Topics in Modern Geometry - Final Assessment

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Part I: topology and affine varieties

Exercise 1

(a)

Claim. *Let $(X, \mathcal{T})$ be a compact topological space, and let Y be any closed set of X. Prove that Y with the subspace topology $\mathcal{T}_Y$ is compact.*

Proof. May $\mathcal{U} \subseteq \mathcal{T}_Y$ be a collection of open sets of Y that cover Y. Then, as Y is closed in X, we have that $X \setminus Y$ is open, as it is the complementary of a closed set, and we have that every open $U \in \mathcal{U}$ is of the form $V_U \cap Y$ with $V_U \in \mathcal{T}$ for being the subspace topology. Therefore, $X = (X \setminus Y) \cup \bigcup_{U \in \mathcal{U}} V_U$. We have a collection of open subsets that covers X, therefore, as X is compact, we have a finite cover: $X = (X \setminus Y) \cup \bigcup_{i=1}^{m} V_{U_i}$. Thus, as $Y \subseteq X$ and no element of Y is in $X \setminus Y$, we have that $Y \subseteq \bigcup_{i=1}^{m} V_{U_i}$, and intersecting those V's with Y we get that $Y \subseteq \bigcup_{i=1}^{m} U_i$, having found a finite subcollection of $\mathcal{U}$ such that $Y = \bigcup_{i=1}^{m} U_i$ (note that $\supseteq$ is trivial as every U is in Y). $\square$

(b)

Claim. *Prove that $Spec(\mathbb{C}[x_1, ..., x_n])$ is compact.*

Proof. For notation simplicity, we are going to call $X = Spec(\mathbb{C}[x_1, ..., x_n])$. Let $\{U_\alpha\}_{\alpha \in \Gamma}$ be a cover of open sets of X. We know that these open sets, in the Zariski topology, are the complementary sets of the closed sets V_I. Therefore, we have

$$X = \bigcup_{\alpha \in \Gamma} U_\alpha = \bigcup_{\alpha \in \Gamma} X \setminus V_{I_\alpha} = X \setminus \bigcap_{\alpha \in \Gamma} V_{I_\alpha} \iff \bigcap_{\alpha \in \Gamma} V_{I_\alpha} = \emptyset$$

Proceeding as in the lecture notes in Lecture 2, we know that

$$\bigcap_{\alpha \in \Gamma} V_{I_\alpha} = V_J, \quad J := \sum_{\alpha \in \Gamma} I_\alpha$$

Thus we have that $V_J = \emptyset$. Now, if $J \neq \mathbb{C}[x_1, ..., x_n]$, then there would be a maximal (and therefore prime) ideal M containing J, which would mean that $M \in V_J$, but that can't be true as $V_J = \emptyset$. Therefore, $J = \mathbb{C}[x_1, ..., x_n]$, which means that $1 \in J$. Hence 1 can be written as a finite sum of elements belonging to a finite number of ideals in our collection, that is:

$$\exists \ \alpha_1, ..., \alpha_m \in \Gamma \ \ \& \ \ \forall 1 \leq j \leq m \ \exists i_j \in I_{\alpha_j} \ \ : \ \ i_1 + ... + i_m = 1$$

Therefore, as 1 belongs to the finite sum of those ideals:

$$\sum_{j=1}^{m} I_{\alpha_j} = \mathbb{C}[x_1, ..., x_n] \Longrightarrow V_K = \emptyset, \ \ K := \sum_{j=1}^{m} I_{\alpha_j}$$

And thus

$$V_K = \bigcap_{j=1}^{m} V_{I_{\alpha_j}} = \emptyset \Longrightarrow X \setminus \bigcap_{j=1}^{m} V_{I_{\alpha_j}} = \bigcup_{j=1}^{m} X \setminus V_{I_{\alpha_j}} = \bigcup_{j=1}^{m} U_{\alpha_j} = X$$

So we have found a finite subcollection of open sets of the original collection such that they cover the whole space, and thus $Spec(\mathbb{C}[x_1, ..., x_n])$ is a compact space with the Zariski topology. □

Exercise 2

(a)

Claim. *Fix $f \in \mathbb{C}[x_1, ..., x_n]$. Find an isomorphism $\mathbb{A}^n \to G_f$, and prove that it is an isomorphism.*

Proof. Let $\Phi : \mathbb{A}^n \to G_f$, $(x_1, ..., x_n) \mapsto (x_1, ..., x_n, f(x_1, ..., x_n))$. First of all, it is obvious that Φ_i is a polynomial $\forall 1 \leq i \leq n+1$.

It is trivial to see that Φ is an injection as $x \neq x' \Rightarrow x_i \neq x'_i$ for some $1 \leq i \leq n$ which means $\Phi_i(x) \neq \Phi_i(x') \Rightarrow \Phi(x) \neq \Phi(x')$.

Due to the definition of G_f, it is also obvious that Φ is surjective, as for any element $x = (x_1, \ldots, x_{n+1}) \in G_f$ we have that $x_{n+1} = f(x_1, ..., x_n)$, and so $x = \Phi(x_1, ..., x_n)$.

The inverse of Φ consists on projecting the first n elements of the points of G_f, and so it is trivial to see:

$$(\Phi \circ \Pi_{x_1,...,x_n})(x_1, ..., x_n, f(x)) = (x_1, ..., x_n, f(x))$$

$$(\Pi_{x_1,...,x_n} \circ \Phi)(x_1, ..., x_n) = (x_1, ..., x_n)$$

Trivially, the projection is a morphism, with a polynomial in each component ($\Pi_i(x) = x_i$). And therefore we have found an isomorphism between $\mathbb{A}^n$ and G_f. □

(b)

Claim. *Show that it does not hold true for rational functions F.*

Proof. Let $F(x) = \frac{1}{x} \in \mathbb{C}(x)$. We now have that $G_F = \{(x, \frac{1}{x}) : x \in \mathbb{C}\}$. But then this graph is the affine variety of a hyperbola: $G_F = \{xy = 1\}$. Therefore, if G_F were isomorphic to $\mathbb{A}^1$, then there would be an isomorphism between their coordinate rings, which are $\mathbb{C}[x, x^{-1}]$ and $\mathbb{C}[x]$, respectively. If there were such a ring isomorphism f, we would have that $f(xx^{-1}) = f(x)f(x^{-1}) = 1$, and thus $f(x)$ would be a unit of $C[x]$, which is only possible if it is a constant, so $f(x) = c \in \mathbb{C}$. But then, $f(c) = f(c * 1) = c * f(1) = c$. We would have therefore found two different polynomials mapping to the same polynomial by f, contradicting the fact that f was a ring isomorphism. Therefore G_F and $\mathbb{A}^1$ are not isomorphic either. □

Exercise 3

Let $J = (x^2y, x - z - 1) \leq \mathbb{C}[x, y, z]$ and $W = \mathbb{V}(J)$.

(a)

Let's find $a, b \in \mathbb{C}$ such that $f(x, y, z) = (x - 2)(x - z) + x^2(y + 1) + 5x - 2z + 3$ is equal in $\mathbb{C}[W]$ to $x^2 + ax + b$.

We know by Prop 1.2 in Lecture 6 that $\mathbb{C}[W] = \frac{\mathbb{C}[x,y,z]}{\mathbb{I}(W)}$.

Thus, by expanding the expression of f, we get

$$x^2 - xz - 2x + 2z + x^2y + x^2 + 5x - 2z + 3$$

. Now we will regroup the first two terms:

$$x(x - z) + 3x + x^2y + x^2 + 3$$

Considering that in $\mathbb{I}(W)$ $x^2y = 0$ and $x - z = 1$, we obtain that in $\mathbb{C}[W]$ f maps into

$$x^2 + 4x + 3$$

(b)

We want to decompose W in two irreducible varieties irredundantly. For that purpose, we consider the two cases when $x^2y = 0$. Either $x = 0$, in which case we are dealing with points which fulfill $z+1 = 0$, or $y = 0$, for which $x - z - 1 = 0$. Therefore, we have $W = \mathbb{V}(x, z+1) \cup \mathbb{V}(y, x-z-1) =: W_1 \cup W_2$. Now let's check that W_1 is not a subvariety of W_2 and vice versa:

$$(0, 1, -1) \in W_1 \setminus W_2 \Rightarrow W_1 \nsubseteq W_2$$

$$(1, 0, 0) \in W_2 \setminus W_1 \Rightarrow W_2 \nsubseteq W_1$$

We still need to prove that W_1, W_2 are indeed irreducible.

Firstly, we have that that $\mathbb{I}(\mathbb{V}(x, z+1)) = (x, z+1)$ and $\mathbb{I}(\mathbb{V}(y, x-z-1)) = (y, x-z-1)$, as both are radical ideals.

$\frac{\mathbb{C}[x,y,z]}{(x,z+1)} \cong \mathbb{C}[y]$, as x vanishes and z takes a constant value. And $\mathbb{C}[y]$ is a domain, so $(x, z+1) = \mathbb{I}(W_1)$ is a prime ideal, which means that W_1 is an irreducible variety.

$\frac{\mathbb{C}[x,y,z]}{(x,x-z-1)} \cong \mathbb{C}[x]$, as y vanishes and z is written in terms of x and constants. As $\mathbb{C}[x]$ is a domain, we again deduce that W_2 is irreducible.

(c)

From the expression of **(a)** it is obvious that f is not constant on W, as it corresponds to a parabola. However, in the irreducible component $W_1 = \mathbb{V}(x, z+1)$, as we have that $x = 0$, we get $f(x, y, z) = 3$, which is clearly constant.

Exercise 4

We are going to decompose $X = \mathbb{V}(y^2 - xz, z^2 - y^3) \subset \mathbb{A}^3$ into irreducible components irredundantly.

If either y or z vanishes then the other one does too. If they are both zero, we have the subvariety $\mathbb{V}(y, z)$.

Now we assume y and z to be nonzero.

$$z^2 = y^3,\ \ y^2 - xz = 0 \Longrightarrow y^4 = x^2z^2 = x^2y^3 \overset{y \neq 0}{\Longrightarrow} y = x^2$$

Now, as $z^2 - y^3 = 0$, we have that $z = \pm\sqrt{y^3} = \pm\sqrt{x^6} = \pm x^3$. But in order to $y^2 - xz = 0$ be fulfilled when $y - x^2 = 0$, it has to be $z = x^3$.

Therefore we have

$$X = \mathbb{V}(y, z) \cup \mathbb{V}(y - x^2, z - x^3) := X_1 \cup X_2$$

Now let's prove that both subvarieties are birationally equivalent to $\mathbb{A}^1$.

We define:

$$\Phi_1 : X_1 \longrightarrow \mathbb{A}^1,\ \ \Phi_1(x, 0, 0) = x, \qquad \Psi_1 : \mathbb{A}^1 \longrightarrow X_1,\ \ \Psi_1(t) = (t, 0, 0)$$

$$\Phi_2 : X_2 \longrightarrow \mathbb{A}^1,\ \ \Phi_2(x, x^2, x^3) = x, \qquad \Psi_2 : \mathbb{A}^1 \longrightarrow X_2,\ \ \Psi_2(t) = (t, t^2, t^3)$$

They are all rational maps as each component of each function is a polynomial on the input variables. Checking the conditions is quite straightforward:

$$(\Phi_1 \circ \Psi_1)(t) = \Phi_1(t, 0, 0) = t, \qquad (\Psi_1 \circ \Phi_1)(x, 0, 0) = \Psi_1(x) = (x, 0, 0)$$

$$(\Phi_2 \circ \Psi_2)(t) = \Phi_2(t, t^2, t^3) = t, \qquad (\Psi_2 \circ \Phi_2)(x, x^2, x^3) = \Psi_2(x) = (x, x^2, x^3)$$

Therefore the rational maps Φ_i and Ψ_i are inverse to each other, and thus X_1 and X_2 are birationally equivalent to $\mathbb{A}^{\mathbb{1}}$

Note: When taking the square root of a complex number c, as there are exactly two numbers a fulfilling $a^2 = c$, we can define $\sqrt{\cdot}$ to map c to the one solution which has a smaller argument in the interval $[-\pi, \pi)$.

Exercise 5

Let $X \subseteq \mathbb{A}^m_{x_1,\dots,x_m}$ and $Y \subseteq \mathbb{A}^n_{y_1,\dots,y_n}$ be irreducible varieties. Let $Z = X \times Y \subseteq \mathbb{A}^{m+n}$.

(a)

Claim. *Z is an affine variety.*

Proof. A point $z \in Z$ fulfills $(z_1, ..., z_m) \in X$ and $(z_{m+1}, ..., z_{m+n}) \in Y$. Thus, $(z_1, ..., z_m)$ vanishes all the polynomials generating X and $(z_{m+1}, ..., z_{m+n})$ vanishes all the polynomials generating Y.

We can now take the projections

$$\Pi_X : \mathbb{A}^{m+n} \to \mathbb{A}^m, (z_1, ..., z_{m+n}) \mapsto (z_1, ..., z_m)$$

$$\Pi_Y : \mathbb{A}^{m+n} \to \mathbb{A}^n, (z_1, ..., z_{m+n}) \mapsto (z_{m+1}, ..., z_{m+n})$$

We can assume that X and Y are generated by finitely generated ideals because $\mathbb{C}[x_1, ..., x_l]$ is Noetherian for any natural l. Thus, calling those finitely generated ideals $I_X = (f_1, ..., f_{r_X}), I_Y = (g_1, ..., g_{r_Y})$, we write

$$K = \{f_i \circ \Pi_X : 1 \leq i \leq r_X\} \cup \{g_j \circ \Pi_Y : 1 \leq j \leq r_Y\} \subset \mathbb{C}[z_1, ..., z_{m+n}]$$

. It is now straightforward to check that $Z = \mathbb{V}((K))$. A point belongs in Z if and only if its first m components vanish all the generating polynomials of I_X and its last n components vanish all the generating polynomials of I_Y, and thus Z is an affine variety given by the polynomials of the ideal generated by the elements of K. □

(b)

Claim. *If X and Y are irreducible, Z is irreducible.*

Proof. Let X and Y be irreducible, and suppose $Z = Z_1 \cup Z_2$ for two varieties Z_1, Z_2 such that $Z_1 \subsetneq Z$, $Z_2 \subsetneq Z$.

For $y \in Y$, we define $X_y = X \times \{y\}$. We have that, as Y is irreducible, X_y is irreducible as well, as it is homeomorphic with the product topology to X. Thus, as $X_y = (Z_1 \cap X_y) \cup (Z_2 \cap X_y)$, and it is irreducible, we have that either $X_y \subseteq Z_1$ or $X_y \subseteq Z_2$ (note that the Z_i's are varieties, and so are the X_y for being homeomorphic to X).

We now define $Y_1 = \{y \in Y : X_y \subseteq Z_1\}, Y_2 = \{y \in Y : X_y \subseteq Z_2\}$. Note that $y \in Y_1$ iff $X_y \subseteq Z_1$, which means that (x, y) vanishes all polynomials generating Z_1 for any $x \in X$.

Thus, if $Z_1 = \mathbb{V}(I_{Z_1})$, for each $h \in I_{Z_1}$ we consider the polynomials $h_x(y_1, ..., y_n) = h(x, y) = h(x_1, ..., x_m, y_1, ..., y_n)$, and we have that $Y_1 = \mathbb{V}((\{h_x | h \in I_{Z_1},\ x \in X\}))$.

We have that $Y = Y_1 \cup Y_2$, and Y is irreducible and Y_1 and Y_2 are varieties, so either $Y = Y_1$ or $Y = Y_2$.

If $Y = Y_1$

$$X \times Y = \bigcup_{y \in Y} X_y = Z_1$$

If $Y = Y_2$

$$X \times Y = \bigcup_{y \in Y} X_y = Z_2$$

In both cases we get a contradiction, as neither of the Z_i can contain the whole space Z. □

Part II: topology and affine varieties

Exercise 6

We will analyse the projective closure, points at infinity and singular points of:

(a)

$2x^2y^2 = x^2 + y^2$

First we get a homogenised polynomial where all terms have degree four:

$$f(x, y, z) = 2x^2y^2 - x^2z^2 - y^2z^2$$

Thus the projective closure is given by $X = \mathbb{V}(f) \subseteq \mathbb{P}^n$.
The points at infinity correspond to

$$X \cap \{z = 0\} = \mathbb{V}(2x^2y^2) \cap \mathbb{V}(z) = \{(0 : 1 : 0), (1 : 0 : 0)\}$$

Let's analyse singularities:

$\frac{\partial f}{\partial x} = 4xy^2 - 2xz^2$ (1) $\qquad$ $\frac{\partial f}{\partial y} = 4x^2y - 2yz^2$ (2) $\qquad$ $\frac{\partial f}{\partial z} = -2z(x^2 + y^2)$ (3)

To vanish (3) either $z = 0$ or $x = \pm iy$.

- $z = 0 \Longrightarrow 4xy^2 = 0$ and $4x^2y = 0 \Longrightarrow (0 : 1 : 0), (1 : 0 : 0) \in Sing(X)$ (it is trivial to check they belong in X).
- $x = \pm iy \Longrightarrow 4iy^3 - 2iyz^2 = 0$ and $-4y^3 - 2yz^2$. If we multiply the first equation by i and add both equations, we get $y^3 = 0 \Rightarrow y = 0$, thus yielding the singularity $(0 : 0 : 1) \in Sing(X)$

Therefore $Sing(X) = \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$

(b)

$y^2 = x^4 + 4ax + 3 \quad a \in \mathbb{C}$

We proceed similarly to **(a)**. First, we define: $g(x, y, z) = y^2z^2 - x^4 - 4axz^3 - 3z^4$ Hence the projective closure of the variety is given by $Y = \mathbb{V}(g)$.
The points at infinity are determined by $z = 0$, where we get

$$g(x, y, 0) = x^4 \Longrightarrow x = 0 \Longrightarrow Y \cap \{z = 0\} = \{(0 : 1 : 0)\}$$

So the only point at infinity is $(0 : 1 : 0)$.
We now analyse the singularities:

$$\frac{\partial g}{\partial x} = -4x^3 - 4az^3 \qquad \frac{\partial g}{\partial y} = 2yz^2 \qquad \frac{\partial g}{\partial x} = 2y^2z - 12axz^2 - 12z^3$$

The second equation forces either y or z to be zero:

- $y = 0 \Longrightarrow x^3 + az^3 = 0$ and $z^2(ax + z) = 0$. z being zero would force x being zero, leading to the origin, which is not a projective point. So, $z \neq 0$, and then $a = -\frac{x^3}{z^3}$ and $z = -ax$. Thus $a = \frac{-x^3}{-a^3x^3}$ has to stand. Note that x is nonzero as $z = -ax$. Thus $a^4 = 1$ is a solution to this. So for the four fourth roots of 1, which we can call ξ_i $1 \leq i \leq 4$ we get that $z = -\xi_i x$ vanishes the gradient, and $g(x, 0, -\xi_i x) = -x^4 + 4x^4 - 3x^4 = 0$.
- $z = 0 \Longrightarrow x = 0$. Thus $(0 : 1 : 0) \in Sing(Y)$ for all $a \in \mathbb{C}$.

Therefore, if $a^4 = 1$, then $Sing(Y) = \{(0 : 1 : 0), (1 : 0 : -a)\}$. Otherwise, $Sing(Y) = \{(0 : 1 : 0)\}$

Exercise 7

Claim. *Let $X = \mathbb{V}(f) \subset \mathbb{P}^n$ be a projective hypersurface defined by a nonconstant homogeneous polynomial $f \in \mathbb{C}[x_0, ..., x_n]$ and let $L \subset \mathbb{P}^n$ be a projective line (i.e. L is defined by $n-1$ linearly independent homogeneous polynomials of degree 1). Then $X \cap L \neq \emptyset$*

Proof. L is determined by a system of linearly independent equations of the form

$$a_{i0}x_0 + ... + a_{in}x_n = 0 \qquad 1 \leq i \leq n-1$$

We then have the matrix $A \in \mathfrak{M}_{(n-1)\times(n+1)}$ such that $A_{ij} = a_{ij} \quad 1 \leq i \leq n-1, 0 \leq j \leq n$. Considering the linear transformation that A describes from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n-1}$, and bearing in mind that A has rank $n-1$ because its rows are linearly independent, we get:

$$dim(kerA) = n+1-rank(A) = n+1-(n-1) = 2 \quad [1]$$

Thus we know there are two linearly independent vectors $u, v \in \mathbb{C}^{n+1}$ such that $\forall \mu, \lambda \in \mathbb{C} : \mu u + \lambda v \in L$

Therefore, the points of the intersection $L \cap X$ are given by the zeroes of $g(\mu, \lambda) := f(\mu u + \lambda v)$, which is a homogeneous polynomial in two variables. Note that each monomial of $f(\mu u + \lambda v)$ can be written as $c(\mu u_0 + \lambda v_0)^{i_0} \dots (\mu u_n + \lambda v_n)^{i_n}$ such that $\sum i_j = d$. By Newton's Binomial formula, each of the addends of the expressions $(\mu u_j + \lambda v_j)^{i_j}$ has degree i_j, and so f expands into terms in which λ and μ have exponents adding $\sum i_j = d$. Thus $g(\mu, \lambda)$ is indeed a homogeneous polynomial of degree $d > 0$. If all the monomials cancelled out then we would have $g(\mu, \lambda) = 0 \ \forall \mu, \forall \lambda$ and we would be finished.

Now we use that every nonconstant homogeneous polynomial in two variables can be factored in linear homogeneous terms, as proved in [2]:

Let $q(x, y)$ be a homogeneous polynomial in two variables of degree $d > 0$. We can assume q is not divisible by y, as otherwise we can just analyse $q(x, y) = y^k q(x, y)'$ where q' is not divisible by y and is a homogeneous polynomial of degree $d-k$. Note that if $d = k$ we are over, and otherwise we are in the case of our assumption. Now we write

$$q(x, y) = \sum_{i=0}^{d} a_i x^i y^{d-i} = y^d \sum_{i=0}^{d} a_i (\frac{x}{y})^i \overset{(*)}{=} y^d \prod_{i=1}^{d} (\alpha_i (\frac{x}{y}) + \beta_i) = \prod_{i=1}^{d} (\alpha_i x + \beta_i y)$$

$(*)$: Apply the Fundamental Theorem of Algebra to the polynomial over the one variable $\frac{x}{y}$.

Hence, this result together with the fact that $g(\mu, \lambda)$ is a nonconstant two-variable homogeneous polynomial helps us conclude that there is at least a linear factor $\alpha\mu + \beta\lambda$ with $(\alpha, \beta) \neq (0, 0)$.

Thus there exists a pair (μ_0, λ_0) such that $\alpha\mu_0 + \beta\lambda_0 = 0$, and it turns out $\mu_0 u + \lambda_0 v \in X \cap L$, as (μ_0, λ_0) vanishes $g(\mu, \lambda)$ and $A(\mu_0 u + \lambda_0 v) = 0$

□

Exercise 8

Let $C = \mathbb{V}(f(x, y)) \subset \mathbb{A}^2$ be the affine curve given by the equation

$$f(x, y) = (x^2 - 3)^2 + (y^2 - 3)^2 - 8$$

(a)

Let's check $(1, 1)$ is a nonsingular point of C. $\nabla f = (4x(x^2 - 3), 4y(y^2 - 3))$

$$\nabla f \mid_{(1,1)} = (-8, -8) \neq (0, 0)$$

$$f(1, 1) = (-2)^2 + (-2)^2 - 8 = 8 - 8 = 0$$

Thus $(1, 1)$ is a nonsingular point of C.

(b)

We now rewrite f as

$$f(x,y) = (x-1)^4 + 4(x-1)^3 - 8(x-1) + (y-1)^4 + 4(y-1)^3 - 8(y-1) \quad (1)$$

Let's prove that the rational function $\Phi(x,y) = \frac{y-1}{x-1}$ is regular at $(1,1)$ and find the value at that point.

We are looking for two polynomials $p(x,y), q(x,y)$ such that $\frac{y-1}{x-1} \sim \frac{p(x,y)}{q(x,y)}$, so we need p and q to be such that $(y-1)q(x,y) - (x-1)p(x,y) \in \mathbb{I}(C)$.

We have that in points of C

$$f(x,y) = (x-1)^4 + 4(x-1)^3 - 8(x-1) + (y-1)^4 + 4(y-1)^3 - 8(y-1) = 0$$
$$\implies (x-1)\left[(x-1)^3 + 4(x-1)^2 - 8\right] = -(y-1)\left[(y-1)^3 + 4(y-1)^2 - 8\right]$$

Therefore, if we choose

$$p(x,y) = (x-1)^3 + 4(x-1)^2 - 8 \qquad\qquad q(x,y) = -(y-1)^3 - 4(y-1)^2 + 8$$

we have that indeed $(y-1)q(x,y) - (x-1)p(x,y) \in \mathbb{I}(C)$.

Thus we can safely say that ϕ is regular at $(1,1)$ now, as we have found an expression which is in the same equivalence class and whose denominator is nonzero in $(1,1)$: $q(1,1) = 8$. We conclude then that $\Phi(1,1) = \frac{p(1,1)}{q(1,1)} = \frac{-8}{8} = -1$

(c)

By using $x-1$ as a uniformiser, we want to see that $\psi(x,y) = x+y-2$ is a is a regular function at $(1,1)$ with order of vanishing $v_{(1,1)}(\psi) = 4$.

It is clear that ψ is regular at $(1,1)$, as it is well defined in that given expression. Let's check it's order of vanishing is four, which is equivalent to saying that the order of vanishing of $\frac{\psi(x,y)}{(x-1)^4}$ is zero, as we have that $v_{(1,1)}(\psi(x,y) - v_{(1,1)}((x-1)^4) = v_{(1,1)}(\frac{\psi(x,y)}{(x-1)^4})$, and obviously $v_{(1,1)}((x-1)^4) = 4$ as $\frac{(x-1)^4}{(x-1)^4} = 1$ which is a regular nonzero function (**Lemma 32** in Lecture 15).

In X, the expression in (1) is fulfilled, and so we have that

$$8(x-1+y-1) = (x-1)^4 + 4(x-1)^3 + (y-1)^4 + (y-1)^4 + 4(y-1)^3$$
$$\implies \frac{\psi(x,y)}{(x-1)^4} = \frac{x+y-2}{(x-1)^4} = \frac{1}{8} + \frac{1}{2(x-1)} + \frac{1}{8}(\Phi(x,y))^4 + \frac{1}{2}(\Phi(x,y))^3\frac{1}{x-1}$$

We then have that

$$v_{(1,1)}\Big(\frac{\psi(x,y)}{(x-1)^4}\Big) \geq min\{v_{(1,1)}(\frac{1}{8} + \frac{1}{8}(\Phi(x,y))^4), v_{(1,1)}(\frac{1}{2(x-1)} + \frac{1}{2}(\Phi(x,y))^3\frac{1}{x-1})\} \ (2)$$

But now, $\frac{1}{8}(1 + (\Phi(x,y))^4)$ is regular and nonzero in $(1,1)$ as we can evaluate it and get the value $1/4$ (as in **(b)**). Thus that has order of vanishing zero.

The other term we can write as (using the the expression of Φ of **(b)**:

$$\Big(\frac{(x-1)^3 + 4(x-1)^2 - 8}{-(y-1)^3 - 4(y-1)^2 + 8}\Big)^3 \frac{1}{2(x-1)} + \frac{1}{2(x-1)}$$
$$= \frac{((x-1)^3 + 4(x-1)^2 - 8)^3 + (-(y-1)^3 - 4(y-1)^2 + 8)^3}{(-(y-1)^3 - 4(y-1)^2 + 8)^3 2(x-1)}$$

In that last expression, the numerator is regular (it's a polynomial) and it is zero in $(1,1)$ and so it has, by **Lemma 32**, a strictly positive order of vanishing.

The denominator's order of vanishing is 1, as it is the sum of its three factor's orders: 0 for the first one, for being a regular nonzero function on $(1,1)$, 0 for the second one for the same reason, and obviously $v_{(1,1)}(x-1) = 1$, as it is a uniformiser. Therefore this rational function has a vanishing order of $\lambda - 1$ with $\lambda > 0$, and thus in the expression (2) we get that $v_{(1,1)}\Big(\frac{\psi(x,y)}{(x-1)^4}\Big) \geq 0$. However, I failed to prove it is an equality.

Exercise 9

Claim. *Suppose that $p_1, ..., p_6$ are the vertices of a hexagon H drawn inside a conic $C \subset \mathbb{P}^2$, with edges $E_1 = \overline{p_1p_2}$, $E_2 = \overline{p_2p_3}, \ldots$, $E_6 = \overline{p_6p_1}$. Then the three points q_1, q_2, q_3 are collinear, where $q_i = E_i \cap E_{i+3}$ is the intersection point for a pair of opposite sides of H.*

Proof. First, we define two cubic curves containing all the p_i points and the q_j points.

We can take $C_1 = E_1 \cup E_3 \cup E_5$ and $C_2 = E_2 \cup E_4 \cup E_6$. First of all, as each of the edges is a line, we have that each line is given by an equation $ax + by + cz = 0$. Thus, each C_i, as the union of three of those lines, is the variety given by the zero locus of the product of three equations like that, thus being a cubic curve.

Secondly, we have that $p_1, p_2 \in E_1$, $p_3, p_4 \in E_3$, $p_5, p_6 \in E_5$, $q_1 \in E_1$, $q_2 \in E_5$ and $q_3 \in E_3$. Thus $\{p_1, p_2, p_3, p_4, p_5, p_6, q_1, q_2, q_3\} \subset C_1$. Furthermore, as $p_1, p_6 \in E_6$, $p_2, p_3 \in E_2$, $p_4, p_5 \in E_4$, and $q_1 \in E_4$, $q_2 \in E_2$, $q_3 \in E_6$, it follows that $\{p_1, p_2, p_3, p_4, p_5, p_6, q_1, q_2, q_3\} \subset C_2$.

We will rule out the case in which C is a degenerate conic, as then H wouldn't be a proper hexagon, having at least three points on one of the lines that would define C. Anyway, if C is degenerate, and we take three points of each of its lines, this reduces to the Pappus' Theorem. From now on we take C not degenerate, and so each q_i is different from the points in the conic. Otherwise C and a line E_i would have 3 points of intersection, which by Bézout's Theorem would mean that the line E_i would be in C, a contradiction as C is not degenerate. We can also argue that a point q_i and a point q_j with $i \neq j$ have to be different, as if they weren't, we would have $E_i \cap E_{i+3} = E_j \cap E_{j+3} = q_i = q_j$. But E_i shares a point with E_j or E_{j+3}, say they share p_k, and thus we would have $q_i \in E_i \cap E_{i+3} \cap E_j \cap E_{j+3} \subset \{p_k\}$, and thus $q_i = p_k$, which we proved to be impossible.

Thus C_1 and C_2 are two cubic curves that intersect at those 9 distinct points. Therefore, by the Cayley-Bacharach theorem, if we take the cubic curve $C \cup \overline{q_1q_2}$ –which contains $p_1, \ldots, p_6, q_1$ and q_2– then it must contain the ninth point q_3. And so we have that $q_3 \in C \cup \overline{q_1q_2}$. q_3 cannot lie on C, as if it did, then the conic C and the cubic curve C_1 would intersect in seven distinct points $\{p_1, \ldots, p_6, q_3\}$, which would contradict Bézout's Theorem. Thus we have that $q_3 \in \overline{q_1q_2}$, and so q_1, q_2 and q_3 are collinear.

Note that $C \cup \overline{q_1q_2}$ is also a cubic curve as it is given by the product of a conic's expression (which is a homogeneous polynomial of degree two) and a line of the form $ax + by + cz = 0$. □

References

[1] Rank-nullity theorem

[2] Stack Exchange Query
`https://math.stackexchange.com/questions/1463537/do-polynomials-in-two-variables-always-factor-in-linear-terms`