## Student Information

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## Answer 1

**a)**
(i) D = A $\cap$ ( B $\cup$ C )
(ii) E = (A $\cap$ B ) $\cup$ C
(iii) F = ( A - B ) $\cup$ (A $\cap$ C )

**b)**
**Let A = {e,n,t}, B = {t,e,r}, and C = {s,e,r}**

(i) ( A X B ) X C = {(e,t),(e,e),(e,r),(n,t),(n,e),(n,r),(t,t),(t,e),(t,r)} X {s,e,r} = {(e,t,s),(e,e,s),(e,r,s),(n,t,s), (n,e,s),(n,r,s),(t,t,s),(t,e,s),(t,r,s),(e,t,e),(e,e,e),(e,r,e),(n,t,e),(n,e,e),(n,r,e),(t,t,e),(t,e,e),(t,r,e),(e,t,r),(e,e,r), (e,r,r),(n,t,r),(n,e,r),(n,r,r),(t,t,r),(t,e,r),(t,r,r)} = {e,n,t} X {(t,s),(t,e),(t,r),(e,s),(e,e),(e,r),(r,s),(r,e),(r,s)} = A X ( B X C )

(ii) (A $\cap$ B ) $\cap$ C = {e,t}$\cap$ {s,e,r} = {e} = {e,n,t}$\cap${e,r} = A $\cap$ ( B $\cap$ C )

(iii) ( A$\oplus$ B ) $\oplus$ C ={n,r}$\oplus${s,e,r} ={s,e,n} = {e,n,t}$\oplus$ {t,s} = {e,n,s} = A $\oplus$ ( B $\oplus$ C )

## Answer 2

**a)**
Take any x $\in$ $A_0$, then we have $\boldsymbol{f}$(x) $\in$ $\boldsymbol{f}(A_0)$. By definition, $\boldsymbol{f}^{-1}(\boldsymbol{f}(A_0))$ = {a | $\boldsymbol{f}$(a) $\in$ $\boldsymbol{f}(A_0)$ }. Since $\boldsymbol{f}$(x) $\in$ $\boldsymbol{f}(A_0)$, x satisfies the condition(1): $\boldsymbol{f}$(a) $\in$ $\boldsymbol{f}(A_0)$; we have x $\in$ $\boldsymbol{f}^{-1}(\boldsymbol{f}(A_0))$. We have proved x $\in$ $A_0$ $\rightarrow$ x $\in$ $\boldsymbol{f}^{-1}(\boldsymbol{f}(A_0))$. That means $A_0 \subseteq \boldsymbol{f}^{-1}(\boldsymbol{f}(A_0))$. Now, suppose $\boldsymbol{f}$ injective. We want to prove that $A_0 = \boldsymbol{f}^{-1}(\boldsymbol{f}(A_0))$. Suppose not; that means we can find an x $\in$ $\boldsymbol{f}^{-1}(\boldsymbol{f}(A_0))$ such that x $\notin$ $A_0$. Because we already know $A_0 \subseteq \boldsymbol{f}^{-1}(\boldsymbol{f}(A_0))$, x $\in$ $\boldsymbol{f}^{-1}(\boldsymbol{f}(A_0))$ $\rightarrow$ x satisfies the contidion(1) then $\boldsymbol{f}$(x) = u $\in$ $\boldsymbol{f}(A_0)$. u $\in$ $\boldsymbol{f}(A_0)$ means, there exists a y $\in$ $A_0$ with $\boldsymbol{f}$(y) = u. Since $\boldsymbol{f}$ is injective and $\boldsymbol{f}$(x) = u = $\boldsymbol{f}$(y), we must have x=y. But x $\notin$ $A_0$ and y $\in$ $A_0$; contradiction.

**b)**
Take any y $\in$ $\boldsymbol{f}^{-1}(\boldsymbol{f}(B_0))$ $\rightarrow$ y = $\boldsymbol{f}$(x) for some x $\in$ $\boldsymbol{f}^{-1}(B_0)$. x $\in$ $\boldsymbol{f}^{-1}(B_0)$ = {a | $\boldsymbol{f}$(a) $\in$ $B_0$} means x satisfies the condition(1); $\boldsymbol{f}$(x) $\in$ $B_0$ $\rightarrow$ y $\in$ $B_0$. So, $\boldsymbol{f}(\boldsymbol{f}^{-1}(B_0)) \subseteq B_0$. Suppose $\boldsymbol{f}$ is surjective, but $\boldsymbol{f}(\boldsymbol{f}^{-1}(B_0)) \neq B_0$. That means we can find a y $\in$ $B_0$ such that y $\notin$ $\boldsymbol{f}(\boldsymbol{f}^{-1}(B_0))$. By surjectivity, there exists an x such that $\boldsymbol{f}$(x) = y $\rightarrow$ x satisfies the condition(1); $\boldsymbol{f}$(x) $\in$ $B_0$ $\rightarrow$ x $\in$ $\boldsymbol{f}^{-1}(B_0)$ $\rightarrow$ $\boldsymbol{f}$(x) $\in$ $\boldsymbol{f}(\boldsymbol{f}^{-1}(B_0))$ $\rightarrow$ y $\in$ $\boldsymbol{f}(\boldsymbol{f}^{-1}(B_0))$; hence contradiction.

## Answer 3

Let A be a non-empty set.
(i) A is countable
(ii) There is a surjective function $\boldsymbol{f} : \mathrm{Z}^+ \rightarrow \mathrm{A}$
(iii) There is an injective function $\boldsymbol{g} : \mathrm{A} \rightarrow \mathrm{Z}^+$

(i) $\rightarrow$ (ii): If A is countably infinite, then there exists a bijection $\boldsymbol{f} : \mathrm{Z}^+ \rightarrow \mathrm{A}$ and then (ii) follows. If A is finite, then there is bijection $h : \{1,2,...,n\} \rightarrow \mathrm{A}$ for some n. Then the function $\boldsymbol{f} : \mathrm{Z}^+ \rightarrow \mathrm{A}$ defined by;

$$f(t) = \begin{cases} h(t) & 1 \leq \mathrm{t} \leq \mathrm{n}, \\ h(n) & t > n \end{cases} \quad \text{is a surjection.}$$

(ii) $\rightarrow$ (iii): Assume that $\boldsymbol{f} : \mathrm{Z}^+ \rightarrow \mathrm{A}$ is a surjection. We claim that there is an injection $\boldsymbol{g} : \mathrm{A} \rightarrow \mathrm{Z}^+$. To define $\boldsymbol{g}$ note that if $\mathrm{a} \in \mathrm{A}$, then $\boldsymbol{f}^{-1}(\{\mathrm{a}\}) \neq \emptyset$. Hence we set $\boldsymbol{g}(\mathrm{a}) = min\boldsymbol{f}^{-1}(\{\mathrm{a}\})$. Now note that if $\mathrm{a} \neq \mathrm{a}'$, then the sets $\boldsymbol{f}^{-1}(\{\mathrm{a}\}) \cap \boldsymbol{f}^{-1}(\{\mathrm{a}'\}) = \emptyset$ which implies $\mathrm{min}^{-1}(\{\mathrm{a}\}) \neq \mathrm{min}^{-1}(\{\mathrm{a}'\})$. Hence $\boldsymbol{g}^{-1}(\mathrm{a}) \neq \boldsymbol{g}^{-1}(\mathrm{a}')$ and $\boldsymbol{g} : \mathrm{A} \rightarrow \mathrm{Z}^+$ is an injective function.

(iii) $\rightarrow$ (i): Assume that $\boldsymbol{g} : \mathrm{A} \rightarrow \mathrm{Z}^+$ is an injection. We want to show that A is countable. Since $\boldsymbol{g} : \mathrm{A} \rightarrow \mathrm{g(A)}$ is a bijection and $\mathrm{g(A)} \subset \mathrm{Z}^+$, proposition "Any subset of a countable set is countable." implies that A is countable.

## Answer 4

**a)**
Let **F** be the set containing finite binary strings and $\epsilon$ denote the empty sequence(the sequence with no terms). Then, the sequence $\mathbf{F} = \{\epsilon,0,1,00,01,10,11,000,001,010,011,...\}$ in which the binary sequences of length 0 are listed, then the binary sequences of length 1 are listed in increasing numeric order, then the binary sequences of length 2 are listed in increasing numeric order, and so on, contains every finite binary sequence exactly once.

**b)**
Let **I** be the set of all infinite sequences of 0s and 1s. We use Cantor's diagonal argument. So we assume (toward a contradiction) that we have an enumeration of the elements of **I**, say as $\mathbf{I} = \{\mathrm{i}_1\mathrm{i}_2\mathrm{i}_3...\}$ where each $\mathrm{s}_\mathrm{n}$ is an infinite sequence of 0s and 1s. We will write $\mathrm{s}_1 = \mathrm{s}_{1,1}\mathrm{s}_{1,2}\mathrm{s}_{1,3} \cdots$, $\mathrm{s}_2 = \mathrm{s}_{2,1}\mathrm{s}_{2,2}\mathrm{s}_{2,3} \cdots$, and so on; so $\mathrm{s}_\mathrm{n} = \mathrm{s}_{\mathrm{n},1}\mathrm{s}_{\mathrm{n},2}\mathrm{s}_{\mathrm{n},3} \cdots$. So we denote the $\boldsymbol{m}$th element of $\mathrm{s}_\mathrm{n}$ by $\mathrm{s}_{\mathrm{n,m}}$. Now we create a new sequence $\mathrm{t} = \mathrm{t}_1\mathrm{t}_2\mathrm{t}_3\mathrm{t}_4 \cdots$ of 0s and 1s as follows: $\mathrm{t}_\mathrm{n} = \mathrm{s}_{\mathrm{n,n}}$ - 1 (so $\mathrm{t}_\mathrm{n} = 1$ if $\mathrm{s}_{\mathrm{n,n}} = 0$ and $\mathrm{t}_\mathrm{n} = 0$ if the $\mathrm{s}_{\mathrm{n,n}}$ is 1). It is clear that t is an element of **I** - it is an infinite sequence of 0s and 1s. However, we will now see that t is not in the list above. Suppose that $\mathrm{t} = \mathrm{i}_\mathrm{k}$ for some value of k. Then $\mathrm{t}_\mathrm{k} = \mathrm{t}_{\mathrm{k,k}}$, but by the construction, $\mathrm{t}_\mathrm{k} \neq \mathrm{i}_{\mathrm{k,k}}$, so this is not possible. We conclude that **I** is not countable, is uncountable.

## Answer 5

**a)**

f(n) is $\Theta$(g(n)), where f(n)=nlog(n), g(n)=log(n!) and there exist constants c and k, if:

(i) f(n) is O(g(n)) $\rightarrow$ f(n) $\leq$ cg(n) whenever $n > k$

$$\begin{aligned} \text{log(n!)} &= \text{log(1)+log(2)+...+log(n)} \\ &\leq \text{log(n)+log(n)+...+log(n)} \\ \text{log(n!)} &\leq \text{nlog(n); where c} = 1 \text{ and } n > 1 \end{aligned}$$

(ii) f(n) is $\Omega$(g(n)) $\rightarrow$ f(n) $\geq$ cg(n)

$$\begin{aligned} \text{log(n!)} &= \textstyle\sum_{i=1}^{n} log(i) \\ &\geq \textstyle\sum_{i=n/2}^{n} log(i) \\ &\geq \textstyle\sum_{i=n/2}^{n} log(n/2) \\ &\geq \tfrac{n}{2}\text{log(n/2)} \\ &= \tfrac{n}{2}\text{(log(n) - log(2)); where c} = \tfrac{1}{2} \text{ and } n > 1 \\ &= \Omega \text{ (nlog(n))} \end{aligned}$$

**b)**

$$2^n = \underbrace{2 * 2 * ... * 2 * 2}_{\text{n times}}$$

$$n! = \underbrace{1 * 2 * ... * (n-1) * n}_{\text{n elements}}$$

$$\frac{2^{\text{n}}}{n!} = \frac{2}{1}*\frac{2}{2}*\frac{2}{3}*...*\frac{2}{n-1}*\frac{2}{n}$$

$$\frac{2^{\text{n}}}{n!} \leq \frac{2}{1}*\frac{2}{2}*1*...*1*\frac{2}{n} = \frac{4}{n}$$

Hence $2^{\text{n}}$ is O(n!) such that n! grows faster than $2^{\text{n}}$ ($|2^{\text{n}}| \leq$ c|n!|); where c = 1 and n $\geq$ 4