LECTURE 2: BASIC TOPOLOGY

1. The order topology

Here we give an important example of a non-metric topology.

Definition 1.1. *A **partially ordered set**, or **poset**, is a pair $(P, \leq)$, where P is a set and $\leq$ is a binary relation on P. The binary relation must satisfy the following properties:*

(1) *(reflexivity) $x \leq x$.*
(2) *(antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$.*
(3) *(transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$.*

Exercise 1.2. *Condition (2) in definition 1.1 is actually unneccessary, because it follows from (1) and (3).*

WARNING: There are several inequivalent topologies that can be defined in a natural way from a partial order. What we will call the *order topology* is this unit is not universally called the order topology everywhere in the mathematical literature. It is also not consistent with the most common definition of the order topology of a totally ordered set.

Definition 1.3. *Let $(P, \leq)$ be a poset. The **order topology** on P is defined to be the smallest topology on P where the sets $C_x = \{y \in P : x \leq y\}$ are closed. The topology generated by the basis of open sets*

$$\mathcal{B} = \{P \setminus (C_{x_1} \cup \cdots \cup C_{x_n}) : x \in X\}. \tag{1.1}$$

Example 1.4. *If $(\mathbb{R}, \leq)$ is given the order topology, the closed sets are the intervals $[a, \infty)$ for $a \in \mathbb{R}$, and the open sets are the intervals $(-\infty, a)$. This is NOT the same as the usual Euclidean topology on $\mathbb{R}$ (which is also often called "the order topology").*

Example 1.5. *The set 2^S of subsets of a set S form a poset under inclusion. We work out this example for the set $S = \{1, 2, 3\}$.*

2. The Zariski topology

Let R be a commutative ring with unity. We put a topology on $\operatorname{Spec}(R)$ as follows. For every ideal I, let

$$V_I := \{P \in \operatorname{Spec}(R) : I \leq P\}. \tag{2.1}$$

Proposition 2.1. *The V_I are the closed sets of a topology on $\operatorname{Spec}(R)$.*

Proof. (1) We have $V_R = \emptyset$ and $V_{(0)} = \operatorname{Spec}(R)$.

(2) If $\mathcal{I}$ is any collection of ideals, then we can take $J = \sum_{I \in iI} I$ to be the ideal generated by all the the ideals in $\mathcal{I}$. Hence,

$$\begin{aligned} \bigcap_{I \in \mathcal{I}} V_I &= \{P \in \operatorname{Spec}(R) : I \leq P \text{ for all } I \in \mathcal{I}\} && (2.2) \\ &= V_J. && (2.3) \end{aligned}$$

(3) If $\{I_1, \ldots, I_n\}$ is a finite collection of ideals, then their intersection is also an ideal $L = \bigcap_{j=1}^n I_j$, and

$$\bigcup_{j=1}^{n} V_{I_j} = \{P \in \operatorname{Spec}(R) : I_j \leq P \text{ for some } 1 \leq j \leq n\} \tag{2.4}$$
$$= V_L. \tag{2.5}$$

We've now proven that $\operatorname{Spec}(R)$ satisfies the "closed set"-variants of the three axioms of a topological space. □

3. Continuous functions

Definition 3.1. *Let X and Y be topological spaces, and let $\phi : X \to Y$ be a function. The function ϕ is **continuous** if the inverse image of any open set is open. In other words, if $V \subseteq Y$ is open, then*

$$\phi^{-1}(V) = \{x \in X : f(x) \in V\} \textit{ is open}. \tag{3.1}$$

Note that $\phi^{-1}(Y \setminus V) = X \setminus \phi^{-1}(V)$, so the condition that ϕ is continuous is also equivalent to, "the inverse image of any closed set is closed".

Now consider a ring homomorphism $f : R \to S$, where R and S are commutative rings with unity. Then, there is an induced map

$$f^* : \operatorname{Spec}(S) \to \operatorname{Spec}(R) \tag{3.2}$$

given by $f^*(Q) = f^{-1}(Q)$. We see that $f^*(Q)$ is prime, because for any $a, b \in R$,

$$ab \in f^*(Q) \implies f(ab) \in Q \tag{3.3}$$
$$\implies f(a)f(b) \in Q \tag{3.4}$$
$$\implies f(a) \in Q \text{ or } f(b) \in Q \tag{3.5}$$
$$\implies a \in f^*(Q) \text{ or } b \in f^*(Q). \tag{3.6}$$

Moreover, it turns out that f^* is continuous.

Proposition 3.2. *The induced map $f^* : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is continuous.*

Proof. For any ideal $I \leq R$, define $f_*(I)$ to be the ideal generated by all $f(r)$ such that $r \in R$. (Note that $f(I)$ itself may not be an ideal, unlike $f^{-1}(I)$).

$$(f^*)^{-1}(V_I) = \{Q \in \operatorname{Spec}(S) : f^*(Q) \in V_I\} \tag{3.7}$$
$$= \{Q \in \operatorname{Spec}(S) : I \leq f^*(Q)\}. \tag{3.8}$$

The statement that $I \leq f^*(Q)$ is saying, "if $r \in I$, then $f(r) \in Q$". This is equivalent to saying, "the ideal generated by $f(r)$ for $r \in I$ is contained in Q", that is, $f_*(I) \leq Q$. Hence,

$$(f^*)^{-1}(V_I) = \{Q \in \operatorname{Spec}(S) : f_*(I) \leq Q\} \tag{3.9}$$
$$= V_{f_*(I)}. \tag{3.10}$$

We've shown that the inverse image of any closed set is closed; thus, f^* is continuous. □