Fields, Forms and Flows 3/34

Problem Sheet 5
Due: Wednesday 14 November (2 weeks)
To hand in: FFF3: 1, 2 FFF34: 1, 2

 Problems marked * are not examinable (but are still worth trying!)

1. An nth order ODE can be reduced to a system of n first-order ODE's (we had examples in Problem Sheet 2). In a similar way, an nth order PDE can be reduced to a system of first-order PDE's. You might wonder whether such a system can be solved using the Frobenius theorem (provided the necessary condition is satisfied), but, as the following examples show, this is generally not possible.

   (a) Consider the one-dimensional wave equation,

   $$u_{tt} - u_{xx} = 0,$$

   where $u_{tt} = \partial^2 u/\partial t^2$, etc. Write the wave equation as a system of first-order partial differential equations for $\phi(x,t) = (\phi^1(x,t), \phi^2(x,t), \phi^3(x,t))$, where $\phi^1 = u$, $\phi^2 = u_x$ and $\phi^3 = u_t$. Explain why this system cannot be expressed in the form $\phi^\alpha_x = f^\alpha(x,t,\phi)$, $\phi^\alpha_t = g^\alpha(x,t,\phi)$.

   (b) Same as the preceding question but for the one-dimensional heat equation,

   $$u_t - u_{xx} = 0.$$

2. Consider the system of coupled first-order nonlinear partial differential equations,

   $$\frac{\partial u}{\partial x} = f^1(x,y,u,v), \quad \frac{\partial v}{\partial x} = f^2(x,y,u,v),$$
   $$\frac{\partial u}{\partial y} = g^1(x,y,u,v), \quad \frac{\partial v}{\partial y} = g^2(x,y,u,v),$$

   with initial data

   $$u(0,0) = r_0, \quad v(0,0) = s_0.$$

   (a) Derive necessary and sufficient conditions on f and g in order for this system to have a solution, and give an expression for u and v in terms of the flows of two vector fields $\mathbb{X}$ and $\mathbb{Y}$ on $\mathbb{R}^4$.

(b) For the case

$$f(x, y, r, s) = (a, b),$$

where a and b are constants, show that the condition is satisfied if and only if g is of the form

$$g(x, y, r, s) = (G^1(y, r - ax, s - bx), G^2(y, r - ax, s - bx)),$$

where G^1 and G^2 are arbitrary functions of three variables. (Remark: If you find the "only if" part difficult just show the "if" part.)

(c) For

$$G^1 = -G^2 = (r - ax)(s - bx)$$

and $r_0 = s_0 = 1$, find $u(x, y)$ and $v(x, y)$ explicitly in a neighbourhood of $(0, 0)$.

3. Consider the following system of coupled first-order nonlinear partial differential equations for the scalar-valued function $u = u(x, y)$, which is a generalization of the system we considered in Section 1.11.2 of the Notes:

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f(x, y, u),$$
$$c\frac{\partial u}{\partial x} + d\frac{\partial u}{\partial y} = g(x, y, u). \qquad (*)$$

Here a, b, c and d are given smooth functions of x and y and we assume that

$$ad - bc \neq 0 \text{ for all } (x, y) \in \mathbb{R}^2.$$

Show that the system has a unique solution for all initial data $u(x_0, y_0) = u_0$ if and only if

$$[\mathbb{V}, \mathbb{W}] = r\mathbb{V} + s\mathbb{W}, \qquad (**)$$

where $\mathbb{V}$ and $\mathbb{W}$ are the vector fields on $\mathbb{R}^3$ given by

$$\mathbb{V}(x, y, z) = (a(x, y), b(x, y), f(x, y, z)), \quad \mathbb{W}(x, y, z) = (c(x, y), d(x, y), g(x, y, z)),$$

for some functions $r(x, y, z)$ and $s(x, y, z)$. ((**) is equivalent to saying the Jacobi bracket of $\mathbb{W}$ and $\mathbb{V}$ lies in the plane spanned by $\mathbb{W}$ and $\mathbb{V}$.) (Suggestion: show that the problem can be reduced to one to which the Frobenius theorem applies. A more general version of this result is discussed in Section 1.12 of the notes, which you can consult, but you can also answer this question without referring to Section 1.12.)