Divide-and-Conquer

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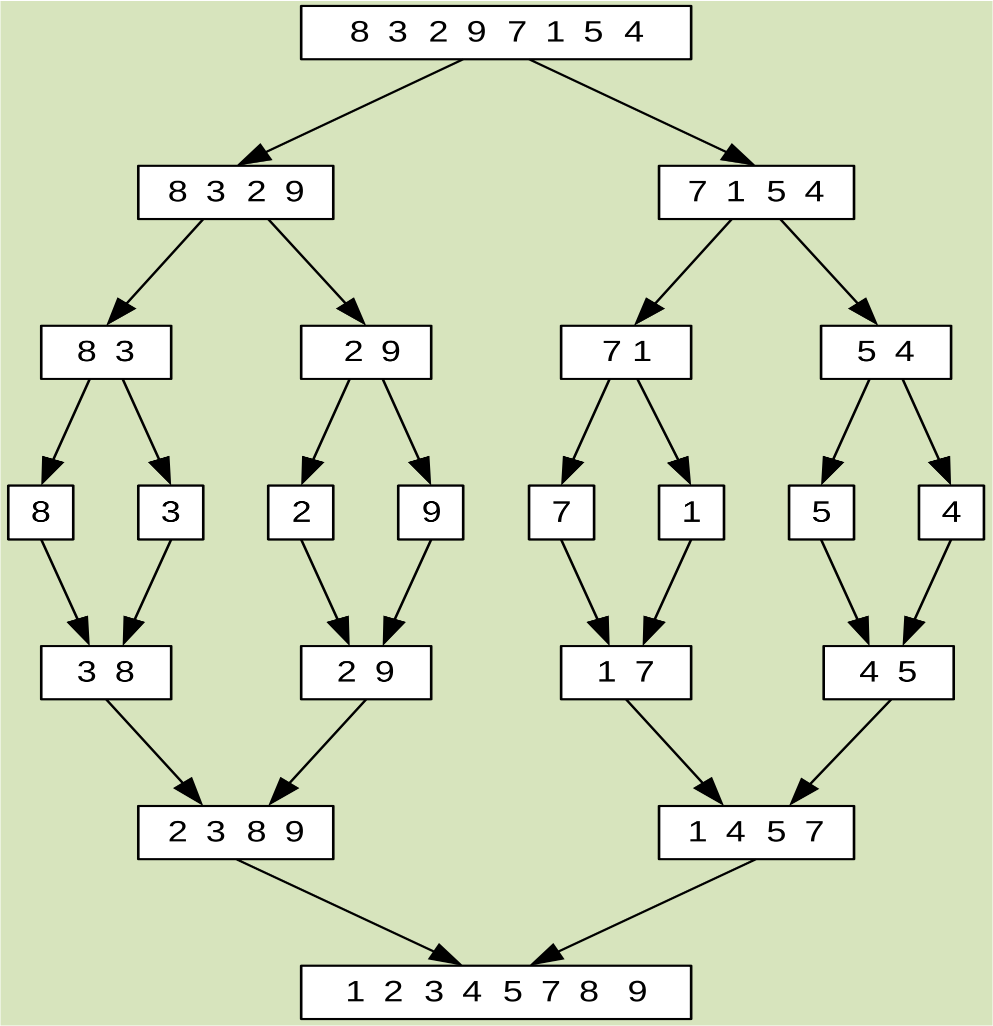
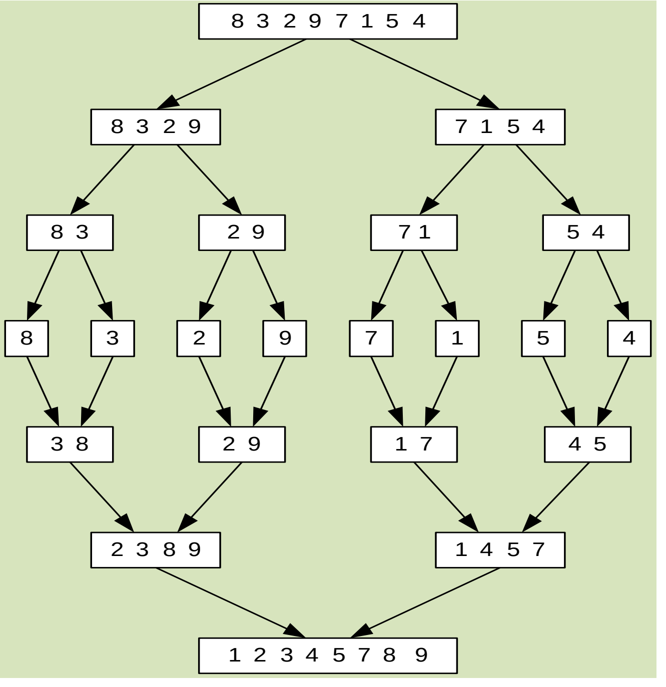
Mergesort



A. Levitin "Introduction to the Design & Analysis of Algorithms," 2nd ed., Ch. 4

Divide-and-Conquer
Mergesort



Merge-sort. In Merge-Sort, we divide the sequence into equal halves, sort them recursively, and then merge them together. Merging two sorted sequences of length $n/2$ takes n comparisons. So the recurrence is:

$$T(n) \quad = \quad 2T(n/2) + n,$$

and, say, for $n = 1$ assume $T(1) = 1$.

Divide-and-Conquer Mergesort

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 2[2T(n/4) + n/2] + n \end{aligned}$$

We can repeat this substitution again, and again, up to $\log n$ times:

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 4T(n/4) + 2n \\ &= 8T(n/8) + 3n \\ &\ldots \\ &= 2^j T(n/2^j) + jn \\ &\ldots \\ &= nT(1) + n\log n \\ &= \Theta(n\log n). \end{aligned}$$

Example

Find the maximum and minimum of a sequence

If n=1, the number is itself min or max

If n>1, divide the numbers into two lists.

Decide the min & max in the first list.

Choose the min & max in the second list.

Decide the min & max of the entire list.

$$T(n)=2T(n/2)+2$$

$$T(n) = \Theta(n)$$ *(proof - later)*

Can you give another algorithm?

Let's solve the following recurrence in general:

$$T(n) = aT(n/b) + n$$

where a > 0, b > 1, T(1) = 1

do repeated substitutions:

$$\begin{aligned}
T(n) &= aT(n/b) + n \\
&= a[aT(n/b^2) + n/b] + n \\
&= a^2T(n/b^2) + (a/b)n + n \\
&\dots \\
&= a^jT(n/b^j) + n[(a/b)^{j-1} + \dots + (a/b)^2 + (a/b) + 1] \\
&\dots \\
&= a^{\log_b n}T(1) + n \cdot \sum_{i=0}^{\log_b n-1} (a/b)^i \\
&= n^{\log_b a} + n \cdot \sum_{i=0}^{\log_b n-1} (a/b)^i
\end{aligned}$$

check

$$n^{\log_b a} + n \cdot \sum_{i=0}^{\log_b n-1} (a/b)^i$$

Case 1: $a = b$. The first term is n. In the summation, we have $\log_b n$ terms and they are all equal $a/b = 1$, so the second term is $n \log_b n$. Thus we get $T(n) = \Theta(n \log n)$.

Case 2: $a < b$. The second term is now a geometric series with the ratio smaller than 1, so $\sum_{i=0}^{\log_b n-1}(a/b)^i = \Theta(1)$. The first term is $n^{\log_b a}$ with $\log_b a < 1$, so we get $T(n) = \Theta(n)$.

Case 3: $a > b$. Summing the geometric series in the second term, we get

check

$$\sum_{i=0}^{\log_b n-1} (a/b)^i \quad = \quad \frac{(a/b)^{\log_b n} - 1}{(a/b) - 1} \quad = \quad \tfrac{b}{a-b}(a^{\log_b n}/b^{\log_b n} - 1) \quad = \quad \tfrac{b}{a-b}(n^{\log_b a}/n - 1)$$

So

$$T(n) \quad = \quad n^{\log_b a} + \tfrac{b}{a-b}(n^{\log_b a} - n) \quad = \quad \Theta(n^{\log_b a}).$$

For $r \neq 1$, the sum of the first n terms of a geometric series is

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\left(\frac{1-r^n}{1-r}\right)$$

Divide-and-Conquer Master Theorem

Theorem **(Master Theorem)**

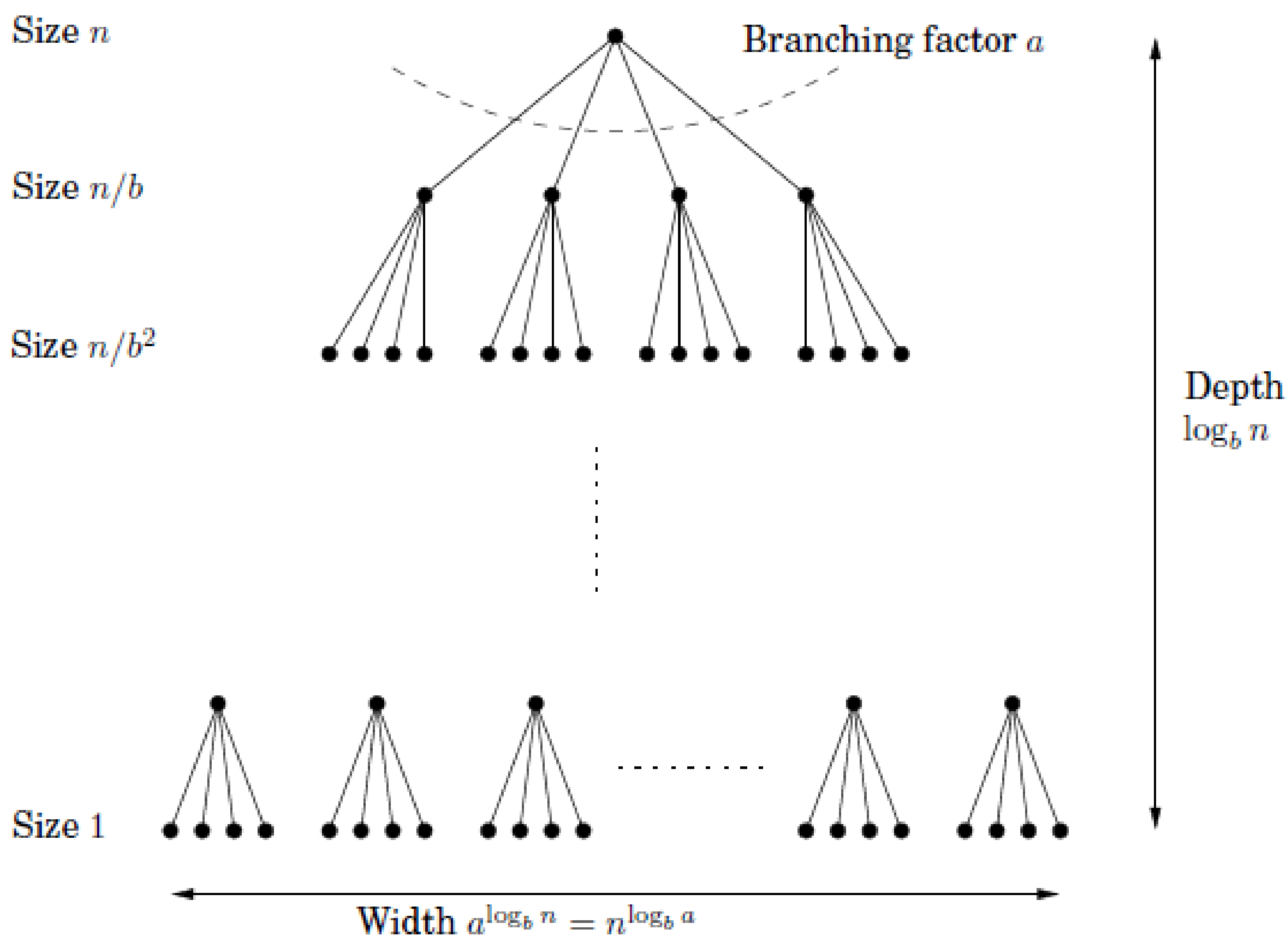
Let $a \geq 1$, $b > 1$, $c > 0$ *and* $d \geq 0$. *If* $T(n)$ *satisfies the recurrence*

$$T(n) = aT(n/b) + cn^d,$$

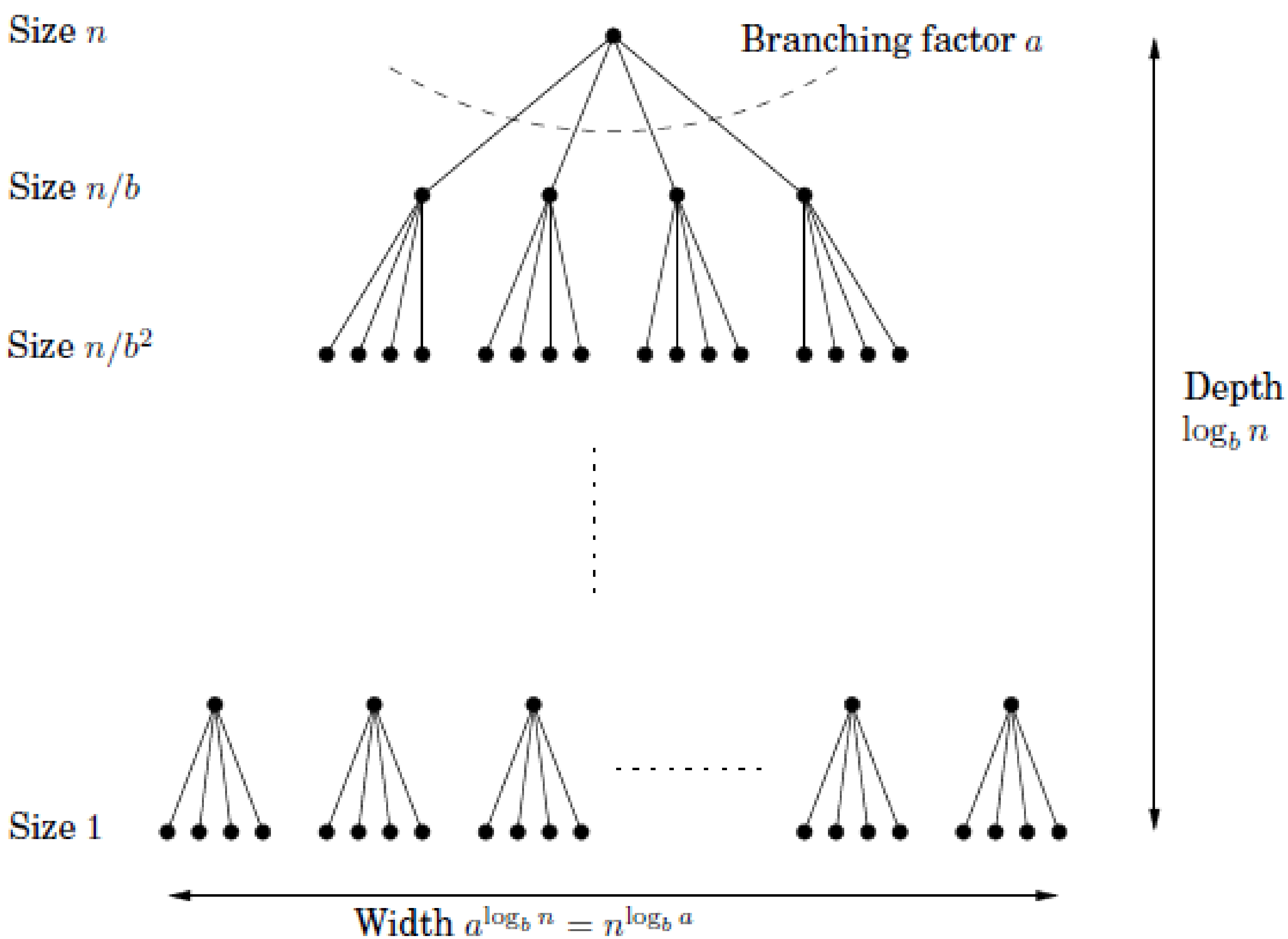
then

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{for } a > b^d \\ \Theta(n^d \log n) & \text{for } a = b^d \\ \Theta(n^d) & \text{for } a < b^d \end{cases}$$

Divide-and-Conquer Master Theorem



Algorithms, S. Dasgupta



$$T(n) = aT(n/b) + cn^d. \qquad T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{for } a > b^d \\ \Theta(n^d \log n) & \text{for } a = b^d \\ \Theta(n^d) & \text{for } a < b^d \end{cases}$$

Divide-and-Conquer

(a) **Algorithm** $\textsc{PrintXs}$ (n : **integer**)

    if $n < 3$

        print("X")

    else

        $\textsc{PrintXs}(\lceil n/3 \rceil)$

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        for $i \leftarrow 1$ **to** $2n$ **do** print("X")

Divide-and-Conquer

(a) **Algorithm** PrintXs (n : **integer**)

```
if n < 3
    print("X")
else
    PrintXs(⌈n/3⌉)
    PrintXs(⌈n/3⌉)
    PrintXs(⌈n/3⌉)
    for i ← 1 to 2n do print("X")
```

There are 3 recursive calls, each with parameter $\lceil n/3 \rceil$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) \quad = \quad 3X(n/3) + 2n.$$

We apply the Master Theorem with $a = 3$, $b = 3$, $c = 2$, $d = 1$. Here, we have $a = b^d$, so the solution is $\Theta(n \log n)$.

Divide-and-Conquer

(b) **Algorithm** PRINTYS (n : **integer**)

```
if n < 2
    print("Y")
else
    for j ← 1 to 16 do PrintYs(⌊n/2⌋)
    for i ← 1 to n^3 do print("Y")
```

Divide-and-Conquer

(b) **Algorithm** $\textsc{PrintYs}$ (n : **integer**)

  if $n < 2$

    print("Y")

  else

    for $j \leftarrow 1$ **to** 16 **do** $\textsc{PrintYs}(\lfloor n/2 \rfloor)$

    for $i \leftarrow 1$ **to** n^3 **do** print("Y")

(b)
There are 16 recursive calls, each with parameter $\lfloor n/2 \rfloor$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) \quad = \quad 16X(n/2) + n^3.$$

We apply the Master Theorem with $a = 16$, $b = 2$, $c = 1$, $d = 3$. Here, we have $a > b^d$, so the solution is $\Theta(n^{\log_2 16})$.

Divide-and-Conquer

(c) **Algorithm** PRINTZS (n : **integer**)

    if $n < 3$

        print("Z")

    else

        PRINTZS($\lceil n/3 \rceil$)

        PRINTZS($\lceil n/3 \rceil$)

        for $i \leftarrow 1$ **to** $7n$ **do** print("Z")

Divide-and-Conquer

(c) **Algorithm** PrintZs (n : **integer**)

  if $n < 3$

    print("Z")

  else

    PrintZs($\lceil n/3 \rceil$)

    PrintZs($\lceil n/3 \rceil$)

    for $i \leftarrow 1$ **to** $7n$ **do** print("Z")

(c)
There are 2 recursive calls, each with parameter $\lceil n/3 \rceil$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) = 2X(n/3) + 7n.$$

We apply the Master Theorem with $a = 2$, $b = 3$, $c = 7$, $d = 1$. Here, we have $a < b^d$, so the solution is $\Theta(n)$.

Divide-and-Conquer

(d) **Algorithm** $\textsc{PrintUs}$ (n : **integer**)

    if $n < 4$

        print("U")

    else

        $\textsc{PrintUs}(\lceil n/4 \rceil)$

        $\textsc{PrintUs}(\lfloor n/4 \rfloor)$

        for $i \leftarrow 1$ **to** 11 **do** print("U")

Divide-and-Conquer

(d) **Algorithm** PRINTUS (n : **integer**)

    if $n < 4$

        print("U")

    else

        PRINTUS($\lceil n/4 \rceil$)

        PRINTUS($\lfloor n/4 \rfloor$)

        for $i \leftarrow 1$ **to** 11 **do** print("U")

(d)
There are 2 recursive calls, each with parameter $\lceil n/4 \rceil$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) \quad = \quad 2X(n/4) + 11.$$

We apply the Master Theorem with $a = 2$, $b = 4$, $c = 11$, $d = 0$. Here, we have $a > b^d$, so the solution is $\Theta(n^{\log_4 2})$.

Divide-and-Conquer

(e) **Algorithm** $\textsc{PrintVs}$ (n : **integer**)

    if $n < 3$

        print("V")

    else

        for $j \leftarrow 1$ **to** 9 **do** $\textsc{PrintVs}(\lfloor n/3 \rfloor)$

        for $i \leftarrow 1$ **to** $2n^3$ **do** print("V")

Divide-and-Conquer

(e) **Algorithm** PrintVs (n : **integer**)

if $n < 3$

  print("V")

else

  for $j \leftarrow 1$ **to** 9 **do** PrintVs($\lfloor n/3 \rfloor$)

  for $i \leftarrow 1$ **to** $2n^3$ **do** print("V")

(e)
There are 9 recursive calls, each with parameter $\lfloor n/3 \rfloor$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) = 9X(n/3) + 2n^3.$$

We apply the Master Theorem with $a = 9$, $b = 3$, $c = 2$, $d = 3$. Here, we have $a < b^d$, so the solution is $\Theta(n^3)$.