☐ Show that any two-qbit state can be written in the basis of Bell states.

Ans: The Bell states are a set of four maximally entangled states for two qubits given by:

① $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

② $|\beta_{01}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$

③ $|\beta_{10}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$

④ $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$

We can rewrite the expression for $|\psi\rangle$ as follows:

$$|\psi\rangle = c_{00}|\beta_{00}\rangle + c_{01}|\beta_{01}\rangle + c_{10}|\beta_{10}\rangle + c_{11}|\beta_{11}\rangle$$

$$= c_{00}\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + c_{01}\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) +$$

$$c_{10}\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) + c_{11}\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$= \frac{1}{\sqrt{2}}\Big\{(c_{00}+c_{10})|00\rangle + (c_{01}+c_{11})|01\rangle$$

$$+ (c_{00}-c_{10})|10\rangle + (c_{01}-c_{11})|11\rangle\Big\}$$

By comparing with original states,

$$= \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

we can see that

$$c_{00} + c_{10} = \alpha \quad \text{——} \; 1$$

$$c_{01} + c_{11} = \beta \quad \text{——} \; 2$$

$$c_{00} - c_{10} = \gamma \quad \text{——} \; 3$$

$$c_{01} - c_{11} = \delta \quad \text{——} \; 4$$

Solving these equation,

$$c_{00} = \frac{\alpha + \gamma}{2}$$

$$c_{01} = \frac{\beta + \delta}{2}$$

$$c_{10} = \frac{\alpha - \gamma}{2}$$

$$c_{11} = \frac{\beta - \delta}{2}$$

Solved