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SYLLABUS FOR JEE ADVANCED

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Logarithms and their Properties

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CHAPTER

01

Complex Numbers

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The square of any real number, whether positive, negative or zero, is always non-negative i.e. $x^2 \geq 0$ for all $x \in R$. Therefore, there will be no real value of x, which when squared, will give a negative number.

Thus, the equation $x^2 + 1 = 0$ is not satisfied for any real value of x. 'Euler' was the first Mathematician to introduce the symbol i (read 'Iota') for the square root of -1 with the property $i^2 = -1$. The theory of complex number was later on developed by Gauss and Hamilton. According to Hamilton, "Imaginary number is that number whose square is a negative number ". Hence, the equation $x^2 + 1 = 0$

$\Rightarrow \qquad x^2 = -1$

or $\qquad x = \pm\sqrt{-1}$

(in the sense of arithmetic, $\sqrt{-1}$ has no meaning).

Symbolically, $\sqrt{-1}$ is denoted by i (the first letter of the word 'Imaginary ').

$\therefore$ Solutions of $x^2 + 1 = 0$ are $x = \pm i$.

Also, i is the unit of complex number, since i is present in every complex number. Generally, if a is positive quantity, then

$$\begin{aligned}\sqrt{-a} \times \sqrt{-a} &= \sqrt{(-1) \times a} \times \sqrt{(-1) \times a} \\ &= \sqrt{-1} \times \sqrt{a} \times \sqrt{-1} \times \sqrt{a} \\ &= i\sqrt{a} \times i\sqrt{a} \\ &= i^2 a = -a\end{aligned}$$

Remark

$\sqrt{-a} = i\sqrt{a}$, where a is positive quantity. Keeping this result in mind, the following computation is correct

$$\sqrt{-a}\sqrt{-b} = i\sqrt{a} \cdot i\sqrt{b} = i^2\sqrt{ab} = -\sqrt{ab}$$

where, a and b are positive real numbers.

But the computation, $\sqrt{-a}\sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{|a||b|}$ is wrong.
Because the property, $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is valid only when atleast one of a and b is non-negative.

If a and b are both negative, then $\sqrt{a}\sqrt{b} = -\sqrt{|a||b|}$.

| Example 1. Is the following computation correct? If not, give the correct computation.

$$\sqrt{-2}\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}$$

Sol. No,

If a and b are both negative real numbers, then $\sqrt{a}\sqrt{b} = -\sqrt{ab}$

Here, $a = -2$ and $b = -3$.

$\therefore \qquad \sqrt{-2}\sqrt{-3} = -\sqrt{(-2)(-3)} = -\sqrt{6}$

| Example 2. A student writes the formula $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Then, he substitutes $a = -1$ and $b = -1$ and finds $1 = -1$. Explain, where he is wrong.

Sol. Since, a and b are both negative, therefore $\sqrt{ab} \neq \sqrt{a}\sqrt{b}$.

Infact a and b are both negative, then we have $\sqrt{a}\sqrt{b} = -\sqrt{ab}$.

| Example 3. Explain the fallacy

$$-1 = i \times i = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1) \times (-1)} = \sqrt{1} = 1.$$

Sol. If a and b are both negative, then

$$\sqrt{a}\sqrt{b} = -\sqrt{|a||b|}$$

$\therefore \qquad \sqrt{-1} \times \sqrt{-1} = -\sqrt{|-1||-1|} = -1$

Session 1

Integral Powers of Iota (i), Switch System Theory

Integral Powers of Iota (i)

(i) If the index of i is whole number, then

$$i^0 = 1, i^1 = i, i^2 = (\sqrt{-1})^2 = -1,$$

$$i^3 = i \cdot i^2 = -i, i^4 = (i^2)^2 = (-1)^2 = 1$$

To find the value of i^n $(n > 4)$ First divide n by 4. Let q be the quotient and r be the remainder.

i.e.
$$\begin{array}{r} 4)\, n\, (q \\ -4q \\ \hline r \end{array}$$

$\Rightarrow \qquad n = 4q + r$

When, $0 \leq r \leq 3$

$\therefore \qquad i^n = i^{4q+r} = (i^4)^q\,(i)^r = (1)^q \cdot (i)^r = i^r$

In general, $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$ for any whole number n.

(ii) If the index of i is a negative integer, then

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i, \; i^{-2} = \frac{1}{i^2} = -1,$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i, \; i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1, \text{ etc.}$$

Example 4. Evaluate.

(i) i^{1998}

(ii) i^{-9999}

(iii) $(-\sqrt{-1})^{4n+3}, n \in N$

Sol. (i) 1998 leaves remainder 2, when it is divided by 4.

i.e. $4\overline{)1998}(499$
$\underline{1996}$
2

$\therefore \quad i^{1998} = i^2 = -1$

Aliter

$$i^{1998} = \frac{i^{2000}}{i^2} = \frac{1}{-1} = -1$$

(ii) 9999 leaves remainder 3, when it is divided by 4.

i.e. $4\overline{)9999}(2499$
$\underline{9996}$
3

$$\therefore \quad i^{-9999} = \frac{1}{i^{9999}} = \frac{1}{i^3} = \frac{i}{i^4} = \frac{i}{1} = i$$

Aliter

$$i^{-9999} = \frac{1}{i^{9999}} = \frac{i}{i^{10000}} = \frac{i}{1} = i$$

(iii) $4n + 3$ leaves remainder 3, when it is divided by 4.

i.e., $4\overline{)4n+3}(n$
$\underline{4n}$
3

$\therefore \quad i^{4n+3} = i^3 = -i$

Now, $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = -(i)^{4n+3}$
$= -(-i)$
$= i$

Aliter $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = -i^{4n+3}$
$= -(i^4)^n \cdot i^3$
$= -(1)^n(-i) = i$

Example 5. Find the value of $1 + i^2 + i^4 + i^6 + \ldots + i^{2n}$, where $i = \sqrt{-1}$ and $n \in N$.

Sol. $\because \quad 1 + i^2 + i^4 + i^6 + \ldots + i^{2n} = 1 - 1 + 1 - 1 + \ldots + (-1)^n$

***Case* I** If n is odd, then

$$1 + i^2 + i^4 + i^6 + \ldots + i^{2n} = 1 - 1 + 1 - 1 + \ldots + 1 - 1 = 0$$

***Case* II** If n is even, then

$$1 + i^2 + i^4 + i^6 + \ldots + i^{2n} = 1 - 1 + 1 - 1 + \ldots + 1 = 1$$

Example 6. If $a = \dfrac{1+i}{\sqrt{2}}$, where $i = \sqrt{-1}$, then find the value of a^{1929}.

Sol. $\because \quad a^2 = \left(\dfrac{1+i}{\sqrt{2}}\right)^2 = \left(\dfrac{1+i^2+2i}{2}\right)$

$$= \left(\frac{1-1+2i}{2}\right) = i$$

$$\therefore \quad a^{1929} = a \cdot a^{1928} = a \cdot (a^2)^{964} = a(i)^{964}$$

$$= a(i)^{4 \times 241} = a \cdot (i^4)^{241} = a$$

Example 7. Dividing $f(z)$ by $z - i$, where $i = \sqrt{-1}$, we obtain the remainder i and dividing it by $z + i$, we get the remainder $1 + i$. Find the remainder upon the division of $f(z)$ by $z^2 + 1$.

Sol. $z - i = 0 \Rightarrow z = i$

Remainder, when $f(z)$ is divided by $(z - i) = i$

i.e. $\quad f(i) = i \quad \ldots$(i)

and remainder, when $f(z)$ is divided by $(z + 1) = 1 + i$

i.e. $\quad f(-i) = 1 + i \quad [\because z + i = 0 \Rightarrow z = -i] \ldots$(ii)

Since, $z^2 + 1$ is a quadratic expression, therefore remainder when $f(z)$ is divided by $z^2 + 1$, will be in general a linear expression. Let $g(z)$ be the quotient and $az + b$ (where a and b are complex numbers) be the remainder, when $f(z)$ is divided by $z^2 + 1$.

Then, $\quad f(z) = (z^2 + 1)g(z) + az + b \quad \ldots$(iii)

$\therefore \quad f(i) = (i^2 + 1)g(i) + ai + b = ai + b$

or $\quad ai + b = i \quad$ [from Eq. (i)] ... (iv)

and $\quad f(-i) = (i^2 + 1)g(-i) - ai + b = -ai + b$

or $\quad -ai + b = 1 + i \quad$ [from Eq. (ii)] ...(v)

From Eqs. (iv) and (v), we get

$$b = \frac{1}{2} + i \quad \text{and} \quad a = \frac{i}{2}$$

Hence, required remainder $= az + b$

$$= \frac{1}{2}iz + \frac{1}{2} + i$$

The Sum of Four Consecutive Powers of *i* (Iota) is Zero

If $n \in I$ and $i = \sqrt{-1}$, then

$$i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n(1 + i + i^2 + i^3)$$
$$= i^n(1 + i - 1 - i) = 0$$

Remark

opposite sign

1. $\sum_{r=p}^{m} f(r) = \sum_{r=1}^{m-p+1} f(r + p - 1)$

2. $\sum_{r=-p}^{m} f(r) = \sum_{r=1}^{m+p+1} f(r - p - 1)$

★

Example 8. Find the value of $\sum_{n=1}^{13}(i^n + i^{n+1})$ (where, $i = \sqrt{-1}$)

Sol. $\because \sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13} i^n + \sum_{n=1}^{13} i^{n+1} = (i+0) + (i^2+0)$

$$= i - 1 \left[\because \sum_{n=2}^{13} i^n = 0 \text{ and } \sum_{n=2}^{13} i^{n+1} = 0 \text{ (three sets of four consecutive powers of } i) \right]$$

any four consecutive

Example 9. Find the value of $\sum_{n=0}^{100} i^{n!}$ (where, $i = \sqrt{-1}$).

Sol. $n!$ is divisible by 4, $\forall\, n \ge 4$.

$$\therefore \sum_{n=4}^{100} i^{n!} = \sum_{n=1}^{97} i^{(n+3)!}$$

$$= i^0 + i^0 + i^0 + \dots 97 \text{ times } = 97 \qquad \dots\text{(i)}$$

$$\therefore \sum_{n=0}^{100} i^{n!} = \sum_{n=0}^{3} i^{n!} + \sum_{n=4}^{100} i^{n!}$$

$$= i^{0!} + i^{1!} + i^{2!} + i^{3!} + 97 \qquad \text{[from Eq. (i)]}$$

$$= i^1 + i^1 + i^2 + i^6 + 97 = i + i - 1 - 1 + 97$$

$$= 95 + 2i$$

Example 10. Find the value of $\sum_{r=1}^{4n+7} i^r$ (where, $i = \sqrt{-1}$).

Sol. $\sum_{r=1}^{4n+7} i^r = i^1 + i^2 + i^3 + \sum_{r=4}^{4n+7} i^r = i - 1 - i + \sum_{r=1}^{4n+4} i^{r+3}$

$$= -1 + 0 \quad [(n+1) \text{ sets of four consecutive powers of } i]$$

$$= -1$$

Example 11. Show that the polynomial $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$ is divisible by $x^3 + x^2 + x + 1$, where $p, q, r, s \in N$.

Sol. Let $f(x) = x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$

and $x^3 + x^2 + x + 1 = (x^2+1)(x+1)$

$$= (x+i)(x-i)(x+1),$$

where $i = \sqrt{-1}$

Now, $f(i) = i^{4p} + i^{4q+1} + i^{4r+2} + i^{4s+3} = 1 + i + i^2 + i^3 = 0$

[sum of four consecutive powers of i is zero]

$$f(-i) = (-i)^{4p} + (-i)^{4q+1} + (-i)^{4r+2} + (-i)^{4s+3}$$

$$= 1 + (-i)^1 + (-i)^2 + (-i)^3 = 1 - i - 1 + i = 0$$

and $f(-1) = (-1)^{4p} + (-1)^{4q+1} + (-1)^{4r+2} + (-1)^{4s+3}$

$$= 1 - 1 + 1 - 1 = 0$$

Hence, by division theorem, $f(x)$ is divisible by $x^3 + x^2 + x + 1$.

Switch System Theory

(Finding Digit in the Unit's Place)

We can determine the digit in the unit's place in a^b, where $a, b \in N$. If last digit of a are 0, 1, 5 and 6, then digits in the unit's place of a^b are 0, 1, 5 and 6 respectively, for all $b \in N$.

Powers of 2

$2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, \dots$ the digits in unit's place of different powers of 2 are as follows :

2, 4, 8, 6, 2, 4, 8, 6, 2,... (period being 4)

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

①②③⓪①②③⓪①... (switch number)

(The remainder when b is divided by 4, can be 1 or 2 or 3 or 0). Then, press the switch number and then we get the digit in unit's place of a^b (just above the switch number) i.e. 'press the number and get the answer'.

Example 12. What is the digit in the unit's place of $(5172)^{11327}$?

Sol. Here, last digit of a is 2.

The remainder when 11327 is divided by 4, is 3. Then, press switch number 3 and then we get 8.

Hence, the digit in the unit's place of $(5172)^{11327}$ is 8.

Powers of 3

$3^1, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, \dots$ the digits in unit's place of different powers of 3 are as follows:

3, 9, 7, 1, 3, 9, 7, 1,... (period being 4)

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

①②③⓪①②③⓪ ... (switch number)

The remainder when b is divided by 4, can be 1 or 2 or 3 or 0. Now, press the switch number and get the unit's place digit (just above).

Example 13. What is the digit in the unit's place of $(143)^{86}$?

Sol. Here, last digit of a is 3.

The remainder when 86 is divided by 4, is 2.

Then, press switch number 2 and then we get 9.

Hence, the digit in the unit's place of $(143)^{86}$ is 9.

Powers of 4

$4^1, 4^2, 4^3, 4^4, 4^5, \ldots$ the digits in unit's place of different powers of 4 are as follows:

4, 6, 4, 6, 4, ... (period being 2)

↑ ↑ ↑ ↑ ↑

①⓪①⓪① ... (switch number)

The remainder when b is divided by 2, can be 1 or 0. Now, press the switch number and get the unit's place digit (just above the switch number).

Example 14. What is the digit in unit's place of $(1354)^{22222}$?

Sol. Here, last digit of a is 4.

The remainder when 22222 is divided by 2, is 0. Then, press switch number 0 and then we get 6.

Hence, the digit in the unit's place of $(1354)^{22222}$ is 6.

Powers of 7

$7^1, 7^2, 7^3, 7^4, 7^5, 7^6, 7^7, 7^8, \ldots$ the digits in unit's place of different powers of 7 are as follows:

7, 9, 3, 1, 7, 9, 3, 1, ... (period being 4)

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

①②③⓪①②③⓪ ... (switch number)

(The remainder when b is divided by 4, can be 1 or 2 or 3 or 0). Now, press the switch number and get the unit's place digit (just above).

Example 15. What is the digit in the unit's place of $(13057)^{941120579}$?

Sol. Here, last digit of a is 7.

The remainder when 941120579 is divided by 4, is 3. Then, press switch number 3 and then we get 3.

Hence, the digit in the unit's place of $(13057)^{941120579}$ is 3.

Powers of 8

$8^1, 8^2, 8^3, 8^4, 8^5, 8^6, 8^7, 8^8, \ldots$ the digits in unit's place of different powers of 8 are as follows:

8, 4, 2, 6, 8, 4, 2, 6, ... (period being 4)

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

①②③⓪①②③⓪ ... (switch number)

The remainder when b is divided by 4, can be 1 or 2 or 3 or 0.

Now, press the switch number and get the unit's place digit (just above the switch number).

Example 16. What is the digit in the unit's place of $(1008)^{786}$?

Sol. Here, last digit of a is 8.

The remainder when 786 is divided by 4, is 2. Then, press switch number 2 and then we get 4.

Hence, the digit in the unit's place of $(1008)^{786}$ is 4.

Powers of 9

$9^1, 9^2, 9^3, 9^4, 9^5, \ldots$ the digits in unit's place of different powers of 9 are as follows:

9, 1, 9, 1, 9, ... (period being 2)

↑ ↑ ↑ ↑ ↑

①⓪①⓪① ... (switch number)

The remainder when b is divided by 2, can be 1 or 0.

Now, press the switch number and get the unit's place digit (just above the switch number).

Example 17. What is the digit in the unit's place of $(2419)^{111213}$?

Sol. Here, last digit of a is 9.

The remainder when 111213 is divided by 2, is 1. Then, press switch number 1 and then we get 9.

Hence, the digit in the unit's place of $(2419)^{111213}$ is 9.

Exercise for Session 1

by putting options.

1. If $(1+i)^{2n}+(1-i)^{2n}=-2^{n+1}$ (where, $i=\sqrt{-1}$) for all those n, which are
(a) even
(b) odd
(c) multiple of 3
(d) None of these

2. If $i=\sqrt{-1}$, the number of values of i^n+i^{-n} for different $n \in I$ is
(a) 1
(b) 2
(c) 3
(d) 4

3. If $a>0$ and $b<0$, then $\sqrt{a}\,\sqrt{b}$ is equal to (where, $i=\sqrt{-1}$)
(a) $-\sqrt{a\cdot|b|}$
(b) $\sqrt{a\cdot|b|}\,i$
(c) $\sqrt{a\cdot|b|}$
(d) None of these

4. Consider the following statements.
$S_1: -6=2i\times 3i=\sqrt{(-4)}\times\sqrt{(-9)}$ (where, $i=\sqrt{-1}$)
$S_2: \sqrt{(-4)}\times\sqrt{(-9)}=\sqrt{(-4)\times(-9)}$
$S_3: \sqrt{(-4)\times(-9)}=\sqrt{36}$
$S_4: \sqrt{36}=6$
Of these statements, the incorrect one is
(a) S_1 only
(b) S_2 only
(c) S_3 only
(d) None of these

5. The value of $\sum_{n=0}^{50} i^{(2n+1)!}$ (where, $i=\sqrt{-1}$) is
(a) i
(b) $47-i$
(c) $48+i$
(d) 0

6. The value of $\sum_{r=-3}^{1003} i^r$ (where $i=\sqrt{-1}$) is
(a) 1
(b) -1
(c) i
(d) $-i$

7. The digit in the unit's place of $(153)^{98}$ is
(a) 1
(b) 3
(c) 7
(d) 9

8. The digit in the unit's place of $(141414)^{12121}$ is
(a) 4
(b) 6
(c) 3
(d) 1

Session 2

Definition of Complex Number, Conjugate Complex Numbers, Representation of a Complex Number in Various Forms

Definition of Complex Number

A number of the form $a+ib$, where $a, b \in R$ and $i=\sqrt{-1}$, is called a **complex number.** It is denoted by z i.e. $z=a+ib$.

A complex number may also be defined as an ordered pair of real numbers; and may be denoted by the symbol (a, b). If we write $z=(a, b)$, then a is called the real part and b is the imaginary part of the complex number z and may be denoted by Re (z) and Im (z), respectively i.e., $a=\text{Re}(z)$ and $b=\text{Im}(z)$.

Two complex numbers are said to be equal, if and only if their real parts and imaginary parts are separately equal.

Thus, $\quad a+ib=c+id$

$\Leftrightarrow \quad a=c$ and $b=d$

where, $a, b, c, d \in R$ and $i=\sqrt{-1}$.

i.e. $\quad z_1=z_2$

$\Leftrightarrow \quad \text{Re}(z_1)=\text{Re}(z_2)$ and $\text{Im}(z_1)=\text{Im}(z_2)$

Important Properties of Complex Numbers

1. The complex numbers do not possess the property of order, i.e., $(a+ib) >$ or $< (c+id)$ is not defined. *For example*, $9+6i > 3+2i$ makes no sense.
2. A real number a can be written as $a+i \cdot 0$. Therefore, every real number can be considered as a complex number, whose imaginary part is zero. Thus, the set of real numbers (R) is a proper subset of the complex numbers (C) i.e. $R \subset C$. Hence, the complex number system is $N \subset W \subset I \subset Q \subset R \subset C$.
3. A complex number z is said to be purely real, if $\text{Im}(z)=0$; and is said to be purely imaginary, if Re $(z)=0$. The complex number $0=0+i \cdot 0$ is both purely real and purely imaginary.
4. In real number system, $a^2+b^2=0 \Rightarrow a=0=b$.

   But if z_1 and z_2 are complex numbers, then $z_1^2+z_2^2=0$ does not imply $z_1=z_2=0$.

   For example, $z_1=1+i$ and $z_2=1-i$

   Here, $z_1 \neq 0, z_2 \neq 0$

   But $z_1^2+z_2^2=(1+i)^2+(1-i)^2=1+i^2+2i+1+i^2-2i$
   $=2+2i^2=2-2=0$

   However, if product of two complex numbers is zero, then atleast one of them must be zero, same as in case of real numbers.

   If $z_1 z_2=0$, then $z_1=0, z_2 \neq 0$ or $z_1 \neq 0, z_2=0$

   or $\quad z_1=0, z_2=0$

Algebraic Operations on Complex Numbers

Let two complex numbers be $z_1=a+ib$ and $z_2=c+id$, where $a, b, c, d \in R$ and $i=\sqrt{-1}$.

1. **Addition** $z_1+z_2=(a+ib)+(c+id)$
   $=(a+c)+i(b+d)$
2. **Subtraction** $z_1-z_2=(a+ib)-(c+id)$
   $=(a-c)+i(b-d)$
3. **Multiplication** $z_1 \cdot z_2=(a+ib) \cdot(c+id)$
   $=ac+iad+ibc+i^2 bd$
   $=ac+i(ad+bc)-bd$
   $=(ac-bd)+i(ad+bc)$
4. **Division** $\dfrac{z_1}{z_2}=\dfrac{(a+ib)}{(c+id)} \cdot \dfrac{(c-id)}{(c-id)}$

   [multiplying numerator and denominator by $c-id$ where atleast one of c and d is non-zero]

$$=\frac{ac-iad+ibc-i^2 bd}{(c)^2-(id)^2}=\frac{ac+i(bc-ad)+bd}{c^2-i^2 d^2}$$

$$=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}=\frac{(ac+bd)}{(c^2+d^2)}+i \frac{(bc-ad)}{(c^2+d^2)}$$

Remark

$\dfrac{1+i}{1-i}=i$ and $\dfrac{1-i}{1+i}=-i$, where $i=\sqrt{-1}$.

Properties of Algebraic Operations on Complex Numbers

Let z_1, z_2 and z_3 be any three complex numbers. Then, their algebraic operations satisfy the following properties :

Properties of Addition of Complex Numbers

(i) **Closure law** z_1+z_2 is a complex number.

(ii) **Commutative law** $z_1+z_2=z_2+z_1$

(iii) **Associative law** $(z_1+z_2)+z_3=z_1+(z_2+z_3)$

(iv) **Additive identity** $z+0=z=0+z$, then 0 is called the additive identity.

(v) **Additive inverse** $-z$ is called the additive inverse of z, i.e. $z+(-z)=0$.

Properties of Multiplication of Complex Numbers

(i) **Closure law** $z_1 \cdot z_2$ is a complex number.

(ii) **Commutative law** $z_1 \cdot z_2 = z_2 \cdot z_1$

(iii) **Associative law** $(z_1 \cdot z_2) z_3 = z_1 (z_2 \cdot z_3)$

(iv) **Multiplicative identity** $z \cdot 1 = z = 1 \cdot z$, then 1 is called the multiplicative identity.

(v) **Multiplicative inverse** If z is a non-zero complex number, then $\frac{1}{z}$ is called the multiplicative inverse of z i.e. z. $\frac{1}{z} = 1 = \frac{1}{z} \cdot z$

(vi) **Multiplication is distributive with respect to addition** $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

Conjugate Complex Numbers

The complex numbers $z=(a, b)=a+ib$ and $\bar{z}=(a,-b)=a-ib$, where a and b are real numbers, $i=\sqrt{-1}$ and $b \neq 0$, are said to be complex conjugate of each other (here, the complex conjugate is obtained by just changing the sign of i).

Note that, sum $=(a+ib)+(a-ib)=2a$, which is real.

And product $=(a+ib)(a-ib)=a^2-(ib)^2$

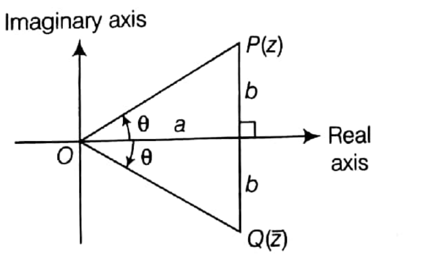
$$= a^2 - i^2 b^2 = a^2 - (-1) b^2$$

$$= a^2 + b^2, \text{ which is real.}$$

Geometrically, $\bar{z}$ is the mirror image of z along real axis on argand plane.

Remark

Let $z=-a-ib, a>0, b>0=(-a,-b)$ (III quadrant)



Then, $\bar{z}=-a+ib=(-a, b)$ (II quadrant). Now,

(i) If z lies in I quadrant, then $\bar{z}$ lies in IV quadrant and *vice-versa.*

(ii) If z lies in II quadrant, then $\bar{z}$ lies in III quadrant and *vice-versa.*

Properties of Conjugate Complex Numbers

Let z, z_1 and z_2 be complex numbers. Then,

(i) $\overline{(\bar{z})} = z$

(ii) $z+\bar{z}=2 \operatorname{Re}(z)$

(iii) $z-\bar{z}=2 \operatorname{Im}(z)$

(iv) $z+\bar{z}=0 \Rightarrow z=-\bar{z} \Rightarrow z$ is purely imaginary.

(v) $z-\bar{z}=0 \Rightarrow z=\bar{z} \Rightarrow z$ is purely real.

(vi) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ Ingeneral,

$$\overline{z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n} = \overline{z_1} \pm \overline{z_2} \pm \overline{z_3} \pm \ldots \pm \overline{z_n}$$

(vii) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

In general, $\overline{z_1 \cdot z_2 \cdot z_3 \ldots z_n} = \overline{z_1} \cdot \overline{z_2} \cdot \overline{z_3} \ldots \overline{z_n}$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(ix) $\overline{z^n} = (\bar{z})^n$

(x) $z_1 \overline{z_2} + \overline{z_1} z_2 = 2 \operatorname{Re}(z_1 \overline{z_2}) = 2 \operatorname{Re}(\overline{z_1} z_2)$

(xi) If $z=f(z_1, z_2)$, then $\bar{z}=f(\overline{z_1}, \overline{z_2})$

| Example 18. If $\frac{x-3}{3+i}+\frac{y-3}{3-i}=i$, where $x, y \in R$ and $i=\sqrt{-1}$, find the values of x and y.

Sol. $\because \quad \frac{x-3}{3+i}+\frac{y-3}{3-i}=i$

$\Rightarrow \quad (x-3)(3-i)+(y-3)(3+i)=i(3+i)(3-i)$

$\Rightarrow (3x-xi-9+3i)+(3y+yi-9-3i)=10i$

$\Rightarrow \quad (3x+3y-18)+i(y-x)=10i$

On comparing real and imaginary parts, we get

$$3x+3y-18=0$$

$\Rightarrow \quad x+y=6 \quad$ ...(

and $\quad y-x=10 \quad$ ...(i

On solving Eqs. (i) and (ii), we get

$$x=-2, \ y=8$$

| Example 19. If $(a+ib)^5=p+iq$, where $i=\sqrt{-1}$, prove that $(b+ia)^5=q+ip$.

Sol. $\because \quad (a+ib)^5 = p+iq$

$\therefore \quad \overline{(a+ib)^5} = \overline{p+iq} \Rightarrow (a-ib)^5=(p-iq)$

$\Rightarrow \quad (-i^2a-ib)^5=(-i^2p-iq) \quad$ [$\because i^2=-$

$\Rightarrow \quad (-i)^5(b+ia)^5=(-i)(q+ip)$

$\Rightarrow \quad (-i)(b+ia)^5=(-i)(q+ip)$

$\therefore \quad (b+ia)^5=(q+ip)$

| Example 20. Find the least positive integral value of n, for which $\left(\dfrac{1-i}{1+i}\right)^n$, where $i=\sqrt{-1}$, is purely imaginary with positive imaginary part.

Sol. $\left(\dfrac{1-i}{1+i}\right)^n=\left(\dfrac{1-i}{1+i}\times\dfrac{1-i}{1-i}\right)^n=\left(\dfrac{1+i^2-2i}{2}\right)^n=\left(\dfrac{1-1-2i}{2}\right)^n$

$= (-i)^n =$ Imaginary

$\Rightarrow$ $n = 1, 3, 5, \ldots$ for positive imaginary part $n = 3$.

| Example 21. If the multiplicative inverse of a complex number is $(\sqrt{3}+4i)/19$, where $i=\sqrt{-1}$, find complex number.

Sol. Let z be the complex number.

Then, $z\cdot\left(\dfrac{\sqrt{3}+4i}{19}\right)=1$

or $$z=\frac{19}{(\sqrt{3}+4i)}\times\frac{(\sqrt{3}-4i)}{(\sqrt{3}-4i)}$$

$$=\frac{19(\sqrt{3}-4i)}{19}=(\sqrt{3}-4i)$$

| Example 22. Find real θ, such that $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$, where $i=\sqrt{-1}$, is

(i) purely real. (ii) purely imaginary.

Sol. Let $z=\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$

On multiplying numerator and denominator by conjugate of denominator,

$$z=\frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}=\frac{(3-4\sin^2\theta)+8i\sin\theta}{(1+4\sin^2\theta)}$$

$$=\frac{(3-4\sin^2\theta)}{(1+4\sin^2\theta)}+i\frac{(8\sin\theta)}{(1+4\sin^2\theta)}$$

(i) For purely real, $\text{Im}(z)=0$

$\Rightarrow$ $$\frac{8\sin\theta}{1+4\sin^2\theta}=0 \text{ or } \sin\theta=0$$

$\therefore$ $$\theta=n\pi,\ n\in I$$

(ii) For purely imaginary, $\text{Re}(z)=0$

$\Rightarrow$ $$\frac{(3-4\sin^2\theta)}{(1+4\sin^2\theta)}=0 \text{ or } 3-4\sin^2\theta=0$$

or $$\boxed{\sin^2\theta=\frac{3}{4}=\left(\frac{\sqrt{3}}{2}\right)^2=\left(\sin\frac{\pi}{3}\right)^2}$$

$\therefore$ $$\theta=n\pi\pm\frac{\pi}{3},\ n\in I$$

| Example 23. Find real values of x and y for which the complex numbers $-3+ix^2y$ and x^2+y+4i, where $i=\sqrt{-1}$, are conjugate to each other.

Sol. Given, $-3+ix^2y=\overline{x^2+y+4i}$

$\Rightarrow$ $-3-ix^2y=x^2+y+4i$

On comparing real and imaginary parts, we get

$$x^2+y=-3 \quad \text{...(i)}$$

and $$-x^2y=4 \quad \text{...(ii)}$$

From Eq. (ii), we get $x^2=-\dfrac{4}{y}$

Then, $-\dfrac{4}{y}+y=-3$ $\left[\text{putting } x^2=-\dfrac{4}{y} \text{ in Eq. (i)}\right]$

$$y^2+3y-4=0 \Rightarrow (y+4)(y-1)=0$$

$\therefore$ $y=-4, 1$

For $y=-4,\ x^2=1 \Rightarrow x=\pm 1$

For $y=1,\ x^2=-4$ [impossible]

$\therefore$ $x=\pm 1,\ y=-4$

| Example 24. If $x=-5+2\sqrt{-4}$, find the value of $x^4+9x^3+35x^2-x+4$.

Sol. Since, $x=-5+2\sqrt{-4} \Rightarrow x+5=4i$

$\Rightarrow$ $(x+5)^2=(4i)^2 \Rightarrow x^2+10x+25=-16$

$\therefore$ $$x^2+10x+41=0 \quad \text{...(i)}$$

Now,

$$
\begin{array}{l|l|l}
x^2+10x+41 & x^4+9x^3+35x^2-x+4 & x^2-x+4 \\
 & x^4+10x^3+41x^2 & \\
 & \underline{-\quad\ -\quad\ -} & \\
 & -x^3-6x^2-x+4 & \\
 & -x^3-10x^2-41x & \\
 & \underline{+\quad\ +\quad\ +} & \\
 & 4x^2+40x+4 & \\
 & 4x^2+40x+164 & \\
 & \underline{-\quad\ -\quad\ -} & \\
 & -160 &
\end{array}
$$

$\therefore$ $x^4+9x^3+35x^2-x+4$

$$=(x^2+10x+41)(x^2-x+4)-160$$

$$=0-160=-160 \quad \text{[from Eq. (i)]}$$

| Example 25. Let z be a complex number satisfying the equation $z^2-(3+i)z+\lambda+2i=0$, where $\lambda\in R$ and $i=\sqrt{-1}$. Suppose the equation has a real root, find the non-real root.

Sol. Let α be the real root. Then,

$$\alpha^2-(3+i)\alpha+\lambda+2i=0$$

$\Rightarrow \quad (\alpha^2 - 3\alpha + \lambda) + i(2 - \alpha) = 0$

On comparing real and imaginary parts, we get

$$\alpha^2 - 3\alpha + \lambda = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad 2 - \alpha = 0 \quad \text{...(ii)}$$

From Eq. (ii), $\alpha = 2$

Let other root be β.

Then, $\alpha + \beta = 3 + i \Rightarrow 2 + \beta = 3 + i$

$\therefore \quad \beta = 1 + i$

Hence, the non-real root is $1 + i$.

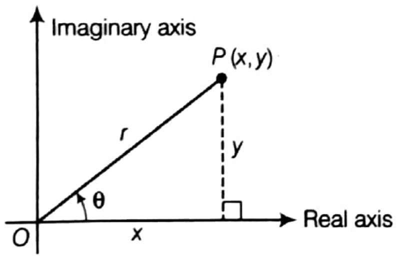
Representation of a Complex Number in Various Forms

Cartesian Form
(Geometrical Representation)

Every complex number $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, can be represented by a point in the cartesian plane known as complex plane (Argand plane) by the ordered pair (x, y).

Modulus and Argument of a Complex Number

Let $\quad z = x + iy = (x, y)$ for all $x, y \in R$ and $i = \sqrt{-1}$.



The length OP is called modulus of the complex number z denoted by $|z|$,

i.e. $\quad OP = r = |z| = \sqrt{(x^2 + y^2)}$

and if $(x, y) \neq (0, 0)$, then θ is called the argument or amplitude of z,

i.e. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ [angle made by OP with positive X-axis]

or $\quad \arg(z) = \tan^{-1}(y/x)$

Also, argument of a complex number is not unique, since if θ is a value of the argument, so also is $2n\pi + \theta$, where $n \in I$. But usually, we take only that value for which $0 \leq \theta < 2\pi$. Any two arguments of a complex number differ by $2n\pi$.

Argument of z will be $\theta, \pi - \theta, \pi + \theta$ and $2\pi - \theta$ according as the point z lies in I, II, III and IV quadrants respectively, where $\theta = \tan^{-1}\left|\frac{y}{x}\right|$.

| Example 26. Find the arguments of $z_1 = 5 + 5i$, $z_2 = -4 + 4i$, $z_3 = -3 - 3i$ and $z_4 = 2 - 2i$, where $i = \sqrt{-1}$.

Sol. Since, z_1, z_2, z_3 and z_4 lies in I, II, III and IV quadrants respectively. The arguments are given by

$$\arg(z_1) = \tan^{-1}\left|\frac{5}{5}\right| = \tan^{-1} 1 = \pi/4$$

$$\arg(z_2) = \pi - \tan^{-1}\left|\frac{4}{-4}\right| = \pi - \tan^{-1} 1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\arg(z_3) = \pi + \tan^{-1}\left|\frac{-3}{-3}\right| = \pi + \tan^{-1} 1 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

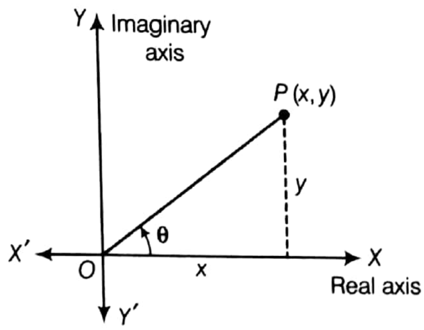
and $\arg(z_4) = 2\pi - \tan^{-1}\left|\frac{-2}{2}\right|$

$$= 2\pi - \tan^{-1} 1 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Principal Value of the Argument

The value θ of the argument which satisfies the inequality $-\pi < \theta \leq \pi$ is called the **principal value** of the argument.

If $z = x + iy = (x, y), \forall x, y \in R$ and $i = \sqrt{-1}$, then $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ always gives the principal value. It depends on the quadrant in which the point (x, y) lies.



(i) $(x, y) \in$ first quadrant $x > 0, y > 0$.

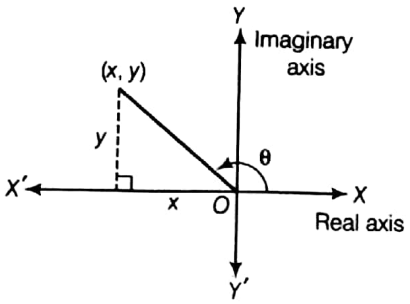
The principal value of $\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

It is an acute angle and positive.

(ii) $(x, y) \in$ second quadrant $x < 0, y > 0$.

The principal value of $\arg(z) = \theta$

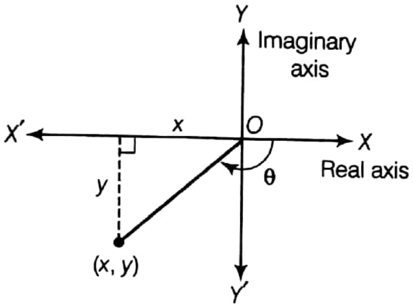
$$= \pi - \tan^{-1}\left(\frac{y}{|x|}\right)$$



It is an obtuse angle and positive.

(iii) $(x, y) \in$ third quadrant $x < 0, y < 0$.

The principal value of $\arg(z) = \theta = -\pi + \tan^{-1}\left(\dfrac{y}{x}\right)$

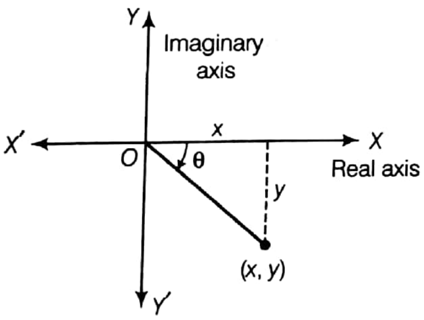


It is an obtuse angle and negative.

(iv) $(x, y) \in$ fourth quadrant $x > 0, y < 0$.

The principal value of $\arg(z) = \theta$

$$= -\tan^{-1}\left(\frac{|y|}{x}\right)$$



It is an acute angle and negative.

| Example 27. Find the principal values of the arguments of $z_1 = 2+2i$, $z_2 = -3+3i$, $z_3 = -4-4i$ and $z_4 = 5-5i$, where $i = \sqrt{-1}$.

all bye cartesian plane graph

Sol. Since, z_1, z_2, z_3 and z_4 lies in I, II, III and IV quadrants respectively. The principal values of the arguments are given by

$$\tan^{-1}\left(\frac{2}{2}\right),\ \pi - \tan^{-1}\left(\frac{3}{|-3|}\right), -\pi + \tan^{-1}\left(\frac{-4}{-4}\right),$$

$$-\tan^{-1}\left(\frac{|-5|}{5}\right)$$

or $\quad \tan^{-1} 1, \pi - \tan^{-1} 1, -\pi + \tan^{-1} 1, \ -\tan^{-1} 1$

or $\quad \dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, -\pi + \dfrac{\pi}{4}, -\dfrac{\pi}{4}$ or $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}$

Hence, the principal values of the arguments of z_1, z_2, z_3 and z_4 are $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}$, respectively.

Remark

1. Unless otherwise stated, amp z implies principal value of the argument.
2. Argument of the complex number 0 is not defined.
3. If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and $\arg(z_1) = \arg(z_2)$.
4. If $\arg(z) = \pi/2$ or $-\pi/2$, z is purely imaginary.
5. If $\arg(z) = 0$ or π, z is purely real.

| Example 28. Find the argument and the principal value of the argument of the complex number $z = \dfrac{2+i}{4i + (1+i)^2}$, where $i = \sqrt{-1}$.

Sol. Since, $z = \dfrac{2+i}{4i+(1+i)^2} = \dfrac{2+i}{4i+1+i^2+2i} = \dfrac{2+i}{6i} = \dfrac{1}{6} - \dfrac{1}{3}i$

$\therefore$ z lies in IV quadrant.

Here, $\theta = \tan^{-1}\left|\dfrac{-\frac{1}{3}}{\frac{1}{6}}\right| = \tan^{-1} 2$

$\therefore \arg(z) = 2\pi - \theta = 2\pi - \tan^{-1} 2$

Hence, principal value of $\arg(z) = -\theta = -\tan^{-1} 2$.

Properties of Modulus

(i) $|z| \geq 0 \Rightarrow |z| = 0$, iff $z = 0$ and $|z| > 0$, iff $z \neq 0$

(ii) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1||z_2|$

In general, $|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3|\ldots|z_n|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\ (z_2 \neq 0)$

(vii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In general, $|z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n| \leq |z_1| + |z_2| + |z_3| + \ldots + |z_n|$

(viii) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$

(ix) $|z^n| = |z|^n$

(x) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Thus, $|z_1|+|z_2|$ is the greatest possible value of $|z_1+z_2|$ and $||z_1|-|z_2||$ is the least possible value of $|z_1+z_2|$.

(xi) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$

or $|z_1|^2 + |z_2|^2 \pm 2\,\text{Re}\,(z_1\bar{z}_2)$

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

(xiii) $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

(xiv) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\{|z_1|^2+|z_2|^2\}$

(xv) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2+b^2)(|z_1|^2+|z_2|^2)$, where $a, b \in R$

(xvi) Unimodular i.e., unit modulus

If z is unimodular, then $|z|=1$. In case of unimodular, let $z = \cos\theta + i\sin\theta, \theta \in R$ and $i=\sqrt{-1}$.

Remark

1. If $f(z)$ is unimodular, then $|f(z)|=1$ and let $f(z) = \cos\theta + i\sin\theta, \theta \in R$ and $i=\sqrt{-1}$.
2. $\dfrac{z}{|z|}$ is always a unimodular complex number, if $z \neq 0$.

(xvii) The multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{\bar{z}}{|z|^2}$.

Example 29. If $\theta_i \in [0, \pi/6], i = 1,2,3,4,5$ and $\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5 = 2$, show that $\dfrac{3}{4} < |z| < 1$.

Sol. Given that,

$$\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5 = 2$$

or $2 = |\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5|$

$2 \le |\sin\theta_1 z^4| + |\sin\theta_2 z^3| + |\sin\theta_3 z^2| + |\sin\theta_4 z| + |\sin\theta_5|$ [by property (vii)]

$\Rightarrow 2 \le |\sin\theta_1||z^4| + |\sin\theta_2||z^3| + |\sin\theta_3||z^2| + |\sin\theta_4||z| + |\sin\theta_5|$ [by property (v)]

$\Rightarrow 2 \le |\sin\theta_1||z|^4 + |\sin\theta_2||z|^3 + |\sin\theta_3||z|^2 + |\sin\theta_4||z| + |\sin\theta_5|$ [by property (ix)] ...(i)

But given, $\theta_i \in [0, \pi/6]$

$\therefore \quad \sin\theta_i \in \left[0, \dfrac{1}{2}\right]$,

i.e. $\quad 0 \le \sin\theta_i \le \dfrac{1}{2}$

$\therefore$ Inequality Eq. (i) becomes,

$$2 \le \frac{1}{2}|z|^4 + \frac{1}{2}|z|^3 + \frac{1}{2}|z|^2 + \frac{1}{2}|z| + \frac{1}{2}$$

$\Rightarrow \quad 3 \le |z|^4 + |z|^3 + |z|^2 + |z|$

$\Rightarrow \quad 3 \le |z| + |z|^2 + |z|^3 + |z|^4 < |z| + |z|^2 + |z|^3 + |z|^4 + \dots + \infty$

$\Rightarrow \quad 3 < |z| + |z|^2 + |z|^3 + |z|^4 + \dots + \infty$

$\Rightarrow \quad 3 < \dfrac{|z|}{1-|z|}$ [here, $|z|<1$]

$\Rightarrow \quad 3 - 3|z| < |z| \Rightarrow 3 < 4|z|$

$\therefore \quad |z| > \dfrac{3}{4}$

Hence, $\quad \dfrac{3}{4} < |z| < 1$ [$\because |z| < 1$]

Example 30. If $|z-2+i| \le 2$, find the greatest and least values of $|z|$, where $i=\sqrt{-1}$.

Sol. Given that, $|z-2+i| \le 2$ ...(i)

$\because \quad |z-2+i| \ge ||z| - |2-i||$ [by property (x)]

$\therefore \quad |z-2+i| \ge ||z| - \sqrt{5}|$ ...(ii)

From Eqs. (i) and (ii), we get

$$||z|-\sqrt{5}| \le |z-2+i| \le 2$$

$\therefore \quad ||z|-\sqrt{5}| \le 2$

$\Rightarrow \quad -2 \le |z| - \sqrt{5} \le 2$

$\Rightarrow \quad \sqrt{5}-2 \le |z| \le \sqrt{5}+2$

Hence, greatest value of $|z|$ is $\sqrt{5}+2$ and least value of $|z|$ is $\sqrt{5}-2$.

Example 31. If z is any complex number such that $|z+4| \le 3$, find the greatest value of $|z+1|$.

Sol. $\because |z+1| = |(z+4)-3|$

$= |(z+4)+(-3)| \le |z+4| + |-3|$

$= |z+4| + 3$

$\le 3 + 3 = 6$ [$\because |z+4| \le 3$]

$\therefore \quad |z+1| \le 6$

Hence, the greatest value of $|z+1|$ is 6.

| Example 32. If $|z_1|=1, |z_2|=2, |z_3|=3$ and $|9z_1z_2+4z_3z_1+z_2z_3|=6$, find the value of $|z_1+z_2+z_3|$.

Sol. $\because \quad |z_1|=1 \Rightarrow |z_1|^2=1$

$\Rightarrow \quad z_1\bar{z}_1=1 \Rightarrow \dfrac{1}{z_1}=\bar{z}_1$

$|z_2|=2 \Rightarrow |z_2|^2=4 \Rightarrow z_2\bar{z}_2=4$

$\Rightarrow \quad \dfrac{4}{z_2}=\bar{z}_2$ and $|z_3|=3 \Rightarrow |z_3|^2=9$

$\Rightarrow \quad z_3\bar{z}_3=9 \Rightarrow \dfrac{9}{z_3}=\bar{z}_3$

and given $|9z_1z_2+4z_3z_1+z_2z_3|=6$

$\Rightarrow \quad |z_1z_2z_3|\left|\dfrac{9}{z_3}+\dfrac{4}{z_2}+\dfrac{1}{z_1}\right|=6$

$\Rightarrow \quad |z_1||z_2||z_3||\bar{z}_3+\bar{z}_2+\bar{z}_1|=6$

$\left[\because \dfrac{1}{z_1}=\bar{z}_1, \dfrac{4}{z_2}=\bar{z}_2 \text{ and } \dfrac{9}{z_3}=\bar{z}_3\right]$

$\Rightarrow \quad 1\cdot2\cdot3\,|\overline{z_1+z_2+z_3}|=6$

$\therefore \quad |z_1+z_2+z_3|=1 \qquad [\because |\bar{z}|=|z|]$

| Example 33. Prove that

$$|z_1|+|z_2|=\left|\frac{1}{2}(z_1+z_2)+\sqrt{z_1z_2}\right|+\left|\frac{1}{2}(z_1+z_2)-\sqrt{z_1z_2}\right|.$$

Sol. RHS $=\left|\dfrac{1}{2}(z_1+z_2)+\sqrt{z_1z_2}\right|+\left|\dfrac{1}{2}(z_1+z_2)-\sqrt{z_1z_2}\right|$

$=\left|\dfrac{z_1+z_2+2\sqrt{z_1z_2}}{2}\right|+\left|\dfrac{z_1+z_2-2\sqrt{z_1z_2}}{2}\right|$

$=\dfrac{1}{2}\{|\sqrt{z_1}+\sqrt{z_2}|^2+|\sqrt{z_1}-\sqrt{z_2}|^2\}$

$=\dfrac{1}{2}\cdot2\{|\sqrt{z_1}|^2+|\sqrt{z_2}|^2\} \qquad$ [by property (xiv)]

$=|z_1|+|z_2|=$ LHS

| Example 34. z_1 and z_2 are two complex numbers, such that $\dfrac{z_1-2z_2}{2-z_1\cdot\bar{z}_2}$ is unimodular, while z_2 is not unimodular. Find $|z_1|$.

Sol. Here, $\left|\dfrac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1$

$\Rightarrow \quad \dfrac{|z_1-2z_2|}{|2-z_1\bar{z}_2|}=1 \qquad$ [by property (vi)]

$\Rightarrow \quad |z_1-2z_2|=|2-z_1\bar{z}_2|$

$\Rightarrow \quad |z_1-2z_2|^2=|2-z_1\bar{z}_2|^2$

$\Rightarrow (z_1-2z_2)(\overline{z_1-2z_2})=(2-z_1\bar{z}_2)(\overline{2-z_1\bar{z}_2})$

[by property (xv)]

$\Rightarrow \quad (z_1-2z_2)(\bar{z}_1-2\bar{z}_2)=(2-z_1\bar{z}_2)(2-\bar{z}_1z_2)$

$\Rightarrow z_1\bar{z}_1-2z_1\bar{z}_2-2z_2\bar{z}_1+4z_2\bar{z}_2$

$=4-2z_1\bar{z}_2-2\bar{z}_1z_2+z_1\bar{z}_1z_2\bar{z}_2$

$\Rightarrow \quad |z_1|^2+4|z_2|^2=4+|z_1|^2|z_2|^2$

$\Rightarrow |z_1|^2-|z_1|^2\cdot|z_2|^2+4|z_2|^2-4=0$

$\Rightarrow \quad (|z_1|^2-4)(1-|z_2|^2)=0$

But $\quad |z_2|\neq1 \qquad$ [given]

$\therefore \quad |z_1|^2=4$

Hence, $\quad |z_1|=2$

Properties of Arguments

(i) $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)+2k\pi, k\in I$

In general, $\arg(z_1z_2z_3\ldots z_n)$

$=\arg(z_1)+\arg(z_2)+\arg(z_3)+\ldots+\arg(z_n)+2k\pi$, $k\in I$.

(ii) $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)+2k\pi, k\in I$

(iii) $\arg\left(\dfrac{z}{\bar{z}}\right)=2\arg(z)+2k\pi, k\in I$

(iv) $\arg(z^n)=n\arg(z)+2k\pi, k\in I$, where proper value of k must be chosen, so that RHS lies in $(-\pi,\pi]$.

(v) If $\arg\left(\dfrac{z_2}{z_1}\right)=\theta$, then $\arg\left(\dfrac{z_1}{z_2}\right)=2n\pi-\theta$, where $n\in I$.

(vi) $\arg(\bar{z})=-\arg(z)$

| Example 35. If $\arg(z_1)=\dfrac{17\pi}{18}$ and $\arg(z_2)=\dfrac{7\pi}{18}$, find the principal argument of z_1z_2 and (z_1/z_2).

Sol. $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)+2k\pi$

$=\dfrac{17\pi}{18}+\dfrac{7\pi}{18}+2k\pi$

$=\dfrac{4\pi}{3}+2k\pi$

$=\dfrac{4\pi}{3}-2\pi=-\dfrac{2\pi}{3} \qquad [\text{for } k=-1]$

and $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)+2k\pi$

$=\dfrac{17\pi}{18}-\dfrac{7\pi}{18}+2k\pi=\dfrac{10\pi}{18}+2k\pi$

$=\dfrac{5\pi}{9}+0=\dfrac{5\pi}{9} \qquad [\text{for } k=0]$

Example 36. If z_1 and z_2 are conjugate to each other, find the principal argument of $(-z_1z_2)$.

Sol. $\because$ z_1 and z_2 are conjugate to each other i.e., $z_2 = \bar{z}_1$, therefore, $z_1z_2 = z_1\bar{z}_1 = |z_1|^2$

$\therefore \arg(-z_1z_2) = \arg(-|z_1|^2) = \arg$ [negative real number]

$= \pi$

Example 37. Let z be any non-zero complex number, then find the value of $\arg(z) + \arg(\bar{z})$.

Sol. $\arg(z) + \arg(\bar{z}) = \arg(z\bar{z})$

$= \arg(|z|^2) = \arg$ [positive real number]

$= 0$

(a) Mixed Properties of Modulus and Arguments

(i) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$

(ii) $|z_1 + z_2| = |z_1| - |z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi$

Proof (i) Let $\arg(z_1) = \theta$ and $\arg(z_2) = \phi$

$\therefore$ $|z_1 + z_2| = |z_1| + |z_2|$

On squaring both sides, we get

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\Rightarrow |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta - \phi) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$\Rightarrow$ $\cos(\theta - \phi) = 1$

$\therefore$ $\theta - \phi = 0$ or $\theta = \phi$

$\therefore$ $\arg(z_1) = \arg(z_2)$

(ii) $\because$ $|z_1 + z_2| = |z_1| - |z_2|$

On squaring both sides, we get

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$\Rightarrow |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta - \phi) = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$\Rightarrow$ $\cos(\theta - \phi) = -1$

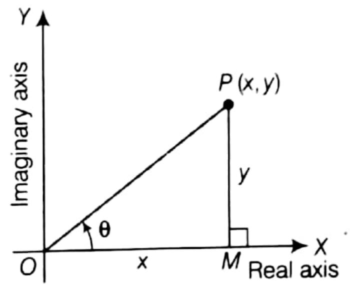
$\therefore$ $\theta - \phi = \pi$ or $\arg(z_1) - \arg(z_2) = \pi$

Remark

1. $|z_1 - z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$
2. $|z_1 - z_2| = |z_1| - |z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi$
3. $|z_1 - z_2| = |z_1 + z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pm\frac{\pi}{2}$, $\bar{z}_1 z_2$ and $\frac{z_1}{z_2}$ are purely imaginary.

(b) Trigonometric or Polar or Modulus Argument Form of a Complex Number

Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, z is represented by $P(x, y)$ in the argand plane.



By geometrical representation,

$$OP = \sqrt{(x^2 + y^2)} = |z|$$

$$\angle POM = \theta = \arg(z)$$

In ΔOPM, $x = OP\cos(\angle POM) = |z|\cos(\arg z)$

and $y = OP\sin(\angle POM) = |z|\sin(\arg z)$

$\because$ $z = x + iy$

$\therefore$ $z = |z|(\cos(\arg z) + i\sin(\arg z))$

or $z = r(\cos\theta + i\sin\theta)$

$\bar{z} = r(\cos\theta - i\sin\theta)$

where, $r = |z|$ and $\theta =$ principal value of $\arg(z)$.

Remark

1. $\cos\theta + i\sin\theta$ is also written as $CiS\ \theta$.
2. **Remember**

$1 = \cos 0 + i\sin 0 \Rightarrow -1 = \cos\pi + i\sin\pi$

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \Rightarrow -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

Example 38. Write the polar form of $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$ (where, $i = \sqrt{-1}$).

Sol. Let $z = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$. Since, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ lies in III quadrant

$\therefore$ Principal value of $\arg(z) = -\pi + \tan^{-1}\left|\frac{-\sqrt{3}/2}{-1/2}\right|$

$$= -\pi + \tan^{-1}\sqrt{3} = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$$

and $|z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\left(\frac{1}{4} + \frac{3}{4}\right)} = \sqrt{1} = 1$

$\therefore$ Polar form of $z = |z|[\cos(\arg z) + i\sin(\arg z)]$

i.e. $-\frac{1}{2} - \frac{i\sqrt{3}}{2} = \left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

(c) Euler's Form

If $\theta \in R$ and $i = \sqrt{-1}$, then $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler's identity.

Now, $e^{-i\theta} = \cos\theta - i\sin\theta$

Let $z = e^{i\theta}$

$\therefore$ $|z| = 1$ and $\arg(z) = \theta$

Also, $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

and if $\theta, \phi \in R$ and $i = \sqrt{-1}$, then

(i) $e^{i\theta} + e^{i\phi} = e^{i\left(\frac{\theta+\phi}{2}\right)} \cdot 2\cos\left(\frac{\theta-\phi}{2}\right)$

$\therefore$ $|e^{i\theta} + e^{i\phi}| = 2\left|\cos\left(\frac{\theta-\phi}{2}\right)\right|$

and $\arg(e^{i\theta} + e^{i\phi}) = \left(\frac{\theta+\phi}{2}\right)$

(ii) $e^{i\theta} - e^{i\phi} = e^{i\left(\frac{\theta+\phi}{2}\right)} \cdot 2i\sin\left(\frac{\theta-\phi}{2}\right)$

$\therefore$ $|e^{i\theta} - e^{i\phi}| = 2\left|\sin\left(\frac{\theta-\phi}{2}\right)\right|$

and $\arg(e^{i\theta} - e^{i\phi}) = \frac{\theta+\phi}{2} + \frac{\pi}{2}$ $\quad [\because i = e^{i\pi/2}]$

Remark

1. $e^{i\theta} + 1 = e^{i\theta/2} \cdot 2\cos(\theta/2)$ (Remember)
2. $e^{i\theta} - 1 = e^{i\theta/2} \cdot 2i\sin(\theta/2)$ (Remember)
3. $\frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i\tan(\theta/2)$ (Remember)
4. If $z = re^{i\theta}$; $|z| = r$, then $\arg(z) = \theta$, $\bar{z} = re^{-i\theta}$
5. If $|z - z_0| = 1$, then $z - z_0 = e^{i\theta}$

I Example 39. Given that $|z - 1| = 1$, where z is a point on the argand plane, show that $\frac{z-2}{z} = i\tan(\arg z)$, where $i = \sqrt{-1}$.

Sol. Given, $|z - 1| = 1$

$\therefore$ $z - 1 = e^{i\theta} \Rightarrow z = e^{i\theta} + 1 = e^{i\theta/2} \cdot 2\cos(\theta/2)$

$\therefore$ $\arg(z) = \theta/2$ ...(i)

$$\text{LHS} = \frac{z-2}{z} = \frac{1 + e^{i\theta} - 2}{1 + e^{i\theta}} = \frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i\tan(\theta/2)$$

$= i\tan(\arg z) = \text{RHS}$ [from Eq. (i)]

I Example 40. Let z be a non-real complex number lying on $|z| = 1$, prove that $z = \frac{1 + i\tan\left(\frac{\arg(z)}{2}\right)}{1 - i\tan\left(\frac{\arg(z)}{2}\right)}$ (where, $i = \sqrt{-1}$).

Sol. Given, $|z| = 1$

$\therefore$ $z = e^{i\theta}$ ...(i)

$\Rightarrow$ $\arg(z) = \theta$ ...(ii)

$$\text{RHS} = \frac{1 + i\tan\left(\frac{\arg(z)}{2}\right)}{1 - i\tan\left(\frac{\arg(z)}{2}\right)} = \frac{1 + i\tan(\theta/2)}{1 - i\tan(\theta/2)} \quad \text{[from Eq. (ii)]}$$

$$= \frac{\cos\theta/2 + i\sin\theta/2}{\cos\theta/2 - i\sin\theta/2} = \frac{e^{i\theta/2}}{e^{-i\theta/2}}$$

$= e^{i\theta} = z = \text{LHS}$ [from Eq. (i)]

I Example 41. Prove that $\tan\left(i\ln\left(\frac{a-ib}{a+ib}\right)\right) = \frac{2ab}{a^2 - b^2}$ (where $a, b \in R^+$ and $i = \sqrt{-1}$).

By taking $a-ib = e^{-i\theta}$, $a+ib = e^{i\theta}$

Sol. $\because$ $\left|\frac{a-ib}{a+ib}\right| = \frac{|a-ib|}{|a+ib|} = 1$ $\quad [\because |\bar{z}| = |z|]$

Let $\frac{a-ib}{a+ib} = e^{i\theta}$ ...(i)

By componendo and dividendo, we get

$$\frac{(a-ib)-(a+ib)}{(a-ib)+(a+ib)} = \frac{e^{i\theta}-1}{e^{i\theta}+1} - \frac{b}{a}i = i\tan(\theta/2)$$

or $\tan\left(\frac{\theta}{2}\right) = -\frac{b}{a}$ ...(ii)

$\therefore$ $\text{LHS} = \tan\left(i\ln\left(\frac{a-ib}{a+ib}\right)\right)$

$= \tan(i\ln(e^{i\theta}))$ [from Eq. (i)]

$= \tan(i \cdot i\theta) = -\tan\theta$

$= -\frac{2\tan\theta/2}{1 - \tan^2\theta/2}$

$= -\frac{2(-b/a)}{1-(-b/a)^2}$ [from Eq. (ii)]

$= \frac{2ab}{a^2 - b^2} = \text{RHS}$

Applications of Euler's Form

If $x, y, \theta \in R$ and $i = \sqrt{-1}$, then

let $z = x + iy$ [cartesian form]

$= |z|(\cos\theta + i\sin\theta)$ [polar form]

$= |z|e^{i\theta}$ [Euler's form]

(i) Product of Two Complex Numbers

Let two complex numbers be

$$z_1 = |z_1|e^{i\theta_1} \text{ and } z_2 = |z_2|e^{i\theta_2},$$

where $\theta_1, \theta_2 \in R$ and $i = \sqrt{-1}$

$$\therefore \quad z_1 \cdot z_2 = |z_1| e^{i\theta_1} \cdot |z_2| e^{i\theta_2} = |z_1||z_2| e^{i(\theta_1 + \theta_2)}$$

$$= |z_1||z_2| (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

Thus, $\quad |z_1 z_2| = |z_1||z_2|$

and $\quad \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

(ii) Division of Two Complex Numbers

Let two complex numbers be

$$z_1 = |z_1| e^{i\theta_1} \quad \text{and} \quad z_2 = |z_2| e^{i\theta_2},$$

where $\theta_1, \theta_2 \in R$ and $i = \sqrt{-1}$

$$\therefore \quad \frac{z_1}{z_2} = \frac{|z_1| e^{i\theta_1}}{|z_2| e^{i\theta_2}} = \frac{|z_1|}{|z_2|} e^{i(\theta_1 - \theta_2)}$$

$$= \frac{|z_1|}{|z_2|} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

Thus, $\quad \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, (z_2 \neq 0)$

and $\arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

(iii) Logarithm of a Complex Number

$$\log_e(z) = \log_e(|z| e^{i\theta}) = \log_e |z| + \log_e(e^{i\theta})$$

$$= \log_e |z| + i\theta = \log_e |z| + i \arg(z)$$

So, the general value of $\log_e(z)$

$$= \log_e(z) + 2n\pi i \, (-\pi < \arg z < \pi).$$

| Example 42. If m and x are two real numbers and $i = \sqrt{-1}$, prove that $e^{2mi\cot^{-1}x}\left(\dfrac{xi+1}{xi-1}\right)^m = 1.$

Sol. Let $\cot^{-1} x = \theta$, then $\cot \theta = x$

$$\therefore \text{LHS} = e^{2mi\cot^{-1}x}\left(\frac{xi+1}{xi-1}\right)^m = e^{2mi\theta}\left(\frac{i\cot\theta + 1}{i\cot\theta - 1}\right)^m$$

$$= e^{2mi\theta}\left(\frac{i(\cot\theta - i)}{i(\cot\theta + i)}\right)^m = e^{2mi\theta}\left(\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right)^m$$

$$= e^{2mi\theta} \cdot \left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^m = e^{2mi\theta} \cdot (e^{-2i\theta})^m$$

$$= e^{2mi\theta} \cdot e^{-2mi\theta} = e^0 = 1 = \text{RHS}$$

| Example 43. If z and w are two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$, prove that $z = -\bar{w}$.

Sol. Let $\arg(w) = \theta$, then $\arg(z) = \pi - \theta$

$$\therefore \quad z = |z| (\cos(\arg z) + i \sin(\arg z))$$

$$= |z| (\cos(\pi - \theta) + i \sin(\pi - \theta))$$

$$= |z| (-\cos\theta + i\sin\theta) = -|z| (\cos\theta - i\sin\theta)$$

$$= -|w| (\cos(\arg w) - i \sin(\arg w))$$

$$= -|w| (\cos(-\arg w) + i \sin(-\arg w))$$

$$= -|\bar{w}| (\cos(\arg \bar{w}) + i \sin(\arg \bar{w})) = -\bar{w}$$

| Example 44. Express $(1+i)^{-i}$, (where, $i = \sqrt{-1}$) in th form $A + iB$.

Sol. Let $A + iB = (1+i)^{-i}$

On taking logarithm both sides, we get

$$\log_e(A + iB) = -i \log_e(1+i)$$

$$= -i \log_e\left(\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right)$$

$$= -i \log_e\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)$$

$$= -i \log_e(\sqrt{2} e^{i\pi/4}) = -i(\log_e \sqrt{2} + \log_e e^{i\pi/4})$$

$$= -i\left(\frac{1}{2}\log_e 2 + \frac{i\pi}{4}\right) = -\frac{i}{2}\log_e 2 + \frac{\pi}{4}$$

$$\therefore A + iB = e^{-\frac{i}{2}\log_e 2 + \frac{\pi}{4}} = e^{\pi/4} \cdot e^{i\log_e 2^{-1/2}}$$

$$= e^{\pi/4} \cdot (\cos(\log_e 2^{-1/2}) + i \sin(\log_e 2^{-1/2}))$$

$$= e^{\pi/4} \cdot \cos\left(\log_e\left(\frac{1}{\sqrt{2}}\right)\right) + i e^{\pi/4} \sin\left(\log_e\left(\frac{1}{\sqrt{2}}\right)\right)$$

| Example 45. If $\sin(\log_e i^i) = a + ib$, where $i = \sqrt{-1}$, find a and b, hence and find $\cos(\log_e i^i)$.

Sol. $a + ib = \sin(\log_e i^i) = \sin(i \log_e i)$

$$= \sin(i(\log_e |i| + i \arg i))$$

$$= \sin(i(\log_e 1 + (i\pi/2)))$$

$$= \sin(i(0 + (i\pi/2))) = \sin(-\pi/2) = -1$$

$\therefore \quad a = -1, b = 0$

Now, $\quad \cos(\log_e i^i) = \sqrt{1 - \sin^2(\log_e i^i)}$

$$= \sqrt{1 - (-1)^2} = \sqrt{(1-1)} = 0$$

Aliter

$\because \quad i^i = (e^{i\pi/2})^i = e^{-\pi/2}$

$\therefore \quad \sin(\log_e i^i) = \sin(\log_e e^{-\pi/2}) = \sin\left(-\dfrac{\pi}{2}\log_e e\right)$

$$= \sin(-\pi/2) = -1 = a + ib \quad \text{[given]}$$

$\therefore \quad a = -1, b = 0$

and $\quad \cos(\log_e i^i) = \cos(\log_e e^{-\pi/2})$

$$= \cos\left(-\frac{\pi}{2}\log_e e\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

| Example 46. Find the general value of $\log_2(5i)$, where $i = \sqrt{-1}$.

Sol. $\log_2 5i = \dfrac{\log_e 5i}{\log_e 2} = \dfrac{1}{\log_e 2}\{\log_e |5i| + i \arg(5i) + 2n\pi i\}$

$$= \frac{1}{\log_e 2}\left\{\log_e 5 + \frac{i\pi}{2} + 2n\pi i\right\}, n \in I$$

Exercise for Session 2

1 If $\frac{1-ix}{1+ix}=a-ib$ and $a^2+b^2=1$, where $a,b\in R$ and $i=\sqrt{-1}$, then x is equal to

(a) $\frac{2a}{(1+a)^2+b^2}$ (b) $\frac{2b}{(1+a)^2+b^2}$ (c) $\frac{2a}{(1+b)^2+a^2}$ (d) $\frac{2b}{(1+b)^2+a^2}$

2 The least positive integer n for which $\left(\frac{1+i}{1-i}\right)^n=\frac{2}{\pi}\left(\sec^{-1}\frac{1}{x}+\sin^{-1}x\right)$ (where, $x\neq 0$, $-1\leq x\leq 1$ and $i=\sqrt{-1}$), is

(a) 2 (b) 4 (c) 6 (d) 8

3 If $z=(3+4i)^6+(3-4i)^6$, where $i=\sqrt{-1}$, then $\text{Im}(z)$ equals to

(a) -6 (b) 0 (c) 6 (d) None of these

4 If $(x+iy)^{1/3}=a+ib$, where $i=\sqrt{-1}$, then $\left(\frac{x}{a}+\frac{y}{b}\right)$ is equal to

(a) $4a^2b^2$ (b) $4(a^2-b^2)$ (c) $4a^2-b^2$ (d) a^2+b^2

5 If $\frac{3}{2+\cos\theta+i\sin\theta}=a+ib$, where $i=\sqrt{-1}$ and $a^2+b^2=\lambda a-3$, the value of λ is

(a) 3 (b) 4 (c) 5 (d) 6

6 If $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) $\sqrt{2}$ (d) 2

7 The complex numbers $\sin x+i\cos 2x$ and $\cos x-i\sin 2x$, where $i=\sqrt{-1}$, are conjugate to each other, for

(a) $x=n\pi, n\in I$ (b) $x=0$ (c) $x=\left(n+\frac{1}{2}\right)n\in I$ (d) no value of x

8 If α and β are two different complex numbers with $|\beta|=1$, then $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

9 If $x=3+4i$ (where, $i=\sqrt{-1}$), the value of $x^4-12x^3+70x^2-204x+225$, is

(a) -45 (b) 0 (c) 35 (d) 15

10 If $|z_1-1|\leq 1, |z_2-2|\leq 2, |z_3-3|\leq 3$, the greatest value of $|z_1+z_2+z_3|$ is

(a) 6 (b) 12 (c) 17 (d) 23

11 The principal value of $\arg(z)$, where $z=\left(1+\cos\frac{8\pi}{5}\right)+i\sin\frac{8\pi}{5}$ (where, $i=\sqrt{-1}$) is given by

(a) $-\frac{\pi}{5}$ (b) $-\frac{4\pi}{5}$ (c) $\frac{\pi}{5}$ (d) $\frac{4\pi}{5}$

12 If $|z_1|=2$, $|z_2|=3$, $|z_3|=4$ and $|z_1+z_2+z_3|=5$, then $|4z_2z_3+9z_3z_1+16z_1z_2|$ is

(a) 24 (b) 60 (c) 120 (d) 240

13 If $|z-i|\leq 5$ and $z_1=5+3i$ (where, $i=\sqrt{-1}$), the greatest and least values of $|iz+z_1|$ are

(a) 7 and 3 (b) 9 and 1 (c) 10 and 0 (d) None of these

14 If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_4}\right)+\arg\left(\frac{z_2}{z_3}\right)$ equals to

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{2}$