Hanzhe Teng SID 500653093
Zhenxiao Qi SID 500654348

CS 111 ASSIGNMENT 4

due Thursday, November 16 (8AM)

Problem 1: Give the asymptotic value (using the Θ-notation) for the number of letters that will be printed by the algorithms below. In each algorithm the argument n is a positive integer. Your solution needs to consist of an appropriate recurrence equation and its solution. You also need to give a brief justification for the recurrence (at most 10 words each).

(i) **Algorithm** PRINTAS (n : **integer**)
    if $n < 5$
        print("A")
    else
        PRINTAS($\lceil n/4 \rceil$)
        PRINTAS($\lceil n/4 \rceil$)
        PRINTAS($\lceil n/4 \rceil$)
        PRINTAS($\lceil n/4 \rceil$)
        for $i \leftarrow 1$ **to** 5 **do** print("A")

(ii) **Algorithm** PRINTBS (n : **integer**)
    if $n < 2$
        print("B")
    else
        for $j \leftarrow 1$ **to** 10 **do** PRINTBS($\lfloor n/2 \rfloor$)
        for $i \leftarrow 1$ **to** $6n^3$ **do** print("B")

(iii) **Algorithm** PRINTCS (n : **integer**)
    if $n < 4$
        print("C")
    else
        PRINTCS($\lceil n/3 \rceil$)
        PRINTCS($\lceil n/3 \rceil$)
        PRINTCS($\lceil n/3 \rceil$)
        PRINTCS($\lceil n/3 \rceil$)
        for $i \leftarrow 1$ **to** $20n^2$ **do** print("C")

Solution 1:
(i) $T(n) = 4T(n/4) + 5$

In this algorithm, it calls itself recursively for 4 times with $n = n/4$, and then calls print for 5 times, where each call takes linear time. So the recurrence is $T(n) = 4T(n/4) + 5$. We can solve this recurrence by the Master Theorem. Compared with $T(n) = aT(n/b) + cn^d$, we have $a = 4, b = 4, c = 5, d = 0$. Since $a = 4 > b^d = 1$, then the asymptotic value of this recurrence is $\Theta(n^{\log_b a}) = \Theta(n^{\log_4 4}) = \Theta(n)$.

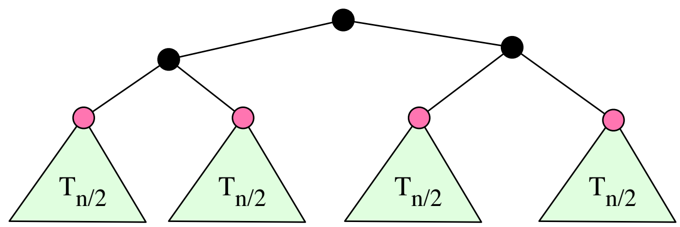
(ii) $T(n) = 10T(n/2) + 6n^3$

In this algorithm, it calls itself recursively for 10 times with $n = n/2$, and then calls print for $6n^3$ times, where each call takes linear time. So the recurrence is $T(n) = 10T(n/2) + 6n^3$. We can solve this recurrence by the Master Theorem. Compared with $T(n) = aT(n/b) + cn^d$, we have $a = 10, b = 2, c = 6, d = 3$. Since $a = 10 > b^d = 8$, then the asymptotic value of this recurrence is $\Theta(n^{\log_b a}) = \Theta(n^{\log_2 10})$.

(iii) $T(n) = 4T(n/3) + 20n^2$

In this algorithm, it calls itself recursively for 4 times with $n = n/3$, and then calls print for $20n^2$ times, where each call takes linear time. So the recurrence is $T(n) = 4T(n/3) + 20n^2$. We can solve this recurrence by the Master Theorem. Compared with $T(n) = aT(n/b) + cn^d$, we have $a = 4, b = 3, c = 20, d = 2$. Since $a = 4 < b^d = 9$, then the asymptotic value of this recurrence is $\Theta(n^d) = \Theta(n^2)$.

Problem 2: For each integer $n \geq 1$ we define a tree T_n, recursively, as follows. For $n = 1$, T_1 is a single node. For $n > 1$, T_n is obtained from four copies of $T_{\lceil n/2 \rceil}$ and three additional nodes, by connecting them as follows:



(In this figure, the subtrees are denoted $T_{n/2}$, without rounding, to reduce clutter.) Let $h(n)$ be the number of nodes in T_n. Give a recurrence equation for $h(n)$ and justify it. Then give the solution of this recurrence using the $\Theta()$ notation.

Solution 2:

For each iteration, we have 4 copies and 3 additional nodes, and the size of the input is divided by 2, then $h(n) = 4h(n/2) + 3$. By Master Theorem, we can compare the formula $T(n) = aT(n/b) + cn^d$ and then have $a = 4, b = 2, c = 3, d = 0$. Since $a = 4 > b^d = 1$, then the asymptotic value of this recurrence is $\Theta(n^{\log_b a}) = \Theta(n^{\log_2 4}) = \Theta(n^2)$.

Problem 3: Bill is buying his wife a bouquet of carnations, daises, roses and tulips. The bouquet will have 26 flowers, with

- between 2 and 11 carnations,
- at most 6 daises,
- at least 3 roses, and
- between 3 and 9 tulips.

How many different combinations of flowers satisfy these requirements? You need to use the counting method for integer partitions and show your work.

Solution 3:

Let's denote carnations by x, daises by y, roses by k, and tulips by m, then $x+y+k+m = 26$. According to the problem, we have the following set of inequalities:

$$
\begin{aligned}
&2 \leq x \leq 11 \\
&0 \leq y \leq 6 \\
&3 \leq k \leq 21 \ (= 26 - 2 - 3) \\
&3 \leq m \leq 9
\end{aligned}
$$

Let $x' = x-2$, $k' = k-3$, $m' = m-3$, from which we can rewrite the equation as $x'+y+k'+m' = 26-8 = 18$ and the constraints as follow:

$$\begin{aligned} 0 &\le x' \le 9 \\ 0 &\le y \le 6 \\ 0 &\le k' \le 18 \\ 0 &\le m' \le 6 \end{aligned}$$

Let $S(x' \le 9 \cap y \le 6 \cap k' \le 18 \cap m' \le 6)$ be the number of combinations of flowers which satisfy the constraints and S_{total} be the number of combinations of flowers without any constraint. Then we have $S(x' \le 9 \cap y \le 6 \cap k' \le 18 \cap m' \le 6) = S_{total} - S(x' \ge 10 \cup y \ge 7 \cup k' \ge 19 \cup m' \ge 7)$. Since

$$S_{total} = \binom{18+4-1}{3} = \binom{21}{3} = 1330$$

and

$$\begin{aligned} &S(x' \ge 10 \cup y \ge 7 \cup k' \ge 19 \cup m' \ge 7) \\ &= S(x' \ge 10) + S(y \ge 7) + S(k' \ge 19) + S(m' \ge 7) \\ &\quad - S(x' \ge 10 \cap y \ge 7) - S(x' \ge 10 \cap k' \ge 19) - S(x' \ge 10 \cap m' \ge 7) \\ &\quad - S(y \ge 7 \cap k' \ge 19) - S(y \ge 7 \cap m' \ge 7) - S(k' \ge 19 \cap m' \ge 7) \\ &\quad + S(x' \ge 10 \cap y \ge 7 \cap k' \ge 19) + S(x' \ge 10 \cap y \ge 7 \cap m' \ge 7) \\ &\quad + S(x' \ge 10 \cap k' \ge 19 \cap m' \ge 7) + S(y \ge 7 \cap k' \ge 19 \cap m' \ge 7) \\ &\quad - S(x' \ge 10 \cap y \ge 7 \cap k' \ge 19 \cap m' \ge 7) \\ &= \binom{18-10+3}{3} + \binom{18-7+3}{3} + \binom{18-19+3}{3} + \binom{18-7+3}{3} \\ &\quad - \binom{18-17+3}{3} - \binom{18-29+3}{3} - \binom{18-17+3}{3} - \binom{18-26+3}{3} \\ &\quad - \binom{18-14+3}{3} - \binom{18-26+3}{3} + \binom{18-36+3}{3} + \binom{18-24+3}{3} \\ &\quad + \binom{18-36+3}{3} + \binom{18-33+3}{3} + \binom{18-43+3}{3} \\ &= \binom{11}{3} + \binom{14}{3} + 0 + \binom{14}{3} \\ &\quad - \binom{4}{3} - 0 - \binom{4}{3} - 0 \\ &\quad - \binom{7}{3} - 0 + 0 + 0 \\ &\quad + 0 + 0 + 0 \\ &= 165 + 364 + 364 - 4 - 4 - 35 \\ &= 850, \end{aligned}$$

then we have $S(x' \le 9 \cap y \le 6 \cap k' \le 18 \cap m' \le 6) = 1330 - 850 = 480$, which is the number of combinations.

Problem 4: We have three sets P, Q, R with the following properties:

(a) $|Q| = 2|P|$ and $|R| = 4|P|$,

(b) $|P \cap Q| = 11$, $|P \cap R| = 7$, $|Q \cap R| = 10$,

(c) $1 \leq |P \cap Q \cap R| \leq 11$,

(d) $|P \cup Q \cup R| = 121$.

Use the inclusion-exclusion principle to determine the number of elements in P. Show your work. (Hint: You may get an equation with two unknowns, but one of them has only a few possible values.)

Solution 4:

Let $|P| = x$, then $|Q| = 2x, |R| = 4x$; let $y = |P \cap Q \cap R|$. By Inclusion-Exclusion Principle, we have $|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|$.

Apply all the properties (b) and (d), we have $121 = x + 2x + 4x - 11 - 7 - 10 + y$. Thus, $149 = 7x + y$. Since $1 \leq y \leq 11$, we can solve by enumeration. When $x = 20, y = 9$, the equation is true. So the number of elements in P is 20.
