Simple Harmonic Motion 简谐运动

1.harmonic oscillator example: ①a mass oscillating at the end of a spring ②a swinging pendulum ③movement of charge in an oscillating electrical circuit
2.Characteristics: ①periodic motion ②an equilibrium position ③a restoring force that is directed towards this equilibrium position ④inertia causing overshoots ⑤a continuous flow of energy between potential and kinetic

Restoring Force $F=-kx$ Hooke Law of elasticity

$F=m\vec{a}=-k\vec{x}$ Newton 2 Law ※mx''+kx=0 (找x,x',x''等式) $\frac{d^2x}{dt^2}=-\omega^2 x$ **General Solution** $x=A\cos(\omega t+\phi)$

Natural Frequency $\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{g}{l}}$ **Period** T=2π/ω **Frequency** f=1/T=2πω phase ϕ
spring constant/stiffness k

>> $x=A\cos(\omega t)$ $v=-\omega A\sin(\omega t)$ $a=-\omega^2 A\cos(\omega t)=-\omega^2 x$

Energy: $E=K+U=\frac{1}{2}kx^2+\frac{1}{2}mv^2=\frac{1}{2}kA^2$
(spring holds elastic PE, mass holds inertial KE) exchange of storage of energy

Phase-Space (x-p) Diagram: $p=m\vec{v}=m\dot{\vec{x}}=-Am\omega_0\sin(\omega_0 t+\varphi)$ $\frac{x}{A^2}+\frac{p}{A^2mk}=1$

>> $\frac{x}{2E/k}+\frac{p}{2mE}=1$ >> $\frac{PE}{E}+\frac{KE}{E}=1$ Phase Space trajectory is an ellipse with constant energy (amplitude A).
Electrical LC Circuit

Force vs PE: $F=-\frac{dU}{dx}=-x\left(\frac{d^2U}{dx^2}\right)_{x=0}$ $U(x)=U(0)+x\left.\frac{dU}{dx}\right|_{x=0}+\frac{x^2}{2}\left(\frac{d^2U}{dx^2}\right)_{x=0}+\ldots\approx\frac{x^2}{2}\left(\frac{d^2U}{dx^2}\right)_{x=0}$

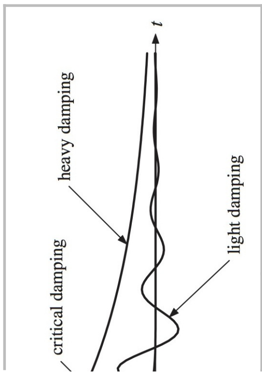
Damped Simple Harmonic Motion 阻尼简谐运动

1. amplitude of oscillation steadily reduces and system energy loses due to dissipative/damping forces (usually linearly proportional to velocity) and system eventually comes to rest.
2. Characteristics: ① amplitude of oscillation decays exponentially with time

※Damping Force: $F_d=-b\vec{v}=-b\dot{\vec{x}}$ b: damping constant [kg/s] (b≥0)
Other kind of damping force also exist: turbulent drag F=-γv|v|∝v²

$m\vec{a}=-k\vec{x}-b\vec{v}$ ※ mx''+bx'+kx=0 **General Solution** $x=A_0e^{-\beta t}\cos(\omega t)$

characteristic equation: $\Delta=b^2-4ac=\gamma^2-4\omega_0^2$ ω_0: **natural frequency** $\omega_0=\sqrt{\frac{k}{m}}$ [s^{-1}] γ: $\gamma=\frac{b}{m}$ [s^{-1}]



Three Case of Damping ω: angular frequency < ω_0

①Δ<0-under/light damping: $\gamma^2/4<\omega_0^2$ $x=A_0e^{-\gamma t/2}\cos(\omega t)$ (damped oscillations)

※Envelope: $\pm e^{-\gamma t/2}$ ※Logarithmic Decrement: $\frac{A_n}{A_{n+1}}=e^{\gamma T/2}$
As damping increases, period T->∞ (γ->2ω₀)

②Δ=0-critical damping: $\gamma^2/4=\omega_0^2$ $x=Ae^{-\gamma t/2}+Bte^{-\gamma t/2}$
(quickest return to equilibrium position without oscillation)

	③Δ>0-over/heavy damping: $\gamma^2/4>\omega_0^2$ $x = Ae^{\left(-\gamma/2+\sqrt{\gamma^2/4-\omega_0^2}\right)t} + Be^{\left(-\gamma/2-\sqrt{\gamma^2/4-\omega_0^2}\right)t}$ (exponential decay of displacement) ignore ω_0 >> $x = Ae^{\gamma t} + Be^{-\gamma t}$

when gives at t=0,v=C it implies x=0 (two initial conditions)

※**Angular Frequency** ω: $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ derived from $\frac{d^2x}{dt^2} + \frac{\omega_0}{Q}\frac{dx}{dt} + \omega_0^2 x = 0$

※**Energy Loss** for Light Damping (other two loss so fast):

$$E(t) = K + U = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA_0^2$$

x=... v=...

for light damping $\gamma/2 \ll \omega_0$, so v can be approximated as $-A_0\omega_0E_0e^{(-\gamma t/2)}\sin(\omega_0 t)$.

$$E(t) = \frac{1}{2}kA_0^2e^{-\gamma t} = E_0e^{-\gamma t} = E_0e^{(-t/\tau)}$$

decay time/ time constant/ lifetime: τ

Note: 善于运用 A≫B A≫1 1≫A的关系, ignore higher order 简化运算(approximation).

Energy dissipation is due to oscillator does work against damping force at the rate(damping force × velocity) $\frac{dE}{dt} = \frac{d}{dt}(\frac{1}{2}mv^2 + \frac{1}{2}kx^2) = (ma+kx)v = (-bv)v$

Rate of (averaged) energy loss: γ $\frac{d}{dt}E = -\gamma E_0e^{-\gamma t} = -\gamma E$

※**Quality Factor**: (∝1/degree of damping=rate at which the oscillator loses energy

① $Q = \frac{\omega_0}{\gamma}$ [dimensionless] (Q large, dissipation small, degree of damping small)

Q=½ critical damping Q>½ light-damping (Q>300 almost no damping) Q<½ heavy damping

② $Q = \frac{\text{energy stored in the oscillator}}{\text{energy dissipated per radian}}$

$$\frac{E_{n+1}}{E_n} = e^{-\gamma T} \approx 1-\gamma T$$

(e^x series expansion, ignore higher order term)

$$\frac{E_{n+1}-E_n}{E_n} = \gamma T = \gamma\frac{2\pi}{\omega_0} = \frac{2\pi}{Q}$$

(fractional change in energy per cycle)

fraction change in energy per radian = 1/Q

Forced Oscillation 强制振荡；受迫振动

1.**Free oscillator**: a system disturbed from rest and then oscillates around equilibrium position with steadily decreasing amplitude.
Forced oscillator: a periodic driving force applied to the system
2. Characteristics: ① system oscillates at applied force frequency after an initial transient response ② ③

※undamped forced oscillator: $m\vec{a} + k\vec{x} = F_0\cos\omega t$ **General Solution** $x = A(\omega)\cos(\omega t - \delta)$

※damped forced oscillator: $m\vec{a} + b\vec{v} + k\vec{x} = F_0\cos\omega t$ $\vec{a} + \gamma\vec{v} + \omega_0^2\vec{x} = \frac{F_0}{m}\cos\omega t$

General Solution $x = A(\omega)\cos(\omega t - \delta)$

Note: driving frequency ω natural frequency ω₀=根号下 k/m phase δ resonance ω≈ω₀。

	Amplitude	Velocity	Power
Frequency	$\omega = \omega' = \omega_0\sqrt{1-1/(2Q^2)}$	$\omega = \omega_0$	$\omega = \omega_0$
Peak Value	$a_{\max} = \frac{a_0 Q}{\sqrt{1-\frac{1}{4Q^2}}}$	$v_{\max} = a_0\omega_0 Q$	$P_{\max} = \frac{1}{2}ma_0^2\omega_0^3 Q$
General	$a(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}}$	$v(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2/\omega^2+\gamma^2}}$	$<P(\omega)>= P_{\max}\frac{\gamma^2}{(\omega_0^2-\omega^2)^2/\omega^2+\gamma^2}$
	$\tan\delta = \frac{\omega\gamma}{(\omega_0^2-\omega^2)}$		$<P>= P_{\max}\frac{\gamma^2/4}{(\omega_0-\omega)^2+\gamma^2/4}$ $Q \gg 1$

a_0: amplitude causing by driving force $=F_0/k$ phase angle δ

power absorbed:
Transient Phenomena:

Coupled Oscillators 耦合振子

normal modes: different ways the system can oscillate normal frequencies: associated frequencies

According to Newton 2 Law: $m\ddot{\vec{x}} = F(\vec{x})$ spring compress-/ extend+

so $[M]\ddot{\vec{x}}+[S]\vec{x}=0$ and assume harmonic response $\ddot{\vec{x}} = -\omega^2\vec{x}$

① normal frequencies [eigenvalues]: $\det\left|-\omega^2[M]+[S]\right| = 0$

② normal mode [eigenvectors]: $\vec{\xi}_i$

$$\vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{\xi}_1\sin(\omega_1 t)+\vec{\xi}_1\cos(\omega_1 t)+\vec{\xi}_2\sin(\omega_2 t)+\vec{\xi}_2\cos(\omega_2 t)$$

③ state vector

④ center of mass motion $q_1=x_1+x_2$ relative motion $q_2=x_1-x_2$

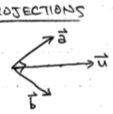
then $[M]\ddot{\vec{x}}+[S]\vec{x}=0$ only involves one variable for each row.

⑤ Orthogonality of Solutions: $[M]\ddot{\vec{x}}+[S]\vec{x}=0$ when [M] is diagonal matrix $\vec{\xi}_i\cdot\vec{\xi}_j = 0$

normal mode are orthogonal In general, $\vec{\xi}_i[M]\vec{\xi}_j = 0$ or $\vec{\xi}_i[K]\vec{\xi}_j = 0$

⑥ Initial Value Problem: n modes

PROJECTIONS



when $\vec{a}\perp\vec{b}$, $\vec{a}\cdot\vec{b}=0$.

$\Rightarrow$ straight forward to write $\vec{u} = f_a\vec{a} + f_b\vec{b}$ by projecting $\vec{u}$ onto $\vec{a}$ and $\vec{b}$.

$\vec{a}\cdot\vec{u} = f_a\,\vec{a}\cdot\vec{a} + f_b\,\cancel{\vec{a}\cdot\vec{b}}^{\,0}$

$= f_a\|\vec{a}\|^2$

Projection: $\Rightarrow f_a = \frac{\vec{a}\cdot\vec{u}}{\|\vec{a}\|^2}$ similarly $f_b = \frac{\vec{b}\times\vec{u}}{\|\vec{b}\|}$

$$\vec{x}(t) = \sum_1^n A_i\vec{\xi}_i\sin(\omega_i t)+B_i\vec{\xi}_i\cos(\omega_i t)$$

with $A_i = \frac{1}{\omega_i}\frac{\vec{\xi}_i\vec{v}_0}{\left\|\vec{\xi}_i\right\|^2}$ & $B_i = \frac{\vec{\xi}_i\vec{x}_0}{\left\|\vec{\xi}_i\right\|^2}$

Example:

1. coupled pendulums: frequency $\omega_0 = \sqrt{\frac{g}{l}}$ $\omega = \sqrt{\frac{g}{l} + \frac{2k}{m}}$

① displace both pendulums same amount in same direction $m\ddot{\vec{x}} = -mg\tan\theta \approx -\frac{mg}{l}\vec{x}$

② displace both pendulums same amount in opposite direction $m\ddot{\vec{x}} = -mg\tan\theta - 2k\vec{x} \approx -\frac{mg}{l}\vec{x} - 2k\vec{x}$

③ displace just one and leave the other at equilibrium position (superposition of 2)

$$E = \frac{1}{2}m\left(\frac{dx_a}{dt}\right)^2 + \frac{1}{2}m\left(\frac{dx_b}{dt}\right)^2 + \frac{1}{2}\frac{mg}{l}(x_a^2 + x_b^2) + \frac{1}{2}k(x_a - x_b)^2$$

2. Longitudinal Oscillation: periodic displacements of mass take place along the line connecting them
oscillating masses coupled by springs (horizontally) oscillating masses coupled by springs (vertically)

3. Transverse Oscillation: periodic displacements of mass take place perpendicular to the line connecting them

Resonant Oscillations
Power Transfer at Resonance

Wave Equation: $\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2}$ where v is speed of wave along string $v = \sqrt{\frac{T}{\mu}}$

T:tension μ:mass per unit length[kg/m]

Standing Waves 驻波

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2 \frac{\partial^2 y(x,t)}{\partial x^2} = 0$$

separability: y(x,t)=f(x)g(t)

$\frac{\partial^2}{\partial t^2} y = f(x)\frac{\partial^2}{\partial t^2} g(t)$ and $\frac{\partial^2}{\partial x^2} y = \frac{\partial^2}{\partial x^2} f(x)g(t)$ so $g(t) = A\cos(\omega t + \alpha)$ $f(x) = A\cos(\frac{\omega}{c}x) + B\sin(\frac{\omega}{c}x)$

Fixed End boundary condition: $y(0,t) = 0$ (B=0) & $y(L,t) = 0$ (Asin(w/c)L=0 >> (w/c)L=nπ)

※ **Normal Modes of a vibrating String**: (∞ of modes)

single frequency response $\omega_n = \frac{n\pi c}{L}$ [s^{-1}] **wave number** k=ω/c [m^{-1}]

$y(x,t) = \sum_1^\infty [A_i \sin(k_i x)\cos(\omega_n t + \alpha_n)] = \sum_1^\infty [C_i \sin(k_i x)\cos(\omega_n t) + D_i \sin(k_i x)\sin(\omega_n t)]$ where

$$C_n = \frac{\int_0^L \sin(k_n x)s(x)dx}{\int_0^L \sin^2(k_n x)dx} \quad \text{and} \quad D_n = \frac{1}{\omega_n}\frac{\int_0^L \sin(k_n x)v(x)dx}{\int_0^L \sin^2(k_n x)dx}$$

※ Normal Modes: $y(x,t) = \sin(n\frac{\pi}{L}x)\cos(nc\frac{\pi}{L}t + \alpha_n)$

① fundamental mode: n=1 frequency ω1 ② second mode: n=2 frequency ω2=2ω1 ③...

Conclusion: standing waves are normal modes of a continuous system.

※ Orthogonality of Standing Waves:

※ Initial Value Problem for a Vibrating String: $y(x_0,0) = s(x_0)$ & $\dot{y}(x_0,0) = v(x_0)$

Fourier Series

Travelling Waves 行波

Standing Wave = 2 Travelling Waves

$y = f(x-ct) + g(x+ct)$ f: traveling wave to right g: to left c:speed of prorogation of wave

$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x-ct)\right) = A\sin(k(x-ct)) = A\sin(kx-\omega t) = A\sin\left(2\pi(\frac{1}{\lambda}x - ft\right)$

Reflection & Transmission:

When a incident wave hits a boundary/interface, part of it reflected, part of it transmitted.

incident(from left) $y_i = A\sin(\omega_1 t - k_1 x)$ A+B=D ω1=ω2=ω

transmitted(to right) $y_t = D\sin(\omega_2 t - k_2 x)$ reflected(back to left) $y_r = B\sin(\omega_1 t + k_1 x)$

※ **impedance** $Z = \frac{T}{c} = \sqrt{T\mu}$ z≥0 (property of string)

※ **Amplitude**: >Transmission D/A $t = \frac{2Z_1}{Z_1+Z_2}$ >Reflection B/A $r = \frac{Z_1-Z_2}{Z_1+Z_2}$ t-r=1

Case:

① hard boundary ($Z_2=\infty$) t=0 r=-1 reflected wave inverted

② soft boundary ($Z_2=0$) r=1

③ one string ($Z_1=Z_2$) t=1 r=0

④ ($Z_1>Z_2$) t>1 r>0 ⑤ ($Z_1<Z_2$) t<1 r<0

Note : Inverted-- phase > out of 180 amplitude原来往上，reflect就是往下了

※ Energy Flux: = energy density × velocity

$$E = \frac{1}{2}\mu_i c\omega^2 A_i^2 = \frac{1}{2}Z_i\omega^2 A_i^2$$

※ Energy

>Transmission D^2/A^2 $T = \frac{\textit{transmitted flux}}{\textit{incident flux}} = \frac{4Z_1Z_2}{(Z_1+Z_2)^2}$

>Reflection B^2/A^2 $R = \frac{\textit{reflected flux}}{\textit{incident flux}} = \frac{(Z_1-Z_2)^2}{(Z_1+Z_2)^2}$

Energy Conservation: T+R=1

Dispersion of Waves 波的弥散 & Wave Packet 波包

non-dispersive media: velocity c constant ($c=\sqrt{T/\mu}$) independent of frequency. and ω=ck for sin(kx-ωt)
Dispersion: velocity c depends on frequency ω. ω(k): dispersion relation

Formula

1.**Simple Harmonic Motion**: F=ma=-kx mx”+kx=0 (去找x,x’,x”等式)

$\frac{d^2x}{dt^2}=-\omega^2 x$ Natural Frequency $\omega=\sqrt{\frac{k}{m}}$ =(g/l)^0.5 >> $x=A\cos\omega t$

Period T=2π/ω Frequency f=1/T=2πω

Energy: E=K+U=½kx²+ ½mv²= ½kA² Force vs PE: F=- dU/dx

2.General Solution without Damping:

$x=A\cos(\omega t+\phi)$ Note: A,Φ depends on initial conditions $x(t_0)$=l1 $x'(t_0)$=l2

Recall: Taylor Expansion at x=a of f(x):

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots$$

3.Potential Energy Approximation: $F=-\frac{dU}{dx}=-x(\frac{d^2U}{dx^2})_{x=x_0}$

4.Given U(x) find k:

Taylor Expand at $$f(x_0+\Delta x)=f(x_0)+\Delta x\left(\frac{df}{dx}\right)_{x=x_0}+\Delta x^2\left(\frac{d^2f}{dx^2}\right)_{x=x_0}+\ldots$$

Ignore higher term and $f(x_0)=0$ $$f(x_0+\Delta x)=\Delta x\left(\frac{df}{dx}\right)_{x=x_0}$$

5.Simple Pendulum with small angle approximation: $\frac{d^2\theta}{dt^2}=-\frac{g}{l}\theta=-\omega^2\theta$ $\omega=\sqrt{\frac{g}{l}}$

General Solution: $\theta=\theta_0\cos(\omega t+\phi)$

7.Energy of Simple Pendulum SHM: $E=\frac{1}{2}mv^2+\frac{1}{2}\frac{mg}{l}x^2$ $v=\frac{dx}{dt}=\sqrt{\frac{g(A^2-x^2)}{l}}$

Energy of all SHM: $E=\frac{1}{2}\alpha v^2+\frac{1}{2}\beta x^2$ $\omega=\sqrt{\frac{\beta}{\alpha}}$

8.LC Circuit SHM: $\frac{d^2q}{dt^2}=-\frac{1}{LC}q$ $\omega=\sqrt{\frac{1}{LC}}$ $q(t)=A_0\cos(\omega q+\phi)$

LC Circuit Energy: $E=\frac{1}{2}LI^2+\frac{1}{2}\frac{q^2}{c}$

9.More General Energy Equation for SHM: $E=\frac{1}{2}\alpha\left(\frac{dz}{dt}\right)^2+\frac{1}{2}\beta z^2$ z: oscillating term

Formula

1.Damped SHM:

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2x=0 \qquad \omega=\sqrt{\omega_0^2-\frac{\gamma^2}{4}} \qquad \gamma=\frac{b}{m}$$

b:damping coefficient

①under/light Damping: $\omega_0^2>\frac{\gamma^2}{4}$

$$x=Ae^{-\frac{\gamma t}{2}}\cos\omega t$$

②Critical Damping: $\omega_0^2=\frac{\gamma^2}{4}$

$$x=(A+Bt)e^{-\frac{\gamma t}{2}}$$

③over/heavy Damping: $\omega_0^2<\frac{\gamma^2}{4}$

$$x=Ae^{\left(-\frac{\gamma t}{2}+\sqrt{\frac{\gamma^2}{4}-\omega_0^2}\right)t}+Be^{\left(-\frac{\gamma t}{2}-\sqrt{\frac{\gamma^2}{4}-\omega_0^2}\right)t}$$

2.Energy Dissipated in Damped SHM:

$$E(t)=E_0e^{-\gamma t} \qquad E(t)=E_0e^{-\gamma t}=E_0e^{(\frac{-t}{\tau})}$$

τ :time constant/life time

3.Dissipated Rate:

$$\frac{dE}{dt}=\frac{d}{dt}\left(\frac{1}{2}mv^2+\frac{1}{2}kx^2\right)=mv\frac{dv}{dt}+kx\frac{dx}{dt}=(ma+kx)v=(-bv)v$$

4.Q-value:

$$Q=\frac{\omega_0}{\gamma}=\frac{energy\_stored}{energy\_dissipated\_per\_cycle}$$

$$\frac{d^2x}{dt^2}+\frac{\omega_0}{Q}\frac{dx}{dt}+\omega_0^2x=0$$

alternate form of Damped SHM ODE & angular frequency

$$\omega=\omega_0\sqrt{1-\frac{1}{4Q^2}}$$

5.Damped RLC Circuit:

$$\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=0 \qquad \omega^2=\frac{1}{LC}-\frac{R^2}{4L^2} \qquad Q=\frac{1}{R}\sqrt{\frac{L}{C}}$$