Algebraic geometry, Sheets 1, 2: selected solutions

For a polynomial f, write $[f]_d$ for the degree d homogeneous component of f. So

$$[f+g]_d = [f]_d + [g]_d$$

and

$$[fg]_d = \sum_{i+j=d} [f]_i [g]_j.$$

We will frequently be making use of the fact that the polynomial ring $k[x_1, x_2, \ldots, x_n]$ over a field k is a unique factorization domain. In particular, if a polynomial f is divisible by two distinct irreducible polynomials p and q, then it is divisible by pq. Also, p is irreducible if and only if (p) is a prime ideal if and only if $k[x_1, x_2, \ldots, x_n]/(p)$ is an integral domain.

(1) Prove that for any ideal J, the radical $\sqrt{J}$ is also an ideal.

This result holds in any commutative ring R. Let $a, b \in \sqrt{J}$. Then $a^n, b^m \in J$ for some $n, m \in \mathbb{N}$, so that $a^i \in J$ and $b^j \in J$ for any $i \geq n$ and $j \geq m$. Let $N = 2\max\{n, m\}$. Then

$$(a+b)^N = \sum_i \binom{N}{i} a^i b^{N-i}.$$

For each summand $a^i b^{N-i}$, we clearly have either $i \geq N/2 \geq n$ or $N - i \geq N/2 \geq m$, so that $(a+b)^N \in J$. Therefore, $a + b \in \sqrt{J}$. Obviously, $0 \in \sqrt{J}$ and $-a \in \sqrt{J}$. Finally, if $r \in R$, then $(ra)^n = r^n a^n \in J$, so that $ra \in \sqrt{J}$.

(2) Let $V := V(I) \subset \mathbb{A}^3$ be the algebraic set corresponding to the ideal $I = (x^2 - yz, xz - x)$. Decompose V into its irreducible components.

Clearly the equations

$$x^2 - yz = 0; xz - x = 0$$

are equivalent to

$$x^2 - yz = 0; x = 0$$

or

$$x^2 - yz = 0; z - 1 = 0.$$

That is,

$$V(I) = V(x^2 - yz, x) \cup V(x^2 - yz, z - 1).$$

But

$$k[x, y, z]/(x^2 - yz, z - 1) \simeq k[x, y]/(x^2 - y) \simeq k[x],$$

which is an integral domain. Therefore, $(x^2 - yz, z - 1)$ is a prime ideal, and $V(x^2 - yz, z - 1)$ is irreducible. On the other hand,

$$V(x^2 - yz, x) = V(x, y) \cup V(x, z)$$

and $k[x, y, z]/(x, y) \simeq k[z]$, $k[x, y, z]/(x, z) \simeq k[y]$. So both $V(x, y)$ and $V(x, z)$ are irreducible. Therefore,

$$V(I) = V(x, y) \cup V(x, z) \cup V(x^2 - yz, z - 1)$$

is the decomposition of $V(I)$ into its irreducible components.

(3) In this exercise, we investigate the relation between ideals and varieties for the following ideals in $\mathbb{C}[x, y, z]$:

$$I_1 := (xy + y^2, xz + yz);$$

$$I_2 := (xy + y^2, xz + yz + xyz + y^2 z);$$
$$I_3 := (xy^2 + y^3, xz + yz).$$

(a) Does $I_k = I_l$ for some $k \neq l$?

Note that $I_1 \supset I_2$ and $I_1 \supset I_3$. But $xz+yz+xyz+y^2z = xz+yz+z(xy+y^2)$, so that $xz+yz \in I_2$. Hence, $I_2 = I_1$. But we claim that $xy + y^2 \notin I_3$. To see this, assume

$$xy + y^2 = a(xy^2 + y^3) + b(xz + yz)$$

for $a, b \in \mathbb{C}[x, y, z]$. Then

$$xy + y^2 = [xy + y^2]_2 = [a(xy^2 + y^3)]_2 + [b(xz + yz)]_2 = [b(xz + yz)]_2 = [b]_0(xz + yz).$$

Since $[b]_0$ is constant, this cannot hold for any b.

(b) Does $V(I_l) = V(I_k)$ for some $i \neq k$?

Of course $V(I_1) = V(I_2)$. The points in $V(I_3)$ satisfy $xz + yz = 0$ and $xy^2 + y^3 = 0$. This is the same as $[x + y = 0$ or $z = 0]$ and $[x + y = 0$ or $y = 0]$. Clearly, this is the same zero set as $V(I_1)$. Therefore, all zero sets are the same.

(6) Show that the hyperbola $\{(x, y) \in \mathbb{A}^2 \mid xy = 1\}$ is not isomorphic to $\mathbb{A}^1$.

Let

$$\phi : k[x, y]/(xy - 1) \rightarrow k[t]$$

be a homomorphism of k-algebras. Then $\phi(x), \phi(y) \in k[t]$ must be invertible. Therefore, they must both be constant. Hence, the coordinate rings of the two varieties are not isomorphic as k-algebras. So the affine varieties are not isomorphic.

(9) Give an example to show that the image of a polynomial map

$$f : \mathbb{C}^n \rightarrow \mathbb{C}^m$$

is not necessarily an algebraic set.

Consider the map $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by

$$(x, y) \mapsto (x, xy).$$

If the image of this map were contained in the zero set of any polynomial $p(s, t)$, then the polynomial $p(x, xy)$ would be identically zero. Writing

$$p(s, t) = \sum_i a_i(s) t^i,$$

we get

$$\sum_i a_i(x) x^i y^i = 0$$

so $a_i(x)x^i$ is identically zero for each i. Hence, $a_i(x) = 0$ for each i, and $p(s, t) = 0$. Therefore, the image of f is not contained in any proper Zariski closed subset of $\mathbb{C}^2$. On the other hand any $(0, t)$ for $t \neq 0$ is not in the image of ϕ. Therefore, the image of ϕ is not the whole space $\mathbb{C}^2$. Therefore, the image cannot be Zariski closed.

You might try to show that this example is in some sense 'the simplest possible.' That is, try to show that that for polynomial maps $\mathbb{C} \rightarrow \mathbb{C}$, $\mathbb{C}^2 \rightarrow \mathbb{C}$, and $\mathbb{C} \rightarrow \mathbb{C}^2$, the image is always Zariski closed.

(12) Show that the Zariski topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ is not the product topology.

Notice that a basis for the product topology are sets of the form $U \times V$ where U and V are open in the first and second $\mathbb{A}^1$-factors.

But since the diagonal $\Delta := V(x-y)$ is a closed set, its complement $\mathbb{A}^2 \setminus \Delta$ is an open set. This set cannot contain any non-empty $U \times V$.

(13)

Let $X = V(x_1, x_2)$ and $Y = V(x_3, x_4)$ in $\mathbb{A}^4$. Show that $I(X \cup Y)$ cannot be generated by two elements.

In fact, $I(X \cup Y)$ cannot even be generated by three elements. Suppose $f \in I(X \cup Y)$. Then $f \in I(X) \cap I(Y)$, so that f can be written $f = ax_1 + bx_2$ as well as $f = cx_3 + dx_4$. Therefore, every monomial occurring in f is divisible by x_1 or x_2 and is divisible by x_3 or x_4. Hence, every monomial in f is divisible by x_1x_3 or x_1x_4 or x_2x_3 or x_2x_4. Therefore,

$$I(X \cup Y) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4).$$

Now, let $\{v_j\}_{j \in J}$ be any set of generators for $I(X \cup Y)$. Then there would be polynomials c_j^{kl} such that $x_kx_l = \sum c_j^{kl} v_j$ for each of the monomials x_kx_l generating the ideal. Since each v_j is in $I(X \cup Y)$, and hence, can be written in terms of the x_kx_l, they have no homogeneous components of degree less than 2. Therefore, we find

$$x_kx_l = \sum [c^{kl}]_0 [v_j]_2$$

for each k, l. That is, x_1x_3, x_1x_4, x_2x_3, x_2x_4 will be linear combinations of the $[v_j]_2$ with *constant coefficients*. Since these four monomials are linearly independent over $\mathbb{C}$, there must be at least four $[v_j]_2$.