Numerical Analysis 23 — Sheet 4

Aitken's Δ^2 Method, Newton-Raphson in higher dimensions, Steepest Decent

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Please hand in questions 1, 2(b), 3(a) by 11am on Thursday 28 February.

1. (a) Rewriting the cubic equation $x^3 + 4x^2 - 10 = 0$ as $x = \frac{1}{2}\sqrt{10 - x^3}$ and using an initial guess $x_0 = 1.5$, use fixed point iteration 6 times to estimate a root of the cubic equation.

   (b) By using Aitken extrapolation, generate 5 better estimates and thereby the root to 4 decimal places.

2. (a) Suppose that A is any non singular matrix. Show that one iteration of the Newton Raphson method will give the correct solution to the problem $A\mathbf{x} = \mathbf{b}$ for *any initial guess.*

   (b) If

   $$f(x, y) = ax^2 + by + c\,, \qquad g(x, y) = dx + e\,,$$

   where a, b, c, d and e are constants, show that the Newton-Raphson method will find a solution ($b, d \neq 0$) in *two* iterations for any initial guess.

   (c) Suppose that

   $$f(x, y) = x^2 - y^2\,, \qquad g(x, y) = 1 + xy\,.$$

   Find two real roots of the system. Show that if the initial guess is $(x_0, y_0) = (\alpha, \alpha)$ then the Newton Raphson method can *never* converge to a root.

3. (a) Let $g(x, y) = x^2 + y^2 + axy$ where $0 < |a| < 2$ is a constant. Show that starting from the point $(x, y) = (\beta, \beta)$, only one step of the Steepest Descent algorithm is needed to find a minimum. Is this still true for a more general starting point?

   (b) Show that if $g(x, y)$ is actually just a function of $x^2 + y^2$, that is, $g(x, y) = G(x^2 + y^2)$, and has a minimum at the origin, then one step of the Steepest Descent algorithm will always be enough to ensure convergence to the minimum from *any* starting point.

4. Consider using the Steepest Descent method to find the root of $f(x, y) = g(x, y) = 0$ where $f(x, y) = x + y$ and $g(x, y) = x + 1$. Prove inductively that starting from the initial guess $x = y = 0$, successive estimates take the following form for $n \geq 0$,

   $$(x_{2n}, y_{2n}) = (-1 + 2^{-n}, 1 - 2^{-n})$$

   and

   $$(x_{2n+1}, y_{2n+1}) = (-1 + 2^{-n-1}, 1 - 2^{-n}).$$