TOPICS IN MODERN GEOMETRY: HOMEWORK 2

DUE 22 OCTOBER AT 11AM

Problems (3), (4), and (5) will be marked for credit (5 marks each), so only those must be handed in. However, you may hand in any problems to receive comments on your work.

(1) Let $R = \mathbb{C}[x_1, \ldots, x_n]$, and let $I, J \leq R$.
   (a) Prove that $\mathbb{V}(I + J) = \mathbb{V}(I) \cap \mathbb{V}(J)$.
   (b) Prove that $\mathbb{V}(I \cap J) = \mathbb{V}(IJ) = \mathbb{V}(I) \cup \mathbb{V}(J)$.

(2) If $X = \mathbb{V}(J)$ for some ideal $J \leq \mathbb{C}[x_1, \ldots, x_n]$, does it follow that $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/J$? Prove or give a counterexample.

(3) Let $R = \mathbb{C}[x, y, z]$. Consider the ideals $I = (xy+y^2, xz+yz)$ and $J = (xy^2+y^3, xz+yz)$.
   (a) Show that $J \subseteq I$ but $J \neq I$.
   (b) Show that $\mathbb{V}(I) = \mathbb{V}(J)$.
   (c) Prove that J is not a radical ideal.

(4) Consider the variety $X \subseteq \mathbb{A}^4_{w,x,y,z}$ with four defining equations given via the following matrix equations:

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}^2 = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

Decompose X as a union of finitely many irreducible varieties.

(5) Prove that the hyperbola $H = \{(x, y) \in \mathbb{A}^2 : xy = 1\}$ is not isomorphic to $\mathbb{A}^1$.

(6) Find a rational parametrisation of the curve $C = \{(x, y) \in \mathbb{A}^2 : y^2 = x^3 + x^2\}$.

(7) Let R be any commutative ring with unity, and let $I \leq R$. Prove that

$$\operatorname{rad}(I) = \bigcap_{\substack{P \geq I \\ P \in \operatorname{Spec}(R)}} P.$$

(8) Find an example of an irreducible curve $C \subseteq \mathbb{A}^3$ such that the ideal $\mathbb{I}(C)$ cannot be expressed using only two generators.