Formalization of Pure Type System

1. Definition

(i) A *pure type system* (*PTS*) is a triple tuple $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ where

(a) $\mathcal{S}$ is a set of *sorts*;

(b) $\mathcal{A} \subseteq \mathcal{S} \times \mathcal{S}$ is a set of *axioms*;

(c) $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{S}$ is a set of *rules*.

(ii) *Raw expressions* A and *raw environments* Γ are defined by

$$
\begin{aligned}
A &::= x \mid s \mid AA \mid \lambda x : A.A \mid \Pi x : A.A \\
\Gamma &::= \varnothing \mid \Gamma, x : A
\end{aligned}
$$

where we use s, t, u, etc. to range over sorts, x, y, z, etc. to range over variables, and A, B, C, a, b, c, etc. to range over expressions.

(iii) Π and λ are used to bind variables. Let $\mathrm{FV}(A)$ denote free variable set of A. Let $A[x := B]$ denote the substitution of x in A with B. Standard notational conventions are applied here. Besides we also define $A \to B$ as $(\Pi x : A.B)$ when $x \notin \mathrm{FV}(B)$.

(iv) The relation $\to_\beta$ on raw expressions is the compatible closure of

$$
(\lambda x : A.M)N \to_\beta M[x := N],
$$

which can be used to define the notation $\twoheadrightarrow_\beta$ and $=_\beta$ by convention.

(v) Type assignment rules for $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ are given in Figure 1.

$$\text{(Ax)} \qquad \frac{}{\vdash s : t} \qquad (s,t) \in \mathcal{A}$$

$$\text{(Var)} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \qquad x \notin \mathrm{dom}(\Gamma)$$

$$\text{(Weak)} \qquad \frac{\Gamma \vdash b : B \qquad \Gamma \vdash A : s}{\Gamma, x : A \vdash b : B} \qquad x \notin \mathrm{dom}(\Gamma)$$

$$\text{(App)} \qquad \frac{\Gamma \vdash f : (\Pi x : A.B) \qquad \Gamma \vdash a : A}{\Gamma \vdash fa : B[x := a]}$$

$$\text{(Lam)} \qquad \frac{\Gamma, x : A \vdash b : B \qquad \Gamma \vdash (\Pi x : A.B) : t}{\Gamma \vdash (\lambda x : A.b) : (\Pi x : A.B)}$$

$$\text{(Pi)} \qquad \frac{\Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : t}{\Gamma \vdash (\Pi x : A.B) : u} \qquad (s,t,u) \in \mathcal{R}$$

$$\text{(Conv)} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash B : s \qquad A =_\beta B}{\Gamma \vdash a : B}$$

Figure 1. Type rules for pure type system

2. Examples of PTSs

References

[1] Simon Peyton Jones and Erik Meijer. Henk: a typed intermediate language. *TIC*, 97, 1997.

[2] Morten Heine Sørensen and Pawel Urzyczyn. *Lectures on the Curry-Howard isomorphism*, volume 149. Elsevier, 2006.