Fields, Forms and Flows 3/34

Solution Sheet 8

1. (a) $f(x) = \sin(x^1)x^2$ is a 0-form, ie a function, and

$$df = \cos(x^1)x^2\,dx^1 + \sin(x^1)\,dx^2.$$

(b) $\alpha(x) = \cos x^1\,dx^2 + dx^3$ is a 1-form, and

$$d\alpha = -\sin x^1\,dx^1 \wedge dx^2.$$

(c) $\beta = f\alpha$ is a 1-form, and

$$d\beta = df \wedge \alpha + f d\alpha.$$

Using the preceding results,

$$\begin{aligned} d\beta &= (\cos(x^1)x^2dx^1 + \sin(x^1)dx^2) \wedge (\cos x^1\,dx^2 + dx^3) + (\sin(x^1)x^2)(-\sin x^1\,dx^1 \wedge dx^2) \\ &= \cos^2(x^1)x^2\,dx^1 \wedge dx^2 + \cos(x^1)x^2\,dx^1 \wedge dx^3 + \sin(x^1)\,dx^2 \wedge dx^3 - \sin^2(x^1)x^2\,dx^1 \wedge dx^2 \\ &= (\cos^2(x^1) - \sin^2(x^1))x^2\,dx^1 \wedge dx^2 + \cos(x^1)x^2\,dx^1 \wedge dx^3 + \sin(x^1)\,dx^2 \wedge dx^3. \end{aligned}$$

(d) $\gamma = df \wedge \alpha$ is a 2-form, and

$$d\gamma = d^2 f \wedge \alpha - df \wedge d\alpha = -df \wedge d\alpha.$$

Using the preceding results,

$$d\gamma = -(\cos(x^1)x^2dx^1 + \sin(x^1)dx^2) \wedge (-\sin x^1\,dx^1 \wedge dx^2) = 0,$$

since $dx^1 \wedge dx^1 \wedge dx^2 = dx^2 \wedge dx^1 \wedge dx^2 = 0$.

2. (a) We have that

$$d(\alpha \wedge d\alpha) = d\alpha \wedge d\alpha + (-1)^k \alpha \wedge d^2\alpha = d\alpha \wedge d\alpha,$$

since $d^2\alpha = 0$. From the (anti)commutativity rule,

$$d\alpha \wedge d\alpha = (-1)^{(k+1)^2} d\alpha \wedge d\alpha,$$

as $d\alpha$ is a $(k+1)$-form. If k is even, $(-1)^{(k+1)^2} = -1$, and we get

$$d\alpha \wedge d\alpha = 0.$$

(b) Let (u, v, x, y) be coordinates on $\mathbb{R}^4$. Let

$$\alpha = u\,dv + x\,dy.$$

Then

$$d\alpha = du \wedge dv + dx \wedge dy,$$

and

$$d\alpha \wedge d\alpha = 2\,du \wedge dv \wedge dx \wedge dy.$$

3. (a) We have that

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta, \tag{1}$$

for α a k-form and β an l-form. Take $n = 3$. For $k = l = 0$, α and β correspond to functions f and g, and $d\alpha$, $d\beta$ correspond to vector fields ∇f and ∇g. Equation (1) corresponds to

$$\nabla(fg) = g\nabla f + f\nabla g.$$

For $k = 0$, $l = 1$, β corresponds to a vector field $\mathbf{E}$, and $d\beta$ to $\nabla \times \mathbf{E}$. $d\alpha \wedge \beta$ corresponds to $\nabla f \times \mathbf{E}$. Equation (1) corresponds to

$$\nabla \times (f\mathbf{E}) = \nabla f \times \mathbf{E} + f\nabla \times \mathbf{E}.$$

For $k=0$ and $l=2$, α corresponds to a function f, β to a vector field $\mathbf{B}$, $d\beta$ to $\nabla\cdot\mathbf{B}$, $\alpha\wedge\beta$ to the vector field $f\mathbf{B}$, and $d(\alpha\wedge\beta)$ to $\nabla\cdot(f\mathbf{B})$. Equation (1) corresponds to

$$\nabla\cdot(f\mathbf{B}) = \nabla f\cdot\mathbf{B} + f\,\nabla\cdot\mathbf{B}.$$

For $k=l=1$, α and β correspond to vector fields $\mathbf{E}$ and $\mathbf{F}$, $\alpha\wedge\beta$ to $\mathbf{E}\times\mathbf{F}$, and $d(\alpha\wedge\beta)$ to $\nabla\cdot(\mathbf{E}\times\mathbf{F})$. Equation (1) corresponds to

$$\nabla\cdot(\mathbf{E}\times\mathbf{F}) = (\nabla\times\mathbf{E})\cdot\mathbf{F} - (\mathbf{E}\cdot(\nabla\times\mathbf{F}).$$

(b) Under the correspondence between vector fields and forms on $\mathbb{R}^3$, we have that $df\leftrightarrow\nabla f$ and $d(df)\leftrightarrow\nabla\times\nabla f$, so that

$$d(df)=0\leftrightarrow\nabla\times\nabla f=0.$$

Next, let ϵ be a one-form on $\mathbb{R}^3$ and $\epsilon\leftrightarrow\mathbf{E}=(\epsilon_1,\epsilon_2,\epsilon_3)$. Then $d\epsilon\leftrightarrow\nabla\times\mathbf{E}$ and $d(d\epsilon)\leftrightarrow\nabla\cdot\nabla\times\mathbf{E}$ so that

$$d(d\epsilon)=0\leftrightarrow\nabla\cdot\nabla\times\mathbf{E}=0.$$

4. Let $r=(x^2+y^2+z^2)^{1/2}$ Then

$$\omega=\frac{x}{r^3}dy\wedge dz+\frac{y}{r^3}dz\wedge dx+\frac{z}{r^3}dx\wedge dy,$$

and

$$d\omega = d\frac{x}{r^3}\wedge dy\wedge dz + d\frac{y}{r^3}\wedge dz\wedge dx + d\frac{z}{r^3}\wedge dx\wedge dy.$$

We have that

$$d\frac{x}{r^3}=\frac{1}{r^3}dx-3\frac{x}{r^4}dr,$$

and

$$dr=\frac{x}{r}dx+\frac{y}{r}dy+\frac{z}{r}dz,$$

so that

$$d\frac{x}{r^3}=\frac{1}{r^5}((r^2-3x^2)dx-3xydy-3xzdz).$$

Then the first term in $d\omega$ is given by

$$d\frac{x}{r^3}\wedge dy\wedge dz=\frac{r^2-3x^2}{r^5}dx\wedge dy\wedge dy;$$

the terms in dy and dz in $d(x/r^3)$ do not contribute, since $dy\wedge dy\wedge dz = dz\wedge dy\wedge dz=0$. The other terms in $d\omega$ are similarly computed, with the result

$$d\omega=\frac{(r^2-3x^2)+(r^2-3y^2)+(r^2-3z^2)}{r^5}dx\wedge dy\wedge dz=\frac{3r^2-3(x^2+y^2+z^2)}{r^5}dx\wedge dy\wedge dz\ =0,$$

as required.

5. * Let $A(x)$ be an $n\times n$ matrix whose components $A_{ij}(x)$ are smooth functions on $\mathbb{R}^n$. Recall the following formula for the determinant of A,

$$\det A=\sum_{\sigma\in S_n}\operatorname{sgn}\sigma A_{1,\sigma(1)}\cdots A_{n,\sigma(n)}.$$

(a) Let $\alpha^{(i)}$ be the 1-form on $\mathbb{R}^n$ given by

$$\alpha^{(i)}=\sum_{j=1}^n A_{ij}dx^j.$$

Then

$$\alpha^{(1)}\wedge\cdots\wedge\alpha^{(n)}=A_{1,j_1}\cdots A_{n,j_n}\,dx^{j_1}\wedge\cdots\wedge dx^{j_n}.$$

The only terms which contribute to the (implied) sum over the j_l's are those for which the j_l's are all distinct. In this case, they determine a unique permutation $\sigma\in S_n$ via

$$\sigma(l)=j_l.$$

Every $\sigma \in S_n$ is realised once in this way. Therefore,

$$\alpha^{(1)} \wedge \cdots \wedge \alpha^{(n)} = \sum_{\sigma \in S_n} A_{1,\sigma(1)} \cdots A_{n,\sigma(n)}\, dx^{\sigma(1)} \wedge \cdots \wedge dx^{\sigma(n)}.$$

But

$$dx^{\sigma(1)} \wedge \cdots \wedge dx^{\sigma(n)} = \operatorname{sgn}\sigma\, dx^1 \wedge \cdots \wedge dx^n.$$

Therefore,

$$\alpha^{(1)} \wedge \cdots \wedge \alpha^{(n)} = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma\, A_{1,\sigma(1)} \cdots A_{n,\sigma(n)}\, dx^1 \wedge \cdots \wedge dx^n = \det A\, dx^1 \wedge \cdots \wedge dx^n,$$

as required.

(b) The rs-th minor of A is given by

$$a_{rs} = (-1)^{s-r} \sum_{\sigma \in S_n \,|\, \sigma(r)=s} \operatorname{sgn}\sigma\, A_{1,\sigma(1)} \cdots A_{r-1,\sigma(r-1)} A_{r+1,\sigma(r+1)} \cdots A_{n,\sigma(n)},$$

where the sum is taken over permutations $\sigma \in S_n$ for which $\sigma(r) = s$. To confirm that the sign is correct, note that, for $\sigma = \sigma_0$ given by

$$\sigma_0 = \begin{pmatrix} 1 & \cdots & r-1 & r & r+1 & \cdots & s & s+1 & \cdots & n \\ 1 & \cdots & r-1 & s & r & \cdots & s-1 & s+1 & \cdots & n \end{pmatrix}$$

(where, for definiteness, we have taken $s \geq r$), the corresponding term in the sum is given by

$$(-1)^{s-r} \operatorname{sgn}\sigma A_{11} \cdots A_{r-1,r-1} A_{r+1,r} \cdots A_{s,s-1} A_{s+1,s+1} \cdots A_{n,n},$$

which is the product of the diagonal elements of the $(n-1) \times (n-1)$ submatrix obtained by removing the rth row and the sth column from A. Since $\operatorname{sgn}\sigma = (-1)^{s-r}$ (σ cyclically permutes the elements r through s and leaves everything else unchanged), the net sign is $((-1)^{s-r})^2 = 1$, as it should be for the product of the diagonal elements of the submatrix.

We have that

$$\begin{aligned} \alpha^{(1)} \wedge \cdots \wedge \alpha^{(n)} &= (-1)^{p-1} \alpha^{(p)} \wedge \left(\alpha^{(1)} \wedge \cdots \wedge \alpha^{(p-1)} \wedge \alpha^{(p+1)} \wedge \cdots \wedge \alpha^{(n)}\right) \\ &= (-1)^{p-1} \sum_{r=1}^{n} A_{pr} dx^r \wedge \left(\alpha^{(1)} \wedge \cdots \wedge \alpha^{(p-1)} \wedge \alpha^{(p+1)} \wedge \cdots \wedge \alpha^{(n)}\right) \\ &= (-1)^{p-1} \sum_{r=1}^{n} A_{pr} dx^r \wedge \left(A_{1,j_1} \cdots A_{p-1,j_{p-1}} A_{p+1,j_{p+1}} \cdots A_{n,j_n} dx^{j_1} \wedge \cdots dx^{j_{p-1}} \wedge dx^{j_{p+1}} \wedge \cdots \wedge dx^{j_n}\right). \end{aligned}$$

The only terms which contribute to the (implied) sum over the j_l's are those for which the j_l's are given by a permutation σ of the integers 1 through n with r omitted (terms with one of the j_l's equal to r vanish because $dx^r \wedge dx^r = 0$). For these non-vanishing terms, there is a unique $\sigma \in S_n$ such that

$$\sigma(l) = \begin{cases} j_l, & l \neq p, \\ r, & l = p. \end{cases}$$

Moreover, all σ's with $\sigma(p) = r$ are realised in this way. Therefore,

$$\begin{aligned} &A_{1,j_1} \cdots A_{p-1,j_{p-1}} A_{p+1,j_{p+1}} \cdots A_{n,j_n} dx^{j_1} \wedge \cdots dx^{j_{p-1}} \wedge dx^{j_{p+1}} \wedge \cdots \wedge dx^{j_n} \\ &\quad = \sum_{\sigma \in S_n | \sigma(p)=r} A_{1,\sigma(1)} \cdots A_{p-1,\sigma(p-1)} A_{p+1,\sigma(p+1)} \cdots A_{n,\sigma(n)} \\ &\qquad\qquad \times dx^{\sigma(1)} \wedge \cdots dx^{\sigma(p-1)} \wedge dx^{\sigma(p+1)} \wedge \cdots \wedge dx^{\sigma(n)} \end{aligned}$$

We have that

$$dx^{\sigma(1)} \wedge \cdots dx^{\sigma(p-1)} \wedge dx^{\sigma(p+1)} \wedge \cdots \wedge dx^{\sigma(n)} = (-1)^{r-p} \operatorname{sgn}\sigma\, dx^1 \wedge \cdots dx^{r-1} \wedge dx^{r+1} \wedge \cdots \wedge dx^n$$

(you can verify that the sign is correct by taking $\sigma = \sigma_0$, where σ_0 is given above). Substitute to obtain

$$\sum_{\sigma \in S_n \mid \sigma(p)=r} A_{1,\sigma(1)} \cdots A_{p-1,\sigma(p-1)} A_{p+1,\sigma(p+1)} \cdots A_{n,\sigma(n)}$$

$$\times\, dx^{\sigma(1)} \wedge \cdots dx^{\sigma(p-1)} \wedge dx^{\sigma(p+1)} \wedge \cdots \wedge dx^{\sigma(n)}$$

$$= \left(\sum_{\sigma \in S_n \mid \sigma(p)=r} (-1)^{r-p} \operatorname{sgn} \sigma\, A_{1,\sigma(1)} \cdots A_{p-1,\sigma(p-1)} A_{p+1,\sigma(p+1)} \cdots A_{n,\sigma(n)} \right)$$

$$\times\, dx^1 \wedge \cdots dx^{r-1} \wedge dx^{r+1} \wedge \cdots \wedge dx^n$$

$$= a_{pr}\, dx^1 \wedge \cdots dx^{r-1} \wedge dx^{r+1} \wedge \cdots \wedge dx^n,$$

where we have used the formula for the (p, r)th minor. Then

$$\alpha^{(1)} \wedge \cdots \wedge \alpha^{(n)} = (-1)^{p-1} \sum_{r=1}^{n} A_{pr} a_{pr}\, dx^r \wedge dx^1 \wedge \cdots dx^{r-1} \wedge dx^{r+1} \wedge \cdots \wedge dx^n$$

$$= \sum_{r=1}^{n} (-1)^{p-1} (-1)^{r-1} A_{pr} a_{pr}\, dx^1 \wedge \cdots \wedge dx^n$$

$$= \sum_{r=1}^{n} (-1)^{p-r} A_{pr} a_{pr}\, dx^1 \wedge \cdots \wedge dx^n.$$

Since, from the first part of the question,

$$\alpha^{(1)} \wedge \cdots \wedge \alpha^{(n)} = \det A\, dx^1 \wedge \cdots \wedge dx^n,$$

it follows that

$$\sum_{r=1}^{n} (-1)^{p-r} A_{pr} a_{pr} = \det A.$$

Given $s \neq p$, let B be the matrix obtained by replacing the sth column of A by the pth column of A. Since the pth and sth columns of B are the same, it follows that $\det B = 0$. Therefore,

$$\sum_{r=1}^{n} B_{pr} b_{rp} = 0.$$

But $B_{pr} = A_{pr}$ while $b_{rp} = \pm a_{rs}$ (the submatrix obtained by deleting the rth row and sth column of A and the submatrix obtained by deleting rth row and pth column of B are the same up to a permutation of columns). Therefore,

$$\sum_{r=1}^{n} A_{pr} a_{rs} = 0, \quad s \neq p.$$

The two preceding results combine to give

$$\sum_{r=1}^{n} (-1)^{p-r} A_{pr} a_{sr} = \delta_{p,s} \det A.$$

(c) Let $u : \mathbb{R}^n \to \mathbb{R}^n$, and let

$$A^i_j(x) = \frac{\partial u^i}{\partial x^j}.$$

Let

$$\beta = \frac{1}{n} \sum_{i=1}^{n} (-1)^{i-1} u^i\, du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^n.$$

Then

$$d\beta = \frac{1}{n} \sum_{i=1}^{n} (-1)^{i-1} du^i \wedge du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^n = \frac{1}{n} \sum_{i=1}^{n} du^1 \cdots \wedge \cdots du^n = du^1 \cdots \wedge \cdots du^n.$$

But

$$du^i = \frac{\partial u^i}{\partial x^j} dx^j = A^i_j dx^j.$$

Use the result of the preceding question to conclude that

$$d\beta = du^1 \cdots \wedge \cdots du^n = \det A\, dx^1 \wedge \cdots \wedge dx^n.$$

6. (a) In this case, we have that

$$\alpha = \tfrac{1}{2}(dx^1 \wedge dx^2 - dx^2 \wedge dx^1) = dx^1 \wedge dx^2,$$

which implies that $\operatorname{pf} A = 1$.

(b) A is given by

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 5 & 0 \end{pmatrix}.$$

In this case, we have that

$$\alpha = \tfrac{1}{2}(3\,dx^1 \wedge dx^2 - 3\,dx^2 \wedge dx^1 - 5\,dx^3 \wedge dx^4 + 5\,dx^4 \wedge dx^3) = 3\,dx^1 \wedge dx^2 - 5\,dx^3 \wedge dx^4.$$

Then

$$\begin{aligned} \frac{1}{2}\alpha \wedge \alpha &= \frac{1}{2}(3\,dx^1 \wedge dx^2 - 5\,dx^3 \wedge dx^4) \wedge (3\,dx^1 \wedge dx^2 - 5\,dx^3 \wedge dx^4) \\ &= \frac{1}{2}\left(-15\,dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 - 15\,dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2\right) = -15 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4, \end{aligned}$$

which implies that $\operatorname{pf} A = -15$.

(c) Under the substitution $A \mapsto \lambda A$, we have that $\alpha \mapsto \lambda\alpha$, so that $\underbrace{\alpha \wedge \cdots \wedge \alpha}_{n \text{ times}} \mapsto \lambda^n \underbrace{\alpha \wedge \cdots \wedge \alpha}_{n \text{ times}}$, so that $\operatorname{pf} A \mapsto \lambda^n \operatorname{pf} A$.

(d) Let $B = M^T A M$. Note that since A is antisymmetric, so is B. Let

$$\beta = \frac{1}{2} B_{ij} dx^i \wedge dx^j.$$

Then $\operatorname{pf} B$ is given by

$$\frac{1}{n!}\underbrace{\beta \wedge \cdots \wedge \beta}_{n \text{ times}} = \operatorname{pf} B\, dx^1 \wedge \cdots \wedge dx^{2n}. \tag{2}$$

We want to express $\operatorname{pf} B$ in terms of $\operatorname{pf} A$. To start, we express β in terms of A and M, as follows:

$$\beta = \frac{1}{2} B_{ij}\, dx^i \wedge dx^j = \frac{1}{2}\sum_{k,l} M^T_{ik} A_{kl} M_{lj}\, dx^i \wedge dx^j = \frac{1}{2}\sum_{k,l} M_{ki} A_{kl} M_{lj}\, dx^i \wedge dx^j$$

(the reason I'm writing in the sums over k and l explicitly is that k and l appear only as lower indices, so that the summation convention wouldn't apply to them). Let

$$dy^l := \sum_l M_{lj}\, dx^j.$$

Then

$$\sum_k M_{ki}\, dx^i = dy^k,$$

and we may write that

$$\beta = \tfrac{1}{2} A_{kl}\, dy^k \wedge dy^l.$$

Then from the definition of the Pfaffian,

$$\frac{1}{n!}\underbrace{\beta \wedge \cdots \wedge \beta}_{n \text{ times}} = \operatorname{pf} A\, dy^1 \wedge \cdots \wedge dy^{2n}.$$

But from Problem **??**,

$$dy^1 \wedge \cdots \wedge dy^{2n} = \det M\, dx^1 \wedge \cdots \wedge dx^{2n}.$$

Therefore,

$$\frac{1}{n!}\underbrace{\beta \wedge \cdots \wedge \beta}_{n \text{ times}} = \operatorname{pf} A \det M\, dx^1 \wedge \cdots \wedge dx^{2n}. \tag{3}$$

Comparing (2) and (3), we get that

$$\operatorname{pf} B = \det M \operatorname{pf} A,$$

as required.

(e) Let $T = A^T A = -A^2$. Clearly T is symmetric. This implies that T has real eigenvalues and that T has a complete set of eigenvectors which can be chosen to have unit length and to be mutually orthogonal. Moreover, for any $\mathbf{w} \in \mathbb{R}^{2n}$,

$$\mathbf{w} \cdot T\mathbf{w} = \mathbf{w} \cdot A^T A\mathbf{w} = (A\mathbf{w}) \cdot (A\mathbf{w}) = ||A\mathbf{w}||^2 \geq 0,$$

which implies that the eigenvalues of T cannot be negative. Suppose T has s distinct nonnegative eigenvalues, which we denote by $\lambda_1^2, \ldots, \lambda_s^2$, where we take the λ_j's to be nonnegative.

If $A = 0$, then $T = 0$, and T has a single distinct eigenvalue, namely $\lambda^2 = 0$. In this case, the result is immediate; for any orthonormal basis $\mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(n)}$, $\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(n)}$, we have that

$$A\mathbf{u}_{(j)} = \lambda_{r(j)}\mathbf{v}_{(j)} = 0, \quad A\mathbf{v}_{(j)} = -\lambda_{r(j)}\mathbf{u}_{(j)} = 0, \quad 1 \leq j \leq n.$$

If $A \neq 0$, then T has at least one strictly positive eigenvalue. Choose one of these – call it $\lambda_*^2 > 0$, and let $V_* \subset \mathbb{R}^{2n}$ denote the subspace of eigenvectors of T with eigenvalue λ_*^2. That is, $\mathbf{w} \in V_*$ if and only if $T\mathbf{w} = \lambda_*^2\mathbf{w}$. Take $\mathbf{u}_{(1)} \in V_*$. WLOG, we may assume that $\mathbf{u}_{(1)}$ is normalised, i.e. $||\mathbf{u}_{(1)}|| = 1$. Let

$$\mathbf{v}_{(1)} = \frac{1}{\lambda_*}A\mathbf{u}_{(1)}.$$

We claim that $||\mathbf{v}_{(1)}|| = 1$, since

$$||\mathbf{v}_{(1)}||^2 = \frac{1}{\lambda_*^2}(A\mathbf{u}_{(1)}) \cdot (A\mathbf{u}_{(1)}) = \frac{1}{\lambda_*^2}\mathbf{u}_{(1)} \cdot A^T A\mathbf{u}_{(1)} = \frac{1}{\lambda_*^2}\mathbf{u}_{(1)} \cdot T\mathbf{u}_{(1)} = \frac{1}{\lambda_*^2}\lambda_*^2\mathbf{u}_{(1)} \cdot \mathbf{u}_{(1)} = 1.$$

Also, we claim that $\mathbf{u}_{(1)}$ and $\mathbf{v}_{(1)}$ are orthogonal. Indeed,

$$\mathbf{u}_{(1)} \cdot \mathbf{v}_{(1)} = \frac{1}{\lambda_*}\mathbf{u}_{(1)} \cdot A\mathbf{u}_{(1)} = 0,$$

since A is antisymmetric (note that $\mathbf{w} \cdot A\mathbf{w} = 0$ for any vector $\mathbf{w}$). We have that

$$A\mathbf{u}_{(1)} = \lambda_*\mathbf{v}_{(1)},$$

by definition. Also,

$$A\mathbf{v}_{(1)} = \frac{1}{\lambda_*}A^2\mathbf{u}_{(1)} = -\frac{1}{\lambda_*}T\mathbf{u}_{(1)} = -\lambda_*\mathbf{u}_{(1)},$$

as required.

It may happen that T has just two linearly independent eigenvectors with eigenvalue λ_*^2, so that V_* is spanned by $\mathbf{u}_{(1)}$ and $\mathbf{v}_{(1)}$. If this is the case, the next part of the argument can be skipped.

Otherwise, choose a vector $\mathbf{u}_{(2)}$ in V_* which is orthogonal to both $\mathbf{u}_{(1)}$ and $\mathbf{v}_{(1)}$ and has length equal to one, and let

$$\mathbf{v}_{(2)} = \frac{1}{\lambda_*}A\mathbf{u}_{(2)}.$$

Arguing as above, we may conclude that

$$A\mathbf{u}_{(2)} = \lambda_*\mathbf{v}_{(2)}, \quad A\mathbf{v}_{(2)} = -\lambda_*\mathbf{u}_{(2)}.$$

Moreover, we claim that $\mathbf{v}_{(2)}$, like $\mathbf{u}_{(2)}$, is orthogonal to both $\mathbf{u}_{(1)}$ and $\mathbf{v}_{(1)}$. Indeed, we have that

$$\mathbf{v}_{(2)} \cdot \mathbf{u}_{(1)} = \frac{1}{\lambda_*}(A\mathbf{u}_{(2)}) \cdot \mathbf{u}_{(1)} = \frac{1}{\lambda_*}\mathbf{u}_{(2)} \cdot A^T\mathbf{u}_{(1)} = -\frac{1}{\lambda_*}\mathbf{u}_{(2)} \cdot A\mathbf{u}_{(1)} = -\mathbf{u}_{(2)} \cdot \mathbf{v}_{(1)} = 0,$$

and similarly

$$\mathbf{v}_{(2)} \cdot \mathbf{v}_{(1)} = \frac{1}{\lambda_*}(A\mathbf{u}_{(2)}) \cdot \mathbf{v}_{(1)} = \frac{1}{\lambda_*}\mathbf{u}_{(2)} \cdot A^T\mathbf{v}_{(1)} = -\frac{1}{\lambda_*}\mathbf{u}_{(2)} \cdot A\mathbf{v}_{(1)} = \mathbf{u}_{(2)} \cdot \mathbf{u}_{(1)} = 0.$$

In this way, we have constructed orthonormal vectors $\mathbf{u}_{(1)}$, $\mathbf{u}_{(2)}$, $\mathbf{v}_{(1)}$, $\mathbf{v}_{(2)}$ in V_* satisfying

$$A\mathbf{u}_{(j)} = \lambda_*\mathbf{v}_{(j)}, \quad A\mathbf{v}_{(j)} = -\lambda_*\mathbf{u}_{(j)}, \quad j = 1, 2.$$

If V_* is not spanned by $\mathbf{u}_{(1)}$, $\mathbf{u}_{(2)}$, $\mathbf{v}_{(1)}$, $\mathbf{v}_{(2)}$, we can find additional orthonormal vectors $\mathbf{u}_{(3)}$, $\mathbf{v}_{(3)}$ in V_* perpendicular to all four of $\mathbf{u}_{(1)}$, $\mathbf{u}_{(2)}$, $\mathbf{v}_{(1)}$, $\mathbf{v}_{(2)}$ and satisfying the preceding (through the same construction as for $\mathbf{u}_{(2)}$ and $\mathbf{v}_{(2)}$).

Preceding in this way, we construct an orthonormal basis $\mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(p)}$, $\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(p)}$ for V_* such that

$$A\mathbf{u}_{(j)} = \lambda_* \mathbf{v}_{(j)}, \quad A\mathbf{v}_{(j)} = -\lambda_* \mathbf{u}_{(j)}, \quad 1 \leq j \leq p.$$

If λ_*^2 is the only eigenvalue of T, i.e. if $T = \lambda_*^2 I$, then V_* is equal to all of $\mathbb{R}^{2n}$, and we are done.

Otherwise, suppose T has a different nonzero eigenvalue λ_{**}^2. Let $V_{**} \subset \mathbb{R}^{2n}$ denote the subspace of eigenvectors of T with eigenvalue λ_{**}^2. As two eigenvectors of a symmetric matrix with distinct eigenvalues are necessarily orthogonal, it follows that every vector in V_* is perpendicular to every vector in V_{**}. Proceeding as with V_*, we can construct an orthonormal basis for V_{**} of the form $\mathbf{u}_{(p+1)}, \ldots, \mathbf{u}_{(p+q)}$, $\mathbf{v}_{(p+1)}, \ldots, \mathbf{v}_{(p+q)}$ such that

$$A\mathbf{u}_{(j)} = \lambda_{**} \mathbf{v}_{(j)}, \quad A\mathbf{v}_{(j)} = -\lambda_{**} \mathbf{u}_{(j)}, \quad p+1 \leq j \leq p+q.$$

We may repeat this procedure for every distinct nonzero eigenvalue of T. In this way, we construct an orthonormal basis $\mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(t)}$, $\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(t)}$ for the subspace spanned by the eigenvectors of T associated with strictly positive eigenvalues, such that

$$A\mathbf{u}_{(j)} = \lambda_{r(j)} \mathbf{v}_{(j)}, \quad A\mathbf{v}_{(j)} = -\lambda_{r(j)} \mathbf{u}_{(j)}, \quad 1 \leq j \leq t,$$

where $\lambda_{r(j)}^2$ is a strictly positive eigenvalue of T.

Finally, we need to consider the case where 0 is an eigenvalue of T. Let $V_0 \subset \mathbb{R}^{2n}$ denote the subspace of null vectors of T. We note that V_0 is necessarily orthogonal to the space spanned by eigenvectors of T associated with strictly positive eigenvalues. Next, we claim that for all $\mathbf{w} \in V_0$, we have that $A\mathbf{w} = 0$. Indeed,

$$||A\mathbf{w}||^2 = (A\mathbf{w}) \cdot (A\mathbf{w}) = \mathbf{w} \cdot A^T A\mathbf{w} = \mathbf{w} \cdot T\mathbf{w} = 0,$$

since $T\mathbf{w} = 0$. It follows that $A\mathbf{w} = 0$. Next, let us note that V_0 is necessarily even dimensional. This is because the dimension of V_0 is equal to the dimension of $\mathbb{R}^{2n}$, namely $2n$, minus the dimension of the subspace spanned by the eigenvectors of T associated with strictly positive eigenvalues. As denoted above, this space is spanned by $\mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(t)}$, $\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(t)}$, and therefore has dimension $2t$. Therefore, $\dim V_0 = 2(n-t)$.

Let $\mathbf{u}_{(t+1)}, \ldots, \mathbf{u}_{(n)}$, $\mathbf{v}_{(t+1)}, \ldots, \mathbf{v}_{(n)}$ denote any orthonormal basis for V_0. Then

$$A\mathbf{u}_{(j)} = 0, \quad A\mathbf{v}_{(j)} = 0, \quad t+1 \leq j \leq n.$$

In this way, we have constructed the required orthonormal basis for $\mathbb{R}^{2n}$, denoted $\mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(n)}$, $\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(n)}$, such that

$$A\mathbf{u}_{(j)} = \lambda_{r(j)} \mathbf{v}_{(j)}, \quad A\mathbf{v}_{(j)} = -\lambda_{r(j)} \mathbf{u}_{(j)}, \quad 1 \leq j \leq n,$$

where $\lambda_{r(j)}^2$ is an eigenvalue of $A^T A$.

(f) Let $\mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(n)}$, $\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(n)}$ be an orthonormal basis for $\mathbb{R}^{2n}$ such that

$$A\mathbf{u}_{(j)} = \lambda_{r(j)} \mathbf{v}_{(j)}, \quad A\mathbf{v}_{(j)} = -\lambda_{r(j)} \mathbf{u}_{(j)}, \quad 1 \leq j \leq n,$$

where $\lambda_{r(j)}^2$ is an eigenvalue of $A^T A$. Let M be the orthogonal matrix whose columns are, taken in sequence from left to right, $\mathbf{u}_{(1)}$, $\mathbf{v}_{(1)}$, $\mathbf{u}_{(2)}$, $\mathbf{v}_{(2)}$, …, $\mathbf{u}_{(n)}$, $\mathbf{v}_{(n)}$. Then the preceding relations can be expressed as

$$AM = MB,$$

where B is the antisymmetric block diagonal matrix with 2×2 blocks of the form

$$\begin{pmatrix} 0 & \lambda_{r(j)} \\ -\lambda_{r(j)} & 0 \end{pmatrix}.$$

Equivalently, the only nonzero elements of B are given by

$$B_{2j-1,2j} = -B_{2j,2j-1} = \lambda_{r(j)}, \quad 1 \le j \le n.$$

For $n = 2$, B looks like this:

$$B = \begin{pmatrix} 0 & \lambda_{r(1)} & 0 & 0 \\ -\lambda_{r(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{r(2)} \\ 0 & 0 & -\lambda_{r(2)} & 0 \end{pmatrix}.$$

Since M is orthogonal, i.e. $M^T M = I$, we may write that

$$A = MBM^T.$$

Therefore,

$$\det A = \det B. \tag{4}$$

The determinant of a block diagonal matrix is the product of the determinants of the blocks, and

$$\det \begin{pmatrix} 0 & \lambda_{r(j)} \\ -\lambda_{r(j)} & 0 \end{pmatrix} = \lambda_{r(j)}^2.$$

Therefore,

$$\det B = \lambda_{r(1)}^2 \lambda_{r(2)}^2 \cdots \lambda_{r(n)}^2. \tag{5}$$

The fact that $A = MBM^T$ coupled with the preceding part (d) of this question implies that

$$\operatorname{pf} A = \det M \ \operatorname{pf} B = \pm \operatorname{pf} B,$$

since $\det M = \pm 1$, so that

$$(\operatorname{pf} A)^2 = (\operatorname{pf} B)^2. \tag{6}$$

Next, we compute $\operatorname{pf} B$ explicitly. We have that

$$\beta := \frac{1}{2} B_{ij} dx^i \wedge dx^j = \lambda_{r(1)}\, dx^1 \wedge dx^2 + \lambda_{r(2)}\, dx^3 \wedge dx^4 + \cdots + \lambda_{r(n)}\, dx^{(2n-1)} \wedge dx^{2n}.$$

It follows that

$$\frac{1}{n!} \underbrace{\beta \wedge \cdots \wedge \beta}_{n \text{ times}} = \lambda_{r(1)} \lambda_{r(2)} \cdots \lambda_{r(n)}\, dx^1 \wedge \cdots \wedge dx^{2n}.$$

From the definition of the pfaffian, we get that

$$\operatorname{pf} B = \lambda_{r(1)} \lambda_{r(2)} \cdots \lambda_{r(n)}.$$

Therefore,

$$(\operatorname{pf} B)^2 = \lambda_{r(1)}^2 \lambda_{r(2)}^2 \cdots \lambda_{r(n)}^2. \tag{7}$$

From (4), (5), (6) and (7), it follows that

$$(\operatorname{pf} A)^2 = (\operatorname{pf} B)^2 = \det B = \det A,$$

as required.

7. Let A denote the map from spherical polar to cartesian coordinates, ie

$$\begin{aligned} A^1(r, \theta, \phi) &= x(r, \theta, \phi) = r \sin\theta \cos\phi, \\ A^2(r, \theta, \phi) &= y(r, \theta, \phi) = r \sin\theta \sin\phi, \\ A^3(r, \theta, \phi) &= z(r, \theta, \phi) = r \cos\theta. \end{aligned}$$

We have that

$$(A^* x)(r, \theta, \phi) = x(r, \theta, \phi) = r \sin\theta \ \cos\phi.$$

Then

$$A^*(dx) = d(A^* x) = d(r \sin\theta \cos\phi) = \sin\theta \cos\phi\, dr + r \cos\theta \cos\phi\, d\theta - r \sin\theta \sin\phi\, d\phi.$$

Similarly,

$$\begin{aligned} A^*(dy) &= d(A^*y) = d(r\sin\theta\sin\phi) \\ &= \sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi, \\ A^*(dz) &= d(A^*z) = d(r\cos\theta) = \cos\theta\,dr - r\sin\theta\,d\theta. \end{aligned}$$

Then

$$A^*(dx\wedge dy\wedge dz) =$$
$$(\sin\theta\cos\phi\,dr + r\cos\theta\cos\phi\,d\theta - r\sin\theta\sin\phi\,d\phi)\wedge(\sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi)\wedge(\cos\theta\,dr - r\sin\theta\,d\theta).$$

The only surviving contributions are proportional to $dr\wedge d\theta\wedge d\phi$. Combining these contributions, we get that

$$\begin{aligned} &A^*(dx\wedge dy\wedge dz) \\ &\quad = \left(r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi\right) dr\wedge d\theta\wedge d\phi \\ &\quad = r^2\sin\theta\,dr\wedge d\theta\wedge d\phi, \end{aligned}$$

which we recognize as the volume element in spherical polar coordinates.

8. Let $\mathbb{R}^2 = \{(P,V)\}$. Suppose there is a 1-form $q = q_1 dP + q_2 dV$ and functions $U = U(P,V)$, $T = T(P,V)$ such that

$$\begin{aligned} dU &= q - P\,dV, \\ d\left(\frac{1}{T}q\right) &= 0. \end{aligned}$$

Then applying d to the first equation, we get that

$$dq = dP\wedge dV,$$

while the second equation gives that

$$-\frac{dT}{T^2}\wedge q + \frac{1}{T}dq = 0,$$

or

$$dq = \frac{1}{T}dT\wedge q.$$

Combinding the two previous equations, we get that

$$\frac{1}{T}dT\wedge q = dP\wedge dV.$$

9. Let a, b and c be three smooth functions on $\mathbb{R}^2 = \{(x,y)\}$ such that any two of the 1-forms da, db and dc are linearly independent. Starting with the 2-form $da\wedge db$, we express db in terms of dc and da to get a 2-form proportional to $da\wedge dc$. Then we express da in terms of db and dc to get a 2-form proportional to $db\wedge dc$. Finally we express dc in terms of da and db to get a 2-form proportional to $da\wedge db$. Comparing the first and last expression, we get the required identity. In detail,

$$\begin{aligned} da\wedge db &= da\wedge\left(\left(\frac{\partial b}{\partial c}\right)_a dc + \left(\frac{\partial b}{\partial a}\right)_c da\right) \\ &= \left(\frac{\partial b}{\partial c}\right)_a da\wedge dc = \left(\frac{\partial b}{\partial c}\right)_a \left(\left(\frac{\partial a}{\partial b}\right)_c db + \left(\frac{\partial a}{\partial c}\right)_b dc\right)\wedge dc \\ &= \left(\frac{\partial b}{\partial c}\right)_a \left(\frac{\partial a}{\partial b}\right)_c db\wedge dc = \left(\frac{\partial b}{\partial c}\right)_a \left(\frac{\partial a}{\partial b}\right)_c db\wedge\left(\left(\frac{\partial c}{\partial a}\right)_b da + \left(\frac{\partial c}{\partial b}\right)_a db\right) \\ &= \left(\frac{\partial b}{\partial c}\right)_a \left(\frac{\partial a}{\partial b}\right)_c \left(\frac{\partial c}{\partial a}\right)_b db\wedge da, \end{aligned}$$

which implies that

$$\left(\frac{\partial a}{\partial b}\right)_c \left(\frac{\partial b}{\partial c}\right)_a \left(\frac{\partial c}{\partial a}\right)_b = -1.$$