# Fields, Forms and Flows 3/34

Solution Sheet 4



1. Lie derivative of a vector field.

   (a) (5 marks) For $n = 1$, the required formula is just Proposition. 1.8.10. Now use induction – suppose it's true for $n$, and verify for $n + 1$ as follows:

   $$\begin{aligned}\frac{\partial^{n+1}}{\partial s^{n+1}}\Psi_{s*}\mathbb{X} = \frac{\partial}{\partial s}\frac{\partial^n}{\partial s^n}\Psi_{s*}\mathbb{X} = \frac{\partial}{\partial s}(-1)^n L_{\mathbb{Y}}^n \Psi_{s*}\mathbb{X} \text{ (by induction hypothesis)}\\ = (-1)^n L_{\mathbb{Y}}^n \frac{\partial}{\partial s}\Psi_{s*}\mathbb{X} \text{ (as } \mathbb{Y} \text{ does not depend on } s)\\ = (-1)^n L_{\mathbb{Y}}^n(-L_{\mathbb{Y}}\Psi_{s*}\mathbb{X}) = (-1)^{n+1}L_{\mathbb{Y}}^{n+1}\Psi_{s*}\mathbb{X},\end{aligned}$$

   as required.

   (b) (5 marks) We have the formal power series

   $$\Psi_{s*}\mathbb{X} = \sum_{n=0}^{\infty} \frac{\partial^n}{\partial s^n}\Psi_{s*}\mathbb{X}\bigg|_{s=0} \frac{s^n}{n!}.$$

   From part (a), since $\Psi_{0*\mathbb{X}} = \mathbb{X}$,

   $$\frac{\partial^n}{\partial s^n}\Psi_{s*}\mathbb{X}\bigg|_{s=0} = (-1)^n L_{\mathbb{Y}}^n \Psi_{s*}\mathbb{X}|_{s=0} = (-1)^n L_{\mathbb{Y}}^n\mathbb{X}.$$

   Substitute to get

   $$\Psi_{s*}\mathbb{X} = \left(\sum_{n=0}^{\infty}\frac{s^n}{n!}(-1)^n L_{\mathbb{Y}}^n\right)\mathbb{X} = e^{-sL_{\mathbb{Y}}}\mathbb{X},$$

   as required. Since $L_{\mathbb{Y}}\mathbb{Y} = [\mathbb{Y},\mathbb{Y}] = 0$, it follows that $L_{\mathbb{Y}}^n\mathbb{Y} = 0$ for $n \geq 1$, so that

   $$\Psi_{s*}\mathbb{Y} = \mathbb{Y}.$$

   (We had a different, flow-based proof of this fact in Proposition 1.8.11).

   (c) (5 marks) Just go from $L_{\mathbb{X}}$-notation to bracket-notation and back again, as follows:

   $$\begin{aligned}(L_{\mathbb{X}}L_{\mathbb{Y}} - L_{\mathbb{Y}}L_{\mathbb{X}})\mathbb{Z} = L_{\mathbb{X}}[\mathbb{Y},\mathbb{Z}] - L_{\mathbb{Y}}[\mathbb{X},\mathbb{Z}] = [\mathbb{X},[\mathbb{Y},\mathbb{Z}]] - [\mathbb{Y},[\mathbb{X},\mathbb{Z}]]\\ = [\mathbb{X},[\mathbb{Y},\mathbb{Z}]] + [[\mathbb{X},\mathbb{Z}],\mathbb{Y}]\\ = [[\mathbb{X},\mathbb{Y}],\mathbb{Z}] \text{ (from the Jacobi identity)} = L_{[\mathbb{X},\mathbb{Y}]}\mathbb{Z},\end{aligned}$$

   as required.

2. Parallel parking.

   (a) (5 marks) Let $\mathbb{S}$ be given by

   $$\mathbb{S} = (0, 0, 0, \omega), \tag{1}$$

   where $\omega$ is a constant. The flow of $\mathbb{S}$ satisfies the system of differential equations

   $$\dot{x} = 0,\ \dot{y} = 0,\ \dot{\theta} = 0,\ \dot{\alpha} = \omega,$$

   with solutions $x(t) = x_0$, $y(t) = y_0$, $\theta(t) = \theta_0$, $\alpha(t) = \alpha_0 + \omega t$. Thus, $\Phi_t$ is given by

   $$\Phi_t(x, y, \theta, \alpha) = (x, y, \theta, \alpha + \omega t). \tag{2}$$

(b) (10 marks) $P = (x, y)$ specifies the centre of the back axle, and $\theta$ specifies the inclination of the back axle to the vertical. If the back wheels roll forward a distance $v\delta t$, $P$ undergoes a displacement $\delta P$ given by

$$\delta P = v\delta t(\sin\theta, \cos\theta). \tag{3}$$

$Q = (x + L\sin\theta, y + L\cos\theta)$ specifies the centre of the front axle, and $\theta + \alpha$ specifies the inclination of the front axle to the vertical. If the front wheels roll forward a distance $\delta\sigma$, $Q$ undergoes a displacement $\delta Q$ given by

$$\delta Q = \delta\sigma(\sin(\theta + \alpha), \cos(\theta + \alpha)). \tag{4}$$

The magnitude of this displacement, $\delta\sigma$, is determined by requiring that the distance between $P+\delta P$ and $Q + \delta Q$ is $L$ (the car can't stretch). We have that

$$|(P + \delta P) - (Q + \delta Q)|^2 = |P - Q|^2 + 2(P - Q) \cdot (\delta P - \delta Q) + O(2).$$

Since $|P - Q| = L$, we must have $2(P - Q) \cdot (\delta P - \delta Q) = 0$, or

$$L(\sin\theta, \cos\theta) \cdot (v\delta t \sin\theta - \delta\sigma \sin(\theta + \alpha), v\delta t \cos\theta - \delta\sigma \cos(\theta + \alpha)) = 0.$$

This simplifies to

$$v\delta t - \delta\sigma(\cos\theta \cos(\theta + \alpha) + \sin\theta \sin(\theta + \alpha)) = v\delta t - \delta\sigma \cos\alpha = 0,$$

or

$$\delta\sigma = \frac{v}{\cos\alpha}\delta t.$$

Note the singularity at $\alpha = \pm\pi/2$. If the front wheels are perpendicular to the car, $P$ can't move (at least not without the front wheels skidding).

As $\theta$ specifies the orientation of the car with respect to the vertical, it is given by, eg, $(Q-P)\cdot(1, 0) = L\sin\theta$. Thus $((Q + \delta Q) - (P + \delta P)) \cdot (1, 0) = L\sin(\theta + \delta\theta)$. At first order in the displacements, we get

$$(\delta Q - \delta P) \cdot (1, 0) = L\cos\theta\delta\theta.$$

Using the expressions (3) and (4) for $\delta P$ and $\delta Q$, we get

$$\delta\theta = \frac{v\delta t}{L\cos\theta}\left(\frac{\sin(\theta + \alpha)}{\cos\alpha} - \sin\theta\right) = \frac{v}{L}\tan\alpha\delta t.$$

The vector field $\mathbb{D}$ which describes the displacement $P \to P + \delta P$, $\theta \to \theta + \delta\theta$ is given by

$$\mathbb{D} = (v\sin\theta, v\cos\theta, \frac{v}{L}\tan\alpha, 0). \tag{5}$$

The flow of $\mathbb{D}$ satisfies the system of differential equations

$$\dot{x} = v\sin\theta,\ \dot{y} = v\cos\theta,\ \dot{\theta} = (v/L)\tan\alpha,\ \dot{\alpha} = 0.$$

The solutions are $\alpha(t) = \alpha_0$, $\theta(t) = \theta_0 + (vt/L)\tan\alpha$, $x(t) = x_0 - L(\cos\theta(t) - \cos\theta_0)/\tan\alpha$, and $y(t) = y_0 + L(\sin\theta(t) - \sin\theta_0)/\tan\alpha$. Thus, the flow is given by

$$\begin{aligned}\Psi_t(x, y, \theta, \alpha) \quad = \quad & \Bigg(x - \frac{L}{\tan\alpha}\left(\cos\left(\theta + \frac{vt}{L}\tan\alpha\right) - \cos\theta\right), \\ & y + \frac{L}{\tan\alpha}\left(\sin\left(\theta + \frac{vt}{L}\tan\alpha\right) - \sin\theta\right), \\ & \theta + \frac{vt}{L}\tan\alpha, \\ & \alpha\Bigg).\end{aligned} \tag{6}$$

Under $\Psi_t$, the point $P$ describes a circle centred at $(x + L\cos\theta/\tan\alpha, y - L\sin\theta/\tan\alpha)$ with radius $L/\tan\alpha$. Note that when $\alpha = 0$ (the front wheels are not turned), the turning radius is infinite, and the car moves in a straight line.

(c) (5 marks) Compute $\Psi_\epsilon(x, y, \theta, \epsilon\Omega)$ through order $\epsilon^2$ by expanding in (6). The $x$-component is given by

$$x - \frac{L}{\tan \epsilon\Omega}\left(\cos\left(\theta + \frac{v\epsilon}{L}\tan\epsilon\Omega\right) - \cos\theta\right).$$

Note that $\tan \epsilon\Omega = \epsilon\Omega + O(\epsilon^3)$, so that

$$\cos\left(\theta + \frac{v\epsilon}{L}\tan\epsilon\Omega\right) - \cos\theta = \cos\left(\theta + \frac{\epsilon^2\Omega v}{L} + O(\epsilon^4)\right) - \cos\theta = -\epsilon^2\frac{\Omega v}{L}\sin\theta + O(\epsilon^4).$$

Substitute into the preceding expression to get the following expression for the $x$-component:

$$x + \frac{L}{\epsilon\Omega}\left(\epsilon^2\frac{\Omega v}{L}\sin\theta + O(\epsilon^4)\right) = x + \epsilon v \sin\theta + O(\epsilon^3).$$

A similar calculation gives the $y$-component as

$$y + \epsilon v\cos\theta + O(\epsilon^3).$$

Next, the $\theta$-component is given by

$$\theta + \frac{v\epsilon}{L}\tan(\epsilon\Omega) = \theta + \epsilon^2\frac{\Omega v}{L} + O(\epsilon^4).$$

Finally, the $\alpha$-component is just $\epsilon\Omega$, since $\alpha$ is unchanged by $\Psi_t$. Collecting results we have that

$$\Psi_\epsilon(x, y, \theta, \epsilon\Omega) = (x + \epsilon v\sin\theta, y + \epsilon v\cos\theta, \theta + \epsilon^2\Omega v/L, \epsilon\Omega) + O(\epsilon^3). \tag{7}$$

(d) (10 marks)

$$\mathbb{A} = [\mathbb{S}, \mathbb{D}] = (\mathbb{S}\cdot\nabla)\mathbb{D} - (\mathbb{D}\cdot\nabla)\mathbb{S}.$$

As $\mathbb{S}$ is constant, $(\mathbb{D}\cdot\nabla)\mathbb{S} = 0$. Also, $(\mathbb{S}\cdot\nabla)\mathbb{D} = \omega\partial\mathbb{D}/\partial\alpha$, so that

$$\mathbb{A} = \left(0, 0, \frac{\omega v}{L}\sec^2\alpha, 0\right).$$

Thus, $\mathbb{A}$ has a component in the $\theta$-direction only. $\Gamma_a$, the flow of $\mathbb{A}$, is given by

$$\Gamma_a(x, y, \theta, \alpha) = (x, y, \theta + (\omega v\sec^2\alpha/L)a, \alpha). \tag{8}$$

Next,

$$\mathbb{B} = [[\mathbb{S}, \mathbb{D}], \mathbb{D}] = [\mathbb{A}, \mathbb{D}] = (\mathbb{A}\cdot\nabla)\mathbb{D} - (\mathbb{D}\cdot\nabla)\mathbb{A}.$$

Since $\mathbb{A}$ has a non-zero component in the $\theta$-direction only, we have that

$$(\mathbb{A}\cdot\nabla)\mathbb{D} = \frac{\omega v}{L}\sec^2\alpha\frac{\partial\mathbb{D}}{\partial\theta} = \frac{\omega v^2}{L}\sec^2\alpha(\cos\theta, -\sin\theta, 0, 0),$$

and, since $\mathbb{A}$ depends only $\alpha$ and the $\alpha$-component of $\mathbb{D}$ vanishes, $(\mathbb{D}\cdot\nabla)\mathbb{A} = 0$. Therefore,

$$\mathbb{B} = \frac{\omega v^2}{L}\sec^2\alpha(\cos\theta, -\sin\theta, 0, 0).$$

Thus, $\mathbb{B}$ has components in the $x$- and $y$-directions only; it is a pure translation. Its flow, $\Delta_b$, is given by

$$\Delta_b(x, y, \theta, \alpha) = \left(x + \frac{\omega v^2}{L}\sec^2\alpha\cos\theta\, b, y - \frac{\omega v^2}{L}\sec^2\alpha\sin\theta\, b, \theta, \alpha\right). \tag{9}$$

Note that the direction of the translation is perpendicular to the axis of the car.

Linear independence: construct a $4\times 4$ matrix whose rows are the components of the vector fields $\mathbb{D}$, $\mathbb{B}$, $\mathbb{A}$ and $\mathbb{S}$ (in that order),

$$\begin{pmatrix} v\sin\theta & v\cos\theta & (v/L)\tan\alpha & 0 \\ \frac{\omega v^2}{L}\sec^2\alpha\cos\theta & -\frac{\omega v^2}{L}\sec^2\alpha\sin\theta & 0 & 0 \\ 0 & 0 & \frac{\omega v}{L}\sec^2\alpha & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

Linear independence of $\mathbb{A}$, $\mathbb{B}$, $\mathbb{S}$ and $\mathbb{D}$ is equivalent to this matrix being nonsingular, ie having nonzero determinant. The determinant is given by $-\omega^3 v^4\sec^4\alpha/L^2$, which is well defined and nonzero provided $\cos\alpha \neq 0$.

(e) (10 marks)
Let $M_\epsilon = \Psi_{-\epsilon} \circ \Phi_{-\epsilon} \circ \Psi_\epsilon \circ \Phi_\epsilon$ (ie, the sequence steer right, drive forward, steer left, drive back, for a time $\epsilon$). Let

$$p = \Phi_\epsilon(x, y, \theta, 0), \quad q = \Psi_\epsilon(p), \quad r = \Phi_{-\epsilon}(q), \quad s = \Psi_{-\epsilon}(r) = M_\epsilon(x, y, \theta, 0).$$

The flows $\Phi_{\pm\epsilon}$ and $\Psi_{\pm\epsilon}$ are given (through $O(\epsilon^3)$) by (2) and (7). Use these expressions to compute $p$, $q$, $r$ and $s$, as follows:

$$\begin{aligned}
p &= (x, & y, &\theta, & \epsilon\omega) + O(\epsilon^3),\\
q &= (x + \epsilon v \sin\theta, & y + \epsilon v\cos\theta, &\theta + \epsilon^2\omega v/L, & \epsilon\omega) + O(\epsilon^3),\\
r &= (x + \epsilon v \sin\theta, & y + \epsilon v\cos\theta, &\theta + \epsilon^2\omega v/L, & 0) + O(\epsilon^3),\\
s &= (x, & y, &\theta + \epsilon^2\omega v/L, & 0) + O(\epsilon^3)\\
&= \Gamma_{\epsilon^2}(x, y, \theta, 0) + O(\epsilon^3).
\end{aligned}$$

Thus,

$$M_\epsilon(x, y, \theta, 0) = \Gamma_{\epsilon^2}(x, y, \theta, 0) + O(\epsilon^3), \tag{10}$$

as required.
Let $(M_\epsilon)^j$ denote $M_\epsilon$ repeated $j$ times, ie

$$(M_\epsilon)^j = \underbrace{M_\epsilon \circ \cdots \circ M_\epsilon}_{j \text{ times}}.$$

From (10), it follows that

$$(M_\epsilon)^j(x, y, \theta, 0) = \Gamma_{j\epsilon^2}(x, y, \theta, 0) + O(j\epsilon^3),$$

so that

$$(M_\epsilon)^{[\epsilon^{-2}a]}(x, y, \theta, 0) = \Gamma_a(x, y, \theta, 0) + O(\epsilon), \tag{11}$$

where $[x]$ denotes the integer part of $x$.
Suppose your car initially has configuration $(x, y, \theta, 0)$, and you want to turn it so that it points in the direction $\theta + \Theta$, all the while keeping $x$ and $y$ to within $O(\epsilon)$ of their initial values. In view of (10) and (11), it suffices to execute the maneuver $M_\epsilon$ $j$ times in succession, where

$$j = \left[\frac{L}{\epsilon^2\omega v}\Theta\right].$$

$\epsilon$ can be made arbitrarily small, but the number of maneuvers $j$ grows as $\epsilon^{-2}$.

(f) (10 marks) Let

$$P_\delta = \Psi_{-\delta} \circ \Gamma_{-\delta T} \circ \Psi_\delta \circ \Gamma_{\delta T}.$$

Let

$$p = \Gamma_{\delta T}(x, y, \theta, 0), \quad q = \Psi_\delta(p), \quad r = \Gamma_{-\delta T}(q), \quad s = \Psi_{-\delta}(r) = P_\delta(x, y, \theta, 0).$$

The flows $\Psi_\delta$ and $\Gamma_{\delta T}$ are given (through $O(\delta^3)$) by (7) and (8). Use these expressions to compute $p$, $q$, $r$ and $s$ as follows, letting $\nu = \omega v T/L$:

$$\begin{aligned}
p &= (x, & y, &\theta + \delta\nu, & 0),\\
q &= (x + \delta v\sin\theta + \delta^2 v\nu\cos\theta, & y + \delta v\cos\theta - \delta^2 v\nu\sin\theta, &\theta + \delta\nu & , 0) + O(\delta^3),\\
r &= (x + \delta v\sin\theta + \delta^2 v\nu\cos\theta, & y + \delta v\cos\theta - \delta^2 v\nu\sin\theta, &\theta, & 0) + O(\delta^3),\\
s &= (x + \delta^2 v\nu\cos\theta, & y - \delta^2 v\nu\sin\theta, &\theta, & 0) + O(\delta^3)\\
&= \Delta_{\delta^2 T}(x, y, \theta, 0) + O(\delta^3).
\end{aligned}$$

Thus,

$$P_\delta(x, y, \theta, 0) = \Delta_{\delta^2 T}(x, y, \theta, 0) + O(\delta^3), \tag{12}$$

as required.
Let $(P_\delta)^k$ denote $P_\delta$ repeated $k$ times, ie

$$(P_\delta)^k = \underbrace{P_\delta \circ \cdots \circ P_\delta}_{k \text{ times}}.$$

Fom (12) it follows that

$$(P_\delta)^k(x,y,\theta,0) = \Delta_{k\delta^2 T}(x,y,\theta,0) + O(k\delta^3),$$

so that

$$(P_\delta)^{[\delta^{-2}b/T]}(x,y,\theta,0) = \Delta_b(x,y,\theta,0) + O(\delta). \tag{13}$$

From part (e) above, we have that

$$\Gamma_{\delta T}(x,y,\theta,0) = (M_{(\delta T)^2})^{[(\delta T)^{-3}]}(x,y,\theta,0) + O(\delta^3).$$

Therefore, $\Gamma_{\delta T}(x,y,\theta,0)$ can be realised, to within $O(\delta^3)$, by a sequence of $M_{\delta^2 T^2}$ maneuvers, ie by a sequence of steers and drives. Therefore, $P_\delta(x,y,\theta,0)$ can also be realised, to within $O(\delta^3)$, by a sequence of steers and drives.

Suppose your car initially has configuration $(x,y,\theta,0)$, and you want to shift it a distance $d$ perpendicular to its length while keeping its direction $\theta$ to within $O(\delta)$ of its original value. In view of (12) and (13), it suffices to execute the maneuver $P_\delta$ $k$ times in succession, where

$$k = \left[\frac{Ld}{\delta^2 T\omega v^2}\Theta\right].$$

3. Noncommutativity of rotations in $\mathbb{R}^3$.

   (a) (5 marks) One way to proceed is to use the result of Question 4 of Problem Sheet 3 and then verify explicitly that

$$\begin{aligned}
[A,B] &= AB - BA = C,\\
[B,C] &= BC - CB = A,\\
[C,A] &= CA - AC = B.
\end{aligned}$$

   Alternatively, we can compute the Jacobi brackets directly. We have that

$$X_A(r) = (0,-z,y), \quad X_B(r) = (z,0,-x), \quad X_C(r) = (-y,x,0).$$

   Then

$$\begin{aligned}
[X_A, X_B] &= \left(\left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)(z,0,-x) - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)(0,-z,y)\right)\\
&= (y,0,0) - (0,x,0) = (y,-x,0) = -X_C.
\end{aligned}$$

   The other relations are similarly proved.

   (b) (5 marks) The system $\dot r = X_C(r)$ yields

$$\dot x = -y, \quad \dot y = x, \quad \dot z = 0.$$

   $\dot z = 0$ implies that $z(t) = z_0$, and the equations for $x$ and $y$ can be solved as follows. We have that $\ddot x = -\dot y = -x$, which has solution $x(t) = \cos t\, x_0 + \sin t\, \dot x_0$. Similarly, $\ddot y = \dot x = -y$, which has solution $y(t) = \cos t\, y_0 + \sin t\, \dot y_0$. But the original differential equations imply that $\dot x_0 = -y_0$ and $\dot y_0 = x_0$. Therefore,

$$\begin{aligned}
x(t) &= \cos t\, x_0 - \sin t\, y_0,\\
y(t) &= \sin t\, x_0 + \cos t\, y_0.
\end{aligned}$$

   Collectively the solutions are described by the matrix equation

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix}(t) = \begin{pmatrix} \cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} x_0\\ y_0\\ z_0\end{pmatrix}.$$

   Therefore, we can write

$$\Phi_{Ct}(r) = \mathcal{R}_C(t)\cdot r,$$

   where

$$\mathcal{R}_C(t) = \begin{pmatrix} \cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}.$$

(c) (5 marks) Similarly,

$$\Phi_{At}(r) = \mathcal{R}_A(t) \cdot r, \quad \Phi_{Bt}(r) = \mathcal{R}_B(t) \cdot r,$$

where

$$\mathcal{R}_A(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}, \quad \mathcal{R}_B(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}.$$

(d) (5 marks) We have that

$$(\Phi_{B\theta} \circ \Phi_{A\theta} \circ \Phi_{-B\theta} \circ \Phi_{-A\theta})(r) = \mathcal{R}_B(\theta)\mathcal{R}_A(\theta)\mathcal{R}_B(-\theta)\mathcal{R}_A(-\theta) \cdot r.$$

Up to and including terms of second order in $\theta$, $\sin\theta = \theta$ and $\cos\theta = 1 - \theta^2/2$, so that

$$\mathcal{R}_A(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\theta^2/2 & -\theta \\ 0 & \theta & 1-\theta^2/2 \end{pmatrix}, \quad \mathcal{R}_B(\theta) \begin{pmatrix} 1-\theta^2/2 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1-\theta^2/2 \end{pmatrix}.$$

To second order in $\theta$,

$$\begin{aligned} &\mathcal{R}_B(-\theta)\mathcal{R}_A(-\theta) \\ &\qquad = \begin{pmatrix} 1-\theta^2/2 & 0 & -\theta \\ 0 & 1 & 0 \\ \theta & 0 & 1-\theta^2/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\theta^2/2 & \theta \\ 0 & -\theta & 1-\theta^2/2 \end{pmatrix} \\ &\qquad\qquad = \begin{pmatrix} 1-\theta^2/2 & \theta^2 & -\theta \\ 0 & 1-\theta^2/2 & \theta \\ \theta & -\theta & 1-\theta^2 \end{pmatrix}. \end{aligned}$$

Then multiply the preceding by $\mathcal{R}_A(\theta)$ on the left and keep terms through $\theta^2$ to get

$$\begin{aligned} &\mathcal{R}_A(\theta)\mathcal{R}_B(-\theta)\mathcal{R}_A(-\theta) \\ &\qquad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\theta^2/2 & -\theta \\ 0 & \theta & 1-\theta^2/2 \end{pmatrix} \begin{pmatrix} 1-\theta^2/2 & \theta^2 & -\theta \\ 0 & 1-\theta^2/2 & \theta \\ \theta & -\theta & 1-\theta^2 \end{pmatrix} \\ &\qquad\qquad = \begin{pmatrix} 1-\theta^2/2 & \theta^2 & -\theta \\ -\theta^2 & 1 & 0 \\ \theta & 0 & 1-\theta^2/2 \end{pmatrix}. \end{aligned}$$

Multiply the preceding by $\mathcal{R}_B(\theta)$ on the left and keep terms through $\theta^2$ to get

$$\begin{aligned} &\mathcal{R}_B(\theta)\mathcal{R}_A(\theta)\mathcal{R}_B(-\theta)\mathcal{R}_A(-\theta) \\ &\qquad = \begin{pmatrix} 1-\theta^2/2 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1-\theta^2/2 \end{pmatrix} \begin{pmatrix} 1-\theta^2/2 & \theta^2 & -\theta \\ -\theta^2 & 1 & 0 \\ \theta & 0 & 1-\theta^2/2 \end{pmatrix} \\ &\qquad\qquad = \begin{pmatrix} 1 & \theta^2 & 0 \\ -\theta^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

But this last expression is equal to $\mathcal{R}_C(-\theta^2)$ through terms of order $\theta^2$.