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Tools for Analysis of Dynamic Systems: Lyapunov's Methods

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A. M. Lyapunov’s (1857--1918) Thesis

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ОБЩАЯ ЗАДАЧА

ОБЪ

УСТОЙЧИВОСТИ ДВИЖЕНІЯ.

РАЗСУЖДЕНІЕ

А. ЛЯПУНОВА.

Lyapunov’s Thesis

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1892.

Lyapunov's Thesis Translated

INT. J. CONTROL, 1992, VOL. 55, NO. 3, 531–773

The general problem of the stability of motion

A. M. LYAPUNOV

Translated from Russian into French by Édouard Davaux, Marine Engineer at Toulon.†
Translated from French into English by A. T. Fuller.‡

Preface

In this work some methods are expounded for the resolution of questions concerning the properties of motion and, in particular, of equilibrium, which are known by the terms *stability* and *instability*.

The ordinary questions of this kind, those to which this work is devoted, lead to the study of differential equations of the form

$$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \ldots, \quad \frac{dx_n}{dt} = X_n,$$

Some Details About Translation

† Mr Lyapunov has very graciously authorized the publication in French of his memoir *Obshchaya zadacha ob ustoichivosti dvizheniya* printed in 1892 by the Mathematical Society of Kharkov. The [French] translation has been reviewed and corrected by the author [Lyapunov], who has added a note based on an article which appeared in 1893 in *Communications de la Société mathématique de Kharkow*.

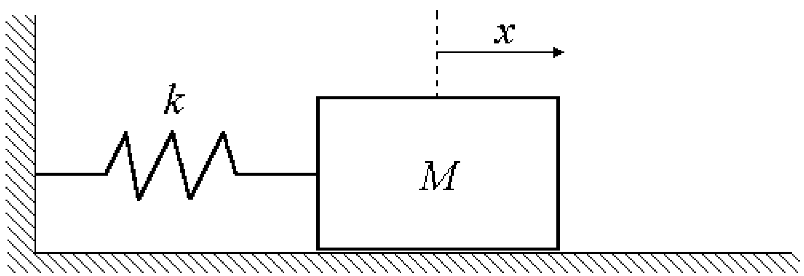
‡ [Comments in square brackets are by A.T.F.]

§ We have in mind the cases where there applies the known theorem of Lagrange on the maxima of the force-function [this is minus the potential energy function], relating to the stability of equilibrium; also, the cases where there applies a more general theorem of Routh on the maxima and minima of the integrals of the equations of motion, allowing the resolution of certain questions relative to the stability of motion (see *The advanced part of A Treatise on the Dynamics of a System of Rigid Bodies*, fourth edition, 1884, pp. 52, 53).

Outline

- Notation using simple examples of dynamical system models
- Objective of analysis of a nonlinear system
- Equilibrium points
- Lyapunov functions
- Stability
- Barbalat's lemma

A Spring-Mass Mechanical System



x---displacement of the mass from
the rest position

Modeling the Mass-Spring System

- Assume a linear mass, where *k* is the linear spring constant
- Apply Newton's law to obtain

$$M\ddot{x} + kx = 0$$

- Define state variables: x_1=x and x_2=dx/dt
- The model in state-space format:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{M}x_1 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

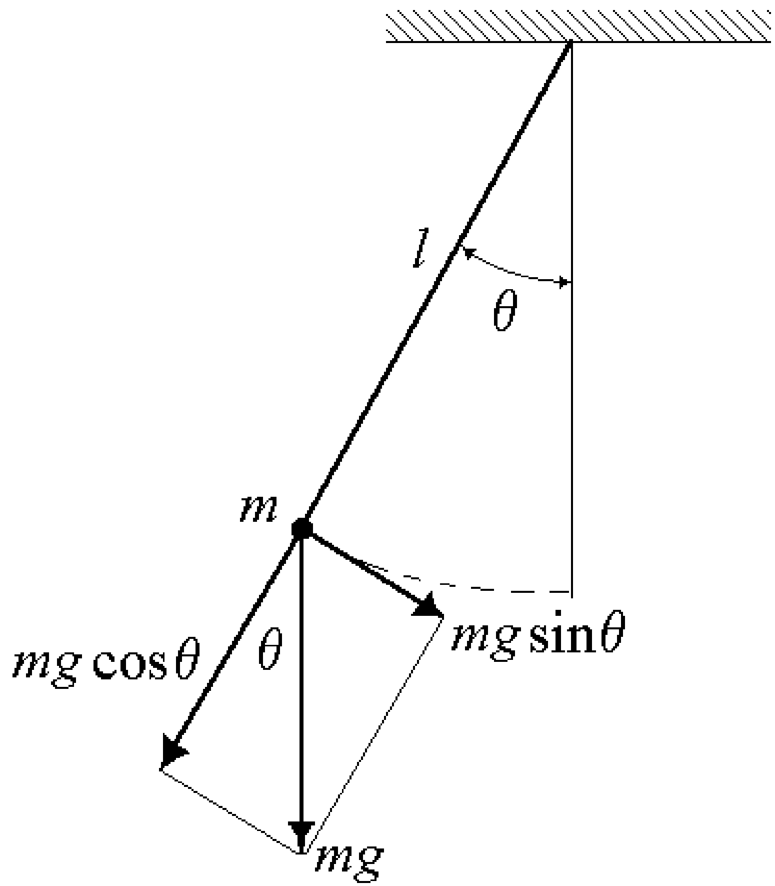
Analysis of the Spring-Mass System Model

- The spring-mass system model is linear time-invariant (LTI)
- Representing the LTI system in standard state-space format

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\
&= \begin{bmatrix} x_2 \\ -\frac{k}{M} x_1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \boldsymbol{A}\boldsymbol{x}
\end{aligned}
$$

Modeling of the Simple Pendulum

The simple pendulum



The Simple Pendulum Model

- Apply Newton's second law

$$J\ddot{\theta} = -mgl\sin\theta$$

where J is the moment of inertia,

$$J = ml^2$$

- Combining gives

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

State-Space Model of the Simple Pendulum

- Represent the second-order differential equation as an equivalent system of two first-order differential equations
- First define state variables,
  $x_1=\theta$ and $x_2=d\theta/dt$
- Use the above to obtain state–space model (nonlinear, time invariant)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

Objectives of Analysis of Nonlinear Systems

- Similar to the objectives pursued when investigating complex linear systems
- Not interested in detailed solutions, rather one seeks to characterize the system behavior---equilibrium points and their stability properties

☞ A device needed for nonlinear system analysis summarizing the system behavior, suppressing detail

Summarizing Function (D.G. Luenberger, 1979)

- A function of the system state vector
- As the system evolves in time, the summarizing function takes on various values conveying some information about the system

Summarizing Function as a First-Order Differential Equation

- The behavior of the summarizing function describes a first-order differential equation
- Analysis of this first-order differential equation in some sense a summary analysis of the underlying system

Dynamical System Models

- Linear time-invariant (LTI) system model

$$\dot{x} = Ax, \quad A \in \Re^{n\times n}$$

- Nonlinear system model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, \ldots, x_n) \\ f_2(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{bmatrix}$$

- Shorthand notation of the above model

$$\dot{x} = f(t, x), \quad x \in \Re^n$$

More Notation

- System model

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

- Solution

$$x(t) = x(t; t_0, x_0)$$

- Example: LTI model,

$$\dot{x} = Ax, \quad x(0) = x_0$$

- Solution of the LTI modeling equation

$$x(t) = e^{At} x_0$$

Equilibrium Point

A vector x_e is an equilibrium point for a dynamical system model

$$\dot{x}(t) = f\big(t, x(t)\big)$$

if once the state vector equals to x_e it remains equal to x_e for all future time. The equilibrium point satisfies

$$f\big(t, x(t)\big) = 0$$

Formal Definition of Equilibrium

- A point $\boldsymbol{x}_e$ is called an equilibrium point of d$\boldsymbol{x}$/dt=$\boldsymbol{f}(t,\boldsymbol{x})$, or simply an equilibrium, at time t_0 if for all $t \geq t_0$,

$$\boldsymbol{f}(t, \boldsymbol{x}_e)=0$$

- Note that if $\boldsymbol{x}_e$ is an equilibrium of our system at t_0, then it is also an equilibrium for all $\tau \geq t_0$

Equilibrium Points for LTI Systems

For the time invariant system

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t=\boldsymbol{f}(\boldsymbol{x})$$

a point is an equilibrium at some time τ if and only if it is an equilibrium at all times

Equilibrium State for LTI Systems

- LTI model

$$\dot{x} = f(t,x) = Ax$$

- Any equilibrium state x_e must satisfy

$$Ax_e = 0$$

- If A^{-1} exist, then we have unique equilibrium state

$$x_e = 0$$

Equilibrium States of Nonlinear Systems

- A nonlinear system may have a number of equilibrium states
- The origin, $\boldsymbol{x}=\mathbf{0}$, may or may not be an equilibrium state of a nonlinear system

Translating the Equilibrium of Interest to the Origin

- If the origin is not the equilibrium state, it is always possible to translate the origin of the coordinate system to that state
- So, no loss of generality is lost in assuming that the origin is the equilibrium state of interest

Example of a Nonlinear System with Multiple Equilibrium Points

- Nonlinear system model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 - x_2 - x_1^2 \end{bmatrix}$$

- Two isolated equilibrium states

$$x_e^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_e^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Isolated Equilibrium

An equilibrium point $\boldsymbol{x}_e$ in R^n is an isolated equilibrium point if there is an $r>0$ such that the r-neighborhood of $\boldsymbol{x}_e$ contains no equilibrium points other than $\boldsymbol{x}_e$

Neighborhood of $\boldsymbol{x}_e$

The *r*-neighborhood of $\boldsymbol{x}_e$ can be a set of points of the form

$$B_r(\boldsymbol{x}_e) = \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{x}_e\| < r\}$$

where ||.|| can be any *p*-norm on R^n

Remarks on Stability

- Stability properties characterize the system behavior if its initial state is close but not at the equilibrium point of interest
- When an initial state is close to the equilibrium pt., the state may remain close, or it may move away from the equilibrium point

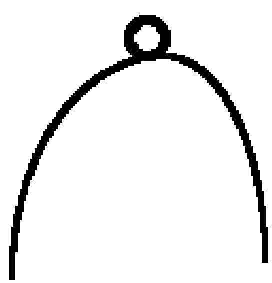
An Informal Definition of Stability

An equilibrium state is stable if whenever the initial state is near that point, the state remains near it, perhaps even tending toward the equilibrium point as time increases

Stability Intuitive Interpretation

(1)

(2)



(3)

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Formal Definition of Stability

An equilibrium state x_{eq} is stable, in the sense of Lyapunov, if for any given t_0 and any positive scalar ε there exist a positive scalar

$$\delta = \delta(t_0, \varepsilon)$$

such that if

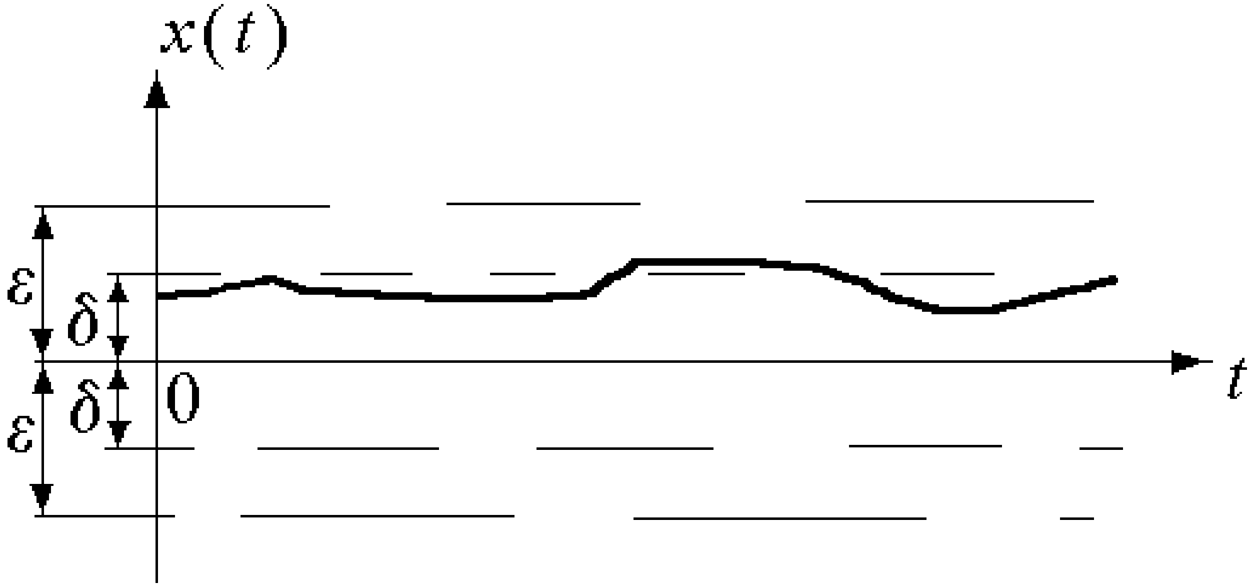
$$\left\| x(t_0) - x_e \right\| < \delta$$

then

$$\left\| x(t; t_0, x_0) - x_e \right\| < \varepsilon$$

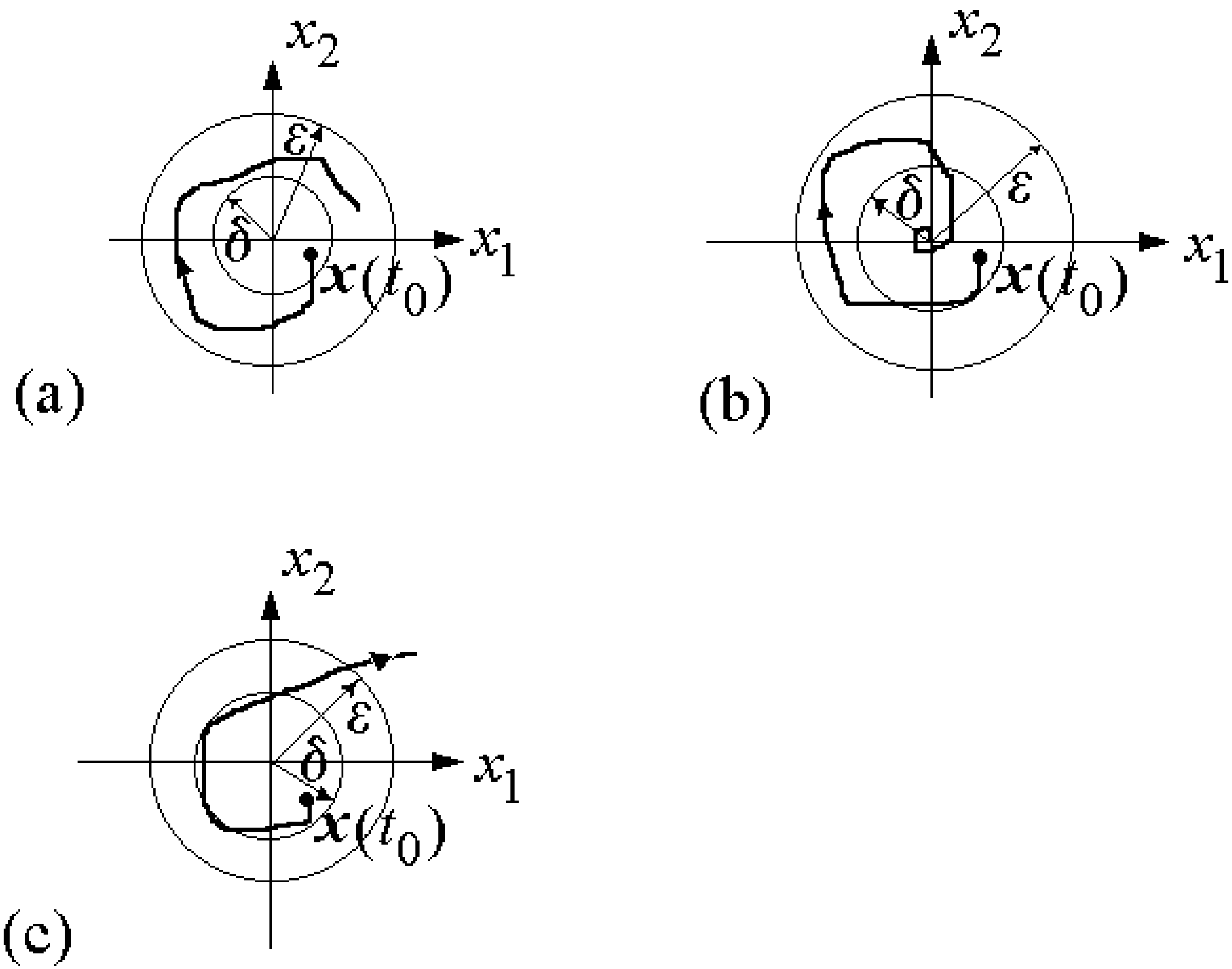
for all $t \geq t_0$

Stability Concept in 1D



(a)

Stability Concepts in 2D



Further Discussion of Lyapunov Stability

Think of a contest between you, the control system designer, and an adversary (nature?)---B. Friedland (ACSD, p. 43, Prentice-Hall, 1996)

Lyapunov Stability Game

- The adversary picks a region in the state space of radius ε
- You are challenged to find a region of radius δ such that if the initial state starts out inside your region, it remains in his region---if you can do this, your system is stable, in the sense of Lyapunov

Lyapunov Stability---Is It Any Good?

- Lyapunov stability is weak---it does not even imply that $\boldsymbol{x}(t)$ converges to $\boldsymbol{x}_e$ as t approaches infinity
- The states are only required to hover around the equilibrium state
- The stability condition bounds the amount of wiggling room for $\boldsymbol{x}(t)$

Asymptotic Stability i.s.L

The property of an equilibrium state of a differential equation that satisfies two conditions:

- (stability) small perturbations in the initial condition produce small perturbations in the solution;

Second Condition for Asymptotic Stability of an Equilibrium

- (attractivity of the equilibrium point) there is a domain of attraction such that whenever the initial condition belongs to this domain the solution approaches the equilibrium state at large times

Asymptotic Stability in the sense of Lyapunov (i.s.L.)

- The equilibrium state is asymptotically stable if
  - it is stable, and
  - convergent, that is,

$$x(t;t_0,x_0) \to x_e \; as \; t \to \infty$$

Convergence Alone Does Not Guarantee Asymptotic Stability

Note: it is not sufficient that just

$$x(t;t_0,x_0) \to x_e \; as \; t \to \infty$$

for asymptotic stability. We need stability too! Why?

How Long to the Equilibrium?

- Asymptotic stability does not imply anything about how long it takes to converge to a prescribed neighborhood of $\boldsymbol{x}_e$
- Exponential stability provides a way to express the rate of convergence

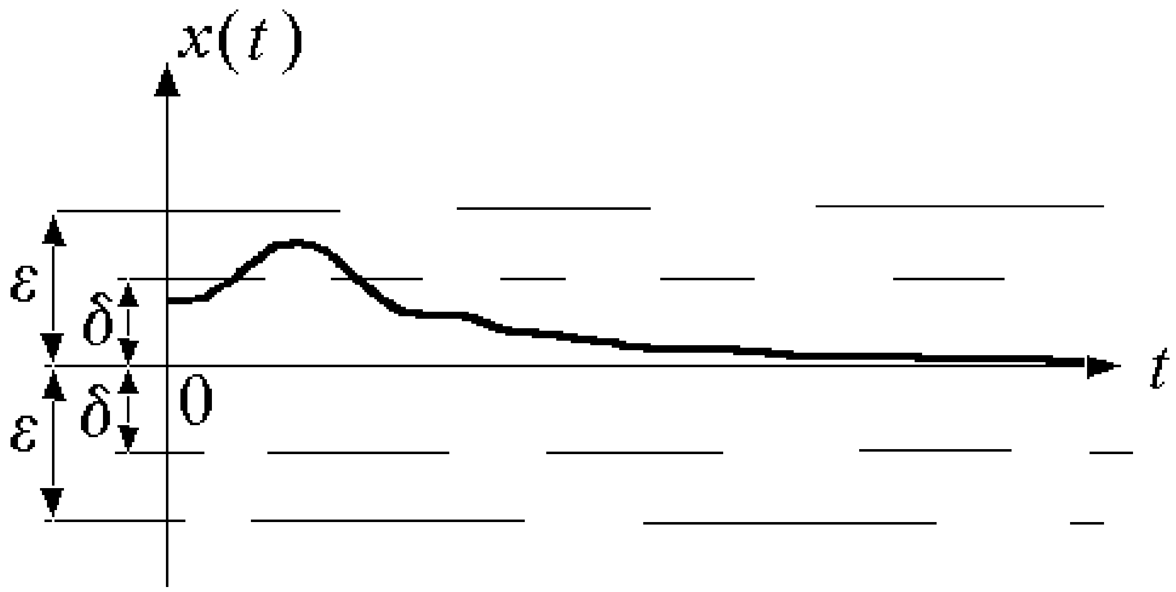
Asymptotic Stability of Linear Systems

- An LTI system is asymptotically stable, meaning, the equilibrium state at the origin is asymptotically stable, if and only if the eigenvalues of $\boldsymbol{A}$ have negative real parts
- For LTI systems asymptotic stability is equivalent with convergence (stability condition automatically satisfied)

Asymptotic Stability of Nonlinear Systems

- For LTI systems asymptotic stability is equivalent with convergence (stability condition automatically satisfied)
- For nonlinear systems the state may initially tend away from the equilibrium state of interest but subsequently may return to it

Asymptotic Stability in 1D



(b)

Convergence Does Not Mean Asymptotic Stability (W. Hahn, 1967)

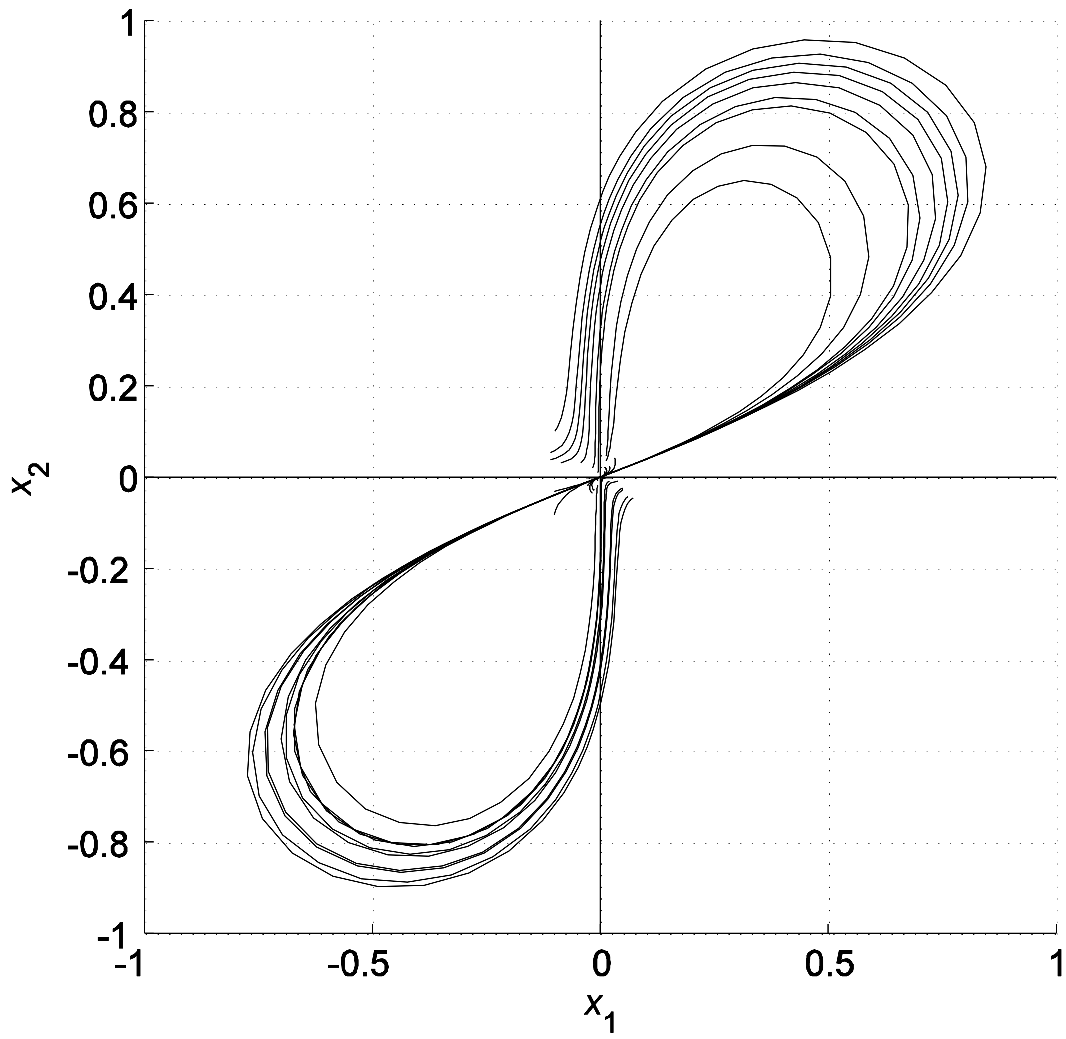
Hahn's 1967 Example---A system whose all solutions are approaching the equilibrium, $\boldsymbol{x}_e=\mathbf{0}$, without this equilibrium being asymptotically stable (Antsaklis and Michel, Linear Systems, 1997, p. 451)

Convergence Does Not Mean Asymptotic Stability (W. Hahn, 1967)

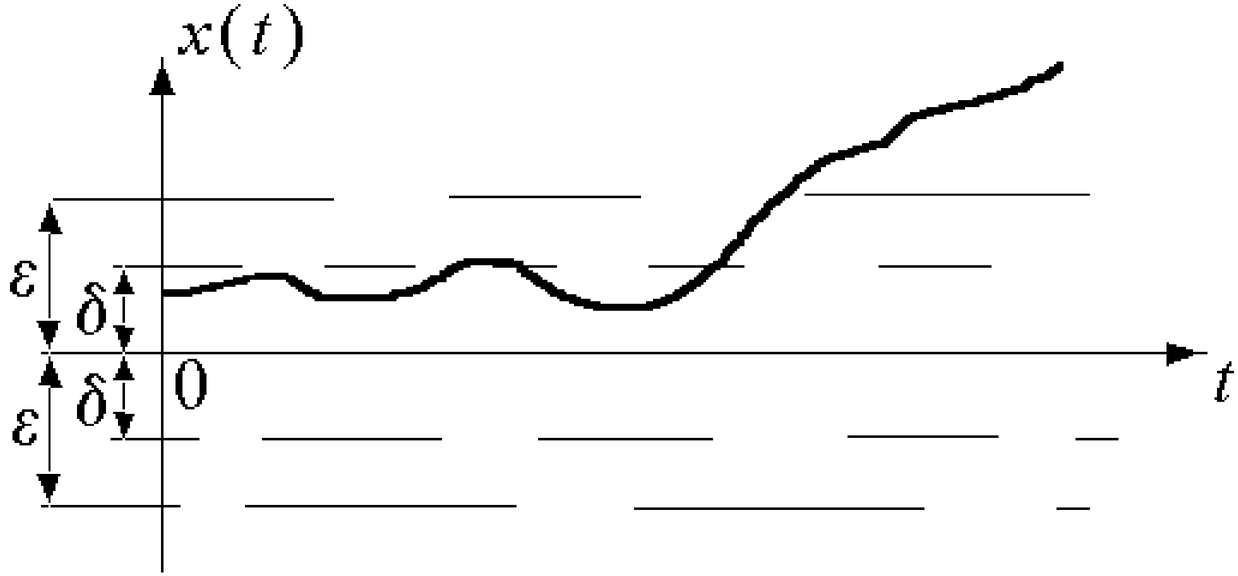
Nonlinear system of Hahn where the origin is attractive but not a.s.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{x_1^2(x_2-x_1)+x_2^5}{(x_1^2+x_2^2)\left(1+(x_1^2+x_2^2)^2\right)} \\ \dfrac{x_2^2(x_2-2x_1)}{(x_1^2+x_2^2)\left(1+(x_1^2+x_2^2)^2\right)} \end{bmatrix}$$

Phase Portrait of Hahn's 1967 Example



Instability in 1D



(c)

Lyapunov Functions---Basic Idea

- Seek an aggregate summarizing function that continually decreases toward a minimum
- For mechanical systems---energy of a free mechanical system with friction always decreases unless the system is at rest, equilibrium

Lyapunov Function Definition

A function that allows one to deduce stability is termed a Lyapunov function

Lyapunov Function Properties for Continuous Time Systems

- Continuous-time system

$$\dot{x}(t) = f\left(x(t)\right)$$

- Equilibrium state of interest

$$x_e$$

Three Properties of a Lyapunov Function

We seek an aggregate summarizing function *V*

- *V* is continuous
- *V* has a unique minimum with respect to all other points in some neighborhood of the equilibrium of interest
- Along any trajectory of the system, the value of *V* never increases

Lyapunov Theorem for Continuous Systems

- Continuous-time system

$$\dot{x}(t) = f(x(t))$$

- Equilibrium state of interest

$$x_e = 0$$

Lyapunov Theorem---Negative Rate of Increase of V

- If $\boldsymbol{x}(t)$ is a trajectory, then $V(\boldsymbol{x}(t))$ represents the corresponding values of V along the trajectory
- In order for $V(\boldsymbol{x}(t))$ not to increase, we require

$$\dot{V}\left(x\left(t\right)\right) \leq 0$$

The Lyapunov Derivative

- Use the chain rule to compute the derivative of $V(\boldsymbol{x}(t))$

$$\dot{V}(x(t)) = \nabla V(x)^T \dot{x}$$

- Use the plant model to obtain

$$\dot{V}(x(t)) = \nabla V(x)^T f(x)$$

- Recall

$$\nabla V(x) = \begin{bmatrix} \dfrac{\partial V}{\partial x_1} & \dfrac{\partial V}{\partial x_2} & \cdots & \dfrac{\partial V}{\partial x_2} \end{bmatrix}^T$$

Lyapunov Theorem for LTI Systems

The system $\mathrm{d}\boldsymbol{x}/\mathrm{d}t=\boldsymbol{A}\boldsymbol{x}$ is asymptotically stable, that is, the equilibrium state $\boldsymbol{x}_e=\mathbf{0}$ is asymptotically stable (a.s), if and only if any solution converges to $\boldsymbol{x}_e=\mathbf{0}$ as t tends to infinity for any initial $\boldsymbol{x}_0$

Lyapunov Theorem Interpretation

- View the vector $\boldsymbol{x}(t)$ as defining the coordinates of a point in an n-dimensional state space
- In an a.s. system the point $\boldsymbol{x}(t)$ converges to $\boldsymbol{x}_e=\mathbf{0}$

Lyapunov Theorem for n=2

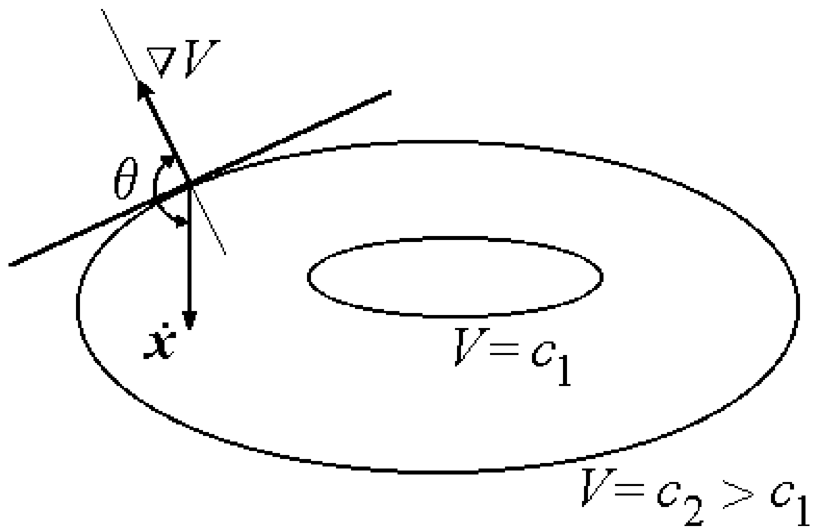
If a trajectory is converging to $\boldsymbol{x}_e=\mathbf{0}$, it should be possible to find a nested set of closed curves $V(x_1,x_2)=c$, $c\geq 0$, such that decreasing values of c yield level curves shrinking in on the equilibrium state $\boldsymbol{x}_e=\mathbf{0}$

Lyapunov Theorem and Level Curves

- The limiting level curve $V(x_1,x_2)=V(\mathbf{0})=0$ is 0 at the equilibrium state $\boldsymbol{x}_e=\mathbf{0}$
- The trajectory moves through the level curves by cutting them in the inward direction ultimately ending at $\boldsymbol{x}_e=\mathbf{0}$

The trajectory is moving in the direction of decreasing V

Note that $\dot{V} = \|\nabla V\| \|\dot{\boldsymbol{x}}\| \cos\theta < 0$



Level Sets

- The level curves can be thought of as contours of a cup-shaped surface

- For an a.s. system, that is, for an a.s. equilibrium state $\boldsymbol{x}_e = \mathbf{0}$, each trajectory falls to the bottom of the cup

Positive Definite Function---General Definition

The function V is positive definite in S, with respect to $\boldsymbol{x}_e$, if V has continuous partials, $V(\boldsymbol{x}_e)=0$, and $V(\boldsymbol{x})>0$ for all $\boldsymbol{x}$ in S, where $\boldsymbol{x} \neq \boldsymbol{x}_e$

Positive Definite Function With Respect to the Origin

Assume, for simplicity, $\boldsymbol{x}_e=\mathbf{0}$, then the function V is positive definite in S if V has continuous partials, $V(\mathbf{0})=0$, and $V(\boldsymbol{x})>0$ for all $\boldsymbol{x}$ in S, where $\boldsymbol{x}\neq\mathbf{0}$

Example: Positive Definite Function

Positive definite function of two variables

$$
\begin{aligned}
V(x_1, x_2) &= 2x_1^2 + 3x_2^2 \\
&= \left[\begin{array}{cc} x_1 & x_2 \end{array}\right] \left[\begin{array}{cc} 2 & 0 \\ 0 & 3 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \\
&= \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \\
&> 0 \quad \text{for all} \quad \boldsymbol{x} \neq \mathbf{0}
\end{aligned}
$$

Positive Semi-Definite Function--- General Definition

The function V is positive semi-definite in S, with respect to $\boldsymbol{x}_e$, if V has continuous partials, $V(\boldsymbol{x}_e)=0$, and $V(\boldsymbol{x})\geq 0$ for all $\boldsymbol{x}$ in S

Positive Semi-Definite Function With Respect to the Origin

Assume, for simplicity, $\boldsymbol{x}_e=\mathbf{0}$, then the function V is positive semi-definite in S if V has continuous partials, $V(\mathbf{0})=0$, and $V(\boldsymbol{x})\geq 0$ for all $\boldsymbol{x}$ in S

Example: Positive Semi-Definite Function

An example of positive semi-definite function of two variables

$$
\begin{aligned}
V(x_1, x_2) &= x_1^2 \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \\
&\geq 0 \quad \text{in} \quad R^2
\end{aligned}
$$

Quadratic Forms

- $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$
- If $\boldsymbol{P}$ not symmetric, need to symmetrize it
- First observe that because the transposition of a scalar equals itself, we have

$$\left(x^T \boldsymbol{P} x\right)^T = x^T \boldsymbol{P}^T x = x^T \boldsymbol{P} x$$

Symmetrizing Quadratic Form

- Perform manipulations

$$\begin{aligned} \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} &= \frac{1}{2}\boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \\ &= \frac{1}{2}\boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T \boldsymbol{P}^T \boldsymbol{x} \\ &= \boldsymbol{x}^T \left(\frac{\boldsymbol{P} + \boldsymbol{P}^T}{2} \right) \boldsymbol{x} \end{aligned}$$

- Note that

$$\left(\frac{\boldsymbol{P} + \boldsymbol{P}^T}{2} \right)^T = \frac{\boldsymbol{P} + \boldsymbol{P}^T}{2}$$

Tests for Positive and Positive Semi-Definiteness of Quadratic Form

- $V=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}$, is positive definite if and only if all eigenvalues of $\boldsymbol{P}$ are positive
- $V=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}$, is positive semi-definite if and only if all eigenvalues of $\boldsymbol{P}$ are non-negative

Comments on the Eigenvalue Tests

- These tests are only good for the case when $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}$. You must symmetrize $\boldsymbol{P}$ before applying the above tests
- Other tests, the Sylvester’s criteria, involve checking the signs of principal minors of $\boldsymbol{P}$

Negative Definite Quadratic Form

$V=\boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}$ is negative definite if and only if

$$-\boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}=\boldsymbol{x}^{\top}(-\boldsymbol{P})\boldsymbol{x}$$

is positive definite

Negative Semi-Definite Quadratic Form

$V=\boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}$ is negative semi-definite if and only if

$$-\boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}=\boldsymbol{x}^{\top}(-\boldsymbol{P})\boldsymbol{x}$$

is positive semi-definite

Example: Checking the Sign Definiteness of a Quadratic Form

- Is ***P***, equivalently, is the associated quadratic form, $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, pd, psd, nd, nsd, or neither?

$$\boldsymbol{P} = \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix}$$

- The associated quadratic form

$$V = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} = 2x_1^2 - 6x_1x_2 + 2x^2$$

Example: Symmetrizing the Underlying Matrix of the Quadratic Form

- Applying the eigenvalue test to the given quadratic form would seem to indicate that the quadratic form is pd, which turns out to be false
- Need to symmetrize the underlying matrix first and then can apply the eigenvalue test

Example: Symetrized Matrix

- Symmetrizing manipulations

$$
\begin{aligned}
\frac{1}{2}\left(\boldsymbol{P}+\boldsymbol{P}^{T}\right) &= \frac{1}{2}\left(\begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix}+\begin{bmatrix} 2 & 0 \\ -6 & 2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}
\end{aligned}
$$

- The eigenvalues of the symmetrized matrix are: 5 and -1
- The quadratic form is indefinite!

Example: Further Analysis

- Direct check that the quadratic form is indefinite
- Take ***x***=[1 0]^T. Then

$$x^T P x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 > 0$$

- Take ***x***=[1 1]^T. Then

$$x^T P x = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2 < 0$$

Stability Test for $\boldsymbol{x}_e=\mathbf{0}$ of d$\boldsymbol{x}$/dt=$\boldsymbol{A}\boldsymbol{x}$

- Let $V=\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ where $\boldsymbol{P}=\boldsymbol{P}^T>0$
- For V to be a Lyapunov function, that is, for $\boldsymbol{x}_e=\mathbf{0}$ to be a.s.,

$$\dot{V}(x(t)) < 0$$

- Evaluate the time derivative of V on the solution of the system d$\boldsymbol{x}$/dt=$\boldsymbol{A}\boldsymbol{x}$---Lyapunov derivative

Lyapunov Derivative for d$\boldsymbol{x}$/dt=$\boldsymbol{Ax}$

- Note that $V(\boldsymbol{x}(t))=\boldsymbol{x}(t)^{\mathsf{T}}\boldsymbol{P}\boldsymbol{x}(t)$
- Use the chain rule

$$
\begin{aligned}
\dot{V}(x(t)) &= \dot{x}^T \boldsymbol{P} x + x^T \boldsymbol{P} \dot{x} \\
&= x^T \boldsymbol{A}^T \boldsymbol{P} x + x^T \boldsymbol{P} \boldsymbol{A} x \\
&= x^T \left(\boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A} \right) x
\end{aligned}
$$

- We used $\dot{x}^T = x^T \boldsymbol{A}^T$

Lyapunov Matrix Equation

- Denote

$$\boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}=-\boldsymbol{Q}$$

- Then the Lyapunov derivative can be represented as

$$\dot{V}=\frac{d}{dt}V=-\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$$

where

$$\boldsymbol{Q}=\boldsymbol{Q}^T>0$$

Terms to Our Vocabulary

- Theorem---a major result of independent interest
- Lemma---an auxiliary result that is used as a stepping stone toward a theorem
- Corollary---a direct consequence of a theorem, or even a lemma

Lyapunov Theorem

The real matrix ***A*** is a.s., that is, all eigenvalues of ***A*** have negative real parts if and only if for any $Q = Q^T > 0$ the solution $\boldsymbol{P}$ of the continuous matrix Lyapunov equation

$$\boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A} = -Q$$

is (symmetric) positive definite

How Do We Use the Lyapunov Theorem?

- Select an arbitrary symmetric positive definite $\boldsymbol{Q}$, for example, an identity matrix, $\boldsymbol{I}_n$
- Solve the Lyapunov equation for $\boldsymbol{P}=\boldsymbol{P}^T$
- If $\boldsymbol{P}$ is positive definite, the matrix $\boldsymbol{A}$ is a.s. If $\boldsymbol{P}$ is not p.d. then $\boldsymbol{A}$ is not a.s.

How NOT to Use the Lyapunov Theorem

- It would be no use choosing ***P*** to be positive definite and then calculating ***Q***
- For unless ***Q*** turns out to be positive definite, nothing can be said about a.s. of ***A*** from the Lyapunov equation

Example: How NOT to Use the Lyapunov Theorem

- Consider an a.s. matrix

$$A = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$$

- Try

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Compute $\boldsymbol{Q} = -\left(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right)$

Example: Computing $\boldsymbol{Q}$

$$
\begin{aligned}
\boldsymbol{Q} &= -\left(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right) \\
&= -\left(\begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}
\end{aligned}
$$

The matrix $\boldsymbol{Q}$ is indefinite!--- recall the previous example

Solving the Continuous Matrix Lyapunov Equation Using MATLAB

- Use the MATLAB's command lyap
- Example:

$$A = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$$

- $\boldsymbol{Q}=\boldsymbol{I}_2$
- P=lyap(A,Q)

$$\boldsymbol{P} = \begin{bmatrix} 2.75 & 0.75 \\ 0.75 & 0.50 \end{bmatrix}$$

- Eigenvalues of ***P*** are positive: 0.2729 and 2.9771; ***P*** is positive definite

Limitations of the Lyapunov Method

- Usually, it is challenging to analyze the asymptotic stability of time-varying systems because it is very difficult to find Lyapunov functions with negative definite derivatives
- When can one conclude asymptotic stability when the Lyapunov derivative is only negative semi-definite?

Some Properties of Time-Varying Functions

- $\dot{f}(t) \to 0$ does not imply that *f*(*t*) has a limit as $t \to \infty$

- *f*(*t*) has a limit as $t \to \infty$ does not imply that $\dot{f}(t) \to 0$

More Properties of Time-Varying Functions

- If $f(t)$ is lower bounded and decreasing ($\dot{f}(t) \leq 0$), then it converges to a limit. (A well-known result from calculus.)

- But we do not know whether $\dot{f}(t) \to 0$ or not as $t \to \infty$

Preparation for Barbalat's Lemma

- Under what conditions

$$\dot{f}(t) \to 0 \quad \text{as} \quad t \to \infty$$

- We already know that the existence of the limit of *f*(*t*) as $t \to \infty$ is not enough for

$$\dot{f}(t) \to 0 \quad \text{as} \quad t \to \infty$$

Continuous Function

- A function $f(t)$ is continuous if small changes in t result in small changes in $f(t)$
- Intuitively, a continuous function is a function whose graph can be drawn without lifting the pencil from the paper

Continuity on an Interval

- Continuity is a *local* property of a function—that is, a function *f* is continuous, or not, at a particular point
- A function being continuous on an interval means only that it is continuous at each point of the interval

Uniform Continuity

- A function *f*(*t*) is uniformly continuous if it is continuous and, in addition, the *size* of the changes in *f*(*t*) depends only on the size of the changes in *t* but not on *t* itself
- The slope of an uniformly continuous function slope is bounded, that is, $\dot{f}$ is bounded
- Uniform continuity is a *global* property of a function

Properties of Uniformly Continuous Function

- Every uniformly continuous function is continuous, but the converse is not true
- A function is uniformly continuous, or not, on an entire interval
- A function may be continuous at each point of an interval without being uniformly continuous on the entire interval

Examples

- Uniformly continuous:

$$f(t) = \sin(t)$$

  Note that the slope of the above function is bounded

- Continuous, but not uniformly continuous on positive real numbers:

$$f(t) = 1/t$$

  Note that as t approaches 0, the changes in $f(t)$ grow beyond any bound

State of an a.s. System With Bounded Input is Bounded

- Example of Slotine and Li, "*Applied Nonlinear Control*," p. 124, Prentice Hall, 1991
- Consider an a.s. stable LTI system with bounded input

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

- The state ***x*** is bounded because ***u*** is bounded and ***A*** is a.s.

Output of a.s. System With Bounded Input is Uniformly Continuous

- Because ***x*** is bounded and u is bounded, $\dot{x}$ is bounded
- Derivative of the output equation is

$$\dot{y} = C\dot{x}$$

- The time derivative of the output is bounded
- Hence, ***y*** is uniformly continuous

Barbalat's Lemma

If *f*(*t*) has a finite limit as $t \to \infty$

and if $\dot{f}(t)$ is uniformly continuous

(or $\ddot{f}$ is bounded), then $\dot{f}(t) \to 0$

as $t \to \infty$

Lyapunov-Like Lemma

Given a real-valued function $W(t,\boldsymbol{x})$ such that

- $W(t,\boldsymbol{x})$ is bounded below
- $W(t,\boldsymbol{x})$ is negative semi-definite
- $\dot{W}(t,\boldsymbol{x})$ is uniformly continuous in t (or $\ddot{W}$ bounded) then

$$\dot{W}(t,\boldsymbol{x}) \to 0 \quad \text{as} \quad t \to \infty$$

Lyapunov-Like Lemma---Example; see p. 211 of the Text

- Interested in the stability of the origin of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 u \\ -x_1 u \end{bmatrix}$$

where *u* is bounded

- Consider the Lyapunov function candidate

$$V = \|\boldsymbol{x}\|^2 = x_1^2 + x_2^2$$

Stability Analysis of the System in the Example

- The Lyapunov derivative of V is

$$\begin{aligned}\dot{V} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2\\ &= 2x_1(-2x_1 + x_2u) - 2x_2x_1u\\ &= -4x_1^2\\ &\leq 0\end{aligned}$$

- The origin is stable; cannot say anything about asymptotic stability
- Stability implies that x_1 and x_2 are bounded

Example: Using the Lyapunov-Like Lemma

- We now show that

$$x_1(t) \to 0 \quad \text{as} \quad t \to \infty$$

- Note that $V = x_1^2 + x_2^2$ is bounded from below and non-increasing as $t \to \infty$
- Thus V has a limit as $t \to \infty$
- Need to show that $\dot{V}$ is uniformly continuous

Example: Uniform Continuity of $\dot{V}$

- Compute the derivative of $\dot{V}$ and check if it is bounded

$$\ddot{V} = -8x_1\dot{x}_1 = -8x_1(-2x_1 + x_2 u)$$

- The function $\dot{V}$ is uniformly continuous because $\ddot{V}$ is bounded
- Hence $\dot{V} \to 0$ as $t \to \infty$
- Therefore $x_1(t) \to 0$ as $t \to \infty$

Benefits of the Lyapunov Theory

- Solution to differential equation are not needed to infer about stability properties of equilibrium state of interest
- Barbalat's lemma complements the Lyapunov Theorem

☞ **Lyapunov functions** are useful in designing robust and adaptive controllers