Fields, Forms and Flows 3/34

Solution Sheet 5

1. (a) (5 marks) The one-dimensional wave equation is given by

$$u_{tt} - u_{xx} = 0.$$

Let $\phi^1(x,t) = u(x,t)$, $\phi^2(x,t) = u_x(x,t)$ and $\phi^3(x,t) = u_t(x,t)$. Then the wave equation is equivalent to

$$\begin{aligned} \phi^1_x &= \phi^2, \\ \phi^1_t &= \phi^3, \\ \phi^2_t - \phi^3_x &= 0, \\ \phi^2_x - \phi^3_t &= 0. \end{aligned}$$

The first two equations are just the definitions of ϕ^1 and ϕ^2. The third expresses the equality of mixed partials, $u_{xt} = u_{tx}$, while the last expresses the wave equation itself. We cannot express these equations in the form

$$\begin{aligned} \frac{\partial \phi^\alpha}{\partial x}(x,t) &= f^\alpha(x,t,\phi), \\ \frac{\partial \phi^\alpha}{\partial t}(x,t) &= g^\alpha(x,t,\phi), \end{aligned}$$

as this would require six equations (three for the components of the partial derivative of ϕ with respect to x, and three for the components of the partial derivative of ϕ with respect to t), and there are only four.

(b) (5 marks) The one-dimensional heat equation is given by

$$u_t - u_{xx} = 0.$$

Let $\phi^1(x,t) = u(x,t)$ and $\phi^2(x,t) = u_x(x,t)$. Then the heat equation is equivalent to

$$\begin{aligned} \phi^1_x &= \phi^2, \\ \phi^1_t - \phi^2_x &= 0. \end{aligned}$$

The first equation is just the definition of ϕ^2, while the second expresses the heat equation itself. We cannot write these equations in the form

$$\begin{aligned} \frac{\partial \phi^\alpha}{\partial x}(x,t) &= f^\alpha(x,t,\phi), \\ \frac{\partial \phi^\alpha}{\partial t}(x,t) &= g^\alpha(x,t,\phi), \end{aligned}$$

as this would require four equations (two for the components of the partial derivative of ϕ with respect to x, and two for the components of the partial derivative of ϕ with respect to t), and there are only two.

2. (a) (15 marks) Write $f^\alpha = f^\alpha(x, y, r, s)$ and $g^\alpha = g^\alpha(x, y, r, s)$; that is, denote the third and fourth arguments of f^α and g^α by r and s. This corresponds to the case $p = 2$ and $q = 2$ in the general statement of the Frobenius Theorem in Section 1.11.3 of the Notes. The reason for writing f^α, g^α rather than f_i^α is to avoid having so many indices. In this solution, to make the expressions more compact, I'll write u_x for $\partial u/\partial x$, etc.

First, let's assume that a solution $u(x, y)$, $v(x, y)$ exists. Then the equality of mixed partials of u,

$$\frac{\partial}{\partial y} u_x = \frac{\partial}{\partial x} u_y,$$

implies that

$$f_y^1 + f_r^1 u_y + f_s^1 v_y = g_x^1 + g_r^1 u_x + g_s^1 v_x.$$

Replacing u_x by f^1, v_x by f^2, u_y by g^1, and v_y by g^2, we get that

$$f_y^1 + f_r^1 g^1 + f_s^1 g^2 = g_x^1 + g_r^1 f^1 + g_s^1 f^2. \tag{1}$$

Similarly, the equality of mixed partials of v,

$$\frac{\partial}{\partial y} v_x = \frac{\partial}{\partial x} v_y,$$

implies that

$$f_y^2 + f_r^2 g^1 + f_s^2 g^2 = g_x^2 + g_r^2 f^1 + g_s^2 f^2. \tag{2}$$

Thus, (1) and (2) must be satisfied if a solution $u(x, y)$, $v(x, y)$ is to exist.

Next, we show that (1) and (2) are sufficient for (**??**) to have a solution (at least locally). Define vector fields $\mathbb{X}$ and $\mathbb{Y}$ on $\mathbb{R}^4$ via

$$\mathbb{X} = (1, 0, f^1, f^2), \quad \mathbb{Y} = (0, 1, g^1, g^2). \tag{3}$$

We compute the Jacobi bracket of $\mathbb{X}$ and $\mathbb{Y}$. Since their first two components are constants, the first two components of their bracket must vanish. Consider the third and fourth components of their bracket.

$$\begin{aligned}[]
[\mathbb{X}, \mathbb{Y}]^3 = (\mathbb{X} \cdot \nabla)\mathbb{Y}^3 - (\mathbb{Y} \cdot \nabla)\mathbb{X}^3 = &\\
\left(\frac{\partial}{\partial x} + f^1 \frac{\partial}{\partial r} + f^2 \frac{\partial}{\partial s}\right) g^1 - \left(\frac{\partial}{\partial y} + g^1 \frac{\partial}{\partial r} + g^2 \frac{\partial}{\partial s}\right) f^1 = &\\
= (g_x^1 + f^1 g_r^1 + f^2 g_s^1) - (f_y^1 + g^1 f_r^1 + g^2 f_s^1).&
\end{aligned} \tag{4}$$

Comparing this expression to (1), we see that

$$[\mathbb{X}, \mathbb{Y}]^3 = \frac{\partial}{\partial x} u_y - \frac{\partial}{\partial y} u_x.$$

A similar calculation shows that

$$\begin{aligned}[]
[\mathbb{X}, \mathbb{Y}]^4 = (\mathbb{X} \cdot \nabla)\mathbb{Y}^4 - (\mathbb{Y} \cdot \nabla)\mathbb{X}^4 = &\\
\left(\frac{\partial}{\partial x} + f^1 \frac{\partial}{\partial r} + f^2 \frac{\partial}{\partial s}\right) g^2 - \left(\frac{\partial}{\partial y} + g^1 \frac{\partial}{\partial r} + g^2 \frac{\partial}{\partial s}\right) f^2 = &\\
= (g_x^2 + f^1 g_r^2 + f^2 g_s^2) - (f_y^2 + g^1 f_r^2 + g^2 f_s^2),&
\end{aligned} \tag{5}$$

so that

$$[\mathbb{X}, \mathbb{Y}]^4 = \frac{\partial}{\partial x} v_y - \frac{\partial}{\partial y} v_x.$$

Therefore, given that (1) and (2) hold, it follows that

$$[\mathbb{X}, \mathbb{Y}] = 0. \tag{6}$$

Let Φ_t and Ψ_q denote the flows of $\mathbb{X}$ and $\mathbb{Y}$ respectively. Given the form of $\mathbb{X}$ and $\mathbb{Y}$, the first two components of these flows are easily determined. Consider Φ_t first, and write its components as

$$\Phi_t(x_0, y_0, r_0, s_0) = (x_t, y_t, \Phi_t^3, \Phi_t^4)(x_0, y_0, r_0, s_0).$$

From (3), $\dot{x}_t = 1$ and $\dot{y}_t = 0$, so that $x_t = x_0 + t$ and $y_t = y_0$. Therefore,

$$\Phi_t(x, y, r, s) = (x + t, y, \Phi_t^3(x, y, r, s), \Phi_t^4(x, y, r, s))$$

where I've dropped the subscript $_0$ from the arguments of Φ_t, just to make the writing simpler. A similar calculation gives

$$\Psi_q(x, y, r, s) = (x, y + q, \Psi_q^3(x, y, r, s), \Psi_q^4(x, y, r, s)).$$

It follows that

$$(\Phi_t \circ \Psi_q)^1(x, y, r, s) = x + t, \quad (\Phi_t \circ \Psi_q)^2(x, y, r, s) = y + q.$$

Define $u(x, y)$ and $v(x, y)$ via

$$(\Phi_x \circ \Psi_y)(0, 0, r_0, s_0) = (x, y, u(x, y), v(x, y)). \tag{7}$$

We claim that $u(x, y)$, $v(x, y)$ satisfies the system (**??**).

It is clear that the initial data is satisfied; letting $x = y = 0$ in (7), we get that

$$(0, 0, r_0, s_0) = (0, 0, u(0, 0), v(0, 0)),$$

since Φ_0 and Ψ_0 are the identity maps. Next we show that the first two of the partial differential equations in (**??**) is satisfied. From the definition of the flow,

$$\frac{\partial}{\partial x}\Phi_x(a, b, c, d) = \mathbb{X}(\Phi_x(a, b, c, d)), \quad \forall (a, b, c, d) \in \mathbb{R}^4.$$

Therefore,

$$\begin{aligned} u_x = \frac{\partial}{\partial x}\Phi_x^3(\Psi_y(0, 0, r_0, s_0)) &= \mathbb{X}^3(\Phi_x(\Psi_y(0, 0, r_0, s_0))) \\ &= \mathbb{X}^3(x, y, u(x, y), v(x, u)) = f^1(x, y, u(x, y), v(x, y)), \end{aligned}$$

and similarly

$$\begin{aligned} v_x = \frac{\partial}{\partial x}\Phi_x^4(\Psi_y(0, 0, r_0, s_0)) &= \mathbb{X}^4(\Phi_x(\Psi_y(0, 0, r_0, s_0))) \\ &= \mathbb{X}^4(x, y, u(x, y), v(x, u)) = f^2(x, y, u(x, y), v(x, y)). \end{aligned}$$

The fact that $[\mathbb{X}, \mathbb{Y}] = 0$ implies that

$$\Phi_x \circ \Psi_q = \Psi_q \circ \Phi_x.$$

Therefore, writing

$$(\Psi_y \circ \Phi_x)(0, 0, r_0, s_0) = (x, y, u(x, y), v(x, y))$$

and taking partial derivatives with respect to y, one can establish the last two partial differential equations in (**??**).

(b) (15 marks) Next, we consider the particular case $f^1(x, y, r, s) = a$, $f^2(x, y, r, s) = b$, where a, b are constants, so that

$$\mathbb{X} = (1, 0, a, b).$$

It follows that

$$[\mathbb{X}, \mathbb{Y}] = (\mathbb{X} \cdot \nabla)\mathbb{Y} = (0, 0, (\mathbb{X} \cdot \nabla)g^1, (\mathbb{X} \cdot \nabla g^2)).$$

Therefore, $[\mathbb{X}, \mathbb{Y}] = 0$ if and only if

$$(\mathbb{X} \cdot \nabla)g^1 = 0, \quad (\mathbb{X} \cdot \nabla)g^2 = 0. \tag{8}$$

Let's consider the equation for g^1; the equation for g^2 is handled similarly. For $a = b = 0$, $\mathbb{X} = (1, 0, 0, 0)$, and the general solution of $(\mathbb{X} \cdot \nabla)g^1 = g_x^1 = 0$ is obviously any (smooth) function of y, r and s only. The case of nonzero a and b is similar. (8) implies that g^1 is invariant along $\mathbb{X}$, so it can be expressed in terms of three independent functions, or coordinates (the analogues of y, r and s), which are similarly invariant.

We can formalise the preceding argument as follows. Define functions (new coordinates if you like)

$$R(x, r) = r - ax, \quad S(x, s) = s - bx.$$

Define $G^1(x, y, R, S)$ via

$$g^1(x, y, r, s) = G^1(x, y, R(x, r), S(x, s)).$$

Then

$$\begin{aligned}(\mathbb{X} \cdot \nabla)g^1 &= \left(\frac{\partial}{\partial x} + a\frac{\partial}{\partial r} + b\frac{\partial}{\partial s}\right) G^1(x, y, R(x, r), S(x, s)) = \\ &= \left((G^1_x - aG^1_R - bG^1_S) + aG^1_R + bG^1_S\right)(x, y, R(x, r), S(x, s)) = \\ &= G^1_x(x, y, R(x, r), S(x, s)),\end{aligned}$$

so that

$$(\mathbb{X} \cdot \nabla)g^1 = 0 \iff G^1_x = 0 \iff g^1(x, y, r, s) = G^1(y, R(x, r), S(x, s)).$$

A similar argument applies to g^2. Summarising, the general solution to (8) is given by

$$g^1(x, y, r, s) = G^1(y, r - ax, s - bx), \quad g^2(x, y, r, s) = G^2(y, r - ax, s - bx). \tag{9}$$

(c) Finally, consider the particular case

$$g^1 = -g^2 = (r - ax)(s - bx)$$

with initial data

$$r_0 = s_0 = 1.$$

From (7), u, v are given by

$$\Phi_x(\Psi_y(0, 0, 1, 1)) = (x, y, u(x, y), v(x, y)).$$

For $\mathbb{X} = (1, 0, a, b)$ (constant), it is easily seen that Φ_t is given by

$$\Phi_t(x, y, r, s) = (x + t, y, r + at, s + bt).$$

As for Ψ_y, let's write

$$\Psi_y(0, 0, 1, 1) = (0, y, r(y), s(y)).$$

It follows that

$$u(x, y) = r(y) + ax, \quad v(x, y) = s(y) + bx. \tag{10}$$

The functions $r(y)$ and $s(y)$ satisfy the first-order system

$$\begin{aligned}r' &= g^1(0, y, r, s) = rs, \quad r(0) = 1,\\ s' &= g^2(0, y, r, s) = -rs, \quad s(0) = 1.\end{aligned}$$

Since $r' = -s'$, it follows that $r(y) + s(y)$ is a constant, which, from the initial conditions, must be equal to 2. Therefore,

$$s(y) = 2 - r(y).$$

Substitute above to get the differential equation

$$r' = r(2 - r), \quad r(0) = 1.$$

This first-order ODE is easily solved, eg using partial fractions:

$$\begin{aligned}\frac{dr}{r(2-r)} &= \tfrac{1}{2}\left(\frac{dr}{r} + \frac{dr}{2-r}\right) = dy,\\ \tfrac{1}{2}\int_1^r \left(\frac{dr}{r} + \frac{dr}{2-r}\right) &= \tfrac{1}{2}(\log r - \log(2 - r)) = y,\\ \frac{r}{2-r} &= e^{2y},\\ r(y) &= \frac{2e^{2y}}{1 + e^{2y}}.\end{aligned}$$

It follows that

$$s(y) = \frac{2}{1 + e^{2y}}$$

Substitute into (10) to get

$$\begin{aligned}u(x, y) &= \frac{2e^{2y}}{1 + e^{2y}} + ax,\\ v(x, y) &= \frac{2}{1 + e^{2y}} + by,\end{aligned}$$

which is the required solution.

3. (15 marks) We show that the system has a unique solution for all initial data $u(x_0, y_0) = u_0$ if and only if

$$[\mathbb{V}, \mathbb{W}] = r\mathbb{V} + s\mathbb{W}, \qquad (**)$$

where $\mathbb{V}$ and $\mathbb{W}$ are the vector fields on $\mathbb{R}^3$ given by

$$\mathbb{V}(x,y,z) = (a(x,y), b(x,y), f(x,y,z)), \quad \mathbb{W}(x,y,z) = (c(x,y), d(x,y), g(x,y,z)),$$

for some functions $r(x,y,z)$ and $s(x,y,z)$.

We proceed by converting the system (*) into the standard form. This is accomplished by first writing (*) in matrix form,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

We can multiply both sides on the left by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

since, by assumption, $ad - bc \neq 0$. We obtain the equivalent system

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{df - bg}{ad - bc}, \\ \frac{\partial u}{\partial y} &= \frac{-cf + ag}{ad - bc}. \qquad (***) \end{aligned}$$

The Frobenius theorem may be applied to the system (***). It follows that (***) has a unique solution for all initial data if and only if the vector fields

$$\mathbb{A} = \left(1, 0, \frac{df - bg}{ad - bc}\right), \quad \mathbb{B} = \left(0, 1, \frac{-cf + ag}{ad - bc}\right)$$

satisfy

$$[\mathbb{A}, \mathbb{B}] = 0. \qquad (****)$$

We want to show that (****) and (**) are equivalent. In fact, we shall prove a more general fact. Suppose two pairs of vector fields on $\mathbb{R}^3$, $\mathbb{A}, \mathbb{B}$ and $\mathbb{V}, \mathbb{W}$, are related by an invertible linear transformation,

$$\begin{pmatrix} \mathbb{V} \\ \mathbb{W} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbb{A} \\ \mathbb{B} \end{pmatrix}.$$

(Note that this relation is satisfied by $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{V}, \mathbb{W}$ as they are defined above.) We show that

$$[\mathbb{A}, \mathbb{B}] \text{ is a linear combination of } \mathbb{A}, \mathbb{B} \Leftrightarrow [\mathbb{V}, \mathbb{W}] \text{ is a linear combination of } \mathbb{V}, \mathbb{W}. \qquad (11)$$

Let us assume that

$$[\mathbb{A}, \mathbb{B}] = r\mathbb{A} + s\mathbb{B}$$

for some functions r and s. Then

$$\begin{aligned} [\mathbb{V}, \mathbb{W}] &= [a\mathbb{A} + b\mathbb{B}, c\mathbb{A} + d\mathbb{B}] \\ &= (ad - bc)[\mathbb{A}, \mathbb{B}] + (aL_{\mathbb{A}}c + bL_{\mathbb{B}}c - cL_{\mathbb{A}}a - dL_{\mathbb{B}}a)\mathbb{A} + (aL_{\mathbb{A}}d + bL_{\mathbb{B}}d - cL_{\mathbb{A}}b - dL_{\mathbb{B}}b)\mathbb{B} \\ &= ((ad - bc)r + aL_{\mathbb{A}}c + bL_{\mathbb{B}}c - cL_{\mathbb{A}}a - dL_{\mathbb{B}}a)\mathbb{A} + ((ad - bc)s + aL_{\mathbb{A}}d + bL_{\mathbb{B}}d - cL_{\mathbb{A}}b - dL_{\mathbb{B}}b)\mathbb{B}. \end{aligned}$$

In the second line, we have used the linearity of the Jacobi bracket, while in the third line we have used the assumption $[\mathbb{A}, \mathbb{B}] = r\mathbb{A} + s\mathbb{B}$. Thus, $[\mathbb{V}, \mathbb{W}]$ may be expressed as a linear combination of $\mathbb{A}$ and $\mathbb{B}$. But $\mathbb{A}$ and $\mathbb{B}$ may be expressed in terms of $\mathbb{V}$ and $\mathbb{W}$, since

$$\begin{pmatrix} \mathbb{A} \\ \mathbb{B} \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \mathbb{V} \\ \mathbb{W} \end{pmatrix}.$$

Thus $[\mathbb{V}, \mathbb{W}]$ can be expressed as a linear combination of $\mathbb{V}$ and $\mathbb{W}$, as was to be shown. The argument for the converse proceeds in exactly the same way.

Given the particular form of $\mathbb{A}$ and $\mathbb{B}$, it turns out that $[\mathbb{A}, \mathbb{B}]$ is a linear combination of $\mathbb{A}$ and $\mathbb{B}$ if and only if $[\mathbb{A}, \mathbb{B}] = 0$. The 'if' part is obvious; 0 is trivially a linear combination of $\mathbb{A}$ and $\mathbb{B}$ (take both with zero coefficient). On the other hand, as we have shown in lectures, the first two components of $[\mathbb{A}, \mathbb{B}]$ necessarily vanish, since the first two components of $\mathbb{A}$ and $\mathbb{B}$ are constant. Therefore, if $[\mathbb{A}, \mathbb{B}] = r\mathbb{A} + s\mathbb{B}$, then, since the first two components of $r\mathbb{A} + s\mathbb{B}$ are just r and s respectively, it follows that $r = s = 0$, ie $[\mathbb{A}, \mathbb{B}] = 0$.

Combining the previous conclusions, we may conclude that (*) has a solution for all initial data if and only if $[\mathbb{V}, \mathbb{W}]$ can be expressed as linear combinations of $\mathbb{V}$ and $\mathbb{W}$.