14-2 Stabilizer Codes

3-qubit bit-flip code

- $|\psi_L\rangle = \alpha|0_L\rangle + |1_L\rangle = \alpha|000\rangle + \beta|111\rangle$
- $[3,1]$ stabilizer code $C(S)$ defined by stabilizer $S = \langle Z_1Z_2, Z_2Z_3\rangle$.
- $|\psi_L\rangle$ is stabilized by S.
- 3 qubits, 2 generators $\Rightarrow$ 3-2=1 encoded qubit (dimension=2)
- Let β_i = measurement result of g_i (generator), changing $1 \to 0$, $-1 \to 1$
  $\beta = \beta_1\beta_2$ is the error syndrome.
  With no error, $\beta = 00$.
- Suppose bit-flip error X_1 occurred.
  $g_1X_1 = -X_1g_1, \quad g_2X_1 = X_1g_2$
  $g_1(X_1|\psi_L\rangle) = -X_1g_1|\psi_L\rangle = -(X_1|\psi_L\rangle)$
  $g_2(X_1|\psi_L\rangle) = X_1g_2|\psi_L\rangle = (X_1|\psi_L\rangle)$
  $\Rightarrow \quad \beta = 10$
  X_2 error: $g_1X_2 = -X_2g_1$, $g_2X_2 = -X_2g_2 \quad \Rightarrow \quad \beta = 11$
  X_3 error: $g_1X_3 = X_3g_1$, $g_2X_3 = -X_3g_2 \quad \Rightarrow \quad \beta = 01$

HW14-1 Find the stabilizer S for the Shor code

$$|0_L\rangle = \tfrac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle),$$
$$|1_L\rangle = \tfrac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle).$$

[7,4,3] Hamming code

parity check matrix $H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$

codeword $x \xrightarrow{\text{error}} x' = x + e$

error syndrome $\beta = Hx' = He$ (3-bit string)

$S = \langle g_1, g_2, g_3\rangle = \langle Z_4Z_5Z_6Z_7, Z_2Z_3Z_6Z_7, Z_1Z_3Z_5Z_7\rangle$

Single bit-flip error X_i anticommutes with at least one of the generators.

β is given by the measurements of g_1, g_2, g_3.

CSS (Calderbank-Shor-Steane) code

$S = \langle g_1, \ldots, g_6\rangle = \langle Z_4Z_5Z_6Z_7, Z_2Z_3Z_6Z_7, Z_1Z_3Z_5Z_7, X_4X_5X_6X_7, X_2X_3X_6X_7, X_1X_3X_5X_7\rangle$

Note that $[g_i, g_j] = 0$ for any i, j.

Single bit-flip error is detected and located by g_1, g_2, g_3.

Single phase-flip error is detected and located by g_4, g_5, g_6.