CHAPTER

03

Sequences and Series

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The word "Sequence" in Mathematics has same meaning as in ordinary English. A collection of objects listed in a sequence means it has identified first member, second member, third member and so on. The most common examples are depreciate values of certain commodity like car, machinery and amount deposits in the bank for a number of years.

Session 1

Sequence, Series, Progression

Sequence

A succession of numbers arranged in a definite order or arrangement according to some well-defined law is called a sequence.

Or

A sequence is a function of natural numbers (N) with codomain is the set of real numbers (R) [complex numbers (C)]. If range is subset of real numbers (complex numbers), it is called a real sequence (complex sequence).

Or

A mapping $f : N \to C$, then $f(n) = t_n$, $n \in N$ is called a sequence to be denoted it by $\{f(1), f(2), f(3), \dots\} = \{t_1, t_2, t_3, \dots\} = \{t_n\}$.

The nth term of a sequence is denoted by $T_n, t_n, a_n, a(n), u_n$, etc.

Remark

The sequence $a_1, a_2, a_3, \dots$ is generally written as $\{a_n\}$.

For example,

(i) 1, 3, 5, 7, ... is a sequence, because each term (except first) is obtained by adding 2 to the previous term and $T_n = 2n - 1, n \in N$.

Or

If $T_1 = 1, T_{n+1} = T_{n+2}, n \geq 1$

(ii) 1, 2, 3, 5, 8, 13, ... is a sequence, because each term (except first two) is obtained by taking the sum of preceding two terms.

Or

If $T_1 = 1, T_2 = 2, T_{n+2} = T_n + T_{n+1}, n \geq 1$

(iii) 2, 3, 5, 7, 11, 13, 17, 19, ... is a sequence.

Here, we cannot express $T_n, n \in N$ by an algebraic formula.

Recursive Formula

A formula to determine the other terms of the sequence in terms of its preceding terms is known as recursive formula.

For example,

If $T_1 = 1$ and $T_{n+1} = 6\,T_n, n \in N$.

Then, $T_2 = 6\,T_1 = 6 \cdot 1 = 6$

$T_3 = 6\,T_2 = 6 \cdot 6 = 36$

$T_4 = 6\,T_3 = 6 \cdot 36 = 216 \dots$

Then, sequence is 1, 6, 36, 216,...

Types of Sequences

There are two types of sequences

1. Finite Sequence

A sequence is said to be finite sequence, if it has finite number of terms. A finite sequence is described by $a_1, a_2, a_3, \dots, a_n$ or $T_1, T_2, T_3, \dots, T_n$, where $n \in N$.

For example

(i) 3, 5, 7, 9, ..., 37

(ii) 2, 6, 18, 54, ..., 4374

2. Infinite Sequence

A sequence is said to be an infinite sequence, if it has infinite number of terms. An infinite sequence is described by $a_1, a_2, a_3, \dots$ or $T_1, T_2, T_3, \dots$

For example,

(i) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$

(ii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

Series

In a sequence, the sum of the directed terms is called a series.

For example, If 1, 4, 7, 10, 13, 16,... is a sequence, then its sum i.e., $1+4+7+10+13+16+\ldots$ is a series.

In general, if $T_1, T_2, T_3, \ldots, T_n, \ldots$ denote a sequence, then the symbolic expression $T_1+T_2+T_3+\ldots+T_n+\ldots$ is called a series associated with the given sequence.

Each member of the series is called its term.

In a series $T_1+T_2+T_3+\ldots+T_r+\ldots$, the sum of first n terms is denoted by S_n. Thus,

$$S_n = T_1+T_2+T_3+\ldots+T_n = \sum_{r=1}^{n} T_r = \sum T_n$$

If S_n denotes the sum of n terms of a sequence.

Then, $S_n - S_{n-1} = (T_1+T_2+T_3+\ldots+T_n) - (T_1+T_2+\ldots+T_{n-1}) = T_n$

Thus, $T_n = S_n - S_{n-1}$

Types of Series

There are two types of series

1. Finite Series

A series having finite number of terms is called a finite series.

For example,

(i) $3+5+7+9+\ldots+21$

(ii) $2+6+18+54+\ldots+4374$

2. Infinite Series

A series having an infinite number of terms is called an infinite series.

For example,

(i) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

(ii) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

Progression

If the terms of a sequence can be described by an explicit formula, then the sequence is called a progression.

Or

A sequence is said to be progression, if its terms increases (respectively decreases) numerically.

For example, The following sequences are progression :

(i) $1, 3, 5, 7, \ldots$

(ii) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots$

(iii) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$

(iv) $1, 8, 27, 256, \ldots$

(v) $8, -4, 2, -1, \frac{1}{2}, \ldots$

The sequences (iii) and (v) are progressions, because

$$|1| > \left|-\frac{1}{3}\right| > \left|\frac{1}{9}\right| > \left|-\frac{1}{27}\right| > \ldots$$

i.e. $$1 > \frac{1}{3} > \frac{1}{9} > \frac{1}{27} > \ldots$$

and $$|8| > |-4| > |2| > |-1| > \left|\frac{1}{2}\right| > \ldots$$

i.e. $$8 > 4 > 2 > 1 > \frac{1}{2} > \ldots$$

Remark

★ All the definitions and formulae are valid for complex numbers in the theory of progressions but it should be assumed (if not otherwise stated) that the terms of the progressions are real numbers.

Example 1. If $f: N \to R$, where $f(n) = a_n = \frac{n}{(2n+1)^2}$, write the sequence in ordered pair form.

Sol. Here, $a_n = \frac{n}{(2n+1)^2}$

On putting $n = 1, 2, 3, 4, \ldots$ successively, we get

$$a_1 = \frac{1}{(2\cdot 1+1)^2} = \frac{1}{9}, \quad a_2 = \frac{2}{(2\cdot 2+1)^2} = \frac{2}{25}$$

$$a_3 = \frac{3}{(2\cdot 3+1)^2} = \frac{3}{49}, \quad a_4 = \frac{4}{(2\cdot 4+1)^2} = \frac{4}{81}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Hence, we obtain the sequence $\frac{1}{9}, \frac{2}{25}, \frac{3}{49}, \frac{4}{81}, \ldots$

Now, the sequence in ordered pair form is

$$\left\{\left(1, \frac{1}{9}\right), \left(2, \frac{2}{25}\right), \left(3, \frac{3}{49}\right), \left(4, \frac{4}{81}\right), \ldots\right\}$$

Example 2. The Fibonacci sequence is defined by $a_1 = 1 = a_2, a_n = a_{n-1} + a_{n-2}, n > 2$. Find $\frac{a_{n+1}}{a_n}$ for $n = 1, 2, 3, 4, 5$.

Sol. $\because \quad a_1 = 1 = a_2$

$\therefore \quad a_3 = a_2 + a_1 = 1 + 1 = 2,$

$a_4 = a_3 + a_2 = 2 + 1 = 3$

$a_5 = a_4 + a_3 = 3 + 2 = 5$

and $\quad a_6 = a_5 + a_4 = 5 + 3 = 8$

$\therefore \quad \frac{a_2}{a_1} = 1, \frac{a_3}{a_2} = \frac{2}{1} = 2, \frac{a_4}{a_3} = \frac{3}{2}, \frac{a_5}{a_4} = \frac{5}{3}$ and $\frac{a_6}{a_5} = \frac{8}{5}$

Example 3. If the sum of n terms of a series is $2n^2 + 5n$ for all values of n, find its 7th term.

Sol. Given, $\quad S_n = 2n^2 + 5n$

$\Rightarrow \quad S_{n-1} = 2(n-1)^2 + 5(n-1) = 2n^2 + n - 3$

$\therefore \quad T_n = S_n - S_{n-1} = (2n^2 + 5n) - (2n^2 + n - 3) = 4n + 3$

Hence, $\quad T_7 = 4 \times 7 + 3 = 31$

Example 4.

(i) Write $\sum_{r=1}^{n}(r^2 + 2)$ in expanded form.

(ii) Write the series $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots + \frac{n}{n+2}$ in sigma form.

Sol. (i) On putting $r = 1, 2, 3, 4, \ldots, n$ in $(r^2 + 2)$, we get $3, 6, 11, 18, \ldots, (n^2 + 2)$

Hence, $\sum_{r=1}^{n}(r^2 + 2) = 3 + 6 + 11 + 18 + \ldots + (n^2 + 2)$

(ii) The rth term of series $= \frac{r}{r+2}$.

Hence, the given series can be written as

$$\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots + \frac{n}{n+2} = \sum_{r=1}^{n}\left(\frac{r}{r+2}\right)$$

c, d, b, c, a

Exercise for Session 1

1. First term of a sequence is 1 and the $(n+1)$th term is obtained by adding $(n+1)$ to the nth term for all natural numbers n, the 6th term of the sequence is
   (a) 7
   (b) 13
   (c) 21
   (d) 27

2. The first three terms of a sequence are 3, 3, 6 and each term after the second is the sum of two terms preceding it, the 8th term of the sequence is
   (a) 15
   (b) 24
   (c) 39
   (d) 63

3. If $a_n = \sin\left(\frac{n\pi}{6}\right)$, the value of $\sum_{n=1}^{6} a_n^2$ is
   (a) 2
   (b) 3
   (c) 4
   (d) 7

4. If for a sequence $\{a_n\}$, $S_n = 2n^2 + 9n$, where S_n is the sum of n terms, the value of a_{20} is
   (a) 65
   (b) 75
   (c) 87
   (d) 97

5. If $a_1 = 2, a_2 = 3 + a_1$ and $a_n = 2a_{n-1} + 5$ for $n > 1$, the value of $\sum_{r=2}^{5} a_r$ is
   (a) 130
   (b) 160
   (c) 190
   (d) 220

Session 2

Arithmetic Progression (AP)

Types of Progression

Progressions are various types but in this chapter we will studying only three special types of progressions which are following :

1. Arithmetic Progression (AP)
2. Geometric Progression (GP)
3. Harmonic Progression (HP)

Arithmetic Progression (AP)

An arithmetic progression is a sequence in which the difference between any term and its just preceding term (i.e., term before it) is constant throughout. This constant is called the common difference (abbreviated as CD) and is generally denoted by 'd'.

Or

An arithmetic progression is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference of the AP.

A finite or infinite sequence $\{t_1, t_2, t_3, \ldots, t_n\}$ or $\{t_1, t_2, t_3, \ldots\}$ is said to be an arithmetic progression (AP), if $t_k - t_{k-1} = d$, a constant independent of k, for $k = 2, 3, 4, \ldots, n$ or $k = 2, 3, 4, \ldots$ as the case may be :

The constant d is called the common difference of the AP.

i.e. $$d = t_2 - t_1 = t_3 - t_2 = \ldots = t_n - t_{n-1}$$

Remarks

1. If a be the first term and d be the common difference, then AP can be written as
$$a, a + d, a + 2d, \ldots, a + (n-1)d, \ldots, \forall\, n \in N.$$
2. If we add the common difference to any term of AP, we get the next following term and if we subtract it from any term, we get the preceding term.
3. The common difference of an AP may be **positive**, **zero**, **negative** or **imaginary.**
4. **Constant AP** common difference of an AP is equal to zero.
5. **Increasing AP** common difference of an AP is greater than zero.
6. **Decreasing AP** common difference of an AP is less than zero.
7. **Imaginary AP** common difference of an AP is imaginary.

Algorithm to determine whether a sequence is an AP or not

***Step* I** Obtain t_n (the nth term of the sequence).

***Step* II** Replace n by $n-1$ in t_n to get t_{n-1}.

***Step* III** Calculate $\boxed{t_n - t_{n-1}.}$

If $t_n - t_{n-1}$ is independent of n, the given sequence is an AP otherwise it is not an AP.

| Example 5.

(i) 1, 3, 5, 7, ... (ii) $\pi, \pi + e^{\pi}, \pi + 2e^{\pi}, \ldots$

(iii) $a, a-b, a-2b, a-3b, \ldots$

Sol. (i) Here, 2nd term − 1st term = 3rd term − 2nd term = ...

$\Rightarrow$ $3 - 1 = 5 - 3 = \ldots = 2$, which is a common difference.

(ii) Here, 2nd term − 1st term = 3rd term − 2nd term = ...

$\Rightarrow$ $(\pi + e^{\pi}) - \pi = (\pi + 2e^{\pi}) - (\pi + e^{\pi}) = \ldots$

$= e^{\pi}$, which is a common difference.

(iii) Here, 2nd term − 1st term = 3rd term − 2nd term = ...

$\Rightarrow (a - b) - a = (a - 2b) - (a - b) = \ldots$

$= -b$, which is a common difference.

| Example 6. Show that the sequence $< t_n >$ defined by $t_n = 5n + 4$ is an AP, also find its common difference.

Sol. We have, $t_n = 5n + 4$

On replacing n by $(n-1)$, we get

$$t_{n-1} = 5(n-1) + 4$$

$$\Rightarrow \quad t_{n-1} = 5n - 1$$

$$\therefore \quad t_n - t_{n-1} = (5n + 4) - (5n - 1) = 5$$

Clearly, $t_n - t_{n-1}$ is independent of n and is equal to 5. So, the given sequence is an AP with common difference 5.

| Example 7. Show that the sequence $< t_n >$ defined by $t_n = 3n^2 + 2$ is not an AP.

Sol. We have, $t_n = 3n^2 + 2$

On replacing n by $(n-1)$, we get

$$t_{n-1} = 3(n-1)^2 + 2$$

$$\Rightarrow \quad t_{n-1} = 3n^2 - 6n + 5$$

$$\therefore \quad t_n - t_{n-1} = (3n^2 + 2) - (3n^2 - 6n + 5)$$

$$= 6n - 3$$

Clearly, $t_n - t_{n-1}$ is not independent of n and therefore it is not constant. So, the given sequence is not an AP.

Remark

If the nth term of a sequence is an expression of first degree in n. For example, $t_n = An + B$, where A, B are constants, then that sequence will be in AP for $t_n - t_{n-1} = (An + B) - [A(n-1) + B]$ $= A[n-(n-1)] = A =$ constant $=$ Common difference or coefficient of n in t_n. Students are advised to consider the above point as a behaviour of standard result.

General Term of an AP

Let 'a' be the first term, 'd' be the common difference and 'l' be the last term of an AP having 'n' terms, where $n \in N$. Then, AP can be written as $a, a+d, a+2d, \ldots, l-2d, l-d, l$

(i) nth Term of an AP from Beginning

1st term from beginning $= t_1 = a = a + (1-1)\,d$

2nd term from beginning $= t_2 = a + d = a + (2-1)\,d$

3rd term from beginning $= t_3 = a + 2d = a + (3-1)\,d$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

n th term from beginning $= t_n = a + (n-1)\,d, \forall\, n \in N$

Hence, n th term of an AP from beginning

$$= t_n = a + (n-1)\,d = l \quad \text{[last term]}$$

(ii) nth Term of an AP from End

1st term from end $= t'_1 = l = l - (1-1)\,d$

2nd term from end $= t'_2 = l - d = l - (2-1)\,d$

3rd term from end $= t'_3 = l - 2d = l - (3-1)\,d$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

nth term from end $= t'_n = \boxed{l - (n-1)\,d,}\ \forall\, n \in N$

Hence, n th term of an AP from end

$t'_n = l - (n-1)\,d = a$ [first term]

Now, it is clear that

$$t_n + t'_n = a + (n-1)\,d + l - (n-1)\,d = a + l$$

or

$$t_n + t'_n = a + l$$

i.e. In a finite AP, the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.

Remark

1. nth term is also called the general term.
2. If last term of AP is t_n and common difference be d, then terms of AP from end are $t_n, t_n - d, t_n - 2d, \ldots$
3. If in a sequence, the terms an alternatively positive and negative, then it cannot be an AP.
4. Common difference of AP $= \dfrac{l-a}{n-1}$, where, $a =$ first term of AP, $l =$ last term of AP and $n =$ number of terms of AP.
5. If t_n, t_{n+1}, t_{n+2} are three consecutive terms of an AP, then $2t_{n+1} = t_n + t_{n+2}$. In particular, if a, b and c are in AP, then $2b = a + c$.

Example 8. Find first negative term of the sequence

$$20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$$

Sol. The given sequence is an AP in which first term, $a = 20$ and common difference, $d = -\dfrac{3}{4}$. Let the nth term of the given AP be the first negative term. Then,

$$t_n < 0 \Rightarrow a + (n-1)\,d < 0$$

$$\Rightarrow \quad 20 + (n-1)\left(-\frac{3}{4}\right) < 0$$

$$\Rightarrow \quad 80 - 3n + 3 < 0$$

$$\Rightarrow \quad n > \frac{83}{3} \text{ or } n > 27\frac{2}{3}$$

$$\Rightarrow \quad n = 28$$

Thus, 28th term of the given sequence is the first negative term.

Example 9. If the mth term of an AP is $\dfrac{1}{n}$ and the nth term is $\dfrac{1}{m}$, then find mnth term of an AP.

Sol. If A and B are constants, then rth term of AP is

$$\boxed{t_r = Ar + B}$$

Given, $t_m = \dfrac{1}{n} \Rightarrow Am + B = \dfrac{1}{n}$ ...(i)

and $t_n = \dfrac{1}{m} \Rightarrow An + B = \dfrac{1}{m}$ ...(ii)

From Eqs. (i) and (ii), we get $A = \dfrac{1}{mn}$ and $B = 0$

mn th term $= t_{mn} = Amn + B = \dfrac{1}{mn} \cdot mn + 0 = 1$

Hence, mn th term of the given AP is 1.

Example 10. If $|x-1|$, 3 and $|x-3|$ are first three terms of an increasing AP, then find the 6th term of on AP.

Sol. *Case* I For $x < 1$,

$$|x-1| = -(x-1)$$

and $$|x-3| = -(x-3)$$

$\therefore 1 - x$, 3 and $3 - x$ are in AP.

$$\Rightarrow \quad 6 = 1 - x + 3 - x$$

$$\Rightarrow \quad x = -1$$

Then, first three terms are 2, 3, 4, which is an increasing AP.

$\therefore$ 6th term is 7. $[\because d = 1]$

Case II For $1 < x < 3$,

$$|x-1| = x-1$$

and $$|x-3| = -(x-3) = 3-x$$

$\therefore x - 1$, 3 and $3 - x$ are in AP.

$$\Rightarrow \quad 6 = x - 1 + 3 - x$$

$$\Rightarrow \quad 6 = 2 \quad \text{[impossible]}$$

Case III For $x > 3$, $|x-1| = x-1$ and $|x-3| = x-3$

$\therefore x-1, 3$ and $x-3$ are in AP.

$\Rightarrow \quad 6 = x - 1 + x - 3 \Rightarrow x = 5$

Then, first three terms are 4, 3, 2, which is a decreasing AP.

by observing values.

| Example 11. In the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ..., where n consecutive terms have the value n, find the 150th term of the sequence.

Sol. Let the 150th term $= n$

Then, $1+2+3+\ldots+(n-1) < 150 < 1+2+3+\ldots+n$

$$\Rightarrow \frac{(n-1)n}{2} < 150 < \frac{n(n+1)}{2}$$

$$\Rightarrow n(n-1) < 300 < n(n+1)$$

Taking first two members

$$n(n-1) < 300 \Rightarrow n^2 - n - 300 < 0$$

$$\Rightarrow \left(n - \frac{1}{2}\right)^2 < 300 + \frac{1}{4}$$

$$\Rightarrow 0 < n < \frac{1}{2} + \frac{\sqrt{1201}}{2}$$

$$\Rightarrow 0 < n < 17.8 \quad \ldots(\text{i})$$

and taking last two members,

$$n(n+1) > 300$$

$$\Rightarrow \left(n + \frac{1}{2}\right)^2 > 300 + \frac{1}{4}$$

$$\therefore n > -\frac{1}{2} + \frac{\sqrt{1201}}{2}$$

$$\Rightarrow n > 16.8 \quad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get

$$16.8 < n < 17.8$$

$$\Rightarrow n = 17$$

| Example 12. If a_1, a_2, a_3, a_4 and a_5 are in AP with common difference $\neq 0$, find the value of $\sum_{i=1}^{5} a_i$ when $a_3 = 2$.

Sol. $\because a_1, a_2, a_3, a_4$ and a_5 are in AP, we have

$$a_1 + a_5 = a_2 + a_4 = a_3 + a_3 \quad [\because t_n + t'_n = a + l]$$

$$a_1 + a_5 = a_2 + a_4 = 4 \quad [\because a_3 = 2]$$

$$a_1 + a_2 + a_3 + a_4 + a_5 = 4 + 2 + 4 = 10$$

$$\Rightarrow \sum_{i=1}^{5} a_i = 10$$

Sum of a Stated Number of Terms of an Arithmetic Series

More than 200 yr ago, a class of German School Children was asked to find the sum of all integers from 1 to 100 inclusive. One boy in the class, an eight year old named **Carl Fredrick Gauss** (1777-1855) who later established his reputation as one of the greatest Mathematicians announced the answer almost at once. The teacher overawed at this asked Gauss to explain how he got this answer. Gauss explained that he had added these numbers in pairs as follows

$$(1+100), (2+99), (3+98), \ldots$$

There are $\frac{100}{2} = 50$ pairs. The answer can be obtained by multiplying 101 by 50 to get 5050.

Sum of *n* Terms of an AP

Let 'a' be the first term, 'd' be the common difference, 'l' be the last term of an AP having n terms and S_n be the sum of n terms, then

$$S_n = a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l \quad \ldots(\text{i})$$

Reversing the right hand terms

$$S_n = l + (l-d) + (l-2d) + \ldots + (a+2d) + (a+d) + a \quad \ldots(\text{ii})$$

On adding Eqs. (i) and (ii), we get

$$2S_n = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) + (a+l)$$

$$= (a+l) + (a+l) + \ldots \text{ upto } n \text{ terms} = n(a+l)$$

$$\therefore \quad \boldsymbol{S_n = \frac{n}{2}(a+l)} \quad \ldots(\text{iii})$$

Now, if we substitute the value of l *viz.*, $l = a + (n-1)d$, in this formula, we get

$$S_n = \frac{n}{2}[a + a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore \quad S_n = \frac{n}{2}[2a + (n-1)d]$$

If we substitute the value of a *viz.*,

$$l = a + (n-1)d$$

or $\quad a = l - (n-1)d$ in Eq. (iii), then

$$S_n = \frac{n}{2}[2l - (n-1)d]$$

If we substitute the value of $a + l$ *viz.*,

$t_n + t'_n = a + l$ in Eq. (iii), then

$$\boldsymbol{S_n = \frac{n}{2}(t_n + t'_n)}$$

***Corollary* I** Sum of first n natural numbers

i.e. $1 + 2 + 3 + 4 + \ldots + n$

Here, $\quad a = 1$ and $d = 1$

$$\therefore \quad S = \frac{n}{2}[2 \cdot 1 + (n-1) \cdot 1]$$

$$= \frac{n(n+1)}{2}$$

***Corollary* II** Sum of first n odd natural numbers

i.e., $1+3+5+\ldots$

Here, $a=1$

and $d=2$

$\therefore$ $S=\frac{n}{2}[2\cdot 1+(n-1)\cdot 2]=n^2$

***Corollary* III** If sum of first n terms is S_n, then sum of next m terms is $\boxed{S_{m+n}-S_n}$

Important Results with Proof

1. If S_n, t_n and d are sum of n terms, nth term and common difference of an AP respectively, then

$$d=t_n-t_{n-1} \quad [n\geq 2]$$
$$t_n=S_n-S_{n-1} \quad [n\geq 2]$$
$$d=S_n-2S_{n-1}+S_{n-2} \quad [n\geq 3]$$

Proof

$\because$ $S_n=t_1+t_2+t_3+\ldots+t_{n-1}+t_n$

$\Rightarrow$ $S_n=S_{n-1}+t_n$

$\therefore$ $t_n=S_n-S_{n-1}$

but $d=t_n-t_{n-1}$

$=(S_n-S_{n-1})-(S_{n-1}-S_{n-2})$

$\therefore$ $d=S_n-2S_{n-1}+S_{n-2}$

2. A sequence is an AP if and only if the sum of its n terms is of the form An^2+Bn, where A and B are constants independent of n.

In this case, the nth term and common difference of the AP are $A(2n-1)+B$ and $2A$, respectively.

Proof As $S_n=An^2+Bn$

$\therefore$ $S_{n-1}=A(n-1)^2+B(n-1)$

$\therefore$ $t_n=S_n-S_{n-1}$

$=(An^2+Bn)-[A(n-1)^2+B(n-1)]$

$=A[n^2-(n-1)^2]+B$

$t_n=A(2n-1)+B$

$\Rightarrow$ $t_{n-1}=A[2(n-1)-1]+B$

$=A(2n-3)+B$

Now, $t_n-t_{n-1}=[A(2n-1)+B]-[A(2n-3)+B]$

$=2A$ [a constant]

Hence, the sequence is an AP.

Conversely, consider an AP with first term a and common difference d.

Sum of first n terms $=\frac{n}{2}[2a+(n-1)d]$

$=\frac{dn^2}{2}+\left(a-\frac{d}{2}\right)n=An^2+Bn$,

where, $A=\frac{d}{2}$, $B=a-\frac{d}{2}$

Hence, $S_n=An^2+Bn$, where A and B are constants independent of n.

Hence, the converse is true.

Corollary $\because$ $S_n=An^2+Bn$

$\therefore$ $t_n=A(2n-1)+B$

$t_n=A$ (replacing n^2 by $2n-1$) + coefficient of n

and $d=2A$

i.e. $d=2$ [coefficient of n^2]

	S_n	t_n	d
1.	$5n^2+3n$	$5(2n-1)+3=10n-2$	10
2.	$-7n^2+2n$	$-7(2n-1)+2$ $=-14n+9$	-14
3.	$-9n^2-4n$	$-9(2n-1)-4$ $=-18n+5$	-18
4.	$4n^2-n$	$4(2n-1)-1=8n-5$	8

3. If $S_n=an^2+bn+c$, where S_n denotes the sum of n terms of a series, then whole series is not an AP. It is AP from the second term onwards.

Proof As $S_n=an^2+bn+c$ for $n\geq 1$, we get

$S_{n-1}=a(n-1)^2+b(n-1)+c$ for $n\geq 2$

Now, $t_n=S_n-S_{n-1}$

$\Rightarrow$ $t_n=a(2n-1)+b$, $n\geq 2$

$\therefore$ $t_{n-1}=a[2(n-1)-1]+b$, $n\geq 3$

$\Rightarrow$ $t_{n-1}=a(2n-3)+b$, $n\geq 3$

$\therefore$ $t_n-t_{n-1}=2a=$ constant, $n\geq 3$

$\therefore$ $t_3-t_2=t_4-t_3=t_5-t_4=\ldots$

But $t_2-t_1=(S_2-S_1)-S_1=S_2-2S_1$

$=(4a+2b+c)-2(a+b+c)$

$=(2a-c)$ $[\because S_1=t_1]$

$\therefore$ $t_2-t_1\neq t_3-t_2$

Hence, the whole series is not an AP. It is AP from the second term onwards.

Ratio of Sums is Given

1. If ratio of the sums of m and n terms of an AP is given by

$$\frac{S_m}{S_n} = \frac{Am^2 + Bm}{An^2 + Bn}$$

where A, B are constants and $A \neq 0$.

$$\therefore \quad S_m = (Am^2 + Bm)\,k,$$

$$S_n = (An^2 + Bn)\,k$$

$$\Rightarrow \quad t_m = S_m - S_{m-1} = [A(2m-1) + B]k$$

$$t_n = S_n - S_{n-1} = [A(2n-1) + B]k$$

$$\therefore \quad \frac{t_m}{t_n} = \frac{A(2m-1)+B}{A(2n-1)+B}$$

$n \Rightarrow 2n-1$

| Example 13. The ratio of sums of m and n terms of an AP is $m^2 : n^2$. The ratio of the mth and nth terms is

(a) $(2m+1):(2n-1)$ (b) $m:n$
(c) $(2m-1):(2n-1)$ (d) None of these

Sol. (c) Here, $\dfrac{S_m}{S_n} = \dfrac{m^2}{n^2}$ $\quad [\because A = 1, B = 0]$

$$\therefore \quad \frac{t_m}{t_n} = \frac{(2m-1)}{(2n-1)}$$

$$\Rightarrow \quad t_m : t_n = (2m-1):(2n-1)$$

2. If ratio of the sums of n terms of two AP's is given by

$$\frac{S_n}{S'_n} = \frac{An + B}{Cn + D}$$

Both are n

where, A, B, C, D are constants and $A, C \neq 0$

$$\therefore \quad S_n = n(An+B)\,k,\ S'_n = n(Cn+D)\,k$$

$$\Rightarrow \quad t_n = [A(2n-1)+B]k,\ t'_n = [C(2n-1)+D]k$$

$$\Rightarrow \quad d = t_n - t_{n-1} = 2A,\ d' = t'_n - t'_{n-1} = 2C$$

$$\therefore \quad \frac{t_n}{t'_n} = \frac{A(2n-1)+B}{C(2n-1)+D} \text{ and } \frac{d}{d'} = \frac{A}{C}$$

Note If $A = 0, C = 0$

Then, $\dfrac{S_n}{S'_n} = \dfrac{B}{D} \Rightarrow \dfrac{t_n}{t'_n} = \dfrac{B}{D}$ and $\dfrac{d}{d'} = \dfrac{0}{0}$ = not defined

Remark

If
$$\frac{t_n}{t'_n} = \frac{an+b}{cn+d}$$

$n \rightarrow \left(\frac{n+1}{2}\right)$

where, a, b, c, d are constants and $a, c \neq 0$, then

$$\frac{S_n}{S'_n} = \frac{a\left(\frac{n+1}{2}\right)+b}{c\left(\frac{n+1}{2}\right)+d}$$

| Example 14. The sums of n terms of two arithmetic progressions are in the ratio $(7n+1):(4n+17)$. Find the ratio of their nth terms and also common differences.

Sol. Given, $S_n : S'_n = (7n+1):(4n+17)$

Here, $A = 7, B = 1, C = 4$ and $D = 17$

$$\therefore \quad \frac{t_n}{t'_n} = \frac{7(2n-1)+1}{4(2n-1)+17} = \frac{14n-6}{8n+13}$$

and
$$\frac{d}{d'} = \frac{A}{C} = \frac{7}{4}$$

Hence, $t_n : t'_n = (14n-6):(8n+13)$ and $d : d' = 7:4$

| Example 15. The sums of n terms of two AP's are in the ratio $(3n-13):(5n+21)$. Find the ratio of their 24th terms.

Sol. Given, $S_n : S'_n = (3n-13):(5n+21)$

Here, $A = 3, B = -13, C = 5$ and $D = 21$

$$\therefore \quad \frac{t_{24}}{t'_{24}} = \frac{3(2\times 24-1)-13}{5(2\times 24-1)+21} = \frac{128}{256} = \frac{1}{2}$$

$$\therefore \quad t_{24} : t'_{24} = 1:2$$

| Example 16. How many terms of the series $20 + 19\frac{1}{3} + 18\frac{2}{3} + \ldots$ must be taken to make 300? Explain the double answer.

Sol. Here, given series is an AP with first term $a = 20$ and the common difference, $d = -\frac{2}{3}$.

Let the sum of n terms of the series be 300.

Then, $\quad S_n = \dfrac{n}{2}\{2a + (n-1)d\}$

$$\Rightarrow \quad 300 = \frac{n}{2}\left\{2\times 20 + (n-1)\left(-\frac{2}{3}\right)\right\}$$

$$\Rightarrow \quad 300 = \frac{n}{3}\{60 - n + 1\}$$

$$\Rightarrow \quad n^2 - 61n + 900 = 0$$

$$\Rightarrow \quad (n-25)(n-36) = 0$$

$$\Rightarrow \quad n = 25 \text{ or } n = 36$$

$\therefore$ Sum of 25 terms = Sum of 36 terms = 300

Explanation of double answer

Here, the common difference is negative, therefore terms go on diminishing and $t_{31} = 20 + (31-1)\left(\dfrac{-2}{3}\right) = 0$ i.e., 31st term becomes zero. All terms after 31st term are negative. These negative terms $(t_{32}, t_{33}, t_{34}, t_{35}, t_{36})$ when added to positive terms $(t_{26}, t_{27}, t_{28}, t_{29}, t_{30})$, they cancel out each other i.e., sum of terms from 26th to 36th terms is zero. Hence, the sum of 25 terms as well as that of 36 terms is 300.

Example 17. Find the arithmetic progression consisting of 10 terms, if the sum of the terms occupying the even places is equal to 15 and the sum of those occupying the odd places is equal to $12\frac{1}{2}$.

Sol. Let the successive terms of an AP be $t_1, t_2, t_3, \ldots, t_9, t_{10}$.

By hypothesis,

$$t_2 + t_4 + t_6 + t_8 + t_{10} = 15$$

$$\Rightarrow \frac{5}{2}(t_2 + t_{10}) = 15$$

$$\Rightarrow t_2 + t_{10} = 6$$

$$\Rightarrow (a + d) + (a + 9d) = 6$$

$$\Rightarrow 2a + 10d = 6 \quad \ldots(i)$$

and

$$t_1 + t_3 + t_5 + t_7 + t_9 = 12\frac{1}{2}$$

$$\Rightarrow \frac{5}{2}(t_1 + t_9) = \frac{25}{2}$$

$$\Rightarrow t_1 + t_9 = 5$$

$$\Rightarrow a + a + 8d = 5$$

$$\Rightarrow 2a + 8d = 5 \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get $d = \frac{1}{2}$ and $a = \frac{1}{2}$

Hence, the AP is $\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots$

must

Example 18. If N, the set of natural numbers is partitioned into groups $S_1 = \{1\}, S_2 = \{2, 3\}, S_3 = \{4, 5, 6\}, \ldots$, find the sum of the numbers in S_{50}.

Sol. The number of terms in the groups are 1, 2, 3, ...

$\because$ The number of terms in the 50th group = 50

$\therefore$ The last term of 1st group = 1

The last term of 2nd group = 3 = 1 + 2

The last term of 3rd group = 6 = 1 + 2 + 3

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

The last term of 49th group = 1 + 2 + 3 + ... + 49

$\therefore$ First term of 50th group = 1 + (1 + 2 + 3 + ... + 49)

$$= 1 + \frac{49}{2}(1 + 49) = 1226$$

$$\therefore \quad S_{50} = \frac{50}{2}\{2 \times 1226 + (50 - 1) \times 1\}$$

$$= 25 \times 2501 = 62525$$

Example 19. Find the sum of first 24 terms of on AP $t_1, t_2, t_3, \ldots$, if it is known that

$$t_1 + t_5 + t_{10} + t_{15} + t_{20} + t_{24} = 225.$$

Sol. We know that, in an AP the sums of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.

Then, $t_1 + t_{24} = t_5 + t_{20} = t_{10} + t_{15}$

but given

$$t_1 + t_5 + t_{10} + t_{15} + t_{20} + t_{24} = 225$$

$$\Rightarrow (t_1 + t_{24}) + (t_5 + t_{20}) + (t_{10} + t_{15}) = 225$$

$$\Rightarrow 3(t_1 + t_{24}) = 225$$

$$\Rightarrow t_1 + t_{24} = 75$$

$$\therefore S_{24} = \frac{24}{2}(t_1 + t_{24}) = 12 \times 75 = 900$$

Example 20. If $(1+3+5+\ldots+p)+(1+3+5+\ldots+q) = (1+3+5+\ldots+r)$, where each set of parentheses contains the sum of consecutive odd integers as shown, then find the smallest possible value of $p+q+r$ (where, $p > 6$). $p = 1 + (n-1)2$

Sol. We know that, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ $\Rightarrow$ $\boxed{n = \frac{p+1}{2}}$

Thus, the given equation can be written as

$$\left(\frac{p+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^2 = \left(\frac{r+1}{2}\right)^2$$

$$\Rightarrow (p + 1)^2 + (q + 1)^2 = (r + 1)^2$$

Therefore, $(p + 1, q + 1, r + 1)$ form a Pythagorean triplet as $p > 6 \Rightarrow p + 1 > 7$

The first Pythagorean triplet containing a number > 7 is (6, 8, 10).

$$\Rightarrow p + 1 = 8, q + 1 = 6, r + 1 = 10$$

$$\Rightarrow p + q + r = 21$$

Properties of Arithmetic Progression

1. If $a_1, a_2, a_3, \ldots$ are in AP with common difference d, then $a_1 \pm k, a_2 \pm k, a_3 \pm k, \ldots$ are also in AP with common difference d.
2. If $a_1, a_2, a_3, \ldots$ are in AP with common difference d, then $a_1k, a_2k, a_3k, \ldots$ and $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots$ are also in AP $(k \neq 0)$ with common differences kd and $\frac{d}{k}$, respectively.
3. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two AP's with common differences d_1 and d_2, respectively. Then, $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are also in AP with common difference $(d_1 \pm d_2)$.
4. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two AP's with common differences d_1 and d_2 respectively, then $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are not in AP.
5. If $a_1, a_2, a_3, \ldots, a_n$ are in AP, then

$$a_r = \frac{a_{r-k} + a_{r+k}}{2}, \forall\, k, 0 \leq k \leq n - r$$

6. If three numbers in AP whose sum is given are to be taken as $\alpha-\beta, \alpha, \alpha+\beta$ and if five numbers in AP whose sum is given, are to be taken as $\alpha-2\beta, \alpha-\beta, \alpha, \alpha+\beta, \alpha+2\beta$, etc.

In general, If $(2r+1)$ numbers in AP whose sum is given, are to be taken as $(r \in N)$.
$\alpha-r\beta, \alpha-(r-1)\beta, \dots, \alpha-\beta, \alpha, \alpha+\beta, \dots, \alpha+(r-1)\beta, \alpha+r\beta$

Remark

1. Sum of three numbers $= 3\alpha$
   Sum of five numbers $= 5\alpha$
   ⋮ ⋮ ⋮ ⋮ ⋮
   Sum of $(2r+1)$ numbers $= (2r+1)\alpha$
2. From given conditions, find two equations in α and β and then solve them. Now, the numbers in AP can be obtained.

7. If four numbers in AP whose sum is given, are to be taken as
$\alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta$ and if six numbers in AP, whose sum is given are to be taken as $\alpha-5\beta, \alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta, \alpha+5\beta$, etc.

In general If $2r$ numbers in AP whose sum is given, are to be taken as $(r \in N)$.
$\alpha-(2r-1)\beta, \alpha-(2r-3)\beta, \dots, \alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta, \dots, \alpha+(2r-3)\beta, \alpha+(2r-1)\beta$

Remark

1. Sum of four numbers $= 4\alpha$
   Sum of six numbers $= 6\alpha$
   ⋮ ⋮ ⋮ ⋮ ⋮
   Sum of $2r$ numbers $= 2r\alpha$
2. From given conditions, find two conditions in α and β and then solve them. Now, the numbers in AP can be obtained.

| Example 21. If $S_1, S_2, S_3, \dots, S_p$ are the sums of n terms of p AP's whose first terms are $1, 2, 3, \dots, p$ and common differences are $1, 2, 3, \dots, (2p-1)$ respectively, show that $\quad S_1+S_2+S_3+\dots+S_p=\frac{1}{2}np(np+1)$.

Sol. $\because 1, 2, 3, \dots, p$ are in AP.

Then, $2\cdot1, 2\cdot2, 2\cdot3, \dots, 2p$ are also in AP. ...(i)
[multiplying 2 to each term]

and $1, 3, 5, \dots, (2p-1)$ are in AP.

Then, $(n-1)\cdot1, (n-1)\cdot3, (n-1)\cdot5, \dots, (n-1)(2p-1)$ are also in AP. ...(ii)
[multiplying $(n-1)$ to each term]

From Eqs. (i) and (ii), we get
$2\cdot1+(n-1)\cdot1, 2\cdot2+(n-1)\cdot3, 2\cdot3+(n-1)\cdot5, \dots, 2p+(n-1)(2p-1)$ are also in AP. ...(iii)
[adding corresponding terms of Eqs. (i) and (ii)]

From Eq. (iii),
$\frac{n}{2}\{2\cdot1+(n-1)\cdot1\}, \frac{n}{2}\{2\cdot2+(n-1)\cdot3\}, \frac{n}{2}\{2\cdot3+(n-1)\cdot5\}, \dots,$
$\frac{n}{2}\{2p+(n-1)(2p-1)\}$ are also in AP
[multiplying $\frac{n}{2}$ to each term]

i.e. $S_1, S_2, S_3, \dots, S_p$ are in AP.

$$\therefore \quad S_1+S_2+S_3+\dots+S_p=\frac{p}{2}\{S_1+S_p\}$$
$$=\frac{p}{2}\left\{\frac{n}{2}[2\cdot1+(n-1)\cdot1]+\frac{n}{2}[2\cdot p+(n-1)(2p-1)]\right\}$$
$$=\frac{np}{4}\{2+(n-1)+2p+(n-1)(2p-1)\}$$
$$=\frac{np}{4}(2np+2)=\frac{1}{2}np(np+1)$$

Aliter

Here, $\quad S_1=1+2+3+\dots$ upto n terms $=\frac{n(n+1)}{2}$

$S_2=2+5+8+\dots$ upto n terms $=\frac{n}{2}[2\cdot2+(n-1)3]$
$=\frac{n(3n+1)}{2}$

Similarly, $S_3=3+8+13+\dots$ upto n terms $=\frac{n(5n+1)}{2}$, etc.

Now, $S_1+S_2+S_3+\dots+S_p$
$$=\frac{n(n+1)}{2}+\frac{n(3n+1)}{2}+\frac{n(5n+1)}{2}+\dots \text{ upto } p \text{ terms}$$
$$=\frac{n}{2}[(n+3n+5n+\dots \text{ upto } p \text{ terms}) + (1+1+1+\dots \text{ upto } p \text{ terms})]$$
$$=\frac{n}{2}\left[\frac{p}{2}(2n+(p-1)2n)+p\right]$$
$$=\frac{np}{2}[n+n(p-1)+1]=\frac{1}{2}np(np+1)$$

| Example 22. Let α and β be roots of the equation $x^2-2x+A=0$ and let γ and δ be the roots of the equation $x^2-18x+B=0$. If $\alpha<\beta<\gamma<\delta$ are in arithmetic progression, then find the values of A and B.

Sol. $\because \alpha, \beta, \gamma, \delta$ are in AP.

Let $\quad \beta=\alpha+d, \gamma=\alpha+2d, \delta=\alpha+3d, d>0$
[here, sum of $\alpha, \beta, \gamma, \delta$ is not given]

Given, $\quad \alpha+\beta=2, \alpha\beta=A$
$\Rightarrow \quad 2\alpha+d=2, \alpha\beta=A$ ...(i)
and $\quad \gamma+\delta=18, \gamma\delta=B$
$\Rightarrow \quad 2\alpha+5d=18, \gamma\delta=B$ ...(ii)

From Eqs. (i) and (ii), we get

$$d = 4, \alpha = -1$$

$\therefore$ $\beta = 3, \gamma = 7, \delta = 11$

$\Rightarrow$ $A = \alpha\beta = (-1)(3) = -3$

and $B = \gamma\delta = (7)(11) = 77$

Example 23. The digits of a positive integer having three digits are in AP and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.

Sol. Let the digit in the unit's place be $a - d$, digit in the ten's place be a and the digit in the hundred's place be $a + d$.

Sum of digits $= a - d + a + a + d = 15$ [given]

$\Rightarrow$ $3a = 15$

$\therefore$ $a = 5$ ...(i)

$\therefore$ Original number $= (a - d) + 10a + 100(a + d)$

$= 111a + 99d = 555 + 99d$

and number formed by reversing the digits

$= (a + d) + 10a + 100(a - d)$

$= 111a - 99d = 555 - 99d$

Given, $(555 + 99d) - (555 - 99d) = 594 \Rightarrow 198d = 594$

$\therefore$ $d = 3$

Hence, original number $= 555 + 99 \times 3 = 852$

Example 24. If three positive real numbers are in AP such that $abc = 4$, then find the minimum value of b.

Sol. $\because$ a, b, c are in AP.

Let $a = A - D, b = A, c = A + D$

Then, $a = b - D, c = b + D$

Now, $abc = 4$

$(b - D)\, b\, (b + D) = 4$

$\Rightarrow$ $b(b^2 - D^2) = 4$

$\Rightarrow$ $b^2 - D^2 < b^2$

$\Rightarrow$ $b(b^2 - D^2) < b^3 \Rightarrow 4 < b^3$

$\therefore$ $b > (4)^{1/3}$ or $b > (2)^{2/3}$

Hence, the minimum value of b is $(2)^{2/3}$.

Example 25. If a, b, c, d are distinct integers form an increasing AP such that $d = a^2 + b^2 + c^2$, then find the value of $a + b + c + d$.

Sol. Here, sum of numbers i.e., $a + b + c + d$ is not given.

Let $b = a + D, c = a + 2D, d = a + 3D, \forall D \in N$

According to hypothesis,

$$a + 3D = a^2 + (a + D)^2 + (a + 2D)^2$$

$\Rightarrow$ $5D^2 + 3(2a - 1)D + 3a^2 - a = 0$ ...(i)

$$\therefore \quad D = \frac{-3(2a - 1) \pm \sqrt{9(2a - 1)^2 - 20(3a^2 - a)}}{10}$$

$$= \frac{-3(2a - 1) \pm \sqrt{(-24a^2 - 16a + 9)}}{10}$$

Now, $-24a^2 - 16a + 9 \geq 0$

$\Rightarrow$ $24a^2 + 16a - 9 \leq 0$

$\Rightarrow$ $-\frac{1}{3} - \frac{\sqrt{70}}{3} \leq a \leq -\frac{1}{3} + \frac{\sqrt{70}}{12}$

$\Rightarrow$ $a = -1, 0$ $[\because a \in I]$

When $a = 0$ from Eq. (i), $D = 0, \frac{3}{5}$ (not possible $\because D \in N$) and for $a = -1$

From Eq. (i), $D = 1, \frac{4}{5}$

$\therefore$ $D = 1$ $[\because D \in N]$

$\therefore$ $a = -1, b = 0, c = 1, d = 2$

Then, $a + b + c + d = -1 + 0 + 1 + 2 = 2$

b, a, a, b, c, c

Exercise for Session 2

1. If nth term of the series $25 + 29 + 33 + 37 + \ldots$ and $3 + 4 + 6 + 9 + 13 + \ldots$ are equal, then n equals
   (a) 11 (b) 12 (c) 13 (d) 14

2. The rth term of the series $2\frac{1}{2} + 1\frac{7}{13} + 1\frac{1}{9} + \frac{20}{23} + \ldots$ is
   by putting r=1
   (a) $\frac{20}{5r+3}$ (b) $\frac{20}{5r-3}$ (c) $20(5r+3)$ (d) $\frac{20}{5r^2+3}$

3. In a certain AP, 5 times the 5th term is equal to 8 times the 8th term, its 13th term is
   (a) 0 (b) -1 (c) -12 (d) -13

4. If the 9th term of an AP is zero, the ratio of its 29th and 19th terms is
   (a) 1:2 (b) 2:1 (c) 1:3 (d) 3:1

5. If the pth, qth and rth terms of an AP are a, b and c respectively, the value of $a(q-r) + b(r-p) + c(p-q)$ is
   (a) 1 (b) -1 (c) 0 (d) $\frac{1}{2}$

6. The 6th term of an AP is equal to 2, the value of the common difference of the AP which makes the product $a_1a_4a_5$ least is given by
   (a) $\frac{8}{5}$ (b) $\frac{5}{4}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

7. The sum of first $2n$ terms of an AP is α and the sum of next n terms is β, its common difference is
   (a) $\frac{\alpha-2\beta}{3n^2}$ (b) $\frac{2\beta-\alpha}{3n^2}$ (c) $\frac{\alpha-2\beta}{3n}$ (d) $\frac{2\beta-\alpha}{3n}$

8. The sum of three numbers in AP is -3 and their product is 8, then sum of squares of the numbers is
   (a) 9 (b) 10 (c) 12 (d) 21

9. Let S_n denote the sum of n terms of an AP, if $S_{2n} = 3S_n$, then the ratio $\frac{S_{3n}}{S_n}$ is equal to
   (a) 9 (b) 6 (c) 16 (d) 12

10. The sum of the products of the ten numbers $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ taking two at a time, is
    (a) -65 (b) 165 (c) -55 (d) 95

11. If $a_1, a_2, a_3, \ldots, a_n$ are in AP, where $a_i > 0$ for all i, the value of
    $$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} \text{ is}$$
    (a) $\frac{1}{\sqrt{a_1}+\sqrt{a_n}}$ (b) $\frac{1}{\sqrt{a_1}-\sqrt{a_n}}$ (c) $\frac{n}{\sqrt{a_1}-\sqrt{a_n}}$ (d) $\frac{(n-1)}{\sqrt{a_1}+\sqrt{a_n}}$

Session 3

Geometric Sequence or Geometric Progression (GP)

Geometric Sequence or Geometric Progression (GP)

A geometric progression is a sequence, if the ratio of any term and its just preceding term is constant throughout. This constant quantity is called the common ratio and is generally denoted by 'r'.

Or

A geometric progression (GP) is a sequence of numbers, whose first term is non-zero and each of the term is obtained by multiplying its just preceding term by a constant quantity. This constant quantity is called common ratio of the GP.

Let $t_1, t_2, t_3, \dots, t_n; t_1, t_2, t_3, \dots$ be respectively a finite or an infinite sequence. Assume that none of t_n''s is 0 and that $\dfrac{t_k}{t_{k-1}} = r$, a constant (i.e., independent of k).

For $k = 2, 3, 4, \dots, n$ or $k = 2, 3, 4, \dots$ as the case may be. We then call $\{t_k\}_{k=1}^{n}$ or $\{t_k\}_{k=1}^{\infty}$ as the case may be a geometric progression (GP). The constant ratio r is called the common ratio (CR) of the GP.

i.e., $$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \dots = \frac{t_n}{t_{n-1}}$$

If $t_1 = a$ is the first term of a GP, then

$$t_2 = ar,\ t_3 = t_2 r = ar^2,\ t_4 = t_3 r = ar^3, \dots,$$
$$t_n = t_{n-1} r = ar^{n-1}$$

It follows that, given that first term a and the common ratio r, the GP can be rewritten as

$a, ar, ar^2, \dots, ar^{n-1}$ (standard GP) or $a, ar, ar^2, \dots, ar^{n-1}, \dots$ (standard GP)
according as it is finite or infinite.

Important Results

1. In a GP, neither $a = 0$ nor $r = 0$.
2. In a GP, no term can be equal to '0'.
3. If in a GP, the terms are alternatively positive and negative, then its common ratio is always negative.
4. If we multiply the common ratio with any term of GP, we get the next following term and if we divide any term by the common ratio, we get the preceding term.
5. The common ratio of GP may be positive, negative or imaginary.
6. If common ratio of GP is equal to unity, then GP is known as **Constant GP**.
7. If common ratio of GP is imaginary or real, then GP is known as **Imaginary GP.**
8. **Increasing and Decreasing GP**
For a GP to be increasing or decreasing $r > 0$. If $r < 0$, terms of GP are alternatively positive and negative and so neither increasing nor decreasing.

a	$a > 0$	$a > 0$	$a < 0$	$a < 0$
r	$0 < r < 1$	$r > 1$	$r > 1$	$0 < r < 1$
Result	Decreasing	Increasing	Decreasing	Increasing

Example 26.

(i) $1, 2, 4, 8, 16, \dots$ (ii) $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \dots$

(iii) $-2, -6, -18, \dots$ (iv) $-8, -4, -2, -1, -\frac{1}{2}, \dots$

(v) $5, -10, 20, \dots$ (vi) $5, 5, 5, 5, \dots$

(vii) $1, 1+i, 2i, -2+2i, \dots; i = \sqrt{-1}$

Sol. (i) Here, $a = 1$

and $r = \frac{2}{1} = \frac{4}{2} = \frac{8}{4} = \frac{16}{8} = \dots = 2$ i.e. $a = 1$ and $r = 2$

Increasing GP ($a > 0, r > 1$)

(ii) Here, $a = 9$

and $r = \frac{3}{9} = \frac{1}{3} = \frac{\frac{1}{3}}{1} = \frac{\frac{1}{9}}{\frac{1}{3}} = \dots = \frac{1}{3}$ i.e. $a = 9, r = \frac{1}{3}$

Decreasing GP ($a > 0, 0 < r < 1$)

(iii) Here, $a = -2$

and $r = \frac{-6}{-2} = \frac{-18}{-6} = \dots = 3$

i.e. $a = -2, r = 3$

Decreasing GP ($a < 0, r > 1$)

(iv) Here, $a = -8$

and $r = \frac{-4}{-8} = \frac{-2}{-4} = \frac{-1}{-2} = \frac{-\frac{1}{2}}{-1} = \dots = \frac{1}{2}$

i.e. $a = -8, r = \frac{1}{2}$

Increasing GP ($a < 0, 0 < r < 1$)

(v) Here, $a = 5$

and $r = \frac{-10}{5} = \frac{20}{-10} = \dots = -2$ i.e., $a = 5, r = -2$

Neither increasing nor decreasing $(r < 0)$

(vi) Here, $a = 5$

and $r = \frac{5}{5} = \frac{5}{5} = \frac{5}{5} = \dots = 1$ i.e., $a = 5, r = 1$

Constant GP $(r = 1)$

(vii) Here, $a = 1$

and $r = \frac{1+i}{1} = \frac{2i}{1+i} = \frac{-2+2i}{2i} = \dots$

$$= (1+i) = \frac{2i(1-i)}{(1+i)(1-i)} = \frac{(-1+i)i}{i^2} = \dots$$

$$= (1+i) = (i+1) = (1+i) = \dots$$

i.e., $a = 1, r = 1 + i$

Imaginary GP $(r = \text{imaginary})$

Example 27. Show that the sequence $<t_n>$ defined by $t_n = \frac{2^{2n-1}}{3}$ for all values of $n \in N$ is a GP. Also, find its common ratio.

Sol. We have, $t_n = \frac{2^{2n-1}}{3}$

On replacing n by $n - 1$, we get

$$t_{n-1} = \frac{2^{2n-3}}{3} \Rightarrow \frac{t_n}{t_{n-1}} = \frac{\frac{2^{2n-1}}{3}}{\frac{2^{2n-3}}{3}} = 2^2 = 4$$

Clearly, $\frac{t_n}{t_{n-1}}$ is independent of n and is equal to 4. So, the given sequence is a GP with common ratio 4.

Example 28. Show that the sequence $<t_n>$ defined by $t_n = 2 \cdot 3^n + 1$ is not a GP.

Sol. We have, $t_n = 2 \cdot 3^n + 1$

On replacing n by $(n-1)$ in t_n, we get

$$t_{n-1} = 2 \cdot 3^{n-1} + 1$$

$$\Rightarrow \quad t_{n-1} = \frac{(2 \cdot 3^n + 3)}{3}$$

$$\therefore \quad \frac{t_n}{t_{n-1}} = \frac{(2 \cdot 3^n + 1)}{\frac{(2 \cdot 3^n + 3)}{3}} = \frac{3(2 \cdot 3^n + 1)}{(2 \cdot 3^n + 3)}$$

Clearly, $\frac{t_n}{t_{n-1}}$ is not independent of n and is therefore not constant. So, the given sequence is not a GP.

General Term of a GP

Let 'a' be the first term, 'r' be the common ratio and 'l' be the last term of a GP having 'n' terms. Then, GP can be written as $a, ar, ar^2, \dots, \frac{l}{r^2}, \frac{l}{r}, l$

(i) nth Term of a GP from Beginning

1st term from beginning $= t_1 = a = ar^{1-1}$

2nd term from beginning $= t_2 = ar = ar^{2-1}$

3rd term from beginning $= t_3 = ar^2 = ar^{3-1}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

nth term from beginning $= t_n = ar^{n-1}, \forall n \in N$

Hence, n th term of a GP from beginning

$$t_n = ar^{n-1} = l \qquad \text{[last term]}$$

(ii) nth Term of a GP from End

1st term from end $= t'_1 = l = \frac{l}{r^{1-1}}$

2nd term from end $= t'_2 = \frac{l}{r} = \frac{l}{r^{2-1}}$

3rd term from end $= t'_3 = \frac{l}{r^2} = \frac{l}{r^{3-1}}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

nth term from end $= t'_n = \frac{l}{r^{n-1}}, \forall n \in N$

Hence, nth term of a GP from end

$$= t'_n = \boxed{\frac{l}{r^{n-1}} = a} \qquad \text{[first term]}$$

Now, it is clear that $t_k \times t'_k = ar^{k-1} \times \frac{l}{r^{k-1}} = a \times l$

or $\quad t_k \times t'_k = a \times l, \forall 1 \le k \le n$

i.e. in a finite GP of n terms, the product of the k th term from the beginning and the k th term form the end is independent of k and equals the product of the first and last terms.

Remark

1. nth term is also called the general term.
2. If last term of GP be t_n and CR is r, then terms of GP from end are $t_n, \frac{t_n}{r}, \frac{t_n}{r^2}, \dots$
3. If in a GP, the terms are alternatively positive and negative, then its common ratio is always negative.

4. If a and l represent first and last term of a GP respectively, then common ratio of GP $= r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$

5. If t_n, t_{n+1}, t_{n+2} are three consecutive terms of a GP, then

$$\frac{t_{n+1}}{t_n} = \frac{t_{n+2}}{t_{n+1}} \Rightarrow t_{n+1}^2 = t_n t_{n+2}$$

In particular, if a, b, c are in GP, then $\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$

On squaring, $\frac{b^2}{a^2} = \frac{c^2}{b^2}$

Hence, a^2, b^2, c^2 are also in GP.

Example 29. If first term of a GP is a, third term is b and $(n+1)$th term is c. The $(2n+1)$th term of a GP is

(a) $c\sqrt{\frac{b}{a}}$ (b) $\frac{bc}{a}$ (c) abc (d) $\frac{c^2}{a}$

Sol. Let common ratio $= r$

$$\therefore \quad b = ar^2 \Rightarrow r = \sqrt{\frac{b}{a}}$$

Also, $$c = ar^n \Rightarrow r^n = \frac{c}{a}$$

$$\therefore \quad t_{2n+1} = ar^{2n} = a(r^n)^2 = a\left(\frac{c}{a}\right)^2 = \frac{c^2}{a}$$

Hence, (d) is the correct answer.

Example 30. The $(m+n)$th and $(m-n)$th terms of a GP are p and q, respectively. Then, the mth term of the GP is

(a) $p\left(\frac{q}{p}\right)^{\frac{m}{2n}}$ (b) $\sqrt{pq}$

(c) $\sqrt{\frac{p}{q}}$ (d) None of these

Sol. Let a be the first term and r be the common ratio, then

$$t_{m+n} = p \Rightarrow ar^{m+n-1} = p \quad \text{...(i)}$$

$$t_{m-n} = q \Rightarrow ar^{m-n-1} = q \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$ar^{m+n-1} \times ar^{m-n-1} = p \times q$$

$$\Rightarrow \quad a^2 r^{2m-2} = pq \Rightarrow ar^{m-1} = \sqrt{pq}$$

$$\Rightarrow \quad t_m = \sqrt{pq}$$

Hence, (b) is the correct answer.

Example 31. If $\sin\theta, \sqrt{2}(\sin\theta + 1), 6\sin\theta + 6$ are in GP, then the fifth term is

(a) 81 (b) $81\sqrt{2}$ (c) 162 (d) $162\sqrt{2}$

Sol. $[\sqrt{2}(\sin\theta + 1)]^2 = \sin\theta(6\sin\theta + 6)$

$\Rightarrow [(\sin\theta + 1)\, 2(\sin\theta + 1) - 6\sin\theta] = 0$

We get, $$\sin\theta = -1, \frac{1}{2}$$

$$\therefore \quad \sin\theta = \frac{1}{2} \quad [\sin\theta = -1 \text{ is not possible}]$$

then first term $= a = \sin\theta = \frac{1}{2}$ and common ratio

$$= r = \frac{\sqrt{2}\left(\frac{1}{2} + 1\right)}{\left(\frac{1}{2}\right)} = 3\sqrt{2}$$

$$\therefore \quad t_5 = ar^4 = \frac{1}{2}(3\sqrt{2})^4 = 162$$

Hence, (c) is the correct answer.

Example 32. The 1025th term in the sequence 1, 22, 4444, 88888888, ... is 4 starts from 4th term also, 8 starts from 8th term

(a) 2^9 (b) 2^{10}

(c) 2^{11} (d) 2^{12}

Sol. The number of digits in each term of the sequence are 1, 2, 4, 8, ... which are in GP. Let 1025th term is 2^n.

Then, So, 1024 starts from 1024th term.

$$1 + 2 + 4 + 8 + \ldots + 2^{n-1} < 1025 \le 1 + 2 + 4 + 8 + \ldots + 2^n$$

$$\Rightarrow \quad \frac{(2-1)(1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1})}{(2-1)} < 1025$$

$$\le \frac{(2-1)(1 + 2 + 2^2 + 2^3 + \ldots + 2^n)}{(2-1)}$$

$$\Rightarrow \quad 2^n - 1 < 1025 \le 2^{n+1} - 1 \Rightarrow 2^n < 1026 \le 2^{n+1} \quad \text{...(i)}$$

or $$2^{n+1} \ge 1026 > 1024$$

$$\Rightarrow \quad 2^{n+1} > 2^{10} \Rightarrow n + 1 > 10$$

$$\therefore \quad n > 9 \quad \therefore n = 10$$

[which is always satisfy Eq. (i)]

Hence, (b) is the correct answer.

Example 33. If a, b, c are real numbers such that $3(a^2 + b^2 + c^2 + 1) = 2(a + b + c + ab + bc + ca)$, then a, b, c are in

(a) AP only (b) GP only

(c) GP and AP (d) None of these

Sol. Given, $3(a^2 + b^2 + c^2 + 1) = 2(a + b + c + ab + bc + ca)$

$$\Rightarrow \quad 2(a^2 + b^2 + c^2 - ab - bc - ca) + (a^2 + b^2 + c^2 - 2a - 2b - 2c + 3)$$

$$\Rightarrow \quad \{(a-b)^2 + (b-c)^2 + (c-a)^2\} + \{(a-1)^2 + (b-1)^2 + (c-1)^2\} = 0$$

$\Rightarrow a - b = b - c = c - a = 0$ and $a - 1 = b - 1 = c - 1 = 0$

$$\Rightarrow \quad a = b = c = 1$$

$\Rightarrow a, b, c$ are in GP and AP.

Hence, (c) is the correct answer.

Sum of a Stated Number of Terms of a Geometric Series

The game of chess was invented by **Grand Vizier Sissa Ben Dhair** for the Indian king **Shirham.** Pleased with the game, the king asked the **Vizier** what he would like as reward. The **Vizier** asked for one grain of wheat to be placed on the first square of the chess, two grains on the second, four grains on the third and so on (each time doubling the number of grains). The king was surprised of the request and told the **vizier** that he was fool to ask for so little.

The inventor of chess was no fool. He told the king "What I have asked for is more wheat that you have in the entire kingdom, in fact it is more than there is in the whole world" He was right. There are 64 squares on a chess board and on the nth square he was asking for 2^{n-1} grains, if you add the numbers

i.e., $$S = 1 + 2 + 2^2 + 2^3 + \dots + 2^{62} + 2^{63} \quad \dots(i)$$

On multiplying both sides by 2, then

$$2S = 2 + 2^2 + 2^3 + 2^4 + \dots + 2^{63} + 2^{64} \quad \dots(ii)$$

On subtracting Eq. (i) from Eq. (ii), we get

$S = 2^{64} - 1 = 18,446,744,073,709,551,615$ grains i.e., represent more wheat that has been produced on the Earth.

Sum of n Terms of a GP

Let a be the first term, r be the common ratio, l be the last term of a GP having n terms and S_n the sum of n terms, then

$$S_n = a + ar + ar^2 + \dots + \frac{l}{r^2} + \frac{l}{r} + l \quad \dots(i)$$

On multiplying both sides by r (the common ratio)

$$r S_n = ar + ar^2 + ar^3 + \dots + \frac{l}{r} + l + lr \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we have

$$S_n - r S_n = a - lr \text{ or } S_n(1-r) = a - lr$$

$$\therefore \quad \boldsymbol{S_n = \frac{a - lr}{1 - r}}, \text{ when } r < 1$$

$$\boldsymbol{S_n = \frac{lr - a}{r - 1}}, \text{ when } r > 1$$

Now, $$l = t_n = ar^{n-1}$$

Then, above formula can be written as

$$\boldsymbol{S_n = \frac{a(1 - r^n)}{(1 - r)}} \text{ when } \boldsymbol{r < 1}, \boldsymbol{S_n = \frac{a(r^n - 1)}{(r - 1)}},$$

when $\boldsymbol{r > 1}$

If $r = 1$, the above formulae cannot be used. But, then the GP reduces to $a, a, a, \dots$

$\therefore \quad S_n = a + a + a + \dots n \text{ times} = na$

Sum to Infinity of a GP, when the Numerical Value of the Common Ratio is Less than Unity, i.e. It is a Proper Fraction

If a be the first term, r be the common ratio of a GP, then

$$S_n = \frac{a(1 - r^n)}{(1 - r)} = \frac{a}{(1 - r)} - \frac{ar^n}{(1 - r)}$$

Let $-1 < r < 1$ i.e. $|r| < 1$, then $\lim_{n \to \infty} r^n \to 0$

Let S_∞ denote the sum to infinity of the GP, then

$$S_\infty = \frac{a}{(1 - r)},$$

where $\quad -1 < r < 1$

Recurring Decimal

Recurring decimal is a very good example of an infinite geometric series and its value can be obtained by means of infinite geometric series as follows

$0.3\dot{2}\dot{7} = 0.327272727\dots$ to infinity

$$= 0.3 + 0.027 + 0.00027 + 0.0000027 + \dots \text{ upto infinity}$$

$$= \frac{3}{10} + \frac{27}{10^3} + \frac{27}{10^5} + \frac{27}{10^7} + \dots \text{ upto infinity}$$

$$= \frac{3}{10} + \frac{27}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \dots \text{ upto infinity}\right)$$

$$= \frac{3}{10} + \frac{27}{10^3}\left(\frac{1}{1 - \frac{1}{10^2}}\right)$$

$$= \frac{3}{10} + \frac{27}{990} = \frac{297 + 27}{990}$$

$$= \frac{324}{990} \quad \text{[rational number]}$$

Aliter (Best method)

Let P denotes the figure which do not recur and suppose them p in number, Q denotes the recurring period consisting of q figures. Let R denotes the value of the recurring decimal.

Then, $$R = 0 \cdot PQQQ \dots$$

$\therefore$ $$10^p \times R = P \cdot QQQ \dots$$

and $$10^{p+q} \times R = PQ \cdot QQQ \dots$$

$\therefore$ Therefore, by subtraction $R = \dfrac{PQ - P}{(10^{p+q} - 10^p)}$.

Corollary I If $R = 0 \cdot QQQ \ldots$

Then, $R = \dfrac{Q}{10^q - 1}$ (when Q denote the recurring period consisting of q figures)

For example, If $R = 0.\dot{3}$, then $R = \dfrac{3}{10^1 - 1} = \dfrac{1}{3}$

Corollary II The value of recurring decimal is always rational number.

Example 34. Find the value of $0.32\dot{5}\dot{8}$.

Sol. Let $R = 0.32\dot{5}\dot{8}$

$\Rightarrow$ $R = 0.3258585858\ldots$ ...(i)

Here, number of figures which are not recurring is 2 and number of figures which are recurring is also 2.

Then, $100R = 32.58585858\ldots$ ...(ii)

and $10000R = 3258.58585858\ldots$ ...(iii)

On subtracting Eq. (ii) from Eq. (iii), we get

$$9900R = 3226$$

$\therefore$ $R = \dfrac{3226}{9900}$

Hence, $R = \dfrac{1613}{4950}$

Shortcut Methods for Recurring Decimals

1. The numerator of the vulgar fraction is obtained by subtracting the non-recurring figure from the given figure.
2. The denominator consists of as many 9's as there are recurring figure and as many zero as there are non-recurring figure.

For example,

(i) $0.36\dot{5}\dot{4} = \dfrac{3654 - 36}{9900} = \dfrac{3618}{9900}$

(ii) $1.3\dot{2}\dot{7} = 1 + 0.3\dot{2}\dot{7} = 1 + \dfrac{327 - 3}{990} = \dfrac{1314}{990}$

(iii) $0.\dot{3} = \dfrac{3 - 0}{9} = \dfrac{1}{3}$

Example 35. Find the sum upto n terms of the series $a + aa + aaa + aaaa + \ldots$, $\forall\, a \in N$ and $1 \le a \le 9$.

Sol. Let $S = a + aa + aaa + aaaa + \ldots$ upto n terms

$= a(1 + 11 + 111 + 1111 + \ldots$ upto n terms)

$= \dfrac{a}{9}(9 + 99 + 999 + 9999 + \ldots$ upto n terms)

$= \dfrac{a}{9}\{(10^1 - 1) + (10^2 - 1) + (10^3 - 1) + (10^4 - 1) + \ldots$ upto n terms$\}$

$= \dfrac{a}{9}\{(10 + 10^2 + 10^3 + \ldots$ upto n terms$) - (1 + 1 + 1 + \ldots n$ times$)\}$

$$= \frac{a}{9}\left\{\frac{10(10^n - 1)}{10 - 1} - n\right\} = \frac{a}{9}\left\{\frac{10}{9}(10^n - 1) - n\right\}$$

[Remember]

In Particular

(i) For $a = 1$, $1 + 11 + 111 + \ldots = \dfrac{1}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(ii) For $a = 2$, $2 + 22 + 222 + \ldots = \dfrac{2}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(iii) For $a = 3$, $3 + 33 + 333 + \ldots = \dfrac{3}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(iv) For $a = 4$, $4 + 44 + 444 + \ldots = \dfrac{4}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(v) For $a = 5$, $5 + 55 + 555 + \ldots = \dfrac{5}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(vi) For $a = 6$, $6 + 66 + 666 + \ldots = \dfrac{6}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(vii) For $a = 7$, $7 + 77 + 777 + \ldots = \dfrac{7}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(viii) For $a = 8$, $8 + 88 + 888 + \ldots = \dfrac{8}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(ix) For $a = 9$, $9 + 99 + 999 + \ldots = \dfrac{9}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

Example 36. Find the sum upto n terms of the series $0.b + 0.bb + 0.bbb + 0.bbbb + \ldots$, $\forall\, b \in N$ and $1 \le b \le 9$.

Sol. Let $S = 0.b + 0.bb + 0.bbb + 0.bbbb + \ldots$ upto n terms

$= b(0.1 + 0.11 + 0.111 + 0.1111 + \ldots$ upto n terms)

$= \dfrac{b}{9}(0.9 + 0.99 + 0.999 + 0.9999 + \ldots$ upto n terms)

$= \dfrac{b}{9}\{(1 - 0.1) + (1 - 0.01) + (1 - 0.001) + (1 - 0.0001) + \ldots$ upto n terms$\}$

$= \dfrac{b}{9}\{(1 + 1 + 1 + 1 + \ldots$ upto n times$) - (0.1 + 0.01 + 0.001 + 0.0001 + \ldots$ upto n terms$)\}$

$$= \frac{b}{9}\left\{n - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots \text{ upto } n \text{ terms}\right)\right\}$$

$$= \frac{b}{9}\left\{n - \frac{\frac{1}{10}\left(1 - \left(\frac{1}{10}\right)^n\right)}{1 - \frac{1}{10}}\right\} = \frac{b}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

[Remember]

In Particular

(i) For $b=1$,

$$0.1+0.11+0.111+\ldots=\frac{1}{9}\left\{n-\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$

(ii) For $b=2$,

$$0.2+0.22+0.222+\ldots=\frac{2}{9}\left\{n-\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$

(iii) For $b=3$,

$$0.3+0.33+0.333+\ldots=\frac{3}{9}\left\{n-\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$

(iv) For $b=4$,

$$0.4+0.44+0.444+\ldots=\frac{4}{9}\left\{n-\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$

(v) For $b=5$,

$$0.5+0.55+0.555+\ldots=\frac{5}{9}\left\{n-\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$

(vi) For $b=6$,

$$0.6+0.66+0.666+\ldots=\frac{6}{9}\left\{n-\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$

(vii) For $b=7$,

$$0.7+0.77+0.777+\ldots=\frac{7}{9}\left\{n-\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$

(viii) For $b=8$,

$$0.8+0.88+0.888+\ldots=\frac{8}{9}\left\{n-\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$

(ix) For $b=9$,

$$0.9+0.99+0.999+\ldots=\frac{9}{9}\left\{n-\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$

Example 37. If N, the set of natural numbers is partitioned into groups $S_1=\{1\}, S_2=\{2,3\}$, $S_3=\{4,5,6,7\}$, $S_4=\{8,9,10,11,12,13,14,15\},\ldots$, then find the sum of the numbers in S_{50}.

Sol. The number of terms in the groups are $1, 2, 2^2, 2^3, \ldots$

$\therefore$ The number of terms in the 50th group $=2^{50-1}=2^{49}$

$\because$ The first term of 1st group $=1=2^0=2^{1-1}$

The first term of 2nd group $=2=2^1=2^{2-1}$

The first term of 3rd group $=4=2^2=2^{3-1}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

The first term of 50th group $=2^{50-1}=2^{49}$

$$\therefore \quad S_{50}=\frac{2^{49}}{2}\{2\times 2^{49}+(2^{49}-1)\times 1\}$$
$$=2^{48}(2^{50}+2^{49}-1)$$
$$=2^{48}[2^{49}(2+1)-1]=2^{48}(3\cdot 2^{49}-1)$$

Example 38. If $S_n=1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$, then calculate the least value of n such that $2-S_n<\frac{1}{100}$.

Sol. Given, $S_n=1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}=\dfrac{1\cdot\left[1-\left(\frac{1}{2}\right)^n\right]}{\left(1-\frac{1}{2}\right)}$

$$\Rightarrow \quad S_n=2-\frac{1}{2^{n-1}}$$

$$\Rightarrow \quad 2-S_n=\frac{1}{2^{n-1}}<\frac{1}{100} \qquad \left[\because 2-S_n<\frac{1}{100}\right]$$

$$\Rightarrow \quad 2^{n-1}>100>2^6$$

$$\Rightarrow \quad 2^{n-1}>2^6$$

$$\therefore \quad n-1>6 \Rightarrow n>7$$

Hence, the least value of n is 8.

Example 39. If $x=1+a+a^2+a^3+\ldots+\infty$ and $y=1+b+b^2+b^3+\ldots+\infty$ show that $1+ab+a^2b^2+a^3b^3+\ldots+\infty=\frac{xy}{x+y-1}$, where $0<a<1$ and $0<b<1$.

Sol. Given, $x=1+a+a^2+a^3+\ldots+\infty=\frac{1}{1-a}$

$$\Rightarrow \quad x-ax=1$$

$$\therefore \quad a=\left(\frac{x-1}{x}\right) \qquad \ldots(\text{i})$$

and $\quad y=1+b+b^2+b^3+\ldots+\infty$

Similarly, $\quad b=\left(\frac{y-1}{y}\right) \qquad \ldots(\text{ii})$

Since, $\quad 0<a<1, 0<b<1$

$\therefore \quad 0<ab<1$

Now, $1+ab+a^2b^2+a^3b^3+\ldots+\infty=\frac{1}{1-ab}$

$$=\frac{1}{1-\left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)} \qquad [\text{from Eqs. (i) and (ii)}]$$

$$=\frac{xy}{xy-xy+x+y-1}$$

Hence, $1+ab+a^2b^2+a^3b^3+\ldots+\infty=\frac{xy}{x+y-1}$

Properties of Geometric Progression

1. If $a_1, a_2, a_3, \ldots$ are in GP with common ratio r, then $a_1k, a_2k, a_3k, \ldots$ and $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots$ are also in GP $(k \neq 0)$ with common ratio r.
2. If $a_1, a_2, a_3, \ldots$ are in GP with common ratio r, then $a_1 \pm k, a_2 \pm k, a_3 \pm k, \ldots$ are not in GP $(k \neq 0)$.
3. If $a_1, a_2, a_3, \ldots$ are in GP with common ratio r, then
   (i) $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are also in GP with common ratio $\frac{1}{r}$.
   (ii) $a_1^n, a_2^n, a_3^n, \ldots$ are also in GP with common ratio r^n and $n \in Q$.
   (iii) $\log a_1, \log a_2, \log a_3, \ldots$ are in AP $(a_i > 0, \forall i)$
   In this case, the converse also holds good.
4. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two GP's with common ratios r_1 and r_2, respectively. Then,
   (i) $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are also in GP with common ratios r_1r_2 and $\frac{r_1}{r_2}$, respectively.
   (ii) $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are not in GP.
5. If $a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}, a_n$ are in GP. Then,
   (i) $a_1a_n = a_2a_{n-1} = a_3a_{n-2} = \ldots$
   (ii) $a_r = \sqrt{a_{r-k}\, a_{r+k}}, \forall\, k, 0 \le k \le n-r$
   (iii) $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = \frac{a_n}{a_{n-1}}$

$$\Rightarrow \quad a_2^2 = a_3a_1, a_3^2 = a_2a_4, \ldots$$

Also, $a_2 = a_1r, a_3 = a_1r^2,$

$a_4 = a_1r^3, \ldots, a_n = a_1r^{n-1}$

where, r is the common ratio of GP.

6. If three numbers in GP whose product is given are to be taken as $\frac{a}{r}, a, ar$ and if five numbers in GP whose product is given are to be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$, etc.

In general If $(2m+1)$ numbers in GP whose product is given are to be taken as $(m \in N)$

$$\frac{a}{r^m}, \frac{a}{r^{m-1}}, \ldots, \frac{a}{r}, a, ar, \ldots, ar^{m-1}, ar^m$$

Remark

1. Product of three numbers $= a^3$
   Product of five numbers $= a^5$
   $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$
   Product of $(2m+1)$ numbers $= a^{2m+1}$
2. From given conditions, find two equations in a and r and then solve them. Now, the numbers in GP can be obtained.

7. If four numbers in GP whose product is given are to be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ and if six numbers in GP whose product is given are to be taken as $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$, etc.

In general If $(2m)$ numbers in GP whose product is given are to be taken as $(m \in N)$

$$\frac{a}{r^{2m-1}}, \frac{a}{r^{2m-3}}, \ldots, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, \ldots, ar^{2m-3}, ar^{2m-1}$$

Remark

1. Product of four numbers $= a^4$
   Product of six numbers $= a^6$
   $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$
   Product of $(2m)$ numbers $= a^{2m}$
2. From given conditions, find two equations in a and r and then solve them. Now, the numbers in GP can be obtained.

| Example 40. If $S_1, S_2, S_3, \ldots, S_p$ are the sum of infinite geometric series whose first terms are 1, 2, 3, ..., p and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{p+1}$ respectively, prove that

$$S_1 + S_2 + S_3 + \ldots + S_p = \frac{p(p+3)}{2}.$$

Sol. $\because$
$$S_p = \frac{p}{1 - \frac{1}{p+1}} = (p+1)$$

$\therefore \quad S_1 = 2, S_2 = 3, S_3 = 4, \ldots$

$\therefore \quad \text{LHS} = S_1 + S_2 + S_3 + \ldots + S_p$

$$= 2 + 3 + 4 + \ldots + (p+1) = \frac{p}{2}(2 + p + 1)$$

$$= \frac{p(p+3)}{2} = \text{RHS}$$

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| Example 41. Let x_1 and x_2 be the roots of the equation $x^2 - 3x + A = 0$ and let x_3 and x_4 be the roots of the equation $x^2 - 12x + B = 0$. It is known that the numbers x_1, x_2, x_3, x_4 (in that order) form an increasing GP. Find A and B.

Sol. $\because x_1, x_2, x_3, x_4$ are in GP.

Let $x_2 = x_1 r,\ x_3 = x_1 r^2,\ x_4 = x_1 r^3$

[here, product of x_1, x_2, x_3, x_4 are not given]

Given, $x_1 + x_2 = 3,\ x_1 x_2 = A$

$\Rightarrow \quad x_1(1 + r) = 3,\ x_1^2 r = A \quad$ ...(i)

and $\quad x_3 + x_4 = 12,\ x_3 x_4 = B$

$\Rightarrow x_1 r^2 (1 + r) = 12,\ x_1^2 r^5 = B \quad$ ...(ii)

From Eqs. (i) and (ii),

$$r^2 = 4 \Rightarrow r = 2 \quad \text{[for increasing GP]}$$

From Eq. (i), $x_1 = 1$

Now, $\quad A = x_1^2 r = 1^2 \cdot 2 = 2 \quad$ [from Eq. (i)]

and $\quad B = x_1^2 r^5 = 1^2 \cdot 2^5 = 32 \quad$ [from Eq. (ii)]

Example 42. Suppose a, b, c are in AP and a^2, b^2, c^2 are in GP, if $a > b > c$ and $a + b + c = \frac{3}{2}$, then find the values of a and c.

Sol. Since, a, b, c are in AP and sum of a, b, c is given.

Let $\quad a = b - D,\ c = b + D \quad [D < 0][\because a > b > c]$

and given $\quad a + b + c = \frac{3}{2}$

$$\Rightarrow b - D + b + b + D = \frac{3}{2}$$

$$\therefore \quad b = \frac{1}{2}$$

Then, $a = \frac{1}{2} - D$ and $c = \frac{1}{2} + D$

Also, given a^2, b^2, c^2 are in GP, then $(b^2)^2 = a^2 c^2$

$$\Rightarrow \quad \pm b^2 = ac \Rightarrow \pm \frac{1}{4} = \frac{1}{4} - D^2$$

$$\Rightarrow \quad D^2 = \frac{1}{4} \pm \frac{1}{4} = \frac{1}{2} \quad [\because D \neq 0]$$

$$\therefore \quad D = \pm \frac{1}{\sqrt{2}} \Rightarrow D = -\frac{1}{\sqrt{2}} \quad [\because D < 0]$$

Hence, $a = \frac{1}{2} + \frac{1}{\sqrt{2}}$ and $c = \frac{1}{2} - \frac{1}{\sqrt{2}}$

Example 43. If the continued product of three numbers in GP is 216 and the sum of their products in pairs is 156, then find the sum of three numbers.

Sol. Here, product of numbers in GP is given.

$\therefore$ Let the three numbers be $\frac{a}{r}, a, ar$.

Then, $\quad \frac{a}{r} \cdot a \cdot ar = 216$

$$\Rightarrow \quad a^3 = 216$$

$$\therefore \quad a = 6$$

Sum of the products in pairs = 156

$$\Rightarrow \quad \frac{a}{r} \cdot a + a \cdot ar + ar \cdot \frac{a}{r} = 156$$

$$\Rightarrow \quad a^2 \left(\frac{1}{r} + r + 1 \right) = 156 \Rightarrow 36 \left(\frac{1 + r^2 + r}{r} \right) = 156$$

$$\Rightarrow \quad 3 \left(\frac{1 + r + r^2}{r} \right) = 13 \Rightarrow 3r^2 - 10r + 3 = 0$$

$$\Rightarrow \quad (3r - 1)(r - 3) = 0 \Rightarrow r = \frac{1}{3} \text{ or } r = 3$$

Putting the values of a and r, the required numbers are 18, 6, 2 or 2, 6, 18. Hence, the sum of numbers is 26.

Example 44. Find a three-digit number whose consecutive digits form a GP. If we subtract 792 from this number, we get a number consisting of the same digits written in the reverse order. Now, if we increase the second digit of the required number by 2, then the resulting digits will form an AP.

Sol. Let the three digits be a, ar, ar^2, then according to hypothesis

$$100a + 10ar + ar^2 - 792 = 100ar^2 + 10ar + a$$

$$\Rightarrow \quad 99a(1 - r^2) = 792$$

$$\Rightarrow \quad a(1 + r)(1 - r) = 8 \quad \text{...(i)}$$

and $a, ar + 2, ar^2$ are in AP.

Then, $\quad 2(ar + 2) = a + ar^2$

$$\Rightarrow \quad a(r^2 - 2r + 1) = 4 \Rightarrow a(r - 1)^2 = 4 \quad \text{...(ii)}$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{r + 1}{r - 1} = -2 \Rightarrow r = \frac{1}{3}$$

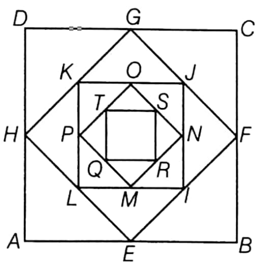
From Eq. (ii), $a = 9$

Thus, digits are 9, 3, 1 and so the required number is 931.

Examples on Application of Progression in Geometrical Figures

Example 45. A square is drawn by joining the mid-points of the sides of a given square. A third square is drawn inside the second square in the same way and this process continues indefinitely. If a side of the first square is 16 cm, then determine the sum of the areas of all the squares.

Sol. Let a be the side length of square, then



$AB = BC = CD = DA = a$

$\because E, F, G, H$ are the mid-points of AB, BC, CD and DA, respectively.

$$\therefore \quad EF = FG = GH = HE = \frac{a}{\sqrt{2}}$$

and I, J, K, L are the mid-points of EF, FG, GH and HE, respectively.

$$\therefore \quad IJ = JK = KL = LI = \frac{a}{2}$$

Similarly, $MN = NO = OP = PM = \frac{a}{2\sqrt{2}}$ and

$$QR = RS = ST = TQ = \frac{a}{4}, \ldots$$

S = Sum of areas

$$= ABCD + EFGH + IJKL + MNOP + QRST + \ldots$$

$$= a^2 + \left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2\sqrt{2}}\right)^2 + \ldots$$

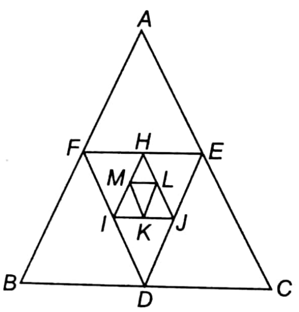
$$= a^2\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$$

$$= a^2\left(\frac{1}{1-\frac{1}{2}}\right) = 2a^2 = 2(16)^2 \qquad [\because a = 16 \text{ cm}]$$

$= 512$ sq cm

Example 46. One side of an equilateral triangle is 24 cm. The mid-points of its sides are joined to form another triangle whose mid-points, in turn, are joined to form still another triangle. This process continues, indefinitely. Find the sum of the perimeters of all the triangles.

Sol. Let a be the side length of equilateral triangle, then $AB = BC = CA = a$



$\because D, E, F$ are the mid-points of BC, CA and AB, respectively.

$$\therefore \quad EF = FD = DE = \frac{a}{2}$$

and H, I, J are the mid-points of EF, FD and DE, respectively.

$$\therefore \quad IJ = JH = HI = \frac{a}{4}$$

Similarly, $KL = ML = KM = \frac{a}{8}, \ldots$

$$P = \text{Sum of perimeters} = 3\left(a + \frac{a}{2} + \frac{a}{4} + \frac{a}{8} + \ldots\right)$$

$$= 3\left(\frac{a}{1-\frac{1}{2}}\right) = 6a = 6 \times 24 = 144 \text{ cm} \qquad [\because a = 24 \text{ cm}]$$

Example 47. Let $S_1, S_2, \ldots$ be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm and the area of S_n less than 1 sq cm. Then, find the value of n.

Sol. We have, length of a side of

S_n = length of diagonal of S_{n+1}

$\Rightarrow$ Length of a side of $S_n = \sqrt{2}$ (length of a side of S_{n+1})

$$\Rightarrow \quad \frac{\text{Length of a side of } S_{n+1}}{\text{Length of a side of } S_n} = \frac{1}{\sqrt{2}}, \text{ for all } n \geq 1$$

$\Rightarrow$ Sides of $S_1, S_2, S_3, \ldots$ form a GP with common ratio $\frac{1}{\sqrt{2}}$ and first term 10.

$$\therefore \quad \text{Side of } S_n = 10\left(\frac{1}{\sqrt{2}}\right)^{n-1} = \frac{10}{2^{\frac{(n-1)}{2}}}$$

$$\Rightarrow \text{Area of } S_n = (\text{Side})^2 = \frac{100}{2^{n-1}}$$

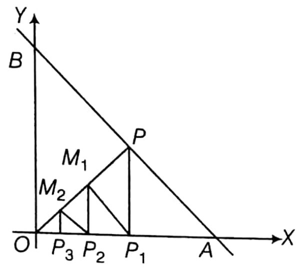
Now, given area of $S_n < 1$

$$\Rightarrow \quad \frac{100}{2^{n-1}} < 1 \Rightarrow 2^{n-1} > 100 > 2^6$$

$$\Rightarrow \quad 2^{n-1} > 2^6 \Rightarrow n - 1 > 6$$

$$\therefore \quad n > 7 \text{ or } n \geq 8$$

Example 48. The line $x + y = 1$ meets X-axis at A and Y-axis at B, P is the mid-point of AB, P_1 is the foot of perpendicular from P to OA, M_1 is that of P_1 from OP; P_2 is that of M_1 from OA, M_2 is that of P_2 from OP; P_3 is that of M_2 from OA and so on. If P_n denotes the nth foot of the perpendicular on OA, then find OP_n.



Sol. We have,

$$(OM_{n-1})^2 = (OP_n)^2 + (P_n M_{n-1})^2$$

$$= (OP_n)^2 + (OP_n)^2 = 2(OP_n)^2 = 2\alpha_n^2 \quad [\text{say}]$$

Also, $(OP_{n-1})^2 = (OM_{n-1})^2 + (P_{n-1}M_{n-1})^2$

$$\Rightarrow \quad \alpha_{n-1}^2 = 2\alpha_n^2 + \frac{1}{2}\alpha_{n-1}^2 \Rightarrow \alpha_n^2 = \frac{1}{4}\alpha_{n-1}^2$$

$$\Rightarrow \quad \alpha_n = \frac{1}{2}\alpha_{n-1}$$

$$\Rightarrow \quad OP_n = \alpha_n = \frac{1}{2}\alpha_{n-1} = \frac{1}{2^2}\alpha_{n-2} = \ldots = \frac{1}{2^n}$$

$$\therefore \quad OP_n = \left(\frac{1}{2}\right)^n$$

Use of GP in Solving Practical Problems

In this part, we will see how the formulae relating to GP can be made use of in solving practical problems.

Example 49. Dipesh writes letters to four of his friends. He asks each of them to copy the letter and mail to four different persons with the request that they continue the chain similarly. Assuming that the chain is not broken and that it costs 25 paise to mail one letter, find the total money spent on postage till the 8th set of letters is mailed.

Sol. Number of letters in the 1st set = 4 (These are letters sent by Dipesh)

Number of letters in the 2nd set = 4 + 4 + 4 + 4 = 16

Number of letters in the 3rd set

= 4 + 4 + 4 + ... + 16 terms = 64

⋮ ⋮ ⋮ ⋮

The number of letters sent in the 1st set, 2nd set, 3rd set, ... are respectively 4, 16, 64, ... which is a GP with $a = 4$,

$$r = \frac{16}{4} = \frac{64}{16} = 4$$

∴ Total number of letters in all the first 8 sets

$$= \frac{4(4^8 - 1)}{4 - 1} = 87380$$

$$\therefore \text{ Total money spent on letters} = 87380 \times \frac{25}{100} = ₹21845$$

Example 50. An insect starts from a point and travels in a straight path 1 mm in the first second and half of the distance covered in the previous second in the succeeding second. In how much time would it reach a point 3 mm away from its starting point.

Sol. Distance covered by the insect in the 1st second = 1 mm

Distance covered by it in the 2nd second $= 1 \times \frac{1}{2} = \frac{1}{2}$ mm

Distance covered by it in the 3rd second $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ mm

⋮ ⋮ ⋮ ⋮

The distance covered by the insect in 1st second, 2nd second, 3rd second, ... are respectively $1, \frac{1}{2}, \frac{1}{4}, \ldots$, which are in GP with $a = 1, r = \frac{1}{2}$. Let time taken by the insect in covering 3 mm be n seconds.

$$\therefore 1 + \frac{1}{2} + \frac{1}{4} + \ldots + n \text{ terms} = 3$$

$$\Rightarrow \quad \frac{1 \cdot \left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}} = 3$$

$$\Rightarrow \quad 1 - \left(\frac{1}{2}\right)^n = \frac{3}{2}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n = -\frac{1}{2}$$

$$\Rightarrow \quad 2^n = -2$$

which is impossible because $2^n > 0$

∴ Our supposition is wrong.

∴ There is no $n \in N$, for which the insect could never 3 mm in n seconds.

Hence, it will never to able to cover 3 mm.

Remark

The maximum distance that the insect could cover is 2 mm.

i.e., $$1 + \frac{1}{2} + \frac{1}{4} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2$$

Example 51. The pollution in a normal atmosphere is less than 0.01%. Due to leakage of a gas from a factory, the pollution is increased to 20%. If every day 80% of the pollution is neutralised, in how many days the atmosphere will be normal?

Sol. Let the pollution on 1st day = 20

The pollution on 2nd day = 20 × 20% = 20 (0.20)

The pollution on 3rd day = $20(0.20)^2$

⋮ ⋮ ⋮ ⋮

Let in n days the atmosphere will be normal

$$\therefore \quad 20(0.20)^{n-1} < 0.01$$

$$\Rightarrow \quad \left(\frac{2}{10}\right)^{n-1} < \frac{1}{2000}$$

Taking logarithm on base 10, we get

$$(n-1)(\log 2 - \log 10) < \log 1 - \log 2000$$

$$\Rightarrow \quad (n-1)(0.3010 - 1) < 0 - (0.3010 + 3)$$

$$\Rightarrow \quad n - 1 > \frac{3.3010}{0.6990}$$

$$\Rightarrow \quad n > 5.722$$

Hence, the atmosphere will be normal in 6 days.

Exercise for Session 3

1. The fourth, seventh and the last term of a GP are 10, 80 and 2560, respectively. The first term and number of terms in GP are

(a) $\frac{4}{5}$, 12 (b) $\frac{4}{5}$, 10 (c) $\frac{5}{4}$, 12 (d) $\frac{5}{4}$, 10

2. If the first and the nth terms of a GP are a and b respectively and if P is the product of the first n terms, then P^2 is equal to

(a) ab (b) $(ab)^{n/2}$ (c) $(ab)^n$ (d) None of these

3. If $a_1, a_2, a_3, (a_1 > 0)$ are three successive terms of a GP with common ratio r, the value of r for which $a_3 > 4a_2 - 3a_1$ holds is given by

(a) $1 < r < 3$ (b) $-3 < r < -1$ (c) $r < 1$ or $r > 3$ (d) None of these

4. If $x, 2x + 2, 3x + 3$ are in GP, the fourth term is

(a) 27 (b) -27 (c) 13.5 (d) -13.5

5. In a sequence of 21 terms the first 11 terms are in AP with common difference 2 and the last 11 terms are in GP with common ratio 2, if the middle term of the AP is equal to the middle term of GP, the middle term of the entire sequence is

(a) $-\frac{10}{31}$ (b) $\frac{10}{31}$ (c) $-\frac{32}{31}$ (d) $\frac{32}{31}$

6. Three distinct numbers x, y, z form a GP in that order and the numbers $7x + 5y, 7y + 5z, 7z + 5x$ form an AP in that order. The common ratio of GP is

(a) -4 (b) -2 (c) 10 (d) 18

7. The sum to n terms of the series $11 + 103 + 1005 + \dots$ is

(a) $\frac{1}{9}(10^n - 1) + n^2$ (b) $\frac{1}{9}(10^n - 1) + 2n$ (c) $\frac{10}{9}(10^n - 1) + n^2$ (d) $\frac{10}{9}(10^n - 1) + 2n$

8. In an increasing GP, the sum of the first and last term is 66, the product of the second and the last but one is 128 and the sum of the sum of the terms is 126, then the number of terms in the series is

(a) 6 (b) 8 (c) 10 (d) 12

9. If S_1, S_2, S_3 be respectively the sum of $n, 2n$ and $3n$ terms of a GP, then $\frac{S_1(S_3 - S_2)}{(S_2 - S_1)^2}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

10. If $|a| < 1$ and $|b| < 1$, then the sum of the series $1 + (1 + a)b + (1 + a + a^2)b^2 + (1 + a + a^2 + a^3)b^3 + \dots$ is

(a) $\frac{1}{(1-a)(1-b)}$ (b) $\frac{1}{(1-a)(1-ab)}$ (c) $\frac{1}{(1-b)(1-ab)}$ (d) $\frac{1}{(1-a)(1-b)(1-ab)}$

11. If the sides of a triangle are in GP and its larger angle is twice the smallest, then the common ratio r satisfies the inequality

(a) $0 < r < \sqrt{2}$ (b) $1 < r < \sqrt{2}$ (c) $1 < r < 2$ (d) $r > \sqrt{2}$

12. If $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$, then a, b, c, d are in

(a) AP (b) GP (c) HP (d) None of these

13. If $(r)_n$ denotes the number $r\,r\,r \dots$ (n digits), where $r = 1, 2, 3, \dots, 9$ and $a = (6)_n, b = (8)_n, c = (4)_{2n}$, then

(a) $a^2 + b + c = 0$ (b) $a^2 + b - c = 0$ (c) $a^2 + b - 2c = 0$ (d) $a^2 + b - 8c = 0$

14. $0.4\dot{2}\dot{7}$ represents the rational number

(a) $\frac{47}{99}$ (b) $\frac{47}{110}$ (c) $\frac{47}{999}$ (d) $\frac{49}{99}$

15. If the product of three numbers in GP be 216 and their sum is 19, then the numbers are

(a) 4, 6, 9 (b) 4, 7, 8 (c) 3, 7, 9 (d) None of these