2-1 Linear Algebra

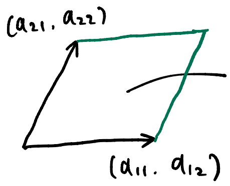
determinant

$$n \times n \text{ matrix } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{11} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}$$

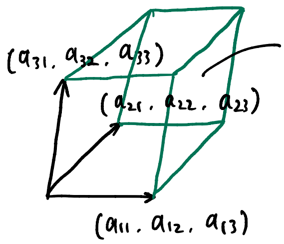
$= $ (volume of the n-dim parallelepiped formed by the row vectors of A) $\times$ ($+1$ or -1)

$n = 2$



area = |det(A)| $\quad |A| = a_{11}a_{12} - a_{21}a_{22}$

$n = 3$



volume = |det(A)|

(−) (+)

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$|A| = \sum_{i_1, i_2, \cdots, i_n} \epsilon_{i_1 i_2 \cdots i_n} a_{1,i_1} a_{2,i_2} \cdots a_{n,i_n}$$ 💬 see Wikipedia 😛

ex) $\left|\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right|$, $\left|\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right|$, $\left|\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right|$, $\left|\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\right|$

eigenvectors and eigenvalues

$$n \times n \text{ matrix } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad n\text{-dimensional vector } |\psi\rangle = \begin{pmatrix} x_1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

(A is given and $|\psi\rangle$ is variable.)

characteristic equation $\boxed{A|\psi\rangle = \lambda|\psi\rangle}$ $\quad \lambda \in \mathbb{C}$

$$A|\psi\rangle = \lambda|\psi\rangle = \lambda I|\psi\rangle$$

$$(A - \lambda I)|\psi\rangle = 0 = \underbrace{\begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix}}_{A - \lambda I} \begin{pmatrix} x_1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

n solutions (in general): $(\lambda_1, |\psi_1\rangle), (\lambda_2, |\psi_2\rangle), \cdots, (\lambda_n, |\psi_n\rangle)$ (excluding $|\psi\rangle = 0\,!$)

λ_i: eigenvalue, $|\psi_i\rangle$: eigenvector

eigenvalues and eigenvectors always appear in pairs

solution: $\boxed{|A - \lambda I| = 0}$

n-th order polynomial $\Rightarrow$ n solutions λ_1, λ_2, $\cdots$, λ_n

every λ_j makes (volume of the parallelepiped formed by the row vectors of $A - \lambda I$) = 0

$\Rightarrow$ (dimension of the space from by n row vectors) $< n$

ex) $\vec{v}_1 = \hat{x}$, $\vec{v}_2 = \hat{y}$, $\vec{v}_3 = \hat{x} + \hat{y} = \vec{v}_1 + \vec{v}_2$

$$A - \lambda I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$\Rightarrow$ n row vectors are not independent

$$(A - \lambda_j I)|\psi_j\rangle = 0 \qquad \underbrace{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}_{A - \lambda_j I} \underbrace{\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}}_{|\psi_j\rangle} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$|\psi_j\rangle$ is orthogonal to every row vector of $A - \lambda_j I$

ex) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad (X - \lambda I)|\psi\rangle = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}|\psi\rangle = 0$

$|X - \lambda I| = \lambda^2 - 1 = 0 \qquad \lambda = \pm 1 \qquad \lambda_1 = 1,\ \lambda_2 = -1$

$\lambda_1 = 1 \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}|\psi_1\rangle = 0 \qquad |\psi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}|\psi_2\rangle = 0 \qquad |\psi_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

normalization

💬 an eigenvector multiplied by a constant is also an eigenvector!

(the characteristic equation is still satisfied)

normalization: multiplying a constant to make the length of an eigenvector one

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{normalization}} |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \xrightarrow{\text{normalization}} |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

diagonalization

$n \times n$ matrix A (square matrix)

You can write $\boxed{A = PDP^{-1}}$ if A is diagonalizable

diagonal

ex) 2×2 matrix A

$$A|\psi_1\rangle = \lambda_1|\psi_1\rangle \qquad A|\psi_2\rangle = \lambda_1|\psi_2\rangle$$

$$\underset{A}{\begin{pmatrix} * & * \\ * & * \end{pmatrix}} \underset{P}{\begin{array}{c} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \\ \psi_1 \ \psi_2 \end{array}} = \begin{array}{c} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \\ \lambda_1\psi_1 \ \lambda_2\psi_2 \end{array} \qquad P^{-1}P = \begin{array}{cc} & \phi_1 \\ \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \\ & \phi_2 \end{array} \begin{array}{c} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \\ \psi_1 \ \psi_2 \end{array} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{array}{c} \phi_1 \\ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \\ \phi_2 \end{array} \begin{array}{c} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \\ \lambda_1\psi_1 \ \lambda_2\psi_2 \end{array} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \equiv D$$

$$P(P^{-1}AP)P^{-1} = PDP^{-1}$$

$$A = PDP^{-1}$$

$$P = \begin{array}{c} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \\ \psi_1 \ \psi_2 \end{array} \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

ex) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$