# Fields, Forms and Flows 3/34

Problem Sheet 7
Due: Wednesday 28 November
To hand in: FFF3: 1, 2, 3(a)     FFF34: 1, 2, 3(a)

 Problems marked * are not examinable (but are still worth trying!)

1. Determine whether $s$ should be $+1$ or $-1$ in the formula

$$f^{(3)} \wedge f^{(8)} \wedge f^{(6)} \wedge f^{(4)} \wedge f^{(1)} = s f^{(1)} \wedge f^{(3)} \wedge f^{(4)} \wedge f^{(6)} \wedge f^{(8)}.$$

(Suggestion: effect the required permutation by a sequence of transpositions between adjacent factors $f^{(i_j)}$ and $f^{(i_{j+1})}$. That is, start by moving $f^{(1)}$ on the left-hand side to the "front", as follows:

$$\begin{aligned} f^{(3)} \wedge f^{(8)} \wedge f^{(6)} \wedge f^{(4)} \wedge f^{(1)} = -f^{(3)} \wedge f^{(8)} \wedge f^{(6)} \wedge f^{(1)} \wedge f^{(4)} = \\ = +f^{(3)} \wedge f^{(8)} \wedge f^{(1)} \wedge f^{(6)} \wedge f^{(4)} = \cdots \end{aligned}$$

Once $f^{(1)}$ is "where it belongs", move $f^{(3)}$ to the left in a similar fashion to where it belongs. Keep track of sign changes.)

2. Let $a = a_i f^{(i)}$ be an algebraic 1-form on $\mathbb{R}^n$ and $b = \frac{1}{2} b_{jk} f^{(j)} \wedge f^{(k)}$ be an algebraic 2-form on $\mathbb{R}^n$ with

$$a_i = \begin{cases} \alpha, & i = 1, \\ 0, & \text{otherwise} \end{cases}, \quad b_{jk} = \begin{cases} \beta, & j = 2, k = 3, \\ -\beta, & j = 3, k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $c = a \wedge b$. From the component formula for the wedge product,

$$c = \frac{1}{2!} a_i b_{jk} f^{(i)} \wedge f^{(j)} \wedge f^{(k)}. \tag{1}$$

We also have that

$$c = \frac{1}{3!} c_{ijk} f^{(i)} \wedge f^{(j)} \wedge f^{(k)}, \tag{2}$$

where $c_{ijk}$ is defined in the usual way, ie

$$c_{ijk} = c(e_{(i)}, e_{(j)}, e_{(k)}).$$

Show that

$$c_{123} = a_1 b_{23},$$

but that

$$c_{231} \neq a_2 b_{31}$$

so that, in general, $c_{ijk} \neq a_i b_{jk}$. Explain why Eqs. (1) and (2) are nevertheless compatible.

3. (a) Let $a$ be a nonvanishing $(n-1)$-form on $\mathbb{R}^n$. Find $(n-1)$ linearly independent algebraic 1-forms $g^{(1)}, g^{(2)}, \ldots, g^{(n-1)}$ on $\mathbb{R}^n$ such that $g^{(j)} \wedge a = 0$. (Hint: show that $f^{(j)} \wedge a$ cannot vanish for at least one $j$. WLOG, assume $j = n$. $g^{(j)}$ can be taken to be an appropriate linear combination of $f^{(j)}$ and $f^{(n)}$, whose coefficients will depend on the components of $a$.) Hence show that

$$a = \alpha g^{(1)} \wedge g^{(2)} \wedge \cdots \wedge g^{(n-1)}$$

for some $\alpha \in \mathbb{R}$, and give a formula for $\alpha$ in terms of the components of $a$. (For $n = 3$, this is equivalent to showing that an arbitrary vector $\mathbf{u}$ can be expressed as the cross-product $\mathbf{v} \times \mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$.)

(b) Suppose that $a$, $b$, $c$ and $d$ are linearly independent algebraic 1-forms on $\mathbb{R}^n$ (hence $n \geq 4$). Let

$$r = a \wedge b + c \wedge d.$$

Show that $r \wedge r \neq 0$. Hence show that one cannot find one-forms $g^{(1)}$ and $g^{(2)}$ for which $r = g^{(1)} \wedge g^{(2)}$.

4. Suppose that $v = v^1 e_{(1)} + v^2 e_{(2)} + v^3 e_{(3)} \in \mathbb{R}^3$.

(a) Let $a = a_1 f^{(1)} + a_2 f^{(2)} + a_3 f^{(3)}$ be a one-form on $\mathbb{R}^3$. Compute $i_v a$. Associating $v$ and $a$ to conventional vectors on $\mathbb{R}^3$ and $i_v a$ to a scalar, what operation in vector algebra does $i_v a$ correspond to?

(b) Let $b = b_{12} f^{(1)} \wedge f^{(2)} + b_{23} f^{(2)} \wedge f^{(3)} + b_{31} f^{(3)} \wedge f^{(1)}$ be a two-form on $\mathbb{R}^3$. Associating $v$, $b$ and $i_v b$ to conventional vectors on $\mathbb{R}^3$, what operation in vector algebra does $i_v b$ correspond to?

(c) Let $\rho = \rho_{123} f^{(1)} \wedge f^{(2)} \wedge f^{(3)}$ be a three-form on $\mathbb{R}^3$. Associating $v$ and $i_v \rho$ to conventional vectors on $\mathbb{R}^3$, show that these vectors are related by scalar multiplication.