# 1 Declarative System

| | |
|---|---|
| Types | $A, B ::= \mathsf{Int} \mid a \mid A \to B \mid \forall a.\, A \mid \star \mid \mathcal{S} \mid \mathcal{G}$ |
| Monotypes | $\tau, \sigma ::= \mathsf{Int} \mid a \mid \tau \to \sigma \mid \mathcal{S} \mid \mathcal{G}$ |
| Castable Types | $\mathbb{C} ::= \mathsf{Int} \mid a \mid \mathbb{C}_1 \to \mathbb{C}_2 \mid \forall a.\, \mathbb{C} \mid \star \mid \mathcal{G}$ |
| Castable Monotypes | $t ::= \mathsf{Int} \mid a \mid t_1 \to t_2 \mid \mathcal{G}$ |
| Contexts | $\Psi ::= \bullet \mid \Psi, x : A \mid \Psi, a$ |
| Colored Types | $A, B ::= {\color{red}\mathsf{Int}} \mid {\color{blue}\mathsf{Int}} \mid {\color{red}a} \mid {\color{blue}a} \mid A \to B \mid \forall {\color{red}a}.A \mid \forall {\color{blue}a}.A \mid {\color{blue}\star} \mid {\color{red}\mathcal{S}} \mid {\color{red}\mathcal{G}} \mid {\color{blue}\mathcal{G}}$ |
| Blue Castable Types | ${\color{blue}\mathbb{C}} ::= {\color{blue}\mathsf{Int}} \mid {\color{blue}a} \mid {\color{blue}\mathbb{C}_1} \to {\color{blue}\mathbb{C}_2} \mid \forall {\color{blue}a}.{\color{blue}\mathbb{C}} \mid {\color{blue}\star} \mid {\color{blue}\mathcal{G}}$ |
| Blue Monotypes | ${\color{blue}\tau} ::= {\color{blue}\mathsf{Int}} \mid {\color{blue}a} \mid {\color{blue}\tau} \to {\color{blue}\sigma} \mid {\color{blue}\mathcal{G}}$ |
| Red Monotypes | ${\color{red}\tau} ::= {\color{red}\mathsf{Int}} \mid {\color{red}a} \mid {\color{red}\tau} \to {\color{red}\sigma} \mid {\color{red}\tau} \to {\color{blue}\sigma} \mid {\color{blue}\tau} \to {\color{red}\sigma} \mid {\color{red}\mathcal{S}} \mid {\color{red}\mathcal{G}}$ |

$\boxed{A \sim B}$ *(Type Consistent)*

$$\frac{}{A \sim A} \qquad \frac{}{{\color{blue}\mathbb{C}} \sim \star} \qquad \frac{}{\star \sim {\color{blue}\mathbb{C}}} \qquad \frac{A_1 \sim B_1 \qquad A_2 \sim B_2}{A_1 \to A_2 \sim B_1 \to B_2} \qquad \frac{A \sim B}{\forall a.\, A \sim \forall a.\, B}$$

$\boxed{\Psi \vdash A <: B}$ *(Subtyping)*

$$\frac{\Psi, a \vdash A <: B}{\Psi \vdash A <: \forall a.\, B}\ \textsc{s-forallR} \qquad \frac{\Psi \vdash {\color{red}\tau} \qquad \Psi \vdash A[{\color{red}a} \mapsto {\color{red}\tau}] <: B}{\Psi \vdash \forall {\color{red}a}.A <: B}\ \textsc{s-forallLr}$$

$$\frac{\Psi \vdash {\color{blue}\tau} \qquad \Psi \vdash A[{\color{blue}a} \mapsto {\color{blue}\tau}] <: B}{\Psi \vdash \forall {\color{blue}a}.A <: B}\ \textsc{s-forallLb} \qquad \frac{{\color{red}a} \in \Psi}{\Psi \vdash {\color{red}a} <: {\color{red}a}}\ \textsc{s-tvarR}$$

$$\frac{{\color{blue}a} \in \Psi}{\Psi \vdash {\color{blue}a} <: {\color{blue}a}}\ \textsc{s-tvarB} \qquad \frac{}{\Psi \vdash {\color{red}\mathsf{Int}} <: {\color{red}\mathsf{Int}}}\ \textsc{s-intR} \qquad \frac{}{\Psi \vdash {\color{blue}\mathsf{Int}} <: {\color{blue}\mathsf{Int}}}\ \textsc{s-intB}$$

$$\frac{\Psi \vdash B_1 <: A_1 \qquad \Psi \vdash A_2 <: B_2}{\Psi \vdash A_1 \to A_2 <: B_1 \to B_2}\ \textsc{s-arrow} \qquad \frac{}{\Psi \vdash {\color{blue}\star} <: {\color{blue}\star}}\ \textsc{s-unknown}$$

$$\frac{}{\Psi \vdash {\color{red}\mathcal{S}} <: {\color{red}\mathcal{S}}}\ \textsc{s-SparR} \qquad \frac{}{\Psi \vdash {\color{red}\mathcal{G}} <: {\color{red}\mathcal{G}}}\ \textsc{s-GparR} \qquad \frac{}{\Psi \vdash {\color{blue}\mathcal{G}} <: {\color{blue}\mathcal{G}}}\ \textsc{s-GparB}$$

$\boxed{\Psi \vdash A \lesssim B}$ *(Consistent Subtyping)*

$$\frac{\Psi, a \vdash A \lesssim B}{\Psi \vdash A \lesssim \forall a.\, B}\ \textsc{cs-forallR} \qquad \frac{\Psi \vdash \tau \qquad \Psi \vdash A[a \mapsto \tau] \lesssim B}{\Psi \vdash \forall a.\, A \lesssim B}\ \textsc{cs-forallL}$$

$$\frac{\Psi \vdash B_1 \lesssim A_1 \qquad \Psi \vdash A_2 \lesssim B_2}{\Psi \vdash A_1 \to A_2 \lesssim B_1 \to B_2}\ \textsc{cs-arrow} \qquad \frac{a \in \Psi}{\Psi \vdash a \lesssim a}\ \textsc{cs-tvar}$$

$$\frac{}{\Psi \vdash \mathsf{Int} \lesssim \mathsf{Int}}\ \textsc{cs-int} \qquad \frac{}{\Psi \vdash \star \lesssim \mathbb{C}}\ \textsc{cs-unknownL} \qquad \frac{}{\Psi \vdash \mathbb{C} \lesssim \star}\ \textsc{cs-unknownR}$$

$$\frac{}{\Psi \vdash \mathcal{S} \lesssim \mathcal{S}}\ \textsc{cs-spar} \qquad \frac{}{\Psi \vdash \mathcal{G} \lesssim \mathcal{G}}\ \textsc{cs-gpar}$$

$\boxed{\Psi \vdash e : A \rightsquigarrow s}$ *(Typing)*

$$\frac{(x : A) \in \Psi}{\Psi \vdash x : A \rightsquigarrow x}\ \textsc{var} \qquad \frac{}{\Psi \vdash n : \mathsf{Int} \rightsquigarrow n}\ \textsc{int} \qquad \frac{\Psi, a \vdash e : A \rightsquigarrow s}{\Psi \vdash e : \forall a.\, A \rightsquigarrow \Lambda a.\, s}\ \textsc{gen}$$

$$\frac{\Psi, x : A \vdash e : B \rightsquigarrow s}{\Psi \vdash \lambda x : A.\, e : A \to B \rightsquigarrow \lambda x : A.\, s}\ \textsc{lamann}$$

$$\frac{\Psi, x : \tau \vdash e : B \rightsquigarrow s}{\Psi \vdash \lambda x.\, e : \tau \to B \rightsquigarrow \lambda x : \tau.\, s}\ \textsc{lam}$$

$$\frac{\begin{array}{cc}\Psi \vdash e_1 : A \rightsquigarrow s_1 & \Psi \vdash A \triangleright A_1 \to A_2 \\ \Psi \vdash e_2 : A_3 \rightsquigarrow s_2 & \Psi \vdash A_3 \lesssim A_1\end{array}}{\Psi \vdash e_1\, e_2 : A_2 \rightsquigarrow (\langle A \hookrightarrow A_1 \to A_2 \rangle s_1)\, (\langle A_3 \hookrightarrow A_1 \rangle s_2)}\ \textsc{app}$$

$\boxed{\Psi \vdash A \triangleright A_1 \to A_2}$ *(Matching)*

$$\frac{\Psi \vdash \tau \qquad \Psi \vdash A[a \mapsto \tau] \triangleright A_1 \to A_2}{\Psi \vdash \forall a.\, A \triangleright A_1 \to A_2}\ \textsc{m-forall} \qquad \frac{}{\Psi \vdash A_1 \to A_2 \triangleright A_1 \to A_2}\ \textsc{m-arr}$$

$$\frac{}{\Psi \vdash \star \triangleright \star \to \star}\ \textsc{m-unknown}$$

## 2 Target: PBC

$$\text{Terms } s ::= x \mid n \mid \lambda x : A.\, s \mid \Lambda a.\, s \mid s_1\, s_2 \mid \langle A \hookrightarrow B \rangle s$$

## 3 Metatheory

**Definition 1 (Substitution).**

1. *Gradual type parameter substitution* $S^{\mathcal{G}} :: \mathcal{G} \to t$
2. *Static type parameter substitution* $S^{\mathcal{S}} :: \mathcal{S} \to \tau$
3. *Type parameter Substitution* $S^{\mathcal{P}} = S^{\mathcal{G}} \cup S^{\mathcal{S}}$

Ningning: Note substitution ranges are monotypes.

**Definition 2 (Translation Pre-order).** *Suppose* $\Psi \vdash e : A \rightsquigarrow s_1$ *and* $\Psi \vdash e : A \rightsquigarrow s_2$*, we define* $s_1 \leq s_2$ *to mean* $s_2 = S^{\mathcal{P}}(s_1)$ *for some* $S^{\mathcal{P}}$*.*

**Proposition 1.** *If* $s_1 \leq s_2$ *and* $s_2 \leq s_1$*, then* $s_1$ *and* $s_2$ *are equal up to* $\alpha$*-renaming of type parameters.*

**Definition 3 (Representative Translation).** $s$ *is a representative translation of a typing derivation* $\Psi \vdash e : A \rightsquigarrow s$ *if and only if for any other translation* $\Psi \vdash e : A \rightsquigarrow s'$ *such that* $s' \leq s$*, we have* $s \leq s'$*. From now on we use* $r$ *to denote a representative translation.*

**Definition 4 (Measurements of Translation).** *There are three measurements of a translation* $s$*,*

1. $[\![s]\!]_{\mathcal{E}}$*, the size of the expression*
2. $[\![s]\!]_{\mathcal{S}}$*, the number of distinct static type parameters in* $s$
3. $[\![s]\!]_{\mathcal{G}}$*, the number of distinct gradual type parameters in* $s$

*We use* $[\![s]\!]$ *to denote the lexicographical order of the triple* $([\![s]\!]_{\mathcal{E}}, -[\![s]\!]_{\mathcal{S}}, -[\![s]\!]_{\mathcal{G}})$*.*

**Definition 5 (Size of types).**

$$\begin{aligned}
[\![\mathsf{Int}]\!] &= 1 \\
[\![a]\!] &= 1 \\
[\![A \to B]\!] &= [\![A]\!] + [\![B]\!] + 1 \\
[\![\forall a.\, A]\!] &= [\![A]\!] + 1 \\
[\![\star]\!] &= 1 \\
[\![\mathcal{S}]\!] &= 1 \\
[\![\mathcal{G}]\!] &= 1
\end{aligned}$$

**Definition 6 (Size of expressions).**

$$
\begin{aligned}
[\![x]\!]_{\mathcal{E}} &= 1 \\
[\![n]\!]_{\mathcal{E}} &= 1 \\
[\![\lambda x : A.\, s]\!]_{\mathcal{E}} &= [\![A]\!] + [\![s]\!]_{\mathcal{E}} + 1 \\
[\![\Lambda a.\, s]\!]_{\mathcal{E}} &= [\![s]\!]_{\mathcal{E}} + 1 \\
[\![s_1\, s_2]\!]_{\mathcal{E}} &= [\![s_1]\!]_{\mathcal{E}} + [\![s_2]\!]_{\mathcal{E}} + 1 \\
[\![\langle A \hookrightarrow B \rangle s]\!]_{\mathcal{E}} &= [\![s]\!]_{\mathcal{E}} + [\![A]\!] + [\![B]\!] + 1
\end{aligned}
$$

**Lemma 1.** *If $\Psi \vdash e : A \rightsquigarrow s$ then $[\![s]\!]_{\mathcal{E}} \geq [\![e]\!]_{\mathcal{E}}$.*

*Proof.* Immediate by inspecting each typing rule.

**Corollary 1.** *If $\Psi \vdash e : A \rightsquigarrow s$ then $[\![s]\!] > ([\![e]\!]_{\mathcal{E}}, -[\![e]\!]_{\mathcal{E}}, -[\![e]\!]_{\mathcal{E}})$.*

*Proof.* By Lemma 1 and note that $[\![s]\!]_{\mathcal{E}} > [\![s]\!]_{\mathcal{S}}$ and $[\![s]\!]_{\mathcal{E}} > [\![s]\!]_{\mathcal{G}}$

**Lemma 2.** $[\![A]\!] \leq [\![S^{\mathcal{P}}(A)]\!]$.

*Proof.* By induction on the structure of $A$. The interesting cases are $A = \mathcal{S}$ and $A = \mathcal{G}$. When $A = \mathcal{S}$, $S^{\mathcal{P}}(A) = \tau$ for some monotype $\tau$ and it is immediate that $[\![\mathcal{S}]\!] \leq [\![\tau]\!]$ (note that $[\![\mathcal{S}]\!] < [\![\mathcal{G}]\!]$ by definition).

**Lemma 3 (Substitution Decreases Measurement).** *If $s_1 \leq s_2$, then $[\![s_1]\!] \leq [\![s_2]\!]$; unless $s_2 \leq s_1$ also holds, otherwise we have $[\![s_1]\!] < [\![s_2]\!]$.*

*Proof.* Since $s_1 \leq s_2$, we know $s_2 = S^{\mathcal{P}}(s_1)$ for some $S^{\mathcal{P}}$. By induction on the structure of $s_1$.

- Case $s_1 = \lambda x : A.\, s$. We have $s_2 = \lambda x : S^{\mathcal{P}}(A).\, S^{\mathcal{P}}(s)$. By Lemma 2 we have $[\![A]\!] \leq [\![S^{\mathcal{P}}(A)]\!]$. By i.h., we have $[\![s]\!] \leq [\![S^{\mathcal{P}}(s)]\!]$. Therefore $[\![\lambda x : A.\, s]\!] \leq [\![\lambda x : S^{\mathcal{P}}(A).\, S^{\mathcal{P}}(s)]\!]$.
- Case $s_1 = \langle A \hookrightarrow B \rangle s$. We have $s_2 = \langle S^{\mathcal{P}}(A) \hookrightarrow S^{\mathcal{P}}(B) \rangle S^{\mathcal{P}}(s)$. By Lemma 2 we have $[\![A]\!] \leq [\![S^{\mathcal{P}}(A)]\!]$ and $[\![B]\!] \leq [\![S^{\mathcal{P}}(B)]\!]$. By i.h., we have $[\![s]\!] \leq [\![S^{\mathcal{P}}(s)]\!]$. Therefore $[\![\langle A \hookrightarrow B \rangle s]\!] \leq [\![\langle S^{\mathcal{P}}(A) \hookrightarrow S^{\mathcal{P}}(B) \rangle S^{\mathcal{P}}(s)]\!]$.
- The rest of cases are immediate.

**Lemma 4 (Representative Translation for Typing).** *For any typing derivation that $\Psi \vdash e : A$, there exists at least one representative translation $r$ such that $\Psi \vdash e : A \rightsquigarrow r$.*

*Proof.* We already know that at least one translation $s = s_1$ exists for every typing derivation. If $s_1$ is a representative translation then we are done. Otherwise there exists another translation $s_2$ such that $s_2 \leq s_1$ and $s_1 \not\leq s_2$. By Lemma 3, we have $[\![s_2]\!] < [\![s_1]\!]$. We continue with $s = s_2$, and get a strictly decreasing sequence $[\![s_1]\!], [\![s_2]\!], \ldots$. By Corollary 1, we know this sequence cannot be infinite long. Suppose it ends at $[\![s_n]\!]$, by the construction of the sequence, we know that $s_n$ is a representative translation of $e$.

*Conjecture 1 (Property of Representative Translation).* If $\bullet \vdash e : A \boxed{\rightsquigarrow s}$, $|s| \Downarrow v$, then we have $\bullet \vdash e : A \boxed{\rightsquigarrow r}$, and $|r| \Downarrow v'$.

Ningning: shall we focus on values of type integer?

**Definition 7 (Erasure of Type Parameters).**

$$\begin{array}{ll} |\mathsf{Int}| = \mathsf{Int} & |a| = a \\ |A \to B| = |A| \to |B| & |\forall a. A| = \forall a. |A| \\ |\star| = \star & |\mathcal{S}| = \mathsf{Int} \\ |\mathcal{G}| = \star & \end{array}$$

**Corollary 2 (Coherence up to cast errors).** *Suppose* $\bullet \vdash e : \mathsf{Int} \boxed{\rightsquigarrow s_1}$ *and* $\bullet \vdash e : \mathsf{Int} \boxed{\rightsquigarrow s_2}$, *if* $|s_1| \Downarrow n$ *then either* $|s_2| \Downarrow n$ *or* $|s_2| \Downarrow \mathsf{blame}$.

Jeremy: maybe Conjecture 1 is enough to prove it?

*Conjecture 2 (Dynamic Guarantee).* Suppose $e' \sqsubseteq e$,

1. If $\bullet \vdash e : A \boxed{\rightsquigarrow r}$, $|r| \Downarrow v$, then for some $B$ and $r'$, we have $\bullet \vdash e' : B \boxed{\rightsquigarrow r'}$, and $B \sqsubseteq A$, and $|r'| \Downarrow v'$, and $v' \sqsubseteq v$.
2. If $\bullet \vdash e' : B \boxed{\rightsquigarrow r'}$, $|r'| \Downarrow v'$, then for some $A$ and $r$, we have $\bullet \vdash e : A \boxed{\rightsquigarrow r}$, and $B \sqsubseteq A$. Moreover, $|r| \Downarrow v$ and $v' \sqsubseteq v$, or $|r| \Downarrow \mathsf{blame}$.

## 4 Efficient (Almost) Typed Encodings of ADTs

- Scott encodings of simple first-order ADTs (e.g. naturals)
- Parigot encodings improves Scott encodings with recursive schemes, but occupies exponential space, whereas Church encoding only occupies linear space.
- An alternative encoding which retains constant-time destructors but also occupies linear space.
- Parametric ADTs also possible?
- Typing rules

*Example 1 (Scott Encoding of Naturals).*

$$\begin{aligned} \mathsf{Nat} &\triangleq \forall a.\, a \to (\star \to a) \to a \\ \mathsf{zero} &\triangleq \lambda x.\, \lambda f.\, x \\ \mathsf{succ} &\triangleq \lambda y : \mathsf{Nat}.\, \lambda x.\, \lambda f.\, f\ y \end{aligned}$$

Scott encodings give constant-time destructors (e.g., predecessor), but one has to get recursion somewhere. Since our calculus admits untyped lambda calculus, we could use a fixed point combinator.

*Example 2 (Parigot Encoding of Naturals).*

$$\begin{aligned} \mathsf{Nat} &\triangleq \forall a.\, a \to (\star \to a \to a) \to a \\ \mathsf{zero} &\triangleq \lambda x.\, \lambda f.\, x \\ \mathsf{succ} &\triangleq \lambda y : \mathsf{Nat}.\, \lambda x.\, \lambda f.\, f\; y\; (y\; x\; f) \end{aligned}$$

Parigot encodings give primitive recursion, apart form constant-time destructors, but at the cost of exponential space complexity (notice in $\mathsf{succ}$ there are two occurances of $y$).

Both Scott and Parigot encodings are typable in System F with positive recursive types, which is strong normalizing.

*Example 3 (Alternative Encoding of Naturals).*

$$\begin{aligned} \mathsf{Nat} &\triangleq \forall a.\, a \to (\star \to (\star \to a) \to a) \to a \\ \mathsf{zero} &\triangleq \lambda x.\, \lambda f.\, x \\ \mathsf{succ} &\triangleq \lambda y : \mathsf{Nat}.\, \lambda x.\, \lambda f.\, f\; y\; (\lambda g.\, g\; x\; f) \end{aligned}$$

This encoding enjoys constant-time destructors, linear space complexity, and primitive recursion. The static version is $\mu b.\, \forall a.\, a \to (b \to (b \to a) \to a) \to a$, which can only be expressed in System F with general recursive types (notice the second $b$ appears in a negative position).

## 5 Algorithmic System

| | |
|---|---|
| Expressions | $e ::= x \mid n \mid \lambda x : A.\, e \mid \lambda x.\, e \mid e_1\, e_2 \mid e : A$ |
| Existential variables | $\widehat{a} ::= \widehat{a}_S \mid \widehat{a}_G$ |
| Types | $A, B ::= \mathsf{Int} \mid a \mid \widehat{a} \mid A \to B \mid \forall a.\, A \mid \star \mid \mathcal{S} \mid \mathcal{G}$ |
| Static Types | $T ::= \mathsf{Int} \mid a \mid \widehat{a} \mid T_1 \to T_2 \mid \forall a.\, T \mid \mathcal{S} \mid \mathcal{G}$ |
| Monotypes | $\tau, \sigma ::= \mathsf{Int} \mid a \mid \widehat{a} \mid \tau \to \sigma \mid \mathcal{S} \mid \mathcal{G}$ |
| Castable Monotypes | $t ::= \mathsf{Int} \mid a \mid \widehat{a} \mid t_1 \to t_2 \mid \mathcal{G}$ |
| Castable Types | $\mathbb{G} ::= \mathsf{Int} \mid a \mid \widehat{a} \mid \mathbb{G}_1 \to \mathbb{G}_2 \mid \forall a.\, \mathbb{G} \mid \star \mid \mathcal{G}$ |
| Static Castable Types | $\mathbb{S} ::= \mathsf{Int} \mid a \mid \widehat{a} \mid \mathbb{S}_1 \to \mathbb{S}_2 \mid \forall a.\, \mathbb{S} \mid \mathcal{G}$ |
| Contexts | $\Gamma, \Delta, \Theta ::= \bullet \mid \Gamma, x : A \mid \Gamma, a \mid \Gamma, \widehat{a} \mid \Gamma, \widehat{a} = \tau$ |
| Complete Contexts | $\Omega ::= \bullet \mid \Omega, x : A \mid \Omega, a \mid \Omega, \widehat{a} = \tau$ |

**Definition 8 (Existential variable contamination).**

$$\begin{aligned} [A]\bullet &= \bullet \\ [A](\Gamma, x : A) &= [A]\Gamma, x : A \\ [A](\Gamma, a) &= [A]\Gamma, a \\ [A](\Gamma, \widehat{a}_S) &= [A]\Gamma, \widehat{a}_G, \widehat{a}_S = \widehat{a}_G \quad \textit{if}\ \widehat{a}_S \in \mathrm{FV}(A) \\ [A](\Gamma, \widehat{a}_G) &= [A]\Gamma, \widehat{a}_G \\ [A](\Gamma, \widehat{a} = \tau) &= [A]\Gamma, \widehat{a} = \tau \end{aligned}$$

$\boxed{\Gamma \vdash A}$ *(Well-formedness of types)*

$$\frac{}{\Gamma \vdash \mathsf{Int}}\ \textsc{ad-int} \qquad \frac{}{\Gamma \vdash \star}\ \textsc{ad-unknown} \qquad \frac{}{\Gamma \vdash \mathcal{S}}\ \textsc{ad-static} \qquad \frac{}{\Gamma \vdash \mathcal{G}}\ \textsc{ad-gradual}$$

$$\frac{}{\Gamma[a] \vdash a}\ \textsc{ad-tvar} \qquad \frac{}{\Gamma[\widehat{a}] \vdash \widehat{a}}\ \textsc{ad-evar} \qquad \frac{}{\Gamma[\widehat{a} = \tau] \vdash \widehat{a}}\ \textsc{ad-solvedEvar}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \to B}\ \textsc{ad-arrow} \qquad \frac{\Gamma, a \vdash A}{\Gamma \vdash \forall a.\, A}\ \textsc{ad-forall}$$

$\boxed{\vdash \Gamma}$ *(Well-formedness of algorithmic contexts)*

$$\frac{}{\vdash \bullet}\ \textsc{wf-empty} \qquad \frac{\vdash \Gamma \qquad \Gamma \vdash A}{\vdash \Gamma, x : A}\ \textsc{wf-var} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, a}\ \textsc{wf-tvar} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \widehat{a}}\ \textsc{wf-evar}$$

$$\frac{\vdash \Gamma \qquad \Gamma \vdash \tau}{\vdash \Gamma, \widehat{a} = \tau}\ \textsc{wf-solvedEvar}$$

$\boxed{\Gamma \vdash A \lesssim B \dashv \Delta}$ *(Algorithmic Consistent Subtyping)*

$$\frac{}{\Gamma[a] \vdash a \lesssim a \dashv \Gamma[a]}\ \textsc{as-tvar} \qquad \frac{}{\Gamma[\widehat{a}] \vdash \widehat{a} \lesssim \widehat{a} \dashv \Gamma[\widehat{a}]}\ \textsc{as-evar}$$

$$\frac{}{\Gamma \vdash \mathsf{Int} \lesssim \mathsf{Int} \dashv \Gamma}\ \textsc{as-int} \qquad \frac{\begin{array}{c}\Gamma \vdash B_1 \lesssim A_1 \dashv \Theta \\ \Theta \vdash [\Theta]A_2 \lesssim [\Theta]B_2 \dashv \Delta\end{array}}{\Gamma \vdash A_1 \to A_2 \lesssim B_1 \to B_2 \dashv \Delta}\ \textsc{as-arrow}$$

$$\frac{\Gamma, a \vdash A \lesssim B \dashv \Delta, a, \Theta}{\Gamma \vdash A \lesssim \forall a.\, B \dashv \Delta}\ \textsc{as-forallR} \qquad \frac{\Gamma, \widehat{a}_S \vdash A[a \mapsto \widehat{a}_S] \lesssim B \dashv \Delta}{\Gamma \vdash \forall a.\, A \lesssim B \dashv \Delta}\ \textsc{as-forallL}$$

$$\frac{}{\Gamma \vdash \mathcal{S} \lesssim \mathcal{S} \dashv \Gamma}\ \textsc{as-spar} \qquad \frac{}{\Gamma \vdash \mathcal{G} \lesssim \mathcal{G} \dashv \Gamma}\ \textsc{as-gpar}$$

$$\frac{}{\Gamma \vdash \star \lesssim \mathbb{G} \dashv [\mathbb{G}]\Gamma}\ \textsc{as-unknownL} \qquad \frac{}{\Gamma \vdash \mathbb{G} \lesssim \star \dashv [\mathbb{G}]\Gamma}\ \textsc{as-unknownR}$$

$$\frac{\Gamma[\widehat{a}] \vdash \widehat{a} \lessapprox A \dashv \Delta}{\Gamma[\widehat{a}] \vdash \widehat{a} \lesssim A \dashv \Delta}\ \textsc{as-instL} \qquad \frac{\Gamma[\widehat{a}] \vdash A \lessapprox \widehat{a} \dashv \Delta}{\Gamma[\widehat{a}] \vdash A \lesssim \widehat{a} \dashv \Delta}\ \textsc{as-instR}$$

$\boxed{\Gamma \vdash \widehat{a} \lessapprox A \dashv \Delta}$ *(Instantiation I)*

$$\frac{\Gamma \vdash \tau}{\Gamma[\widehat{a}_S] \vdash \widehat{a}_S \lessapprox \tau \dashv \Gamma[\widehat{a}_S = \tau]}\ \textsc{instl-solveS}$$

$$\frac{\Gamma \vdash t}{\Gamma[\widehat{a}_G] \vdash \widehat{a}_G \lessapprox t \dashv \Gamma[\widehat{a}_G = t]}\ \textsc{instl-solveG}$$

$$\frac{}{\Gamma[\widehat{a}_S] \vdash \widehat{a}_S \lessapprox \star \dashv \Gamma[\widehat{a}_G, \widehat{a}_S = \widehat{a}_G]}\ \textsc{instl-solveUS}$$

$$\frac{}{\Gamma[\widehat{a}_G] \vdash \widehat{a}_G \lessapprox \star \dashv \Gamma[\widehat{a}_G]}\ \textsc{instl-solveUG}$$

$$\frac{}{\Gamma[\widehat{a}_S][\widehat{b}_G] \vdash \widehat{a}_S \lessapprox \widehat{b}_G \dashv \Gamma[\widehat{a}_G, \widehat{a}_S = \widehat{a}_G][\widehat{b}_G = \widehat{a}_G]}\ \textsc{instl-reachSG1}$$

$$\frac{}{\Gamma[\widehat{b}_S][\widehat{a}_G] \vdash \widehat{a}_G \lessapprox \widehat{b}_S \dashv \Gamma[\widehat{b}_G, \widehat{b}_S = \widehat{b}_G][\widehat{a}_G = \widehat{b}_G]}\ \textsc{instl-reachSG2}$$

$$\frac{}{\Gamma[\widehat{a}][\widehat{b}] \vdash \widehat{a} \lessapprox \widehat{b} \dashv \Gamma[\widehat{a}][\widehat{b} = \widehat{a}]}\ \textsc{instl-reachOtherwise}$$

$$\frac{\begin{array}{c}\Gamma[\widehat{a}_2, \widehat{a}_1, \widehat{a} = \widehat{a}_1 \to \widehat{a}_2] \vdash A_1 \lessapprox \widehat{a}_1 \dashv \Theta \\ \Theta \vdash \widehat{a}_2 \lessapprox [\Theta]A_2 \dashv \Delta\end{array}}{\Gamma[\widehat{a}] \vdash \widehat{a} \lessapprox A_1 \to A_2 \dashv \Delta}\ \textsc{instl-arr}$$

$$\frac{\Gamma[\widehat{a}], b \vdash \widehat{a} \lessapprox B \dashv \Delta, b, \Theta}{\Gamma[\widehat{a}] \vdash \widehat{a} \lessapprox \forall b.\, B \dashv \Delta}\ \textsc{instl-forallR}$$

$\boxed{\Gamma \vdash A \lessapprox \widehat{a} \dashv \Delta}$ *(Instantiation II)*

$$\frac{\Gamma \vdash \tau}{\Gamma[\widehat{a}_S] \vdash \tau \lessapprox \widehat{a}_S \dashv \Gamma[\widehat{a}_S = \tau]}\ \textsc{instr-solveS}$$

$$\frac{\Gamma \vdash t}{\Gamma[\widehat{a}_G] \vdash t \lessapprox \widehat{a}_G \dashv \Gamma[\widehat{a}_G = t]}\ \textsc{instr-solveG}$$

$$\frac{}{\Gamma[\widehat{a}_S] \vdash \star \lessapprox \widehat{a}_S \dashv \Gamma[\widehat{a}_G, \widehat{a}_S = \widehat{a}_G]}\ \textsc{instr-solveUS}$$

$$\frac{}{\Gamma[\widehat{a}_G] \vdash \star \lessapprox \widehat{a}_G \dashv \Gamma[\widehat{a}_G]}\ \textsc{instr-solveUG}$$

$$\frac{}{\Gamma[\widehat{a}_S][\widehat{b}_G] \vdash \widehat{b}_G \lessapprox \widehat{a}_S \dashv \Gamma[\widehat{a}_G, \widehat{a}_S = \widehat{a}_G][\widehat{b}_G = \widehat{a}_G]}\ \textsc{instr-reachSG1}$$

$$\frac{}{\Gamma[\widehat{b}_S][\widehat{a}_G] \vdash \widehat{b}_S \lessapprox \widehat{a}_G \dashv \Gamma[\widehat{b}_G, \widehat{b}_S = \widehat{b}_G][\widehat{a}_G = \widehat{b}_G]}\ \textsc{instr-reachSG2}$$

$$\frac{}{\Gamma[\widehat{a}][\widehat{b}] \vdash \widehat{b} \lessapprox \widehat{a} \dashv \Gamma[\widehat{a}][\widehat{b} = \widehat{a}]}\ \textsc{instr-reachOtherwise}$$

$$\frac{\begin{array}{c}\Gamma[\widehat{a}_2, \widehat{a}_1, \widehat{a} = \widehat{a}_1 \to \widehat{a}_2] \vdash \widehat{a}_1 \lessapprox A_1 \dashv \Theta \\ \Theta \vdash [\Theta]A_2 \lessapprox \widehat{a}_2 \dashv \Delta\end{array}}{\Gamma[\widehat{a}] \vdash A_1 \to A_2 \lessapprox \widehat{a} \dashv \Delta}\ \textsc{instr-arr}$$

$$\frac{\Gamma[\widehat{a}], \widehat{b}_S \vdash B[b \mapsto \widehat{b}_S] \lessapprox \widehat{a} \dashv \Delta}{\Gamma[\widehat{a}] \vdash \forall b.\, B \lessapprox \widehat{a} \dashv \Delta}\ \textsc{instr-forallL}$$

$\boxed{\Gamma \vdash e \Rightarrow A \dashv \Delta}$ *(Inference)*

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A \dashv \Gamma}\ \textsc{inf-var} \qquad \frac{}{\Gamma \vdash n \Rightarrow \mathsf{Int} \dashv \Gamma}\ \textsc{inf-int}$$

$$\frac{\Gamma, x : A \vdash e \Rightarrow B \dashv \Delta, x : A, \Theta}{\Gamma \vdash \lambda x : A.\, e \Rightarrow A \to B \dashv \Delta}\ \textsc{inf-lamann}$$

$$\frac{\Gamma, \widehat{a}_S, \widehat{b}_S, x : \widehat{a}_S \vdash e \Leftarrow \widehat{b}_S \dashv \Delta, x : \widehat{a}_S, \Theta}{\Gamma \vdash \lambda x.\, e \Rightarrow \widehat{a}_S \to \widehat{b}_S \dashv \Delta}\ \textsc{inf-lam}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash e : A \Rightarrow A \dashv \Delta}\ \textsc{inf-anno} \qquad \frac{\begin{array}{c}\Gamma \vdash e_1 \Rightarrow A \dashv \Theta_1 \\ \Theta_1 \vdash [\Theta_1]A \triangleright A_1 \to A_2 \dashv \Theta_2 \\ \Theta_2 \vdash e_2 \Leftarrow [\Theta_2]A_1 \dashv \Delta\end{array}}{\Gamma \vdash e_1\, e_2 \Rightarrow A_2 \dashv \Delta}\ \textsc{inf-app}$$

$\boxed{\Gamma \vdash e \Leftarrow A \dashv \Delta}$ *(Checking)*

$$\frac{\Gamma, x : A \vdash e \Leftarrow B \dashv \Delta, x : A, \Theta}{\Gamma \vdash \lambda x.\, e \Leftarrow A \to B \dashv \Delta}\ \textsc{chk-lam} \qquad \frac{\Gamma, a \vdash e \Leftarrow A \dashv \Delta, a, \Theta}{\Gamma \vdash e \Leftarrow \forall a.\, A \dashv \Delta}\ \textsc{chk-gen}$$

$$\frac{\Gamma \vdash e \Rightarrow A \dashv \Theta \qquad \Theta \vdash [\Theta]A \lesssim [\Theta]B \dashv \Delta}{\Gamma \vdash e \Leftarrow B \dashv \Delta}\ \textsc{chk-sub}$$

$\boxed{\Gamma \vdash A \triangleright A_1 \to A_2 \dashv \Delta}$ *(Algorithmic Matching)*

$$\frac{\Gamma, \widehat{a}_S \vdash A[a \mapsto \widehat{a}_S] \triangleright A_1 \to A_2 \dashv \Delta}{\Gamma \vdash \forall a.\, A \triangleright A_1 \to A_2 \dashv \Delta}\ \textsc{am-forall}$$

$$\frac{}{\Gamma \vdash A_1 \to A_2 \triangleright A_1 \to A_2 \dashv \Gamma}\ \textsc{am-arr} \qquad \frac{}{\Gamma \vdash \star \triangleright \star \to \star \dashv \Gamma}\ \textsc{am-unknown}$$

$$\frac{}{\Gamma[\widehat{a}] \vdash \widehat{a} \triangleright \widehat{a}_1 \to \widehat{a}_2 \dashv \Gamma[\widehat{a}_1, \widehat{a}_2, \widehat{a} = \widehat{a}_1 \to \widehat{a}_2]}\ \textsc{am-var}$$

$\boxed{\Gamma \longrightarrow \Delta}$ *(Context extension)*

$$\frac{}{\bullet \longrightarrow \bullet}\ \textsc{ext-id} \qquad \frac{\Gamma \longrightarrow \Delta \qquad [\Delta]A = [\Delta]A'}{\Gamma, x : A \longrightarrow \Delta, x : A'}\ \textsc{ext-var} \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma, a \longrightarrow \Delta, a}\ \textsc{ext-tvar}$$

$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \widehat{a} \longrightarrow \Delta, \widehat{a}}\ \textsc{ext-evar} \qquad \frac{\Gamma \longrightarrow \Delta \qquad [\Delta]\tau = [\Delta]\tau'}{\Gamma, \widehat{a} = \tau \longrightarrow \Delta, \widehat{a} = \tau'}\ \textsc{ext-solvedEvar}$$

$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \widehat{a}_S \longrightarrow \Delta, \widehat{a}_S = \tau}\ \textsc{ext-solveS} \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma, \widehat{a}_G \longrightarrow \Delta, \widehat{a}_G = t}\ \textsc{ext-solveG}$$

$$\frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \widehat{a}}\ \textsc{ext-add} \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \widehat{a}_S = \tau}\ \textsc{ext-addSolveS}$$

$$\frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \widehat{a}_G = t}\ \textsc{ext-addSolveG}$$

## 6 Metatheory

**Theorem 1 (Instantiation Soundness)** *Given* $\Delta \longrightarrow \Omega$ *and* $[\Gamma]A = A$ *and* $\widehat{a} \notin \textsc{fv}(A)$*:*

1. *If* $\Gamma \vdash \widehat{a} \lessapprox A \dashv \Delta$ *then* $[\Omega]\Delta \vdash [\Omega]\widehat{a} \lesssim [\Omega]A$.
2. *If* $\Gamma \vdash A \lessapprox \widehat{a} \dashv \Delta$ *then* $[\Omega]\Delta \vdash [\Omega]A \lesssim [\Omega]\widehat{a}$.

**Theorem 2 (Soundness of Algorithmic Consistent Subtyping)** *If* $\Gamma \vdash A \lesssim B \dashv \Delta$ *where* $[\Gamma]A = A$ *and* $[\Gamma]B = B$ *and* $\Delta \longrightarrow \Omega$ *then* $[\Omega]\Delta \vdash [\Omega]A \lesssim [\Omega]B$.

**Theorem 3 (Soundness of Algorithmic Typing)** *Given* $\Delta \longrightarrow \Omega$*:*

1. *If* $\Gamma \vdash e \Rightarrow A \dashv \Delta$ *then* $\exists e'$ *such that* $[\Omega]\Delta \vdash e' : [\Omega]A$ *and* $\lfloor e \rfloor = \lfloor e' \rfloor$.
2. *If* $\Gamma \vdash e \Leftarrow A \dashv \Delta$ *then* $\exists e'$ *such that* $[\Omega]\Delta \vdash e' : [\Omega]A$ *and* $\lfloor e \rfloor = \lfloor e' \rfloor$.

**Theorem 4 (Instantiation Completeness)** *Given* $\Gamma \longrightarrow \Omega$ *and* $A = [\Gamma]A$ *and* $\widehat{a} \notin \textsc{unsolved}(\Gamma)$ *and* $\widehat{a} \notin \textsc{fv}(A)$*:*

1. *If* $[\Omega]\Gamma \vdash [\Omega]\widehat{a} \lesssim [\Omega]A$ *then there are* $\Delta, \Omega'$ *such that* $\Omega \longrightarrow \Omega'$ *and* $\Delta \longrightarrow \Omega'$ *and* $\Gamma \vdash \widehat{a} \lessapprox A \dashv \Delta$.
2. *If* $[\Omega]\Gamma \vdash [\Omega]A \lesssim [\Omega]\widehat{a}$ *then there are* $\Delta, \Omega'$ *such that* $\Omega \longrightarrow \Omega'$ *and* $\Delta \longrightarrow \Omega'$ *and* $\Gamma \vdash A \lessapprox \widehat{a} \dashv \Delta$.

**Theorem 5 (Generalized Completeness of Consistent Subtyping)** *If* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$ *and* $\Gamma \vdash B$ *and* $[\Omega]\Gamma \vdash [\Omega]A \lesssim [\Omega]B$ *then there exist* $\Delta$ *and* $\Omega'$ *such that* $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash [\Gamma]A \lesssim [\Gamma]B \dashv \Delta$.

**Theorem 6 (Matching Completeness)** *Given* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$*, if* $[\Omega]\Gamma \vdash [\Omega]A \triangleright A_1 \to A_2$ *then there exist* $\Delta$*,* $\Omega'$*,* $A_1'$ *and* $A_2'$ *such that* $\Gamma \vdash [\Gamma]A \triangleright A_1' \to A_2' \dashv \Delta$ *and* $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $A_1 = [\Omega']A_1'$ *and* $A_2 = [\Omega']A_2'$.

**Theorem 7 (Completeness of Algorithmic Typing)** *Given* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$*, if* $[\Omega]\Gamma \vdash e : A$ *then there exist* $\Delta$*,* $\Omega'$*,* $A'$ *and* $e'$ *such that* $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash e' \Rightarrow A' \dashv \Delta$ *and* $A = [\Omega']A'$ *and* $\lfloor e \rfloor = \lfloor e' \rfloor$.