Lecture 12: Rational functions and morphisms

Recall. We can write a projective algebraic subset $X = \mathbb{V}(I)$ as the locus where some ideal of homogeneous polynomials $I \subset \mathbb{C}[x_0, \ldots, x_n]$ vanishes:

$$\mathbb{V}(I) = \{p \in \mathbb{P}^n : f(p) = 0,\ \forall f \in I\}.$$

However, it is important to note that homogeneous polynomials do *not*, in general, give well-defined functions on X. Indeed, if f is well-defined on $\mathbb{P}^n$ of degree d, then we must have

$$f(p_0, \ldots, p_n) = f(\lambda p_0, \ldots, \lambda p_n) = \lambda^d f(p_0, \ldots, p_n) \quad \forall \lambda \in \mathbb{C}^\times.$$

If $f(p_0, \ldots, p_n) \neq 0$ then this implies that $d = 0$ and that f is a constant function! Therefore, in order to work with functions on X we have to work with *rational functions*.

1 Rational functions

Let $X \subset \mathbb{P}^n$ be a projective algebraic variety (i.e. irreducible in the Zariski topology).

Definition 13. A *rational function* on X is a partially defined function $f \colon X \dashrightarrow \mathbb{C}$ given by $f = \frac{g}{h}$, where $g, h \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous polynomials *of the same degree* d.

As long as $h(p_0, \ldots, p_n) \neq 0$, this *does* give a well-defined function on X since

$$f(\lambda p_0, \ldots, \lambda p_n) = \frac{\lambda^d g(p_0, \ldots, p_n)}{\lambda^d h(p_0, \ldots, p_n)} = \frac{g(p_0, \ldots, p_n)}{h(p_0, \ldots, p_n)} = f(p_0, \ldots, p_n) \quad \forall \lambda \in \mathbb{C}^\times.$$

Lemma 14. *If $h, h' \notin \mathbb{I}(X)$, then two rationals functions $f = \frac{g}{h}$ and $f' = \frac{g'}{h'}$ define the same rational function on X if and only if $gh' - g'h \in \mathbb{I}(X)$.*

Proof. Since $h, h' \notin \mathbb{I}(X)$ it follows that $U = X \setminus \mathbb{V}(h, h')$ is a non-empty Zariski open subset of X, and therefore that $\mathbb{I}(X) = \mathbb{I}(U)$. The two functions f and f' are the same if and only if $(f - f')(p) = 0$ for all $p \in U$. But since $f - f' = \frac{g}{h} - \frac{g'}{h'} = \frac{gh' - g'h}{hh'}$ and $hh' \neq 0$ on U, we have that $(f - f')(p) = 0$ for all $p \in U$ if and only if $(gh' - g'h) \in \mathbb{I}(U)$. □

Example 15. If $X = \mathbb{V}(y^2 z - x(x^2 + z^2) \subset \mathbb{P}^2$ then it looks like the rational function $f = \frac{z}{x}$ is not defined at the two points where $x = 0$, $X \cap \mathbb{V}(x) = \{(0 : 1 : 0), (0 : 0 : 1)\}$. However, it follows from Lemma 14 that $\frac{z}{x}$ and $\frac{x^2+z^2}{y^2}$ determine the same rational function on X because $y^2 z - x(x^2 + z^2) \in \mathbb{I}(X)$. Now $y^2 \neq 0$ at $(0 : 1 : 0)$, so we can evaluate f at this point using the new expression to get

$$f = \frac{z}{x} = \frac{x^2 + z^2}{y^2} \implies f(0 : 1 : 0) = \frac{0^2 + 0^2}{1^2} = 0.$$

We find that f actually *is* well-defined at the first point $(0 : 1 : 0)$, although still not at the second point $(0 : 0 : 1)$.

Note that Lemma 14 gives an equivalence relation for rational functions on X:

$$\frac{g}{h} \sim \frac{g'}{h'} \iff gh' - g'h \in \mathbb{I}(X).$$

We can define rational functions on X using this equivalence relation $\sim$.

Definition 16.

1. The *rational function field* of X is

$$\mathbb{C}(X) = \left\{ \frac{g}{h} : \begin{array}{c} g, h \in \mathbb{C}[x_0, \dots, x_n] \text{ are hgs of the} \\ \text{same degree, and } h \notin I(X) \end{array} \right\} \Big/ \sim$$

2. We say that $p \in X$ is a *regular point* of $f \in \mathbb{C}(X)$ if we can write f as $f = \frac{g}{h}$, for some g, h such that $h(p) \neq 0$ (and an *indeterminate point* of f if not).

3. The *domain of definition of* $f \in \mathbb{C}(X)$ is

$$\mathrm{dom}(f) = \{p \in X : f \text{ is regular at } p\}$$

and the *ideal of denominators of* f is

$$\mathrm{denom}(f) = \left\langle h \in \mathbb{C}[x_0, \dots, x_n] : f = \frac{g}{h} \text{ for some choice of } g, h \right\rangle.$$

Note that $\mathrm{dom}(f) = X \setminus \mathbb{V}(\mathrm{denom}(f))$ is a non-empty Zariski open subset of X for any $f = \frac{g}{h} \in \mathbb{C}(X)$. (It is non-empty since $h \notin \mathbb{I}(X)$ implies that there is at least one point $p \in X$ such that $h(p) \neq 0$, and hence p is a regular point of f).

2 Rational maps and morphisms

We can use rational functions to define rational maps.

Definition 17. Suppose $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are two projective algebraic varieties.

1. A *rational map* $f \colon X \dashrightarrow Y$ is a partially defined map given by

$$f(p) = (f_0(p) : \dots : f_n(p)) \qquad \text{for some } f_0, \dots, f_n \in \mathbb{C}(X)$$

where $f(p) \in Y \subset \mathbb{P}^n$ when $f(p)$ is defined. We call f *birational* if there exists a rational map $g \colon Y \dashrightarrow X$ such that $f \circ g = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$.

2. A rational map f is *regular* at $p \in X$, if there exists an expression $f = (f_0 : \dots : f_m)$ such that each f_i is regular at p and at least one $f_i(p) \neq 0$.

3. A *morphism* $f \colon X \to Y$ is a rational map which is regular at all points $p \in X$. We call f an *isomorphism* if there exists a morphism $g \colon Y \to X$ such that $f \circ g = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$.

Note that (as in the affine case) two projective varieties X and Y are birational if and only if $\mathbb{C}(X) = \mathbb{C}(Y)$. We call a variety X *rational* if it is birational to a projective space $\mathbb{P}^n$ (or equivalently if $\mathbb{C}(X) \cong \mathbb{C}(x_0, \dots, x_n) = \mathbb{C}(\mathbb{P}^n)$).

Example 18.

1. Suppose that $X = \mathbb{V}(x_0x_1 - x_2x_3) \subset \mathbb{P}^3$ is a quadric hypersurface and consider the birational map $\pi\colon X \dashrightarrow \mathbb{P}^2$ with $\pi(p_0 : p_1 : p_2 : p_3) = (p_1 : p_2 : p_3)$, given by forgetting the x_0 coordinate. The map $\phi(p_1 : p_2 : p_3) = \left(\frac{p_2p_3}{p_1} : p_1 : p_2 : p_3\right)$ is a birational inverse to π, so that X is birational to $\mathbb{P}^2$ and hence rational. Is π an isomorphism? What is $\operatorname{dom}(\pi) \subset X$ and $\operatorname{dom}(\phi) \subset \mathbb{P}^2$?

2. **Product of projective varieties.** Since the product of affine spaces is an affine space $\mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{n+m}$ it is easy to see that the product of two affine algebraic sets $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ is an affine algebraic set:

$$X \times Y = \left\{ p \in \mathbb{A}^{m+n} \;:\; \begin{array}{c} f(z_1, \ldots, z_m) = 0,\ \forall f \in \mathbb{I}(X) \text{ and} \\ g(z_{m+1}, \ldots, z_{m+n}) = 0,\ \forall g \in \mathbb{I}(Y) \end{array} \right\}.$$

However for projective varieties it is no longer clear, since $\mathbb{P}^m \times \mathbb{P}^n \ncong \mathbb{P}^{n+m}$ (unless either $m = 0$ or $n = 0$). First we must show that $\mathbb{P}^m \times \mathbb{P}^n$ is a projective variety by finding an embedding[1] $\phi\colon \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ for some N. In fact, we can take $N = mn + n + m = (m+1)(n+1) - 1$, where we think of coordinates z_{ij} on $\mathbb{P}^N$ as being the entries of a $(m+1) \times (n+1)$ matrix. Now take $\phi\colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ to be the map:

$$\phi\Big((x_0 : \ldots : x_m) \times (y_0 : \ldots : y_m)\Big) = \begin{pmatrix} x_0y_0 & x_0y_1 & \cdots & x_0y_n \\ x_1y_0 & x_1y_1 & \cdots & x_1y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_0 & x_my_1 & \cdots & x_my_n \end{pmatrix}$$

and show that ϕ maps $\mathbb{P}^m \times \mathbb{P}^n$ onto the subvariety $Z \subset \mathbb{P}^N$ defined by

$$Z = \mathbb{V}\left(\det\begin{pmatrix} z_{ik} & z_{il} \\ z_{jk} & z_{jl} \end{pmatrix} = z_{ik}z_{jl} - z_{il}z_{jm} : 1 \le i, j \le m, 1 \le k, l \le n\right).$$

If you have done qu. 6 on Homework sheet 3 you can try generalising your solution to show that $\mathbb{P}^m \times \mathbb{P}^n \cong Z$ in the case of arbitrary n and m. Now we can take the product of two arbitrary projective varieties $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ by considering the image $\phi(X \times Y) \subset \mathbb{P}^N$.

3 Lack of regular functions on a projective variety

The following result may at first sight might appear surprising, but cf. Liouville's theorem in complex analysis (which says that any bounded holomorphic function on $\mathbb{C}$ must be constant).

Theorem 19. *If X is a projective variety and $f \in \mathbb{C}(X)$ is regular at all points of X, then f is a constant function.*

It easily follows that a projective variety has no non-trivial morphisms to an affine variety.

Corollary 20. *If $f\colon X \to Y$ is a morphism from a projective variety $X \subset \mathbb{P}^n$ to an affine variety $Y \subset \mathbb{A}^m$ then f is a constant map (i.e. f maps X to a point).*

Proof. If $f\colon X \to Y$ is a morphism then in coordinates $f = (f_1, \ldots, f_m)$ where $f_i \in \mathbb{C}(X)$ is a regular at p for each i and for all $p \in X$. By the Theorem, f_i is constant for all i and hence f is constant. $\square$

[1] i.e. a morphism $\phi\colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ which is an isomorphism onto the image $\phi(\mathbb{P}^m \times \mathbb{P}^n)$.