Set Algebra for Probability Theory– II
[Listen to uploaded audio files too if needed]

In last study material, we were introduced to the concepts of propositions and sets along with set algebra preliminaries.

In this material, we will get introduced to Cartesian products and functions along with relations, which are needed for understanding probability theory fundamentals.

First let us get introduced to functions (note that probability is actually a function of special kind!!).

First note that, here we will use A^c to understand the complement of set *A*. Recall that a complement is a set of elements outside of that given set.

-- Functions

Let, A and B are two sets. A function from *A* to *B* is defined as:

$f : A \to B$

Here we need to understand few terminologies. Set *A* is called the domain of function and set *B* is called co-domain of the function. A function actually means (generally, but not always in special cases in higher mathematics) that $\forall x \in A, \exists y \in B, y = f(x)$. It means the function maps elements of set in domain to elements in co-domain.

In such case, the preimage of a function is given by f^{-1} where it is given as:

$f^{-1} : B \to A$

It means the preimage maps from co-domain to domain such that, $x = f^{-1}(y)$.

There are three main classes of functions such as:

Injective function:

An injective function is defined as: $f : A \to (E \subset B)$ such that $\forall x \in A, \exists y \in E, y = f(x)$ and if $f(x) = f(w)$ then $x = w$.

It means that injective function maps into subset and it is one-to-one type.

Surjection function:

A surjective function is defined as: $f : A \to B$ such that $\forall x \in A, \exists y \in B, y = f(x)$ and $\exists x, w \in A, f(x) = f(w)$.

It means that a surjective function maps whole co-domain and there can be multiple elements mapping to same element in co-domain.

Bijection function:

A function is bijection iff it is both injective and surjective.

Let us look into some properties of functions:

(a) $$f^{-1}(A'^{c}) = (f^{-1}(A'))^{c}$$

(b) $$f^{-1}\Big(\bigcup_i A'_i\Big) = \bigcup_i f^{-1}(A'_i)$$

(c) $$f^{-1}\Big(\bigcap_i A'_i\Big) = \bigcap_i f^{-1}(A'_i).$$

-- Cartesian product

The Cartesian product is named after great mathematician and philosopher Rene DesCartes.

If we are given two sets *A* and *B* then the Cartesian product is defined as:

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$$

The important thing to note here is that the pair of elements (*a*, *b*) are ordered. What does it mean? It means: (*a*, *b*) and (*b*, *a*) are not same or equivalent elements. Here order of appearance matters!

-- Power set

The power set if a very important concept, which is heavily used in probability theory. A power set of a set S is given by:

$$\mathscr{P}(S) = \{A \mid A \subset S\}$$

It means that power set is a set of subsets of a given set. Naturally we have ϕ, S as elements of the power set of S.

-- Sequence of sets

A sequence of sets is given by:

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

Note that this is an increasing sequence because we are moving into bigger and bigger sets slowly.

However, the opposite is also possible where we move into smaller and smaller subsets slowly. That sequence is a decreasing sequence of sets.

Interestingly, the sequences of sets have effects on probability computations as we will see later.

Let us now look into a few important properties of sets and functions together.

Cardinality of set:

The cardinality of a set is the number of elements present in a set. It is denoted by $|A|$ to understand the cardinality of set A. Caution: the notation looks like mod function but it is not that.

Countability of set:

A set is called countable if there is a function $f : N \rightarrow A$ which is a bijection function where N is set of natural numbers. If no such function is found then the set A is called uncountable.

Note that not all sets are countable!! For example, set of real numbers is *not* countable.

Assignments:

Q1. Prove that, if $A_i \subset R$ (R is set of real numbers) and $S = \{A_i : i \geq 0\}$ then $\forall A_i \in S, \bigcap_{i \in I} A_i \neq \phi$ if $A_i \subset A_k, i > k$.

Q2. Prove that a surjective function does not have preimage in traditional sense.

Q3. Prove that if $E \subset A \times B$ and $f : E \to R$ such that $f((a,b)) = a + b$ then f may not have inverse. Here R is a set of real numbers.

Q4. The time counted by humans everyday is a continuous real number system which appears as circular. However, the set of real numbers is not countable and the cosmological time does not appear as circular. Do you think we have a contradiction here? If so, then what is wrong? Detail with mathematical justification/proof. Otherwise, if you think that time-counting system by humans are correct then mathematically prove/justify it showing that there is no contradiction in time-counting system.