Fields, Forms and Flows 3/34

Problem Sheet 9
Due: Wednesday 12 December
To hand in: FFF3: 1, 2(a), 3(b), 4(a)   FFF34: 2(a), 3(b), 4(a)(b)

 Problems marked * are not examinable (but are still worth trying!)

1. Let $A : \mathbb{R}^2 = \{(u, v)\} \to \mathbb{R}^3 = \{(x, y, z)\}$ be given by
$$A(u, v) = (u, v, F(u, v)).$$
Compute A^*dx, A^*dy and A^*dz, and find a function f on $\mathbb{R}^3$ such that $A^*df = 0$.

2. Coordinate-independent formula for d.

   (a) Let ω be a 1-form on $\mathbb{R}^n$. Show that
   $$d\omega(\mathbb{X}, \mathbb{Y}) = L_{\mathbb{X}}(\omega(\mathbb{Y})) - L_{\mathbb{Y}}(\omega(\mathbb{X})) - \omega([\mathbb{X}, \mathbb{Y}])$$
   for any smooth vector fields $\mathbb{X}, \mathbb{Y}$. (Hint: it suffices (explain why) to consider the case $\omega = f\, dg$.)

   (b) Show by induction that, for ω a k-form on $\mathbb{R}^n$,
   $$\begin{aligned} d\omega(\mathbb{X}_{(1)}, \dots, \mathbb{X}_{(k+1)}) = &\sum_{i=1}^{k+1} (-1)^{i+1} L_{\mathbb{X}_{(i)}} \left(\omega(\mathbb{X}_{(1)}, \dots, \widehat{\mathbb{X}_{(i)}}, \dots, \mathbb{X}_{(k+1)}) \right) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega \left([\mathbb{X}_{(i)}, \mathbb{X}_{(j)}], \mathbb{X}_{(1)}, \dots, \widehat{\mathbb{X}_{(i)}}, \dots, \widehat{\mathbb{X}_{(j)}}, \dots, \mathbb{X}_{(k+1)} \right), \end{aligned}$$
   where the caret denotes an argument which is to be omitted. (Hint: for the induction step, consider k-forms of the form $df \wedge \alpha$, where α is a $(k-1)$-form.)

   (c) Let ω be a nonvanishing 1-form on $\mathbb{R}^n$. Show that $\omega = f\, dg$ for some functions f and g if and only if, for any two vector fields $\mathbb{X}$ and $\mathbb{Y}$ for which $\omega(\mathbb{X}) = \omega(\mathbb{Y}) = 0$, we have that $\omega([\mathbb{X}, \mathbb{Y}]) = 0$. (Hint: For "only if", note that $L_{[\mathbb{X},\mathbb{Y}]}g = L_{\mathbb{X}}L_{\mathbb{Y}}g - L_{\mathbb{Y}}L_{\mathbb{X}}g$. For "if", use the Frobenius theorem to produce a function g such that $\omega(\mathbb{X}) = 0 \iff dg(\mathbb{X}) = 0$ – Section 1.12.2 of the lecture notes on alternative versions of the Frobenius theorem may be useful.)

3. Let μ be a nonvanishing n-form on $\mathbb{R}^n$. Given a smooth vector fields $\mathbb{X}$ on $\mathbb{R}^n$, the *divergence* of $\mathbb{X}$ with respect to μ, denoted $\operatorname{div}_\mu \mathbb{X}$, is the function on $\mathbb{R}^n$ defined by

$$L_{\mathbb{X}}\mu = (\operatorname{div}_\mu \mathbb{X})\,\mu.$$

   (a) Let Φ_t denote the flow of $\mathbb{X}$. Show that

$$\Phi_t^*\mu = \mu \quad \forall t \iff \operatorname{div}_\mu \mathbb{X} = 0.$$

   (b) For

$$\mu = dx^1 \wedge \cdots \wedge dx^n,$$

   show that

$$\operatorname{div}_\mu \mathbb{X} = \frac{\partial \mathbb{X}^i}{\partial x^i}.$$

   (Hint: There are different ways to show this. You can use $L_{\mathbb{X}} dx^j = dL_{\mathbb{X}} x^j = d\mathbb{X}^j$, or start from the homotopy formula and show that $L_{\mathbb{X}}\mu = d(i_{\mathbb{X}}\mu)$.)

4. Kelvin-Helmholtz Theorem.

   (a) Let $\omega = \omega_i dx^i$ be a 1-form and $\mathbb{X} = \mathbb{X}^i e_{(i)}$ a vector field on $\mathbb{R}^n$. Show that

$$L_{\mathbb{X}}\omega = \left(\frac{\partial \mathbb{X}^j}{\partial x^i}\omega_j + \mathbb{X}^j \frac{\partial \omega_i}{\partial x^j}\right) dx^i.$$

   (b) Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{v}_t}{\partial t} + (\mathbf{v}_t \cdot \nabla)\mathbf{v}_t = -\nabla p_t,$$

   where $\mathbf{v}_t = \mathbf{v}_t(r)$ is the fluid velocity and $p_t = p_t(r)$ is the pressure. Let $\nu_t(r)$ be the 1-form associated to $\mathbf{v}_t(r)$, ie

$$\nu_t = \sum_{i=1}^{3} v_t^i dr^i.$$

   Show that

$$\frac{\partial \nu_t}{\partial t} + L_{\mathbf{v}_t}\nu_t = -d(p_t - \tfrac{1}{2}v_t^2),$$

   where $v_t^2 = \mathbf{v}_t \cdot \mathbf{v}_t$.

   (c) Let

$$\omega_t = d\nu_t.$$

   The vector field associated to ω_t, namely $\nabla \times \mathbf{v}_t$, is called the vorticity. Show that

$$\frac{\partial \omega_t}{\partial t} + L_{\mathbf{v}_t}\omega_t = 0.$$

(d) Suppose that $\hat{\Phi}_t : \mathbb{R}^3 \to \mathbb{R}^3$ is a one-parameter family of diffeomorphisms (not necessarily a subgroup) which satisfies the system of differential equations

$$\frac{\partial \hat{\Phi}_t}{\partial t} = \mathbf{v}_t \circ \hat{\Phi}_t.$$

$\hat{\Phi}_t(r)$ gives the position at time t of a fluid element which started at r at $t = 0$. Show that

$$\frac{\partial}{\partial t}\left(\hat{\Phi}_t^* \omega_t\right) = 0.$$

[This result is called the Helmholtz-Kelvin circulation theorem. A consequence is that vortex rings – domains isolating regions of nonvanishing vorticity – are carried along (or advected) by the fluid while remaining intact (like smoke rings). This discovery formed the basis of Maxwell's vortex theory of atoms (he proposed that atoms were knotted vortex rings in the aether - distinct knots to correspond to distinct chemical elements). This in turn initiated, through the work of Tait, the discipline of knot theory.]

5. Poincaré Lemma examples.

(a) Let $\mathbf{B}(r)$ be a smooth vector field on $\mathbb{R}^3$ and suppose that $\nabla \cdot \mathbf{B} = 0$. Write down an expression for a vector field $\mathbf{A}(r)$ for which

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Find $\mathbf{A}$ for the case where $\mathbf{B}$ is constant.

(b) Let $\rho(r)$ be a smooth function on $\mathbb{R}^3$. Find a vector field $\mathbf{E}(r)$ on $\mathbb{R}^3$ such that

$$\rho = \nabla \cdot \mathbf{E}.$$

Find $\mathbf{E}$ for the case where ρ is constant.