Numerical Analysis 23 — Sheet 6

Romberg Integration and Gaussian Quadrature

Please hand in questions 2 and 3 by 11am on Thursday 14 March

1. Use Romberg integration to estimate
$$\int_0^1 \sin(x)\,\mathrm{d}x$$
as accurately as possible starting with the Trapezoidal estimates T_1, T_2 and T_4. Compare your best estimate with the exact result.

   Do the same for the integral
$$\int_0^1 x\ln(x)\,\mathrm{d}x\,.$$
You will find that the approximation is less accurate in the second case. Can you think of a reason why?

2. An approximation to the definite integral
$$\int_{-1}^1 x(t)\,\mathrm{d}t$$
is sought that is of the form $a\,x(-1/3) + b\,x(0) + c\,x(1/3)$ where a, b and c are constants. How should they be chosen to ensure that the resulting formula is exact for an arbitrary quadratic polynomial? Show that this formula is also exact for any odd function $x(t)$.

3. The Legendre polynomials are orthogonal polynomials on the interval $[-1, 1]$ with weight function $w(x) = 1$. Derive the first three Legendre polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$ by requiring the orthogonality conditions
$$\int_{-1}^1 P_0(x)\,P_1(x)\,\mathrm{d}x = \int_{-1}^1 P_0(x)\,P_2(x)\,\mathrm{d}x = \int_{-1}^1 P_1(x)\,P_2(x)\,\mathrm{d}x = 0\,,$$
and the standardization conditions
$$P_0(1) = P_1(1) = P_2(1) = 1\,.$$
Hence determine the weights w_j and the positions x_j in the Gauss Legendre quadrature formula
$$\int_{-1}^1 f(x)\,\mathrm{d}x \approx \sum_{j=1}^n w_j f(x_j)$$
for $n = 2$. Apply your result to approximate the integral
$$\int_{-1}^1 \frac{1}{x+3}\,\mathrm{d}x$$
by making only two evaluations of the integrand. Hence derive an estimate for $\ln(2)$.