Fields, Forms and Flows 3/34

Problem Sheet 6
Due: Wednesday 21 November
To hand in: FFF3: 1, 3(b), 6, 7 FFF34: 1, 3(b), 6, 7

 Problems marked * are not examinable (but are still worth trying!)

1. (a) Let $e_{(1)}$ and $e_{(2)}$ be a basis for a two-dimensional vector space V, and let $f^{(1)}$ and $f^{(2)}$ be the associated dual basis on V^*. Let

$$\bar{e}_{(1)} = e_{(1)}, \quad \bar{e}_{(2)} = e_{(1)} + e_{(2)}.$$

Let $\bar{f}^{(1)}$ and $\bar{f}^{(2)}$ denote the dual basis to the $\bar{e}_{(j)}$'s. Express $\bar{f}^{(1)}$ and $\bar{f}^{(2)}$ as linear combinations of $f^{(1)}$ and $f^{(2)}$.

(b) Let $e_{(1)}, \ldots, e_{(n)}$ and $\bar{e}_{(1)}, \ldots, \bar{e}_{(n)}$ be two bases for V, and suppose

$$\bar{e}_{(i)} = \sum_{j=1}^{n} M_{ij} e_{(j)},$$

where M is an $n \times n$ matrix. Let $f^{(j)}$ and $\bar{f}^{(j)}$ denote the dual bases of $e_{(i)}$ and $\bar{e}_{(i)}$ respectively, and let

$$\bar{f}^{(i)} = \sum_{j=1}^{n} N_{ij} f^{(j)}.$$

Show that

$$N = M^{T^{-1}}.$$

2. Let V and W denote vector spaces of dimensions n and m, respectively, and let $L(V, W)$ denote the set of linear maps from V to W.

(a) Show that $L(V, W)$ is a vector space.

(b) Show that $L(V, W)$ may be identified with the space of functions on $V \times W^*$ which are linear in each argument.

3. (a) Show that every permutation $\sigma \in S_n$ can be expressed as a product of transpositions.

(b) Write

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix}$$

as a product of transpositions.

4. Show that
$$\tau_{ij} = \sigma \tau_{12} \sigma^{-1},$$
where σ is any permutation for which $\sigma(1) = i$ and $\sigma(2) = j$.

5. Let $(i_1, \ldots, i_k)$ denote an (ordered) k-tuple of distinct integers in $\{1, \ldots, N\}$. Define a permutation $\sigma \in S_n$ by
$$\sigma(i_1) = i_2, \quad \sigma(i_2) = i_3, \quad \ldots, \quad \sigma(i_{k-1}) = i_k, \quad \sigma(i_k) = i_1,$$
$$\sigma(j) = j \ \text{ if } j \neq i_1, \ldots, i_k.$$
σ is called a k-*cycle*. Show that
$$\operatorname{sgn} \sigma = (-1)^{k-1}.$$
(Note that a transposition is a 2-cycle.)

6. Show that the permutation matrix $P(\sigma)$ is orthogonal, ie
$$P(\sigma)^{-1} = P(\sigma)^T,$$
where $P(\sigma)^T$ denotes the transpose of $P(\sigma)$. Hence show that $(\det P(\sigma))^2 = 1$, so that $\det P(\sigma) = \pm 1$. (It then follows that $\operatorname{sgn} \det P(\sigma) = \det P(\sigma)$.)

7. Given $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w} \in \mathbb{R}^3$, let
$$a(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$
Show that a is an algebraic 3-form on $\mathbb{R}^3$.