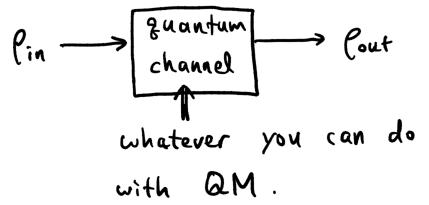
11-2 Quantum Channel

§8.2 Quantum operations



$\rho_{\text{out}} = \mathcal{E}(\rho_{\text{in}})$

ex) unitary evolution

$$\mathcal{E}(\rho) = U\rho U^\dagger$$

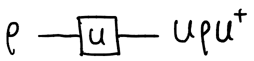
ex) projective masurement

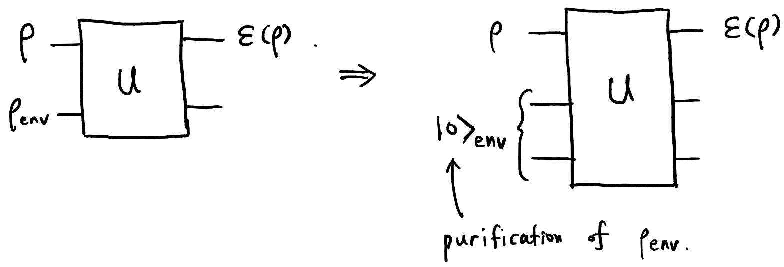
$$\mathcal{E}(\rho) = \sum_i |i\rangle\langle i|\rho|i\rangle\langle i| = \sum_i \langle i|\rho|i\rangle|i\rangle\langle i| = \sum_i p(i)|i\rangle\langle i|$$

$$\rho = (\alpha|0\rangle + \beta|1\rangle)(\alpha^*\langle 0| + \beta^*\langle 1|)$$

$$\mathcal{E}(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1| = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

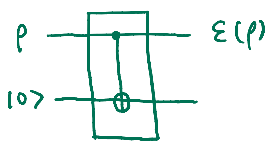
environment and quantum operations

system only: 

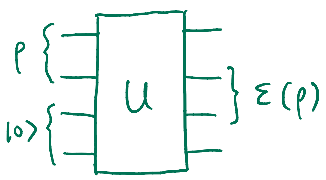
system + environment: 

$$\mathcal{E}(\rho) = \mathrm{Tr}_{\text{env}}\left[U\,(\rho \otimes |0\rangle_{\text{env}}\langle 0|)\,U^\dagger\right]$$

ex) $\mathcal{E}(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$



We also assume $\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}}$.

ex) 

$\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}}$

operator-sum representation

basis states for env： $|e_0\rangle, |e_1\rangle, |e_2\rangle, \cdots$

$$\mathcal{E}(\rho) = \mathrm{Tr}_{\mathrm{env}}\left[U\,(\rho \otimes |e_0\rangle\langle e_0|)\,U^\dagger\right]$$

$$= \sum_k \langle e_k|U\,(\rho \otimes |e_0\rangle\langle e_0|)\,U^\dagger|e_k\rangle$$

$$= \sum_k \langle e_k|U|e_0\rangle\rho\langle e_0|U^\dagger|e_k\rangle$$

$$\boxed{\mathcal{E}(\rho) = \sum_k E_k\rho E_k^\dagger}$$

$E_k \equiv \langle e_k|U|e_0\rangle$ Kraus operator

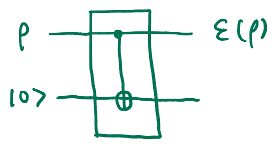
$$\sum_k E_k^\dagger E_k = \sum_k \langle e_0|U^\dagger|e_k\rangle\langle e_k|U|e_0\rangle = I$$

$$\boxed{\sum_k E_k^\dagger E_k = I}$$

trace-preserving

$$\mathrm{Tr}[\mathcal{E}(\rho)] = \mathrm{Tr}\Big[\sum_k E_k\rho E_k^\dagger\Big] = \mathrm{Tr}\Big[\sum_k E_k^\dagger E_k\rho\Big] = \mathrm{Tr}(\rho) = I$$

ex)



$$U = |0\rangle_s\langle 0| \otimes I_e + |1\rangle_s\langle 1| \otimes X_e = U^\dagger$$

$$E_0 = {}_e\langle 0|U|0\rangle_e = |0\rangle_s\langle 0|\,{}_e\langle 0|I_e|0\rangle_e + |1\rangle_s\langle 1|\,{}_e\langle 0|X_e|0\rangle_e = |0\rangle_s\langle 0|$$

$$E_1 = {}_e\langle 1|U|0\rangle_e = |0\rangle_s\langle 0|\,{}_e\langle 1|I_e|0\rangle_e + |1\rangle_s\langle 1|\,{}_e\langle 1|X_e|0\rangle_e = |1\rangle_s\langle 1|$$

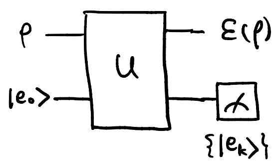
$$\mathcal{E}(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$$

💬 operator-sum representation & general measurement

general measurement： $\sum_k M_k^\dagger M_k = 0$ $\quad M_k$： measurement operator

$$\rho \to \frac{M_k\rho M_k^\dagger}{\mathrm{Tr}(\cdots)} \text{ with probability } \mathrm{Tr}(M_k\rho M_k^\dagger)$$

$$\rho \to \sum_k \mathrm{Tr}(M_k\rho M_k^\dagger)\frac{M_k\rho M_k^\dagger}{\mathrm{Tr}(M_k\rho M_k^\dagger)} = \sum_k E_k\rho E_k^\dagger$$



$$P(|e_k\rangle) = \mathrm{Tr}\left[I \otimes |e_k\rangle\langle e_k|\,\{U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger\}\right]$$

$$= \mathrm{Tr}_s\,[\mathrm{Tr}_{\mathrm{env}}(\cdots)]$$

$$= \mathrm{Tr}_s\left[E_k\rho E_k^\dagger\right]$$

$$\mathcal{E}(\rho) = \sum_k P(|e_k\rangle)\frac{E_k\rho E_k^\dagger}{P(|e_k\rangle)}$$

randomly replacing ρ by $\dfrac{E_k\rho E_k^\dagger}{\mathrm{Tr}(E_k\rho E_k^\dagger)}$ with probability $\mathrm{Tr}(E_k\rho E_k^\dagger)$