Measure Theoretic Probability
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In this study material, we will show how measure theory plays a vital role in probability theory.

To be honest, actually probability theory stands on the shoulder of the gigantic notion of Measure Theory. In other words, probability theory is *a kind of* measure in limited interval in reals following some probability-specific axioms. That means, measure theory is a more generalized body of knowledge.

-- Brief encounter with Measure Theory

Let X be any set and $W = P(X)$ be a power set. The measure is a function $f : W \to R^+$ such that the following conditions are satisfied considering $\{0\} \subset R^+$:

$$A, A_i, A_j \in W,$$
$$f(A) \geq 0,$$
$$[i \neq j] \Rightarrow [A_i \cap A_j = \phi],$$
$$f(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} f(A_i).$$

Explanations: The first condition says that in any case the measure of some subset of X should be always positive inclusive of zero. The second condition says that if we take partition of set X and find individual measure, then the total measure of union will equal to the sum of each measure.

-- Measure Theoretic Probability

Let Σ be a $\sigma - a\lg ebra$ on X and a function is given as: $p : \Sigma \to R$. This function is a probability measure if it follows standard probability axioms in [0,1] and also the followings:

1. *If, for any* $\{A_i\}_{i=1}^{n} \in \Sigma^n$ *that is pairwise disjoint,* $p(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} p(A_i)$ *then we say* p *is finitely additive*

2. *If, for any* $\{A_i\}_{i=1}^{\infty} \in \Sigma^{\infty}$ *that is pairwise disjoint,* $p(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} p(A_i)$ *then we say* p *is* $\sigma-$*additive*

Explanations: The first condition talks about measure condition in finite sense (because $n < \infty$). The second condition is a more generalized version for positive additivity and it is called sigma-additive. Both are actually following measure theory conditions.

-- Subtle distinctions

Now let us look at function $p(.)$ in two different lights. First one is NOT a probability measure but a finite measure. The second one is a restriction so that $p(.)$ becomes a probability measure.

$$If\ p\ is\ a\ measure\ such\ that\ p(X) < \infty,\ then\ it\ is\ a\ finite\ measure$$

$$If\ p(X) = 1\ then\ it\ is\ a\ probability\ measure$$

-- Boole Inequality Theorem

For any probability space $\{X, \Sigma, p\}$

$$p\{\cup_{i=1}^{\infty} A_i\} \leq \sum_{i=1}^{\infty} p(A_i) \text{ for any } \{A_m\} \in \Sigma^{\infty}$$

Explanation: This is saying that if we take any arbitrary subsets of X then as a generalization the probability measure will follow the above mentioned inequality in sigma-algebra on X. One way to look at this is that: it is a generalized version of measure theory condition considering non-disjoint subsets too.

-- Almost Sure Equality of RVs

Let H_1, H_2 be two random variables (RVs) and (X, Σ, p) be a probability measure space. The RVs are called as Almost-Surely-Equal if the following condition is satisfied:

$p(w \in X : H_1(w) = H_2(w)) = 1$

The above definition consider at least one such element in probability measure space. However, there can be subset $A \subset X, |A| > 1$ where the above statement can be valid.

The Almost-Surely-Equal RVs are often denoted as: $H_1 =_{\bullet} H_2$.

-- Relationship to Expectations

Suppose $H_1 =_{\bullet} H_2$ in the probability measure space (X, Σ, p) and E denotes expected values.
Then we can derive two conclusions as given below:

1. If $H_1 \geq_{\bullet} H_2$ then $E(H_1) \geq E(H_2)$
2. If $H_1 =_{\bullet} H_2$ then $E(H_1) = E(H_2)$

Explanations: The second condition is saying that $=_{\bullet}$ relation between RVs maintains = relation between expected values. The first condition is a more generalized version saying that if two RVs maintain $\geq_{\bullet}$ relation then, in ordering relation the expected values will maintain $\geq$ relation.

Assignments:

Q1. Prove the following claim:

Let $\{X, \Sigma, p\}$ be a probability space and $A, B \in \Sigma$, such that $A \subseteq B$. Then $p(A) \leq p(B)$

Q2. Prove the statements

1. If $H_1 \geq_{\bullet} H_2$ then $E(H_1) \geq E(H_2)$
2. If $H_1 =_{\bullet} H_2$ then $E(H_1) = E(H_2)$
