Fields, Forms and Flows 3/34

Solution Sheet 6

1. (a) In general, we may write that

$$\bar{f}^{(1)} = af^{(1)} + bf^{(2)}, \quad \bar{f}^{(2)} = cf^{(1)} + df^{(2)}.$$

First we determine $\bar{f}_{(1)}$. We have that

$$\bar{f}^{(1)}(\bar{e}_{(1)}) = 1 \implies (af^{(1)} + bf^{(2)})(e_{(1)}) = 1 \implies a = 1.$$
$$\bar{f}^{(1)}(\bar{e}_{(2)}) = 0 \implies (af^{(1)} + bf^{(2)})(e_{(1)} + e_{(2)}) = 0 \implies a + b = 0.$$

It follows that $a = 1$ and $b = -1$, or

$$\bar{f}^{(1)} = f^{(1)} - f^{(2)}.$$

Next we determine $\bar{f}_{(2)}$. We have that

$$\bar{f}^{(2)}(\bar{e}_{(1)}) = 0 \implies (cf^{(1)} + df^{(2)})(e_{(1)}) = 0 \implies c = 0.$$
$$\bar{f}^{(2)}(\bar{e}_{(2)}) = 1 \implies (cf^{(1)} + df^{(2)})(e_{(1)} + e_{(2)}) = 1 \implies c + d = 1.$$

It follows that $c = 0$ and $d = 1$, or

$$\bar{f}^{(2)} = f^{(2)}.$$

(b) From the definition of the dual basis,

$$\bar{f}^{(i)}(\bar{e}_{(j)}) = \delta^i_j.$$

Substituting the definitions of $\bar{f}^{(i)}$ and $\bar{e}_{(j)}$, we get that

$$\delta^i_j = \left(\sum_{k=1}^n N_{ik} f^{(k)}\right)\left(\sum_{l=1}^n M_{jl} e_{(l)}\right) = \sum_{k=1}^n \sum_{l=1}^n N_{ik} M_{jl} f^{(k)}(e_{(l)}) = \sum_{k=1}^n \sum_{l=1}^n N_{ik} M_{jl} \delta^k_l = \sum_{k=1}^n N_{il} M_{jl} = (NM^T)_{ij}.$$

Thus, $NM^T = I$, or

$$N = (M^T)^{-1}.$$

2. (a) Let $f \in L(V, W)$. Given $\alpha \in \mathbb{R}$, we define $\alpha f : V \to W$ by

$$(\alpha f)(v) := \alpha f(v).$$

We claim that $\alpha f \in L(V, W)$, i.e. that af is linear. To see this, let $u, v \in V$ and $\beta, \gamma \in \mathbb{R}$. Then

$$(\alpha f)(\beta u + \gamma v) = \alpha f(\beta u + \gamma v) = \alpha\beta f(u) + \alpha\gamma f(v) = \beta(\alpha f)(u) + \gamma(\alpha f)(v).$$

Next, let $f, g \in L(V, W)$, and define $f + g : V \to W$ by

$$(f + g)(v) = f(v) + g(v).$$

We claim that $f + g \in L(V, W)$, i.e. that $f + g$ is linear. To see this, let $u, v \in V$ and $\beta, \gamma \in \mathbb{R}$. Then

$$(f+g)(\beta u+\gamma v) = f(\beta u+\gamma v)+g(\beta u+\gamma v) = \beta f(u)+\gamma f(v)+\beta g(u)+\gamma g(v) = \beta(f+g)(u)+\gamma(f+g)(v).$$

(b) Given $f \in L(V, W)$, we define a function $\phi : V \times W^* \to \mathbb{R}$ as follows: given $v \in V$ and $\xi \in W^*$, we define

$$\phi(v, \xi) = \xi(f(v)).$$

We claim that ϕ is linear in v. Let $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\phi(\alpha u + \beta v, \xi) = \xi(f(\alpha u + \beta v)) = \xi(\alpha f(u) + \beta f(v)) = \alpha\xi(f(u)) + \beta\xi(f(v)) = \alpha\phi(u, \xi) + \beta\phi(v, \xi).$$

We claim that ϕ is linear in ξ as well. Let $\xi, \eta \in W^*$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\phi(v, \alpha\xi + \beta\nu) = (\alpha\xi + \beta\nu)(f(v)) = \alpha\xi(f(v)) + \beta\nu(f(v)) = \alpha\phi(u, \xi) + \beta\phi(v, \nu)).$$

Finally, given $\phi : V \times W^* \to \mathbb{R}$ which is linear in each argument, we define a map $f : V \to W$ as follows: given $v \in V$, we define $f(v) \in W$ by

$$\xi(f(v)) = \phi(v, \xi)$$

for all $\xi \in W^*$. Since $(W^*)^* = W$, this definition makes sense, and it is straightforward to verify that f is linear (for brevity, the argument is omitted, but it is similar to the preceding).

3. (a) To show:

$$\sigma \in S_n \ \text{ can be expressed as a product of transpositions.} \tag{1}$$

Given $\sigma \in S_n$, define

$$A(\sigma) = \left\{ r \in \{1, \ldots, N\} \,\middle|\, \sigma(r) \neq r \right\}.$$

That is, $A(\sigma)$ contains all the integers between 1 and n which are mapped to different integers by σ. We proceed by induction on $|A(\sigma)|$, the number of elements of $|A(\sigma)|$. The assertion (1) is trivially true for $|A(\sigma)| = 0$; in this case σ is the identity e, and e can be written as a (null) product of permutations.

Hence, assume that (1) is true for all τ with $|A(\tau)| \leq M$. Let $\sigma \in S_n$ be a permutation for which $|A(\sigma)| = M + 1$. Take $r \in A(\sigma)$ and let $s = \sigma(r)$. Then $s \in A(\sigma)$ (otherwise, we would have $\sigma(r) = \sigma(s) = s$, and σ would not be 1-1). Let

$$\tau = \tau_{rs}\sigma.$$

Claim that

$$A(\tau) \subsetneq A(\sigma),$$

that is, $A(\tau)$ is proper subset of $A(\sigma)$. First, suppose $t \in A(\tau)$ and $t \neq r, s$. Then

$$\tau(t) \neq t \implies \tau_{rs}\sigma(t) \neq t \implies \sigma(t) \neq \tau_{rs}(t) \implies \sigma(t) \neq t,$$

so that $t \in A(\sigma)$. But $A(\sigma)$ contains both r and s, so it follows that $A(\tau)$ is a subset of $A(\sigma)$. However, $r \notin A(\tau)$, since

$$\tau(r) = \tau_{rs}(\sigma(r)) = \tau_{rs}(s) = r.$$

Therefore, $A(\tau)$ is a proper subset of $A(\sigma)$, so that $|A(\tau)| \leq M$.

By the induction hypothesis, it follows that τ can be expressed as a product of transpositions. Therefore, $\sigma = \tau_{rs}\tau$ can be expressed as a product of transpositions.

(b) We have that

$$\begin{aligned}
\tau_{13}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 5 & 3 & 2 \end{pmatrix}, \\
\tau_{26}\tau_{13}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 3 & 6 \end{pmatrix}, \\
\tau_{34}\tau_{26}\tau_{13}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 4 & 6 \end{pmatrix}, \\
\tau_{45}\tau_{34}\tau_{26}\tau_{13}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = e.
\end{aligned}$$

Therefore,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix} = \tau_{13}\tau_{26}\tau_{34}\tau_{45}.$$

4. Let $\tau_* = \sigma\tau_{12}\sigma^{-1}$, as above. We note that, since $\sigma(1) = i$ and $\sigma(2) = j$, it follows that

$$\sigma^{-1}(i) = 1, \quad \sigma^{-1}(j) = 2.$$

Then

$$\tau_*(i) = \sigma(\tau_{12}(\sigma^{-1}(i))) = \sigma(\tau_{12}(1)) = \sigma(2) = j,$$

and

$$\tau_*(j) = \sigma(\tau_{12}(\sigma^{-1}(j))) = \sigma(\tau_{12}(2)) = \sigma(1) = i.$$

Now suppose $k \neq i, j$. Then $\sigma^{-1}(k) \neq 1, 2$. It follows that $\tau_{12}(\sigma^{-1}(k)) = \sigma^{-1}(k)$, so that

$$\tau_*(k) = \sigma(\tau_{12}(\sigma^{-1}(k))) = \sigma(\sigma^{-1}(k)) = k.$$

Thus $\tau_* = \tau_{ij}$.

5. Given $(i_1, \ldots, i_k)$, an (ordered) k-tuple of distinct integers in $\{1, \ldots, N\}$, define $\sigma \in S_n$ by

$$\sigma(i_1) = i_2, \quad \sigma(i_2) = i_3, \quad \ldots, \quad \sigma(i_{k-1}) = i_k, \quad \sigma(i_k) = i_1,$$
$$\sigma(j) = j \ \text{ if } j \neq i_1, \ldots, i_k.$$

It is easily verified that

$$\sigma = \tau_{i_1 i_k}\tau_{i_1 i_{k-1}} \cdots \tau_{i_1 i_4}\tau_{i_1 i_3}\tau_{i_1 i_2}.$$

As σ is a product of $k-1$ transpositions, it follows that

$$\operatorname{sgn}\sigma = (-1)^{k-1}.$$

6. Let $M = P(\sigma)P(\sigma)^T$. We will show that $M = I$, as follows:

$$M_{ik} = \sum_{j=1}^{n} P_{ij}(\sigma)P_{jk}(\sigma)^T.$$

But

$$P_{ij}(\sigma) = \delta_{i,\sigma(j)},$$

so that

$$P_{jk}(\sigma)^T = P_{kj}(\sigma) = \delta_{k,\sigma(j)}.$$

Therefore

$$M_{ik} = \sum_{j=1}^{n} \delta_{i,\sigma(j)}\delta_{k,\sigma(j)} = \delta_{ik},$$

since the sum vanishes unless $i = k$, in which case the only nonzero contribution is from the term with $\sigma(j) = i = k$. Thus $M = I$, as claimed, and $P(\sigma)$ is orthogonal. It follows further that

$$1 = \det M = \det[P(\sigma)\, P(\sigma)^T] = \det P(\sigma)\ \det P(\sigma)^T = (\det P(\sigma))^2\,.$$

7. Let us verify the antisymmetry property first. It is clear that $a(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ changes sign if $\mathbf{u}$ and $\mathbf{v}$ are interchanged, by the antisymmetry of the cross product. Also, from the triple product rule,

$$a(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = a(\mathbf{v}, \mathbf{w}, \mathbf{u}).$$

Together with the preceding result, this shows that $a(\mathbf{u}, \mathbf{v}, \mathbf{w})$ changes sign if $\mathbf{v}$ and $\mathbf{w}$ are exchanged. A similar argument using $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ demonstrates that $a(\mathbf{u}, \mathbf{v}, \mathbf{w})$ changes sign if $\mathbf{w}$ and $\mathbf{u}$ are exchanged. Antisymmetry is established.

Let us verify linearity with respect to $\mathbf{w}$. Since the dot product is linear, we have that

$$a(\mathbf{u}, \mathbf{v}, \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2) = (\mathbf{u} \times \mathbf{v}) \cdot (\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2) = \alpha_1 (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}_1 + \alpha_2 (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}_2 = \alpha_1 a(\mathbf{u}, \mathbf{v}, \mathbf{w}_1) + \alpha_2 a(\mathbf{u}, \mathbf{v}, \mathbf{w}_2).$$

Using the triple product rule, a similar argument can be given to show that $a(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is linear in $\mathbf{u}$ and $\mathbf{v}$. Alternatively, one could use the linearity of the cross product.