Topics in Modern Geometry

Problems class 13

1 Points of clarification

1. The finiteness of the generating set for Qu. 4 of Homework 3 follows from the fact that $\mathbb{C}[x_0, \ldots, x_n]$ is Noetherian, which in turn follows from Hilbert's basis theorem.
2. The *Zariski closure* of a set $X \subset \mathbb{P}^n$ is the smallest Zariski closed set $\overline{X} \subset \mathbb{P}^n$ (i.e. set of the form $\overline{X} = \mathbb{V}(I)$ for a homogenenous ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$) such that $X \subset \overline{X}$. Equivalently, $\overline{X} = \bigcap_{X \subset Z} Z$ where the intersection runs over all *Zariski closed* subsets $Z \subset \mathbb{P}^n$.
3. A *regular function* on a projective variety $X \subset \mathbb{P}^n$ is a rational function $f \in \mathbb{C}(X)$ which is regular at all points $p \in X$.
4. An *embedding* $f \colon X \to \mathbb{P}^n$ is a morphism from X into $\mathbb{P}^n$ which is an isomorphism from X onto the image $f(X) \subset \mathbb{P}^n$.

2 More problems

In addition to Homework sheet 3, you can use these problems as further practice questions. They are mostly taken from Reid's *Undergraduate algebraic geometry.*

Extension problems to Homework 3

1. (**Qu. 6**+)
   (a) Show that $\mathbb{P}^1 \times \mathbb{P}^1 \ncong \mathbb{P}^2$ by finding two lines $L_1, L_2 \subset \phi(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$ with $L_1 \cap L_2 = \emptyset$. (Compare with Bézout's theorem later on in the course.)
   (b) Try generalising your solution to the case of $\mathbb{P}^m \times \mathbb{P}^n$.
2. (**Qu. 7**+) Compare the closed sets $\mathbb{V}(I'), \mathbb{V}(\widetilde{I}) \subset \mathbb{P}^3$. What are their irreducible components?
3. (**Qu. 10**+) Find the equation of the image $\overline{\phi(L)} \subset \mathbb{P}^2$ where L is the line given by:

$$x + y + z = 0, \qquad x + y = 0, \qquad x = 0$$

Additional problems

4. Write down the three standard affine charts of $C \subset \mathbb{P}^2$ and find the intersection points of C with the coordinate axes of $\mathbb{P}^2$, where C is one of the following curves:
   (a) $y^2 z = x^3 + axz^2 + bz^3$ where $a, b \in \mathbb{C}$,
   (b) $x^2y^2 + y^2z^2 + z^2x^2 = 2xyz(x + y + z)$,
   (c) $xz^3 = (x^2 + z^2)y^2$.
5. Find $\operatorname{dom} \pi$ and $\operatorname{dom} \psi$ in Example 18.1 of the lecture notes.
6. Show that the map $f \colon \mathbb{P}^1 \to \mathbb{P}^d$ given by $f(u : v) = (u^d : u^{d-1}v : \ldots : uv^{d-1} : v^d)$ is an embedding. The image $f(\mathbb{P}^1) \subset \mathbb{P}^d$ is called a *rational normal curve of degree d.*
7. The *second Veronese embedding of* $\mathbb{P}^2$ is the map $v \colon \mathbb{P}^2 \to \mathbb{P}^5$ given by

$$v(a : b : c) = (a^2 : ab : ac : b^2 : bc : c^2).$$

   Show that v is an embedding of $\mathbb{P}^2$. Show that v sends lines in $\mathbb{P}^2$ to conics in $\mathbb{P}^5$, and conics in $\mathbb{P}^2$ to rational normal curves of degree 4 in $\mathbb{P}^5$.