Vector

Vector Algebra

Scalar: sign+magnitude+unit   Vector: direction(tip-tail)+magnitude+unit   (2D/3D)

1.Addition: (parallelogram rule, head/tip-to-tail rule) $\vec{v}+\vec{w}=\begin{bmatrix} a \\ b \end{bmatrix}+\begin{bmatrix} c \\ d \end{bmatrix}=\begin{bmatrix} a+c \\ b+d \end{bmatrix}$

>Property of Addition: commutative v+w=w+v   associative (v+w)+z=v+(w+z)

Zero: $\vec{0}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ u+0=u

Negative: Given v, there is some u that satisfies v+u=0. we will denote this particular u by (-v)

$-\vec{v}=\begin{bmatrix} -a \\ -b \end{bmatrix}$

2.Subtraction: $\vec{v}-\vec{w}=\vec{v}+(-\vec{w})=\begin{bmatrix} a-c \\ b-d \end{bmatrix}$

3.Scalar Multiplication/Product: $k\vec{v}=\begin{bmatrix} ka \\ kb \end{bmatrix}$

If 2 vectors are multiples of each other, they are parallel.   u=tv

Magnitude of a vector: Euclidean Norm   2D $|\vec{v}|=\sqrt{a^2+b^2}$ 3D $|\vec{v}|=\sqrt{a^2+b^2+c^2}$

Triangle Inequality: $|\vec{v}+\vec{w}| \le |\vec{v}|+|\vec{w}|$

4.Dot/Scalar Product: 2D $\vec{v}\cdot\vec{w}=ac+bd$ 3D $\vec{v}\cdot\vec{w}=ac+bd+cf$ $\vec{v}\cdot\vec{w}=vw\cos\theta$

dot product is the scalar $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+...+a_nb_n$

IkuI=IkIIuI   IuI=0 <==>u=0

I.Property of Dot Product: ① v·v=a^2+b^2=Ivl^2 ②distributive u·(v+w)=u·v+u·w ③commutative u·v=v·u ④ (ku)·v=k(u·v) not associative ⑤ 0·a=0

II. Geometric Interpretation: (cosine law) θ: smaller angle between v&w (less than 180°)

$\vec{v}\cdot\vec{w}=|\vec{v}||\vec{w}|\cos\theta$ v·w>0 iff θ∈[0,π/2)   v·w<0 iff θ∈(π/2,π]   v·w=0 iff θ=π/2 (v w perpendicular)

v·w≤vw   v·w can be + or - v·w=v·Iprojection of w on vI

※Orthogonality: For two non-zero vectors v & w, iff v·w=0, v and w is orthogonal.

Zero vector is perpendicular to every vector.

III.Projection:

①scalar projection of w onto v $comp_{\vec{v}}(\vec{w})=\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|}$ ②vector projection of w onto v $proj_{\vec{v}}(\vec{w})=\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{v}=\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|}\cdot\frac{\vec{v}}{|\vec{v}|}$

Proof:

IV.Inner Product:(abstract scalar field)

5.Cross/Vector Product: $\vec{v}\times\vec{w}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix}=\hat{i}\begin{vmatrix} b & c \\ e & f \end{vmatrix}-\hat{j}\begin{vmatrix} a & c \\ d & f \end{vmatrix}+\hat{k}\begin{vmatrix} a & b \\ d & e \end{vmatrix}=\begin{bmatrix} bf-ec \\ -af+dc \\ ae-bd \end{bmatrix}$

$\vec{v}\times\vec{w}=\vec{a}_n|vw\sin\theta|$ a_n: right-hand rule

cross product is the vector $\vec{a}\times\vec{b}=\langle a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1\rangle$ (only defined for vectors in R^3)

u×v orthogonal to both u & v.

Two vectors pointing same direction, there are infinite number of vectors orthogonal to them.

I.Property of Cross Product: ①anti-commutative u×v=-v×u ②k(u×v)=(ku)×v=u×(kv) ③distributive u×(v+w)=u×v+u×w (u+v)×w=u×w+v×w

④u×0=0×u=0   ⑤not associative u×(v×w)≠(u×v)×w

Right Hand Rule: direction of u×v

①4 fingers pointing at the first vector(u) then curl to the second(v), thumb points to their cross product.

②index-v, middle-u, thumb-u×v

II.Geometric Interpretation:(Lagrange's identity)   angle between 2 vectors θ

$|\vec{v}\times\vec{w}|=|\vec{v}||\vec{w}|\sin\theta$

$|u\times v|^2=|u|^2|v|^2-(u\cdot v)^2=|u|^2|v|^2-|u|^2|v|^2\cos^2\theta=|u|^2|v|^2\sin^2\theta$ so, $|u\times v|=|u||v|\sin\theta$

※Parallel: For two non-zero vectors v & w, iff v×w=0, v and w is parallel.

III.①Area of parallelogram determined by u and v is length of cross product u×v

Area of Parallelogram=height×base=$|u||v|\sin\theta$ (two sides u & v, height=$|v|\sin\theta$)

②Volume of parallelepiped determined by u,v,w is absolute value of scalar triple product

Volume of Parallelepiped=height×area of base=$|u\cdot(v\times w)|$

Scalar Triple Product $u\cdot(v\times w)=w\cdot(u\times v)=v(w\times u)=-u\cdot(w\times v)=-w\cdot(v\times u)=-v(u\times w)$

Vector Triple Product(Lagrange Formula/back-cab rule) $u\times(v\times w)=v(u\cdot w)-w(u\cdot v)$

6.Division by a vector is not defined.

Lines & Planes

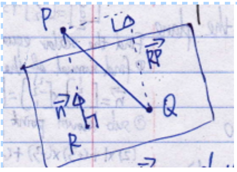
Vector Equation of a line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}+t\vec{d}$ $\vec{P}=\vec{P}_0+t\vec{d}$

$P_0(x_0,y_0,z_0)$ known point on the line   d: direction vector

Vector Equation of a plane: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}+t\vec{d}_1+s\vec{d}_2$ $\vec{P}=\vec{P}_0+t\vec{d}_1+s\vec{d}_2$ d1&d2 not parallel

Scalar Equation of a plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ $ax+by+cz=ax_0+by_0+cz_0$

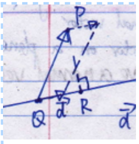
n: normal vector $\vec{n}=\vec{d}_1\times\vec{d}_2=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\overrightarrow{P_0P}=\begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix}$ and $\overrightarrow{P_0P}\cdot\vec{n}=0$



P arbitrary point in space, Q arbitrary point on plane, R projection of P on plane

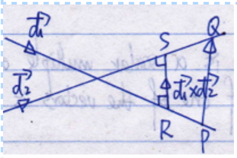
Distance from Point to Plane: $|\overrightarrow{RP}|=\dfrac{|\vec{n}\cdot\overrightarrow{QP}|}{|\vec{n}|}$

$$\overrightarrow{RP}=proj_{\vec{n}}(\overrightarrow{QP})=\frac{\vec{n}\cdot\overrightarrow{QP}}{|\vec{n}|^2}\cdot\vec{n}=\frac{\vec{n}\cdot\overrightarrow{QP}}{|\vec{n}|}\cdot\frac{\vec{n}}{|\vec{n}|}$$



Distance from Point to Line: $|\overrightarrow{RP}|=\dfrac{|\overrightarrow{QP}\times\vec{d}|}{|\vec{d}|}$

Distance between 2 points: $\overrightarrow{P_1P_2}=\overrightarrow{OP_2}-\overrightarrow{OP_1}$



Distance between 2 lines: $|\overrightarrow{RS}|=proj_{\vec{d}_1\times\vec{d}_2}(\overrightarrow{PQ})=\dfrac{|(\vec{d}_1\times\vec{d}_2)\cdot\overrightarrow{PQ}|}{|\vec{d}_1\times\vec{d}_2|}$

How we see/draw:
central projection
parallel projection
orthogonal projection

Matrix

m×n Matrix m rows n columns $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\begin{bmatrix} a_{ij} \end{bmatrix}$ where i=1,…m and j=1,…n

A vector is a matrix with only one column. (all vectors are inherently column vectors)
1.Addition:
2.Scalar Multiplication:

3.Multiplication: ${}^{n}A^{m} \times {}^{m}B^{p} = {}^{n}C^{p}$ inner dimensions must match $c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$

Properties: A(B+C)=AB+AC (AB)C=A(BC) AB≠BA
Linear Transformation: a function that maps a vector to a vector with properties, L(kv)=kL(v) and L(u+v)=L(u)+L(v)
identity transformation: I(u)=u zero transformation: 0(u)=0
Composition of two transformation: linear transformation C as A followed by B,C(u)=B(A(u))
Mc=Mb·Ma

4.Inverse and Determinant: C=AB then $C^{-1}=B^{-1}A^{-1}$

$M^{-1}M=1$ $M^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ determinant IMI or det(M)=ad-bc when IMI=0, no inverse

5.Transpose: C=AB then $C^T=B^TA^T$

Note that for scalar $\alpha^T=\alpha$

6.Partitioned Matrices:

Let A be m×n and $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ where B is $m_1 \times n_1$, E is $m_2 \times n_2$, C is $m_1 \times n_2$, D is $m_2 \times n_1$ and $m_1+m_2=m$ and $n_1+n_2=n$

Let A be a square, non-singular matrix of order m, partition A as $A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A11 and A22 are nonsingular, then $A^{-1}=\begin{bmatrix} \left(A_{11}-A_{12}A_{22}^{-1}A_{21}\right)^{-1} & -A_{11}^{-1}A_{12}\left(A_{22}-A_{21}A_{11}^{-1}A_{12}\right)^{-1} \\ -A_{22}^{-1}A_{21}\left(A_{11}-A_{12}A_{22}^{-1}A_{21}\right)^{-1} & \left(A_{22}-A_{21}A_{11}^{-1}A_{12}\right)^{-1} \end{bmatrix}$

7.Matrix Differentiation: (Jacobian)

y=Ψ(x) where y is an m-element vector and x is an n-element vector: $\frac{\partial y}{\partial x}=\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_3} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}_{m\times n}$

If y=Ax, then ∂y/∂x=A
Let scalar $\alpha=y^TAx$, then $\partial\alpha/\partial x=y^TA$ and $\partial\alpha/\partial y=x^TA^T$.

Proof: ①Define $w^T=y^TA$, then $\alpha=w^Tx$, with definition of Jacobian, $\partial\alpha/\partial x=w^T=y^TA$ ②Since α is scalar, $\alpha=\alpha^T=x^TA^Ty$, then $\partial\alpha/\partial y=\alpha^T/\partial y=x^TA^T$

Quadratic Form of scalar $\alpha = x^T Ax$ where x is n×1 A is n×n and A not depend on x, then $\frac{\partial \alpha}{\partial x} = x^T\left(A + A^T\right)$

Proof:

Reduced Normal Form(RNF)

A matrix is in RNF if: the first non-zero entry in each row is a 1; the other entries in the columns containing these leading 1's are zero; the leading 1's move to the right as we move down the rows; any zero rows are collected at the bottom.

Elementary Operations--Gaussian Elimination
Interchange two equations (two rows of augmented matrix)
Multiply one equation(row) by a non-zero constant
Add a multiple of one equation to another equation(row)

Elementary Matrices: we can associate matrices with three elementary operations.

A linear system:

①Consistent System: inverse
i) Unique Solution(every variable in RNF is a leading variable)
ii) Infinite Solution(at least one free variable)
Rank: number of leading 1's
Solutions to AX=b with m equations and n variables(total number of variables), if rank is r(number of leading variables), the number of free variables is n-r (number of free variables).

Homogeneous System: AX=0 trivial solution X=0

②Inconsistent System: No solution least square problem

error vector $\vec{E} = \overline{A}\vec{x} - \overline{b} = \begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ e_m \end{bmatrix}$ (when AX≠B) $\left\|\vec{E}\right\| = \sqrt{e_1^2 + ...e_m^2}$

Normal Equations: $\overline{A}^T(\overline{b} - \overline{A}\vec{x}) = \vec{0}$ $\overline{A}^T\overline{b} - \overline{A}^T\overline{A}\vec{x} = 0$ $\overline{A}^T\overline{b} = \overline{A}^T\overline{A}\vec{x}$

least squares solution $\vec{x} = (\overline{A}^T\overline{A})^{-1}\overline{A}^T\overline{b}$

Curve Fitting

Curve Fitting: take a set of points and fit a function to those points
Interpolation: take a curve fi and construct new, related points on the curve
Extrapolation: Take a curve, extend the graph to generate new data points i.e. new data point outside the original data interval

Vector Space

[Definition: Vector Space]

A vector space V over a field Γ of elements {α,β,γ...},called scalars, is a set of elements {u,v,w..},called vectors, such that the following axioms are satisfied:

①There exists an operation of vector addition, denoted u+v, such that for all u,v,w ∈V :
(AI)-Closure: u+v ∈V
(AII)-Associativity: (u+v)+w=u+(v+w)
(AIII)-Zero: There exists a zero or null vector 0 ∈V such that v+0=v. Note: v+0 not 0+v
(AIV)-Negative: There exists an negative -v ∈V such that v+(-v)=0.
②There exists an operation of scalar multiplication, denoted αv, such that for all u,v∈V and α,β∈R:
(MI)-Closure: αv ∈V
(MII)-Associativity: α(βv)=(αβ)v
(MIII)-Distributivity: a)under scalar addition: αv+βv=(α+β)v
b)under vector addition:αv+αu=α(v+u)
(MIV)-Unitary: For the identity element 1∈RΓ such that 1u=u.
[Proposition: I] For every u,-u ∈V,-u+u=0
[Proposition: II] For every u ∈V,0+u=u
[Theorem: Cancellation Theorem] If u+w=v+w then u=v for any u,v,w ∈V
[Proposition: III] Let u ∈V.Then:
1. the zero vector 0 ∈V is unique
2. the negative -u of u is unique
3. -(-u)=u
[Definition: Subtraction/Difference]
If u,v ∈V, then the subtraction of v from u, denoted by u-v is u-v=u+(-v)
[Proposition: IV Commutativity] For u,v ∈V, u+v=v+u
[Proposition: V Properties of Zero] For all v ∈V and all α ∈Γ,
1.v0=0 2.0v=0 3.If αv=0, then either α=0 or v=0.
[Proposition: VI] For all v ∈V and all α ∈RΓ, (-α)v=-(αv)=α(-v)
Note: (-1)v=-v. Used to calculating the negative of v.

[Space of Column] ^nR [Space of matrices] $^nR^m$ (all n×m matrices over the field $\mathbb{R}$)
[Space of Function] F(X) = {f | f:X→R} R^2: ordered pairs

Proof: “only one” “uniquely”
Suppose another one exists, then find contradiction or prove these two are same.

Subspace

[Definition: Subspace]

A subset U of a vector space V is a subspace of V if and only if U is itself a vector space over the same field with the same vector addition and scalar multiplication of V

[Theorem: I Subspace Test]

Let $\mathscr{U}$ be a subset of a vector space V.Then U is a subspace of V, over the same field Γ with the same vector addition and scalar multiplication as V, if and only if for all u,v∈V and all α∈R:
(SI)-Zero: There exists a zero or null vector 0∈U
(SII)-Closure under vector addition: u+v ∈U
(SIII)-Closure under scalar multiplication: αu ∈ U
[Image Space] (image/range of A) A∈ mRn im A={y | y= Ax, x∈ nR} ⊂mR
[Null Space] (solution space of A) A∈ mRn null A={x | Ax=0, x∈ nR} ⊂nR

(the set of all solutions to the homogeneous system)

[Subspace of Function] U ={f I f(0)=0} ⊂ F[0,1]

If V is any vector space, zero subspace {0} and V are subspaces of V.

Any subspace of Rn other than {0} or the vector space itself Rn is called proper subspace.

Planes and Lines through the origin in R3 are all subspaces of R3.

Eigenspace: Eλ(A)={X in Rn I AX=λX}

A vector X is in Eλ(A) if and only if (λI-A)X=0. Eλ(A)=null(λI-A)

Eλ(A) is called eigenspace of A corresponding to λ. λ is an eigenvalue of A if Eλ(A)≠{0}.

the nonzero vectors in Eλ(A) are called the eigenvectors of A corresponding to λ.

Spanning Set

[Definition: Linear Combination]

A vector v ∈V is a linear combination of {v1,v2...vn} ⊂V if and only if it can be written as v=∑λjvj=λ1v1+...+λnvn for some λj ∈Γ.

[Definition: Spanning Set]

The span of {v1,v2...vn} ⊂ V, denoted span {v1,v2...vn} is

span {v1,v2...vn}={v I v=∑λjvj , ∀λj∈Γ}

[Proposition: I]

The span of {v1,v2...vn} ⊂ V is a subspace of the vector space V.

Theorem: Let U=span {v1,v2...vn} in Rn.Then,

1.U is a subspace of Rn containing each vi.

2.If W is a subspace of Rn and each vi is in W, then U ⊆ W.

Proof: two span is equal: span{v_1,v_2,v_3} = span{u_1,u_2,u_3}

First,Prove span{v_1,v_2,v_3} ⊆ span{u_1,u_2,u_3} >> v_1,v_2,v_3 ∈span{u_1,u_2,u_3}

Then,Prove span{u_1,u_2,u_3} ⊆ span{v_1,v_2,v_3} >> u_1,u_2,u_3 ∈span{v_1,v_2,v_3}

If $a_1,a_2...a_k$ are nonzero scalars,show that span{$a_1v_1,a_2v_2...a_nv_n$} = span{$v_1,v_2...v_n$}.

Proof: ①Since $a_i$vi is in span{vi} for each i.

Let U=span {$v_1,v_2...v_n$} in R^n.Then, ①U is a subspace of R^n containing each v_i. ②If W is a subspace of R^n and each v_i is in W, then U ⊆ W.

Theorem above shows that span{aivi}⊆span{vi}.

Since vi=(1/ai)(aivi) is in span{aivi},we get span{vi}⊆span{aivi}.

Ø is the basis for {0} span Ø={0} span {0}=?

Linear Independence

[Definition: Linear Independence]

A set of vectors {v1,v2...vn} in a vector space V is linearly independent if and only if ∑λjvj=λ1v1+...+λnvn=0 implies that all λj=0.

(trivial linear combination--every coefficients zero)

(The only way to express the zero vector as a linear combination of the vectors in the set is to have all coefficients be zero)

[Proposition: I]

If {v1,v2...vn} ⊂V is linearly independent and v=∑λjvj for all v ∈V, then λj are uniquely determined.

[Theorem: Minimal Spanning Set]

Let {v1,v2...vn} ⊂V ,a vector space.For every vk(k=1...n),

span{v1...vk-1,vk+1...vn} ⊊ span{v1...vn} if and only if {v1,v2...vn} is linearly independent.

[Corollary:] Prove by Contraposition

Let {v1,v2...vn} ⊂V,a vector space.For at least one vk (1≤k≤n),span {v1...vk-1,vk+1...vn} = span{v1...vn} if and only if {v1,v2...vn} is linearly dependent.

[Theorem: Fundamental Theorem of Linear Algebra] Prove by Contraposition

Let V be a vector space spanned by n vectors.If a set of m vectors from V is linearly independent, then m≤n.

If v≠0 in V, then {v} is an independent set.
Proof:let tv=0.If t≠0 then v=1v=(1/t)(tv)=(1/t)0=0,contrary to assumption.So t=0.

Proof of Fundamental Theorem Page267

Bases

[Definition: Basis]
A set of vector {e1,e2...en} ∈V is a basis for the vector space V if and only if
①{e1,e2...en} is linearly independent and ② {e1,e2...en} spans V (generation)
[Theorem: Invariance Theorem]
Every basis for a given vector space contains the same number of vectors
[Definition: Dimension]
The dimension of a vector space V, denoted dim V, is the number of vectors in any of its bases.
[Proposition: II]
Let V be a finite-dimensional vector space with dim V=n.Then
1.A linearly independent set of vectors in V can at most contain n vectors.
2.A spanning set for V must at least contain n vectors.

Example:
Matrix: dim ${}^{n}R^{m}$=n*m polynomial: dim ℙn=n+1 standard basis{1,x,x2,x3...xn}
By convention, dim{0}=0 >>a basis containing no vectors
({0} is not linearly independent, Zero vector does not belong to any independent set.)
The space Mmn has dimension mn, and one basis consists of all m×n matrices with exactly one entry equal to 1 and all other entries equal to 0.--Standard Basis of Mmn.

Existence of Bases

A vector space V is called finite dimensional if it is spanned by a finite set of vectors.
We regard the zero vector space {0} as finite dimensional (it has an empty basis).
[Theorem: IV] Prove by Contraposition (Contradiction)
Let {v1,v2...vn} ⊂V be linearly independent.Then for a vector v ∈V, {v,v1,v2...vn} is linearly independent if and only if v ≠ span {v1,v2...vn}
[Theorem: V:Existence of Bases] Prove by Construction
Let V be a vector space spanned by a finite set of vectors.Then every linearly independent set of vectors in V can be extended to a basis for V.
[Theorem: VI]
Let U and W be subspaces of a finite-dimensional vector space V.It follows that
1. U is finite-dimensional and dim U ≤ dim V.
2.If U ⊆ W, then dim U ≤ dim W
3.If U ⊆ W, and dim U = dim W, then U = W
[Theorem: VII] Prove by Construction
Any spanning set for a vector space V contains a basis for V.
[Theorem: VIII]
Let V be a vector space and dim V=n.Then
1.Any set {v1,v2...vn} ⊂V that is linearly independent is a basis for V.
2.Any set {v1,v2...vn} ⊂V that spans V is a basis for V.

Rephrase:
Let V (V≠{0} needed?) be a finite dimensional vector space.Then:
1--V. Any independent set in V can be enlarged(by adding vectors) to a basis of V.
2.--VII. Any spanning set for V can be cut down(by deleting vectors) to a basis of V.

Let V be a vector space,and let $\{v_1,v_2...v_n\}$ be a set of vectors in V,where dim V=n. Then:
$\{v_1,v_2...v_n\}$ is independent if and only if $\{v_1,v_2...v_n\}$ spans V.

No set of more than n vectors in V can be independent && No set of fewer than n vectors in V can span V.

span {∅}={0}

symmetric matrix
[b a]
[a c]
V={0} only have one subspace
If U∩W=U, and dim U=dim W, then U=W.
(U∩W=U, U is a subspace of W, dimU=dim W, so U=W)

Dimension&Rank

[Definition: Pivot] A column in rref(A) which contains a leading one has a corresponding column in A, called a pivot column of A
[Definition: Basis]
[Definition: Rank] Number of leading ones in rref(A)
[Definition: Nullity] Number of columns of A - rank(A)

[Definition: Dimension]

[Theorem: The Pivot Theorem] ①The Pivot columns of a matrix are linearly independent.
②A non-Pivot column of A is a linear combination of the pivot columns of A.

Proof:

Consider the fundamental frame sequence identity $\mathbf{rref}(A) = EA$ where $E = E_k \cdots E_2 E_1$ is a product of elementary matrices. Let $B = E^{-1}$. Then

$$\mathrm{col}(\mathbf{rref}(A), j) = E\,\mathrm{col}(A, j)$$

implies that a pivot column j of A satisfies

$$\mathrm{col}(A, j) = B\,\mathrm{col}(I, j).$$

Because the columns of I are independent, then also the pivot columns of A are independent, by the Lemma.

Proof of Non-Pivot Column Dependence

A non-pivot column j of A corresponds to a free variable x_j. Assign $x_j = -1$ and all other free variables zero. Then all lead variables are uniquely determined in the general solution of $Ax = 0$. The vector x so defined is a solution of $Ax = 0$. The equation $Ax = 0$ can be written as a linear combination of the columns of A:

$$\sum_{k=1}^{n} x_k\,\mathrm{col}(A, k) = 0.$$

Move the term $x_j\,\mathrm{col}(A, j)$ across the equal sign, use $x_j = -1$, and then swap sides to obtain the equality

$$\mathrm{col}(A, j) = \sum_{k \neq j} x_k\,\mathrm{col}(A, k).$$

The right side of this relation contains zero terms for all non-pivot columns. Therefore, the right side is a linear combination of the pivot columns of A, which implies the result.

[Lemma: 1] Let B invertible and v1,...vp independent, then Bv1,...Bvp are independent.

Rank(A)+Nullity(A)=column dimension of A
[Theorem:] The number of independent rows of a matrix equals the number of independent columns.
rank(A)=rank(AT)

Proof:

Proof that $\mathbf{rank}(A) = \mathbf{rank}(A^T)$

Let S denote the set of all linear combinations of the rows of A. Then S is a subspace, known as the row space of A. A frame sequence from A to $\mathbf{rref}(A)$ consists of combination, swap and multiply operations on the rows of A. Therefore, each nonzero row of $\mathbf{rref}(A)$ is a linear combination of the rows of A. Because these rows are independent and span S, then they are a basis for S. The size of the basis is $\mathbf{rank}(A)$.

The pivot theorem applied to A^T implies that each vector in S is a linear combination of the pivot columns of A^T. Because the pivot columns of A^T are independent and span S, then they are a basis for S. The size of the basis is $\mathbf{rank}(A^T)$.

The two competing bases for S have sizes $\mathbf{rank}(A)$ and $\mathbf{rank}(A^T)$, respectively. But the size of a basis is unique, called the dimension of the subspace S, hence the equality

$$\mathbf{rank}(A) = \mathbf{rank}(A^T).$$

[Theorem: Pivot Method] Let A be the augmented matrix of v1,...vk. Let the leading ones in rref(A) occur in columns i1,...ip. Then a largest independent subset of k vectors v1,...vk is the set vi1,vi2…vip.

Row & Column space

1.Null Space: kernel(A)=nullspace(A) = {x: Ax=0}
compute rref(A),write out general solution x to Ax=0, where free variables are assigned parameter names t1,...tk. Report the basis for nullspace(A) as the list ∂t1x,... ∂tkx
2.Column Space: image(A)=colspace(A) = {y: y=Ax for some x}
compute rref(A), identify pivot columns i1...ik. Report the basis for colspace(A) as the list of columns i1...ik
3.Row Space: rowspace(A)=colspace(AT)={w: w =ATy for some y}
compute rref(AT), identify pivot columns j1...jk of AT. Report the basis for rowspace(A) as the list of rows j1...jk of A
compute rref(A), rowspace(A) has a different basis consisting of the list of nonzero rows of rref(A)
[Theorem: Dimension] Number of elements in a basis
nullity(A)=dim(nullspace(A))=dim(kernel(A))
rank(A)=dim(colspace(A))=dim(image(A))   =dim(rowspace(A))= dim(kernel(AT))

Eigen Value Problem

$\underline{Ax=b}$ $\underline{x'=Ax}$ $\underline{A}\in{}^nR^n$ $\underline{x}\in{}^nF[R]$ $\underline{x}=[xi]\in{}^nR$

1.Characteristic Polynomial/Eigen polynomial/Eigen Equation:(of $\underline{A}$) $C_{\underline{A}}(\lambda)=\det(\lambda\underline{1}-\underline{A})$

Eigen Problem: $\underline{Ap}=\lambda\underline{p}$ find λ such that $\underline{Ap}=\lambda\underline{p}$ or $(\lambda\underline{1}-\underline{A})\underline{p}=0$

2.Eigen Space: $\varepsilon_\lambda=\text{null}(\lambda\underline{1}-\underline{A})=\{\underline{p}\in{}^nR \mid \underline{Ap}=\lambda\underline{p}\}$

Eigen Vector: $\underline{p}$ is called eigen vector Eigen Matrix: $\underline{p}=[\underline{p_1}, \underline{p_2},..\underline{p_n}]\in{}^nR^n$

3.Multiplicity: Let $\underline{A}\in{}^nR^n$, with eigenvalues $\lambda_i(i=1,2...r)$, If $C_{\underline{A}}(\lambda)=(\lambda-\lambda_1)^{n1}(\lambda-\lambda_2)^{n2}...(\lambda-\lambda_r)^{nr}$ For λ_i **algebraic multiplicity** is n_i and **geometric multiplicity** is $m_i=\dim(\varepsilon_{\lambda i})$.

Recall: Dimension--number of vectors in any bases of a vector space.

Base-linear independent+span

①Multiplicity Theorem: Let λ be eigenvalue of $\underline{A}\in{}^nR^n$,

then $1\le m\le n$, where m is geometric multiplicity and n is algebraic multiplicity.

If n=1, then m=n=1.

②Diagonalization Test: Let $\underline{A}\in{}^nR^n$, with distinct eigenvalues $\lambda_i(i=1,2...r)$,

A is diagonalizable iff the algebraic and geometric multiplicity are same for all λ_i. ($n_i=m_i$)

$$\begin{aligned}\dot{x}_1 &= 4x_1 \qquad + x_3\\ \dot{x}_2 &= 2x_1+3x_2+2x_3\\ \dot{x}_3 &= x_1 \qquad + 4x_3\end{aligned}$$

1.Cast differential equation in matrix form $\underline{x'=Ax}$ $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ $A=\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix}$ $x_0=\begin{bmatrix}0\\2\\2\end{bmatrix}$

2.Eigen Value of the System: $\det(\lambda I-A)=\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda&0&1\\2&3-\lambda&2\\1&0&4-\lambda\end{bmatrix}$

Eigen Equation: $C(\lambda)=\det(\lambda\underline{1}-\underline{A})=(\lambda-3)^2(\lambda-5)=0$

Eigen Value : $\lambda_1=3$ (repeated eigenvalue once again) $\lambda_2=5$

3.Basis for each eigen space (eigenvectors):

①For $\lambda_1=3$, $\overline{\overline{3\underline{1}-\underline{A}}}=\begin{bmatrix}-1&0&-1\\0&0&0\\0&0&0\end{bmatrix}$ -x-z=0 so $\begin{aligned}x&=-z\\y&=y\\z&=z\end{aligned}$ --> $\begin{bmatrix}x\\y\\z\end{bmatrix}=y\begin{Bmatrix}0\\1\\0\end{Bmatrix}+z\begin{Bmatrix}-1\\0\\1\end{Bmatrix}$

$p=p_2\begin{bmatrix}0\\1\\0\end{bmatrix}+p_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$ eigen space $\varepsilon_{\lambda=3}=span\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$

②similarly, For $\lambda_2=5$, $\overline{\overline{5\underline{1}-\underline{A}}}=\begin{bmatrix}1&0&-1\\0&1&-2\\0&0&0\end{bmatrix}$ $p=p_3\begin{bmatrix}1\\2\\1\end{bmatrix}$ eigen space $\varepsilon_{\lambda=5}=span\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$

Basis for $\lambda_1=3$, $E_{\lambda=3}=\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}=\{p_1,p_2\}$ for $\lambda_2=5$, $E_{\lambda=5}=\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}=\{p_3\}$

4.Why A is **diagonalizable**:

For $\lambda_1=3$, $m_1=\dim\varepsilon_{\lambda=3}=2=n_1$ For $\lambda_2=5$, multiplicity theorem gives $m_2=n_2=1$

5.Unique Solution to the system subject to/given the initial conditions $x_1(0)=0$, $x_2(0)=2$, $x_3(0)=2$

The unique solution to the system of linear differential equation x'=Ax is $x(t)=C_1e^{3t}\underline{p_1}+C_2e^{3t}\underline{p_2}+C_3e^{5t}\underline{p_3}$

where $\eta_0 = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$ is the solution to $\underline{p}\underline{z} = \underline{x_0} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ so z_1 $\begin{array}{rcrcl} z_2 & + & z_3 & = & 0 \\ & + & 2z_3 & = & 2 \\ -z_2 & + & z_3 & = & 2 \end{array}$

$z_1=0$ $z_2=-1$ $z_3=-1$ so $\eta_0 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ Solution $x(t)=0e^{3t}\underline{p_1}-1e^{3t}\underline{p_2}-1e^{5t}\underline{p_3} = \begin{bmatrix} -e^{3t}+e^{5t} \\ 2e^{5t} \\ e^{3t}+e^{5t} \end{bmatrix}$

Column j of the n×n identity matrix In is denoted Ej and called the jth coordinate vector in Rn and the set {E1,E2...En} is called the standard basis of Rn.

Let U and W be subspace of .
Intersection U ∩W = {X in Rn l X belongs to both U and W }
Sum U+W = {X in Rn l X is a sum of a vector in U and a vector in W }

linear combination/independence
A linear combination vanishes if it equals the zero vector.
A linear combination is trivial if every coefficient is zero.
Standard Basis {E1,E2...Ek} of is independent.
A set of vectors in Rn is linearly dependent if some nontrivial linear combination vanishes.