Probability Theory– V

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In this study material, we will study Borel-Cantelli Lemma, Expectation of a random variable and probability distributions.

First we will study the Borel-Cantelli lemma. The Borel-Cantelli lemma deals with algebraic estimation of *additive* probability measures in a sequence of sets (recall sequence of sets from earlier study material about set algebra). The Borel-Cantelli lemma considers that additive probability measure is finite (we will see this when we will state the Borel-Cantelli lemma).

Suppose $< A_k >_{k \in I}$ is a sequence of event sets. The main question Borel-Cantelli lemma ask is: whether infinitely many probabilistic events occur or only finitely many events occur in a sequence ?? Suppose we denote the every event space in the infinite sequence as:

$$F_n = \bigcup_{k=n}^{\infty} A_k$$

Next, we denote the *infinitely-many-times-occurring-events* as:

$$E = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

Now, we are ready to state Borel-Cantelli lemma.

-- Borel-Cantelli lemma

If $\sum_{n=1}^{\infty} P(A_n) < \infty$ *then it holds that* $P(E) = 0$.

Explanation: It means that if additive probability measures in a sequence of events sets is finite then there are only finite number of event sets where probability is 1 (i.e. not infinitely many 1 in probability measure of event sets in sequence).

Proof of lemma uses decreasing sequence of event sets.

Proof One notes that F_n is a decreasing set of events. This is simply so because

$$F_n = \bigcup_{k=n}^{\infty} A_k = A_n \bigcup \left(\bigcup_{k=n+1}^{\infty} A_k \right) = A_n \bigcup F_{n+1}$$

and thus

$$F_n \supset F_{n+1}.$$

$$\mathbf{P}(E) = \mathbf{P}(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mathbf{P}(F_n) = \lim_{n\to\infty} \mathbf{P}(\bigcup_{k=n}^{\infty} A_k).$$

We have, however, by subadditivity that

$$\mathbf{P}(\bigcup_{k=n}^{\infty} A_k) \le \sum_{k=n}^{\infty} \mathbf{P}(A_k)$$

and this sum $\to 0$ as $n \to \infty$, if the sum $\sum_1^{\infty} \mathbf{P}(A_k)$ converges. Thus we have shown the proposition, as claimed.

Explanation about proof: As we have shown that $\exists l \in R^+, \sum_{n=1}^{\infty} P(A_n) = l$ due to convergence, hence it is true that $\exists m \in Z^+, \alpha \in R^+, m + \alpha = l$ (because $\sum_1^m 1 = m$, loosely speaking).

-- Expected value of random variable

Suppose $X : \Omega \to R$ is a random variable and the random variable takes the values in real number set given by:

$\{x_1, \ldots, x_m\}$

The expected value of the random variable is defined as:

$$E[X] \stackrel{\text{def}}{=} \sum_{i=1}^{m} x_i \mathbf{P}(X = x_i)$$

Note that, here the probability symbol means this:

$$\mathbf{P}(X = x_i) \equiv \mathbf{P}(\{\omega \in \Omega \mid X(\omega) = x_i\})$$

---(Another version)---

We can define expected value of a random variable in another way (*with exactly same meaning*) as:

$$E[X] = \sum_{\forall e_i \in \Omega} P(e_i \in \Omega).X(e_i)$$

Here we have considered discrete probability and the entire event set.

In case of the continuous probability measure (*not discrete*), it is given as ($\omega \in \Omega$):

$$E[X] = \int_{\Omega} X(\omega) d\mathbf{P}(\omega)$$

Reason for integration is that: now the random variable and the associated probability function are two continuous functions.

-- Probability distributions of random variable *X*, p.d.f and p.m.f

We start by defining the continuous random variables. Let first $f_X(x)$ be function such that

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1, \quad f_X(x) \geq 0, \quad \text{for all } x \text{ in } \mathbf{R}.$$

Then the function

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

is a probability of continuous random variable *X*. Note here that: $f_X(.)$ is continuous.

This means two things here: (1) $d/dx(F_X) = f_X$ and (2) $P(a \leq X \leq b) = \int_a^b f_X(x)dx$ (Meaning: it is probability measure of random variable *X* in interval [a,b]).

The function $f_X(x)$ is called probability density function (p.d.f.) of *X*.

Considering continuity we can estimate probability of any arbitrary *A* using p.d.f. as:

$$\mathbf{P}(X \in A) = \int_A f_X(x) dx.$$

Now we consider the discrete probability measures. In discrete case, the distribution is called probability mass function (p.m.f.). It is given as:

$$F_X(x) = \sum_{x_k \le x} p_X(x_k),$$

where

$$p_X(x_k) = \mathbf{P}(X = x_k).$$

The function $p_X(x_k)$ is called the **probability mass function** (p.m.f.) of X. Then it must hold that

$$\sum_{k=-\infty}^{\infty} p_X(x_k) = 1, p_X(x_k) \ge 0.$$

So if we look back to random variables (r.v.) then we can compute expectations using p.d.f. and p.m.f. as:

$$E[X] = \begin{cases} \sum\limits_{k=-\infty}^{\infty} x_k p_X(x_k) & \text{discrete r.v.,} \\ \int\limits_{-\infty}^{\infty} x f_X(x)\, dx & \text{continuous r.v.} \end{cases}$$

-- Condition of expected value to exist

In the statement of condition below, the function $F(.)$ is the probability distribution function.

For a real valued random variable, its expectation exists if $E[X^+] := \int_0^\infty xF(dx) < +\infty$ and $E[X^-] := -\int_{-\infty}^0 xF(dx) < +\infty$. Then the expectation is given by

$$E[X] = E[X^+] - E[X^-].$$

If we encounter a case where $E[X^+] = \infty$ and $E[X^-] = \infty$, the expected value is not defined.

Assignments:

Q1. Prove that if $\sum_{n=1}^{\infty} P(A_n) \to +\infty$ then $P(E) \ne 0$ in Borel-Cantelli lemma.

Q2. If $g : \Omega \to (A \subset \Omega)$ is an identity function then prove that $E[X] \ge E[X(g)]$, where $X : \Omega \to R$ is a random variable (discrete or continuous).