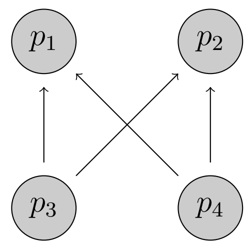
# TOPICS IN MODERN GEOMETRY: HOMEWORK 1 MODEL SOLUTIONS

(1) Let $P = \{p_1, p_2, p_3, p_4\}$ be the following partially ordered set, considered as a topological space with the order topology. (Here, $p \to q$ means $p \leq q$.)




(a) List the open sets of $P$.

The open sets of $P$ are the complements of the closed sets from part (b); that is, $\{p_1, p_2, p_3, p_4\}$, $\{p_2, p_3, p_4\}$, $\{p_1, p_3, p_4\}$, $\{p_3, p_4\}$, $\{p_4\}$, $\{p_3\}$, and $\emptyset$.

(b) List the closed sets of $P$.

The basic closed sets of $P$ are $\{p_1\}$, $\{p_2\}$, $\{p1, p2, p3\}$, and $\{p1, p3, p4\}$. So, all closed sets are $\emptyset$, $\{p_1\}$, $\{p_2\}$, $\{p1, p2\}$, $\{p1, p2, p3\}$, $\{p1, p3, p4\}$, and $\{p_1, p_2, p_3, p_4\}$.

(2) Let $f$ be the bijection from $\mathbb{C}^n$ to $\mathrm{mSpec}(\mathbb{C}[x_1, \ldots, x_n])$ given by

$$f(a_1, \ldots, a_n) = (x_1 - a_1, \ldots, x_n - a_n).$$

Consider $\mathbb{C}^n$ to have the Euclidean topology and $\mathrm{mSpec}(\mathbb{C}[x_1, \ldots, x_n])$ to have the Zariski topology. Prove that $f$ is continuous.

Let $I$ be any ideal of $\mathbb{C}[x_1, \ldots, x_n]$, and consider the associated closed set $V_I$. It suffices to show that $f^{-1}(V_I)$ is closed. We have

$$f^{-1}(V_I) = \{(a_1, \ldots, a_n) \in \mathbb{C}^n : I \leq (x_1 - a_1, \ldots, x_n - a_n)\}. \tag{0.1}$$

The ideal $I$ is finitely generated because $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian, so we have

$$I = (f_1, \ldots, f_m) \tag{0.2}$$

for some generators $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$. Thus,

$$f^{-1}(V_I) = \{(a_1, \dots, a_n) \in \mathbb{C}^n : I \leq f_j(a_1, \dots, a_n) = 0 \text{ for } 1 \leq j \leq m\} \tag{0.3}$$

$$= \bigcap_{j=1}^{m} \{(a_1, \dots, a_n) \in \mathbb{C}^n : I \leq f_j(a_1, \dots, a_n) = 0\}. \tag{0.4}$$

The vanishing set of a polynomial is a closed set in $\mathbb{C}^n$, and an intersection of closed sets is closed, so $f^{-1}(V_I)$ is closed.

(3) Let $f$ be the map from (2). Show that (for every $n \geq 1$) $f^{-1}$ is not continuous.

First, consider the case $n = 1$. The set $\mathbb{Z}$ of integers is a closed set of $\mathbb{C}$. Any ideal of $\mathbb{C}[x]$ is principal, so the closed sets of $\mathrm{mSpec}(\mathbb{C}[x])$ are the sets $V_{(g(x))} \cap \mathrm{mSpec}(\mathbb{C}[x]) = \{(x - a) : g(a) = 0\}$. A polynomial $g(x) \in \mathbb{C}[x]$ vanishes at finitely many points—unless it is identically zero—so the closed sets of $\mathrm{mSpec}(\mathbb{C}[x])$ are all either finite or the whole space. Thus, the infinite set $f(\mathbb{Z})$ is not closed in $\mathrm{mSpec}(\mathbb{C}[x])$. Therefore, $f^{-1}$ is not continuous.

Now, consider the general case. Let $f_n = f$ be the map from $\mathbb{C}^n \to \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n])$ and $f_1$ be the map from $\mathbb{C} \to \mathrm{Spec}(\mathbb{C}[x])$. There is a ring homomorphism $\pi : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x]$ defined by $\pi(g(x_1, x_2, \dots, x_n)) = g(x_1, 0, \dots, 0)$. As proven in Lecture 3, Proposition 3.2, the induced map $\pi^* : \mathrm{Spec}(\mathbb{C}[x]) \to \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n])$ is continuous. Note that $\pi^*((x - a)) = (x_1 - a, x_2, \dots, x_n)$, so $\pi^*$ restricts to a map $j : \mathrm{mSpec}(\mathbb{C}[x])$ to $\mathrm{mSpec}(\mathbb{C}[x_1, \dots, x_n])$. If $f_n(\mathbb{Z} \times \mathbb{C}^{n-1})$ were closed, then $j^{-1}(f(\mathbb{Z} \times \mathbb{C}^{n-1})) = f_1(\mathbb{Z})$ would be closed, but we just showed that it isn't. So $f_n(\mathbb{Z} \times \mathbb{C}^{n-1})$ is not closed, and thus $f_n^{-1} = f^{-1}$ is not continuous.

**(4)** For questions (4)–(6), we will need the definitions of some properties that a topological space $X$ can have.

**T1** If $x \in X$, then $\{x\}$ is closed.

**T2** If $x, y \in X$ and $x \neq y$, then there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. (A space $X$ with this property is called *Hausdorff*.)

Prove that, if $X$ has property **T2**, then $X$ has property **T1**.

Let $X$ be a topological space with property **T2**, and consider any point $x \in X$. Then, for any $y \in X$ such that $x \neq y$, there exist open sets $U_x, V_y \subseteq X$ such that $x \in U_x$, $y \in V_y$, and $U_x \cap V_y = \emptyset$. Then, the set

$$X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} V_y \tag{0.5}$$

is open, so $\{x\}$ is closed.

**(5)** Prove that any metric space has property **T2**.

Let $X$ be a metric space with distance function $d : X \times X \to \mathbb{R}_{\geq 0}$. Suppose $x, y \in X$ such that $x \neq y$, and let $\delta = d(x, y)$. Consider the open sets

$$U = B_x\left(\frac{\delta}{2}\right) \tag{0.6}$$

$$V = B_y\left(\frac{\delta}{2}\right). \tag{0.7}$$

Clearly, $x \in U$ and $y \in V$. If $z \in U \cap V$, then $\delta = d(x, y) \leq d(x, z) + d(z, y) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$, which is impossible. Thus, $U \cap V = \emptyset$.

**(6)** Let $(P, \leq)$ be a partially ordered set, considered as a topological space with the order topology. Prove that, if $P$ has property **T2**, then $x \leq y \implies x = y$.

First, we show that, in any partially ordered set, the smallest closed set containing $x$ is $C_x = \{y : x \leq y\}$. To do so, let $C$ be any closed set containing $x$. Then, $C$ can be written as an intersection of finite unions of sets $C_s$:

$$C = \bigcap_{S \in \mathcal{S}} \bigcup_{y \in S} C_z, \tag{0.8}$$

where $\mathcal{S}$ is a collection of finite sets $S \subseteq P$. Thus, for all $S \in \mathcal{S}$,

$$x \in \bigcup_{z \in S} C_z, \tag{0.9}$$

so for each $S$, there is some $z_S$ such that $x \in C_{z_S}$. That is, $z_S \leq x$. If $x \leq y$, then $z_S \leq y$; that is, $C_x \subseteq X_{z_S}$ for every $S \in \mathcal{S}$. By eq. (0.8), $C_x \subseteq C$.

Suppose $P$ has property **T2**. By **(3)**, $P$ also has property **T1**. That is, for any $x \in P$, $\{x\}$ is closed, so by the above, $C_x \subseteq \{x\}$, and in fact $C_x = \{x\}$ because $x \in C_x$. If $x \leq y$, then $y \in C_x = \{x\}$, so $x = y$.

(7) Use (6) to prove that, if $R$ is a commutative Noetherian ring with unity satisfying $\operatorname{rad}((0)) = (0)$, and $\operatorname{Spec}(R)$ has property **T2**, then $R$ is a direct product of finitely many fields.

This problem required more commutative algebra knowledge than I initially thought it did—sorry about that! By known results in algebra, a Noetherian ring has finitely many minimal prime ideals, and $\operatorname{rad}((0))$ is the intersection of all the minimal prime ideals of $R$. But by property **T2** and **(6)**, every prime ideal of $R$ is minimal; write $\operatorname{Spec}(R) = \{I_1, \ldots, I_n\}$. It follows that, for any $k \neq \ell$, $I_k + I_\ell = R$. Thus, the Chinese Remainder Theorem gives an isomorphism

$$R/\operatorname{rad}((0)) = R/(I_1 \cap \cdots \cap I_n) \cong R/(I_1 \cdots I_n) \cong R/I_1 \times \cdots \times R/I_n. \tag{0.10}$$

We have $R = R/\operatorname{rad}((0))$ because $\operatorname{rad}((0)) = (0)$. The $I_k$ are maximal as well as minimal, so the $R/I_k$ are fields.

(8) Describe $\operatorname{Spec}(\mathbb{R}[x])$.

Because $\mathbb{R}[x]$ is a domain, $(0)$ is the unique minimal prime ideal. Because $\mathbb{R}[x]$ is a principal ideal domain, all the nonzero prime ideals are maximal and generated by a single irreducible polynomial in $\mathbb{R}[x]$. Up to multiplication by units, the irreducible polynomials in $\mathbb{R}[x]$ are either monic linear polynomials, or monic quadratic polynomials of negative discriminant.

To summarise,

$$\operatorname{Spec}(\mathbb{R}[x]) = \{(0)\} \cup \{(x-a) : a \in \mathbb{R}\} \cup \{(x^2+bx+c) : b, c \in \mathbb{R} \text{ and } b^2-4c<0\}. \quad (0.11)$$

The closed sets of $\operatorname{Spec}(\mathbb{R}[x])$ are all of $\operatorname{Spec}(\mathbb{R}[x])$ and any finite subset not containing $(0)$.

(9) Give a rational parametrisation of the hyperbola $H = \{(x,y) : x^2 - y^2 = 1\}$.

Consider the point $P = (1,0) \in H$. Draw a line $\ell_t$ of slope $t$ through $P$; the equation of $\ell_t$ is $y = t(x-1)$. Find the points of intersection between $H$ and $\ell_t$:

$$x^2 - y^2 = 1 \text{ and } y = t(x-1) \quad (0.12)$$

$$\implies x^2 - t^2(x-1)^2 = 1 \quad (0.13)$$

$$\implies (1-t^2)x^2 + 2t^2x - (1+t^2) = 0 \quad (0.14)$$

$$\implies (1-t^2)(x-1)\left(x + \frac{1+t^2}{1-t^2}\right) = 0. \quad (0.15)$$

When $x = 1$, we recover the point $P$. When $x = -\frac{1+t^2}{1-t^2} = \frac{t^2+1}{t^2-1}$, we find the point of intersection

$$P_t = \left(\frac{t^2+1}{t^2-1}, \frac{2t}{t^2-1}\right). \quad (0.16)$$

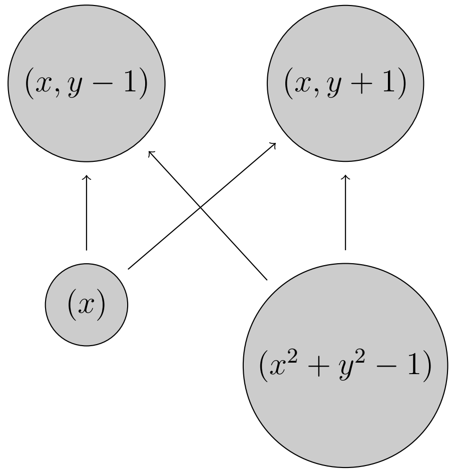
We may then verify that, for $t \neq \pm 1$,

$$\left(\frac{t^2+1}{t^2-1}\right)^2 - \left(\frac{2t}{t^2-1}\right)^2 = 1. \quad (0.17)$$

To check that $\varphi(t) := \left(\frac{t^2+1}{t^2-1}, \frac{2t}{t^2-1}\right)$ is a rational parametrisation, one may construct the rational inverse $\psi(x,y) := \frac{y}{x-1}$. One then checks algebraically that $\psi(\varphi(t)) = t$ for $t \in \mathbb{A}^1 \setminus \{1,-1\}$, and $\varphi(\psi(x,y)) = (x,y)$ for $(x,y) \in H \setminus \{(0,1),(0,-1)\}$.

(10) Let $P$ be the partially ordered set from (1). Find an example of a commutative ring $R$ with unity such that $\operatorname{Spec}(R) \cong P$ as a topological space.

In $\mathbb{C}[x,y]$, we have the following inclusions of ideals:




We use quotients and localisation to "snip off" the rest of $\operatorname{Spec}(\mathbb{C}[x,y])$, leaving only these ideals.

Let $R_1 = \mathbb{C}[x,y]$. Let $R_2 = R_1/(x(x^2+y^2-1))$; then, $\operatorname{Spec}(R_2)$ has two minimal prime ideals $(x)$ and $(x^2+y^2-1)$, maximal prime ideals corresponding to the points on the affine variety $\{x(x^2+y^2-1)=0\}$, and no other prime ideals. To get rid of all the maximal ideals of $R_2$ except for $(x,y-1)$ and $(x,y+1)$, let

$$S = \{f \in R_2 : f(0,1) \neq 0 \text{ and } f(0,-1) \neq 0\}. \tag{0.18}$$

Let $R$ be the localisation $R = S^{-1}R_2$. Then, $\operatorname{Spec}(R) \cong P$.