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Review

A tutorial on linear and bilinear matrix inequalities

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Abstract

This is a tutorial on the mathematical theory and process control applications of linear matrix inequalities (LMIs) and bilinear matrix inequalities (BMIs). Many convex inequalities common in process control applications are shown to be LMIs. Proofs are included to familiarize the reader with the mathematics of LMIs and BMIs. LMIs and BMIs are applied to several important process control applications including control structure selection, robust controller analysis and design, and optimal design of experiments. Software for solving LMI and BMI problems is reviewed.

1. Introduction

A linear matrix inequality (LMI) is a convex constraint. Consequently, optimization problems with convex objective functions and LMI constraints are solvable relatively efficiently with off-the-shelf software. The form of an LMI is very general. Linear inequalities, convex quadratic inequalities, matrix norm inequalities, and various constraints from control theory such as Lyapunov and Riccati inequalities can all be written as LMIs. Further, multiple LMIs can always be written as a single LMI of larger dimension. Thus, LMIs are a useful tool for solving a wide variety of optimization and control problems. Most control problems of interest that cannot be written in terms of an LMI can be written in terms of a more general form known as a bilinear matrix inequality (BMI). Computations over BMI constraints are fundamentally more difficult than those over LMI constraints, and there does not exist off-the-shelf algorithms for solving BMI problems. However, algorithms are being developed for BMI problems, the best of which can be applied to process control problems of modest complexity.

The many ‘‘nice’’ theoretical properties of LMIs and BMIs have made them the emerging paradigm for formulating optimization and control problems. While LMI/BMIs are gaining wide acceptance in academia, they have had little impact in process control practice. One of the main reasons for this is that process control engineers are generally unfamiliar with the mathematics of LMI/BMIs, and there is no introductory text available to aid the control engineer in learning these mathematics. As of the writing of this paper, the only text that covers LMIs in any depth is the research monograph of Boyd and co-workers [22]. Although this monograph is a useful roadmap for locating LMI results scattered throughout the electrical engineering literature, it is not a textbook for teaching the concepts of LMIs to process control engineers. Furthermore, no existing text covers BMIs in any detail.

This tutorial is an extension of a document used to train process control engineers at the University of Illinois on the mathematical theory and applications of LMIs and BMIs. Besides training graduate students, the tutorial is also intended for industrial process control engineers who wish to understand the literature or use LMI software, and experts from other fields (for example, process optimization) who wish to initiate investigations into LMI/BMIs. The only assumed background is basic calculus, a course in state space control theory [74,37], and a solid foundation in matrix theory [16,66].

The tutorial includes the proofs of several main results on LMIs. These are included for several reasons. First, many of the proofs are difficult to locate in the literature in the form that is most useful for applications to modern control problems. Second, the simplicity of the proofs provides some insights into the underlying geometry that manifests itself in terms of properties of the LMIs. Third, working through these proofs is the only way to become sufficiently experienced in the algebraic

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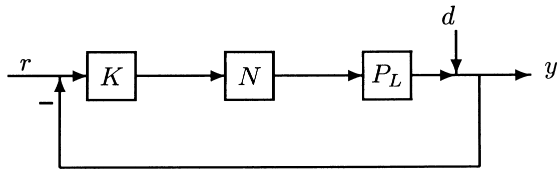

Fig. 1. Reactive ion etcher in classical feedback form.

manipulations necessary to be able to formulate the LMI/BMIs to solve new process control problems. Finally, students learn far more by working through problems or proofs than from reading theorem after theorem.

This paper is organized as follows. First an example is used to motivate studies in LMI/BMIs. The second section defines the LMI and discusses some of its basic properties. The third section shows how inequalities of many different types can be written as LMIs or BMIs. The fourth section discusses optimization problems over LMI or BMI constraints, and why such optimization problems can be efficiently solved numerically. The fifth section reviews algorithms and software packages used to solve LMI/BMI optimization problems, and the sixth section lists LMI/BMI problems that are important in process control applications. This is followed by concluding remarks.

2. Motivating example: a reactive ion etcher

A large number of control problems can be written in terms of LMIs or BMIs that cannot be solved using Lyapunov equations, Riccati equations, spectral factorization, or other classical techniques. The following is an industrial process control problem in which the only tractable solution is via an optimization over LMI and BMI constaints.

Etching is known to be a highly nonlinear multivariable process that is strongly dependent on reactor geometry. Attempts to control etch characteristics usually manipulate the reactor pressure, gas flow rate, and the power applied to the electrodes. However, due to many disturbances, complicated reaction dynamics, and the general lack of detailed fundamental understanding of the plasma behavior, it is impossible to predict etch performance for a system given a set of inputs. In many cases, it is impossible to even predict etch performance for the same system on two different runs. For this reason, it is impossible to maintain consistent etch quality without the use of feedback control. The feedback controller must be designed to be robust to the variability in process behavior as well as the nonlinear nature of the reactive ion etching process.

Here, we consider the laboratory reactive ion etcher studied by Vincent et al. [146]. The manipulated variables were the power of the applied rf voltage and the throttle valve position which specifies the input gas flowrate, and the controlled variables were the fluorine concentration and the bias voltage. Like many other chemical processes described in the literature, the plasma dynamics of a reactive ion etching process were reasonably well described as a static input nonlinearity N followed by a linear time-invariant (LTI) plant P_L (see Fig. 1), which is the well known Hammerstein model structure [51,106,134]. This nonlinear model was identified using an iterative least squares algorithm with data obtained from an experimental system by exciting it with a pseudo-random binary signal with varying amplitude [146]. The identified LTI plant for their experimental process was

$$P_L \begin{bmatrix} \dfrac{-1.89e^{-.5s}(s-38.2)}{(s+5.37)(s+0.160)} & \dfrac{-35.9(s-37.8)}{s^2+6.5s+20.2} \\[2ex] \dfrac{0.0239e^{-.5s}(s-9.6)}{(s+1.05)(s+0.214)} & \dfrac{-0.143(s-38.9)}{s^2+3.28s+4.14} \end{bmatrix} \tag{1}$$

The natural controller structure to use has the form $K=\hat{N}^{-1}K_L$ where K_L is designed to stabilize the linear portion of the plant P_L and $\hat{N}^{-1}$ is an approximate inverse of the static nonlinearity N. If the input nonlinearity N were identified perfectly then $\hat{N}^{-1}$ would be an exact inverse of N and there would be an identity mapping from K_L to P_L. However, in practice the identification is not perfect, and there is a nonlinear mapping from K_L to P_L. Furthermore, it is highly unlikely that the system is nonlinear only at the process input. Output nonlinearity is also a probability.

Nonlinearities in both the input and the output can be rigorously accounted for by the uncertainty description shown in Fig. 2. The operators Δ_I and Δ_O can vary within set bounds as functions of time, and can achieve an identical input-output mapping for any possible

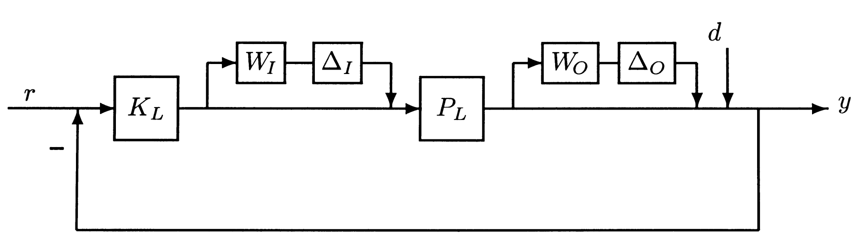

Fig. 2. Reactive ion etcher with input and output nonlinearities modeled as uncertainty.

nonlinearity within the magnitude of the bounds set by the uncertainty weights W_I and W_O.

The only known method for designing a globally optimal robust nonlinear controller for this process is by formulating the controller design as an optimization over LMI and BMI contraints (see Section 7.5 for details). The formulation allows a direct optimization of the worst-case closed loop performance over the set of plants described by the nonlinear uncertainty description. The BMI-based controller responded more than twice as fast to set point changes than a carefully tuned classical controller (linear quadratic control, whose computation was via a Ricatti equation), while at the same time providing guaranteed robustness [143].

3. The linear matrix inequality

Here we define the LMI and some of its basic properties. We will use upper case Roman to refer to matrices, lower case Roman to refer to vectors or scalars, lower case Greek to refer to scalars, and upper case calligraphic to refer to sets. The symbol $\forall$ should be read "for all" and the symbol $\in$ should be read "is an element of". The notation $\mathcal{R}^m$ denotes the set of real vectors of length m, and $\mathcal{R}^{n\times n}$ denotes the set of real $n \times n$ matrices.

3.1. Definition

A linear matrix inequality (LMI) has the form:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{2}$$

where $x \in \mathcal{R}^m, F_i \in \mathcal{R}^{n\times n}$. The inequality means that $F(x)$ is a positive definite matrix, that is,

$$z^T F(x) z > 0, \forall z \neq 0, z \in \mathcal{R}^n. \tag{3}$$

The symmetric matrices $F_i, i = 0, 1, \ldots, m$ are fixed and x is the variable. Thus, $F(x)$ is an affine function of the elements of x.

Eq. (2) is a *strict* LMI. Requiring only that $F(x)$ be positive semidefinite is referred to as a *nonstrict* LMI. The strict LMI is feasible if the set $\{x|F(x) > 0\}$ is nonempty (a similar definition applies to nonstrict LMIs). Any feasible nonstrict LMI can be reduced to an equivalent strict LMI that is feasible by eliminating implicit equality constraints and then reducing the resulting LMI by removing any constant nullspace ([22], page 19). We will therefore focus our attention on strict LMIs.

3.2. LMI equivalence to polynomial inequalities

It is informative to represent the LMI in terms of scalar inequalities. More specifically, the LMI (2) is equivalent to n polynomial inequalities. To see this, consider the well-known result in matrix theory (e.g. page 951 of [154]) that an n by n real symmetric matrix A is positive definite if and only if all of its principal minors are positive. Let A_{ij} be the ijth element of A. Recall that the principal minors of A are

$$A_{11}, \det\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right), \det\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\right),$$

$$\ldots, \det\left(\begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \ldots & A_{nn} \end{bmatrix}\right) \tag{4}$$

We apply this result to give that the LMI (2) is equivalent to:

$$F_{0,11} + \sum_{i=1}^{m} x_i F_{i,11} > 0 \quad \text{(a linear inequality)}$$

$$\left(F_{0,11} + \sum_{i=1}^{m} x_i F_{i,11}\right)\left(F_{0,22} + \sum_{i=1}^{m} x_i F_{i,22}\right) - \left(F_{0,12} + \sum_{i=1}^{m} x_i F_{i,12}\right)\left(F_{0,21} + \sum_{i=1}^{m} x_i F_{i,21}\right) > 0$$

(a quadratic inequality)

$$\vdots$$

$$\det\left(\begin{bmatrix} F(x)_{11} & \ldots & F(x)_{1k} \\ \vdots & & \vdots \\ F(x)_{k1} & \ldots & F(x)_{kk} \end{bmatrix}\right) > 0$$

(kth order polynomial inequality)

$$\vdots$$

$$\det(F(x)) > 0 \quad (n\text{th order polynomial inequality})$$

The n polynomial inequalities in x range from order 1 to order n.

3.3. Convexity

A set C is said to be *convex* if $\lambda x + (1 - \lambda)y \in C$ for all $x, y \in C$ and $\lambda \in (0, 1)$ [107]. An important property of LMIs is that the set $\{x|F(x) > 0\}$ is convex, that is, the LMI (2) forms a convex constraint on x. To see this, let x and y be two vectors such that $F(x) > 0$ and $F(y) > 0$, and let $\lambda \in (0,1)$. Then

$$\begin{aligned} F(\lambda x + (1-\lambda)y) &= F_0 + \sum_{i=1}^{m}(\lambda x_i + (1-\lambda)y_i)F_i \\ &= \lambda F_0 + (1-\lambda)F_0 + \lambda\sum_{i=1}^{m} x_i F_i \\ &\quad + (1-\lambda)\sum_{i=1}^{m} y_i F_i \\ &= \lambda F(x) + (1-\lambda)F(y) \\ &> 0. \end{aligned} \tag{5}$$

3.4. *LMIs are not unique*

The same set of variables x can be represented as the feasible set of different LMIs. For instance, if $A(x)$ is positive definite then $A(x)$ subject to a congruence transformation (see section 14.7 of [154]) is also positive definite:

$$A > 0 \iff x^T A x > 0, \forall x \neq 0 \tag{6}$$

$$\iff z^T M^T A M z > 0, \forall z \neq 0, M \text{ nonsingular} \tag{7}$$

$$\iff M^T A M > 0 \tag{8}$$

This implies, for example, that some rearrangements of matrix elements do not change the feasible set of the LMI.

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} > 0 &\iff \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} > 0 \\ &\iff \begin{bmatrix} D & C \\ B & A \end{bmatrix} > 0 \end{aligned} \tag{9}$$

3.5. *Multiple LMIs can be expressed as a single LMI*

One of the advantages of representing process control problems with LMIs is the ability to consider multiple control requirements by appending additional LMIs. Consider a set defined by q LMIs:

$$F^1(x) > 0; F^2(x) > 0; \ldots; F^q(x) > 0 \tag{10}$$

Then an equivalent single LMI is given by

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i = \operatorname{diag}\{F^1(x), F^2(x), \ldots, F^q(x)\} > 0, \tag{11}$$

where

$$F_i = \operatorname{diag}\{F_i^1, F_i^2, \ldots, F_i^q\}, \forall i = 0, \ldots, m \tag{12}$$

and $\operatorname{diag}\{X_1, X_2, \ldots, X_q\}$ is a block diagonal matrix with blocks $X_1, X_2, \ldots, X_q$. This result can be proved from the fact that the eigenvalues of a block diagonal matrix are equal to the union of the eigenvalues of the blocks, or from the definition of positive definiteness.

4. The generality of LMIs and BMIs

This section shows how many common inequalities can be written as LMIs. In addition, it shows how many control properties of interest can be written exactly in terms of the feasibility of an LMI. Such a problem is referred to as an *LMI feasibility problem*.

4.1. *Linear constraints can be expressed as an LMI*

Linear constraints are ubiquitous in process control applications. Model Predictive Control has become the most popular multivariable controller design method in many industries precisely because of its ability to address linear constraints on process variables [32,48,61,95,110, 114]. The standard linear programming and quadratic programming model predictive control formulations can be written in terms of LMIs. Here we show the first step, which is to write the linear constraints on process variables as LMI constraints.

Consider the general linear constraint $Ax < b$ written as n scalar inequalities:

$$b_i - \sum_{j=1}^{m} A_{ij} x_j > 0, \quad i = 1, \ldots, n \tag{13}$$

where $b \in \mathcal{R}^n$, $A \in \mathcal{R}^{n\times m}$, and $x \in \mathcal{R}^m$. Each of the n scalar inequalities is an LMI. Since multiple LMIs can be written as a single LMI, the linear inequalities (13) can be expressed as a single LMI.

4.2. *Stability of linear systems*

Stability is one of the most basic needs for any closed loop system. Some methods for analyzing the stability of linear systems are covered in undergraduate process control textbooks [102,133]. Moreover, some nonlinear processes can be analyzed (at least to some degree) with linear techniques by performing a change of variables, such as in binary distillation [91] and pH neutralization [68,101].

The Lyapunov method for analyzing stability is described in most texts on process dynamics [70,108]. The basic idea is to search for a positive definite function of the state (called the Lyapunov function) whose time derivative is negative definite. A necessary and sufficient condition for the linear system

$$\dot{x} = Ax \tag{14}$$

to be stable is the existence of a Lyapunov function $V(x) = x^T P x$ where P is a symmetric positive definite matrix such that the time derivative of V is negative for all $x \neq 0$ [108]:

$$\begin{aligned}\frac{\mathrm{d}V(x)}{\mathrm{d}t} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T\left(A^T P + PA\right)x < 0, \forall x \neq 0\end{aligned} \tag{15}$$

$$\iff A^T P + PA < 0 \tag{16}$$

This is an LMI, where P is the variable. To see this, select a basis for symmetric $n \times n$ matrices.

As an example basis, for $i \geqslant j$ define E^{ij} as the matrix with its (i, j) and (j, i) elements equal to one, and all of its other elements equal to zero. There are $m = n(n+1)/2$ linearly independent matrices E^{ij} and any symmetric matrix P can be written uniquely as

$$P = \sum_{j=1}^{n} \sum_{i \geqslant j}^{n} P_{ij} E^{ij}, \tag{17}$$

where P_{ij} is the (i, j) element of P. Thus the matrices E^{ij} form a basis for symmetric $n \times n$ matrices (in fact, if the columns of each E^{ij} are stacked up as vectors, then the resulting vectors form an orthogonal basis, which could be made orthonormal by scaling).

Substituting for P in terms of its basis matrices gives the alternative form for the Lyapunov inequality

$$\begin{aligned}A^T P + PA &= A^T\left(\sum_{j=1}^{n}\sum_{i \geqslant j}^{n} P_{ij}E^{ij}\right) + \left(\sum_{j=1}^{n}\sum_{i \geqslant j}^{n} P_{ij}E^{ij}\right)A \\ &= \sum_{j=1}^{n}\sum_{i \geqslant j}^{n} P_{ij}\left(A^T E^{ij} + E^{ij}A\right) < 0\end{aligned} \tag{18}$$

which is in the form of an LMI (2), with $F_0 = \underline{\underline{0}}$ and $F_k = -A^T E^{ij} - E^{ij}A$, for $k = 1, \ldots, m$. The elements of the vector x in (2) are the P_{ij}, $i \geqslant j$, stacked up on top of each other.

4.3. Stability of nonlinear and time varying systems

Many of the processes commonly encountered in process control applications can be adequately modeled as being linear time invariant (LTI). However, many chemical processes cannot be adequately analyzed using LTI techniques, including reactive ion etching [140], packed bed reactors [46], and most batch processes [9].

In Section 4.2, we showed how testing the stability of a linear system could be posed as an LMI feasibility problem. Now let us consider a generalization of that problem to testing the stability of a set of linear time varying systems that are described by a convex hull of matrices (a matrix polytope):

$$\dot{x} = A(t)x, \quad A(t) \in \mathrm{Co}\{A_1, \ldots, A_L\} \tag{19}$$

An alternative way of writing this is [105]:

$$\dot{x} = A(t)x, \quad A(t) = \sum_{i=1}^{L}\lambda_i A_i, \quad \forall \lambda_i \geqslant 0, \sum_{i=1}^{L}\lambda_i = 1. \tag{20}$$

A necessary and sufficient condition for the existence of a quadratic Lyapunov function $V(x) = x^T P x$ that proves the stability of (20) is the existence of $P = P^T > 0$ that satisfies:

$$\begin{aligned}&\frac{\mathrm{d}V(x)}{\mathrm{d}t} = \dot{x}^T P x + x^T P \dot{x} < 0, \quad \forall x \neq 0, \\ &\forall A(t) \in \mathrm{Co}\{A_1, \ldots, A_L\}\end{aligned} \tag{21}$$

$$\begin{aligned}&\iff x^T\left[A(t)^T P + PA(t)\right]x < 0, \quad \forall x \neq 0, \\ &\forall A(t) \in \mathrm{Co}\{A_1, \ldots, A_L\}\end{aligned} \tag{22}$$

$$\iff A(t)^T P + PA(t) < 0, \quad \forall A(t) \in \mathrm{Co}\{A_1, \ldots, A_L\} \tag{23}$$

$$\iff \left(\sum_{i=1}^{L}\lambda_i A_i\right)^T P + P\left(\sum_{i=1}^{L}\lambda_i A_i\right) < 0, \forall \lambda_i \geqslant 0, \sum_{i=1}^{L}\lambda_i = 1 \tag{24}$$

$$\iff \sum_{i=1}^{L}\lambda_i\left(A_i^T P + PA_i\right) < 0, \forall \lambda_i \geqslant 0, \sum_{i=1}^{L}\lambda_i = 1 \tag{25}$$

$$\iff A_i^T P + PA_i < 0, \forall i = 1, \ldots, L \tag{26}$$

The search for P that satisfies these inequalities is an LMI feasibility problem. This condition is also a sufficient condition for the stability of nonlinear time varying systems where the Jacobian of the nonlinear system is contained within the convex hull in (20) [84]. There are several difficulties in applying the LMI condition for analyzing stability of nonlinear systems. First, it is very difficult to construct a convex hull for which the Jacobian of a nonlinear system is provably contained within.

Second, such a description will usually be highly conservative, since the convex hull overbounds the Jacobian of the real nonlinear system. Third, each new vertex adds another matrix inequality to the LMI feasibility problem (26). For a system with a large number of states (which is equal to the dimension of A) and vertices (L), solving the LMI feasibility problem (26) can become computationally prohibitive. The strength of the approach is that LMIs for controller synthesis for systems of the form (20) are relatively easy to construct [22,77,131].

4.4. The Schur complement lemma

The Schur complement lemma converts a class of convex nonlinear inequalities that appears regularly in control problems to an LMI. The convex nonlinear inequalities are

$$R(x) > 0, \quad Q(x) - S(x)R(x)^{-1}S(x)^T > 0, \tag{27}$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ depend affinely on x. The Schur complement lemma converts this set of convex nonlinear inequalities into the equivalent LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0. \tag{28}$$

A proof of the Schur complement lemma using only elementary calculus is given in the Appendix. In what follows, the Schur complement lemma is applied to several inequalities that appear in process control.

4.5. Maximum singular value

The maximum singular value measures the maximum gain of a multivariable system, where the magnitude of the input and output vector is quantified by the Euclidean norm [130]. It is also very useful for quantifying frequency-domain performance and robustness for multivariable systems [96,130]. Process applications are provided in many popular undergraduate process control textbooks [102,126].

The maximum singular value of a matrix A which affinely depends on x is denoted by $\overline{\sigma}(A(x))$, which is the square root of the largest eigenvalue of $A(x)A(x)^T$. The inequality $\overline{\sigma}(A(x)) < 1$ is a nonlinear convex constraint on x that may be written as an LMI using the Schur complement lemma:

$$\overline{\sigma}(A(x)) < 1 \iff A(x)A(x)^T < I \tag{29}$$

$$\iff I - A(x)I^{-1}A(x)^T > 0 \tag{30}$$

$$\iff \begin{bmatrix} I & A(x) \\ A(x)^T & I \end{bmatrix} > 0 \tag{31}$$

Here $A(x)$ corresponds to $S(x)$ in the LMI (28), and $Q(x)$ and $R(x)$ correspond to I.

4.6. Ellipsoidal inequality

Ellipsoid constraints are important in process identification, parameter estimation, and statistics [15,27, 41,85]; as well as certain fast model predictive control algorithms [138,139]. Applications recently described in the literature include crystallization processes [88,93], polymer film extruders [54], and paper machines [138, 139].

An ellipsoid described by

$$(x - x_c)^T P^{-1}(x - x_c) < 1, \quad P = P^T > 0 \tag{32}$$

can be expressed as an LMI using the Schur complement lemma with $Q(x) = 1$, $R(x) = P$, and $S(x) = (x - x_c)^T$:

$$\begin{bmatrix} 1 & (x - x_c)^T \\ (x - x_c) & P \end{bmatrix} > 0. \tag{33}$$

4.7. Algebraic Riccati inequality

Algebraic Riccati equations are used extensively in optimal control, as described in textbooks on advanced process control [111,130], which describe applications to chemical reactors, distillation columns, and other processes. A result involving a Riccati equation can be replaced with an equivalent result where the equality is replaced by an inequality [151]. More specifically, these optimal controllers can be constructed by computing a positive definite symmetric matrix P that satisfies the algebraic Riccati inequality:

$$A^T P + PA + PBR^{-1}B^T P + Q < 0 \tag{34}$$

where A and B are fixed, Q is a fixed symmetric matrix, and R is a fixed symmetric positive definite matrix.

The Riccati inequality is quadratic in P but can be expressed as a linear matrix inequality by applying the Schur complement lemma:

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} > 0. \tag{35}$$

The next two sections provide examples of algebraic Riccati inequalities for analyzing the properties of linear or nonlinear systems.

4.8. Bounded real lemma

The Bounded real lemma forms the basis for LMI approaches to robust process control which have been applied to reactive ion etching [140,143], polymer extruders [141], and paper machines [141], and gain scheduling which has been applied to chemical reactors [12,13]. Although the Bounded real lemma has application to the control of both linear and nonlinear processes, the actual result is based on the state space system representation of a linear system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x(0) = 0, \tag{36}$$

where $A \in \mathcal{R}^{n\times n}, B \in \mathcal{R}^{n\times p}, C \in \mathcal{R}^{p\times n}$, and $D \in \mathcal{R}^{p\times p}$ are given data. Assume that A is stable and that (A, B, C) is minimal [74]. The transfer function matrix is

$$G(s) = C(sI - A)^{-1}B + D. \tag{37}$$

The worst-case performance of a system measured in terms of the integral squared errors of the input and output is quantified by the H_∞ norm [157]:

$$\| G(s) \|_\infty = \sup_{\mathrm{Re}(s)>0} \bar{\sigma}(G(s)) = \sup_{\omega\in\mathcal{R}} \bar{\sigma}(G(j\omega)). \tag{38}$$

The H_∞ norm can be written in terms of an LMI. To see this, we will use a result from the literature [158] that the H_∞ norm of $G(s)$ is less than γ if and only if $\gamma^2 I - D^T D > 0$ and there exists $P = P^T > 0$ such that

$$\left(A^T P + PA + C^T C\right) + \left(PB + C^T D\right) \times \left(\gamma^2 I - D^T D\right)^{-1}\left(B^T P + D^T C\right) < 0 \tag{39}$$

The Schur complement lemma implies that this Riccati inequality is equivalent to the existence of $P = P^T > 0$ such that the following LMI holds:

$$\begin{bmatrix} -\left[A^T P + PA + C^T C\right] & -\left[PB + C^T D\right] \\ -\left[B^T P + D^T C\right] & \gamma^2 I - D^T D \end{bmatrix} > 0 \tag{40}$$

which is equivalent to

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} < 0. \tag{41}$$

It is common to incorporate weights on the input u and output y so that the condition of interest is whether the H_∞ norm of $W_1(s)G(s)W_2(s)$ is less than 1. A system with an H_∞ norm less than one is said to be *strictly bounded real*. This condition is checked by testing the feasibility of the LMI using the state-space matrices for the product $W_1(s)G(s)W_2(s)$.

4.9. Positive real lemma

Robustness analysis has been widely applied in the process control literature. Examples include distillation columns [129], packed bed reactors [46] and a reactive ion etching [143]. A property that is regularly exploited in the development of robustness analysis tools [14,72] for linear systems subject to linear or nonlinear perturbations is *passivity*. The linear system (36) is said to be *passive* if

$$\int_0^\tau u(t)^T y(t)\mathrm{d}t \geqslant 0 \tag{42}$$

for all u and $\tau \geqslant 0$. This property is equivalent to the existence of $P = P^T > 0$ such that [22]

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D^T - D \end{bmatrix} \leqslant 0. \tag{43}$$

It is instructive to show the connection between the bounded real lemma and the positive real lemma [5], especially since it is often referred to in the robust control literature. A standard result from network theory [10,45,125] is that passivity is equivalent to $G(s)$ in (37) being positive real, that is,

$$G(s)^* + G(s) \geqslant 0 \quad \forall \mathrm{Re}\{s\} > 0 \tag{44}$$

where $G(s)^*$ is the complex conjugate transpose of $G(s)$.

The relationship between bounded real and positive real is that $[I - G(s)][I + G(s)]^{-1}$ is strictly positive real if and only if $G(s)$ is strictly bounded real. This follows from [87]

$$\bar{\sigma}(A) < 1 \iff A^* A < I \tag{45}$$

$$\iff (I + A^*)^{-1}(2I - 2A^* A)(I + A)^{-1} > 0 \tag{46}$$

$$\iff (I + A^*)^{-1}[(I - A^*)(I + A) + (I + A^*)(I - A)] \times (I + A)^{-1} > 0 \tag{47}$$

$$\iff (I + A^*)^{-1}(I - A^*) + (I - A)(I + A)^{-1} > 0 \tag{48}$$

$$\iff \left[(I - A)(I + A)^{-1}\right]^* + (I - A)(I + A)^{-1} > 0 \tag{49}$$

4.10. The S procedure

The S procedure greatly extends the usefulness of LMIs by allowing non-LMI conditions that commonly arise in nonlinear systems analysis to be represented as LMIs (although with some conservatism). This technique has been applied to the analysis of pH neutralization processes [119] and crystallization processes [116].

First we will describe the S procedure as it applies to quadratic functions, and then discuss its application to quadratic forms. Let $\alpha_0, \ldots, \alpha_p$ be quadratic scalar functions of $x \in \mathcal{R}^n$:

$$\alpha_i(x) = x^T T_i x + 2u_i^T x + \beta_i, \quad i = 0, \ldots, p; \quad T_i = T_i^T \tag{50}$$

The existence of $\tau_1 \geqslant 0, \ldots, \tau_p \geqslant 0$ such that

$$\alpha_0(x) - \sum_{i=1}^{p} \tau_i \alpha_i(x) \geqslant 0, \forall x, \tag{51}$$

implies that

$$\alpha_0(x) \geqslant 0, \quad \forall x \text{ such that } \alpha_i(x) \geqslant 0, i = 1, \ldots, p. \tag{52}$$

To see why this is true, assume there exists $\tau_1 \geqslant 0, \ldots, \tau_p \geqslant 0$ such that (51) holds for all $\alpha_i(x) \geqslant 0, i = 1, \ldots, p$. Then

$$\alpha_0(x) \geqslant \sum_{i=1}^{p} \tau_i \alpha_i(x) \geqslant 0, \forall x. \tag{53}$$

Note that (51) is equivalent to

$$\begin{bmatrix} T_0 & u_0 \\ u_0^T & \beta_0 \end{bmatrix} - \sum_{i=1}^{p} \tau_i \begin{bmatrix} T_i & u_i \\ u_i^T & \beta_i \end{bmatrix} \geqslant 0 \tag{54}$$

since

$$x^T T x + 2u^T x + \beta \geqslant 0, \forall x \tag{55}$$

$$\iff \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} T & u \\ u^T & \beta \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \geqslant 0, \forall x \tag{56}$$

$$\iff \begin{bmatrix} \xi x \\ \xi \end{bmatrix}^T \begin{bmatrix} T & u \\ u^T & \beta \end{bmatrix} \begin{bmatrix} \xi x \\ \xi \end{bmatrix} \geqslant 0, \forall x, \xi \tag{57}$$

$$\iff \begin{bmatrix} T & u \\ u^T & \beta \end{bmatrix} \geqslant 0. \tag{58}$$

Hence the above S procedure can be equivalently written in terms of quadratic forms. Instead of writing the above version which is completely in terms of non-strict inequalities, we will provide here a version that applies to the case where the main inequality is strict (the proof is similar). Let $T_0, \ldots, T_p$ be symmetric matrices. If there exists $\tau_1 \geqslant 0, \ldots, \tau_p \geqslant 0$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i > 0, \tag{59}$$

then

$$x^T T_0 x > 0 \forall x \neq 0 \text{ such that } x^T T_i x \geqslant 0, i = 1, \ldots, p. \tag{60}$$

4.11. *Stability of linear systems with nonlinear perturbations*

The derivation of LMI feasibility problems to analyze the stability or performance of linear systems subject to linear/nonlinear time invariant/varying perturbations is rather straightforward conceptually [22], although the algebra can be messy for more complex systems [115,118]. For continuous time systems, the basic approach is to postulate a positive definite Lyapunov function of the state and some undetermined matrices, and then apply the S procedure (if necessary) to derive LMI conditions on the undetermined matrices which imply that the time derivative of the Lyapunov function is negative definite. For discrete time systems, the divided difference of the Lyapunov function is used instead of the time derivative. Here we show how this approach is applied to a system of especial relevance to process control applications.

Consider a discrete time system subject to slope-restricted static nonlinearities:

$$\begin{aligned} x(k+1) &= Ax(k) + B\phi(q(k)) \\ q(k) &= Cx(k) \end{aligned} \tag{61}$$

with the nonlinearities described by

$$\phi_i(q_i(k))[\phi_i(q_i(k)) - q_i(k)] \leqslant 0, \text{for } i = 1, \ldots, m \tag{62}$$

with the local slope restrictions

$$0 < \frac{\phi_i(q_i(k+1)) - \phi_i(q_i(k))}{q_i(k+1) - q_i(k)} < T_{ii}, \text{ for } i = 1, \ldots, m \tag{63}$$

where T_{ii} is the maximum slope of the i^{th} nonlinearity. This can be used to represent a linear process with actuator limitation nonlinearities which is controlled by an antiwindup compensator [36,78,30], or a closed loop system with each component being either a linear system or a dynamic artificial neural network [117,120]. Both of these types of closed loop systems have been extensively studied in the process control literature (see the above references and citations therein).

The Lur'e-Lyapunov function is defined by

$$V(x(k)) = x^T(k) P x(k) + 2 \sum_{i=1}^{m} \int_0^{q_i(k)} \phi_i(\sigma) Q_{ii} \mathrm{d}\sigma \tag{64}$$

where P is positive definite and the Q_{ii} are nonnegative so that the Lyapunov function is positive definite. The first term is the standard quadratic Lyapunov function which is discussed in many state space systems textbooks [100] and in textbooks on process analysis [70,108], which describe applications to polymerization and other chemical reactors. The second term was introduced by Lur'e [86] to include the nonlinearities (62) explicitly in the Lyapunov function.

The method of Lyapunov for discrete time systems is to write the divided difference for the Lyapunov function:

$$V(x(k+1)) - V(x(k)) \tag{65}$$

with the state vector substituted in using (61). The overall system is globally asymptotically stable if the undetermined matrices P and Q_{ii} can be computed so that $V(x(k+1)) - V(x(k))$ is less than zero. The nonlinearities are bounded in the divided difference using (63) and the mean value theorem, and the S procedure is used to convert the divided difference subject to the inequalities (62) to an LMI. With some algebra to collect the terms [115,118], it is found that a sufficient condition for the global asymptotic stability of (61)–(63) is the existence of a positive-definite matrix P and diagonal positive semidefinite matrices Q and $R \in \mathcal{R}^{h\times h}$ such that

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} > 0 \tag{66}$$

where

$$M_{1,1} = -A^TPA + P - (A-I)^TC^TTQC(A-I) \tag{67}$$

$$M_{1,2} = -A^TPB - (A-I)^TC^TTQCB - (A-I)^TC^TQ - C^TR \tag{68}$$

$$M_{2,1} = -B^TPA - B^TC^TTQC(A-I) - QC(A-I) - RC \tag{69}$$

$$M_{2,2} = -B^TPB - B^TC^TTQCB - QCB - B^TC^TQ + 2R \tag{70}$$

and $T = \mathrm{diag}\{T_{ii}\}$. The new matrix R is introduced by the S procedure. This is an LMI feasibility problem that has been applied to the analysis of pH neutralization processes [119] and crystallization processes [116] under nonlinear feedback control.

4.12. Variable reduction lemma

The variable reduction lemma allows the solution of algebraic Riccati inequalities that involve a matrix of unknown dimension. This often arises when finding the controller that minimizes the H_∞ norm (see Section 7.5 for an example).

Given a symmetric matrix $A \in \mathcal{R}^{n\times n}$ and two matrices P and Q of column dimension n, consider the problem of finding some matrix Θ of compatible dimensions such that

$$A + P^T\Theta^TQ + Q^T\Theta P < 0 \tag{71}$$

This equation is solvable for some Θ if and only if the following two conditions hold:

$$W_P^TAW_P < 0 \tag{72}$$

$$W_Q^TAW_Q < 0 \tag{73}$$

where W_P and W_Q are matrices whose columns are bases for the null spaces of P and Q, respectively. A proof of this result is given in [58].

4.13. Bilinear matrix inequality

Bilinear inequalities arise in pooling and blending problems [147], systems analysis [140], and nonlinear programming. A bilinear matrix inequality (BMI) is of the form:

$$F(x,y) = F_0 + \sum_{i=1}^{m} x_iF_i + \sum_{j=1}^{n} y_jG_j + \sum_{i=1}^{m}\sum_{j=1}^{n} x_iy_jH_{ij} > 0 \tag{74}$$

where G_j and H_{ij} are symmetric matrices of the same dimension as F_i, and $y \in \mathcal{R}^n$. Bilinear matrix inequalities were popularized by Safonov and co-workers in a series of proceedings papers [63–65,125], and first applied to a nontrivial process description (i.e., a chemical reactive ion etcher) by VanAntwerp and Braatz [140], and was later applied to paper machines [141].

A BMI is an LMI in x for fixed y and an LMI in y for fixed x, and so is convex in x and convex in y. The bilinear terms make the set not *jointly convex* in x and y. To see this, consider the simplest BMI which is the bilinear inequality

$$1 - xy > 0, \tag{75}$$

where x and y are scalar variables. One way to see that this set is nonconvex is to graph the set in the xy-plane and apply the definition of convexity. Another way to see this is to consider two elements of the set that contradict the definition of convexity. For example, consider (x,y) equal to the values $(0.1, 7.9)$ and $(7.9, 0.1)$. Both values satisfy the bilinear inequality since $1-(0.1)(7.9) = 1-(7.9)(0.1) = 0.21 > 0$. But the point on the line half way between the two values $(1/2(0.1,7.9) + 1/2(7.9,0.1) = (4,4))$ does not satisfy the bilinear inequality: $1 - 4{\cdot}4 = -15 < 0$.

Besides bilinear and general quadratic inequalities

$$x^TQx + c^Tx + p > 0, \tag{76}$$

general polynomial inequalities can also be written as BMIs. Consider, for example, the nonlinear inequality

$$x^3 + yz < 1 \tag{77}$$

By defining $x^2 = w$, and $x = v$, this inequality is equivalent to:

$$1 - xw - yz > 0 \tag{78}$$

$$x - v \geqslant 0$$

$$v - x \geqslant 0$$

$$w - vx \geqslant 0$$

$$vx - w \geqslant 0$$

Since a BMI describes sets that are not necessarily convex, they can describe much wider classes of constraint sets than LMIs, and can be used to represent more types of optimization and control problems. The main drawback of BMIs is that they are much more difficult to handle computationally than LMIs.

5. Optimization problems

Many optimization and control problems can be written in terms of finding a feasible solution to a set of LMIs or BMIs. Most problems, however, are best written in terms of optimizing a simple objective function over a set of LMIs or BMIs. There is a fundamental difference between the computational requirements for optimization problems over LMIs, and those over BMIs. This section begins with an introduction to convex optimization and computational complexity, which provides a fundamental framework for understanding the relative complexities of optimization problems. This is followed by the definition of some optimization problems that appear when formulating and solving control problems using LMIs/BMIs.

5.1. Computational complexity and convexity

Optimization problems are generally characterized as being in one of two classes: P and NP-hard [62,104]. The class P refers to problems in which the time needed to exactly solve the problem can always be bounded by a single function which is polynomial in the amount of data needed to define the problem. Such problems are said to be solvable in *polynomial time*. Although the exact consequences of a problem being NP-hard is still a fundamental open question in the theory of computational complexity, it is generally accepted that a problem being NP-hard means that its solution cannot be computed in polynomial time in the worst case. It is important to understand that being NP-hard is a property of the problem itself, not of any particular algorithm. It is also important to understand that having a problem be NP-hard does not imply that practical algorithms are not possible. Practical algorithms for NP-hard problems exist and typically involve approximation, heuristics, branch-and-bound, or local search [35,62,104]. Determining whether a problem is polynomial time or NP-hard informs the systems engineer what kind of accuracy and speed can be expected by the best algorithms, and what kinds of algorithms to investigate for providing practical solutions to the problem.

Suppose that a real valued function $f(x)$ is defined on a convex set $C \in \mathcal{R}^n$. The function $f(x)$ is *convex* on C if [107]

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) \tag{79}$$

for all $x, y \in C$ and $\lambda \in [0, 1]$. A *convex optimization problem* has the form

$$\inf_{x \in C} f(x), \tag{80}$$

where $f(x)$ is a convex function in x, C is a convex set, and inf refers to the infimum over C. If the infimum is achieved by an element in C, then the minimization problem will be written as min.

Well known problems that can be formulated as convex optimization problems include linear programming and convex quadratic programming. The advantage of formulating control problems in terms of convex optimization problems (when possible) is that wide classes of convex optimization problems are in the class P [97]. Being in P means that these problems can be provably solved efficiently on a computer. This makes convex optimization problems desirable for solving large scale systems problems. Convex optimization problems often occur in engineering practice and many can be written as LMIs. This is the strength of using LMI formulations. Convex optimizations over LMIs are solvable in polynomial time.

Other systems engineering problems cannot be written in terms of LMIs, but can be written in terms of BMIs. Nearly every control problem of interest can be written in terms of an optimization problems over BMIs. These optimization problems, however, are NP-hard [135], which implies that it is highly unlikely that there exists a polynomial-time algorithm for solving these problems. This means that algorithms for solving optimization problems over BMIs are currently limited to problems of modest size. Algorithms and their expected performance will be discussed in more detail in Section 6. For the rest of this section we review the most common LMI and BMI optimization problems that appear in control applications.

5.2. Semidefinite programming

The following optimization problem is commonly referred to as a semidefinite program (SDP) [4]:

$$\inf_{\substack{x \\ F(x)>0}} c^T x \tag{81}$$

One SDP which often arises in control applications is the LMI eigenvalue problem (EVP). It is the minimization of the maximum eigenvalue of a matrix that depends affinely on the variable x, subject to an LMI constraint on x. Many performance analysis tests, such as computing the H_∞ norm in (38), can be written in terms of an EVP [144]. Two common forms of the EVP are presented so that readers will recognize them:

$$\inf_{\substack{x,\lambda \\ \lambda I - A(x)>0 \\ B(x)>0}} \lambda \tag{82}$$

$$\inf_{\substack{x,\lambda \\ A(x,\lambda)>0}} \lambda \tag{83}$$

where $A(x, \lambda)$ is affine in x and λ.

The equivalence of (81), (82), and (83) will now be demonstrated. The LMI eigenvalue problem (82) can be written in the form (83) by defining $A(x, \lambda) = \operatorname{diag}\{\lambda I - A(x), B(x)\}$ (recall that multiple LMIs can be written as a single LMI of larger dimension). To show that a problem in the form (83) can be written in the form (81), define $\hat{x} = [x^T \lambda]^T$, $F(\hat{x}) = A(\hat{x})$, and $c^T = [0^T 1]^T$ where 0 is a vector of zeros. To see that (81) transforms to (82) consider

$$\inf_{F(x)>0} c^T x = \inf_{\substack{c^T x<\lambda \\ F(x)>0}} \lambda = \inf_{\substack{1\lambda - c^T x>0 \\ F(x)>0}} \lambda = \inf_{\substack{\lambda I - A(x)>0 \\ F(x)>0}} \lambda. \tag{84}$$

QED.

5.3. Generalized eigenvalue problems

A large number of the control properties can be computed as a generalized eigenvalue problem (GEVP), including many robustness margins and the minimized condition number discussed in Section 7. A GEVP is, given square matrices A and B, $B > 0$, to find scalars λ and nonzero vectors y such that

$$Ay = \lambda By \tag{85}$$

The computation of the largest generalized eigenvalue can be written in terms of an optimization problem with LMI-like constraints. Consider that the positive definiteness of B implies that for sufficiently large λ, $\lambda B - A > 0$. As λ is reduced from some sufficiently high value, at some point the matrix $\lambda B - A$ will lose rank, at which point there exists a nonzero vector y that solves (85), implying that this value of λ is the largest generalized eigenvalue. Hence

$$\lambda_{\max} = \min_{\lambda B - A \geqslant 0} \lambda = \inf_{\lambda B - A > 0} \lambda \tag{86}$$

Often it is desired to minimize the largest generalized eigenvalue of two symmetric matrices which depend affinely on a variable x, subject to an LMI constraint on x.

$$\inf_{\substack{B(x)>0 \\ C(x)>0}} \lambda_{\max}(A(x), B(x)). \tag{87}$$

Here $\lambda_{\max}(A(x), B(x))$ is the largest generalized eigenvalue of two matrices, A and B, each of which depend affinely on x. From (86) this optimization problem is equivalent to

$$\inf_{\substack{\lambda B(x) - A(x)>0 \\ B(x)>0 \\ C(x)>0}} \lambda. \tag{88}$$

The problem of minimizing the maximum generalized eigenvalue is a *quasiconvex* objective function subject to a convex constraint, where quasiconvexity means that

$$\lambda_{\max}(A(\theta x + (1-\theta)z), B(\theta x + (1-\theta)z)) \leqslant \max\left\{\lambda_{\max}(A(x), B(x)), \lambda_{\max}(A(z), B(z))\right\} \tag{89}$$

for all $\theta \in [0, 1]$ and all feasible x and z. To see that this is true, first define $\hat{\lambda}$ equal to the right hand side of (89). Then

$$\hat{\lambda} \geqslant \lambda_{\max}(A(x), B(x)) \text{ and } \hat{\lambda} \geqslant \lambda_{\max}(A(z), B(z)). \tag{90}$$

From (86), this implies that

$$\hat{\lambda} B(x) - A(x) \geqslant 0 \text{ and } \hat{\lambda} B(z) - A(z) \geqslant 0. \tag{91}$$

It follows that, for all $\theta \in [0, 1]$,

$$\theta\left[\hat{\lambda} B(x) - A(x)\right] + (1-\theta)\left[\hat{\lambda} B(z) - A(z)\right] \geqslant 0 \tag{92}$$

$$\iff \hat{\lambda} B(\theta x + (1-\theta)z) - A(\theta x + (1-\theta)z) \geqslant 0. \tag{93}$$

This and (86) imply that

$$\hat{\lambda} \geqslant \lambda_{\max}(A(\theta x + (1-\theta)z), B(\theta x + (1-\theta)z)). \tag{94}$$

QED.

5.4. Convex determinant optimization problem

We will refer to the following as a convex determinant optimization problem (CDOP):

$$\inf_{\substack{A(x)>0\\ B(x)>0}} \log\ \det\left(A(x)^{-1}\right) \tag{95}$$

where A and B are symmetric matrices which are affine functions of x. As we will see in Section 7, this problem appears in a variety of ellipsoidal approximation problems associated with state and parameter estimation problems, and in optimal experimental design. The proof that $\log\ \det\left(A(x)^{-1}\right) = -\log(\det(A))$ is convex, which implies that a CDOP can be solved relatively efficiently on a computer, is given in the appendix.

5.5. BMI problem

An optimization over BMI constraints is called a BMI problem:

$$\inf_{\substack{x,y\\ A\left([x^T y^T]^T\right)>0\\ F(x,y)>0}} \quad c^T x + d^T y \tag{96}$$

where $F(x, y)$ is defined in (74).

Many important problems in control that cannot be stated in terms of LMIs can be stated in terms of BMIs. Examples include robustness analysis [43,109], a large number of robust controller synthesis problems including low order and decentralized control [125,63], bilinear programming, and linear complementarity problems [2,3,42]. Additionally, a large number of process design problems can be written exactly or approximately in this form [124,147,148].

In the same way that the EVP (82) is an optimization over LMI constraints, there is a corresponding optimization over BMI constraints called the BMI eigenvalue problem (BEVP):

$$\inf_{\substack{x,y,\gamma\\ A\left([x^T y^T]^T\right)>0\\ \lambda\max(F(x,y))<\gamma}} \quad \gamma \tag{97}$$

where $\lambda_{\max}$ is the maximum eigenvalue of $F(x, y)$.

Using algebra similar as for the LMI eigenvalue problem, it can be shown that this is a special case of the BMI optimization problem.

6. Solving optimizations over LMI or BMI constraints

Here we outline the algorithms and review software used to solve optimization problems over LMIs and BMIs.

6.1. Solving LMI problems

The easiest algorithm to implement for solving LMI problems is the ellipsoid algorithm (see Fig. 3) [18]. It solves a convex objective function with convex constraints. In the first step, an ellipsoid is computed that contains the optimum point. Often this means computing an ellipsoid that covers the constraint set (see Fig. 3a). The next step is to compute a plane that passes through the center of the ellipsoid such that the solution is guaranteed to lie on one side of the plane (Fig. 3b). Boyd et al. [22] gives analytical expressions for this *cutting plane* for each of the LMI problems. The main point is that for each of the LMI problems there is a half space which is definitely "uphill," so that any points in that half space can be discarded. The remaining half ellipsoid is itself covered by an ellipsoid of minimal volume (Fig. 3c) and the process is repeated (Fig. 3d) until the algorithm converges to the optimal solution.

A more computationally efficient algorithm for solving LMI problems is the interior point method [97]. The interior point method uses the constraints to define a barrier function which is convex within the feasible region and infinite outside it. This barrier function is incorporated into an objective function, which allows the constrained optimization problem to be replaced with an unconstrained optimization problem which can be solved using Newton's method. The *analytic center* is defined to be the point which minimizes the unconstrained optimization problem. A scalar in the objective to the unconstrained optimization problem is iterated until the analytic center is optimal for the original problem.

The interior point method is, in some ways, similar to the penalty function method [107]. In both cases, the constraint set is incorporated into the objective function of an unconstrained optimization problem which can be solved using Newton's method. Also, in both cases a scalar in the objective is iterated until the solution to the unconstrained optimization problem is equal to the solution to the original problem. However, both the objective functions and the scalar that is iterated are different in the two methods. The ellipsoid algorithm, on the other hand, works more like a standard branch and bound algorithm [90], in that it is continually discarding infeasible regions from the search. For an optimization over a single scalar variable, the ellipsoid algorithm is equivalent to the bisection algorithm [44].

A modification to the LMIs that can be critical for obtaining convergence of these algorithms is to include a constraint that keeps the numerics well conditioned and the variables bounded. It is simplest to illustrate this modification with an example. Consider, for example, the search for a $P = P^T > 0$ that satisfies the LMI feasibility problem

$$A_i^T P + P A_i < 0, \forall i = 1, \ldots, L \tag{98}$$

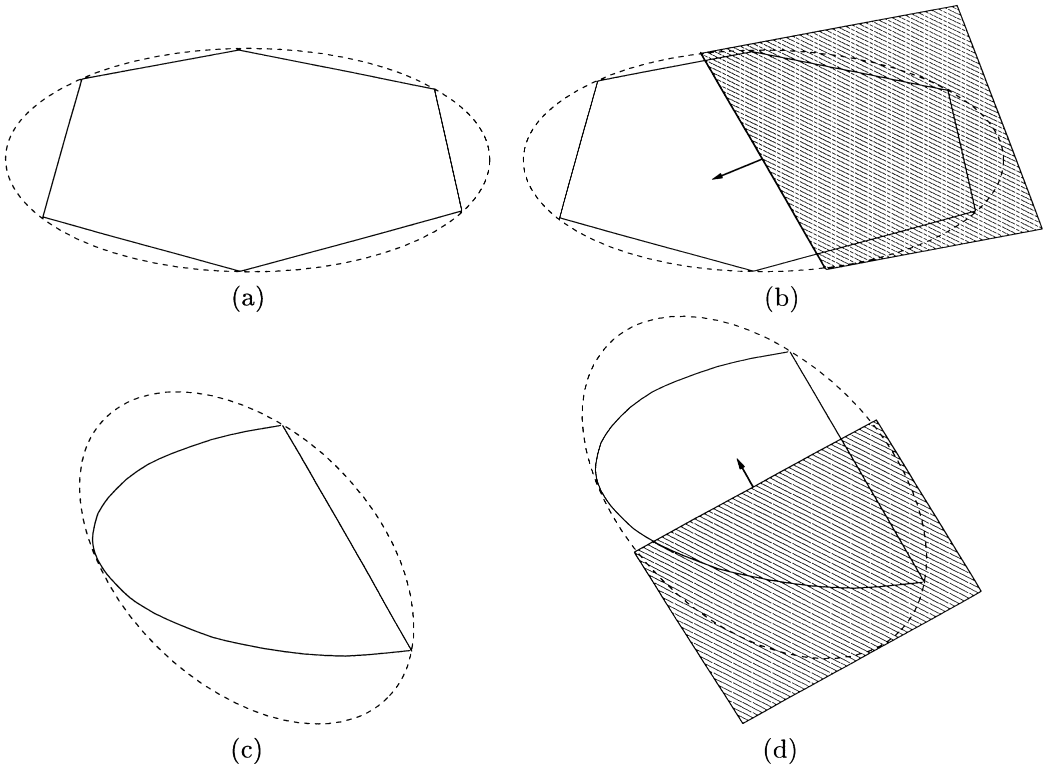

Fig. 3. The ellipsoid algorithm algorithm (the vectors shown in (b) and (d) are perpendicular to the half spaces).

This was a sufficient condition for stability of a class of systems described earlier (26). Now consider augmenting the above LMIs with

$$\mu I < P < I \tag{99}$$

This limits the condition number of P to $1/\mu$ while bounding the set of feasible matrices P. The bounding of P does not affect the feasibility of the original problem, and the condition number limit does not appreciably restrict P provided that μ is small. The advantage of the condition number limit is that it will prevent the LMI solution algorithm from converging to a P that could lead to roundoff problems [145].

6.2. *Numerical software for solving LMI problems*

Several research groups have produced publicly available software packages for solving LMI problems. Gahinet and Nemirovskii wrote a software package called LMI-Lab [59] which evolved into the Matlab's LMI Control Toolbox [60]. The LMI Control Toolbox accepts problem statements in a high level mathematical form and solves the problem with a projective interior point algorithm. Kojima, Shindoh, and Hara wrote SDPA (Semi-Definite Programming Algorithm) [57], which is based on a Mehrotra type predictor-corrector infeasible primal-dual interior-point method. It does not allow the user to state LMI problems in a high level language.

Vandenberghe and Boyd produced the code SP [23] which is an implementation of Nesterov and Todd's primal-dual potential reduction method for semidefinite programming (this is an interior point algorithm). SP can be called from within Matlab [94]. Boyd and Wu extended the usefulness of the SP program by writing SDPSOL [25,26], which is a parser/solver that calls SP. The advantages of SDPSOL are that the problem can be specified in a high level language, and SDPSOL can run without Matlab. SDPSOL can, in addition to linear objective functions, handle trace and convex determinant objective functions.

LMITOOL is another software package for solving LMI problems that uses the SP solver for its computations [50]. LMITOOL interfaces with Matlab, and there is an associated graphical user interface known as TKLMITOOL [49]. The Induced-Norm Control Toolbox [17] is a Matlab toolbox for robust and optimal control based on LMITOOL.

The 1996 IEEE International Symposium on Computer Aided Control System Design in Dearborn, Michigan [1] included a session on algorithms and software for LMI problems. The presentations were focused more on algorithms than providing comparisons between software packages or other computational results. The numerical results that were presented showed that the currently available software can handle problems with $F(x)$ in (2) up to size 100×100. The solvers that call SP are the easiest to use and can handle bigger problems than the other software. As of the publication of this tutorial, none of the above LMI solvers exploit matrix sparsity to a high degree.

6.3. Solving BMI problems

Consider the following BMI problem, where we have defined l and r as those variables which appear in the BMI constraint:

$$\inf_{\substack{A(l,r,x,y,\gamma)>0 \\ F_0+\sum_i\sum_j l_i r_j F_{ij}>0 \\ \underline{l}_i \leqslant l_i \leqslant \bar{l}_i \\ \underline{r}_j \leqslant r_j \leqslant \bar{r}_j}} \gamma \tag{100}$$

where A is jointly affine in l, r, x, y, and γ.

A global optimization approach such as branch and bound is required for guaranteed convergence to the global optimum of a BMI problem because the BMI problem is not convex. While several branch and bound algorithms have been developed for solving BMI problems [64,155,156], what appears to be the most efficient algorithm was developed relatively recently [136,140,141, 142,143]. The art to developing an efficient branch and bound algorithm is to derive tight upper and lower bounds for the objective function over any given part of the domain. Reducing the ranges of all problem variables as much as possible is frequently the key to tight objective function bounding. The approach uses LMI relaxations as lower bounds for the BMI.

$$\inf_{\substack{A(l,r,x,y,\gamma)>0 \\ F_0+\sum_i\sum_j l_i r_j F_{ij}>0 \\ \underline{l}_i \leqslant l_i \leqslant \underline{l}_i \\ \underline{r}_j \leqslant r_j \leqslant \bar{r}_j}} \gamma = \inf_{\substack{A(l,r,x,y,\gamma)>0 \\ F_0+\sum_i\sum_j w_{ij} F_{ij}>0 \\ \underline{l}_i \leqslant l_i \leqslant \underline{l}_i \\ \bar{r}_j \leqslant r_j \leqslant \bar{r}_j \\ w_{ij}=l_i r_j}} \gamma \geqslant \inf_{\substack{A(l,r,x,y,\gamma)>0 \\ F_0+\sum_i\sum_j w_{ij} F_{ij}>0 \\ \underline{l}_i \leqslant l_i \leqslant \bar{l}_i \\ \underline{r}_i \leqslant r_i \leqslant \bar{r}_i \\ w_{ij}\in\left[\underline{w}_{ij},\bar{w}_{ij}\right]}} \gamma \tag{101}$$

where the overbar (underbar) indicates the upper (lower) bound for a variable and

$$\underline{w}_{ij} = \min\left\{\underline{l}_i\underline{r}_j, \bar{l}_i\underline{r}_j, \underline{l}_i\bar{r}_j, \bar{l}_i\bar{r}_j\right\} \tag{102}$$

$$\bar{w}_{ij} = \max\left\{\underline{l}_i\underline{r}_j, \bar{l}_i\underline{r}_j, \underline{l}_i\bar{r}_j, \bar{l}_i\bar{r}_j\right\}. \tag{103}$$

Further, because w_{ij} is a bilinear term the following additional constraints may be included in the lower bound (101) [90]:

$$\begin{array}{l} w_{ij} \leqslant \underline{r}_j l_i + \bar{l}_i r_j - \underline{r}_j \bar{l}_i \\ w_{ij} \leqslant \underline{l}_i r_j + \bar{r}_j l_i - \underline{l}_i \bar{r}_j \\ w_{ij} \geqslant \bar{l}_i r_j + \bar{r}_j l_i - \bar{r}_j \bar{l}_i \\ w_{ij} \geqslant \underline{l}_i r_j + \underline{r}_j l_i - \underline{l}_i \underline{r}_j \end{array}. \tag{104}$$

An LMI upper bound is derived by local optimization or by fixing some of the variables. For instance:

$$\inf_{\substack{A(l,r,x,y,\gamma)>0 \\ \underline{l}_i \leqslant l_i \leqslant \bar{l}_i \\ \underline{r}_j \leqslant r_j \leqslant \bar{r}_j \\ F_0+\sum_i\sum_j l_i r_j F_{ij}>0}} \gamma \leqslant \inf_{\substack{A(l,r,x,y,\gamma)>0 \\ l_i=\underline{l}_i \\ \underline{r}_j \leqslant r_j \leqslant \bar{r}_j \\ F_0+\sum_i\sum_j l_i r_j F_{ij}>0}} \gamma \tag{105}$$

With these polynomial-time computable LMI upper and lower bounds, the nonconvex optimization (100) is ideal for the application of the branch and bound algorithm. Interested readers are referred to [137] for more details.

7. Applications

This section lists a variety of LMI and BMI problems that have been or should be studied in process control.

7.1. Control structure selection

Assume that the matrix $M \in \mathcal{R}^{n\times m}$, $n \geqslant m$ is full rank. The condition number of M is the ratio of its largest singular value to its smallest

$$\kappa(M) = \frac{\bar{\sigma}(M)}{\underline{\sigma}(M)}. \tag{106}$$

The condition number appears rather naturally in many control problems, including control structure selection [121,96,102,130,149] and model identification [121,54,83]. It is certainly the one of the most used (and misused [82,28,29]) matrix functions in process control. Its application to chemical processes such as distillation columns is described in many undergraduate process control textbooks [102,126].

Another matrix function that is more relevant to many applications is the minimized condition number:

$$\inf_{R,L} \kappa(LMR) \tag{107}$$

where $L \in \mathcal{R}^{n\times n}$ and $R \in \mathcal{R}^{m\times m}$ are diagonal and nonsingular. The minimized condition number (107) is used for integral controllability tests based on steady-state information [67,96] and for the selection of sensors and actuators using dynamic information [38,112,38,99,98]. The sensitivity of stability to uncertainty in control systems is given in terms of the minimized condition number in [127,128]. The minimized condition number is applied regularly in the process industries, as part of the Robust Multivariable Predictive Control Technology sold by Honeywell [89]. The application to a fractionator and a paper machine is described in [89].

It was shown in [30] how to pose the minimized condition number as a GEVP (88). Here we provide an alternative derivation that follows the later derivation in [22]. Note that the definition of the condition number implies that it is greater than or equal to 1. For $\gamma \geqslant 1$, we have that

$$\kappa(LMR) \leqslant \gamma \iff \mu I \leqslant (LMR)^T(LMR) \leqslant \mu\gamma^2 I \tag{108}$$

$$\iff I \leqslant \left(\hat{L}MR\right)^T\left(\hat{L}MR\right) \leqslant \gamma^2 I \tag{109}$$

$$\iff \left(RR^T\right)^{-1} \leqslant M^T\left(\hat{L}^T\hat{L}\right)M \leqslant \gamma^2\left(RR^T\right)^{-1} \tag{110}$$

$$\iff Q \leqslant M^T PM \leqslant \gamma^2 Q \tag{111}$$

for diagonal $P > 0 \in \mathcal{R}^{n\times n}$ and diagonal $Q > 0 \in \mathcal{R}^{m\times m}$. Therefore, solving the minimized condition number problem (107) is equivalent to solving the GEVP (88):

$$\inf_{\substack{P>0\\ Q>0\\ Q\leqslant M^T PM\leqslant \gamma^2 Q}} \gamma^2 \tag{112}$$

where P and Q are diagonal.

7.2. Parameter estimation and model predictive control

The approximation of polytopes with ellipsoids have numerous applications, including parameter estimation [39,56,80] and model predictive control [34,138,139]. The model predictive control application has been implemented on paper machine models constructed from industrial data [139].

An ellipsoid has the form

$$\varepsilon = \left\{By + d \mid \parallel y \parallel \leqslant 1\right\} \tag{113}$$

where $B = B^T > 0$. This ellipsoid is centered at d and has volume proportional to $\det(B)$. Consider the polytope

$$\mathcal{P} = \left\{x \mid A_i^T x \leqslant b_i, i = 1, \ldots, L\right\} \tag{114}$$

where A_i^T is the ith row of the matrix A. An ellipsoid $\mathcal{E}$ is contained inside the polytope $\mathcal{P}$ if

$$A_i^T(By + d) \leqslant b_i \forall y, \parallel y \parallel \leqslant 1 \tag{115}$$

$$\iff \max_{\|y\|\leqslant 1} A_i^T By + A_i^T d \leqslant b_i \tag{116}$$

$$\iff \parallel BA_i \parallel + A_i^T d \leqslant b_i \tag{117}$$

Thus, the maximum volume ellipsoid $\mathcal{E}$ contained in the polytope $\mathcal{P}$ is given by

$$\max_{\substack{B=B^T>0,d\\ \|BA_i\|+A_i^T d\leqslant b_i}} \log \quad \det(B) \tag{118}$$

This optimization is convex in the variables B and d. For the case where the center of the ellipsoid is known (e.g. $d = 0$ when $Ax \leqslant b$ defines a symmetric polytope), (117) can be written as an LMI using the Schur complement lemma:

$$\parallel BA_i \parallel + A_i^T d \leqslant b_i \iff b_i - A_i^T d \geqslant 0 \text{ and } \parallel BA_i \parallel^2 \leqslant \left(b_i - A_i^T d\right)^2 \tag{119}$$

$$\iff b_i - A_i^T d \geqslant 0 \text{ and } \left(b_i - A_i^T d\right)^2 - A_i^T BI^{-1} BA_i \geqslant 0 \tag{120}$$

$$\iff \begin{bmatrix} \left(b_i - A_i^T d\right)^2 & A_i^T B \\ BA_i & I \end{bmatrix} \geqslant 0, \forall i \in [1, L] \tag{121}$$

Hence in this case (118) can be written as the CDOP (95):

$$\max_{\substack{B=B^T>0,d\\ \begin{bmatrix} (b_i-A_i^T d)^2 & A_i^T B \\ BA_i & I\end{bmatrix}\geqslant 0}} \log \quad \det(B) \tag{122}$$

In the case where d is unknown, (118) is not an LMI but is still a convex program that can be solved, for instance, by interior point methods [76,97].

A related problem of interest is to determine the smallest ellipsoid which encloses a given polytope. First define the *convex hull* of a given set of points $\mathcal{T}$ in $\mathcal{R}^n$ as the set of all convex combinations of points in $\mathcal{T}$. An equivalent definition is the smallest convex set containing $\mathcal{T}$ [105]. Let the polytope be described as the convex hull of its vertices

$$\mathcal{P} = \mathrm{Co}\{v_1, \ldots, v_L\} \tag{123}$$

and write the ellipsoid

$$\mathcal{E} = \left\{x \mid \parallel Ax - b \parallel \leqslant 1, A = A^T > 0\right\}, \tag{124}$$

where its center is $A^{-1}b$ and its volume is proportional to $\det\left(A^{-1}\right)$. Then the minimum volume ellipsoid which encloses the polytope is given by

$$\inf_{\substack{A=A^T>0\\ \|Av_i-b\|\leqslant 1}} \log \quad \det\left(A^{-1}\right). \tag{125}$$

This problem is convex in A and b, and can be written as a CDOP (95) by applying the results of Section 4.6:

$$\inf_{\substack{A=A^T>0 \\ \begin{bmatrix} 1 & (Av_i-b)^T \\ Av_i-b & I \end{bmatrix}>0}} \log \det\left(A^{-1}\right) \tag{126}$$

7.3. Optimal design of experiments

The goal of optimal experimental design is to maximize the informativeness of data collected from the process [11]. Optimal experimental design algorithms have been applied to chemical kinetics [20,19,21,73,113], synthetic fiber manufacture [75], petroleum fractionation [132], crystallization [92], distillation [79], and polymer film extrusion [53]. While most formulations require the solution of nonconvex optimization problems [69,150], here is presented a formulation for linear parameter estimation which requires only the solution of a CDOP (95).

The goal is to estimate a vector of parameters x from some measurement $y = Ax + w$ where A is a matrix of inputs and w is zero-mean white measurement noise. The error covariance of the minimum variance estimator is $\left(A^TA\right)^{-1}$. If the rows of $A = [a_1, \ldots, a_L]^T$ are chosen from a set of possible test vectors,

$$a_i \in \{v_1, \ldots, v_L\}, \quad i = 1, \ldots, L, \tag{127}$$

the goal of D-optimal experimental design is to select the vectors so that the determinant of the error covariance is minimized.

We can write

$$A^TA = \sum_{i=1}^{L} \lambda_i v_i v_i^T \tag{128}$$

where $\lambda_i \geqslant 0$ is the fraction of rows equal to the vector v_i, which implies that $\sum_{i=1}^{L} \lambda_i = 1$. When L is a large number, the λ_i can be treated as continuous variables instead of integer multiples of $1/L$.

Then the D-optimal design problem is the CDOP (95) [153]:

$$\inf_{\substack{\lambda_i \geqslant 0 \\ \sum_{i=1}^{L}\lambda_i=1}} \log \det\left(\sum_{i=1}^{L} \lambda_i v_i v_i^T\right)^{-1} \tag{129}$$

7.4. Robust control system analysis

The robustness margin for a variety of linear systems subject to linear or nonlinear perturbations [31,71,96,123,130] can be computed by solving

$$\upsilon = \inf_{D\in\mathcal{D}} \bar{\sigma}\left(DMD^{-1}\right) \tag{130}$$

where M is a complex matrix and $\mathcal{D}$ is the set of complex nonsingular block diagonal matrices with some blocks possibly being repeated. This robustness margin has been applied to numerous processes over the past 15 years, including distillation columns [29,40,96,130], pH neutralization [119], packed bed reactors [47], paper machines [81,122,33], polymer film extrusion [52,55], and reactive ion etching [143].

This problem can be written in terms of a GEVP (88):

$$\upsilon^2 = \inf_{\substack{D\in\mathcal{D} \\ (DMD^{-1})^*(DMD^{-1}) \leqslant \gamma^2 I}} \gamma^2 \tag{131}$$

$$= \inf_{\substack{D\in\mathcal{D} \\ M^*D^*DM \leqslant \gamma^2 D^*D}} \gamma^2 \tag{132}$$

$$= \inf_{\substack{P\in\mathcal{P} \\ M^*PM \leqslant \gamma^2 P}} \gamma^2 \tag{133}$$

where $\mathcal{P}$ is the set of complex symmetric positive definite $n \times n$ block diagonal matrices with the corresponding blocks from $\mathcal{D}$ being repeated.

7.5. Robust nonlinear controller synthesis

BMI formulations arise naturally in the design of robust optimal inversion-based controllers for nonlinear processes. Here we present the BMI formulation which was applied to the nonlinear simulation model of a reactive ion etcher constructed from experimental data presented in Section 2. The BMI-based controller demonstrated substantially improved performance and robustness over a traditional nonlinear controller [140,143].

After the nonlinear inversion technique removed the most significant nonlinearities, the control synthesis problem consisted of designing a linear controller for a linear plant subject to norm-bounded nonlinear time varying perturbations. The state space realization for the plant transfer function $G(s) = C(sI - A)^{-1}B + D$ was represented by

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{134}$$

where $D_{22} = 0$ without loss of generality [157]. The controller that optimizes the induced 2-norm performance objective subject to the constraint of stability of the closed loop system with norm-bounded nonlinear time varying perturbations was computed from the solution of the BEVP (97) [140,143]:

$$\gamma^* = \inf_{\substack{(L,R,X,Y)\in\mathcal{B} \\ \lambda_{\max}(L_1R_1) \leqslant 1}} \gamma \tag{135}$$

where $\mathcal{B}$ is the set such that L, R, X and Y are symmetric matrices,

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & \gamma I \end{bmatrix}; \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & \gamma I \end{bmatrix}; \quad L_1, R_1 \in \mathcal{D}; \tag{136}$$

and

$$\begin{bmatrix} \begin{pmatrix} B_2 \\ D_{12} \end{pmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} AX + XA^T & XC_1^T & B_1 \\ C_1X & -L & D_{11} \\ B_1^T & D_{11}^T & -R \end{bmatrix} \times \begin{bmatrix} \begin{pmatrix} B_2 \\ D_{12} \end{pmatrix}^{\perp^T} & 0 \\ 0 & I \end{bmatrix} < 0; \tag{137}$$

$$\begin{bmatrix} \begin{pmatrix} C_2^T \\ D_{21}^T \end{pmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} YA + A^TY & YB_1 & C_1^T \\ B_1^TY & -R & D_{11} \\ C_1 & D_{11}^T & -L \end{bmatrix} \times \begin{bmatrix} \begin{pmatrix} C_2^T \\ D_{21}^T \end{pmatrix}^{\perp^T} & 0 \\ 0 & I \end{bmatrix} < 0; \tag{138}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{139}$$

Here $A^{\perp}$ is a matrix whose rows form a basis for the null space of A^T. The only nonconvexity in (135) is the constraint $\lambda_{\max}(L_1R_1) \leqslant 1$ (which is a BMI).

As the algebra of this derivation is lengthy and involved, only a summary is given here. The state space equations for the closed loop system are written as functions of the state space matrices of the plant and the controller. A version of the Bounded real lemma is used to write the induced 2-norm performance objective in terms of matrix inequalities, and the variable reduction lemma of Section 4.12 is used to remove explicit dependence of the matrix inequalities on the controller state space matrices. Finally, D and D^{-1} are replaced with L and R and the additional constraint that $\lambda_{\max}(R_1L_1) < 1$. Readers interested in a detailed derivation are referred to [137].

A closely related formulation was used in the design of linear controllers that optimize the robust performance for large scale sheet and film processes, such as polymer film extruders and paper machines [141]. In those particular applications, the only process nonlinearities were perturbations about the nominal linear dynamics.

7.6. Robust model predictive control

Here we describe an LMI-based robust model predictive control algorithm which applied to a non-isothermal nonadiabatic continuous stirred tank reactor (CSTR) [77]. The LMI approach provided similar performance as a non-LMI-based model predictive control algorithm, while having the capability of providing robustness to model uncertainty.

Consider the discrete time time varying linear system

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{140}$$

$$y(k) = C(k)x(k) \tag{141}$$

where each state space matrix is arbitrarily time varying and lies within a polytope (see Section 4.3). Define $x(k|k)$ as the state of the uncertain system measured at sampling time k, $x(k+i|k)$ as the state of the system at time $k+i$ predicted at time k, $u(k+i|k)$ as the control move at time $k+i$ computed at time k, and W and R are positive definite weighting matrices. For this control problem, the objective was to compute the state feedback matrix F:

$$u(k+i|k) = Fx(k+i|k) \tag{142}$$

so as to minimize an upper bound on the infinite horizon quadratic objective:

$$\sum_{i=0}^{\infty} x(k+i|k)^T Wx(k+i|k) + u(k+i|k)Ru(k+i|k). \tag{143}$$

at sampling time k. This state feedback matrix is given by [77]:

$$F = YQ^{-1} \tag{144}$$

where $Q > 0$ and Y are solutions to the following EVP (82):

$$\inf_{\gamma,Q,Y} \gamma \tag{145}$$

subject to

$$\begin{bmatrix} 1 & x(k|k)^T \\ x(k|k) & Q \end{bmatrix} \geqslant 0 \tag{146}$$

$$\begin{bmatrix} Q & QA_i^T + Y^TB_i^T & QW^{1/2} & Y^TR^{1/2} \\ A_iQ + B_iY & Q & 0 & 0 \\ W^{1/2}Q & 0 & \gamma I & 0 \\ R^{1/2}Y & 0 & 0 & \gamma I \end{bmatrix} \geqslant 0, \quad \forall i = 1, 2, \ldots, L. \tag{147}$$

The derivation uses a positive definite quadratic function of the state to bound the performance objective (143), and then uses the Schur complement lemma to manipulate this inequality into the form of the LMI constraint (147). Input and output constraints can also be handled by augmenting the EVP with additional LMI constraints (see [77] for further details).

The state feedback matrix F is computed at each sampling instance k, and used to compute the control move $u(k) = u(k|k)$ to be implemented. When a new measurement is taken, F and the control move are recomputed. The model predictive controller can be shown to be stabilizing for all matrices within the matrix polytope [77].

7.7. Gain-scheduled/linear parameter varying systems

Gain scheduling is discussed in undergraduate process control textbooks [102,126]. A relatively new approach to the design of gain-scheduled controllers is to represent the process as being linear parameter varying (LPV):

$$x(k+1) = A(p(k))x(k) + B(p(k))u(k) \tag{148}$$

$$y(k) = C(p(k))x(k) + D(p(k))u(k) \tag{149}$$

where the state space matrices are explicit functions of a time varying parameter vector $p(k)$. An LPV process reduces to a linear time varying process for a given trajectory, and reduces to a linear time invariant system for a constant parameter vector $p(k)$. This model representation forms the basis for a solid theoretical framework for the design of gain-scheduled controllers using LMIs [103,152,8,7].

It is common to assume that the state space matrices are affine functions of $p(k)$ and that the time varying parameter $p(k)$ varies within a polytope. Then the gain-scheduled (or LPV) controller has a form

$$\hat{x}(k+1) = \hat{A}(p(k))\hat{x}(k) + \hat{B}(p(k))y(k) \tag{150}$$

$$u(k) = \hat{C}(p(k))\hat{x}(k) + \hat{D}(p(k))y(k) \tag{151}$$

similar to that for the process. The controller is assumed to be able to measure or estimate $p(k)$ on-line, so this information can be used by the controller to provide improved performance over controllers which do not exploit such information. The controller matrices that guarantee global asymptotic stability and minimize an induced 2-norm performance objective can be computed as an EVP. The LMIs are derived using a quadratic Lyapunov function and a generalization of the Bounded real lemma. The EVP is somewhat similar to the BEVP in Section 7.5, but with $L = R = I$, and so will not be given here (see [6,7] for the exact form of the LMIs).

An interesting variation on the LPV approach is to treat the parameters as validity functions for linear models used to represent the nonlinear process dynamics locally [9,12,13]. Each local model is assigned to an element of the vector $p(k)$ which approaches 1 when the plant moves into the local region of the model and approaches 0 as the plant moves into other regions. The elements of $p(k)$ sum up to 1 at each time instance.

While [13] applies an induced 2-norm approach similar to that described above, [9] proposes an LPV-based model predictive control (MPC) design procedure which is an extension of the approach discussed in Section 7.6 (the LMIs have a similar structure as those in Section 7.6). The LPV-based MPC control algorithm is shown to asymptotically stabilize the closed loop LPV process. The algorithm was applied to a continuous stirred tank reactor with output multiplicity, and to a semibatch reactor for free-radical polymerization of polymethyl methacrylate. Although the closed loop performance of the LMI-based LPV-MPC algorithm was not quite as good as an LPV-based quadratic programming algorithm (similar to traditional MPC), it had the advantage of guaranteeing closed loop stability.

8. Conclusions

A tutorial was provided on the mathematical theory and process control applications of linear and bilinear matrix inequalities. Many common convex inequalities occurring in nonlinear programming and several tests for the stability of linear and nonlinear systems were written in terms of LMI feasibility problems. Algorithms for solving optimization problems with LMI or BMI constraints and publicly available software were reviewed. This was followed by a survey of applications of LMIs and BMIs to control problems associated with chemical and mechanical processes. These included control structure selection, parameter estimation, experimental design, and optimal linear and nonlinear feedback control.

The authors believe that LMIs and BMIs form a set of mathematical tools which are fundamental to the background of a process control engineer. It is hoped that the many examples provided throughout the paper provide a convincing justification for this belief.

Appendix

Proof of the Schur complement lemma

($\Rightarrow$) Assume

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \tag{152}$$

and define

$$F(u,v)=\begin{bmatrix}u\\v\end{bmatrix}^T\begin{bmatrix}Q(x) & S(x)\\ S(x)^T & R(x)\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} \tag{153}$$

then

$$F(u,\ v)>0 \quad \forall[u\,v]\neq 0 \tag{154}$$

First, consider $u=0$. Then

$$F(0,v)=v^TR(x)v>0, \forall v\neq 0 \ \Rightarrow\ R(x)>0.$$

Next consider

$$\upsilon=-R(x)^{-1}S(x)^Tu, \text{with } \ u\neq 0.$$

Then

$$F(u,v)=u^T\big(Q(x)-S(x)R(x)^{-1}S(x)^T\big)u>0, \forall u\neq 0$$
$$\Rightarrow\ Q(x)-S(x)R(x)^{-1}S(x)^T>0.$$

($\Leftarrow$) Now assume

$$Q(x)-S(x)R(x)^{-1}S(x)^T>0, \quad R(x)>0. \tag{155}$$

with $F(u,v)$ defined as in (153).

We will fix u and optimize over v.

$$\nabla_vF^T=2Rv+2S^Tu=0 \tag{156}$$

Since $R>0$, (156) gives a single extrema $v=-R^{-1}S^Tu$. Plugging this into (153) gives $F(u)=u^T\big(Q-SR^{-1}S^T\big)u$. Since $\big(Q-SR^{-1}S^T\big)>0$ the minimum of $F(u)$ occurs for $u=0$, which also implies that $v=0$. Thus the minimum of $F(u,v)$ occurs at $(0,0)$ and is equal to zero. Therefore, $F(u,v)$ is positive definite. QED.

Proof that $-\log(\det(A))$ is convex in A for $A=A^T>0$. We will use the following lemma [24].

Lemma 1. *A function $f(x)$ is convex in $x\in S$ if and only if $f(t)=f(x_0+th)$ is convex in t for all x_0, h, and t such that $x_0+th\in S$ and $x_0\in S$.*

Define $S=\{A|A=A^T>0\}$. Lemma 1 implies that $-\log(\det(A))$ is convex in A on $A=A^T>0$ if and only if $-\log(\det(A_0+tH))$ is convex in t for all $A_0=A_0^T>0$ and H which satisfy $A_0+tH=(A_0+tH)^T>0$. Note that

$$-\log(\det(A_0+tH)) \tag{157}$$

$$=-\log(\det(A_0))-\log\Big(\det\Big(I+tA_0^{-1/2}HA_0^{-1/2}\Big)\Big) \tag{158}$$

$$=-\log(\det(A_0))-\sum_i\log\Big(1+t\lambda_i\Big(A_0^{-1/2}HA_0^{-1/2}\Big)\Big). \tag{159}$$

The last step follows because the determinant is the product of the eigenvalues. The condition $A_0^T=A_0>0$ implies that its matrix square root exists, and $A_0+tH>0$ implies that $I+tA_0^{-1/2}HA_0^{-1/2}>0$. Hence

$$1+t\lambda_i(A_0^{-1/2}HA_0^{-1/2})>0.$$

The first and second derivatives of

$$-\log(1+t\lambda_i(A_0^{-1/2}HA_0^{-1/2}))$$

are

$$-\frac{\mathrm{d}}{\mathrm{d}t}\log\Big(1+t\lambda_i\Big(A_0^{-1/2}HA_0^{-1/2}\Big)\Big)$$
$$=-\lambda_i\Big(A_0^{-1/2}HA_0^{-1/2}\Big)/\Big(1+t\lambda_i\Big(A_0^{-1/2}HA_0^{-1/2}\Big)\Big); \tag{160}$$

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\Big(1+t\lambda_i\Big(A_0^{-1/2}HA_0^{-1/2}\Big)\Big)$$
$$=\lambda_i^2\Big(A_0^{-1/2}HA_0^{-1/2}\Big)/\Big(1+t\lambda_i\Big(A_0^{-1/2}HA_0^{-1/2}\Big)\Big)^2>0. \tag{161}$$

The second derivative greater than zero implies that $-\log\Big(1+t\lambda_i\Big(A_0^{-1/2}HA_0^{-1/2}\Big)\Big)$ is convex in t. The convexity of a constant and the sum of convex functions implies that $-\log(\det(A_0+tH))$ is convex in t for all allowable t. QED.

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