Student Information

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Answer 1

a) For every 1 to be followed immediately by a 0, length l of the string must be greater than or equal to 2. When l=2, there is only 1 bit string that 1 followed immediately by 0, such that a_2=1.
When l=3, there are 2 bit strings that every 1 followed immediately by 0, such that a_3=2.
When l=4, there are 4 bit strings that every 1 followed immediately by 0, such that a_4=4.
When l=5, there are 7 bit strings that every 1 followed immediately by 0, such that a_5=7.
When l=6, there are 12 bit strings that every 1 followed immediately by 0, such that a_6=12.
When l=7, there are 20 bit strings that every 1 followed immediately by 0, such that a_7=20.
And so on.It is now clear that we have a recurrence relation with the variable l, length of the bit string, such that $a_l = a_{l-1} + a_{l-2} + 1$; with initial conditions a_2=1, and a_3=2.

When we plug in l=9 to the recurrence relation:
$a_9 = a_8 + a_7 + 1 = (a_7 + a_6 + 1) + (a_6 + a_5 + 1) + 1 = (20 + 12 + 1) + (12 + 7 + 1) + 1$
$= 33 + 20 + 1 = 54$

b) Since the arrangement does not constrained the solution, we use combinations of 10 from 8 to 10.

$$\sum_{n=8}^{10} C(10,n) = C(10,8) + C(10,9) + C(10,10)$$

$$= 45 + 10 + 1 = 56$$

c) From the theorem at page 561 from the textbook.

$$3^4 - \binom{3}{2} \cdot 2^4 + \binom{3}{1} \cdot 1^4 = 81 - 3 \cdot 16 - 3 \cdot 1 = 36$$

d) We first choose 1 Discrete Mathematics textbook and 1 Signals and Systems textbook. Then out of remaining 10 books(4 Discrete Mathematics textbooks and 6 Signals and Systems textbook) we chose 2 books.

$$C(5,1) \cdot C(7,1) \cdot (\frac{10!}{2! \cdot 4! \cdot 6!}) = 7 \cdot 5 \cdot 105 = 3625$$

Answer 2

a) When n=1, our set has 2 subsets that do not contain two consecutive numbers, such that $a_1 = 2$. When n=2, our set has 4 subsets since {1,2} contain two consecutive numbers we do not count that one, such that $a_2 = 3$.

$$\text{n=1: } \varnothing, \{1\} \text{ ; } a_1 = 2$$

$$\text{n=2: } \varnothing, \{1\}, \{2\} \text{ ; } a_2 = 3$$

$$\text{n=3: } \varnothing, \{1\}, \{2\}, \{3\}, \{1,3\} \text{ ; } a_3 = 5$$

$$\text{n=4: } \varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,4\} \text{ ; } a_4 = 8$$

$$\text{n=5: } \varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,5\}, \{1,3,5\} \text{ ; } a_5 = 13$$

$$\text{n=6: } \varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,5\}, \{3,6\},$$

$$\{4,6\}, \{1,3,5\}, \{1,3,6\}, \{1,4,6\}, \{2,4,6\} \text{ ; } a_6 = 21$$

And so on. Therefore, our recurrence relation is represented as follows:

$$a_n = a_{n-1} + a_{n-2} \text{ for n} \geq 3; \text{ with initial conditions } a_1 = 2 \text{ and } a_2 = 3$$

b)
By using the definitions of **Generating function for the sequence** $a_0, a_1, ..., a_k, ...$ of real numbers is the infinite series

$$\mathbf{G(x)} = a_0 + a_1 x + a_2 x^2 + ... + a_k x^k + ... = a_0 + \sum_{k=1}^{\infty} a_k x^k$$

Extended binomial theorem

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} \cdot x^k$$

Our recursive relation is $a_k = a_{k-1} + a_{k-2}$, for k$\geq$ 3 and has the initial conditions $a_1 = 2$ and $a_2 = 3$

To make our work with generating functions simpler, we extend this sequence by setting $a_0 =1$; when we assign this value to a_0 and use the recurrence relation, we have $3 = a_2 = a_1 + a_0 = 2 + 1$, which is consistent with our original initial condition.

$$\textit{Let } \mathbf{G(x)} = a_0 + \sum_{k=1}^{\infty} a_k x^k$$

$$G(x) - a_0 - a_1 x - a_2 x^2 = \sum_{k=3}^{\infty} a_k x^k$$

$$= \sum_{k=3}^{\infty} (a_{k-1} + a_{k-2}) x^k$$

$$= x \sum_{k=3}^{\infty} (a_{k-1}) x^{k-1} + x^2 \sum_{k=3}^{\infty} (a_{k-2}) x^{k-2}$$

$$= x \sum_{m=2}^{\infty} a_m x^m + x^2 \sum_{n=1}^{\infty} a_n x^n$$

$$= x(G(x) - a_0 - a_1 x) + x^2(G(x) - a_0)$$

We thus obtain the equation:

$$G(x) - a_0 - a_1 x - a_2 x^2 = x(G(x) - a_0 - a_1 x) + x^2(G(x) - a_0)$$

By substituting $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$

$$G(x) - 1 - 2x - 3x^2 = x(G(x) - 1 - 2x) + x^2(G(x) - 1)$$

$$G(x)(1-x-x^2) = x+1$$

$$G(x) = \frac{x+1}{1-x-x^2}$$

We found an expression for G(x), now we need to term the sequence $\{a_k\}$.
First we will determine the **partial fractions**

$$\frac{x+1}{1-x-x^2} = \frac{A}{x+\frac{1-\sqrt{5}}{2}} + \frac{B}{x+\frac{1+\sqrt{5}}{2}}$$

A + B = -1 and A - B = $\frac{-1}{\sqrt{5}}$

$$Therefore\ A = \frac{-5-\sqrt{5}}{10}\ and\ B = \frac{-5+\sqrt{5}}{10}$$

$$G(x) = \frac{(\frac{5+3\sqrt{5}}{10})}{1-\frac{(1+\sqrt{5})x}{2}} + \frac{(\frac{5-3\sqrt{5}}{10})}{1-\frac{(1-\sqrt{5})x}{2}}$$

By using the identity $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$, we have

$$G(x) = (\frac{5+3\sqrt{5}}{10})\sum_{k=0}^{\infty}(\frac{(1+\sqrt{5}}{2})^k x^k + (\frac{5-3\sqrt{5}}{10})\sum_{k=0}^{\infty}(\frac{(1-\sqrt{5}}{2})^k x^k$$

$$Consequently, a_k = (\frac{5+3\sqrt{5}}{10})(\frac{(1+\sqrt{5}}{2})^k + (\frac{5-3\sqrt{5}}{10})(\frac{(1-\sqrt{5}}{2})^k$$

Answer 3

$$a_n = 4a_{n-1} + a_{n-2} - 4a_{n-3}\ implies\ c_1 = 4, c_2 = 1, and c_3 = -4.$$

Then we can obtain the characteristic equation as $r^3 - 4r^2 - r + 4 = (r-4)(r+1)(r-1)$

Roots of this equation as follows, $r_1 = 4, r_2 = -1,$ and $r_3 = 1$

Therefore our recurrence relation is $a_n = \alpha_1 4^n + \alpha_2(-1)^n + \alpha_3 1^n$ and now we can use our initial conditions.

$$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 4$$

$$a_1 = 4\alpha_1 - \alpha_2 + \alpha_3 = 8$$

$$a_3 = 16\alpha_1 + \alpha_2 + \alpha_3 = 34$$

$\alpha_1 = 2, \alpha_2 = 1,$ and $\alpha_3 = 1$; from the Theory at page 545 we can write our recurrence relation as follows

$$a_n = 2 \cdot 4^n + 1 \cdot (-1)^n + 1 \cdot 1^n$$

Answer 4

From the definition of "Equivalence Relation"; for R to be an equivalence relation it needs to be reflexive, symmetric, and transitive.
Reflexive: (x,y)R(x,y)

$$\forall x \forall y((3\text{x - 2y}) = (3\text{x - 2y}))$$

Symmetric:

$$\forall x \forall y \forall z \forall t(((\text{x,y})\text{R}(\text{z,t})) \rightarrow ((\text{z,t})\text{R}(\text{x,y})))$$

$$\Longleftrightarrow$$

$$\forall x \forall y \forall z \forall t (((3x - 2y) = (3z - 2t)) \rightarrow ((3z - 2t) = (3x - 2y)))$$

Transitive:

$$\forall x \forall y \forall z \forall t \forall k \forall p ((((x,y)R(z,t)) \wedge ((z,t)R(k,p))) \rightarrow ((x,y)R(k,p)))$$

$$\Longleftrightarrow$$

$$\forall x \forall y \forall z \forall t \forall k \forall p (((3x - 2y = 3z - 2t) \wedge (3z - 2t = 3k - 2p)) \rightarrow (3x - 2y = 3k - 2p))$$

Therefore, we have proved that the defined binary relation R is an equivalence relation.

Graphical representations of [(2, 3)] and [(2, 3)] in the Cartesian coordinate system as follows:
Red line indicates [(2, 3)]; with 3x - 2y = 0
Blue line indicates [(2, -3)]; with 3x - 2y = 12