QUADRATIC EQUATION

1. INTRODUCTION :

The algebraic expression of the form $ax^2 + bx + c$, $a \neq 0$ is called a quadratic expression, because the highest order term in it is of second degree. Quadratic equation means, $ax^2 + bx + c = 0$. In general whenever one says zeroes of the expression $ax^2+bx + c$, it implies roots of the equation $ax^2 + bx + c = 0$, unless specified otherwise.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

2. SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS :

(a) The general form of quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

The roots can be found in following manner :

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \Rightarrow \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This expression can be directly used to find the two roots of a quadratic equation.

(b) The expression $b^2 - 4\,ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If $\alpha \,\&\, \beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$, then :

(i) $\alpha + \beta = -\,b/a$ (ii) $\alpha\beta = c/a$ (iii) $|\alpha - \beta| = \sqrt{D}/|a|$

(d) A quadratic equation whose roots are $\alpha \,\&\, \beta$ is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)\,x + \alpha\beta = 0$ i.e. x^2 – (sum of roots) x + product of roots = 0.

Illustration 1 : If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is -

(A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$ (C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Solution : Since α, β are the roots of equation $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\beta^2 - 3\beta + 5 = 0$

$\therefore$ $\alpha^2 - 3\alpha = -5$

$\beta^2 - 3\beta = -5$

Putting in $(\alpha^2 - 3\alpha + 7)$ & $(\beta^2 - 3\beta + 7)$ ....(i)

$-5 + 7, -5 + 7$

$\therefore$ 2 and 2 are the roots.

$\therefore$ The required equation is

$x^2 - 4x + 4 = 0$. **Ans. (B)**

Illustration 2 : If α and β are the roots of $ax^2 + bx + c = 0$, find the value of $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$.

Solution : We know that $\alpha + \beta = -\dfrac{b}{a}$ & $\alpha\beta = \dfrac{c}{a}$

$$(a\alpha + b)^{-2} + (a\beta + b)^{-2} = \frac{1}{(a\alpha + b)^2} + \frac{1}{(a\beta + b)^2}$$

$$= \frac{a^2\beta^2 + b^2 + 2ab\beta + a^2\alpha^2 + b^2 + 2ab\alpha}{(a^2\alpha\beta + ba\beta + ba\alpha + b^2)^2} = \frac{a^2(\alpha^2 + \beta^2) + 2ab(\alpha + \beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha + \beta) + b^2)^2}$$

$(\alpha^2 + \beta^2$ can always be written as $(\alpha + \beta)^2 - 2\alpha\beta)$

$$= \frac{a^2\left[(\alpha+\beta)^2 - 2\alpha\beta\right] + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2} = \frac{a^2\left[\frac{b^2-2ac}{a^2}\right] + 2ab\left(-\frac{b}{a}\right) + 2b^2}{\left(a^2\frac{c}{a} + ab\left(-\frac{b}{a}\right) + b^2\right)^2} = \frac{b^2-2ac}{a^2c^2}$$

Alternatively :

Take $b = -(\alpha+\beta)a$

$$(a\alpha + b)^{-2} + (a\beta + b)^{-2} = \frac{1}{a^2}\left[\frac{1}{(\alpha-\alpha-\beta)^2} + \frac{1}{(\beta-\alpha-\beta)^2}\right]$$

$$= \frac{1}{a^2}\left[\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}\right] = \frac{1}{a^2}\left[\frac{b^2-2ac}{a^2 . \frac{c^2}{a^2}}\right] = \frac{b^2-2ac}{a^2c^2}$$

Do yourself - 1 :

(i) Find the roots of following equations :

(a) $x^2 + 3x + 2 = 0$ (b) $x^2 - 8x + 16 = 0$ (c) $x^2 - 2x - 1 = 0$

(ii) Find the roots of the equation $a(x^2 + 1) - (a^2 + 1)x = 0$, where $a \neq 0$.

(iii) Solve : $\frac{6-x}{x^2-4} = 2 + \frac{x}{x+2}$

(iv) If the roots of $4x^2 + 5k = (5k + 1)x$ differ by unity, then find the values of k.

3. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b , c $\in$ R & $a \neq 0$ then ;

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + i\,q$ is one root of a quadratic equation, then the other root must be the conjugate $p - i\,q$ & vice versa. $(p, q \in R$ & $i = \sqrt{-1})$.

(b) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, c $\in$ Q & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

Illustration 3 : If the coefficient of the quadratic equation are rational & the coefficient of x^2 is 1, then find the equation one of whose roots is $\tan\frac{\pi}{8}$.

Solution : We know that $\tan\frac{\pi}{8} = \sqrt{2} - 1$

Irrational roots always occur in conjugational pairs.

Hence if one root is $(-1 + \sqrt{2})$, the other root will be $(-1 - \sqrt{2})$. Equation is

$(x - (-1+\sqrt{2}))\ (x - (-1-\sqrt{2})) = 0 \qquad \Rightarrow \qquad x^2 + 2x - 1 = 0$

Illustration 4 : Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Solution : Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow \quad D = (a - 10)^2 - 4 \quad \Rightarrow \quad 4 = (a - 10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow \quad (a - 10) = \pm 2 \quad \Rightarrow \quad a = 12, 8$ **Ans.**

Do yourself - 2 :

(i) If $2+\sqrt{3}$ is a root of the equation $x^2 + bx + c = 0$, where $b, c \in Q$, find b, c.

(ii) For the following equations, find the nature of the roots (real & distinct, real & coincident or imaginary).

(a) $x^2 - 6x + 10 = 0$

(b) $x^2-(7+\sqrt{3})x+6(1+\sqrt{3})=0$

(c) $4x^2 + 28x + 49 = 0$

(iii) If ℓ, m are real and $\ell \neq m$, then show that the roots of $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are real and unequal.

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4. ROOTS UNDER PARTICULAR CASES :

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(a) If $b = 0$ $\Rightarrow$ roots are equal in magnitude but opposite in sign

(b) If $c = 0$ $\Rightarrow$ one root is zero other is $-b/a$

(c) If $a = c$ $\Rightarrow$ roots are reciprocal to each other

(d) If $\left.\begin{matrix} a>0 \ c<0 \\ a<0 \ c>0 \end{matrix}\right\}$ $\Rightarrow$ roots are of opposite signs

(e) If $\left.\begin{matrix} a>0, b>0, c>0 \\ a<0, b<0, c<0 \end{matrix}\right\}$ $\Rightarrow$ both roots are negative.

(f) If $\left.\begin{matrix} a>0, b<0, c>0 \\ a<0, b>0, c<0 \end{matrix}\right\}$ $\Rightarrow$ both roots are positive.

(g) If sign of a = sign of b $\neq$ sign of c $\Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of c $\neq$ sign of a $\Rightarrow$ Greater root in magnitude is positive.

(i) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a or $(-b-a)/a$.

Illustration 5 : If equation $\frac{x^2-bx}{ax-c}=\frac{k-1}{k+1}$ has roots equal in magnitude & opposite in sign, then the value of k is -

(A) $\frac{a+b}{a-b}$ (B) $\frac{a-b}{a+b}$ (C) $\frac{a}{b}+1$ (D) $\frac{a}{b}-1$

Solution : Let the roots are α & $-\alpha$.

given equation is

$(x^2 - bx)(k + 1) = (k - 1)(ax - c)$ {Considering, $x \neq c/a$ & $k \neq -1$}

$\Rightarrow \quad x^2(k + 1) - bx(k + 1) = ax (k - 1) - c(k - 1)$

$\Rightarrow \quad x^2(k + 1) - bx(k + 1) - ax (k - 1) + c(k - 1) = 0$

Now sum of roots = 0 $\quad (\because \ \alpha - \alpha = 0)$

$\therefore \quad b(k + 1) + a(k - 1) = 0 \quad \Rightarrow \quad k = \frac{a-b}{a+b}$ **Ans. (B)**

***Illustration 6** : If roots of the equation $(a-b)x^2 + (c-a)x + (b-c) = 0$ are equal, then a, b, c are in

(A) A.P. (B) H.P. (C) G.P. (D) none of these

***Solution** :* $(a-b)x^2 + (c-a)x + (b-c) = 0$

As roots are equal so

$B^2 - 4AC = 0 \Rightarrow (c-a)^2 - 4(a-b)(b-c) = 0 \Rightarrow (c-a)^2 - 4ab + 4b^2 + 4ac - 4bc = 0$

$\Rightarrow (c-a)^2 + 4ac - 4b(c+a) + 4b^2 = 0 \Rightarrow (c+a)^2 - 2 \,.\, (2b)(c+a) + (2b)^2 = 0$

$\Rightarrow [c+a-2b]^2 = 0 \Rightarrow c+a-2b = 0 \Rightarrow c+a = 2b$

Hence a, b, c are in A. P.

Alternative method :

$\because$ Sum of the coefficients = 0

Hence one root is 1 and other root is $\dfrac{b-c}{a-b}$.

Given that both roots are equal, so

$1 = \dfrac{b-c}{a-b} \Rightarrow a-b = b-c \Rightarrow 2b = a+c$

Hence a, b, c are in A.P. **Ans. (A)**

Do yourself - 3 :

(i) Consider $f(x) = x^2 + bx + c$.

(a) Find c if $x = 0$ is a root of $f(x) = 0$.

(b) Find c if $\alpha, \dfrac{1}{\alpha}$ are roots of $f(x) = 0$.

(c) Comment on sign of b & c, if $\alpha < 0 < \beta$ & $|\beta| > |\alpha|$, where α, β are roots of $f(x) = 0$.

5. IDENTITY :

An equation which is true for every value of the variable within the domain is called an identity , for example : $5(a-3) = 5a - 15$, $(a+b)^2 = a^2 + b^2 + 2ab$ for all $a, b \in R$.

Note : A quadratic equation cannot have three or more roots & if it has , it becomes an identity.

If $ax^2 + bx + c = 0$ is an identity $\Leftrightarrow a = b = c = 0$

***Illustration 7** :* If the equation $(\lambda^2 - 5\lambda + 6)x^2 + (\lambda^2 - 3\lambda + 2)x + (\lambda^2 - 4) = 0$ has more than two roots, then find the value of λ ?

***Solution** :* As the equation has more than two roots so it becomes an identity. Hence

$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2, 3$

and $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$

and $\lambda^2 - 4 = 0 \Rightarrow \lambda = 2, -2$

So $\lambda = 2$ **Ans. $\lambda = 2$**

6. COMMON ROOTS OF TWO QUADRATIC EQUATIONS :

(a) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then

$a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\dfrac{\alpha^2}{bc' - b'c} = \dfrac{\alpha}{a'c - ac'} = \dfrac{1}{ab' - a'b}$

Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b} = \dfrac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

(b) If both roots are same then $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$.

Illustration 8 : Find p and q such that $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots in common.

Solution : $a_1 = p,\ b_1 = 5,\ c_1 = 2$

$a_2 = 3,\ b_2 = 10,\ c_2 = q$

We know that :

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2} \Rightarrow \frac{p}{3}=\frac{5}{10}=\frac{2}{q} \qquad \Rightarrow \qquad p=\frac{3}{2}\ ;\ q=4$$

****Illustration 9 :*** The equations $5x^2 + 12x + 13 = 0$ and $ax^2 + bx + c = 0$ $(a,b,c \in R)$ have a common root, where a, b, c are the sides of the Δ ABC. Then find $\angle C$.

(A) 45 (B) 60 (C) 90 (D) 30

Solution : As we can see discriminant of the equation $5x^2 + 12x + 13 = 0$ is negative so roots of the equation are imaginary. We know that imaginary roots always occurs in pair. So this equation can not have single common roots with any other equation having real coefficients. So both roots are common of the given equations.

Hence $\frac{a}{5}=\frac{b}{12}=\frac{c}{13}=\lambda(\text{let})$

then $a = 5\lambda,\ b = 12\lambda\ ,\ c = 13\lambda$

$$\text{Now } \cos C = \frac{a^2+b^2-c^2}{2ab}=\frac{25\lambda^2+144\lambda^2-169\lambda^2}{2(5\lambda)(12\lambda)}=0$$

$\therefore \quad \angle C = 90$ **Ans. (C)**

Do yourself - 4 :

(i) If $x^2 + bx + c = 0$ & $2x^2 + 9x + 10 = 0$ have both roots, find b & c.

(ii) If $x^2 - 7x + 10 = 0$ & $x^2 - 5x + c = 0$ have a common root, find c.

(iii) Show that $x^2 + (a^2 - 2)x - 2a^2 = 0$ and $x^2 - 3x + 2 = 0$ have exactly one common root for all $a \in R$.

7. REMAINDER THEOREM :

If we divide a polynomial f(x) by $(x - \alpha)$ the remainder obtained is $f(\alpha)$. If $f(\alpha)$ is 0 then $(x - \alpha)$ is a factor of f(x).

Consider $f(x) = x^3 - 9x^2 + 23x - 15$

$f(1) = 0 \quad \Rightarrow \quad (x - 1)$ is a factor of f(x).

$f(x) = (x - 2)(x^2 - 7x + 9) + 3$. Hence $f(2) = 3$ is remainder when f(x) is divided by $(x - 2)$.

8. SOLUTION OF RATIONAL INEQUALITIES :

Let $y=\frac{f(x)}{g(x)}$ be an expression in x where f(x) & g(x) are polynomials in x. Now, if it is given that $y > 0$ (or < 0 or ≥ 0 or ≤ 0), this calls for all the values of x for which y satisfies the constraint. This solution set can be found by following steps :

Step I : Factorize f(x) & g(x) and generate the form :

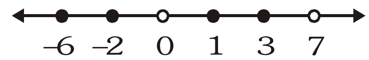
$$y=\frac{(x-a_1)^{n_1}(x-a_2)^{n_2}\ldots\ldots..(x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots\ldots.(x-b_p)^{m_p}}$$

where $n_1 n_2 \ldots\ldots\ldots n_k$, $m_1, m_2 \ldots\ldots. m_p$ are natural numbers and $a_1, a_2 \ldots\ldots. a_k$, $b_1, b_2 \ldots\ldots b_p$ are real numbers. Clearly, here $a_1, a_2 \ldots\ldots a_k$ are roots of f(x)=0 & $b_1, b_2, \ldots\ldots b_p$ are roots of g(x) = 0.

Step II : Here y vanishes (becomes zero) for $a_1, a_2, \ldots\ldots\ldots a_k$. These points are marked on the number line with a black dot. They are solution of y=0.

If g(x)=0, $y = \dfrac{f(x)}{g(x)}$ attains an undefined form, hence $b_1, b_2 \ldots\ldots b_k$ are excluded from the solution. These points are marked with white dots.

e.g. $f(x) = \dfrac{(x-1)^3 (x+2)^4 (x-3)^5 (x+6)}{x^2 (x-7)^3}$



Step-III : Check the value of y for any real number greater than the right most marked number on the number line. If it is positive, then y is positive for all the real numbers greater than the right most marked number and vice versa.

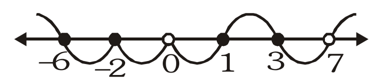
Step-IV : If the exponent of a factor is odd, then the point is called simple point and if the exponent of a factor is even, then the point is called double point

$$\frac{(x-1)^3 (x+2)^4 (x-3)^5 (x+6)}{x^2 (x-7)^3}$$

Here 1,3,–6 and 7 are simple points and –2 & 0 are double points.

From right to left, beginning above the number line (if y is positive in step 3 otherwise from below the line), a wavy curve should be drawn which passes through all the marked points so that when passing through a simple point, the curve intersects the number line and when passing through a double point, the curve remains on the same side of number line.

$f(x) = \dfrac{(x-1)^3 (x+2)^4 (x-3)^5 (x+6)}{x^2 (x-7)^3}$



As exponents of (x + 2) and x are even, the curve does not cross the number line. This method is called wavy curve method.

Step-V : The intervals where the curve is above number line, y will be positive and the intervals where the curve is below the number line, y will be negative. The appropriate intervals are chosen in accordance with the sign of inequality & their union represents the solution of inequality.

Note :

(i) Points where denominator is zero will never be included in the answer.

(ii) If you are asked to find the intervals where f(x) is non-negative or non-positive then make the intervals closed corresponding to the roots of the numerator and let it remain open corresponding to the roots of denominator.

(iii) Normally we cannot cross-multiply in inequalities. But we cross multiply if we are sure that quantity in denominator is always positive.

(iv) Normally we cannot square in inequalities. But we can square if we are sure that both sides are non negative.

(v) We can multiply both sides with a negative number by changing the sign of inequality.

(vi) We can add or subtract equal quantity to both sides of inequalities without changing the sign of inequality.

Illustration 10 : Find x such that $3x^2 - 7x + 6 < 0$

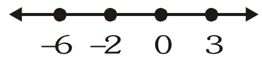
Solution : $D = 49 - 72 < 0$

As $D < 0$, $3x^2 - 7x + 6$ will always be positive. Hence $x \in \phi$.

Illustration 11 : $(x^2 - x - 6)(x^2 + 6x) \geq 0$

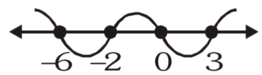
Solution : $(x-3)(x+2)(x)(x+6) \geq 0$

Consider $E = x(x-3)(x+2)(x+6)$, $E = 0 \Rightarrow x = 0, 3, -2, -6$ (all are simple points)



For $x \geq 3$ $\quad E = \underbrace{x}_{+ve}\ \underbrace{(x-3)}_{+ve}\ \underbrace{(x+2)}_{+ve}\ \underbrace{(x+6)}_{+ve}$

$=$ positive



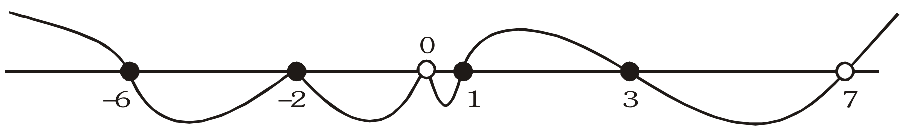
Hence for $x(x-3)(x+2)(x+6) \geq 0$

$x \in (-\infty,-6] \cup [-2,0] \cup [3,\infty)$

Illustration 12 : Let $f(x) = \dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$. Solve the following inequality

(i) $f(x) > 0$ (ii) $f(x) \geq 0$ (iii) $f(x) < 0$ (iv) $f(x) \leq 0$

Solution : We mark on the number line zeros of the function : 1, –2, 3 and –6 (with black circles) and the points of discontinuity 0 and 7 (with white circles), isolate the double points : –2 and 0 and draw the wavy curve :



From graph, we get

(i) $x \in (-\infty, -6) \cup (1, 3) \cup (7, \infty)$

(ii) $x \in (-\infty, -6] \cup \{-2\} \cup [1, 3] \cup (7, \infty)$

(iii) $x \in (-6, -2) \cup (-2, 0) \cup (0, 1) \cup (3, 7)$

(iv) $x \in [-6, 0) \cup (0, 1] \cup [3, 7)$

Do yourself - 5 :

(i) Find range of x such that

(a) $(x - 2)(x + 3) \geq 0$ (b) $\dfrac{x}{x+1} > 2$

(c) $\dfrac{3x-1}{4x+1} \leq 0$ (d) $\dfrac{(2x-1)(x+3)(2-x)(1-x)^2}{x^4(x+6)(x-9)(2x^2+4x+9)} < 0$

(e) $\dfrac{7x-17}{x^2-3x+4} \geq 1$ (f) $x^2 + 2 \leq 3x \leq 2x^2 - 5$

9. QUADRATIC EXPRESSION AND IT'S GRAPHS :

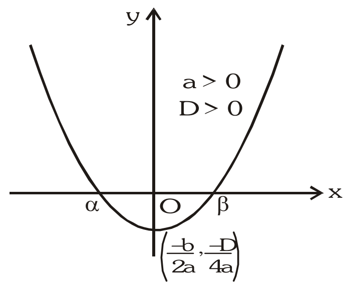
Consider the quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ & $a, b, c \in R$ then ;

(a) The graph between x, y is always a parabola. If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.

(b) The graph of $y = ax^2 + bx + c$ can be divided in 6 broad categories which are as follows :

(Let the real roots of quadratic equation $ax^2 + bx + c = 0$ be α & β where $\alpha \leq \beta$).

Fig. 1

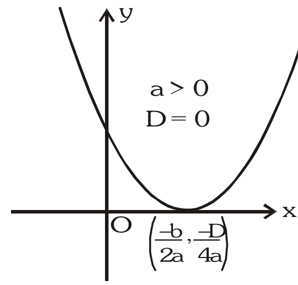


Roots are real & distinct

$ax^2 + bx + c > 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

$ax^2 + bx + c < 0 \ \forall \ x \in (\alpha, \beta)$

Fig. 2

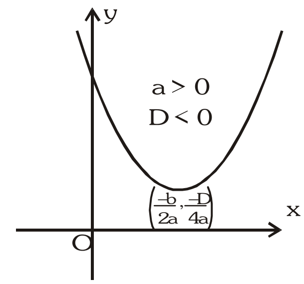


Roots are coincident

$ax^2 + bx + c > 0 \ \forall \ x \in R - \{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

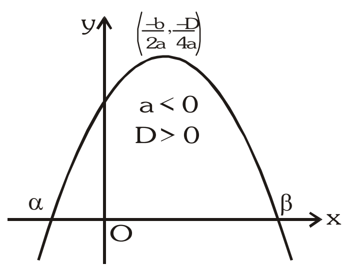
Fig.3



Roots are complex conjugate

$ax^2 + bx + c > 0 \ \forall \ x \in R$

Fig. 4

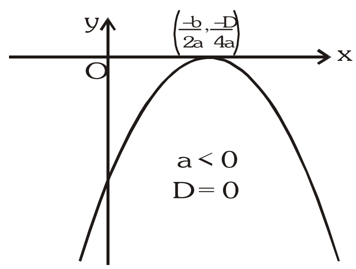


Roots are real & distinct

$ax^2 + bx + c > 0 \ \forall \ x \in (\alpha, \beta)$

$ax^2 + bx + c < 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

Fig. 5

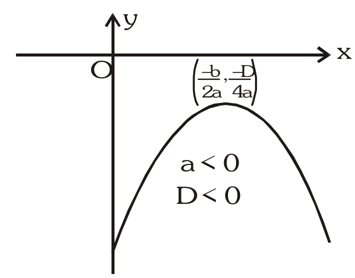


Roots are coincident

$ax^2 + bx + c < 0 \ \forall \ x \in R - \{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

Fig.6



Roots are complex conjugate

$ax^2 + bx + c < 0 \ \forall \ x \in R$

Important Note :

(i) The quadratic expression $ax^2 + bx + c > 0$ for each $x \in R \Rightarrow a > 0, D < 0$ & vice-versa (Fig. 3)

(ii) The quadratic expression $ax^2 + bx + c < 0$ for each $x \in R \Rightarrow a < 0, D < 0$ & vice-versa (Fig. 6)

10. MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSIONS : $y = ax^2 + bx + c$:

We know that $y = ax^2 + bx + c$ takes following form : $y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a^2}\right]$,

which is a parabola. $\therefore$ vertex $= \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

When $a > 0$, y will take a minimum value at vertex; $x = \frac{-b}{2a}$; $y_{min} = \frac{-D}{4a}$

When $a < 0$, y will take a maximum value at vertex; $x = \frac{-b}{2a}$; $y_{max} = \frac{-D}{4a}$.

If quadratic expression ax^2+bx+c is a perfect square, then $a > 0$ and $D = 0$

Illustration13 : If f(x) is a quadratic expression such that $f(x) > 0 \ \forall \ x \in R$, and if $g(x) = f(x) + f'(x) + f''(x)$, then prove that $g(x) > 0 \ \forall \ x \in R$.

Solution : Let $f(x) = ax^2 + bx + c$

Given that $f(x) > 0$ so $a > 0$, $b^2 - 4ac < 0$

Now $g(x) = ax^2 + bx + c + 2ax + b + 2a \ = ax^2 + (b + 2a)x + (b + c + 2a)$

For this quadratic expression $a > 0$ and discriminant

$D = (b + 2a)^2 - 4a(b + c + 2a) = b^2 + 4a^2 + 4ab - 4ab - 4ac - 8a^2 \ = b^2 - 4ac - 4a^2 < 0$

So $g(x) > 0 \ \forall \ x \in R$.

Illustration 14 : The value of the expression $x^2+2bx+c$ will be positive for all real x if -

(A) $b^2-4c>0$ (B) $b^2-4c<0$ (C) $c^2<b$ (D) $b^2<c$

Solution : As a > 0, so this expression will be positive if D < 0

so $4b^2 - 4c < 0$

$b^2 < c$ **Ans. (D)**

Illustration 15 : The minimum value of the expression $4x^2 + 2x + 1$ is -

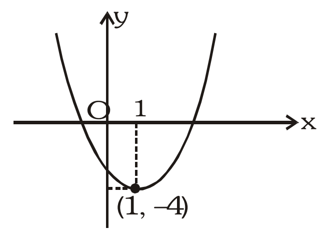
(A) 1/4 (B) 1/2 (C) 3/4 (D) 1

Solution : Since a = 4 > 0 therefore its minimum value is $= \dfrac{4(4)(1)-(2)^2}{4(4)} = \dfrac{16-4}{16} = \dfrac{12}{16} = \dfrac{3}{4}$ **Ans. (C)**

**Illustration16* : If $y = x^2 - 2x - 3$, then find the range of y when :

(i) $x \in R$ (ii) $x \in [0,3]$ (iii) $x \in [-2,0]$

Solution : We know that minimum value of y will occur at



$$x=-\frac{b}{2a}=-\frac{(-2)}{2\times 1}=1$$

$$y_{min}= -\frac{D}{4a}=\frac{-(4+3\times 4)}{4} = -4$$

(i) $x \in R$;

$y \in [-4,\infty)$ **Ans.**

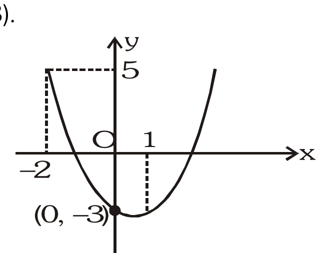
(ii) $x \in [0, 3]$

$f(0) = -3,\ f(1) = -4, f(3) = 0$

$\because$ $f(3) > f(0)$

$\therefore$ y will take all the values from minimum to f(3).

$y \in [-4, 0]$ **Ans.**



(iii) $x \in [-2, 0]$

This interval does not contain the minimum value of y for $x \in R$.

y will take values from f(0) to f(–2)

$f(0) =-3$

$f(-2) = 5$

$y \in [-3, 5]$ **Ans.**

Illustration 17 : If $ax^2 + bx + 10 = 0$ does not have real & distinct roots, find the minimum value of 5a–b.

Solution : Either $f(x) \ge 0\ \forall\ x \in R$ or $f(x) \le 0\ \forall\ x \in R$

$\because$ $f(0) = 10 > 0 \Rightarrow f(x) \ge 0\ \forall\ x \in R$

$\Rightarrow$ $f(-5) = 25a - 5b + 10 \ge 0$

$\Rightarrow$ $5a - b \ge -2$ **Ans.**

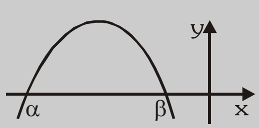
Do yourself - 6

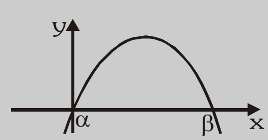
(i) Find the minimum value of :

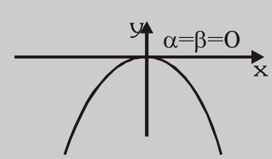
(a) $y = x^2 + 2x + 2$ (b) $y = 4x^2 - 16x + 15$

(ii) For following graphs of $y = ax^2 + bx + c$ with $a,b,c \in R$, comment on the sign of :

(i) a (ii) b (iii) c (iv) D (v) $\alpha + \beta$ (vi) $\alpha\beta$

(1) 

(2) 

(3) 

(iii) Given the roots of equation $ax^2 + bx + c = 0$ are real & distinct, where $a,b,c \in R^+$, then the vertex of the graph will lie in which quadrant.

***(iv)** Find the range of 'a' for which :

(a) $ax^2 + 3x + 4 > 0\quad \forall\ x \in R$ (b) $ax^2 + 4x - 2 < 0\quad \forall\ x \in R$

11. INEQUALITIES INVOLVING MODULUS FUNCTION :

Properties of modulus function :

(i) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$, where a is positive.

(ii) $|x| \leq a \Rightarrow x \in [-a, a]$, where a is positive

(iii) $|x| > |y| \Rightarrow x^2 > y^2$

(iv) $\left||a|-|b|\right| \leq |a \pm b| \leq |a|+|b|$

(v) $|x + y| = |x| + |y| \Rightarrow xy \geq 0$

(vi) $|x - y| = |x| + |y| \Rightarrow xy \leq 0$

Illustration 18 : If x satisfies $|x - 1| + |x - 2| + |x - 3| \geq 6$, then

(A) $0 \leq x \leq 4$ (B) $x \leq -2$ or $x \geq 4$

(C) $x \leq 0$ or $x \geq 4$ (D) none of these

Solution : **Case I :** $x \leq 1$, then

$1 - x + 2 - x + 3 - x \geq 6 \Rightarrow x \leq 0$

Hence $x \leq 0$ ...(i)

Case II : $1 < x \leq 2$, then

$x - 1 + 2 - x + 3 - x \geq 6 \Rightarrow x \leq -2$

But $1 < x \leq 2 \Rightarrow$ No solution. ...(ii)

Case III : $2 < x \leq 3$, then

$x - 1 + x - 2 + 3 - x \geq 6 \Rightarrow x \geq 6$

But $2 < x \leq 3 \Rightarrow$ No solution. ...(iii)

Case IV : $x > 3$, then

$x - 1 + x - 2 + x - 3 \geq 6 \Rightarrow x \geq 4$

Hence $x \geq 4$ ...(iv)

From (i), (ii), (iii) and (iv) the given inequality holds for $x \leq 0$ or $x \geq 4$.

Illustration 19 : Solve for x : (a) $||x - 1| + 2| \leq 4$. (b) $\frac{x-4}{x+2} \leq \left|\frac{x-2}{x-1}\right|$

Solution : (a) $||x - 1| + 2| \leq 4 \Rightarrow -4 \leq |x - 1| + 2 \leq 4$

$\Rightarrow -6 \leq |x - 1| \leq 2$

$\Rightarrow |x - 1| \leq 2 \Rightarrow -2 \leq x - 1 \leq 2$

$\Rightarrow -1 \leq x \leq 3 \Rightarrow x \in [-1, 3]$

(b) **Case 1 :** Given inequation will be statisfied for all x such that

$\frac{x-4}{x+2} \leq 0 \Rightarrow x \in (-2, 4] - \{1\}$ ....(i)

(Note : {1} is not in domain of RHS)

Case 2 : $\frac{x-4}{x+2} > 0 \Rightarrow x \in (-\infty, -2) \cup (4, \infty)$ ....(ii)

Given inequation becomes

$\frac{x-2}{x-1} \geq \frac{x-4}{x+2}$ or $\frac{x-2}{x-1} \leq -\frac{x-4}{x+2}$

on solving we get $x \in \left(-2, 4/5\right) \cup (1, \infty)$; on solving we get $x \in (-2, 0] \cup (1, 5/2]$

taking intersection with (ii) we get $x \in (4, \infty)$ ....(iii); taking intersection with (ii) we get $x \in \phi$

Hence, solution of the original inequation : $x \in (-2,\infty) - \{1\}$ (taking union of (i) & (iii))

Illustration 20 : The equation $|x|+\left|\frac{x}{x-1}\right|=\frac{x^2}{|x-1|}$ is always true for x belongs to

(A) $\{0\}$ (B) $(1, \infty)$ (C) $(-1, 1)$ (D) $(-\infty, \infty)$

Solution : $\frac{x^2}{|x-1|}=\left|x+\frac{x}{x-1}\right|$

$\because$ $|x|+\left|\frac{x}{x-1}\right|=\left|x+\frac{x}{x-1}\right|$ is true only if $\left(x.\frac{x}{x-1}\right)\geq 0 \Rightarrow x \in \{0\} \cup (1, \infty)$. **Ans (A, B)**

12. IRRATIONAL INEQUALITIES :

Illustration 21 : Solve for x, if $\sqrt{x^2-3x+2} > x-2$

Solution :

$$\left[\begin{array}{l}\left\{\begin{array}{l}x^2-3x+2\geq 0\\ x-2\geq 0\\ (x^2-3x+2)>(x-2)^2\end{array}\right.\\ \quad\text{or}\\ \left\{\begin{array}{l}x^2-3x+2\geq 0\\ x-2<0\end{array}\right.\end{array}\right. \Rightarrow \left[\begin{array}{l}\left\{\begin{array}{l}(x-1)(x-2)\geq 0\\ (x-2)\geq 0\\ x-2>0\end{array}\right. \Rightarrow x>2\\ \quad\text{or}\\ \left\{\begin{array}{l}(x-1)(x-2)\geq 0\\ x-2<0\end{array}\right. \Rightarrow x\leq 1\end{array}\right.$$

Hence, solution set of the original inequation is $x \in R - (1,2]$

Do yourself - 7 :

(i) Solve for x if $\frac{|x^2-4|}{x^2+x-2}>1$

(ii) Solve for x if $\sqrt{x^2-x}>(x-1)$

13. LOGARITHMIC INEQUALITIES :

Points to remember :

(i) $\log_a x < \log_a y \Leftrightarrow \left[\begin{array}{lll}x<y & \text{if} & a>1\\ x>y & \text{if} & 0<a<1\end{array}\right.$

(ii) If $a > 1$, then (a) $\log_a x < p \Rightarrow 0 < x < a^p$ (b) $\log_a x > p \Rightarrow x > a^p$

(iii) If $0 < a < 1$, then (a) $\log_a x < p \Rightarrow x > a^p$ (b) $\log_a x > p \Rightarrow 0 < x < a^p$

Illustration 22 : Solve for x : (a) $\log_{0.5}(x^2 - 5x + 6) \geq -1$ (b) $\log_{1/3}(\log_4(x^2 - 5)) > 0$

Solution : (a) $\log_{0.5}(x^2 - 5x + 6) \geq -1 \Rightarrow 0 < x^2 - 5x + 6 \leq (0.5)^{-1}$

$\Rightarrow 0 < x^2 - 5x + 6 \leq 2$

$$\left\{\begin{array}{l}x^2-5x+6>0\\ x^2-5x+6\leq 2\end{array}\right. \Rightarrow x\in[1,2)\cup(3,4]$$

Hence, solution set of original inequation : $x \in [1,2) \cup (3,4]$

(b) $\log_{1/3}(\log_4(x^2-5)) > 0 \quad\Rightarrow\quad 0 < \log_4(x^2-5) < 1$

$$\begin{cases} 0 < \log_4(x^2-5) \Rightarrow x^2-5 > 1 \\ \log_4(x^2-5) < 1 \Rightarrow 0 < x^2-5 < 4 \end{cases} \quad\Rightarrow\quad 1 < (x^2-5) < 4$$

$$\Rightarrow \quad 6 < x^2 < 9 \quad\Rightarrow\quad x \in \left(-3,-\sqrt{6}\right)\cup\left(\sqrt{6},3\right)$$

Hence, solution set of original inequation : $x \in \left(-3,-\sqrt{6}\right)\cup\left(\sqrt{6},3\right)$

Illustration 23 : Solve for x : $\log_2 x \le \dfrac{2}{\log_2 x - 1}$.

Solution : Let $\log_2 x = t$

$$t \le \frac{2}{t-1} \Rightarrow t - \frac{2}{t-1} \le 0$$

$$\Rightarrow \quad \frac{t^2-t-2}{t-1} \le 0 \Rightarrow \frac{(t-2)(t+1)}{(t-1)} \le 0$$

$\Rightarrow \quad t \in (-\infty, -1] \cup (1,2]$

or $\quad \log_2 x \in (-\infty,-1] \cup (1,2]$

or $\quad x \in \left(0,\frac{1}{2}\right] \cup (2,4]$

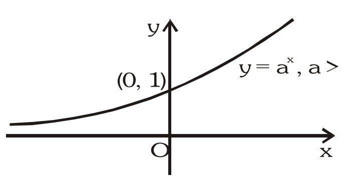
Illustration 24 : Solve the inequation : $\log_{2x+3} x^2 < \log_{2x+3}(2x+3)$

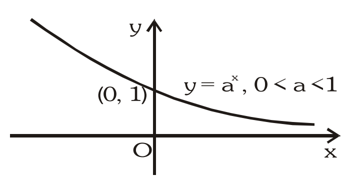
Solution : This inequation is equivalent to the collection of the systems

$$\left[\begin{array}{l} \begin{cases} 2x+3>1 \\ 0<x^2<2x+3 \end{cases} \\ \text{or} \\ \begin{cases} 0<2x+3<1 \\ x^2>2x+3>0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x>-1 \\ (x-3)(x+1)<0 \ \ \& \ \ x \ne 0 \end{cases} \\ \text{or} \\ \begin{cases} -\frac{3}{2}<x<-1 \\ (x-3)(x+1)>0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x>-1 \\ -1<x<3 \end{cases} \Rightarrow -1<x<3 \ \ \& \ \ x\ne 0 \\ \text{or} \qquad\qquad\qquad\qquad \text{or} \\ \begin{cases} -\frac{3}{2}<x<-1 \\ x<-1 \text{ or } x>3 \end{cases} \Rightarrow -\frac{3}{2}<x<-1 \end{array}\right.$$

Hence, solution of the original inequation is $x \in \left(-\frac{3}{2}, -1\right) \cup (-1,\ 0) \cup (0,\ 3)$

14. EXPONENTIAL INEQUATIONS :





If $a^{f(x)} > b \Rightarrow \left[\begin{array}{ll} f(x) > \log_a b & \text{when } a>1 \\ f(x) < \log_a b & \text{when } 0<a<1 \end{array}\right.$

Illustration 25 : Solve for x : $2^{x+2} > \left(\dfrac{1}{4}\right)^{\frac{1}{x}}$

Solution : We have $2^{x+2} > 2^{-2/x}$. Since the base $2 > 1$, we have $x + 2 > -\frac{2}{x}$ (the sign of the inequality is retained).

Now $x+2+\frac{2}{x}>0 \quad \Rightarrow \quad \frac{x^2+2x+2}{x}>0$

$\Rightarrow \quad \frac{(x+1)^2+1}{x}>0 \quad \Rightarrow \quad x \in (0, \infty)$

Illustration 26 : Solve for x : $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$

Solution : We have $\left(\frac{5}{4}\right)^{1-x} < \left(\frac{16}{25}\right)^{2(1+\sqrt{x})}$ or $\left(\frac{4}{5}\right)^{x-1} < \left(\frac{4}{5}\right)^{4(1+\sqrt{x})}$

Since the base $0<\frac{4}{5}<1$, the inequality is equivalent to the inequality $x - 1 > 4(1+\sqrt{x})$

$\Rightarrow \quad \frac{x-5}{4} > \sqrt{x}$

Now, R.H.S. is positive

$\Rightarrow \quad \frac{x-5}{4} > 0 \quad \Rightarrow \quad x > 5$ ....(i)

we have $\frac{x-5}{4} > \sqrt{x}$

both sides are positive, so squaring both sides

$\Rightarrow \quad \frac{(x-5)^2}{16} > x$ or $\frac{(x-5)^2}{16} - x > 0$

or $x^2 - 26x + 25 > 0$ or $(x - 25)(x - 1) > 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (25, \infty)$ ....(ii)

intersection (i) & (ii) gives $x \in (25, \infty)$

Do yourself-8 :

(i) Solve for x : (a) $\log_{0.3}(x^2 + 8) > \log_{0.3}(9x)$ (b) $\log_7\left(\frac{2x-6}{2x-1}\right) > 0$

(ii) Solve for x : $\left(\frac{2}{3}\right)^{\frac{|x|-1}{|x|+1}} > 1$

15. MAXIMUM & MINIMUM VALUES OF RATIONAL ALGEBRAIC EXPRESSIONS :

$$y = \frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}, \ \frac{1}{ax^2+bx+c}, \ \frac{a_1x+b_1}{a_2x^2+b_2x+c_2}, \ \frac{a_1x^2+b_1x+c_1}{a_2x+b_2} :$$

Sometime we have to find range of expression of form $\frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$. The following procedure is used :

Step 1 : Equate the given expression to y i.e. $y = \frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$

Step 2 : By cross multiplying and simplifying, obtain a quadratic equation in x.
$(a_1 - a_2y)x^2 + (b_1 - b_2y)x + (c_1 - c_2y) = 0$

Step 3 : Put Discriminant ≥ 0 and solve the inequality for possible set of values of y.

Illustration 27 : For $x \in R$, find the set of values attainable by $\frac{x^2-3x+4}{x^2+3x+4}$.

Solution : Let $y = \frac{x^2-3x+4}{x^2+3x+4}$

$x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0$

Case- I : $y \neq 1$

For $y \neq 1$ above equation is a quadratic equation.

So for $x \in R$, $D \geq 0$

$\Rightarrow \quad 9(y + 1)^2 - 16(y - 1)^2 \geq 0 \quad \Rightarrow \quad 7y^2 - 50y + 7 \leq 0$

$\Rightarrow \quad (7y - 1)(y - 7) \leq 0 \quad \Rightarrow \quad y \in \left[\frac{1}{7}, 7\right] - \{1\}$

Case II : when $y = 1$

$\Rightarrow \quad 1 = \frac{x^2-3x+4}{x^2+3x+4}$

$\Rightarrow \quad x^2 + 3x + 4 = x^2 - 3x + 4$

$\Rightarrow \quad x = 0$

Hence $y = 1$ for real value of x.

so range of y is $\left[\frac{1}{7}, 7\right]$

Illustration 28 : Find the values of a for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.

Solution : Let $y = \frac{ax^2+3x-4}{3x-4x^2+a}$

$x^2(a + 4y) + 3(1 - y)x - (4 + ay) = 0$

If $x \in R$, $D \geq 0$

$\Rightarrow \quad 9(1 - y)^2 + 4(a + 4y)(4 + ay) \geq 0 \quad \Rightarrow \quad (9 + 16a)y^2 + (4a^2 + 46)y + (9 + 16a) \geq 0$

for all $y \in R$, $(9 + 16a) > 0$ & $D \leq 0$

$\Rightarrow \quad (4a^2 + 46)^2 - 4(9 + 16a)(9 + 16a) \leq 0 \Rightarrow \quad 4(a^2 - 8a + 7)(a^2 + 8a + 16) \leq 0$

$\Rightarrow \quad a^2 - 8a + 7 \leq 0 \quad \Rightarrow \quad 1 \leq a \leq 7$

$9 + 16a > 0$ & $1 \leq a \leq 7$

Taking intersection, $a \in [1, 7]$

Now, checking the boundary values of a

For $a = 1$

$$y = \frac{x^2+3x-4}{3x-4x^2+1} = -\frac{(x-1)(x+4)}{(x-1)(4x+1)}$$

$\because \quad x \neq 1 \Rightarrow y \neq -1$

$\Rightarrow \quad a = 1$ is not possible.

if $\quad a = 7$

$$y = \frac{7x^2+3x-4}{3x-4x^2+7} = \frac{(7x-4)(x+1)}{(7-4x)(x+1)} \qquad \because x \neq -1 \Rightarrow y \neq -1$$

So y will assume all real values for some real values of x.

So $\quad a \in (1,7)$

Do yourself - 9 :

(i) Prove that the expression $\frac{8x-4}{x^2+2x-1}$ cannot have values between 2 and 4, in its domain.

(ii) Find the range of $\frac{x^2+2x+1}{x^2+2x+7}$, where x is real

16. LOCATION OF ROOTS :

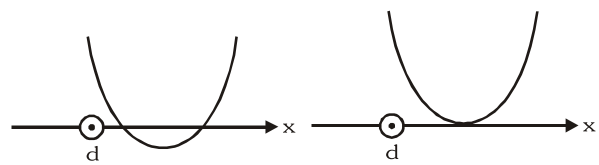
This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :

Consider the quadratic equation $ax^2 + bx + c = 0$ with $a > 0$ and let $f(x) = ax^2 + bx + c$

Type-1 : Both roots of the quadratic equation are greater than a specific number (say d).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\frac{b}{2a} > d$



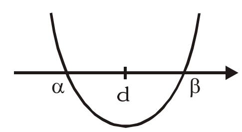
Note : When both roots of the quadratic equation are less than a specific number d than the necessary and sufficient condition will be :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\frac{b}{2a} < d$

Type-2 :

Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than 'd' and other root less than 'd' or 'd' lies between the roots of the given equation.

The necessary and sufficient condition for this are : $f(d) < 0$



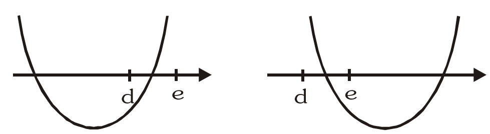
Note : Consideration of discriminant is not needed.

Type-3 :

Exactly one root lies in the interval (d, e).

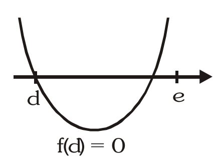
The necessary and sufficient condition for this are :

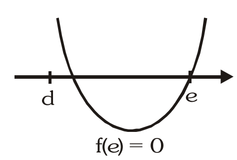
$f(d) \cdot f(e) < 0$



Note : The extremes of the intervals found by given conditions give 'd' or 'e' as the root of the equation.

Hence in this case also check for end points.





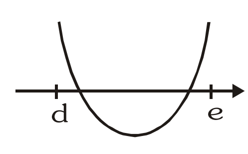
Type-4 :

When both roots are confined between the number d and e (d < e).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $f(e) > 0$

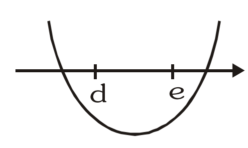
(iv) $d < -\frac{b}{2a} < e$



Type-5 :

One root is greater than e and the other roots is less than d (d < e).

The necessary and sufficient condition for this are : $f(d) < 0$ and $f(e) < 0$



Note : If $a < 0$ in the quadratic equation $ax^2 + bx + c = 0$ then we divide the whole equation by 'a'. Now assume $x^2 + \frac{b}{a}x + \frac{c}{a}$ as f(x). This makes the coefficient of x^2 positive and hence above cases are applicable.

Illustration 29 : Find the values of the parameter 'a' for which the roots of the quadratic equation $x^2 + 2(a - 1)x + a + 5 = 0$ are

(i) real and distinct
(ii) equal
(iii) opposite in sign
(iv) equal in magnitude but opposite in sign
(v) positive
(vi) negative
(vii) greater than 3
(viii) smaller than 3
(ix) such that both the roots lie in the interval (1, 3)

Solution : Let $f(x) = x^2 + 2(a - 1)x + a + 5 = Ax^2 + Bx + C$ (say)

$\Rightarrow$ $A = 1$, $B = 2(a - 1)$, $C = a + 5$.

Also $D = B^2 - 4AC = 4(a - 1)^2 - 4(a + 5) = 4(a + 1)(a - 4)$

(i) $D > 0$

$\Rightarrow$ $(a + 1)(a - 4) > 0$ $\Rightarrow$ $a \in (-\infty, -1)\cup(4, \infty)$.

(ii) $D = 0$

$\Rightarrow$ $(a + 1)(a - 4) = 0$ $\Rightarrow$ $a = -1, 4$.

(iii) This means that 0 lies between the roots of the given equation.

$\Rightarrow$ $f(0) < 0$ and $D > 0$ i.e. $a \in (-\infty, -1) \cup (4, \infty)$

$\Rightarrow$ $a + 5 < 0 \Rightarrow a < -5$ $\Rightarrow$ $a \in (-\infty, -5)$.

(iv) This means that the sum of the roots is zero

$\Rightarrow$ $-2(a - 1) = 0$ and $D > 0$ i.e. $a \in -(-\infty, -1) \cup (4, \infty)$ $\Rightarrow$ $a = 1$

which does not belong to $(-\infty, -1)\cup(4, \infty)$

$\Rightarrow$ $a \in \phi$.

(v) This implies that both the roots are greater than zero

$\Rightarrow$ $-\dfrac{B}{A} > 0,\ \dfrac{C}{A} > 0, D \geq 0.$ $\Rightarrow$ $-(a - 1) > 0$, $a + 5 > 0$, $a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow$ $a < 1$, $-5 < a$, $a \in (-\infty, -1]\cup[4, \infty)$ $\Rightarrow$ $a \in (-5, -1]$.

(vi) This implies that both the roots are less than zero

$\Rightarrow$ $-\dfrac{B}{A} < 0,\ \dfrac{C}{A} > 0, D \geq 0.$ $\Rightarrow$ $-(a - 1) < 0$, $a + 5 > 0$, $a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow$ $a > 1$, $a > -5$, $a \in (-\infty, -1]\cup[4, \infty)$ $\Rightarrow$ $a \in [4, \infty)$.

(vii) In this case

$-\dfrac{B}{2a} > 3$, A.f(3) > 0 and $D \geq 0$.

$\Rightarrow$ $-(a - 1) > 3$, $7a + 8 > 0$ and $a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow$ $a < -2$, $a > -8/7$ and $a \in (-\infty, -1]\cup[4, \infty)$

Since no value of 'a' can satisfy these conditions simultaneously, there can be no value of a for which both the roots will be greater than 3.

(viii) In this case

$-\dfrac{B}{2a} < 3$, A.f(3) > 0 and $D \geq 0$.

$\Rightarrow$ $a > -2$, $a > -8/7$ and $a \in (-\infty, -1]\cup[4, \infty)$ $\Rightarrow$ $a \in (-8/7, -1]\cup[4, \infty)$

(ix) In this case

$1 < -\dfrac{B}{2A} < 3$, A.f(1) > 0, A.f(3) > 0, $D \geq 0$.

$\Rightarrow$ $1 < -1(a - 1) < 3$, $3a + 4 > 0$, $7a + 8 > 0$, $a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow$ $-2 < a < 0$, $a > -4/3$, $a > -8/7$, $a \in (-\infty, -1]\cup[4, \infty)$ $\Rightarrow$ $a \in \left(-\dfrac{8}{7}, -1\right]$

Illustration 30 : Find value of k for which one root of equation $x^2 - (k+1)x + k^2 + k - 8 = 0$ exceeds 2 & other is less than 2.

Solution : $4-2(k+1) + k^2 + k-8 < 0 \quad \Rightarrow k^2 - k - 6 < 0$

$(k-3)(k+2) < 0 \quad \Rightarrow \quad -2 < k < 3$

Taking intersection, $k \in (-2, 3)$.

Illustration 31 : Find all possible values of a for which exactly one root of $x^2 -(a+1)x + 2a = 0$ lies in interval (0,3).

Solution : $f(0)\ .\ f(3) < 0$

$\Rightarrow \quad 2a\,(9- 3(a + 1)+2a) < 0 \quad \Rightarrow \quad 2a\,(-a + 6) < 0$

$\Rightarrow \quad a(a - 6) > 0 \quad \Rightarrow \quad a< 0$ or $a > 6$

Checking the extremes.

If $a = 0$, $\quad x^2 - x = 0$

$x = 0, 1$

$1 \in (0, 3)$

If $a = 6$, $\quad x^2 - 7x + 12 = 0$

$x = 3, 4 \qquad$ But $4 \notin (0, 3)$

Hence solution set is

$a \in (-\infty,0] \cup (6,\infty)$

Illustration 32 : If α is a root of the equation $ax^2 + bx + c = 0$ and β is a root of the equation $-ax^2 + bx + c = 0$, then prove that there will be a root of the equation $\frac{a}{2}x^2 + bx + c = 0$ lying between α and β.

Solution : Let $f(x) = \frac{a}{2}x^2 + bx + c$

$f(\alpha) = \frac{a}{2}\alpha^2 + b\alpha + c = a\alpha^2 + b\alpha + c - \frac{a}{2}\alpha^2$

$= -\frac{a}{2}\alpha^2 \qquad$ (As α is a root of $ax^2 + bx + c = 0$)

And $\quad f(\beta) = \frac{a}{2}\beta^2 + b\beta + c = -a\beta^2 + b\beta + c + \frac{3a}{2}\beta^2$

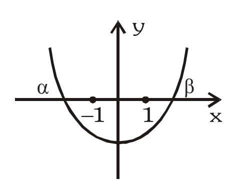
$= \frac{3a}{2}\beta^2 \qquad$ (As β is a root of $-ax^2 + bx + c = 0$)

Now $\quad f(\alpha)\ .\ f(\beta) = \frac{-3}{4}a^2\alpha^2\beta^2 < 0$

$\Rightarrow \quad f(x) = 0$ has one real root between α and β.

Illustration 33 : Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β where $\alpha < -1$ and $\beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$.

Solution : Let $f(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$



from graph $f(-1) < 0$ and $f(1) < 0$

$\Rightarrow \quad 1 + \frac{c}{a} - \frac{b}{a} < 0$ and $1 + \frac{c}{a} + \frac{b}{a} < 0$

Multiplying these two, we get $\left(1 + \frac{c}{a}\right)^2 - \frac{b^2}{a^2} > 0$

$\Rightarrow \quad \left|1 + \frac{c}{a}\right| > \left|\frac{b}{a}\right| \qquad \left\{\alpha\beta < -1 \Rightarrow \frac{c}{a} < -1\right\}$

$\Rightarrow \quad 1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$

Do yourself - 10 :

(i) If α, β are roots of $7x^2 + 9x - 2 = 0$, find their position with respect to following $(\alpha < \beta)$:

(a) -3 (b) 0 (c) 1

(ii) If $a > 1$, roots of the equation $(1 - a)x^2 + 3ax - 1 = 0$ are -

(A) one positive one negative (B) both negative

(C) both positive (D) both non-real

(iii) Find the set of value of a for which the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are less than 3.

(iv) If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.

(v) If α, β are roots of $4x^2 - 16x + \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then find the range of λ.

17. GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES :

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Illustration 34 : If $x^2 + 2xy + 2x + my - 3$ have two linear factor then m is equal to -

(A) 6, 2 (B) -6, 2 (C) 6, -2 (D) -6, -2

Solution : Here $a = 1$, $h = 1$, $b = 0$, $g = 1$, $f = m/2$, $c = -3$

$$\text{So} \quad \Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & m/2 \\ 1 & m/2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow -\frac{m^2}{4} - (-3 - m/2) + m/2 = 0 \qquad \Rightarrow -\frac{m^2}{4} + m + 3 = 0$$

$$\Rightarrow m^2 - 4m - 12 = 0 \qquad \Rightarrow m = -2, 6$$

Ans. (C)

Do yourself - 11 :

(i) Find the value of k for which the expression $x^2 + 2xy + ky^2 + 2x + k = 0$ can be resolved into two linear factors.

18. THEORY OF EQUATIONS :

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ are roots of the equation, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots.\ a_{n-1}x + a_n = 0$,

where $a_0, a_1, \ldots\ldots\ldots a_n$ are constants and $a_0 \neq 0$.

$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots\ldots\ldots (x - \alpha_n)$

$\therefore \quad a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n)$

Comparing the coefficients of like powers of x, we get

$$\sum \alpha_i = -\frac{a_1}{a_0} = S_1 \quad \text{(say)}$$

$$\text{or} \quad S_1 = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$$

$$S_2 = \sum_{i \neq j} \alpha_i \alpha_j = (-1)^2 \frac{a_2}{a_0}$$

$$S_3 = \sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k = (-1)^3 \frac{a_3}{a_0}$$

$\vdots$

$$S_n = \alpha_1 \alpha_2 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coefficient of } x^n}$$

where S_k denotes the sum of the product of root taken k at a time.

Quadratic equation : If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Cubic equation : If α, β, γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}, \ \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \quad \text{and} \quad \alpha\beta\gamma = -\frac{d}{a}$$

Note :

(i) If α is a root of the equation f (x) = 0, then the polynomial f(x) is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of f(x) and conversely.

(ii) Every equation of nth degree ($n \geq 1$) has exactly n root & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation f (x) = 0 are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs.**

(iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x)=0 must have atleast one real root between 'a' and 'b'.

(vi) Every equation f(x) = 0 of degree odd has atleast one real root of a sign opposite to that of its last term.

Descartes rule of signs :

The maximum number of positive real roots of polynomial equation f (x) = 0 is the number of changes of signs in f (x).

Consider $x^3 + 6x^2 + 11x - 6 = 0$

The signs are : + + + –

As there is only one change of sign, the equation has atmost one positive real root.

The maximum number of negative real roots of a polynomial equation f (x) = 0 is the number of changes of signs in f (–x)

Consider $f(x) = x^4 + x^3 + x^2 - x - 1 = 0$

$f(-x) = x^4 - x^3 + x^2 + x - 1 = 0$

3 sign changes, hence atmost 3 negative real roots.

Illustration 35 : If two roots are equal, find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$.

Solution : Let roots be α, α and β

$$\therefore \quad \alpha + \alpha + \beta = -\frac{20}{4} \quad \Rightarrow \quad 2\alpha + \beta = -5 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

$$\therefore \quad \alpha \cdot \alpha + \alpha\beta + \alpha\beta = -\frac{23}{4} \quad \Rightarrow \quad \alpha^2 + 2\alpha\beta = -\frac{23}{4} \ \& \ \alpha^2\beta = -\frac{6}{4}$$

from equation (i)

$$\alpha^2 + 2\alpha(-5 - 2\alpha) = -\frac{23}{4} \quad \Rightarrow \quad \alpha^2 - 10\alpha - 4\alpha^2 = -\frac{23}{4} \quad \Rightarrow 12\alpha^2 + 40\alpha - 23 = 0$$

$$\therefore \quad \alpha = 1/2, \ -\frac{23}{6}$$

when $\alpha = \frac{1}{2}$

$\alpha^2\beta = \frac{1}{4}(-5-1) = -\frac{3}{2}$

when $\alpha = -\frac{23}{6} \Rightarrow \alpha^2\beta = \frac{23\times 23}{36}\left(-5-2\text{x}\left(-\frac{23}{6}\right)\right) \neq -\frac{3}{2} \quad \Rightarrow \quad \alpha = \frac{1}{2} \qquad \beta = -6$

Hence roots of equation $= \frac{1}{2}, \frac{1}{2}, -6$ **Ans.**

Illustration 36 : If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find :

(i) $\sum \alpha^3$ (ii) $\alpha^2(\beta+\gamma)+\beta^2(\gamma+\alpha)+\gamma^2(\alpha+\beta)$

Solution : We know that $\alpha+\beta+\gamma = p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = r$

(i) $\alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma + (\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma)^2 - 3(\alpha\beta+\beta\gamma+\gamma\alpha)\}$

$= 3r + p\{p^2 - 3q\} = 3r + p^3 - 3pq$

(ii) $\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta) = \alpha^2(p-\alpha)+\beta^2(p-\beta)+\gamma^2(p-\gamma)$

$= p(\alpha^2+\beta^2+\gamma^2) - 3r - p^3 + 3pq = p(p^2 - 2q) - 3r - p^3 + 3pq = pq - 3r$

Illustration 37 : If $b^2 < 2ac$ and a, b, c, d $\in$R, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

Solution : Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

Then $\alpha+\beta+\gamma = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$\alpha\beta\gamma = \frac{-d}{a}$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$

$\Rightarrow$ $\alpha^2 + \beta^2 + \gamma^2 < 0$, which is not possible if all α, β, γ are real. So atleast one root is non-real, but complex roots occurs in pair. Hence given cubic equation has two non-real and one real roots.

Illustration 38 : If q, r, s are positive, show that the equation $f(x) \equiv x^4 + qx^2 + rx - s = 0$ has one positive, one negative and two imaginary roots.

Solution : Product = $-s < 0$

let roots be $\alpha,\beta,\gamma,\delta$

$\Rightarrow$ $\alpha\beta\gamma\delta < 0$

this is possible when -

(i) one root is negative & three are positive

(ii) three roots are negative & one is positive

(iii) one root negative, one positive & two roots imaginary.

$f(x) \equiv x^4 + qx^2 + rx - s$

As there is only one change of sign, the equation has atmost one positive root.

$f(-x) \equiv x^4 + qx^2 - rx - s$

Again there is only one change of sign, the equation has atmost only one negative root.

so (i), (ii) can't be possible.

Hence there is only one negative root, one positive root & two imaginary roots.

Do yourself - 12 :

(i) Let α, β be two of the roots of the equation $x^3 - px^2 + qx - r = 0$. If $\alpha + \beta = 0$, then show that $pq = r$

(ii) If two roots of $x^3 + 3x^2 - 9x + c = 0$ are equal, then find the value of c.

(iii) If α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$, then find the value of

(a) $\sum \alpha^2$ (b) $\sum \frac{1}{\alpha}$ (c) $\sum \alpha^2(\beta + \gamma)$

19. TRANSFORMATION OF THE EQUATION :

Let $ax^2 + bx + c = 0$ be a quadratic equation with two roots α and β. If we have to find an equation whose roots are $f(\alpha)$ and $f(\beta)$, i.e. some expression in α & β, then this equation can be found by finding α in terms of y. Now as α satisfies given equation, put this α in terms of y directly in the equation.

$y = f(\alpha)$

By transformation, $\alpha = g(y)$

$a(g(y))^2 + b(g(y)) + c = 0$

This is the required equation in y.

Illustration 39 : If the roots of $ax^2 + bx + c = 0$ are α and β, then find the equation whose roots are :

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$ (b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (c) α^2, β^2

Solution : (a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$

put, $y = \frac{-2}{\alpha} \Rightarrow \alpha = \frac{-2}{y}$

$a\left(-\frac{2}{y}\right)^2 + b\left(\frac{-2}{y}\right) + c = 0 \quad \Rightarrow \quad cy^2 - 2by + 4a = 0$

Required equation is $cx^2 - 2bx + 4a = 0$

(b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$

put, $y = \frac{\alpha}{\alpha+1} \Rightarrow \alpha = \frac{y}{1-y}$

$\Rightarrow \quad a\left(\frac{y}{1-y}\right)^2 + b\left(\frac{y}{1-y}\right) + c = 0 \quad \Rightarrow \quad (a + c - b)y^2 + (-2c + b)y + c = 0$

Required equation is $(a + c - b)x^2 + (b - 2c)x + c = 0$

(c) α^2, β^2

put $y = \alpha^2 \Rightarrow \alpha = \sqrt{y}$

$ay + b\sqrt{y} + c = 0$

$b^2y = a^2y^2 + c^2 + 2acy$

$\Rightarrow \quad a^2y^2 + (2ac - b^2)y + c^2 = 0$

Required equation is $a^2x^2 + (2ac - b^2)x + c^2 = 0$

Illustration 40 : If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ then find equation whose roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$.

Solution : Put $y = \frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = -\frac{a\gamma}{d} \quad (\because \alpha\beta\gamma = -\frac{d}{a})$

Put $x = -\dfrac{dy}{a}$

$$\Rightarrow \quad a\left(-\frac{dy}{a}\right)^3 + b\left(-\frac{dy}{a}\right)^2 + c\left(-\frac{dy}{a}\right) + d = 0$$

Required equation is $d^2x^3 - bdx^2 + acx - a^2 = 0$

Do yourself - 13 :

(i) If α, β are the roots of $ax^2 + bx + c = 0$, then find the equation whose roots are

(a) $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}$ (b) $\dfrac{1}{a\alpha+b}, \dfrac{1}{a\beta+b}$ (c) $\alpha+\dfrac{1}{\beta}, \quad \beta+\dfrac{1}{\alpha}$

(ii) If α, β are roots of $x^2 - px + q = 0$, then find the quadratic equation whose root are $(\alpha^2-\beta^2)(\alpha^3-\beta^3)$ and $\alpha^2\beta^3+\alpha^3\beta^2$.

Miscellaneous Illustrations :

Illustrations 41 : A polynomial in x of degree greater than three, leaves remainders 2, 1 and –1 when divided, respectively, by $(x - 1)$, $(x + 2)$ and $(x + 1)$. What will be the remainder when it is divided by $(x - 1)(x + 2)(x + 1)$.

Solution : Let required polynomial be $f(x) = p(x)(x - 1)(x + 2)(x + 1) + a_0x^2 + a_1x + a_2$

By remainder theorem, $f(1) = 2$, $f(-2) = 1$, $f(-1) = -1$.

$$\Rightarrow \quad a_0 + a_1 + a_2 = 2$$
$$4a_0 - 2a_1 + a_2 = 1$$
$$a_0 - a_1 + a_2 = -1$$

Solving we get, $a_0 = \dfrac{7}{6}$, $a_1 = \dfrac{3}{2}$, $a_2 = \dfrac{2}{3}$

Remainder when f(x) is divided by $(x - 1)(x + 2)(x + 1)$

will be $\dfrac{7}{6}x^2 + \dfrac{3}{2}x + \dfrac{2}{3}$.

Illustrations 42 : If α, β are the roots of $x^2 + px + q = 0$, and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equations have a common root.

Solution : α, β are the roots of $x^2 + px + q = 0$

$\therefore \quad \alpha + \beta = -p, \ \alpha\beta = q$ ....(1)

and γ, δ are the roots of $x^2 + rx + s = 0$

$\therefore \quad \gamma + \delta = -r, \quad \gamma\delta = s$ ....(2)

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$$= [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$$
$$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$$
$$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s(\alpha^2 + \beta^2) + sr(\alpha + \beta) + s^2$$
$$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s((\alpha + \beta)^2 - 2\alpha\beta)) + sr(\alpha + \beta) + s^2$$
$$= q^2 - pqr + r^2q + s(p^2 - 2q) + sr(-p) + s^2$$
$$= (q - s)^2 - rpq + r^2q + sp^2 - prs$$
$$= (q - s)^2 - rq(p - r) + sp(p - r)$$
$$= (q - s)^2 + (p - r)(sp - rq)$$

For a common root (Let $\alpha = \gamma$ or $\beta = \delta$) ....(3)

then $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$ ....(4)

from (3) and (4), we get

$$(q - s)^2 + (p - r)(sp - rq) = 0$$

$\Rightarrow \quad (q - s)^2 = (p - r)(rq - sp)$, which is the required condition.

Illustrations 43 : If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then find the interval in which y lies.

Solution : $(y^2 - 5y + 3)(x^2 + x + 1) < 2x, \ \forall \ x \in R$

$\Rightarrow \quad y^2 - 5y + 3 < \dfrac{2x}{x^2 + x + 1}$

Let $\dfrac{2x}{x^2 + x + 1} = P$

$\Rightarrow \quad px^2 + (p - 2)x + p = 0$

(1) Since x is real, $(p - 2)^2 - 4p^2 \geq 0$

$\Rightarrow \quad -2 \leq p \leq \dfrac{2}{3}$

(2) The minimum value of $2x/(x^2 + x + 1)$ is -2. So,

$y^2 - 5y + 3 < -2 \quad \Rightarrow \quad y^2 - 5y + 5 < 0$

$\Rightarrow \quad y \in \left(\dfrac{5-\sqrt{5}}{2}, \dfrac{5+\sqrt{5}}{2}\right)$

ANSWERS FOR DO YOURSELF

1 : (i) (a) $-1, -2$; (b) 4; (c) $1 \pm \sqrt{2}$; (ii) $a, \dfrac{1}{a}$; (iii) $\dfrac{7}{3}$ (iv) $3, -\dfrac{1}{5}$

2 : (i) $b = -4, c = 1$; (ii) (a) imaginary; (b) real & distinct; (c) real & coincident

3 : (i) (a) $c = 0$; (b) $c = 1$; (c) $b \rightarrow$ negative, $c \rightarrow$ negative

4 : (i) $b = \dfrac{9}{2}, c = 5$; (ii) $c = 0, 6$

5 : (i) (a) $x \in (-\infty, -3] \cup [2, \infty)$; (b) $x \in (-2, -1)$; (c) $\left(-\dfrac{1}{4}, \dfrac{1}{3}\right]$;

(d) $x \in (-6, -3) \cup \left(\dfrac{1}{2}, 2\right) - \{1\} \cup (9, \infty)$; (e) $[3,7]$; (f) ϕ

6 : (i) (a) $1, x = -1$; (b) $-1, x = 2$

(ii) (1) (i) $a < 0$ (ii) $b < 0$ (iii) $c < 0$ (iv) $D > 0$ (v) $\alpha + \beta < 0$ (vi) $\alpha\beta > 0$

(2) (i) $a < 0$ (ii) $b > 0$ (iii) $c = 0$ (iv) $D > 0$ (v) $\alpha + \beta > 0$ (vi) $\alpha\beta = 0$

(3) (i) $a < 0$ (ii) $b = 0$ (iii) $c = 0$ (iv) $D = 0$ (v) $\alpha + \beta = 0$ (vi) $\alpha\beta = 0$

(iii) Third quadrant

(iv) (a) $a > 9/16$ (b) $a < -2$

7 : (i) $x \in (-\infty, -2) \cup (1, 3/2)$ (ii) $x \in R - (0,1]$

8 : (i) (a) $x \in (1,8)$ (b) $x \in (-\infty, 1/2)$ (ii) $x \in (-1,1)$

9 : (ii) least value = 0, greatest value = 1.

10 : (i) $-3 < \alpha < 0 < \beta < 1$; (ii) C; (iii) $a < 2$; (iv) $a < 2$; (v) $12 < \lambda < 16$

11 : (i) 0, 2

12 : (ii) $-27, 5$; (iii) (a) $\dfrac{1}{a^2}(b^2 - 2ac)$, (b) $-\dfrac{c}{d}$, (c) $\dfrac{1}{a^2}(3ad - bc)$

13 : (i) (a) $c^2y^2 + y(2ac - b^2) + a^2 = 0$; (b) $acx^2 - bx + 1 = 0$; (c) $acx^2 + (a + c)bx + (a + c)^2 = 0$

(ii) $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$