Topics in Modern Geometry

Solutions to problem sheet 3

Warm-up problems

1. Verify that the maps $\phi_i \colon U_i \to \mathbb{A}^n$ and $\psi_i \colon \mathbb{A}^n \to U_i$ (as defined in lectures for the standard affine open sets $U_i \subset \mathbb{P}^n$) are well-defined, bijective and are inverse to each other.

The map ϕ_i is well-defined on $\mathbb{P}^n$ since

$$\phi_i(p) = \left(\frac{p_0}{p_i}, \dots, \frac{p_{i-1}}{p_i}, \frac{p_{i+1}}{p_i}, \dots, \frac{p_n}{p_i}\right) = \left(\frac{\lambda p_0}{\lambda p_i}, \dots, \frac{\lambda p_{i-1}}{\lambda p_i}, \frac{\lambda p_{i+1}}{\lambda p_i}, \dots, \frac{\lambda p_n}{\lambda p_i}\right) = \phi_i(\lambda p)$$

where $p_i \neq 0$ for all $p \in \mathbb{P}^n$ and $\lambda \in \mathbb{C}^\times$. The map ψ_i is well-defined since

$$\psi_i(a_1, \dots, a_n) = (a_1 : \dots : a_{i-1} : 1 : a_{i+1} : \dots : a_n) \neq (0 : \dots : 0)$$

for any $a \in \mathbb{A}^n$.

The map ϕ_i is injective since $\phi_i(p) = \phi_i(p')$ if and only if $p_j = \frac{p_i}{p_i'} p_j'$ for all j, where $\frac{p_i}{p_i'} \in \mathbb{C}^\times$, and surjective since $\phi_i(a_1 : \dots, a_{i-1} : 1 : a_i : \dots : a_n) = (a_1, \dots, a_n)$ for any $a \in \mathbb{A}^n$.

The map ψ_i is injective since $\phi_i(a) = \phi_i(a')$ if and only if $a_j = a_j'$ for all j, and surjective since $\psi_i\left(\frac{p_0}{p_i}, \dots \frac{p_{i-1}}{p_i}, \frac{p_{i+1}}{p_i}, \dots, \frac{p_n}{p_i}\right) = (p_0 : \dots : p_{i-1} : p_i : p_{i+1} : \dots : p_n)$ for any $p \in \mathbb{P}^n$.

They are inverse, since clearly $\phi_i(\psi_i(a)) = a$ and, for $p \in U_i$, rescaling coordinates by $p_i \in \mathbb{C}^\times$ gives

$$\psi_i(\phi_i(p)) = \left(\frac{p_0}{p_i} : \dots : \frac{p_{i-1}}{p_i} : 1 : \frac{p_{i+1}}{p_i} : \dots : \frac{p_n}{p_i}\right) = p.$$

2. Consider $z \in \mathbb{C}$ as the point $(z : 1) \in \mathbb{P}^1$ and ∞ as the point $(1 : 0) \in \mathbb{P}^1$. Extend the usual addition and multiplication on $\mathbb{C}$ to $\mathbb{P}^1$ by setting $z + \infty = \infty$ if $z \neq \infty$, $z \cdot \infty = \infty$ if $z \neq 0$, $\infty^{-1} = 0$ and $0^{-1} = \infty$. Write down formulae for $x + y$, xy and x^{-1} in terms of the homogeneous coordinates $x = (x_0 : x_1)$ and $y = (y_0 : y_1)$.

We can write

$$(x_0 : x_1) + (y_0 : y_1) = \left(\frac{x_0}{x_1} : 1\right) + \left(\frac{y_0}{y_1} : 1\right) = \left(\frac{x_0}{x_1} + \frac{y_0}{y_1} : 1\right) = (x_0 y_1 + x_1 y_0 : x_1 y_1)$$

$$(x_0 : x_1)(y_0 : y_1) = \left(\frac{x_0}{x_1} : 1\right)\left(\frac{y_0}{y_1} : 1\right) = \left(\frac{x_0 y_0}{x_1 y_1} : 1\right) = (x_0 y_0 : x_1 y_1)$$

$$(x_0 : x_1)^{-1} = \left(\frac{x_0}{x_1} : 1\right)^{-1} = \left(\frac{x_1}{x_0} : 1\right) = (x_1 : x_0)$$

Note $x + y$ is defined as long as we don't have $x = y = (1 : 0)$ and xy is defined as long as we don't have $x = (0 : 1)$ and $y = (1 : 0)$, or vice versa.

3. Which of the following ideals are homogeneous?

$$\langle x + y^2 \rangle \subset \mathbb{C}[x, y], \quad \langle x^3 + 2y^2 z, z^2 + 3xy \rangle \subset \mathbb{C}[x, y, z], \quad \langle t + yz, x^4 + yz, t \rangle \subset \mathbb{C}[x, y, z, t]$$

The first one isn't, since $x + y^2 \in \langle x + y^2 \rangle$, but $x, y^2 \notin \langle x + y^2 \rangle$.

The second one is by exercise 5.

The last one is, since $\langle t + yz, x^4 + yz, t \rangle = \langle yz, x^4, t \rangle$ and we use exercise 5 again.

Assessed problems

4. (**5 marks**) Show that an ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$ is homogeneous if and only if there is a finite generating set $I = \langle f_1, \dots, f_k \rangle$ where each f_i is homogeneous.

Suppose $I \subset \mathbb{C}[x_0, \dots, x_n]$ is a homogeneous ideal. We can find a finite generating set $I = \langle f_1, \dots, f_k \rangle$, since $\mathbb{C}[x_0, \dots, x_n]$ is Noetherian. Since I is homogeneous we can write f_i as a sum of homogeneous parts $f_i = \sum_{j=1}^{d_i} f_{ij}$ with $f_{ij} \in I$ for all i, j. Now let $J = \langle f_{ij} : \forall i, \forall j \rangle$. Clearly $I \subseteq J$ since $f_i \in J$ for all i, and $J \subseteq I$ since $f_{ij} \in I$ for all i, j, so $I = J$. Therefore I is generated by the polynomials f_{ij}, which are a finite homogeneous generating set.

Conversely suppose $I = \langle f_1, \dots, f_k \rangle$ has a finite homogeneous generating set. Then any $g \in I$ can be written as $g = \sum_{i=1}^{k} g_i f_i$ for some $g_i \in \mathbb{C}[x_0, \dots, x_n]$. Now write $g_i = \sum_{j=1}^{d_i} g_{ij}$ as a sum of homogeneous terms. Then $g = \sum_{i,j} g_{ij} f_i$ is an expression for g where each term $g_{ij} f_i$ is homogeneous and $g_{ij} f_i \in I$. Therefore I is a homogeneous ideal.

5. (**5 marks**) Suppose that $X \subset \mathbb{A}^n$ is an affine algebraic variety. Prove that the projective closure $\widetilde{X} \subset \mathbb{P}^n$ is equal to $\overline{X} \subset \mathbb{P}^n$, the Zariski closure of $X \subset \mathbb{P}^n$, where we think of X as a subset of $\mathbb{P}^n$ in the 0th standard affine chart $X \subset (\mathbb{A}^n \cong U_0) \subset \mathbb{P}^n$.

The Zariski closure $\overline{X}$ is by definition the smallest Zariski closed subset of $\mathbb{P}^n$ which contains X. Since $X \subset \widetilde{X}$ and $\widetilde{X}$ is Zariski closed it follows that $\overline{X} \subseteq \widetilde{X}$.

Now suppose there is a point $p \in \widetilde{X} \setminus \overline{X}$. It follows that there is a polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ such that $f(q) = 0$ for all $q \in X$, but $f(p) \neq 0$. But now $p \notin X = \widetilde{X} \cap U_0$ so $p = (0 : p_1 : \dots : p_n)$. Now let $\deg f = d$ and write $f = \sum_{i=0}^{d} x_0^{d-i} f_i$ where $f_i \in \mathbb{C}[x_1, \dots, x_n]$ is homogeneous of degree i. It follows that $f_d(p) \neq 0$ and that f_d is a nonzero polynomial. But now, setting $f_{(0)} = f(1, x_1, \dots, x_n)$, we still have $f_{(0)}(q) = 0$ for all $q \in X$ and therefore $f_{(0)} \in \mathbb{I}(X)$. Since $\deg f = \deg f_{(0)} = d$ it follows that $f = \widetilde{f_{(0)}} \in \mathbb{I}(\widetilde{X})$. But now $f \in \mathbb{I}(\widetilde{X})$ and $f(p) \neq 0$ implies that $p \notin \widetilde{X}$, which is a contradiction.

6. (**5 marks**) Show that the product variety $X = \mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the quadric hypersurface $Y = \mathbb{V}(z_0 z_3 - z_1 z_2) \subset \mathbb{P}^3$ under the morphism $\phi \colon X \to Y$

$$\phi\big((x_0 : x_1) \times (y_0 : y_1)\big) = (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1)$$

and describe the inverse morphism $\psi \colon Y \to X$. (The map ϕ is usually called the *Segre embedding* of $\mathbb{P}^1 \times \mathbb{P}^1$.)

First we remark that ϕ is well-defined. Note that

$$\phi\big((\lambda x_0 : \lambda x_1) \times (\mu y_0 : \mu y_1)\big) = \phi\big((x_0 : x_1) \times (y_0 : y_1)\big)$$

for all $\lambda, \mu \in \mathbb{C}^\times$ and that no point of $X = \mathbb{P}^1 \times \mathbb{P}^1$ is sent to $(0 : 0 : 0 : 0)$. Now we check that ϕ is a morphism. Note that at any point of X at least one x coordinate and at least one y coordinate is nonzero. Assume that $x_0, y_0 \neq 0$. Then

$$\phi\big((x_0 : x_1) \times (y_0 : y_1)\big) = \left(1 : \frac{y_1}{y_0} : \frac{x_1}{x_0} : \frac{x_1 y_1}{x_0 y_0}\right)$$

is an expression for ϕ in terms of rational functions on X which is regular for all points with $x_0, y_0 \neq 0$ in X. We proceed similarly for the other cases, in which $x_1 \neq 0$ or $y_1 \neq 0$. Thus ϕ is a morphism. Moreover, the image of ϕ is clearly contained in $Y \subset \mathbb{P}^3$ since $(x_0 y_0)(x_1 y_1) - (x_0 y_1)(x_1 y_0) = 0$.

To show that ϕ is an isomorphism we only need to write down an inverse map $\psi \colon Y \to X$ and check that it is a morphism. Let

$$\psi(z_0 : z_1 : z_2 : z_3) = \begin{cases} (1 : z_2 z_0^{-1}) \times (1 : z_1 z_0^{-1}) & \text{if } z_0 \neq 0 \\ (1 : z_3 z_1^{-1}) \times (z_0 z_1^{-1} : 1) & \text{if } z_1 \neq 0 \\ (z_0 z_2^{-1} : 1) \times (1 : z_3 z_2^{-1}) & \text{if } z_2 \neq 0 \\ (z_1 z_3^{-1} : 1) \times (z_2 z_3^{-1} : 1) & \text{if } z_3 \neq 0 \end{cases}$$

This is clearly well-defined and defines a morphism in each of the affine charts $Y \cap \{z_i \neq 0\}$, and these all agree with each other by the identity $z_0 z_3 = z_1 z_2$. Therefore we have that ψ is a morphism.

Finally we check that it is an inverse. If $x_0, y_0 \neq 0$ we see that

$$\psi\phi\big((x_0 : x_1) \times (y_0 : y_1)\big) = \psi(x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1) = (1 : x_1x_0^{-1}) \times (1 : y_1y_0^{-1}) = (x_0 : x_1) \times (y_0 : y_1)$$

and we can proceed similarly if one of $x_1 \neq 0$ or $y_1 \neq 0$ instead. If $z_0 \neq 0$ then we see that

$$\phi\psi(z_0 : z_1 : z_2 : z_3) = \phi((1 : z_2z_0^{-1}) \times (1 : z_1z_0^{-1})) = (1 : z_1z_0^{-1} : z_2z_0^{-1} : z_1z_2z_0^{-2}) = (z_0 : z_1 : z_2 : z_3)$$

and we can proceed similarly if one of the other $z_i \neq 0$ instead.

Additional problems

7. Let $I = \langle x_2 - x_1^2, x_3 - x_1x_2 \rangle$, let $\widetilde{I}$ be the homogenisation of I with respect to x_0, and let $I' = \langle x_0x_2 - x_1^2, x_0x_3 - x_1x_2 \rangle$ be the ideal obtained by the homogenisation of the generators. Show that $I' \subsetneq \widetilde{I}$. (*Hint*: consider the polynomial $x_1x_3 - x_2^2$.)

Since $x_1x_3 - x_2^2 = x_1(x_3 - x_1x_2) - x_2(x_2 - x_1^2) \in I$ it follows that $x_1x_3 - x_2^2 \in \widetilde{I}$ (since it is already a homogeneous polynomial). However if $x_1x_3 - x_2^2 \in I'$ then we would have to have $\alpha, \beta \in \mathbb{C}$ such that $x_1x_3 - x_2^2 = \alpha(x_0x_3 - x_1x_2) + \beta(x_0x_2 - x_1^2) \in I'$ which is clearly impossible. Therefore $I' \subsetneq \widetilde{I}$.

8. Prove that any irreducible factor of a homogeneous polynomial is homogeneous.

Suppose $f \in \mathbb{C}[x_0, \ldots, x_n]$ is homogeneous and that f factors as $f = gh$. We will show that g and h are both homogeneous. Let $g = \sum_{i=d_1}^{d_2} g_i$ be a representation of g where g_{d_1} is the nonzero homogeneous part containing all the terms of least degree in g, and g_{d_2} is the nonzero homogeneous part containing all the terms of highest degree in g. Write $h = \sum_{i=e_1}^{e_2} h_i$ similarly. Now the degree $d_1 + e_1$ part of f is the nonzero term $g_{d_1}h_{e_1}$ and the degree $d_2 + e_2$ part of f is the nonzero term $g_{d_2}h_{e_2}$. Since f is homogeneous we must have $d_1 + e_1 = d_2 + e_2$ and since $d_1 \leq d_2$ and $e_1 \leq e_2$ this implies that $d_1 = d_2$ and $e_1 = e_2$. In other words, g and h are homogeneous.

9. Prove that a regular function on $\mathbb{P}^1$ is constant. (*Hint*: Suppose $f \in \mathbb{C}(\mathbb{P}^1)$ is a regular function. Show that the restriction to $U_0 \simeq \mathbb{A}^1$ must be a polynomial $f|_{U_0} = p(\frac{x_1}{x_0})$. Now what does $f|_{U_1}$ look like?)

Pick a regular function $f \in \mathbb{C}(\mathbb{P}^1)$. If we restrict f to the affine chart U_0 and let $t = \frac{x_1}{x_0}$ be the coordinate on $U_0 \cong \mathbb{A}^1$, then we get that f is regular for all $t \in \mathbb{A}^1$ so we must have that $f(t) \in \mathbb{C}[t]$ is a polynomial in t. Now if we restrict to the affine chart U_1, the coordinate on U_1 is given by $\frac{x_0}{x_1} = \frac{1}{t}$ and we must also have that $f \in \mathbb{C}[\frac{1}{t}]$. But now $f \in \mathbb{C}[t] \cap \mathbb{C}[\frac{1}{t}] = \mathbb{C}$, so f must be constant.

10. Show that the rational map $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $\phi(x : y : z) = (yz : zx : xy)$ is birational. What is ϕ^{-1}? What is the locus where ϕ is an isomorphism? What is the locus where ϕ is defined? (The map ϕ is usually called the *Cremona transformation*.)

We have that $\phi(x : y : z) = (yz : zx : xy) = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$, so clearly $\phi^2 = \mathrm{id}_{\mathbb{P}^2}$. We have that $\phi = \phi^{-1}$, so ϕ is a rational map with a rational inverse and is therefore is biratonal.

Note that $\phi\phi(x : y : z) = \phi(yz : zx : xy) = (x^2yz : xy^2z : xyz^2)$, and this gives back $(x : y : z)$ as long as $xyz \neq 0$. Therefore ϕ is an isomorphism for the locus $\mathbb{P}^2 \setminus \mathbb{V}(xyz)$.

Along one of the coordinate lines, say $\mathbb{V}(x)$, we have that $\phi(0 : y : z) = (yz : 0 : 0)$ which gives the point $(1 : 0 : 0)$ as long as $yz \neq 0$. So ϕ is defined, but not injective, on points $(0 : y : z)$ with $yz \neq 0$. Similarly for $\mathbb{V}(y)$ and $\mathbb{V}(z)$. Therefore the locus where ϕ is defined is $\mathbb{P}^2 \setminus \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$.

For one of the coordinate points, e.g. $(1 : 0 : 0)$, since $\phi = \phi^{-1}$ we see that $\phi^{-1}(1 : 0 : 0)$ contains all of the points $(0 : y : z)$ with $yz \neq 0$, so ϕ cannot be defined at this point. Similarly for $(0 : 1 : 0)$ and $(0 : 0 : 1)$.