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Lecture Notes for A Geometrical Introduction to Robotics and Manipulation

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4.1 Introduction

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Definition: Dynamics

Physical laws governing the motions of bodies and aggregates of bodies.

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Definition: Dynamics

Physical laws governing the motions of bodies and aggregates of bodies.

□ A short history:



“Everything happens for a reason.”

Aristotle (384 BC - 322 BC)

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Definition: Dynamics

Physical laws governing the motions of bodies and aggregates of bodies.

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Experiments with cannon balls from the tower of Pisa.

G. Galilei (1564 - 1642)

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G. Galilei (1564 - 1642)



Laws of motion.

I. Newton (1642 - 1726)

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Laws of motion from particles to rigid bodies.

L. Euler (1707 - 1783)

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Laws of motion from particles to rigid bodies.

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Calculus of Variation and the Principles of least action.

J. Lagrange (1736 - 1813)

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Laws of motion from particles to rigid bodies.

L. Euler (1707 - 1783)



Calculus of Variation and the Principles of least action.

J. Lagrange (1736 - 1813)



Quaternions and Hamilton's Principle.

W. Hamilton (1805 - 1865)

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4.2 Lagrange's Equations

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4.2 Lagrange Equations

□ A simple example:

Newton's Equation:

$$m\ddot{x} = F_x$$

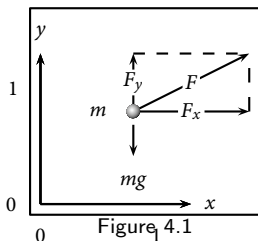
$$m\ddot{y} = F_y - mg$$

Lagrangian Equation:

Momentum: $P_x = m\dot{x}$

$$P_y = m\dot{y}$$

$$\frac{d}{dt}P_x = F_x, \quad \frac{d}{dt}P_y = F_y - mg$$



4.2 Lagrange Equations

□ A simple example:

Newton's Equation:

$$m\ddot{x} = F_x$$

$$m\ddot{y} = F_y - mg$$

Momentum: $P_x = m\dot{x}$

$$P_y = m\dot{y}$$

$$\frac{d}{dt}P_x = F_x, \frac{d}{dt}P_y = F_y - mg$$

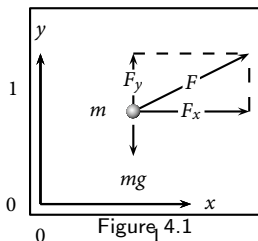


Figure 4.1

Lagrangian Equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F_x$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = F_y$$

Lagrangian function:

$$L = T - V, P_x = \frac{\partial L}{\partial \dot{x}}, P_y = \frac{\partial L}{\partial \dot{y}}$$

Kinetic energy:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Potential energy:

$$V = mgy$$

4.2 Lagrange's Equations

□ Generalization to multibody systems:

$q_i, i = 1, \dots, n$: generalized coordinates

Kinetic energy:

$$T = T(q, \dot{q})$$

Potential energy:

$$V = V(q)$$

Lagrangian:

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

$\tau_i, i = 1, \dots, n$: external force on q_i

Lagrangian Equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i, i = 1, \dots, n$$

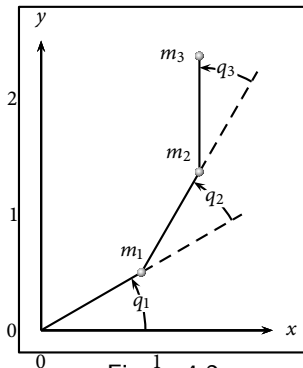


Figure 4.2

4.2 Lagrange's Equations

◇ Example: Pendulum equation

Generalized coordinate:

$$\theta \in S^1$$

Kinematics:

$$x = l \sin \theta, y = -l \cos \theta$$

$$\dot{x} = l \cos \theta \cdot \dot{\theta}, \dot{y} = l \sin \theta \cdot \dot{\theta}$$

Kinetic energy:

$$T(\theta, \dot{\theta}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m l^2 \dot{\theta}^2$$

Potential energy:

$$V = mgl(1 - \cos \theta)$$

Lagrangian function:

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos \theta), \Rightarrow \frac{\partial L}{\partial \theta} = m l^2 \dot{\theta}, \frac{\partial L}{\partial \dot{\theta}} = -mgl \sin \theta$$

Equation of motion:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \tau \Rightarrow m l^2 \ddot{\theta} + mgl \sin \theta = \tau$$

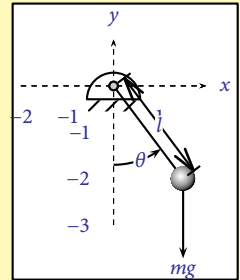


Figure 4.3

4.2 Lagrange's Equations

◇ Example: A spherical pendulum

Generalized coordinate:

$$r(\theta, \phi) = \begin{bmatrix} l \sin \theta \cos \phi \\ l \sin \theta \sin \phi \\ -l \cos \theta \end{bmatrix}$$

Kinetic energy:

$$T = \frac{1}{2} m \|\dot{r}\|^2 = \frac{1}{2} m l^2 (\dot{\theta}^2 + (1 - \cos^2 \theta) \dot{\phi}^2)$$

Potential energy:

$$V = -mgl \cos \theta$$

Lagrangian function:

$$L(q, \dot{q}) = \frac{1}{2} m l^2 (\dot{\theta}^2 + (1 - \cos^2 \theta) \dot{\phi}^2) + mgl \cos \theta$$

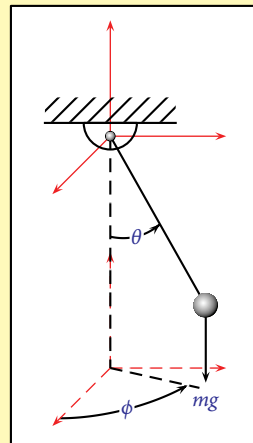


Figure 4.4

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4.2 Lagrange's Equations

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$$\begin{cases} \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt} (ml^2 \dot{\theta}) = ml^2 \ddot{\theta}, \\ \frac{\partial L}{\partial \theta} = ml^2 \sin \theta \cos \theta \dot{\phi}^2 - mgl \sin \theta \end{cases}$$

$$\begin{cases} \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt} (ml^2 \sin^2 \theta \dot{\phi}) = ml^2 \sin^2 \theta \ddot{\phi} + 2ml^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi}, \\ \frac{\partial L}{\partial \phi} = 0 \end{cases}$$

$$\begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 \sin^2 \theta \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} -ml^2 s_{\theta} c_{\theta} \dot{\phi}^2 \\ 2ml^2 s_{\theta} c_{\theta} \dot{\theta} \dot{\phi} \end{bmatrix} + \begin{bmatrix} mgl s_{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



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□ Kinetic energy of a rigid body:

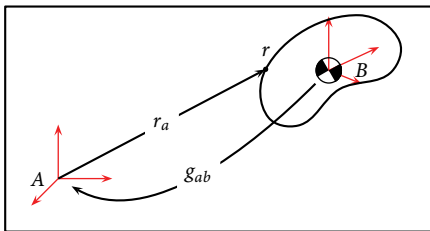


Figure 4.5

Volume occupied by the body:	V
Mass density:	$\rho(r)$
Mass:	$m = \int_V \rho(r) dV$
Mass center:	$\bar{r} \triangleq \frac{1}{m} \int_V \rho(r) r dV$
Relative to frame at the mass center:	$\bar{r} = 0$

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4.2 Lagrange's Equations

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Kinetic energy (In A-frame):

$$\begin{aligned}
 T &= \frac{1}{2} \int_V \rho(r) \|\dot{p} + \dot{R}r\|^2 dV = \frac{1}{2} \int_V \rho(r) (\|\dot{p}\|^2 + 2\dot{p}^T \dot{R}r + \|\dot{R}r\|^2) dV \\
 &= \frac{1}{2} m \|\dot{p}\|^2 + \underbrace{\dot{p}^T \dot{R} \int_V \rho(r) r dV}_{=0} + \frac{1}{2} \int_V \rho(r) \|\dot{R}r\|^2 dV
 \end{aligned}$$

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Kinetic energy (In A-frame):

$$\begin{aligned}
 T &= \frac{1}{2} \int_V \rho(r) \|\dot{p} + \dot{R}r\|^2 dV = \frac{1}{2} \int_V \rho(r) (\|\dot{p}\|^2 + 2\dot{p}^T \dot{R}r + \|\dot{R}r\|^2) dV \\
 &= \frac{1}{2} m \|\dot{p}\|^2 + \underbrace{\dot{p}^T \dot{R} \int_V \rho(r) r dV}_{=0} + \boxed{\frac{1}{2} \int_V \rho(r) \|\dot{R}r\|^2 dV}
 \end{aligned}$$

$$\begin{aligned}
 &\downarrow \\
 &\frac{1}{2} \int_V \rho(r) \|\dot{R}r\|^2 dV \\
 &= \frac{1}{2} \int \rho(r) \|R^T \dot{R}r\|^2 dV = \frac{1}{2} \int \rho(r) \|\hat{\omega}r\|^2 dV = \frac{1}{2} \int \rho(r) \|\hat{r}\omega\|^2 dV \\
 &= \frac{1}{2} \int \rho(r) (-\omega^T \hat{r}^2 \omega) dV = \frac{1}{2} \omega^T \left(- \int \rho(r) \hat{r}^2 dV \right) \omega \triangleq \frac{1}{2} \omega^T \mathcal{I} \omega
 \end{aligned}$$

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where

$$\mathcal{I} = - \int \rho(r) \hat{r}^2 dV \triangleq \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}$$

is the *Inertia tensor* with

$$\begin{aligned} \mathcal{I}_{xx} &= \int \rho(r)(y^2 + z^2) dx dy dz, & \mathcal{I}_{xy} &= - \int \rho(r) xy dx dy dz \\ T &= \frac{1}{2} m \|\dot{p}\|^2 + \frac{1}{2} (\omega^b)^T \mathcal{I} \omega^b = \frac{1}{2} m \|R^T \dot{p}\|^2 + \frac{1}{2} (\omega^b)^T \mathcal{I} \omega^b \\ &= \frac{1}{2} (V^b)^T \underbrace{\begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix}}_{M^b} V^b \end{aligned}$$

M^b : Generalized inertia matrix in B -frame.



4.2 Lagrange's Equations

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◇ Example: M^b for a rectangular object

$$\rho = \frac{m}{l\omega h}$$

$$\mathcal{I}_{xx} = \int_V \rho(y^2 + z^2) dx dy dz$$

$$= \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} (y^2 + z^2) dx dy dz$$

$$= \rho \left(\frac{1}{12} (l\omega^3 h + l\omega h^3) \right) = \frac{m}{12} (\omega^2 + h^2)$$

$$\mathcal{I}_{xy} = - \int_V \rho xy dV = -\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} xy dx dy dz$$

$$= -\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{y}{2} x^2 \Big|_{-\frac{l}{2}}^{\frac{l}{2}} dy dz = 0$$

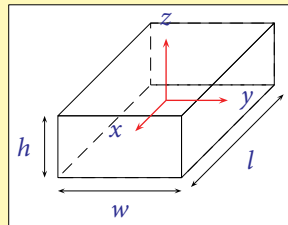


Figure 4.6

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$$\mathcal{I} = \begin{bmatrix} \frac{m}{12}(w^2 + h^2) & 0 & 0 \\ 0 & \frac{m}{12}(l^2 + h^2) & 0 \\ 0 & 0 & \frac{m}{12}(w^2 + l^2) \end{bmatrix}, M^b = \begin{bmatrix} mI_{3 \times 3} & 0 \\ 0 & \mathcal{I} \end{bmatrix}$$

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$$\mathcal{I} = \begin{bmatrix} \frac{m}{12}(w^2 + h^2) & 0 & 0 \\ 0 & \frac{m}{12}(l^2 + h^2) & 0 \\ 0 & 0 & \frac{m}{12}(w^2 + l^2) \end{bmatrix}, M^b = \begin{bmatrix} mI_{3 \times 3} & 0 \\ 0 & \mathcal{I} \end{bmatrix}$$

□ M^b under change of frames:

$$\hat{V}_1 = g_1^{-1} \cdot \dot{g}_1, \quad T = \frac{1}{2} V_1^T M_1^b V_1$$

$$V_1 = \text{Ad}_{g_0} V_2$$

$$\begin{aligned} T &= \frac{1}{2} (\text{Ad}_{g_0} V_2)^T M_1^b (\text{Ad}_{g_0} V_2) \\ &= \frac{1}{2} V_2^T \text{Ad}_{g_0}^T M_1^b \text{Ad}_{g_0} V_2 \triangleq \frac{1}{2} V_2^T M_2^b V_2 \end{aligned}$$

$$M_2^b = \text{Ad}_{g_0}^T M_1^b \text{Ad}_{g_0}$$

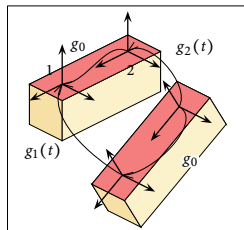


Figure 4.7

4.2 Lagrange's Equations

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◇ Example: Dynamics of a 2-dof planar robot

$$\mathcal{I}_i = \begin{bmatrix} \mathcal{I}_{xx_i} & 0 & 0 \\ 0 & \mathcal{I}_{yy_i} & 0 \\ 0 & 0 & \mathcal{I}_{zz_i} \end{bmatrix}, i = 1, 2$$

$$T(\theta, \dot{\theta}) = \frac{1}{2} m_1 \|v_1\|^2 + \frac{1}{2} \omega_1^T \mathcal{I}_1 \omega_1 \\ + \frac{1}{2} m_2 \|v_2\|^2 + \frac{1}{2} \omega_2^T \mathcal{I}_2 \omega_2$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \quad \omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

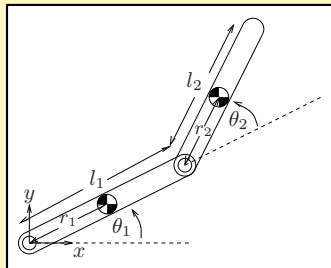


Figure 4.8

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4.2 Lagrange's Equations

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$$P_i = \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ 0 \end{bmatrix} : \text{Mass center}$$

γ_i : Distance from joint i to mass center

Change of Coordinates:

$$\begin{cases} \bar{x}_1 = r_1 c_1 \\ \bar{y}_1 = r_1 s_1 \end{cases} \Rightarrow \begin{cases} \dot{\bar{x}}_1 = -r_1 s_1 \dot{\theta}_1 \\ \dot{\bar{y}}_1 = r_1 c_1 \dot{\theta}_1 \end{cases}$$

$$\begin{cases} \bar{x}_2 = l_1 c_1 + r_2 c_{12} \\ \bar{y}_2 = l_1 s_1 + r_2 s_{12} \end{cases} \Rightarrow \begin{cases} \dot{\bar{x}}_2 = -(l_1 s_1 + r_2 s_{12}) \dot{\theta}_1 - r_2 s_{12} \dot{\theta}_2 \\ \dot{\bar{y}}_2 = (l_1 c_1 + r_2 c_{12}) \dot{\theta}_1 + r_2 c_{12} \dot{\theta}_2 \end{cases}$$

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4.2 Lagrange's Equations

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Kinetic energy:

$$\begin{aligned}
 T(\theta, \dot{\theta}) &= \frac{1}{2}m_1(\dot{\tilde{x}}_1^2 + \dot{\tilde{y}}_1^2) + \frac{1}{2}\mathcal{I}_{z_1}\dot{\theta}_1^2 + \frac{1}{2}m_2(\dot{\tilde{x}}_2^2 + \dot{\tilde{y}}_2^2) + \frac{1}{2}\mathcal{I}_{z_2}(\dot{\theta}_1^2 + \dot{\theta}_2^2) \\
 &= \frac{1}{2}[\dot{\theta}_1 \ \dot{\theta}_2] \underbrace{\begin{bmatrix} \alpha + 2\beta c_2 & \delta + \beta c_2 \\ \delta + \beta c_2 & \delta \end{bmatrix}}_{M(\theta)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
 \end{aligned}$$

$$\alpha = \mathcal{I}_{z_1} + \mathcal{I}_{z_2} + m_1 r_1^2 + m_2 (l_1^2 + r_2^2),$$

$$\beta = m_2 l_1 r_2, \delta = \mathcal{I}_{z_2} + m_2 r_2^2, L = T$$

Equation of motion:

$$M(\theta) \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -\beta s_2 \dot{\theta}_2 & -\beta s_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ \beta s_2 \dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

† End of Section †

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□ Dynamics of an open-chain manipulator:

Definition:

L_i : frame at mass center of link i , $g_{sl_i}(\theta) = e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_i \theta_i} g_{sl_i}(o)$

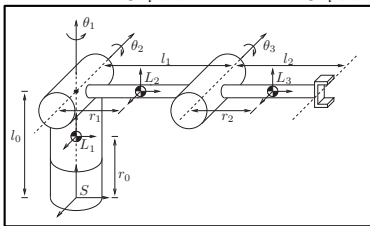


Figure 4.9

$$V_{sl_i}^b = J_{sl_i}^b(\theta) \dot{\theta} = [\xi_1^\dagger \ \xi_2^\dagger \ \dots \ \xi_i^\dagger \ 0 \ \dots \ 0] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_i \\ \dot{\theta}_{i+1} \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = J_i(\theta) \dot{\theta}$$

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$$\xi_j^\dagger = \text{Ad}^{-1}(e^{\hat{\xi}_{j+1}\theta_{j+1}} \dots e^{\hat{\xi}_i\theta_i} g_{sl_i}(0)) \xi_j, \quad j \leq i$$

$$T_i(\theta, \dot{\theta}) = \frac{1}{2} (V_{sl_i}^b)^T M_i^b V_{sl_i}^b = \frac{1}{2} \dot{\theta}^T J_i^T(\theta) M_i^b J_i(\theta) \dot{\theta}$$

$$T(\theta) = \sum_{i=1}^n T_i(\theta, \dot{\theta}) = \frac{1}{2} \dot{\theta}^T M(\theta) \dot{\theta},$$

$$M(\theta) = \sum_i J_i^T(\theta) M_i^b J_i(\theta) = \frac{1}{2} \sum_{i,j=1}^n M_{ij}(\theta) \dot{\theta}_i \dot{\theta}_j$$

$$h_i(\theta): \text{Height of } L_i, \quad V_i(\theta) = m_i g h_i(\theta), \quad V(\theta) = \sum_{i=1} m_i g h_i(\theta)$$

Lagrange's Equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i, \quad i = 1, \dots, n,$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} = \frac{d}{dt} \left(\sum_{j=1}^n M_{ij} \dot{\theta}_j \right) = \sum_{j=1}^n M_{ij} \ddot{\theta}_j + \dot{M}_{ij} \dot{\theta}_j$$

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$$\begin{aligned}
 \frac{\partial L}{\partial \theta_i} &= \frac{1}{2} \sum_{j,k=1}^n \frac{\partial M_{kj}}{\partial \theta_i} \dot{\theta}_k \dot{\theta}_j - \frac{\partial V}{\partial \theta_i}, & \dot{M}_{ij} &= \sum_k \frac{\partial M_{ij}}{\partial \theta_k} \dot{\theta}_k \\
 &\Rightarrow \sum_{j=1}^n M_{ij} \ddot{\theta}_j + \sum_{j,k=1}^n \left(\frac{\partial M_{ij}}{\partial \theta_k} \dot{\theta}_j \dot{\theta}_k - \frac{1}{2} \frac{\partial M_{kj}}{\partial \theta_i} \dot{\theta}_k \dot{\theta}_j \right) + \frac{\partial V}{\partial \theta_i} = \tau_i \\
 &\Rightarrow \sum_{j=1}^n M_{ij} \ddot{\theta}_j + \sum_{j,k=1}^n \Gamma_{ijk} \dot{\theta}_k \dot{\theta}_j + \frac{\partial V}{\partial \theta_i} = \tau_i \\
 \Gamma_{ijk} &\triangleq \frac{1}{2} \left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right)
 \end{aligned}$$

$\dot{\theta}_i \cdot \dot{\theta}_j, i \neq j$: Coriolis term, $\dot{\theta}_i^2$: Centrifugal term

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$$\begin{aligned}
 \frac{\partial L}{\partial \theta_i} &= \frac{1}{2} \sum_{j,k=1}^n \frac{\partial M_{kj}}{\partial \theta_i} \dot{\theta}_k \dot{\theta}_j - \frac{\partial V}{\partial \theta_i}, & \dot{M}_{ij} &= \sum_k \frac{\partial M_{ij}}{\partial \theta_k} \dot{\theta}_k \\
 &\Rightarrow \sum_{j=1}^n M_{ij} \ddot{\theta}_j + \sum_{j,k=1}^n \left(\frac{\partial M_{ij}}{\partial \theta_k} \dot{\theta}_j \dot{\theta}_k - \frac{1}{2} \frac{\partial M_{kj}}{\partial \theta_i} \dot{\theta}_k \dot{\theta}_j \right) + \frac{\partial V}{\partial \theta_i} = \tau_i \\
 &\Rightarrow \sum_{j=1}^n M_{ij} \ddot{\theta}_j + \sum_{j,k=1}^n \Gamma_{ijk} \dot{\theta}_k \dot{\theta}_j + \frac{\partial V}{\partial \theta_i} = \tau_i \\
 \Gamma_{ijk} &\triangleq \frac{1}{2} \left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right)
 \end{aligned}$$

$\dot{\theta}_i \cdot \dot{\theta}_j, i \neq j$: Coriolis term, $\dot{\theta}_i^2$: Centrifugal term

Define: $C_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^n \Gamma_{ijk} \dot{\theta}_k = \frac{1}{2} \sum_{k=1}^n \left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right) \dot{\theta}_k$

$$\Rightarrow \boxed{M(\theta) \ddot{\theta} + C(\theta, \dot{\theta}) \dot{\theta} + N(\theta) = \tau}$$

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Property 1:

- 1 $M(\theta) = M^T(\theta), \dot{\theta}^T M(\theta) \dot{\theta} \geq 0, \dot{\theta}^T M(\theta) \dot{\theta} = 0 \Leftrightarrow \dot{\theta} = 0$
- 2 $\dot{M} - 2C \in \mathbb{R}^{n \times n}$ is skew symmetric

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Property 1:

- 1 $M(\theta) = M^T(\theta)$, $\dot{\theta}^T M(\theta) \dot{\theta} \geq 0$, $\dot{\theta}^T M(\theta) \dot{\theta} = 0 \Leftrightarrow \dot{\theta} = 0$
- 2 $\dot{M} - 2C \in \mathbb{R}^{n \times n}$ is skew symmetric

Proof :

$$\begin{aligned}
 (\dot{M} - 2C)_{ij} &= \dot{M}_{ij} - 2C_{ij}(\theta) \\
 &= \sum_{k=1}^n \frac{\partial M_{ij}}{\partial \theta_k} \dot{\theta}_k - \frac{\partial M_{ij}}{\partial \theta_k} \dot{\theta}_k - \frac{\partial M_{ik}}{\partial \theta_j} \dot{\theta}_k + \frac{\partial M_{kj}}{\partial \theta_i} \dot{\theta}_k \\
 &= \sum_{k=1}^n \frac{\partial M_{kj}}{\partial \theta_i} \dot{\theta}_k - \frac{\partial M_{ik}}{\partial \theta_j} \dot{\theta}_k
 \end{aligned}$$

Switching i and j shows $(\dot{M} - 2C)^T = -(\dot{M} - 2C)$



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◇ Example: Planar 2-DoF Robot (continued)

$$m_{11}(\theta) = \alpha + 2\beta \cos \theta_2, m_{22} = \delta$$

$$m_{12}(\theta) = m_{21}(\theta) = \delta + \beta \cos \theta_2$$

$$c_{11}(\theta, \dot{\theta}) = -\beta \sin \theta_2 \cdot \dot{\theta}_2, c_{12}(\theta, \dot{\theta}) = -\beta \sin \theta_2 (\dot{\theta}_1 + \dot{\theta}_2)$$

$$c_{21}(\theta, \dot{\theta}) = \beta \sin \theta_2 \cdot \dot{\theta}_1, c_{22}(\theta, \dot{\theta}) = 0$$

$$\Gamma_{111} = \frac{1}{2} \left(\frac{\partial M_{11}}{\partial \theta_1} + \frac{\partial M_{11}}{\partial \theta_1} - \frac{\partial M_{11}}{\partial \theta_1} \right) = \frac{1}{2} \frac{\partial M_{11}}{\partial \theta_1} = 0$$

$$\Gamma_{112} = \frac{1}{2} \left(\frac{\partial M_{11}}{\partial \theta_2} + \frac{\partial M_{12}}{\partial \theta_1} - \frac{\partial M_{21}}{\partial \theta_1} \right) = \frac{1}{2} \frac{\partial M_{11}}{\partial \theta_2} = -\beta \sin \theta_2$$

$$\Gamma_{121} = \frac{1}{2} \left(\frac{\partial M_{12}}{\partial \theta_1} + \frac{\partial M_{11}}{\partial \theta_2} - \frac{\partial M_{12}}{\partial \theta_1} \right) = \frac{1}{2} \frac{\partial M_{11}}{\partial \theta_2} = -\beta \sin \theta_2$$

$$\Gamma_{122} = \frac{1}{2} \left(\frac{\partial M_{12}}{\partial \theta_2} + \frac{\partial M_{12}}{\partial \theta_2} - \frac{\partial M_{22}}{\partial \theta_1} \right) = \frac{\partial M_{12}}{\partial \theta_2} - \frac{1}{2} \frac{\partial M_{22}}{\partial \theta_1} = -\beta \sin \theta_2$$

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$$\Gamma_{211} = \frac{1}{2} \left(\frac{\partial M_{21}}{\partial \theta_1} + \frac{\partial M_{21}}{\partial \theta_1} - \frac{\partial M_{11}}{\partial \theta_2} \right) = \frac{\partial M_{21}}{\partial \theta_1} - \frac{1}{2} \frac{\partial M_{11}}{\partial \theta_2} = \beta \sin \theta_2$$

$$\Gamma_{212} = \frac{1}{2} \left(\frac{\partial M_{21}}{\partial \theta_2} + \frac{\partial M_{22}}{\partial \theta_1} - \frac{\partial M_{21}}{\partial \theta_2} \right) = \frac{1}{2} \frac{\partial M_{22}}{\partial \theta_1} = 0$$

$$\Gamma_{221} = \frac{1}{2} \left(\frac{\partial M_{22}}{\partial \theta_1} + \frac{\partial M_{21}}{\partial \theta_2} - \frac{\partial M_{12}}{\partial \theta_2} \right) = \frac{1}{2} \frac{\partial M_{22}}{\partial \theta_1} = 0$$

$$\Gamma_{222} = \frac{1}{2} \left(\frac{\partial M_{22}}{\partial \theta_2} + \frac{\partial M_{22}}{\partial \theta_2} - \frac{\partial M_{22}}{\partial \theta_2} \right) = \frac{1}{2} \frac{\partial M_{22}}{\partial \theta_2} = 0$$

$$\begin{aligned} \dot{M} - 2C &= \begin{bmatrix} -2\beta \sin \theta_2 \cdot \dot{\theta}_2 & -\beta \sin \theta_2 \cdot \dot{\theta}_2 \\ -\beta \sin \theta_2 \cdot \dot{\theta}_2 & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} -2\beta \sin \theta_2 \cdot \dot{\theta}_2 & -2\beta \sin \theta_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ 2\beta \sin \theta_2 \cdot \dot{\theta}_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \beta \sin \theta_2 (2\dot{\theta}_1 + \dot{\theta}_2) \\ -\beta \sin \theta_2 (2\dot{\theta}_1 + \dot{\theta}_2) & 0 \end{bmatrix} \Leftarrow \text{skew-symmetric} \end{aligned}$$



4.3 Dynamics of Open-chain Manipulators

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◇ Example: Dynamics of a 3-dof robot

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_2 = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$g_{sl_1}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ 0 \\ r_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix},$$

$$g_{sl_2}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ r_1 \\ l_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix},$$

$$g_{sl_3}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ l_1 + r_2 \\ l_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix}$$

$$M_i = \left[\begin{array}{ccc|ccc} m_i & 0 & 0 & 0 & 0 & 0 \\ 0 & m_i & 0 & \mathcal{I}_{x_i} & 0 & 0 \\ 0 & 0 & m_i & 0 & \mathcal{I}_{y_i} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}_{z_i} \end{array} \right]$$

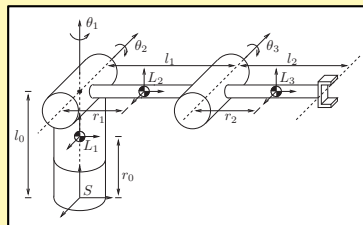


Figure 4.9

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m_i : The mass of the object

\mathcal{I}_{x_i} : The moment of inertia about the x axis

$$\Gamma_{112} = (\mathcal{I}_{y_2} - \mathcal{I}_{z_2} - m_2 r_1^2) c_2 s_2 + (\mathcal{I}_{y_2} - \mathcal{I}_{z_3}) c_{23} s_{23} - t(l_1 s_2 + r_2 s_{23})$$

$$\Gamma_{113} = (\mathcal{I}_{y_3} - \mathcal{I}_{z_3}) c_{23} s_{23} - t r_2 s_{23}$$

$$\Gamma_{121} = (\mathcal{I}_{y_2} - \mathcal{I}_{z_2} - m_2 r_1^2) c_2 s_2 + (\mathcal{I}_{y_3} - \mathcal{I}_{z_3}) c_{23} s_{23} - t(l_1 s_2 + r_2 s_{23})$$

$$\Gamma_{131} = (\mathcal{I}_{y_3} - \mathcal{I}_{z_3}) c_{23} s_{23} - t r_2 s_{23}$$

$$\Gamma_{211} = (\mathcal{I}_{z_2} - \mathcal{I}_{y_2} + m_2 r_1^2) c_2 s_2 + (\mathcal{I}_{z_3} - \mathcal{I}_{y_3}) c_{23} s_{23} + t(l_1 s_2 + r_2 s_{23})$$

$$\Gamma_{223} = -l_1 m_3 r_2 s_3$$

$$\Gamma_{232} = -l_1 m_3 r_2 s_3$$

$$\Gamma_{233} = -l_1 m_3 r_2 s_3$$

$$\Gamma_{311} = (\mathcal{I}_{z_3} - \mathcal{I}_{y_3}) c_{23} s_{23} + t r_2 s_{23}$$

$$\Gamma_{322} = l_1 m_3 r_2 s_3$$

where $t = m_3(l_1 c_2 + r_2 c_{23})$.

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4.3 Dynamics of Open-chain Manipulators

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$$N(\theta, \dot{\theta}) = \frac{\partial V}{\partial \dot{\theta}}, V(\theta) = m_1 g h_1(\theta) + m_2 g h_2(\theta) + m_2 g h_3(\theta)$$

$$g_{sl_i}(\theta) = e^{\dot{\xi}_1 \theta_1} \dots e^{\dot{\xi}_i \theta_i} g_{sl_i}(0) \Rightarrow$$

$$h_1(\theta) = r_0, h_2(\theta) = l_0 - r_1 \sin \theta, h_3(\theta)$$

$$= l_0 - l_1 \sin \theta_2 - r_2 \sin(\theta_2 + \theta_3)$$

$$J_1 = J_{sl_1}^b(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_2 = J_{sl_2}^b(\theta) = \begin{bmatrix} -r_1 c_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -r_1 & 0 \\ 0 & -1 & 0 \\ -s_2 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix}$$

$$J_3 = J_{sl_3}^b(\theta) = \begin{bmatrix} -l_2 c_2 - r_2 c_{23} & 0 & 0 \\ 0 & l_1 s_3 & 0 \\ 0 & -r_2 - l_1 c_3 & -r_2 \\ 0 & -1 & -1 \\ -s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \end{bmatrix}$$

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4.3 Dynamics of Open-chain Manipulators

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$$M(\theta) = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = J_1^T M_1 J_1 + J_2^T M_2 J_2 + J_3^T M_3 J_3$$

$$M_{11} = I_{y2}s_2^2 + I_{y3}s_{23}^2 + I_{z1} + I_{z2}c_2^2 + I_{z3}c_{23}^2 + m_2r_1^2c_2^2 + m_3(l_1c_2 + r_2c_{23})^2$$

$$M_{12} = M_{13} = M_{21} = M_{31} = 0$$

$$M_{22} = I_{x2} + I_{x3} + m_2l_1^2 + M_2r_1^2 + m_3r_2^2 + 2m_3l_1r_2c_3$$

$$M_{23} = I_{x3} + m_3r_2^2 + m_3l_1r_2c_3$$

$$M_{32} = I_{x3} + m_3r_2^2 + m_3l_1r_2c_3$$

$$M_{33} = I_{x3} + m_3r_2^2$$

$$C_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^n \Gamma_{ijk} \dot{\theta}_k = \frac{1}{2} \sum_{k=1}^n \left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right) \dot{\theta}_k \quad \diamond$$

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□ Additional Properties of the dynamics in terms of POE:

Define:

$$A_{ij} = \begin{cases} \text{Ad}_{e^{\xi_{j+1}\theta_{j+1}} \dots e^{\xi_i\theta_i}}^{-1} & i > j \\ I & i = j \\ 0 & i < j \end{cases}$$

$$J_i(\theta) = \text{Ad}_{g_{sl_i}(0)}^{-1} [A_{i1}\xi_1 \dots A_{ii}\xi_i \ 0 \dots 0]$$

$$M'_i = \text{Ad}_{g_{sl_i}(0)}^{-T} M_i \text{Ad}_{g_{sl_i}(0)}^{-1} \quad (\text{inertia of } i^{\text{th}} \text{ link in } S)$$

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□ Additional Properties of the dynamics in terms of POE:

Define:

$$A_{ij} = \begin{cases} \text{Ad}_{e^{\xi_{j+1}\theta_{j+1}} \dots e^{\xi_i\theta_i}}^{-1} & i > j \\ I & i = j \\ 0 & i < j \end{cases}$$

$$J_i(\theta) = \text{Ad}_{g_{sl_i}^{-1}(0)} [A_{i1} \xi_1 \dots A_{ii} \xi_i \ 0 \dots 0]$$

$$M'_i = \text{Ad}_{g_{sl_i}^{-1}(0)}^T M_i \text{Ad}_{g_{sl_i}(0)}^{-1} \quad (\text{inertia of } i^{\text{th}} \text{ link in } S)$$

Property 2:

$$M_{ij}(\theta) = \sum_{l=\max(i,j)}^n \xi_i^T A_{li}^T M'_l A_{lj} \xi_j, \quad C_{ij}(\theta, \dot{\theta}) = \frac{1}{2} \sum_{k=1}^n \left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right) \dot{\theta}_k$$

where

$$\frac{\partial M_{ij}}{\partial \theta_k} = \sum_{l=\max(i,j)}^n ([A_{k-1,i} \xi_i, \xi_k]^T A_{lk}^T M'_l A_{lj} \xi_j + \xi_i^T A_{li}^T M'_l A_{lk} [A_{k-1,j} \xi_j, \xi_k])$$

† End of Section †

4.4 Coordinate Invariant Algorithms

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- 2 Lagrange's Equations
- 3 Dynamics of Open-chain Manipulators
- 4 Coordinate Invariant Algorithms**
- 5 Lagrange's Equations with Constraints
 - General formulation
 - Unified geometric approach (Liu G.F., Li Z.X.)
 - Gauge invariant formulation (Aghili, F.)
- 6 References

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□ Newton-Euler equations in spatial frame:

Newton's Equation:

$$f^s = \frac{d}{dt}(m\dot{p}) = m\ddot{p}$$

Spatial angular momentum:

$$\mathcal{I}^s \cdot \omega^s = R(\mathcal{I} \cdot \omega^b) = \underbrace{R \cdot \mathcal{I} \cdot R^T}_{\mathcal{I}^s} \cdot \omega^s$$

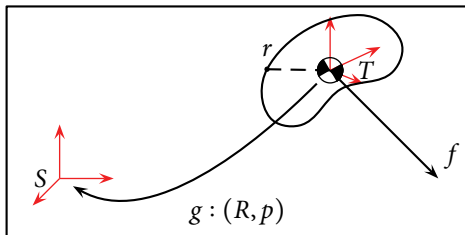


Figure 4.10

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$$\begin{aligned}\tau^s &= \frac{d}{dt}(\mathcal{I}^s \omega^s) = \frac{d}{dt}(R\mathcal{I}R^T \omega^s) = \mathcal{I}^s \dot{\omega}^s + \dot{R}\mathcal{I}R^T \omega^s + R\mathcal{I}\dot{R}^T \omega^s \\ &= \mathcal{I}^s \dot{\omega}^s + \underbrace{\dot{R}R^T}_{\hat{\omega}^s} \mathcal{I}^s \omega^s - R\mathcal{I}R^T \hat{\omega}^s \omega^s = \mathcal{I}^s \dot{\omega}^s + \omega^s \times (\mathcal{I}^s \omega^s)\end{aligned}$$

□ Transform equations to body frame:

$$\frac{d}{dt}(m\dot{p}) = \frac{d}{dt}(mRv^b) = m\dot{R}v^b + mR\dot{v}^b, R^T f^s = mR^T \dot{R}v^b + m\dot{v}^b$$

$$\Rightarrow f^b = m\omega^b \times v^b + m\dot{v}^b,$$

$$\tau^b = R^T \tau^s = R^T \frac{d}{dt}(R\mathcal{I}\omega^b) = \mathcal{I}\dot{\omega}^b + \omega^b \times \mathcal{I}\omega^b$$

$$\Rightarrow \underbrace{\begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix}}_{M^b} \underbrace{\begin{bmatrix} \dot{v}^b \\ \dot{\omega}^b \end{bmatrix}}_{V^b} + \begin{bmatrix} \omega^b \times m v^b \\ \omega^b \times \mathcal{I} \omega^b \end{bmatrix} = \begin{bmatrix} f^b \\ \tau^b \end{bmatrix} = F^b \quad (*)$$

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4.4 Coordinate Invariant Algorithms

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Define:

$$[\cdot, \cdot] : se(3) \times se(3) \mapsto se(3), [\hat{\xi}_1, \hat{\xi}_2] \triangleq \hat{\xi}_1 \hat{\xi}_2 - \hat{\xi}_2 \hat{\xi}_1$$

if

$$[\hat{\xi}_1, \hat{\xi}_2] = \begin{bmatrix} \hat{\omega}_i & v_i \\ 0 & 0 \end{bmatrix}, i = 1, 2$$

then

$$\hat{\xi} = \begin{bmatrix} (\omega_1 \times \omega_2)^\wedge & \hat{\omega}_1 v_2 - \hat{\omega}_2 v_1 \\ 0 & 0 \end{bmatrix} = \text{ad}_{\hat{\xi}_1} \cdot \hat{\xi}_2$$

where

$$\text{ad}_{\hat{\xi}_1} = \begin{bmatrix} \hat{\omega}_1 & \hat{v}_1 \\ 0 & \hat{\omega}_1 \end{bmatrix}$$

It is straightforward computation to see that:

$$(*) \Leftrightarrow M^b \dot{V}^b - \text{ad}_{V^b}^T M^b V^b = F^b$$

Property 3:

$$\text{Ad}_g[\hat{\xi}_1, \hat{\xi}_2] = [\text{Ad}_g \hat{\xi}_1, \text{Ad}_g \hat{\xi}_2] \Rightarrow \text{Ad}_g \text{ad}_{\hat{\xi}_1} = \text{ad}_{\text{Ad}_g \hat{\xi}_1} \text{Ad}_g$$

4.4 Coordinate Invariant Algorithms

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□ Coordinate invariance of Newton-Euler equations:

$$M_1 \dot{V}_1 - \text{ad}_{V_1}^T M_1 V_1 = F_1$$

$$\begin{cases} V_1 = \text{Ad}_{g_0} V_2, \\ F_1 = \text{Ad}_{g_0^{-1}}^T F_2 \end{cases} \Rightarrow$$

$$F_1 = (\text{Ad}_{g_0}^T)^{-1} F_2 = (\text{Ad}_{g_0}^{-1})^T F_2$$

$$M_1 = \text{Ad}_{g_0}^{-T} M_2 \text{Ad}_{g_0}^{-1}$$

$$\Rightarrow \text{Ad}_{g_0}^{-T} M_2 \text{Ad}_{g_0}^{-1} \text{Ad}_{g_0} \dot{V}_2 - \text{ad}_{(\text{Ad}_{g_0} \dot{V}_2)}^T \text{Ad}_{g_0}^{-T} M_2 \text{Ad}_{g_0}^{-1} \text{Ad}_{g_0} V_1 = \text{Ad}_{g_0}^{-T} F_2$$

Since $\text{ad}_{\text{Ad}_{g_0} V} = \text{Ad}_{g_0} \text{ad}_V \text{Ad}_{g_0}^{-1}$ by Property 3, we have

$$M_2 \dot{V}_2 - \text{ad}_{V_2}^T M_2 V_2 = F_2$$

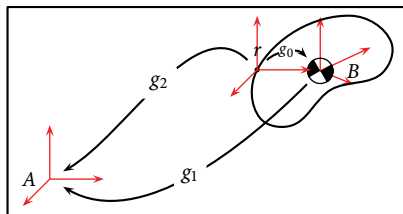


Figure 4.11

4.4 Coordinate Invariant Algorithms

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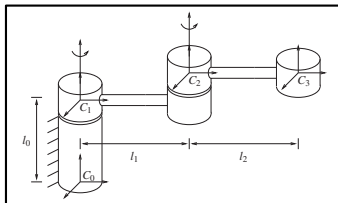


Figure 4.12

- C_i : Frame fixed to link i , located along the i^{th} axis
 F_i : Generalized force link $i-1$ exerting on link i , expressed in C_i
 τ_i : Joint torque of link i
 $g_{i-1,i}$: Transformation of C_i relative to C_{i-1}

$$g_{i-1,i}(\theta_i) = e^{\xi_i' \theta_i} \cdot g_{i-1,i}(0) = g_{i-1,i}(0) e^{\xi_i \theta_i}$$

$$\xi_i = \text{Ad}_{g_{i-1,i}^{-1}(0)} \cdot \xi_i': \quad i^{th} \text{ axis in } C_i \text{ frame.}$$

$$\xi_i = \begin{cases} \begin{bmatrix} 0 \\ z_i \end{bmatrix} : & \text{Revolute joint.} \\ \begin{bmatrix} z_i \\ 0 \end{bmatrix} : & \text{Prismatic joint.} \end{cases}$$

4.4 Coordinate Invariant Algorithms

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$$\Rightarrow g_{i-1,i}^{-1} \cdot \dot{g}_{i-1,i} = \hat{\xi}_i \cdot \dot{\theta}_i$$

M_i : Moment of inertia in C_i

$$M_i = \begin{bmatrix} m_i I & -m_i \hat{r}_i \\ m_i \hat{r}_i & \mathcal{I}_i - m_i \hat{r}_i^2 \end{bmatrix} \quad \begin{array}{ll} m_i: & \text{Mass of link } i \\ \mathcal{I}_i: & \text{inertia tensor} \end{array}$$

$$g_i = g_{i-1} g_{i-1,i}$$

$$\hat{V}_i = g_i^{-1} \cdot \dot{g}_i = g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + \hat{\xi}_i \dot{\theta}_i$$

$$V_i = \text{Ad}_{g_{i-1,i}^{-1}} V_{i-1} + \hat{\xi}_i \dot{\theta}_i$$

$$\begin{aligned} \dot{\hat{V}}_i &= \dot{g}_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + g_{i-1,i}^{-1} \hat{V}_{i-1} \dot{g}_{i-1,i} + g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + \hat{\xi}_i \ddot{\theta}_i \\ &= -\dot{g}_{i-1,i}^{-1} g_{i-1,i} \dot{g}_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} \dot{g}_{i-1,i}^{-1} g_{i-1,i} \\ &\quad + g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + \hat{\xi}_i \ddot{\theta}_i \end{aligned}$$

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$$\begin{aligned}
 &= -\dot{\xi}_i \dot{\theta}_i (\text{Ad}_{g_{i-1,i}^{-1}} V_{i-1})^\wedge + (\text{Ad}_{g_{i-1,i}^{-1}} V_{i-1})^\wedge \dot{\xi}_i \dot{\theta}_i + (\text{Ad}_{g_{i-1,i}^{-1}} \dot{V}_{i-1})^\wedge + \dot{\xi}_i \ddot{\theta}_i \\
 &\Rightarrow \dot{V}_i = \xi_i \ddot{\theta}_i + \text{Ad}_{g_{i-1,i}^{-1}} \dot{V}_{i-1} - \text{ad}_{\xi_i \dot{\theta}_i} (\text{Ad}_{g_{i-1,i}^{-1}} V_{i-1})
 \end{aligned}$$

□ Forward Recursion (kinematics):

$$\left\{ \begin{array}{l} \text{init. : } V_0 = 0, \dot{V}_0 = \begin{bmatrix} g \\ 0 \end{bmatrix} \text{ (gravity vector)} \\ g_{i-1,i} = g_{i-1,i}(0) e^{\xi_i \theta_i} \\ V_i = \text{Ad}_{g_{i-1,i}^{-1}} V_{i-1} + \xi_i \dot{\theta}_i \\ \dot{V}_i = \xi_i \ddot{\theta}_i + \text{Ad}_{g_{i-1,i}^{-1}} \dot{V}_{i-1} - \text{ad}_{\xi_i \dot{\theta}_i} (\text{Ad}_{g_{i-1,i}^{-1}} V_{i-1}) \end{array} \right.$$

□ Backward Recursion (inverse dynamics):

F_{n+1} : End-effector wrench, $g_{n,n+1}$: transform from tool frame to C_n

$$\begin{aligned}
 F_i &= \text{Ad}_{g_{i,i+1}^{-1}}^T \cdot F_{i+1} + M_i \dot{V}_i - \text{ad}_{V_i}^T \cdot M_i V_i \\
 \tau_i &= \xi_i^T \cdot F_i
 \end{aligned}$$

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$$V_1 = \text{Ad}_{g_{0,1}^{-1}} \cdot V_0 + \xi_1 \dot{\theta}_1$$

$$F_n = \text{Ad}_{g_{n,n+1}^{-1}}^T \cdot F_{n+1} + M_n \dot{V}_n - \text{ad}_{V_n}^T \cdot (M_n V_n)$$

Define:

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \in \mathbb{R}^{6n \times 1}, \dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} \in \mathbb{R}^n, \xi = \begin{bmatrix} \xi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \xi_n \end{bmatrix} \in \mathbb{R}^{6n \times n}$$

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} \in \mathbb{R}^{6n \times 1}, \tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} \in \mathbb{R}^n, P_0 = \begin{bmatrix} \text{Ad}_{g_{0,1}^{-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n \times 6}$$

$$P_t = \begin{bmatrix} 0 \cdots 0 & \text{Ad}_{g_{n,n+1}^{-1}} \end{bmatrix} \in \mathbb{R}^{6 \times 6n}$$

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$$V_1 = \text{Ad}_{g_{0,1}}^{-1} \cdot V_0 + \xi_1 \dot{\theta}_1$$

$$V_2 = \text{Ad}_{g_{1,2}}^{-1} V_1 + \xi_2 \dot{\theta}_2$$

$$\vdots$$

$$V_n = \text{Ad}_{g_{n-1,n}}^{-1} V_{n-1} + \xi_n \dot{\theta}_n$$

$$\Rightarrow \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ -\text{Ad}_{g_{1,2}}^{-1} & I & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & -\text{Ad}_{g_{n-1,n}}^{-1} & I \end{bmatrix}}_{G^{-1}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}}_V$$

$$= \underbrace{\begin{bmatrix} \text{Ad}_{g_{0,1}}^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{P_0} V_0 + \underbrace{\begin{bmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{bmatrix}}_{\xi} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}}_{\dot{\theta}}$$

Thus $V = GP_0 V_0 + G\xi\dot{\theta}$

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where

$$G = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ \text{Ad}_{g_{1,2}}^{-1} & I & 0 & \cdots & 0 \\ \text{Ad}_{g_{1,3}}^{-1} & \text{Ad}_{g_{2,3}}^{-1} & I & \ddots & \vdots \\ \vdots & \vdots & \ddots & I & 0 \\ \text{Ad}_{g_{1,n}}^{-1} & \text{Ad}_{g_{2,n}}^{-1} & \cdots & \text{Ad}_{g_{n-1,n}}^{-1} & I \end{bmatrix} \in \mathbb{R}^{6n \times 6n}$$

$$\dot{V}_1 = \xi_1 \ddot{\theta}_1 + \text{Ad}_{g_{0,1}}^{-1} \dot{V}_0 - \text{ad}_{\xi_1 \dot{\theta}_1} (\text{Ad}_{g_{0,1}}^{-1} V_0)$$

$$\dot{V}_2 - \text{Ad}_{g_{1,2}}^{-1} \dot{V}_1 = \xi_2 \ddot{\theta}_2 - \text{ad}_{\xi_2 \dot{\theta}_2} (\text{Ad}_{g_{1,2}}^{-1} V_1)$$

$$\dot{V}_n - \text{Ad}_{g_{n-1,n}}^{-1} \dot{V}_{n-1} = \xi_n \ddot{\theta}_n - \text{ad}_{\xi_n \dot{\theta}_n} (\text{Ad}_{g_{n-1,n}}^{-1} V_{n-1})$$

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$$\underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ -\text{Ad}_{g_{1,2}}^{-1} I & \ddots & & \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & -\text{Ad}_{g_{n-1,n}}^{-1} I & I \end{bmatrix}}_{G^{-1}} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \vdots \\ \dot{V}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \text{Ad}_{g_{0,1}}^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{P_0} \dot{V}_0 + \underbrace{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}}_{\xi} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \vdots \\ \ddot{\theta}_n \end{bmatrix} +$$

$$\underbrace{\begin{bmatrix} -\text{ad}_{\xi_1 \dot{\theta}_1} & 0 & \cdots & 0 \\ 0 & -\text{ad}_{\xi_2 \dot{\theta}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\text{ad}_{\xi_n \dot{\theta}_n} \end{bmatrix}}_{\text{ad}_{\xi \dot{\theta}}} \begin{bmatrix} \text{Ad}_{g_{0,1}}^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} V_0 +$$

$$\underbrace{\begin{bmatrix} -\text{ad}_{\xi_1 \dot{\theta}_1} & 0 & \cdots & 0 \\ 0 & -\text{ad}_{\xi_2 \dot{\theta}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\text{ad}_{\xi_n \dot{\theta}_n} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \text{Ad}_{g_{1,2}}^{-1} & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \text{Ad}_{g_{n-1,n}}^{-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}}_{\Gamma}$$

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Thus

$$\dot{V} = G \cdot \xi \ddot{\theta} + G \cdot P_0 \dot{V}_0 + G \cdot \text{ad}_{\xi \dot{\theta}} P_0 V_0 + G \cdot \text{ad}_{\xi \dot{\theta}} \Gamma V$$

Finally the backward recursion:

$$\begin{aligned} F_n &= \text{Ad}_{g_{n,n+1}}^T F_{n+1} + M_n \dot{V}_n - \text{ad}_{V_n}^T \cdot (M_n V_n) \\ F_{n-1} &= \text{Ad}_{g_{n-1,n}}^T F_n + M_{n-1} \dot{V}_{n-1} - \text{ad}_{V_{n-1}}^T \cdot (M_{n-1} V_{n-1}) \\ &\vdots \\ F_1 &= \text{Ad}_{g_{1,2}}^T F_2 + M_1 \dot{V}_1 - \text{ad}_{V_1}^T (M_1 V_1) \end{aligned}$$

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$$\Rightarrow \begin{bmatrix} I - \text{Ad}_{g_{1,2}}^T & 0 & 0 \\ 0 & I & \ddots & 0 \\ \vdots & \ddots & \ddots & -\text{Ad}_{g_{n-1,n}}^T \\ 0 & \dots & 0 & I \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \underbrace{\begin{bmatrix} M_1 & M_2 & \dots & M_n \end{bmatrix}}_M \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \vdots \\ \dot{V}_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \text{Ad}_{g_{n,n+1}}^T \end{bmatrix}}_{P_t^T} F_{n+1} +$$

$$\underbrace{\begin{bmatrix} -\text{ad}_{V_1}^T & \dots & \dots \\ & & -\text{ad}_{V_n}^T \end{bmatrix}}_{\text{ad}_V^T} \begin{bmatrix} M_1 & \dots & M_n \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} I - \text{Ad}_{g_{1,2}}^T & 0 & 0 \\ 0 & I & \ddots & 0 \\ \vdots & \ddots & \ddots & -\text{Ad}_{g_{n-1,n}}^T \\ 0 & \dots & 0 & I \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \underbrace{\begin{bmatrix} M_1 & M_2 & \dots & M_n \end{bmatrix}}_M \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \vdots \\ \dot{V}_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \text{Ad}_{g_{n,n+1}}^T \end{bmatrix}}_{P_t^T} F_{n+1} +$$

$$\underbrace{\begin{bmatrix} -\text{ad}_{V_1}^T & \dots & -\text{ad}_{V_n}^T \end{bmatrix}}_{\text{ad}_V^T} \begin{bmatrix} M_1 & \dots & M_n \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

$$F = G^T M \dot{V} + G^T P_t^T \underbrace{F_{n+1}}_{F_t} + G^T \cdot \text{ad}_V^T M V$$

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$$\tau = \xi^T \cdot F$$

$$\tau = \xi^T G^T M \dot{V} + \xi^T G^T P_t^T F_t + \xi^T G^T \cdot \text{ad}_V^T M V$$

$$\begin{aligned} &= \xi^T G^T M (G \xi \ddot{\theta} + G P_0 \dot{V}_0 + G \cdot \text{ad}_{\xi \dot{\theta}} P_0 V_0 + G \cdot \text{ad}_{\xi \dot{\theta}} \Gamma V) + \\ \Rightarrow &\xi^T G^T P_t^T F_t + \xi^T G^T \cdot \text{ad}_V^T M V \\ &= \xi^T G^T M G \xi \ddot{\theta} + \xi^T G^T M G P_0 \dot{V}_0 + \xi^T G^T M G \cdot \text{ad}_{\xi \dot{\theta}} \Gamma V + \\ &\xi^T G^T P_t^T F_t + \xi^T G^T \cdot \text{ad}_V^T M V \end{aligned}$$

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$$\tau = \xi^T \cdot F$$

$$\tau = \xi^T G^T M \dot{V} + \xi^T G^T P_t^T F_t + \xi^T G^T \cdot \text{ad}_V^T M V$$

$$\begin{aligned} &= \xi^T G^T M (G \xi \ddot{\theta} + G P_0 \dot{V}_0 + G \cdot \text{ad}_{\xi \dot{\theta}} P_0 V_0 + G \cdot \text{ad}_{\xi \dot{\theta}} \Gamma V) + \\ \Rightarrow \quad &\xi^T G^T P_t^T F_t + \xi^T G^T \cdot \text{ad}_V^T M V \\ &= \xi^T G^T M G \xi \ddot{\theta} + \xi^T G^T M G P_0 \dot{V}_0 + \xi^T G^T M G \cdot \text{ad}_{\xi \dot{\theta}} \Gamma V + \\ &\xi^T G^T P_t^T F_t + \xi^T G^T \cdot \text{ad}_V^T M V \end{aligned}$$

Finally we get (recall that $V_0 = 0$, we have $V = G \xi \dot{\theta}$):

$$\begin{aligned} &\underbrace{\xi^T G^T M G \xi \ddot{\theta}}_{M(\theta)} + \underbrace{\xi^T G^T (M G \cdot \text{ad}_{\xi \dot{\theta}} \Gamma + \text{ad}_V^T M) G \xi \dot{\theta} +}_{C(\theta, \dot{\theta})} \\ &\underbrace{\xi^T G^T M G P_0 \dot{V}_0}_{\phi(\theta)} + \underbrace{\xi^T G^T P_t^T F_t}_{J_t^T(\theta)} = \tau \end{aligned}$$

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$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + \phi(\theta) + J_t^T(\theta)F_t = \tau$$

$$M(\theta) = \xi^T G^T M G \xi$$

$$C(\theta, \dot{\theta}) = \xi^T G^T (M \text{Gad}_{\xi\dot{\theta}} \Gamma + \text{ad}_V^T M) G \xi \dot{\theta}$$

$$\phi(\theta) = \xi^T G^T M G P_0 \dot{V}_0$$

$$J_t = P_t G \xi$$

$$M = \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_n \end{bmatrix}$$

Property 4:

$$\Gamma^n = 0,$$

$$G = (I - \Gamma)^{-1} = I + \Gamma + \Gamma^2 + \dots + \Gamma^{n-1}$$

$$I + \Gamma G = G$$

† End of Section †

4.4 Coordinate Invariant Algorithms

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□ Square root factorization of M :

Definition: Articulated body inertia algorithm (Featherstone)

$$\hat{M}_n = M_n, b_n = -\text{ad}_{V_n}^T (M_n V_n)$$

$$F_i = \hat{M}_i \dot{V}_i + b_i, i = n, \dots, 1$$

$$b_i = b_i(V_i, V_{i+1}, \xi_{i+1}, \hat{M}_{i+1}, \tau_{i+1}) \text{ (bias force)}$$

$$\left\{ \begin{array}{l} F_i = \text{Ad}_{g_{i,i+1}}^T F_{i+1} + M_i \dot{V}_i - \text{ad}_{V_i}^T M_i V_i \\ F_{i+1} = \hat{M}_{i+1} \dot{V}_{i+1} + b_{i+1} \\ \dot{V}_{i+1} = \text{Ad}_{g_{i,i+1}}^{-1} \dot{V}_i + \xi_{i+1} \ddot{\theta}_{i+1} - \text{ad}_{\xi_{i+1} \dot{\theta}_{i+1}} \text{Ad}_{g_{i,i+1}}^{-1} V_i \end{array} \right. \Rightarrow$$

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4.4 Coordinate Invariant Algorithms

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□ Square root factorization of M :

$$F_{i+1} = \hat{M}_{i+1}(\text{Ad}_{g_{i,i+1}}^{-1} \dot{V}_i + \xi_{i+1} \ddot{\theta}_{i+1} - \text{ad}_{\xi_{i+1} \dot{\theta}_{i+1}} \text{Ad}_{g_{i,i+1}}^{-1} V_i) + b_{i+1} \Rightarrow$$

$$\tau_{i+1} = \xi_{i+1}^T F_{i+1} = \xi_{i+1}^T (\hat{M}_{i+1}(\text{Ad}_{g_{i,i+1}}^{-1} \dot{V}_i + \xi_{i+1} \ddot{\theta}_{i+1} - \text{ad}_{\xi_{i+1} \dot{\theta}_{i+1}} \text{Ad}_{g_{i,i+1}}^{-1} V_i) + b_{i+1}) \Rightarrow$$

$$\ddot{\theta}_{i+1} = \frac{\tau_{i+1} + \xi_{i+1}^T (\hat{M}_{i+1}(-\text{Ad}_{g_{i,i+1}}^{-1} \dot{V}_i + \text{ad}_{\xi_{i+1} \dot{\theta}_{i+1}} \text{Ad}_{g_{i,i+1}}^{-1} V_i) - b_{i+1})}{\xi_{i+1}^T \hat{M}_{i+1} \xi_{i+1}}$$

$$\Rightarrow$$

$$\begin{aligned} F_i &= \text{Ad}_{g_{i,i+1}}^T (\hat{M}_{i+1}(\text{Ad}_{g_{i,i+1}}^{-1} \dot{V}_i + \\ &\xi_{i+1} \left(\frac{\tau_{i+1} + \xi_{i+1}^T (\hat{M}_{i+1}(-\text{Ad}_{g_{i,i+1}}^{-1} \dot{V}_i + \text{ad}_{\xi_{i+1} \dot{\theta}_{i+1}} \text{Ad}_{g_{i,i+1}}^{-1} V_i) - b_{i+1})}{\xi_{i+1}^T \hat{M}_{i+1} \xi_{i+1}} \right. \\ &\left. - \text{ad}_{\xi_{i+1} \dot{\theta}_{i+1}} \text{Ad}_{g_{i,i+1}}^{-1} V_i) + b_{i+1}) + M_i \dot{V}_i - \text{ad}_{V_i}^T M_i V_i \right. \\ &\Rightarrow \end{aligned}$$

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□ Square root factorization of M :

$$\hat{M}_i = M_i + \text{Ad}_{g_{i,i+1}}^T \hat{M}_{i+1} \text{Ad}_{g_{i,i+1}}^{-1} - \frac{\text{Ad}_{g_{i,i+1}}^T \hat{M}_{i+1} \xi_{i+1} \xi_{i+1}^T \hat{M}_{i+1} \text{Ad}_{g_{i,i+1}}^{-1}}{\xi_{i+1}^T \hat{M}_{i+1} \xi_{i+1}}$$

$$b_i = \text{Ad}_{g_{i,i+1}}^T b_{i+1} - \text{ad}_{V_i}^T M_i V_i - \text{Ad}_{g_{i,i+1}}^T \hat{M}_{i+1} \text{ad}_{\xi_{i+1} \dot{\theta}_{i+1}} \text{Ad}_{g_{i,i+1}}^{-1} V_i +$$

$$\frac{\text{Ad}_{g_{i,i+1}}^T \hat{M}_{i+1} \xi_{i+1} (\tau_{i+1} - \xi_{i+1}^T b_{i+1} + \xi_{i+1}^T \hat{M}_{i+1} \text{ad}_{\xi_{i+1} \dot{\theta}_{i+1}} \text{Ad}_{g_{i,i+1}}^{-1} V_i)}{\xi_{i+1}^T \hat{M}_{i+1} \xi_{i+1}}$$

$$\Rightarrow \hat{M} = M + \Gamma^T \hat{M} \Gamma - \Gamma^T \hat{M} \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} \Gamma$$

$$\Rightarrow M = \hat{M} - \Gamma^T \hat{M} \Gamma + \Gamma^T \hat{M} \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} \Gamma$$

$$\Rightarrow M(\theta) = \xi^T G^T (\hat{M} - \Gamma^T \hat{M} \Gamma + \Gamma^T \hat{M} \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} \Gamma) G \xi$$

$$\Rightarrow M(\theta) = \xi^T G^T \hat{M}^T \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} G \xi \triangleq W W^T$$

$$W = \xi^T G^T \hat{M}^T \xi (\xi^T \hat{M} \xi)^{-\frac{1}{2}}$$

† End of Section †

4.5 Lagrange's Equations with Constraints

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 - Unified geometric approach (Liu G.F., Li Z.X.)
 - Gauge invariant formulation (Aghili, F.)
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1 General formulation

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Definition: Holonomic constraints

Let $q = (q_1, \dots, q_n) \in E$, Ambient Space. A *holonomic constraint* is a set of constraint equations:

$$h_i(q) = 0, i = 1, \dots, k$$

If $\{dh_i(q)\}_{i=1}^k$ is linearly independent, then $Q = h^{-1}(0)$ is a manifold of $\dim m \triangleq n - k$.

$$T_q Q : \{V \in T_q E | dh_i \cdot V = 0, \forall i = 1, \dots, k\} \subset T_q E :$$

subspace of permissible velocities.

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4.5 Lagrange's Equations with Constraints

1 General formulation

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$$\begin{aligned} T_q^* Q^\perp &: \{f \in T_q^* E \mid \langle f, v \rangle = 0, \forall V \in T_q Q\} \\ &= \text{span}\{dh_1, \dots, dh_k\} \subset T_q^* E: \end{aligned}$$

the subspace of constraint forces (Annihilator of $T_q Q$).

$$\dim(T_q Q) = m, \dim(T_q^* Q^\perp) = n - m = k$$

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$$\begin{aligned} T_q^* Q^\perp &: \{f \in T_q^* E \mid \langle f, v \rangle = 0, \forall V \in T_q Q\} \\ &= \text{span}\{dh_1, \dots, dh_k\} \subset T_q^* E: \end{aligned}$$

the subspace of constraint forces (Annihilator of $T_q Q$).

$$\dim(T_q Q) = m, \dim(T_q^* Q^\perp) = n - m = k$$

Definition: Constraint forces

$$\Gamma = \frac{\partial h^T}{\partial q} \cdot \lambda$$

$\lambda \in \mathbb{R}^k$: the vector of relative magnitudes of constraint forces.

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4.5 Lagrange's Equations with Constraints

1 General formulation

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Definition: Pfaffian Constraints

A Pfaffian constraint has the form:

$$A(q)\dot{q} = 0, A(q) \in \mathbb{R}^{k \times n}$$

Given a Pfaffian constraint,

$$\Delta_q = \{v_q \in T_q E | A(q) \cdot v_q = 0\} \subset T_q E :$$

distribution of permissible velocities.

4.5 Lagrange's Equations with Constraints

1 General formulation

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Definition: Pfaffian Constraints

A Pfaffian constraint has the form:

$$A(q)\dot{q} = 0, A(q) \in \mathbb{R}^{k \times n}$$

Given a Pfaffian constraint,

$$\Delta_q = \{v_q \in T_q E | A(q) \cdot v_q = 0\} \subset T_q E :$$

distribution of permissible velocities.

Definition:

$A(q)\dot{q} = 0$ is holonomic (or integrable) iff Δ_q is an involutive distribution, iff

$$\exists h_i : E \mapsto \mathbb{R}, i = 1, \dots, k \text{ s.t. } \Delta_q = T_q Q, Q = h^{-1}(0)$$

4.5 Lagrange's Equations with Constraints

1 General formulation

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§ Constraint forces:

$$\Gamma = A^T(q) \cdot \lambda, \lambda \in \mathbb{R}^k$$

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1 General formulation

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§ Constraint forces:

$$\Gamma = A^T(q) \cdot \lambda, \lambda \in \mathbb{R}^k$$

§ Kinetic energy:

$$T(q, \dot{q}) = \frac{1}{2} \dot{q}^T \cdot M(q) \cdot \dot{q}$$

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§ Potential energy:

$$V(q)$$

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§ Kinetic energy:

$$T(q, \dot{q}) = \frac{1}{2} \dot{q}^T \cdot M(q) \cdot \dot{q}$$

§ Potential energy:

$$V(q)$$

§ Lagrangian:

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

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□ Lagrange's equations with constraints:

$$M(q)\ddot{q} + C(q, \dot{q}) + N(q) + A^T(q)\lambda = F$$

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1 General formulation

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□ Lagrange's equations with constraints:

$$M(q)\ddot{q} + C(q, \dot{q}) + N(q) + A^T(q)\lambda = F$$

□ Explicit solution for constraint forces:

$$A(q)\ddot{q} + \dot{A}(q)\dot{q} = 0$$

$$(AM^{-1}A^T)\lambda = AM^{-1}(F - C - N) + \dot{A}\dot{q}$$

$$\lambda = (AM^{-1}A^T)^{-1}(AM^{-1}(F - C - N) + \dot{A}\dot{q})$$

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◇ Example:

$$x^2 + y^2 = l^2$$

$$\underbrace{\begin{bmatrix} x & y \end{bmatrix}}_{A(q)} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 0$$

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ mg \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \lambda = 0$$

$$\begin{aligned} \lambda &= (AM^{-1}A^T)^{-1}(AM^{-1}(F - C - N) + \dot{A}\dot{q}) \\ &= -\frac{m}{l^2}(gy + \dot{x}^2 + \dot{y}^2) \end{aligned}$$

$$\text{Tension} = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| \lambda = \frac{mg}{l}y + \frac{m}{l}(\dot{x}^2 + \dot{y}^2)$$

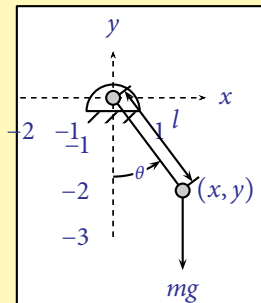


Figure 4.13

4.5 Lagrange's Equations with Constraints

1 General formulation

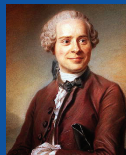
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□ Lagrange-d'Alembert formulation:

Given the Pfaffian constraint $A(q)\dot{q} = 0$ and virtual displacement $\delta q \in \mathbb{R}^k$, we have:

Theorem 1 (D'Alembert Principle):
Forces of constraints do no virtual work!

$$(A^T(q)\lambda) \cdot \delta q = 0 \text{ for } A(q)\delta q = 0$$



1717–1783

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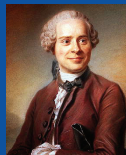
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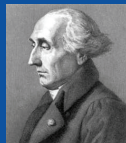
$$(A^T(q)\lambda) \cdot \delta q = 0 \text{ for } A(q)\delta q = 0$$



1717–1783

Theorem 2 (Lagrange-d'Alembert Equation):

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - \tau \right) \cdot \delta q = 0, A(q)\delta q = 0$$



1736–1813

4.5 Lagrange's Equations with Constraints

1 General formulation

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Let $A(q) = [A_1(q) \ A_2(q)]$, and $A_2(q) \in \mathbb{R}^{k \times k}$ is invertible, then $\delta q_1 \in \mathbb{R}^{n-k}$ are free variables:

$$\begin{aligned}
 \delta q_2 &= -A_2^{-1}(q)A_1(q)\delta q_1 \\
 &\Rightarrow \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - \tau \right) \cdot \delta q \\
 &= \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} - \tau_1 \right) \cdot \delta q_1 + \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} - \tau_2 \right) \cdot \delta q_2 \\
 &= \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} - \tau_1 \right) \cdot \delta q_1 + \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} - \tau_2 \right) \cdot (-A_2^{-1}A_1)\delta q_1
 \end{aligned}$$

(Continues next slide)

4.5 Lagrange's Equations with Constraints

1 General formulation

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As $\delta q_1 \in \mathbb{R}^{n-k}$ is free,

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} - \tau_1 \right) - A_1^T A_2^{-T} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} - \tau_2 \right) = 0$$

Lagrange-d'Alembert equation

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◇ Example: Dynamics of a rolling disk

Pfaffian constraint:

$$\begin{cases} \dot{x} - \rho \cos \theta \dot{\phi} = 0 \\ \dot{y} - \rho \sin \theta \dot{\phi} = 0 \end{cases}$$

$$\Rightarrow A(q)\dot{q} = \begin{bmatrix} 1 & 0 & 0 & -\rho \cos \theta \\ 0 & 1 & 0 & -\rho \sin \theta \end{bmatrix} \dot{q} = 0$$

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\mathcal{I}_1\ddot{\theta}^2 + \frac{1}{2}\mathcal{I}_2\dot{\phi}^2$$

Lagrange-d'Alembert equation:

$$\left(\begin{bmatrix} m & & & \\ & m & & \\ & & \mathcal{I}_1 & \\ & & & \mathcal{I}_2 \end{bmatrix} \ddot{q} - \begin{bmatrix} 0 \\ 0 \\ \tau_\theta \\ \tau_\phi \end{bmatrix} \right) \cdot \delta q = 0$$

where $A(q) \cdot \delta q = 0$.

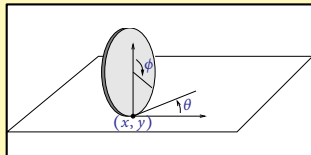


Figure 4.14

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4.5 Lagrange's Equations with Constraints

1 General formulation

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As

$$\begin{cases} \delta x = \rho \cos \theta \cdot \delta \phi \\ \delta y = \rho \sin \theta \cdot \delta \phi \end{cases}$$

the equation of motion is:

$$\left(\begin{bmatrix} 0 & 0 \\ m\rho c_\theta & m\rho s_\theta \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} - \begin{bmatrix} \tau_\theta \\ \tau_\phi \end{bmatrix} \right) \cdot \begin{bmatrix} \delta\theta \\ \delta\phi \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ m\rho c_\theta & m\rho s_\theta \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \tau_\theta \\ \tau_\phi \end{bmatrix}$$

As

$$\begin{cases} \ddot{x} = \rho \cos \theta \cdot \ddot{\phi} - \rho \sin \theta \cdot \dot{\theta} \dot{\phi} \\ \ddot{y} = \rho \sin \theta \cdot \ddot{\phi} + \rho \cos \theta \cdot \dot{\theta} \dot{\phi} \end{cases}$$

$$\Rightarrow \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 + m\rho^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \tau_\theta \\ \tau_\phi \end{bmatrix}$$

Solve for $(\theta(t), \phi(t))$, and then solve for $(x(t), y(t))$ from:

$$\begin{cases} \dot{x} = \rho \cos \theta \cdot \dot{\phi} \\ \dot{y} = \rho \sin \theta \cdot \dot{\phi} \end{cases} \leftarrow \text{1st order differential equation}$$

4.5 Lagrange's Equations with Constraints

1 General formulation

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□ Nature of nonholonomic constraints:

Consider $q = (r, s) \in \mathbb{R}^2 \times \mathbb{R}$ with Pfaffian constraint,

$$\dot{s} + a^T(r)\dot{r} = 0, a(r) \in \mathbb{R}^2$$

Lagrangian:

$$L = L(r, \dot{r}, \dot{s})$$

and **constrained Lagrangian:**

$$L_c(r, \dot{r}) = L(r, \dot{r}, -a^T(r)\dot{r})$$

\Rightarrow **Lagrange's equation:**

$$\frac{d}{dt} \frac{\partial L_c}{\partial \dot{r}_i} - \frac{\partial L_c}{\partial r_i} = 0, i = 1, 2$$

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}_i} - a_i(r) \frac{\partial L}{\partial \dot{s}} \right) - \left(\frac{\partial L}{\partial r_i} - \frac{\partial L}{\partial \dot{s}} \sum_j \frac{\partial a_j}{\partial r_i} \dot{r}_j \right) = 0$$

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$$\Rightarrow \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i} \right) - a_i(r) \underbrace{\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} \right)}_{\neq 0} = \frac{\partial L}{\partial \dot{s}} \left(\dot{a}_i(r) - \sum_j \frac{\partial a_j}{\partial r_i} \dot{r}_j \right) (*)$$

If the constraint is holonomic, i.e.

$$a_i(r) = \frac{\partial h}{\partial r_i} \text{ for some } h : E \mapsto \mathbb{R}$$

then RHS (right hand side) of (*) equals

$$\frac{\partial L}{\partial \dot{s}} \left(\sum_j \frac{\partial^2 h}{\partial r_i \partial r_j} \dot{r}_j - \sum_j \frac{\partial^2 h}{\partial r_j \partial r_i} \dot{r}_j \right) = 0$$

4.5 Lagrange's Equations with Constraints

2 Unified geometric approach (Liu G.F., Li Z.X.)

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□ Metric, duality and orthogonality on $T_q E$:

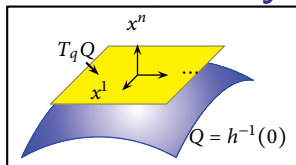


Figure 4.15

$$\begin{aligned}\mathcal{K} &\triangleq \frac{1}{2} \ll \dot{q}, \dot{q} \gg_M \\ &= \frac{1}{2} \dot{q}^T M(q) \dot{q}\end{aligned}$$

$$T_q Q^\perp = \{V_1 \in T_q E \mid \ll V_1, V_2 \gg_M \triangleq V_1^T M V_2 = 0, \forall V_2 \in T_q Q\}$$

$$T_q^* Q^\perp = \{f \in T_q^* E \mid \langle f, V \rangle = 0, \forall V \in T_q Q\} : \text{constraint forces}$$

$$T_q E = T_q Q \oplus T_q Q^\perp, T_q^* E = T_q^* Q \oplus T_q^* Q^\perp$$

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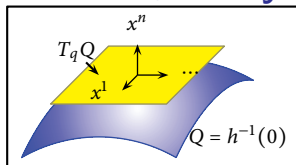


Figure 4.15

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$$T_q^* Q^\perp = \{f \in T_q^* E \mid \langle f, V \rangle = 0, \forall V \in T_q Q\} : \text{constraint forces}$$

$$T_q E = T_q Q \oplus T_q Q^\perp, T_q^* E = T_q^* Q \oplus T_q^* Q^\perp$$

Definition:

$$M^b : T_q E \mapsto T_q^* E, \langle M^b V_1, V_2 \rangle = V_1^T M V_2 = \ll V_1, V_2 \gg_M$$

$$M^\sharp : T_q^* E \mapsto T_q E, M^\sharp = M^b{}^{-1}$$

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2 Unified geometric approach (Liu G.F., Li Z.X.)

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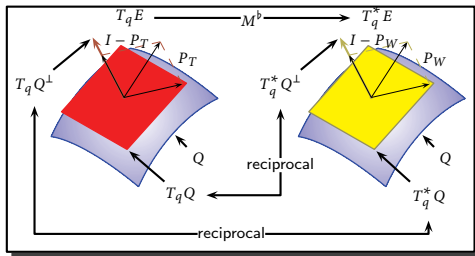


Figure 4.16

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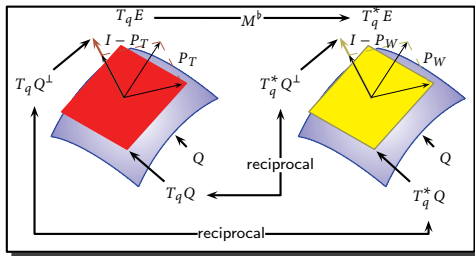


Figure 4.16

Property 5:

Under the basis $\frac{\partial}{\partial q_i}$ and $dq_i, i = 1, \dots, n$ of $T_q E$ and $T_q^* E$ respectively, the matrix representation of M^b and M^\sharp is M and M^{-1} respectively.

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2 Unified geometric approach (Liu G.F., Li Z.X.)

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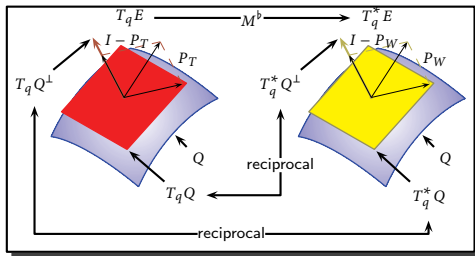


Figure 4.16

Property 5:

Under the basis $\frac{\partial}{\partial q_i}$ and $dq_i, i = 1, \dots, n$ of $T_q E$ and $T_q^* E$ respectively, the matrix representation of M^b and M^\sharp is M and M^{-1} respectively.

Property 6:

$$M^\sharp(T_q^* Q) = T_q Q$$

$$M^\sharp(T_q^* Q^\perp) = T_q Q^\perp$$

4.5 Lagrange's Equations with Constraints

2 Unified geometric approach (Liu G.F., Li Z.X.)

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Given

$$h : E \mapsto \mathbb{R}^k, m = n - k$$

$$h_* \triangleq T_q h : T_q E \mapsto T_{h(q)} \mathbb{R}^k$$

$$h^* \triangleq T_q^* h : T_{h(q)}^* \mathbb{R}^k \mapsto T_q^* E$$

we have:

Property 7:

$$\ker h_* = T_q Q, h_*(T_q Q^\perp) = T_{h(q)} \mathbb{R}^k, h^*(T_{h(q)}^* \mathbb{R}^k) = T_q^* Q^\perp$$

$$\begin{array}{ccc}
 T_q^* E & \xleftarrow{h^*} & T_q^* \mathbb{R}^k \\
 \downarrow M^\sharp & & \downarrow M_2^\sharp \\
 T_q E & \xrightarrow{h_*} & T_{h(q)} \mathbb{R}^k
 \end{array}
 \quad M_2^\sharp = h_* \circ M^\sharp \circ h^*$$

4.5 Lagrange's Equations with Constraints

2 Unified geometric approach (Liu G.F., Li Z.X.)

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Lemma 1:

The map $(I - P_\omega) : T_q^*E \mapsto T_q^*Q^\perp$ given by

$$(I - P_\omega) = h^* \circ M_2^b \circ h_* \circ M^\sharp$$

is a well-defined projection map, with the property:

$$(I - P_\omega)f_1 = 0, \forall f_1 \in T_q^*Q$$

$$(I - P_\omega)f_2 = f_2, \forall f_2 \in T_q^*Q^\perp$$

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$$(I - P_\omega) = h^* \circ M_2^b \circ h_* \circ M^\sharp$$

is a well-defined projection map, with the property:

$$(I - P_\omega)f_1 = 0, \forall f_1 \in T_q^*Q$$

$$(I - P_\omega)f_2 = f_2, \forall f_2 \in T_q^*Q^\perp$$

Proof :

Given $f_1 \in T_q^*Q, M^\sharp(f_1) \in T_qQ = \ker h_*$, then $(I - P_\omega)f_1 = 0$. For $f_2 \in T_q^*Q^\perp, \exists \lambda \in \mathbb{R}^{n-m}$ s.t. $f_2 = h^*\lambda$, and

$$(I - P_\omega)f_2 = h^*M_2^bh_*M^\sharp h^*\lambda = h^*\lambda = f_2$$

thus $P_\omega : T_q^*E \mapsto T_q^*Q$ is a well-defined projection map. Similarly,

$$P_T : T_qE \mapsto T_qQ, P_T = I - M^\sharp h^*M_2^bh_*$$

and

$$(I - P_T) : T_qE \mapsto T_qQ^\perp$$

are projection maps.



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2 Unified geometric approach (Liu G.F., Li Z.X.)

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Lemma 2:

$$P_{\omega}M = MP_T$$

$$P_{\omega}h^* = h_*P_T = 0$$

$$P_T = P_{\omega}^T$$

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Lemma 2:

$$P_{\omega}M = MP_T$$

$$P_{\omega}h^* = h_*P_T = 0$$

$$P_T = P_{\omega}^T$$

For nonholonomic constraints:

$$h_* \leftarrow A(q)$$

$$h^* \leftarrow A^*(q)$$

$$T_q Q \leftarrow \Delta_q$$

$$T_q^* Q^{\perp} \leftarrow \text{span}\{a_i(q), i = 1, \dots, k\}$$

application in hybrid velocity/force control.

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2 Unified geometric approach (Liu G.F., Li Z.X.)

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□ Lagrange's equations of motion:

$$M(q)\ddot{q} + C(q, \dot{q}) + N = \tau + A^T(q)\lambda \Rightarrow$$

$$\lambda = (AM^{-1}A^T)^{-1}AM^{-1}(M\ddot{q} + (C + N - \tau))$$

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□ Lagrange's equations of motion:

$$M(q)\ddot{q} + C(q, \dot{q}) + N = \tau + A^T(q)\lambda \Rightarrow$$

$$\lambda = (AM^{-1}A^T)^{-1}AM^{-1}(M\ddot{q} + (C + N - \tau))$$

Define $P_\omega = I - A^T(AM^{-1}A^T)^{-1}AM^{-1}$, then:

$$P_\omega M\ddot{q} + P_\omega C + P_\omega N = P_\omega \tau$$

Denote $\tilde{C} = P_\omega C$, $\tilde{N} = P_\omega N$, $\tilde{\tau} = P_\omega \tau$, $P_\omega M\ddot{\theta} \triangleq \tilde{M}\ddot{\theta}$ is the inertia force in T_q^*Q .

4.5 Lagrange's Equations with Constraints

2 Unified geometric approach (Liu G.F., Li Z.X.)

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□ Lagrange's equations of motion:

$$M(q)\ddot{q} + C(q, \dot{q}) + N = \tau + A^T(q)\lambda \Rightarrow$$

$$\lambda = (AM^{-1}A^T)^{-1}AM^{-1}(M\ddot{q} + (C + N - \tau))$$

Define $P_\omega = I - A^T(AM^{-1}A^T)^{-1}AM^{-1}$, then:

$$P_\omega M\ddot{q} + P_\omega C + P_\omega N = P_\omega \tau$$

Denote $\tilde{C} = P_\omega C$, $\tilde{N} = P_\omega N$, $\tilde{\tau} = P_\omega \tau$, $P_\omega M\ddot{\theta} \triangleq \tilde{M}\ddot{\theta}$ is the inertia force in T_q^*Q .

Definition: Dynamics in T_q^*Q

$$\tilde{M}\ddot{\theta} + \tilde{C} + \tilde{N} = \tilde{\tau}$$

(Continues next slide)

4.5 Lagrange's Equations with Constraints

2 Unified geometric approach (Liu G.F., Li Z.X.)

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Similarly

$$(I - P_\omega)(M\ddot{q} + C + N) = (I - P_\omega)\tau + A^T\tau$$

Let

$$P_T = I - M^{-1}A^T(AM^{-1}A^T)^{-1}A$$

then since $P_\omega M = MP_T$, we have:

Definition: Dynamics in $T_q^*Q^\perp$

$$M(I - P_T)(\ddot{q} + M^{-1}C) = (I - P_\omega)(\tau - N) + A^T\lambda$$

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2 Unified geometric approach (Liu G.F., Li Z.X.)

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Similarly

$$(I - P_\omega)(M\ddot{q} + C + N) = (I - P_\omega)\tau + A^T\tau$$

Let

$$P_T = I - M^{-1}A^T(AM^{-1}A^T)^{-1}A$$

then since $P_\omega M = MP_T$, we have:

Definition: Dynamics in $T_q^*Q^\perp$

$$M(I - P_T)(\ddot{q} + M^{-1}C) = (I - P_\omega)(\tau - N) + A^T\lambda$$

□ Geometric Interpretation:

$$\nabla \leftrightarrow M$$

$$M\ddot{q} + C + N = \tau + A^T\lambda \Leftrightarrow M\nabla_{\dot{q}}\dot{q} = \tau - N + A^T\lambda$$

$$\tilde{\nabla} \leftrightarrow \text{induced metric on } T_qQ$$

$$S : TQ \otimes TQ \mapsto N(Q) : 2^{\text{nd}} \text{ fundamental form}$$

$$TQ : \text{tangent vector field}$$

$$N(Q) : \text{normal vector field}$$

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$$\nabla_X Y = \tilde{\nabla}_X Y + S(X, Y)$$

$$\underbrace{M(I - P_T)(\ddot{q} + M^{-1}C)}_{S(\dot{q}, \dot{q})} = (I - P_\omega)(\tau - N) + A^T \lambda$$

$MS(\dot{q}, \dot{q})$: centrifugal force due to curvature of Q in E

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$$\nabla_X Y = \tilde{\nabla}_X Y + S(X, Y)$$

$$M \underbrace{(I - P_T)(\ddot{q} + M^{-1}C)}_{S(\dot{q}, \dot{q})} = (I - P_\omega)(\tau - N) + A^T \lambda$$

$MS(\dot{q}, \dot{q})$: centrifugal force due to curvature of Q in E

Definition: Hybrid position/force control

$$M\tilde{\nabla}_{\dot{q}}\dot{q} = \tilde{\tau} - \tilde{N}$$

$$MS(\dot{q}, \dot{q}) = (I - P_\omega)(\tau - N) + A^T \lambda$$

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◇ Example: Dynamics of a Spherical Pendulum

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}\dot{q}^T M \dot{q}$$

$$q = (x, y, z)^T, M = mI$$

$$h(q) = q^T q - r^2 = 0$$

$$A = (x, y, z), M_2^\# = AM^\#A^T = r^2/m$$

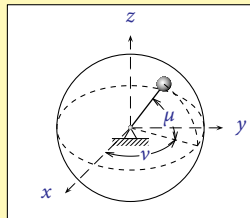


Figure 4.17

$$P_\omega = \frac{1}{r^2} \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yz & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix}$$

$$I - P_\omega = \frac{1}{r^2} \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}$$

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$$P_T = P_\omega^T = P_\omega$$

(μ, ν) : Spherical coordinates

$$q = (r \cos \mu \cos \nu, r \cos \mu \sin \nu, r \sin \mu)^T$$

$$\tilde{\nabla}_{\dot{q}} \dot{q} = P_T (\nabla_{\dot{q}} \dot{q})$$

$$= \begin{bmatrix} -r \sin \mu \cos \nu & -r \sin \nu \\ -r \sin \mu \sin \nu & r \cos \nu \\ r \cos \mu & 0 \end{bmatrix} \begin{bmatrix} \dot{\nu}_1 \\ \dot{\nu}_2 \end{bmatrix}$$

$$S(\dot{q}, \dot{q}) = (I - P_T)(\nabla_{\dot{q}} \dot{q})$$

$$= (-\ddot{\mu}^2 - \cos^2 \mu \dot{\nu}^2) \begin{bmatrix} r \cos \mu \cos \nu \\ r \cos \mu \sin \nu \\ r \sin \mu \end{bmatrix}$$

where

$$\nu_1 = \ddot{\mu} + \sin \mu \cos \mu \dot{\nu}^2$$

$$\nu_2 = \cos \mu \ddot{\nu} - 2 \sin \mu \dot{\mu} \dot{\nu}$$



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2 Unified geometric approach (Liu G.F., Li Z.X.)

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□ Control Algorithm:

1 holonomic constraints:

\tilde{q} : coordinates of Q

$$q = \psi(\tilde{q}) \Rightarrow \dot{q} = J \cdot \dot{\tilde{q}}$$

$$\tau = MJ(\ddot{\tilde{q}}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + C_1 + N + A^T(-\lambda_d + K_I \int (\lambda - \lambda_d))$$

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1 holonomic constraints:

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$$q = \psi(\tilde{q}) \Rightarrow \dot{q} = J \cdot \dot{\tilde{q}}$$

$$\tau = MJ(\ddot{\tilde{q}}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + C_1 + N + A^T(-\lambda_d + K_I \int (\lambda - \lambda_d))$$

2 nonholonomic constraints:

Let $J(q) \in \mathbb{R}^{n \times m}$ be s.t. $AJ = 0$. Write $\dot{q} = J \cdot u$ for some u

$$\tau = MJ(\dot{u}_d - K_p(u - u_d)) + M\dot{J}u + C + N + A^T(-\lambda_d + K_I \int (\lambda - \lambda_d))$$

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◇ Example: 6-DoF manipulator on a sphere with frictionless point contact

■ Contact constraint:

$$v_z = 0 \Leftrightarrow [0 \ 0 \ 1 \ 0 \ 0 \ 0] \text{Ad}_{g_{ff}^{-1}} V_{of} = 0$$

\Rightarrow Holonomic constraint:

$\eta = (\alpha_o^T, \alpha_f^T, \psi) : \text{Parametrization of } Q$

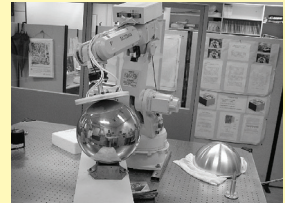
$$P_\omega = \text{diag}(1, 1, 0, 1, 1, 1)$$

■ Newton-Euler Equations of motion:

$$M\dot{V}_{of} - \text{ad}_{V_{of}}^T M V_{of} = F_m + G + A^T \lambda$$

$$V_{of} = \begin{bmatrix} R_\psi M_o & -M_f & 0 \\ 0 & 0 & 0 \\ R_\psi R_o K_o M_o & -R_o K_f M_f & 0 \\ -T_o M_o & -T_f M_f & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_o \\ \dot{\alpha}_f \\ \dot{\psi} \end{bmatrix} \triangleq J \dot{\eta}$$

$$MJ\ddot{\eta} + C_1 = F_m + G + A^T \lambda \quad (*)$$



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4.5 Lagrange's Equations with Constraints

2 Unified geometric approach (Liu G.F., Li Z.X.)

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$P_w(*) :$

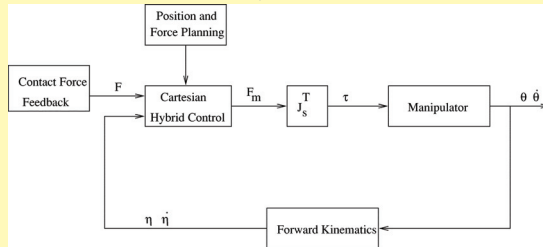
$$\tilde{M}\ddot{\eta} + \tilde{C}_1 = B_1 F_m + B_1 G$$

$$-\phi_3 - \lambda = b_2 F_m + b_2 G$$

$$\hat{F} = [f_1 \ f_2 \ f_4 \ f_5 \ f_6]^T = \tilde{M}(\ddot{\eta}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + \tilde{C}_1 - B_1 G$$

$$f_3 = -\phi_3 - \lambda_d + K_I \int (\lambda - \lambda_d) - b_2 G$$

$$\tau = J_s^T F_m$$



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2 Unified geometric approach (Liu G.F., Li Z.X.)

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◇ Example: 6-DoF manipulator rolling on a sphere

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\dot{A}_1(q)} V_{l_o l_f} = 0$$

$$f_c = A_1^T \lambda, \lambda \in \mathbb{R}^4$$

$$\begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} = -R_0 (K_f + R_\psi K_o R_\psi) M_f \dot{\alpha}_f$$

$$V_{of} = \text{Ad}_{g_{f l_f}} \cdot V_{l_o l_f}$$

$$= \text{Ad}_{g_{f l_f}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -R_o (K_f + R_\psi K_o R_\psi) M_f \\ 0 \end{bmatrix} \dot{\alpha}_f$$

$$\triangleq J_f \dot{\alpha}_f$$

$\text{span}\{J_f\}$: Not involutive

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2 Unified geometric approach (Liu G.F., Li Z.X.)

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$$\begin{aligned}
 MJ_f \ddot{\alpha}_f + (MJ_f \dot{\alpha}_f - \text{ad}_{J_f \dot{\alpha}_f}^T MJ_f \dot{\alpha}_f) &= F_m + G + A^T \lambda \\
 F_m &= MJ_f (\ddot{\alpha}_{fd} - K_p (\dot{\alpha}_f - \dot{\alpha}_{fd})) + (MJ_f \dot{\alpha}_f - \text{ad}_{J_f \dot{\alpha}_f}^T MJ_f \dot{\alpha}_f) \\
 &\quad + A^T (-\lambda_d + \int (\lambda - \lambda_d)) - G
 \end{aligned}$$



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$$\begin{aligned}
 MJ_f \ddot{\alpha}_f + (MJ_f \dot{\alpha}_f - \text{ad}_{J_f \dot{\alpha}_f}^T MJ_f \dot{\alpha}_f) &= F_m + G + A^T \lambda \\
 F_m &= MJ_f (\ddot{\alpha}_{fd} - K_p (\dot{\alpha}_f - \dot{\alpha}_{fd})) + (MJ_f \dot{\alpha}_f - \text{ad}_{J_f \dot{\alpha}_f}^T MJ_f \dot{\alpha}_f) \\
 &\quad + A^T (-\lambda_d + \int (\lambda - \lambda_d)) - G
 \end{aligned}$$

◇

◇ Example: Redundant parallel manipulator

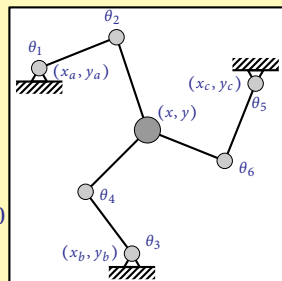
$$\theta = (\theta_1, \dots, \theta_6) \in E$$

$$\theta_a = (\theta_1, \theta_3, \theta_5)$$

$$\theta_p = (\theta_2, \theta_4, \theta_6)$$

$$H(\theta) = \begin{bmatrix} x_a + lc_1 + lc_{12} - x_b - lc_3 - lc_{34} \\ y_a + ls_1 + ls_{12} - y_b - ls_3 - ls_{34} \\ x_a + lc_1 + lc_{12} - x_c - lc_5 - lc_{56} \\ y_a + ls_1 + ls_{12} - y_c - ls_5 - ls_{56} \end{bmatrix} = 0$$

where $c_{ij} = \cos(\theta_i + \theta_j)$, $s_{ij} = \sin(\theta_i + \theta_j)$.



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$$M_i(\theta) \in \mathbb{R}^{2 \times 2} : i^{th} \text{ chain}$$

$$M(\theta) = \text{diag}(M_1(\theta), \dots, M_3(\theta))$$

$$M(\theta)\ddot{\theta} + C + N = \tau + A^T \lambda$$

If all joints are actuated, we can achieve:
 Position control of end-effector
 +
 internal grasping force

As $\tau_2, \tau_4, \tau_6 = 0$,

$\tilde{\theta} \in \mathbb{R}^2$: local parametrization of $Q = H^{-1}(0)$

$\theta = \psi(\tilde{\theta})$: embedding of Q in E

$$\dot{\theta} = J\dot{\tilde{\theta}}$$

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4.5 Lagrange's Equations with Constraints

2 Unified geometric approach (Liu G.F., Li Z.X.)

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Given $P_\omega : T_\theta^*E \mapsto T_\theta^*Q$, the dynamics in T_θ^*Q is given by:

$$P_\omega MJ\ddot{\theta} + P_\omega(C_1 + N) = P_\omega\tau$$

$$\tilde{\tau} = (\tau_1, \tau_3, \tau_5)$$

$$\tilde{P}_\omega = (P_1, P_3, P_5)$$

$$\hat{\tau} = \hat{P}_\omega \tilde{\tau} = P_\omega \tau \in \mathbb{R}^6$$

$$\hat{\tau} = P_\omega MJ(\ddot{\theta}_d - K_v\dot{\tilde{e}} - K_p\tilde{e}) + P_\omega(C_1 + N)$$



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3 Gauge invariant formulation (Aghili, F.)

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□ Gauge-invariant Formulation (Aghili):

◇ Square root factorization of inertia matrix:

$M = WW^T$ (square root factorization)

$$\begin{cases} v \triangleq W^T \dot{q} \in \mathbb{R}^n \\ u \triangleq W^{-1} \tau \in \mathbb{R}^n \end{cases} \quad T = \frac{1}{2} \dot{q}^T M \dot{q} = \frac{1}{2} v^T v$$

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$$M = WW^T \text{ (square root factorization)}$$

$$\begin{cases} v \triangleq W^T \dot{q} \in \mathbb{R}^n \\ u \triangleq W^{-1} \tau \in \mathbb{R}^n \end{cases} \quad T = \frac{1}{2} \dot{q}^T M \dot{q} = \frac{1}{2} v^T v$$

◇ Lagrange's Equation:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = \tau \Rightarrow \frac{d}{dt} (Wv) - \frac{\partial v^T}{\partial q} v = \tau \Rightarrow$$

$$W\dot{v} + \dot{W}v - \frac{\partial v^T}{\partial q} v = \tau \Rightarrow \dot{v} + W^{-1}(\dot{W} - \frac{\partial v^T}{\partial q})v = W^{-1}\tau = u$$

Define $\Gamma \triangleq W^{-1}(\dot{W} - \frac{\partial v^T}{\partial q})$, then:

$$\dot{v} + \Gamma v = u$$

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4.5 Lagrange's Equations with Constraints

3 Gauge invariant formulation (Aghili, F.)

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□ Gauge-invariant Formulation (Aghili):

◇ Change of coordinates:

$$\bar{\mathbf{v}} = V^T \mathbf{v}, \bar{\mathbf{u}} = V^T \mathbf{u}, V \in U(n) \Rightarrow \frac{d}{dt}(V\bar{\mathbf{v}}) + \Gamma(V\bar{\mathbf{v}}) = V\bar{\mathbf{u}} \Rightarrow$$

$$V\dot{\bar{\mathbf{v}}} + \dot{V}\bar{\mathbf{v}} + \Gamma V\bar{\mathbf{v}} = V\bar{\mathbf{u}} \Rightarrow \dot{\bar{\mathbf{v}}} + V^T(\Gamma + \dot{V}V^T)V\bar{\mathbf{v}} = \bar{\mathbf{u}}$$

$$VV^T = I \Rightarrow \dot{V}V^T + V\dot{V}^T = 0 \Rightarrow \dot{V}V^T = -(\dot{V}V^T)^T =: -\Omega \Rightarrow$$

$$\bar{\Gamma} \triangleq V^T(\Gamma - \Omega)V, \Rightarrow \boxed{\dot{\bar{\mathbf{v}}} + \bar{\Gamma}\bar{\mathbf{v}} = \bar{\mathbf{u}}}$$

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3 Gauge invariant formulation (Aghili, F.)

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$$V\dot{\bar{v}} + \dot{V}\bar{v} + \Gamma V\bar{v} = V\bar{u} \Rightarrow \dot{\bar{v}} + V^T(\Gamma + \dot{V}V^T)V\bar{v} = \bar{u}$$

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$$\bar{\Gamma} \triangleq V^T(\Gamma - \Omega)V, \Rightarrow \boxed{\dot{\bar{v}} + \bar{\Gamma}\bar{v} = \bar{u}}$$

◇ Pfaffian constraint:

$$A(q)\dot{q} = 0, A(q) \in \mathbb{R}^{m \times n}, \Lambda \triangleq AW^{-T} \Rightarrow \Lambda v = 0,$$

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$$V\dot{\bar{v}} + \dot{V}\bar{v} + \Gamma V\bar{v} = V\bar{u} \Rightarrow \dot{\bar{v}} + V^T(\Gamma + \dot{V}V^T)V\bar{v} = \bar{u}$$

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$$A(q)\dot{q} = 0, A(q) \in \mathbb{R}^{m \times n}, \Lambda \triangleq AW^{-T} \Rightarrow \Lambda v = 0,$$

◇ Lagrange's equation with constraint:

$$\boxed{\dot{v} + \Gamma v = u + \Lambda^T \lambda}$$

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4.5 Lagrange's Equations with Constraints

3 Gauge invariant formulation (Aghili, F.)

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□ Gauge-invariant Formulation (Aghili):

◇ SVD of Λ :

$$\Lambda = U \Sigma V^T, \bar{v} = V^T v, \bar{u} = V^T u, \bar{\Lambda} \triangleq \Lambda V$$

where $\bar{\Lambda} \bar{v} = 0$ and:

$$\sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, S = \text{diag}(\sigma_1, \dots, \sigma_r), \sigma_1 \geq \dots \geq \sigma_r, r \leq m$$

$$\begin{cases} U = [U_1 \ U_2], U_1 \in \mathbb{R}^{m \times r}, U_2 \in \mathbb{R}^{m \times (m-r)} \\ V = [V_1 \ V_2], V_1 \in \mathbb{R}^{n \times r}, V_2 \in \mathbb{R}^{n \times (n-r)} \end{cases} \Rightarrow$$

$$\bar{\Lambda} = [\Lambda_r \ 0_{m \times (n-r)}], \Lambda_r \triangleq U_1 S, \bar{\Lambda} \bar{v} = 0 \Rightarrow$$

$$\bar{v} = \begin{bmatrix} 0_{r \times 1} \\ v_r \end{bmatrix}, v_r \triangleq V_2^T v = V_2^T W^T \dot{q}$$

(Continues next slide)

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3 Gauge invariant formulation (Aghili, F.)

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□ Gauge-invariant Formulation (Aghili):

$$\bar{u} = \begin{bmatrix} u_o \\ u_r \end{bmatrix}, \begin{cases} u_o = V_1^T W^{-1} \tau \\ u_r = V_2^T W^{-1} \tau \end{cases}$$

$$\bar{\Gamma}_{ij} \triangleq V_i^T (\Gamma - \Omega) V_j, i, j = 1, 2, \Gamma_r \triangleq \bar{\Gamma}_{22}, \Gamma_o \triangleq \bar{\Gamma}_{12}$$

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$$\bar{\Gamma}_{ij} \triangleq V_i^T (\Gamma - \Omega) V_j, i, j = 1, 2, \Gamma_r \triangleq \bar{\Gamma}_{22}, \Gamma_o \triangleq \bar{\Gamma}_{12}$$

◇ Decoupled equation of motion/constrained force:

$$\begin{aligned} \dot{v}_r + \Gamma_r v_r &= u_r \\ \Gamma_o v_r &= u_o + \Gamma_r^T \lambda \end{aligned}$$

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$$\bar{\Gamma}_{ij} \triangleq V_i^T (\Gamma - \Omega) V_j, i, j = 1, 2, \Gamma_r \triangleq \bar{\Gamma}_{22}, \Gamma_o \triangleq \bar{\Gamma}_{12}$$

◇ Decoupled equation of motion/constrained force:

$$\begin{aligned} \dot{v}_r + \Gamma_r v_r &= u_r \\ \Gamma_o v_r &= u_o + \Gamma_r^T \lambda \end{aligned}$$

◇ Combined equation of motion (Kane's equation):

$$\frac{d}{dt} \begin{bmatrix} q \\ v_r \end{bmatrix} = \begin{bmatrix} W^{-T} V_2 \\ -\Gamma_r \end{bmatrix} v_r + \begin{bmatrix} 0 \\ I \end{bmatrix} u_r$$

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4.5 Lagrange's Equations with Constraints

3 Gauge invariant formulation (Aghili, F.)

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□ Gauge-invariant Formulation (Aghili):

◇ Composite error vector ε

$q = q(\theta), \theta \in \mathbb{R}^{n-r}$: generalized coordinate (of $Q \subset \mathbb{R}^n$)

$$v_r = B(\theta)\dot{\theta}, B(\theta) \triangleq V_2^T W^T J, J \triangleq \frac{\partial q}{\partial \theta}, \tilde{\theta} \triangleq \theta - \theta_d, \tilde{v}_r \triangleq v_r - v_{r_d} \Rightarrow$$

$$\varepsilon \triangleq \tilde{v}_r + BK_p \tilde{\theta} = B(\dot{\tilde{\theta}} + K_p \tilde{\theta}) : \text{composite error}$$

$$s \triangleq v_{r_d} - BK_p \tilde{\theta} = v_r - \varepsilon = B(\dot{\theta}_d - K_p \tilde{\theta}), \tilde{\lambda} \triangleq \lambda - \lambda_d$$

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$$v_r = B(\theta)\dot{\theta}, B(\theta) \triangleq V_2^T W^T J, J \triangleq \frac{\partial q}{\partial \theta}, \tilde{\theta} \triangleq \theta - \theta_d, \tilde{v}_r \triangleq v_r - v_{r_d} \Rightarrow$$

$$\varepsilon \triangleq \tilde{v}_r + BK_p \tilde{\theta} = B(\dot{\tilde{\theta}} + K_p \tilde{\theta}) : \text{composite error}$$

$$s \triangleq v_{r_d} - BK_p \tilde{\theta} = v_r - \varepsilon = B(\dot{\theta}_d - K_p \tilde{\theta}), \tilde{\lambda} \triangleq \lambda - \lambda_d$$

◇ Hybrid position/force control

$$u_r = \dot{s} + \Gamma_r s - K_d \varepsilon$$

$$u_o = -\Lambda_r^T \lambda_d + \Gamma_o v_r$$

Note: integration term $K_I \int (\lambda - \lambda_d)$ is missing from u_o .
(Continues next slide)

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3 Gauge invariant formulation (Aghili, F.)

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□ Gauge-invariant Formulation (Aghili):

$$\dot{v}_r + \Gamma_r v_r = u_r = \frac{d}{dt}(v_r - \varepsilon) + \Gamma_r(v_r - \varepsilon) - K_d \varepsilon \Rightarrow \dot{\varepsilon} = -(\Gamma_r + K_d)\varepsilon$$

$$\Lambda_r^T \lambda + \Gamma_o v_r = u_o = -\Lambda_r^T \lambda_d + \Gamma_o v_r \Rightarrow \Lambda_r^T \tilde{\lambda} + K_I \int \tilde{\lambda} = 0$$

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4.5 Lagrange's Equations with Constraints

3 Gauge invariant formulation (Aghili, F.)

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□ Gauge-invariant Formulation (Aghili):

$$\dot{v}_r + \Gamma_r v_r = u_r = \frac{d}{dt}(v_r - \varepsilon) + \Gamma_r(v_r - \varepsilon) - K_d \varepsilon \Rightarrow \dot{\varepsilon} = -(\Gamma_r + K_d)\varepsilon$$

$$\Lambda_r^T \lambda + \Gamma_o v_r = u_o = -\Lambda_r^T \lambda_d + \Gamma_o v_r \Rightarrow \Lambda_r^T \tilde{\lambda} + K_I \int \tilde{\lambda} = 0$$

◇ Equivalence to the Geometric approach

$$\boxed{\begin{matrix} P_\omega = W V_2 V_2^T W^{-1} \\ I - P_\omega = W V_1 V_1^T W^{-1} \end{matrix}} \Rightarrow \begin{matrix} V^T W^{-1} P_\omega \tau = \begin{bmatrix} 0 \\ V_2^T W^{-1} \tau \end{bmatrix} = \begin{bmatrix} 0 \\ u_r \end{bmatrix} \\ V^T W^{-1} (I - P_\omega) \tau = \begin{bmatrix} V_1^T W^{-1} \tau \\ 0 \end{bmatrix} = \begin{bmatrix} u_o \\ 0 \end{bmatrix} \end{matrix}$$

(Continues next slide)

4.5 Lagrange's Equations with Constraints

3 Gauge invariant formulation (Aghili, F.)

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□ Gauge-invariant Formulation (Aghili):

1. Geometric approach (K_p, K_I, K_d):

$$u = \underbrace{\begin{bmatrix} V_1^T W^{-1} A^T (-\lambda_d + K_I \int (\lambda - \lambda_d)) \\ V_2^T W^T J (\ddot{\theta}_d - K_d \dot{\tilde{\theta}} - K_p \tilde{\theta}) \end{bmatrix}}_{fb} + \underbrace{V^T W^{-1} C_1}_{ff}$$

2. Gauge-invariant formulation (\tilde{K}_p, \tilde{K}_d):

$$u = \begin{bmatrix} u_o \\ u_r \end{bmatrix} = \underbrace{\begin{bmatrix} V_1^T W^{-1} A^T (-\lambda_d) \\ V_2^T W^T J (\ddot{\theta}_d - \tilde{K}'_d \dot{\tilde{\theta}} - \tilde{K}'_p \tilde{\theta}) \end{bmatrix}}_{fb} + \underbrace{C'_1}_{ff} \Rightarrow$$

$$K'_d \triangleq \tilde{K}_p + (V_2^T W^T J)^{-1} \tilde{K}_d (V_2^T W^T J)$$

$$K'_p \triangleq (V_2^T W^T J)^{-1} \tilde{K}_d (V_2^T W^T J) \tilde{K}_p$$

† End of Section †

4.6 References

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