Fields, Forms and Flows 3/34

Solution Sheet 3

1. We have that $A^2 = -I$, so that

$$A^{2n} = (-1)^n I, \quad A^{2n+1} = (-1)^n A.$$

Then

$$e^{tA} = \sum_{j=0}^{\infty} \frac{t^j A^j}{j!} = \sum_{n=0}^{\infty} \left[\frac{t^{2n} A^{2n}}{(2n)!} + \frac{t^{2n+1} A^{2n+1}}{(2n+1)!} \right] \quad \text{(splitting the sum into even and odd terms)}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n t^{2n}}{(2n!)} I + \frac{(-1)^n t^{2n+1}}{(2n+1)!} A \right] = \cos t I + \sin t A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

2. Action-angle variables for the simple harmonic oscillator (25 marks).

(a)

$$x(t) = x_0 \cos(\omega t) + \frac{\dot{x}_0}{\omega} \sin(\omega t).$$

(b) Let $q = x$. $p = \dot{x}$ and $z = (q, p)$. Then

$$\dot{z} = \mathbb{X}(z), \quad \text{where } \mathbb{X}(z) = (p, -\omega^2 q).$$

(c) We have that

$$\mathbb{Y}(\theta, I) = F'(F^{-1}(\theta, I)))\mathbb{X}(F^{-1}(\theta, I)).$$

$F^{-1}(\theta, I)$ is obtained by expressing q and p as functions of θ and I. We get

$$q(\theta, I) = \sqrt{\frac{2I}{\omega}} \sin\theta, \quad p(\theta, I) = \sqrt{2I\omega} \cos\theta,$$

so

$$F^{-1}(\theta, I) = \left(\sqrt{\frac{2I}{\omega}} \sin\theta, \quad \sqrt{2I\omega} \cos\theta \right).$$

We have that

$$F'(q, p) = \begin{pmatrix} \partial\theta/\partial q & \partial\theta/\partial p \\ \partial I/\partial q & \partial I/\partial p \end{pmatrix} = \begin{pmatrix} \omega p/(p^2 + \omega^2 q^2) & -\omega q/(p^2 + \omega^2 q^2) \\ \omega q & p/\omega \end{pmatrix}.$$

Therefore,

$$F'(F^{-1}(\theta, I))) = \begin{pmatrix} \sqrt{\omega/2I} \cos\theta & -\sqrt{1/2\omega I} \sin\theta \\ \sqrt{2\omega I} \sin\theta & \sqrt{2I/\omega} \cos\theta \end{pmatrix}.$$

Also,

$$\mathbb{X}(q(\theta, I), p(\theta, I)) = \left(\sqrt{2I\omega} \cos\theta, -\omega\sqrt{2I\omega} \sin\theta \right).$$

Thus,

$$\mathbb{Y}(\theta, I) = \begin{pmatrix} \sqrt{\omega/2I} \cos\theta & -\sqrt{1/2\omega I} \sin\theta \\ \sqrt{2\omega I} \sin\theta & \sqrt{2I/\omega} \cos\theta \end{pmatrix} \begin{pmatrix} \sqrt{2I\omega} \cos\theta \\ -\omega\sqrt{2I\omega} \sin\theta \end{pmatrix} = \begin{pmatrix} \omega \\ 0 \end{pmatrix}.$$

Thus, in terms of (θ, I), the system of ODE's is given by

$$\dot{\theta} = \omega, \quad \dot{I} = 0.$$

It is clear that $\mathbb{Y}(\theta, I)$ is constant, i.e. independent of θ and I.

(d) Since $\mathbb{Y}$ is constant, it is easy to solve the system $(\dot{\theta}, \dot{I}) = \mathbb{Y}(\theta, I)$. We obtain

$$\theta(t) = \theta_0 + \omega t, \quad I(t) = I_0,$$

so that

$$\Psi_t(\theta_0, I_0) = (\theta_0 + \omega t, I_0).$$

(e) We have that

$$\begin{aligned} F^{-1} \circ \Psi_t \circ F(q_0, p_0) &= F^{-1}\Psi_t(\tan^{-1}(\omega q_0/p_0), (p_0^2 + \omega^2 q_0^2)/2\omega)) \\ &= F^{-1}(\tan^{-1}(\omega q_0/p_0) + \omega t, (p_0^2 + \omega^2 q_0^2)/2\omega) \\ &= (q_0 \cos\omega t + p_0/\omega \sin\omega t, -\omega q_0 \sin\omega t + p_0 \cos\omega t), \end{aligned}$$

using

$$\sin\left(\tan^{-1} a\right) = \frac{a}{\sqrt{1+a^2}}, \quad \cos\left(\tan^{-1} a\right) = \frac{1}{\sqrt{1+a^2}}.$$

3. (15 marks) Pushforward example.

(a) It is clear that F is smooth so long as $u \neq 1$ (the denominator in the expression for F^1, the first component of F, blows up at $u = 1$). Thus F is a smooth map of X into $\mathbb{R}^2$. Let $(x, y) = F(u, v)$ and solve for u and v, as follows:

$$x = \frac{1+u}{1-u}, \quad y = u + v.$$

From the first equation,

$$u = \frac{x-1}{x+1},$$

which is well defined provided $x \neq -1$. Then the second equation, $v = y - u$, combined with the preceding gives

$$v = y - \frac{x-1}{x+1},$$

which again is well defined provided $x \neq -1$. Thus the image of F is the set of points (x, y) with $x \neq 1$, and

$$F^{-1}(x, y) = \left(\frac{x-1}{x+1}, \ y - \frac{x-1}{x+1}\right),$$

which is smooth for $x \neq -1$.

(b) Let $\mathbb{X}(u, v) = (v, 1)$. We have the general formula for the pushforward,

$$(F_*\mathbb{X})(F(u, v)) = F'(u, v) \cdot \mathbb{X}(u, v).$$

Letting $F(u, v) = (x(u, v), y(u, v))$, the derivative $F'(u, v)$ is given by

$$F'(u, v) = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}(u, v) = \begin{pmatrix} 2/(u-1)^2 & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore,

$$(F_*\mathbb{X})(F(u, v)) = \begin{pmatrix} 2/(1-u)^2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} 2v/(1-u)^2 \\ v+1 \end{pmatrix}.$$

(I've written the answer as a column vector to make it easier to follow the matrix multiplication.)

Next, we want to evaluate $F_*\mathbb{X}$ at (x, y) rather than at $F(u, v)$. So, in the preceding, we have to replace (u, v) by $F^{-1}(x, y)$, ie replace u by $(x-1)/(x+1)$ and v by $y-(x-1)/(x+1)$ to get (after some algebra)

$$(F_*\mathbb{X})(x, y) = \begin{pmatrix} (x+1)^2 y/2 - (x^2-1)/2 \\ y + 2/(x+1) \end{pmatrix}.$$

(c) Let $\mathbb{Y}(u,v) = (u^2+v^2, uv)$. We compute $[\mathbb{X},\mathbb{Y}]$ as follows:

$$\begin{aligned}
[\mathbb{X},\mathbb{Y}] &= (\mathbb{X}\cdot\nabla)\mathbb{Y} - (\mathbb{Y}\cdot\nabla)\mathbb{X} \\
&= \left(v\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(u^2+v^2, uv) - \left((u^2+v^2)\frac{\partial}{\partial u} + uv\frac{\partial}{\partial v}\right)(v,1) \\
&= \left(\left(v\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(u^2+v^2) - \left((u^2+v^2)\frac{\partial}{\partial u} + uv\frac{\partial}{\partial v}\right)v,\right. \\
&\qquad \left.\left(v\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)uv - \left((u^2+v^2)\frac{\partial}{\partial u} + uv\frac{\partial}{\partial v}\right)1\right) \\
&= (2uv + 2v - uv, v^2 + u) = ((u+2)v, u+v^2),
\end{aligned}$$

as required. (Note: the second and third lines contain expressions for the first and second components of $[\mathbb{X},\mathbb{Y}]$ respectively.

4. (10 marks) It is important to keep track of terms which arise when partial derivatives act on products. When you work out $[[\mathbb{X},\mathbb{Y}],\mathbb{Z}]$, some terms contain the product of the component of one vector field with the first partial derivatives of components of the two others. Others contain the product of the components of two of the vector fields with the second partial derivatives of components of the third. Explicitly, the kth component of $[[\mathbb{X},\mathbb{Y}],\mathbb{Z}]^k$ is given by

$$\begin{aligned}
[[\mathbb{X},\mathbb{Y}],\mathbb{Z}]^k = [(\mathbb{X}\cdot\nabla)\mathbb{Y} - (\mathbb{Y}\cdot\nabla)\mathbb{X},\mathbb{Z}]^k &= \left(\mathbb{X}^i\frac{\partial\mathbb{Y}^j}{\partial x^i} - \mathbb{Y}^i\frac{\partial\mathbb{X}^j}{\partial x^i}\right)\frac{\partial\mathbb{Z}^k}{\partial x^j} - \mathbb{Z}^i\left(\frac{\partial\mathbb{X}^j}{\partial x^i}\frac{\partial\mathbb{Y}^k}{\partial x^j} + \mathbb{X}^j\frac{\partial^2\mathbb{Y}^k}{\partial x^i\partial x^j}\right. \\
&\qquad \left.-\frac{\partial\mathbb{Y}^j}{\partial x^i}\frac{\partial\mathbb{X}^k}{\partial x^j} - \mathbb{Y}^j\frac{\partial^2\mathbb{X}^k}{\partial x^i\partial x^j}\right),
\end{aligned}$$

where, as usual, the summation convention applied to the indices i and j above. Similarly,

$$\begin{aligned}
[[\mathbb{Y},\mathbb{Z}],\mathbb{X}]^k = [(\mathbb{Y}\cdot\nabla)\mathbb{Z} - (\mathbb{Z}\cdot\nabla)\mathbb{Y},\mathbb{X}]^k &= \left(\mathbb{Y}^i\frac{\partial\mathbb{Z}^j}{\partial x^i} - \mathbb{Z}^i\frac{\partial\mathbb{Y}^j}{\partial x^i}\right)\frac{\partial\mathbb{X}^k}{\partial x^j} - \mathbb{X}^i\left(\frac{\partial\mathbb{Y}^j}{\partial x^i}\frac{\partial\mathbb{Z}^k}{\partial x^j} + \mathbb{Y}^j\frac{\partial^2\mathbb{Z}^k}{\partial x^i\partial x^j}\right. \\
&\qquad \left.-\frac{\partial\mathbb{Z}^j}{\partial x^i}\frac{\partial\mathbb{Y}^k}{\partial x^j} - \mathbb{Z}^j\frac{\partial^2\mathbb{Y}^k}{\partial x^i\partial x^j}\right),
\end{aligned}$$

and

$$\begin{aligned}
[[\mathbb{Z},\mathbb{X}],\mathbb{Y}]^k = [(\mathbb{Z}\cdot\nabla)\mathbb{X} - (\mathbb{X}\cdot\nabla)\mathbb{Z},\mathbb{Y}]^k &= \left(\mathbb{Z}^i\frac{\partial\mathbb{X}^j}{\partial x^i} - \mathbb{Y}^i\frac{\partial\mathbb{Z}^j}{\partial x^i}\right)\frac{\partial\mathbb{Y}^k}{\partial x^j} - \mathbb{Y}^i\left(\frac{\partial\mathbb{Z}^j}{\partial x^i}\frac{\partial\mathbb{X}^k}{\partial x^j} + \mathbb{Z}^j\frac{\partial^2\mathbb{X}^k}{\partial x^i\partial x^j}\right. \\
&\qquad \left.-\frac{\partial\mathbb{X}^j}{\partial x^i}\frac{\partial\mathbb{Z}^k}{\partial x^j} - \mathbb{X}^j\frac{\partial^2\mathbb{Z}^k}{\partial x^i\partial x^j}\right).
\end{aligned}$$

Add up the three sets of terms to see that each term appears twice (possibly after changing the names of the dummy indices of summation) with opposite sign.

5. (10 marks) Jacobi bracket for vector fields in $\mathbb{R}^3$.

Let's start with $\boldsymbol{\nabla}\times(\mathbf{f}\times\mathbf{g})$ and commute component wise, using the Levi-Cevita symbol for the components of the cross-product. Since we are working in $\mathbb{R}^3$, let's not use superscripts for indices, in keeping with standard vector-calculus notation. We get that

$$(\boldsymbol{\nabla}\times(\mathbf{f}\times\mathbf{g}))_i = \sum_{j,k=1}^{3}\epsilon_{ijk}\frac{\partial}{\partial r_j}(\mathbf{f}\times\mathbf{g})_k = \sum_{j,k=1}^{3}\epsilon_{ijk}\frac{\partial}{\partial r_j}\left(\sum_{l,m=1}^{3}\epsilon_{klm}f_l g_m\right).$$

Use the identity

$$\sum_{k=1}^{3}\epsilon_{ijk}\epsilon_{klm} = \sum_{k=1}^{3}\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

and the product rule

$$\frac{\partial}{\partial r_j} f_l g_m = f_l \frac{\partial g_m}{\partial r_j} + g_m \frac{\partial f_l}{\partial r_j}$$

to obtain

$$\begin{aligned}
(\boldsymbol{\nabla} \times (\mathbf{f} \times \mathbf{g}))_i &= \sum_{j,l,m=1}^{3} (\delta_{il}\delta jm - \delta_{im}\delta_{jl}) \left(f_l \frac{\partial g_m}{\partial r_j} + g_m \frac{\partial f_l}{\partial r_j} \right) \\
&= \sum_{j=1}^{3} \left(f_i \frac{\partial g_j}{\partial r_j} + g_j \frac{\partial f_i}{\partial r_j} - g_i \frac{\partial f_j}{\partial r_j} - f_j \frac{\partial g_i}{\partial r_j} \right).
\end{aligned}$$

This last equation can be written more concisely as the vector equation

$$\boldsymbol{\nabla} \times (\mathbf{f} \times \mathbf{g}) = (\boldsymbol{\nabla} \cdot \mathbf{g})\mathbf{f} + (\mathbf{g} \cdot \boldsymbol{\nabla})\mathbf{f} - (\boldsymbol{\nabla} \cdot \mathbf{f})\mathbf{g} - (\mathbf{f} \cdot \boldsymbol{\nabla})\mathbf{g}.$$

(Alternatively, you can get the preceding equation using the "BAC CAB" rule, but be careful to allow the gradient to act on both **f** and **g**.) Rearranging, we get

$$[\mathbf{f}, \mathbf{g}] = (\mathbf{f} \cdot \boldsymbol{\nabla})\mathbf{g} - (\mathbf{g} \cdot \boldsymbol{\nabla})\mathbf{f} = (\boldsymbol{\nabla} \cdot \mathbf{g})\mathbf{f} - (\boldsymbol{\nabla} \cdot \mathbf{f})\mathbf{g} - \boldsymbol{\nabla} \times (\mathbf{f} \times \mathbf{g}).$$

6. Linear vector fields (15 marks)

   (a)
   $$[\mathbb{X}, \mathbb{Y}] = (\mathbb{X} \cdot \nabla)\mathbb{Y} - (\mathbb{Y} \cdot \nabla)\mathbb{X}.$$

   We have that

   $$(\mathbb{X} \cdot \nabla)\mathbb{Y}^i = \left(\sum_{j,k=1}^{n} A_{jk} x^k \frac{\partial}{\partial x_j} \right) \left(\sum_{l=1}^{n} B_{il} x^l \right) = \sum_{j,k,l=1}^{n} A_{jk} x^k B_{il} \delta_{jl} = \sum_{j,k} A_{jk} x^k B_{ij} = (BAx)^i.$$

   Similarly,
   $$(\mathbb{Y} \cdot \nabla)\mathbb{X}^i = (ABx)^i.$$

   Thus,
   $$[\mathbb{X}, \mathbb{Y}] = (BA - AB)x.$$

   (b) For $F(x) = Sx$ and $\mathbb{Z}(x) = Cx$, we have that $F_*\mathbb{Z}(x) = SCS^{-1}x$. Thus,

   $$F_*[\mathbb{X}, \mathbb{Y}](x) = S(BA - AB)S^{-1}x.$$

   On the other hand,

   $$[F_*\mathbb{X}, F_*\mathbb{Y}] = (SBS^{-1}\, SAS^{-1} - SAS^{-1}\, SBS^{-1})x = (SBAS^{-1} - SABS^{-1})x = S(BA - AB)S^{-1}x.$$

   So the two expressions agree.

   (c) This is just a matter of calculation,

   $$\begin{aligned}
   [A, [B, C]] = [A, BC - CB] &= [A, BC] - [A, CB] = ABC - BCA - ACB + CBA, \\
   [B, [C, A]] = [B, CA - AC] &= [B, CA] - [B, AC] = BCA - CAB - BAC + ACB, \\
   [C, [A, B]] = [C, AB - BA] &= [C, AB] - [C, BA] = CAB - ABC - CBA + BAC.
   \end{aligned}$$

   Add the three equations to find that the right hand side vanishes.
   From above, if $\mathbb{X}(x) = Ax$, $\mathbb{Y}(x) = Bx$ and $\mathbb{Z}(x) = Cx$, then $[\mathbb{X}, \mathbb{Y}](x) = [B, A]x$, and $[\mathbb{Z}, [\mathbb{X}, \mathbb{Y}]] = [[B, A], C]\, x$. Thus, the preceding calculation coincides with the Jacobi identity.

7. Spherical polar coordinates (25 marks)

(a) Let R and Θ be vector fields on $\mathbb{R}^3 = \{(x, y, z)\}$ given by

$$R(x, y, z) = (x, y, z),$$
$$\Theta(x, y, z) = (xz, yz, -(x^2 + y^2)).$$

Then

$$[R, \Theta] =$$
$$(x\partial/\partial x + y\partial/\partial y + z\partial/\partial z)(xz, yz, -(x^2+y^2)) - (xz\partial/\partial x + yz\partial/\partial y - (x^2+y^2)\partial/\partial z)(x, y, z)$$
$$= (2xz, 2yz, -2(x^2 + y^2)) - (xz, yz, -(x^2 + y^2)) = 2\Theta - \Theta = \Theta,$$

as required.

(b) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$F(x, y, z) = (r(x, y, z), \theta(x, y, z), \phi(x, y, z)),$$

where

$$r(x, y, z) = (x^2 + y^2 + z^2)^{1/2},$$
$$\theta(x, y, z) = \cos^{-1}\left(\frac{z}{(x^2 + y^2 + z^2)^{1/2}}\right),$$
$$\phi(x, y, z) = \tan^{-1}\left(\frac{y}{x}\right).$$

Let's compute F_*R first. We have that

$$(F_*R)(r, \theta, \phi) = F'(x, y, z) \cdot R(x, y, z),$$

where (r, θ, ϕ) and (x, y, z) are related as above. Straightforward computation gives

$$F'(x, y, z) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{pmatrix} = \begin{pmatrix} x/r & y/r & z/r \\ xz/(r^2\rho) & yz/(r^2\rho) & -\rho/r^2 \\ -y/\rho^2 & x/\rho^2 & 0 \end{pmatrix},$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ and $\rho = (x^2 + y^2)^{1/2} = r \sin\theta$. Then

$$(F_*R)(r, \theta, \phi) = (r, 0, 0).$$

Similarly,

$$(F_*\Theta)(r, \theta, \phi) = (0, \rho, 0) = (0, r\sin\theta, 0).$$

Then

$$[F_*R, F_*\Theta](r, \theta, \phi) = \left(r\frac{\partial}{\partial r}(0, r\sin\theta, 0) - r\sin\theta\frac{\partial}{\partial \theta}(r, 0, 0)\right)$$
$$= (0, r\sin\theta, 0) = F_*\Theta(r, \theta, \phi).$$

8. * (10 marks) The vector fields $\mathbb{X}$ and $\mathbb{Y}$ are smooth, so to establish completeness it suffices to show that any solution of the differential equations $\dot{x} = \mathbb{X}(x)$ and $\dot{x} = \mathbb{Y}(x)$ remains bounded for all time. Let's consider the second,

$$\dot{x}(t) = \sin x^3(t), \quad x(0) = x_0.$$

It is clear that if $x_0 = \pm(n\pi)^{1/3}$ for n a non-negative integer, we have that $x(t) = x_0$ is a solution (since $\sin x_0^3 = 0$ in this case). Suppose $x_0 \neq (n\pi)^{1/3}$. For definiteness, take $x_0 > 0$ (the argument for $x_0 < 0$ is similar). Then choose n so that

$$n\pi < x_0^3 < (n+1)\pi.$$

It is clear that we must have

$$n\pi < x^3(t) < (n+1)\pi, \forall t,$$

since if $x^3(t_*)$ were equal to $n\pi$ or $(n+1)\pi$ for some t_*, this would contradict the uniqueness of solutions (we already know that $x^3(t) = n\pi$ and $x^3(t) = (n+1)\pi$ are solutions for all t). Thus, solutions are bounded. A similar argument applies to solutions of $\dot{x} = \mathbb{X}(x)$.

Computation gives

$$[\mathbb{X}, \mathbb{Y}] = 3x^2 \cos^2 x^2 + 3x^2 \sin^2 x^2 = 3x^2.$$

We have seen (Problem Sheet 2.5) that solutions of $\dot{x} = x^2$ can have singularities in finite time. Hence the flow of $[\mathbb{X}, \mathbb{Y}]$ is not complete.