UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level H)

DIFFERENTIABLE MANIFOLDS 34

MATH M2900

(Paper Code MATH-M2900)

January 2015, 2 hours 30 minutes

This paper contains **five** *questions*
A candidate's **FOUR** *best answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Unless stated otherwise, all functions, vector fields, differentiable forms and maps are taken to be smooth.

$$\cosh(s+t) = \cosh s \cosh t + \sinh s \sinh t$$

$$(F_*\mathbb{X})\,(F(x)) = F'(x) \cdot \mathbb{X}(x)$$

$$L_{\mathbb{Y}}\omega = \left.\frac{\partial}{\partial s}\right|_{s=0} \Psi_s^*\omega$$

$$L_{\mathbb{X}} = i_{\mathbb{X}}d + di_{\mathbb{X}}$$

$$\partial c = \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} c_{(j,\alpha)}$$

Do not turn over until instructed.

1. (25 marks)

   (a) (6 marks)
   Let $\mathbb{X}(x, y) = (y, x)$ be a vector field on $\mathbb{R}^2$. Show that its flow $\Phi_t(x, y)$ is given by
   $$\Phi_t(x_0, y_0) = M(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$
   where
   $$M(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},$$
   and verify that $\Phi_s \circ \Phi_t = \Phi_{s+t}$.

   (b) (4 marks)
   Let $\mathbb{X} = (y, x)$ and $\mathbb{Y} = (\sin x, \sin y)$ be vector fields on $\mathbb{R}^2$. Compute $[\mathbb{X}, \mathbb{Y}]$.

   (c) (3 marks)
   Let σ be the permutation given by
   $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix},$$
   ie $\sigma(1) = 4$, $\sigma(2) = 1$, etc. Write σ as a product of transpositions and determine whether it is even or odd.

   (d) (12 marks)
   Let α be the 1-form and β the 2-form on $\mathbb{R}^3$ given by
   $$\alpha = xdy - ydx,$$
   $$\beta = xz\, dx \wedge dz,$$
   and let $\mathbb{X}$ be the vector field on $\mathbb{R}^3$ given by
   $$\mathbb{X} = (-x, y, 0).$$

   i. Compute $\alpha \wedge \beta$, combining terms where possible.
   ii. Compute $d\alpha$, combining terms where possible.
   iii. Compute $i_{\mathbb{X}}\alpha$, combining terms where possible.
   iv. Compute $L_{\mathbb{X}}\alpha$.

Continued...

2. (25 marks)

(a) Let A, B be the open sets in $\mathbb{R}^2$ given by

$$A = \{(x, y) \,|\, y \neq -1\}, \quad B = \{(u, v) \,|\, u \neq 1\}.$$

Let $F : A \to B$ be the diffeomorphism given by

$$F(x, y) = \left(\frac{y}{1+y}, x - y \right),$$

and let $\mathbb{X}$ be the vector field on U given by

$$\mathbb{X}(x, y) = (y, x).$$

i. (4 marks) Find F^{-1}.

ii. (5 marks) Compute $(F_*\mathbb{X})(u, v)$.

iii. (6 marks) Let $\mathbb{Y}$ be the vector field on B given by

$$\mathbb{Y}(u, v) = \big((1 - u)(u + v - uv), -v\big),$$

and let Ψ_t be the flow of $\mathbb{Y}$. Given $(u_0, v_0) \in B$ with $(u_0, v_0) \neq (0, 0)$, let $(u(t), v(t)) = \Psi_t(u_0, v_0)$. Show that

$$\lim_{t\to\infty} \left(\frac{u + v - uv}{u} \right)(t) = 1.$$

(b) Let $\mathbb{X}$ be a vector field on $\mathbb{R}^n$.

i. (4 marks) Let α and β be differential forms on $\mathbb{R}^n$. Show that

$$L_{\mathbb{X}}(\alpha \wedge \beta) = (L_{\mathbb{X}}\alpha) \wedge \beta + \alpha \wedge L_{\mathbb{X}}\beta.$$

If you wish, you may use the fact that

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$$

for a diffeomorphism F on $\mathbb{R}^n$,

ii. (6 marks) Let Δ be the set of differential forms on $\mathbb{R}^n$ (of any degree) such that if $\alpha \in \Delta$, then

$$L^r_{\mathbb{X}}\alpha = 0$$

for some sufficiently large positive integer r. Show that if $\alpha, \beta \in \Delta$, then $\alpha \wedge \beta \in \Delta$. (*Suggestion: use induction.*)

Continued...

3. (25 marks total).

Let

$$u : \mathbb{R}^2 \to \mathbb{R}^n; \quad (x, y) \mapsto u(x, y) = \big(u^1(x, y), \ldots, u^n(x, y)\big)$$

be a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^n$. Let $A(x, y)$ and $B(x, y)$ be $n \times n$ matrices depending smoothly on x and y. Consider the system of first-order partial differential equations given by

$$\begin{aligned} \frac{\partial u}{\partial x} &= A \cdot u, \\ \frac{\partial u}{\partial y} &= B \cdot u, \\ u(x_0, y_0) &= u_0 \in \mathbb{R}^n. \end{aligned} \tag{1}$$

(a) (5 marks) Show that a necessary condition for (1) to have a solution for all (x_0, y_0, u_0) is that

$$\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} = [B, A], \text{ where } [B, A] := BA - AB. \tag{2}$$

(b) (5 marks) Assuming that A depends only on y, find the flow Φ_t of the vector field $\mathbb{X}$ on $\mathbb{R}^{n+2}$ given by

$$\mathbb{X}(x, y, u) = (1, 0, A(y) \cdot u).$$

(c) (9 marks) Suppose that A_0 and B_0 are fixed $n \times n$ matrices such that

$$[B_0, A_0] = \lambda B_0$$

for some $\lambda \in \mathbb{R}$. Letting

$$A(x, y) := A_0 + f(y)B_0, \quad B(x, y) := g(y)B_0,$$

show that the condition (2) may be satisfied provided that f and g satisfy a certain condition, which you should state. Hence find the solution of (1) when $g(y) := 2y$ and $f(0) := 0$. You should express the solution in two distinct forms, namely

$$\text{i) } u(x, y) = P(x)Q(y) \cdot u_0, \text{ and ii) } u(x, y) = S(y)R(x) \cdot u_0,$$

where $P(x)$, $Q(y)$, $R(x)$ as $S(y)$ are $n \times n$ matrices.

Question 3 continued overleaf...

(d) (6 marks) Using the preceding results, show that (2) is satisfied for

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $A(x, y) := A_0 + 2y^2 B_0$, $B(x, y) := 2yB_0$. Letting $x_0 = y_0 = 0$, find $u(x, y)$, expressing your solution in the form

$$u = M(x, y) \cdot u_0.$$

The elements of the 2×2 matrix M should be expressed as explicit functions of x and y.

Continued...

4. (25 marks total)

You are given the following statement of the Poincaré Lemma: If $\hat{\Phi}_t$ is a one-parameter family of diffeomorphisms on $U \subset \mathbb{R}^n$ and $\hat{\mathbb{X}}_t$ the time-dependent vector field defined by

$$\hat{\mathbb{X}}_t \circ \hat{\Phi}_t = \frac{\partial}{\partial t}\hat{\Phi}_t \,,$$

and if β is a closed k-form on U such that

$$\hat{\Phi}_1^* \beta = \beta, \quad \lim_{\epsilon \to 0} \hat{\Phi}_\epsilon^* \beta = 0,$$

then $\beta = d\alpha$, where

$$\alpha = \int_0^1 \hat{\Phi}_t^*(i_{\hat{\mathbb{X}}_t}\beta)\, dt. \tag{3}$$

In what follows, let U be the open set in $\mathbb{R}^3$ given by

$$U = \{(x, y, z) \,|\, z \neq 0\}.$$

Let β be the two-form on U given by

$$\beta = \frac{xz\, dx \wedge dy - xy\, dx \wedge dz}{z^2}.$$

(a) (4 marks) Show that $d\beta = 0$.

(b) (6 marks) Let $\hat{\Phi}_t : U \to U$ be given by

$$\hat{\Phi}_t(x, y, z) = (tx, t^2 y, z).$$

Find $\hat{\mathbb{X}}_t$ as defined above, and show that

$$\lim_{t \to 0} \hat{\Phi}_t^* \beta = 0.$$

(c) (10 marks) Using the formula (3) above, find a one-form α on U such that

$$\beta = d\alpha.$$

Question 4 continued overleaf...

(d) (5 marks) Suppose that $\hat{\Psi}_t$ is another one-parameter family of diffeomorphisms on U such that

$$\hat{\Psi}_1^* \beta = \beta, \quad \lim_{\epsilon \to 0} \hat{\Psi}_\epsilon^* \beta = 0,$$

and that $\hat{\mathbb{Y}}_t$ is the time-dependent vector field defined by

$$\hat{\mathbb{Y}}_t \circ \hat{\Psi}_t = \frac{\partial}{\partial t} \hat{\Psi}_t.$$

Without using the Poincaré Lemma, show that

$$d \int_0^1 \left(\hat{\Phi}_t^* (i_{\hat{\mathbb{X}}_t} \beta) - \hat{\Psi}_t^* (i_{\hat{\mathbb{Y}}_t} \beta) \right) dt = 0. \tag{4}$$

Continued...

5. (25 marks total)

   (a) Let $c : I^3 \to \mathbb{R}^3$ be the singular 3-cube given by
   $$c(u, v, w) = (u, uv, uvw)\,.$$
   Let $x = (x, y, z)$ denote cartesian coordinates on $\mathbb{R}^3$, and let ω be the 2-form on $\mathbb{R}^3$ given by
   $$\omega = 2yz\, dz \wedge dx + z^2\, dy \wedge dx.$$

      i. (3 marks) Compute $c^*\omega$.
      ii. (3 marks) Compute $c^*d\omega$.
      iii. (2 marks) Compute $\int_c d\omega$.
      iv. (5 marks) Without using Stokes' theorem, compute $\int_{\partial c} \omega$.

   (b) (12 marks) Let ω be a closed k-form, $\mathbb{X}$ a smooth vector field and f a smooth function on $\mathbb{R}^n$. Let $\Psi_t : \mathbb{R}^n \to \mathbb{R}^n$ denote the flow of the vector field $\mathbb{Y} = f\mathbb{X}$, and let $c : I^k \to \mathbb{R}^n$ denote a singular k-cube in $\mathbb{R}^n$.

   Suppose that

   - $i_{\mathbb{X}}\omega = d\rho$ for some $(k-2)$-form ρ,
   - $L_{\mathbb{X}} f = 0$, and
   - $f \circ c_{(j,\alpha)} = 0$ for all $1 \le j \le k$ and $\alpha = 0$ or 1.

   Show that
   $$\frac{d}{dt} \int_c \Psi_t^* \omega = 0$$
   for all t.

End of examination.