# Fields, Forms and Flows 3/34

Solution Sheet 1

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1. i) For $F(x) = A \cdot x$ to be 1-1, we require that $x_1 = x_2$ whenever $A \cdot x_1 = A \cdot x_2$. Since the map is linear, is equivalent to requiring that $x = 0$ whenever $A \cdot x = 0$. This condition holds if and only if the $m$ columns of $A$ are linearly independent. For this to be true, it is necessary (but not sufficient) that $m \le n$ (ie, $A$ can't have more columns than rows). As example, $F$ is 1-1 for

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \text{but not for} \qquad A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

One can express the condition also by saying that $F(x)$ is 1-1 if the *rank* of the matrix $A$ is $m$.

ii) For $F(x) = A \cdot x$ to be onto, we require that, for any $y \in \mathbb{R}^n$, the equation $y = A \cdot x$ always has at least one solution $x$ ($x$ needn't be unique). This condition holds if and only if the $m$ columns of $A$ span $\mathbb{R}^n$. For this to be true, it is necessary (but not sufficient) that $m \ge n$. (ie, $A$ must have at least as many columns as rows). As example, $F$ is onto for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \qquad \text{but not for} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

One can express the condition also by saying that $F(x)$ is onto if the *rank* of the matrix $A$ is $n$.

2. We'll construct the map from $\mathbb{R}^2$ to $\mathbb{R}$ in 3 stages. First, we construct an invertible map $f$ from the interval $(0,1)$ to $\mathbb{R}$. Next, using $f$, we'll construct an invertible map $G$ from $\mathbb{R}^2 = \{(u,v)\}$ to $U = \{(u,v) | 0 < u, v < 1\}$ ($U$ is the open unit square). Then we'll construct a map $H$ from $U$ to $(0,1)$. Our required map $F : \mathbb{R}^2 \to \mathbb{R}$ may then be taken to be

$$F = f \circ H \circ G.$$

If $H$ is $1-1$, then $F$ is $1-1$, and if $H$ is invertible, then $F$ is invertible with inverse

$$F^{-1} = G^{-1} \circ H^{-1} \circ f^{-1}.$$

First, we'll construct $H$, and therefore $F$, which is $1-1$.

(a) Step 1. Take $f : (0,1) \to \mathbb{R}$ to be

$$f(x) = \ln \frac{x}{1-x},$$

for example. We can show $f$ is invertible by writing down the inverse explicitly, $f^{-1} : \mathbb{R} \to (0,1)$, which is given by

$$f^{-1}(y) = \frac{e^y}{1+e^y}.$$

(b) Step 2. Define $G : \mathbb{R}^2 \to U$ by

$$G(u,v) = (f^{-1}(u), f^{-1}(v)).$$

Then $G^{-1} : U \to \mathbb{R}^2$ is given by $G^{-1}(r,s) = (f(r), f(s))$.

(c) Step 3. Given $(u, v) \in U$, so that $0 < u, v < 1$, we can write $u$ and $v$ as decimals,

$$u = 0.u_1 u_2 \cdots, \quad v = 0.v_1 v_2 \cdots,$$

where $u_j$ and $v_j$ are sequences of decimal digits between 0 and 9. Decicmal expansions aren't always unique; for example, $.200 \cdots = .1999 \cdots$. However, if we specify that neither $u_j$ nor $v_j$ can end with an infinite sequence of consecutive 9's, then $u_j$ and $v_j$ are uniquely determined by $u$ and $v$.

We define $H : U \to (0, 1); (u, v) \mapsto H(u, v)$ by interleaving the decimal expansions of $u$ and $v$ to obtain a single decimal, as follows:

$$H(u, v) = 0.u_1 v_1 u_2 v_2 \cdots$$

It is clear that $H$ is 1-1, ie $H(u, v) = H(u', v')$ implies that $u = u'$ and $v = v'$. However, $H$ is not onto, since, for example, $x = .x_1 9 x_3 9 x_5 9 \cdots$ is not contained in its image.

An invertible map from $\mathbb{R}^2$ to $\mathbb{R}$ can also be constructed. We'll just give a sketch of one possible construction. From the preceding, it suffices to construct a mapping from the open unit square $U$ to the open unit interval $I = (0, 1)$.

(a) Let $U^*$ denote the set of points $(u, v) \in U$ with $u$ and $v$ both irrational, and similarly let $I^*$ denote the irrational points in $I$. An irrational number $u$ between 0 and 1 can be uniquely expressed as a continued fraction,

$$u = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}},$$

where $a_1, a_2, \ldots$ is an infinite sequence of positive integers. We write $u = [a_1, a_2, \ldots]$, and similarly $v = [b_1, b_2, \ldots]$. Then define $Q : U^* \to I^*$ by $P(u, v) = x = [a_1, b_1, a_2, b_2, \ldots]$. $Q$ is invertible.

(b) We also need to construct invertible maps $P : U \to U^*$ and $R : I \to I^*$. For $R$, note that the interval $I$ can be divided into the algebraic numbers A, ie numbers which are roots of a polynomial with integer coefficients, and the transcendental numbers $\mathcal{T}$, which are not the roots of such polynomials. Algebraic numbers include the rational numbers $\mathbb{Q}$, and both $\mathbb{Q}$ and $\mathcal{A}$ are countable. Hence there is an invertible map – let's call it $r$ – from $\mathcal{A}$ to $\mathcal{A} - \mathbb{Q}$ (ie, the irrational algebraic numbers). Define $R$ by $R(x) = x$ if $x \in \mathcal{T}$ and $R(x) = r(x)$ if $x \notin \mathcal{T}$. $R$ is then an invertible map from $I$ to $I^*$, and $P$ can be taken to be $P(u, v) = (R(u), R(v))$.

3. Take $x \in U$. Then $F(x) \in F(U)$, so that $x \in F^{-1}(F(U))$, as required. Next, take $y \in F(F^{-1}(V))$. Then $y = F(x)$ for some $x \in F^{-1}(V)$. But $x \in F^{-1}(V)$ implies that $F(x) \in V$. Since $y = F(x)$, it follows that $y \in V$, as required. Let $F(x) = x^2$. Let $U = (0, 1)$. Then $F(U) = (0, 1)$, while $F^{-1}(F(U)) = (-1, 0) \cup (0, 1)$. Let $V = (-1, 0]$. Then $F^{-1}(V) = \{0\}$, so that $F(F^{-1}(V)) = \{0\}$.

4. Given $y \in B_\epsilon(x)$. We must show that there exists a $\delta > 0$ such that $B_\delta(y)$ is contained in $B_\epsilon(x)$ (in other words, any $z$ within a distance $\delta$ of $y$ is within a distance $\epsilon$ of $x$). In fact, any positive $\delta$ with

$$\delta < \epsilon - ||y - x||$$

will work. For suppose $z \in B_\delta(y)$. Then

$$||z - x|| = ||(z - y) + (y - x)|| \leq ||z - y|| + ||y - x||,$$

using the triangle inequality. But $||z - y|| < \delta$ and $||y - x|| < \epsilon - \delta$, so $||z - x|| < \epsilon$, as required.

5. (a) Let $U_\alpha$ be a family of open sets in $\mathbb{R}^m$. The index $\alpha$ could range over a finite set, a countable infinite set (eg, the natural numbers) or an uncountable set (eg, the real numbers). Let

$$V = \bigcup_\alpha U_\alpha,$$

where the union is taken over all $\alpha$. We must show that $V$ is open, ie, for all $x \in V$ there exists an $\epsilon > 0$ such that $B_\epsilon(x)$ is contained in $V$.

Let $x \in V$. Then $x \in U_\alpha$ for some $\alpha$. Since $U_\alpha$ is open, $\exists \epsilon > 0$ such that $B_\epsilon(x) \subset U_\alpha$. Since $U_\alpha$ is contained in $V$, it follows that $B_\epsilon(x) \subset V$, as required.

(b) Let $U_1$ and $U_2$ be open sets in $\mathbb{R}^m$. Let

$$V = U_1 \cap U_2.$$

We must show that $V$ is open.

If $V$ is empty, there is nothing to show; the empty set is open. If $V$ is not empty, let $x \in V$. Then $x \in U_1$ and $x \in U_2$. Since $U_1$ is open, there $\exists \epsilon_1 > 0$ such that $B_{\epsilon_1}(x) \subset U_1$. Similarily, since $U_2$ is open, $\exists \epsilon_2 > 0$ such that $B_{\epsilon_2}(x) \subset U_2$. Let $\epsilon$ be the smaller of $\epsilon_1$ and $\epsilon_2$. Then $B_\epsilon(x)$ is contained in both $U_1$ and $U_2$, so $B_\epsilon(x) \subset V$, as required.

Note that this argument extends to the intersection of a finite collection of open sets. However, the intersection of an infinite collection of open sets need not be open. For example, the intersection of the open intervals $I_n = (-1/n, 1/n)$ over all positive integers $n$ contains only 0, and therefore is not open.

(c) Let $U \subset \mathbb{R}$ be open. Given $x \in U$, choose $\delta(x) > 0$ such that $B_{\delta(x)}(x) \subset U$. Then

$$U = \bigcup_x B_{\delta(x)}(x),$$

for it is clear that every point in $U$ is contained in at least one of the intervals on the right-hand side (namely, the interval around that point), while the intervals on the right-hand side are contained in $U$ by assumption.

6. First, suppose that $X$ is closed. This means that $X$ contains all its boundary points. Let $x \in \tilde{X}$. Then $x \notin X$, so that $x$ is not a boundary point of $X$. This means there exists an $\epsilon > 0$ such that $B_\epsilon(x) \cap X$ is empty. Then $B_\epsilon(x) \subset \tilde{X}$. But this shows that $\tilde{X}$ contains a neighbourhood of each of its points, so that $\tilde{X}$ is open.

Next, suppose that $\tilde{X}$ is open. We want to show that $X$ is closed, i.e. that $X$ contains each of its boundary points. Let $x$ be a boundary point of $X$. Then for all $\epsilon > 0$, $B_\epsilon(x) \cap X$ is not empty. This means that, for all $\epsilon > 0$, $B_\epsilon(x)$ is not wholly contained in $\tilde{X}$. Thus, since $\tilde{X}$ is open, we may conclude that $x \notin \tilde{X}$, which means that $x \in X$.

7. First, suppose that $F$ is continuous. Let $Y \subset V$ be open. We wish to show that $X := F^{-1}(Y)$ is open. Therefore, given $x \in X$, we must find $\delta > 0$ such that $B_\delta(x) \subset X$. Here is how: Let $y = F(x)$. Since $x \in X$, it follows that $y \in Y$ (this is just what it means for $x$ to belong to the inverse image of $Y$). Since $Y$ is open, $\exists \epsilon > 0$ such that $B_\epsilon(y) \subset Y$. Now we use the fact the $F$ is continuous: Since $F$ is continuous, it follows that $\exists \delta > 0$ such that $||x' - x|| < \delta$ implies that $||F(x') - y|| < \epsilon$. Another way to say this is that if $x' \in B_\delta(x)$, then $F(x') \in B_\epsilon(y)$. But this is just saying that $B_\delta(x)$ is contained in $F^{-1}(B_\epsilon(y))$, which in turn is contained in $F^{-1}(Y) = X$. This is what we wanted to show.

Next, suppose that the inverse image of every open set $Y$ in $V$ is an open set $X$ in $U$. We wish to show that for all $x \in X$ and for all $\epsilon > 0$, there exists $\delta > 0$ such that $||x' - x|| < \delta$ implies that $||F(x') - F(x)|| < \epsilon$, or, in other words, $F(B_\delta(x)) \subset B_\epsilon(F(x))$. Here is how. Let $y = F(x)$. Since $Y$ is open, $\exists \epsilon > 0$ such that $B_\epsilon(y) \subset Y$. By assumption, since $B_\epsilon(y)$ is open, then $F^{-1}(B_\epsilon)$ is open. Clearly $x \in F^{-1}(B_\epsilon)$. Therefore, $\exists \delta > 0$ such that $B_\delta(x) \subset F^{-1}(B_\epsilon)$. But this implies that $F(B_\delta(x)) \subset B_\epsilon(F(x))$, which is what we wanted to show.

8. i) Let $A = (\frac{1}{2}, \frac{3}{2})$, for example. Then $f^{-1}(A) = [0, \infty)$, which is not open. ii) Let $B = (-\frac{1}{2}, \frac{1}{2})$. Then $g^{-1}(B)$ is not open. In particular, while $0 \in g^{-1}(B)$ (since $g(0) = 0$, and $0 \in (-\frac{1}{2}, \frac{1}{2})$), no $\delta$-neighbourhood of 0 is contained in $g^{-1}(B)$. This is because, arbitrarily close to 0, there are points - let's call them $x_n$ – for which, say, $|\sin 1/x_n| = 1$. Just let $x_n = 1/((n + \frac{1}{2})\pi)$.

9. Let $f(x) = 1$ for all $x$. Then the image of any set, open or otherwise, is $\{1\}$, which is closed.