Introduction to Robotics (CS223A) Homework #1

(Winter 2017/2018) Due: **Wednesday, January 24, 4:00pm**

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Instructions:

- **Print this problem set and fill in your answers in the dedicated white space right below the question statement. Alternatively you may also submit a typewritten writeup. The assignments should be submitted via Gradescope.**
- Setup instructions for programming portions are detailed in the README file included with the code.
- Use abbreviations for trigonometric functions (e.g. $c\theta$ for $\cos(\theta)$, s_1 or $s\theta_1$ for $\sin(\theta_1)$) in situations where it would be tedious to repeatedly write sin,cos, etc.
- Unless instructed otherwise, leave square roots in symbolic form rather than writing out their decimal values.
- If you give a vector as an answer, make sure that you specify what frame it is given in (if it is not clear from context). The same rule applies to rotation and transformation matrices.

1. The vector ${}^A\mathbf{P}$ is first rotated about $\hat{X}_A$ by θ degrees and then subsequently rotated about $\hat{Y}_A$ by ϕ degrees.

   (a) Give the 3×3 rotation matrix that accomplishes these rotations in the given order. You may leave your answer as a product of matrices.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \Rightarrow \boxed{R_y(\phi)R_x(\theta) = \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}}$$

$$R_y(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

   (b) What is the rotated vector if ${}^A\mathbf{P} = [2, 1, 1]^T$, $\theta = 60°$, and $\phi = 45°$? Be sure to specify the frame of your vector representation.

      i. Implement the appropriate functions in *rotations.py* (where it says *HW1 Q1b: Rotation operators*). This part of the homework will be graded with Gradescope Autograder.

      ii. Use the above functions to compute the answer and give your answer on your solution paper.

$${}^A P = \begin{bmatrix} 2.380 \\ -0.366 \\ -0.448 \end{bmatrix}$$

2. A frame {B} is initially coincident with a frame {A}. First, we rotate {B} about $\hat{Z}_B$ by θ degrees. Next, we rotate the resulting frame {B} about $\hat{Y}_B$ by ϕ. Finally, we rotate the resulting frame {B} about $\hat{X}_B$ by θ again.

(a) Determine the 3×3 rotation matrix, ${}^A_BR$, that will change the description of a vector P in frame {B}, ${}^B\mathbf{P}$, to frame {A}, ${}^A\mathbf{P}$. You may leave your answer as a product of matrices.

$${}^A_BR = R_z(\theta)R_y(\phi)R_x(\theta)$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

(b) What is the value of ${}^A_BR$ if $\theta = 45°$, $\phi = 60°$? Implement the appropriate functions in *rotations.py* (where it says *HW1 Q2b: Euler and Fixed angles*) and use them to compute the answer. Handle representation singularities explicitly in your code. Similar to Q1, please give your answer on your solution paper.

$${}^A_BR = \begin{bmatrix} 0.3536 & -0.067 & 0.933 \\ 0.3536 & 0.933 & -0.067 \\ -0.866 & 0.3536 & 0.3536 \end{bmatrix}$$

(c) Compute the rotation matrix if $\phi = 90°$? Can you achieve the same final {B} using a different set of Euler Angles? Explain why.

$${}^A_BR = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Also achieved when $\theta = 0$, so $(0, 90, 0)$.
This is because of Gimbal Lock: since $\phi = 90°$ the other 2 rotations act about an effectively equivalent direction

3. Frame {A} and frame {B} are fixed with respect to an inertial ground frame.

(a) Consider a velocity vector in frame {A}, ${}^A\mathbf{V}$. How will it change if we express it in frame {B}? Are ${}^A\mathbf{V}$ and ${}^B\mathbf{V}$ the same vector? Comment upon their magnitudes and directions. If they are different how can you transform one into the other?

They should have equal magnitude but different directions. You can transform using the rotation matrix ${}^A_B R$ and translation vector ${}^A P_B$, combined into the homogenous tran. mat. ${}^A_B T$

(b) Given

$$ {}^A\mathbf{V} = \begin{bmatrix} -4 \\ z \\ -6 \end{bmatrix}, {}^B\mathbf{V} = \begin{bmatrix} -6 \\ 4 \\ 1 \end{bmatrix}, $$

where ${}^A\mathbf{V}$, ${}^B\mathbf{V}$ have the same meaning as in part (a). Assume that transforming frame {A} into frame {B} requires translating {A} by $(0, 2, 1)$ and then rotating it 90° about $\hat{X}_{A'}$, and rotating the resulting frame θ about $\hat{Z}_{A''}$ (Note that A' and A'' are intermediate frames). Determine the value of θ and z.

$$ {}^A_{A'}H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{A'}_{A''}H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{A''}_{B}H = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^A V = {}^A_B H \, {}^B V $$

$$ = {}^A_{A'}H \; {}^{A'}_{A''}H \; {}^{A''}_{B}H \; {}^B V $$

$$ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^A V $$

$$ = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & -1 & 2 \\ \sin\theta & \cos\theta & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^A V $$

$$ |{}^A V| = |{}^B V| $$

$$ 4^2 + z^2 + 6^2 = 6^2 + 4^2 + 1^2 $$

$$ z = \pm 1 $$

$$ \begin{bmatrix} -4 \\ 1 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & -1 & 2 \\ \sin\theta & \cos\theta & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 4 \\ 1 \\ 1 \end{bmatrix} $$

$$ \begin{bmatrix} -4 \\ 1 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} -6\cos\theta - 4\sin\theta \\ 1 \\ -6\sin\theta + 4\cos\theta \\ 1 \end{bmatrix} $$

$$ -8 = -12\cos\theta - 8\sin\theta $$
$$ -18 = 12\cos\theta - 18\sin\theta $$
$$ -26 = -26\sin\theta $$
$$ 1 = \sin\theta $$
$$ \theta = 90° $$

$$ \boxed{\Rightarrow \theta = 90°, \quad z = 1} $$

4. Given the following transformation matrices:

$$T1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T2 = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$T3 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T4 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Are T1, T2, T3 and T4 valid transformation matrices? Explain why or why not, and if there are multiple reasons why a matrix is invalid, include each. (We define a transformation matrix as a rotation and a translation, i.e. a "rigid body" transformation)

T1 is invalid. Its determinant $= 0.5 \neq 1$.

T2 is not valid. Bottom row $\neq$ [0 0 0 1]

T3 is valid. Bottom = [0 0 0 1]. Det = 1, mag of rows + columns are 1.

T4 is invalid. Det = -1

(b) Implement function *mat_to_quat* in *rotations.py* and find the Euler parameters that represent the rotations for the correct matrix (or matrices).

T3 ⇒ (0, 0, -0.5, 0.866)

(c) Also find the unit vector that defines the axis of rotation, and the angle of rotation for the correct matrix (or matrices).

Axis: 0, 0, -1

Angle: 60°

5. (a) Prove the following Lemma :

For all rotations, at least one of the Euler parameters is greater than or equal to 1/2.

Let all parameters be less than 0.5 :

Then for all parameters $\varepsilon_i^2 < 0.25$

And their sum $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 < 1$

Which violates the Normality Condition

$\therefore$ by contradiction, one of the parameters must be greater than or equal to 0.5

(b) When possible, determine the Euler parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ for the following matrices:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad R_4 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$(0,0,0,1)$ $\quad$ $(0, 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ $\quad$ Not valid $\quad$ $(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, 0)$

(c) **[Extra Credit]** Prove that 3D infinitesimal rotations commute (clearly state the assumptions you make and their consequences).

assume $\cos\theta \approx 1 - \frac{\theta^2}{2}$
$\sin\theta \approx \theta$

$$R_x(\theta) R_z(\phi) = \begin{bmatrix} c\phi & s\phi & 0 \\ -c\theta s\phi & c\theta c\phi & s\theta \\ s\theta s\phi & -s\theta c\phi & c\theta \end{bmatrix}$$

$$\approx \begin{bmatrix} 1-\frac{\phi^2}{2} & \phi & 0 \\ -(1-\frac{\theta^2}{2})\phi & (1-\frac{\theta^2}{2})(1-\frac{\phi^2}{2}) & \theta \\ \theta\phi & -\theta(1-\frac{\phi^2}{2}) & 1-\frac{\theta^2}{2} \end{bmatrix}$$

$$R_z(\phi) R_x(\theta) = \begin{bmatrix} c\phi & s\phi & s\phi c\theta \\ -s\phi & c\phi c\theta & c\phi s\theta \\ 0 & -s\theta & c\theta \end{bmatrix}$$

$$\approx \begin{bmatrix} 1-\frac{\phi^2}{2} & \phi & \phi\theta \\ -\phi & (1-\frac{\phi^2}{2})(1-\frac{\theta^2}{2}) & (1-\frac{\phi^2}{2})\theta \\ 0 & -\theta & 1-\frac{\theta^2}{2} \end{bmatrix}$$

Which means: $\phi\theta = 0$

$\theta \doteq \theta(1-\frac{\phi^2}{2})$

$1-\frac{\theta^2}{2} = 1$

Stricter assumption: (very small angles)

$\sin\theta = 0$

$\cos\theta = 1$

$$R_x(\theta) R_z(\phi) \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\phi) R_x(\theta) \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Any infinitesimal rotation is effectively the identity matrix

6. (a) Prove that 2D rotations commute.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Show $R_\theta R_\phi = R_\phi R_\theta$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$

$$\begin{bmatrix} c\theta c\phi - s\theta s\phi & -c\theta s\phi - s\theta c\phi \\ s\theta c\phi + c\theta s\phi & -s\theta s\phi + c\theta c\phi \end{bmatrix} = \begin{bmatrix} c\theta c\phi - s\theta s\phi & -s\theta c\phi - c\theta s\phi \\ c\theta s\phi + s\theta c\phi & -s\theta s\phi + c\theta c\phi \end{bmatrix}$$

(b) Prove that 3D rotations do not necessarily commute.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \qquad R_z(\phi) = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Show $R_x(\theta) R_z(\phi) \neq R_z(\phi) R_x(\theta)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix}$$

$$\begin{bmatrix} c\phi & s\phi & 0 \\ -c\theta s\phi & c\theta c\phi & s\theta \\ s\theta s\phi & -s\theta c\phi & c\theta \end{bmatrix} = \begin{bmatrix} c\phi & s\phi & s\phi s\theta \\ -s\phi & c\phi c\theta & c\phi s\theta \\ 0 & -s\theta & c\theta \end{bmatrix}$$

$-c\theta s\phi = -s\phi$

$c\theta = 1$

$\theta = 0$

$s\theta = s\theta c\phi$

$c\phi = 1$

$\phi = 0$

only true if $\theta = 0$ and $\phi = 0$

$\Rightarrow$ Do not necessarily commute