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Answer 1

By assuming that all the balls are equally likely to be chosen from the each box, we can show our calculations base on that assumption. Let the events L and M defined as follows:

L = {Chosen box}
M = {Color of the chosen ball}

a)

$$\mathrm{P(M = G \mid L = X)} = \frac{\text{number of outcomes in } \{M = G \cap L = X\}}{\text{number of outcomes in } \{L = X\}} = \frac{2}{6} = 0.33\overline{3}$$

b)

$$\mathrm{P(M = R) = P(M = R \mid L = X)P(L = X) + P(M = R \mid L = Y)P(L = Y)}$$

$$= \frac{2}{6}\,\frac{2}{5} + \frac{1}{5}\,\frac{3}{5} = 0.25\overline{3}$$

c)

From the Law of Total Probability and Bayes Rule for two events, we can simplify our calculations to obtain the probability that we had chosen the box Y given that the ball we picked is blue.

$$\mathrm{P(L = Y \mid M = B)} = \frac{P(M = B \mid L = Y)P(L = Y)}{P(M = B)}$$

By applying the Law of Total Probability;

$$\mathrm{P(M = B) = P(M = B \mid L = X)P(L = X) + P(M = B \mid L = Y)P(L = Y)}$$

$$= \frac{2}{6}\frac{2}{5} + \frac{2}{5}\frac{3}{5} = 0.37\overline{3}$$

$$\mathrm{P(M = B \mid L = Y)P(L = Y)} = \frac{2}{5}\frac{3}{5} = 0.24$$

Then our result becomes:

$$\mathrm{P(L = Y \mid M = B)} = \frac{0.24}{0.37\overline{3}} = 0.643$$

Answer 2

a)

We need to prove or disprove the following:

$$(\mathrm{A} \cap \mathrm{B} = \emptyset) \iff (\overline{A} \cup \overline{B} = \Omega)$$

Then:

$$\begin{aligned}\text{A and B are mutually exclusive} &\iff \mathrm{P(A \cap B)} = 0\\ &\iff \mathrm{P}(\overline{A \cap B}) = 1\\ &\iff \mathrm{P}(\overline{A} \cup \overline{B}) = 1\\ &\iff \overline{A} \text{ and } \overline{B} \text{ are exhaustive.}\end{aligned}$$

b)

We need to give a counter example to disprove the statement.For that, let's define the events A, B, and C as the output of rolling a die:

A = {1,2,3}, B = {4,5,6}, C = {1,4}

$\overline{A}$, $\overline{B}$, and $\overline{C}$ are exhaustive because $\overline{A} \cup \overline{B} \cup \overline{C} = \{1, 2, 3, 4, 5, 6\}$, i.e. $\mathrm{P}(\overline{A} \cup \overline{B} \cup \overline{C}) = 1$.

A, B, C are not mutually exclusive because $P(A \cap C) = P(\{1\}) \neq 0$

Answer 3

By using the definition of the Binomial Distribution and defining the event K as the number of successes:
$K = \{\text{Number of Successes}\}$

$$P(k) = P(K = k) = \binom{5}{k} \left(\frac{2}{6}\right)^k \left(\frac{4}{6}\right)^{5-k}$$

a)

$$P(2) = P(K = 2) = \binom{5}{2} \left(\frac{2}{6}\right)^2 \left(\frac{4}{6}\right)^3 = \frac{80}{243} = 0.329$$

b)

$$P(K \geq 2) = \sum_{k=2}^{5} \binom{5}{k} \left(\frac{2}{6}\right)^k \left(\frac{4}{6}\right)^{5-k}$$

$$= \binom{5}{2} \left(\frac{2}{6}\right)^2 \left(\frac{4}{6}\right)^3 + \binom{5}{3} \left(\frac{2}{6}\right)^3 \left(\frac{4}{6}\right)^2 + \binom{5}{4} \left(\frac{2}{6}\right)^4 \left(\frac{4}{6}\right)^1 + \binom{5}{5} \left(\frac{2}{6}\right)^5 \left(\frac{4}{6}\right)^0$$

$$\frac{80}{243} + \frac{40}{243} + \frac{10}{243} + \frac{1}{243} = \frac{131}{243} = 0.539$$

Answer 4

a)

$$P(A = 1, C = 0) = \sum_{b \in B} P(A = 1, B = b, C = 0) = P(A = 1, B = 0, C = 0) + P(A = 1, B = 1, C = 0)$$

$$= 0.06 + 0.09 = 0.15$$

b)

$$P(B=1) = \sum_{c \in C} \sum_{a \in A} P(A=a, B=1, C=c) = \sum_{c \in C} P(A=0, B=1, C=c) + P(A=1, B=1, C=c)$$

$$= P(A=0, B=1, C=0) + P(A=0, B=1, C=1) + P(A=1, B=1, C=0) + P(A=1, B=1, C=1)$$

$$= 0.21 + 0.02 + 0.09 + 0.08 = 0.40$$

c)

From the definition, for the random variables A and B to be independent below equality must hold for all values of a and b such that a' ∈ A and b' ∈ B;

P(A = a', B = b') = P(A = a') P(B = b')

Let us, now, try to give a counter example to prove that random variables A and B are not independent. For convenience, let's choose the values a' = 0 and b' = 1, since we found P(B = 1) on part (b).We will examine if the below equality holds:

P(A = 0, B = 1) = P(A = 0) P(B = 1)

$$P(A=0, B=1) = \sum_{c \in C} P(A=0, B=1, C=c) = P(A=0, B=1, C=0) + P(A=0, B=1, C=1)$$

$$= 0.21 + 0.02 = 0.23$$

$$P(A=0) = \sum_{c \in C} \sum_{b \in B} P(A=0, B=b, C=c) = \sum_{c \in C} P(A=0, B=0, C=c) + P(A=0, B=1, C=c)$$

$$= P(A=0, B=0, C=0) + P(A=0, B=0, C=1) + P(A=0, B=1, C=0) + P(A=0, B=1, C=1)$$

$$= 0.14 + 0.08 + 0.21 + 0.02 = 0.45$$

$$0.23 = \text{P(A = 0, B = 1)} \neq \text{P(A = 0) P(B = 1)} = (0.45)(0.40) = 0.18$$

Since we have given a counter example, we can simply say random variables A and B are **not independent**.

d)

From the definition of conditionally independence, for the random variables A and B to be conditionally independent given C = 1, below equality must hold for all values of a and b such that a' ∈ A and b' ∈ B;

$$\text{P(A = a', B = b' | C = 1) = P(A = a' | C = 1) P(B = b' | C = 1)}$$

Let us, now, try to give a counter example to prove that random variables A and B are not conditionally independent given C = 1. We will examine if the above equality holds for all combinations of a' and b':

(i) P(A = 0, B = 0 | C = 1) = P(A = 0 | C = 1) P(B = 0 | C = 1)

$$\text{P(A = 0, B = 0 | C = 1)} = \frac{P(A=0, B=0, C=1)}{P(C=1)} = \frac{0.08}{0.50} = 0.16$$

$$\text{P(A = 0 | C = 1)} = \frac{P(A=0, C=1)}{P(C=1)} = \frac{0.10}{0.50} = 0.20$$

$$\text{P(B = 0 | C = 1)} = \frac{P(B=0, C=1)}{P(C=1)} = \frac{0.40}{0.50} = 0.80$$

$$\text{L = P(A = 0, B = 0 | C = 1) = 0.16}$$

$$\text{M = P(A = 0 | C = 1) P(B = 0 | C = 1)} = (0.20)(0.80) = 0.16$$

Since L = M, equality of conditionally independence of random variables A and B given C = 1 holds for A = 0 and B = 0.

(ii) P(A = 0, B = 1 | C = 1) = P(A = 0 | C = 1) P(B = 1 | C = 1)

$$\mathrm{P(A = 0,\ B = 1 \mid C = 1)} = \frac{P(A=0, B=1, C=1)}{P(C=1)} = \frac{0.02}{0.50} = 0.04$$

$$\mathrm{P(A = 0 \mid C = 1)} = \frac{P(A=0, C=1)}{P(C=1)} = \frac{0.10}{0.50} = 0.20$$

$$\mathrm{P(B = 1 \mid C = 1)} = \frac{P(B=1, C=1)}{P(C=1)} = \frac{0.10}{0.50} = 0.20$$

$$\mathrm{L = P(A = 0,\ B = 1 \mid C = 1) = 0.04}$$

$$\mathrm{M = P(A = 0 \mid C = 1)\ P(B = 1 \mid C = 1) = (0.20)(0.80) = 0.04}$$

Since L = M, equality of conditionally independence of random variables A and B given C = 1 holds for A = 0 and B = 1.

(iii) $\mathrm{P(A = 1,\ B = 1 \mid C = 1) = P(A = 1 \mid C = 1)\ P(B = 1 \mid C = 1)}$

$$\mathrm{P(A = 1,\ B = 1 \mid C = 1)} = \frac{P(A=1, B=1, C=1)}{P(C=1)} = \frac{0.08}{0.50} = 0.16$$

$$\mathrm{P(A = 1 \mid C = 1)} = \frac{P(A=1, C=1)}{P(C=1)} = \frac{0.40}{0.50} = 0.80$$

$$\mathrm{P(B = 1 \mid C = 1)} = \frac{P(B=1, C=1)}{P(C=1)} = \frac{0.10}{0.50} = 0.20$$

$$\mathrm{L = P(A = 0,\ B = 0 \mid C = 1) = 0.16}$$

$$\mathrm{M = P(A = 0 \mid C = 1)\ P(B = 0 \mid C = 1) = (0.20)(0.80) = 0.16}$$

Since L = M, equality of conditionally independence of random variables A and B given C = 1 holds for A = 1 and B = 1.

$$\textbf{(iv)}\ \mathrm{P(A = 1,\ B = 0 \mid C = 1) = P(A = 1 \mid C = 1)\ P(B = 0 \mid C = 1)}$$

$$\mathrm{P(A = 1,\ B = 0 \mid C = 1)} = \frac{P(A = 1, B = 0, C = 1)}{P(C = 1)} = \frac{0.32}{0.50} = 0.64$$

$$\mathrm{P(A = 1 \mid C = 1)} = \frac{P(A = 1, C = 1)}{P(C = 1)} = \frac{0.40}{0.50} = 0.80$$

$$\mathrm{P(B = 0 \mid C = 1)} = \frac{P(B = 0, C = 1)}{P(C = 1)} = \frac{0.40}{0.50} = 0.80$$

$$\mathrm{L = P(A = 1,\ B = 0 \mid C = 1) = 0.64}$$

$$\mathrm{M = P(A = 1 \mid C = 1)\ P(B = 0 \mid C = 1) = (0.80)(0.80) = 0.64}$$

Since L = M, equality of conditionally independence of random variables A and B given C = 1 holds for A = 1 and B = 0.

Since the conditionally independence equalities hold for all the values of random variables A and B, given C = 1; we can suggest that random variables A and B are **conditionally independent**.