Lecture 16: Bézout's theorem

In this lecture we will see how to define the *intersection multiplicity* $m_p(C, D)$ of two plane curves C and D at a point $p \in \mathbb{P}^2$. We will then prove *Bézout's theorem*:

Theorem 39 (Bézout's theorem)**.** *Suppose that $C, D \subset \mathbb{P}^2$ are two projective plane curves of degrees $\deg C = d$, $\deg D = e$, which have no common components. Then C and D intersect in precisely de points when counted with multiplicity, i.e.*

$$\sum_{p \in C \cap D} m_p(C, D) = de.$$

Note that C and D are not necessarily nonsingular or irreducible and may intersect in a horribly complicated way (e.g. see Figure 1). The trick to proving the theorem is to come up with the right definition for $m_p(C, D)$.

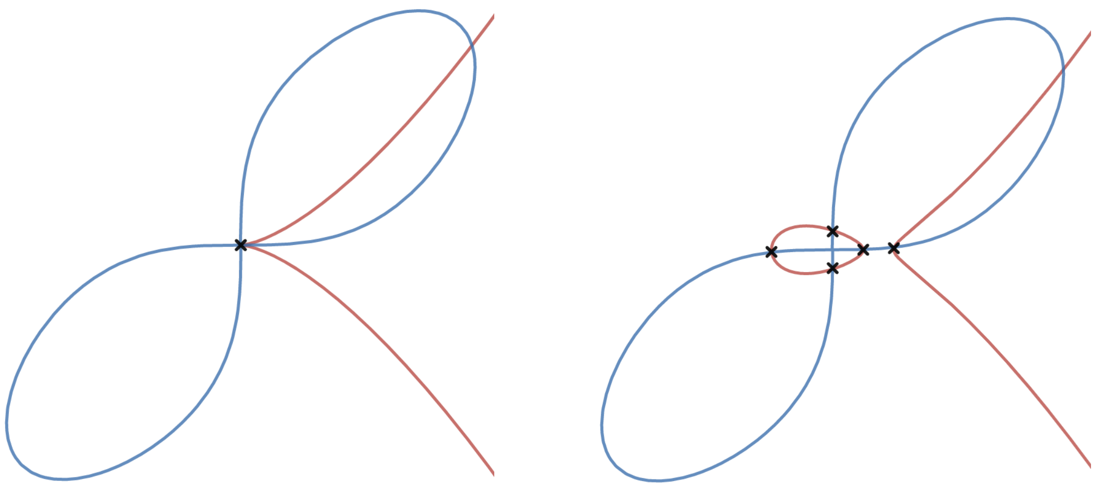

Figure 1: How many times do the curves $C = \mathbb{V}(x^3 - y^2)$ and $D = \mathbb{V}(x^4 + y^4 - xy)$ intersect at the origin $0 \in \mathbb{A}^2$? By considering $C_\epsilon = \mathbb{V}(x^3 - \epsilon x + \epsilon^2 - y^2)$ as $\epsilon \to 0$, we suspect that $m_0(C, D) \geq 5$. (In fact $m_0(C, D) = 5$, as you should soon be able to show using the resultant.)

Remark. For the purpose of counting intersection multiplicities correctly, it is convenient to allow a plane curve C to have multiple components (i.e. if $C = \mathbb{V}(f) \subset \mathbb{P}^2$ and f factors into irreducibles as $f = f_1^{a_1} \cdots f_n^{a_n}$, then the irreducible component $C_i = \mathbb{V}(f_i)$ is counted a_i times).

1 Some easy cases

1.1 A line and a curve

Suppose that $C = \mathbb{V}(f) \subset \mathbb{P}^2$ is a (not necessarily irreducible) curve of degree d and that L is the line $L = \mathbb{V}(ax + by + cz) \subset \mathbb{P}^2$ with $a, b, c \in \mathbb{C}$, not all zero. Without loss of generality we can assume that $a \neq 0$. If L is an irreducible component of C then we set $m_p(C, L) = \infty$.

Suppose L is not an irreducible component of C. Then $f_L(y,z) := f\left(-\frac{by+cz}{a}, y, z\right) \in \mathbb{C}[y,z]$ is a nonzero polynomial of degree d. A root $f_L(y_0, z_0) = 0$ corresponds to an intersection point $p = (x_0 : y_0 : z_0) \in C \cap L$, where $x_0 = -\frac{by_0+cz_0}{a}$. We define the intersection multiplicity to be $m_p(C, L) = m$, where m is the multiplicity of the root $(yz_0 - y_0 z)$ of f_L. Clearly in this case $\sum_{p \in C \cap L} m_p(C, L) = \deg f$, so Bézout's theorem holds.

1.2 Two conics

We can write a conic $f(x, y, z) = ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$ as a matrix product

$$f(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{x}^T M_f \boldsymbol{x}$$

This gives a bijection $M_f \leftrightarrow f$, between 3×3 symmetric matrices over $\mathbb{C}$ and homogeneous quadratic polynomials in $\mathbb{C}[x, y, z]$.

Definition 40. Suppose that $f, g \in \mathbb{C}[x, y, z]$ are two linearly independent homogeneous quadratic polynomials, and let $C_1 = \mathbb{V}(f)$ and $C_2 = \mathbb{V}(g)$ be the corresponding conics. The *pencil* $|C_1, C_2|$ is the set of conics

$$|C_1, C_2| = \left\{ \mathbb{V}(\lambda f + \mu g) \subset \mathbb{P}^2 : (\lambda : \mu) \in \mathbb{P}^1 \right\}.$$

Proposition 41.

1. *The conic $C = \mathbb{V}(f) \subset \mathbb{P}^2$ is singular, if and only if* $\det(M_f) = 0$.

2. *The pencil of conics $|C_1, C_2|$ contains either* 1, 2 *or* 3 *singular conics.*

Proof.

1. Check for yourself that $C = \mathbb{V}(f)$ is singular at $(x_0 : y_0 : z_0) \in \mathbb{P}^2$ if and only if the vector $(x_0\ y_0\ z_0) \in \ker M_f$.

2. A conic $C \in |C_1, C_2|$ is given by $C = \mathbb{V}(\lambda f_1 + \mu f_2)$. Therefore the singular conics are given by the roots of the (nonzero) cubic polynomial $\det(\lambda M_{f_1} + \mu M_{f_2}) = 0$. $\square$

Intersection of two conics. We can use a singular conic $C_0 \in |C_1, C_2|$ to find the four intersection points $C_1 \cap C_2$. Note that if $f(p) = g(p) = 0$ then $\lambda f(p) + \mu g(p) = 0$ for all $(\lambda : \mu) \in \mathbb{P}^1$. Therefore $p \in C_1 \cap C_2$ if and only if $p \in C$ for all $C \in |C_1, C_2|$. Since C_0 is singular we have $C_0 = L_1 \cup L_2$ (allowing for the case $L_1 = L_2$) and we combine multiplicities by adding them, i.e. $m_p(C_0, D) = m_p(L_1, D) + m_p(L_2, D)$. Therefore Bézout's theorem holds:

$$\sum_{p \in C_1 \cap C_2} m_p(C_1, C_2) = \sum_{p \in L_1 \cap C_2} m_p(L_1, C_2) + \sum_{p \in L_2 \cap C_2} m_p(L_2, C_2) = 2 + 2 = 4.$$

2 The resultant

In order to correctly define the intersection multiplicity of two plane curves at a point, we first need to introduce the resultant. Suppose $f, g \in R[x]$ are two polynomials with coefficients in a commutative ring R. Write them as

$$f = \sum_{i=0}^{d} f_i x^i, \quad g = \sum_{i=0}^{e} g_i x^i$$

where $f_i, g_i \in R$ and $f_d, g_e \neq 0$.

Definition 42. The *resultant of f and g (with respect to x)* is $R_{f,g} = \det M_{f,g}$, where $M_{f,g}$ is the following $(d+e) \times (d+e)$ matrix:

$$M_{f,g} = \begin{pmatrix} f_0 & f_1 & \cdots & f_{d-1} & f_d & & & & \\ & f_0 & f_1 & \cdots & f_{d-1} & f_d & & & \\ & & \ddots & & & & \ddots & & \\ & & & f_0 & f_1 & \cdots & f_{d-1} & f_d \\ g_0 & g_1 & \cdots & g_{e-1} & g_e & & & & \\ & g_0 & g_1 & \cdots & g_{e-1} & g_e & & & \\ & & \ddots & & & & \ddots & & \\ & & & g_0 & g_1 & \cdots & g_{e-1} & g_e \end{pmatrix}$$

where the coefficients of f are written in the first e rows, the coefficients of g are written in the next d rows and all of the remaining entries are 0.

The point of the resultant is that it gives us a criteron to check if f and g share a common root, without having to find the root explicitly.

Proposition 43. *If $R = \mathbb{C}$, then $f, g \in \mathbb{C}[x]$ share a common root if and only if $R_{f,g} = 0$.*

Proof. If f and g have some common root α, then we can write $f(x) = b(x)(x-\alpha)$ and $g(x) = a(x)(x-\alpha)$ for some polynomials $a, b \in \mathbb{C}[x]$ with $\deg a = e-1$ and $\deg b = d-1$. This happens if and only if we have two such polynomials $a, b \in \mathbb{C}[x]$, with

$$a(x)f(x) = b(x)g(x) \implies \sum_{i=0}^{d+e-1} \sum_{j=0}^{i} a_i f_j x^{i+j} = \sum_{i=0}^{d+e-1} \sum_{j=0}^{i} b_i g_j x^{i+j}.$$

Writing this expression out as a matrix product gives $\boldsymbol{x}^t M_{f,g} \boldsymbol{v}$, where $\boldsymbol{x} = (1, x, \cdots, x^{d+e-1})$ and $\boldsymbol{v}$ is the (nonzero) vector

$$\boldsymbol{v} = (a_0, a_1, \cdots, a_{e-1}, -b_0, -b_1, \cdots, -b_{d-1}) \in \mathbb{C}^{d+e}.$$

Since this holds for all $x \in \mathbb{C}$, we must have $\boldsymbol{v} \in \ker M_{f,g} \implies R_{f,g} = 0$. $\square$

We are interested in the case that $R = \mathbb{C}[y,z]$ and $f, g \in \mathbb{C}[y,z][x] = \mathbb{C}[x,y,z]$ are homogeneous polynomials, in which case $R_{f,g}(y,z) \in \mathbb{C}[y,z]$ is a polynomial in y and z.

Proposition 44 (Properties of the resultant)**.**

1. *Suppose $f, g \in \mathbb{C}[x,y,z]$ have $f(1,0,0) \neq 0$ and $g(1,0,0) \neq 0$. Then f and g have a nontrivial common factor if and only if $R_{f,g}(y,z) = 0$.*

2. *$R_{f,g}(y_0, z_0) = 0$ for some $y_0, z_0 \in \mathbb{C}$ if and only if there exists some $x_0 \in \mathbb{C}$ such that $f(x_0, y_0, z_0) = g(x_0, y_0, z_0) = 0$.*

3. *If $f, g \in \mathbb{C}[x,y,z]$ are homogeneous polynomials of degrees d, e, then $R_{f,g}(y,z) \in \mathbb{C}[y,z]$ is a homogeneous polynomial of degree de.*

Remark. In (1) we suppose that $f(1,0,0) \neq 0$ so that f has a x^d term, i.e. the degree of $f(x,y,z)$ is the same as the degree of f regarded as a polynomial in terms of x. Similarly for g.

3 Sketch proof of Bézout's Theorem 39

Theorem 45 (Weak version of Bézout's theorem)**.** *Suppose that* $C, D \subset \mathbb{P}^2$ *are plane curves of degrees* $\deg C = d$, $\deg D = e$ *with no common components. Then* $\#(C \cap D) \leq de$.

Proof. If $\#(C \cap D) > de$, let $\{p_1, \ldots, p_{de+1}\} \subset C \cap D$ be a set of distinct points. Choose a point $q \in \mathbb{P}^2$ such that q does not lie on C, D or any of the lines $\overline{p_i p_j} \subset \mathbb{P}^2$. By translating, we can assume $q = (1 : 0 : 0)$ and therefore that $C = \mathbb{V}(f)$ and $D = \mathbb{V}(g)$ where f, g are homogeneous polynomials of degrees d, e with $f(1, 0, 0), g(1, 0, 0) \neq 0$.

Now $R_{f,g}(y, z)$ is a homogeneous polynomial of degree de by Proposition 44(3), and if $p_i = (x_i : y_i : z_i)$ then $R_{f,g}(y_i, z_i) = 0$ by Proposition 44(2). Moreover $(y_i : z_i) \neq (y_j : z_j)$ for any i, j since otherwise the points p_i, p_j, q would be collinear in $\mathbb{P}^2$. But now $R_{f,g}(y, z)$ has at least $de + 1$ distinct roots, so we must have $R_{f,g}(y, z) = 0$ is identically zero. By Proposition 44(1), C and D share a common component. □

Definition 46. Suppose we have two plane curves C, D satisfying the following conditions

1. $(1 : 0 : 0) \notin C \cap D$,
2. $(1 : 0 : 0) \notin \overline{p_i, p_j}$ for any $p_i, p_j \in C \cap D$,
3. $(1 : 0 : 0) \notin T_p C$ and $(1 : 0 : 0) \notin T_p D$ for any $p \in C \cap D$.

We define the *intersection multiplicity* $m_p(C, D)$ to be the multiplicity of the root $(y_0 : z_0)$ of $R_{f,g}(y, z)$ for any $p = (x_0 : y_0 : z_0) \in C \cap D$ (and $m_p(C, D) = 0$ for any $p \notin C \cap D$).

Using this definition of $m_p(C, D)$ a careful adjustment of the proof of Theorem 45 can be used to prove Bézout's Theorem 39.