Fields, Forms and Flows 3/34

Problem Sheet 10
Not to be handed in
FFF3 and FFF34: 1(a)(c), 3(c)(d)

 Problems marked * are not examinable (but are still worth trying!)

1. Consider the singular two-cube in $\mathbb{R}^2$,

$$c : [0,1]^2 \to \mathbb{R}^2; \quad c(x^1, x^2) = (x^1 x^2, x^2).$$

   (a) Draw the image of c in $\mathbb{R}^2$ (it is a triangle).

   (b) Compute ∂c and show explicitly that $\partial(\partial c) = 0$.

   (c) Let

$$\omega = \frac{1}{2}(y^1 dy^2 - y^2 dy^1)$$

   be a one-form on $\mathbb{R}^2$. Compute $\int_c d\omega$ and $\int_{\partial c} \omega$.

2. Classical Stokes' theorem in $\mathbb{R}^3$. In this question, instead of using generic notation, ie, coordinates $(x^1, \ldots, x^k)$ on the unit k-cube and $(y^1, \ldots, y^n)$ on $\mathbb{R}^n$, we'll use coordinates appropriate to three-dimensional space. Instead of writing $c : [0,1] \to \mathbb{R}^3$ for a singular one-cube, we'll write $\mathbf{r}(t)$, $0 \le t \le 1$, to emphasise that a singular one-cube is a parameterised curve. Similarly, we'll write singular two-cubes as $\mathbf{S}(u,v)$, $0 \le u, v \le 1$ (ie, as parameterised surfaces).

   (a) Let $\mathbf{r}(t)$, $0 \le t \le 1$, denote a smooth parameterised curve and $F(\mathbf{r}) = F_x dx + F_y dy + F_z dz$ a one-form on $\mathbb{R}^3$. Show that

$$(\mathbf{r}^* F)(t) = \mathbf{F}(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t)\, dt,$$

   where $\mathbf{F} = (F_x, F_y, F_z)$.

   (b) Let $\mathbf{S}(u,v)$, $0 \le u, v \le 1$, denote a smooth parameterised surface and $B(\mathbf{r}) = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$ a two-form on $\mathbb{R}^3$. Show that

$$(\mathbf{S}^* B)(u,v) = \mathbf{B}(\mathbf{S}(u,v)) \cdot \mathbf{N}(u,v)\, du \wedge dv,$$

   where $\mathbf{B} = (B_x, B_y, B_z)$ and

$$\mathbf{N} = \frac{\partial \mathbf{S}}{\partial u} \times \frac{\partial \mathbf{S}}{\partial v}.$$

   ($\mathbf{N}$ is normal to the surface $\mathbf{S}(u,v)$, but is not necessarily of unit length.)

(c) Let $S(u, v)$ be a smooth parameterised surface, as above. Regarding $S(u, v)$ as a singular two-cube, show that

$$\partial S = \mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3 - \mathbf{r}_4,$$

where

$$\mathbf{r}_1(t) = \mathbf{S}(t, 0), \quad \mathbf{r}_2(t) = \mathbf{S}(1, t), \quad \mathbf{r}_3(t) = \mathbf{S}(t, 1), \quad \mathbf{r}_4(t) = \mathbf{S}(0, t).$$

(d) Show that Stokes' theorem implies that

$$\begin{aligned} &\int_0^1 \int_0^1 (\nabla \times F)(\mathbf{S}(u, v)) \cdot \mathbf{N}(u, v)\, du dv \\ &= \int_0^1 \left(\mathbf{F}(\mathbf{r}_1(t)) \cdot \dot{\mathbf{r}}_1(t) + \mathbf{F}(\mathbf{r}_2(t)) \cdot \dot{\mathbf{r}}_2(t) - \mathbf{F}(\mathbf{r}_3(t)) \cdot \dot{\mathbf{r}}_3(t) - \mathbf{F}(\mathbf{r}_4(t)) \cdot \dot{\mathbf{r}}_4(t)\right)\, dt. \end{aligned}$$

3. (a) Let ω be a k-form on $\mathbb{R}^n$ and suppose that $d\omega = 0$. Show that, for any singular $(k+1)$-cube c on $\mathbb{R}^n$,

$$\int_{\partial c} \omega = 0.$$

(b) Let ω be a k-form on $\mathbb{R}^n$ and suppose that $\omega = d\alpha$ for some $(k-1)$-form α. Show that, for any singular k-cube c on $\mathbb{R}^n$ with $\partial c = 0$,

$$\int_c \omega = 0.$$

(c) Let

$$\omega = \frac{ydx - xdy}{x^2 + y^2}$$

be a one-form on the punctured plane $\mathbb{R}^2 - \{0\}$. Show that

$$d\omega = 0.$$

(d) Let c be the singular one-cube on $\mathbb{R}^2 - \{0\}$ given by

$$c(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Show that

$$\partial c = 0.$$

Show that

$$\int_c \omega = -2\pi.$$

Using Stokes' theorem, conclude that $\omega \neq df$ for any function (ie, 0-form) f on $\mathbb{R}^2 - \{0\}$, and that $c \neq \partial c_2$ for an singular two-cube c_2 on $\mathbb{R}^2 - \{0\}$.