Topics in Modern Geometry 3 2018

Final assessment

Due Thursday 6 December at 12pm

There are two parts to this assessment, each worth 25 marks. Please answer both parts and hand you solutions into the locked box by the deadline above.

Part I: topology and affine varieties

1. (**5 marks**) Let $(X, \mathcal{T})$ be a topological space. We say that X is **compact** if, for any collection $\mathcal{U} \subseteq \mathcal{T}$ of open sets that "cover" X, in the sense that
$$X = \bigcup_{U \in \mathcal{U}} U,$$
there exists a finite subcollection $\{U_1, \ldots, U_k\} \subseteq \mathcal{U}$ that also cover X, that is,
$$X = \bigcup_{j=1}^{k} U_j.$$
   (a) Let X be a compact topological space, and let Y be any closed set of X. Prove that Y with the subspace topology is compact.
   (b) Prove that $\operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n])$ is compact.

2. (**5 marks**) Consider a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$. The graph of f is defined to be the variety in $\mathbb{A}^{n+1}$ defined by
$$G_f := \{x_{n+1} = f(x_1, \ldots, x_n)\} \subseteq \mathbb{A}^{n+1}_{x_1,\ldots,x_{n+1}}.$$
More generally, for a rational function $F = \frac{f}{g} \in \mathbb{C}(x_1, \ldots, x_n)$ written so that f and g have no common factor, the graph of F is
$$G_F := \{g(x_1, \ldots, x_n)x_{n+1} = f(x_1, \ldots, x_n)\} \subseteq \mathbb{A}^{n+1}_{x_1,\ldots,x_{n+1}}.$$
   (a) Fix $f \in \mathbb{C}[x_1, \ldots, x_n]$. Find an isomorphism $\mathbb{A}^n \to G_f$, and prove that it is an isomorphism.
   (b) Show that (a) does not hold true for rational functions by finding an example of a rational function $F \in \mathbb{C}(x)$ such that G_F is not isomorphic to $\mathbb{A}^1$.

3. (**5 marks**) Let $J = (x^2y, x - z - 1) \leq \mathbb{C}[x, y, z]$, and let $W = \mathbb{V}(J) \subseteq \mathbb{A}^3_{x,y,z}$.
   (a) Show that the polynomial
$$f(x, y, z) = (x - 2)(x - z) + x^2(y + 1) + 5x - 2z + 3$$
   is equal in $\mathbb{C}[W]$ to $x^2 + ax + b$ for some $a, b \in \mathbb{C}$, and determine a and b.
   (b) Decompose W into irreducible components irredundantly (that is, in such a way that no variety in the decomposition is a subvariety of any of the others).
   (c) Show that f is not constant on W, but f is constant on one of the irreducible components of W.

4. (**5 marks**) Consider the affine variety
$$X = \mathbb{V}(y^2 - xz, z^2 - y^3) \subset \mathbb{A}^3.$$
Decompose X into irreducible components irredundantly. Prove that all the irreducible components are birationally equivalent to $\mathbb{A}^1$.

5. (**5 marks**) Let $X \subseteq \mathbb{A}^m_{x_1,\ldots,x_m}$ and $Y \subseteq \mathbb{A}^n_{y_1,\ldots,y_n}$ be irreducible varieties. Consider the direct product $Z = X \times Y \subseteq \mathbb{A}^{m+n}$.
   (a) Prove that Z is an affine variety.
   (b) Prove that, if X and Y are irreducible, then Z is irreducible.

Part II: projective varieties and curves

6. For the affine curves defined by the following polynomials write down their projective closure, determine their points at infinity and find all of their singular points:

   (a) (**3 marks**) $2x^2y^2 = x^2 + y^2$,

   (b) (**3 marks**) $y^2 = x^4 + 4ax + 3$ for some constant $a \in \mathbb{C}$.

7. (**5 marks**) Let $X = \mathbb{V}(f) \subset \mathbb{P}^n$ be a projective hypersurface defined by a nonconstant homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_m]$ and let $L \subset \mathbb{P}^n$ be a projective line (i.e. L is defined by $n-1$ linearly independent homogeneous polynomials of degree 1). Show that $X \cap L \neq \emptyset$.

8. Let $C = \mathbb{V}(f(x,y)) \subset \mathbb{A}^2$ be the affine curve given by the equation

$$f(x,y) = (x^2-3)^2 + (y^2-3)^2 - 8.$$

   (a) (**2 marks**) Show that $(1,1)$ is a nonsingular point of C.

   (b) (**3 marks**) By rewriting f as

$$f(x,y) = (x-1)^4 + 4(x-1)^3 - 8(x-1) + (y-1)^4 + 4(y-1)^3 - 8(y-1),$$

   or otherwise, show that the rational function $\phi(x,y) = \frac{y-1}{x-1}$ is regular at $(1,1)$ and find the value of $\phi(1,1)$.

   (c) (**4 marks**) By using $x-1$ as a uniformiser at $(1,1)$, show that $\psi(x,y) = x + y - 2$ is a regular function at $(1,1)$ with order of vanishing $v_{(1,1)}(\psi) = 4$.

9. (**5 marks**) Use the Cayley–Bacharach theorem to prove the following result:

   Suppose that $p_1, \ldots, p_6$ are the vertices of a hexagon H drawn inside a conic $C \subset \mathbb{P}^2$, with edges $E_1 = \overline{p_1p_2}$, $E_2 = \overline{p_2p_3}, \ldots, E_6 = \overline{p_6p_1}$. Then the three points q_1, q_2, q_3 are collinear, where $q_i = E_i \cap E_{i+3}$ is the intersection point for a pair of opposite sides of H.