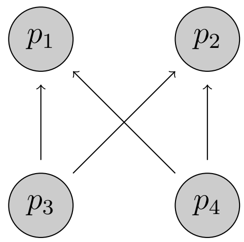
TOPICS IN MODERN GEOMETRY: PROBLEMS CLASS 4

Definition 0.1. *Let* $(X, \mathcal{T})$ *be a topological space and* $Y \subseteq X$*. The* ***subspace topology*** *on* Y *is defined by the collection of open sets*

$$\mathcal{S} = \{U \cap Y : U \in \mathcal{T}\}.$$

Definition 0.2. *Let* R *be a commutative ring with unity. Recall that* $\mathrm{mSpec}(R) \subseteq \mathrm{Spec}(R)$ *is the set of maximal ideals of* R*. We may consider* $\mathrm{mSpec}(R)$ *to be a topological space by endowing it with the subspace topology.*

(1) Let $P = \{p_1, p_2, p_3, p_4\}$ be the following partially ordered set, considered as a topological space with the order topology. (Here, $p \to q$ means $p \le q$.)



  (a) List the open sets of P.
  (b) List the closed sets of P.

(2) Describe $\mathrm{Spec}(R)$ and $\mathrm{mSpec}(R)$ for the following rings R, as explicitly as you can. In each case, say what $\mathrm{Spec}(R)$ and $\mathrm{mSpec}(R)$ as sets, then classify the closed sets of each.
  (a) $R = \mathbb{Z}$
  (b) $R = \mathbb{C}[x] \times \mathbb{C}[x]$
  (c*) $R = \mathbb{Z}[x]$
  (d*) $R = \mathbb{C}[x, y]$

(3) Let R be a commutative ring with unity, and let I be an ideal of R.
  (a) Prove that
$$\mathrm{Spec}(R/I) \cong \{P \in \mathrm{Spec}(R) : P \ge I\}.$$
  (b) Can you prove a similar statement about $\mathrm{Spec}(R_I)$, where $R_I = (R \setminus I)^{-1}R$ is the localisation?