EECS 106B/206B
Discussion 3

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Action Items

- BCourses/Piazza - is everyone on?
- Concurrent Enrollment students fill out both forms:
  - Student accounts
  - Lab access
- Does everyone else have Lab access?
- Remember, all students who did not take 106a must finish lab 0 with a GSI before you're allowed to use the robots
  - M 1-2, W 4-5, Th 2-4
- Homework: Due Sunday

Linear vs Non-Linear Systems

$$\dot{x} = Ax + Bu \qquad \dot{x} = f(x, u)$$

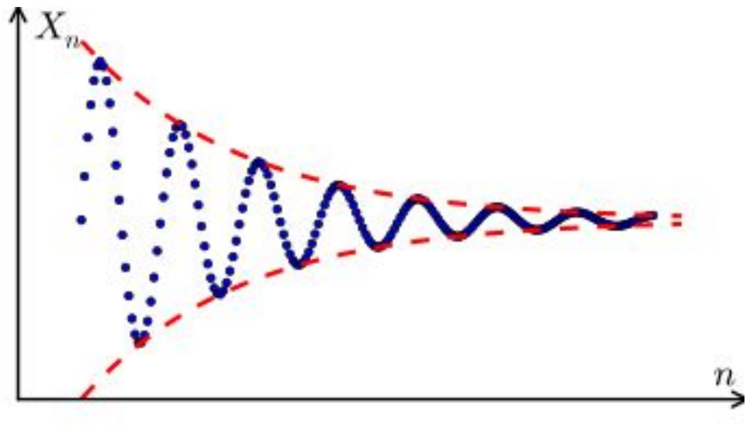
LTI Systems Review

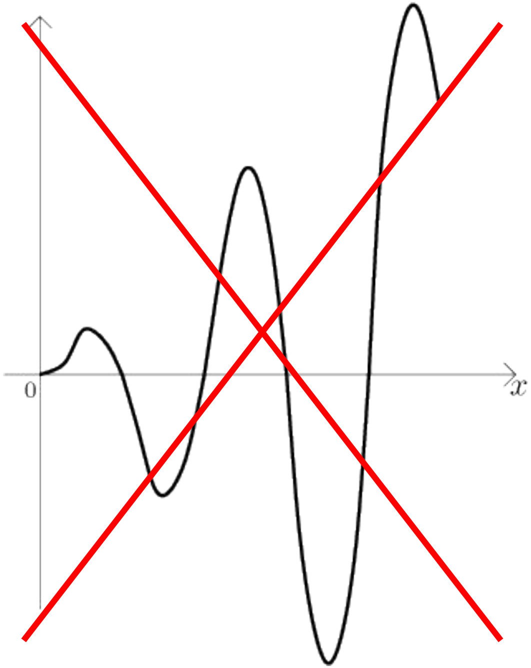
- Linearity
  - Linear relationship between input and output
  - Can be scaled and summed
- Time Invariance
  - Output does not depend on when input happens
- Impulse response
  - Function to characterize system

Defining Stability

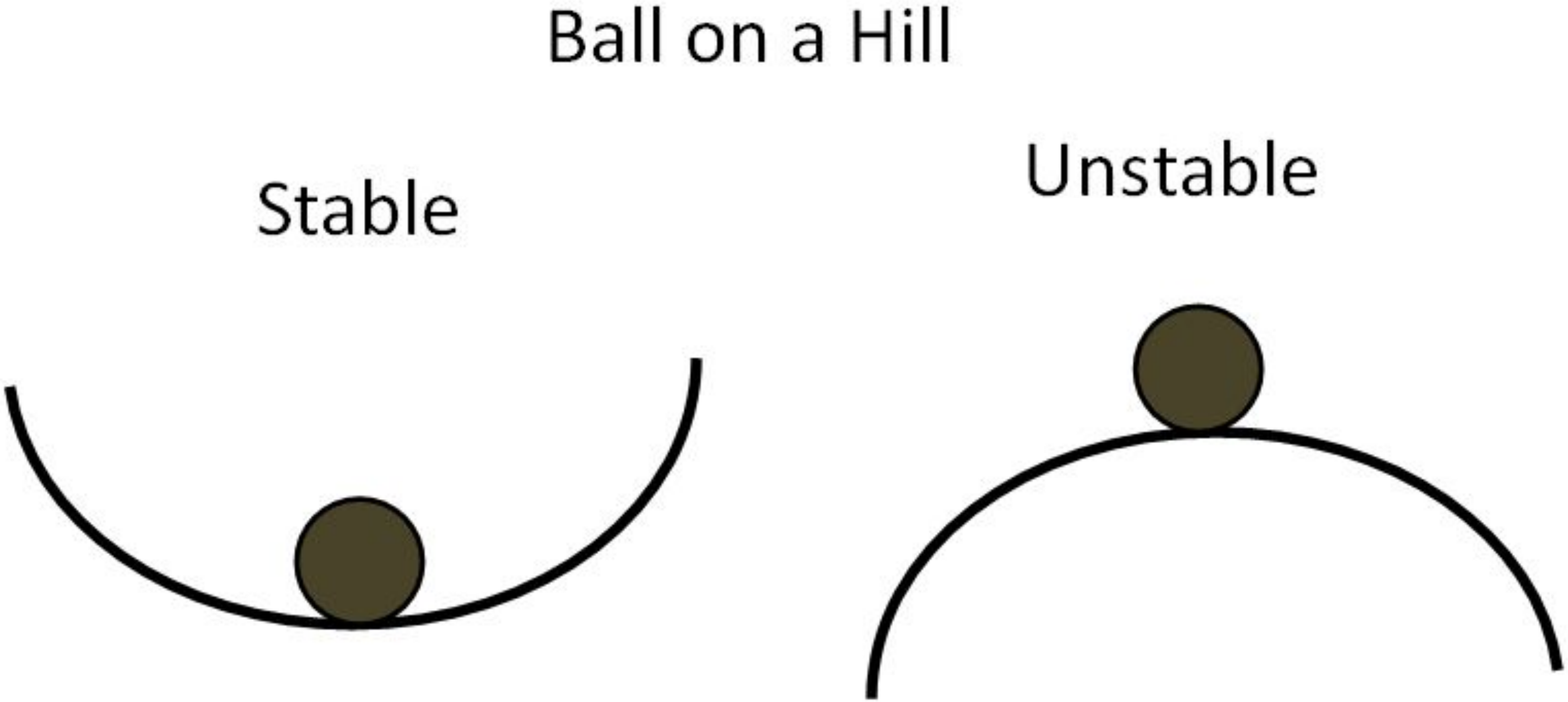
A system is stable (over some range of inputs) if the outputs to those inputs don’t go to infinity

- Linear Systems: BIBO Stability, Hurwitz
- Nonlinear Systems: Lyapunov Stability (+ more)





Example



BIBO Stability

- If given a bounded input, the output will be bounded

A signal is bounded if there is a finite value $B > 0$ such that the signal magnitude never exceeds B, that is

$|y[n]| \leq B \quad \forall n \in \mathbb{Z}$ for discrete-time signals, or

$|y(t)| \leq B \quad \forall t \in \mathbb{R}$ for continuous-time signals.

Hurwitz Stability (standard method)

- If the eigenvalues of the A matrix are

  $$Re\{\lambda_i\} < 0, \quad \forall i$$

  Then then A is exponentially stable

- You can test this with the Routh-Hurwitz Test
- If eigenvalues are on the imaginary axis (exactly one of which can be at the origin), A is *marginally stable*, or *stable in the sense of Lyapunov (SISL)*

- If your system is discrete-time, the A matrix must instead be Schur. We will not be covering discrete time systems in this class.

Lyapunov Stability

Definition 4.1. Stability in the sense of Lyapunov
The equilibrium point $x^* = 0$ of (4.31) is *stable (in the sense of Lyapunov)* at $t = t_0$ if for any $\epsilon > 0$ there exists a $\delta(t_0, \epsilon) > 0$ such that

$$\|x(t_0)\| < \delta \quad \Longrightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq t_0. \tag{4.32}$$

Equilibrium point: $\dot{x} = 0$ (shift origin so that this occurs at $x = 0$)

Essentially BIBO stability for nonlinear functions

Asymptotic Stability

An equilibrium point $x^* = 0$ of (4.31) is *asymptotically stable* at $t = t_0$ if

1. $x^* = 0$ is stable, and
2. $x^* = 0$ is locally attractive; i.e., there exists $\delta(t_0)$ such that

$$\|x(t_0)\| < \delta \quad \Longrightarrow \quad \lim_{t \to \infty} x(t) = 0. \tag{4.33}$$

The system is stable and converges to zero (since $x^* = 0$)

Exponential Stability

Definition 4.3. Exponential stability, rate of convergence
The equilibrium point $x^* = 0$ is an *exponentially stable* equilibrium point of (4.31) if there exist constants $m, \alpha > 0$ and $\epsilon > 0$ such that

$$\|x(t)\| \leq m e^{-\alpha(t-t_0)} \|x(t_0)\| \tag{4.34}$$

for all $\|x(t_0)\| \leq \epsilon$ and $t \geq t_0$. The largest constant α which may be utilized in (4.34) is called the *rate of convergence*.

Positive-Definite functions

Definition 13.2 Let V be a continuous map from $\mathbb{R}^n$ to $\mathbb{R}$. We call $V(x)$ a *locally positive definite* (lpd) function around $x = 0$ if

1. $V(0) = 0$.

2. $V(x) > 0, \ 0 < \|x\| < r$ for some r.

Positive-Definite matrix

- Positive Definite: $A \succ 0$
- Positive Semidefinite: $A \succeq 0$

A positive-definite (symmetric) matrix:

- $x^T A x > 0, \;\; A \in \mathbb{R}^{n \times n}, \;\; \forall x \in \mathbb{R}^n, \;\; x \neq 0$
- All eigenvalues > 0
- $A^* A$ is positive definite and hermitian

Comparison between linear and nonlinear stability

	Linear	Nonlinear
Unstable	At least one Re{EV} > 0	Not SISL. Sometimes hard to prove
Stability in the Sense of Lyapunov	One or more Re{EV} = 0, no EV's in open right half plane.	There exists some locally PD V whose derivative is negative semi-definite
Asymptotic Stability	All EVs in open left half plane	There exists some PD V whose derivative is ND
Exponential Stability	Same as asymptotic	x is bounded by an exponential function m*exp(alpha t) x0

Lyapunov's Direct Method

- Find some energy function V(x)   (a common one is $x^T P x$)
- Take the Lie derivative with respect to f(x)

$$\dot{V}|_{\dot{x}=f(x)} = L_f V = \frac{\partial V}{\partial x} f$$

- If V is locally PD and Vdot <= 0 locally, then f is SISL
- If V is locally PD and -Vdot is locally PD, then f is locally asymptotically stable
- If V is globally PD and -Vdot is globally PD, then f is globally stable

Indirect Lyapunov's Method

A function is *locally* stable at a point if the *linearization* of the function at that point is *globally* (BIBO) stable.

$$\dot{x} = f(x,t)$$

$$A(t) = \left.\frac{\partial f(x,t)}{\partial x}\right|_{x=0}$$

$$f_1(x,t) = f(x,t) - A(t)x$$

If: $$\lim_{\|x\|\to 0} \sup_{t\geq 0} \frac{\|f_1(x,t)\|}{\|x\|} = 0.$$

Then Linearization: $\dot{z} = A(t)z$

If A(t)x is stable, then f(x,t) is locally stable

Lyapunov Equation

A way to explicitly find Lyapunov functions for linear systems

- We assume V is of the form $V = x^T P x, \ P \succ 0$
- Differentiating yields: $\dot{V} = x^T (A^T P + PA) x$
- Note for a symmetric P: $\dot{V} = 2x^T PAx$
- To ensure stability, we require $\dot{V} \prec 0$
- Thus, $A^T P + PA \prec 0$

Since this is a linear system, Hurwitz stability is easier for us to check. However:

- By setting $A^T P + PA = -Q, \quad Q \succ 0$ we can easily determine how fast we want the system to decay, and optimize some set of requirements (LQR)