Topics in Modern Geometry

Problem sheet 4: Tangent spaces, singularities & curves

Due Thurs 8 November at 12pm

Problems (4), (5), and (6) will be marked for credit (5 marks each), so it is only necessary to hand in solutions to these problems. However, you are strongly advised to attempt all problems and you may also hand in solutions to these to receive comments on your work.

Warm-up problems

1. Show that the hypersurface $X_d = \mathbb{V}(x_0^d + x_1^d + \ldots + x_n^d) \subset \mathbb{P}^n$ is nonsingular $\forall n \geq 1$ and $\forall d \geq 1$.
2. Find the equation of T_pC, where $C = \mathbb{V}(y - x^2, z - x^3) \subset \mathbb{A}^3$ and $p = (\lambda, \lambda^2, \lambda^3)$ for some $\lambda \in \mathbb{C}$.
3. Find the tangent line $T_{p_1}C$ to the curve $C = \mathbb{V}(y^2+y-x^3+x^2) \subset \mathbb{A}^2$ at the point $p_1 = (0,0) \in C$. Now find the other point of intersection p_2 in $T_{p_1}C \cap C$. Find p_3, p_4, p_5 by repeating this procedure, using $T_{p_i}C \cap C$ to find p_{i+1}. What do you notice about p_5? Sketch the curve C in $\mathbb{R}^2$, mark the points $p_1, \ldots, p_5$ and draw all the tangent lines that you've found.

Assessed problems

4. Given a nonsingular plane cubic curve $C = \mathbb{V}(y^2z - x^3 - axz^2 - bz^3) \subset \mathbb{P}^2$ and the point $p = (0:1:0) \in C$, show that there are exactly four tangent lines to C which pass through p.
5. For which values of $a \in \mathbb{C}$ is the following hypersurface singular? Find $\operatorname{sing}(X)$ in each case.
$$X = \mathbb{V}\left(y^2 - x^3 + 3a^2x - 2a(a-2)\right) \subset \mathbb{A}^2_{x,y}$$
6. Verify that Bézout's theorem holds for the following two plane curves $C_1, C_2 \subset \mathbb{P}^2$, where
$$C_1 = \mathbb{V}(y^2z - x^2(x+z)) \quad \text{and} \quad C_2 = \mathbb{V}(y^2 - (x+y)(x+z)).$$

Additional problems

7. Following on from Qu. 2, show that all of the tangent lines to C that you have found are contained in the hypersurface $S = \mathbb{V}(3x^2y^2 - 4x^3z - 4y^3 + 6xyz - z^2)$ and show that $\operatorname{sing}(S) = C$.
8. Find $X_d = \{p \in X : \dim T_pX \geq d\}$ for $d = 1,2,3,4$, where X is the affine variety:
$$X = \mathbb{V}(xy - z^2 - t^3, tz - x^5) \subset \mathbb{A}^4_{x,y,z,t}$$
(Note that the Jacobian matrix $J = \operatorname{Jac}(\mathbb{I}(X))$ has rank $< r$ if and only if all of the $r \times r$-sized submatrices of J have determinant 0.) From this description, what are $\dim(X)$ and $\operatorname{sing}(X)$?
9. Given the nonsingular projective plane conic $C = \mathbb{V}(xz - y^2) \subset \mathbb{P}^2$ find the equation of another nonsingular plane conic $C' \subset \mathbb{P}^2$ such that C and C' intersect in exactly
   (a) 3 points with multiplicities $1, 1, 2$,
   (b) 2 points with multiplicities $1, 3$,
   (c) 1 point with multiplicity 4.

   (*Hint*: You could first try finding a singular conic C'' with the right intersection multiplicities and then consider curves in the pencil $|C, C''|$.)
10. Let $C = \mathbb{V}(f) \subset \mathbb{P}^2$ be a nonsingular projective plane curve. By following a similar argument to the derivation of the formula for a tangent line in Lecture 14, show that the *inflection points* of C (i.e. the points $p \in C$ for which the tangent line T_pC intersects C with multiplicity ≥ 3) are given by $C \cap \mathbb{V}(H_f)$, where H_f is the *Hessian*:
$$H_f = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{0 \leq i,j \leq 2}$$