Electrostatic Boundary-value problems:

- Poisson's Equation: $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$
- Laplace's Equation: $\nabla^2 V = 0$

In Cartesian coordinates

$$\nabla^2 V = \nabla \cdot \nabla V = (\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}) \cdot (\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z})$$

$$= (\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2})$$

In cylindrical coordinates

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

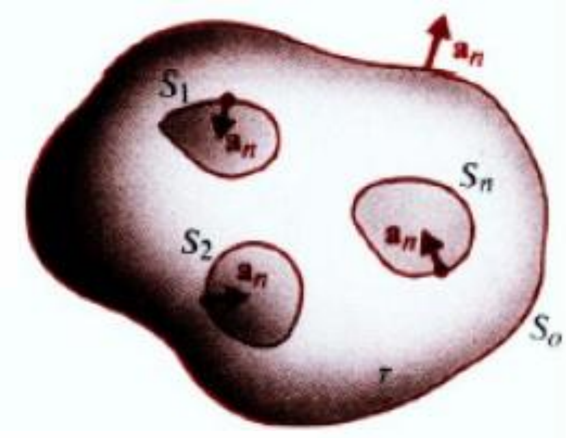
In spherical coordinates

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\frac{\partial V}{\partial R}) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{R^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Uniqueness of Electrostatic Solutions

Uniqueness Theorem: a solution of Poisson's equation (of which Laplace's equation is a special case) that satisfies the given boundary conditions is a unique solution.

$$\nabla^2 V_1 = -\frac{\rho}{\varepsilon}, \qquad \nabla^2 V_2 = -\frac{\rho}{\varepsilon}$$



Also assume that both V_1 and V_2 satisfy the same boundary conditions on $S_1, S_2, \ldots, S_n$

new difference potential: $V_d = V_1 - V_2$

$$\nabla^2 V_d = 0 \quad \text{in } \tau\text{, and } V_d = 0 \quad \text{on conducting boundaries}$$

Using the vector identity: $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$

$f = V_d$ and $\mathbf{A} = \nabla V_d;$

$$\oint_{S_0}(V_d\nabla V_d)\cdot\mathbf{a}_n ds+\oint_{S_1}(V_d\nabla V_d)\cdot\mathbf{a}_n ds+\ldots+\oint_{S_n}(V_d\nabla V_d)\cdot\mathbf{a}_n ds=\int_\tau|\nabla V_d|^2 dv$$

$$\oint_{S_0}(V_d\nabla V_d)\cdot\mathbf{a}_n ds = 0$$

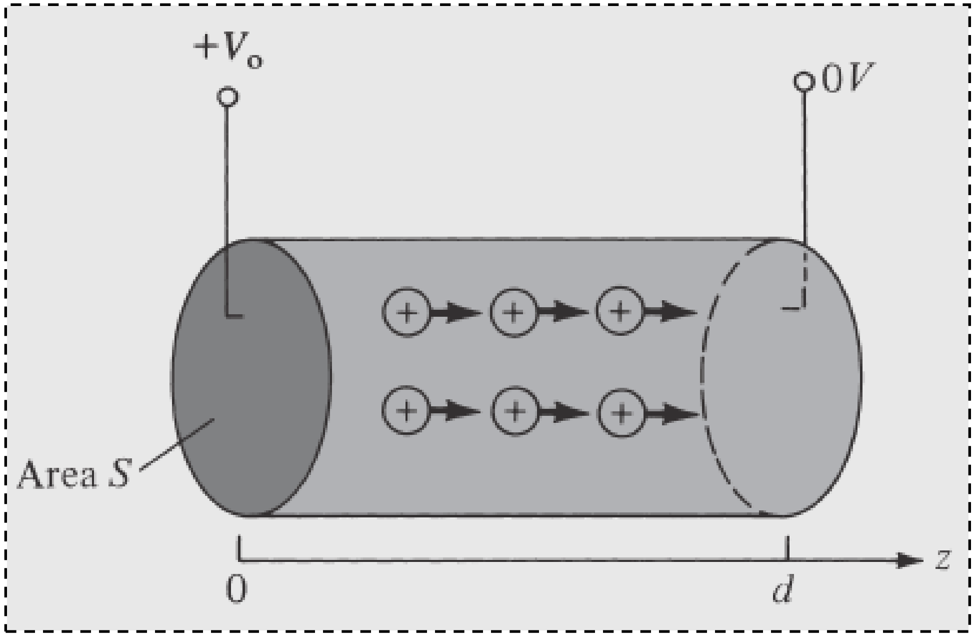
$$\int_\tau|\nabla V_d|^2 dv=0 \quad\Longrightarrow\quad V_1=V_2 \quad \text{everywhere in } \tau$$

- General procedure:
  - Solve the equation (Poisson/Laplace) (in appropriate coordinate system)
  - Apply the boundary conditions
  - Calculate V
  - **E** = -Grad V, **D** = ε**E** and **J** = σ**E**

Consider an example of parallel plate capacitor to illustrate these steps

Example: Electrohydrodynamic Pump (EHD)

Pumping is based on the force transmitted to the cooling fluid by charges in an electric field.



Let charge density be ρ_o

Calculate the pressure of the pump

Example: Semi-infinite conducting planes

Calculate V and **E** in the region between the planes