Fields, Forms and Flows 3/34

Solution Sheet 10

1. Consider the singular two-cube in $\mathbb{R}^2$,

$$c : [0,1]^2 \to \mathbb{R}^2; \quad c(x^1, x^2) = (x^1x^2, x^2).$$

(a) The image of c is the right triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$.

(b) First, let's compute the faces of c. We have that

$$\begin{aligned} c_{(1,0)}(x) &= c(0,x) = (0,x), \\ c_{(1,1)}(x) &= c(1,x) = (x,x), \\ c_{(2,0)}(x) &= c(x,0) = (0,0), \\ c_{(2,1)}(x) &= c(x,1) = (x,1). \end{aligned}$$

Then

$$\partial c = -c_{(1,0)} + c_{(1,1)} + c_{(2,0)} - c_{(2,1)}.$$

To evaluate $\partial(\partial c)$, we first compute the boundaries of the faces. We have that

$$\begin{aligned} \partial c_{(1,0)}(x) &= c_{(1,0)}(1) - c_{(1,0)}(0) = (0,1) - (0,0), \\ \partial c_{(1,1)}(x) &= c_{(1,1)}(1) - c_{(1,1)}(0) = (1,1) - (0,0), \\ \partial c_{(2,0)}(x) &= c_{(2,0)}(1) - c_{(2,0)}(0) = (0,0) - (0,0) = 0, \\ \partial c_{(2,1)}(x) &= c_{(2,1)}(1) - c_{(2,1)}(0) = (1,1) - (0,1). \end{aligned}$$

Add up the terms on the right-hand side with the appropriate signs to get

$$\partial(\partial c) = -\partial c_{(1,0)} + \partial c_{(1,1)} + \partial c_{(2,0)} - \partial c_{(2,1)} = +((0,0)-(0,1)) - (0,0) + (1,1) + ((0,1)-(1,1)) = 0.$$

(c) Let

$$\omega = \frac{1}{2}(y^1 dy^2 - y^2 dy^1).$$

First, let's compute

$$\int_c d\omega = \int_{[0,1]^2} c^* d\omega.$$

We have that

$$d\omega = dy^1 \wedge dy^2.$$

Then

$$c^* d\omega = \left(\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2} - \frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1} \right) dx^1 \wedge dx^2 = x^2 dx^1 \wedge dx^2.$$

This is equivalent to noting that $c^*y^1 = x^1x^2$ and $c^*y^2 = x^2$, so that $c^*dy^1 = x^1dx^2 + x^2dx^1$ and $c^*y^2 = dx^2$, and therefore

$$c^*\omega = (x^1dx^2 + x^2dx^1) \wedge dx^2 = x^2 dx^1 \wedge dx^1.$$

It follows that

$$\int_c d\omega = \int_0^1 \int_0^1 x^2 \, dx^1 dx^2 = \frac{1}{2}$$

Next we compute

$$\int_{\partial c} \omega = \sum_{j=1}^{2} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{[0,1]} c^*_{(j,\alpha)}\omega.$$

From above,

$$(c^*_{(2,1)}\omega)(x) = \frac{1}{2}\left(y^1(x)\frac{\partial y^2}{\partial x}(x) - y^2(x)\frac{\partial y^1}{\partial x}(x) \right) dx,$$

where

$$(y^1(x), y^2(x)) = c_{(2,1)}(x) = (x, 1).$$

Therefore,

$$(c^*_{(2,1)}\omega)(x) = -\frac{1}{2}dx.$$

Similar calculations give

$$(c^*_{(1,0)}\omega)(x) = (c^*_{(1,1)}\omega)(x) = (c^*_{(2,0)}\omega)(x) = 0.$$

Therefore,

$$\int_{\partial c} \omega = -\int_{[0,1]} c^*_{(2,1)}\omega = \int_0^1 \frac{1}{2}dx = \frac{1}{2}.$$

Thus, $\int_c d\omega = \int_{\partial c} \omega$, in accord with Stokes' theorem.

2. (a) Let $\mathbf{r}(t)$, $0 \le t \le 1$, denote a smooth parameterised curve and $F(\mathbf{r}) = F_x dx + F_y dy + F_z dz$ a one-form on $\mathbb{R}^3$. Then

$$(\mathbf{r}^* F)(t) = \left(F_x(\mathbf{r}(t))\frac{dx}{dt}(t) + F_y(\mathbf{r}(t))\frac{dy}{dt}(t) + F_z(\mathbf{r}(t))\frac{dz}{dt}(t) \right) dt = \mathbf{F}(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t)\, dt,$$

where $\mathbf{F} = (F_x, F_y, F_z)$.

(b) Let $\mathbf{S}(u,v)$, $0 \le u, v \le 1$, denote a smooth parameterised surface and $B(\mathbf{r}) = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$ a two-form on $\mathbb{R}^3$. Then

$$\begin{aligned}(\mathbf{S}^* B)(u,v) = B_x(\mathbf{S}(u,v)) &\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u} \right) du \wedge dv \\ &+ B_y(\mathbf{S}(u,v)) \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u} \right) du \wedge dv \\ &+ B_z(\mathbf{S}(u,v)) \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) du \wedge dv.\end{aligned}$$

Let

$$\mathbf{N} = \frac{\partial \mathbf{S}}{\partial u} \times \frac{\partial \mathbf{S}}{\partial v}.$$

Then

$$\begin{aligned} N_x &= \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}, \\ N_y &= \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}, \\ N_z &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}. \end{aligned}$$

Substituting this in the expression above, we can write

$$(\mathbf{S}^* B)(u,v) = \mathbf{B}(\mathbf{S}(u,v)) \cdot \mathbf{N}(u,v)\, du \wedge dv.$$

(c) Let $S(u,v)$ be a smooth parameterised surface, as above. Let

$$\begin{aligned} \mathbf{r}_1(t) &= \mathbf{S}_{(2,0)}(t) = \mathbf{S}(t,0), \\ \mathbf{r}_2(t) &= \mathbf{S}_{(1,1)}(t) = \mathbf{S}(1,t), \\ \mathbf{r}_3(t) &= \mathbf{S}_{(2,1)}(t) = \mathbf{S}(t,1), \\ \mathbf{r}_4(t) &= \mathbf{S}_{(1,0)}(t) = \mathbf{S}(0,t). \end{aligned}$$

Then

$$\partial S = \mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3 - \mathbf{r}_4.$$

(d) According to Stokes' theorem,

$$\int_{\mathbf{S}} dF = \int_{\partial \mathbf{S}} F.$$

We want to write this in vector notation. The two-form dF corresponds to the vector $\nabla \times \mathbf{F}$, so, using the preceding results, the left-hand side can be written as

$$\int_0^1 \int_0^1 (\nabla \times F)(\mathbf{S}(u,v)) \cdot \mathbf{N}(u,v)\, dudv.$$

The right-hand side is just

$$\int_0^1 (\mathbf{F}(\mathbf{r}_1(t)) \cdot \dot{\mathbf{r}}_1(t) + \mathbf{F}(\mathbf{r}_2(t)) \cdot \dot{\mathbf{r}}_2(t) - \mathbf{F}(\mathbf{r}_3(t)) \cdot \dot{\mathbf{r}}_3(t) - \mathbf{F}(\mathbf{r}_4(t)) \cdot \dot{\mathbf{r}}_4(t))\, dt.$$

3. (a) If ω is a k-form on $\mathbb{R}^n$ with $d\omega = 0$, then, for any singular $(k+1)$-cube c on $\mathbb{R}^n$, Stokes' theorem implies that

$$\int_{\partial c} \omega = \int_c d\omega = 0 \quad (\text{since } d\omega = 0).$$

(b) If ω is a k-form on $\mathbb{R}^n$ and $\omega = d\alpha$ for some $(k-1)$-form α, then for any singular k-cube c on $\mathbb{R}^n$ with $\partial c = 0$, Stokes' theorem implies that

$$\int_c \omega = \int_c d\alpha = \int_{\partial c} \alpha = 0 \quad (\text{since } \partial c = 0).$$

(c) Consider the one-form

$$\omega = \frac{ydx - xdy}{x^2 + y^2}$$

on the punctured $\mathbb{R}^2 - \{0\}$. We have that

$$d\omega = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) dy \wedge dx - \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy.$$

Substituting the formulas

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}, \quad \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2},$$

into the preceding, we get that

$$d\omega = \left(\frac{(x^2+y^2) - 2y^2 + (x^2+y^2) - 2x^2}{(x^2+y^2)^2}\right) dy \wedge dx = 0.$$

(d) Let c be the singular one-cube on $\mathbb{R}^2 - \{0\}$ given by

$$c(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Since $c(0) = c(1)$, we have that

$$\partial c = -c(0) + c(1) = 0.$$

We have that

$$\int_c \omega = \int_{[0,1]} c^*\omega.$$

Also,

$$c^*\omega = \left(\frac{ydx/dt - xdy/dt}{x^2+y^2} dt\right)\Bigg|_{x=\cos 2\pi t, y = \sin 2\pi t} = -2\pi(\sin^2(2\pi t) + \cos^2(2\pi t))\, dt = -2\pi dt.$$

Alternative calculation: $c^*x = \cos 2\pi t$, so that $c^*dx = dc^*x = -2\pi \sin 2\pi t\, dt$. Similarly, $c^*y = \sin 2\pi t$, so that $c^*dy = 2\pi \cos 2\pi t\, dt$. Therefore,

$$c^*\omega = c^*\left(\frac{y}{x^2+y^2}\right) c^*\, dx - c^*\left(\frac{x}{x^2+y^2}\right) c^*\, dy = -2\pi dt.$$

Either way, we get that

$$\int_c \omega = -\int_0^1 2\pi dt = -2\pi. \tag{1}$$

We cannot have $\omega = df$ for any function f on $\mathbb{R}^2 - \{0\}$, for this would imply that

$$\int_c \omega = \int_c df = \int_{\partial c} f = 0 \quad (\text{since } \partial c = 0),$$

in contradiction to the preceding. Similarly, we cannot have $c = \partial c_2$ for a singular two-cube c_2 on $\mathbb{R}^2 - \{0\}$, for this would imply that

$$\int_c \omega = \int_{\partial c_2} \omega = \int_{c_2} d\omega = 0 \quad (\text{since } d\omega = 0),$$

in contradiction to the preceding.