Lecture 15: Curves

For the rest of the course we will focus on curves—i.e. algebraic varieties of dimension 1. (*Note*: although these are 1-dimensional over $\mathbb{C}$, they are 2-dimensional over $\mathbb{R}$.) Much of the theory discussed in this lecture is true for any curve, but for simplicity we will restrict to the case of plane curves.

Recall. The *field of rational functions* on an affine algebraic variety $X \subset \mathbb{A}^n$ is given by

$$\mathbb{C}(X) = \left\{ \tfrac{g}{h} : g, h \in \mathbb{C}[x_1, \dots, x_n] \right\} / \left(\tfrac{g}{h} = \tfrac{g'}{h'} \iff gh' - g'h \in \mathbb{I}(X) \right)$$

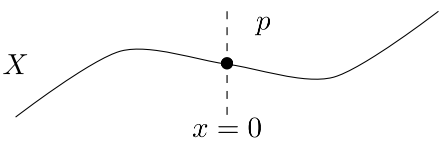
and that $\phi \in \mathbb{C}(X)$ is *regular at* p if there exists a representation $\phi = \frac{g}{h}$ where $h(p) \neq 0$.

1 Local geometry

Suppose $X = \mathbb{V}(f) \subset \mathbb{A}^2$ is an irreducible plane affine curve which is nonsingular at $p \in X$. Wlog we can translate p to the origin $(0,0)$ and assume that the line $\mathbb{V}(x)$ is *not* tangent to $p \in X$ (else reflect X in the diagonal of $\mathbb{A}^2$ to switch $x \leftrightarrow y$). Now f must be of the form

$$f(x, y) = ax + by + \cdots \quad \text{(terms of degree} \geq 2) \tag{$*$}$$

where $a, b \in \mathbb{C}$. At least one of $a, b \neq 0$, since $p \in X$ is nonsingular, and in fact $b \neq 0$, since $\mathbb{V}(x)$ is not the tangent line to $p \in X$. We have the following picture:



1.1 Order of vanishing of a regular function

Definition 31. The *order of vanishing* $v_p(\phi)$ of a regular function ϕ at p is

$$v_p(\phi) = \max \left\{ n \geq 0 : \tfrac{\phi}{x^n} \text{ is regular at } p \right\}.$$

By definition we also set $v_p(0) = \infty$.

Lemma 32.

1. *If ϕ is regular at p and $\phi(p) = 0$ then $v_p(\phi) > 0$.*
2. *If $v_p(\phi) = n$ then $\frac{\phi}{x^n}$ is regular and nonzero at p. This property determines $v_p(\phi)$ uniquely.*
3. *$v_p(\phi\psi) = v_p(\phi) + v_p(\psi)$*
4. *$v_p(\phi + \psi) \geq \min\{v_p(\phi), v_p(\psi)\}$ and equality holds if $v_p(\phi) \neq v_p(\psi)$.*

Proof.

1. We need to show that if ϕ is regular at p and $\phi(p) = 0$, then $\frac{\phi}{x}$ is also regular at p. But if $\phi(p) = 0$ then $\phi = \frac{g}{h}$ where $g(p) = 0$, so we can write $g(x,y) = xg_1 + yg_2$ for some $g_1, g_2 \in \mathbb{C}[x,y]$. Now $\frac{\phi}{x} = \frac{g_1}{h} + \frac{y}{x}\frac{g_2}{h}$ where $\frac{g_1}{h}, \frac{g_2}{h}$ are regular at p, so the result will follow if we can show $\frac{y}{x}$ is regular at p. From $(*)$ we can write $f(x,y) = xf_1 + yf_2$ with $f_1, f_2 \in \mathbb{C}[x,y]$ where $f_1(p) = a$ and $f_2(p) = b \neq 0$. As a rational function on X, we have $\frac{y}{x} = -\frac{f_1}{f_2}$, since $f = xf_1 + yf_2 \in \mathbb{I}(X)$. Since $f_2(p) \neq 0$ this shows that $\frac{y}{x}$ is regular at p.

2. By definition $\phi' = \frac{\phi}{x^n}$ is regular at p and, if $\phi'(p) = 0$, then $\frac{\phi'}{x} = \frac{\phi}{x^{n+1}}$ is regular at p by (1), contradicting the definition of $\nu_p(\phi)$. Clearly there can be at most one n such that $\frac{\phi}{x^n}$ is regular and nonzero at p.

3. If $v_p(\phi) = m$ and $v_p(\psi) = n$ then $\frac{\phi\psi}{x^{m+n}} = \frac{\phi}{x^m}\frac{\psi}{x^n}$ is regular and nonzero at p, hence $v_p(\phi\psi) = m + n$ by (2).

4. Let $v_p(\phi) = m$, $v_p(\psi) = n$ and (wlog) assume $m \leq n$. Then $\frac{\phi+\psi}{x^m}$ is regular at p, so $v_p(\phi+\psi) \geq \min\{m,n\}$. If $m < n$ then $\frac{\phi}{x^m}$ is nonzero at p whereas $\frac{\psi}{x^m}$ is zero at p, so $\frac{\phi+\psi}{x^m}$ must be nonzero at p. Hence $v_p(\phi+\psi) = \min\{m,n\}$ by (2). □

For an irreducible projective curve $X \subset \mathbb{P}^2$ we can define v_p at any nonsingular point $p \in X$ by restricting to an affine patch containing p and following this construction. It can be shown that v_p is independent of any choices made.

1.2 Order of vanishing of a rational function

Definition 33.

1. A rational function $\phi = \frac{g}{h} \in \mathbb{C}(X)$ has *order of vanishing* $v_p(\phi) = v_p(g) - v_p(h)$ at p. If $v_p(\phi) = -n < 0$ we say that ϕ has a *pole of order* n at $p \in X$.

2. A rational function $t \in \mathbb{C}(X)$ with $v_p(t) = 1$ is called a *uniformiser* at $p \in X$.

It can be shown that $v_p(\phi)$ is independent of the choice of g and h. Note that in the previous discussion, $x \in \mathbb{C}(X)$ was a uniformiser at p. Given any uniformiser t and any $0 \neq \phi \in \mathbb{C}(X)$, we have $v_p(\phi) = n \iff \frac{\phi}{t^n}$ is regular and nonzero at p.

Lemma 34. *A rational function $\phi \in \mathbb{C}(X)$ is regular at p if and only if $v_p(\phi) \geq 0$.*

Proof. Clearly if ϕ is regular at p then $v_p(\phi) \geq 0$. Conversely, write $\phi = \frac{g}{h}$ where $v_p(g) = m$ and $v_p(h) = n$. Pick a uniformiser t and write $g = g't^m$ and $h = h't^n$ where g', h' are regular and nonzero at p. Then $\phi = \frac{g'}{h'}t^{m-n}$. If $v_p(\phi) = m - n \geq 0$ then ϕ is regular at p. □

Example 35. Suppose $X = \mathbb{A}^1_x$. At the point $\lambda \in \mathbb{A}^1$ the function $x - \lambda$ is a uniformiser. For a regular function $f \in \mathbb{C}[x]$ we have $v_\lambda(f) = \max\{m \geq 0 : (x-\lambda)^m \text{ divides } f(x)\}$, or in other words the multiplicity of λ as a root of f. In particular, summing over all $\lambda \in \mathbb{A}^1$ we get $\deg(f) = \sum_{\lambda \in \mathbb{A}^1} v_\lambda(f)$. Similarly, for a rational function $\phi = \frac{g}{h}$, we have $\sum_{\lambda\in\mathbb{A}^1} v_\lambda(\phi) = \deg g - \deg h$.

Now suppose that[1] $m := \deg g - \deg h \geq 0$, and consider $\phi(x)$ as a rational function on $\mathbb{P}^1$ by taking the homogenisation $\widetilde{\phi}(x,y) = \frac{\widetilde{g}}{y^m \widetilde{h}}$ with respect to y. Note that at $\infty = (1:0)$, the rational function $\frac{y}{x} \in \mathbb{C}(\mathbb{P}^1)$ is a uniformiser, and we have $\widetilde{g}(1,0), \widetilde{h}(1,0) \neq 0$. Therefore $v_\infty(\widetilde{\phi}) = -m$ and $\sum_{\lambda\in\mathbb{P}^1} v_\lambda(\widetilde{\phi}) = \sum_{\lambda\in\mathbb{A}^1} v_\lambda(\widetilde{\phi}) + v_\infty(\widetilde{\phi}) = 0$. So a rational function $\phi \in \mathbb{C}(\mathbb{P}^1)$ always has the *same number of zeroes as poles* (counted with multiplicity).

[1] Or come up with a similar argument if $\deg g - \deg h < 0$.

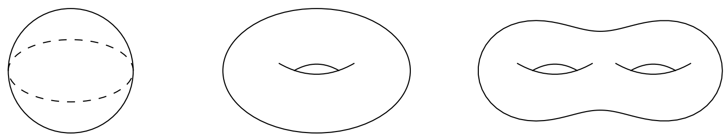
1.3 Extending rational maps from plane curves

Proposition 36. *Given an irreducible plane curve X, a rational map $f\colon X \dashrightarrow \mathbb{P}^n$ and a nonsingular point $p \in X$, then f can always be defined at p. In particular, if X is nonsingular then f can always be extended to a morphism $f\colon X \to \mathbb{P}^n$.*

Proof. We want to show that f is defined at $p \in X$. Pick a uniformiser $t \in \mathbb{C}(X)$ at p. Then we can write $f = (f_1 t^{a_1} : \ldots : f_m t^{a_m})$, where $a_i \in \mathbb{Z}$ and the f_i are all regular and nonzero at p. Now suppose that $a = \min_{i=0,\ldots,m} a_i$ and let $b_i = a_i - a$. Multiplying all coordinates of f by t^{-a} gives $f = (f_1 t^{b_1} : \ldots : f_m t^{b_m})$ where $b_i \geq 0$ for all i and at least one $b_i = 0$. This expression for f is a well-defined at p since the ith coordinate of $f(p)$ is either 0 if $b_i > 0$ or $f_i(p) \neq 0$ if $b_i = 0$, and there is at least one nonzero coordinate. $\square$

2 Global geometry—the genus

The main global invariant that distinguishes a nonsingular curve is called the *genus*. From the topological point of view, the genus $g \in \mathbb{Z}_{\geq 0}$ is the number of 'holes' that the curve has:



$\cdots$ etc.

$g = 0$ $\qquad$ $g = 1$ $\qquad$ $g = 2$

Giving a rigorous algebraic definition of the genus would take too long, so we will just consider some examples.

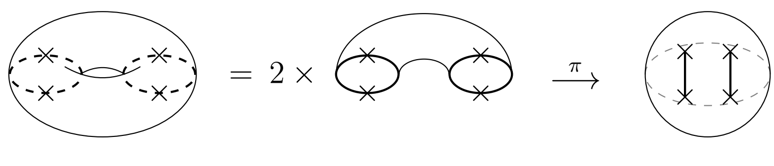
Fact 37. A nonsingular plane curve $C \subset \mathbb{P}^2$ of degree d has genus $g(C) = \frac{(d-1)(d-2)}{2}$.

Lines and conics. We already know that a line or a plane conic is isomorphic to $\mathbb{P}^1$, which we have already seen is the *Riemann sphere* (cf. Lecture 10). Therefore, these curves have genus 0. (In fact, *any* curve of genus 0 must *always* be isomorphic to $\mathbb{P}^1$!)

Cubic curves. *Sketch that a nonsingular cubic curve C has $g(C) = 1$.*

Fact 38. An irreducible plane cubic $C \subset \mathbb{P}^2$ is isomorphic to a plane cubic in *Weierstrass normal form* $\mathbb{V}(y^2z - x^3 - axz^2 - bz^3)$ for some $a, b \in \mathbb{C}$.

A cubic C in Weierstrass form has a projection $\pi\colon C \to \mathbb{P}^1$ with $\pi(x : y : z) = (x : z) \in \mathbb{P}^1$ and $\pi(0 : 1 : 0) = (1 : 0)$. The morphism π is generally 2-to-1, given by considering $y = \pm\sqrt{\frac{x^3+axz^2+bz^3}{z}}$ as a two-valued function on $\mathbb{P}^1$. However there are four *ramification points* where y is single valued, given by $(1 : 0)$ and $(\alpha_i : 1)$ for $\alpha_1, \alpha_2, \alpha_3$ the roots of $x^3 + ax + b = 0$. If we cut $\mathbb{P}^1$ along two branch curves each joining two of the four points, we get two 'sheets' isomorphic to a twice-punctured copy of $\mathbb{P}^1$, where y is single-valued. We can join the sheets up together to get a torus.



Remark. Note, not every value of $g(C) \in \mathbb{Z}_{\geq 0}$ can occur for a nonsingular plane curve $C \subset \mathbb{P}^2$! In particular, there are no plane curves with $g(C) = 2$. Curves of genus 2 do exist, but they can't be embedded in $\mathbb{P}^2$. (In fact 'most' curves can't be embedded in $\mathbb{P}^2$.)