UNIVERSITY OF BRISTOL

Examination for the Degree of M.Sci. (Level M)

DIFFERENTIABLE MANIFOLDS 34

MATH M2900

(Paper Code MATH-M2900)

January 2016, 2 hours 30 minutes

This paper contains **FOUR** *questions, all of which will be used for assessment.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Unless stated otherwise, all functions, vector fields, differentiable forms and maps are taken to be smooth.

$$\left(F_*\mathbb{X}\right)\left(F(x)\right) = F'(x)\cdot\mathbb{X}(x)$$

$$L_{\mathbb{Y}}\omega = \left.\frac{\partial}{\partial s}\right|_{s=0}\Psi_s^*\omega$$

$$\left.\frac{\partial}{\partial t}\hat{\Phi}_t^*\omega\right|_{t=\tau} = \hat{\Phi}_\tau^* L_{\hat{\mathbb{X}}_\tau}\omega$$

$$L_{\mathbb{X}} = i_{\mathbb{X}}d + di_{\mathbb{X}}$$

$$\partial c = \sum_{j=1}^{k}\sum_{\alpha=0,1}(-1)^{j+\alpha}c_{(j,\alpha)}$$

Do not turn over until instructed.

1. (25 marks)

   (a) (5 marks) Let $\Phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

   $$\Phi_t(x, y, z) = (x, y + tx, z + t(1 + t)x + aty).$$

   Find the unique value of $a \in \mathbb{R}$ for which Φ_t is a one-parameter subgroup of diffeomorphisms. For this value of a, compute the vector field $\mathbb{X}$ which generates Φ_t.

   (b) (12 marks) Let A, B be the open sets in $\mathbb{R}^2$ given by

   $$A = \{(x, y) \,|\, y > 0\}, \quad B = \{(u, v) \,|\, v^2 > 2u\}.$$

   Let $F : A \to B$ be the diffeomorphism given by

   $$F(x, y) = \left(\tfrac{1}{2}(x^2 - y^2), x\right),$$

   and let $\mathbb{X}$, $\mathbb{Y}$ be the vector fields on A given by

   $$\mathbb{X}(x, y) = (1, 1), \quad \mathbb{Y}(x, y) = (y, -x).$$

   i. Compute $[\mathbb{X}, \mathbb{Y}]$.

   ii. Find F^{-1}.

   iii. Compute $(F_*\mathbb{X})(u, v)$ and $(F_*\mathbb{Y})(u, v)$.

   iv. Let $\mathbb{W}$ and $\mathbb{Z}$ be the vector fields on B given by

   $$\mathbb{W}(u, v) = (v - f, 1), \quad \mathbb{Z}(u, v) = (2vf, f),$$

   where $f(u, v) = (v^2 - 2u)^{1/2}$. Compute $[\mathbb{W}, \mathbb{Z}]$.

   (c) (8 marks) Let $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ be a one-parameter subgroup of diffeomorphisms generated by a vector field $\mathbb{X}$, and let f be a function on $\mathbb{R}^n$.

   Show that $\Phi_t^* f = (1 + t)f$ for all t if and only if $f = 0$.

Continued...

2. (25 marks)

Consider the second-order partial differential equation,

$$\frac{\partial \phi}{\partial x} - \frac{\partial^2 \phi}{\partial y^2} = f - h, \tag{1}$$

where $\phi = \phi(x, y)$ and f, h are functions of x, y, ϕ and $\partial\phi/\partial y$.

(a) (2 marks) By letting $u = \phi$ and $v = \partial\phi/\partial y$, show that Eq. (1) can be written as the following pair of first-order partial differential equations:

$$\begin{aligned} \frac{\partial u}{\partial y} &= v, \\ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= f(x, y, u, v) - h(x, y, u, v). \end{aligned} \tag{2}$$

(b) (3 marks) Suppose we augment the system Eq. (2) with the following two equations,

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= f(x, y, u, v) + h(x, y, u, v), \\ \frac{\partial v}{\partial x} &= g(x, y, u, v), \end{aligned} \tag{3}$$

to obtain a system of four first-order partial differential equations for u and v. Find two vector fields $\mathbb{X}(x, y, u, v)$, $\mathbb{Y}(x, y, u, v)$ of the form

$$\begin{aligned} \mathbb{X} &= (1, 0, A, B), \\ \mathbb{Y} &= (0, 1, C, D), \end{aligned}$$

such that a necessary condition for the system Eqs. (2)–(3) to have a solution for all initial data is given by

$$[\mathbb{X}, \mathbb{Y}] = 0.$$

You should express the components A, B, C, and D explicitly in terms of x, y, u, v, f, g and/or h.

Question 2 continued overleaf...

(c) (8 marks) Let

$$f = p(y)v, \quad g = -v, \quad h = q(y)v.$$

Assuming that $v \neq 0$, show that the necessary condition $[\mathbb{X}, \mathbb{Y}] = 0$ may be reduced to a single equation involving p, p' and q, which you should find. Given that $q = \alpha/y$, where α is a constant, find the general solution p of this equation. *(Hint: The answer is given by*

$$p = ay^{-\alpha} - \frac{y}{1+\alpha}, \tag{4}$$

where a is a constant, but to receive full marks you should derive this result.)

(d) (12 marks) Taking $a = 0$ and $\alpha = 1$ in Eq. (4) above, find the solution $u(x, y)$, $v(x, y)$ of the system Eqs. (2)–(3) with initial data

$$u(0, 1) = 0, \quad v(0, 1) = 1.$$

Continued...

3. (25 marks)

   (a) (12 marks) Let α be the 1-form and β the 2-form on $\mathbb{R}^3$ given by

   $$\alpha = y^2\,dx - x^2\,dy,$$
   $$\beta = z\,dx \wedge dz,$$

   and let $\mathbb{X}$ be the vector field on $\mathbb{R}^3$ given by

   $$\mathbb{X} = (e^x, e^y, 0).$$

   i. Compute $\alpha \wedge \beta$, combining terms where possible.
   ii. Compute $d\alpha$, combining terms where possible.
   iii. Compute $i_{\mathbb{X}}\alpha$, combining terms where possible.
   iv. Compute $L_{\mathbb{X}}\alpha$, combining terms where possible.

Question 3 continued overleaf...

(b) (13 marks) You are given the following statement of the Poincaré Lemma: If $\hat{\Phi}_t$ is a one-parameter family of diffeomorphisms on $U \subset \mathbb{R}^n$ and $\hat{\mathbb{X}}_t$ the time-dependent vector field defined by

$$\hat{\mathbb{X}}_t \circ \hat{\Phi}_t = \frac{\partial}{\partial t} \hat{\Phi}_t \,,$$

and if β is a closed k-form on U such that

$$\hat{\Phi}_1^* \beta = \beta, \quad \lim_{\epsilon \to 0} \hat{\Phi}_\epsilon^* \beta = 0,$$

then $\beta = d\alpha$, where

$$\alpha = \int_0^1 \hat{\Phi}_t^* (i_{\hat{\mathbb{X}}_t} \beta) \, dt. \tag{5}$$

In what follows, let $U = \mathbb{R}^4 = \{(w, x, y, z\}$, and let β be the three-form on $\mathbb{R}^4$ given by

$$\beta = w(x+y)z \, dw \wedge dx \wedge dy + wxy \, dw \wedge (dx - dy) \wedge dz$$

i. Show that $d\beta = 0$.

ii. Let $\hat{\Phi}_t : U \to U$ be given by

$$\hat{\Phi}_t(w, x, y, z) = (w, tx, y, z).$$

Find $\hat{\mathbb{X}}_t$ as defined above, and show that

$$\lim_{t \to 0} \hat{\Phi}_t^* \beta = 0.$$

iii. Using the formula (5) above, find a two-form α on $\mathbb{R}^4$ such that

$$\beta = d\alpha.$$

4. (25 marks total)

   (a) (13 marks) Let $c : I^2 \to \mathbb{R}^2$ be the singular 2-cube given by

   $$c(s,t) = (st, s^m t^n)\,,$$

   where m and n are positive integers. Let (x, y) denote cartesian coordinates on $\mathbb{R}^2$, and let ω be the 1-form on $\mathbb{R}^2$ given by

   $$\omega = x\,dy.$$

   i. Compute $c^*\omega$.
   ii. Compute $c^*d\omega$.
   iii. Compute $\int_c d\omega$.
   iv. Without using Stokes' theorem, compute $\int_{\partial c} \omega$.

   (b) (12 marks) Let $\mathbb{X}$ and $\mathbb{Y}$ be vector fields on $\mathbb{R}^n$ such that $[\mathbb{X}, \mathbb{Y}] = 0$, and let Φ_t and Ψ_s denote their respective flows. Let $c : I^2 \to \mathbb{R}^n$ denote the singular 2-cube given by

   $$c(s,t) = \Psi_s(\Phi_t(0)),$$

   where 0 denotes the origin in $\mathbb{R}^n$. Let $\omega = \omega_i\,dx^i$ be a 1-form on $\mathbb{R}^n$.

   i. Show that

   $$\frac{\partial c}{\partial t}(s,t) = \mathbb{X}(c(s,t)), \quad \frac{\partial c}{\partial s}(s,t) = \mathbb{Y}(c(s,t)).$$

   ii. Suppose that $i_{\mathbb{X}}\omega = i_{\mathbb{Y}}\omega = 0$. Show that

   $$\int_c d\omega = 0.$$