Partial Differential Equation

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Partial Differential Equation

[**Ordinary Differential Equation**] dependent on 1 variable x
[**Partial Differential Equation**] mathematical equation containing partial derivatives
[**Boundary Condition**] specify value of solution u(or its derivatives) at a fixed point
[**Initial Condition**]
[**Order**] The order of a DE is the highest derivative that appears.
[**Linear**] A DE is linear if the solution u and its derivative only appear raised to the power of one. (every derivative of u and itself only appear in first order.)
[**Homogeneous**] A linear PDE is homogeneous if it is in the form L(u)=0, where L is a Linear Operator (a map that maps functions to functions) acting on u: L(λu+v)=λL(u)+L(v) for all u,v.
Any linear combination of linear operators is a linear operator.
Linear Homogeneous iff every term contains exactly one instance of u or one of its derivative.
Linear if L(λu+v)=λL(u)+L(v) for all u,v
Ex. $\partial^2u/\partial x^2+\partial u/\partial t=x^2$ (linear) $\partial^2u/\partial x^2+\partial u/\partial t=(\partial u/\partial y)^2$ (no linear)
A PDE is linear if there exists a linear map L such that the PDE can be written L(u)=f(x,y), a linear PDE is homogeneous if f(x,y)=0.

(Superposition of Solutions)
If u_1,u_2 satisfy the same linear homogeneous PDE, then so does $\lambda_1u_1+\lambda_2u_2$ (linear combination of solutions) for $\lambda_1,\lambda_2\in\mathbb{R}$

Proof:

Linearity and Homogeneity can apply to BC or IC.
Ex. $u(0,y)=y^2$ (linear, not homo u≠0) $\partial u(x,0)/\partial x=0$ (linear, homo) $u^2(0,x)=3$ (no linear, no homo)

Let d≥1 ①$\mathbb{R}^d$ ②basis for $\mathbb{R}^d=\{e_1,\ldots e_d\}$ $e_j=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}$ j^{th} position so $\vec{x}=\begin{pmatrix}x_1\\ \vdots\\ x_d\end{pmatrix}=\sum_{j=1}^{d}x_je_j$

Let f: $\mathbb{R}^d\to\mathbb{R}$ be twice differentiate:

①Gradient $\nabla f=\sum_{j=1}^{d}\left(\frac{\partial f}{\partial x_j}\right)e_j$ $\in\mathbb{R}^d$ (vector) $\nabla=\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}$

②Laplacian $\nabla^2f=\Delta f=\sum_{j=1}^{d}\frac{\partial^2 f}{\partial^2 x_j}$ (scalar)

$$\Delta=\underbrace{\nabla\cdot\nabla}_{inner\ product}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

In polar coordinate: $$\Delta=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}\right)=\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}\right)$$

③Directional Derivative of f in direction $x\in\mathbb{R}^d$: $$\vec{x}\cdot\nabla f=\sum_{j=1}^{d}x_j\frac{\partial f}{\partial x_j}$$

Show that $e_j\nabla f=\frac{\partial f}{\partial x_j}$

Proof:

2. Method of Characteristic

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+c(x,y)u(x,y)=\begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix}\cdot\nabla u+c(x,y)u(x,y)=g(x,y)$$

(First-Order Linear PDE in 2 variables)
if homo g=0
$a(x,y)\cdot(\partial u/\partial x)+b(x,y)\cdot(\partial u/\partial y)+c(x,y)\cdot U(x,y)=g(x,y)$ initial condition $u(x_0,y)=f(y)$ for all $y\in R$
Known: a b c g f & x_0 Find: u(x,y) by constructing solution surface $S=\{(x,y,u(x,y)): x,y\in R\}$

①Solve system of ODE initial-value problems:
$X'(t)=a(X(t),Y(t))$ and $X(0)=x_0$ $Y'(t)=b(X(t),Y(t))$ and $Y(0)=y_0$ (free parameter now)
②Solve ODE IV problems: $U'(t)=g(X(t),Y(t)) - c(X(t),Y(t))U(t)$ subject to $U(0)=U(x_0,y_0)=f(y_0)$
Note: 若$g(X(t),Y(t))-c(X(t),Y(t))U(t)$ 包含x y, 参照①的结果 替换为t 或y_0.
③Given $x,y\in R$, find t such that $X(t)=x$ and $Y(t)=y$. Note: sometimes not explicitly
④Plug back and Check $u(x,y)=U(X(t),Y(t))$ satisfies original PDE.

Note: error function-- integral cannot be computed in closed form

Method of Characteristic for PDE with 3 variables:

$$a(x,y,z)\frac{\partial u}{\partial x}+b(x,y,z)\frac{\partial u}{\partial y}+c(x,y,z)\frac{\partial u}{\partial z}+d(x,y,z)u(x,y,z)=g(x,y,z)$$

①Parametrize u(x(t),y(t),z(t)) so dx/dt=a(x,y,z) dy/dt=b(x,y,z) dz/dt=c(x,y,z) $x(0)=x_0$ $y(0)=y_0$ $z(0)=z_0$
②du/dt=-d(x,y,z)u we can get u(t)=... with IC $u(0)=f(x_0,y_0,z_0)$
③use BC to get f(x,y,z), u(x,y,z)=f(x,y,z)·.....

Example: $$\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=x^2$$ with boundary condition u(1,y)=y
① a=1 b=3 c=0 $g=x^2$ so x'=a=1 y'=b=3
$x=t+x_0=t+1$ since $x_0=1$ $y=3t+y_0$ ② $u'=x^2=(t+1)^2$
③so $u=t^3/3+t^2+t+y_0$ $= (x-1)^3/3+(x-1)^2+(x-1)+y_0$
Note: If b=y, then Y'=y, $y=y_0e^t$

Example: Wave Equation (vibrating strings and membranes)

Wave Equation

Vibrating String with Fixed Ends:
Boundary Condition: u(0,t)=0 u(L,t)=0
Initial Condition: u(x,0)=f(x) ∂u(x,0)/∂t=g(x)

Example: Heat Equation

1. **Heat Conduction 1-D rod**:
thermal energy density e(x,t)=amount of thermal energy per unit volume
Heat Energy= e(x,t)AΔx
Heat flux ϕ(x,t)= amount of thermal energy per unit time flowing to the right per unit surface area
Heat Source Q(x,t)=heat energy per unit volume generated per unit time

Conservation of heat energy:

thin slice: $$\frac{\partial}{\partial t}\left[e(x,t)A\Delta x\right] \approx \varphi(x,t)A - \varphi(x+\Delta x,t)A + Q(x,t)A\Delta x$$ Δx→0 ∂e/∂t=-∂ϕ/∂x+Q

Rate of change of heat energy in time	heat energy flowing across boundaries per unit time	heat energy generated inside per unit time

exact $$\int_a^b \frac{\partial e(x,t)}{\partial t}dx = \frac{d}{dt}\int_a^b e(x,t)dx = \varphi(a,t)-\varphi(b,t)+\int_a^b Q(x,t)dx = -\int_a^b \frac{\partial \varphi(x,t)}{\partial x}dx + \int_a^b Q(x,t)dx$$

Again, ∂e/∂t=-∂ϕ/∂x+Q

I> u(x,t)=temperature c=specific heat ρ(x)=mass density
total thermal energy= e(x,t)AΔx=c(x)u(x,t)ρAΔx so e(x,t)=c(x)u(x,t)ρ
II> Fourier Law of conduction: $\phi=-K_0(\partial u/\partial x)$

With I and II, heat equation becomes $$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)+Q$$

no source of thermal energy so Q=0 $$\frac{\partial u}{\partial t} = \frac{K_0}{c\rho}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

②Initial Condition: u(x,0)=f(x)
③Boundary Condition: at each end of the rod (x=0 & x=L)
i> Prescribed Temperature: $u(0,t)=u_B(t)$=temperature of fluid bath/reservoir with which the rod is in contact.
ii> Insulated Boundary: $$-K_0(x=0)\frac{\partial u(0,t)}{\partial x}=\varphi(t)$$ iii> Newton Law of Cooling: $$-K_0(x=0)\frac{\partial u(0,t)}{\partial x} = -H[u(0,t)-u_B(t)]$$

$$y_{1,2} = (K+T\pm\sqrt{K^2-KT+T^2})/3,$$

2. **Heat Conduction multidimension**:

$$c\rho\frac{\partial u}{\partial t} = K_0\nabla^2 u + Q$$ Q=0 $$\frac{\partial u}{\partial t} = \frac{K_0}{c\rho}\left(\nabla^2 u\right)$$

Heat energy Conservation: $$\frac{d}{dt}\iiint_R c\rho u\,dV = -\oiint \varphi\cdot\hat{n}\,dS + \iiint_R Q\,dV$$

by divergence Theorem $$c\rho\frac{\partial u}{\partial t} = -\nabla\cdot\varphi + Q$$ Fourier Law of Heat Conduction: $$\varphi = -K_0\nabla u$$

3. **Heat Equation with zero temperature at finite ends**: eigenfunction= sin
Boundary Condition: u(0,t)=0 u(L,t)=0
u(0,t)=0 → ψ(0)G(t)=0 → <u>ψ(0)=0</u> or G(t)=0 (trivial solution since u(t,x)=0 then)

u(L,t)=0 → ψ(L)G(t)=0 → ψ(L)=0 or G(t)=0 (trivial solution since u(t,x)=0 then)

①λ>0: ψ(x)={ sin (nπ/L)x } ②λ=0 trivial ③λ<0 trivial

so $$u(t,x)=\left\{\sin\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right\}$$ Superposition ~~ $$u(t,x)=\sum_{n=0}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Initial Condition: u(x,0)=f(x)

at t=0, u(0,x)=Σ$A_n$sin( ) f(x) must be in similar form as u(0,x)

$$u(0,x)=f(x)=\sum_{n=0}^{M}A_n\sin\left(\frac{n\pi}{L}x\right)$$

i) function f(x) can be approximated by a finite linear combination of eigenfunctions

ii) approximation goes better as M increases iii) consider limit M→∞, resulting infinite series converges to f(x)

4. **Heat Equation with insulated ends**: eigenfunction=cos

Boundary Condition: ∂u(0,t)/∂x=0 ∂u(L,t)/∂x=0

∂u(0,t)/∂x=0 → ∂ψ(0)/∂x=0

∂u(L,t)/∂x=0 → ∂ψ(L)/∂x=0

①λ>0: ψ(x)={ cos (nπ/L)x } ②λ=0 ψ(x)=Constant ③λ<0 trivial

so $$u(t,x)=\left\{\cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right\}$$ Superposition ~~ $$u(t,x)=\sum_{n=0}^{\infty}A_n\cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Initial Condition: u(x,0)=f(x)=A_0+Σ$A_n$cos( )

at t=0, u(0,x)=Σ$A_n$cos( )

5. **Heat Conduction in a thin circular ring**: eigenfunction = sin+cos

Boundary Condition: Ψ(-L)=Ψ(L) ∂Ψ(-L)/∂x=∂Ψ(L)/∂x mixed periodic BC non-homogeneous

①λ>0: ψ(x)={ cos (nπ/L)x }

②λ=0 ψ(x)=Constant

Note: only one independent eigenfunction corresponding to λ=0 but λ=$(n\pi/L)^2$>0, there are 2 independent eigenfunctions sin( ) and cos( ).

③λ<0 trivial

After superposition: $$u(x,t)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}+\sum_{n=0}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Initial Condition: u(x,0)=f(x)=A_0+Σ$A_n$cos( )+ Σ$B_n$sin( )

at t=0, u(0,x)=Σ$A_n$cos( )

6. **Laplace Equation**: Heat flow for a steady state

①inside a Rectangle [0,L]×[0,H]:

(steady-state heat conduction in a 2D region, no heat source, boundary temperature is a prescribed function of position):

Find equilibrium temperature inside rectangle

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Boundary Condition: u(0,y)=g_1(y) u(L,y)=g_2(y) u(x,0)=f_1(x) u(x,H)=f_2(x)

②**in a Thin Disk**: Poisson Equation

(Steady State, no heat source, prescribed temperature on boundary):

Find equilibrium temperature inside disk

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right)=0$$

Boundary Condition: u(a,θ)=f(θ) u(r,-π)=u(r,π)

separation of variable: u(r,θ)=G(r)h(θ)

in this case, separation constant is +λ, or G(0)=∞, non-physically.

(a) h(θ) can be found in the similar way previously.
(b) G(r), set $G(r)=r^p$, then find p , p=p(λ)

Separation of Variable

Heat Equation $\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$ (t>0, 0<x<L) linear homogeneous PDE

Separation of Variable/Product Method: u(x,t)=Ψ(x)G(t)

∂u/∂t=Ψ(x)G'(t) $\partial^2 u/\partial x^2$ =Ψ''(x)G(t) Heat Equation: Ψ(x)G'(t)=k Ψ''(x)G(t)

$\frac{1}{k}\frac{G'(t)}{G(t)}=\frac{\varphi''(x)}{\varphi(x)}=-\lambda$ so $G'(t)=-\lambda kG(t)$ $\varphi''(x)=-\lambda\varphi(x)$

Note: separation constant ±λ, it depends on physical meaning of Ψ(x) and G(t). Sometimes, Ψ(x) must satisfy Ψ(0)=0, which means it is oscillating, then +λ cannot satisfy.

I. **Time-Dependent Equation** G(t): $G'(t)=-\lambda kG(t)$ $G(t)=\left\{e^{-k\lambda t}\right\}$

①λ>0 exponentially decay at t+ ②λ<0 exponentially increase at t+ ③λ=0 remain constant

II. **Position-Dependent Equation** Ψ(x): $\varphi''(x)=-\lambda\varphi(x)$

Note: if solutions of PDE+BC UNIQUE, then only trivial solution Ψ(x)=0; not able to obtain nontrivial solutions of a linear homogeneous PDE by product method.

Non-Trivial Solution Ψ(x): eigenvalue λ of Boundary Value Problem (a certain value of λ)

Note: eigenfunction Ψ(x) corresponding to eigenvalue λ only exists for certain value of λ.

①λ>0 (2 complex conjugate roots): $r=\pm i(\lambda)^{1/2}$ oscillating

Eigenfunction: $\varphi(x)=\left\{e^{i\sqrt{\lambda}x},e^{-i\sqrt{\lambda}x}\right\}=\{\cos\sqrt{\lambda}x,\sin\sqrt{\lambda}x\}$ Note: cosx and sinx are linear combinations of $e^{\pm ix}$.

②λ=0 (1 root): r=0 eigenfunction: $\varphi(x)=\{x,Constant\}$

③λ<0 (2 real unequal roots): $r=\pm(-\lambda)^{1/2}$ one positive one negative (一般都没有solution) exponential

Eigenfunction: $\varphi(x)=\left\{e^{\sqrt{-\lambda}x},e^{-\sqrt{-\lambda}x}\right\}=\{\cosh\sqrt{-\lambda}x,\sinh\sqrt{-\lambda}x\}$

Note: $\cosh z=\frac{1}{2}(e^z+e^{-z})$ $\sinh z=\frac{1}{2}(e^z-e^{-z})$ (coshz)'=sinhz (sinhz)'= coshz

In a polar

Summary of Separation of Variable Method:

1.linear homogeneous PDE with linear homogeneous BC
2.temporarily ignore nonzero IC
3.separate variable and separation constant
4.Determine separation constants as eigenvalue of BVP
5.Solve other DE, product all solutions of PDE
6.principle of superposition(linear combination of all product solutions)
7.attempt to satisfy IC by determine Fourier Coefficient by Orthogonality of Eigenfunctions.
Note: do not apply IC u(x,0)=f(x) before superposition!

Boundary Condition and Initial Condition: (homogeneous)
Boundary Condition and Initial Condition: (non-homogeneous) Heat Equation Q5

Complex Differentiable and Harmonic Functions

$z = x + iy \qquad x = \frac{z + \bar{z}}{2} \qquad y = \frac{z - \bar{z}}{2i}$

Harmonic and analytic function:

Cauchy-Riemann: $u_x = v_y \quad u_y = -v_x$

f is complex differentiable, Wirtinger Equation $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$

Mean Value Property:

Let D be a domain in R2, u be continuous on closure D and harmonic in D, for any z=(x,y)∈D and r>0, such that B(z,r)=B((x,y),r)⊂D: $u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\, d\theta = \frac{1}{2\pi}\int_{\partial B(z,r)} u(z + e^{it})\, dt$

Proof:

Maximum Principle:

If u is continuous on the domain D and harmonic in D, then if u has a maximum at (x,y)⊂closure D, then (x,y)∈∂D (maximum on the boundary)

Proof:

Cauchy Theorem:

Fourier Series

Fourier Series: $f(x)=\sum_{n=0}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$

A function f(x) is <u>piecewise smooth</u> (on some interval) if the interval can be broken up into pieces/ sections such that in each piece the f(x) and its derivative df/dx is continuous.

$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx$ $\quad$ $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$ $\quad$ $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$

$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx$ $\quad$ $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos(\frac{n\pi x}{L})\,dx$ $\quad$ $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin(\frac{n\pi x}{L})\,dx$

Note: Odd Function $A_0=0$ $A_n=0$ Even Function $B_n=0$

Fourier Convergence Theorem:
Let f be piecewise continuously differentiable(smooth) on [-L,L], then Fourier Series of f converges to periodic extension of f(x) where periodic extension of f(x) is continuous, and to ½(f(x+)+f(x-)) where periodic extension of f(x) has a jump discontinuity.

If f is continuous at x, then $f(x^+)=f(x^-)=f(x)$
$f(-L^-)=f(L)$ $\quad$ $f(L^+)=f(-L)$

I.**Odd Function**: $A_0=0$ $A_n=0$
Let f:[0,L]→R, then <u>odd extension</u> of f is function f_o:[-L,L]→R where $f_o=f(x)$ if x≥0 and $f_o=-f(-x)$ if x<0.
Fourier Sine Series of f on [0,L]:

$$f(x)\sim\sum_{n=0}^{\infty}B_n(f)\sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad B_n(f)=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

II. **Even Function**: $B_n=0$
Let f:[0,L]→R, then <u>even extension</u> of f is function f_e:[-L,L]→R where $f_e=f(x)$ if x≥0 and $f_e=f(-x)$ if x<0.

Fourier Cosine Series of f on [0,L]:

$$f(x)\sim\sum_{n=0}^{\infty}A_n(f)\cos\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad A_0(f)=\frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad A_n(f)=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

f(x)=½[f(x)+f(-x)]+½[f(x)-f(-x)] $\quad$ $f_e(x)$=½[f(x)+f(-x)] $\quad$ $f_o(x)$=½[f(x)-f(-x)]

Complex Fourier Series:

$$f(x)=\sum_{m=-\infty}^{\infty}C_m e^{i\frac{m\pi x}{L}} \quad where \quad C_m=\begin{cases}\frac{A_m(f)-iB_m(f)}{2} & m>0\\ A_0(f) & m=0\\ \frac{A_{-m}(f)+iB_{-m}(f)}{2} & m<0\end{cases}$$

$$C_m=\frac{1}{2L}\int_{-L}^{L}f(x)e^{-i\frac{m\pi x}{L}}\,dx$$

Proof:

$$f(x)=A_0+\sum_{n=1}^{\infty}A_n(f)\cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty}B_n(f)\sin\left(\frac{n\pi x}{L}\right)=A_0+\sum_{n=1}^{\infty}A_n(f)\left(\frac{e^{i\frac{n\pi x}{L}}+e^{-i\frac{n\pi x}{L}}}{2}\right)+\sum_{n=1}^{\infty}B_n(f)\left(\frac{e^{i\frac{n\pi x}{L}}-e^{-i\frac{n\pi x}{L}}}{2i}\right)$$

$$=A_0+\sum_{n=1}^{\infty}\left(\frac{1}{2}A_n(f)-\frac{i}{2}B_n(f)\right)e^{i\frac{n\pi x}{L}}+\sum_{n=1}^{\infty}\left(\frac{1}{2}A_n(f)+\frac{i}{2}B_n(f)\right)e^{-i\frac{n\pi x}{L}}$$

$$=A_0+\sum_{n=1}^{\infty}\left(\frac{1}{2}A_n(f)-\frac{i}{2}B_n(f)\right)e^{i\frac{n\pi x}{L}}+\sum_{-\infty}^{n=-1}\left(\frac{1}{2}A_{-n}(f)+\frac{i}{2}B_{-n}(f)\right)e^{i\frac{n\pi x}{L}}$$

$$=\sum_{m=-\infty}^{\infty}C_m e^{i\frac{m\pi x}{L}}$$

Differentiate Fourier Series:
A Fourier Series(continuous) can be differentiated term by term if f'(x) is piecewise smooth.

If f:[-L,L]→R is continuously differentiable and f(L)=f(-L), then its Fourier Series can be differentiable term-by-term, and the result is the Fourier Series of f'.

If f:[0,L]→R, then its Fourier Cosine Series can be differentiated if f is continuously differentiable and f(L)=f(-L), then its Fourier Series can be differentiable term-by-term, and the result is the Fourier Series of f'. (even function, endpoints always match)

If f:[0,L]→R is continuously differentiable and f(0)=f(L)=0, then its Fourier Sine Series can be differentiable, and the result is the Fourier Series of f'. (odd function, has to match zero to be continuous)

Orthogonality: functions A(x) and B(x) are orthogonal if $\int_0^L A(x)B(x)\,dx=0$ over [0,L]

For trigonometric functions:

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}=\begin{cases}0 & n\neq m\\ L/2 & n=m\neq 0\\ L & n=m=0\end{cases}\qquad \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}=\begin{cases}0 & n\neq m, m=n=0\\ L/2 & n=m\end{cases}$$

Integral over any number of complete periods of square of a sine or cosine is one half of the interval length.

Application: For $f(x)=\Sigma B_n\sin(n\pi x/L)$ n:dummy index $f(x)\sin(m\pi x/L)=\Sigma B_n\sin(n\pi x/L)\sin(m\pi x/L)$
Integrate from 0 to L: $\int f(x)\sin(m\pi x/L)=\int\Sigma B_n\sin(n\pi x/L)\sin(m\pi x/L)=\Sigma B_n\int\sin(n\pi x/L)\sin(m\pi x/L)$
Summing over n, only m=n term contributes to infinite sum: $\int f(x)\sin(m\pi x/L)=B_m\int\sin(m\pi x/L)\sin(m\pi x/L)=B_m\int\sin^2(m\pi x/L)$
so $B_m=\int f(x)\sin(m\pi x/L)\div\int\sin^2(m\pi x/L)=(2/L)\int f(x)\sin(m\pi x/L)$

Orthogonal Set of Functions: a set of functions each member of which is orthogonal to every other member

odd function, Fourier sine series, $A_n=0$, $B_n\neq0$ (sin term)	even function, Fourier cosine series, $A_n\neq0$, $B_n=0$ (cos term)
$f(x)\sim\sum_{n=0}^{\infty}B_n(f)\sin\left(\frac{n\pi x}{L}\right)$ with $B_n(f)=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$f(x)\sim\sum_{n=0}^{\infty}A_n(f)\cos\left(\frac{n\pi x}{L}\right)$ with $A_0(f)=\frac{1}{L}\int_0^L f(x)dx$ $A_n(f)=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$
Differentiate term by term when f(0)=0, f(L)=f(-L)=0	Differentiate term by term when f'(0)=0

Uniqueness of Solution Proof:
suppose solutions u_1, u_2 so $v=u_1-u_2$ also a solution by superposition
① maximum principle for harmonic functions:
② method of energy functionals: E[w](t)
show E[v](0)=0 and $\partial E/\partial t=0$ so E=0 forever

Maximize or Minimize Problem:
Consider about dX/dt=0 t is the variable

Sturm-Liouville Eigenvalue Problem

Sturm-Liouville Eigenvalue Problem: $\frac{d}{dx}\left(p\frac{d\varphi}{dx}\right)+q\varphi+\lambda\sigma\varphi=0$ where λ is eigenvalue, x∈(a,b)

Regular Sturm-Liouville: $\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right)+q(x)\varphi+\lambda(x)\sigma(x)\varphi=0$ x∈(a,b)

Boundary Condition:

$$\beta_1\varphi(a)+\beta_2\frac{d\varphi}{dx}(a)=0$$

$$\beta_3\varphi(b)+\beta_4\frac{d\varphi}{dx}(b)=0$$

	Heat Flow	Vibrating String	
$\phi=0$	fixed(zero) temperature	fixed(zero) displacement	First Kind, Dirichlet Condition
$d\phi/dx=0$	Insulated	Free	Second Kind, Neumann Condition
$d\phi/dx=\pm h\phi$ (left end+, right end-)	(Homogeneous) Newton's law of cooling 0° outside temperature, $h=H/K_0>0$ (physical)	(Homogeneous) elastic boundary condition, $h=k/T_0>0$ (physical)	Third Kind, Robin Condition
$\phi(-L)=\phi(L)$ $d\phi(-L)/dx=\phi(L)/dx$ $\lvert\phi(0)\rvert<\infty$	Perfect Thermal Contact, Bounded Temperature	---	Periodicity Condition Singularity Condition

Regular: coefficient p q σ real and continuous everywhere (including endpoints) p>0 σ>0 everywhere(including endpoints)

※ **Theorem**:

①all eigenvalues λ are real

②There exist an infinite number of eigenvalues: $\lambda_1<\lambda_2<\ldots<\lambda_n<\lambda_{n+1}<\ldots$ (bounded below)
there is a smallest eigenvalue, denoted λ_1, there is not a largest eigenvalue and $\lambda_n\to\infty$ as $n\to\infty$

③Corresponding to each eigenvalue λ_n, there is an eigenfunction $\phi_n(x)$, which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly n-1 zeros for a<x<b.

④the eigenfunction $\phi_n(x)$ form a "complete" set, any piecewise smooth function f(x) can be represented by a **generalized Fourier series** of eigenfunctions (**eigenfunction expansion**): $f(x)=\sum_{n=1}^{\infty}c_n\varphi_n(x)$

where <u>coefficient of eigenfunction expansion/generalized Fourier coefficient</u>: $c_n=\frac{\int_a^b f(x)\varphi_n(x)\sigma(x)dx}{\int_a^b \varphi_n^2(x)\sigma(x)dx}$

this infinite series converges to [f(x+)+f(x-)]/2 for a<x<b (if coefficients a_n are properly chosen)

⑤Eigenfunctions belonging to different eigenvalues are **orthogonal** relative to the weight function σ(x):

$\int_a^b \varphi_n(x)\varphi_m(x)\sigma(x)dx=0$ if $\lambda_n\neq\lambda_m$.

⑥Any eigenvalue can be related to its eigenfunction by the **Rayleigh Quotient**:

$$\lambda = \frac{\int \varphi L(\varphi)}{\int \varphi^2 \sigma} = \frac{[-p\varphi \frac{d\varphi}{dx}]_a^b + \int_a^b [p\left(\frac{d\varphi}{dx}\right)^2 - q\varphi^2]dx}{\int_a^b [\varphi^2 \sigma]dx}$$

where boundary conditions may somewhat simplify this expression.

Note: apply RQ to eliminate possibility of negative eigenvalues.

※ Define **linear differential operator** L: V→C^2[a,b] by $L(f) = -\lambda \sigma f = \frac{d}{dx}(p\frac{df}{dx}) + qf$
(operator acts on a function and yields another function)

If λ and ϕ are an eigenvalue-eigenfunction pair of SL problem L(ϕ)+λσϕ=0

※ **Lagrange Identity**: $uL(v) - vL(u) = \frac{d}{dx}[p(uv' - vu')]$

※ **Green's Formula**: $\int_a^b [uL(v) - vL(u)] = [p(uv' - vu')]_a^b$

Self-Adjointness:

For functions u, v satisfying the same set of homogeneous Boundary Conditions (regular SL type), then $\int_a^b [uL(v) - vL(u)] = 0$. Operator L is self-adjoint.

※ Speed of Convergence

Parsevals Relation

In higher dimensions, solutions to SL problem have the following properties:

1.all the eigenvalues are real 2.$\lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 \le \ldots$ <or≤ ? lim $\lambda_n = \infty$ as $n \to \infty$

3.λ_1 is a simple eigenvalue but λ_n, n≥2 might not be. There are only finitely many eigenfunctions(up to constant multiple)

4.The eigenfunctions form a complete basis for any f∈C^2 satisfying the BC. We can write $f(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$ for some $c_n(f) \in R$

5.For any n≠m, orthogonal

Green's Function

When the PDE is not homogeneous

Green Function for Boundary Value Problems for ordinary differential equantions
1-Dimensional steady state heat equation

①Method of Variation of Parameters:
Wronskian:

②**Eigenfunction Expansion**:

$$G(x,x_0)=\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(x_0)}{-\lambda_n\int_a^b \varphi_n^2\sigma\,dx}$$

③**Dirac Delta Function**:

$$\delta(x-x_i)=\begin{cases}0 & x\neq x_i\\ \infty & x=x_i\end{cases}$$

Properties: $\int_{-\infty}^{\infty}\delta(x-x_i)dx_i=1$ even function $\delta(x-x_i)=\delta(x_i-x)$

$$H(x-x_i)=\begin{cases}0 & x<x_i\\ 1 & x>x_i\end{cases}$$

Heaviside Unit Step Function

$\delta(x-x_i)=\frac{d}{dx}H(x-x_i)$ and $H(x-x_i)=\int_{-\infty}^{x}\delta(x_0-x_i)dx_0$ Scaling: $\delta[c(x-x_i)]=\frac{1}{|c|}\delta(x-x_i)$

Define $f(x)=\int_{-\infty}^{\infty}f(x_i)\delta(x-x_i)dx_i$

$$\frac{d^2G(x,x_0)}{dx^2}=\delta(x-x_0)$$

Jump Condition: Green function G(x,x_s) is continuous at x=x_s (source), but dG/dx is not continuous at x=x_s.(discontinuity=1)

Non-homogeneous Boundary Conditions:

Fredhom Alternative:

Number of Solutions	Eigenvalue	$\int_a^b f(x)\varphi_n(x)dx$	
1	λ≠0 for all n	0	force function orthogonal to eigen functions
infinite	λ=0 exist n	0	
0	λ=0 exist n	≠0	

If nontrivial homogeneous solutions solving the same homogeneous boundary conditions are equivalent to eigenfunctions corresponding to zero eigenvalue.

General **first order linear equation** : $\frac{dy}{dt}+p(t)y=g(t)$

Step:

①Multiply the DE by an integrating factor μ(t)=$e^{\int p(t)dt}$.

μ(t)y'+μ(t)p(t)y= μ(t)g(t) So, μ(t)'= μ(t)p(t) Separable >>dμ(t)/μ(t)=p(t)dt ln μ(t)=∫p(t) +k choose k=0, μ(t)=$e^{\int p(t)dt}$.

Remind: $e^{\ln x}=x$ $n\ln x=\ln x^n$.

②d(μ(t)y)/dt= (μ(t)y)'= μ(t)g(t) ③ μ(t)y= ∫μ(t)g(t)=*XXX*+C Note: 注意+C的!!!

④y= [∫μ(t)g(t)]/μ(t) =[*XXX*+C]/μ(t)

when ∫μ(t)g(t) hard to evaluate >>> $\mu(t)=\exp\left[\int_{t_0}^{t} p(s)ds\right]$

$$y=\frac{1}{\mu(t)}\int_{t_0}^{t}\mu(z)q(z)dz+\frac{C}{\mu(t)}=\frac{\int_{t_0}^{t}\mu(z)q(z)dz}{\exp\left[\int_{t_0}^{t}p(s)ds\right]}+\frac{C}{\exp\left[\int_{t_0}^{t}p(s)ds\right]}=\int_{t_0}^{t}q(z)\exp\left[\int_{t}^{z}p(s)ds\right]+C\exp\left[-\int_{t_0}^{t}p(s)ds\right]$$

Infinite Domain Problems: Fourier Transform

1. **Fourier Transform**: $F(\omega)=F[f](\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{i\omega x}\,dx$ 第一个F 是function 第二个 F 是transform operator

Inverse Fourier Transform: $f(x)=F^{-1}[g](x)=\int_{-\infty}^{\infty}g(\omega)e^{-i\omega x}\,d\omega$

Fourier Transform & its inverse is linear transformation of function.

wave number ω=nπ/L

Linearity: $F\{\alpha g(x)+\beta h(x)\}=\alpha G(\omega)+\beta H(\omega)$

Shift Theorem: $f_a(x)=f(x-a)$ then $F[f_a(x)](\omega)=e^{i\omega a}F[f(x-a)](\omega)$ ($F[f_a]=e^{i\omega a}F[f]$)

Scaling: $f_a(x)=\frac{1}{a}f(\frac{x}{a})$ then $F[f_a(x)](\omega)=a\omega F[f(\frac{x}{a})](\omega)$ ($F[f_a]=a\omega F[f]$)???

Multiplication by x: $F[xf(x)](\omega)=-i\frac{dF[f(x)](\omega)}{d\omega}$

Derivative:

$F[\frac{\partial^n f}{\partial x^n}](\omega)=(-i\omega)^n F[f](\omega)$ $F[\frac{\partial^n f}{\partial t^n}](\omega)=\frac{\partial^n F[f](\omega)}{\partial t^n}$

A spatial Fourier Transform of a time derivative equals time derivative of the Fourier Transform.

Dirac Delta Function: $F[\delta(x-x_0)](\omega)=\frac{1}{2\pi}e^{i\omega x_0}$

Convolution: $F^{-1}[FG]=F^{-1}[F]\times F^{-1}[G]$ convolution of f,g is defined by $f*g=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x-y)g(y)dy$

Gaussian: $g(x)=\sqrt{\frac{\pi}{\alpha}}e^{-\frac{x^2}{4\alpha}}$ Fourier Transform: $F[g](\omega)=G(\omega)=e^{-\alpha\omega^2}$

Lemma: $f(x)=e^{-\beta\omega^2}$ Fourier Transform: $F[f](\omega)=\frac{1}{\sqrt{4\pi\beta}}e^{-\frac{\omega^2}{4\beta}}$

Burger's Equation (non-linear)