$\mathcal{C} = \mathbb{V}(y^2z - x^3 - axz^2 - bz^3) \subset \mathbb{P}^2$

By simple calculations we deduce that the tangent line through p is $z = 0$ and that $\mathcal{C}$ is nonsingular iff $(a, b) \neq (0, 0)$.

We are interested in drawing the lines that go through $p = (0 : 1 : 0)$ and seeing when those lines intersect $\mathcal{C}$ with multiplicity 2, to get tangent lines.

The lines that go through p are parameterised by $rx + tz = 0$, with $(r, t) \neq (0, 0)$. We have already found the line $z = 0$ so we are interested in cases where $r \neq 0$. We can then assume $r = 1$, and analyse the lines of the form $x + tz = 0$. We then have $x = -tz$, and we can plug that expression into the polynomial generating $\mathcal{C}$.

We have that $t^3z^3 + atz^3 - bz^3 + y^2 = z[z^2(t^3 + at - b) + y^2] = 0$ We want this expression to show points with double multiplicity, which is achieved when $y = \pm i|z|\sqrt{t^3 + at - b}$ and $t^3 + at - b = 0$, which yields that $(-t : 0 : 1)$ is a point on $\mathbb{V}(x + tz) \cap \mathcal{C}$ of multiplicity two, and so the line V(x+tz) is a tangent to $\mathcal{C}$ that passes through p for each solution t.

If we had that the polynomial $t^3 + at - b$ always has 3 **different** roots then we would have finished. However, this is not always the case, for example if $a = -3, b = 2$, it has a double root and we are only able to find two different tangent lines apart from $\mathbb{V}(z)$