LECTURE 6: COORDINATE RINGS AND FUNCTION FIELDS OF AFFINE VARIETIES

1. THE COORDINATE RING

Let $X \subseteq \mathbb{A}^n$ be an affine variety.

Definition 1.1. *A function $f : X \to \mathbb{C}$ is **regular** if there's a polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$ such that $f(x) = F(x)$ for all $x \in X$. The set of regular functions form a ring, which we call the **coordinate ring** $\mathbb{C}[X]$.*

There is a surjective homomorphism π from $\mathbb{C}[x_1, \dots, x_n]$ to $\mathbb{C}[X]$, given by restriction of a polynomial $\pi(F) = F|_X$. This allows us to give a give an algebraic description of the coordinate ring.

Proposition 1.2. $\mathbb{C}[X] \cong \mathbb{C}[x_1, \dots, x_n]/\mathbb{I}(X)$.

Proof. For any $F \in \mathbb{C}[x_1, \dots, x_n]$,

$$\pi(F) = 0 \iff f(a) = 0 \text{ for all } a \in X \iff f \in \mathbb{I}(X). \tag{1.1}$$

Thus, $\mathbb{C}[X] \cong \mathbb{C}[x_1, \dots, x_n]/\mathbb{I}(X)$. $\square$

Here are some examples of coordinate rings.

- $\mathbb{C}[A^n] = \mathbb{C}[x_1, \dots, x_n]$.
- $\mathbb{C}[\emptyset] = \{0\}$.
- $\mathbb{C}[\{a\}] = \mathbb{C}$.
- If $Y = \{y = 0\} \subseteq \mathbb{A}^2_{x,y}$, then $\mathbb{C}[X] \cong \mathbb{C}[x]$.
- If $H = \{xy = 1\} \subseteq \mathbb{A}^2_{x,y}$, then $\mathbb{C}[H] \cong \mathbb{C}[x, x^{-1}]$.

2. IRREDUCIBLE VARIETIES

Definition 2.1. *Let $X \subseteq \mathbb{A}^n$ be a nonempty affine variety. Then, X is **reducible** if it can be written in the form $X = X_1 \cup X_2$, where X_1 and X_2 are strict subvarieties of X (that is, $X_j \subseteq X$ is a variety and $X_j \neq X$, for $j = 1, 2$). The variety X is **irreducible** if it is not reducible.*

For example, consider the variety $X = \mathbb{V}((xy)) = \{(x, y) \in \mathbb{A}^2 : xy = 0\}$. This variety is reducible because it may be written as a union of the two coordinate axes:

$$X = \{(x, y) \in \mathbb{A}^2 : x = 0\} \cup \{(x, y) \in \mathbb{A}^2 : y = 0\}. \tag{2.1}$$

On the other hand, the varieties $\{(x, y) \in \mathbb{A}^2 : x = 0\}$ and $\{(x, y) \in \mathbb{A}^2 : y = 0\}$ cannot be decomposed further—that is, they are irreducible. Any affine variety X may be decomposed uniquely as a union of finitely many distinct irreducible varieties.

If you're thinking that irreducibility of varieties should be related to primality of ideals, you'd be correct. In fact, we have the following proposition.

Proposition 2.2. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then, X is irreducible if and only if $\mathbb{I}(X)$ is prime.*

Proof. We proceed by proving the contrapositive.

Suppose that X is reducible. Write $X = X_1 \cup X_2$ for some strict subvarieties X_1 and X_2. Because X_1 and X_2 are the vanishing sets of some ideal in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, there must exist some $F_1, F_2 \in \mathbb{C}[x_1, \ldots, x_n]$ such that F_1 is identically zero on X_1 but not on X_2, and F_2 is identically zero on X_2 but not on X_1. Thus, the product F_1F_2 is identically zero on X—that is, zero in the coordinate ring $\mathbb{C}[X]$. Therefore, $F_1F_2 \in \mathbb{I}(X)$ but $F_1, F_2 \notin \mathbb{I}(X)$, so $\mathbb{I}(X)$ is not prime.

Conversely, suppose that $\mathbb{I}(X)$ is not prime. Then, there exists some $F_1, F_2 \in \mathbb{C}[x_1, \ldots, x_n]$ such that $F_1F_2 \in \mathbb{I}(X)$, but $F_1 \notin \mathbb{I}(X)$ and $F_2 \notin \mathbb{I}(X)$. Let $X_1 = \mathbb{V}((F_1)) \cap \mathbb{I}(X)$ and $X_2 = \mathbb{V}((F_2)) \cap \mathbb{I}(X)$. Then, $X = X_1 \cup X_2$. $\square$

3. The function field

Let $X \subseteq \mathbb{A}^n$ be an irreducible affine variety. By proposition 2.2, the coordinate ring $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(X)$ is an integral domain.

Definition 3.1. *If X is an irreducible affine variety, then the* ***function field*** *(field of rational functions) $\mathbb{C}(X)$ is the field of fractions of $\mathbb{C}[X]$.*

Note that $Y = \{y = 0\} \subseteq \mathbb{A}^2_{x,y}$ and $H = \{xy = 1\} \subseteq \mathbb{A}^2_{x,y}$ have non-isomorphic coordinate rings (exercise), but their function fields are both $\mathbb{C}(x)$.