Fields, Forms and Flows 3/34

Solution Sheet 7

1. There are many ways to do this. The suggestion is to start with $f^{(3)} \wedge f^{(8)} \wedge f^{(6)} \wedge f^{(4)} \wedge f^{(1)}$, then move $f^{(1)}$ to the left by a sequence of transpositions with its neighbours until it gets to the front. $f^{(1)}$ has to move through 4 factors, so 4 transpositions are required, and we get no net sign:
$$f^{(3)} \wedge f^{(8)} \wedge f^{(6)} \wedge f^{(4)} \wedge f^{(1)} = +f^{(1)} \wedge f^{(3)} \wedge f^{(8)} \wedge f^{(6)} \wedge f^{(4)}.$$
Next, move $f^{(4)}$ to the left to where it should be, namely just after $f^{(3)}$. It has to move through two factors, so two transpositions are required, and there is no sign change.
$$f^{(1)} \wedge f^{(3)} \wedge f^{(8)} \wedge f^{(6)} \wedge f^{(4)} = +f^{(1)} \wedge f^{(3)} \wedge f^{(4)} \wedge f^{(8)} \wedge f^{(6)}.$$
Finally, swap $f^{(8)}$ and $f^{(6)}$ (which produces a sign change) to get
$$f^{(1)} \wedge f^{(3)} \wedge f^{(4)} \wedge f^{(8)} \wedge f^{(6)} = -f^{(1)} \wedge f^{(3)} \wedge f^{(4)} \wedge f^{(6)} \wedge f^{(8)}.$$
So
$$f^{(3)} \wedge f^{(8)} \wedge f^{(6)} \wedge f^{(4)} \wedge f^{(1)} = -f^{(1)} \wedge f^{(3)} \wedge f^{(4)} \wedge f^{(6)} \wedge f^{(8)}.$$
So $s = -1$.

2. Let $a = a_i f^{(i)}$ be a 1-form on $\mathbb{R}^n$ and $b = \frac{1}{2} b_{jk} f^{(j)} \wedge f^{(k)}$ be a 2-form on $\mathbb{R}^n$ with
$$a_i = \begin{cases} \alpha, & i = 1, \\ 0, & \text{otherwise} \end{cases}, \quad b_{jk} = \begin{cases} \beta, & j = 2, k = 3, \\ -\beta, & j = 3, k = 2, \\ 0, & \text{otherwise.} \end{cases}$$
Let $c = a \wedge b$. We have that
$$c_{123} = c(e_{(1)}, e_{(2)}, e_{(3)}) = \tfrac{1}{2}\left(a(e_{(1)})b(e_{(2)}, e_{(3)}) - a(e_{(1)})b(e_{(3)}, e_{(2)})\right) \\ + (\text{terms involving } a(e_{(j)}),\ j \neq 1)$$
But $a(e_{(j)})$ vanishes unless $j = 1$, so that
$$c_{123} = \tfrac{1}{2}(\alpha\beta - \alpha(-\beta)) = \alpha\beta.$$
Clearly this is equal to $a_1 b_{23} = \alpha\beta$.
On the other hand,
$$c_{231} = c_{123} = \alpha\beta,$$
since 231 is obtained from 123 by an even permutation, and, in general,
$$w_{i_{\sigma(1)} \cdots i_{\sigma(k)}} = \operatorname{sgn}(\sigma) w_{i_1 \cdots i_k},$$
while $a_2 b_{31} = 0$. Thus, in general,
$$c_{ijk} \neq a_i b_{jk}.$$
It is still nevertheless true that
$$\frac{1}{3!} c_{ijk} f^{(i)} \wedge f^{(j)} \wedge f^{(k)} = \frac{1}{2!} a_i b_{jk} f^{(i)} \wedge f^{(j)} \wedge f^{(k)}. \tag{1}$$
The reason is that, in general, $w_{i_1 \cdots i_k} f^{(i_1)} \wedge \cdots \wedge f^{(i_k)} = 0$ (where a sum over the i_j's is implied) does not imply that the coefficients $w_{i_1 \cdots i_k}$ vanish. The basis k-forms $f^{(i_1)} \wedge \cdots \wedge f^{(i_k)}$ are not linearly independent, although the restricted set of basis k-forms in which $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ are linearly independent.

3. (a) Let a be a nonzero $(n-1)$-form on $\mathbb{R}^n$. Let us show that there is at least one basis 1-form $f^{(r)}$ for which $f^{(r)} \wedge a \neq 0$. Introduce the shorthand

$$F^{(j)} = f^{(1)} \wedge \cdots f^{(j-1)} \wedge f^{(j+1)} \wedge \cdots \wedge f^{(n)}$$

for the basis $(n-1)$-form given by the consecutive wedge product of the $f^{(i)}$'s with $f^{(j)}$ omitted. We expand a as

$$a = a_j F^{(j)}.$$

Then

$$f^{(j)} \wedge a = (-1)^{j-1} a_j f^{(1)} \wedge \cdots \wedge f^{(n)},$$

which vanishes if and only if $a_j = 0$. Since $a \neq 0$, at least one of the a_j's mustn't vanish. For definiteness, let's assume

$$a_n \neq 0.$$

Let

$$g^{(j)} = \left((-1)^{j-1} f^{(j)} - (-1)^{n-1} \frac{a_j}{a_n} f^{(n)}\right), \quad 1 \leq j \leq n-1.$$

Then

$$g^{(j)} \wedge a = \left(a_j - \frac{a_j}{a_n} a_n\right) f^{(1)} \wedge \cdots \wedge f^{(n)} = 0.$$

The $g^{(j)}$'s are clearly linearly independent.

Claim that

$$a = (-1)^{(n-1)(n-2)/2} a_n \, g^{(1)} \wedge g^{(2)} \wedge \ldots \wedge g^{(n-1)}.$$

Indeed, we have that

$$g^{(1)} \wedge g^{(2)} \wedge \cdots \wedge g^{(n-1)} = \left(f^{(1)} - (-1)^{n-1} \frac{a_1}{a_n} f^{(n)}\right) \wedge \left(-f^{(2)} - (-1)^{n-1} \frac{a_2}{a_n} f^{(n)}\right) \wedge \cdots$$
$$\cdots \wedge \left((-1)^{n-2} f^{(n-1)} - (-1)^{n-1} \frac{a_{n-1}}{a_n} f^{(n)}\right).$$

All terms in the wedge product which involve two or more factors of $f^{(n)}$ vanish, so the surviving terms include the one term in which no factor of $f^{(n)}$ appears, namely

$$\begin{aligned} f^{(1)} \wedge \left(-f^{(2)}\right) \wedge \cdots \wedge \left((-1)^{n-2} f^{(n-1)}\right) &= (-1)^{(n-1)(n-2)/2} f^{(1)} \wedge \cdots \wedge f^{(n-1)} \\ &= (-1)^{(n-1)(n-2)/2} F^{(n)}, \end{aligned}$$

and the $(n-1)$-terms in which a single factor of $f^{(n)}$ appears. For example the term in which the jth factor in the wedge product is taken to be $f^{(n)}$ (times a scalar factor) is given by

$$\begin{aligned} & f^{(1)} \wedge \left(-f^{(2)}\right) \wedge \cdots \wedge \left((-1)^{j-2} f^{(j-1)}\right) \wedge \left(-(-1)^{n-1} \frac{a_j}{a_n} f^{(n)}\right) \wedge \left((-1)^j f^{(j+1)}\right) \wedge \cdots \\ & \qquad \cdots \wedge \left((-1)^{n-2} f^{(n-1)}\right) \\ &= -(-1)^{(n-1)(n-2)/2+(n-1)-(j-1)} \frac{a_j}{a_n} f^{(1)} \wedge \cdots f^{(j-1)} \wedge f^{(n)} \wedge f^{(j+1)} \wedge \cdots \wedge f^{(n-1)} \\ &= -(-1)^{(n-1)(n-2)/2+(n-1)-(j-1)+(n-j-1)} \frac{a_j}{a_n} F^{(j)} \\ &= (-1)^{(n-1)(n-2)/2} \frac{a_j}{a_n} F^{(j)}. \end{aligned}$$

Therefore,

$$g^{(1)} \wedge g^{(2)} \wedge \cdots \wedge g^{(n-1)} = (-1)^{(n-1)(n-2)/2} \left(F^{(n)} + \sum_{j=1}^{n-1} \frac{a_j}{a_n} F^{(j)}\right) = (-1)^{(n-1)(n-2)/2} \frac{1}{a_n} a_j F^{(j)},$$

where in this last expression a summation over j is to be understood. Therefore,

$$(-1)^{(n-1)(n-2)/2} a_n\, g^{(1)} \wedge g^{(2)} \wedge \cdots \wedge g^{(n-1)} = a,$$

as required.

(b) We have that

$$r \wedge r = 2\, a \wedge b \wedge c \wedge d,$$

since the terms $(a \wedge b) \wedge (a \wedge b)$ and $(c \wedge d) \wedge (c \wedge d)$ vanish by antisymmetry, and $(a \wedge b) \wedge (c \wedge d) = (c \wedge d) \wedge (a \wedge b)$. Then

$$r \wedge r(e_{(i_1)}, e_{(i_2)}, e_{(i_3)}, e_{(i_4)}) = M(i_1, i_2, i_3, i_4),$$

where $M(i_1, i_2, i_3, i_4)$ is the determinant of the 4×4 matrix whose rth row consists of the i_rth components of a, b, c, d. This may be regarded as a minor of the $n \times 4$ rectangular matrix whose columns are a, b, c, d. If all of these minors vanish, then a, b, c, d are linearly dependent, contrary to assumption. Therefore, $r \wedge r \neq 0$. This implies that we cannot find one-forms $g^{(1)}$ and $g^{(2)}$ for which $r = g^{(1)} \wedge g^{(2)}$, for this would imply that $r \wedge r = 0$. Thus, the property of $(n-1)$-forms on $\mathbb{R}^n$ explored in the first part of this question does not extend to 2-forms on $\mathbb{R}^4$ (but it holds, of course, for 2-forms on $\mathbb{R}^3$).

4. (a) We have that

$$i_v a = v^j a_j.$$

If we associate v and a with conventional vectors, $i_v a$ corresponds to the dot product of v and a.

(b) We have that

$$i_v b = (-v^2 b_{12} + v^3 b_{31}) f^{(1)} + (v^1 b_{12} - v^3 b_{23}) f^{(2)} + (v^2 b_{23} - v^1 b_{31}) f^{(3)}.$$

If we associate v with the vector $\mathbf{V} = (v^1, v^2, v^3)$ and b with the vector $\mathbf{B} = (b_{23}, b_{31}, b_{12})$ and $i_v b = (-v^2 b_{12} + v^3 b_{31}, v^1 b_{12} - v^3 b_{23}, v^2 b_{23} - v^1 b_{31})$, then $i_v b$ corresponds to $\mathbf{B} \times \mathbf{V}$.

(c) We have that

$$i_v \rho = \rho_{123} \left(v^3 f^{(1)} \wedge f^{(2)} + v^1 f^{(2)} \wedge f^{(3)} + v^2 f^{(3)} \wedge f^{(1)} \right).$$

If we associate v with the vector $\mathbf{V} = (v^1, v^2, v^3)$ and $i_v \rho$ with the vector $\mathbf{W} = \rho_{123}(v^1, v^2, v^3)$, then $\mathbf{W} = \rho_{123} \mathbf{V}$.