Robot Dynamics

Introduction to Robotics
EECS 106B/206B
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Dynamics

- ***Dynamics***: relation between ***accelerations*** and ***forces***
- Important things to know to solve dynamics…
  - Mass and inertia of all components
  - Relevant forces and torques
    - Actuation, gravity, friction, interaction, etc.
- ***Lagrangian Dynamics***
  - An energy approach to finding equations of motion

Newtonian Dynamics

- ***Newton's Second Law*** for unconstrained point masses:

$$f = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

where $\dot{x} \in \mathbb{R}^3$ is the inertial velocity

- We cannot simply write

$$\tau = m\ddot{\theta}$$

- Why?
  - Mechanism is not a point mass
  - Configuration space is not Euclidean or inertial

Basic Formulation

- Instead of a single point mass or a set of independent point masses, we are interested in rigid bodies
- ***Rigid bodies*** are composed of points that are interconnected!
  - Position (***holonomic***) constraints
- ***Generalized coordinates*** (q) describe the configuration of a system using fewer variables
  - Manipulator: joint angles (θ)

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Dynamics Using Generalized Coordinates

- Dynamics for systems with generalized coordinates (q)
  - Newton's Law does not apply
  - Use the ***Euler-Lagrange equations*** instead

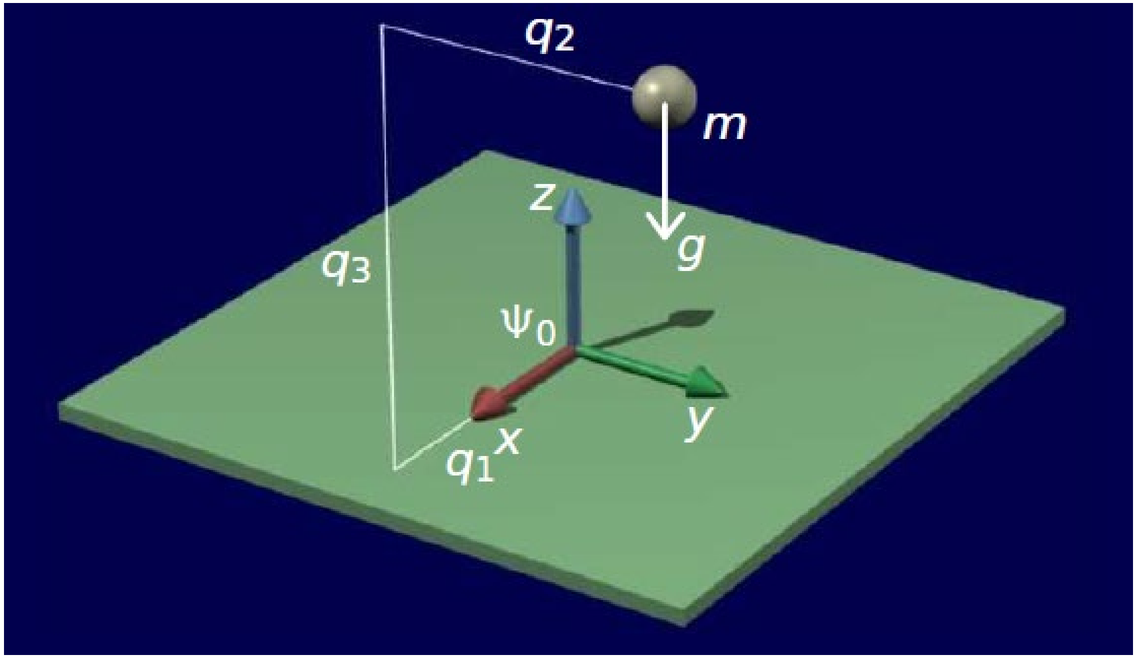
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \Upsilon$$

  - Where $L(q, \dot{q})$ is the ***Lagrangian*** given by

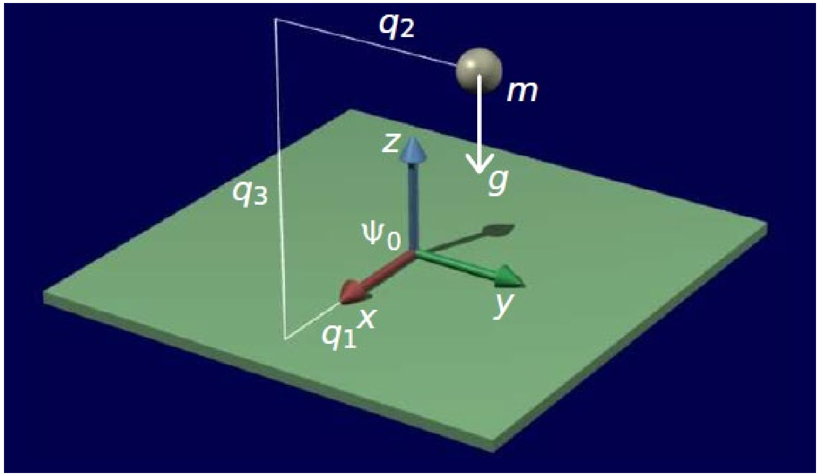
$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

  - And Υ is the generalized force associated with $\dot{q}$

Example: Point Mass



Example: Point Mass



- Kinetic energy: $T(q,\dot{q}) = \frac{1}{2} m \dot{q}^T \dot{q}$
- Potential energy: $V(q) = mgq_3$
- Lagrangian: $L(q,\dot{q}) = T(q,\dot{q}) - V(q)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = m\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = 0$$

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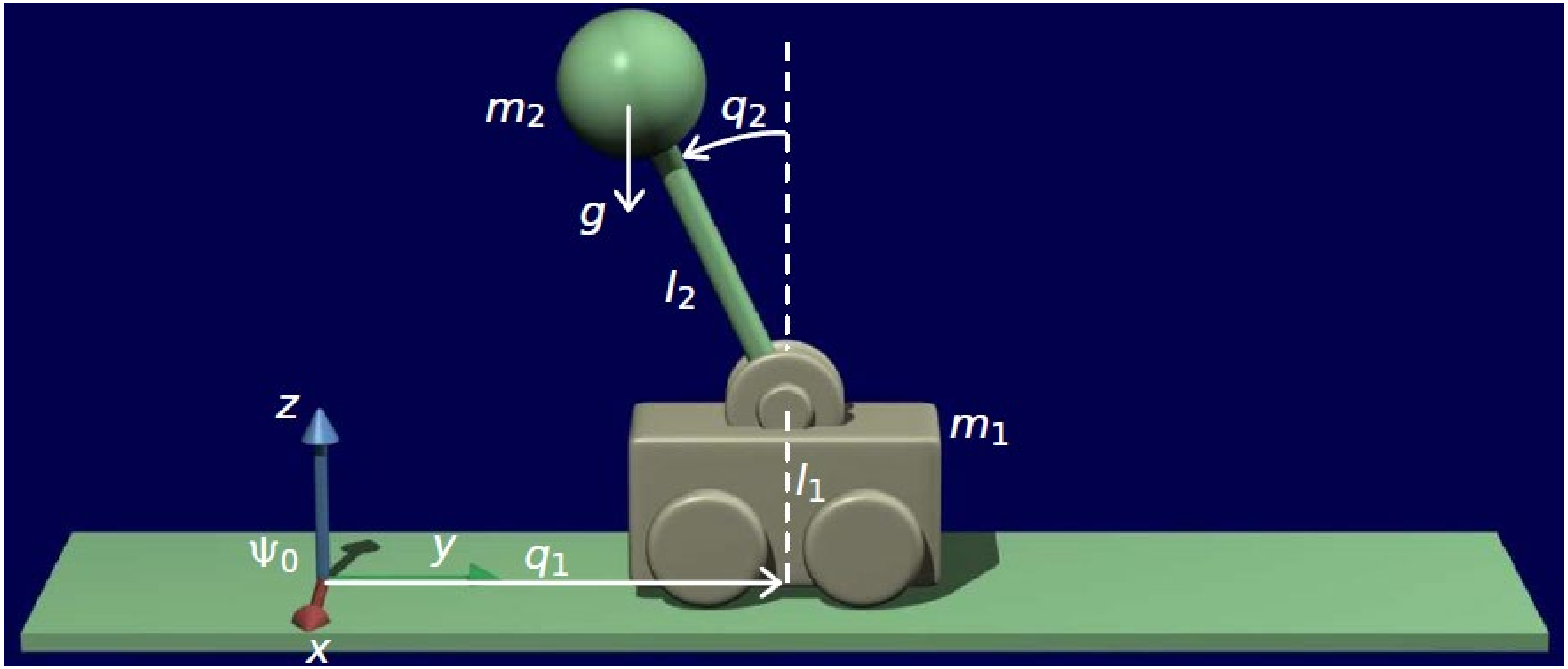
Example: Pendulum Cart



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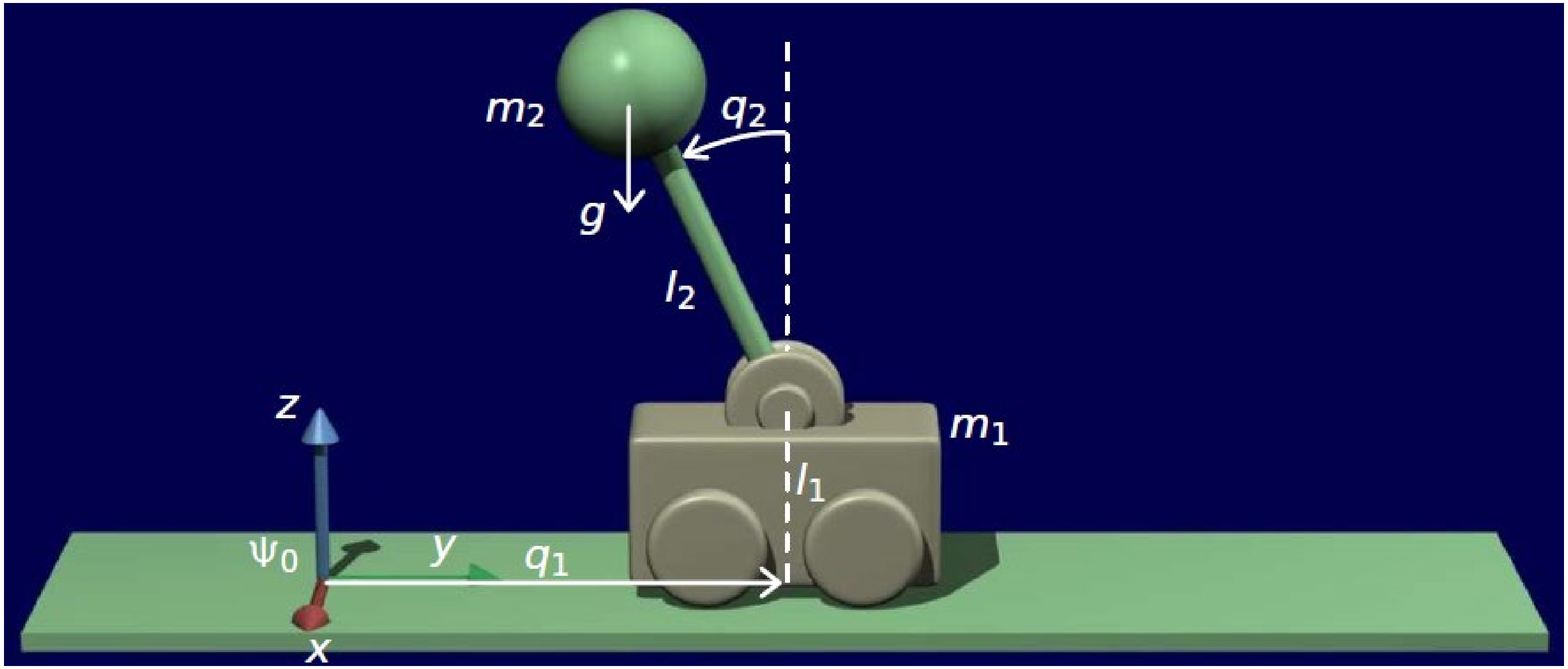
Example: Pendulum Cart



- Kinetic energy:

$$T(q,\dot{q}) = \frac{1}{2}m_1\dot{q}_1{}^2 + \frac{1}{2}m_2(\dot{q}_1{}^2 + l_2{}^2\dot{q}_2{}^2 - 2l_2\cos(q_2)\dot{q}_1\dot{q}_2)$$

- Potential energy: $V(q) = m_2 g(l_1 + l_2\cos(q_2))$

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Applying to Mechanisms…

- In order to apply these techniques to robotic mechanisms, we need:
  - ***Generalized coordinates*** q: choose joint angles θ
  - ***Kinetic energy***: energies due to motion of the links
  - ***Potential energy***: e.g. gravity on links

Center of Mass (CoM)

- ***Center of mass***
  - Balancing point on a rigid body
  - If suspended from that point, body will tend to not rotate
- Compute CoM position as a weighted average

$$\begin{bmatrix} x_{CoM} \\ y_{CoM} \\ z_{CoM} \end{bmatrix} = \int_V \rho(r)\, r\, dV = \frac{1}{m} \iiint\limits_{x,y,z} \rho(x,y,z) \begin{bmatrix} x \\ y \\ z \end{bmatrix} dx dy dz$$

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Mass vs. Inertia

- Translation energy vs. rotation energy
  - Point mass m with $\dot{x} = v$: $T = \frac{1}{2} m v^2$
  - Point mass m with $\dot{x} = \omega r$: $T = \frac{1}{2} m(\omega r)^2 = \frac{1}{2}(m r^2)\omega^2$
- Both energies are the same kinetic energy, just expressed in different coordinates!
- Inertia
  - ***Inertia*** J of a mass m: 'angular weight' s.t. $T = \frac{1}{2} J \omega^2$
  - $\omega \in \mathbb{R}$: angular velocity around a specific axis
  - Inertia depends on axis direction and location

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Kinetic Energy of Rigid Body

- Velocity of a point (CoM) in an inertial frame given by

$$\dot{p} + \dot{R}r$$

where $r \in \mathbb{R}^3$ is the point coordinates in body frame

- Then we can express the kinetic energy using a volume integral in the following form

$$T = \frac{1}{2}\int_V \rho(r)\left\|\dot{p} + \dot{R}r\right\|^2 dV$$

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Kinetic Energy of Rigid Body

- Expanding the product and simplifying (see MLS 4.2)

$$T = \underbrace{\frac{1}{2} m \|\dot{p}\|^2}_{\textit{translational}} + \underbrace{\frac{1}{2} \omega^T \boldsymbol{I} \omega}_{\textit{rotational}}$$

where $\boldsymbol{I}$ is the ***inertia tensor*** expressed in the body frame such that

$$\boldsymbol{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Kinetic Energy of Rigid Body

- Represented in terms of a body velocity V^b we have the following equation

$$T = \frac{1}{2}\left(V^b\right)^T \begin{bmatrix} mI & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} V^b = \frac{1}{2}\left(V^b\right)^T \boldsymbol{\mathcal{M}} V^b$$

where $\boldsymbol{\mathcal{M}}$ is the ***generalized inertia matrix*** expressed in the body frame

Newton-Euler Equation

- A Lagrangian approach to dynamics can be limited by the ***local*** parametrization of a given configuration
  - E.g. Euler angles representing orientation

- ***Newton-Euler Equations*** globally characterize the dynamics of a rigid body subject to external forces and torques (derivation in MLS 4.2)

$$\begin{bmatrix} mI & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \dot{v}_b \\ \dot{\omega}_b \end{bmatrix} + \begin{bmatrix} \omega_b \times m v_b \\ \omega_b \times \boldsymbol{I} \omega_b \end{bmatrix} = F^b$$

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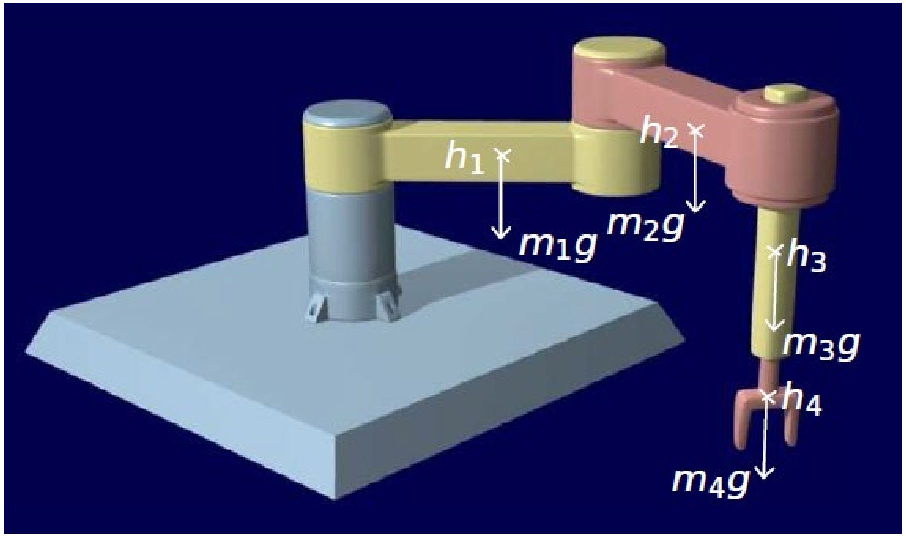
Potential Energy of a Manipulator

- Consider the following manipulator



- The potential energy can be summed for each link using the center of mass of each link

$$V(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta)$$

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Kinetic Energy of a Manipulator

- Let's first look at the kinetic energy of a body

$$T = \frac{1}{2}\left(V^b\right)^T \boldsymbol{\mathcal{M}} V^b$$

- This can be expressed in terms of joint velocities $\dot{\theta}$ using what we know about Jacobians

$$V^b = J^b(\theta)\dot{\theta}$$

$$T(\theta) = \frac{1}{2}\dot{\theta}^T J^b(\theta)^T \boldsymbol{\mathcal{M}} J^b(\theta)\dot{\theta}$$

Kinetic Energy of a Manipulator

- The kinetic energy of all bodies summed is now

$$T = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}$$

where is the ***manipulator inertia matrix*** defined as

$$M(\theta) = \sum_{i=1}^{n} J_i^b(\theta)^T \boldsymbol{\mathcal{M}}_i J_i^b(\theta)$$

  - $M(\theta) \in \mathbb{R}^{n\times n}$ is symmetric, positive definite

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Equations of Motion for an Open-Chain Manipulator

- We can now put things together into the Lagrangian

$$L(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} - V(\theta)$$

- When substituted into the Euler-Lagrange equation and simplified, we get the following form

$$\boxed{M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) = \tau}$$

$$N(\theta) = \frac{\partial V}{\partial \theta} \qquad C_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \Gamma_{ijk}\dot{\theta}_k = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ki}}{\partial \theta_j} - \frac{\partial M_{jk}}{\partial \theta_i}\right)\dot{\theta}_k$$

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Equations of Motion for an Open-Chain Manipulator

$$\boxed{M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) = \tau}$$

- Multiply by M^{-1} to find $\ddot{\theta}$ as a function of τ

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ -M^{-1}C\dot{\theta} - M^{-1}N\end{bmatrix} + \begin{bmatrix}0 \\ M^{-1}\end{bmatrix}\tau$$

- We now have a first order ODE in the form

$$\dot{x} = f(x) + g(x)u$$

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