LECTURE 2: BASIC TOPOLOGY

1. Definitions

Recall the definition of a metric space.

Definition 1.1. *A* ***metric space*** *is a pair* (X, d)*, where* X *is a set, and* $d : X \times X \to \mathbb{R}_{\geq 0}$ *is a distance function. The distance function must have the following properties:*

(1) $d(x, y) = 0 \iff x = y$.
(2) $d(x, y) = d(y, x)$.
(3) *(Triangle inequality)* $d(x, z) \leq d(x, y) + d(y, z)$.

In a metric space, the **open ball** of radius r about x is the set of all points of distance less than r from x.

$$B_r(x) := \{y \in X : d(x, y) < r\}. \tag{1.1}$$

An **open set** is any union of (possibly infinitely many) open balls.

A topological space is a way to generalise the notion of a metric space by *specifying which sets are open* instead of specifying a distance function. This is a useful notion even when talking about very pedestrian spaces, such as the Euclidean plane $\mathbb{R}^2$. The two metrics

$$d_1(x, y) = |x| + |y| \text{ and } d_2(x, y) = \sqrt{x^2 + y^2} \tag{1.2}$$

on $\mathbb{R}^2$ yield the same set of open sets, and topology gives us a way to say that they are "equivalent". Even more importantly, although all metric spaces will be topological spaces, not all topological spaces will be metric spaces. And (maybe surprisingly) non-metric topologies are central to the theory of algebraic geometry.

Definition 1.2. *A* ***topological space*** *is a pair* $(X, \mathcal{T})$*, where* X *is a set and* $\mathcal{T}$ *is a set of subsets of* X *called a* ***topology on*** X*. The topology* $\mathcal{T}$ *must satisfy the following properties:*

(1) $\emptyset, X \in \mathcal{T}$.
(2) *(Closure under unions) If* $\mathcal{S} \subseteq \mathcal{T}$*, then* $\displaystyle\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.
(3) *(Closure under finite intersections) If* $U_1, \ldots, U_n \in \mathcal{T}$*, then* $\displaystyle\bigcap_{j=1}^{n} U_j \in \mathcal{T}$.

Any set X has two "trivial" topologies. The **indiscrete** topology on X is the smallest possible topology, $\mathcal{T} = \{\emptyset, X\}$. The **discrete** topology is the largest possible topology, that is, the full power set $\mathcal{T} = 2^X = \{S \subseteq X\}$.

If $(X, \mathcal{T})$ is a topological space, a subset $C \subseteq X$ is called **closed** if its complement of an open set, that is, if $X \setminus C \in \mathcal{T}$. It is straightforward to see that specifying which sets are closed is equivalent to specifying which sets are open.

In practice, it is often useful to define a topology not by specifying all the open sets (nor all the closed sets), but rather by specifying a subset of the open sets that "generates" the topology. Such a subset is called a *base*.

Definition 1.3. *A* ***base*** *for a topology on a set X is a set $\mathcal{B}$ of subsets of X, with the following properties:*

(1*) $\bigcup_{B\in\mathcal{B}} B = X.$

(2*) *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

The ***topology $\mathcal{T}$ generated by*** *$\mathcal{B}$ is the set of all unions of sets in $\mathcal{B}$.*

$$\mathcal{T} = \left\{ \bigcup_{A\in\mathcal{A}} A : \mathcal{A} \subseteq \mathcal{B} \right\}. \tag{1.3}$$

Proposition 1.4. *Let $\mathcal{B}$ be a base for a topology on X, and let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Then $(X, \mathcal{T})$ is, in fact, a topological space.*

Proof. We prove each of the properties listed in definition 1.2 in turn.

To prove (1), note that $\emptyset \in \mathcal{T}$ because it is the union over $\mathcal{S} = \emptyset$, whereas $X \in \mathcal{T}$ by (1*).

To prove (2), simply note that a union of unions of sets in $\mathcal{B}$ is a union of sets in $\mathcal{B}$.

To prove (3), it suffices to show that the intersection $U_1 \cap U_2$ of two open sets

$$U_1 = \bigcup_{B\in\mathcal{S}_1} B \text{ and } U_2 = \bigcup_{B\in\mathcal{S}_2} B \tag{1.4}$$

is open. For each $x \in U_1 \cap U_2$, there exists $B_{1,x} \in \mathcal{S}_1$ such that $x \in B_{1,x}$ and $B_{2,x} \in \mathcal{S}_2$ such that $x \in B_{2,x}$. By (2*), there exists $B_{3,x} \in \mathcal{B}$ such that $x \in B_{3,x} \subseteq B_{1,x} \cap B_{2,x}$. It follows that

$$U_1 \cap U_2 = \bigcup B_{3,x} \in \mathcal{T}. \tag{1.5}$$

We've now shown that $\mathcal{T}$ is a topology on X. □

Remark 1.5. *In fact, any set $\mathcal{S}$ of subsets of X may be used to define a topology, by first enlarging it to the base*

$$\mathcal{B} = \{X\} \cup \{\textit{finite intersection of sets in } \mathcal{S}\}. \tag{1.6}$$

Proposition 1.6. *Let (X, d) be a metric space, and let $\mathcal{B}$ be the set of all open balls in X. Then, $\mathcal{B}$ is a base for a topology on X.*

Proof. We have

$$\bigcup_{x\in X} B_1(x) = X, \tag{1.7}$$

so (1*) is satisfied. To prove (2*), consider $x \in B_{r_1}(x_1) \cap B_{r_2}(x_2)$. Let

$$\delta = \min\{r_1 - d(x, x_1), r_2 - d(x, x_2)\} > 0. \tag{1.8}$$

If $y \in B_\delta(x)$, then

$$d(y, x_1) \leq d(y, x) + d(x, x_1) < \delta + d(x, x_1) \leq r_1, \tag{1.9}$$

and by a similar argument, $d(y, x_2) \leq r_2$. We've now prove (2*). □

We call the topology defined in proposition 1.6 the **metric topology**.