Sequences of Random Variables

[Listen to uploaded audio files too if needed]

In this study material, we will start with sequence of random variables. We will also study the convergences of such sequences.

-- Definition of a sequence

First, this definition is about any sequence (i.e. general sequence), not specifically related to random variable.

A sequence is given by $f : N \to A$, where N is set of natural numbers and A is any arbitrary set.
As a result a sequence is generated, which is given as: $(a_i)_{i=1}^{+\infty}, a_i \in A$. Note that, here $i \in N$.
Two very important things are associated to a sequence: either a sequence converges OR a sequence never converges.

-- Sequence of Random Variables

A sequence of random variables is given by $(X_i)_{i=1}^{+\infty}$, where each X_i is a random variable and as usual it is given as: $X_i : \Omega \to R$. One very important condition is: $\forall X_i, -\infty < X_i < +\infty$.

For simplicity of writing, the sequence $(X_i)_{i=1}^{+\infty}$ is often denoted as X_n without altering any meaning of sequence of random variables. There are three main important concepts, which are associated to X_n.

-- Concept 1: Probabilistic convergence of sequence X_n

$$X_n \textit{ converges in probability to } X \textit{ if for all } \varepsilon > 0,\ \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0.$$

Explanations:

The above convergence criteria says that the random variable sequence X_n effectively converges to *another* random variable X such that the probability of difference between X_n at infinity and X is always within a band of small positive real number ε.

That is why the probability of difference between X_n and X at infinity to be more than band ε is 0.

-- Concept 2: Almost-Sure-Convergence of sequence X_n

X_n *converges almost surely to* X *if* $P(\lim_{n\to\infty} X_n = X) = 1.$

Explanations:

It says that effectively at infinity the sequence of random variables X_n will converge to another random variable X for sure. That is the reason that, the probability of X_n becoming equal to X is certain (=1) at infinity.

-- Concept 3: Convergence distribution of sequence X_n

X_n *converges in distribution to* X *if for all bounded, continuous mappings* $f : \mathbb{R} \to \mathbb{R}$, $\lim_{n\to\infty} Ef(X_n) = Ef(X)$.

Explanations:

It says that, there is another random variable X and a real-valued function f available such that at infinity the expected value of $f(X_n)$ = expected value of $f(X)$. So, in another way, we can say that expected values of a sequence of random variables effectively converges into expected value of another random variable.

This concept results in following lemma.

-- Lemma: Weak Convergence

If X_n is a sequence of random variables with probability distributions μ_n and X is another random variable with probability distribution μ then $\lim_{n\to+\infty}(Ef(X_n)) = Ef(X)$ *if and only if* $\lim_{n\to+\infty} \mu_n = \mu$.

The proof is classical and very compact as stated below:

Proof. We have $Ef(X_n) = \int f \circ X_n \,\mathrm{d}P = \int f \,\mathrm{d}X_n(P) = \int f \,\mathrm{d}\mu_n$, and by similar arguments, $Ef(X) = \int f \,\mathrm{d}\mu$. From these observations, the result follows. $\square$

Explanations:

Proof is considering continuous probability distribution case. So, we can compute $Ef(X_n)$ as an integral over function composition. This effectively must converge to $\int fd\mu$ (considering this integral is convergent).

-- A few very important probability distributions

Binomial distribution: $P(n,e,k) = C_k^n .e^k .(1-e)^{n-k}$, where first factor is combinatorial (*n* choose *k*). *n* is number of times experiment done, *e* is probability of an event to happen, *k* is confirmed outcome of desired event. This distribution is discrete type.

Geometric distribution: $P(X = n) = (1-q)^{n-1}.q$, here probability of success in each experiment is *q* and *n* is number of experiments done. This is also discrete type distribution.

Poisson distribution: $P(X = n) = (\lambda^n .e^{-\lambda})/n!$, here e = 2.71828….. is Euler number, *n* is number of successful event outcome from experiments, λ is positive real number representing expected value of *X*. This is also discrete type distribution.

Gauss-Laplace normal distribution: This is continuous type distribution. The p.d.f. is given by: $f(x) = [1/(\sigma(\sqrt[2]{2\pi}))][e^{-(x-\mu)^2/2\sigma^2}]$, here μ is a mean of distribution, σ is standard deviation, σ^2 is variance.

Another important continuous probability distribution is Chi-square distribution (omitted here for complexity, it needs gamma function).

Assignments:

Q1. Compare geometric and Poisson distributions by using numerical computations and graphs.

Q2. Check whether convergence distribution of X_n (i.e. Concept 3) holds or not if *f* is a real valued constant function. Prove your result.

Q3. Start from probabilistic convergence (Concept 1) and derive to Almost-Sure-Convergence (Concept 2) by using limiting value $\varepsilon \to 0$.