Products of Varieties

Mathematics 683, Fall 2013

In this note we prove the existence of the product of two varieties, by solving problems 3.15 and 3.16 of Hartshorne. In problem 3.15, the case of two affine varities is considered, and in 3.16, the general case is done. The affine case is used in the general case, and we will see that it is used by taking appropriate open covers of a variety consisting of affine varieties.

3.15. *Products of Affine Varieties.* Let $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ be affine varieties.

(a) Show that $X \times Y \subseteq \mathbf{A}^{n+m}$ with its induced topology is irreducible. The affine variety $X \times Y$ is called the product of X and Y. Note that its topology is in general not equal to the product topology.

(b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties, i.e., show (i) the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms, and (ii) given a variety Z, and the morphisms $Z \to X$, $Z \to Y$, there is a unique morphism $Z \to X \times Y$ making the appropriate diagram commutative.

(d) Show that $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Proof. We view $I(X) \subseteq k[x_1, \dots, x_n]$ and $I(Y) \subseteq k[y_1, \dots, y_m]$. To see that $X \times Y$ is closed, we point out that $X \times Y$ is the zero set of the ideal of $k[x_1, \dots, x_n, y_1, \dots, y_m]$ generated by I and I'. Suppose that $X \times Y$ is reducible, and say $X \times Y = A \cup B$ with A and B closed. For $y \in Y$, let $X_y = X \times \{y\}$. We have that X is homeomorphic to X_y via the map $x \mapsto (x, y)$, so X_y is irreducible. Since $X_y = (A \cap X_y) \cup (B \cap X_y)$, we have either $X_y \subseteq A$ or $X_y \subseteq B$. Let $V_1 = \{y \in Y \mid X_y \subseteq A\}$, and define V_2 accordingly. We have $Y = V_1 \cup V_2$, so since Y is irreducible, $Y = V_1$ or $Y = V_2$. If $Y = V_1$, then $X \times Y = \bigcup_y X_y = A$. In either case we see that $X \times Y$ is irreducible. Therefore, $X \times Y$ is an affine variety.

To see that $X \times Y$ is a product in the category of varieties, note that the two projection maps are morphisms since they are defined by polynomial functions. If $\varphi : Z \to X$ and $\psi : Z \to Y$ are morphisms of varieties, then the unique map making the appropriate diagram commute is $\sigma(P) = (\varphi(P), \psi(P))$ for $P \in Z$. This is a morphism since it is given by polynomial functions (as φ and ψ are given by polynomial equations). Thus, there is a unique morphism $\sigma : Z \to X \times Y$ making the following diagram commute.

For (b), note that the projections from $X \times Y$ to X and Y are morphisms since they are defined by polynomial functions. These projections induce ring homomorphisms from $A(X)$ and $A(Y)$ to $A(X \times Y)$. This then induces a ring homomorphism $\theta : A(X) \otimes_k A(Y) \to A(X \times Y)$, given on generators by $\theta(f \otimes g)(P, Q) = f(P)g(Q)$. This map is surjective since the coordinate functions $\overline{x_1}, \dots, \overline{y_m}$ get mapped to the coordinate functions of $X \times Y$, which generate $A(X \times Y)$. For injectivity, suppose that $\sum_{i=1}^{l} f_i \otimes g_i$ is in $\ker(\theta)$, with l minimal. Then $f_l \neq 0$. Pick $P \in X$ with $f_l(P) \neq 0$. We have $\sum f_i(P)g_i(Q) = 0$ for each $Q \in Y$. Therefore, $g_l = -f_l(P)^{-1} \sum f_i(P) g_i$. Using this relation, we obtain a shorter expression for $\sum f_i \otimes g_i$, a contradiction to the minimality of l. Therefore, θ is an isomorphism.

Finally, to verify that $\dim(X \times Y) = \dim(X) + \dim(Y)$, we show that $\dim(A(X \times Y)) = \dim(A(X)) + \dim(A(Y))$. By Noether normalization, $A(X)$ is integral over a polynomial ring $k[t_1, \dots t_u]$ and $A(Y)$ is integral over $k[s_1, \dots s_v]$ for some u and v. If $a \in A(X)$, then $a^r + \alpha_{r-1} a^{r-1} + \dots + \alpha_0 = 0$ for some $\alpha_i \in k[t_1, \dots, t_u]$. If $b \in A(Y)$, then

$$(a \otimes b)^r + (\alpha_{r-1} \otimes b)(a \otimes b)^{r-1} + \dots + \alpha_0 \otimes b^n = 0,$$

so $a \otimes b$ is integral over $k[t_1, \dots, t_u] \otimes A(Y)$. Since every element of $A(X) \otimes A(Y)$ is a sum of such elements, $A(X) \otimes A(Y)$ is integral over $k[t_1, \dots, t_u] \otimes A(Y)$. A similar calculation shows that $k[t_1, \dots, t_u] \otimes A(Y)$ is integral over $k[t_1, \dots, t_u] \otimes k[s_1, \dots, s_v]$. By transitivity, $A(X) \otimes A(Y)$ is integral over this ring. However, $k[t_1, \dots, t_u] \otimes k[s_1, \dots, s_v]$ is isomorphic to a polynomial ring in $u + v$ variables. Consequently, the quotient field of $A(X) \otimes A(Y)$ is algebraic over a purely transcendental extension $k(t_1, \dots, t_u, s_1, \dots, s_v)$ of k, so its transcendence degree over k is $u + v$. Therefore, by the dimension theorem we get that $\dim(X \times Y) = u + v = \dim X + \dim(Y)$.

3.16. *Products of Quasi–Projective Varieties.* Use the Segre embedding (Ex. 2.14) to identify $\mathbf{P}^n \times \mathbf{P}^m$ with its image and hence give it a structure of a projective variety. Now for any two quasi–projective varieties $X \subseteq \mathbf{P}^n$ and $Y \subseteq \mathbf{P}^m$, consider $X \times Y \subseteq \mathbf{P}^n \times \mathbf{P}^m$.

(a) Show that $X \times Y$ is a quasi–projective variety.

(b) If X, Y are both projective, show that $X \times Y$ is projective.

(c) Show that $X \times Y$ is a product in the category of varieties.

Proof. For (a) we have to show that $X \times Y$ is locally closed and irreducible. Let us call W the image of $X \times Y$ under the Segre embedding ψ. We then need to show that W is locally closed and irreducible. Let π_i be the two projection maps from W to X and Y respectively. That is, if $P \in X$ and $Q \in Y$, then π_1 is defined by $\pi_1(\psi(P, Q)) = P$, and π_2 is defined similarly. These maps are morphisms, since they are well defined, and we will see that they are defined locally by polynomials. If $P = (x_{ij}) \in W$, then $\pi_1(P) = (x_{0l}, x_{1l}, \dots, x_{nl})$ for any l for which some $x_{kl} \neq 0$, so π_1 is indeed a morphism. Note that $W = \pi_1^{-1}(X) \cap \pi_2^{-1}(Y)$. Each of these two sets are locally closed in $\mathbf{P}^n \times \mathbf{P}^m$ since X (resp. Y) is locally closed in $\mathbf{P}^n$ (resp. $\mathbf{P}^m$), and the π_i are continuous. Therefore, W is the intersection of a closed and an open set, so W is locally closed in $\mathbf{P}^n \times \mathbf{P}^m$. For irreducibility, suppose that $W = C_1 \cup C_2$ with each C_i closed in W. We verify that $W = C_i$ for some i by following the affine argument. Define a map $\varphi_y : X \to W$ by $\varphi_y(x) = \psi(x, y)$, for $y \in Y$ fixed, and let $X_y = \mathrm{im}(\varphi_y)$. Since $X_y = \pi_2^{-1}(\{y\})$, X_y is a closed subset of W, hence X_y is closed in $\mathbf{P}^n \times \mathbf{P}^m$. The map φ_y is defined by polynomials, so φ_y is a morphism. Furthermore, $\pi_1(\varphi_y(x)) = x$ and $\varphi_y(\pi_1(x, y)) = (x, y)$, so φ_y is an isomorphism. Therefore, X_y is a variety, and, in particular, X_y is irreducible. Since $X_y = (C_1 \cap X_y) \cup (C_2 \cap X_y)$, we have $X_y \subseteq C_i$ for some i. Let $Y_i = \{y \in Y \mid X_y \subseteq C_i\}$. Then $Y = Y_1 \cup Y_2$. Once we show that each Y_i is closed, irreducibility of Y will give $Y = Y_i$ for some i. This will then give $W = C_i$, and will finish the proof that W is irreducible. To see that Y_1 is irreducible, we have $Y_1 = \bigcap_{x \in X} \{y \in Y \mid \psi(x, y) \in C_1\}$. Let $T_x = \{y \in Y \mid \psi(x, y) \in C_1\}$, so $Y_1 = \bigcap_{x \in X} T_x$. For $x \in X$, define φ_x in a similar way as for φ_y. Then $\varphi_x(T_x) = \psi(\{x\} \times T_x) = C_1 \cap Y_x$ is closed in Y_x, so since φ_x^{-1} is a morphism, T_x is closed in Y. Thus, Y_1 is an intersection of closed sets, so Y is closed.

Suppose that X and Y are projective, so that X is closed in $\mathbf{P}^n$ and Y is closed in $\mathbf{P}^m$. Then, as above, we have $X \times Y = \pi_1^{-1}(X) \cap \pi_2^{-1}(Y)$. Each of these pieces is closed in $\mathbf{P}^n \times \mathbf{P}^m$ since the π_i are continuous, $X \subseteq \mathbf{P}^n$ is closed, and $Y \subseteq \mathbf{P}^m$ is closed. Therefore, this intersection is closed, so $X \times Y$ is a projective variety.

We now show that $X \times Y$ is a product in the category of varieties. Let Z be a variety, and suppose that there are morphisms $\sigma : Z \to X$ and $\tau : Z \to Y$. We want to show there is a unique morphism $\rho : Z \to X \times Y$ with $\pi_1 \circ \rho = \sigma$ and $\pi_2 \circ \rho = \tau$. We note that if ρ is any function with these two relations, then $\rho(P) = \psi(\sigma(P), \tau(P))$, so ρ is unique. We define ρ by this formula. What is more difficult is to show that ρ is a morphism. We verify that ρ is a morphism by looking at a convenient open cover of Z. Let $\{U_i\}$ and $\{V_i\}$ be open affine covers of X and Y. Let $Z_{ij} = \sigma^{-1}(U_i) \cap \tau^{-1}(V_j)$, an open subset of Z since σ and τ are continuous. Furthermore, the Z_{ij} cover Z since the U_i (resp. V_j) cover X (resp. Y). It suffices to show that $\rho|_{Z_{ij}}$ is a morphism for each i, j. We have the diagram

$$Z_{ij} \overset{\sigma\times\tau}{\to} U_i \times V_j \overset{\psi}{\to} \psi(U_i, V_j) \subseteq W.$$

The map $\sigma \times \tau$ is a morphism by the affine product result. Therefore, the composition is a morphism. But this is ρ on Z_{ij}. Therefore, ρ is a morphism.