ME/AE8112 Problem Set PS2 **Due Wednesday June 10th at 4:00PM**

Instructions:
For each problem, state all your assumptions, explain your approach and solution method, include a good copy of any preparation work that needs to be done by hand (derivations, discretizations… etc.) in your report.

If the problem requires writing a computer program, include your code and its output in your report.

In addition, email your codes to Prof. Dworkin by the due date and time. Your codes may be run to verify output, or put them through a plagiarism checker. All codes that you email should be as separate attachments, titled PSnum_Qnum_Lastname_Firstname_Studentnum.dat (or .f90 or .f95 or .c etc.)
Example: PS2_Q3_Dworkin_Seth_0000001.f90. Make sure that they can be opened and read in a text editor such as Notepad or equivalent.

Email your codes and report in the same email. Your email should be timestamped by Dr. Dworkin's email provider by 3:59:59 PM on the due date or else your assignment will be considered late. All of your report must be contained in a single PDF file. (I.e., your email should have one PDF report which contains your codes, and in addition it should have each code that you wrote as a separate attachment.)

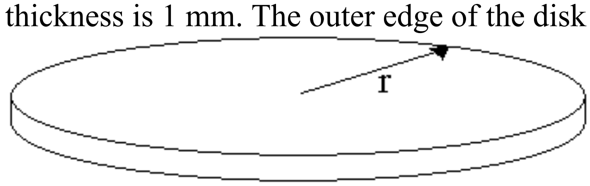
For all programs that you write, include a header with your full name as registered at the university, student number, date, assignment number, question number, description of the program, and programming language used.

Before you hand in your assignment, refer to the document on assignment preparation instructions posted on D2L under Content – Problem Sets.

Questions
1. Consider the problem of heat conduction in a thin disk, governed by the equation:

$$\nabla\cdot(\lambda\nabla T)-\frac{\varepsilon\sigma A\left(T^4-T_\infty^4\right)}{V}=0$$

Heat transfer occurs via radiation from the top and bottom of the disk, where A is the area of the radiating surface, V is the volume of the radiating element, $\lambda = 1000$ [W/mK], $\varepsilon = 1$, $\sigma = 5.67 \times 10^{-8}$ [W/m^{2}K^4], and $T_\infty = 0$ [K]. The radius of the disk is 10 cm and the thickness is 1 mm. The outer edge of the disk is maintained at 800 K.



a) Write a computer program to solve for temperature in the plate as a function of radius using finite volumes and an equispaced grid. You will see in the derivation of your discretizations that you will get non-linear relationships between the temperatures in your control volumes. Therefore, you will need to solve iteratively using Newton's method where each Newton iteration takes the form,

$$[J][\Delta T]=-[F]$$

Assume an initial guess of T = 800 K and use a numerically evaluated Jacobian. Iterate until the 2-norm of the update vector,

$$\|[\Delta T]\|_2=\frac{\sqrt{\sum_i(\Delta T_i)^2}}{N}$$

is less than a Tol. Start with Tol = 10^{-9}. Calculate the power in through the heated edge (P_{in}), and the power lost through radiation (P_{out}). Run your code with 100, 200, and 300 control volumes and calculate the % heat loss error each time.

$$\%\text{ heat loss error} = 100\times\left|P_{out}-P_{in}\right|/P_{in}$$

Make a plot of temperature as a function of radius.

b) By increasing the number of control volumes beyond 300, determine the maximum number of control volumes your code can handle before crashing or not compiling due to either exceeding computer memory limitations or taking more than ten minutes to converge. Your control volume limit may not be the same as that of your colleagues.

c) Show, analytically, that the Jacobian matrix at the *i*th iteration is tri-diagonal. Modify your program to solve the linear system at each iteration using the Thomas algorithm (if you did not do so originally). Jacobian storage in your modified program must now be done with three vectors, rather than a two dimensional array. Run your program with 500, 1000, 3000, 8000, and 10,000 control volumes and a Newton Tolerance of 10^{-8}.

d) Calculate % heat loss error as defined in part a). Determine the execution time of your program. Discuss in two sentences or less how % heat loss error and execution time vary with the number of control volumes.
****

2. Perform a complete Fourier Stability Analysis on the transient 1D heat conduction equation:

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \qquad \alpha = 1$$

discretized using a finite difference Crank Nicholson formulation. State all of your assumptions. Check for consistency between the formulation and the conservation equation. Check to see if there are conditions for which the formulation is or is not convergent.
****

3. a) Return to question 2b on problem set 1. Compare your solution to that posted online and make sure that your program runs correctly. Replace the linear system solver that you used with one that performs Gauss-Seidel iteration. Gauss-Seidel iteration requires an initial guess. Use an initial guess analogous to that used in the "Jacobi_handout.pdf" file that is posted on D2L in Content – Lecture Content. Run your program with $N = 8$, 16, 32, and 64 control volumes. Use 3000 Gauss-Seidel iterations each time. Calculate the same three error norms from question 2b on problem set 1. Also, at the end of the Gauss-Seidel iterations, calculate the linear system residual, defined as

$$\left|\vec{R}\right| = \left|b - Ax\right| = \frac{\sqrt{\sum_{i=1}^{N} R_i^{\,2}}}{N}$$

Compare your solution generated with $N = 64$ to those calculated for question 2b on problem set 1. Compare the error norms calculated with Gauss-Seidel to those calculated for question 2b on problem set 1. Discuss your results in five sentences or less.

b) Now replace the linear system solver with one that performs preconditioned Bi-CGSTAB. (Hint: I found it easiest to use the pseudo code from the 1992 Van der Vorst paper.)
Use $K = K_1 = \text{diag}(A)$. Run your program with $N = 64$. Note that in practice, you do not need to use a symmetric matrix. With the same initial guess, how many iterations of preconditioned Bi-CGSTAB are needed to achieve the same linear system residual that was achieved with 3000 Gauss-Seidel iterations and $N = 64$? For both Gauss-Seidel and preconditioned Bi-CGSTAB, plot the linear system residual as a function of iteration number. Discuss your results in three sentences or less.
****

4. Consider the problem of unsteady thermal diffusion in a flat plate or rod:

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

in the domain [0 cm < x < 25 cm], initially at a temperature of $T(x,0) = T_i = 0$°C. At time $t = 0$, the surface of the plate at $x = 0$ is suddenly placed in water at a temperature of $T_\infty = 25$°C. Convective heat transfer between the water to the plate is characterized by:

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = h\left(T_\infty - T(0,t)\right)$$

Assume that at 25 cm, it is a sufficient distance that the temperature gradient at that point is negligible.
The material properties of the plate and water are:

$$\rho = 8933 \quad kg/m^3, \; c_p = 385 \quad J/kg \cdot K$$

$$\lambda = 12 \quad W/m \cdot K, \; h = 50 \quad W/m^2 \cdot K$$

a) Solve this problem using finite volumes and $\theta = 1$. Use an equispaced grid with 10 control volumes per cm and a TDMA solver. Continue iterating until $T(0,t) = 0.99^* T_\infty$. Calculate the time taken, t_{total}, to reach this point to within one second. Choose your own timestep and show that your timestep is small enough so that t_{total} is independent of timestep size.

Plot T(x) at $t = 0$, $t = .25^*t_{total}$, $t = .5^*t_{total}$, $t = .75^*t_{total}$, and $t = t_{total}$
Plot $T(x = 0)$, $T(x = 12.5$ cm$)$, and $T(x = 25$ cm$)$ as functions of time.

Comment on your results and on the suitability of the assumptions you have made in five sentences or less.