Topics in Modern Geometry

Problem sheet 3: Projective varieties

Due 1 November at 12pm

Problems (4), (5), and (6) will be marked for credit (5 marks each), so it is only necessary to hand in solutions to these problems. However, you are strongly advised to attempt all problems and you may also hand in solutions to these to receive comments on your work.

Warm-up problems

1. Verify that the maps $\phi_i \colon U_i \to \mathbb{A}^n$ and $\psi_i \colon \mathbb{A}^n \to U_i$ (as defined in lectures for the standard affine open sets $U_i \subset \mathbb{P}^n$) are well-defined, bijective and are inverse to each other.

2. Consider $z \in \mathbb{C}$ as the point $(z : 1) \in \mathbb{P}^1$ and ∞ as the point $(1 : 0) \in \mathbb{P}^1$. Extend the usual addition and multiplication on $\mathbb{C}$ to $\mathbb{P}^1$ by setting $z + \infty = \infty$ if $z \neq \infty$, $z \cdot \infty = \infty$ if $z \neq 0$, $\infty^{-1} = 0$ and $0^{-1} = \infty$. Write down formulae for $x + y$, xy and x^{-1} in terms of the homogeneous coordinates $x = (x_1 : x_2)$ and $y = (y_1 : y_2)$.

3. Which of the following ideals are homogeneous?
$$\langle x + y^2 \rangle \subset \mathbb{C}[x, y], \quad \langle x^3 + 2y^2 z, z^2 + 3xy \rangle \subset \mathbb{C}[x, y, z], \quad \langle t + yz, x^4 + yz, t \rangle \subset \mathbb{C}[x, y, z, t]$$

Assessed problems

4. (**5 marks**) Show that an ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$ is homogeneous if and only if there is a finite generating set $I = \langle f_1, \dots, f_k \rangle$ where each f_i is homogeneous.

5. (**5 marks**) Suppose that $X \subset \mathbb{A}^n$ is an affine algebraic variety. Prove that the projective closure $\widetilde{X} \subset \mathbb{P}^n$ is equal to $\overline{X} \subset \mathbb{P}^n$, the Zariski closure of $X \subset \mathbb{P}^n$, where we think of X as a subset of $\mathbb{P}^n$ in the 0th standard affine chart $X \subset (\mathbb{A}^n \cong U_0) \subset \mathbb{P}^n$.

6. (**5 marks**) Show that the product variety $X = \mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the quadric hypersurface $Y = \mathbb{V}(z_0 z_3 - z_1 z_2) \subset \mathbb{P}^3$ under the morphism $\phi \colon X \to Y$
$$\phi\big((x_0 : x_1) \times (y_0 : y_1)\big) = (x_0 y_0 : x_0 y_1 : x_1 y_0 : y_0 y_1)$$
and describe the inverse morphism $\psi \colon Y \to X$. (The map ϕ is usually called the *Segre embedding* of $\mathbb{P}^1 \times \mathbb{P}^1$.)

Additional problems

7. Let $I = \langle x_2 - x_1^2, x_3 - x_1 x_2 \rangle$, let $\widetilde{I}$ be the homogenisation of I with respect to x_0, and let $I' = \langle x_0 x_2 - x_1^2, x_0 x_3 - x_1 x_2 \rangle$ be the ideal obtained by the homogenisation of the generators. Show that $I' \subsetneq \widetilde{I}$. (*Hint*: consider the polynomial $x_1 x_3 - x_2^2$.)

8. Prove that any irreducible factor of a homogeneous polynomial is homogeneous.

9. Prove that a regular function on $\mathbb{P}^1$ is constant. (*Hint*: Suppose $f \in \mathbb{C}(\mathbb{P}^1)$ is a regular function. Show that the restriction to $U_0 \simeq \mathbb{A}^1$ must be a polynomial $f|_{U_0} = p(\frac{x_1}{x_0})$. Now what does $f|_{U_1}$ look like?)

10. Show that the rational map $\phi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $\phi(x : y : z) = (yz : zx : xy)$ is birational. What is ϕ^{-1}? What is the locus where ϕ is an isomorphism? What is the locus where ϕ is defined? (The map ϕ is usually called the *Cremona transformation*.)