Summary notes for Fields, Forms and Flows 3 and 34

These summary notes are intended to present the main results in outline form. They may be helpful for revision, but they are not a substitute for going through the lecture notes. In particular, you can't assume that all you need to know is in these summary notes.

1. **Diffeomorphisms (Section 1.5).** Let U and V be open subsets of $\mathbb{R}^n$. A smooth map $F \in C^\infty(U, V)$ is a *diffeomorphism* if F is invertible with $F^{-1} \in C^\infty(V, U)$. Let $\text{Diff}(U, V)$ denote the set of diffeomorphisms between U and V. If U and V are the same, we write $\text{Diff}(U)$ for the set of diffeomorphisms of U to itself. The set of diffeomorphisms $\text{Diff}(U)$ on U forms a group under composition (Proposition 1.5.3).

2. **ODEs, vector fields and flows (Section 1.6).** Let $\mathbb{X} : U \to \mathbb{R}^n$ be a smooth vector field on an open set $U \subset \mathbb{R}^n$. Then the first-order system

$$\dot{x}(t) = \mathbb{X}(x(t)), \quad x(0) = x_0,$$

has a unique solution $x(t, x_0)$ for $-T < t < T$ for some $T > 0$ (which may depend on x_0) (Theorem 1.6.3). As a function of initial conditions, $x(t, x_0)$ is smooth (Theorem 1.6.7). A vector field $\mathbb{X}$ is *complete* if $x(t, x_0)$ is defined for all t and $x_0 \in U$. Suppose $\mathbb{X}$ is smooth and complete. We define a map

$$\Phi : \mathbb{R} \times U \to U; (t, x_0) \mapsto \Phi_t(x_0) = x(t, x_0).$$

Φ is called the *flow* of the vector field $\mathbb{X}$. Φ has the following properties (Proposition 1.6.9):

   (a) $\Phi_0 = Id_U$.

   (b) $\Phi_t \circ \Phi_s = \Phi_{t+s}$.

   (c) $\Phi_t : U \to \Phi_t(U)$ is a diffeomorphism.

   (d) $\Phi \in C^\infty(\mathbb{R} \times U, \mathbb{R}^n)$.

A map with properties (a)–(d) is called a *one-parameter subgroup of diffeomorphisms*.

Conversely, if Φ is a one-parameter subgroup of diffeomorphisms on $U \subset \mathbb{R}^n$, we define a vector field $\mathbb{X}$ on U by

$$\mathbb{X}(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} \Phi_t(x).$$

Then Φ is the flow of $\mathbb{X}$ (**Proposition 1.6.12**).

Matrix exponential (Examples 1.6.10, 1.6.13.) Let $A \in \mathbb{R}^{n\times n}$ be an $n \times n$ matrix. Define

$$e^{tA} = (I + tA + \tfrac{1}{2}t^2A^2 + \cdots) = \sum_{j=0}^{\infty} \frac{t^j A^j}{j!}.$$

Proposition1.6.9 says that

$$e^{tA}e^{sA} = e^{(s+t)A}.$$

Proposition 1.6.12 says that

$$\left.\frac{d}{dt}\right| e^{tA} = A.$$

3. **Pushforward of vector fields (Section 1.7)** Let $U \subset \mathbb{R}^n$ be open, and let $\mathcal{X}(U)$ denote the set of smooth vector fields on U. Furthermore, let $V \subset R^n$ be open, and let $F \in \mathrm{Diff}(U, V)$. We define a map $F_* : \mathcal{X}(U) \to \mathcal{X}(V) : \mathbb{X} \mapsto F_*\mathbb{X}$, by either of the following equivalent formulas:

$$\begin{aligned} F_*\mathbb{X}(y) &= F'(F^{-1}(y)) \cdot \mathbb{X}(F^{-1}(y)), \\ F_*\mathbb{X}(F(x)) &= F'(x) \cdot \mathbb{X}(x), \end{aligned} \tag{1}$$

or without arguments

$$F_*\mathbb{X} = (F' \cdot \mathbb{X}) \circ F^{-1} \qquad \text{or} \qquad F_*\mathbb{X} \circ F = F' \cdot \mathbb{X}.$$

$F_*\mathbb{X}$, a smooth vector field on V, is called the *pushforward* of $\mathbb{X}$ by F. The definition (1) is motivated by changing variables in the system of ODE's described by $\mathbb{X}$. That is, if

$$\dot{x} = \mathbb{X}(x)$$

and we define $y(t) = F(x(t))$ and $\mathbb{Y} = F_*\mathbb{X}$, then $y(t)$ satisfies the system

$$\dot{y} = \mathbb{Y}(y).$$

At the level of flows, we have the following (**Proposition 1.7.3**): Let Φ_t be the flow of $\mathbb{X}$. Then Ψ_t, the flow of $F_*\mathbb{X}$, is given by

$$\Psi_t = F \circ \Phi_t \circ F^{-1}$$

For a linear vector field $\mathbb{X}(x) = Ax$, where $A \in \mathbb{R}^{n\times n}$, and a linear diffeomorphism $F(x) = Sx$, where $S \in \mathbb{R}^{n\times n}$ and S is invertible,

$$(F_*\mathbb{X})(y) = SAS^{-1}y.$$

(See Example 1.7.4).

The pushforward by a composition of two maps is given by **Proposition 1.7.5**: Let $F \in \mathrm{Diff}(U, V)$, $G \in \mathrm{Diff}(V, W)$, and $\mathbb{X} \in \mathcal{X}(U)$ be a smooth vector field. Then

$$(G \circ F)_*\mathbb{X} = G_*F_*\mathbb{X}.$$

4. **Jacobi bracket. Commuting flows (Section 1.8).** Let $U \subset \mathbb{R}^n$ be open and let $\mathbb{X}, \mathbb{Y} \in \mathcal{X}(U)$. The *Jacobi bracket* of $\mathbb{X}$ and $\mathbb{Y}$, denoted $[\mathbb{X}, \mathbb{Y}]$, is the vector field in $\mathcal{X}(U)$ given by

$$[\mathbb{X}, \mathbb{Y}] = (\mathbb{X} \cdot \nabla)\mathbb{Y} - (\mathbb{Y} \cdot \nabla)\mathbb{X}.$$

Here, $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Let Ψ_s be the flow of $\mathbb{Y}$. Then (**Proposition 1.8.2**)

$$[\mathbb{X}, \mathbb{Y}] = \left.\frac{\partial}{\partial s}\right|_{s=0} \Psi_{s*}\mathbb{X}.$$

The Jacobi bracket has the following properties (**Proposition 1.8.5**):

(a) Linearity. $[a\mathbb{X} + b\mathbb{Y}, \mathbb{Z}] = a[\mathbb{X}, \mathbb{Z}] + b[\mathbb{Y}, \mathbb{Z}]$, where $a, b \in \mathbb{R}$.

(b) Antisymmetry. $[\mathbb{X}, \mathbb{Y}] = -[\mathbb{Y}, \mathbb{X}]$.

(c) Product rule. $[\mathbb{X}, f\mathbb{Y}] = f[\mathbb{X}, \mathbb{Y}] + (\mathbb{X} \cdot \nabla f)Y$, where $f : U \to \mathbb{R}$ is a smooth function.

Proposition 1.8.6: The pushforward of the Jacobi bracket is given by

$$F_*[\mathbb{X}, \mathbb{Y}] = [F_*\mathbb{X}, F_*\mathbb{Y}].$$

Proposition 1.8.8 : Let $\mathbb{Z} \in \mathcal{X}(U)$ be a third vector field. The Jacobi identity states that

$$[[\mathbb{X}, \mathbb{Y}], \mathbb{Z}] + [[\mathbb{Y}, \mathbb{Z}], \mathbb{X}] + [[\mathbb{Z}, \mathbb{X}], \mathbb{Y}] = 0.$$

Proposition 1.8.10. The derivative of the pushforward by a flow at arbitrary time. Let $U \subset \mathbb{R}^n$ be open, let $\mathbb{X}, \mathbb{Y} \in \mathcal{X}(U)$ and let Ψ_s denote the flow of $\mathbb{Y}$. Then

$$\frac{\partial}{\partial s}\Psi_{s*}\mathbb{X} = [\Psi_{s*}\mathbb{X}, \mathbb{Y}].$$

Proposition 1.8.11. A vector field is invariant under pushforward by its flow. Let $\mathbb{X} \in \mathcal{X}(U)$ be a vector field with flow Φ_t. Then

$$\Phi_{t*}\mathbb{X} = \mathbb{X}.$$

Theorem 1.8.12. Commutativity of flows and Jacobi bracket. Let $\mathbb{X}$ and $\mathbb{Y}$ be smooth vector fields on an open set $U \subset \mathbb{R}^n$ with flows Φ_t and Ψ_s, respectively. Then

$$\Phi_t \circ \Psi_s = \Phi_s \circ \Phi_t \iff [\mathbb{X}, \mathbb{Y}] = 0.$$

5. **Pullback and Lie derivative on smooth functions. Noncommuting flows. (Sections 1.9 and 1.10.)**

   Let $U, V \subset \mathbb{R}^n$ be open, and let $F \in \mathrm{Diff}(U, V)$. We define a map $F^* : C^\infty(V) \to C^\infty(U)$, called the *pullback by* F, which maps smooth functions on V into smooth functions on U. (Later we'll extend the definition to differential forms.) Given $f \in C^\infty(V)$, the pullback of f by F is defined by

   $$F^*f = f \circ F, \quad \text{i.e.} \quad F^*f(x) = f(F(x)).$$

   The pullback is a linear map; that is, if $f, g \in C^\infty(V)$ and $a, b \in \mathbb{R}$, then

   $$F^*(af + bg) = aF^*f + bF^*g.$$

   Proposition 1.9.2. The pullback by a composition. Let $U, V, W \subset \mathbb{R}^n$ be open, and $F \in \mathrm{Diff}(U, V)$ and $G \in \mathrm{Diff}(V, W)$. Then

   $$(G \circ F)^* = F^*G^*.$$

   Let $\mathbb{X} \in \mathcal{X}(U)$. We define a map $L_\mathbb{X} : C^\infty(U) \to C^\infty(U)$, called the *Lie derivative with respect to* $\mathbb{X}$, which maps smooth functions into smooth functions. (Later we will extend the definition to differential forms.) Given $f \in C^\infty(U)$, we define the Lie derivative of f by $\mathbb{X}$ as

   $$L_\mathbb{X} f = (\mathbb{X} \cdot \nabla f) = X^i \frac{\partial f}{\partial x^i}.$$

   That is, $L_\mathbb{X} f$ is the directional derivative of f along $\mathbb{X}$.

   Proposition 1.9.4 Let $\mathbb{X} \in \mathcal{X}(U)$ and let Φ be the flow of $\mathbb{X}$. Then

   $$\Phi_t^* f = \sum_{j=0}^\infty \frac{t^j}{j!} L_\mathbb{X}^j f = e^{tL_\mathbb{X}} f.$$

Here it is assumed that $\Phi_t^* f(x)$ is analytic in t. Then the series for $\Phi_t^* f(x)$ converges in some neighbourhood.

Proposition 1.9.6. Differential-operator form of Jacobi bracket. Let $U \subset \mathbb{R}^n$ be open and let $\mathbb{X}, \mathbb{Y} \in \mathcal{X}(U)$, and $f \in C^\infty(U)$. Then

$$L_{\mathbb{X}} L_{\mathbb{Y}} f - L_{\mathbb{Y}} L_{\mathbb{X}} f = L_{[\mathbb{X},\mathbb{Y}]} f.$$

Theorem 1.10.1. Noncommuting flows. Let $\mathbb{X}$ and $\mathbb{Y}$ be vector fields with flows Φ_t and Ψ_s respectively, and let

$$\Gamma_{s,t} = \Psi_{-s} \circ \Phi_{-t} \circ \Psi_s \circ \Phi_t.$$

Let f be a smooth function. Then

$$\Gamma_{s,t}^* f = f + st L_{[\mathbb{X},\mathbb{Y}]} f + O(3),$$

where $O(3)$ denotes terms of third and higher order in s and t. Equivalently,

$$\Gamma_{s,t}(x) = x + st[\mathbb{X}, \mathbb{Y}](x) + O(3).$$

6. **Frobenius theorem. (Section 1.11.)** Let x denote coordinates on $\mathbb{R}^p$ and z coordinates on $\mathbb{R}^q$. Let $U \subset \mathbb{R}^p$ and $V \subset \mathbb{R}^q$ be open. Let f_i^α denote smooth functions on $U \times V$,

$$f_i^\alpha : U \times V \to \mathbb{R}; \quad (x, z) \mapsto f_i^\alpha(x, z),$$

where $1 \le i \le p$ and $1 \le \alpha \le q$. Consider the system of first-order partial differential equations for $u : U \to V$ given by

$$\begin{aligned} \frac{\partial u^\alpha}{\partial x^i}(x) &= f_i^\alpha(x, u(x)), \\ u(x_0) &= u_0, \qquad x_0 \in U,\ u_0 \in V. \end{aligned} \tag{2}$$

Define p vector fields $\mathbb{X}_{(i)}$, $1 \le i \le p$, on $U \times V$ as follows:

$$\mathbb{X}_{(i)}^j(x, z) = \delta_i^j, 1 \le j \le p, \quad \mathbb{X}_{(i)}^{p+\alpha}(x, z) = f_i^\alpha(x, z), 1 \le \alpha \le q.$$

That is, among the first p components of $\mathbb{X}_{(i)}$, there is a single nonzero component, namely the ith, which is equal to one, while the last q components of $\mathbb{X}_{(i)}$ are given by $f_i^1, \ldots, f_i^q$. Suppose the vector fields $\mathbb{X}_{(i)}$ are complete.

Frobenius theorem (Theorem 1.11.2.) For all $(x_0, u_0) \in U \times V$, the system (2) has a solution $u(x)$ if and only if

$$[\mathbb{X}_{(i)}, \mathbb{X}_{(j)}] = 0, \quad 1 \le i, j \le p.$$

Moreover, if a solution exists, then it is unique.

Explicit construction of solutions: Let $\Phi_{(i)t}$ denote the flow of $\mathbb{X}_{(i)}$. The first p components of $\Phi_{(i)t}$ are given by

$$\Phi_{(i)t}^j(x_0) = x_0^j + \delta_i^j t, \quad 1 \le j \le p.$$

For simplicity, suppose $x_0 = 0$ (it is easy to generalise to $x_0 \neq 0$). Then $u(x)$ is obtained from

$$(x, u(x)) = \left(\Phi_{(1)x^1} \circ \cdots \circ \Phi_{(p)x^p}\right)(0, u_0).$$

7. **Dual space**

Definition 2.1.1. Let V denote an n-dimensional vector space. The *dual space* of V, denoted V^*, is the vector space consisting of *linear functions* on V.

Let $e_{(1)}, \ldots, e_{(n)}$ denote a basis for V. We define a set of n elements of V^*, denoted $f^{(1)}, \ldots, f^{(n)}$, by

$$f^{(j)}(e_{(i)}) = \delta_i^j.$$

Proposition 2.1.3. $f^{(1)}, \ldots, f^{(n)}$ constitute a basis for V^*, called the *dual basis*.

Under a linear transformation, vectors in V and V^* transform differently. Let $e_{(1)}, \ldots, e_{(n)}$ and $\bar{e}_{(1)}, \ldots, \bar{e}_{(n)}$ be two bases for V. Then one set of basis vectors can be expressed as linear combinations of the others, e.g.

$$\bar{e}_{(i)} = \sum_{j=1}^{n} M_{ij} e_{(j)},$$

where M is an $n \times n$ matrix. Let $f^{(j)}$ and $\bar{f}^{(j)}$ denote the dual bases of $e_{(i)}$ and $\bar{e}_{(i)}$ respectively. Here, too, one set a basis vectors can be expressed as linear combinations of the others,

$$\bar{f}^{(i)} = \sum_{j=1}^{n} N_{ij} f^{(j)}.$$

Proposition 2.1.4.

$$N = (M^T)^{-1}.$$

8. **Permutations**

Proposition 2.2.1. Every permutation can be written as a product (composition) of transpositions.

Proposition 2.2.2. For all $\sigma, \tau \in S_n$,

$$P(\sigma\tau) = P(\sigma)P(\tau),$$

where $P(\sigma)$ is the permutation matrix with elements $P_{ij}(\sigma) = \delta_{i,\sigma(j)}$.

Definition 2.2.3. The *sign* of a permutation, denoted $\operatorname{sgn}\sigma$, is defined by

$$\operatorname{sgn}\sigma = \operatorname{sgn}\det P(\sigma).$$

Proposition 2.2.4. For all $\sigma, \tau \in S_n$,

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau).$$

Proposition 2.2.5.

$$\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma).$$

Proposition 2.2.6. If τ_{rs} is a transposition, then $\operatorname{sgn}\tau_{rs} = -1$.

Proposition 2.2.7. If σ is a product of k transpositions, then $\operatorname{sgn}\sigma = (-1)^k$.

Proposition 2.2.8. Let $f : S_n \to \mathbb{R}$ be a function on S_n. Then for all $\alpha, \beta \in S_n$,

$$\sum_{\sigma \in S_n} f(\sigma) = \sum_{\sigma \in S_n} f(\alpha\sigma\beta).$$

Also,

$$\sum_{\sigma \in S_n} f(\sigma) = \sum_{\sigma \in S_n} f(\sigma^{-1}).$$

9. **Algebraic k-forms**

An *algebraic k-form on V* is a function on V^k that is linear in each argument and which changes sign if two arguments are interchanged. That is, letting a denote an algebraic k-form, we have that

$$a : V^k \to \mathbb{R}; \left(\mathbf{v_{(1)}}, \dots, \mathbf{v_{(k)}}\right) \mapsto a\left(\mathbf{v_{(1)}}, \dots, \mathbf{v_{(k)}}\right).$$

Linearity with respect to each argument means that, for $u, w \in V$ and $\alpha, \beta \in \mathbb{R}$,

$$a\left(\mathbf{v_{(1)}}, \dots, \alpha u + \beta w, \dots, \mathbf{v_{(k)}}\right) = \alpha a\left(\mathbf{v_{(1)}}, \dots, u, \dots, \mathbf{v_{(k)}}\right) + \beta a\left(\mathbf{v_{(1)}}, \dots, w, \dots, \mathbf{v_{(k)}}\right).$$

Changing sign under the interchange of two arguments means that, for any j, l with $1 \le j < l \le k$,

$$a\left(\mathbf{v_{(1)}}, \dots, , \mathbf{v_{(j)}}, \dots, \mathbf{v_{(l)}}, \dots, \mathbf{v_{(k)}}\right) = -a\left(\mathbf{v_{(1)}}, \dots, , \mathbf{v_{(l)}}, \dots, \mathbf{v_{(j)}}, \dots, \mathbf{v_{(k)}}\right).$$

Denote the set of algebraic k-forms by $\Lambda^k(V)$. By convention, $\Lambda^0(V)$ is given by $\mathbb{R}$. Also, $\Lambda^1(V)$ is identified with the dual space V^* (the antisymmetry requirement is empty for $k = 1$).

Proposition 2.3.1. $\Lambda^k(V)$ is a vector space.

Proposition 2.3.2. For $a \in \Lambda^k(V)$ and for $\sigma \in S_k$,

$$a(v_{(\sigma(1))}, \dots, v_{(\sigma(k))}) = \operatorname{sgn}\sigma\, a(v_{(1)}, \dots, v_{(k)}).$$

Let $e_{(1)}, \dots, e_{(n)}$ denote a basis for V. Given $v \in V$, we write

$$v = v^i e_{(i)},$$

where we use the summation convention: if an index appears twice on one side of an equation, once as an upper index and once as a lower index, then we sum over that index. We introduce some notation. Let $I = (i_1, \dots, i_k)$ denote an ordered k-tuple of indices, where $1 \le i_r \le n$ (I is also called a multi-index). We introduce a Kronecker delta for pairs of k-tuples of indices, defined by

$$\delta(I, J) = \begin{cases} 1, & i_1 = j_1, \dots, i_k = j_k, \\ 0, & \text{otherwise.} \end{cases}$$

Given $\sigma \in S_k$, define

$$\sigma(I) = \left(i_{\sigma^{-1}(1)}, \dots, i_{\sigma^{-1}(k)}\right).$$

That, is $\sigma(I)$ is a permutation of the indices comprising I.

Proposition 2.4.1. Let $\sigma, \tau \in S_k$. Then

$$\sigma(\tau(I)) = (\sigma\tau)(I).$$

Let $E_{(I)} \in V^k$ denote the k-tuple of basis vectors given by

$$E_{(I)} = \left(e_{(i_1)}, \ldots, e_{(i_k)}\right).$$

Given $a \in \Lambda^k(V)$, we write

$$a_I = a(E_{(I)}).$$

We will call the a_I's the *coefficients* of a with respect to the basis $e_{(i)}$. An alternative notation for the coefficients is

$$a_{i_1 \cdots i_k} = a\left(e_{(i_1)}, \ldots, e_{(i_k)}\right).$$

Definition 2.4.2. Let $J = (j_1, \ldots, j_k)$. The *basis k-form* $F^{(J)}$ is the algebraic k-form on V defined by

$$F_I^{(J)} := F^{(J)}(E_{(I)}) = \begin{cases} 0, & \text{if } j_r = j_s \text{ for some } r \neq s, \\ \operatorname{sgn} \sigma, & J = \sigma(I), \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.4.3. Suppose J consists of distinct indices. Then

$$F^{(J)}(E_{(I)}) = \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \; \delta(\sigma(I), J).$$

Proposition 2.4.5.

$$F^{(J)}(v_{(1)}, \ldots v_{(k)}) = \det \begin{pmatrix} {v_{(1)}}^{j_1} & {v_{(2)}}^{j_1} & \cdots & {v_{(k)}}^{j_1} \\ {v_{(1)}}^{j_2} & {v_{(2)}}^{j_2} & \cdots & {v_{(k)}}^{j_2} \\ \vdots & \vdots & \cdots & \vdots \\ {v_{(1)}}^{j_k} & {v_{(2)}}^{j_k} & \cdots & {v_{(k)}}^{j_k} \end{pmatrix}.$$

Proposition 2.4.7. For $\alpha \in S_k$,

$$F^{(\alpha(J))} = \operatorname{sgn} \alpha \, F^{(J)}.$$

Proposition 2.4.9. Let $a \in \Lambda^k(V)$. Then

$$a = \frac{1}{k!} a_J F^{(J)},$$

where we use the summation convention for J (that is, there is a sum over each index j_r in $J = (j_1, \ldots, j_k)$).

Let J_* denote a k-tuple of indices which are distinct and in ascending order. That is, $J_* = (j_1, \ldots, j_k)$ with $j_1 < j_2 < \cdots < j_k$.

Proposition 2.4.10. The $F^{(J_*)}$'s form a basis for $\Lambda^k(V)$, and for all $a \in \Lambda^k(V)$,

$$a = \sum_{J_*} a_{J_*} F^{(J_*)}.$$

10. **Wedge product**

Proposition 2.5.1. Let $a \in \Lambda^k(V)$ be an algebraic k-form and $b \in \Lambda^l(V)$ be an algebraic l-form. Their *wedge product*, denoted $a \wedge b$, is the algebraic $(k+l)$-form defined by

$$a \wedge b\,(v_{(1)}, \dots, v_{(k+l)}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn} \sigma\, a(v_{(\sigma(1))}, \dots, v_{(\sigma(k))}) b(v_{(\sigma(k+1))}, \dots, v_{(\sigma(k+l))}). \tag{3}$$

Proposition 2.5.3.

i) $a \wedge b$ is an algebraic $(k+l)$-form. That is, $a \wedge b$, as defined by (3), is linear in each argument and changes sign under interchange of any pair of arguments.

ii) Linearity. If a is an algebraic k-form and b and c are algebraic l-forms, then

$$a \wedge (b + c) = a \wedge b + a \wedge c.$$

iii) (Anti)commutativity. If a is an algebraic k-form and b is an algebraic l-form, then

$$a \wedge b = (-1)^{kl} b \wedge a.$$

In other words, if either k or l is even, then $a \wedge b = b \wedge a$. If both k and l are odd, then $a \wedge b = -b \wedge a$.

iv) Associativity. If a is an algebraic k-form, b an algebraic l-form, and c an algebraic m-form, then

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c.$$

v) Basis k-forms. Let $J = (j_1, \dots, j_k)$. Then

$$F^{(J)} = f^{(j_1)} \wedge \cdots \wedge f^{(j_k)}.$$

11. **Contraction mapping**

Definition 2.7.1. Let $v \in V$. We define $i_v : \Lambda^k(V) \to \Lambda^{k-1}(V)$, the contraction with v, by

$$i_v c = 0, \text{ for } c \in \Lambda^0(V),$$
$$(i_v a)(w_{(1)}, \dots, w_{(k-1)}) = a(v, w_{(1)}, \dots, w_{(k-1)}), \text{ for } a \in \Lambda^k(V),\ k > 0,$$

where $w_{(1)}, \dots, w_{(k-1)} \in V$. Thus, i_v map a k-form a to a $(k-1)$-form $i_v a$ by fixing the first argument of the k-form to be v. The contraction on any zero-form is defined to be zero.

Proposition 2.7.2.

$$i_v(a + b) = i_v a + i_v b, \quad a, b \in \Lambda^k(V),$$
$$i_v(a \wedge b) = (i_v a) \wedge b + (-1)^k a \wedge (i_v b), \quad a \in \Lambda^k(V), b \in \Lambda^l(V).$$

Proposition 2.7.5.

Let

$$a = \frac{1}{k!} a_{i_1 \cdots i_k} f^{(i_1)} \wedge \cdots \wedge f^{(i_k)} \in \Lambda^k(V).$$

Then

$$i_v a = \frac{1}{(k-1)!} v^j a_{j i_2 \cdots i_k} f^{(i_2)} \cdots \wedge f^{(i_k)}.$$

12. **Differential forms.**

Definition 3.1.1. Let U be an open subset of $\mathbb{R}^n$. A **differential k-form** on U, or **k-form** for short, is a smooth map

$$\alpha : U \to \Lambda^k(\mathbb{R}^n); \quad x \mapsto \alpha(x) = \frac{1}{k!}\alpha_J(x)F^{(J)}.$$

Here, "smooth" means that the coefficient functions are smooth, i.e. $\alpha_J(x) \in \mathbb{C}^\infty(U)$. Zero-forms are just smooth functions on U.

Definition 3.1.2. Given $\alpha, \beta \in \Omega^k(U)$, then $\alpha(x)$ and $\beta(x)$ are both algebraic k-forms, and it makes sense to add them. Thus, we define $\alpha + \beta \in \Omega^k(U)$ by

$$(\alpha + \beta)(x) := \alpha(x) + \beta(x).$$

Definition 3.1.3. If α is a k-form on U and β is an l-form, then we define the $(k+l)$-form $\alpha \wedge \beta$ by

$$(\alpha \wedge \beta)(x) := \alpha(x) \wedge \beta(x).$$

Definition 3.1.4. Let $U \subset \mathbb{R}^n$ be open. Given a differential k-form $\omega \in \Omega^k$ and a vector field $\mathbb{X} \in \mathcal{X}(U)$, the **contraction** of ω with $\mathbb{X}$, denoted $i_{\mathbb{X}}\omega$, is the differential $(k-1)$-form defined by

$$(i_{\mathbb{X}}\omega)(x) := i_{\mathbb{X}(x)}\omega(x).$$

13. **Exterior derivative.**

Definition 3.2.1. Let $U \subset \mathbb{R}^n$ be open. The **exterior derivative**, denoted $d : \Omega^k(U) \to \Omega^{k+1}(U)$, is a map from differential k-forms to differential $(k+1)$-forms defined as follows:

- $k = 0$. For $g \in \Omega^0(U) = C^\infty(U)$,

$$dg := \frac{\partial g}{\partial x^i} f^{(i)}.$$

- $k > 0$. For $\omega \in \Omega^k(U)$, we may write that

$$\omega = \frac{1}{k!}\omega_{i_1 \dots i_k} f^{(i_1)} \wedge \cdots \wedge f^{(i_k)},$$

where $\omega_{i_1 \dots i_k}(x) \in C^\infty(U)$ (cf Proposition 2.4.10). Then

$$d\omega(x) := \frac{1}{k!} d\omega_{i_1 \dots i_k} \wedge f^{(i_1)} \wedge \cdots \wedge f^{(i_k)}.$$

Equivalently,

$$d\omega(x) := \frac{1}{k!}\frac{\partial \omega_{i_1 \dots i_k}}{\partial x^j} f^{(j)} \wedge f^{(i_1)} \wedge \cdots \wedge f^{(i_k)}.$$

Notation. It is conventional to write dx^i instead of $f^{(i)}$.

Proposition 3.2.2. Let $\alpha, \beta \in \Omega^k(U)$. Then

$$d(\alpha + \beta) = d\alpha + d\beta.$$

Proposition 3.2.3. Let $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^l(U)$. Then

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

Proposition 3.2.4. For all $\omega \in \Omega^k(U)$,

$$d^2\omega = 0.$$

14. **Pullback.**

Definition 3.3.1. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open, and let $F : U \to V$ be a smooth map. The **pullback**, denoted F^*, is a map $F^* : \Omega^k(V) \to \Omega^k(U)$ – that is, the pullback maps differential forms on V back to differential forms on U. Given $\beta \in \Omega^k(V)$, $F^*\beta$ is defined as follows. We note that as $F^*\beta$ is a differential k-form on U, $F^*\beta(x)$ is an algebraic k-form on $\mathbb{R}^m$, which may defined by specifying its value when applied to k arbitrary vectors in $\mathbb{R}^m$. Denoting these vectors by $u_{(1)}, \ldots, u_{(k)}$, the definition is given by

$$(F^*\beta)(x; \mathbf{u_{(1)}}, \ldots, \mathbf{u_{(k)}}) = \beta(F(x); F'(x)\mathbf{u_{(1)}}, \ldots, F'(x)\mathbf{u_{(k)}}).$$

For 0-forms, i.e. functions, $F^*f = f \circ F$, in accord with Definition 1.9.1.

Proposition 3.3.4.

$$F^*(\beta \wedge \gamma) = F^*\beta \wedge F^*\gamma.$$

Proposition 3.3.7. Let $\beta \in \Omega^k(V)$. Then

$$F^*d\beta = dF^*\beta.$$

Proposition 3.3.9. Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^p$ be open. Let $F : U \to V$ and $G : V \to W$ be smooth maps. Then

$$(G \circ F)^* = F^*G^*.$$

15. **Lie derivative and Poincaré Lemma**

Definition 3.4.1. Let $U \subset \mathbb{R}^n$ be open, and let $\mathbb{X} \in \mathcal{X}(U)$ be a smooth vector field on U with flow Φ_t. Let $\omega \in \Omega^k(U)$ be a differential k-form. The **Lie derivative** of ω with respect to $\mathbb{X}$, denoted $L_{\mathbb{X}}\omega$, is given by

$$L_{\mathbb{X}}\omega = \frac{\partial}{\partial t}\bigg|_{t=0} \Phi_t^*\omega.$$

Proposition 3.4.2.

i) $L_{\mathbb{X}}(\alpha + \beta) = L_{\mathbb{X}}\alpha + L_{\mathbb{X}}\beta$

ii) $L_{\mathbb{X}}(\alpha \wedge \beta) = (L_{\mathbb{X}}\alpha) \wedge \beta + \alpha \wedge L_{\mathbb{X}}\beta$

iii) $L_{\mathbb{X}}d = dL_{\mathbb{X}}$.

Proposition 3.4.3. For $\omega \in \Omega^k(U)$,

$$L_{\mathbb{X}}\omega = i_{\mathbb{X}}d\omega + di_{\mathbb{X}}\omega.$$

Definition 3.5.1. A differential k-form ω is **closed** if $d\omega = 0$. ω is **exact** if $\omega = d\alpha$ for some $(k-1)$-form α.

Definition 3.5.2. Let $U, V \subset \mathbb{R}^n$ be open, and let I be an open interval in $\mathbb{R}$. A **one-parameter family of diffeomorphisms** is a smooth family of maps

$$\hat{\Phi} : I \times U \to V; \quad (t, x) \mapsto \hat{\Phi}_t(x)$$

such that $\hat{\Phi}_t$ is a diffeomorphism onto its image. That is, letting $U_t = \hat{\Phi}_t(U) \subset V$, then $\hat{\Phi}_t : U \to U_t$ is a diffeomorphism.

Proposition 3.5.4.

$$\frac{\partial}{\partial t}\hat{\Phi}_t^*\bigg|_{t=t_0}\omega = \hat{\Phi}_{t_0}^* L_{\hat{\mathbb{X}}_{t_0}}\omega.$$

Theorem 3.5.5. (Poincaé Lemma.) Let $\hat{\Phi}_t : U \to U$ be a one-parameter family of diffeomorphisms defined for $0 < t \leq 1$. Let $\beta \in \Omega^k(U)$ be a closed k-form. Suppose that

$$\hat{\Phi}_1^*\beta = \beta, \quad \lim_{t\to 0}\hat{\Phi}_t^*\beta = 0.$$

Then

$$\beta = d\alpha,$$

where

$$\alpha = \int_0^1 \hat{\Phi}_t^*\left(i_{\hat{\mathbb{X}}_t}\beta\right)\,dt,$$

and $\hat{\mathbb{X}}_t$ is defined as above by

$$\frac{\partial}{\partial t}\hat{\Phi}_t(x) = \hat{\mathbb{X}}_t(\hat{\Phi}_t(x)).$$

16. **Singular k-cubes and integration of differential forms.**

Definition 4.1.1. Let $U \subset \mathbb{R}^n$ be open. A **singular k-cube** on U is a smooth map

$$c : I^k \to U.$$

Definition 4.1.2. The **integral of a k-form ω over a singular k-cube** c, denoted $\int_c \omega$, is defined by

$$\int_c w := \int_{I^k} c^*\omega := \int_{I^k} f(t)\,dt^1\cdots dt^k.$$

Definition 4.1.4. Let $U, V \subset \mathbb{R}^n$ be open. Let $G : U \to V$ be a diffeomorphism. Then $\det G'(x) \neq 0$ for all $x \in U$. We say that G is **orientation-preserving** if $\det G' > 0$ on U.

Proposition 4.1.6. Let $B \subset \mathbb{R}^k$ be an open set which contains the k-cube I^k. Let $G : B \to B$ be an orientation-preserving diffeomorphism, and suppose that $G(I^k) = I^k$. Let $c : I^k \to U$ be a singular k-cube, and $\omega \in \Omega^k(U)$. Then

$$\int_c \omega = \int_{c\circ G}\omega.$$

17. **Boundaries.**

Definition 4.2.1. A **singular k-chain** on U, denoted $\mathcal{C}$, is a formal sum of a finite number of singular k-cubes $c_r : I^k \to U$ with integer coefficients, i.e.

$$\mathcal{C} = a_1c_1 + \cdots a_sc_s, \quad a_r \in \mathbb{Z}.$$

Definition 4.2.2. Let $c : I^k \to U$ be a singular k-cube on U. Take j such that $1 \leq j \leq k$ and $\alpha = 0$ or 1. The **(j,α)-th face** of c, denoted $c_{(j,\alpha)}$, is the singular $(k-1)$-cube given by

$$c_{(j,\alpha)} : I^{k-1} \to U,$$

where

$$c_{(j,\alpha)}(t^1, \ldots, t^{k-1}) = c(t^1, \ldots, t^{j-1}, \alpha, t^j, \ldots, t^{k-1}).$$

Definition 4.2.3. Let $c : I^k \to U$ be a singular k-cube. The **boundary** of a singular k-cube c, denoted ∂c, is the singular $(k-1)$-chain given by

$$\partial c = \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} c_{(j,\alpha)}.$$

Let $\omega \in \Omega^{k-1}(U)$ be a $(k-1)$-form, and $c : I^k \to U$ a singular k-cube. Then

$$\int_{\partial c} \omega = \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{c_{(j,\alpha)}} \omega.$$