13-1 Quantum Error-Correcting Codes

digitizing error

single qubit $|\psi\rangle \xrightarrow{\text{error}} \mathcal{E}(|\psi\rangle\langle\psi|)$

$\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_k E_k|\psi\rangle\langle\psi|E_k^\dagger$

Environments measure the qubits (Lecture Note 11-2)

$$|\psi\rangle \to \frac{E_k|\psi\rangle}{\sqrt{p(k)}} \quad \text{with probability} \quad p(k) = \langle\psi|E_k^\dagger E_k|\psi\rangle$$

Any 2×2 matrix is a linear sum of I, X, Y, Z.

$E_k|\psi\rangle = \sqrt{1-p_k}I|\psi\rangle + a_kX|\psi\rangle + b_kY|\psi\rangle + c_kZ|\psi\rangle \qquad |a_k|^2, |b_k|^2, |c_k|^2 \sim$ error rate

X: bit-flip error, Z: phase-flip error, $Y = -iZX$: both

But, error is continuous. How to handle?

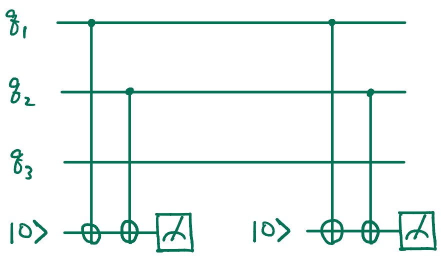
syndrome measurement

ex) Suppose only bit-flip errors occur.

$|0_L\rangle = |000\rangle, \quad |1_L\rangle = |111\rangle$

$X_1(\alpha|000\rangle + \beta|111\rangle) = \alpha|100\rangle + \beta|011\rangle$

How to know the error without changing the state?



syndrome measurement

$(\alpha|100\rangle + \beta|011\rangle)|0\rangle|0\rangle \to (\alpha|100\rangle + \beta|011\rangle)|1\rangle|0\rangle$

$X_2(\alpha|000\rangle + \beta|111\rangle) = \alpha|010\rangle + \beta|101\rangle \to (\alpha|010\rangle + \beta|101\rangle)|1\rangle|1\rangle$

$X_3(\alpha|000\rangle + \beta|111\rangle) = \alpha|001\rangle + \beta|110\rangle \to (\alpha|001\rangle + \beta|110\rangle)|0\rangle|1\rangle$

However,

$X_1X_2(\alpha|000\rangle + \beta|111\rangle) = \alpha|110\rangle + \beta|001\rangle \to (\alpha|110\rangle + \beta|001\rangle)|0\rangle|1\rangle$

$\xrightarrow{\text{correction}} \alpha|111\rangle + \beta|000\rangle \qquad$ wrong state

Suppose we use an $[n, 1]$ error-correcting code, correcting single-qubit bit-flip & phase-flip errors.

Suppose the errors are uncorrelated. (independent errors on each qubit)

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\downarrow$ encoding

$|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$

$\downarrow$ $\mathcal{E}(|\psi_L\rangle\langle\psi_L|) = \sum_k E_k|\psi_L\rangle\langle\psi_L|E_k^\dagger$

sum of various terms: $|\psi_L\rangle, X_1|\psi_L\rangle, \ldots, X_n|\psi_L\rangle, \ldots, Y_1|\psi_L\rangle, \ldots, Z_1|\psi_L\rangle, \ldots, X_1X_2|\psi_L\rangle, X_1Y_2|\psi_L\rangle, \ldots$

$\downarrow$ syndrome measurement & collapse of the state

probability $1 - O(p)$: $|\psi_L\rangle|0\rangle_X|0\rangle_Z$ measuring where X, Z errors occurred

probability $O(p)$: $(X_1|\psi_L\rangle)|1\rangle_X|0\rangle_Z$, $(X_2|\psi_L\rangle)|2\rangle_X|0\rangle_Z$, ...,

$(Y_1|\psi_L\rangle)|1\rangle_X|1\rangle_Z$, $(Y_2|\psi_L\rangle)|2\rangle_X|2\rangle_Z$, ...,

$(Z_1|\psi_L\rangle)|0\rangle_X|1\rangle_Z$, $(Z_2|\psi_L\rangle)|0\rangle_X|2\rangle_Z$, ...

probability $O(p^2)$: $(X_1X_2|\psi_L\rangle)$|some syndrome⟩, $(X_1Y_2|\psi_L\rangle)$|some syndrome⟩, ...

$\downarrow$ correcting error according to the error syndrome

probability $1 - O(p^2)$: $|\psi_L\rangle$

probability $O(p^2)$: |wrong state⟩

The error rate is now $O(p^2)$.

quantum Hamming bound

Consider $[n, 1]$ quantum error-correcting code, correcting single-qubit errors.

What is minimum n?

n qubits: 2^n-dimensional Hilbert space (2^n orthogonal vectors)

correctable errors: $X_1, ..., X_n, Y_1, ..., Y_n, Z_1, ..., Z_n$

$\Rightarrow$ $|0_L\rangle, |1_L\rangle, X_1|0_L\rangle, X_1|1_L\rangle, ..., Y_1|0_L\rangle, Y_1|1_L\rangle, ...$ should be distinguishable, i.e., orthogonal.

$\Rightarrow$ $2(1 + 3n) \leq 2^n$

$\Rightarrow$ $n \geq 5$

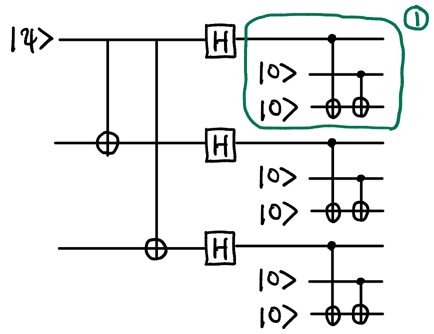
💬 최소 5개 이상의 qubit에 encoding을 해야 quantum error-correction을 할 수 있음.

§10.2 Shor code

$[9, 1]$ code correcting single-qubit errors

$$|0_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$



$\alpha|0\rangle + \beta|1\rangle \to \alpha|000\rangle + \beta|111\rangle \to \alpha|+++\rangle + \beta|---\rangle$

①: $|+\rangle = |0\rangle + |1\rangle \to |000\rangle + |111\rangle$

$|-\rangle = |0\rangle - |1\rangle \to |000\rangle - |111\rangle$

- How to measure $\{|0_L\rangle, |1_L\rangle\}$?

  $X_1X_2X_3|0_L\rangle = \frac{1}{2\sqrt{2}}(|111\rangle + |000\rangle)(\cdots)(\cdots) = |0_L\rangle$

  $X_1X_2X_3|1_L\rangle = \frac{1}{2\sqrt{2}}(|111\rangle - |000\rangle)(\cdots)(\cdots) = -\frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(\cdots)(\cdots) = -|1_L\rangle$

  You measure $X_1X_2X_3$ or $X_4X_5X_6$ or $X_7X_8X_9$.

- How to measure X_i error?

  $X_1|0_L\rangle = \frac{1}{2\sqrt{2}}(|100\rangle + |011\rangle)(\cdots)(\cdots)$

  $X_1|1_L\rangle = \frac{1}{2\sqrt{2}}(|100\rangle - |011\rangle)(\cdots)(\cdots)$

  $Z_1Z_2|00\rangle = |00\rangle, \quad Z_1Z_2|11\rangle = |11\rangle, \quad Z_1Z_2|01\rangle = -|01\rangle, \quad Z_1Z_2|10\rangle = -|10\rangle$

  You measure Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9.

- How to measure Z_i error?

  $Z_1|0_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$

  $Z_1|1_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$

  $X_1X_2X_3X_4X_5X_6(Z_1|0_L\rangle) = -Z_1|0_L\rangle$

  $X_1X_2X_3X_4X_5X_6(Z_1|1_L\rangle) = -Z_1|1_L\rangle$

  $X_1X_2X_3X_4X_5X_6|0_L\rangle = |0_L\rangle$

  $X_1X_2X_3X_4X_5X_6|1_L\rangle = |1_L\rangle$

  You measure $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.