10-1 Mixed States

§2.4 Density operators (Density matrices)

ex) ① $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

② $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

③ $|0\rangle$ with probability $\frac{1}{2}$, $|1\rangle$ with probability $\frac{1}{2}$

$\{|0\rangle, |1\rangle\}$-basis measurement vs $|\pm\rangle$-basis measurement

$|\psi_1\rangle$ $|\psi_2\rangle$ $|\psi_3\rangle$ ⋯ $|\psi_i\rangle$ with prob. p_i.

$$\boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|} \qquad \sum_i p_i = 1, \ \ p_i \geq 0$$

ensemble : collection of events

when you have ρ, you have only a single event (one of $|\psi_i\rangle$'s).

$|0\rangle$ $|1\rangle$ $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}I$ $\qquad$ $|+\rangle$ $|-\rangle$ $\rho = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{2}I$

two states are exactly the same! (no way to distinguish)

pure states vs mixed states (1/2)

$|\psi\rangle$: pure state $(\rho = |\psi\rangle\langle\psi|)$

ρ : mixed state (# of non-zero $p_i \geq 2$)

time evolution $\quad |\psi(t_2)\rangle = U|\psi(t_1)\rangle$

$$\rho(t_2) = U\rho(t_1)U^\dagger \qquad U\sum_i p_i|\psi(t_1)\rangle\langle\psi(t_1)|U^\dagger = \sum_i p_i|\psi(t_2)\rangle\langle\psi(t_2)|$$

expectation value $\quad \langle A\rangle = \langle\psi|A|\psi\rangle$

$\langle A\rangle = \mathrm{Tr}(\rho A)$ $\qquad$ trace?

trace

$$A = \sum_{i,j} A_{ij}|i\rangle\langle j| = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

$$\mathrm{Tr}(A) \equiv \sum_{i=1}^{n} A_{ii} = \sum_i \langle i|A|i\rangle \qquad \text{sum of the diagonal elements}$$

$$\mathrm{Tr}(AB) = \sum_i \langle i|A\,B|i\rangle = \sum_i\sum_j \langle i|A|j\rangle\langle j|B|i\rangle = \sum_i\sum_j \langle j|B|i\rangle\langle i|A|j\rangle = \sum_j \langle j|BA|j\rangle = \mathrm{Tr}(BA)$$

$$I = \sum_j |j\rangle\langle j|$$

$$\boxed{\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)} \qquad \text{cyclic}$$

$\boxed{\mathrm{Tr}(UAU^\dagger) = \mathrm{Tr}(A)}$ The trace is unitarily invariant.

$A = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$ λ_i: eigenvalue

$\boxed{\mathrm{Tr}(A) = \sum_i \lambda_i}$ sum of eigenvalues

ex) Show $\langle A\rangle = \mathrm{Tr}(\rho A)$.

$\mathrm{Tr}(\rho) = 1$

pure states vs mixed states (2/2)

measurement $\{M_m\}$, $\sum_m M_m^\dagger M_m = I$

pure states: $P(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$

$$|\psi\rangle \to \frac{M_m|\psi\rangle}{\sqrt{P(m)}}$$

mixed states: $P(m) = \mathrm{Tr}(M_m \rho M_m^\dagger)$

$$\rho \to \frac{M_m \rho M_m^\dagger}{P(m)}$$

ex) $\rho = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$

measurement in the computational basis

$M_1 = |0\rangle\langle 0|$, $M_2 = |1\rangle\langle 1|$

$P(|0\rangle) = \mathrm{Tr}(M_1 \rho M_1^\dagger) = \mathrm{Tr}(\frac{1}{3}|0\rangle\langle 0|) = \frac{1}{3}$

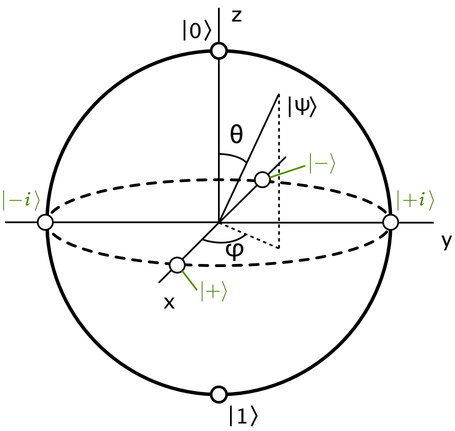
purity

purity $\equiv \mathrm{Tr}(\rho^2)$

pure states: purity $= 1$

mixed states: purity < 1

Bloch sphere



Pauli matrices

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\vec{\sigma} \equiv (X, Y, Z)$

$\vec{r} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\phi)$

$$\boxed{\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}}$$

$$\rho(r,\theta,\phi) = \begin{pmatrix} \frac{1+r\cos\theta}{2} & \frac{r}{2}\sin\theta e^{-i\phi} \\ \frac{r}{2}\sin\theta e^{i\phi} & \frac{1-r\cos\theta}{2} \end{pmatrix} = \frac{1-r}{2}I + r\begin{pmatrix} \frac{1+\cos\theta}{2} & \frac{1}{2}\sin\theta e^{-i\phi} \\ \frac{1}{2}\sin\theta e^{i\phi} & \frac{1-\cos\theta}{2} \end{pmatrix}$$

$$= \frac{1-r}{2}I + r\begin{pmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

$\rho(r,\theta,\phi) = \frac{1-r}{2}I + r|\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)|$

$$\begin{cases} r=1 & \rho = |\psi\rangle\langle\psi| \quad \text{pure state} \\ 0<r<1 & \text{mixed state} \\ r=0 & \rho = \frac{I}{2} \quad \text{maximally mixed state} \end{cases}$$

HW10-1 Show that the eigenvalues of $\rho(\vec{r}) = \frac{I+\vec{r}\cdot\vec{\sigma}}{2}$ are $\frac{1+r}{2}$ and $\frac{1-r}{2}$, and the corresponding eigenvectors are $|\psi(\theta,\phi)\rangle$ and $|\psi(\pi-\theta,\pi+\phi)\rangle$.

(hint) just show $\rho(\vec{r})|\psi(\theta,\phi)\rangle = \frac{1+r}{2}|\psi(\theta,\phi)\rangle$ and $\rho(\vec{r})|\psi(\pi-\theta,\pi+\phi)\rangle = \frac{1-r}{2}|\psi(\pi-\theta,\pi+\phi)\rangle$.

criteria for valid quantum states

1. $\rho = \rho^\dagger$ Hermitian
2. $\rho \geq 0$ positive-semidefinite $\Leftrightarrow$ all eigenvalues of ρ are non-negative.

   $\Leftrightarrow$ $\langle\psi|\rho|\psi\rangle \geq 0$ for $\forall|\psi\rangle$
3. $\mathrm{Tr}(\rho) = 1$ normalization