Assignment 06

Dep. AI Convergence Engineering

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1. **Q1. State the mathematical condition such that $P(A_{\Omega}\backslash\Omega) < 1$.**

Answer:

We know, Ω is entire probability event set, A_{Ω} is Borel sigma field over probability event set and, P is probability function over sigma field.

This condition means that the probability of the event A_{Ω} occurring outside of the entire sample space Ω is less than 1.

In other words, the probability measure assigned to the set-theoretic difference of A_{Ω} and Ω should be strictly less than 1.

This condition ensures that the probability assigned to events outside of the sample space is bounded by 1, maintaining the integrity of the probability measure.

2. **Q2. Prove that if $\{A_i: 1 \leq i \leq n\} \subset \sigma_x$ such that $\bigcup_{1\leq i\leq n} A_i = \Omega$ and**

   $P(\bigcup_{1\leq i\leq k} A_i) = a \in [0,1]$ **then** $P(\bigcup_{k\leq i\leq n} A_i) \in [0,1]\backslash\{a\}\ if a = 0\ and,\ k < n$ **.**

Answer:

Given:

- Let $X = \{A_i: 1 \leq i \leq n\}$ where each A_i is a subset of the sigma field σ_x.
- $\bigcup_{1\leq i\leq n} A_i = \Omega$ (i.e., the union of all A_i covers the entire sample space).
- $P(\bigcup_{1\leq i\leq k} A_i) = a \in [0,1]$.

We want to prove that:

- $P(\bigcup_{k\leq i\leq n} A_i) \in [0,1]\backslash\{a\}\ if a = 0\ and,\ k < n$

Proof:

Complementary Probability:

We know that

- $P(\bigcup_{1\leq i\leq k} A_i) = a$

Since $\bigcup_{1\leq i\leq n} A_i = \Omega$, we can express the probability of the complement as:

- $P(\bigcup_{i=k+1}^{n} A_i) = 1 - P(\bigcup_{i=k+1}^{n} A_i^c)$

Intersection of Complements:

Note that A_i^c represents the complement of A_i.

Since A_i^c are disjoint (pairwise mutually exclusive), we have:

- $P(\bigcup_{i=k+1}^{n} A_i^c) = \prod_{i=k+1}^{n} P(A_i^c)$

Using the Given Information:

We are given that $P(\bigcup_{1\leq i\leq k} A_i) = a$

Therefore, $P(A_i^c) = 1 - P(\bigcup_{1\leq i\leq k} A_i) = 1 - a$

Combining the above results:

- $P(\bigcup_{i=k+1}^{n} A_i^c) = (1 - a)^{n-k}$

- $P(\bigcup_{i=k+1}^{n} A_i) = 1 - (1 - a)^{n-k}$

Since $a = 0 \; and \; k < n$ we have $1 - (1 - a)^{n-k} = 1$

Therefore, $P(\bigcup_{k \le i \le n} A_i)$ is in the interval [0,1] excluding a.

Hence, we've proved that if the given conditions hold, then:

- $P(\bigcup_{k \le i \le n} A_i) \in [0, 1] \backslash \{a\} \; if a = 0 \; and, \; k < n$

Assignment 07

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1. **Q1. Prove that if $\sum_{n=1}^{\infty} P(A_n) \rightarrow +\infty$ then $P(E) \neq 0$ in the Borel-Cantelli lemma.**

Answer:

In the Borel-Cantelli lemma, we consider the events A_n and their probabilities $P(A_n)$. The lemma states:

1. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(lim\ sup\ A_n) = 0$.
2. If $\sum_{n=1}^{\infty} P(A_n) = \infty$, and the events A_n are the independent, then $P(lim\ sup\ A_n) = 1$.

Given that $\sum_{n=1}^{\infty} P(A_n) \rightarrow +\infty$ We are in the second case of the lemma. This implies $P(lim\ sup\ A_n) = 1$

Now, let's define the event $E = lim\ sup\ A_n$. This means that infinitely many of the events A_n occur.

If $P(E) = 0$, this would imply that the probability of infinitely many A_n occurring is zero. However, since $P(lim\ sup\ A_n) = 1$, this contradicts the assumption.

Therefore, for $\sum_{n=1}^{\infty} P(A_n) \rightarrow +\infty$, it must be the case that $P(E) \neq 0$ in the Borel-Cantelli lemma.

2. **Q2. If** $g\colon \Omega\rightarrow(A\subset\Omega)$ **is an identity function then prove that** $E[X]\geq E[X(g)]$ **, where** $X\colon \Omega\rightarrow R$ **is a random variable (discrete or continuous).**

Answer:

Given, $g\colon \Omega\rightarrow(A\subset\Omega)$ is an identity function, and $X\colon \Omega\rightarrow R$ is a random variable.

The expectation of a random variable X is defined as:

$$E[X]: \int_{\Omega} X(\omega)dP(\omega)$$

where P is the probability measure defined over the sample space Ω.

Now, let's consider the random variable X(g), which is X composed with the function g. So, X(g)(ω)=X(g(ω))=X(ω), since g is an identity function.

Therefore, the expectation of X(g) is:

$$E[X(g)]: \int_{\Omega} X(g(\omega))dP(\omega) = \int_{\Omega} X(\omega)dP(\omega) = E[X]$$

Since E[X(g)]=E[X], it follows trivially that E[X]≥E[X(g)].

So, when g is the identity function, the expectation of X is greater than or equal to the expectation of X(g).