1.2 VECTOR ALGEBRA

1.2.1 Basic Operations of Vector Algebra

A **scalar** quantity is characterized by its magnitude. Temperature, charge and voltage are scalars. Ordinary type is used for scalars, e.g., V for voltage.

A **vector** has both magnitude and direction. Velocity and the electric field are vectors. Boldface type is used for vectors, e.g., **E** for electric field. A vector **A** may be represented as a *directed line segment* or arrow (Figure 1-1). The length of the arrow represents the magnitude of the vector, denoted by | **A** | or A, and the arrow head indicates its direction. Switching the head and tail of arrow changes **A** to **–A.** A vector is changed by rotation, but not by translation.

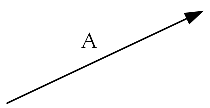


Figure 1-1. ***The vector* A *as a directed line segment.***

Vector Addition

To form the vectorial sum **A + B,** place the tail of **B** at the head of **A** as shown in Figure 1-2(a). The sum **A + B** is the arrow from the tail of **A** to the head of **B.** One may also reverse the order and place the tail of **A** at the head of **B** (Figure 1-2(b)). Note that both procedures yield the same result and one which is also identical to the parallelogram method (Figure 1-2(c)) with which you may be familiar.

Vector addition is commutative as seen in Figures 1-2(a), (b):

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1-1}$$

It is also associative

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{1-2}$$

To subtract **B** from **A,** we add the negative of **B** to **A** (see Figure 1-2(d)):

$$\mathbf{A} - \mathbf{B}) = \mathbf{A} + (-\mathbf{B}) \tag{1-3}$$

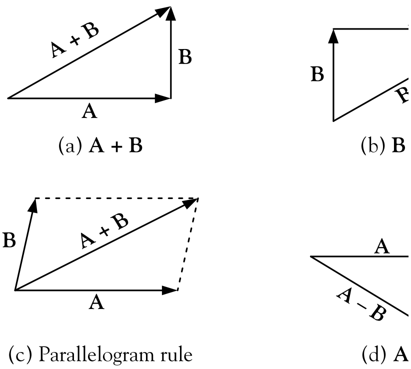


Figure 1-2. ***Addition of vectors.***

Multiplication of a Vector by a Scalar

If a vector is multiplied by a scalar a its magnitude is multiplied by the magnitude of a. The direction of the vector is unchanged if the scalar is positive and real. Scalar multiplication is distributive:

$$a(\mathbf{A}+\mathbf{B}) = a\mathbf{A} + a\mathbf{B} \tag{1-4}$$

Scalar division by a corresponds to multiplication by the inverse of a:

$$\frac{\mathbf{B}}{a} = \left(\frac{1}{a}\right)\mathbf{B} \tag{1-5}$$

If we divide a vector **A** by its magnitude A, we obtain a vector of unit length pointing in the direction of **A,** i.e., a *unit vector* **a.**

$$\mathbf{a} = \frac{\mathbf{A}}{A}, \quad |\mathbf{a}| = 1 \tag{1-6}$$

All unit vectors have the same magnitude; they differ only in *direction.*

Dot or Scalar Product

The dot product of two vectors is a scalar and it is defined as

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta_{AB} \quad (0 \le \theta_{AB} \le \pi) \tag{1-7}$$

where A = | **A** |, B = | **B** |, θ_{AB} is the angle between **A** and **B** (Figure 1-3).

The dot product is A times the projection of **B** on **A** (B cos θ_{AB}) or B times the projection of **A** on **B** (A cos θ_{AB}). A projection is negative for $\theta_{AB} > \pi/2$.

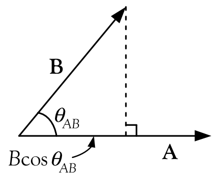


Figure 1-3. ***The dot product.***

The dot product is commutative:

$$\mathbf{B}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{B} \tag{1-8}$$

and distributive:

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} \tag{1-9}$$

Special cases are:

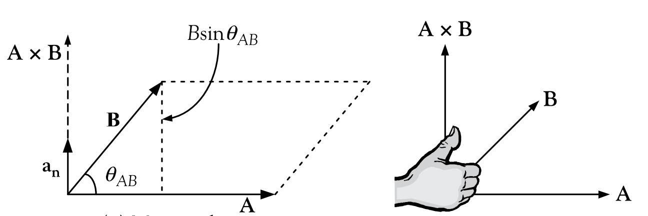
$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= AB \ (\text{for } \theta_{AB} = 0)\\ &= 0 \ (\text{for } \theta_{AB} = \pi/2)\\ \mathbf{A}\cdot\mathbf{A} &= A^2\end{aligned}$$

Cross or Vector Product

The cross product of two vectors is a vector and it is defined as follows:

$$\mathbf{A}\times\mathbf{B} = \mathbf{a}_n \ AB \sin\theta_{AB} (0 \le \theta_{AB} \le \pi) \tag{1-10}$$

Note that the magnitude of **A** × **B** is AB sin θ_{AB}, which is the area (base times height) of the parallelogram formed by **A** and **B** (see Figure 1-4(a)). Either A or B may be taken as the base. The direction of the cross product is that of $\mathbf{a}_n$ (Figure 1-4(a)) which is a unit vector perpendicular to the plane of the parallelogram (the plane formed by **A** and **B**). The



(a) Magnitude. (b) Direction by right-hand rule.

Figure 1-4. ***The cross product.***

direction of the normal $\mathbf{a}_n$ is determined by the right hand rule (Figure 1-4(b)). Note that $\mathbf{A} \times \mathbf{B}$ is perpendicular to both **A** and **B.**

The cross product is distributive:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \qquad (1\text{-}11)$$

The cross product is *not commutative*:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \qquad (1\text{-}12)$$

Special cases are:

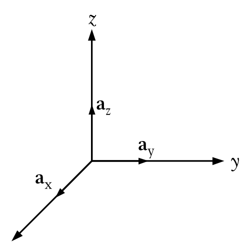
$$|\mathbf{A} \times \mathbf{B}| = AB \text{ (for } \theta_{AB} = \pi/2)$$
$$= 0 \text{ (for } \theta_{AB} = 0)$$
$$\mathbf{A} \times \mathbf{A} = 0$$

1.2.2 Vector Algebra in Rectangular Coordinates

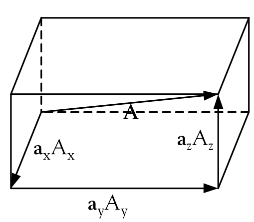
A vector may be represented in terms of its components in any coordinate system; we choose the rectangular (Cartesian) coordinate system for this section. Figure 1-5(a) shows the Cartesian coordinate system with its unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$. Figure 1-5(b) shows a vector **A** and its projections $\mathbf{a}_x A_x$, $\mathbf{a}_y A_y$, $\mathbf{a}_z A_z$ on the x, y, z axes. By translation, we can place the projections head to tail and show that

$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z \qquad (1\text{-}13)$$

A_x, A_y, A_z are the components of **A** or the magnitudes of its projections. The magnitudes may either precede or follow the unit vectors.



(a) Unit vectors.



(b) Components of vector **A**.

Figure 1-5. ***Representation of a vector in rectangular coordinate system.***

Vector Addition

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) + (\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z) \\ &= \mathbf{a}_x (A_x + B_x) + \mathbf{a}_y (A_y + B_y) + \mathbf{a}_z (A_z + B_z)\end{aligned} \tag{1-14}$$

Multiplication by a Scalar

$$\begin{aligned}c\mathbf{A} &= c(\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) \\ &= \mathbf{a}_x (cA_x) + \mathbf{a}_y (cA_y) + \mathbf{a}_z (cA_z)\end{aligned} \tag{1-15}$$

Dot Product

Unit vector relationships:

$$\mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1$$
$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_x \cdot \mathbf{a}_z = \mathbf{a}_y \cdot \mathbf{a}_z = 0$$

Thus

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) + (\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z) \\ &= A_x B_x + A_y B_y + A_z B_z \ \text{(using the distributive property)}\end{aligned} \tag{1-16}$$

Special case:

$$\mathbf{A}\cdot\mathbf{A} = A_x^2 + A_y^2 + A_z^2 = |\mathbf{A}|^2$$

Thus

$$|\mathbf{A}| = \sqrt{A_x^2 + A_z^2 + A_z^2}$$

Cross Product

Unit vector relations:

$$\begin{aligned}\mathbf{a}_x \times \mathbf{a}_y &= \mathbf{a}_z & \mathbf{a}_y \times \mathbf{a}_x &= -\mathbf{a}_z \\ \mathbf{a}_y \times \mathbf{a}_z &= \mathbf{a}_x & \mathbf{a}_z \times \mathbf{a}_y &= -\mathbf{a}_x \\ \mathbf{a}_z \times \mathbf{a}_x &= \mathbf{a}_y & \mathbf{a}_x \times \mathbf{a}_z &= -\mathbf{a}_y\end{aligned}$$

(Simply use the right-hand rule or consider the cycle xyzxyz to determine the cross product)

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) + (\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z)\\ &= \mathbf{a}_x(A_y B_z - A_z B_y) + \mathbf{a}_y(A_z B_x - A_x B_z) + \mathbf{a}_z(A_z B_y - A_y B_x)\end{aligned} \tag{1-17a}$$

The results above are identical to the convenient determinant form of the cross product:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix} \tag{1-17b}$$

1.2.3 Triple Products

Scalar Triple Product **A · (B × C)**

A · (B × C) is a scalar whose magnitude is equal to the volume of the parallelepiped (box) formed by the three vectors. Figure 1-6(b) shows vectors **A, B, C** and the box which they form. We note that **B** × **C** is a vector with direction $\mathbf{a}_n$ and magnitude equal to the area of the base of the parallelepiped: (see Figure 1-6(a))

$$\mathbf{B}\times\mathbf{C} = \mathbf{a}_n \ (\text{area of Base}_{CB})$$

Then

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= (\mathbf{a}_n\cdot\mathbf{A})\ (\text{area of Base}_{CB})\\ &= \pm h_a(\text{area of Base}_{CB}) = \pm(\text{volume of the box})\end{aligned} \tag{1-18}$$

($\mathbf{a}_n$ · **A**) is positive in the case of Figure 1-6. If **A** were reversed, ($\mathbf{a}_n$ · **A**) would be negative.

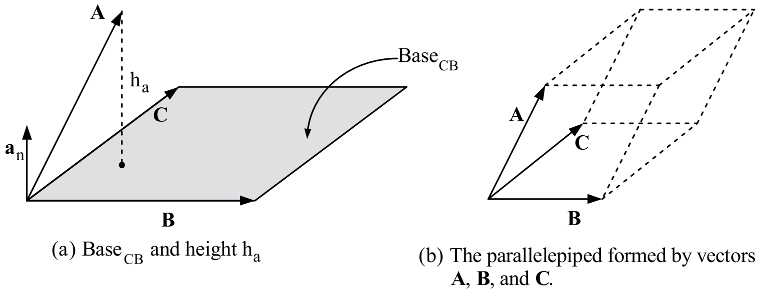


Figure 1-6. ***The Triple Product* A · (B × C).**

Since the volume of the box can be obtained from three different bases and heights:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$
$$= -\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B}) = -\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = -\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A}) \quad (1\text{-}19)$$

Thus we can change the order so long as we retain the cycle ABCABC. We may also interchange dot and cross products in Eq. (1-19) to obtain:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \quad (1\text{-}20)$$

Note that all the parentheses of this section may be removed since there is only one possible location.

There is also a determinant form of the scalar triple product:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad (1\text{-}21)$$

This identity may be shown by laboriously expanding both sides in rectangular coordinates.

Vector Triple Product A × (B × C)

A × (**B** × **C**) is a vector. We note that the direction of **B** × **C** is that of $\mathbf{a}_n$ (Figure 1-6). Then **A** × (**B** × **C**) is perpendicular to (**B** × **C**), i.e., normal to $\mathbf{a}_n$. Thus **A** × (**B** × **C**) lies in the plane formed by **B** and **C** and has components in the **B** and **C** directions. The "BAC-CAB" rule gives these components:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{1-22}$$

The identity above can be shown by expanding in rectangular components (see Problem 1-5). The parenthesis in $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ *is* necessary since it differs from $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

Example 1-1

Let $\mathbf{A} = \mathbf{a}_y\, 2 + \mathbf{a}_z\, 2$

$\mathbf{B} = \mathbf{a}_y\, 3$

$\mathbf{C} = \mathbf{a}_x\,(-4) + \mathbf{a}_y(4)$

Find $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.
Solution:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 0 & 2 & 2 \\ 0 & 3 & 0 \\ -4 & 4 & 0 \end{vmatrix} = (-4)\{2\cdot 0 - 2\cdot 3\} = 24$$

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \\ &= \mathbf{a}_y 3\cdot(8) - (-\mathbf{a}_x 4 + \mathbf{a}_y 4)\cdot 6 = \mathbf{a}_x 24\end{aligned}$$

Sketch the vectors and the parallelepiped (box). Show by geometrical considerations that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ has only an $\mathbf{a}_x$ component.

Example 1-2

Let $\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$

Find $\mathbf{A} \cdot \mathbf{a}_z$ and $\mathbf{A} \times \mathbf{a}_z$
Solution:

$$\mathbf{A}\cdot\mathbf{a}_z = \mathbf{a}_z\cdot(\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) = 0 + 0 + A_z = A_z$$