UNIVERSITY OF BRISTOL

Examination for the Degree of M.Sci. (Level M)

DIFFERENTIABLE MANIFOLDS 34

MATH M2900J

(Paper Code MATH-M2900J)

January 2017, 2 hours 30 minutes

This paper contains **FOUR** *questions, all of which will be used for assessment.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Unless stated otherwise, all functions, vector fields, differential forms and maps are taken to be smooth.

$$L_{\mathbb{X}} = i_{\mathbb{X}} d + d i_{\mathbb{X}}$$

$$(F_* \mathbb{X})\,(F(x)) = F'(x) \cdot \mathbb{X}(x)$$

$$\frac{\partial \Phi_t}{\partial t}(x) = \mathbb{X}(\Phi_t(x))$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\int_0^1 \frac{dt}{(a^2 + b^2 t^2)^{3/2}} = \frac{1}{a^2 (a^2 + b^2)^{1/2}}$$

$$\int_0^1 \frac{dt}{(a^2 + b^2 t^2)^{5/2}} = \frac{2}{3a^4 (a^2 + b^2)^{1/2}} + \frac{1}{3a^2 (a^2 + b^2)^{3/2}}$$

$$\partial c = \sum_{j=1}^{k} \sum_{\alpha = 0,1} (-1)^{j+\alpha} c_{(j,\alpha)}$$

$$i_{\mathbb{X}}(\alpha \wedge \beta) = (i_{\mathbb{X}} \alpha) \wedge \beta + (-1)^k \alpha \wedge (i_{\mathbb{X}} \beta), \text{ where } \alpha \text{ is a } k\text{-form.}$$

Do not turn over until instructed.

1. (25 marks)

   (a) (12 marks) Let α be the 1-form and β the 2-form on $\mathbb{R}^3$ given by

   $$\alpha = y^2\, dx - x^2\, dy,$$
   $$\beta = z\, dy \wedge dz,$$

   and let $\mathbb{X}$ be the vector field on $\mathbb{R}^3$ given by

   $$\mathbb{X} = (e^y, e^x, 0).$$

   i. Compute $\alpha \wedge \beta$, combining terms where possible.
   ii. Compute $d\alpha$, combining terms where possible.
   iii. Compute $i_{\mathbb{X}}\alpha$, combining terms where possible.
   iv. Compute $L_{\mathbb{X}}\alpha$, combining terms where possible.

   (b) (13 marks) Let $\mathbb{X}$ be the vector field on $\mathbb{R}^3$ given by

   $$\mathbb{X}(x, y, z) = (1, x, xy).$$

   Let $U = \{(x, y, z) \,|\, y \neq 0, z \neq 0\} \subset \mathbb{R}^3$, and let $F : U \to U$ be the diffeomorphism given by

   $$F(x, y, z) = (xyz, yz, z).$$

   i. Compute the flow of $\mathbb{X}$.
   ii. Compute $F_*\mathbb{X}$.
   iii. Let Ψ_t denote the flow of $F_*^j\mathbb{X}$, where F_*^j denotes F_* applied j times. Compute $\Psi_2(0, -1, 1)$.

Continued...

2. (25 marks)

(a) (4 marks) Let σ be the permutation given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix},$$

ie $\sigma(1) = 4$, $\sigma(2) = 1$, etc. Write σ as a product of transpositions and determine whether it is even or odd.

(b) Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ denote the standard basis vectors in $\mathbb{R}^3$, and let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ denote three fixed but otherwise arbitrary vectors in $\mathbb{R}^3$. Consider the following system of first-order partial differential equations for $\mathbf{A}(x, y) = A_1(x, y)\mathbf{e}_1 + A_2(x, y)\mathbf{e}_2 + A_3(x, y)\mathbf{e}_3$:

$$\begin{aligned} \frac{\partial \mathbf{A}}{\partial x} &= \mathbf{e}_3 \times \mathbf{A}, \\ \frac{\partial \mathbf{A}}{\partial y} &= \mathbf{a} + (\mathbf{b} \cdot \mathbf{A})\mathbf{b} + (\mathbf{c} \cdot \mathbf{A})\mathbf{A}, \end{aligned} \tag{1}$$

with initial data

$$\mathbf{A}(x_0, y_0) = \mathbf{A}_0. \tag{2}$$

i. (9 marks) Derive a necessary and sufficient condition on $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ for the system (1)–(2) to have a solution defined in a neighbourhood of (x_0, y_0) for any $(x_0, y_0) \in \mathbb{R}^2$ and for any $\mathbf{A}_0 \in \mathbb{R}^3$.

ii. (12 marks) Let $\mathbf{a} = \mathbf{b} = 0$, $\mathbf{c} = \mathbf{e}_3$. Find the solution $\mathbf{A}(x, y)$ of (1) with initial data

$$\mathbf{A}(0, 0) = \mathbf{e}_1 + \mathbf{e}_3.$$

3. (25 marks) You are given the following statement of the Poincaré Lemma: If $\hat{\Phi}_t$ is a one-parameter family of diffeomorphisms on $U \subset \mathbb{R}^n$ and $\hat{\mathbb{X}}_t$ the time-dependent vector field defined by

$$\hat{\mathbb{X}}_t \circ \hat{\Phi}_t = \frac{\partial}{\partial t} \hat{\Phi}_t \,,$$

and if β is a closed k-form on U such that

$$\hat{\Phi}_1^* \beta = \beta, \quad \lim_{\epsilon \to 0} \hat{\Phi}_\epsilon^* \beta = 0,$$

then $\beta = d\alpha$, where

$$\alpha = \int_0^1 \hat{\Phi}_t^* (i_{\hat{\mathbb{X}}_t} \beta) \, dt. \tag{3}$$

In what follows, let $U = \{(x, y, z) \,|\, z \neq 0\} \subset \mathbb{R}^3$, and let β be the two-form on U given by

$$\beta = \frac{1}{r^5} \left[(x^2 + y^2 - 2z^2) \, dx \wedge dy - 3zy \, dz \wedge dx - 3zx \, dy \wedge dz \right] ,$$

where $r = (x^2 + y^2 + z^2)^{1/2}$.

(a) Show that $d\beta = 0$.

(b) Let $\hat{\Phi}_t : U \to U$ be given by

$$\hat{\Phi}_t(x, y, z) = (t^2 x, t^2 y, tz),$$

where $t > 0$. Find $\hat{\mathbb{X}}_t$ as defined above.

(c) Show that

$$\lim_{t \to 0} \hat{\Phi}_t^* \beta = 0.$$

(d) Using the formula (3) above, find a two-form α on U such that

$$\beta = d\alpha.$$

Continued...

4. (25 marks total)

   (a) (13 marks) Let $c : I^2 \to \mathbb{R}^2$ be the singular 2-cube given by

   $$c(s,t) = (s+t, s^m t^n)\,,$$

   where m and n are positive integers. Let (x,y) denote cartesian coordinates on $\mathbb{R}^2$, and let ω be the 1-form on $\mathbb{R}^2$ given by

   $$\omega = (x+y)\,dx.$$

   i. Compute $c^*\omega$.
   ii. Compute $c^*d\omega$.
   iii. Compute $\int_c d\omega$.
   iv. Without using Stokes' theorem, compute $\int_{\partial c} \omega$.

   (b) (12 marks) Let μ be a nonvanishing n-form on $\mathbb{R}^n$. Given a smooth vector field $\mathbb{X}$ on $\mathbb{R}^n$, the *divergence* of $\mathbb{X}$ with respect to μ, denoted $\mathrm{div}_\mu \mathbb{X}$, is the function on $\mathbb{R}^n$ defined by

   $$L_{\mathbb{X}}\mu = (\,\mathrm{div}_\mu \mathbb{X})\,\mu.$$

   i. Let Φ_t denote the flow of $\mathbb{X}$. Show that

   $$\Phi_t^*\mu = \mu \iff \mathrm{div}_\mu \mathbb{X} = 0.$$

   ii. Let

   $$\mu = \rho(x)dx^1 \wedge \cdots \wedge dx^n,$$

   where $\rho(x) > 0$, and compute $\mathrm{div}_\mu \mathbb{X}$. Hence show that $\mathrm{div}_\mu \mathbb{X} = 0$ for $\mathbb{X}(x) = (x^2, x^3, \ldots, x^n, x^1)/\rho(x)$.

End of examination.