1-3 Linear Algebra

linear? linear map f: (vector) → (vector)

$f(v+w) = f(v) + f(w)$ v, w: vectors

$f(cv) = cf(v)$

$$\left.\right\} f(v) \to \underbrace{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}_{f} \underbrace{\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}}_{v}$$ matrix operation

We will consider vectors in a Hilbert space.

Hilber space: a set of vectors with complex coefficients (계수가 복소수인 벡터의 모음)

cf) Euclidean space $\vec{v} = v_x\hat{x} + v_y\hat{y}$ $v_x, v_y \in \mathbb{R}$ (real numbers)

	Hilbert space	Euclidean space
ket vectors	$\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$ $\alpha, \beta \in \mathbb{C}$	$\vec{A} = a\hat{x} + b\hat{y}$ $a, b \in \mathbb{R}$
basis vectors	$\|0\rangle, \|1\rangle, \ldots$	$\hat{x}, \hat{y}, \ldots$
bra vectors	$\langle\psi\|$	$\vec{A}\cdot$
inner product	$\langle\psi\|\phi\rangle$	$\vec{A}\cdot\vec{B}$

complex conjugate $z = x + iy$ $z^* = x - iy$ $(z^*)^* = z$

ket vectors $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $|\phi\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$ $\underbrace{|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{basis vectors}}$ column vectors

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$

dual vectors $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \xleftrightarrow{\text{dual}} \langle\psi| = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^\dagger$ adjoint

bra vectors $\underbrace{\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}}_{\text{basis vectors}}$ row vectors

$\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$

inner product $\langle\psi|\phi\rangle = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \alpha^*\gamma + \beta^*\delta$

$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 =$ (length of $|\psi\rangle$)^2

$\||\psi\rangle\| \equiv \sqrt{\langle\psi|\psi\rangle}$ norm

$\langle 0|0\rangle = \langle 1|1\rangle = 1$, $\langle 0|1\rangle = \langle 1|0\rangle = 0$ orthonomal basis

$\Rightarrow \langle j|k\rangle = \delta_{jk}$ delta function

$\Rightarrow \langle\psi|\phi\rangle = (\alpha^*\langle 0| + \beta^*\langle 1|)(\gamma|0\rangle + \gamma|1\rangle) = \alpha^*\gamma + \beta^*\delta$

If $\langle\psi|\phi\rangle = 0$, then $|\psi\rangle$ and $|\phi\rangle$ are orthogonal.

tensor product

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ \beta \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix} \end{array} \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$$

$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) = \alpha\gamma|0\rangle \otimes |0\rangle + \alpha\delta|0\rangle \otimes |1\rangle + \beta\gamma|1\rangle \otimes |0\rangle + \beta\gamma|1\rangle \otimes |1\rangle$

$|0\rangle \otimes |0\rangle \equiv |0\rangle|0\rangle \equiv |00\rangle$

ex) $|\psi_1\rangle = |0\rangle + 2|1\rangle, \quad |\psi_2\rangle = 2|0\rangle + 3|1\rangle$

$\||\psi_1\rangle\| = ?, \quad \||\psi_2\rangle\| = ?, \quad |\psi_1\rangle \otimes |\psi_1\rangle = ?, \quad \||\psi_1\rangle \otimes |\psi_1\rangle\| = ?$

linear operators

$|j\rangle\langle k| : \; |k\rangle \to |j\rangle$

$(|1\rangle\langle 0|)|0\rangle \equiv |1\rangle\langle 0|0\rangle = |1\rangle$

$(a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle) = a\alpha|0\rangle + b\beta|0\rangle + c\alpha|1\rangle + d\beta|1\rangle$

A linear operator is a matrix!

$|0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \cdots$

$|j\rangle\langle k|$: j-th row, k-th column

$$A = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1| = \begin{pmatrix}a & b\\c & d\end{pmatrix}$$

$$A|\psi\rangle = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}a\alpha + b\beta\\c\alpha + d\beta\end{pmatrix}$$

ex) $A = \begin{pmatrix}1 & 2\\-2 & 3\end{pmatrix}, \quad |\psi\rangle = \begin{pmatrix}2\\1\end{pmatrix}$

$A|\psi\rangle$ using the bra, ket notation?

$I \equiv |0\rangle\langle 0| + |1\rangle\langle 1|$ identity operator

$I|\psi\rangle = |\psi\rangle$

$X \equiv |0\rangle\langle 1| + |1\rangle\langle 0|$

$X(\alpha|0\rangle + \beta|1\rangle) = \beta|0\rangle + \alpha|1\rangle$

adjoint $(|j\rangle\langle k|)^\dagger = |k\rangle\langle j|$

$\begin{pmatrix}a & b\\c & d\end{pmatrix}^\dagger = \begin{pmatrix}a^* & c^*\\b^* & d^*\end{pmatrix}$ transposition + complex conjugation

$(\alpha|\psi\rangle)^\dagger = \alpha^*\langle\psi| \qquad \alpha \in \mathbb{C}$

$(A|\psi\rangle)^\dagger = \langle\psi|A^\dagger$

$(AB)^\dagger = B^\dagger A^\dagger$

$\langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle$

$A = \begin{pmatrix}a & b\\c & d\end{pmatrix}, \quad B = \begin{pmatrix}q & r\\s & t\end{pmatrix}$

$$A \otimes B = \begin{pmatrix}a & b\\c & d\end{pmatrix} \otimes \begin{pmatrix}q & r\\s & t\end{pmatrix} = \begin{pmatrix}a\begin{pmatrix}q & r\\s & t\end{pmatrix} & b\begin{pmatrix}q & r\\s & t\end{pmatrix}\\ c\begin{pmatrix}q & r\\s & t\end{pmatrix} & d\begin{pmatrix}q & r\\s & t\end{pmatrix}\end{pmatrix} = \begin{pmatrix}aq & ar & bq & br\\as & at & bs & bt\\cq & cr & dq & dr\\cs & ct & ds & dt\end{pmatrix}$$

$$A \otimes B = (a|0\rangle\langle 0| + b|0\rangle\langle 1| + \dots) \otimes (q|0\rangle\langle 0| + r|0\rangle\langle 1| + \dots)$$

$$= aq|00\rangle\langle 00| + ar|00\rangle\langle 01| + \cdots + bq|00\rangle\langle 10| + br|00\rangle\langle 11| + \dots$$

$(A \otimes B)|\psi\rangle \otimes |\phi\rangle = A|\psi\rangle \otimes B|\phi\rangle$

ex) $X = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad Z = |0\rangle\langle 0| - |1\rangle\langle 1|, \quad (X \otimes Z)(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) = ?$