TOPICS IN MODERN GEOMETRY: HOMEWORK 2 MODEL SOLUTIONS TO MARKED PROBLEMS

(3) Let $R = \mathbb{C}[x, y, z]$. Consider the ideals $I = (xy+y^2, xz+yz)$ and $J = (xy^2+y^3, xz+yz)$.

(a) Show that $J \subseteq I$ but $J \neq I$.

If $f \in J$, then

$$\begin{aligned} f &= (xy^2 + y^3)f_1(x, y) + (xz + yz)f_2(x, y) \\ &= (xy + y^2)(yf_1(x, y)) + (xz + yz)f_2(x, y) \\ &\in I. \end{aligned}$$

Thus, $J \subseteq I$. On the other hand, whereas $xy + y^2 \in I$ we can show that $xy + y^2 \notin J$. Any element of J is of the form $(xy^2+y^3)f_1(x, y)+(xz+yz)f_2(x, y)$, so in particular, any degree 2 polynomial in J must be a constant multiple $xz + yz$. But $xy + y^2$ is not a constant multiple of $xz + yz$. Therefore, $J \neq I$.

(b) Show that $\mathbb{V}(I) = \mathbb{V}(J)$.

We compute

$$\begin{aligned} \mathbb{V}(I) &= \{(x + y)y = 0 \text{ and } (x + y)z = 0\} \\ &= \{x = -y\} \cup \{y = z = 0\}; \\ \mathbb{V}(J) &= \{(x + y)y^2 = 0 \text{ and } (x + y)z = 0\} \\ &= \{x = -y\} \cup \{y = z = 0\}. \end{aligned}$$

Thus, $\mathbb{V}(I) = \mathbb{V}(J)$.

(c) Prove that J is not a radical ideal.

SOLUTION 1: By Hilbert's Nullstellensatz, $\mathbb{I}(\mathbb{V}(I)) = \mathrm{rad}(I)$ and $\mathbb{I}(\mathbb{V}(J)) = \mathrm{rad}(J)$. By (b), $\mathbb{V}(I) = \mathbb{V}(J)$, so we can take the vanishing ideal of both sides to obtain $\mathrm{rad}(I) = \mathrm{rad}(J)$. By (a), $J \subsetneq I \subseteq \mathrm{rad}(I) = \mathrm{rad}(J)$, so J is not a radical ideal.

SOLUTION 2: As shown in part (a), $xy + y^2 \notin J$. However, $(xy + y^2)^2 = (x + y)(xy^2 + y^3) \in J$. Thus, by the definition of the radical, $xy + y^2 \in \mathrm{rad}(J)$; therefore, $J \neq \mathrm{rad}(J)$ and J is not a radical ideal.

(4) Consider the variety $X \subseteq \mathbb{A}^4_{w,x,y,z}$ with four defining equations given via the following matrix equations:

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}^2 = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

Decompose X as a union of finitely many irreducible varieties.

The points of X are described by the matrix equation

$$\begin{pmatrix} w^2 + xy & wx + xz \\ wy + yz & xy + z^2 \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

We rewrite this condition as

$$w^2 + xy = w \text{ and } x(w + z - 1) = 0 \text{ and } y(w + z - 1) = 0 \text{ and } xy + z^2 = z.$$

If $\begin{pmatrix} w & x \\ y & z \end{pmatrix} \in X$, then either $x = y = 0$ or $w + z = 1$.

In the case where $x = y = 0$, we have

$$w^2 = w \text{ and } z^2 = z,$$

so $w \in 0, 1$ and $z \in 0, 1$. We obtain the following four points $P_1, P_2, P_3, P_4 \in X$:

$$P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad P_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad P_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the case where $w + z = 1$, we observe that

$$\begin{aligned} w^2 + xy = w &\implies (1 - z)^2 + xy = 1 - z \\ &\implies 1 - 2z + z^2 + xy = 1 - z \\ &\implies z^2 + xy = z. \end{aligned}$$

Thus, the condition that $z^2 + xy = z$ is redundant, and $\mathbb{I}(X \cap \{w + z = 1\}) = (w + z - 1, w^2 + xy - w)$. Furthermore, note that $P_2, P_3 \in \mathbb{V}((w + z - 1, w^2 + xy - w))$, but $P_1, P_4 \notin \mathbb{V}((w + z - 1, w^2 + xy - w))$.

We have the following irredundant decomposition:

$$X = \mathbb{V}((w + z - 1, w^2 + xy - w)) \cup \{P_1\} \cup \{P_4\}.$$

The point varieties $\{P_1\}$ and $\{P_4\}$ are irreducible because $\mathbb{C}[P_1] \cong \mathbb{C}$ and $\mathbb{C}[P_4] \cong \mathbb{C}$ are domains. Moreover,

$$\mathbb{C}[w, x, y, z]/(w + z - 1, w^2 + xy - w) \cong \mathbb{C}[w, x, y]/(w^2 + xy - w)$$

is a domain because $w^2 + xy - w$ is an irreducible polynomial (and so $(w^2 + xy - w)$ is a prime ideal). Thus, $\mathbb{V}((w + z - 1, w^2 + xy - w))$ an irreducible variety.

(5) Prove that the hyperbola $H = \{(x, y) \in \mathbb{A}^2 : xy = 1\}$ is not isomorphic to $\mathbb{A}^1$.

Any isomorphism between affine varieties induces a $\mathbb{C}$-linear isomorphism between the coordinate rings. Thus, it suffices to show that the coordinate rings $\mathbb{C}[H]$ and $\mathbb{C}[A^1]$ are not isomorphic as $\mathbb{C}$-algebras. We have $\mathbb{C}[H] = \mathbb{C}[x, y]/(xy - 1) = \mathbb{C}[x, x^{-1}]$ and $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[x]$. Under any $\mathbb{C}$-linear ring homomorphism

$$f : \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x],$$

$f(x)$ must be an invertible element of $\mathbb{C}[x]$, but $\mathbb{C}[x]^\times = \mathbb{C}$, so $f(x) = a$ is constant. By $\mathbb{C}$-linearity, $f(a) = af(1) = a$, so $f(x) = f(a)$ and f is not injective. In particular, f cannot be an isomorphism, and the two coordinate rings are not isomorphic.