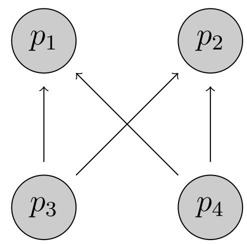
TOPICS IN MODERN GEOMETRY: HOMEWORK 1

DUE 12 OCTOBER AT 2PM

Problems (4), (5), and (6) will be marked for credit (5 marks each), so only those must be handed in. However, you may hand in any problems to receive comments on your work.

(1) Let $P = \{p_1, p_2, p_3, p_4\}$ be the following partially ordered set, considered as a topological space with the order topology. (Here, $p \to q$ means $p \leq q$.)



  (a) List the open sets of P.
  (b) List the closed sets of P.

(2) Let f be the bijection from $\mathbb{C}^n$ to $\mathrm{mSpec}(\mathbb{C}[x_1, \ldots, x_n])$ given by

$$f(a_1, \ldots, a_n) = (x_1 - a_1, \ldots, x_n - a_n).$$

Consider $\mathbb{C}^n$ to have the Euclidean topology and $\mathrm{mSpec}(\mathbb{C}[x_1, \ldots, x_n])$ to have the Zariski topology. Prove that f is continuous.

(3) Let f be the map from (2). Show that (for every $n \geq 1$) f^{-1} is not continuous.

(4) For questions (4)–(6), we will need the definitions of some properties that a topological space X can have.
T1 If $x \in X$, then $\{x\}$ is closed.
T2 If $x, y \in X$, then there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. (A space X with this property is called *Hausdorff*.)
Prove that, if X has property **T2**, then X has property **T1**.

(5) Prove that any metric space has property **T2**.

(6) Let $(P, \leq)$ be a partially ordered set, considered as a topological space with the order topology. Prove that, if P has property **T2**, then $x \leq y \implies x = y$.

(7) Use (6) to prove that, if R is a commutative ring with unity and $\mathrm{Spec}(R)$ has property **T2**, then R is a direct product of fields.

(8) Describe $\mathrm{Spec}(\mathbb{R}[x])$.

(9) Give a rational parametrisation of the hyperbola $H = \{(x, y) : x^2 - y^2 = 1\}$.

(10) Let P be the partially ordered set from (1). Find an example of a commutative ring R with unity such that $\mathrm{Spec}(R) \cong P$ as a topological space.