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Question: Write $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ in the form of $Y = UDU^{+}$

Sol^n: The characteristic equation for Y is given by:

$$|Y - \lambda I| = 0$$

given, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

I = identity matrix

$$= \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0$$

Eigenvalues: $\lambda^2 - 1 = 0$; $\lambda = \pm 1$

Eigenvectors for $\lambda = 1$

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad |\psi_1\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Eigenvectors for $\lambda = -1$

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad |\psi_2\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad U = (\psi_1, \psi_2) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$

$$U^{\dagger} = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

Now

$$UU^{\dagger} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

So,

$$UDU^{\dagger} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/2 & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Proved