Fields, Forms and Flows 3/34

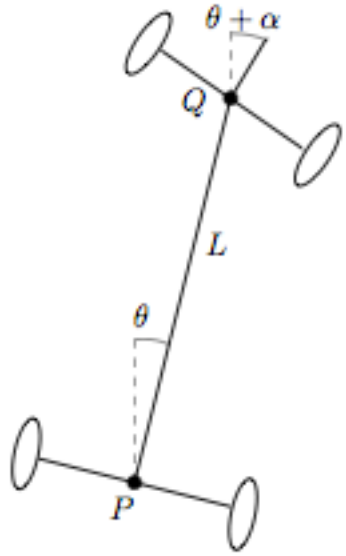
Problem Sheet 4
Due: Wednesday 31 October
To hand in: FFF3: 1, 3 FFF34: 1, 3

 Problems marked * are not examinable (but are still worth trying!)

1. We define the Lie derivative of a vector field $\mathbb{Y}$ with respect to a vector field $\mathbb{X}$ as follows:
$$L_{\mathbb{X}}\mathbb{Y} := [\mathbb{X}, \mathbb{Y}],$$
where $\mathbb{X}$ and $\mathbb{Y}$ are vector fields. (The connection with the Lie derivative on functions is as follows: We have that
$$L_{\mathbb{X}} f = \mathbb{X} \cdot \nabla f = \left. \frac{\partial}{\partial t} \Phi_t^* f \right|_{t=0},$$
where Φ_t is the flow of $\mathbb{X}$. We therefore define
$$L_{\mathbb{X}}\mathbb{Y} := - \left. \frac{\partial}{\partial t} \Phi_{t*} \mathbb{Y} \right|_{t=0} = -[\mathbb{Y}, \mathbb{X}] = [\mathbb{X}, \mathbb{Y}].$$
The reason for the minus sign in the definition is that the push-forward goes in the opposite direction to the pull-back.) Then
$$L_{\mathbb{X}}^2 \mathbb{Y} = L_{\mathbb{X}}(L_{\mathbb{X}}\mathbb{Y}) = [\mathbb{X}, [\mathbb{X}, \mathbb{Y}]], \quad L_{\mathbb{X}}^3 \mathbb{Y} = [\mathbb{X}, [\mathbb{X}, [\mathbb{X}, \mathbb{Y}]]], \quad \text{etc}$$
(it's the ability to write the nested brackets in a compact way that makes this notation useful here).

   (a) Let Ψ_s be the flow of $\mathbb{Y}$. Show that
$$\frac{\partial^n}{\partial s^n} \Psi_{s*} \mathbb{X} = (-1)^n L_{\mathbb{Y}}^n \Psi_{s*} \mathbb{X}.$$
   (Suggestion: recall the derivation of equation (10) following Definition 1.9.3 in the notes. Use induction.)

   (b) Derive the (formal) power series
$$\Psi_{s*} \mathbb{X} = e^{-sL_{\mathbb{Y}}} \mathbb{X}.$$
   (Suggestion: follow the model of Proposition 1.9.4.) Use this expression to show that $\Psi_{s*}\mathbb{Y} = \mathbb{Y}$.



(c) Let $\mathbb{Z}$ be a vector field. Show that the Jacobi identity can be written in the form

$$(L_{\mathbb{X}}L_{\mathbb{Y}} - L_{\mathbb{Y}}L_{\mathbb{X}})\mathbb{Z} = L_{[\mathbb{X},\mathbb{Y}]}\mathbb{Z}.$$

2. Parallel parking. Consider the model car shown above. The centre of the back axle is at P, with coordinates (x, y). The direction of the back wheels makes an angle θ with the vertical. The length of the body is L. The centre of the front axle is at Q, and the direction of the front wheels makes an angle $\theta+\alpha$ with the vertical. The configuration of the car is determined by x, y, θ and α, which collectively we regard as a point (x, y, θ, α) in $\mathbb{R}^4$ (of course, changing θ or α by 2π does not change the configuration of the car).

   (a) Let $\mathbb{S}$ be the smooth vector field on $\mathbb{R}^4$ given by

   $$\mathbb{S}(x, y, \theta, \alpha) = (0, 0, 0, \omega),$$

   where ω is a constant ($\mathbb{S}$ stands for 'steer'). Determine the flow Φ_t of $\mathbb{S}$.

   (b) Determine the changes in x, y and θ if the back wheels roll forward a small distance $v\delta t$ in the direction θ (the front wheels roll forward, too, in the direction $\theta+\alpha$, by a distance you should determine). Show that this infinitesimal change in the configuration is described by the vector field $\mathbb{D}$ given by

   $$\mathbb{D}(x, y, \theta, \alpha) = (v\sin\theta, v\cos\theta, (v/L)\tan\alpha, 0),$$

   where v is a constant ($\mathbb{D}$ stands for 'drive'). (Hint: the car can't stretch! Its length is fixed.) Compute the flow Ψ_s of $\mathbb{D}$. (If you find the derivation of $\mathbb{D}$ difficult just continue with the computation of its flow.)

(c) Compute $\Psi_\epsilon(x, y, \theta, \epsilon\Omega)$ up to and including terms of order ϵ^2. If ϵ is regarded as a small parameter, this represents driving for a short time ϵ with the front wheels only slightly turned by angle $\epsilon\Omega$.

(d) Compute $\mathbb{A} = [\mathbb{S}, \mathbb{D}]$ and $\mathbb{B} = [[\mathbb{S}, \mathbb{D}], \mathbb{D}]$. Show that for $\cos\alpha \neq 0$, $\mathbb{A}$, $\mathbb{B}$, $\mathbb{S}$ and $\mathbb{D}$ are linearly independent. Compute the flows, Γ_a and Δ_b, of $\mathbb{A}$ and $\mathbb{B}$ respectively. (It turns out that Γ_a describes a pure rotation in θ, while Δ_b describes a pure translation of the car perpendicular to its direction.)

(e) Verify Theorem 1.10.1 by showing explicitly that

$$\begin{aligned}(\Psi_{-\epsilon} \circ \Phi_{-\epsilon} \circ \Psi_\epsilon \circ \Phi_\epsilon)(x, y, \theta, 0) &= \Gamma_{\epsilon^2}(x, y, \theta, 0) + O(\epsilon^3) \\ &= (x, y, \theta, 0) + \epsilon^2 \mathbb{A}(x, y, \theta, 0) + O(\epsilon^3).\end{aligned}$$

Explain how, by a sequence of small steer and drive maneuvers, you can change the orientation θ of your car by as much as you like while changing x, y and α as little as you like.

(f) Verify Theorem 1.10.1 by showing explicitly that

$$\begin{aligned}(\Psi_{-\delta} \circ \Gamma_{-\delta T} \circ \Psi_\delta \circ \Gamma_{\delta T})(x, y, \theta, 0) &= \Delta_{\delta^2 T}(x, y, \theta, 0) + O(\delta^3) \\ &= (x, y, \theta, 0) + \delta^2 T \mathbb{B}(x, y, \theta, 0) + O(\delta^3).\end{aligned}$$

Here T may be regarded as a fixed constant with dimensions of time (if you are keeping track of dimensions, this makes the expression dimensionally sensible), or else may be set equal to 1 (if you prefer to ignore dimensional considerations). Explain how, by a sequence of small steer and drive maneuvers, you can translate your car in a direction perpendicular to its length by as much as you like while changing θ and α as little as you like. (This is parallel parking into a tight space.)

3. Noncommutativity of rotations in $\mathbb{R}^3$.

(a) Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and let

$$\mathbb{X}_A(r) = A \cdot r, \quad \mathbb{X}_B(r) = B \cdot r, \quad \mathbb{X}_C(r) = C \cdot r$$

be linear vector fields on $\mathbb{R}^3$, with $r = (x, y, z)$. Show that

$$[\mathbb{X}_A, \mathbb{X}_B] = -\mathbb{X}_C, \quad [\mathbb{X}_B, \mathbb{X}_C] = -\mathbb{X}_A, \quad [\mathbb{X}_C, \mathbb{X}_A] = -\mathbb{X}_B.$$

(It may be helpful to refer to Problem 3.6.)

(b) Show that Φ_{Ct}, the flow of $\mathbb{X}_C$, is given by

$$\Phi_{Ct}(r) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(c) Derive similar formulas for the flows Φ_{At} and Φ_{Bt} of $\mathbb{X}_A$ and $\mathbb{X}_B$.

(d) Show explicitly that, for small θ,

$$(\Phi_{B\theta} \circ \Phi_{A\theta} \circ \Phi_{B(-\theta)} \circ \Phi_{A(-\theta)})(r) = \Phi_{C(-\theta^2)}(r) + O(\theta^3).$$

(A related fact: Hold a bank note in front of you so that the Queen is upright on your right looking at you. Take the z-direction to be pointing up and y to be pointing in front of you. Rotate the bank note about y by -90° (that is, anticlockwise from your perspective), then about z by 180°, then about y by 90° (that is, clockwise), then about z by 180°. The Queen should be looking at you again, but upside-down on your left.)