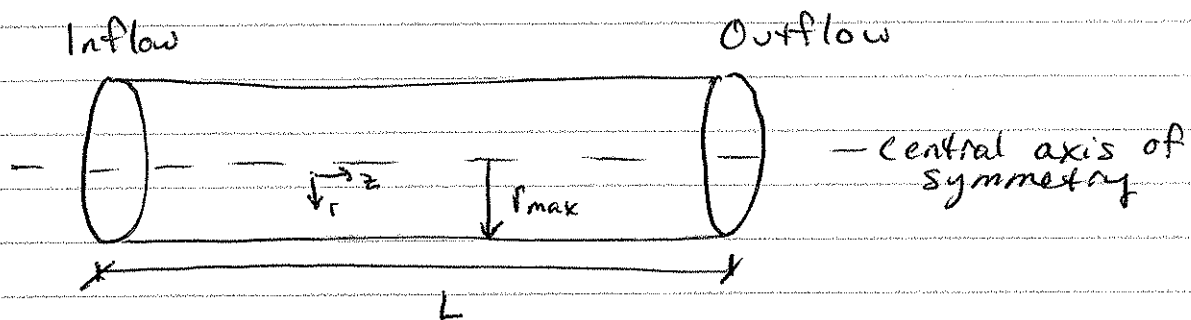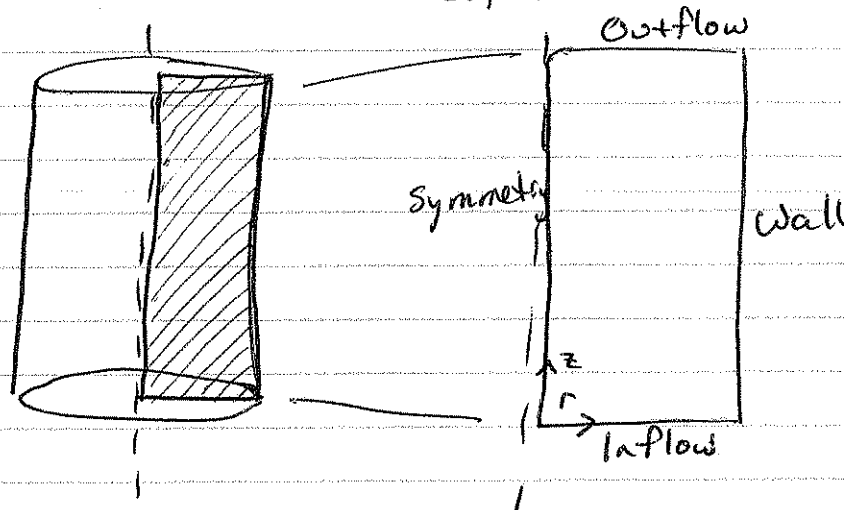Consider axisymmetric pipe flow

If the inflow boundary condition is axisymmetric (i.e. $\vec{u}_{inflow} \neq f(\theta)$) then the problem can be modelled in a 2D (r,z) domain

The vorticity velocity equations for incompressible constant density, constant viscosity flow are

$$\rho \frac{\partial \omega}{\partial t} = \mu \nabla^2 \omega - \rho \left[\nabla \times (\omega \times \vec{u}) \right]$$

$$\omega = \nabla \times \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

$$\nabla^2 \vec{u} = - \nabla \times \omega$$

In axisymmetric cylindrical coordinates
(all $\phi_\theta = 0$, all $\frac{\partial \phi}{\partial \theta} = 0$)

these equations are written as: $\vec{u} = (V_r, V_z)$

$$\rho \frac{\partial \omega}{\partial t} = \mu \left(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{\omega}{r}\right) \right) - \rho \left(V_r \frac{\partial \omega}{\partial r} + V_z \frac{\partial \omega}{\partial z} - \frac{V_r \omega}{r} \right)$$

$$\omega = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{\partial V_z}{\partial z} = 0$$

Poisson $\Bigg\{$

$$\frac{\partial^2 V_r}{\partial r^2} + \frac{\partial^2 V_r}{\partial z^2} = \frac{\partial \omega}{\partial z} - \frac{\partial}{\partial r}\left(\frac{V_r}{r}\right)$$

$$\frac{\partial^2 V_z}{\partial r^2} + \frac{\partial^2 V_z}{\partial z^2} = -\frac{\partial \omega}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial z}$$

Vorticity

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} = \frac{\mu}{\rho}\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{\omega}{r}\right)\right) - V_r \frac{\partial \omega}{\partial r} - V_z \frac{\partial \omega}{\partial z} + \frac{V_r \omega}{r}$$

$$\omega^{n+1} = \omega^n + \frac{\Delta t \mu}{\rho}\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{\omega}{r}\right)\right) - \Delta t \left(V_r \frac{\partial \omega}{\partial r} + V_z \frac{\partial \omega}{\partial z} - \frac{V_r \omega}{r}\right)$$

along the axis of symmetry $\omega^{n+1} = 0$

along the inflow $\omega^{n+1} = \frac{\partial V_r^n}{\partial z} - \frac{\partial V_z^n}{\partial r}$

along the wall $\omega^{n+1} = \frac{\cancel{\partial V_r^n}^{0}}{\partial z} - \frac{\partial V_z^n}{\partial r}$

since $V_r (wall) = 0$

at the outflow $\frac{\partial \omega}{\partial z} = 0$

at the axis of symmetry $\omega = 0$

Radial Velocity

$$\frac{\partial^2 V_r}{\partial r^2} + \frac{\partial^2 V_r}{\partial z^2} = \frac{\partial \omega}{\partial z} - \frac{\partial}{\partial r}\left(\frac{V_r}{r}\right)$$

$$\frac{\partial^2 V_r}{\partial r^2} + \frac{\partial^2 V_r}{\partial z^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} = \frac{\partial \omega}{\partial z}$$

$$\frac{V_{r_{i+1}} - 2V_{r_i} + V_{r_{i-1}}}{\Delta r^2} + \frac{V_{r_{j+1}} - 2V_{r_j} + V_{r_{j-1}}}{\Delta z^2} + \frac{V_{r_{i+1}} - V_{r_{i-1}}}{2r\Delta r} - \frac{V_{r_{i,j}}}{r^2} = \frac{\omega_{j+1}^{n+1} - \omega_{j-1}^{n+1}}{2\Delta z}$$

$$V_{r_{j-1}}\left(\frac{1}{\Delta z^2}\right) + V_{r_{i-1}}\left(\frac{1}{\Delta r^2} - \frac{1}{2r\Delta r}\right) + V_{r_{ij}}\left(\frac{-2}{\Delta r^2} - \frac{2}{\Delta z^2} - \frac{1}{r^2}\right)$$

$$+ V_{r_{i+1}}\left(\frac{1}{\Delta r^2} + \frac{1}{2r\Delta r}\right) + V_{r_{j+1}}\left(\frac{1}{\Delta z^2}\right) = \frac{\omega_{j+1}^{n+1} - \omega_{j-1}^{n+1}}{2\Delta z}$$

at the inflow, wall, and centerline, $V_r = 0$

at the outflow, $\frac{\partial V_r}{\partial z} = 0$

$$V_{r_{i,j}} = V_{r_{i,j-1}}$$

Axial Velocity

$$\frac{\partial^2 V_z}{\partial r^2} + \frac{\partial^2 V_z}{\partial z^2} = -\frac{\partial \omega}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial z}$$

$$\frac{\partial^2 V_z}{\partial r^2} + \frac{\partial^2 V_z}{\partial z^2} = -\frac{1}{r}\frac{\partial V_r}{\partial z} - \frac{\partial \omega}{\partial r}$$

$$\frac{V_{z_{i+1}} - 2V_{z_i} + V_{z_{i-1}}}{\Delta r^2} + \frac{V_{z_{j+1}} - 2V_{z_j} + V_{z_{j-1}}}{\Delta z^2} = -\frac{V_{r_{j+1}} - V_{r_{j-1}}}{r\, 2\Delta z}$$

$$\frac{W_{i+1} - W_{i-1}}{2\Delta r}$$

$$V_{z_{j-1}}\left(\frac{1}{\Delta z^2}\right) + V_{z_{i-1}}\left(\frac{1}{\Delta r^2}\right) + V_{z_{ij}}\left(\frac{-2}{\Delta r^2} - \frac{2}{\Delta z^2}\right) + V_{z_{i+1}}\left(\frac{1}{\Delta r^2}\right)$$

$$+ V_{z_{j+1}}\left(\frac{1}{\Delta z^2}\right) = -\frac{V_{r_{j+1}} - V_{r_{j-1}}}{2r\Delta z} - \frac{W_{i+1} - W_{i-1}}{2\Delta r}$$

at the inflow $V_z = V_{z_{in}}$
at the wall $V_z = 0$
at the centerline (see "Axial Velocity 2")
      or use $\frac{\partial V_z}{\partial r} = 0$ (1st order)

At the outflow $\frac{\partial V_z}{\partial z} = 0$

Axial Velocity 2 Page 1

$$\frac{\partial^2 V_z}{\partial r^2} + \frac{\partial^2 V_z}{\partial z^2} = -\frac{\partial w}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial z} \quad \text{①} \quad \text{is the gov eq.}$$

$$\frac{\partial V_z}{\partial r} = 0 \quad \text{is the B.C.}$$

For a higher order B.C.

$$V_z^{i+1} = V_z^i + \Delta r \frac{\partial V_z}{\partial r} + \frac{\Delta r^2}{2}\frac{\partial^2 V_z}{\partial r^2} + \cdots$$

$$\frac{\partial V_z}{\partial r} = 0 = \frac{V_z^{i+1} - V_z^i}{\Delta r} - \frac{\Delta r}{2}\frac{\partial^2 V_z}{\partial r^2}$$

$$\frac{\partial^2 V_z}{\partial r^2} = \frac{2}{\Delta r}\left(V_z^{i+1} - V_z^i\right) \quad \text{②}$$

sub ① into ②

$$-\frac{\partial^2 V_z}{\partial z^2} - \frac{\partial w}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial z} - \frac{2}{\Delta r^2}\left(V_z^{i+1} - V_z^i\right) = 0$$

what about $\frac{1}{r}\frac{\partial V_r}{\partial z}$?

$$\lim_{r \to 0} \frac{\frac{\partial V_r}{\partial z}}{r} = \lim_{r \to 0} \frac{\frac{\partial}{\partial r}\left(\frac{\partial V_r}{\partial z}\right)}{\frac{\partial}{\partial r}(r)} = \frac{\partial^2 V_r}{\partial r \partial z}$$

$$-\frac{\partial^2 V_z}{\partial z^2} - \frac{\partial w}{\partial r} - \frac{\partial^2 V_r}{\partial r \partial z} - \frac{2}{\Delta r^2}\left(V_z^{i+1} - V_z^i\right) = 0$$

$$\frac{\partial^2 V_z}{\partial z^2} + \frac{2}{\Delta r^2}\left(V_z^{i+1} - V_z^i\right) = -\frac{\partial w}{\partial r} - \frac{\partial^2 V_r}{\partial r \partial z}$$

Axial Velocity 2   Page 2.

$$\frac{V_{z,j+1} - 2V_{z_{i,j}} + V_{z,j-1}}{\Delta z^2} + \frac{2}{\Delta r^2}\left(V_{z,i+1} - V_{z_{i,j}}\right) = -\frac{\partial w}{\partial r} - \frac{\partial^2 V_r}{\partial r \partial z}$$

$$V_{z,j-1}\left(\frac{1}{\Delta z^2}\right) + V_{z_{i,j}}\left(\frac{-2}{\Delta z^2} - \frac{2}{\Delta r^2}\right) + V_{z,i+1}\left(\frac{2}{\Delta r^2}\right) + V_{z,j+1}\left(\frac{1}{\Delta z^2}\right)$$

$$= -\frac{\partial w}{\partial r} - \frac{\partial^2 V_r}{\partial v \partial z}$$

Axial Velocity (from Continuity)

$$\frac{\partial V_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r V_r)$$

$$\frac{\partial V_z}{\partial z} = -\frac{1}{r} r \frac{\partial V_r}{\partial r} - \frac{1}{r} V_r \frac{\partial r}{\partial r}$$

$$\frac{\partial V_z}{\partial z} = -\frac{\partial V_r}{\partial r} - \frac{V_r}{r}$$

Option 1

$$\frac{V_{z_{i,j}} - V_{z_{i,j-1}}}{\Delta z} = -\frac{V_{r_{i,j}}}{r(i)} - \frac{V_{r_{i+1,j}} - V_{r_{i-1,j}}}{2\Delta r}$$

$$V_{z_{i,j}} = \left[-\frac{V_{r_{i,j}}}{r(i)} - \frac{V_{r_{i+1,j}} - V_{r_{i-1,j}}}{2\Delta r} \right]\Delta z + V_{z_{i,j-1}}$$