CHAPTER

11

Sets, Relations and Functions

Learning Part

Session 1

- Definition of Set
- Representation of Set
- Different Types of Sets
- Laws and Theorems
- Venn Diagrams (Euler-Venn Diagrams)

Session 2

- Ordered Pair
- Definition of Relation
- Ordered Relation
- Composition of Two Relations

Session 3

- Definition of Functions
- Domain, Codomain and Range
- Composition of Mapping
- Equivalence Classes
- Partition of Set
- Congruences

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Definition of Set, Representation of Set, Different Types of Sets, Laws and Theorems, Venn Diagram (Euler-Venn Diagrams)

Introduction

The concept of set is fundamental in modern Mathematics. Today this concept is being used in different branches of Mathematics and widely used in the foundation of relations and functions. The theory of sets was developed by German Mathematician **Georg Cantor** (1845-1918).

Definition of Set

A set is well-defined collection of distinct objects. Sets are usually denoted by capital letters $A, B, C, X, Y, Z, \ldots$.

Examples of sets

(i) The set of all complex numbers.
(ii) The set of vowels in the alphabets of English language.
(iii) The set of all natural numbers.
(iv) The set of all triangles in a plane.
(v) The set of all states in India.
(vi) The set of all months in year which has 30 days.
(vii) The set of all stars in space.

Elements of the Set

The elements of the set are denoted by small letters in the alphabets of English language, i.e. $a, b, c, x, y, z, \ldots$.
If x is an element of a set A, we write $x \in A$ (read as 'x belongs to A').

If x is not an element of A, then we write $x \notin A$ (read as 'x does not belong to A').

For example,

If $A = \{1, 2, 3, 4, 5\}$, then $3 \in A, 6 \notin A$.

Representation of a Set

There are two methods for representing a set.

1. Tabulation or Roster or Enumeration Method

Under this method, the elements are enclosed in curly brackets or braces { } after separating them by commas.

Remark

1. The order of writing the elements of a set is immaterial, so $\{a, b, c\}, \{b, a, c\}, \{c, a, b\}$ all denote the same set.
2. An element of a set is not written more than once, i.e. the set {1, 2, 3, 4, 3, 3, 2, 1, 2, 1, 4} is identical with the set {1, 2, 3, 4}.

For example,

1. If A is the set of prime numbers less than 10, then
$$A = \{2, 3, 5, 7\}$$
2. If A is the set of all even numbers lying between 2 and 20, then
$$A = \{4, 6, 8, 10, 12, 14, 16, 18\}$$

2. Set Builder Method

Under this method, the stating properties which its elements are to satisfy, then we write
$$A = \{x\, P(x)\} \quad \text{or} \quad A = \{x : P(x)\}$$
and read as 'A is the set of elements x, such that x has the property P'.

Remark

1. " : " or " | " means 'such that'.
2. The other names of this method are property method, rule method and symbolic method.

For example,

1. If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, then we can write
$$A = \{x \in N : x \leq 8\}.$$
2. A is the set of all odd integers lying between 2 and 51, then
$$A = \{x : 2 < x < 51, x \text{ is odd}\}.$$

Some Standard Sets

- N denotes set of all natural numbers $= \{1, 2, 3, \ldots\}$.
- Z or I denotes set of all integers
$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- Z_0 or I_0 denotes set of all integers excluding zero
$= \{\ldots, -3, -2, -1, 1, 2, 3, \ldots\}$.
- Z^+ or I^+ denotes set of all positive integers
$= \{1, 2, 3, \ldots\} = N$.
- E denotes set of all even integers
$= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$.
- O denotes set of all odd integers
$= \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$.
- W denotes set of all whole numbers $= \{0, 1, 2, 3, \ldots\}$.
- Q denotes set of all rational numbers $= \{x : x = p/q$, where p and q are integers and $q \neq 0\}$.
- Q_0 denotes set of all non-zero rational numbers $\{x : x = p/q$, where p and q are integers and $p \neq 0$ and $q \neq 0\}$.
- Q^+ denotes set of all positive rational numbers $= \{x : x = p/q$, where p and q are both positive or negative integers$\}$
- R denotes set of all real numbers.
- R_0 denotes set of all non-zero real numbers.
- R^+ denotes set of all positive real numbers.
- $R - Q$ denotes set of all irrational numbers.
- C denotes set of all complex numbers
$= \{a + ib : a, b \in R$ and $i = \sqrt{-1}\}$.
- C_0 denotes set of all non-zero complex numbers
$= \{a + ib : a, b \in R_0$ and $i = \sqrt{-1}\}$.
- N_a denotes set of all natural numbers which are less than or equal to a, where a is positive integer
$= \{1, 2, 3, \ldots, a\}$.

Different Types of Sets

1. Null Set or Empty Set or Void Set

A set having no element is called a null set or empty set or void set. It is denoted by ϕ or $\{\ \}$.

Remark

1. ϕ is called the null set.
2. ϕ is unique.
3. ϕ is a subset of every set.
4. ϕ is never written within braces i.e., $\{\phi\}$ is not the null set.
5. $\{0\}$ is not an empty set as it contains the element 0 (zero).

For example,

1. $\{x : x \in N, 4 < x < 5\} = \phi$
2. $\{x : x \in R, x^2 + 1 = 0\} = \phi$
3. $\{x : x^2 = 25, x$ is even number$\} = \phi$

2. Singleton or Unit Set

A set having one and only one element is called singleton or unit set.

For example, $\{x : x - 3 = 4\}$ is a singleton set.

Since, $x - 3 = 4 \Rightarrow x = 7$

$\therefore \quad \{x : x - 3 = 4\} = \{7\}$

3. Subset

If every element of a set A is also an element of a set B, then A is called the subset of B, we write $A \subseteq B$ (read as A is subset of B or A is contained in B).

Thus, $A \subseteq B \Leftrightarrow [x \in A \Rightarrow x \in B]$

Remark

1. Every set is a subset of itself
i.e., $A \subseteq A$.
2. If $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$.

For example,

1. If $A = \{2, 3, 4\}$ and $B = \{5, 4, 2, 3, 1\}$, then $A \subseteq B$.
2. The sets $\{a\}, \{b\}, \{a, b\}, \{b, c\}$ are the subsets of the set $\{a, b, c\}$.

4. Total Number of Subsets

If a set A has n elements, then the number of subsets of $A = 2^n$.

Example 1. Write the letters of the word ALLAHABAD in set form and find the number of subsets in it and write all subsets.

Sol. There are 5 different letters in the word ALLAHABAD i.e., A, L, H, B, D, then set is {A, B, D, H, L}, then number of subsets $= 2^5 = 32$ and all subsets are

ϕ, {A}, {B}, {D}, {H}, {L}, {A, B}, {A, D}, {A, H}, {A, L}, {B, D}, {B, H}, {B, L}, {D, H}, {D, L}, {H, L}, {A, B, D}, {A, B, H}, {A, B, L}, {A, D, H}, {A, D, L}, {A, H, L}, {B, D, H}, {B, D, L}, {B, H, L}, {D, H, L}, {A, B, D, H}, {A, B, D, L}, {B, D, H, L}, {A, D, H, L}, {A, B, H, L}, {A, B, D, H, L}.

5. Equal Sets

Two sets A and B are said to be equal, if every element of A is an element of B, and every element of B is an element of A. If A and B are equal, we write $A = B$.

It is clear that $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$.

For example,

1. The sets $\{1, 2, 5\}$ and $\{5, 2, 1\}$ are equal.
2. $\{1, 2, 3\} = \{x : x^3 - 6x^2 + 11x - 6 = 0\}$

6. Power Set

The set of all the subsets of a given set A is said to be the power set A and is denoted by $P(A)$ or 2^A.

Symbolically, $P(A) = \{x : x \subseteq A\}$

Thus, $x \in P(A) \Leftrightarrow x \subseteq A.$

Remark

1. ϕ and A are both elements of $P(A)$.
2. If $A = \phi$, then $P(\phi) = \{\phi\}$, a singleton but ϕ is a null set.
3. If $A = \{a\}$, then $P(A) = \{\phi, \{a\}\}$

   For example, If $A = \{a, b, c\}$, then

   $P(A)$ or $2^A = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$

   Also, $n(P(A))$ or $n(2^A) = 2^3 = 8$
4. Since, $P(\phi) = \{\phi\}$

   $\therefore$ $P(P(\phi)) = \{\phi, \{\phi\}\}$

   and $P(P(P(\phi))) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$
5. If A has n elements, then $P(A)$ has 2^n elements.

7. Super Set

The statement $A \subseteq B$ can be rewritten as $B \supseteq A$, then B is called the super set of A and is written as $B \supset A$.

8. Proper Subset

A set A is said to be proper subset of a set B, if every element of A is an element of B and B has atleast one element which is not an element of A and is denoted by $A \subset B$ (read as "A is a proper subset of B").

For example,

1. If $A = \{1, 2, 4\}$ and $B = \{5, 1, 2, 4, 3\}$, then $A \subset B$

   Since, $3, 5 \notin A$.
2. If $A = \{a, b, c\}$ and $B = \{c, b, a\}$, then $A \not\subset B$ (since, B does not contain any element which is not in A).
3. $N \subset I \subset Q \subset R \subset C$

9. Finite and Infinite Sets

A set in which the process of counting of elements comes to an end is called a finite set, otherwise it is called an infinite set.

For example,

1. Each one of the following sets is a finite set.
   (i) Set of universities in India.
   (ii) Set of Gold Medalist students in Civil Branch, sec A in A.M.I.E. (India).
   (iii) Set of natural numbers less than 500.
2. Each one of the following is an infinite set.
   (i) Set of all integers.
   (ii) Set of all points in a plane.
   (iii) $\{x : x \in R, 1 < x < 2\}$
   (iv) Set of all concentric circles with centre as origin.

10. Cardinal Number of a Finite Set

The number of distinct elements in a finite set A is called cardinal number and the cardinal number of a set A is denoted by $n(A)$.

For example,

If $A = \{-3, -1, 8, 9, 13, 17\}$, then $n(A) = 6$.

11. Comparability of Sets

Two sets A and B are said to be comparable, if either $A \subset B$ or $B \subset A$ or $A = B$, otherwise A and B are said to be incomparable.

For example,

1. The sets $A = \{1,2,3\}$ and $B = \{1,2,4,6\}$ are incomparable (since $A \not\subset B$ or $B \not\subset A$ or $A \neq B$)
2. The sets $A = \{1, 2, 4\}$ and $B = \{1, 4\}$ are comparable (since $B \subset A$).

12. Universal Set

All the sets under consideration are likely to be subsets of a set is called the universal set and is denoted by Ω or S or U.

For example,

1. The set of all letters in alphabet of English language $U = \{a, b, c, \dots, x, y, z\}$ is the universal set of vowels in alphabet of English language.

   i.e., $A = \{a, e, i, o, u\}$.
2. The set of all integers $I = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ is the universal set of all even integers

   i.e., $\{0, \pm 2, \pm 4, \pm 6, \dots\}$

13. Union of Sets

The union of two sets A and B is the set of all those elements which are either in A or in B or in both. This set is denoted by $A \cup B$ or $A + B$ (read as 'A union B' or 'A cup B' or 'A join B').

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

or $A \cup B = \{x : x \in A \vee x \in B\}$

Clearly, $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

For example,

1. If $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$, then $A \cup B = \{1, 2, 3, 4, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$
2. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, $C = \{7, 8\}$, then $A \cup B \cup C = \{1, 2, 3, 4, 5, 7, 8\}$

Remark

The union of a finite number of sets $A_1, A_2, A_3, \ldots, A_n$ is represented by $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$.

Symbolically, $\bigcup_{i=1}^{n} A_i = \{x : x \in A_i$ for atleast one $i\}$

14. Intersection of Sets

The intersection of two sets A and B is the set of all elements which are common in A and B. This set is denoted by $A \cap B$ or AB (read as 'A intersection B' or 'A cap B' or 'A meet B').

Symbolically, $A \cap B = \{x : x \in A$ and $x \in B\}$

or $\quad A \cap B = \{x : x \in A \wedge x \in B\}$

Clearly, $\quad x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

For example,

1. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$, then $A \cap B = \{3\}$.
2. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$, then $A \cap B \cap C = \{3\}$

Remark

The intersection of a finite number of sets $A_1, A_2, A_3, \ldots, A_n$ represented by

$$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n \quad \text{or} \quad \bigcap_{i=1}^{n} A_i$$

Symbolically, $\bigcap_{i=1}^{n} A_i = \{x : x \in A_i$ for all $i\}$

15. Disjoint Sets

If the two sets A and B have no common element. i.e., $A \cap B = \phi$, then the two sets A and B are called disjoint or mutually exclusive events.

For example, If $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, then $A \cap B = \phi$

Hence, A and B are disjoint sets.

Remark

If $S = \{a_1, a_2, a_3, \ldots, a_n\}$, so number of ordered pairs of disjoint sets of S is $\frac{3^n + 1}{2}$.

($\because$ each element in either (A) or (B) or neither

$\therefore$ Total ways $= 3^n$ i.e., $A = B$, iff $A = B = \phi$ (1 case) otherwise A and B are interchangeable.

$\therefore$ Number of ordered pairs of disjoint sets of

$S = 1 + \frac{3^n - 1}{2} = \frac{3^n + 1}{2}$)

16. Difference of Sets

If A and B be two given sets, then the set of all those elements of A which do not belong to B is called difference of sets A and B. It is written as $A - B$. It is also denoted by $A \sim B$ or $A \backslash B$ or $C_A B$ (complement of B in A).

Symbolically, $A - B = \{x : x \in A$ and $x \notin B\}$

Clearly, $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$.

Remark

1. $A - B \neq B - A$
2. The sets $A - B$, $B - A$ and $A \cap B$ are disjoint sets.
3. $A - B \subseteq A$ and $B - A \subseteq B$
4. $A - \phi = A$ and $A - A = \phi$

For example,

If $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$, then $A - B = \{1, 2, 3\}$.

17. Symmetric Difference of Two Sets

Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ or $(A \cup B) - (A \cap B)$ and is denoted by $A \Delta B$ or $A \oplus B$ (A direct sum B).

i.e., $\quad A \oplus B$ or $A \Delta B = (A - B) \cup (B - A)$

and $\quad A \oplus B$ or $A \Delta B = (A \cup B) - (A \cap B)$

Remark

1. $A \Delta B = \{x : x \in A$ and $x \notin B\}$

   or $\quad A \Delta B = \{x : x \in B$ and $x \notin A\}$
2. $A \Delta B = B \Delta A$ (commutative)

For example,

Let $\quad A = \{1,2,3,4,5\}$ and $B = \{1,3,5,7\}$,

then $\quad A - B = \{2,4\}$, $B - A = \{7\}$

$\therefore \quad A \Delta B = (A - B) \cup (B - A) = \{2,4,7\}$

18. Complement Set

Let U be the universal set and A be a set, such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^c or $C(A)$ or $U - A$.

Symbolically, A' or A^c or $C(A) = \{x : x \in U$ and $x \notin A\}$.

Clearly, $\quad x \in A' \Leftrightarrow x \notin A$.

Remark

1. $U' = \phi$ and $\phi' = U$
2. $A \cup A' = U$ and $A \cap A' = \phi$

For example,

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{1, 3, 5, 7\}$.

Then, $A' = U - A = \{2, 4, 6\}$

Laws and Theorems

1. **Idempotent Laws**

   For **any set A,**

   (i) $A \cup A = A$ (ii) $A \cap A = A$

   Proof

   (i) Let $x \in A \cup A \Leftrightarrow x \in A$ or $x \in A$

   $\Leftrightarrow x \in A$

   Hence, $A \cup A = A$

   (ii) Let $x \in A \cap A \Leftrightarrow x \in A$ and $x \in A$

   $\Leftrightarrow x \in A$

   Hence, $A \cap A = A$

2. **Identity Laws**

   For **any set A,**

   (i) $A \cup \phi = A$ (ii) $A \cap \phi = \phi$

   (iii) $A \cup U = U$ (iv) $A \cap U = A$

   Proof

   (i) Let $x \in A \cup \phi \Leftrightarrow x \in A$ and $x \in \phi$

   $\Leftrightarrow x \in A$

   Hence, $A \cup \phi = A$

   (ii) Let $x \in A \cap \phi \Leftrightarrow x \in A$ and $x \in \phi$

   $\Leftrightarrow x \in \phi$

   Hence, $A \cap \phi = \phi$

   (iii) Let $x \in A \cup U \Leftrightarrow x \in A$ or $x \in U$

   $\Leftrightarrow x \in U$

   Hence, $A \cup U = U$

   (iv) Let $x \in A \cap U \Leftrightarrow x \in A$ and $x \in U$

   $\Leftrightarrow x \in A$

   Hence, $A \cap U = A$

3. **Commutative Laws**

   For any two sets A and B, we have

   (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

   Proof

   (i) Let $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

   $\Leftrightarrow x \in B$ or $x \in A$

   $\Leftrightarrow x \in B \cup A$

   $\therefore x \in A \cup B \Leftrightarrow x \in B \cup A$

   Hence, $A \cup B = B \cup A$

   (ii) Let $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

   $\Leftrightarrow x \in B$ and $x \in A$

   $\Leftrightarrow x \in B \cap A$

   $\therefore x \in A \cap B \Leftrightarrow x \in B \cap A$

   Hence, $A \cap B = B \cap A$.

4. **Associative Laws**

   For any three sets A, B and C, we have

   (i) $A \cup (B \cup C) = (A \cup B) \cup C$

   (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

   Proof

   (i) Let $x \in A \cup (B \cup C) \Leftrightarrow x \in A$ or $x \in B \cup C$

   $\Leftrightarrow x \in A$ or ($x \in B$ or $x \in C$)

   $\Leftrightarrow$ ($x \in A$ or $x \in B$) or $x \in C$

   $\Leftrightarrow x \in A \cup B$ or $x \in C$

   $\Leftrightarrow x \in (A \cup B) \cup C$

   $\therefore x \in A \cup (B \cup C) \Leftrightarrow x \in (A \cup B) \cup C$

   Hence, $A \cup (B \cup C) = (A \cup B) \cup C$.

   (ii) Let $x \in A \cap (B \cap C) \Leftrightarrow x \in A$ and $x \in B \cap C$

   $\Leftrightarrow x \in A$ and ($x \in B$ and $x \in C$)

   $\Leftrightarrow$ ($x \in A$ and $x \in B$) and $x \in C$

   $\Leftrightarrow x \in A \cap B$ and $x \in C$

   $\Leftrightarrow x \in (A \cap B) \cap C$

   Hence, $A \cap (B \cap C) = (A \cap B) \cap C$.

5. **Distributive Laws**

   For any three sets A, B and C, we have

   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   Proof

   (i) Let $x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $x \in B \cap C$

   $\Leftrightarrow x \in A$ or ($x \in B$ and $x \in C$)

   $\Leftrightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

   $\Leftrightarrow x \in A \cup B$ and $x \in A \cup C$

   $\Leftrightarrow x \in [(A \cup B) \cap (A \cup C)]$

   $\therefore x \in A \cup (B \cap C) \Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

   Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

   (ii) Let $x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $x \in B \cup C$

   $\Leftrightarrow x \in A$ and ($x \in B$ or $x \in C$)

   $\Leftrightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

   $\Leftrightarrow x \in A \cap B$ or $x \in A \cap C$

   $\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

   $\therefore x \in A \cap (B \cup C) \Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

   Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

6. **For any two sets A and B, we have**

   (i) $P(A) \cap P(B) = P(A \cap B)$

   (ii) $P(A) \cup P(B) \subseteq P(A \cup B)$

   where, $P(A)$ is the power set of A.

   Proof

   (i) Let $x \in P(A) \cap P(B) \Leftrightarrow x \in P(A)$ or $x \in P(B)$

   $\Leftrightarrow x \subseteq A$ or $x \subseteq B$

   $\Leftrightarrow x \subseteq A \cap B$

   $\Leftrightarrow x \in P(A \cap B)$

   Hence, $P(A) \cap P(B) = P(A \cap B)$

(ii) Let $x \in P(A) \cup P(B) \Leftrightarrow x \in P(A)$ or $x \in P(B)$

$\Leftrightarrow x \subseteq A$ or $x \subseteq B$

$\Leftrightarrow x \subseteq A \cup B$

$\Leftrightarrow x \in P(A \cup B)$

Hence, $P(A) \cup P(B) \subseteq P(A \cup B)$

7. If A is any set, then $(A')' = A$

Proof Let $x \in (A')' \Leftrightarrow x \notin A' \Leftrightarrow x \in A$

Hence, $(A')' = A$

8. **De-Morgan's Laws**

For any three sets A, B and C, we have

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

(iii) $A - (B \cup C) = (A - B) \cap (A - C)$

(iv) $A - (B \cap C) = (A - B) \cup (A - C)$

Proof

(i) Let $x \in (A \cup B)' \Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A$ and $x \notin B$

$\Leftrightarrow x \in A'$ and $x \in B'$

$\Leftrightarrow x \in A' \cap B'$

$\therefore\ x \in (A \cup B)' \Leftrightarrow x \in A' \cap B'$

Hence, $(A \cup B)' = A' \cap B'$.

(ii) Let $x \in (A \cap B)' \Leftrightarrow x \notin A \cap B$

$\Leftrightarrow x \notin A$ or $x \notin B$

$\Leftrightarrow x \in A'$ or $x \in B'$

$\Leftrightarrow x \in A' \cup B'$

$\therefore\ x \in (A \cap B)' \Leftrightarrow x \in A' \cup B'$

Hence, $(A \cap B)' = A' \cup B'$.

(iii) Let $x \in A - (B \cup C) \Leftrightarrow x \in A$ and $x \notin B \cup C$

$\Leftrightarrow x \in A$ and ($x \notin B$ and $x \notin C$)

$\Leftrightarrow$ ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$)

$\Leftrightarrow x \in (A - B)$ and $x \in (A - C)$

$\Leftrightarrow x \in (A - B) \cap (A - C)$

Hence, $A - (B \cup C) = (A - B) \cap (A - C)$.

(iv) Let $x \in A - (B \cap C) \Leftrightarrow x \in A$ and $x \notin (B \cap C)$

$\Leftrightarrow x \in A$ and ($x \notin B$ or $x \notin C$)

$\Leftrightarrow$ ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$)

$\Leftrightarrow x \in (A - B)$ or $x \in (A - C)$

$\Leftrightarrow x \in (A - B) \cup (A - C)$

Hence, $A - (B \cap C) = (A - B) \cup (A - C)$.

Aliter

$A - (B \cap C) = A \cap (B \cap C)'$ $\quad [\because A - B = A \cap B']$

$= A \cap (B' \cap C)'$

$= (A \cap B') \cup (A \cap C')$

$= (A - B) \cup (A - C)$

More Results on Operations on Sets

For any two sets A and B, we have

1. $A \subseteq A \cup B$, $B \subseteq A \cup B$, $A \cap B \subseteq A$, $A \cap B \subseteq B$

2. $A - B = A \cap B'$

Proof

Let $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow x \in A$ and $x \in B'$

$\Leftrightarrow x \in A \cap B'$

Hence, $A - B = A \cap B'$

3. $(A - B) \cup B = A \cup B$

Proof $(A - B) \cup B = (A \cap B') \cup B$

$= (A \cup B) \cap (B' \cup B)$ [from distributive law]

$= (A \cup B) \cap U$

$= A \cup B$

Hence, $(A - B) \cup B = A \cup B$

4. $(A - B) \cap B = \phi$

Proof $(A - B) \cap B = (A \cap B') \cap B$

$= A \cap (B' \cap B)$ [from associative law]

$= A \cap \phi = \phi$

Hence, $(A - B) \cap B = \phi$

5. $A \subseteq B \Leftrightarrow B' \subseteq A'$

Proof Only if part Let $A \subseteq B$ ...(i)

To prove $B' \subseteq A'$

Let $x \in B' \Rightarrow x \notin B$

$\Rightarrow x \notin A$ $\quad [\because A \subseteq B]$

$\Rightarrow x \in A'$

Thus, $x \in B' \Rightarrow x \in A'$ $\quad [\because B \subseteq A]$

Hence, $B' \subseteq A'$ ...(ii)

If part Let $B' \subseteq A'$ ...(iii)

To prove $A \subseteq B$

Let $x \in A \Rightarrow x \notin A'$

$\Rightarrow x \notin B'$ [from Eq.(iii)]

$\Rightarrow x \in B$

Hence, $A \subseteq B$ ...(iv)

From Eqs. (ii) and (iv), we get $A \subseteq B \Leftrightarrow B' \subseteq A'$

6. $A - B = B' - A'$

Proof $A - B = (A \cap B')$

$= B' \cap A = B' \cap (A')' = B' - A'$

Hence, $A - B = B' - A'$

7. $(A \cup B) \cap (A \cup B') = A$

Proof $(A \cup B) \cap (A \cup B') = A \cup (B \cap B')$

[by distributive law]

$= A \cup \phi = A$

Hence, $(A \cup B) \cap (A \cup B') = A$

8. $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

Proof $(A - B) \cup (B - A) \cup (A \cap B)$

$= [(A \cup B) - (A \cap B)] \cup (A \cap B)$

$=[(A \cup B) \cap (A \cap B)'] \cup (A \cap B)$
$=[(A \cup B) \cup (A \cap B)] \cap [(A \cap B)' \cup (A \cap B)]$
$=(A \cup B) \cap U = A \cup B$

Hence, $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

9. $\boldsymbol{A - (A - B) = A \cap B}$

Proof $A - (A - B) = A - (A \cap B')$
$= A \cap (A \cap B')'$
$= A \cap (A' \cup B)$
$= (A \cap A') \cup (A \cap B)$
$= \phi \cup (A \cap B) = A \cap B$

Hence, $A - (A - B) = A \cap B$

10. $\boldsymbol{A - B = B - A \Leftrightarrow A = B}$

Proof Only if part Let $A - B = B - A$ ...(i)

To prove $A = B$

Let $x \in A \Leftrightarrow (x \in A \text{ and } x \notin B)$ or $(x \in A \text{ and } x \in B)$
$\Leftrightarrow x \in (A - B)$ or $x \in (A \cap B)$
$\Leftrightarrow x \in (B - A)$

or $x \in A \cap B$ [from Eq. (i)]
$\Leftrightarrow (x \in B \text{ and } x \notin A)$ or $(x \in B \text{ and } x \in A)$
$\Leftrightarrow x \in B$

Hence, $A = B$

If part Let $A = B$

To prove $A - B = B - A$

Now, $A - B = A - A = \phi$ $[\because B = A]$

and $B - A = A - A = \phi$ $[\because B = A]$

$\therefore$ $A - B = B - A$

Hence, $A = B \Rightarrow A - B = B - A$

11. $\boldsymbol{A \cup B = A \cap B \Leftrightarrow A = B}$

Proof Only if part Let $A \cup B = A \cap B$

Now, $x \in A \Rightarrow x \in A \cup B$

$\Rightarrow$ $x \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow$ $x \in B$

Thus, $A \subseteq B$ ...(i)

Again, $y \in B \Rightarrow y \in A \cup B$

$\Rightarrow$ $y \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow$ $y \in A$

Thus, $B \subseteq A$ ...(ii)

From Eqs. (i) and (ii), we have $A = B$

Thus, $A \cup B = A \cap B \Rightarrow A = B$.

If part Let $A = B$ ...(iii)

To prove $A \cup B = A \cap B$

Now, $A \cup B = A \cup A = A$ $[\because B = A]$...(iv)

and $A \cap B = A \cap A = A$ $[\because B = A]$...(v)

From Eqs. (iv) and (v), we have $A \cup B = A \cap B$

Hence, $A \cup B = A \cap B \Leftrightarrow A = B$

| Example 2. Let A, B and C be three sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.

Sol. Given, $A \cup B = A \cup C$ ...(i)

and $A \cap B = A \cap C$ ...(ii)

To prove $B = C$.

From Eq. (i), $(A \cup B) \cap C = (A \cup C) \cap C$

$\Rightarrow$ $(A \cap C) \cup (B \cap C) = (A \cap C) \cup (C \cup C)$

$\Rightarrow$ $(A \cap B) \cup (B \cap C) = (A \cap C) \cup C$
$[\because A \cap C = A \cap B]$

$\Rightarrow$ $(A \cap B) \cup (B \cap C) = C$ $[\because A \cap C \subseteq C]$

Thus, $C = (A \cap B) \cup (B \cap C)$ ...(iii)

Again, from Eq. (i), $(A \cup B) \cap B = (A \cup C) \cap B$

$\Rightarrow$ $(A \cap B) \cup (B \cap B) = (A \cap B) \cup (C \cap B)$

$\Rightarrow$ $(A \cap B) \cup B = (A \cap B) \cup (B \cap C)$

$\Rightarrow$ $B = (A \cap B) \cup (B \cap C)$
$[\because A \cap B \subseteq B]$

Thus, $B = (A \cap B) \cup (B \cap C)$ ...(iv)

From Eqs. (iii) and (iv), we have $B = C$.

| Example 3. Let A and B be any two sets. If for some set X, $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$, prove that $A = B$.

Sol. Given, $A \cap X = B \cap X = \phi$ ...(i)

and $A \cup X = B \cup X$ ...(ii)

From Eq. (ii), $A \cap (A \cup X) = A \cap (B \cup X)$

$\Rightarrow$ $A = (A \cap B) \cup (A \cap X)$
$[\because A \subseteq A \cup X \therefore A \cap (A \cup X) = A]$

$\Rightarrow$ $A = (A \cap B) \cup \phi$ $[\because A \cap X = \phi]$

$\Rightarrow$ $A = (A \cap B)$

$\Rightarrow$ $A \subseteq B$ ...(iii)

Again, $A \cup X = B \cup X$

$\Rightarrow$ $B \cap (A \cup X) = B \cap (B \cup X)$

$\Rightarrow$ $(B \cap A) \cup (B \cap X) = B$
$[\because B \subseteq B \cup X \therefore B \cap (B \cup X) = B]$

$\Rightarrow$ $(B \cap A) \cup \phi = B$ $[\because B \cap X = \phi]$

$\Rightarrow$ $B \cap A = B$

$\Rightarrow$ $B \subseteq A$ ...(iv)

From Eqs. (iii) and (iv), we have $A = B$.

| Example 4. If A and B are any two sets, prove that $P(A) = P(B) \Rightarrow A = B$.

Sol. Given, $P(A) = P(B)$ ...(i)

To prove $A = B$

Let $x \in A \Rightarrow$ there exists a subset X of A such that $x \in X$.

Now, $X \subseteq A \Rightarrow X \in P(A)$

$\Rightarrow \quad X \subseteq B$

$\Rightarrow \quad x \in B \quad [\because x \in X]$

Thus, $\quad x \in A \Rightarrow x \in B$

$\therefore \quad A \subseteq B \quad$ ...(ii)

Let $y \in B \Rightarrow$ there exists a subset Y of B such that $y \in Y$.

Now, $\quad Y \subseteq B \Rightarrow Y \in P(B)$

$\Rightarrow \quad Y \in P(A) \quad [\because P(B) = P(A)]$

$\Rightarrow \quad Y \subseteq A$

$\Rightarrow \quad y \in A \quad [\because y \in Y]$

Thus, $\quad y \in B \Rightarrow y \in A$

$\therefore \quad B \subseteq A \quad$ ...(iii)

From Eqs. (ii) and (iii), we have $A = B$

Use of Sets in Logical Problems

M = Set of students which have Mathematics.

P = Set of students which have Physics.

C = Set of students which have Chemistry.

Applying the different operations on the above sets, then we get following important results.

M' = Set of students which have no Mathematics.

P' = Set of students which have no Physics.

C' = Set of students which have no Chemistry.

$M \cup P$ = Set of students which have atleast one subject Mathematics or Physics.

$P \cup C$ = Set of students which have atleast one subject Physics or Chemistry.

$C \cup M$ = Set of students which have atleast one subject Chemistry or Mathematics.

$M \cap P$ = Set of students which have both subjects Mathematics and Physics.

$P \cap C$ = Set of students which have both subjects Physics and Chemistry.

$C \cap M$ = Set of students which have both subjects Chemistry and Mathematics.

$M \cap P'$ = Set of students which have Mathematics but not Physics.

$P \cap C'$ = Set of students which have Physics but not Chemistry.

$C \cap M'$ = Set of students which have Chemistry but not Mathematics.

$(M \cup P)'$ = Set of students which have not both subjects Mathematics and Physics.

$(P \cup C)'$ = Set of students which have not both subjects Physics and Chemistry.

$(C \cup M)'$ = Set of students which have not both subjects Chemistry and Mathematics.

$(M \cap P \cap C)$ = Set of students which have all three subjects Mathematics, Physics and Chemistry.

$(M \cup P \cup C)$ = Set of all students which have three subjects.

Cardinal Number of Some Sets

If A, B and C are finite sets and U be the finite universal set, then

(i) $n(A') = n(U) - n(A)$

(ii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(iii) $n(A \cup B) = n(A) + n(B)$, if A and B are disjoint non-void sets.

(iv) $n(A \cap B') = n(A) - n(A \cap B)$

(v) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

(vi) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

(vii) $n(A - B) = n(A) - n(A \cap B)$

(viii) $n(A \cap B) = n(A \cup B) - n(A \cap B') - n(A' \cap B)$

(ix) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

(x) If $A_1, A_2, A_3, \ldots, A_n$ are disjoint sets, then
$n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = n(A_1) + n(A_2) + n(A_3) + \ldots + n(A_n)$

Example 5. If A and B be two sets containing 6 and 3 elements respectively, what can be the minimum number of elements in $A \cup B$? Also, find the maximum number of elements in $A \cup B$.

Sol. We have, $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$,

$n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum, respectively.

***Case* I** If $n(A \cap B)$ is minimum i.e., $n(A \cap B) = 0$ such that

$$A = \{a, b, c, d, e, f\} \text{ and } B = \{g, h, i\}$$

$$\therefore \quad n(A \cup B) = n(A) + n(B) = 6 + 3 = 9$$

***Case* II** If $n(A \cap B)$ is maximum i.e., $n(A \cap B) = 3$, such that

$$A = \{a, b, c, d, e, f\} \text{ and } B = \{d, a, c\}$$

$$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B) = 6 + 3 - 3 = 6$$

Example 6. Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each with five elements and $B_1, B_2, \ldots, B_n$ are n sets each with three elements.

Let $\quad \bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$

Assume that each element of S belongs to exactly ten of the A_i's and exactly to nine of the B_j's. Find n.

Sol. Given, A's are thirty sets with five elements each, so

$$\sum_{i=1}^{30} n(A_i) = 5 \times 30 = 150 \quad \text{...(i)}$$

If the m distinct elements in S and each element of S belongs to exactly 10 of the A_i's, so we have

$$\sum_{i=1}^{30} n(A_i) = 10m \quad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii), we get $10m = 150$

$$\therefore \quad m = 15 \quad \text{...(iii)}$$

Similarly, $\sum_{j=1}^{n} n(B_j) = 3n$ and $\sum_{j=1}^{n} n(B_j) = 9m$

$$\therefore \quad 3n = 9m \Rightarrow n = \frac{9m}{3} = 3m$$

$$= 3 \times 15 = 45 \quad \text{[from Eq. (iii)]}$$

Hence, $\quad n = 45$

| Example 7. In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only? How many can speak Bengali only? How many can speak both Hindi and Bengali?

Sol. Let H and B be the set of those people who can speak Hindi and Bengali respectively, then according to the problem, we have

$$n(H \cup B) = 1000,$$
$$n(H) = 750, n(B) = 400$$

We know that,

$$n(H \cup B) = n(H) + n(B) - n(H \cap B)$$
$$1000 = 750 + 400 - n(H \cap B)$$
$$\therefore \quad n(H \cap B) = 150$$

$\therefore$ Number of people speaking Hindi and Bengali both is 150.

Also, $\quad n(H \cap B') = n(H) - n(H \cap B)$
$$= 750 - 150$$
$$= 600$$

Thus, number of people speaking Hindi only is 600.

Again, $\quad n(B \cap H') = n(B) - n(B \cap H) = 400 - 150 = 250$

Thus, number of people speaking Bengali only is 250.

| Example 8. A survey of 500 television watchers produced the following information, 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games. How many watch all the three games? How many watch exactly one of the three games?

Sol. Let F, H and B be the sets of television watchers who watch Football, Hockey and Basketball, respectively.

Then, according to the problem, we have

$$n(U) = 500, \; n(F) = 285, \; n(H) = 195,$$
$$n(B) = 115, \; n(F \cap B) = 45,$$
$$n(F \cap H) = 70, \; n(H \cap B) = 50$$

and $\quad n(F' \cup H' \cup B') = 50,$

where U is the set of all the television watchers.

Since, $\quad n(F' \cup H' \cup B') = n(U) - n(F \cup H \cup B)$

$$\Rightarrow \quad 50 = 500 - n(F \cup H \cup B)$$
$$\Rightarrow \quad n(F \cup H \cup B) = 450$$

We know that,

$$n(F \cup H \cup B) = n(F) + n(H) + n(B) - n(F \cap H) - n(H \cap B) - n(B \cap F) + n(F \cap H \cap B)$$

$$\Rightarrow \quad 450 = 285 + 195 + 115 - 70 - 50 - 45 + n(F \cap H \cap B)$$

$$\therefore \quad n(F \cap H \cap B) = 20$$

which is the number of those who watch all the three games. Also, number of persons who watch football only

$$= n(F \cap H' \cap B')$$
$$= n(F) - n(F \cap H) - n(F \cap B) + n(F \cap H \cap B)$$
$$= 285 - 70 - 45 + 20 = 190$$

The number of persons who watch hockey only

$$= n(H \cap F' \cap B')$$
$$= n(H) - n(H \cap F) - n(H \cap B) + n(H \cap F \cap B)$$
$$= 195 - 70 - 50 + 20 = 95$$

and the number of persons who watch basketball only

$$= n(B \cap H' \cap F')$$
$$= n(B) - n(B \cap H) - n(B \cap F) + n(H \cap F \cap B)$$
$$= 115 - 50 - 45 + 20 = 40$$

Hence, required number of those who watch exactly one of the three games

$$= 190 + 95 + 40 = 325$$

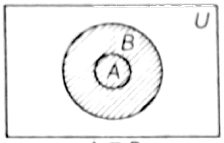
Venn Diagrams

(Euler-Venn Diagrams)

The diagram drawn to represent sets are called Venn diagrams or Euler Venn diagrams. Here, we represent the universal set U by points within rectangle and the subset A of the set U is represented by the interior of a circle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then to represent A and B by two intersecting circles.

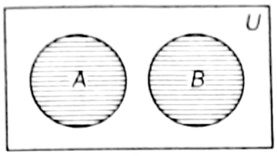
Venn Diagrams in Different Situations

1. Subset

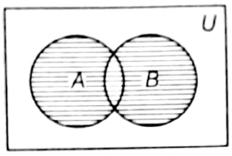


$A \subseteq B$

2. Union of sets

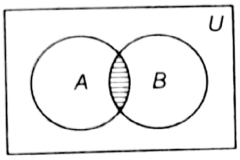


$A \cup B$, when $A \cap B = \phi$

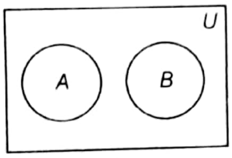


$A \cup B$, when neither $A \subseteq B$ nor $B \subseteq A$

3. Intersection of sets

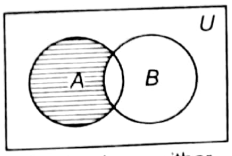


$A \cap B$, when neither $A \subseteq B$ nor $B \subseteq A$

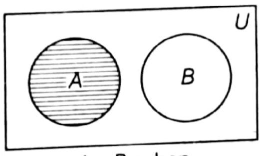


$A \cap B$, when $A \cap B = \phi$ (no shaded one)

4. Difference of sets

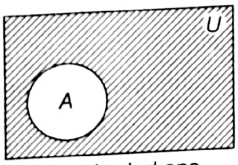


$A - B$, when neither $A \subseteq B$ nor $B \subseteq A$



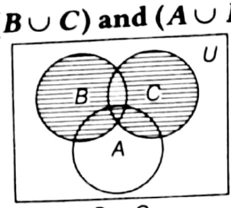
$A - B$, when $A - B = \phi$

5. Complement set

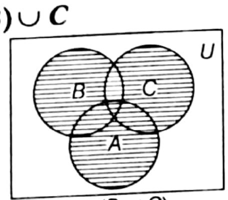


A' = shaded one

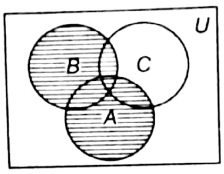
6. $A \cup (B \cup C)$ and $(A \cup B) \cup C$



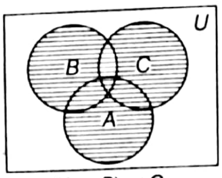
$B \cup C$



$A \cup (B \cup C)$



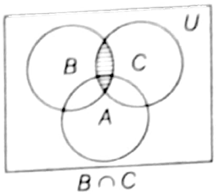
$A \cup B$



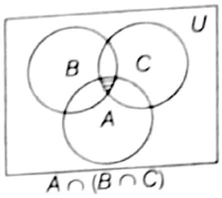
$(A \cup B) \cup C$

Hence, $A \cup (B \cup C) = (A \cup B) \cup C$ which is associative law for union.

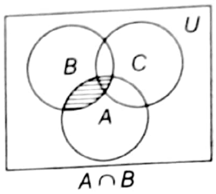
7. $A \cap (B \cap C)$ and $(A \cap B) \cap C$



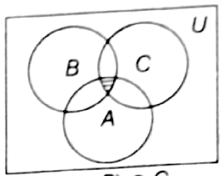
$B \cap C$



$A \cap (B \cap C)$



$A \cap B$



$(A \cap B) \cap C$

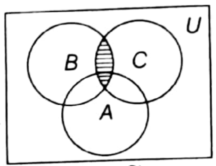
Hence, $A \cap (B \cap C) = (A \cap B) \cap C$ which is associative law for intersection.

8. Distributive law

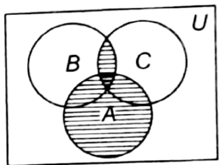
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

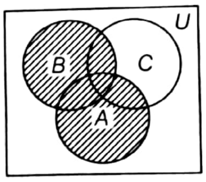
(i)



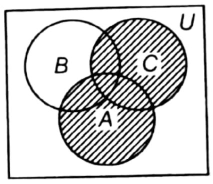
$(B \cap C)$



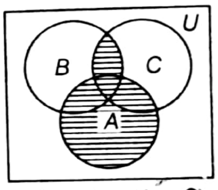
$A \cup (B \cap C)$



$(A \cup B)$



$(A \cup C)$

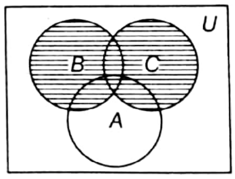


$(A \cup B) \cap (A \cup C)$

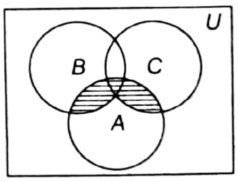
It is clear from diagrams that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

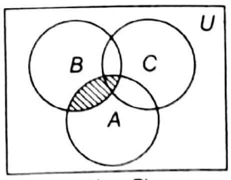
(ii)



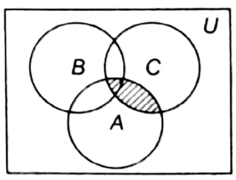
$(B \cup C)$



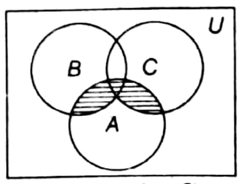
$A \cap (B \cup C)$



$(A \cap B)$



$(A \cap C)$

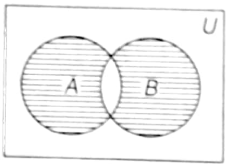


$(A \cap B) \cup (A \cap C)$

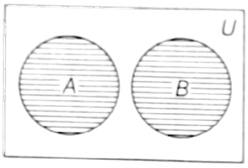
It is clear from diagrams that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

9. Symmetric difference



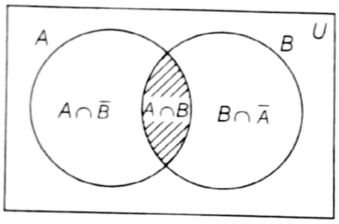
$A \Delta B$, when neither $A \subseteq B$ nor $B \subseteq A$ and $A \cap B \neq \phi$

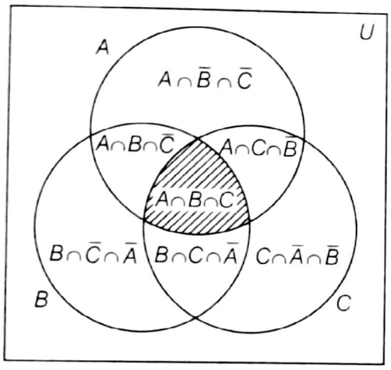


$A \Delta B$, when $A \cap B = \phi$

Remark

Remember with the help of figures.





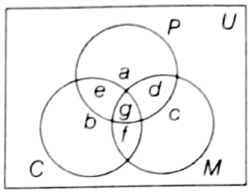
I Example 9. A class has 175 students. The following table shows the number of students studying one or more of the following subjects in this case.

Subjects	Number of students
Mathematics	100
Physics	70
Chemistry	46
Mathematics and Physics	30
Mathematics and Chemistry	28
Physics and Chemistry	23
Mathematics, Physics and Chemistry	18

How many students are enrolled in Mathematics alone, Physics alone and Chemistry alone? Are there students who have not offered any one of these subjects?

Sol. Let P, C and M denotes the sets of students studying Physics, Chemistry and Mathematics, respectively.

Let a, b, c, d, e, f, g denote the elements (students) contained in the bounded region as shown in the diagram.



Then,
$$a + d + e + g = 170$$
$$c + d + f + g = 100$$
$$b + e + f + g = 46$$
$$d + g = 30$$
$$e + g = 23$$
$$f + g = 28$$
$$g = 18$$

After solving, we get $g = 18$, $f = 10$, $e = 5$, $d = 12$, $a = 35$, $b = 13$ and $c = 60$

$$\therefore \quad a + b + c + d + e + f + g = 153$$

So, the number of students who have not offered any of these three subjects $= 175 - 153 = 22$

Number of students studying Mathematics only, $c = 60$

Number of students studying Physics only, $a = 35$

Number of students studying Chemistry only, $b = 13$

Aliter

Let P, C and M be the sets of students studying Physics, Chemistry and Mathematics, respectively. Then, we are given that

$$n(P) = 70, n(C) = 46, n(M) = 100$$
$$n(M \cap P) = 30, n(M \cap C) = 28$$
$$n(P \cap C) = 23$$
and $$n(M \cap P \cap C) = 18$$

$\therefore$ The number of students enrolled in Mathematics only

$$= n(M \cap P' \cap C') = n(M \cap (P \cup C)')$$
[by De-Morgan's law]
$$= n(M) - n(M \cap (P \cup C))$$
$$= n(M) - \{n[(M \cap P) \cup (M \cap C)]\}$$
[by distributive law]
$$= n(M) - n(M \cap P) - (M \cap C) + n(M \cap P \cap C)$$
$$= 100 - 30 - 28 + 18 = 60$$

Similarly, the number of students enrolled in Physics only, $n(P \cap M' \cap C')$

$$= n(P) - n(P \cap M) - n(P \cap C) + n(P \cap M \cap C)$$
$$= 70 - 30 - 23 + 18 = 35$$

and the number of students enrolled in Chemistry only,

$$n(C \cap M' \cap P') = n(C) - n(C \cap M) - n(C \cap P) + n(C \cap M \cap P)$$

$= 46 - 28 - 23 + 18 = 13$

and the number of students who have not offered any of the three subjects,

$n(M' \cap P' \cap C') = n(M \cap P \cap C)'$ [by De-Morgan's law]

$= n(U) - n(M \cup P \cup C)$

$= n(U) - \{n(M) + n(P) + n(C) - n(M \cap P)$

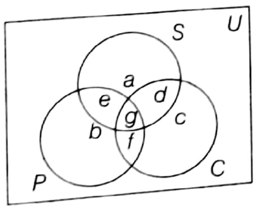
$\quad - n(M \cap C) - n(P \cap C) + n(P \cap C \cap M)\}$

$= 175 - \{100 + 70 + 46 - 30 - 28 - 23 + 8\}$

$= 175 - 153 = 22$

Example 10. In a pollution study of 1500 Indian rivers the following data were reported. 520 were polluted by sulphur compounds, 335 were polluted by phosphates, 425 were polluted by crude oil, 100 were polluted by both crude oil and sulphur compounds, 180 were polluted by both sulphur compounds and phosphates, 150 were polluted by both phosphates and crude oil and 28 were polluted by sulphur compounds, phosphates and crude oil. How many of the rivers were polluted by atleast one of the three impurities?
How many of the rivers were polluted by exactly one of the three impurities?

Sol. Let S, P and C denote the sets of rivers polluted by sulphur compounds, by phosphates and by crude oil respectively, and let a, b, c, d, e, f, g denote the elements (impurities) contained in the bounded region as shown in the diagram.



Then, $a + d + e + g = 520$

$c + d + f + g = 425$

$b + e + f + g = 335 \Rightarrow d + g = 100$

$e + g = 180 \Rightarrow f + g = 150$

$g = 28$

After solving, we get

$g = 28, f = 122, e = 152, b = 33, d = 72, c = 203$ and $a = 268$

The number of rivers were polluted by atleast one of the three impurities

$= (a + b + c + d + e + f + g) = 878$

and the number of rivers were polluted by exactly one of the three impurities,

$a + b + c = 268 + 33 + 203 = 504$

Aliter

Let S, P and C denote the sets of rivers polluted by sulphur compounds, by phosphates and by crude oil, respectively.

Then, we are given that

$n(S) = 520, n(P) = 335, n(C) = 425, n(C \cap S) = 100,$

$n(S \cap P) = 180, n(P \cap C) = 150$ and $n(S \cap P \cap C) = 28.$

The number of rivers polluted by atleast one of the three impurities,

$n(S \cup P \cup C)$

$= n(S) + n(P) + n(C) - n(S \cap P)$

$\quad - n(P \cap C) - n(C \cap S) + n(S \cap P \cap C)$

$= 520 + 335 + 425 - 180 - 150 - 100 + 28 = 878$

and the number of rivers polluted by exactly one of the three impurities,

$n\{(S \cap P' \cap C') \cup (P \cap C' \cap S') \cup (C \cap P' \cap S')\}$

$= n\{(S \cap (P \cup C)'\} \cup \{P \cap (C \cup S)'\} \cup \{C \cap P \cup S)'\}$

$= n(S \cap (P \cup C)') + n(P \cap (C \cup S)') + n(C \cap (P \cup S)')$

$= n(S) - n(S \cap P) - n(S \cap C)$

$\quad + n(S \cap P \cap C) + n(P) - n(P \cap C) - n(P \cap S)$

$\quad + n(S \cap P \cap C)$

$\quad + n(C) - n(C \cap P) - n(C \cap S) + n(S \cap P \cap C)$

$= n(S) + n(P) + n(C) - 2n(S \cap P) - 2n(S \cap C)$

$\quad - 2n(P \cap C) + 3n(S \cap P \cap C)$

$= 520 + 335 + 425 - 360 - 200 - 300 + 84 = 504$

Exercise for Session 1

1. If $X = \{4^n - 3n - 1 : n \in N\}$ and $y = \{9(n-1) : n \in N\}$, then $X \cup Y$ equals
(a) X (b) Y (c) N (d) None of these

2. If $N_a = \{an : n \in N\}$, then $N_5 \cap N_7$ equals
(a) N (b) N_5 (c) N_7 (d) N_{35}

3. If A and B are two sets, then $A \cap (A \cup B)'$ equals
(a) A (b) B (c) ϕ (d) None of these

4. If U be the universal set and $A \cup B \cup C = U$, then $[(A - B) \cup (B - C) \cup (C - A)']$ equals
(a) $A \cup B \cup C$ (b) $A \cap B \cap C$ (d) $A \cup (B \cap C)$ (d) $A \cap (B \cup C)$

5. If A and B are two sets, then $(A - B) \cup (B - A) \cup (A \cap B)$ equals
(a) $A \cup B$ (b) $A \cap B$ (c) A (d) B'

6. If $A = \{x : x$ is a multiple of 4$\}$ and $B = \{x : x$ is a multiple of 6$\}$, then $A \subset B$ consists of all multiple of
(a) 4 (b) 8 (c) 12 (d) 16

7. A set contains $2n + 1$ elements. The number of subsets of this set containing more than n elements equals
(a) 2^{n-1} (b) 2^n (c) 2^{n+1} (d) 2^{2n}

8. If $A = \{\phi, \{\phi\}\}$, then the power set of A is
(a) A (b) $\{\phi, \{\phi\}, A\}$
(c) $\{\phi, \{\phi\}, \{\{\phi\}\}, A\}$ (d) None of these

9. Given $n(U) = 20$, $n(A) = 12$, $n(B) = 9$, $n(A \cap B) = 4$, where U is the universal set, A and B are subsets of U, then $n((A \cup B)')$ equals
(a) 3 (b) 9 (c) 11 (d) 17

10. A survey shows that 63% of the Indians like cheese, whereas 76% like apples. If x % of the Indians like both cheese and apples, then x can be
(a) 40 (b) 65 (c) 39 (d) None of these

11. In a class of 55 students, the number of students studying different subjects are 23 in Mathematics, 24 in Physics, 19 in Chemistry, 12 in Mathematics and Physics, 9 in Mathematics and Chemistry, 7 in Physics and Chemistry and 4 in all the three subjects. The number of students who have taken exactly one subject is
(a) 6 (b) 7 (c) 9 (d) 22

Session 2

Ordered Pair, Definition of Relation, Ordered Relation, Composition of Two Relations

Ordered Pair

If A be a set and $a, b \in A$, then the ordered pair of elements a and b in A denoted by (a, b), where a is called the first coordinate and b is called the second coordinate.

Remark

1. Ordered pairs (a, b) and (b, a) are different.
2. Ordered pairs (a, b) and (c, d) are equal iff $a = c$ and $b = d$.

Cartesian Product of Two Sets

The cartesian product to two sets A and B is the set of all those ordered pairs whose first coordinate belongs to A and second coordinate belongs to B. This set is denoted by $A \times B$ (read as 'A cross B' or 'product set of A and B').

Symbolically, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

or $\quad A \times B = \{(a, b) : a \in A \wedge b \in B\}$

Thus, $\quad (a, b) \in A \times B \Leftrightarrow a \in A \wedge b \in B$

Similarly, $\quad B \times A = \{(b, a) : b \in B \wedge a \in A\}$

Remark

1. $A \times B \neq B \times A$
2. If A has p elements and B has q elements, then $A \times B$ has pq elements.
3. If $A = \phi$ and $B = \phi$, then $A \times B = \phi$.
4. Cartesian product of n sets $A_1, A_2, A_3, \ldots, A_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ $a_i \in A_i$, $i = 1, 2, 3 \ldots, n$ and is denoted by $A_1 \times A_2 \times \ldots \times A_n$ or $\prod_{i=1}^{n} A_i$.

| Example 11. If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, find $A \times B$, $B \times A$ and show that $A \times B \neq B \times A$.

Sol. $A \times B = \{1, 2, 3\} \times \{4, 5\} = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

and $B \times A = \{4, 5\} \times \{1, 2, 3\} = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$

It is clear that $A \times B \neq B \times A$.

| Example 12. If A and B be two sets and $A \times B = \{(3, 3), (3, 4), (5, 2), (5, 4)\}$, find A and B.

Sol. A = First coordinates of all ordered pairs = $\{3, 5\}$

and B = Second coordinates of all ordered pairs = $\{2, 3, 4\}$

Hence, $A = \{3, 5\}$ and $B = \{2, 3, 4\}$

Important Theorems on Cartesian Product

If A, B and C are three sets, then

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

(iv) $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$, where S and T are two sets.

(v) If $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$

(vi) If $A \subseteq B$, then $(A \times B) \cap (B \times A) = A^2$ = $A \times A$

(vii) If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$

| Example 13. If A and B are two sets given in such a way that $A \times B$ consists of 6 elements and if three elements of $A \times B$ are $(1, 5)$, $(2, 3)$ and $(3, 5)$, what are the remaining elements?

Sol. Since, $(1, 5), (2, 3), (3, 5) \in A \times B$, then clearly $1, 2, 3 \in A$ and $3, 5 \in B$.

$A \times B = \{1, 2, 3\} \times (3, 5)$

$= (1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)$

Hence, the remaining elements are $(1, 3)$, $(2, 5)$, $(3, 3)$.

Relations

Introduction of Relation

We use sentences depending upon the relationship of an object to the other object in our daily life such as

(i) 'Ram, Laxman, Bharat, Shatrughan' were the sons of Dashrath.
(ii) 'Sita' was the wife of Ram.
(iii) 'Laxman' was the brother of Ram.
(iv) 'Dashrath' was the father of Ram.
(v) 'Kaushaliya' was the mother of Ram.

If Ram, Laxman, Bharat, Shatrughan, Sita, Kaushaliya and Dashrath are represented by a, b, c, d, e, f and y respectively and A represents the set, then

$$A = \{a, b, c, d, e, f, y\}$$

Here, we see that any two elements of set A are related many ways, i.e. a, b, c, d are sons of y. 'a' is the son of y is represented by aRy. Similarly, b is the son of y, c is the son of y and d is also son of y are represented as $b\,R\,y$, $c\,R\,y$ and $d\,R\,y$, respectively.

If we write here $y\,R\,a$ it means that y is the son of a which is impossible, since a is the son of y. Hence, y and a cannot be related like this. Its generally represented as $y \not R a$. Hence, we can say that a and y are in definite order. a comes before R and y after R. Therefore, aRy may be represented as a order pair (a, y). Similarly, bRy, cRy and dRy are represented by (b, y), (c, y) and (d, y), respectively. If all pairs will represented by a set, then we see that first element of each pair is the son of second element. Hence, the set of these pairs may be represented by set R, then

$$R = \{(a, y), (b, y), (c, y), (d, y)\}$$

Symbolically, $R = \{(x, y) : x, y \in A$, where x is son of $y\}$

It is clear that R is subset of $A \times A$

i.e., $R \subseteq A \times A$

Corollary In above example, if

$A = \{a, b, c, d\}$ and $B = \{e, f, y\}$, then

$R = \{(x, z) : x \in A, z \in B$, where x is son of $z\}$

It is clear that $R \subseteq A \times B$.

Definition of Relation

A relation (or binary relation) R, from a non-empty set A to another non-empty set B, is a subset of $A \times B$.

i.e., $R \subseteq A \times B$ or $R \subseteq \{(a, b) : a \in A, b \in B\}$

Now, if (a, b) be an element of the relation R, then we write aRb (read as 'a is related to b')

i.e., $(a, b) \in R \Leftrightarrow aRb$

and if (a, b) is not an element of the relation R, then we write $a \not R b$ (read as 'a is not related to b'),

i.e. $(a, b) \notin R \Leftrightarrow a \not R b$.

Remark

1. Any subset of $A \times A$ is said to be a relation on A.
2. If A has m elements and B has n elements, then $A \times B$ has $m \times n$ elements and total number of different relations from A to B is 2^{mn}.
3. If $R = A \times B$, then Domain $R = A$ and Range $R = B$.
4. The domain as well as range of the empty set ϕ is ϕ.
5. If A = Dom R and B = Ran R, then we write $B = R[A]$.

For example,

Let $A = \{1, 2, 3\}$ and $B = \{3, 5, 7\}$, then
$A \times B = \{(1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 3), (3, 5), (3, 7)\}$.
But $R \subseteq A \times B$
i.e., every subset of $A \times B$ is a relation from A to B. If we consider the relation, $R = \{(1, 5), (1, 7), (3, 5), (3, 7)\}$
Then, $1\,R\,5; 1\,R\,7; 3\,R\,5; 3\,R\,7$
Also, $1 \not R\, 3; 2 \not R\, 3; 2 \not R\, 5; 2 \not R\, 7; 3 \not R\, 3$;
Clearly, Domain $R = \{1, 3\}$ and Range $R = \{5, 7\}$

For example,

Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then number of different relations from A to B is $2^{3\times2} = 2^6 = 64$ because A has 3 elements and B has 2 elements.

Types of Relations from One Set to Another Set

1. Empty Relation

A relation R from A to B is called an empty relation or a void relation from A to B if $R = \phi$.

For example,

Let $A = \{2, 4, 6\}$ and $B = \{7, 11\}$
Let $R = \{(a, b) : a \in A, b \in B$ and $a - b$ is even$\}$
As, none of the numbers $2 - 7, 2 - 11, 4 - 7, 4 - 11, 6 - 7, 6 - 11$ is an even number, $R = \phi$.
Hence, R is an empty relation.

2. Universal Relation

A relation R from A to B is said to be the universal relation, if $R = A \times B$.

For example, Let $A = \{1, 2\}$, $B = \{1, 3\}$ and
$R = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$
Here, $R = A \times B$
Hence, R is the universal relation from A to B.

Types of Relations on a Set

1. Empty Relation

A relation R on a set A is said to be an empty relation or a void relation, if $R = \phi$.

For example,

Let $A = \{1, 3\}$ and $R = \{(a, b) : a, b \in A \text{ and } a + b \text{ is odd}\}$

Hence, R contains no element, therefore R is an empty relation on A.

2. Universal Relation

A relation R on a set A is said to be universal relation on A, if $R = A \times A$.

For example,

Let $A = \{1, 2\}$

and $R = [(1, 1), (1, 2), (2, 1), (2, 2)]$

Here, $R = A \times A$

Hence, R is the universal relation on A.

3. Identity Relation

A relation R on a set A is said to be the identity relation on A, if

$$R = [(a, b) : a \in A, b \in A \text{ and } a = b]$$

Thus, identity relation, $R = [(a, a) : \forall\, a \in A]$

Identity relation on set A is also denoted by I_A.

Symbolically, $I_A = [(a, a) : a \in A]$

For example,

Let $A = \{1, 2, 3\}$

Then, $I_A = \{(1, 1), (2, 2), (3, 3)\}$

Remark

In an identity relation on A every element of A should be related to itself only.

4. Inverse Relation

If R is a relation from a set A to a set B, then inverse relation of R to be denoted by R^{-1}, is a relation from B to A.

Symbolically, $R^{-1} = \{(b, a) : (a, b) \in R\}$

Thus, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}, \forall\, a \in A, b \in B.$

Remark

1. $\text{Dom}(R^{-1}) = \text{Range}(R)$ and $\text{Range}(R^{-1}) = \text{Dom}(R)$
2. $(R^{-1})^{-1} = R$

For example,

If $R = \{(1, 2), (3, 4), (5, 6)\}$, then

$R^{-1} = \{(2, 1), (4, 3), (6, 5)\}$

$\therefore (R^{-1})^{-1} = \{(1, 2), (3, 4), (5, 6)\} = R$

Here, $\text{dom}(R) = \{1, 3, 5\}$, $\text{range}(R) = \{2, 4, 6\}$

and $\text{dom}(R^{-1}) = \{2, 4, 6\}$, $\text{range}(R^{-1}) = \{1, 3, 5\}$

Clearly, $\text{dom}(R^{-1}) = \text{range}(R)$

and $\text{range}(R^{-1}) = \text{dom}(R)$.

Various Types of Relations

1. Reflexive Relation

A relation R on a set A is said to be reflexive, if $a\,R\,a, \forall\, a \in A$

i.e., if $(a, a) \in R, \forall\, a \in A$

For example,

Let $A = \{1, 2, 3\}$

$R_1 = \{(1, 1), (2, 2), (3, 3)\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3)\}$

and $R_3 = \{(1, 1), (2, 2), (2, 3), (3, 2)\}$

Here, R_1 and R_2 are reflexive relations on A, R_3 is not a reflexive relation on A as $(3, 3) \notin R_3$, i.e. $3 \not R_3\, 3$.

Remark

The identity relation is always a reflexive relation but a reflexive relation may or may not be the identity relation. It is clear in the above example given, R_1 is both reflexive and identity relation on A but R_2 is a reflexive relation on A but not an identity relation on A.

2. Symmetric Relation

A relation R on a set A is said to be symmetric relation, if

$$a\,R\,b \Rightarrow b\,R\,a, \forall\, a, b \in A$$

i.e., if $(a, b) \in R \Rightarrow (b, a) \in R, \forall\, a, b \in A$

For example,

Let $A = \{1, 2, 3\}$

$R_1 = \{(1, 2), (2, 1)\}$

$R_2 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$

and $R_3 = \{(2, 3), (1, 3), (3, 1)\}$

Here, R_1 and R_2 are symmetric relations on A but R_3 is not a symmetric relation on A because $(2, 3) \in R_3$ and $(3, 2) \notin R_3$.

3. Anti-symmetric Relation

A relation R on a set A is said to be anti-symmetric, if $a\,R\,b, b\,R\,a \Rightarrow a = b, \forall\, a, b \in A$

i.e., $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b, \forall\, a, b \in A$

For example,

Let R be the relation in N (natural number) defined by, "x is divisor of y", then R is anti-symmetric because x divides y and y divides $x \Rightarrow x = y$

4. Transitive Relation

A relation R on a set A is said to be a transitive relation, if $a\,R\,b$ and $b\,R\,c \Rightarrow aRc, \forall\ a, b, c \in A$

i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall\ a, b, c \in A$

For example,

Let $A = \{1, 2, 3\}$

$R_1 = \{(1, 2), (2, 3), (1, 3), (3, 2)\}$

$R_2 = \{(2, 3), (3, 1)\}$

$R_3 = \{(1, 3), (3, 2), (1, 2)\}$

Then, R_1 is not transitive relation on A because $(2, 3) \in R_1$ and $(3, 2) \in R_1$ but $(2, 2) \notin R_1$. Again, R_2 is not transitive relation on A because $(2, 3) \in R_2$ and $(3, 1) \in R_2$ but $(2, 1) \notin R_2$. Finally R_3 is a transitive relation.

Example 14. Let $A = \{1, 2, 3\}$ and $R = \{(a, b) : a, b \in A$, a divides b and b divides $a\}$. Show that R is an identity relation on A.

Sol. Given, $A = \{1, 2, 3\}$

$a \in A, b \in B$, a divides b and b divides a.

$\Rightarrow \quad a = b$

$\therefore \quad R = \{(a, a), a \in A\} = \{(1, 1), (2, 2), (3, 3)\}$

Hence, R is the identity relation on A.

Example 15. Let $A = \{3, 5\}$, $B = \{7, 11\}$.
Let $R = \{(a, b) : a \in A, b \in B, a - b \text{ is even}\}$.
Show that R is an universal relation from A to B.

Sol. Given, $A = \{3, 5\}$, $B = \{7, 11\}$.

Now, $R = \{(a, b) : a \in A, b \in B \text{ and } a - b \text{ is even}\}$

$= \{(3, 7), (3, 11), (5, 7), (5, 11)\}$

Also, $A \times B = \{(3, 7), (3, 11), (5, 7), (5, 11)\}$

Clearly, $R = A \times B$

Hence, R is an universal relation from A to B.

Example 16. Prove that the relation R defined on the set N of natural numbers by $xRy \Leftrightarrow 2x^2 - 3xy + y^2 = 0$ is not symmetric but it is reflexive.

Sol. (i) $2x^2 - 3x \cdot x + x^2 = 0, \forall\ x \in N$.

$\therefore\ x\,R\,x, \forall\ x \in N$, i.e. R is reflexive.

(ii) For $x = 1, y = 2, 2x^2 - 3xy + y^2 = 0$

$\therefore\ 1R2$

But $2 \cdot 2^2 - 3 \cdot 2 \cdot 1 + 1^2 = 3 \neq 0$

So, 2 is not related to 1 i.e., $2 \not R 1$

$\therefore\ R$ is not symmetric.

Example 17. Let N be the set of natural numbers and relation R on N be defined by $x\,R\,y \Leftrightarrow x$ divides y, $\forall\ x, y \in N$.
Examine whether R is reflexive, symmetric, anti-symmetric or transitive.

Sol. (i) x divides x i.e., $x\,R\,x, \forall\ x \in N$

$\therefore\ R$ is reflexive.

(ii) 1 divides 2 i.e., $1\ R\ 2$ but $2 \not R 1$ as 2 does not divide 1.

(iii) x divides y and y divides $x \Rightarrow x = y$

i.e., $x\,Ry$ and $y\,R\,x \Rightarrow x = y$

$\therefore\ R$ is anti-symmetric relation.

(iv) $x\,Ry$ and $y\,Rz \Rightarrow x$ divides y and y divides z.

$\Rightarrow kx = y$ and $k'y = 2$, where k, k' are positive integers.

$\Rightarrow\ kk'x = z\ \Rightarrow\ x$ divides $z\ \Rightarrow\ x\,Rz$

$\therefore\ R$ is transitive.

Equivalence Relation

A relation R on a set A is said to be an equivalence relation on A, when R is (i) reflexive (ii) symmetric and (iii) transitive. The equivalence relation denoted by ~.

Example 18. N is the set of natural numbers. The relation R is defined on $N \times N$ as follows
$(a, b) R (c, d) \Leftrightarrow a + d = b + c$
Prove that R is an equivalence relation.

Sol. (i) $(a, b)\ R\ (a, b)\ \Rightarrow a + b = b + a$

$\therefore\ R$ is reflexive.

(ii) $(a, b)\ R\ (c, d)\ \Rightarrow a + d = b + c$

$\Rightarrow \quad c + b = d + a\ \Rightarrow\ (c, d)\ R\ (a, b)$

$\therefore\ R$ is symmetric.

(iii) $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow a + d = b + c$ and $c + f = d + e$

$\Rightarrow\ a + d + c + f = b + c + d + e$

$\Rightarrow \quad a + f = b + e\ \Rightarrow (a, b)\ R\ (e, f)$

$\therefore\ R$ is transitive.

Thus, R is an equivalence relation on $N \times N$.

Example 19. A relation R on the set of complex numbers is defined by $z_1 R z_2 \Leftrightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real, show that R is an equivalence relation.

Sol. (i) $z_1Rz_1 \Rightarrow \dfrac{z_1 - z_1}{z_1 + z_1}, \forall z_1 \in C\ \Rightarrow\ 0$ is real

$\therefore\ R$ is reflexive.

(ii) $z_1 R z_2 \Rightarrow \frac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow -\left(\frac{z_2 - z_1}{z_1 + z_2}\right)$ is real $\Rightarrow \left(\frac{z_2 - z_1}{z_1 + z_2}\right)$ is real

$\Rightarrow z_2 R z_1, \forall z_1, z_2 \in C$

$\therefore$ R is symmetric.

(iii) $\because z_1 R z_2 \Rightarrow \frac{z_1 - z_2}{z_1 + z_2}$ is real

$$\Rightarrow \overline{\left(\frac{z_1 - z_2}{z_1 + z_2}\right)} = -\left(\frac{z_1 - z_2}{z_1 + z_2}\right)$$

$$\Rightarrow \left(\frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_1 + \bar{z}_2}\right) + \left(\frac{z_1 - z_2}{z_1 + z_2}\right) = 0$$

$$\Rightarrow 2(z_1\bar{z}_1 - z_2\bar{z}_2) = 0 \Rightarrow |z_1|^2 = |z_2|^2 \quad \text{...(i)}$$

Similarly, $z_2 R z_2 \Rightarrow |z_2|^2 = |z_3|^2$ ...(ii)

From Eqs. (i) and (ii), we get

$z_1 R z_2, z_2 R z_3$

$\Rightarrow |z_1|^2 = |z_3|^2$

$\Rightarrow z_1 R z_3$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

Ordered Relation

A relation R is called ordered, if R is transitive but not an equivalence relation.

Symbolically, $a R b, b R c \Rightarrow a R c, \forall a, b, c \in A$

For example,

Let $R = \{(1, 2), (2, 1), (2, 3), (3, 2), (1, 3)\}$.

Here, R is symmetric.

Since, $(1, 2) \in R \Rightarrow (2, 1) \in R$, $(2, 3) \in R \Rightarrow (3, 2) \in R$

and R is transitive.

Since, $(1, 2) \in R$, $(2, 3) \in R \Rightarrow (1, 3) \in R$

but R is not reflexive.

Since, $(1, 1) \notin R$, $(2, 2) \notin R$, $(3, 3) \notin R$

Hence, R is not an equivalence relation.

$\therefore$ R is an ordered relation.

Partial Order Relation

A relation R is called partial order relation, if R is reflexive, transitive and anti-symmetric at the same time.

For example,

Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$

$\therefore$ $R^{-1} = \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 2), (3, 1)\}$

$R \cap R^{-1} = \{(1, 1), (2, 2), (3, 3)\} = \text{Identity}$

$\therefore$ R is anti-symmetric.

It is clear that R is reflexive.

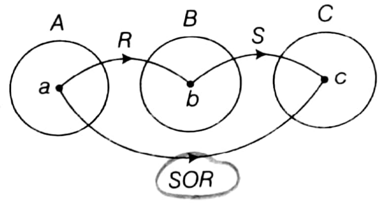
Since, $(1, 1) \in R$, $(2, 2) \in R$, $(3, 3) \in R$ and R is transitive.

Since, $(1, 2) \in R$ and $(2, 3) \in R \Rightarrow (1, 3) \in R$

Hence, R is partial order relation.

Composition of Two Relations

If A, B and C are three sets such that $R \subseteq A \times B$ and $S \subseteq B \times C$, then $(SOR)^{-1} = R^{-1}OS^{-1}$. It is clear that aRb, bSc $\Rightarrow aSORc$.



More generally,

$$(R_1 O R_2 O R_3 O \ldots O R_n)^{-1} = R_n^{-1} O \ldots O R_3^{-1} O R_2^{-1} O R_1^{-1}$$

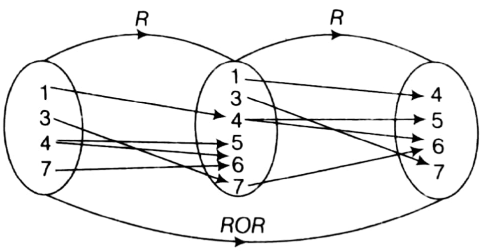
Example 20. Let R be a relation such that $R = \{(1, 4), (3, 7), (4, 5), (4, 6), (7, 6)\}$, find

(i) $R^{-1}OR^{-1}$ and (ii) $(R^{-1}OR)^{-1}$.

Sol. (i) We know that, $(ROR)^{-1} = R^{-1}OR^{-1}$

Dom $(R) = \{1, 3, 4, 7\}$

Range $(R) = \{4, 5, 6, 7\}$



We see that,

$1 \longrightarrow 4 \longrightarrow 5 \Rightarrow (1, 5) \in ROR$

$1 \longrightarrow 4 \longrightarrow 6 \Rightarrow (1, 6) \in ROR$

$3 \longrightarrow 7 \longrightarrow 6 \Rightarrow (3, 6) \in ROR$

$\therefore$ $ROR = \{(1, 5), (1, 6), (3, 6)\}$

Then, $R^{-1}OR^{-1} = (ROR)^{-1}$

$= \{(5, 1), (6, 1), (6, 3)\}$

(ii) We know that, $(R^{-1}OR)^{-1} = R^{-1}O(R^{-1})^{-1} = R^{-1}OR$

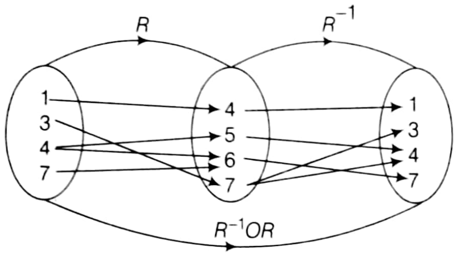
Since,

$R = \{(1, 4), (3, 7), (4, 5), (4, 6), (7, 6)\}$

$\therefore$ $R^{-1} = \{(4, 1), (7, 3), (5, 4), (6, 4), (6, 7)\}$

$\therefore$ Dom $(R) = \{1, 3, 4, 7\}$, Range $(R) = \{4, 5, 6, 7\}$

Dom $(R^{-1}) = \{4, 5, 6, 7\}$, Range $(R^{-1}) = \{1, 3, 4, 7\}$



We see that,

$$1 \xrightarrow{R} 4 \xrightarrow{R^{-1}} 1 \Rightarrow (1, 1) \in R^{-1}OR$$

$$3 \xrightarrow{R} 7 \xrightarrow{R^{-1}} 3 \Rightarrow (3, 3) \in R^{-1}OR$$

$$4 \xrightarrow{R} 5 \xrightarrow{R^{-1}} 4 \Rightarrow (4, 4) \in R^{-1}OR$$

$$4 \xrightarrow{R} 6 \xrightarrow{R^{-1}} 4 \Rightarrow (4, 4) \in R^{-1}OR$$

$$4 \xrightarrow{R} 6 \xrightarrow{R^{-1}} 7 \Rightarrow (4, 7) \in R^{-1}OR$$

$$7 \xrightarrow{R} 6 \xrightarrow{R^{-1}} 4 \Rightarrow (7, 4) \in R^{-1}OR$$

$$7 \xrightarrow{R} 6 \xrightarrow{R^{-1}} 7 \Rightarrow (7, 7) \in R^{-1}OR$$

$$\therefore \; R^{-1}OR = \{(1, 1), (3, 3), (4, 4), (7, 7), (4, 7), (7, 4)\}$$

Hence, $(R^{-1}OR)^{-1} = R^{-1}OR = \{(1, 1), (3, 3)$
$(4, 4), (7, 7), (4, 7), (7, 4)\}$

Theorems on Binary Relations

If R is a relation on a set A, then

(i) R is reflexive $\Rightarrow R^{-1}$ is reflexive.

(ii) R is symmetric $\Rightarrow R^{-1}$ is symmetric.

(iii) R is transitive $\Rightarrow R^{-1}$ is transitive.

Exercise for Session 2

1. If $A = \{2, 3, 5\}$, $B = \{2, 5, 6\}$, then $(A - B) \times (A \cap B)$ is

(a) $\{(3, 2), (3, 3), (3, 5)\}$ (b) $\{(3, 2), (3, 5), (3, 6)\}$
(c) $\{(3, 2), (3, 5)\}$ (d) None of these

2. If $n(A) = 4$, $n(B) = 3$, $n(A \times B \times C) = 24$, then $n(C)$ equals

(a) 1 (b) 2 (c) 17 (d) 288

3. The relation R defined on the set of natural numbers as $\{(a, b) : a$ differs from b by 3$\}$ is given by

(a) $\{(1, 4), (2, 5), (3, 6), \ldots\}$ (b) $\{(4, 1), (5, 2), (6, 3), \ldots\}$ (c) $\{(1, 3), (2, 6), (3, 9), \ldots\}$ (d) None of these

4. Let A be the non-void set of the children in a family. The relation 'x is a brother of y' on A, is

(a) reflexive (b) anti-symmetric (c) transitive (d) equivalence

5. Let $n(A) = n$, then the number of all relations on A, is

(a) 2^n (b) $2^{n!}$ (c) 2^{n^2} (d) None of these

6. If $S = \{1, 2, 3, \ldots, 20\}$, $K = \{a, b, c, d\}$, $G = \{b, d, e, f\}$. The number of elements of $(S \times K) \cup (S \times G)$ is

(a) 40 (b) 100 (c) 120 (d) 140

7. The relation R is defined on the set of natural numbers as $\{(a, b) : a = 2b\}$, then R^{-1} is given by

(a) $\{(2, 1) (4, 2) (6, 3), \ldots\}$ (b) $\{(1, 2) (2, 4) (3, 6), \ldots\}$ (c) R^{-1} is not defined (d) None of these

8. The relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on set $A = \{1, 2, 3\}$ is

(a) reflexive but not symmetric (b) reflexive but not transitive
(c) symmetric and transitive (d) Neither symmetric nor transitive

9. The number of equivalence relations defined in the set $S = \{a, b, c\}$ is

(a) 5 (b) 3! (c) 2^3 (d) 3^3

10. If R be a relation < from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$, i.e. $(a, b) \in R \Leftrightarrow a < b$, then ROR^{-1}, is

(a) $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$ (b) $\{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$
(c) $\{(3, 3), (3, 5), (5, 3), (5, 5)\}$ (d) $\{(3, 3), (3, 4), (4, 5)\}$

Session 3

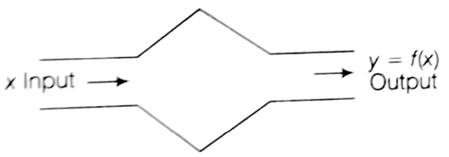
Definition of Functions, Domain, Codomain and Range, Composition of Mapping, Equivalence Classes, Partition of Set, Congruences

Functions

Introduction

If two variable quantities x and y according to some law are so related that corresponding to each value of x (considered only real), which belongs to set E, there corresponds one and only one finite value of the quantity y (i.e., unique value of y). Then, y is said to be a function (single valued) of x, defined by $y = f(x)$, where x is the **argument** or **independent variable** and y is the **dependent variable** defined on the set E.

For example, If r is the radius of the circle and A its area, then r and A are related by $A = \pi r^2$ or $A = f(r)$. Then, we say that the area A of the circle is the function of the radius r. **Graphically**,



Where, y is the image of x and x is the pre-image of y under f.

Remark

1. If to each value of x, which belongs to set E there corresponds one or more than one values of the quantity y. Then, y is called the multiple valued function of x defined on the set E.
2. The word 'FUNCTION' is used only for single valued function. *For example*, $y = \sqrt{x}$ is single valued functions but $y^2 = x$ is a multiple valued function.
   $\therefore \; y^2 = x \Rightarrow y = \pm\sqrt{x}$ for one value of x, y gives two values.

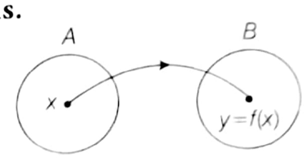
Definition of Functions

If A and B be two non-empty sets, then a function from A to B associates to each element x in A, a unique element $f(x)$ in B and is written as

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

which is read as f is a mapping from A to B.

The other terms used for functions are **operators** or **transformations.**



Remark

1. If $x \in A$, $y = [f(x)] \in B$, then $(x, y) \in f$.
2. If $(x_1, y_1) \in f$ and $(x_2, y_2) \in f$, then $y_1 = y_2$.

Domain, Codomain and Range

Domain The set of A is called the domain of f (denoted by D_f).

Codomain The set of B is called the codomain of f (denoted by C_f).

Range The range of f denoted by R_f is the set consisting of all the images of the elements of the domain A.

Range of $f = [f(x) : x \in A]$

The range of f is always a subset of codomain B.

Onto and Into Mappings

In the mapping $f : A \to B$ such

$$f(A) = B$$

i.e., Range = Codomain

Then, the function is **Onto** and if $f(A) \subset B$, i.e. Range $\subset$ Codomain, then the function is **Into**.

Remark

Onto functions is also known as **surjective.**

Method to Test Onto or Into Mapping

Let $f : A \to B$ be a mapping. Let y be an arbitrary element in B and then $y = f(x)$, where $x \in A$. Then, express x in terms of y.

Now, if $x \in A, \forall\, y \in B$, then f is onto
and if $x \notin A, \forall\, y \in B$, then f is into.

For into mapping Find an element of B which is not f-image of any element of A.

One-one and Many-one Mapping

(i) The mapping $f : A \to B$ is called one-one mapping, if no two different elements of A have the same image in B. Such a mapping is also known as **injective mapping** or an **injection** or **monomorphism.**

Method to Test One-one If $x_1, x_2 \in A$,

then $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 = x_2$ and $x_1 \neq x_2$

$\Rightarrow$ $f(x_1) \neq f(x_2)$

(ii) The mapping $f : A \to B$ is called many-one mapping, if two or more than two different elements in A have the same image in B.

Method to Test Many-one

If $x_1, x_2 \in A$, then $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 \neq x_2$

From above classification, we conclude that function is of four types

(i) One-one onto (bijective)
(ii) One-one into
(iii) Many-one onto
(iv) Many-one into

Number of Functions (Mappings) at One Place in a Table

Let $f : A \to B$ be a mapping such that A and B are finite sets having m and n elements respectively, then

Description of mappings	
(i) Total number of mappings from A to B	n^m
(ii) Total number of one-one mappings from A to B	${}^nC_m \times m!$. $n \geq m$
(iii) Total number of many-one mappings from A to B	
(iv) Total number of onto (surjective) mappings from A to B	$\sum_{r=1}^{n} (-1)^{n-r}$ $({}^nC_r\, r^m)$ $m \geq n$
(v) Total number of one-one onto (bijective) mappings from A to B	$n!$ $m = n$
(vi) Total number of into mappings from A to B	

Example 21. Let N be the set of all natural numbers. Consider $f : N \to N : f(x) = 2x, \forall\, x \in N$. Show that f is one-one into.

Sol. Let $x_1, x_2 \in N$, then

$$f(x_1) = f(x_2)$$

$\Rightarrow$ $2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Let $y = 2x$, then $x = \frac{y}{2}$

Now, if we put $y = 5$, then $x = \frac{5}{2} \notin N$.

This show that $5 \in N$ has no pre-image in N. So, f is into.

Hence, f is one-one and into.

Example 22. Show that the mapping $f : R \to R : f(x) = \cos x, \forall\, x \in R$ is neither one-one nor onto.

Sol. Let $x_1, x_2 \in R$.

Then, $f(x_1) = f(x_2) \Rightarrow \cos x_1 = \cos x_2$

$\Rightarrow$ $x_1 = 2n\pi \pm x_2 \Rightarrow x_1 \neq x_2$

$\therefore$ f is not one-one.

Let $y = \cos x$, but $-1 \leq \cos x \leq 1$

$\therefore$ $y \in [-1, 1]$

$[-1, 1] \subset R$

So, f is into (not onto).

Hence, f is neither one-one nor onto.

Constant Mapping

The mapping $f : A \to B$ is known as a constant mapping, if the range of B has only one element.

For all $x \in A$, $f(x) = a$, where as $a \in B$.

Identity Mapping

The mapping $f : A \to B$ is known as an identity mapping, if $f(a) = a, \forall\, a \in A$ and it is denoted by I_A.

Remark

I_A is bijective or bijection.

Equal Mapping

Let A and B be two mappings are $f : A \to B$ and $g : A \to B$ such that

$$f(x) = g(x), \forall\, x \in A$$

Then, the mappings f and g are equal and written as $f = g$.

Inclusion Mapping

The mapping $f : A \to B$ is known as inclusion mapping.

If $A \subseteq B$, then $f(a) = a, \forall\, a \in A$.

Equivalent or Equipotent or Equinumerous Set

The mapping $f : A \to B$ is known as equivalent sets, if A and B are both one-one and onto and written as $A \sim B$ which is read as 'A wiggle B'.

Inverse Mapping

If $f : A \to B$ be one-one and onto mapping, let $b \in B$, then there exist exactly one element $a \in A$ such that $f(a) = b$, so we may define

$$f^{-1} : B \to A : f^{-1}(b) = a$$

$$\Leftrightarrow \qquad f(a) = b$$

The function f^{-1} is called the inverse of f. A functions is invertible iff f is one-one onto.

Remark

1. $f^{-1}(b) \subseteq A$
2. If $f : A \to B$ and $g : B \to A$ then f and g are said to be invertible.

| Example 23. Let $f : R \to R$ be defined by $f(x) = \cos(5x + 2)$. Is f invertible? Justify your answer.

Sol. For invertible of f, f must be bijective (i.e., one-one onto).

If $\quad x_1, x_2 \in R$,

then $\quad f(x_1) = f(x_2)$

$\Rightarrow \quad \cos(5x_1 + 2) = \cos(5x_2 + 2)$

$\Rightarrow \quad 5x_1 + 2 = 2n\pi \pm (5x_2 + 2)$

$\Rightarrow \quad x_1 \neq x_2$

$\therefore$ f is not one-one.

But $\quad -1 \leq \cos(5x + 2) \leq 1$

$\therefore \quad -1 \leq f(x) \leq 1$

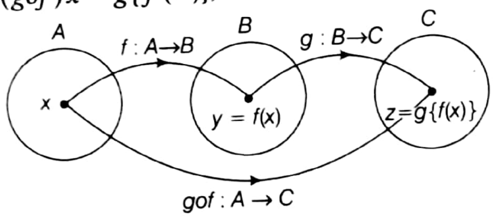
Range $= [-1, 1] \subset R$

$\therefore$ f is into mapping.

Hence, the function $f(x)$ is no bijective and so it is not invertible.

Composition of Mapping

Let A, B and C be three non-empty sets. Let $f : A \to B$ and $g : B \to C$ be two mappings, then $gof : A \to C$. This function is called the product or composite of f and g, given by $(gof)x = g\{f(x)\}$, $\forall\, x \in A$.



Important Remarks

1. (i) $(fog)x = f\{g(x)\}$ (ii) $(fof)x = f\{f(x)\}$
(iii) $(gog)x = g\{g(x)\}$ (iv) $(fg)x = f(x) \cdot g(x)$
(v) $(f \pm g)x = f(x) \pm g(x)$ (vi) $\left(\dfrac{f}{g}\right)x = \dfrac{f(x)}{g(x)}$; $g(x) \neq 0$
2. Let $h : A \to B$, $g : B \to C$ and $f : C \to D$ be any three functions. Then, $(fog)oh = fo(goh)$.
3. Let $f : A \to B$, $g : B \to C$ be two functions, then
(i) f and g are injective $\Rightarrow gof$ is injective.
(ii) f and g are surjective $\Rightarrow gof$ is surjective.
(iii) f and g are bijective $\Rightarrow gof$ is bijective.
4. An injective mapping from a finite set to itself in bijective.

| Example 24. If $f : R \to R$ and $g : R \to R$ be two mapping such that $f(x) = \sin x$ and $g(x) = x^2$, then

(i) prove that $fog \neq gof$.

(ii) find the values of $(fog)\dfrac{\sqrt{\pi}}{2}$ and $(gof)\left(\dfrac{\pi}{3}\right)$.

Sol. (i) Let $x \in R$

$\therefore \quad (fog)x = f\{g(x)\}$ $\qquad [\because g(x) = x^2]$

$= f\{x^2\} = \sin x^2$ $\qquad$ ...(i)

$[\because f(x) = \sin x]$

and $(gof)x = g\{f(x)\}$

$= g(\sin x)$ $\qquad [\because f(x) = \sin x]$

$= \sin^2 x$ $\qquad$ ...(ii)

$[\because g(x) = x^2]$

From Eqs. (i) and (ii), we get $(fog)x \neq (gof)x$, $\forall\, x \in R$

Hence, $fog \neq gof$

(ii) From Eq. (i), $(fog)x = \sin x^2$

$$\therefore \quad (fog)\frac{\sqrt{\pi}}{2} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and from Eq. (ii), $(gof)x = \sin^2 x$

$$\therefore \quad (gof)\frac{\pi}{3} = \sin^2\frac{\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$$

| Example 25. If the mapping f and g are given by

$f = \{(1, 2), (3, 5), (4, 1)\}$

and $\quad g = \{(2, 3), (5, 1), (1, 3)\}$,

write down pairs in the mapping fog and gof.

Sol. Domain $f = \{1, 3, 4\}$, Range $f = \{2, 5, 1\}$

Domain $g = \{2, 5, 1\}$, Range $g = \{1, 3\}$

$\because \quad$ Range f = Dom $g = \{(2, 5, 1)\}$

$\therefore$ gof mapping is defined.

Then, gof mapping defined following way

$$\begin{tikzcd} \{1, 3, 4\} \arrow[r, "f"] \arrow[rr, bend right, "gof"'] & \{2, 5, 1\} \arrow[r, "g"] & \{1, 3\} \end{tikzcd}$$

We see that, $\quad f(1) = 2, f(3) = 5, f(4) = 1$

and $\quad g(2) = 3, g(5) = 1, g(1) = 3$

$\therefore \quad (gof)(1) = g\{f(1)\} = g(2) = 3$

$(gof)(3) = g\{f(3)\} = g(5) = 1$

$$(gof)(4) = g\{f(4)\} = g(1) = 3$$

Hence, $gof = \{(1, 3), (3, 1), (4, 3)\}$

Now, since Range of $f \subset$ Dom f

$\therefore$ fog is defined.

Then, fog mapping defined following way

$$\{2, 5, 1\} \xrightarrow{g} \{1, 3, 4\} \xrightarrow{f} \{2, 5, 1\}$$

fog

We see that, $g(2) = 3, g(5) = 1, g(1) = 3$

$f(1) = 2, f(3) = 5, f(4) = 1$

$\therefore$ $(fog)(2) = f\{g(2)\} = f(3) = 5$

$(fog)(5) = f\{g(5)\} = f(1) = 2$

$(fog)(1) = f\{g(1)\} = f(3) = 5$

Hence, $fog = \{(2,5),(5,2),(1,5)\}$

Equivalence Classes

If R be an equivalence relation on a set A, then $[a]$ is equivalence class of a with respect to R.

Symbolically, X_a or $[a] = \{x : x \in X, x\, R\, a\}$.

Remark

1. Square brackets[ ] are used to denote the equivalence classes.
2. $a \in [a]$ and $a \in [b] \Rightarrow [a] = [b]$
3. Either $[a] = [b]$ or $[a] \cap [b] = \phi$
4. Equivalence class of a also denoted by $E(a)$ or $\bar{a}$.
5. If $a \sim b$, $\dfrac{(a-b)}{m} = k$, the total number of equivalence class is m.

I Example 26. Let $I = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$ and $R = \{(a,b) : (a-b)/4 = k, k \in I\}$ is an equivalence relation, find equivalence class.

Sol. Given, $\dfrac{a-b}{4} = k$

$\Rightarrow$ $a = 4k + b$, where $0 \le b < 4$

It is clear b has only value in 0, 1, 2, 3.

(i) Equivalence class of $[0] = \{x : x \in I$ and $x \sim 0\}$
$= \{x : x - 0 = 4k\} = \{0, \pm 4, \pm 8, \pm 12, \ldots\}$
where, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$

(ii) Equivalence class of $[1] = \{x : x \in I$ and $x \sim 1\}$
$= \{x : x - 1 = 4k\} = \{x : x = 4k + 1\}$
$= \{\ldots, -11, -7, -3, 1, 5, 9, \ldots\}$

(iii) Equivalence class of $[2] = \{x : x \in I$ and $x \sim 2\}$
$= \{x : x - 2 = 4k\} = \{x : x = 4k + 2\}$
$= \{\ldots, -10, -6, -2, 2, 6, 10, \ldots\}$

(iv) Equivalence class of $[3] = \{x : x \in I$ and $x \sim 3\}$
$= \{x : x - 3 = 4k\} = \{x : x = 4k + 3\}$
$= \{\ldots, -9, -5, -1, 5, 9, 13, \ldots\}$

Continue this process, we see that the equivalence class

$$[4] = [0], [5] = [1], [6] = [2], [7] = [3], [8] = [0]$$

Hence, total equivalence relations are [0], [1], [2], [3] and also clear

(i) $I = [0] \cup [1] \cup [2] \cup [3]$

(ii) every equivalence is a non-empty.

(iii) for any two equivalence classes $[a] \cap [b] = \phi$.

Partition of a Set

If A be a non-empty set, then a partition of A, if

(i) A is a collection of non-empty disjoint subsets of A.

(ii) union of collection of non-empty sets is A.

i.e., If A be a non-empty set and A_1, A_2, A_3, A_4 are subsets of A, then the set $\{A_1, A_2, A_3, A_4\}$ is called partition, if

(i) $A_1 \cup A_2 \cup A_3 \cup A_4 = A$

(ii) $A_1 \cap A_2 \cap A_3 \cap A_4 = \phi$

For example,

If $A = \{0, 1, 2, 3, 4\}$ and $A_1 = \{0\}, A_2 = \{1\}, A_3 = \{4\}$ and $A_4 = \{2, 3\}$, then we see that for $P = \{A_1, A_2, A_3, A_4\}$

(i) all A_1, A_2, A_3, A_4 are non-empty subset of A

(ii) $A_1 \cup A_2 \cup A_3 \cup A_4 = \{0, 1, 2, 3, 4\} = A$ and

(iii) $A_i \cap A_j \neq \phi, \forall\, i \neq j\, (i, j = 1, 2, 3, 4)$

Hence, from definition $P = \{A_1, A_2, A_3, A_4\}$ is partition of A.

Congruences

Let m be a positive integer, then two integers a and b are said to be congruent modulo m, if $a - b$ is divisible by m.

i.e., $m \overline{)a-b}\ (\lambda$

$a - b$

$\underline{-\ \ +}$

0

$\therefore$ $a - b = m\lambda$, where λ is a positive integer.

The congruent modulo 'm' is defined on all $a\, b \in I$ by $a \equiv b \pmod m$, if $a - b = m\lambda$, $\lambda \in I_+$.

I Example 27. Find congruent solutions of $155 \equiv 7 \pmod 4$.

Sol. Since, $\left(\dfrac{155-7}{4} = \dfrac{148}{4} = 37\right)$

and $a = 155, b = 7, m = 4$

$\therefore$ $\lambda = \dfrac{a-b}{4} = \dfrac{155-7}{4} = \dfrac{148}{4}$ [here, $a = 155, b = 7$]

$= 37$ (integer)

| Example 28. Find all congruent solutions of $8x \equiv 6 \pmod{14}$.

Sol. Given, $8x \equiv 6 \pmod{14}$

$\therefore \quad \lambda = \dfrac{8x - 6}{14}$, where $\lambda \in I_+$

$\therefore \quad 8x = 14\lambda + 6$

$\Rightarrow \quad x = \dfrac{14\lambda + 6}{8}$

$\Rightarrow \quad x = \dfrac{7\lambda + 3}{4} = \dfrac{4\lambda + 3(\lambda + 1)}{4}$

$x = \lambda + \dfrac{3}{4}(\lambda + 1)$, where $\lambda \in I_+$

and here greatest common divisor of 8 and 14 is 2, so there are two required solutions.

For $\lambda = 3$ and 7, $x = 6$ and 13.

Exercise for Session 3

1. The values of b and c for which the identity $f(x + 1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$, are

(a) $b = 2, c = 1$ (b) $b = 4, c = -1$ (c) $b = -1, c = 4$ (d) $b = -1, c = 1$

2. If $f(x) = \dfrac{x - 1}{x + 1}$, then $f(ax)$ in terms of $f(x)$ is equal to

(a) $\dfrac{f(x) + a}{1 + af(x)}$ (b) $\dfrac{(a - 1)f(x) + a + 1}{(a + 1)f(x) + a - 1}$ (c) $\dfrac{(a + 1)f(x) + a - 1}{(a - 1)f(x) + a + 1}$ (d) None of these

3. If f be a function satisfying $f(x + y) = f(x) + f(y)$, $\forall\, x, y \in R$. If $f(1) = k$, then $f(n)$, $n \in N$ is equal to

(a) k^n (b) nk (c) k (d) None of these

4. If $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ is a function described by the formula $g(x) = \alpha x + \beta$, what values should be assigned to α and β?

(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 2, \beta = -1$ (c) $\alpha = 1, \beta = -2$ (d) $\alpha = -2, \beta = -1$

5. The values of the parameter α for which the function $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itself, is

(a) -2 (b) -1 (c) 1 (d) 2

6. If $f(x) = (a - x^n)^{1/n}$, where $a > 0$ and $n \in N$, then $fof(x)$ is equal to

(a) a (b) x (c) x^n (d) a^n

by prop-29

7. If $f(x) = (ax^2 + b)^3$, the function g such that $f(g(x)) = g(f(x))$, is given by

(a) $g(x) = \left(\dfrac{b - x^{1/3}}{a}\right)^{1/2}$ (b) $g(x) = \dfrac{1}{(ax^2 + b)^3}$ (c) $g(x) = (ax^2 + b)^{1/3}$ (d) $g(x) = \left(\dfrac{x^{1/3} - b}{a}\right)^{1/2}$

8. Which of the following functions from I to itself are bijections?

(a) $f(x) = x^3$ (b) $f(x) = x + 2$ (c) $f(x) = 2x + 1$ (d) $f(x) = x^2 + x$

9. Let $f : R - \{n\} \to R$ be a function defined by $f(x) = \dfrac{x - m}{x - n}$, where $m \neq n$. Then,

(a) f is one-one onto (b) f is one-one into (c) f is many-one onto (d) f is many-one into

10. If $f(x + 2y, x - 2y) = xy$, then $f(x, y)$ equals

(a) $\dfrac{x^2 - y^2}{8}$ (b) $\dfrac{x^2 - y^2}{4}$ (c) $\dfrac{x^2 + y^2}{4}$ (d) $\dfrac{x^2 - y^2}{2}$

Shortcuts and Important Results to Remember

1 Every set is a subset of itself.

2 Null set is a subset of every set.

3 The set $\{0\}$ is not an empty set as it contains one element 0. The set $\{\phi\}$ is not an empty set as it contains one element ϕ.

4 The order of finite set A of n elements is denoted by $O(A)$ or $n(A)$.

5 Number of subsets of a set containing n elements is 2^n.

6 Number of proper subsets of a set containing n elements is $2^n - 1$.

7 If $A = \phi$, then $P(A) = \phi$; $\therefore n(P(A)) = 1$.

8 The order of an infinite set is undefined.

9 A natural number p is a prime number, if p is greater than one and its factors are 1 and p only.

10 Finite sets are equivalent sets only, when they have equal number of elements.

11 Equal sets are equivalent sets but equivalent sets may not be equal sets.

12 If A is any set, then $A \subseteq A$ is true but $A \subset A$ is false.

13 If $A \subseteq B$, then $A \cup B = B$

14 $A \subset B \Leftrightarrow A \subseteq B$ and $A \neq B$

15 $x \notin A \cup B \Leftrightarrow x \notin A$ ~~and~~ or $x \notin B$

16 $x \notin A \cap B \Leftrightarrow x \notin A$ ~~or~~ and $x \notin B$

17 If $A_1, A_2, \ldots, A_n$ is a finite family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ or $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$.

18 If $A_1, A_2, A_3, \ldots, A_n$ is a finite family of sets, then their intersection is denoted by $\bigcap_{i=1}^{n} A_i$ or $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$.

19 $R - Q$ is the set of all irrational numbers.

20 Total number of relations from set A to set B is equal to $2^{n(A)n(B)}$.

21 The universal relation on a non-empty set is always reflexive, symmetric and transitive.

22 The identity relation on a non-empty set is always anti-symmetric.

23 The identity relation on a set is also called the diagonal relation on A.

24 For two relations R and S, the composite relations RoS, SoR may be void relations.

25 Every polynomial function $f : R \to R$ of degree odd is ONTO.

26 Every polynomial function $f : R \to R$ of degree even is INTO.

27 (i) The number of onto functions that can be defined from a finite set A containing n elements onto a finite set B containing 2 elements $= 2^n - 2$

(ii) The number of onto functions that can be defined from a finite set A containing n elements onto a finite set B containing 3 elements $= 3^n - 3 \cdot 2^n + 3$

28 If a set A has n elements, then the number of binary relations on $A = n^{n^2}$.

29 If $fog = gof$, then either $f^{-1} = g$ or $g^{-1} = f$.

30 If f and g are bijective functions such that $f : A \to B$ and $g : B \to C$, then $gof : A \to C$ is bijective. Also, $(gof)^{-1} = f^{-1}og^{-1}$.

31 Let $f : A \to B$, $g : B \to C$ be two functions, then

(i) f and g are injective $\Rightarrow gof$ is injective

(ii) f and g are surjective $\Rightarrow gof$ is surjective

(iii) f and g are bijective $\Rightarrow gof$ is bijective

32 Let $f : A \to B$, $g : B \to C$ be two functions, then

(i) $gof : A \to C$ is injective $\Rightarrow f : A \to B$ is injective

(ii) $gof : A \to C$ is surjective $\Rightarrow g : B \to C$ is surjective

(iii) $gof : A \to C$ is injective and $g : B \to C$ is surjective $\Rightarrow$ $f : A \to B$ is injective

(iv) $gof : A \to C$ is surjective and $g : B \to C$ is injective $\Rightarrow$ $f : A \to B$ is surjective

33 An injective mapping from a finite set to itself is bijective.

JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **6 multiple choice examples**. Each example has four choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct.

• Ex. 1 *Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. The values of m and n are*

(a) 7, 6 (b) 6, 3 (c) 5, 1 (d) 8, 7

Sol. (*b*) Since, $2^m - 2^n = 56 = 8 \times 7 = 2^3 \times 7$

$$\Rightarrow \quad 2^n(2^{m-n} - 1) = 2^3 \times 7$$

$$\Rightarrow \quad n = 3 \text{ and } 2^{m-n} = 8 = 2^3 \Rightarrow n = 3 \text{ and } m - n = 3$$

$$\Rightarrow \quad n = 3 \text{ and } m - 3 = 3 \quad \Rightarrow n = 3 \text{ and } m = 6$$

• Ex. 2 *If* $aN = \{ax : x \in N\}$ *and* $bN \cap cN = dN$, *where* b, $c \in N$ *are relatively prime, then*

by taking examples $b=2$, $c=3$

(a) $d = bc$ (b) $c = bd$
(c) $b = cd$ (d) None of these

Sol. (a) bN = The set of positive integral multiples of b
cN = The set of positive integral multiples of c
$\therefore$ $bN \cap cN$ = The set of positive integral multiples of bc
$= bc\,N$ [$\because$ b and c are prime]
$\therefore \quad d = bc$

• Ex. 3 *In a town of 10000 families, it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C, 5% families buy newspapers A and B, 3% buy newspapers B and C and 4% buy newspapers A and C. If 2% families buy all the three newspapers, then number of families which buy A only is*

(a) 3100 (b) 3300 (c) 2900 (d) 1400

Sol. (*b*) $n(A) = 40\%$ of $10000 = 4000$
$n(B) = 20\%$ of $10000 = 2000$
$n(C) = 10\%$ of $10000 = 1000$
$n(A \cap B) = 5\%$ of $10000 = 500$
$n(B \cap C) = 3\%$ of $10000 = 300$
$n(C \cap A) = 4\%$ of $10000 = 400$
$n(A \cap B \cap C) = 2\%$ of $10000 = 200$

We want to find $n(A \cap B^c \cap C^c) = n[A \cap (B \cup C)^c]$
$= n(A) - n[A \cap (B \cup C)] = n(A) - n[(A \cap B) \cup (A \cap C)]$
$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$
$= 4000 - [500 + 400 - 200] = 4000 - 700 = 3300$

• Ex. 4 *Let R be the relation on the set R of all real numbers defined by aRb iff* $|a - b| \le 1$. *Then, R is*

(a) reflexive and symmetric (b) symmetric only
(c) transitive only (d) anti-symmetric only

Sol. (a) $\because$ $|a - a| = 0 < 1 \Rightarrow a\,R\,a, \forall\, a \in R$
$\therefore$ R is reflexive.
Again, $aRb \Rightarrow |a - b| \le 1$
$\Rightarrow \quad |b - a| \le 1 \Rightarrow bRa$
$\therefore$ R is symmetric.
Again, $1\,R\,2$ and $2R1$ but $2 \ne 1$
$\therefore$ R is not anti-symmetric.
Further, $1R2$ and $2R\,3$ but $1 \not R 3$ [$\because$ $|1 - 3| = 2 > 1$]
$\therefore$ R is not transitive.

• Ex. 5 *The relation R defined on* $A = \{1, 2, 3\}$ *by aRb, if* $|a^2 - b^2| \le 5$. *Which of the following is false?*

(a) $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)\}$
(b) $R^{-1} = R$
(c) Domain of $R = \{1, 2, 3\}$
(d) Range of $R = \{5\}$

Sol. (*d*) Let $a = 1$
Then, $|a^2 - b^2| \le 5 \Rightarrow |1 - b^2| \le 5$
$\Rightarrow \quad |b^2 - 1| \le 5 \Rightarrow b = 1, 2$
Let $a = 2$
Then, $|a^2 - b^2| \le 5$
$\Rightarrow \quad |4 - b^2| \le 5 \Rightarrow |b^2 - 4| \le 5$
$\therefore \quad b = 1, 2, 3$
Let $a = 3$
Then, $|a^2 - b^2| \le 5$
$\Rightarrow \quad |9 - b^2| \le 5 \Rightarrow |b^2 - 9| \le 5 \Rightarrow b = 2, 3$
$\therefore \quad R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$
$R^{-1} = \{(y, x) : (x, y) \in R\}$
$= \{(1, 1), (2, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\} = R$
Domain of $R = \{x : (x, y) \in R\} = \{1, 2, 3\}$
Range of $R = \{y : (x, y) \in R\} = \{1, 2, 3\}$

• Ex. 6 *If* $f(x) = \dfrac{1}{(1 - x)}$, $g(x) = f\{f(x)\}$ *and* $h(x) = f[f\{f(x)\}]$. *Then the value of* $f(x) \cdot g(x) \cdot h(x)$ *is*

(a) 6 (b) −1 (c) 1 (d) 2

Sol. (*b*) $\because$ $g(x) = f\{f(x)\} = f\left(\dfrac{1}{1 - x}\right) = \dfrac{1}{1 - \dfrac{1}{1 - x}} = \dfrac{x - 1}{x}$

and $h(x) = f[f\{f(x)\}] = f(g(x))$

$$= \frac{1}{1 - g(x)} = \frac{1}{1 - \dfrac{x - 1}{x}} = x$$

$$\therefore \quad f(x) \cdot g(x) \cdot h(x) = \frac{1}{(1 - x)} \cdot \frac{(x - 1)}{x} \cdot x = -1$$

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **3 multiple choice examples**. Each example has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

• Ex. 7 *If I is the set of integers and if the relation R is defined over I by aRb, iff a − b is an even integer, a, b ∈ I, the relation R is*

(a) reflexive (b) anti-symmetric
(c) symmetric (d) equivalence

Sol. (a, c, d)

$aRb \Leftrightarrow a - b$ is an even integer, $a, b \in I$

$a - a = 0$ (even integer)

$\therefore \quad (a, a) \in R, \ \forall \ a \in I$

$\therefore$ R is reflexive relation.

Let $(a, b) \in R \Rightarrow (a - b)$ is an even integer.

$\Rightarrow \quad -(b - a)$ is an even integer.

$\Rightarrow \quad (b - a)$ is an even integer.

$\Rightarrow \quad (b, a) \in R$

$\therefore$ R is symmetric relation.

Now, let $(a, b) \in R$ and $(b, c) \in R$

Then, $(a - b)$ is an even integer and $(b - c)$ is an even integer.

So, let $\quad a - b = 2x_1, x_1 \in I$

and $\quad b - c = 2x_2, x_2 \in I$

$\therefore \quad (a - b) + (b - c) = 2(x_1 + x_2)$

$\Rightarrow \quad (a - c) = 2(x_1 + x_2) \Rightarrow a - c = 2x_3$

$\therefore$ $(a - c)$ is an even integer.

$\therefore$ aRb and $bRc \Rightarrow aRc$ So, R is transitive relation.

Hence, R is an equivalence relation.

A relation R on given set A is said to be anti-symmetric iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b, \forall \ a, b \in A$.

$\therefore$ Given relation is not anti-symmetric relation.

• Ex. 8 *If* $f(x) = \dfrac{a - x}{a + x}$, *the domain of* $f^{-1}(x)$ *contains*

(a) $(-\infty, \infty)$ (b) $(-\infty, -1)$
(c) $(-1, \infty)$ (d) $(0, \infty)$

Sol. (b, c, d)

Let $\quad y = f(x) = \dfrac{a - x}{a + x} \Rightarrow ay + xy = a - x$

$\therefore \quad x = \dfrac{a(1 - y)}{(1 + y)} = f^{-1}(y) \Rightarrow f^{-1}(x) = \dfrac{a(1 - x)}{(1 + x)}$

$\therefore$ $f^{-1}(x)$ is not defined for $x = -1$.

Domain of $f^{-1}(x)$ belongs to $(-\infty, -1) \cup (-1, \infty)$.

Now, for $a = -1$, given function $f(x) = -1$, which is constant.

Then, $f^{-1}(x)$ is not defined.

$\therefore \quad a \neq -1$

• Ex. 9 *If* $f(x) = \dfrac{\sin([x]\pi)}{x^2 + x + 1}$, *where* $[\cdot]$ *denotes the greatest integer function, then*

(a) f is one-one
(b) f is not one-one and non-constant
(c) f is constant function (d) f is zero function

Sol. (c, d)

$\because \quad \sin([x]\pi) = 0$

$\therefore \quad f(x) = 0 \quad [\because [x]$ is an integer$]$

$\Rightarrow$ $f(x)$ is a constant function and also $f(x)$ is a zero function.

JEE Type Solved Examples : Passage Based Questions

- This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 10 to 12)

If $A = \{x : |x| < 2\}$, $B = \{x : |x - 5| \le 2\}$, $C = \{x : |x| > x\}$ *and* $D = \{x : |x| < x\}$

• 10. *The number of integral values in* $A \cup B$ *is*

(a) 4 (b) 6
(c) 8 (d) 10

• 11. *The number of integral values in* $A \cap C$ *is*

(a) 1 (b) 2
(c) 3 (d) 0

• 12. *The number of integral values in* $A \cap D$ *is*

(a) 2 (b) 4
(c) 6 (d) 0

Sol. (Ex. Nos. 10 to 12)

$A = \{x : |x| < 2\} = \{x : -2 < x < 2\} = (-2, 2)$

$B = \{x : |x - 5| \le 2\} = \{x : -2 \le x - 5 \le 2\}$

$= \{x : 3 \le x \le 7\} = [3, 7]$

$C = \{x : |x| > x\} = \{x : x < 0\} = (-\infty, 0)$

and $D = \{x : |x| < x\} = \phi$

10. (c) $A \cup B = (-2, 2) \cup [3, 7]$

Integral values in $A \cup B$ are $-1, 0, 1, 3, 4, 5, 6, 7$.

$\therefore$ Number of integral values in $A \cup B$ is 8.

11. (a) $A \cap C = (-2, 2) \cap (-\infty, 0) = (-2, 0)$

Integral value in $A \cap C$ is -1.

$\therefore$ Number of integral values in $A \cap C$ is 1.

12. (d) $A \cap D = (-2, 2) \cap \phi = \phi$

$\therefore$ Number of integral values in $A \cap D$ is 0.

Passage II
(Ex. Nos. 13 to 15)

If $A = \{x : x^2 - 2x + 2 > 0\}$ *and* $B = \{x : x^2 - 4x + 3 \le 0\}$

13. *$A \cap B$ equals*

(a) $[1, \infty)$ (b) $[1, 3]$

(c) $(-\infty, 3]$ (d) $(-\infty, 1) \cup (3, \infty)$

14. *$A - B$ equals*

(a) $(-\infty, \infty)$ (b) $(1, 3)$

(c) $(3, \infty)$ (d) $(-\infty, 1) \cup (3, \infty)$

15. *$A \cup B$ equals*

(a) $(-\infty, 1)$ (b) $(3, \infty)$

(c) $(-\infty, \infty)$ (d) $(1, 3)$

Sol. (Ex. Nos. 13 to 15)

$A = \{x : x^2 - 2x + 2 > 0\} = \{x : (x - 1)^2 + 1 > 0\} = (-\infty, \infty)$

$B = \{x : x^2 - 4x + 3 \le 0\} = \{x : (x - 1)(x - 3) \le 0\}$

$= \{x : 1 \le x \le 3\} = [1, 3]$

13. (b) $A \cap B = (-\infty, \infty) \cap [1, 3] = [1, 3]$

14. (d) $A - B = (-\infty, \infty) - [1, 3] = (-\infty, 1) \cup (3, \infty)$

15. (c) $A \cup B = (-\infty, \infty) \cup [1, 3] = (-\infty, \infty)$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

Ex. 16 *If $f : R^+ \to A$, where $A = \{x : -5 < x < \infty\}$ is defined by $f(x) = x^2 - 5$ and if $f^{-1}(13) = \{-\lambda\sqrt{(\lambda - 1)}, \lambda\sqrt{(\lambda - 1)}\}$, the value of λ is*

Sol. (3) $f^{-1}(13) = \{x : f(x) = 13\} = \{x : x^2 - 5 = 13\}$

$= \{x : x^2 = 18\} = \{x : x = \pm 3\sqrt{2}\}$

$= \{-3\sqrt{2}, 3\sqrt{2}\}$

$= \{-\lambda\sqrt{(\lambda - 1)}, \lambda\sqrt{(\lambda - 1)}\}$ [given]

$\therefore \quad \lambda = 3$

Ex. 17 *If $A = \{2, 3\}$, $B = \{4, 5\}$ and $C = \{5, 6\}$, then $n\{(A \times B) \cup (B \times C)\}$ is*

Sol. (8) $\because A \times B = \{2, 3\} \times \{4, 5\}$

$= \{(2, 4), (2, 5), (3, 4), (3, 5)\}$

and $B \times C = \{4, 5\} \times \{5, 6\}$

$= \{(4, 5), (4, 6), (5, 5), (5, 6)\}$

$\therefore \quad (A \times B) \cup (B \times C) = \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (4, 6), (5, 5), (5\ 6)\}$

Now, $n\{(A \times B) \cup (B \times C)\} = 8$

JEE Type Solved Examples : Matching Type Questions

- This section contains **1 examples.** Example 18 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

• *Ex. 18*

	Column I		Column II
(A)	$R = \{(x, y) : x < y ; x, y \in N\}$	(p)	Reflexive
(B)	$S = \{(x, y) : x + y = 10 ; x, y \in N\}$	(q)	Symmetric
(C)	$T = \{(x, y) : x = y$ or $x - y = 1 ; x, y \in N\}$	(r)	Transitive
(D)	$U = \{(x, y) : x^y = y^x ; x, y \in N\}$	(s)	Equivalence

Sol. (A) → (r); (B) → (q); (C) → (p); (D) → (p, q, r, s)

(A) $\because \quad R = \{(x, y) : x < y; x, y \in N\}$

$x \nless x \therefore (x, x) \notin R$

So, R is not reflexive.

Now, $(x, y) \in R \Rightarrow x < y \nRightarrow y < x \Rightarrow (y, x) \notin R$

$\therefore$ R is not symmetric.

Let $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad x < y$ and $y < z \Rightarrow x < z \Rightarrow (x, z) \in R$

$\therefore R$ is transitive.

(B) $\because S = \{(x, y) : x + y = 10 ; x, y \in N\}$

$\therefore \quad x + x = 10 \Rightarrow 2x = 10 \Rightarrow x = 5$

So, each element of N is not related to itself by the relation $x + y = 10$.

$\therefore$ S is not reflexive.

Now, $(x, y) \in S \Rightarrow x + y = 10 \Rightarrow y + x = 10$

$\Rightarrow \quad (y, x) \in S$

$\therefore$ S is symmetric relation.

Now, let $(3, 7) \in S$ and $(7, 3) \in S \Rightarrow (3, 3) \notin S$

$\therefore$ S is not transitive.

(C) $\because \quad T = \{(x, y) : x = y$ or $x - y = 1; x, y \in N\}$

$\therefore \quad x = x$

So, $\quad (x, x) \in T, \forall x \in N$

$\therefore \quad T$ is reflexive.

Let $\quad (3, 2) \in T$ and $3 - 2 = 1$

$\nRightarrow \quad 2 - 3 = -1 \Rightarrow (2, 3) \notin T$

$\therefore \quad T$ is not symmetric.

Now, let $\quad (3, 2) \in T$ and $(2, 1) \in T$

$\therefore \quad 3 - 2 = 1$ and $2 - 1 = 1$

Then, $\quad (3, 1) \notin T \qquad [\because 3 - 1 = 2 \neq 1]$

$\therefore \quad T$ is not transitive.

(D) $U = \{(x, y) : x^y = y^x; x, y \in N\}$

$\because \quad x^x = x^x$

$\therefore \quad (x, x) \in U$

$\therefore$ U is reflexive.

Now, $(x, y) \in U \Rightarrow x^y = y^x$

$\Rightarrow \quad y^x = x^y \Rightarrow (y, x) \in U$

$\therefore$ U is symmetric.

Now, let $(x, y) \in U$ and $(y, z) \in U$

$\Rightarrow \quad x^y = y^x$ and $y^z = z^y$

Now, $\quad (x^y)^z = (y^x)^z$

$\Rightarrow \quad (x^z)^y = (y^z)^x \Rightarrow (x^z)^y = (z^y)^x$

$\Rightarrow \quad (x^z)^y = (z^x)^y \Rightarrow x^z = z^x \Rightarrow (x, z) \in U$

$\therefore \quad U$ is transitive.

Hence, U is an equivalence relation.

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** Example numbers 19 and 20 are Assertion-Reason type examples. Each of these examples contains two statements:

Statement-1 (Assertion) and

Statement-2 (Reason)

Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below:

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true; Statement-2 is false

(d) Statement-1 is false; Statement-2 is true

• *Ex. 19* Statement-1 *If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.*

Statement-2 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Sol. (a) We have, $B = B \cup (A \cap B)$

$= B \cup (A \cap C) \qquad [\because A \cap B = A \cap C]$

$= (A \cup C) \cap (B \cup C)$ $\quad [\because A \cup B = A \cup C]$
$= (A \cap B) \cup C$
$= (A \cap C) \cup C$ $\quad [\because A \cap B = A \cap C]$
$= C$

Hence, Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation of Statement-1.

• **Ex. 20 Statement-1** *If U is universal set and $B = U - A$, then $n(B) = n(U) - n(A)$.*
Statement-2 *For any three arbitrary sets A, B and C, if $C = A - B$, then $n(C) = n(A) - n(B)$.*

Sol. (c) $\because$ $\quad B = U - A = A'$

$\therefore$ $\quad n(B) = n(A') = n(U) - n(A)$

So, Statement-1 is true.

But for any three arbitrary sets A, B and C, we cannot always have

$$n(C) = n(A) - n(B)$$

if $\quad C = A - B$

As it is not specified A is universal set or not. In case not conclude

$$n(C) = n(A) - n(B)$$

Hence, Statement-2 is false.

Sets, Relations and Functions Exercise 8 :
Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

75. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9)$ be a relation on the set $A = \{3, 6, 9, 12\}$.
The relation is **[AIEEE 2005, 3M]**
(a) an equivalence relation
(b) reflexive and symmetric only
(c) reflexive and transitive only
(d) reflexive only

76. Let W denotes the words in the English dictionary. Define the relation R by $R = \{(x, y) \in W \times W\}$ the words x and y have atleast one letter in common, then R is **[AIEEE 2006, 3M]**
(a) not reflexive, symmetric and transitive
(b) reflexive, symmetric and not transitive
(c) reflexive, symmetric and transitive
(d) reflexive, not symmetric and transitive

77. Let R be the real line, consider the following subsets of the plane $R \times R$ such that **[AIEEE 2008, 3M]**
$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$
$T = \{(x, y) : x - y \text{ is an integer}\}$.
Which one of the following is true?
(a) Both S and T are equivalence relations on R
(b) S is an equivalence relation on R but T is not
(c) T is an equivalence relation on R but S is not
(d) Neither S nor T is an equivalence relations on R

78. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then **[AIEEE 2009, 4M]**
(a) $A \cap B = \phi$ (b) $A = B$
(c) $A = C$ (d) $B = C$

79. Let $S = \{1, 2, 3, 4\}$. The total number of unordered pair of disjoint subsets of S is equal to **[IIT-JEE 2010, 5M]**
(a) 25 (b) 34 (c) 42 (d) 41

80. Consider the following relations.
$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$
$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right) \middle| m, n, p \text{ and } q \text{ are integers such that } n, q \neq 0 \text{ and } qm = pn\right\}$, then **[AIEEE 2010, 4M]**
(a) neither R nor S is an equivalence relation
(b) S is an equivalence relation but R is not an equivalence relation
(c) R and S both are equivalence relations
(d) R is an equivalence relation but S is not an equivalence relation

81. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then, **[IIT-JEE 2011, 3M]**
(a) $P \subset Q$ and $A - P \neq \phi$ (b) $Q \not\subset P$
(c) $P \not\subset Q$ (d) $P = Q$

82. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then, the set of all x satisfying $(fogogof)(x) = (gogof)(x)$, where $(fog)(x) = f(g(x))$ is **[IIT-JEE 2011, 3M]**
(a) $\pm\sqrt{n\pi}, n \in \{0, 1, 2, \ldots\}$
(b) $\pm\sqrt{n\pi}, n \in \{1, 2, 3, \ldots\}$
(c) $\frac{\pi}{2} + 2n\pi, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
(d) $2n\pi, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

83. Let R be the set of real numbers.
Statement-1 $A = \{(x, y) \in R \times R : y - x \text{ is an integer}\}$ is an equivalence relation on R.
Statement-2 $B = \{(x, y) \in R \times R : x = \alpha y \text{ for some rational number } \alpha\}$ is an equivalence relation on R. **[AIEEE 2011, 4M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is false
(c) Statement-1 is false, Statement-2 is true
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

84. Let A and B be two sets containing 2 elements and 4 elements, respectively. The number of subsets of $A \times B$ having 3 or more elements, is **[JEE Main 2013, 4M]**
(a) 220 (b) 219 (c) 211 (d) 256

85. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n = N\}$, where N is the set of natural numbers, then $X \cup Y$ is equal to **[JEE Main 2014, 4M]**
(a) X (b) Y (c) N (d) $Y - X$

86. Let A and B be two sets containing four and two elements, respectively. Then, the number of subsets of the set $A \times B$, each having atleast three elements is **[JEE Main 2015, 4M]**
(a) 275 (b) 510 (c) 219 (d) 256

Answers

Exercise for Session 1

1. (b)	2. (d)	3. (c)	4. (b)	5. (a)	6. (c)
7. (d)	8. (c)	9. (a)	10. (c)	11. (d)	

Exercise for Session 2

1. (c)	2. (b)	3. (b)	4. (c)	5. (c)	6. (c)
7. (b)	8. (a)	9. (a)	10. (c)		

Exercise for Session 3

1. (b)	2. (c)	3. (b)	4. (b)	5. (b)	6. (b)
7. (d)	8. (b)	9. (b)	10. (a)		

Chapter Exerises

1. (a)	2. (a)	3. (a)	4. (d)	5. (a)	6. (c)
7. (b)	8. (c)	9. (b)	10. (d)	11. (c)	12. (b)
13. (c)	14. (d)	15. (b)	16. (c)	17. (b)	18. (d)
19. (c)	20. (d)	21. (c)	22. (b)	23. (d)	24. (b)
25. (d)	26. (b)	27. (b)	28. (a)	29. (c)	30. (b)
31. (b)	32. (d)	33. (c)	34. (d)	35. (a)	36. (d)
37. (c)	38. (d)	39. (d)	40. (a,b,c,d)		41. (a,b,c)
42. (c,d)	43. (b)	44. (a)	45. (d)	46. (a)	47. (b)
48. (a)	49. (5)	50. (1)	51. (3)	52. (2)	53. (9)

54. (A) $\rightarrow$ (r); (B) $\rightarrow$ (q); (C) $\rightarrow$ (p); (D) $\rightarrow$ (s)
55. (A) $\rightarrow$ (s); (B) $\rightarrow$ (r); (C) $\rightarrow$ (p); (D) $\rightarrow$ (q)
56. (b) 57. (c) 58. (c) 59. (a)
60. (i) B (ii) C (iii) $\{2\}$ (iv) $\{x : x$ is an odd prime, natural number$\}$
61. (i) $S \cap W$ (ii) $T' \cap W'$ (iii) $(M \cup T \cup S)'$
62. $X \cup Y = \{(x, y) : x^2 + y^2 \le 1$ or $0 \le x \le 1$ and $-1 \le y \le 1\}$
$X \cap Y = \{(x, y) : x^2 + y^2 \le 1$ and $x \ge 0\}$
63. 5, 2, 12, 60 64. 43 65. 20 66. 45, 110
67. 44
70. (i) Not reflexive, symmetric, not transitive
(ii) Reflexive, symmetric, transitive
(iii) Reflexive, not symmetric, transitive
(iv) Reflexive, symmetric, not transitive
(v) Not reflexive, not symmetric, transitive
71. (i) Injective (ii) Injective
(iii) Bijective (iv) Not injective
(v) Surjective
72. $(fog)x = e^{3x-2}$; $x \in R$ $(gof)x = 3e^x - 2$; $x \in R$
Domain of $(fog)^{-1}(x) = (0, \infty)$.
Domain of $(gof)^{-1}(x) = (-2, \infty)$.
73. $\frac{3}{(1-x)^2}$, $R - \{1\}$
$\frac{df^{-1}(x)}{dx} = \frac{3}{(1-x)^2}$, Domain of $\frac{df^{-1}(x)}{dx} = R - \{1\}$
74. $(hofog)\,x = \begin{cases} 0, & x^2 \le 0 \\ x^2, & x^2 \ge 0 \end{cases}$, h is not an identity function and fog is not invertible.

75. (c)	76. (b)	77. (c)	78. (d)	79. (d)	80. (b)
81. (d)	82. (a)	83. (a)	84. (b)	85. (b)	86. (c)