3-1 Linear Algebra

unitary transformation

$n \times n$ matrix U

U is unitary if $\boxed{U^\dagger U = UU^\dagger = I}$

U preserves the inner product.

$$\left.\begin{array}{l} |\psi\rangle \xrightarrow{U} |\psi'\rangle = U|\psi\rangle \\ |\phi\rangle \xrightarrow{U} |\phi'\rangle = U|\phi\rangle \end{array}\right) \Rightarrow \langle\psi'|\phi'\rangle = \langle\psi|U^\dagger U|\phi\rangle = \langle\psi|\phi\rangle$$

$\Rightarrow$ U preserves the orthogonality.

U preserves the norm.

$$\||\psi\rangle\|^2 = \langle\psi|\psi\rangle = \langle\psi|U^\dagger U|\psi\rangle = \|U|\psi\rangle\|^2$$

$\Rightarrow$ $\boxed{U \text{ is a rotation.}}$ U is also called a unitary rotation.

{basis vectors} $\xrightarrow{U}$ {rotated basis vectors} basis transformation

ex) two orthonomal bases: $\{|v_i\rangle\}, \{|w_i\rangle\}$

let $U = \sum_i |w_i\rangle\langle v_i|$

$U|v_i\rangle = \sum_j |w_j\rangle\langle v_j|v_i\rangle = |w_i\rangle, \;\; U^\dagger|w_i\rangle = |v_i\rangle$ (basis change)

you can show $U^\dagger U = UU^\dagger = I$ (unitary)

some definitions

$n \times n$ matrix A

A is Hermitian if $A^\dagger = A$

A is anti-Hermitian if $A^\dagger = -A$

A is unitary is $A^\dagger A = AA^\dagger = I$

A is normal if $A^\dagger A = AA^\dagger$

Hermitian, anti-Hermian, and unitary matrices are all normal.

normal matrices

A normal matrix A can be written as

$A = UDU^\dagger$, U: unitary, D: diagonal

$A = UDU^\dagger \quad \Rightarrow \quad AU = UD$

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \quad \cdots$$

Let $|\psi_j\rangle = U|j\rangle$

$A|\psi_j\rangle = AU|j\rangle = UD|j\rangle = \lambda_j U|j\rangle = \lambda_j|\psi_j\rangle$, $\quad\therefore |\psi_j\rangle$: eigenvector

$\langle\psi_j|\psi_k\rangle = \langle j|U^\dagger U|k\rangle = \langle j|k\rangle = \delta_{jk}$, $\quad\therefore \{\psi_j\}$: orthonormal basis

$$\boxed{A = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|} = U\left(\sum_j \lambda_j |j\rangle\langle j|\right)U^\dagger = UDU^\dagger$$

ex) Find A^n. (n: integer)

operator functions (for normal matrices)

$$A = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$$

Taylor series function $f(x)$

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \Rightarrow f(A) = c_0 I + c_1 A + c_2 A^2 + c_3 A^3 + \ldots$$

$$A^n = \sum_j \lambda_j^n |\psi_j\rangle\langle\psi_j| = UD^nU^\dagger$$

$$\begin{aligned} f(A) &= c_0 UIU^\dagger + c_1 UDU^\dagger + c_2 UD^2U^\dagger + \ldots \\ &= U(c_0 I + c_1 D + c_2 D^2 + \ldots)U^\dagger \\ &= U\begin{pmatrix} f(\lambda_1) & 0 & \cdots \\ 0 & f(\lambda_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}U^\dagger = \sum_j f(\lambda_j)|\psi_j\rangle\langle\psi_j| \end{aligned}$$

normal matrices *

normal matrices $N^\dagger N = NN^\dagger$

$\Leftrightarrow$ $\boxed{N = UDU^\dagger}$ U: unitary, D: diagonal

Hermitian matrices $H^\dagger = H$

$\boxed{\text{Eigenvalues of } H \text{ are real.}}$

unitary matrices $U^\dagger U = UU^\dagger = I$

$\boxed{\text{Eigenvalues of } U \text{ have magnitude 1.}}$ $\lambda = e^{i\theta}$ ex) why?

$\boxed{U = e^{iH}}$ H: Hermitian

HW3-1 Write $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ in the form of $Y = UDU^\dagger$.