CHAPTER

04

Logarithms and Their Properties

Learning Part

Session 1

- Definition
- Characteristic and Mantissa

Session 2

- Principle Properties of Logarithm

Session 3

- Properties of Monotonocity of Logarithm
- Graphs of Logarithmic Functions

Practice Part

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The technique of logarithms was introduced by **John Napier** (1550-1617). The logarithm is a form of indices which is used to simplify the algebraic calculations. The operations of multiplication, division of a very large number becomes quite easy and get converted into simple operations of addition and subtraction, respectively. The results obtained are correct upto some decimal places.

Session 1

Definition, Characteristic and Mantissa

Definition

The logarithm of any positive number, whose base is a number (>0) different from 1, is the index or the power to which the base must be raised in order to obtain the given number.

i.e. if $a^x = b$ (where $a > 0, \neq 1$), then x is called the logarithm of b to the base a and we write $\log_a b = x$, clearly $b > 0$. Thus, $\log_a b = x \Leftrightarrow a^x = b, a > 0, a \neq 1$ and $b > 0$.

If $a = 10$, then we write $\log b$ rather than $\log_{10} b$. If $a = e$, we write $\ln b$ rather than $\log_e b$. Here, 'e' is called as **Napier's base** and has numerical value equal to 2.7182. Also, $\log_{10} e$ is known as **Napierian constant**.

i.e. $$\log_{10} e = 0.4343$$

$\therefore$ $$\ln b = 2.303 \log_{10} b$$

$$\left[\text{since, } \ln b = \log_{10} b \times \log_e 10 = \frac{1}{\log_{10} e} \times \log_{10} b = \frac{1}{0.4343} \log_{10} b = 2.303 \log_{10} b\right]$$

Remember

(i) $\log 2 = \log_{10} 2 = 0.3010$

(ii) $\log 3 = \log_{10} 3 = 0.4771$

(iii) $\ln 2 = 2.303 \log 2 = 0.693$

(iv) $\ln 10 = 2.303$

***Corollary* I** From the definition of the logarithm of the number b to the base a, we have an identity

$$a^{\log_a b} = b, a > 0, a \neq 1 \text{ and } b > 0$$

which is known as the **Fundamental Logarithmic Identity**.

***Corollary* II** The function defined by $f(x) = \log_a x, a > 0, a \neq 1$ is called logarithmic function. Its domain is $(0, \infty)$ and range is R (set of all real numbers).

***Corollary* III** $a^x > 0, \forall x \in R$

(i) If $a > 1$, then a^x is monotonically increasing.

For example, $5^{2.7} > 5^{2.5}, 3^{222} > 3^{111}$

(ii) If $0 < a < 1$, then a^x is monotonically decreasing.

For example, $\left(\frac{1}{5}\right)^{2.7} < \left(\frac{1}{5}\right)^{2.5}, (0.7)^{222} < (0.7)^{212}$

***Corollary* IV**

(i) If $a > 1$, then $a^{-\infty} = 0$

$\therefore$ $\log_a 0 = -\infty$ (if $a > 1$)

(ii) If $0 < a < 1$, then $a^{\infty} = 0$

$\therefore$ $\log_a 0 = +\infty$ (if $0 < a < 1$)

***Corollary* V** (i) $\log_a b \to \infty$, if $a > 1, b \to \infty$

(ii) $\log_a b \to -\infty$, if $0 < a < 1, b \to \infty$

Remark

1. 'log' is the abbreviation of the word 'logarithm'.
2. **Common logarithm** (Brigg's logarithms) The base is 10.
3. If $x < 0, a > 0$ and $a \neq 1$, then $\log_a x$ is an imaginary.
4. If $a > 1$, $\log_a x = \begin{cases} +\text{ve}, & x > 1 \\ 0, & x = 1 \\ -\text{ve}, & 0 < x < 1 \end{cases}$

   And if $0 < a < 1$, $\log_a x = \begin{cases} +\text{ve}, & 0 < x < 1 \\ 0, & x = 1 \\ -\text{ve}, & x > 1 \end{cases}$
5. $\log_a 1 = 0$ ($a > 0, a \neq 1$)

   $\log_a a = 1$ ($a > 0, a \neq 1$) and $\log_{(1/a)} a = -1$ ($a > 0, a \neq 1$)

Example 1. Find the value of the following :

(i) $\log_9 27$ (ii) $\log_{3\sqrt{2}} 324$

(iii) $\log_{1/9}(27\sqrt{3})$ (iv) $\log_{(5+2\sqrt{6})}(5-2\sqrt{6})$

(v) $\log_{0.2} 0.008$ (vi) $2^{2\log_4 5}$

(vii) $(0.4)^{-\log_{2.5}\left\{\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\right\}}$ (viii) $(0.05)^{\log_{\sqrt{20}}(0.\bar{3})}$

Sol. (i) Let $x = \log_9 27$

$\Rightarrow \quad 9^x = 27 \Rightarrow 3^{2x} = 3^3 \Rightarrow 2x = 3$

$\therefore \quad x = \dfrac{3}{2}$

(ii) Let $x = \log_{3\sqrt{2}} 324$

$\Rightarrow (3\sqrt{2})^x = 324 = 2^2 \cdot 3^4 \Rightarrow (3\sqrt{2})^x = (3\sqrt{2})^4$

$\therefore \quad x = 4$

(iii) Let $x = \log_{1/9}(27\sqrt{3})$

$\Rightarrow \left(\dfrac{1}{9}\right)^x = 27\sqrt{3} \Rightarrow 3^{-2x} = 3^{7/2} \Rightarrow -2x = 7/2$

$\therefore \quad x = -\dfrac{7}{4}$

(iv) $\because (5+2\sqrt{6})(5-2\sqrt{6}) = 1$

or $5 + 2\sqrt{6} = \dfrac{1}{5-2\sqrt{6}}$ ...(i)

Now, let $x = \log_{(5+2\sqrt{6})}(5-2\sqrt{6})$

$= \log_{1/(5-2\sqrt{6})} 5 - 2\sqrt{6} = -1$ [from Eq. (i)]

(v) Let $x = \log_{0.2} 0.008$

$\Rightarrow (0.2)^x = 0.008 \Rightarrow (0.2)^x = (0.2)^3 \Rightarrow x = 3$

(vi) Let $x = 2^{2\log_4 5} = 4^{\log_4 5} = 5$

(vii) Let $x = (0.4)^{-\log_{2.5}\left\{\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\right\}}$

$= \left(\dfrac{4}{10}\right)^{-\log_{2.5}\left\{\frac{\frac{1}{3}}{1-\frac{1}{3}}\right\}} = \left(\dfrac{2}{5}\right)^{-\log_{2.5}\left(\frac{1}{2}\right)} = \left(\dfrac{5}{2}\right)^{\log_{5/2}\left(\frac{1}{2}\right)} = \dfrac{1}{2}$

(viii) Let $x = (0.05)^{\log_{\sqrt{20}}(0.\bar{3})} = (0.05)^{\log_{\sqrt{20}}(\lambda)}$ ...(i)

where, $\lambda = 0.\bar{3}$ ...(ii)

Then, $\lambda = 0.33333\ldots$ ...(iii)

$\Rightarrow \quad 10\lambda = 3.33333\ldots$

On subtracting Eq. (ii) from Eq. (iii), we get

$9\lambda = 3 \Rightarrow \lambda = \dfrac{1}{3}$

Now, from Eq. (i), $x = (0.05)^{\log_{\sqrt{20}}\left(\frac{1}{3}\right)}$

$= \left(\dfrac{1}{20}\right)^{\log_{(20)^{1/2}}(3)^{-1}} = \left(\dfrac{1}{20}\right)^{-\frac{1}{1/2}\log_{20} 3}$

$= 20^{(2\log_{20} 3)} = 20^{\log_{20} 3^2} = 3^2 = 9$

Example 2. Find the value of the following:

(i) $\log_{\tan 45^\circ}\cot 30^\circ$ (ii) $\log_{(\sec^2 60^\circ - \tan^2 60^\circ)}\cos 60^\circ$

(iii) $\log_{(\sin^2 30^\circ + \cos^2 30^\circ)} 1$ (iv) $\log_{30} 1$

Sol. (i) Here, base $= \tan 45^\circ = 1$ tan

$\therefore$ log is not defined.

(ii) Here, base $= \sec^2 60^\circ - \tan^2 60^\circ = 1$

$\therefore$ log is not defined.

(iii) $\because \log_{(\sin^2 30^\circ + \cos^2 30^\circ)} 1 = \log_1 1 \neq 1$

$\because$ Here, base = 1

$\therefore$ log is not defined.

(iv) $\log_{30} 1 = 0$

Characteristic and Mantissa

The integral part of a logarithm is called the **characteristic** and the fractional part (decimal part) is called **mantissa**.

i.e., $\log N = \text{Integer} + \text{Fractional or decimal part } (+\text{ve})$

$\qquad\qquad\downarrow \qquad\qquad\qquad \downarrow$

$\qquad$ Characteristic $\qquad$ Mantissa

The mantissa of log of a number is always kept positive.

i.e., if $\log 564 = 2.751279$, then 2 is the characteristic and 0.751279 is the mantissa of the given number 564.

And if $\log 0.00895 = -2.0481769$

$= -2 - 0.0481769$

$= (-2-1) + (1 - 0.0481769)$

$= -3 + 0.9518231$

Hence, -3 is the characteristic and 0.9518231 (not 0.0481769) is mantissa of log 0.00895.

In short, $-3 + 0.9518231$ is written as $\bar{3}.9518231$.

Remark

1. If $N > 1$, the characteristic of $\log N$ will be one less than the number of digits in the integral part of N.
   For example, If $\log 235.68 = 2.3723227$
   Here, $N = 235.68$
   $\therefore$ Number of digits in the integral part of $N = 3$
   $\Rightarrow$ Characteristic of $\log 235.68 = N - 1 = 3 - 1 = 2$
2. If $0 < N < 1$, the characteristic of $\log N$ is negative and numerically it is one greater than the number of zeroes immediately after the decimal part in N.
   For example, If $\log 0.0000279 = \bar{5}.4456042$
   Here, four zeroes immediately after the decimal point in the number 0.0000279 is $(4+1)$, i.e. $\bar{5}$.
3. If the characteristics of $\log N$ be n, then number of digits in N is $(n+1)$ (Here, $N > 1$).
4. If the characteristics of $\log N$ be $-n$, then there exists $(n-1)$ number of zeroes after decimal part of N (here, $0 < N < 1$).

Example 3. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the number of digits in 6^{20}.

Sol. Let $P = 6^{20} = (2 \times 3)^{20}$

$$\therefore \quad \log P = 20 \log(2 \times 3) = 20\{\log 2 + \log 3\}$$
$$= 20\{0.301 + 0.477\}$$
$$= 20 \times 0.778 = 15.560$$

Since, the characteristic of $\log P$ is 15, therefore the number of digits in P will be 15 + 1, i.e. 16.

Example 4. Find the number of zeroes between the decimal point and first significant digit of $(0.036)^{16}$, where $\log 2 = 0.301$ and $\log 3 = 0.477$.

Sol. Let $P = (0.036)^{16} \Rightarrow \log P = 16 \log(0.036)$

$$= 16 \log\left(\frac{36}{1000}\right) = 16 \log\left(\frac{2^2 \cdot 3^2}{1000}\right)$$
$$= 16\{\log 2^2 + \log 3^2 - \log 10^3\}$$
$$= 16\{2\log 2 + 2\log 3 - 3\}$$
$$= 16\{2 \times 0.301 + 2 \times 0.477 - 3\}$$
$$= 16\{1.556 - 3\} = 24.896 - 48$$
$$= -48 + 24 + 0.896$$
$$= -24 + 0.896 = \overline{24} + 0.896$$

$\therefore$ The required number of zeroes $= 24 - 1 = 23$.

a, b, b, b, b

Exercise for Session 1

1. The value of $\log_{2\sqrt{3}} 1728$ is
   (a) 6
   (b) 8
   (c) 3
   (d) 5

2. The value of $\log_{(8-3\sqrt{7})}(8 + 3\sqrt{7})$ is
   (a) −2
   (b) −1
   (c) 0
   (d) Not defined

3. The value of $(0.16)^{\log_{2.5}\left\{\frac{1}{3} + \frac{1}{3^2} + \dots\right\}}$ is
   (a) 2
   (b) 4
   (c) 6
   (d) 8

4. If $\log 2 = 0.301$, the number of integers in the expansion of 4^{17} is
   (a) 9
   (b) 11
   (c) 13
   (d) 15

5. If $\log 2 = 0.301$, then the number of zeroes between the decimal point and the first significant figure of 2^{-34} is
   (a) 9
   (b) 10
   (c) 11
   (d) 12

Session 2

Principle Properties of Logarithm

Principle Properties of Logarithm

Let m and n be arbitrary positive numbers, $a>0, a\neq 1, b>0, b\neq 1$ and α, β be any real numbers, then

(i) $\log_a (mn) = \log_a m + \log_a n$

In general, $\log_a (x_1 x_2 x_3 \dots x_n) = \log_a x_1 + \log_a x_2 + \log_a x_3 + \dots + \log_a x_n$ (where, $x_1, x_2, x_3, \dots, x_n > 0$)

Or

$$\boxed{\log_a \left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \log_a x_i, \forall x_i > 0}$$

Use when ∏ convert to Σ

where, $i = 1, 2, 3, \dots, n$.

(ii) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^\alpha = \alpha \log_a m$

(iv) $\log_{a^\beta} m = \frac{1}{\beta} \log_a m$

(v) $\log_b m = \frac{\log_a m}{\log_a b}$

Remark

1. $\log_b a \cdot \log_a b = 1 \Leftrightarrow \log_b a = \frac{1}{\log_a b}$
2. $\boxed{\log_b a \cdot \log_c b \cdot \log_a c = 1}$
3. $\log_y x \cdot \log_z y \cdot \log_a z = \log_a x$
4. $e^{\ln a^x} = a^x$

Extra Properties of Logarithm

(i) $a^{\log_b x} = x^{\log_b a}$, $b \neq 1$, a, b, x are positive numbers.

(ii) $a^{\log_a x} = x, a > 0, a \neq 1, x > 0$

(iii) $\log_{a^k} x = \frac{1}{k} \log_a x, \ a > 0, a \neq 1, x > 0$

(iv) $\log_a x^{2k} = 2k \log_a |x|, \ a > 0, a \neq 1, k \in I$

(v) $\log_{a^{2k}} x = \frac{1}{2k} \log_{|a|} x, \ x > 0, a > 0, a \neq \pm 1$ and $k \in I \sim \{0\}$

(vi) $\log_{a^\alpha} x^\beta = \frac{\beta}{\alpha} \log_a x, \ x > 0, a > 0, a \neq 1, \alpha \neq 0$

(vii) $\log_a x^2 \neq 2 \log_a x, \ a > 0, a \neq 1$

Since, domain of $\log_a (x^2)$ is $R \sim \{0\}$ and domain of $\log_a x$ is $(0, \infty)$ are not same.

(viii) $a^{\log_b a} = \sqrt{a}$, if $b = a^2, a > 0, b > 0, b \neq 1$

(ix) $a^{\log_b a} = a^2$, if $b = \sqrt{a}, a > 0, b > 0, b \neq 1$

| Example 5. Solve the equation $3 \cdot x^{\log_5 2} + 2^{\log_5 x} = 64$.

Sol. $\because \quad 3 \cdot x^{\log_5 2} + 2^{\log_5 x} = 64$

$\Rightarrow \quad 3 \cdot 2^{\log_5 x} + 2^{\log_5 x} = 64$ [by extra property (i)]

$\Rightarrow \quad 4 \cdot 2^{\log_5 x} = 64$

$\Rightarrow \quad 2^{\log_5 x} = 4^2 = 2^4$

$\therefore \quad \log_5 x = 4$

$\Rightarrow \quad x = 5^4 = 625$

| Example 6. If $4^{\log_{16} 4} + 9^{\log_3 9} = 10^{\log_x 83}$, find x.

Sol. $\because \quad 4^{\log_{16} 4} = \sqrt{4} = 2$ [by extra property (ix)]

and $\quad 9^{\log_3 9} = 9^2 = 81$ [by extra property (viii)]

$\therefore \quad 4^{\log_{16} 4} + 9^{\log_3 9} = 2 + 81 = 83 = 10^{\log_x 83}$

$\Rightarrow \quad \log_{10} 83 = \log_x 83$

$\therefore \quad x = 10$

| Example 7. Prove that $a^{\sqrt{\log_a b}} - b^{\sqrt{\log_b a}} = 0$.

Sol. $\because \quad a^{\sqrt{(\log_a b)}} = a^{\sqrt{\log_a b} \times \sqrt{\log_a b} \times \sqrt{\log_b a}}$

$= a^{\log_a b \cdot \sqrt{(\log_b a)}}$

$= b^{\sqrt{\log_b a}}$ [by extra property (ii)]

Hence, $a^{\sqrt{(\log_a b)}} - b^{\sqrt{(\log_b a)}} = 0$

| Example 8. Prove that $\frac{\log_2 24}{\log_{96} 2} - \frac{\log_2 192}{\log_{12} 2} = 3$.

Sol. LHS $= \frac{\log_2 24}{\log_{96} 2} - \frac{\log_2 192}{\log_{12} 2}$

$= \log_2 24 \times \log_2 96 - \log_2 192 \times \log_2 12$

Now, let $12 = \lambda$, then

LHS $= \log_2 2\lambda \times \log_2 8\lambda - \log_2 16\lambda \times \log_2 \lambda$

$= (\log_2 2 + \log_2 \lambda)(\log_2 8 + \log_2 \lambda)$
$\qquad - (\log_2 16 + \log_2 \lambda)\log_2 \lambda$
$= (\log_2 2 + \log_2 \lambda)(\log_2 2^3 + \log_2 \lambda)$
$\qquad - (\log_2 2^4 + \log_2 \lambda)\log_2 \lambda$
$= (1 + \log_2 \lambda)(3\log_2 2 + \log_2 \lambda)$
$\qquad - (4\log_2 2 + \log_2 \lambda)\log_2 \lambda$
$= (1 + \log_2 \lambda)(3 + \log_2 \lambda) - \log_2 \lambda(4 + \log_2 \lambda)$
$= 3$
$=$ RHS

Example 9. Solve for a, λ, if

$$\log_\lambda a \cdot \log_5 \lambda \cdot \log_\lambda 25 = 2.$$

Sol. Here, $\lambda > 0, \lambda \neq 1$

We have, $\log_\lambda a \cdot \{\log_5 \lambda \cdot \log_\lambda 25\} = 2$

$\Rightarrow (\log_\lambda a)\{\log_5 25\} = 2$

$\Rightarrow (\log_\lambda a)\{\log_5 5^2\} = 2$

$\Rightarrow (\log_\lambda a)\{2\log_5 5\} = 2$

$\Rightarrow (\log_\lambda a)\{2\} = 2$

$\therefore \log_\lambda(a) = 1$ or $a = \lambda$

b, b, d, c, c

Exercise for Session 2

1. If $a = \log_{24} 12$, $b = \log_{48} 36$ and $c = \log_{36} 24$, $1 + abc$ is equal to
   (a) $2ab$ (b) $2bc$ (c) $2ca$ (d) $ba + bc$
2. The value of $\log_4[\log_2\{\log_2(\log_3 81)\}]$ is equal to
   (a) -1 (b) 0 (c) 1 (d) 2
3. $\log_2 \log_2 (\underbrace{\sqrt{\sqrt{\ldots\sqrt{\sqrt{2}}}}}_{n \text{ times}})$ is equal to
   (a) 0 (b) 1 (c) n (d) $-n$
4. If $a = \log_3 5$, $b = \log_{17} 25$, which one of the following is correct?
   (a) $a < b$ (b) $a = b$ (c) $a > b$ (d) None of these
5. The value of $\log_{0.75} \log_2 \sqrt{\sqrt[-2]{(0.125)}}$ is equal to
   (a) -1 (b) 0 (c) 1 (d) None of these

Session 3

Properties of Monotonocity of Logarithm, Graphs of Logarithmic Functions

Properties of Monotonocity of Logarithm

1. Constant Base

(i) $\log_a x > \log_a y \Leftrightarrow \begin{cases} x > y > 0, \text{ if } a > 1 \\ 0 < x < y, \text{ if } 0 < a < 1 \end{cases}$

(ii) $\log_a x < \log_a y \Leftrightarrow \begin{cases} 0 < x < y, \text{ if } a > 1 \\ x > y > 0, \text{ if } 0 < a < 1 \end{cases}$

(iii) $\log_a x > p \Leftrightarrow \begin{cases} x > a^p, \text{ if } a > 1 \\ 0 < x < a^p, \text{ if } 0 < a < 1 \end{cases}$

(iv) $\log_a x < p \Leftrightarrow \begin{cases} 0 < x < a^p, \text{ if } a > 1 \\ x > a^p, \text{ if } 0 < a < 1 \end{cases}$

2. Variable Base

(i) $\log_x a$ is defined, if $a > 0, x > 0, x \neq 1$.

(ii) If $a > 1$, then $\log_x a$ is monotonically decreasing in $(0, 1) \cup (1, \infty)$.

$\log_a x$

(iii) If $0 < a < 1$, then $\log_x a$ is monotonically increasing in $(0, 1) \cup (1, \infty)$.

Very Important Concepts

(i) If $a > 1, p > 1,$ then $\log_a p > 0$

(ii) If $0 < a < 1, p > 1,$ then $\log_a p < 0$

(iii) If $a > 1, 0 < p < 1,$ then $\log_a p < 0$

(iv) If $p > a > 1,$ then $\log_a p > 1$

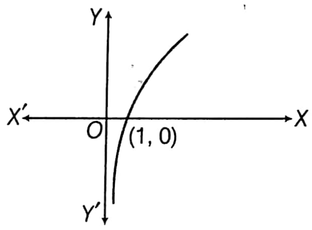
(v) If $a > p > 1,$ then $0 < \log_a p < 1$

(vi) If $0 < a < p < 1,$ then $0 < \log_a p < 1$

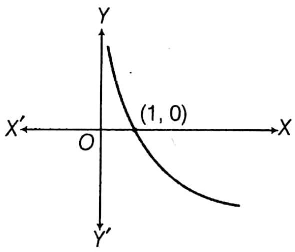
(vii) If $0 < p < a < 1,$ then $\log_a p > 1$

Graphs of Logarithmic Functions

1. **Graph of $y = \log_a x$, if $a > 1$ and $x > 0$**



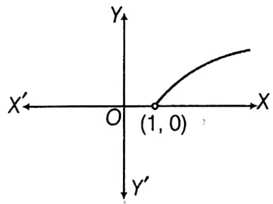
2. **Graph of $y = \log_a x$, if $0 < a < 1$ and $x > 0$**

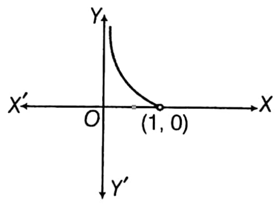


Remark

1. If the number x and the base 'a' are on the same side of the unity, then the logarithm is positive.

Case I $y = \log_a x, a > 1, x > 1$ **Case II** $y = \log_a x, 0 < a < 1, 0 < x < 1$

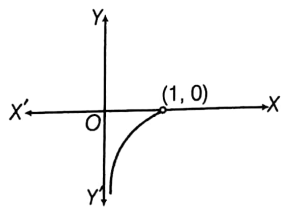


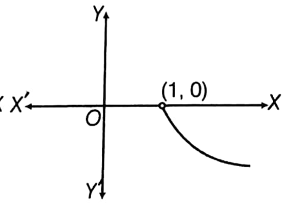


2. If the number x and the base a are on the opposite sides of the unity, then the logarithm is negative.

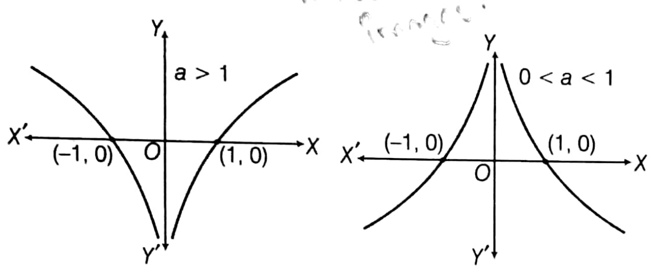
Case I $y = \log_a x, a > 1, 0 < x < 1$

Case II $y = \log_a x, 0 < a < 1, x > 1$





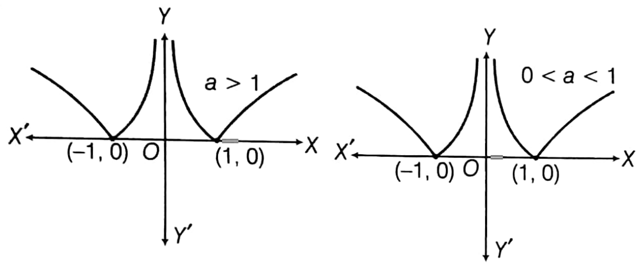
3. Graph of $y = \log_a |x|$



Remark

Graphs are symmetrical about Y-axis.

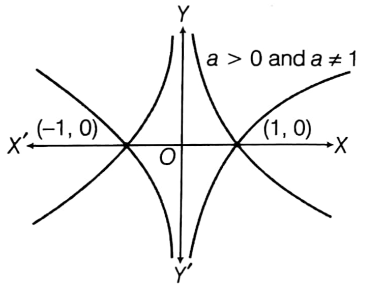
4. Graph of $y = |\log_a |x||$



Remark

Graphs are same in both cases i.e., $a > 1$ and $0 < a < 1$.

5. Graph of $|y| = \log_a |x||$



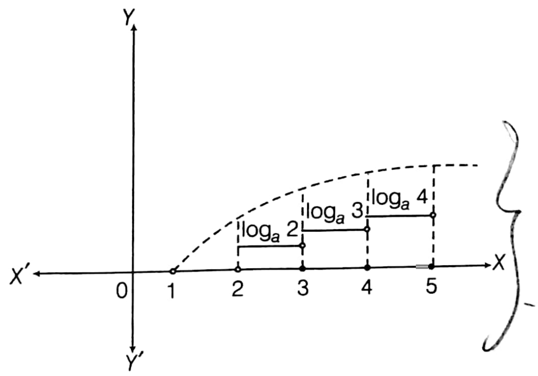
6. Graph of $y = \log_a [x]$, $a > 1$ and $x \geq 1$

(where $[\cdot]$ denotes the greatest integer function)

Since, when $1 \leq x < 2, [x] = 1 \Rightarrow \log_a [x] = 0$

when $2 \leq x < 3, [x] = 2 \Rightarrow \log_a [x] = \log_a 2$

when $3 \leq x < 4, [x] = 3 \Rightarrow \log_a [x] = \log_a 3$ and so on.



| Example 10. Arrange in ascending order $\log_2(x), \log_3(x), \log_e(x), \log_{10}(x)$, if

(i) $x > 1$ (ii) $0 < x < 1$.

Sol. $\because 2 < e < 3 < 10$

(i) For $x > 1$, $\log_x 2 < \log_x e < \log_x 3 < \log_x 10$

$$\Rightarrow \quad \frac{1}{\log_2(x)} < \frac{1}{\log_e(x)} < \frac{1}{\log_3(x)} < \frac{1}{\log_{10}(x)}$$

$$\Rightarrow \quad \log_2(x) > \log_e(x) > \log_3(x) > \log_{10}(x)$$

Hence, ascending order is

$$\log_{10}(x) < \log_3(x) < \log_e(x) < \log_2(x)$$

(ii) For $0 < x < 1$, $\log_x 2 > \log_x e > \log_x 3 > \log_x 10$

$$\Rightarrow \quad \frac{1}{\log_2(x)} > \frac{1}{\log_e(x)} > \frac{1}{\log_3(x)} > \frac{1}{\log_{10}(x)}$$

$$\therefore \quad \log_2(x) < \log_e(x) < \log_3(x) < \log_{10}(x)$$

which is in ascending order.

| Example 11. If $\log 11 = 1.0414$, prove that $10^{11} > 11^{10}$.

Sol. $\because \quad \log 10^{11} = 11 \log 10 = 11$

and $\log 11^{10} = 10 \log 11 = 10 \times 1.0414 = 10.414$

It is clear that, $\quad 11 > 10.414$

$\Rightarrow \quad \log 10^{11} > \log 11^{10} \quad$ [$\because$ here, base = 10]

$\Rightarrow \quad 10^{11} > 11^{10}$

| Example 12. If $\log_2(x - 2) < \log_4(x - 2)$, find the interval in which x lies.

Sol. Here, $x - 2 > 0$

$\Rightarrow \quad x > 2 \quad$ ...(i)

and $\quad \log_2(x-2) < \log_{2^2}(x-2) = \frac{1}{2}\log_2(x-2)$

$\Rightarrow \quad \log_2(x-2) < \frac{1}{2}\log_2(x-2)$

$\Rightarrow \quad \frac{1}{2}\log_2(x-2) < 0 \Rightarrow \log_2(x-2) < 0$

$\Rightarrow \quad x - 2 < 2^0 \Rightarrow x - 2 < 1$

$\Rightarrow \quad x < 3 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$2 < x < 3 \text{ or } x \in (2, 3)$$

| Example 13. Prove that $\log_n(n+1) > \log_{(n+1)}(n+2)$ for any natural number $n > 1$.

Sol. Since, $\frac{n+1}{n} = 1 + \frac{1}{n} > 1 + \frac{1}{n+1} = \left(\frac{n+2}{n+1}\right)$

For $\quad n > 1$,

$$\log_n\left(\frac{n+1}{n}\right) > \log_{n+1}\left(\frac{n+1}{n}\right) > \log_{n+1}\left(\frac{n+2}{n+1}\right)$$

$$\Rightarrow \quad \log_n(n+1) - \log_n n > \log_{(n+1)}(n+2) - \log_{(n+1)}(n+1)$$

$$\Rightarrow \quad \log_n(n+1) - 1 > \log_{(n+1)}(n+2) - 1$$

$$\therefore \quad \log_n(n+1) > \log_{(n+1)}(n+2) \quad \textbf{Hence proved.}$$

How to Find Minimum Value of $\lambda_1 \log_a x + \lambda_2 \log_x a, a > 0, x > 0, a \neq 1, x \neq 1$ and $\lambda_1, \lambda_2 \in R_+$

$$\because \quad AM \geq GM$$

$$\Rightarrow \quad \frac{\lambda_1 \log_a x + \lambda_2 \log_x a}{2} \geq \sqrt{(\lambda_1 \log_a x)(\lambda_2 \log_x a)} = \sqrt{\lambda_1 \lambda_2}$$

$$\Rightarrow \quad \lambda_1 \log_a x + \lambda_2 \log_x a \geq 2\sqrt{\lambda_1 \lambda_2}$$

Hence, the minimum value of $\lambda_1 \log_a x + \lambda_2 \log_x a$ is $2\sqrt{\lambda_1 \lambda_2}$.

| Example 14. Find the least value of the expression $2\log_{10} x - \log_x 0.01$, where $x > 0, x \neq 1$.

Sol. Let $P = 2\log_{10} x - \log_x 0.01 = 2\log_{10} x - \log_x(10^{-2})$

$$= 2(\log_{10} x + \log_x 10)$$

$$\geq 2 \cdot 2 = 4 \qquad \text{[by above article]}$$

$$\therefore \quad P \geq 4$$

Hence, the least value of P is 4.

| Example 15. Which is smaller 2 or $(\log_\pi 2 + \log_2 \pi)$?

Sol. Let $P = \log_\pi 2 + \log_2 \pi > 2$ [by above article] [$\because \pi \neq 2$]

$$\therefore \quad P > 2$$

$$\Rightarrow \quad (\log_\pi 2 + \log_2 \pi) > 2$$

Hence, the smaller number is 2.

Exercise for Session 3

1 If $\log_{0.16}(a+1) < \log_{0.4}(a+1)$, then a satisfies

(a) $a > 0$ (b) $0 < a < 1$ (c) $-1 < a < 0$ (d) None of these

2 The value of x satisfying the inequation $x^{\frac{1}{\log_{10} x}} \cdot \log_{10} x < 1$, is

(a) $0 < x < 10$ (b) $0 < x < 10^{10}$ (c) $0 < x < 10^{1/10}$ (d) None of these

3 If $\log_{\operatorname{cosec} x} \sin x > 0$, then

(a) $x > 0$ (b) $x < 0$ (c) $-1 < x < 1$ (d) None of these

4 The value of $\log_{10} 3$ lies in the interval

(a) $\left(\frac{2}{5}, \frac{1}{2}\right)$ (b) $\left(0, \frac{1}{2}\right)$ (c) $\left(0, \frac{2}{5}\right)$ (d) None of these

5 The least value of n in order that the sum of first n terms of the infinite series $1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots$, should differ from the sum of the series by less than 10^{-6}, is (given, $\log 2 = 0.30103, \log 3 = 0.47712$)

(a) 14 (b) 27 (c) 53 (d) 57

Shortcuts and Important Results to Remember

1 For a non-negative number 'a' and $n \geq 2, n \in N$, $\sqrt[n]{a} = a^{1/n}$.

2 The number of positive integers having base a and characteristic n is $a^{n+1} - a^n$.

3 Logarithm of zero and negative real number is not defined.

4 $|\log_b a + \log_a b| \geq 2, \forall a > 0, a \neq 1, b > 0, b \neq 1$.

5 $\log_2 \log_2 \underbrace{\sqrt{\sqrt{\sqrt{\sqrt{\ldots\sqrt{2}}}}}}_{n \text{ times}} = -n$

6 $a^{\sqrt{\log_a b}} = b^{\sqrt{\log_b a}}$

7 Logarithms to the base 10 are called common logarithms (Brigg's logarithms).

8 ~~If x~~ $= \log_c b + \log_b c, y = \log_a c + \log_c a, z = \log_a b + \log_b a$, then $x^2 + y^2 + z^2 - 4 = xyz$.

JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **8 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

• Ex. 1 *The expression* $\log_2 5 - \sum_{k=1}^{4} \log_2\left(\sin\left(\frac{k\pi}{5}\right)\right)$ *reduces to* $\frac{p}{q}$, *where p and q are co-prime, the value of* $p^2 + q^2$ *is*

(a) 13 (b) 17 (c) 26 (d) 29

Sol. (b) Let $p = \log_2 5 - \sum_{k=1}^{4} \log_2\left(\sin\left(\frac{k\pi}{5}\right)\right)$

$$= \log_2 5 - \left\{\log_2\left(\sin\left(\frac{\pi}{5}\right)\right) + \log_2\left(\sin\left(\frac{2\pi}{5}\right)\right) + \log_2\left(\sin\left(\frac{3\pi}{5}\right)\right) + \log_2\left(\sin\left(\frac{4\pi}{5}\right)\right)\right\}$$

$$= \log_2 5 - \log_2\left\{\sin\frac{\pi}{5}\cdot\sin\frac{2\pi}{5}\cdot\sin\frac{3\pi}{5}\cdot\sin\frac{4\pi}{5}\right\}$$

$$= \log_2 5 - \log_2\left\{\sin^2\left(\frac{\pi}{5}\right)\cdot\sin^2\left(\frac{2\pi}{5}\right)\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{(1-\cos 72^\circ)(1-\cos 144^\circ)}{4}\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{(1-\sin 18^\circ)(1+\cos 36^\circ)}{4}\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{\left(1-\frac{\sqrt{5}-1}{4}\right)\left(1+\frac{\sqrt{5}+1}{4}\right)}{4}\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{(5-\sqrt{5})(5+\sqrt{5})}{64}\right\} = \log_2 5 - \log_2\left(\frac{5}{16}\right)$$

$$= \log_2\left(5 \times \frac{16}{5}\right) = \log_2 2^4 = \frac{4}{1} = \frac{p}{q} \quad \text{[given]}$$

$\therefore \quad p = 4, q = 1$

Hence, $p^2 + q^2 = 4^2 + 1^2 = 17$

• Ex. 2 *If* $3 \le a \le 2015, 3 \le b \le 2015$ *such that* $\log_a b + 6\log_b a = 5$, *the number of ordered pairs (a, b) of integers is*

(a) 48 (b) 50 (c) 52 (d) 54

Sol. (c) Let $x = \log_a b$ ...(i)

$\Rightarrow \quad x + \frac{6}{x} = 5 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

From Eq. (i), we get $\log_a b = 2, 3$

$\Rightarrow \quad b = a^2$ or a^3

The pairs (a, b) are

$(3, 3^2), (4, 4^2), (5, 5^2), (6, 6^2), \ldots, (44, 44^2)$ and $(3, 3^3), (4, 4^3), (5, 5^3), \ldots, (12, 12^3)$.

Hence, there are $42 + 10 = 52$ pairs.

• Ex. 3 *The lengths of the sides of a triangle are* $\log_{10} 12$, $\log_{10} 75$ *and* $\log_{10} n$, *where* $n \in N$. *If a and b are the least and greatest values of n respectively, the value of* $b - a$ *is divisible by*

(a) 221 (b) 222 (c) 223 (d) 224

Sol. (c) In a triangle,

$\log_{10} 12 + \log_{10} 75 > \log_{10} n \Rightarrow n < 12 \times 75 = 900$

$\therefore \quad n < 900$ ...(i)

and $\quad \log_{10} 12 + \log_{10} n > \log_{10} 75$

$\Rightarrow \quad n > \frac{75}{12} = \frac{25}{4}$

$\therefore \quad n > \frac{25}{4}$ ...(ii)

From Eqs. (i) and (ii), we get $\frac{25}{4} < n < 900$

$\therefore \quad n = 7, 8, 9, 10, \ldots, 899$

Hence, $\quad a = 7, b = 899$

$\therefore \quad b - a = 892 = 4 \times 223$

Hence, $b - a$ is divisible by 223.

• Ex. 4 *If* $5\log_{abc}(a^3 + b^3 + c^3) = 3\lambda\left(\frac{1+\log_3(abc)}{\log_3(abc)}\right)$ *and* $(abc)^{a+b+c} = 1$ *and* $\lambda = \frac{m}{n}$, *where m and n are relative primes, the value of* $|m+n| + |m-n|$ *is*

(a) 8 (b) 10 (c) 12 (d) 14

Sol. (b) $\because \quad (abc)^{a+b+c} = 1 = (abc)^0$

$\therefore \quad a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

Now, LHS $= 5\log_{abc}(a^3 + b^3 + c^3) = 5\log_{abc}(3abc)$ ...(i)

and RHS $= 3\lambda\left(\frac{1+\log_3(abc)}{\log_3(abc)}\right) = 3\lambda\left(\frac{\log_3(3abc)}{\log_3(abc)}\right)$

$= 3\lambda\log_{abc}(3abc)$ ...(ii)

From Eqs. (i) and (ii), we get

$5\log_{abc}(3abc) = 3\lambda\log_{abc}(3abc)$

$\therefore \quad \lambda = \frac{5}{3} = \frac{m}{n}$ [given]

$\Rightarrow \quad m = 5, n = 3$

Hence, $|m+n| + |m-n| = 8 + 2 = 10$

• ***Ex. 5*** *If* $a^{\log_b c} = 3 \cdot 3^{\log_4 3} \cdot 3^{\log_4 3^{\log_4 3}} \cdot 3^{\log_4 3^{\log_4 3^{\log_4 3}}} \ldots \infty$, *where* $a, b, c \in Q$, *the value of* abc *is*

(a) 9 (b) 12 (c) 16 (d) 20

Sol. (c) $a^{\log_b c} = 3^{1 + \log_4 3 + (\log_4 3)^2 + (\log_4 3)^3 + \ldots \infty}$

$$= 3^{1/(1 - \log_4 3)} = 3^{1/\log_4 (4/3)} = 3^{\log_{4/3} 4}$$

$$\therefore \quad a = 3, b = \frac{4}{3}, c = 4$$

Hence, $abc = 3 \cdot \frac{4}{3} \cdot 4 = 16$

• ***Ex. 6*** *Number of real roots of equation*

$$3^{\log_3 (x^2 - 4x + 3)} = (x - 3) \text{ is}$$

(a) 0 (b) 1 (c) 2 (d) infinite

Sol. (a) $\because \quad 3^{\log_3 (x^2 - 4x + 3)} = (x - 3)$ ...(i)

Eq. (i) is defined, if $x^2 - 4x + 3 > 0$

$\Rightarrow \quad (x - 1)(x - 3) > 0$

$\Rightarrow \quad x < 1 \text{ or } x > 3$ ...(ii)

Eq. (i) reduces to $x^2 - 4x + 3 = x - 3 \Rightarrow x^2 - 5x + 6 = 0$

$\therefore \quad x = 2, 3$ ...(iii)

From Eqs. (ii) and (iii), use get $x \in \phi$

$\therefore$ Number of real roots = 0

• ***Ex. 7*** *If* $\log_6 a + \log_6 b + \log_6 c = 6$, *where* $a, b, c \in N$ *and* a, b, c *are in GP and* $b - a$ *is a square of an integer, then the value of* $a + b - c$ *is*

(a) 21 (b) 15 (c) 9 (d) 3

Sol. (b) $\because \quad \log_6 a + \log_6 b + \log_6 c = 6$

$\Rightarrow \quad \log_6 (abc) = 6$

$\Rightarrow \quad abc = 6^6$

$\Rightarrow \quad b^3 = 6^6 \quad [\because b^2 = ac]$

$\Rightarrow \quad b = 36$

Also, $b - a = 36 - a$ is a square for $a = 35, 32, 27, 20, 11$

Now, $c = \frac{b^2}{a} = \frac{36^2}{a}$ is an integer for $a = 27$

$\therefore \quad a = 27, b = 36, c = 48$

Hence, $a + b - c = 27 + 36 - 48 = 15$

• ***Ex. 8*** *If* $x = \log_{2a} \left(\frac{bcd}{2} \right), y = \log_{3b} \left(\frac{acd}{3} \right)$, $z = \log_{4c} \left(\frac{abd}{4} \right)$ *and* $w = \log_{5d} \left(\frac{abc}{5} \right)$ *and* $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{w+1} = \log_{abcd} N + 1$, *the value of* N *is*

(a) 40 (b) 80
(c) 120 (d) 160

Sol. (c) $\because \quad x = \log_{2a} \left(\frac{bcd}{2} \right)$

$\Rightarrow \quad x + 1 = \log_{2a} \left(\frac{2abcd}{2} \right) = \log_{2a} (abcd)$

$\therefore \quad \frac{1}{x+1} = \log_{abcd} 2a$

Similarly, $\frac{1}{y+1} = \log_{abcd} 3b, \frac{1}{z+1} = \log_{abcd} 4c$

and $\frac{1}{w+1} = \log_{abcd} 5d$

$\therefore \quad \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{w+1} = \log_{abcd} (2a \cdot 3b \cdot 4c \cdot 5d)$

$= \log_{abcd} (120 abcd)$

$= \log_{abcd} 120 + 1$

$= \log_{abcd} N + 1$ [given]

Hence, $N = 120$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **4 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

• ***Ex. 9*** *The equation*
$(\log_{10} x + 2)^3 + (\log_{10} x - 1)^3 = (2 \log_{10} x + 1)^3$ *has*

(a) no natural solution (b) two rational solutions
(c) no prime solution (d) one irrational solution

Sol. (b, c, d) Let $\log_{10} x + 2 = a$ and $\log_{10} x - 1 = b$

$\therefore$ $a + b = 2\log_{10} x + 1$, then given equation reduces to

$$a^3 + b^3 = (a + b)^3$$

$\Rightarrow \quad 3ab(a + b) = 0 \Rightarrow a = 0 \text{ or } b = 0 \text{ or } a + b = 0$

$\Rightarrow \quad \log_{10} x + 2 = 0 \text{ or } \log_{10} x - 1 = 0$

or $\quad 2\log_{10} x + 1 = 0$

$\Rightarrow \quad x = 10^{-2} \text{ or } x = 10 \text{ or } x = 10^{-1/2}$

Hence, $x = \frac{1}{100} \text{ or } x = 10 \text{ or } x = \frac{1}{\sqrt{10}}$

• ***Ex. 10*** *The value of* $\frac{\log_5 9 \cdot \log_7 5 \cdot \log_3 7}{\log_3 \sqrt{6}} + \frac{1}{\log_4 \sqrt{6}}$ *is co-prime with*

(a) 1 (b) 3 (c) 4 (d) 5

Sol. (*a, b, d*) Let $P = \dfrac{\log_5 9 \cdot \log_7 5 \cdot \log_3 7}{\log_3 \sqrt{6}} + \dfrac{1}{\log_4 \sqrt{6}}$

$$= \frac{\log_3 9}{\log_3 \sqrt{6}} + \log_{\sqrt{6}} 4 = \log_{\sqrt{6}} 9 + \log_{\sqrt{6}} 4$$

$$= \log_{\sqrt{6}}(36) = \log_{\sqrt{6}}(\sqrt{6})^4 = 4 \Rightarrow P = 4$$

which is co-prime with 1, 3, 4 and 5.

• **Ex. 11** *Which of the following quantities are irrational for the quadratic equation*
$(\log_{10} 8)x^2 - (\log_{10} 5)x = 2(\log_2 10)^{-1} - x$?

(a) Sum of roots (b) Product of roots
(c) Sum of coefficients (d) Discriminant

Sol. (*c, d*) $\because$ $(\log_{10} 8)x^2 - (\log_{10} 5)x = 2(\log_2 10)^{-1} - x$

$\Rightarrow$ $(3\log_{10} 2)x^2 + (1 - \log_{10} 5)x - 2\log_{10} 2 = 0$

$\Rightarrow$ $(3\log_{10} 2)x^2 + (\log_{10} 2)x - 2\log_{10} 2 = 0$

Now, Sum of roots $= -\dfrac{1}{3}$ = Rational

Product of roots $= -\dfrac{2}{3}$ = Rational

Sum of coefficients $= 3\log_{10} 2 + \log_{10} 2 - 2\log_{10} 2$
$= 2\log_{10} 2$ = Irrational

Discriminant $= (\log_{10} 2)^2 + 24(\log_{10} 2)^2$
$= 25(\log_{10} 2)^2$ = Irrational

• **Ex. 12** *The system of equations*
$\log_{10}(2000xy) - \log_{10} x \cdot \log_{10} y = 4$
$\log_{10}(2yz) - \log_{10} y \cdot \log_{10} z = 1$
and $\log_{10}(zx) - \log_{10} z \cdot \log_{10} x = 0$
has two solutions (x_1, y_1, z_1) *and* (x_2, y_2, z_2), *then*

(a) $x_1 + x_2 = 101$ (b) $y_1 + y_2 = 25$
(c) $x_1 x_2 = 100$ (d) $z_1 z_2 = 100$

Sol. (*a, b, c, d*) Let $\log_{10} x = a$, $\log_{10} y = b$ and $\log_{10} z = c$

Then, given equations reduces to

$a + b - ab = 4 - \log_{10} 2000 = \log_{10} 5$ ...(i)

$b + c - bc = 1 - \log_{10} 2 = \log_{10} 5$ ...(ii)

and $c + a - ca = 0$ ...(iii)

From Eqs. (i) and (ii), we get

$$a + b - ab = b + c - bc$$

$\Rightarrow$ $(c - a) - b(c - a) = 0$

$\Rightarrow$ $(c - a)(1 - b) = 0$

$1 - b \neq 0, c - a = 0 \Rightarrow c = a$

From Eq. (iii), we get

$$2a - a^2 = 0 \Rightarrow a = 0, 2$$

Then, $c = a \Rightarrow c = 0, 2$

and $b = \log_{10} 5, 2 - \log_{10} 5$

$\therefore$ $\log_{10} x = 0, 2 \Rightarrow x = 10^0, 10^2$

$\Rightarrow$ $x = 1, 100$

$\Rightarrow$ $x_1 = 1, x_2 = 100$

and $\log_{10} y = \log_{10} 5, 2 - \log_{10} 5$
$= \log_{10} 5, \log_{10} 20$

$\Rightarrow$ $y = 5, 20$

$\Rightarrow$ $y_1 = 5, y_2 = 20$

and $\log_{10} z = 0, 2 \Rightarrow z = 10^0, 10^2$

$\Rightarrow$ $z = 1, 100$

$\Rightarrow$ $z_1 = 1, z_2 = 100$

Finally, $x_1 + x_2 = 1 + 100 = 101$, $y_1 + y_2 = 5 + 20 = 25$,
$x_1 x_2 = 1 \times 100 = 100$ and $z_1 z_2 = 1 \times 100 = 100$

JEE Type Solved Examples : Passage Based Questions

- This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 13 to 15)

Suppose that $\log_{10}(x - 2) + \log_{10} y = 0$ and $\sqrt{x} + \sqrt{(y - 2)} = \sqrt{(x + y)}$.

13. The value of x is
(a) $2 + \sqrt{2}$ (b) $1 + \sqrt{2}$ (c) $2\sqrt{2}$ (d) $4 - \sqrt{2}$

14. The value of y is
(a) 2 (b) $2\sqrt{2}$ (c) $1 + \sqrt{2}$ (d) $2 + 2\sqrt{2}$

15. If $x^{2t^2 - 6} + y^{6 - 2t^2} = 6$, the value of $t_1 t_2 t_3 t_4$ is
(a) 1 (b) 2 (c) 4 (d) 8

Sol. (Ex. Nos. 13-15)

$\because$ $\log_{10}(x - 2) + \log_{10} y = 0$

$\therefore$ $x - 2 > 0, y > 0$

$\Rightarrow$ $x > 2, y > 0$ ...(i)

and $\log_{10}\{(x - 2)y\} = 0$

$\Rightarrow$ $(x - 2)y = 10^0 = 1$

$\therefore$ $(x - 2)y = 1$ ...(ii)

Also, given that $\sqrt{x} + \sqrt{(y - 2)} = \sqrt{(x + y)}$

$\therefore$ $x \geq 0, y - 2 \geq 0, x + y \geq 0$

$\Rightarrow \qquad x \geq 0, y \geq 2$ ...(iii)

On squaring both sides, we get

$$x + y - 2 + 2\sqrt{x}\sqrt{(y-2)} = x + y$$

$\Rightarrow \qquad \sqrt{x}\sqrt{y-2} = 1$

$\Rightarrow \qquad x(y-2) = 1$ ...(iv)

From Eqs. (i) and (iii), we get

$$x > 2, y \geq 2$$

and from Eqs. (ii) and (iv), we get $y = x$

From Eq. (ii), $(x-2)x = 1$

$\Rightarrow \qquad x^2 - 2x - 1 = 0$

$\therefore \qquad x = \dfrac{2 \pm \sqrt{4+4}}{2}$ [neglect –ve sign, since $x > 2$]

13. (*b*) $x = (\sqrt{2} + 1)$.

14. (*c*) $y = x = \sqrt{1+1}$

15. (*d*) $\because \quad x^{2t^2-6} + y^{6-2t^2} = 6$

$\Rightarrow \qquad x^{2t^2-6} + (x^{-1})^{2t^2-6} = 6$

$\Rightarrow \qquad (x^2)^{t^2-3} + (x^{-2})^{t^2-3} = 6$

$\Rightarrow \qquad (3+2\sqrt{2})^{t^2-3} + (3-2\sqrt{2})^{t^2-3} = 6$

Now, we get $\quad t^2 - 3 = \pm 1$

$\Rightarrow \qquad t^2 = 4, 2$

$\therefore \qquad t = \pm 2, \pm\sqrt{2}$

$\therefore \qquad t_1 t_2 t_3 t_4 = (2)(-2)(\sqrt{2})(-\sqrt{2}) = 8$

Passage II

(Ex. Nos. 16 to 18)

If $10^{\log_p \{\log_q (\log_r x)\}} = 1$ and $\log_q \{\log_r (\log_p x)\} = 0$.

16. The value of x is

(a) q^r (b) r^q (c) r^p (d) rq

Sol. (*b*) $\because \quad 10^{\log_p [\log_q (\log_r x)]} = 1 = 10^0$

$\Rightarrow \qquad \log_p \{\log_q (\log_r x)\} = 0$

$\Rightarrow \qquad \log_q (\log_r x) = 1 \Rightarrow \log_r x = q$

$\Rightarrow \qquad x = r^q$ ...(i)

and $\qquad \log_q \{\log_r (\log_p x)\} = 0$

$\Rightarrow \qquad \log_r (\log_p x) = 1 \Rightarrow \log_p x = r$

$\therefore \qquad x = p^r$ ...(ii)

From Eqs. (i) and (ii), we get $\quad x = r^q = p^r$

17. The value of p is

(a) $r^{q/r}$ (b) rq (c) 1 (d) $r^{r/q}$

Sol. (*a*) $\because \qquad r^q = p^r$ ...(iii)

$\Rightarrow \qquad p = r^{q/r}$

18. The value of q is

(a) $r^{p/r}$ (b) $p \log_p r$ (c) $r \log_r p$ (d) $r^{r/p}$

Sol. (*c*) From Eq. (iii),

$$q \log r = r \log p \Rightarrow q = r\left(\frac{\log p}{\log r}\right) = r \log_r p$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

• Ex. 19 *If x_1 and x_2 are the solutions of the equation $x^{\log_{10} x} = 100x$ such that $x_1 > 1$ and $x_2 < 1$, the value of $\dfrac{x_1 x_2}{2}$ is*

Sol. (5) $\because \quad x^{\log_{10} x} = 100x$

Taking logarithm on both sides on base 10, then we get

$$\log_{10} x \cdot \log_{10} x = \log_{10} 100 + \log_{10} x$$

$\Rightarrow \qquad (\log_{10} x)^2 - \log_{10} x - 2 = 0$

$\Rightarrow \qquad (\log_{10} x - 2)(\log_{10} x + 1) = 0$

$\therefore \qquad \log_{10} x = 2, -1 \Rightarrow x = 10^2, 10^{-1}$

$\therefore \qquad x_1 = 100, x_2 = \dfrac{1}{10}$

$\therefore \qquad \dfrac{x_1 x_2}{2} = 5$

• Ex. 20 *If $(31.6)^a = (0.0000316)^b = 100$, the value of $\dfrac{1}{a} - \dfrac{1}{b}$ is*

Sol. (3) $\because \quad (31.6)^a = (0.0000316)^b = 100$

$\Rightarrow \qquad a \log_{10}(31.6) = b \log_{10}(0.0000316) = \log_{10} 100$

$\Rightarrow \qquad a \log_{10}(31.6) = b \log_{10}(31.6 \times 10^{-6}) = 2$

$\Rightarrow \qquad a \log_{10}(31.6) = b \log_{10}(31.6) - 6b = 2$

$\Rightarrow \qquad \dfrac{2}{a} = \log_{10}(31.6)$

and $\qquad \dfrac{2}{b} = \log_{10}(31.6) - 6$

$\therefore \qquad \dfrac{2}{a} - \dfrac{2}{b} = 6$

$\Rightarrow \qquad \dfrac{1}{a} - \dfrac{1}{b} = 3$

JEE Type Solved Examples : Matching Type Questions

- This section contains **2 examples.** Examples 24 and 25 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

• Ex. 21

	Column I		Column II
(A)	If x_1 and x_2 satisfy the equation $(x+1)^{\log_{10}(x+1)} = 100(x+1)$, then the value of $(x_1+1)(x_2+1)+5$ is	(p)	irrational
(B)	The product of all values of x which make the following statement true $(\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$, is	(q) (r)	rational prime
(C)	If $\log_b a = -3$, $\log_b c = 4$ and if the value of x satisfying the equation $a^{3x} = c^{x-1}$ is expressed in the form p/q, where p and q are relatively prime, then q is	(s)	composite
		(t)	twin prime

Sol. A → (q, s, t), B → (p), C → (q, r)

(A) $(x+1)^{\log_{10}(x+1)} = 100(x+1)$

Taking logarithm on both sides on base 10, then we get

$$\log_{10}(x+1)\cdot\log_{10}(x+1) = \log_{10}100 + \log_{10}(x+1)$$

$$\Rightarrow \quad \{\log_{10}(x+1)\}^2 = 2 + \log_{10}(x+1)$$

$$\Rightarrow \{\log_{10}(x+1)\}^2 - \log_{10}(x+1) - 2 = 0$$

$$\Rightarrow \{\log_{10}(x+1) - 2\}\{\log_{10}(x+1)+1\} = 0$$

$$\therefore \quad \log_{10}(x+1) = 2, -1$$

$$\Rightarrow \quad (x+1) = 10^2, 10^{-1}$$

$$\therefore \quad (x_1+1)(x_2+1) = 10^2 \times 10^{-1} = 10$$

$$\Rightarrow \quad (x_1+1)(x_2+1) + 5 = 10 + 5 = 15 = 3\times 5$$

(B) $\because (\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$

$$\Rightarrow \quad 2\log_5 x - 2\log_x 5 = \log_3 54 - \log_3 2 = \log_3(27) = 3$$

Let $\log_5 x = \lambda$, then

$$2\lambda - \frac{2}{\lambda} = 3$$

$$\Rightarrow \quad 2\lambda^2 - 3\lambda - 2 = 0$$

$$\Rightarrow \quad 2\lambda^2 - 4\lambda + \lambda - 2 = 0$$

$$2\lambda(\lambda-2) + 1(\lambda-2) = 0 \Rightarrow \lambda = -\frac{1}{2}, 2$$

$$\therefore \quad \log_5 x = -\frac{1}{2}, 2$$

$$\Rightarrow \quad x = 5^{-1/2}, 5^2 \text{ or } x = \frac{1}{\sqrt{5}}, 25$$

$\therefore$ Product of the values of $x = \frac{1}{\sqrt{5}} \times 25 = 5\sqrt{5}$

(C) $\because \quad \log_b a = -3$ and $\log_b c = 4$

$$\therefore \quad \log_c a = -\frac{3}{4} \qquad \text{...(i)}$$

and $\quad a^{3x} = c^{x-1}$

$$\Rightarrow \quad 3x\log a = (x-1)\log c$$

$$\Rightarrow \quad 3x\log_c a = x - 1$$

$$\Rightarrow \quad 3x \times -\frac{3}{4} = x - 1 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad -9x = 4x - 4 \text{ or } x = \frac{4}{13}$$

$$\therefore \quad q = 13 \qquad \text{[prime and rational]}$$

• Ex. 22

	Column I		Column II
(A)	If α and β are the roots of $ax^2 + bx + c = 0$, where $a = 2^{\log_2 3} - 3^{\log_3 2}$, $b = 1 + 2^{\sqrt{\log_2 3}} - 3^{\sqrt{\log_3 2}}$ and $c = \log_2 \log_2 \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{2}}}}}}$, then HM of α and β is	(p) (q)	divisible by 2 divisible by 4
(B)	The sum of the solutions of the equation $\lvert x-1\rvert^{\log_2 x^2 - 2\log_x 4} = (x-1)^7$ is	(r) (s)	divisible by 6 divisible by 8
(C)	If $5(\log_y x + \log_x y) = 26$, $xy = 64$, then the value of $\lvert x - y\rvert$ is	(t)	divisible by 10

Sol. A → (p, q, r), B → (p, r), C → (p, r, t)

(A) $\because a = 3 - 2 = 1, b = 1, c = \log_2 \log_2 2^{2^{-6}}$

$$= \log_2(2^{-6}) = -6$$

The equation reduces to $x^2 + x - 6 = 0$

$$\therefore \quad \alpha + \beta = -1, \ \alpha\beta = -6$$

$$\therefore \quad \text{HM} = \frac{2\alpha\beta}{\alpha+\beta} = \frac{2(-6)}{(-1)} = 12$$

(B) Obviously, $x = 2$ is a solution. Since, LHS is positive, $x - 1 > 0$. The equation reduces to

$$\log_2 x^2 - 2\log_x 4 = 7$$

$$\Rightarrow \quad 2\lambda - \frac{4}{\lambda} = 7, \text{ where } \lambda = \log_2 x$$

$\Rightarrow \quad 2\lambda^2 - 7\lambda - 4 = 0 \Rightarrow \lambda = 4, -\frac{1}{2}$

$\therefore \quad \log_2 x = 4, -\frac{1}{2} \Rightarrow x = 2^4, 2^{-1/2}$

$\Rightarrow \quad x = 16, \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = 16, x \neq \frac{1}{\sqrt{2}} \quad [\because x > 1]$

$\therefore$ Solutions are $x = 2, 16$

$\therefore$ Sum of solutions $= 2 + 16 = 18$

(C) If $\alpha = \log x, \beta = \log y$

$\therefore \quad \log_y x + \log_x y = \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$\therefore \quad 5(\log_y x + \log_x y) = 26$

$\Rightarrow \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{26}{5}$

Let $\frac{\alpha}{\beta} = \lambda$, then $\lambda + \frac{1}{\lambda} = \frac{26}{5}$

$\Rightarrow \quad 5\lambda^2 - 26\lambda + 5 = 0$

$\Rightarrow \quad 5\lambda^2 - 25\lambda - \lambda + 5 = 0$

$\Rightarrow \quad (\lambda - 5)(5\lambda - 1) = 0$

$\Rightarrow \quad \lambda = 5, \frac{1}{5}$

$\therefore \quad \frac{\alpha}{\beta} = 5, \frac{1}{5} \Rightarrow \frac{\alpha}{\beta} = 5$

$\Rightarrow \quad \alpha = 5\beta \quad$ ...(i)

and $\alpha + \beta = \log x + \log y = \log(xy) = \log(64)$

$\therefore \quad \alpha + \beta = 6\log 2 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$\beta = \log 2$ and $\alpha = 5\log 2$

$\Rightarrow \quad y = 2, x = 32$ or $y = 32, x = 2$

$\therefore \quad |x - y| = 30$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** Example numbers 23 to 24 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

- ***Ex. 23*** **Statement-1** *If* $N = \left(\frac{1}{0.4}\right)^{20}$, *then N contains 7 digits before decimal.*

Statement-2 *Characteristic of the logarithm of N to the base 10 is 7.*

Sol. (*d*) $\because \quad N = \left(\frac{1}{0.4}\right)^{20} = \left(\frac{10}{2^2}\right)^{20}$

$\Rightarrow \log_{10} N = 20(1 - 2\log_{10} 2) = 20(1 - 2 \times 0.3010)$

$= 20 \times 0.3980 = 7.9660$

Since, characteristic of $\log_{10} N$ is 7, therefore the number of digits in N will be $7 + 1$, i.e. 8.

Hence, Statement-1 is false and Statement-2 is true.

- ***Ex. 24*** **Statement-1** *If* $p, q \in N$ *satisfy the equation* $x^{\sqrt{x}} = (\sqrt{x})^x$ *and* $q > p$, *then q is a perfect number.*

Statement-2 *If a number is equal to the sum of its factor, then number is known as perfect number.*

Sol. (*d*) $\because \quad x^{\sqrt{x}} = (\sqrt{x})^x$

Taking logarithm on both sides on base e, then

$\ln(x)^{\sqrt{x}} = \ln(\sqrt{x})^x$

$\Rightarrow \quad \sqrt{x}\ln x = x\ln\sqrt{x} \Rightarrow \sqrt{x}\ln x = \frac{x}{2}\ln x$

$\Rightarrow \quad \ln x\left(\sqrt{x} - \frac{x}{2}\right) = 0$

$\Rightarrow \quad \ln x \cdot \sqrt{x} \cdot \left(1 - \frac{\sqrt{x}}{2}\right) = 0$

$\Rightarrow \quad \ln x = 0, \sqrt{x} = 0, 1 - \frac{\sqrt{x}}{2} = 0$

$\therefore \quad x = 1, 0, 4$

$\because \quad x \in N$

$\therefore \quad x = 1, 4 \Rightarrow p = 1$ and $q = 4$

$\because \quad 4 = 1 \times 2 \times 2 \Rightarrow 4 \neq 1 + 2 + 2$

$\therefore$ q is not a perfect number.

Hence, Statement-1 is false and Statement-2 is true.

Logarithms and Their Properties Exercise 1 : Single Option Correct Type Questions

This section contains **20 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $\log_{10} 2 = 0.3010\ldots$, the number of digits in the number 2000^{2000} is
(a) 6601 (b) 6602 (c) 6603 (d) 6604

2. There exist a positive number λ, such that $\log_2 x + \log_4 x + \log_8 x = \log_\lambda x$, for all positive real numbers x.
If $\lambda = \sqrt[b]{a}$, where $a, b \in N$, the smallest possible value of $(a+b)$ is equal to
(a) 12 (b) 63 (c) 65 (d) 75

3. If a, b and c are the three real solutions of the equation
$$x^{\log_{10}^2 x + \log_{10} x^3 + 3} = \frac{2}{\dfrac{1}{\sqrt{x+1}-1} - \dfrac{1}{\sqrt{x+1}+1}}$$
where, $a > b > c$, then a, b, c are in
(a) AP (b) GP
(c) HP (d) $a^{-1} + b^{-1} = c^{-1}$

4. If $f(n) = \prod_{i=2}^{n-1} \log_i (i+1)$, the value of $\sum_{k=1}^{100} f(2^k)$ equals
(a) 5010 (b) 5050 (c) 5100 (d) 5049

5. If $\log_3 27 \cdot \log_x 7 = \log_{27} x \cdot \log_7 3$, the least value of x, is
(a) 7^{-3} (b) 3^{-7} (c) 7^3 (d) 3^7

6. If $x = \log_5(1000)$ and $y = \log_7(2058)$, then
(a) $x > y$ (b) $x < y$
(c) $x = y$ (d) None of these

7. If $\log_5 120 + (x-3) - 2\log_5(1 - 5^{x-3}) = -\log_5(0.2 - 5^{x-4})$, then x is
(a) 1 (b) 2 (c) 3 (d) 4

8. If $x_n > x_{n-1} > \ldots > x_2 > x_1 > 1$, the value of
$$\log_{x_1} \log_{x_2} \log_{x_3} \ldots \log_{x_n} x_n^{x_{n-1}^{\cdot^{\cdot^{x_1}}}} \text{ is}$$
(a) 0 (b) 1
(c) 2 (d) undefined

9. If $\dfrac{x(y+z-x)}{\log x} = \dfrac{y(z+x-y)}{\log y} = \dfrac{z(x+y-z)}{\log z}$,
then $x^y y^x = z^y y^z$ is equal to
(a) $z^x x^z$ (b) $x^z y^x$ (c) $x^y y^z$ (d) $x^x y^y$

10. If $y = a^{\frac{1}{1-\log_a x}}$ and $z = a^{\frac{1}{1-\log_a y}}$, then x is equal to
(a) $a^{\frac{1}{1+\log_a z}}$ (b) $a^{\frac{1}{2+\log_a z}}$ (c) $a^{\frac{1}{1-\log_a z}}$ (d) $a^{\frac{1}{2-\log_a z}}$

11. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval
(a) $(-\infty, 1)$
(b) $(1, 2)$
(c) $(2, \infty)$
(d) None of the above

12. The value of $a^x - b^y$ is (where $x = \sqrt{\log_a b}$ and $y = \sqrt{\log_b a}$, $a > 0$, $b > 0$ and $a, b \neq 1$)
(a) 1 (b) 2
(c) 0 (d) -1

13. If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$, $z = 1 + \log_c ab$, then $\dfrac{xyz}{xy + yz + zx}$ is equal to
(a) 0 (b) 1
(c) -1 (d) 2

14. The value of $a^{\frac{\log_b(\log_b N)}{\log_b a}}$ is
(a) $\log_a N$ (b) $\log_b N$
(c) $\log_N a$ (d) $\log_N b$

15. The value of $49^A + 5^B$, where $A = 1 - \log_7 2$ and $B = -\log_5 4$ is
(a) 10.5 (b) 11.5
(c) 12.5 (d) 13.5

16. The number of real values of the parameter λ for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} \lambda = 0$ with real coefficients will have exactly one solution is
(a) 1 (b) 2
(c) 3 (d) 4

17. The number of roots of the equation $x^{\log_x (x+3)^2} = 16$ is
(a) 1 (b) 0
(c) 2 (d) 4

18. The point on the graph $y = \log_2 \log_6 \{2^{\sqrt{(2x+1)}} + 4\}$, whose y-coordinate is 1 is
(a) $(1, 1)$ (b) $(6, 1)$
(c) $(8, 1)$ (d) $(12, 1)$

19. Given, $\log 2 = 0.301$ and $\log 3 = 0.477$, then the number of digits before decimal in $3^{12} \times 2^8$ is
(a) 7 (b) 8
(c) 9 (d) 11

20. The number of solution(s) for the equation $2\log_x a + \log_{ax} a + 3\log_{a^2 x} a = 0$, is
(a) one (b) two
(c) three (d) four

Logarithms and Their Properties Exercise 2 : More than One Correct Option Type Questions

- This section contains **9 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

21. If $x^{(\log_2 x)^2 - 6\log_2 x + 11} = 64$, then x is equal to

(a) 2 (b) 4 (c) 6 (d) 8

22. If $\log_\lambda x \cdot \log_5 \lambda = \log_x 5$, $\lambda \neq 1, \lambda > 0$, then x is equal to

(a) λ (b) 5 (c) $\frac{1}{5}$ (d) None of these

23. If $S = \{x : \sqrt{\log_x \sqrt{3x}}$, where $\log_3 x > -1\}$, then

(a) S is a finite set (b) $S \in \phi$

(c) $S \subset (0, \infty)$ (d) S properly contains $\left(\frac{1}{3}, \infty\right)$

24. If x satisfies $\log_2(9^{x-1} + 7) = 2 + \log_2(3^{x-1} + 1)$, then

(a) $x \in Q$

(b) $x \in N$

(c) $x \in \{x \in Q : x < 0\}$

(d) $x \in N_e$ (set of even natural numbers)

25. $\log_p \log_p \underbrace{\sqrt[p]{\sqrt[p]{\sqrt[p]{\ldots \sqrt[p]{p}}}}}_{n \text{ times}}$, $p > 0$ and $p \neq 1$ is equal to

(a) n (b) $-n$

(c) $\frac{1}{n}$ (d) $\log_{1/p}(p^n)$

26. If $\log_a x = \alpha$, $\log_b x = \beta$, $\log_c x = \gamma$ and $\log_d x = \delta$, $x \neq 1$ and $a, b, c, d \neq 0, > 1$, then $\log_{abcd} x$ equals

(a) $\leq \frac{\alpha + \beta + \gamma + \delta}{16}$ (b) $\geq \frac{\alpha + \beta + \gamma + \delta}{16}$

(c) $\frac{1}{\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}}$ (d) $\frac{1}{\alpha\beta\gamma\delta}$

27. If $\log_{10} 5 = a$ and $\log_{10} 3 = b$, then

(a) $\log_{10} 8 = 3(1 - a)$ (b) $\log_{40} 15 = \frac{(a + b)}{(3 - 2a)}$

(c) $\log_{243} 32 = \left(\frac{1 - a}{b}\right)$ (d) All of these

28. If x is a positive number different from 1, such that $\log_a x$, $\log_b x$ and $\log_c x$ are in AP, then

(a) $\log b = \frac{2(\log a)(\log c)}{(\log a + \log c)}$ (b) $b = \frac{a + c}{2}$

(c) $b = \sqrt{ac}$ (d) $c^2 = (ac)^{\log_a b}$

29. If $|a| < |b|$, $b - a < 1$ and a, b are the real roots of the equation $x^2 - |\alpha| x - |\beta| = 0$, the equation $\log_{|b|}\left|\frac{x}{a}\right| - 1 = 0$ has

(a) one root lying in interval $(-\infty, a)$

(b) one root lying in interval (b, ∞)

(c) one positive root

(d) one negative root

Logarithms and Their Properties Exercise 3 : Passage Based Questions

- This section contains **4 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Q. Nos. 30 to 32)

Let $\log_2 N = a_1 + b_1$, $\log_3 N = a_2 + b_2$ *and* $\log_5 N = a_3 + b_3$, *where* $a_1, a_2, a_3 \in I$ *and* $b_1, b_2, b_3 \in [0, 1)$.

30. If $a_1 = 5$ and $a_2 = 3$, the number of integral values of N is

(a) 16 (b) 32 (c) 48 (d) 64

31. If $a_1 = 6$, $a_2 = 4$ and $a_3 = 3$, the largest integral value of N is

(a) 124 (b) 63

(c) 624 (d) 127

32. If $a_1 = 6$, $a_2 = 4$ and $a_3 = 3$, the difference of largest and smallest integral values of N, is

(a) 2 (b) 8

(c) 14 (d) 20

Passage II

(Q. Nos. 33 to 35)

Let 'S' denotes the antilog of 0.5 to the base 256 and 'K' denotes the number of digits in 6^{10} *(given* $\log_{10} 2 = 0.301$, $\log_{10} 3 = 0.477$*) and G denotes the number of positive integers, which have the characteristic 2, when the base of logarithm is 3.*

33. The value of G is

(a) 18 (b) 24 (c) 30 (d) 36

34. The value of KG is

(a) 72 (b) 144 (c) 216 (d) 288

35. The value of SKG is
(a) 1440
(b) 17280
(c) 2016
(d) 2304

Passage III
(Q. Nos. 36 to 38)

Suppose 'U' denotes the number of digits in the number $(60)^{100}$ and 'M' denotes the number of cyphers after decimal, before a significant figure comes in $(8)^{-296}$. If the fraction U/M is expressed as rational number in the lowest term as p/q (given $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$).

36. The value of p is
(a) 1
(b) 2
(c) 3
(d) 4

37. The value of q is
(a) 5
(b) 2
(c) 3
(d) 4

38. The equation whose roots are p and q, is
(a) $x^2 - 3x + 2 = 0$
(b) $x^2 - 5x + 6 = 0$
(c) $x^2 - 7x + 12 = 0$
(d) $x^2 - 9x + 20 = 0$

Passage IV (Q. Nos. 39 to 41)

Let G, O, E and L be positive real numbers such that $\log(G \cdot L) + \log(G \cdot E) = 3, \log(E \cdot L) + \log(E \cdot O) = 4,$ $\log(O \cdot G) + \log(O \cdot L) = 5$ *(base of the log is 10).*

39. If the value of the product $(GOEL)$ is λ, the value of $\sqrt{\log\lambda\sqrt{\log\lambda\sqrt{\log\lambda \ldots}}}$ is
(a) 3
(b) 4
(c) 5
(d) 7

40. If the minimum value of $3G + 2L + 2O + E$ is $2^{\lambda}3^{\mu}5^{\nu}$, where λ, μ and ν are whole numbers, the value of $\sum(\lambda^{\mu} + \mu^{\lambda})$ is
(a) 7
(b) 13
(c) 19
(d) None of these

41. If $\log\left(\frac{G}{O}\right)$ and $\log\left(\frac{O}{E}\right)$ are the roots of the equation
(a) $x^2 + x = 0$
(b) $x^2 - x = 0$
(c) $x^2 - 2x + 3 = 0$
(d) $x^2 - 1 = 0$

Logarithms and Their Properties Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

42. If $x, y \in R^+$ and $\log_{10}(2x) + \log_{10} y = 2$ and $\log_{10} x^2 - \log_{10}(2y) = 4$ and $x + y = \frac{m}{n}$, where m and n are relative prime, the value of $m - 3n^6$ is

43. A line $x = \lambda$ intersects the graph of $y = \log_5 x$ and $y = \log_5(x + 4)$. The distance between the points of intersection is 0.5. Given $\lambda = a + \sqrt{b}$, where a and b are integers, the value of $(a + b)$ is

44. If the left hand side of the equation $a(b - c)x^2 + b(c - a)xy + c(a - b)y^2 = 0$ is a perfect square, the value of
$$\left\{\frac{\log(a + c) + \log(a - 2b + c)}{\log(a - c)}\right\}^2, (a, b, c \in R^+, a > c) \text{ is}$$

45. Number of integers satisfying the inequality $\left(\frac{1}{3}\right)^{\frac{|x+2|}{2-|x|}} > 9$ is

46. If $x > 2$ is a solution of the equation $|\log_{\sqrt{3}} x - 2| + |\log_3 x - 2| = 2$, then the value of x is

47. Number of integers satisfying the inequality $\log_2 \sqrt{x} - 2\log_{1/4}^2 x + 1 > 0$, is

48. The value of $b (> 0)$ for which the equation $2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2)$ has coincident roots, is

49. The value of $\dfrac{2^{\log_{2^{1/4}} 2} - 3^{\log_{27} 125} - 4}{7^{4\log_{49} 2} - 3}$ is

50. If x_1 and x_2 $(x_2 > x_1)$ are the integral solutions of the equation $(\log_5 x)^2 + \log_{5x}\left(\frac{5}{x}\right) = 1$, the value of $|x_2 - 4x_1|$ is

51. If $x = \log_\lambda a = \log_a b = \frac{1}{2}\log_b c$ and $\log_\lambda c = nx^{n+1}$, the value of n is

Logarithms and Their Properties Exercise 5 : Matching Type Questions

- This section contains **3 questions**. Questions 52 to 54 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

52.

	Column I		Column II
(A)	$\dfrac{\log_3 243}{\log_2 \sqrt{32}}$	(p)	positive integer
(B)	$\dfrac{2\log 6}{(\log 12 + \log 3)}$	(q)	negative integer
(C)	$\log_{1/3}\left(\dfrac{1}{9}\right)^{-2}$	(r)	rational but not integer
(D)	$\dfrac{\log_5 16 - \log_5 4}{\log_5 128}$	(s)	prime

53.

	Column I		Column II
(A)	The expression $\sqrt{\log_{0.5}^2 8}$ has the value equal to	(p)	1
(B)	The value of the expression $(\log_{10} 2)^3 + \log_{10} 8 \cdot \log_{10} 5 + (\log_{10} 5)^3 + 3$, is	(q)	2
(C)	Let $N = \log_2 15 \cdot \log_{1/6} 2 \cdot \log_3\left(\dfrac{1}{6}\right)$. The value of $[N]$ is (where $[\cdot]$ denotes the greatest integer function)	(r)	3
(D)	If $(52.6)^a = (0.00526)^b = 100$, the value of $\dfrac{1}{a} - \dfrac{1}{b}$ is	(s)	4

54.

	Column I		Column II
(A)	If $\log_{1/x}\left\{\dfrac{2(x-2)}{(x+1)(x-5)}\right\} \ge 1$, then x can belongs to	(p)	$\left(0, \dfrac{1}{3}\right]$
(B)	If $\log_3 x - \log_3^2 x \le \dfrac{3}{2}\log_{(1/2\sqrt{2})} 4$, then x can belongs to	(q)	$(1, 2]$
(C)	If $\log_{1/2}(4-x) \ge \log_{1/2} 2 - \log_{1/2}(x-1)$, then x belongs to	(r)	$[3, 4)$
(D)	Let α and β are the roots of the quadratic equation $(\lambda^2 - 3\lambda + 4)x^2 - 4(2\lambda - 1)x + 16 = 0$, if α and β satisfy the condition $\beta > 1 > \alpha$, then p can lie in	(s)	$(3, 8)$

Logarithms and Their Properties Exercise 6 : Statement I and II Type Questions

- **Directions** Question numbers 55 to 60 are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
  (c) Statement-1 is true, Statement-2 is false
  (d) Statement-1 is false, Statement-2 is true

55. Statement-1 $\log_{10} x < \log_3 x < \log_e x < \log_2 x$ $(x > 0, x \ne 1)$.

Statement-2 If $0 < x < 1$, then $\log_x a > \log_x b \Rightarrow 0 < a < b$.

56. Statement-1 The equation $7^{\log_7(x^3+1)} - x^2 = 1$ has two distinct real roots.

Statement-2 $a^{\log_a N} = N$, where $a > 0, a \ne 1$ and $N > 0$.

57. Statement-1 $\left(\dfrac{1}{3}\right)^7 < \left(\dfrac{1}{3}\right)^4$

$$\Rightarrow \quad 7\log\left(\frac{1}{3}\right) < 4\log\left(\frac{1}{3}\right) \Rightarrow 7 < 4$$

Statement-2 If $ax < ay$, where $a < 0$, $x, y > 0$, then $x > y$.

58. Statement-1 The equation $x^{\log_x (1-x)^2} = 9$ has two distinct real solutions.

Statement-2 $a^{\log_a b} = b$, when $a > 0, a \ne 1, b > 0$.

59. Statement-1 The equation $(\log x)^2 + \log x^2 - 3 = 0$ has two distinct solutions.

Statement-2 $\log x^2 = 2\log x$.

60. Statement-1 $\log_x 3 \cdot \log_{x/9} 3 = \log_{81}(3)$ has a solution.
Statement-2 Change of base in logarithms is possible.

Logarithms and Their Properties Exercise 7 : Subjective Type Questions

- In this section, there are **27 subjective** questions.

61. (i) If $\log_7 12 = a$, $\log_{12} 24 = b$, then find value of $\log_{54} 168$ in terms of a and b.

(ii) If $\log_3 4 = a$, $\log_5 3 = b$, then find the value of $\log_3 10$ in terms of a and b.

62. If $\dfrac{\ln a}{b-c} = \dfrac{\ln b}{c-a} = \dfrac{\ln c}{a-b}$, prove the following.

(i) $abc = 1$

(ii) $a^a \cdot b^b \cdot c^c = 1$

(iii) $a^{b^2+bc+c^2} \cdot b^{c^2+ca+a^2} \cdot c^{a^2+ab+b^2} = 1$

(iv) $a + b + c \geq 3$

(v) $a^a + b^b + c^c \geq 3$

(vi) $a^{b^2+bc+c^2} + b^{c^2+ca+a^2} + c^{a^2+ab+b^2} \geq 3$

63. Prove that $\log_{10} 2$ lies between $\dfrac{1}{3}$ and $\dfrac{1}{4}$.

64. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the number of integers in

(i) 5^{200} (ii) 6^{20}

(iii) the number of zeroes after the decimal is 3^{-500}.

65. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the value of $\log(3.375)$.

66. Find the least value of $\log_2 x - \log_x(0.125)$ for $x > 1$.

67. Without using the tables, prove that

$$\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} > 2.$$

68. Solve the following equations.

(i) $x^{1+\log_{10} x} = 10x$

(ii) $\log_2(9 + 2^x) = 3$

(iii) $2 \cdot x^{\log_4 3} + 3^{\log_4 x} = 27$

(iv) $\log_4 \log_3 \log_2 x = 0$

(v) $x^{\frac{\log_{10} x + 5}{3}} = 10^{5+\log_{10} x}$

(vi) $\log_3\left(\log_9 x + \dfrac{1}{2} + 9^x\right) = 2x$

(vii) $4^{\log_{10} x+1} - 6^{\log_{10} x} - 2 \cdot 3^{\log_{10} x^2+2} = 0$

(viii) $\dfrac{\log_{10}(x-3)}{\log_{10}(x^2-21)} = \dfrac{1}{2}$

(ix) $x^{\log_2 x+4} = 32$

(x) $\log_a x = x$, where $a = x^{\log_4 x}$

(xi) $\log_{\sqrt{2}\sin x}(1 + \cos x) = 2$

69. Find a rational number, which is 50 times its own logarithm to the base 10.

70. Find the value of the expression

$$\frac{2}{\log_4 (2000)^6} + \frac{3}{\log_5 (2000)^6}.$$

71. Find the value of x satisfying
$\log_a\{1 + \log_b\{1 + \log_c(1 + \log_p x)\}\} = 0$.

72. Find the value of $4^{5\log_{4\sqrt{2}}(3-\sqrt{6}) - 6\log_8(\sqrt{3}-\sqrt{2})}$.

73. Solve the following inequations.

(i) $\log_{(2x+3)} x^2 < 1$

(ii) $\log_{2x}(x^2 - 5x + 6) < 1$

(iii) $\log_2(2 - x) < \log_{1/2}(x + 1)$

(iv) $\log_{x^2}(x + 2) < 1$

(v) $3^{\log_3 \sqrt{(x-1)}} < 3^{\log_3(x-6)} + 3$

(vi) $\log_{1/2}(3x - 1)^2 < \log_{1/2}(x + 5)^2$

(vii) $\log_{10} x + 2 \leq \log_{10}^2 x$

(viii) $\log_{10}(x^2 - 2x - 2) \leq 0$

(ix) $\log_x\left(2x - \dfrac{3}{4}\right) > 2$

(x) $\log_{1/3} x < \log_{1/2} x$

(xi) $\log_{2x+3} x^2 < \log_{2x+3}(2x + 3)$

(xii) $\log_2^2 x + 3\log_2 x \geq \dfrac{5}{2}\log_{4\sqrt{2}} 16$

(xiii) $(x^2 + x + 1)^x < 1$

(xiv) $\log_{(3x^2+1)} 2 < \dfrac{1}{2}$

(xv) $x^{(\log_{10} x)^2 - 3\log_{10} x + 1} > 1000$

(xvi) $\log_4\{14 + \log_6(x^2 - 64)\} \leq 2$

(xvii) $\log_2(9 - 2^x) \leq 10^{\log_{10}(3-x)}$

(xviii) $\log_a\left(\dfrac{2x+3}{x}\right) \geq 0$ for

(a) $a > 1$, (b) $0 < a < 1$

(xix) $1 + \log_2(x - 1) \leq \log_{x-1} 4$

(xx) $\log_{5x+4}(x^2) \leq \log_{5x+4}(2x + 3)$

74. Solve $\sqrt{\log_x (ax)^{1/5} + \log_a (ax)^{1/5}} + \sqrt{\log_a\left(\dfrac{x}{a}\right)^{1/5} + \log_x\left(\dfrac{a}{x}\right)^{1/5}} = a$.

75. It is known that $x = 9$ is root of the equation,

$$\log_\pi (x^2 + 15a^2) - \log_\pi (a-2) = \log_\pi \frac{8ax}{a-2}$$

find the other roots of this equation.

76. Solve $\log_4(\log_3 x)+\log_{1/4}(\log_{1/3} y)=0$ and
$x^2+y^2=\frac{17}{4}$.

77. Find the real value(s) of x satisfying the equation $\log_{2x}(4x)+\log_{4x}(16x)=4$.

78. Find the sum and product of all possible values of x which makes the following statement true

$$\log_6 54+\log_x 16=\log_{\sqrt{2}} x-\log_{36}\left(\frac{4}{9}\right).$$

79. Solve the equation

$$\frac{3}{2}\log_4(x+2)^3+3=\log_4(4-x)^3+\log_4(x+6)^3.$$

80. Solve $\log_2(4^{x+1}+4)\cdot\log_2(4^x+1)=\log_{1/\sqrt{2}}\left(\frac{1}{\sqrt{8}}\right)$.

81. Solve the system of equations $2^{\sqrt{x}+\sqrt{y}}=256$ and
$\log_{10}\sqrt{xy}-\log_{10}\left(\frac{3}{2}\right)=1$.

82. Solve the system of equations
$\log_2 y=\log_4(xy-2)$, $\log_9 x^2+\log_3(x-y)=1$.

83. Find the solution set of the inequality

$$2\log_{1/4}(x+5)>\frac{9}{4}\log_{\frac{1}{3\sqrt{3}}}(9)+\log_{\sqrt{(x+5)}}(2).$$

84. Solve $\log_3(\sqrt{x}+|\sqrt{x}-1|)=\log_9(4\sqrt{x}-3+4|\sqrt{x}-1|)$

85. In the inequality

$$(\log_2 x)^4-\left(\log_{1/2}\frac{x^5}{4}\right)^2-20\log_2 x+148<0$$

holds true in (a, b), where $a, b\in N$. Find the value of $ab(a+b)$.

86. Find the value of x satisfying the equation

$$\sqrt{(\log_3\sqrt[3]{3x}+\log_x\sqrt[3]{3x})\cdot\log_3 x^3}+\sqrt{\left(\log_3\sqrt[3]{\left(\frac{x}{3}\right)}+\log_x\sqrt[3]{\left(\frac{3}{x}\right)}\right)\log_3 x^3}=2.$$

87. If P is the number of natural numbers whose logarithm to the base 10 have the characteristic P and Q is the number of natural numbers reciprocals of whose 3 logarithms to the base 10 have the characteristic $-q$, show that $\log_{10}P-\log_{10}Q=p-q+1$.

Logarithms and Their Properties Exercise 8 : Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

88. Let $a=\log_3\log_3 2$ and an integer k satisfying $1<2^{(-k+3^{-a})}<2$, then k equals to **[IIT-JEE 2008, 1.5M]**

(a) 0 (b) 1
(c) 2 (d) 3

89. Let (x_0, y_0) be solution of the following equations $(2x)^{\ln 2}=(3y)^{\ln 3}$ and $3^{\ln x}=2^{\ln y}$, then x_0 is **[IIT-JEE 2011, 3M]**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 6

90. The value of

$$6+\log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\cdots}}}\right)$$ is

[IIT-JEE 2012, 4M]

91. If $3^x=4^{x-1}$, then x equals **[JEE Advanced 2013, 3M]**

(a) $\frac{2\log_3 2}{2\log_3 2-1}$ (b) $\frac{2}{2-\log_2 3}$

(c) $\frac{1}{1-\log_4 3}$ (d) $\frac{2\log_2 3}{2\log_2 3-1}$

Answers

Exercise for Session 1

1. (a) 2. (b) 3. (b) 4. (b) 5. (b)

Exercise for Session 2

1. (b) 2. (b) 3. (d) 4. (c) 5. (c)

Exercise for Session 3

1. (c) 2. (c) 3. (d) 4. (a) 5. (c)

Chapter Exercises

1. (c) 2. (d) 3. (b) 4. (b) 5. (a) 6. (a)
7. (a) 8. (b) 9. (a) 10. (c) 11. (c) 12. (c)
13. (b) 14. (b) 15. (c) 16. (b) 17. (b) 18. (d)
19. (c) 20. (b)
21. (a, b, d) 22. (b, c) 23. (c, d) 24. (a, b) 25. (b, d)
26. (a, c) 27. (a, b, c, d) 28. (a, d) 29. (c, d)
30. (b) 31. (d) 32. (a) 33. (a) 34. (b) 35. (d)
36. (b) 37. (c) 38. (b) 39. (b) 40. (a) 41. (d)
42. (9) 43. (6) 44. (4) 45. (3) 46. (9) 47. (3)
48. (4) 49. (7) 50. (1) 51. (2)
52. (A) $\rightarrow$ (p, s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (r)
53. (A) $\rightarrow$ (r), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (q)
54. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q, r), (D) $\rightarrow$ (s)
55. (d) 56. (d) 57. (d) 58. (d) 59. (c) 60. (d)
61. (i) $\dfrac{ab+1}{a(8-5b)}$ (ii) $\dfrac{ab+2}{2b}$ 64. (i) 140 (ii) 16 (iii) 238 65. 0.528
66. $2\sqrt{3}$ 68. (i) $10, \dfrac{1}{10}$ (ii) $x \in \phi$
(iii) $x = 16$ (iv) $x = 8$ (v) $\{10^{-5}, 10^3\}$
(vi) $x = \dfrac{1}{3}$ (vii) $x = \dfrac{1}{100}$ (viii) $x = 5$ (ix) $x = 2$ or $\dfrac{1}{32}$
(x) $x = 2$ (xi) $x = \dfrac{\pi}{3}$

69. 100 70. $\dfrac{1}{6}$ 71. 1 72. 9
73. (i) $x \in \left(-\dfrac{3}{2}, 3\right) \cup \{-1, 0\}$ (ii) $x \in \left(0, \dfrac{1}{2}\right) \cup (1, 2) \cup (3, 6)$
(iii) $x \in \left(-1, \dfrac{1-\sqrt{5}}{2}\right) \cup \left(\dfrac{1+\sqrt{5}}{2}, 2\right)$
(iv) $x \in (-2, 1) \cup (2, \infty) \sim \{-1, 0\}$ (v) $x > 6$
(vi) $x \in (-\infty, -5) \cup (-5, -1) \cup (3, \infty)$
(vii) $x \in (0, 10^{-1}] \cup [10^2, \infty)$
(viii) $x \in [-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$
(ix) $x \in \left(\dfrac{3}{8}, \dfrac{1}{2}\right) \cup \left(1, \dfrac{3}{2}\right)$ (x) $x \in (0, 1)$
(xi) $x \in \left(-\dfrac{3}{2}, -1\right) \cup (-1, 3)$ (xii) $x \in \left(0, \dfrac{1}{16}\right] \cup [2, \infty)$
(xiii) $x \in (-\infty, -1)$ (xiv) $x \in (-\infty, -1) \cup (1, \infty)$
(xv) $x \in (1000, \infty)$ (xvi) $x \in [-10, -8) \cup (8, 10]$
(xvii) $x \in (-\infty, 0]$
(xviii) (a) $x \in (-\infty, -3] \cup (0, \infty)$ (b) $x \in \left[-3, -\dfrac{3}{2}\right)$
(xix) $x \in (2, 3]$ (xx) $x \in \left(-\dfrac{3}{5}, -\dfrac{3}{2}\right) \cup [-1, 0) \cup (0, 3]$
74. $x = a^{4/5_{a2}}$ 75. $x = 15$ for $a = 3$
76. $x = 2$ or $\dfrac{1}{2}$, $y = \dfrac{1}{2}$ or 2
77. $x = 1, 2^{-3/2}$ 78. Sum $= \dfrac{9}{2}$, Product $= 2$
79. $x = 2$ 80. $x = 0$ 81. (9, 25) and (25, 9)
82. $x = 3,\ y = 2$ 83. $x \in (-5, -4) \cup (-3, -1)$
84. $x = \dfrac{25}{64}$ 85. 3456 86. $x \in (1, 3]$ 88. (b)
89. (c) 90. (4) 91. (a, b, c)