Probability Theory– IV

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In this study material, we will deal with Borel sigma algebra, probability distribution functions and random variables. We will also introduce to Doob-Dynkin theorem.

We will first develop the definition of Borel sigma algebra (named after mathematician E. Borel).

-- Borel sigma algebra

The Borel sigma algebra or sigma field on event set Ω is a collection of subsets A_Ω if it satisfies following axioms:

1. $\Omega \in A_\Omega$
2. $A \in A_\Omega \Rightarrow A^c \in A_\Omega$
3. $\{A_i : 1 \le i \le n\} \subset A_\Omega \Rightarrow \bigcup_{1 \le i \le n} A_i \in A_\Omega$

Explanations:

The first axiom says that the entire event set is a member of Borel sigma field. The reason is that total probability measures of entire event space should be 1 in any case and so we need to include it to maintain upper bound.

The second axiom says that if an event subset is a member then the complement of it should also be a member of sigma field. Sure, because the logic is that probability assignments (discrete or continuous) should address every events in the event set. None can be left outside.

The third axiom says that finite union of event subsets is a member of Borel sigma field if each subset is a member of the algebraic field. This is to maintain consistency of probability measures over the Borel sigma field such that probability measures of an event subset does not cross the probability of bigger event set including it.

REMARK: Note that, this definition of Borel sigma field does *not* change the definition of probability function. It can be defined as:

$$P : A_\Omega \rightarrow [0,1]$$

So, we are now measuring probability over the Borel sigma field of event set.

-- Probability measure space

Now we can prepare probability measure space based on Borel sigma field. It is given as:

$$\wp = (\Omega, A_\Omega, P)$$

Recall that, here Ω is entire probability event set, A_Ω is Borel sigma field over probability event set and, P is probability function over sigma field.

Now we are capable to handle the random variables appropriately. Conceptually, a random variable is a function that assigns events to values in a range set. Theoretically, the range set need not be subset of real numbers always, although for computational simplicity it is considered so. Let us now define a random variable.

-- Random variables

Consider R be a set of real numbers. A real-valued random variable is a real-valued function $X:\Omega \to R$ such that $\forall A \in B_R$ the following condition is satisfied:

$$X^{-1}(A) = \{w : X(w) \in A\} \in A_\Omega$$

NOTE: Here B_R is Borel sigma algebra over real numbers.

Explanations:

Clearly the above definition says that a random variable actually connects Borel sigma fields between probability event set and the real numbers. Also note that, random variable is measurable in the sense that, $\forall w \in \Omega, -\infty < X(w) < +\infty$.

-- Doob-Dynkin Theorem

Let X, Y be two random variables such that X is real valued random variable and Y maintains following condition (σ_X is sigma field over Ω for X):

$$Y^{-1}(A \in B_R) = \{w : Y(w) \in A\} \in \sigma_X$$

Then, there is a Borel function $f_{Borel} : R \to R$ such that $Y = f_{Borel}(X(\sigma_X))$.

Proof: Non trivial and so omitted for simplicity.

Explanation: The Doob-Dynkin theorem says that: if two random variables assign values of events in real sigma field such that, one random variable maintains sigma field of another random variable, then there is a function from real sigma field to real sigma field connecting the two sigma fields of two random variables.

-- Probability Distribution Function of Random Variable

The random variable X can have probability distribution denoted by $F(x \in R) = P(\{X \leq x\})$ such that:

$$F(x) = P(\{X \leq x\}) \equiv P(\{w \in \Omega : X(w) \in (-\infty, x]\})$$

Explanation: It says that given a random variable, we can find out the probability distribution of that random variable within a given upper bound by looking at probabilities assigned to events such that random variable assigns those events within that fixed given upper bound.

Assignments:

Q1. State the mathematical condition such that $P(A_\Omega \setminus \Omega) < 1$.

Q2. Prove that if $\{A_i : 1 \leq i \leq n\} \subset \sigma_X$ such that $\bigcup_{1 \leq i \leq n} A_i = \Omega$ and $P(\bigcup_{1 \leq i < k} A_i) = a \in [0,1]$ then $P(\bigcup_{k \leq i \leq n} A_i) \in [0,1] \setminus \{a\}$ if $a = 0$ and, $k < n$.