Topics in Modern Geometry

Solutions to problem sheet 4

Warm-up problems

1. Show that the hypersurface $X_d = \mathbb{V}(x_0^d + x_1^d + \ldots + x_n^d) \subset \mathbb{P}^n$ is nonsingular $\forall n \geq 1$ and $\forall d \geq 1$.

The singular locus of a projective variety $X_d \subset \mathbb{P}^n$ is given by the vanishing of all the partial derivatives. Therefore

$$\operatorname{sing}(X_d) = \mathbb{V}\left(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right) = \mathbb{V}(x_0^{d-1}, \ldots, x_n^{d-1}) = \mathbb{V}(x_0, \ldots, x_n) = \emptyset,$$

since not all of the coordinates can vanish at a point of $\mathbb{P}^n$.

2. Find the equation of T_pC, where $C = \mathbb{V}(y - x^2, z - x^3) \subset \mathbb{A}^3$ and $p = (\lambda, \lambda^2, \lambda^3)$ for some $\lambda \in \mathbb{C}$.

If $f = y - x^2$ and $g = z - x^3$, then at $p = (\lambda, \lambda^2, \lambda^3)$ we have

$$\left.\frac{\partial f}{\partial x}\right|_p (x - \lambda) + \left.\frac{\partial f}{\partial y}\right|_p (y - \lambda^2) + \left.\frac{\partial f}{\partial z}\right|_p (z - \lambda^3) = -2\lambda(x - \lambda) + (y - \lambda^2) = 0$$
$$\left.\frac{\partial g}{\partial x}\right|_p (x - \lambda) + \left.\frac{\partial g}{\partial y}\right|_p (y - \lambda^2) + \left.\frac{\partial g}{\partial z}\right|_p (z - \lambda^3) = -3\lambda^2(x - \lambda) + (z - \lambda^3) = 0$$

so the tangent line is given by $T_pC = \mathbb{V}(y - 2\lambda x + \lambda^2, z - 3\lambda^2 x + 2\lambda^3)$.

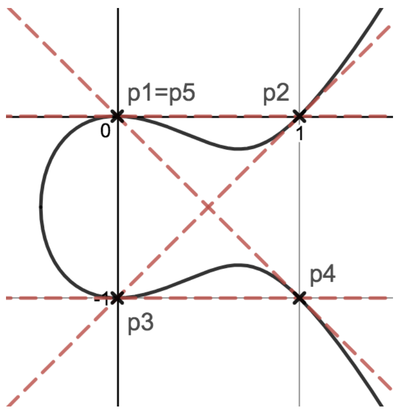
3. Find the tangent line $T_{p_1}C$ to the curve $C = \mathbb{V}(y^2 + y - x^3 + x^2) \subset \mathbb{A}^2$ at the point $p_1 = (0, 0) \in C$. Now find the other point of intersection p_2 in $T_{p_1}C \cap C$. Find p_3, p_4, p_5 by repeating this procedure, using $T_{p_i}C \cap C$ to find p_{i+1}. What do you notice about p_5? Sketch the curve C in $\mathbb{R}^2$, mark the points $p_1, \ldots, p_5$ and draw all the tangent lines that you've found.

Setting $f(x, y) = y^2 + y - x^3 + x^2$, we have $\frac{\partial f}{\partial x} = -3x^2 + 2x$ and $\frac{\partial f}{\partial y} = 2y + 1$.

We have $p_1 = (0, 0)$ and we calculate that $T_{p_1}C = \mathbb{V}(0(x - 0) + 1(y - 0)) = \mathbb{V}(y)$.

Now $C \cap \mathbb{V}(y) = \mathbb{V}(f(x, 0), y) = \mathbb{V}(x^2(1 - x), y) = \mathbb{V}(x, y) \cup \mathbb{V}(x - 1, y)$, so $p_2 = (1, 0)$.

By similar calculations we get $T_{p_2}C = \mathbb{V}(y - x + 1)$, $p_3 = (0, -1)$, $T_{p_3}C = \mathbb{V}(y + 1)$, $p_4 = (1, -1)$, $T_{p_4}C = \mathbb{V}(x + y)$ and $p_5 = (0, 0)$. We notice that $p_1 = p_5$.



(*Aside note*: In terms of the group law on a cubic curve that we saw in Lecture 18, if we let the origin $o \in C$ be the point at ∞ (which is $(0:1:0)$ after taking the projective closure $\overline{C} \subset \mathbb{P}^2$) then $o' = o$, since o is an inflection point of $\overline{C}$, and $p_1 + p_3 = o' = o \implies p_3 = -p_1$, since $\overline{C} \cap \mathbb{V}(x) = \{o, p_1, p_3\}$. Similarly, the tangent line $T_{p_i}C$ tells us that $2p_i = -p_{i+1}$ in this group. In particular $4p_1 = -2p_2 = p_3 = -p_1 \implies 5p_1 = o$, i.e. p_1 has order 5.)

Assessed problems

4. **Given a nonsingular plane cubic curve $C = \mathbb{V}(y^2z - x^3 - axz^2 - bz^3) \subset \mathbb{P}^2$ and the point $p = (0:1:0) \in C$, show that there are exactly four tangent lines to C which pass through p.**

Let $C = \mathbb{V}(f)$ where $f = y^2z - x^3 - axz^2 - bz^3$ for some $a, b \in \mathbb{C}$. Let us factor the cubic term $x^3 + axz^2 + bz^3 = (x - \alpha_1 z)(x - \alpha_2 z)(x - \alpha_3 z)$ for some complex numbers $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. We must have that the α_i are all distinct since, if $\alpha_1 = \alpha_2$ say, then C is singular at $(\alpha_1 : 0 : 1) \in C$.

Solution 1. The lines through p are the lines of the form $\mathbb{V}(ax + cz)$ for some $a, c \in \mathbb{C}$ and a point $q \in C$ has a tangent line T_qC of this form if and only if $\frac{\partial f}{\partial y}\Big|_q = 0$. Therefore the set of points of C whose tangent line goes through p is given by

$$\mathbb{V}\left(f, \frac{\partial f}{\partial y}\right) = \mathbb{V}\left(y^2z - x^3 - axz^2 - bz^3, 2yz\right) = \mathbb{V}\left(x^3 + axz^2 + bz^3, y\right) \cup \mathbb{V}(x, z)$$

which is the set of four points $\{(\alpha_1 : 0 : 1), (\alpha_2 : 0 : 1), (\alpha_3 : 0 : 1), (0 : 1 : 0)\}$. As above, the roots α_i are distinct or else C is singular.

Solution 2. Alternatively, we note that the tangent line $T_pC = \mathbb{V}(0.x + 0.y + 1.z) = \mathbb{V}(z)$ certainly passes through p, so one such line is $\mathbb{V}(z)$. All other lines through p are of the form $\mathbb{V}(ax + cz)$ for some $a, c \in \mathbb{C}$, and we can assume $a \neq 0$, since we have already considered the line $\mathbb{V}(z)$. Now we consider lines of the form $L_\lambda = \mathbb{V}(x - \lambda z)$ for $\lambda \in \mathbb{C}$, so

$$\begin{aligned} C \cap L_\lambda &= \mathbb{V}(y^2z - x^3 - axz^2 - bz^3, x - \lambda z) = \mathbb{V}(y^2z - \lambda^3 z^3 - a\lambda z^3 - bz^3, x - \lambda z) \\ &= \mathbb{V}(x, z) \cup \mathbb{V}(y^2 - (\lambda^3 + a\lambda + b)z^2, x - \lambda z) \end{aligned}$$

The first factor $\mathbb{V}(x, z)$ corresponds to the point $(0:1:0)$ and the second factor to the two other points of intersection $(\lambda : \pm\sqrt{\lambda^3 + a\lambda + b} : 1)$. Now L_λ is a tangent line to C when $L_\lambda \cap C$ contains a point of multiplicity 2, which happens precisely when $\lambda^3 + a\lambda + b = 0$, i.e. when $\lambda = \alpha_i$ for some i. Since the roots are distinct this gives three more tangent lines.

5. **For which values of $a \in \mathbb{C}$ is the following hypersurface singular? Find $\operatorname{sing}(X)$ in each case.**

$$X = \mathbb{V}\left(y^2 - x^3 + 3a^2x - 2a(a-2)\right) \subset \mathbb{A}^2_{x,y}$$

A singular point of X is the solution to the following system of equations

$$f = y^2 - x^3 + 3a^2x - 2a(a-2) = 0, \quad \frac{\partial f}{\partial x} = -3x^2 + 3a^2 = 0, \quad \frac{\partial f}{\partial y} = 2y = 0$$

so we have

$$\begin{aligned} \operatorname{sing} X &= \mathbb{V}(-x^3 + 3a^2x - 2a(a-2), x^2 - a^2, y) \\ &= \mathbb{V}(a(a^2 - a + 2), x - a, y) \cup \mathbb{V}(a(a^2 + a - 2), x + a, y). \end{aligned}$$

Therefore X is singular with $\operatorname{sing}(X) = \{(a, 0)\}$ if a is a root of $a(a^2 - a + 2) = 0$, singular with $\operatorname{sing}(X) = \{(-a, 0)\}$ if a is a root of $a(a^2 + a - 2) = 0$ and is nonsingular otherwise. If $a(a^2 + a - 2) = a(a+2)(a-1) = 0$ then either $a = 0, 1, -2$. If $a(a^2 - a + 2) = 0$ then, as we've already considered $a = 0$, we must have $a = \frac{1 \pm \sqrt{-7}}{2}$.

6. Verify that Bézout's theorem holds for the following two plane curves $C_1, C_2 \subset \mathbb{P}^2$, where

$$C_1 = \mathbb{V}(y^2z - x^2(x+z)) \quad \textbf{and} \quad C_2 = \mathbb{V}(y^2 - (x+y)(x+z)).$$

Solution 1. As explained in Problems Class 17 we can use the following property of the intersection multiplicity: $m_p(\mathbb{V}(f), \mathbb{V}(g)) = m_p(\mathbb{V}(f+ag), \mathbb{V}(g))$ for all $a \in \mathbb{C}[x,y,z]$ homogeneous of degree $\deg a = \deg f - \deg g$. If we write $f = y^2z - x^2(x+z)$, $g = y^2 - (x+y)(x+z)$ and $a = -z$, then

$$\mathbb{V}(f - zg) = \mathbb{V}(z(x+y)(x+z) - x^2(x+z)) = \mathbb{V}(x+z) \cup \mathbb{V}(z(x+y) - x^2)$$

so let $C_3 = \mathbb{V}(x+z)$ be a line and $C_4 = \mathbb{V}(z(x+y) - x^2)$ be a conic. Then

$$m_p(C_1, C_2) = m_p(C_3 \cup C_4, C_2) = m_p(C_3, C_2) + m_p(C_4, C_2).$$

Now in lectures we saw how to find the intersection multiplicity between a line and a curve and also how to find the intersection multiplicity between two conics. Following the line/curve procedure for $C_2 \cap C_3$ we find $g(-z, y, z) = y^2$ has a double root, so at the (unique) intersection point $(-1:0:1) \in C_2 \cap C_3$ we have $m_{(-1:0:1)}(C_3, C_2) = 2$. For C_2 and C_4 (conic/conic case) we use Proposition 41(2) to find a singular conic in the pencil $|C_2, C_4|$ by finding a root of $\det(\lambda M_{C_2} + \mu M_{C_4}) = 0$, i.e.

$$\det \begin{pmatrix} -\mu - \lambda & -\frac{1}{2}\lambda & \frac{1}{2}\mu - \frac{1}{2}\lambda \\ -\frac{1}{2}\lambda & \lambda & \frac{1}{2}\mu - \frac{1}{2}\lambda \\ \frac{1}{2}\mu - \frac{1}{2}\lambda & \frac{1}{2}\mu - \frac{1}{2}\lambda & 0 \end{pmatrix} = 0.$$

By observation $\det = 0$ if $\lambda = \mu$ since then all entries in the last row become zero, so we can replace C_4 by $\mathbb{V}(y^2 - (x+y)(x+z) + z(x+y) - x^2) = \mathbb{V}(y^2 - xy - 2x^2) = \mathbb{V}(y - 2x) \cup \mathbb{V}(y + x)$. Now

$$m_p(C_2, C_4) = m_p(C_2, C_5) + m_p(C_2, C_6)$$

where $C_5 = \mathbb{V}(y - 2x)$ and $C_6 = \mathbb{V}(y + x)$. Since we have $g(x, 2x, z) = 4x^2 - 3x(x+z) = x(x - 3z)$ we see that $C_2 \cap C_5$ contains the points $(0:0:1), (3:6:1)$ with multiplicity 1 each, and since $g(x, -x, z) = x^2$ we see that $C_2 \cap C_6$ consists of the point $(0:0:1)$ with multiplicity 2. Adding these all up together we find

$$m_p(C_1, C_2) = \begin{cases} 1 & \text{if } p = (3:6:1), \\ 2 & \text{if } p = (-1:0:1), \\ 3 & \text{if } p = (0:0:1), \\ 0 & \text{otherwise.} \end{cases}$$

Since $1 + 2 + 3 = 6 = 2 \times 3$ we see that Bézout holds.

Solution 2. Alternatively, since the original problem was not well-suited to using the resultant, the following alternative version of the problem was suggested at Problems Class 17:

$$C_1' = \mathbb{V}(x^2(x-z) - y^2(x+y-z)), \qquad C_2' = \mathbb{V}(x^2 - (y-x)(x+y-z)).$$

The intersection multiplicity can now be calculated by finding the resultant of

$$f' = x^3 - zx^2 - y^2x - y^2(y-z), \quad g' = 2x^2 - zx - y(y-z)$$

with respect to x. We can calculate $R(f',g')$ by hand using repeated row-and-column operations:

$$\begin{vmatrix} 1 & -z & -y^2 & -y^2(y-z) & 0 \\ 0 & 1 & -z & -y^2 & -y^2(y-z) \\ 2 & -z & -y(y-z) & 0 & 0 \\ 0 & 2 & -z & -y(y-z) & 0 \\ 0 & 0 & 2 & -z & -y(y-z) \end{vmatrix} = \begin{vmatrix} 1 & -z & -y^2 & -y^2(y-z) & 0 \\ 0 & 1 & -2y-z & yz-y^2 & 0 \\ 2 & -z & -y(y-z) & 0 & 0 \\ 0 & 2 & -z & -y(y-z) & 0 \\ 0 & 0 & 2 & -z & -y(y-z) \end{vmatrix} \qquad (\mathrm{row}_2 \mapsto \mathrm{row}_2 - y\cdot \mathrm{row}_5)$$

$$= -y(y-z)\begin{vmatrix} 1 & -z & -y^2 & -y^2(y-z) \\ 0 & 1 & -2y-z & -y(y-z) \\ 2 & -z & -y(y-z) & 0 \\ 0 & 2 & -z & -y(y-z) \end{vmatrix}$$

$$= -y(y-z)\begin{vmatrix} 1 & -2y-z & yz-y^2 & 0 \\ 0 & -1 & -2y & 0 \\ 2 & -z & -y(y-z) & 0 \\ 0 & 2 & -z & -y(y-z) \end{vmatrix} \qquad \begin{pmatrix} \mathrm{row}_2 \mapsto \mathrm{row}_2 - \mathrm{row}_4 \\ \mathrm{row}_1 \mapsto \mathrm{row}_1 - y\cdot \mathrm{row}_5 \end{pmatrix}$$

$$= y^2(y-z)^2\begin{vmatrix} 1 & -2y-z & yz-y^2 \\ 0 & -1 & -2y \\ 2 & -z & -y(y-z) \end{vmatrix}$$

$$= y^2(y-z)^2\begin{vmatrix} 1 & -2y-z & -y(y-z) \\ 0 & -1 & -2y \\ 0 & 4y+z & y(y-z) \end{vmatrix} \qquad (\mathrm{row}_3 \mapsto \mathrm{row}_3 - 2\mathrm{row}_1)$$

from which we can obtain that

$$R(f',g') = y^2(y-z)^2\begin{vmatrix} -1 & -2y \\ 4y+z & y(y-z) \end{vmatrix} = y^3(y-z)^2\begin{vmatrix} -1 & -2 \\ 4y+z & y-z \end{vmatrix} = y^3(y-z)^2(7y+3z).$$

Now plugging the roots of the resultant back into $C_1' \cap C_2'$ we find that the intersection points are

$$C_1' \cap C_2' = \begin{cases} p_1 = (0:0:1) \text{ with } m_{p_1}(C_1',C_2') = 3 & \text{for the root } y=0 \implies x=0 \\ p_2 = (0:1:1) \text{ with } m_{p_2}(C_1',C_2') = 2 & \text{for the root } y-z=0 \implies x=0 \\ p_3 = (6:-3:7) \text{ with } m_{p_3}(C_1',C_2') = 1 & \text{for the root } 7y+3z=0 \implies x=-2y. \end{cases}$$

Note that $(1:0:0) \notin C_1' \cap C_2'$ and the calculation shows that there are three intersection points, no two of which lie on a line through $(1:0:0)$. Moreover $(1:0:0)$ does not belong to any of the following tangent lines: $T_{p_2}C_1' = \mathbb{V}(x+y-z)$, $T_{p_3}C_2' = \mathbb{V}(5x-11y-3z)$, $T_{p_1}C_2' = \mathbb{V}(x-y)$, $T_{p_2}C_2' = \mathbb{V}(x+y-z)$ and $T_{p_3}C_2' = \mathbb{V}(17x+13y-9z)$. At p_1 we find that C_1' is singular and we need to check the tangent lines as explained in Problems Class 17—i.e. we write $C_1' = \mathbb{V}(y^2z - x^2z + x^3 - xy^2 - y^3)$ and see that when $z=1$ the terms of least degree are $y^2 - x^2 = (y-x)(y+x)$, so the tangent lines we have to consider at this point are $\mathbb{V}(y-x)$ and $\mathbb{V}(y+x)$. Note $(1:0:0)$ does not belong to these lines either. Therefore the calculation of $m_p(C_1',C_2')$ using the resultant is valid (as in Definition 46 of the lecture notes). Since $1+2+3 = 6 = 2\times 3$ we see that Bézout holds.

Additional problems

7. **Following on from Qu. 2, show that all of the tangent lines to C that you have found are contained in the hypersurface $S = \mathbb{V}(3x^2y^2 - 4x^3z - 4y^3 + 6xyz - z^2)$ and show that $\mathrm{sing}(S) = C$.**

The tangent lines we found were given by the equation $T_pC = \mathbb{V}(y - 2\lambda x + \lambda^2, z - 3\lambda^2 x + 2\lambda^3)$, for $p = (\lambda, \lambda^2, \lambda^3)$. If $f(x,y,z) = 3x^2y^2 - 4x^3z - 4y^3 + 6xyz - z^2$, then to check that $T_pC \subset S$ we need to verify that

$$(*) = f\left(x, \lambda(2x-\lambda), \lambda^2(3x-2\lambda)\right) = 0.$$

We get

$$\begin{aligned}(*) &= \lambda^2[3x^2(2x-\lambda)^2 - 4x^3(3x-2\lambda) - 4\lambda(2x-\lambda)^3 + 6x\lambda(2x-\lambda)(3x-2\lambda) - \lambda^2(3x-2\lambda)^2] \\ &= \lambda^2(2x-\lambda)[3x^2(2x-\lambda) - 4\lambda(2x-\lambda)^2 + 6x\lambda(3x-2\lambda) - (3x-2\lambda)(2x^2+x\lambda+2\lambda^2)] \\ &= \lambda^2(2x-\lambda)(3x-2\lambda)[(2x-\lambda)(x-2\lambda) + 6x\lambda - (2x^2+x\lambda+2\lambda^2)] \\ &= \lambda^2(2x-\lambda)(3x-2\lambda)[0] = 0.\end{aligned}$$

To calculate the singular locus of S, note that

$$\frac{\partial f}{\partial z} = 0 \implies -4x^3 + 6xy - 2z = 0 \implies z = x(3y - 2x^2)$$

and that

$$\frac{\partial f}{\partial y} = 0 \implies 6x^2y - 12y^2 + 6xz = 0 \implies x^2y - 2y^2 + x^2(3y - 2x^2) = 0 \implies -2(y - x^2)^2 = 0.$$

So all singular points satisfy $y = x^2$ and $z = x(3y - 2x^2) = x^3$. Now it is easy to check that f and $\frac{\partial f}{\partial x}$ also vanish after substituting $y = x^2$ and $z = x^3$. Therefore $\mathrm{sing}(S) = \mathbb{V}(y - x^2, z - x^3) = C$.

8. Find $X_d = \{p \in X : \dim T_pX \geq d\}$ for $d = 1, 2, 3, 4$, where X is the affine variety:

$$X = \mathbb{V}(xy - z^2 - t^3, tz - x^5) \subset \mathbb{A}^4_{x,y,z,t}$$

(Note that the Jacobian matrix $J = \mathrm{Jac}\,(\mathbb{I}(X))$ has rank $< r$ if and only if all of the $r \times r$-sized submatrices of J have determinant 0.) From this description, what are $\dim(X)$ and $\mathrm{sing}(X)$?

If $f = xy - z^2 - t^3$ and $g = tz - x^5$, then the Jacobian matrix is given by

$$J := \mathrm{Jac}(\mathbb{I}(X)) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} & \frac{\partial f}{\partial t} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} & \frac{\partial g}{\partial t} \end{pmatrix} = \begin{pmatrix} y & x & -2z & -3t^2 \\ -5x^4 & 0 & t & z \end{pmatrix}.$$

If $d = \dim T_p(X) = \dim\ker J|_p$, then $d = 2$ if there is a nonzero 2×2 minor, $d = 3$ if there is a nonzero entry (i.e. 1×1 minor) and $d = 4$ if all entries of J are zero. In particular $\dim T_p(X)$ is always at least 2. Therefore $X_1 = X_2 = X$. Now X_3 is defined by the equations of X and the 2×2 minors of J. Let m_{ij} be the 2×2 minor obtained by picking the i, jth columns of J. Now $m_{12} = 5x^5 = 0 \implies x = 0$. If $x = 0$ we have $m_{13} = yt = 0$ and $m_{14} = yz = 0$, so either $y = 0$ or $z = t = 0$. If $y = 0$ the only nonzero minor of J left is m_{34} and if $z = t = 0$ all the other minors now vanish. Therefore

$$\begin{aligned} X_3 &= \mathbb{V}(xy - z^2 - t^3, tz - x^5, x, y, 3t^3 - 2z^2) \cup \mathbb{V}(xy - z^2 - t^3, tz - x^5, x, z, t) \\ &= \mathbb{V}(x, y, -z^2 - t^3, tz, 3t^3 - 2z^2) \cup \mathbb{V}(x, z, t) = \mathbb{V}(x, z, t) \end{aligned}$$

where the first term reduces to $\mathbb{V}(x, y, z, t)$, which is already contained in $\mathbb{V}(x, z, t)$, so we ignore it. It is easy to see that $X_4 = \mathbb{V}(x, y, z, t)$. Therefore $X_1 = X_2 = X$, and $\dim(X) = 2$ since it is defined to be the largerst d such that $X_d = X$. The singular locus is $\mathrm{sing}(X) = X_{d+1}$, therefore $\mathrm{sing}(X) = X_3 = \mathbb{A}^1_y$ is the y-axis.

9. Given the nonsingular projective plane conic $C = \mathbb{V}(xz - y^2) \subset \mathbb{P}^2$ find the equation of another nonsingular plane conic $C' \subset \mathbb{P}^2$ such that C and C' intersect in exactly

(a) 3 points with multiplicities $1, 1, 2$,

(b) 2 points with multiplicities $1, 3$,

(c) 1 point with multiplicity 4.

(*Hint*: You could first try finding a singular conic C'' with the right intersection multiplicities and then consider curves in the pencil $|C, C''|$.)

It is easy to write down a singular conic which has the required intersection multiplicities. For instance $L_1 = \mathbb{V}(x)$ is the tangent line to C at $(0:0:1)$, $L_2 = \mathbb{V}(y)$ is a line with $L_2 \cap C = \{(1:0:0), (0:0:1)\}$ and $L_3 = \mathbb{V}(x - z)$ is a line with $L_3 \cap C = \{(1:1:1), (1:-1:1)\}$.

Now for (a) we can take $C'' = L_1 \cup L_3 = \mathbb{V}(x(x - z))$. Then $C_\lambda = \mathbb{V}(xz - y^2 + \lambda x(x - z))$ has the right intersection multiplicities with C for all λ, since C'' does. We just need to find one value of $\lambda \neq 0$ such that C_λ is nonsingular (e.g. any value of $\lambda \neq 0, 1, \infty$ will do). Similarly for (b) we can take $C'' = L_1 \cup L_2 = \mathbb{V}(xy)$ and for (c) we can take $C'' = L_1 \cup L_1 = \mathbb{V}(x^2)$ and repeat.

10. Let $C = \mathbb{V}(f) \subset \mathbb{P}^2$ be a nonsingular projective plane curve. By following a similar argument to the derivation of the formula for a tangent line in Lecture 14, show that the *inflection points* of C (i.e. the points $p \in C$ for which the tangent line T_pC intersects C with multiplicity ≥ 3) are given by $C \cap \mathbb{V}(H_f)$, where H_f is the *Hessian*:

$$H_f = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{0 \leq i,j \leq 2}$$

The line L is an inflection line at $p \in C = \mathbb{V}(f)$ if L intersects C at p with multiplicity ≥ 3. Therefore, parameterising L as $L = \{(p_0 + m_0 t : p_1 + m_1 t : p_2 + m_2 t) \in \mathbb{P}^2 : t \in \mathbb{C}\}$ where m_i is the slope of L in the x_i-direction, and substituting into $f(x_0, x_1, x_2)$ gives

$$f = f(p_0, p_1, p_2) + \sum_{i=0}^{2} \left.\frac{\partial f}{\partial x_i}\right|_p m_i t + \sum_{i=0}^{2}\sum_{j=0}^{2} \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_p m_i m_j t^2 + \cdots$$

Note that the third term can be written

$$\begin{pmatrix} m_0 & m_1 & m_2 \end{pmatrix} \left.\begin{pmatrix} \frac{\partial^2 f}{\partial x_0^2} & \frac{\partial^2 f}{\partial x_0 \partial x_1} & \frac{\partial^2 f}{\partial x_0 \partial x_2} \\ \frac{\partial^2 f}{\partial x_0 \partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_0 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}\right|_p \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} = 0$$

where $(m_0\, m_1\, m_2)$ is a nonzero vector, since C is a nonsingular curve. This says that p is an inflection point if and only if the matrix in the middle has a nontrivial kernel, iff the determinant is zero, i.e. $H_f(p) = 0$. Therefore the inflection points of C are given by $\mathbb{V}(f, H_f)$.