UNIVERSITY OF BRISTOL

School of Mathematics

Differentiable Manifolds 34

MATH M2900

(Paper code MATH–M2900J)

January 2018   2 hours 30 minutes

This paper contains FOUR questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Unless stated otherwise, all functions, vector fields, differential forms and maps are taken to be smooth.

$$L_{\mathbb{X}} = i_{\mathbb{X}} d + d i_{\mathbb{X}}$$

$$\int \frac{dz}{z^2+1} = \arctan z + \text{constant}$$

$$\frac{\partial \Phi_t}{\partial t}(x) = \mathbb{X}(\Phi_t(x))$$

$$\left.\frac{\partial}{\partial t} \hat{\Phi}_t^* \,\omega\right|_{t=\tau} = \hat{\Phi}_\tau^* L_{\hat{\mathbb{X}}_\tau} \omega$$

$$\partial c = \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} c_{(j,\alpha)}$$

$$i_{\mathbb{X}}(\alpha \wedge \beta) = (i_{\mathbb{X}}\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_{\mathbb{X}}\beta), \text{ where } \alpha \text{ is a } k\text{-form.}$$

Do not turn over until instructed.

1. (a) (**5 marks**) Let $\mathbb{X}(x, y) = (2x, y)$ and $\mathbb{Y} = (y, x)$ be vector fields on $\mathbb{R}^2$.

   i. Find the flow of $\mathbb{X}$.

   ii. Compute $[\mathbb{X}, \mathbb{Y}]$.

   (b) Let $\mathbb{X}$ and $\mathbb{Y}$ be vector fields on $\mathbb{R}^n$ with flows Φ_t and Ψ_s respectively.

   i. (**6 marks**) Given that $L_{\mathbb{X}} L_{\mathbb{Y}} f - L_{\mathbb{Y}} L_{\mathbb{X}} f = L_{[\mathbb{X},\mathbb{Y}]} f$ for all functions $f \in C^\infty(\mathbb{R}^n)$, show that

   $$L_{\mathbb{X}} L_{\mathbb{Y}} \omega - L_{\mathbb{Y}} L_{\mathbb{X}} \omega = L_{[\mathbb{X},\mathbb{Y}]} \omega$$

   for all k-forms $\omega \in \Omega^k(\mathbb{R}^n)$. (Hint: one approach is to use induction on the degree of ω.)

   ii. (**10 marks**) Derive that $\Phi_t^* \,\omega = e^{tL_{\mathbb{X}}}\,\omega$ for all k-forms $\omega \in \Omega^k(\mathbb{R}^n)$. Given that similarly $\Psi_s^* \,\omega = e^{sL_{\mathbb{Y}}}\,\omega$ show that

   $$(\Psi_{-s} \circ \Phi_{-t} \circ \Psi_s \circ \Phi_t)^* \,\omega = \omega + st L_{[\mathbb{X},\mathbb{Y}]} \omega + O(3),$$

   where $O(3)$ denotes terms of third and higher order in s and t.

   iii. (**4 marks**) With $\mathbb{X}$ and $\mathbb{Y}$ given by the expressions in part (a) above and $\omega \in \Omega^1(\mathbb{R}^2)$ given by $\omega = y\,dx - x\,dy$, show that

   $$\frac{\partial}{\partial s}\frac{\partial}{\partial t} (\Psi_{-s} \circ \Phi_{-t} \circ \Psi_s \circ \Phi_t)^* \,\omega = O(1),$$

   where $O(1)$ denotes terms of first and higher order in s and t.

Continued...

2. Consider the following system of second-order partial differential equations for $u = u(x, y)$:

$$\begin{aligned} \frac{\partial u}{\partial x} &= u^l, \\ \frac{\partial^2 u}{\partial y^2} &= e^{-x}\, u\, \frac{\partial u}{\partial y}, \end{aligned} \tag{1}$$

where $l > 0$, with initial data

$$u(x_0, y_0) = u_0, \qquad \frac{\partial u}{\partial y}(x_0, y_0) = v_0. \tag{2}$$

(a) (**4 marks**) Write (1) as an equivalent system of four first-order partial differential equations.

(b) (**7 marks**) Show that (1) – (2) has a solution defined in a neighbourhood of (x_0, y_0) for all $(x_0, y_0) \in \mathbb{R}^2$ and $(u_0, v_0) \in \mathbb{R}^2$ if and only if l takes a specific value, and determine what this value is. (You may invoke the Frobenius theorem without proof.)

(c) (**10 marks**) For $l = 1$, $(x_0, y_0) = (0, 0)$ and $(u_0, v_0) = (1, 1)$, find the solution $u(x, y)$ of (1) – (2).

(d) (**4 marks**) What is the radius of the largest open disk in $\mathbb{R}^2$ centred at the origin in which $u(x, y)$ is smooth?

Continued...

3. (a) You are given the following statement of the Poincaré Lemma: If $\hat{\Phi}_t$ is a one-parameter family of diffeomorphisms on $\mathbb{R}^n$ and $\hat{\mathbb{X}}_t$ is the time-dependent vector field defined by

$$\hat{\mathbb{X}}_t \circ \hat{\Phi}_t = \frac{\partial}{\partial t}\hat{\Phi}_t,$$

and if β is a closed k-form on $\mathbb{R}^n$ such that

$$\hat{\Phi}_1^*\beta = \beta, \qquad \lim_{\epsilon \to 0} \hat{\Phi}_\epsilon^*\beta = 0,$$

then $\beta = d\alpha$, where

$$\alpha = \int_0^1 \hat{\Phi}_t^*(i_{\hat{\mathbb{X}}_t}\beta)\, dt. \tag{3}$$

In what follows, let β be the two-form on $\mathbb{R}^3$ given by

$$\beta = (y\, dx + x\, dy) \wedge dz.$$

i. (**3 marks**) Show that $d\beta = 0$.

ii. (**6 marks**) Let $\hat{\Phi}_t$: $\mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\hat{\Phi}_t(x, y, z) = (t^2 x, t^2 y, t(z + t - 1)),$$

where $t > 0$. Find $\hat{\mathbb{X}}_t$ as defined above.

iii. (**3 marks**) Show that

$$\lim_{t \to 0} \hat{\Phi}_t^*\beta = 0.$$

iv. (**6 marks**) Using the formula (3) with $\hat{\mathbb{X}}_t$ as defined above, find a one-form α on $\mathbb{R}^3$ such that

$$\beta = d\alpha.$$

(b) (**7 marks**) Let $\hat{\Psi}_s$ be a one-parameter family of diffeomorphisms on $\mathbb{R}^n$ with $\hat{\Psi}_0 = \hat{\Psi}_1$, and let $\hat{\mathbb{Y}}_s$ be defined by

$$\hat{\mathbb{Y}}_s \circ \hat{\Psi}_s = \frac{\partial}{\partial s}\hat{\Psi}_s.$$

If β is a closed k-form on $\mathbb{R}^n$, show that the $(k-1)$-form α defined by

$$\alpha := \int_0^1 \hat{\Psi}_s^*(i_{\hat{\mathbb{Y}}_s}\beta)\, ds$$

is closed.

Continued...

4. (a) Let $c\colon I^2 \to \mathbb{R}^3$ be the singular 2-cube given by

$$c(s,t) = (s+t, s^m t^n, s-t),$$

where m and n are positive integers. Let (x, y, z) denote Cartesian coordinates on $\mathbb{R}^3$, and let ω be the one-form on $\mathbb{R}^3$ given by

$$\omega = (x+y+z)\,dx.$$

   i. (**3 marks**) Compute $c^*\omega$.
   ii. (**3 marks**) Compute $c^*d\omega$.
   iii. (**3 marks**) Compute $\int_c d\omega$.
   iv. (**4 marks**) Without using Stokes' theorem, compute $\int_{\partial c} \omega$.

   (b) Let α be the one-form on $\mathbb{R}^3$ and β the two-form on $\mathbb{R}^3$ given by

$$\alpha = yz\,dx + xz\,dy,$$
$$\beta = z\,dy \wedge dz,$$

and let $\mathbb{X}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbb{X} = (x, -y, 1).$$

   i. (**3 marks**) Compute $\alpha \wedge \beta$, combining terms where possible.
   ii. (**3 marks**) Compute $d\alpha$, combining terms where possible.
   iii. (**3 marks**) Compute $i_{\mathbb{X}} d\alpha$, combining terms where possible.
   iv. (**3 marks**) Compute $L_{\mathbb{X}}\alpha$, combining terms where possible.

End of examination.