Fields, Forms and Flows 3/34

Solution Sheet 2

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1. f' is just the usual derivative, ie $f'(x) = 2ax + b$. f' vanishes for $x = -b/2a$ ($a \neq 0$). For any other x, $f(x)$ is invertible in a small neighbourhood around x. Eg, take x_0 such that $x_0 - (-b/2a) = \delta > 0$. Let $V = f(B_\delta(x_0))$. Then there exists a $g : V \to B_\delta(x_0)$ with $g = f^{-1}$. In fact, the inverse is given by the quadratic formula,
$$x(y) = \frac{-b \pm \sqrt{b^2 - 4a(c-y)}}{2a},$$
where the quantity inside the square root is necessarily positive for $y \in V$ (check), and the sign in front of the square root is determined by requiring that $|x(y) - x_0| < \delta$.

2. Let $F(x, y) = (u(x, y), v(x, y))$, so that $u = x^2 - y^2$ and $v = 2xy$. F' is given by
$$F'(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix},$$
and $\det F'(x, y) = 4(x^2 + y^2)$, which vanishes if and only if $x = y = 0$. Next, solve for x and y in terms of u and v. We have $y = v/2x$; substitute this into the expression for u to get $u = x^2 - v^2/4x^2$, which leads to $x^4 - ux^2 - v^2/4$. We can solve for x^2 (there is only one solution for which x^2 is positive), and hence x, in terms of u and v. To get y in terms of u and v, use $y^2 = x^2 - u$ and substitute the expression for x^2. The result is
$$x = \pm\left(\frac{\sqrt{u^2+v^2}+u}{2}\right)^{1/2}, \quad y = \pm\left(\frac{\sqrt{u^2+v^2}-u}{2}\right)^{1/2}.$$
The expressions for x and y in terms of u and v are smooth away from $u = v = 0$ (in particular, they are smooth for $v = 0, u \neq 0$, where one of the arguments of the square root goes through zero). Thus, given $(x, y) \neq 0$, there is an open set U containing (x, y) and an open set $V = F(U)$ containing $(u(x, y), v(x, y))$ such that $F : U \to V$ is invertible and the inverse map $G : V \to U$ is smooth. However, even excluding the origin from the xy-plane and the uv-plane, F is not 1-1 on $\mathbb{R}^2 - \{(0, 0)\}$; (x, y) and $(-x, -y)$ get mapped to the same point.

3. (a) Suppose $f(a) = f(b)$, with $a \neq b$. Without loss of generality, assume $a < b$. Then Rolle's theorem implies that there exists some $c \in [a, b]$ such that $f'(c) = 0$, contrary to assumption. (b) f' is given by
$$f'(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix},$$
and $\det f'(x, y) = e^{2x} \neq 0$. Clearly $f(x, y)$ is not 1-1, since $f(x, y + 2\pi) = f(x, y)$. Thus, in more than one dimension, $\det f'(x) \neq 0$ is not enough to ensure that f is globally invertible. In general, the inverse of f may only be defined locally (ie, not on all of $\mathbb{R}^n$).

4. $f'(0)$ is the limit as $h \to 0$ of $(f(h) - f(0))/h$. Since $f(0) = 0$ by definition, $f'(0) = \lim_{h\to 0} f(h)/h = \lim_{h\to 0} \frac{1}{2} + h \sin(1/h) = \frac{1}{2}$. For $x \neq 0$, $f'(x) = \frac{1}{2} + 2x \sin(1/x) - \cos(1/x)$. Thus $f'(x)$ is not continuous at $x = 0$, as arbitrarily close to $x = 0$, $f'(x)$ assumes any value in $[-\frac{1}{2}, 1\frac{1}{2}]$. In particular, at $x_n = 1/(n\pi)$, f' is either positive (if n is odd) or negative (if n is even). This implies that f cannot be 1-1 in any neighbourhood of 0. For if a continuous function is $1 - 1$, then it is either strictly increasing or strictly decreasing; if the function is differentiable, its derivative cannot change sign.

5. If $\partial f/\partial y$ vanishes everywhere, then f depends only on x, and therefore is not 1-1. So we can assume there exists a point $(a, b) \in \mathbb{R}^2$ for which $\partial f/\partial y(a, b) \neq 0$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x, y) = (x, f(x, y))$. Then
$$F'(x,y) = \begin{pmatrix} 1 & 0 \\ \partial f/\partial x & \partial f/\partial y \end{pmatrix}(x,y).$$
Thus $\det F'(a, b) = \partial f/\partial y(a, b) \neq 0$. Let $f(a, b) = c$, so that $F(a, b) = (a, c)$. By the inverse function theorem, there exist open sets $U, V \subset \mathbb{R}^2$ with $(a, b) \in U$ and $(a, c) \in V$ such that $F : U \to V$ is 1-1 with smooth inverse. Since V is open, there is a $\delta > 0$ so that $(a + t, c) \in V$ for $|t| < \delta$. Let $(x(t), y(t)) = F^{-1}(a + t, c)$. Then $F(x(t), y(t)) = (a + t, c)$, so that $f(x(t), y(t)) = c$. Since F^{-1} is 1-1, the points $(x(t), y(t))$ are distinct for different values of t. Thus f is not 1-1.

6. (a) We can identify the space of 2×2 matrices with $\mathbb{R}^4$,

$$x = (a, b, c, d) \longleftrightarrow X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For the purposes of this solution, we'll denote vectors by small letters, eg, x, y, etc, and the corresponding matrices by capital letters, eg X, Y, etc.
Squaring a matrix, ie

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix},$$

corresponds to the map $F : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$F((a, b, c, d)) = (a^2 + bc, b(a+d), c(a+d), bc + d^2).$$

The derivative of this map is given by

$$F'((a, b, c, d)) = \begin{pmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{pmatrix}.$$

Let $x_0 = (2, 0, 0, -1)$, so that

$$X_0 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $Y_0 = F(x_0) = (4, 0, 0, 1)$, so that

$$Y_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have that

$$F'(x_0) = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

which is clearly nonsingular. Therefore, by the inverse function theorem, there exists an open neighbourhood U of x_0 and an open neighbourhood V of y_0 such that $F : U \to V$ is invertible with smooth inverse $G : V \to U$. U can be made arbitrarily small (so that points in U are arbitrarily close to x_0). Let K be the arbitrary 2×2 matrix in the problem. For ϵ sufficiently small, $y_0 + \epsilon k$ belongs to V, and $x = G(y_0 + \epsilon k)$ belongs to U, and therefore is close to x_0. The fact that $F(G(y_0 + \epsilon k)) = y_0 + \epsilon k$ implies that $X^2 = Y_0 + \epsilon K$, so X is the required square root.

(b) Let $x_0 = (1, 0, 0, -1)$, so that

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$F'(x_0) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

which is clearly singular. Therefore, the inverse function theorem does not apply in this case.
Let's try to find the asked-for square root directly. Taking K as in the suggestion with $\epsilon \neq 0$, we want to find (a, b, c, d) small such that

$$\begin{pmatrix} 1+a & b \\ c & -1+d \end{pmatrix}^2 = \begin{pmatrix} (1+a)^2 + bc & b(a+d) \\ c(a+d) & (-1+d)^2 + bc \end{pmatrix} = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$$

('small' means that (a, b, c, d) should go to zero as ϵ goes to zero).
Equating the off-diagonal elements we get that $c(a+d) = 0$ and $b(a+d) = \epsilon$, so that

$$c = 0.$$

Equating the diagonal elements and substituting $c = 0$ gives $(1+a)^2 = 1 = (-1+d)^2$, with solutions $a = 0$ or -2, $d = 0$ or 2. Since $a + d$ cannot vanish (since $b(a+d) = \epsilon$), we conclude that

$$\text{Either } a = 0, d = 2, b = \epsilon/2 \text{ or } a = -2, d = 0, b = -\epsilon/2.$$

In neither case is (a, b, c, d) small, so a square root near X_0 cannot be found.

7. Let $y(t) \in \mathbb{R}^4$ be given by $(x(t), \dot{x}(t), \ddot{x}(t), t)$. Then

$$\dot{y} = (\dot{x}, \ddot{x}, d^3x/dt^3, 1) = (y^2, y^3, \sin y^4(y^1y^3/(y^2)^2), 1).$$

That is, $y(t)$ satisfies the first-order autonomous system

$$\dot{y} = Y(y),$$

with

$$Y((y^1, y^2, y^3, y^4)) = (y^2, y^3, \sin y^4(y^1y^3/(y^2)^2), 1).$$

8. Given $\ddot{q} = 1$, let $x = (u, v) = (q, \dot{q})$. Then $\dot{x} = (\dot{u}, \dot{v}) = \mathbb{X}(x) = (v, 1)$. The equation $\dot{v} = 1$ has solution $v(t) = v_0 + t$. The equation $\dot{u} = v = v_0 + t$ has solution $u_0 + v_0t + t^2/2$. Therefore, the flow is given by

$$\Phi_t(u_0, v_0) = (u_0 + v_0t + t^2/2, v_0 + t),$$

or

$$\Phi_t(u, v) = (u + vt + t^2/2, v + t).$$

(It doesn't matter what we call the argument of Φ_t.)

Now verify the composition law:

$$\begin{aligned}(\Phi_s \circ \Phi_t)((u, v)) &= \Phi_s(\Phi_t((u, v)))\\ &= \Phi_s((u + vt + t^2/2, v + t))\\ &= (u + vt + t^2/2 + (v + t)s + s^2/2, v + t + s)\\ &= (u + v(s + t) + (s + t)^2/2, v + (s + t)) = \Phi_{s+t}((u, v)),\end{aligned}$$

as required.

9. Let $\dot{x} = x^2$. $x(t) = 0$ is the unique solution satisfying the initial condition $x(0) = x_0 = 0$ (since x^2 is smooth, solutions are unique). If $x_0 \neq 0$, then

$$\int_{x_0}^{x} \frac{dx'}{x'^2} = \int_0^t dt,$$

or

$$\frac{1}{x_0} - \frac{1}{x} = t.$$

Solve to get

$$x(t) = \frac{x_0}{1 - x_0t}.$$

The solution has a singularity at $T = 1/x_0$. For $x_0 < 0$, the solution is defined for $t \in (1/x_0, \infty)$ (the singularity is in the past). For $x_0 > 0$, the solution is defined for $t \in (-\infty, 1/x_0)$ (the singularity is in the future). If $x_0 = 0$, then $x(t) = 0$, and the solution is defined for all times.

Next, let $\dot{x} = x^2/(1 + x^2)$. If $x_0 = 0$, then $x(t) = 0$ is the unique solution (since $x^2/(1 + x^2)$ is smooth, solutions are unique). Integrate to get

$$\int_{x_0}^{x} \frac{(1 + x'^2)dx'}{x'^2} = \int_0^t dt,$$

or

$$\frac{1}{x_0} - \frac{1}{x} + x - x_0 = t.$$

Multiplying by x we obtain a quadratic equation in x which may be solved to obtain

$$x(t) = \Phi_t(x_0) = \tfrac{1}{2}\left(x_0 + t - 1/x_0 \pm \sqrt{(x_0 + t - 1/x_0)^2 + 4}\right).$$

Note that the sign of the square root is determined by requiring that $x(0) = x_0$ (the sign is equal to sign of x_0). This solution has no singularities and therefore is defined for all t.

10. Let

$$V(q) = \begin{cases} -\frac{3}{4}q^{4/3}, & q > 0, \\ 0, & q \leq 0. \end{cases}$$

Then $\ddot{q} = -V'(q)$ is equivalent to the first-order system

$$(\dot{x}^1, \dot{x}^2) = (x^2, -V'(x^1)),$$

where $x^1 = q$ and $x^2 = \dot{q}$. Let $q(t) = 0$. Clearly $q(t)$ satisfies the differential equation, as $\ddot{q} = 0 = V'(0)$, with initial conditions $q(0) = \dot{q}(0) = 0$. Now look for another solution of the form $q(t) = at^b$. Substituting into the differential equation gives $ab(b-1)t^{b-2} = a^{1/3}t^{b/3}$ for $t > 0$. Equating exponents and coefficients of t, we get that $b/3 = b-2$, or $b = 3$, and $ab(b-1) = a^{1/3}$, or $a = 6^{-3/2}$. What about $t < 0$? The preceding analysis does not apply, because while we still have that $\ddot{q} = t/\sqrt{6}$, we also have that $q(t) = 6^{-3/2}t^3 < 0$ for $t < 0$. But $V'(q) = 0$ for $q < 0$. Therefore, for $t < 0$, $q(t) = 6^{-3/2}t^3$ does not satisfy the differential equation $\ddot{q} = -V'(q)$. On the other hand, $q(t) = 0$ is obviously a solution for $t < 0$. Thus,

$$q(t) = \begin{cases} 0, & t \leq 0, \\ 6^{-3/2}t^3, & t > 0 \end{cases}$$

satisfies the differential equation. The initial conditions are $q(0) = \dot{q}(0) = 0$, the same as those satisfied by the trivial solution $q(0)$. In fact, there is a family of solutions parameterised by $t_0 \geq 0$ given by

$$q(t) = \begin{cases} 0, & t \leq t_0, \\ \left(\frac{1}{6}\right)^{3/2}(t-t_0)^3, & t > t_0, \end{cases}$$

all of which satisfy the initial conditions. Thus solutions of Newton's equation are not unique in this case. A particle at rest at the top of the hill with height $V(q)$ could start rolling down at any time; Newtonian mechanics does not tell us when. Note that the force $F(q) = -V'(q)$ is not smooth at $q = 0$. The singularity in the derivative $F'(q)$ at $q = 0$ is responsible for the nonuniqueness of the solution.

More generally, consider

$$\ddot{q} = \begin{cases} 0, & q \leq 0, \\ q^\alpha, & q > 0, \end{cases}$$

where $\alpha > 0$, with initial conditions $q(0) = \dot{q}(0) = 0$. We take $\alpha > 0$ to ensure continuity of $\ddot{q}$ at $q = 0$. Let $q = at^b$ for $t \geq 0$. Then we get that $ab(b-1)t^{b-2} = a^\alpha t^{\alpha b}$. Equating exponents yields $b = 2/(1-\alpha)$. To have $\dot{q}(0) = 0$, we must have $b > 1$. This implies that $2/(1-\alpha) > 1$, or $-1 < \alpha < 1$ (but we also have $\alpha > 0$). Equating coefficients yields $ab(b-1) = a^\alpha$, or

$$a = \left(\frac{(1-\alpha)^2}{2(1+\alpha)}\right)^{1/(1-\alpha)},$$

which is well defined for $0 < \alpha < 1$. Thus, for $0 < \alpha < 1$, we get a family of solutions parameterised by $t_0 > 0$,

$$q(t) = \begin{cases} 0, & t \leq t_0, \\ \left(\frac{(1-\alpha)^2}{2(1+\alpha)}\right)^{1/(1-\alpha)}(t-t_0)^{2/(1-\alpha)}, & t > t_0. \end{cases}$$