Lecture 11: Examples & the projective Nullstellensatz

Recap.

- In the last lecture we defined projective space
$$\mathbb{P}^n = \Big\{(p_0 : \ldots : p_n) : (p_0 : \ldots : p_n) = (\lambda p_0 : \ldots : \lambda p_n),\ \forall \lambda \in \mathbb{C}^\times\Big\}$$
and introduced the standard affine charts $U_i = \{p \in \mathbb{P}^n : p_i \neq 0\} \cong \mathbb{A}^n$.
- An projective algebraic set $X = \mathbb{V}(I) \subset \mathbb{P}^n$ is the subset of $\mathbb{P}^n$ defined by the vanishing of a homogeneous ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$.

1 Affine charts on a projective variety

We can use the charts $U_i \subset \mathbb{P}^n$ to obtain affine charts for any projective algebraic set $X \subset \mathbb{P}^n$.

Definition 7. If $X \subset \mathbb{P}^n$ is a projective algebraic set then the ith affine chart on X is:
$$X_{(i)} := X \cap U_i = \{p \in X : p_i \neq 0\}.$$

Note that $X_{(i)}$ is an affine algebraic set in $\mathbb{A}^n$. We have an isomorphism $\phi_i \colon U_i \to \mathbb{A}^n$, where the coordinates on $\mathbb{A}^n$ are $y_j = \frac{x_j}{x_i}$ for $j = 0, \ldots, \widehat{i}, \ldots, n$,[1]. Therefore if $X = \mathbb{V}(I) \subset \mathbb{P}^n$ for a homogeneous ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$, then $X_{(i)} = \mathbb{V}(I_{(i)}) \subset \mathbb{A}^n$ is given by the vanishing of the (inhomogeneous) ideal
$$I_{(i)} = \langle f(y_0, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n) : f \in I \rangle \subset \mathbb{C}[y_0, \ldots, \widehat{y_i}, \ldots, y_n].$$
The ideal $I_{(i)}$ is called the *dehomogenisation of I with respect to x_i*.[2]

2 Projective closure of an affine variety

Homogenisation. We can reverse this process and turn inhomogeneous polynomials/ideals into homogeneous polynomials/ideals by adding a new variable.

Definition 8.

1. If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a polynomial of degree d, then the polynomial
$$\widetilde{f} = x_0^d f\Big(\tfrac{x_1}{x_0}, \ldots, \tfrac{x_n}{x_0}\Big) \in \mathbb{C}[x_0, x_1, \ldots, x_n]$$
is the *homogenisation of f with respect to x_0*.

[1] *Note:* the hat notation $\widehat{\cdot}$ in a sequence means that we omit that term.

[2] In practice, we often don't bother changing notation from x_j to y_j and simply just set $x_i = 1$. This is usually harmless, but you have to be careful when comparing coordinates on two different charts $X_{(i)}$ and $X_{(j)}$.

2. If $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is an ideal, then the ideal

$$\widetilde{I} = \left\langle \widetilde{f} \in \mathbb{C}[x_0, x_1, \ldots, x_n] : \forall f \in I \right\rangle$$

is the *homogenisation of I with respect to x_0*.

Warning. We really have to homogenise *all* elements of the ideal. Only homogenising a generating set for I may not give a generating set for $\widetilde{I}$ (see Problem Sheet 3, qu. 7).

We can use homogenisation to obtain projective varieties from affine ones.

Definition 9. If $X = \mathbb{V}(I) \subset \mathbb{A}^n$ is an affine algebraic set for some ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, then the *projective closure of X* is the projective algebraic set $\widetilde{X} = \mathbb{V}(\widetilde{I}) \subset \mathbb{P}^n$ defined by the homogenisation $\widetilde{I} \subset \mathbb{C}[x_0, x_1, \ldots, x_n]$.

Taking the projective closure of an affine variety gives the same result as embedding the variety in one of the standard affine charts of projective space and then taking the Zariski closure (see Problem Sheet 3, qu. 5).

3 Examples

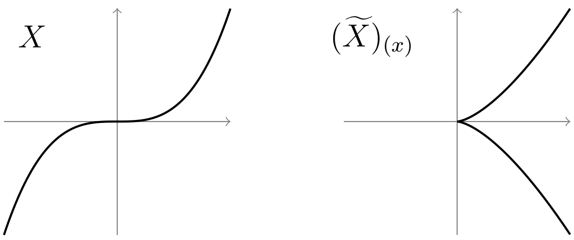
Example 10.

1. As we discussed in the warm-up to the last lecture, we can now check that two parallel lines in the plane meet at one point at ∞. Suppose $L_1, L_2 \subset \mathbb{A}^2_{x,y}$ are two parallel lines given by $L_1 = \mathbb{V}(ax + by + c)$ and $L_2 = \mathbb{V}(ax + by + c')$ for some $a, b, c, c' \in \mathbb{C}$ with $c \neq c'$ and not both of $a, b = 0$. Then the projective closures are $\widetilde{L_1} = \mathbb{V}(ax + by + cz)$ and $\widetilde{L_2} = \mathbb{V}(ax + by + c'z)$ and their intersection $\widetilde{L_1} \cap \widetilde{L_2} \subset \mathbb{P}^2_{(x:y:z)}$ is given by

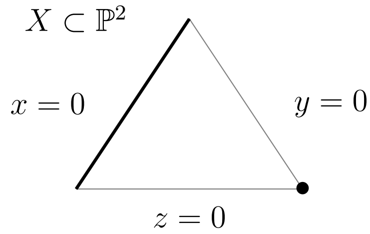
$$\widetilde{L_1} \cap \widetilde{L_2} = \mathbb{V}(ax + by + cz, ax + by + c'z) = \mathbb{V}(ax + by, z) = \{(b : -a : 0)\}.$$

   (In fact, given that the point $(b : -a : 0)$ is independent of c and c', we have just shown that all lines parallel to L_1 and L_2 meet in the same point at ∞.)

2. Suppose we take the affine variety $X = \mathbb{V}(x - y^3) \subset \mathbb{A}^2_{x,y}$. What does X look like "at ∞"? The projective closure is $\widetilde{X} = \mathbb{V}(xz^2 - y^3) \subset \mathbb{P}^2_{(x:y:z)}$, and the intersection of $\widetilde{X}$ with the hyperplane at ∞ is given by $\widetilde{X} \cap \mathbb{V}(z) = \{(1 : 0 : 0)\}$. To see what $\widetilde{X}$ looks like near the point $(1 : 0 : 0) \in \mathbb{P}^2$ we can consider the affine chart $U_x = \{x \neq 0\}$ to find that $(\widetilde{X})_{(x)} = \mathbb{V}(z^2 - y^3)$ has a cusp!



3. Suppose $X = \mathbb{V}(xy, xz) \subset \mathbb{P}^2_{(x:y:z)}$ so that $X = \mathbb{V}(x) \cup \mathbb{V}(y, z)$ is the union of the line $\mathbb{V}(x) = \{(0 : y : z)\}$ and the point $\mathbb{V}(y, z) = \{(1 : 0 : 0)\}$.



Restricting to the affine patch $X_{(x)} = \mathbb{V}(y, z) \subset \mathbb{A}^2_{y,z}$, and taking the projective closure $\widetilde{X_{(x)}}$ doesn't give back X—we are missing the line $\mathbb{V}(x) \subset (\mathbb{P}^2 \setminus U_x)$! Similarly, for either $\widetilde{X_{(y)}}$ or $\widetilde{X_{(z)}}$ we lose the point $\mathbb{V}(y, z)$.

4 The projective Nullstellensatz

There is a projective version of the Nullstellensatz. As in the affine case, it is easy to prove that $\mathbb{I}(\mathbb{V}(I)) \subseteq \sqrt{I}$ for a homogeneous ideal I and that $X = \mathbb{V}(\mathbb{I}(X))$ for a projective algebraic set X. However it is no longer true that $\mathbb{V}(I) = \emptyset \iff I = \langle 1 \rangle$, due to the ideal $\langle x_0, \ldots, x_n \rangle$ which defines the point $(0, \ldots, 0) \in \mathbb{A}^{n+1}$. We call $\langle x_0, \ldots, x_n \rangle$ the *irrelevant ideal* of $\mathbb{C}[x_0, \ldots, x_n]$.

Theorem 11 (Projective Nullstellensatz)**.** *Suppose that $I \subset \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous ideal and that $\mathbb{V}(I) \subset \mathbb{P}^n$ is the corresponding projective algebraic set.*

1. *$\mathbb{V}(I) = \emptyset$ if and only if $\langle x_0, \ldots, x_n \rangle \subseteq \sqrt{I}$.*
2. *If $\mathbb{V}(I) \neq \emptyset$ then $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.*

Proof. For $I \subset \mathbb{C}[x_0, \ldots, x_n]$ we consider the affine algebraic set $\mathbb{V}_{\mathbb{A}^{n+1}}(I) \subset \mathbb{A}^{n+1}$, which has the property that

$$(p_0, \ldots, p_n) \in \mathbb{V}_{\mathbb{A}^{n+1}}(I) \iff (\lambda p_0, \ldots, \lambda p_n) \in \mathbb{V}_{\mathbb{A}^{n+1}}(I) \quad \forall p \in \mathbb{A}^{n+1}, \forall \lambda \in \mathbb{C}^\times.$$

Since $\mathbb{V}_{\mathbb{P}^n}(I) = (\mathbb{V}_{\mathbb{A}^{n+1}}(I) \setminus 0)/\sim$, we have

$$\mathbb{V}_{\mathbb{P}^n}(I) = \emptyset \iff \mathbb{V}_{\mathbb{A}^{n+1}}(I) \subseteq \{0\} \iff \langle x_0, \ldots, x_n \rangle \subseteq \sqrt{I}$$

where the second $\iff$ comes from using the affine Nullstellensatz for $\mathbb{V}_{\mathbb{A}^{n+1}}(I)$. If $\mathbb{V}_{\mathbb{P}^n}(I) \neq \emptyset$, then we have

$$f \in \mathbb{I}(\mathbb{V}_{\mathbb{P}^n}(I)) \iff f \in \mathbb{I}(\mathbb{V}_{\mathbb{A}^{n+1}}(I)) \iff f \in \sqrt{I}.$$

where, again, the second $\iff$ comes from using the affine Nullstellensatz for $\mathbb{V}_{\mathbb{A}^{n+1}}(I)$. $\square$

Corollary 12. *The correspondences $\mathbb{I}$ and $\mathbb{V}$ give the following inverse bijections between graded rings and projective geometry:*

Graded rings		*Projective geometry*
$\mathbb{C}[x_0, \ldots, x_n]$	$\longleftrightarrow$	*projective space* $\mathbb{P}^n$
{ *hgs radical ideals* }	$\longleftrightarrow$	{ *proj algebraic sets* }
$\cup$		$\cup$
{ *hgs prime ideals* }	$\longleftrightarrow$	{ *proj varieties* }
$\cup$		$\cup$
{ *hgs maximal ideals* }	$\longleftrightarrow$	{ *points in* $\mathbb{P}^n$ }