Complex Numbers

Complex Number: $\mathbb{C}=\{(x,y): x,y\in R\}$ ordered pair of real numbers (point in complex plane with rectangular coordinates x and y) real number(x,0) pure imaginary number(0,y) $y\neq 0$

z=(x,y) Cartesian form**=x+*i*y** rectangular form**=r(cosθ+θ*i*sinθ)** polar form**=re$^{i\theta}$** exponential form

x: a real number, x=*Re*z real part, y: a real number, y=*Im*z imaginary part $Rez\le |Rez| \le z$ $Imz\le |Imz| \le z$

$z=(x,y)=(x,0)+(0,y)=(1,0)(x,0)+(0,1)(0,y)=x+iy$ $i^2=(0,1)(0,1)=(-1,0)=-1$

Two complex number are equal when they have the same real parts and the same imaginary parts.

Algebra

1.Basic Operations:

①**Sum**: Abelian group $(\mathbb{C},+)$ with unit element (0,0)

$z_1+z_2=(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)=(x_1+x_2)+i(y_1+y_2)$

(a)commutative (b)associative (c)distributive

(d)additive inverse $-z=(-x,-y)$ additive identity/neutral element $0=(0,0)$ $z+0=z$

(e)Closure: $z_1\in\mathbb{C}$ $z_2\in\mathbb{C}$, then $z_1+z_2\in\mathbb{C}$

②**Product**: Abelian group $(\mathbb{C}-\{(0,0)\},\cdot)$ with unit element (0,0)

$z_1z_2=(x_1x_2-y_1y_2,y_1x_2+x_1y_2)=(x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2)$

$|z_1\cdot z_2|=|z_1|\cdot|z_2|$ $\arg(z_1\cdot z_2)=\arg(z_1)+\arg(z_2)$

(a)commutative (b)associative (c)any complex number times zero is zero $z\cdot 0=(x+iy)\cdot(0+i0)=0+i0=0$

(d)multiplicative inverse(for nonzero complex number) $zz^{-1}=1$

$$z^{-1}=\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)=\frac{\bar{z}}{|z|^2}=\frac{x-iy}{x^2+y^2}=\frac{x-iy}{(x+iy)(x-iy)}=\frac{1}{r}e^{i(-\theta)}$$

multiplicative identity/neutral/unit element $1=(1,0)$ $z\cdot 1=z$ $1/z=z^{-1}$

(e)A product z_1z_2 is zero iff at least one of the factors z_1 and z_2 is zero.

(f)Closure: $z_1\in\mathbb{C}$ $z_2\in\mathbb{C}$, then $z_1\cdot z_2\in\mathbb{C}$

③**Subtraction & Division**: $z_1-z_2=z_1+(-z_2)$ $z_1/z_2=z_1(z_2)^{-1}$ $(z_2\neq 0)$

④other properties: $(z_1+z_2)/z_3=(z_1+z_2)(z_3)^{-1}=z_1(z_3)^{-1}+z_2(z_3)^{-1}=z_1/z_3+z_2/z_3$

$$(z_1+z_2)^n=\sum_{k=0}^{n}\binom{n}{k}z_1^k z_2^k$$

$(z_1/z_3)(z_2/z_4)=(z_1z_2)/(z_3z_4)$

2.Vector and Modulus:

①**Modulus/Absolute Value**: $r=|z|=(x^2+y^2)^{1/2}=((Rez)^2+(Imz)^2)^{1/2}$ distance of z from the origin 0.

distance between point (x,y) and origin; length pf radius vector representing z.

Note: $z_1<z_2$ is meaningless unless both real numbers

②**Argument**: value of θ $\tan\theta=y/x$ principal value of argument Arg $z\in(-\pi,\pi]$, $\arg z=\text{Arg } z+2n\pi$

$z_1=r_1e^{i\theta_1}$ $z_2=r_2e^{i\theta_2}$ $z_1z_2=r_1r_2e^{i(\theta_1+\theta_2)}$ $\arg(z_1z_2)=\theta_1+\theta_2$

Euler Formula: $e^{i\theta}=\cos\theta+i\sin\theta$ de Moivre Formula: $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$

Note: $e^{i\theta_1}e^{i\theta_2}=e^{i(\theta_1+\theta_2)}$ $1/e^{i\theta}=e^{i(-\theta)}$ $e^{i(\theta+2\pi)}=e^{i\theta}$ $|e^{i\theta}|=1$ $d(e^{i\theta})/d\theta=i(e^{i\theta})$

3.Complex Conjugate: $z^*=(x+iy)^*=x-iy$ Note: $z^*=\bar{z}$(overbar) reflection of z in the real axis $i^*=-i$

4.Power & Root: $z=re^{i\theta}$

①**Power**: $z^n=\left(re^{i\theta}\right)^n=r^ne^{i(n\theta)}$ Two nonzero complex numbers are equal iff $r_1=r_2$ and $\theta_1=\theta_2+2k\pi$

②**Root**: $Re^{i\phi}=w=z^n=(re^{i\theta})^n=(r^ne^{in\theta})$ $z=\sqrt[n]{R}e^{i(\frac{\varphi}{n}+\frac{2k\pi}{n})}$

$r^n=R$ ($R\geq 0$, unique positive n[th] root) $n\theta=\phi \pmod{2\pi}$

③**Root of Unity**:($w=1=z^n$) $z=e^{i(\frac{2k\pi}{n})}$ $1-z^{n+1}=(1-z)(1+z+\ldots+z^n)$

$z^{**}=z$ $|z^*|=|z|$ $\overline{z_1\pm z_2}=\bar{z}_1\pm\bar{z}_2$ $\overline{z_1z_2}=\bar{z}_1\bar{z}_2$ $\overline{z_1/z_2}=\bar{z}_1/\bar{z}_2$ $z\bar{z}=|z|^2$ $Rez=(z+z^*)/2$ $Imz=(z+z^*)/2i$

$|z_1z_2|=|z_1||z_2|$ $(e^{i\theta})^*=e^{i(-\theta)}$ Note: $z\cdot z^*=x^2+y^2=|z|^2$ $z\cdot z=(x^2-y^2)+2(ixy)=z^2$

5.**Complex solutions or inequality Region**:

$w^n - 1 = (w-1)(1+w+w^2+...w^{n-1})$

Lagrange Trigonometric Identity: $1+\cos\theta+\cos 2\theta+...+\cos n\theta=\frac{1}{2}+\frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)}$

Geometric

$|\mathrm{Re}z|\leq|z|$ $|\mathrm{Im}z|\leq|z|$

1.**Triangle Inequality**: $\big||z_1|-|z_2|\big|\leq|\pm z_1\pm z_2|\leq|z_1|+|z_2|$ for sums $|z_1+...+z_n|\leq|z_1|+...+|z_n|$

Proof: $-|z|\leq\mathrm{Re}z\leq|z|$ $-|z|\leq\mathrm{Im}z\leq|z|$
$|z+w|^2=(z+w)(z^*+w^*)=|z|^2+|w|^2+zw^*+z^*w=|z|^2+|w|^2+2\mathrm{Re}(zw^*)\leq|z|^2+|w|^2+2|zw^*|=|z|^2+|w|^2+2|z||w^*|=|z|^2+|w|^2+2|z||w|=(|z|+|w|)^2$

$|z+w|=|z|+|w|$ when $\mathrm{Re}(zw^*)=|zw^*|$ (zw^* is real and nonnegative)

2.**Curves in ℂ & Sketch**:

①.**Distance**: $|z-w|$ distance between points z and w in the Argand diagram

②.**Line**: $\beta z+\bar{\beta}\bar{z}+\gamma=0$ $\beta\in\mathbb{C}^\times=\mathbb{C}\setminus\{0\}$ $\gamma\in\mathbb{R}$ ($\beta=0$ not a line)

Two lines are orthogonal when $\mathrm{Re}(\beta_1\bar{\beta}_2)=0$

Proof: ① direction vector of the line $<\mathrm{Im}\beta,\mathrm{Re}\beta>$ $<\mathrm{Im}\beta_1,\mathrm{Re}\beta_1>\cdot<\mathrm{Im}\beta_2,\mathrm{Re}\beta_2>=\mathrm{Im}\beta_1\mathrm{Im}\beta_2+\mathrm{Re}\beta_1\mathrm{Re}\beta_2=\mathrm{Re}(\beta_1\beta_2^*)=0$
②let $\beta_1=a+ib$ $\beta_2=c+id$ $z=x+iy$ plug back, when $ac+bd=0$, L1&L2 are orthogonal, $\mathrm{Re}(\beta_1\beta_2^*)=ac+bd$

Line Segment with end point α,β: $[\alpha,\beta]=\{(1-t)\alpha+t\beta: 0\leq t\leq1, \alpha,\beta\in\mathbb{C}\}$

Real Axis: $\mathrm{Im}z=0$ $z=z^*$ $|z-\alpha|=|z-\alpha^*|$ ($\mathrm{Im}\alpha\neq0$) Imaginary Axis: $\mathrm{Re}z=0$ $|z-\alpha|=|z+\alpha|$ ($\mathrm{Re}\alpha\neq0$)

vertical Line: $\mathrm{Re}(z+C)=0$ horizontal Line: $\mathrm{Im}(z+C)=0$ slant Line: $\mathrm{Re}[Az+B]=0$ angle of slant arg(A)

Perpendicular Bisector: $|z-\alpha|=|z-\beta|$ equidistant point

③.**Circle**: $\alpha z\bar{z}+\beta z+\bar{\beta}\bar{z}+\gamma=0$ $\alpha\in\mathbb{R}^\times$ $\beta\in\mathbb{C}^\times$ $\gamma\in\mathbb{R}$ $|\beta|^2>\alpha\gamma$ ($\alpha=0$ a line, $|\beta|^2=\alpha\gamma$ one solution)

$|z-\alpha|=r$ ($\alpha\in\mathbb{C}$) Circle of Apollonius: $|(z-\alpha)/(z-\beta)|=\lambda$ ($\alpha,\beta\in\mathbb{C}$, $\lambda\in\mathbb{R}^+\setminus\{1\}$)

※Ellipse: $|z-\alpha|+|z-\beta|=C$ ($C>|\alpha-\beta|$) foci α and β, major axis length C.
($C=|\alpha-\beta|$ points on the line segment from α to β $C<|\alpha-\beta|$ no solution)
※Hyperbola $\mathrm{Re}(z^2)=C$ $|z-\alpha|-|z-\beta|=C$ ($C\in\mathbb{R}+$) ($C=0$ a line)
※Parabola

④.Circular Arc:

⑤.Circline: circles through infinity=lines $|(z-\alpha)/(z-\beta)|=\lambda$ ($\lambda\in\mathbb{R}>0$) when $\lambda=1$ lines; else,circles.

Segment of a circline(arc):
i) a circle joining points α and β
ii) line segment joining points α and β
iii) a half-line/ray from $\alpha\in\mathbb{C}$ to infinity(any direction) $\arg(z-\alpha)=\mu$

⑥.Inverse Point:

⑦.Sketch: boundary included--full line boundary excluded--dash line

3.**Regions in complex plane(Argand Diagram)/Topology**:

1.**ε-Neighborhood/Disc of point z_0**: $D_\varepsilon(z_0)=\{z\in\mathbb{C}: |z-z_0|<\varepsilon\}$ open disk with center z_0 and radius ε
(all points z lying inside but not on a circle centered at z_0 with specified positive radius ε)

Closed Disk/Closure: $\bar{D}_\varepsilon(z_0)=\{z\in\mathbb{C}: |z-z_0|\leq\varepsilon\}=D_\varepsilon(z_0)\cup\{\text{boundary circle } |z-z_0|=\varepsilon\}$

Deleted/Punctured Disc: $0<|z-z_0|<\varepsilon$, $D^\times_\varepsilon(z_0)=D_\varepsilon(z_0)\setminus\{z_0\}$(all points z in an ε neighborhood of z_0 except for point z_0 itself)

2.(Open) **Annuli**: $\{z\in\mathbb{C}: \varepsilon_1<|z-z_0|<\varepsilon_2, 0\leq\varepsilon_1<\varepsilon_2\}$ or $\{z\in\mathbb{C}: \varepsilon<|z-z_0|, 0\leq\varepsilon\}$ $\varepsilon_1=0$, punctured disc $Đ_{\varepsilon2}(z_0)$

3.**Half-Plane**: open upper half-plane $H^{+}=\{z\in\mathbb{C}: \mathrm{Im}z>0\}$ closed $\{\mathrm{Im}z\geq 0\}$ open lower half-plane $H^{-}=\{z\in\mathbb{C}: \mathrm{Im}z<0\}$

4.**Sector**: $S_{\alpha,\beta}=\{z\in\mathbb{C}: 0\neq z=|z|e^{i\theta}\in\mathbb{C}$ with $\alpha<\theta<\beta\}$ quadrant if $\beta-\alpha=\pi/2$ half-plane if $\beta-\alpha=\pi$

5.Point (P):

①**Interior**: there is some neighborhood of z_0 that contains only points of set S.
=some open disk with center at P lies in set S.

②**Exterior**: there exists a neighborhood of z_0 containing no points of S.

③**Boundary**: a point all of whose neighborhoods contain at least one point in S and at least one point not in S. =every open disk center at P contains a point in S and a point not in S.

Boundary of S, ∂S: totality of all boundary points.

④**Accumulation Point**: each deleted neighborhood of z_0 contains at least one point of S.
=every open disk centered at P contains a point of S different from P.

<u>If a set S is closed, it contains all of its accumulation points.</u>

origin is the only accumulation points of set $z_n=i/n$

⑤**Isolated Point**: P lies in S and some open disk centered at P contains no point of S other than P.

6.Set (S):

①(a)**Open**: it contains none of its boundary points (open iff each of its points is interior)

(b)**Closed**: it contains all of its boundary points e.g: $|z-z_0|<R$ open $|z-z_0|>R$ open $|z-z_0|\leq R$ closed

Closure of set S, $\bar{S}$(overbar): closed set consisting of all points in S together with the boundary of S.

(c)**Neither open Nor closed**: not open-a boundary point that is contained in the set; not closed-exists a boundary point not contained in the set. e.g. $0<|z|\leq 1$

(d)**Both open And closed**: $\mathbb{C}$ and empty set ∅

②**Connected**: an open set is connected if each pair of points z_1 and z_2 in it can be joined by a polygonal line/path consisting of a finite number of line segments joined end to end, that lies entirely in S.

Domain: nonempty open connected set (any neighborhood is a domain)

Region: a domain together with some, none or all of its boundary points

③**Bounded**: every point of S lies inside some circle $|z|=R$

4.Transformation:

①**Affine Linear Transformation**: $f(z)=az+b$ $a,b\in\mathbb{C}$

Translation $w=z+\alpha$ Dilation $w=rZ$ $r\in R+$ Rotation $w=(e^{i\alpha})z$

Aff($\mathbb{C}$) is nonabelian/non-commutative

②**Linear Fractional Transformation**:

1. $f(z)=\dfrac{az+b}{cz+d}$ where $a,b,c,d\in\mathbb{C}$ when $ad-bc\neq 0$ **Möbius Transformation**

2.Bijection: inverse of f $f^{-1}=(dz-b)/(-cz+a)$ f: $\mathbb{C}\backslash\{-d/c\}\rightarrow\mathbb{C}\backslash\{a/c\}$

①injection:(one-to-one,into) suppose $f(z_1)=f(z_2)$ ~~~> $z_1=z_2$
②surjection:(onto) check f^{-1} is one-to-one

3.<u>Every linear fractional transformation is a composition of translations $f(z)=z+b$, dilations $f(z)=az$, inversions $f(z)=1/z$.</u>

Proof: $c=0$ $f(z)=(a/d)z+b/d$ $c\neq 0$ $f(z)=[(bc-ad)/c^2][1/z+(d/c)]+a/c$

4.<u>Möbius Transformations map circles and lines into circles and lines(circline)</u>.

translation and dilation maps c and l into c and l.
Prove for inversion:

5.In complex plane, $-\infty=\infty$. (extended accumulation point)

Extended complex plane: $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ also called Riemann Sphere, complex projective line CP.

$z\pm\infty=\pm\infty+z=\infty+\infty=\infty$

Note: $\infty+\infty$=not defined $z+(-z)=0$ while $z=\infty$ and $-z=\infty$

When $z\neq0$, $\infty\cdot z=z\cdot\infty=\infty\cdot\infty=\infty$ $z/\infty=0$ $z/0=\infty$ $\infty^*=\infty$

$\infty-\infty$ and ∞/∞ not defined Note: $(z-\infty)+\infty=\infty+\infty=\infty$ while $z+(-\infty+\infty)=z+0$ so not defined

Now, **Möbius Transformation** is defined for all $z\in\mathbb{C}$. If $c=0$ then $f(\infty)=\infty$; or $f(\infty)=a/c$ and $f(-d/c)=\infty$.

$$Cross\ Ratio=\frac{z-z_0}{z_1-z_0}\frac{z_1-z_2}{z-z_2}$$ $f(z_0)=0$ $f(z_1)=1$ $f(z_2)=\infty$

Proof: uniqueness

③**Circles in Riemann Sphere Σ & Stereographic Projection**: a map $\Phi: \Sigma\to\mathbb{C}$

$\mathbb{C}=\{(x,y,0)\in R^3\}$ as x-y plane in $R^3=\{(x,y,z)\}$ unit sphere $\Sigma=\{(x,y,z)\in R^3 \mid x^2+y^2+z^2=1\}$ North Pole $N=(0,0,1)$

1.For point $P\in\Sigma\backslash\{N\}$, the line NP intersects $\mathbb{C}$ in exactly one point Q, $\Phi(P)=Q$; $\Phi(N)=\infty\in\mathbb{C}$

$z=x+iy=a/(1-c)+ib/(1-c)$ $a=2x/(x^2+y^2+1)$ $b=2y/(x^2+y^2+1)$ $c=(x^2+y^2-1)/(x^2+y^2+1)$

$\Phi(0,0,1)=\infty$ $\Phi^{-1}(\infty)=(0,0,1)$

2.Bijection:

Proof: bijection ①For map Φ: $S=(a,b,c)\in\Sigma/\{N\}$ so $NS=N+t(S-N)=(0,0,1)+t(a,b,c-1)=(ta,tb,t(c-1)+1)$
NS hits x-y plane ($\mathbb{C}$ Plane) when $t(c-1)+1=0$ $t=1/(1-c)$ so $x=a/1-c$ $y=b/1-c$
②For inverse map Φ^{-1}: a point $C=(x,y,0)\in\mathbb{C}$, $\Phi(S=(a,b,c)\in\Sigma/\{N\})=C$, so $x^2+y^2=(a^2+b^2)/(1-c)^2=(1+c)/(1-c)$
so $c=(x^2+y^2-1)/(x^2+y^2+1)$ plug back to $x=a/1-c$ $y=b/1-c$, we can get $a=2x/(x^2+y^2+1)$ and $b=2y/(x^2+y^2+1)$

3.Stereographic Projection map circles and lines in $\mathbb{C}$(extended) into circles R in Σ.

Note: For a line in $\mathbb{C}$, when $R\in\Sigma$, $\infty\in\Phi^{-1}(\infty)=(0,0,1)$; means $N\in R$

Proof:

Analytic Functions

1.**Function**: Let S be a set of complex numbers, a function f(the mapping) defined on S is a rule that assigns to each z in S(domain of definition of f) a complex number w (value of f at z/image of z under f,f(z)).

$z=x+iy=re^{i\theta}$ $w=u+iv$ $f(z)=f(x+iy)=w=u+iv=u(x,y)+iv(x,y)=u(r,\theta)+iv(r,\theta)$ $z\to w=f(z)$ z is mapped to w by f.

Polynomial of degree n: $P(z)=a_0+a_1z+a_2z^2+...a_nz^n$ (a_i complex constants & $a_n\neq0$)

Rational Function: $P(z)/Q(z)$ $(Q(z)\neq0)$

Multiple-valued function: assign more than one value to a point z in the domain of definition.

Multifunction: a rule that assigning a non-empty subset of $\mathbb{C}$(finite or infinite) to each element of its domain set S.

2.**Limit & Continuity**:

①Limit: Suppose f is a complex function with domain G and z_0 is an accumulation point of G. There is a complex number w_0 such that for every $\varepsilon>0$ we can find $\delta>0$ so that for all $z\in G$, satisfying $0<|z-z_0|<\delta$, we have $|f(z)-w_0|<\varepsilon$. Then w_0 is the limit of f as z approaches z_0: $\lim_{z\to z_0} f(z)=w_0$

$\lim f(z)+c\lim g(z)=\lim[f(z)+cg(z)]$ $\lim f(z)\cdot\lim g(z)=\lim[f(z)\cdot g(z)]$ $\lim f(z)/\lim g(z)=\lim[f(z)/g(z)]$

$\lim_{z\to z_0} f(z)=w_0$	**Finite Limit at Finite Points**	$\forall\varepsilon>0\ \exists\delta>0$, $0<\lvert z-z_0\rvert<\delta$, then $\lvert f(z)-w_0\rvert<\varepsilon$ $\forall\varepsilon>0\ \exists\delta>0$, $z\in D^\times_\delta(z_0)$, then $f(z)\in D_\varepsilon(w_0)$	∞--->0
$\lim_{z\to\infty} f(z)=w_0$	**Finite Limit at Infinity ∞**	$\forall\varepsilon>0\ \exists\delta>0$, $\lvert z\rvert>1/\delta$, then $\lvert f(z)-w_0\rvert<\varepsilon$ $\forall\varepsilon>0\ \exists\delta>0$, $z\in D^\times_\delta(\infty)$, then $f(z)\in D_\varepsilon(w_0)$	$\xi=1/z$ $\lim f(1/\xi)=w_0$ when $\xi\to0$
$\lim_{z\to z_0} f(z)=\infty$	**Infinite Limit at a Finite Point**	$\forall\varepsilon>0\ \exists\delta>0$, $0<\lvert z-z_0\rvert<\delta$, then $\lvert f(z)\rvert>1/\varepsilon$ $\forall\varepsilon>0\ \exists\delta>0$, $z\in D^\times_\delta(z_0)$, then $f(z)\in D_\varepsilon(\infty)$	$\lim 1/f(z)=0$ when $z\to z_0$
$\lim_{z\to\infty} f(z)=\infty$	**Infinite Limite at Infinity ∞**	$\forall\varepsilon>0\ \exists\delta>0$, $\lvert z\rvert>1/\delta$, then $\lvert f(z)\rvert>1/\varepsilon$ $\forall\varepsilon>0\ \exists\delta>0$, $z\in D^\times_\delta(\infty)$, then $f(z)\in D_\varepsilon(\infty)$	$\xi=1/z$ $\lim 1/f(1/\xi)=0$ when $\xi\to0$

②Continuity: Suppose f is a complex function. If z_0 is in the domain of function and either z_0 is an isolated point of the domain or $\lim_{z\to z_0}f(z)=f(z_0)$, then f is continuous at z_0.

(i)limit exist (ii)f is defined at z0 (iii)limit=f(z0)

f is continuous on $G\subseteq\mathbb{C}$, if f is continuous at every point $z\in G$.

If f is continuous at an accumulation point w_0 and $\lim_{z\to z_0}g(z)=w_0$, then $\lim_{z\to z_0}f(g(z))=f(\lim_{z\to z_0}g(z))=f(w_0)$

3.**Derivative**:

①Differentiability: Suppose f:G→ℂ is a complex-valued function and z_0 is an interior point of G.

Derivative of f at z_0:
$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

differentiable at z_0: limiting value independent of the manner in which h→0
not differentiable at z_0: limit differs from different path.
holomorphic at z_0: f is differentiable for all points in an open disk centered at z_0.
function f is holomorphic on the open set G⊆ℂ, if it is differentiable(hence holomorphic) at every point in G.
functions which are differentiable(hence holomorphic) in the whole complex plane are entire.

[f(z)+cg(z)]'=f'(z)+cg'(z) [f(z)·g(z)]'=f'(z)g(z)+f(z)g'(z) $[f(z)/g(z)]'=[f'(z)g(z)-f(z)g'(z)]/[g(z)]^2$
$(z^n)'=nz^{n-1}$ [h(g(z))]'=h'(g(z))·g'(z) $[f^{-1}(z)]'=1/f'(f^{-1}(z))$

Differentiability ensures continuity

②**Cauchy-Riemann Equations**: Complex-valued function f defined on an open set G and be differentiable at z=x+iy∈G, Let f=u(x,y)+iv(x,y), u and v have first order partial derivatives at (x,y) and $u_x=v_y \quad u_y=-v_x$

Proof:

Limitation of CR Equation: contrapositive of CR is useful to prove non-differentiability, but CR is not on its own sufficient to guarantee differentiability.
Example: f(z)=f(x,y)= 1 if x or y is zero =0 otherwise. (CR holds but $\lim_{\Delta z\to 0}(f(\Delta z)-f(0))/\Delta z$ not exist)

Let f=u(x,y)+iv(x,y) for z=x+iy∈G (open subset of ℂ), Assume u and v have continuous first-order partial derivatives in G and at point z, $u_x=v_y$ $u_y=-v_x$, then f'(z) exists.

Proof: P 59

CR Equation in polar form: $u_r=\frac{1}{r}v_\theta \quad \frac{1}{r}u_\theta=-v_r$ at points $z_0=r_0e^{i\theta 0}$ with $r_0\neq 0$

③**Wirtinger Operator/Derivative**:

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) \quad \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

Note: $\partial z/\partial z^*=0$ $\partial z^*/\partial z=0$

If $\partial f/\partial z^*=0$, f is differentiable and $\partial f/\partial z=u_x+iv_x=v_y-iu_y=df/dz$

Differentiable: CR satisfies(Wirtinger Operator=0) and partial derivatives continuous(derivatives exist around them).

4.**Holomorphic Functions**:

A complex-valued function f which is differentiable at every point of G is holomorphic in G.
i.e. f'(z) exists for each z∈G.
A complex-valued function f is holomorphic at point a∈ℂ if there exists r>0 such that f is defined and holomorphic in D(a;r).
A function f of complex variable z is analytic/holomorphic at a point z_0 if it has a derivative (f'(z) exists) at each point in an ε-neighborhood of z_0. (regular&holomorphic=analyticity)
If f is holo at z_0, it is also holo in neighborhood of z_0.
① f,g holo in G and λ∈ℂ, then λf, f+g, fg(all defined pointwise in the usual way) are holo in G:
(λf)'(z)= λf'(z) (f+g)'(z)=f'(z)+g'(z) (fg)'(z)=f'(z)g(z)+f(z)g'(z)
② chain rule: f holo in G and g holo in an open set containing f(G): (g∘f)z=g(f(z)) (g∘f)'z=g'(f(z))f'(z)
③ let f holo in G and suppose f(z)≠0 for all z∈G. Then 1/f is holo in G and for any z∈G, $(1/f)'(z)=f'(z)/(f(z))^2$
Example: any polynomial (finite sum of c_nz^n) is holomorphic in ℂ
rational function p(z)/q(z) is holo in any open set in which q(z) is never zero.

Domains of holomorphicity: the biggest subset of ℂ where function f is holomorphic.
Note: Domains of holomorphicity is an open set. (z0 holo then its neighborhood holo)
Note: use Wirtinger Operator $\partial f/\partial z^*=0$.
Holomorphic functions are infinitely differentiable: C^∞ C:continuity ∞:infinite differentiable

If f is holomorphic at z_0, then it has infinitely many derivatives at z_0. All of its derivatives are holomorphic at z_0. all partial derivatives of u and v exist and are continuous. $f^{(n)}(z)=\frac{\partial^n}{\partial x^n}u+i\frac{\partial^n}{\partial y^n}v$

If f is holomorphic at z_0, then Re f and Im f are harmonic in a neighborhood of z_0.

If f and g holo, so are f+g, fg, f/g (g≠0), f∘g.

5.**Harmonic Functions**: Let $G\subseteq\mathbb{C}$ be a region. A function u: G→R is harmonic in G if it has continuous second partials in G and satisfies Laplace Equation: $u_{xx}+u_{yy}=0$ in G.

- Definition of harmonic functions
- Harmonic functions from holomorphic functions

Harmonic Conjugate:

v is a harmonic conjugate of/to u if u+iv is holomorphic on Ω: $u_x=v_y$ and $u_y=-v_x$, or $u_r=(1/r)v_\theta$ and $(1/r)u_\theta=-v_r$.

Note: use CR and initial condition to find v first, then f=u+iv=f(z) a function of z

- Existence of harmonic conjugates on simply-connected domains

6.**Conformal Transformation**:

A function f is conformal at z_0, it preserves angles and orientation at that point.

A transformation function f is **conformal** at z_0, if it is analytic/holomorphic at z_0 and $f'(z_0)\neq0$

take a path $C:[a,b]\to\mathbb{C}$ in z-plane, and its image $f\circ C:[a,b]\to\mathbb{C}$ in w-plane.

a point z_0 in z-plane that C goes through at time $\lambda\in[a,b]$, $C(\lambda)=z_0$ and its image $w_0=f(z_0)=f(C(\lambda))$.

then, $(f\circ C)'(\lambda)=f'(z_0)C'(\lambda)$.

<rotation> by angle of rotation of f at z_0 --$\arg f'(z_0)$ <dilation> by scale factor --$|f'(z_0)|$

7.**Order of Zero**:

f is holomorphic at z_0, the point z_0 is a zero of f if $f(z_0)=0$.

Zero z_0 of f is of order m if $f(z_0)=f'(z_0)=\ldots f^{(m-1)}(z_0)=0$ but $f^{(m)}(z_0)\neq0$.

Note: f has a zero of order 0 at z_0, if f is holomorphic at z_0 and $f(z_0)\neq0$. Zeros of order 1,2... are called simple, double...

Example: (z-a)m has a zero of order m, zeros of sinz are simple.

Theorem: f has a Taylor expansion $f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n$, the following are equivalent

(i) $f(a)=f'(a)=\ldots f^{(m-1)}(a)=0$ and $f^{(m)}(a)\neq0$. (ii) $f(z)=\sum_{n=m}^{\infty}c_n(z-a)^n$ where $c_m\neq0$

(iii)$f(z)=(z-a)^m g(z)$ where $g(a)\neq0$

(iv)there exists a non-zero constant $C\in\mathbb{C}$ such that $\lim_{z\to a}(z-a)^m f(z)$ exists and equals C.

Compound Zeros:

If f and g have zeros of order m≥0 and n≥0 respectively at z_0, then fg has a zero of order m+n.

$z^2\sin^4 z$ has a zero of order 2+4=6 at z=0 and zeros of order 0+4=4 at z=kπ

Critical Point:

Holomorphic function f has a **critical point** of order m at z_0, if f' has a zero of order m at z_0.

Find critical point: (i) $f'(z_0)=0$ find value of z_0 (ii) find n when $f^n(z_0)\neq0$, (n-1) is order of critical point

(iii) behave near critical point : Locally, $f(z)-f(z_0)=\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+O[(z-z_0)^{n+1}]$

behaves like z^n or z^n followed by a rotation of π

Elementary Functions

1.**Exponential**: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^x e^{iy} = e^x(\cos y + i \sin y)$ $e^{z_1+z_2} = e^{z_1}e^{z_2}$ $e^{z_1-z_2} = e^{z_1}/e^{z_2}$ $e^0 = 1$ $e^{-z} = 1/e^z$

Holomorphy:(hence continupus) $\frac{d}{dz}e^z = e^z$ e^z is entire (differentiable for all z)

$e^z \neq 0$ (for all complex number z) but it can be negative

$|e^z| = e^{Rez} = e^x$ $|e^{iy}| = 1$ $\arg(e^z) = y + 2n\pi$ e^z is periodic with a pure imaginary period $2\pi i$. not one-to-one

2.**Logarithm**:(inverse of exponential) $e^w = z$ (z: nonzero complex number) $z = re^{i\theta}$ log(e)=1

$\log z = \log|z| + i\arg z = \ln r + i(\theta + 2\pi n)$ (multi-valued)

Principal value/branch of logz (n=0): $Log\ z = \log|z| + iArg\ z = \ln r + i\theta$ θ∈(-π,π] (uni/single-valued)

Example: log(1)=2iπn Log(1)=0 log(-1)=i(2n+1)π Log(-1)=πi

$\log_{(\alpha)} z = \ln|z| + i\theta$ where θ∈(α,α+2π) single-valued, continuous, analytic

(not defined on the ray θ=α, not continuous on the ray)

$\frac{d}{dz}\log_{(\alpha)} z = \frac{1}{z}$ $\frac{d}{dz}Log\ z = \frac{1}{z}$

A **branch** of a multiple-valued function f is any single-valued function F that analytic in some domain at each point z of which the value F(z) is one of the values of f.

Branch Cut: a portion of a line/curve introduced to define a branch F of a multi-valued function f.

Points on the branch cut for F are singular points of F. (origin + ray θ=α)

upper edge & lower edge of the cut

Branch Point: any point common to all branch cuts of f. (origin)

$nLog\ z \overset{maybe}{\neq} Log\ z^n$

$\log z_1 z_2 = \log z_1 + \log z_2$ $\log z_1/z_2 = \log z_1 - \log z_2$ $Log\ z_1 z_2 \overset{maybe}{\neq} Log\ z_1 + Log\ z_2$

Power: If z is a nonzero complex number (z≠0), $z^n = e^{n\log z} = e^{n(\log|z| + i\arg z)}$

$$z^{1/n} = \exp(\frac{1}{n}\log z) = \exp\left[\frac{1}{n}\ln r + \frac{i(\theta+2nk)}{n}\right] = \sqrt[n]{r}\exp\left[\frac{i(\theta+2nk)}{n}\right]$$

$e^{\log z} \overset{maybe}{=} z$

3.**Trigonometric and Hyperbolic Functions**:

Trigonometric	Hyperbolic
$\sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{e^{iz}-e^{-iz}}{2i}$ 2i!!!!!! odd $\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = \frac{e^{iz}+e^{-iz}}{2}$ even Period 2π, $\sin^2 z + \cos^2 z = 1$	$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \frac{e^z - e^{-z}}{2}$ odd $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \frac{e^z+e^{-z}}{2}$ even Period 2πi $\cosh^2 z - \sinh^2 z = 1$
functions are entire (linear combinations of entire functions e^{iz} and e^{-iz}) $\frac{d}{dz}\sin z = \cos z$ $\frac{d}{dz}\cos z = -\sin z$ $(e^{iz})' = i(e^{iz})$ $(e^{-iz})' = -i(e^{-iz})$	functions are entire (linear combinations of entire functions e^z and e^{-z}) $\frac{d}{dz}\sinh z = \cosh z$ $\frac{d}{dz}\cosh z = \sinh z$
$\sin(-z) = -\sin z$ $\cos(-z) = \cos z$ $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$	$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$ $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

		$\sin(z+\frac{\pi}{2})=\cos z$	$\sin(z-\frac{\pi}{2})=-\cos z$
Osborn Rules: $\sin iz = i\sinh z$ $\quad \cos iz=\cosh z$ $\sin(iy)=i\sinh y$ $\quad \sin z=\sin x\cosh y+i\cos x\sinh y$ $\cos(iy)=\cosh y$ $\quad \cos z=\cos x\cosh y-i\sin x\sinh y$ $\lvert\sin z\rvert^2=\sin^2 x+\sinh^2 y$ $\quad \lvert\cos z\rvert^2=\cos^2 x+\sinh^2 y$ **Unboundness**: sinhy tends to ∞ when y tends to ∞, so complex sine sinz and cosine cosz are not bounded on the complex plane		$-i\sinh(iz)=\sin z$ $\quad -i\sin(iz)=\sinh z$ $\cosh(iz)=\cos z$ $\quad \cos(iz)=\cosh z$ $\sinh z=\sinh x\cos y+i\cosh x\sin y$ $\cosh z=\cosh x\cos y+i\sinh x\sin y$ $\lvert\sinh z\rvert^2=\sinh^2 x+\sin^2 y$ $\quad \lvert\cosh z\rvert^2=\sinh^2 x+\cos^2 y$	
Zeros of sinz: z=nπ	Zeros of cosz: z=π/2+nπ	Zeros of sinhz: z=nπi	Zeros of coshz: z=(π/2+nπ)i
tanz=sinz/cosz (analytic everywhere expect at singularities) period π. singularities of tanz: zeros of cosz,z=π/2+nπ			

Zero of a given function: number of z_0 such that $f(z_0)=0$

Inverse: (multivalued function)

$$\sin^{-1}z=-i\log[iz\pm\sqrt{1-z^2}] \qquad \cos^{-1}z=-i\log[z\pm i\sqrt{1-z^2}] \qquad \tan^{-1}z=\frac{i}{2}\log\frac{i+z}{i-z}$$

$$\left(\sin^{-1}z\right)'=\frac{1}{\sqrt{1-z^2}} \qquad \left(\cos^{-1}z\right)'=\frac{-1}{\sqrt{1-z^2}} \qquad \left(\tan^{-1}z\right)'=\frac{1}{1+z^2}$$

$$\sinh^{-1}z=\log[z\pm\sqrt{z^2+1}] \qquad \cosh^{-1}z=\log[z\pm\sqrt{z^2-1}] \qquad \tanh^{-1}z=\frac{1}{2}\log\frac{1+z}{1-z}$$

Derivation: $w=\sin^{-1}z$ → $z=\sin w=(e^{iw}-e^{-iw})/2i$ → $(e^{iw})^2-2iz(e^{iw})-1=0$ $\quad e^{iw}=iz\pm(1-z^2)^{1/2}$
take logarithm: $w=\sin^{-1}z=-i\log[iz\pm(1-z^2)^{1/2}]$ multivalued function

4.**Complex Exponents**: $z\neq0$, exponen c is any complex number $z^c=e^{c\log z}$ (multivalued)

$$\frac{1}{z^c}=\frac{1}{e^{c\log z}}=e^{-c\log z}=z^{-c} \qquad \frac{d}{dz}z^c=cz^{c-1}$$

Derivation: chain rule

Principal Value/Branch of z^c: $z^c=e^{cLog\,z}$ on domain |z|>0, -π<Argz<π

Exponential Function with base c(nonzero complex constant): $c^z=e^{z\log c}$ (multivalued) $\quad \frac{d}{dz}c^z=c^z\log c$

Principal Value of c^z at z: $c^z=e^{zLog\,c}$

Derivation:

5.**Combining Branch Cuts**:

$\log_{(\alpha)}z$ branch cut at z=0 & arg z=α+2π discontinuity: -2πi

If $-\log_{(\alpha)}z=i^2\log_{(\alpha)}z$ branch cut at z=0 & arg z=α+3π i^2:rotated ccw by π

If $\frac{1}{2}\log_{(\alpha)}z$ discontinuity: -πi

$z^{1/2}=e^{\frac{1}{2}Logz}$ discontinuity: $\cdot e^{-2\pi i}$

Mapping/Transformation

Image of a point z in the domain of definition S is the point w=f(z).
Set of images of all points in a set T that is contained in S is image of T.
Image of entire domain of definition S is range of f.
Inverse image of a point w is the set of all points z in the domain of definition of f that have w as their image.

Translation: w=f(z)=z+C C: complex number
Rotation: $w=f(z)=Cz=e^{i\theta}z$ (rotated by θ ccw)
Reflection: $w=f(z)=z^*$ (in real axis)

1.$w=z^2$ ($z=x+iy$ ~~> $w=u+iv$ $u=x^2-y^2$ $v=2xy$)
①Grid Transformation: $x=x_0$ $y=y_0$ (real axis& imaginary axis as special case)
Vertical Line: $x=x_0$ and parametrization $y=t$ ~~> $u=x_0^2-t^2$ $v=2x_0t$, $u=x_0^2-(v/2x_0)^2$ [parabola,开口左]
when $x_0=0$, $u=-t^2$ $v=0$ [lines on negative real axis rebounce once]
Horizontal Line: $y=y_0$ and parametrization $x=t$ ~~> $u=t^2-y_0^2$ $v=2ty_0$, $u=(v/2y_0)^2-y_0^2$ [parabola,开口右]
when $y_0=0$, $u=t^2$ $v=0$ [lines on positive real axis rebounce once]
②Quadrant Circle: ¼D
$z=re^{i\theta}$ $\theta\in[0,\pi/2]$ ~~> $w=z^2=r^2e^{i2\theta}$ [half circle with radius r^2]
③Half Plane: H
$z=re^{i\theta}$ $\theta\in[0,\pi]$ $r>0$ ~~> $w=z^2=r^2e^{i2\theta}$ [entire w plane]
④vertical Strip: $[0,1]\times[0,\infty]$ ~~> closed semi-parabolic region

2.$w=e^z$ ($z=x+iy$ ~~> $w=e^xe^{iy}$)
①vertical and horizontal line segments are mapped onto portions of circles and rays:
Vertical Line: $x=x_0$ ~~> $w=e^{x0}e^{iy}$ [circle with radius e^{x0} transversed ccw]
Horizontal Line: $y=y_0$ ~~> $w=e^xe^{iy0}$ [ray with arg(y_0)]
②Rectangular Region: $[a,b]\times[c,d]$ ~~> $w=\rho e^{i\phi}$ with $\rho=e^x\in[e^a,e^b]$ $\phi=y\in[c,d]$ [portion of circular ring]
③Infinite Strip: ~~> sector consider vertical (horizontal ?) lines transformation, regard strip as combination of lines
3.$w=\sin z=\sin x\cosh y+i\cos x\sinh y$ ($z=x+iy$ ~~> $u=\sin x\cosh y$ $v=\cos x\sinh y$)
①Vertical Lines : $x\in[-\pi/2,\pi/2]$ $x=x_0$ ~~> $u=\sin x_0\cosh y$ $v=\cos x_0\sinh y$
so $(u/\sin x_0)^2-(v/\cos x_0)^2=1$ [hyperbola with foci $w=\pm((\sin x_0)^2+(\cos x_0)^2)^{1/2}=\pm1$]
$x_0\in(0,\pi/2)$ right branch of hyperbola $x_0\in(-\pi/2,0)$ left branch of hyperbola
$x_0=\pi/2$ ~~> $u=\cosh y$ $v=0$ $x_0=-\pi/2$ ~~> $u=-\cosh y$ $v=0$
when $x_0=0$ (imaginary axis) ~~> $u=0$ $y=\sinh y$ [positive imaginary axis in w-plane]

interior: $(-\pi/2,\pi/2)\times(0,\infty)$ vertical lines ~~~>
②Semi-Infinite Strip: ~~> upper half plane $[-\pi/2,\pi/2]\times[0,\infty]$ ~~> $v\geq0$
right-hand half of strip ~~> first quadrant

③Horizontal Lines: $x\in[-\pi,\pi]$ $y=y_0$ ~~> $u=\sin x\cosh y_0$ $v=\cos x\sinh y_0$
so $(u/\cosh y_0)^2+(v/\sinh y_0)^2=1$ [whole ellipse with foci $w=\pm((\cosh y_0)^2+(\sinh y_0)^2)^{1/2}=\pm1$]
when $y_0=0$ (real axis) ~~> $u=\sin x$ $y=0$
④rectangular region ~~> half ellipse
⑤$w=\cos z=\sin(z+\pi/2)$ (translation $\pi/2$ to right) $w=\sinh z=-i\sin(iz)$ ($Z=iz$ $W=\sin Z$ $w=-iW$)

4.$w=$branch of $z^{1/2}=e^{1/2 \log z}$ ($z=x+iy$ ~~> $u=\sin x\cosh y$ $v=\cos x\sinh y$)
Choose principal branch $z^{1/2}=e^{1/2 \mathrm{Log} z}=r^{1/2}e^{(i\theta/2)}$
quarter disk ~~> sector with 45°
5.Square Root of Polynomial:

6.$w=\log_{(\alpha)}z$ wedge? problem 7.3

Riemann Surface: complex plane consisting more than one sheet.

Mapping:
1.linear transformation:

$$\left.\begin{array}{l} w=\mathbb{C}z \quad (\mathbb{C}\neq0) \\ w=z+\mathbb{C} \end{array}\right\} \Rightarrow w=Az+B \text{ (expansion/contraction+rotation+translation)}$$

Note: $w=z+c$ $c=a+ib$ ~~>$u=x+a$ $v=y+b$ a>0 to the right b>0 upwards, tricky.....

2.$w=1/z=z^*/|z|^2$: on extended ℂ plane $z=0$ ~~> $w=\infty$ $z=\infty$ ~~> $w=0$

$w=1/z$ maps circles and lines to circles and lines: $u=x/(x^2+y^2)$ $v=-y/(x^2+y^2)$

General Form for circles($A\neq0$) and lines($A=0$): $A(x^2+y^2)+Bx+Cy+D=0$ ($B^2+C^2>4AD$)

~~> $D(u^2+v^2)+Bu-Cv+A=0$

Circle($A\neq0$) not passing through origin($D\neq0$) in z-plane ~~> circle not passing through origin in w-plane

Circle($A\neq0$) passing through origin($D=0$) in z-plane ~~> line not passing through origin in w-plane

Line($A=0$) not passing through origin($D\neq0$) in z-plane ~~> circle passing through origin in w-plane

Line($A=0$) not passing through origin($D=0$) in z-plane ~~> line passing through origin in w-plane

3.Bilinea/Linear Fractional Transformation: one-to-one

$f(z)=\dfrac{az+b}{cz+d}$ (a,b,c,d∈ℂ) when $ad-bc\neq0$, **Möbius Transformation**

Möbius Transformation transforms circles and lines into circles and lines.

if $c=0$ $f(\infty)=\infty$ if $c\neq0$ $f(\infty)=a/c$ $f(-d/c)=\infty$

Inverse Transformation: $f^{-1}(w)=z=(-dw+b)/(cw-a)$

upper half plane H ~~> unit disk D Im $z=0$ ~~> $|w|=1$

Find conformal transformation: think step by step and use composition, break up transformation

Note: Transformation is conformal at every point (holomorphic& its derivative vanishes nowhere on it)

Integral

• Reminder on the Riemann integral

• Integrals of complex-valued functions on the real line/of a real variable

Green's theorem

• Complex integrals: definition and basic properties

※ Derivative of function: $\dfrac{d}{dt}w(t)=w'(t)=u'(t)+iv'(t)$ $\quad [z_0w(t)]'=z_0w'(t)$ $\quad [e^{z_0t}]'=z_0e^{z_0t}$

w(t) is continuous on interval [a,b], so does its component functions u(t) and v(t).

"*Mean Value Theorem for derivatives*" false ==> Even w'(t) exists in (a,b) and suppose a number c∈(a,b), but it is possible $w'(c)\neq[w(b)-w(a)]/(b-a)$. e.g $w(t)=\exp(it)$ $|w'|=1$ but $b=2\pi$ $a=0$ $w(b)-w(a)=0$

※ Integral of function: $\int_a^b w(t)dt=\int_a^b u(t)dt+i\int_a^b v(t)dt$ $\quad \mathrm{Re}\left[\int_a^b w(t)dt\right]=\int_a^b \mathrm{Re}[w(t)]dt$ $\quad \mathrm{Im}\left[\int_a^b w(t)dt\right]=\int_a^b \mathrm{Im}[w(t)]dt$

Note: w(t) is **piecewise continuous** (continuous everywhere except for a finite number of discontinuous points (but it has one-sided limits)) on interval [a,b].

Fundamental Theorem of Calculus: Suppose $w=u+iv$ and $W=U+iV$ continuous on [a,b]. If $W'=w$, then $U'=u$ and $V'=v$.

Proof: $\int_a^b w(t)dt=U(t)|_a^b+iV(t)|_a^b=[U(b)-U(a)]+i[V(b)-V(a)]=W(b)-W(a)=W(t)|_a^b$

"*Mean Value Theorem for Integrals*" false ==> suppose a number c∈(a,b), but it is possible $\int_a^b w(t)dt\neq w(c)\times(b-a)$. e.g $w(t)=\exp(it)$ $|\int_a^b w(t)dt|=0$ but $b=2\pi$ $a=0$ $|\exp(ic)|\times2\pi=2\pi$

Integral of complex-valued functions of a complex variable is defined on curves in the complex plane.

※ path/arc: $\gamma:[a,b]\to\mathbb{C}$ counterclockwise (positive oriented)

$x=x(t)$, $y=y(t)$ with $t\in[a,b]$, a set of points $z=(x,y)=z(t)$ in the complex plane is arc

-Simple: no cross/intersect itself; if $z(t_1)\neq z(t_2)$, when $t_1=t_2$.

-Simple Closed arc (Jordan arc): simple and $z(a)=z(b)$

-Positively Oriented: in the counterclockwise direction

-Differentiable arc: $z'=x'+iy'$ continuous Length of Arc$=\int_a^b|z'(t)|dt$

-Smooth arc: z'(t) continuous on [a,b] and nonzero on (a,b) unit tangent vector no turns.

Jordan curve theorem: points on any simple closed contour C are boundary points of two distinct domain.(interior of C and bounded, exterior of C and unbounded)

contour: piecewise smooth path (arc consisting of a finite number of smooth arcs (Legs) joined end to end)

1.**Contour/Line Integral**:

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)dt = \int_a^b (u\,dx - v\,dy) + i\int_a^b (v\,dx + u\,dy)$$

$\int_C z_0 f(z)dz = z_0\int_C f(z)dz$ $\quad\int_C f(z)+g(z)dz = \int_C f(z)dz + \int_C g(z)dz$ $\quad\int_{-C} f(z) = -\int_C f(z)dz$

concatenation of path: $\int_{C_1+C_2} f(z) = \int_{C_1} f(z) + \int_{C_2} f(z)$

Example: ① $\oint_C \frac{dz}{z} = 2\pi i$ Let $z=e^{i\theta}$, $\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-i\theta} ie^{i\theta}\,d\theta = 2\pi i = \oint_C \frac{|z|^2}{z}dz = \oint_C \frac{dz}{z}$ ② $\oint_C z^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$

Parametrization of Contour: $\gamma(t)$ two point in z-plane (x_1,y_1) (x_2,y_2)

2.Upper Bounds for Modulus of contour integrals:
①Two paths with same initial and final points may have different integral values(dependent on path).

$$\left|\int_\gamma f(z)dz\right| \leq \int_\gamma |f(z)|\,dz$$

Proof:

Note: contour integration independent of path, then integral around a closed path has zero value.

②Bounding Complex Integral: $\left|\int_\gamma f(z)dz\right| \leq ML$

Proof:

3.Antiderivative:
Suppose function f continuous on a domain D, following statements are equivalent:
(i) f(z) has an antiderivative F(z) throughout D. (F(z) is unique except for an additive constant)
(ii) integrals of f(z) along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value: $\int_{z_1}^{z_2} f(z)dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)$
(iii) integrals of f(z) around closed contours lying entirely in D all have zero.

Proof:

Green Theorem: $\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dx\,dy$

4.①Cauchy Theorem: f is holomorphic on and inside simple closed contour C and f' continuous on C, $\oint_C f(z)dz = 0$

②**Cauchy-Goursat Theorem**: f is holomorphic on and inside simple closed contour C, $\oint_C f(z)dz = 0$

Proof: by∫f=∫(udx-vdy)+i∫(vdx+udy)=∬(-v_x-u_y)dxdy+i∬(u_x-v_y)dxdy (Green Theorem) and Cauchy-Riemann equations $u_x=v_y$ $-v_x=u_y$

(a)Simple Connected Domain: a domain such that every simple closed contour within it encloses only points of D.
A function holomorphic in simply connected domain D,it must have an antiderivative everywhere in D.

Proof: holomorphic means continuous, so f has an antiderivative in D ?????

(b)Multiply Connected Domain:
Suppose that C is a simple closed contour described in ccw direction, and C_k are simple closed contours interior to C, all in ccw direction, that are disjoint and whose interiors have no points in common. If function f is holomorphic on all of these contours and throughout the multiply connected domain consisting of points inside C and exterior to each C_k, then $\int_C f(z)dz + \sum_k \int_{C_k} f(z)dz = 0$

Principle of Deformation of paths: Let C_1 and C_2 denote positively oriented simple closed contours, where C1 is interior to C2. If function f is holomorphic in closed region consisting of those contours and all points between them, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

use of holes: $\int_{\gamma_{total}} = \sum \int_{\gamma_{holes}}$

③**Cauchy Integral Formula**: f holomorphic on and inside C (simple closed contour, oriented positively)

z_0 is point interior to C, then $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$ used to evaluate integrals: $\int_C \frac{f(z)}{z - z_0}dz = 2\pi i f(z_0)$

Proof:

Extension: $$\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(z_0)$$

Proof:

5. Consequences of Cauchy Integral Formula:

Existence of all derivatives of holomorphic functions: If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.

Proof: f holo at z_0→ f holo around z_0→ for sufficiently small ρ, f(z) holo on and inside $C_\rho(z_0)$→ derivative of f at z_0 exists→f' holo at z_0

If a function f(z)=u(x,y)+iv(x,y) is analytic at a point z=(x,y), then the component functions u and v have continuous partial derivatives of all orders at that point.

①**Morera's theorem**: Let f be continuous on a region D and suppose $\int_C f(z)dz=0$ for every closed contour C lying in D. Then f is holomorphic on D.

②**Cauchy Inequality**: let f be analytic on and inside circle C centered at z_0 with radius R.

Suppose $|f(z)| \le M$ for all z on C_R, Then $$\left|f^{(n)}(z_0)\right| \le \frac{n!}{R^n}M$$

Proof: Cauchy-Goursat

③**Liouville's theorem**: function f is entire and bounded in the complex plane, then f(z) is constant throughout the plane

Proof: Cauchy Inequality at n=1: $|f'(z_0)| \le M/R = 0$ as $(R\to\infty)$ $f'(z_0)=0$ since z_0 is arbitrary, f'(z)=0 everywhere in the complex plane

④**Fundamental Theorem of Algebra**: Polynomial $P(z)=a_0+a_1z+...a_nz_n$ ($a_n \neq 0$) of degree n(≥1) has at least one zero; at least one point z_0 such that $P(z_0)=0$

Proof: By Contradiction. Suppose P(z) has no complex roots, $P(z)\neq 0$ for every $z\in\mathbb{C}$, then 1/P(z) entire, Also f(z) is bounded on $\mathbb{C}$: (Proof @PS9.3) By Liouville Theorem, f(z) is constant, then P(z) is constant. Contract.

⑤Maximum Principle: D domain, f holo on D, then |f(z)| has no maximum on D. f(z) achieves absolute maximum on the boundary ∂D.

⑥**Gauss Mean Value Theorem**: If function analytic within and on a circle, its value at the center is the arithmetic mean of its values on the circle.

$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho(z_0)} \frac{f(z)dz}{z - z_0} \xrightarrow{z=z_0+\rho e^{i\theta}} \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta$$

Proof: By Cauchy-Goursat

⑦Suppose that $|f(z)| \le |f(z_0)|$ at each point z in some neighborhood $|z-z_0| < \varepsilon$ in which f is analytic. Then f(z) has constant value $f(z_0)$ throughout $|z-z_0| < \varepsilon$.

Proof:

⑧**Maximum Modulus Principle**:

If function f is analytic and not constant in domain D, then |f(z)| has no maximum value in D; there is no point z_0 in the domain such that $|f(z)| \le f(z_0)$ for all points z in it.

Suppose a function is continuous on a closed, connected, bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of |f(z)| in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.

Proof:

Series

1.**Convergence of Sequences**:
An infinite sequence $z_1, z_2 .. z_n, ...$ of complex numbers has a limit z, if for each positive ε, there exists a positive integer n_0 such that $|z_n - z| < \varepsilon$ whenever $n > n_0$.
The limit of a sequence of complex numbers is unique if it exists. When the limit exists, the sequence is converge to z, $\lim_{n\to\infty} z_n = z$

Suppose zn=xn+iyn and z=x+iy. Then $\lim_{n\to\infty} z_n = z$ iff $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

Proof:

2.**Convergence of Series**:
An infinite series

Proof:

Convergence and Absolute Convergence

Proof:

3.**Power Series**: $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

Infinite Geometric Series: $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (\lvert z\rvert < 1)$	$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (\lvert z\rvert < \infty)$	$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad (\lvert z\rvert < \infty)$ sinhz=-isin(iz)
$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (\lvert z\rvert < \infty)$	$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad (\lvert z\rvert < \infty)$	$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad (\lvert z\rvert < \infty)$

4.**Taylor Series**: function is analytic at z_0, then it can have Taylor expansion at z_0
Taylor Theorem: Suppose a function f is analytic throughout the disk $|z-z_0|<R_0$. Then f(z) has the power series representation: Taylor series expansion about z_0 $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$.
Taylor series converges to f(z) when z lies in the open disk $|z-z_0|<R_0$.
Maclaurin Series: when $z_0=0$ circle of convergence: $|z-z_0|<R_0$ radius of convergence: R_0.

Proof:

5.**Laurent Series**: if function f fails to be analytic at point z_0, we cannot apply Taylor theorem, but we can find a series representation of f involving both positive and **negative powers** of $(z-z_0)$.
Laurent Theorem: Suppose a function f is analytic throughout an annular domain $R_1<|z-z_0|<R_2$, centered at z_0. Then at each point in the domain, f(z) has the series representation:

Laurent Series Expansion about z_0 $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$ and $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$,C is any positively oriented simple closed contour around z_0 lying in domain $R_1<|z-z_0|<R_2$. Note: R_1 could be 0 and R_2 could be ∞.

Laurent Series: $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$, $R_1<|z-z_0|<R_2$, where $c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

Proof:

6.If a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z=z_1$ ($z_1 \neq z_0$), then it is absolutely convergent at each point z in the open disk $|z-z_0|<R_1$, where $R_1=|z_1-z_0|$.
Circle of Convergence: largest circle centered at z_0 such that the power series converges at each point inside.
If z_1 is inside circle of convergence $|z-z_0|=R$ of a power series $\Sigma a_n(z-z_0)^n$ and $R_1=|z_1-z_0|$, then the power series is uniformly convergent in the closed disk $|z-z_0|<R_1$.

Proof:

7.**Uniqueness** of Series Representations: Taylor and Laurent series representations of functions are unique
If a series $\Sigma a_n(z-z_0)^n$ converges to f(z) at all points interior to some circle $|z-z_0|=R$, then it is the Taylor series expansion for f about z_0 in that domain.
If a series $\Sigma a_n(z-z_0)^n+\Sigma b_n(z-z_0)^{-n}$ converges to f(z) at all points in some region $R_1<|z-z_0|<R_2$, then it is the Laurent series expansion for f about z0 in that domain.

Sum integration and differentiation

※ Multiplication and Division of power series:
Multiplication: **Cauchy Product**

$$f(z)g(z)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_k b_{n-k}(z-z_0)^n=a_0b_0+(a_0b_1+a_1b_0)(z-z_0)+(a_0b_2+a_1b_1+a_2b_0)(z-z_0)^2+\ldots+\left(\sum_{k=0}^{n}a_k b_{n-k}\right)(z-z_0)^n+\ldots$$

Leibniz Rule: $$[f(z)g(z)]^{(n)}=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(z)g^{(n-k)}(z)$$

Division: find all negative power terms (principal part) plus one positive term and then $O(z^n)$

Residue

1.**Isolated Singular Point** (singularities):
z_0 is a singular point of function f if f is not analytic at z_0, but is analytic at some point in every neighborhood of z_0.
A singular point is isolated if, in addition, there is a deleted neighborhood $0<|z-z_0|<\varepsilon$ of throughout which f is analytic.

Example: (i) Logz singular point z=0 not isolated --because every deleted ε neighborhood of z=0 contains points on branch cut(negative real axis)
(ii) 1/sin(π/z) singular point z=0(not isolated) z=1/n (isolated)

①Removable Singularity: $b_n=0$ for all n
②Essential Singularity: an infinite number of b_n are nonzero
③Pole of order N: $b_N\neq0$ and $b_{N+1}=b_{N+2}=\ldots=0$ simple pole: first order pole (only $b_1\neq0$)
Pichard Theorem: in each neighborhood of an essential singular point, a function assumes every finite value, with one possible exception, an infinite number of times.

Proof:

Principal Part (of f at z_0): portion of series involving negative powers of $(z-z_0)$

2.**Residue** (of f at isolated singular point z_0): Coefficient of $(z-z_0)^{-1}$ in Laurent expansion $b_1=\operatorname*{Res}_{z=z_0} f(z)$

$$\int_C f(z)dz=2\pi i b_1=2\pi i \operatorname*{Res}_{z=z_0} f(z)$$

Proof: CB231

Note: analyticity of a function within and on a simple closed contour C is a sufficient condition for the value of integral around C to be zero; but not necessary. e.g $\int \exp(1/z^2)dz=0$ when C: $|z|=1$ singularity at z=0 but $\int=0$

If f is holomorphic inside and on a simple closed contour C except for a finite number of singular points, then those singular points must be isolated.

3.**Cauchy Residue Theorem**: Let C be a positively oriented simple closed contour. If f is analytic inside C except for a finite number of singular points z_k inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{\substack{singular \\ points\ z_k}} \operatorname*{Res}_{z=z_k} f(z)$$

Proof: Cauchy-Goursat Theorem

4.Calculation of Residue

① Write Laurent Series Expansion and find b_{-1}.

② **Residues at Poles**:

An isolated singular point z_0 of a function f is a pole of order m iff f(z) can be written in the form:

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m}$$ where $\phi(z)$ is analytic at z_0 and $\phi(z_0)\neq 0$.

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z)}{(m-1)!}$$ Note: 0!=1 and $\phi^{(0)}(z)=\phi(z)$

Proof:

5.**Zeros** of Analytic Function: function f is analytic at z_0, all of the derivatives $f^{(n)}(z)$ exist at z_0.

point z_0 is a zero(of f) of order m if $f(z_0)=f'(z_0)=...f^{(m-1)}(z_0)=0$ but $f^{(m)}(z_0)\neq 0$. (all derivatives of lower order vanishes at z_0)

Order of Zero: m

Note: f has a zero of order 0 at z_0, if f is holomorphic at z_0 and $f(z_0)\neq 0$. Zeros of order 1,2... are called simple, double...

Example: $(z-a)^m$ has a zero of order m, zeros of sinz are simple.

Let f be a function analytic at point z_0, f has a zero of order m at z_0 iff there exists a function g analytic at z_0 with $g(z_0)\neq 0$ such that $f(z)=(z-z_0)^m g(z)$

Proof:

Given a function f and a point z_0, suppose that f is analytic at z_0, $f(z_0)=0$ but f(z) is not identically equal to zero in any neighborhood of z_0. Then, $f(z)\neq 0$ throughout some deleted neighborhood $0<|z-z_0|<\varepsilon$.

Given a function f and a point z_0, suppose that f is analytic throughout a neighborhood N_0 of z_0, f(z)=0 at each point z of a domain D or line segment L containing z_0. Then, f(z)=0 in throughout N_0 (f(z) identically equals to zero throughout N_0)

Proof:

6.**Zeros and Poles**: determine poles for quotients of analytic functions

Suppose two functions p and q are analytic at z_0, $p(z_0)\neq 0$ and q has a zero of order m at z_0. Then f(z)=p(z)/q(z) has a pole of order m at z_0. (zeros of order m creates poles of order m)

Proof:

Suppose two functions p and q are analytic at z_0. If $p(z_0)\neq 0$, $q(z_0)=0$ and $q'(z_0)\neq 0$, then z_0 is a simple pole of p(z)/q(z) and Residue of p(z)/q(z) at z_0 is $p(z_0)/q'(z_0)$.

Proof: Point z_0 is a zero of order m=1 so $q(z)=(z-z_0)g(z)$, where g(z) is analytic and nonzero at z_0.
If z_0 is a simple pole of p(z)/q(z), then $p(z)/q(z)=\phi(z)/(z-z_0)$, where $\phi(z)$ is analytic and nonzero at z_0. Thus, $\phi(z)=p(z)/g(z)$
Residue of p(z)/q(z) at $z_0=\phi(z_0)=p(z_0)/g(z_0)$ Residues of Poles where $g(z_0)= d[(z-z_0)g(z)]/dz_{@z=z_0} =q'(z_0)$

7.**Behavior of functions near isolated singular points**: (behavior depends on type of pole)

If z_0 is a pole of function f, then $\lim_{z\to z_0} f(z)=\infty$.

Proof: assume f has a pole of order m at z_0, then $f(z)=\phi(z)/(z-z_0)^m$, where $\phi(z)$ is analytic and $\phi(z_0)\neq 0$.

$$\lim_{z\to z_0}\frac{1}{f(z)} = \lim_{z\to z_0}\frac{(z-z_0)^m}{\varphi(z)} = \frac{\lim_{z\to z_0}(z-z_0)^m}{\lim_{z\to z_0}\varphi(z)} = \frac{0}{\varphi(z_0)} = 0$$

If z_0 is a removable singular point of a function f, then f is analytic and bounded in some deleted neighborhood $0<|z-z_0|<\varepsilon$.

Proof:

Riemann Theorem: Suppose function f is analytic and bounded in $0<|z-z_0|<\varepsilon$, if f is not analytic at z_0, then it has a removable singularity there.

Proof:

Casorati-Weierstrass Theorem: Suppose z_0 is an essential singularity of function f, and let w_0 be any complex number. Then, for any positive number ε, inequality $|f(z)-w_0|<\varepsilon$ is satisfied at some point z in each deleted neighborhood $0<|z-z_0|<\delta$.

Proof: By Contradiction

Residue Application

1.**Improper Integral**: if limit exists, improper integral converges to that limit

$$\int_0^\infty f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx \quad \text{and} \quad \int_{-\infty}^0 f(x)dx = \lim_{R\to\infty}\int_{-R}^0 f(x)dx$$

If both limits exist, integral converges to sum of the values of these limits:

$$\int_{-\infty}^\infty f(x)dx = \lim_{R_1\to\infty}\int_0^{R_1} f(x)dx + \lim_{R_2\to\infty}\int_{-R_2}^0 f(x)dx \quad (❀)$$

Cauchy Principal Value: $P.V.\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty}\int_{-R}^R f(x)dx$

Note: PV existence does not guarantee integral ❀ always converges

Suppose f(x) is even, and assume Cauchy principal value exists, then $\int_0^\infty f(x)dx = \frac{1}{2}P.V.\int_{-\infty}^\infty f(x)dx$.

Proof:

2.Calculating Improper Integral:

$$P.V.\int_{-\infty}^\infty f(x)dx = 2\pi i \sum_{\substack{z_k \\ singularities}} \operatorname*{Res}_{z=z_k} f(z)$$

Examples:

Improper integrals involving trigonometric functions

3.**Jordan Lemma**: Suppose that

a function f(z) is analytic at all points in the upper half plane y≥0 that are exterior to a circle $|z|=R_0$,

C_R denotes a semicircle $z=Re^{i\theta}$ ($0\le\theta\le\pi$), where R>R0,

for all points z on C_R, there is a positive constant M_R such that $|f(z)|\le M_R$ and lim $M_R=0$ when R approaches to ∞.

Then for every positive constant a, $\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}dz = 0$

Proof: $\int_{C_R} f(z)e^{iaz}dz = \int_0^\pi f(Re^{i\theta})e^{iaRe^{i\theta}}\cdot Rie^{i\theta}d\theta$ with $|f(Re^{i\theta})|\le M_R$ and $|Rie^{i\theta}|=R$

and $\left|e^{iaRe^{i\theta}}\right| = \left|e^{iaR(\cos\theta+i\sin\theta)}\right| = \left|e^{iaR\cos\theta}\right|\left|e^{-aR\sin\theta}\right| \le e^{-aR\sin\theta}$ So $\left|\int_{C_R} f(z)e^{iaz}dz\right| \le M_R R\int_0^\pi e^{-aR\sin\theta}d\theta < \frac{M_R\pi}{a} \xrightarrow{R\to\infty} 0$

Jordan Inequality: $\int_0^\pi e^{-R\sin\theta}d\theta < \frac{\pi}{R}$

Proof: $\sin\theta\ge 2\theta/\pi$ when $\theta\in[0,\pi/2]$ If R>0, $\exp(-R\sin\theta)\le\exp(-2R\theta/\pi)$

From 0 to π/2 integration, $\int\exp(-R\sin\theta)\le\int\exp(-2R\theta/\pi)=(\pi/2R)(1-e^{-R})\le(\pi/2R)$

Definite Trigonometric Integral: $\int_0^{2\pi} F(\cos\theta,\sin\theta)d\theta = \oint_{C_1(0)} F(\frac{z+z^{-1}}{2},\frac{z-z^{-1}}{2i})\frac{dz}{iz}$ $z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Proof: application of residue theorem

Variant:

$$\int_0^{2\pi} F(\cos m\theta, \sin n\theta)\, d\theta = \oint_{C_1(0)} F\left(\frac{z^m + z^{-m}}{2}, \frac{z^n - z^{-n}}{2i}\right)\frac{dz}{iz}$$

Sum-Difference	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ $\quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$	$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$
Double Angle	$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$	$\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$
Half Angle	$\sin^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$ $\quad \cos^2\dfrac{\alpha}{2} = \dfrac{1+\cos\alpha}{2}$ $\quad \tan^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{1+\cos\alpha}$	
Product and Sum	$\sin\alpha\cos\beta = \dfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ $\cos\alpha\sin\beta = \dfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$ $\cos\alpha\cos\beta = \dfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$ $\sin\alpha\sin\beta = \dfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$	$\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$ $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$ $\cos\beta + \cos\alpha = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$ $\cos\beta - \cos\alpha = 2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$