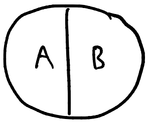
11-1 Schmidt Decomposition and Purification



A: dimension d_A. basis: $\{|0\rangle_A, \cdots, |d_A - 1\rangle_A\}$

B: dimension d_B. basis: $\{|0\rangle_B, \cdots, |d_B - 1\rangle_A\}$

AB: dimension $d_A d_B$

$$|\psi\rangle_{AB} = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} c_{ij} |i\rangle_A |j\rangle_B \qquad \text{\# of basis states} = d_A d_B$$

Schmidt decomposition (for pure states)

Suppose $d_A \le d_B$.

You can always write

$$\boxed{|\psi\rangle_{AB} = \sum_{i=0}^{d_A-1} \lambda_i |\phi_i\rangle_A |\phi'_i\rangle_B}$$

$|\phi_i\rangle_A, |\phi'_i\rangle_B$: orthonormal basis states

Note that $d_A \le d_B$.

λ_i: Schmidt coefficient

\# of non-zero λ_i's: Schmidt rank / Schmidt number

ex) Two-qubit state $|\psi\rangle = a|0\rangle|0\rangle + b|0\rangle|1\rangle + c|1\rangle|0\rangle + d|1\rangle|1\rangle$

You can always write $|\psi\rangle = \lambda_0|\phi_0\rangle|\phi'_0\rangle + \lambda_1|\phi_1\rangle|\phi'_1\rangle$.

proof $|\psi\rangle = \sum_{n,m} c_{nm}|n\rangle|m\rangle$

c_{nm} is a matrix $\xrightarrow{\text{singular value decomposition (SVD)}}$ $c_{nm} = (udv^\dagger)_{nm} = \sum_l u_{nl} d_{ll} v^\dagger_{lm}$

u, v: unitary, d: real diagonal

$$|\psi\rangle = \sum_{l,n,m} u_{nl} d_{ll} v^\dagger_{lm} |n\rangle|m\rangle$$

Let $|\phi_l\rangle = \sum_n u_{nl}|n\rangle = \sum_n u^\dagger_{ln}|n\rangle$ (basis transformation)

$|\phi'_l\rangle = \sum_m v^\dagger_{lm}|m\rangle$

$\lambda_l = d_{ll}$

Then, $|\psi\rangle = \sum_l \lambda_l |\phi_l\rangle|\phi'_l\rangle$

$$\rho_{AB} = \sum_i \sum_j \lambda_i \lambda_j^* |\phi_i\rangle_A\langle\phi_j| \otimes |\phi'_i\rangle_B\langle\phi'_j|$$

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_i |\lambda_i|^2 |\phi_i\rangle_A\langle\phi_i|$$

$$\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \sum_i |\lambda_i|^2 |\phi'_i\rangle_B\langle\phi'_i|$$ note that λ_i's are the same

Schmidt rank $= 1$: separable state

> 1: entangled state (much more complicated for mixed ρ_{AB}!)

ex) $|\psi\rangle = \frac{1}{2}|0\rangle_A|0\rangle_B + \frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B + \frac{1}{\sqrt{6}}|1\rangle_A|0\rangle_B - \frac{1}{2\sqrt{3}}|1\rangle_A|1\rangle_B$

$$\rho = (\tfrac{1}{2}|00\rangle + \tfrac{1}{\sqrt{2}}|01\rangle + \tfrac{1}{\sqrt{6}}|10\rangle - \tfrac{1}{2\sqrt{3}}|11\rangle)(\tfrac{1}{2}\langle 00| + \tfrac{1}{\sqrt{2}}\langle 01| + \tfrac{1}{\sqrt{6}}\langle 10| - \tfrac{1}{2\sqrt{3}}\langle 11|)$$

$$\rho_A = \tfrac{3}{4}|0\rangle_A\langle 0| + \tfrac{1}{4}|1\rangle_A\langle 1|$$

$$\Rightarrow \quad \lambda_0 = \tfrac{\sqrt{3}}{2},\ |\phi_0\rangle_A = |0\rangle,\ \lambda_1 = \tfrac{1}{2},\ |\phi_1\rangle_A = |1\rangle$$

$$\begin{aligned}|\psi\rangle &= \tfrac{\sqrt{3}}{2}|0\rangle_A|\phi_0\prime\rangle_B + \tfrac{1}{2}|1\rangle_A|\phi_1\prime\rangle_B \\ &= \tfrac{\sqrt{3}}{2}|0\rangle_A(\tfrac{1}{\sqrt{3}}|0\rangle + \tfrac{\sqrt{2}}{\sqrt{3}}|1\rangle)_B + \tfrac{1}{2}|1\rangle_A(\tfrac{\sqrt{2}}{\sqrt{3}}|0\rangle - \tfrac{1}{\sqrt{3}}|1\rangle)_B\end{aligned}$$

von Neumann entropy (§11.3)

$$\boxed{S(\rho) \equiv -\mathrm{Tr}(\rho \log \rho) = -\sum_i p_i \log p_i}$$

p_i: eigenvalue of ρ

pure state: $S(|\psi\rangle\langle\psi|) = 0$

fully mixed state $\rho = \sum_{i=0}^{d-1} \frac{1}{d}|i\rangle\langle i|$: $S(\rho) = \log d$

n-qubit states: $S(\rho) \leq n \qquad \because\ d = 2^n$

If a composite system AB is in a pure state,

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_i |\lambda_i|^2 |\phi_i\rangle_A\langle\phi_i|$$

$$\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \sum_i |\lambda_i|^2 |\phi_i'\rangle_B\langle\phi_i'|$$

$$\therefore S(\rho_A) = S(\rho_B) = -\sum_i |\lambda_i|^2 \log|\lambda_i|^2$$

purification

mixed state $\rho_A \rightarrow |\psi\rangle_{AR}$ s.t. $\rho_A = \mathrm{Tr}_R|\psi\rangle_{AR}\langle\psi|$ $\qquad R$: reference system

$\rho_A = \rho_A^\dagger \rightarrow$ diagonalizable

$\rho_A = \sum_i p_i|\phi_i\rangle_A\langle\phi_i| \rightarrow |\psi\rangle_{AR} = \sum_i \sqrt{p_i}|\phi_i\rangle_A|i\rangle_R$ $\qquad$ you can choose any orthonormal basis $\{|i\rangle_R\}$