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Answer 1

Before getting started we need to recall a definition. To get the marginal pmf of one variable, we add the joint probabilities over all values of the other variable.

$$P_X(x) = P\{X = x\} = \sum_y P_{(X,Y)}(x, y)$$

a)

$$P_X(0) = P\{X = 0\} = \sum_y P_{(X,Y)}(0, y) = P_{(X,Y)}(0, 0) + P_{(X,Y)}(0, 2) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12}$$

$$P_X(1) = P\{X = 1\} = \sum_y P_{(X,Y)}(1, y) = P_{(X,Y)}(1, 0) + P_{(X,Y)}(1, 2) = \frac{4}{12} + \frac{2}{12} = \frac{6}{12}$$

$$P_X(2) = P\{X = 2\} = \sum_y P_{(X,Y)}(2, y) = P_{(X,Y)}(2, 0) + P_{(X,Y)}(2, 2) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12}$$

$$\mu_X = E(X) = \sum_x xP(x) = 0 \times \frac{3}{12} + 1 \times \frac{6}{12} + 2 \times \frac{3}{12} = 1$$

$$Var(X) = E(X-E(X))^2 = \sum_x (x-\mu_X)^2 P(x) = (0-1)^2 \times \frac{3}{12} + (1-1)^2 \times \frac{6}{12} + (2-1)^2 \times \frac{3}{12} = \frac{6}{12} = 0.5$$

b)

Let's call the probability mass function Z, such that Z = X + Y.

$$P_z(0) = P\{X + Y = 0\} = P_{(X,Y)}(0, 0) = \frac{1}{12}$$

$$P_z(1) = P\{X + Y = 1\} = P_{(X,Y)}(1, 0) = \frac{4}{12}$$

$$P_z(2) = P\{X + Y = 2\} = P_{(X,Y)}(0, 2) + P_{(X,Y)}(2, 0) = \frac{2}{12} + \frac{1}{12} = \frac{3}{12}$$

$$P_z(3) = P\{X+Y=3\} = P_{(X,Y)}(1,2) = \frac{2}{12}$$
$$P_z(4) = P\{X+Y=4\} = P_{(X,Y)}(2,2) = \frac{2}{12}$$

c)

$$Cov(X,Y) = E\{(X - E(X))(Y - E(Y))\} = E(XY) - E(X)E(Y)$$

Before we determine Cov(X,Y), we first need to do some calculations.

$$P_Y(0) = P\{Y=0\} = \sum_x P_{(X,Y)}(x,0) = P_{(X,Y)}(0,0)+P_{(X,Y)}(1,0)+P_{(X,Y)}(2,0) = \frac{1}{12}+\frac{4}{12}+\frac{1}{12} = \frac{6}{12}$$
$$P_Y(2) = P\{Y=2\} = \sum_x P_{(X,Y)}(x,2) = P_{(X,Y)}(0,2)+P_{(X,Y)}(1,2)+P_{(X,Y)}(2,2) = \frac{2}{12}+\frac{2}{12}+\frac{2}{12} = \frac{6}{12}$$
$$\mu_Y = E(Y) = \sum_y yP(y) = 0 \times \frac{6}{12} + 2 \times \frac{6}{12} = 1$$
$$\mu_{XY} = E(XY) = \sum_x \sum_y xyP_{(X,Y)}(x,y)$$
$$= (1)(2)P_{(X,Y)}(1,2) + (2)(2)P_{(X,Y)}(2,2) = 2 \times \frac{2}{12} + 4 \times \frac{2}{12} = 1$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \mu_{XY} - \mu_X\mu_Y = 1 - 1 \times 1 = 0$$

d)

$$\mu_A = E(A) = \sum_a aP(a)$$
$$\mu_B = E(B) = \sum_b bP(b)$$

$$\begin{aligned} Cov(A,B) &= E[(A - E(A))(B - E(B))] = E[AB - \mu_A B - \mu_B A + \mu_A\mu_B] \\ &= E[AB] - \mu_A E(B) - \mu_B E(A) + \mu_A\mu_B \\ &= E[AB] - \mu_A\mu_B - \mu_B\mu_A + \mu_A\mu_B \end{aligned}$$

$$= E[AB] - \mu_A \mu_B$$

Then we need to find an expression for the expectation of AB, such that E(AB):

$$E[AB] = \sum_a \sum_b ab P_{(A,B)}(a,b)$$

If the random variables A and B are independent of each other we can rewrite as:

$$\begin{aligned} E[AB] &= \sum_a \sum_b ab P_A(a) P_B(b) \\ &= (\sum_a a P_A(a))(\sum_b b P_B(b)) \\ &= E[A]E[B] = \mu_A \mu_B \end{aligned}$$

Then we can find the covariance of random variables A and B, such that A and B are independent Cov(A,B):

$$Cov(A,B) = E[AB] - \mu_A \mu_B = \mu_A \mu_B - \mu_A \mu_B = 0$$

e)

Since the statement "If the random variables A and B are independent, Cov(A,B)" is not an if and only if statement, we can not relate what we have found in part (c) to the independence of the random variables X and Y. Instead, we need to check whether the following holds for all values of x and y.

$$P_{(X,Y)}(x,y) = P_X(x) P_Y(y)$$

Let us first try the values X = 0 and Y = 0:

$$P_{(X,Y)}(0,0) = \frac{1}{12},\ P_X(0) = \frac{3}{12},\ P_Y(0) = \frac{6}{12}$$
$$P_{(X,Y)}(0,0) \neq P_X(0) P_Y(0)$$

Since 1 counter example is enough, we can clearly say that random variables X and Y are **not independent**.

Answer 2

a)

By using the definition of the Binomial Distribution and defining the random variable B as the number of broken pens among 12 pens:
B = {Number of Broken Pens Among 12 Pens}

$$\mathrm{P}_B(\mathrm{b}) = \mathrm{P}(\mathrm{B} = \mathrm{b}) = \binom{12}{b}\ (0.2)^b\ (0.8)^{12-b}$$

$$\mathrm{P}(\mathrm{B} \geq 3) = 1 \text{ - } \mathrm{P}(\mathrm{B} \leq 2) = 1 \text{ - } \sum_{k=0}^{2} \binom{12}{b}\ (0.2)^b\ (0.8)^{12-b}$$

$$= 1 - [\binom{12}{0}(0.8)^{12} + \binom{12}{1}(0.2)^1(0.8)^{11} + \binom{12}{2}(0.2)^2(0.8)^{10}]$$

$$= 1 - [(0.8)^{12} + 12(0.2)^1(0.8)^{11} + 66(0.2)^2(0.8)^{10}]$$

$$= 1 - [0.5583] = 0.4417$$

b)

For fifth pen we test to be second broken pen, we only have to 1 broken pen among the first 4 pens. By using the definition of the Binomial Distribution and defining the random variable D as the number of broken pens among 4 pens:
D = {Number of Broken Pens Among 4 Pens}

$$\mathrm{P}_D(\mathrm{d}) = \mathrm{P}(\mathrm{D} = \mathrm{d}) = \binom{4}{d}\ (0.2)^d\ (0.8)^{4-d}$$

Then:

$$\mathrm{P}_D(1) = \mathrm{P}(\mathrm{D} = 1) = \binom{4}{1}\ (0.2)^1\ (0.8)^3 = 4(0.2)(0.8)^3 = 0.4096$$

Since we have found the probability that there is only one broken pen among first 4, we now need to multiply that with (0.2) to obtain the probability that fifth pen we test to be second broken pen:

$$(0.4096)(0.2) = 0.8192$$

c)

We already know the Binomial probability density function:

$$P(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Then we need to find an expression for the expected value:

$$\begin{aligned}
E[X] &= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} \\
&= \sum_{x=1}^{n} x \binom{n}{x} p^x q^{n-x} \\
&= \sum_{x=1}^{n} \frac{xn}{x} \binom{n-1}{x-1} p^x q^{n-x} \\
&= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x} \\
&= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{(n-1)-(x-1)} \\
&= np(p+q)^{n-1} = np
\end{aligned}$$

The p, the probability of a pen is broken, is 0.2. Therefore, on average, to find 4 broken pens we need to test n pens, such that:

$$n = \frac{4}{p} = \frac{4}{0.2} = \mathbf{20}$$

Answer 3

From the definition we know that exponential distribution has the density and expectation:

$$f(x) = \lambda e^{-\lambda x}$$

$$E[X] = \int t f(t) dt = \int_0^{\infty} t \lambda e^{-\lambda t} dt = \frac{1}{\lambda}$$

Since Bob gets a phone call every 4 hours on average, E(X) = 4, $\lambda = \frac{1}{4}$.

Also, let us define the random variable X, where X is the number of events during the time interval [0, t]. This X has Poisson distribution with parameter λt:

$$P_X(x) = e^{-\lambda t} \frac{(\lambda t)^x}{x!}$$

a)

Probability of " Bob does not get a phone call for at least 2 hours" is equal to 1 minus the probability of "Bob gets a phone call in first 2 hours".
The probability of getting a phone call at least once:

$$P_X\{X \geq 1\} = 1 - P_X\{X = 0\} = 1 - P_X(0) = 1 - e^{-\lambda t}\frac{(\lambda t)^0}{0!} = 1 - e^{-\lambda t}$$

The probability of getting a phone call at least once in the time interval [0,t]:

$$F_T(t) = 1 - P_X\{X \geq 1\} = e^{-\lambda t}$$

With the $\lambda = \frac{1}{4}$ and t = 2:

$$F_T(2) = 1 - P_X\{X \geq 1\} = e^{-\frac{1}{2}} = 0.6065$$

Alternatively:

$$\int_2^{\infty} \lambda e^{-\lambda t} dt = [1 - e^{-\lambda t}]_2^{\infty} \xrightarrow{For \lambda = \frac{1}{4}} [1 - e^{\frac{t}{4}}]_2^{\infty} = e^{-\frac{1}{2}} = 0.6065$$

b)

The probability of getting 3 phone calls at most:

$$P_X\{X \leq 3\} = P_X\{X = 0\} + P_X\{X = 1\} + P_X\{X = 2\} + P_X\{X = 3\}$$

$$= e^{-\lambda t} + \lambda t e^{-\lambda t} + \frac{(\lambda t)^2}{2} e^{-\lambda t} + \frac{(\lambda t)^3}{6} e^{-\lambda t}$$

$$= \frac{e^{-\lambda t}}{6}(6 + 6\lambda t + 3\lambda^2 t^2 + \lambda^3 t^3)$$

The probability of getting 3 phone calls at most in the time interval [0,t]:

$$F_T(t) = P_X\{X \leq 3\} = \frac{e^{-\lambda t}}{6}(6 + 6\lambda t + 3\lambda^2 t^2 + \lambda^3 t^3)$$

With the $\lambda = \frac{1}{4}$ and t = 10:

$$F_T(10) = P_X\{X \leq 3\} = \frac{443e^{-2.5}}{48} = 0.7575$$

c)

Here, let us define two random variables K and M such that:
K = "Getting 3 phone calls at most in first 16 hours"
M = "Getting 3 phone calls at most in first 10 hours"

And we are asked to find, given M the probability of K, such that:

$$P(K|M) = \frac{P(K \cap M)}{P(M)}$$

Here we can clearly see that:

$$P(K \cap M) = P(K)$$

Therefore, our equation becomes:

$$P(K|M) = \frac{P(K)}{P(M)}$$

We have found the probability of getting 3 phone calls at most at part (b), now we use that probability individually for the events K and M with $\lambda = \frac{1}{4}$ and t is equal to 16 and 10 respectively.

$$F_T(10) = P_X\{X \leq 3\} = \frac{443e^{-2.5}}{48} = 0.7575$$

$$F_T(16) = P_X\{X \leq 3\} = \frac{142e^{-4}}{6} = 0.4335$$

$$P(K|M) = \frac{P(K)}{P(M)} = \frac{F_T(16)}{F_T(10)} = \frac{0.4335}{0.7575} = 0.5722$$