Set Algebra for Probability Theory– I
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The original concept of probability theory was originated from analytic set theory or commonly called as set algebra. In this study material, we will cover basic notions of set algebra with analytic view.

-- Mathematical Logic Quantifiers

There are two quantifiers in mathematical logic called: for all elements (denoted by $\forall$) and there exists one (denoted by $\exists$).

For example:

If $x^2+1=0$ then $\forall r$ not a complex number, r is not a root of $x^2+1=0$.
However, $\exists r$ on unit circle on Gauss plane such that, r is a root of $x^2+1=0$.

-- Proposition $P(x_1, x_2, ...x_n)$

Let $< x_i >_{i=1}^{n}$ be a point in *n*-dimensional space, where n is positive non-zero integer. We consider x_i is a variable in *i*-th dimension such that $1 \le i \le n$. The unambiguous statement evaluator $P(x_1, x_2, ...x_n)$ over the *n*-variables is called a proposition if and only if $P(x_1, x_2, ...x_n)$ can take only binary values 0 or 1 discretely at a time for a given combination of values of x_i and nothing in between.

For example:

If $x^2-1=0$ and y is a real number such that $P(y) \equiv [y = \pm 1]$ then $P(y)$ is a proposition determining $x^2-1=0$.

Sometime for simplicity, one can write new proposition $P_x(x) \equiv [x^2 = 1]$ and in this case $P_x(x) \cong P(y)$.

If $P_x(x)$ *always* indicate $P(y)$ then we say, $P_x(x) \Rightarrow P(y)$. This means left proposition implies right proposition.

-- Tautology and Contradiction

A proposition $P(x_1, x_2, ...x_n)$ is called a tautology if for any combinatorial values of $< x_i >_{i=1}^{n}$ the $P(x_1, x_2, ...x_n) = 1$.

On the other hand, if $P(x_1, x_2, ...x_n) = 0$ for any combinatorial values of $< x_i >_{i=1}^{n}$ then the proposition is a contradiction.

-- Set

Suppose $P(x)$ is a defined proposition. A set S is an unambiguous collection of elements (discrete or continuous) such that following axiom is satisfied:

$$S = \{x : P(x)\}$$

Note that, $P(x)$ is discrete (always), however set S may *not* be necessarily discrete.

For example:

If $P(x) = 1 \quad x > 1$ then $S = \{x : P(x)\}$ can be discrete or continuous based on nature of element x. If x is a real variable then S is continuous. Otherwise, S is discrete.

Note that, set S can be constructed in following ways too, where *i* is an integer:

$$S = \{x : P_i(x), 1 \le i \le N\}$$

If x is an element of set S then we write: $x \in S$, otherwise $x \notin S$.
The set $S = \phi$ is called empty set and it is an axiomatically valid set.

-- Set algebra

The set algebraic (main) properties are summarized below:

Let *A* and *B* be two sets. The algebraic properties of these two sets are:

Intersection:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Union:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

These algebraic expressions can be generalized further. Let I be an index set.
The set $S = \{A_i : i \ge 0\}$ is a family of sets.

The generalized union and intersection are given by:

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

Note here that, in the above expression the union and intersections are infinite.

Subset:

A subset A of a set S is denoted by $A \subset S$ if $\forall x(x \in A) \Rightarrow (x \in S)$.

Pair-wise Disjoint-ness:

In set $S = \{A_i : i \geq 0\}$ if

$$A_i \cap A_j = \emptyset$$

whenever $i \neq j$ then these two sets are pair-wise disjoint.

Set-difference:

The set-difference between two sets is given as:

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Complement of set:

The complement of a set A is given by:

$$A^{\mathrm{c}} = S \setminus A = \{x \in S \mid x \notin A\}$$

-- De Morgan Theorem:

The De Morgan theorem allows the algebraic interchanges between set union and intersections. The theorem states that:

Let $(A_i)_{i \in I}$ be a non-empty indexed collection of subsets $A_i \subset S$ of a fixed set S. Then we have

$$\Big(\bigcup_{i \in I} A_i\Big)^{\mathrm{c}} = \bigcap_{i \in I} A_i^{\mathrm{c}} \qquad \textit{and} \qquad \Big(\bigcap_{i \in I} A_i\Big)^{\mathrm{c}} = \bigcup_{i \in I} A_i^{\mathrm{c}}.$$

-- Properties of set algebraic operations:

- Commutative: $A \cap B = B \cap A$ and $A \cup B = B \cup A$.
  (But $A - B \neq B - A$).
- Associative: $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$. (also for $\cup$)
- Distributive:

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}$$

Assignments:

Q1. Algebraically prove that if $A \subset B_1$ and $C \subset B_2$ then $A \cap C = \phi$ if $A \cap B_1 \cap B_2 = \phi$, where $B_1 \cap B_2 \neq \phi$ and $B_1 \not\subset B_2, B_2 \not\subset B_1$.

Q2. Assume that $P(x_1, x_2)$ is a proposition. Prepare a set S using $P(x_1, x_2)$ such that $S = \phi$.

Q3. Prove that if A and B are two sets such that, propositions $P(x \in A)$ and $Q(x \in B)$ are tautologies whereas $P(x \in B)$ and $Q(x \in A)$ are contradictions, then $A \cap B = \phi$.