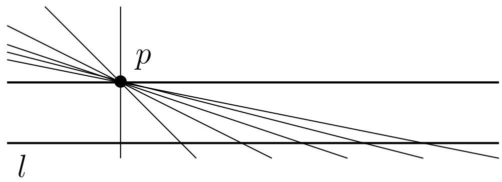
Lecture 10: Projective varieties

Why projective geometry?

Almost all pairs of lines drawn in the plane meet at precisely one point. Unfortunately this isn't always true due to the existence of *parallel lines*. However parallel lines are the exception rather than the rule. Indeed, one version Euclid's fifth axiom (due to the Scottish mathematician John Playfair) states that:

> In a plane, given a line l and a point p not on l, at most one line parallel to l can be drawn through p.

As the lines drawn through p approach the parallel line it is clear that the point of intersection moves further and further away, in either direction.



We would like to solve our problem by saying that parallel lines actually *do* intersect, and their intersection is a point at ∞. Projective space provides a way to make this rigourous.

1 Projective space

Definition 1. *The n-dimensional projective space* $\mathbb{P}^n$ is

$$\mathbb{P}^n = \{l \subset \mathbb{A}^{n+1} : 0 \in l\},$$

the set of all lines in $\mathbb{A}^{n+1}$ that pass through the origin $0 \in \mathbb{A}^{n+1}$.

1.1 Projective space as a quotient of $\mathbb{A}^{n+1} \setminus 0$.

Given a line l and any *nonzero* point $p \in l$ with coordinates $(p_0, \dots, p_n) \in \mathbb{A}^{n+1}$, then all other points on l are given by $(\lambda p_0, \dots, \lambda p_n)$ as $\lambda \in \mathbb{C}$ varies. Therefore we can identify $\mathbb{P}^n$ with the set of equivalence classes

$$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/ \sim$$

where $\sim$ is the equivalence relation

$$(p_0, \dots, p_n) \sim (p_0', \dots, p_n') \iff \exists \lambda \in \mathbb{C}^\times \text{ such that } p_i = \lambda p_i' \quad \forall i$$

(i.e. two points of $\mathbb{A}^{n+1}$ are considered to give the same point in $\mathbb{P}^n$ if they lie on the same line).

We write points in $\mathbb{P}^n$ in coordinates as $(p_0 : \dots : p_n)$, subject to the rescaling rule $\sim$. The coordinates p_i can take any value in $\mathbb{C}$, except $(0 : \dots : 0)$ which is *not* a point of $\mathbb{P}^n$.

1.2 The standard affine charts.

We can parameterise almost all of the lines in $\mathbb{A}^{n+1}$ if we assume one of our coordinates is nonzero.

Definition 2. The *ith standard affine chart* for $\mathbb{P}^n$ is the subset

$$U_i = \{(p_0 : \ldots : p_n) \in \mathbb{P}^n : p_i \neq 0\}$$

where the ith coordinate is nonzero for $i = 0, \ldots, n$.

To see why it is called an affine chart we define maps

$$\phi_i \colon U_i \to \mathbb{A}^n \qquad \phi_i(p_0 : \cdots : p_n) = (\tfrac{p_0}{p_i}, \cdots, \tfrac{p_{i-1}}{p_i}, \tfrac{p_{i+1}}{p_i}, \cdots, \tfrac{p_n}{p_i}),$$

$$\psi_i \colon \mathbb{A}^n \to U_i \qquad \psi_i(a_1, \cdots, a_n) = (a_1 : \cdots : a_i : 1 : a_{i+1} : \cdots : a_n).$$

These maps are well-defined on their image and domain, and are inverses. (Why? See Problem sheet 3, qu. 1.) Therefore these maps provide isomorphisms $U_i \cong \mathbb{A}^n$ and, since at least one coordinate is nonzero $\forall p \in \mathbb{P}^n$, we have

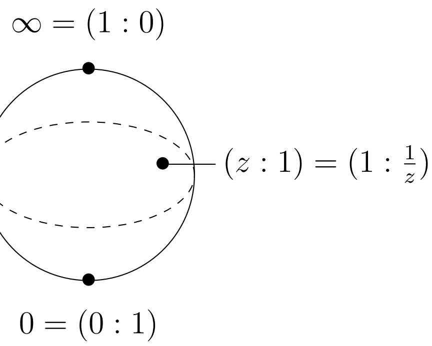
$$\mathbb{P}^n = U_0 \cup U_1 \cup \ldots \cup U_n.$$

1.3 The projective line as the Riemann sphere.

In the case of the projective line $\mathbb{P}^1$ we have a covering by two affine charts $\mathbb{P}^1 = U_0 \cup U_1$, with $U_0, U_1 \cong \mathbb{A}^1$, $U_0 \cap U_1 = \mathbb{A}^1 \setminus 0$ and which are related by the map

$$\phi_1 \circ \psi_0 \colon (\mathbb{A}^1 \setminus 0) \to (\mathbb{A}^1 \setminus 0), \qquad (\phi_1 \circ \psi_0)(z) = \phi_1(1 : z) = \tfrac{1}{z}.$$

This is the glueing description of $\mathbb{P}^1$ as *the Riemann sphere*:



1.4 Decomposition of $\mathbb{P}^n$ into affine pieces.

What does the complement $\mathbb{P}^n \setminus U_i$ look like? It is the subset of lines in $\mathbb{A}^{n+1}$ contained in the linear subspace $\{a_i = 0\} \simeq \mathbb{A}^n$. Therefore $\mathbb{P}^n \setminus U_i \simeq \mathbb{P}^{n-1}$. Continuing inductively we can write $\mathbb{P}^n$ as a disjoint union of sets

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1} = \cdots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0$$

where $\mathbb{A}^0$ is an affine space of dimension 0 (i.e. a point).

2 Homogeneous polynomials

Definition 3. A polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ is *homogeneous of degree* d if every monomial appearing in f has the same degree d.

Note that any $f \in \mathbb{C}[x_0, \dots, x_n]$ of degree d has a unique expression $f = \sum_{i=0}^{d} f_i$ where f_i is homogeneous of degree i for all $i = 0, \dots, d$.

Lemma 4. *If f is homogeneous of degree d then*

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_1, \dots, x_n)$$

for any $\lambda \in \mathbb{C}$.

Proof. Exercise. $\square$

Definition 5. An ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$ is *homogeneous* if every $f \in I$ can be written as a sum of homogeneous parts $f = \sum_{i=1}^{d} f_i$ with $f_i \in I$ for all i.

An ideal I is homogeneous, if and only if there is a finite generating set $I = \langle f_1, \dots, f_n \rangle$ such that each f_i is homogeneous (see Problem sheet 3, qu. 4).

2.1 Graded rings.

The ring $R = \mathbb{C}[x_0, \dots, x_n]$ is a *graded ring.* This means that it has an additive decomposition $R = \bigoplus_{d \geq 0} R_d$, as a $\mathbb{C}$-vector space, where multiplication obeys the rule: if $f \in R_d$ and $g \in R_e$ then $fg \in R_{d+e}$. In this case, R_d is the set of all homogeneous polynomials of degree d.

3 Projective varieties

By Lemma 4, if $f \in \mathbb{C}[x_0, \dots, x_n]$ is a homogeneous polynomial then the condition $f(p) = 0$ is well-defined for points $p \in \mathbb{P}^n$. (But not for general f!)

Definition 6.

1. A *projective algebraic set* is a subset

   $$\mathbb{V}(I) = \{p \in \mathbb{P}^n : f(p) = 0, \ \forall f \in I\}$$

   defined by the vanishing of a homogeneous ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$.

2. Given any subset $X \subset \mathbb{P}^n$, the *ideal of functions vanishing on* X is:

   $$\mathbb{I}(X) = \langle f \in \mathbb{C}[x_0, \dots, x_n] : f(p) = 0, \ \ \forall p \in X \rangle \, .$$

3. The *Zariski topology* on $\mathbb{P}^n$ is the topology for which the closed sets are given by the projective algebraic sets.[1]

4. A *projective variety* X is a projective algebraic set which is irreducible in the Zariski topology, i.e. X cannot be written as $X = X_1 \cup X_2$—a union of two non-trivial projective algebraic sets.

We make the same definitions for projective varieties as we did for affine varieties (*Zariski closure*, *irreducibility* etc.). All the usual statements for $\mathbb{V}$ and $\mathbb{I}$ carry over from the affine case, e.g. $X \subseteq \mathbb{V}(\mathbb{I}(X))$. Note that the standard affine patch $U_i \subset \mathbb{P}^n$ is an open subset in the Zariski topology, since it is the complement to a Zariski closed set $U_i = \mathbb{P}^n \setminus \mathbb{V}(x_i)$.

[1] *Note*: $\mathbb{P}^n$ also has the usual topology as a complex (or real) manifold coming from the standard open affine charts $U_i \cong \mathbb{A}^n \subset \mathbb{P}^n$. However the Zariski topology still makes sense if we replace $\mathbb{C}$ by a general field k.