10-2 Reduced Density Matrix

partial trace

composite system $A + B$

basis states: $|i\rangle_A \otimes |j\rangle_B \equiv |i, j\rangle$

X_{AB}: operator acting on $\mathcal{H}_A \otimes \mathcal{H}_B$

$$X_{AB} = \sum_{i,j,i',j'} |i, j\rangle\langle i, j|X_{AB}|i', j'\rangle\langle i', j'| = \sum_{i,j,i',j'} X_{ij,i'j'}|i, j\rangle\langle i', j'| \qquad X_{ij,i'j'} = \langle i, j|X_{AB}|i', j'\rangle$$

$$= \sum_{i,j,i',j'} X_{ij,i'j'}|i\rangle_A\langle i'| \otimes |j\rangle_B\langle j'|$$

partial trace $$\mathrm{Tr}_A(X_{AB}) = \sum_{i,j,i',j'} X_{ij,i'j'}\mathrm{Tr}\Big(|i\rangle_A\langle i'|\Big)|j\rangle_B\langle j'|$$

$$= \sum_{i,j,j'} X_{ij,ij'}|j\rangle_B\langle j'|: \text{ operator acting on } \mathcal{H}_B$$

$$\mathrm{Tr}_B(X_{AB}) = \sum_{i,j,i',j'} X_{ij,i'j'}|i\rangle_A\langle i'|\,\mathrm{Tr}\Big(|j\rangle_B\langle j'|\Big)$$

$$= \sum_{i,i',j} X_{ij,i'j}|i\rangle_A\langle i'|: \text{ operator acting on } \mathcal{H}_A$$

ex) $|\psi\rangle = \alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B$, $\rho = |\psi\rangle\langle\psi|$, $\mathrm{Tr}_A(\rho) = ?$, $\mathrm{Tr}_B(\rho) = ?$

$$\boxed{\mathrm{Tr}(X_{AB}) = \mathrm{Tr}_A(\mathrm{Tr}_B(X_{AB})) = \mathrm{Tr}_B(\mathrm{Tr}_A(X_{AB}))}$$

reduced density matrix

state ρ_{AB} of composite system $A + B$

$\Rightarrow$ state of system A (ignoring B): $\rho_A = \mathrm{Tr}_B(\rho_{AB})$

state of system B (ignoring A): $\rho_B = \mathrm{Tr}_A(\rho_{AB})$

why? Given ρ_{AB}, what is the state ρ_A of system A such that $\langle X_A\rangle = \mathrm{Tr}(\rho_A X_A)$ for any operator X_A on system A?

$$\langle X_A\rangle = \langle X_A \otimes I_B\rangle = \mathrm{Tr}[(X_A \otimes I_B)\rho_{AB}] = \mathrm{Tr}_A[\mathrm{Tr}_B\{(X_A \otimes I_B)\rho_{AB}\}] = \mathrm{Tr}_A[X_A\mathrm{Tr}_B(\rho_{AB})]$$
$$= \mathrm{Tr}_A(X_A\rho_A)$$

$\boxed{\text{Any local operation on } A \text{ (unitary, measurement, whatever) cannot alter } \rho_B!}$

$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|11\rangle_{AB}$$
$$\rho_A = |\alpha|^2|0\rangle_A\langle 0| + |\beta|^2|1\rangle_A\langle 1|$$
$$\rho_B = |\alpha|^2|0\rangle_B\langle 0| + |\beta|^2|1\rangle_B\langle 1|$$

$\boxed{A + B: \text{ pure } \rightarrow A \text{ or } B \text{ alone: mixed! (when } A, B \text{ are entangled)}}$