14-1 Stabilizer Formalism (§10.5)

- If $[A, B] = AB - BA = 0$, there exists a common eigenstate $|\psi\rangle$ of A and B.

  $$A|\psi\rangle = a|\psi\rangle, \quad B|\psi\rangle = b|\psi\rangle$$

- $[X, Z] \neq 0 \quad \Rightarrow \quad$ no common eigenstate of X and Z
- $g_1 = X \otimes X, \quad g_2 = Z \otimes Z$: eigenvalues $= \pm 1 \qquad \because g_1^2 = g_2^2 = I \otimes I$

  $[g_1, g_2] = 0$

  You can define $|\alpha_1, \alpha_2\rangle, \quad \alpha_1, \alpha_2 = \pm 1,$

  $$g_1|\alpha_1, \alpha_2\rangle = \alpha_1|\alpha_1, \alpha_2\rangle, \quad g_2|\alpha_1, \alpha_2\rangle = \alpha_2|\alpha_1, \alpha_2\rangle.$$

  $|+1, +1\rangle, |+1, -1\rangle, |-1, +1\rangle, |-1, -1\rangle$ are Bell states!

  $(X \otimes X)(|00\rangle + |11\rangle) = |00\rangle + |11\rangle$

  $(Z \otimes Z)(|00\rangle + |11\rangle) = |00\rangle + |11\rangle$

  $|+1, +1\rangle = |00\rangle + |11\rangle$

  $(X \otimes X)(|01\rangle - |10\rangle) = |10\rangle - |01\rangle = -(|01\rangle - |10\rangle)$

  $(Z \otimes Z)(|01\rangle - |10\rangle) = -|01\rangle + |10\rangle = -(|01\rangle - |10\rangle)$

  $|-1, -1\rangle = |01\rangle - |10\rangle$

- $S = \langle X \otimes X, Z \otimes Z\rangle$ is called a stabilizer.

  $X \otimes X$ and $Z \otimes Z$ are called the generators of S.

  $|\psi\rangle = |00\rangle + |11\rangle$ satisfies $g|\psi\rangle = |\psi\rangle$ for $\forall g \in S$.

  We say $|\psi\rangle$ is stabilized by S.

- $S = \langle Z \otimes Z\rangle$ is a stabilizer.

  $Z \otimes Z$ stabilizes $|00\rangle, |11\rangle$.

  S stabilizes 2-dimensional vector space $V_S = \{\alpha|00\rangle + \beta|11\rangle \,|\, \alpha, \beta \in \mathbb{C}\}$, not a single vector.

- $S = \langle Z_1Z_2, Z_2Z_3\rangle$

  $|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle$ is stabilized by S.

  S is generated by 2 generators.

  There are 3 qubits.

  $\Rightarrow \quad S$ stabilizes 2^{3-2}-dimensional space.

  $S = \langle Z_1Z_2, Z_2Z_3\rangle, \quad g_1 = Z_1Z_2, \quad g_2 = Z_2Z_3$

  3 qubits: $2^3 = 8$ dimensional space

  $\rightarrow \quad (g_1 = +1) \oplus (g_1 = -1) \qquad 4+4$

  $\rightarrow \quad (g_1 = +1, g_2 = +1) \oplus (g_1 = +1, g_2 = -1) \oplus (g_1 = -1, g_2 = +1) \oplus (g_1 = -1, g_2 = -1) \qquad 2+2+2+2$

  If S has $(n-k)$ generators made of n-qubit Pauli operators, S stabilizes 2^k-dimensional Hilbert space. (related to $[n, k]$ codes!)