# Fields, Forms and Flows 3/34

Problem Sheet 3
Due: Wednesday 24 October
To hand in: FFF3: 2, 3, 6(a)(b)  FFF34: 2, 3, 7

 Problems marked * are not examinable (but are still worth trying!)

1. Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   Show that

$$e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

2. Action-angle variables for the simple harmonic oscillator.

   (a) Find the general solution of the simple harmonic oscillator equation,

$$\ddot{x} = -\omega^2 x.$$

   (b) Write the simple harmonic oscillator as a first-order autonomous system for $z = (q, p)$, where $q = x$ and $p = \dot{x}$, of the form

$$\dot{z} = \mathbb{X}(z).$$

   Verify that $\mathbb{X}$ is linear.

   (c) Consider the change of coordinates

$$Z = (\theta, I) = F(z),$$

   where

$$\theta(z) = \tan^{-1}\left(\frac{\omega q}{p}\right), \quad I(z) = \frac{p^2 + \omega^2 q^2}{2\omega}.$$

   The coordinates $(\theta, I)$ are called *action-angle variables* ($\theta$ is the angle, and $I$ is the action). Compute the push forward $\mathbb{Y} = F_*\mathbb{X}$, and find thereby the first-order system

$$(\dot{\theta}, \dot{I}) = \mathbb{Y}(\theta, I)$$

   satisfied by $(\theta, I)$. Verify that $\mathbb{Y}$ is constant.

   (d) Solve the preceding system for $\theta(t)$ and $I(t)$ to obtain the flow $\Psi_t$ of $\mathbb{Y}$,

$$(\theta(t), I(t)) = \Psi_t(\theta_0, I_0).$$

(e) Compute

$$\Phi_t = F^{-1} \circ \Psi_t \circ F,$$

and show this agrees with the solution you calculated in a) above.

3. Pushforward example.

(a) Let $X$ and $Y$ be open sets in $\mathbb{R}^2$ given by

$$X = \{(u,v) \in \mathbb{R}^2 \,|\, u \neq 1\}, \quad Y = \{(x,y) \in \mathbb{R}^2 \,|\, x \neq -1\}.$$

(While it's not necessary, for the sake of clarity we are using different coordinates in $X$ and $Y$, namely $(u,v)$ for points in $X$ and $(x,y)$ for points in $Y$.) Show that $F : X \to Y$ given by

$$F(u,v) = \left(\frac{1+u}{1-u},\ u+v\right)$$

is a diffeomorphism with inverse $F^{-1} : Y \to X$ given by

$$F^{-1}(x,y) = \left(\frac{x-1}{x+1},\ y - \frac{x-1}{x+1}\right).$$

(b) Let $\mathbb{X}(u,v) = (v,1)$ be a vector field on $X$. Show that

$$(F_*\mathbb{X})(F(u,v)) = \left(\frac{2v}{(1-u)^2},\ v+1\right).$$

Hence show that

$$(F_*\mathbb{X})(x,y) = \left((x+1)^2 y/2 - (x^2-1)/2,\ y + \frac{2}{x+1}\right).$$

(c) Let $\mathbb{Y}(u,v) = (u^2+v^2, uv)$ be another vector field on $X$. Show that

$$[\mathbb{X}, \mathbb{Y}](u,v) = ((u+2)v, u+v^2).$$

[Some remarks, just for information: Transformations of the form $x \mapsto (ax+b)/(cx+d)$ are called *fractional linear transformations*, or *Möbius transformations*. They can be extended to maps of complex numbers (in which case $a$, $b$, $c$ and $d$ can be complex as well). In fact, by identifying the extended complex plane ($\mathbb{C}$ along with "the point at infinity") with the two-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1\}$ via *stereographic projection*,

$$z \mapsto (x,y,z) = \frac{1}{1+z\bar{z}}\,(2\mathrm{Re}\,z, 2\mathrm{Im}\,z, 1 - z\bar{z}),$$

these transformations can be regarded as diffeomophisms on $S^2$. As an exercise, you might like to verify that composition of two fractional linear transformations, say $z \mapsto (az+b)/(cz+d)$ and $z \mapsto (Az+B)/(Cz+D)$ is another fractional linear transformation $z \mapsto (\alpha z+\beta)/(\gamma z+\delta)$ whose coefficients are obtained given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.]$$

4. Starting from the formula $[\mathbb{X}, \mathbb{Y}] = (\mathbb{X} \cdot \nabla)\mathbb{Y} - (\mathbb{Y} \cdot \nabla)\mathbb{X}$, prove the Jacobi identity in the form
$$[[\mathbb{X}, \mathbb{Y}], \mathbb{Z}] + [[\mathbb{Z}, \mathbb{X}], \mathbb{Y}] + [[\mathbb{Y}, \mathbb{Z}], \mathbb{X}] = 0$$
(Suggestion: show explicitly that certain pairs of representative terms cancel, and then argue that all terms come in such pairs.)

5. Jacobi bracket for vector fields in $\mathbb{R}^3$.

   Let $\mathbf{f}(\mathbf{r})$, $\mathbf{g}(\mathbf{r})$ be vector fields on $\mathbb{R}^3$. Show that
$$[\mathbf{f}, \mathbf{g}] = -\boldsymbol{\nabla} \times (\mathbf{f} \times \mathbf{g}) + (\boldsymbol{\nabla} \cdot \mathbf{g})\mathbf{f} - (\boldsymbol{\nabla} \cdot \mathbf{f})\mathbf{g}.$$

6. Linear vector fields.

   (a) Let $\mathbb{X}(x) = Ax$, $\mathbb{Y}(x) = Bx$, where $A, B$ are $n \times n$ matrices. Show that $[\mathbb{X}, \mathbb{Y}] = Cx$, where $C = BA - AB$.

   (b) Let $F(x) = Sx$, where $S$ is an invertible $n \times n$ matrix, so that $F$ is a diffeomorphism. Verify Proposition 9.2, i.e.
$$F_*[\mathbb{X}, \mathbb{Y}] = [F_*\mathbb{X}, F_*\mathbb{Y}]$$
   by using the explicit formulas for the push forward and Jacobi bracket of linear vector fields.

   (c) The commutator of matrices $A$ and $B$, denoted $[A, B]$, is defined as
$$[A, B] = AB - BA.$$
   Verify that
$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$
   Show that this is equivalent to the Jacobi identity for linear vector fields.

7. Spherical polar coordinates.

   A diffeomorphism $F : U \to V$ can be viewed in two ways. The first, the active point of view, is that $F$ maps points in $U$ to points in $V$. The second, the passive point of view, is that $F$ describes a change of coordinates. That is, $F(x)$ associates new coordinates, namely $(F_1(x), \ldots, F_n(x))$, to a given point $x$ whose old coordinates were just $(x_1, \ldots, x_n)$. We take the passive point of view in this question, which concerns the transformation between cartesian and spherical polar coordinates in $\mathbb{R}^3$.

   (a) Let $R$ and $\Theta$ be vector fields on $\mathbb{R}^3 = \{(x, y, z)\}$ given by
$$R(x, y, z) = (x, y, z),$$
$$\Theta(x, y, z) = (xz, yz, -(x^2 + y^2)).$$
   Show that
$$[R, \Theta] := (R \cdot \nabla)\Theta - (\Theta \cdot \nabla)R = \Theta, \tag{1}$$
   where the symbol := means "defined", and
$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

(b) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$F(x, y, z) = (r(x, y, z), \theta(x, y, z), \phi(x, y, z)),$$

where

$$\begin{aligned} r(x, y, z) &= (x^2 + y^2 + z^2)^{1/2}, \\ \theta(x, y, z) &= \cos^{-1}\left(\frac{z}{(x^2 + y^2 + z^2)^{1/2}}\right), \\ \phi(x, y, z) &= \tan^{-1}\left(\frac{y}{x}\right). \end{aligned}$$

From the passive point of view, $F$ describes the transformation to spherical polar coordinates. $F$ is not a diffeomorphism (spherical polar coordinates are not good coordinates on all of $\mathbb{R}^3$ – the singularities lie along the $z$-axis), but we will ignore this fact here (the formulas we derive will be valid away from the $z$-axis). Show that

$$\begin{aligned} F_*R(r, \theta, \phi) &= (r, 0, 0), \\ F_*\Theta(r, \theta, \phi) &= (0, r \sin\theta, 0). \end{aligned}$$

Hence show by direct calculation (and thereby confirm the general formula $[F_*\mathbb{X}, F_*\mathbb{Y}] = F_*[\mathbb{X}, \mathbb{Y}]$) that

$$[F_*R, F_*\Theta] := (F_*R \cdot \nabla)F_*\Theta - (F_*\Theta \cdot \nabla)F_*R = F_*\Theta, \tag{2}$$

where

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right).$$

Remark: Let me try to anticipate and resolve some possible confusion (and hopefully not create confusion where none existed). You might be asking, "Why in (1) do we take $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, while in (2) we take $\nabla = (\partial/\partial r, \partial/\partial \theta, \partial/\partial \phi)$?" In (1), $\partial R^j/\partial y$ means, "Take the partial derivative of the $j$th component of $R$ with respect to its second argument, which we choose to call $y$". In (2), $\partial(F_*R)^k/\partial\theta$ means, "Take the partial derivative of the $k$th component of $F_*R$ with respect to its second argument, which we choose to call $\theta$". This is an illustration of the fact that the formalism we are developing treats all coordinates systems on the same footing; formulas have the same structure in all coordinate systems. Indeed, this is what the formula $[F_*\mathbb{X}, F_*\mathbb{Y}] = F_*[\mathbb{X}, \mathbb{Y}]$ is saying from the passive point of view; the expression for the Jacobi bracket looks the same in all coordinate systems.

8. * Let

$$\mathbb{X}(x) = \cos x^3, \quad \mathbb{Y}(x) = \sin x^3$$

be vector fields on $\mathbb{R}$. Show that $\mathbb{X}$ and $\mathbb{Y}$ are complete, but that $[\mathbb{X}, \mathbb{Y}]$ is not complete.