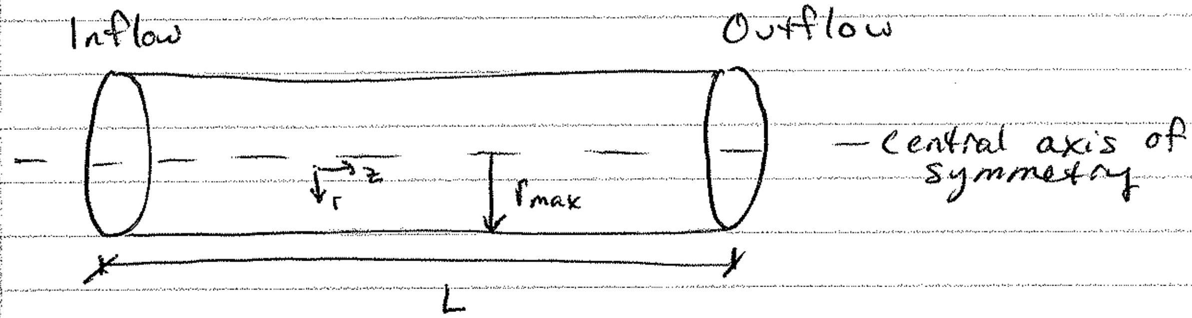
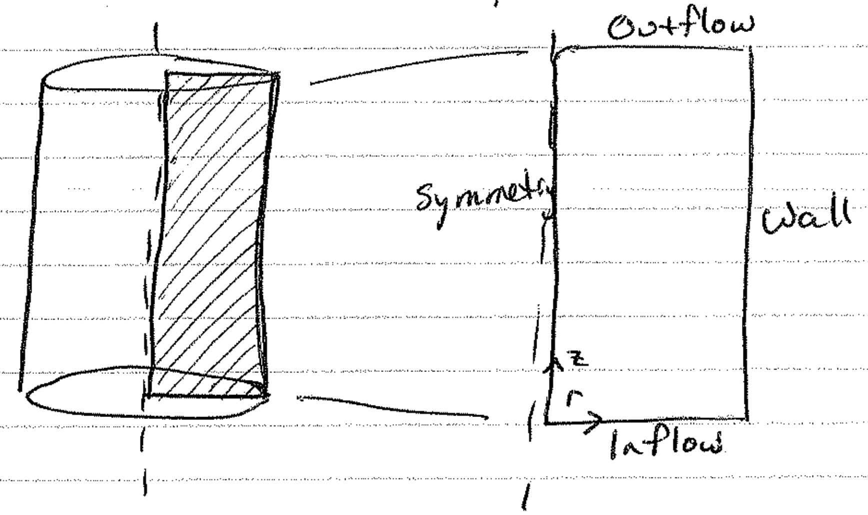
Consider axisymmetric pipe flow



If the inflow boundary condition is axisymmetric (i.e. $\vec{u}_{inflow} \neq f(\theta)$) then the problem can be modelled in a 2D (r, z) domain



The vorticity velocity equations for incompressible constant density, constant viscosity flow are

$$\rho \frac{\partial \omega}{\partial t} = \mu \nabla^2 \omega - \rho\left[\nabla \times (\omega \times \vec{u})\right]$$

$$\omega = \nabla \times \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

$$\nabla^2 \vec{u} = -\nabla \times \omega$$

In axisymmetric cylindrical coordinates
(all $\phi_\theta = 0$, all $\frac{\partial \phi}{\partial \theta} = 0$)

these equations are written as: $\vec{u} = (V_r, V_z)$

$$\rho \frac{\partial \omega}{\partial t} = \mu\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{\omega}{r}\right)\right) - \rho\left(V_r \frac{\partial \omega}{\partial r} + V_z \frac{\partial \omega}{\partial z} - \frac{V_r \omega}{r}\right)$$

$$\omega = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{\partial V_z}{\partial z} = 0$$

$$\text{Poisson}\begin{cases} \dfrac{\partial^2 V_r}{\partial r^2} + \dfrac{\partial^2 V_r}{\partial z^2} = \dfrac{\partial \omega}{\partial z} - \dfrac{\partial}{\partial r}\left(\dfrac{V_r}{r}\right) \\[2ex] \dfrac{\partial^2 V_z}{\partial r^2} + \dfrac{\partial^2 V_z}{\partial z^2} = -\dfrac{\partial \omega}{\partial r} - \dfrac{1}{r}\dfrac{\partial V_r}{\partial z} \end{cases}$$

Vorticity

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} = \frac{\mu}{\rho}\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{\omega}{r}\right)\right) - V_r \frac{\partial \omega}{\partial r} - V_z \frac{\partial \omega}{\partial z} + \frac{V_r \omega}{r}$$

$$\omega^{n+1} = \omega^n + \frac{\Delta t \mu}{\rho}\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{\omega}{r}\right)\right) - \Delta t\left(V_r \frac{\partial \omega}{\partial r} + V_z \frac{\partial \omega}{\partial z} - \frac{V_r \omega}{r}\right)$$

along the axis of symmetry $\omega^{n+1} = 0$

along the inflow $\omega^{n+1} = \frac{\partial V_r^n}{\partial z} - \frac{\partial V_z^n}{\partial r}$

along the wall $\omega^{n+1} = \cancelto{0}{\frac{\partial V_r^n}{\partial z}} - \frac{\partial V_z^n}{\partial r}$

since $V_r(\text{wall}) = 0$

at the outflow $\frac{\partial \omega}{\partial z} = 0$

at the axis of symmetry $\omega = 0$

Radial Velocity

$$\frac{\partial^2 V_r}{\partial r^2} + \frac{\partial^2 V_r}{\partial z^2} = \frac{\partial \omega}{\partial z} - \frac{\partial}{\partial r}\left(\frac{V_r}{r}\right)$$

$$\frac{\partial^2 V_r}{\partial r^2} + \frac{\partial^2 V_r}{\partial z^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} = \frac{\partial \omega}{\partial z}$$

$$\frac{Vr_{i+1} - 2Vr_i + Vr_{i-1}}{\Delta r^2} + \frac{Vr_{j+1} - 2Vr_j + Vr_{j-1}}{\Delta z^2} + \frac{Vr_{i+1} - Vr_{i-1}}{2r\Delta r} - \frac{Vr_{i,j}}{r^2} = \frac{\omega_{j+1}^{n+1} - \omega_{j-1}^{n+1}}{2\Delta z}$$

$$Vr_{j-1}\left(\frac{1}{\Delta z^2}\right) + Vr_{i-1}\left(\frac{1}{\Delta r^2} - \frac{1}{2r\Delta r}\right) + Vr_{ij}\left(\frac{-2}{\Delta r^2} - \frac{2}{\Delta z^2} - \frac{1}{r^2}\right)$$

$$+ Vr_{i+1}\left(\frac{1}{\Delta r^2} + \frac{1}{2r\Delta r}\right) + Vr_{j+1}\left(\frac{1}{\Delta z^2}\right) = \frac{\omega_{j+1}^{n+1} - \omega_{j-1}^{n+1}}{2\Delta z}$$

at the inflow, wall, and centerline, $V_r = 0$

at the outflow, $\frac{\partial V_r}{\partial z} = 0$

$$Vr_{i,j} = Vr_{i,j-1}$$

Axial Velocity

$$\frac{\partial^2 V_z}{\partial r^2} + \frac{\partial^2 V_z}{\partial z^2} = -\frac{\partial \omega}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial z}$$

$$\frac{\partial^2 V_z}{\partial r^2} + \frac{\partial^2 V_z}{\partial z^2} = -\frac{1}{r}\frac{\partial V_r}{\partial z} - \frac{\partial \omega}{\partial r}$$

$$\frac{V_{z\,i+1} - 2V_{z\,i} + V_{z\,i-1}}{\Delta r^2} + \frac{V_{z\,j+1} - 2V_{z\,j} + V_{z\,j-1}}{\Delta z^2} = -\frac{V_{r\,j+1} - V_{r\,j-1}}{r\,2\Delta z}$$

$$-\frac{\omega_{i+1} - \omega_{i-1}}{2\Delta r}$$

$$V_{z\,j-1}\left(\frac{1}{\Delta z^2}\right) + V_{z\,i-1}\left(\frac{1}{\Delta r^2}\right) + V_{z\,ij}\left(\frac{-2}{\Delta r^2} - \frac{2}{\Delta z^2}\right) + V_{z\,i+1}\left(\frac{1}{\Delta r^2}\right)$$

$$+ V_{z\,j+1}\left(\frac{1}{\Delta z^2}\right) = -\frac{V_{r\,j+1} - V_{r\,j-1}}{2r\Delta z} - \frac{\omega_{i+1} - \omega_{i-1}}{2\Delta r}$$

at the inflow $V_z = V_{z\,in}$
at the wall $V_z = 0$
at the centerline (see "Axial Velocity 2")
or use $\frac{\partial V_z}{\partial r} = 0$ (1st order)

At the outflow $\frac{\partial V_z}{\partial z} = 0$

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Axial Velocity 2 Page 1

$$\frac{\partial^2 V_z}{\partial r^2} + \frac{\partial^2 V_z}{\partial z^2} = -\frac{\partial \omega}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial z} \quad (1)$$ is the gov eq.

$$\frac{\partial V_z}{\partial r} = 0$$ is the B.C.

For a higher order B.C.

$$V_z^{i+1} = V_z^{i} + \Delta r \frac{\partial V_z}{\partial r} + \frac{\Delta r^2}{2}\frac{\partial^2 V_z}{\partial r^2} + \dots$$

$$\frac{\partial V_z}{\partial r} = 0 = \frac{V_z^{i+1} - V_z^{i}}{\Delta r} - \frac{\Delta r}{2}\frac{\partial^2 V_z}{\partial r^2}$$

$$\frac{\partial^2 V_z}{\partial r^2} = \frac{2}{\Delta r}\left(V_z^{i+1} - V_z^{i}\right) \quad (2)$$

sub (1) into (2)

$$-\frac{\partial^2 V_z}{\partial z^2} - \frac{\partial \omega}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial z} - \frac{2}{\Delta r^2}\left(V_z^{i+1} - V_z^{i}\right) = 0$$

What about $\frac{1}{r}\frac{\partial V_r}{\partial z}$?

$$\lim_{r\to 0} \frac{\frac{\partial V_r}{\partial z}}{r} = \lim_{r \to 0} \frac{\frac{\partial}{\partial r}\left(\frac{\partial V_r}{\partial z}\right)}{\frac{\partial}{\partial r}(r)} = \frac{\partial^2 V_r}{\partial r \partial z}$$

$$-\frac{\partial^2 V_z}{\partial z^2} - \frac{\partial \omega}{\partial r} - \frac{\partial^2 V_r}{\partial r \partial z} - \frac{2}{\Delta r^2}\left(V_z^{i+1} - V_z^{i}\right) = 0$$

$$\frac{\partial^2 V_z}{\partial z^2} + \frac{2}{\Delta r^2}\left(V_z^{i+1} - V_z^{i}\right) = -\frac{\partial \omega}{\partial r} - \frac{\partial^2 V_r}{\partial r \partial z}$$

Axial Velocity Z Page 2.

$$\frac{Vz_{j+1} - 2Vz_{i,j} + Vz_{j-1}}{\Delta z^2} + \frac{2}{\Delta r^2}\left(Vz_{,i+1} - Vz_{i,j}\right) = -\frac{\partial \omega}{\partial r} - \frac{\partial^2 Vr}{\partial r \partial z}$$

$$Vz_{j-1}\left(\frac{1}{\Delta z^2}\right) + Vz_{i,j}\left(\frac{-2}{\Delta z^2} - \frac{2}{\Delta r^2}\right) + Vz_{,i+1}\left(\frac{2}{\Delta r^2}\right) + Vz_{j+1}\left(\frac{1}{\Delta z^2}\right)$$

$$= -\frac{\partial \omega}{\partial r} - \frac{\partial^2 Vr}{\partial v \partial z}$$

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Axial Velocity (from Continuity)

$$\frac{\partial V_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r V_r)$$

$$\frac{\partial V_z}{\partial z} = -\frac{1}{r} r \frac{\partial V_r}{\partial r} - \frac{1}{r} V_r \frac{\partial r}{\partial r}$$

$$\frac{\partial V_z}{\partial z} = -\frac{\partial V_r}{\partial r} - \frac{V_r}{r}$$

<u>Option 1</u>

$$\frac{V_{z\,i,j} - V_{z\,i,j-1}}{\Delta z} = -\frac{V_{r\,i,j}}{r(i)} - \frac{V_{r\,i+1,j} - V_{r\,i-1,j}}{2\Delta r}$$

$$V_{z\,i,j} = \left[-\frac{V_{r\,i,j}}{r(i)} - \frac{V_{r\,i+1,j} - V_{r\,i-1,j}}{2\Delta r}\right]\Delta z + V_{z\,i,j-1}$$