equations of BC, CA and AB.

If $P(x_1, y_1)$ lies inside the triangle, then P and A must be on the same side of BC, P and B must be on the same side of AC, P and C must be on the same side of AB, then

$$\frac{a_1x_1 + b_1y_1 + c_1}{a_1x' + b_1y' + c_1} > 0 \quad ...(i)$$

$$\frac{a_2x_1 + b_2y_1 + c_2}{a_2x'' + b_2y'' + c_2} > 0 \quad ...(ii)$$

and $$\frac{a_3x_1 + b_3y_1 + c_3}{a_3x''' + b_3y''' + c_3} > 0 \quad ...(iii)$$

The required values of $P(x_1, y_1)$ must be intersection of these inequalities Eqs. (i), (ii) and (iii).

...hod) : First draw the exact diagram of ...he point $P(x_1, y_1)$ ...ne $y = ax + b$ for all x_1, then

$$P \equiv (x_1, ax_1 + b)$$

and the portion DE of the line $y = ax + b$ (Excluding D and E) lies within the triangle. Now line $y = ax + b$ cuts any two sides out of three sides, then find coordinates of D and E.

$D \equiv (\alpha, \beta)$

and $E \equiv (\gamma, \delta)$ (say)

then $\alpha < x_1 < \gamma$

and $\beta < ax_1 + b < \delta$

| Example 49. For what values of the parameter t does the point $P(t, t+1)$ lies inside the triangle ABC where $A \equiv (0,3), B \equiv (-2,0)$ and $C \equiv (6,1)$.

Sol. Equations of sides

$BC : x - 8y + 2 = 0$

$CA : x + 3y - 9 = 0$

and $AB : 3x - 2y + 6 = 0$

Since, $P(t, t+1)$ lies inside the triangle ABC, then P and A must be on the same side of BC

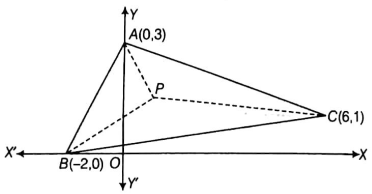
$\therefore \dfrac{\text{value of } (x - 8y + 2) \text{ at } P(t, t+1)}{\text{value of } (x - 8y + 2) \text{ at } A(0, 3)} > 0$

i.e. $\dfrac{t - 8(t+1) + 2}{0 - 24 + 2} > 0$

or $\dfrac{-7t - 6}{-22} > 0$

or $7t + 6 > 0$

$\therefore t > -\dfrac{6}{7}$ ...(i)



and P, B must be on the same side of CA

$\therefore \dfrac{\text{value of } (x + 3y - 9) \text{ at } P(t, t+1)}{\text{value of } (x + 3y - 9) \text{ at } B(-2, 0)} > 0$

i.e. $\dfrac{t + 3(t+1) - 9}{-2 + 0 - 9} > 0$

or $\dfrac{4t - 6}{-11} > 0$

or $4t - 6 < 0$

$\therefore t < \dfrac{3}{2}$ ...(ii)

and P, C must be on the same side of AB

$\therefore \dfrac{\text{value of } (3x - 2y + 6) \text{ at } P(t, t+1)}{\text{value of } (3x - 2y + 6) \text{ at } C(6, 1)} > 0$

i.e. $\dfrac{3t - 2(t+1) + 6}{18 - 2 + 6} > 0$

or $\dfrac{t + 4}{22} > 0$

or $t + 4 > 0$

$\therefore t > -4$ ...(iii)

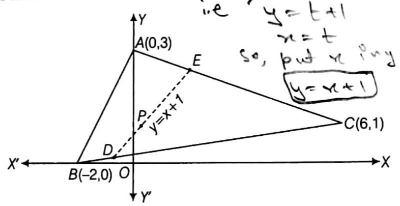
From Eqs. (i), (ii) and (iii), we get

$$-\frac{6}{7} < t < \frac{3}{2}$$

i.e. $t \in \left(-\dfrac{6}{7}, \dfrac{3}{2}\right)$

Aliter : First draw the exact diagram of ΔABC, the point $P(t, t+1)$ move on the line $y = x + 1$ for all t.

Points ke hisab se eqn bonao
i.e $y = t+1$
$x = t$
So, put x in y
$\boxed{y = x + 1}$



Now, D and E are the intersection of

$y = x + 1, x - 8y + 2 = 0$

and $y = x + 1, \; x + 3y - 9 = 0$

respectively.

$\therefore D \equiv \left(-\dfrac{6}{7}, \dfrac{1}{7}\right)$

and $E \equiv \left(\dfrac{3}{2}, \dfrac{5}{2}\right)$

Thus, the points on the line $y = x + 1$ whose x-coordinates lies between $-\dfrac{6}{7}$ and $\dfrac{3}{2}$ lie within the triangle ABC.

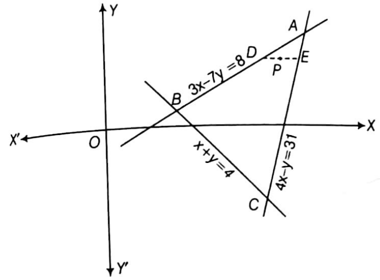
Hence, $-\dfrac{6}{7} < t < \dfrac{3}{2}$

i.e. $t \in \left(-\dfrac{6}{7}, \dfrac{3}{2}\right)$

| Example 50. Find λ if $(\lambda, 2)$ is an interior point of ΔABC formed by $x + y = 4, 3x - 7y = 8$ and $4x - y = 31$.

Sol. Let $P \equiv (\lambda, 2)$

First draw the exact diagram of ΔABC, the point $P(\lambda, 2)$ move on the line $y = 2$ for all λ.



Now, D and E are the intersection of

$3x - 7y = 8, y = 2$

and $4x - y = 31, y = 2$ respectively.

$\therefore D \equiv \left(\dfrac{22}{3}, 2\right)$ and $E \equiv \left(\dfrac{33}{4}, 2\right)$

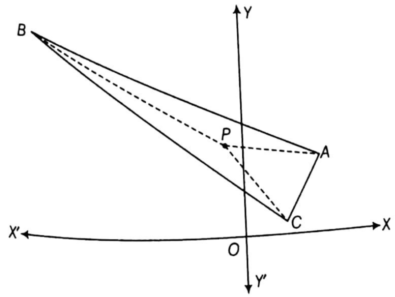
Thus, the points on the line $y = 2$ whose x-coordinates lies between $\dfrac{22}{3}$ and $\dfrac{33}{4}$ lie within the ΔABC.

Hence, $\dfrac{22}{3} < \lambda < \dfrac{33}{4}$

i.e. $\lambda \in \left(\dfrac{22}{3}, \dfrac{33}{4}\right)$

| Example 51. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$.

Sol. The coordinates of the vertices are



$$A\left(\frac{5}{4}, \frac{7}{8}\right), B(-7, 5) \text{ and } C\left(\frac{1}{3}, \frac{1}{9}\right)$$

$\because P(\alpha, \alpha^2)$ lies inside the ΔABC, then

(i) A and P must lie on the same side of BC

(ii) B and P must lie on the same side of CA

(iii) C and P must lie on the same side of AB, then

$$\frac{\frac{5}{2} + \frac{21}{8} - 1}{2\alpha + 3\alpha^2 - 1} > 0$$

$\Rightarrow \dfrac{33}{2\alpha + 3\alpha^2 - 1} > 0$

or $3\alpha^2 + 2\alpha - 1 > 0$

$\Rightarrow (\alpha + 1)\left(\alpha - \dfrac{1}{3}\right) > 0$

$\Rightarrow \alpha \in (-\infty, -1) \cup \left(\dfrac{1}{3}, \infty\right)$ ...(i)

and $\dfrac{-35 - 30 - 1}{5\alpha - 6\alpha^2 - 1} > 0$

$\Rightarrow 5\alpha - 6\alpha^2 - 1 < 0$

$\Rightarrow \left(\alpha - \dfrac{1}{3}\right)\left(\alpha - \dfrac{1}{2}\right) > 0$

$\therefore \alpha \in (-\infty, 1/3) \cup (1/2, \infty)$ ..(ii)

and $\dfrac{\frac{1}{3} + \frac{2}{9} - 3}{\alpha + 2\alpha^2 - 3} > 0$

$\Rightarrow \alpha + 2\alpha^2 - 3 < 0$

$\Rightarrow (2\alpha + 3)(\alpha - 1) < 0$

$\therefore \alpha \in (-3/2, 1)$ ...(iii)

From Eq. (i), Eq. (ii) and Eq. (iii), we get

$$\alpha \in (-3/2, -1) \cup (1/2, 1).$$

Aliter : Let $P(\alpha, \alpha^2)$ first draw the exact diagram of ΔABC.

The point $P(\alpha, \alpha^2)$ move on the curve $y = x^2$ for all α.

Now, intersection of $y = x^2$

and $2x + 3y - 1 = 0$

or $2x + 3x^2 - 1 = 0$

$\therefore x = -1, x = \dfrac{1}{3}$

Let intersection points

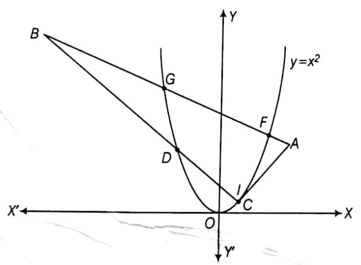
$D \equiv (-1, 1)$ and $E \equiv \left(\dfrac{1}{3}, \dfrac{1}{9}\right)$

intersection of $y = x^2$

and $x + 2y - 3 = 0$

or $x + 2x^2 - 3 = 0$

$\therefore x = 1, x = -3/2$



Let intersection points

$$F \equiv (1, 1) \text{ and } G \equiv \left(-\frac{3}{2}, \frac{9}{4}\right)$$

and intersection of $y = x^2$ and $5x - 6y - 1 = 0$

or $\quad 5x - 6x^2 - 1 = 0$

$\therefore \quad x = \frac{1}{3}, x = \frac{1}{2}$

Let intersection points

$$H \equiv \left(\frac{1}{3}, \frac{1}{9}\right) \text{ and } I \equiv \left(\frac{1}{2}, \frac{1}{4}\right)$$

Thus the points on the curve $y = x^2$ whose x-coordinate lies between $-3/2$ & -1 and $\frac{1}{2}$ & 1 lies within the triangle ABC.

Hence, $\quad -\frac{3}{2} < \alpha < -1$ or $\frac{1}{2} < \alpha < 1$

i.e. $\quad \alpha \in \left(-\frac{3}{2}, -1\right) \cup \left(\frac{1}{2}, 1\right)$

Equations of Lines Parallel and Perpendicular to a Given Line

Theorem 1: The equation of line parallel to $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is some constant.

Proof : Let the equation of any line parallel to

$$ax + by + c = 0 \quad \text{...(i)}$$

be

$$a_1x + b_1y + c_1 = 0 \quad \text{...(ii)}$$

then

$$\frac{a_1}{a} = \frac{b_1}{b} = k \quad \text{(say)}$$

$\therefore$

$$a_1 = ak, b_1 = bk$$

Then from Eq. (ii),

$$akx + bky + c = 0$$

Dividing it by k, then

$$ax + by + \frac{c}{k} = 0$$

or

$$ax + by + \lambda = 0 \quad \left(\text{writing } \lambda \text{ for } \frac{c}{k}\right)$$

Hence, any line parallel to $ax + by + c = 0$ is

$$ax + by + \lambda = 0$$

where λ is some constant.

Aliter : The given line is

$$ax + by + c = 0 \quad \text{...(i)}$$

Its slope $= -\frac{a}{b}$

Thus, any line parallel to Eq. (i) is given by

$$y = \left(-\frac{a}{b}\right)x + \lambda_1$$

$\Rightarrow \quad ax + by - b\lambda_1 = 0$

$\Rightarrow \quad ax + by + \lambda = 0 \quad$ (writing λ for $-b\lambda_1$)

where, λ is some constant.

Corollary : The equation of the line parallel to $ax + by + c = 0$ and passing through (x_1, y_1) is

$$a(x - x_1) + b(y - y_1) = 0$$

Working Rule :

(i) Keep the terms containing x and y unaltered.

(ii) Change the constant.

(iii) The constant λ is determined from an additional condition given in the problem.

Theorem 2 : The equation of the line perpendicular to the line $ax + by + c = 0$ is

$bx - ay + \lambda = 0$, where λ is some constant.

Proof : Let the equation of any line perpendicular to

$$ax + by + c = 0 \quad \text{... (i)}$$

be

$$a_1x + b_1y + c_1 = 0 \quad \text{...(ii)}$$

then

$$aa_1 + bb_1 = 0$$

or

$$aa_1 = -bb_1$$

$\Rightarrow$

$$\frac{a_1}{b} = \frac{b_1}{-a} = k \quad \text{(say)}$$

$\therefore$

$$a_1 = bk, b_1 = -ak$$

Then, from Eq. (ii), $bkx - aky + c_1 = 0$ dividing it by k, then

$$bx - ay + \frac{c_1}{k} = 0$$

or

$$bx - ay + \lambda = 0 \quad \left(\text{writing } \lambda \text{ for } \frac{c_1}{k}\right)$$

Hence, any line perpendicular to $ax + by + c = 0$ is

$$bx - ay + \lambda = 0$$

where, λ is some constant.

Aliter : The given line is

$$ax + by + c = 0 \quad \text{...(i)}$$

Its slope $= -\frac{a}{b}$

Slope of perpendicular line of Eq. (i) is $\frac{b}{a}$.

Thus any line perpendicular to Eq. (i) is given by

$$y = \left(\frac{b}{a}\right)x + \lambda_1$$

$\Rightarrow \quad bx - ay + a\lambda_1 = 0$

or $\quad bx - ay + \lambda = 0 \quad$ (writing λ for $a\lambda_1$)

where, λ is some constant.

Corollary 1 : The equation of the line through (x_1, y_1) and perpendicular to $ax + by + c = 0$ is

$$b(x - x_1) - a(y - y_1) = 0$$

Corollary 2 : Also equation of the line perpendicular to $ax + by + c = 0$ is written as

$$\frac{x}{a} - \frac{y}{b} + k = 0, \text{ where } k \text{ is some constant.}$$

Working Rule :

(i) Interchange the coefficients of x and y and changing sign of one of these coefficients.

(ii) Changing the constant term.

(iii) The value of λ can be determined from an additional condition given in the problem.

Example 52. Find the general equation of the line which is parallel to $3x - 4y + 5 = 0$. Also find such line through the point $(-1, 2)$.

Sol. Equation of any parallel to $3x - 4y + 5 = 0$ is

$$3x - 4y + \lambda = 0 \quad \text{...(i)}$$

which is general equation of the line.

Also Eq. (i) passes through $(-1, 2)$, then

$$3(-1) - 4(2) + \lambda = 0$$

$\therefore \quad \lambda = 11$

Then from Eq. (i) required line is

$$3x - 4y + 11 = 0$$

Example 53. Find the general equation of the line which is perpendicular to $x + y + 4 = 0$. Also find such line through the point $(1, 2)$.

Sol. Equation of any line perpendicular to $x + y + 4 = 0$ is

$$x - y + \lambda = 0 \quad \text{...(i)}$$

which is general equation of the line.

Also Eq. (i) passes through $(1, 2)$, then

$$1 - 2 + \lambda = 0$$

$\therefore \quad \lambda = 1$

Then from Eq. (i), required line is

$$x - y + 1 = 0$$

Example 54. Show that the equation of the line passing through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line

$x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ is

$x\cos\theta - y\sin\theta = a\cos 2\theta$

Sol. The given equation $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ can be written as

$$x\sin\theta + y\cos\theta = a\sin\theta\cos\theta \quad \text{...(i)}$$

$\therefore$ equation of perpendicular line of Eq. (i) is

$$x\cos\theta - y\sin\theta = \lambda \quad \text{...(ii)}$$

Also it is pass through $(a\cos^3\theta, a\sin^3\theta)$

$\therefore \quad a\cos^3\theta \cdot \cos\theta - a\sin^3\theta \cdot \sin\theta = \lambda$

$\Rightarrow \quad \lambda = a(\cos^4\theta - \sin^4\theta)$

$$= a(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$

$$= a \cdot 1 \cdot \cos 2\theta = a\cos 2\theta$$

From Eq. (ii), the required equation of the line is

$$x\cos\theta - y\sin\theta = a\cos 2\theta$$

Aliter : (From corollary (2) of Theorem (2)

Equation of any line perpendicular to the line $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$, is

$$\frac{x}{\sec\theta} - \frac{y}{\mathrm{cosec}\,\theta} = k$$

or $\quad x\cos\theta - y\sin\theta = k \quad$ ...(iii)

Also, it pass through $(a\cos^3\theta, a\sin^3\theta)$

$\therefore \quad a\cos^3\theta \cdot \cos\theta - a\sin^3\theta \cdot \sin\theta = k$

or $\quad k = a(\cos^4\theta - \sin^4\theta)$

$$= a(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$

$$= a \cdot 1 \cdot \cos 2\theta$$

$$= a\cos 2\theta$$

From Eq. (iii), the required equation of the line is

$$x\cos\theta - y\sin\theta = a\cos 2\theta$$

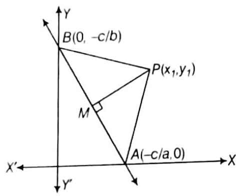
Distance of a Point From a Line

Theorem : The length of perpendicular from a point (x_1, y_1) to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Proof : Given line is $ax + by + c = 0$

$$\Rightarrow \quad \frac{x}{\left(-\frac{c}{a}\right)} + \frac{y}{\left(-\frac{c}{b}\right)} = 1$$



Let the given line intersects the X-axis and Y-axis at A and B respectively, then coordinates of A and B are $\left(-\frac{c}{a}, 0\right)$ and $\left(0, -\frac{c}{b}\right)$ respectively.

Draw PM perpendicular to AB.

Now, Area of $\Delta\, PAB$

$$= \frac{1}{2}\left| x_1\left(0 + \frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0\,(y_1 - 0)\right|$$

$$= \frac{1}{2}\left|\frac{c}{ab}\right| |ax_1 + by_1 + c| \quad \text{...(i)}$$

Let $PM = p$

Also, area of $\Delta\, PAB$

$$= \frac{1}{2}\cdot AB \cdot PM = \frac{1}{2}\sqrt{\left\{\left(-\frac{c}{a} - 0\right)^2 + \left(0 + \frac{c}{b}\right)^2\right\}}\cdot p$$

$$= \frac{1}{2}\left|\frac{c}{ab}\right| \sqrt{(a^2 + b^2)}\cdot p \quad \text{... (ii)}$$

From Eqs. (i) and (ii), we have

$$\frac{1}{2}\left|\frac{c}{ab}\right| \sqrt{(a^2 + b^2)}\cdot p = \frac{1}{2}\left|\frac{c}{ab}\right| |ax_1 + by_1 + c|$$

or $$p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Aliter I : Let PM makes an angle θ with positive direction of X-axis.

Then, equation of PM in distance form will be

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = p \quad (\because PM = p)$$

Therefore coordinates of M will be

$$(x_1 + p\cos\theta,\ y_1 + p\sin\theta)$$

Since, M lies on $ax + by + c = 0$, then

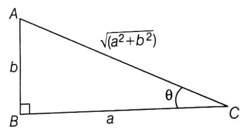
$$a(x_1 + p\cos\theta) + b(y_1 + p\sin\theta) + c = 0$$

or $$p(a\cos\theta + b\sin\theta) = -(ax_1 + by_1 + c) \quad \text{...(iii)}$$

Since, slope of $AB = -\frac{a}{b}$

$\therefore$ Slope of $PM = \frac{b}{a}$

$\therefore$ $\tan\theta = \frac{b}{a}$ ($\because$ PM makes an angle θ with positive direction of X-axis)



then $$\sin\theta = \frac{b}{\sqrt{(a^2 + b^2)}}$$

and $$\cos\theta = \frac{a}{\sqrt{(a^2 + b^2)}}$$

Now, from Eq. (iii),

$$p\left(\frac{a^2}{\sqrt{(a^2 + b^2)}} + \frac{b^2}{\sqrt{(a^2 + b^2)}}\right) = -(ax_1 + by_1 + c)$$

or $$p = -\frac{(ax_1 + by_1 + c)}{\sqrt{(a^2 + b^2)}}$$

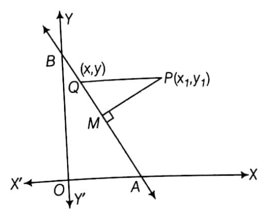
Since, p is positive

$\therefore$ $$p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Aliter II : Let $Q\,(x, y)$ be any point on the line

$$ax + by + c = 0$$

Hence, the length of perpendicular from P on AB will be least value of PQ.



Let $$z = (PQ)^2$$

$$= (x - x_1)^2 + (y - y_1)^2 \quad \text{...(iv)}$$

$$= (x - x_1)^2 + \left(-\frac{c}{b} - \frac{ax}{b} - y_1\right)^2$$

$$\left(\begin{array}{l} \because ax + by + c = 0 \\ \therefore y = -\frac{c}{b} - \frac{ax}{b} \end{array}\right)$$

$\therefore$ $$\frac{dz}{dx} = 2(x - x_1) + 2\left(-\frac{c}{b} - \frac{ax}{b} - y_1\right)\left(-\frac{a}{b}\right)$$

and $$\frac{d^2z}{dx^2} = 2 + 2\left(-\frac{a}{b}\right)\left(-\frac{a}{b}\right) = 2\left(1 + \frac{a^2}{b^2}\right) = \text{positive}$$

$\because$ z is minimum

$\therefore$ PQ is also minimum.

For maximum or minimum, $\frac{dz}{dx} = 0$

$$2(x - x_1) + 2\left(-\frac{c}{b} - \frac{ax}{b} - y_1\right)\left(-\frac{a}{b}\right) = 0$$

or $$2(x - x_1) + 2(y - y_1)\left(-\frac{a}{b}\right) = 0 \quad \left(\because y = -\frac{c}{b} - \frac{ax}{b}\right)$$

or $$\frac{(x - x_1)}{a} = \frac{(y - y_1)}{b} = \frac{a(x - x_1) + b(y - y_1)}{a\cdot a + b\cdot b}$$

$$= \frac{(ax + by + c) - (ax_1 + by_1 + c)}{(a^2 + b^2)}$$

(by law of proportion)

$$= \frac{0 - (ax_1 + by_1 + c)}{(a^2 + b^2)} \quad (\because ax + by + c = 0)$$

$$\Rightarrow \quad (x - x_1) = -\frac{a(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

and $$(y - y_1) = -\frac{b(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

$\therefore$ From Eq. (iv),

$$PQ = \sqrt{(ax_1 + by_1 + c)^2\,\frac{(a^2 + b^2)}{(a^2 + b^2)^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

$\because$ Least value of PQ is PM

$\therefore$ $$p = PM = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Aliter III : Let $M \equiv (h, k)$

Since, $M\,(h, k)$ lies on AB,

$$ah + bk + c = 0 \quad \text{..(v)}$$

Now, AB and PM are perpendicular to each other, then

(slope of PM) $\times$ (slope of AB) $= -1$

$$\Rightarrow \quad \frac{y_1 - k}{x_1 - h} \times \left(-\frac{a}{b}\right) = -1$$

$$\Rightarrow \quad \frac{x_1 - h}{a} = \frac{y_1 - k}{b} = \frac{a(x_1 - h) + b(y_1 - k)}{a\,.\,a + b\,.\,b}$$

(by law of proportion)

$$= \frac{(ax_1 + by_1 + c) - (ah + bk + c)}{a^2 + b^2}$$

$$= \frac{ax_1 + by_1 + c}{a^2 + b^2} \quad \text{[from Eq. (v)]}$$

$\therefore$ $$(PM)^2 = (x - x_1)^2 + (y - y_1)^2$$

$$= \left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)^2 (a^2 + b^2)$$

$\therefore$ Length of perpendicular

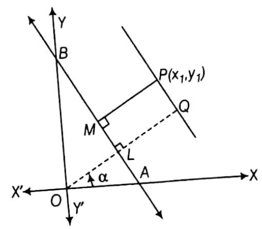
$$PM = \pm\frac{(ax_1 + by_1 + c)}{\sqrt{(a^2 + b^2)}}$$

Hence, $$PM = p = \frac{|ax_1 + by_1 + c)}{\sqrt{(a^2 + b^2)}}$$

Aliter IV : Equation of AB in normal form is

$$\frac{a}{\sqrt{(a^2 + b^2)}}\,x + \frac{b}{\sqrt{(a^2 + b^2)}}\,y = \frac{-c}{\sqrt{(a^2 + b^2)}}$$

$$\Rightarrow \quad OL = -\frac{c}{\sqrt{(a^2 + b^2)}}$$



Equation of line parallel to AB and passes through (x_1, y_1) is

$$a(x - x_1) + b(y - y_1) = 0$$

or $$ax + by = ax_1 + by_1$$

Normal form is

$$\frac{a}{\sqrt{(a^2+b^2)}}x + \frac{b}{\sqrt{(a^2+b^2)}}y = \frac{ax_1 + by_1}{\sqrt{(a^2+b^2)}}$$

$$\Rightarrow \quad OQ = \frac{ax_1 + by_1}{\sqrt{(a^2+b^2)}}$$

$$\therefore \quad PM = QL = OQ - OL = \frac{ax_1 + by_1 + c}{\sqrt{(a^2+b^2)}}$$

Hence, required perpendicular distance

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2+b^2)}}$$

Aliter V : The equation of line through $P(x_1, y_1)$ and perpendicular to $ax + by + c = 0$ is

$$b(x - x_1) - a(y - y_1) = 0 \qquad \text{...(vi)}$$

If this perpendicular meet the line $ax + by + c = 0$ in $M(x_2, y_2)$ then (x_2, y_2) lie on both the lines $ax + by + c = 0$ and Eq. (vi), then

$b(x_2 - x_1) - a(y_2 - y_1) = 0, ax_2 + by_2 + c = 0$

$ax_2 + by_2 + c = a(x_2 - x_1) + b(y_2 - y_1) + ax_1 + by_1 + c = 0$

or $\quad b(x_2 - x_1) - a(y_2 - y_1) = 0 \qquad$ ...(vii)

and $\quad a(x_2 - x_1) + b(y_2 - y_1) = -(ax_1 + by_1 + c)$ ...(viii)

On squaring and adding Eqs. (vii) and (viii), we get

$$(a^2 + b^2)((x_2 - x_1)^2 + (y_2 - y_1)^2) = (ax_1 + by_1 + c)^2$$

or $$PM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2+b^2)}}$$

Hence, length of perpendicular

$$PM = p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2+b^2)}}$$

Corollary 1 : The length of perpendicular from the origin to the line $ax + by + c = 0$ is

$$\frac{|a \cdot 0 + b \cdot 0 + c|}{\sqrt{(a^2+b^2)}} \text{ i.e. } \frac{|c|}{\sqrt{(a^2+b^2)}}$$

Corollary 2 : The length of perpendicular from (x_1, y_1) to the line $x\cos\alpha + y\sin\alpha = p$ is

$$\frac{|x_1\cos\alpha + y_1\sin\alpha - p|}{\sqrt{(\cos^2\alpha + \sin^2\alpha)}} = |x_1\cos\alpha + y_1\sin\alpha - p|$$

Working Rule :

(i) Put the point (x_1, y_1) for (x, y) on the LHS while the RHS is zero.

(ii) Divide LHS after Eq. (i) by $\sqrt{(a^2+b^2)}$, where a and b are the coefficients of x and y respectively.

Example 55. Find the sum of the abscissas of all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$.

Sol. Any point on the line $x + y = 4$ can be taken as $(x_1, 4 - x_1)$. As it is at a unit distance from the line $4x + 3y - 10 = 0$, we get

$$\frac{|4x_1 + 3(4 - x_1) - 10|}{\sqrt{(4^2 + 3^2)}} = 1$$

$\Rightarrow \quad |x_1 + 2| = 5 \Rightarrow x_1 + 2 = \pm 5$

$\Rightarrow \quad x_1 = 3$ or -7

$\therefore$ Required sum $= 3 - 7 = -4$.

Example 56. If p and p' are the length of the perpendiculars from the origin to the straight lines whose equations are $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$, then find the value of $4p^2 + p'^2$.

Sol. We have, $p = \dfrac{|-a|}{\sqrt{(\sec^2\theta + \mathrm{cosec}^2\theta)}}$

$$\therefore \quad p^2 = \frac{a^2}{\sec^2\theta + \mathrm{cosec}^2\theta} = \frac{a^2\sin^2\theta\cos^2\theta}{1}$$

$$\Rightarrow \quad 4p^2 = a^2\sin^2 2\theta \qquad \text{...(i)}$$

and $$p' = \frac{|-a\cos 2\theta|}{\sqrt{(\cos^2\theta + \sin^2\theta)}} = |-a\cos 2\theta|$$

$$\therefore \quad (p')^2 = a^2\cos^2 2\theta \qquad \text{...(ii)}$$

$\therefore$ Adding Eqs. (i) and (ii), we get

$$4p^2 + p'^2 = a^2$$

Example 57. If p is the length of the perpendicular from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, then prove that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}.$$

Sol. p = length of perpendicular from origin to

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$= \frac{|0 + 0 - 1|}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}} = \frac{1}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}}$$

or $$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} \text{ or } \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$$

Example 58. Prove that no line can be drawn through the point $(4, -5)$ so that its distance from $(-2, 3)$ will be equal to 12.

Sol. Suppose, if possible.
Equation of line through $(4, -5)$ with slope of m is

$$y + 5 = m(x - 4)$$

$\Rightarrow \quad mx - y - 4m - 5 = 0$

Then, $$\frac{|m(-2) - 3 - 4m - 5|}{\sqrt{m^2 + 1}} = 12$$

$\Rightarrow \quad |-6m - 8| = 12\sqrt{(m^2 + 1)}$

On squaring, $\quad (6m + 8)^2 = 144(m^2 + 1)$

$\Rightarrow \quad 4(3m + 4)^2 = 144(m^2 + 1)$

$\Rightarrow \quad (3m + 4)^2 = 36(m^2 + 1)$

$\Rightarrow \quad 27m^2 - 24m + 20 = 0 \qquad$ ...(i)

Since, the discriminant of Eq. (i) is $(-24)^2 - 4 \cdot 27 \cdot 20 = -1584$ which is negative, there is no real value of m. Hence no such line is possible.

Distance between Two Parallel Lines

Let the two parallel lines be

$$ax + by + c = 0 \quad \text{and} \quad ax + by + c_1 = 0$$

The distance between the parallel lines is the perpendicular distance of any point on one line from the other line.

Let (x_1, y_1) be any point on $ax + by + c = 0$

$$\therefore \quad ax_1 + by_1 + c = 0 \qquad \text{...(i)}$$

Now, perpendicular distance of the point (x_1, y_1) from the line $ax + by + c_1 = 0$ is

$$\frac{|ax_1 + by_1 + c_1|}{\sqrt{(a^2+b^2)}} = \frac{|c_1 - c|}{\sqrt{(a^2+b^2)}} \quad \text{[from Eq. (i)]}$$

This is required distance between the given parallel lines.

Aliter I : The distance between the lines is

$$d = \frac{\lambda}{\sqrt{(a^2+b^2)}}$$

(i) $\lambda = |c_1 - c|$, if both the lines are on the same side of the origin.

(ii) $\lambda = |c_1| + |c|$, if the lines are on the opposite side of the origin.

Aliter II : Find the coordinates of any point on one of the given lines, preferably putting $x = 0$ or $y = 0$. Then the perpendicular distance of this point from the other line is the required distance between the lines.

Example 59. Find the distance between the lines $5x - 12y + 2 = 0$ and $5x - 12y - 3 = 0$.

Sol. The distance between the lines
$5x - 12y + 2 = 0$ and $5x - 12y - 3 = 0$ is

$$\frac{|2 - (-3)|}{\sqrt{(5)^2 + (-12)^2}} = \frac{5}{13}$$

Aliter I : The constant term in both equations are 2 and -3 which are of opposite sign. Hence origin lies between them.

$\therefore$ Distance between lines is $\dfrac{|2| + |-3|}{\sqrt{(5)^2 + (-12)^2}} = \dfrac{5}{13}$

Aliter II : Putting $y = 0$ in $5x - 12y - 3 = 0$ then $x = \frac{3}{5}$

$\therefore \quad \left(\frac{3}{5}, 0\right)$ lie on $5x - 12y - 3 = 0$

Hence, distance between the lines
$5x - 12y + 2 = 0$ and $(5x - 12y - 3 = 0)$
$=$ Distance from $\left(\frac{3}{5}, 0\right)$ to the line $5x - 12y + 2 = 0$

$$= \frac{\left|5 \times \frac{3}{5} - 0 + 2\right|}{\sqrt{5^2 + (-12)^2}} = \frac{5}{13}$$

Example 60. Find the equations of the line parallel to $5x - 12y + 26 = 0$ and at a distance of 4 units from it.

Sol. Equation of any line parallel to $5x - 12y + 26 = 0$ is

$$5x - 12y + \lambda = 0 \qquad \text{...(i)}$$

Since, the distance between the parallel lines is 4 units, then

$$\frac{|\lambda - 26|}{\sqrt{(5)^2 + (-12)^2}} = 4$$

or $\quad |\lambda - 26| = 52$ or $\lambda - 26 = \pm 52$

or $\quad \lambda = 26 \pm 52 \quad \therefore \lambda = -26$ or 78

Substituting the values of λ in Eq. (i), we get

$5x - 12y - 26 = 0$

and $\quad 5x - 12y + 78 = 0$

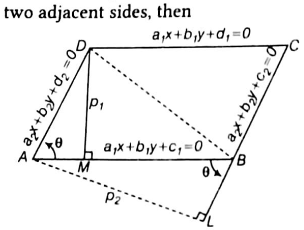
Area of Parallelogram

Theorem : Area of parallelogram $ABCD$ whose sides AB, BC, CD and DA are represented by $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$ is

$$\frac{p_1 p_2}{\sin\theta} \quad \text{or} \quad \frac{|c_1 - d_1||c_2 - d_2|}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

where, p_1 and p_2 are the distances between parallel sides and θ is the angle between two adjacent sides.

Proof : Since, p_1 and p_1 are the distances between the pairs of parallel sides of the parallelogram and θ is the angle between two adjacent sides, then



Area of parallelogram $ABCD$

$$= 2 \times \text{Area of } \Delta ABD$$

$$= 2 \times \frac{1}{2} \times AB \times p_1$$

$$= AB \times p_1$$

$$= \frac{p_2}{\sin\theta} \times p_1 \qquad \left(\because \text{ in } \Delta ABL, \sin\theta = \frac{p_2}{AB}\right)$$

$$= \frac{p_1 p_2}{\sin\theta} \qquad \text{...(i)}$$

Now, p_1 = Distance between parallel sides AB and DC

$$= \frac{|c_1 - d_1|}{\sqrt{(a_1^2 + b_1^2)}}$$

and p_2 = Distance between parallel sides AD and BC

$$= \frac{|c_2 - d_2|}{\sqrt{(a_2^2 + b_2^2)}}$$

Also, $$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{a_1}{b_1} - \left(-\frac{a_2}{b_2}\right)}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}\right|$$

$$\left(\begin{array}{l}\because m_1 = \text{slope of } AB = -\frac{a_1}{b_1} \\ \text{and } m_2 = \text{slope of } AD = -\frac{a_2}{b_2}\end{array}\right)$$

$$= \left|\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right|$$

$$\sin\theta = \frac{|a_1 b_2 - a_2 b_1|}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

Now, substitute the values of p_1, p_2 and $\sin\theta$ in Eq. (i)

$$\therefore \text{ Area of parallelogram } ABCD = \frac{|c_1 - d_1||c_2 - d_2|}{|a_1 b_2 - a_2 b_1|} = \frac{|c_1 - d_1||c_2 - d_2|}{\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|}$$

Corollaries :

1. If $p_1 = p_2$, then $ABCD$ becomes a rhombus

$$\therefore \text{ Area of rhombus } ABCD = \frac{p_1^2}{\sin\theta}$$

$$= \frac{(c_1 - d_1)^2}{|a_1 b_2 - a_2 b_1| \sqrt{\left(\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}\right)}}$$

2. If d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus, then

$$\boxed{\text{Area of rhombus} = \frac{1}{2} d_1 d_2}$$

3. Area of the parallelogram whose sides are $y = mx + a$, $y = mx + b$, $y = nx + c$ and $y = nx + d$ is $\dfrac{|a - b||c - d|}{|m - n|}$.

| Example 61. Show that the area of the parallelogram formed by the lines $x + 3y - a = 0$, $3x - 2y + 3a = 0$, $x + 3y + 4a = 0$ and $3x - 2y + 7a = 0$ is $\frac{20}{11}a^2$ sq units.

Sol. Required area of the parallelogram

$$= \frac{|-a - 4a||3a - 7a|}{\left|\begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix}\right|} = \frac{20}{11} a^2 \text{ sq units}$$

| Example 62. Show that the area of the parallelogram formed by the lines

$$x\cos\alpha + y\sin\alpha = p,\ x\cos\alpha + y\sin\alpha = q,$$
$$x\cos\beta + y\sin\beta = r,\ x\cos\beta + y\sin\beta = s \text{ is}$$
$$|(p - q)(r - s)\operatorname{cosec}(\alpha - \beta)|.$$

Sol. The equation of sides of the parallelogram are

$$x\cos\alpha + y\sin\alpha - p = 0,$$
$$x\cos\alpha + y\sin\alpha - q = 0,$$
$$x\cos\beta + y\sin\beta - r = 0,$$
and
$$x\cos\beta + y\sin\beta - s = 0$$

$\therefore$ Required area of the parallelogram

$$= \frac{|-p - (-q)||-r - (-s)|}{\left|\begin{vmatrix} \cos\alpha & \sin\alpha \\ \cos\beta & \sin\beta \end{vmatrix}\right|} = \frac{|p - q||r - s|}{|\sin(\beta - \alpha)|}$$

$$= |(p - q)(r - s)\operatorname{cosec}(\alpha - \beta)|$$

| Example 63. Prove that the diagonals of the parallelogram formed by the lines $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{b} + \frac{y}{a} = 1$, $\frac{x}{a} + \frac{y}{b} = 2$ and $\frac{x}{b} + \frac{y}{a} = 2$ are at right angles. Also find its area ($a \ne b$).

Sol. The distance between the parallel sides

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ and } \frac{x}{a} + \frac{y}{b} = 2$$

is $$\frac{|2 - 1|}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}} = \frac{1}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}} = p_1 \qquad \text{(say)}$$

and the distance between the parallel sides

$$\frac{x}{b} + \frac{y}{a} = 1 \text{ and } \frac{x}{b} + \frac{y}{a} = 2$$

is $$\frac{|2 - 1|}{\sqrt{\left(\frac{1}{b^2} + \frac{1}{a^2}\right)}} = \frac{1}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}} = p_2 \qquad \text{(say)}$$

Here, $p_1 = p_2$.

$\therefore$ Parallelogram is a rhombus.

But we know that diagonals of rhombus are perpendicular to each other.

$$\therefore \quad \text{Area of the rhombus} = \frac{|(-1 + 2)(-1 + 2)|}{\left|\begin{vmatrix} \frac{1}{a} & \frac{1}{b} \\ \frac{1}{b} & \frac{1}{a} \end{vmatrix}\right|}$$

$$= \frac{a^2 b^2}{|b^2 - a^2|} (a \ne b)$$

| Example 64. Show that the four lines $ax \pm by \pm c = 0$ enclose a rhombus whose area is $\dfrac{2c^2}{|ab|}$.

Sol. The given lines are

$$ax + by + c = 0 \qquad \text{...(i)}$$
$$ax + by - c = 0 \qquad \text{...(ii)}$$
$$ax - by + c = 0 \qquad \text{...(iii)}$$
and
$$ax - by - c = 0 \qquad \text{...(iv)}$$

Distance between the parallel lines Eqs. (i) and (ii) is $\dfrac{2c}{\sqrt{(a^2 + b^2)}} = p_1$ (say) and distance between the parallel lines Eqs. (iii) and (iv) is

$$\frac{2c}{\sqrt{(a^2 + b^2)}} = p_2 \qquad \text{(say)}$$

Here, $p_1 = p_2$

$\therefore$ it is a rhombus.

$$\therefore \quad \text{Area of rhombus} = \frac{|(c + c)(c + c)|}{\left|\begin{vmatrix} a & b \\ a & -b \end{vmatrix}\right|} = \frac{4c^2}{|-2ab|} = \frac{2c^2}{|ab|}$$

Exercise for Session 2

1. The number of lines that are parallel to $2x + 6y - 7 = 0$ and have an intercept 10 between the coordinate axes is
   (a) 1 (b) 2 (c) 4 (d) infinitely many

2. The distance between the lines $4x + 3y = 11$ and $8x + 6y = 15$ is
   (a) $\frac{7}{2}$ (b) $\frac{7}{5}$ (c) $\frac{7}{10}$ (d) $\frac{9}{10}$

3. If the algebraic sum of the perpendicular distances from the points (2, 0), (0, 2) and (1, 1) to a variable straight line is zero, then the line passes through the point
   (a) (1, 1) (b) (–1, 1) (c) (–1, –1) (d) (1, –1)

4. If the quadrilateral formed by the lines $ax + by + c = 0$, $a'x + b'y + c' = 0$, $ax + by + c' = 0$ and $a'x + b'y + c' = 0$ have perpendicular diagonals, then
   (a) $b^2 + c^2 = b'^2 + c'^2$ (b) $c^2 + a^2 = c'^2 + a'^2$ (c) $a^2 + b^2 = a'^2 + b'^2$ (d) None of these

5. The area of the parallelogram formed by the lines $3x - 4y + 1 = 0$, $3x - 4y + 3 = 0$, $4x - 3y - 1 = 0$ and $4x - 3y - 2 = 0$, is
   (a) $\frac{1}{7}$ sq units (b) $\frac{2}{7}$ sq units (c) $\frac{3}{7}$ sq units (d) $\frac{4}{7}$ sq units

6. Area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$ equals

(a) $\frac{|m+n|}{(m-n)^2}$ (b) $\frac{2}{|m+n|}$ (c) $\frac{1}{|m+n|}$ (d) $\frac{1}{|m-n|}$

7. The coordinates of a point on the line $y = x$ where perpendicular distance from the line $3x + 4y = 12$ is 4 units, are

(a) $\left(\frac{3}{7}, \frac{5}{7}\right)$ (b) $\left(\frac{3}{2}, \frac{3}{2}\right)$ (c) $\left(-\frac{8}{7}, -\frac{8}{7}\right)$ (d) $\left(\frac{32}{7}, -\frac{32}{7}\right)$

8. A line passes through the point (2, 2) and is perpendicular to the line $3x + y = 3$, then its y-intercept is

(a) $-\frac{2}{3}$ (b) $\frac{2}{3}$ (c) $-\frac{4}{3}$ (d) $\frac{4}{3}$

9. If the points (1, 2) and (3, 4) were to be on the opposite side of the line $3x - 5y + a = 0$, then

(a) $7 < a < 11$ (b) $a = 7$ (c) $a = 11$ (d) $a < 7$ or $a > 11$

10. The lines $y = mx$, $y + 2x = 0$, $y = 2x + k$ and $y + mx = k$ form a rhombus if m equals

(a) -1 (b) $\frac{1}{2}$ (c) 1 (d) 2

11. The points on the axis of x, whose perpendicular distance from the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is a

(a) $\frac{b}{a}\left(a \pm \sqrt{(a^2 + b^2)}, 0\right)$ (b) $\frac{a}{b}\left(b \pm \sqrt{(a^2 + b^2)}, 0\right)$

(c) $\frac{b}{a}(a + b, 0)$ (d) $\frac{a}{b}\left(a \pm \sqrt{(a^2 + b^2)}, 0\right)$

12. The three sides of a triangle are given by $(x^2 - y^2)(2x + 3y - 6) = 0$. If the point $(-2, a)$ lies inside and $(b, 1)$ lies outside the triangle, then

(a) $a \in \left(2, \frac{10}{3}\right)$; $b \in (-1, 1)$ (b) $a \in \left(-2, \frac{10}{3}\right)$; $b \in \left(-1, \frac{9}{2}\right)$

(c) $a \in \left(1, \frac{10}{3}\right)$; $b \in (-3, 5)$ (d) None of these

13. Are the points (3, 4) and (2, –6) on the same or opposite sides of the line $3x - 4y = 8$?

14. If the points (4, 7) and $(\cos\theta, \sin\theta)$, where $0 < \theta < \pi$, lie on the same side of the line $x + y - 1 = 0$, then prove that θ lies in the first quadrant.

15. Find the equations of lines parallel to $3x - 4y - 5 = 0$ at a unit distance from it.

16. Show that the area of the parallelogram formed by the lines $2x - 3y + a = 0$, $3x - 2y - a = 0$, $2x - 3y + 3a = 0$ and $3x - 2y - 2a = 0$ is $\frac{2a^2}{5}$ sq units.

17. A line 'L' is drawn from $P(4, 3)$ to meet the lines $L_1 : 3x + 4y + 5 = 0$ and $L_2 : 3x + 4y + 15 = 0$ at point A and B respectively. From 'A' a line, perpendicular to L is drawn meeting the line L_2 at A_1. Similarly from point 'B' a line, perpendicular to L is drawn meeting the line L_1 at B_1. Thus a parallelogram AA_1BB_1 is formed. Find the equation (s) of 'L' so that the area of the parallelogram AA_1BB_1 is least.

18. The vertices of a ΔOBC are $O(0, 0)$, $B(-3, -1)$, $C(-1, -3)$. Find the equation of the line parallel to BC and intersecting the sides OB and OC and whose perpendicular distance from the origin is $\frac{1}{2}$.