

INVERSE TRIGONOMETRIC FUNCTION

1. INTRODUCTION :

The inverse trigonometric functions, denoted by $\sin^{-1}x$ or $(\arcsin x)$, $\cos^{-1}x$ etc., denote the angles whose sine, cosine etc, is equal to x . The angles are usually the numerically smallest angles, except in the case of $\cot^{-1}x$, and if positive & negative angles have same numerical value, the positive angle has been chosen.

It is worthwhile noting that the functions $\sin x$, $\cos x$ etc are in general not invertible. Their inverse is defined by choosing an appropriate domain & co-domain so that they become invertible. For this reason the chosen value is usually the simplest and easy to remember.

2. DOMAIN & RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS :

S.No	$f(x)$	Domain	Range
(1)	$\sin^{-1}x$	$ x \leq 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(2)	$\cos^{-1}x$	$ x \leq 1$	$[0, \pi]$
(3)	$\tan^{-1}x$	$x \in \mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(4)	$\sec^{-1}x$	$ x \geq 1$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$ or $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
(5)	$\operatorname{cosec}^{-1}x$	$ x \geq 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(6)	$\cot^{-1}x$	$x \in \mathbb{R}$	$(0, \pi)$

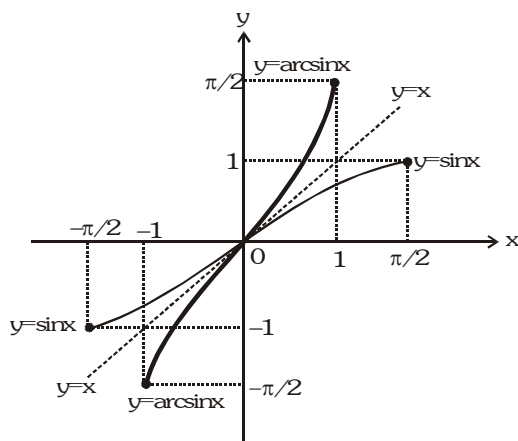
3. GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$

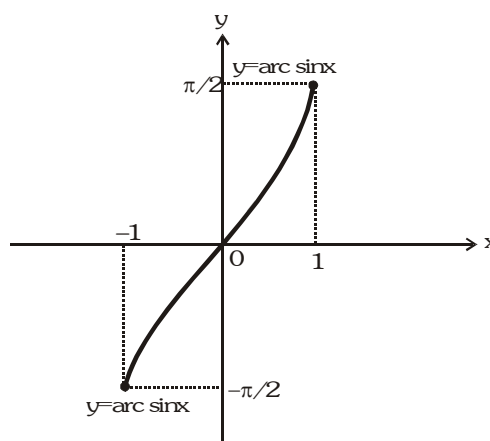
$f(x) = \sin x$

$f^{-1} : [-1, 1] \rightarrow [-\pi/2, \pi/2]$

$f^{-1}(x) = \sin^{-1}(x)$



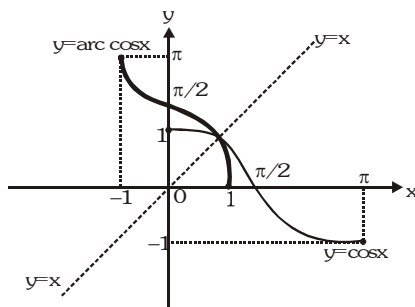
(Taking image of $\sin x$ about $y = x$ to get $\sin^{-1}x$)



$(y = \sin^{-1}x)$

(b) $f : [0, \pi] \rightarrow [-1, 1]$

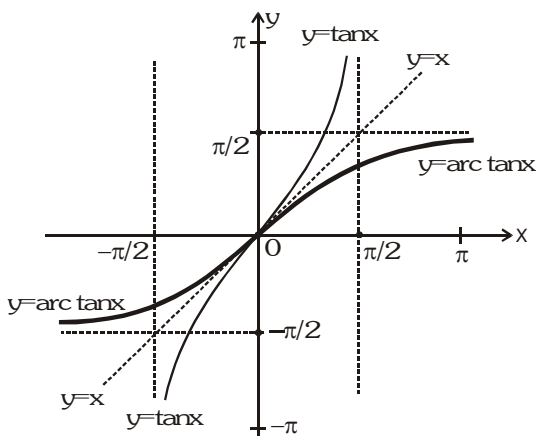
$f(x) = \cos x$



(Taking image of $\cos x$ about $y = x$)

(c) $f : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$

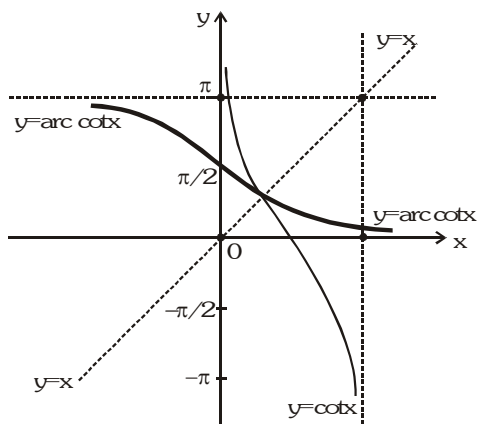
$f(x) = \tan x$



(Taking image of $\tan x$ about $y = x$)

(d) $f : (0, \pi) \rightarrow \mathbb{R}$

$f(x) = \cot x$



(Taking image of $\cot x$ about $y = x$)

(e) $f : [0, \pi/2) \cup (\pi/2, \pi] \rightarrow (-\infty, -1] \cup [1, \infty)$

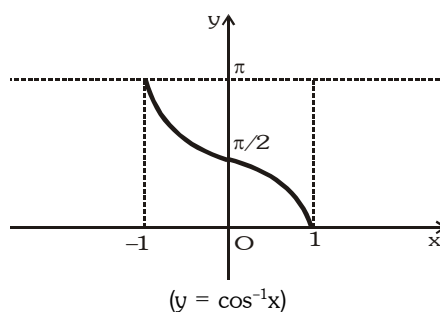
$f(x) = \sec x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow [0, \pi/2) \cup (\pi/2, \pi]$

$f^{-1}(x) = \sec^{-1} x$

$f^{-1} : [-1, 1] \rightarrow [0, \pi]$

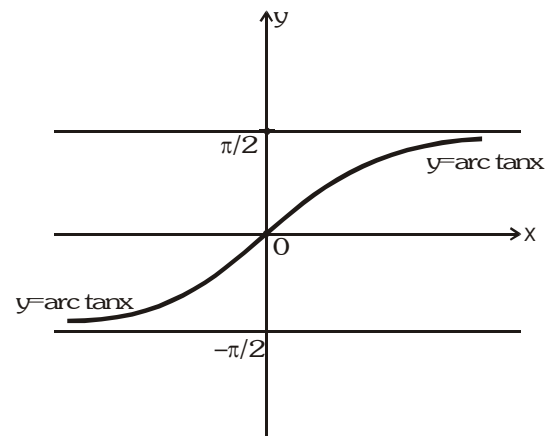
$f^{-1}(x) = \cos^{-1} x$



($y = \cos^{-1} x$)

$f^{-1} : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$

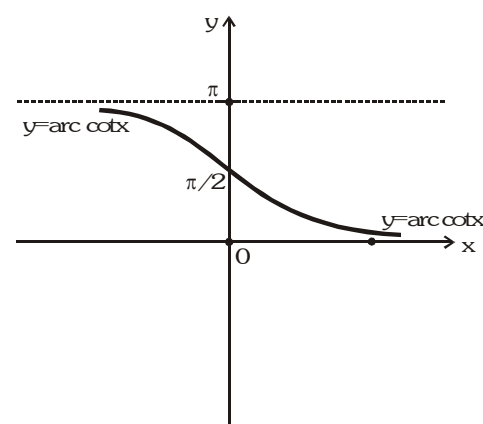
$f^{-1}(x) = \tan^{-1} x$



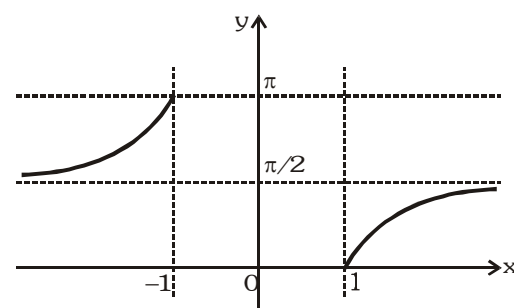
($y = \tan^{-1} x$)

$f^{-1} : \mathbb{R} \rightarrow (0, \pi)$

$f^{-1}(x) = \cot^{-1} x$



($y = \cot^{-1} x$)

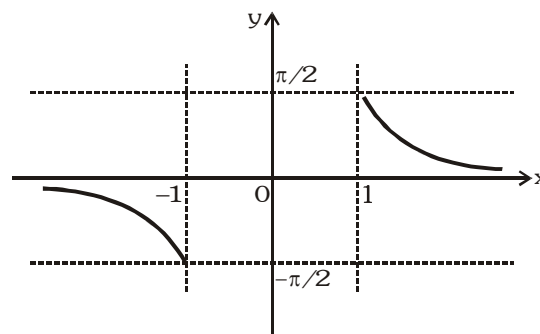


(f) $f : [-\pi/2, 0) \cup (0, \pi/2] \rightarrow (-\infty, -1] \cup [1, \infty)$

$f(x) = \operatorname{cosec} x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow [-\pi/2, 0) \cup (0, \pi/2]$

$f^{-1}(x) = \operatorname{cosec}^{-1} x$



From the above discussions following IMPORTANT points can be concluded.

- All the inverse trigonometric functions represent an angle.
- If $x \geq 0$, then all six inverse trigonometric functions viz $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\operatorname{cosec}^{-1} x$, $\cot^{-1} x$ represent an acute angle.
- If $x < 0$, then $\sin^{-1} x$, $\tan^{-1} x$ & $\operatorname{cosec}^{-1} x$ represent an angle from $-\pi/2$ to 0 (IVth quadrant)
- If $x < 0$, then $\cos^{-1} x$, $\cot^{-1} x$ & $\sec^{-1} x$ represent an obtuse angle. (IInd quadrant)
- IIIrd quadrant is never used in inverse trigonometric function.

Illustration 1 : The value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is equal to

(A) $\frac{\pi}{4}$

(B) $\frac{5\pi}{12}$

(C) $\frac{3\pi}{4}$

(D) $\frac{13\pi}{12}$

Solution : $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$

Ans. (C)

Illustration 2 : If $\sum_{i=1}^{2n} \cos^{-1} x_i = 0$ then find the value of $\sum_{i=1}^{2n} x_i$

Solution : We know, $0 \leq \cos^{-1} x \leq \pi$

Hence, each value $\cos^{-1} x_1, \cos^{-1} x_2, \cos^{-1} x_3, \dots, \cos^{-1} x_{2n}$ are non-negative their sum is zero only when each value is zero.

i.e., $\cos^{-1} x_i = 0$ for all i

$\Rightarrow x_i = 1$ for all i

$\therefore \sum_{i=1}^{2n} x_i = x_1 + x_2 + x_3 + \dots + x_{2n} = \underbrace{\{1 + 1 + 1 + \dots + 1\}}_{2n \text{ times}} = 2n$ { using (i) }

$\Rightarrow \sum_{i=1}^{2n} x_i = 2n$

Ans.

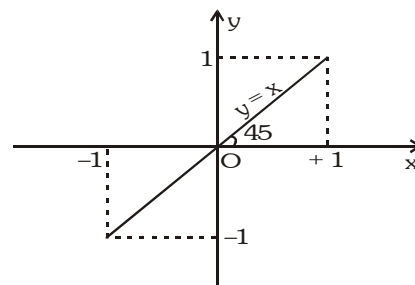
Do yourself - 1 :

- If α, β are roots of the equation $6x^2 + 11x + 3 = 0$, then
 (A) both $\cos^{-1} \alpha$ and $\cos^{-1} \beta$ are real (B) both $\operatorname{cosec}^{-1} \alpha$ and $\operatorname{cosec}^{-1} \beta$ are real
 (C) both $\cot^{-1} \alpha$ and $\cot^{-1} \beta$ are real (D) none of these
- If $\sin^{-1} x + \sin^{-1} y = \pi$ and $x = ky$, then find the value of $39^{2k} + 5^k$.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :

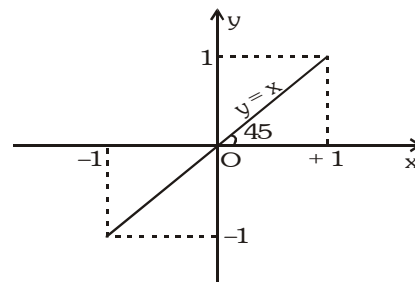
P-1 (i) $y = \sin (\sin^{-1} x) = x$

$x \in [-1,1], y \in [-1,1]$



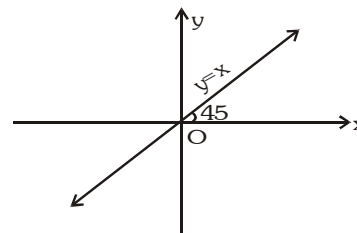
(ii) $y = \cos (\cos^{-1} x) = x$

$x \in [-1,1], y \in [-1,1]$



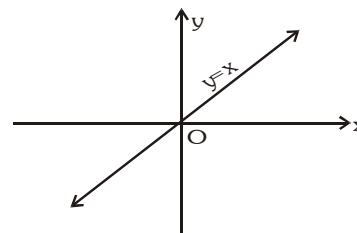
(iii) $y = \tan(\tan^{-1} x) = x$

$x \in \mathbb{R}, y \in \mathbb{R}$



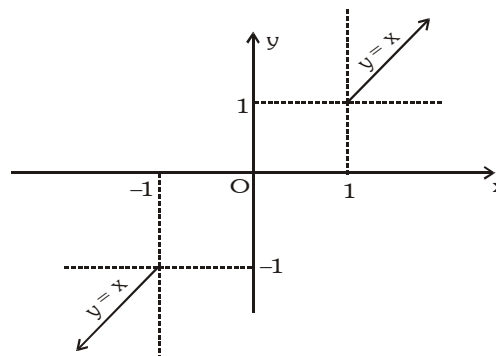
(iv) $y = \cot(\cot^{-1} x) = x,$

$x \in \mathbb{R}; y \in \mathbb{R}$



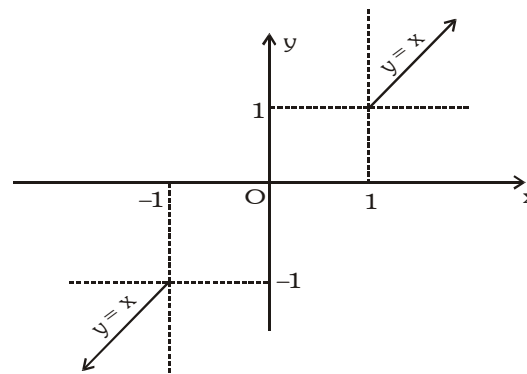
(v) $y = \operatorname{cosec} (\operatorname{cosec}^{-1} x) = x,$

$|x| \geq 1, |y| \geq 1$



(vi) $y = \sec(\sec^{-1} x) = x$

$|x| \geq 1 ; |y| \geq 1$



Note : All the above functions are aperiodic.

Illustration 3 : Evaluate the following :

(i) $\sin(\cos^{-1} 3/5)$ (ii) $\cos(\tan^{-1} 3/4)$ (iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Solution :

(i) Let $\cos^{-1} 3/5 = \theta$. Then,
 $\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5$
 $\therefore \sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$. Then,
 $\tan\theta = 3/4$
 $\Rightarrow \cos\theta = \frac{4}{5} \quad \left\{ \because \text{as } \cos^2\theta = \frac{1}{1 + \tan^2\theta} \right\}$
 $\therefore \cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

(iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

Ans.

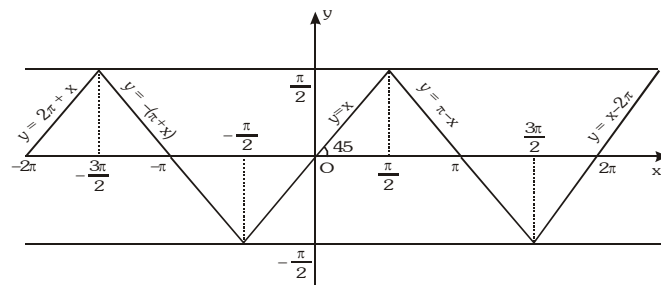
Do yourself - 2 :

Evaluate the following :

(i) $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right)$ (ii) $\sin\left(\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right)$ (iii) $\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$
 (iv) $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right)$ (v) $\cos\left(\sin^{-1}\frac{1}{2}\right)$ (vi) $\sin\left(\cos^{-1}\frac{3}{5}\right)$

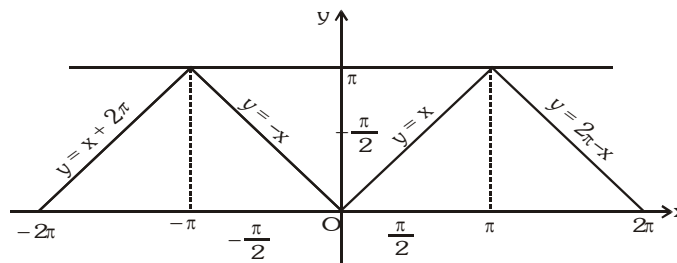
P-2 (i) $y = \sin^{-1}(\sin x)$, $x \in \mathbb{R}$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ periodic with period 2π and it is an odd function.

$$\sin^{-1}(\sin x) = \begin{cases} -\pi - x, & -\pi \leq x \leq -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$



(ii) $y = \cos^{-1}(\cos x)$, $x \in \mathbb{R}$, $y \in [0, \pi]$, periodic with period 2π and it is an even function.

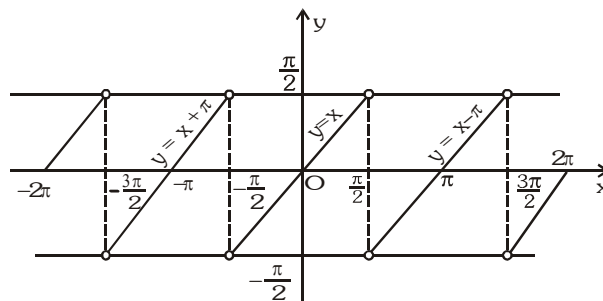
$$\cos^{-1}(\cos x) = \begin{cases} -x, & -\pi \leq x \leq 0 \\ x, & 0 < x \leq \pi \end{cases}$$



(iii) $y = \tan^{-1}(\tan x)$

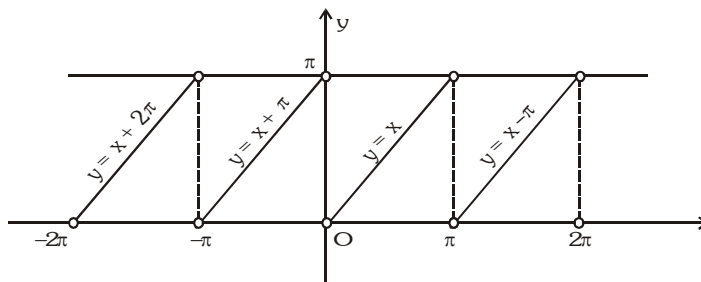
$x \in \mathbb{R} - \left\{(2n-1)\frac{\pi}{2}, n \in \mathbb{I}\right\}$; $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, periodic with period π and it is an odd function.

$$\tan^{-1}(\tan x) = \begin{cases} x + \pi, & -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x - \pi, & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

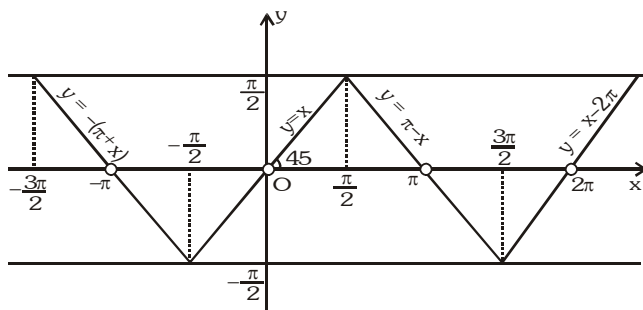


- (iv) $y = \cot^{-1}(\cot x)$, $x \in \mathbb{R} - \{n\pi, n \in \mathbb{I}\}$, $y \in (0, \pi)$, periodic with period π and neither even nor odd function.

$$\cot^{-1}(\cot x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \\ x - \pi, & \pi < x < 2\pi \end{cases}$$



- (v) $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$, $x \in \mathbb{R} - \{n\pi, n \in \mathbb{I}\}$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, is periodic with period 2π and it is an odd function.



- (vi) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π and it is an even function.

$$x \in \mathbb{R} - \left\{(2n-1)\frac{\pi}{2}, n \in \mathbb{I}\right\}, y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

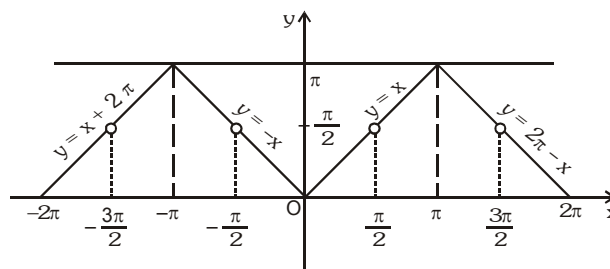


Illustration 4 : The value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$ is -

- (A) $5\pi/6$ (B) $\pi/2$ (C) $3\pi/2$ (D) none of these

Solution : $\sin^{-1}(-\sqrt{3}/2) = -\sin^{-1}(\sqrt{3}/2) = -\pi/3$

and $\cos^{-1}(\cos(7\pi/6)) = \cos^{-1}(\cos(2\pi - 5\pi/6)) = \cos^{-1}(\cos(5\pi/6)) = 5\pi/6$

Hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6)) = -\frac{\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{2}$

Ans.(B)

Illustration 5 : Evaluate the following :

$$(i) \quad \sin^{-1}(\sin \pi/4) \qquad (ii) \quad \cos^{-1}\left(\cos \frac{7\pi}{6}\right)$$

Solution : (i) $\sin^{-1}(\sin \pi/4) = \frac{\pi}{4}$

(ii) $\cos^{-1}\left(\cos \frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$, because $\frac{7\pi}{6}$ does not lie between 0 and π .

$$\begin{aligned} \text{Now, } \cos^{-1}\left(\cos \frac{7\pi}{6}\right) &= \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right) \qquad \left\{ \because \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6} \right\} \\ &= \cos^{-1}\left(\cos \frac{5\pi}{6}\right) = \frac{5\pi}{6} \end{aligned}$$

Ans.

Illustration 6 : Evaluate the following :

$$(i) \quad \sin^{-1}(\sin 10) \qquad (ii) \quad \tan^{-1}(\tan (-6)) \qquad (iii) \quad \cot^{-1}(\cot 4)$$

Solution : (i) We know that $\sin^{-1}(\sin \theta) = \theta$, if $-\pi/2 \leq \theta \leq \pi/2$

Here, $\theta = 10$ radians which does not lie between $-\pi/2$ and $\pi/2$

But, $3\pi - \theta$ i.e., $3\pi - 10$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

$$\text{Also, } \sin(3\pi - 10) = \sin 10$$

$$\therefore \sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = (3\pi - 10)$$

(ii) We know that,

$\tan^{-1}(\tan \theta) = \theta$, if $-\pi/2 < \theta < \pi/2$. Here, $\theta = -6$, radians which does not lie between $-\pi/2$ and $\pi/2$. We find that $2\pi - 6$ lies between $-\pi/2$ and $\pi/2$ such that;

$$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$$

$$\therefore \tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = (2\pi - 6)$$

(iii) $\cot^{-1}(\cot 4) = \cot^{-1}(\cot(\pi + (4 - \pi))) = \cot^{-1}(\cot(4 - \pi)) = (4 - \pi)$

Ans.

Illustration 7 : Prove that $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3) = 15$

Solution :

We have,

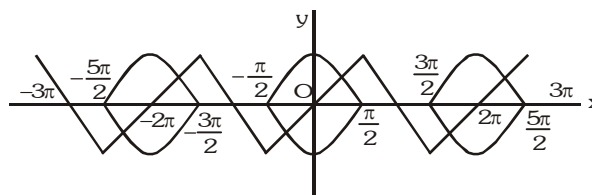
$$\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$$

$$= \left\{ \sec\left(\tan^{-1} 2\right) \right\}^2 + \left\{ \operatorname{cosec}\left(\cot^{-1} 3\right) \right\}^2 = \left\{ \sec\left(\tan^{-1} \frac{2}{1}\right) \right\}^2 + \left\{ \operatorname{cosec}\left(\cot^{-1} \frac{3}{1}\right) \right\}^2$$

$$= \left\{ \sec\left(\sec^{-1} \sqrt{5}\right) \right\}^2 + \left\{ \operatorname{cosec}\left(\operatorname{cosec}^{-1} \sqrt{10}\right) \right\}^2 = (\sqrt{5})^2 + (\sqrt{10})^2 = 15$$

Illustration 8 : Find the number of solutions of (x, y) which satisfy $|y| = \cos x$ and $y = \sin^{-1}(\sin x)$, where $|x| \leq 3\pi$.

Solution : Graphs of $y = \sin^{-1}(\sin x)$ and $|y| = \cos x$ meet exactly six times in $[-3\pi, 3\pi]$.



Do yourself - 3 :

Evaluate the following :

- (i) $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ (ii) $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$ (iii) $\sin^{-1}(\sin 2)$ (iv) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$
(v) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ (vi) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ (vii) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

P-3 (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad -1 \leq x \leq 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \quad x \in \mathbb{R}$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2} \quad |x| \geq 1$

P-4 (i) $\sin^{-1}(-x) = -\sin^{-1} x \quad , \quad -1 \leq x \leq 1$

(ii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, x \leq -1 \text{ or } x \geq 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x \quad , \quad x \in \mathbb{R}$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x \quad , \quad x \in \mathbb{R}$

(v) $\cos^{-1}(-x) = \pi - \cos^{-1} x \quad , \quad -1 \leq x \leq 1$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x \quad , \quad x \leq -1 \text{ or } x \geq 1$

P-5 (i) $\operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x} ; \quad x \leq -1, x \geq 1$

(ii) $\sec^{-1} x = \cos^{-1} \frac{1}{x} ; \quad x \leq -1, x \geq 1$

(iii) $\cot^{-1} x = \begin{cases} \tan^{-1} \frac{1}{x} & ; x > 0 \\ \pi + \tan^{-1} \frac{1}{x} & ; x < 0 \end{cases}$

Illustration 9 : Prove that $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \begin{cases} \pi/2 & , \text{ if } x > 0 \\ -\pi/2 & , \text{ if } x < 0 \end{cases}$

Solution : We have , $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & , \text{ for } x > 0 \\ -\pi + \cot^{-1} x & , \text{ for } x < 0 \end{cases}$

$\therefore \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \tan^{-1} x + \cot^{-1} x = \pi/2 & , \text{ if } x > 0 \\ \tan^{-1} x + \cot^{-1} x - \pi = \pi/2 - \pi = -\pi/2 & , \text{ if } x < 0 \end{cases}$

Do yourself - 4 :

(i) Prove the following :

(a) $\cos^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{12}{5}\right)$ (b) $\sin^{-1}\left(-\frac{4}{5}\right) = \tan^{-1}\left(-\frac{4}{3}\right) = \cos^{-1}\left(-\frac{3}{5}\right) - \pi$

(ii) Find the value of $\sin(\tan^{-1} a + \tan^{-1} \frac{1}{a})$; $a \neq 0$

P-6 (i) (a) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \frac{x+y}{1-xy} & \text{where } x > 0, y > 0 \text{ \& } xy < 1 \\ \pi + \tan^{-1} \frac{x+y}{1-xy} & \text{where } x > 0, y > 0 \text{ \& } xy > 1 \\ \frac{\pi}{2}, & \text{where } x > 0, y > 0 \text{ \& } xy = 1 \end{cases}$

(b) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$ where $x > 0, y > 0$

(c) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \left[\frac{x+y+z-xyz}{1-xy-yz-zx} \right]$ if $x > 0, y > 0, z > 0$ \& $xy + yz + zx < 1$

(ii) (a) $\sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1} [x\sqrt{1-y^2} + y\sqrt{1-x^2}] & \text{where } x > 0, y > 0 \text{ \& } (x^2 + y^2) \leq 1 \\ \pi - \sin^{-1} [x\sqrt{1-y^2} + y\sqrt{1-x^2}] & \text{where } x > 0, y > 0 \text{ \& } x^2 + y^2 > 1 \end{cases}$

(b) $\sin^{-1} x - \sin^{-1} y = \sin^{-1} [x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ where $x > 0, y > 0$

(iii) (a) $\cos^{-1} x + \cos^{-1} y = \cos^{-1} [xy - \sqrt{1-x^2}\sqrt{1-y^2}]$ where $x > 0, y > 0$

(b) $\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1} (xy + \sqrt{1-x^2}\sqrt{1-y^2}) & ; x < y, x, y > 0 \\ -\cos^{-1} (xy + \sqrt{1-x^2}\sqrt{1-y^2}) & ; x > y, x, y > 0 \end{cases}$

Note : In the above results x \& y are taken positive. In case if these are given as negative, we first apply P-4 and then use above results.

Illustration 10 : Prove that

(i) $\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{13} = \tan^{-1} \frac{2}{9}$ (ii) $\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$

Solution : (i) L.H.S. = $\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{13}$

$$= \tan^{-1} \left\{ \frac{\frac{1}{7} + \frac{1}{13}}{1 - \frac{1}{7} \times \frac{1}{13}} \right\} \quad \left\{ \because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right); \text{ if } xy < 1 \right\}$$

$$= \tan^{-1} \left(\frac{20}{90} \right) = \tan^{-1} \left(\frac{2}{9} \right) = \text{R.H.S.}$$

(ii) $\left(\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} \right) + \left(\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} \right)$

$$= \tan^{-1} \left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \times \frac{1}{7}} \right) + \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \times \frac{1}{8}} \right) = \tan^{-1} \left(\frac{6}{17} \right) + \tan^{-1} \left(\frac{11}{23} \right)$$

$$= \tan^{-1} \left(\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}} \right) = \tan^{-1} \left(\frac{325}{325} \right) = \tan^{-1} (1) = \frac{\pi}{4}$$

Ans.

Illustration 11 : Prove that $\sin^{-1} \frac{12}{13} + \cot^{-1} \frac{4}{3} + \tan^{-1} \frac{63}{16} = \pi$

Solution :

We have,

$$\begin{aligned} & \sin^{-1} \frac{12}{13} + \cos^{-1} \frac{4}{5} + \tan^{-1} \frac{63}{16} \\ &= \tan^{-1} \frac{12}{5} + \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{63}{16} \quad \left[\because \sin^{-1} \frac{12}{13} = \tan^{-1} \frac{12}{5} \text{ and } \cos^{-1} \frac{4}{5} = \tan^{-1} \frac{3}{4} \right] \\ &= \pi + \tan^{-1} \left\{ \frac{\frac{12}{5} + \frac{3}{4}}{1 - \frac{12}{5} \times \frac{3}{4}} \right\} + \tan^{-1} \frac{63}{16} \quad \left[\because \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), \text{ if } xy > 1 \right] \\ &= \pi + \tan^{-1} \left(\frac{63}{-16} \right) + \tan^{-1} \left(\frac{63}{16} \right) \\ &= \pi - \tan^{-1} \frac{63}{16} + \tan^{-1} \frac{63}{16} \quad \left[\because \tan^{-1}(-x) = -\tan^{-1} x \right] \\ &= \pi \end{aligned}$$

Illustration 12 : Prove that : $\cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$

Solution :

We have, $\cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{5}{13} + \sin^{-1} \frac{3}{5}$ $\left[\because \cos^{-1} \frac{12}{13} = \sin^{-1} \frac{5}{13} \right]$

$$= \sin^{-1} \left\{ \frac{5}{13} \times \sqrt{1 - \left(\frac{3}{5} \right)^2} + \frac{3}{5} \times \sqrt{1 - \left(\frac{5}{13} \right)^2} \right\} = \sin^{-1} \left\{ \frac{5}{13} \times \frac{4}{5} + \frac{3}{5} \times \frac{12}{13} \right\} = \sin^{-1} \frac{56}{65}$$

Illustration 13 : If $x = \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a))))))$ and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1}a))))))$, where $a \in [0, 1]$. Find the relationship between x and y in terms of 'a'

Solution :

Here, $x = \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a))))))$ $\left\{ \begin{array}{l} \text{Let } \sin \theta = a \Rightarrow \sec \theta = \frac{1}{\sqrt{1-a^2}} \end{array} \right.$

$$= \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec \theta))))$$

$$\Rightarrow x = \operatorname{cosec} \left(\tan^{-1} \left(\cos \left(\cot^{-1} \left(\frac{1}{\sqrt{1-a^2}} \right) \right) \right) \right) \quad \left\{ \begin{array}{l} \text{Let } \cot^{-1} \left(\frac{1}{\sqrt{1-a^2}} \right) = \phi \Rightarrow \cot \phi = \frac{1}{\sqrt{1-a^2}} \\ \text{therefore } \cos \phi = \frac{1}{\sqrt{2-a^2}} \end{array} \right.$$

$$= \operatorname{cosec}(\tan^{-1}(\cos \phi))$$

$$\Rightarrow x = \operatorname{cosec} \left(\tan^{-1} \left(\frac{1}{\sqrt{2-a^2}} \right) \right) \quad \left\{ \begin{array}{l} \text{Let, } \tan^{-1} \left(\frac{1}{\sqrt{2-a^2}} \right) = \psi \Rightarrow \tan \psi = \frac{1}{\sqrt{2-a^2}} \\ \text{therefore } \operatorname{cosec} \psi = \sqrt{3-a^2} \end{array} \right.$$

$$= \operatorname{cosec} \psi$$

$$\Rightarrow x = \sqrt{3-a^2} \quad \dots (i)$$

$$\begin{aligned}
 &\text{and } y = \sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1} a)))))) \quad \left\{ \begin{array}{l} \text{Let } \cos^{-1} a = \alpha \Rightarrow \cos \alpha = a \Rightarrow \operatorname{cosec} \alpha = \frac{1}{\sqrt{1-a^2}} \\ \Rightarrow y = \sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec} \alpha)))) \\ \Rightarrow y = \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right) \quad \left\{ \begin{array}{l} \text{Let, } \tan^{-1} \frac{1}{\sqrt{1-a^2}} = \beta \Rightarrow \tan \beta = \frac{1}{\sqrt{1-a^2}} \\ \Rightarrow \sin \beta = \frac{1}{\sqrt{2-a^2}} \\ \Rightarrow y = \sec\left(\cot^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right) \quad \left\{ \begin{array}{l} \text{Let } \cot^{-1} \frac{1}{\sqrt{2-a^2}} = \gamma \Rightarrow \cot \gamma = \frac{1}{\sqrt{2-a^2}} \Rightarrow \sec \gamma = \sqrt{3-a^2} \\ \Rightarrow y = \sec \gamma \\ \Rightarrow y = \sqrt{3-a^2} \end{array} \right. \\ \text{from (i) and (ii), } x = y = \sqrt{3-a^2} . \end{array} \right. \end{array} \right.
 \end{aligned}$$

Ans.

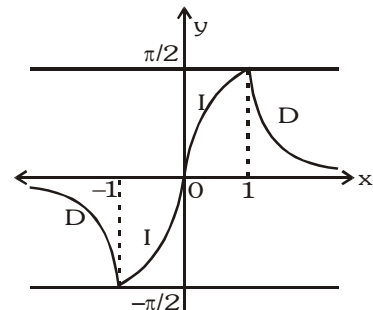
Do yourself - 5 :

Prove the following :

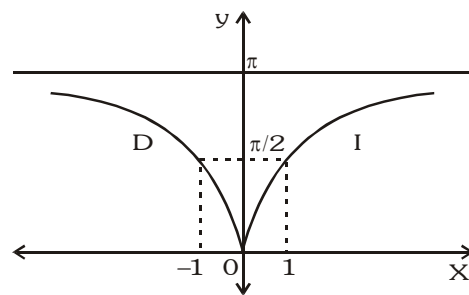
$$\begin{aligned}
 \text{(i)} \quad &\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \cos^{-1}\left(\frac{36}{85}\right) & \text{(ii)} \quad &\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4} \\
 \text{(iii)} \quad &\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left(\frac{1}{2}\right)
 \end{aligned}$$

4. SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS :

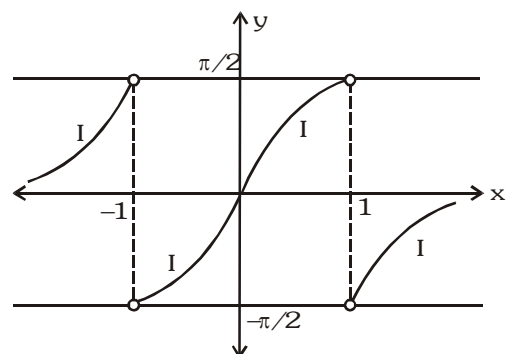
$$\text{(a)} \quad y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2 \tan^{-1} x & \text{if } |x| \leq 1 \\ \pi - 2 \tan^{-1} x & \text{if } x > 1 \\ -(\pi + 2 \tan^{-1} x) & \text{if } x < -1 \end{cases}$$



$$\text{(b)} \quad y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2 \tan^{-1} x & \text{if } x \geq 0 \\ -2 \tan^{-1} x & \text{if } x < 0 \end{cases}$$

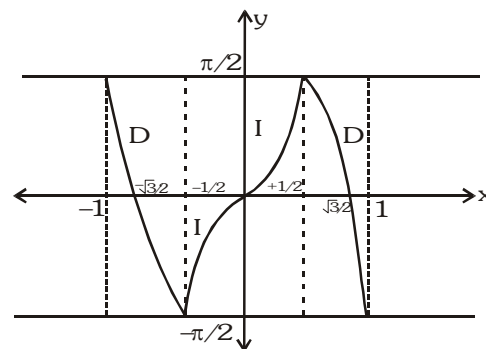


$$\text{(c)} \quad y = f(x) = \tan^{-1} \frac{2x}{1-x^2} = \begin{cases} 2 \tan^{-1} x & \text{if } |x| < 1 \\ \pi + 2 \tan^{-1} x & \text{if } x < -1 \\ -(\pi - 2 \tan^{-1} x) & \text{if } x > 1 \end{cases}$$



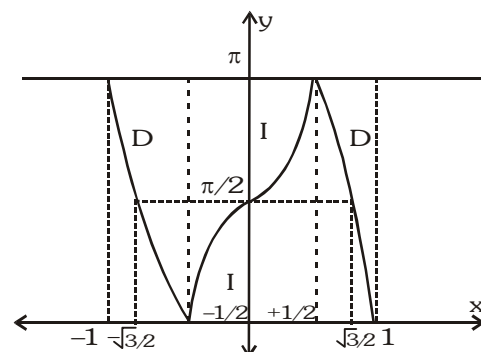
(d) $y = f(x) = \sin^{-1}(3x - 4x^3)$

$$= \begin{cases} -(\pi + 3\sin^{-1} x) & \text{if } -1 \leq x \leq -\frac{1}{2} \\ 3\sin^{-1} x & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \pi - 3\sin^{-1} x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

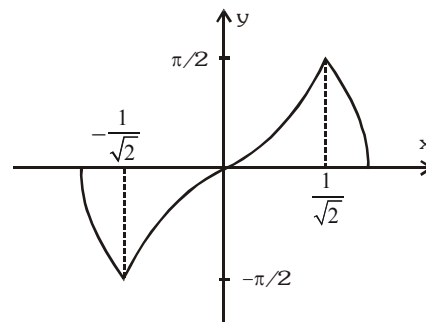


(e) $y = f(x) = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1} x - 2\pi & \text{if } -1 \leq x \leq -\frac{1}{2} \\ 2\pi - 3\cos^{-1} x & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 3\cos^{-1} x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$



(f) $\sin^{-1}(2x\sqrt{1-x^2}) = \begin{cases} -(\pi + 2\sin^{-1} x) & -1 \leq x \leq -\frac{1}{\sqrt{2}} \\ 2\sin^{-1} x & -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1} x & \frac{1}{\sqrt{2}} \leq x \leq 1 \end{cases}$



(g) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1} x & 0 \leq x \leq 1 \\ 2\pi - 2\cos^{-1} x & -1 \leq x \leq 0 \end{cases}$

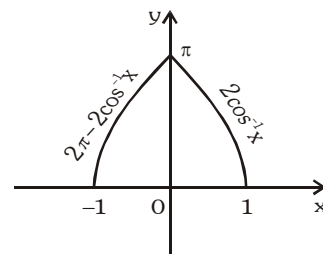


Illustration 14 : Evaluate : (i) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$ (ii) $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$

Solution : (i) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\} = \tan\left\{\tan^{-1}\left(\frac{2 \times \frac{1}{5}}{1 - \frac{1}{25}}\right) - \tan^{-1}1\right\} \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } |x| < 1\right]$

$$= \tan\left\{\tan^{-1}\frac{5}{12} - \tan^{-1}1\right\} = \tan\left\{\tan^{-1}\left(\frac{\frac{5}{12} - 1}{1 + \frac{5}{12}}\right)\right\} = \tan\left\{\tan^{-1}\left(\frac{-7}{17}\right)\right\} = \frac{-7}{17}$$

(ii) Let $\cos^{-1} \frac{\sqrt{5}}{3} = \theta$. Then, $\cos \theta = \frac{\sqrt{5}}{3}$, $0 < \theta < \pi/2$

Now, $\tan \left(\frac{1}{2} \cos^{-1} \frac{\sqrt{5}}{3} \right)$

$$= \tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \sqrt{\frac{1 - \frac{\sqrt{5}}{3}}{1 + \frac{\sqrt{5}}{3}}} = \sqrt{\frac{3 - \sqrt{5}}{3 + \sqrt{5}}} = \sqrt{\frac{(3 - \sqrt{5})^2}{(3 + \sqrt{5})(3 - \sqrt{5})}} = \sqrt{\frac{(3 - \sqrt{5})^2}{9 - 5}} = \frac{3 - \sqrt{5}}{2}$$

Illustration 15 : Prove that : $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

Solution : We have, $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7}$

$$= \tan^{-1} \left\{ \frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2} \right)^2} \right\} + \tan^{-1} \frac{1}{7} \quad \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1 - x^2} \right), \text{ if } -1 < x < 1 \right]$$

$$= \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{1}{7} = \tan^{-1} \left\{ \frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \times \frac{1}{7}} \right\} = \tan^{-1} \frac{31}{17}$$

Illustration 16 : Prove that $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right)$, $x \in [0, 1]$

Solution : We have, $\frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} \cos^{-1} \left\{ \frac{1 - (\sqrt{x})^2}{1 + (\sqrt{x})^2} \right\} = \frac{1}{2} \times 2 \tan^{-1} \sqrt{x} = \tan^{-1} \sqrt{x}$.

Alter : Putting $\sqrt{x} = \tan \theta$, we have $\Rightarrow \theta \in \left[0, \frac{\pi}{4} \right]$

$$\text{RHS} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) = \frac{1}{2} \cos^{-1} (\cos 2\theta) = \theta \quad \because \left(2\theta \in \left[0, \frac{\pi}{2} \right] \right)$$

$$= \tan^{-1} \sqrt{x} = \text{LHS}$$

Illustration 17 : Prove that : (i) $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99} = \frac{\pi}{4}$

(ii) $2 \tan^{-1} \frac{1}{5} + \sec^{-1} \frac{5\sqrt{2}}{7} + 2 \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$

Solution : (i) $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99} = 2 \left\{ 2 \tan^{-1} \frac{1}{5} \right\} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99}$

$$\begin{aligned}
 &= 2 \left\{ \tan^{-1} \frac{2 \times 1/5}{1 - (1/5)^2} \right\} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99} \quad \left[\begin{array}{l} \because 2 \tan^{-1} x \\ = \tan^{-1} \frac{2x}{1-x^2}, \text{ if } |x| < 1 \end{array} \right] \\
 &= 2 \tan^{-1} \frac{5}{12} - \left\{ \tan^{-1} \frac{1}{70} - \tan^{-1} \frac{1}{99} \right\} = \tan^{-1} \left\{ \frac{2 \times 5/12}{1 - (5/12)^2} \right\} - \tan^{-1} \left\{ \frac{\frac{1}{70} - \frac{1}{99}}{1 + \frac{1}{70} \times \frac{1}{99}} \right\} \\
 &= \tan^{-1} \frac{120}{119} - \tan^{-1} \frac{29}{6931} = \tan^{-1} \frac{120}{119} - \tan^{-1} \frac{1}{239} = \tan^{-1} \left\{ \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}} \right\} = \tan^{-1} 1 = \frac{\pi}{4} \\
 \text{(ii)} \quad &2 \tan^{-1} \frac{1}{5} + \sec^{-1} \frac{5\sqrt{2}}{7} + 2 \tan^{-1} \frac{1}{8} = 2 \left\{ \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8} \right\} + \sec^{-1} \frac{5\sqrt{2}}{7} \\
 &= 2 \tan^{-1} \left\{ \frac{\frac{1}{5} + \frac{1}{8}}{1 - \frac{1}{5} \times \frac{1}{8}} \right\} + \tan^{-1} \sqrt{\left(\frac{5\sqrt{2}}{7} \right)^2 - 1} \quad \left[\because \sec^{-1} x = \tan^{-1} \sqrt{x^2 - 1} \right] \\
 &= 2 \tan^{-1} \frac{13}{39} + \tan^{-1} \frac{1}{7} = 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \left\{ \frac{2 \times 1/3}{1 - (1/3)^2} \right\} + \tan^{-1} \frac{1}{7} \quad \left[\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, \text{ if } |x| < 1 \right] \\
 &= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} = \tan^{-1} \left\{ \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \times \frac{1}{7}} \right\} = \tan^{-1} 1 = \frac{\pi}{4}
 \end{aligned}$$

Do yourself - 6 :

Prove the following results :

(i) $2 \tan^{-1} \left(\frac{1}{5} \right) + \tan^{-1} \left(\frac{1}{8} \right) = \tan^{-1} \left(\frac{4}{7} \right)$

(ii) $2 \sin^{-1} \left(\frac{3}{5} \right) - \tan^{-1} \left(\frac{17}{31} \right) = \frac{\pi}{4}$

6. EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS :

Illustration 18 : The equation $2 \cos^{-1} x + \sin^{-1} x = \frac{11\pi}{6}$ has

- (A) no solution (B) only one solution (C) two solutions (D) three solutions

Solution : Given equation is $2 \cos^{-1} x + \sin^{-1} x = \frac{11\pi}{6}$

$$\Rightarrow \cos^{-1} x + (\cos^{-1} x + \sin^{-1} x) = \frac{11\pi}{6} \Rightarrow \cos^{-1} x + \frac{\pi}{2} = \frac{11\pi}{6} \Rightarrow \cos^{-1} x = 4\pi / 3$$

which is not possible as $\cos^{-1} x \in [0, \pi]$

Ans. (A)

Illustration 19 : If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = 5\pi^2 / 8$, then x is equal to-

- (A) -1 (B) 0 (C) 1 (D) none of these

Solution : The given equation can be written as $(\tan^{-1} x + \cot^{-1} x)^2 - 2 \tan^{-1} x \cot^{-1} x = 5\pi^2 / 8$

Since $\tan^{-1} x + \cot^{-1} x = \pi/2$ we have

$$(\pi/2)^2 - 2 \tan^{-1} x (\pi/2 - \tan^{-1} x) = 5\pi^2 / 8$$

$$\Rightarrow 2(\tan^{-1} x)^2 - 2(\pi/2) \tan^{-1} x - 3\pi^2 / 8 = 0 \Rightarrow \tan^{-1} x = -\pi / 4 \Rightarrow x = -1 \quad \text{Ans. (A)}$$

Illustration 20 : Solve the equation : $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$

Solution : $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$
taking tangent on both sides

$$\Rightarrow \tan \left(\tan^{-1} \left(\frac{x-1}{x-2} \right) + \tan^{-1} \left(\frac{x+1}{x+2} \right) \right) = 1 \Rightarrow \frac{\tan \left(\tan^{-1} \left(\frac{x-1}{x-2} \right) \right) + \tan \left(\tan^{-1} \left(\frac{x+1}{x+2} \right) \right)}{1 - \tan \left(\tan^{-1} \left(\frac{x-1}{x-2} \right) \right) \tan \left(\tan^{-1} \left(\frac{x+1}{x+2} \right) \right)} = 1$$

$$\Rightarrow \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2} \cdot \frac{x+1}{x+2}} = 1 \Rightarrow \frac{(x-1)(x+2) + (x-2)(x+1)}{x^2 - 4 - (x^2 - 1)} = 1 \Rightarrow 2x^2 - 4 = -3 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

Now verify $x = \frac{1}{\sqrt{2}}$

$$= \tan^{-1} \left(\frac{\frac{1}{\sqrt{2}} - 1}{\frac{1}{\sqrt{2}} - 2} \right) + \tan^{-1} \left(\frac{\frac{1}{\sqrt{2}} + 1}{\frac{1}{\sqrt{2}} + 2} \right) = \tan^{-1} \left(\frac{\sqrt{2} - 1}{2\sqrt{2} - 1} \right) + \tan^{-1} \left(\frac{\sqrt{2} + 1}{2\sqrt{2} + 1} \right)$$

$$= \tan^{-1} \left(\frac{(2\sqrt{2} + 1)(\sqrt{2} - 1) + (2\sqrt{2} - 1)(\sqrt{2} + 1)}{(2\sqrt{2} - 1)(2\sqrt{2} + 1) - (\sqrt{2} - 1)(\sqrt{2} + 1)} \right) = \tan^{-1} \left(\frac{6}{6} \right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$x = -\frac{1}{\sqrt{2}}$$

$$= \tan^{-1} \left(\frac{-\frac{1}{\sqrt{2}} - 1}{-\frac{1}{\sqrt{2}} - 2} \right) + \tan^{-1} \left(\frac{-\frac{1}{\sqrt{2}} + 1}{-\frac{1}{\sqrt{2}} + 2} \right) = \tan^{-1} \left(\frac{\sqrt{2} + 1}{2\sqrt{2} + 1} \right) + \tan^{-1} \left(\frac{\sqrt{2} - 1}{2\sqrt{2} - 1} \right)$$

{same as above}

$$= \tan^{-1}(1) = \frac{\pi}{4}$$

$$\therefore x = \pm \frac{1}{\sqrt{2}} \text{ are solutions}$$

Ans.

Illustration 21 : Solve the equation : $2 \tan^{-1}(2x + 1) = \cos^{-1}x$.

Solution : Here, $2 \tan^{-1}(2x + 1) = \cos^{-1}x$

$$\text{or } \cos(2 \tan^{-1}(2x + 1)) = x \quad \left\{ \text{We know } \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right\}$$

$$\begin{aligned} \therefore \frac{1 - (2x + 1)^2}{1 + (2x + 1)^2} &= x \Rightarrow (1 - 2x - 1)(1 + 2x + 1) = x(4x^2 + 4x + 2) \\ \Rightarrow -2x \cdot 2(x + 1) &= 2x(2x^2 + 2x + 1) \Rightarrow 2x(2x^2 + 2x + 1 + 2x + 2) = 0 \\ \Rightarrow 2x(2x^2 + 4x + 3) &= 0 \\ \Rightarrow x = 0 \text{ or } 2x^2 + 4x + 3 &= 0 \quad \{\text{No solution}\} \end{aligned}$$

Verify $x = 0$

$$2\tan^{-1}(1) = \cos^{-1}(1) \Rightarrow \frac{\pi}{2} = \frac{\pi}{2}$$

$\therefore x = 0$ is only the solution

Ans.

Do yourself - 7 :

Solve the following equation for x :

(i) $\sin\left[\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}x\right] = 1$

(ii) $\cos^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$

(iii) $\cot^{-1}x - \cot^{-1}(x + 2) = \frac{\pi}{12}$, where $x > 0$.

7. INEQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTION :

Illustration 22 : Find the complete solution set of $\sin^{-1}(\sin 5) > x^2 - 4x$.

Solution :

$$\begin{aligned} \sin^{-1}(\sin 5) > x^2 - 4x &\Rightarrow \sin^{-1}[\sin(5 - 2\pi)] > x^2 - 4x \\ \Rightarrow x^2 - 4x < 5 - 2\pi &\Rightarrow x^2 - 4x + (2\pi - 5) < 0 \\ \Rightarrow 2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi} &\Rightarrow x \in (2 - \sqrt{9 - 2\pi}, 2 + \sqrt{9 - 2\pi}) \end{aligned}$$

Ans.

Illustration 23 : Find the complete solution set of $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \leq 0$, where $[.]$ denotes the greatest integer function.

Solution :

$$\begin{aligned} [\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 &\leq 0 \\ \Rightarrow ([\cot^{-1}x] - 3)^2 &\leq 0 \Rightarrow [\cot^{-1}x] = 3 \Rightarrow 3 \leq \cot^{-1}x < 4 \Rightarrow x \in (-\infty, \cot 3] \end{aligned}$$

Illustration 24 : If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in \mathbb{N}$, then the maximum value of n is -

- (A) 1 (B) 5 (C) 9 (D) none of these

Solution :

$$\begin{aligned} \cot^{-1}\frac{n}{\pi} &> \frac{\pi}{6} \\ \Rightarrow \cot\left(\cot^{-1}\left(\frac{n}{\pi}\right)\right) &< \cot\left(\frac{\pi}{6}\right) \Rightarrow \frac{n}{\pi} < \sqrt{3} \\ \Rightarrow n < \pi\sqrt{3} &\Rightarrow n < 5.5 \text{ (approx)} \\ \Rightarrow n = 5 &\because (n \in \mathbb{N}) \end{aligned}$$

Ans. (B)

Do yourself - 8 :

(i) Solve the inequality $\tan^{-1}x > \cot^{-1}x$.

(ii) Complete solution set of inequation $(\cos^{-1}x)^2 - (\sin^{-1}x)^2 > 0$, is

- (A) $\left[0, \frac{1}{\sqrt{2}}\right)$ (B) $\left[-1, \frac{1}{\sqrt{2}}\right)$ (C) $(-1, \sqrt{2})$ (D) none of these

8. SUMMATION OF SERIES :

Illustration 25 : Prove that :

$$\tan^{-1} \left(\frac{c_1 x - y}{c_1 y + x} \right) + \tan^{-1} \left(\frac{c_2 - c_1}{1 + c_2 c_1} \right) + \tan^{-1} \left(\frac{c_3 - c_2}{1 + c_3 c_2} \right) + \dots + \tan^{-1} \left(\frac{c_n - c_{n-1}}{1 + c_n c_{n-1}} \right) + \tan^{-1} \left(\frac{1}{c_n} \right) = \tan^{-1} \left(\frac{x}{y} \right)$$

Solution :

$$\begin{aligned} \text{L.H.S. } & \tan^{-1} \left(\frac{c_1 x - y}{c_1 y + x} \right) + \tan^{-1} \left(\frac{c_2 - c_1}{1 + c_2 c_1} \right) + \tan^{-1} \left(\frac{c_3 - c_2}{1 + c_3 c_2} \right) + \dots + \tan^{-1} \left(\frac{c_n - c_{n-1}}{1 + c_n c_{n-1}} \right) + \tan^{-1} \left(\frac{1}{c_n} \right) \\ &= \tan^{-1} \left(\frac{\frac{x}{y} - \frac{1}{c_1}}{1 + \frac{x}{y} \cdot \frac{1}{c_1}} \right) + \tan^{-1} \left(\frac{\frac{1}{c_1} - \frac{1}{c_2}}{1 + \frac{1}{c_1} \cdot \frac{1}{c_2}} \right) + \tan^{-1} \left(\frac{\frac{1}{c_2} - \frac{1}{c_3}}{1 + \frac{1}{c_2} \cdot \frac{1}{c_3}} \right) + \dots + \tan^{-1} \left(\frac{\frac{1}{c_{n-1}} - \frac{1}{c_n}}{1 + \frac{1}{c_{n-1}} \cdot \frac{1}{c_n}} \right) + \tan^{-1} \left(\frac{1}{c_n} \right) \\ &= \left(\tan^{-1} \frac{x}{y} - \tan^{-1} \frac{1}{c_1} \right) + \left(\tan^{-1} \frac{1}{c_1} - \tan^{-1} \frac{1}{c_2} \right) + \left(\tan^{-1} \frac{1}{c_2} - \tan^{-1} \frac{1}{c_3} \right) + \dots \\ & \quad + \left(\tan^{-1} \frac{1}{c_{n-1}} - \tan^{-1} \frac{1}{c_n} \right) + \tan^{-1} \left(\frac{1}{c_n} \right) \\ &= \tan^{-1} \left(\frac{x}{y} \right) = \text{R.H.S.} \end{aligned}$$

Do yourself - 9 :

(i) Evaluate : $\sum_{r=1}^{\infty} \tan^{-1} \left(\frac{2}{1 + (2r+1)(2r-1)} \right)$

Miscellaneous Illustrations :

Illustration 26 : If $\tan^{-1} y = 4 \tan^{-1} x$, $\left(|x| < \tan \frac{\pi}{8} \right)$, find y as an algebraic function of x and hence prove that

$$\tan \frac{\pi}{8} \text{ is a root of the equation } x^4 - 6x^2 + 1 = 0.$$

Solution : We have $\tan^{-1} y = 4 \tan^{-1} x$

$$\Rightarrow \tan^{-1} y = 2 \tan^{-1} \frac{2x}{1-x^2} \quad (\text{as } |x| < 1)$$

$$= \tan^{-1} \frac{\frac{4x}{1-x^2}}{1 - \frac{4x^2}{(1-x^2)^2}} = \tan^{-1} \frac{4x(1-x^2)}{x^4 - 6x^2 + 1} \quad \left(\text{as } \left| \frac{2x}{1-x^2} \right| < 1 \right)$$

$$\Rightarrow y = \frac{4x(1-x^2)}{x^4 - 6x^2 + 1}$$

$$\text{If } x = \tan \frac{\pi}{8}$$

$$\Rightarrow \tan^{-1} y = 4 \tan^{-1} x = \frac{\pi}{2} \Rightarrow y \text{ is not defined} \Rightarrow x^4 - 6x^2 + 1 = 0$$

Ans.

Illustration 27 : If $A = 2 \tan^{-1}(2\sqrt{2}-1)$ and $B = 3 \sin^{-1}(1/3) + \sin^{-1}(3/5)$, then show $A > B$.

Solution : We have, $A = 2 \tan^{-1}(2\sqrt{2}-1) = 2 \tan^{-1}(1.828)$

$$\Rightarrow A > 2 \tan^{-1}(\sqrt{3}) \Rightarrow A > \frac{2\pi}{3} \quad \dots\dots (i)$$

also we have, $\sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{1}{2}\right) \Rightarrow \sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{6}$

$\Rightarrow 3\sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{2}$

also, $3\sin^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(3 \cdot \frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right) = \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$

$\Rightarrow 3\sin^{-1}(1/3) < \sin^{-1}(\sqrt{3}/2) \Rightarrow 3\sin^{-1}(1/3) < \pi/3$

also, $\sin^{-1}(3/5) = \sin^{-1}(0.6) < \sin^{-1}(\sqrt{3}/2) \Rightarrow \sin^{-1}(3/5) < \pi/3$

Hence, $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5) < \frac{2\pi}{3}$ (ii)

From (i) and (ii), we have $A > B$.

Illustration 28 : Solve for x : If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$, where $[.]$ denotes the greatest integer function.

Solution : We have, $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$

$\Rightarrow 1 \leq \sin^{-1} \cdot \cos^{-1} \cdot \sin^{-1} \cdot \tan^{-1}x \leq \frac{\pi}{2} \Rightarrow \sin 1 \leq \cos^{-1} \cdot \sin^{-1} \cdot \tan^{-1}x \leq 1$

$\Rightarrow \cos \sin 1 \geq \sin^{-1} \cdot \tan^{-1}x \geq \cos 1 \Rightarrow \sin \cos \sin 1 \geq \tan^{-1}x \geq \sin \cos 1$

$\Rightarrow \tan \sin \cos \sin 1 \geq x \geq \tan \sin \cos 1$

Hence, $x \in [\tan \sin \cos 1, \tan \sin \cos \sin 1]$

Ans.

Illustration 29 : If $\theta = \tan^{-1}(2 \tan^2 \theta) - \frac{1}{2}\sin^{-1}\left(\frac{3 \sin 2\theta}{5 + 4 \cos 2\theta}\right)$ then find the sum of all possible values of $\tan \theta$.

Solution : $\theta = \tan^{-1}(2 \tan^2 \theta) - \frac{1}{2}\sin^{-1}\left(\frac{3 \sin 2\theta}{5 + 4 \cos 2\theta}\right) \Rightarrow \theta = \tan^{-1}(2 \tan^2 \theta) - \frac{1}{2}\sin^{-1}\left(\frac{6 \tan \theta}{9 + \tan^2 \theta}\right)$

$\Rightarrow \theta = \tan^{-1}(2 \tan^2 \theta) - \frac{1}{2}\sin^{-1}\left[\frac{2\left(\frac{1}{3}\tan \theta\right)}{1 + \left(\frac{1}{3}\tan \theta\right)^2}\right] \Rightarrow \theta = \tan^{-1}(2 \tan^2 \theta) - \frac{2}{2}\tan^{-1}\left(\frac{1}{3}\tan \theta\right)$

$\Rightarrow \theta = \tan^{-1}(2 \tan^2 \theta) - \tan^{-1}\left(\frac{1}{3}\tan \theta\right)$ (i)

taking tangent on both sides

$\Rightarrow \tan \theta = \frac{6 \tan^2 \theta - \tan \theta}{3 + 2 \tan^3 \theta} \Rightarrow 2 \tan^4 \theta - 6 \tan^2 \theta + 4 \tan \theta = 0$

$\Rightarrow 2 \tan \theta (\tan^3 \theta - 3 \tan \theta + 2) = 0 \Rightarrow 2 \tan \theta (\tan \theta - 1)^2 (\tan \theta + 2) = 0$

$\Rightarrow \tan \theta = 0, 1, -2$ which satisfy equation (i)

$\therefore \text{sum} = 0 + 1 - 2 = -1$

Ans.

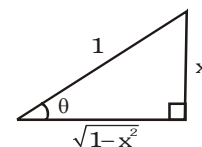
Illustration 30 : Transform $\sin^{-1}x$ in other inverse trigonometric functions, where $x \in (-1, 1) - \{0\}$

Solution : **Case-I :** $0 < x < 1$

Let $\sin^{-1}x = \theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$

Now, $\cos \theta = \sqrt{1 - \sin^2 \theta} \Rightarrow \theta = \cos^{-1} \sqrt{1 - x^2}$

$\Rightarrow \sin^{-1}x = \cos^{-1} \sqrt{1 - x^2} = \sec^{-1}\left(\frac{1}{\sqrt{1 - x^2}}\right)$



$$\tan \theta = \frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow \theta = \tan^{-1} \frac{x}{\sqrt{1-x^2}} \Rightarrow \sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow \sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}} = \cot^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right)$$

$$\text{Hence, } \sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$$

$$= \sec^{-1} \frac{1}{\sqrt{1-x^2}} = \tan^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right) = \cot^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) = \operatorname{cosec}^{-1} \left(\frac{1}{x} \right), 0 < x < 1$$

Case-II : $-1 < x < 0$

$$\text{Let } \sin^{-1} x = \theta \quad \theta \in \left(-\frac{\pi}{2}, 0 \right)$$

$$\text{Then } x = \sin \theta$$

$$\Rightarrow \cos \theta = \sqrt{1-x^2} \quad \Rightarrow \cos(-\theta) = \sqrt{1-x^2}$$

$$\Rightarrow \theta = -\cos^{-1} \sqrt{1-x^2} \quad \Rightarrow \sin^{-1} x = -\cos^{-1} \sqrt{1-x^2} = -\sec^{-1} \left(\frac{1}{\sqrt{1-x^2}} \right)$$

$$\text{Again, } \tan \theta = \frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow \theta = \tan^{-1} \frac{x}{\sqrt{1-x^2}} \Rightarrow \sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow \sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}} = -\pi + \cot^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) \quad \left[\because \tan^{-1} x = -\pi + \cot^{-1} \left(\frac{1}{x} \right), x < 0 \right]$$

$$\text{Hence, } \sin^{-1} x = -\cos^{-1} \sqrt{1-x^2}$$

$$= -\sec^{-1} \frac{1}{\sqrt{1-x^2}} = \tan^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right) = -\pi + \cot^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) = \operatorname{cosec}^{-1} \left(\frac{1}{x} \right), -1 < x < 0$$

ANSWERS FOR DO YOURSELF

- | | | | | |
|--|----------------------------|------------------------|----------------------|---|
| 1 : (i) C | (ii) 1526 | | | |
| 2 : (i) $\frac{15}{8}$ | (ii) $\frac{1}{\sqrt{10}}$ | (iii) $\frac{4}{5}$ | (iv) 1 | (v) $\frac{\sqrt{3}}{2}$ (vi) $\frac{4}{5}$ |
| 3 : (i) $\frac{\pi}{6}$ | (ii) $\frac{\pi}{6}$ | (iii) $\pi - 2$ | (iv) $\frac{\pi}{6}$ | |
| (v) $-\frac{\pi}{3}$ | (vi) $-\frac{\pi}{4}$ | (vii) $\frac{2\pi}{3}$ | | |
| 4 : (ii) $\begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \end{cases}$ | | | | |
| 7 : (i) $\frac{1}{5}$ | (ii) 1 | (iii) $\sqrt{3}$ | | |
| 8 : (i) $(1, \infty)$ | (ii) B | | | |
| 9 : (i) $\pi/4$ | | | | |

EXERCISE - 01**CHECK YOUR GRASP****SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)**

- The value of $\sin^{-1}(-\sqrt{3}/2)$ is -
 (A) $-\pi/3$ (B) $-2\pi/3$ (C) $4\pi/3$ (D) $5\pi/3$
- $\cos\left(2\tan^{-1}\left(\frac{1}{7}\right)\right)$ equals -
 (A) $\sin(4\cot^{-1}3)$ (B) $\sin(3\cot^{-1}4)$ (C) $\cos(3\cot^{-1}4)$ (D) $\cos(4\cot^{-1}4)$
- The value of $\sec\left[\sin^{-1}\left(-\sin\frac{50\pi}{9}\right) + \cos^{-1}\cos\left(-\frac{31\pi}{9}\right)\right]$ is equal to -
 (A) $\sec\frac{10\pi}{9}$ (B) $\sec\frac{\pi}{9}$ (C) 1 (D) -1
- $\cos\left(\cos^{-1}\cos\left(\frac{8\pi}{7}\right) + \tan^{-1}\tan\left(\frac{8\pi}{7}\right)\right)$ has the value equal to -
 (A) 1 (B) -1 (C) $\cos\frac{\pi}{7}$ (D) 0
- $(\sin^{-1}x)^2 + (\sin^{-1}y)^2 + 2(\sin^{-1}x)(\sin^{-1}y) = \pi^2$, then $x^2 + y^2$ is equal to -
 (A) 1 (B) $3/2$ (C) 2 (D) $1/2$
- $\cot^{-1}[(\cos\alpha)^{1/2}] - \tan^{-1}[(\cos\alpha)^{1/2}] = x$, then $\sin x =$
 (A) $\tan^2\left(\frac{\alpha}{2}\right)$ (B) $\cot^2\left(\frac{\alpha}{2}\right)$ (C) $\tan\alpha$ (D) $\cot\left(\frac{\alpha}{2}\right)$
- $\tan(\cos^{-1}x)$ is equal to
 (A) $\frac{x}{1+x^2}$ (B) $\frac{\sqrt{1+x^2}}{x}$ (C) $\frac{\sqrt{1-x^2}}{x}$ (D) $\sqrt{1-2x}$
- If $x = 2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \tan^{-1}(\sqrt{3})$ and $y = \cos\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{x}{2}\right)\right)$ then which of the following statements holds good?
 (A) $y = \cos\frac{3\pi}{16}$ (B) $y = \cos\frac{5\pi}{16}$ (C) $x = 4\cos^{-1}y$ (D) none of these
- If $x = \tan^{-1}1 - \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\frac{1}{2}$; $y = \cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)$ then -
 (A) $x = \pi y$ (B) $y = \pi x$ (C) $\tan x = -(4/3)y$ (D) $\tan x = (4/3)y$
- $\tan^{-1}2 + \tan^{-1}3 = \operatorname{cosec}^{-1}x$, then x is equal to -
 (A) 4 (B) $\sqrt{2}$ (C) $-\sqrt{2}$ (D) none of these
- The number k is such that $\tan\{\arctan(2) + \arctan(20k)\} = k$. The sum of all possible values of k is -
 (A) $-\frac{19}{40}$ (B) $-\frac{21}{40}$ (C) 0 (D) $\frac{1}{5}$
- If $\sin^{-1}x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is -
 (A) 0 (B) $\frac{1}{\sqrt{5}}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{3}}{2}$

13. If $\tan(\cos^{-1}x) = \sin(\cot^{-1} 1/2)$ then x is equal to -
 (A) $1/\sqrt{5}$ (B) $2/\sqrt{5}$ (C) $3/\sqrt{5}$ (D) $\sqrt{5}/3$
14. $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$ is true if -
 (A) $x \in [0,1]$ (B) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (C) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (D) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$
15. Domain of the explicit form of the function y represented implicitly by the equation $(1+x)\cos y - x^2 = 0$ is -
 (A) $(-1,1)$ (B) $\left(-1, \frac{1-\sqrt{5}}{2}\right]$ (C) $\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$ (D) $\left[0, \frac{1+\sqrt{5}}{2}\right]$
16. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to -
 (A) $-4\sin^2\alpha$ (B) $4\sin^2\alpha$ (C) 4 (D) $2\sin 2\alpha$
17. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then -
 (A) $x^2 + y^2 + z^2 + xyz = 0$ (B) $x^2 + y^2 + z^2 + xyz = 1$ (C) $x^2 + y^2 + z^2 + 2xyz = 0$ (D) $x^2 + y^2 + z^2 + 2xyz = 1$
18. If $\tan^{-1}\frac{x}{\pi} < \frac{\pi}{3}$, $x \in \mathbb{N}$, then the maximum value of x is -
 (A) 2 (B) 5 (C) 7 (D) none of these
19. The solution of the inequality $(\tan^{-1}x)^2 - 3\tan^{-1}x + 2 \geq 0$ is -
 (A) $(-\infty, \tan 1] \cup [\tan 2, \infty)$ (B) $(-\infty, \tan 1]$ (C) $(-\infty, -\tan 1] \cup [\tan 2, \infty)$ (D) $[\tan 2, \infty)$
20. The set of values of x , satisfying the equation $\tan^2(\sin^{-1}x) > 1$ is -
 (A) $[-1,1]$ (B) $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ (C) $(-1,1) - \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ (D) $[-1,1] - \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

21. If numerical value of $\tan\left\{\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right\}$ is $\frac{a}{b}$, then -
 (A) $a + b = 23$ (B) $a - b = 11$ (C) $3b = a + 1$ (D) $2a = 3b$
22. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left(\cos\left(-\frac{14\pi}{5}\right)\right)\right]$ is/are -
 (A) $\cos\left(-\frac{7\pi}{5}\right)$ (B) $\sin\left(\frac{\pi}{10}\right)$ (C) $\cos\left(\frac{2\pi}{5}\right)$ (D) $-\cos\left(\frac{3\pi}{5}\right)$
23. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$ equals to
 (A) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (C) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (D) $\tan^{-1}\left(\frac{1}{2}\right)$
24. $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1}x$, then roots of the equation are -
 (A) 0 (B) 1 (C) -1 (D) -2

CHECK YOUR GRASP						ANSWER KEY				EXERCISE-1					
Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	A	A	D	B	C	A	C	A	C	D	A	B	D	B	C
Que.	16	17	18	19	20	21	22	23	24						
Ans.	B	D	B	B	C	A,B,C	B,C,D	A,D	A,B,C						

EXERCISE - 02**BRAIN TEASERS****SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)**

1. $\cos^{-1}x = \tan^{-1}x$ then -

(A) $x^2 = \left(\frac{\sqrt{5}-1}{2}\right)$

(B) $x^2 = \left(\frac{\sqrt{5}+1}{2}\right)$

(C) $\sin(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$

(D) $\tan(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$

2. The value of $\sin\left(\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)\right) + \cos\left(\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)\right)$ is/are equal to -

(A) 1

(B) $\frac{3\sqrt{2}}{\sqrt{10}}$

(C) $\sqrt{2} \sin\left(\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right) + \cot^{-1}(1)\right)$

(D) $\sqrt{2} \sin\left(\pi - \tan^{-1}(1) - \frac{1}{2}\tan^{-1}\frac{4}{3}\right)$

3. The value of $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$ for $0 < A < (\pi/4)$ is -

(A) $4 \tan^{-1}(1)$

(B) $2 \tan^{-1}(2)$

(C) 0

(D) none

4. For the equation $2x = \tan(2\tan^{-1}a) + 2\tan(\tan^{-1}a + \tan^{-1}a^3)$, which of the following is/are invalid ?

(A) $a^2x + 2a = x$

(B) $a^2 + 2ax + 1 = 0$

(C) $a \neq 0$

(D) $a \neq -1, 1$

5. The value of $\left[\tan\left\{\frac{\pi}{4} + \frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right\}\right]^{-1}$, where $(0 < a < b)$, is -

(A) $\frac{b}{2a}$

(B) $\frac{a}{2b}$

(C) $\frac{\sqrt{b^2 - a^2}}{2b}$

(D) $\frac{\sqrt{b^2 - a^2}}{2a}$

6. Identify the pair(s) of functions which are identical -

(A) $y = \tan(\cos^{-1}x)$; $y = \frac{\sqrt{1-x^2}}{x}$

(B) $y = \tan(\cot^{-1}x)$; $y = \frac{1}{x}$

(C) $y = \sin(\arctan x)$; $y = \frac{x}{\sqrt{1+x^2}}$

(D) $y = \cos(\arctan x)$; $y = \sin(\arccot x)$

7. Which of the following, satisfy the equation $2\cos^{-1}x = \cot^{-1}\left(\frac{2x^2-1}{\sqrt{4x^2-4x^4}}\right)$

(A) $(-1, 0)$

(B) $(0, 1)$

(C) $\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

(D) $[-1, 1]$

8. The solution set of the equation $\sin^{-1}\sqrt{1-x^2} + \cos^{-1}x = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) - \sin^{-1}x$ is -

(A) $[-1, 1] - \{0\}$

(B) $(0, 1] \cup \{-1\}$

(C) $[-1, 0] \cup \{1\}$

(D) $[-1, 1]$

9. If $0 < x < 1$, then $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ is equal to -

(A) $\frac{1}{2}\cos^{-1}x$

(B) $\cos^{-1}\sqrt{\frac{1+x}{2}}$

(C) $\sin^{-1}\sqrt{\frac{1-x}{2}}$

(D) $\frac{1}{2}\tan^{-1}\sqrt{\frac{1+x}{1-x}}$

10. The number of real solutions of $\tan^{-1} \sqrt{x(x+1)} + \sin^{-1} \sqrt{x^2 + x + 1} = \frac{\pi}{2}$ is - [JEE 99]
 (A) zero (B) one (C) two (D) infinite
11. If $[\sin^{-1}x] + [\cos^{-1}x] = 0$, where 'x' is a non negative real number and [.] denotes the greatest integer function, then complete set of values of x is -
 (A) $(\cos 1, 1)$ (B) $(-1, \cos 1)$ (C) $(\sin 1, 1)$ (D) $(\cos 1, \sin 1)$
12. Value of k for which the point $(\alpha, \sin^{-1}\alpha)$ ($\alpha > 0$) lies inside the triangle formed by $x + y = k$ with co-ordinate axes is -
 (A) $\left(1 + \frac{\pi}{2}, \infty\right)$ (B) $\left(-\left(1 + \frac{\pi}{2}\right), \left(1 + \frac{\pi}{2}\right)\right)$ (C) $\left(-\infty, 1 + \frac{\pi}{2}\right)$ (D) $(-1 - \sin 1, 1 + \sin 1)$
13. Solution set of the inequality $\sin^{-1} \left(\sin \frac{2x^2 + 3}{x^2 + 1} \right) \leq \pi - \frac{5}{2}$ is -
 (A) $(-\infty, 1) \cup (1, \infty)$ (B) $[-1, 1]$ (C) $(-1, 1)$ (D) $(-\infty, -1] \cup [1, \infty)$
14. Consider two geometric progressions $a_1, a_2, a_3, \dots, a_n$ & $b_1, b_2, b_3, \dots, b_n$ with $a_r = \frac{1}{b_r} = 2^{r-1}$ and another sequence $t_1, t_2, t_3, \dots, t_n$ such that $t_r = \cot^{-1} (2a_r + b_r)$ then $\lim_{n \rightarrow \infty} \sum_{r=1}^n t_r$ is -
 (A) 0 (B) $\pi/4$ (C) $\tan^{-1} 2$ (D) $\pi/2$
15. The sum of the infinite terms of the series -
 $\cot^{-1} \left(1^2 + \frac{3}{4} \right) + \cot^{-1} \left(2^2 + \frac{3}{4} \right) + \cot^{-1} \left(3^2 + \frac{3}{4} \right) + \dots$ is equal to -
 (A) $\tan^{-1}(1)$ (B) $\tan^{-1}(2)$ (C) $\tan^{-1}(3)$ (D) $\frac{3\pi}{4} - \tan^{-1} 3$

BRAIN TEASERS					ANSWER KEY			EXERCISE-2		
Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A,C	B,C,D	A	B,C	C	A,B,C,D	B	C	A,B,C	C
Que.	11	12	13	14	15					
Ans.	D	A	B	B	B,D					

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

FILL IN THE BLANKS

- $\tan \left[\cos^{-1} \frac{1}{2} + \tan^{-1} \left(\frac{-1}{\sqrt{3}} \right) \right] = \dots\dots\dots$
- $\cos (\tan^{-1} 2) = \dots\dots\dots$
- $\tan \left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{2} \right) = \dots\dots\dots$
- $\cos \left[\cos^{-1} \left(\frac{-\sqrt{3}}{2} \right) + \frac{\pi}{6} \right] = \dots\dots\dots$
- $\sin^{-1} \frac{3}{\sqrt{73}} + \cos^{-1} \frac{11}{\sqrt{146}} + \cot^{-1} \sqrt{3} = \dots\dots\dots$
- $\tan^{-1} \left(\frac{1}{\sqrt{2}} \right) + \sin^{-1} \left(\frac{1}{\sqrt{5}} \right) - \cos^{-1} \left(\frac{1}{\sqrt{10}} \right) - \cot^{-1} \left(\frac{1+\sqrt{2}}{1-\sqrt{2}} \right) = \dots\dots\dots$
- $\sin \left[\frac{\pi}{2} - \sin^{-1} \left(\frac{-\sqrt{3}}{2} \right) \right] = \dots\dots\dots$
- $-\left(\cos^{-1} \frac{1}{\sqrt{3}} + \cos^{-1} \frac{1}{\sqrt{6}} \right) - \cos^{-1} \left(\frac{\sqrt{10}-1}{3\sqrt{2}} \right) + 4 \cot^{-1} 1 = \dots\dots\dots$
- $\tan^{-1} \left[\frac{3 \sin 2\alpha}{5+3 \cos 2\alpha} \right] + \tan^{-1} \left[\frac{\tan \alpha}{4} \right], \text{ where } -\frac{\pi}{2} < \alpha < \frac{\pi}{2} = \dots\dots\dots$
- The number of roots of the equation $\sqrt{\sin x} = \cos^{-1}(\cos x)$ is $\dots\dots\dots$

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

- | Column-I | | Column-II | |
|----------|---|-----------|-----------|
| (A) | $\sin^{-1} \left(\sin \frac{33\pi}{7} \right)$ | (p) | $-2\pi/7$ |
| (B) | $\cos^{-1} \left(\cos \frac{46\pi}{7} \right)$ | (q) | $2\pi/7$ |
| (C) | $\tan^{-1} \left(\tan \left(\frac{-33\pi}{7} \right) \right)$ | (r) | $3\pi/7$ |
| (D) | $\cot^{-1} \left(\cot \left(\frac{-46\pi}{7} \right) \right)$ | (s) | $4\pi/7$ |
- | Column-I | | Column-II | |
|----------|---|-----------|--------------------------|
| (A) | $\sin(\tan^{-1} x)$ | (p) | x |
| (B) | $\cos(\tan^{-1} x)$ | (q) | $\frac{x}{\sqrt{x^2+1}}$ |
| (C) | $\sin(\cot^{-1}(\tan(\cos^{-1} x))) ; x \in (0, 1]$ | (r) | $\frac{1}{\sqrt{x^2+1}}$ |
| (D) | $\sin(\operatorname{cosec}^{-1}(\cot(\tan^{-1} x))) ; x \in (0, 1]$ | (s) | $\sqrt{1-x^2}$ |

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

3. $x \geq 0, y \geq 0, z \geq 0$ and $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = k$, the possible value(s) of k, if

Column-I		Column-II	
(A)	$xy + yz + zx = 1$, then	(p)	$k = \frac{\pi}{2}$
(B)	$x + y + z = xyz$, then	(q)	$k = \pi$
(C)	$x^2 + y^2 + z^2 = 1$ and $x + y + z = \sqrt{3}$, then	(r)	$k = 0$
(D)	$x = y = z$ and $xyz \geq 3\sqrt{3}$, then	(s)	$k = \frac{7\pi}{6}$

ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

- (A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.
 (B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.
 (C) Statement-I is true, Statement-II is false.
 (D) Statement-I is false, Statement-II is true.

1. **Statement-I** : Range of $\cos\left(\sec^{-1}\frac{1}{x} + \operatorname{cosec}^{-1}\frac{1}{x} + \tan^{-1}x\right)$ is $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Because

Statement-II : Range of $\sin^{-1}x + \tan^{-1}x + \cos^{-1}x$ is $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

- (A) A (B) B (C) C (D) D

2. **Statement-I** : If r, s & t be the roots of the equation : $x(x-2)(3x-7) = 2$, then $\tan^{-1}r + \tan^{-1}s + \tan^{-1}t = 3\pi/4$.

Because

Statement-II : The roots of the equation $x(x-2)(3x-7) = 2$ are real & negative.

- (A) A (B) B (C) C (D) D

3. **Statement-I** : If $\sum_{i=1}^{2n} \sin^{-1}x_i = n\pi, n \in \mathbb{N}$. Then $\sum_{i=1}^n x_i = \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i^3$

Because

Statement-II : $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}, \forall x \in [-1, 1]$.

- (A) A (B) B (C) C (D) D

4. Let $f : \mathbb{R} \rightarrow [0, \pi/2)$ defined by $f(x) = \tan^{-1}(x^2 + x + a)$, then

Statement-I : The set of values of a for which f(x) is onto is $\left[\frac{1}{4}, \infty\right)$.

Because

Statement-II : Minimum value of $x^2 + x + a$ is $a - \frac{1}{4}$.

- (A) A (B) B (C) C (D) D

5. **Statement-I** : $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{9}{5}\right) = \pi - \frac{9}{5}$.

Because

Statement-II : $\operatorname{cosec}^{-1}(\operatorname{cosec}x) = \pi - x ; \forall x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$

- (A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

Consider the two equations in x ; (i) $\sin\left(\frac{\cos^{-1}x}{y}\right) = 1$ (ii) $\cos\left(\frac{\sin^{-1}x}{y}\right) = 0$

The sets $X_1, X_2 \subseteq [-1, 1]$; $Y_1, Y_2 \subseteq I - \{0\}$ are such that

X_1 : the solution set of equation (i)

X_2 : the solution set of equation (ii)

Y_1 : the set of all integral values of y for which equation (i) possess a solution

Y_2 : the set of all integral values of y for which equation (ii) possess a solution

Let : C_1 be the correspondence : $X_1 \rightarrow Y_1$ such that $x C_1 y$ for $x \in X_1, y \in Y_1$ & (x, y) satisfy (i).

C_2 be the correspondence : $X_2 \rightarrow Y_2$ such that $x C_2 y$ for $x \in X_2, y \in Y_2$ & (x, y) satisfy (ii).

On the basis of above information, answer the following questions :

- The number of ordered pair (x, y) satisfying correspondence C_1 is
(A) 1 (B) 2 (C) 3 (D) 4
- The number of ordered pair (x, y) satisfying correspondence C_2 is
(A) 1 (B) 2 (C) 3 (D) 4
- $C_1 : X_1 \rightarrow Y_1$ is a function which is -
(A) one-one (B) many-one (C) onto (D) into

Comprehension # 2

Let $h_1(x) = \sin^{-1}(3x - 4x^3)$; $h_2(x) = \cos^{-1}(4x^3 - 3x)$ & $f(x) = h_1(x) + h_2(x)$

when $x \in [-1, \frac{-1}{2}]$; let $f(x) = a \cos^{-1}x + b\pi$; $a, b \in \mathbb{Q}$

$h_1(x) = p \sin^{-1}x + q\pi$; $p, q \in \mathbb{Q}$

$h_2(x) = r \cos^{-1}x + s\pi$; $r, s \in \mathbb{Q}$

Let C_1 be the circle with centre (p, q) & radius 1 & C_2 be the circle with centre (r, s) & radius 1.

On the basis of above information, answer the following questions :

- $p + r + 2q - s =$
(A) 0 (B) 1 (C) 2 (D) 4
- If $b \cdot \log_{|s|} |p + q| = k \cdot a$, then value of k is -
(A) $\frac{9}{2}$ (B) 6 (C) $\frac{-3}{2}$ (D) none of these
- Radical axis of circle C_1 & C_2 is -
(A) $12x - 2y - 3 = 0$ (B) $12x + 2y - 3 = 0$ (C) $-12x + 2y - 3 = 0$ (D) none of these

MISCELLANEOUS TYPE QUESTION

ANSWER KEY

EXERCISE -3

Fill in the Blanks

- $\frac{1}{\sqrt{3}}$
- $\frac{1}{\sqrt{5}}$
- $\frac{17}{6}$
- 1
- $\frac{5\pi}{12}$
- $-\pi$
- $\frac{1}{2}$
- 0
- α
- infinite many solutions

Match the Column

- (A) \rightarrow (q), (B) \rightarrow (s), (C) \rightarrow (q), (D) \rightarrow (r)
- (A) \rightarrow (q), (B) \rightarrow (r), (C) \rightarrow (p), (D) \rightarrow (p)
- (A) \rightarrow (p), (B) \rightarrow (q,r), (C) \rightarrow (p), (D) \rightarrow (q,s)

Assertion & Reason

- D
- C
- A
- D
- A

Comprehension Based Questions

Comprehension # 1 : 1. B 2. D 3. A,C Comprehension # 2 : 1. A 2. C 3. A

EXERCISE - 04 [A]

CONCEPTUAL SUBJECTIVE EXERCISE

1. Find the domain of definition the following functions.

(Read the symbols $[*]$ and $\{ * \}$ as greatest integers and fractional part functions respectively)

(a) $f(x) = \cos^{-1} \frac{2}{2 + \sin x}$

(b) $f(x) = \frac{1}{x} + 2^{\arcsin x} + \frac{1}{\sqrt{x-2}}$

(c) $e^{\cos^{-1} x} + \cot^{-1} \left[\frac{x}{2} - 1 \right] + \frac{1}{2} \ln \{x\}$

(d) $f(x) = \sin^{-1} \left(\frac{x-3}{2} \right) - \log_{10} (4-x)$

(e) $f(x) = \frac{\sqrt{1-\sin x}}{\log_5 (1-4x^2)} + \cos^{-1} (1-\{x\})$

(f) $f(x) = \sqrt{3-x} + \cos^{-1} \left(\frac{3-2x}{5} \right) + \log_6 (2|x|-3) + \sin^{-1} (\log_2 x)$

2. Find the domain and range of the following functions.

(Read the symbols $[*]$ and $\{ * \}$ as greatest integers and fractional part function respectively)

(a) $y = \cot^{-1} (2x - x^2)$

(b) $f(x) = \sec^{-1} (\log_3 \tan x + \log_{\tan x} 3)$

(c) $f(x) = 2^{\cos^{-1} \left(\sin \left(x + \frac{\pi}{3} \right) \right)} + \left[\frac{\sqrt{1-2\cos x}}{2} \right]$

(d) $f(x) = \tan^{-1} \left(\log_4 (5x^2 - 8x + 4) \right)$

3. Draw the graph of the following functions :

(a) $f(x) = \sin^{-1}(x+2)$

(b) $g(x) = [\cos^{-1}x]$, where $[]$ denotes greatest integer function.

(c) $h(x) = -|\tan^{-1}(3x)|$

4. Express $f(x) = \arccos x + \arccos \left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2} \right)$ in simplest form and hence find the values of

(a) $f\left(\frac{2}{3}\right)$

(b) $f\left(\frac{1}{3}\right)$

5. If $\cos^{-1} \frac{x}{a} + \cos^{-1} \frac{y}{b} = \alpha$ then prove that $\frac{x^2}{a^2} - \frac{2xy}{ab} \cos \alpha + \frac{y^2}{b^2} = \sin^2 \alpha$.

6. Prove that : $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} = \sin^{-1} \frac{77}{85}$

7. Prove that : $\cos^{-1} x = 2 \sin^{-1} \sqrt{\frac{1-x}{2}} = 2 \cos^{-1} \sqrt{\frac{1+x}{2}}$

8. Prove that : $\tan^{-1} \frac{2}{3} = \frac{1}{2} \tan^{-1} \frac{12}{5}$

9. Prove that : $3 \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{20} = \frac{\pi}{4} - \tan^{-1} \frac{1}{1985}$

10. If $\sin^2 x + \sin^2 y < 1$ for all $x, y \in \mathbb{R}$ then prove that $\sin^{-1} (\tan x \cdot \tan y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

11. Prove that : $\cot^{-1} \left(\frac{1+ab}{a-b} \right) + \cot^{-1} \left(\frac{1+bc}{b-c} \right) + \cot^{-1} \left(\frac{1+ca}{c-a} \right) = \pi$, ($a > b > c > 0$)

12. Let $\cos^{-1} x + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$. If x satisfies the cubic $ax^3 + bx^2 + cx - 1 = 0$, then find the value of $a + b + c$.

13. If $\alpha = 2 \tan^{-1} \left(\frac{1+x}{1-x} \right)$ & $\beta = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right)$ for $0 < x < 1$ then prove that $\alpha + \beta = \pi$. What is the value of $\alpha + \beta$ will be if $x > 1$?

14. Solve the following equations :

- (a) $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$
 (b) $\tan^{-1} \frac{1}{1+2x} + \tan^{-1} \frac{1}{1+4x} = \tan^{-1} \frac{2}{x^2}$
 (c) $\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}(3x)$
 (d) $\sin^{-1} x = \cos^{-1} x + \sin^{-1}(3x-2)$
 (e) $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$
 (f) $2 \tan^{-1} x = \cos^{-1} \frac{1-a^2}{1+a^2} - \cos^{-1} \frac{1-b^2}{1+b^2} \quad a > 0, b > 0$
 (g) $\cos^{-1} \frac{x^2-1}{x^2+1} + \tan^{-1} \frac{2x}{x^2-1} = \frac{2\pi}{3}$

15. Find the sum of the series :

- (a) $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{2}{9} + \dots + \tan^{-1} \frac{2^{n-1}}{1+2^{2n-1}} + \dots \infty$
 (b) $\cot^{-1} 7 + \cot^{-1} 13 + \cot^{-1} 21 + \cot^{-1} 31 + \dots$ to n terms.
 (c) $\tan^{-1} \frac{1}{x^2+x+1} + \tan^{-1} \frac{1}{x^2+3x+3} + \tan^{-1} \frac{1}{x^2+5x+7} + \tan^{-1} \frac{1}{x^2+7x+13} + \dots$ to n terms.

CONCEPTUAL SUBJECTIVE EXERCISE

ANSWER KEY

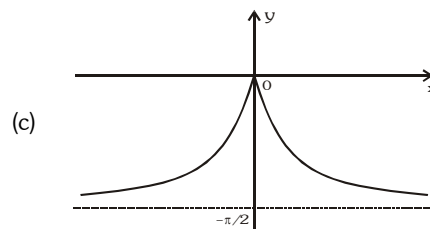
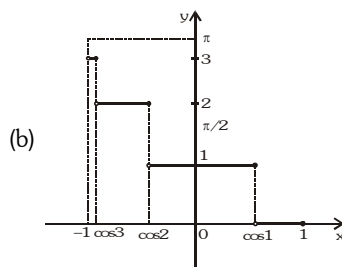
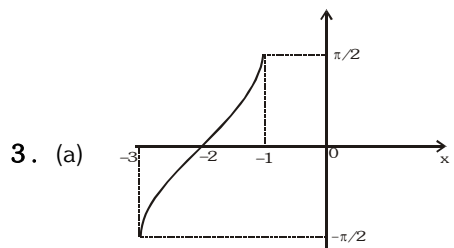
EXERCISE-4(A)

1. (a) $[2n\pi, (2n+1)\pi]; n \in \mathbb{I}$ (b) ϕ (not defined for any real x) (c) $(-1, 1) - \{0\}$ (d) $1 \leq x < 4$
 (e) $x \in (-1/2, 1/2), x \neq 0$ (f) $\left(\frac{3}{2}, 2\right]$

2. (a) $D : x \in \mathbb{R} \quad R : [\pi/4, \pi)$

- (b) $D : x \in \left(2n\pi, 2n\pi + \frac{\pi}{2}\right) \cup \left((2n+1)\pi, 2n\pi + \frac{3\pi}{2}\right) - \left\{x \mid x = 2n\pi + \frac{\pi}{4} \text{ or } 2n\pi + \frac{5\pi}{4}\right\} n \in \mathbb{I}; R : \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] - \left[\frac{\pi}{2}\right]$
 or $n\pi < x < \pi/2 + n\pi; x \neq \pi/4 + n\pi$

- (c) $[2^{\pi/6}, 2^{\pi}]$ (d) $D : x \in \mathbb{R}; R : \left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$



4. (a) $\frac{\pi}{3}$ (b) $2\cos^{-1}\left(\frac{1}{3}\right) - \frac{\pi}{3}$

12. 26

13. $-\pi$

14. (a) $x = \frac{1}{2}\sqrt{\frac{3}{7}}$ (b) $x = 3$ (c) $x = 0, \frac{1}{2}, -\frac{1}{2}$ (d) $x = 1, \frac{1}{2}$ (e) $x = 0, \frac{1}{2}$ (f) $x = \frac{a-b}{1+ab}$

- (g) $x = 2 - \sqrt{3}$ या $\sqrt{3}$

15. (a) $\frac{\pi}{4}$ (b) $\arccot \cot \left[\frac{2n+5}{n} \right]$ (c) $\arctan(x+n) - \arctan x$

EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. Find the domain of definition the following functions.

$$(a) \quad f(x) = \log_{10}(1 - \log_7(x^2 - 5x + 13)) + \cos^{-1}\left(\frac{3}{2 + \sin \frac{9\pi x}{2}}\right)$$

$$(b) \quad f(x) = \sqrt{\sin(\cos x)} + \ln(-2 \cos^2 x + 3 \cos x + 1) + e^{\cos^{-1}\left(\frac{2 \sin x + 1}{2\sqrt{2} \sin x}\right)}$$

2. Prove that :

$$(a) \quad \sin^{-1}[\cos(\sin^{-1} x)] + \cos^{-1}[\sin(\cos^{-1} x)] = \frac{\pi}{2}, |x| \leq 1 \quad (b) \quad \tan^{-1}\left(\frac{m}{n}\right) + \tan^{-1}\left(\frac{n-m}{n+m}\right) = \begin{cases} \frac{\pi}{4} & \frac{m}{n} > -1 \\ -\frac{3\pi}{4} & \frac{m}{n} < -1 \end{cases}$$

3. Prove that : $\sin^{-1} \frac{1}{\sqrt{2}} + \sin^{-1} \frac{\sqrt{2}-1}{\sqrt{6}} + \dots + \sin^{-1} \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} + \dots = \frac{\pi}{2}$

4. If $\arcsin x + \arcsin y + \arcsin z = \pi$ then prove that : $(x, y, z > 0)$

$$(a) \quad x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz \quad (b) \quad x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$$

5. Find the integral values of K for which the system of equations ;

$$\begin{cases} \arcsin x + (\arcsin y)^2 = \frac{K\pi^2}{4} \\ (\arcsin y)^2 (\arcsin x) = \frac{\pi^4}{16} \end{cases} \text{ possesses solutions \& find those solutions.}$$

6. Express $\frac{\beta^3}{2} \operatorname{cosec}^2\left[\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right] + \frac{\alpha^3}{2} \sec^2\left[\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right]$ as an integral polynomial in α & β .

7. Solve the following inequalities :

$$(a) \quad \arcsin \cot^2 x - 5 \arcsin \cot x + 6 > 0 \quad (b) \quad \arcsin x > \arcsin \cos x$$

$$(c) \quad 4 \arcsin \tan^2 x - 8 \arcsin \tan x + 3 < 0 \quad \& \quad 4 \arcsin \cot x - \arcsin \cot^2 x - 3 \geq 0$$

8. Find all the positive integral solutions of, $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$.

9. Let $f(x) = \cot^{-1}(x^2 + 4x + \alpha^2 - \alpha)$ be a function defined $R \rightarrow \left(0, \frac{\pi}{2}\right]$ then find the complete set of real values of α for which $f(x)$ is onto.

10. Find all values of k for which there is a triangle whose angles have measure $\tan^{-1}\left(\frac{1}{2}\right), \tan^{-1}\left(\frac{1}{2}+k\right)$ and $\tan^{-1}\left(\frac{1}{2}+2k\right)$.

11. Find the range of the function $f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$.

12. Find the number of roots of the following equations :

$$(a) \quad \sqrt{1 + \cos 2x} = \sqrt{2} \sin^{-1}(\sin x) \quad (b) \quad \sin\left(\sin^{-1}(\log_{\frac{1}{2}} x)\right) + 2 \cos\left(\sin^{-1}\left(\frac{x}{2} - 1\right)\right) = 0$$

$$(c) \quad |y| = \cos x \quad \text{and} \quad y = \cot^{-1}(\cot x) \quad \text{in} \quad \left(-\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

BRAIN STORMING SUBJECTIVE EXERCISE		ANSWER KEY	EXERCISE-4(B)
1. (a) $\frac{21}{9}, \frac{25}{9}$	(b) $2n\pi + \frac{\pi}{6}; n \in I$	5. $K = 2; \cos \frac{\pi^2}{4}, 1 \& \cos \frac{\pi^2}{4}, -1$	6. $(\alpha^2 + \beta^2)(\alpha + \beta)$
7. (a) $(\cot 2, \infty) \cup (-\infty, \cot 3)$	(b) $\left(\frac{\sqrt{2}}{2}, 1\right]$	(c) $\left(\tan \frac{1}{2}, \cot 1\right]$	8. $x = 1; y = 2 \& x = 2; y = 7$
9. $\frac{1 \pm \sqrt{17}}{2}$			
10. $k = \frac{11}{4}$	11. $\left[\frac{\pi^3}{32}, \frac{7\pi^3}{8}\right]$	12. (a) Infinite ; (b) zero ; (c) 2	

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. The value of $\cos^{-1}(-1) - \sin^{-1}(1)$ is- [AIEEE-2002]
 (1) π (2) $\frac{\pi}{2}$ (3) $\frac{3\pi}{2}$ (4) $-\frac{3\pi}{2}$
2. The trigonometric equation $\sin^{-1} x = 2 \sin^{-1} a$, has a solution for- [AIEEE-2003]
 (1) $|a| \leq \frac{1}{\sqrt{2}}$ (2) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$ (3) all real values of a (4) $|a| < \frac{1}{2}$
3. If $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$, then $4x^2 - 4xy \cos \alpha + y^2$ is equal to - [AIEEE-2005]
 (1) $2 \sin 2\alpha$ (2) 4 (3) $4 \sin^2 \alpha$ (4) $-4 \sin^2 \alpha$
4. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$ then a value of x is- [AIEEE-2007]
 (1) 1 (2) 3 (3) 4 (4) 5
5. The value of $\cot\left(\operatorname{cosec}^{-1} \frac{5}{3} + \tan^{-1} \frac{2}{3}\right)$ is equal to- [AIEEE-2008]
 (1) $\frac{6}{17}$ (2) $\frac{3}{17}$ (3) $\frac{4}{17}$ (4) $\frac{5}{17}$
6. If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then [JEE (Main)-2013]
 (1) $x = y = z$ (2) $2x = 3y = 6z$ (3) $6x = 3y = 2z$ (4) $6x = 4y = 3z$

PREVIOUS YEARS QUESTIONS

ANSWER KEY

EXERCISE-5 [A]

Que.	1	2	3	4	5	6
Ans	2	1	3	2	1	1

EXERCISE - 05 [B]

JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. Prove that $\cos \tan^{-1} \sin \cot^{-1} x = \sqrt{\frac{x^2 + 1}{x^2 + 2}}$ [JEE 2002 (Mains), 5]

2. Domain of $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ is -

- (A) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (B) $\left[-\frac{1}{4}, \frac{3}{4}\right]$ (C) $\left[-\frac{1}{4}, \frac{1}{4}\right]$ (D) $\left[-\frac{1}{4}, \frac{1}{2}\right]$

[JEE 2003 (screening), 3]

3. If $\sin(\cot^{-1}(x+1)) = \cos(\tan^{-1}x)$, then $x =$ [JEE 2004 (screening)]

- (A) $-\frac{1}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) 1

4. $\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then find the value of $\tan(t)$. [JEE 2006, 1½]

5. Let (x, y) be such that $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$ [JEE 2007, 6]

Match the statements in column-I with statements in column-II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column-I

Column-II

- | | |
|--|--|
| (A) If $a = 1$ and $b = 0$, then (x, y) | (p) lies on the circle $x^2 + y^2 = 1$ |
| (B) If $a = 1$ and $b = 1$, then (x, y) | (q) lies on $(x^2 - 1)(y^2 - 1) = 0$ |
| (C) If $a = 1$ and $b = 2$, then (x, y) | (r) lies on $y = x$ |
| (D) If $a = 2$ and $b = 2$, then (x, y) | (s) lies on $(4x^2 - 1)(y^2 - 1) = 0$ |

6. If $0 < x < 1$, then $\sqrt{1+x^2} [\{x \cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$ [JEE 2008, 3]

- (A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

7. The value of $\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^n 2k\right)\right)$ is [JEE (Advanced) 2013, 2]

- (A) $\frac{23}{25}$ (B) $\frac{25}{23}$ (C) $\frac{23}{24}$ (D) $\frac{24}{23}$

8. Match List-I with List-II and select the correct answer using the code given below the lists.

List-I

List-II

P. $\left(\frac{1}{y^2} \left(\frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) - \tan(\sin^{-1} y)} \right)^2 + y^4 \right)^{1/2}$ takes value

1. $\frac{1}{2} \sqrt{\frac{5}{3}}$

Q. If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then

2. $\sqrt{2}$

possible value of $\cos \frac{x-y}{2}$ is

R. If $\cos\left(\frac{\pi}{4} - x\right) \cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x +$

3. $\frac{1}{2}$

$\cos\left(\frac{\pi}{4} + x\right) \cos 2x$ then possible value of $\sec x$ is

S. If $\cot(\sin^{-1} \sqrt{1-x^2}) = \sin(\tan^{-1}(x\sqrt{6}))$, $x \neq 0$,

4. 1

then possible value of x is

Codes :

	P	Q	R	S
(A)	4	3	1	2
(B)	4	3	2	1
(C)	3	4	2	1
(D)	3	4	1	2

[JEE-Advanced 2013, 3, (-1)]

PREVIOUS YEARS QUESTIONS

ANSWER KEY

EXERCISE-5 [B]

2. D 3. A 4. 1 5. (A) - p, (B) - q, (C) - p, (D) - s 6. C 7. B 8. B