Linear Algebra

01-3

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(7) Example: $|\psi_1\rangle = |0\rangle + 2|1\rangle$, $|\psi_2\rangle = 2|0\rangle + 3|1\rangle$

$\| |\psi_1\rangle \| = ?$, $\| |\psi_2\rangle \| = ?$, $|\psi_1\rangle \oplus |\psi_1\rangle = ?$

Tensor product

$\| |\psi_1\rangle \oplus |\psi_1\rangle \| = ?$

Solution (i) $\| |\psi_1\rangle \| = ? \to$ Norm of $|\psi_1\rangle$

$$\| |\psi_1\rangle \| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

(ii) $\| |\psi_2\rangle \| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

(iii) $|\psi_1\rangle \otimes |\psi_1\rangle$

$$= (|0\rangle + 2|1\rangle) \otimes (|0\rangle + 2|1\rangle)$$

$= |0\rangle \otimes |0\rangle + 2|0\rangle \otimes |1\rangle + 2|1\rangle \otimes |0\rangle + 4|1\rangle \otimes |1\rangle$

$= |00\rangle + 2|01\rangle + 2|10\rangle + 4|11\rangle$

(iv) $\| |\psi_1\rangle \otimes |\psi_1\rangle \|$

$= \sqrt{1^2 + 2^2 + 2^2 + 4^2}$

$= \sqrt{1 + 4 + 4 + 16}$

$= 5$

② Example:

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} \quad |\psi\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A|\psi\rangle = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$= (2 + 3$

$$-\begin{pmatrix} a\alpha + b\beta \\ (-4) + (-2) \end{pmatrix}$$

$$c\alpha \qquad d\beta$$

$$\therefore\ 2|0\rangle + 2|0\rangle - 4|1\rangle - 2|1\rangle$$

Proved

③ **Example** $X = |0\rangle\langle 1| + |1\rangle\langle 0|$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$(X \otimes Z)\left(a|00\rangle + b|00\rangle + c|10\rangle + d|11\rangle\right) = ?$$

Sol^n: Let's find,

$$X \otimes Z|00\rangle = |10\rangle$$

$$X \otimes Z|01\rangle = -|11\rangle$$

$$X \otimes Z|10\rangle = |00\rangle$$

$$X \otimes Z|11\rangle = -|01\rangle$$

$$\therefore\ a|10\rangle - b|11\rangle + c|00\rangle - d|01\rangle$$

Proved

④ Example: Performing Diagonalization.

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$\lambda = \pm 1$ Eigenvalues

Eigenvectors for 1

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-x_1 - x_2 = 0$$

$$x_1 = -x_2$$

Lets $x_2 = 1$, $x_1 = -1$

Now for $\lambda = -1$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

Lets $x_2 = 1$, $\therefore x_1 = 1$

$\therefore D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$,

swap

multiply by -1

$$x^{-1} = \frac{1}{\det(x)} \operatorname{Adj}(x) = \frac{1}{-2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Now

$$X D X^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \frac{1}{2}$$

$$= \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+1 & 1+1 \\ 1+1 & -1+1 \end{pmatrix} \frac{1}{2}$$

$$= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{\underline{\text{Proved}}}$$

From basis $\{|v_i\rangle, |w_i\rangle\}$

let $U = \sum_i |v_i\rangle\langle w_i| \quad \{\text{Unitary}\}$

$$UU^\dagger = \left(\sum_i |v_i\rangle\langle w_i|\right)\left(\sum_j |w_j\rangle\langle v_j|\right)$$

$$= \sum_{ij} |v_i\rangle\langle w_i|w_j\rangle\langle v_j|$$

$$= \sum_j |v_j\rangle\langle v_j|$$

$$= I \quad \underline{\underline{\text{Proved}}}$$

Ex: $A^n = ?$

$$A^2 = \left(\sum_j \lambda_j |\psi_j\rangle\langle\psi_j|\right)^2$$

$$= \sum_j \lambda_j |\psi_j\rangle\langle\psi_j| \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$$

$$= \sum_j \lambda_j^2 |\psi_j\rangle\langle\psi_j|$$

$$A^2 = U D \underline{U^\dagger U} D U^\dagger$$

$$= U D^2 U^\dagger$$

$$= \sum_j \lambda_j^2 |\psi_j\rangle\langle\psi_j|$$

$\lambda = e^{i\theta}$

$U = e^{iH}$

$$U = V \begin{pmatrix} e^{i\theta} & & \\ & e^{i\theta} & \\ & & \ddots \end{pmatrix} V^\dagger$$

$e^{iH} \quad \big(e^{i\theta_1} \quad \big) \quad {}^\dagger$

$$= V \begin{pmatrix} \ddots & & \\ & e^{i\theta_2} & \\ & & \ddots \end{pmatrix} V^{+}$$

$$H = V \begin{pmatrix} \theta_1 & & \\ & \theta_2 & \\ & & \ddots \end{pmatrix} \underline{V^{+}}$$

$= U$ Proved

$\underline{\underline{en}}$ $$\left(\langle \psi(\theta_1, \phi) | \psi(\theta_2, \phi) \rangle \right)^2$$

$$= \left| \left(\cos\theta_1/2 \langle 0| + e^{i\phi} \sin\theta_1/2 \langle 1| \right) \left(\cos\theta_2/2 |0\rangle + e^{i\phi} \sin\theta_2/2 |1\rangle \right) \right|$$

$$= \left| \cos\theta_1/2 \cos\theta_2/2 + \sin\theta_1/2 \sin\theta_2/2 \right|^2$$

$$= \left| \cos \frac{\theta_1 - \theta_2}{2} \right|^2$$