4-2 Quantum Gates

§1.3 Quantum computation

single-qubit gates

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ bit-flip gate

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$X(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle$$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ phase-flip gate

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad Z|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

$$X(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ Hadamard gate

$$H|0\rangle = H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$H|1\rangle = H\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\boxed{\text{eigenvectors of } Z \xleftrightarrow{H} \text{eigenvectors of } X}$$

diagrams

x —▷o— $\bar{x}$ NOT. (classical)

$\alpha|0\rangle + \beta|1\rangle$ —$\boxed{X}$— $\alpha|0\rangle + \beta|1\rangle$

$\alpha|0\rangle + \beta|1\rangle$ —$\boxed{Z}$— $\alpha|0\rangle - \beta|1\rangle$

$\alpha|0\rangle + \beta|1\rangle$ —$\boxed{H}$— $\alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle-|1\rangle}{\sqrt{2}}$

💬 beware the direction!

circuit identities

HW4-1 Show —$\boxed{X}$— = —$\boxed{H}$—$\boxed{Z}$—$\boxed{H}$—

—$\boxed{Z}$— = —$\boxed{H}$—$\boxed{X}$—$\boxed{H}$—

——— = —$\boxed{X}$—$\boxed{X}$— = —$\boxed{Z}$—$\boxed{Z}$— = —$\boxed{H}$—$\boxed{H}$— $\quad I = X^2 = Z^2 = H^2$

💬 General single-qubit operations are a unitary rotation of the Bloch vector about a certain rotation axis. $\Rightarrow$ reversible! (one-to-one mapping)

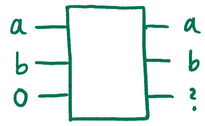
ex) An operation like $|0\rangle \to |0\rangle$, $|1\rangle \to |0\rangle$ is impossible!

two-qubit gates

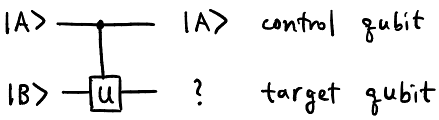
(classical) two-bit gates

irreversible!

how to make these reversible?



controlled-U (controlled-unitary) gate

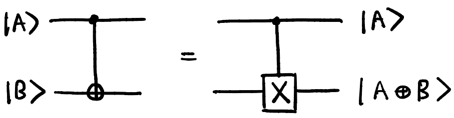


$$|A\rangle \to |A\rangle$$

$$|B\rangle \to \begin{cases} |B\rangle & \text{if } \ |A\rangle = |0\rangle \\ U|B\rangle & \text{if } \ |A\rangle = |1\rangle \end{cases}$$

$$(\alpha|0\rangle + \beta|1\rangle)|B\rangle \to \alpha|0\rangle|B\rangle + \beta|1\rangle(U|B\rangle)$$

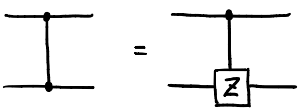
CNOT (controlled-NOT) gate



$$|B\rangle \to \begin{cases} |B\rangle & \text{if } \ |A\rangle = |0\rangle \\ X|B\rangle & \text{if } \ |A\rangle = |1\rangle \end{cases} \qquad \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} \to \begin{matrix} |00\rangle \\ |01\rangle \\ |11\rangle \\ |10\rangle \end{matrix}$$

$$\text{CNOT} = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

CZ (controlled-Z) gate



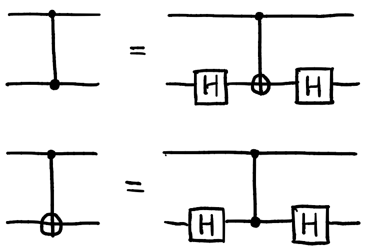
(no distinction between the control and target)

$$|B\rangle \to \begin{cases} |B\rangle & \text{if } \ |A\rangle = |0\rangle \\ Z|B\rangle & \text{if } \ |A\rangle = |1\rangle \end{cases} \qquad \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} \to \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ -|11\rangle \end{matrix}$$

$$\text{CZ} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z$$

circuit identities

HW4-2 Show



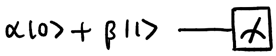
universal set of quantum gates

example of a universal set: $\{H, T, \mathrm{CNOT}\}$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{pmatrix}$$

Any unitary operation on any number of qubits can be decomposed into the gates from the universal set.

measurements

measurement in the computational basis: 

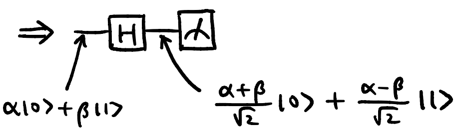
$$P(0) = |\alpha|^2, \quad P(1) = |\beta|^2$$

ex) measurement in the X basis

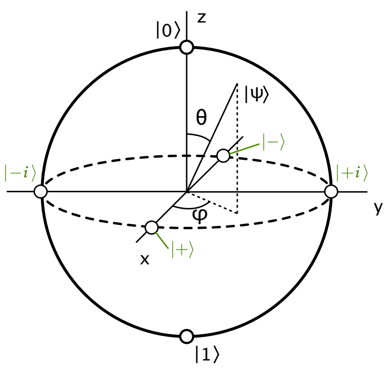
$$\left\{ |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \right\}$$

$$\alpha|0\rangle + \beta|1\rangle = \alpha\frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta\frac{|+\rangle - |-\rangle}{\sqrt{2}}$$
$$= \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$$

$$P(|\pm\rangle) = \left|\frac{\alpha \pm \beta}{\sqrt{2}}\right|^2$$



measurement in an arbitrary basis



① unitary rotation

② measurement in the computational basis

ex) measurement in Y basis

① rotation by $\pi/2$ along the x axis

$|+i\rangle \to |0\rangle, \; |-i\rangle \to |1\rangle$

② measurement in the computational basis