Topics in Modern Geometry

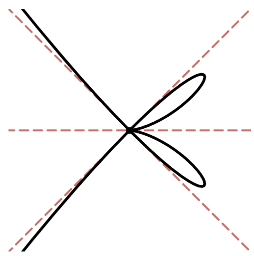
Problems class 17

1 Points of clarification

1. Definition 46(3) for the intersection multiplicity $m_p(C, D)$ needs to be stated more carefully when one (or both) of the curves C, D are singular at p.

   If $p \in C$ is singular say, then we can pick an affine patch of C where $p = (p_1, p_2) \in \mathbb{A}^2$ and then change coordinates by $(x, y) \mapsto (x - p_1, y - p_2)$ to assume that $p = (0, 0)$. Now let $C = \mathbb{V}(f(x, y))$, where $f(0, 0) = 0$. There exists a unique integer $m \geq 1$ such that $f(x, y) = f_m(x, y) + \cdots$, where $f_m \in \mathbb{C}[x, y]$ is a nonzero homogeneous polynomial of degree m, and $\cdots$ is a sum of terms of higher degree. Now we can factor f_m as a product of linear terms $f_m = \prod_{i=1}^{m} l_i$ and treat the lines $L_i = \mathbb{V}(l_i)$ as the tangent lines to $p \in C$, for $i = 1, \ldots, m$. When $p \in C$ is singular, Definition 46(3) should say that $(1 : 0 : 0) \notin L_i$ for $i = 1, \ldots, m$.

   Example: If $C = \mathbb{V}(x^5 - x^2y^2 + y^4)$ then $f(x, y) = y^2(y + x)(y - x) + \cdots$ and near $(0, 0)$ the curve has three tangent lines $\mathbb{V}(y)$, $\mathbb{V}(y + x)$ and $\mathbb{V}(y - x)$:

2. Another important property of the intersection multiplicity is the following: suppose that we have two plane curves $C = \mathbb{V}(f)$ and $D = \mathbb{V}(g)$ where $m := \deg f - \deg \geq 0$. Then for all $a \in \mathbb{C}[x, y, z]$ homogeneous of degree m, we have that $m_p(C, D) = m_p(C, E)$, where $E = \mathbb{V}(f + ag)$.

2 More problems

In addition to Homework sheet 4, you can use these problems as further practice questions.

1. Do the exercises from Lecture 14, Example 23(2):

   (a) Prove Euler's formula: that $\sum_{i=0}^{n} \frac{\partial f}{\partial x_i} x_i = \deg(f) f$ for a homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$.

   (b) Suppose $U_i \subset \mathbb{P}^n$ is a standard affine chart containing $p \in X$, and $T_p X_{(i)}$ is the affine tangent space of $p \in X_{(i)}$ in $U_i \cong \mathbb{A}^n$. Show that $T_p X_{(i)} = (T_p^{\text{proj}} X)_{(i)}$.

2. Let $C = \mathbb{V}(x^2 + y^2 - 1) \subset \mathbb{A}^2_{x,y}$ be the standard affine circle. Show that we can take the regular function x to be a uniformiser at all points $p \in C$ except $(1, 0)$. Write down a different regular function which is a uniformiser at $(1, 0)$. Show that the rational function $\frac{y-1}{x}$ is regular at $p = (0, 1)$ and calculate $v_p\left(\frac{y-1}{x}\right)$.

3. Let $C = \mathbb{V}(xz - y^2) \subset \mathbb{P}^2$ and let ϕ be given by the rational map

$$\phi \colon C \dashrightarrow \mathbb{P}^1, \qquad \phi(x : y : z) = (x^2 - xy : yz).$$

   Find the points of C where this formula for ϕ is not defined and work out how to extend ϕ to a morphism at these points.

3 Fun fact

This is an extension of question 10 on Homework sheet 4. Try checking all the details in the following construction for the plane cubic curve $C = \mathbb{V}(x^3 + y^3 + z^3)$.

Qu. Can you find twelve lines in the plane $\mathbb{P}^2$ all of which intersect at nine points, such that each point lies in the intersection of exactly four lines and each line contains exactly three points?

Ans. The answer is *no* in the real plane $\mathbb{RP}^2$—the Sylvester–Gallai theorem says that for any finite set of points in $\mathbb{RP}^2$, we can always find a line in $\mathbb{RP}^2$ containing exactly two of the points. However, the answer is *yes* in the complex plane $\mathbb{CP}^2$!

Construction. A nonsingular plane cubic curve $C = \mathbb{V}(f) \subset \mathbb{P}^2$ intersects its Hessian $\mathbb{V}(H_f)$ in nine points with multiplicity 1. Therefore C has exactly nine inflection points $p_1, \ldots, p_9$. Any line that passes through two inflection points p_i, p_j intersects C at a third point, which must also be an inflection point p_k. There are exactly twelve lines you can obtain this way. Each line contains three of the points p_i, p_j, p_k and exactly four of these lines pass through each p_i. Such a configuration of lines is called a *Hesse configuration*.

As already mentioned, it is impossible to draw these lines in $\mathbb{RP}^2$. This is the best you can do:

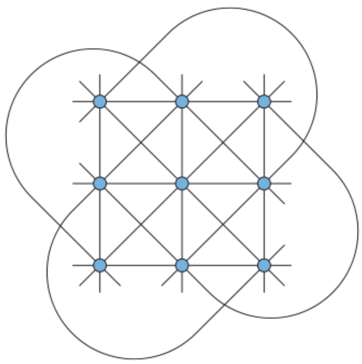


Image taken from `https://en.wikipedia.org/wiki/Hesse_configuration`. (The 12 'lines' only cross at the 9 blue dots—the other crossing points aren't really supposed to be there.)