METHODS OF DIFFERENTIATION

The process of calculating derivative is called differentiation.

1. DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE :

Obtaining the derivative using the definition $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = f'(x) = \frac{dy}{dx}$ is called calculating derivative using first principle or ab initio or delta method.

Illustration 1 : Differentiate each of following functions by first principle :

(i) $f(x) = \tan x$ (ii) $f(x) = e^{\sin x}$

Solution : (i) $f'(x) = \lim\limits_{h \to 0} \frac{\tan(x+h) - \tan x}{h} = \lim\limits_{h \to 0} \frac{\tan(x+h-x)[1 + \tan x \tan(x+h)]}{h}$

$= \lim\limits_{h \to 0} \frac{\tanh}{h} . (1 + \tan^2 x) = \sec^2 x.$ **Ans.**

(ii) $f'(x) = \lim\limits_{h \to 0} \frac{e^{\sin(x+h)} - e^{\sin x}}{h} = \lim\limits_{h \to 0} e^{\sin x} \frac{[e^{\sin(x+h) - \sin x} - 1]}{\sin(x+h) - \sin x}\left(\frac{\sin(x+h) - \sin x}{h}\right)$

$= e^{\sin x} \lim\limits_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = e^{\sin x} \cos x$ **Ans.**

Do yourself -1 :

(i) Differentiate each of following functions by first principle:

(a) $f(x) = \ell nx$ (b) $f(x) = \frac{1}{x}$

2. DERIVATIVE OF STANDARD FUNCTIONS :

	f(x)	f'(x)		f(x)	f'(x)
(i)	x^n	nx^{n-1}	(ii)	e^x	e^x
(iii)	a^x	$a^x \ell na$, $a > 0$	(iv)	ℓnx	$1/x$
(v)	$\log_a x$	$(1/x) \log_a e$, $a > 0$, $a \neq 1$	(vi)	sinx	cosx
(vii)	cosx	− sinx	(viii)	tanx	$\sec^2 x$
(ix)	secx	secx tanx	(x)	cosecx	− cosecx . cotx
(xi)	cotx	$-\text{cosec}^2 x$	(xii)	constant	0
(xiii)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$	(xiv)	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
(xv)	$\tan^{-1} x$	$\frac{1}{1+x^2}, x \in R$	(xvi)	$\sec^{-1} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$
(xvii)	$\text{cosec}^{-1} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$	(xviii)	$\cot^{-1} x$	$\frac{-1}{1+x^2}, x \in R$

3. FUNDAMENTAL THEOREMS :

If f and g are derivable functions of x, then,

(a) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$ (b) $\frac{d}{dx}(cf) = c\frac{df}{dx}$, where c is any constant

(c) $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$ known as "PRODUCT RULE"

(d) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$ where $g \neq 0$ known as "QUOTIENT RULE"

(e) If y = f(u) & u = g (x) then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ known as "CHAIN RULE"

Note : In general if y = f(u) then $\frac{dy}{dx} = f'(u).\frac{du}{dx}$.

Illustration 2 : If $y = e^x \tan x + x\log_e x$, find $\frac{dy}{dx}$.

Solution : $y = e^x.\tan x + x \quad \log_e x$

On differentiating we get,

$$\frac{dy}{dx} = e^x \quad \tan x + e^x \quad \sec^2 x + 1 \quad \log x + x \quad \frac{1}{x}$$

Hence, $\frac{dy}{dx} = e^x(\tan x + \sec^2 x) + (\log x + 1)$ **Ans.**

Illustration 3 : If $y = \frac{\log x}{x} + e^x \sin 2x + \log_5 x$, find $\frac{dy}{dx}$.

Solution : On differentiating we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{x}\right) + \frac{d}{dx}(e^x \sin 2x) + \frac{d}{dx}(\log_5 x) = \frac{\frac{1}{x}x - \log x\,.\,1}{x^2} + e^x \sin 2x + 2e^x\,.\cos 2x + \frac{1}{x\log_e 5}$$

Hence, $\frac{dy}{dx} = \left(\frac{1 - \log x}{x^2}\right) + e^x(\sin 2x + 2\cos 2x) + \frac{1}{x\log_e 5}$ **Ans.**

Illustration 4 : If $x = \exp\left(\tan^{-1}\left(\frac{y - x^2}{x^2}\right)\right)$, then $\frac{dy}{dx}$ equals -

(A) $x[1 + \tan(\log x) + \sec^2 x]$

(B) $2x[1 + \tan(\log x)] + \sec^2 x$

(C) $2x[1 + \tan(\log x)] + \sec x$

(D) $2x + x[1 + \tan(\log x)]^2$

Solution : Taking log on both sides, we get

$$\log x = \tan^{-1}\left(\frac{y - x^2}{x^2}\right) \Rightarrow \tan(\log x) = (y - x^2) / x^2$$

$\Rightarrow \quad y = x^2 + x^2 \tan(\log x)$

On differentiating, we get

$$\therefore \frac{dy}{dx} = 2x + 2x\tan(\log x) + x\sec^2(\log x) \Rightarrow 2x[1 + \tan(\log x)] + x\sec^2(\log x)$$

$= 2x + x[1 + \tan(\log x)]^2$ **Ans. (D)**

Illustration 5 : If $y = \log_e(\tan^{-1}\sqrt{1+x^2})$, find $\frac{dy}{dx}$.

Solution : $y = \log_e\ (\tan^{-1}\sqrt{1+x^2})$

On differentiating we get,

$$= \frac{1}{\tan^{-1}\sqrt{1+x^2}} \cdot \frac{1}{1+(\sqrt{1+x^2})^2} \cdot \frac{1}{2\sqrt{1+x^2}} .2x$$

$$= \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)\left\{1+\left(\sqrt{1+x^2}\right)^2\right\}\sqrt{1+x^2}} = \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)\left(2+x^2\right)\sqrt{1+x^2}}$$ **Ans.**

> **Do yourself -2 :**
>
> (i) Find $\frac{dy}{dx}$ if -
>
> (a) $y = (x + 1)(x + 2)(x + 3)$ (b) $y = e^{5x}\tan(x^2 + 2)$

4. LOGARITHMIC DIFFERENTIATION :

To find the derivative of a function :

(a) which is the product or quotient of a number of functions or

(b) of the form $[f(x)]^{g(x)}$ where f & g are both derivable.

It is convenient to take the logarithm of the function first & then differentiate.

Illustration 6 : If $y = (\sin x)^{\ell n\, x}$, find $\frac{dy}{dx}$

Solution : $\ell n\ y = \ell n\ x.\ \ell n\ (\sin x)$

On differentiating we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ \ell n\ (\sin x) + \ell n\ x.\ \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^{\ell n\, x}\left[\frac{\ell n(\sin x)}{x} + \cot x\ \ell n\, x\right]$$ **Ans.**

Illustration 7 : If $y = \frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$ find $\frac{dy}{dx}$

Solution : $\ell n\ y = \frac{1}{2}\ \ell n\ x + \frac{2}{3}\ \ell n\ (1 - 2x) - \frac{3}{4}\ \ell n\ (2 - 3x) - \frac{4}{5}\ell n\ (3 - 4x)$

On differentiating we get,

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}\right)$$ **Ans.**

> **Do yourself -3 :**
>
> **(i)** Find $\frac{dy}{dx}$ if $y = x^x$ **(ii)** Find $\frac{dy}{dx}$ if $y = e^x.e^{x^2}.e^{x^3}.e^{x^4}$

5. DIFFERENTIATION OF IMPLICIT FUNCTIONS : $\phi(x, y) = 0$

(a) To find dy /dx of implicit functions, we differentiate each term w.r.t. x regarding y as a function of x & then collect terms with dy/dx together on one side.

(b) In the case of implicit functions, generally, both x & y are present in answers of dy/dx.

Illustration 8 : If $x^y + y^x = 2$, then find $\frac{dy}{dx}$.

Solution : Let $u = x^y$ and $v = y^x$

$u + v = 2 \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$

Now $u = x^y$ and $v = y^x$

$\Rightarrow \ell n\, u = y\, \ell nx$ and $\ell n\, v = x\, \ell n\, y$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \ell nx \frac{dy}{dx}$ and $\frac{1}{v}\frac{dv}{dx} = \ell n\, y + \frac{x}{y}\frac{dy}{dx}$

$\Rightarrow \frac{du}{dx} = x^y\left(\frac{y}{x} + \ell nx\frac{dy}{dx}\right)$ and $\frac{dv}{dx} = y^x\left(\ell n\, y + \frac{x}{y}\frac{dy}{dx}\right)$

$\Rightarrow x^y\left(\frac{y}{x} + \ell n\, x\frac{dy}{dx}\right) + y^x\left(\ell ny + \frac{x}{y}\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{\left(y^x \ell ny + x^y.\frac{y}{x}\right)}{\left(x^y \ell nx + y^x.\frac{x}{y}\right)}$ **Ans.**

Illustration 9 : If $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cos x.....}}}$, prove that $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$.

Solution : Given function is $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1+y}} = \frac{(1+y)\sin x}{1 + y + \cos x}$

or $y + y^2 + y\cos x = (1 + y)\sin x$

Differentiate both sides with respect to x,

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x = (1 + y)\cos x + \frac{dy}{dx}\sin x$$

$$\frac{dy}{dx}(1 + 2y + \cos x - \sin x) = (1 + y)\cos x + y\sin x$$

or $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$ **Ans.**

Do yourself -4 :

(i) Find $\frac{dy}{dx}$, if $x + y = \sin(x - y)$

(ii) If $x^2 + xe^y + y = 0$, find y', also find the value of y' at point (0,0).

6. PARAMETRIC DIFFERENTIATION :

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Illustration 10 : If y = a cos t and x = a(t − sint) find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$

Solution : $\frac{dy}{dx} = \frac{-a\sin t}{a(1-\cos t)} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1$ **Ans.**

Illustration 11 : Prove that the function represented parametrically by the equations. $x = \frac{1+t}{t^3}$; $y = \frac{3}{2t^2} + \frac{2}{t}$

satisfies the relationship : $x(y')^3 = 1 + y'$ (where $y' = \frac{dy}{dx}$)

Solution : Here $x = \frac{1+t}{t^3} = \frac{1}{t^3} + \frac{1}{t^2}$

Differentiating w.r. to t

$$\frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3}$$

$$y = \frac{3}{2t^2} + \frac{2}{t}$$

Differentiating w.r. to t

$$\frac{dy}{dt} = -\frac{3}{t^3} - \frac{2}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t = y'$$

Since $x = \frac{1+t}{t^3} \Rightarrow x = \frac{1+y'}{(y')^3}$ or $x(y')^3 = 1 + y'$ **Ans.**

Do yourself -5 :

(i) Find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ if $y = \cos^4 t$ & $x = \sin^4 t$.

(ii) Find the slope of the tangent at a point P(t) on the curve $x = at^2$, $y=2at$.

7. DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION :

Let $y= f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

8. DERIVATIVE OF A FUNCTION AND ITS INVERSE FUNCTION :

If g is inverse of f, then

(a) $g\{f(x)\} = x$ (b) $f\{g(x)\} = x$

$g'\{f(x)\}f'(x)=1$ $f'\{g(x)\}g'(x) = 1$

Illustration 12 : Differentiate $\log_e(\tan x)$ with respect to $\sin^{-1}(e^x)$.

Solution : $\frac{d(\log_e \tan x)}{d(\sin^{-1}(e^x))} = \frac{\frac{d}{dx}(\log_e \tan x)}{\frac{d}{dx}\sin^{-1}(e^x)} = \frac{\cot x.\sec^2 x}{e^x.1/\sqrt{1-e^{2x}}} = \frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x \cos x}$ **Ans.**

Illustration 13 : If g is inverse of f and $f'(x) = \dfrac{1}{1+x^n}$, then g'(x) equals :-

(A) $1 + x^n$ (B) $1 + [f(x)]^n$ (C) $1 + [g(x)]^n$ (D) none of these

Solution : Since g is the inverse of f. Therefore

$f(g(x)) = x$ for all x

$\Rightarrow \dfrac{d}{dx} f(g(x)) = 1$ for all x

$\Rightarrow f'(g(x))\, g'(x) = 1 \Rightarrow g'(x) = \dfrac{1}{f'(g(x))} = 1 + (g(x))^n$ **Ans. (C)**

Do yourself -6 :

(i) Differentiate $x^{\ell nx}$ with respect to ℓnx.

(ii) If g is inverse of f and $f(x) = 2x + \sin x$; then g'(x) equals:

(A) $-\dfrac{3}{x^2} + \dfrac{1}{\sqrt{1-x^2}}$ (B) $2 + \sin^{-1}x$ (C) $2 + \cos g(x)$ (D) $\dfrac{1}{2+\cos(g(x))}$

9. HIGHER ORDER DERIVATIVES :

Let a function $y = f(x)$ be defined on an interval (a, b). If $f(x)$ is differentiable function, then its derivative $f'(x)$ [or (dy/dx) or y'] is called the first derivative of y w.r.t. x. If $f'(x)$ is again differentiable function on (a, b), then its derivative $f''(x)$ [or d^2y/dx^2 or y''] is called second derivative of y w.r.t. x. Similarly, the 3rd order derivative of y w.r.to x, if it exists, is defined by $\dfrac{d^3y}{dx^3} = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$ and denoted by $f'''(x)$ or y''' and so on.

Note : If $x = f(\theta)$ and $y = g(\theta)$ where 'θ' is a parameter then $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$ & $\dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)\Big/\dfrac{dx}{d\theta}$

In general $\dfrac{d^ny}{dx^n} = \dfrac{d}{d\theta}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)\Big/\dfrac{dx}{d\theta}$

Illustration 14 : If $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in R$. Then find f(x) independent of f'(1), f''(2) and f'''(3).

Solution : Here, $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$

put $f'(1) = a,\ f''(2) = b,\ f'''(3) = c$ ....(i)

$\therefore\ f(x) = x^3 + ax^2 + bx + c$

$\Rightarrow f'(x) = 3x^2 + 2ax + b$ or $f'(1) = 3 + 2a + b$ ....(ii)

$\Rightarrow f''(x) = 6x + 2a$ or $f''(2) = 12 + 2a$ ....(iii)

$\Rightarrow f'''(x) = 6$ or $f'''(3) = 6$ ....(iv)

from (i) and (iv), c = 6

from (i), (ii) and (iii) we have, a = –5, b = 2

$\therefore\ f(x) = x^3 - 5x^2 + 2x + 6$ **Ans.**

Illustration 15 : If x = a (t + sin t) and y = a(1 − cos t), find $\frac{d^2y}{dx^2}$.

Solution : Here x = a (t + sin t) and y = a (1−cos t)

Differentiating both sides w.r.t. t, we get :

$$\frac{dx}{dt} = a(1 + \cos t) \quad \text{and} \quad \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\left(\frac{t}{2}\right)$$

Again differentiating both sides, we get,

$$\frac{d^2y}{dx^2} = \sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dt}{dx} = \frac{1}{2}\sec^2(t/2)\cdot\frac{1}{a(1+\cos t)} = \frac{1}{2a}\cdot\frac{\sec^2\left(\frac{t}{2}\right)}{2\left(\cos^2\frac{t}{2}\right)}$$

Hence, $$\frac{d^2y}{dx^2} = \frac{1}{4a}\cdot\sec^4\left(\frac{t}{2}\right)$$ **Ans.**

Illustration 16 : y = f(x) and x = g(y) are inverse functions of each other then express g'(y) and g''(y) in terms of derivative of f(x).

Solution : $$\frac{dy}{dx} = f'(x) \quad \text{and} \quad \frac{dx}{dy} = g'(y)$$

$$\Rightarrow \quad g'(y) = \frac{1}{f'(x)} \qquad \text{..........(i)}$$

Again differentiating w.r.t. to y

$$g''(y) = \frac{d}{dy}\left(\frac{1}{f'(x)}\right) = \frac{d}{dx}\left(\frac{1}{f'(x)}\right).\frac{dx}{dy} = -\frac{f''(x)}{(f'(x))^2}.\left(\frac{1}{f'(x)}\right)$$

$$\Rightarrow \quad g''(y) = -\frac{f''(x)}{(f'(x))^3} \qquad \text{..........(ii)}$$

Which can also be remembered as $$\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$ **Ans.**

Do yourself : 7

(i) If $y = xe^{x^2}$ then find y''. **(ii)** Find y'' at x = π/4, if y = x tan x.

(iii) Prove that the function $y= e^x \sin x$ satisfies the relationship y'' − 2y' + 2y = 0.

10. DIFFERENTIATION OF DETERMINANTS :

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h. l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Illustration 17 : If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, find f'(x).

Solution : Here, $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

On differentiating, we get,

$$\Rightarrow f'(x) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

or $$f'(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

As we know if any two rows or columns are equal, then value of determinant is zero.

$$= 0 + 0 + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix} \quad \therefore \quad f'(x) = 6\,(2x^2 - x^2)$$

Therefore, $f'(x) = 6x^2$ **Ans.**

Do yourself : 8

(i) If $f(x) = \begin{vmatrix} e^x & x^2 \\ \ell nx & \sin x \end{vmatrix}$, then find f '(1). **(ii)** If $f(x) = \begin{vmatrix} 2x & x^2 & x^3 \\ x^2+2x & 1 & 3x+1 \\ 2x & 1-3x^2 & 5x \end{vmatrix}$ then find f ' (1).

11. L'HÕPITAL'S RULE :

(a) This rule is applicable for the indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point 'a', except, may be, at the point 'a' itself and $g'(x) \neq 0$, and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty,$$

then $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists (L' Hôpital's rule). The point 'a' may be either finite or improper ($+\infty$ or $-\infty$).

(b) Indeterminate forms of the type $0.\ \infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(c) Indeterminate forms of the type 1^{∞}, ∞^0 or 0^0 are reduced to forms of the type $0 \quad \infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x).\,\ell nf(x)}$.

Illustration 18 : Evaluate $\lim\limits_{x\to 0}|x|^{\sin x}$

Solution : $\lim\limits_{x\to 0}|x|^{\sin x} = \lim\limits_{x\to 0} e^{\sin x \log_e|x|} = e^{\lim\limits_{x\to 0}\frac{\log_e|x|}{\operatorname{cosec} x}}$

$= e^{\lim\limits_{x\to 0}\frac{1/x}{-\operatorname{cosec} x \cot x}}$ (applying L'Hôpital's rule)

$= e^{\lim\limits_{x\to 0}-\frac{\sin^2 x}{x\cos x}} = e^{\lim\limits_{x\to 0}-\left(\frac{\sin x}{x}\right)^2\cdot\left(\frac{x}{\cos x}\right)} = e^{-(1)^2\cdot(0)} = e^0 = 1$ **Ans.**

Illustration 19 : Solve $\lim\limits_{x\to 0^+}\log_{\sin x}\sin 2x$.

Solution : Here $\lim\limits_{x\to 0^+}\log_{\sin x}\sin 2x$

$$= \lim_{x\to 0^+}\frac{\log\sin 2x}{\log\sin x} \qquad \left(\frac{-\infty}{-\infty}\text{form}\right)$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{\sin 2x}\cdot 2\cos 2x}{\frac{1}{\sin x}\cdot\cos x} \qquad \{\text{applying L'Hôpital's rule}\}$$

$$= \lim_{x\to 0^+}\frac{\left(\frac{(2x)}{\sin(2x)}\right)\cos 2x}{\left(\frac{x}{\sin x}\right)\cos x} = \lim_{x\to 0^+}\frac{\cos 2x}{\cos x} = 1$$ **Ans.**

Illustration 20 : Evaluate $\lim\limits_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$.

Solution : Here, $A = \lim\limits_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$ $(\infty^0$ form)

$$\therefore \quad \log A = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{e^n}{\pi}\right) = \lim_{n\to\infty}\frac{n\log e - \log\pi}{n} \quad \left(\frac{\infty}{\infty}\text{form}\right)$$

$$= \lim_{n\to\infty}\frac{\log e - 0}{1} \qquad \{\text{applying L'Hôpital's rule}\}$$

$\log A = 1 \Rightarrow A = e^1$ or $\lim\limits_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n} = e$ **Ans.**

Do yourself : 9

(i) Using L'Hõpital's rule find (a) $\lim\limits_{x\to 0}\frac{\tan x - x}{x^3}$ (b) $\lim\limits_{x\to 0}\frac{e^x - x - 1}{x^2}$

(ii) Using L'Hõpital's rule verify that : (a) $\lim\limits_{x\to 0}\frac{\sin x - \tan x}{x^3} = -\frac{1}{2}$ (b) $\lim\limits_{x\to 0}\frac{\ell n(1+x)}{x} = 1$

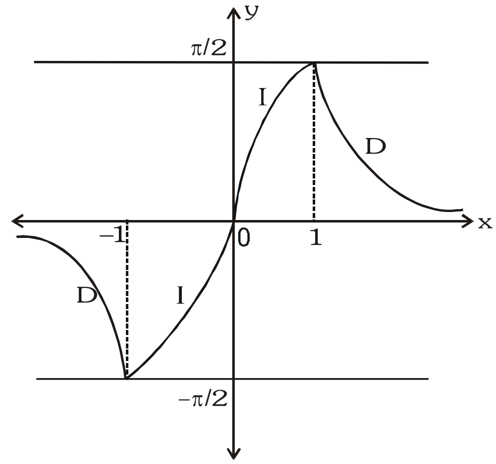
INTERESTING FACT :

In 1694 John Bernoulli agreed to accept a retainer of 300 pounds per year from his former student L'Hôpital to solve problems for him and keep him up to date on calculus. One of the problems was the so-called 0/0 problem, which Bernoulli solved as agreed. When L'Hôpital published his notes on calculus in book form in 1696, the 0/0 rule appeared as a theorem. L'Hôpital acknowledged his debt to Bernoulli and, to avoid claiming authorship of the book's entire contents, had the book published anonymously. Bernoulli nevertheless accused L'Hôpital of plagiarism, an accusation inadvertently supported after L'Hôpital's death in 1704 by the publisher's promotion of the book as L'Hôpital's. By 1721, Bernoulli, a man so jealous he once threw his son Daniel out of the house for accepting a mathematics prize from the French Academy of Sciences, claimed to have been the author of the entire work. As puzzling and fickle as ever, history accepted Bernoulli's claim (until recently), but still named the rule after L'Hôpital.

12. ANALYSIS AND GRAPHS OF SOME INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $$y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases}$$

Important points :



(i) Domain is $x \in R$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

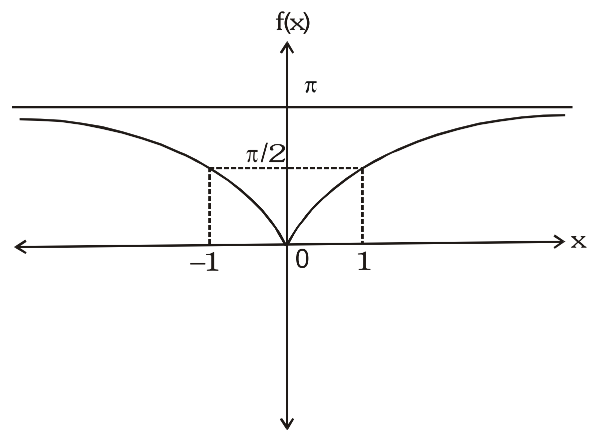
(ii) f is continuous for all x but not differentiable at x = 1, –1

(iii) $$\frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & \text{for } |x| < 1 \\ \text{non existent} & \text{for } |x| = 1 \\ \dfrac{-2}{1+x^2} & \text{for } |x| > 1 \end{cases}$$

(iv) Increasing in (–1 , 1) & Decreasing in $(-\infty, -1) \cup (1, \infty)$

(b) Consider $$y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$$

Important points :



(i) Domain is $x \in R$ & range is $[0, \pi)$

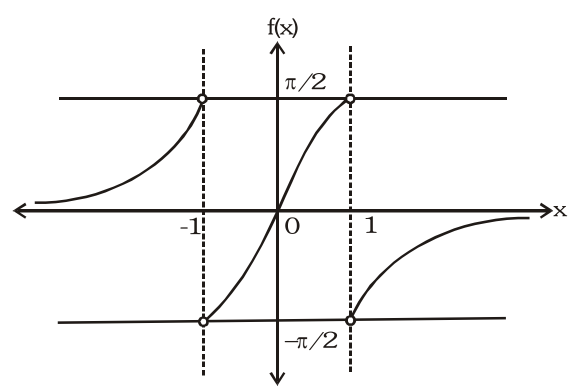
(ii) Continuous for all x but not differentiable at x = 0

(iii) $$\frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & \text{for } x > 0 \\ \text{non existent} & \text{for } x = 0 \\ -\dfrac{2}{1+x^2} & \text{for } x < 0 \end{cases}$$

(iv) Increasing in $(0, \infty)$ & Decreasing in $(-\infty, 0)$

(c) $$y = f(x) = \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & |x| < 1 \\ \pi + 2\tan^{-1}x & x < -1 \\ -(\pi - 2\tan^{-1}x) & x > 1 \end{cases}$$

Important points :



(i) Domain is R – {1, – 1} & range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(ii) It is neither continuous nor differentiable at x = 1, –1

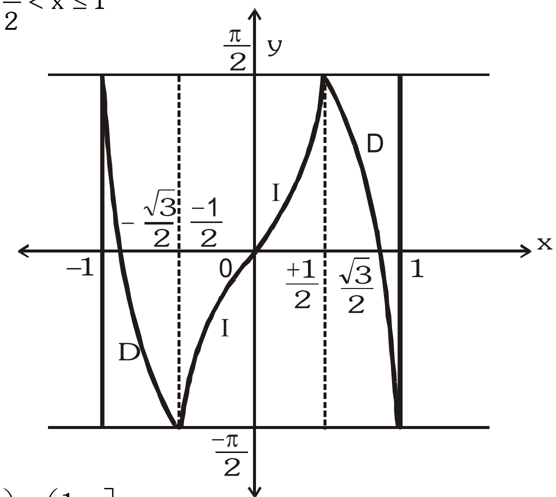
(iii) $$\frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & |x| \ne 1 \\ \text{non existent} & |x| = 1 \end{cases}$$

(iv) Increasing $\forall$ x in its domain

(v) It is bounded for all x

(d) $y = f(x) = \sin^{-1}(3x-4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if } -1 \le x < -\frac{1}{2} \\ 3\sin^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

Important points :



(i) Domain is $x \in [-1, 1]$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) Continuous everywhere in its domain

(iii) Not derivable at $x = -\frac{1}{2}, \frac{1}{2}$

(iv) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in (-\frac{1}{2}, \frac{1}{2}) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1) \end{cases}$

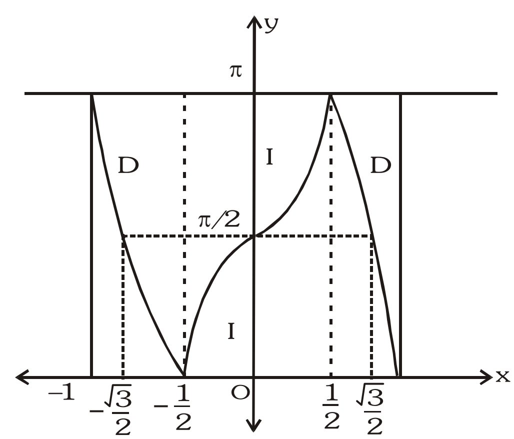
(v) Increasing in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and Decreasing in $\left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$

(e) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x < -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

Important points :

(i) Domain is $x \in [-1, 1]$ & range is $[0, \pi]$

(ii) Continuous everywhere in its domain



(iii) Not derivable at $x = -\frac{1}{2}, \frac{1}{2}$

(iv) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

(v) Increasing in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ &

Decreasing in $\left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$

GENERAL NOTE :

Concavity is decided by the sign of 2^{nd} derivative as :

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ Concave upwards ; $\frac{d^2y}{dx^2} < 0 \Rightarrow$ Concave downwards

Illustration 21 : $\frac{d}{dx}\left\{\sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)\right\} =$

(A) $-\frac{1}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) -1

Solution : Let $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$. Put $x = \cos 2\theta$ $\theta \in \left(0, \frac{\pi}{2}\right]$

$\therefore$ $y = \sin^2 \cot^{-1}\left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right) = \sin^2 \cot^{-1}(\cot\theta)$

$\therefore$ $y = \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-x}{2} = \frac{1}{2}-\frac{x}{2}$

$\therefore$ $\frac{dy}{dx} = -\frac{1}{2}$. Ans (A)

Illustration 22 : If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ then find

(i) $f'(2)$ (ii) $f'\left(\frac{1}{2}\right)$ (iii) $f'(1)$

Solution : $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\Rightarrow$ $y = \sin^{-1}(\sin 2\theta)$

$$y = \begin{cases} \pi - 2\theta & \frac{\pi}{2} < 2\theta < \pi \\ 2\theta & \frac{-\pi}{2} \le 2\theta \le \frac{\pi}{2} \\ -(\pi + 2\theta) & -\pi < 2\theta < -\frac{\pi}{2} \end{cases} \Rightarrow f(x) = \begin{cases} \pi - 2\tan^{-1}x & x > 1 \\ 2\tan^{-1}x & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\frac{2}{1+x^2} & x > 1 \\ \frac{2}{1+x^2} & -1 < x < 1 \\ \frac{-2}{1+x^2} & x < -1 \end{cases}$$

(i) $f'(2) = -\frac{2}{5}$ (ii) $f'\left(\frac{1}{2}\right) = \frac{8}{5}$ (iii) $f'(1^+) = -1$ and $f'(1^-) = +1$ $\Rightarrow$ $f'(1)$ does not exist Ans.

Do yourself : 10

(i) If $y = \cos^{-1}(4x^3 - 3x)$

Then find (a) $f'\left(-\frac{\sqrt{3}}{2}\right)$, (b) $f'(0)$, (c) $f'\left(\frac{\sqrt{3}}{2}\right)$.

Miscellaneous Illustrations :

Illustration 23 : If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then prove that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$ -

Solution : Put $x = \sin\alpha \Rightarrow \alpha = \sin^{-1}(x)$

$y = \sin\beta \Rightarrow \beta = \sin^{-1}(y)$

$\Rightarrow \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$

$$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=2a\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \cot\left(\frac{\alpha-\beta}{2}\right)=a$$

$$\Rightarrow \alpha-\beta=2\cot^{-1}(a)$$

$$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}(a)$$

differentiating w.r.t. to x.

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$ hence proved **Ans.**

Illustration 24 : Find second order derivative of $y = \sin x$ with respect to $z = e^x$.

Solution :

$$\frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{\cos x}{e^x}$$

$$\Rightarrow \frac{d^2y}{dz^2}=\frac{d}{dx}\left(\frac{\cos x}{e^x}\right).\frac{dx}{dz} = \frac{-e^x\sin x-\cos x e^x}{\left(e^x\right)^2}.\frac{1}{e^x}$$

$$\Rightarrow \frac{d^2y}{dz^2}=-\frac{(\sin x+\cos x)}{e^{2x}}$$ **Ans.**

Illustration 25 : If $y = (\tan^{-1}x)^2$ then prove that $(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx} = 2$

Solution : $y = (\tan^{-1}x)^2$

Differentiating w.r.t. x

$$\frac{dy}{dx}=\frac{2\tan^{-1}x}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)\frac{dy}{dx}=2\tan^{-1}(x)$$

Again differentiating w.r.t. x

$$\Rightarrow \left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=\frac{2}{\left(1+x^2\right)} \Rightarrow \left(1+x^2\right)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2$$ **Ans.**

Illustration 26 : Obtain differential coefficient of $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ with respect to $\cos^{-1}\sqrt{\dfrac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

Solution : Assume $u = \tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$, $v = \cos^{-1}\sqrt{\dfrac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

The function needs simplification before differentiation Let $x = \tan\theta$

$$\therefore \quad u = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$v = \cos^{-1}\sqrt{\frac{1+\sec\theta}{2\sec\theta}} = \cos^{-1}\sqrt{\frac{1+\cos\theta}{2}} = \cos^{-1}\left(\cos\frac{\theta}{2}\right) = \frac{\theta}{2} \Rightarrow u = v$$

$$\therefore \quad \frac{du}{dv} = 1.$$

Ans.

ANSWERS FOR DO YOURSELF

1 :	(i) (a) $\dfrac{1}{x}$	(b) $-\dfrac{1}{x^2}$		
2 :	(i) (a) $3x^2 + 12x + 11$	(b) $5e^{5x}\tan(x^2+2) + 2xe^{5x}\sec^2(x^2+2)$		
3 :	(i) $x^x(\ell nx + 1)$	(ii) $y(1 + 2x + 3x^2 + 4x^3)$		
4 :	(i) $\dfrac{\cos(x-y)-1}{\cos(x-y)+1}$	(ii) $y' = -\left(\dfrac{2x+e^y}{xe^y+1}\right)$, -1		
5.	(i) -1	(ii) $\dfrac{1}{t}$		
6 :	(i) $2(x^{\ell nx})(\ell nx)$	(ii) D		
7 :	(i) $y'' = 4y + 2xy'$	(ii) $\pi + 4$		
8 :	(i) $e(\sin 1 + \cos 1) - 1$	(ii) 9		
9.	(i) (a) $\dfrac{1}{3}$	(b) $\dfrac{1}{2}$		
10 :	(i) (a) -6	(b) 3	(c) -6	

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. If $y=\dfrac{\sec\ x-\tan\ x}{\sec\ x+\tan\ x}$ then $\dfrac{dy}{dx}$ equals -

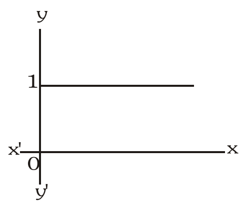
(A) $2\sec x(\sec x-\tan x)$ (B) $-2\sec x(\sec x-\tan x)^2$

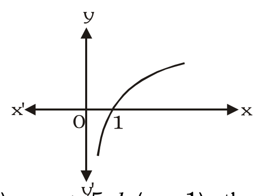
(C) $2\sec x(\sec x+\tan x)^2$ (D) $-2\sec x(\sec x+\tan x)^2$

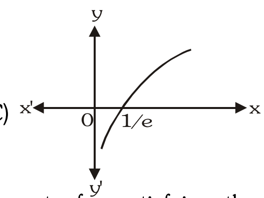
2. If $y=\dfrac{1+x^2+x^4}{1+x+x^2}$ and $\dfrac{dy}{dx}=ax+b$, then values of a & b are -

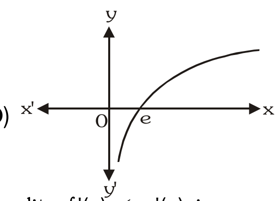
(A) a = 2 , b = 1 (B) a = –2 , b = 1 (C) a = 2 , b = –1 (D) a = –2 , b = –1

3. Which of the following could be the sketch graph of $y=\dfrac{d}{dx}\left(x\ell nx\right)$?

(A) 

(B) 

(C) 

(D) 

4. Let f(x) = x +3 $\ln(x-2)$ & g(x) = x + 5 $\ln(x-1)$, then the set of x satisfying the inequality $f'(x) < g'(x)$ is -

(A) $\left(2,\ \dfrac{7}{2}\right)$ (B) $(1,\ 2)\cup\left(\dfrac{7}{2},\ \infty\right)$ (C) $(2,\ \infty)$ (D) $\left(\dfrac{7}{2},\ \infty\right)$

5. Differential coefficient of $\left(x^{\frac{\ell+m}{m-n}}\right)^{\frac{1}{n-\ell}}.\left(x^{\frac{m+n}{n-\ell}}\right)^{\frac{1}{\ell-m}}.\left(x^{\frac{n+\ell}{\ell-m}}\right)^{\frac{1}{m-n}}$ w.r.t. x is -

(A) 1 (B) 0 (C) –1 (D) x^{lmn}

6. If $y=\dfrac{1}{1+x^{n-m}+x^{p-m}}+\dfrac{1}{1+x^{m-n}+x^{p-n}}+\dfrac{1}{1+x^{m-p}+x^{n-p}}$ then $\dfrac{dy}{dx}$ at $x=e^{m^{n^p}}$ is equal to -

(A) e^{mnp} (B) $e^{mn/p}$ (C) $e^{np/m}$ (D) none of these

7. If $\cos^{-1}\left(\dfrac{x^2-y^2}{x^2+y^2}\right)=\log a$ then $\dfrac{dy}{dx}=$

(A) $-\dfrac{x}{y}$ (B) $-\dfrac{y}{x}$ (C) $\dfrac{y}{x}$ (D) $\dfrac{x}{y}$

8. If $f(x)=\prod\limits_{n=1}^{100}(x-n)^{n(101-n)}$; then $\dfrac{f(101)}{f'(101)}=$

(A) 5050 (B) $\dfrac{1}{5050}$ (C) 10010 (D) $\dfrac{1}{10010}$

9. If $f(x)=(|\,x\,|)^{|\sin x|}$, then $f'\left(-\dfrac{\pi}{4}\right)$ is -

(A) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}\log\dfrac{4}{\pi}-\dfrac{2\sqrt{2}}{\pi}\right)$ (B) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}\log\dfrac{4}{\pi}+\dfrac{2\sqrt{2}}{\pi}\right)$

(C) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}\log\dfrac{\pi}{4}-\dfrac{2\sqrt{2}}{\pi}\right)$ (D) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}\log\dfrac{\pi}{4}+\dfrac{2\sqrt{2}}{\pi}\right)$

10. If $y = \frac{x}{a+}\frac{x}{b+}\frac{x}{a+}\frac{x}{b+}\frac{x}{a+}\frac{x}{b+}.........\infty$ then $\frac{dy}{dx}$ -

(A) $\frac{a}{ab+2ay}$ (B) $\frac{b}{ab+2by}$ (C) $\frac{a}{ab+2by}$ (D) $\frac{b}{ab+2ay}$

11. If $y = x^{x^2}$ then $\frac{dy}{dx} =$

(A) $2\,\ell n\, x.x^{x^2}$ (B) $(2\,\ell n\, x+1).x^{x^2}$ (C) $(\ell n\, x+1).x^{x^2+1}$ (D) $x^{x^2+1}.\ell n\,(ex^2)$

12. If $x^m \cdot y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is -

(A) xy (B) $\frac{x}{y}$ (C) $\frac{y}{x}$ (D) $\frac{x+y}{xy}$

13. If $x\sqrt{(1+y)} + y\sqrt{(1+x)} = 0$, then $\frac{dy}{dx}$ equals -

(A) $\frac{1}{(1+x)^2}$ (B) $-\frac{1}{(1+x)^2}$ (C) $-\frac{1}{(1+x)}+\frac{1}{(1+x)^2}$ (D) none of these

14. If $x^2 e^y + 2xye^x + 13 = 0$, then $\frac{dy}{dx}$ equals -

(A) $-\frac{2xe^{y-x}+2y(x+1)}{x(xe^{y-x}+2)}$ (B) $\frac{2xe^{x-y}+2y(x+1)}{x(xe^{y-x}+2)}$ (C) $-\frac{2xe^{x-y}+2y(x+1)}{x(xe^{y-x}+2)}$ (D) none of these

15. If $x = e^{y+e^{y+.........to\ \infty}}$, x > 0 then $\frac{dy}{dx}$ is -

(A) $\frac{x}{1+x}$ (B) $\frac{1+x}{x}$ (C) $\frac{1-x}{x}$ (D) $\frac{1}{x}$

16. If $x = \theta - \frac{1}{\theta}$ and $y = \theta + \frac{1}{\theta}$, then $\frac{dy}{dx} =$

(A) $\frac{x}{y}$ (B) $\frac{y}{x}$ (C) $\frac{-x}{y}$ (D) $\frac{-y}{x}$

17. The derivative of $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, (x > 0) is -

(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

18. Let g is the inverse function of f & $f'(x) = \frac{x^{10}}{(1+x^2)}$. If g (2) = a then g'(2) is equal to -

(A) $\frac{5}{2^{10}}$ (B) $\frac{1+a^2}{a^{10}}$ (C) $\frac{a^{10}}{1+a^2}$ (D) $\frac{1+a^{10}}{a^2}$

19. Let f(x) = sinx ; g(x) = x^2 & h(x)=$\log_e x$ & F(x) = h[g(f(x))] then $\frac{d^2F}{dx^2}$ is equal to -

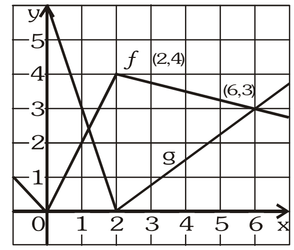
(A) $2\,\text{cosec}^3 x$ (B) $2\cot(x^2) - 4x^2\,\text{cosec}^2(x^2)$

(C) $2x\cot x^2$ (D) $-2\,\text{cosec}^2 x$

20. If $f(x) = \sqrt{x^2+1}$, $g(x) = \frac{x+1}{x^2+1}$ and h(x) = 2x – 3, then $f'(h'(g'(x))) =$

(A) 0 (B) $\frac{1}{\sqrt{x^2+1}}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{x}{\sqrt{x^2+1}}$

21. If f & g are the functions whose graphs are as shown, let $u(x) = f(g(x))$; $w(x) = g(g(x))$, then the value of $u'(1) + w'(1)$ is -



(A) $-\frac{1}{2}$ (B) $-\frac{3}{2}$

(C) $-\frac{5}{4}$ (D) does not exist

22. $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all real x and $f(5) = 2 = f'(5)$ then $f^2(10) + g^2(10)$ is -

(A) 2 (B) 4 (C) 8 (D) none of these

23. If $f(x) = x^n$, then the value of $f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots\ldots + \frac{(-1)^n f^{''''''''\ldots\ldots(n\ times)}(1)}{n!}$ -

(A) 2^{n-1} (B) 0 (C) 1 (D) 2^n

24. A function $y = f(x)$ has second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is -

(A) $(x + 1)^3$ (B) $(x + 1)^2$ (C) $(x - 1)^2$ (D) $(x - 1)^3$

25. If $f(x) = x + \frac{x^2}{1!} + \frac{x^3}{2!} + \ldots\ldots \frac{x^n}{(n-1)!}$, then $f(0) + f'(0) + f''(0) + \ldots\ldots + f^{''''\ldots\ldots\ n\ times}(0)$ is equal to -

(A) $\frac{n(n+1)}{2}$ (B) $\frac{(n^2+1)}{2}$ (C) $\left(\frac{n(n+1)}{2}\right)^2$ (D) $\frac{n(n+1)(2n+1)}{6}$

26. Let $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$. Then $\underset{x\to 0}{\text{Limit}} \frac{f'(x)}{x} =$

(A) 2 (B) –2 (C) –1 (D) 1

27. If f is differentiable in (0, 6) & $f'(4) = 5$ then $\underset{x\to 2}{\text{Lim}} \frac{f(4) - f(x^2)}{2 - x} =$

(A) 5 (B) 5/4 (C) 10 (D) 20

28. If $f(4) = g(4) = 2$; $f'(4) = 9$; $g'(4) = 6$ then $\underset{x\to 4}{\text{Limit}} \frac{\sqrt{f(x)} - \sqrt{g(x)}}{\sqrt{x} - 2}$ is equal to -

(A) $3\sqrt{2}$ (B) $\frac{3}{\sqrt{2}}$ (C) 0 (D) none of these

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

29. The slope(s) of common tangent(s) to the curves $y = e^{-x}$ & $y = e^{-x}\sin x$ can be -

(A) $-e^{-\pi/2}$ (B) $-e^{-\pi}$ (C) $\frac{\pi}{2}$ (D) 1

30. If $y + \ell n(1 + x) = 0$, which of the following is true ?

(A) $e^y = xy' + 1$ (B) $y' = -\frac{1}{(x+1)}$ (C) $y' + e^y = 0$ (D) $y' = e^y$

31. If $y = 2^{3^x}$, then y' equals -

(A) $3^x \ell n3\ \ell n2$ (B) $y(\log_2 y)\ \ell n3\ \ell n2$ (C) $2^{3^x} . 3^x\ \ell n6$ (D) $2^{3^x} . 3^x\ \ell n3\ \ell n2$

32. If $y = 3t^2$ & $x = 2t$ then $\frac{d^2y}{dx^2}$ equals-

(A) 3t (B) 3 (C) $\frac{3}{2}$ (D) None of these

33. If g is inverse of f and $f(x) = x^2 + 3x - 3$ $(x > 0)$ then g'(1) equals-

(A) $\frac{1}{2g(1)+3}$ (B) -1 (C) $\frac{1}{5}$ (D) $\frac{-f'(1)}{(f(1))^2}$

CHECK YOUR GRASP **ANSWER KEY** EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	B	C	C	D	B	D	C	B	A	D	D	C	B	A	C
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Ans.	A	C	B	D	C	B	C	B	D	A	B	D	A	A,B	A,B,C
Que.	31	32	33												
Ans.	B,D	C	A,C												

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. If y = fofof (x) and f (0) = 0, f '(0) = 2, then find y'(0) -

(A) 6 (B) 7 (C) 8 (D) 9

2. If $y^2 = p(x)$ is a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ is equal to -

(A) p'''(x) . p'(x) (B) p''(x) . p'''(x) (C) p(x) . p'''(x) (D) none of these

3. If y is a function of x then $\frac{d^2y}{dx^2}+y\frac{dy}{dx}=0$. If x is a function of y then the equation becomes -

(A) $\frac{d^2x}{dy^2}+x\frac{dx}{dy}=0$ (B) $\frac{d^2x}{dy^2}+y\left(\frac{dx}{dy}\right)^3=0$ (C) $\frac{d^2x}{dy^2}-y\left(\frac{dx}{dy}\right)^2=0$ (D) $\frac{d^2x}{dy^2}-x\left(\frac{dx}{dy}\right)^2=0$

4. If y = tanx tan 2x tan 3x then $\frac{dy}{dx}$ is equal to-

(A) 3 $\sec^2$ 3x tan x tan 2x + $\sec^2$x tan 2x tan 3x + 2 $\sec^2$ 2x tan 3x tan x

(B) 2y (cosec 2x + 2 cosec 4x + 3 cosec 6x)

(C) 3 $\sec^2$ 3x − 2 $\sec^2$ 2x − $\sec^2$ x

(D) $\sec^2$x + 2 $\sec^2$ 2x + 3 $\sec^2$ 3x

5. If $y=e^{\sqrt{x}}+e^{-\sqrt{x}}$ then $\frac{dy}{dx}$ equals -

(A) $\frac{e^{\sqrt{x}}-e^{-\sqrt{x}}}{2\sqrt{x}}$ (B) $\frac{e^{\sqrt{x}}-e^{-\sqrt{x}}}{2x}$ (C) $\frac{1}{2\sqrt{x}}\sqrt{y^2-4}$ (D) $\frac{1}{2\sqrt{x}}\sqrt{y^2+4}$

6. Let $y=\sqrt{x+\sqrt{x+\sqrt{x+......\infty}}}$ then $\frac{dy}{dx}$ -

(A) $\frac{1}{2y-1}$ (B) $\frac{x}{x-2y}$ (C) $\frac{1}{\sqrt{1+4x}}$ (D) $\frac{y}{2x+y}$

7. If $2^x + 2^y = 2^{x+y}$ then $\frac{dy}{dx}$ has the value equal to -

(A) $-\frac{2^y}{2^x}$ (B) $\frac{1}{1-2^x}$ (C) $1-2^y$ (D) $\frac{2^x(1-2^y)}{2^y(2^x-1)}$

8. The functions $u = e^x \sin x$; $v = e^x \cos x$ satisfy the equation -

(A) $v\frac{du}{dx}-u\frac{dv}{dx}=u^2+v^2$ (B) $\frac{d^2u}{dx^2}=2v$ (C) $\frac{d^2v}{dx^2}=-2u$ (D) none of these

9. Two functions f & g have first & second derivatives at x = 0 & satisfy the relations,

$f(0)=\frac{2}{g(0)}$, f'(0) = 2 g'(0) = 4g (0), g''(0) = 5 f''(0) = 6 f(0) = 3 then -

(A) if h (x) = $\frac{f(x)}{g(x)}$ then h'(0) = $\frac{15}{4}$ (B) if k(x) = f(x) . g(x) sinx then k'(0) = 2

(C) $\underset{x\to 0}{\text{Limit}}\frac{g'(x)}{f'(x)}=\frac{1}{2}$ (D) none of these

10. If $y^2 + b^2 = 2xy$, then $\frac{dy}{dx}$ equals -

(A) $\frac{1}{xy-b^2}$ (B) $\frac{y}{y-x}$ (C) $\frac{xy-b^2}{(y-x)^2}$ (D) $\frac{xy-b^2}{y}$

11. If $\sqrt{y+x}+\sqrt{y-x}=c$, then $\frac{dy}{dx}$ is equal to -

(A) $\frac{2x}{c^2}$ (B) $\frac{x}{y+\sqrt{y^2-x^2}}$ (C) $\frac{y-\sqrt{y^2-x^2}}{x}$ (D) $\frac{c^2}{2y}$

12. $\lim\limits_{x\to 0^+}\left(\left(x^{x^x}\right)-x^x\right)$ is -

(A) equal to 0 (B) equal to 1 (C) equal to -1 (D) non existent

13. Select the correct statements -

(A) The function f defined by $f(x) = \begin{cases} 2x^2+3 & \text{for } x \le 1 \\ 3x+2 & \text{for } x > 1 \end{cases}$ is neither differentiable nor continuous at $x = 1$.

(B) The function $f(x) = x^2|x|$ is twice differentiable at $x = 0$

(C) If f is continuous at $x = 5$ and $f(5) = 2$ then $\lim\limits_{x\to 2} f(4x^2-11)$ exists.

(D) If $\lim\limits_{x\to a}(f(x)+g(x)) = 2$ and $\lim\limits_{x\to a}(f(x) - g(x)) = 1$ then $\lim\limits_{x\to a} f(x).\ g(x)$ may not exist.

14. Let $\ell = \lim\limits_{x\to 0} x^m (\ell nx)^n$ where $m, n \in N$ then -

(A) ℓ is independent of m and n (B) ℓ is independent of m and depends on m

(C) ℓ is independent of n and depends on m (D) ℓ is dependent on both m and n

15. $\lim\limits_{x\to 0}\frac{\log_{\sin^2 x}\cos x}{\log_{\sin^2\frac{x}{2}}\cos\frac{x}{2}}$ has the value equal to -

(A) 1 (B) 2 (C) 4 (D) none of these

BRAIN TEASERS			ANSWER KEY					EXERCISE-2	
Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C	C	A,B,C	A,C	A,C,D	A,B,C,D	A,B,C	A,B,C	B,C
Que.	11	12	13	14	15					
Ans.	A,B,C	C	B,C	A	C					

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

TRUE / FALSE

1. Let u(x) and v(x) are differentiable functions such that $\frac{u}{v}(x) = 7$ If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$ then $\frac{p+q}{p-q} = 1$

2. If $f(x) = |x - 2|$, then $f'(f(x)) = 1$ for $x > 20$

3. If $f(0) = a$, $f'(0) = b$, $g(0) = 0$ and $(fog)'(0) = c$, then $g'(0) = \frac{c}{b}$

4. The differential coefficient of f(logx) w.r.t. logx where f(x) = logx is $\frac{1}{\log x}$

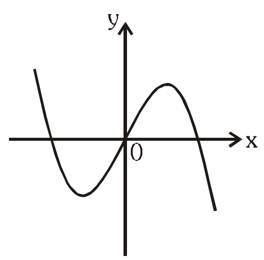
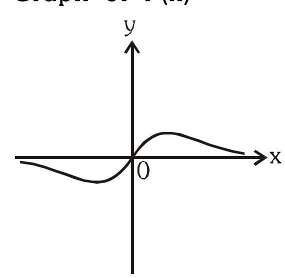
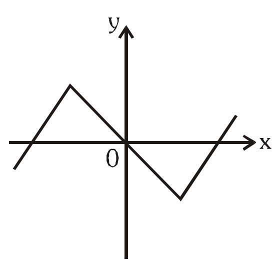
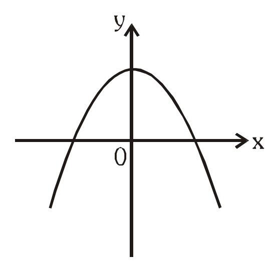
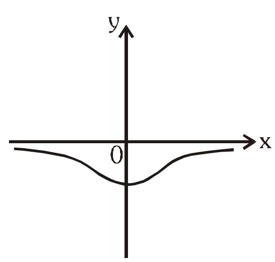
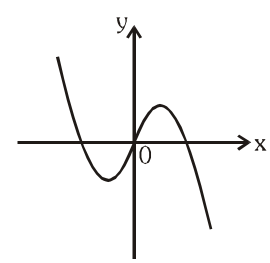
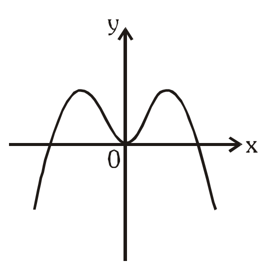
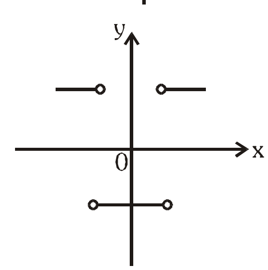
5. $f'(\sin x) = (f(\sin x))'$

6. If $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$, then $\frac{dy}{dx}$ at (2, –1) is $\frac{6}{7}$

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1.

	Column-I		Column-II
	Graph of f(x)		Graph of f'(x)
(A)		(p)	
(B)		(q)	
(C)		(r)	
(D)		(s)	

2.

	Column-I		Column-II
(A)	If $f(x) = x^3 + x + 1$, then $f'(x^2 + 1)$ at $x = 0$ is	(p)	1
(B)	If $f(x) = \log_{x^2}(\log x)$, then $f'(e^e)$ is equal to	(q)	0
(C)	For the function $y = \ell n \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ if $\frac{dy}{dx}$ = secx + p, then p is equal to	(r)	28
(D)	If $f(x) = \lvert x^3 - x^2 + x - 1\rvert \sin x$, then $4f'(28f(f(\pi)))$ is equal to	(s)	4

ASSERTION & REASON

These questions contain Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : Let f(x) is a continuous function defined from R to Q and f(5) = 3 then differential coefficient of f(x) w.r.t. x will be 0.
Because
Statement-II : Differentiation of constant function is always zero.
(A) A (B) B (C) C (D) D

2. **Statement-I** : Derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is 1 for $0 < x < 1$.
Because
Statement-II : $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $-1 \le x \le 1$.
(A) A (B) B (C) C (D) D

3. Consider $f(x) = \frac{x}{x^2-1}$ & $g(x) = f''(x)$.
Statement-I : Graph of g(x) is concave up for $x > 1$.
Because
Statement-II : $\frac{d^n}{dx^n}(f(x)) = \frac{(-1)^n n!}{2}\left\{\frac{1}{(x+1)^{n+1}}+\frac{1}{(x-1)^{n+1}}\right\}$, $n \in N$
(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

Let $\frac{f(x+y)-f(x)}{2} = \frac{f(y)-1}{2} + xy$,, $xy \in R$. f(x) is differentiable and $f'(0) = 1$. Let g(x) be a derivable function at $x = 0$ and follows the functional rule $g\left(\frac{x+y}{k}\right) = \frac{g(x)+g(y)}{k}$ $(k \in R,\ k \neq 0,\ 2)$

Let $g'(0) = \lambda \neq 0$

On the basis of above information, answer the following questions :

1. Domain of $\ell n(f(x))$ is-

(A) R^+ (B) $R - \{0\}$ (C) R (D) R^-

2. Range of $y = \log_{3/4}(f(x))$

(A) $(-\infty, 1]$ (B) $\left[\frac{3}{4}, \infty\right)$ (C) $(-\infty, \infty)$ (D) R

3. If the graphs of $y = f(x)$ and $y = g(x)$ intersect in coincident points the λ can take values-

(A) 3 (B) 1 (C) -1 (D) 4

Comprehension # 2

Limits that lead to the indeterminate forms 1^∞, 0^0, ∞^0 can sometimes be solved taking logarithm first and then using L' Hõpital 's rule

Let $\operatorname*{Lim}_{x\to a}(f(x))^{g(x)}$ is in the form of ∞^0, it can be written as $e^{\lim\limits_{x\to a} g(x)\ell nf(x)} = e^L$

where $L = \lim\limits_{x\to a}\dfrac{\ell nf(x)}{1/g(x)}$ is $\dfrac{\infty}{\infty}$ form and can be solved using L' Hõpital 's rule.

On the basis of above information, answer the following questions :

1. $\operatorname*{Lim}_{x\to 1^+} x^{1/(1-x)}$ -

(A) -1 (B) e^{-1} (C) -2 (D) e^{-2}

2. $\operatorname*{Lim}_{x\to\infty}\left[(\ell nx)^{1/2x} + x^{1/x^n}\right] \forall\, n \in N$ -

(A) 2 (B) 0 (C) $e^{1/2}$ (D) e

3. $\operatorname*{Lim}_{x\to 0^+}(\sin x)^{2\sin x}$

(A) 1 (B) 0 (C) 2 (D) does not exist

Comprehension # 3

Left hand derivative and right hand derivative of a function f(x) at a point x = a are defined as

$$f'(a^-) = \lim_{h\to 0^+}\frac{f(a)-f(a-h)}{h} = \lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h} \text{ and}$$

$$f'(a^+) = \lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-}\frac{f(a)-f(a-h)}{h} = \lim_{x\to a^+}\frac{f(a)-f(x)}{a-x} \text{ respectively}$$

Let f be a twice differentiable function. We also know that derivative of an even function is odd function and derivative of an odd function is even function.

On the basis of above information, answer the following questions :

1. If f is odd, which of the following is Left hand derivative of f at x = $-a$

(A) $\lim\limits_{h\to 0^-}\dfrac{f(a-h)-f(a)}{-h}$ (B) $\lim\limits_{h\to 0^-}\dfrac{f(h-a)-f(a)}{h}$ (C) $\lim\limits_{h\to 0^+}\dfrac{f(a)+f(a-h)}{-h}$ (D) $\lim\limits_{h\to 0^-}\dfrac{f(-a)-f(-a-h)}{-h}$

2. If f is even, which of the following is Right hand derivative of f' at x = a

(A) $\lim\limits_{h\to 0^-}\dfrac{f'(a)+f'(-a+h)}{h}$ (B) $\lim\limits_{h\to 0^+}\dfrac{f'(a)+f'(-a-h)}{h}$

(C) $\lim\limits_{h\to 0^-}\dfrac{-f'(-a)+f'(-a-h)}{-h}$ (D) $\lim\limits_{h\to 0^+}\dfrac{f'(a)+f'(-a+h)}{-h}$

3. The statement $\lim\limits_{h\to 0}\dfrac{f(-x)-f(-x-h)}{h} = \lim\limits_{h\to 0}\dfrac{f(x)-f(x-h)}{-h}$ implies that for all $x \in R$

(A) f is odd (B) f is even

(C) f is neither odd nor even (D) nothing can be concluded

Comprehension # 4

An operator Δ is defined to operate on differentiable functions defined as follows.

If $f(x)$ is a differentiable function then $\Delta(f(x)) = \lim\limits_{h\to 0}\dfrac{f^3(x+h)-f^3(x)}{h}$

g(x) is a differentiable function such that the slope of the tangent to the curve y = g(x) at any point (a, g(a)) is equal to $2e^a(a+1)$ also $g(0)=0$.

On the basis of above information, answer the following questions :

1. $\Delta(g(x))$ at $x=\ell n2$ is –

(A) $24\ell n2\,\{2\ell n2+2\}$ (B) $\ell n(4e^2)\ell n^2 2$ (C) $96\ell n(4e^2)\ell n^2 2$ (D) $192\ell n(4e)\,\ell n^2 2$

2. $\Delta(\Delta(x+2))_{x=0}$

(A) $2^5\,3^9$ (B) $2^9\,3^5$ (C) $2^4\,3^5$ (D) $2^6\,3^4$

3. $\lim\limits_{x\to 0}\dfrac{\Delta g(x)}{\ell n(\cos 2x)}$

(A) –12 (B) 12 (C) 24 (D) –24

MISCELLANEOUS TYPE QUESTION	ANSWER KEY	EXERCISE-3

- ***True / False***

1. T 2. T 3. T 4. T 5. F 6. T

- ***Match the Column***

1. (A) $\to$ (q); (B) $\to$ (s); (C) $\to$ (p); (D) $\to$ (r) 2. (A) $\to$ (s); (B) $\to$ (q); (C) $\to$ (q); (D) $\to$ (s)

- ***Assertion & Reason***

1. A 2. C 3. A

- ***Comprehension Based Questions***

Comprehension # 1 : 1. C 2. A 3. A,C

Comprehension # 2 : 1. B 2. A 3. A

Comprehension # 3 : 1. A 2. A 3. B

Comprehension # 4 : 1. C 2. D 3. A

EXERCISE - 04 [A]

CONCEPTUAL SUBJECTIVE EXERCISE

1. If $y = \frac{a + bx^{3/2}}{x^{5/4}}$ and $\frac{dy}{dx}$ vanishes at x = 5 then find $\frac{a}{b}$.

2. If $y = \frac{x^4 + 4}{x^2 - 2x + 2}$ then find $\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}}$

3. If f'(x) = $\sqrt{2x^2 - 1}$ and y = f(x^2) then find $\frac{dy}{dx}$ at x = 1.

4. If x = $\frac{1+\ell nt}{t^2}$ and y = $\frac{3+2\ell nt}{t}$. Show that $y\frac{dy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 1$.

5. If $f_n(x) = e^{f_{n-1}(x)}$ for all n ∈ N and $f_0(x) = x$ then show that $\frac{d}{dx}\{f_n(x)\} = f_1(x).f_2(x)\ldots\ldots\ldots f_n(x)$.

6. If $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ell n\sqrt{x + \sqrt{x^2+1}}$ prove that 2y = xy' + ℓny', where y' denotes the derivative of y w.r.t. x.

7. Let $f(x) = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + \ldots\ldots\infty}}}$

Compute the value of f(100).f'(100).

8. If $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ & $x = \sec^{-1}\frac{1}{2u^2-1}$, $u \in \left(0, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$; prove that $2\frac{dy}{dx} + 1 = 0$

9. If y = $\tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$, then find $\frac{dy}{dx}$ for x ∈ (–1, 1).

10. If x = 2 cost – cos2t & y = 2 sint – sin2t, find the value of (d^2y / dx^2) when $t = (\pi/2)$.

11. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{(x-c)} + 1$, Prove that $\frac{y'}{y} = \frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$ [JEE 98]

12. If $\sqrt{x^2+y^2} = e^{\arcsin\frac{y}{\sqrt{x^2+y^2}}}$. Prove that $\frac{d^2y}{dx^2} = \frac{2(x^2+y^2)}{(x-y)^3}$, x > 0.

13. Let $f(x) = x^2 - 4x - 3,\ x > 2$ and let g be the inverse of f. Find the value of g' at f(x) = 9

14. If y = xℓn[$(ax)^{-1} + a^{-1}$], prove that $x(x+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y - 1$

15. If $x = \sec\theta - \cos\theta$; $y = \sec^n\theta - \cos^n\theta$, then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2+4) = 0$.

16. (a) Differentiate $y = \cos^{-1}\frac{1-x^2}{1+x^2}$ w. r. t. $\tan^{-1}x$, stating clearly where function is not differentiable.

(b) If $y = \sin^{-1}(3x - 4x^3)$ find dy/dx stating clearly where the function is not derivable in (–1,1).

17. Suppose f and g are two functions such that f, g : R $\rightarrow$ R,

$f(x) = \ell n\left(1+\sqrt{1+x^2}\right)$ and $g(x) = \ell n\left(x+\sqrt{1+x^2}\right)$

then find the value of $x.e^{g(x)}\left(f\left(\frac{1}{x}\right)\right)' + g'(x)$ at x = 1.

18. Determine the values of a, b and c so that $\lim\limits_{x\to 0}\frac{(a+b\cos x)x - c\sin x}{x^5} = 1$

Solve using L'Hõpital's rule or series expansion. (Q.18 – Q.21)

19. $\lim\limits_{x\to 0}\frac{x\cos x - \ell n(1+x)}{x^2}$

20. $\lim\limits_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{\sin^2 x}\right]$

21. If $\lim\limits_{x\to a}\frac{a^x - x^a}{x^x - a^a} = -1$ find 'a'.

22. $\lim\limits_{x\to 0}\log_{\tan^2 x}(\tan^2 2x)$

23. If $f(x) = \begin{vmatrix} (x-a)^4 & (x-a)^3 & 1 \\ (x-b)^4 & (x-b)^3 & 1 \\ (x-c)^4 & (x-c)^3 & 1 \end{vmatrix}$ then $f'(x) = \lambda.\begin{vmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{vmatrix}$ Find the value of λ .

CONCEPTUAL SUBJECTIVE EXERCISE — **ANSWER KEY** — EXERCISE-4(A)

1. $\sqrt{5}$ **2.** 3 **3.** 2 **7.** 100 **9.** $\frac{1-2x}{2\sqrt{1-x^2}}$ **10.** $-\frac{3}{2}$ **13.** $\frac{1}{8}$

16. (a) Not differentiable at x = 0 (b) Not derivable at $x = \pm 1/2$ **17.** zero

18. a = 120; b = 60; c = 180 **19.** $\frac{1}{2}$ **20.** $-\frac{1}{3}$ **21.** a = 1 **22.** 1 **23.** 3

EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. If $x = \frac{1}{z}$ and $y = f(x)$, show that : $\frac{d^2f}{dx^2} = 2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$

2. Prove that if $|a_1 \sin x + a_2 \sin 2x + ... + a_n \sin nx| \le |\sin x|$ for $x \in R$, then $|a_1 + 2a_2 + 3a_3 + ... + na_n| \le 1$

3. Show that the substitution $z = \ell n\left(\tan\frac{x}{2}\right)$ changes the equation $\frac{d^2y}{dx^2} + \cot x\frac{dy}{dx} + 4y\,\text{cosec}^2\,x = 0$ to $(d^2y/dz^2) + 4y = 0$

4. Find a polynomial function f(x) such that f(2x) = f'(x) f''(x).

5. If Y = sX and Z = tX, where all the letters denotes the function of x and suffixes denotes the differentiation w.r.t. x then prove that $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3\begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$

6. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3.(x^3 - y^3)$, prove that $\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$.

7. If α be a repeated root of a quadratic equation f(x) = 0 & A(x), B(x), C(x) be the polynomials of degree 3, 4 & 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by f(x), where dash denotes the derivative.

8. If $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13} + ...$ upto n terms. Find dy/dx, expressing your answer in 2 terms.

9. Let g(x) be a polynomial, of degree one & f(x) be defined by $f(x) = \begin{cases} g(x) & , \quad x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x} & , \quad x > 0 \end{cases}$

Find the continuous function f(x) satisfying f'(1) = f(–1)

10. Let $\frac{f(x+y) - f(x)}{2} = \frac{f(y) - a}{2} + xy$ for all real x and y. If f(x) is differentiable and f'(0) exists for all real permissible values of 'a' and is equal to $\sqrt{5a - 1 - a^2}$. Prove that f(x) is positive for all real x.

11. Find the value f(0) so that the function $f(x) = \frac{1}{x} - \frac{2}{e^{2x} - 1}, x \ne 0$ is continuous at x = 0 & examine the differentiability of f(x) at x = 0.

12. If $\lim_{x\to 0}\frac{a\sin x - bx + cx^2 + x^3}{2x^2.\ell n(1+x) - 2x^3 + x^4}$ exists & is finite, find the value of a, b, c & the limit.

BRAIN STORMING SUBJECTIVE EXERCISE **ANSWER KEY** EXERCISE-4(B)

4. $\frac{4x^3}{9}$ **8.** $\frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$ **9.** $f(x) = \begin{cases} -\frac{2}{3}\left[\frac{1}{6} + \ell n\frac{3}{2}\right]x & \text{if} \quad x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x} & \text{if} \quad x > 0 \end{cases}$

11. f(0) = 1, differentiable at x = 0, $f'(0^+) = -(1/3)$; $f'(0^-) = -(1/3)$ **12.** a = 6, b = 6, c = 0 ; 3/40

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. If f(1) = 1, f'(1) = 2, then $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1} =$ [AIEEE - 2002]

(1) 2 (2) 1 (3) 3 (4) 4

2. $\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]}$, $n \in N$, (where [x] denotes greatest integer less than or equal to x)- [AIEEE - 2002]

(1) Has value -1 (2) Has values 0 (3) Has value 1 (4) Does not exist

3. If $y = \log_y x$, then $\frac{dy}{dx} =$ [AIEEE-2002]

(1) $\frac{1}{x+\log y}$ (2) $\frac{1}{\log x(1+y)}$ (3) $\frac{1}{x(1+\log y)}$ (4) $\frac{1}{y+\log x}$

4. If $x = 3\cos\theta - 2\cos^3\theta$ and $y = 3\sin\theta - 2\sin^3\theta$, then $\frac{dy}{dx} =$ [AIEEE-2002]

(1) $\sin\theta$ (2) $\cos\theta$ (3) $\tan\theta$ (4) $\cot\theta$

5. If $y = \left(x+\sqrt{1+x^2}\right)^n$ then $(1 + x^2)y_2 + xy_1 =$ [AIEEE-2002]

(1) ny^2 (2) n^2y (3) n^2y^2 (4) None of these

6. If $f(x) = x^n$, then the values of $f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n f^n(1)}{n!}$ is- [AIEEE-2003]

(1) 1 (2) 2^n (3) 2^{n-1} (4) 0

7. Let f(x) be a polynomial function of second degree. If f(1) = f(–1) and a, b, c are in A.P. then f'(a), f'(b) and f'(c) are in- [AIEEE-2003]

(1) Arithmetic-Geometric Progression (2) Arithmetic progression (A.P.)

(3) Geometric progression (G.P.) (4) Harmonic progression (H.P.)

8. If $x = e^{y+e^{y+\ldots..to\infty}}$, $x > 0$, then $\frac{dy}{dx}$ is - [AIEEE-2004]

(1) $\frac{x}{1+x}$ (2) $\frac{1}{x}$ (3) $\frac{1-x}{x}$ (4) $\frac{1+x}{x}$

9. If $x^m.y^n = (x + y)^{m+n}$, then $\frac{dy}{dx}$ is- [AIEEE-2006]

(1) $\frac{x+y}{xy}$ (2) xy (3) $\frac{x}{y}$ (4) $\frac{y}{x}$

10. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then y'(1) equals :- [AIEEE-2009]

(1) log 2 (2) –log2 (3) –1 (4) 1

11. Let f : (–1, 1) → R be a differentiable function with f(0) = – 1 and f'(0) = 1. Let $g(x) = [f(2f(x) + 2)]^2$. Then g'(0) :- [AIEEE-2010]

(1) 4 (2) –4 (3) 0 (4) –2

12. $\dfrac{d^2x}{dy^2}$ equals :- **[AIEEE-2011]**

(1) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (2) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$ (3) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (4) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$

13. If $y = \sec(\tan^{-1}x)$, then $\dfrac{dy}{dx}$ at $x = 1$ is equal to : **[JEE-(Main)-2013]**

(1) $\dfrac{1}{\sqrt{2}}$ (2) $\dfrac{1}{2}$ (3) 1 (4) $\sqrt{2}$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	1	1	3	4	2	4	2	3	4	3
Que.	11	12	13							
Ans.	2	2	1							

EXERCISE - 05 [B]

JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. (a) If $\ln(x + y) = 2xy$, then $y'(0) =$ [JEE 2004 (Scr.)]

(A) 1 (B) –1 (C) 2 (D) 0

(b) $$f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right), & -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2}, & \text{at } x = 0 \\ \dfrac{e^{ax/2}-1}{x}, & 0 < x < \dfrac{1}{2} \end{cases}$$

If f(x) is differentiable at x = 0 and $|c| < 1/2$ then find the value of 'a' and prove that $64b^2 = 4 - c^2$.

[JEE 2004, 4]

2. (a) If $y = y(x)$ and it follows the relation $x\cos y + y\cos x = \pi$, then $y''(0)$:-

(A) 1 (B) –1 (C) π (D) $-\pi$

(b) If P(x) is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that P(1) =1, P(0) = 0 and $P'(x) > 0\ \forall\ x \in [0, 1]$, then :-

(A) $S = \phi$ (B) $S = (1 - a)x^2 + ax,\ 0 < a < 2$

(C) $(1 - a)x^2 + ax,\ a \in (0, \infty)$ (D) $S = (1 - a)x^2 + ax,\ 0 < a < 1$

(c) If f(x) is a continuous and differentiable function and $f(1/n) = 0,\ \forall\ n \ge 1$ and $n \in I$, then :-

(A) $f(x) = 0,\ x \in (0, 1]$ (B) $f(0) = 0,\ f'(0) = 0$

(C) $f'(x) = 0 = f''(x),\ x \in (0, 1]$ (D) $f(0) = 0$ and $f'(0)$ need not to be zero

[JEE 2005 (Scr.)]

(d) If $f(x - y) = f(x)\ \ g(y) - f(y)\ \ g(x)$ and $g(x - y) = g(x)\ \ g(y) + f(x)\ \ f(y)$ for all $x, y \in R$. If right hand derivative at x = 0 exists for f(x). Find derivative of g(x) at x = 0. [JEE 2004 (Scr.)]

3. For $x > 0$, $\lim\limits_{x\to 0}\left((\sin x)^{1/x} + (1/x)^{\sin x}\right)$ is :- [JEE 2006, 3]

(A) 0 (B) –1 (C) 1 (D) 2

4. $\dfrac{d^2x}{dy^2}$ equals :- [JEE 2007, 3]

(A) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$ (C) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (D) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

5. (a) Let $g(x) = \ell n\ f(x)$ where f(x) is a twice differentiable positive function on $(0, \infty)$ such that $f(x + 1) = x\ f(x)$. Then for N = 1, 2, 3

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(A) $-4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots\ldots\dfrac{1}{(2N-1)^2}\right\}$ (B) $4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots\ldots\dfrac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots\ldots\dfrac{1}{(2N+1)^2}\right\}$ (D) $4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots\ldots\dfrac{1}{(2N+1)^2}\right\}$

(b) Let f and g be real valued functions defined on interval (–1, 1) such that g"(x) is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x) \sin x$.

Statement-1 : $\lim\limits_{x\to 0} [g(x) \cot x - g(0)\operatorname{cosec} x] = f''(0)$

and

Statement-2 : $f'(0) = g(0)$

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation of statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true. **[JEE 2008, 3+3]**

6. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is **[JEE 2009, 4]**

7. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

[JEE 2011, 4]

PREVIOUS YEARS QUESTIONS	ANSWER KEY	EXERCISE-5 [B]	
1. (a) A ; (b) a = 1	2. (a) C ; (b) B ; (c) B, (d) g' (0) = 0	3. C	4. D
5. (a) A ; (b) A	6. 2	7. 1	