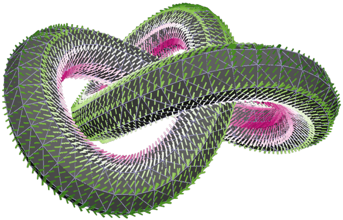
An HP-Hierarchical Framework for the Finite Element Exterior Calculus with Applications to Elasticity Problems

WCCM 2018, MS906, New York, NY, USA



Robert L. Gates, Maximilian R. Bittens

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Institut für Baumechanik und Numerische Mechanik
Leibniz Universität Hannover

Goal: an elegant, general *hp*-adaptive framework for structure-preserving discretization in higher dimensions

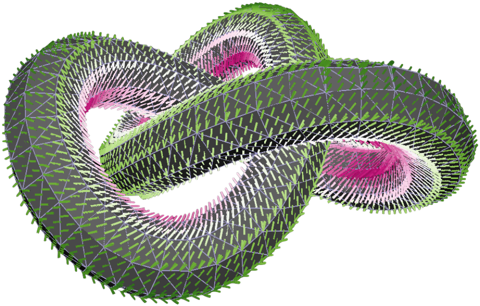
Supported theories

- Finite Element Exterior Calculus (Arnold, Falk, Winther)
- Discrete Exterior Calculus (Hirani)
- ...

Supported applications (in principle)

- **Mixed linearized elasticity**, weak symm. (Arnold et al.)
- Mixed linearized elastodynamics, weak symm. (Arnold, Lee)
- **Mixed nonlinear elasticity** (Angoshtari, Yavari)
- Parabolic Finite Element Exterior Calculus (Arnold, Chen)
- **Riemannian manifolds**
- Non-simply-connected domains, **computational homology via de Rham's theorem**

Theoretical basis



A *scalar-valued* **differential** k**-form** is a totally antisymmetric $\binom{0}{k}$-tensor, defined by the alternating multilinear map

$$\underbrace{T_P\mathcal{M} \times \ldots \times T_P\mathcal{M}}_{k\text{-fold}} \quad \to \quad \mathbb{R}\,.$$

The **space** Λ^k **of differential** k**-forms** is a vector space, together with the **exterior product** $\wedge : \Lambda^k \times \Lambda^p \to \Lambda^{k+p}$ such that

$$\boldsymbol{\omega} \wedge \boldsymbol{\nu} = (-1)^{kp} \boldsymbol{\nu} \wedge \boldsymbol{\omega}\,,$$

and for k odd, $\boldsymbol{\omega} \wedge \boldsymbol{\omega} = 0$. A basis for Λ^k is given by the k-fold wedges $\{\mathbf{d}x^{\sigma(1)} \wedge \ldots \wedge \mathbf{d}x^{\sigma(k)} \mid \boldsymbol{\sigma} \in \Sigma(k,n)\}$. Hence

$$\boldsymbol{\omega} = \sum_{\boldsymbol{\sigma} \in \Sigma(k,n)} \omega_{\boldsymbol{\sigma}} \mathbf{d}x^{\boldsymbol{\sigma}}\,, \quad \mathbf{d}x^i \in T_P^*\mathcal{M}\,.$$

The **exterior derivative** is the map $\mathbf{d} : \Lambda^k \to \Lambda^{k+1}$ such that for $\omega \in \Lambda^0$

$$\mathbf{d}\omega = \frac{\partial \omega}{\partial x^1}\mathbf{d}x^1 + \ldots + \frac{\partial \omega}{\partial x^n}\mathbf{d}x^n\,.$$

and for $\boldsymbol{\omega} \in \Lambda^k$, $\mathbf{d}\boldsymbol{\omega} = \mathbf{d}\omega_\sigma \wedge \mathbf{d}x^{\boldsymbol{\sigma}}$.

The **exterior coderivative** $\boldsymbol{\delta} = (-1)^{n(k+1)+1+s} \star \boldsymbol{d} \star$ is the formal adjoint of $\boldsymbol{d}$.

Hodge star $\star_g : \Lambda^k \xrightarrow{g} \Lambda^{n-k}$, $\star\star\boldsymbol{\omega} = (-1)^{k(n-k)+s}\boldsymbol{\omega}$, $\langle \boldsymbol{\omega}, \boldsymbol{\nu} \rangle = \boldsymbol{\omega} \wedge \star_g \boldsymbol{\nu}$

Spaces of polynomial differential forms

Space of **multivariate polynomials of total degree at most** r on $m \leq n$-simplex f:

$$\mathcal{P}_r(f) = \{\boldsymbol{\lambda}^{\boldsymbol{\alpha}} = \lambda_0^{\alpha_0} \cdots \lambda_m^{\alpha_m} \mid \boldsymbol{\alpha} \in \mathbb{N}_0^{0:m}, |\boldsymbol{\alpha}| = r\}$$

Homogeneous polynomials of degree at most r:

$$\mathcal{H}_r(\mathbb{R}^m) = \{\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \mid \boldsymbol{\alpha} \in \mathbb{N}_0^{1:m}, |\boldsymbol{\alpha}| = r\}$$

Space of **polynomial differential forms**:

$$\mathcal{P}_r\Lambda^k = \{p_{\boldsymbol{\sigma}} \mathbf{d}x^{\boldsymbol{\sigma}} \mid p_{\boldsymbol{\sigma}} \in \mathcal{P}_r\}$$

Koszul differential $\kappa : \mathcal{P}_r\Lambda^k \to \mathcal{P}_{r+1}\Lambda^{k-1}$

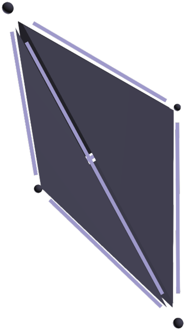
$$\kappa\boldsymbol{\omega} = \sum_{\boldsymbol{\sigma} \in \Sigma(k,n)} \sum_{i=1}^{k} (-1)^{i+1} \omega_{\boldsymbol{\sigma}} x^{\sigma(i)} \mathbf{d}x^{\sigma(1)} \wedge \ldots \wedge \widehat{\mathbf{d}x^{\sigma(i)}} \wedge \ldots \wedge \mathbf{d}x^{\sigma(k)}$$

Reduced space (warning: $\dim \kappa\mathcal{H}_r\Lambda^k \neq \dim \mathcal{H}_r\Lambda^k$):

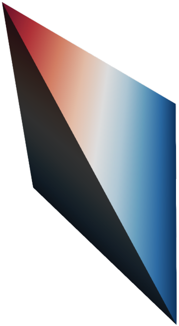
$$\mathcal{P}_r^-\Lambda^k(f) = \mathcal{P}_{r-1}\Lambda^k(f) + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$$

Source: Arnold, Falk, Winther (2006)

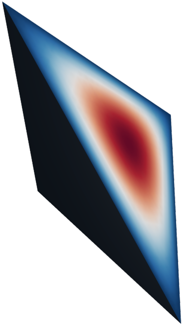
Geometric decomposition



Subsimplices of T^3



$\mathcal{P}_1\Lambda^0(T^3, v)$

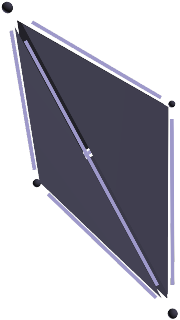


$\mathcal{P}_3\Lambda^0(T^3, f)$

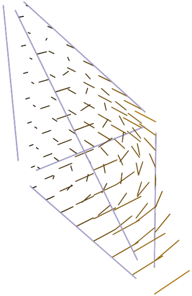
Let T^n be the **reference** ***n*-simplex** associated to the **finite element** $(T^n, \mathcal{P}, \Sigma)$ and $f \in \Delta^m(T^n)$ one of its ***m*-subsimplices**. Then associate to each $m = 0 \dots n$ a weight space W^m such that the degrees of freedom Σ read

$$\boldsymbol{u} \in P^n \mapsto \int_f Tr_f \boldsymbol{u} \wedge \boldsymbol{q}\,, \quad \boldsymbol{q} \in W^m\,, \quad f \in \Delta^m(T^n)\,.$$

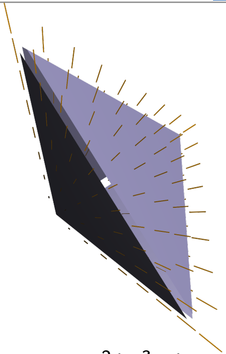
Geometric decomposition



Subsimplices of T^3



$\mathcal{P}_1^- \Lambda^1(T^3, e)$



$\star\mathcal{P}_1^- \Lambda^2(T^3, f)$

The set of finite element basis functions $\mathcal{P}$ is dual to Σ:

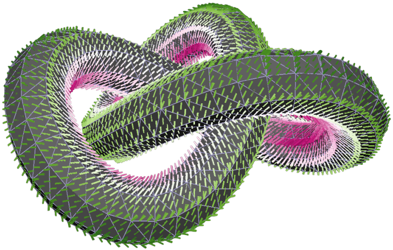
$$\sigma^i(\boldsymbol{u}_j) = \delta^i_j\,, \quad \boldsymbol{u} \in \mathcal{P}\,, \quad \sigma \in \Sigma\,.$$

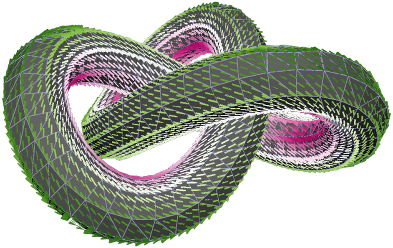
Hence, identify each element of $\mathcal{P}$ with an element of one of the weight spaces W^m and a subsimplex f. Then $\mathcal{P}(T^n)$ admits the decomposition

$$\mathcal{P}(T^n) = \bigoplus_{\substack{f \in \Delta^m(T^n)\\ m=0\dots n}} \mathcal{P}(T^n, f)\,.$$

Verification of basis and assembly

Computing the kernel of the discrete Hodge Laplacian $\delta \mathrm{d} + \mathrm{d}\delta$ for $k = 1$ gives an approximation to the basis for the cohomology class $\mathfrak{H}^1$ of the torus:





The dimension of this basis corresponds to the **first Betti number, or the number of 1-holes, of the torus, $b_1 = 2 = \dim \mathfrak{H}^1$.**

In this case, $\mathfrak{H}^1$ **can intuitively be associated with the curl- and (divergence-)free fluid flows** on the surface of the torus knot.

P-Hierarchical decomposition

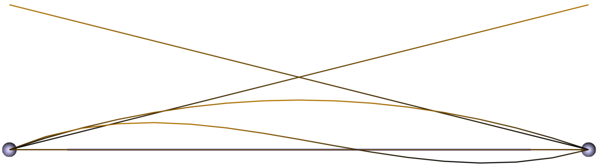
The subspaces associated to m-subsimplices f are nested:

$$\mathcal{P}_r^- \Lambda^k(T^n, f) \subseteq \mathcal{P}_r \Lambda^k(T^n, f) \subseteq \mathcal{P}_{r+1}^- \Lambda^k(T^n, f)\,.$$

Thus, $\mathcal{P}_r^- \Lambda^k(T^n, f)$ **admits the p-hierarchical decomposition**

$$\mathcal{P}_r^- \Lambda^k = L \oplus \Delta\mathcal{P}_2^- \Lambda^k \oplus \ldots \oplus \Delta\mathcal{P}_r^- \Lambda^k$$

into its "lowest-order" space L (the Whitney forms $\mathcal{P}_1^- \Lambda^k$) and the algebraic complements $\Delta\mathcal{P}_r^- \Lambda^k$.



Similarly, $\mathcal{P}_r \Lambda^k(T^n, f)$ **admits the p-hierarchical decomposition**

$$\mathcal{P}_r \Lambda^k = L \oplus \Delta\mathcal{P}_1^{\pm} \Lambda^k \oplus \Delta\mathcal{P}_2^+ \Lambda^k \oplus \ldots \oplus \Delta\mathcal{P}_r^+ \Lambda^k\,.$$

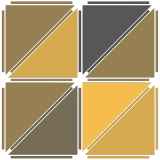
These decompositions can be computed numerically.

Putting it all together: p-refinement

Given the geometric and p-hierarchical decompositions of the bases, it is *in principle* possible to **assign a polynomial order to edges, faces, volumes, etc. independently**.

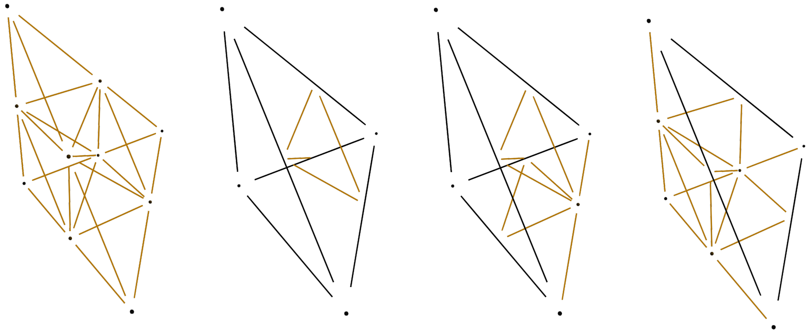
Lemma Let Ω_d represent the set of top-dimensional simplices in a given simplicial representation of $\Omega \subset \mathbb{R}^n$ with polynomial orders $r > 0$ assigned by the surjection $\mathfrak{O} : \Omega_d \to \mathbb{N}$. To each subsimplex $f \in \Delta_k \overline{T}$ of dimension $0 < k \leq n$, the maximum allowed polynomial order assignable is determined by the map

$$f \mapsto \min \left\{ \mathfrak{O}(\overline{T}) \,:\, f \subset \overline{T},\; \overline{T} \in \Omega_d \right\} .$$



Then the approximation on Ω_d is invariant under all possible choices of decomposable bases for the spaces $\mathcal{P}\Lambda^k$ and $\mathcal{P}^-\Lambda^k$.

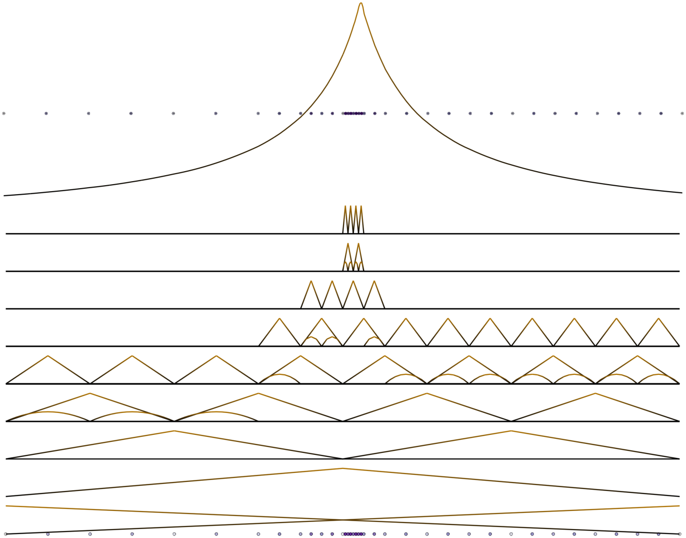
Freudenthal's regular simplex subdivision partitions an n-simplex into s^n simplices with the same n-volume. In the present case, the grading is $s = 2$.



Freudenthal subdivision | face refinement | edge refinement | vertex refinement

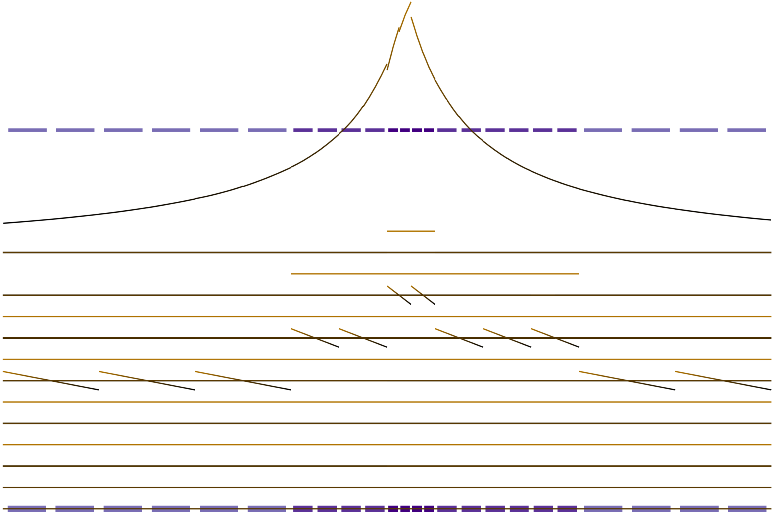
We generalize the scheme for all n by decomposing it into successive refinements of m-subsimplices of T^n, as shown above. This strategy entirely avoids the introduction of "hanging mesh entities".

Hierarchical basis for $\mathcal{P}^{-}\Lambda^{0}(\Omega^{1})$



Linear independence by construction for $k = 0$.

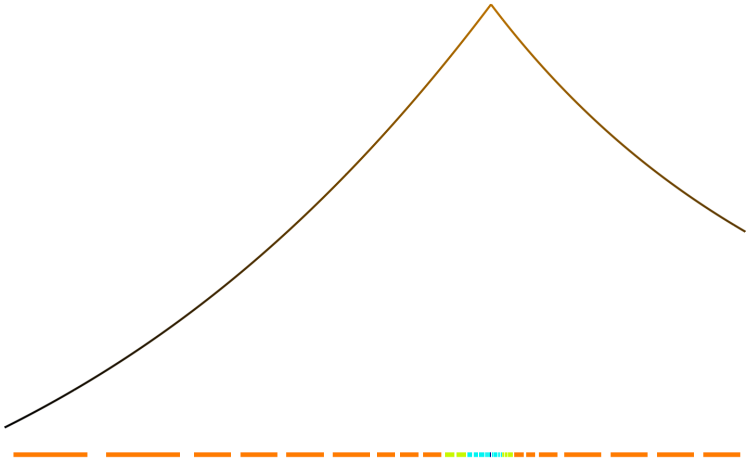
Hierarchical basis for $\mathcal{P}^{-}\Lambda^{1}(\Omega^{1})$

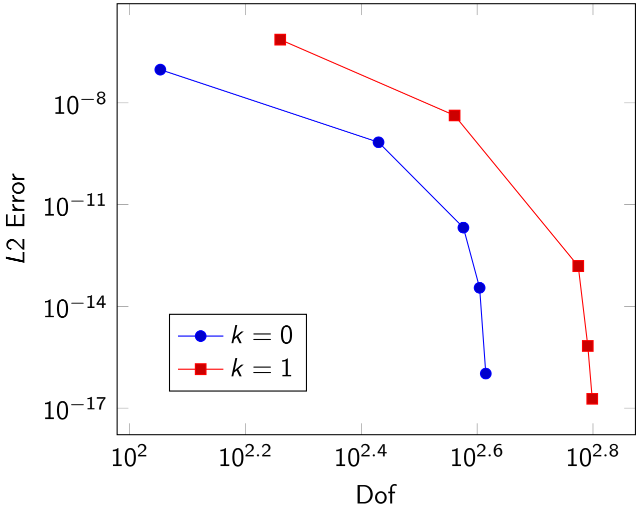


For all $k > 0$, linear independence must be ensured, per hierarchical parent-child relationship, by a linear constraint on "child" Whitney forms $\mathcal{P}_1^{-}\Lambda^{k}$.

Function with singularity in 1d

$$f(x) = \frac{1 - e^{-|2-(x-\frac{\pi}{2})^2|+0.1}}{|2-(x-\frac{\pi}{2})^2|+0.1}$$





L^2 projection, maximum polynomial order $r = 6$

Putting it all together: *h*- and *p*-refinement

Let the polynomial order of relevant n-simplices propagate hierarchically upwards from a yet to be defined set Ω_d^* of n-simplices (across all levels) which cover the base complex Ω_d without mutual intersection. It is now necessary that the polynomial order for leaf-type subsimplices of dimension $0 < k \leq n$ be uniquely defined by their upwards-adjacent n-simplices *without reference to the hierarchical descendants thereof*. There are two possible limits of choice in the definition of order-defining n-simplices Ω_d^*: (1) they are the roots of the hierarchical trees $\Omega_d^* = \Omega_d$ or (2):

Definition An n-simplex is called *order-defining* iff.

(a) at least one of the $f \in \Delta \overline{T}$ is of leaf type and

(b) none of its hierarchical ancestors fulfill condition (a).

Lemma The maximum polynomial order assignable to a leaf subsimplex f of dimension $0 < k \leq n$ is determined by the map

$$f \mapsto \min \left\{ \mathfrak{O}^*(\overline{T}) \, : \, f \subset \overline{T}, \; \overline{T} \in \Omega_d^* \right\} , \qquad (1)$$

where Ω_d^* is the set of all order-defining n-simplices with roots in Ω_d and $\mathfrak{O}^* : \Omega_d^* \to \mathbb{N}$. *Then the approximation on* $\cup_{l=1}^{L} \Omega_d^l$ *is invariant under all possible choices of decomposable bases for the spaces* $\mathcal{P}\Lambda^k$ *and* $\mathcal{P}^-\Lambda^k$.

Obtain polynomial eigenfunctions from the problem

$$\mathcal{L}\phi = \omega\phi$$

on the reference simplex $\hat{T}$, where the differential operator (Braess, 2005) is symmetric in the sense that

$$\mathcal{L} = \sum_{f_{\boldsymbol{\sigma}} \in \Delta_{\mathbf{1}} \hat{T}} \lambda_{\sigma(1)}\lambda_{\sigma(2)} \left(\frac{\partial}{\partial\lambda_{\sigma(2)}} - \frac{\partial}{\partial\lambda_{\sigma(1)}} \right)^2 .$$

The eigenvalue problem has the distinct eigenvalues $\omega_p = p(p+n)$ for the polynomial orders $p = 0, 1, 2, \ldots, r$.

Let $a \in \mathcal{P}_r(\hat{T})$ of order r, the spectral coefficients a_{pi} of a may be obtained such that

$$a(\boldsymbol{\xi}) = \sum_{p=0,\ldots,r} \sum_{i=1,\ldots,\#\Phi_p} a_{pi}\phi_{pi}(\boldsymbol{\xi}) \quad \text{and} \quad a_p = \frac{1}{\#\Phi_p} \sum_{i=1,\ldots,\#\Phi_p} a_{pi} \,, \quad p = 0, \ldots, r \,.$$

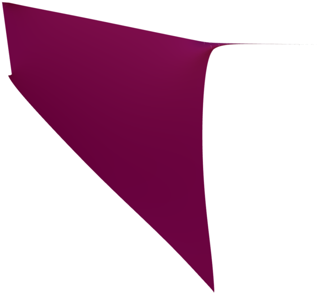
Following Mavriplis (1989), fit the sequence $\{a_p\}_{p=1}^{r}$ to an exponential decay of the form

$$\hat{a}(p) = c \exp(-\sigma p)$$

using an L^1-minimization. Explicit solution of their continuation result leads to the **problem-agnostic error and spectral decay indicator**

$$\varepsilon(c,\sigma) = \sqrt{\frac{2a_r^2}{2r+1} + c^2 \exp(\sigma)\, E_1((2r+3)\sigma)} \,.$$

Computational problems



The mixed version of the Hodge Laplacian is

$$\boldsymbol{\sigma} = \boldsymbol{\delta} \boldsymbol{u}, \quad \mathbf{d}\boldsymbol{\sigma} + \boldsymbol{\delta}\mathbf{d}\boldsymbol{u} + \boldsymbol{p} = \boldsymbol{f}, \quad P_{\mathfrak{H}_k} \boldsymbol{u} = 0\,,$$

and the corresponding weak form is the problem: find $\boldsymbol{u} \in \Lambda^k, \boldsymbol{\sigma} \in \Lambda^{k-1}, \boldsymbol{p} \in \mathfrak{H}^k$ such that

$$\int_\Omega \boldsymbol{\tau} \wedge \star\boldsymbol{\sigma} - \int_\Omega \mathbf{d}\boldsymbol{\tau} \wedge \star\boldsymbol{u} = -\int_{\partial\Omega} \mathrm{Tr}\,(\boldsymbol{\tau} \wedge \star\boldsymbol{u}) \qquad \forall \boldsymbol{\tau} \in \Lambda^{k-1}\,,$$

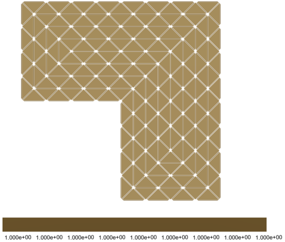
$$\int_\Omega \boldsymbol{v} \wedge \star\mathbf{d}\boldsymbol{\sigma} + \int_\Omega \mathbf{d}\boldsymbol{v} \wedge \star\mathbf{d}\boldsymbol{u} + \int_\Omega \boldsymbol{v} \wedge \star\boldsymbol{p} = \int_\Omega \boldsymbol{v} \wedge \star\boldsymbol{f} + \int_{\partial\Omega} \mathrm{Tr}\,(\boldsymbol{v} \wedge \star\mathbf{d}\boldsymbol{u}) \qquad \forall \boldsymbol{v} \in \Lambda^k\,,$$

$$\int_\Omega \boldsymbol{q} \wedge \star\boldsymbol{u} = 0 \qquad \forall \boldsymbol{q} \in \mathfrak{H}^k\,.$$

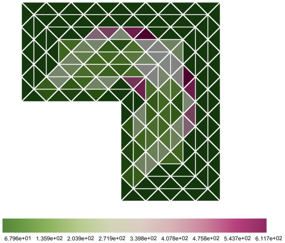
For the example in $n = 2$ in the following $k = 2$, $\star\boldsymbol{u} = 0$ on $\partial\Omega$, $\dim\mathfrak{H}^2 = 0$ and $\boldsymbol{f} \in C^0\Lambda^k(\Omega)$ non-differentiable at radius 0.25 and 0.75 about the origin.

For the example in $n = 3$ in the following $k = 3$, $\star\boldsymbol{u} = \bar{u}$ on $\partial\Omega$, $\dim\mathfrak{H}^3 = 0$, and $\boldsymbol{f} = 1$.

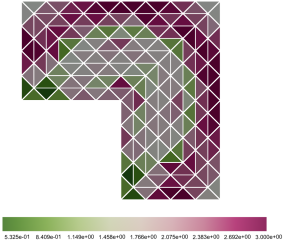
Hodge Laplacian ($k = 2$, $\boldsymbol{\sigma} \in \mathcal{P}_r^- \Lambda^1$, $\boldsymbol{u} \in \mathcal{P}_r^- \Lambda^2$)



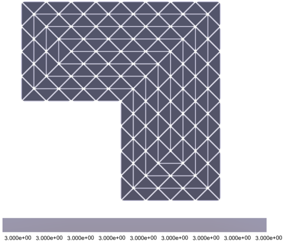
Subsimplices supporting dofs



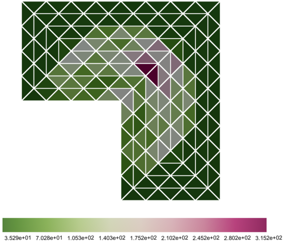
Error for $\boldsymbol{\sigma} \in \mathcal{P}^- \Lambda^1$



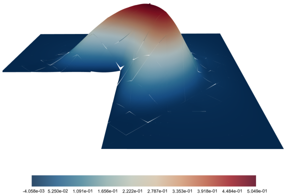
Cumulative decay rate



Polynomial order distribution

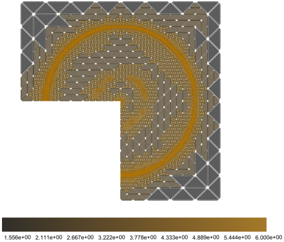


Error for $\boldsymbol{u} \in \mathcal{P}^- \Lambda^2$

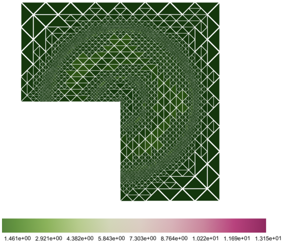


Solution field $\boldsymbol{u} \in \mathcal{P}^- \Lambda^2$

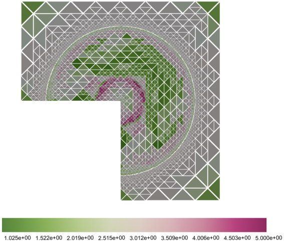
Hodge Laplacian ($k = 2$, $\boldsymbol{\sigma} \in \mathcal{P}_r^- \Lambda^1$, $\boldsymbol{u} \in \mathcal{P}_r^- \Lambda^2$)



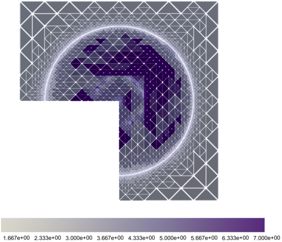
Subsimplices supporting dofs



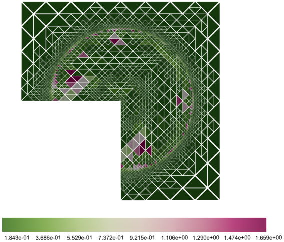
Error for $\boldsymbol{\sigma} \in \mathcal{P}^- \Lambda^1$



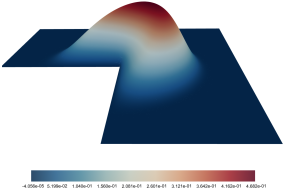
Cumulative decay rate



Polynomial order distribution



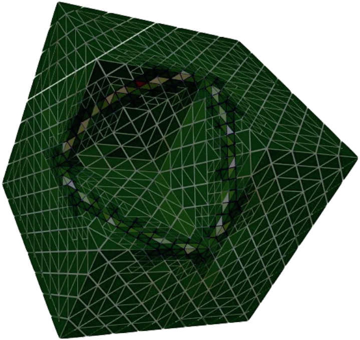
Error for $\boldsymbol{u} \in \mathcal{P}^- \Lambda^2$



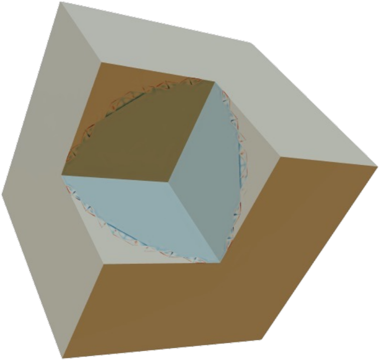
Solution field $\boldsymbol{u} \in \mathcal{P}^- \Lambda^2$

Fichera corner

Hodge Laplacian for $k = n = 3$,
$\boldsymbol{\sigma} \in \mathcal{P}^{-}\Lambda^{2}$, $\boldsymbol{u} \in \mathcal{P}^{-}\Lambda^{3}$, $\boldsymbol{f} = 0$,
$\star\boldsymbol{u} = 1$ inside the intersection of the unit ball with $\partial\Omega$, zero otherwise.



Refined mesh with error in $\boldsymbol{u}$



$\boldsymbol{u} \in \mathcal{P}^{-}\Lambda^{3}$

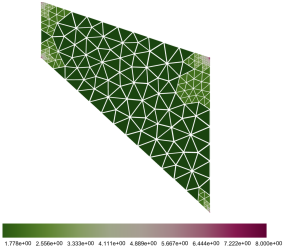
The weak form of the elasticity problem is: find
$(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{p}) \in H\Lambda^{n-1}(\Omega, \mathbb{R}^n) \times L_2\Lambda^n(\Omega, \mathbb{R}^n) \times L_2\Lambda^n(\Omega, \mathbb{R}^n \wedge \mathbb{R}^n)$ such that

$$
\begin{array}{ccccccll}
\langle \boldsymbol{\tau}, \mathbb{D}\boldsymbol{\sigma} \rangle & - & \langle \mathrm{d}\boldsymbol{\tau}, \boldsymbol{u} \rangle & - & \langle S\boldsymbol{\tau}, \boldsymbol{p} \rangle & = & -\langle \boldsymbol{\tau}, \boldsymbol{u} \rangle_{\partial\Omega}\,, & \forall \boldsymbol{\tau} \in H\Lambda^{n-1}(\Omega, \mathbb{R}^n)\,, \\
 & & & & -\langle \boldsymbol{v}, \mathrm{d}\boldsymbol{\sigma} \rangle & = & \langle \boldsymbol{v}, \boldsymbol{b} \rangle\,, & \forall \boldsymbol{v} \in L_2\Lambda^n(\Omega, \mathbb{R}^n)\,, \\
 & & & & -\langle \boldsymbol{q}, S\boldsymbol{\sigma} \rangle & = & 0\,, & \forall \boldsymbol{q} \in L_2\Lambda^n(\Omega, \mathbb{R}^n \wedge \mathbb{R}^n)\,.
\end{array}
$$

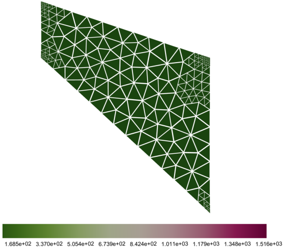
According to Arnold et al. $\boldsymbol{\sigma} \in \mathcal{P}_{r+1}\Lambda^1$, $\boldsymbol{u} \in \mathcal{P}_r\Lambda^2$, $\boldsymbol{p} \in \mathcal{P}_r\Lambda^2$ is a stable discretization for this problem.

For the plane strain Cook's membrane problem: $\star\boldsymbol{u} = 0$ (left), $(\star\boldsymbol{\sigma})(\boldsymbol{n}) = 0$ (top & bottom), $(\star\boldsymbol{\sigma})(\boldsymbol{n}) = (0, 100/16)^{\mathrm{T}}$ (right), $E = 250$, $\nu = 0.5$.

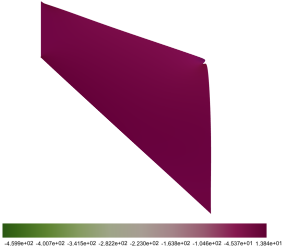
Hellinger-Reißner Elasticity ($\nu = 0.5$, weak symm.)



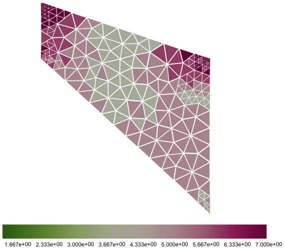
Subsimplices supporting dofs



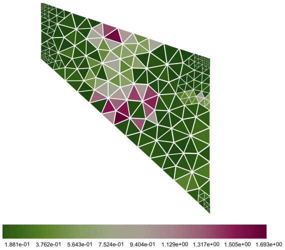
Error for $\boldsymbol{\sigma} \in \mathcal{P}\Lambda^1$



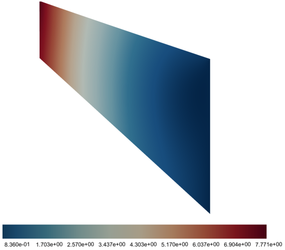
Solution field $\mathrm{tr}\boldsymbol{\sigma}$



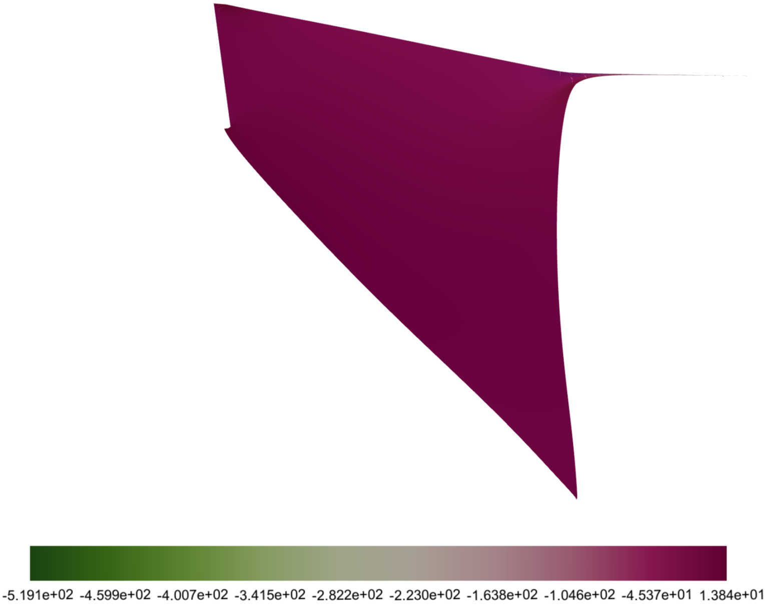
Polynomial order distribution

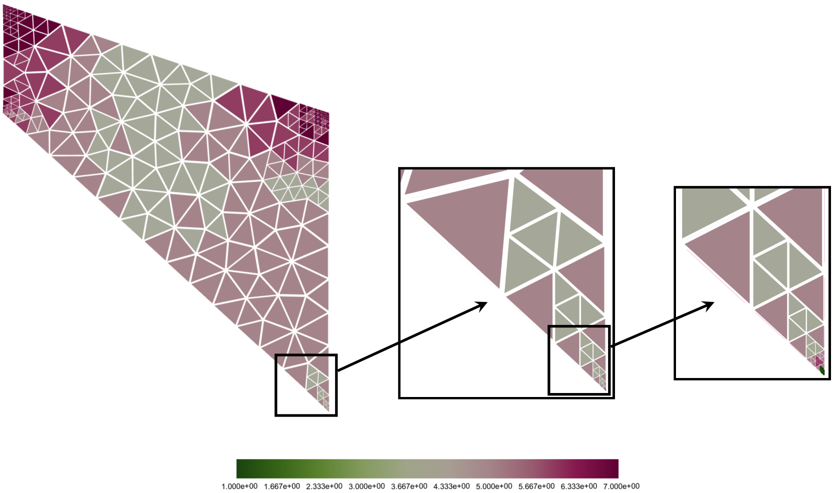


Error for $\boldsymbol{u} \in \mathcal{P}\Lambda^2$



Solution field $\boldsymbol{u} \in \mathcal{P}\Lambda^2$


-5.191e+02 -4.599e+02 -4.007e+02 -3.415e+02 -2.822e+02 -2.230e+02 -1.638e+02 -1.046e+02 -4.537e+01 1.384e+01


1.000e+00 1.667e+00 2.333e+00 3.000e+00 3.667e+00 4.333e+00 5.000e+00 5.667e+00 6.333e+00 7.000e+00

Implementation

julia

Features of the math kernel

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Much of the code is templated (using julia's macros and generated functions) and optimized for minimal heap allocation. As a result, most low-level loops are unrolled.

Dense polynomial kernel

- Lexicographic storage for all n and total degree p
- Entirely templated multivariate Horner scheme
- Operations: evaluation, chaining, multiplication, addition

Algebraic kernel

- Piecewise-polynomial immersions $\varphi : \mathcal{M} \to \mathcal{N}$, non-affine FEEC to all n and k
- Highly optimized pointwise operations for vector-space-valued differential forms $\mathrm{Alt}^k(T_P^*\mathcal{M}; W)$ and homogeneous multivectors $\mathrm{Alt}^k(T_P\mathcal{M}; W)$:

  $\wedge,\ \dot{\wedge},\ \overset{\otimes}{\wedge},\ \star_{\boldsymbol{g}},\ S,\ \mathrm{Alt}^k(V; W) \to \mathbb{R}$
- and for W-valued polynomial forms: $\mathbf{d}$, $\boldsymbol{\kappa}$, φ^*, φ_*, Tr_f
- Derived operations: $\boldsymbol{\delta}$, div, covariant grad, etc.
- Yes, tensors are part of this: $\mathfrak{T}^1_1 = \mathrm{Alt}^1(T_P^*\mathcal{M}; T_P\mathcal{M})$ and more...
- **Coefficients are stored in statically-sized $\binom{n}{k}$ arrays and allocated on the stack.**

Integration over codimension 0, codimension 1 (think: local Stokes theorem)

Features of the FE kernel

Basis generation

- Ciarlet-triplet representation $(T, \mathcal{P}, \Sigma)$
- Generation of the basis for $\mathcal{P}$ in BigFloat from Σ on the reference element
- Gram-Schmidt hierarchicalization with optional orthogonalization schemes

Mesh kernel

- Hierarchical simplicial complex,
- Allows for component subdomains, boundary subdomains,
- Currently reads gmsh files.

Weak form kernel and interface

- Define weak forms by interfacing,
- Multiple fields (Bubnov & Petrov), compat. ensured during refinement,
- User-defined geometry computations (think: material laws, coefficients, known fields),
- Strong boundary conditions via user-defined projectors,
- Full control without going deep into the code.

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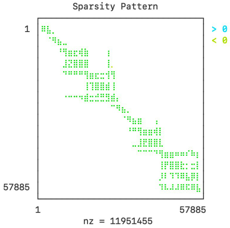
Integration and assembly

- Hierarchical numerical integration schemes,
- Updating assembly using dynamic graph structure, performs minimal degree permutation of structurally-symmetric system, and converts to SparseMatrixCSC,
- Each subdomain can be assigned a different weak form,
- Can assemble at least a 2.5M $\times$ 2.5M system for a mixed *hp*-problem efficiently in serial on a consumer-level laptop computer.



Solver

- Currently, we use only lufact from SuiteSparse for reliability and testing purposes,
- We know that this does not scale and therefore are working on an ILU0-preconditioned gmres with projected initial guess.

Avenues for parallelization and distribution

The code is currently at the stage linear-serial, however everything was designed with nonlinear-heterogeneous-distributed in mind

- Using julia's Distributed.Future and Distributed.RemoteChannel infrastructure, **all datastructures can be simply distributed across nodes along the lines of aforementioned component subdomains while using a coordinating master node**.
- Smaller, shared-memory operations can be multithreaded on workers belonging to a node.
- Each node can assemble its contribution to a system matrix or its contribution to an RHS (matrix-free) using shared-memory parallel reduction.
- Distributed parallel reduction can then be done in a similar fashion.
- **Heterogeneous compute architectures can be employed, GPU/Coprocessor $\rightarrow$ CPU $\rightarrow$ Node $\rightarrow$ Cluster (entirely in julia).**
- Distributed solvers can be interfaced, e.g. those of PETSc.

Outlook

Improvements to the **algebraic kernel**

- **Lorentzian manifolds & FE Clifford Algebra** (Gilette, Holst)
- Relativistic problems on $\mathcal{M} \subseteq \mathbb{R}^{n+1}$ dimensions (Salamon, et al.)

Improvements to the **refinement & assembly kernel**

- Intensify stack vs. heap allocation
- Parallelization & distribution

More applications to **real-world problems**

- Heterogeneous materials
- Complex material laws

Publish the software under an appropriate open-source license