RATIONAL FUNCTIONS AND PARTIAL FRACTIONS.

Given this expression $f(x) = \frac{g(x)}{h(x)}$, where $h(x)$ and $g(x)$ are Polynomials and $h(x) = 0$ it becomes undefined. but it must not be equal to 0.

The degree of $g(x) <$ degree of $h(x)$; $f(x)$ is called proper fraction. However if the degree of $g(x)$ is greater than the degree of $h(x)$; $f(x)$ is called improper fraction.

PARTIAL FRACTION.

Consider this function;

$$f(x) = \frac{5x+23}{x^2+9x+20} = \frac{A}{x+5} + \frac{B}{x+4}$$

Partial fraction.

Whole fraction

Proper fractions:

i) If $h(x)$ has a non-repeating linear factor of the form $ax+b$ then the partial fraction decomposition of $\frac{g(x)}{h(x)}$ is $\frac{A}{ax+b}$

Example: $\frac{5x+23}{(x+5)(x+4)} = \frac{A}{x+5} + \frac{B}{(x+4)}$

$5x+23 = A(x+4) + B(x+5)$

when $x = -5$

$5(-5) + 23 = A(-5+4) + B(-5+5)$

$-25 + 23 = -A$

when $x = 0$,

$7 = A - 3C$

$\Rightarrow 3C = A - 7$

$C = -6/3 = -2$

$C = -2$

when $x = 1$

$6 = 2A - 2B - 2C$

$6 = 2(1) - 2B - 2(-2)$

$2B = $ [illegible]

$B = $ [illegible]

$\therefore x^4 - 2x^3 - x^2 - 4x + 4 = (x+1) + \frac{A}{(x-3)} + \frac{Bx + C}{x^2+1}$

$= (x+1) + \frac{1}{x-3} - \frac{2}{x^2+1}$

$3x^2+7x-8 = A(x+2)(x-3) + B(x+1)(x-3) + C(x+1)(x+2)$

when $x = -1$

$3(-1)^2+7(-1)-8 = A(-1+2)(-1-3)$

$3-7-8 = A(1)(-4)$

$\frac{-12}{-4} = \frac{-4A}{-4}$

$A = 3$

when $x = -2$

$3(-2)^2+7(-2)-8 = B(-2+1)(-2-3)$

$3(4) - 14 - 8 = B(-1)(-5)$

$12 - 14 - 8 = 5B$

$-10 = 5B$

$B = -2$

when $x = 3$

$40 = 20C$

$C = 2$

2) Express $\frac{x^2+3x+4}{(x+3)^3}$ in P.F

$\frac{x^2+3x+4}{(x+2)^3} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+3)^3}$

$x^2+3x+4 = A(x+2)^2 + B(x+2) + C$

when $x = -2$

$(-2)^2 + 3(-2) + 4 = C$

$4 - 6 + 4$ $\therefore C = 2$

(b) If $h(x)$ has a k-repeating linear factor of the form $(ax+b)^k$, then the partial fraction decomposition of $\frac{g(x)}{h(x)}$ is $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}$

Example

$$\frac{x-1}{(x+1)(x-2)^2} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

c) If $h(x)$ has a non-repeating quadratic factor of the form ax^2+bx+c which is irreducible or cannot be factored, the P.F decomposition is $\frac{Ax+B}{ax^2+bx+c}$, where A, B are constant

Example:

$$\frac{x^2-5x+1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}.$$

d) If $h(x)$ has a k-repeating quadratic factor of the form $(ax^2+bx+c)^k$, the P.F decomposition is

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}.$$

Example: Resolve into P.F

$$\frac{3x^2+7x-8}{(x+1)(x+2)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-3)}$$

Express $\dfrac{x^4-2x^3-x^2-4x+4}{x^3-3x^2+x-3}$ in Pf.

$$
\begin{array}{r|l}
 & x \\
\hline
x^3-3x^2+x+3 & \cancel{x^4}-2x^3-x^2-\cancel{4x}+4 \\
 & -\ x^4-3x^2+\cancel{x^2}+3x \\
\hline
 & x^2-2x^2-x
\end{array}
$$

$$
\begin{array}{r|l}
 & x+1 \\
\hline
x^3-3x^2+x-3 & x^4-2x^3-x^2-4x+4 \\
 & x^4-3x^3+x^2-3x \\
\hline
 & x^3-2x^2-x+4 \\
 & x^3-3x^2+x-3 \\
\hline
 & x^2-2x+7
\end{array}
$$

$\therefore x^4-2x^3-x^2-4x+4 = x+1+\dfrac{x^2-2x+7}{x^3-2x^2+x-3}$

$= x+1+\dfrac{x^2-2x+7}{(x-3)(x^2+1)}$

Now; $\dfrac{x^2-2x+7}{(x-3)(x^2+1)} = \dfrac{A}{x-3}+\dfrac{Bx+C}{x^2+1}$

$x^2-2x+7 = A(x^2+1)+(Bx+C)(x-3)$

when $x=3$

$10 = 10A$

$A = 1$

$\boxed{0, -1 \; \& \; 1}$

When $x = 0$

$4 = 4A + 2B + C$

$4 = 4A + 2B + 2$

$4A + 2B = 2$

When $x = -1$

$2 = A + B + 2$

$2 - 2 = A + B$

$A + B = 0$

$4A + 2B = 2 \quad \times 1$

$A + B = 0 \quad \times 2$

$4A + 2B = 2$

$-2A + 2B = 0$

$2A = 2$

$A = 1$

$4(1) + 2B = 2$

$4 + 2B = 2$

$2B = 2 - 4$

$2B = -2$

$B = -1$

$$\frac{x^2 + 3x + 4}{(x+3)^3} = \cancel{A}\frac{1}{x+2} - \frac{1}{(x+2)^2} + \frac{2}{(x+3)^3}$$