Assignment 04

Dep. AI Convergence Engineering

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1. **Q1. Prove the chain rule using set algebra.**

Answer:

Let's consider two functions f(x) and g(y), where y = f(x) and z = g(y). We want to find the derivative of the composition z = g(f(x)) with respect to x.

Define the sets:

- X be the set of input values for f.
- Y be the set of output values for f, and Y' be the set of output values for g.
- Z be the set of output values for the composition g(f(x)).

Define the mappings:

- f : X → Y
- g : Y' → Z

Express the composition z = g(f(x)) using sets:

- Z = {z∣z = g(y), y∈Y}
- y=f(x), we have Y = {y∣y = f(x), x∈X}

Consider the change in z as x changes:

- Δz = g(y + Δy) − g(y), where Δy = f(x + Δx) − f(x)

Rewrite Δz using set notation:

- Δz = $\{z' \mid z'$ = g(y' + $\Delta y'$), y' ∈Y, $\Delta y'$ ∈ $Y'\}$ − {z∣z = g(y), y ∈ Y}

Substitute Δz and Δx using set notation:

- $\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\{z' \mid z' = g(y' + \Delta y'), \Delta y' \in Y'\} - \{z|z = g(y), y \in Y\}}{\Delta x}$

Simplify the expression and apply the limit to get the derivative of the composition:

- $\frac{dz}{dx} = \{ z^{''} | z^{''} = g^{'}(g^{''}) . f^{'}(x),\ y^{''} \in Y^{'},\ x \in X \}$

This result shows that the derivative of the composition g(f(x)) with respect to x is the product of the derivative of g with respect to its input and the derivative of f with respect to its input, evaluated at their respective points. This is essentially the chain rule expressed in terms of set algebra.

2. **Q2. Prove that if $A_1,\ A_2 \subset \Omega$ and $A_1 \cap A_2 \neq \varphi$ such that $P(A_1) < P(A_2 \backslash A_1 \cap A_2)$ then $\exists \varepsilon > 0$ such that $P(A_1 \backslash A_1 \cap A_2) \geq \varepsilon^{-1} . P(A_2)$.**

Answer:

Given:

- $A_1,\ A_2 \subset \Omega$
- $A_1,\ A_2 \neq \varphi$ and
- $P(A_1) < P(A_2 \backslash A_1 \cap A_2)$

Let's denote $B = P(A_1 \cap A_2)$

Given $P(A_1) < P(A_2 \backslash B)$ from this we can define,

- $P(A_2 \backslash B) - P(A_1) > 0$
- $P(A_2) - P(B) - P(A_1) > 0$
- $P(A_2) > P(A_1) + P(B)$

We are aim to find $\exists \varepsilon > 0$

From $P(A_1 \backslash A_1 \cap A_2) \geq \varepsilon^{-1} . P(A_2)$,

$\Rightarrow P(A_1) - P(B) \geq \varepsilon^{-1} . P(A_2)$

$\Rightarrow P(A_1) \geq \varepsilon^{-1} . P(A_2) + P(B)$

$\Rightarrow P(A_1) \geq \varepsilon^{-1} . P(A_2) + P(B)$

$\Rightarrow \varepsilon . P(A_1) \geq P(A_2) + P(B)$ (iff ε > 0 this is true)

$\Rightarrow \varepsilon \geq \frac{P(A_2) + P(B)}{P(A_1)}$ (as $A_1 \neq \varphi$ given so P(A_1) > 0)

From the equation numerator and denominator both are positive so ε has a value which is greater than 0.

3. **Q3. Prove that if Ω is discrete uniform probability space and $A_1 \subset \Omega$, $A_2 \subset \Omega$ such that $A_1 \cup A_2 = \Omega$ then $[P(A_1) = P(A_2)] \Leftrightarrow [|A_1| = |A_2|]$**

Answer:

Given:

- Ω is a discrete uniform probability space.
- A1 and A2 are subsets of Ω such that $A_1 \cup A_2 = \Omega$.

We want to prove:

- $P(A_1) = P(A_2)$
- $|A_1| = |A_2|$

For any event A in a discrete uniform probability space Ω:

- $P(A_1) = \frac{|A_1|}{|\Omega|}$
- $P(A_2) = \frac{|A_2|}{|\Omega|}$

Now, let's prove each statement:

Proving Statement 1: $P(A_1) = P(A_2)$

Given that $A_1 \cup A_2 = \Omega$, we have:

- $P(\Omega) = P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

Since $P(A_1)$ and $P(A_2)$ partition Ω, $P(A_1 \cap A_2) = 0$

So we have,

- $P(A_1) + P(A_2) = 1$

Given the symmetry of the situation, it implies that $P(A_1) = P(A_2)$

Proving Statement $|A_1| = |A_2|$

We're given that $A_1 \cup A_2 = \Omega$., so every outcome in Ω belongs to either A_1 or A_2 (or both).

Mathematically, this can be expressed as:

- $|\Omega| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Since $|A_1|$ and $|A_2|$ partition Ω, $|A_1 \cap A_2| = 0$

So we have,

- $|\Omega| = |A_1| + |A_2| = 1$

Given that Ω is a discrete uniform probability space, every outcome is equally likely. Therefore, $|A_1| = |A_2|$