# Lecture 14: Tangent space, singularities and dimension

Let us start by supposing that $X = \mathbb{V}(f) \subset \mathbb{A}^n$ is an *affine hypersurface*—i.e. an affine variety defined by a single irreducible nonconstant equation $f \in \mathbb{C}[x_1, \dots, x_n]$.

## 1 The tangent space $T_pX$

**Definition 21.** A *tangent line* to $p \in X$ is a line $L \subset \mathbb{A}^n$ which intersects $X$ at $p$ with multiplicity $\geq 2$.

**What does this condition for $L$ to be tangent to $p \in X$ mean?** We can parameterise the line as $L = \{(p_1 + m_1 t, \dots, p_n + m_n t) : t \in \mathbb{C}\}$, where $m_i$ is the slope of $L$ in the $x_i$-direction. Now substitute $x_i = p_i + m_i t$ into $f$ and expand out as a polynomial in terms of $t$ to get:

$$f(p_1 + m_1 t, \dots, p_n + m_n t) = f(p) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_p m_i t + O(t^2)$$

where $\left.\frac{\partial f}{\partial x_i}\right|_p \in \mathbb{C}$ is the constant obtained by evaluating $\frac{\partial f}{\partial x_i}$ at $p$.[1] The condition for $L$ to be a tangent line at $p \in X$ means that $t = 0$ is a double root of this expression (i.e. that the coefficients of the $t^0$ and $t^1$ terms vanish). Therefore, for $L$ to be tangent we require

$$\sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_p m_i t = 0 \implies \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_p (x_i - p_i) = 0$$

where the condition on the right is independent of the choice of $m_i$, and hence independent of the choice of $L$.

**Definition 22.** The *tangent space to $p \in X$* is the subspace $T_pX \subset \mathbb{A}^n$ defined by this linear equation, i.e.

$$T_pX = \left\{ q \in \mathbb{A}^n : \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_p (q_i - p_i) = 0 \right\}$$

**Example 23.**

1. Suppose $X = \mathbb{V}(x^2 - 3xy + y^2 + 2x - y + 1) \subset \mathbb{A}^2_{x,y}$. Then at the point $p = (3, 2)$ we have

$$\left.\frac{\partial f}{\partial x}\right|_p = (2x - 3y + 2)|_p = 2, \qquad \left.\frac{\partial f}{\partial y}\right|_p = (-3x + 2y - 1)|_p = -6,$$

so the tangent space $T_pX \subset \mathbb{A}^2$ is the line defined by the equation:

$$2(x-3) - 6(y-2) = 0 \implies x - 3y + 3 = 0.$$

[1] In this setting differentiation is understood purely formally—i.e. $\frac{\partial}{\partial x_i}$ is an operation on polynomials which satisfies the Leibniz rule $\frac{\partial (fg)}{\partial x_i} = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}$ and sends $x_i^n \mapsto n x_i^{n-1}$, $x_j \mapsto 0$ if $j \neq i$, and $\lambda \mapsto 0$ if $\lambda \in \mathbb{C}$. It is defined for arbitrary polynomials by extending $\mathbb{C}$-linearly.

2. **Projective hypersurfaces.** For projective hypersurface $X = \mathbb{V}(f) \subset \mathbb{P}^n$ we can define the (projective) tangent space as a linear subspace of $\mathbb{P}^n$, by the formula:

$$T_p^{\text{proj}} X = \left\{ q \in \mathbb{P}^n : \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x_i} \right|_p q_i = 0 \right\}$$

To check that we actually have $p \in T_p^{\text{proj}} X$ we need to use *Euler's formula*, which says that $\sum_{i=0}^n \frac{\partial f}{\partial x_i} x_i = \deg(f) f$ for a homogeneous polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$.

**Exercises**: (1) Prove Euler's formula. (2) Suppose $U_i \subset \mathbb{P}^n$ is a standard affine chart containing $p \in X$, and $T_p X_{(i)}$ is the affine tangent space of $p \in X_{(i)}$ in $U_i \cong \mathbb{A}^n$. Show that $T_p X_{(i)} = (T_p^{\text{proj}} X)_{(i)}$.

# 2 Singularities

Note that the equation defining the tangent space $T_p X$ is nonzero as long as at least one of the partial derivatives $\frac{\partial f}{\partial x_i}$ does not vanish at $p$.

**Definition 24.** The point $p \in X$ is *singular* if $\left. \frac{\partial f}{\partial x_i} \right|_p = 0$ for all $i = 1, \dots, n$, and *nonsingular* (or *smooth*) otherwise. The *singular locus of* $X$ is the set of all singular points of $X$

$$\operatorname{sing}(X) = \left\{ p \in X : \left. \frac{\partial f}{\partial x_i} \right|_p = 0, \quad \forall i = 1, \dots, n \right\}$$

A hypersurface $X$ is *nonsingular* if $\operatorname{sing}(X) = \emptyset$.

**Remark.** The reason that we call nonsingular points 'smooth' is that, by the inverse function theorem, the nonsingular points of $X$ are precisely the points where $X$ is a *manifold*.

In the case we are considering of a hypersurface $X \subset \mathbb{A}^n$, the tangent space is either $T_p X \cong \mathbb{A}^{n-1}$ if $p$ is nonsingular or $T_p X \cong \mathbb{A}^n$ (i.e. the whole space) if $p$ is singular.

**Proposition 25.** *The nonsingular locus* $X \setminus \operatorname{sing}(X)$ *is a dense Zariski open subset of* $X$.

*Proof.* The singular locus $\operatorname{sing}(X) \subset X$ is a Zariski closed subset since it is defined by the vanishing of the following polynomials

$$\operatorname{sing}(X) = \mathbb{V}\left( f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right) \subset \mathbb{A}^n.$$

Therefore we only need to show that there is at least one nonsingular point $p \in X$. If there is no nonsingular point then $\frac{\partial f}{\partial x_i} \in \mathbb{I}(X) = \langle f \rangle$ for all $i = 1, \dots, n$. But, thinking of $\frac{\partial f}{\partial x_i}$ as a polynomial in $x_i$, we have $\deg_{x_i} \frac{\partial f}{\partial x_i} = \deg_{x_i} f - 1 < \deg_{x_i} f$. Since $f$ is irreducible, if $0 \neq g \in \langle f \rangle$ then $\deg_{x_i} g \geq \deg_{x_i} f$. Therefore we must have $\frac{\partial f}{\partial x_i} = 0$ for all $i$. Over $\mathbb{C}$ only possibility is that $f$ is a constant function, which contradicts our assumptions on $X$. $\square$

**Example 26.** The affine variety $X = \mathbb{V}(x^3 + 3x^2 - y^2) \subset \mathbb{A}^3$ is singular at the point $(0,0)$ and nonsingular elsewhere, since

$$\operatorname{sing}(X) = \mathbb{V}\left(x^3 + 3x^2 - y^2, 3x^2 + 6x, -2y\right) = \mathbb{V}(x, y) = \{(0,0)\} \subset \mathbb{A}^2.$$

(*Note*: if we had just considered the ideal $\mathbb{V}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, and forgotten to include the equation $f(x,y) = x^3 + 3x^2 - y^2$ defining $X$, then we may end up thinking that $(-2,0)$ is also a singular point of $X$. However $(-2,0) \notin X$ since $f(-2,0) = 4 \neq 0$.)

## 3 The general case

Now suppose that $X \subset \mathbb{A}^n$ is any affine algebraic variety. The general definition of the tangent space is similar to the hypersurface case.

**Definition 27.** The *tangent space to* $p \in X$ is the subspace $T_pX \subset \mathbb{A}^n$ defined by the linear equations

$$T_pX = \left\{ q \in \mathbb{A}^n : \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_p (q_i - p_i) = 0, \quad \forall f \in \mathbb{I}(X) \right\}.$$

## 4 Dimension

**Definition 28.** The *Jacobian matrix* $\mathrm{Jac}(I)$ of an ideal $I = \langle f_1, \dots, f_r \rangle \subset \mathbb{C}[x_1, \dots, x_n]$ is the $r \times n$ matrix of partial derivatives

$$\mathrm{Jac}(I) = \left( \frac{\partial f_i}{\partial x_j} \right)_{i=1,\dots,r,\, j=1,\dots,n}$$

The following proposition lets us define the *dimension* of $X$.

**Proposition 29.** *The function* $\dim T_\bullet X \colon X \mapsto \mathbb{Z}$*, where* $(\dim T_\bullet X)(p) = \dim T_pX$*, is an upper-semicontinuous function on* $X$ *with respect to the Zariski topology (which is just a fancy way of saying that the sets* $X_d = \{p \in X : \dim T_pX \geq d\}$ *are all Zariski closed).*

*Proof.* The dimension of tangent space at a point $p \in X$ is given by

$$\begin{aligned}
\dim T_pX &= \dim \left\{ q \in \mathbb{A}^n : \sum_{j=1}^{n} \left.\frac{\partial f_i}{\partial x_j}\right|_p (q_j - p_j) = 0, \quad \forall i = 1, \dots, r \right\} \\
&= \dim \left\{ q \in \mathbb{A}^n : \left.\mathrm{Jac}(\mathbb{I}(X))\right|_p \cdot \begin{pmatrix} q_1 - p_1 \\ \vdots \\ q_n - p_n \end{pmatrix} = 0 \right\} \\
&= \dim \ker \left.\mathrm{Jac}(\mathbb{I}(X))\right|_p .
\end{aligned}$$

Now we have $p \in X_d \iff \dim T_pX \geq d \iff \mathrm{rank}\, \left.\mathrm{Jac}(\mathbb{I}(X))\right|_p \leq n - d$, and this happens if and only if every $(n-d+1) \times (n-d+1)$ minor of $\left.\mathrm{Jac}(\mathbb{I}(X))\right|_p$ vanishes. Each of these minors is a determinant of a matrix with polynomial entries, and hence a polynomial. Therefore $X_d \subseteq X$ is Zariski closed, since $X_d$ is defined by the vanishing of some polynomials. $\square$

This means that there is a well-defined lowest value $d_{\min}$ of $\dim T_\bullet X$ on a dense open subset of $X$, i.e. $d_{\min}$ is the value of $d$ such that $X_d = X$ and $X_{d+1} \subsetneq X$.

**Definition 30.**

1. This lower bound $d_{\min}$ for $\dim T_\bullet X$ is called the *dimension* of $X$, and denoted $\dim(X)$.
2. We call a point $p \in X$ *nonsingular* if $\dim T_pX = \dim(X)$ and *singular* otherwise. The *singular locus of* $X$ is the (Zariski closed) subset $\mathrm{sing}(X) \subset X$ of all singular points of $X$. A variety $X$ is *nonsingular* if $\mathrm{sing}(X) = \emptyset$.

**Other ways of defining dimension.** It is important to know that there are other ways of defining the dimension of an algebraic variety which are more algebraic. For example, the *transcendence degree* $\mathrm{trdeg}_{\mathbb{C}} F$ of a field extension $F/\mathbb{C}$ is defined to be the size of a maximal set of algebraically independent elements $\{t_1, \dots, t_k\} \subset F$ (i.e. $t_1, \dots, t_k$ do not satisfy any polynomial equation with coefficients in $\mathbb{C}$). For an algebraic variety $X$ it turns out that $\dim(X) = \mathrm{trdeg}_{\mathbb{C}}\, \mathbb{C}(X)$. See Reid's *Undergraduate Algebraic Geometry* §6 for a discussion.