Chapter - 7

Time - Varying Fields And Maxwell's Equations

Static Electric Fields

$$\nabla \cdot \mathbf{D} = \rho_v$$

- the electric flux density emerging from a point equals to the volume charge density

$$\nabla \cdot \mathbf{B} = 0$$

- Magnetic sources exists in pair (North and South pole)

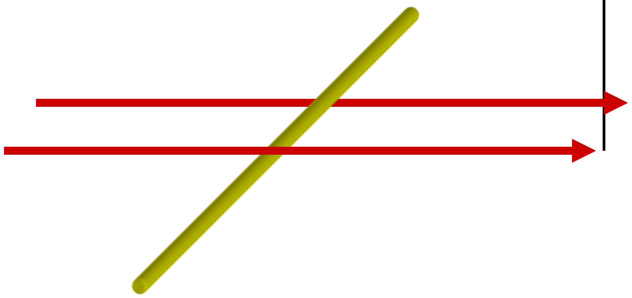
$$\nabla \times \mathbf{E} = 0$$

- energy used for moving an electric charge around a closed loop is equal to zero

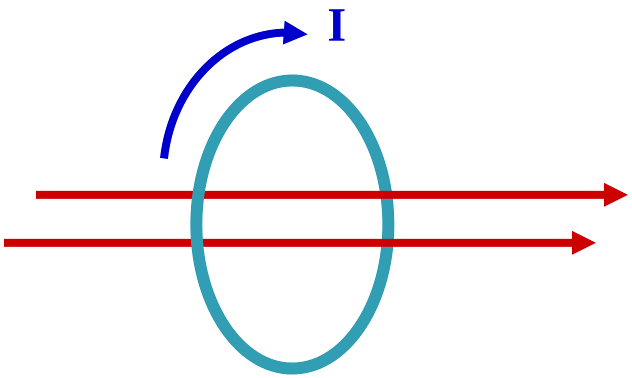
$$\nabla \times \mathbf{H} = \mathbf{J}$$

- magnetic field around a closed path equals to the current inside

It is observed experimentally that **changes** in magnetic flux induce an emf in a conductor.

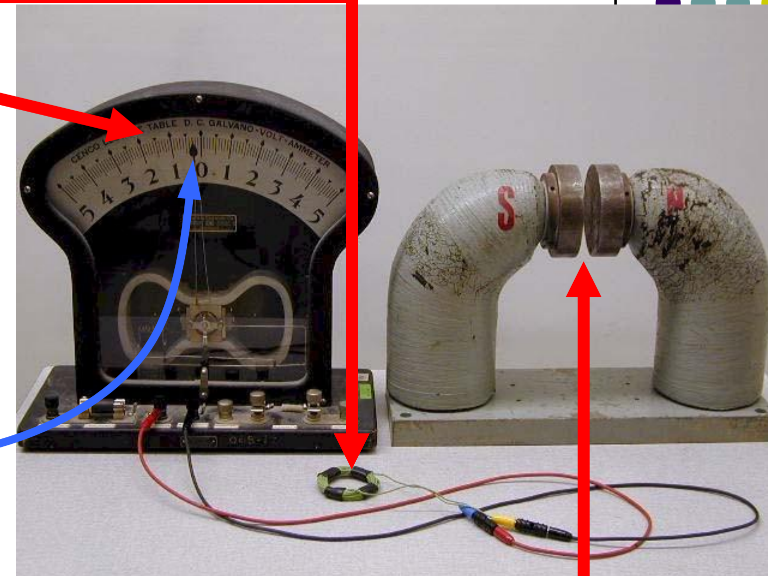


An electric current is induced if there is a closed circuit (e.g., loop of wire) in the **changing** magnetic flux.



A constant magnetic flux does not induce an emf—it takes a changing magnetic flux.

Passing the coil through the magnet would induce an emf in the coil.



Faraday's Law of Electromagnetic Induction

- Experimentally discovered: A current was induced in a conducting loop when the magnetic flux linking the loop changed.

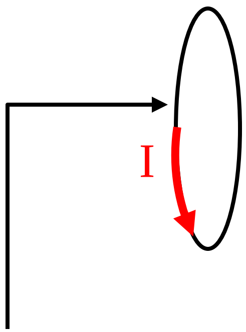
Fundamental Postulate for Electromagnetic Induction

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.}$$

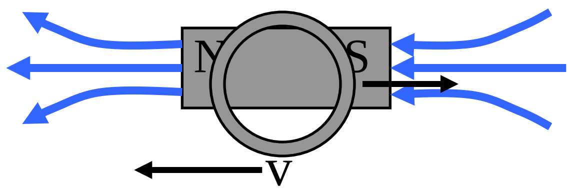
It says that the electric field intensity in a region of time-varying magnetic flux density is therefore non-conservative and cannot be expressed as the gradient of a scalar potential.

- **Stationary Loop in Time-varying B Field**
- **Moving Loop in Static B Field**
- **Moving Loop in Time-Varying Field.**

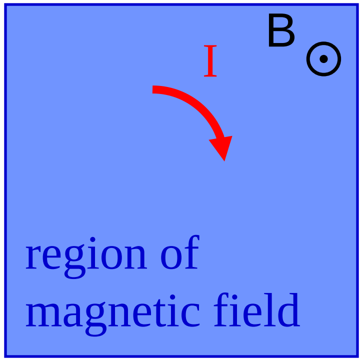
  - A magnet may move through a loop of wire, or a loop of wire may be moved through a magnetic field. These involve observable motion.



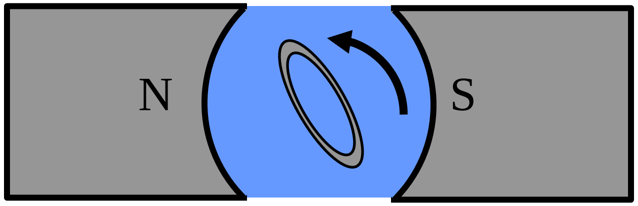
this part of the loop is
closest to your eyes



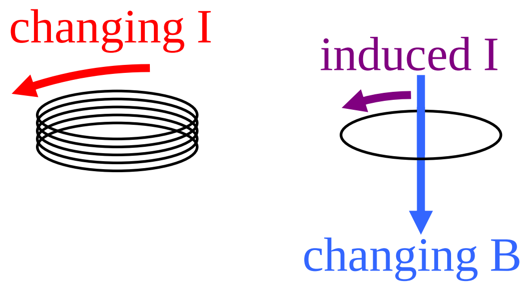
move magnet toward coil



change area of loop inside magnetic field



rotate coil in magnetic field



• A changing current in a loop of wire gives rise to a changing magnetic field (predicted by Ampere's law) which can induce a current in another nearby loop of wire.

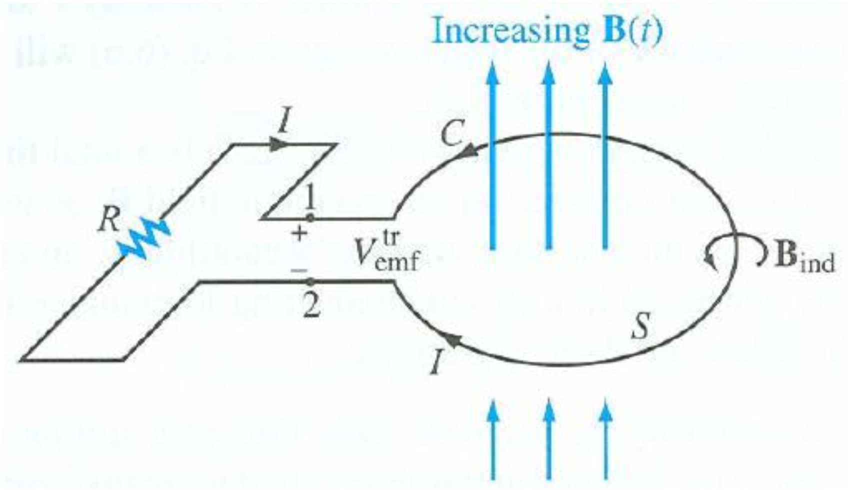
In this case, nothing observable (to your eye) is moving, although, of course microscopically, electrons are in motion.

Induced emf is produced by a changing magnetic flux.

Stationary Loop in a Time-varying Magnetic Field

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}.$$

Lenz's Law: The current in the loop is always in such a direction as to oppose the change of magnetic flux that produced it



Example: Inductor in a changing Magnetic Field

A inductor is formed by winding N turns of a thin conducting wire into a circular loop of radius *a*. The inductor loop is in the x-y plane with its centre at the origin, and it is connected to a resistor R. In the presence of a magnetic field given by

$$\mathbf{B} = B_0\left(\mathbf{a}_y 2 + \mathbf{a}_z 3\right)\sin\omega t$$

Find:

(a) The magnetic flux linking a single turn of the inductor

(b) The transformer emf, given that N = 10, B_0 = 0.2 T, a = 10 cm and Omega = 1000rad/s

(c) Induced current in the circuit for R = 1 kOhms

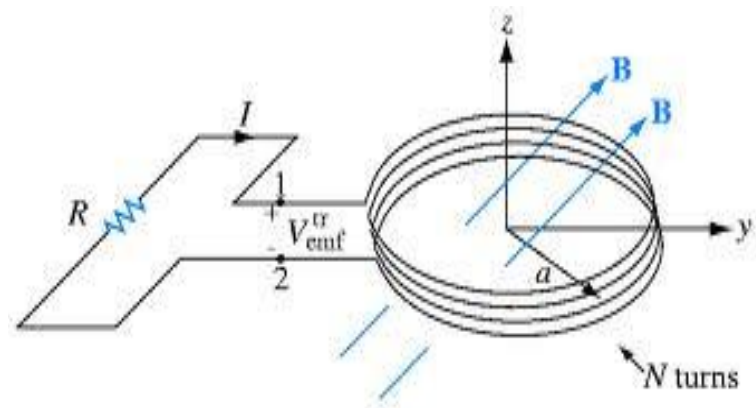
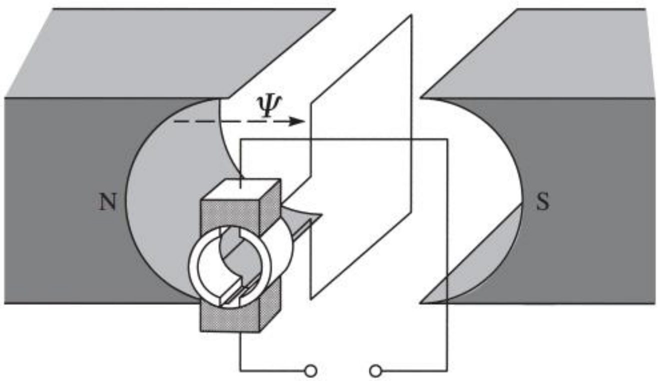
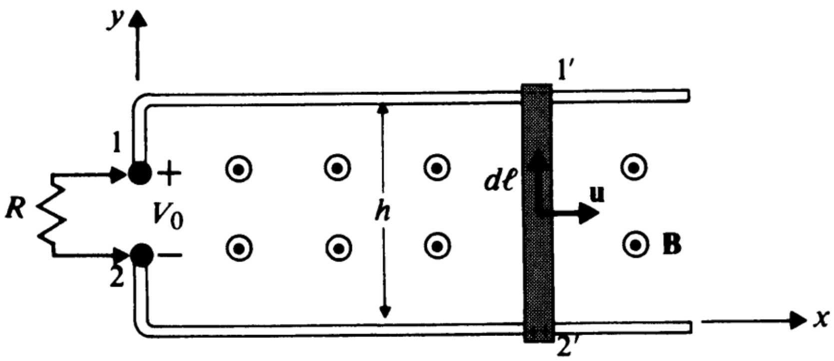


Figure 6-3

Moving Loop in Static B Field

$$\boxed{\mathcal{V}' = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \qquad \text{(V).}}$$



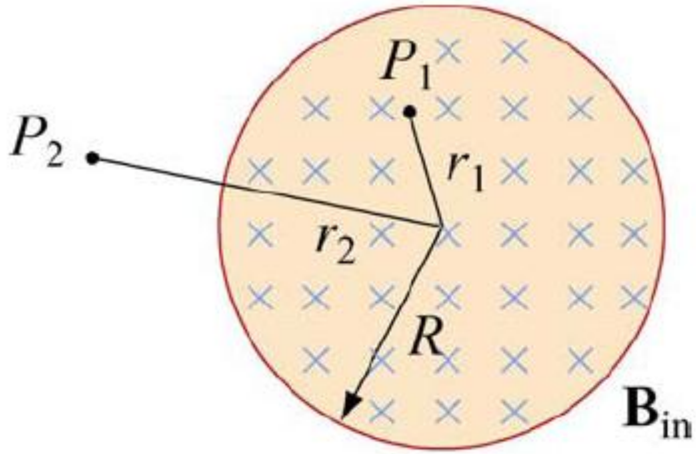


Moving Loop in Time – varying B Field

$$\boxed{\oint_C \mathbf{E}' \cdot d\ell = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\ell \qquad \text{(V)}.}$$

Home-Assignment

For the situation described in the figure below,



the magnetic field changes with time according to the expression

$$B = \left(2.00t^3 - 4.00t^2 + 0.800\right)\ \text{T}$$

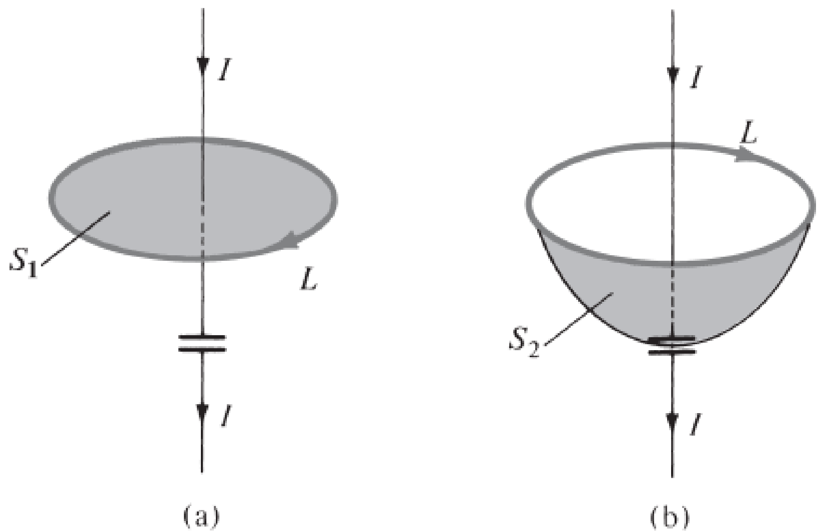
and $r_2 = 2R = 5.00\ \text{cm}$.

(a) Calculate the magnitude and direction of the force exerted on an electron located at point P_2 when $t = 2.00\ \text{s}$.

(b) At what time is the force equal to zero?

Displacement Current

$$\nabla \times \overline{H} = \overline{J} + \overline{J}_d$$



Final Form of Maxwell's Equations

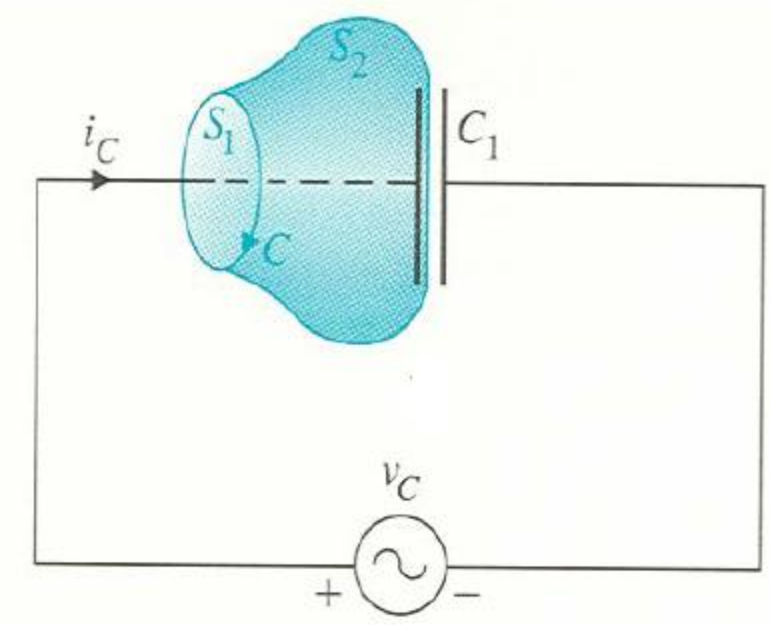
	Differential	**Integral**	**Remarks**
1	$\nabla \cdot D = \rho_V$	$\oint_S D \cdot dS = \int_V \rho_V dv$	Gauss' Law
2	$\nabla \cdot B = 0$	$\oint_S B \cdot dS = 0$	Nonexistence of isolated charge
3	$\nabla x E = -\frac{\partial B}{\partial t}$	$\oint_L E \cdot dl = -\frac{\partial}{\partial t}\int_S B \cdot dS$	Faraday's Law
4	$\nabla x H = J + \frac{\partial D}{\partial t}$	$\oint_L H \cdot dl = \int_S \left(J + \frac{\partial D}{\partial t} \right) \cdot dS$	Ampere's circuital law

In 1 and 2, S is a closed surface enclosing the volume V

In 2 and 3, L is a closed path that bounds the surface S

Example:

Show that $i_c = i_d$



$$v_c = V_0 \sin \omega t$$

Potential Functions

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \longrightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \longleftarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \longleftarrow \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

$$\longrightarrow \boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

We can see that B and **E** are coupled through the vector and scalar potentials.

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho}{R} dv' \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R} dv'$$

no longer valid for time-varying EM fields

When charge and current vary slowly with time (at a very low frequency) and the range of interest R is small in comparison with the wavelength, the above two expressions would be considered as quasi-static fields approximation.

However, at high source frequency and longer range → **Time-retardation effects**

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} + \mu\varepsilon \frac{\partial}{\partial t}(-\nabla V - \frac{\partial \mathbf{A}}{\partial t})$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \nabla(\mu\varepsilon \frac{\partial V}{\partial t}) - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t})$$

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t}\right)$$

If we choose $\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0$

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

Vector wave equation for **A**

Lorentz-gauge condition

$$\nabla \cdot \mathbf{D} = \rho$$

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

$$-\nabla \cdot \varepsilon\left(\nabla V + \frac{\partial \mathbf{A}}{\partial t}\right) = \rho$$

For a constant ε

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon}$$

Scalar wave equation for V

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Use Lorentz-gauge condition

Solution of Wave Equations for Potentials

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$ Scalar wave equation for V

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$ Vector wave equation for **A**

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \cancel{\frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)} + \cancel{\frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}} - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0$$

Because of spherical symmetry, potential V is independent of θ or ϕ

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0$$

Introducing a new variable $V(R,t) = \frac{1}{R}U(R,t)$ $\qquad \frac{\partial^2 U}{\partial R^2} - \mu\varepsilon \frac{\partial^2 U}{\partial t^2} = 0$

with its solution $U(R,t) = f(t - R\sqrt{\mu\varepsilon})$ or $V(R,t) = \frac{1}{R} f(t - R/u)$

- $$V(R,t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(t - R/u)}{R} dv'$$ Retarded scalar potential
- $$\mathbf{A}(R,t) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J}(t - R/u)}{R} dv'$$ Retarded vector potential

Wave Equations of *E* & *H* in Source-Free Region

Wave in a simple nonconducting medium

$$\nabla^2\mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2\mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Wave speed

$$u = 1/\sqrt{\mu\varepsilon}$$

Electromagnetic Boundary conditions

$$\boxed{E_{1t} = E_{2t} \qquad (\text{V/m});}$$

$$\boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad (\text{A/m}).}$$

$$\boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \qquad (\text{C/m}^2);}$$

$$\boxed{B_{1n} = B_{2n} \qquad (\text{T}).}$$

Interface Between Two Lossless Linear Media

$$\sigma = 0$$

$$\rho_s = 0 \text{ and } \mathbf{J}_s = 0$$

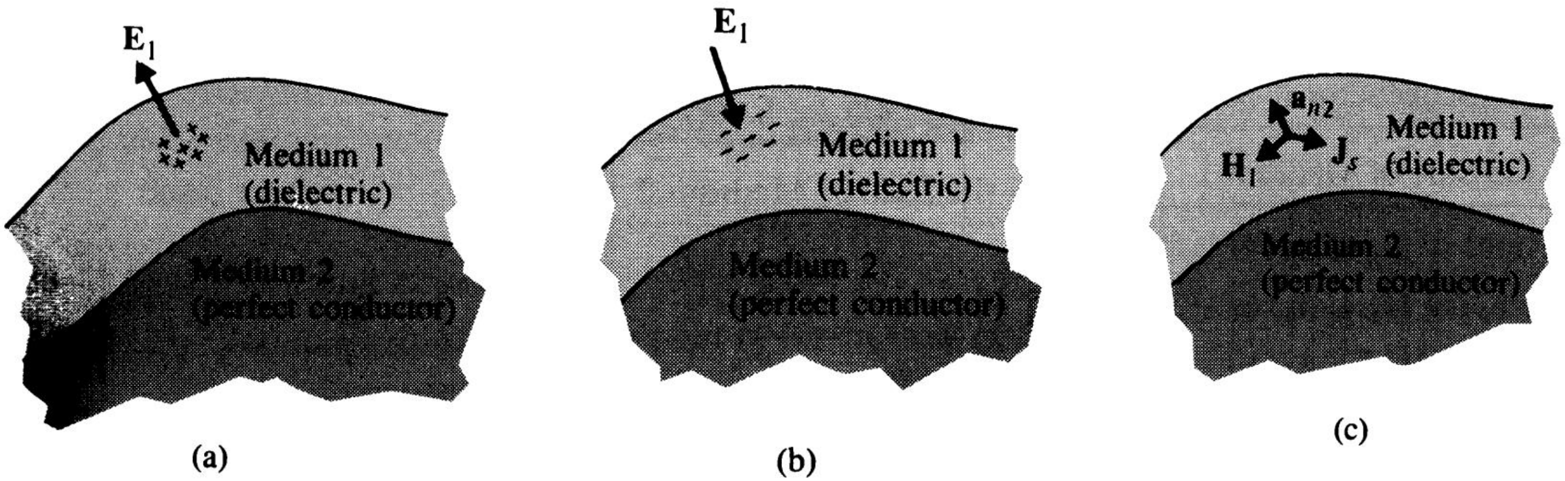
$$E_{1t} = E_{2t} \rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\epsilon_1}{\epsilon_2}$$

$$H_{1t} = H_{2t} \rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}$$

$$D_{1n} = D_{2n} \rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

$$B_{1n} = B_{2n} \rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

Interface Between Dielectric and Perfect Conductor



B and **H** are also zero in the interior of a conductor in a time-varying situation

Boundary Conditions between a Dielectric (Medium 1) and a Perfect Conductor (Medium 2) (Time-Varying Case)

On the Side of Medium 1	On the Side of Medium 2
$E_{1t} = 0$	$E_{2t} = 0$
$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$	$H_{2t} = 0$
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$

Time-Harmonic Fields

Maxwell's equations are *linear* differential equations hence sinusoidal time variations of source functions of a given frequency will produce sinusoidal variations of **E** and **H** with the *same frequency* in the steady state.

A time-harmonic field is one that varies periodically or sinusoidally with time

For time-harmonic fields it is convenient to use a phasor notation

Time-Harmonic Electromagnetics

- Time-harmonic Maxwell's Equations (for a simple medium)
- Time harmonic wave equations for scalar potential *V* and vector potential **A**
- Lorentz condition

$$\boxed{V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho e^{-jkR}}{R}\, dv' \qquad \text{(V)},}$$

$$\boxed{\mathbf{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J} e^{-jkR}}{R}\, dv' \qquad \text{(Wb/m)}.}$$

$$e^{-jkR} = 1 - jkR + \frac{k^2R^2}{2} + \cdots,$$

1. Find phasors $V(R)$ and $\mathbf{A}(R)$ from
2. Find phasors $\mathbf{E}(R) = -\nabla V - j\omega\mathbf{A}$ and $\mathbf{B}(R) = \nabla \times \mathbf{A}$.
3. Find instantaneous $\mathbf{E}(R, t) = \mathcal{R}e[\mathbf{E}(R)e^{j\omega t}]$ and $\mathbf{B}(R, t) = \mathcal{R}e[\mathbf{B}(R)e^{j\omega t}]$ for a cosine reference.

Source-Free Fields in Simple Media

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},$$
$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E},$$
$$\nabla \cdot \mathbf{E} = 0,$$
$$\nabla \cdot \mathbf{H} = 0.$$

$$\left.\begin{array}{l} \boxed{\nabla^2\mathbf{E} + k^2\mathbf{E} = 0} \\ \\ \boxed{\nabla^2\mathbf{H} + k^2\mathbf{H} = 0,} \end{array}\right\}$$

Homogeneous vector Helmholtz's Equations

If the simple medium is **conducting**, a current **J** will flow

Concept of complex permittivity and loss tangent

Example:2 [Use time-domain approach]

Suppose the Electric field in a source free (i.e. ρ_v=0) region is given by a wave travelling in the z-direction

$$\mathbf{E} = E_o \sin(\omega t - \beta z)\mathbf{a_x}$$

Find the value of the magnetic field present. What must be the value of β so that both fields satisfy Maxwell's equations?

$$\nabla x E = -\frac{\partial B}{\partial t}$$

$$= -(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}) \times \left(E_o \sin(\omega t - \beta z)\mathbf{a_x} \right)$$

$$= \beta E_o \cos(\omega t - \beta z)\mathbf{a_y}$$

$$\mathbf{B} = \int \beta E_o \cos(\omega t - \beta z)\mathbf{a_y}\, dt$$

$$= \frac{\beta E_o}{\omega} \sin(\omega t - \beta z)\mathbf{a_y}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_o} = \frac{\beta E_o}{\omega \mu_o} \sin(\omega t - \beta z)\mathbf{a_y}$$

This shows that an associated time-varying H-field must **co-exist**.

$$\varepsilon_o \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} = \frac{\beta^2 E_o}{\omega \mu_o} \cos(\omega t - \beta z)\mathbf{a_x}$$

$$\mathbf{E} = \frac{\beta^2 E_o}{\omega^2 \mu_o \varepsilon_o} \sin(\omega t - \beta z)\mathbf{a_x} \qquad \mathbf{E} = E_o \sin(\omega t - \beta z)\mathbf{a_x}$$

$\beta = \omega\sqrt{\mu_o \varepsilon_o} \quad \Rightarrow$ phase velocity $u_p = \omega / \beta = 1/\sqrt{\mu_o \varepsilon_o} = 3 \times 10^8\, m/s$

Solve this example using phasors approach