Fields, Forms and Flows 3/34

Problem Sheet 8
Due: Wednesday 5 December
To hand in: FFF3: 1, 2, 4, 7   FFF34: 1, 2, 7, 8

 Problems marked * are not examinable (but are still worth trying!)

1. Identify the degree, k, of the following forms on $\mathbb{R}^n$, and compute their exterior derivatives.
   (a) $f(x) = \sin(x^1)x^2$
   (b) $\alpha(x) = \cos x^1\, dx^2 + dx^3$
   (c) $\beta = f\alpha$, where f and α are given above.
   (d) $\gamma = df \wedge \alpha$, where f and α are given above.

2. (a) Let α be a k-form. Show that
$$d(\alpha \wedge d\alpha) = 0$$
   if k is even.
   (b) Construct a one-form α for which $d(\alpha \wedge d\alpha) \neq 0$. (Note that since $d(\alpha \wedge d\alpha)$ is a 4-form in this case, you will need $n \geq 4$. In fact, you can take $n = 4$.)

3. (a) Show that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, where k is the degree of α, implies, for scalar functions f, g and vector fields $\mathbf{E}$ and $\mathbf{B}$ on $\mathbb{R}^3$, that
$$\begin{aligned}
\nabla\,(fg) &= f\,\nabla\, g + g\,\nabla\, f,\\
\nabla\times\,(f\mathbf{E}) &= \nabla\, f\times\mathbf{E} + f\,\nabla\times\,\mathbf{E},\\
\nabla\cdot\,(f\mathbf{B}) &= \nabla\, f\cdot\mathbf{B} + f\,\nabla\cdot\,\mathbf{B},\\
\nabla\cdot\,(\mathbf{E}\times\mathbf{B}) &= \nabla\times\,\mathbf{E}\cdot\mathbf{B} - \mathbf{E}\cdot\,\nabla\times\,\mathbf{B}.
\end{aligned}$$
   (b) Show that $d^2 = 0$ implies that $\nabla\times\nabla f = 0$ and $\nabla\cdot\nabla\times\mathbf{E} = 0$ for functions f and vector fields $\mathbf{E}$ on $\mathbb{R}^3$.

4. Let
$$\omega = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}.$$
   be a two-form on $\mathbb{R}^3 - 0$ (ie, $\mathbb{R}^3$ with the origin removed). Show that $d\omega = 0$.

5. * Let $A(x)$ be an $n \times n$ matrix whose components $A_{ij}(x)$ are smooth functions on $\mathbb{R}^n$.

   (a) Let $\alpha_{(i)}$ be the 1-form on $\mathbb{R}^n$ given by

   $$\alpha_{(i)} = \sum_{j=1}^{n} A_{ij} dx^j.$$

   Show that

   $$\alpha_{(1)} \wedge \cdots \wedge \alpha_{(n)} = \det A \, dx^1 \wedge \cdots \wedge dx^n.$$

   (b) The rs-th minor of A, denoted a_{rs}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the rth row and the sth column from A. Use the preceding to obtain the formula

   $$\sum_{r=1}^{n} A_{pr} a_{sr} = (-1)^{p-s} \det A \, \delta_{p,s}.$$

   (In fact, this calculation can be generalised to obtain a formula for the determinant of A as a sum of products of determinants of $k \times k$ and $(n-k) \times (n-k)$ submatrices of A. This is called the Laplace expansion of the determinant.)

   (c) Let $u : \mathbb{R}^n \to \mathbb{R}^n$, and let

   $$A^i_j(x) = \frac{\partial u^i}{\partial x^j}.$$

   Let

   $$\beta = \frac{1}{n} \sum_{i=1}^{n} (-1)^{i-1} u^i \, du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^n.$$

   Show that

   $$d\beta = \det A \, dx^1 \wedge \cdots \wedge dx^n.$$

6. * Let $A \in \mathbb{R}^{2n \times 2n}$ be an antisymmetric $2n \times 2n$ matrix. Consider the 2-form α on $\mathbb{R}^{2n}$ given by

   $$\alpha = \frac{1}{2} A_{ij} dx^i \wedge dx^j,$$

   where we are employing the summation convention. The *pfaffian* of A, denoted $\operatorname{pf} A$, is defined by the following relation:

   $$\frac{1}{n!} \underbrace{\alpha \wedge \cdots \wedge \alpha}_{n \text{ times}} = \operatorname{pf} A \, dx^1 \wedge \cdots \wedge dx^{2n}.$$

   Here, the left-hand side contains the wedge product of α with itself n times, which is a $2n$-form (as is the right-hand side).

   (a) Let A be the 2×2 antisymmetric matrix whose nonzero elements are given by $A_{12} = 1$ and $A_{21} = -1$. Show that $\operatorname{pf} A = 1$.

   (b) Let A be the 4×4 antisymmetric matrix whose nonzero elements are given by $A_{12} = 3$, $A_{34} = -5$, $A_{21} = -3$, and $A_{43} = 5$. Show that $\operatorname{pf} A = -15$.

(c) Show that

$$\operatorname{pf}(\lambda A) = \lambda^n \operatorname{pf} A.$$

(d) Let M be a $2n \times 2n$ matrix. Show that

$$\operatorname{pf}(M^T A M) = \det M \operatorname{pf} A.$$

The result of the previous question 5a may be helpful.

(e) Show that the eigenvalues of $A^T A$ are real and nonnegative, and that there exists an orthonormal basis for $\mathbb{R}^{2n}$, denoted $\mathbf{u}_{(1)}, \dots, \mathbf{u}_{(n)}$, $\mathbf{v}_{(1)}, \dots, \mathbf{v}_{(n)}$, such that

$$A\mathbf{u}_{(j)} = \lambda_{r(j)} \mathbf{v}_{(j)}, \quad A\mathbf{v}_{(j)} = -\lambda_{r(j)} \mathbf{u}_{(j)}, \quad 1 \le j \le n,$$

where $\lambda^2_{r(j)}$ is an eigenvalue of $A^T A$, which can depend on j but might be the same for different j's (note that the number of distinct eigenvalues of $A^T A$ is less than $2n$; in fact, the number of distinct eigenvalues of $A^T A$ is at most n). (Suggestion: Show that $T := A^T A = -A^2$ is positive semidefinite symmetric, i.e. i) T is symmetric, ii) $\mathbf{w} \cdot T\mathbf{w} \ge 0$ for all $\mathbf{w} \in \mathbb{R}^{2n}$. Use the fact that the eigenvalues of T are real and nonnegative and that there exists an orthonormal basis of eigenvectors of T. Show that the subspace of eigenvectors of T with given eigenvalue λ_j is necessarily even dimensional.)

(f) Show that

$$(\operatorname{pf} A)^2 = \det A.$$

7. Write down the pull-backs of the differential forms dx, dy and dz in $\mathbb{R}^3$ under the map

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta. \end{aligned}$$

from polar to cartesian coordinates. (In effect, this is expressing dx, dy and dz in terms of dr, $d\theta$ and $d\phi$.) Also, compute the pull-back of $dx \wedge dy \wedge dz$.

8. Let $\mathbb{R}^2 = \{(P, V)\}$. Suppose there is a 1-form $q = q_1 dP + q_2 dV$ and functions $U = U(P, V)$, $T = T(P, V)$ such that

$$dU = q - P\,dV,$$

$$d\left(\frac{1}{T} q\right) = 0.$$

Show that

$$\frac{1}{T} dT \wedge q = dP \wedge dV.$$

[Note: If P,V,T, U and q represent the pressure, volume, temperature, internal energy, and heat of a homogeneous substance (eg, a gas), the first two relations above correspond to the first and second laws of thermodynamics. The third relation, which you are asked to derive, has been called Carnot's formula. It was known to Sadi Carnot in 1832 before either the first or second law had been formulated, and it turns out that many of the basic results of thermodynamics can be derived from it alone. See JH Hannay, American Journal of Physics **74**, pp. 134-140, 2006.]

9. Let a, b and c be three smooth functions on $\mathbb{R}^2 = \{(x, y)\}$ such that any two of the 1-forms da, db and dc are linearly independent. Then we can expand any one of these 1-forms in terms of the other two. For example, for da we write

$$da = \left(\frac{\partial a}{\partial b}\right)_c db + \left(\frac{\partial a}{\partial c}\right)_b dc.$$

The coefficient $(\partial a/\partial b)_c$ may be interpreted as the derivative of a with respect to b if c is held fixed. Similarly, db can be expanded in terms of da and dc, and dc expanded in terms of da and db. Show that

$$\left(\frac{\partial a}{\partial b}\right)_c \left(\frac{\partial b}{\partial c}\right)_a \left(\frac{\partial c}{\partial a}\right)_b = -1.$$