LECTURE 5: AFFINE SPACE

1. AFFINE SPACE AND THE IDEAL-VARIETY CORRESPONDENCE

Fix $n \in \mathbb{N}$, and let $R = \mathbb{C}[x_1, \ldots, x_n]$. We will fix this notation for the remainder or this lecture.

Definition 1.1 (Affine space as a set)**.** *Define **affine** n**-space** $\mathbb{A}^n := \mathbb{C}^n$ as a set. Points of $\mathbb{A}^n$ will be denoted as $a = (a_1, \ldots, a_n)$.*

Definition 1.2. *If $J \leq R$, let*

$$\mathbb{V}(J) := \{a \in \mathbb{A}^n : (\forall f \in J) f(a) = 0\}. \tag{1.1}$$

*A subset of $\mathbb{A}^n$ of the form $\mathbb{V}(J)$ is called an **affine variety**.*

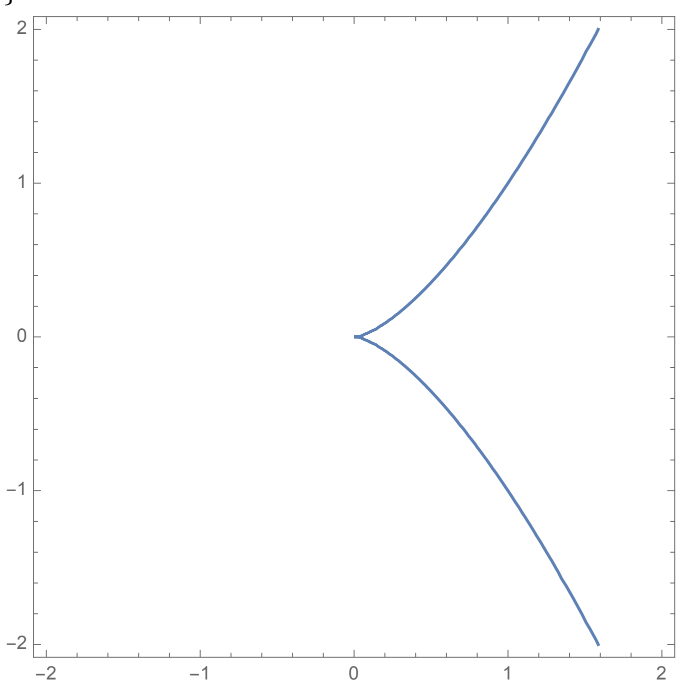
Definition 1.3. *If $X \subseteq \mathbb{A}^n$ is an affine variety, then define*

$$\mathbb{I}(X) := \{f \in R : (\forall a \in X) f(a) = 0\}. \tag{1.2}$$

Example 1.4. *If $J = (x) \leq \mathbb{C}[x, y]$, then the variety of J is a line, $\mathbb{V}(J) = \{(x, y) \in \mathbb{A}^2 : x = 0\}$.*

Example 1.5. *If $J = (x, y - 1) \leq \mathbb{C}[x, y]$, then the variety of J is a point, $\mathbb{V}(J) = \{(0, 1)\}$.*

Example 1.6. *If $J = (y^2 - x^3) \leq \mathbb{C}[x, y]$, then the variety of J is the **cuspidal cubic** $\mathbb{V}(J) = \{(x, y) \in \mathbb{A}^2 : y^2 = x^3\}$.*



The pair of maps $J \mapsto \mathbb{V}(J)$ and $X \mapsto \mathbb{I}(X)$ is known as the "ideal-variety correspondence". It follows from the definitions that both maps are order-reversing:

- If $J_1 \subseteq J_2$, then $\mathbb{V}(J_1) \supseteq \mathbb{V}(J_2)$.

- If $X_1 \subseteq X_2$, then $\mathbb{I}(X_1) \supseteq \mathbb{I}(X_2)$.

Proposition 1.7. *If X is any affine variety, then $\mathbb{V}(\mathbb{I}(X)) = X$.*

Proof. We have

$$\mathbb{V}(\mathbb{I}(X)) = \{a \in \mathbb{A}^n : (\forall f \in \mathbb{I}(X)) f(a) = 0\} \tag{1.3}$$
$$= \{a \in \mathbb{A}^n : \text{ If } f(b) = 0 \text{ for all } b \in X, \text{ then } f(a) = 0\}, \tag{1.4}$$

so if $a \in X$, then certainly $a \in \mathbb{V}(\mathbb{I}(X))$. Thus, $X \subseteq \mathbb{V}(\mathbb{I}(X))$.

Write $X = \mathbb{V}(J)$ for some ideal $J \leq R$. By a similar argument to the above, $J \subseteq \mathbb{I}(\mathbb{V}(J))$. Applying $\mathbb{V}$ to both sides of the inclusion, $\mathbb{V}(J) \supseteq \mathbb{V}(\mathbb{I}(\mathbb{V}(J)))$; in other words, $X \supseteq \mathbb{V}(\mathbb{I}(X))$. We've proved both inclusions; therefore, $\mathbb{V}(\mathbb{I}(X)) = X$. ☐

2. Hilbert's Nullstellensatz

Looking at proposition 1.7, one might guess that $\mathbb{V}$ and $\mathbb{I}$ are inverse functions; that is, not only does $\mathbb{V}(\mathbb{I}(X)) = X$, but also $\mathbb{I}(\mathbb{V}(J)) = J$. This is false (in general). For example, consider the ideal $J = (x^2) \leq \mathbb{C}[x, y]$. We compute

$$\mathbb{V}(J) = \{(x, y) \in \mathbb{A}^2 : x^2 = 0\} = \{(x, y) \in \mathbb{A}^2 : x = 0\}. \tag{2.1}$$

Thus,

$$\mathbb{I}(\mathbb{V}(J)) = \{f(x, y) \in \mathbb{C}[x, y] : (\forall y) f(0, y) = 0\}. \tag{2.2}$$

Writing $f(x, y) = \displaystyle\sum_{j=0}^{d}\sum_{k=0}^{e} c_{jk}x^j y^k$, we see that the condition that $f \in \mathbb{I}(\mathbb{V}(J))$ is equivalent to the condition that $\displaystyle\sum_{k=0}^{d} c_{0k}y^k = 0$ for all $y \in \mathbb{C}$, that is, each $c_{0k} = 0$. In turn, that is equivalent to the condition that $f(x, y)$ is divisible by x. Hence $\mathbb{I}(\mathbb{V}(J)) = (x) \neq J$.

This failure may be salvaged with the following theorem.

Theorem 2.1 (Hilbert's Nullstellensatz)**.** *Let $J \leq R = \mathbb{C}[x_1, \ldots, x_n]$. Then, $\mathbb{I}(\mathbb{V}(J)) = \mathrm{rad}(J)$, where*

$$\mathrm{rad}(J) = \{f \in R : f^n \in J \textit{ for some } n\}. \tag{2.3}$$

If the ideal $J = \mathrm{rad}(J)$, then J is called **radical**. If P is a prime ideal, it's not difficult to see that P is radical (exercise).

Unlike the proof of proposition 1.7, the proof of theorem 2.1 is difficult. We skip the proof for now and will return to it in Lecture 9.

Corollary 2.2. $\mathrm{mSpec}(\mathbb{C}[x_1, \ldots, x_n]) = \{(x_1 - a_1, \ldots, x_n - a_n) : (a_1, \ldots, a_n) \in \mathbb{A}^n\}$.

Proof. Let $R = \mathbb{C}[x_1, \ldots, x_n]$. First, we show that the ideals $(x_1 - a_1, \ldots, x_n - a_n)$ are indeed maximal ideals of R. Let $J = (x_1 - a_1, \ldots, x_n - a_n)$. Then,

$$\mathbb{V}(J) = \{(b_1, \ldots, b_n) \in \mathbb{A}^n : f(\tag{2.4}$$

Let $\mathfrak{m}$ be a maximal ideal of R. By the Nullstellensatz, $\mathbb{I}(\mathbb{V}(\mathfrak{m})) = \mathrm{rad}(\mathfrak{m}) = \mathfrak{m}$. First consider the case when $\mathbb{V}(\mathfrak{m}) = \emptyset$. In this case, $\mathfrak{m} = \mathbb{I}(\emptyset) = R$, which is impossible because R is not a maximal ideal (by definition). Thus, $\mathbb{V}(\mathfrak{m}) \neq \emptyset$; choose any $a = (a_1, \ldots, a_n) \in \mathbb{V}(\mathfrak{m})$. Then,

$$\mathfrak{m} = \mathbb{I}(\mathbb{V}(\mathfrak{m})) \leq \mathbb{I}(\{a\}), \tag{2.5}$$

so by maximality, $\mathfrak{m} = \mathbb{I}(\{a\})$. But $\mathbb{I}(\{a\})$ is precisely the set of polynomials that vanish at a. By expanding a polynomial $f \in R$ about the point $x = a$,

$$f(x_1, \ldots, x_n) = \sum_{j_1,\ldots,j_n} c_{j_1,\ldots,j_n}(x_1 - a_1)^{j_1} \cdots (x_n - a_n)^{j_n}, \tag{2.6}$$

we see that $f \in \mathbb{I}(\{a\})$ if and only if the constant term $c_{0,\ldots,0} = 0$, which is equivalent to saying that $f \in (x_1 - a_1, \ldots, x_n - a_n)$. Thus, $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$.

It remains to show that $\mathbb{I}(\{a\}) = (x_1 - a_1, \ldots, x_n - a_n)$ is actually a maximal ideal. If J is any ideal containing $\mathbb{I}(\{a\})$, then $\mathbb{V}(J)$ is either $\emptyset$ or $\{a\}$. If $\mathbb{V}(J) = \emptyset$, then $\mathrm{rad}(J) = R$, so $J = R$. If $\mathbb{V}(J) = \{a\}$, then $\mathrm{rad}(J) = \mathbb{I}(\{a\})$, so $J \leq \mathbb{I}(\{a\})$; but we already know $J \geq \mathbb{I}(\{a\})$, so $J = \mathbb{I}(\{a\})$. Thus, $\mathbb{I}(\{a\})$ is a maximal ideal. $\square$

Definition 2.3 (Zariski topology on $\mathbb{A}^n$)**.** *Affine space $\mathbb{A}^n$ may be identified with $\mathrm{mSpec}(\mathbb{C}[x_1, \ldots, x_n])$ via the bijection*

$$(a_1, \ldots, a_n) \mapsto (x_1 - a_1, \ldots, x_n - a_n). \tag{2.7}$$

The Zariski topology on $\mathbb{A}^n$ is the unique topology that makes this bijection a homeomorphism; that is, a set is defined to be closed if and only if it is the image of a closed set. Specifically, the closed sets are those of the form $\mathbb{V}(J)$ for $J \leq R$, that is, affine varieties.