Numerical Analysis 23 — Sheet 5

Differentiation and Integration

Please hand in questions 1 and 2(a)-(c) by Thursday 7 March.

1. Consider calculating $f^{'}(1)$ for $f(x) = e^x$ using the forward difference formula
$$f^{'}(x) \approx \frac{f(x+h) - f(x)}{h}.$$
Derive the truncation error incurred by using this formula. Now pretend you can only work to 4 decimal figures and use various $h = 10^{-n}$ with $n = 0, 1, \ldots, 5$ to generate estimates of $f^{'}(1)$. What is the optimal choice of n which gives greatest accuracy and explain how you can predict this from the sizes of the truncation and round-off errors.

2. You are told that a function $f(x)$ has the following values
$$\begin{aligned} f(1.0) &= 2.287355, \\ f(1.1) &= 2.677335, \\ f(1.2) &= 3.094479, \\ f(1.3) &= 3.535581, \\ f(1.4) &= 3.996196. \end{aligned}$$
   (a) Using forward difference and central difference formulae numerically approximate the derivative $f^{'}(1.2)$ with $h = 0.2$ and $h = 0.1$.

   (b) Derive a forward 3-point formula to approximate $f^{'}(x_0)$ using $f(x_0)$, $f(x_0 + h)$ and $f(x_0 + 2h)$ which is accurate to $O(h^2)$. Use this to find $f^{'}(1.2)$.

   (c) Produce a better estimate of $f^{'}(1.2)$ by using Richardson's extrapolation for the central difference formula. (The actual value of $f'(1.2)$ is 4.297549.)

   (d) Use the composite Trapezoidal Rule to estimate the integral
$$\int_1^{1.4} f(x)\,dx.$$

3. Given $f(x_0)$, $f^{'}(x_0 + h)$ and $f(x_0 - \lambda h)$ where $\lambda = O(1)$ and $h \ll 1$, build a formula for estimating $f^{''}(x_0)$. What is the error incurred by using this formula. Show that for $\lambda^2 = 3$, the error is $O(h^2)$.