Fields, Forms and Flows 3/34

Solution Sheet 9

1. We have that $A(u, v) = (u, v, F(u, v))$. Then $A^*x = u$, $A^*y = v$ and $A^*z = F(u, v)$. It follows that

$$A^*\,dx = dA^*x = du, \quad A^*\,dy = dA^*y = dv, \quad A^*\,dz = dA^*z = dF = \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial v}dv.$$

Let

$$f(x, y, z) = z - F(x, y).$$

Then

$$A^*f = F(u, v) - F(u, v) = 0,$$

so that

$$A^*\,df = 0.$$

2. Coordinate-independent formula for d.

(a) First, we show that

$$(d(fdg))(\mathbb{X}, \mathbb{Y}) = L_{\mathbb{X}}(fdg(\mathbb{Y})) - L_{\mathbb{Y}}(fdg(\mathbb{X})) - fdg([\mathbb{X}, \mathbb{Y}]), \tag{1}$$

for arbitrary smooth vector fields $\mathbb{X}$ and $\mathbb{Y}$. Since $d(fdg) = df \wedge dg$, the left-hand side of (1) is

$$d(fdg)(\mathbb{X}, \mathbb{Y}) = df(\mathbb{X})dg(\mathbb{Y}) - df(\mathbb{Y})dg(\mathbb{X}) = L_{\mathbb{X}}fL_{\mathbb{Y}}g - L_{\mathbb{Y}}fL_{\mathbb{X}}g,$$

where we used $df(\mathbb{X}) = \frac{\partial f}{\partial x^j}\,dx^j(\mathbb{X}) = \frac{\partial f}{\partial x^j}\,\mathbb{X}^j = \mathbb{X} \cdot \nabla f = L_{\mathbb{X}}f$, and similar.
On the right-hand side, we have that

$$L_{\mathbb{X}}(fdg(\mathbb{Y})) = L_{\mathbb{X}}(fL_{\mathbb{Y}}(g)) = L_{\mathbb{X}}(f)L_{\mathbb{Y}}(g) + fL_{\mathbb{X}}L_{\mathbb{Y}}g.$$

Similarly,

$$L_{\mathbb{Y}}(fdg(\mathbb{X})) = L_{\mathbb{Y}}(f)L_{\mathbb{X}}(g) + fL_{\mathbb{Y}}L_{\mathbb{X}}g,$$

while

$$fdg([\mathbb{X}, \mathbb{Y}]) = fL_{[\mathbb{X},\mathbb{Y}]}g = fL_{\mathbb{X}}L_{\mathbb{Y}}g - fL_{\mathbb{Y}}L_{\mathbb{X}}g.$$

Therefore, the right-hand side of (1) is given by

$$L_{\mathbb{X}}(f)L_{\mathbb{Y}}(g) - L_{\mathbb{Y}}(f)L_{\mathbb{X}}(g),$$

and (1) is confirmed.

An arbitrary 1-form ω on $\mathbb{R}^n$ can be expressed as a sum of terms of the form fdg, where f and g are smooth functions (in particular, we have that $\omega = \omega_i dx^i$). Therefore, (1) implies that, in general,

$$d\omega(\mathbb{X}, \mathbb{Y}) = L_{\mathbb{X}}(\omega(\mathbb{Y})) - L_{\mathbb{Y}}(\omega(\mathbb{X})) - \omega([\mathbb{X}, \mathbb{Y}])$$

for any smooth vector fields $\mathbb{X}, \mathbb{Y}$.

(b) We show by induction that, for ω a k-form on $\mathbb{R}^n$,

$$d\omega(\mathbb{X}_{(1)}, \dots, \mathbb{X}_{(k)}) = \sum_{i=1}^{k} (-1)^{i+1} L_{\mathbb{X}_{(i)}} \left(\omega(\mathbb{X}_{(1)}, \dots, \widehat{\mathbb{X}_{(i)}}, \dots, \mathbb{X}_{(k)})\right) + \sum_{i<j} (-1)^{i+j} \omega\left([\mathbb{X}_{(i)}, \mathbb{X}_{(j)}], \mathbb{X}_{(1)}, \dots, \widehat{\mathbb{X}_{(i)}}, \dots, \widehat{\mathbb{X}_{(j)}}, \dots, \mathbb{X}_{(k)}\right), \tag{2}$$

where the caret denotes an argument which is to be omitted.

In part (a), we showed that (2) holds for $k = 1$. We assume that it holds for k. Then it suffices to show that it holds for $(k + 1)$-forms ω of the form

$$\omega = df \wedge \alpha, \tag{3}$$

where f is a smooth function and α a k-form. (Note that we can write $g\,dx^1 \wedge \cdots \wedge dx^{k+1}$ as $df \wedge \alpha$ by letting $f = x^1$ and $\alpha = g\,dx^2 \wedge \cdots\cdots \wedge dx^{k+1}$, and an arbitrary $(k+1)$-form can be written as a linear combination of such terms.)

It is perhaps simplest to start from the rhs of (2) and work towards the lhs. We write the rhs as a sum of terms $T_1 + T_2$, where

$$T_1 = \sum_{i=1}^{k+1} (-1)^{i+1} L_{\mathbb{X}_{(i)}} \left(df \wedge \alpha \left(\mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \mathbb{X}_{(k+1)} \right) \right), \tag{4}$$

$$T_2 = \sum_{i<l} (-1)^{i+l} df \wedge \alpha \left([\mathbb{X}_{(i)}, \mathbb{X}_{(l)}], \mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \widehat{\mathbb{X}_{(l)}}, \ldots, \mathbb{X}_{(k+1)} \right), \tag{5}$$

and for future convenience, we have replaced j by l in (5). Since $d(df \wedge \alpha) = -df \wedge d\alpha$, our aim is to show that

$$T_1 + T_2 = -(df \wedge d\alpha)(\mathbb{X}_{(1)}, \ldots, \mathbb{X}_{(k+1)}). \tag{6}$$

Consider T_1 first. From the formula for the wedge product,

$$T_1 = \sum_{i=1}^{k+1} (-1)^{i+1} L_{\mathbb{X}_{(i)}} \Bigg(\Bigg(\sum_{j<i} - \sum_{j>i} \Bigg) (-1)^{j+1} df(\mathbb{X}_j) \alpha(\mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{\{(i),(j)\}}}, \ldots, \widehat{\mathbb{X}_{\{(i),(j)\}}}, \ldots, \mathbb{X}_{(k+1)}) \Bigg).$$

The notation $\widehat{\mathbb{X}_{\{(i),(j)\}}}$ means that the arguments $\mathbb{X}_{(i)}$ and $\mathbb{X}_{(j)}$ are to be omitted (we don't know which one appears first, as the j sum contains terms with $j < i$ as well as $j > i$). The extra minus sign in the $j > i$ terms reflects the fact that $\mathbb{X}_{(i)}$ is missing from the arguments of α, so that $X_{(j)}$ is the $(j-1)$th argument of α, rather than the jth. Using the product rule for $L_{\mathbb{X}_{(i)}}$ (which in the preceding is applied to the product of two functions) and noting that $df(\mathbb{X}_{(j)}) = L_{\mathbb{X}_{(j)}} f$, we get that

$$\begin{aligned} T_1 = &- \sum_{i<j} (-1)^{i+j} \left(\left(L_{\mathbb{X}_{(i)}} L_{\mathbb{X}_{(j)}} - L_{\mathbb{X}_{(j)}} L_{\mathbb{X}_{(i)}} \right) f \right) \alpha(\mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \widehat{\mathbb{X}_{(j)}}, \ldots, \mathbb{X}_{(k+1)}) \\ &- \sum_{j=1}^{k+1} (-1)^{j+1} df(\mathbb{X}_{(j)}) \Bigg(\sum_{i<j} - \sum_{i>j} \Bigg) (-1)^{i+1} L_{\mathbb{X}_{(i)}} \left(\alpha(\mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \widehat{\mathbb{X}_{(j)}}, \ldots, \mathbb{X}_{(k+1)}) \right), \end{aligned} \tag{7}$$

where in the second line we have interchanged the i and j sums (you should check that each term has the correct sign).

Next consider T_2 in (5). From the formula for the wedge product,

$$\begin{aligned} T_2 = \sum_{i<l} (-1)^{i+l} \Bigg[& df([\mathbb{X}_{(i)}, \mathbb{X}_{(l)}]) \alpha \left(\mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \widehat{\mathbb{X}_{(l)}}, \ldots, \mathbb{X}_{(k+1)} \right) \\ & - \Bigg(\sum_{j<i<l} - \sum_{i<j<l} + \sum_{i<l<j} \Bigg) (-1)^{j+1} df(\mathbb{X}_{(j)}) \\ & \times \alpha \left([\mathbb{X}_{(i)}, \mathbb{X}_{(l)}], \mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \widehat{\mathbb{X}_{(l)}}, \ldots, \widehat{\mathbb{X}_{(j)}}, \ldots, \mathbb{X}_{(k+1)} \right) \Bigg]. \end{aligned}$$

Note that the terms in the first line are those where $[\mathbb{X}_{(i)}, \mathbb{X}_{(l)}]$ is taken to be the argument of df, whereas those in the second and third lines are where $[\mathbb{X}_{(i)}, \mathbb{X}_{(l)}]$ is taken to be one of the arguments of α. The sums in the second line are taken over j with i and l fixed; you should verify that the terms in these sums have the correct sign.

We re-write the preceding expression for T_2, noting that, in the first line, $df([\mathbb{X}_{(i)}, \mathbb{X}_{(l)}])$ is just

$(L_{\mathbb{X}_{(i)}} L_{\mathbb{X}_{(l)}} - L_{\mathbb{X}_{(l)}} L_{\mathbb{X}_{(i)}})f$, and in the second line rearranging the sums over i, j and l to get

$$
\begin{aligned}
T_2 = \sum_{i<l} (-1)^{i+l} ((L_{\mathbb{X}_{(i)}} L_{\mathbb{X}_{(l)}} - L_{\mathbb{X}_{(l)}} L_{\mathbb{X}_{(i)}})f) \alpha \left(\mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \widehat{\mathbb{X}_{(l)}}, \ldots, \mathbb{X}_{(k+1)}\right) \\
- \sum_{j=1}^{k+1} (-1)^{j+1} df(\mathbb{X}_{(j)}) \left(\sum_{j<i<l} - \sum_{i<j<l} + \sum_{i<l<j} \right) (-1)^{i+l} \\
\times \alpha \left([\mathbb{X}_{(i)}, \mathbb{X}_{(l)}], \mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \widehat{\mathbb{X}_{(l)}}, \ldots, \widehat{\mathbb{X}_{(j)}}, \ldots, \mathbb{X}_{(k+1)} \right). \quad (8)
\end{aligned}
$$

From (7) and (8),

$$
\begin{aligned}
T_1 + T_2 = - \sum_{j=1}^{k+1} (-1)^{j+1} df(\mathbb{X}_{(j)}) \\
\times \left[\left(\sum_{i<j} - \sum_{i>j} \right) (-1)^{i+1} L_{\mathbb{X}_{(i)}} \left(\alpha(\mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \widehat{\mathbb{X}_{(j)}}, \ldots, \mathbb{X}_{(k+1)}) \right) + \right. \\
\left. \left(\sum_{i<l<j} - \sum_{i<j<l} + \sum_{i<l<j} \right) (-1)^{i+l} \alpha \left([\mathbb{X}_{(i)}, \mathbb{X}_{(l)}], \mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(i)}}, \ldots, \widehat{\mathbb{X}_{(l)}}, \ldots, \widehat{\mathbb{X}_{(j)}}, \ldots, \mathbb{X}_{(k+1)} \right) \right].
\end{aligned}
$$

By the induction hypothesis, the expression in square brackets is just $d\alpha(\mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(j)}}, \ldots, \mathbb{X}_{(k+1)})$. Therefore,

$$
T_1 + T_2 = - \sum_{j=1}^{k+1} (-1)^{j+1} df(\mathbb{X}_{(j)})\, d\alpha(\mathbb{X}_{(1)}, \ldots, \widehat{\mathbb{X}_{(j)}}, \ldots, \mathbb{X}_{(k+1)}) = -(df \wedge d\alpha)(\mathbb{X}_{(1)}, \ldots, \mathbb{X}_{(k+1)}),
$$

yielding the required result (6).

(c) Let ω be a nonvanishing 1-form on $\mathbb{R}^n$. First, suppose that $\omega = f\, dg$, where f and g are smooth functions on $\mathbb{R}^n$. Then $d\omega = df \wedge dg$. From the formula for the exterior derivative obtained in (a) above,

$$
\omega([\mathbb{X}, \mathbb{Y}]) = d\omega(\mathbb{X}, \mathbb{Y}) - L_{\mathbb{X}}(\omega(\mathbb{Y})) + L_{\mathbb{Y}}(\omega(\mathbb{X})). \quad (9)
$$

If $\omega(\mathbb{X}) = \omega(\mathbb{Y}) = 0$, it follows that

$$
\omega([\mathbb{X}, Y]) = (df \wedge dg)(\mathbb{X}, \mathbb{Y}).
$$

But $\omega(\mathbb{X}) = \omega(\mathbb{Y}) = 0$ implies that $dg(\mathbb{X}) = dg(\mathbb{Y}) = 0$, which implies that

$$
(df \wedge dg)(\mathbb{X}, \mathbb{Y}) = df(\mathbb{X})\, dg(\mathbb{Y}) - df(\mathbb{Y})\, dg(\mathbb{X}) = 0.
$$

Therefore,

$$
\omega([\mathbb{X}, \mathbb{Y}]) = 0,
$$

as required.

Establishing the converse result is more difficult, and the proof below makes use of the Frobenius Theorem. Given a 1-form ω, let $\mathcal{V}_\omega$ denote the space of vector fields V for which $\omega(V) = 0$. $\mathcal{V}_\omega$ is spanned by $n-1$ linearly independent vector fields (the equation $\omega(V) = 0$ constitutes a single relation amongst the n components of V). Assuming that

$$
\mathbb{X}, \mathbb{Y} \in \mathcal{V}_\omega \implies [\mathbb{X}, \mathbb{Y}] \in \mathcal{V}_\omega, \quad (10)
$$

we must show that $\omega = f\, dg$ for some functions f and g.

In the “Alternative Version of the Frobenius Theorem” notes, it is shown (see Lemma 1 and the surrounding discussion) that, locally at least, and with a linear change of coordinates if necessary, we can choose a basis $\mathbb{X}_{(1)}, \ldots, \mathbb{X}_{(n-1)}$ for $\mathcal{V}$ such that

$$
[\mathbb{X}_{(i)}, \mathbb{X}_{(j)}] = 0 \text{ for all } 1 \le i, j \le n-1,
$$

and $\mathbb{X}_{(i)}$ is of the form

$$
\mathbb{X}_{(i)j}(x) = \begin{cases} \delta_{ij}, & j < n \\ r_{(i)}(x), & j = n \end{cases}.
$$

That is, $\mathbb{X}_{(i)}$ is the sum of the unit vector in the ith direction and a component (with variable coefficient) in the nth direction.

Then consider the associated system of $n-1$ partial differential equations given by

$$\frac{\partial g}{\partial x^i}(x) = r_{(i)}(x^1, \ldots, x^{n-1}, g(x)), \quad 1 \le i \le n-1,$$
$$g(0, \ldots, 0, x^n) = x^n. \tag{11}$$

That is, the system (11) specifies the derivatives of the unknown function g with respect to the first $n-1$ components of x, while the last component of x gives the value of g at $x^1 = \ldots = x^{n-1} = 0$ – therefore, $g(x)$ is a family of solutions parameterised by x^n. Another way to think about this: the equation $x^n = g(x^1, \ldots, x^{n-1}, a)$ specifies a $(n-1)$-dimensional surface in $\mathbb{R}^n$ which intersects the x^n-axis at $x^n = a$. Letting a vary, we get a family of nonintersecting surfaces.

Since $[\mathbb{X}_{(i)}, \mathbb{X}_{(j)}] = 0$, it follows from the Frobenius theorem that, at least in a neighbourhood of the origin, the system (11) has a unique solution $g(x)$. (11) can be written as

$$(dg(\mathbb{X}_{(i)})(x^1, \ldots, x^{n-1}, g(x)) = 0,$$

since the vector fields $\mathbb{X}_{(i)}$ are tangent to the surfaces $x^n = g(x^1, \ldots, x^{n-1}, a)$ for all a (in a suitable neigbhourhood of the origin). By the inverse function theorem, the map $x \mapsto (x^1, \ldots, x^{n-1}, g(x))$, restricted to a suitable neighbourhood of the origin, is a diffeomorphism (you can verify that the Jacobian matrix of the map, evaluated at $x = 0$, is the identity matrix, and therefore is invertible). It follows that, in a suitable neighbourhood of the origin,

$$dg(\mathbb{X}_{(i)}) = 0.$$

Therefore, any 1-form ω which satisfies $\omega(\mathbb{X}_{(i)}) = 0$ is proportional to dg, ie

$$\omega = f dg$$

for some function f. This is the required result.

3. Let μ be a nonvanishing n-form on $\mathbb{R}^n$. Given a smooth vector field $\mathbb{X}$ on $\mathbb{R}^n$, the *divergence* of $\mathbb{X}$ with respect to μ, denoted $\operatorname{div}_\mu \mathbb{X}$, is the function on $\mathbb{R}^n$ defined by

$$L_{\mathbb{X}}\mu = (\operatorname{div}_\mu \mathbb{X})\, \mu.$$

(a) Let Φ_t denote the flow of $\mathbb{X}$. Then

$$\frac{\partial}{\partial t}\Phi_t^*\mu = \Phi_t^* L_{\mathbb{X}}\mu.$$

Hence

$$\Phi_t^*\mu = \mu \iff \frac{\partial}{\partial t}\Phi_t^*\mu = 0 \iff \Phi_t^* L_{\mathbb{X}}\mu = 0 \iff L_{\mathbb{X}}\mu = 0 \iff \operatorname{div}_\mu \mathbb{X} = 0,$$

where the third equivalence follows from the fact that Φ_t^* is invertible, so that if $\Phi_t^*\omega = 0$ for any differential form ω, it follows that $\omega = 0$.

(b) We give here one solution. Let

$$\mu = dx^1 \wedge \cdots \wedge dx^n.$$

Then

$$L_{\mathbb{X}}\mu = d(i_{\mathbb{X}}\mu),$$

since $i_{\mathbb{X}} d\mu = 0$. Then

$$d(i_{\mathbb{X}}\mu) = \sum_{i=1}^n (-1)^{i-1} d\left(\mathbb{X}^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n\right),$$

where $\widehat{\ }$ indicates a factor which is to be omitted from the product. We have that

$$\sum_{i=1}^n (-1)^{i-1} d(\mathbb{X}^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n) = \sum_{i=1}^n (-1)^{i-1} d\mathbb{X}^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$
$$= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \mathbb{X}^i}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n = \sum_{i=1}^n \frac{\partial \mathbb{X}^i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n = \sum_{i=1}^n \frac{\partial \mathbb{X}^i}{\partial x^i}\mu.$$

Therefore,

$$\mathrm{div}_\mu \mathbb{X} = \frac{\partial \mathbb{X}^i}{\partial x^i},$$

as required.

4. Kelvin-Helmholtz Theorem.

(a) Let $\omega = \omega_i dx^i$ be a 1-form and $\mathbb{X} = \mathbb{X}^i e_{(i)}$ a vector field on $\mathbb{R}^n$. Then

$$L_\mathbb{X}\omega = L_\mathbb{X}(\omega_i dx^i) = L_\mathbb{X}(\omega_i)dx^i + \omega_i L_\mathbb{X}(dx^i).$$

We have that

$$L_\mathbb{X}\omega_i = \mathbb{X}^j \frac{\partial \omega_i}{\partial x^j}$$

and

$$L_\mathbb{X} dx^i = d(L_\mathbb{X} x^i) = d\mathbb{X}^i = \frac{\partial \mathbb{X}^i}{\partial x^j} dx^j$$

(note that $L_\mathbb{X} x^i = \mathbb{X}^i$). Therefore,

$$L_\mathbb{X}\omega = \mathbb{X}^j \frac{\partial \omega_i}{\partial x^j} dx^i + \omega_i \frac{\partial \mathbb{X}^i}{\partial x^j} dx^j = \left(\mathbb{X}^j \frac{\partial \omega_i}{\partial x^j} + \omega_j \frac{\partial \mathbb{X}^j}{\partial x^i}\right) dx^i.$$

(b) Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{v}_t}{\partial t} + (\mathbf{v}_t \cdot \nabla)\mathbf{v}_t = -\nabla p_t,$$

where $\mathbf{v}_t = \mathbf{v}_t(r)$ is the fluid velocity and $p_t = p_t(r)$ is the pressure. Let $\nu_t(r)$ be the 1-form associated to $\mathbf{v}_t(r)$, ie

$$\nu_t = \sum_{i=1}^{3} v_t^i dr^i.$$

Then, using (a) above for $L_{\mathbf{v}_t}\nu_t$ and the Euler equation, we have that

$$\begin{aligned}\frac{\partial \nu_t}{\partial t} + L_{\mathbf{v}_t}\nu_t &= \sum_{i=1}^{3}\left(-\frac{\partial p_t}{\partial r^i} - \sum_{j=1}^{3} v_t^j \frac{\partial v_t^i}{\partial r^j} + \sum_{j=1}^{3}\left(v_t^j \frac{\partial v_t^i}{\partial r^j} + v_t^j \frac{\partial v_t^j}{\partial r^i}\right)\right) dr^i \\ &= \sum_{i=1}^{3}\left(-\frac{\partial p_t}{\partial r^i} + \sum_{j=1}^{3} v_t^j \frac{\partial v_t^j}{\partial r^i}\right) dr^i = \sum_{i=1}^{3} \frac{\partial}{\partial r^i}\left(-p_t + \tfrac{1}{2}v_t^2\right) dr^i = -d(p_t - \tfrac{1}{2}v_t^2).\end{aligned}$$

(c) Let

$$\omega_t = d\nu_t.$$

Then, since d commutes with $\partial/\partial t$ (d involves only spatial derivatives) and with $L_{\mathbf{v}_t}$ (d commutes with Lie derivatives), we have that

$$\frac{\partial \omega_t}{\partial t} + L_{\mathbf{v}_t}\omega_t = d\left(\frac{\partial \nu_t}{\partial t} + L_{\mathbf{v}_t}\nu_t\right).$$

Using the result from (b), we have that

$$d\left(\frac{\partial \nu_t}{\partial t} + L_{\mathbf{v}_t}\nu_t\right) = -d^2(p_t - \tfrac{1}{2}v_t^2) = 0,$$

since $d^2 = 0$. Therefore,

$$\frac{\partial \omega_t}{\partial t} + L_{\mathbf{v}_t}\omega_t = 0.$$

(d) Suppose that $\hat{\Phi}_t : \mathbb{R}^3 \to \mathbb{R}^3$ is a one-parameter family of diffeomorphisms (not necessarily a subgroup) which satisfies the system of differential equations

$$\frac{\partial \hat{\Phi}_t}{\partial t} = \mathbf{v}_t \circ \hat{\Phi}_t. \tag{12}$$

Then, using the product rule, we have that

$$\frac{\partial}{\partial t}\left(\hat{\Phi}_t^*\omega_t\right) = \frac{\partial}{\partial t}\hat{\Phi}_t^*\omega_s\bigg|_{s=t} + \hat{\Phi}_t^*\frac{\partial \omega_t}{\partial t}.$$

In the first term, we are differentiating with respect to the t-dependence in $\hat{\Phi}_t$ only. For a general k-form α we know from the lectures

$$\frac{\partial}{\partial t}\left(\hat{\Phi}_t^*\alpha\right) = \hat{\Phi}_t^*\left(L_{\mathbf{v}_t}\alpha\right),$$

where $\mathbf{v}_t$ is the vector field defined in (12). Combining this with the result of (c), we get that

$$\frac{\partial}{\partial t}\left(\hat{\Phi}_t^*\omega_t\right) = \hat{\Phi}_t^*\left(L_{\mathbf{v}_t}\omega_t + \frac{\partial \omega_t}{\partial t}\right) = 0,$$

as required.

5. Poincaré Lemma examples. In our proof of the Poincaré Lemma (Theorem 3.5.5), we showed that, for k-forms on $\mathbb{R}^n$, if $d\omega = 0$, then $\omega = d\alpha$, where

$$\alpha = \int_0^1 \hat{\Phi}_t^*\left(i_{\hat{\mathbb{X}}_t}\omega\right)\,dt, \tag{13}$$

and

$$\hat{\mathbb{X}}_t(x) = \frac{x}{t}, \quad \hat{\Phi}_t(x) = tx.$$

In what follows, we evaluate α for $n = 3$ and $k = 2, 3$. We write $\mathbf{r}$ in place of x.

(a) Let $\mathbf{B}(\mathbf{r})$ be a smooth vector field on $\mathbb{R}^3$ and suppose that $\nabla \cdot \mathbf{B} = 0$. Let β be the associated 2-form. Then $d\beta = 0$. We evaluate (13) as follows:

$$i_{\hat{\mathbb{X}}_t}\beta = \hat{\mathbb{X}}_t^i\beta_{ij}\,dr^j = \frac{1}{t}r^i\beta_{ij}\,dx^j = \frac{1}{t}(\mathbf{r}\times\mathbf{B})^j\,dr^j,$$

where we have used the fact that $\beta_{12} = -\beta_{21} = B_3$, etc. Then

$$(\hat{\Phi}_t^*(i_{\hat{\mathbb{X}}_t}\beta))(\mathbf{r}) = \left(\hat{\Phi}_t^*\left(\frac{1}{t}(\mathbf{B}\times\mathbf{r})^j\,dr^j\right)\right)(\mathbf{r}) = (\mathbf{B}(t\mathbf{r})\times\mathbf{r})^j(\hat{\Phi}_t^*dr^j)(\mathbf{r}),$$

since $(\hat{\Phi}_t^*\mathbf{B})(\mathbf{r}) = \mathbf{B}(t\mathbf{r})$ and $(\hat{\Phi}_t^*\mathbf{r})(\mathbf{r}) = t\mathbf{r}$, so that the factors $1/t$ and t cancel out. To proceed, note that

$$\hat{\Phi}_t^*dr^j = d(\hat{\Phi}_t^*r_j) = d(tr_j) = t\,dr_j.$$

Therefore,

$$\left(\hat{\Phi}_t^*\left(i_{\hat{\mathbb{X}}_t}\beta\right)\right)(\mathbf{r}) = t(\mathbf{B}(t\mathbf{r})\times\mathbf{r})^j\,dr^j.$$

Let

$$\alpha_j = \int_0^1 t(\mathbf{B}(t\mathbf{r})\times\mathbf{r})^j\,dt.$$

From the proof of the Poincaré Lemma, $\beta = d\alpha$, where $\alpha = \alpha_j\,dr^j$.

Letting $\mathbf{A}$ be the vector field associated to the 1-form α, we have that

$$\mathbf{A}(\mathbf{r}) = \int_0^1 t\mathbf{B}(t\mathbf{r})\times\mathbf{r}\,dt.$$

Then $\beta = d\alpha$ implies that

$$\mathbf{B} = \nabla\times\mathbf{A}.$$

Let us verify the preceding formula. Using the vector identity

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = (\nabla\cdot\mathbf{b})\mathbf{a} + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\nabla\cdot\mathbf{a})\mathbf{b} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

and the facts that

$$\nabla\cdot\mathbf{B}(t\mathbf{r}) = t(\nabla\cdot\mathbf{B})(t\mathbf{r}) = 0 \text{ (by assumption)}$$
$$\nabla\cdot\mathbf{r} = 3, \quad (\mathbf{B}\cdot\nabla)\,\mathbf{r} = \mathbf{B},$$

we get that

$$t\nabla \times (\mathbf{r} \times \mathbf{B}(t\mathbf{r})) = 2t\mathbf{B}(t\mathbf{r}) + t^2 \frac{d}{dt}\mathbf{B}(t\mathbf{r}) = \frac{d}{dt}\left(t^2\mathbf{B}(t\mathbf{r})\right).$$

Then

$$(\nabla \times \mathbf{A})(\mathbf{r}) = \int_0^1 \frac{d}{dt}\left(t^2\mathbf{B}(t\mathbf{r})\right)\, dt = t^2\mathbf{B}(t\mathbf{r})\big|_{t=0}^1 = \mathbf{B}(\mathbf{r}).$$

If $\mathbf{B}$ is constant, then

$$\mathbf{A}(\mathbf{r}) = \mathbf{B} \times \mathbf{r} \int_0^1 t\, dt = \tfrac{1}{2}\mathbf{B} \times \mathbf{r},$$

and it is easy to verify that $\mathbf{B} = \nabla \times \mathbf{A}$ in this case.

(b) Let $\rho(\mathbf{r})$ be a smooth function on $\mathbb{R}^3$. Let

$$\mu = \rho\, dr^1 \wedge dr^2 \wedge dr^3$$

denote the associated 3-form. We evaluate (13) as follows:

$$i_{\hat{\mathbb{X}}_t}\mu = \frac{1}{t}\rho(\mathbf{r})\left(r^1 dr^2 \wedge dr^3 + r^2 dr^3 \wedge dr^1 + r^3 dr^1 \wedge dr^2\right),$$

and

$$(\hat{\Phi}_t^*(i_{\hat{\mathbb{X}}_t}\mu))(\mathbf{r}) = t^2\rho(t\mathbf{r})\left(r^1 dr^2 \wedge dr^3 + r^2 dr^3 \wedge dr^1 + r^3 dr^1 \wedge dr^2\right),$$

using the expressions for $\hat{\Phi}^* dr^j$ obtained in (a) above. Let

$$\alpha = \left(\int_0^1 t^2(\rho(t\mathbf{r})\, dt\right)\left(r^1 dr^2 \wedge dr^3 + r^2 dr^3 \wedge dr^1 + r^3 dr^1 \wedge dr^2\right).$$

Then according to the Poincaré Lemma, $\mu = d\alpha$.

Letting $\mathbf{E}$ be the vector field associated to the 2-form α, we have that

$$\mathbf{E}(\mathbf{r}) = \left(\int_0^1 t^2\rho(t\mathbf{r})\, dt\right)\mathbf{r}.$$

Then $\mu = d\alpha$ implies that

$$\rho = \nabla \cdot \mathbf{E}.$$

Let us verify the preceding formula. We have that

$$t^2\nabla \cdot (\rho(t\mathbf{r})\mathbf{r}) = t^2\mathbf{r} \cdot \nabla\rho(t\mathbf{r}) + 3t^2\rho(t\mathbf{r}) = t^3\frac{d}{dt}\rho(t\mathbf{r}) + 3t^2\rho(t\mathbf{r}) = \frac{d}{dt}\left(t^3\rho(t\mathbf{r})\right).$$

Therefore,

$$(\nabla \cdot \mathbf{E})(\mathbf{r}) = \int_0^1 \frac{d}{dt}\left(t^3\rho(t\mathbf{r})\right)\, dt = t^3\rho(t\mathbf{r})\big|_{t=0}^1 = \rho(\mathbf{r}).$$

If ρ is constant, then

$$\mathbf{E}(\mathbf{r}) = \frac{\rho}{3}\mathbf{r},$$

and it is immediately evident that $\nabla \cdot \mathbf{E} = \rho$ in this case.