October 6, 2017

Divide and Conquer

Chapter 2 of Dasgupta *et al.*

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Divide and Conquer

- *Divide*: If the input size is too large to deal with in a straightforward manner, divide the data into two or more disjoint subsets
- *Recur*: Use divide and conquer to solve the sub-problems associated with the data subsets
- *Conquer*: Take the solutions to the sub-problems and "merge" these solutions into a solution for the original problem

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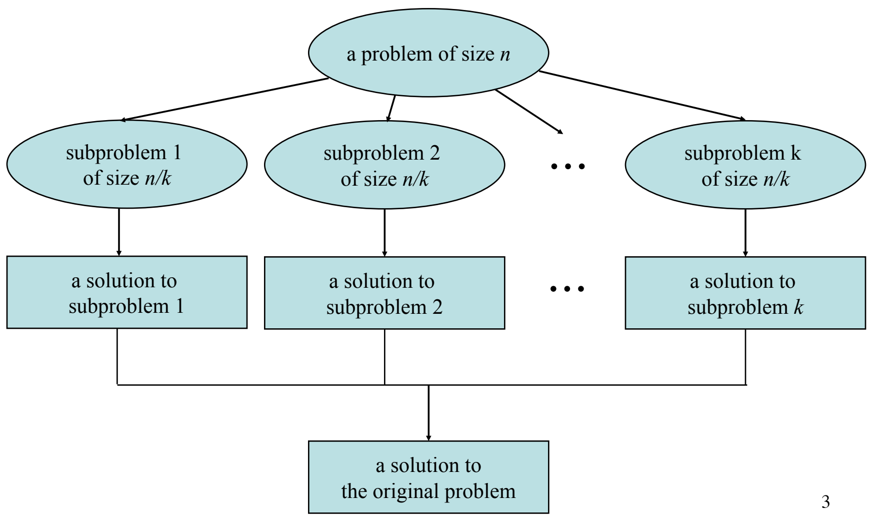
Divide and Conquer



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Outline

- Already covered/known
  - Sorting: Mergesort
  - Searching: Binary Search
- Integer Multiplication (Karatsuba)
- Matrix Multiplication (Strassen)
- Closest Pair
- Linear-time selection

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Integer multiplication (Karatsuba)

Integer multiplication

- Given positive integers y, z, compute $x=y*z$
- A naïve multiplication algorithm is below

```
def naive_mul(y,z):
    x = 0
    while z > 0:
        if z % 2 == 1:
             x += y
        y *= 2
        z /= 2
    return x
```

Remark: these two operations can be implemented as $O(1)$ shifts

Integer multiplication

Addition takes $O(n)$ bit operations, where n is the number of bits in y and z. The naive multiplication algorithm takes $O(n)$ n-bit additions. Therefore, the naive multiplication algorithm takes $O(n^2)$ bit operations.

Can we multiply using fewer bit operations?

Integer multiplication

Suppose n is a power of 2. Divide y and z into two halves, each with $n/2$ bits.

y	a	b
z	c	d

Integer multiplication

Then

$$\begin{aligned} y &= a2^{n/2} + b \\ z &= c2^{n/2} + d \end{aligned}$$

and so

$$\begin{aligned} yz &= (a2^{n/2} + b)(c2^{n/2} + d) \\ &= ac2^n + (ad + bc)2^{n/2} + bd \end{aligned}$$

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Integer multiplication

This computes yz with 4 multiplications of $n/2$ bit numbers, and some additions and shifts. Running time given by $T(1) = c$, $T(n) = 4T(n/2) + dn$, which has solution $O(n^2)$ by the General Theorem. No gain over naive algorithm!

Example 5.7: *Consider the recurrence*

$$T(n) = 4T(n/2) + n.$$

In this case, $n^{\log_b a} = n^{\log_2 4} = n^2$. *Thus, we are in Case 1, for* $f(n)$ *is* $O(n^{2-\varepsilon})$ *for* $\varepsilon = 1$. *This means that* $T(n)$ *is* $\Theta(n^2)$ *by the master method.*

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Integer multiplication (Karatsuba algorithm)

- Consider the product
  $(a-b)(d-c) = (ad + bc) - (ac + bd)$
- It contains two of the products we need (*ad* and *bc*)
- Then
  $yz=ac2^n+[(a-b)(d-c)+(ac+bd)]2^{n/2} + bd$
- We need three multiplications of *n/2* bits and *O(n)* additional work

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Integer multiplication (Karatsuba algorithm)

Therefore,

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 3T(n/2) + dn & \text{otherwise} \end{cases}$$

where c, d are constants.

Therefore, by our general theorem, the divide and conquer multiplication algorithm uses

$$T(n) = O(n^{\log 3}) = O(n^{1.59})$$

bit operations.

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Karatsuba algorithm

```
def multiply(y,z):
    l = max(len(y), len(z))
    if l == 1:
        return [y[0] * z[0]]
    y = [0 for i in range(len(y), l)] + y;
    z = [0 for i in range(len(z), l)] + z;
    m0 = (l + 1) / 2
    a = y[:m0]
    b = y[m0:]
    c = z[:m0]
    d = z[m0:]
```

Remark: pad *y* and *z* so that they have the same length

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Karatsuba algorithm (continued)

```
    p0 = multiply(a, c)
    p1 = multiply(add(a, b), add(c, d))
    p2 = multiply(b, d)

    z0 = p0
    z1 = subtract(p1, add(p0, p2))
    z2 = p2

    z0prod = z0 + [0 for i in range(0, l)]
    z1prod = z1 + [0 for i in range(0, l / 2)]

    return add(add(z0prod, z1prod), z2)
```

Remark: compute z1 = p1 – p0 – p2

Remark: compute z0 b^l + z1 b^(l/2) + z2

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Matrix multiplication (Strassen)

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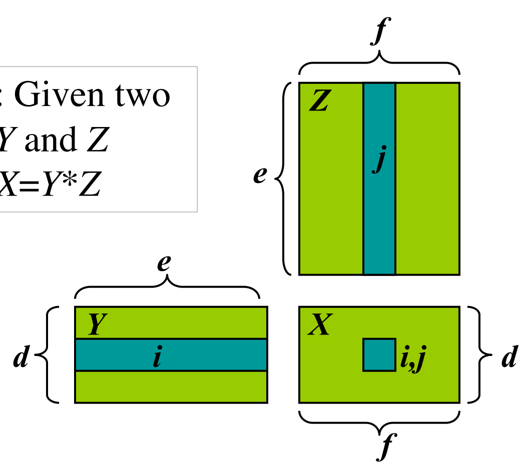
Matrix multiplication

Problem: Given two matrices *Y* and *Z* compute *X*=*Y***Z*



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Matrix multiplication

```
def mult(Y,Z):
    X = zero(len(Y),len(Z[0]))

    for i in range(len(Y)):
        for j in range(len(Z[0])):
            for k in range(len(Z)):
                X[i][j] += Y[i][k]*Z[k][j]

    return X
```

Algorithm **mult(Y,Z)** is $O(n^3)$, can we do better? 17

Matrix multiplication

Divide X, Y, Z each into four $(n/2)\times(n/2)$ matrices.

$$X = \begin{bmatrix} I & J \\ K & L \end{bmatrix}$$
$$Y = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
$$Z = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

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Matrix multiplication

Then

$$\begin{aligned} I &= AE + BG \\ J &= AF + BH \\ K &= CE + DG \\ L &= CF + DH \end{aligned}$$

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Matrix multiplication

Let $T(n)$ be the time to multiply two $n \times n$ matrices.

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 8T(n/2) + dn^2 & \text{otherwise} \end{cases}$$

where c, d are constants.

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Matrix multiplication

Therefore,

$$
\begin{aligned}
T(n) &= 8T(n/2) + dn^2 \\
&= 8(8T(n/4) + d(n/2)^2) + dn^2 \\
&= 8^2T(n/4) + 2dn^2 + dn^2 \\
&= 8^3T(n/8) + 4dn^2 + 2dn^2 + dn^2 \\
&= 8^iT(n/2^i) + dn^2 \sum_{j=0}^{i-1} 2^j \\
&= 8^{\log n}T(1) + dn^2 \sum_{j=0}^{\log n-1} 2^j \\
&= cn^3 + dn^2(n-1) \\
&= O(n^3)
\end{aligned}
$$

Master Thorem case 1:

$f(n) \in O(n^{\log_2 8-\varepsilon})$?

$dn^2 \in O(n^{3-\varepsilon})$? true for ε=1

Then $T(n) \in \Theta(n^3)$

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Matrix multiplication

- The naïve Divide and Conquer algorithm is no better than the straightforward algorithm
- However, it gives us an insight on the next algorithm
- Strassen's algorithm uses only 7 multiplications instead of 8

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Strassen algorithm

Compute

$$
\begin{aligned}
M_1 &:= (A+C)(E+F) \\
M_2 &:= (B+D)(G+H) \\
M_3 &:= (A-D)(E+H) \\
M_4 &:= A(F-H) \\
M_5 &:= (C+D)E \\
M_6 &:= (A+B)H \\
M_7 &:= D(G-E)
\end{aligned}
$$

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Strassen algorithm

Then,

$$
\begin{aligned}
I &:= M_2+M_3-M_6-M_7 \\
J &:= M_4+M_6 \\
K &:= M_5+M_7 \\
L &:= M_1-M_3-M_4-M_5
\end{aligned}
$$

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Strassen algorithm

$$
\begin{aligned}
I &:= M_2 + M_3 - M_6 - M_7 \\
&= (B+D)(G+H) + (A-D)(E+H) \\
&\quad - (A+B)H - D(G-E) \\
&= (BG + BH + DG + DH) \\
&\quad + (AE + AH - DE - DH) \\
&\quad + (-AH - BH) + (-DG + DE) \\
&= BG + AE
\end{aligned}
$$

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Strassen algorithm

$$
\begin{aligned}
J &:= M_4 + M_6 \\
&= A(F-H) + (A+B)H \\
&= AF - AH + AH + BH \\
&= AF + BH
\end{aligned}
$$

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Strassen algorithm

$$\begin{aligned} K &:= M_5 + M_7 \\ &= (C+D)E + D(G-E) \\ &= CE + DE + DG - DE \\ &= CE + DG \end{aligned}$$

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Strassen algorithm

$$\begin{aligned} L &:= M_1 - M_3 - M_4 - M_5 \\ &= (A+C)(E+F) - (A-D)(E+H) \\ &\quad - A(F-H) - (C+D)E \\ &= AE + AF + CE + CF - AE - AH \\ &\quad + DE + DH - AF + AH - CE - DE \\ &= CF + DH \end{aligned}$$

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```
def strassen(Y,Z):
    if len(Y) <= 2:
        return mult(Y,Z)
    else:
        A,B,C,D = partition(Y)
        E,F,G,H = partition(Z)
        M1 = strassen(add(A,C),add(E,F))
        M2 = strassen(add(B,D),add(G,H))
        M3 = strassen(sub(A,D),add(E,H))
        M4 = strassen(A,sub(F,H))
        M5 = strassen(add(C,D),E)
        M6 = strassen(add(A,B),H)
        M7 = strassen(D,sub(G,E))
        I = sub(sub(add(M2,M3),M6),M7)
        J = add(M4,M6)
        K = add(M5,M7)
        L = sub(sub(sub(M1,M3),M4),M5)
        return recompose(I,J,K,L)
```

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Analysis of Strassen algorithm

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 7T(n/2) + dn^2 & \text{otherwise} \end{cases}$$

where c, d are constants.

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Analysis of Strassen algorithm

$$
\begin{aligned}
T(n) &= 7T(n/2) + dn^2 \\
&= 7(7T(n/4) + d(n/2)^2) + dn^2 \\
&= 7^2T(n/4) + 7dn^2/4 + dn^2 \\
&= 7^3T(n/8) + 7^2dn^2/4^2 + 7dn^2/4 + dn^2 \\
&= 7^iT(n/2^i) + dn^2 \sum_{j=0}^{i-1} (7/4)^j \\
&= 7^{\log n}T(1) + dn^2 \sum_{j=0}^{\log n-1} (7/4)^j \\
&= cn^{\log 7} + dn^2 \frac{(7/4)^{\log n} - 1}{7/4 - 1} \\
&= cn^{\log 7} + \frac{4}{3}dn^2\left(\frac{n^{\log 7}}{n^2} - 1\right) \\
&= O(n^{\log 7}) \\
&\approx O(n^{2.8})
\end{aligned}
$$

Master Thorem case 1:

$f(n) \in O(n^{\log_2 7-\varepsilon})$?

$dn^2 \in O(n^{2.8-\varepsilon})$? true for ε=0.5

Then $T(n) \in \Theta(n^{\log_2 7})$

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Discussion

- There is a large constant hidden which makes Strassen impractical, unless the matrices are large (*n>45*) and dense
- For sparse matrices there are faster methods
- Strassen is not as *numerically stable* as the naïve
- Sub-matrices at each level consume space
- FYI: the current best algorithm for dense matrices runs in $O(n^{2.376})$
- Lower bound $\Omega(n^2)$ [for dense matrices]

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Closest Pair

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Closest Pair Problem

- Let $P_1=(x_1,y_1),\ldots,P_n=(x_n,y_n)$ be a set S of n points in the plane
- Problem: Find the two closest points in S
- Assumptions:
  - n is a power of two
  - points are ordered by their x coordinate (if not, we can sort them in $O(n \log n)$ time)

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Closest-Pair Problem: Brute-force

- Compute the distance between every pair of distinct points
- Return the indexes of the points for which the distance is the smallest

Time complexity?

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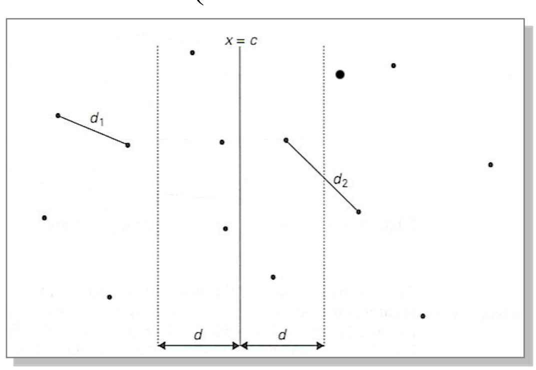
Closest-Pair: Divide and Conquer

Step 1. Divide the points in S into two subsets S_1 and S_2 by a vertical line $x = c$ so that half the points lie to the left or on the line and half the points lie to the right or on the line (c is the median of the x coord)



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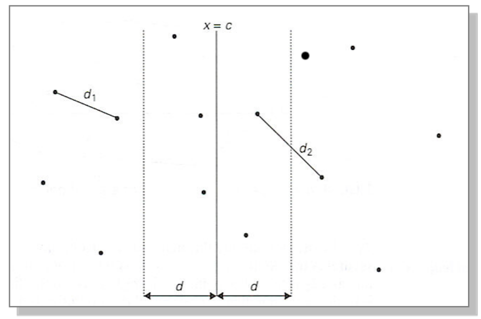
Closest-Pair: Divide and Conquer

Step 2. Find recursively the closest pairs for the left and right subsets. Let d_1, d_2 be the distances of the two closest pairs.

Set $d = \min\{d_1, d_2\}$



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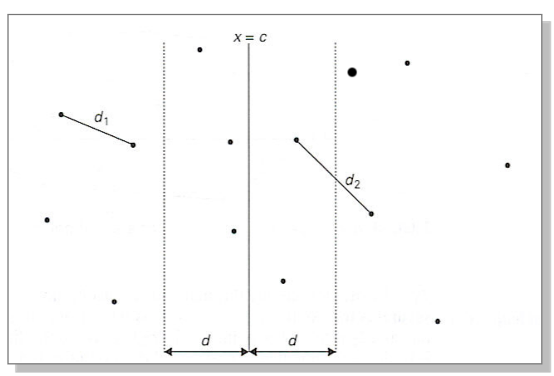
Closest Pair: Divide and Conquer

Step 3. Consider the vertical strip *2d*-wide centered at *x=c*. Let *Y* be the subset of points in this vertical strip of width 2*d*



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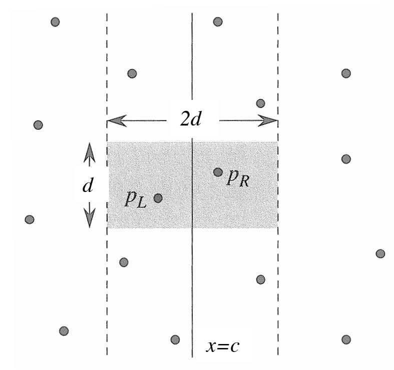
Closest Pair: Divide and Conquer

- **Observation 1:** if a pair of points p_L,p_R has distance less than *d*, both points of the pair **must** be within *Y*



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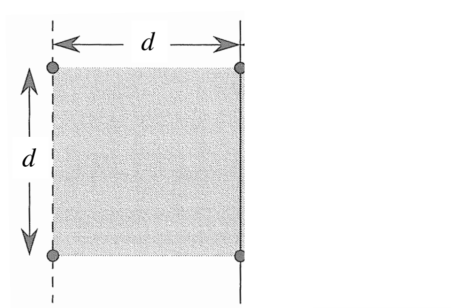
Closest Pair: Divide and Conquer

Observation 2: Since all the points within S_1 are at least *d* units apart, at most 4 points can reside within the *d* x *d* square



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Closest Pair: Divide and Conquer

Proof: Let's suppose (for sake of contradiction) that five or more points are found in a square of size *d* x *d*. Divide the square into four smaller squares of size *d/2* x *d/2*. At least one pair of points must fall within the same smaller square: these two points will be at a distance *d*/sqrt(2) < *d*, which leads to a contradiction.

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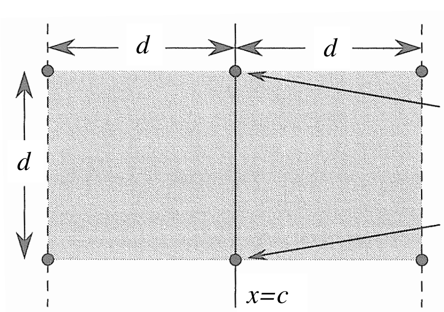
Closest Pair: Divide and Conquer

Consequence: At most 8 points can reside within the *d* x 2*d* rectangle, because on each side all points are at least *d* unit apart



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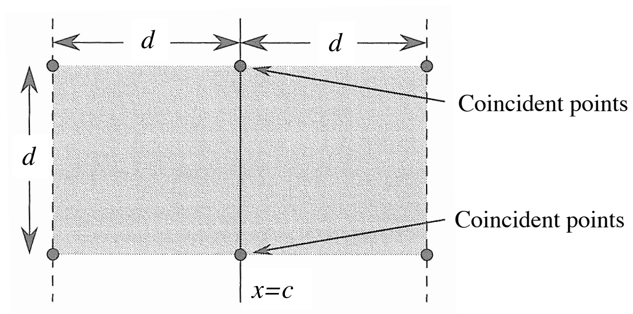
Closest Pair: Divide and Conquer

Step 4. For each point *p* in *Y*, try to find points in *Y* that are within *d* units of *p*. Only *7* points in *Y* that follow *p* need to be considered



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Closest pair in Python

```
def closestPair(xP, yP):
    n = len(xP)
    if n <= 3:
        return bruteForceClosestPair(xP)
    Xl = xP[:n/2]
    Xr = xP[n/2:]
    Yl, Yr = [], []
    median = Xl[-1].x
    for p in yP:
        if p.x <= median:
            Yl.append(p)
        else:
            Yr.append(p)
```

Remark: **xP** and **yP** is the same of input points (*x*,*y*), but **xP** is sorted by *x* and **yP** is sorted by *y*

Remark: **Xl** is the first half of the points sorted by *x*, and **Xr** is the second half

Remark: **Yl** contains the points (sorted by *y*) which have a *x* coordinate smaller than the median

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```
dl, pairl = closestPair(Xl, Yl)
dr, pairr = closestPair(Xr, Yr)
dm, pairm = (dl, pairl) if dl < dr else (dr, pairr)

st = [p for p in yP if abs(p.x - median) < dm]
n_st = len(st)
closest = (dm, pairm)
if n_st > 1:
    for i in range(n_st-1):
        for j in range(i+1,min(i+8, n_st)):
            if d(st[i],st[j]) < closest[0]:
               closest = (d(st[i],st[j]),(st[i],st[j]))
return closest
```

Remark: variable **st** contains the points in the strip [median-dm, median+dm] sorted by *y*

Remark: **d**(x,y) returns the distance between x and y

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Analysis of the Closest-Pair Algorithm

- We can keep the points in Y stored in increasing order of their y coordinates, which is maintained by merging during the execution of step 4
- We can process the points in Y sequentially in linear time
- Running time is described by $T(n)=2T(n/2)+O(n)$
- By the Master Theorem, $T(n)$ is $O(n \log n)$

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Linear-time selection

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Linear-time selection

- Problem: Select the *i*-th smallest element in an unsorted array of size *n* (assume distinct elements)
- Trivial solution: sort *A*, select *A[i]* time complexity is *O(n log n)*

- Can we do it in linear time? Yes, thanks to Blum, Floyd, Pratt, Rivest, and Tarjan

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Linear-time selection

Select (A, $start$, end, i) /* i is the i-th order statistic */

1. divide input array A into $\lceil n/5 \rceil$ groups of size 5 (and one leftover group if n % 5 is not 0)
2. find the median of each group of size 5 by sorting the groups of 5 and then picking the middle element
3. call Select recursively to find x, the median of the $\lceil n/5 \rceil$ medians
4. partition array around x, splitting it into two arrays L (elements smaller than x) and R (elements bigger than x)
5. $k \Leftarrow |L|+1$
   if $(i = k)$ **then return** x
   else if $(i < k)$ **then** Select (L, i)
   else Select (R, $i - k$)

$\lceil r \rceil$ means the *ceiling* (rounding to the next integer) of real number r

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Python linear-time selection

```
def selection(a, rank):
    n = len(a)
    if n <= 5:
        return rank_by_sorting(a, rank)
    medians = [rank_by_sorting(a[i:i+5], 3)
               for i in range(0, n-4, 5)]
    median = selection(medians, (len(medians) + 1) // 2)
    L, R = [], []
    for x in a:
        if x < median:
            L += [x]
        else:
            R += [x]
    if rank <= len(L):
        return selection(L, rank)
    else:
        return selection(R, rank - len(L))
```

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Example

Let us run Select(A, 1, 28, 11), where

A={12, 34, 0, 3, 22, 4, 17, 32, 3, 28, 43, 82, 25, 27, 34, 2, 19, 12, 5, 18, 20, 33, 16, 33, 21, 30, 3, 47}

Note that the elements in this example are not distinct.

Example

First make groups of 5

12	4	43	2	20	30
34	17	82	19	33	3
0	32	25	12	16	47
3	3	27	5	33	
22	28	34	18	21	

Example

Then find medians in each group

0	4	25	2	20	3
3	3	27	5	16	30
12	17	34	12	21	47
34	32	43	19	33	
22	28	82	18	33	

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Example

Then find median of medians

0	4	25	2	20	3
3	3	27	5	16	30
12	17	34	12	21	47
34	32	43	19	33	
22	28	82	18	33	

12, 12, 17, 21, 30, 34

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Example

Use 17 as the pivot value and partition original array

0	4	25	2	20	3
3	3	27	5	16	30
12	17	34	12	21	47
34	32	43	19	33	
22	28	82	18	33	

12, 12, 17, 21, 30, 34

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Example

After partitioning

$L = \{12, 0, 3, 4, 3, 2, 12, 5, 16, 3\}$

L contains 10 elements smaller than 17

$\{17\}$ *this is the 11-th smallest*

$R = \{34, 22, 32, 28, 43, 82, 25, 27, 34, 19, 18, 20, 33, 33, 21, 30, 47\}$

R contains 17 elements bigger than 17

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Linear-time selection

- Finding the median of medians guarantees that *x* causes a "good split"
- At least a constant fraction of the *n* elements $\leq x$ and a constant fraction $> x$
- Analysis: we need to find the worst case for the size of *L* and *R*

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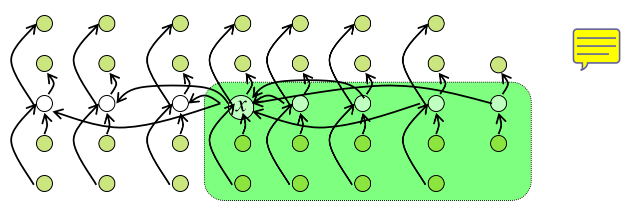
Linear-time selection: analysis

Observation: At least 1/2 of the medians found in step 2 are greater than the median of medians *x*. So at least half of the *[n/5]* groups contribute 3 elements that are bigger than *x*, except for the one group with less than 5 elements and the group with *x* itself



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Linear-time selection: analysis

- Therefore there are
  $$3(\lceil 1/2 \lceil n/5 \rceil \rceil - 2) \geq (3n/10) - 6$$
  elements are $> x$ (or $< x$)
- So worst-case split has at most $(7n/10) + 6$ elements in "big" section of the problem, that is:
  $$max\{|L|,|R|\} < (7n/10) + 6$$

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Linear-time selection: analysis

Running Time:

1. $O(n)$ (break into groups of 5)
2. $O(n)$ (sorting 5 numbers and finding median is $O(1)$ time)
3. $T(\lceil n/5 \rceil)$ (recursive call to find median of medians)
4. $O(n)$ (partition is linear time)
5. $T(7n/10 + 6)$ (maximum size of subproblem)

Recurrence relation

$$\begin{aligned} T(n) &= T(\lceil n/5 \rceil) + T(7n/10 + 6) + O(n) & n > 80 \\ &= \Theta(1) & n \leq 80 \end{aligned}$$

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Linear-time selection: analysis

Fact: $T(n) = T([n/5])+T(7n/10 + 6)+O(n)$ is $O(n)$

Proof:

Base case: easy (omitted).

$T(n) = T([n/5]) + T(7n/10 + 6) + O(n)$
$\leq c[n/5] + c(7n/10 + 6) + O(n)$
$\leq c((n/5) + 1) + 7cn/10 + 6c + O(n)$
$= cn - [c(n/10 - 7) - dn]$
$\leq cn$

This step holds since $n \geq 80$ implies $(n/10 - 7)$ is positive.

Choosing c big enough makes $c(n/10 - 7) - dn$ positive, so last line holds.

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Reading assignment on Chapter 4

- Mergesort (section 2.3)
- Binary Search (page 50, box)
- Integer Multiplication (Karatsuba, section 2.1)
- Matrix Multiplication (Strassen, section 2.5)
- Closest pair (problem 2.32)
- Medians (section 2.4 covers randomized)
- Skip FFT

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