AE/ME8112 Problem Set 5 **Due Friday, July 24th at 4:00PM**

Instructions:
For each problem, state all your assumptions, explain your approach and solution method, include a good copy of any preparation work that needs to be done by hand (derivations, discretizations… etc.) in your report.

If the problem requires writing a computer program, include your code and its output in your report.

In addition, email your codes to Prof. Dworkin by the due date and time. Your codes may be run to verify output, or put through a plagiarism checker. All codes that you email should be as separate attachments, titled PSnum_Qnum_Lastname_Firstname_Studentnum.dat (or .f90 or .f95 or .c etc.)
Example: PS4_Q2_Dworkin_Seth_0000001.f90. Make sure that they can be opened and read in a text editor such as Notepad or equivalent.

Email your codes and report in the same email. Your email should be timestamped by Dr. Dworkin's email provider by 3:59:59 PM on the due date or else your assignment will be considered late. All of your report must be contained in a single PDF file. (I.e., your email should have one PDF report which contains your codes, and in addition it should have each code that you wrote as a separate attachment.)

For all programs that you write, include a header with your full name as registered at the university, student number, date, assignment number, question number, description of the program, and programming language used.

Before you hand in your assignment, refer to the document on assignment preparation instructions posted on D2L under Content – Problem Sets.

Questions
1. Consider laminar gaseous flow through a cylindrical pipe of length, L and radius R with a centerline axial velocity of $CLVZ$.

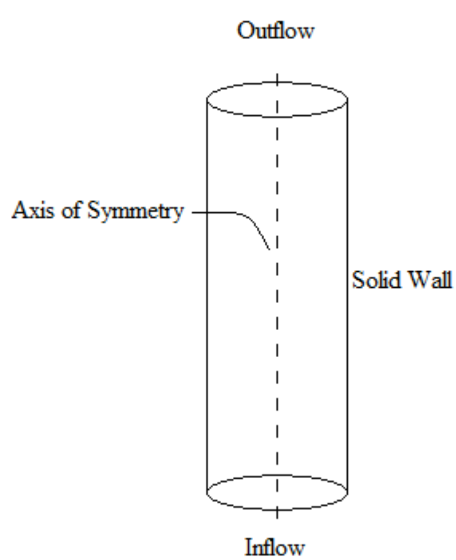
$CLVZ = 5\ cm/s$

$\rho = 8.3e-4\ \ g/cm^3$

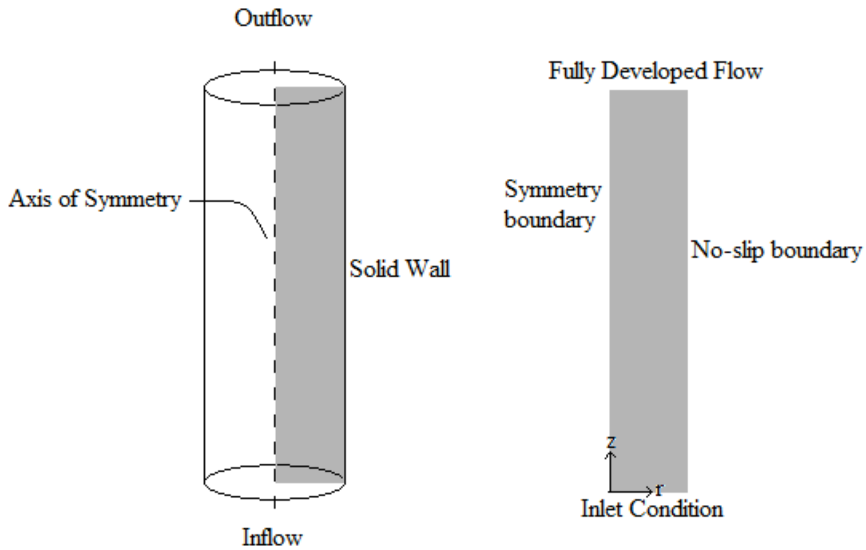
$R = 1.0\ \ cm$

$L = 10.0\ cm$

$\mu = 1.4e-4\ \ g/cm*s$



Because the flow is axisymmetric, the problem can be modelled as two-dimensional flow in r and z.



a) Derive an explicit finite difference discretization of the vorticity-velocity equations:

vorticity transport:
$$\rho\frac{\partial \omega}{\partial t} = \mu\frac{\partial^2 \omega}{\partial r^2} + \mu\frac{\partial^2 \omega}{\partial z^2} + \mu\frac{\partial}{\partial r}\left(\frac{\omega}{r}\right) - \rho v_r\frac{\partial \omega}{\partial r} - \rho v_z\frac{\partial \omega}{\partial z} + \frac{\rho v_r \omega}{r}.$$

Poisson radial velocity:
$$\frac{\partial^2 v_r}{\partial r^2} + \frac{\partial^2 v_r}{\partial z^2} = \frac{\partial \omega}{\partial z} - \frac{\partial}{\partial r}\left(\frac{v_r}{r}\right),$$

Continuity:
$$\frac{\partial v_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r v_r).$$

where ω is the vorticity, v_r is the radial velocity component, and v_z is the axial velocity component. Note that the above equations have been derived under the assumptions of axisymmetry, constant density, and constant viscosity. Note that the definition of vorticity, in axisymmetric cylindrical coordinates, which is used to impose vorticity boundary conditions is,

$$\omega = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}.$$

The Boundary conditions are:

<u>Inflow:</u>
v_z is parabolic, decreasing from *CLVZ* at the centerline to 0 at the wall.
v_r is zero at the inflow.
$\omega = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$. Implementation of this BC will require discretizing $\frac{\partial v_r}{\partial z}$ and $\frac{\partial v_z}{\partial r}$. You can use a first order one-sided difference for $\frac{\partial v_r}{\partial z}$ and a second-order centered difference for $\frac{\partial v_z}{\partial r}$.

<u>Wall:</u>
v_z is zero
v_r is zero
$\omega = -\frac{\partial v_z}{\partial r}$, since v_r, and therefore also $\frac{\partial v_r}{\partial z} = 0$ at the wall.

Outflow:
At the outflow, you can assume fully developed flow, *i.e.* the flow is no longer changing in the axial direction.
$\frac{\partial \omega}{\partial z} = \frac{\partial v_r}{\partial z} = \frac{\partial v_z}{\partial z} = 0$.

Symmetry
At the axis of symmetry, you may assume that the flow is identical on either side of the axis, and you may also assume that v_r is identically zero (from fluids theory) and ω is identically zero (from fluids theory). This gives some options as to which boundary conditions to apply. I recommend,
ω is zero
v_r is zero
$\frac{\partial v_z}{\partial z} = 0$. This can be shown from continuity, since $\frac{\partial v_r}{\partial r} = 0$ from symmetry and $\frac{v_r}{r} = 0$ since $\lim_{r \to 0}\left(\frac{v_r}{r}\right) = 0$.

alternatively, $\frac{\partial v_z}{\partial r} = 0$ could also be used.
b) Write a program to solve the vorticity velocity equations for pipe flow. Each timestep should proceed explicitly in three (plus one) main steps.

Once an initial guess is generated for ω, v_r, and v_z at each grid point,
Step 1: Solve for vorticity and the current time level based on vorticity and velocity at the previous time level. This can be done directly, grid point by grid point, without solving a linear system.
Step 2: Solve for v_r at each grid point from the Poisson radial velocity equation. You should generate and solve a pentadiagonal linear system at each timestep. Use your preconditioned Bi-CGSTAB subroutine with a linear system tolerance of 10^{-8}.
Step 3. Solve for v_z from continuity based on the newly calculated radial velocity field. Use a first order backward difference for the axial velocity derivative. That way, you can solve for axial velocity directly, without solving a linear system, since axial velocity will only be a function of the recently calculated radial velocity field, and the axial velocity at the previous grid row.
Step 4. check for convergence and increment the timestep.

After each timestep, calculate the axial velocity flow at axial height *j*, defined as

$$\dot{m}_j = 2\pi \int_0^R \rho v_z r dr$$

via numerical integration. Conservation of mass would dictate that the axial mass flow through any pipe cross section should be the same as the axial mass flow at the inlet. Calculate the absolute value of the difference between mass flow at the inlet and mass flow at each other axial height. Iterate until the percentage maximum mass flow difference,

$$\frac{\underset{j=1,NZ}{Max}\left|\dot{m}_j - \dot{m}_1\right|}{\dot{m}_1} \le Tol$$

Start with *Tol* = 1e-5, but try different tolerances and look at the effect. You will want to adjust this tolerance based on your desired accuracy and computational cost. I recommend a timestep of 10^{-6}, *NR* = 21 grid points in the radial direction, and *NZ* = 201 grid points in the axial direction. The interplay between timestep size (constant or adaptive), computation time, and accuracy, is an important one. You may wish to investigate it and make some comments in your report.

c) Run your program with an initial guess of $v_r = v_z = \omega = 0$ and plot $\frac{\underset{j=1,NZ}{Max}\left|\dot{m}_j - \dot{m}_1\right|}{\dot{m}_1}$ and $\frac{\underset{j=1,NZ}{Avg}\left|\dot{m}_j - \dot{m}_1\right|}{\dot{m}_1}$ as a function of iteration number. Make a vector plot of the flow field and a contour plot of the vorticity field. Was your tolerance sufficiently small? Discuss.

d) Perform a test of the sensitivity to the initial guess. For this test (and for debugging) note that the following initial guess is very close to the actual solution:

$$v_r(r,z) = 0, \qquad v_z(r,z) = CLVZ * \left[1 - \left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2\right], \qquad \omega(r,z) = 2 * CLVZ * \left(\frac{\mathrm{r}}{\mathrm{R}^2}\right)$$