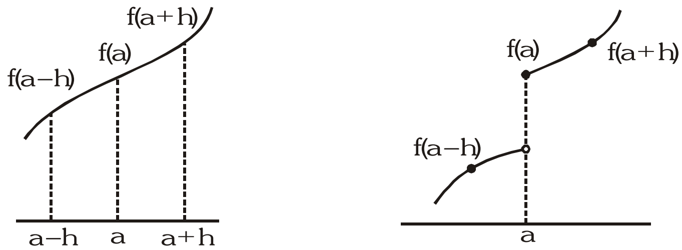
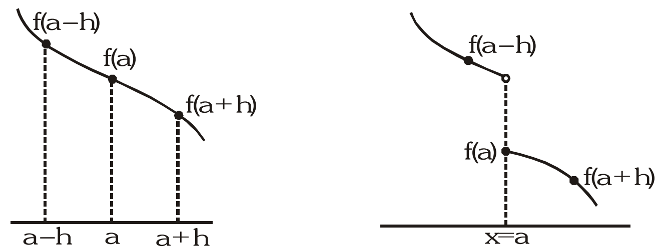
MONOTONICITY

1. MONOTONICITY AT A POINT :

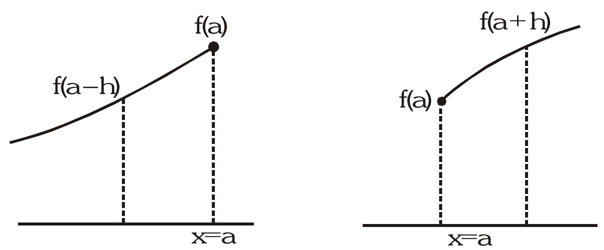
(a) A function f(x) is called an increasing function at point x = a, if in a sufficiently small neighbourhood of x = a ; $f(a - h) < f(a) < f(a + h)$



(b) A function f(x) is called a decreasing function at point x = a, if in a sufficiently small neighbourhood of x= a ; $f(a - h) > f(a) > f(a + h)$



Note : If x = a is a boundary point, then use the appropriate one sides inequality to test Monotonicity of f(x).



$f(a) > f(a-h)$
increasing at $x = a$

$f(a+h) > f(a)$
decreasing at $x = a$

(c) **Testing of monotonicity of differentiable function at a point.**

(i) If $f'(a) > 0$, then f(x) is increasing at x = a.

(ii) If $f'(a) < 0$, then f(x) is decreasing at x = a.

(iii) If $f'(a) = 0$, then examine the sign of $f'(a^+)$ and $f'(a^-)$.

(1) If $f'(a^+) > 0$ and $f'(a^-) > 0$, then increasing

(2) If $f'(a^+) < 0$ and $f'(a^-) < 0$, then decreasing

(3) otherwise neither increasing nor decreasing.

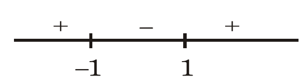
Illustration 1 : Let $f(x) = x^3 - 3x + 2$. Examine the nature of function at points x = 0, 1 & 2.

Solution :

$f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1) = 0$

$\Rightarrow\ x = \pm 1$



(i) $f'(0) = -3 \quad \Rightarrow \quad$ decreasing at x = 0

(ii) $f'(1) = 0$

also, $f'(1^+)$ = positive and $f'(1^-)$ = negative

$\Rightarrow$ neither increasing nor decreasing at x = 0.

(iii) $f'(2) = 9 \quad \Rightarrow \quad$ increasing at x = 2

Do yourself - 1 :

(i) If function $f(x) = x^3 + \lambda x^2 - \lambda x + 1$ is increasing at $x = 0$ & decreasing at $x = 1$, then find the greatest integral value of λ.

2. MONOTONICITY OVER AN INTERVAL :

(a) A function $f(x)$ is said to be monotonically increasing (MI) in (a, b) if $f'(x) \geq 0$ where equality holds only for discrete values of x i.e. $f'(x)$ does not identically become zero for $x \in (a, b)$ or any sub interval.

(b) $f(x)$ is said to be monotonically decreasing (MD) in (a, b) if $f'(x) \leq 0$ where equality holds only for discrete values of x i.e. $f'(x)$ does not identically become zero for $x \in (a, b)$ or any sub interval.

- By discrete points, we mean that points where $f'(x) = 0$ does not form an interval.

Note :

(i) A function is said to be monotonic if it's either increasing or decreasing.

(ii) If a function is invertible it has to be either increasing or decreasing.

Illustration 2 : Prove that the function $f(x) = \log\left(x^3 + \sqrt{x^6+1}\right)$ is entirely increasing.

Solution : Now, $f(x) = \log\left(x^3 + \sqrt{x^6+1}\right) \Rightarrow f'(x) = \dfrac{1}{x^3+\sqrt{x^6+1}}\left(3x^2 + \dfrac{6x^5}{2\sqrt{x^6+1}}\right) = \dfrac{3x^2}{\sqrt{x^6+1}} > 0$

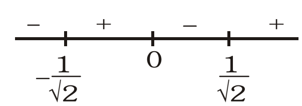
$\Rightarrow$ $f(x)$ is increasing.

Illustration 3 : Find the intervals of monotonicity of the function $y = x^2 - \log_e|x|$, $(x \neq 0)$.

Solution : Let $y = f(x) = x^2 - \log_e|x|$

$$f(x) = \begin{cases} x^2 - \log_e(-x), & x < 0 \\ x^2 - \log_e(x), & x > 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x - \dfrac{1}{(-x)}(-1), & x < 0 \\ 2x - \dfrac{1}{x}, & x > 0 \end{cases}$$

$\therefore f'(x) = 2x - \dfrac{1}{x}$; for all $x(x \neq 0)$



$$f'(x) = \frac{2x^2-1}{x} \quad \Rightarrow \quad f'(x) = \frac{\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right)}{x}$$

So $f'(x) > 0$ when $x \in \left(-\dfrac{1}{\sqrt{2}}, 0\right) \cup \left(\dfrac{1}{\sqrt{2}}, \infty\right)$ and $f'(x) < 0$ when $x \in \left(-\infty, -\dfrac{1}{\sqrt{2}}\right) \cup \left(0, \dfrac{1}{\sqrt{2}}\right)$

$\therefore$ $f(x)$ is increasing when $x \in \left(-\dfrac{1}{\sqrt{2}}, 0\right) \cup \left(\dfrac{1}{\sqrt{2}}, \infty\right)$

and decreasing when $x \in \left(-\infty, -\dfrac{1}{\sqrt{2}}\right) \cup \left(0, \dfrac{1}{\sqrt{2}}\right)$ **Ans.**

Illustration 4 : If a, b, c are real, then $f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$ is decreasing in

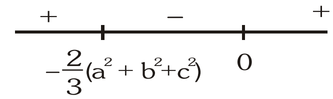
(A) $\left(-\dfrac{2}{3}(a^2+b^2+c^2), 0\right)$ (B) $\left(0, \dfrac{2}{3}(a^2+b^2+c^2)\right)$

(C) $\left(0, \dfrac{a^2+b^2+c^2}{3}\right)$ (D) no where

Solution : $f'(x)=\begin{vmatrix}1 & 0 & 0\\ ab & x+b^2 & bc\\ ac & bc & x+c^2\end{vmatrix}+\begin{vmatrix}x+a^2 & ab & ac\\ 0 & 1 & 0\\ ac & bc & x+c^2\end{vmatrix}+\begin{vmatrix}x+a^2 & ab & ac\\ ab & x+b^2 & bc\\ 0 & 0 & 1\end{vmatrix}$

$=\left(x+b^2\right)\left(x+c^2\right)-b^2c^2+\left(x+a^2\right)\left(x+c^2\right)-a^2c^2+\left(x+a^2\right)\left(x+b^2\right)-a^2b^2$

$=3x^2+2x\left(a^2+b^2+c^2\right)$



f(x) will be decreasing when $f'(x)<0$

$\Rightarrow 3x^2+2x\left(a^2+b^2+c^2\right)<0 \Rightarrow x\in\left(-\frac{2}{3}\left(a^2+b^2+c^2\right),0\right)$ **Ans. (A)**

Illustration 5 : Prove the following

(i) $y = e^x + \sin x$ is increasing in $x \in R^+$

(ii) $y = 2x - \sin x - \tan x$ is decreasing in $x \in (0, \pi/2)$

Solution : (i) $f(x) = e^x + \sin x,\ x \in R^+ \Rightarrow f'(x) = e^x + \cos x$

Clearly $f'(x) > 0\ \forall\ x \in R^+$ (as $e^x > 1$, $x\in R^+$ and $-1 \le \cos x \le 1$, $x \in R^+$)

Hence f(x) is increasing.

(ii) $f(x) = 2x - \sin x - \tan x \quad x \in (0, \pi/2)$

$\Rightarrow \quad f'(x) = 2 - \cos x - \sec^2 x$

$\Rightarrow \quad f'(x) = \cos^2 x - \cos x - (\cos^2 x + \sec^2 x - 2)$

$= \cos^2 x - \cos x - (\cos x - \sec x)^2$

$\therefore \quad f'(x) < 0,\ x \in (0, \pi/2) \qquad \because \quad \cos^2 x < \cos x,\ x \in (0, \pi/2)$

Hence f(x) is decreasing in $(0, \pi/2)$

Do yourself - 2 :

(i) If $f(x) = \sin x + \ell n\ |\sec x + \tan x| - 2x$ for $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ then check the monotonicity of $f(x)$

(ii) The function $f(x) = 2\log(x - 2) - x^2 + 4x + 1$ increases in the interval.

(A) (1,2) (B) (2,3) (C) $\left(\frac{5}{2},3\right)$ (D) (2,4)

3. GREATEST AND LEAST VALUE OF A FUNCTION :

(a) If a continuous function $y = f(x)$ is increasing in the closed interval [a, b], then $f(a)$ is the least value and $f(b)$ is the greatest value of $f(x)$ in [a, b] (figure-1)

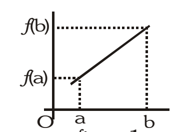


figure-1

(b) If a continuous function $y = f(x)$ is decreasing in [a, b], then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in [a, b]. (figure-2)

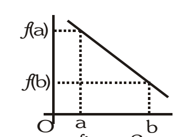
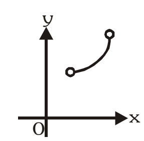


figure-2



(c) If a continuous function $y = f(x)$ is increasing/decreasing in the (a,b), then no greatest and least value exist.

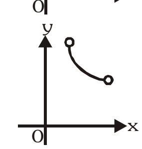


Illustration 6 : Show that $f(x) = \sin^{-1}\frac{x}{\sqrt{1+x^2}} - \ell nx$ is decreasing in $x \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$. Also find its range.

Solution : $f(x) = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} - \ell nx = \tan^{-1}x - \ell nx \Rightarrow f'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{x} = \dfrac{-(1+x^2-x)}{x(1+x^2)}$

$\therefore \quad f'(x) < 0 \quad \forall\ x \in \left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$

$\Rightarrow$ f(x) is decreasing $f(x)|_{max} = f\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6} + \dfrac{1}{2}\ell n3$ & $f(x)|_{min} = f(\sqrt{3}) = \dfrac{\pi}{3} - \dfrac{1}{2}\ell n3$

$\therefore$ Range of $f(x) = \left[\dfrac{\pi}{3} - \dfrac{1}{2}\ell n3, \dfrac{\pi}{6} + \dfrac{1}{2}\ell n3\right]$ **Ans.**

Do yourself - 3 :

(i) Let $f(x) = x - \dfrac{1}{x}$. Find the greatest and least value of f(x) for $x \in (0, 4)$.

4. SPECIAL POINTS :

(a) **Critical points :** The points of domain for which f'(x) is equal to zero or doesn't exist are called critical points.

(b) **Stationary points:** The stationary points are the points of domain where f'(x) = 0 .

Note : Every stationary point is a critical point but vice-versa is not true.

Illustration 7 : Find the critical point(s) & stationary point(s) of the function $f(x) = (x-2)^{2/3}(2x+1)$

Solution : $f(x) = (x-2)^{2/3}(2x+1)$

$$f'(x) = (x-2)^{2/3} \cdot 2 + (2x+1)\frac{2}{3}(x-2)^{-1/3} = 2(x-2)^{2/3} + \frac{2}{3}(2x+1)\frac{1}{(x-2)^{1/3}}$$

$$= \left[2(x-2) + \frac{2}{3}(2x+1)\right]\frac{1}{(x-2)^{1/3}} = \frac{2(5x-5)}{3(x-2)^{1/3}}$$

f'(x) does not exist at x = 2 and f'(x) = 0 at x = 1

$\therefore$ x = 1, 2 are critical points and x = 1 is stationary point.

Do yourself - 4 :

(i) Find the critical points and stationary point of the function $f(x) = \dfrac{e^x}{x}$

5. PROVING INEQUALITIES USING MONOTONICITY :

Comparison of two functions f(x) and g(x) can be done by analysing their monotonic behaviour.

Illustration 8 : For $x \in \left(0, \dfrac{\pi}{2}\right)$ prove that $\sin x < x < \tan x$

Solution : Let $f(x) = x - \sin x \Rightarrow f'(x) = 1 - \cos x$

$f'(x) > 0$ for $x \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow$ f(x) is M.I. $\Rightarrow$ $f(x) > f(0)$

$\Rightarrow$ $x - \sin x > 0 \Rightarrow x > \sin x$

Similarly consider another function $g(x) = x - \tan x \Rightarrow g'(x) = 1 - \sec^2 x$

$g'(x) < 0$ for $x \in \left(0, \dfrac{\pi}{2}\right) \Rightarrow$ g(x) is M.D.

Hence $g(x) < g(0)$

$x - \tan x < 0 \Rightarrow x < \tan x$

$\sin x < x < \tan x$ Hence proved

Illustration 9 : For $x \in (0, 1)$ prove that $x - \dfrac{x^3}{3} < \tan^{-1}x < x - \dfrac{x^3}{6}$ & hence or otherwise find $\lim\limits_{x\to 0}\left[\dfrac{\tan^{-1}x}{x}\right]$

Solution : Let $f(x) = x - \dfrac{x^3}{3} - \tan^{-1}x \Rightarrow f'(x) = 1 - x^2 - \dfrac{1}{1+x^2} = -\dfrac{x^4}{1+x^2}$

$f'(x) < 0$ for $x \in (0, 1) \Rightarrow f(x)$ is M.D. $\Rightarrow f(x) < f(0)$

$\Rightarrow x - \dfrac{x^3}{3} - \tan^{-1}x < 0 \Rightarrow x - \dfrac{x^3}{3} < \tan^{-1}x$ .... (i)

Similarly $g(x) = x - \dfrac{x^3}{6} - \tan^{-1}x$

$$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2} = \frac{x^2(1-x^2)}{2(1+x^2)}$$

$g'(x) > 0$ for $x \in (0, 1) \Rightarrow g(x)$ is M.I.

$x - \dfrac{x^3}{6} - \tan^{-1}x > 0 \Rightarrow x - \dfrac{x^3}{6} > \tan^{-1}x$ .... (ii)

from (i) and (ii), we get $x - \dfrac{x^3}{3} < \tan^{-1}x < x - \dfrac{x^3}{6}$ Hence Proved

Now, $1 - \dfrac{x^2}{3} < \dfrac{\tan^{-1}x}{x} < 1 - \dfrac{x^2}{6} \Rightarrow \lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x} = 1$

but $\lim\limits_{x\to 0}\left[\dfrac{\tan^{-1}x}{x}\right] = 0$

Do yourself - 5 :

(i) Find the larger of $\ell n(1 + x)$ & $\dfrac{\tan^{-1}x}{1+x}$, for $x > 0$.

(ii) Show that : $\sin x < x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$, for $x > 0$.

(iii) For $x > 1$, $y = \ell nx$ satisfies

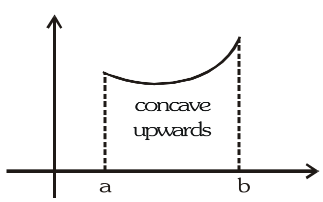
(A) $x - 1 > y$ (B) $x^2 - 1 > y$ (C) $y > x - 1$ (D) $\dfrac{x-1}{x} < y$

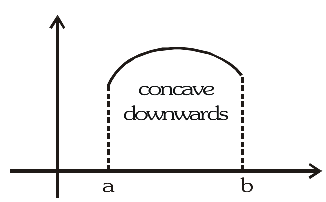
6. SIGNIFICANCE OF THE SIGN OF IInd ORDER DERIVATIVE :

The sign of the 2^{nd} order derivative determines the concavity of the curve.

If $f''(x) > 0\ \forall\ x \in (a, b)$ then graph of $f(x)$ is concave upward in (a, b).

Similarly if $f''(x) < 0\ \forall\ x \in (a, b)$ then graph of $f(x)$ is concave downward in (a, b).





7. PROVING INEQUALITIES USING GRAPHS & CONCAVITY :

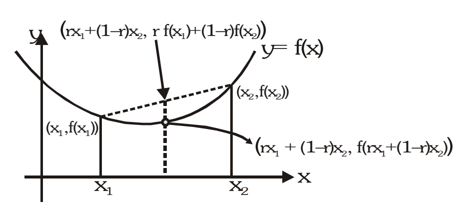
Generally these inequalities involve comparison between values of two function at some particular points.

Note :

(i) If function f(x) is concave upward, then

$f\left(rx_1 + (1-r)x_2\right) < rf(x_1) + (1-r)f(x_2)\ \forall$ distinct

x_1 & $x_2 \in$ domain of f(x) and $r > 0$.



(fig. 1)

(ii) If function f(x) is concave downward, then

$$f\left(rx_1+(1-r)x_2\right) > rf(x_1)+(1-r)f(x_2) \quad \forall \text{ distinct}$$

x_1 & $x_2 \in$ domain of f(x) and r > 0.

Note : Equality hold when x_1 and x_2 coincide.

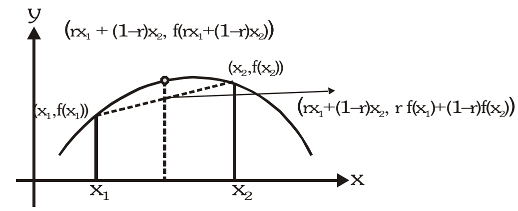


Illustration 10 : Prove that for any two numbers x_1 & x_2, $\dfrac{3e^{x_1}+e^{x_2}}{4} > e^{\frac{3x_1+x_2}{4}}$. (fig. 2)

Solution : Assume f(x) = e^x and let x_1 & x_2 be two points on the curve y = e^x. Let R be another point which divides line segment PQ in ratio 1 : 3.

y coordinate of point R is $\dfrac{3e^{x_1}+e^{x_2}}{4}$ and

y coordinate of point S is $e^{\frac{3x_1+x_2}{4}}$. Since f(x) = e^x is always concave up, hence point R will always be above point S.

$$\Rightarrow \quad \frac{3e^{x_1}+e^{x_2}}{4} > e^{\frac{3x_1+x_2}{4}}$$

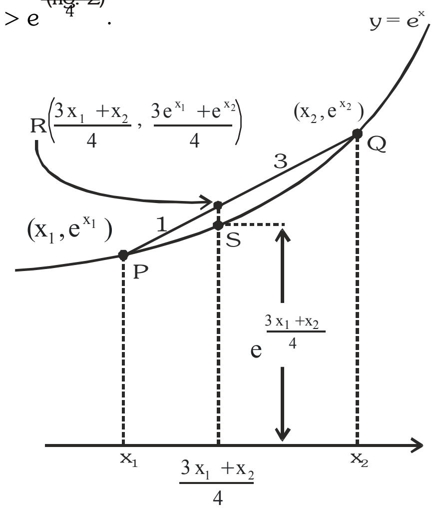


Illustration 11: In any ΔABC prove that sinA + sinB + sinC ≤ $\dfrac{3\sqrt{3}}{2}$.

Solution : Co-ordinates of centroid G are $\left(\dfrac{A+B+C}{3}, \dfrac{\sin A+\sin B+\sin C}{3}\right)$

from figure we have

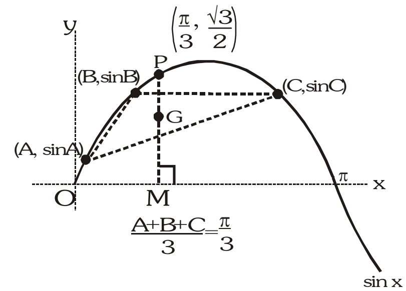
PM ≥ GM

$$\sin\left(\frac{A+B+C}{3}\right) \ge \frac{\sin A+\sin B+\sin C}{3}$$

$$\sin\left(\frac{\pi}{3}\right) \ge \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \quad \sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$$

equality holds when triangle is equilateral.



Do yourself - 6 :

(i) In any triangle ABC, prove that $\cos A+\cos B+\cos C \le \dfrac{3}{2}$.

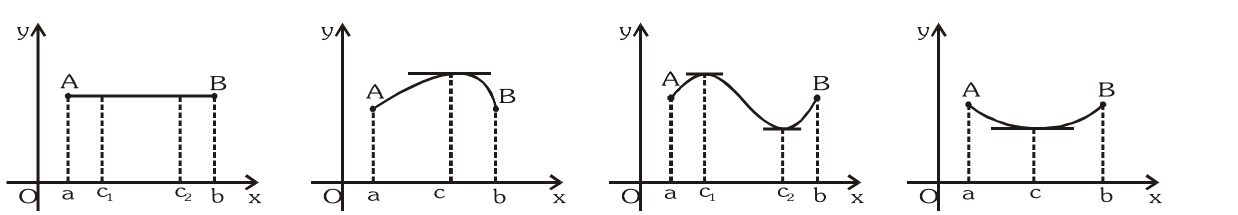
8. ROLLE'S THEOREM :

Let f be a function that satisfies the following three hypotheses :

(a) f is continuous on the closed interval [a, b]. **(b)** f is differentiable on the open interval (a, b)

(c) f(a) = f(b)

Then there exist at least one number c in (a, b) such that f'(c) = 0.



(a) (b) (c) (d)

Note : If f is differentiable function then between any two consecutive roots of f(x) = 0, there is atleast one root of the equation f'(x) = 0.

(d) Geometrical Interpretation :

Geometrically, the Rolle's theorem says that somewhere between A and B the curve has at least one tangent parallel to x-axis.

Illustration 12 : Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2].

Solution : Here we observe that

(a) f(x) is polynomial and since polynomial are always continuous, as well as differentiable. Hence f(x) is continuous in the [0,2] and differentiable in the (0, 2).

&

(b) $f(0) = 0,\ f(2) = 2^3 - 3.\ (2)^2 + 2(2) = 0$

$\therefore \quad f(0) = f(2)$

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exists some $c \in (0, 2)$ such that $f'(c) = 0$

$$\Rightarrow \quad f'(c) = 3c^2 - 6c + 2 = 0 \quad \Rightarrow \quad c = 1 \pm \frac{1}{\sqrt{3}}$$

where both $c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$ thus Rolle's theorem is verified.

Illustration 13 : Show that between any two roots of $e^{-x} - \cos x = 0$ there exists at least one root of $\sin x - e^{-x} = 0$.

Solution : If x = a and x = b are two distinct roots of $e^{-x} - \cos x = 0$

then $e^{-a} - \cos a = 0$ and $e^{-b} - \cos b = 0$ .... (1)

and let $f(x) = e^{-x} - \cos x$

We observe that

(i) $\therefore$ e^{-x} and cos x are continuous as well as differentiable in [a, b] then f(x) is also continuous in [a, b] & differentiable in (a,b).

(ii) $$\left.\begin{aligned} f(a) &= e^{-a} - \cos a = 0 \\ \text{and } f(b) &= e^{-b} - \cos b = 0 \end{aligned}\right\} \quad \{\text{from } (1)\}$$

i.e. f(a) = f(b) = 0

Thus f satisfies all the three conditions of Rolle's theorem in [a, b]. Hence there is at least one value of x in (a, b), say c such that f'(c) = 0.

Now $f'(c) = 0 \quad \Rightarrow \quad -e^{-c} + \sin c = 0 \quad \Rightarrow \quad \sin c - e^{-c} = 0$

$\Rightarrow$ c is a root of the equation $\sin x - e^{-x} = 0$.

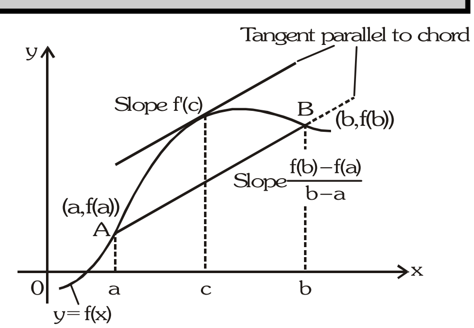
Hence between any two roots of the equation $e^{-x} - \cos x = 0$ there is at least one root of the equation $\sin x - e^{-x} = 0$.

Do yourself - 7 :

(i) Verify Rolle's theorem for $y = 1 - x^{4/3}$ on the interval [–1,1]

(ii) Show that between any two roots of tanx = 1 there exists atleast one root of tanx = –1.

9. LAGRANGE'S MEAN VALUE THEOREM (LMVT) :



Let f be a function that satisfies the following hypotheses:

(i) f is continuous in [a, b]

(ii) f is differentiable in (a, b).

Then there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

(a) **Geometrical Interpretation :**

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

(b) **Physical Interpretations :**

If we think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over [a, b] and f'(c) as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Illustration 14 : Find c of the Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval [1, 3].

Solution : Given $f(x) = 3x^2 + 5x + 7$ ...... (i)

$\Rightarrow \quad f(1) = 3 + 5 + 7 = 15$ and $f(3) = 27 + 15 + 7 = 49$

Again $f'(x) = 6x + 5$

Here $a = 1,\ b = 3$

Now from Lagrange's mean value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow \quad 6c + 5 = \frac{f(3)-f(1)}{3-1} = \frac{49-15}{2} = 17 \text{ or } c = 2.$$

Illustration 15 : If f(x) is continuous and differentiable over [–2, 5] and $-4 \le f'(x) \le 3$ for all x in (–2, 5), then the greatest possible value of $f(5) - f(-2)$ is -

(A) 7 (B) 9 (C) 15 (D) 21

Solution : Apply LMVT

$$f'(x) = \frac{f(5)-f(-2)}{5-(-2)} \text{ for some x in } (-2, 5)$$

$$\text{Now, } -4 \le \frac{f(5)-f(-2)}{7} \le 3$$

$-28 \le f(5) - f(-2) \le 21$

$\therefore$ Greatest possible value of $f(5) - f(-2)$ is 21.

Illustration 16 : If functions f(x) and g(x) are continuous in [a, b] and differentiable in (a, b), show that there will be at least one point c, a < c < b such that $\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$

Solution : Let $F(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix} = f(a)\, g(x) - g(a)\, f(x)$ ......(i)

$\Rightarrow \quad F'(x) = f(a)\, g'(x) - g(a)\, f'(x)$ ......(ii)

Since f(x) and g(x) are continuous in [a, b] and differentiable in (a, b), therefore, from (i) and (ii) it follows that F(x) is continuous in [a, b] and differentiable in (a, b).

Also from (i), $F(a) = f(a)g(a) - g(a)f(a) = 0$

And $F(b) = f(a)\, g(b) - g(a)\, f(b)$

Now by mean value theorem for F(x) in [a, b], there will be at least one point c, a < c < b

such that $F'(c) = \dfrac{F(b)-F(a)}{b-a}$

$$\Rightarrow \quad f(a)\, g'(c) - g(a)\, f'(c) = \frac{f(a)g(b)-g(a)f(b)-0}{b-a}$$

or $\quad f(a)g(b) - g(a)f(b) = (b - a)\{f(a)g'(c) - g(a)f'(c)\}$ or $\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$

Do yourself - 8 :

(i) If $f(x) = x^2$ in [a, b], then show that there exist atleast one c in (a, b) such that a, c, b are in A.P.

(ii) Using LMVT, prove that $\frac{x}{1+x} < \ell n(1+x) < x$ for $x > 0$

10. SPECIAL NOTE :

Use of Monotonicity in identifying the number of roots of the equation in a given interval. Suppose a and b are two real numbers such that,

(a) Let f(x) is differentiable & either MI or MD for $0 \le x \le b$.

&

(b) f(a) and f(b) have opposite signs.

Then there is one & only one root of the equation $f(x) = 0$ in (a, b).

Miscellaneous Illustrations :

Illustration 17 : If $g(x) = f(x) + f(1 - x)$ and $f''(x) < 0$; $0 \le x \le 1$, show that g(x) increasing in $x \in [0, 1/2]$ and decreasing in $x \in [1/2, 1]$

Solution : $\therefore$ $f''(x) < 0 \Rightarrow f'(x)$ is decreasing function.

Now, $g(x) = f(x) + f(1 - x)$ $\therefore$ $g'(x) = f'(x) - f'(1 - x)$ .... (i)

Case I : If $x \ge (1 - x) \Rightarrow x \ge 1/2$

$\therefore$ $f'(x) \le f'(1 - x)$ $\Rightarrow$ $f'(x) - f'(1 - x) \le 0$

$\Rightarrow$ $g'(x) \le 0$

$\therefore$ g(x) decreases in $x \in \left[\frac{1}{2}, 1\right]$

Case II : If $x \le (1 - x) \Rightarrow x \le 1/2$

$\therefore$ $f'(x) \ge f'(1 - x)$ $\Rightarrow$ $f'(x) - f'(1 - x) \ge 0$

$\Rightarrow$ $g'(x) \ge 0$

$\therefore$ g(x) increases in $x \in [0, 1/2]$

Illustration 18 : Prove that if $2a_0^2 < 15a$, all roots of $x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d = 0$ can't be real. It is given that a_0, a, b, c, d $\in$ R.

Solution : Let $f(x) = x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d$

$f'(x) = 5x^4 - 4a_0x^3 + 9ax^2 + 2bx + c$

$f''(x) = 20x^3 - 12a_0x^2 + 18ax + 2b$

$f'''(x) = 60x^2 - 24a_0x + 18a = 6(10x^2 - 4a_0x + 3a)$

Now, discriminant $= 16a_0^2 - 4.\ 10.\ 3a = 8(2a_0^2 - 15a) < 0$

as $2a_0^2 - 15a < 0$ is given.

Hence the roots of $f'''(x) = 0$ can not be real.

$\therefore$ $f''(x)$ have one real root and $f'(x) = 0$ have at most two real roots so $f(x) = 0$ have at most three real roots.

Therefore all the roots of $f(x) = 0$ will not be real.

Illustration 19 : Show by using mean value theorem that $\frac{\beta-\alpha}{1+\beta^2} < \tan^{-1}\beta - \tan^{-1}\alpha < \frac{\beta-\alpha}{1+\alpha^2}$ where $\beta > \alpha > 0$.

Solution : Take $f(x) = \tan^{-1}x$

$\Rightarrow$ $f'(x) = \frac{1}{1+x^2}$. By mean value theorem for $f(x)$ in $[\alpha, \beta]$

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(c) = \frac{1}{1+c^2} \quad \text{where } \alpha < c < \beta \qquad \text{........ (i)}$$

Now, $c > \alpha \Rightarrow \frac{1}{1+c^2} < \frac{1}{1+\alpha^2}$

as $c < \beta \Rightarrow \frac{1}{1+c^2} > \frac{1}{1+\beta^2}$

$\therefore$ from (i), $\frac{1}{1+\beta^2} < \frac{f(\beta)-f(\alpha)}{\beta-\alpha} < \frac{1}{1+\alpha^2}$

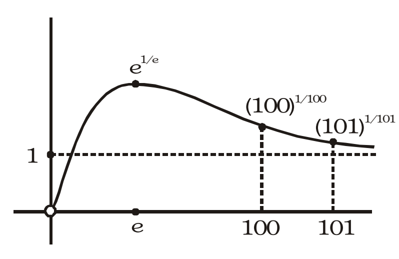
or $\frac{\beta-\alpha}{1+\beta^2} < f(\beta)-f(\alpha) < \frac{\beta-\alpha}{1+\alpha^2}$

Hence, $\frac{\beta-\alpha}{1+\beta^2} < \tan^{-1}\beta - \tan^{-1}\alpha < \frac{\beta-\alpha}{1+\alpha^2}$

Illustration 20 : Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

Solution : Assume $f(x) = x^{1/x}$ and let us examine monotonic nature of $f(x)$, $(x > 0)$

$f'(x) = x^{1/x} \cdot \left(\frac{1-\ell nx}{x^2}\right)$



$f'(x) > 0 \quad \Rightarrow \quad x \in (0, e)$

and $f'(x) < 0 \Rightarrow \quad x \in (e, \infty)$

Hence $f(x)$ is M.D. for $x \geq e$

and since $100 < 101$

$\Rightarrow$ $f(100) > f(101)$

$\Rightarrow$ $(100)^{1/100} > (101)^{1/101}$

Illustration 21 : For $\forall\ x \in [0, 1]$, let the second derivative $f''(x)$ of a function $f(x)$ exist and satisfy $|f''(x)| \leq 1$. If $f(0) = f(1)$ then show that $|f'(x)| < 1$ for all x in $[0, 1]$.

Solution : Since $f''(x)$ exists for all x in [0, 1]

$\therefore$ $f(x)$ and $f'(x)$ are differentiable as well as continuous for all x in [0, 1]

Now $f(x)$ is continuous in [0, 1] and differentiable in (0, 1) and $f(0) = f(1)$

$\therefore$ By Rolle's theorem there is at least one c such that $f'(c) = 0$, where $0 < c < 1$.

Case I : Let $x = c$ then $f'(x) = f'(c) = 0$

$\therefore$ $|f'(x)| = |0| = 0 < 1$

Case II : Let $x > c$. By Largrange's mean value theorem for $f'(x)$ in [c, x]

$$\frac{f'(x) - f'(c)}{x - c} = f''(\alpha) \text{ for at least one } \alpha,\ c < \alpha < x$$

or $f'(x) = (x - c)\, f''(\alpha)$ $\quad (\because\ f'(c) = 0)$

or $|f'(x)| = |x - c|\,|f''(\alpha)|$

But $x \in [0, 1]$, $c \in (0, 1)$

$\therefore$ $|x - c| < 1 - 0$ or $|x - c| < 1$

and given $|f''(x)| \le 1 \ \forall\ x \in [0, 1]$

$\therefore$ $|f''(\alpha)| \le 1$

$\therefore$ $|f'(x)| < 1.1.$ $\quad (\because\ |f'(x)| = |x - c|\,|f''(\alpha)|$

or $|f'(x)| < 1\ \forall\ x \in [0, 1]$

Case III: Let $x < c$ then

$$\frac{f'(x) - f'(c)}{x - c} = f''(\alpha)$$

$\therefore$ $|-f'(x)| = |c - x|\,|f''(\alpha)|$

$\Rightarrow$ $|f'(x)| < 1.1$ or $|f''(x)| < 1$

Combining all cases, we get $|f'(x)| < 1\ \forall\ x \in [0, 1]$

ANSWERS FOR DO YOURSELF

1 : (i) -1
2 : (i) Increasing **(ii)** B
3 : (i) Not defined
4 : (i) $x = 1$ is a critical point as well as stationary point (Note $x = 0$ is not in the domain of $f(x)$)
5 : (i) $\ell n(1+x)$ **(iii)** A,B,D
7 : (i) Rolles theorem is valid

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. Function $f(x) = x^2(x-2)^2$ is -

(A) increasing in $(0,1)\cup(2,\infty)$
(B) decreasing in $(0,1)\cup(2,\infty)$
(C) decreasing function
(D) increasing function

2. The function $f(x) = \tan x - x$

(A) always increases
(B) always decreases
(C) never decreases
(D) sometimes increases and sometimes decreases

3. The function f, defined by $f(x) = (x+2)e^{-x}$ is -

(A) decreasing for all x
(B) decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$
(C) increasing for all x
(D) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$

4. Function $f(x) = x^3 + 6x^2 + (9+2k)x + 1$ is increasing function if

(A) $k \ge \frac{3}{2}$
(B) $k > \frac{3}{2}$
(C) $k < \frac{3}{2}$
(D) $k \le \frac{3}{2}$

5. The function $f(x) = \cos x - 2px$ is monotonically decreasing for

(A) $p < \frac{1}{2}$
(B) $p > \frac{1}{2}$
(C) $p < 2$
(D) $p > 2$

6. The value of 'a' for which the function f(x) = sinx − cosx − ax + b decreases for all real values of x, is -

(A) $a \ge -\sqrt{2}$
(B) $a \le -\sqrt{2}$
(C) $a \le \sqrt{2}$
(D) $a \ge \sqrt{2}$

7. Let f(x) be a quadratic expression which is positive for all real values of x. If g(x) = f(x) + f'(x) +f''(x), then for any real x -

(A) $g(x) < 0$
(B) $g(x) > 0$
(C) $g(x) = 0$
(D) $g(x) \ge 0$

8. $f(x) = \int\left(2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}\right)dx$ then f is -

(A) increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$
(B) increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$
(C) increasing in $(-\infty, \infty)$
(D) decreasing in $(-\infty, \infty)$

9. Let $f(x) = \begin{cases} \dfrac{4-x}{2-\sqrt{x}} & \text{for } 0 < x < 4 \\ 4 & \text{for } x = 4 \\ 16-3x & \text{for } 4 < x < 6 \end{cases}$

Which of the following properties does f have on the interval (0, 6) ?

I. $\ell n\, f(x)$ exists ;
II. f is continuous
III. f is monotonic

(A) I only
(B) II only
(C) III only
(D) none

10. The length of largest continuous interval in which function f (x) = 4x − tan2x is monotonic, is -

(A) $\pi/2$ (B) $\pi/4$ (C) $\pi/8$ (D) $\pi/16$

11. The largest set of real values of x for which $ln\,(1 + x) \le x$ is

(A) $(-1, \infty)$ (B) $(-1, 0) \cup (0, \infty)$ (C) $[0, \infty)$ (D) $(0, \infty)$

12. The true set of real values of x for which the function, $f(x) = x\,\ell n\,x - x + 1$ is positive is

(A) $(1, \infty)$ (B) $(1/e, \infty)$ (C) $[e, \infty)$ (D) $(0,1) \cup (1, \infty)$

13. Number of solution of the equation $3\tan x + x^3 = 2$ in $\left(0, \frac{\pi}{4}\right)$ is

(A) 0 (B) 1 (C) 2 (D) 3

14. Rolle's theorem in the indicated intervals will not be valid for which of the following function-

(A) $f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$; $x \in [-1, 1]$

(B) $g(x) = \begin{cases} \dfrac{1-\cos x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$; $x \in [-2\pi, 2\pi]$

(C) $h(x) = \begin{cases} \dfrac{1-\cos x}{x^2} & x \neq 0 \\ \dfrac{1}{2} & x = 0 \end{cases}$; $x \in [-2\pi, 2\pi]$

(D) $k(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$; $x \in \left[-\dfrac{1}{\pi}, \dfrac{1}{2\pi}\right]$

15. Consider the function for x = [−2, 3], $f(x) = \begin{cases} \dfrac{x^3 - 2x^2 - 5x + 6}{x - 1} & \text{if} \quad x \neq 1 \\ -6 & \text{if} \quad x = 1 \end{cases}$, then

(A) f is discontinuous at x = 1 ⇒ Rolle's theorem is not applicable in [−2, 3]

(B) $f(-2) \neq f(3)$ ⇒ Rolle's theorem is not applicable in [−2, 3]

(C) f is not derivable in (−2, 3) ⇒ Rolle's theorem is not applicable

(D) Rolle's theorem is applicable as f satisfies all the conditions and c of Rolle's theorem is 1/2

16. If the function $f(x) = 2x^2 + 3x + 5$ satisfies LMVT at x = 2 on the closed interval [1, a] then the value of 'a' is equal to -

(A) 3 (B) 4 (C) 6 (D) 1

17. Consider the function $f(x) = 8x^2 - 7x + 5$ on the interval [−6, 6]. The value of c that satisfies the conclusion of the mean value theorem, is -

(A) −7/8 (B) −4 (C) 7/8 (D) 0

18. Let f be a function which is continuous and differentiable for all real x. If f(2) = − 4 and $f'(x) \ge 6$ for all $x \in [2, 4]$ then

(A) f(4) < 8 (B) $f(4) \ge 8$ (C) $f(4) \ge 12$ (D) none of these

19. If $f(x) = x^3 + 7x - 1$ then f(x) has a zero between x = 0 and x = 1. The theorem which best describes this, is -

(A) Rolle's theorem (B) mean value theorem

(C) maximum-minimum value theorem (D) intermediate value theorem

20. Consider $f(x) = |1 - x|$, $1 \le x \le 2$ and $g(x) = f(x) + b \sin\frac{\pi}{2}x$, $1 \le x \le 2$

then which of the following is correct ?

(A) Rolles theorem is applicable to both f, g and $b = \frac{3}{2}$

(B) LMVT is not applicable to f and Rolles theorem if applicable to g with $b = \frac{1}{2}$

(C) LMVT is applicable to f and Rolles theorem is applicable to g with b = 1

(D) Rolles theorem is not applicable to both f, g for any real b.

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

21. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then. **[JEE 98]**

(A) h is increasing whenever f is increasing
(B) h is increasing whenever f is decreasing
(C) h is decreasing whenever f is decreasing
(D) nothing can be said in general

22. Let $f(x) = \int e^x(x-1)(x-2)dx$. Then 'f' increases in the interval -

(A) $(-\infty, -2)$ (B) $(-2, -1)$ (C) $(1, 2)$ (D) $(2, \infty)$

23. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$ then -

(A) f(x) is increasing on (–1, 2)
(B) f(x) is continuous on [–1, 3]
(C) f'(2) does not exist
(D) f(x) has the maximum value at x = 2

24. Let $f(x) = 8x^3 - 6x^2 - 2x + 1$, then -

(A) f(x) = 0 has no root in (0, 1)
(B) f(x) = 0 has at least one root in (0, 1)
(C) f'(c) vanishes for some $c \in (0,1)$
(D) none

25. Let f and g be two functions defined on an interval I such that $f(x) \ge 0$ and $g(x) \le 0$ for all $x \in I$ and f is strictly decreasing on I while g is strictly increasing on I then -

(A) the product function fg is strictly increasing on I
(B) the product function fg is strictly decreasing on I
(C) fog(x) is monotonically increasing on I
(D) fog(x) is monotonically decreasing on I

CHECK YOUR GRASP — **ANSWER KEY** — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A	A	D	A	B	D	B	C	B	B
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	A	D	B	D	D	A	D	B	D	C
Que.	21	22	23	24	25					
Ans.	A,C	A,B,D	A,B,C,D	B,C	A,D					

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. A differentiable function f(x) is strictly increasing $\forall\ x \in R$, Then -

(A) $f'(x) > 0\ \ \forall\ x \in R$.

(B) $f'(x) \geq 0\ \ \forall\ x \in R$, provided it vanishes at finite number of points.

(C) $f'(x) \geq 0\ \ \forall\ x \in R$ provided it vanishes at discrete points though the number of these discrete points may not be finite.

(D) $f'(x) \geq 0\ \ \forall\ x \in R$ provided it vanishes at discrete points and the number of these discrete points must be infinite.

2. The function $y = \dfrac{2x-1}{x-2}$ $(x \neq 2)$ -

(A) is its own inverse
(B) decreases for all values of x
(C) has a graph entirely above x-axis
(D) is bound for all x

3. If f(0) = f(1) = f(2) = 0 & function f(x) is twice differentiable in (0, 2) and continuous in [0, 2]. Then which of the following is/are definitely true -

(A) $f''(c) = 0;\ \ \forall\ c \in (0, 2)$
(B) $f'(c) = 0$; for atleast two $c \in (0, 2)$
(C) $f'(c) = 0$; for exactly one $c \in (0, 2)$
(D) $f''(c) = 0$; for atleast one $c \in (0, 2)$

4. Consider the function $f(x) = \begin{cases} x\sin\dfrac{\pi}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$ then the number of points in (0, 1) where the derivative f'(x) vanishes, is -

(A) 0
(B) 1
(C) 2
(D) infinite

5. Let $\phi(x) = (f(x))^3 - 3(f(x))^2 + 4f(x) + 5x + 3\sin x + 4\cos x\ \forall\ x \in R$, then -

(A) ϕ is increasing whenever f is increasing
(B) ϕ is increasing whenever f is decreasing
(C) ϕ is decreasing whenever f is decreasing
(D) ϕ is decreasing if $f'(x) = -11$

6. If $f(x) = \dfrac{x}{\sin x}$ and $g(x) = \dfrac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval -

(A) both f(x) and g(x) are increasing functions
(B) both f(x) and g(x) are decreasing functions
(C) f(x) is an increasing function
(D) g(x) is an increasing function

7. $f(x) > x\ ;\forall\ x \in R$. Then the equation $f(f(x)) - x = 0$ has–

(A) Atleast one real root

(B) More than one real root

(C) no real root if f(x) is a polynomial & one real root if f(x) is not a polynomial

(D) no real root at all

8. A function $y = f(x)$ is given by $x = \dfrac{1}{1+t^2}$ & $y = \dfrac{1}{t(1+t^2)}$ for all t > 0 then f is :

(A) increasing in (0,3/2) & decreasing in (3/2, ∞)
(B) increasing in (0,1)
(C) increasing in (0,∞)
(D) decreasing in (0,1)

9. Suppose that f is differentiable for all x such that $f'(x) \le 2$ for all x. If $f(1) = 2$ and $f(4) = 8$ then $f(2)$ has the value equal to -

(A) 3 (B) 4 (C) 6 (D) 8

10. The function $f(x) = \dfrac{\ell n(\pi + x)}{\ell n(e + x)}$ is -

(A) increasing on $(0, \infty)$
(B) decreasing on $(0, \infty)$
(C) increasing on $(0, \pi/e)$, decreasing on $(\pi/e, \infty)$
(D) decreasing on $(0, \pi/e)$, increasing on $(\pi/e, \infty)$

11. Number of solution(s) satisfying the equation, $3x^2 - 2x^3 = \log_2 (x^2 + 1) - \log_2 x$ is -

(A) 1 (B) 2 (C) 3 (D) none

12. Let $g(x) = 2f(x/2) + f(1 - x)$ and $f''(x) < 0$ in $0 \le x \le 1$ then g (x) -

(A) decreases in [0,2/3) (B) decreases in (2/3,1] (C) increases in [0,2/3) (D) increases in (2/3,1]

13. If $f(x) = a^{\{a^{|x|}\,\mathrm{sgn}\,x\}}$; $g(x) = a^{[a^{|x|}\,\mathrm{sgn}\,x]}$ for $a > 0$, $a \ne 1$ and $x \in R$, where { } & [ ] denote the fractional part and integral part functions respectively, then which of the following statements holds good for the function h (x), where $(\ell n\, a)h(x) = (\ell n\ f(x) + \ell n\, g(x))$ -

(A) 'h' is even and increasing for a > 1
(B) 'h' is odd and decreasing for a < 1
(C) 'h' is even and decreasing for a < 1
(D) 'h' is odd and increasing for a > 1

14. Number of roots of the equation $x^2 . e^{2-|x|} = 1$ is -

(A) 2 (B) 4 (C) 6 (D) infinite

15. Equation $\dfrac{1}{(x+1)^3} - 3x + \sin x = 0$ has -

(A) no real root
(B) two real and distinct roots
(C) exactly one negative root
(D) exactly one root between –1 and 1

16. The values of p for which the function $f(x) = \left(\dfrac{\sqrt{p+4}}{1-p} - 1\right) x^5 - 3x + \ell n\ 5$ decreases for all real x is

(A) $(-\infty, \infty)$ (B) $\left[-4, \dfrac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$ (C) $\left[-3, \dfrac{5-\sqrt{27}}{2}\right] \cup (2, \infty)$ (D) $[1, \infty)$

BRAIN TEASERS — **ANSWER KEY** — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A,B	B,D	D	A,D	C	D	B	B	B
Que.	11	12	13	14	15	16				
Ans.	A	B,C	D	B	B,C,D	B				

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

MATCH THE COLUMN

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

1.

	Column-I		Column-II
(A)	The equation $x \log x = 3 - x$ has at least one root in	(p)	[0, 1]
(B)	If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one root in	(q)	[1, 3]
(C)	If $c = \sqrt{3}$ & $f(x) = x + \frac{1}{x}$ then interval in which LMVT is applicable for f(x) is	(r)	[0, 3]
(D)	If $c = \frac{1}{2}$ & $f(x) = 2x - x^2$, then interval in which LMVT is applicable for f(x) is	(s)	[–1, 1]

ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is truez ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : The quadratic equation $10x^2-28x + 17 = 0$ has atleast one root in [1,2].

Because :

Statement-II : $f(x) = e^{10x}(x - 1)(x - 2)$ satisfies all the conditions for Rolle's theorem in [1,2]

(A) A (B) B (C) C (D) D

2. Let $f : R \to R$, $f(x) = x^3 + x^2 + 3x + \sin x$.

Statement-I : f(x) is one-one.

Because :

Statement-II : f(x) is decreasing function.

(A) A (B) B (C) C (D) D

3. **Statement-I** : The greatest of the numbers 1, $2^{1/2}$, $3^{1/3}$, $4^{1/4}$, $5^{1/5}$, $6^{1/6}$, $7^{1/7}$ is $3^{1/3}$.

Because :

Statement-II : $x^{1/x}$ is increasing for $0 < x < e$ and decreasing for $x > e$.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

$f : (0, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be defined as, $f(x) = \text{arc}\tan(\ell nx)$

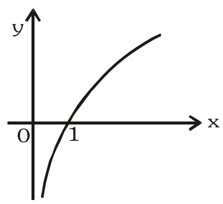
On the basis of above information, answer the following questions :

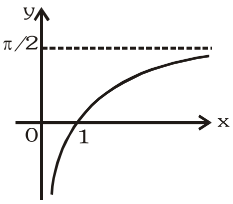
1. The above function can be classified as

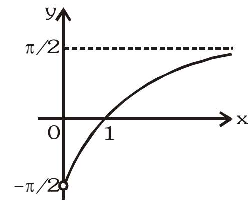
(A) injective but not surjective (B) surjective but not injective

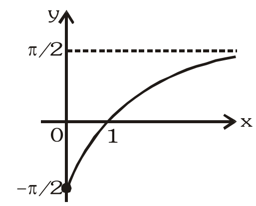
(C) neither injective nor surjective (D) both injective as well as surjective

2. The graph of $y = f(x)$ is best represented as -

(A) 

(B) 

(C) 

(D) 

3. If x_1, x_2 and x_3 are the points at which $g(x) = [f(x)]$ is discontinuous where [.] denotes greatest integer function, then $x_1 + x_2 + x_3$ is -

(A) equal to 2 (B) equal to 3

(C) greater than 3 (D) greater than 2 but less than 3

Comprehension # 2

Consider the polynomial function

$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. This function has monotonicity as given below :

in $(-\infty, a_1)$ decreasing

in (a_1, a_2) increasing

in (a_2, a_3) decreasing

in (a_3, ∞) increasing

A rectangle ABCD is formed such that

$\ell(AB)$ = portion of the tangent to the curve $y = f(x)$ at $x = a_1$, intercepted between the lines $x = a_1$ & $x = a_3$.

$\ell(BC)$ = portion of the line $x = a_3$ intercepted between the curve & x-axis.

On the basis of above information, answer the following questions :

1. Triplet (a_1, a_2, a_3) is given by -

(A) $(-1, 0, 2)$ (B) $(0, -1, 2)$ (C) $(2, -1, 0)$ (D) $(2, 0, -1)$

2. Area of rectangle ABCD -

(A) 51 (B) 57 (C) 87 (D) 81

3. The equation $f(x) = 0$ has -

(A) 2 real, 2 imaginary roots (B) 2 complex, 2 irrational roots

(C) 4 real & distinct roots (D) 2 real coincident roots & 2 irrational roots

MISCELLANEOUS TYPE QUESTION **ANSWER KEY** EXERCISE-3

- ***Match the Column***

1. (A) → (q,r), (B) → (r), (C) → (q), (D) → (p)

- ***Assertion & Reason***

1. A **2.** C **3.** A

- ***Comprehension Based Questions***

Comprehension # 1 : **1.** D **2.** C **3.** C

Comprehension # 2 : **1.** A **2.** D **3.** D

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

1. Find the intervals of monotonicity for the following functions & represent your solution set on the number line.

(a) $f(x) = 2.e^{x^2-4x}$ (b) $f(x) = e^x/x$ (c) $f(x) = x^2e^{-x}$

2. Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$

3. Show that the function $y = \arctan x - x$ decreases everywhere.

4. Find the intervals of monotonicity of the function : $y = \ell n(x + \sqrt{1+x^2})$

5. Find the intervals of monotonicity of the function : $y = \dfrac{10}{4x^3 - 9x^2 + 6x}$

6. Find the value of x > 1 for which the function $F(x) = \int_x^{x^2} \frac{1}{t}\ell n\left(\frac{t-1}{32}\right)dt$ is increasing and decreasing.

7. Find the range of values of 'a' for which the function $f(x) = x^3 + (2a + 3)x^2 + 3(2a + 1)x + 5$ is monotonic in R. Hence find the set of values of 'a' for which f(x) is invertible.

8. If $f(x) = \left(\frac{a^2-1}{3}\right)x^3 + (a-1)x^2 + 2x + 1$ is monotonic increasing for every $x \in R$, then find the range of values of 'a'.

9. Find the set of all values of the parameter 'a' for which the function

$f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in R$ and has no critical points for all $x \in R$.

10. Find the greatest & the least values of the following functions in the given interval if they exist.

(a) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$ (b) $y = x^5 - 5x^4 + 5x^3 + 1$ in $[-1, 2]$

(c) $y = \sin 2x - x$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $y = 2\tan x - \tan^2 x$ $\left[0, \frac{\pi}{2}\right)$

11. Prove the following inequalitities :

(a) $x^2 - 1 > 2x\, \ell nx$ for $x > 1$ (b) $2x\ell nx > 4(x-1) - 2\,\ell n\, x$ for $x > 1$.

(c) $\tan^2 x + 6\,\ell n \sec x + 2\cos x + 4 > 6\sec x$ for $x \in \left(\frac{3\pi}{2}, 2\pi\right)$

12. Let f, g be differentiable on R and suppose that $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all $x \ge 0$. Show that $f(x) \le g(x)$ for all $x \ge 0$.

13. Using monotonicity prove that $\dfrac{\tan x_2}{\tan x_1} > \dfrac{x_2}{x_1}$ for $0 < x_1 < x_2 < \dfrac{\pi}{2}$

14. Identify which is greater : $\dfrac{1+e^{2/3}}{e^{1/3}}$ or $\dfrac{1+\pi^{2/3}}{\pi^{1/3}}$

15. Verify Rolles theorem for $f(x) = (x - a)^m(x - b)^n$ on [a, b]; m, n being positive integer.

16. Check the validity of Rolle's theorem for the function $y = x^3 + 4x^2 - 7x - 10$ in the interval [–1, 2].

17. Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, $0 < c < 1$ such that $f(c) = 0$.

18. If the equation $a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x = 0$ has a positive root α, prove that the equation $na_0x^{n-1} + (n-1)a_1x^{n-2} + \ldots\ldots + a_{n-1} = 0$ also has a positive root smaller that α.

19. f(x) and g(x) are differentiable functions for $0 \le x \le 2$ such that $f(0) = 5$, $g(0) = 0$, $f(2) = 8$, $g(2) = 1$.

Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.

20. If f, ϕ, ψ are continuous in [a,b] and derivable in]a,b[then show that there is a value of c lying between a & b such that, $\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix} = 0$

21. The function $y = \dfrac{2-x^2}{x^4}$ takes on equal values at the end-points of the interval [–1, 1]. Make sure that the derivative of this function does not vanish at any point of the interval [–1, 1], and explain this deviation from Rolle's theorem.

22. A (0, 1), $B\left(\dfrac{\pi}{2}, 1\right)$ are two points on the graph given by y = 2 sinx + cos2x. Prove that there exists a point P on the curve between A & B such that tangent at P is parallel to AB. Find the co-ordinates of P.

23. With the aid of Lagrange's formula prove the inequalities $\dfrac{a-b}{a} \le \ell n\dfrac{a}{b} \le \dfrac{a-b}{b}$, for the condition $0 < b \le a$.

24. Let f : [a, b] $\to$ R be continuous on [a, b] and differentiable on (a, b). If f(a) < f(b), then show than f'(c) > 0 for some c $\in$ (a, b).

25. Show that the function $f(x) = x^n + px + q$ cannot have more than two real roots if n is even and more than three if n is odd.

Investigate the behaviour of the following functions and construct their graph : (Q. 26 to 34)

26. $y = x^3 - 3x + 2$ **27.** $y = x^4 - 10x^2 + 9$ **28.** $y = (x-1)^2(x-2)^3$

29. $y = (x + 3)/(x - 1)$ **30.** $y = x + \sin x$ **31.** $y = 2.e^{x^2-4x}$

32. $y = e^x/x$ **33.** $y = x^2e^{-x}$ **34.** $y = \dfrac{x^2-3x+4}{x^2+3x+4}$

35. Investigate the behaviour of the function $y = (x^3 + 4)/(x + 1)^3$ and construct its graph. How many solutions does the equation $(x^3 + 4)/(x + 1)^3 = c$ possess ?

CONCEPTUAL SUBJECTIVE EXERCISE **ANSWER KEY** EXERCISE-4(A)

1. (a) I in $(2, \infty)$ & D in $(-\infty, 2)$ (b) I in $(1, \infty)$ & D in $(-\infty, 0) \cup (0, 1)$ (c) I in (0, 2) & D in $(-\infty, 0) \cup (2, \infty)$

2. $(-2, 0) \cup (2, \infty)$ **4.** Increases monotonically.

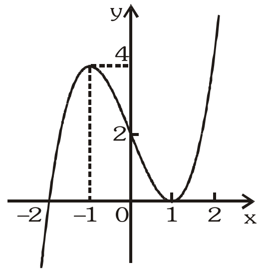
5. decreases in $(-\infty, 0) \cup \left(0, \dfrac{1}{2}\right) \cup (1, \infty)$ increases in $\left(\dfrac{1}{2}, 1\right)$ **6.** I in $(3, \infty)$ and D in (1,3)

7. $0 \le a \le \dfrac{3}{2}$ **8.** $a \in (-\infty, -3] \cup [1, \infty)$ **9.** $a < -(2+\sqrt{5})$ or $a > \sqrt{5}$

10. (a) Maximum at x = –1 and f(–1) = 18; Minimum at x = 1/8 and f(1/8) = –9/4 (b) 2 & –10

(c) $\dfrac{\pi}{2}$ and $-\dfrac{\pi}{2}$ (d) The greatest value is equal to 1, no least value.

14. $\dfrac{1+\pi^{2/3}}{\pi^{1/3}}$ **15.** $c = \dfrac{mb+na}{m+n}$ which lies between a & b **22.** $\left(\dfrac{\pi}{6}, \dfrac{3}{2}\right)$

26.


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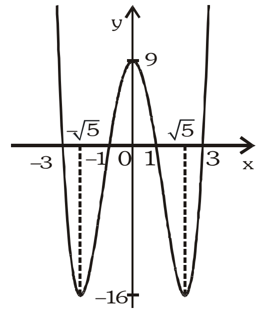

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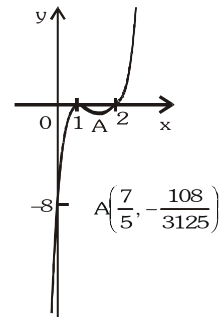

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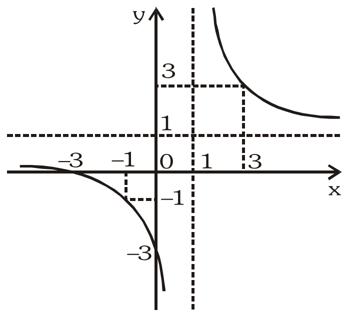

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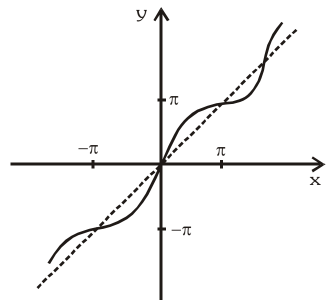

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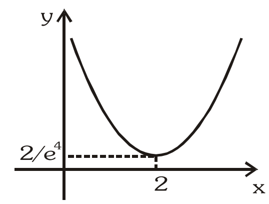
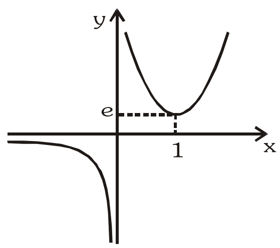
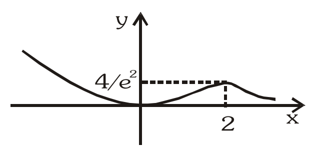
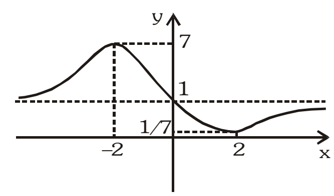


32.


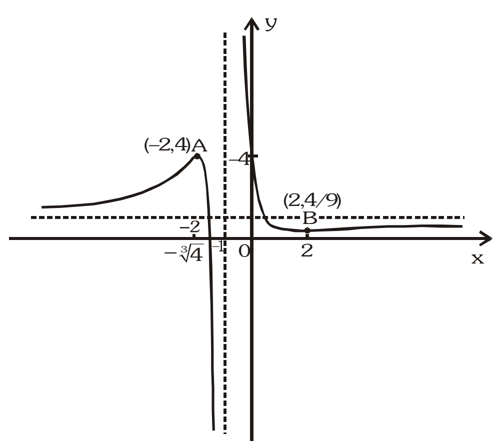
33.


34.


35. One root for $c \in \left(-\infty, \frac{4}{9}\right) \cup (4, \infty)$;

two roots for $c \in \left\{\frac{4}{9}, 1, 4\right\}$;

three roots for $c \in \left(\frac{4}{9}, 1\right) \cup (1, 4)$



EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. Let a + b = 4, where a < 2 and let g(x) be a differentiable function. If $\frac{dg}{dx}>0$ for all x, prove that $\int_0^a g(x)dx + \int_0^b g(x)dx$ increases as (b – a) increases.

2. Find all the values of the parameter 'a' for which the function;
$f(x) = 8ax - a\sin 6x - 7x - \sin 5x$ increases & has no critical points for all $x \in R$.

3. Prove that if f is differentiable on [a, b] and if f(a) = f(b) = 0 then for any real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$.

4. Let f be continuous on [a, b] and differentiable on (a, b). If f(a) = a and f(b) = b, show that there exist distinct c_1, c_2, in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

5. Prove the inequality $e^x > (1 + x)$ using LMVT for all $x \in R_0$ and use it to determine which of the two numbers e^π and π^e is greater.

6. For $x \in \left(0, \frac{\pi}{2}\right)$ identify which is greater (2sinx + tanx) or (3x). Hence find $\lim\limits_{x\to 0}\left[\frac{3x}{2\sin x + \tan x}\right]$, where [.] denote the greatest integer function.

7. Suppose that on the interval [–2, 4] the function f is differentiable, f(–2) = 1 and $|f'(x)| \le 5$. Find the bounding function of f on [–2, 4], using LMVT.

8. Using LMVT prove that : (a) tan x > x in $\left(0, \frac{\pi}{2}\right)$, (b) sin x < x for x > 0.

9. If $ax^2 + \frac{b}{x} \ge c$ for all positive x where a > 0 and b > 0, then show that $27ab^2 \ge 4c^3$.

10. Show that $1 + x\ell n\left(x + \sqrt{x^2+1}\right) \ge \sqrt{1+x^2}$ for all $x \in R$.

11. Let $f(x) = \begin{cases} xe^{ax}, & x \le 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$, where a is positive constant. Find the interval in which f'(x) is increasing.

12. If b > a, find the minimum value of $|(x - a)^3| + |(x - b)^3|$, $x \in R$.

13. f(x) is a differentiable function and g(x) is a double differentiable function such that $|f(x)| \le 1$ and $f'(x) = g(x)$. If $f^2(0) + g^2(0) = 9$. Prove that there exists some $c \in (-3, 3)$ such that $g(c).\ g''(c) < 0$

BRAIN STORMING SUBJECTIVE EXERCISE	ANSWER KEY	EXERCISE-4(B)

2. $(6, \infty)$ 5. e^π 6. 2sinx + tanx, limit = 0 7. y = –5x – 9 and y = 5x + 11

11. $\left[\frac{-2}{a}, \frac{a}{3}\right]$ 12. $(b - a)^3/4$

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. f(x) and g(x) are two differentiable function in [0, 2] such that $f''(x) - g''(x) = 0$, $f'(1) = 2$, $g'(1) = 4$, $f(2) = 3$, $g(2) = 9$, then $f(x) - g(x)$ at $x = 3/2$ is- [AIEEE-2002]

(1) 0 (2) 2 (3) 10 (4) –5

2. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2,1) and at that point the tanget to the graph is $y = 3x - 5$, then the function, is- [AIEEE-2004]

(1) $(x - 1)^2$ (2) $(x - 1)^3$ (3) $(x + 1)^3$ (4) $(x + 1)^2$

3. A function is matched below against an interval where it is supposed to be increasing. which of the following pairs is incorrectly matched ? [AIEEE-2005]

	interval	function
(1)	$(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(2)	$[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(3)	$\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(4)	$(-\infty, -4)$	$x^3 + 6x^2 + 6$

4. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in- [AIEEE-2007]

(1) $(\pi/4, \pi/2)$ (2) $(-\pi/2, \pi/4)$ (3) $(0, \pi/2)$ (4) $(-\pi/2, \pi/2)$

5. Let $f : R \to R$ be a continuous function defined by $f(x) = \dfrac{1}{e^x + 2e^{-x}}$

Statement–1 : $f(c) = \dfrac{1}{3}$, for some $c \in R$.

Statement–2 : $0 < f(x) \le \dfrac{1}{2\sqrt{2}}$, for all $x \in R$. [AIEEE-2010]

(1) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is false, Statement–2 is true.

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

1. 4 2. 2 3. 3 4. 2 5. 1

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. (a) For all $x \in (0, 1)$:

(A) $e^x < 1 + x$ (B) $\log_e(1+x) < x$ (C) $\sin x > x$ (D) $\log_e x > x$

(b) Consider the following statements S and R : [JEE 2000 (Screening) 1+1M out of 35]

S : Both sin x & cos x are decreasing functions in the interval $(\pi/2, \pi)$.

R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b)

Which of the following is true ?

(A) both S and R are wrong

(B) both S and R are correct, but R is not the correct explanation for S

(C) S is correct and R is the correct explanation for S

(D) S is correct and R is wrong.

2. Let $f(x) = \int e^x (x-1)(x-2)\,dx$ then f decreases in the interval - [JEE 2000, Screening, 1M out of 35]

(A) $(-\infty, 2)$ (B) $(-2, -1)$ (C) $(1, 2)$ (D) $(2, \infty)$

3. If $f(x) = xe^{x(1-x)}$, then f(x) is - [JEE 2001 (Screening) 1M out of 35]

(A) increasing on $\left(-\frac{1}{2}, 1\right)$ (B) decreasing on R (C) increasing on R (D) decreasing on $\left[-\frac{1}{2}, 1\right]$

4. Let $-1 \le p \le 1$. Show that the equations $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it. [JEE 2001 (Mains), 5M]

5. The length of a longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is - [JEE 2002 (Screening), 3]

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{3\pi}{2}$ (D) π

6. (a) Using the relation $2(1 - \cos x) < x^2$, $x \ne 0$ or otherwise, prove that $\sin(\tan x) \ge x \;\; \forall\, x \in \left[0, \frac{\pi}{4}\right]$.

(b) Let $f : [0, 4] \to R$ be a differentiable function. [JEE 2003 (Mains), 4+4M out of 60]

(i) Show that there exist $a, b \in [0, 4]$, $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$

(ii) Show that there exist α, β with $0 < \alpha < \beta < 2$ such that $\int_0^4 f(t)dt = 2(\alpha f(\alpha^2) + \beta f(\beta^2))$

7. Let $f(x)=\begin{cases} x^\alpha \ell n x, & x > 0 \\ 0, & x = 0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if α = [JEE 2004, Screening]

(A) –2 (B) –1 (C) 0 (D) 1/2

8. If $p(x) = 51x^{101} - 2323x^{100} - 45x + 1035$, using Rolle's theorem prove that atleast one root of p(x) = 0 lies between. $\left(45^{\frac{1}{100}}, 46\right)$. [JEE 2004 (Mains), 2M out of 60]

9. Prove that $\sin x + 2x \ge \frac{3x\;\;(x+1)}{\pi} \;\; \forall \;\; x \in \left[0, \frac{\pi}{2}\right]$ (Justify the inequality, If any used.)

[JEE 2004 (Mains), 4M out of 60]

10. If $f(x) = x^3 + bx^2 + cx + d,\ 0 < b^2 < c$, then f(x) **[JEE 2004, Screening]**

(A) is strictly increasing (B) has a local maxima (C) has a local minima (D) has bounded area

11. If f(x) is twice differentiable function f(1) = 1, f(2) = 4, f(3) = 9 **[JEE 2005, Screening]**

(A) $f''(x) = 2$, for atleast one in $x \in (1, 3)$ (C) $f''(x) = f'(x) = 5$, for some $x \in (2, 3)$

(C) $f''(x) = 3,\ \forall \in (2, 3)$ (D) $f''(x) = 2$, for some $x \in (1, 2)$

12. If f(x) is a twice differentiable function such that f(a) = 0, f(b) = 2, f(c) = –1, f(d) = 2 , f(e) = 0 where $a < b < c < d < e$ then the minimum number of zeros of $g(x) = (f'(x))^2 + f''(x) f(x)$ in the interval [a, e] is.

[JEE 2006, 6M out of 184]

13. Let f(x) = 2 + cos x for all real x.

Statement-1 : For each real t, there exists a point c in [t, t + π] such that f'(c) = 0 because

Statement-2 : f(t) = f(t + 2π) for each real t. **[JEE 2007, 3M]**

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

Comprehension (Q. 14 to 16)

If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation f(x) = 0 has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation f(x) = 0 has a root in R.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

14. The line y = x meets $y = ke^x$ for $k \le 0$ at :- **[2007, 4M]**

(A) no point (B) one point (C) two point (D) more than two points

15. The positive value of k for which $ke^x - x = 0$ has only one root is :- **[2007, 4M]**

(A) $\frac{1}{e}$ (B) 1 (C) e (D) $\log_e 2$

16. For k > 0, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is :- **[2007, 4M]**

(A) $\left(0,\frac{1}{e}\right)$ (B) $\left(\frac{1}{e},1\right)$ (C) $\left(\frac{1}{e},\infty\right)$ (D) (0, 1)

17. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is -

(A) even and is strictly increasing in (0, ∞) **[JEE 2008, 3M, –1M]**

(B) odd and is strictly decreasing in (–∞, ∞)

(C) odd and is strictly increasing in (–∞, ∞)

(D) neither even nor odd, but is strictly increasing in (–∞, ∞)

18. Let $f(x)$ be a non-constant twice differentiable function defined on (–∞, ∞) such that $f(x) = f(1 - x)$ and $f'\left(\frac{1}{4}\right)=0$. Then, **[JEE 2008, 4M]**

(A) $f''(x)$ vanishes at least twice on [0, 1] (B) $f'\left(\frac{1}{2}\right) = 0$

(C) $\int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\,dx = 0$ (D) $\int_{0}^{1/2} f(t)e^{\sin \pi t}dt = \int_{1/2}^{1} f(1-t)e^{\sin \pi t}dt$

19. For the function $f(x) = x\cos\frac{1}{x},\ x \geq 1$, [JEE 2009, 4M, –1M]

(A) for at least one x in the interval $[1,\infty)$, $f(x+2) - f(x) < 2$

(B) $\lim\limits_{x\to\infty} f'(x) = 1$

(C) for all x in the interval $[1,\infty)$, $f(x+2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1,\infty)$

20. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ell nx + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true? [JEE 10, 3M]

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)|+|f'(x)| \leq \beta$ for all $x \in (0, \infty)$

21. Let $f : (0,1) \to R$ be defined by $f(x) = \frac{b-x}{1-bx}$,

where b is a constant such that $0 < b < 1$. Then

(A) f is not invertible on $(0,1)$

(B) $f \neq f^{-1}$ on $(0,1)$ and $f'(b) = \frac{1}{f'(0)}$

(C) $f = f^{-1}$ on $(0,1)$ and $f'(b) = \frac{1}{f'(0)}$

(D) f^{-1} is differentiable on $(0,1)$

[JEE 2011, 4M]

22. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is [JEE 2011, 4M]

Paragraph for Question 23 and 24

Let $f(x) = (1 - x)^2\sin^2x + x^2$ for all $x \in \mathbb{R}$, and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ell nt\right) f(t)dt$ for all $x \in (1,\infty)$.

23. Consider the statements :

P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$

Then [JEE 2012, 3M, –1M]

(A) both **P** and **Q** are true

(B) **P** is true and **Q** is false

(C) **P** is false and **Q** is true

(D) both **P** and **Q** are false

24. Which of the following is true ? [JEE 2012, 3M, –1M]

(A) g is increasing on $(1,\infty)$

(B) g is decreasing on $(1,\infty)$

(C) g is increasing on (1,2) and decreasing on $(2,\infty)$

(D) g is decreasing on (1,2) and increasing on $(2,\infty)$

25. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0,\infty)$, then - **[JEE 2012, 4M]**

(A) f has a local maximum at x = 2

(B) f is decreasing on (2,3)

(C) there exists some c ∈ (0,∞) such that $f''(c) = 0$

(D) f has a local minimum at x = 3

26. The number of points in (–∞, ∞), for which $x^2 - x\sin x - \cos x = 0$, is **[JEE 2013, 2M]**

(A) 6 (B) 4 (C) 2 (D) 0

27. Let $f(x) = x \sin \pi x$, x > 0. Then for all natural numbers n, $f'(x)$ vanishes at - **[JEE 2013, 4M, –1M]**

(A) a unique point in the interval $\left(n, n+\frac{1}{2}\right)$

(B) a unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

(C) a unique point in the interval (n, n + 1)

(D) two points in the interval (n, n + 1)

PREVIOUS YEARS QUESTIONS	ANSWER KEY	EXERCISE-5 [B]

1. (a) B (b) D **2.** C **3.** A **4.** $\cos\left(\frac{1}{3}\cos^{-1} p\right)$ **5.** A **7.** D **10.** A **11.** A **12.** 6

13. B **14.** B **15.** A **16.** A **17.** C **18.** A,B,C,D **19.** B,C,D **20.** B,C

21. A **22.** 2 **23.** C **24.** B **25.** A,B,C,D **26.** C **27.** B,C