Student Information

Full Name : mergen

Answer 1

Fermat's Little Theorem states that if p is prime and a is an integer not divisible by p, then

$$a^{p-1} \equiv 1(\boldsymbol{mod}\ p)$$

Furthermore, for every integer a we have

$$a^p \equiv a(\boldsymbol{mod}\ p)$$

By applying that theorem where p is a prime, x is a positive integer which is not divisible by p, and y is the smallest positive integer that

$$x^y \equiv 1(\boldsymbol{mod}\ p)$$

We can imply that y = *(p-1)*c for some constant $c \in$ Z^+. However, as indicated y is the smallest positive integer that can ensure the given equivalence, then c be 1 for which y has its smallest value. Therefore, since we obtained y = *(p-1)*, we can simply say y divides *(p-1)* and denote that with y | *(p-1)*.

Answer 2

Proof by Contradiction: Let's assume that 169 divides ($2n^2$+10n-7), such that 169 | ($2n^2$+10n-7). Then, it is clear that ($2n^2$+10n-7) is 169k, since ∀n(n ∈ Z^+ ∧ ($2n^2$+10n-7) > 0) k is a positive integer that is ∀k ∈ Z^+. By subtracting 169k from both sides, we obtain

$$2n^2 + 10n - (7 + 169k) = 0$$

We can now find the roots of the equation by first finding that discriminant(Δ = b^2 - 4ac), where a = 2, b = 10, and c = -(7+169k).

$$\Delta = 10^2 - 4 * 2 * (-7 - 169k) = 156 + 8 * 169k$$

At this point, since Δ consist of integer multipliers, remember that ∀k ∈ Z^+, Δ must be an integer. For that equation to have some integer valued roots, $\sqrt{\Delta}$ must be a rational number. Therefore, for $\sqrt{\Delta}$ to be a rational number $\sqrt{\Delta}$ must be a perfect square.

$$\Delta = 156 + 8 * 169k = 2^2 * 13 * (3 + 26k)$$

For Δ to be a perfect square (3+26k) part must contain at least one 13 multiplier so that 13|(3+26k). However, ∀k((3+26k) ≡ 3 (***mod*** 13)) contradicts with 13|(3+26k). Since it is a contradiction, our assumption 169 divides ($2n^2$+10n-7) was incorrect and by contradiction it is to be concluded as

$$169 \not| (2n^2 + 10n - 7)$$

Answer 3

If a ≡ b (***mod*** m), by the definition of congruence, we know that $m|$*(a-b)*. This means that there is an integer $k \in$ Z, such that *(a-b)* = *km*. Then, n divides *km*. Since ***gcd*(m,n)** = 1, we have n divides k, so $k = nt$ for some t ∈ Z. Therefore, *(a-b)* = *km* = *ntm* such that *nm* divides *(a-b)*. Hence,

$$a \equiv b(\boldsymbol{mod}\ nm)$$

Answer 4

Solution: Let *P(n)* be the proposition that the sum of the first n terms of j(j + 1)(j + 2)$\cdots$ (j+k-1) for j ∈ {1,2,...,n} is

$$(\boldsymbol{a}) \qquad \frac{n(n+1)(n+2)\cdots(n+k)}{(k+1)}$$

We must do two things to prove that *P(n)* is true for n = 1,2,3,... . Namely, we must show that *P(1)* is true and that the conditional statement *P(t) implies P(t+1)* is true for t = 1,2,3,... .

Basis Step: *P(1)* is true, because

$$1(2)(3)\cdots(k) = (k!) = \frac{1(1+1)(1+2)\cdots(k)(1+k)}{(k+1)}$$

The leftmost side of this equation is factorial of k because (k!) is the sum of the first term. The rightmost side of this equation is found by substituting 1 for n in the equation (***a***).

Inductive Step: For the inductive hypothesis we assume that *P(t)* holds for an arbitrary positive integer t. That is, we assume that

$$\sum_{j=1}^{t} j(j+1)(j+2)\cdots(j+k-1) = \frac{t(t+1)(t+2)\cdots(t+k)}{(k+1)}$$

Under this assumption, it must be shown that *P(t+1)* is true, namely, that

$$\sum_{j=1}^{t+1} j(j+1)(j+2)\cdots(j+k-1) = \frac{(t+1)(t+2)\cdots(t+k)(t+k+1)}{(k+1)}$$

is also true.

We observe that the summation of the right-hand side of *P(t+1)* is (t+1)(t+2)$\cdots$(t+k) more than the summation of the right-hand side of *P(t)*. Our strategy will be to add (t+1)(t+2)$\cdots$(t+k) to the both sides of of the equation in *P(t)* with and simplify the result algebraically to complete the inductive step.

$$\begin{aligned}
\Big(\sum_{j=1}^{t} j(j+1)(j+2)\cdots(j+k-1)\Big)+(t+1)(t+2)\cdots(t+k) &\overset{\text{IH}}{=} \frac{t(t+1)(t+2)\cdots(t+k)}{(k+1)}+(t+1)(t+2)\cdots(t+k) \\
&= (1+\frac{t}{k+1})*((t+1)(t+2)(t+3)\cdots(t+k)) \\
&= (\frac{t+k+1}{k+1})*((t+1)(t+2)(t+3)\cdots(t+k)) \\
&= \frac{(t+1)(t+2)\cdots(t+k)(t+k+1)}{(k+1)}
\end{aligned}$$

This last equation shows that *P(t+1)* is true under the assumption that *P(t)* is true. This completes the inductive step.

We have completed the *basis step* and the *inductive step*, so by mathematical induction we know that ***P(n)*** is true for all positive integers k and n.

$$(P(1) \wedge \forall t(P(t) \rightarrow P(t+1))) \rightarrow \forall n P(n)$$

Answer 5

Solution: Let *P(n)* be the proposition that $H_n \leq 7^n$ for $n \geq 0$.

Basis Step: For base cases note that
- For (n=0), P(0) is true because $H_0 = 1 \leq 7^0 = 1$,
- For (n=1), P(1) is true because $H_1 = 3 \leq 7^1 = 7$,
- For (n=2), P(0) is true because $H_2 = 5 \leq 7^2 = 49$.

Inductive Step: Let $n > 2$. Assume that $H_i \leq 7^i$ for all integers i with $0 \leq i < n$. Consider H_n. By our inductive hypothesis, we know that

$$H_n = 5 * H_{n-1} + 5 * H_{n-2} + 63 * H_{n-3}$$

$$\leq 5 * 7^{n-1} + 5 * 7^{n-2} + 63 * 7^{n-3}$$

$$= 5 * 7^2 * 7^{n-3} + 5 * 7^1 * 7^{n-3} + 63 * 7^0 * 7^{n-3}$$

$$= 245 * 7^{n-3} + 35 * 7^{n-3} + 63 * 7^{n-3}$$

$$= 343 * 7^{n-3} = 7^3 * 7^{n-3} = 7^n$$

Therefore, $H_n \leq 7^n$ and result holds by strong induction.