Fields, Forms and Flows 3/34

Problem Sheet 1
Due: Wednesday 10 October
To hand in: DM3: 1, 4, 5(a)(b) DM34: 5(a)(b), 6, 8

 Problems marked * are not examinable (but are still worth trying!)

1. Consider the linear map $F : \mathbb{R}^m \to \mathbb{R}^n$ given by $F(x) = A \cdot x$, where A is an $n \times m$ matrix. Determine the conditions on m, n and A for F to be i) 1-1, ii) onto.

2. * Construct a 1-1 map from $\mathbb{R}^2$ to $\mathbb{R}$. Can you construct a map which is invertible?

3. * Given a map $F : X \to Y$. Given $U \subset X$, show that $U \subset F^{-1}(F(U))$. Given $V \subset Y$, show that $F(F^{-1}(V)) \subset V$. Give an example of a function $F : \mathbb{R} \to \mathbb{R}$ and subsets $U, V \subset \mathbb{R}$ for which $F^{-1}(F(U)) \neq U$ and $F(F^{-1}(V)) \neq V$.

4. * For $x \in \mathbb{R}^m$ and $\epsilon > 0$, show that $B_\epsilon(x)$ is open. (Hint: Recall the triangle inequality, $||x + y|| \leq ||x|| + ||y||$, which holds for all $x, y \in \mathbb{R}^m$.)

5. * Prove Proposition 1.2.3. a) Show that the union of (any number of) open sets in $\mathbb{R}^m$ is open. b) Show that the intersection of two open sets in $\mathbb{R}^m$ is open (note: the empty set is regarded as open). c) Show that an open set in $\mathbb{R}$ can be expressed as a union of open intervals.

6. * Prove Proposition 1.2.11. Let $X \subset \mathbb{R}^n$. Show that X is closed if and only if $\tilde{X}$ is open ($\tilde{X}$ is the complement of X).

7. * Prove Proposition 1.3.6: Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets, and let $F : U \to V$ be a map from U to V. Then F is continuous if and only if for all open sets $Y \subset V$, $F^{-1}(Y)$ is open. That is, F is continuous if and only if the inverse image of every open set is open.

8. * The functions

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}, \quad g(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

are not continuous. Find an open set $A \subset \mathbb{R}$ such that $f^{-1}(A)$ is not open, and an open set $B \subset \mathbb{R}$ such that $g^{-1}(B)$ is not open.

9. * Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $U \subset \mathbb{R}$ such that $f(U)$ is not open.