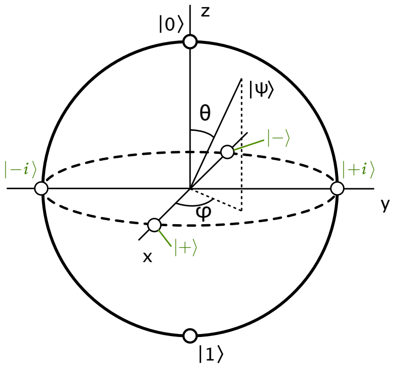
5-1 Single-Qubit Operations (§4.2)



$$|\psi(\theta,\phi)\rangle = \cos\tfrac{\theta}{2}|0\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|1\rangle$$

Pauli operators

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X^2 = Y^2 = Z^2 = I$$

$$\boxed{R_x(\theta) \equiv e^{-i\frac{\theta}{2}X}}$$

$$\begin{aligned} R_x(\theta) &= I - i\tfrac{\theta}{2}X + \tfrac{1}{2!}\left(-i\tfrac{\theta}{2}X\right)^2 + \tfrac{1}{3!}\left(-i\tfrac{\theta}{2}X\right)^3 + \dots \\ &= I\left(1 - \tfrac{1}{2!}\left(\tfrac{\theta}{2}\right)^2 + \tfrac{1}{4!}\left(\tfrac{\theta}{2}\right)^4 + \dots\right) - iX\left(\tfrac{\theta}{2} - \tfrac{1}{3!}\left(\tfrac{\theta}{2}\right)^3 + \tfrac{1}{5!}\left(\tfrac{\theta}{2}\right)^5 + \dots\right) \\ &= \cos\tfrac{\theta}{2}\,I - i\sin\tfrac{\theta}{2}\,X \end{aligned}$$

$$\boxed{R_x(\theta) = \cos\tfrac{\theta}{2}\,I - i\sin\tfrac{\theta}{2}\,X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}}$$

$$\begin{aligned} R_x(\alpha)|\psi(\theta,90^\circ)\rangle &= (\cos\tfrac{\alpha}{2}I - i\sin\tfrac{\alpha}{2}X)(\cos\tfrac{\theta}{2}|0\rangle + i\sin\tfrac{\theta}{2}|1\rangle) \\ &= (\cos\tfrac{\alpha}{2}\cos\tfrac{\theta}{2} + \sin\tfrac{\alpha}{2}\sin\tfrac{\theta}{2})|0\rangle + i(\cos\tfrac{\alpha}{2}\sin\tfrac{\theta}{2} - \sin\tfrac{\alpha}{2}\cos\tfrac{\theta}{2})|1\rangle \\ &= \cos\tfrac{\theta-\alpha}{2}|0\rangle + i\sin\tfrac{\theta-\alpha}{2}|1\rangle = |\psi(\theta-\alpha, 90^\circ)\rangle \end{aligned}$$

$$\boxed{\begin{aligned} R_y(\theta) &\equiv e^{-i\frac{\theta}{2}Y} = \cos\tfrac{\theta}{2}\,I - i\sin\tfrac{\theta}{2}\,Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \\ R_z(\theta) &\equiv e^{-i\frac{\theta}{2}Z} = \cos\tfrac{\theta}{2}\,I - i\sin\tfrac{\theta}{2}\,Z = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} \end{aligned}}$$

$$R_y(\theta)|0\rangle = \cos\tfrac{\theta}{2}|0\rangle + \sin\tfrac{\theta}{2}|1\rangle = |\psi(\theta,0)\rangle$$

$$\begin{aligned} R_z(\phi)|\psi(\theta,0)\rangle &= \cos\tfrac{\phi}{2}(\cos\tfrac{\theta}{2}|0\rangle + \sin\tfrac{\theta}{2}|1\rangle) - i\sin\tfrac{\phi}{2}(\cos\tfrac{\theta}{2}|0\rangle - \sin\tfrac{\theta}{2}|1\rangle) \\ &= e^{-i\phi/2}\cos\tfrac{\theta}{2}|0\rangle + e^{i\phi/2}\sin\tfrac{\theta}{2}|1\rangle = e^{-i\phi/2}(\cos\tfrac{\theta}{2}|0\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|1\rangle) \\ &= e^{-i\phi/2}|\psi(\theta,\phi)\rangle \end{aligned}$$

$$\boxed{|\psi(\theta,\phi)\rangle = R_z(\phi)R_y(\theta)|0\rangle}$$

$$\boxed{HR_x(\theta)H = R_z(\theta) \qquad HR_z(\theta)H = R_x(\theta)}$$

rotation along an arbitrary axis

$\vec{\sigma} = (X, Y, Z)$

$\hat{n} = (n_x, n_y, n_z)$ $\quad |\hat{n}| = 1$ $\quad$ unit vector

$\hat{n}\cdot\vec{\sigma} = n_xX + n_yY + n_zZ$

HW5-1 Show $(\hat{n}\cdot\vec{\sigma})^2 = I$.

(hint) See lecture note 04-1 for the algebra of Pauli operators.

$$\boxed{R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}} = \cos\tfrac{\theta}{2}\, I - i\sin\tfrac{\theta}{2}\,\hat{n}\cdot\vec{\sigma}}$$

HW5-2 Let $\hat{n}_H = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. Show $R_{\hat{n}_H}(180^\circ) = -iH$.

💬 $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8}\begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} = e^{i\pi/8} R_z(\tfrac{\pi}{4})$ $\quad\frac{\pi}{8}$ gate

💬 $\{T, H\}$ is a universal set for single-qubit operation. intuition?