LECTURE 7: MORPHISMS OF AFFINE VARIETIES

1. Morphisms

Definition 1.1. *Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieites. A **morphism** from X to Y is a function $\varphi : X \to Y$ that is the restriction of a polynomial map from $\mathbb{A}^m$ to $\mathbb{A}^n$. An **isomorphism** is a morphism that is bijective and whose inverse is also a morphism.*

Remark 1.2. *Morphisms $\varphi : X \to Y$ are in one-to-one bijection with ring homomorphisms $f : \mathbb{C}[Y] \to \mathbb{C}[X]$.*

If $\varphi : X \to Y$ and φ is the restriction of $\Phi : \mathbb{A}^m \to A^n$, define the homomorphism $F : \mathbb{C}[y_1, \ldots, y_n] \to \mathbb{C}[x_1, \ldots, x_m]$ by

$$F(y_j) := \Phi_j(x_1, \ldots, x_n). \tag{1.1}$$

Define $f : \mathbb{C}[Y] \to \mathbb{C}[X]$ to be the restriction of F to $\mathbb{C}[Y]$ composed with the quotient map $\mathbb{C}[x_1, \ldots, x_m] \to \mathbb{C}[X]$.

If $f : \mathbb{C}[Y] \to \mathbb{C}[X]$ and f is the restriction of $F : \mathbb{C}[y_1, \ldots, y_n] \to \mathbb{C}[x_1, \ldots, x_m]$, define the homomorphism $\Phi : \mathbb{A}^m \to A^n$ by

$$\Phi_j(x_1, \ldots, x_n) = F(y_j). \tag{1.2}$$

Define $\varphi : X \to Y$ to be the restriction of Φ.

Exercise: Prove this is a well-defined bijection.

Proposition 1.3. *Two varieties X and Y are isomorphic if and only if $\mathbb{C}[X] \cong \mathbb{C}[Y]$.*

Proof. Exercise. □

Example 1.4. *Let $\varphi : \mathbb{A}^2_{x,y} \to \mathbb{A}^1_x$ be the projection map given by $\varphi(x, y) = x$.*

The corresponding ring homomorphism $f : \mathbb{C}[x] \to \mathbb{C}[x, y]$ is the unique homomorphism satisfying $f(x) = x$; that is, the inclusion map defined by $f(p(x)) = p(x)$ for any $p(x) \in \mathbb{C}[x]$.

Example 1.5. *A morphism from A^1_t to cuspidal cubic $Y = \{y^2 = x^3\} \subseteq \mathbb{A}^2_{x,y}$ is given by $t \mapsto (t^2, t^3)$.*

The corresponding ring homomorphism $f : \mathbb{C}[Y] \to \mathbb{C}[t]$ is defined on the generators of $\mathbb{C}[Y] = \mathbb{C}[x, y]/(y^2 - x^3)$ by $f(x) = t^2$ and $f(y) = t^3$.

Example 1.6. *Consider the points of $\mathbb{A}^{n^2}$ as $n \times n$ matrices with entries in $\mathbb{C}$. The special linear group may be regarded as a subvariety:*

$$\mathrm{SL}_2(\mathbb{C}) = \{M \in \mathbb{A}^{n^2} : \det(M) = 1\} \tag{1.3}$$

Then, the squaring map $\sigma(M) = M^2$ is a morphism of affine varieties from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{SL}_2(\mathbb{C})$.

Two varieties X and Y are isomorphic if and only if $\mathbb{C}[X] \cong \mathbb{C}[Y]$.

2. RATIONAL MAPS

Definition 2.1. *A **rational map** from X to Y is a pair (f, U), where U is a nonempty Zariski open subset of X, and f is a function from U to Y defined by rational functions (ratios of polynomials).*

Definition 2.2. *A **birational map** from X to Y is a rational map f from X to Y which has a "rational inverse", that is, a rational map f^{-1} from Y to X such that $f^{-1}(f(x)) = x$ for all $x \in U$ and $f(f^{-1}(y)) = y$ for all $y \in V$, for some nonempty Zariski open subsets $U \subseteq X$ and $V \subseteq Y$.*

If there's a birational map from X to Y, we say that X and Y are **birationally equivalent**. In the particular case of a birational map from A^m to Y, we say that Y has a **rational parametrisation**.

Example 2.3. *Consider the "unit circle" $C = \{(x, y) \in \mathbb{A}^2 : x^2 + y^2 = 1\}$. As shown in lecture 1, the function*

$$p(t) = \left(\frac{-2t}{t^2+1}, \frac{-t^2+1}{t^2+1}\right) \tag{2.1}$$

defines a birational map from $\mathbb{A}^2$ to C. Note that $p(t)$ is defined on $\mathbb{A}^1 \setminus \{i, -i\}$.

Exercise: show that $p(t)$ is birational by constructing a rational inverse.

Proposition 2.4. *Two irreducible varieties X and Y are birationally equavalent if and only if $\mathbb{C}(X) \cong \mathbb{C}(Y)$.*

Proof. Exercise. □

Example 2.5. *Consider the cuspidal cubic $Y = \{y^2 = x^3\} \subseteq \mathbb{A}^2_{x,y}$. This curve has the rational parametrisation $\varphi(t) = (t^2, t^3)$. To show that $\varphi(t)$ is a rational parametrisation, consider the map $\psi(x, y) = \frac{y}{x}$ from $Y \setminus \{(0, 0)\}$ to $\mathbb{A}^1_t$. If $t \in \mathbb{A}^1_t \setminus \{0\}$, then $\psi(\varphi(t)) = \frac{t^3}{t^2} = t$; if $(x, y) \in Y \setminus \{(0, 0)\}$, then $\varphi(\psi(x, y)) = \left(\frac{y^2}{x^2}, \frac{y^3}{x^3}\right) = \left(\frac{x^3}{x^2}, \frac{y^3}{y^2}\right) = (x, y)$. Thus, ψ is a birational inverse to φ.*