Numerical Analysis 23 **Sheet 3**

Fixed-Point Iteration, Newton-Raphson Method

Please hand in questions 1, 2(a)(b)(c) and 5 by 11am on Thursday 21 February.

1. Show that the function $g(x) = 2^{-x}$ has a unique fixed point x^* in the interval $[1/3, 1]$, and that the fixed point iteration $x_{n+1} = g(x_n)$ with initial point $x_0 \in [1/3, 1]$ converges to this fixed point x^*.

   By using the mean value theorem find an upper bound for the number of iterations of the fixed point method with initial point $x_0 \in [1/3, 1]$ which would be required to find the fixed point x^* to an accuracy of 10^{-4}.

2. Consider the following iterating schemes for calculating $21^{1/3}$. Show that each of the iteration schemes has a fixed point which is a solution of the equation $x^3 = 21$. Rank the methods in order based upon the speed of convergence assuming that $x_0 = 1$ in each case. (You may assume that all four iteration schemes converge to one of their fixed points if $x_0 = 1$.)

   (a) $x_{n+1} = 20x_n/21 + 1/x_n^2$

   (b) $x_{n+1} = x_n - (x_n^3 - 21)/(3x_n^2)$

   (c) $x_{n+1} = (21/x_n)^{1/2}$

   (d) $x_{n+1} = x_n - (x_n^4 - 21x_n)/(x_n^2 - 21)$

3. Using a graphical method investigate the behaviour of the Newton Raphson iteration for calculating the unique root of $f(x) = (4x - 7)/(x - 2)$ given the following starting points i) 1.625 ii) 1.875 iii) 1.5 iv) 1.95 and v) 3.

4. Consider the following algorithm

$$x_n = x_{n-1} - f(x_{n-1})/f'(x_{n-1}) - \frac{1}{2}f''(x_{n-1})(f(x_{n-1}))^2/(f'(x_{n-1}))^3$$

   for finding a root of the function $f(x)$. Show that in general this scheme will have at least cubic convergence.

5. Derive the useful estimate

$$\frac{1}{2}\left(x + \frac{a}{x}\right)$$

   for approximating $\sqrt{a}$ with an initial guess x by applying Newton Raphson to $f(x) = x^2 - a$. Generalise this to an equivalent formula for $\sqrt[n]{a}$ and use it to estimate $\sqrt[3]{9}$ using an initial guess of $x = 2$.