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数字信号处理

Digital Signal Processing

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第三章 离散傅里叶变换

本章主要内容

- 傅里叶变换的几种形式
- 离散傅里叶级数
- 离散傅里叶变换的定义和性质
- 频域采样
- **DFT**的应用



数字信号处理 (Digital Signal Processing)

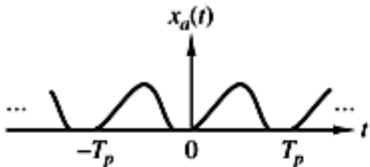
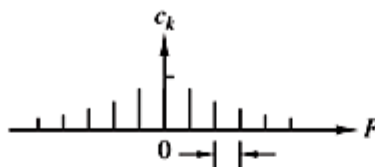
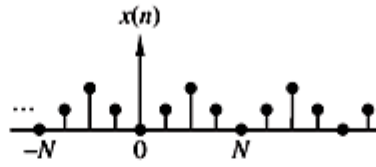
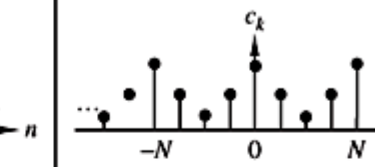
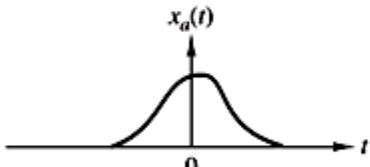
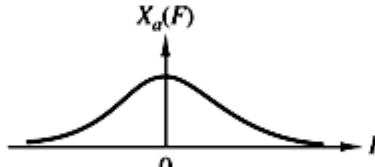
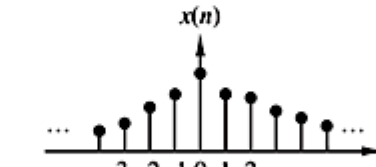
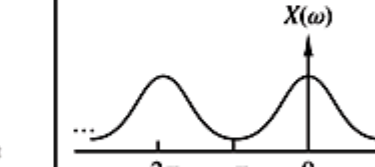
		Continuous-time signals		Discrete-time signals	
		Time-domain	Frequency-domain	Time-domain	Frequency-domain
Periodic signals	Fourier series	 $c_k = \frac{1}{T_p} \int_{T_p} x_a(t) e^{-j2\pi k F_0 t} dt$ $F_0 = \frac{1}{T_p}$ $x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$		 $c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}$ $x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$	
		Continuous and periodic	Discrete and aperiodic	Discrete and periodic	Discrete and periodic
Aperiodic signals	Fourier transforms	 $X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft} dt$ $x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi Ft} dF$		 $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$ $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$	
		Continuous and aperiodic	Continuous and aperiodic	Discrete and aperiodic	Continuous and periodic

Figure 4.3.1 Summary of analysis and synthesis formulas.



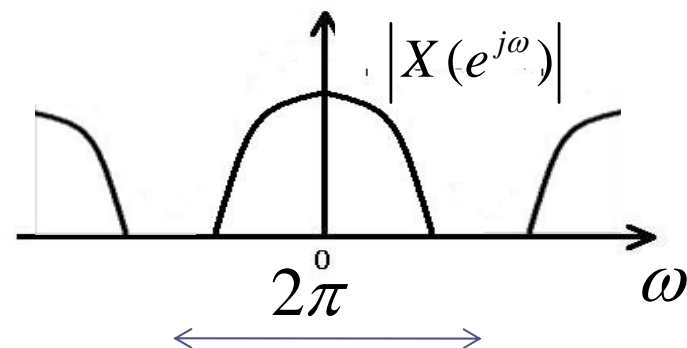
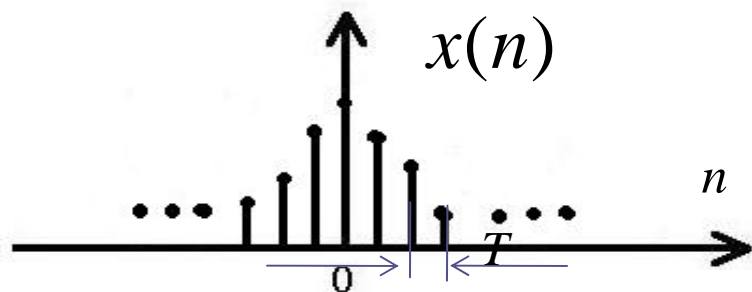
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§ 3-3 离散傅里叶级数(DFS)

DFS变换的推导

由DTFT
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

由DTFT推导DFS

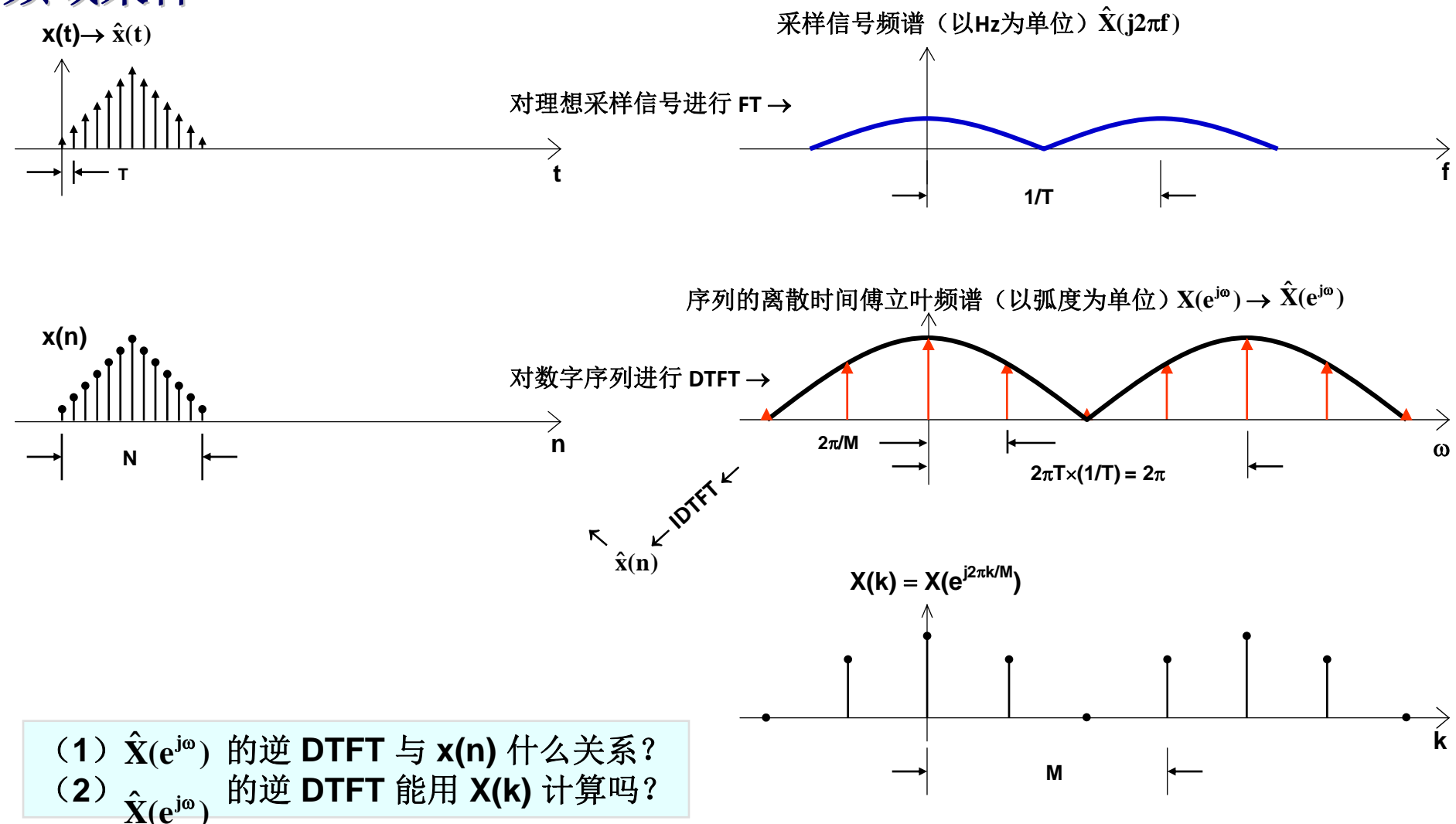


连续频谱



数字信号处理 (Digital Signal Processing)

频域采样



- (1) $\hat{X}(e^{j\omega})$ 的逆 DTFT 与 $x(n)$ 什么关系?
- (2) $\hat{X}(e^{j\omega})$ 的逆 DTFT 能用 $X(k)$ 计算吗?

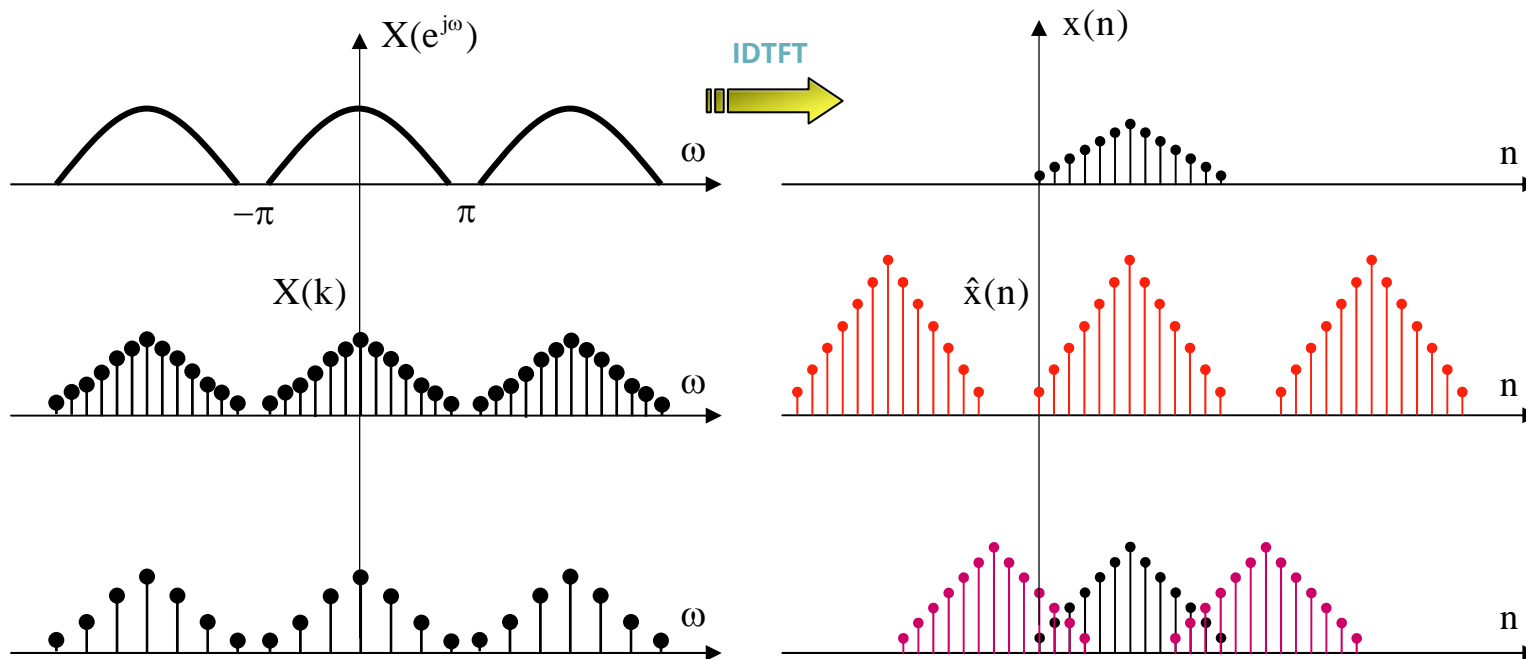


数字信号处理 (Digital Signal Processing)

$$\begin{aligned}\hat{x}(n) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{X}(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{k=-\infty}^{\infty} \underbrace{X(e^{j\frac{2\pi}{M}k})}_{X(k)} \delta(\omega - 2\pi k / M) \right] e^{j\omega n} d\omega = \boxed{\frac{1}{2\pi} \sum_{k=0}^{M-1} X(k) e^{j\frac{2\pi}{M}kn}} \\&= \frac{1}{2\pi} \int_0^{2\pi} [X(e^{j\omega}) p_{\delta}(\omega)] e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) \left(\frac{M}{2\pi} \sum_{m=-\infty}^{\infty} e^{jMm\omega} \right) e^{j\omega n} d\omega \\&= \frac{M}{2\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) e^{j\omega(n+mM)} d\omega = \boxed{\frac{M}{2\pi} \sum_{m=-\infty}^{\infty} x(n+mM)}\end{aligned}$$



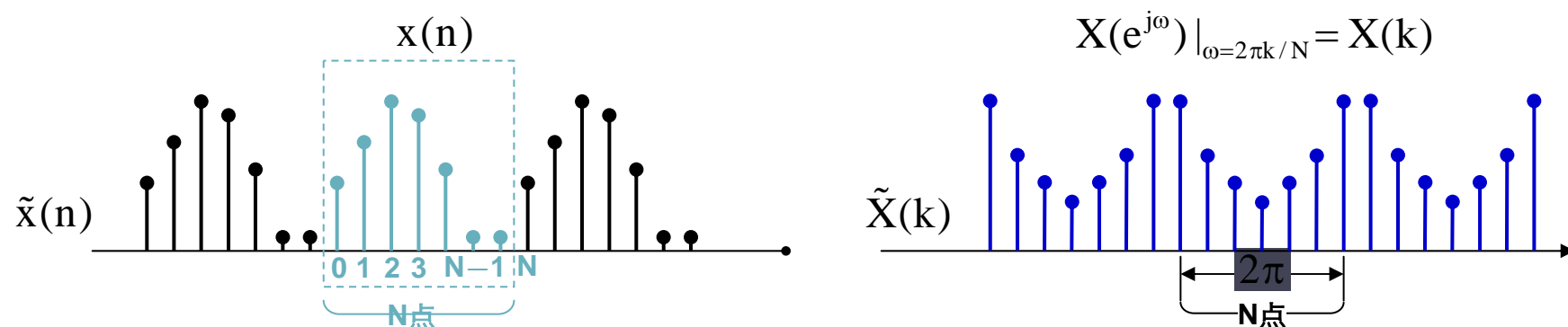
数字信号处理 (Digital Signal Processing)



- 频域过采样会造成时域“稀疏延拓”： $M > N$
- 频域欠采样会造成时域“折叠延拓”： $M < N$
- 时频域不可能同时存在冗余！



数字信号处理 (Digital Signal Processing)



$$\hat{x}(n) = \frac{M}{2\pi} \sum_{m=-\infty}^{\infty} x(n + mM) = \frac{1}{2\pi} \sum_{k=0}^{M-1} X(e^{j\frac{2\pi}{M}k}) e^{j\frac{2\pi}{M}kn}$$

$$\Downarrow \quad \boxed{M = N}$$

$$\tilde{x}(n) = \sum_{m=-\infty}^{\infty} x(n + mN) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j\frac{2\pi}{N}k}) e^{j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi}{N}kn}$$

DFS
↑
↓

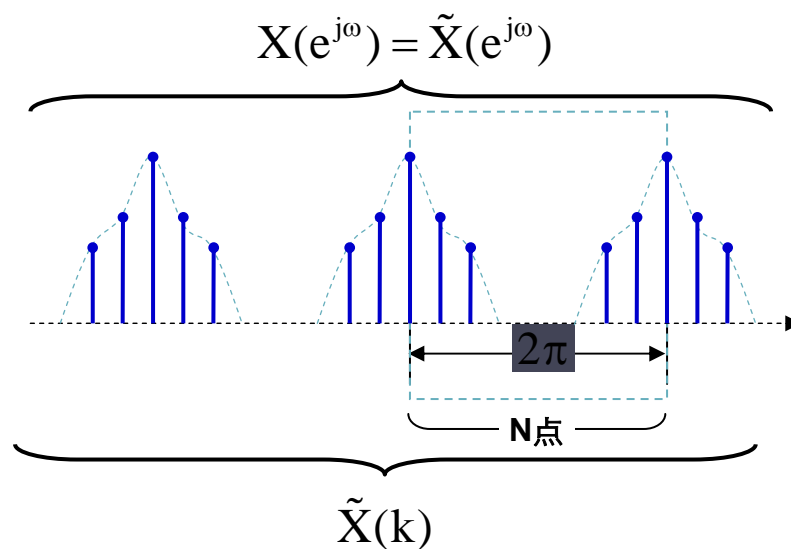
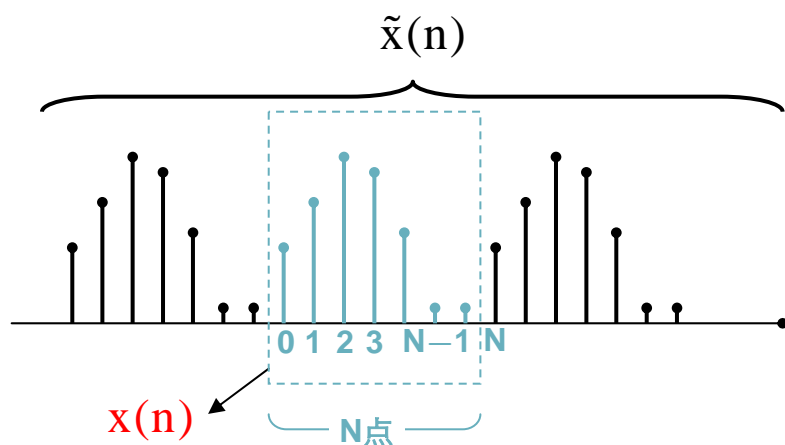
$$= \frac{2\pi}{N} \text{IDTFT}[\hat{X}(e^{j\omega})]$$

$$\tilde{X}(k) = X(e^{j\frac{2\pi}{N}k}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi}{N}kn}$$



数字信号处理 (Digital Signal Processing)

$\tilde{x}(n) \rightarrow \tilde{X}(k)$?



$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}(n)e^{-j\omega n} \quad -\infty < \omega < \infty$$

$$\tilde{X}(k) = \tilde{X}(e^{j\omega})|_{\omega=k\frac{2\pi}{N}} = \sum_{n=0}^{N-1} \tilde{x}(n)e^{-j\frac{2\pi}{N}kn} \quad -\infty < k < \infty$$



数字信号处理 (Digital Signal Processing)

$\tilde{X}(k) \rightarrow \tilde{x}(n)$?

$$\begin{aligned}\sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi}{N}kr} &= \sum_{k=0}^{N-1} \left(\sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi}{N}kn} \right) e^{j\frac{2\pi}{N}kr} \\ &= \sum_{k=0}^{N-1} \left(\sum_{n=0}^{N-1} \tilde{x}(n) e^{j\frac{2\pi}{N}k(r-n)} \right) = \sum_{n=0}^{N-1} \tilde{x}(n) \left(\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(r-n)} \right) \\ &= \sum_{k=0}^{N-1} \tilde{x}(r) e^{j\frac{2\pi}{N}k \cdot 0} = N\tilde{x}(r) \\ \therefore \tilde{x}(n) &= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi}{N}k \cdot n} \xrightarrow{W_N = e^{-j\frac{2\pi}{N}}} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-kn}\end{aligned}$$

DFS变换对

$$\tilde{X}(k) = \text{DFS}[\tilde{x}(n)] = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn}$$

$$\tilde{x}(n) = \text{IDFS}[\tilde{X}(k)] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-kn}$$



- $W_N = e^{-j2\pi/N}$ 性质

Symmetry (对称性)

$$(W_N^{nk})^* = W_N^{-nk} = W_N^{(N-n)k} = W_N^{n(N-k)}$$

Periodicity (周期性)

$$W_N^{k+N} = W_N^k$$

Reducibility (可约性)

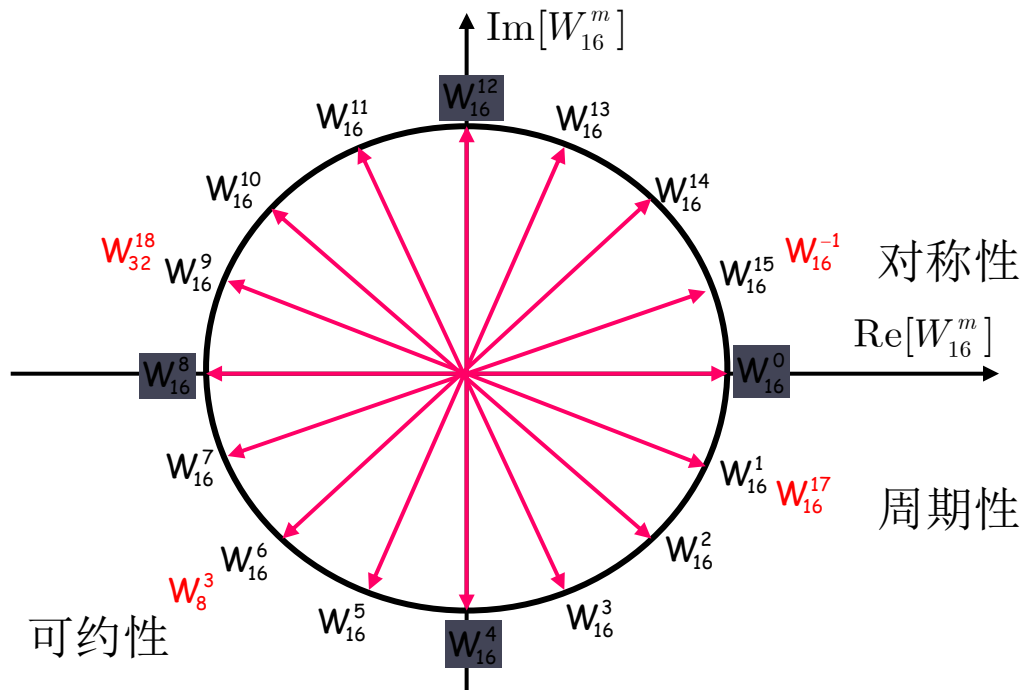
$$W_N^k = W_{mN}^{mk} = W_{N/m}^{k/m}$$



数字信号处理 (Digital Signal Processing)

正交性

$$\sum_{k=0}^{N-1} W_N^{(n-m)k} = \begin{cases} N & n-m = iN \\ 0 & n-m \neq iN \end{cases}$$



数字信号处理 (Digital Signal Processing)

例 3.1 设 $\tilde{x}(n)$ 为周期脉冲串, 求其 DFS 系数。

$$\tilde{x}(n) = \sum_{r=-\infty}^{\infty} \delta(n+rN) \quad (r \text{ 为整数})$$

解 将周期脉冲串 $\tilde{x}(n)$ 展开成离散傅里叶级数

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-kn}$$

$\tilde{X}(k)$ 就是其 k 次谐波系数。

因为对于 $0 \leq n \leq N-1$, $\tilde{x}(n) = \delta(n)$, 所以利用式(3.5)求出 $\tilde{x}(n)$ 的 DFS 系数为

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{nk} = \sum_{n=0}^{N-1} \delta(n) W_N^{nk} = 1$$



数字信号处理 (Digital Signal Processing)

例 3.2 已知周期序列 $\tilde{x}(n)$ 如图 3.2 所示, 其周期 $N=10$, 试求解 $\tilde{x}(n)$ 的傅里叶级数系数 $\tilde{X}(k)$ 。

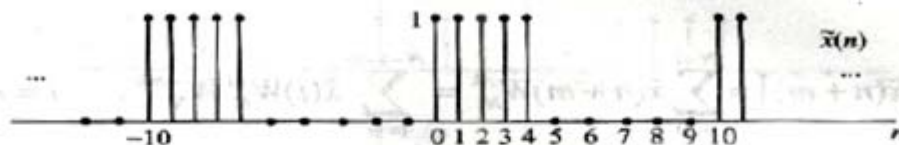


图 3.2 周期序列 $\tilde{x}(n)$ (周期 $N=10$)

解 由式(3.5)有

$$\begin{aligned}\tilde{X}(k) &= \sum_{n=0}^{10-1} \tilde{x}(n) W_{10}^{nk} = \sum_{n=0}^4 e^{-j\frac{2\pi}{10}nk} \\ \tilde{X}(k) &= \frac{1 - e^{-j\frac{2\pi}{10} \cdot 5}}{1 - e^{-j\frac{2\pi}{10}}} = e^{-j\frac{4\pi k}{10}} \frac{\sin(5\pi k/10)}{\sin(\pi k/10)}\end{aligned}\quad (3.7)$$

图3.3为离散傅里叶级数系数 $\tilde{X}(k)$ 的幅值示意图。

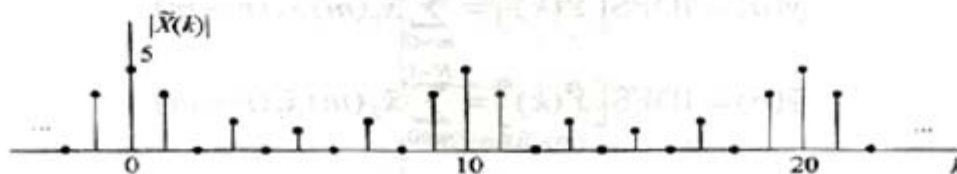


图 3.3 周期序列 $\tilde{x}(n)$ 的傅里叶级数系数 $\tilde{X}(k)$ 的幅值

DFS的主要性质

$\tilde{x}_1(n), \tilde{x}_2(n)$ 都是周期为N的周期序列

$$\tilde{X}_1(k) \stackrel{\Delta}{=} DFS[\tilde{x}_1(n)]$$

$$\tilde{X}_2(k) \stackrel{\Delta}{=} DFS[\tilde{x}_2(n)]$$

1. 线性特性

迭加原理

$$\tilde{x}_3(n) = a\tilde{x}_1(n) + b\tilde{x}_2(n)$$

$$\tilde{X}_3(k) = DFS[a\tilde{x}_1(n) + b\tilde{x}_2(n)] = a\tilde{X}_1(k) + b\tilde{X}_2(k)$$



数字信号处理 (Digital Signal Processing)

$\tilde{x}(n)$ 是周期为N的周期序列

$$\tilde{X}(k) \triangleq DFS[\tilde{x}(n)]$$

2. 序列位移

(1) 时域移位

$$\begin{aligned} \text{若 } \tilde{x}(n) &\xleftrightarrow{DFS} \tilde{X}(k), \text{ 则 } \tilde{x}(n-m) \xleftrightarrow{DFS} W_N^{mk} \tilde{X}(k) \\ \tilde{x}(n+m) &\xleftrightarrow{DFS} W_N^{-mk} \tilde{X}(k) \end{aligned}$$

(2) 频域移位

$$\begin{aligned} \text{若 } \tilde{X}(k) &\xleftrightarrow{IDFS} \tilde{x}(n), \text{ 则 } \tilde{X}(k-l) \xleftrightarrow{IDFS} W_N^{-nl} \tilde{x}(n) \\ \tilde{X}(k+l) &\xleftrightarrow{IDFS} W_N^{nl} \tilde{x}(n) \end{aligned}$$



数字信号处理 (Digital Signal Processing)

证明

$$\begin{aligned} DFS[\tilde{x}(n+m)] &= \sum_{n=0}^{N-1} \tilde{x}(n+m) W_N^{nk} & i = n+m \\ &= \sum_{i=m}^{N-1+m} \tilde{x}(i) W_N^{ki} W_N^{-mk} \\ &= W_N^{-mk} \sum_{i=m}^{N-1+m} \tilde{x}(i) W_N^{ki} \\ &= W_N^{-mk} \tilde{X}(k) \end{aligned}$$

$$\sum_{i=m}^{N-1+m} \tilde{x}(i) W_N^{ki} = \sum_{i=0}^{N-1} \tilde{x}(i) W_N^{ki} = \tilde{X}(k)$$



数字信号处理 (Digital Signal Processing)

3. 周期卷积特性

(1) 时域

$$\forall \tilde{x}_1(n) \xleftrightarrow{DFS} \tilde{X}_1(k), \quad \tilde{x}_2(n) \xleftrightarrow{DFS} \tilde{X}_2(k)$$

$$\tilde{X}_3(k) = \tilde{X}_1(k) \tilde{X}_2(k)$$

IDFS

$$\tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m) \tilde{x}_2(n-m)$$

$$= \sum_{m=0}^{N-1} \tilde{x}_2(m) \tilde{x}_1(n-m)$$

$$= \tilde{x}_1(n) \tilde{\otimes} \tilde{x}_2(n) \rightarrow \text{周期卷积}$$

比较:

$$* \quad \sum_{m=-\infty}^{+\infty}$$

$$\tilde{\otimes} \quad \sum_{m=0}^{N-1} \quad \text{仅一个周期}$$

$$\tilde{x}_3(n) = \tilde{x}_3(n+N)$$

时域周期卷积 \longleftrightarrow 频域相乘



数字信号处理 (Digital Signal Processing)

证明 $\tilde{x}_3(n) = IDFS[\tilde{X}_3(k)] = IDFS[\tilde{X}_1(k)\tilde{X}_2(k)]$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_1(k) \tilde{X}_2(k) W_N^{-nk}$$

$$\tilde{X}_1(k) = \sum_{m=0}^{N-1} \tilde{x}_1(m) W_N^{mk}$$

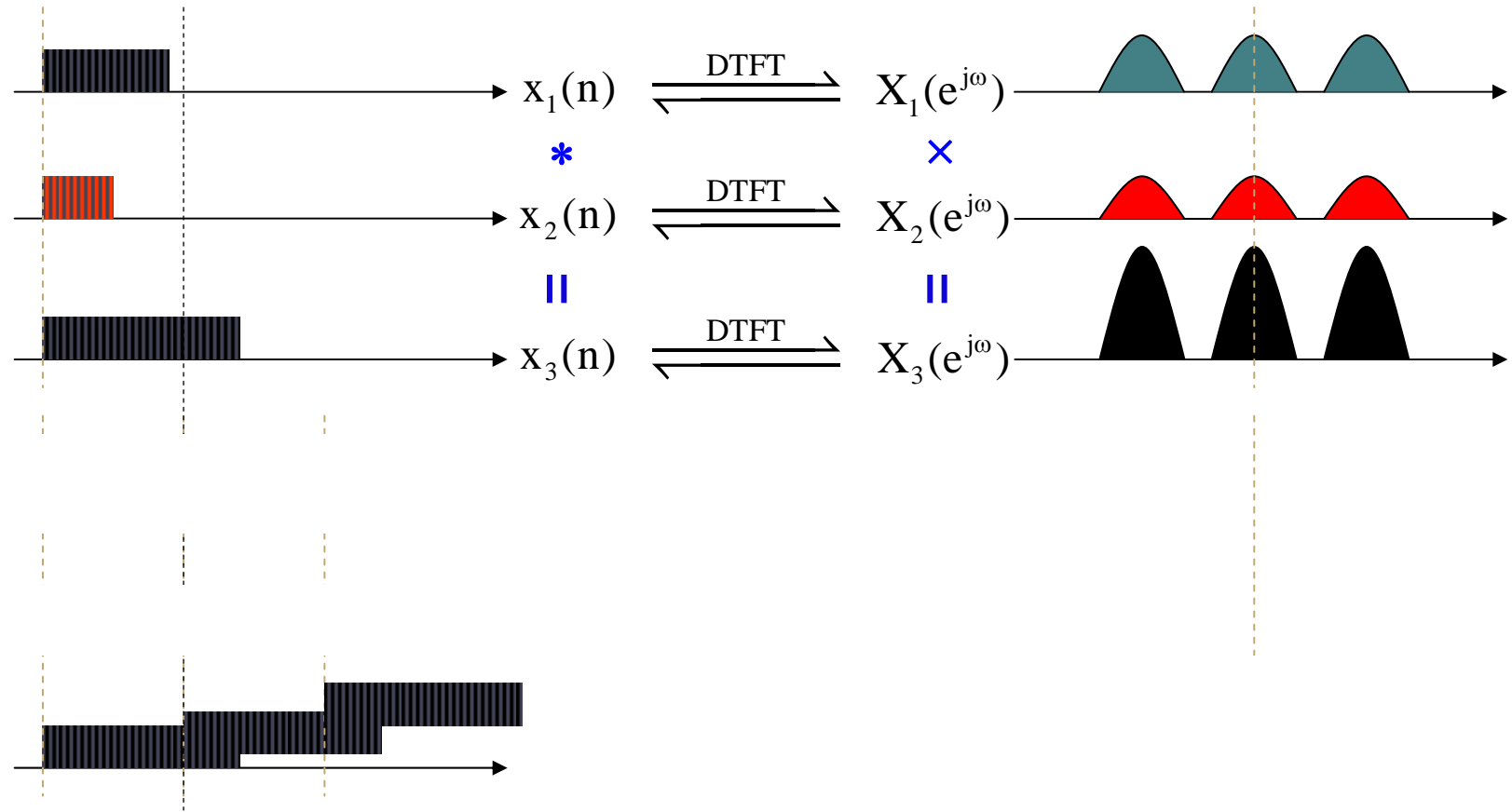
$$\begin{aligned} \tilde{x}_3(n) &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \tilde{x}_1(m) \tilde{X}_2(k) W_N^{-(n-m)k} \\ &= \sum_{m=0}^{N-1} \tilde{x}_1(m) \left[\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_2(k) W_N^{-(n-m)k} \right] \\ &= \sum_{m=0}^{N-1} \tilde{x}_1(m) \tilde{x}_2(n-m) \end{aligned}$$



数字信号处理 (Digital Signal Processing)

时域周期卷积定理

从线性卷积定理想起



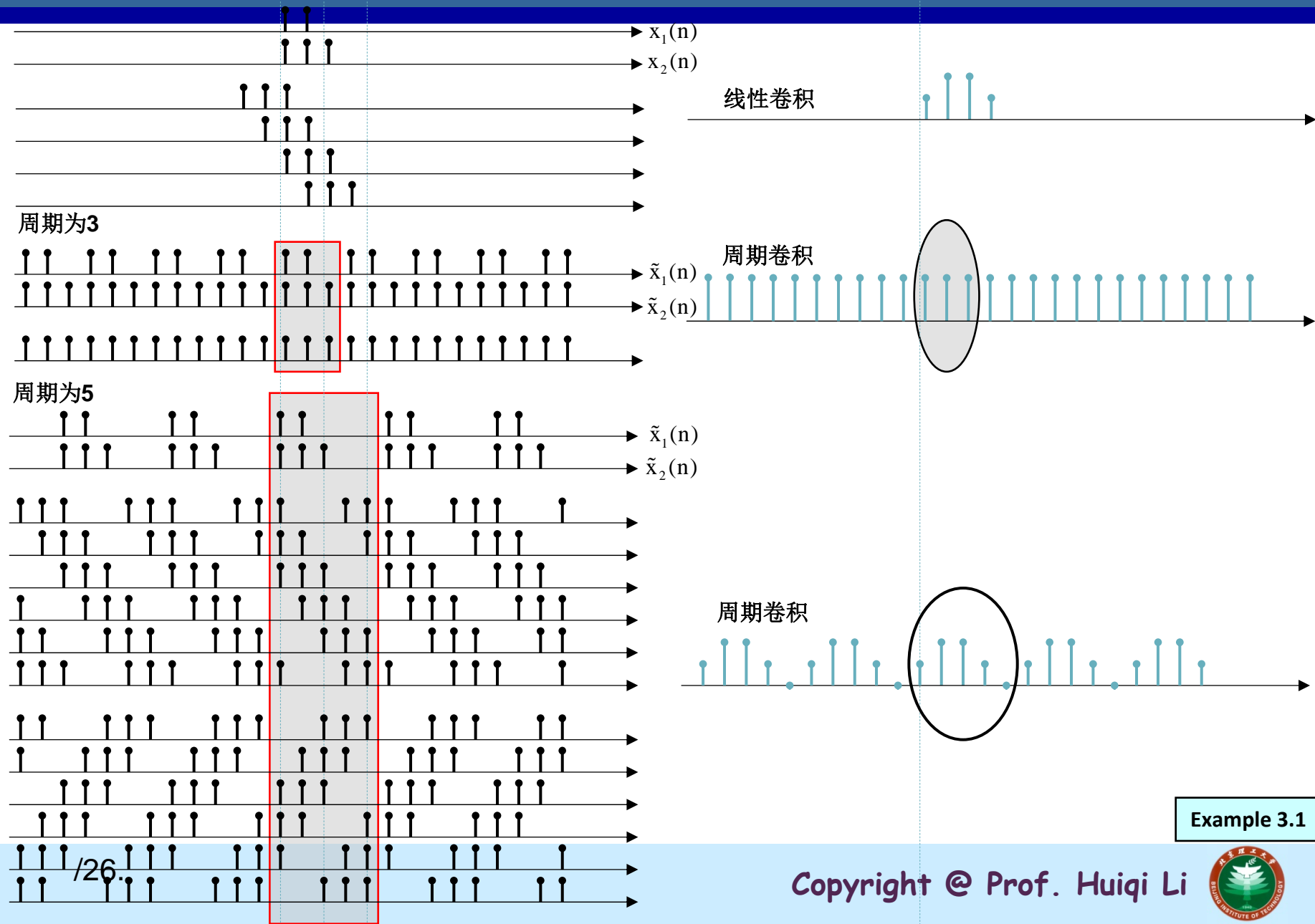
(2) 频域

$$\tilde{x}_3(n) = \tilde{x}_1(n)\tilde{x}_2(n) \xleftrightarrow{DFS} \tilde{X}_3(k) = \frac{1}{N} \tilde{X}_1(k) \tilde{\otimes} \tilde{X}_2(k)$$

时域相乘 \longleftrightarrow 频域周期卷积



数字信号处理 (Digital Signal Processing)



Example 3.1



4. 对称特性

$$(1) \quad \forall \tilde{x}(n) \xleftrightarrow{DFS} \tilde{X}(k)$$

$$\text{则} \quad \tilde{x}^*(n) \xleftrightarrow{DFS} \tilde{X}^*(-k)$$

$$\tilde{x}^*(-n) \xleftrightarrow{DFS} \tilde{X}^*(k)$$

$$DTFT[x(n)] = X(e^{j\omega})$$

$$DTFT[x^*(n)] = X^*(e^{-j\omega})$$

$$DTFT[x^*(-n)] = X^*(e^{j\omega})$$

$$x(n) \leftrightarrow X(e^{j\omega})$$

$$x^*(n) \leftrightarrow X^*(e^{-j\omega})$$

$$x^*(-n) \leftrightarrow X^*(e^{j\omega})$$



数字信号处理 (Digital Signal Processing)

$$(2) \quad \forall \tilde{x}(n) \xleftrightarrow{DFS} \tilde{X}(k)$$

则 $\text{Re}[\tilde{x}(n)] \xleftrightarrow{DFS} \tilde{X}_e(k) = \frac{1}{2} [\tilde{X}(k) + \tilde{X}^*(-k)]$

$$j \text{Im}[\tilde{x}(n)] \xleftrightarrow{DFS} \tilde{X}_o(k) = \frac{1}{2} [\tilde{X}(k) - \tilde{X}^*(-k)]$$

$$\tilde{x}_e(n) = \frac{1}{2} [\tilde{x}(n) + \tilde{x}^*(-n)] \xleftrightarrow{DFS} \text{Re}[\tilde{X}(k)] \quad \text{共轭对称}$$

$$\tilde{x}_o(n) = \frac{1}{2} [\tilde{x}(n) - \tilde{x}^*(-n)] \xleftrightarrow{DFS} j \text{Im}[\tilde{X}(k)] \quad \text{共轭反对称}$$

$$\text{Re}\{x(n)\} = \frac{1}{2} [x(n) + x^*(n)] \leftrightarrow \frac{1}{2} [X(e^{j\omega}) + X^*(e^{-j\omega})] = X_e(e^{j\omega})$$

$$j \text{Im}\{x(n)\} = \frac{1}{2} [x(n) - x^*(n)] \leftrightarrow \frac{1}{2} [X(e^{j\omega}) - X^*(e^{-j\omega})] = X_o(e^{j\omega})$$

$$x_e(n) = \frac{1}{2} [x(n) + x^*(-n)] \leftrightarrow \frac{1}{2} [X(e^{j\omega}) + X^*(e^{j\omega})] = \text{Re}\{X(e^{j\omega})\}$$

$$x_o(n) = \frac{1}{2} [x(n) - x^*(-n)] \leftrightarrow \frac{1}{2} [X(e^{j\omega}) - X^*(e^{j\omega})] = j \text{Im}\{X(e^{j\omega})\}$$



数字信号处理 (Digital Signal Processing)

(3)

$$\forall \tilde{x}(n) = \tilde{x}^*(n) \quad \text{实序列}$$

$$\begin{array}{c} \downarrow \qquad \downarrow \\ \tilde{X}(k) = \tilde{X}^*(-k) \end{array} \quad \text{共轭对称}$$

$$\begin{array}{c} \downarrow \\ |\tilde{X}(k)| = |\tilde{X}(-k)| \end{array} \quad \text{偶函数}$$

$$\arg[\tilde{X}(k)] = -\arg[\tilde{X}(-k)] \quad \text{奇函数}$$

实序列: $x(n) \leftrightarrow X(e^{j\omega})$

$$1. x(n) = x^*(n) \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$2. \begin{cases} X(e^{j\omega}) = \operatorname{Re}\{X(e^{j\omega})\} + j \operatorname{Im}\{X(e^{j\omega})\} \\ X^*(e^{-j\omega}) = \operatorname{Re}\{X(e^{-j\omega})\} - j \operatorname{Im}\{X(e^{-j\omega})\} \end{cases}$$

$$\Rightarrow \begin{cases} \operatorname{Re}\{X(e^{j\omega})\} = \operatorname{Re}\{X(e^{-j\omega})\} \\ \operatorname{Im}\{X(e^{j\omega})\} = -\operatorname{Im}\{X(e^{-j\omega})\} \end{cases}$$

$X(e^{j\omega})$ 实部是偶函数, 虚部是奇函数

$$3. \text{极坐标形式: } X(e^{j\omega}) = |X(e^{j\omega})| e^{j \arg[X(e^{j\omega})]}$$

$$\text{幅度是 } \omega \text{ 的偶函数 } |X(e^{j\omega})| = |X(e^{-j\omega})|$$

$$\text{相位是 } \omega \text{ 的奇函数 } \arg[X(e^{j\omega})] = -\arg[X(e^{-j\omega})]$$

