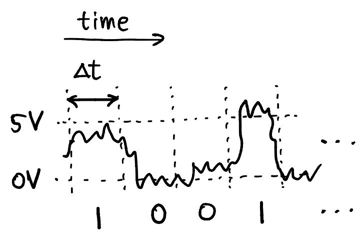
12-2 Threshold Theorem

Q) Can we execute quantum algorithms with arbitrary precision using imperfect machines in a noisy environment?

A) YES, in principle.

Outline of the idea (classical version)

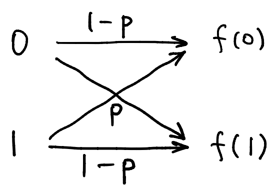
electric signal → bit stream



clock speed $= \dfrac{1}{\Delta t}$

There exist various ways of encoding.

error model



p: error rate (probability of error per operation)

Error is caused by external noise, imperfect operations, etc.

All such effects → single parameter p

QC: a finite number of quantum gates (universal set) ⇒ digial computer (NOT analog)

suppose an error-free operation $|\psi\rangle \to U|\psi\rangle$

noisy operator $\rho \to \mathcal{E}_{\text{noisy}}(\rho) = \mathcal{E}_{\text{error}}(U\rho U^\dagger)$ (exact U followed by $\mathcal{E}_{\text{noisy}}$)

$$\mathcal{E}_{\text{error}}(\rho) = (1-p_0)\rho + \sum_{k\neq 0} p_k E_k \rho E_k^\dagger \qquad p_k: \text{ error rates}$$

various error models

error-correcting code

logical bits $0_L = 000,\ 1_L = 111$ repetition code

majority voting $100,\ 010,\ 001 \to$ interpreted as "0"

$110,\ 101,\ 011 \to$ interpreted as "1"

error rate $p_1 = 3p^2(1-p) + p^3 = 3p^2 - 2p^3 \sim p^2$

$p_1 < p$ for $p < 1/2$

QC: You need to consider both bit-flip and phase-flip errors.

You need to consider no-cloning theorem and the state colapse by measurement.

You need to use fault-tolerant operations (no propagation of error by error correction).

concatenation of codes

encoding	# of bits	error rate
$0_{L_1} = 000, \quad 1_{L_1} = 111$	3	$\sim p^2$
$0_{L_2} = 0_{L_1}0_{L_1}0_{L_1}, \quad 1_{L_2} = 1_{L_1}1_{L_1}1_{L_1}$	3^2	$\sim p^4$
$0_{L_3} = 0_{L_2}0_{L_2}0_{L_2}, \quad 1_{L_3} = 1_{L_2}1_{L_2}1_{L_2}$	3^3	$\sim p^8$
ℓ-th level	3^ℓ	$\sim p^{2^\ell}$

fault-tolerance threshold

p_c : maximum p satisfying $p_1 < p$ (depending on error models, error-correcting codes, etc.)

If $p < p_c$, you can do quantum computation with any precision in principle.

But, concatenated codes are NOT efficient!