Fields, Forms and Flows 3/34

Problem Sheet 2
Due: Wednesday 17 October
To hand in: DM3: 3, 4, 8, 9   DM34: 6(a), 8, 9

 Problems marked * are not examinable (but are still worth trying!)

1. Let $f(x) = ax^2 + bx + c$ be a function on $\mathbb{R}$, where a, b, c are constants and $a \neq 0$. According to the Inverse Function Theorem, which values of x have a neigbourhood $I_\delta(x)$ on which f is invertible?

2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(x, y) \mapsto (x^2 - y^2, 2xy)$. Compute F', and show that $\det F' = 0$ if and only if $x = y = 0$. Compute the inverse of F explicitly, and discuss how the result is consistent with the Inverse Function Theorem.

3. (a) If $f : \mathbb{R} \to \mathbb{R}$ (say f is smooth) satisfies $f'(a) \neq 0$ for all $a \in \mathbb{R}$, show that f is 1-1 on $\mathbb{R}$. (You may use Rolle's theorem – if f has a continuous derivative and $f(a) = f(b)$ for some $a \neq b$, then f' must vanish somewhere between a and b). (b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that $\det f'(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$, but that f is not 1-1.

4. * Let $f(x) = \frac{1}{2}x + x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(0) = \frac{1}{2}$, but that $f'(x)$ is not continuous at $x = 0$. Show that f is not 1-1 in any neighbourhood of 0. (Thus, the Inverse Function Theorem need not hold if f' is not continuous.)

5. * Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function on $\mathbb{R}^2$. Use the Inverse Function Theorem to show that f is not 1-1.

6. (a) Let
$$Y_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$
and let K be an arbitrary 2×2 matrix. Show that, for ϵ sufficiently small, the matrix $Y_0 + \epsilon K$ has a square root that is close to the matrix
$$X_0 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$
That is, for sufficiently small ϵ, there exists a matrix $X(\epsilon)$ depending smoothly on ϵ such that $X^2(\epsilon) = Y_0 + \epsilon K$ and $X(0) = X_0$. (Suggestion: identify the space of 2×2 matrices with $\mathbb{R}^4$, and apply the inverse function theorem to an appropriately defined map $F : \mathbb{R}^4 \to \mathbb{R}^4$.)

(b) Show that, for ϵ small and K an arbitrary 2×2 matrix, the matrix $I + \epsilon K$ does not necessarily have a real square root that is close to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(Suggestion: Let $K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.) This is in contrast to what happens for 1×1 matrices, ie real numbers. Let $y_0 > 0$, and let $x_0^2 = y_0$ (x_0 could be the positive or negative square root of y_0). Then for ϵ small, $y_0 + \epsilon$ always has a square root that is close to x_0.

7. Write the third-order nonautonomous ODE

$$\frac{d^3x}{dt^3} = \sin t \, \frac{x\ddot{x}}{\dot{x}^2}$$

as a first-order autonomous system on $\mathbb{R}^4$.

8. For $q \in \mathbb{R}$, write $\ddot{q} = 1$ as a first-order system, find the flow $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$, and verify that $\Phi_t \circ \Phi_s = \Phi_{t+s}$.

9. Let $\dot{x} = x^2$. Solve this (autonomous) differential equation, and show that the flow $\Phi_t(x_0)$ has a singularity at $t = T$, where T depends on x_0. Over what interval is the flow defined? Next, let $\dot{x} = x^2/(1 + x^2)$. Solve the differential equation, and show that the flow $\Phi_t(x_0)$ is defined for all t.

10. * Let

$$V(q) = \begin{cases} -\frac{3}{4} q^{4/3}, & q > 0, \\ 0, & q \leq 0. \end{cases}$$

Write Newton's second law, $\ddot{q} = -V'(q)$, as a first-order system in $x = (q, \dot{q})$. Show that $q(t) = 0$ is a solution to the equation with initial conditions $q(0) = 0$, $\dot{q}(0) = 0$. Find another solution which satisfies these initial conditions of the form $q(t) = at^b$. Consider more generally the differential equation

$$\ddot{q} = \begin{cases} 0, & q \leq 0, \\ q^\alpha, & q > 0. \end{cases}$$

For which values of $\alpha > 0$ are there two solutions, $q(0)$ and $q = at^b$ (for appropriate a and b), which satisfy the initial conditions $q(0) = \dot{q}(0) = 0$?