Probability Theory– III

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In this study material, we will deal with Bays theorem, conditions and concept of independence of probability measures.

-- Bays theorem

The Bays theorem proposes an inherent property of conditional probability.

The conditional probability is the probability of some event provided some other event happened. For example: probability of eating a noodle dish given that chosen noodle dish probabilistically contains red-pepper.

The simplified version of Bays theorem states that:

$$P(A \mid B) = [P(B \mid A).P(A)] / P(B)$$

Proof: We will use set algebra to prove it because that makes proof very compact and logically tight.

Recall that, $P(A \mid B).P(B) = P(A \cap B)$ and $P(B \mid A).P(A) = P(B \cap A)$.
However, due to commutativity, $P(A \cap B) = P(B \cap A)$
Hence, $P(A \mid B) = [P(B \mid A).P(A)] / P(B)$.

-- Generalized version of Bays theorem

The generalized version of Bays theorem looks complicated but it is actually a straight extension of the above theorem. It is as follows.

Let $T = \{A_i \subset S : 1 \le i \le N\}$ is a partition of Ω.

$$\mathbf{P}(A_l \mid B) = \frac{\mathbf{P}(B \mid A_l)\,\mathbf{P}(A_l)}{\sum_{i=1}^{k} \mathbf{P}(B \mid A_i)\,\mathbf{P}(A_i)}, \quad l = 1, 2, \ldots, k.$$

considering $N = k$.

-- Mutual independence of probability

The mutual independence of probability says that one event is not correlated to another event. The events can jointly occur with independence.

That means, the following condition is required to be satisfied:

$$P(A \cap B) = P(A)P(B)$$

Explanations:

This means that elements in intersection of two event sets are independent in assigning probabilities and no one provide any information about another element (uncorrelated).

In case of conditional probability with disjoint event sets, the condition to be satisfied is given as:

$$P(A|B) = P(A)$$

Clearly, the conditional probability of *A* is independent of *B* in this case.

-- Theorem: Independence events with of non-zero probability do not imply mutually exclusive event sets.

Proof: We will use proof by contradiction.
Let $A \subset \Omega$ and $B \subset \Omega$ be two event sets and independent with non-zero probability assignment such that $A \cup B = \Omega$. Let us assume events sets are independent implying they are mutually exclusive.

Hence, due to mutual exclusiveness, $A \cap B = \phi$ and $P(\phi) = 0$.
However, if the event sets are independent then $P(A \cap B) = P(A).P(B)$.
In this case, if independence imply mutual exclusion then $P(A).P(B) = 0$.
Thus, either $P(A) = 0$ or $P(B) = 0$.
Suppose, $P(A) = 0$ and $P(B) \neq 0$. In this case, $P(B) = 1$ because, $A \cup B = \Omega$ and $P(\Omega) = 1$. We arrive to a contradiction because, $P(A) = 0$ is not possible (i.e. probability assignments are non-zero). Similarly, $P(B) = 0$ is also invalid.
Thus, the condition $P(A).P(B) = 0$ is invalid and $P(A).P(B) > 0$.
Hence, $P(A \cap B) = 0 \neq P(A).P(B)$.

-- Conditional independence

The conditional independence law says that, if event sets *A* and *B* are independent given event set *C* then:

$$P(A \cap B|C) = P(A|C)P(B|C)$$

Explanation: It means that if two event sets are independent with respect to a third event set, then the joint independent probability of two independent event sets are separable in product form with respect to the third event set (because, none are dependent on anybody).

Assignments:

Q1. Prove generalized Bays theorem using set algebra.

Q2. Prove that $\lim_{i\to+\infty} P(A_i \cap B_i \mid C_k) \to 0$ if $i,k \in Z^+$ and k is a positive fixed integer and A_i, B_i are conditionally independent.

Q3. Compute the total probability of $\lim_{i\to k} P(A_i \cap B_i \mid C_k) \to 0$, where $i,k \in Z^+$ and $\forall i > 0, P(A_i \mid C_k) = 2^{-m} P(A_{i+1} \mid C_k)$. Assume $k < +\infty$.