Inclusion – Exclusion Principle

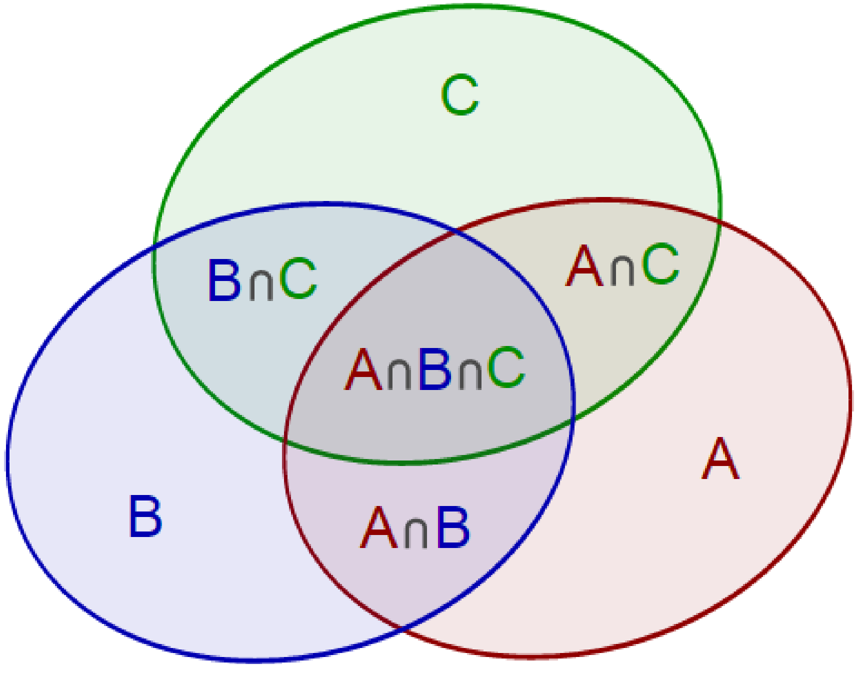
Inclusion-Exclusion Principle

In its most basic form, inclusion-exclusion is a way of counting the membership of a union of sets.

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Inclusion-Exclusion Principle

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$



Inclusion-Exclusion Principle

$$
\begin{aligned}
|S_1 \cup S_2 \cup S_3 \cup S_4| = {} & |S_1| + |S_2| + |S_3| + |S_4| \\
& -|S_1 \cap S_2| - |S_1 \cap S_3| - |S_1 \cap S_4| - |S_2 \cap S_3| - |S_2 \cap S_4| - |S_3 \cap S_4| \\
& +|S_1 \cap S_2 \cap S_3| + |S_1 \cap S_2 \cap S_4| + |S_1 \cap S_3 \cap S_3| + |S_2 \cap S_3 \cap S_4| \\
& -|S_1 \cap S_2 \cap S_3 \cap S_4|
\end{aligned}
$$

Theorem: For any finite sets $S_1, S_s, \ldots, S_k$ we have

$$\left|\bigcup_{i=1}^{k} S_i\right| = \sum_{j=1}^{k} (-1)^{j+1} \sum_{l_1 < l_2 < \cdots < l_j} \left|S_{l_1} \cap S_{l_2} \cap \cdots \cap S_{l_j}\right|$$

Inclusion – Exclusion Principle

Theorem 1 (Inclusion-Exclusion) *For any finite sets $S_1, S_2, ..., S_k$, we have*

$$\left|\bigcup_{i=1}^{k} S_i\right| = \sum_{j=1}^{k}(-1)^{j+1} \sum_{l_1<l_2<\ldots<l_j} |S_{l_1} \cap S_{l_2} \cap \ldots \cap S_{l_j}|$$

Proof. (Using Binomial theorem)

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

Let x = 1, and y = -1: *$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \ldots \binom{n}{n}$, but $\binom{n}{0} = 1$*

$$0 = 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \ldots \binom{n}{n}$$

$$0 = 1 - \left(\binom{n}{1} - \binom{n}{2} + \binom{n}{3} \ldots \binom{n}{n}\right)$$

Then $\binom{n}{1} - \binom{n}{2} + \binom{n}{3} \ldots \binom{n}{n} = 1$.

Suppose, some element A belongs to n sets. Then the number of times that it is counted on the right is

$$\binom{n}{1} - \binom{n}{2} + \binom{n}{3} \ldots \binom{n}{n} = 1.$$

Example

Example: Things are not going well on Capitol Hill. The special prosecutor indicted **49** congressmen on various charges.

- Among them: **26** congressmen were indicted for bribery, **29** for fraud, and **19** for perjury.

- We further know that: **12** congressmen received indictments for bribery and fraud, **11** congressmen received indictments for bribery and perjury, and **9** congressmen received indictments for fraud and perjury.

Determine how many congressmen were indicted on all three charges?

Example

Let the following sets be the set of congressmen that were indicted

- Set X for bribery
- Set Y for fraud
- Set Z for perjury

$$|X|=26,\ |Y|=29,\ |Z|=19$$

$$|X \cap Y|=12,\ |X \cap Z|=11,\ |Y \cap Z|=9$$

$$|X \cup Y \cup Z|=49$$

$$|X \cap Y \cap Z| \text{ -- } ?$$

Example

$$|X \cup Y \cup Z| = |X| + |Y| + |Z|$$

$$-|X \cap Y| - |X \cap Z| - |Y \cap Z|$$

$$+|X \cap Y \cap Z|$$

Let $|X \cap Y \cap Z| = x$. Then we have:

$$49 = 26 + 29 + 19 - 12 - 11 + x, \text{ and}$$

$$x = 49 - 26 - 29 - 19 + 12 + 11 + 9$$

$$= 7$$

Answer:

There were 7 congressmen indicted on all three charges.

Inclusion-Exclusion Principle

Example: We have a group of athletes, consisting of runners, swimmers, and cyclists. Half of the athletes in this group are runners, 33 are swimmers and 54 are cyclists. 12 athletes run and swim, 15 run and cycle, 21 swim and cycle, and 6 of them do all three sports. How many athletes are in this group?

Inclusion – Exclusion Principle
Application to Computing Euler Totient

If n is a positive integer, then $\varphi(n)$ is the number of integers k in the range $1 \le k \le n$ for which $\gcd(n, k) = 1$

Theorem 2 *Let $p_1, p_2, \ldots, p_k$ be all different prime factors of n. Then*

$$\phi(n) = n \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

Use Inclusion – Exclusion Principle to prove.

Computing the Number of Integer Partitions

Integer partitions;

Partitions with lower bounds,

Or’s of lower bounds;

Partitions with upper bounds;

Partitions with lower and upper bounds.

Integer Partitions

We need to equip the US cycling team with 10 bicycles. All bicycles are identical, except possibly for color. Each bicycle can be painted in one of three colors: red, green and blue. How may ways are there to do that?

Denote by r, g, and b the numbers of bicycles of each color.

Then $r + g + b = 10$, where $0 \leq r, g, b \leq 10$.

So our goal is to compute the number of solutions (r, g, b) of this equation.

Integer Partitions

Each solution (r, g, b) of this equation is called a **partition** of 10.

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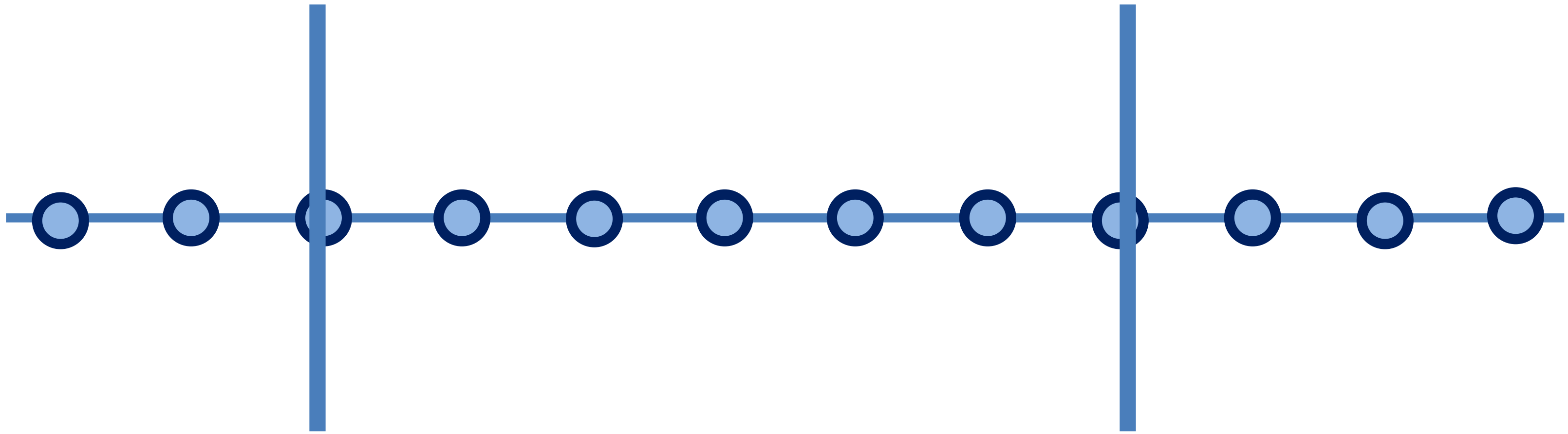
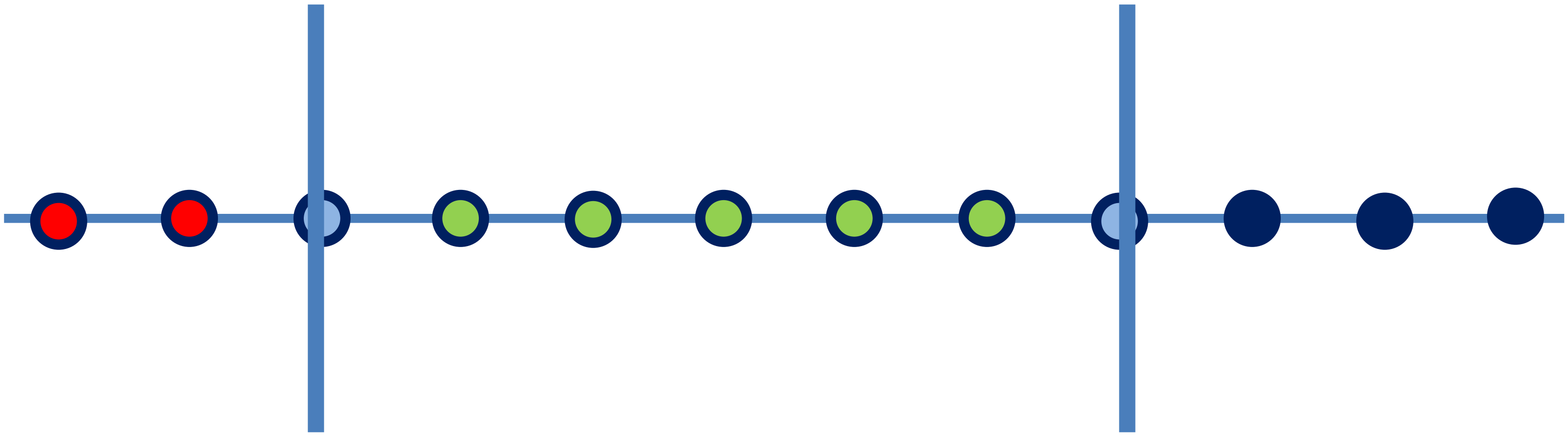


Integer Partitions

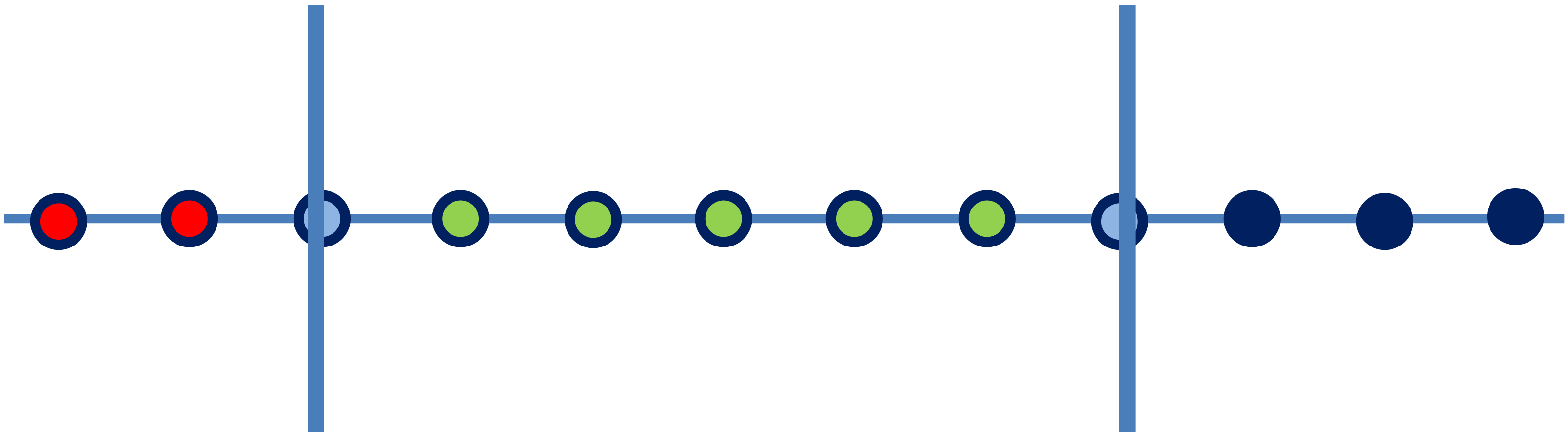
Each solution (r, g, b) of this equation is called a **partition** of 10.



One possible solution: (2, 5, 3).

Integer Partitions

Each solution (r, g, b) of this equation is called a **partition** of 10.



The total number of partitions of 13 is

$$S_{Total} = \binom{12}{2} = 60$$

Theorem: The number of solutions of

$$x_1 + x_2 + \cdots + x_k = m,$$

where $0 \le x_i \le m$ for all i, is $S_{Total} = \binom{m+k-1}{k-1}$.

Integer Partitions

Partitions with lower bounds. Suppose we have an extra constraint that there can't be less then 3 red bicycles. That is

$$r + g + b = 10,$$

$$\text{where } 3 \leq r \leq 10 \text{ and } 0 \leq g, b \leq 10.$$

Partitions with lower bounds

We can write r as r = r ′ + 3, where r ′ ≥ 0. Substituting, our equation reduces to

r ′ + g + b = 7, where 0 ≤ r ′, g, b ≤ 7

The number of solutions is

$$S(r \geq 3) = \binom{10 - 3 + 3 - 1}{3 - 1} = \binom{9}{2} = 36$$

Theorem: Let $a_1, \ldots, a_n$ be non-negative integers such that $A = \sum_{i=1}^{k} a_1 \leq m$. The number of solutions of

$$x_1 + x_2 + \cdots + x_k = m,$$

where $a_1 \leq x_1 \leq m$ for all i, is $\binom{m - A + k - 1}{k - 1}$

Or's of lower bounds

Or's of Lower Bounds: Suppose we have more constrains:

- there can not be less then 3 red bicycles color

OR

- there can not be less then 2 green bicycles

OR

- there can not be less then 6 blue bicycles

How many solutions will the equation

$$r + g + b = 10$$

have, if

$$r \geq 3 \quad \textbf{OR} \quad g \geq 2 \quad \textbf{OR} \quad b \geq 6?$$

Or’s of lower bounds

r + g + b = 10 , where r ≥ 3 **OR** g ≥ 2 **OR** b ≥ 6

Find $S(r \geq \mathbf{3} \cup g \geq \mathbf{2} \cup b \geq \mathbf{6})$

Using inclusion-exclusion principle:

$$S(r \geq 3 \cup g \geq 2 \cup b \geq 6) = S(r \geq 3) + S(g \geq 2) + S(b \geq 6)$$

$$- S(r \geq 3 \cap g \geq 2) - S(r \geq 3 \cap b \geq 6) - S(g \geq 2 \cap b \geq 6)$$

$$+ S(r \geq 3 \cap g \geq 2 \cap b \geq 6)$$

$$= \binom{10 - 3 + 3 - 1}{2} + \binom{10 - 2 + 3 - 1}{2} + \binom{13 - 6 + 3 - 1}{2}$$

$$- \binom{13 - (3 + 2) + 3 - 1}{2} - \binom{13 - (3 + 6) + 3 - 1}{2} -$$

$$\binom{13 - (2 + 6) + 3 - 1}{2} + \binom{13 - (3 + 2 + 6) + 3 - 1}{2} = \dots$$

Integer Partition

Partitions with upper bounds: Let us compute the number of solutions of

$$r + g + b = 10,$$

if $0 \leq r \leq 2$ **and** $0 \leq g \leq 1$ **and** $0 \leq b \leq 5$.

Find $S(r \leq 2 \cap g \leq 1 \cap b \leq 1)$

Note:

$S(r \leq 2 \cap g \leq 1 \cap b \leq 1)$ =

$$S_{Total} - S(r \geq 3 \cup g \geq 2 \cup b \geq 6)$$

Where S_{Total} is the number of solutions of r + g + b = 10 without any constraints.

Computing the Number of Integer Partitions with Mixed Bounds

Alexander is planning an 18-day trip. He wants to spend

less than 5 days in Italy,

between 2 and 8 days in France,

at least 1 but no more than 8 days in England.

Determine the number of ways to plan his trip.

less than 5 days in Italy,
between 2 and 8 days in France,
at least 1 but no more than 8 days in England

$$\begin{aligned} 0 \leq \quad & x & \leq 4 \\ 2 \leq \quad & y & \leq 8 \\ 1 \leq \quad & z & \leq 8 \end{aligned}$$

x + y + z = 18

Let $y' = y - 2$ and $z' = z - 1$, from these we can rewrite the equation as $x + y' + z' = 15$ and the constraints as:

$$\begin{aligned} 0 \leq \quad & x & \leq 4 \\ 0 \leq \quad & y' & \leq 6 \\ 0 \leq \quad & z' & \leq 7 \end{aligned}$$

$$\begin{aligned} S(x \leq 4 \wedge y' \leq 6 \wedge z' \leq 7) &= S - S(x \geq 5 \vee y' \geq 7 \vee z' \geq 8) \\ &= S - (S(x \geq 5) + S(y' \geq 7)S(z' \geq 8) \\ &\qquad - S(x \geq 5 \wedge y' \geq 7) - S(x \geq 5 \wedge z' \geq 8) - S(y' \geq 7 \wedge z' \geq 8) \\ &\qquad + S(x \geq 5 \wedge y' \geq 7 \wedge z' \geq 8)) \\ &= \binom{17}{2} - (\binom{12}{2} + \binom{10}{2} + \binom{9}{2} - \binom{5}{2} - \binom{4}{2} - \binom{2}{2} + \binom{0}{2}) \\ &= 136 - (66 + 45 + 36 - 10 - 6 - 1 + 0) \\ &= 136 - 130 \\ &= 6 \end{aligned}$$

Example

Little Red Riding Hood is assembling a fruit basket for her sick grandmother. The basket will contain 26 fruit, including apples, bananas, mangos, and strawberries (and no other fruit). The basket must contain:

at least 6, but no more than 2, apples,

at least 4, bananas,

at least 5, but no more than 1, mango,

at least 3 strawberries.

Determine the number of ways to assemble the fruit basket.