4-1 Qubits

§1.2 Quantum bits (qubits)

💬 qubits – may mean real systems, may mean mathematical objects

A qubit is made of two orthogonal states $|0\rangle$ and $|1\rangle$.

💬 meaning of being "orthogonal"? completely distinguishable

$|0\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}$ are called computational basis states.

state of a qubits $|\psi\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha|0\rangle + \beta|1\rangle$

We say $|\psi\rangle$ is a superposition of $|0\rangle$ and $|1\rangle$.

💬 $|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ vs $|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$P(0) = P(1) = 1/2$, but $\langle\psi_1|\psi_2\rangle = 0$

They are completely different states. Beware the phase factor!

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \alpha = |\alpha|e^{i\phi_\alpha},\ \beta = |\beta|e^{i\phi_\beta}$$
$$= e^{i\phi_\alpha}\left(|\alpha|\,|0\rangle + e^{i(\phi_\beta-\phi_\alpha)}|\beta|\,|1\rangle\right)$$

$e^{i(\phi_\beta-\phi_\alpha)}$: phase factor

Bloch sphere

single-qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \alpha, \beta \in \mathbb{C}$

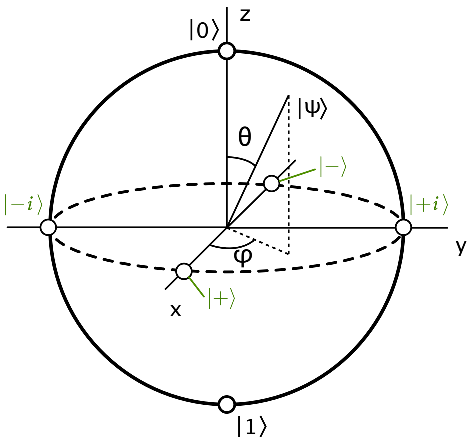
$$\alpha = |\alpha|e^{i\phi_\alpha},\ \ \beta = |\beta|e^{i\phi_\beta} \ \Rightarrow\ \text{4 real parameters}$$
$$|\alpha|^2 + |\beta|^2 = 1 \ \Rightarrow\ |\alpha| = \cos\frac{\theta}{2},\ |\beta| = \sin\frac{\theta}{2},\ \ 0 \le \theta \le \pi$$

Why $\frac{\theta}{2}$? You'll see.

$$\phi = \phi_\beta - \phi_\alpha$$

$$|\psi(\theta,\phi)\rangle = \cancel{e^{i\phi_\alpha}}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right) \quad \Rightarrow \quad \text{two real parameters } \theta,\ \phi$$

no physical meaning
(this does not change anything in what happens!)



Pauli operators

$$X = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix},\ Y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix},\ Z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

All Pauli operators have eigenvalues ± 1.

$$X|+\rangle = |+\rangle,\ X|-\rangle = -|-\rangle,\ |\pm\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$
$$Y|+i\rangle = |+i\rangle,\ Y|-i\rangle = -|-i\rangle,\ |\pm i\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$$
$$Z|0\rangle = |0\rangle,\ Z|1\rangle = -|1\rangle$$

$$X^2 = Y^2 = Z^2 = I$$

$$XY = iZ,\ YZ = iX,\ ZX = iY$$

$$YX = -iZ,\ ZY = -iX,\ XZ = -iY$$

Any 2×2 Hermitian matrix A can be written as

$$A = c_0 I + c_x X + c_y Y + c_z Z \qquad c_0, c_x, c_y, c_z \in \mathbb{R}$$

ex) Find $|\langle\psi(\theta_1, \phi)|\psi(\theta_2, \phi)\rangle|^2$.

ex) Find $|\langle\psi(\frac{\pi}{2}, \phi_1)|\psi(\frac{\pi}{2}, \phi_2)\rangle|^2$.

multiple qubits

ex) two bits: 00, 01, 10, 11

two qubits: $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$ $\qquad 4 \times 2 \underbrace{-2}$ real numbers

normalization condition
global phase

n-qubit state $\quad |\psi\rangle = \displaystyle\sum_{i_1, i_2, \ldots, i_n \in \{0,1\}} \alpha_{i_1, i_2, \ldots, i_n} |i_1 i_2 \cdots i_n\rangle$

$n = 500 \;\Rightarrow\; 2^n >$ # of atoms in the universe!

Classical computers can not store this data.