November 17, 2017

Weighted Graphs

Outline

- (single-source) shortest path
  - Dijkstra (Section 4.4)
  - Bellman-Ford (Section 4.6)
- (all-pairs) shortest path
  - Floyd-Warshall (Section 6.6)
- minimum spanning tree
  - Kruskal (Section 5.1.3)
  - Prim (Section 5.1.5)

Shortest Path

- Let G be a weighted graph ($w(e)$ is the weight of the edge e)
- The length of a path P is the sum of the weights of the edges of P
- If $P=e_0,e_1,\ldots,e_{k-1}$ then the length of P is $\sum w(e_i)$

Single-Source Shortest Path

- The *distance* from a vertex u to vertex v, denoted by $d(u,v)$ is the length of a minimum length path (also called *shortest-path*) from u to v, if such a path exists
- If the path does not exists, $d(u,v)=+\infty$
- Note that if there is a negative cycle, then the distance may not be defined

Optimal Substructure

- Fact: subpaths of shortest paths are shortest paths
- Proof: decompose a shortest path $p=<v_1,v_2,\ldots,v_k>$ into $v_1 \rightarrow v_i \rightarrow v_j \rightarrow v_k$. Then $w(p)=w(v_1,v_i)+w(v_i,v_j)+w(v_j,v_k)$. If $v_i \rightarrow v_j$ is not optimal, then we could make the path $v_1 \rightarrow v_k$ shorter, which contradicts the optimality of p.

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Shortest-Path Problems

- Single-source (single-destination). Find a shortest path from a given source (vertex s) to all the other vertices
  positive weights → greedy algorithm
  pos. & neg. weights → dynamic programming
- All-pairs. Find shortest-paths for every pair of vertices
  pos. & neg. weights → dynamic programming

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Dijkstra's algorithm

Dijkstra's algorithm

- Dijkstra's algorithm finds shortest paths from a start vertex *s* to all the other vertices
- It works on a simple graph with non-negative weights (i.e., it works only if $w(e) \geq 0$, for all edges *e*)

Dijkstra's algorithm

- The algorithm computes for each vertex *u* the *distance* to *u* from the start vertex *s*, that is, the weight of a shortest path between *s* and *u*
- The algorithm keeps track of the set of vertices for which the distance has been computed, called the *cloud* *S*

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Dijkstra's algorithm

- Every vertex has a *label* associated with it
- For any vertex *u*, we can refer to its *d* label as *d[u]*
- *d[u]* stores an approximation of the distance between *s* and *u*
- The algorithm will update a *d[u]* value when it finds a shorter path from *s* to *u*

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Dijkstra’s algorithm

- When a vertex u is added to the cloud, its label $d[u]$ is equal to the actual (final) distance between the starting vertex s and vertex u
- Initially, we set
  - $d[s]=0$ ... the distance from s to itself is 0 ...
  - $d[u]=\infty$ for $u \neq s$ ... these will change ...

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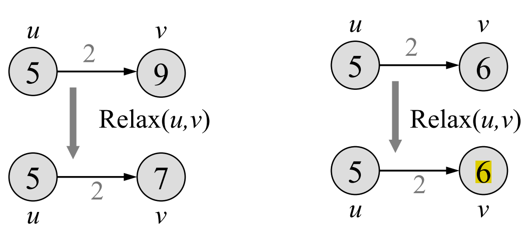
Edge relaxation

- For each vertex v in the graph, we maintain in $d[v]$ the estimate of the shortest path from s
- Relaxing an edge (u,v) means testing whether we can improve the shortest path to v found so far by going through u



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Expanding the Cloud

- Repeat until all vertices have been put in the cloud
  - let u be a vertex not in the cloud that has smallest $d[u]$ (on the first iteration, the starting vertex will be chosen)
  - we add u to the cloud S
  - we update $d[.]$ of the adjacent vertices of u as follows (*edge relaxation*)

for each vertex z adjacent to u **do**
  if z is not in the cloud S **then**
    if $d[u] + weight(u,z) < d[z]$ **then**
      $d[z] \Leftarrow d[u] + weight(u,z)$

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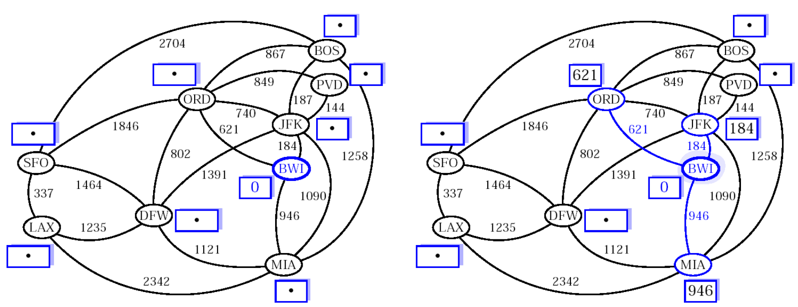
Example *s*=BWI



(a) (b)

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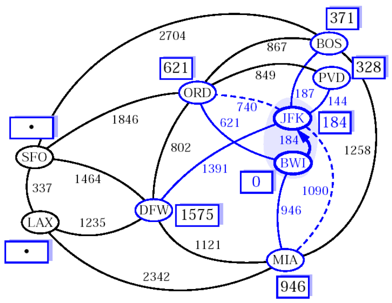
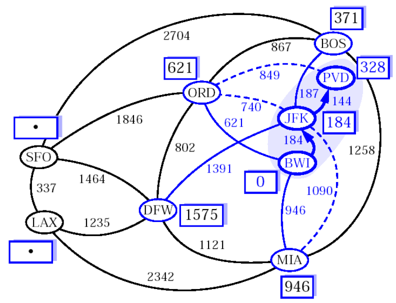
Example



(c)



(d)

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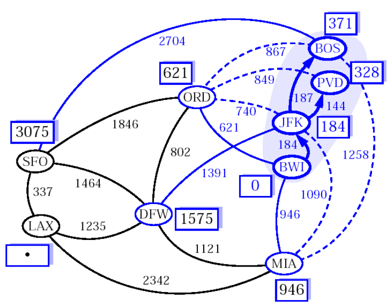
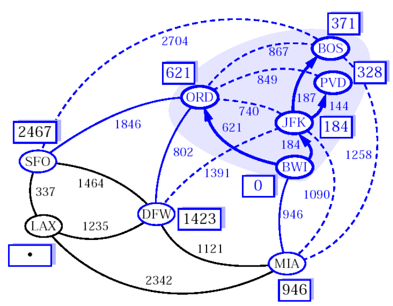
Example



(e)



(f)

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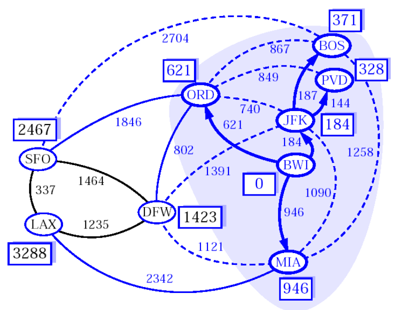
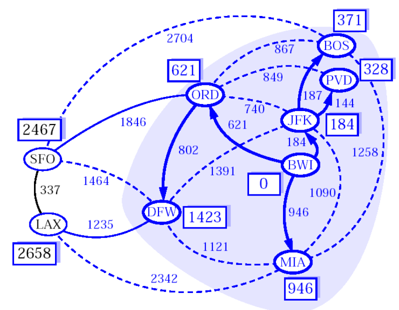
Example



(g)



(h)

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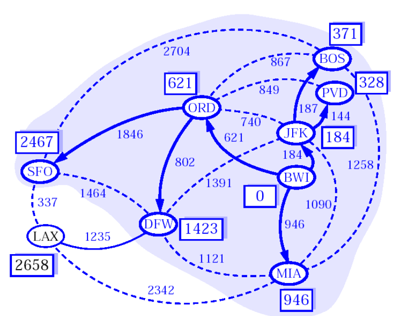
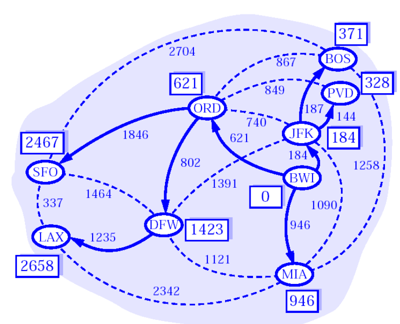
Example



(i)



(j)

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Dijkstra's algorithm

Algorithm ShortestPath(G, v):

Input: A simple undirected weighted graph G with nonnegative edge weights, and a distinguished vertex v of G

Output: A label $D[u]$, for each vertex u of G, such that $D[u]$ is the distance from v to u in G

```
Initialize D[v] ← 0 and D[u] ← +∞ for each vertex u ≠ v.
Let a priority queue Q contain all the vertices of G using the D labels as keys.
while Q is not empty do
   {pull a new vertex u into the cloud}
   u ← Q.removeMin()
   for each vertex z adjacent to u such that z is in Q  do
      {perform the relaxation procedure on edge (u,z)}
      if D[u] + w((u,z)) < D[z] then
         D[z] ← D[u] + w((u,z))
         Change to D[z] the key of vertex z in Q.
return the label D[u] of each vertex u
```

D[.] is d[.]

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Time complexity

- Use a *heap-based priority queue Q* to store the vertices not in the cloud, where *d[u]* is the key of a vertex *u* in *Q*
- Insert all vertices in *Q*, takes $O(n \log n)$
- Each iteration of the while, we spend $O(\log n)$ time to remove vertex *u* from *Q* and $O(deg(u) \log n)$ to perform the relaxation step
- Overall, $O(n \log n + \Sigma_v(deg(v) \log n))$ which is $O((n+m) \log n)$ [using binary heaps]

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Greedy choice

- <u>Theorem</u>: In Dijkstra's algorithm, whenever a vertex *u* is pulled into *S*, the label *d[u]* is equal to *d(s,u)* (the length of a shortest path from *s* to *u),* and the equality is maintained thereafter
- Proof: (by contradiction) omitted

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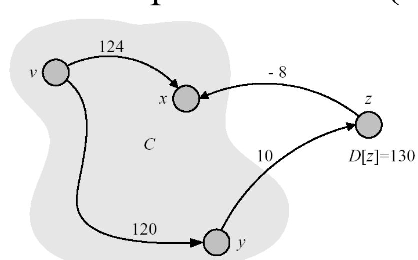
Negative weights

- Dijkstra fails on graphs with negative edges
- <u>Example</u>: Bringing *z* into *C* and performing edge relaxation invalidates the previously computed shorted path distance (124) to *x*



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Bellman-Ford’s algorithm

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Bellman-Ford’s algorithm

- Dijkstra’s algorithm does not work when the weighted graph contains negative edges
  – we cannot be greedy anymore on the assumption that the lengths of paths will not decrease in the future
- Bellman-Ford’s algorithm detects negative cycles (returns *false*) or returns the shortest path-tree

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Bellman-Ford's algorithm

- Use *d[]* labels (like in Dijkstra's and Prim's)
- Initialize *d[s]=0*, *d[]=* ∞ otherwise
- Perform |*V*|-*1* rounds
- In each round, we attempt an edge relation for **all** the edges in the graph (arbitrary order)
- An extra round of edge relaxation can tell the presence of a negative cycle

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Bellman-Ford's algorithm

```
Algorithm Bellman-Ford(G(V,E),s)
   for each u in V
      d[u] ⇦ ∞
   d[s] ⇦ 0
   for i ⇦ 1 to |V|-1 do
       for each (u,v) in E do
        if d[v] > d[u] + w(u,v) then
             d[v] ⇦ d[u] + w(u,v)
   for each (u,v) in E do
       if d[v] > d[u] + w(u,v) then
          return FALSE
   return d[],TRUE
```

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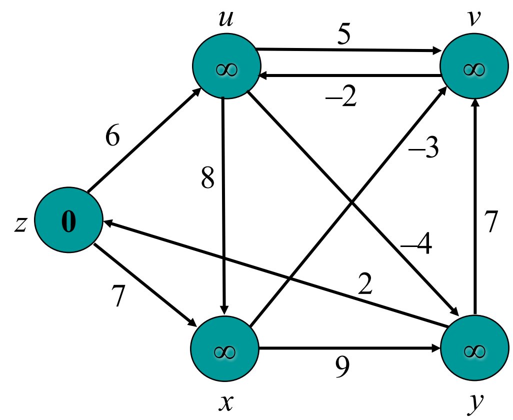
Iteration 0



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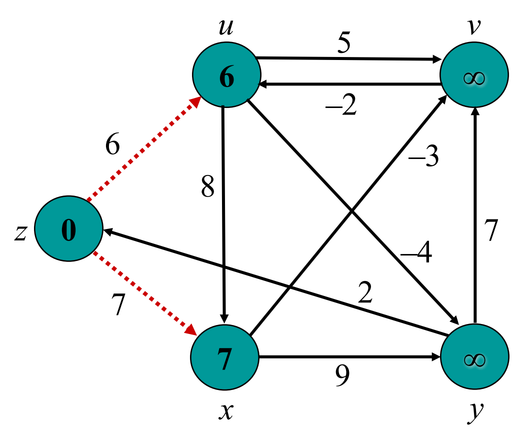
Iteration 1



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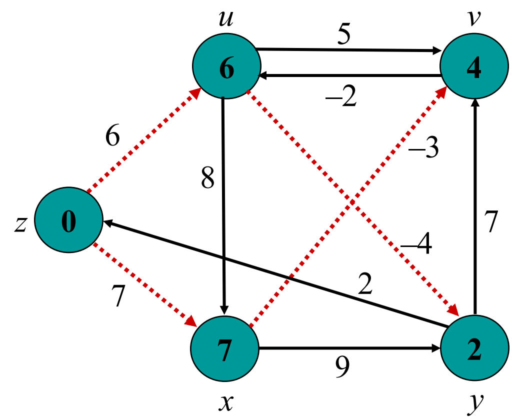
Iteration 2



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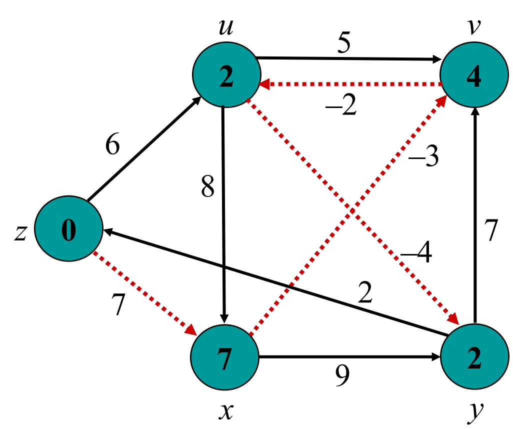
Iteration 3



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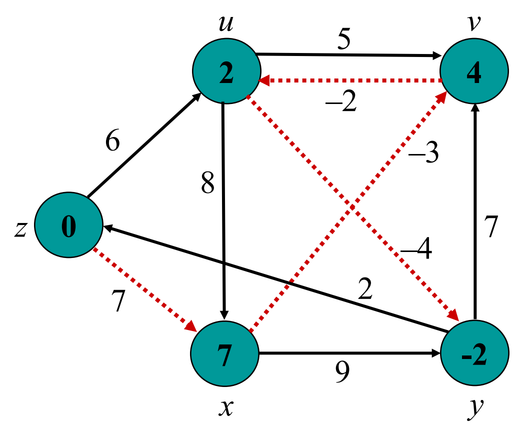
Iteration 4



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Observe that BF is essentially dynamic programming. Let $d(i, j)$ = "cost of the shortest path from s to i that uses at most j edges/hops"

$$d(i,j) = \begin{cases} 0 & \textbf{if } i = s \ \&\ j = 0 \\ \infty & \textbf{if } i \neq s \ \&\ j = 0 \\ \min_{(k,i) \text{ in } E}\{d(k, j-1) + w(k,i),\ d(i, j-1)\} & \textbf{if } j > 0 \end{cases}$$

$i \rightarrow$ / $j \downarrow$

	z	u	v	x	y
	1	2	3	4	5
0	0	∞	∞	∞	∞
1	0	6	∞	7	∞
2	0	6	4	7	2
3	0	2	4	7	2
4	0	2	4	7	–2

Why $O(nm)$?

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Bellman-Ford's correctness

Theorem 7.4: *If after performing the above computation there is an edge (u,z) that can be relaxed (that is, $D[u]+w((u,z)) < D[z]$), then the graph G contains a negative-weight cycle. Otherwise, $D[u] = d(v,u)$ for each vertex u in G.*

- Works for negative-weight edges
- Can detect the presence of negative-weight cycles
- Running time is *O(nm)*

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Floyd-Warshall's algorithm

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All-pairs shortest path

- We want to compute the shortest path distance between every pair of vertices in a directed graph *G* (*n* vertices, *m* edges)
- We want to know *D[i,j]* for all *i,j*, where *D[i,j]*=shortest distance from v_i to v_j

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All-pairs shortest path

- If *G* has no negative-weight edges, we could use Dijkstra's algorithm repeatedly from each vertex
- It would take $O(n\ (m+n)\ log\ n)$ time, that is $O(n^2\ log\ n + nm\ log\ n)$ time, which could be as large as $O(n^3\ log\ n)$

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All-pairs shortest path

- If *G* has negative-weight edges (but no negative-weight cycles) we could use Bellman-Ford's algorithm repeatedly from each vertex
- Recall that Bellman-Ford's algorithm runs in $O(nm)$
- It would take $O(n^2m)$ time, which could be as large $O(n^4)$ time

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All-pairs shortest path

- We now see an algorithm to solve the all-pairs shortest path in $O(n^3)$ time
- The graph can contain negative-weight edges (but no negative-weight cycles)

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All-pairs shortest path

- Let $G=(V,E)$ a weighted directed graph
- Let $V=(v_1,v_2,\ldots,v_n)$
- Define cost function $D^k_{i,j}$ = "the shortest distance from v_i to v_j using only vertices $\{v_1,v_2,\ldots,v_k\}$"

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A dynamic programming shortest-path

Initially we set

$$D^0_{i,j} = \begin{cases} 0 & \text{if } i = j \\ w((v_i,v_j)) & \text{if } (v_i,v_j) \in E \\ +\infty & \text{otherwise} \end{cases}$$

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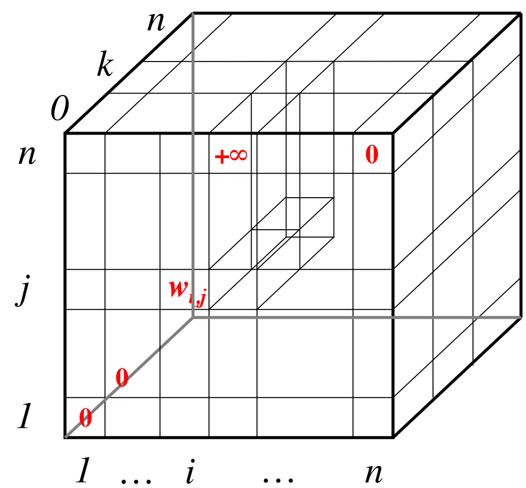
A dynamic programming shortest-path



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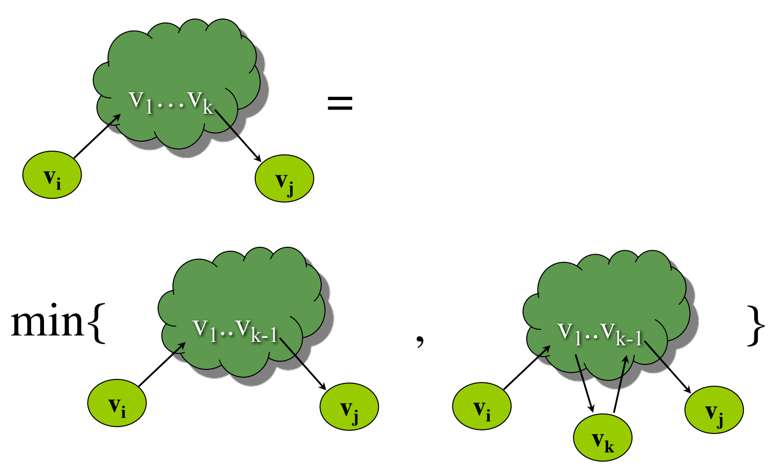
A dynamic programming shortest-path



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A dynamic programming shortest-path

- The cost of going from v_i to v_j using vertices *1,…,k* is the shorter between
  - (do not to use v_k) The shortest path from v_i to v_j using vertices *1,…,k-1*
  - (use v_k) The shortest path from v_i to v_k using *1,…,k-1* plus the cost of the shortest path from v_k to v_j using *1,…,k-1*

Then $$D_{i,j}^{k} = \min\left\{ D_{i,j}^{k-1}, D_{i,k}^{k-1} + D_{k,j}^{k-1} \right\}.$$

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All-pairs shortest path

Algorithm AllPairs($\vec{G}$):

Input: A weighted directed graph $\vec{G}$ with n vertices numbered $v_1, v_2, \ldots, v_n$
Output: A matrix D such that $D[i,j]$ is distance from v_i to v_j in $\vec{G}$

```
for i ← 1 to n do
   for j ← 1 to n do
      if i = j then
         Set D^0[i,i] ← 0 and continue looping
      if (v_i, v_j) is an edge in G then
         Set D^0[i,j] ← w((v_i, v_j))
      else
         Set D^0[i,j] ← +∞
for i ← 1 to n do
   for j ← 1 to n do
      for k ← 1 to n do
         Set D^k[i,j] ← min{D^{k-1}[i,j], D^{k-1}[i,k] + D^{k-1}[k,j]}
Return D^n
```

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All-pairs shortest path

- Floyd-Warshall's algorithm computes the shortest path distance between each pair of vertices of *G* in $O(n^3)$ time

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Minimum Spanning Tree

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Minimum Spanning Tree

- Given a weighted undirected graph G, find a tree T that spans all the vertices of G and minimizes the sum of the weights on the edges, that is

$$w(T) = \Sigma_{e\in T}\, w(e)$$

- We want a spanning tree of minimum cost

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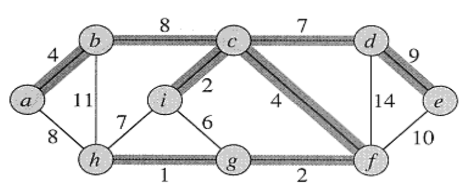
Example



$w(T)=4+8+7+9+2+4+2+1=37$

Note that the MST is not necessarily unique

For example, add *(a,h),* delete *(b,c)*

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Growing a MST: Generic algorithm

- Grow MST one edge at a time
- Manage a set of edges *A*, maintaining the following invariant
  - prior to each iteration, *A* is a subset of some MST
- At each iteration, we determine an edge *(u,v)* that can be added to *A* without violating this invariant
- If $A \cup \{(u, v)\}$ is also a subset of a MST, then *(u, v)* is called a *safe edge* for *A*

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Generic MST algorithm

```
GENERIC-MST(G, w)
1  A ← Ø
2  while A does not form a spanning tree
3       do find an edge (u, v) that is safe for A
4            A ← A ∪ {(u, v)}
5  return A
```

• Loop in lines 2-4 is executed $|V| - 1$ times because any MST tree contains $|V| - 1$ edges

• The overall execution time depends on how to find a safe edge (step 3)

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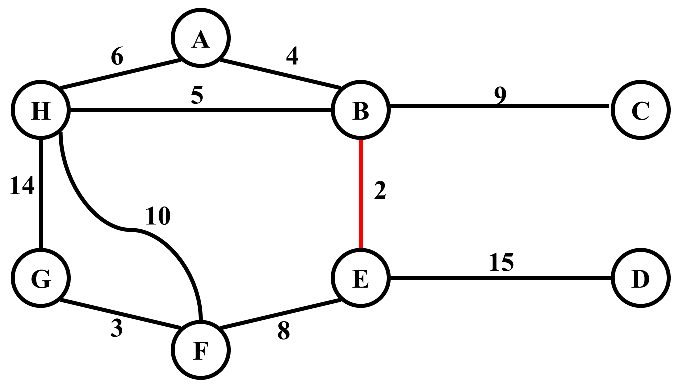
First Edge

- Which edge is clearly safe? Is the "shortest edge" safe?



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Greedy Choice

- Definitions
  - Cut *(S, V-S):* a partition of *V*
  - Crossing edge: one endpoint in *S* and the other in *V-S*
  - A cut respects a set of *A* of edges if no edges in *A* crosses the cut
  - A light edge crossing a partition if its weight is the minimum of any edge crossing the cut
- Theorem. Let *A* be a subset of *E* that is included in some MST of *G=(V,E)*. Let *(S, V-S)* be any cut of *G* that respects *A*, and let *(u, v)* be a light edge crossing *(S, V-S)*. Then, edge *(u, v)* is safe for *A*.

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Examples of Cuts and light edges

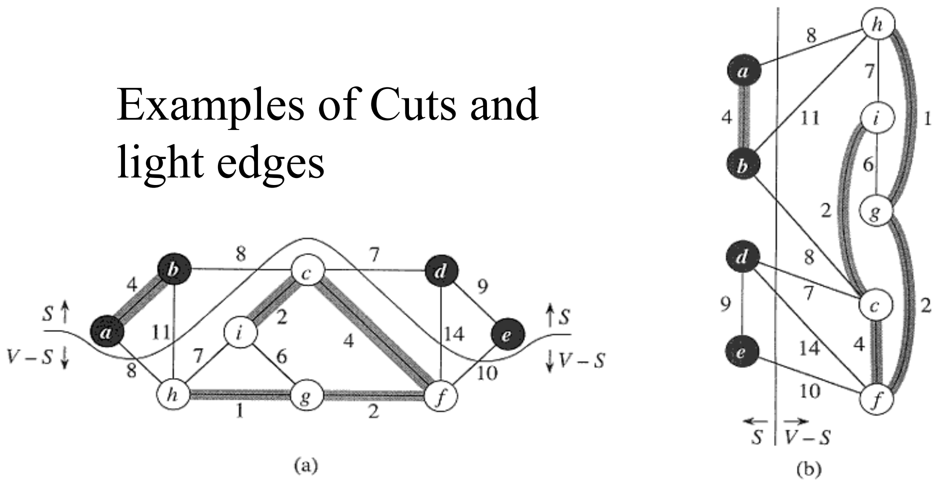

Figure 23.2 Two ways of viewing a cut $(S, V - S)$ of the graph from Figure 23.1. **(a)** The vertices in the set S are shown in black, and those in $V - S$ are shown in white. The edges crossing the cut are those connecting white vertices with black vertices. The edge (d, c) is the unique light edge crossing the cut. A subset A of the edges is shaded; note that the cut $(S, V - S)$ respects A, since no edge of A crosses the cut. **(b)** The same graph with the vertices in the set S on the left and the vertices in the set $V - S$ on the right. An edge crosses the cut if it connects a vertex on the left with a vertex on the right.

Proof of Greedy Choice Thm

- Let T be a MST that includes A, and assume T does not contain the light edge (u, v). [If it does, we are done.]
- First, we construct another MST T' that includes $A \cup \{(u, v)\}$
  Adding (u,v) to T induces a cycle
  - Let (x,y) be the edge on the cycle crossing $(S, V\text{-}S)$, then $w(u,v) \leq w(x,y)$
  - $T' = T - (x,y) \cup (u,v)$
  - T' is also a MST because it is a spanning tree of G and $w(T') = w(T) - w(x,y) + w(u,v) \leq w(T)$
- Second, we prove that (u,v) is safe for A
  - Since $A \subseteq T$ and $(x, y) \notin A$ then $A \subseteq T'$. Therefore $A \cup \{(u, v)\} \subseteq T'$. Since T' is a MST, (u,v) is safe for A

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Optimal substructure property

- Let T be an MST of G. Let (u,v) be an edge in T
- Removing (u,v) partitions T into two trees T_1 and T_2
- Let $(S,V\text{-}S)$ be a cut that respect T_1, let E_1 be the subset of edges incident to S, and E_2 be the subset of edges incident to $V\text{-}S$
- <u>Claim</u>: T_1 is an MST of $G_1 = (S,E_1)$, and T_2 is an MST of $G_2 = (V\text{-}S,E_2)$
  - Note that $w(T) = w(u,v) + w(T_1) + w(T_2)$
  - A "cheaper" tree than T_1 or T_2 cannot exists, otherwise T would not be optimal

Generic MST algorithm

GENERIC-MST(G, w)

$A \leftarrow \emptyset$
while A does not form a spanning tree
    do find an edge (u, v) that is safe for A
        $A \leftarrow A \cup \{(u, v)\}$
return A

Kruskal's algorithm

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Kruskal's algorithm

- Consider the edges one at a time, by increasing weight
- Accept an edge if it connects two different trees

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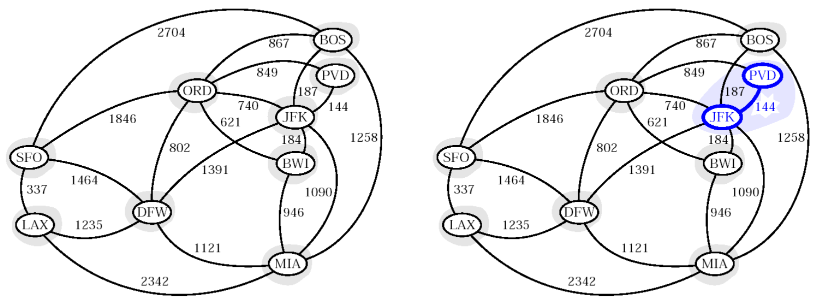
Example



(a) (b)

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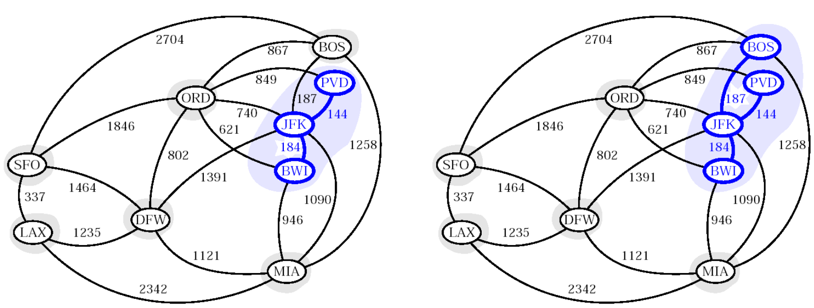
Example



(c) (d)

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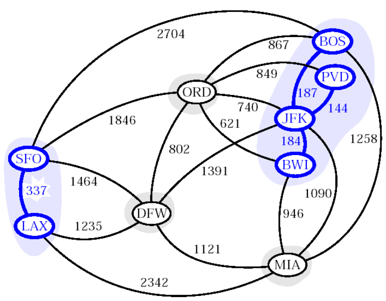
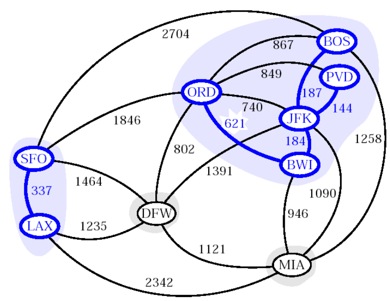
Example



(e)



(f)

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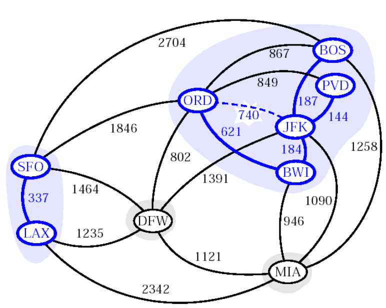
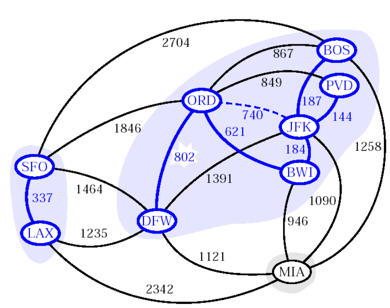
Example



(g)



(h)

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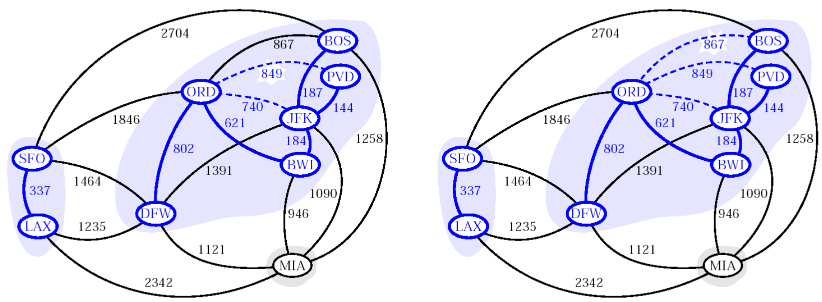
Example



(i) (j)

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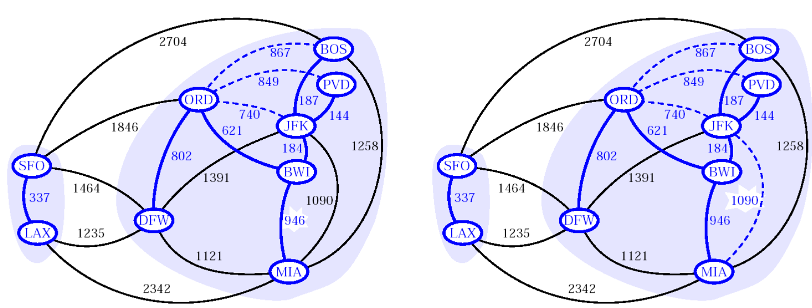
Example



(k) (l)

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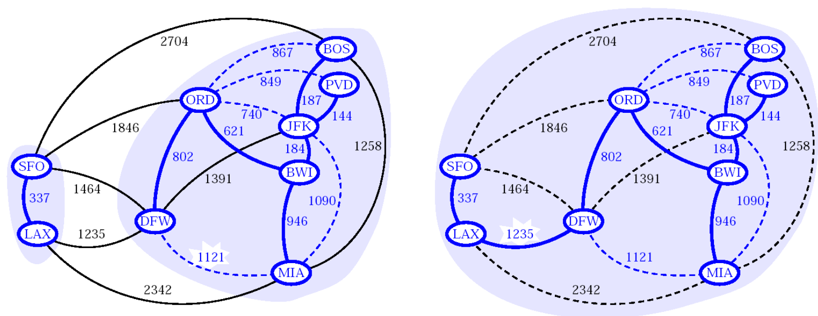
Example



(m) (n)

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Kruskal's algorithm

Algorithm Kruskal(G):

Input: A simple connected weighted graph G with n vertices and m edges

Output: A minimum spanning tree T for G

for each vertex v in G **do**
  Define an elementary cluster $C(v) \leftarrow \{v\}$.
Initialize a priority queue Q to contain all edges in G, using the weights as keys.
$T \leftarrow \emptyset$ $\{T$ will ultimately contain the edges of the MST$\}$
while T has fewer than $n-1$ edges **do**
  $(u,v) \leftarrow Q$.removeMin()
  Let $C(v)$ be the cluster containing v, and let $C(u)$ be the cluster containing u.
  if $C(v) \neq C(u)$ **then**
    Add edge (v,u) to T.
    Merge $C(v)$ and $C(u)$ into one cluster, that is, union $C(v)$ and $C(u)$.
return tree T

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Data Structure for Kruskal's algorithm

- The data structure maintains a forest of trees
- We need a data structure that maintains a partition, i.e., a collection of disjoint sets, with the following operations
  - *find(u):* return the set storing *u*
  - *union(u,v):* replace the sets storing *u* and *v* with their union

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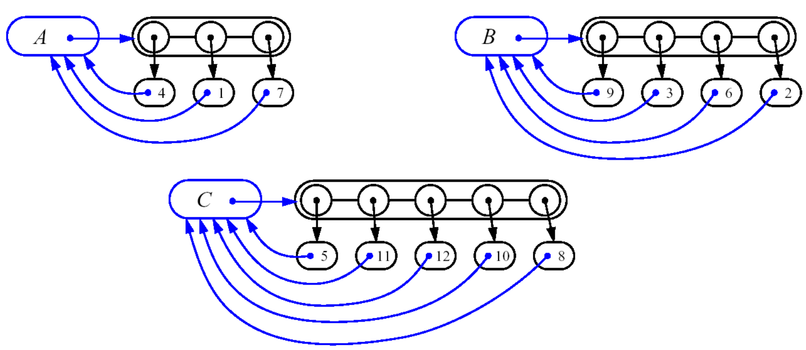
Data structure for sets



A={1,4,7} B={2,3,6,9} C={5,8,10,11,12}

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Representation of a Partition

- Each set is stored in a sequence (list)
- Each element has a reference back to the set
  - operation *find(u)* takes *O(1)* time, and returns the set of which u is a member
  - in operation *union(u,v),* we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation *union(u,v)* is $min(n_u,n_v)$, where n_u and n_v are the sizes of the sets storing u and v

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Kruskal's algorithm running time

- Whenever a vertex is added to a tree, the size of the tree containing the vertex at least double
- Each vertex is moved to a new tree at most *log n* times
- Total time merging trees is *O(n log n)*
- Cost of creating the priority queue *O(m log m)* which is *O(m log n)*
- Overall running time is *O((n+m) log n)*

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Prim's algorithm

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Prim's algorithm

- The edges in the set *A* always forms a single tree
- The tree starts from an arbitrary vertex and grows until the tree spans all the vertices in *V*
- At each step, a light edge is added to the tree *A* that connects *A* to an isolated vertex of $G_A=(V, A)$
- "Greedy" because the tree is augmented at each step with an edge that contributes the minimum amount possible to the tree's weight

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Prim's vs. Dijkstra's

- Prim's strategy similar to Dijkstra's
- Grows the MST *T* one edge at a time
- Cloud covering the portion of *T* already computed
- Label *D[u]* associated with each vertex *u* outside the cloud (distance to the cloud)

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Prim's algorithm

- For any vertex *u*, *D[u]* represents the weight of the current best edge for joining *u* to the rest of the tree in the cloud (as opposed to the total sum of edge weights on a path from start vertex to *u*)
- Use a priority queue *Q* whose keys are *D* labels, and whose elements are vertex-edge pairs

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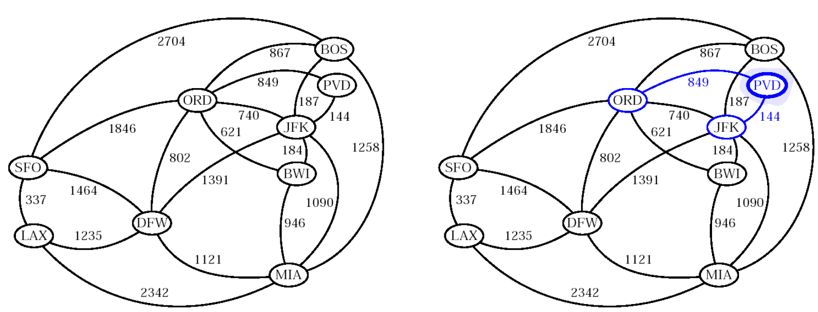
Prim’s algorithm

- Any vertex *v* can be the starting vertex
- We still initialize *D[v]=0* and all the *D[u]* values to +∞

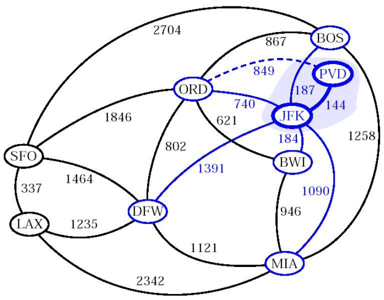
- We can reuse code from Dijkstra’s, just change a few things

Example

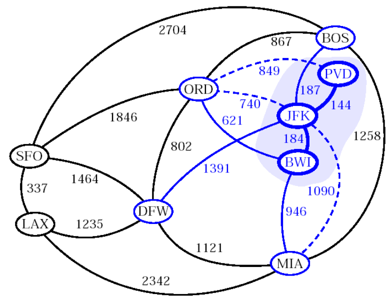


(a) (b)

Example



(c)



(d)

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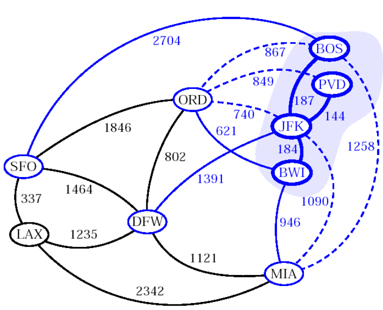
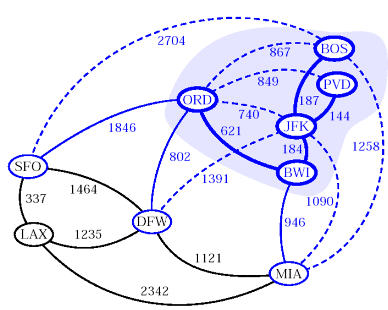
Example



(e)



(f)

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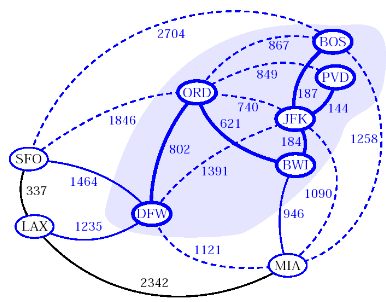
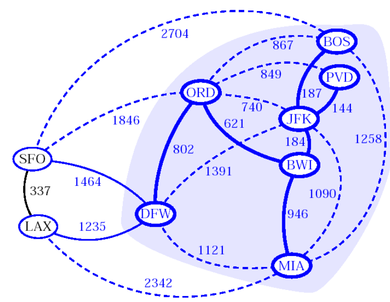
Example



(g)



(h)

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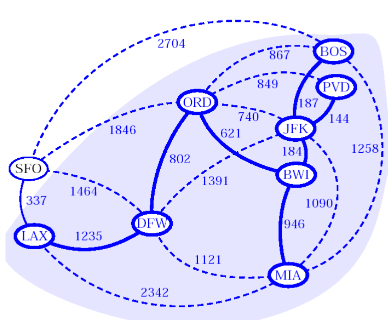
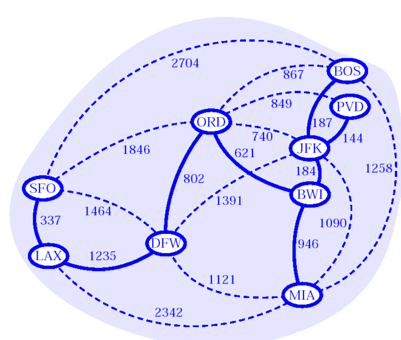
Example



(i)



(j)

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Pseudo Code

Algorithm PrimJarnik(G):
  Input: A weighted connected graph G with n vertices and m edges
  Output: A minimum spanning tree T for G

```
Pick any vertex v of G
D[v] ← 0
for each vertex u ≠ v do
    D[u] ← +∞
Initialize T ← ∅.
Initialize a priority queue Q with an item ((u,null), D[u]) for each vertex u,
where (u,null) is the element and D[u]) is the key.
while Q is not empty do
    (u,e) ← Q.removeMin()
    Add vertex u and edge e to T.
    for each vertex z adjacent to u such that z is in Q do
        {perform the relaxation procedure on edge (u,z)}
        if w((u,z)) < D[z] then
            D[z] ← w((u,z))
            Change to (z,(u,z)) the element of vertex z in Q.
            Change to D[z] the key of vertex z in Q.
return the tree T
```

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Time complexity

- Initializing the queue takes *O(n log n)* [binary heap]
- Each iteration of the while, we spend *O(log n)* time to remove vertex *u* from *Q* and *O(deg(u) log n)* to perform the relaxation step
- Overall, $O(n \log n + \Sigma_v(deg(v) \log n))$ which is $O((n+m) \log n)$ [if using a binary heap]

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Summary

	Time complexity	*Notes*
Dijkstra	$O((n+m) \log n)$ using p.q. $O(n^2+m)$ using array	Non-negative weights
Bellman-Ford	$O(m\ n)$	Negative weights ok
All-pairs	$O(n^3)$	Negative weights ok
Kruskal	$O((n+m) \log n)$ using p.q.	
Prim-Jarnik	$O((n+m) \log n)$ using p.q.	

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Reading Assignment

- Dasgupta
  - single-source shortest path (4.4, 4.6 and 4.7)
  - all-pairs shortest path (6.6)
  - minimum spanning tree (5.1.3, 5.1.5)

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