All Modules Have Flat Covers

M.Sc. Thesis

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CONTENTS

ABSTRACT

In this thesis, we demonstrate detailed proof of Bican, Bachir, and Enochs's results so that each module has a flat cover. Historically, the notion of flat modules was introduced by J-P Serre in 1955-1956. A few years later, when the injective envelopes had already been studied, the dual notion of the injective envelopes, the projective covers, was investigated. H. Bass in his 1959 thesis introduced the projective covers and he described the apt rings, in which each module has a projective cover (left/right perfect rings). After this result arose the question of when a module has a **flat** cover. After many years, the significant progress which had shown by J. Xu, brought to the fore again this open problem, which was finally solved and presented in Enochs's paper " All modules have flat covers ", in which was crucial the use of useful lemmas proved by Eklof and stated at his paper " How to make Ext vanish ".

- In the first chapter, we introduce the notion of extension of modules $E(A, B)$, we prove some useful lemmas and we present the relation between extension $E(A, B)$ and the abelian group $\mathrm{Ext}^1_R(A, B)$.
- In the second chapter, we give a summary of envelopes (cover), special envelopes (special covers), and their connection with the concept of cotorsion theories.
- In Chapter 3, we introduce the meaning of flat modules, pure submodules, and the relation between them and we end the chapter with some characterizations for pure exact sequences. In the chapter, we give an extended review of ordinal numbers and we focus on transfinite induction, which we use a lot in the rest materials of the chapter concurrent with chapter 5. Also, in the fourth chapter we provide a detailed presentation of Eklof's paper results, which includes essential techniques for vanishing the Ext functor.
- Finally, in Chapter 5, we present Xu's result, which is the key to proving the desired result, we define the meaning of flat cotorsion theory and we prove the central theorem of the thesis.

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CHAPTER 1

EXTENSIONS OF MODULES

1.1 Extensions

Let A, B be two R-modules. We want to consider all R - modules E such that B is a submodule of E and $E/B \cong A$.

Definition 1.1. A short exact sequence

$$0 \to B \xrightarrow{\kappa} E \xrightarrow{\nu} A \to 0 \tag{1.1}$$

is called an **extension** of A by B.

Definition 1.2. Two extensions $0 \to B \to E_1 \to A \to 0$ and $0 \to B \to E_2 \to A \to 0$ are called **equivalent** if there exists a homomorphism $\xi : E_1 \to E_2$ such that the following diagram is commutative:

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, equal] & E_1 \arrow[r, two heads] \arrow[d, "\xi"] & A \arrow[d, equal] \\ B \arrow[r, tail] & E_2 \arrow[r, two heads] & A \end{tikzcd}$$

Notation 1.1. From the last diagram, we observe that ξ is an isomorphism, so it is easy to show that equivalence of extensions defines an equivalence relation on the set of extensions of A by B. We denote this set by $E(A, B)$.

Notation 1.2. The set $E(A, B)$ contains at least one element, the extension

$$0 \to B \xrightarrow{i_B} A \oplus B \xrightarrow{\pi_A} A \to 0.$$

Any extension of A by B equivalent to the above extension is called a **split extension** or **trivial extension** of A by B. We denote the trivial extension by 0.

Lemma 1.1. An R - module A is projective if and only if $E(A,B) = \{0\}$ for each R - module B.

Proof. We assume that A is projective. Let B be an arbitrary module and let

$$0 \to B \xrightarrow{\kappa} E \xrightarrow{\nu} A \to 0$$

an extension of A by B. Since A is projective there is $\sigma \in \mathrm{Hom}_R(A, E)$ s.t. the following diagramm commutes

$$\begin{tikzcd} E \arrow[r, "\nu"] & A \arrow[r] & 0 \\ A \arrow[u, "\sigma"] \arrow[ur, "\mathrm{id}"'] & & \end{tikzcd}$$

Therefore the above extension splits. Conversely, we assume that $E(A,B) = 0$, for every module B. Let $\mu\colon M \to N$ an epimorphism and $\varphi\colon A \to N$. We consider the pullback

$$P = \{(x,y) \in M \oplus A \mid \mu(x) = \varphi(y)\}$$

of diagramm

$$\begin{tikzcd} M \arrow[r, "\mu"] & N \arrow[r] & 0 \\ & A \arrow[u, "\varphi"'] & \end{tikzcd}$$

Thus we have the following short exact sequence

$$0 \to \ker\mu \to P \xrightarrow{\pi_A} A \to 0$$

which splits, therefore exists $\sigma\colon A \to P$ s.t. $\pi_A \circ \sigma = \mathrm{id}_A$. If ψ is the following composition

$$A \xrightarrow{\sigma} P \xrightarrow{\pi_M} M$$

then $\varphi = \mu \circ \psi$. ☐

Lemma 1.2. The square

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & X \end{tikzcd} \tag{1.2}$$

is a pullback diagram if and only if the following sequence is exact:

$$0 \to Y \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{\langle\varphi,-\psi\rangle} X.$$

Proof. Show that the universal property of the pullback diagram of (φ, ψ) is the same as the universal property of the kernel of $\langle\varphi, -\psi\rangle$. ☐

Lemma 1.3. If diagram 1.2 is a pullback diagram, then

(a) $\ker \alpha \xrightarrow{\cong} \ker \psi$ via β

(b) if ψ is an epimorphism, then α is also an epimorphism.

Proof. (a)
- Let $y \in \ker \alpha$. By commutativity of the diagram, we have $\psi\beta(y)$, i.e., $\beta(y) \in \ker \psi$, therefore $\beta|_{\ker \alpha} \colon \ker \alpha \to \ker \psi$.
- Since $\{\alpha, \beta\}$ is injective, we conclude that $\beta\big|_{\ker \alpha}$ is a monomorphism.
- For surjectivity, if $b \in \ker \psi$, then $\langle \varphi, -\psi \rangle (0, b) = 0$. Therefore, exists $y \in Y$ such that $\beta(y) = b$ and $\alpha(y) = 0$, which proves the desired result.

(b) Let $a \in A$. Then, exists $b \in B$ such that $\psi(b) = \varphi(a)$, since that $(a, b) \in \ker \langle \varphi, -\psi \rangle = \operatorname{Im} \{\alpha, \beta\}$. The desired result implied immediately from the last notation.

$\square$

Lemma 1.4. Consider the diagram

$$\begin{tikzcd} B \arrow[r, tail, "\kappa'"] \arrow[d, equal] & E' \arrow[r, two heads, "\nu'"] \arrow[d, "\xi"] & A' \arrow[d, "\alpha"] \\ B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

be a commutative diagram with exact rows. Then, the right - hand square is a pullback diagram.

Proof.
- Consider the pullback diagram

$$\begin{tikzcd} P \arrow[r, "\varepsilon"] \arrow[d, "\varphi"] & A' \arrow[d, "\alpha"] \\ E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

From Lemma 1.4, we have that ε is an epimorphism and $\ker \varepsilon \cong B$.

- Therefore, we obtain the extension $0 \to B \to P \to A' \to 0$. Showing that the extensions $0 \to B \xrightarrow{\kappa} E' \xrightarrow{\nu} A'$ and $0 \to B \to P \to A' \to 0$ are equivalent implies $Y \cong P$, and the desired result follows.

$\square$

The reader is left to prove the dual results of the above lemmas.

1.2 $E(A, B)$ and $\mathrm{Ext}^1(A, B)$

Let A be an arbitrary R - module. We can take P a projective module and a short exact sequence

$$0 \to K \xrightarrow{i} P \xrightarrow{\pi} A \to 0$$

For every R - module B we can apply then functor $\mathrm{Hom}_R(-, B)$ to the above sequence

$$0 \to \mathrm{Hom}_R(A, B) \xrightarrow{\pi^*} \mathrm{Hom}_R(P, B) \xrightarrow{i^*} \mathrm{Hom}_R(K, B)$$

We define

$${}_{\pi}\mathrm{Ext}^1(A, B) := \mathrm{coker}\left(\mathrm{Hom}_R(P, B) \xrightarrow{i^*} \mathrm{Hom}_R(K, B)\right)$$

Proposition 1.1. The isomophism class of ${}_{\pi}\mathrm{Ext}^1(A, B)$ is independent from the choice of π, for every R - module B. Therefore for simplicity we write $\mathrm{Ext}^1(A, B)$ for ${}_{\pi}\mathrm{Ext}^1(A, B)$.

Theorem 1.1. There is a bijection $E(A, B) \cong \mathrm{Ext}^1(A, B)$ preserving 0.

Proof. Let $\varphi\colon K \to B$. We shall show that this morphism corresponds to an extension

$$0 \to B \to Q_\varphi \to A \to 0$$

With the above denotation consider the pushout

$$Q_\varphi = P \oplus B / \{(i(k), -\varphi(k)) \mid k \in K\}$$

of diagramm

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & B \\ \downarrow{\scriptstyle i} & & \\ P & & \end{array}$$

If we define $\psi\colon Q_\varphi \to A$, $\psi([p, b]) = \pi(p)$, can be easily seen that the following sequence

$$0 \to B \xrightarrow{\pi_B} Q_\varphi \xrightarrow{\psi} A \to 0$$

if an extension of A by B. Is left to the reader to show that if $\varphi, \varphi' \in \mathrm{Im} i^*$, then the induced extensions

$$0 \to B \xrightarrow{\pi_B} Q_\varphi \xrightarrow{\psi} A \to 0 \quad \text{and} \quad 0 \to B \xrightarrow{\pi_B} Q_{\varphi'} \xrightarrow{\psi'} A \to 0$$

are equivalent. Conversely, let

$$0 \to B \xrightarrow{\kappa} E \xrightarrow{\nu} A \to 0$$

be an extension of A by B. By projectivity of P there are $\varphi\colon P \to E$ and $\psi\colon K \to B$ s.t. the following diagramm commutes

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "\psi"] & P \arrow[r, "\pi"] \arrow[d, dashed, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & B \arrow[r, "\kappa"] & E \arrow[r, "\nu"] & A \arrow[r] & 0
\end{tikzcd}$$

Is left to the reader to show that if two extensions are equivalent, then for the induced maps $\psi, \psi'\colon K \to B$ is true that $\psi - \psi' \in \mathrm{Im} i^*$.

Finally, it can be easily seen that if $\varphi\colon K \to B$ s.t. there if $\Phi\colon P \to B$ and $\Phi|_K = \Phi \circ i = \varphi$, then the induced extension is the trivial extension. This shows that the above bijection preserves 0. $\square$

Corollary 1.1. The set $E(A,B)$ has the structure of an abelian group

Proof. See [1] section 7.2.1 for an explicit description of the group structure in terms of extensions. $\square$

CHAPTER 2

APPROXIMATIONS OF MODULES

2.1 Preenvelopes and Envelopes

Definition 2.1. Let M be an R-module, and $\mathscr{C}$ a class of R-modules closed under isomorphic images and direct sums. A map $f \in \operatorname{Hom}(M, C)$, where $C \in \mathscr{C}$, is called a $\mathscr{C}$ - **preenvelope** if, for every $f' \in \operatorname{Hom}(M, C')$ with $C' \in \mathscr{C}$, there exists $g \in \operatorname{Hom}(C, C')$ making the following diagram commutative:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f'"'] & C \arrow[d, "g"] \\ & C' \end{tikzcd}$$

The $\mathscr{C}$ -preenvelope f is a $\mathscr{C}$ - **envelope** of M if g is an automorphism, whenever $g \in \operatorname{Hom}_R(C, C)$ and $f = gf$.

Example 2.1. Let $\mathscr{I}_0$ be the class of injective R-modules. If M is an R-module and $E(M)$ is the injective hull of M, then the inclusion map $i : M \hookrightarrow E(M)$ is a $\mathscr{C}$ - **envelope**.

Proof. Let $f' \colon M \to I$ be an R - homomorphism, where I is an injective R - module. Since I is injective and i is monomorphism, then there is $g \in \operatorname{Hom}_R(E(M), I)$ s.t. the following diagram commutes

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[dr, "f'"'] & E(M) \arrow[d, "g"] \\ & I \end{tikzcd}$$

Now let $g\colon E(M) \to E(M)$ be an R - homomorphism s.t. $i = gi$. Since $E(M)$ is an essensial extension of M, then g is monomorphism. Then there is $g' \in \mathrm{End}_R(E(M))$ s.t.

$$\begin{tikzcd} E(M) \arrow[r, hook, "g"] \arrow[dr, "\mathrm{id}"'] & E(M) \arrow[d, "g'"] \\ & E(M) \end{tikzcd}$$

It's obvious that $g'i = i$ and similarly we conclude that $g' \in \mathrm{Aut}_R(E(M))$. Hence, this implies that $g\colon E(M) \to E(M)$ is an R - automorphism. $\square$

Notation 2.1. In general, given an R-module M, there may be several different $\mathscr{C}$ - preenvelopes for M but no $\mathscr{C}$ - envelopes. However, the next lemma demonstrates that if a $\mathscr{C}$ - envelope exists, it is recognized as the minimal $\mathscr{C}$ -preenvelope in the sense of the following lemma:

Lemma 2.1. Let $f : M \to C$ be a $\mathscr{C}$ - envelope, and $f' : M \to C'$ be a $\mathscr{C}$ - preenvelope. Then,

(a) $C' = D \oplus D'$, $\mathrm{Im}f' \subseteq D$, and the map $f' : M \to D$ is a $\mathscr{C}$ - envelope of M.

(b) The map f' is a $\mathscr{C}$ - envelope if and only if it has no proper direct summand contained in $\mathrm{Im}f'$.

Proof. (a) Since f, f' are $\mathscr{C}$ - , there exist $g : C \to C'$ and $g' : C' \to C$ such that $gf = f'$ and $g'f' = f$. Therefore, the following commutative diagram arises:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f"'] & C \arrow[d, "g'g"] \\ & C \end{tikzcd}$$

Since f is a $\mathscr{C}$ - envelope, $g'g$ is an automorphism. Thus, g is a monomorphism, g' is an epimorphism, and let $D = \mathrm{Im}g \cong C$ and $\mathrm{Im}f' \subseteq D$. If $D' = C'/\mathrm{Im}g$, then the short exact sequence

$$0 \to D \hookrightarrow C' \to D' \to 0$$

splits since $g'g$ is an automorphism, and hence, $C' \cong D \oplus D'$.

(b) The desired result follows directly from (a).

$\square$

2.2 Precovers and Covers

Definition 2.2. Let $\mathscr{C} \subseteq \mathrm{Mod} - R$ be closed under isomorphic images and direct summands. Let $M \in \mathrm{Mod}-R$. Then $f \in \mathrm{Hom}_R(C, M)$, with $C \in \mathscr{C}$, is called a $\mathscr{C}$ **- precover** of M, if for each $C' \in \mathscr{C}$ and for each $f' \in \mathrm{Hom}_R(C', M)$, there is $g \in \mathrm{Hom}_R(C', C)$ s.t. the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "f"] & M \arrow[d, "f'"] \\ & C' \arrow[ul, "g"] \end{tikzcd}$$

A $\mathscr{C}$ - precover $f \in \mathrm{Hom}_R(C, M)$ is called $\mathscr{C}$ - **cover** of M provided that $fg = f$ and $g \in \mathrm{End}_R(C)$ implies that $g \in \mathrm{Aut}_R(C)$.

Notation 2.2. By preceding definition it's obvious that $f \in \mathrm{Hom}_R(C, M)$ is a $\mathscr{C}$ - precover if and only if f induces a surjective abelian group homomorphism

$$\mathrm{Hom}_R(C', C) \xrightarrow{f_*} \mathrm{Hom}_R(C', M).$$

Example 2.2. Let $\mathscr{P}_0$ the category of projective R - modules. Then each $M \in \mathrm{Mod} - R$ has a $\mathscr{P}_0$ - precover. Many questions can arise here. Is it true that each module has a $\mathscr{P}_0$ - cover ? *In addition, if we want to generalized this, is it true that each module has a flat cover ?* To answer this question we need to approach things in a different way

Lemma 2.2. Let $f\colon C \to M$ be a $\mathscr{C}$ - cover of M. Let $f'\colon C' \to M$ be any $\mathscr{C}$ - precover of M. Then

(a) $C' = D \oplus D'$, where $D \subseteq \ker f'$ and $f' \restriction D'$ is a $\mathscr{C}$ - cover of M ;

(b) f' is a $\mathscr{C}$ - cover of M iff C' has no non - zero direct summands contained in $\ker f'$.

Proof. Dual to the proof of Lemma 2.1. □

2.3 Cotorsion Theories

Definition 2.3. Let $\mathscr{C} \subseteq \mathrm{Mod} - R$. Define

$$\mathscr{C}^{\perp} = \left\{N \in \mathrm{Mod} - R \mid \mathrm{Ext}^1_R(C, N) = 0,\ \text{for all } C \in \mathscr{C}\right\}$$

and

$$^{\perp}\mathscr{C} = \left\{N \in \mathrm{Mod} - R \mid \mathrm{Ext}^1_R(N, C) = 0,\ \text{for all } C \in \mathscr{C}\right\}$$

Definition 2.4. Let $M \in \mathrm{Mod} - R$. A $\mathscr{C}$ - preenvelope $f\colon M \to C$ is called **special** if the following sequence is exact

$$0 \to M \xrightarrow{f} C \to \mathrm{coker} f \to 0$$

and $\mathrm{coker} f \in^{\perp} \mathscr{C}$. Dually, a $\mathscr{C}$ - precover $f\colon C \to M$ is called **special** if the following sequence is exact

$$0 \to \ker f \hookrightarrow C \xrightarrow{f} M \to 0$$

and $\ker f \in \mathscr{C}^{\perp}$.

Lemma 2.3. Let $M \in \mathrm{Mod} - R$ and $\mathscr{C} \subseteq \mathrm{Mod} - R$ closed under extensions.

(a) If $\mathscr{I}_0 \subseteq \mathscr{C}$ and $f\colon M \to C$ a $\mathscr{C}$ - envelope, then f is special.

(b) If $\mathscr{P}_0 \subseteq \mathscr{C}$ and $f\colon C \to M$ a $\mathscr{C}$ - cover, then f is special.

Proof. (a) We want to show that f is injective and that $\mathrm{coker} f \in^{\perp} \mathscr{C}$.

- For injectivity, let $M \xrightarrow{i} E(M)$ the injective hull of M. Then there are $k\colon C \to E(M)$ s.t. the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "i"'] & C \arrow[d, "k"] \\ & E(M) \end{tikzcd}$$

  Since i in injective and $i = k \circ f$, then f is injective.

- We shall show that $D = \mathrm{coker} f \in^{\perp} \mathscr{C}$. Let $C' \in \mathscr{C}$. Showing that $E(D, C') = 0$ is equivalent showing that $\mathrm{Ext}^1_R(D, C') = 0$. Hence we must show that every s.e.q.

$$0 \to C' \to X \xrightarrow{h} D \to 0$$

  splits. We consider the following pullback diagramm

$$\begin{tikzcd} P \arrow[r, "\beta"] \arrow[d, "\gamma"] & X \arrow[d, "h"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

From Lemma 1.4, we have the following commutative diagram :

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & C' \arrow[r, equal] \arrow[d] & C' \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, equal] & P \arrow[r, "\beta"] \arrow[d, "\gamma"] & X \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & M \arrow[r, "f"] & C \arrow[r, "g"] \arrow[d] \arrow[u, bend left, "\delta"] & D \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

- Since $\mathscr{C}$ is closed under extensions and $C, C' \in \mathscr{C}$, then $P \in \mathscr{C}$.
- Since f is a $\mathscr{C}$-envelope, there exists $\delta\colon C \to P$ such that $\alpha = \delta f$. Then, we have $f = \gamma\alpha = \gamma\delta f$, hence $\gamma\delta$ is an automorphism of C.
- We define $i\colon D \to X$ by $i(g(c)) = \beta\delta(\gamma\delta)^{-1}(c)$. The map i is well-defined via the above commutative diagram. It is straightforward that $hi = \mathrm{id}_D$, and we have shown the desired result.

□

Definition 2.5. Let $\mathscr{A}, \mathscr{B} \subseteq \mathrm{Mod} - R$. The pair $(\mathscr{A}, \mathscr{B})$ is called a **cotorsion theory** if

$$\mathscr{A} =^{\perp} \mathscr{B} \quad \text{and} \quad \mathscr{B} = \mathscr{A}^{\perp}.$$

If $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ a cotorsion theory, then then class $\mathscr{K}_{\mathfrak{C}} = \mathscr{A} \cap \mathscr{B}$ is called the **kernel** of $\mathfrak{C}$.

Example 2.3. Let $\mathscr{C}$ be any class of modules. Then the pair

$$\mathfrak{S}_{\mathscr{C}} = \left({}^{\perp}\mathscr{C}, \left({}^{\perp}\mathscr{C}\right)^{\perp}\right)$$

it's a cotorsion theory and it's called the cotorsion theory **generated** be the class $\mathscr{C}$.

Proof. For any class of modules $\mathscr{A}$ it's obvious that $\mathscr{A} \subseteq^{\perp} \left(\mathscr{A}^{\perp}\right)$, therefore ${}^{\perp}\mathscr{C} \subseteq^{\perp} \left(\left({}^{\perp}\mathscr{C}\right)^{\perp}\right)$. For the converse relation, if $M \in^{\perp} \left(\left({}^{\perp}\mathscr{C}\right)^{\perp}\right)$, implied that

$$\mathrm{Ext}^1_R(M, X) = 0, \quad \text{whenever } \mathrm{Ext}^1_R(B, X) = 0, \quad \forall\, B \in^{\perp} \mathscr{C}.$$

Since $\mathrm{Ext}^1_R(M, X) = 0$ for each $X \in \mathscr{C}$, then $M \in^{\perp} \mathscr{C}$. □

Example 2.4. Let $\mathscr{C}$ be any class of modules. Then the pair

$$\mathfrak{C}_{\mathscr{C}} = \left({}^{\perp}\left(\mathscr{C}^{\perp}\right), \mathscr{C}^{\perp} \right)$$

similarly to the above example it's a cotorsion theory and it's called the **cotorsion theory cogenerated** by the class $\mathscr{C}$.

Example 2.5. By the Lemma 1.1 it's easily seen that $\mathfrak{S}_{\mathrm{Mod}-R} = (\mathrm{Mod} - R, \mathscr{I}_0)$ and $\mathfrak{C}_{\mathrm{Mod}-R} = (\mathscr{P}_0, \mathrm{Mod} - R)$, where $\mathscr{I}_0$ and $\mathscr{P}_0$ are the classes of injectives and projectives R - modules respectively. These cotorsion theories are called **trivial cotorsion theories** .

Motivation 1. The main reason that we introduce and investigate the $\mathscr{C}$ - special preenvelopes and $\mathscr{C}$ - special precovers is their close relation to cotorsion theories.

Definition 2.6. A cotorsion theory $(\mathscr{A}, \mathscr{B})$ is said to have **enough injectives** (resp. **enough projectives**) if every module M has a special $\mathscr{B}$ - preenvelope (has a special $\mathscr{A}$ - precover).

Motivation 2. Although the two concepts above look completely different, we will prove that if a cotorsion theorem satisfies one, then it satisfies the other and vice versa. In this case the cotorsion theorem is called **complete**.

Lemma 2.4. Let R be a ring and $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ be a cotorsion theory of modules. Then the following are equivalent :

(a) Each module has a special $\mathscr{A}$ - precover.

(b) Each module has a special $\mathscr{B}$ - preenvelope.

Proof. We assume that each module has a special $\mathscr{A}$ - precover. Let $M \in \mathrm{Mod} - R$. We will secure the existense of an $\mathscr{B}$ - preenvelope of M. There is a short exact sequence

$$0 \to M \to I \xrightarrow{\pi} F \to 0$$

where I is an injective module. Thus, there is a special $\mathscr{A}$ - precover

$$0 \to B \to A \xrightarrow{\rho} F \to 0, \quad A \in \mathscr{A},\ B \in \mathscr{B}$$

and we consider the pullback diagramm of π, ρ

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\gamma"] & I \arrow[d, "\pi"] \\
A \arrow[r, "\rho"] & F
\end{tikzcd}$$

From Lemma 1.4 induced the following commuting diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M \arrow[r, equal] \arrow[d] & M \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "\gamma"'] & I \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & B \arrow[r] & A \arrow[r, "\rho"] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $B, I \in \mathscr{B}$ and $\mathscr{B}$ is closed under extensions (it's an easy application of long exact sequence), then $P \in \mathscr{B}$. So the s.e.s.

$$0 \to M \to P \xrightarrow{\gamma} A \to 0$$

is a special $\mathscr{B}$ - preenvelope. $\square$

CHAPTER 3

FLAT MODULES AND PURITY

3.1 Flat Modules

Definition 3.1. A right module P_R is called **flat** if the functor $P \otimes_R -$ is exact on ${}_R\mathfrak{M}$. Equivalently, whenever $A \to B$ is injective in ${}_R\mathfrak{M}$, so is $P \otimes_R A \to P \otimes_R B$ in the category of abelian groups.

Proposition 3.1. Let $\varphi\colon R \to S$ be a ring homomorphism, whereby S can be viewed as left R - module. If P_R is R - flat, then the right S - module $P' := P \otimes_R S$ is S - flat.

Proof. Let $A \xrightarrow{i} B$ be an injective homomorphism in ${}_S\mathfrak{M}$. We want to show that

$$P' \otimes_S A \xrightarrow{-\otimes i} P' \otimes_S B$$

is injective. We can identify $S \otimes_S A \cong A$ and $S \otimes_S B \cong B$. Since A, B can be viewed as left R - modules via φ, then $P \otimes_R A \to P \otimes_R B$. Combining these remarks implies the desired result. $\square$

Proposition 3.2. If $P = \bigoplus_{i\in I} P_i \in \mathfrak{M}_R$, then P is flat if and only if each P_i is flat.

Proof. The crucial remark to prove this proposition is the fact that $- \otimes_R A$ is left adjoint functor of $\mathrm{Hom}_R(A, -)$. Thus $- \otimes_R A$ preserves colimits. Therefore, there is an abelian group isomorphism

$$\left(\bigoplus_{i\in I} P_i\right) \otimes_R A \cong \bigoplus_{i\in I} (P_i \otimes_R A)$$

Hence, the desired result implied immediately. $\square$

Corollary 3.1. Any projective (right) - module is flat.

Proof. It can be easily seen that the free (right) R is flat. Therefore, each free module is flat, being a direct sum of copies of R. Since every flat module is a direct summand of a free module, by preceding proposition we're done. □

Proposition 3.3. Let $\{P_i \mid i \in I\}$ be a direct system of right modules over any ring R, where I is a directed set. If each P_i ($i \in I$) is flat, then so is the direct limit module $P := \varinjlim P_i$

Proof. Let $A \xrightarrow{j} B$ an injection in ${}_R\mathfrak{M}$. For each $i \in I$, $P_i \otimes_R A \to P_i \otimes_R B$ is injective. By the construction of direct limits it follows easily that

$$\varinjlim (P_i \otimes_R A) \longrightarrow \varinjlim (P_i \otimes_R B)$$

is injective. Since $- \otimes_R A, - \otimes_R B$ preserves direct limits, then $P \otimes_R A \longrightarrow P \otimes_R B$ is injective as desired. □

Notation 3.1. Each module is the direct limit of its finitely generated submodules.

Proof. Let M be an R - module and $\mathscr{A} = \{N \leq M \mid N \text{ is f.g}\}$. Then $(\mathscr{A}, \subseteq)$ is a direct system, of which direct limit is equal to

$$\varinjlim N = \bigcup \mathscr{A} = M.$$

□

Corollary 3.2. Let $P \in \mathfrak{M}_R$, of which each f.g. submodule is flat. Then P is flat.

Proof. The result implied immediately by Proposition 3.3 and Notation 3.1. □

3.2 Pure Exact Sequences

Definition 3.2. A (short) exact sequence

$$\mathscr{E}\colon 0 \to A \xrightarrow{\varphi} B \to C \to 0$$

in $\mathfrak{M}_R$ is said to be **pure** (exact) if $\mathscr{E} \otimes_R C'$ is exact, for each $C' \in_R \mathfrak{M}$. In this case, we say that $\varphi(A)$ is a **pure submodule** of B.

Example 3.1. Let $\mathscr{E}\colon 0 \to A \xrightarrow{\varphi} B \to C \to 0$ a split short exact sequence. Then $\mathscr{E}$ is pure.

Proof. Let $C' \in_R \mathfrak{M}$. Since $\mathscr{E}$ splits, then exists $\psi \in \mathrm{Hom}_R(B, A)$ s.t. $\psi \circ \varphi = \mathrm{id}_A$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi"] & B \arrow[l, bend left, "\psi"] \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

If we apply the functor $- \otimes_R C'$ on last relation the injectivity of $\varphi \otimes_R C'$ implied immediately. □

Example 3.2. Direct sum of pure exact sequences is pure exact.

Proof. Let $B_i \leq A_i$ pure submodules, $i \in I$. We shall show that $B := \bigoplus_{i\in I} B_i$ is a pure submodule of $A := \bigoplus_{i\in I} A_i$. Equivalently, if $C \in_R \mathfrak{M}$ and $B \xrightarrow{j} A$ the inclusion map, then $B \otimes_R C \xrightarrow{j\otimes_R C} A \otimes_R C$ is injective. Since each square is commutative

$$\begin{tikzcd} B_i \otimes_R C \arrow[r, tail, "j_i \otimes_R C"] \arrow[d, tail] & A_i \otimes_R C \arrow[d, tail] \\ \bigoplus_{i\in I} (B_i \otimes_R C) \arrow[r, "\bigoplus_{i\in I} j_i \otimes_R C"] & \bigoplus_{i\in I} (A_i \otimes_R C) \end{tikzcd}$$

then $\bigoplus_{i\in I} j_i \otimes_R C$ is injective and since $- \otimes_R C$ preserves direct sums it's obvious that $j \otimes_R C$ is injective. □

Example 3.3. More general, similar to the above example, since $-\otimes_R C$ preserves direct limits, then direct limit of any system of pure short exact sequences is pure exact.

Example 3.4. For any family of right R - modules $\{B_i\}_{i\in I}$, $\bigoplus_{i\in I} B_i$ is pure submodule of $\prod_{i\in I} B_i$.

Proof. Let $C \in_R \mathfrak{M}$. We set $\bigoplus_{i\in I} B_i \xrightarrow{j} \prod_{i\in I} B_i$ the inclusion map. We define

$$\varepsilon\colon \left(\prod_{i\in I} B_i\right) \otimes_R C \to \prod_{i\in I} (B_i \otimes_R C), \quad \varepsilon\left(\{b_i\}_{i\in I} \otimes_R c\right) = \{b_i \otimes_R c\}_{i\in I}.$$

Thus the following diagramm commutes

$$\begin{tikzcd} \left(\bigoplus_{i\in I} B_i\right) \otimes_R C \arrow[r, "j\otimes_R C"] \arrow[d, "\cong"] & \left(\prod_{i\in I} B_i\right) \otimes_R C \arrow[d, "\varepsilon"] \\ \bigoplus_{i\in I} (B_i \otimes_R C) \arrow[r, tail] & \prod_{i\in I} (B_i \otimes_R C) \end{tikzcd}$$

it's easily seen that $\varepsilon \circ j \otimes_R C$ is injective and therefore $j \otimes_R C$ is injective. $\square$

Example 3.5. Let $A \subseteq B \subseteq C$ be right R - modules. If $A \subseteq C$ is pure submodule, then $A \subseteq B$ is pure submodule. Conversely, if $A \subseteq B$ pure and $B \subseteq C$ pure, then $A \subseteq C$ is pure.

Example 3.6. Let $\varphi\colon R \to S$ be a ring homomorphism. Then S can be viewed as left R - module via f. If $\mathscr{E}$ is an exact pure sequence in $\mathfrak{M}_R$, then $\mathscr{E} \otimes_R S$ is pure in $\mathfrak{M}_S$.

Notation 3.2. Let $C \in \mathfrak{M}_R$ and

$$\mathscr{E}\colon 0 \to K \xrightarrow{\varphi} P \to C \to 0$$

a projective resolution of C. Let $X \in_R \mathfrak{M}$. Then $\mathscr{E} \otimes_R X$ is exact iff

$$\operatorname{Tor}_1^R(C, X) = \ker\left(\varphi \otimes_R X\colon K \otimes_R X \to P \otimes_R X\right) = 0.$$

Theorem 3.1 (Characterization of Flat Modules)**.** A right R - module C if flat if and only if every short exact sequence

$$\mathscr{E}\colon 0 \to A \to B \to C \to 0$$

in $\mathfrak{M}_R$ is pure.

Proof. First, we assume that C is flat. Therefore, $\operatorname{Tor}_n^R(C, X) = 0$, for all $n \in \mathbb{N}$ and $X \in_R \mathfrak{M}$. By homology long exact sequence theorem it's obvious that every short exact sequence $\mathscr{E}\colon 0 \to A \to B \to C \to 0$ is pure.

$$\begin{tikzcd}
0 & C \otimes X \arrow[l] & B \otimes X \arrow[l] & A \otimes X \arrow[l] \\
& \mathrm{Tor}_1^R(C,X) = 0 \arrow[urr, "\delta^0"] & \mathrm{Tor}_1^R(B,X) \arrow[l] & \mathrm{Tor}_1^R(A,X) \arrow[l] \\
& \mathrm{Tor}_2^R(C,X) \arrow[urr, "\delta^1"] & \cdots\cdots \arrow[l] & \cdots\cdots
\end{tikzcd}$$

The inverse result implied immediately from Notation 3.2. $\square$

Corollary 3.3. Let $\mathscr{E}\colon 0 \to A \to B \to C \to 0$ be exact in $\mathfrak{M}_R$.

(a) Assume B is flat. Then $\mathscr{E}$ is pure iff C if flat.

(b) Assume C is flat. Then B is flat iff A is flat.

Proof. (a) Let $X \in_R \mathfrak{M}$. We apply homology long exact sequence theorem in short exact sequence $\mathscr{E}\colon 0 \to A \to B \to C \to 0$, so that

$$\begin{tikzcd}
0 & C \otimes X \arrow[l] & B \otimes X \arrow[l] & A \otimes X \arrow[l] \\
& \mathrm{Tor}_1^R(C,X) \arrow[urr, "\delta^0"] & \mathrm{Tor}_1^R(B,X) = 0 \arrow[l] & \mathrm{Tor}_1^R(A,X) \arrow[l] \\
& \mathrm{Tor}_2^R(C,X) \arrow[urr, "\delta^1"] & \cdots\cdots \arrow[l] & \cdots\cdots
\end{tikzcd}$$

From the above diagram, immediately implied that $\mathscr{E}$ is pure iff $\mathrm{Tor}_1^R(C,X) = 0$, for each $X \in_R \mathfrak{M}$ iff C if flat.

(b) If C if flat, then $\mathrm{Tor}_n^R(C,X) = 0$, for all $n \in \mathbb{N}$ and $X \in_R \mathfrak{M}$. By homology long exact sequence theorem

$$\begin{tikzcd}
0 & C \otimes X \arrow[l] & B \otimes X \arrow[l] & A \otimes X \arrow[l] \\
& \mathrm{Tor}_1^R(C,X) = 0 \arrow[urr, "\delta^0"] & \mathrm{Tor}_1^R(B,X) \arrow[l] & \mathrm{Tor}_1^R(A,X) \arrow[l] \\
& \mathrm{Tor}_2^R(C,X) = 0 \arrow[urr, "\delta^1"] & \cdots\cdots \arrow[l] & \cdots\cdots
\end{tikzcd}$$

From the above diagramm $\mathrm{Tor}_n^R(B,X) \cong \mathrm{Tor}_n^R(A,X)$, for each $X \in_R \mathfrak{M}$ and therefore B if flat iff A is flat.

$\square$

Definition 3.3. A module P_R is said to be **finitely presented** (f.p.) if exists a short exact sequence

$$0 \to K \to F \to P \to 0$$

in $\mathfrak{M}_R$, where K is finitely generated (f.g.) and F is a free module of finite rank. Equivalently, exists an exact sequence in $\mathfrak{M}_R$ as follows :

$$R^m \to R^n \to P \to 0$$

Theorem 3.2 (Characterization of Pure Exact Sequences). For any short exact sequence $\mathscr{E}: 0 \to A \to B \to C \to 0$ in $\mathfrak{M}_R$, the following are equivalent :

(a) $\mathscr{E}$ is pure exact.

(b) $\mathscr{E} \otimes_R C'$ is exact for any f.p. left R - module C'.

(c) If $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_m \in B$ and $s_{ij} \in R$ ($1 \le i \le m$ and $1 \le j \le n$) are give such that

$$a_j = \sum_i b_i s_{ij}$$

then there are $a'_1, \ldots, a'_m \in A$ s.t.

$$a_j = \sum_i a'_i s_{ij}$$

(d) Given any commutative digram in $\mathfrak{M}_R$

$$\begin{tikzcd} R^n \arrow[r, "\sigma"] \arrow[d, "\alpha"] & R^m \arrow[d, "\beta"] \\ A \arrow[r, hook] & B \end{tikzcd}$$

there exists $\vartheta \in \mathrm{Hom}_R(R^m, A)$ s.t. $\vartheta\sigma = \alpha$.

$$\begin{tikzcd} R^n \arrow[r, "\sigma"] \arrow[d, "\alpha"] & R^m \arrow[d, "\beta"] \arrow[dl, dashed, "\vartheta"] \\ A \arrow[r, hook] & B \end{tikzcd}$$

(e) The sequence $\mathrm{Hom}_R(M, \mathscr{E})$ is exact for every M f.p. right R - module.

(f) $\mathscr{E}$ is the direct limit of a direcy system of split exact sequences

$$0 \to A \to B_i \to C_i \to 0$$

where the C_i's are f.p. right R - modules.

Proof. • (a) $\to$ (b) The desired result implied immediately from definition of pure exact sequence.

- (b) $\to$ (c) We consider the submodule of $R^n = \bigoplus_{j=1}^n Re_j$

$$K = \left\langle \left\{ \sum_{j=1}^{m} s_{ij} e_j \mid 1 \leq i \leq m \right\} \right\rangle$$

Thus the left R - module R^n/K is f.p. and therefore $A \otimes_R (R^n/K) \xrightarrow{\rho} B \otimes_R (R^n/K)$ is injective. Since the sequence is exact

$$A \otimes_R K \to A \otimes (R^n) \to A \otimes (R^n/K) \to 0$$

we can identify

$$A \otimes (R^n/K) \equiv A \otimes (R^n) / \mathrm{Im}(A \otimes_R K)$$

Similarly, we can identify

$$B \otimes (R^n/K) \equiv B \otimes (R^n) / \mathrm{Im}(B \otimes_R K)$$

Then

$$\rho\left(\left[\sum_j a_j \otimes e_j\right]\right) = \sum_j \sum_i [b_i s_{ij} \otimes e_j] = \sum_i \left[b_i \otimes \left(\sum_j s_{ij} e_j\right)\right] = 0$$

Since ρ is injective, then $\sum_j a_j \otimes e_j \in \mathrm{Im}(A \otimes_R K)$. Equivalenlty, there are $a'_1, \ldots, a'_m \in A$ s.t.

$$\sum_j a_j \otimes e_j = \sum_i a'_i \otimes \left(\sum_j s_{ij} e_j\right) = \sum_j \left(\sum_i a'_i s_{ij}\right) \otimes e_j$$

Since $A \otimes_R -$ preserves direct sums, then $A \otimes_R R^n \equiv \bigoplus_{j=1}^n A \otimes Re_j$. Therefore, for each $j \in \{1, \ldots, n\}$, we have that

$$a_j = \sum_i a'_i s_{ij}$$

- (c) $\to$ (d) We denote $R^n = \bigoplus_{j=1}^n Re_j$ and $R^m = \bigoplus_{j=1}^m R\tilde{e_j}$. Then, for each $1 \leq i \leq m$ and $1 \leq j \leq n$ we set $a_j = \alpha(e_j)$ and $b_i = \beta(\tilde{e_i})$. Since $\sigma(e_j) \in R^m$, then there are $s_{ij} \in R$ s.t.

$$\sigma(e_j) = \sum_i \tilde{e_i} s_{ij}$$

Then

$$a_j = \alpha(e_j) = \beta\sigma(e_j) = \sum_i b_i s_{ij}$$

By (c), there are $a'_1, \ldots, a'_m \in A$ s.t.

$$a_j = \sum_i a'_i s_{ij}$$

We define $\vartheta \in \operatorname{Hom}_R(R^m, A)$, where $\vartheta(\tilde{e}_i) = a'_i$. Then $\vartheta\sigma = \alpha$, as desired.

- (d) $\to$ (e) Let M in $\mathfrak{M}_R$ a f.p. module Then exists exact sequence

$$R^n \xrightarrow{\sigma} R^m \xrightarrow{\tau} M \to 0$$

We will show that exactness of $\operatorname{Hom}_R(M, \mathscr{E})$, showing that

$$\psi_* \colon \operatorname{Hom}_R(M, B) \to \operatorname{Hom}_R(M, C)$$

is surjective. Let $\gamma \in \operatorname{Hom}_R(M, C)$. By freebess of R^m, exists $\beta \in \operatorname{Hom}_R(R^m, B)$ s.t. the following diagram commutes

$$\begin{tikzcd} R^m \arrow[r, "\tau"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "\gamma"] & 0 \\ B \arrow[r, "\psi"] & C \arrow[r] & 0 \end{tikzcd}$$

We notice that

$$\psi\beta\sigma = \gamma\tau\sigma = 0$$

and therefore $\operatorname{Im}(\beta\sigma) \subseteq \ker\psi = A$. We set $\alpha = \beta\sigma$. Then the following diagram commutes

$$\begin{tikzcd} R^n \arrow[r, "\sigma"] \arrow[d, "\alpha"] & R^m \arrow[r, "\tau"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "\gamma"] & 0 \\ A \arrow[r, hook] & B \arrow[r, "\psi"] & C \arrow[r] & 0 \end{tikzcd}$$

By (d), there exists $\vartheta \in \operatorname{Hom}_R(R^m, A)$ s.t.

$$\alpha = \vartheta\sigma \Rightarrow \beta\sigma = \vartheta\sigma \Rightarrow (\beta - \vartheta)\,\sigma = 0$$

From the above relation, since (M, τ) is the cokernel of σ, then exists $\lambda \in \operatorname{Hom}_R(M, B)$ s.t. the following diagram commutes

$$\begin{tikzcd} R^n \arrow[r, "\sigma"] & R^m \arrow[r, "\tau"] \arrow[d, "\beta-\vartheta"'] & M \arrow[dl, "\lambda"] \\ & B & \end{tikzcd}$$

Therefore

$$\psi\lambda\tau = \psi(\beta - \vartheta) = \psi\beta - \psi\vartheta = \gamma\tau$$

Since τ is right invertible then $\psi\lambda = \gamma$, as we desire.

- (e) $\to$ (f) It's true that every module is direct limit of f.p. modules. For the proof of this statement we refer the reader to [2] (Lazard, Govorov Theorem). Therefore C is the direct limit of some direct system $(C_i, \gamma_i)_{i\in I}$, where C_i's are f.p. modules.
  - We consider the pullback diagram of γ_i, ψ

$$\begin{array}{ccccc} B_i & \xrightarrow{\psi_i} & C_i & \longrightarrow & 0 \\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \gamma_i} & & \\ B & \xrightarrow{\psi} & C & \longrightarrow & 0 \end{array}$$

    where $B_i = \{(x,y) \in C_i \oplus B \mid \gamma_i(x) = \psi(y)\}$ and $\psi_i = \pi_1,\ \beta_i = \pi_2$.
  - By Lemma 1.4,the induced following diagram commutes

$$\begin{array}{lccccccccc} \mathscr{E}_i: & 0 & \longrightarrow & A & \longrightarrow & B_i & \xrightarrow{\psi_i} & C_i & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \gamma_i} & & \\ \mathscr{E} & 0 & \longrightarrow & A & \hookrightarrow & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \end{array}$$

  - By (e), there exists $\lambda_i \in \mathrm{Hom}_R(C_i, B)$ s.t. $\psi\lambda_i = \gamma_i$. Let $\rho_i \colon C_i \to B_i$ defined by $\rho_i(x) = (x, \lambda_i(x))$. Then it's obvious that $\psi_i\rho_i = \mathrm{id}_{C_i}$ and therefore $\mathscr{E}_i$ splits. I can be easily seen that $\mathscr{E} = \varinjlim \mathscr{E}_i$.
- (f) $\to$ (a) The result implied immediately by Example 3.3.

$\square$

Corollary 3.4. Let M be an R - module and $S \subseteq M$ submodule. Then S is pure if and only if for every $m \geq 1$, for every $T \subseteq R^m$ finitely generated submodule and for every $h\colon T \to S$, whenever h can be extended to $\tilde{h}\colon R^m \to M$, then can be extended to a $\vartheta\colon R^m \to S$, so that the following diagramm commutes

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, "h"'] & R^m \arrow[d, "\tilde{h}"] \arrow[dl, dashed, "\vartheta"] \\ S \arrow[r, hook] & M \end{tikzcd}$$

Proof. We assume that S is pure. Since T if f.g., there exists an epimorphism, for some $n \in \mathbb{N}$,

$$R^n \xrightarrow{\pi} T$$

We consider a morphism $h\colon T \to S$ s.t. can be extended to $\tilde{h}\colon R^m \to T$. Therefore the following diagram commutes

$$\begin{tikzcd} T \arrow[r, hook, "i"] \arrow[d, "h"'] & R^m \arrow[d, "\tilde{h}"] \\ S \arrow[r, hook, "j"'] & M \end{tikzcd}$$

The existence of morphism π implies the following commutating diagram :

$$\begin{tikzcd} R^n \arrow[r, "i\pi"] \arrow[d, "h\pi"'] & R^m \arrow[d, "\tilde{h}"] \\ S \arrow[r, hook, "j"'] & M \end{tikzcd}$$

and from Theorem 3.2 there exists $\varphi\colon R^m \to S$ s.t. the following diagramm commutes

$$\begin{tikzcd} R^n \arrow[r, "i\pi"] \arrow[d, "h\pi"'] & R^m \arrow[d, "\tilde{h}"] \arrow[dl, dashed, "\vartheta"] \\ S \arrow[r, hook, "j"'] & M \end{tikzcd}$$

Thus

$$\vartheta i\pi = h\pi \Rightarrow \vartheta i = h$$

so ϑ is the desired extension.

Conversely, consider a commutative diagram

$$\begin{tikzcd} R^n \arrow[r, "f"] \arrow[d, "h"'] & R^m \arrow[d, "\tilde{h}"] \\ S \arrow[r, hook, "j"'] & M \end{tikzcd}$$

and we will show that there is $\vartheta\colon R^m \to S$ s.t. the following diagram commutes

$$\begin{tikzcd} R^n \arrow[r, "f"] \arrow[d, "h"'] & R^m \arrow[d, "\tilde{h}"] \arrow[dl, dashed, "\vartheta"] \\ S \arrow[r, hook, "j"'] & M \end{tikzcd}$$

If $T = \mathrm{Im} f \subseteq R^m$, then T if f.g. and there is a commutative diagram

$$\begin{tikzcd} R^n \arrow[r, "f"] \arrow[d, "h"] & T \arrow[r, hook, "i"] \arrow[d, "\tilde{h}i"] & R^m \arrow[d, "\tilde{h}"] \\ S \arrow[r, equal] & S \arrow[r, hook, "j"'] & M \end{tikzcd}$$

By our assumptions, there exists $\vartheta\colon R^m \to S$ s.t. the following diagram commutes

$$\begin{tikzcd} R^n \arrow[r, "f"] \arrow[d, "h"] & T \arrow[r, hook, "i"] \arrow[d, "\tilde{h}i"] & R^m \arrow[d, "\tilde{h}"] \arrow[dl, dashed, "\vartheta"] \\ S \arrow[r, equal] & S \arrow[r, hook, "j"'] & M \end{tikzcd}$$

From the above diagram it can be easily seen that $\vartheta f = h$ $\square$

Corollary 3.5. Let $S \subseteq M$ submodule. Let $P = \bigoplus_{(T,R^m)} R^m$ and $U = \bigoplus_{(T,R^m)} T$, summed over the set of all (T, R^m), where $m \geq 1$ and T is f.g.

Then S is pure if and only if for any $h\colon U \to S$ homomorphism, which can be extended to a homomophism $\tilde{h}\colon P \to M$, then can be extended to a homomorphism $\tilde{h}\colon P \to S$.

Proof. Use Corollary 3.4. □

CHAPTER 4

VANISHING OF Ext FUNCTOR

4.1 A Review of Ordinal Numbers

Consider the class WO of all well-ordered sets; if we denote by $\cong$ the relation "being isomorphic to" between ordered structures, then $\cong$ defines an equivalence relation on WO. An ordinal can be thought of as an equivalence class of WO under the relation $\cong$; more precisely, the class Ord of all ordinals satisfy the property that, for any well-ordered set A, there exists exactly one ordinal isomorphic to A.

Notation 4.1. If A and B are ordered sets, $A \hookrightarrow B$ means that A is embeddable into B, i.e there exists an order-preserving injective map from A to B.

Theorem 4.1 (Transfinite Induction)**.** Let $(A, <)$ be a well-ordered set and $P(x)$ a property defined on A satisfying :

$$\forall a \in A,\ [(\forall b < a\, P(b)) \Rightarrow P(a)]$$

Then $P(a)$ is true for every $a \in A$.

Proof.

- Consider $B \coloneqq \{a \in A \mid\ P(a)\ \text{ is not true}\}$. For the sake of contradiction we assume that $B \neq \emptyset$.
- Since A is well-ordered, we can consider $a = \min(B)$. Then $P(b)$ is true for every $b < a$, but $P(a)$ is false, which contradicts the hypothesis of the theorem. Thus, $B = \emptyset$.

□

Definition 4.1 (Initial Segment)**.** Let $(A,<)$ be a well - ordered set and $a \in A$. The **initial segment** of A determined by a is the subset of A of the form $A_a = \{b \in A \mid b \leq a\}$.

Proposition 4.1. Let $(A,<)$ be a well-ordered set. If B is a proper initial segment of A, then there is no embedding $f\colon A \to B$. In particular, A and B are not isomorphic.

Proof. For the sake of contradiction, we assume that there is an embedding $f\colon A \to B$.

- We shall prove by induction on A that for all $x \in A$ it holds that $f(x) \geq x$. Let $a \in A$ and assume that for all $b < a$, $f(b) \geq b$. Let $b \in A$ such that $b < a$. Since f preserves the order, we have $f(b) < f(a)$, and by induction hypothesis we also have $b \leq f(b) \geq b$ hence $b < f(a)$. The latter relation implies that $f(a) > b$, for all $b < a$, hence $f(a) \geq a$.
- Since B is a proper subset of A, there exists $a \in A \setminus B$, and since B is an initial segment of A, we then have $a > b$, for all $b \in B$. In particular, we have $a > f(a)$, hence a contradiction.

□

Definition 4.2. A set A is called **transitive** if every element of A is also a subset of A. Equivalently, A is transitive if and only if for each $a \in A$ and $x \in A$, then $x \in A$.

Lemma 4.1. Let A be a transitive set. Then $\in$ is a transitive relation on A if and only if for every $a \in A$, a is a transitive set.

Proof.

- First, we assume that $\in$ is transitive. Let $a \in A$. We want to prove that a is a transitive set. Let $y \in a$ and $x \in y$. We want to show that $x \in a$. Since $\in$ is transitive relation it suffices to show that $x, y \in A$. Since $a \in A$ and A is transitive, then $y \in A$ and therefore $y \subseteq A$. Thus $x \in A$ and we're done.
- Conversely, assume that a is a transitive set, for all $a \in A$. Let $a, b, c \in A$ such that $a \in b \in c$. Since c is a transitive set, this relation implies $a \in c$.

□

Lemma 4.2. A union of transitive sets is a transitive set.

Proof. Let $\{A_i\}_{i \in I}$ be a family of transitive sets and set $A \coloneqq \bigcup_{i \in I} A_i$. We want to show that A is transitive. Let $a \in A$ and $x \in a$. There exists $i \in I$ such that $a \in A_i$. Since A_i is a transitive set, the relation $x \in a \in A_i$ implies $x \in A_i$, hence $x \in A$. □

Definition 4.3. A set α is called **ordinal** if it's transitive and the pair $(\alpha, \in)$ is a well ordered set.

Remark 1.
- The class Ord of all ordinals is not a set in the sense of axiomatic set theory.
- The definition above implies in particular that $\in$ is an well-order on α, so it is a transitive relation. According to Lemma 4.1, this means that any element of α is a transitive set.

Example 4.1. Each natural number $n+1 = \{0, \ldots, n-1\} \cup \{n\}$ is ordinal. In addition, $\omega = \bigcup_{n \in \mathbb{N}} n$ is ordinal.

Definition 4.4. Let $\mathscr{R}$ be a relation on a set S. Then $\mathscr{R}$ is a **strict ordering** (on S) if and only if $\mathscr{R}$ satisfies the strict ordering axioms:

(a) *Asymmetry* :
$$\forall a, b \in S : (a\mathscr{R}b) \implies \neg(b\mathscr{R}a)$$

(b) *Transitivity* :
$$\forall a, b, c \in S : (a\mathscr{R}b) \wedge (b\mathscr{R}c) \implies a\mathscr{R}c$$

Proposition 4.2. The binary relation $\in$ defines a strict order on Ord.

Proof.
- $\in$ *is transitive* : Let $\alpha \in \beta \in \gamma$ all in Ord. Since γ is a transitive set, we have $\alpha \in \gamma$.
- $\in$ *is antisymmetric* : Assume there exists $\alpha, \beta \in \mathrm{Ord}$ such that $\beta \in \alpha \in \beta$.
- Since β is a transitive set, we have $\beta \in \beta$, and since $(\beta, \in)$ is well-ordered, this is contradiction.

□

Remark 2. The order we consider on Ord will always be the one given by $\in$; thus, if α, β are ordinals, $\alpha < \beta$ means $\alpha \in \beta$.

Proposition 4.3. Let α be an ordinal. Then

$$\alpha = \{\beta \mid \ \beta \text{ is an ordinal and } \beta < \alpha\}$$

Proof. Let $\beta \in \alpha$, we shall show that β is an ordinal.

- By remark 1 we know that β is a transitive set.
- Since α is a transitive set, we have $\beta \subseteq \alpha$, so the relation $\in$ defined on β is the restriction of the relation $\in$ defined on α. Since $(\alpha, \in)$ is well-ordered, this implies that $(\beta, \in)$ is well-ordered. Thus, β is an ordinal.

$\square$

Corollary 4.1. Let $\alpha, \beta \in \mathrm{Ord}$. Then

(a) $\alpha \subseteq \beta$ if and only if forall $\delta \in \mathrm{Ord}: \ \delta < \alpha \Rightarrow \delta < \beta$.

(b) $\alpha = \beta$ if and only if forall $\delta \in \mathrm{Ord}: \delta < \alpha \Leftrightarrow \delta < \beta$.

Corollary 4.2. Let $\alpha, \beta \in \mathrm{Ord}$ such that $\alpha < \beta$. Then α is a proper initial segment of β.

Lemma 4.3. Let α, β be ordinals such that $\beta \not< \alpha$. Then $\gamma = \min(\beta \setminus \alpha)$ exists and is included in α. If moreover $\alpha \subsetneqq \beta$, then $\gamma = \alpha$, and so $\alpha \in \beta$.

Proof.
- The existence of γ comes from the fact that $\beta \setminus \alpha \neq \emptyset$ and that β is well-ordered.
- Note that since $\gamma \in \beta$, γ is an ordinal and $\gamma < \beta$. Let ordinal $\delta < \gamma$. Since $\gamma < \beta$, we have $\delta \in \beta$. However, since $\delta < \gamma$, we have by minimality of γ that $\delta \in \alpha$. This proves that $\gamma \subseteq \alpha$.
- Now assume that $\alpha \subset \beta$ and let $\delta < \alpha$; we also have $\delta \in \beta$. If $\delta > \gamma$, we would have $\alpha > \gamma$, i.e., $\gamma \in \alpha$, which by definition of γ is impossible. Since $\delta, \gamma \in \beta$, and β is totally ordered, this implies $\delta < \gamma$. This proves that $\alpha \subseteq \gamma$, hence $\gamma = \alpha$.

$\square$

Lemma 4.4. Let α, β be ordinals. Then $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.

Proof.
- If $\alpha = \beta$ there is nothing to prove. If $\alpha < \beta$, the fact that β is a transitive set implies $\alpha \subseteq \beta$.
- We assume that $\alpha \subset \beta$. In that case, Lemma 4.3 implies that $\alpha \in \beta$ and we're done.

$\square$

Proposition 4.4. The order $<$ (which is also $\in$) is a total order on Ord.

Proof.
- Let α, β be ordinals such that $\beta \not< \alpha$. By Lemma 4.4, we have $\beta \not\subseteq \alpha$, which by Lemma 4.3 implies $\gamma = \min(\beta \setminus \alpha) \subseteq \alpha$.
- By Lemma 4.4, we have $\gamma \leq \alpha$. however, by definition of γ, we can't have $\gamma \in \alpha$, hence $\gamma = \alpha$, hence $\alpha \in \beta$.

□

Proposition 4.5. If $\alpha \neq \beta$, then α and β are not isomorphic.

Proof. Since $<$ is a total order, we can assume $\alpha < \beta$. Then α is a proper initial segment of β, which by Proposition 4.1 implies that α and β are not isomorphic. □

Proposition 4.6. The pair $(\mathrm{Ord}, <)$ is well-ordered.

Proof. Since the order is total, we just have to show that there is no strictly decreasing infinite sequence of ordinals

$$\alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_n > \cdots$$

If such a sequence existed, then $\alpha_n \in \alpha_0$ for every $n \geq 0$, so $(\alpha_n)_{n>0}$ would be an infinite decreasing sequence of elements of α_0, which would contradict the fact that α_0 is well-ordered. □

Proposition 4.7.
(a) If α is an ordinal, then so is $\alpha \cup \{\alpha\}$. The ordinal $\alpha + 1 := \alpha \cup \{\alpha\}$ is called the **successor** of α.

(b) If A is a set of ordinals, then $\sup(A) = \bigcup A$ is an ordinal.

Proof.
- Set $\delta = \bigcup A$. Then δ is a union of transitive sets so by Lemma 4.2 it is a transitive set.
- To show that δ is well-ordered, just note that $\delta \subset \mathrm{Ord}$, and that Ord is well-ordered.
- We will show that δ is the supremum of A. Clearly, $\alpha < \delta$, for any $\alpha \in A$. Let $\gamma \in \mathrm{Ord}$ such that $\gamma > \alpha$, for all $\alpha \in A$.

- Let $\beta \in \delta$; there exists $\alpha \in A$ such that $\beta \in \alpha < \gamma$, hence $\beta \in \gamma$. This proves that $\delta \subseteq \gamma$, hence $\delta \leq \gamma$.

$\square$

Remark 3.
- The definition of the successor of an ordinal is consistent with the usual definition of the successor of an integer: indeed, if $n \in \omega$, then $n + 1 = \{0, 1, \ldots, n\} = n \cup \{n\}$.
- $\alpha + 1$ is the smallest ordinal strictly bigger than α.
- $\sup(A)$ is not necessarily a max: take $A := \{2n \mid n \in \omega\}$, then $\sup(A) = \omega$, but A has no max.
- However, if we take $A = \{0, 1, 3\}$, then $\sup(A) = \max(A) = 3$.
- If α is an ordinal, then in particular it is a set of ordinals, and in that case we have $\sup \alpha = \alpha$.

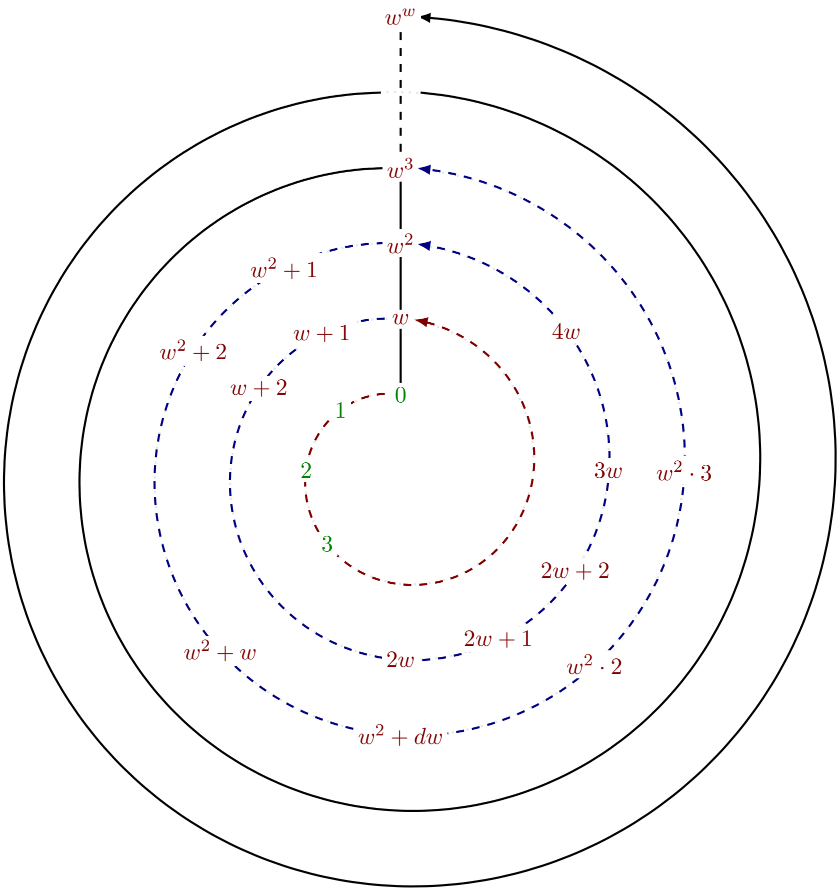
Definition 4.5. An ordinal which is not a successor and is not 0 is called a **limit ordinal** .

Example 4.2. ω is a limit ordinal (it is actually the smallest one).

Corollary 4.3.
- Thus, we can say that there are three kinds of ordinals: 0, successor ordinals, and limit ordinals.
- The distinction between limit and successor ordinals is an important one, since they have different properties; for example, a successor ordinal has a max, but a limit ordinal does not. We will also see that we usually separate the case of successor and limit ordinal when making a proof by induction on ordinals.
- Proposition 4.7 gives us the tools to inductively construct ordinals. Remember that natural numbers are constructed by starting with 0 and by then repeatedly applying the successor map: we define 1 as the successor of 0, 2 as the successor of 1, and so on. Ordinals are constructed by alternately applying these two operations:
  - Taking the successor of the last ordinal defined.
  - Once the successor operation has been repeated ω times, take the supremum of all the already defined ordinals.

– More precisely: we start by defining 0, then apply the successor operation ω times to construct the set of natural numbers. We then define ω as the supremum of all natural numbers. We then repeat the same process: after ω comes its successor $\omega + 1 := \omega \cup \{\omega\}$, then $\omega + 2 := (\omega + 1) \cup \{\omega + 1\}$, and so on; after applying the successor operation ω times, we arrive at $\omega + \omega := \sup\{n \in \omega\}(\omega + n)$.

– By repeating this process indefinitely, we construct the class of ordinals.



Motivation 3. We conclude our review of ordinal numbers with the most important result. This is a generalization of ordinary induction, which is a very useful tool for applying induction to continuous chains of sets in which the index set is a set of ordinals.

Theorem 4.2 (Transfinite induction on Ord)**.** Let $\mathscr{P}(x)$ be a property defined on ordinals such that :

- $\mathscr{P}(0)$ is true.
- If $\mathscr{P}(\alpha)$ is true, then $\mathscr{P}(\alpha)$ is true.
- If α is a limit ordinal and if $\mathscr{P}(\beta)$ is true for every $\beta < \alpha$, then $\mathscr{P}(\alpha)$ is true.

Then $\mathscr{P}(\alpha)$ is true for every $\alpha \in \mathrm{Ord}$.

Theorem 4.3 (Transfinite induction on an ordinal)**.** Let $\alpha \in \mathrm{Ord}$ and $\mathscr{P}(x)$ a property defined on α such that

- $\mathscr{P}(0)$ is true.
- If $\beta + 1 < \alpha$ and $\mathscr{P}(\beta)$ is true, then $\mathscr{P}(\beta + 1)$ is true.
- If $\beta < \alpha$ is a limit ordinal and if $\mathscr{P}(\gamma)$ is true for every $\gamma < \beta$, then $\mathscr{P}(\beta)$ is true.

Then $\mathscr{P}(\beta)$ is true for every $\beta < \alpha$.

4.2 Vanishing of Ext Functor

Lemma 4.5. Let N be a module. Let $\{M_\alpha \mid \alpha < \kappa\}$ be a continuous chain of modules. Put $M = \bigcup_{\alpha<\kappa} M_\alpha$. We assume that

- $\mathrm{Ext}^1_R(M_0, N) = 0$ and
- $\mathrm{Ext}^1_R\,(M_{\alpha+1}/M_\alpha, N) = 0$, whenever $\alpha + 1 < \kappa$

Then $\mathrm{Ext}^1_R(M, N) = 0$.

Proof. Put $M = M_k$. By Theorem 4.3, we will prove the desired result by using induction on $\alpha \leq \kappa$, that is $\mathrm{Ext}^1_R(M_\alpha, N) = 0$, for each $a \leq \kappa$.

(a) *Zero Case.* By assumptions, is true that $\mathrm{Ext}^1_R(M_\alpha, N) = 0$, for $\alpha = 0$.

(b) *Successor Case.* Let $\alpha = \beta + 1 < \kappa$. We assume that $\mathrm{Ext}^1_R\,(M_\beta, N) = 0$. If we apply the functor $\mathrm{Ext}^1_R(-, N)$ to the short exact sequence

$$0 \to M_\beta \hookrightarrow M_\alpha \to M_\alpha/M_\beta \to 0$$

then implied

$$0 = \mathrm{Ext}^1_R(M_\alpha/M_\beta, N) \to \mathrm{Ext}^1_R(M_\alpha, N) \to \mathrm{Ext}^1_R(M_\beta, N) = 0$$

(c) *Limit Case* Let $\alpha < \kappa$ a limit ordinal. We consider short exact sequence

$$0 \to N \to I \xrightarrow{\pi} I/N$$

where I is injective. We want to show that $\mathrm{Ext}^1_R(M_\alpha, N) = 0$, so it suffices to show that

$$\pi_* \colon \mathrm{Hom}(M_\alpha, I) \to \mathrm{Hom}(M_\alpha, I/C)$$

Let $\varphi \in \mathrm{Hom}(M_\alpha, I/C)$. We're looking for $\psi \in \mathrm{Hom}(M_\alpha, I)$ s.t. $\pi\psi = \varphi$.

- If we construct a continuous chain of homomorphisms $\{\psi_\beta \colon M_\beta \to I\}_{\beta<\alpha}$ s.t.

$$\varphi \restriction_{M_\beta} = \pi\psi_\beta \quad \text{and} \quad \psi_\beta \restriction_{M_\gamma} = \psi_\gamma, \quad \forall \gamma < \beta < \alpha$$

  then if we set $\psi := \bigcup_{\beta<\alpha} \psi_\beta$, then $\psi \in \operatorname{Hom}(M_\alpha, I)$ and $\varphi = \pi\psi$.
- We shall construct this chain with induction on $\beta < \alpha$.
  - We assume that ψ_β is already defined. Since I is injective, there $\eta \in \operatorname{Hom}(M_{\beta+1}, I)$, which extends ψ_β

$$\begin{tikzcd} M_\beta \arrow[r, "i"] \arrow[d, "\psi_\beta"'] & M_{\beta+1} \arrow[dl, "\eta"] \\ I & \end{tikzcd}$$

  - If $\delta = \varphi \restriction_{M_{\beta+1}} - \pi\eta \in \operatorname{Hom}(M_{\beta+1}, I/N)$ it can be easily seen that $\delta \restriction_{M_\beta} = 0$. Hence there is $\tilde{\delta} \colon M_{\beta+1}/M_\beta \to I/N$ s.t. the following diagramm commutes.

$$\begin{tikzcd} M_{\beta+1} \arrow[r, "\delta"] \arrow[d, "\mathrm{pr}"'] & I/N \\ M_{\beta+1}/M_\beta \arrow[ur, dashed, "\tilde{\delta}"'] & \end{tikzcd}$$

    Since $\operatorname{Ext}^1_R(M_{\beta+1}/M_\beta, N) = 0$, then there is $\tilde{\varepsilon} \colon M_{\beta+1}/M_\beta \to I$ s.t. $\pi\tilde{\varepsilon} = \tilde{\delta}$. We set $\varepsilon = \tilde{\varepsilon} \circ \mathrm{pr}$. Notice that $\varepsilon \restriction_{M_\beta} = 0$

$$\begin{tikzcd} M_{\beta+1} \arrow[r, "\delta"] \arrow[d, "\mathrm{pr}"'] & I/C \\ M_{\beta+1}/M_\beta \arrow[ur, dashed, "\tilde{\delta}"'] \arrow[r, "\tilde{\varepsilon}"'] & I \arrow[u, "\pi"'] \end{tikzcd}$$

  - Thus $\pi\varepsilon = \delta = \varphi \restriction_{M_{\beta+1}} - \pi\eta$. Therefore if we set $\psi_{\beta+1} = \varepsilon + \eta$, then $\psi_{\beta+1}$ satisfy the desired properties.
  - For a limit ordinal $\beta < \alpha$, we put $\psi_\beta = \bigcup_{\gamma<\beta} \psi_\gamma$.

$\square$

4.3 Sets of Modules and Complete Cotorsion Theories

Remark 4. In general, given a class of modules $\mathscr{S}$, we don't have specific criteria for testing if the cotorsion theory cogenerated by $\mathscr{S}$ is complete or not. A useful application of the preceding lemma is that whenever $\mathscr{S}$ is a **set** of modules, then the cotorsion theory cogenerated by $\mathscr{S}$ is complete.

Lemma 4.6. Let $\mathscr{S}$ be a set of modules. If $X = \bigoplus_{S\in\mathscr{S}} S$, then $X^\perp = S^\perp$.

Proof.

- Let $P \in \mathscr{S}^{\perp}$. We consider an injective resolution of P

$$0 \to P \xrightarrow{i} I \xrightarrow{\varepsilon} I/P \to 0$$

We want to show that $\operatorname{Ext}^1_R(X, P) = 0$, equivalently we want to show that

$$\varepsilon_* \colon \operatorname{Hom}(X, I) \to \operatorname{Hom}(X, I/P)$$

is surjective. Let $\varphi \colon X \to I/P$. If we denote with i_S the embedding of S on X, then for each $S \in \mathscr{S}$, there is $\psi_S \in \operatorname{Hom}(S, I)$ s.t. the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "i_s"] \arrow[drr, "\psi_s"'] & X \arrow[r, "\varphi"] & I/P \\ & & I \arrow[u, "\pi"'] \end{tikzcd}$$

By universal property of direct sums, there is a unique $\psi \colon X \to I$ such that $\psi_s = \psi \circ i_S$. It can be easily seen that $\varphi = \pi \circ \psi$.

$$\begin{tikzcd} S \arrow[r, "i_s"] \arrow[drr, bend right, "\psi_s"'] & X \arrow[r, "\varphi"] \arrow[dr, dashed, "\psi"] & I/P \\ & & I \arrow[u, "\pi"'] \end{tikzcd}$$

- The converse relation implied immediately from the projection map $\pi_S \colon X \to S$.

□

Theorem 4.4. Let $\mathscr{S}$ be a set of modules. Let M be a module. There is a short exact sequence

$$0 \to M \hookrightarrow P \to N \to 0$$

where $P \in \mathscr{S}^{\perp}$, $N \in {}^{\perp}\left(\mathscr{S}^{\perp}\right)$ and P is the union of a continuous chain of submodules such that

- $P_0 = M$ and
- P_{a+1}/P_a is isomorphic to a direct sum of copies of element of $\mathscr{S}$, for each $\alpha+1 < \lambda$.

In particular, $M \hookrightarrow P$ is a special $\mathscr{S}^{\perp}$ - preenvelope of M.

Proof. By Lemma 4.6 w.l.o.g. we can assume that $\mathscr{S}$ consists of a single element S.

- Let $0 \to K \xrightarrow{\mu} F \to S \to 0$, a short exact sequence with F be a free module. Let λ be an infinite cardinal such that K be < λ - generated.

- We shall construct inductively a continuous chain of modules $\{P_\alpha \mid \alpha < \lambda\}$, which satisfies the assumptions of the theorem. We set $P_0 = M$.

- If $\alpha + 1 < \lambda$, we assume that we have already construct P_β, for each $\beta \leq \alpha$. If $X_\alpha = \operatorname{Hom}(K, P_\alpha)$ we define

$$\mu_\alpha = \bigoplus_{X_\alpha} \mu \in \operatorname{Hom}\left(K^{(X_\alpha)}, F^{(X_\alpha)}\right)$$

  that is μ_α is the direct sum of X_α - copies of μ. From definition of μ_α it's obvious that implied short exact sequence

$$0 \to K^{(X_\alpha)} \xrightarrow{\mu_\alpha} F^{(X_\alpha)} \to S^{(X_\alpha)} \to 0$$

  therefore μ_α is monomoprhism and $\operatorname{coker}\mu_\alpha$ is isomorphic to a direct sum of copies of S.

- Let $\varphi_\alpha \in \operatorname{Hom}(K^{(X_\alpha)}, P_\alpha)$ be the canonical morphism, where

$$\varphi_\alpha\left(\{k_\eta\}_{\eta \in X_\alpha}\right) = \sum_{\eta \in X_\alpha} \eta(k_\eta).$$

  For each $\eta \in X_\alpha$ we denote with ν_η and ν'_η the canonical embeddings

$$\nu_\eta \colon K \to K^{(X_\alpha)} \quad \text{and} \quad \nu'_\eta \colon F \to F^{(X_\alpha)}.$$

  Some trivial but in the same time important remarks are that $\eta = \varphi_\alpha \circ \nu_\eta$ and $\nu_{\eta'} \circ \mu = \mu_\alpha \circ \nu_\eta$.

- We consider the pushout diagram of μ_α and φ_α

$$\begin{tikzcd} K^{(X_\alpha)} \arrow[r, "\mu_\alpha"] \arrow[d, "\varphi_\alpha"'] & F^{(X_\alpha)} \arrow[d, "\psi_\alpha"] \\ P_\alpha \arrow[r, "i"] & P_{\alpha+1} \end{tikzcd}$$

  By dual result of Lemma 1.4 we have that i is monomorphism and $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of S.

- If $\alpha \leq \lambda$ is a limit ordinal we define $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$. We put $P = \bigcup_{\alpha<\lambda} P_\alpha$.

- First, we will prove that $P \in S^\perp$. Equivalently, it suffices to show that

$$\mu^* \colon \operatorname{Hom}(F, P) \to \operatorname{Hom}(K, P)$$

  is surjective. Let $\varphi \in \operatorname{Hom}(K, P)$. Since, K is < λ generated, then exists $\alpha < \lambda$ and $\eta \in X_\alpha$ s.t. $\eta(k) = \varphi(k)$, for each $k \in K$.

- If we combine the above relations it's true that

  $$\psi_\alpha \nu'_\eta \mu = \psi_\alpha \mu_\alpha \nu_\eta = i\varphi_\alpha \mu_\eta = i\eta.$$

  If we define $\psi\colon F \to P$ s.t. $\psi(f) = \psi_\alpha \nu'_\eta(f)$, for each $f \in F$, then $\varphi = \psi\mu$.

$$\begin{tikzcd}
K \arrow[rr, "\mu"] \arrow[dr, "\eta"] \arrow[ddr, "i\eta"'] \arrow[dddr, bend right, "\varphi"'] & & F \arrow[ddl, "\psi_\alpha \nu'_\eta"] \arrow[dddl, bend left, "\psi"] \\
& P_\alpha \arrow[d, "i"] & \\
& P_{\alpha+1} & \\
& P &
\end{tikzcd}$$

- It remains to show that $N = P/M \in^{\perp} \left(\mathscr{S}^{\perp}\right)$. Since $N = P/M$, then N is the union of the continuous chain $\{N_\alpha \mid \alpha < \lambda\}$, where $N_\alpha = P_\alpha/M$. Let $X \in S^{\perp}$.
  - Since $P_0 = M$, then $N_0 = 0$. Thus it's obvious that $\operatorname{Ext}^1_R(N_0, X) = 0$.
  - If $\alpha + 1 < \lambda$, then we have shown that P_{a+1}/P_α is isomorphic to a direct sum of copies of S and since $X \in S^{\perp}$ it's true that $\operatorname{Ext}^1_R\left(P_{\alpha+1}/P_\alpha, X\right) = 0$.

  By Lemma 4.5 we have that $\operatorname{Ext}^1_R(N, X) = 0$ and therefore $N \in^{\perp} \left(S^{\perp}\right)$.

□

Corollary 4.4. Let $\mathscr{S}$ be a set of modules. Then the cotorsion theory

$$\mathfrak{C}_{\mathscr{S}} = \left({}^{\perp}\left(\mathscr{S}^{\perp}\right), \mathscr{S}^{\perp}\right)$$

is complete.

CHAPTER 5

ALL MODULES HAVE FLAT COVERS

5.1 Existence of Special Precovers Implies Cover's Existence

Motivation 4. We aim to prove that each module has a flat cover. A key point to prove this result is to prove that the existence of a special flat precover induces the existence of a flat cover. If we combine this result with the main result of Chapter 5 (flat cotorsion theory is complete) will have prove the desired result.

Theorem 5.1. Let R be a ring and M be a module. Let $\mathscr{C}$ be a class of modules closed under extensions and arbitrary direct limits. Assume that M has a special $\mathscr{C}^{\perp}$ - preenvelope φ, with $\operatorname{coker}\varphi \in \mathscr{C}$. Then M has a $\mathscr{C}^{\perp}$ envelope.

Definition 5.1. With the above assumptions, an exact sequence

$$0 \to M \to F \to C \to 0, \quad C \in \mathscr{C}$$

is called an **Ext - generator** if for each exact sequence

$$0 \to M \to F' \to C' \to 0, \quad C \in \mathscr{C}$$

there exist $f \in \operatorname{Hom}_R(F', F)$ and $g \in \operatorname{Hom}_R(C', C)$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Lemma 5.1. With the assumptions of the Theorem 5.1 assume that $0 \to M \to F \to C \to 0$ is an Ext - generator. Then there exists an Ext - generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

such that $\ker f = \ker f'f$ in any commutative diagram whose rows are Ext - generators

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

Proof. We assume, for that sake of contradiction, that the above result is not true. Then for any Ext generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

there exists a commutative diagram whose rows are Ext - generators

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

such that $\ker f \subsetneqq \ker(f'f)$. With induction, we will construct for each ordinal α a stricly increasing chain of submodules of F $\{\ker f_{0\beta} \mid \beta < \alpha\}$ and that is contradiction.

- **Zero Case.** We put $F' = F_0 = F$, $C' = C_0 = C$ and $f = \mathrm{id}_F$, $g = \mathrm{id}_C$. Then there exist $F_1 = F''$ and $C_1 = C'' \in \mathscr{C}$ with a pair of morphisms $f_{01} = f'$, $g_{01} = g'$ s.t. the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

its rows are Ext - generators and $0 = \ker f \subsetneqq \ker f_{01}$.

- **Successor Case.** We assume that the Ext - generator

$$0 \to M \to F_\alpha \to C_\alpha \to 0$$

is defined together with $f_{\beta\alpha} \in \mathrm{Hom}(F_\beta, F_\alpha)$ and $g_{\beta\alpha} \in \mathrm{Hom}(C_\beta, C_\alpha)$ s.t. for any $\beta \leq \alpha$ the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f_{0\beta}"] & C \arrow[r] \arrow[d, "g_{0\beta}"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_\beta \arrow[r] \arrow[d, "f_{\beta\alpha}"] & C_\beta \arrow[r] \arrow[d, "g_{\beta\alpha}"] & 0 \\
0 \arrow[r] & M \arrow[r] & F_\alpha \arrow[r] & C_\alpha \arrow[r] & 0
\end{tikzcd}$$

and $\ker(f_{0\beta}) \subsetneqq \ker(f_{0\beta'})$ for each $\beta \leq \beta' \leq \alpha$. Then there exist an Ext - generator $0 \to M \to F_{\alpha+1} \to C_{\alpha+1} \to 0$ and $f_{\alpha,\alpha+1}$ and $g_{\alpha,\alpha+1}$ s.t. the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f_{0\alpha}"] & C \arrow[r] \arrow[d, "g_{0\alpha}"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_\alpha \arrow[r] \arrow[d, "f_{\alpha,\alpha+1}"] & C_\alpha \arrow[r] \arrow[d, "g_{\alpha,\alpha+1}"] & 0 \\
0 \arrow[r] & M \arrow[r] & F_{\alpha+1} \arrow[r] & C_{\alpha+1} \arrow[r] & 0
\end{tikzcd}$$

its rows are Ext - generators and $\ker f_{0\alpha} \subsetneqq \ker f_{0\alpha+1}$, where $f_{\beta,\alpha+1} = f_{\alpha,\alpha+1} f_{\beta\alpha}$ and $g_{\beta,\alpha+1} = g_{\alpha,\alpha+1} g_{\beta\alpha}$, for all $\beta \leq \alpha$.

- **Limit Case.** We assume that α is a limit ordinal and that

$$0 \to M \to F_\beta \to C_\beta \to 0$$

is defined for each $\beta < \alpha$ together with $f_{\beta,\beta'} \in \mathrm{Hom}_R(F_\beta, F_{\beta'})$ and $g_{\beta,\beta'} \in \mathrm{Hom}_R(C_\beta, C_{\beta'})$, whenever $\beta \leq \beta' < \alpha$. Then the " triad "

$$\left(0 \to M \to F_\beta \to C_\beta \to 0, \left(f_{\beta\beta'}, g_{\beta,\beta'}\right)_{\beta \leq \beta' < \alpha}\right)_{\beta < \alpha}$$

is a directed system and let

$$(0 \to M \to F_\alpha \to C_\alpha \to 0, f_{\beta\alpha}, g_{\beta\alpha})_{\beta < \alpha}$$

be the direct limit of the above system, where $F_\alpha = \varinjlim F_\beta$ and $C_\alpha = \varinjlim C_\beta \in \mathscr{C}$, since $\mathscr{C}$ is closed under arbitrary direct limits. Then it can be easily seen that

$$0 \to M \to F_\alpha \to C_\alpha \to 0$$

is an Ext - generator and that $\ker(f_{0\beta}) \subsetneqq \ker(f_{0\alpha})$, for all $\beta < \alpha$.

□

Lemma 5.2. With the assumptions of the Theorem 5.1 assume $0 \to M \to F \to C \to 0$ is an Ext - generator. Then there exists an Ext - generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

such that $\ker(f') = 0$ in any commutative diagram whose rows are Ext - generators

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

Proof. We will inductively construct a countable system $\mathbb{D}$ of Ext - generators using the Lemma 5.1 and we shall show that the direct limit of this system satisfies the desired property.

- **Zero Case.** We set $0 \to M \to F \to C \to 0$ be the 0-th term of $\mathbb{D}$. By Lemma 5.1, there exists an Ext - generator $0 \to M \to F_1 \to C_1 \to 0$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f_0^1"] & C \arrow[r] \arrow[d, "g_0^1"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F_1 \arrow[r] & C_1 \arrow[r] & 0 \end{tikzcd}$$

  such that for any commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f_0^1"] & C \arrow[r] \arrow[d, "g_0^1"] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_1 \arrow[r] \arrow[d, "f'"] & C_1 \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

  whose rows are Ext - generators, then $\ker\left(f' f_0^1\right) = \ker f_0^1$.

- **Inductive Step.** We assume that for some $m \in \mathbb{N}$ we have constructed a directed system $\mathbb{D}_m$ of Ext - generators

$$(0 \to M \to F_i \to C_i \to 0)_{i=1}^{m}, \quad (f_{ij} \in \operatorname{Hom}(F_i, F_j),\ g_{ij} \in \operatorname{Hom}(C_i, C_j))_{i \leq j \leq m}$$

where f_i^{i+1}, g_i^{i+1} defined as the zero case and for each $i \leq j$ we define

$$f_i^j = f_{j-1}^j \circ \cdots \circ f_i^{i+1} \quad \text{and} \quad g_i^j = g_{j-1}^j \circ \cdots \circ g_i^{i+1}$$

By Lemma 5.1 there exists an Ext - generator $0 \to M \to F_{n+1} \to C_{n+1} \to 0$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_n \arrow[r] \arrow[d, "f_n^{n+1}"] & C \arrow[r] \arrow[d, "g_n^{n+1}"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F_{n+1} \arrow[r] & C_{n+1} \arrow[r] & 0 \end{tikzcd}$$

such that for any commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_n \arrow[r] \arrow[d, "f_n^{n+1}"] & C_n \arrow[r] \arrow[d, "g_n^{n+1}"] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_{n+1} \arrow[r] \arrow[d, "f'"] & C_{n+1} \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

So we have constructed a countable direct system $\mathbb{D}$ of Ext - generators

$$(0 \to M \to F_n \to C_n \to 0)_{n\in\mathbb{N}}, \quad (f_{ij} \in \mathrm{Hom}(F_i, F_j),\ g_{ij} \in \mathrm{Hom}(C_i, C_j))_{i\leq j}$$

with the above properties. We consider the direct limit of $\mathbb{D}$

$$\big(0 \to M \to F' \to C' \to 0, (\varphi_n, \psi_n)_{n\in\mathbb{N}}\big)$$

so $F' = \varinjlim F_n$ and $C' = \varinjlim C_n \in \mathscr{C}$. We will show that this direct limit satisfied the desired property.

We consider a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

whose rows are Ext - generators. We assume, for the sake of contradiction, that exists $[x] \in \ker(f')$ and $[x] \neq [0]$. There is a $n \in \mathbb{N}$ s.t. $x \in F_n$, since $\varphi_n(x) = [x] \neq 0$, thus for each $m \geq n$ then $f_n^m(x) \neq 0$, therefore $x \notin \ker(f_n^m)$. By construction of the direct system, the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_n \arrow[r] \arrow[d, "f_n^{n+1}"] & C_n \arrow[r] \arrow[d, "g_n^{n+1}"] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_{n+1} \arrow[r] \arrow[d, "f'\varphi_{n+1}"] & C_1 \arrow[r] \arrow[d, "g'\varphi_{n+1}"] & 0 \\ 0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

then

$$\ker\left(f'\varphi_{n+1}f_n^{n+1}\right) = \ker(f'\varphi_n) = \ker\left(f_n^{n+1}\right)$$

and that is contraction. □

Lemma 5.3. With the assumptions of Theorem 5.1 let $0 \to M \xrightarrow{\varphi} F' \xrightarrow{\pi} C' \to 0$ be the Ext - generator constructed in Lemma 5.2. Then $\varphi\colon M \to F'$ is a $\mathscr{C}^{\perp}$ - envelope of M.

Proof. Firstly, we will show that for any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f'} & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

then f' is an automorphism. Since f' is injective by previous lemma, it suffices to show that f' is surjective. We assume that this is not true. We set

$$(0 \to M \to F_0 \to C_0 \to 0) = (0 \to M \to F_1 \to C_1 \to 0) = (0 \to M \to F \to C \to 0)$$

and $f_{01} = f'$. Then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_0 & \longrightarrow & C_0 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_{01}} & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F_1 & \longrightarrow & C_1 & \longrightarrow & 0 \end{array}$$

Again we set

$$(0 \to M \to F_2 \to C_2 \to 0) = (0 \to M \to F \to C \to 0)$$

Then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_{12}} & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

where f_{12} is injective but is not surjective. If $f_{02} = f_{12}f_{01}$, then $\mathrm{Im} f_{02} \subsetneqq \mathrm{Im} f_{01}$ In general, for each $n \in \mathbb{N}$ there are

$$(0 \to M \to F_n \to C_n \to 0) \quad \text{and } f_{n-1,n} \in \mathrm{Hom}\left(F_{n-1}, F_n\right)$$

where $f_{n-1,n}$ is injective but not surjective s.t.

$$\mathrm{Im} f_{0,n} \subsetneqq \mathrm{Im} f_{1,n} \subsetneqq \cdots \subsetneqq \mathrm{Im} f_{n-1,n} \subsetneqq F'$$

Therefore we have that $\mathrm{card}\,(F') \geq n$, forall $n \in \mathbb{N}$, so $\mathrm{card}\,(F') \geq \omega$. We shall show that $\mathrm{card}\,(F') \geq \beta$, forall β ordinal and that is contradiction. Let β be an arbitrary ordinal. We assume that there are

$$(0 \to M \to F_\lambda \to C_\lambda \to 0) = (0 \to M \to F \to C \to 0)\,, \quad \forall\, \lambda < \beta$$

and for each $\lambda < \beta$ we have

$$(f_{\lambda,\lambda+1} \colon F_\lambda \to F_{\lambda+1}) = \left(f' \colon F' \to F'\right)$$

and for each $\kappa < \lambda < \nu < \beta$

$$f_{\kappa,\nu} = f_{\lambda,\nu} f_{\kappa,\lambda}$$

- If $\beta = \gamma + 1$, then we set

  $$(0 \to M \to F_\beta \to C_\beta \to 0) = (0 \to M \to F \to C \to 0)$$

  and $f_{\gamma,\beta} = f'$. Then we set $f_{\lambda,\beta} = f_{\gamma,\beta} f_{\lambda,\gamma}$. Then each there is an stricly increasing chain of submodules of F'

  $$\{\mathrm{Im} f_{\lambda,\beta} \mid \lambda < \beta\}$$

  thus $\mathrm{card}\,(F') \geq \beta$.

- If β is limit ordinal we take the direct limit of the above direct system

  $$\left(0 \to M \to F'_\beta \to C'_\beta \to 0\right) = \left(0 \to M \to \varinjlim F_\lambda \to \varinjlim C_\lambda \to 0\right)$$

  and we set $g_{\lambda,\beta}$ be the canonical induced maps, for all $\lambda < \beta$. Since the sequence $(0 \to M \to F' \to C' \to 0)$ is an Ext - generator, then there exists a pair of morphisms (h, p) and commutative diagram

  $$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F'_\beta & \longrightarrow & C'_\beta & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

  and we set

  $$(0 \to M \to F_\beta \to C_\beta \to 0) = \left(0 \to M \to F' \to C' \to 0\right) \quad \text{and} \quad f_{\lambda,\beta} = h g_{\lambda,\beta}$$

  From the above relations it can be easily shown that

  $$\{\mathrm{Im} f_{\lambda,\beta} \mid \lambda < \beta\}$$

  is a strickly increasing chain consisting submodules of F', therefore $\mathrm{card}(F') \geq \beta$.

$\square$

Motivation 5. We have shown that in a module class $\mathscr{C}$ closed under extensions and direct limits, the existence of a special preenvelope yields to the existence of a $\mathscr{C}^{\perp}$ - envelope. Can you claim that the dual result is true ? Equivalently, the existence of a $\mathscr{L}$ - precover implies the existence a $\mathscr{L}$ - cover ? Yes! The proof of this result follows if we slightly modify the above lemmas and their proofs. For this reason we just state the lemmas necessary for the theorem's proof.

Theorem 5.2 (Xu). Assume $\mathscr{C}$ is closed under direct limits. If M has an $\mathscr{C}$ - precover then M has an $\mathscr{C}$ - cover $L \to M$. Furthermore, if $\mathscr{C}$ is closed under extensions then $\ker(L \to M) \in \mathscr{C}^{\perp}$.

Lemma 5.4. Assume $\mathscr{C}$ is closed under direct limits. If $L \to M$ is an $\mathscr{C}$- precover of M, then there is a precover $\overline{L} \to M$ and a commutative diagram

$$\begin{array}{ccc} L & \longrightarrow & M \\ \downarrow{\scriptstyle f} & & \| \\ \overline{L} & \longrightarrow & M \end{array}$$

s.t. for any precover $L^* \to M$ and any commutative diagram

$$\begin{array}{ccc} \overline{L} & \longrightarrow & M \\ \downarrow{\scriptstyle g} & & \| \\ L^* & \longrightarrow & M \end{array}$$

then $\ker(gf) = \ker(f)$.

Lemma 5.5. Assume $\mathscr{C}$ is closed under direct limits. If $L \to M$ is an $\mathscr{C}$- precover of M, then there is a precover $\overline{L} \to M$ and a commutative diagram

$$\begin{array}{ccc} L & \longrightarrow & M \\ \downarrow{\scriptstyle f} & & \| \\ \overline{L} & \longrightarrow & M \end{array}$$

s.t. for any precover $L^* \to M$ and any commutative diagram

$$\begin{array}{ccc} \overline{L} & \longrightarrow & M \\ \downarrow{\scriptstyle g} & & \| \\ L^* & \longrightarrow & M \end{array}$$

then $\ker(g) = 0$.

Lemma 5.6. Assume $\mathscr{C}$ is closed under direct limits. If $L \to M$ is an $\mathscr{C}$- precover of M. If $\overline{L} \to M$ be the precover defined in the previous lemma, then $\overline{L} \to M$ is an $\mathscr{L}$ - cover.

5.2 Flat Cotorsion Theory

Definition 5.2. For a class (right resp. left) of R modules $\mathscr{C}$ we put

$$\mathscr{C}^{\top} = \left\{N \in R - \mathrm{Mod} \mid \mathrm{Tor}_1^R(C, N) = 0 \text{ for all } C \in \mathscr{C}\right\}$$

$$^{\top}\mathscr{C} = \left\{N \in R - \mathrm{Mod} \mid \mathrm{Tor}_1^R(N, C) = 0 \text{ for all } C \in \mathscr{C}\right\}$$

Definition 5.3. Let $\mathscr{A}, \mathscr{B} \subseteq \mathrm{Mod} - R$. The pair $(\mathscr{A}, \mathscr{B})$ is called a **Tor - torsion theory** if

$$\mathscr{A} =^{\top} \mathscr{B} \quad \text{and} \quad \mathscr{B} = \mathscr{A}^{\top}.$$

Example 5.1. Using Chapter 3 it can be easily seen that the pair $(\mathscr{F}\mathscr{L}(R), R - \mathrm{Mod})$, where $\mathscr{F}\mathscr{L}(R)$ is the class of flat R - modules, is a Tor - torsion theory.

Lemma 5.7. Let R be a ring and $(\mathscr{A}, \mathscr{B})$ be a Tor - torsion theory. Then $= \left(\mathscr{A}, \mathscr{A}^{\perp}\right)$ is a cotorsion theory.

Proof. We want to show that $\mathscr{A} =^{\perp} \left(\mathscr{A}^{\perp}\right)$. The relation $\mathscr{A} \subseteq^{\perp} \left(\mathscr{A}^{\perp}\right)$ is obvious. Conversely, let $M \in^{\perp} \left(\mathscr{A}^{\perp}\right)$ and we want to show that $M \in \mathscr{A} =^{\top} \mathscr{B}$. Let $B \in \mathscr{B}$ and we consider the canonical isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Tor}_1^R(M, B), \mathbb{Q}/\mathbb{Z}\right) \cong \mathrm{Ext}_R^1\left(M, \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z})\right)$$

It suffices to show that $\mathrm{Hom}\left(B, \mathbb{Q}/\mathbb{Z}\right)) \in \mathscr{A}^{\perp}$. Similarly, if $A \in \mathscr{A}$ then

$$\mathrm{Ext}_R^1\left(A, \mathrm{Hom}\left(B, \mathbb{Q}/\mathbb{Z}\right)\right) \cong \mathrm{Hom}_{\mathbb{Z}}\left(\underbrace{\mathrm{Tor}_1^R(A, B)}_{0}, \mathbb{Q}/\mathbb{Z}\right)$$

By [2] (4.7) we deduce that $\mathrm{Tor}_1^R(M, B) = 0$. □

Definition 5.4. By previous lemma it can be easily seen that $(\mathscr{F}\mathscr{L}(R), R - \mathrm{Mod})$ is a Tor - torsion pair, therefore the pair $(\mathscr{F}\mathscr{L}(R), \mathscr{E}(R))$ is a cotorsion pair, where $\mathscr{E}(R) = \mathscr{F}\mathscr{L}(R)^{\perp}$ is called the class of all **Enochs cotorsion** modules .

5.3 The Class of Flat Modules is a Cover Class

Lemma 5.8. If R is a ring and if $|R| \leq \kappa$ for a infinite ordinal κ (for example $\kappa = \max\{\aleph_0, |R|\}$), then if $x \in M$ for a left R - module M, there is a pure submodule $S \subseteq M$ with $x \in S$ and $|S| \leq \kappa$.

Proof. We construct a sequence $S_0 \subseteq S_1 \subseteq \cdots \subseteq \cdots M$ of M. We set $S_0 = Rx$. We assume that we have already defined S_n and we consider the set

$$X = \left\{ h\colon U \to S_n \mid h \text{ hom. and can be extended to } \tilde{h}\colon P \to M \right\}$$

and $\tilde{X}$ the set of those $\tilde{h}$, for each $h \in X$. We set

$$S_{n+1} \coloneqq S_n + \sum_{\tilde{h} \in \tilde{X}} \tilde{h}(P)$$

Let $S = \bigcup_{n=0}^{\infty} S_n$. It remains to show that S is pure submodule of M. We observe that each $h\colon U \to S_n$, which can be extended to $\tilde{h}\colon P \to M$, then can be extended to $\tilde{h}\colon P \to S_{n+1}$.

With denotation of corollary 3.5, let $h\colon T \to S$ homomorphism, which can be extended to $\tilde{h}\colon P \to M$. Since T is f.g. there is $n \in \mathbb{N}$ s.t. h can be factored to $T \to S_n \to S$ then h can be extended to

$$P \xrightarrow{\tilde{h}} S_{n+1} \to S \to M$$

therefore $S \subseteq M$ is pure. Finally, by the construction of S we have that $|S| \leq \kappa$. □

Proposition 5.1. Let M be an R - module and κ be an infite cardinal s.t. $|R| < \kappa$. Then there is an ordinal λ and a continuous increasing family of pure submodules $\{M_\alpha \mid a < \lambda\}$, such that

- $M_0 = 0$
- $M = M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha$
- $|M_{\alpha+1}/M_\alpha| \leq \kappa$, for all $\alpha + 1 < \lambda$.

Proof. We enumerate M as $M = \{x_\alpha \mid \alpha < \lambda\}$, where $|M| = \lambda$. We will define a family $\{M_\alpha \mid a < \lambda\}$ which satisfies the desired properties with use of induction.

- **Zero Case.** $M_0 = 0$

- **Successive Case.** Let $\alpha = \gamma + 1$. We assume that M_β have defined for every $\beta < \alpha$ s.t. M_β is pure
$$|M_{\beta+1}/M_\beta| \leq \kappa, \quad \text{for every } \beta + 1 < \alpha$$
and $M_\beta = \bigcup_{\beta' < \beta} M_{\beta'}$, whenever β is limit. If $M_\gamma = M$, then the procedure stop and we're done. Otherwise, let $x \in M \setminus M_\gamma$, therefore
$$0 \neq R\overline{x} \subseteq M/M_\gamma$$
Since $|R\overline{x}| \leq |R| \leq \kappa$, then by Lemma 5.8 there is a pure $N \subseteq M/M_\gamma$ s.t. $R \cdot \overline{x} \subseteq N$. Then there exists $M_{\gamma+1} \subseteq M$ submodule s.t. $M_\gamma \subseteq M_{\gamma+1}$ and
$$N = M_{\gamma+1}/M_\gamma \Rightarrow |M_{\gamma+1}/M_\gamma| = |N| \leq \kappa$$
Finally, we will show that $M_{\gamma+1}$ is pure, which implied immediately since
$$N = M_{\gamma+1}/M_\gamma \subseteq M/M_\gamma \quad \text{and} \quad M_\gamma \subseteq M$$
are pure.
- **Limit Case.** If α is a limit ordinal, then we define $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ and M_α is pure.

$\square$

Corollary 5.1. If $M \in \mathscr{FL}(R)$, then there is a continuous, increasing family of flat submodules

- $M_0 = 0$
- $M_{\alpha+1}/M_\alpha$ is flat and $|M_{\alpha+1}/M_\alpha| \leq \kappa$
- $M = \bigcup_\alpha M_\alpha$

Proof. We consider a pure family $\{M_\alpha\}_\alpha$ as previous proposition. Since $M_\alpha \subseteq M$ pure and M is flat by Corollary 3.3, then M_α is flat. Since $M_\alpha \subseteq M_{\alpha+1}$ pure, then is we consider the following pure exact sequence
$$0 \to M_\alpha \to M_{\alpha+1} \to M_{\alpha+1}/M_\alpha \to 0$$
by 3.3 the quotient $M_{\alpha+1}/M_\alpha$ is flat. $\square$

Theorem 5.3. The flat cotorsion theory $(\mathscr{FL}(R), \mathscr{E}(R))$ is complete. Since $\mathscr{FL}(R)$ is closed under direct limits, by Theorem 5.2, $\mathscr{FL}(R)$ is a cover class.

Proof. We consider the set

$$\mathscr{S} = \{M \text{ is flat} \mid |M| \leq \kappa\}$$

We will show that the flat cotorsion theory $(\mathscr{F}\mathscr{L}(R), \mathscr{E}(R))$ is cogenerated by $\mathscr{S}$, equivalently

$$\mathscr{E}(R) = (\mathscr{F}\mathscr{L}(R))^{\perp} = \mathscr{S}^{\perp}$$

The first relation it is obvious since

$$\mathscr{S} \subseteq \mathscr{F}\mathscr{L}(R) \Rightarrow \mathscr{E}(R) = (\mathscr{F}\mathscr{L}(R))^{\perp} \subseteq \mathscr{S}^{\perp}$$

Let $C \in \mathscr{S}$. We shall show that $\mathrm{Ext}^1(F, C) = 0$, for every $F \in \mathscr{F}\mathscr{L}(R)$. Let $F \in \mathscr{F}\mathscr{L}(R)$ and $\{F_\alpha \mid \alpha < \lambda\}$ a flat family of submodules with the properties of the above corollary. Then

$$F_{\alpha+1}/F_\alpha \in \mathscr{F}\mathscr{L}(R) \quad \text{and} \quad |F_{\alpha+1}/F_\alpha| \leq \kappa$$

therefore $F_{\alpha+1}/F_\alpha \in \mathscr{S}$. Since

$$\mathrm{Ext}^1(F_0, C) = 0 \quad \text{and} \quad \mathrm{Ext}^1(F_{\alpha+1}/F_\alpha, C) = 0 \; \forall \alpha + 1 < \lambda$$

by Lemma ?? we deduce that

$$\mathrm{Ext}^1(F, C) = \mathrm{Ext}^1\left(\bigcup_{\alpha<\lambda} F_\alpha, C\right) = 0$$

$\square$

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