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CS 111 ASSIGNMENT 3

due Tuesday, November 7 (8AM)

Problem 1: We want to tile a $2 \times n$ strip with unit size tiles that are either green, red or yellow, such that no two red tiles are adjacent. (Adjacent means here that they belong to the same or two consecutive columns, so even their corners cannot touch.) Here is an example of a correct tiling:

Y	Y	G	G	Y	G	R	Y	Y	G	R	G	Y	R
G	R	Y	R	G	Y	G	Y	R	G	G	Y	Y	Y

Give a formula for the number of such tilings. Your solution must include a recurrence equation, with a full justification, and its step-by-step solution.

Solution 1:

First, let's clarify the restriction in this problem. That two red tiles cannot be adjacent implies that if there is a red tile in the present 2×1 strip, the previous and the next strip cannot contain a red tile. Apparently, two red tiles cannot be in a same 2×1 strip as well.

Then let's figure out how many conditions are there in a 2×1 strip. For the strip with a red tile, it can be RG, RY, GR, YR, where we use **R** to denote; for the strip without red tiles, it can be GG, YY, GY, YG, where we use **NR** to denote.

Let $\mathbf{T_n}$ denotes the $2 \times n$ strip, and T_n denotes the number of the tilings of this strip. Similarly, $\mathbf{T_{n-1}}$ denotes the $2 \times (n-1)$ strip, and T_{n-1} denotes the number of the tilings.

For a $2 \times n$ strip, there are two ways to reach it. If the last 2×1 strip does not have a red strip, we do not need to check the previous strip; it can be $\mathbf{T_{n-1}} + \mathbf{NR}$. (The symbol $+$ here just represents that to put the strips next to each other.) If the last 2×1 strip encompasses a red tile, then we have to check the previous 2×1 strip and make sure that that strip does not have any red tile. Thus, it can only be $\mathbf{T_{n-2}} + \mathbf{NR} + \mathbf{R}$.

Through the discussion above, we can draw the recurrence relation $T_n = 4 \cdot T_{n-1} + 16 \cdot T_{n-2}$. Because there are 4 cases for either a **R** strip or a **NR** strip.

To solve this recurrence relation, we also need the initial conditions. Because this is a second-degree recurrence, we need two conditions $T_1 = 8, T_2 = 48$. Then we can solve it.

The characteristic equation is $r^2 - 4r - 16 = 0$. We have $\Delta = (-4)^{-}4 \cdot 1 \cdot (-16) = 80$. The roots $r = \dfrac{4 \pm 4\sqrt{5}}{2} = 2 \pm 2\sqrt{5}$. So the general solution $T_n = \alpha_1 \cdot (2 + 2\sqrt{5})^n + \alpha_2 \cdot (2 - 2\sqrt{5})^n$. By initial conditions, we have the following equation set.

$$\alpha_1 \cdot (2 + 2\sqrt{5}) + \alpha_2 \cdot (2 - 2\sqrt{5}) = 8 \tag{1}$$

$$\alpha_1 \cdot (2 + 2\sqrt{5})^2 + \alpha_2 \cdot (2 - 2\sqrt{5})^2 = 48 \tag{2}$$

By computing, we have $\alpha_1 = \dfrac{5 + 3\sqrt{5}}{10}$ and $\alpha_2 = \dfrac{5 - 3\sqrt{5}}{10}$.

Thus, the final solution $T_n = \dfrac{5 + 3\sqrt{5}}{10} \cdot (2 + 2\sqrt{5})^n + \dfrac{5 - 3\sqrt{5}}{10} \cdot (2 - 2\sqrt{5})^n (n \geq 1)$.

If we let T_n denotes the $2 \times (n+1)$ strip, which means T_0 denotes the first 2×1 strip, we will have a different equation set.

$$\alpha_1 + \alpha_2 = 8 \tag{3}$$

$$\alpha_1 \cdot (2 + 2\sqrt{5}) + \alpha_2 \cdot (2 - 2\sqrt{5}) = 48 \tag{4}$$

By computing, we have $\alpha_1 = \frac{4\sqrt{5}+8}{\sqrt{5}}$ and $\alpha_2 = \frac{4\sqrt{5}-8}{\sqrt{5}}$.

Under this circumstance, the final solution $T_n = \frac{4\sqrt{5}+8}{\sqrt{5}} \cdot (2+2\sqrt{5})^n + \frac{4\sqrt{5}-8}{\sqrt{5}} \cdot (2-2\sqrt{5})^n (n \geq 0)$.

Problem 2: Solve the following recurrence equations:

$$\begin{aligned} f_n &= 3f_{n-1} - f_{n-2} - 2f_{n-3} + 2n + 1 \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 3 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

Solution 2:

1) We first find the general solution f'_n by solving the associated homogeneous equation $f_{nc} = 3f_{n-1} - f_{n-2} - 2f_{n-3}$. The corresponding characteristic equation is $r^3 - 3r^2 + r + 2 = 0$. By computing, we have the characteristic roots $r_1 = 2, r_2 = \frac{1+\sqrt{5}}{2}, r_3 = \frac{1-\sqrt{5}}{2}$. Hence, the general solution is $f'_n = \alpha_1 2^n + \alpha_2 (\frac{1+\sqrt{5}}{2})^n + \alpha_3 (\frac{1-\sqrt{5}}{2})^n$.

2) Then let's figure out the particular solution f''_n. Since $g(n) = 2n+1$ and $s = 1$ is not a root of the characteristic equation, there is a particular solution of the form $f''_n = p_1 n + p_0$. With this form of the particular solution, the original recurrence $f_n = 3f_{n-1} - f_{n-2} - 2f_{n-3} + 2n + 1$ turns into $p_1 n + p_0 = 3(p_1(n-1) + p_0) - (p_1(n-2) + p_0) - 2(p_1(n-3) + p_0) + 2n + 1$. After rearranging this equation, we have $n(p_1 - 2) + p_0 - 5p_1 - 1 = 0$. Because this equation must be true no matter what the value of n is, then the coefficient of the item associated with n must be zero. Thus, we have one solution $p_1 = 2$. Since $p_0 - 5p_1 - 1$ should also be zero, then we have $p_0 = 11$. So the particular solution $f''_n = 2n + 11$.

3) The final solution $f_n = f'_n + f''_n = \alpha_1 2^n + \alpha_2 (\frac{1+\sqrt{5}}{2})^n + \alpha_3 (\frac{1-\sqrt{5}}{2})^n + 2n + 11$. We solve it by initial conditions. Recall $f_0 = 0, f_1 = 1, f_2 = 3$, then we have the following equation set.

$$\alpha_1 + \alpha_2 + \alpha_3 + 11 = 0 \tag{5}$$

$$\alpha_1 \cdot 2 + \alpha_2 \cdot \frac{1+\sqrt{5}}{2} - \alpha_3 \cdot \frac{1-\sqrt{5}}{2} + 13 = 1 \tag{6}$$

$$\alpha_1 \cdot 4 + \alpha_2 \cdot \frac{6+2\sqrt{5}}{4} + \alpha_3 \cdot \frac{6-2\sqrt{5}}{4} + 15 = 3 \tag{7}$$

By computing, we have $\alpha_1 = 11, \alpha_2 = -11 - \dfrac{23}{\sqrt{5}}$, $\alpha_3 = -11 + \dfrac{23}{\sqrt{5}}$.

Overall, the final solution $f_n = 11 \cdot 2^n + (-11 - \dfrac{23}{\sqrt{5}}) \cdot (\dfrac{1+\sqrt{5}}{2})^n + (-11 + \dfrac{23}{\sqrt{5}}) \cdot (\dfrac{1-\sqrt{5}}{2})^n + 2n + 11 (n \geq 0)$.

Problem 3: Solve the following recurrence equations:

$$\begin{aligned} t_n &= 2t_{n-1} + t_{n-2} + 2^n \\ t_0 &= 0 \\ t_1 &= 2 \end{aligned}$$

Show your work.

Solution 3:

1) We first find the general solution t'_n by solving the associated homogeneous equation $t_n c = 2t_{n-1} + t_{n-2}$. The corresponding characteristic equation is $r^2 - 2r - 1 = 0$. By computing, we have the roots $r_1 = 1 + \sqrt{2}, r_2 = 1 - \sqrt{2}$. Hence, the general solution is $t'_n = \alpha_1(1+\sqrt{2})^n + \alpha_2(1-\sqrt{2})^n$.

2) Then let's figure out the particular solution t''_n. Since $g(n) = 2^n$ and the base 2 is not a root of t_{nc}, there is a particular solution of the form $t''_n = p_0 \cdot 2^n$. By recurrence equation $t_n = 2t_{n-1} + t_{n-2} + 2^n$, we have $p_0 \cdot 2^n = 2 \cdot p_0 \cdot 2^{n-1} + p_0 \cdot 2^{n-2} + 2^n$. Because this equation must be true no matter what the value of n is, then the coefficient of the item associated with n must be zero. Thus, we have the solution $p_0 = -4$. So the particular solution $t''_n = -4 \cdot 2^n$.

3) The final solution $t_n = t'_n + t''_n = \alpha_1(1+\sqrt{2})^n + \alpha_2(1-\sqrt{2})^n - 4 \cdot 2^n$. We solve this by initial conditions. Recall $t_0 = 0$ and $t_1 = 2$, then we have the following equation set.

$$\alpha_1 + \alpha_2 - 4 = 0 \tag{8}$$

$$\alpha_1 \cdot (1+\sqrt{2}) + \alpha_2 \cdot (1-\sqrt{2}) - 4 \cdot 2 = 2 \tag{9}$$

By computing, we have $\alpha_1 = \dfrac{4+3\sqrt{2}}{2}, \alpha_2 = \dfrac{4-3\sqrt{2}}{2}$.

Therefore, the final solution $t_n = \dfrac{4+3\sqrt{2}}{2} \cdot (1+\sqrt{2})^n + \dfrac{4-3\sqrt{2}}{2} \cdot (1-\sqrt{2})^n - 4 \cdot 2^n (n \geq 0)$.
