Problem 1

a.

由定义:

$$L_{n,k}(x)=\prod_{i=0,i\neq k}^{n}\frac{x-x_i}{x_k-x_i}$$

$$P(x)=\sum_{k=0}^{n}f(x_k)L_{n,k}(x)$$

$$L_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P(x)=f(0)L_0(x)+f(0.3)L_1(x)+f(0.6)L_2(x)$$

$$P(x)=-11.22x^2+3.81x+1$$

$$\text{由于}f(x)=P(x)+\frac{f^{n+1}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n)$$

$$\frac{f^{n+1}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n)=\frac{f^3(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$\frac{f^3(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)=-1.5e^{2\xi(x)}sin(3\xi(x))-\frac{23}{3}e^{2\xi(x)}cos(3\xi(x))\ldots(1)$$

$$\text{易得}(1)\text{式的绝对值小于等于}10.941$$

$$\text{考虑}(x-x_0)(x-x_1)(x-x_2)$$

$$\text{当}x\in[0,0.6]\text{时，}|(x-x_0)(x-x_1)(x-x_2)|\leq 0.0104$$

$$\text{综上，最大误差边界为}0.1137$$

b.

$$L_{n,k}(x) = \prod_{i=0,i\neq k}^{n} \frac{x-x_i}{x_k-x_i}$$

$$P(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P(x) = f(2)L_0(x) + f(2.4)L_1(x) + f(2.6)L_2(x)$$

$$P(x) = -0.033x^2 + 0.0003x + 1$$

$$\text{由于} f(x) = P(x) + \frac{f^{n+1}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$$

$$\frac{f^{n+1}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n) = \frac{f^3(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$\frac{f^3(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2) = \frac{3\sin(\ln(\xi(x))) + \cos(\ln(\xi(x)))}{\xi(x)^3}(x-x_0)(x-x_1)(x-x_2)\dots(1)$$

$$\text{易得}, (1)\text{式的绝对值小于等于}0.33576$$

$$|(x-x_0)(x-x_1)(x-x_2)| \leq 0.016901$$

$$\text{最大误差边界为}0.005675$$

Problem 2

$$L_{n,k}(x) = \prod_{i=0,i\neq k}^{n} \frac{x-x_i}{x_k-x_i}$$

$$P(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$P(x) = f(0)L_0(x) + f(0.5)L_1(x) + f(1)L_2(x) + f(2)L_3(x)$$

$$f(0) = 0, f(0.5) = y, f(1) = 3, f(2) = 2$$

$$\text{易得} y = 4.25$$

Problem 3

a.

$$P_{2,3} = \frac{1}{x_3 - x_2}[(x - x_2)P_3 - (x - x_3)P_2] = 2.4$$
$$P_2 = 2.4$$

b.

$$P_{0,1,2,3} = \frac{1}{x_3 - x_0}[(x - x_0)P_{1,2,3} - (x - x_3)P_{0,1,2}]$$
$$P_{0,1,2,3}(2.5) = \frac{1}{x_3 - x_0}[(2.5 - x_0) * 3 - (2.5 - x_3)P_{0,1,2}(2.5)]$$
$$P_{0,1,2} = \frac{1}{x_2 - x_0}[(x - x_0)P_{1,2} - (x - x_2)P_{0,1}]$$
$$P_{0,1} = \frac{1}{x_1 - x_0}[(x - x_0)P_1 - (x - x_1)P_0] = 2x + 1$$
$$P_{0,2} = \frac{1}{x_2 - x_0}[(x - x_0)P_2 - (x - x_2)P_0] = x + 1$$
$$P_{1,2} = \frac{1}{x_2 - x_1}[(x - x_1)P_2 - (x - x_2)P_1]$$
$$P_{0,1,2,3}(2.5) = 2.875$$

Problem 4

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = 10$$
$$f[x_1] = 3$$
$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{50}{7}$$
$$f[x_0, x_1] = 5$$
$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = 5$$
$$f[x_0] = 1$$

Problem 5

natural cubic spline

$$S_0 = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3$$
$$S_1 = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3$$
$$f(0) = 0 = a_0 \quad f(1) = 1 = a_0 + b_0 + c_0 + d_0 = a_1$$
$$f(2) = 2 = a_1 + b_1 + c_1 + d_1$$
$$S_0'(1) = S_1'(1) \quad => \quad b_0 + 2c_0 + 3d_0 = b_1$$
$$S_0''(1) = S_1''(1) \quad => \quad 2c_0 + 6d_0 = 2c_1$$
$$\text{由于是} nataural \ \ cubic \ \ spline$$
$$S_0''(0) = S_1''(2) = 0 \quad => 6d_1 + 2c_1 = 2c_0 = 0$$
$$\text{解得:}$$
$$a_0 = 0, b_0 = 1, c_0 = 0, d_0 = 0$$
$$a_1 = 1, b_1 = 1, c_1 = 0, d_1 = 0$$
$$x \in [0,1]\text{时,} \ S(X) = x$$
$$x \in [1,2]\text{时}, S(x) = x$$

clamped cubic spline

$$S_0 = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3$$
$$S_1 = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3$$
$$f(0) = 0 = a_0 \quad f(1) = 1 = a_0 + b_0 + c_0 + d_0 = a_1$$
$$f(2) = 2 = a_1 + b_1 + c_1 + d_1$$
$$S_0'(1) = S_1'(1) \quad => \quad b_0 + 2c_0 + 3d_0 = b_1$$
$$S_0''(1) = S_1''(1) \quad => \quad 2c_0 + 6d_0 = 2c_1$$
$$\text{由于是} clamped \ boundary$$
$$S_0'(0) = f'(0) \quad S_1'(2) = f'(2)$$
$$S_0'(0) = S'(2) = 1$$
$$b_0 = b_1 + 2c_1 + 6d_1 = 1$$
$$\text{解得:}$$
$$a_0 = 0, b_0 = 1, c_0 = 0, d_0 = 0$$
$$a_1 = 1, b_1 = 1, c_1 = 0, d_1 = 0$$
$$\text{当} x \in [0,1]\text{时}, S(x) = x$$
$$\text{当} x \in [1,2]\text{时} S(x) = x$$

Problem 6

考虑证明AX=0有且只有零解

$$AX = 0$$

设k满足以下条件：

$$0 < |x_k| = max|x_j|$$

$$\sum_{j=1}^{n} a_{ij}x_j = 0$$

取$i = k$ $\quad a_{kk}x_k = -\sum_{j=1,j\neq k}^{n} a_{kj}x_j$

$$|a_{kk}||x_k| \leq \sum_{j=1,j\neq k}^{n} |a_{kj}||x_j|$$

$$|a_{kk}| \leq \sum_{j=1,j\neq k}^{n} |a_{kj}|\frac{|x_j|}{|x_k|} \leq \sum_{j=1,j\neq k}^{n} |a_{kj}|$$

与$|a_{kk}| > |a_{kj}|$矛盾，所以$x = 0$

由于$AX = 0$有且只有0解，A可逆