University of the Philippines Diliman

MATHEMATICS 23
Elementary Analysis III

Course Module

Institute of Mathematics

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Contents

1 Functions of More Than One Variable **1**

1.1 Introduction 1
1.2 Limits and Continuity 9
1.3 Partial Derivatives 17
1.4 Differentiability, Differentials and Local Linear Approximation 25
1.5 Chain Rule 33
1.6 Directional Derivatives and Gradients 39
1.7 Relative Extrema of Functions of Two Variables 47
1.8 Absolute Extrema of Functions of More than One Variable 52
1.9 Parametric Surfaces 61

2 Multiple Integration **69**

2.1 Double Integrals over Rectangular Regions 69
2.2 Double Integrals Over General regions 74
2.3 Double Integrals in Polar Coordinates 84
2.4 Applications of Double Integrals 93
2.5 Triple Integrals 104
2.6 Triple Integrals in Cylindrical Coordinates 114
2.7 Triple Integrals in Spherical Coordinates 122
2.8 Applications of Triple Integrals 130

3 Vector Calculus **137**

3.1 Scalar and Vector Fields 137
3.2 Divergence and Curl 142
3.3 Conservative Vector Fields 144
3.4 Line Integrals of Scalar Fields 152
3.5 Line Integrals of Vector Fields 160
3.6 Fundamental Theorem of Line Integrals 164
3.7 Green's Theorem 171
3.8 Surface Integrals of Scalar Fields 179
3.9 Surface Integrals of Vector Fields 186
3.10 Stokes' Theorem and Gauss's Divergence Theorem 192

Chapter 1

Functions of More Than One Variable

1.1 Introduction

In this section, we extend the notion of functions in one variable to real-valued functions of two or more variables. In particular, we identify the domain and range of some multivariate functions. We also determine the graph of some multivariate functions and introduce the notion of contour plots as an aid to visualize the graph of a multivariate functions.

Definition 1.1.1. Let $D \subseteq \mathbb{R}^2$. A ***function*** f ***of two variables*** x and y is a correspondence from the set D to $\mathbb{R}$ such that each element $(x, y) \in D$ is associated to a unique real number $f(x, y)$.

Example 1.1.1. Suppose a particle, with mass m, moves with acceleration a. Newton's second law of motion tells us that the net force F acting upon the particle can be computed using m and a. Hence, F is a function of m and a. In particular, the formula is

$$F = f(m, a) = ma.$$

The idea of functions of two variables can be extended further to functions depending on three or more variables.

Definition 1.1.2. Let $D \subseteq \mathbb{R}^n$, where $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$.

1. A ***function*** f ***of*** n ***variables*** $x_1, x_2, \ldots, x_{n-1}$ and x_n, is a correspondence from the set D to $\mathbb{R}$ such that each point $(x_1, x_2, \ldots, x_n) \in D$ is associated to a unique real number $f(x_1, x_2, \ldots, x_n)$.
2. The set D is called the ***domain*** of f. The domain of f is denoted by $\operatorname{dom} f$.
3. The set of all resulting values $f(x_1, x_2, \ldots, x_n)$ for $(x_1, x_2, \ldots, x_n) \in \operatorname{dom} f$ is called the ***range*** of f. The range of f is denoted by $\operatorname{ran} f$.

Definition 1.1.3. Let $D_1, D_2 \subseteq \mathbb{R}^n$. Two functions $f : D_1 \to \mathbb{R}$ and $g : D_2 \to \mathbb{R}$ are equal if and only if $D_1 = D_2$ and for all $(x_1, \ldots, x_n) \in D_1$, we have $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$.

Remark. If no restrictions are stated explicitly, then it is understood that the domain comprises all points $(x_1, \ldots, x_n)$ for which $f(x_1, \ldots, x_n)$ is a unique real number.

Example 1.1.2. If a rectangular box has length l, width w and height h units, then its volume V can be computed using these three variables. We say that V is a function of l, w and h, and write this as $V = f(l, w, h)$. In particular, the formula is given by

$$V = f(l, w, h) = lwh.$$

A physical restriction to these variables is that they should be positive. Hence, the domain is

$$\{(l, w, h) \in \mathbb{R}^3 \,:\, l > 0,\ w > 0,\ h > 0\}.$$

Example 1.1.3. Let $f(x, y) = x^2 + \sqrt[3]{xy}$. Find $f(2, -4)$, $f(1, 0)$, $f(t, t^2)$ and $f(2y^2, 4y)$.
Solution: Note that the domain of f is the entire xy-plane. By substitution, we get

- $f(2, -4) = 2^2 + \sqrt[3]{2(-4)} = 4 - 2 = 2$
- $f(1, 0) = 1^2 + \sqrt[3]{1(0)} = 1$
- $f(t, t^2) = t^2 + \sqrt[3]{t(t^2)} = t^2 + t$
- $f(2y^2, 4y) = \left(2y^2\right)^2 + \sqrt[3]{2y^2(4y)} = 4y^4 + 2y$

Example 1.1.4. Identify and sketch the domain of $f(x, y) = \dfrac{1}{\sqrt{x^2 - y}}$.
Solution: For the expression to be a real number, the radicand $x^2 - y$ in the denominator must be positive. That is, $x^2 > y$. Hence, the domain is

$$\{(x, y) \in \mathbb{R}^2 \,:\, y < x^2\}.$$

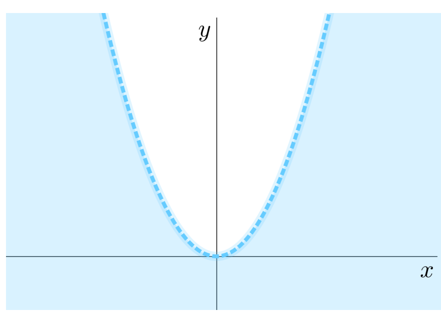


Figure 1.1: Graph of $y < x^2$

The parabola $y = x^2$ divides the xy-plane into two: all points on or above it, and all points below it. The point $(0, -1)$, which is a point below the parabola, satisfies $y < x^2$. In fact, all points below the parabola (and only them) satisfy the inequality $y < x^2$. Hence, the domain includes all points below the parabola. The parabola being a *dashed* curve indicates that points on it are not part of the domain.

Example 1.1.5. Identify and sketch the domain of $f(x, y) = \sin^{-1}(x - 1)$. Identify the range of f.
Solution: The inverse sine function is defined only for values in the interval $[-1, 1]$. Thus, $-1 \leqslant x - 1 \leqslant 1$, or that $0 \leqslant x \leqslant 2$. Hence, the domain is

$$\{(x, y) \in \mathbb{R}^2 \,:\, 0 \leqslant x \leqslant 2\}.$$

Graphically, the domain consists of points between the lines $x = 0$ and $x = 2$, including these lines. Meanwhile, the range of f is the interval $[-\pi/2, \pi/2]$.

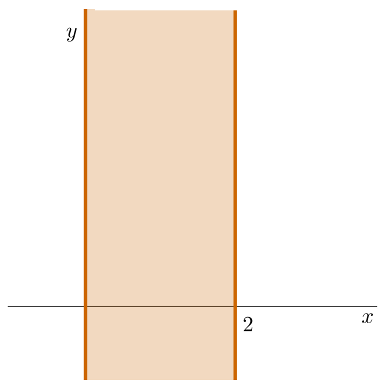


Figure 1.2: Domain of $f(x, y) = \sin^{-1}(x - 1)$

Example 1.1.6. Identify the domain of $g(x, y, z) = \ln(1 - x^2 - y^2 - z^2)$.
Solution: The natural logarithm is defined only for positive values. Thus, $1 - x^2 - y^2 - z^2 > 0$ or that $x^2 + y^2 + z^2 < 1$. Hence, the domain is

$$\{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 < 1\}.$$

Graphically, the domain is the set of points in $\mathbb{R}^3$ inside the unit sphere centered at the origin, excluding the sphere itself.

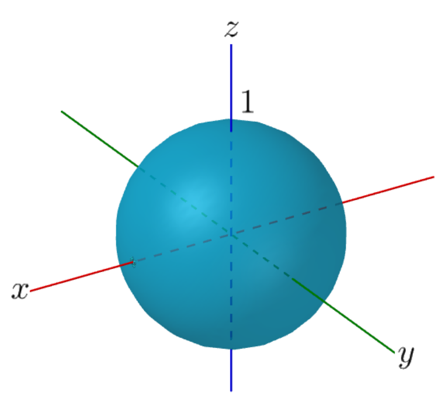


Figure 1.3: Domain of $g(x, y, z) = \ln(1 - x^2 - y^2 - z^2)$

Graphs of Functions of Two Variables

Suppose f is a function of two variables x and y. The ***graph*** of f is the set of all points (x, y, z) in the three-dimensional space such that $(x, y) \in \operatorname{dom} f$ and $z = f(x, y)$. The graph of a function in two variables is a surface in the three-dimensional space $\mathbb{R}^3$.

Example 1.1.7. Sketch the graph of the following functions.

1. $f(x, y) = 2 - 2x - y$

   Solution: Note that $\operatorname{dom} f = \mathbb{R}^2$. The graph of f is the graph of the equation $z = 2 - 2x - y$, which is a plane with normal vector $\langle 2, 1, 1\rangle$. We can deduce from the graph that $\operatorname{ran} f = \mathbb{R}$.

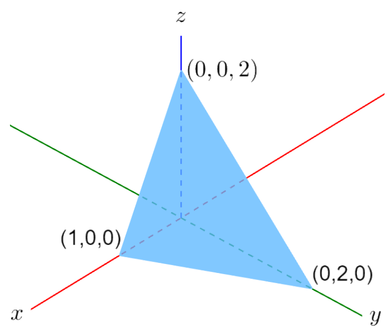


Figure 1.4: Graph of $f(x, y) = 2 - 2x - y$

2. $g(x, y) = \sqrt{x^2 + y^2}$

   Solution: The domain of g is clearly $\mathbb{R}^2$. The graph of g is the graph of $z = \sqrt{x^2 + y^2}$. Squaring both sides, we have

$$z^2 = x^2 + y^2,$$

   which represents a circular cone. Since the formula for g imposes that $z \geq 0$, the graph of g is the upper nappe of the cone. It follows that $\operatorname{ran} g = [0, +\infty)$.

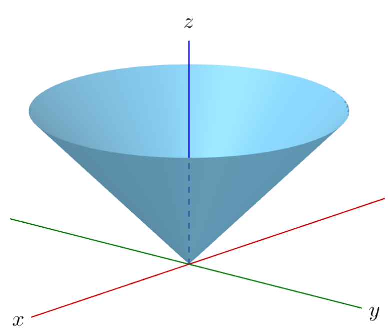


Figure 1.5: Graph of $g(x, y) = \sqrt{x^2 + y^2}$

Level Curves

Given a multivariate function, it is not always easy to visualize its graph. For instance, a function in variables x, y, z has a graph in the 4-dimensional real space. A useful and common method of picturing the graph of a function is by drawing a contour plot. As an example, a contour plot (or contour map) of a mountainous region depicts level curves showing the graduation in mountain elevation.

Let f be a function of two variables x and y, say $f(x,y) = 4x^2 + y^2$. Let us cut across the graph of f by a horizontal plane $z = k$, for several values of $k \in \mathbb{R}$.

$$\begin{array}{lcl@{\qquad\qquad}lcl} k=4 & : & 4x^2+y^2=4 & k=1 & : & 4x^2+y^2=1 \\ k=3 & : & 4x^2+y^2=3 & k=0 & : & \{(0,0)\} \\ k=2 & : & 4x^2+y^2=2 & k<0 & : & \emptyset \end{array}$$

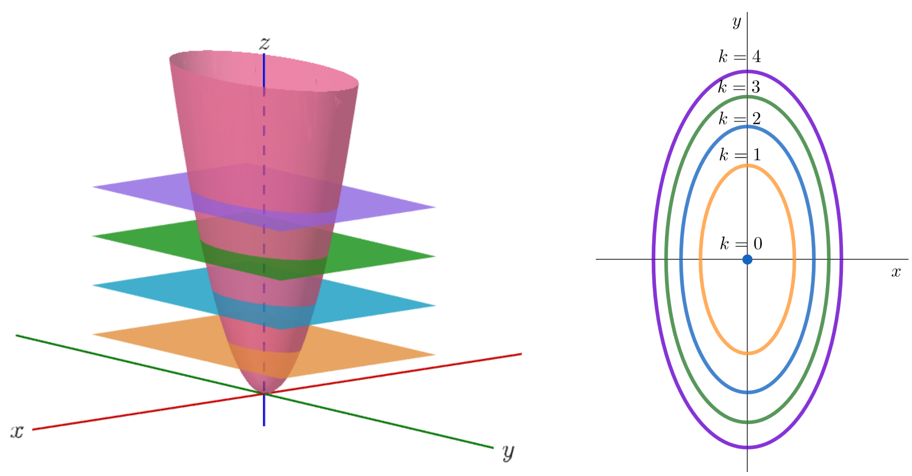


Figure 1.6: Level curves of $f(x,y) = 4x^2 + y^2$

For a specific k, the intersection is a curve on the $z = k$ plane such that $f(x,y) = k$. Projecting the intersection curves onto the xy-plane for several values of k, we get a ***contour plot*** of f. Each curve in the contour plot is called a ***level curve (with constant*** k***)***. The constant k is also called the *elevation* of the level curve. A contour plot of $f(x,y) = 4x^2 + y^2$ for $k = 0, 1, 2, 3, 4$ is shown above.

Question: Can two distinct level curves intersect? (*Answer: No. Why?*)

Example 1.1.8. Sketch a contour map of $f(x,y) = y^2 - x^2$ for $z = -10$, -5, 0, 5 and 10.

Solution: Note that the surface $z = y^2 - x^2$ is a hyperbolic paraboloid. The cross sections of the surface with the planes $z = -10$, -5, 0, 5, and 10 are hyperbolas. A contour map is shown below.

$$\begin{aligned} k = 10 &\ :\ y^2 - x^2 = 10 & \qquad k = -5 &\ :\ x^2 - y^2 = 5 \\ k = 5 &\ :\ y^2 - x^2 = 5 & \qquad k = -10 &\ :\ x^2 - y^2 = 1 \\ k = 0 &\ :\ x^2 - y^2 = 0 \end{aligned}$$

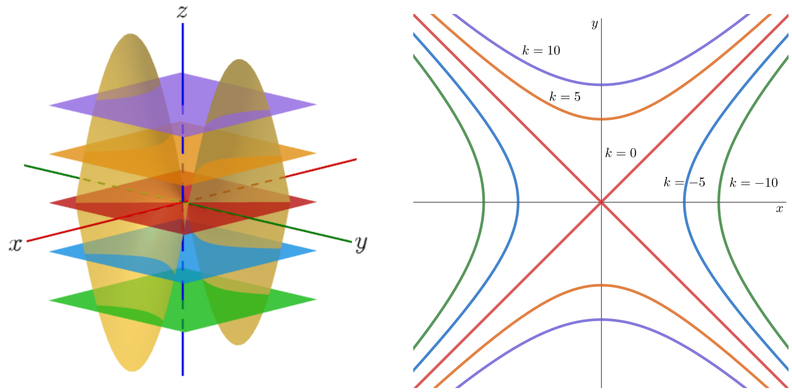


Figure 1.7: Level curves of Level curves of $f(x, y) = y^2 - x^2$

Example 1.1.9. The surface $f(x, y) = \sin^2 x + \frac{1}{4}y^2$ has the following graph.

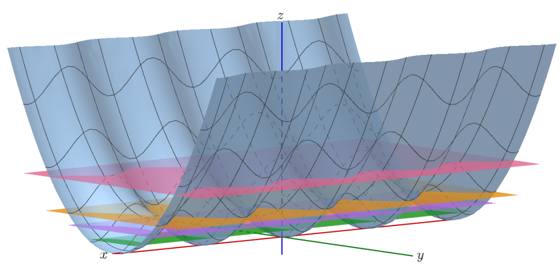


Figure 1.8: Graph of $f(x, y) = \sin^2 x + \frac{1}{4}y^2$

The following are the cross sections of the surface with several horizontal planes. A contour plot of the surface is then.

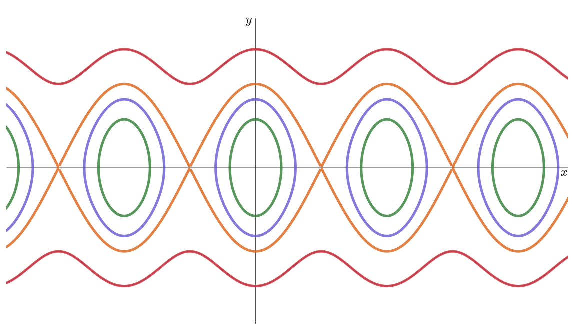


Figure 1.9: Cross sections of $f(x, y) = \sin^2 x + \frac{1}{4}y^2$

Remark. A set of level curves gives us an idea on how fast the function value changes.

Example 1.1.10. A contour map for $f(x, y) = 4x^2 + y^2$ is shown below, where the level curves $f(x, y) = k$ are taken over equal increments of k. Observe that as k becomes larger, so does the elliptic trace, and the level curves become closer together. So $f(x, y)$ changes more rapidly at the points (x, y) on the contour map where the level curves are closer together.

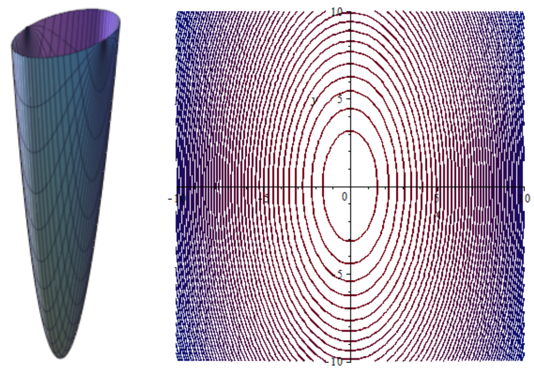


Figure 1.10: Graph and a contour map of $z = 4x^2 + y^2$

Level Surfaces

For functions of three variables, it is not easy to visualize their graphs since the graphs will lie on a four-dimensional space. However, its ***level surfaces*** are obtained by considering several values for $k \in \mathbb{R}$ such that $f(x, y, z) = k$, and these may give us some geometric insight into the function. Note that the term level surface is used as the counterpart of level curve for functions of three or more variables.

Example 1.1.11. Let $f(x, y, z) = x + y + z$. The level surfaces are equations of the form

$$x + y + z = k$$

which are planes with normal vector $\langle 1, 1, 1\rangle$. Illustrated below are a few of the level surfaces of $f(x, y, z)$.

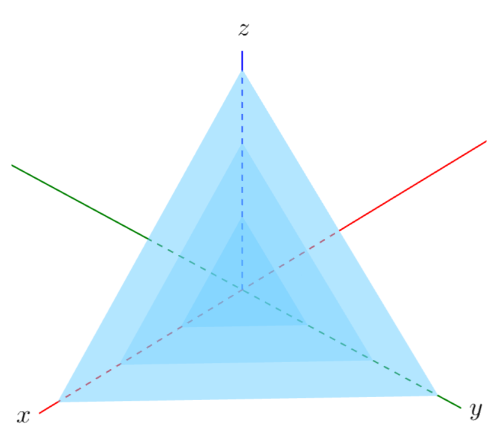


Figure 1.11: Level surfaces of $f(x, y, z) = x + y + z$

Example 1.1.12. Find the level surfaces of $f(x,y,z) = x^2 + y^2 + z^2$ at $k = 1, 4$, and 9.

Solution: The equation of the level surface at k is $x^2 + y^2 + z^2 = k$, which sketches a three-dimensional sphere centered at the origin of radius k when $k > 0$.

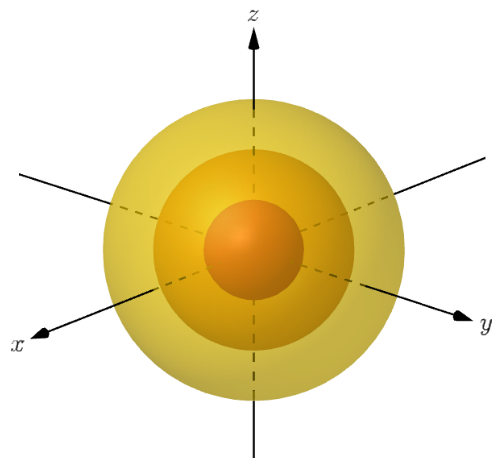


Figure 1.12: Level surfaces of $f(x,y,z) = x^2 + y^2 + z^2$

EXERCISES 1.1

I. Let $f(x,y) = x^2y + \sin x + \cos y$. Evaluate the following.

1. $f(t, 2t)$
2. $f(x+y, x-y)$
3. $f(x+h, y) - f(x,y)$
4. $f\left(\frac{x^2}{y}, x^3y\right)$

II. Let $f(x,y) = \sqrt{1 - x^2 - y^2}$ and $g(x,y) = \ln(4 - x^2 - y^2)$. Evaluate the following.

1. $g(1,-1) + g(-1,1)$
2. $f(2+s, 1-s)$
3. $f(r^2 - s, 2r + s)$
4. $g\left(f(1,0), f(0,1)\right)$

III. Let $f(x) = e^x \cos 2x$ and $g(x,y) = x^2y - \ln(x+y)$. Evaluate the following.

1. $g\left(f\left(\frac{\pi}{4}\right), 1\right)$
2. $f\left(g(0,1)\right)$
3. $g\left(0, f(\pi)\right)$
4. $f\Big(g(f(\frac{\pi}{4}), f(\pi))\Big)$

IV. Identify and sketch the domain of the following functions.

1. $f(x,y) = \sqrt{x+2}$
2. $f(x,y) = \sqrt{xy}$
3. $f(x,y) = \sqrt{y-x}\ln(x+y)$
4. $f(x,y) = \dfrac{1}{x^2y + y^3 - y}$
5. $f(x,y) = \sin^{-1} x + \cos^{-1} 2y$
6. $f(x,y) = \sin^{-1}(x + y - 1)$
7. $f(x,y) = \dfrac{\log(1 - x^2 - y^2)}{x^2 + y^2}$

8. $f(x,y) = \dfrac{\sqrt{4-x^2-y^2}}{\ln x}$

9. $f(x,y) = \sin^{-1}(|x|+|y|)$

10. $f(x,y,z) = e^{\sqrt{x^2+y^2-z}}$

11. $f(x,y,z) = xz\cos^{-1}(y^2-1)$

12. $f(x,y,z) = \cos^{-1}(x-1)\sin^{-1}(y^2-1)$

V. Sketch the graph of the following functions.

1. $f(x,y) = x+y$
2. $f(x,y) = y^3$
3. $f(x,y) = 4-x^2-4y^2$
4. $f(x,y) = \sqrt{1-x^2}$
5. $f(x,y) = 4-\sqrt{x^2+y^2}$

VI. Sketch a contour map for the following surfaces.

1. $f(x,y) = x+y$
2. $f(x,y) = x^2-y$
3. $f(x,y) = y^2-x^2$
4. $f(x,y) = 4-x^2-4y^2$
5. $f(x,y) = 4-\sqrt{x^2+y^2}$
6. $f(x,y,z) = x^2+4y^2-z$

1.2 Limits and Continuity

Recall that we have been introduced to limits and continuity of functions of a single variable. We extend these concepts to functions of two or more variables.

Limits

Let us investigate the values of $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$ as (x,y) approaches $(0,0)$. The table below shows the values of $f(x,y)$ corresponding to the x and y values on the first column and first row, respectively.

$x \backslash y$	-0.05	-0.01	-0.001	0	0.001	0.01	0.05
-0.050	-0.07500	-0.02885	-0.00300	0	0.00300	0.02885	0.07500
-0.010	-0.00577	-0.01500	-0.00297	0	0.00297	0.01500	0.00577
-0.001	-0.00006	-0.00030	-0.00150	0	0.00150	0.00030	0.00006
0.000	0	0	0	undefined	0	0	0
0.001	-0.00006	-0.00030	-0.00150	0	0.00150	0.00030	0.00006
0.010	-0.00577	-0.01500	-0.00297	0	0.00297	0.01500	0.00577
0.050	-0.07500	-0.02885	-0.00300	0	0.00300	0.02885	0.07500

Table 1.1: Values of $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$ as (x,y) approaches $(0,0)$

As (x, y) approaches $(0, 0)$, it seems that $f(x, y)$ approaches 0. As it will turn out later, this observation is correct. We shall say that, as (x, y) approaches to $(0, 0)$, the limit of $f(x, y)$ is 0. We write this as

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0.$$

Definition 1.2.1. Let f be a function of two variables defined on an open disk centered at (a, b), except possibly at (a, b). The ***limit*** of $f(x, y)$ as (x, y) approaches (a, b) is L, written as

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if for every small number $\varepsilon > 0$, there is a corresponding small number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

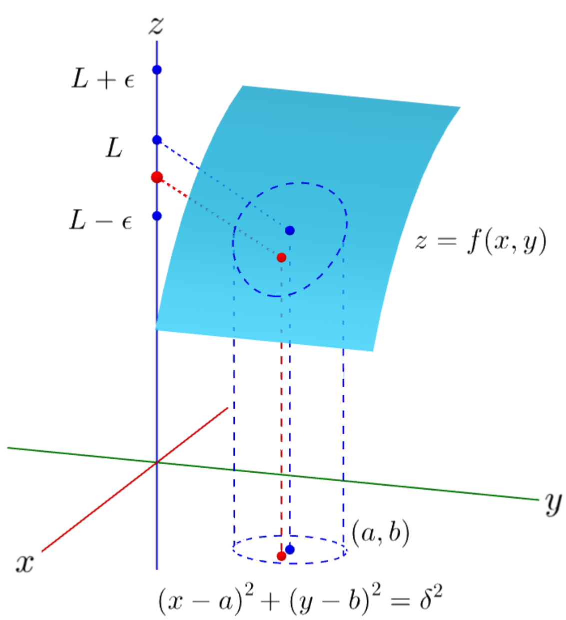


Figure 1.13: Graphical interpretation of $\lim_{(x,y)\to(a,b)} f(x, y) = L$

Example 1.2.1. Prove that $\lim_{(x,y)\to(0,0)} 4\sqrt{x^2 + y^2} = 0$.

Solution: Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{4}$. (This choice of δ will soon be evident from the computations.) The distance between (x, y) and the point $(0, 0)$ is given by $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$. Suppose

$$0 < \sqrt{x^2 + y^2} < \delta.$$

Then

$$\begin{aligned}|f(x,y)-L| &= |4\sqrt{x^2+y^2}-0| \\ &= 4\sqrt{x^2+y^2} \\ &< 4\delta \qquad \text{(By assumption)} \\ &= 4\left(\frac{\varepsilon}{4}\right) \qquad \text{(By the choice of } \delta) \\ &= \varepsilon.\end{aligned}$$

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} 4\sqrt{x^2+y^2}=0$.

Example 1.2.2. Prove that $\displaystyle\lim_{(x,y)\to(1,-1)}(3x+2y)=1$.

Solution: Let $f(x,y)=3x+2y$, and $L=1$. For any small number $\varepsilon>0$ we choose, we want to find a small number $\delta>0$, such that

$$|f(x,y)-L|=|3x+2y-1|<\varepsilon$$

whenever the distance between (x,y) and $(1,-1)$ is less than δ, that is, $\sqrt{(x-1)^2+(y+1)^2}<\delta$.

Note that $|3x+2y-1|=|3(x-1)+2(y+1)|\leqslant 3|x-1|+2|y+1|$, and we want this to be less than ε. Also, notice that

$$|x-1|=\sqrt{(x-1)^2}\leq\sqrt{(x-1)^2+(y+1)^2}<\delta, |y+1| \ = \sqrt{(y-1)^2}\leq\sqrt{(x-1)^2+(y+1)^2}<\delta.$$

Thus,

$$|3x+2y-1|\leq 3|x-1|+2|y+1|<3\delta+2\delta=5\delta.$$

Hence, by taking $\delta=\dfrac{\varepsilon}{5}$, we have

$$|3x+2y-1|<5\delta=5\left(\frac{\varepsilon}{5}\right)=\varepsilon.$$

Therefore, $\displaystyle\lim_{(x,y)\to(1,-1)}(3x+2y)=1$.

Remark. If the limit of $f(x,y)$ as (x,y) approaches (a,b) exists, then that limit is unique.

As in the case of limits for functions of one variable, there are rules that can simplify the computation of limits for multivariate functions. We have the following theorem.

Theorem 1.2.2. Let f and g be functions of two variables such that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = M.$$

Then the following hold:

1. $\displaystyle\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L \pm M$
2. $\displaystyle\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = LM$
3. $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, provided $M \neq 0$.

Note that a polynomial function in two variables x and y is a sum of monomial terms of the form $cx^m y^n$, where c is a real constant, and m and n are nonnegative integers. Meanwhile, a rational function (in two variables) is a quotient of two polynomial functions. We have the following result.

Theorem 1.2.3. If f is a polynomial or rational function and $(a,b) \in \operatorname{dom} f$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Example 1.2.3. Let us illustrate the above theorem with the following computations:

1. $\displaystyle\lim_{(x,y)\to(2,1)} (3x^2y - 2y^2) = 12 - 2 = 10$
2. $\displaystyle\lim_{(x,y)\to(-1,0)} \frac{2+y^2}{xy - 2x^2} = \frac{2}{-2} = -1$

Limit Along a Curve

Recall that on the real number line, one can approach a number from only two directions, from the right and from the left. On the other hand, on xy-plane, there are infinitely many ways one can approach a point (a,b).

In the following definition we shall have the notion of a limit of the function along a curve as (x,y) approaches a point, which is the extension of the notion of one-sided limits for functions of one variable. We also recall that for functions of one variable, if the limit from the left is not equal to the limit from the right, then the limit does not exist. The case for functions of two variables is analogous.

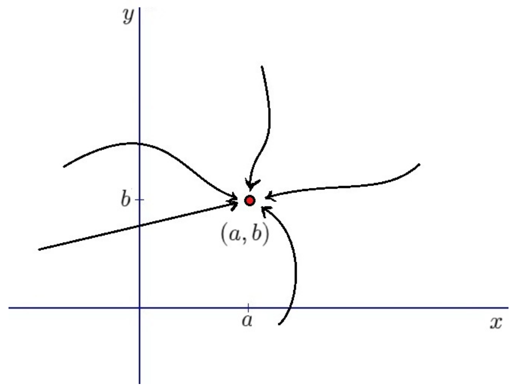


Figure 1.14: The point (a, b) can be approached in infinitely many ways.

Definition 1.2.4. Let C be a smooth curve that passes through the point (x_0, y_0). We define the ***limit of*** $f(x, y)$ ***as*** (x, y) ***approaches*** (x_0, y_0) ***along*** C as follows:

1. If C has equation $y = g(x)$, then

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } C}} f(x, y) = \lim_{x\to x_0} f(x, g(x)).$$

2. If C has equation $x = g(y)$, then

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } C}} f(x, y) = \lim_{y\to y_0} f(g(y), y).$$

Remark. If the limits of $f(x, y)$ as (x, y) approaches a point along two different curves are not equal, or if the limit along a curve does not exist (provided that the curve lies in the domain of f), then the limit of f does not exist.

Example 1.2.4. Determine whether $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x^2 + y^2}$ exists or not.

Solution: Consider the curve C to be the line $y = x$. Along C, we have

$$\frac{1}{x^2 + y^2} = \frac{1}{x^2 + x^2} = \frac{1}{2x^2}.$$

Thus,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}} \frac{1}{x^2 + y^2} = \lim_{x\to 0} \frac{1}{2x^2} = +\infty,$$

and the limit does not exist.

Example 1.2.5. Show that $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 - y^2}{(x-1)^2 + y^2}$ does not exist.

Solution: We consider the limit along the x-axis ($y = 0$) and then the line $x = 1$. It is important to note that both of these curves pass through the point $(1, 0)$. Along $y = 0$, we have.

$$\lim_{\substack{(x,y)\to(1,0)\\ \text{along } x\text{-axis}}} \frac{(x-1)^2 - y^2}{(x-1)^2 + y^2} = \lim_{x\to 1} \frac{(x-1)^2 - 0^2}{(x-1)^2 + 0^2} = \lim_{x\to 1} \frac{(x-1)^2}{(x-1)^2} = 1.$$

On the other hand, along $x = 1$, we have

$$\lim_{\substack{(x,y)\to(1,0)\\ \text{along } x=1}} \frac{(x-1)^2 - y^2}{(x-1)^2 + y^2} = \lim_{y\to 0} \frac{0^2 - y^2}{0^2 + y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1.$$

Since the limits along the two curves are not equal, then $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 - y^2}{(x-1)^2 + y^2}$ does not exist.

Example 1.2.6. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$ does not exist.

Solution: Let C_1 be the line $y = x$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} \frac{xy^2}{x^2 + y^4} = \lim_{x\to 0} \frac{x(x)^2}{x^2 + x^4} = \lim_{x\to 0} \frac{x}{1 + x^2} = 0.$$

Let C_2 be the parabola $x = y^2$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}} \frac{xy^2}{x^2 + y^4} = \lim_{y\to 0} \frac{y^2(y^2)}{y^4 + y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}.$$

Since the limits along C_1 and C_2 are distinct, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$ does not exist.

Example 1.2.7. Determine $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}$.

Solution: Let C_1 be a non-vertical line through the origin. That is, $C_1 : y = mx$, for some $m \in \mathbb{R}$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} \frac{3x^2y}{x^2 + y^2} = \lim_{x\to 0} \frac{3x^2(mx)}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{3mx}{1 + m^2} = 0.$$

Along the curves $C_2 : y = x^2$, $C_3 : x = y^2$, $C_4 : y = x^3$, $C_5 : x = y^3$, it can easily be verified that the limits of the function as $(x, y) \to (0, 0)$ are also zero. These computations seem to indicate that the limit is zero. However, these are not enough to say that the limit is zero. We need to prove that it by definition.

Let us show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0$. Let $\varepsilon > 0$. We find $\delta > 0$ such that

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| = \left| \frac{3x^2y}{x^2 + y^2} \right| < \varepsilon$$

whenever $0 < \sqrt{x^2+y^2} < \delta$. Note that $\left|\dfrac{3x^2y}{x^2+y^2}\right| = \dfrac{3x^2|y|}{x^2+y^2}$ and that $x^2 \leq x^2+y^2$. Thus

$$\frac{3x^2|y|}{x^2+y^2} \leq \frac{3x^2|y|}{x^2} = 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2},$$

which we want to be less than ε. Thus, if we take $\delta = \dfrac{\varepsilon}{3}$, then

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| \leq 3\sqrt{x^2+y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

Therefore, the limit is indeed zero.

Continuity

Recall that a single-variable function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$, is said to be continuous at the point $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

We extend the notion of continuity to real-valued functions of several variables.

Definition 1.2.5. Let $D \subseteq \mathbb{R}^2$ and $(a,b) \in D$. A function $f : D \to \mathbb{R}$ is said to be ***continuous at*** (a,b) if

1. $f(a,b)$ is defined,
2. $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ exists, and
3. $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Remarks.

1. The function f is said to be ***continuous on the set*** $D \subseteq \mathbb{R}^2$ if it is continuous at all points on D.
2. If f is continuous at all points on the xy-plane, we say that f is ***continuous everywhere***.
3. If a function is not continuous at a point, then we say it is ***discontinuous*** at that point. Moreover, if the limit exists, the discontinuity is ***removable***. Otherwise, the discontinuity is ***essential***.

The following theorem illustrates ways of producing continuous functions from known continuous functions.

Theorem 1.2.6. Let $h : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Fix $(a, b) \in \mathbb{R}^2$.

1. Sums, differences and products of continuous functions are continuous.
2. Quotients of continuous functions are continuous, except at the points where the denominator is zero.
3. If h is continuous at (a, b) and g is continuous at $h(a, b)$, then $(g \circ h)(x, y)$ is continuous at (a, b).

Example 1.2.8. Use the above theorem to show that the following functions are continuous everywhere.

1. $f(x, y) = xe^{xy} + y^{2/3}$
2. $g(x, y) = \cos(x^3 + y^3) - |x^3 + y^3|$
3. $h(x, y) = \dfrac{xy}{1 + x^2 + y^2}$

Solution: The polynomials x and xy are continuous. Also, e^x and $y^{2/3}$ are continuous. By Theorem 1.2.6 (3), the composition e^{xy} is continuous. By Theorem 1.2.6 (1), xe^{xy} is continuous since it is a product of continuous functions. Therefore, by Theorem 1.2.6 (1) again, $f(x, y)$ is continuous since it is a sum of continuous functions.

Likewise, g is continuous since it is a difference and composition of continuous functions $\cos x$, $x^3 + y^3$, and $|x|$. Finally, h is continuous being a quotient of continuous functions $1 + x^2 + y^2$ and xy and the denominator $1 + x^2 + y^2$ is never zero.

Example 1.2.9. Evaluate the following limits.

1. $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2 + y^2}$
2. $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^3y^2}{1 - xy}$

Solution:

1. The function $f(x, y) = \dfrac{xy}{x^2 + y^2}$ is continuous at $(-1, 2)$, being a quotient of continuous functions at $(-1, 2)$. Since the denominator is non-zero, the limit of $f(x, y)$ is $f(-1, 2) = -\dfrac{2}{5}$.
2. Similarly, $f(x, y) = \dfrac{x^3y^2}{1 - xy}$ is continuous at $(2, 2)$, being a quotient of continuous functions and the denominator is not zero at $(2, 2)$. Therefore, the limit is $f(2, 2) = -\dfrac{32}{3}$.

EXERCISES 1.2

I. Show that the following limits do not exist.

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2-y^2}{x^2+y^2}$
2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^4+y^2}$
3. $\displaystyle\lim_{(x,y)\to(2,0)} \frac{2(x-2)y^2}{5(x-2)^2+y^4}$
4. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\sin(x-y)}{1-xy}$

II. Determine whether the discontinuity of the following functions at the point P is removable or essential.

1. $f(x,y) = \dfrac{1-e^{-2x^2-2y^2}}{x^2+y^2}$, $P(0,0)$
2. $f(x,y) = (x^2+y^2)\ln(x^2+y^2)$, $P(0,0)$
3. $f(x,y) = \dfrac{2y^2-3xy}{\sqrt{x^2+y^2}}$, $P(0,0)$
   (*hint: convert to polar coordinates*)
4. $f(x,y) = \dfrac{x^3-4xy^2}{x^2+y^2}$, $P(0,0)$

1.3 Partial Derivatives

Recall that, for a function in one variable, its derivative measures the rate of change of the function with respect to the variable. Analogously, we may consider the rates of change of $f(x,y)$ in the x- and y- directions for functions f in variables x and y.

Let $z = f(x,y)$ be defined at (a,b). If we fix $y = b$, then $f(x,b)$ becomes a function of x alone. Geometrically, we obtain the curve of intersection of the plane $y = b$ and the graph of f. This curve on the plane $y = b$ has slope at (a,b) equal to $\left.\dfrac{d}{dx}f(x,b)\right|_{x=a}$.

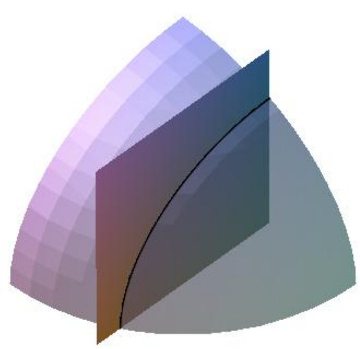


Figure 1.15: Curve of intersection of the plane $y = b$ and the graph of f

Similarly, if we fix $x = a$, then $f(a, y)$ becomes a function of y alone. Geometrically, we obtain the curve of intersection of the plane $x = a$ and the graph of f. This curve on the plane $x = a$ has slope at (a, b) equal to $\left.\dfrac{d}{dy} f(a, y)\right|_{y=b}$.

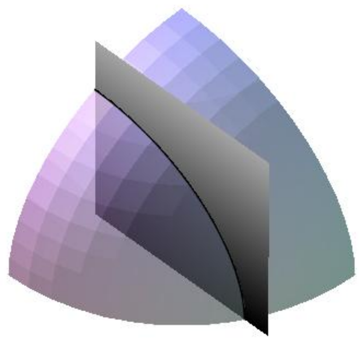


Figure 1.16: Curve of intersection of the plane $x = a$ and the graph of f

For a real-valued function $f(x)$, recall that the derivative of f, with respect to the variable x, is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

provided that this limit exists. The following definition provides an analog for functions of more than one variable.

Definition 1.3.1. Let $z = f(x, y)$.

1. The ***partial derivative of*** f ***with respect to*** x at any point (x, y) is

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

provided that this limit exists.

2. The ***partial derivative of*** f ***with respect to*** y at any point (x, y) is

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y},$$

provided that this limit exists.

Example 1.3.1. Find f_x and f_y at any point given that $f(x, y) = 2x^2y^2 + 2y + 4x$.

Solution: Consider

$$\begin{aligned}
\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x} &= \frac{2(x+\Delta x)^2\,y^2+2y+4(x+\Delta x)-(2x^2y^2+2y+4x)}{\Delta x}\\
&= \frac{2x^2y^2+4x(\Delta x)\,y^2+2(\Delta x)^2\,y^2+2y+4x+4\Delta x-2x^2y^2-2y-4x}{\Delta x}\\
&= \frac{4x(\Delta x)\,y^2+2(\Delta x)^2\,y^2+4\Delta x}{\Delta x}\\
&= 4xy^2+2y^2(\Delta x)+4.
\end{aligned}$$

Thus,

$$\begin{aligned}
f_x(x,y) &= \lim_{\Delta x\to 0}[4xy^2+2y^2(\Delta x)+4]\\
&= 4xy^2+4.
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
f_y(x,y) &= \lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}\\
&= \lim_{\Delta y\to 0}\frac{2x^2(y+\Delta y)^2+2(y+\Delta y)++4x-(2x^2y^2+2y+4x)}{\Delta y}\\
&= \lim_{\Delta y\to 0}\frac{4x^2y(\Delta y)+2x^2(\Delta y)^2+2\Delta y}{\Delta y}\\
&= \lim_{\Delta y\to 0}[4x^2+2x^2(\Delta y)+2]\\
&= 4x^2y+2.
\end{aligned}$$

Remark. The partial derivative with respect to a variable can also be obtained by applying theorems for ordinary differentiation while treating the other variable constant.

In the case of $f(x,y)=2x^2y^2+2y+4x$,

$$\begin{aligned}
f_x(x,y) &= (2y^2)\frac{d}{dx}(x^2)+2y\frac{d}{dx}(1)+4\frac{d}{dx}(x)\\
&= 4xy^2+4.
\end{aligned}$$

On the other hand, differentiating with respect to y by treating x constant yields

$$\begin{aligned}
f_y(x,y) &= 2x^2(2y)+2+0\\
&= 4x^2+2.
\end{aligned}$$

Remark. The partial derivatives of f at (a,b) are also given by:

1. $f_x(a,b)=\lim\limits_{x\to a}\dfrac{f(x,b)-f(a,b)}{x-a}$
2. $f_y(a,b)=\lim\limits_{y\to b}\dfrac{f(a,y)-f(a,b)}{y-b}$

Example 1.3.2. Find $f_x(1,1)$ and $f_y(1,1)$ given that $f(x,y) = \dfrac{3x}{y} + 4xy^2$.

Solution: For $x \neq 1$, we have

$$\begin{aligned} \frac{f(x,1) - f(1,1)}{x-1} &= \frac{(3x+4x) - (3+4)}{x-1} \\ &= \frac{7x-7}{x-1} \\ &= \frac{7(x-1)}{x-1} \\ &= 7. \end{aligned}$$

Thus,

$$f_x(1,1) = \lim_{x \to 1} \frac{f(x,1) - f(1,1)}{x-1} = 7.$$

Similarly, for f_y, if $y \neq 1$, we have

$$\begin{aligned} \frac{f(1,y) - f(1,1)}{y-1} &= \frac{\frac{3}{y} + 4y^2 - 7}{y-1} \\ &= \frac{\frac{3+4y^3-7y}{y}}{y-1} \\ &= \frac{(2y+3)(2y-1)(y-1)}{y(y-1)} \\ &= \frac{(2y+3)(2y-1)}{y}. \end{aligned}$$

Hence,

$$f_y(1,1) = \lim_{y \to 1} \frac{(2y+3)(2y-1)}{y} = 5.$$

□

Notations. Let $z = f(x,y)$. The following denote the partial derivative of f with respect to x:

$$f_x(x,y), \;\; D_x f(x,y), \;\; \frac{\partial}{\partial x} f(x,y), \;\; \frac{\partial z}{\partial x}.$$

Similarly, the partial derivative of f with respect to y is denoted by any of the following:

$$f_y(x,y), \;\; D_y f(x,y), \;\; \frac{\partial}{\partial y} f(x,y), \;\; \frac{\partial z}{\partial y}.$$

Example 1.3.3. Evaluate $\dfrac{\partial}{\partial y} \sin^2(2x - 3y)$.

Solution:

$$\begin{aligned} \frac{\partial}{\partial y} \sin^2(2x-3y) &= \frac{\partial}{\partial y} \left(\sin(2x-3y)\right)^2 \\ &= 2[\sin(2x-3y)][\cos(2x-3y)][-3] \\ &= -6 \sin(2x-3y) \cos(2x-3y) \end{aligned}$$

Example 1.3.4. Find the slope of the tangent line to the curve of intersection of the sphere $x^2 + y^2 + z^2 = 9$ and the plane $y = 2$ at the point $P(1, 2, 2)$.

Solution: Let $z = f(x, y) = \sqrt{9 - x^2 - y^2}$. (*Why is it enough to consider just the upper hemisphere?*) The desired slope is given by $f_x(1, 2)$. Differentiating f with respect to x by treating y constant, we have

$$f_x(x, y) = -\frac{x}{\sqrt{9 - x^2 - y^2}}.$$

Hence, $f_x(1, 2) = -\dfrac{1}{2}$.

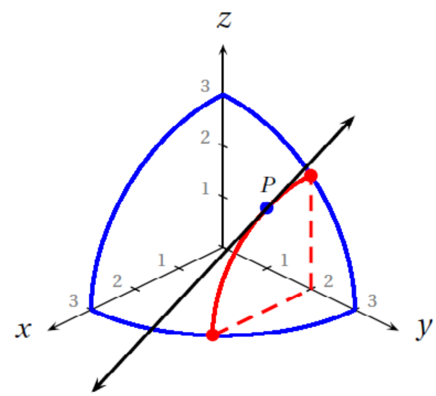


Figure 1.17: Curve of intersection of f and $y = 2$

Remark. The partial derivatives f_x and f_y are also called the derivatives of f in the x- and y-directions, respectively. Moreover, these partial derivatives can also be interpreted as rates of change.

Example 1.3.5. The volume V of a right circular cylinder is given by $V = \pi r^2 h$, where r is the radius of its base and h is its height. Find a formula for (a) the instantaneous rate of change of V with respect to r if h remains constant and (b) the instantaneous rate of change of V with respect to h if r remains constant.

Solution:

(a) $\dfrac{\partial V}{\partial r} = 2\pi r h$ (b) $\dfrac{\partial V}{\partial h} = \pi r^2$

Remark. For a function of n variables $x_1, x_2, \ldots, x_n$, there are n different partial derivatives. The partial derivative with respect to the ith variable is defined by

$$f_{x_i}(x_1, x_2, \ldots, x_n) = \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + \Delta x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta x_i}.$$

As in the case of two variables, f_{x_i} is obtained by performing ordinary differentiation with respect to x_i while treating all the other variables constant.

Notations. Let $u = f(x_1, x_2, \ldots, x_n)$. The following are the other notations for the partial derivative with respect to x_i:

$$D_{x_i}\left[f(x_1, x_2, \ldots, x_n)\right], \quad D_i[f(x_1, x_2, \ldots, x_n)], \quad \frac{\partial u}{\partial x_i}.$$

Example 1.3.6. Let $f(x, y, z) = \dfrac{xz}{y^2} + \sin(2x + 3y + 4z) + e^{x^2 z}$. Find f_x, f_y and f_z.

Solution:

$$\begin{aligned}
f_x(x, y, z) &= \frac{z}{y^2} + 2\cos(2x + 3y + 4z) + 2xze^{x^2 z} \\
f_y(x, y, z) &= \frac{-2xz}{y^3} + 3\cos(2x + 3y + 4z) \\
f_z(x, y, z) &= \frac{x}{y^2} + 4\cos(2x + 3y + 4z) + x^2 e^{x^2 z}
\end{aligned}$$

Higher Order Partial Derivatives

Let $z = f(x, y)$. The functions $f_x(x, y)$ and $f_y(x, y)$ are also functions of x and y. So, we may also consider the partial derivatives of f_x and f_y. These are our notations for various partial derivatives:

$$\begin{aligned}
f_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \\
f_{xy} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \\
f_{yx} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \\
f_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

The above derivatives are called the *second order partial derivatives* of f. In writing the mixed second partial derivatives, take care about the nuances in the notation in terms of the order that we write the variables.

Remark. Analogous definitions can be made for functions of three or more variables and for third or higher order partial derivatives.

Example 1.3.7. Let $w = f(x, y) = \ln(2x + 3y)$. Find (a) $\dfrac{\partial^2 w}{\partial x \partial y}$ and (b) $f_{xy}(x, y)$.

Solution:

(a) $\dfrac{\partial w}{\partial y} = f_y(x, y) = \dfrac{1}{2x + 3y}(3) = \dfrac{3}{2x + 3y}$

$$\frac{\partial^2 w}{\partial x \partial y} = f_{yx}(x, y) = \frac{\partial}{\partial x}\left[3(2x + 3y)^{-1}\right] = (-3)(2x + 3y)^{-2}(2) = \frac{-6}{(2x + 3y)^2}$$

(b) $\dfrac{\partial w}{\partial x} = f_x(x,y) = \dfrac{1}{2x+3y}(2) = \dfrac{2}{2x+3y}$

$\dfrac{\partial^2 w}{\partial y \partial x} = f_{xy}(x,y) = \dfrac{\partial}{\partial x}\left[2(2x+3y)^{-1}\right] = (-2)(2x+3y)^{-2}(3) = \dfrac{-6}{(2x+3y)^2}$

Example 1.3.8. Let $g(x,y,z) = z\ln(x^2 y\cos z)$. Find $\dfrac{\partial^3 g}{\partial x \partial x \partial z}$ and g_{xzx}.

Solution: Computing for $\dfrac{\partial^3 g}{\partial x \partial x \partial z}$, we have

$$\begin{aligned}
\frac{\partial g(x,y,z)}{\partial z} &= \ln(x^2 y\cos z) + z\left(\frac{1}{x^2 y\cos z}\right)\left(x^2 y(-\sin z)\right) \\
&= 2\ln x + \ln y + \ln(\cos z) - z\tan z \\
\frac{\partial^2 g(x,y,z)}{\partial x \partial z} &= \frac{2}{x} \\
\frac{\partial^3 g(x,y,z)}{\partial x \partial x \partial z} &= -\frac{2}{x^2}.
\end{aligned}$$

For g_{xzx}, it can be verified that $g_{zx} = g_{xz}$ for all (x,y,z) in the domain of g. Thus,

$$g_{xzx} = g_{zxx} = -\frac{2}{x^2}.$$

The above examples show that $f_{xy} = f_{yx}$ and $g_{zx} = g_{xz}$. This is not a coincidence. In fact, the second order mixed partial derivatives are equal under certain conditions.

Theorem 1.3.2 (Clairaut's Theorem). Let f be a function of x and y. If f_{xy} and f_{yx} are continuous on some circular region D on a plane, then for all (x,y) on D,

$$f_{xy} = f_{yx}.$$

Example 1.3.9. No function $f(x,y)$ exists such that $f_x(x,y) = 3x+2y$ and $f_y(x,y) = 4x+5y$. Indeed, Clairaut's theorem tells us that if such a function exists, then $f_{xy} = f_{yx}$. But in this case, $f_{xy}(x,y) = 2$ and $f_{yx}(x,y) = 4$, which are both continuous, yet not equal.

Remark. Clairaut's theorem can also be extended to arbitrary open and simply connected domains and to functions of more than two variables.

EXERCISES 1.3

I. Find the indicated partial derivatives by definition.

1. $f(x,y) = \sqrt{x+2y};\ f_y(2,1)$
2. $h(x,y) = 4xy - \dfrac{3x}{y^2};\ h_y(4,2)$
3. $g(x,y) = \dfrac{3y^2}{x^2+1};\ g_x(-3,2)$
4. $f(x,y) = (x^2+y^2)^{1/3};\ f_x(0,0)$
5. $f(x,y) = \begin{cases} \dfrac{x^2-1}{x-y} & \text{if } x \neq y \\ 2y & \text{if } x = y \end{cases};\ f_x(1,1)$

II. Find $\partial f/\partial x$ and $\partial f/\partial y$. For items 10 - 13, recall the *Fundamental Theorem of Calculus.*

1. $f(x,y) = (x^2-1)(y+2)$
2. $f(x,y) = \sqrt{x^2+y^2}$
3. $f(x,y) = \dfrac{1}{x+y}$
4. $f(x,y) = \dfrac{x+y}{xy-1}$
5. $f(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$
6. $f(x,y) = x^y$
7. $f(x,y) = \log_y x$
8. $f(x,y) = \log(xy-y^3)$
9. $f(x,y) = \dfrac{2x^2y}{\sqrt{x^2+y^2}}$
10. $f(x,y) = \displaystyle\int_2^{x^2y} e^{t^2}\,dt$
11. $f(x,y) = \displaystyle\int_{y^2-x^2}^{2} \sec^3(t+1)\,dt$
12. $f(x,y) = \displaystyle\int_y^x \ln(t^3+2)\,dt$
13. $f(x,y) = \displaystyle\int_{x-y}^{x-2y} \cos(t^2-1)\,dt$

III. Find f_x, f_y, and f_z.

1. $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$
2. $f(x,y,z) = \dfrac{x+2y-z}{2x-3y+z}$
3. $f(x,y,z) = \sin^{-1}(xyz)$
4. $f(x,y,z) = \ln(x+2y+3z)$
5. $f(x,y,z) = z^2e^{2z/xy}$

IV. Find all second order partial derivatives.

1. $f(x,y) = x^2y + \cos y + y\sin x$
2. $f(x,y) = \ln(x+y)$
3. $f(x,y) = xy\ln(x^2y^3) + \tan^{-1}\left(\dfrac{y+2}{x+3}\right)$

V. Find the indicated derivatives.

1. $f(x,y) = \sin(x^2+y^2);\quad f_{xy}(x,y)$
2. $f(x,y) = \dfrac{x}{x^2-y};\quad f_{yx}(x,y)$
3. $f(x,y) = e^{xy-1};\quad f_{xy}(1,1)$
4. $f(x,y) = x^2e^{-5y};\quad f_{xxy},\ f_{yyy}$
5. $f(x,y,z) = x\ln(x^2y^3z^4);\quad f_{xyy},\ f_{xyz},\ f_{xzy}$
6. $f(x,y,z) = e^{xy}\sinh 2z - e^{xy}\cosh 2z;\ f_z$

7. $u = \tan^{-1}(xyzw)$; $\dfrac{\partial u}{\partial w}$
8. $f(r, \theta, \phi) = 4r^2 \sin\theta + 5e^r \cos\theta \sin\phi - 2\cos\phi$; f_θ
9. $w = x^2y + y^2z + z^2x$; $\dfrac{\partial^3 w}{\partial x \partial y \partial z}$
10. $f(u, v) = \ln\cos(u - v)$; f_{uuv}
11. $u = \tan^{-1}(xyzw)$; $\dfrac{\partial^2 u}{\partial y \partial w}$
12. $g(x, y, z) = \sin xyz$; g_{xy}, g_{yz}, g_{xz}

VI. Do as indicated.

1. Show that $f(x, y) = \ln(e^{ax} + e^{ay})$ satisfies $f_x(x, y) + f_y(x, y) = a$.
2. If $z = f(x^3 - y^3)$, show that $y^2 \dfrac{\partial z}{\partial x} + x^2 \dfrac{\partial z}{\partial y} = 0$.
3. If $z = y^2 + f(x^2 - 2y)$, where f is differentiable, show that

$$\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 2xy.$$

1.4 Differentiability, Differentials and Local Linear Approximation

Differentiability

Recall that a function of a single variable is differentiable at a point if its derivative at that point exists. Geometrically, this means that the graph of the function has a unique non-vertical tangent line at the given point. However, for functions of two variables, differentiability is not as simple as the existence of all first-order partial derivatives.

For a function $f(x, y)$, consider

$$\Delta f = \Delta f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

This quantity Δf, called the *increment* of f, denotes the change in the value of $f(x, y)$ when (x, y) moves from (x_0, y_0) to $(x_0 + \Delta x, y_0 + \Delta y)$. Here, Δx and Δy are called *increments* of x and y, respectively. We have the following definition of differentiability for functions of two variables.

Definition 1.4.1. A function f of two variables x and y is said to be ***differentiable*** at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.$$

Remarks. Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$.

1. f is said to be ***differentiable on*** $D \subseteq \mathbb{R}^2$ if it is differentiable at all points in $\mathbb{R}$.
2. f is said to be ***differentiable everywhere*** if it is differentiable at all points in $\mathbb{R}^2$.

In this section of the text, we shall not go into the details of the definition of differentiability. Instead, we shall discuss the consequences of the definition.

Theorem 1.4.2.

1. If a function is differentiable at a point, then it is continuous at that point.
2. If all first-order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.

Remark. The graph of a differentiable function is smooth and does not contain any breaks, creases, or cusps. See Figure 1.18.

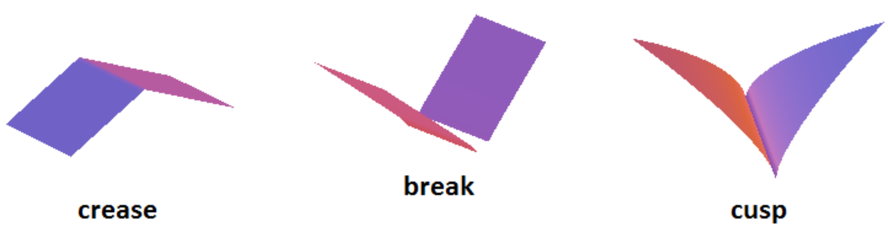


Figure 1.18: Some features of the graph of a function that hinder differentiability

Example 1.4.1. Discuss the differentiability of the following functions at the origin.

1. $f(x, y) = e^{xy}$

   By Theorem 1.4.2 (2), the function f is differentiable everywhere since $f_x(x, y) = ye^{xy}$ and $f_y(x, y) = xe^{xy}$ are continuous everywhere. Thus, it is differentiable at $(0, 0)$.

2. $f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$

   The function f is not differentiable at $(0,0)$ since it is not continuous at $(0,0)$.

3. $f(x,y) = \sqrt{x^2+y^2}$

   By Theorem 1.4.2 (1), the function f is not differentiable at $(0,0)$ since $f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$ is undefined at $(0,0)$.

4. $f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \end{cases}$

   It can be verified, by definition, that f is continuous at $(0,0)$. Moreover, both $f_x(0,0)$ and $f_y(0,0)$ exist. In particular,

$$\begin{aligned} f_x(0,0) &= \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} \\ &= \lim_{x\to 0} \frac{0-0}{x} \\ &= 0. \end{aligned}$$

   Similarly, it can be shown that $f_y(0,0) = 0$. Despite these, f is not differentiable at $(0,0)$. We shall show that the last condition for differentiability is not satisfied.

   Let $(x_0,y_0) = (0,0)$. Consider

$$\begin{aligned} &\frac{\Delta f - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \\ &\quad= \frac{f(0+\Delta x, 0+\Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \\ &\quad= \frac{\dfrac{(\Delta x)(\Delta y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = \frac{(\Delta x)(\Delta y)}{(\Delta x)^2+(\Delta y)^2}. \end{aligned}$$

   Along the line $\Delta x = \Delta y$, the limit is

$$\lim_{\Delta x\to 0} \frac{(\Delta x)^2}{2(\Delta x)^2} = \frac{1}{2}.$$

   Thus, $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \neq 0$, and so f fails to be differentiable at $(0,0)$.

Differentials

In this section, we introduce the notion of differentials for functions of more than one variable. A *differential* is an expression depending on the partial derivatives of a function. It is useful in approximating certain values of a function. First, we recall the definition of the differential dy of a differentiable function f given by $y = f(x)$:

$$dy := f'(x)dx.$$

Now, we extend this notion via the following definition.

Definition 1.4.3. Let $z = f(x, y)$ be differentiable at (x_0, y_0). We define the ***(total) differential*** of z at (x_0, y_0) by

$$dz := [f_x(x_0, y_0)]\, dx + [f_y(x_0, y_0)]\, dy$$

where dx and dy are the changes in the values of x and y, respectively.

Remarks.

1. The differential of $z = f(x, y)$ at (x_0, y_0) approximates Δf, which is the change in the value of $f(x, y)$ when (x, y) changes from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$.
2. At any point (x, y) where $z = f(x, y)$ is differentiable, dz can also be written as
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Example 1.4.2. Let $z = \tan^{-1}(xy)$. Compute dz and compare dz with the change in z when (x, y) changes from $(1, 1)$ to $(0.9, 1.01)$.

Solution: Note that $z = \tan^{-1}(xy)$ is differentiable at $(1, 1)$. The differential of z is given by

$$\begin{aligned} dz &= \left[\frac{\partial}{\partial x}\tan^{-1}(xy)\right] dx + \left[\frac{\partial}{\partial y}\tan^{-1}(xy)\right] dy \\ &= \left(\frac{y}{1 + x^2y^2}\right) dx + \left(\frac{x}{1 + x^2y^2}\right) dy. \end{aligned}$$

Computing the differential at $(1, 1)$, we have

$$\begin{aligned} dx &= 0.9 - 1 = -0.1 \\ dy &= 1.01 - 1 = 0.01 \\ dz &= \left(\frac{1}{1 + 1^2(1)^2}\right)(-0.1) + \left(\frac{1}{1 + 1^2(1)^2}\right)(0.01) = -0.0495. \end{aligned}$$

Using a calculator, the change in z is

$$\Delta z = \tan^{-1}[(0.9)(1.01)] - \tan^{-1}[1(1)] \approx -0.047632\ldots.$$

Hence, the discrepancy between dz and Δz is about $|-0.0495 - (-0.047632)| = 0.001868$. □

Example 1.4.3. The legs of a right triangle are measured to be 3cm and 4cm, with a maximum error of 0.05cm in each measurement. Use differentials to approximate the maximum possible error in the calculated value of (a) the hypotenuse and (b) the area of the triangle.

Solution: Let x and y be the lengths (in cm) of each leg of the right triangle. Then the hypotenuse $h(x, y)$ and area $a(x, y)$ of the triangle are given by

$$h(x, y) = \sqrt{x^2 + y^2}$$
$$a(x, y) = \frac{xy}{2}.$$

The maximum possible errors for the calculated value of the hypotenuse and area are

$$\Delta h = h(3 \pm 0.05, 4 \pm 0.05) - h(3, 4)$$
$$\Delta a = a(3 \pm 0.05, 4 \pm 0.05) - a(3, 4).$$

We use differentials to estimate these errors. Recall that

$$\begin{aligned} \Delta h &\approx dh = \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy \\ \Delta a &\approx da = \frac{1}{2} y dx + \frac{1}{2} x dy. \end{aligned}$$

If $x = 3$ and $y = 4$, then $dx = \pm 0.05$ and $dy = \pm 0.05$. Thus,

$$\begin{aligned} |dh| &\leq \frac{|x|}{\sqrt{x^2 + y^2}} |dx| + \frac{|y|}{\sqrt{x^2 + y^2}} |dy| \\ &= \frac{3}{\sqrt{3^2 + 4^2}} (0.05) + \frac{4}{\sqrt{3^2 + 4^2}} (0.05) \\ &= 0.07. \end{aligned}$$

Hence, the maximum possible error in the computed hypotenuse is approximately 0.07 cm. Likewise,

$$\begin{aligned} |da| &\leq \frac{1}{2} |y||dx| + \frac{1}{2} |x||dy| \\ &= \frac{1}{2} [3(0.05) + 4(0.05)] = 0.175. \end{aligned}$$

Hence, the maximum possible error in the computed area is approximately 0.175 cm^2.

Remark. The concept of differentials can also be extended to functions of n variables. That is, if u is a differentiable function of the variables $x_1, x_2, \ldots, x_n$, then

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \cdots + \frac{\partial u}{\partial x_n} dx_n,$$

where dx_i represents the change in the value of x_i $(i = 1, \ldots, n)$.

Example 1.4.4. Compute the differential of the function $u = \dfrac{t^3 + \cos r}{\sqrt{s}}$.

Solution: We have $du = \dfrac{3t^2}{\sqrt{s}} dt - \dfrac{\sin r}{\sqrt{s}} dr - \dfrac{(t^3 + \cos r)}{2s^{3/2}} ds.$

Local Linear Approximation

Recall that if $z = f(x, y)$ is differentiable at (x_0, y_0), then $\Delta f \approx dz$. That is

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

If we let $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$, the above approximation can be rewritten as

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Since f is differentiable at (x_0, y_0), the error in approximation becomes smaller as (x, y) gets closer to (x_0, y_0). We say that the function f is approximated by a linear function near (x_0, y_0).

Definition 1.4.4. Let $f(x, y)$ be differentiable at (x_0, y_0). The ***local linear approximation of*** f at (x_0, y_0) is defined as

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Remark 1.4.5. The local linear approximation $L(x, y)$ of $f(x, y)$ at a point (x_0, y_0) is also known as the *linearization* of f at that point. For points (x, y) near (x_0, y_0), we have $f(x, y) \approx L(x, y)$.

Example 1.4.5. Find the linearization of $f(x, y) = e^x \sin y$ at the origin.

Solution: Computing the partial derivatives, we have:

$$\begin{aligned} f_x(x, y) &= e^x \sin y &\implies\quad& f_x(0, 0) = 0 \\ f_y(x, y) &= e^x \cos y &\implies\quad& f_y(0, 0) = 1. \end{aligned}$$

Thus, the linearization of $f(x, y)$ at $(0, 0)$ is

$$L(x, y) = f(0, 0) + 0(x - 0) + 1(y - 0) = y.$$

That is, $e^x \sin y \approx y$ for (x, y) close to the origin.

Example 1.4.6. Approximate $\dfrac{1}{\sqrt{(3.92)^2 + (3.01)^2}}$ using local linear approximation.

Solution: First, we consider a suitable function to be used. Let $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$. The problem is then reduced to approximating $f(3.92, 3.01)$. We use the local linear approximation of f at the point where $(x, y) = (4, 3)$, which is relatively close to $(3.92, 3.01)$.

Computing the partial derivatives at $(4, 3)$, we have

$$\begin{aligned} f_x(x, y) &= -\frac{x}{(x^2 + y^2)^{3/2}} &\implies\quad& f_x(4, 3) = -\frac{4}{125} \\ f_y(x, y) = f_y(x, y) &= -\frac{y}{(x^2 + y^2)^{3/2}} &\implies\quad& f_y(4, 3) = -\frac{3}{125}. \end{aligned}$$

Thus, the local linear approximation of f at the point $(4,3)$ is

$$L(x,y) = f(4,3) - \frac{4}{125}(x-4) - \frac{3}{125}(y-3).$$

That is, for all (x,y) close to $(4,3)$, $f(x,y) \approx L(x,y)$. Therefore,

$$\frac{1}{\sqrt{(3.92)^2+(3.01)^2}} = f(3.92, 3.01) \approx \frac{1}{5} - \frac{4}{125}(3.92-4) - \frac{3}{125}(3.01-3) = \frac{2529}{12500}.$$

Remark. The surface represented by the local linear approximation of $f(x,y)$ at a point (a,b) is a plane given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Graphically, if we keep on zooming in at that point, as shown below, the surface appears to coincide with this plane. We shall call this plane the *tangent plane* to the surface $z = f(x,y)$ at the given point.

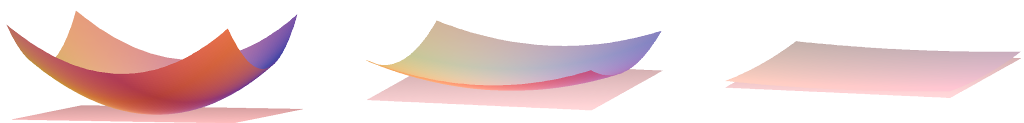


Figure 1.19: The plane of the local linear approximation $L(x,y)$ and the graph of f

EXERCISES 1.4

I. Find the local linear approximation of $f(x,y)$ at the indicated points.

1. $f(x,y) = 3x - 4y + 5$ at $(1,1)$
2. $f(x,y) = e^x \cos y$ at $\left(0, \frac{\pi}{2}\right)$
3. $f(x,y) = xy^2 + y\cos(x-1)$ at $(1,2)$.
4. $f(x,y) = x\sqrt{y}$; $(1,4)$
5. $f(x,y) = \tan^{-1}(x+2y)$; $(1,0)$

II. Using local linear approximation, estimate the following.

1. $\sqrt{20 - (1.95)^2 - (1.08)^2}$
2. $\ln(3(2.01) - 4.98)$
3. $\dfrac{1}{\sqrt{(1.01)^2 + (0.02)^2}}$

III. Given that the function $G(x,y)$ is differentiable at $(-2,-1)$ with $G_x(-2,-1) = -1$, $G_y(-2,-1) = 4$ and $G(-2,-1) = 1$,

1. determine the linearization of G at $(-2,-1)$;
2. estimate $G(-2.01, -0.97)$.

IV. Find the differential of the following functions.

1. $z = x^3 \ln(y^2)$
2. $R = \alpha\beta^2 \cos\gamma$
3. $w = xye^{xz}$
4. $u = e^t \sin\theta$
5. $w = \ln\sqrt{x^2 + y^2 + z^2}$
6. $v = y\cos xy$
7. $u = \dfrac{r}{s + 2t}$

V. Determine the equation of the plane tangent to the surface at the given point.

1. $f(x, y) = x\sin(xy)$; $P\left(1, \frac{\pi}{2}, 1\right)$
2. $f(x, y) = 2\tan^{-1}(x + y^2)$; $P\left(0, 1, \frac{\pi}{2}\right)$

VI. Do as indicated.

1. Show that $f(x, y) = x^2 \sin y$ is differentiable everywhere.
2. Find the local linear approximation of $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at $(2, 1)$ and use it to approximate $f(1.95, 1.08)$.
3. At what point P is the linearization of $f(x, y) = x^2y$ equal to $L(x, y) = -4x + 4y + 8$?
4. If (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz where $z = 5x^2 + y^2$. In particular, compute the error $|\Delta z - dz|$.
5. A boundary strip 3 in wide is painted around a rectangle whose dimensions are 100 ft by 200 ft. Use differentials to approximate the amount (in square feet) of paint needed to cover the strip.
6. Use differentials to estimate the amount of tin in a closed tin can with inner diameter 8cm and inner height 12cm if the tin is 0.04cm thick.
7. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10cm high and 4cm in diameter (on the inside) if the metal in the top and bottom is 0.1cm thick and the metal in the sides is 0.05cm thick.
8. The ideal gas law says that the pressure P, temperature T and volume V of a gas satisfy the equation $P = \dfrac{kT}{V}$ for some constant k. Use differentials to approximate the percentage change in pressure if the temperature is decreased by 4% and the volume is increased by 5%.
9. Find the points on the surface represented by $f(x, y) = x^2 - xy + 3$ where the tangent plane is perpendicular to the plane with equation $2x + y + z = 0$.
10. *True or False.* If the second order partial derivatives of $f(x, y)$ are continuous, then $f_{xyx} = f_{xxy}$.

1.5 Chain Rule

Recall that if y is a differentiable function of x, and, in turn, x is a differentiable function of t, then y is also a differentiable function of t. Moreover,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

For functions of several variables, we have the following analogous formula.

Theorem 1.5.1 (General Chain Rule). Suppose that u is a differentiable function of n variables $x_1, x_2, \ldots, x_n$. For $i = 1, \ldots, n$, suppose x_i is a differentiable function of m variables $t_1, t_2, \ldots, t_m$. Then u is also a differentiable function of the m variables $t_1, \ldots, t_m$. Moreover, for each $j = 1, \ldots, m$, we have

$$\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_j}.$$

Remark. Let $u = f(x_1, x_2, \ldots, x_n)$ be differentiable, where $x_i = g_i(t_1, t_2, \ldots, t_m)$ is differentiable, for $i = 1$ to n. Graphically, the dependence of the variables is given by Figure 1.20.

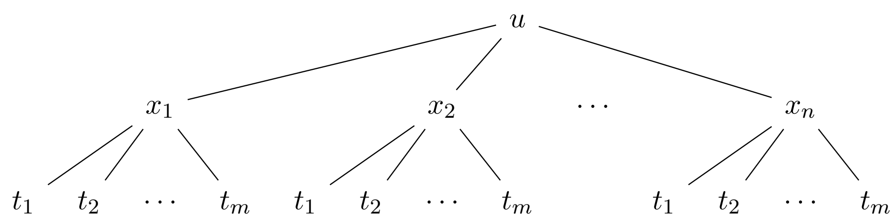


Figure 1.20: The chains from u to each variable t_i via the variables x_j

In particular, if $z = f(x, y)$ is differentiable with respect with x and y, where x and y are both differentiable functions of t. Then z is also a differentiable function of the single variable t. Graphically, the dependence of the variables is given below.

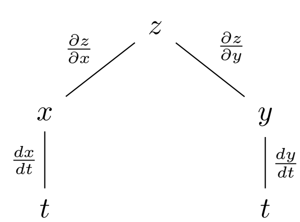


Figure 1.21: The two chains from z to t via the variables x and y

Moreover,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Example 1.5.1. Let $z = x^2y$, where $x = t^2$ and $y = t^3$. Use the chain rule to find $\dfrac{dz}{dt}$, and verify the result using direct substitution.

Solution: We have

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2xy)(2t) + (x^2)(3t^2) \\
&= (2t^5)(2t) + (t^4)(3t^2) \\
&= 7t^6.
\end{aligned}$$

Indeed, by direct substitution, $z = x^2y = (t^2)^2(t^3) = t^7$. Thus, $\dfrac{dz}{dt} = 7t^6$.

Remark. Let $z = f(x, y)$ be differentiable, where x and y are differentiable functions of s and t. Then z is also a differentiable function of s and t. The dependence of the variables is shown below.

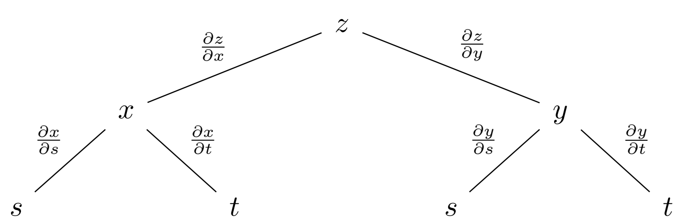


Figure 1.22: The dependence among the variables z, x, y, s, t

Moreover,

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.
\end{aligned}$$

Example 1.5.2. Let $z = e^{xy}$, where $x = 2s + t$ and $y = \dfrac{s}{t}$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ at $(s, t) = (-1, 1)$, using chain rule.

Solution: Using the chain rule we have

$$\begin{aligned}
\frac{\partial z}{\partial s} &= ye^{xy}(2) + xe^{xy}\left(\frac{1}{t}\right) &&= e^{xy}\left(2y + \frac{x}{t}\right) \\
\frac{\partial z}{\partial t} &= ye^{xy}(1) + xe^{xy}\left(-\frac{s}{t^2}\right) &&= e^{xy}\left(y - \frac{xs}{t^2}\right).
\end{aligned}$$

If $(s, t) = (-1, 1)$, then $(x, y) = (-1, -1)$. Thus,

$$\begin{aligned}
\frac{\partial z}{\partial s} &= e^{(-1)(-1)}\left(2(-1) + \frac{-1}{1}\right) &&= -3e \\
\frac{\partial z}{\partial t} &= e^{(-1)(-1)}\left(-1 - \frac{(-1)(-1)}{(1)^2}\right) &&= -2e.
\end{aligned}$$

Example 1.5.3. Write out the chain rule for $z = z(x, y)$, where $x = x(r, s, u, v)$ and $y = y(r, s, u, v)$.

Solution: From the diagram below,

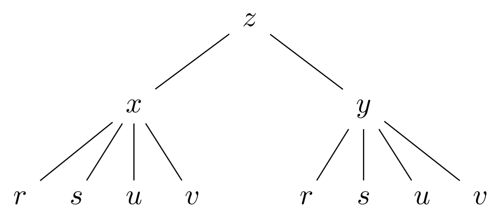


Figure 1.23: The dependence among the variables

we obtain

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} & \qquad \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} & \qquad \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \end{aligned}$$

Example 1.5.4. Let $u = x^2 + yz$, where $x = pr\cos\theta$, $y = pr\sin\theta$, $z = p + r$. Find u_p, u_r and u_θ when $p = 2$, $r = 3$ and $\theta = 0$.

Solution: Applying the chain rule, we obtain

$$\begin{aligned} u_p &= u_x x_p + u_y y_p + u_z z_p \\ &= 2x(r\cos\theta) + z(r\sin\theta) + y(1) \\ &= 2xr\cos\theta + zr\sin\theta + y \\ u_r &= u_x x_r + u_y y_r + u_z z_r \\ &= 2x(p\cos\theta) + z(p\sin\theta) + y(1) \\ &= 2xp\cos\theta + zp\sin\theta + y \\ u_\theta &= u_x x_\theta + u_y y_\theta \\ &= 2x(-pr\sin\theta) + z(pr\cos\theta) \\ &= -2xpr\sin\theta + zpr\cos\theta. \end{aligned}$$

When $p = 2$, $r = 3$, $\theta = 0$, we have

$$\begin{aligned} x &= 2(3)\cos 0 = 6 \\ y &= 2(3)\sin 0 = 0 \\ z &= 2 + 3 = 5. \end{aligned}$$

Thus, we have

$$\begin{aligned} u_p &= 2(6)(3\cos 0) + 5(3\sin 0) + 0 = 36 \\ u_r &= 2(6)(2\cos 0) + 5(2\sin 0) + 0 = 24 \\ u_\theta &= -2(6)(2)(\sin 0) + 5(2)(3)(\cos 0) = 30. \end{aligned}$$

Implicit Differentiation

Consider $x^3 + y^3 = 6xy$, where y is a differentiable function of x. We have learned in the previous calculus course that $\frac{dy}{dx}$ may be obtained by implicit differentiation. We will now illustrate how the two-variable chain rule can simplify the computation. Suppose y is a differentiable function of x, and that $F(x, y) = 0$ gives the implicit relation between y and x.

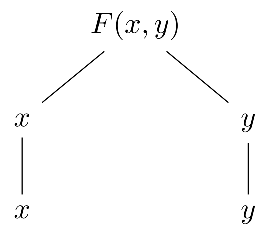


Figure 1.24: The dependence of the variables for $F(x, y)$

Differentiating both sides of $F(x, y) = 0$ with respect to x, we get

$$F_x(x, y)\left(\frac{dx}{dx}\right) + F_y(x, y)\left(\frac{dy}{dx}\right) = 0.$$

Since $\dfrac{dx}{dx} = 1$, we have

$$F_x(x, y) + F_y(x, y)\left(\frac{dy}{dx}\right) = 0.$$

Hence, we have the following theorem.

Theorem 1.5.2. Let y be a differentiable function of x such that $F(x, y) = 0$ is the implicit equation relating x and y. Then,

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

Example 1.5.5. Compute for $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Solution: First, we write $x^3 + y^3 = 6xy$ in the form $F(x, y) = 0$. Let $F(x, y) = x^3 + y^3 - 6xy$. Then,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}.$$

Example 1.5.6. Consider the equation

$$x^3 + y^3 + z^3 = 1 - 6xyz$$

where z is a differentiable function of the two independent variables x and y.

Then $\dfrac{\partial z}{\partial x}$ may be obtained through implicit differentiation as follows:

$$\begin{aligned}\frac{\partial}{\partial x}\left(x^3+y^3+z^3\right) &= \frac{\partial}{\partial x}(1-6xyz)\\ 3x^2+3z^2\frac{\partial z}{\partial x} &= -6yz-6xy\frac{\partial z}{\partial x}.\end{aligned}$$

Thus,

$$\begin{aligned}\left(3z^2+6xy\right)\frac{\partial z}{\partial x} &= -3x^2-6yz\\ \frac{\partial z}{\partial x} &= -\frac{3x^2+6yz}{3z^2+6xy}.\end{aligned}$$

Using a similar computation, we have $\dfrac{\partial z}{\partial y} = -\dfrac{3y^2+6xz}{3z^2+6xy}$.

We have a similar result for three variables. Let z be a differentiable function of two independent variables x and y. Suppose $F(x,y,z)=0$ defines implicitly the relation between x, y and z.

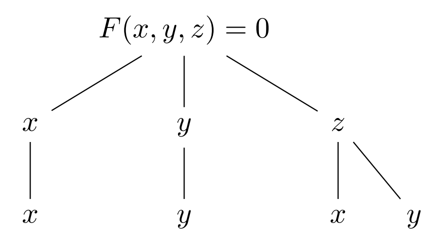


Figure 1.25: The dependence of the variables for $F(x,y,z)$

Differentiating with respect to x, we get

$$F_x(x,y,z)\frac{dx}{dx}+F_z(x,y,z)\frac{\partial z}{\partial x}=0.$$

We may also differentiate with respect to y to obtain

$$F_y(x,y,z)\frac{dy}{dy}+F_z(x,y,z)\frac{\partial z}{\partial y}=0.$$

This gives us the following theorem.

Theorem 1.5.3. Let z be a differentiable function of the independent variables x and y. Suppose $F(x,y,z)=0$ is the implicit equation relating x, y and z. Then

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \quad\text{and}\quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.$$

Example 1.5.7. Let $x^3 + y^3 + z^3 = 1 - 6xyz$, where z is a function of x and y. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Solution: Let us write the equation as $F(x, y, z) = 0$. Let

$$F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1.$$

Using Theorem 1.5.3, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy}.$$

EXERCISES 1.5

I. Find the indicated partial derivatives.

1. $\ln(x^2 + yz) = 2xz - y$; $\dfrac{\partial z}{\partial y}$ and $\dfrac{\partial z}{\partial x}$
2. $e^{9r^2+st} - t\sin(r + s) = 2rs$; $\dfrac{\partial r}{\partial s}$
3. $\sec^2(xz + y) = \tan^{-1}(x - 2yz) + 1$; $\dfrac{\partial z}{\partial y}$ at $(0, 0, 0)$
4. $2xyz + e^{w^2-2xy} - 2zw = 2x + y$; $\dfrac{\partial w}{\partial x}$.

II. Do as indicated.

1. Given that $x^2y + \tan(xy + z) = 3x + yz$, where $x = 2p^2r + t$ and $y = 3\cos(s + t)$, find $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.
2. Let z be a differentiable function of x and y such that $2xyz = H(x, y, z) - 2\cosh(x + y)$, where H is a differentiable function of x, y and z. If $H_x(1, -1, 3) = H_y(1, -1, 3) = 2$ and $H_z(1, -1, 3)$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(1, -1, 3)$.
3. Let z be a differentiable function of x and y that satisfies the equation
$$z + xy = H(y, z),$$
where H is a differentiable function of y and z. Suppose that $x = 1$ if $(y, z) = (4, 3)$. If $H_y(4, 3) = 5$ and $H_z(4, 3) = 3$, find $\dfrac{\partial z}{\partial y}$ at the point where $(x, y, z) = (1, 4, 3)$.
4. Suppose $z = f(x, y)$, where $x = g(u, v)$, $y = h(u, v)$, $g(1, 1) = 2$, $g_u(1, 1) = -3$, $g_v(1, 1) = 4$, $h(1, 1) = -1$, $h_u(1, 1) = -2$, $h_v(1, 1) = -4$, $f_x(2, -1) = -6$, $f_y(2, -1) = 5$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ when $u = v = 1$.
5. Suppose that $w = x\cos(yz^2)$, where $x = \sin t$, $y = t^2$, $z = 2e^t$. Find the rate of change of w with respect to t at $t = 0$.

1.6 Directional Derivatives and Gradients

Recall that the partial derivative of a function $f(x, y)$ with respect to x at a point (a, b) gives the slope of the curve of intersection of the surface $z = f(x, y)$ and the plane $y = b$. We can generalize this concept when we change the plane $y = b$ with another vertical planepassing through (a, b). We have the following definition.

Definition 1.6.1. Let f be a function of x and y. The ***directional derivative*** of f at a point (x, y) along a unit vector $\vec{u} = \langle u_1, u_2 \rangle$, denoted $D_{\vec{u}} f(x, y)$, is defined by

$$D_{\vec{u}} f(x, y) := \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h},$$

if this limit exists.

Example 1.6.1. Use the definition to find the directional derivative of $f(x, y) = x^2 + 3y^2$ along the unit vector $\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$ at the point $(1, -1)$.

Solution:

$$\begin{aligned}
D_{\vec{u}} f(1, -1) &= \lim_{h \to 0} \frac{f(1 + \frac{\sqrt{3}}{2}h, -1 - \frac{h}{2}) - f(1, -1)}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left[\left((1 + \frac{\sqrt{3}}{2} h \right)^2 + 3 \left(-1 - \frac{h}{2} \right)^2 - 4 \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\left(1 + \sqrt{3} h + \frac{3}{4} h^2 \right) + 3 \left(1 + h + \frac{h^2}{4} \right) - 4 \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\sqrt{3} h + \frac{3}{2} h^2 + 3h \right] \\
&= \lim_{h \to 0} \left(\sqrt{3} + \frac{3}{2} h + 3 \right) \\
&= \sqrt{3} + 3.
\end{aligned}$$

Remarks.

1. Let C be the curve of intersection of a surface defined by an equation $z = f(x, y)$ and the vertical plane passing through $P(x_0, y_0, f(x_0, y_0))$ along the direction of a given unit vector $\vec{u} = \langle u_1, u_2 \rangle$, as shown in Figure 1.26. Geometrically, $D_{\vec{u}} f(x_0, y_0)$ is the *rate of change* of f along $\vec{u}$ at the point $P_0(x_0, y_0)$. It is the slope of the tangent line to C at P.

2. The partial derivatives are directional derivatives along the vectors, $\hat{\imath}$ and $\hat{\jmath}$.

   a. If $\hat{\imath} = \langle 1, 0 \rangle$, then $D_{\hat{\imath}} f(x, y) = f_x(x, y)$.

   b. If $\hat{\jmath} = \langle 0, 1 \rangle$, then $D_{\hat{\jmath}} f(x, y) = f_y(x, y)$.

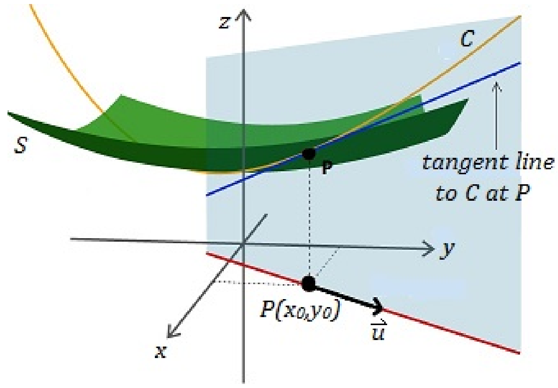


Figure 1.26: Directional derivative as slope

Definition 1.6.2. Let f be a function of x and y such that f_x and f_y exist at the point (x, y). The *gradient* of f at a point (x, y), denoted $\vec{\nabla} f(x, y)$, is the vector defined by

$$\vec{\nabla} f(x, y) := \langle f_x(x, y), f_y(x, y) \rangle.$$

Example 1.6.2. Find the gradient of $f(x, y) = x \cos y - \dfrac{e^y}{x}$ at $(-1, 0)$.

Solution: For any (x, y),

$$\begin{aligned} \vec{\nabla} f(x, y) &= \langle f_x(x, y), f_y(x, y) \rangle \\ &= \left\langle \cos y + \frac{e^y}{x^2}\, ,\, -x \sin y - \frac{e^y}{x} \right\rangle. \end{aligned}$$

Hence,

$$\vec{\nabla} f(-1, 0) = \langle 2, 1 \rangle\, .$$

Theorem 1.6.3. If f is a differentiable function of x and y, then f has a directional derivative along any unit vector $\vec{u}$, and

$$D_{\vec{u}} f(x, y) = \vec{\nabla} f(x, y) \cdot \vec{u}.$$

Proof. Suppose $\vec{u} = \langle u_1, u_2 \rangle$ is a unit vector and $(x, y) \in \mathrm{dom} f$. By definition,

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h}.$$

By letting $g(h) = f(x + hu_1, y + hu_2)$, the above equation becomes

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0). \tag{1.1}$$

Meanwhile, by the Chain Rule,

$$\begin{aligned} g'(h) &= \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}, \\ g'(0) &= f_x(x,y)u_1 + f_y(x,y)u_2 = \vec{\nabla} f(x,y)\cdot \vec{u}. \end{aligned} \tag{1.2}$$

Combining equations (1.1) and (1.2), we get

$$D_{\vec{u}} f(x_0,y_0) = \vec{\nabla} f(x_0,y_0)\cdot \vec{u}.$$

□

Example 1.6.3. Find the rate of change of $f(x,y) = x\cos y - \dfrac{e^y}{x}$ at $(-1,0)$ along $\langle 4,-3\rangle$.

Solution: Note that the vector $\vec{u}$ in the definition and in Theorem 1.6.3 has to be a unit vector. Recall from Example 1.6.2 that $\vec{\nabla} f(-1,0) = \langle 2,1\rangle$. The unit vector parallel to $\langle 4,-3\rangle$ is

$$\vec{u} = \frac{\langle 4,-3\rangle}{\|\langle 4,-3\rangle\|} = \frac{\langle 4,-3\rangle}{\sqrt{4^2+(-3)^2}} = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle .$$

Therefore,

$$\begin{aligned} D_{\vec{u}} f(-1,0) &= \vec{\nabla} f(-1,0)\cdot \vec{u} \\ &= \langle 2,1\rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle = 1. \end{aligned}$$

This means that at $(-1,0)$, f is increasing at a rate of 1 per unit step along the direction of $\vec{u}$.

Remark. If f is a function of x, y and z and $\vec{u} = \langle u_1,u_2,u_3\rangle$ is a unit vector, then

1. $D_{\vec{u}} f(x,y,z) := \lim\limits_{h\to 0} \dfrac{f(x+hu_1, y+hu_2, z+hu_3) - f(x,y,z)}{h}$
2. $\vec{\nabla} f(x,y,z) := \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$
3. $D_{\vec{u}} f(x,y,z) = \vec{\nabla} f(x,y,z)\cdot \vec{u}$, provided f is differentiable.

Example 1.6.4. Determine the rate of change of $f(x,y,z) = x\cos z - ye^z - y^2 + 4$ at $(2,1,0)$ along the vector $\langle 3,-4,5\rangle$.

Solution: Solving for the gradient of f,

$$\begin{aligned} \vec{\nabla} f(x,y,z) &= \langle \cos z, -e^z - 2y, -x\sin z - ye^z\rangle \\ \vec{\nabla} f(2,1,0) &= \langle 1,-3,-1\rangle . \end{aligned}$$

Normalizing the vector $\langle 3,-4,5\rangle$, we get

$$\vec{u} = \frac{\langle 3,-4,5\rangle}{\|\langle 3,-4,5\rangle\|} = \frac{\langle 3,-4,5\rangle}{\sqrt{3^2+(-4)^2+5^2}} = \left\langle \frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle.$$

Thus, the rate of change of f at the point $(2,1,0)$ along $\langle 3,-4,5\rangle$ is

$$\begin{aligned} D_{\vec{u}}f(2,1,0) &= \vec{\nabla} f(2,1,0)\cdot\vec{u} \\ &= \langle 1,-3,-1\rangle\cdot\left\langle \frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = \sqrt{2} \end{aligned}$$

Important information about the function is relayed by the gradient of a function: it gives a way to identify the direction of steepest ascent and steepest descent of the graph. In general, we have the following result.

Theorem 1.6.4. Let f be a function of x and y (or x, y and z).

(i) The maximum rate of change of f at a point (x,y) (resp. (x,y,z)) is $\|\vec{\nabla} f(x,y)\|$ (resp. $\|\vec{\nabla} f(x,y,z)\|$) and this occurs along $\vec{\nabla} f(x,y)$ (resp. $\vec{\nabla} f(x,y,z)$).

(ii) The minimum rate of change of f at a point (x,y) (resp. (x,y,z)) is $-\|\vec{\nabla} f(x,y)\|$ (resp. $-\|\vec{\nabla} f(x,y,z)\|$) and this occurs along $-\vec{\nabla} f(x,y)$ (resp. $-\vec{\nabla} f(x,y,z)$).

Proof. We provide the proof for the case that f is a function of x, y and z. Let $\vec{u}$ be a unit vector and θ be the angle between $\vec{\nabla} f(x,y,z)$ and $\vec{u}$. Then

$$\begin{aligned} D_{\vec{u}}f(x,y,z) &= \vec{\nabla} f(x,y,z)\cdot\vec{u} \\ &= \|\vec{\nabla} f(x,y,z)\|\,\|\vec{u}\|\cos\theta \quad \text{(by a theorem from Math 22)} \\ &= \|\vec{\nabla} f(x,y,z)\|\cos\theta\ . \end{aligned}$$

Thus, $D_{\vec{u}}f(x,y,z)$ is maximum when $\cos\theta=1$, that is, when $\theta=0$. This implies $\vec{\nabla} f(x,y,z)$ and $\vec{u}$ point in the same direction and the maximum value of the directional derivative is $\|\vec{\nabla} f(x,y,z)\|$. On the other hand, $D_{\vec{u}}f(x,y,z)$ is minimum when $\cos\theta=-1$; that is, when $\theta=\pi$. This implies $\vec{\nabla} f(x,y,z)$ and $\vec{u}$ point in opposite directions and the minimum value of the directional derivative is $-\|\vec{\nabla} f(x,y,z)\|$. □

Example 1.6.5. Let $f(x,y)=\dfrac{y}{x}+\ln(2x-y)$. Find the maximum rate of change of f at $(1,-1)$ and give a vector along which it occurs.

Solution: The maximum rate of change of f at $(1,-1)$ is given by $\|\vec{\nabla} f(1,-1)\|$ and this occurs along $\vec{\nabla} f(1,-1)$. We have

$$\begin{aligned} \vec{\nabla} f(x,y) &= \langle f_x(x,y), f_y(x,y)\rangle \\ &= \left\langle -\frac{y}{x^2}+\frac{2}{2x-y}, \frac{1}{x}-\frac{1}{2x-y}\right\rangle \\ \vec{\nabla} f(1,-1) &= \left\langle \frac{5}{3}, \frac{2}{3}\right\rangle \\ \|\vec{\nabla} f(1,-1)\| &= \sqrt{\left(\frac{5}{3}\right)^2+\left(\frac{2}{3}\right)^2} = \frac{\sqrt{29}}{3}. \end{aligned}$$

Thus, the maximum rate of change of f at $(1,-1)$ is $\dfrac{\sqrt{29}}{3}$, which occurs along $\left\langle \dfrac{5}{3}, \dfrac{2}{3} \right\rangle$.

Example 1.6.6. Find the minimum directional derivative of $f(x,y,z) = xy^2 - x^2z + y^3 - 1$ at the point $(3,0,-2)$.

Solution: We first solve for the gradient of f:

$$\begin{aligned} \vec{\nabla} f(x,y,z) &= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle \\ &= \langle y^2 - 2xz, 2xy + 3y^2, -x^2 \rangle \\ \therefore \vec{\nabla} f(3,0,-2) &= \langle 12, 0, -9 \rangle. \end{aligned}$$

The minimum directional derivative of f at $(3,0,-2)$ is

$$-\|\vec{\nabla} f(3,0,-2)\| = -\sqrt{12^2 + (-9)^2} = -15.$$

Tangent Planes

In many situations, it is important to determine an equation of the plane tangent to a surface S described by an equation of the form $F(x,y,z) = k$ at a given point P. Recall that the tangent plane to S at the point P is exactly the plane that shares the same normal vector as S at P. The following theorem helps us to determine this tangent plane.

Theorem 1.6.5. Let S be a surface whose equation is of the form $F(x,y,z) = k$, where k is a constant. Then $\vec{\nabla} F(x_0, y_0, z_0)$ is normal to S at the point (x_0, y_0, z_0).

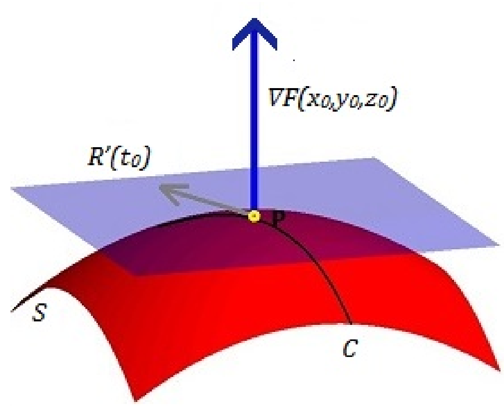


Figure 1.27: Tangent plane to $F(x,y,z) = k$

Proof. Let $C : \vec{R}(t) = \langle x(t), y(t), z(t) \rangle$ be any curve on S passing through $P(x_0, y_0, z_0)$ and such that $\vec{R}(t_0) = \langle x_0, y_0, z_0 \rangle$. Then for any t,

$$F(x(t), y(t), z(t)) = k.$$

Differentiating both sides with respect to t, we get

$$F_x(x(t), y(t), z(t))x'(t) + F_y(x(t), y(t), z(t))y'(t) + F_z(x(t), y(t), z(t))z'(t) = 0 \ ,$$

or equivalently,

$$\vec{\nabla} F(x(t), y(t), z(t)) \cdot \vec{R}'(t) = 0.$$

In particular, if $t = t_0$, we have

$$\vec{\nabla} F(x_0, y_0, z_0) \cdot \vec{R}'(t_0) = 0.$$

Thus, $\vec{\nabla} F(x_0, y_0, z_0)$ is perpendicular to $\vec{R}'(t_0)$.

Note that this holds for every curve C on S through P. Since the representation of $\vec{R}'(t_0)$ with initial point at P is tangent to S at P, then $\vec{\nabla} F(x_0, y_0, z_0)$ is normal to S at P. □

Recall that the tangent plane to a surface S at a point P on S is the plane through P that shares the same normal vector with S at P. The normal line to S at P is the line through P that is parallel to a normal vector to S at P.

Example 1.6.7. Find an equation of the tangent plane and a vector equation of the normal line to the surface defined by the equation $xy^3 - x^2z + yz^2 = 1$ at the point $(2, -1, -3)$.

Solution: Let $F(x, y, z) = xy^3 - x^2z + yz^2$. Then,

$$\vec{\nabla} F(x, y, z) = \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)\rangle = \langle y^3 - 2xz, 3xy^2 + z^2, -x^2 + 2yz\rangle.$$

Thus,

$$\vec{\nabla} F(2, -1, -3) = \langle 11, 15, 2\rangle \, .$$

is a normal vector to the tangent plane to S at $(2, -1, -3)$. Therefore, an equation of the tangent plane is

$$11(x - 2) + 15(y + 1) + 2(z + 3) = 0,$$

and the normal line can be represented by

$$\vec{R}(t) = \langle 2 + 11t, -1 + 15t, -3 + 2t\rangle.$$

Analogous to Theorem 1.6.5, we have for a function of x and y, the following.

Theorem 1.6.6. Let C be a curve with equation $f(x, y) = k$, where k is a constant. Then $\vec{\nabla} f(x_0, y_0)$ is orthogonal to C at (x_0, y_0).

Example 1.6.8. Find a vector orthogonal to the unit circle at the point $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Solution: Let $f(x,y) = x^2 + y^2$. Then the unit circle has equation $f(x,y) = 1$, and $\vec{\nabla} f(x,y) = \langle 2x, 2y\rangle$. Thus,

$$\vec{\nabla} f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left\langle -1, \sqrt{3}\right\rangle,$$

is a vector orthogonal to the unit circle at P.

EXERCISES 1.6

I. Determine the directional derivative of f at the point P along the indicated direction.

1. $f(x,y) = ye^x$, $P(0,2)$, $\vec{v} = \langle -1, 1\rangle$
2. $f(x,y) = \dfrac{xy}{x^2+y^2}$, $P(1,-1)$, $\vec{v} = \sqrt{3}\hat{\imath} - \hat{\jmath}$
3. $f(x,y) = x^3 + xy + y^3$, $P(-1,1)$, $\vec{u}$ is the unit vector that makes an angle $\pi/6$ with the positive $x-$axis
4. $f(x,y) = x^2 - 4xy + y^2$, $P(0,-1)$, $\vec{u}$ is the unit vector that makes an angle $\frac{\pi}{4}$ with the positive $x-$axis
5. $f(x,y,z) = \tan^{-1}\left(\dfrac{xy^2}{2z^2-1}\right)$, $P(0,1,1)$, $\vec{v} = \langle -1, 2, 2\rangle$
6. $f(x,y,z) = x^2y - xy^2 + z^3$, $P(1,1,2)$, $\vec{u}$ is the vector from P to $Q(1,-2,-2)$

II. For each function in I,

1. determine the maximum directional derivative at P and give a vector along which this value is attained.
2. determine the minimum directional derivative at P and give a vector along which this value is attained.

III. Find an equation of the tangent plane and give a vector equation of the normal line to the given surface at the point Q.

1. $z = \sqrt{x^2+y^2}$; $Q(3,4,5)$
2. $x = e^y \cos z$; $Q(e,1,0)$
3. $zx^2 - xy^2 - yz^2 = 18$; $Q(0,-2,3)$
4. $x^y = z^{x^2} + xy$; $Q(1,0,1)$

IV. Do as indicated.

1. Let $g(x,y) = e^{2y}\tan^{-1}\left(\dfrac{y}{3x}\right) - 2$. Find a direction from the point $(1,3)$ for which the value of g does not change.
2. Find a vector equation of the line tangent to the curve of intersection of the surfaces $x^2 + y^2 - z = 8$ and $x - y^2 + z^2 = -2$ at the point $(2,-2,0)$.
3. Find all points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line through the points $(3,-1,0)$ and $(5,3,6)$.

4. Let $f(x,y) = \dfrac{y^2}{2} - x^4$.
   (a) Determine the directional derivative of f at $(1,-2)$ along $\langle -3,4\rangle$.
   (b) Give a direction vector from the point $(\frac{1}{2},3)$ where f decreases most rapidly.
   (c) Give all unit vectors from the point $(\frac{1}{2},3)$ where the rate of change of f is zero.

5. Let $f(x,y) = \dfrac{e^{2y}}{x} - x^3y + 4$.
   (a) Find the directional derivative of f at the point $(1,0)$ along the direction of the vector $\langle 5,12\rangle$.
   (b) Find a vector along which f decreases most rapidly at the point $(1,0)$.

6. Let $f(x,y,z) = (x+y)^2 z$.
   (a) Determine $D_{\vec{u}}f(1,-2,1)$ where $\vec{u}$ is the unit vector in the direction of $\langle 2,-1,1\rangle$.
   (b) Find the maximum possible value of $D_{\vec{u}}f(1,-2,1)$ and the direction in which it occurs.
   (c) Find an equation of the plane tangent to the surface $f(x,y,z) = 1$ at the point $(1,-2,1)$.

7. Let f be a differentiable function of two variables with $f(0,4) = 0$ and $\vec{\nabla} f(0,4) = \langle 2,\sqrt{3}\rangle$.
   (a) Find the maximum rate of change of f at $(0,4)$ and the direction in which it occurs.
   (b) Find the instantaneous rate of change of f at $(0,4)$ in the direction of the vector $\langle -1,\sqrt{3}\rangle$.

8. Let $f(x,y) = y + 4e^{x-y}$. Let O denote the origin and $\vec{V} = \hat{\imath} - 2\hat{\jmath}$.
   (a) Find the directional derivative of f at O in the direction of $\vec{V}$.
   (b) Find a unit vector along which f decreases most rapidly at O.
   (c) Find the greatest rate of increase of f at O.

9. Let S be a surface defined by the function $f(x,y) = \dfrac{y^2}{x}$.
   (a) Find the rate of change of f at $P(2,4)$ in the direction of the vector $\langle 1,-2\rangle$.
   (b) What is the minimum rate of change of f at P?
   (c) What is the vector equation of the normal line to S at $Q(2,4,8)$?

10. Given the function $f(x,y) = xe^y + \dfrac{\sin y}{x}$ and $P(1,0)$.
    (a) Give a unit vector along which f increases most rapidly at P.
    (b) Determine the directional derivative of f at P along the vector $\vec{V} = \langle -3,4\rangle$.

11. Let $f(x,y) = y^2e^{2x} + \cos(xy)$ and $A = (0,2)$.
    (a) Find the directional derivative of f on the point A in the direction of $\vec{v} = \hat{\imath} - \hat{\jmath}$.
    (b) Find the maximum rate of change of f at A.
    (c) Find a unit vector in the direction in which f decreases most rapidly at A.

(d) Find the equation of the tangent plane to surface given by the equation $f(x,y) = \ln(z-1)+5$ at the point $(0,2,5)$.

12. Find an equation of the tangent plane and vector equation of the normal line to the surface $z^2 = \dfrac{y}{x} + \ln(2y-z)$. at the point $(1,1,1)$.
13. Find an equation of the tangent plane and vector equation of the normal line to the surface $z = xy + \sin z$ at the point $(\pi, 1, \pi)$.
14. Determine the symmetric equations of the normal line to the surface S at the point $(0,0,1)$ if S is described the Cartesian equation $\ln(2y+z) = xz^2$.
15. Give the equation of the plane tangent to the ellipsoid $x^2 + 3y^2 + 4z^2 = 8$ at $(1,1,-1)$.

1.7 Relative Extrema of Functions of Two Variables

Recall that given a function $f(x)$, we can obtain its relative extrema by first finding its critical numbers, the values of x in the domain that satisfies $f'(x) = 0$ or make f' non-existent, and then using either the first derivative test or second derivative test on the critical numbers.

In this section, we will extend some of these results of relative maxima and minima for single variable functions to functions of two variables.

Definition 1.7.1. Let f be a function of x and y and let $(x_0, y_0) \in \mathrm{dom} f$.

1. The function f has a ***relative maximum*** or ***local maximum*** at (x_0, y_0) if for all points (x, y) in some disk centered at (x_0, y_0), $f(x_0, y_0) \geq f(x, y)$.
2. The function f has a ***relative minimum*** or ***local minimum*** at (x_0, y_0) if for all points (x, y) in some disk centered at (x_0, y_0), $f(x_0, y_0) \leq f(x, y)$.
3. If f has either a relative maximum or a relative minimum at (x_0, y_0), we say that f has a ***relative extremum*** at (x_0, y_0).

In single variable functions, if f has a relative extrema at $x = a$ and f' is differentiable at $x = a$, then $f'(a) = 0$. We have the following similar result for functions of two variables.

Theorem 1.7.2. If f has a relative extremum at (x_0, y_0) and the first-order partial derivatives of f exist at (x_0, y_0), then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

Proof. Consider the single variable function $g(x) = f(x, y_0)$. Suppose f has a relative extremum at (x_0, y_0). Then, g has a relative extremum (of the same type) at $x = x_0$. Therefore, $g'(x_0) = 0$. Since $g'(x_0) = f_x(x_0, y_0)$, we have $f_x(x_0, y_0) = 0$.

Similarly, if we consider $h(y) = f(x_0, y)$ and suppose that f has a relative extremum at (x_0, y_0), then h has a relative extremum (of the same type) at $y = y_0$. Therefore, $h'(y_0) = 0$. Since $h'(y_0) = f_y(x_0, y_0)$, we have $f_y(x_0, y_0) = 0$. □

We have the following analogous definition for a critical number of a function $f(x)$.

Definition 1.7.3. Let f be a function of x and y. A point $(x_0, y_0) \in \text{dom} f$ is called a ***critical point*** of f if any of the following holds:

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ and/or $f_y(x_0, y_0)$ does not exist.

Remarks

1. Theorem 1.7.2 states that if f has a relative extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.
2. Not all critical points yield a relative extremum.

Example 1.7.1. Find the critical points of $f(x, y) = x^2 - y^2$ and determine if these correspond to relative extrema of f.

Solution: Since f is a polynomial in x and y, then both partial derivatives exist at any point. Therefore, the only critical points are the solutions to the system

$$\begin{cases} f_x(x, y) = 2x = 0 & \implies x = 0 \\ f_y(x, y) = -2y = 0 & \implies y = 0. \end{cases}$$

Thus, $(0, 0)$ is the only critical point of f. Note that the graph of f is a hyperbolic paraboloid as shown in Figure 1.28.

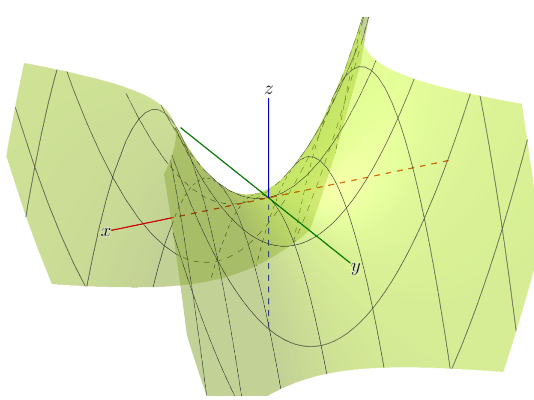


Figure 1.28: Graph of $f(x, y) = x^2 - y^2$

Observe that on the xz-coordinate plane, the trace is a parabola $z = x^2$ opening upward and its minimum is at $(0, 0)$ while on the yz-coordinate plane, the trace is a parabola $z = -y^2$ opening downward with its maximum at $(0, 0)$.

Any open disk centered at $(0, 0)$ contains a portion of each parabola. Hence, there is neither a relative maximum nor a relative minimum at $(0, 0)$. We call the point $(0, 0)$ a *saddle point*, as the surface is shaped like a saddle at $(0, 0)$.

Definition 1.7.4. Let $f(x, y)$ be a function and $(x_0, y_0) \in \text{dom } f$. We say that f has a ***saddle point*** at (x_0, y_0) if (x_0, y_0) is a critical point of f and f does not have a relative extremum at (x_0, y_0).

In most cases, to determine whether or not a function has a relative extremum at a critical point without looking at its graph, we use the following test which is analogous to the Second Derivative Test for functions of one variable.

Theorem 1.7.5 (Second Derivatives Test). Let f be a function of x and y with $\vec{\nabla} f(x_0, y_0) = \vec{0}$ for some $(x_0, y_0) \in \text{dom} f$. Suppose that the second-order partial derivatives of f are continuous on a disk centered at (x_0, y_0). Let

$$D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 .$$

1. If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).
2. If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has relative maximum at (x_0, y_0).
3. If $D(x_0, y_0) < 0$, then f has saddle point at (x_0, y_0).

Remark. No conclusion can be made about a critical point (x_0, y_0) if $D(x_0, y_0) = 0$.

Example 1.7.2. Find all the relative extrema and saddle points of $f(x, y) = 8 - 3y^2 - 6xy - 2x^3$.

Solution: Since f is a polynomial in x and y, its partial derivatives always exist. So the only critical points are the solutions to the system

$$\begin{cases} f_x(x, y) = -6y - 6x^2 = 0 & \text{(i)} \\ f_y(x, y) = -6y - 6x = 0 & \text{(ii)} \end{cases}$$

From (ii), we get $y = -x$. Plugging in to (i), we have $6x - 6x^2 6x(1 - x) = 0$. This means that either $x = 0$ or $x = 1$. Using the fact that $y = -x$, we get the critical points $(0, 0)$ and $(1, -1)$.

Next, we compute for the second-order partial derivatives of f given as follows:

$$f_{xx}(x, y) = -12x \qquad f_{yy}(x, y) = -6 \qquad f_{xy}(x, y) = -6.$$

By the Second Derivatives Test, we have the following conclusions.

(x_0, y_0)	f_{xx}	f_{yy}	f_{xy}	D	conclusion
$(0,0)$	0	-6	-6	-36	saddle point
$(1,-1)$	-12	-6	-6	36	relative maximum

Thus, f has a saddle point at $(0,0)$ and a relative maximum at $(1,-1)$.

Example 1.7.3. Find all relative extrema and saddle points of

$$f(x,y) = x^3 - 3x^2y + y^2 + 2y - 5.$$

Solution: The critical points of f are the solutions to the system

$$\begin{cases} f_x(x,y) = 3x^2 - 6xy = 0 & \text{(i)} \\ f_y(x,y) = -3x^2 + 2y + 2 = 0 & \text{(ii)} \end{cases}$$

From (i), we have

$$3x^2 - 6xy = 3x(x-2y) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = 2y.$$

We then analyze this by cases.
Case 1. If $x = 0$, then (ii) becomes

$$2y + 2 = 0 \quad \Longrightarrow \quad y = -1.$$

Thus, $(0,-1)$ is a critical point of f.
Case 2. If $x = 2y$, then (ii) becomes

$$\begin{aligned} -3(2y)^2 + 2y + 2 &= 0 \\ -12y^2 + 2y + 2 &= 0 \\ 2(3y+1)(2y-1) &= 0 \\ y = -\frac{1}{3} \text{ or } y &= \frac{1}{2} \end{aligned}$$

Using the fact that $x = 2y$, we obtain the critical points $\left(-\frac{2}{3}, -\frac{1}{3}\right)$ and $\left(1, \frac{1}{2}\right)$.

The second-order partial derivatives of f are given as follows:

$$f_{xx}(x,y) = 6x - 6y \qquad f_{yy}(x,y) = 2 \qquad f_{xy}(x,y) = -6x.$$

We now apply the Second Derivatives Test.

(x_0, y_0)	f_{xx}	f_{yy}	f_{xy}	D	*conclusion*
$(0,-1)$	6	2	0	12	relative minimum
$\left(-\frac{2}{3}, -\frac{1}{3}\right)$	-2	2	4	-20	saddle point
$\left(1, \frac{1}{2}\right)$	3	2	-6	-30	saddle point

Hence, $f(0,-1) = -6$ is a relative minimum and f has saddle points at $\left(-\frac{2}{3}, -\frac{1}{3}\right)$ and $\left(1, \frac{1}{2}\right)$.

Example 1.7.4. Consider the functions

$$f(x,y) = x^4 + y^4 \qquad g(x,y) = -x^4 - y^4 \qquad \text{and} \qquad h(x,y) = x^3 - y^3.$$

It is easy to verify that the only critical point of each function is $(0,0)$ and that the Second Derivatives Test fails since $D(0,0) = 0$. However, it can be seen graphically that f has a relative minimum at $(0,0)$, g has a relative maximum at $(0,0)$, and h has a saddle point at $(0,0)$.

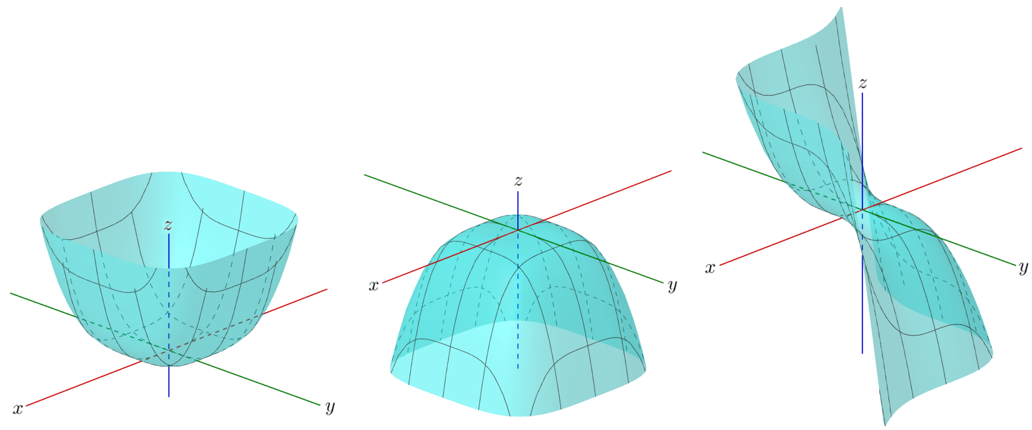


Figure 1.29: Graphs of the functions $f(x,y) = x^4 + y^4$ (left), $g(x,y) = -x^4 - y^4$ (middle), and $h(x,y) = x^3 - y^3$ (right)

EXERCISES 1.7

I. Find all critical points of f and use the Second Derivatives Test to determine whether f has a saddle point or a relative maximum or minimum at each of those points.

1. $f(x,y) = x^2 + y^3 - 6y^2 - 2x + 1$
2. $f(x,y) = \dfrac{1}{y} - \dfrac{64}{x} + xy$
3. $f(x,y) = x^2 - xy - y^2 - 3x - y$
4. $f(x,y) = x^4 - 4x^3 + 2y^2 + 8xy + 1$
5. $f(x,y) = x^3 + y^3 + 3y^2 - 3x - 9y + 2$
6. $f(x,y) = e^{xy} + 2$

II. Determine and classify all relative extrema and saddle points of the following functions using the Second Derivatives Test.

1. $f(x,y) = 3y^2x + x^3 - 3y^2 - 3x^2 + 3$
2. $f(x,y) = -x^2 + 2xy + \dfrac{1}{3}y^3 - 3y$
3. $f(x,y) = x^3 + 3xy - 3y^2 + 2$
4. $f(x,y) = x^4 + 3y^2 - 2x^2 - 6y + 5$
5. $f(x,y) = xy^2 - 6x^2 - 3y^2$
6. $f(x,y) = 4 + x^3 + y^3 - 3xy$
7. $f(x,y) = x^3 + 5y^3 - 3x^2y - 3y + 1$
8. $f(x,y) = 3x^3 + 2y^2 - 6xy + 1$
9. $f(x,y) = x^2 + y^2 + x^2y + 4$
10. $f(x,y) = 8 - 3y^2 + 6xy - 2x^3$
11. $f(x,y) = y^2 - 4xy + x^3 + 4x - 5$
12. $f(x,y) = (x-2)^2 + y - e^{y-1}$
13. $f(x,y) = y^2 + \dfrac{x}{2} - \sin(x + \pi)$

III. Do as indicated.

1. The function h is continuous on $\mathbb{R}^2$ with partial derivatives

$$h_x(x,y) = x^2 - y, \quad h_y(x,y) = y - x.$$

Find the critical points and use the Second Derivatives Test to determine whether there is a relative minimum, relative maximum or saddle point.

2. Suppose $f(x,y)$ is differentiable on $\mathbb{R}^2$ with

$$f_x(x,y) = x^2 + 2y, \quad f_y(x,y) = 2x - y.$$

At each critical point, use the Second Derivatives Test to determine whether there is a relative minimum, relative maximum or saddle point.

3. Given $f_x(x,y) = 2xy - 2y$ and $f_y(x,y) = x^2 + 4y^2 - 2x + 3y$, find the critical points and use the Second Derivatives Test to determine whether there is a relative minimum, relative maximum or saddle point.

1.8 Absolute Extrema of Functions of More than One Variable

We can also extend the concept of absolute or global extrema to functions of more than one variable. We have the following definition.

Definition 1.8.1. Let f be a function of x and y with domain $D \subseteq \mathbb{R}^2$ and let $(x_0, y_0) \in D$.

1. The function f has an ***absolute maximum*** or ***global maximum*** if for all $(x,y) \in D$, $f(x_0, y_0) \geq f(x,y)$.

2. The function f has an ***absolute minimum*** or ***global minimum*** if for all $(x,y) \in D$, $f(x_0, y_0) \leq f(x,y)$.

3. If f has either an absolute maximum or an absolute minimum at (x_0, y_0), we say that f has an ***absolute extremum*** at (x_0, y_0).

Recall that if a function f is continuous on $[a,b]$, the Extreme Value Theorem guarantees existence of the absolute extrema of f on $[a,b]$. We have an analogous result for functions of several variables. First, we define the concept of closed and boundedness on $\mathbb{R}^2$.

Definition 1.8.2. Let D be a region in $\mathbb{R}^2$. The point (x_0, y_0) is said to be a ***boundary point*** of D if every disk centered at (x_0, y_0) contains points in D and points not in D. The set of all boundary points of D is called the ***boundary*** of D.

Example 1.8.1.

1. Let $D_1 = \{(x, y) : x^2 + y^2 \leq 1\}$. Then $(1, 0)$ is a boundary point of D since a disk centered at $(1, 0)$ with radius $r > 0$ will contain $(1 - \frac{r}{2}, 0)$ and $(1 + \frac{r}{2}, 0)$, which are inside and outside D_1, respectively. In fact, the curve $x^2 + y^2 = 1$ is the boundary of D_1.

2. Let $D_2 = \{(x, y) : x^2 + y^2 < 1\}$. The curve $x^2 + y^2 = 1$ is also the boundary of D_2.

3. Let D_3 be the unit circle $x^2 + y^2 = 1$. Each point of D_3 is a boundary point of D_3. In fact, D_3 is the boundary of D_3.

Definition 1.8.3. Let D be a region in $\mathbb{R}^2$.

1. D is said to be ***closed*** if it contains all its boundary points.

2. D is said to be ***bounded*** if it is a subregion of a closed disk.

Example 1.8.2. From Example 1.8.1, D_1 and D_3 are both closed and bounded domains.

The following is the extension of the Extreme Value Theorem to functions of two variables.

Theorem 1.8.4. Let f be a function of x and y. If f is continuous on a closed and bounded region D, then f has an absolute maximum and an absolute minimum on D.

Remark: Steps in finding the absolute extrema of f on a closed and bounded region D.

(i) Find all critical points of f lying inside D.

(ii) Partition the boundary into a union of curves. Restrict f on each boundary curve, thereby getting a function of a single variable. Find all critical numbers of the resulting function.

(iii) Evaluate f at all points obtained in (i) and (ii), and at the intersection points of the boundary curves of D. The largest among these values is the absolute maximum value of f on D while the least is the absolute minimum value of f on D.

Example 1.8.3. Find the absolute extrema of $f(x, y) = x^2 + 3y^2 - 4x - 6y$ on the triangular region with vertices at the points (0,0), (4,0), and (4,4).

Solution: Note that the triangular region D is closed and bounded and f is continuous on D. Hence, f has both absolute maximum and absolute minimum values on D due to the Extreme Value Theorem.

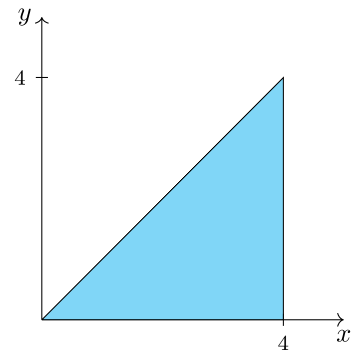


Figure 1.30

(i) First, we look for all critical points of f inside D.

$$\begin{cases} f_x(x,y) = 2x - 4 = 0 & \Longrightarrow \quad x = 2 \\ f_y(x,y) = 6y - 6 = 0 & \Longrightarrow \quad y = 1 \end{cases}$$

So, $(2,1)$ is a critical point of f that lies inside D.

(ii) Next, we look for points on the boundary of D where extrema may occur.

(a) portion of the x-axis: $y = 0$, $0 \le x \le 4$

$$f(x,0) = \alpha(x) = x^2 - 4x$$
$$\alpha'(x) = 2x - 4$$
$$\alpha'(x) = 0 \Rightarrow x = 2$$

Extremum may occur at $(2,0)$.

(b) portion of the line $x = 4$, $0 \le y \le 4$

$$f(4,y) = \beta(y) = 3y^2 - 6y$$
$$\beta'(y) = 6y - 6$$
$$\beta'(y) = 0 \Rightarrow y = 1$$

Extremum may occur at $(4,1)$.

(c) portion of the line $y = x$, $0 \le x \le 4$

$$f(x,x) = \gamma(x) = 4x^2 - 10x$$
$$\gamma'(x) = 8x - 10$$
$$\gamma'(x) = 0 \Rightarrow x = \frac{5}{4}$$

Extremum may occur at $\left(\frac{5}{4}, \frac{5}{4}\right)$.

(iii) Finally, we compare all function values at the points obtained in (i) and (ii), and at the intersection points of the boundary curves.

(x_0, y_0)	$f(x_0, y_0)$
(2,1)	-7
(2,0)	-4
(4,1)	-3
$(\frac{5}{4}, \frac{5}{4})$	$-\frac{25}{4}$
(0,0)	0
(4,0)	0
(4,4)	24

Thus, the absolute minimum value of f on D is -7 and the absolute maximum value of f on D is 24.

Suppose that we are finding the absolute extrema of $f(x, y)$ over all points (x, y) that satisfy $g(x, y) = k$, where k is a constant (see Figure 1.31). In this case, we say that we are *maximizing or minimizing* $f(x, y)$ *subject to the constraint* $g(x, y) = k$. To do this, we use the following theorem.

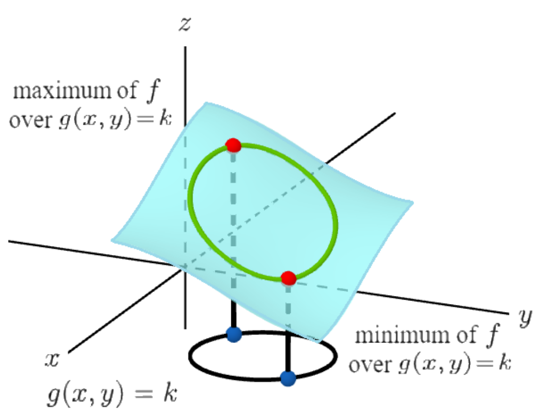


Figure 1.31: Extrema over $g(x, y) = k$

Theorem 1.8.5. Suppose f and g are functions of x and y with continuous first-order partial derivatives. If f has a relative extremum at a point (x_0, y_0) subject to the constraint $g(x, y) = k$, where k is a constant, and $\vec{\nabla} g(x_0, y_0) \neq \vec{0}$, then there exists a constant λ such that

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0). \tag{1.3}$$

Proof. If $\vec{\nabla} f(x_0, y_0) = \vec{0}$ then equation (1.3) holds if $\lambda = 0$. Suppose $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$. We want to show that $\vec{\nabla} g(x_0, y_0)$ is parallel to $\vec{\nabla} f(x_0, y_0)$. Represent the curve defined by $g(x, y) = k$ by a vector function $\vec{R}(t) = \langle x(t), y(t)\rangle$ such that $\vec{R}(t_0) = \langle x_0, y_0\rangle$ and $\vec{R}'(t_0) \neq \vec{0}$. Let $\phi(t) = f(x(t), y(t))$. Since f has a relative extremum at (x_0, y_0), we have

$$\phi'(t_0) = 0. \tag{1.4}$$

Meanwhile,

$$\begin{aligned}\phi'(t) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \vec{\nabla} f(x,y) \cdot \vec{R}'(t). \end{aligned} \tag{1.5}$$

By Equations (1.4) and (1.5), we have

$$\phi'(t_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{R}'(t_0) = 0.$$

This implies

$$\vec{\nabla} f(x_0, y_0) \perp \vec{R}'(t_0).$$

But by Theorem 1.6.6, $\vec{\nabla} g(x_0, y_0) \perp \vec{R}'(t_0)$. Therefore, $\vec{\nabla} g(x_0, y_0)$ is parallel to $\vec{\nabla} f(x_0, y_0)$; that is $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$, for some constant λ. □

Remarks.

1. The constant λ in Theorem 1.8.5 is called a *Lagrange multiplier*.

2. Theorem 1.8.5 also holds if f and g are functions of three variables. Equation (1.3) leads to the following method.

Method of Lagrange Multiplier

To find the maximum and minimum values of a function subject to a constraint (assuming these values exist):

(i) If f is a function of x and y, solve for all points (x, y) satisfying the equations:

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = k \qquad\qquad \text{(constraint)} \end{cases}$$

If f is a function of x,y and z, solve for all points (x, y, z) satisfying the equations:

$$\begin{cases} f_x(x,y,z) = \lambda g_x(x,y,z) \\ f_y(x,y,z) = \lambda g_y(x,y,z) \\ f_z(x,y,z) = \lambda g_z(x,y,z) \\ g(x,y,z) = k \qquad\qquad \text{(constraint)} \end{cases}$$

(ii) Evaluate f at all points obtained in (i). The largest of these values is the absolute maximum value of f subject to the constraint while the least is the absolute minimum value of f subject to the constraint.

Example 1.8.4. Use the method of Lagrange multiplier to find the absolute maximum and absolute minimum values of $f(x, y) = 4x^3 + y^2$ on the ellipse $2x^2 + y^2 = 1$.

Solution: Let $g(x, y) = 2x^2 + y^2$. We solve for all points (x, y) satisfying the system

$$\begin{cases} 12x^2 = 4x\lambda & \text{(i)} \\ 2y = 2y\lambda & \text{(ii)} \\ 2x^2 + y^2 = 1 & \text{(iii)} \end{cases}.$$

Equation (ii) is equivalent to

$$\begin{aligned} 2y - 2y\lambda &= 0 \\ 2y(1-\lambda) &= 0 \\ y = 0 \text{ or } \lambda &= 1. \end{aligned}$$

Case 1. If $y = 0$ then $x = \pm\frac{1}{\sqrt{2}}$ by (iii). Extrema may occur at $\left(\frac{1}{\sqrt{2}}, 0\right)$ and $\left(-\frac{1}{\sqrt{2}}, 0\right)$.

Case 2. If $\lambda = 1$ then from (i),

$$\begin{aligned} 12x^2 - 4x &= 0 \\ 4x(3x-1) &= 0 \\ x = 0 \text{ or } x &= \frac{1}{3}. \end{aligned}$$

From (iii), if $x = 0$ then $y = \pm 1$. If $x = \frac{1}{3}$ then

$$\begin{aligned} y^2 &= 1 - 2\left(\frac{1}{3}\right)^2 \\ &= 1 - \frac{2}{9} = \frac{7}{9} \\ y &= \pm\frac{\sqrt{7}}{3}. \end{aligned}$$

So extrema may occur at $(0, 1)$, $(0, -1)$, $\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right)$ and $\left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$.

The potential absolute extrema of f are then,

(x_0, y_0)	$f(x_0, y_0)$
$\left(\frac{1}{\sqrt{2}}, 0\right)$	$\sqrt{2}$
$\left(-\frac{1}{\sqrt{2}}, 0\right)$	$-\sqrt{2}$
$(0, 1)$	1
$(0, -1)$	1
$\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right)$	$\frac{25}{27}$
$\left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$	$\frac{25}{27}$

Thus, subject to the constraint, the absolute maximum value of f is $\sqrt{2}$ and its absolute minimum value is $-\sqrt{2}$.

Example 1.8.5. Find the dimensions of the rectangular box with an open top with largest volume if the total surface area is 216 cm^2.

Solution: Let x, y, z be the length, width and height, respectively, of the rectangular box with an open top.

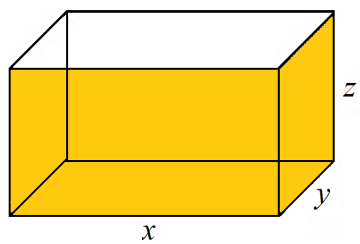


Figure 1.32: Rectangular box with open top

We want to maximize the volume $f(x, y, z) = xyz$ subject to the surface area $xy + 2xz + 2yz = 216$. Let $g(x, y, z) = xy + 2xz + 2yz$. We solve for points (x, y, z) satisfying

$$\begin{cases} yz = (y + 2z)\lambda & \text{(i)} \\ xz = (x + 2z)\lambda & \text{(ii)} \\ xy = (2x + 2y)\lambda & \text{(iii)} \\ xy + 2xz + 2yz = 216. & \text{(iv)} \end{cases}$$

Note that the Lagrange multiplier λ in this problem is nonzero (*Why?*). Multiplying equation (i) by x, equation (ii) by y, and equation (iii) by z, we get

$$\begin{cases} xyz = (y + 2z)x\lambda & \text{(v)} \\ xyz = (x + 2z)y\lambda & \text{(vi)} \\ xyz = (2x + 2y)z\lambda & \text{(vii)} \\ xy + 2xz + 2yz = 216. \end{cases}$$

Subtracting (v) and (vi), we have

$$\begin{aligned} (y + 2z)x\lambda - (x + 2z)y\lambda &= 0 \\ 2xz\lambda - 2yz\lambda &= 0 \\ 2z\lambda(x - y) &= 0 \\ z = 0 \text{ or } x &= y\ . \end{aligned}$$

Since the height $z > 0$, we have $x = y$. From equations (vi) and (vii), we have

$$\begin{aligned}(x+2z)y\lambda - (2x+2y)z\lambda &= 0\\ xy\lambda - 2xz\lambda &= 0\\ x\lambda(y-2z) &= 0\\ x = 0 \text{ or } y &= 2z\ .\end{aligned}$$

Since $x > 0$, we have $y = 2z$. Therefore, the box with an open top whose volume is the largest possible has dimensions $x \times y \times z$ where $x = y = 2z$. Equation (iv) becomes

$$\begin{aligned}4z^2 + 2(2z)z + 2(2z)z &= 216\\ 12z^2 &= 216\\ z^2 &= 18.\end{aligned}$$

Thus, the solutions of the system of equations (i) - (iv) are $(x, y, z) = (6\sqrt{2}, 6\sqrt{2}, 3\sqrt{2})$ and $(x, y, z) = (-6\sqrt{2}, -6\sqrt{2}, -3\sqrt{2})$.

Note that $f(6\sqrt{2}, 6\sqrt{2}, 3\sqrt{2}) > 0$ and $f(-6\sqrt{2}, -6\sqrt{2}, -3\sqrt{2}) < 0$. So subject to the constraint, f has an absolute maximum at $(x, y, z) = (6\sqrt{2}, 6\sqrt{2}, 3\sqrt{2})$, that is, the largest box with an open top whose surface area is 216 cm^2 has length $6\sqrt{2}$ cm, width $6\sqrt{2}$ cm, and height $3\sqrt{2}$ cm.

EXERCISES 1.8

I. Find the absolute maximum and minimum values of f on the set D.

1. $f(x, y) = xy - x - 3y + 1$; D is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(5, 0)$
2. $f(x, y) = xe^y - x^2 - e^y + 2$; D is the rectangular region with vertices $(0, 0)$, $(0, 1)$, $(2, 1)$, and $(2, 0)$
3. $f(x, y) = x^2y$; $D = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$

II. Use the method of Lagrange multiplier to determine the absolute maximum and minimum values of the function subject to the given constraint.

1. $f(x, y) = 2x^2 + 4y^2 + 5$; $x^2 + y^2 - 2x = 0$
2. $f(x, y) = xy^2$; $8x^2 + y^2 = 24$
3. $f(x, y, z) = xyz - 1$; $x^2 + y^2 + z^2 = 1$
4. $f(x, y, z) = x^2 + y^2 + z^2$; $x^4 + y^4 + z^4 = 1$

III. Do as indicated.

1. Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible.
2. Find the point on the parabola $y = x^2$ closest to the point $(-3, 0)$.

3. Suppose that the temperature at a point (x, y) on a metal plate is given by $T(x, y) = 4y^2 - 4xy + x^2 + 2$. An ant, walking on the plate, traverses a circle of radius 5 centered at the origin. Determine the highest and lowest temperatures encountered by the ant.

IV. Use the method of Lagrange multipliers to find the minimum and maximum values of the following functions subject to their respective constraint curves/surfaces.

1. $f(x, y) = xy^2$, subject to the constraint $2x^2 + y^2 = 2$
2. $f(x, y) = x^2 - x + y^2$, subject to the constraint $2x^2 + y^2 = 1$
3. $f(x, y) = xy$, subject to the constraint $x^2 + 3y^2 = 6$
4. $f(x, y) = xy$, subject to the constraint $x^2 + 2y^2 = 4$
5. $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ subject to the constraint $x^2 + y^2 = 4$
6. $f(x, y) = 5x - y + 1$, subject to the constraint $10x^2 + y^2 = 56$
7. $f(x, y) = 25 - x^2 - y^2$, subject to the constraint $x^2 + y^2 - 4y = 0$
8. $g(x, y, z) = 10x - 8y + 6z$, subject to the constraint $x^2 + y^2 + z^2 = 50$
9. $g(x, y, z) = 3x + 6y + 2z$, subject to the constraint $2x^2 + 4y^2 + z^2 = 70$

V. Do as indicated.

1. Use the method of Lagrange multipliers to find the lowest point on the curve of intersection of the plane $4y - x = 5$ and the paraboloid $z = x^2 + y^2$.
2. Use the method of Lagrange multipliers to find the points on the ellipse $2x^2 + y^2 = 18$ which are closest to and farthest from the point $(2, 0)$.

   (*Hint: Consider the square of the distance from* $P(x, y)$ *to* $(2, 0)$*, i.e.,*

   $$f(x, y) = (x - 2)^2 + y^2.)$$

3. Use the method of Lagrange multipliers to find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 5$.

   (*Hint: Consider the square of the distance from* $P(x, y, z)$ *to* $(1, 0, -2)$*, i.e.,*

   $$f(x, y, z) = (x - 1)^2 + y^2 + (z + 2)^2.)$$

4. Determine the dimensions of a rectangular box, open at the top having a volume of 32 cubic ft and requiring the least amount of material for its construction.
5. A circular metal wire with equation $x^2 + y^2 = 1$ is heated so that the temperature at any point (x, y) on the wire is $T(x, y) = x^2 - y^2$. Determine the hottest and coldest points on the wire.
6. Given $f(x, y) = -x^2 - 4y^2$.

(a) Find the extreme values of f on the circle $x^2 + y^2 = 4$ using the method of Lagrange multipliers.

(b) Using the result in (a), determine the extreme values of f on the disk $x^2 + y^2 \leq 4$.

7. Find the absolute extrema of the function $f(x, y) = 1 + xy - x - y$ on the region bounded by the graphs of $y = x^2$ and $y = 4$.

1.9 Parametric Surfaces

Let

$$\vec{R}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k} \tag{1.6}$$

be a vector-valued function of two variables on a region D in the uv-plane. The collection of all points $(x, y, z) \in \mathbb{R}^3$ such that

$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v) \\ z &= z(u, v) \end{aligned} \tag{1.7}$$

as (u, v) varies throughout D is called a *parametric surface* S. Equations (1.6) and (1.7) are respectively called the *vector equation* and *parametric equations* of S. The variables u and v are referred to as *parameters*. The surface S is traced out by the tip of the position vector $\vec{R}(u, v)$ as (u, v) varies throughout the parameter domain D.

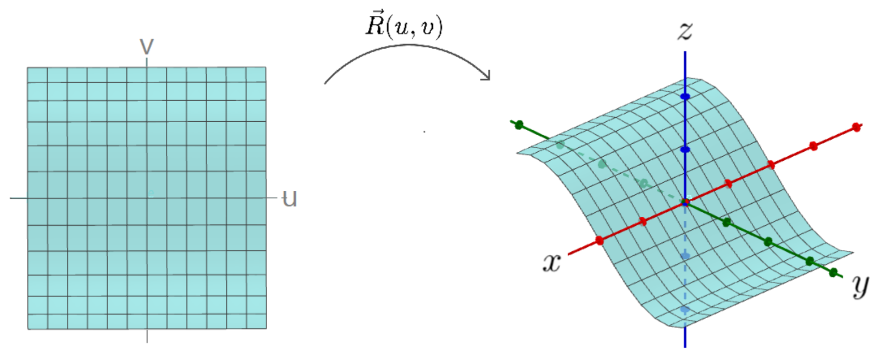


Figure 1.33: Mapping of the region D to the parametric surface S

If u is held constant, say $u = u_0$, and v varies, then $\vec{R}(u_0, v)$ defines a curve on S called a *constant u-curve*. If v is held constant, say $v = v_0$, and u varies, then $\vec{R}(u, v_0)$ defines a curve on S called a *constant v-curve*. Constant u-curves and constant v-curves partition the surface S into portions called *patches*, as shown in Figure 1.34.

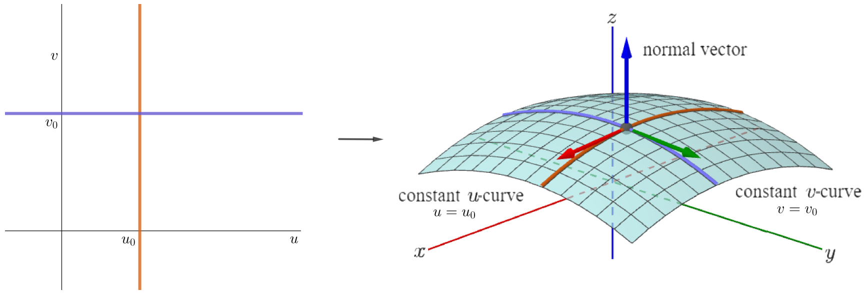


Figure 1.34: Constant u and v curves

Example 1.9.1. Sketch a portion of the parametric surface with vector equation

$$\vec{R}(u,v) = \langle u + 2v, u - v + 3, 2u + v \rangle \ .$$

Solution: The vector equation is equivalent to the parametric equations

$$\begin{cases} x = u + 2v \\ y = u - v + 3 \\ z = 2u + v. \end{cases}$$

Adding the first two equations gives $x + y = 2u + v + 3$. Substituting z in place of $2u + v$, we get $x + y = z + 3$ or $x + y - z = 3$, which is an equation of the plane through $(3, 0, 0)$, $(0, 3, 0)$, and $(0, 0, -3)$.

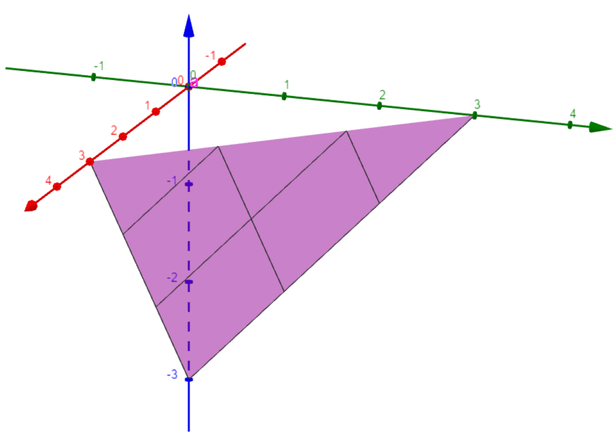


Figure 1.35: The plane $x + y - z = 3$

Example 1.9.2. Describe the parametric surface defined by

$$\vec{R}(u,v) = u \cos v \ \hat{\imath} + u \sin v \ \hat{\jmath} + (4 - u^2)\hat{k}.$$

Identify all constant u-curves and the constant v-curves with $v = 0$ and $v = \frac{\pi}{2}$.

Solution: The parametric equations of the surface are

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = 4 - u^2 \end{cases}.$$

Considering $x^2 + y^2$, we get

$$x^2 + y^2 \;=\; u^2\cos^2 v + u^2\sin^2 v \;=\; u^2 \;=\; 4 - z.$$

Thus, the given parametric surface is the paraboloid

$$z = 4 - x^2 - y^2.$$

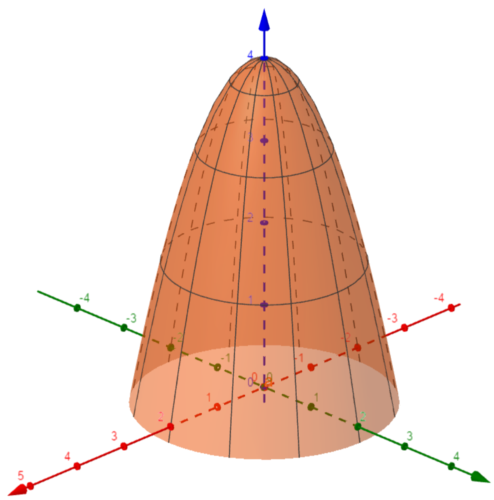


Figure 1.36: The paraboloid $z = 4 - x^2 - y^2$

As for the constant u and v-curves, we get the following.

(i) If $u = u_0$, then z is constant; that is, $z = 4 - u_0^2$. Thus, the constant u-curves are the traces of the surface on planes parallel to the xy-plane. These constant u-curves are circles of the form $x^2 + y^2 = u_0^2$.

(ii) If $v = 0$, then $x = u$, $y = 0$ and $z = 4 - u^2$. This gives the parabola $z = 4 - x^2$ on the xz-plane.

(iii) If $v = \frac{\pi}{2}$, then $x = 0$, $y = u$ and $z = 4 - u^2$. This gives the parabola $z = 4 - y^2$ on the yz-plane.

Remark. Surfaces of revolution may be represented by parametric equations. Let $f(x) \geq 0$ and S be the surface obtained by revolving the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis. Let θ be the

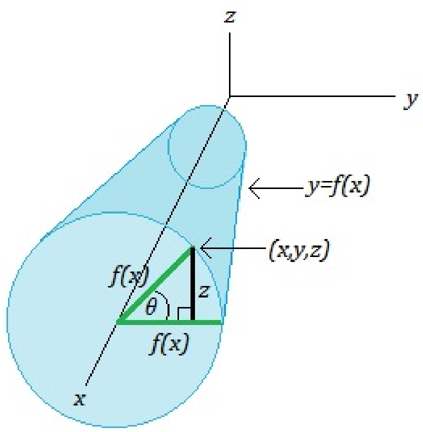


Figure 1.37: Surface of revolution

angle of rotation as shown in Figure 1.37.

Then S has parametric equations

$$\begin{cases} x = x \\ y = f(x)\cos\theta \\ z = f(x)\sin\theta. \end{cases}$$

Similarly, if S is obtained by revolving $y = f(z)$ about the z-axis, then S has parametric equations

$$\begin{cases} x = f(z)\sin\theta \\ y = f(z)\cos\theta \\ z = z. \end{cases}$$

If S is obtained by revolving $x = f(y)$ about the y-axis, then S has parametric equations

$$\begin{cases} x = f(y)\cos\theta \\ y = y \\ z = f(y)\sin\theta. \end{cases}$$

Example 1.9.3. Give a set of parametric equations for the surface obtained by revolving the curve $y = e^{-x}$ about the x-axis.

Solution: If we fix x, we get the parametric equations

$$\begin{cases} x = x \\ y = e^{-x}\cos\theta \\ z = e^{-x}\sin\theta. \end{cases}$$

Example 1.9.4. Give a set of parametric equations for the hyperboloid of one sheet

$$4x^2 - y^2 + 4z^2 = 4.$$

Solution: The given hyperboloid is the surface generated by revolving the hyperbola $4x^2 - y^2 = 4$ on the xy-plane about the y-axis. The generating curve may be written as

$$\begin{aligned} 4x^2 - y^2 = 4 &\Longrightarrow x^2 = \frac{4+y^2}{4} \\ &\Longrightarrow x = \pm\frac{\sqrt{4+y^2}}{2}\ . \end{aligned}$$

But because of the symmetry of $x = \pm\frac{\sqrt{4+y^2}}{2}$ about the y-axis, we may take $x = \frac{\sqrt{4+y^2}}{2}$ as the generator. Hence, by fixing y, the hyperboloid has parametric equations

$$\begin{cases} x = \frac{\sqrt{4+y^2}}{2}\cos\theta \\ y = y \\ z = \frac{\sqrt{4+y^2}}{2}\sin\theta. \end{cases}$$

Definition 1.9.1. Let $\vec{R}$ be a vector valued function of two parameters u and v; that is

$$\vec{R}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle\ .$$

The partial derivatives of $\vec{R}$ with respect to u and v, denoted $\vec{R}_u$ and $\vec{R}_v$ respectively, are the vector-valued functions defined by

$$\begin{aligned} \vec{R}_u(u,v) &:= \lim_{\Delta u\to 0}\frac{\vec{R}(u+\Delta u, v) - \vec{R}(u,v)}{\Delta u} \\ \vec{R}_v(u,v) &:= \lim_{\Delta v\to 0}\frac{\vec{R}(u, v+\Delta v) - \vec{R}(u,v)}{\Delta v} \end{aligned}$$

if these limits exist.

Remark. If the first partials of $x(u,v)$, $y(u,v)$ and $z(u,v)$ exists, then

$$\begin{aligned} \vec{R}_u(u,v) &= \langle x_u(u,v), y_u(u,v), z_u(u,v)\rangle \\ \vec{R}_v(u,v) &= \langle x_v(u,v), y_v(u,v), z_v(u,v)\rangle. \end{aligned}$$

Theorem 1.9.2. Let $S : \vec{R}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ such that $\vec{R}_u(u_0,v_0)$ and $\vec{R}_v(u_0,v_0)$ both exist and are nonzero vectors. Then $\vec{R}_u(u_0,v_0)\times\vec{R}_v(u_0,v_0)$ is normal to S at the point on S corresponding to $u = u_0$ and $v = v_0$.

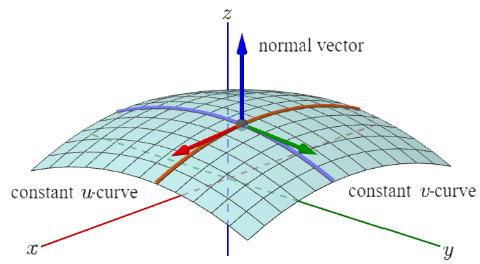


Figure 1.38

Proof. Let $P(x_0, y_0, z_0)$ be the point on S corresponding to $u = u_0$ and $v = v_0$, that is, $\vec{R}(u_0, v_0) = \langle x_0, y_0, z_0 \rangle$.

If $\vec{R_u}(u_0, v_0) \neq \vec{0}$, then $\vec{R_u}(u_0, v_0)$ is a tangent vector to the constant v-curve with $v = v_0$ at P. Similarly, if $\vec{R_v}(u_0, v_0) \neq \vec{0}$, then $\vec{R_v}(u_0, v_0)$ is a tangent vector to the constant u-curve with $u = u_0$ at P (see Figure 1.38).

Then, $\vec{R_u}(u_0, v_0) \times \vec{R_v}(u_0, v_0)$ is orthogonal to the tangent plane to S at P which contains $\vec{R_u}(u_0, v_0)$ and $\vec{R_v}(u_0, v_0)$. This proves that the vector $\vec{R_u}(u_0, v_0) \times \vec{R_v}(u_0, v_0)$ is normal to S at P. □

Example 1.9.5. Find an equation of the tangent plane to the surface S defined by the vector function $\vec{R}(u, v) = \langle \sin v, v - 1, e^u \rangle$ at the point where $u = v = 0$.

Solution: Note that $\vec{R}(0,0) = \langle 0, -1, 1 \rangle$ and so the point of tangency is $P(0, -1, 1)$. We then solve for the partial derivatives $\vec{R_u}(0,0)$ and $\vec{R_v}(0,0)$.

$$\begin{aligned} \vec{R_u}(u,v) &= \langle 0, 0, e^u \rangle & \Longrightarrow \quad \vec{R_u}(0,0) &= \langle 0, 0, 1 \rangle \\ \vec{R_v}(u,v) &= \langle \cos v, 1, 0 \rangle & \Longrightarrow \quad \vec{R_v}(0,0) &= \langle 1, 1, 0 \rangle \end{aligned}$$

A normal vector to S at the point $(0, -1, 1)$ is given by

$$\vec{N} = \vec{R_u}(0,0) \times \vec{R_v}(0,0) = \langle -1, 1, 0 \rangle \ .$$

Thus, an equation of the tangent plane to S at the point $(0, -1, 1)$ is $-x + (y + 1) = 0$.

Example 1.9.6. Find a vector equation of the normal line to the surface S defined by the vector equation $\vec{R}(u, v) = \left\langle u^2, v^2, u + v \right\rangle$ at the point $(1, 4, 3)$.

Solution: First, we solve for the unique ordered pair (u, v) that gives the point $(1, 4, 3)$, that is, solving the following system.

$$\begin{cases} x = u^2 = 1 & \Longrightarrow \quad u = \pm 1 \\ y = v^2 = 4 & \Longrightarrow \quad v = \pm 2 \\ z = u + v = 3. \end{cases}$$

This means that the point $(x, y, z) = (1, 4, 3)$ corresponds to the point $(u, v) = (1, 2)$. Computing for $\vec{R_u}(1, 2)$ and $\vec{R_v}(1, 2)$, we get

$$\vec{R_u}(u, v) = \langle 2u, 0, 1\rangle \quad \Longrightarrow \quad \vec{R_u}(1, 2) = \langle 2, 0, 1\rangle$$
$$\vec{R_v}(u, v) = \langle 0, 2v, 1\rangle \quad \Longrightarrow \quad \vec{R_v}(1, 2) = \langle 0, 4, 1\rangle.$$

A normal vector to S at the point $(1, 4, 3)$ is given by

$$\vec{N} \;=\; \vec{R_u}(1, 2) \times \vec{R_v}(1, 2) \;=\; \langle -4, -2, 8\rangle.$$

Thus, a vector equation of the normal line to S at the point $(1, 4, 3)$ is

$$\vec{R}(t) = \langle 1 - 4t, 4 - 2t, 3 + 8t\rangle.$$

EXERCISES 1.9

I. Find a parametrization of the surface. (There is more than one way to do these.)

1. The paraboloid $x = 4 - y^2 - z^2$, $x \geq 0$
2. The portion of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 3$
3. The portion of the plane $2x + 2y + z = 2$ inside the cylinder $x^2 + y^2 = 9$

II. Identify the given parametric surface by finding a corresponding Cartesian equation.

1. $\vec{R}(u, v) = \langle 2u + v, u - 2v, 2u + 2v\rangle$
2. $\vec{R}(r, \theta) = \langle r\cos\theta, r\sin\theta, 2r\rangle$
3. $\vec{R}(\theta, \phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$

III. Find an equation of the tangent plane and the normal line to the parametric surface at the given point.

1. $\vec{R}(u, v) = \langle u + v, 3u^2, u - v\rangle$; $(u, v) = (1, 1)$
2. $\vec{R}(u, v) = \langle u + v, u\cos v, v\sin u\rangle$; $(u, v) = (1, 0)$
3. $\vec{R}(u, v) = \langle u^2 - v^2, 3u + 2v, v - 2u\rangle$; $P(0, 1, -3)$
4. $\vec{R}(u, v) = \langle uv, ue^v, ve^u\rangle$; $O(0, 0, 0)$

Chapter 2

Multiple Integration

2.1 Double Integrals over Rectangular Regions

In calculus of one variable, we used the area under the curve as the motivation for defining definite integrals. In a similar manner, we use the volume under the surface as the motivation for introducing double integrals.

Volume Problem. Let $R = [a, b] \times [c, d] = \{(x, y) \,|\, a \leq x \leq b, c \leq y \leq d\}$, a rectangular region on the xy-plane, and let f be a function of x and y such that f is continuous on R and $f(x, y) \geq 0$, for all $(x, y) \in R$. Find the volume of the solid under $z = f(x, y)$ over the rectangle R, as in Figure 2.1.

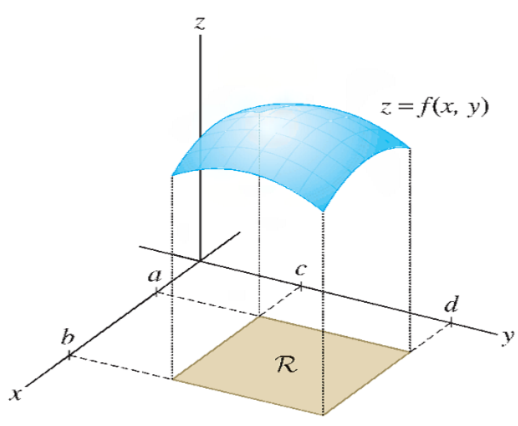


Figure 2.1: Solid over a rectangle

To answer the problem, we proceed as follows.

- Set $x_0 = a, x_m = b, y_0 = c, y_n = d$. Partition R into subrectangles by dividing $[a, b]$ into m subintervals $[x_0, x_1]$, $[x_1, x_2], \ldots, [x_{m-1}, x_m]$, where each subinterval $[x_{i-1}, x_i]$ has length Δx, and $[c, d]$ into n subintervals $[y_0, y_1], [y_1, y_2], \ldots, [y_{n-1}, y_n]$, where each subinterval $[y_{j-1}, y_j]$ has length Δy. Call this partition P. The area of each subrectangle R_k of the partition P is thus $\Delta A_k = \Delta x \Delta y$.

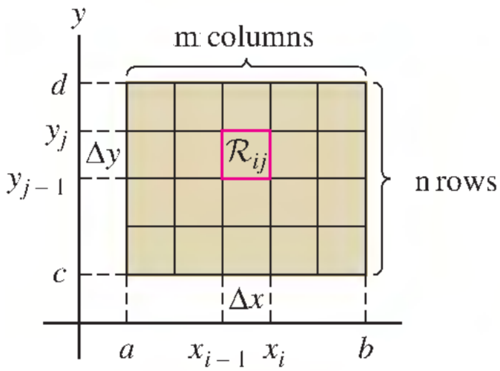


Figure 2.2: Partitioning of the region R

- Take an arbitrary point (x_k^*, y_k^*) in each subrectangle R_k and construct rectangular prisms with bases R_k and height $f(x_k^*, y_k^*)$, as shown in Figure 2.3. Then, the volume of each prism is

$$\Delta V_k = f(x_k^*, y_k^*)\Delta A_k.$$

- A good approximation for the volume of the solid is

$$S_k = \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k, \tag{2.1}$$

where this approximation becomes more accurate as the number of subrectangles $N = mn$ increases.

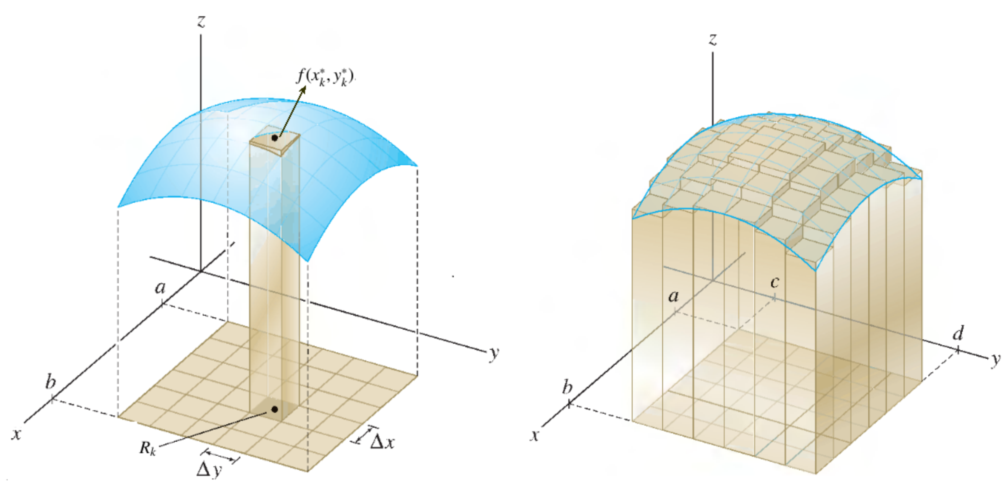


Figure 2.3: Approximation of the solid using rectangular prisms

The sum S_k in equation (2.1) is called a *Riemann sum* and the limit of these Riemann sums as the area of each subrectangle goes to zero will be used to define the double integral of f over the rectangle R. Define the *norm* $\|P\|$ *of a partition* P to be the longest diagonal of any subrectangle in the partition. If the function f is continuous, one can show that the Riemann sums S_k converge to some limit as $\|P\| \to 0$, which we denote by

$$\lim_{\|P\|\to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k.$$

If this limit exists, it is independent of the choice of points (x_k^*, y_k^*) in R_k. As the norm $\|P\|$ tends to zero, the subrectangles R_k get smaller so their number N increases. Hence we may also write the above limit as

$$\lim_{N\to\infty} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k$$

if we know that $\Delta A_k \to 0$ as $N \to \infty$ (For example, this happens when the partition divides the region into congruent partitions.)

Definition 2.1.1. Let f be a function of x and y and $R = [a, b] \times [c, d]$. The ***double integral*** of f over the rectangle R is defined by

$$\iint\limits_R f(x, y)\, dA := \lim_{\|P\|\to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k \ , \tag{2.2}$$

provided this limit exists.

If the double integral of f over R exists, we say that f is *integrable* on R. The following theorem gives a sufficient condition for which a function of two variables is integrable.

Theorem 2.1.2. If a function of two variables is continuous on a closed rectangular region R, then it is integrable on R.

It is worth noting that several discontinuous functions are also integrable, including functions which are discontinuous only at a finite number of points or at smooth curves.

The following theorem gives a practical method for calculating double integrals over rectangular regions.

Theorem 2.1.3 (Fubini's Theorem). If f is continuous on $R = [a, b] \times [c, d]$, then

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy \tag{2.3}$$

$$= \int_a^b \int_c^d f(x, y)\, dy\, dx \ . \tag{2.4}$$

The right hand sides of Equations (2.3) and (2.4) are called *iterated integrals*. To evaluate

$$\int_c^d \int_a^b f(x,y)\,dx\,dy,$$

we first integrate $f(x,y)$ with respect to x from $x=a$ to $x=b$ (holding y fixed). This method is called *partial integration with respect to* x. Then we integrate the resulting function of y from $y=c$ to $y=d$.

Similarly, to evaluate

$$\int_a^b \int_c^d f(x,y)\,dy\,dx,$$

we first integrate $f(x,y)$ with respect to y from $y=c$ to $y=d$ (holding x fixed). This method is called *partial integration with respect to* y. Then we integrate the resulting function of x from $x=a$ to $x=b$.

Example 2.1.1. Evaluate $\displaystyle\iint\limits_R (2xy - x + y)\,dA$, where $R=[2,4]\times[-3,6]$.

Solution: We have

$$\begin{aligned}\iint\limits_R (2xy-x+y)\,dA &= \int_{-3}^6\int_2^4 (2xy-x+y)\,dx\,dy = \int_{-3}^6 \left(x^2y - \frac{x^2}{2} + xy\right)\Bigg|_{x=2}^{x=4} dy\\ &= \int_{-3}^6 \left[(16y-8+4y)-(4y-2+2y)\right]\,dy = \int_{-3}^6 (14y-6)\,dy\\ &= \left(7y^2-6y\right)\Big|_{-3}^6 = (252-36)-(63+18)\\ &= 135.\end{aligned}$$

Another Solution: Using the other order of integration, we get

$$\begin{aligned}\iint\limits_R (2xy-x+y)\,dA &= \int_2^4\int_{-3}^6 (2xy-x+y)\,dy\,dx = \int_2^4 \left(xy^2 - xy + \frac{y^2}{2}\right)\Bigg|_{y=-3}^{y=6} dx\\ &= \int_2^4 \left[(36x-6x+18)-\left(9x+3x+\frac{9}{2}\right)\right]\,dx = \int_2^4 \left(18x+\frac{27}{2}\right)\,dx\\ &= \left(9x^2+\frac{27x}{2}\right)\Bigg|_2^4 = (144+54)-(36+27)\\ &= 135.\end{aligned}$$

Example 2.1.2. Evaluate $\displaystyle\iint\limits_R ye^{xy}\,dA$, where $R=[0,2]\times[0,1]$.

Solution: We have

$$\iint\limits_R ye^{xy}\,dA = \int_0^1\int_0^2 ye^{xy}\,dx\,dy\ .$$

For the inner integral, we let $u = xy$. Then $du = y\,dx$, and as x varies from 0 to 2, u varies from 0 to $2y$. Therefore,

$$\begin{aligned}\iint\limits_R ye^{xy}\,dA &= \int_0^1 \int_0^{2y} e^u\,du\,dy = \int_0^1 e^u \Big|_{u=0}^{u=2y} dy = \int_0^1 \left(e^{2y}-1\right)\,dy \\ &= \left(\frac{e^{2y}}{2} - y\right)\Big|_0^1 = \left(\frac{e^2}{2} - 1\right) - \left(\frac{1}{2} - 0\right) \\ &= \frac{e^2-3}{2}.\end{aligned}$$

Example 2.1.3. Find the volume of the solid under the surface $z = e^x \sin y$ that lies above the rectangle $R = [0,1] \times [0,\pi]$.

Solution: The volume V is equal to a double integral, which can be expressed as an iterated integral:

$$\begin{aligned}V &= \iint\limits_R e^x \sin y\,dA = \int_0^\pi \int_0^1 e^x \sin y\,dx\,dy \\ &= \int_0^\pi \left(e^x \sin y\right)\Big|_{x=0}^{x=1} dy = \int_0^\pi \left(e\sin y - \sin y\right)\,dy \\ &= \left(-e\cos y + \cos y\right)\Big|_0^\pi = (e-1) - (-e+1) \\ &= 2e - 2\ .\end{aligned}$$

EXERCISES 2.1

I. Evaluate the iterated double integral.

1. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^\pi (x^2\cos y - \sin x)\,dx\,dy$
2. $\displaystyle\int_0^1 \int_0^2 xye^y\,dx\,dy$
3. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^1 x\sin(xy)\,dy\,dx$
4. $\displaystyle\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx$

II. Find the volume of the indicated solid

1. under $2x - 3y + z = 6$ and above the rectangle $R = \{(x,y) \mid -1 \le x \le 0, -2 \le y \le 1\}$
2. under $z = x^2 - y^2$ and above the square $R = [1,3] \times [-1,1]$
3. in the first octant bounded by the cylinder $z = 9 - y^2$ and the plane $x = 2$

2.2 Double Integrals Over General regions

Let f be a function of x and y that is continuous on a closed and bounded region R on the xy-plane. Consider a rectangular region $D = [a, b] \times [c, d]$ enclosing R and define a function F by

$$F(x, y) = \begin{cases} f(x, y) & , \text{ if } (x, y) \in R \\ 0 & , \text{ if } (x, y) \in D \setminus R. \end{cases}$$

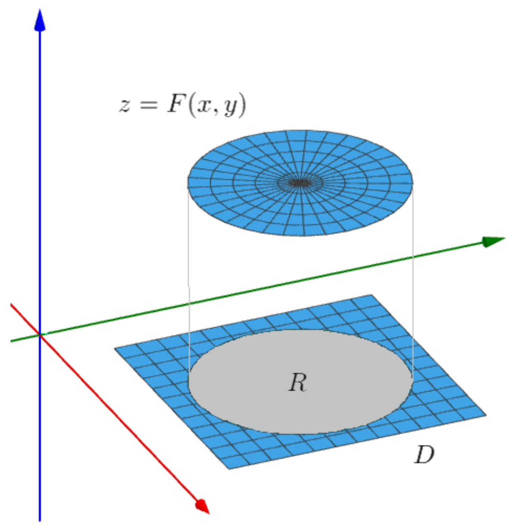


Figure 2.4: Graph of the extension $F(x, y)$

Definition 2.2.1. The ***double integral*** of f over R is defined by

$$\iint\limits_R f(x, y)\, dA := \iint\limits_D F(x, y)\, dA\ .$$

This definition makes sense because D is a rectangle and so $\iint_D F(x, y)\, dA$ has been defined in Section 2.1. Also, it does not matter what rectangle D we use as long as it contains R since $F(x, y) = 0$ when (x, y) is outside R and so they contribute nothing to the integral. Moreover, we can still interpret $\iint_R f(x, y)\, dA$ as the volume of the solid that lies above R and under the surface $z = f(x, y)$.

Remark. The value of the integral $\iint_R f(x, y)\, dA$ is the net signed volume of the solid bounded by the surface $z = f(x, y)$, where $(x, y) \in R$. That is, the difference between the volume of the portion of the solid above R and the portion below R.

Now in order to set-up iterated integrals over general regions, we classify *simple* regions as either type I or type II.

Type I Region

A plane region R is said to be of *type I* if it lies between two continuous curves $y = g_1(x)$ and $y = g_2(x)$; that is, $R = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$.

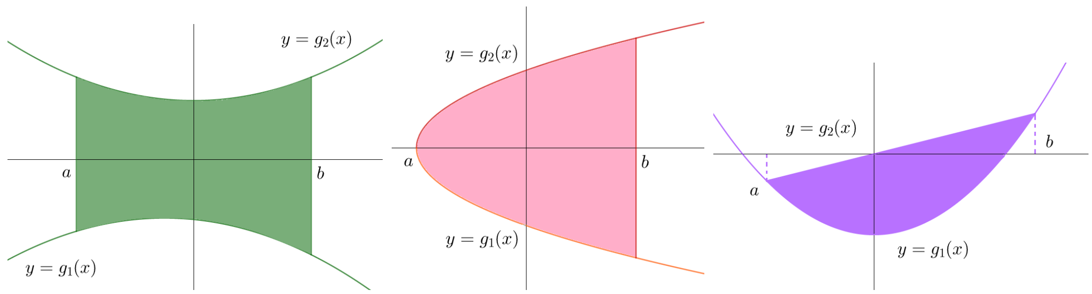


Figure 2.5: Examples of Type I Regions

Let $D = [a, b] \times [c, d]$ be a rectangle that encloses a type I region R (see Figure 2.6). Then for a fixed $x \in [a, b]$, we have $[c, d] = [c, g_1(x)] \cup [g_1(x), g_2(x)] \cup [g_2(x), d]$. Therefore,

$$\begin{aligned}\iint\limits_R f(x,y)\,dA &= \iint\limits_D F(x,y)\,dA \\ &= \int_a^b \int_c^d F(x,y)\,dy\,dx \\ &= \int_a^b \int_c^{g_1(x)} F(x,y)\,dy\,dx + \int_a^b \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy\,dx + \int_a^b \int_{g_2(x)}^{d} F(x,y)\,dy\,dx\ .\end{aligned}$$

Note that when $c \le y < g_1(x)$ or $g_2(x) < y \le d$, $F(x, y) = 0$ since $(x, y) \in D \backslash R$. While $F(x, y) = f(x, y)$ when $g_1(x) \le y \le g_2(x)$ since $(x, y) \in R$. Hence,

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx\ .$$

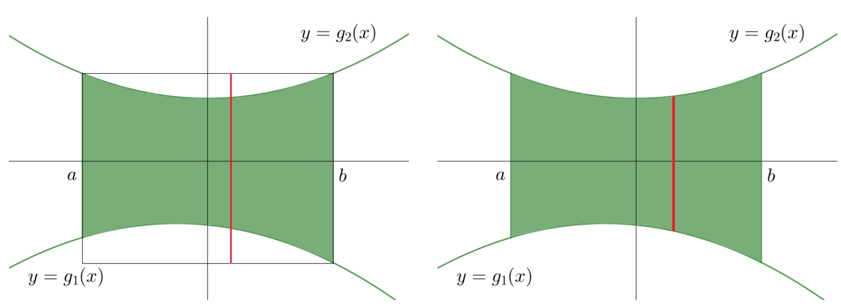


Figure 2.6: Analyzing Type I regions using vertical strips

Theorem 2.2.2. If f is continuous on a type I region R given by

$$R = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx\ .$$

In setting up an iterated integral over a type I region, it is helpful to draw a vertical strip inside R. Notice in Figure 2.6 that the lower boundary $y = g_1(x)$ of the vertical strip gives the lower limit of the inner integral while the upper boundary $y = g_2(x)$ of the strip gives the upper limit.

Type II Region

A plane region R is said to be of *type II* if it lies between two continuous curves $x = h_1(y)$ and $x = h_2(y)$; that is, $R = \{(x, y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$.

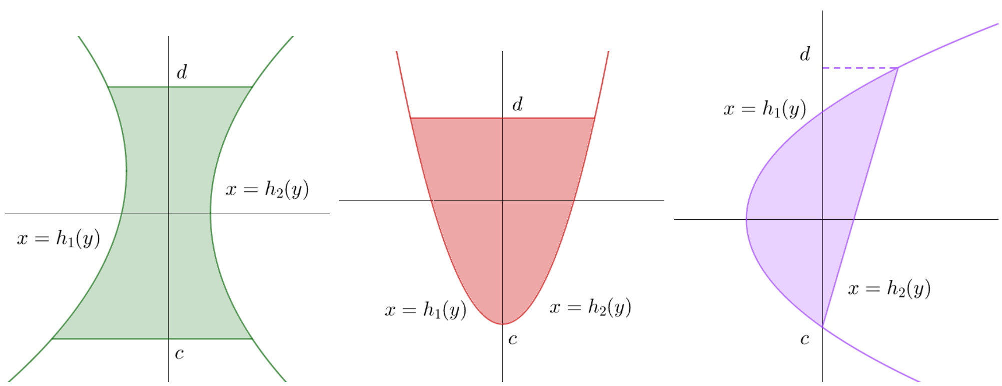


Figure 2.7: Examples of Type II Regions

Suppose R is a type II region and $D = [a, b] \times [c, d]$ is a rectangle enclosing R. Then for a fixed $y \in [c, d]$, we have $[a, b] = [a, h_1(y)] \cup [h_1(y), h_2(y)] \cup [h_2(y), b]$. We then have

$$\begin{aligned}
\iint\limits_R f(x, y)\, dA &= \iint\limits_D F(x, y)\, dA \\
&= \int_c^d \int_a^b F(x, y)\, dx\, dy \\
&= \int_c^d \int_a^{h_1(y)} F(x, y)\, dx\, dy + \int_c^d \int_{h_1(y)}^{h_2(y)} F(x, y)\, dx\, dy + \int_c^d \int_{h_2(y)}^{d} F(x, y)\, dx\, dy\ .
\end{aligned}$$

When $a \le x < h_1(y)$ or $h_2(y) < x \le b$, $F(x, y) = 0$ since $(x, y) \in D \backslash R$. While for $h_1(y) \le x \le h_2(y)$, $F(x, y) = f(x, y)$ since $(x, y) \in R$. Therefore,

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy\ .$$

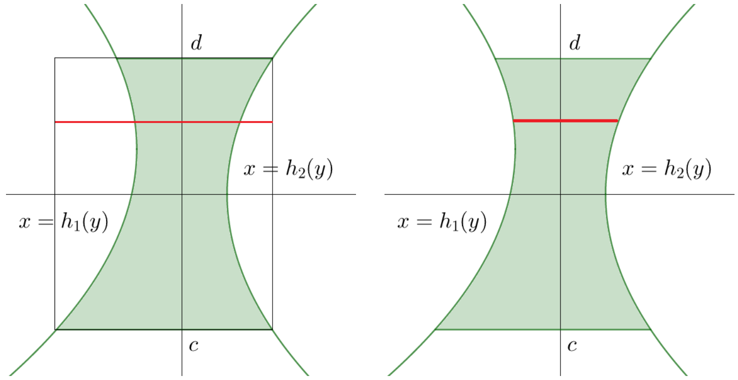


Figure 2.8: Analyzing Type I regions using horizontal strips

Theorem 2.2.3. If f is continuous on a type II region R given by

$$R = \{(x, y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$

then

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \ .$$

For a type II region, it is helpful to draw a horizontal strip inside R. In Figure 2.8, the left boundary $x = h_1(y)$ of the horizontal strip gives the lower limit of the inner integral while the right boundary $x = h_2(y)$ of the strip gives the upper limit.

Example 2.2.1. Evaluate $\iint\limits_R 4x^3 y\, dA$, where R is enclosed by $y = x^2$ and $x = y^2$.

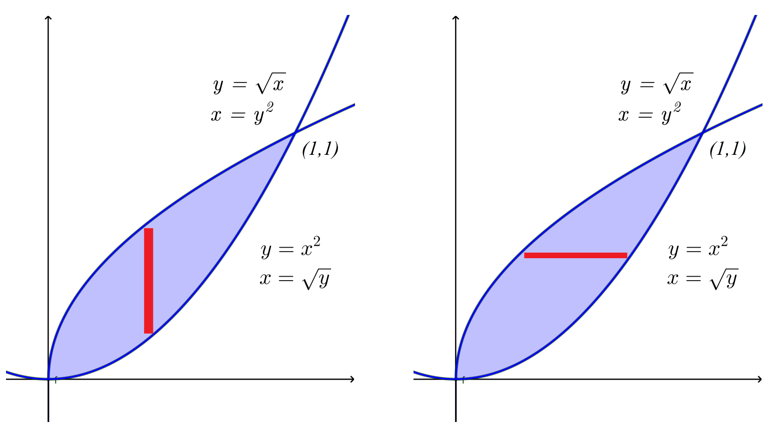


Figure 2.9: The region enclosed by $y = x^2$ and $x = y^2$

Solution: Using type I (vertical strips) classification, we have

$$\begin{aligned}\iint\limits_R 4x^3y\,dA &= \int_0^1\int_{x^2}^{\sqrt{x}} 4x^3y\,dy\,dx = \int_0^1 2x^3y^2\Big|_{y=x^2}^{y=\sqrt{x}}\,dx\\ &= \int_0^1 \left(2x^4-2x^7\right)\,dx = \left(\frac{2}{5}x^3-\frac{1}{4}x^8\right)\Big|_0^1\\ &= \left(\frac{2}{5}-\frac{1}{4}\right)-(0-0)=\frac{3}{20}\,.\end{aligned}$$

Another Solution: Using type II classification (horizontal strips), we get

$$\begin{aligned}\iint\limits_R 4x^3y\,dA &= \int_0^1\int_{y^2}^{\sqrt{y}} 4x^3y\,dx\,dy = \int_0^1 x^4y\Big|_{x=y^2}^{x=\sqrt{y}}\,dx\\ &= \int_0^1 \left(y^3-y^9\right)\,dy = \left(\frac{y^4}{4}-\frac{y^{10}}{10}\right)\Big|_0^1 = \frac{1}{4}-\frac{1}{10}=\frac{3}{20}\,.\end{aligned}$$

Example 2.2.2. Determine the volume of the tetrahedron bounded by $x+y+z=1$ and the three coordinate planes.

Solution: The solid is under the plane $z=1-x-y$ and it lies above the triangular region R shown in Figure 2.10.

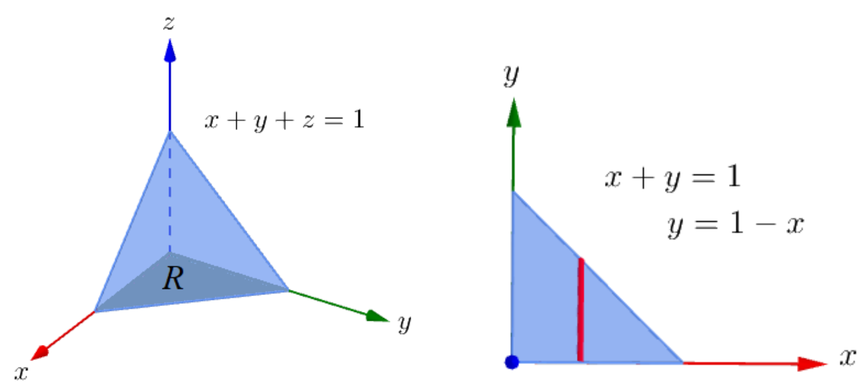


Figure 2.10: The tetrahederon bounded by $x+y+z=1$ and the coordinate planes, and its projection to the xy-plane

Thus, the volume of the tetrahedron will be given by

$$\begin{aligned}V=\iint\limits_R(1-x-y)\,dA &= \int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx\\ &= \int_0^1\left(y-xy-\frac{y^2}{2}\right)\Big|_{y=0}^{y=1-x}\,dx\\ &= \int_0^1\left[(1-x)-x(1-x)-\frac{(1-x)^2}{2}\right]\,dx\\ &= \int_0^1\frac{(1-x)^2}{2}\,dx=\int_0^1\frac{(x-1)^2}{2}\,dx=\frac{(x-1)^3}{6}\Big|_0^1=\frac{1}{6}\,.\end{aligned}$$

Remark. Some properties of the double integral.

1. For any constant c, $\displaystyle\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$.

2. $\displaystyle\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

3. If $R = R_1 \cup R_2$ such that R_1 and R_2 do not overlap, then

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA\ .$$

Example 2.2.3. Evaluate $\displaystyle\iint_R 2x\,dA$ where R is enclosed by $y = x^2$, $x = y^3$ and $x + y = 2$.

Solution: The region R is not a simple region but a union of simple regions. For instance, if we consider vertical strips inside R, then $R = R_1 \cup R_2$ where R_1 and R_2 are both of type I.

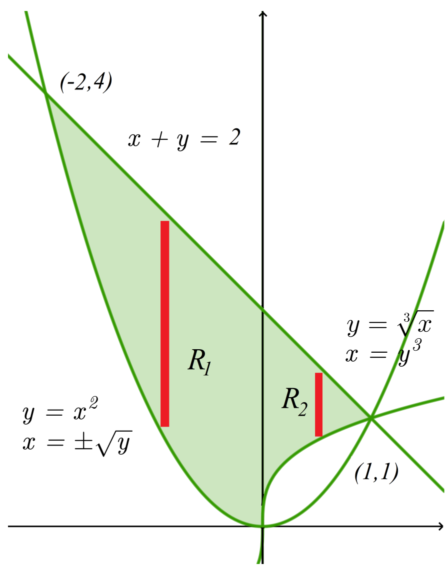


Figure 2.11: Analyzing R as a Type I region

Thus, we have

$$\iint_R 2x\,dA = \iint_{R_1} 2x\,dA + \iint_{R_2} 2x\,dA\ .$$

Note that

$$\begin{aligned}\iint\limits_{R_1} 2x\,dA &= \int_{-2}^{0}\int_{x^2}^{2-x} 2x\,dy\,dx \\ &= \int_{-2}^{0} (2xy)\Big|_{y=x^2}^{y=2-x} dx \\ &= \int_{-2}^{0} \left[2x(2-x) - 2x(x^2)\right] dx \\ &= \int_{-2}^{0} \left(4x - 2x^2 - 2x^3\right) dx \\ &= \left(2x^2 - \frac{2}{3}x^3 - \frac{x^4}{2}\right)\Big|_{-2}^{0} \\ &= 0 - \left(8 + \frac{16}{3} - 8\right) = -\frac{16}{3}\ .\end{aligned}$$

$$\begin{aligned}\iint\limits_{R_2} 2x\,dA &= \int_{0}^{1}\int_{x^{\frac{1}{3}}}^{2-x} 2x\,dy\,dx \\ &= \int_{0}^{1} (2xy)\Big|_{y=x^{\frac{1}{3}}}^{y=2-x} dx \\ &= \int_{0}^{1} \left[2x(2-x) - 2x(x^{\frac{1}{3}})\right] dx \\ &= \int_{0}^{1} \left(4x - 2x^2 - 2x^{\frac{4}{3}}\right) dx \\ &= \left(2x^2 - \frac{2}{3}x^3 - \frac{6}{7}x^{\frac{7}{3}}\right)\Big|_{0}^{1} \\ &= \left(2 - \frac{2}{3} - \frac{6}{7}\right) - 0 = \frac{10}{21}\ .\end{aligned}$$

Therefore,

$$\iint\limits_{R} 2x\,dA = -\frac{16}{3} + \frac{10}{21} = -\frac{34}{7}\ .$$

Another Solution: We may also consider the region R as a union of two type II regions as shown below.

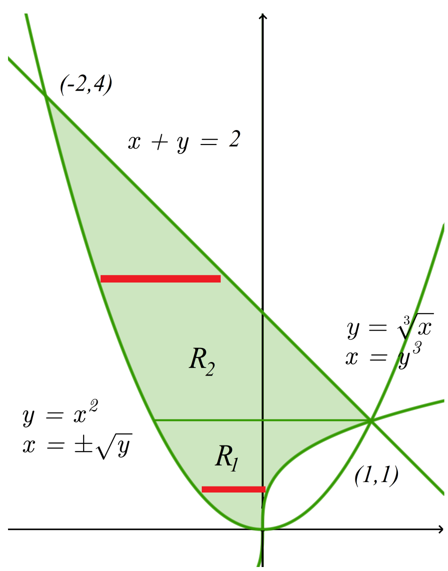


Figure 2.12: Analyzing R as a Type II region

We thus have

$$\iint\limits_{R} 2x\,dA = \iint\limits_{R_1} 2x\,dA + \iint\limits_{R_2} 2x\,dA\ .$$

Note that

$$\begin{aligned}\iint_{R_1} 2x\,dA &= \int_0^1 \int_{-\sqrt{y}}^{y^3} 2x\,dx\,dy \\ &= \int_0^1 x^2\Big|_{x=-\sqrt{y}}^{x=y^3} dy \\ &= \int_0^1 \left[\left(y^3\right)^2 - (-\sqrt{y})^2\right] dy \\ &= \int_0^1 \left(y^6 - y\right) dy \\ &= \left(\frac{y^7}{7} - \frac{y^2}{2}\right)\Bigg|_0^1 \\ &= \left(\frac{1}{7} - \frac{1}{2}\right) - 0 \\ &= -\frac{5}{14}\ .\end{aligned}$$

$$\begin{aligned}\iint_{R_2} 2x\,dA &= \int_1^4 \int_{-\sqrt{y}}^{2-y} 2x\,dx\,dy \\ &= \int_1^4 x^2\Big|_{x=-\sqrt{y}}^{x=2-y} dy \\ &= \int_1^4 \left[(2-y)^2 - (-\sqrt{y})^2\right] dy \\ &= \int_1^4 \left((4 - 5y + y^2\right) dy \\ &= \left(4y - \frac{5}{2}y^2 + \frac{y^3}{3}\right)\Bigg|_1^4 \\ &= \left(16 - 40 + \frac{64}{3}\right) - \left(4 - \frac{5}{2} + \frac{1}{3}\right) \\ &= -\frac{9}{2}\ .\end{aligned}$$

Hence,

$$\iint_R 2x\,dA = -\frac{5}{14} - \frac{9}{2} = -\frac{34}{7}.$$

Reversing the Order of Integration

In some cases, reversing the order of integration is necessary to simplify the evaluation of an iterated double integral.

Example 2.2.4. Evaluate the iterated integral $\displaystyle\int_0^1 \int_0^\pi x\cos(xy)\,dx\,dy$.

Solution: Finding an antiderivative for the function $f(x, y) = x\cos(xy)$ with respect to x would require the use of integration by parts. But notice that we can easily find an antiderivative with respect to y. So to evaluate the integral, we reverse the order of integration. By the Fubini's Theorem, we have

$$\int_0^1 \int_0^\pi x\cos(xy)\,dx\,dy = \int_0^\pi \int_0^1 x\cos(xy)\,dy\,dx\ .$$

Let $u = xy$. Then $du = x\,dy$, and as y varies from 0 to 1, u varies from 0 to x. Hence,

$$\begin{aligned}\int_0^1 \int_0^\pi x\cos(xy)\,dx\,dy &= \int_0^\pi \int_0^x \cos u\,du\,dx \\ &= \int_0^\pi \sin u\Big|_{u=0}^{u=x} dx \\ &= \int_0^\pi \sin x\,dx \\ &= -\cos x\Big|_0^\pi \\ &= -(-1-1) = 2\ .\end{aligned}$$

Example 2.2.5. Evaluate $\displaystyle\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dx\,dy.$

Solution: It is impossible to evaluate the iterated double integral in the given order since the function e^{x^2} has no elementary antiderivative with respect to x. But, this problem can be solved by changing the order of integration. The given iterated integral classifies the region as Type II because of the order $dx\,dy$. To switch the ordering of integration, we sketch the region of integration and re-classify it as Type I.

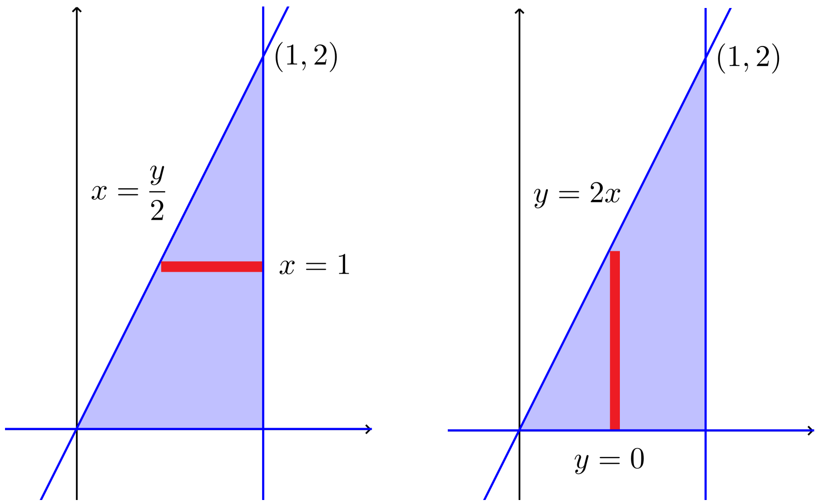


Figure 2.13: Reclassifying a Type II region as a Type I region

As a Type I region,

$$R : \begin{cases} \text{left boundary: } x = \frac{y}{2} \\ \text{right boundary: } x = 1 \\ y\text{-limits: } 0 \le y \le 2 \end{cases}$$

As a Type II region,

$$R : \begin{cases} \text{lower boundary: } y = 0 \\ \text{upper boundary: } y = 2x \\ x\text{-limits: } 0 \le x \le 1 \end{cases}$$

Therefore, using vertical strips,

$$\begin{aligned}
\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dx\,dy &= \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx \\
&= \int_0^1 ye^{x^2}\Big|_{y=0}^{y=2x}\,dx \\
&= \int_0^1 \left(2xe^{x^2} - 0\right)\,dx \\
&= \int_0^1 2xe^{x^2}\,dx \\
&= e^{x^2}\Big|_0^1 = e - 1\ .
\end{aligned}$$

Example 2.2.6. Evaluate $\displaystyle\int_0^2 \int_{-\frac{x}{2}}^0 \cos\left(2y + y^2\right)\, dy\, dx$.

Solution: The above iterated integral classifies the region of integration as of type I.

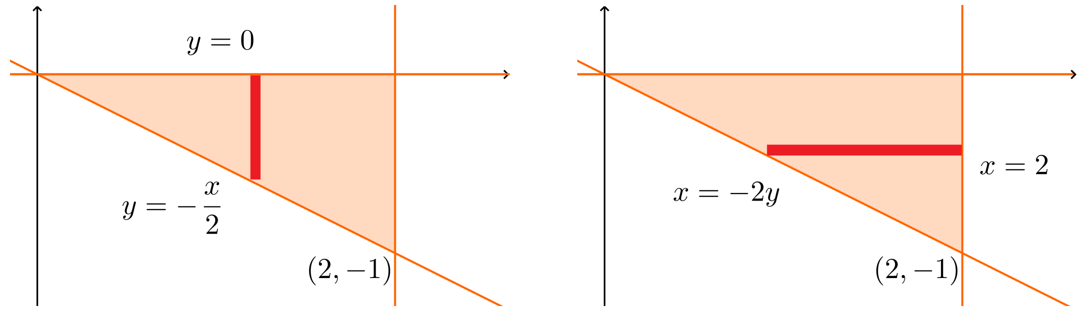


Figure 2.14: Reclassifying a Type I region as a Type II region

As a type I region,

$$R : \begin{cases} \text{lower boundary: } y = -\frac{x}{2} \\ \text{upper boundary: } y = 0 \\ x\text{-limits: } 0 \le x \le 2 \end{cases}$$

As a type II region,

$$R : \begin{cases} \text{left boundary: } x = -2y \\ \text{right boundary: } x = 2 \\ y\text{-limits: } -1 \le y \le 0 \end{cases}$$

Using horizontal strips, we now have

$$\begin{aligned}
\int_0^2 \int_{-\frac{x}{2}}^0 \cos\left(2y + y^2\right)\, dy\, dx &= \int_{-1}^0 \int_{-2y}^2 \cos\left(2y + y^2\right)\, dx\, dy \\
&= \int_{-1}^0 x\cos\left(2y + y^2\right)\Big|_{x=-2y}^{x=2}\, dy \\
&= \int_{-1}^0 \left[2\cos\left(2y + y^2\right) + 2y\cos\left(2y + y^2\right)\right]\, dy \\
&= \int_{-1}^0 (2 + 2y)\cos\left(2y + y^2\right)\, dy \\
&= \sin\left(2y + y^2\right)\Big|_{-1}^0 \\
&= \sin 0 - \sin(-1) \\
&= \sin 1\ .
\end{aligned}$$

EXERCISES 2.2

I. Set up the iterated double integral equivalent to $\displaystyle\iint_R f(x, y)\, dA$ over the given region.

1. Triangular region with vertices at $(-2, 0)$, $(0, 4)$ and $(2, 0)$
2. Region bounded by $y = 4 - x^2$ and the $x-$axis

3. Region enclosed by the circle $x^2 + y^2 = 1$
4. Triangular region with vertices at $(0,0)$, $(2,-4)$ and $(4,2)$
5. Region enclosed by $y = \sqrt{x}$, $2x + y = 10$ and $7x + y = 0$

II. Find the volume of the following solids.

1. Tetrahedron bounded by $2x - y + z = 4$ and the coordinate planes
2. Solid enclosed by the $xy-$plane, $yz-$plane, $x = y$ and the cylinder $z = 4 - y^2$
3. Solid in the first octant under $z = x^2 + y^2$ within the cylinder $x^2 + y^2 = 1$

III. Evaluate the integral by reversing the order of integration.

1. $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}\, dx\, dy$
2. $\displaystyle\int_0^3 \int_{x^2}^9 x^3 e^{y^3}\, dy\, dx$
3. $\displaystyle\int_0^3 \int_{\sqrt{\frac{y}{3}}}^1 \cos x^3\, dx\, dy$
4. $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3 + 1}\, dy\, dx$
5. $\displaystyle\int_0^4 \int_0^{\sqrt{y}} e^{12x - x^3}\, dx\, dy$

IV. Do as indicated.

1. Express
$$\int_0^1 \int_0^1 \frac{1}{\sqrt{1 + e^x}}\, dx\, dy + \int_1^e \int_{\ln y}^1 \frac{1}{\sqrt{1 + e^x}}\, dx\, dy$$
as a single iterated double integral and evaluate.
2. If $f(x,y)$ is continuous on $[a,b] \times [c,d]$ and
$$g(x,y) = \int_a^x \int_c^y f(s,t)\, dt\, ds,$$
for $a < x < b, c < y < d$, show that $g_{xy} = g_{yx} = f(x,y)$.

2.3 Double Integrals in Polar Coordinates

Recall that a point P on the xy-plane with rectangular coordinates (x,y) has corresponding polar coordinates (r,θ), where $|r|$ is the distance of P from the origin, and θ is an angle formed from the positive $x-$axis to the ray $\vec{OP}$ if $r > 0$ and to the extension of the ray $\vec{OP}$ if $r < 0$. The rectangular and polar coordinates are related by the following equations:

$$r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta.$$

A *simple polar region* in polar coordinates is a region enclosed by two rays, $\theta = \alpha$ and $\theta = \beta$, with $0 \leq \beta - \alpha \leq 2\pi$, and two continuous polar curves $r = r_1(\theta)$ and $r = r_2(\theta)$ such that $r_1(\theta)$ and $r_2(\theta)$

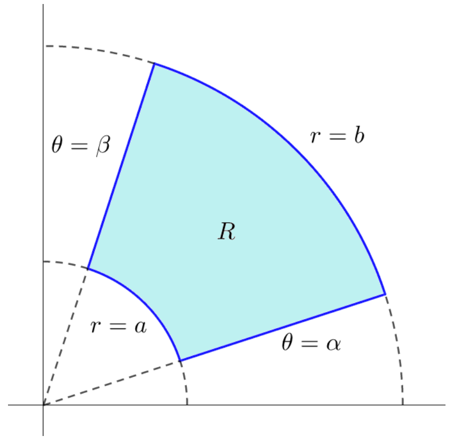


Figure 2.15: A polar rectangle

are nonnegative on $[\alpha, \beta]$. A special case is called a *polar rectangle*; that is, a polar region enclosed by two rays $\theta = \alpha$ and $\theta = \beta$, and two circles $r = a$ and $r = b$.

Let R be a polar rectangle. Divide $[a, b]$ into m subintervals $[r_{i-1}, r_i]$, $i = 1, 2, \ldots, m$, with $r_0 = a$ and $r_m = b$, each of length $\Delta r = \frac{b-a}{m}$. Divide $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$, $j = 1, 2, \ldots, n$, with $\theta_0 = \alpha$ and $\theta_n = \beta$, each of length $\Delta\theta = \frac{\beta-\alpha}{n}$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the region R into small polar rectangles R_{ij} as shown in Figure 2.16.

We renumber the polar rectangles that lie inside R (the order does not matter) and call the rectangles $R_k,\ k = 1, 2, \ldots, N$. Take the center (r_k^*, θ_k^*) of R_k.

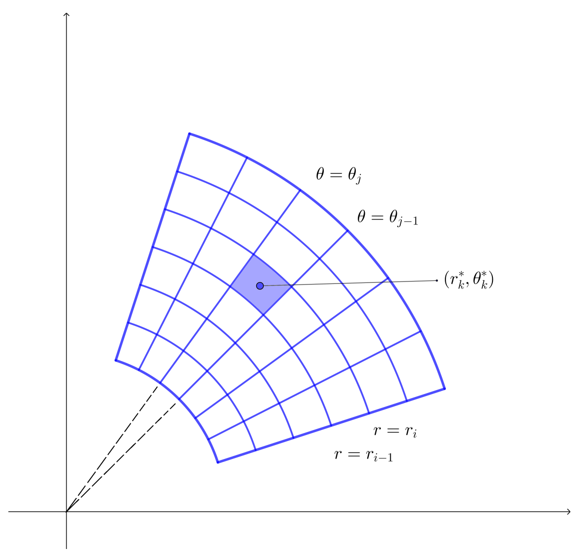


Figure 2.16: Partitioning a polar rectangle

The area ΔA_k of R_k is the difference between the areas of two sectors, so from the formula:

$$\text{area of a sector} = \frac{1}{2}(\text{radius})^2(\text{included angle}),$$

we have

$$\begin{aligned}\Delta A_k &= \frac{1}{2}\left(r_k^* + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r_k^* - \frac{\Delta r}{2}\right)^2 \Delta\theta \\ &= \frac{\Delta\theta}{2}\left[\left(r_k^* + \frac{\Delta r}{2}\right)^2 - \left(r_k^* - \frac{\Delta r}{2}\right)^2\right] \\ &= r_k^* \Delta r \Delta\theta.\end{aligned}$$

We then form the Riemann sum

$$S_N = \sum_{k=1}^{N} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\Delta A_k = \sum_{k=1}^{N} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*) r_k^* \Delta r \Delta\theta.$$

By taking the limit of these sums as N approaches infinity, we get the following:

Theorem 2.3.1. If f is continuous on a polar rectangle $R = \{(r,\theta) \mid 0 \le a \le r \le b,\ \alpha \le \theta \le \beta\}$, then

$$\iint\limits_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta\ .$$

Example 2.3.1. Evaluate $\displaystyle\iint\limits_R 4xy\, dA$, where R is the region in the 2nd quadrant enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution: The circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$ have polar equations $r = 1$ and $r = 2$, respectively. The region R is given below.

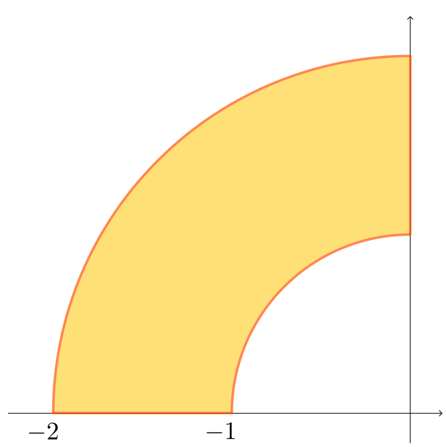


Figure 2.17: Region in the 2nd quadrant enclosed by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

Using polar coordinates, we have

$$\begin{aligned}\iint\limits_R 4xy\,dA &= \int_{\frac{\pi}{2}}^{\pi}\int_1^2 4(r\sin\theta)(r\cos\theta)r\,dr\,d\theta\\ &= \int_{\frac{\pi}{2}}^{\pi}\left(r^4\sin\theta\cos\theta\right)\Big|_{r=1}^{r=2}d\theta\\ &= \int_{\frac{\pi}{2}}^{\pi} 15\sin\theta\cos\theta\,d\theta\\ &= 15\frac{\sin^2\theta}{2}\Big|_{\frac{\pi}{2}}^{\pi}\\ &= -\frac{15}{2}\ .\end{aligned}$$

Example 2.3.2. Evaluate $\displaystyle\int_0^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}}\frac{1}{\sqrt{x^2+y^2+1}}\,dy\,dx.$

Solution: First, we need to sketch the region of integration. The iterated integral classifies R as a type I region because of the order of integration $dy\,dx$.

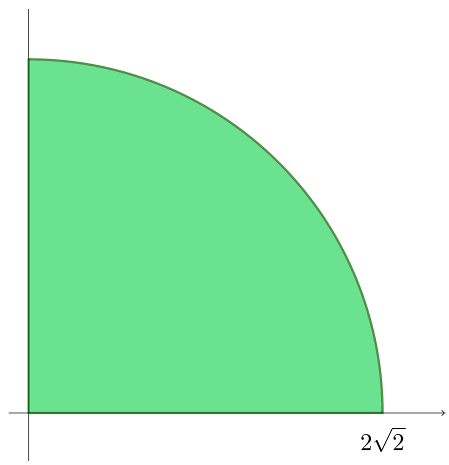


Figure 2.18: Reclassifying a Type I region as a polar rectangle

As a type I region,

$$R:\begin{cases}\text{lower boundary: } y=0\\ \text{upper boundary: } y=\sqrt{8-x^2}\\ \qquad\qquad\Longrightarrow x^2+y^2=8\\ x\text{-limits: } 0\le x\le 2\sqrt{2}\end{cases}$$

As a simple polar rectangle,

$$R:\begin{cases}r\text{-limits: } 0\le r\le 2\sqrt{2}\\ \theta\text{-limits: } 0\le\theta\le\frac{\pi}{2}\end{cases}$$

Using polar coordinates, we get

$$\int_0^{2\sqrt{2}}\int_0^{\sqrt{8-y^2}}\frac{1}{\sqrt{x^2+y^2+1}}\,dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{2\sqrt{2}}\frac{1}{\sqrt{r^2+1}}\,r\,dr\,d\theta\ .$$

Let $u = r^2 + 1$. Then $du = 2r\,dr$ and so $r\,dr = \frac{du}{2}$. Moreover as r varies from 0 to $2\sqrt{2}$, u varies from 1 to 9. Hence,

$$\begin{aligned}
\int_0^{2\sqrt{2}} \int_0^{\sqrt{8-y^2}} \frac{1}{\sqrt{x^2+y^2+1}}\,dx\,dy &= \int_0^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} \frac{1}{\sqrt{r^2+1}}\,r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_1^9 \frac{u^{-\frac{1}{2}}}{2}\,du\,d\theta = \int_0^{\frac{\pi}{2}} u^{\frac{1}{2}}\Big|_{u=1}^{u=9}\,d\theta \\
&= \int_0^{\frac{\pi}{2}} (3-1)\,d\theta = \int_0^{\frac{\pi}{2}} 2\,d\theta = 2\theta\Big|_0^{\frac{\pi}{2}} \\
&= \pi\ .
\end{aligned}$$

Theorem 2.3.1 can be extended to the case where R is any simple polar region.

Theorem 2.3.2. If f is continuous on a simple polar region

$$R = \{(r,\theta) \mid 0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta\}\,,$$

then

$$\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta\ .$$

Remark. To find the limits of integration of $\iint\limits_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$, we follow the steps given below.

(i) Sketch the region of integration R and label the bounding curves.

(ii) To find the r-limits of integration, imagine a ray emanating from the origin intersecting R in the direction of increasing r.

(iii) The r-limits will be the r-values (may depend on θ) of the entry and exit curves of the ray.

(iv) To find the θ-limits of integration, determine the least and the largest θ-values (constant) that bound the region R.

Example 2.3.3. Evaluate $\iint\limits_R \sqrt{x^2+y^2}\,dA$ over the region enclosed by $(x-1)^2+y^2=1$.

Solution: The circle $(x-1)^2+y^2=1$ has polar equation $r = 2\cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
Using polar coordinates and exploiting the fact that the region is symmetric with respect to the

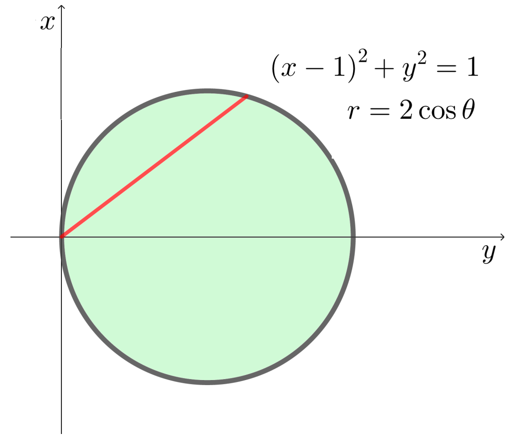


Figure 2.19: Region enclosed by $(x-1)^2 + y^2 = 1$

polar axis, we have

$$\begin{aligned}
\iint\limits_R \sqrt{x^2+y^2}\,dA &= 2\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r(r\,dr\,d\theta) \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\,dr\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \frac{r^3}{3}\Big|_{r=0}^{r=2\cos\theta} d\theta \\
&= \frac{16}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta\,d\theta \\
&= \frac{16}{3}\int_0^{\frac{\pi}{2}} (1-\sin^2\theta)(\cos\theta)\,d\theta.
\end{aligned}$$

Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Moreover, as θ varies from 0 to $\frac{\pi}{2}$, u varies from 0 to 1. Therefore

$$\begin{aligned}
\iint\limits_R \sqrt{x^2+y^2}\,dA &= \frac{16}{3}\int_0^1 (1-u^2)\,du \\
&= \frac{16}{3}\left(u - \frac{u^3}{3}\right)\Big|_0^1 \\
&= \frac{32}{9}.
\end{aligned}$$

Example 2.3.4. Let R be the region in the first quadrant between $x^2+(y-3)^2 = 9$ and $x^2+y^2 = 36$. Evaluate $\displaystyle\iint\limits_R x\,dA$.

Solution: The circles $x^2+(y-3)^2 = 9$ and $x^2+y^2 = 36$ have polar equations $r = 6\sin\theta$ and $r = 6$, respectively. The region R is given below.

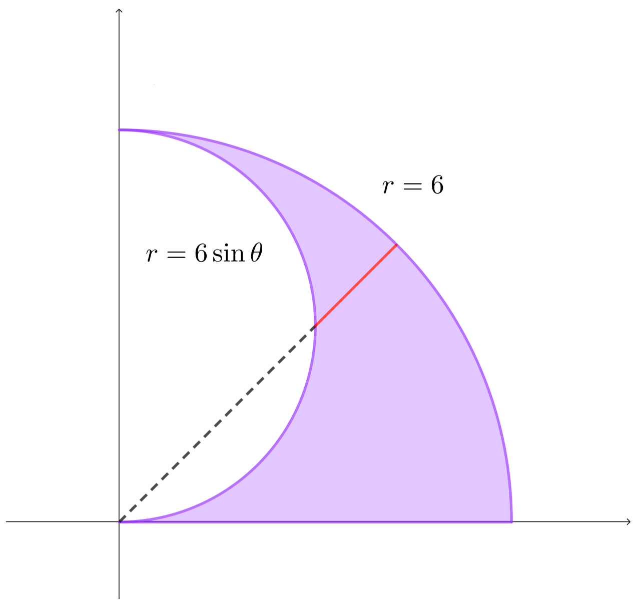


Figure 2.20: Region in the first quadrant between $x^2 + (y-3)^2 = 9$ and $x^2 + y^2 = 36$

$$R : \begin{cases} r\text{-limits: } 6\sin\theta \le r \le 6 \\ \theta\text{-limits: } 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

Thus,

$$\begin{aligned}
\iint\limits_R x\,dA &= \int_0^{\frac{\pi}{2}} \int_{6\sin\theta}^{6} (r\cos\theta)\, r\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_{6\sin\theta}^{6} r^2 \cos\theta\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{r^3}{3}\cos\theta \right) \Bigg|_{r=6\sin\theta}^{r=6} d\theta \\
&= \int_0^{\frac{\pi}{2}} (72\cos\theta - 72\sin^3\theta\cos\theta)\, d\theta \\
&= \left(72\sin\theta - 18\sin^4\theta\right) \Big|_0^{\frac{\pi}{2}} \\
&= (72 - 18) - 0 = 54\ .
\end{aligned}$$

Example 2.3.5. Find the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.

Solution: The projection of the solid onto the xy-plane is the region bounded by the projection of the intersection of the two paraboloids. Solving for the intersection of the paraboloids by equating their z-components, we get $x^2 + y^2 = 8 - x^2 - y^2$ or $x^2 + y^2 = 4$, which in polar coordinates, is equivalent to $r = 2$.

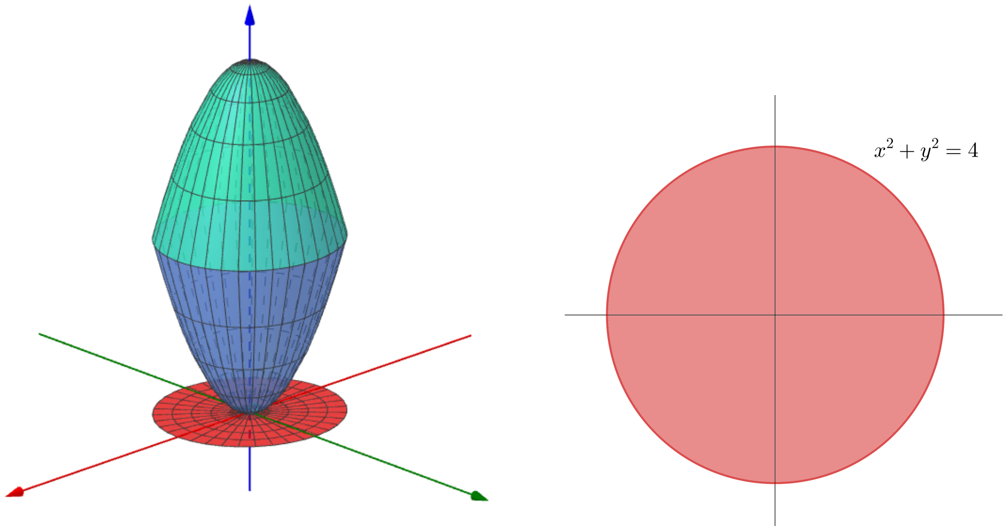


Figure 2.21: Solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$, and its projection to the xy-plane

Let V_1 be the volume of solid that lies above R under $z = 8 - x^2 - y^2$ and let V_2 be the volume solid that lies above R under $z = x^2 + y^2$. Then, the required solid has volume

$$\begin{aligned}
V &= V_1 - V_2 \\
&= \iint\limits_R (8 - x^2 - y^2)\, dA - \iint\limits_R (x^2 + y^2)\, dA \\
&= \iint\limits_R \left[8 - 2(x^2 + y^2)\right] dA \\
&= \int_0^{2\pi} \int_0^2 \left(8 - 2r^2\right) r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 \left(8r - 2r^3\right) dr\, d\theta \\
&= \int_0^{2\pi} \left(4r^2 - \frac{r^4}{2}\right)\Bigg|_{r=0}^{r=2} d\theta \\
&= \int_0^{2\pi} \left[(16 - 8) - 0\right] d\theta \\
&= \int_0^{2\pi} 8\, d\theta \\
&= 8\theta\Big|_0^{2\pi} = 16\pi\ .
\end{aligned}$$

EXERCISES 2.3

I. Convert the following iterated integrals into polar form.

1. $\displaystyle\int_0^{2\sqrt{2}}\int_{-\sqrt{8-y^2}}^{0}\sin(x^2+y^2)\,dx\,dy$
2. $\displaystyle\int_0^{1}\int_{x}^{\sqrt{2-x^2}}(1-x^2-y^2)\,dy\,dx$
3. $\displaystyle\int_0^{2}\int_{0}^{\sqrt{2y-y^2}}\sqrt{x^2+y^2}\,dx\,dy$
4. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{1}^{\sqrt{4-x^2}}xy\,dy\,dx$

II. Evaluate the following integrals using polar coordinates.

1. $\displaystyle\iint_R e^{x^2+y^2}\,dA$, R is the region between the circles $x^2+y^2=1$ and $x^2+y^2=9$
2. $\displaystyle\iint_R \frac{1}{1+x^2+y^2}\,dA$, R is the sector in the first quadrant bounded by $y=0$, $y=x$ and $x^2+y^2=4$
3. $\displaystyle\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dA$, R is the region in the first quadrant inside $x^2+y^2=2y$ and outside $x^2+y^2=2$
4. $\displaystyle\iint_R xy\,dA$, R is the region in the first quadrant bounded by the x-axis and $(x-1)^2+y^2=1$

III. Use polar coordinates to find the volume of the given solids.

1. solid bounded by the cylinders $x^2+y^2=4$ and $x^2+z^2=4$
2. solid above the xy-plane, under the plane $y+z=4$ and within the cylinder $(x-2)^2+y^2=4$
3. solid bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2=8$
4. solid enclosed by the hyperboloid $-x^2-y^2+z^2=1$ and the plane $z=2$

IV. Let $I=\displaystyle\int_0^{+\infty}e^{-x^2}\,dx$.

1. Use the fact that
$$I^2=\left[\int_0^{+\infty}e^{-x^2}\,dx\right]\left[\int_0^{+\infty}e^{-y^2}\,dy\right]=\int_0^{+\infty}\int_0^{+\infty}e^{-(x^2+y^2)}\,dx\,dy$$
and convert the resulting integral into polar coordinates to show that $I=\dfrac{\sqrt{\pi}}{2}$.
2. Using (1.), show that
$$\int_{-\infty}^{+\infty}e^{-x^2}\,dx=\sqrt{\pi}.$$
3. A single random variable X is normally distributed with parameters μ (mean) and σ^2 (variance) if its probability density function is given by $\mathbb{P}[X=x]=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x\in\mathbb{R}$. Using a change of variable and (2.), verify that
$$\int_{-\infty}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx=1.$$

2.4 Applications of Double Integrals

Volume of a Solid

Let f be a continuous function of x and y such that $f(x, y) \geq 0$ for all (x, y) in a closed and bounded region R on the xy-plane. The volume V of the solid that lies above the region R and under the surface $z = f(x, y)$ is given by

$$V = \iint_R f(x, y)\, dA.$$

Area of a Plane Region

If R is a closed and bounded region on the xy-plane, then its area A is given by $A = \displaystyle\iint_R 1\, dA$.

Proof. Consider the solid that lies above R and under $z = 1$. See Figure 2.22.

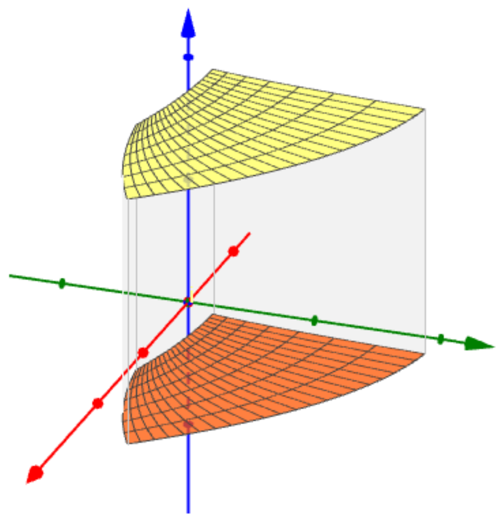


Figure 2.22: Solid under $z = 1$

Since this solid is a cylinder, then its volume is $V =$ base area $\times$ height $= A \times 1 = A$. But using double integral, we have $V = \displaystyle\iint_R 1\, dA$. Therefore, $A = \displaystyle\iint_R 1\, dA$. □

Example 2.4.1. Find the area of the region enclosed by $x = y^2 - 1$ and $x + y = 1$.

Solution: The region R is of type II.

$$R : \begin{cases} x\text{-limits: } y^2 - 1 \leq x \leq 1 - y \\ y\text{-limits: } -2 \leq y \leq 1 \end{cases}$$

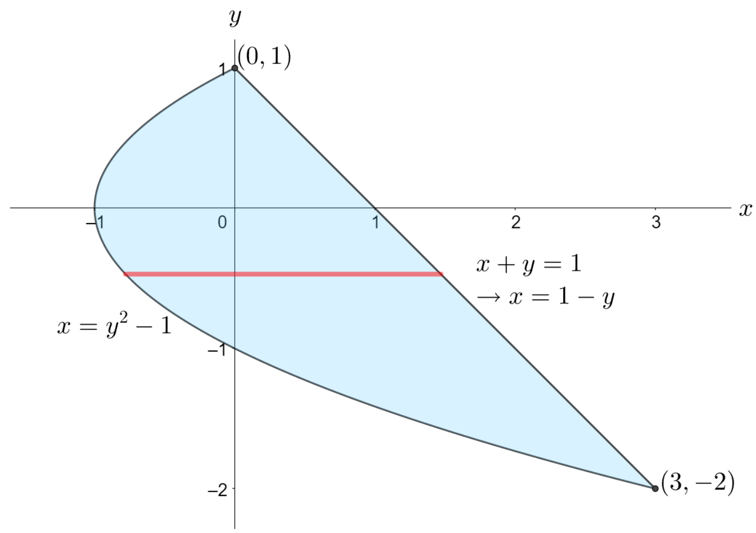


Figure 2.23: Region enclosed by $x = y^2 - 1$ and $x + y = 1$

Using horizontal strips, we have

$$
\begin{aligned}
A &= \iint_R dA \\
&= \int_{-2}^{1} \int_{y^2-1}^{1-y} dx\, dy \\
&= \int_{-2}^{1} x \Big|_{x=y^2-1}^{x=1-y} dy \\
&= \int_{-2}^{1} \left[(1-y) - (y^2-1)\right] dy \\
&= \int_{-2}^{1} \left(2 - y - y^2\right) dy \\
&= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right) \Big|_{-2}^{1} \\
&= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) \\
&= \frac{9}{2}\,.
\end{aligned}
$$

Mass and Center of Mass of a Lamina

A *lamina* is a flat sheet of material whose thickness is negligible. For example, a thin sheet of galvanized iron and a piece of paper are models for a lamina.

Suppose a given lamina is in the shape of a rectangle $R = [a, b] \times [c, d]$ on the xy-plane. Let $f(x, y)$ be the density (mass per unit area) of the lamina at a point (x, y) in R, where f is continuous on R. We partition R into N subrectangles R_k by uniformly dividing $[a, b]$ into m subintervals, each of length Δx and $[c, d]$ into n subintervals, each of length Δy. Let $\Delta A = \Delta x \Delta y$ be the area of each R_k. We pick a point (x_k^*, y_k^*) in each R_k.

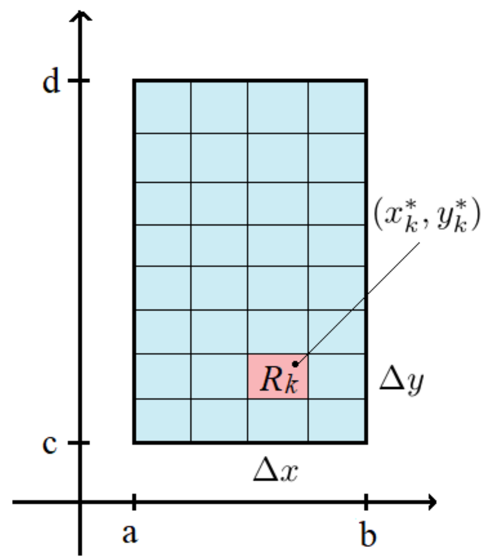


Figure 2.24: Partitioning a lamina R in the xy-plane

If each R_k has sufficiently small area, the continuity of f implies that each R_k is approximately homogenous; that is, each R_k has approximately constant density $f(x_k^*, y_k^*)$. Therefore, the mass of each R_k is approximated by $f(x_k^*, y_k^*)\Delta A$ and the mass of the lamina is approximated by $\sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A$. By taking the limit as N approaches infinity, the mass M of the rectangular lamina is given by

$$M = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A = \iint_R f(x, y)\, dA \ .$$

If a given lamina R is not a rectangular region, we simply enclose R by a rectangle D and define the density $f(x, y)$ to be 0 outside R. Then we proceed with the approximation, as we did for the rectangular case. This method will result to the same formula.

The moment of a particle about an axis is defined to be the product of its mass and its directed distance from the axis. For a given lamina, the moment about the x-axis is approximated by

$$\sum_{k=1}^{N}(f(x_k^*, y_k^*)\Delta A)y_k^* \ .$$

Therefore, the moment of the lamina about the x-axis, denoted M_x, is

$$M_x = \lim_{N\to\infty}\sum_{k=1}^{N} y_k^* f(x_k^*, y_k^*)\Delta A = \iint\limits_R y f(x,y)\, dA \ .$$

Similarly, the moment of the lamina about the y-axis, denoted M_y, is

$$M_y = \lim_{N\to\infty}\sum_{k=1}^{N} x_k^* f(x_k^*, y_k^*)\Delta A = \iint\limits_R x f(x,y)\, dA \ .$$

The center of mass of a lamina is a point $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{M} \text{ and } \overline{y} = \frac{M_x}{M} \ .$$

The physical significance of the center of mass is that the lamina balances horizontally when supported at its center of mass.

Example 2.4.2. Find the center of mass of the triangular lamina with vertices at $(0,0)$, $(2,0)$ and $(2,2)$ if the density at a point (x,y) is $f(x,y) = 6xy$.

Solution: The lamina is enclosed by the x-axis and the lines $x = 2$ and $x = y$.

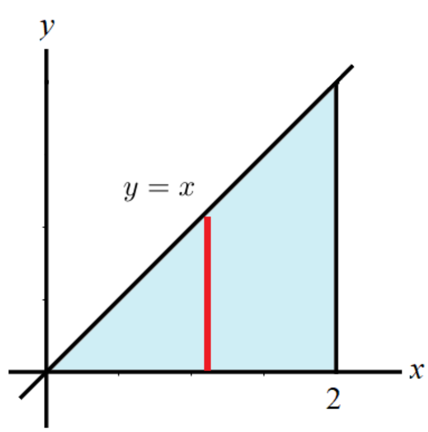


Figure 2.25: Triangular lamina with vertices at $(0,0)$, $(2,0)$ and $(2,2)$

The mass of the lamina is

$$\begin{aligned} M &= \iint_R f(x,y)\,dA = \iint_R 6xy\,dA \\ &= \int_0^2 \int_0^x 6xy\,dy\,dx = \int_0^2 3xy^2\Big|_{y=0}^{y=x} dx \\ &= \int_0^2 3x^3\,dx = \frac{3}{4}x^4\Big|_0^2 \\ &= 12\ . \end{aligned}$$

The moment about the x-axis is

$$\begin{aligned} M_x &= \iint_R y f(x,y)\,dA = \iint_R 6xy^2\,dA \\ &= \int_0^2 \int_0^x 6xy^2\,dy\,dx = \int_0^2 2xy^3\Big|_{y=0}^{y=x} dx \\ &= \int_0^2 2x^4\,dx = \frac{2}{5}x^5\Big|_0^2 \\ &= \frac{64}{5}\ . \end{aligned}$$

The moment about the y-axis is

$$\begin{aligned} M_y &= \iint_R x f(x,y)\,dA = \iint_R 6x^2y\,dA \\ &= \int_0^2 \int_0^x 6x^2y\,dy\,dx = \int_0^2 3x^2y^2\Big|_{y=0}^{y=x} dx \\ &= \int_0^2 3x^4\,dx = \frac{3}{5}x^5\Big|_0^2 \\ &= \frac{96}{5}\ . \end{aligned}$$

So, we have

$$\overline{x} = \frac{M_y}{M} = \frac{\frac{96}{5}}{12} = \frac{8}{5} \quad\text{and}\quad \overline{y} = \frac{M_x}{M} = \frac{\frac{64}{5}}{12} = \frac{16}{15}\ .$$

Hence, the center of mass of the lamina is at $\left(\frac{8}{5}, \frac{16}{15}\right)$.

Surface Area

Let S be a surface defined by a vector equation $\vec{R}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where $(u,v) \in D$ satisfies the following conditions:

- the parameter domain D is bounded
- the surface S is smooth, that is, $\vec{R}_u(u,v) \times \vec{R}_v(u,v) \neq \vec{0}$ everywhere, and
- points in S are traversed precisely once as (u,v) varies in D.

For simplicity, we start by considering a rectangular parameter domain D. Partition D into N subrectangles D_k of dimensions Δu and Δv. In effect, the parametric surface will be partitioned into N patches S_k. Let (u_k, v_k) be the lower left corner of D_k and let P_k be the corner of the patch S_k corresponding to it. The side of S_k determined by $\vec{R}(u_k, v_k)$ and $\vec{R}(u_k + \Delta u, v_k)$ can be approximated by

$$\vec{R}(u_k + \Delta u, v_k) - \vec{R}(u_k, v_k).$$

But from the definition of the tangent vector $\vec{R}_u(u, v)$, we have

$$\vec{R}(u_k + \Delta u, v_k) - \vec{R}(u_k, v_k) \approx \vec{R}_u(u_k, v_k)\Delta u.$$

Hence, the side of S_k determined by $\vec{R}(u_k, v_k)$ and $\vec{R}(u_k + \Delta u, v_k)$ can be approximated by $\vec{R}_u(u_k, v_k)\Delta u$. Likewise, the side of S_k determined by $\vec{R}(u_k, v_k)$ and $\vec{R}(u_k, v_k + \Delta v)$ can be approximated by $\vec{R}_v(u_k, v_k)\Delta v$.

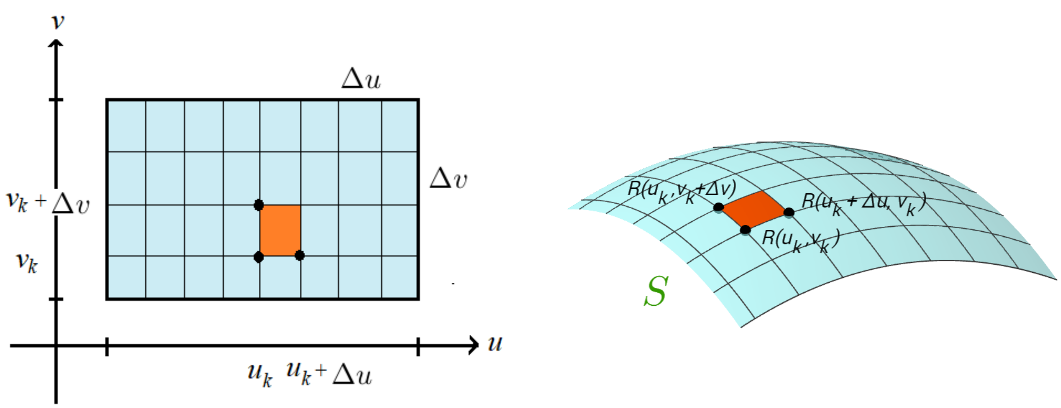


Figure 2.26: The domain D and the surface S

Consequently, the area of the patch S_k has an approximation given by the area of the parallelogram determined by the vectors $\vec{R}_u(u_k, v_k)\Delta u$ and $\vec{R}_v(u_k, v_k)\Delta v$. That is,

$$A(S_k) \approx \|\vec{R}_u(u_k, v_k)\Delta u \times \vec{R}_v(u_k, v_k)\Delta v\| = \|\vec{R}_u(u_k, v_k) \times \vec{R}_v(u_k, v_k)\|\Delta u \Delta v.$$

Therefore, the area $A(S)$ of the surface S is approximated by

$$\sum_{k=1}^{N} \|\vec{R}_u(u_k, v_k) \times \vec{R}_v(u_k, v_k)\|\Delta u \Delta v.$$

As we increase N, the approximation improves. Hence,

$$\begin{aligned} A(S) &= \lim_{N\to\infty} \sum_{k=1}^{N} \|\vec{R}_u(u_k, v_k) \times \vec{R}_v(u_k, v_k)\|\Delta u \Delta v \\ &= \iint\limits_D \|\vec{R}_u(u, v) \times \vec{R}_v(u, v)\| \, dA \, . \end{aligned}$$

Example 2.4.3. Find the area of the portion of the plane $\vec{R}(u,v) = \langle 4u+5v, 1+3v, 2-3u\rangle$ traced by points (u,v) satisfying $0 \le u \le 1$ and $-1 \le v \le 1$.

Solution: The region in the uv-plane is the rectangle $[0,1] \times [-1,1]$.

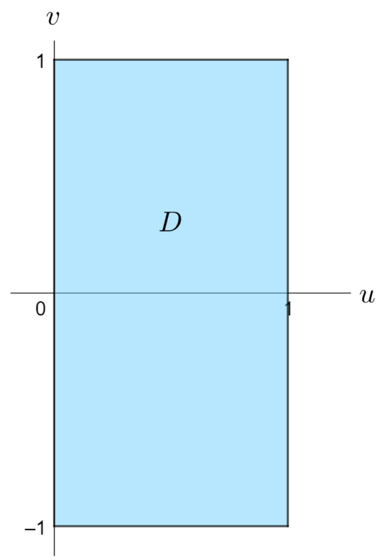


Figure 2.27: The rectangle $[0,1] \times [-1,1]$

If $\vec{R}(u,v) = \langle 4u+5v, 1+3v, 2-3u\rangle$, then

$$\begin{aligned} \vec{R}_u(u,v) &= \langle 4, 0, -3\rangle \\ \vec{R}_v(u,v) &= \langle 5, 3, 0\rangle \\ \vec{R}_u(u,v) \times \vec{R}_v(u,v) &= \langle 9, -15, 12\rangle \\ \|\vec{R}_u(u,v) \times \vec{R}_v(u,v)\| &= \sqrt{(9)^2 + (-15)^2 + (12)^2} = 15\sqrt{2}\ . \end{aligned}$$

Therefore,

$$\begin{aligned} A(S) &= \iint\limits_D 15\sqrt{2}\, dA \\ &= 15\sqrt{2} \iint\limits_D \, dA \\ &= 15\sqrt{2} \cdot (\text{area of } D) \\ &= (15\sqrt{2})(2) = 30\sqrt{2}\ . \end{aligned}$$

Example 2.4.4. Find the area of the portion of the surface defined by the vector function $\vec{R}(u,v) = \langle u+v, uv, u-v\rangle$, for points (u,v) satisfying $u^2 + v^2 \le 6$.

Solution: The region in the uv-plane is the circle centered at the origin of radius $\sqrt{6}$.

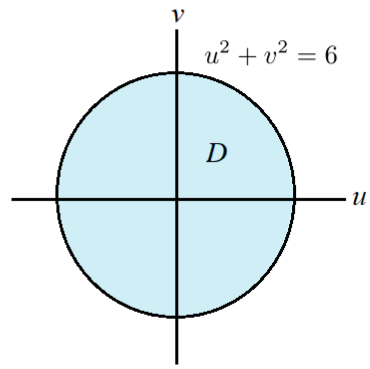


Figure 2.28: Circle centered at the origin of radius $\sqrt{6}$

If $\vec{R}(u,v) = \langle u+v, uv, u-v\rangle$, then

$$\begin{aligned}
\vec{R}_u(u,v) &= \langle 1, v, 1\rangle \\
\vec{R}_v(u,v) &= \langle 1, u, -1\rangle \\
\vec{R}_u(u,v) \times \vec{R}_v(u,v) &= \langle -v-u, 2, u-v\rangle \\
\|\vec{R}_u(u,v) \times \vec{R}_v(u,v)\| &= \sqrt{2u^2 + 2v^2 + 4}\ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
A(S) &= \iint_D \sqrt{2u^2 + 2v^2 + 4}\, dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{6}} \sqrt{2r^2 + 4}\ r\, dr\, d\theta\ .
\end{aligned}$$

Let $u = 2r^2 + 4$. Then $du = 4r\, dr$. Moreover, as r varies from 0 to $\sqrt{6}$, u varies from 4 to 16. Hence

$$\begin{aligned}
A(S) &= \int_0^{2\pi} \int_4^{16} \frac{u^{\frac{1}{2}}}{4}\, du\, d\theta \\
&= \int_0^{2\pi} \left.\frac{u^{\frac{3}{2}}}{6}\right|_{u=4}^{u=16} d\theta \\
&= \int_0^{2\pi} \left(\frac{64}{6} - \frac{8}{6}\right) d\theta \\
&= \int_0^{2\pi} \frac{28}{3}\, d\theta \\
&= \left.\frac{28}{3}\theta\right|_0^{2\pi} = \frac{56\pi}{3}\ .
\end{aligned}$$

Suppose now that a surface S is the graph of a differentiable function $z = f(x,y)$ over a closed and bounded region R. Then, a natural parametrization for S is

$$x = x,\ \ y = y,\ \ z = f(x,y)$$

with corresponding vector equation

$$\vec{R}(x, y) = \langle x, y, f(x, y)\rangle.$$

Hence,

$$\vec{R}_x(x, y) = \langle 1, 0, f_x(x, y)\rangle \quad\text{and}\quad \vec{R}_y(x, y) = \langle 0, 1, f_y(x, y)\rangle.$$

It follows that

$$\vec{R}_x(x, y) \times \vec{R}_y(x, y) = \langle -f_x(x, y), -f_y(x, y), 1\rangle = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right\rangle,$$

Hence,

$$\begin{aligned} A(S) &= \iint_R \|\vec{R}_x(x, y) \times \vec{R}_y(x, y)\|\, dA \\ &= \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA\ . \end{aligned}$$

Example 2.4.5. Let S be the portion of the plane $3x - 2y - z + 6 = 0$ that lies above the region enclosed by the parabola $y = x^2 - 4$ and the line $y = x + 2$. Find the area of S.

Solution: The surface S is the portion of $z = 3x - 2y + 6$ above the region R shown below.

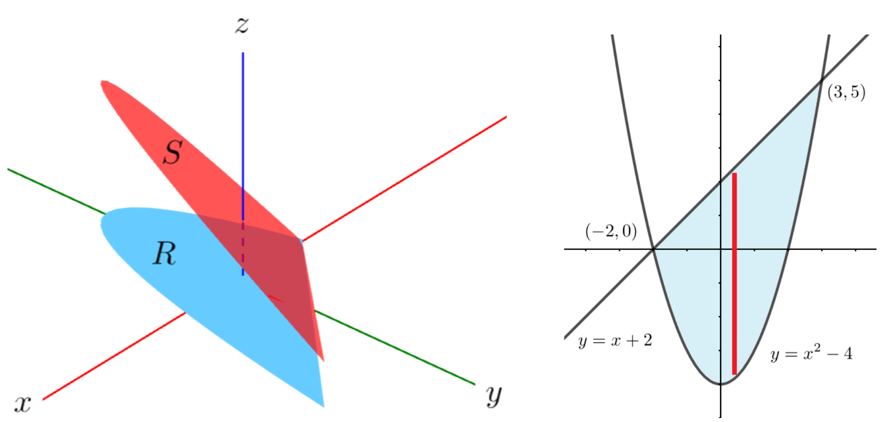


Figure 2.29: The surface S and it projection to the xy-plane

Note that

$$\begin{aligned} A(S) &= \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \iint_R \sqrt{1 + (3)^2 + (-2)^2}\, dA \\ &= \sqrt{14} \iint_R dA. \end{aligned}$$

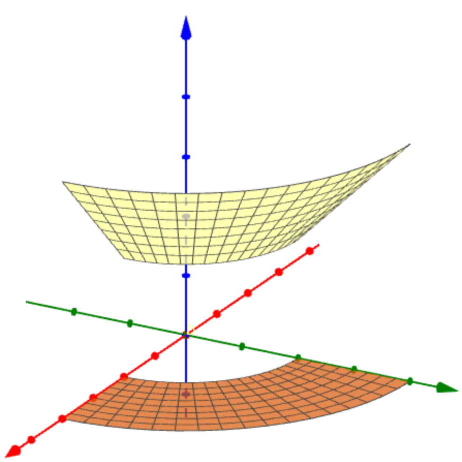
Note that

$$
\begin{aligned}
\iint\limits_R dA &= \int_{-2}^{3}\int_{x^2-4}^{x+2} dy\,dx \\
&= \int_{-2}^{3} y\Big|_{y=x^2-4}^{y=x+2} dx \\
&= \int_{-2}^{3} \left[(x+2)-(x^2-4)\right]\,dx \\
&= \int_{-2}^{3} \left(-x^2+x+6\right)\,dx \\
&= \left(-\frac{x^3}{3}+\frac{x^2}{2}+6x\right)\Big|_{-2}^{3} \\
&= \left[\left(-9+\frac{9}{2}+18\right)-\left(\frac{8}{3}+2-12\right)\right] \\
&= \frac{125}{6}\,.
\end{aligned}
$$

Therefore, the area of S, $A(S)=\dfrac{125}{6}\sqrt{14}$.

Example 2.4.6. Find the surface area of the portion of the cone $z=\sqrt{x^2+y^2}$ in the first octant between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

Solution:



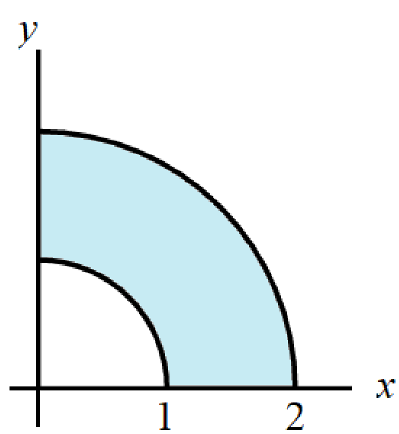


Figure 2.30: The portion of the cone and its projection to the xy-plane

$$\begin{aligned}
A(S) &= \iint\limits_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \iint\limits_R \sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\, dA \\
&= \sqrt{2}\iint\limits_R dA \\
&= \sqrt{2}\cdot(\text{area of } R) \\
&= \sqrt{2}\left[\frac{1}{4}\left(\pi(2)^2-\pi(1)^2\right)\right] \\
&= \frac{3}{4}\sqrt{2}\,\pi\ .
\end{aligned}$$

EXERCISES 2.4

I. Find the area of the region R using double integral.

1. R is enclosed by the parabola $y = 4 - x^2$ and the lines $y = 2x + 4$ and $y = -x - 2$
2. R is the region enclosed by the cardioid with polar equation $r = 2 - 2\cos\theta$

II. Determine the area of the surface S.

1. $S : \vec{R}(u, v) = \langle u + v, uv, u - v\rangle$, with $u^2 + v^2 \le 6$
2. $S : \vec{R}(u, v) = \langle u\cos v, u\sin v, u\rangle$, with $0 \le u \le 2v$ and $0 \le v \le 2\pi$
3. S is the surface cut from the plane $2x + y + z = 4$ by the planes $x = 0$, $x = 1$, $y = 0$ and $y = 1$
4. S is the portion of the sphere $x^2 + y^2 + z^2 = 16$ between the planes $z = 1$ and $z = 2$
5. S is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$

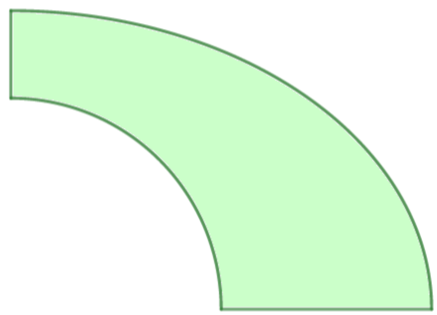
III. Determine the mass and center of mass of the lamina in the shape of the region bounded by $y = x^3$ and $y = \sqrt{x}$ given that the density is $\delta(x, y) = 2x + 6y$.

IV. Set up the double integral that will give the mass of a lamina that occupies the region bounded by $y = e^x$, $y = 0$, $x = 0$ and $x = 1$, and has the density function $\delta(x, y) = y$.

V. Set up an iterated double integral in polar coordinates that gives the mass of a lamina in the shape of the region in the first quadrant inside the circle $x^2 + y^2 = 9$ but outside the circle $x^2 + (y - 1)^2 = 1$, having density $\delta(x, y) = x^3y$.

VI. Set up an iterated double integral equal in polar coordinates equal to the mass of the lamina occupying the region bounded by the curves $y = \sqrt{9 - x^2}$ and $y = |x|$ with density function $\delta(x, y) = x^2 + y$.

VII. A lamina R occupies the region in the first quadrant bounded above by the ellipse $x^2+2y^2=4$ and below by the circle $x^2+y^2=1$ (see region below). If the density of the lamina at any point (x,y) is $\delta(x,y)=1+x$, set up the iterated double integrals(s) in Cartesian coordinates equal to the mass of R.



2.5 Triple Integrals

Triple integrals of functions of three variables are a fairly straightforward generalization of double integrals. To arrive at an analogous definition, we consider the problem of finding the mass of a solid with a given density function.

Mass Problem. Let G be a rectangular box with faces parallel to the three coordinate planes; that is,

$$G=[a,b]\times[c,d]\times[r,s]=\{(x,y,z)\mid a\le x\le b, c\le y\le d, r\le x\le s\},$$

and suppose that the density at a point $(x,y,z)\in G$ is $f(x,y,z)$, a function continuous on G. Find the mass M of G.

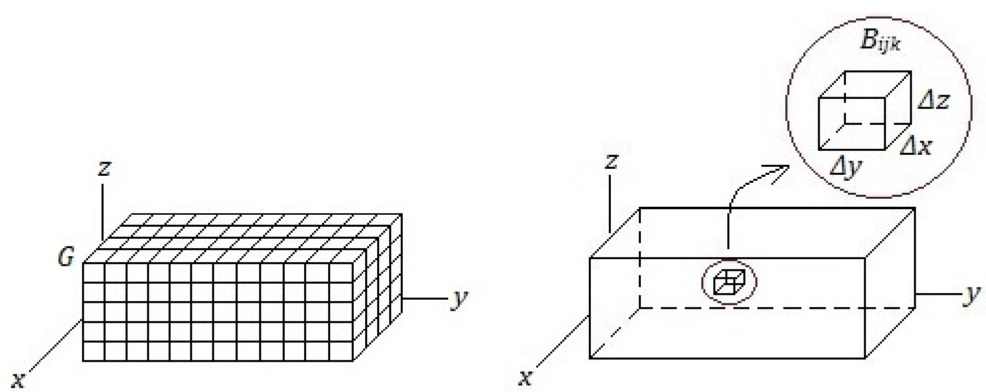


Figure 2.31: Partitioning a rectangular box

To answer the problem, we proceed as follows:

- Partition G into subboxes by dividing $[a, b]$ into l subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, l$, each of length Δx, $[c, d]$ into m subintervals $[y_{j-1}, y_j]$, $j = 1, 2, \ldots, m$, each of length Δy, and $[r, s]$ into n subintervals $[z_{k-1}, z_k]$, $k = 1, 2, \ldots, n$, each of length Δz. Let $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$, as in Figure 2.31. The volume of each B_{ijk} is $\Delta V = \Delta x \Delta y \Delta z$. For simplicity, we renumber the subboxes (the order does not matter) and call them B_ℓ, $\ell = 1, 2, \ldots, N$.

- Take a point $(x_\ell^*, y_\ell^*, z_\ell^*)$ in each B_ℓ and assume that B_ℓ has constant density $f(x_\ell^*, y_\ell^*, z_\ell^*)$. Using the formula mass = density $\times$ volume, the mass of each subbox is approximately

$$f(x_\ell^*, y_\ell^*, z_\ell^*)\Delta V \ .$$

- A good approximation for the mass M of G is

$$M \approx \sum_{\ell=1}^{N} f(x_\ell^*, y_\ell^*, z_\ell^*)\Delta V.$$

  This approximation becomes better as the norm of the partition $\|P\|$, which is defined as the longest diagonal of each subbox, approaches 0. As $\|P\| \to 0$, the total number of subboxes N approaches infinity. Moreover, if the partitioning is uniform (which is the case here), then as N approaches infinity, it also follows that $\|P\|$ approaches 0.

Definition 2.5.1. Let f be a function of x, y and z and $R = [a, b] \times [c, d] \times [r, s]$. The ***triple integral*** of f over the rectangular box G is defined by

$$\iiint\limits_G f(x, y, z)\, dV := \lim_{\|P\| \to 0} \sum_{\ell=1}^{N} f(x_\ell^*, y_\ell^*, z_\ell^*)\Delta V = \lim_{N \to \infty} \sum_{\ell=1}^{N} f(x_\ell^*, y_\ell^*, z_\ell^*)\Delta V \ .$$

provided this limit exists

When a unique limiting value is attained, no matter how the partitions and the points $(x_\ell^*, y_\ell^*, z_\ell^*)$ are chosen, we say f is integrable over G. Similar to double integrals, it can be shown that if f is continuous, then f is integrable.

$$M = \iiint\limits_G f(x, y, z)\, dV.$$

Remark. Some properties of the triple integral.

1. If c is a constant, then $\displaystyle\iiint\limits_G cf(x, y, z)\, dV = c \iiint\limits_G f(x, y, z)\, dV$.

2. $\displaystyle\iiint\limits_G [f(x, y, z) \pm g(x, y, z)]\ dV = \iiint\limits_G f(x, y, z)\, dV \pm \iiint\limits_G g(x, y, z)\, dV$.

3. If $G = G_1 \cup G_2$ such that G_1 and G_2 do not overlap, then

$$\iiint\limits_G f(x,y,z)\,dV = \iiint\limits_{G_1} f(x,y,z)\,dV + \iiint\limits_{G_2} f(x,y,z)\,dV \ .$$

To evaluate a triple integral over a rectangular box, we make use of the extension of the Fubini's Theorem to triple integrals.

Theorem 2.5.2. If f is continuous on the box $G = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint\limits_G f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

There are five other possible orders in which the above integral may be evaluated: $dx\,dz\,dy$, $dy\,dx\,dz$, $dy\,dz\,dx$, $dz\,dx\,dy$ and $dz\,dy\,dx$.

Example 2.5.1. Evaluate $\iiint\limits_G (2x + y\sin z)\,dV$, where $G = [-1,1] \times [0,2] \times [0, \frac{\pi}{2}]$.

Solution: We have

$$\begin{aligned}
\iiint\limits_G (2x + y\sin z)\,dV &= \int_0^{\frac{\pi}{2}} \int_0^2 \int_{-1}^1 (2x + y\sin z)\,dx\,dy\,dz \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 \left(x^2 + xy\sin z\right)\Big|_{x=-1}^{x=1}\,dy\,dz \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 \left[(1 + y\sin z) - (1 - y\sin z)\right]\,dy\,dz \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 2y\sin z\,dy\,dz \\
&= \int_0^{\frac{\pi}{2}} y^2 \sin z\Big|_{y=0}^{y=2}\,dz \\
&= \int_0^{\frac{\pi}{2}} 4\sin z\,dz \\
&= -4\cos z\Big|_0^{\frac{\pi}{2}} = 4 \ .
\end{aligned}$$

Triple Integrals Over General Regions

Definition 2.5.3. Let f be a continuous function of x, y and z on a solid G. Consider any rectangular box $E = [a,b] \times [c,d] \times [r,s]$ that encloses G and define a function F by

$$F(x,y,z) = \begin{cases} f(x,y,z) & , \text{ if } (x,y,z) \in G \\ 0 & , \text{ if } (x,y,z) \in E \setminus G \end{cases} .$$

The ***triple integral*** of f over G is defined by

$$\iiint\limits_G f(x,y,z)\,dV := \iiint\limits_E F(x,y,z)\,dV .$$

To evaluate triple integrals over general solids, we classify simple solids as either type xy, type xz or type yz.

1. A *simple solid G is of type xy* if it lies between two surfaces with equations $z = u_1(x,y)$ and $z = u_2(x,y)$ for all points (x,y) in a region R on the xy-plane; that is,
$$G = \{(x,y,z) \mid u_1(x,y) \le z \le u_2(x,y),\ (x,y) \in R\} .$$

2. A *simple solid G is of type xz* if it lies between two surfaces with equations $y = u_1(x,z)$ and $y = u_2(x,z)$ for all points (x,z) in a region R the xz-plane; that is,
$$G = \{(x,y,z) \mid u_1(x,z) \le y \le u_2(x,z),\ (x,z) \in R\} .$$

3. A *simple solid G is of type yz* if it lies between the surfaces with equations $x = u_1(y,z)$ and $x = u_2(y,z)$ for all points (y,z) in a region R on the yz-plane; that is,
$$G = \{(x,y,z) \mid u_1(y,z) \le x \le u_2(y,z),\ (y,z) \in R\} .$$

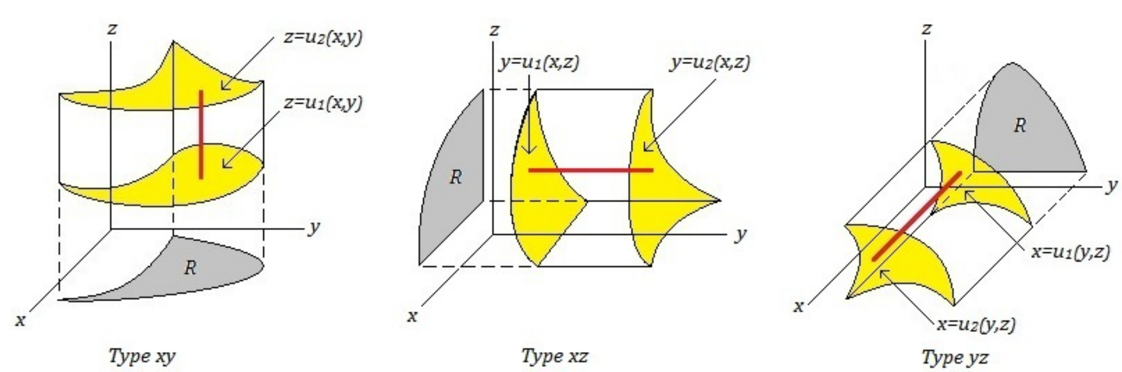


Figure 2.32: Simple Solids of Types xy, xz and yz

By a similar argument that leads to Theorem 2.2.2, we have the following.

Theorem 2.5.4. Let G be a solid.

1. If G is of type xy and R is its projection onto the xy-plane, then
$$\iiint\limits_G f(x,y,z)\,dV = \iint\limits_R \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA\,.$$

2. If G is of type xz and R is its projection onto the xz-plane, then
$$\iiint\limits_G f(x,y,z)\,dV = \iint\limits_R \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,dA\,.$$

3. If G is of type yz and R is its projection onto the yz-plane, then
$$\iiint\limits_G f(x,y,z)\,dV = \iint\limits_R \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dA\,.$$

Example 2.5.2. Evaluate $\displaystyle\iiint\limits_G 6xy\,dV$, where G is the solid bounded by the planes $x=0$, $y=0$, $x+y+z=1$ and $x+y-z=1$.

Solution: The solid G is of type xy bounded below by $z=x+y-1$ and above by $z=1-x-y$. Its projection R onto the xy-plane can be classified as type I.

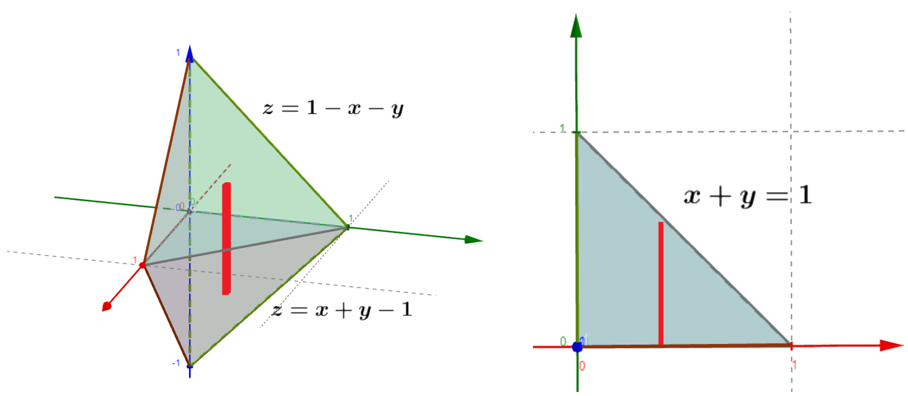


Figure 2.33: Solid bounded by the planes $x=0$, $y=0$, $x+y+z=1$ and $x+y-z=1$

By Theorem 2.5.4, we have
$$\begin{aligned}\iiint\limits_G 6xy\,dV &= \iint\limits_R \int_{x+y-1}^{1-x-y} 6xy\,dz\,dA\\ &= \int_0^1\int_0^{1-x}\int_{x+y-1}^{1-x-y} 6xy\,dz\,dy\,dx.\end{aligned}$$

Computing this we get

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_{x+y-1}^{1-x-y} 6xy\,dz\,dy\,dx &= \int_0^1 \int_0^{1-x} 6xyz\Big|_{z=x+y-1}^{z=1-x-y} dy\,dx \\
&= \int_0^1 \int_0^{1-x} [6xy(1-x-y) - 6xy(x+y-1)]\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left(12xy - 12x^2y - 12xy^2\right)\,dy\,dx \\
&= \int_0^1 \left(6xy^2 - 6x^2y^2 - 4xy^3\right)\Big|_{y=0}^{y=1-x} dx \\
&= \int_0^1 \left[6x(1-x)^2 - 6x^2(1-x)^2 - 4x(1-x)^3\right]\,dy \\
&= \int_0^1 2x(1-x)^3\,dx \\
&= -\int_1^0 2(1-u)u^3\,du, \qquad \text{let } u = 1-x,\ du = -dx \\
&= \int_0^1 (2u^3 - 2u^4)\,du \\
&= \left(\frac{u^4}{2} - \frac{2u^5}{5}\right)\Big|_0^1 = \frac{1}{10}\ .
\end{aligned}$$

Example 2.5.3. Evaluate $\iiint\limits_G x\,dV$, where G is the solid in the first quadrant bounded by the surfaces $y+z=4$ and $y=4-x^2, y=0$.

Solution: The solid G is of type xy bounded below by $z=0$ and above by $z=4-y$. Its projection R onto the xy-plane can be classified as type I.

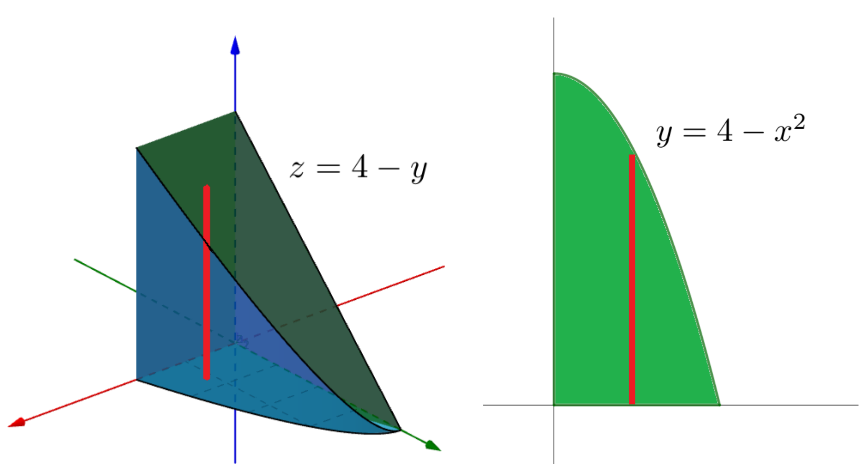


Figure 2.34: Solid in the first quadrant bounded by $y+z=4$ and $y=4-x^2, y=0$

We have

$$\begin{aligned}
\iiint\limits_G x\,dV &= \iint\limits_R \int_0^{4-y} x\,dz\,dA \\
&= \int_0^2 \int_0^{4-x^2} \int_0^{4-y} x\ dzdydx \\
&= \int_0^2 \int_0^{4-x^2} xz\Big|_{z=0}^{z=4-y}\ dydx \\
&= \int_0^2 \int_0^{4-x^2} 4x - xy\ dydx \\
&= \int_0^2 \left(4xy - \frac{xy^2}{2}\right)\Big|_{y=0}^{y=4-x^2}\ dx \\
&= \int_0^2 8x - \frac{x^5}{2}\ dx \\
&= \left(4x^2 - \frac{x^6}{12}\right)\Big|_{x=0}^{x=2} = \frac{32}{3}.
\end{aligned}$$

Example 2.5.4. Evaluate $\iiint\limits_G dV$, where G is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $2x + z = 2$ and $y + 2z = 4$.

Solution: For simpler computation, we view G as a solid of type xz and the projection R onto the xz-plane may be classified as type I region.

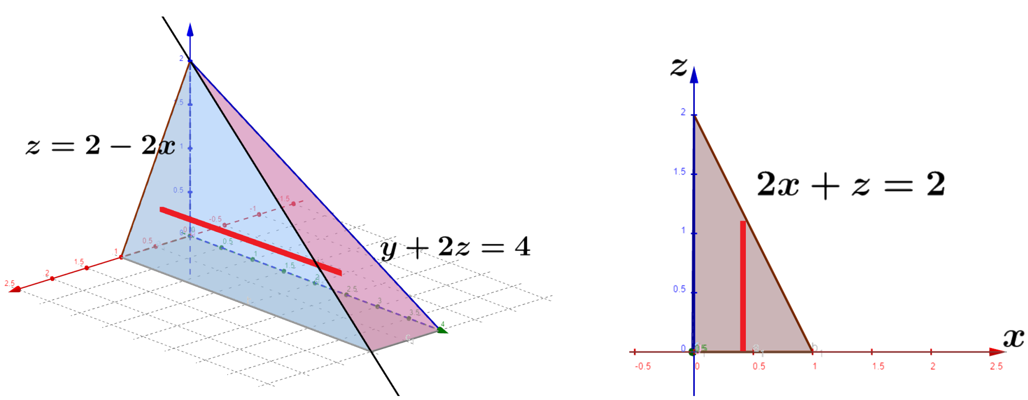


Figure 2.35: Solid bounded by $x = 0$, $y = 0$, $z = 0$, $2x + z = 2$ and $y + 2z = 4$, and its projection to the xz-plane

We have

$$
\begin{aligned}
\iiint\limits_G dV &= \int_0^1 \int_0^{2-2x} \int_0^{4-2z} dy\, dz\, dx \\
&= \int_0^1 \int_0^{2-2x} y\Big|_{y=0}^{y=4-2z} dz\, dx \\
&= \int_0^1 \int_0^{2-2x} (4-2z)\; dz\, dx \\
&= \int_0^1 \left(4z - z^2\right)\Big|_{z=0}^{z=2-2x} dx \\
&= \int_0^1 \left(4(2-2x) - (2-2x)^2\right)\, dx \\
&= \int_0^1 (4-4x^2)\, dx \\
&= \left(4x - \frac{4}{3}x^3\right)\Big|_0^1 \\
&= 4 - \frac{4}{3} = \frac{8}{3}\ .
\end{aligned}
$$

Another Solution: We can view the solid G as a union of two solids (both of type xy) separated by the plane of intersection of $2x+z=2$ and $y+2z=4$.

$$
\begin{aligned}
y + 2z = 4 &\cap 2x + z = 2 \\
y + 2(2-2x) &= 4 \\
y - 4x &= 0 \\
y &= 4x
\end{aligned}
$$

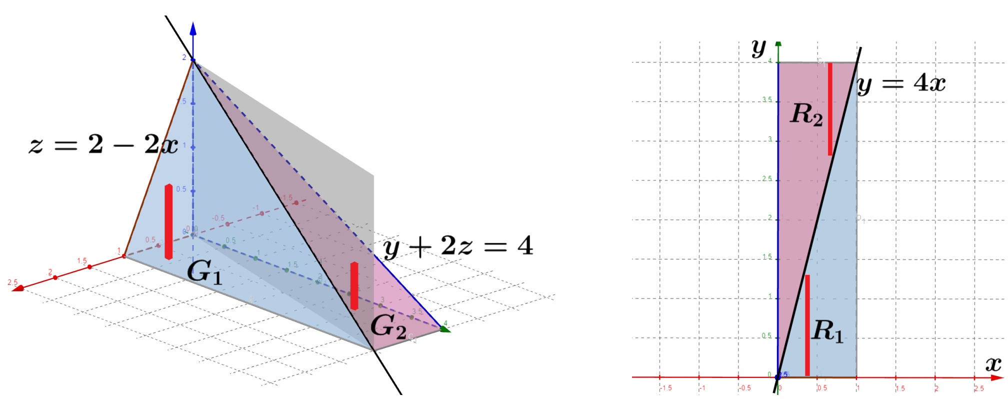


Figure 2.36: Solid bounded by $x=0$, $y=0$, $z=0$, $2x+z=2$ and $y+2z=4$, and its projection to the xy-plane

Therefore, $G = G_1 \cup G_2$, where G_1 is in front of the plane $y=4x$ while G_2 is behind the plane $y=4x$. Let R_1 and R_2 be the projections, respectively, of G_1 and G_2 onto the xy-plane. By the properties of triple integral, we have

$$\begin{aligned}
\iiint\limits_G dV &= \iiint\limits_{G_1} dV + \iiint\limits_{G_2} dV \\
&= \iint\limits_{R_1} \int_0^{2-2x} dz\, dA + \iint\limits_{R_2} \int_0^{\frac{4-y}{2}} dz\, dA \\
&= \int_0^1 \int_0^{4x} \int_0^{2-2x} dz\, dy\, dx + \int_0^1 \int_{4x}^4 \int_0^{\frac{4-y}{2}} dz\, dy\, dx\ .
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\int_0^1 \int_0^{4x} \int_0^{2-2x} dz\, dy\, dx &= \int_0^1 \int_0^{4x} (2-2x)\, dy\, dx \\
&= \int_0^1 (8x - 8x^2)\, dx = \left(4x^2 - \frac{8}{3}x^3\right)\Bigg|_0^1 \\
&= \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}.
\end{aligned}$$

It is left as an exercise to show that

$$\int_0^1 \int_{4x}^4 \int_0^{\frac{4-y}{2}} dz\, dy\, dx = \frac{4}{3}.$$

Hence,

$$\iiint\limits_G dV = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}.$$

Example 2.5.5. Rewrite the triple integral

$$\int_0^4 \int_0^{4-y} \int_0^{\sqrt{z}} f(x,y,z)\, dx\, dz\, dy$$

such that partial integration is performed in the following order: with respect to z first, then to y, and finally to x.

Solution: The solid G is is considered as type yz and it satisfies:

$$G : \begin{cases} 0 \le x \le \sqrt{z}, \\ 0 \le z \le 4 - y, \\ 0 \le y \le 4. \end{cases}$$

As a type xy solid, the points (x, y, z) of G will be bounded as follows:

$$G : \begin{cases} x^2 \le z \le 4 - y, \\ 0 \le y \le 4 - x^2, \\ 0 \le x \le 2. \end{cases}$$

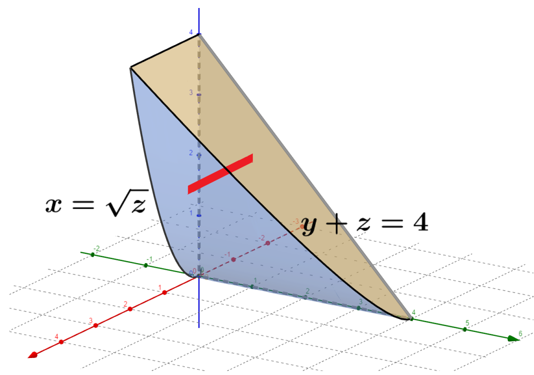
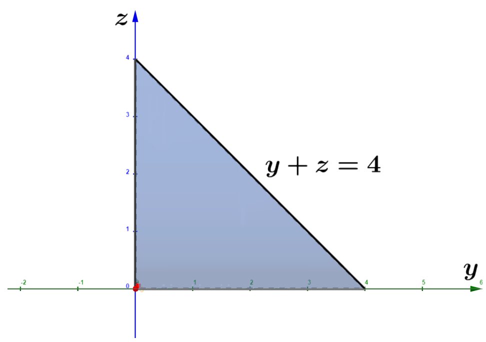


Figure 2.37: Solid in the first octant bounded by $z = x^2$ and $y + z = 4$



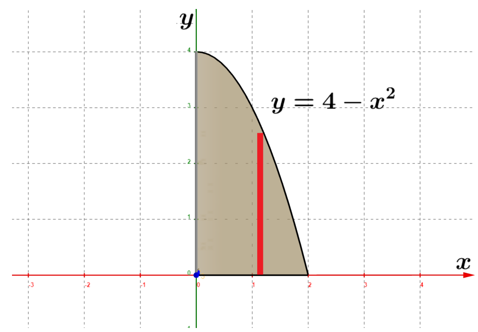


Figure 2.38: Projection of the solid to the yz and xy planes

We then have

$$\int_0^4 \int_0^{4-y} \int_0^{\sqrt{z}} f(x, y, z)\, dx\, dz\, dy = \int_0^2 \int_0^{4-x^2} \int_{x^2}^{4-y} f(x, y, z)\, dz\, dy\, dx.$$

EXERCISES 2.5

I. Evaluate the iterated triple integral.

1. $\displaystyle\int_0^1 \int_0^z \int_0^{y+z} 6yz\, dx\, dy\, dz$
2. $\displaystyle\int_0^3 \int_0^1 \int_0^{\sqrt{1-x^2}} xe^y\, dz\, dx\, dy$
3. $\displaystyle\int_0^1 \int_x^{2x} \int_0^y 2xyz\, dz\, dy\, dx$
4. $\displaystyle\int_0^1 \int_0^x \int_0^z xe^{-z^2}\, dy\, dz\, dx$
5. $\displaystyle\int_1^2 \int_y^{y^2} \int_0^{\ln x} ye^z\, dz\, dx\, dy$

II. Evaluate the triple integral.

1. $\displaystyle\iiint_G (xy + z^2)\, dV$ where $G = \{(x, y, z) : 0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3\}$.

2. $\displaystyle\iiint_G (x+2y)\,dV$, where G is the tetrahedron enclosed by the three coordinate planes and the plane $2x-3y-z=6$.
3. $\displaystyle\iiint_G z\,dV$, where G is bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $y=3x$, and $z=0$ in the first octant.
4. $\displaystyle\iiint_G xyz\,dV$, where G lies below the cylinder $z=1-x^2$ and above the rectangle $R=\{(x,y)\,|\,-1\le x\le 0, 0\le y\le 2\}$.
5. $\displaystyle\iiint_G x\,dV$, where G is the solid bounded by the surfaces $y+z=4$, $y=4-x^2$, $y=0$ and $z=0$.

2.6 Triple Integrals in Cylindrical Coordinates

In this section, we discuss the cylindrical coordinate system which basically transforms a coordinate plane to a polar plane.

The Cylindrical Coordinate System

The three-dimensional cylindrical coordinate system assigns to every point P in space at least one coordinate triple of the form (r,θ,z), as shown in Figure 2.39, where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the rectangular vertical coordinate.

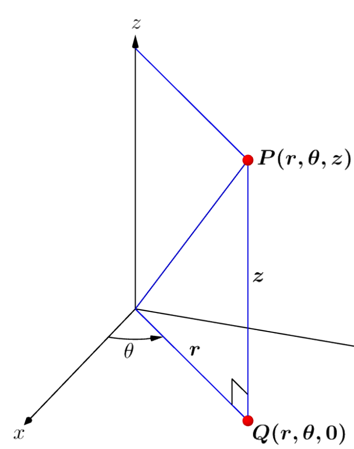


Figure 2.39: Cylindrical coordinates of a point P

The rectangular coordinates and cylindrical coordinates are related by the following equations.

$$x=r\cos\theta,\ \ y=r\sin\theta,\ \ z=z$$

$$r^2=x^2+y^2,\ \ \tan\theta=\frac{y}{x}$$

Remark 2.6.1. In cylindrical coordinates, the equation

1. $r = r_0$, where $r_0 > 0$ describes a circular cylinder about the z-axis with radius r_0, while $r = 0$ represents the z-axis.

2. $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

3. $z = z_0$ describes a plane perpendicular to the z-axis.

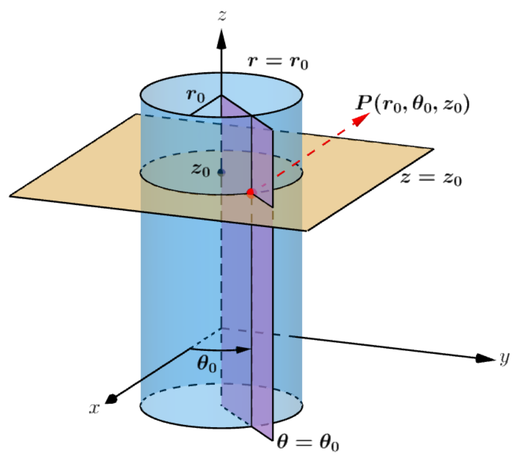


Figure 2.40: Graphs of $r = r_0$, $\theta = \theta_0$ and $z = z_0$

Integration in Cylindrical Coordinates

Let G be a solid of type xy bounded below by $z = u_1(x, y)$ and above by $z = u_2(x, y)$ and whose projection R onto the $xy-$plane is a simple polar region, that is,

$$R = \{(r, \theta) \mid r_1(\theta) \leq r \leq r_2(\theta),\ \alpha \leq \theta \leq \beta\}.$$

Partition G into cylindrical wedges, rather than into rectangular boxes. See Figure 2.41. In each cylindrical wedge, r, θ and z change by Δr, $\Delta\theta$ and Δz, respectively. The volume ΔV_ℓ of such cylindrical wedge is the area A_ℓ of its base times the height Δz.
Let $(r_\ell^*, \theta_\ell^*, z_\ell^*)$ be the center of each wedge. Recall that in polar coordinates,

$$\Delta A_\ell = r_\ell^* \Delta r \Delta\theta.$$

So,

$$\Delta V_\ell = \Delta A_\ell \Delta z = r_\ell^* \Delta z \Delta r \Delta\theta.$$

The triple Riemann sum for f over G is of the form

$$\sum_{\ell=1}^{N} f(r_\ell^* \cos\theta_\ell^*, r_\ell^* \sin\theta_\ell^*, z_\ell^*)\ r_\ell^* \Delta z \Delta r \Delta\theta.$$

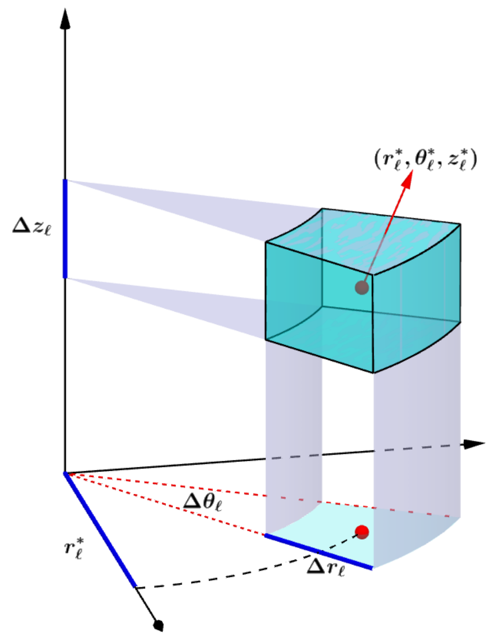


Figure 2.41: A cylindrical wedge

Taking the limit of the Riemann sum as $N \to \infty$, we get the following conversion

$$\iiint_G f(x,y,z)\,dV = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\; r\,dz\,dr\,d\theta\ .$$

Example 2.6.1. Evaluate $\iiint_G dV$, where G is enclosed by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

Solution: The solid G is of type xy and its projection onto the xy-plane is the curve of intersection of the two paraboloids.

$$\begin{aligned} z = 3x^2 + 3y^2 &\cap z = 4 - x^2 - y^2 \\ 3x^2 + 3y^2 &= 4 - x^2 - y^2 \\ x^2 + y^2 &= 1 \end{aligned}$$

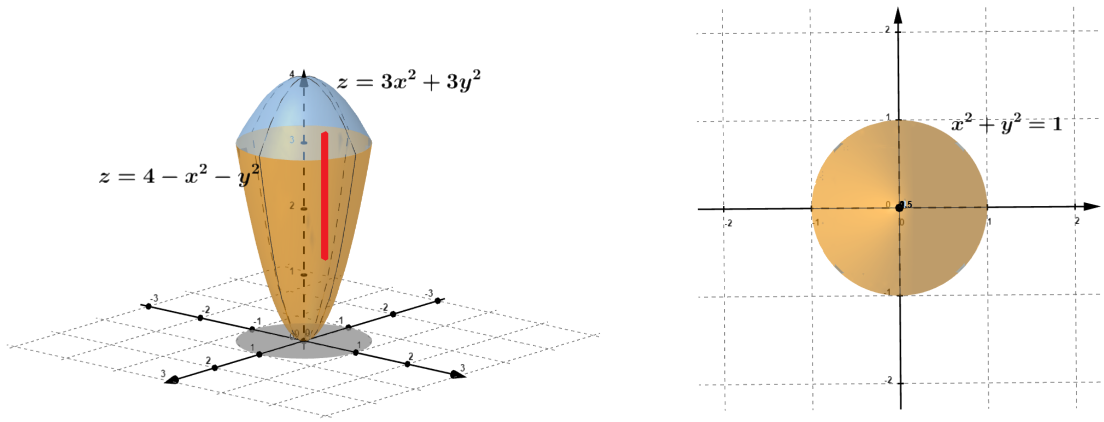


Figure 2.42: Solid enclosed by $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$

We have

$$\begin{aligned}
\iiint\limits_G dV &= \iint\limits_R \int_{3x^2+3y^2}^{4-x^2-y^2} dz\, dA \\
&= \iint\limits_R z\Big|_{z=3x^2+3y^2}^{z=4-x^2-y^2} dA \\
&= \iint\limits_R \left[(4-x^2-y^2)-(3x^2+3y^2)\right] dA \\
&= \iint\limits_R \left(4-4x^2-4y^2\right) dA.
\end{aligned}$$

Now, since R is bounded by a circle centered at the origin, it would be easier if we use polar coordinates for the remaining double integral. We get

$$\begin{aligned}
\iiint\limits_G dV &= \int_0^{2\pi}\int_0^1 \left(4-4r^2\right) r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(4r-4r^3\right) dr\, d\theta \\
&= \int_0^{2\pi} (2r^2-r^4)\Big|_{r=0}^{r=1} d\theta \\
&= \int_0^{2\pi} d\theta \\
&= \theta\Big|_0^{2\pi} = 2\pi\ .
\end{aligned}$$

Example 2.6.2. Evaluate $\displaystyle\iiint\limits_G \frac{2z}{x^2+y^2}\, dV$, where G is the solid above the xy-plane, below the cone $z = \sqrt{x^2+y^2}$ and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

Solution: In cylindrical coordinates, the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ have equations $r = 1$ and $r = 2$, respectively, while the cone $z = \sqrt{x^2 + y^2}$ has equation $z = r$. The projection of G onto the xy-plane is the region between the circles $r = 1$ and $r = 2$.

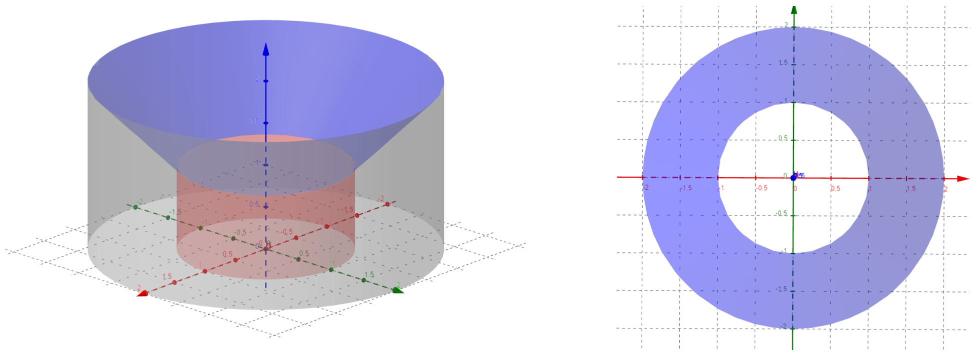


Figure 2.43: Solid above $z = 0$, below $z = \sqrt{x^2 + y^2}$, and between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

We then have

$$\begin{aligned}
\iiint\limits_G \frac{2z}{x^2+y^2}\,dV &= \int_0^{2\pi}\int_1^2\int_0^r \frac{2z}{r^2}\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2\int_0^r \frac{2z}{r}\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 \left.\frac{z^2}{r}\right|_{z=0}^{z=r} dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 r\,dr\,d\theta \\
&= \int_0^{2\pi} \left.\frac{r^2}{2}\right|_{r=1}^{r=2} d\theta \\
&= \int_0^{2\pi} \frac{3}{2}\,d\theta \\
&= \left.\frac{3}{2}\theta\right|_0^{2\pi} = 3\pi\ .
\end{aligned}$$

Example 2.6.3. Evaluate $\iiint\limits_G dV$, where G is enclosed by the paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$.

Solution: The paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$ have cylindrical equations $z = r^2$ and $z = 4 - r^2$, respectively. The projection of the solid on the xy-plane is bounded by the circle of intersection of the two paraboloids. Solving for the intersection of the paraboloids, we get

$$\begin{aligned} z = r^2 \;\cap\; z &= 4 - r^2 \\ r^2 &= 4 - r^2 \\ r^2 &= 2 \\ r &= \sqrt{2}. \end{aligned}$$

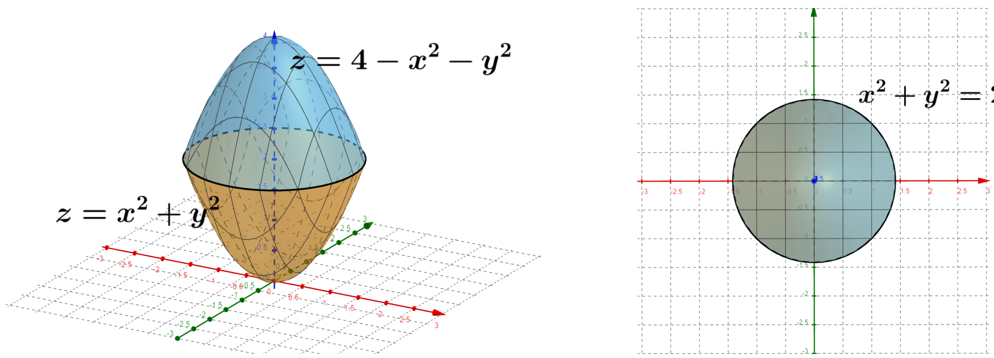


Figure 2.44: Solid enclosed by $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$

We then have

$$\begin{aligned} \iiint\limits_G dV &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} rz\Big|_{z=r^2}^{z=4-r^2} dr\,d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} \left[r(4-r^2) - r(r^2)\right] dr\,d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} (4r - 2r^3)\,dr\,d\theta \\ &= \int_0^{2\pi} \left(2r^2 - \frac{r^4}{2}\right)\Bigg|_{r=0}^{r=\sqrt{2}} d\theta \\ &= \int_0^{2\pi} 2\,d\theta \\ &= 2\theta\Big|_0^{2\pi} = 4\pi\ . \end{aligned}$$

Example 2.6.4. Write an iterated triple integral in cylindrical coordinates equivalent to

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{x^2+y^2}^{2x} y\,dz\,dy\,dx\ .$$

Solution: The given iterated integral tells us that the solid of integration is bounded below by $z = x^2 + y^2$ and above by $z = 2x$ whose respective cylindrical equations are $z = r^2$ and $z = 2r\cos\theta$. Moreover, the outer double integral says that the projection R of the solid onto the xy-plane is of type I bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$ for $0 \le x \le 2$. By completing the squares, it can be shown that $y = \sqrt{2x - x^2}$ is the upper semi-circle of $(x-1)^2 + y^2 = 1$, whose polar equation is $r = 2\cos\theta$.

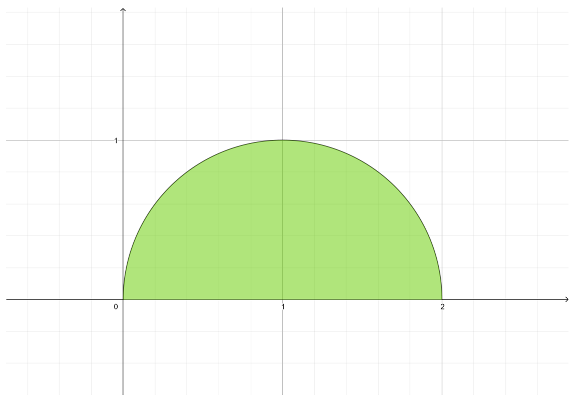


Figure 2.45: Projection of solid to the xy-plane

Therefore, we have

$$\begin{aligned}\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{x^2+y^2}^{2x} y\,dz\,dy\,dx &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{r^2}^{2r\cos\theta} (r\sin\theta) r\,dz\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{r^2}^{2r\cos\theta} r^2 \sin\theta\,dz\,dr\,d\theta\ .\end{aligned}$$

EXERCISES 2.6

I. Evaluate the integral using cylindrical coordinates.

1. $\displaystyle\iiint\limits_G y\,dV$, where G is the solid above the xy-plane , below the plane $z = x + 2$ and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$.

2. $\displaystyle\iiint\limits_G \sqrt{x^2+y^2}\,dV$, where G is the solid above the cone $z = \sqrt{x^2+y^2}$ and below the plane $z = 4$.

3. $\displaystyle\iiint\limits_G \frac{1}{\sqrt{x^2+y^2}}\,dV$, where G is the solid under the paraboloid $z = 9 - x^2 - y^2$ and above the plane $z = 1$.

4. $\displaystyle\iiint\limits_G dV$, where G is the solid above the cone $z = \sqrt{2x^2+2y^2}$ and within the sphere $x^2 + y^2 + z^2 = 12$.

5. $\iiint\limits_G (x-y)\,dV$, where G is the solid in the first octant under $z = x^2+y^2$, between the cylinders $x^2+y^2=4$ and $x^2+y^2=9$ and enclosed by the planes $z=0$, $x=2y$ and $x=0$.

II. Convert the iterated triple integral to cylindrical coordinates.

1. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{\frac{3}{2}}\,dz\,dx\,dy$

2. $\displaystyle\int_{-\sqrt{2}}^{0}\int_{-\sqrt{2-x^2}}^{0}\int_{0}^{\sqrt{x^2+y^2}} xy\,dz\,dy\,dx$

III. Evaluate the following using cylindrical coordinates.

1. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2}(1+x^2+y^2)\,dz\,dy\,dx$

2. $\displaystyle\int_{0}^{\sqrt{2}}\int_{x}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2}dz\,dy\,dx$

IV. In rectangular coordinates, the iterated integral $\displaystyle\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{1}{\cos\theta+\sin\theta}}\int_{0}^{r^2\cos^2\theta} r\,dz\,dr\,d\theta$ has the form $\displaystyle\int_{0}^{1}\int_{0}^{h(x)}\int_{0}^{g(x,y)}dz\,dy\,dx$.

1. Find $g(x,y)$ and $h(x)$.
2. Evaluate the resulting iterated integral.

V. Set up the iterated integral that gives the volume of the solid that is bounded above by the sphere $x^2+y^2+z^2=4$, below by $z=0$ and inside the cylinder $x^2+y^2=1$ using cylindrical coordinates.

VI. Consider the solid that is inside the cylinder $x^2+y^2=4$, above the xy-plane, and below the circular paraboloid $z=5-x^2-y^2$. Set up the iterated integral that is equal to the volume of the solid. Use

1. rectangular coordinates
2. cylindrical coordinates

2.7 Triple Integrals in Spherical Coordinates

The Spherical Coordinate System

The spherical coordinate system assigns to a point P in space an ordered triple of the form (ρ, θ, ϕ), as shown in Figure 2.46. The first coordinate, ρ, is the distance of P from the origin. The second coordinate, θ, is the same angular coordinate as in the cylindrical coordinate system. The third coordinate, ϕ, is the smallest positive angle that $\overrightarrow{OP}$ makes with the positive z-axis. So, $0 \leq \phi \leq \pi$.

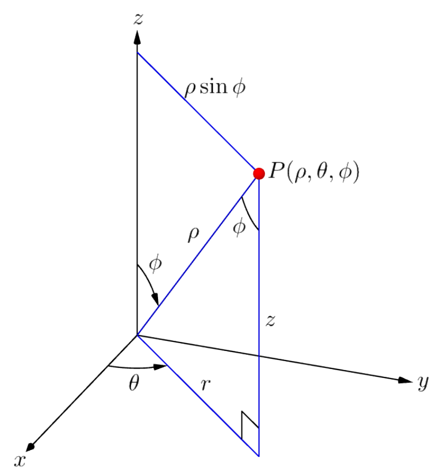


Figure 2.46: Spherical coordinates of a point P

The rectangular and spherical coordinates are related by the following equations.

$$x = \rho \sin \phi \cos \theta, \;\; y = \rho \sin \phi \sin \theta, \;\; z = \rho \cos \phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2}\,, \;\; \tan \theta = \frac{y}{x}\,, \;\; \tan \phi = \frac{\sqrt{x^2 + y^2}}{z}$$

Remark. In spherical coordinates, the equation

1. $\rho = \rho_0$, where $\rho_0 \neq 0$, represents the sphere of radius ρ_0 centered at the origin while $\rho = 0$ represents only the origin.

2. $\theta = \theta_0$ represents the half-plane hinged at the z-axis that makes an angle θ_0 with the positive x-axis.

3. $\phi = \phi_0$ represents

   (a) the positive z-axis if $\phi_0 = 0$ and the negative z-axis if $\phi_0 = \pi$.

   (b) the xy-plane if $\phi_0 = \frac{\pi}{2}$.

   (c) a cone that opens upward with vertex at the origin if $0 < \phi_0 < \frac{\pi}{2}$.

   (d) a cone that opens downward with vertex at the origin if $\frac{\pi}{2} < \phi_0 < \pi$.

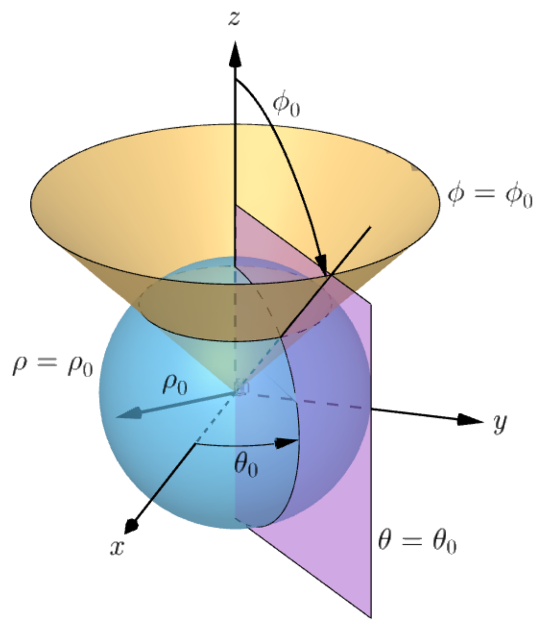


Figure 2.47: Graphs of $\rho = \rho_0$, $\theta = \theta_0$ and $\phi = \phi_0$

Example 2.7.1. Find the rectangular coordinates of the point that has the spherical coordinates $(\rho, \theta, \phi) = (2, \pi, \pi/2)$.

Solution: Recall: $\begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$ $\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \tan\theta = \frac{y}{x} \\ \tan\phi = \frac{\sqrt{x^2+y^2}}{z}. \end{cases}$

Given $(\rho, \theta, \phi) = (2, \pi, \pi/2)$, we then have

$$\begin{cases} x = 2\sin\frac{\pi}{2}\cos\pi = -2 \\ y = 2\sin\frac{\pi}{2}\sin\pi = 0 \\ z = 2\cos\frac{\pi}{2} = 0. \end{cases}$$

Example 2.7.2. Find the spherical coordinates of the point that has rectangular coordinates $(x, y, z) = (1, 1, \sqrt{2})$.

Solution: Given $(x, y, z) = (1, 1, \sqrt{2})$, we have

$$\begin{cases} \rho = \sqrt{1^2 + 1^2 + (\sqrt{2})^2} = 2 \\ \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4} \\ \phi = \tan^{-1}\frac{\sqrt{1^2+1^2}}{(\sqrt{2})^2} = \frac{\pi}{4}. \end{cases}$$

Example 2.7.3. Find a spherical equation for the cone $z = \sqrt{x^2+y^2}$.

Solution: We use the conversion formula $\tan\phi = \dfrac{\sqrt{x^2+y^2}}{z}$ on $z = \sqrt{x^2+y^2}$. We get

$$\tan\phi = 1 \;\Rightarrow\; \phi = \frac{\pi}{4}.$$

Example 2.7.4. Find a spherical equation for the sphere $x^2+y^2+(z-1)^2 = 1$.

Solution: Writing x, y and z in terms of ρ, θ and ϕ, we get

$$\begin{aligned}
x^2+y^2+(z-1)^2 &= 1\\
x^2+y^2+z^2-2z &= 0\\
\rho^2 - 2\rho\cos\phi &= 0\\
\rho(\rho - 2\cos\phi) &= 0\\
\underbrace{\rho = 0}_{\text{origin}} \quad \text{or} \quad \rho &= 2\cos\phi\ .
\end{aligned}$$

Thus, the given sphere has spherical equation $\rho = 2\cos\phi$.

Integration in Spherical Coordinates

When computing integrals in spherical coordinates, we partition the solid of integration into spherical wedges. The size of each spherical wedge is given by the changes $\Delta\rho$, $\Delta\theta$, $\Delta\phi$. Such a spherical wedge has one edge a circular arc of length $\rho^*_\ell\Delta\phi$, another edge a circular arc of length $\rho^*_\ell\sin\phi^*_\ell\Delta\theta$, and thickness $\Delta\rho$, with $(\rho^*_\ell, \theta^*_\ell, \phi^*_\ell)$ as its center. Each spherical wedge approximates a cube for sufficiently large N, and k. It can be shown that the volume ΔV_{ijk} of each spherical wedge is given by

$$\Delta V_\ell = {\rho^*_\ell}^2 \sin\phi^*_\ell \Delta\rho\Delta\theta\Delta\phi\ .$$

So the corresponding Riemann sum for a continuous function $f(x, y, z)$ is

$$\sum_{\ell=1}^{N} f(\rho^*_\ell\sin\phi^*_\ell\cos\theta^*_\ell, \rho^*_\ell\sin\phi^*_\ell\sin\theta^*_\ell, \rho^*_\ell\cos\phi^*_\ell)\ {\rho^*_\ell}^2\sin\phi^*_\ell\Delta\rho\Delta\theta\Delta\phi\ .$$

By taking the limit as $N \to \infty$, we get

$$\iiint\limits_G f(x,y,z)\,dV = \iiint\limits_G f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\ .$$

Note that in spherical coordinates $dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$.

When setting-up an iterated integral in spherical coordinates, we take the following steps.

- *Sketch.* Sketch the solid G of integration and label the boundaries with their corresponding spherical equations. Draw a strip/line through G that emanates from the origin.

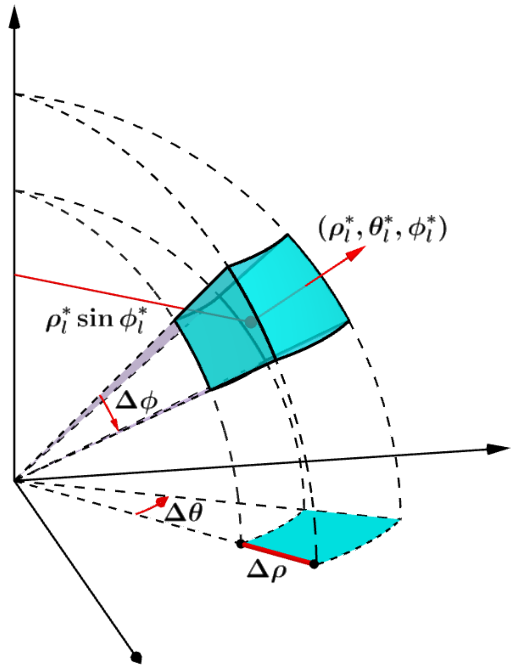


Figure 2.48: A spherical wedge

- *ρ-limits of integration.* The strip enters the solid at $\rho = \rho_1(\theta, \phi)$ and leaves at $\rho = \rho_2(\theta, \phi)$.
- *θ-limits of integration.* Consider the projection R of G on the xy-plane. Draw a ray through R from the origin. As the ray sweeps through R in the counterclockwise direction, the angle it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$.
- *ϕ-limits of integration.* Go back to the strip through G and let it sweep G from its nearest position from the positive z-axis. The angle it makes with the positive z-axis runs from $\phi = \gamma$ to $\phi = \delta$.

Example 2.7.5. Evaluate $\iiint\limits_G z\,dV$, where G is the solid within $z = \sqrt{x^2 + y^2}$ and $x^2+y^2+z^2 = 2$.

Solution: First, note that $z = \rho \cos \phi$ in spherical coordinates. So we are integrating the function $\rho \cos \phi$. Now, the sphere $x^2 + y^2 + z^2 = 2$ in spherical coordinates is $\rho = \sqrt{2}$ while the cone $z = \sqrt{x^2 + y^2}$ has equation $\phi = \frac{\pi}{4}$, from Example **??**.
The strip enters the solid at $\rho = 0$ (origin) and leaves at $\rho = \sqrt{2}$ (sphere). The projection onto the xy-plane covers the four quadrants which gives $0 \leq \theta \leq 2\pi$. The strip swipes out the solid from $\phi = 0$ (positive z-axis) to $\phi = \frac{\pi}{4}$ (cone).

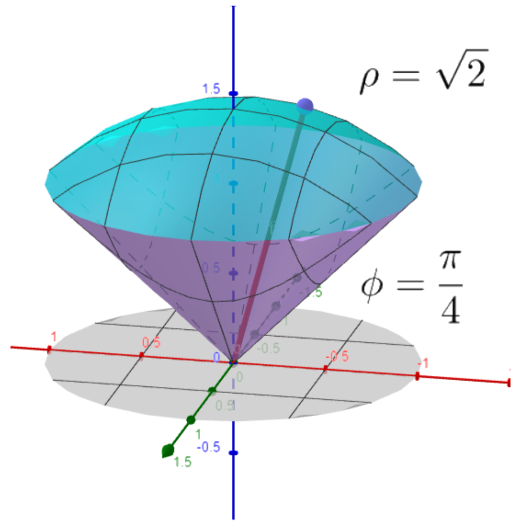


Figure 2.49: Solid within $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = 2$

$$\begin{aligned}
\iiint\limits_G z\,dV &= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sqrt{2}} (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sqrt{2}} \rho^3\cos\phi\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{4}}\int_0^{2\pi} \frac{\rho^4}{4}\cos\phi\sin\phi\Big|_{\rho=0}^{\rho=\sqrt{2}} d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{4}}\int_0^{2\pi} \cos\phi\sin\phi\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{4}} \theta\cos\phi\sin\phi\Big|_{\theta=0}^{\theta=2\pi} d\phi \\
&= \int_0^{\frac{\pi}{4}} 2\pi\cos\phi\sin\phi\,d\phi \\
&= \pi\sin^2\phi\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{2}\,.
\end{aligned}$$

Example 2.7.6. Evaluate $\iiint\limits_G \sqrt{x^2+y^2+z^2}\,dV$, where G is the solid above the xy-plane outside the sphere $x^2+y^2+(z-1)^2 = 1$ and inside the sphere $x^2+y^2+z^2 = 4$.

Solution: In spherical coordinates, the integrand $\sqrt{x^2+y^2+z^2}$ is equivalent to ρ, the sphere $x^2+y^2+(z-1)^2 = 1$ is $\rho = 2\cos\phi$, shown in Example 2.7.4, while $x^2+y^2+z^2 = 4$ is $\rho = 2$. The strip enters the solid at $\rho = 2\cos\phi$ (inner sphere) and leaves at $\rho = 2$ (outer sphere). The projection on the xy-plane covers the four quadrants which gives $0 \le \theta \le 2\pi$. The strip swipes out the solid from $\phi = 0$ (positive z-axis) to $\phi = \frac{\pi}{2}$ (xy-plane). Therefore

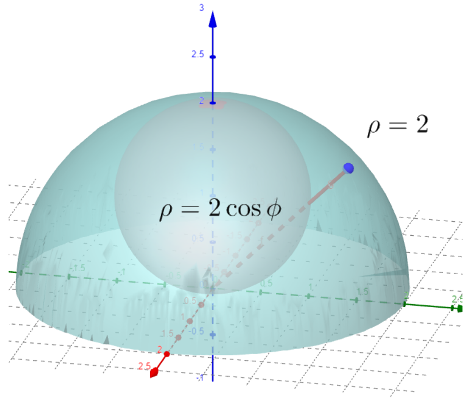


Figure 2.50: Solid above $z = 0$, outside $x^2 + y^2 + (z-1)^2 = 1$, and inside $x^2 + y^2 + z^2 = 4$

$$\begin{aligned}
\iiint\limits_G \sqrt{x^2+y^2+z^2}\,dV &= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_{2\cos\phi}^{2} \rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_{2\cos\phi}^{2} \rho^3\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \frac{\rho^4}{4}\sin\phi\Big|_{\rho=2\cos\phi}^{\rho=2}\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} (4\sin\phi - 4\cos^4\phi\sin\phi)\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}} (4\sin\phi - 4\cos^4\phi\sin\phi)\theta\Big|_{\theta=0}^{\theta=2\pi}\,d\phi \\
&= \int_0^{\frac{\pi}{2}} (8\pi\sin\phi - 8\pi\cos^4\phi\sin\phi)\,d\phi \\
&= \left(-8\pi\cos\phi + \frac{8}{5}\pi\cos^5\phi\right)\Big|_0^{\frac{\pi}{2}} \\
&= 0 - \left(-8\pi + \frac{8}{5}\pi\right) = \frac{32}{5}\pi\ .
\end{aligned}$$

Example 2.7.7. Evaluate $\iiint\limits_G dV$, where G is the solid in the first octant inside the sphere $x^2 + y^2 + z^2 = 1$ bounded by the cone $z = \sqrt{3x^2 + 3y^2}$ and the 3 coordinate planes.

Solution: The sphere $x^2 + y^2 + z^2 = 1$ is equivalent to $\rho = 1$. For the cone, we have

$$\tan\phi = \frac{\sqrt{x^2+y^2}}{z} = \frac{\sqrt{x^2+y^2}}{\sqrt{3x^2+3y^2}} = \frac{1}{\sqrt{3}}$$
$$\implies \phi = \frac{\pi}{6}\ .$$

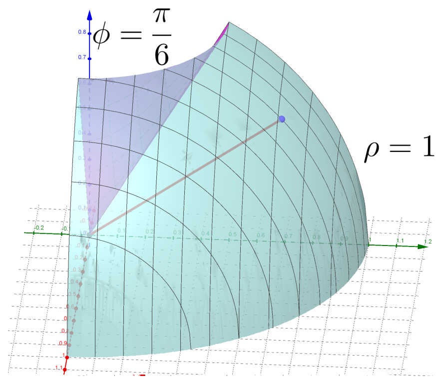


Figure 2.51: Solid in the first octant, inside $x^2 + y^2 + z^2 = 1$, bounded by $z = \sqrt{3x^2 + 3y^2}$ and the 3 coordinate planes

The strip enters the solid at $\rho = 0$ (origin) and leaves at $\rho = 1$ (sphere). The projection on the xy-plane covers the first quadrant which gives $0 \le \theta \le \frac{\pi}{2}$. The strip swipes out the solid from $\phi = \frac{\pi}{6}$ (cone) to $\phi = \frac{\pi}{2}$ (xy-plane). Thus, we have

$$\begin{aligned}
\iiint_G dV &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\rho^3}{3} \sin\phi \Big|_{\rho=0}^{\rho=1} d\theta \, d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{3} \sin\phi \, d\theta \, d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\theta}{3} \sin\phi \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\pi}{6} \sin\phi \, d\phi \\
&= -\frac{\pi}{6} \cos\phi \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= 0 - \left(-\frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{12} \pi \ .
\end{aligned}$$

Example 2.7.8. Use spherical coordinates to evaluate

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx.$$

Solution: The solid G is bounded below by xy-plane and above by the sphere of radius 2, while the projection of G onto xy-plane is a circle of radius 2.

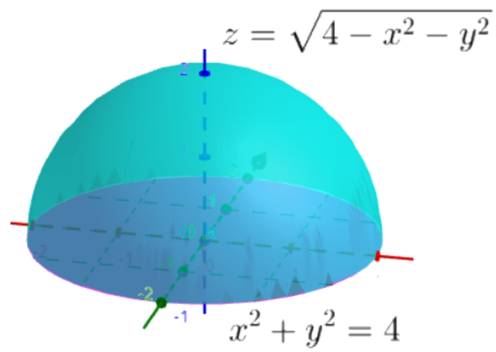


Figure 2.52: Solid bounded below by $z = 0$ and above $x^2 + y^2 + z^2 = 4$

Thus, we have

$$\begin{aligned}
\iiint\limits_G f(x,y,z)\,dV &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dzdydx \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^2 (\rho^2\cos^2\phi\rho)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^2 \rho^5\cos^2\phi\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \frac{32}{3}\cos^2\phi\sin\phi\,d\theta\,d\phi \\
&= \frac{64\pi}{3}\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi \\
&= \frac{64\pi}{3}\left[-\frac{1}{3}\cos^3\phi\Big|_0^{\frac{\pi}{2}}\right] \\
&= \frac{64}{9}\pi\ .
\end{aligned}$$

EXERCISES 2.7

I. Evaluate the integral using spherical coordinates.

1. $\displaystyle\iiint\limits_G \frac{1}{x^2+y^2+z^2}\,dV$, where G lies above the cone $z = \sqrt{x^2+y^2}$ and between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$

2. $\displaystyle\iiint\limits_G dV$, where G is the solid in the first octant under the cone $z = \sqrt{x^2+y^2}$ and within the sphere $x^2+y^2+(z-2)^2 = 4$

3. $\displaystyle\iiint\limits_G z\,dV$, where G lies above $z = 2$ and within $x^2+y^2+z^2 = 16$

II. Convert to an iterated triple integral in spherical coordinates.

1. $\displaystyle\int_0^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_{\sqrt{3x^2+3y^2}}^{3} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$

2. $\displaystyle\int_0^{3} \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} dz\, dx\, dy$

III. Convert the following to spherical coordinates.

1. $\displaystyle\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} (x^2+y^2)\, dz\, dy\, dx$

2. $\displaystyle\int_{-1}^{0} \int_{-\sqrt{1-y^2}}^{0} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} x\, dz\, dx\, dy$

IV. Let G be the solid bounded by the spheres $\rho = 2$, $\rho = 4$ and the cone $\phi = \dfrac{\pi}{4}$. Use spherical coordinates to evaluate $\displaystyle\iiint\limits_G \sqrt{x^2+y^2+z^2}\, dV$.

V. Set up the integral in spherical coordinates that gives the volume of the region inside the sphere $x^2+y^2+z^2=1$ and above the cone $z=\sqrt{3(x^2+y^2)}$.

VI. Set up the iterated triple integral which gives the volume of the solid in the first octant bounded below by the cone $z = 2\sqrt{x^2+y^2}$, within the sphere $x^2+y^2+z^2=9$. (One angle here is not special.)

2.8 Applications of Triple Integrals

Volume of a Solid

Let f be a constant function equal to 1 on a solid G and consider the rectangular box E that encloses G. Define a function F by

$$F(x,y,z) = \begin{cases} 1, & \text{if } (x,y,z) \in G \\ 0, & \text{if } (x,y,z) \in E \setminus G \end{cases} .$$

Then

$$\begin{aligned} \lim_{N\to\infty} \sum_{\ell=1}^{N} F(x_\ell^*, y_\ell^*, z_\ell^*)\delta V &= \iiint\limits_E F(x,y,z)\, dV \\ &= \iiint\limits_{E\setminus G} 0\, dV + \iiint\limits_G 1\, dV \\ &= \iiint\limits_G dV. \end{aligned}$$

We therefore define the volume of the solid G by

$$\iiint\limits_G dV.$$

Example 2.8.1. Find the volume of the solid bounded by the three coordinate planes and the plane $z = 1 - x - y$.

Solution: The solid G is of type xy bounded below by $z = 0$ and above by $z = 1 - x - y$. Its projection R onto the xy-plane can be classified as type I.

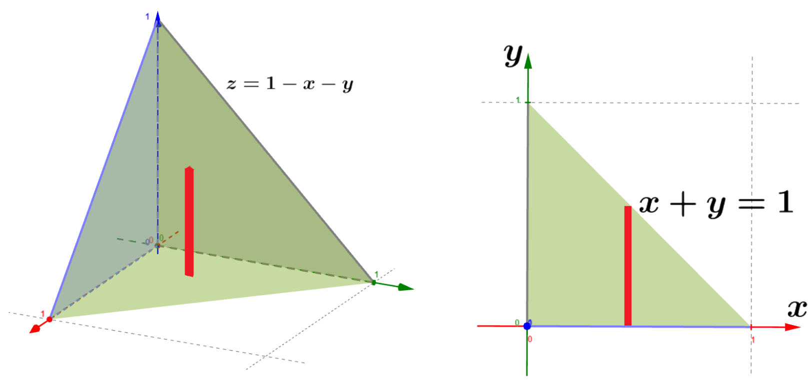


Figure 2.53: Solid bounded by the coordinate planes and $z = 1 - x - y$, and its projection to the xy-plane

We have

$$\begin{aligned}
\iiint\limits_G dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx \\
&= \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx \\
&= \int_0^1 \left(y - xy - \frac{y^2}{2} \right) \Bigg|_{y=0}^{y=1-x} dx \\
&= \int_0^1 \left[(1 - x) - x(1 - x) - \frac{(1 - x)^2}{2} \right] dx \\
&= \frac{1}{2} \int_0^1 (1 - 2x + x^2)\, dx \\
&= \frac{1}{2} \left(x - x^2 + \frac{x^3}{3} \right) \Bigg|_0^1 = \frac{1}{6}\ .
\end{aligned}$$

Example 2.8.2. Set up an iterated triple integral in rectangular coordinates that gives the volume of the solid bounded by the paraboloid $z = 4x^2 + y^2$ and the cylinder $z = 4 - 3y^2$.

Solution: The solid G is of type xy bounded below by $z = 4x^2 + y^2$ and above by $z = 4 - 3y^2$.

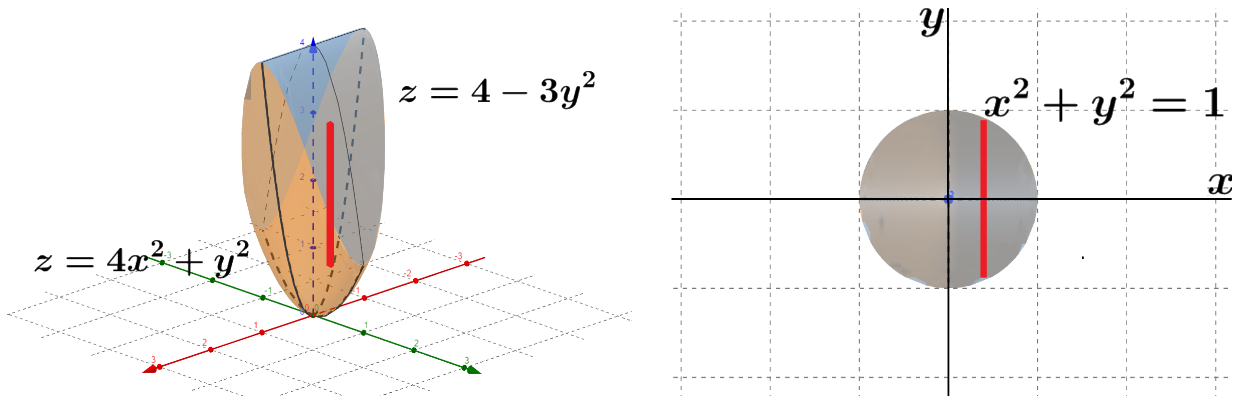


Figure 2.54: Solid bounded by $z = 4x^2 + y^2$ and $z = 4 - 3y^2$, and its projection to the xy-plane

The projection R of G onto the $xy-$plane is determined by the curve of intersection of the 2 surfaces, that is, $x^2 + y^2 = 1$. Thus, the volume of G is given by

$$\iiint_G dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{4x^2+y^2}^{4-3y^2} dz\, dy\, dx.$$

Mass and Center of Mass of a Solid

Let G be a solid with density function $f(x, y, z)$. Then the mass M of G is given by

$$M = \iiint_G f(x, y, z)\, dV \ .$$

The moments about the coordinate planes are given by the formulas

$$M_{xy} = \iiint_G z f(x, y, z)\, dV\,, \quad M_{xz} = \iiint_G y f(x, y, z)\, dV\,, \quad M_{yz} = \iiint_G x f(x, y, z)\, dV$$

The center of mass of G is the point $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{M_{yz}}{M}\ , \qquad \overline{y} = \frac{M_{xz}}{M}\ , \qquad \overline{z} = \frac{M_{xy}}{M}$$

Example 2.8.3. Let G be the solid bounded below by the xy-plane and above by the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and suppose that the density at a point (x, y, z) in G is proportional to the distance of the point from the origin. Set up the iterated triple integrals that give the mass and moments of mass of G using (1) rectangular/Cartesian coordinates, (2) cylindrical coordinates and (3) spherical coordinates.

Solution: The density function of G is $f(x,y,z) = k\sqrt{x^2+y^2+z^2}$, for some positive constant k.

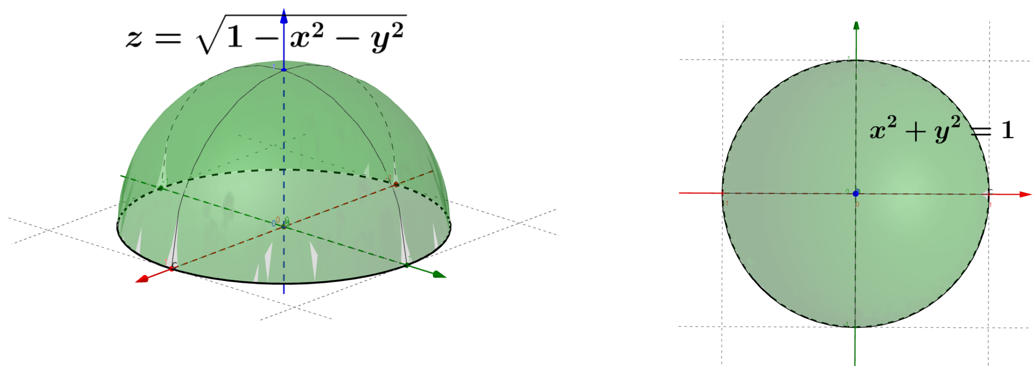


Figure 2.55: Solid bounded below by $z = 0$ and above by $z = \sqrt{1-x^2-y^2}$

1. The solid is of type xy and its projection R onto the xy-plane is bounded by the trace of the hemisphere; that is, the circle $x^2+y^2 = 1$. Now, the region R can be classified as type I, with lower boundary $y = -\sqrt{1-x^2}$ and upper boundary $y = \sqrt{1-x^2}$ for $-1 \le x \le 1$.

   Therefore, in rectangular coordinates we have

$$\begin{aligned}
M &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} k\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx \\
M_{xy} &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} kz\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx \\
M_{xz} &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} ky\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx \\
M_{yz} &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} kx\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx
\end{aligned}$$

2. In cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$ and $x^2+y^2 = r^2$. Thus, the density function is $f(r,\theta,z) = k\sqrt{r^2+z^2}$. The hemisphere $z = \sqrt{1-x^2-y^2}$ has cylindrical equation $z = \sqrt{1-r^2}$. The circle $x^2+y^2 = 1$ has polar equation $r = 1$. Hence,

$$\begin{aligned}
M &= \int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{\sqrt{1-r^2}} k\sqrt{r^2+z^2}\,r\,dz\,dr\,d\theta \\
M_{xy} &= \int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{\sqrt{1-r^2}} kz\sqrt{r^2+z^2}\,r\,dz\,dr\,d\theta \\
M_{xz} &= \int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{\sqrt{1-r^2}} k(r\sin\theta)\sqrt{r^2+z^2}\,r\,dz\,dr\,d\theta \\
M_{yz} &= \int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{\sqrt{1-r^2}} k(r\cos\theta)\sqrt{r^2+z^2}\,r\,dz\,dr\,d\theta.
\end{aligned}$$

3. In spherical coordinates, $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$, and $x^2 + y^2 + z^2 = r^2$. Therefore, the density function is $f(\rho, \theta, \phi) = k\rho$.

   The strip enters the solid at $\rho = 0$ (origin) and leaves at $\rho = 1$ (sphere). The projection of G covers four quadrants so $0 \leq \theta \leq 2\pi$. The strip swipes out the solid from $\phi = 0$ (positive z-axis) to $\phi = \frac{\pi}{2}$ (xy-plane). Therefore,

$$\begin{aligned}
M &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 k\rho\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
M_{xy} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 (\rho\cos\phi)(k\rho)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
M_{xz} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 (\rho\sin\phi\sin\theta)(k\rho)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
M_{yz} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 (\rho\sin\phi\cos\theta)(k\rho)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.
\end{aligned}$$

EXERCISES 2.8

I. Use a triple integral to find the volume of the given solid.

   1. the tetrahedron enclosed by the coordinate planes and the plane $3x - y - 2z = 12$
   2. the solid bounded by the elliptic cylinder $4x^2 + z^2 = 4$ and the planes $y = 0$ and $y = z + 2$
   3. the solid enclosed by the elliptic paraboloids $z = 2x^2 + y^2$ and $z = 12 - x^2 - 2y^2$
   4. the solid in the first octant inside $x^2 + y^2 + z^2 = 4$ and outside $x^2 + y^2 + (z-1)^2 = 1$

II. Use cylindrical or spherical coordinates, whichever seems more appropriate, to find the center of mass of the solid with the given density.

   1. the solid in the first octant within the cylinder $x^2 + y^2 = 4$ and under the paraboloid $z = 4 - x^2 - y^2$, density is $\delta(x, y, z) = 2z$.
   2. the solid above the cone $z = \sqrt{x^2 + y^2}$ and under the hemisphere $z = \sqrt{9 - x^2 - y^2}$, density is $\delta(x, y, z) = x^2 + y^2 + z^2$

III. Do as indicated.

   1. Let G be the solid in the first octant bounded on the sides by the xz-plane and the plane $y = x$, above by the sphere $x^2 + y^2 + z^2 = 16$, and below by the cone $z = \sqrt{3x^2 + 3y^2}$. Express (but do not evaluate) the volume of G as triple iterated integrals in
      (a) cylindrical coordinates; and
      (b) spherical coordinates.
   2. Consider the solid G bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 2$. Suppose that the density of G at any point (x, y, z) is $\delta(x, y, z) = z^3$.

(a) Set up the iterated triple integral which gives the mass of G using rectangular AND cylindrical coordinates.

(b) Find the mass of G using spherical coordinates.

3. Let G be solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 16$, punctured by the cylinder $x^2 + y^2 = 4$. Suppose that at any point (x, y, z) in G, the density is given by

$$\delta(x, y, z) = \frac{z(x^2 + y^2)}{(x^2 + y^2 + z^2)^2}.$$

(a) Set up an iterated triple integral in cylindrical coordinates that gives the mass of G.

(b) Find the mass of G using an iterated triple integral in spherical coordinates.

Chapter 3

Vector Calculus

3.1 Scalar and Vector Fields

Scalar and vector fields are just technical labels for special functions. A scalar field takes in an n-tuple and produces a scalar, while a vector field also takes in an n-tuple but produces an n-dimensional vector. Scalar fields are the same as the multivariate functions that were discussed in the first part of the course.

Definition 3.1.1. A ***scalar field*** on $\mathbb{R}^n$ is any real-valued function whose domain is a subset of $\mathbb{R}^n$. A ***vector field*** on $\mathbb{R}^n$ is a function whose output is an n-dimensional vector where each of the n components is a scalar field on $\mathbb{R}^n$.

We will only be interested in the cases when $n = 2$ and $n = 3$.

Example 3.1.1. The functions $f(x, y) = x^2 - 3xy + 3$ and $\phi(x, y) = \dfrac{2\cos(xy) - x}{e^{x-y} + 1}$ are scalar fields on $\mathbb{R}^2$ while the functions $f(x, y, z) = x^2 - y^3 + z^4$ and $\phi(x, y, z) = \sin(xy - z) + \sqrt{1 + x^2}$ are scalar fields on $\mathbb{R}^3$.

A vector field $\vec{F}$ on $\mathbb{R}^2$ has the form

$$\vec{F}(x, y) = \langle P(x, y)\,,\, Q(x, y)\rangle = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath},$$

where P and Q are scalar fields on $\mathbb{R}^2$. A vector field $\vec{G}$ on $\mathbb{R}^3$ has the form

$$\vec{G}(x, y, z) = \langle P(x, y, z)\,,\, Q(x, y, z)\,,\, R(x, y, z)\rangle = P(x, y, z)\hat{\imath} + Q(x, y, z)\hat{\jmath} + R(x, y, z)\hat{k},$$

where P, Q, and R are scalar fields on $\mathbb{R}^3$. Naturally, the domain of a vector field is the intersection of the domains of its component functions.

Example 3.1.2. The functions $\vec{F}(x, y) = \langle x + y\,,\, xy\rangle$ and $\vec{G}(x, y) = \langle \cos x\,,\, \sin y\rangle$ are examples of vector fields on $\mathbb{R}^2$ while the functions $\vec{F}(x, y, z) = \langle xy\,,\, xz\,,\, yz\rangle$ and $\vec{G}(x, y, z) = \langle z^2\,,\, x - y\,,\, e^{xyz}\rangle$ are examples of vector fields on $\mathbb{R}^3$. On the other hand, the function $\vec{H}(x, y) = \langle x^2\,,\, y - x\,,\, \cos x\rangle$ is *not* a vector field because it takes in an ordered pair (2-tuple) but produces a 3-dimensional vector.

Example 3.1.3. If ϕ is a scalar field on $\mathbb{R}^n$, then its gradient $\vec{\nabla}\phi$ is a (gradient) vector field on $\mathbb{R}^n$. For instance, the gradient of the scalar field $\phi(x, y) = x^2y$ is the vector field $\vec{\nabla}\phi(x, y) = \langle 2xy\,,\, x^2\rangle$.

To visually represent a vector field $\vec{F}(x, y)$ on $\mathbb{R}^2$, we select some points (x, y) on its domain and for each point (x, y), we draw the vector $\vec{F}(x, y)$ with initial point at (x, y).

Example 3.1.4. Sketch the vector field $\vec{F}(x, y) = \langle -y\,,\, x\rangle$.

Solution: We first find some values of $\vec{F}(x, y)$:

(x,y)	$(0,0)$	$(1,0)$	$(1,1)$	$(0,1)$	$(-1,1)$	$(-1,0)$	$(-1,-1)$	$(0,-1)$	$(1,-1)$
$\vec{F}(x,y)$	$\langle 0\,,\,0\rangle$	$\langle 0\,,\,1\rangle$	$\langle -1\,,\,1\rangle$	$\langle -1\,,\,0\rangle$	$\langle -1\,,\,-1\rangle$	$\langle 0\,,\,-1\rangle$	$\langle 1\,,\,-1\rangle$	$\langle 1\,,\,0\rangle$	$\langle 1\,,\,1\rangle$

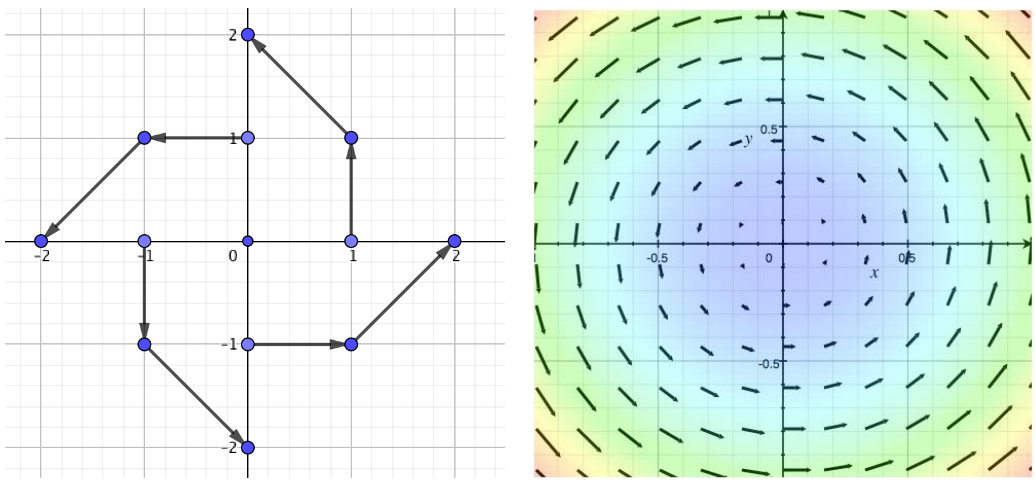


Figure 3.1: Sketch of the vector field $\vec{F}(x, y)\langle -y\,,\, x\rangle$

Using a graphing tool in plotting the vector $\vec{F}(x, y)$ at more points (x, y), we see that the vectors reveal a pattern. If we view the vector $\vec{F}(x, y)$ as the tangent vector to a curve, we see that the curves that $\vec{F}$ is tangent to are concentric circles about the origin. These curves are called *flowlines* or *streamlines.*

Flowlines are curves that are parametrized by $\vec{R}(t) = \langle x(t)\,,\, y(t)\rangle$ such that their tangent vector $\vec{R}'(t) = \langle x'(t)\,,\, y'(t)\rangle$ is equal to the vector field $\vec{F}$ at the point $(x(t), y(t))$, i.e.,

$$\langle x'(t)\,,\, y'(t)\rangle = \vec{F}(x(t), y(t)). \tag{3.1}$$

In our example above where $\vec{F}(x, y) = \langle -y\,,\, x\rangle$, Equation (3.1) becomes

$$\langle x'(t)\,,\, y'(t)\rangle = \langle -y(t)\,,\, x(t)\rangle.$$

Equating corresponding components gives $x'(t) = -y(t)$ and $y'(t) = x(t)$. Techniques in differential equations theory are needed to show that $x(t) = a\cos t$ and $y(t) = a\sin t$, where $a \in \mathbb{R}$. Hence,

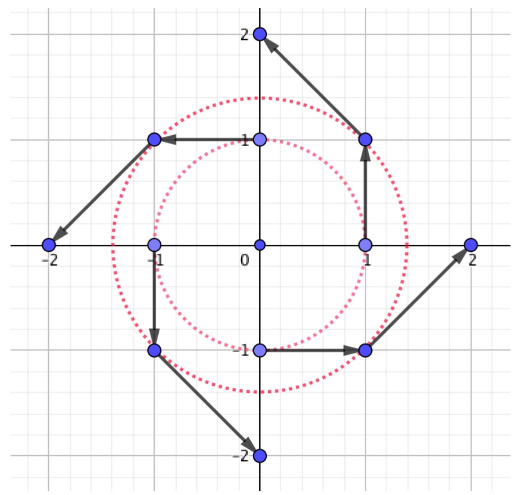


Figure 3.2: Flowlines of $\vec{F}(x, y) = \langle -y\,,\, x\rangle$

the family of flowlines are $\vec{R}(t) = \langle a\cos t\,,\, a\sin t\rangle$, which is a family of concentric circles about the origin. Determination of flowlines is outside the scope of the course.

Several examples of vector fields arise in physics, engineering and meteorology. For instance, the *gravitational force field* is the vector field that assigns to each point (x, y, z) in space the gravitational force acting at (x, y, z). Vector fields are usually named after the kind of vector it is representing. Thus, we can talk about force fields, velocity fields, acceleration fields, etc. Meteorologists and weather news reporters use vector field representations of air velocity to illustrate wind flow and forecast weather. Vector fields are also used in aerodynamics.

Example 3.1.5. The following illustrates the velocity field of the air surrounding a sedan with a rear wing.

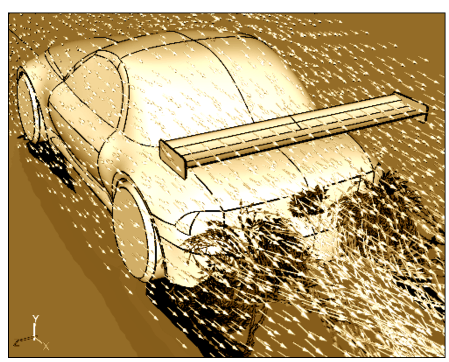


Figure 3.3: Edited version of the original image courtesy of Symscape.com "CFD Study of a Car With and Without Wing" licensed under Creative Commons Attribution 3.0 Unported License.

Example 3.1.6. Let $\phi_1(x, y) = (x^2 + y^2)^{-1/2}$. Then

$$\vec{\nabla}\phi_1(x, y) = -(x^2 + y^2)^{-3/2}\langle x\,,\, y\rangle.$$

A sketch of this gradient vector field is illustrated in Figure 3.4. This can be interpreted as the 2-dimensional version of our gravitational force field. It can also be interpreted as the scaled version of the electric field induced by a single negative point charge.

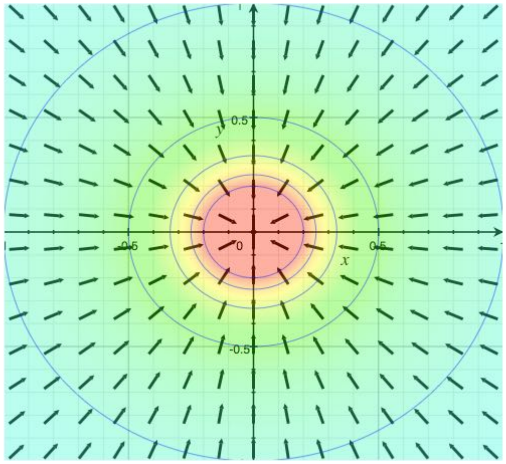


Figure 3.4: Sketch of $\nabla\phi_1(x,y) = -(x^2+y^2)^{-3/2}\langle x\,,\,y\rangle$

Example 3.1.7. Let $\phi_2(x,y) = (x^2+(y+0.5)^2)^{-1/2} - (x^2+(y-0.5)^2)^{-1/2}$. Then its gradient vector field $\vec{\nabla}\phi_2(x,y)$ is equal to

$$\left\langle -\frac{x}{(x^2+(y+0.5)^2)^{3/2}} + \frac{x}{(x^2+(y-0.5)^2)^{3/2}}\,,\, -\frac{y+0.5}{(x^2+(y+0.5)^2)^{3/2}} + \frac{y-0.5}{(x^2+(y-0.5)^2)^{3/2}} \right\rangle.$$

A sketch of which is illustrated in Figure 3.5. This can be interpreted as the electric field induced by two opposite electric charges: a positive point charge at the point $(0,1/2)$ and a negative point charge at $(0,-1/2)$.

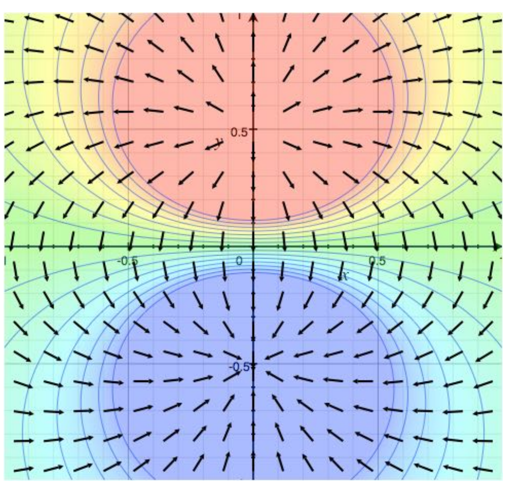


Figure 3.5: Sketch of $\phi_2(x,y) = (x^2+(y+0.5)^2)^{-1/2} - (x^2+(y-0.5)^2)^{-1/2}$

Example 3.1.8. Let $f(x, y) = \sin x \, \sin y$, whose gradient is given by

$$\vec{\nabla} f(x, y) = \langle \cos x \sin y \, , \, \sin x \cos y \rangle.$$

It is easily verified by the Second-Derivatives Test that f attains local maxima at $(\pi/2, \pi/2)$ and at $(-\pi/2, -\pi/2)$ and local minima at $(\pi/2, -\pi/2)$ and at $(-\pi/2, \pi/2)$. Depicted in the picture are the sketch of $\vec{\nabla} f$ and the contour lines (or level curves) of the surface $z = f(x, y)$.

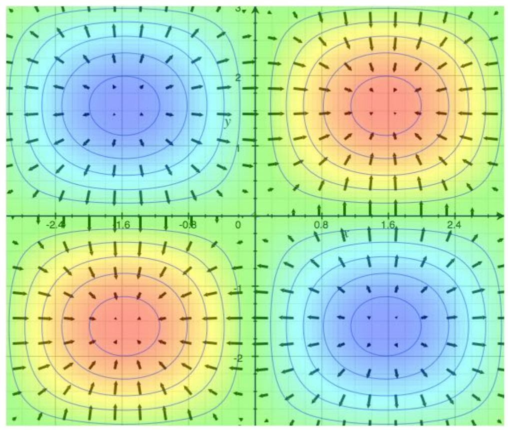


Figure 3.6: Contour map of $f(x, y) = \sin x \sin y$ together with the sketch of $\vec{\nabla} f(x, y) = \langle \cos x \sin y, \sin x \cos y \rangle$

Observe that the picture verifies the following known properties:

- The gradient is zero at the local extrema.
- The arrows always point towards the point where local maximum occurs.
- The gradient is perpendicular to the contour curves.

EXERCISES 3.1 Draw a sketch of the following vector fields on $\mathbb{R}^2$ or $\mathbb{R}^3$ at least at the points (x, y) or (x, y, z) where $x, y, z \in \{-1, 0, 1\}$.

1. $\vec{F}(x, y) = \langle 0 \, , \, \sqrt[3]{y} \rangle$
2. $\vec{F}(x, y) = \left\langle \dfrac{x}{2} \, , \, \dfrac{y}{2} \right\rangle$
3. $\vec{F}(x, y) = \langle x - y \, , \, x + y \rangle$
4. $\vec{F}(x, y, z) = \dfrac{\langle x \, , \, y \, , \, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}, (x, y, z) \neq (0, 0, 0)$

3.2 Divergence and Curl

We present two operators on vector fields: the divergence operator and the curl operator. Aside from being ubiquitous operators in mathematical modeling and the theory of partial differential equations, they also have physical meanings which are important in applications.

Definition 3.2.1. Suppose $\vec{F}$ is a vector field on $\mathbb{R}^3$ defined by

$$F(x,y,z) = \langle P(x,y,z)\,,\, Q(x,y,z)\,,\, R(x,y,z)\rangle,$$

where P, Q, and R are scalar fields on $\mathbb{R}^3$, all of whose first-order partial derivatives exist.

1. The ***divergence*** of $\vec{F}$, denoted by $\operatorname{div}\vec{F}$, is the scalar field given by

$$\operatorname{div}\vec{F} = P_x + Q_y + R_z.$$

2. The ***curl*** of $\vec{F}$, denoted by $\operatorname{curl}\vec{F}$, is the vector field given by

$$\operatorname{curl}\vec{F} = \langle R_y - Q_z\,,\, P_z - R_x\,,\, Q_x - P_y\rangle.$$

Example 3.2.1. Define $\vec{F}(x,y,z) = \langle P(x,y,z)\,,\, Q(x,y,z)\,,\, R(x,y,z)\rangle = \langle x^2y\,,\, y^2z\,,\, xz^2\rangle$. Then

$$\begin{aligned}
\operatorname{div}\vec{F}(x,y,z) &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(xz^2) \\
&= 2xy + 2yz + 2xz, \\
\operatorname{curl}\vec{F}(x,y,z) &= \langle R_y - Q_z\,,\, P_z - R_x\,,\, Q_x - P_y\rangle \\
&= \langle 0 - y^2\,,\, 0 - z^2\,,\, 0 - x^2\rangle = -\langle y^2\,,\, z^2\,,\, x^2\rangle.
\end{aligned}$$

Remark. It should be stressed that the divergence operator takes in a vector field and gives out a scalar field, while the curl operator takes in a vector field and gives out another vector field.

An easier way to remember how to compute the divergence and curl of a vector field uses the following mnemonic device: Define the *del operator* $\vec{\nabla}$ by

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}\,,\, \frac{\partial}{\partial y}\,,\, \frac{\partial}{\partial z}\right\rangle,$$

which we will now treat as a "vector". To see how this works, given a scalar field $f(x,y,z)$ as input, $\vec{\nabla}$ gives out the vector field

$$\left\langle \frac{\partial}{\partial x}\,,\, \frac{\partial}{\partial y}\,,\, \frac{\partial}{\partial z}\right\rangle f = \left\langle \frac{\partial}{\partial x}f\,,\, \frac{\partial}{\partial y}f\,,\, \frac{\partial}{\partial z}f\right\rangle = \vec{\nabla} f(x,y,z),$$

the familiar gradient of f. Now, if $\vec{F}$ is a vector field on $\mathbb{R}^3$, then we have

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} \quad \text{and} \quad \operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F}.$$

Indeed, assuming P, Q and R are scalar fields on $\mathbb{R}^3$ and $\vec{F} = \langle P\,,\,Q\,,\,R\rangle$ and keeping in mind that $\vec{\nabla}$ is a "vector", then

$$\begin{aligned}
\vec{\nabla}\cdot\vec{F} &= \left\langle \frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right\rangle \cdot \left\langle P\,,\,Q\,,\,R\right\rangle \\
&= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \\
&= P_x + Q_y + R_z \\
&= \operatorname{div}\vec{F}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\vec{\nabla}\times\vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\,,\,\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\,,\,\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle \\
&= \langle R_y - Q_z\,,\,P_z - R_x\,,\,Q_x - P_y\rangle \\
&= \operatorname{curl}\vec{F}.
\end{aligned}$$

To get the curl of a vector field $\vec{F}$, it is necessary that $\vec{F}$ is a vector field on $\mathbb{R}^3$. In the case that it has only two components, we treat it as a vector field on $\mathbb{R}^3$ by giving it a zero third component.

Example 3.2.2 (Curl of a two-dimensional vector field)**.** Let P and Q be scalar fields on $\mathbb{R}^2$ and $\vec{F} = \langle P\,,\,Q\rangle$. To compute its curl, we write $\vec{F}(x,y,z) = \langle P(x,y)\,,\,Q(x,y)\,,\,0\,\rangle$ and use the above mnemonic to get

$$\begin{aligned}
\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} \\
&= \left\langle 0\,,\,0\,,\,Q_x - P_y\right\rangle.
\end{aligned}$$

Example 3.2.3. Find the divergence and curl of $\vec{F}(x,y,z) = \langle e^x\sin z\,,\,ye^{-x}\,,\,z\tan y\rangle$ at the origin.

Solution: Using the del operator $\vec{\nabla}$,

$$\begin{aligned}
\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} &= \frac{\partial}{\partial x}(e^x\sin z) + \frac{\partial}{\partial y}(ye^{-x}) + \frac{\partial}{\partial z}(z\tan y) \\
&= e^x\sin z + e^{-x} + \tan y
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x\sin z & ye^{-x} & z\tan y \end{vmatrix} \\
&= \langle z\sec^2 y\,,\,e^x\cos z\,,\,-ye^{-x}\rangle.
\end{aligned}$$

Therefore, at the origin, $\operatorname{div} \vec{F}(0,0,0) = 0 + 1 + 0 = 1$ and $\operatorname{curl} \vec{F}(0,0,0) = \langle 0\,,\,1\,,\,0 \rangle$.

EXERCISES 3.2 Find the divergence and curl of the following vector fields.

1. $\vec{F}(x,y) = \left\langle -\dfrac{y}{x}\,,\,\dfrac{1}{x} \right\rangle$
2. $\vec{F}(x,y) = \langle 2x\,,\,3y \rangle$
3. $\vec{F}(x,y) = \langle 3x^2 y\,,\,-2xy^3 \rangle$
4. $\vec{F}(x,y,z) = \langle y+z\,,\,x+z\,,\,x+y \rangle$
5. $\vec{F}(x,y,z) = \langle xy\,,\,ze^y\,,\,z \rangle$
6. $\vec{F}(x,y,z) = \langle \cos 2x\,,\,\sin 2y\,,\,\tan z \rangle$
7. $\vec{F}(x,y,z) = \langle 3x^2\,,\,-z\cos y\,,\,-\sin y \rangle$
8. $\vec{F}(x,y,z) = \left\langle 10^{10^x} - 2y\,,\,3x + 10y\,,\,5 \right\rangle$
9. $\vec{F}(x,y,z) = \langle \cos^2 x\,,\,\sin^2 x\,,\,-z \rangle$
10. $\vec{F}(x,y,z) = \langle e^{2x}\,,\,3x^2yz\,,\,2y^2z + x \rangle$
11. $\vec{F}(x,y,z) = \langle e^x \sin z, ye^{-x}, z\tan y \rangle$
12. $\vec{F}(x,y,z) = \langle z - y, x - z, x - y \rangle$
13. $\vec{F}(x,y,z) = \langle x^2, xy, yz \rangle$
14. $\vec{F}(x,y,z) = \left\langle zye^{x^2}, xze^{y^2}, xye^{z^2} \right\rangle$
15. $\vec{F}(x,y,z) = (\ln y + 4x)\hat{\imath} + (\sin x - z + 2y)\hat{\jmath} + (3x + 2y + z)\hat{k}$

3.3 Conservative Vector Fields

We have seen in the previous sections how taking the gradient of a scalar field results in a vector field. We now ask: Are all vector fields gradient vector fields? That is, given any vector field $\vec{F}$, can we always find a scalar field ϕ such that $\vec{F} = \vec{\nabla}\phi$? We will see that in general, the answer is no, and so we give a special name to those vector fields satisfying this apparently nice property.

Definition 3.3.1. A vector field $\vec{F}$ is said to be ***conservative*** if it is the gradient of a scalar field, i.e., if there exists a scalar field ϕ such that $\vec{F} = \vec{\nabla}\phi$. In this case, we call ϕ a *potential function for* $\vec{F}$.

We have seen in Section 3.1 that several vector fields in applications such as the gravitational force field and the electric field due to finitely many electric charges illustrated in Example 3.1.4 are conservative. Their potential functions are indicated in that example.

Example 3.3.1. The vector field $\vec{F}(x,y,z) = \langle x\,,\,y\,,\,z \rangle$ is conservative. To see this, let ϕ be the scalar field defined by $\phi(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)$. We see that $\phi_x = x$, $\phi_y = y$ and $\phi_z = z$. That is, $\vec{\nabla}\phi = \vec{F}$, and hence ϕ is a potential function for $\vec{F}$.

The following example illustrates that not all vector fields are conservative.

Example 3.3.2. Show that the vector field $\vec{F}(x,y) = \langle -y\,,\,x \rangle$ is not conservative.

Solution: We proceed by contradiction. Suppose otherwise that $\vec{F}$ is conservative. Then there exists a scalar field ϕ such that $\vec{F} = \vec{\nabla}\phi$. This equality of two vectors means that we have the system

$$\begin{cases} \phi_x(x,y) &= -y \\ \phi_y(x,y) &= x. \end{cases} \tag{3.2}$$

Now to recover the potential function ϕ, we proceed by *partial integration*, the inverse process of partial differentiation. In the first equation of (3.2), we treat y as a constant and integrate with respect to x.

$$\phi(x,y) = \int \phi_x(x,y)\,dx = \int (-y)\,dx = -xy + C(y). \tag{3.3}$$

It is important to observe that the constant of integration is a function of y as we have treated y as a constant.

The next step is to differentiate (3.3) with respect to y to get

$$\phi_y(x,y) = -x + C'(y) \tag{3.4}$$

We then have two expressions both equal to ϕ_y: one in (3.2) and another in (3.4). They must therefore be equivalent to each other, i.e.,

$$x = -x + C'(y). \tag{3.5}$$

That is, $C'(y) = 2x$. This is a contradiction since C is a function of y only and so its derivative C' should be independent of x. This implies that such a potential function ϕ does not exist and therefore $\vec{F}$ is not conservative.

The above example outlines a systematic method of finding potential functions of a given conservative vector field $\vec{F}$ and thereby determining whether $\vec{F}$ is conservative or not. Assuming that the step in (3.5) does not reduce to a contradiction, the last step would be to integrate C' with respect to y so that the resulting antiderivative C could be substituted into the recovered potential function in (3.3). We illustrate this in the next example.

Example 3.3.3. Show that the vector field $\vec{F}(x,y) = \langle e^{2y} - 2x + 1\,,\, 2xe^{2y} + 3y^2 - 2\rangle$ is conservative.

Solution: To show that $\vec{F}$ is conservative, we try to find a function ϕ such that $\vec{\nabla}\phi = \vec{F}$, i.e.,

$$\begin{cases} \phi_x(x,y) &= e^{2y} - 2x + 1 \\ \phi_y(x,y) &= 2xe^{2y} + 3y^2 - 2. \end{cases} \tag{3.6}$$

Partially integrating the first equation in (3.6) with respect to x (remember that we are treating y as constant) immediately yields the potential function

$$\phi(x,y) = \int (e^{2y} - 2x + 1)\,dx = xe^{2y} - x^2 + x + C(y). \tag{3.7}$$

It is left to find out what $C(y)$ is to finish our solution. To do this, we differentiate (3.7) with respect to y so that we get another representation for ϕ_y which we could compare to the second equation of (3.6):

$$\phi_y(x, y) = 2xe^{2y} + C'(y). \tag{3.8}$$

Equating (3.8) and the second equation of (3.6) yields

$$2xe^{2y} + 3y^2 - 2 = 2xe^{2y} + C'(y),$$

which simplifies to

$$C'(y) = 3y^2 - 2.$$

Integrating this with respect to y gives

$$C(y) = \int (3y^2 - 2)\, dy = y^3 - 2y + K. \tag{3.9}$$

Here K is an actual constant. Finally, after substituting (3.9) into (3.7), we obtain

$$\phi(x, y) = xe^{2y} - x^2 + x + y^3 - 2y + K.$$

This shows that a potential function for $\vec{F}$ exists and therefore we conclude that $\vec{F}$ is conservative.

We may ask ourselves if there are other potential functions besides the one found in the above process. It should come as no surprise to the reader that if a potential function ϕ for a vector field $\vec{F}$ exists, then all other potential functions will only differ from ϕ by a constant.

Theorem 3.3.2. Let $\vec{F}$ be a conservative vector field and assume that ϕ and ψ are potential functions for $\vec{F}$. Then $\phi = \psi + K$ for some constant K.

Proof. We only show this for the case when $\vec{F}$ is a vector field on $\mathbb{R}^2$. From the assumption, since ϕ and ψ are potential functions for $\vec{F}$, then by definition,

$$\vec{\nabla}\phi = \vec{F} = \vec{\nabla}\psi.$$

This implies that $\phi_x = \psi_x$ and $\phi_y = \psi_y$ simultaneously. The first of these equalities is equivalent to $\dfrac{\partial}{\partial x}(\phi - \psi) = 0$. Since the partial derivative with respect to x is zero, $\phi - \psi$ is constant with respect to x (but it may be a function of y.) Hence,

$$\phi(x, y) - \psi(x, y) = C(y). \tag{3.10}$$

Taking the partial derivative with respect to y gives $\phi_y - \psi_y = C'(y)$. Since $\phi_y = \psi_y$, it follows that $C'(y) = 0$. Hence $C(y)$ must be constant, say $C(y) \equiv K$. Substituting this into (3.10) gives the desired conclusion. $\square$

A routine extension of the process in Example 3.3.2 would enable us to conclude whether a vector field on $\mathbb{R}^3$ is conservative or not.

Example 3.3.4. Verify that $\vec{F}(x, y, z) = \langle 2xy + 1\,,\, x^2 + 2yz\,,\, y^2 - 2\rangle$ is conservative by finding all of its potential functions.

Solution: We wish to find a function ϕ such that $\vec{\nabla}\phi = \vec{F}$. This equality of vectors translates to equality of its components given by

$$\begin{cases} \phi_x &= \quad 2xy + 1 \\ \phi_y &= \quad x^2 + 2yz \\ \phi_z &= \quad y^2 - 2. \end{cases} \tag{3.11}$$

From the first equation above, by treating y and z as constants and integrating with respect to x we obtain

$$\phi(x, y, z) = \int (2xy + 1)\, dx = x^2y + x + C(y, z). \tag{3.12}$$

We only need to find $C(y, z)$. Taking the partial derivative with respect to y, we obtain $\phi_y = x^2 + C_y(y, z)$. In view of the second equation of (3.11), we conclude that $C_y(y, z) = 2yz$. Integrating this with respect to y gives

$$C(y, z) = \int 2yz\, dy = y^2z + D(z).$$

Substituting this into (3.12), we obtain

$$\phi(x, y, z) = x^2y + x + y^2z + D(z). \tag{3.13}$$

It remains to find the function $D(z)$. We differentiate (3.13) with respect to z to get

$$\phi_z = y^2 + D'(z).$$

Comparing this to (3.11) yields $D'(z) = -2$ which implies that $D(z) = -2z + K$. Finally, substituting this into (3.13) gives all the potential functions for $\vec{F}$:

$$\phi(x, y, z) = x^2y + x + y^2z - 2z + K.$$

We first recall Clairaut's Theorem (Theorem 1.3.2) in Chapter 1. It states that if the second-order partial derivatives of a function are continuous on a domain $D \subseteq \mathbb{R}^2$, then its mixed partial derivatives are equal in D.

The above result extends to functions of several variables. For example, if ϕ is a scalar field on $\mathbb{R}^3$ and has continuous second-order partial derivatives, then by fixing one variable, we can conclude from Clairaut's theorem that

$$\phi_{xy} = \phi_{yx} \quad \phi_{xz} = \phi_{zx} \quad \text{and} \quad \phi_{yz} = \phi_{zy}.$$

The next results relate the divergence and curl of a vector field with the property of the field being conservative. The first of these will become important because it gives a necessary condition for a vector field to be conservative.

Theorem 3.3.3. Let P, Q and R be scalar fields on $\mathbb{R}^3$ with continuous first-order partial derivatives. Let $\vec{F} = \langle P\,,\, Q\,,\, R\rangle$. Then

1. If $\vec{F}$ is conservative, then $\operatorname{curl}\vec{F} = \vec{0}$.
2. If P, Q and R have continuous second-order partial derivatives, then $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$.

Proof.

1. Suppose that $\vec{F}$ is conservative. Then there exists a scalar field ϕ such that $\vec{\nabla}\phi = \vec{F}$, i.e., $\langle \phi_x\,,\, \phi_y\,,\, \phi_z\rangle = \langle P\,,\, Q\,,\, R\rangle$. Hence, we have

$$\phi_x = P, \quad \phi_y = Q \quad \text{and} \quad \phi_z = R.$$

It follows that

$$\begin{aligned}\phi_{xy} &= P_y, \quad \phi_{yx} = Q_x, \quad \phi_{zx} = R_x\\ \phi_{xz} &= P_z, \quad \phi_{yz} = Q_z, \quad \phi_{zy} = R_y.\end{aligned}$$

The continuity of the first-order partial derivatives of P, Q and R translates to the continuity of the second-order partial derivatives of ϕ. Hence, we may now invoke Clairaut's theorem and conclude that $R_y = Q_z$, $P_z = R_x$ and $Q_x = P_y$, that is,

$$\operatorname{curl}\vec{F} = \langle R_y - Q_z\,,\, P_z - R_x\,,\, Q_x - P_y\rangle = \langle 0\,,\, 0\,,\, 0\rangle = \vec{0}.$$

This proves the first assertion.

2. The continuity of the second-order partial derivatives of the component functions of $\vec{F}$ allows us to conclude from Clairaut's theorem that

$$P_{yz} = P_{zy}, \quad Q_{xz} = Q_{zx} \quad \text{and} \quad R_{xy} = R_{yx}. \tag{3.14}$$

Using the definition of the curl and divergence of $\vec{F}$, we have

$$\begin{aligned}\operatorname{div}(\operatorname{curl}\vec{F}) &= \operatorname{div}\big(\langle R_y - Q_z\,,\, P_z - R_x\,,\, Q_x - P_y\rangle\big)\\ &= (R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz})\,.\end{aligned}$$

In view of (3.14), we see that $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$, as needed.

□

Remark 3.3.4. The first result has a powerful contrapositive. It says that whenever $\vec{F}$ has components which have continuous first partial derivatives, then it is not conservative whenever $\operatorname{curl}\vec{F} \neq \vec{0}$. For instance, it was shown in Example 3.3.2 that the vector field $\vec{F}(x, y) = \langle -y\,,\, x\rangle$ has no potential function and therefore is not conservative. However, we can easily arrive at the same conclusion by computing $\operatorname{curl}\vec{F}$, and avoid the quite lengthy process of determining a potential function. Using the result in Example 3.2.2, we have

$$\operatorname{curl}\vec{F} = \langle 0\,,\,0\,,\,1-(-1)\rangle = \langle 0\,,\,0\,,\,2\rangle \neq \vec{0}.$$

The converse of the first result which states that $\operatorname{curl}\vec{F} = \vec{0}$ implies $\vec{F}$ is conservative is *not* true in general. $\vec{F}$ needs to satisfy some conditions on its domain for this to be true and this will be stated in the next theorem. Meanwhile, we first need to define some topological concepts.

Definition 3.3.5. Let C be a plane curve which is parametrized by $\vec{R}(t),\ t \in [a,b]$.

1. C is ***simple*** if it does not intersect itself, except possibly at the endpoints.
2. C is ***closed*** if $\vec{R}(a) = \vec{R}(b)$, i.e., its initial and terminal points coincide.

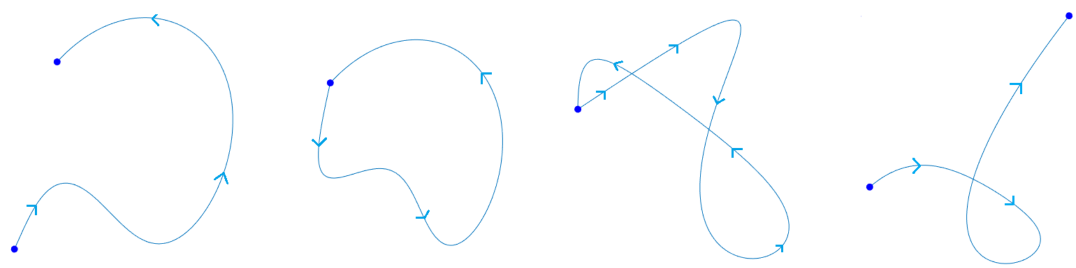


Figure 3.7: Examples of (from the left) a simple curve; a simple closed curve; a non-simple closed curve; a non-simple, non-closed curve

Definition 3.3.6. Let D be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. D is ***connected*** if for any two points $P_1, P_2 \in D$, there exists a continuous path from P_1 to P_2 which lies entirely in the region.
2. A connected region D is ***simply connected*** if every loop in D can be contracted to a point in D without ever leaving D.

Note that a region D in $\mathbb{R}^2$ is simply connected if D has no holes or slits. For example, the unit disk $\{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \le 1\}$ is simply connected while the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ and the "slitted" plane $\mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \,|\, x = y\}$ are not simply connected. Meanwhile in $\mathbb{R}^3$, the solids that are simply connected are those without holes going all the way through it. For example, the unit sphere is simply connected while the torus is not simply connected. The following theorem provides a partial converse to Theorem 3.3.3 (1).

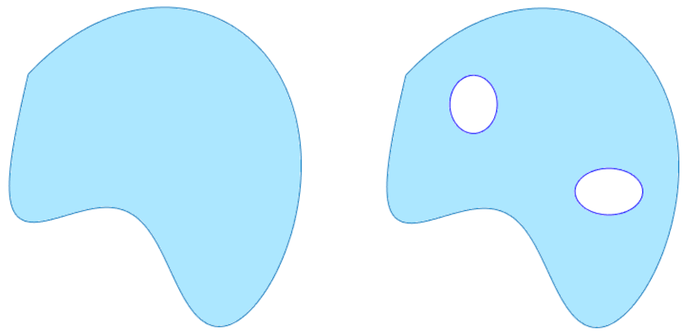


Figure 3.8: Examples of (from the left) a simply connected region; a connected, but not simply connected region

Theorem 3.3.7. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ defined on a simply-connected region, and whose component functions have continuous first-order partial derivatives. Then, $\vec{F}$ is conservative if and only if $\operatorname{curl} \vec{F} = \vec{0}$.

For instance, if the first-order partial derivatives of the component functions of $\vec{F}$ are continuous on the whole plane, then the above equivalence holds. The vector field $\vec{F}(x,y) = \langle x^{-1}\,,\, y^{-1}\rangle$ is not defined on a simply connected domain but we can always avoid where the slits are (the x- and y-axes). For example, if $D = \{(x,y) \in \mathbb{R}^2 \,|\, x > 0 \text{ and } y > 0\}$ is the first quadrant, then $\vec{F}$ is defined on the simply-connected region D.

Using Example 3.2.2 which states that $\operatorname{curl}(\langle P\,,\, Q\,,\, 0\rangle) = \langle 0\,,\, 0\,,\, Q_x - P_y\rangle$, the following corollary is immediate.

Corollary 3.3.8. Let P and Q be scalar fields defined on a simply-connected region in $\mathbb{R}^2$ and having continuous first-order partial derivatives. Let $\vec{F} = \langle P\,,\, Q\rangle$. Then $\vec{F}$ is conservative if and only if $Q_x = P_y$.

Example 3.3.5. Determine whether the given vector field $\vec{F}$ is conservative or not.

1. $\vec{F}(x,y) = \langle 2x + y\,,\, 2x\rangle$
2. $\vec{F}(x,y) = \langle \sin y - y\sin x\,,\, x\cos y + \cos x\rangle$
3. $\vec{F}(x,y,z) = \langle y^2z^3\,,\, 2xyz^3\,,\, 3xy^2z^2\rangle$.

Solution: Observe that all of the component functions are differentiable on the whole space, so the assumptions of the previous theorem are satisfied. Thus, we just need to compute for the curl of the vector field, and check if it is $\vec{0}$. For numbers 1 and 2, we use Corollary 3.3.8.

1. Let $P(x,y) = 2x + y$ and $Q(x,y) = 2x$. Then $P_y = 1$ and $Q_x = 2$. Since $Q_x \neq P_y$, then Corollary 3.3.8 proves that $\vec{F}$ is not conservative.

2. Let $P(x,y) = \sin y - y \sin x$ and $Q(x,y) = x \cos y + \cos x$. Then $P_y(x,y) = \cos y - \sin x$ and $Q_x(x,y) = \cos y - \sin x$. Since $P_y = Q_x$, then Corollary 3.3.8 shows that $\vec{F} = \langle P\,,\,Q\rangle$ is conservative. In fact, a potential for $\vec{F}$ is $\phi(x,y) = x \sin y + y \cos x$.

3. Observe that,

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$
$$= \langle 6xyz^2 - 6xyz^2\,,\, 3y^2z^2 - 3y^2z^2\,,\, 2yz^3 - 2yz^3\rangle = \vec{0}.$$

By the above theorem, we conclude that $\vec{F}$ is conservative. In fact, it is easy to verify that a potential for $\vec{F}$ is $\phi(x,y,z) = xy^2z^3$.

EXERCISES 3.3

I. Find the conservative vector field having the given potential function.

1. $\phi(x,y) = x + y$
2. $\phi(x,y) = 3x^2 + 2y^3$
3. $\phi(x,y) = \tan^{-1}(x^2y)$
4. $\phi(x,y) = -(x^2 + y^2)^{-1}$
5. $\phi(x,y,z) = \sqrt{x^2 + y^2 + z^2}$
6. $\phi(x,y,z) = z \sin(x^2 - y)$

II. Show that the given vector field $\vec{F}$ is conservative, and find a potential function for $\vec{F}$.

1. $\vec{F}(x,y) = \langle 2x + y\,,\, x + 2y - 2\rangle$
2. $\vec{F}(x,y) = \langle y^2 + 2x + 4\,,\, 2xy + 4y - 5\rangle$
3. $\vec{F}(x,y) = \langle ye^x - y\,,\, e^x - x + 5\rangle$
4. $\vec{F}(x,y) = \langle e^{-y} - 2x\,,\, -xe^{-y} - \sin y\rangle$
5. $\vec{F}(x,y) = (1 + 4x^2y^4)^{-1}\langle 2y^2\,,\, 4xy\rangle$
6. $\vec{F}(x,y,z) = \langle 2xy + 1\,,\, x^2 + 2yz\,,\, y^2 - 2\rangle$
7. $\vec{F}(x,y,z) = \langle 6x - 4y\,,\, z - 4x\,,\, y - 8z\rangle$
8. $\vec{F}(x,y,z) = \langle 6xy \sin z + 2x\,,\, 3x^2 \sin z + 5\,,\, 3x^2y \cos z\rangle$
9. $\vec{F}(x,y,z) = \langle z^2 \sec^2 x\,,\, 2ye^{3z}\,,\, 3y^2e^{3z} + 2z \tan x\rangle$
10. $\vec{F}(x,y,z) = \langle e^x \sin z + 2yz\,,\, 2xz + 2y\,,\, e^x \cos z + 2xy + 3z^2\rangle$
11. $\vec{F}(x,y,z) = \langle 2x \cos y, e^z - x^2 \sin y, ye^z\rangle$
12. $\vec{F}(x,y,z) = 3x^2\hat{\imath} - z \cos y\hat{\jmath} - \sin y\hat{k}$

13. $\vec{F}(x,y) = \left(\dfrac{y}{x} + \sin y\right)\hat{\imath} + (\ln x + x\cos y - 4y)\hat{\jmath}$

14. $\vec{F}(x,y) = \langle 2x - y\sin x, \cos x - e^y \rangle$

15. $\vec{F}(x,y) = \langle 2xy^3 + e^x, 3x^2y^2 + 3y^2 \rangle$

16. $\vec{F}(x,y) = \langle e^{-y}\cos x, -e^{-y}\sin x - 1 \rangle$

17. $\vec{F}(x,y,z) = 2ye^{2x}\hat{\imath} + (e^{2x} + 2y)\hat{\jmath} + 3z^2\hat{k}$

18. $\vec{F}(x,y) = \left(3x^2y + \dfrac{1}{xe^y}\right)\hat{\imath} + \left(x^3 - \dfrac{\ln x}{e^y} + 2y\right)\hat{\jmath}$

19. $\vec{F}(x,y) = \left\langle e^x \ln y - y\sin x, \dfrac{e^x}{y} + \cos x + 1 \right\rangle$

III. Prove that if ϕ is a scalar field which has continuous second-order partial derivatives, then $\operatorname{curl}(\vec{\nabla}\phi) = \vec{0}$.

3.4 Line Integrals of Scalar Fields

In this section, we generalize the concept of integration with respect to x or y that we have studied in our first calculus course. Recall that we motivated integration by computing the area of a plane region where the base is always a segment of an axis of $\mathbb{R}^2$.

What happens now if the segment $[a,b]$ is transformed into a plane curve or a space curve? How do we compute the area, then?

Line integrals with respect to the arclength parameter

We first evaluate the line integral of a continuous scalar field f over a piecewise smooth curve C. We motivate our study of line integrals of scalar fields with respect to arclength by the problem of finding the area of a general surface whose base may be curved.

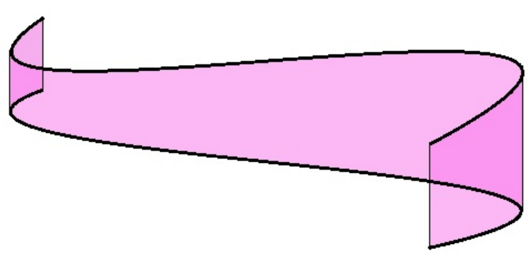


Figure 3.9: A vertical curtain

(Curtain Area Problem) Let C be a smooth plane curve parametrized by $\vec{R}(t) = \langle x(t)\,,\, y(t)\rangle$, $t \in [a,b]$. Let f be a positive continuous function of two variables. What is the area of the "curtain" whose base is C, whose height at any point $(x(t), y(t))$ is given by $z = f(\vec{R}(t))$ and whose lateral surface is perpendicular to the plane containing C?

The Method

1. Partition $[a, b]$ into subintervals of equal lengths via
$$a = t_0 < t_1 < t_2 < \ldots < t_n = b,$$
and for each $i \in \{0, 1, 2, \ldots, n\}$, set $x_i = x(t_i)$ and $y_i = y(t_i)$. The corresponding points $P_i(x_i, y_i)$ (or simply P_i) divide C into n subarcs of lengths Δs_1, $\Delta s_2, \ldots, \Delta s_n$.

2. For each $i \in \{1, 2, \ldots, n\}$, choose $t_i^* \in [t_{i-1}, t_i]$ which in turn corresponds to a point $P_i^*(x_i^*, y_i^*)$ on the subarc $P_{i-1}P_i$.

3. For each $i \in \{1, 2, \ldots, n\}$, draw a cylinder with arc $P_{i-1}P_i$ as the base and with z-coordinate $f(x_i^*, y_i^*)$ as the height.

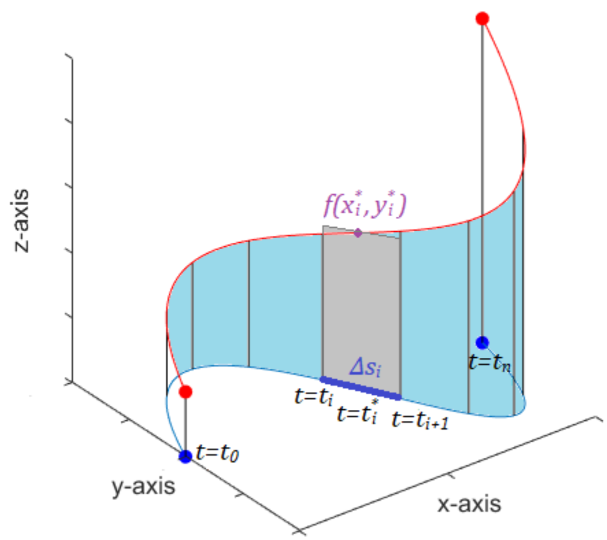


Figure 3.10: Partitioning the curtain

4. The area of each cylinder is approximately $f(x_i^*, y_i^*)\Delta s_i$ and therefore the area of the curtain is approximately
$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i.$$

5. The error in the above approximation vanishes as all the subarcs shrink in length, i.e., as $\Delta t = \dfrac{b-a}{n}$ tends to zero. This happens if n tends to infinity. Thus,
$$A = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i.$$

If this limit exists, we denote it by $\int_C f(x,y)\,ds$.

Definition 3.4.1. Let f be a function of two variables x and y that is continuous on a region containing the smooth plane curve C defined by a vector function $\vec{R}(t) = \langle x(t)\,,\, y(t)\rangle$, $t \in [a,b]$. The ***line integral*** of f along C with respect to the arclength parameter is defined by

$$\int_C f(x,y)\,ds = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i,$$

if this limit exists.

The following important remarks detail guidelines on how to evaluate line integrals.

Remarks.

1. The value of a line integral with respect to arclength does not depend on the direction the curve is traced. This means that if $-C$ denotes the curve C traced in the opposite direction, then
$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds.$$

2. From the previous calculus course, we know that $\dfrac{ds}{dt} = \left\|\vec{R}'(t)\right\|$. Thus, in terms of differentials, $ds = \left\|\vec{R}'(t)\right\| dt$ so that after introducing the parameter t into the definition of the line integral, we have
$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\left\|\vec{R}'(t)\right\| dt = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

3. The line integral with respect to the arclength parameter is independent of the parametrization of C. This means that if C can be parametrized by both $\vec{R}(t) = \langle x(t)\,,\, y(t)\rangle$, $t \in [a,b]$ and $\vec{S}(t) = \langle \xi(t)\,,\, \eta(t)\rangle$, $t \in [c,d]$, then
$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\left\|\vec{R}'(t)\right\| dt = \int_c^d f(\xi(t),\eta(t))\left\|\vec{S}'(t)\right\| dt.$$

4. If C is piecewise smooth, that is, if it is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, then
$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \ldots + \int_{C_n} f(x,y)\,ds.$$

5. Other physical interpretations of the line integral: If $f(x,y)$ denotes the linear density of a wire shaped like the curve C that is parametrized by $\vec{R}(t)$, $t \in [a,b]$, then

   (a) The mass of the wire equals $m = \int_C f(x,y)\,ds$.

(b) The center of mass of the wire is located at $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m}\int_C xf(x,y)\,ds \qquad \text{and} \qquad \bar{y} = \frac{1}{m}\int_C yf(x,y)\,ds.$$

(c) If $f(x,y) = 1$, then $\displaystyle\int_C ds = \int_a^b \left\|\vec{R}'(t)\right\| dt$ gives the arclength of C.

Example 3.4.1. Given $f(x,y) = x^2y + x$ and $C : \vec{R}(t) = \langle 3\sin t\,,\, 3\cos t\rangle$, $t \in [0, \pi/2]$. Find the area of the curtain whose base is C and the height is $f(x,y)$ for every (x,y) in C.

Solution: We first compute for $\left\|\vec{R}'(t)\right\|$. Note that $\vec{R}'(t) = \langle 3\cos t\,,\, -3\sin t\rangle$ so that $\left\|\vec{R}'(t)\right\| = \sqrt{9\cos^2 t + 9\sin^2 t} = 3$. The area of the curtain A is given by

$$\begin{aligned}
A = \int_C f(x,y)\,ds &= \int_0^{\pi/2} f(3\sin t, 3\cos t)\left\|\vec{R}'(t)\right\| dt \\
&= 3\int_0^{\pi/2} (27\sin^2 t\cos t + 3\sin t)\,dt \\
&= 9\int_0^{\pi/2} (9\sin^2 t\cos t + \sin t)\,dt \\
&= 9(3\sin^3 t - \cos t)\Big|_0^{\pi/2} \\
&= 9((3-0) - (0-1)) = 36.
\end{aligned}$$

Many problems involve line integrals along a segment joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$. In such cases, we need to parametrize the segment $\overline{AB}$. Recall from a previous calculus course that one parametrization is given by

$$\vec{R}(t) = \langle x_1 + (x_2 - x_1)t\,,\, y_1 + (y_2 - y_1)t\rangle,$$

where $0 \le t \le 1$. In this parametrization, if $t = 0$, we get A, while if $t = 1$, we get B. That is, this is a parametrization for the the segment traversed from A to B.

Example 3.4.2. Evaluate $\displaystyle\int_C 2x\,ds$, where C consists of the portion of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the line segment from $(1,1)$ to $(3,2)$.

Solution: Let C_1 be the portion of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and let C_2 be the line segment from $(1,1)$ to $(3,2)$. Then $C = C_1 \cup C_2$, and because C_1 and C_2 are smooth curves, we have

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds.$$

The curve C_1 can be parametrized naturally by $\vec{R}_1(t) = \langle t\,,\, t^2\rangle$, $t \in [0,1]$.
So $\left\|\vec{R}_1'(t)\right\| = \left\|\langle 1\,,\, 2t\rangle\right\| = \sqrt{1+4t^2}$. Therefore,

$$\int_{C_1} 2x\,ds = \int_0^1 2t\sqrt{1+4t^2}\,dt = \left.\frac{(1+4t^2)^{3/2}}{6}\right|_0^1 = \frac{5\sqrt{5}-1}{6}.$$

On the other hand, C_2 can be parametrized by

$$\vec{R}_2(t) = \langle 1 + (3-1)t\,,\, 1 + (2-1)t \rangle = \langle 1 + 2t\,,\, 1 + t \rangle,$$

where $t \in [0, 1]$. So, $\left\| \vec{R}_2'(t) \right\| = \left\| \langle 2\,,\, 1 \rangle \right\| = \sqrt{5}$. Therefore,

$$\int_{C_2} 2x\, ds = \int_0^1 2(1+2t)\sqrt{5}\, dt = 2\sqrt{5}\left(t + t^2\right)\Big|_0^1 = 4\sqrt{5}.$$

Finally,

$$\int_C 2x\, ds = \frac{5\sqrt{5}-1}{6} + 4\sqrt{5} = \frac{29\sqrt{5}-1}{6}.$$

Line Integrals with respect to x and y

Aside from line integrals of functions of two variables with respect to the arclength parameter, we can also talk of line integrals with respect to x and y.

Definition 3.4.2. Let f be a function of two variables x and y that is continuous on a region containing the smooth curve C described by the vector function $\vec{R}(t) = \langle x(t)\,,\, y(t) \rangle$, $t \in [a, b]$.

1. The ***line integral of f along C with respect to x*** is
$$\int_C f(x,y)\, dx = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta x_i,$$
if this limit exists.

2. The ***line integral of f along C with respect to y*** is
$$\int_C f(x,y)\, dy = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta y_i,$$
if this limit exists.

Remarks.

1. We evaluate the above integrals using the same trick that we have used for the line integrals with respect to arclength. We just write $dx = \dfrac{dx}{dt}\, dt = x'(t)\, dt$ and $dy = \dfrac{dy}{dt}\, dt = y'(t)\, dt$. Hence, we have the following formulas.
$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$
$$\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt.$$

2. If $-C$ denotes the curve C traced in the opposite direction, then
$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$$
$$\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy.$$

3. The line integral with respect to x or y is independent of parametrization of the curve in the same sense as our previous remark.

4. If C is piecewise smooth curve with disjoint smooth components C_1, $C_2, \ldots, C_n$, then
$$\int_C f(x,y)\,dx = \sum_{i=1}^{n} \int_{C_i} f(x,y)\,dx$$
$$\int_C f(x,y)\,dy = \sum_{i=1}^{n} \int_{C_i} f(x,y)\,dy.$$

5. It often happens that line integrals with respect to x and y occur together. In this case, we will write
$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy.$$

Example 3.4.3. Let C be the curve parametrized by $\vec{R}(t) = \langle 1-t^2\,,\, t\rangle$, where $t \in [-1,2]$. Evaluate $\displaystyle\int_C 3y\,dx + 4xy\,dy$.

Solution: The derivative of $\vec{R}$ is $\vec{R}'(t) = \langle x'(t)\,,\, y'(t)\rangle = \langle -2t\,,\, 1\rangle$. Therefore,
$$\begin{aligned}\int_C 3y\,dx + 4xy\,dy &= \int_{-1}^{2} 3t\,(-2t)\,dt + 4(1-t^2)t\,(1)\,dt \\ &= \int_{-1}^{2} (-6t^2 + 4t - 4t^3)\,dt \\ &= (-2t^3 + 2t^2 - t^4)\Big|_{-1}^{2} \\ &= -27.\end{aligned}$$

Line Integrals along Space Curves

Line integrals of functions of three variables along a three-dimensional curve are defined similarly as those of line integrals of functions of two variables along a plane curve.

Definition 3.4.3. Let f be a function of three variables x, y and z that is continuous on some region containing the smooth curve C described by the vector function $\vec{R}(t) = \langle x(t)\,,\, y(t)\,,\, z(t)\rangle$, $t \in [a, b]$. Then, we define

$$\int_C f(x,y,z)\,ds = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i = \int_a^b f(x(t), y(t), z(t))\,\|\vec{R}'(t)\|\,dt$$
$$\int_C f(x,y,z)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta x_i = \int_a^b f(x(t), y(t), z(t))\,x'(t)\,dt$$
$$\int_C f(x,y,z)\,dy = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta y_i = \int_a^b f(x(t), y(t), z(t))\,y'(t)\,dt$$
$$\int_C f(x,y,z)\,dz = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta z_i = \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt,$$

if these limits exist.

The reader should anticipate that the properties outlined in the previous remark remain true for the last three line integrals along space curves. In particular, we may write

$$\begin{aligned}\int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz \\ = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.\end{aligned}$$

Example 3.4.4. Find the mass of the thin wire shaped in the form of the curve C parametrized by $\vec{R}(t) = \langle \cos t\,,\, \sin t\,,\, t\rangle$, $t \in [0, \pi]$, if the density function is $x^2 \sin z$.

Solution: The magnitude of $\vec{R}'$ is $\|\vec{R}'(t)\| = \|\langle -\sin t\,,\, \cos t\,,\, 1\rangle\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. The mass of the thin wire m is

$$m = \int_C x^2 \sin z\,ds = \sqrt{2}\int_0^{\pi} \cos^2 t \sin t\,dt = -\sqrt{2}\,\frac{\cos^3 t}{3}\bigg|_0^{\pi} = -\frac{\sqrt{2}}{3}(-1-1) = \frac{2\sqrt{2}}{3}.$$

Example 3.4.5. Let C be the line segment from the point $P(3, 0, -1)$ to $Q(2, -3, 1)$. Evaluate $I = \int_C (y - x)\,dx + z\,dy + (x - y)\,dz$.

Solution: A parametrization of the line segment from P to Q is

$$\vec{R}(t) = \langle 3 + (2-3)t\,,\, 0 + (-3-0)t\,,\, -1 + (1-(-1))t\rangle = \langle 3 - t\,,\, -3t\,,\, 2t - 1\rangle,$$

where $t \in [0, 1]$. Hence, $\vec{R}'(t) = \langle -1\,,\, -3\,,\, 2\rangle$. Therefore, we have

$$\begin{aligned} I &= \int_0^1 (-3t - (3-t))\,(-1)\,dt + (2t-1)\,(-3)\,dt + (3 - t - (-3t))\,(2)\,dt \\ &= \int_0^1 12\,dt = 12.\end{aligned}$$

EXERCISES 3.4

I. Evaluate the line integral over the given curve.

1. $\int_C (x-y)\,ds$, where C is the curve parametrized by $\vec{R}(t) = \langle 2t, 3-t\rangle$, $0 \le t \le 1$

2. $\int_C \left(2+x^2y\right)\,ds$, where C is the upper half of the semicircle $x^2+y^2=1$

3. $\int_C y\sin z\,ds$, where C is the curve defined by $\vec{R}(t) = \langle \cos t, \sin t, t\rangle$, $t \in [0, 2\pi]$

4. $\int_C 3y\,dx + 4xy\,dy$, where C is the curve described by $\vec{R}(t) = \left\langle 1-t^2, t\right\rangle$, $t \in [-1, 2]$

5. $\int_C (3x+2y)\,dx + (2x-y)\,dy$, where C is the curve $y = \sin\left(\frac{\pi x}{2}\right)$ from $(0,0)$ to $(1,1)$

6. $\int_C y^2\,dx + x\,dy$, where C is the line segment from $(-5,-3)$ to $(0,2)$

7. $\int_C y^2\,dx + x\,dy$, where C is the line segment from $(0,2)$ to $(-5,-3)$

8. $\int_C y^2\,dx + x\,dy$, where C is the arc of the parabola $x = 4-y^2$ from $(-5,-3)$ to $(0,2)$

9. $\int_C (y-x)\,dx + z\,dy + (x-y)\,dz$, where C is the line segment from $(3,0,-1)$ to $(2,-3,1)$

10. $\int_C y\,dx + z\,dy + x\,dz$, where $C = C_1 \cup C_2$, C_1 is the line segment from $(2,0,0)$ to $(3,4,5)$, C_2 is the line segment from $(3,4,5)$ to $(3,4,0)$

11. $\int_C 2y\,ds$, where C is the curve described by $\vec{R}(t) = \left\langle \frac{t^2}{2}, t\right\rangle$, $t \in [0,1]$

12. $\int_C (xy+y^2)\,ds$, where C is the lower half of the circle $x^2+y^2=9$, described in the counterclockwise direction.

13. $\int_C y\,dx + z\,dy - x\,dz$, where C is the line segment from $(0,1,2)$ to $(1,3,6)$ followed by the line segment from $(1,3,6)$ to $(1,3,2)$.

14. $\int_C (x^2+xy)\,dx + (y^2-xy)\,dy$, where C consists of the line segment $y=x$ from the point $(0,0)$ to the point $(2,2)$ followed by the vertical line from $(2,2)$ to $(2,0)$.

15. $\int_C yz\,dx + xy\,dz$, where C is the path where $x=e^t$, $y=e^{3t}$, and $z=e^{-t}$, $t \in [0,1]$.

16. $\int_C y\,ds$, where C is the arc of the curve $y=x^3$ from $(0,0)$ to $(1,1)$.

17. $\int_C 3xz\,ds$, where C is the oriented curve with vector equation $\vec{R}(t) = \langle 4t, \sin 3t, \cos 3t\rangle$, $0 \le t \le \pi$.

18. $\int_C \frac{z^{1/2}}{x^2+y^2}\,ds$, where C is the curve defined by $\vec{R}(t) = \langle \sin t, \cos t, t^2\rangle$, $t \in [0, \sqrt{2}]$.

19. $\int_C 3(y-2)^{1/2}\,ds$, where $C = C_1 \cup C_2$, where C_1 is the line segment from $(4,3,0)$ to $(2,2,-2)$, and C_2 is given by the vector function $\vec{R}(t) = \left\langle 2\cos t, t^2+2, 2\sin t - 2\right\rangle$, where $t \in [0, \sqrt{3}]$.

20. $\int_C (4z - x + y)\,ds$, where $C = C_1 \cup C_2$ where C_1 is the line segment from $(0,3,-1)$ to $(2,1,-2)$, and C_2 is the curve with vector equation $\vec{R}(t) = \langle -2t^3, 3+2t^3, -1+t^3\rangle$, $t \in [-1,1]$.

II. Use a line integral to find the mass of a wire running along the parabola $y = x^2$ from $(0,0)$ to $(1,1)$, if the density (mass per unit length) of the wire at any point (x,y) is numerically equal to x.

III. Find the mass of a thin wire shaped in the form of the curve $x = 2t$, $y = \ln t$ and $z = 4\sqrt{t}$, $1 \le t \le 4$, if the density function is $\sqrt{x+y}$.

3.5 Line Integrals of Vector Fields

The motivation behind studying line integrals of vector fields is in finding the work done by a continuous force field $\vec{F}$ on an object moving along a piecewise smooth curve C. We immediately consider vector fields on $\mathbb{R}^3$ along space curves since the two-dimensional analog is easy to infer by disregarding the elevation component.

(Generalized Work Problem) Suppose that a particle exerts a force field $\vec{F} = \langle P\,,\,Q\,,\,R\rangle$ along a smooth curve C parametrized by $\vec{R}(t) = \langle x(t)\,,\,y(t)\,,\,z(t)\rangle$, $t \in [a,b]$. How much work is the particle exerting?

We recall that the work done by a constant force $\vec{F}$ along a straight line segment is $W = \vec{F}\cdot\vec{d}$, where $\vec{d}$ is parallel to and has the same length as the line segment.

The Method Assume that P, Q, and R are functions of three variables x, y and z that are continuous on some region containing the curve C.

1. Partition $[a,b]$ into subintervals $a = t_0 < t_1 < \ldots < t_n = b$ of equal lengths Δt. For $i \in \{0,1,\ldots,n\}$ denote $x_i = x(t_i)$, $y_i = y(t_i)$, and $z_i = z(t_i)$. The corresponding points $P_i(x_i, y_i, z_i)$ divide C into n subarcs.

2. If n is large, the ith subarc $P_{i-1}P_i$ is almost a straight segment and by continuity of $\vec{F}$, the force exerted on this subarc is almost constant, say $\vec{F}(x_i^*, y_i^*, z_i^*)$ corresponding to some choice of $t_i^* \in [t_{i-1}, t_i]$.

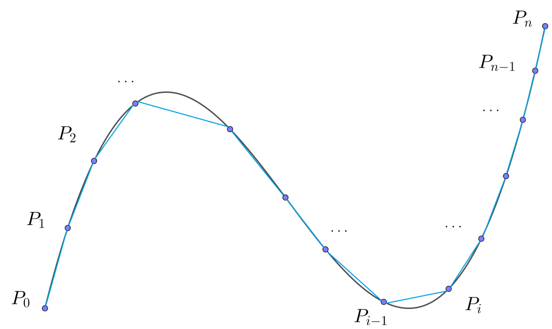


Figure 3.11: Approximating a curve using a sum of line segments

3. Therefore using the formula for work exerted by a constant force along a straight line, we see that the total work done is approximately

$$W \approx \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot d\vec{R}_i,$$

where $d\vec{R}_i = \vec{R}(t_i) - \vec{R}(t_{i-1})$.

4. Error vanishes as the parameter difference Δt vanishes, i.e., as n tends to infinity, and we get the exact work

$$W = \lim_{n \to +\infty} \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot d\vec{R}_i.$$

If this limit exists, we denote it by $\displaystyle\int_C \vec{F} \cdot d\vec{R}$.

Definition 3.5.1. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on some region containing the smooth space curve C parametrized by $\vec{R}(t) = \langle x(t)\,,\, y(t)\,,\, z(t)\rangle$, $t \in [a, b]$. The ***line integral of*** $\vec{F}$ ***along*** C is defined by

$$\int_C \vec{F} \cdot d\vec{R} = \lim_{n \to +\infty} \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot d\vec{R}_i,$$

if this limit exists.

Remarks.

1. To evaluate the above line integral, we just do the usual trick of dividing and multiplying by dt. After introducing the parameter t inside the arguments of $\vec{F}$, we then write $d\vec{R} = \dfrac{d\vec{R}}{dt}\, dt = \vec{R}'(t)\, dt$. Hence

$$\int_C \vec{F} \cdot d\vec{R} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t)\, dt. \tag{3.15}$$

2. There are at least two ways by which line integrals of vector fields can be viewed as line integrals of scalar fields. The first one is easy to see just by expanding (3.15). By the definitions of $\vec{F}$ and $\vec{R}$, we have

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{R} &= \int_a^b \langle P(x(t),y(t),z(t))\,,\,Q(x(t),y(t),z(t))\,,\,R(x(t),y(t),z(t))\rangle \cdot \left\langle \frac{dx}{dt}\,,\,\frac{dy}{dt}\,,\,\frac{dz}{dt}\right\rangle dt\\ &= \int_C \langle P(x,y,z)\,,\,Q(x,y,z)\,,\,R(x,y,z)\rangle \cdot \langle dx\,,\,dy\,,\,dz\rangle\\ &= \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.\end{aligned}$$

Another way of seeing this is to recall that the unit tangent vector $\vec{T}$ is the unit vector parallel to $\vec{R}'$. Since $ds = \left\|\vec{R}'(t)\right\|dt$, it follows that

$$\vec{T}(t)\,ds = \frac{\vec{R}'(t)}{\left\|\vec{R}'(t)\right\|}\left\|\vec{R}'(t)\right\|dt = \vec{R}'(t)\,dt.$$

Hence in view of (3.15), we obtain

$$\int_C \vec{F}\cdot d\vec{R} = \int_C \vec{F}\cdot\vec{T}\,ds.$$

As such, all properties of line integrals of scalar fields are also enjoyed by line integrals of vector fields.

Example 3.5.1. Given $\vec{F}(x,y) = \langle x^2\,,\,-xy\rangle$ and C is the curve parametrized by $\vec{R}(t) = \langle \cos t\,,\,\sin t\rangle$, $t \in [0,\pi]$, evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{R}$.

Solution: Using (3.15), we get

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{R} &= \int_0^\pi \langle \cos^2 t\,,\,-\cos t\sin t\rangle \cdot \langle -\sin t\,,\,\cos t\rangle\,dt\\ &= \int_0^\pi -2\cos^2 t\sin t\,dt \qquad\qquad \text{Let } u=\cos t \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad du = -\sin t\,dt\\ &= \int_1^{-1} 2u^2\,du\\ &= \frac{2}{3}u^3\Big|_1^{-1}\\ &= \frac{2}{3}(-1-1) = -\frac{4}{3}.\end{aligned}$$

Example 3.5.2. Let $\vec{F}(x,y,z) = \langle z\,,\,y\,,\,x\rangle$. Evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{R}$ along the line segment from $A(-2,1,-3)$ to $B(-1,5,0)$.

Solution: The segment C traversed from A to B can be parametrized by

$$\vec{R}(t) = \langle -2+(-1-(-2))t\,,\,1+(5-1)t\,,\,-3+(0-(-3))t\rangle = \langle t-2\,,\,4t+1\,,\,3t-3\rangle,\ \ t\in[0,1].$$

Thus, $\vec{R}'(t) = \langle 1\,,\,4\,,\,3\rangle$ so that

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{R} = \int_0^1 \vec{F}(\vec{R}(t))\cdot\vec{R}'(t)\,dt &= \int_0^1 \langle 3t-3\,,\,4t+1\,,\,t-2\rangle\cdot\langle 1\,,\,4\,,\,3\rangle\,dt\\ &= \int_0^1 (3t-3)+(16t+4)+(3t-6)\,dt\\ &= \int_0^1 (22t-5)\,dt\\ &= \left(11t^2-5t\right)\Big|_0^1 = 6.\end{aligned}$$

Example 3.5.3. Compute the amount of work done by the force field $\vec{F}(x,y) = \langle 2y-x\,,\,xy\rangle$ in moving an object from the point $(0,0)$ to the point $(4,4)$ along the curve defined by $\vec{R}(t) = \langle t^2\,,\,2t\rangle$.

Solution: The points $(0,0)$ and $(4,4)$ correspond respectively to the parameter values $t=0$ and $t=2$. Also, $\vec{R}'(t) = \langle 2t\,,\,2\rangle$. Thus, the total work done equals

$$\begin{aligned}W = \int_C \vec{F}\cdot d\vec{R} &= \int_0^2 \langle 4t-t^2\,,\,2t^3\rangle\cdot\langle 2t\,,\,2\rangle\,dt\\ &= \int_0^2 (8t^2+2t^3)\,dt\\ &= \left(\frac{8t^3}{3}+\frac{t^4}{2}\right)\Bigg|_0^2 = \frac{88}{3}.\end{aligned}$$

EXERCISES 3.5

I. Evaluate the line integral $\displaystyle\int_C \vec{F}\cdot d\vec{R}$.

1. $\vec{F}(x,y) = \left(x^2+y^2\right)\hat{\imath} + 3x^2y\hat{\jmath}$; C is the portion of the parabola $y=x^2$ from $(-1,1)$ to $(2,4)$
2. $\vec{F}(x,y) = x^2\hat{\imath} - xy\hat{\jmath}$; C is the quarter circle $\vec{R}(t) = \cos t\hat{\imath} + \sin t\hat{\jmath}$ from the point $(1,0)$ to $(0,1)$
3. $\vec{F}(x,y) = x^2\hat{\imath} - xy\hat{\jmath}$; C is the quarter circle $\vec{R}(t) = \cos t\hat{\imath} + \sin t\hat{\jmath}$ from the point $(0,1)$ to $(1,0)$
4. $\vec{F}(x,y,z) = \langle xy, yz, xz\rangle$; C is the curve described by $\vec{R}(t) = \left\langle t, t^2, t^3\right\rangle$, $t\in[0,1]$
5. $\vec{F}(x,y,z) = \langle y, 0, yz\rangle$; C is the helix $\vec{R}(t) = \langle \cos t, \sin t, 3t\rangle$, $0\le t\le 2\pi$
6. $\vec{F}(x,y,z) = \langle 2x, 0, -2z\rangle$; C is the curve described by $\vec{R}(t) = \left\langle 2t, \dfrac{t^3}{3}, t^2\right\rangle$, $t\in[0,1]$
7. $\vec{F}(x,y) = (10^{10^x}-2y)\hat{\imath} + (3x+10y)\hat{\jmath}$; C is the directed line segment from $(0,0)$ to $(0,-2)$

II. Find the amount of work done by the force field $\vec{F}$ in moving an object along the given curve.

1. $\vec{F}(x,y,z) = \langle x-y, x+z, y-z\rangle$; $\vec{R}(t) = \langle t^3, t^2, t\rangle$, $-1 \le t \le 1$
2. $\vec{F}(x,y,z) = \langle z-x, 2y, 2xz\rangle$; $\vec{R}(t) = \langle t^2, e^t, t-1\rangle$, $t \in [0,1]$
3. $\vec{F}(x,y,z) = \left\langle \frac{x}{2}, 2\sin^2(2y), \cos^2(z+6)\right\rangle$; the line segment from $(4,3,0)$ to $(2,2,-2)$
4. $\vec{F}(x,y) = \big\langle 2xy - 2y - \tan^{-1}(x-y), x^2 + \tan^{-1}(x-y)\big\rangle$; the line segment from $(-1,-1)$ to $(2,2)$

3.6 Fundamental Theorem of Line Integrals

In comparison to evaluating single integrals $\displaystyle\int_a^b f(x)\,dx$, solving for the value of a line integral seems to be onerous due to the introduction of the parametrization of C. We recall that in the former, it is enough to find an antiderivative F, i.e., one satisfying $D_xF(x) = f(x)$, to invoke the Fundamental Theorem of Calculus which allows us to "cancel out" the integral sign and the derivative operator D_x and evaluate F at the endpoints of $[a,b]$. That is, under some continuity conditions,

$$\int_a^b f(x)\,dx = \int_a^b D_xF(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Now, suppose that a vector field $\vec{F}$ is conservative. Then it can be written as $\vec{F} = \vec{\nabla}\phi$ for some scalar field ϕ. (Here, $\vec{\nabla}$ acts as the analogue to the derivative above.) Do we have the same conclusion too? That is, can we cancel out the integral sign and the differential operator $\vec{\nabla}$ in the sense that

$$\int_C \vec{F}\cdot d\vec{R} = \int_C \vec{\nabla}\phi\cdot d\vec{R} = \phi(B) - \phi(A),$$

where A and B are the endpoints of C? The affirmative answer is provided by the Fundamental Theorem of Line Integrals (FTLI).

Theorem 3.6.1 (Fundamental Theorem of Line Integrals)**.** Let C be a piecewise smooth curve that is parametrized by $\vec{R}(t)$, $a \le t \le b$. Let ϕ be a differentiable function of two or three variables whose gradient $\vec{\nabla}\phi$ is continuous on some region containing C. Then,

$$\int_C \vec{\nabla}\phi\cdot d\vec{R} = \phi\big(\vec{R}(b)\big) - \phi\big(\vec{R}(a)\big).$$

That is, if $\vec{F}$ is a conservative vector field with potential function ϕ, and if C is any piecewise smooth curve with initial point A and terminal point B, then

$$\int_C \vec{F}\cdot d\vec{R} = \phi(B) - \phi(A).$$

Proof. We present the proof for a smooth curve C. For a piecewise smooth curve, we will do the proof for each smooth component of C. We will also only present the proof for the two-dimensional case since the extension to higher dimensions is immediate.

Suppose that C is parametrized by $R(t) = \langle x(t)\,,\, y(t)\rangle$, $t \in [a, b]$. Let the initial point of C be $A(x_1, y_1) = (x(a), y(a))$ and let the terminal point be $B(x_2, y_2) = (x(b), y(b))$. Evaluating the line integral, we get

$$\begin{aligned}\int_C \vec{\nabla}\phi \cdot d\vec{R} = \int_C \langle \phi_x\,,\, \phi_y\rangle \cdot d\vec{R} &= \int_C \phi_x\,(x, y)\; dx + \phi_y\,(x, y)\; dy \\ &= \int_a^b \big(\phi_x\,(x(t), y(t))\, x'(t) + \phi_y\,(x(t), y(t))\, y'(t)\big)\; dt.\end{aligned}$$

Observe that this is already a single integral. Thus, if we could find an antiderivative for the integrand, we may apply the Fundamental Theorem of Calculus. This antiderivative is found by applying the multivariate chain rule. We obtain

$$\begin{aligned}D_t\left[\phi\,(x(t), y(t))\right] &= \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} \\ &= \phi_x\,(x(t), y(t))\, x'(t) + \phi_y\,(x(t), y(t))\, y'(t)\end{aligned}$$

Therefore, substituting this to our previous computation yields

$$\begin{aligned}\int_C \vec{\nabla}\phi \cdot d\vec{R} &= \int_a^b D_t\left[\phi\,(x(t), y(t))\right]\; dt \\ &= \phi\,(x(t), y(t))\,\Big|_a^b \\ &= \phi(x(b), y(b)) - \phi(x(a), y(a)) \\ &= \phi(x_2, y_2) - \phi(x_1, y_1).\end{aligned}$$

This is the desired conclusion. □

Remark. Suppose that $\vec{F}$ is continuous, and C_1 and C_2 are any two paths (piecewise smooth curves) with the same initial point A and the same terminal point B.

1. In general, $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{R} \neq \int_{C_2} \vec{F} \cdot d\vec{R}$.

2. However, if $\vec{F}$ is a conservative vector field ($\vec{F} = \vec{\nabla}\phi$ for some scalar field ϕ), then

$$\int_{C_1} \vec{F} \cdot d\vec{R} = \int_{C_2} \vec{F} \cdot d\vec{R} = \phi(B) - \phi(A).$$

   This means that the value of the line integral is *independent of path* and only depends on the initial and terminal points of C.

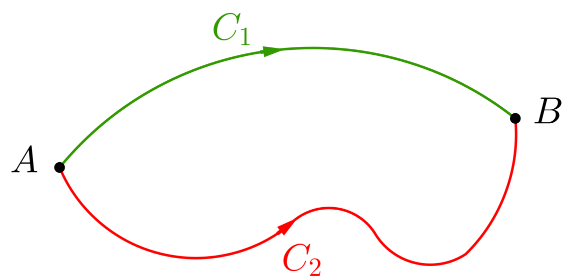


Figure 3.12: Two paths, C_1 and C_2, from A to B

Example 3.6.1. Let $\vec{F}(x,y) = \langle y\,,\, x\rangle$, and C be a piecewise smooth curve from $(0,0)$ to $(1,1)$. Notice that $\vec{F}$ is conservative with potential $\phi(x,y) = xy$. Therefore, it follows from the Fundamental Theorem of Line Integrals that

$$\int_C \vec{F}\cdot d\vec{R} = \phi(x,y)\Big|_{(0,0)}^{(1,1)} = xy\Big|_{(0,0)}^{(1,1)} = 1. \tag{3.16}$$

Let us illustrate the second statement in the previous remark by computing the line integral along several curves from $(0,0)$ to $(1,1)$:

1. $\vec{R}(t) = \langle t^m\,,\, t^n\rangle$, $t\in[0,1]$ and $m,n\in\mathbb{N}$.
2. $\vec{R}(t) = \Big\langle t\,,\, \dfrac{t}{2-t}\Big\rangle,\ t\in[0,1]$
3. $\vec{R}(t) = \Big\langle te^{t-1}\,,\, \dfrac{1-\cos(\pi t)}{2}\Big\rangle,\ t\in[0,1]$

Indeed, we have the following:

1. Using our evaluation procedure,

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{R} &= \int_0^1 \langle t^n\,,\, t^m\rangle\cdot\langle mt^{m-1}\,,\, nt^{n-1}\rangle\, dt\\ &= \int_0^1 mt^{n+m-1} + nt^{m+n-1}\, dt\\ &= (m+n)\frac{t^{n+m}}{n+m}\Big|_0^1 = 1.\end{aligned}$$

2. Similarly,

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{R} &= \int_0^1 \Big\langle \frac{t}{2-t}\,,\, t\Big\rangle\cdot\Big\langle 1\,,\, \frac{2}{(2-t)^2}\Big\rangle\, dt\\ &= \int_0^1 \frac{t}{2-t} + \frac{2t}{(2-t)^2}\, dt\\ &= \int_0^1 \left(\frac{4}{(2-t)^2} - 1\right) dt\\ &= \left(\frac{4}{2-t} - t\right)\Big|_0^1 = 1.\end{aligned}$$

3. This next item is intended to show the reader the advantage of the FTLI over evaluating a line integral from definition in some cases. Observe how easily we got the value of the line integral in (3.16) while we spent more time on the parametrization of C to solve the above line integrals. In some cases, evaluating line integrals without FTLI is impractical or impossible. For instance, the reader is enjoined to verify that after introducing the parametrization of C, we will obtain

$$\int_C \vec{F} \cdot d\vec{R} = \frac{1}{2} \int_0^1 e^{t-1} \left((t+1)(1 - \cos(\pi t)) + (\pi t) \sin(\pi t) \right) \, dt.$$

Although not impossible to solve, it will be more cumbersome to evaluate this.

Example 3.6.2. Compute $\displaystyle\int_C \vec{F} \cdot d\vec{R}$.

1. $\vec{F}(x, y) = \langle \sin y - y \sin x \, , \, x \cos y + \cos x \rangle$; C is the segment from $(0, 0)$ to (π, π).

2. $\vec{F}(x, y, z) = \langle y^2 z^3 \, , \, 2xyz^3 \, , \, 3xy^2 z^2 \rangle$; $C = C_1 \cup C_2$ where C_1 is the line segment from $(0, 0, 0)$ to $(1, 1, 1)$ and C_2 is parametrized by $\vec{R}(t) = \langle t \, , \, e^{t^2-1} \, , \, \sqrt[3]{t} \rangle, \ t \in [1, 2]$.

Solution: The vector fields above have already been shown to be conservative in Example 3.3.5. Their potential functions are also indicated there.

1. Observe that $\vec{F} = \vec{\nabla} \phi$ where $\phi(x, y) = x \sin y + y \cos x$. Hence, by FTLI,

$$\int_C \vec{F} \cdot d\vec{R} = \phi(x, y) \Big|_{(0,0)}^{(\pi,\pi)} = (x \sin y + y \cos x) \Big|_{(0,0)}^{(\pi,\pi)} = -\pi.$$

2. A potential for $\vec{F}$ is $\phi(x, y, z) = xy^2 z^3$ which is continuous. Note that C is piecewise smooth so that FTLI guarantees that the line integral depends only on the value of ϕ at the initial point $(0, 0, 0)$ and the terminal point of $\vec{R}(2)$ which is $(2, e^3, \sqrt[3]{2})$. By the formula of FTLI, we obtain

$$\int_C \vec{F} \cdot d\vec{R} = \phi(x, y, z) \Big|_{(0,0,0)}^{(2, e^3, \sqrt[3]{2})} = xy^2 z^3 \Big|_{(0,0,0)}^{(2, e^3, \sqrt[3]{2})} = 4e^6.$$

Theorem 3.6.2. If C is a piecewise smooth closed curve lying in an open region B and $\vec{F}$ is a conservative vector field that is continuous on B, then $\displaystyle\int_C \vec{F} \cdot d\vec{R} = 0$.

Proof. Since $\vec{F}$ is conservative, we can find a scalar field ϕ so that $\vec{\nabla} \phi = \vec{F}$. Let C be parametrized by $\vec{R}(t), \ t \in [a, b]$. Since C is closed, the initial point $\vec{R}(a)$ coincides with the terminal point $\vec{R}(b)$. By application of FTLI,

$$\int_C \vec{F} \cdot d\vec{R} = \phi(\vec{R}(b)) - \phi(\vec{R}(a)) = 0.$$

□

Remark. The theorem mentioned above implies that the work done by a conservative vector field over a closed path is zero.

Example 3.6.3. Find the work done by the vector field $\vec{F}(x,y) = \langle y^2 e^{xy^2} + 2x\,,\, 2xye^{xy^2} - 5\rangle$ along the unit circle C parametrized by $\vec{R}(t) = \langle \cos t\,,\, \sin t\rangle,\ t \in [0, 2\pi]$.

Solution: Before we do some computations, we first verify if $\vec{F}$ is conservative. To do this, we invoke Corollary 3.3.8 and just check whether $Q_x = P_y$ assuming $\vec{F} = \langle P\,,\, Q\rangle$. Indeed,

$$\frac{\partial}{\partial x}\left(2xye^{xy^2} - 5\right) = 2y(xy^2e^{xy^2} + e^{xy^2}) = 2ye^{xy^2}(xy^2 + 1),$$

and

$$\frac{\partial}{\partial y}\left(y^2e^{xy^2} + 2x\right) = y^2(2xy)e^{xy^2} + 2ye^{xy^2} = 2ye^{xy^2}(xy^2 + 1).$$

Since the two expressions are equal, we conclude that $\vec{F}$ is conservative. Therefore, as the unit circle is a closed curve, Theorem 3.6.2 asserts that the work done by $\vec{F}$ along C is 0.

Independence of the Path

Definition 3.6.3. The line integral $\displaystyle\int_C \vec{F}\cdot d\vec{R}$ is *independent of the path* C if

$$\int_{C_1} \vec{F}\cdot d\vec{R} = \int_{C_2} \vec{F}\cdot d\vec{R}$$

for all piecewise smooth curves C_1 and C_2 having the same initial and terminal points.

Theorem 3.6.4. Let $\vec{F}$ be a vector field whose component functions are continuous on an open connected region D. The following statements are equivalent.

(i) $\vec{F}$ is a conservative vector field on D.

(ii) $\displaystyle\int_C \vec{F}\cdot d\vec{R}$ is independent of the path from any point A in D to any point B in D.

(iii) $\displaystyle\int_C \vec{F}\cdot d\vec{R} = 0$ for every closed path C in D.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) can be shown using FTLI. It is thus left to show that (ii) $\Leftrightarrow$ (iii).

Suppose the line integral is independent of the path. Let C be any closed curve in the domain. This curve can be divided into two subcurves C_1 and C_2 such that $C = C_1 \cup C_2$. Note that C_1 and C_2 will have the same initial and terminal points, interchanged. So C_1 and $-C_2$ will have the same initial and terminal points. Since the line integral is independent of the path, and using a property of line integrals, we obtain

$$\int_{C_1} \vec{F} \cdot d\vec{R} = \int_{-C_2} \vec{F} \cdot d\vec{R} = -\int_{C_2} \vec{F} \cdot d\vec{R}.$$

Therefore,

$$\int_{C} \vec{F} \cdot d\vec{R} = \int_{C_1} \vec{F} \cdot d\vec{R} + \int_{C_2} \vec{F} \cdot d\vec{R} = 0.$$

Conversely, let $\displaystyle\int_C \vec{F} \cdot d\vec{R} = 0$ for any closed curve C in the domain. Let C_1 and C_2 be any two paths having the same initial and terminal points. Then C_1 and $-C_2$ will have the same initial and terminal points, interchanged. Thus, $C = C_1 \cup -C_2$ is closed. We have

$$\begin{aligned} 0 = \int_C \vec{F} \cdot d\vec{R} &= \int_{C_1} \vec{F} \cdot d\vec{R} + \int_{-C_2} \vec{F} \cdot d\vec{R} \\ &= \int_{C_1} \vec{F} \cdot d\vec{R} - \int_{C_2} \vec{F} \cdot d\vec{R} \end{aligned}$$

Therefore, $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{R} = \int_{C_2} \vec{F} \cdot d\vec{R}$. Hence, the line integral is independent of the path. □

Remark. If $\vec{F}$ is conservative, the FTLI and the independence of the path of the line integral involved suggest two alternate methods to compute the line integral $\int_C \vec{F} \cdot d\vec{R}$.

1. Find a potential function for $\vec{F}$ and use the Fundamental Theorem for Line Integrals.
2. Use the definition of the line integral, considering any C (preferably one that will make computations easier) from the initial to the terminal point.

Example 3.6.4. Show that $\displaystyle\int_C \vec{F} \cdot d\vec{R}$, where $\vec{F}(x, y) = \langle 2xy^3\,,\, 1 + 3x^2y^2 \rangle$ is independent of the path, and evaluate the line integral from $A(2, 1)$ to $B(3, 1)$.

Solution: Observe that

$$\frac{\partial}{\partial x}\left(1 + 3x^2y^2\right) = 6xy^2 = \frac{\partial}{\partial y}\left(2xy^3\right).$$

Hence, by Corollary 3.3.8, $\vec{F}$ is conservative and consequently, by Theorem 3.6.4, the line integral is independent of the path. We evaluate this line integral in two ways:

1. Since $\vec{F}$ is conservative, it is possible to use FTLI. A potential function for $\vec{F}$ is $\phi(x, y) = y + x^2y^3$. (Verify!) Hence,

$$\int_C \vec{F} \cdot d\vec{R} = \phi(3, 1) - \phi(2, 1) = (1 + 9) - (1 + 4) = 5.$$

2. Assuming a potential is hard to find, what we can do is to furnish a simple parametrization of C, for instance, $\vec{R}(t) = \langle t\,,\, 1\rangle,\ t \in [2, 3]$. Therefore, introducing this parametrization to the line integral yields

$$\int_C \vec{F} \cdot d\vec{R} = \int_2^3 \langle 2t\,,\, 1 + 3t^2\rangle \cdot \langle 1\,,\, 0\rangle\, dt = \int_2^3 2t\, dt = t^2\Big|_2^3 = 9 - 4 = 5.$$

EXERCISES 3.6

I. Given a force field $\vec{F}(x,y) = \langle y, x\rangle$ that moves a particle from the origin to the point $(4,2)$. Compare the work done on the particle by the force field given:

1. C is the line segment from the origin to $(4,2)$.
2. C is the portion of the parabola $x = y^2$ from the origin to $(4,2)$.

II. Given a force field $\vec{F}(x,y) = (y^2 + 2x + 4)\hat{\imath} + (2xy + 4y - 5)\hat{\jmath}$ that moves a particle from the origin to the point $(1,1)$. Compare the work done on the particle by the force field given:

1. C is the line segment from the origin to $(1,1)$.
2. C is the portion of the parabola $y = x^2$ from the origin to $(1,1)$.
3. C is the portion of the graph of $y = x^3$ from the origin to $(1,1)$.

III. Evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{R}$ if $\vec{F}(x,y) = (e^{-y} - 2x)\,\hat{\imath} - (xe^{-y} + \sin y)\,\hat{\jmath}$ and C is the first quadrant arc of the circle $\vec{R}(t) = \pi\cos t\hat{\imath} + \pi\sin t\hat{\jmath}$, $t \in [0, \pi/2]$ using

1. the definition of the line integral, and
2. the Fundamental Theorem of Line Integrals.

IV. Evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{R}$, where $\vec{F}(x,y) = \langle 2xy^3 + e^x, 3x^2y^2 + 3y^2\rangle$ and C is the curve described by $\vec{R}(t) = \langle -t^2, 1 - t^3\rangle$, $t \in [0,1]$.

V. Show that the following line integrals are independent of the path, and evaluate the line integral from point A to point B.

1. $\displaystyle\int_C 2xe^y\,dx + x^2e^y\,dy$; $A(1,0)$ and $B(3,2)$
2. $\displaystyle\int_C 2x\sin y\,dx + \left(x^2\cos y - 3y^2\right)\,dy$; $A(-1,0)$ and $B(5,1)$
3. $\displaystyle\int_C \vec{F}\cdot d\vec{R}$, where $\vec{F}(x,y) = \frac{1}{y}\hat{\imath} - \frac{x}{y^2}\hat{\jmath}$; $A(5,-1)$ and $B(9,-3)$
4. $\displaystyle\int_C (4x + 2y - z)\,dx + (2x - 2y + z)\,dy + (-x + y + 2z)\,dz$; $A(4,-2,1)$ and $B(-1,2,0)$
5. $\displaystyle\int_C \vec{F}\cdot d\vec{R}$, where $\vec{F}(x,y,z) = \left(\frac{1}{y} - \frac{2z}{x^2}\right)\hat{\imath} - \left(\frac{1}{z} + \frac{x}{y^2}\right)\hat{\jmath} + \left(\frac{2}{x} + \frac{y}{z^2}\right)\hat{k}$; $A(2,-1,1)$ and $B(4,2,-2)$
6. $\displaystyle\int_C \vec{F}\cdot d\vec{R}$, where $\vec{F}(x,y) = \left(\frac{y}{x} + \sin y\right)\hat{\imath} + (\ln x + x\cos y - 4y)\hat{\jmath}$; $A(1,\pi)$ and $B(e,0)$
7. $\displaystyle\int_C \vec{F}\cdot d\vec{R}$, where $\vec{F}(x,y,z) = 2ye^{2x}\hat{\imath} + (e^{2x} + 2y)\hat{\jmath} + 3z^2\hat{k}$; $A(\ln 2,1,1)$ and $B(\ln 2,2,2)$
8. $\displaystyle\int_C \vec{F}\cdot d\vec{R}$, where $\vec{F}(x,y) = \left(3x^2y + \frac{1}{xe^y}\right)\hat{\imath} + \left(x^3 - \frac{\ln x}{e^y} + 2y\right)\hat{\jmath}$; $A(e,0)$ and $B(1,1)$

9. $\displaystyle\int_C \vec{F}\cdot d\vec{R}$, where $\vec{F}(x,y) = \left\langle e^x \ln y - y\sin x, \dfrac{e^x}{y} + \cos x + 1\right\rangle$; $A(\frac{\pi}{2},1)$ and $B(0,e)$

VI. Do as indicated.

1. A particle moves on the circle $\vec{R}(t) = 2\cos t\hat{\imath} + 2\sin t\hat{\jmath}$, $t\in[0,2\pi]$. Find the total work done if the motion is caused by the force field $\vec{F}(x,y) = \left(\dfrac{xe^{2y}}{x^2+2}\right)\hat{\imath} + e^{2y}\ln\left(x^2+2\right)\hat{\jmath}$.
2. Find the work done by $\vec{F}(x,y,z) = \langle 2x\cos y, e^z - x^2\sin y, ye^z\rangle$ on a particle that moves on any smooth curve from the point $(0,1,0)$ to the point $(2,0,3)$.
3. Find the work done by $\vec{F}(x,y,z) = 3x^2\hat{\imath} - z\cos y\hat{\jmath} - \sin y\hat{k}$ on a particle moving along a directed path from $(2,0,3)$ to $(-1,\pi,0)$
4. Find the amount of work done by $\vec{F}(x,y) = \langle 2x - y\sin x, \cos x - e^y\rangle$ in moving a particle along the curve C defined by $\vec{R}(t) = \left\langle \dfrac{\pi t^2}{4}, t\right\rangle$ where $0\le t\le 2$.
5. Find the work done by $\vec{F}(x,y) = \langle e^{-y}\cos x, -e^{-y}\sin x - 1\rangle$ on a particle that moves along the part of the curve $y = \dfrac{1}{2}(5^{\cos x} + \sin x)$ from $(\frac{\pi}{2},1)$ to $(-\frac{\pi}{2},0)$.

3.7 Green's Theorem

In this section, we will see that a line integral along a simple closed plane curve C is related to an ordinary double integral over the plane region R with boundary C. The boundary curve C is said to be *positively oriented* if it is equipped with a parametrization $\vec{R}(t)$ such that the region R remains on the left as $\vec{R}(t)$ traces the curve C. The symbol

$$\oint_C P dx + Q dy$$

then denotes a line integral around C with positive orientation. The following result was published by George Green in 1828.

Theorem 3.7.1 (Green's Theorem)**.** Let C be a positively oriented piecewise-smooth simple closed curve that bounds the region R in the xy-plane. Suppose that P and Q are scalar fields of x and y with continuous first-order partial derivatives on R. Then

$$\oint_C P dx + Q dy = \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Proof. The complete proof of Green's Theorem for general regions is beyond the scope of the course but we can provide a proof for the case in which the region R is of both type I and type II. Recall that if R is of type I, then it has a description of the form $g_1(x)\le y\le g_2(x)$, $a\le x\le b$.

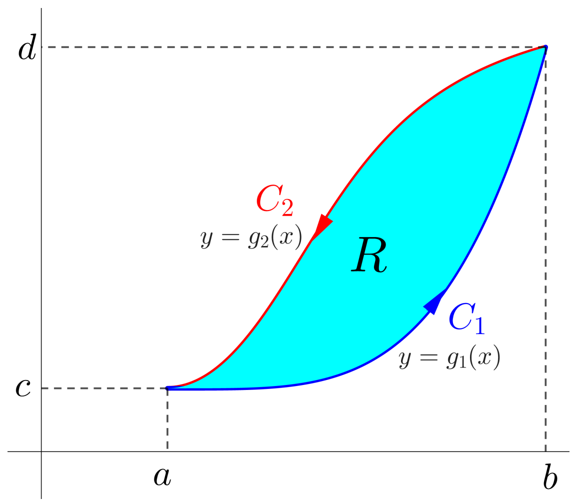


Figure 3.13: Region R bounded by the positively oriented curve $C = C_1 \cup C_2$

Note that C_1 and $-C_2$ can be parametrized as

$$C_1 : \vec{R}(t) = \langle t\,,\ g_1(t)\rangle,\ \ t \in [a,b],$$

$$-C_2 : \vec{R}(t) = \langle t\,,\ g_2(t)\rangle,\ \ t \in [a,b].$$

So,

$$\begin{aligned}
\oint_C P\,dx &= \int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx \\
&= \int_{C_1} P(x,y)\,dx - \int_{-C_2} P(x,y)\,dx \\
&= \int_a^b P(t,g_1(t))\,dt - \int_a^b P(t,g_2(t))\,dt \\
&= \int_a^b \left(P(t,g_1(t)) - P(t,g_2(t))\right)\,dt \\
&= \int_a^b \left(P(x,g_1(x)) - P(x,g_2(x))\right)\,dx.
\end{aligned}$$

Meanwhile,

$$\iint_R \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx = \int_a^b \left(P(x,g_2(x)) - P(x,g_1(x))\right)dx.$$

Thus,

$$\oint_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dA. \tag{3.17}$$

In a similar manner, if R has description $h_1(y) \le x \le h_2(y)$, $c \le y \le d$, it can be shown that

$$\oint_C Q\,dy = \iint_R \frac{\partial Q}{\partial y}\,dA. \tag{3.18}$$

By adding the left-hand sides and right-hand sides of equations (3.17) and (3.18), we get the desired result. □

Example 3.7.1. Use Green's Theorem to evaluate $\oint_C \left(e^{x^2} - 2y\right) dx + \left(4x + e^{2y}\right) dy$, where C is the circle $x^2 + y^2 = 1$ traced counterclockwise.

Solution: With $P(x,y) = e^{x^2} - 2y$ and $Q(x,y) = 4x + e^{2y}$, we see that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4 - (-2) = 6.$$

Let R be the circular disk bounded by $x^2 + y^2 = 1$. It is obvious that P and Q have continuous first-order partial derivatives on R. By Green's Theorem, we have

$$\oint_C \left(e^{x^2} - 2y\right) dx + \left(4x + e^{2y}\right) dy = \iint_R 6\, dA = 6A_R = 6\pi.$$

Example 3.7.2. Suppose $C = C_1 \cup C_2$, where C_1 is the portion of $y = x^2$ from $(0,0)$ to $(1,1)$ and C_2 is the portion of $x = y^2$ from $(1,1)$ to $(0,0)$. Compute the work done by $\vec{F}(x,y) = \langle \sin(\cos x) - y^2, \tan^{-1} y \rangle$ in moving a particle once along C.

Solution: Note that the work done is given by $W = \oint_C \vec{F} \cdot d\vec{R}$. To compute the line integral directly, we would need to parametrize C_1 and C_2 separately. Instead, we apply Green's Theorem with $P(x,y) = \sin(\cos x) - y^2$ and $Q(x,y) = \tan^{-1} y$. It can be verified that P and Q have continuous first-order partial derivatives on the region R enclosed by the positively oriented piecewise-smooth simple closed curve C as shown below.

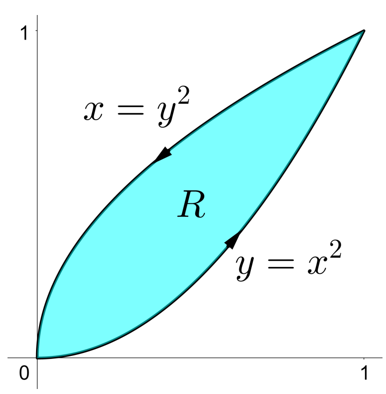


Figure 3.14: Region R bounded by $y = x^2$ and $x = y^2$

Moreover,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - (-2y) = 2y.$$

Therefore,

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{R} &= \iint\limits_R 2y\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 2y\,dy\,dx \\ &= \int_0^1 y^2\Big|_{y=x^2}^{y=\sqrt{x}}\,dx = \int_0^1 \left(x - x^4\right)\,dx \\ &= \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}.\end{aligned}$$

Remark. Green's Theorem can be extended to regions with boundaries that consist of two or more simple closed curves.

Consider the annular region R in the figure on the left with boundary C consisting of two simple closed curves, with one inside the other. The positive direction of C − the direction for which R always lies on the left − is counterclockwise traversal on the outer curve and clockwise traversal on the inner curve.

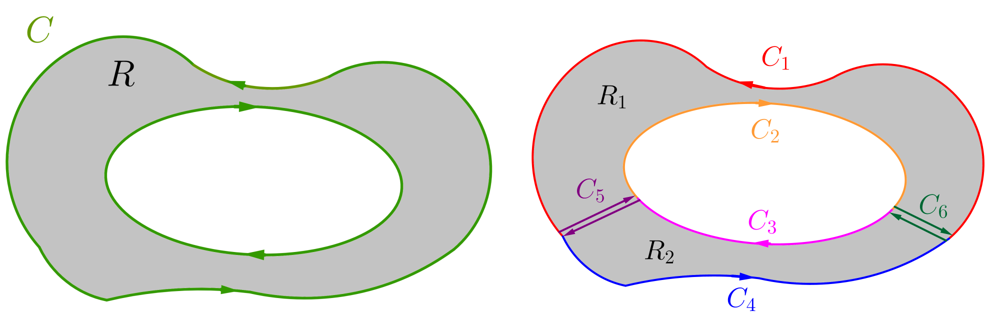


Figure 3.15: Green's Theorem for annular regions

To prove the remark, we divide R into two regions R_1 and R_2 by using the crosscuts $\pm C_5$ and $\pm C_6$, as shown on the right part of the figure above. Then R_1 is enclosed by C_1, C_5, C_2, and C_6 while R_2 is enclosed by C_4, $-C_6$, C_3, and $-C_5$. Applying Green's Theorem in each of these subregions, we get

$$\begin{aligned}\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint\limits_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint\limits_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \int_{C_1} Pdx + Qdy + \int_{C_5} Pdx + Qdy + \int_{C_2} Pdx + Qdy + \int_{C_6} Pdx + Qdy \\ &\quad + \int_{C_3} Pdx + Qdy + \int_{-C_5} Pdx + Qdy + \int_{C_4} Pdx + Qdy + \int_{-C_6} Pdx + Qdy \\ &= \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy + \int_{C_3} Pdx + Qdy + \int_{C_4} Pdx + Qdy \\ &= \int_C Pdx + Qdy.\end{aligned}$$

Example 3.7.3. Let $C = C_1 \cup C_2$, where C_1 is the circle $x^2 + y^2 = 2$ traced clockwise and C_2 is the circle $x^2 + y^2 = 4$ traced counterclockwise. Evaluate $\displaystyle\oint_C \sqrt{x^2+y^2}\,dx + \sqrt{x^2+y^2}\,dy$.

Solution: Let R be the region enclosed by C. Observe that C is positively oriented. Moreover, the functions

$$P(x,y) = Q(x,y) = \sqrt{x^2+y^2}$$

have continuous first order-partial derivatives in R. In fact, the only point of discontinuity of the partial derivatives is $(0,0) \notin R$. Now,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{x-y}{\sqrt{x^2+y^2}}.$$

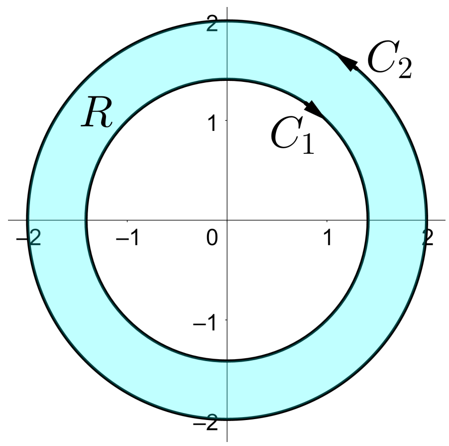


Figure 3.16: Region inside $x^2 + y^2 = 4$ and outside $x^2 + y^2 = 2$

Hence by Green's Theorem,

$$\begin{aligned}
\oint_C \sqrt{x^2+y^2}\,dx + \sqrt{x^2+y^2}\,dy &= \iint_R \frac{x-y}{\sqrt{x^2+y^2}}\,dA \\
&= \int_0^{2\pi}\int_{\sqrt{2}}^{2} (\cos\theta - \sin\theta) r\,dr\,d\theta \\
&= \int_0^{2\pi} (\cos\theta - \sin\theta) \frac{r^2}{2}\bigg|_{r=\sqrt{2}}^{r=2}\,d\theta \\
&= \int_0^{2\pi} (\cos\theta - \sin\theta)\,d\theta \\
&= (\sin\theta + \cos\theta)\bigg|_0^{2\pi} = 0.
\end{aligned}$$

Example 3.7.4. Let $\vec{F}(x,y) = \left\langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$.

1. Evaluate $\displaystyle\oint_{C_a} \vec{F} \cdot d\vec{R}$, where C_a is the circle centered at the origin of radius $a > 0$ traced counterclockwise.

2. Use Green's Theorem to evaluate $\oint_C \vec{F} \cdot d\vec{R}$ for any positively oriented, piecewise smooth, and simple curve C that encloses the origin.

Solution:

1. Observe that $\vec{F}$ is not continuous at $(0,0)$ and so are its first-order partial derivatives. Since C_a encloses the point $(0,0)$, Green's Theorem is not applicable. To evaluate the integral, we use the parametrization $C_a : \vec{R}(t) = \langle a\cos t, a\sin t\rangle$, $t \in [0, 2\pi]$. So, $\vec{R}'(t) = \langle -a\sin t, a\cos t\rangle$. We then have

$$\begin{aligned}\oint_{C_a} \vec{F} \cdot d\vec{R} &= \int_0^{2\pi} \left\langle -\frac{a\sin t}{a^2}, \frac{a\cos t}{a^2}\right\rangle \cdot \langle -a\sin t, a\cos t\rangle\, dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

2. As in the preceding example, Green's Theorem is not applicable on the region bounded by C. Let a be sufficiently small so that C_a lies inside C. Denote by $-C_a$ the circle C_a traced clockwise and consider $C_o = C \cup -C_a$. Then

$$\oint_{C_o} \vec{F} \cdot d\vec{R} = \oint_C \vec{F} \cdot d\vec{R} + \oint_{-C_a} \vec{F} \cdot d\vec{R} = \oint_C \vec{F} \cdot d\vec{R} - \oint_{C_a} \vec{F} \cdot d\vec{R}.$$

Now, let R be the region enclosed by C_o. With $P_y(x,y) = Q_x(x,y) = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$, by Green's Theorem

$$\oint_{C_o} \vec{F} \cdot d\vec{R} = \iint\limits_R 0\, dA = 0.$$

Therefore,

$$\oint_C \vec{F} \cdot d\vec{R} = \oint_{C_a} \vec{F} \cdot d\vec{R} = 2\pi.$$

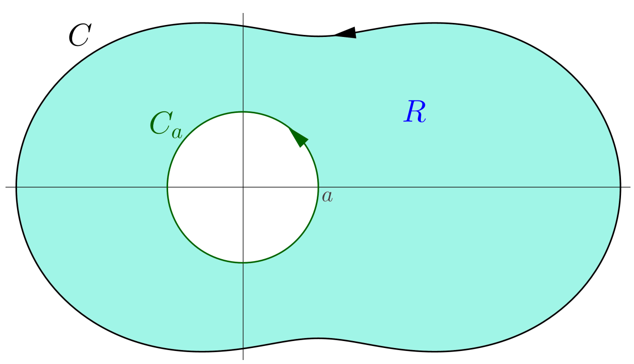


Figure 3.17

Line integrals may also be used to compute areas of closed and bounded regions, as stated in the following corollary.

Corollary 3.7.2. The area of the region R enclosed by a positively oriented piecewise-smooth simple closed curve C is given by

$$A = \frac{1}{2}\oint_C -ydx + xdy = -\oint_C ydx = \oint_C xdy.$$

Proof. With $P(x,y) = -y$ and $Q(x,y) = 0$, Green's Theorem gives

$$-\oint_C ydx = \iint\limits_R 1dA = A.$$

Similarly, With $P(x,y) = 0$ and $Q(x,y) = x$, Green's Theorem gives

$$\oint_C xdy = \iint\limits_R 1dA = A.$$

The third result may be obtained by averaging the right-hand sides of the last two equations. □

Example 3.7.5. Find the area of the region enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution: With parametrization $x = a\cos t$, $y = b\sin t$, $t \in [0, 2\pi]$, the first equation in Corollary 3.7.6 gives

$$A = \frac{1}{2}\int_0^{2\pi} -(b\sin t)(-a\sin t)dt + (a\cos t)(b\cos t)dt = \frac{1}{2}ab\int_0^{2\pi} dt = \pi ab.$$

EXERCISES 3.7

I. Apply Green's Theorem to evaluate $\oint_C Pdx + Qdy$ along the specified positively oriented closed curve C.

1. $P(x,y) = y^2 - \sin^{-1} x^2$, $Q(x,y) = 2xy + 4x$; C is the parallelogram with vertices at $(0,0)$, $(4,0)$, $(1,3)$ and $(5,3)$.
2. $P(x,y) = y + e^x$, $Q(x,y) = 2x^2 + \cos y$; C is the triangle with vertices at $(0,0)$, $(4,0)$ and $(6,4)$.
3. $P(x,y) = x - y^2$, $Q(x,y) = 2xy$; C is the boundary of the region between the x-axis and the graph of $y = \sin x$ for $0 \le x \le \pi$.
4. $P(x,y) = 2xy - 3y^2$, $Q(x,y) = x^2 + y^2$; C is the circle $x^2 + y^2 - 4x = 0$.

5. $P(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$, $Q(x,y) = -\dfrac{x}{\sqrt{x^2+y^2}}$; C is the boundary of the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.

II. Use a line integral to determine the area of the region enclosed by the astroid with parametric equations $x=\cos^3 t$, $y=\sin^3 t$, $0\le t\le 2\pi$.

III. Use Green's Theorem to do the following. Assume C is positively oriented unless otherwise stated.

1. Consider the vector field $\vec{G}(x,y) = (10^{10^x} - 2y)\hat{\imath} + (3x+10y)\hat{\jmath}$. Evaluate $\oint_C \vec{G}\cdot d\vec{R}$, where C is the parallelogram with vertices $(0,-2)$, $(4,0)$, $(4,2)$ and $(0,0)$.

2. Let $\vec{F}(x,y) = \langle \cos^2 x + \sin y, x\cos y + 2x\rangle$ and let C be the boundary of the region in the first quadrant enclosed by $x^2+y^2=1$ and the coordinate axes, oriented positively. Evaluate $\oint_C \vec{F}\cdot d\vec{R}$.

3. Evaluate the line integral $\oint_C (x^2+y^2)\,dx + (x^2y)\,dy$, where C is the closed curve determined by $x=y^2$ and $y=-x$.

4. Find the work done by the force field $\vec{F}(x,y) = \langle e^y - 3y^2, xe^y + 6xy\rangle$ in moving a particle along the line segment $(-1,-1)$ to $(2,2)$ then along the portion of the parabola $x=y^2-2$ from $(2,2)$ to $(-1,-1)$.

5. Evaluate the line integral $\oint_C (x^2+y^2)\,dx + xy^2\,dy$ where C is the closed curve determined by $y^2=x$ and $y=-x$.

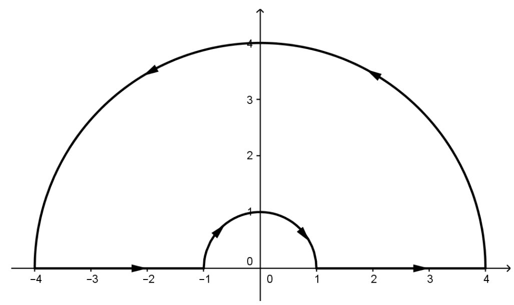
6. Evaluate the line integral $\oint_C y^2\,dx + x^2\,dy$, where C is the closed curve determined by the x-axis, the line $x=1$, and the curve $y=x^2$ traversed in the counterclockwise direction.

7. Evaluate $\oint_C (x^2+xy)\,dx + xy^2\,dy$, where C is the closed path starting from the origin along the x-axis to $(1,0)$, then along the line segment to $(0,1)$, and then back to the origin along the y-axis.

8. Evaluate $\oint_C \left(\tan^{-1}(x^2) - y^2\right)dx + (4xy + \sinh(e^y))\,dy$ on the positively oriented boundary of the region enclosed by $x=4-y^2$, $y=x-4$, and $y=2$.

9. Let $\vec{F}(x,y) = \langle 2xy, xy+x^2\rangle$. Evaluate $\oint_C \vec{F}\cdot d\vec{R}$, where C is the triangular path traced in the counterclockwise direction with vertices at the points $(0,0)$, $(1,0)$ and $(2,1)$.

10. Evaluate $\oint_C \left(xy^3 + \dfrac{y^2}{2} + e^{x^3}\right)dx + \left(\dfrac{3x^2y^2}{2} + xy + x^2\right)dy$ where C is given below:



3.8 Surface Integrals of Scalar Fields

Recall that in Section 2.4, we showed that the mass of a lamina that follows the shape of a region R and whose density is the function $f(x, y)$ is

$$M = \iint\limits_R f(x, y)\, dA.$$

In this section, we will define the integral of a scalar field of three variables over a surface. To motivate the definition, we will consider the problem of finding the mass of a curved or bent lamina with a given density function.

(Mass Problem) Let $\mathcal{S}$ be a smooth curved lamina (a surface) that is parametrized by the vector function

$$\vec{R}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle, \ (u, v) \in D,$$

and whose density at a point $(x, y, z) \in \mathcal{S}$ is $f(x, y, z)$. What is the mass of $\mathcal{S}$?

The Method

1. Partition $\mathcal{S}$ into n small patches $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$, with areas $\Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_n$, respectively.

2. For each $\mathcal{S}_i$, choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$.

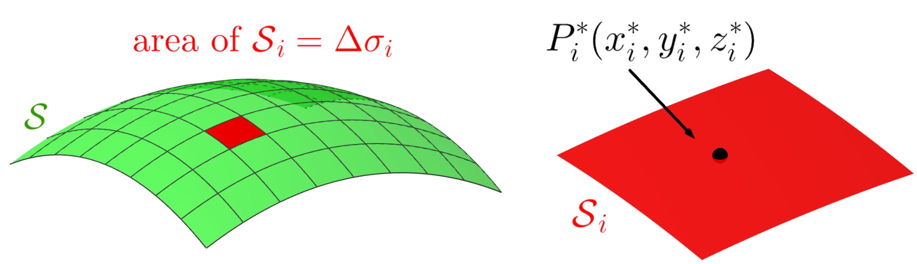


Figure 3.18: Approximating a surface using flat laminas

3. Assume $\mathcal{S}_i$ has constant density $f(P_i^*)$. The mass of each patch is approximately $f(P_i^*)\Delta\sigma_i$ and therefore the mass of $\mathcal{S}$ is approximately
$$\sum_{i=1}^{n} f(P_i^*)\Delta\sigma_i.$$

4. We will use the expression $n \to \infty$ to mean increasing n so that the maximum dimension of each patch approaches 0. Thus,
$$M = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\Delta\sigma_i.$$
If this limit exists, we denote it by $\displaystyle\iint_{\mathcal{S}} f(x,y,z)\,d\sigma$.

Definition 3.8.1. Let f be a scalar field of x, y and z that is continuous on a smooth surface $\mathcal{S}$ parametrized by the vector function $\vec{R}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$. The ***surface integral of*** f ***over*** $\mathcal{S}$ is defined by
$$\iint_{\mathcal{S}} f(x,y,z)\,d\sigma = \lim_{n\to+\infty} \sum_{i=1}^{n} f(P_i^*)\Delta\sigma_i,$$
if this limit exists.

Recall in Section 2.4 that the area of a patch is approximately
$$\Delta\sigma_i \approx \|\vec{R}_u(u_i,v_i) \times \vec{R}_v(u_i,v_i)\|\Delta u_i \Delta v_i.$$

Thus, we have the following formula to evaluate a surface integral.

Remark. If $\mathcal{S}$ is parametrized by $\vec{R}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$, then
$$\iint_{\mathcal{S}} f(x,y,z)\,d\sigma = \iint_{D} f(\vec{R}(u,v))\,\|\vec{R}_u(u,v) \times \vec{R}_v(u,v)\|\,dA.$$

Example 3.8.1. Evaluate the surface integral $\displaystyle\iint_{\mathcal{S}} (x^2+y^2)\,d\sigma$, where $\mathcal{S}$ is the surface defined by $\vec{R}(u,v) = \langle u, v, u-v\rangle$, $u^2+v^2 \le 4$.

Solution: Begin by computing the partial derivatives of $\vec{R}$.
$$\begin{aligned}\vec{R}_u(u,v) &= \langle 1,0,1\rangle\\ \vec{R}_v(u,v) &= \langle 0,1,-1\rangle.\end{aligned}$$

The cross product is $\vec{R}_u(u,v) \times \vec{R}_v(u,v) = \langle -1, 1, 1\rangle$, which implies that $\|\vec{R}_u(u,v) \times \vec{R}_v(u,v)\| = \sqrt{3}$. The domain is given below.

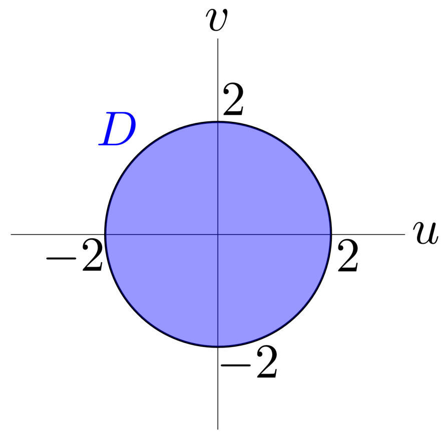


Figure 3.19: Domain in the uv-plane satisfying $u^2 + v^2 \leq 4$

Hence, we have

$$\begin{aligned}
\iint_S (x^2+y^2)\,d\sigma &= \iint_D (u^2+v^2)\,\sqrt{3}\,dA \\
&= \sqrt{3}\int_0^{2\pi}\int_0^2 r^2 \cdot r dr\,d\theta \\
&= \sqrt{3}\int_0^{2\pi}\left(\frac{r^4}{4}\right)\Big|_{r=0}^{r=2} d\theta \\
&= \sqrt{3}\int_0^{2\pi} 4\,d\theta \\
&= 8\sqrt{3}\,\pi.
\end{aligned}$$

Example 3.8.2. Find the mass of the torus parametrized by

$$\vec{R}(u,v) = (2+\cos u)\cos v\,\hat{\imath} + (2+\cos u)\sin v\,\hat{\jmath} + \sin u\,\hat{\mathrm{k}},$$

where $0 \leq u, v \leq 2\pi$, given that the density is $\delta(x,y,z) = x^2+y^2+z^2$.

Solution: First, the density of the surface in terms of the parameters is

$$\begin{aligned}
\delta(\vec{R}(u,v)) &= (2+\cos u)^2\cos^2 v + (2+\cos u)^2 \sin^2 v + \sin^2 u \\
&= (2+\cos u)^2 + \sin^2 u = 4 + 4\cos u + \cos^2 u + \sin^2 u \\
&= 5 + 4\cos u.
\end{aligned}$$

Now, the partial derivatives of $\vec{R}$ are

$$\begin{aligned}
\vec{R}_u(u,v) &= -\sin u\cos v\,\hat{\imath} - \sin u \sin v\,\hat{\jmath} + \cos u\,\hat{\mathrm{k}} \\
\vec{R}_v(u,v) &= -(2+\cos u)\sin v\,\hat{\imath} + (2+\cos u)\cos v\,\hat{\jmath}\,.
\end{aligned}$$

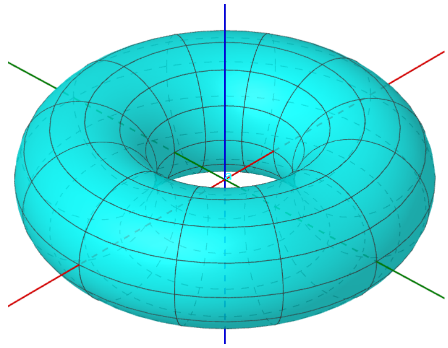


Figure 3.20: Torus with major radius 2 and minor radius 1

The cross product of these vectors is

$$\vec{R}_u(u,v) \times \vec{R}_v(u,v) = -(2+\cos u)(\cos u \cos v\,\hat{\imath} + \cos u \sin v\,\hat{\jmath} + \sin u\,\hat{\mathrm{k}})$$

which implies that

$$\begin{aligned}
\|\vec{R}_u(u,v) \times \vec{R}_v(u,v)\| &= (2+\cos u)\sqrt{(\cos u \cos v)^2 + (\cos u \sin v)^2 + \sin^2 u} \\
&= (2+\cos u)\sqrt{\cos^2 u(\cos^2 v + \sin^2 v) + \sin^2 u} \\
&= (2+\cos u)\sqrt{\cos^2 u + \sin^2 u} \\
&= 2+\cos u.
\end{aligned}$$

Hence, the mass of the torus is

$$\begin{aligned}
M &= \iint\limits_D \delta(\vec{R}(u,v))\,\|\vec{R}_u(u,v) \times \vec{R}_v(u,v)\|\,dA \\
&= \int_0^{2\pi}\int_0^{2\pi} (5+4\cos u)(2+\cos u)\,du\,dv \\
&= \int_0^{2\pi}\int_0^{2\pi} (10+13\cos u+4\cos^2 u)\,du\,dv \\
&= \int_0^{2\pi}\int_0^{2\pi} \left(10+13\cos u+4\left(\frac{1+\cos 2u}{2}\right)\right)\,du\,dv \\
&= \int_0^{2\pi}\int_0^{2\pi} (12+13\cos u+2\cos 2u)\,du\,dv \\
&= \int_0^{2\pi} (12u+13\sin u+\sin 2u)\Big|_{u=0}^{u=2\pi}\,dv \\
&= \int_0^{2\pi} 24\pi\,dv \\
&= 48\pi^2.
\end{aligned}$$

Suppose now that a smooth surface $\mathcal{S}$ is defined by a Cartesian equation $z = g(x,y)$. Then a natural parametrization for $\mathcal{S}$ would be

$$\vec{R}(x,y) = x\,\hat{\imath} + y\,\hat{j} + g(x,y)\,\hat{k}\ .$$

With this representation, our formula will then lead to the following.

Remark. Let $\mathcal{S}$ be a surface with equation $z = g(x, y)$ and let R be its projection onto the xy-plane. If g, g_x and g_y are continuous on R and f is continuous on $\mathcal{S}$, then the surface integral of f over $\mathcal{S}$ is

$$\iint_{\mathcal{S}} f(x, y, z)\, d\sigma = \iint_{R} f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA.$$

Example 3.8.3. Let $f(x, y, z) = 2x - z$ and let $\mathcal{S}$ be the portion of the plane $2x + 2y + z = 4$ in the first octant. Evaluate $\displaystyle\iint_{\mathcal{S}} f(x, y, z)\, d\sigma$.

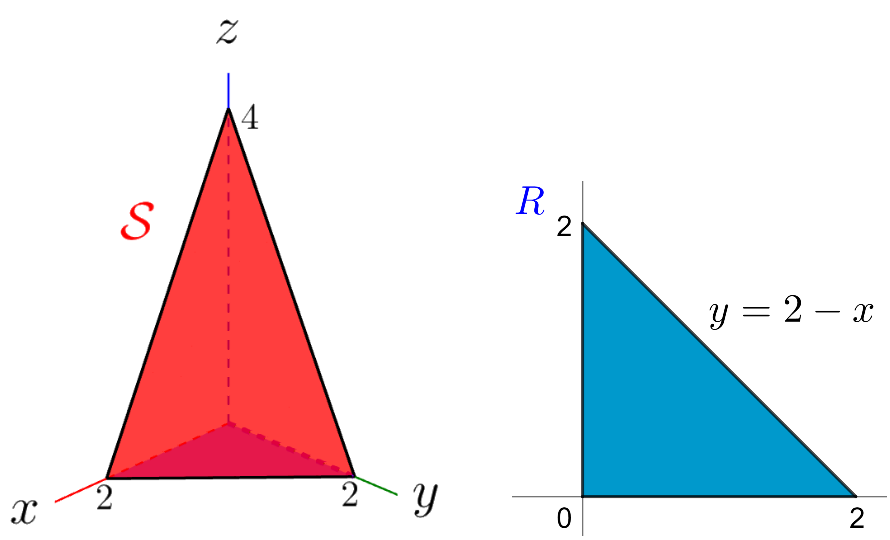


Figure 3.21: Portion of $2x + 2y + z = 4$ in the first octant, and its projection to the xy-plane

Solution: The surface S and its projection to the xy-plane is shown in Figure 3.21. Write the equation of the surface $\mathcal{S}$ as $z = g(x, y) := 4 - 2x - 2y$. Thus,

$$f(x, y, g(x, y)) = 2x - (4 - 2x - 2y) = 4x + 2y - 4$$

and

$$\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} = \sqrt{1 + (-2)^2 + (-2)^2} = 3.$$

Therefore,

$$
\begin{aligned}
\iint\limits_{\mathcal{S}} f(x,y,z)\,d\sigma &= 3\iint\limits_{R} (4x+2y-4)dA \\
&= 3\int_0^2\int_0^{2-x} (4x+2y-4)\,dy\,dx \\
&= 3\int_0^2 \left(4xy+y^2-4y\right)\Big|_{y=0}^{y=2-x}\,dx \\
&= 3\int_0^2 \left[4x(2-x)+(2-x)^2-4(2-x)\right]\,dx \\
&= 3\int_0^2 \left(-3x^2+8x-4\right)\,dx \\
&= 3\left(-x^3+4x^2-4x\right)\Big|_0^2 \\
&= 3\left[(-8+16-8)-0\right] = 0.
\end{aligned}
$$

Example 3.8.4. Suppose that the density of the paraboloid $z = x^2+y^2$ is given by $\delta(x,y,z) = \dfrac{1}{\sqrt{1+4z}}$. Find the mass of the portion of the paraboloid between the planes $z=1$ and $z=2$.

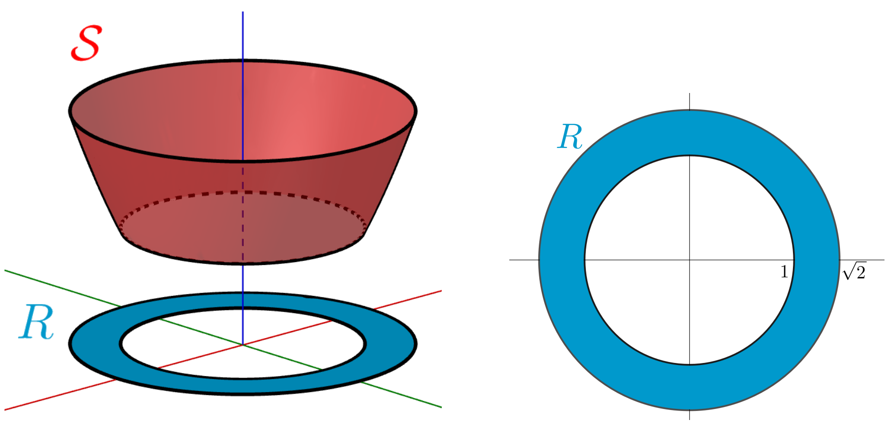


Figure 3.22: Portion of $z = x^2+y^2$ between $z=1$ and $z=2$, and its projection to the xy-plane

Solution: The surface S and its projection to the xy-plane is shown in Figure 3.22. Since $\mathcal{S}$ has equation $z = g(x,y) := x^2+y^2$, then $g_x(x,y) = 2x$ and $g_y(x,y) = 2y$. Moreover,

$$\delta(x,y,g(x,y)) = \frac{1}{\sqrt{1+4(x^2+y^2)}}$$

and

$$\sqrt{1+[g_x(x,y)]^2+[g_y(x,y)]^2} = \sqrt{1+(2x)^2+(2y)^2} = \sqrt{1+4x^2+4y^2}.$$

Thus,

$$\begin{aligned} M &= \iint\limits_{\mathcal{S}} \delta(x,y,z)\,d\sigma \\ &= \iint\limits_{R} \frac{1}{\sqrt{1+4(x^2+y^2)}}\sqrt{1+4x^2+4y^2}\,dA \\ &= \iint\limits_{R} dA = A_R = \pi(\sqrt{2}^2-1) = \pi. \end{aligned}$$

EXERCISES 3.8

I. Evaluate the surface integral $\displaystyle\iint\limits_{\mathcal{S}} f(x,y,z)\,d\sigma$.

1. $f(x,y,z) = x+z$; $\mathcal{S} : \vec{R}(u,v) = u\,\hat{\imath} + 3\cos v\,\hat{\jmath} + 3\sin v\,\hat{k}$, $0 \le u \le 4$, $0 \le v \le \frac{\pi}{2}$
2. $f(x,y,z) = xyz$; $\mathcal{S}$ is the portion of the plane $x+z=6$ between the planes $x=0$, $x=2$, $y=-1$ and $y=2$
3. $f(x,y,z) = z^2$; $\mathcal{S}$ is the portion of the sphere $x^2+y^2+z^2=4$ above $z=1$
4. $f(x,y,z) = \sqrt{x^2+y^2}$; $\mathcal{S} : \vec{R}(u,v) = u\cos v\hat{\imath} + u\sin v\hat{\jmath} + (v+3)\hat{k}$ where $0 \le u \le 1$, $0 \le v \le \pi$
5. $f(x,y,z) = 2x-3y+z$; $\mathcal{S} : \vec{R}(u,v) = \langle 2v, v-u, u\rangle$, $0 \le v \le 1$ and $0 \le v \le 2-2u$
6. $f(x,y,z) = \dfrac{4-z}{\sqrt{1+4x^2+4y^2}}$; $\mathcal{S}$ is the portion of the paraboloid $z = 4-x^2-y^2$ which lies above the xy-plane
7. $f(x,y,z) = \dfrac{x^2-x+z}{\sqrt{2+4y^2}}$; $\mathcal{S}$ is the part of the surface $z = y^2+x$ that lies inside the cylinder $x^2+y^2=1$
8. $f(x,y,z) = x$; $\mathcal{S}$ is the portion of the plane $2x+y+z=4$ in the first octant
9. $f(x,y,z) = \sqrt{1+2z}$; $\mathcal{S}$ is the portion of the surface $x^2+y^2=2z$ enclosed by the planes $x=0$, $y=1$ and $y=x$

II. Find the mass of the curved lamina in the shape of the surface $\mathcal{S}$.

1. $\mathcal{S}$ is the portion of the cylinder $z = 1-x^2$ above the rectangle $R = [0,\sqrt{2}] \times [0,2]$ with density $\delta(x,y,z) = 8xz$
2. $\mathcal{S}$ is the portion of the plane $z = 4+x+y$ that lies inside the cylinder $x^2+y^2=4$ with density $\delta(x,y,z) = x^2z+y^2z$
3. $\mathcal{S} : \vec{R}(u,v) = \left\langle u^2, \dfrac{v^2}{2}, uv \right\rangle$ where $1 \le u \le v$, $1 \le v \le 2$ with density $\delta(x,y,z) = \dfrac{z}{2x+2y}$

3.9 Surface Integrals of Vector Fields

One of the most important applications of surface integrals involves the computation of the flux of a vector field, oftentimes associated to fluid flow. To define the flux across a surface $\mathcal{S}$, we assume that $\mathcal{S}$ has a unit normal vector field that varies continuously from point to point of $\mathcal{S}$. Such a surface is said to be *orientable*. This condition excludes from our consideration one-sided (nonorientable) surfaces, such as a Möbius strip. An orientable surface $\mathcal{S}$ has two sides. So, when you orient a surface, you are selecting one of the two possible normal vectors.

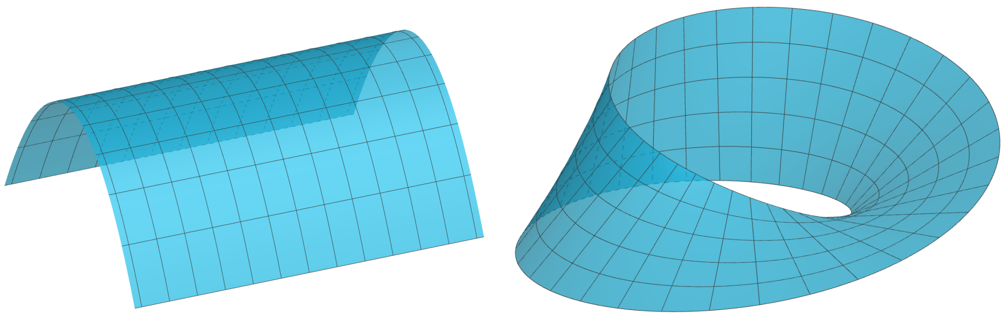


Figure 3.23: Examples of (from the left) an orientable surface (parabolic cylinder); a non-orientable surface (Möbius strip)

Orientation of a Parametric Surface

When a surface is expressed parametrically, the parametric equations create a natural orientation of the surface. In particular, if a smooth parametric surface $\mathcal{S}$ is defined by the vector function

$$\vec{R}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{\mathrm{k}}$$

then the unit normal vector

$$\vec{n} = \frac{\vec{R}_u \times \vec{R}_v}{\|\vec{R}_u \times \vec{R}_v\|} \tag{3.19}$$

is a continuous vector function of the parameters. Thus, Equation (3.19) defines an orientation of the surface. We call this the *positive orientation* of $\mathcal{S}$. The orientation given by $-\vec{n}$ is called the *negative orientation* of $\mathcal{S}$.

Example 3.9.1. Consider the unit sphere parametrized by

$$\vec{R}(u,v) = \sin u \cos v\,\hat{\imath} + \sin u \sin v\,\hat{\jmath} + \cos u\,\hat{\mathrm{k}}\,, u \in [0,\pi], v \in [0, 2\pi].$$

The partial derivatives are

$$\begin{aligned} \vec{R}_u(u,v) &= \cos u \cos v\,\hat{\imath} + \cos u \sin v\,\hat{\jmath} - \sin u\,\hat{\mathrm{k}} \\ \vec{R}_v(u,v) &= -\sin u \sin v\,\hat{\imath} + \sin u \cos v\,\hat{\jmath}\,. \end{aligned}$$

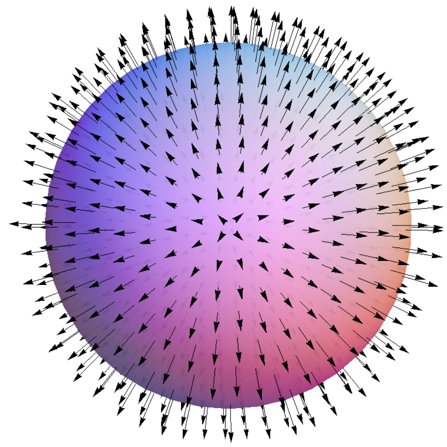


Figure 3.24: The unit sphere $\vec{R}(u, v)$ and its unit vectors

So,

$$\vec{R}_u(u, v) \times \vec{R}_v(u, v) = \sin^2 u \cos v\,\hat{\imath} + \sin^2 u \sin v\,\hat{\jmath} + \sin u \cos u\,\hat{\mathrm{k}}$$

and a simple computation gives

$$\|\vec{R}_u(u, v) \times \vec{R}_v(u, v)\| = |\sin u| = \sin u,$$

where $|\sin u| = \sin u$ since $u \in [0, \pi]$.
Hence, the normal vector that defines the positive orientation of the sphere is

$$\vec{n} = \frac{\vec{R}_u \times \vec{R}_v}{\|\vec{R}_u \times \vec{R}_v\|} = \sin u \cos v\,\hat{\imath} + \sin u \sin v\,\hat{\jmath} + \cos u\,\hat{\mathrm{k}}.$$

Notice that $\vec{n}$ has the same components as $\vec{R}$. This means that the positive orientation of the unit sphere is given by normal vectors pointing outward.

Orientation of a Cartesian-defined Surface

A surface $\mathcal{S}$ with Cartesian equation $z = g(x, y)$ can be expressed parametrically by

$$\vec{R}(x, y) = x\,\hat{\imath} + y\,\hat{\jmath} + g(x, y)\,\hat{\mathrm{k}}.$$

With this parametrization and in view of (3.19), a unit normal vector to $\mathcal{S}$ at any point is given by

$$\vec{n} = \frac{-g_x(x, y)\,\hat{\imath} - g_y(x, y)\,\hat{\jmath} + \hat{\mathrm{k}}}{\sqrt{[g_x(x, y)]^2 + [g_y(x, y)]^2 + 1}} \tag{3.20}$$

which points upward since the third component is positive. This unit normal vector field defines the *positive orientation* of $\mathcal{S}$.

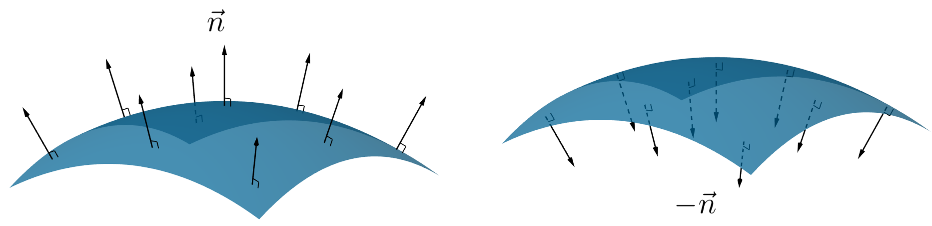


Figure 3.25: Positive (left) and Negative (right) orientation of $z = g(x, y)$

Flux of a Vector Field

(Flux Problem) Suppose a smooth surface $\mathcal{S}$ oriented by the unit normal vector field $\vec{n}$ is submerged in a fluid having continuous velocity field $\vec{F}$. What is the amount of fluid crossing the surface $\mathcal{S}$ per unit of time?

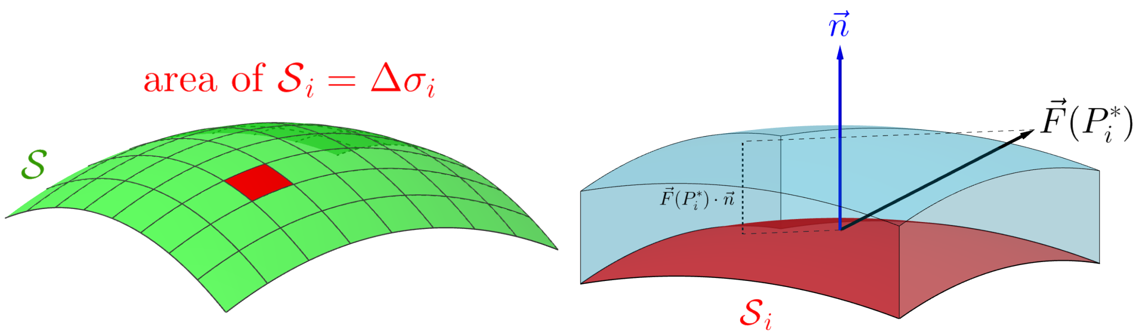


Figure 3.26: Flux per patch $\mathcal{S}_i$

The Method

1. Partition $\mathcal{S}$ into n patches $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$, with respective areas $\Delta\sigma_i$.

2. If the patches are small and the flow is not too chaotic (e.g., when $\vec{F}$ is continuous), we can assume that each patch has constant velocity $\vec{F}(P_i^*)$ for some fixed point $P_i^*(x_i^*, y_i^*, z_i^*)$ on $\mathcal{S}_i$.

3. The amount of fluid crossing the patch $\mathcal{S}_i$ per unit of time is approximated by the volume of the solid as shown in Figure 3.26. That is,
$$\Delta V_i \approx \left(\vec{F}(P_i^*) \cdot \vec{n}\right) \Delta\sigma_i.$$

4. If we now increase n in such a way that the maximum dimension of each patch goes to zero (which we denote by $n \to \infty$), then the error in the above approximation vanishes. Thus,
$$\text{flux } = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\vec{F}(P_i^*) \cdot \vec{n}\right) \Delta\sigma_i.$$
If this limit exists, we denote it by $\displaystyle\iint_{\mathcal{S}} \vec{F}(x, y, z) \cdot \vec{n}\, d\sigma$.

Definition 3.9.1. Let $\mathcal{S}$ be a smooth surface. If the components of the vector field $\vec{F}$ are continuous on $\mathcal{S}$ and if $\vec{n}$ determines the orientation of $\mathcal{S}$, then the *flux of $\vec{F}$ across $\mathcal{S}$* is defined by

$$\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,d\sigma = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\vec{F}(P_i^*)\cdot\vec{n}\right)\,\Delta\sigma_i,$$

if this limit exists.

Observe that the flux of $\vec{F}$ across $\mathcal{S}$ is the surface integral of the *scalar field* $\vec{F}\cdot\vec{n}$. Thus, to compute the flux, we apply the formulas discussed in the preceding section.

Remarks.

1. Let $\mathcal{S} : \vec{R}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{\mathrm{k}}$, where $(u,v)\in D$. Suppose $\mathcal{S}$ has positive orientation. Using Equation (3.19) and a previous remark, we have
$$\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{D} \vec{F}(\vec{R}(u,v))\cdot(\vec{R}_u\times\vec{R}_v)\,dA\,.$$

2. Let $\mathcal{S}$ be the upward oriented surface defined by the Cartesian equation $z = g(x,y)$, and let R be the projection of $\mathcal{S}$ onto the xy-plane. Using Equation (3.20) and a previous remark, we have
$$\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{R} \vec{F}(x,y,g(x,y))\cdot\langle -g_x(x,y), -g_y(x,y), 1\rangle\,dA\,.$$

Example 3.9.2. Compute the flux of $\vec{F}(x,y,z) = \langle x, y, 2z\rangle$ across the positively oriented surface $\mathcal{S}$ defined parametrically by $\vec{R}(u,v) = \langle u, uv, v\rangle$, where $u\in[0,2]$ and $v\in[0,1]$.

Solution: The partial derivatives of $\vec{R}$ are

$$\vec{R}_u(u,v) = \langle 1, v, 0\rangle \text{ and } \vec{R}_v(u,v) = \langle 0, u, 1\rangle.$$

So,

$$\vec{R}_u(u,v)\times\vec{R}_v(u,v) = \langle v, -1, u\rangle.$$

On the surface $\mathcal{S}$, $\vec{F}(\vec{R}(u,v)) = \langle u, uv, 2v\rangle$. Hence,

$$\begin{aligned}
\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,d\sigma &= \iint_{D} \vec{F}(\vec{R}(u,v))\cdot\vec{R}_u\times\vec{R}_v\,dA, \text{ where } D = [0,2]\times[0,1] \\
&= \iint_{D} \langle u, uv, 2v\rangle\cdot\langle v, -1, u\rangle\,dA \\
&= \iint_{D} 2uv\,dA = \int_0^1\int_0^2 2uv\,du\,dv \\
&= \int_0^1 u^2v\Big|_{u=0}^{u=2}dv = \int_0^1 4v\,dv \\
&= 2v^2\Big|_0^1 = 2.
\end{aligned}$$

Example 3.9.3. Let $\mathcal{S}$ be the portion of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 4$ oriented by upward normal vectors. Calculate the flux of the vector field $\vec{F}(x,y,z) = \langle -2x, -2y, z\rangle$ across $\mathcal{S}$.

Solution: Since $\mathcal{S} : z = g(x,y) := \sqrt{x^2+y^2}$, we have

$$g_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad g_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}.$$

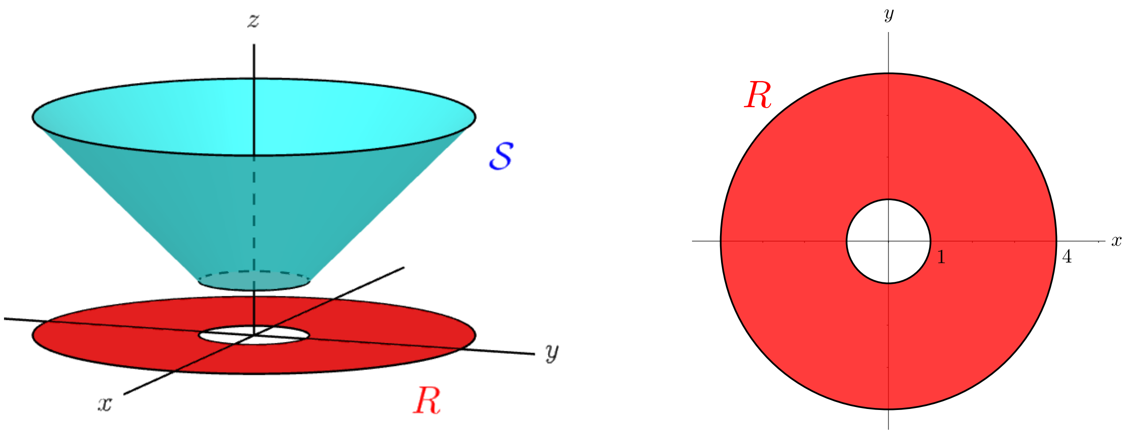


Figure 3.27: Portion of $z = \sqrt{x^2+y^2}$ between $z = 1$ and $z = 4$, and its projection to the xy-plane

Therefore,

$$\begin{aligned}
\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,d\sigma &= \iint_R \vec{F}(x,y,g(x,y))\cdot\langle -g_x(x,y), -g_y(x,y), 1\rangle\,dA \\
&= \iint_R \Big\langle -2x, -2y, \sqrt{x^2+y^2}\Big\rangle \cdot \Big\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\Big\rangle\,dA \\
&= \iint_R \left(\frac{2x^2}{\sqrt{x^2+y^2}} + \frac{2y^2}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\right) dA \\
&= \iint_R 3\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_1^4 3r(r\,dr\,d\theta) \\
&= \int_0^{2\pi} r^3\Big|_{r=1}^{r=4}\,d\theta = \int_0^{2\pi} 63\,d\theta = 126\pi.
\end{aligned}$$

Example 3.9.4. Find the flux of the velocity field $\vec{F}(x,y,z) = \langle x, y, z\rangle$ across the portion $\mathcal{S}$ of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane with an upward orientation.

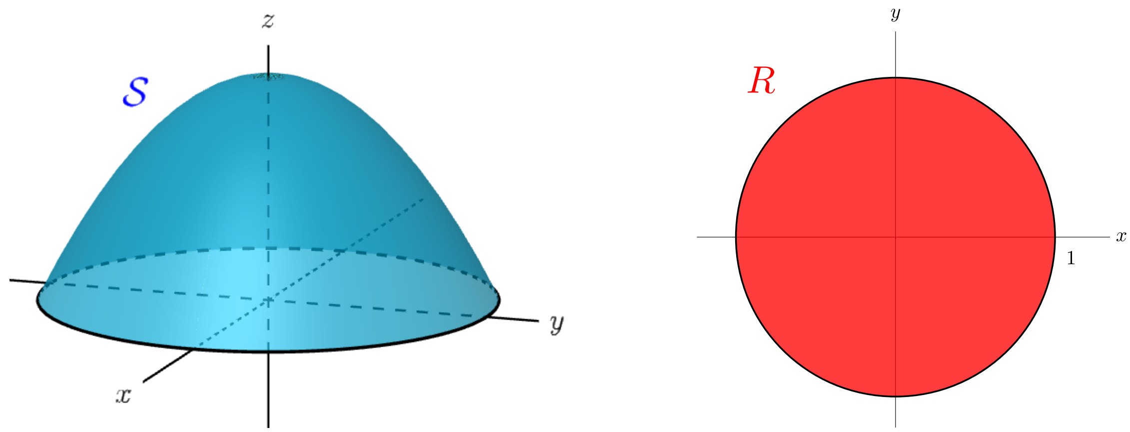


Figure 3.28: Portion of $z = 1 - x^2 - y^2$ above $z = 0$, and its projection to the xy-plane

Solution: Let $g(x,y) = 1 - x^2 - y^2$. Then $g_x = -2x$ and $g_y = -2y$. Hence, we have that

$$\begin{aligned}
\text{flux} &= \iint\limits_R \vec{F}(x,y,g(x,y)) \cdot \langle -g_x\,,\, -g_y\,,\, 1\rangle\, dA \\
&= \iint\limits_R \left\langle x, y, 1 - x^2 - y^2\right\rangle \cdot \langle 2x, 2y, 1\rangle\ dA \\
&= \iint\limits_R \left(2x^2 + 2y^2 + 1 - x^2 - y^2\right) dA \\
&= \iint\limits_R \left(x^2 + y^2 + 1\right) dA \\
&= \iint\limits_R \left(x^2 + y^2\right) dA + \iint\limits_R\ dA,
\end{aligned}$$

where R is the projection of $\mathcal{S}$ onto the xy-plane. In this case, R is the unit disk, so the second term above is equal to $A_R = \pi$. Meanwhile,

$$\begin{aligned}
\iint\limits_R \left(x^2 + y^2\right)\ dA &= \int_0^{2\pi}\int_0^1 r^2 \cdot r\, dr d\theta \\
&= \int_0^{2\pi} \frac{r^4}{4}\bigg|_{r=0}^{r=1} d\theta \\
&= \int_0^{2\pi} \frac{1}{4} d\theta \\
&= \frac{1}{4}(2\pi - 0) = \frac{\pi}{2}
\end{aligned}$$

Finally, the flux of $\vec{F}$ across $\mathcal{S}$ equals $\dfrac{\pi}{2} + \pi = \dfrac{3\pi}{2}$.

EXERCISES 3.9

I. Compute the flux of the vector field $\vec{F}$ across $\mathcal{S}$.

1. $\vec{F}(x, y, z) = \langle z, y, z\rangle$; $\mathcal{S} : \vec{R}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u\rangle$, where $0 \le u \le \pi$ and $0 \le v \le 2\pi$, with positive orientation
2. $\vec{F}(x, y, z) = z\,\hat{\imath} + y\,\hat{\jmath} + x\,\hat{k}$; $\mathcal{S}$ is the portion of the plane $x + 4y + z = 12$ in the first octant oriented by upward unit normal vectors
3. $\vec{F}(x, y, z) = \langle x - y, y - x, 2z\rangle$; $\mathcal{S}$ is the positively oriented portion of the hyperbolic paraboloid $z = x^2 - y^2$ whose projection onto the xy-plane is the region enclosed by the parabola $y = x^2$ and the line $y = 4$
4. $\vec{F}(x, y, z) = x^3\,\hat{\imath} + y^3\,\hat{\jmath} + 4x^2y^2\,\hat{k}$; $\mathcal{S}$ is the portion of the paraboloid $z = 4 - x^2 - y^2$ in the first octant oriented by upward unit normal vectors.
5. $\vec{F}(x, y, z) = \sqrt{x^2 + y^2}\hat{k}$; $\mathcal{S} : \vec{R}(u, v) = \langle u \cos v, u \sin v, 2u\rangle$, where $0 \le u \le 1$ and $0 \le v \le \pi$, with positive orientation
6. $\vec{F}(x, y, z) = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$; $\mathcal{S} : \vec{R}(u, v) = \langle uv, v-u, u-v\rangle$, where $0 \le u \le 1$ and $0 \le v \le 1$, with positive orientation
7. $\vec{F}(x, y, z) = \langle y + z, xy, xz\rangle$; $\mathcal{S} : \vec{R}(u, v) = \langle v, 2 \cos u, 2 \sin u\rangle$, $0 \le u \le 2\pi$, where $0 \le v \le 5$, with positive orientation
8. $\vec{F}(x, y, z) = \langle \cos^2 x, \sin^2 x, -z\rangle$; $\mathcal{S}$ is the portion of $2x + 2y + z = 2$ in the first octant with an upward orientation
9. $\vec{F}(x, y, z) = \langle x, y, z\rangle$; $\mathcal{S}$ is the part of the positively-oriented paraboloid $z = 1-4x^2-4y^2$ above the xy-plane
10. $\vec{F}(x, y, z) = -x\hat{\imath} + (y + 2)\hat{\jmath} + z\hat{k}$; $\mathcal{S}$ is the portion of the plane $3x + 2y + z = 6$ in the first octant with upward orientation

3.10 Stokes' Theorem and Gauss's Divergence Theorem

This section will give a physical meaning to the divergence and curl operators that were defined in the previous sections. The theorem by Stokes relates an integral across a surface to a line integral along its curve boundary while the theorem by Gauss relates an integral over a solid to an integral across its surface boundary.

Stokes' Theorem

We recall that an orientable surface S is one where we can define a continuous vector field $\vec{n}$ that is nonzero and normal to every point in S.

Theorem 3.10.1 (Stokes' Theorem)**.** Let S be an oriented surface whose boundary is a simple, closed curve C with positive orientation, and let $\vec{F}$ be a vector field in $\mathbb{R}^3$ whose components have continuous first order partial derivatives on some region containing S. Then

$$\oint_C \vec{F} \cdot d\vec{R} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, d\sigma$$

where $\vec{n}$ is the normal vector field of S.

Remarks.

1. An intuitive way to visualize the a boundary curve of *positive orientation* relative to a surface is to imagine walking around S with your head pointing in the direction of $\vec{n}$. If the curve C is positively oriented, then S will be on your left.

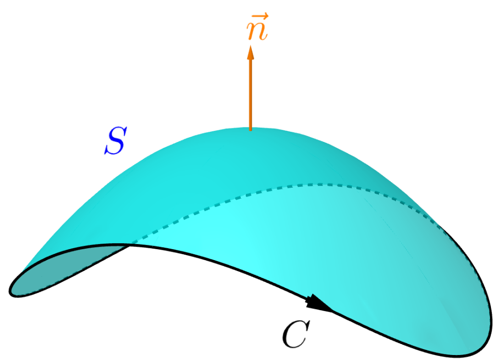


Figure 3.29: Positively oriented boundary curve C relative to an oriented surface S

2. Stokes' theorem relates a line integral (left) and a surface integral (right).

Example 3.10.1. Verify Stokes' Theorem for the vector field $\vec{F}(x, y, z) = \langle -y\,,\, x\,,\, z^2\rangle$ and the hemispherical surface S defined by $x^2 + y^2 + z^2 = 1$, $z \geq 0$, with upward orientation.

Solution: Clearly, the curve boundary C is the unit circle $x^2 + y^2 = 1$; $z = 0$, which can be parametrized by $\vec{R}(t) = \langle \cos t\,,\, \sin t\,,\, 0\rangle$, $t \in [0, 2\pi]$. Hence, to evaluate the line integral on the left-hand side, we put the parametrization inside to get

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{R} &= \int_0^{2\pi} \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t) \, dt \\
&= \int_0^{2\pi} \langle -\sin t\,,\, \cos t\,,\, 0\rangle \cdot \langle -\sin t\,,\, \cos t\,,\, 0\rangle \, dt \\
&= \int_0^{2\pi} 1 \, dt = 2\pi.
\end{aligned}$$

On the right-hand side, note that S has Cartesian equation $z = \sqrt{1-x^2-y^2}$. Thus

$$\langle -z_x\,,\, -z_y\,,\, 1\rangle = \left\langle \frac{x}{\sqrt{1-x^2-y^2}}\,,\, \frac{y}{\sqrt{1-x^2-y^2}}\,,\, 1\right\rangle.$$

Moreover,

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & z^2 \end{vmatrix} = \langle 0\,,\, 0\,,\, 2\rangle.$$

Therefore, since the unit disk $R:\ x^2+y^2 \le 1$ is the projection of S onto the xy-plane,

$$\begin{aligned} \iint\limits_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma &= \iint\limits_R \langle 0\,,\,0\,,\,2\rangle \cdot \left\langle \frac{x}{\sqrt{1-x^2-y^2}}\,,\, \frac{y}{\sqrt{1-x^2-y^2}}\,,\, 1\right\rangle dA \\ &= 2\iint\limits_R 1\,dA \\ &= 2A_R = 2\pi. \end{aligned}$$

For this example, Stokes' Theorem is verified.

Remarks. Revisiting Green's Theorem and the Fundamental Theorem of Line Integrals

1. Stokes' Theorem is the 3D-generalization of Green's Theorem. Recall that in Green's Theorem, if C is a closed curve in $\mathbb{R}^2$ and R is the region enclosed by C, then

$$\oint_C \langle P\,,\, Q\rangle \cdot d\vec{R} = \iint\limits_R \left(Q_x - P_y\right)\,dA.$$

   Indeed, suppose this is in the three-dimensional setting. Then $\vec{F} = \langle P\,,\, Q\,,\, 0\rangle$ and the surface that is enclosed by C is precisely R. Since R is a portion of the xy-plane, $\hat{k}$ is the unit normal vector to R for the positive orientation. Moreover, $\operatorname{curl}\vec{F} = \langle 0\,,\, 0\,,\, Q_x - P_y\rangle$. Hence, on one hand,

$$\oint_C \vec{F}\cdot d\vec{R} = \oint_C \langle P\,,\, Q\,,\, 0\rangle \cdot \langle dx\,,\, dy\,,\, dz\rangle = \oint_C P\,dx + Q\,dy,$$

   and on the other hand,

$$\begin{aligned} \iint\limits_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma &= \iint\limits_D \langle 0\,,\, 0\,,\, Q_x - P_y\rangle \cdot \langle 0\,,\, 0\,,\, 1\rangle\,dA \\ &= \iint\limits_D \left(Q_x - P_y\right)\,dA. \end{aligned}$$

   This illustrates that Stokes' Theorem is the three-dimensional generalization of Green's Theorem.

2. In some sense, Stokes' Theorem also generalizes the principle of the Fundamental Theorem of Calculus and of Line Integrals where the value of the integral depends only at the endpoints and not on the path connecting the endpoints.

   In the case of the surface integral, the Stokes' Theorem states that the integral across a surface depends only on its curve boundary and not on the (relative) interior points of the surface. For instance, if C is the positively oriented circle, then the surface integral of $\operatorname{curl}\vec{F}\cdot\vec{n}$ across any surface with curve boundary C will have the same value.

   This can be exploited to evaluate line integrals along a simple closed curves (especially those which are piecewise-defined). If we want to use Stokes' Theorem to evaluate $\oint_C \vec{F}\cdot d\vec{R}$, we may choose any surface S whose curve boundary is C. For instance, if the points in C are coplanar, then we may take S as the region of a plane enclosed by C.

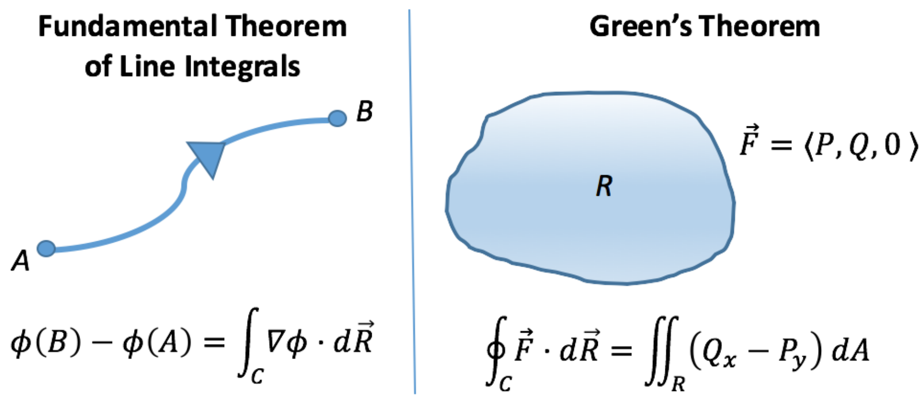


Figure 3.30: FTLI and Green's Theorem relative to Stokes' Theorem

Example 3.10.2. Let C be the closed rectangular path passing through the points $(0,0,0)$, $(2,0,0)$, $(2,1,1)$, and $(0,1,1)$ in that order. Find $\oint_C \vec{F}\cdot d\vec{R}$ if $\vec{F}(x,y,z) = \langle x^2\,,\,4xy^3\,,\,xy^2\rangle$.

Solution: We first remark that since C is segmented, then directly computing this line integral entails evaluating four line integrals along the four sides of the rectangle. We will avoid this by instead using Stokes' Theorem which states that the line integral is equal to the surface integral of $\operatorname{curl} F\cdot\vec{n}$ across any surface with curve boundary C. We choose the simplest – the plane S enclosed by C. Using methods from Math 22, this plane S has equation $z = y$. In this case, $\langle -z_x\,,\,-z_y\,,\,1\rangle = \langle 0\,,\,-1\,,\,1\rangle$. Moreover,

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x & 4y^3 & xy^2. \end{vmatrix} = \langle 2xy\,,\,-y^2\,,\,4y^3\rangle.$$

Finally, by Stokes' Theorem,

$$\oint_C \vec{F}\cdot d\vec{R} = \iint\limits_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma = \iint\limits_R \langle 2xy\,,\,-y^2\,,\,4y^3\rangle\cdot\langle 0\,,\,-1\,,\,1\rangle\,dA$$

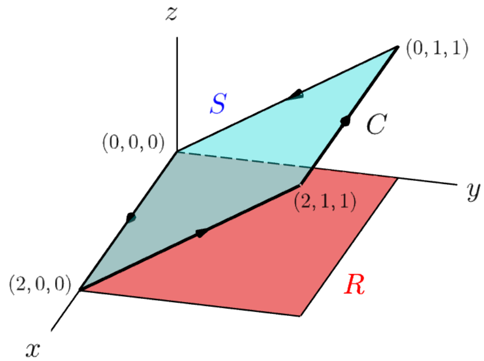


Figure 3.31: Closed rectangle with vertices (0, 0, 0), (2, 0, 0), (2, 1, 1), and (0, 1, 1), and its projection to the xy-plane

where $R = [0, 2] \times [0, 1]$ is the projection of S onto the xy-plane. Therefore,

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{R} &= \int_0^2 \int_0^1 \left(y^2 + 4y^3\right) \, dy dx \\ &= \left(\int_0^2 1 \, dx\right) \left(\int_0^1 (y^2 + 4y^3) \, dy\right) \\ &= 2 \cdot \left(\frac{y^3}{3} + y^4\right)\Bigg|_0^1 \\ &= 2\left(\frac{1}{3} + 1\right) = \frac{8}{3}.\end{aligned}$$

Example 3.10.3. Evaluate the surface integral $\iint\limits_S \operatorname{curl} \vec{F} \cdot \vec{n} \, d\sigma$ where $\vec{F}(x, y, z) = \langle y \, , \, 2x - yz \, , \, z^3 x - y \rangle$ and S is the portion of the paraboloid $f(x, y) = 5 - x^2 - 4y^2$ above the plane $z = 1$, and $\vec{n}$ is the unit normal vector to S with a positive z-component.

Solution: Method 0 First, we attempt to solve this using direct evaluation. Note that

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \langle -1 + y \, , \, -z^3 \, , \, 1 \rangle.$$

Moreover,

$$\langle -f_x \, , \, -f_y \, , \, 1 \rangle = \langle 2x \, , \, 8y \, , \, 1 \rangle.$$

Also, notice that the intersection of the paraboloid and the plane satisfies $5 - x^2 - 4y^2 = 1$, and therefore, it is clear that the projection of S onto the xy-plane is $R = \{(x, y) \mid x^2 + 4y^2 \leq 4\}$. Therefore, using the formula for evaluating surface integrals,

$$\begin{aligned}\iint\limits_S \operatorname{curl} \vec{F} \cdot \vec{n} \, d\sigma &= \iint\limits_R \langle -1 + y \, , \, -(5 - x^2 - 4y^2)^3 \, , \, 1 \rangle \cdot \langle 2x \, , \, 8y \, , \, 1 \rangle \, dA \\ &= \int_{-1}^1 \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} \left(-2x + 2xy - 8y(5 - x^2 - 4y^2)^3 + 1\right) \, dx \, dy.\end{aligned}$$

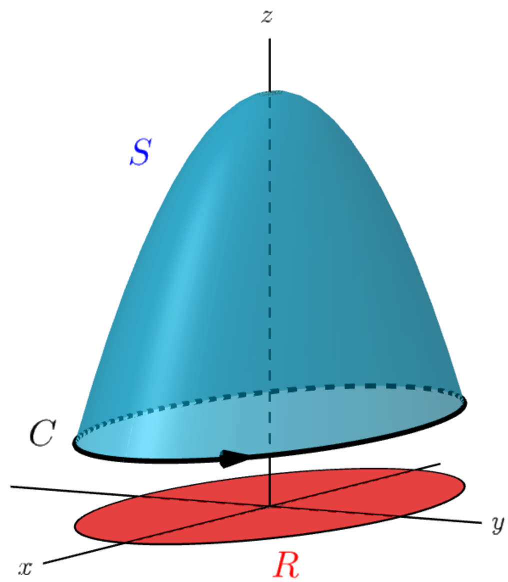


Figure 3.32: Portion of $z = 5 - x^2 - 4y^2$ above $z = 1$

This is clearly a complicated integral to solve. We now use Stokes' Theorem to evaluate this integral. There are two ways to use the theorem. One way is to evaluate the line integral along C of $\vec{F} \cdot d\vec{R}$, as provided for by Stokes' Theorem. Another way is to compute the surface integral across a simpler surface whose curve boundary is C.

Method 1 From the above computation, we deduce that the curve boundary of the surface S has equation

$$C: \; x^2 + 4y^2 = 4; \quad z = 1.$$

Since the first equation is equivalent to $\dfrac{x^2}{4} + y^2 = 1$, then C can be parametrized by $\vec{R}(t) = \langle 2\cos t\,,\, \sin t\,,\, 1\rangle$, $t \in [0, 2\pi]$. Consequently,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{R} &= \int_0^{2\pi} \langle \sin t\,,\, 2(2\cos t) - \sin t\,,\, 2\cos t - \sin t\rangle \cdot \langle -2\sin t\,,\, \cos t\,,\, 0\rangle \, dt \\
&= \int_0^{2\pi} \left(-2\sin^2 t + 4\cos^2 t - \sin t \cos t\right) \, dt \\
&= \int_0^{2\pi} \left(-(1 - \cos 2t) + 2(1 + \cos 2t) - \frac{1}{2}\sin 2t\right) \, dt \\
&= \int_0^{2\pi} \left(1 + 3\cos 2t - \frac{1}{2}\sin 2t\right) \, dt \\
&= \left(t + \frac{3\sin 2t}{2} + \frac{\cos 2t}{4}\right)\Bigg|_0^{2\pi} = 2\pi.
\end{aligned}$$

Method 2 For the second method, since the surface integral does not depend on the whole surface but only on the curve boundary, then we may replace the paraboloid by a simpler surface, say the planar region enclosed by C.

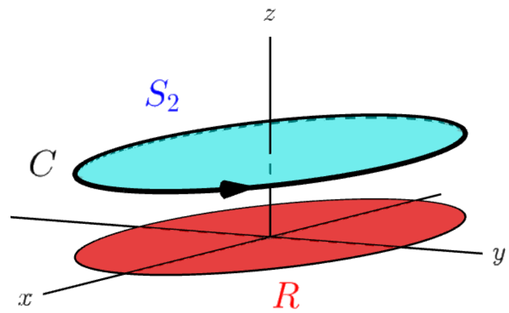


Figure 3.33: Portion of $z = 1$ above R

Clearly, this surface, which we call S_2 is the portion of the plane $z = 1$ inside the elliptic cylinder $x^2 + 4y^2 = 4$. In this case, $\langle -z_x, -z_y, 1\rangle = \langle 0, 0, 1\rangle$, and so

$$\iint\limits_{S_2} \operatorname{curl} \vec{F} \cdot \vec{n}\, d\sigma = \iint\limits_{R} \langle -1 + y\,,\, -(1)^3\,,\, 1\rangle \cdot \langle 0\,,\, 0\,,\, 1\rangle\, dA$$
$$= \iint\limits_{R} 1 dA = A_R = 2\pi.$$

Interpretation of the Curl of a Vector Field

Recall that if $\vec{F}$ is interpreted as a force field, then the line integral $\int_C \vec{F} \cdot d\vec{R}$ is interpreted as the work done by $\vec{F}$ along a curve C. However, if $\vec{F}$ is interpreted as a velocity field and C is a closed curve, then we interpret $\oint_C \vec{F} \cdot d\vec{R}$ as the *total circulation of* $\vec{F}$ *inside* C. To explain this terminology, let us consider some fluid moving across a surface S bounded by a closed curve C. We first establish that the total circulation *inside* C will only depend on the boundary.

Suppose S is partitioned into small rectangles such that each rectangle partition is approximately planar. If we orient each small rectangle with the upward normal vector, we see that the total line integral equals the line integral along the boundary curve C of the surface S. This is because each line boundary in the center is traversed twice but in opposite directions, thereby cancelling its contribution to the sum.

This establishes that the total circulation of $\vec{F}$ inside C does not depend on the surface that is bounded by C, but only depends on the value of the line integral along the boundary.

We now consider a small rectangular partition σ of the mesh of the surface that is bounded by the simple closed curve γ. Illustrated below are two different velocity fields of the fluid. On the left, we have a more laminar flow while on the right, we have a fluid with apparent circulation. For each of the following scenario, let us try to surmise the value of $\oint_{\gamma_i} \vec{F} \cdot d\vec{R}_i$ where $\vec{R}_i$ is a parametrization of a segment γ_i of γ, for $i = 1, 2, 3, 4$.

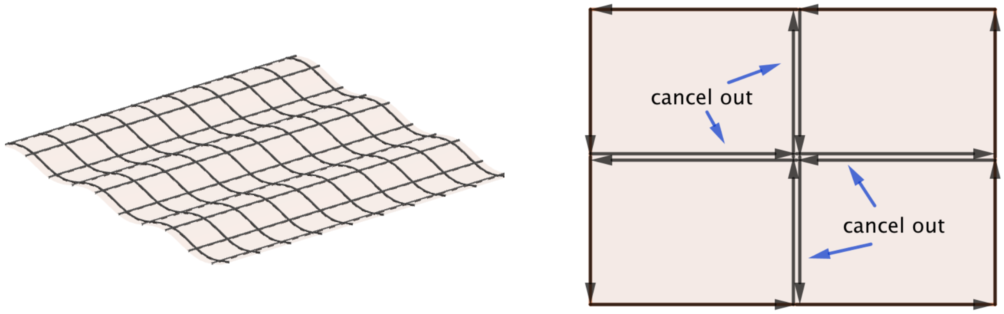


Figure 3.34: (Left) The surface S partitioned by a rectangular mesh. (Right) The sum of all line integrals around each small rectangle produces cancellations of interior line integrals, and thereby reduces to the line integral along the boundary.

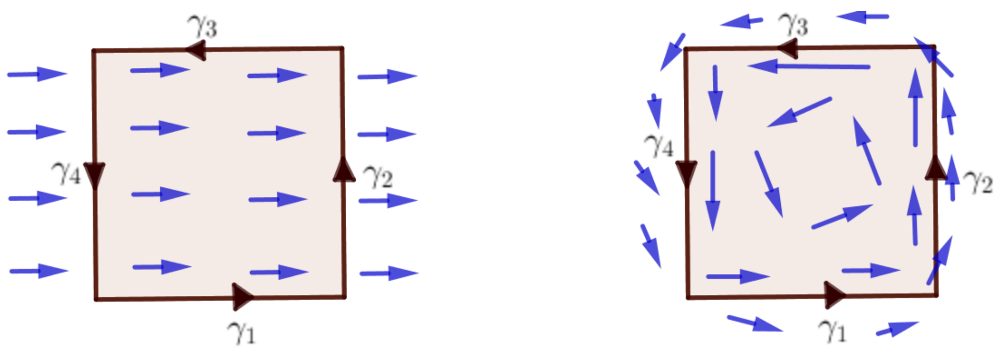


Figure 3.35: Two different velocity fields of fluids moving across a region containing the same curve $\gamma = \bigcup_{i=1}^{4} \gamma_i$

First, we consider the scenario on the left. We notice that along γ_2 and γ_4, either $\vec{R}_2$ or $\vec{R}_4$ is perpendicular to $\vec{F}$. Since the dot product of perpendicular vectors is zero, it follows that $\oint_{\gamma_k} \vec{F} \cdot d\vec{R}_k = 0$ for $k = 2, 4$. Now, for the integral along γ_1, we observe that $\vec{F}$ and $\vec{R}_1$ are in the same direction. This tells us that $\oint_{\gamma_1} \vec{F} \cdot d\vec{R}_1$ gives a positive value. On the other hand, since $\vec{F}$ and $\vec{R}_3$ are in opposite directions, we conclude that $\oint_{\gamma_3} \vec{F} \cdot d\vec{R}_3$ gives a negative value. Therefore, the value along γ_1 cancels out with the value found along γ_3. For the first scenario, we may say that the circulation of $\vec{F}$ inside γ is zero. In effect, if an object is carried by the flow, it has no inclination to spin or rotate about the axis perpendicular to the surface that is bounded by the curve.

Now for the second scenario on the right, we observe that for each $i = 1, 2, 3, 4$, $\vec{F}$ and γ_i either point towards the same direction or create an acute angle between them. This implies that each $\oint_{\gamma_i} \vec{F} \cdot d\vec{R}$ contributes a positive value to the circulation of $\vec{F}$ inside γ. In effect, an object carried through by the fluid's current has the inclination to spin in the positive direction (i.e., in the same direction as γ) while moving across the surface bounded by γ.

Using a small partition σ of the surface, we can therefore observe that inside the curve γ — which is to say across the surface σ bounded by γ — the line integral measures the tendency of the velocity field to spin or rotate in the same direction as γ on the local planar approximation of S. Since this is set in $\mathbb{R}^3$, we might expect that the notion of rotation of $\vec{F}$ is vector valued with x-,y-, and z-components. This is captured by $\operatorname{curl}\vec{F}$. The tendency to rotate about an axis perpendicular to the local planar approximation of σ is thus modeled by the dot product of $\operatorname{curl}\vec{F}$ with the normal vector $\vec{n}$ of σ.

Remarks.

1. In other references, $\operatorname{curl}\vec{F}$ is denoted by $\operatorname{rot}\vec{F}$.

2. If $\operatorname{curl}\vec{F} = \vec{0}$, then the velocity field is said to *irrotational*. This property is a frequent assumption in the study of fluid flow, e.g. solving the Navier-Stokes problem, and in the study of wave motion.

3. The velocity field of a laminar flow does not necessarily have zero circulation inside a closed curve. In Figure 3.35, if the flow near γ_1 has higher velocity (longer arrows) than that of the flow near γ_3, then the line integrals along γ_1 and γ_3 do not cancel out, but will give a positive value for the circulation of $\vec{F}$ inside γ.

Gauss's Theorem

The next theorem by Gauss relates a surface integral that computes the flux of a velocity field across a closed and bounded surface with a triple integral of the divergence of the velocity field.

To indicate that we are computing the integral across a surface that is assumed to be closed (in the sense that it is the surface boundary of a closed and bounded solid), it is customary to put a closed loop across the double integral sign.

Theorem 3.10.2 (Gauss's Divergence Theorem)**.** Let G be a closed and bounded solid in $\mathbb{R}^3$ whose surface boundary S is piecewise smooth. Let $\vec{F}$ be a vector field whose components have continuous first partial derivatives. If $\vec{n}$ represents the outward normal vector of S, then

$$\oiint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_G \operatorname{div}\vec{F}\,dV.$$

Example 3.10.4. Let $\vec{F}(x,y,z) = \langle 0\,,\,0\,,\,z^3\rangle$, and let S be the outward oriented sphere centered at the origin with radius 1. Evaluate $\oiint_S \vec{F}\cdot\vec{n}\,d\sigma$ (1) directly and (2) using Gauss's Theorem.

Solution: (1) Direct Method: First, we parametrize the unit sphere using the spherical coordinates. Note that $\rho = 1$.

$$S: \quad \vec{R}(\phi,\theta) = \langle \sin\phi\cos\theta\,,\,\sin\phi\sin\theta\,,\,\cos\phi\rangle, \quad \phi\in[0,\pi],\ \theta\in[0,2\pi].$$

Thus,

$$\begin{aligned}\vec{R}_\phi \times \vec{R}_\theta &= \langle \cos\phi\cos\theta\,,\ \cos\phi\sin\theta\,,\ -\sin\phi\rangle \times \langle -\sin\phi\sin\theta\,,\ \sin\phi\cos\theta\,,\ 0\rangle \\ &= \sin\phi\langle \cos\theta\sin\phi\,,\ \sin\theta\sin\phi\,,\ \cos\phi\rangle.\end{aligned}$$

Therefore, using the method of evaluating flux we get

$$\begin{aligned}\oiint_S \vec{F}\cdot\vec{n}\,d\sigma &= \int_0^{2\pi}\int_0^{\pi} \vec{F}(\vec{R}(\phi,\theta))\cdot(\vec{R}_\phi\times\vec{R}_\theta)\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} \langle 0\,,\ 0\,,\ \cos^3\phi\rangle\cdot\sin\phi\langle\cos\theta\sin\phi\,,\ \sin\theta\sin\phi\,,\ \cos\phi\rangle\,d\phi\,d\theta \\ &= \int_0^{2\pi} 1\,d\theta\cdot\int_0^{\pi}\sin\phi\cos^4\phi\,d\phi \qquad \begin{aligned}&\text{Let } u=\cos\phi\\ &-du=\sin\phi\,d\phi\end{aligned} \\ &= -2\pi\int_1^{-1} u^4\,du \\ &= -2\pi\left(\frac{1}{5}u^5\right)\Bigg|_1^{-1} = -2\pi\left(-\frac{2}{5}\right) = \frac{4\pi}{5}.\end{aligned}$$

(2) Using Gauss's Theorem: By Gauss, if G is the unit sphere, then

$$\oiint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_G \operatorname{div} F\,dV.$$

First, the divergence of $\vec{F}$ is $\operatorname{div}\vec{F} = 3z^2$. We will then use spherical coordinates in evaluating this triple integral. Accordingly,

$$\begin{aligned}\iiint_G \operatorname{div}\vec{F}\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 3(\rho\cos\phi)^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 3\int_0^{2\pi}1\,d\theta\cdot\int_0^{\pi}\cos^2\phi\sin\phi\,d\phi\cdot\int_0^1\rho^4\,d\rho \qquad \begin{aligned}&\text{Let } u=\cos\phi\\ &-du=\sin\phi\,d\phi\end{aligned} \\ &= 3\cdot 2\pi\cdot\int_1^{-1}(-u^2)\,du\cdot\frac{\rho^5}{5}\Bigg|_0^1 \\ &= \frac{6\pi}{5}\cdot\left(-\frac{u^3}{3}\right)\Bigg|_1^{-1} \\ &= \frac{6\pi}{5}\left(\frac{2}{3}\right) = \frac{4\pi}{5}.\end{aligned}$$

For this example, Gauss's theorem is verified.

Example 3.10.5. Find the outward flux of a fluid moving with velocity $\vec{F}(x,y,z) = \langle x\,,\ y\,,\ z\rangle$ across (i) the sphere $x^2+y^2+z^2=1$ and across (ii) the boundary of the unit cube $[0,1]^3$.

Solution: Using Gauss's Theorem, the outward flux of a fluid across a closed surface equals the triple integral of the divergence of the velocity of the fluid. In this case, $\operatorname{div}\vec{F} = 3$. Hence for (i), if G is the solid inside the unit sphere, then

$$\text{flux} = \iiint\limits_G \operatorname{div}\vec{F}\,dV = \iiint\limits_G 3\,dV = 3\cdot\text{volume}(G) = 3\cdot\frac{4\pi}{3} = 4\pi.$$

For (ii), if E is the unit cube $[0,1]^3$, then

$$\text{flux} = \iiint\limits_E \operatorname{div}\vec{F}\,dV = \iiint\limits_E 3\,dV = 3\cdot\text{volume}(E) = 3\cdot 1^3 = 3.$$

It is remarkable that Gauss's theorem can simplify our calculations for the flux. If we had computed the surface integral directly in (ii), we would have had computed six integrals corresponding to the six faces of the cube.

Interpretation of the Divergence of a Vector Field

We first argue informally why the surface integral of $\vec{F}\cdot\vec{n}$ corresponds to a measurement of some quantity of the solid inside the surface. Suppose G is the solid bounded by the closed surface S. Suppose that we divide G into small cubes. Note that if two cubes have a common surface, then their normal vectors at any point on the common surface would be identical except that they are pointing in opposite directions. This observation leaves out the surface integrals across the boundary. Hence, for any partitioning of G into cubes, the sum of all surface integrals across all the surfaces of the partitioning cubes equals the sum of the surface integrals across the boundary of the partition.

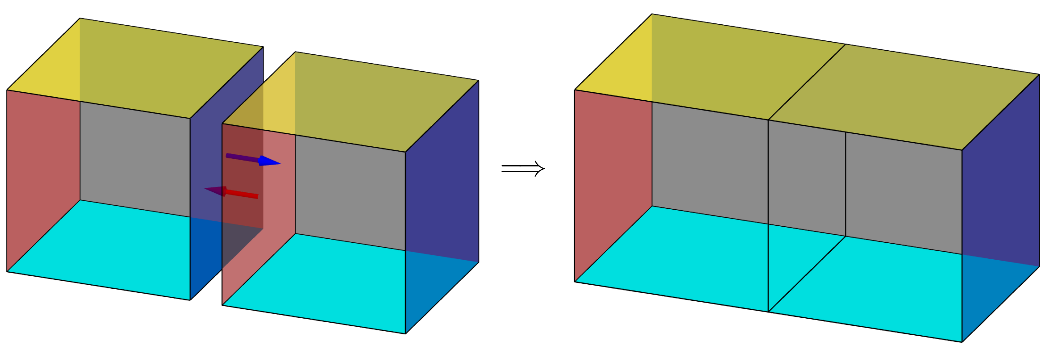


Figure 3.36: The normal vectors of the common surface of two adjacent cubes are antiparallel, and therefore, their surface integrals cancel out.

Now, let us consider a small cube of the partition. To find out what the surface integral of $\vec{F}\cdot\vec{n}$ measure, we imagine how this integral gives a positive value. Since the integrand is a dot product, it is maximized when the two vectors $\vec{F}$ and $\vec{n}$ point in the same direction. This happens when the vectors around a point inside the surface digress or diverge away from the point in a radial manner. This property of the vector field $\vec{F}$ is captured by $\operatorname{div}\vec{F}$.

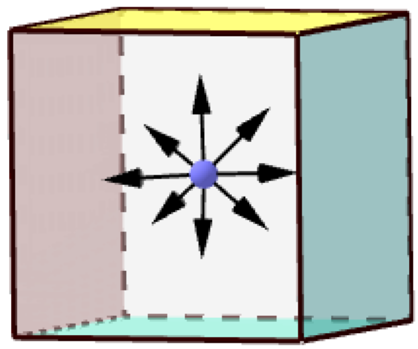


Figure 3.37: The flux integral measures how much the vectors around a point inside the surface digress from the point

Remarks.

1. If $\operatorname{div}\vec{F} > 0$ at a point, then this point is said to be a *source* and implies that the fluid expands about the point. If $\operatorname{div}\vec{F} < 0$ at a point, then this point is called a *sink* implying that the fluid contracts about this point, i.e., the fluid tends to converge to this point.

2. In fluid mechanics, the divergence of $\vec{F}$ equals the negative rate of change of the density of the fluid along the flowlines of $\vec{F}$ with respect to time. Thus if $\operatorname{div}\vec{F} = 0$, the density of the fluid does not change as the fluid moves along the flowlines of $\vec{F}$. This flow is commonly called *incompressible*.

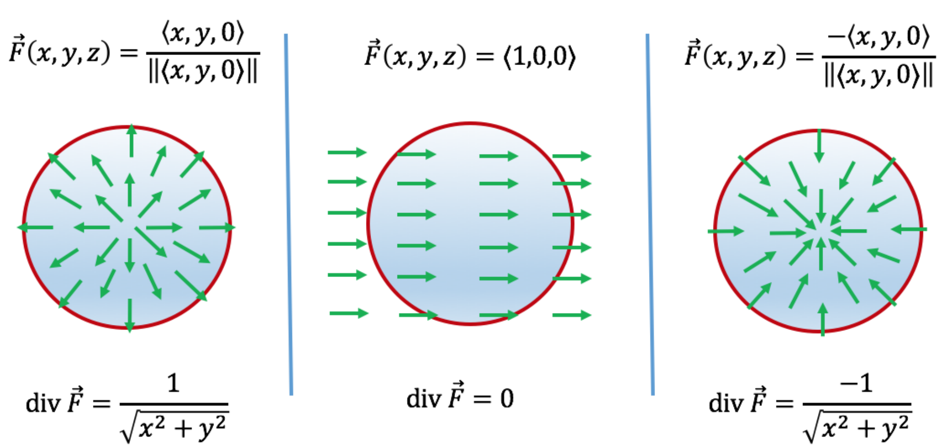


Figure 3.38: Graphical interpretation of the divergence of $\vec{F}$

Gauss's Theorem in Two Dimensions

The following derives the two-dimensional version of Gauss's Divergence Theorem. Here, the flux of a fluid with velocity $\vec{F} = \langle P\,,\, Q\rangle$ along a curve C equals $\displaystyle\int_C \vec{F}\cdot\vec{n}\,ds$ where $\vec{n}$ is the unit normal vector of C.

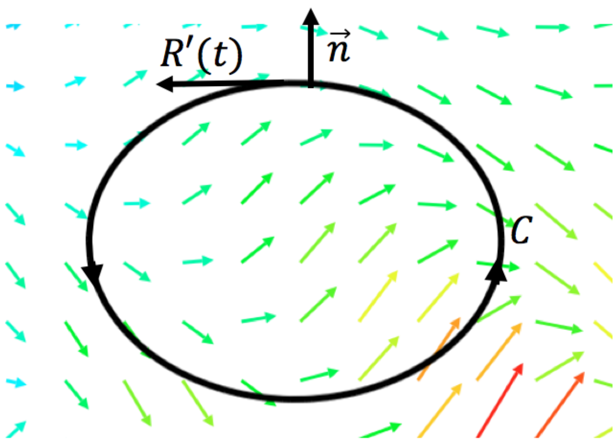


Figure 3.39: Closed curve C with unit vector $\vec{n}$

Suppose that a fluid with velocity $\vec{F}$ is moving along a region containing a simple, closed, piecewise smooth curve C with positive orientation. Assume that the components of $\vec{F} = \langle P\,,\, Q\rangle$ have continuous first partials, and that C is parametrized by $\vec{R}(t) = \langle x(t)\,,\, y(t)\rangle,\ t\in[a,b]$. Since the tangent vector to C is $\vec{R}'(t) = \langle x'(t)\,,\, y'(t)\rangle$, then the outward unit normal vector is

$$\vec{n} = \frac{\langle y'(t)\,,\, -x'(t)\rangle}{\sqrt{y'(t)^2 + x'(t)^2}}.$$

It can be verified that this vector is perpendicular to $\vec{R}'$.

Hence, directly computing flux using the definition yields

$$\begin{aligned}\oint_C \vec{F}\cdot\vec{n}\,ds &= \int_a^b \langle P\,,\, Q\rangle \cdot \frac{\langle y'(t)\,,\, -x'(t)\rangle}{\sqrt{y'(t)^2 + x'(t)^2}}\,\sqrt{x'(t)^2 + y'(t)^2}\,dt \\ &= \int_a^b \big(P\,y'(t) - Q\,x'(t)\big)\,dt \\ &= \int_C -Q\,dx + P\,dy.\end{aligned}$$

If R is the region enclosed by C, then Green's Theorem implies

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint\limits_R P_x + Q_y\,dA = \iint\limits_R \operatorname{div}\vec{F}\,dA.$$

Example 3.10.6. Find the outward flux of the fluid with velocity $\vec{F}(x,y) = \langle x^3\,,\, y^3\rangle$ across the unit circle $x^2+y^2=1$.

Solution: Gauss's Theorem is applicable since the curve is closed. The divergence of the velocity field is $\operatorname{div}\vec{F} = 3(x^2+y^2)$. If R is the unit disk, then using polar coordinates gives

$$\begin{aligned}
\text{flux} &= \iint_R \operatorname{div}\vec{F}\,dA \\
&= 3\iint_R x^2+y^2\,dA \\
&= 3\int_0^{2\pi}\int_0^1 r^2\,r dr d\theta \\
&= 3\int_0^{2\pi} d\theta \cdot \int_0^1 r^3\,dr \\
&= 6\pi\cdot\frac{1}{4} = \frac{3\pi}{2}.
\end{aligned}$$

EXERCISES 3.10

I. Use Stokes' Theorem to compute the total circulation of $\vec{F}$ inside C, i.e. the line integral $\oint_C \vec{F}\cdot d\vec{R}$.

1. $\vec{F}(x,y,z) = \langle z^2\,,\, y^2\,,\, x\rangle$ and C is the boundary of the portion of the plane $x+y+z=1$ in the first octant traced counterclockwise when viewed from above.
2. $\vec{F}(x,y,z) = \langle 2yz\,,\, xz\,,\, xy\rangle$ and C is the positively oriented boundary of the surface parametrized by $\vec{R}(u,v) = \langle u\cos v\,,\, u\sin v\,,\, u^2\sin^2 v\rangle$ where $u\in[0,1]$ and $v\in[0,2\pi]$.
3. $\vec{F}(x,y,z) = \langle x\,,\, z\,,\, 2y\rangle$ and C is the intersection of the plane $z=x$ and the cylinder $x^2+y^2=1$ traced counterclockwise when viewed from above
4. $\vec{F}(x,y,z) = \langle y\,,\, z\,,\, x\rangle$ and C is closed curve consisting of the quarter circle $y^2+z^2=1,\ x=0$ from $A(0,1,0)$ to $B(0,0,1)$, the line segment from B to the origin, and the line segment from the origin to A.
5. $\vec{F}(x,y,z) = \left\langle \sin x - \dfrac{y^3}{3}\,,\, \cos y + \dfrac{x^3}{3}\,,\, xyz\right\rangle$ where C is intersection of the cone $z=\sqrt{x^2+y^2}$ and the plane $z=1$ traced counterclockwise when viewed from above

II. Given the following velocity fields $\vec{F}$ and corresponding surface S, use Stokes' Theorem to compute $\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma$.

1. $\vec{F}(x,y,z) = \langle z\,,\, x\,,\, y\rangle$ and S is the paraboloid $z=x^2+y^2$ that is below the plane $z=1$ with upward orientation.
2. $\vec{F}(x,y,z) = \langle 3y\,,\, 4z\,,\, -6x\rangle$ and S is the paraboloid $z=16-x^2-y^2$ above the xy-plane with upward orientation.

3. $\vec{F}(x,y,z) = \langle x\,,\,x\,,\,y\rangle$ and S is the hemisphere $x^2+y^2+z^2=1$ above the xy-plane with upward orientation.
4. $\vec{F}(x,y,z) = \langle x^2-y^2\,,\,y^2-z^2\,,\,z^2-x^2\rangle$ and S is the part of the plane $x+y+z=1$ in the first octant with upward orientation.
5. $\vec{F}(x,y,z) = \langle -y\,,\,x\,,\,z\rangle$ and S is the part of the sphere $x^2+y^2+z^2=25$ below the plane $z=4$ with outward orientation.
6. $\vec{F}(x,y,z) = \langle x\,,\,1\,,\,y\rangle$ and S is the outward oriented cylinder $x^2+y^2=1$ between the planes $z=0$ and $z=1$. (Hint: Connect the two boundaries by an incision across the cylinder, e.g. at $\{(1,0,z)\,|\,0\le z\le 1\}$. Be careful about the orientation of the curves.)

III. Use Gauss's Divergence Theorem to compute the outward flux of the fluid with the given velocity $\vec{F}$ across the given surface.

1. $\vec{F}(x,y,z) = \langle y\,,\,x\,,\,z\rangle$ across the boundary of the parallelepiped $[-1,1]\times[1,2]\times[0,4]$.
2. $\vec{F}(x,y,z) = \langle 3x^2+4x\,,\,6xy-2yz\,,\,z^2-12xz+y^3\rangle$ across the closed surface S consisting of the paraboloid $z=1-x^2-y^2$ above the xy-plane, and the unit disk $x^2+y^2\le 1$.
3. $\vec{F}(x,y,z) = \langle x+y+z\,,\,y\,,\,2x-y\rangle$ across the closed cylinder $x^2+y^2=1$ between the planes $z=0$ and $z=3$.
4. $\vec{F}(x,y,z) = \langle y\,,\,x\,,\,z^2\rangle$ across the closed surface consisting of the paraboloid $z=x^2+y^2$ below $z=1$ and the enclosing circular disk $x^2+y^2\le 1$; $z=1$.
5. $\vec{F}(x,y,z) = \langle 2\,,\,x\,,\,z^2\rangle$ across the boundary S of the cylindrical shell
$$\{(x,y,z)\,|\,1\le x^2+y^2\le 4,\ 0\le z\le 2\}.$$

IV. Use the two-dimensional version of the Gauss's Divergence Theorem to compute the outward flux of the fluid with the given velocity $\vec{F}=\langle P\,,\,Q\rangle$ along the given closed curve C.

1. $P(x,y) = x^2-\sin^{-1}y^2$, $Q(x,y)=2xy+4y$; C is the parallelogram with vertices at $(0,0)$, $(4,0)$, $(1,3)$ and $(5,3)$.
2. $P(x,y) = x+e^y$, $Q(x,y)=2y^2+\cos x$; C is the triangle with vertices at $(0,0)$, $(4,0)$ and $(6,4)$.
3. $P(x,y)=y-x^2$, $Q(x,y)=2xy$; C is the boundary of the region between the x-axis and the graph of $y=\sin x$ for $0\le x\le\pi$.
4. $P(x,y)=2xy-3x^2$, $Q(x,y)=x^2+y^2$; C is the circle $x^2+y^2-4x=0$.
5. $P(x,y)=\dfrac{x}{\sqrt{x^2+y^2}}$, $Q(x,y)=\dfrac{y}{\sqrt{x^2+y^2}}$; C is the boundary of the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.

V. Do as indicated.

1. If F is a continuously differentiable velocity field and S is a closed surface, what does Stokes' Theorem say about $\oiint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, d\sigma$? (Hint: Divide the surface into two and apply Stokes' Theorem to both surfaces.)

2. What does the Gauss's Divergence theorem reduce to in one-dimensional space? (Hint: The integral along the boundary of an interval is just the sum at the two endpoints. Moreover, at the right endpoint of the interval, $\vec{n} = 1$ while at the left endpoint of the interval, $\vec{n} = -1$.)

3. Let $\rho = \sqrt{x^2 + y^2 + z^2}$ and define $\vec{F}(x, y, z) = \dfrac{1}{\rho^3} \langle x\,,\ y\,,\ z \rangle$.

   (a) Show that $\operatorname{div} \vec{F} = 0$ on any region in $\mathbb{R}^3$ that does not contain the origin.

   (b) Let S be the unit sphere $x^2 + y^2 + z^2 = 1$. Show that $\oiint_S \vec{F} \cdot \vec{n}\, d\sigma = 4\pi$.

   (c) Explain why Gauss's Theorem did not apply to the above surface integral. What is $\oiint_S \vec{F} \cdot \vec{n}\, d\sigma$ if S is any closed surface not encompassing the origin?

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