Week 13: Thursday. 5.6: Similarity Transformations

Course Goals

After studying section 5.6: Similarity Transformations, you should

1. *Understand the concept and properties of similar matrices and their applications.*
2. *Understand the concept and properties of normal matrices.*
3. *Understand Schur's lemma and the ~~Jordan form~~.*

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Course Outcomes

To manifest that you have reached the above course goals, you should be able to

1. *Discuss how determinants, eigenvalues and eigenvectors of two similar matrices are related.*
2. *Show why the following matrices are diagonalizable and discuss the special forms of their diagonalization: Hermitian/symmetric, skew–Hermitian/skew–symmetric, unitary/orthogonal, and normal.*
3. *Find the similarity transformation between two matrices representing the same linear transformation with respect to two different bases.*
4. *Find the Jordan form of a given matrix.*
5. *State Schur's lemma and find the triangular matrix that is similar to a given matrix.*

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Symmetric Matrix

A real-vaued $n \times n$ matrix A is called **symmetric** if:

$$A^T = A.$$

Hermitian Matrix

A $n \times n$ matrix A with (possibly) complex values is called **Hermitian** if:

$$A^H = \overline{A}^T = A.$$

$\overline{a+ib} = a-ib$

$\overline{5} = 5$

Important Properties

- A symmetric (or Hermitian) matrix has **real eigenvalues**.
- Eigenvectors from different eigenvalues for a symmetric (or Hermitian) matrix are **orthogonal**.

A^k

e^{At}

$A = S^{-1} \Lambda S$

$\Rightarrow$ Always diagonalizable

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Skew Symmetric Matrix

A real-vaued $n \times n$ matrix A is called **symmetric** if:

$$A^T = -A.$$

Skew Hermitian Matrix

A $n \times n$ matrix A with (possibly) complex values is called **skew-Hermitian** if:

$$A^H = \overline{A}^T = -A.$$

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Property

If A is Hermitian, show $K = iA$ is skew-Hermitian.

- To be Hermitian,
  $a_{ij} = \overline{a_{ji}} \implies \mathrm{Re}(a_{ij}) = \mathrm{Re}(a_{ji})$ and $\mathrm{Im}(a_{ij}) = -\mathrm{Im}(a_{ji})$.
- To be skew-Hermitian, $a_{ij} = -\overline{a_{ji}}$.
- If $K = iA$ then $k_{ij} = ia_{ij}$. If $z = (a + ib)$ $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$.

$$\mathrm{Re}(k_{ij}) = -\mathrm{Im}(a_{ij}) = \mathrm{Im}(a_{ji}) = -\mathrm{Re}(k_{ji})$$

and

$$\mathrm{Im}(k_{ij}) = \mathrm{Re}(a_{ij}) = \mathrm{Re}(a_{ji}) = \mathrm{Im}(k_{ji})$$

- But together these properties imply:

$$k_{ij} = -\overline{k_{ji}}$$

and thus K is skew-Hermitian.

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Orthogonal Matrix

A real-valued $n \times n$ matrix Q is called **orthogonal** if its columns are orthonormal.

$$Q^{-1} = Q^T.$$

Unitary Matrix

A $n \times n$ matrix U with complex values is called **unitary** if its columns are orthonormal.

$$U^{-1} = U^H.$$

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Examples

11. U is a unitary matrix. Show that U has the following properties.
    a. U preserves inner products, and as a consequence lengths.
    b. All eigenvalues have absolute value 1.
    c. Eigenvectors corresponding to different eigenvalues are orthogonal.

a. To show U preserves inner products, we need to show that $\vec{x}^H\vec{y} = (U\vec{x})^H(U\vec{y})$.

$$(U\vec{x})^H(U\vec{y}) = \vec{x}^H U^H U\vec{y} = \vec{x}^H\vec{y}.$$

This is because $U^H U = I$. This also says that U preserves lengths because:

$$\|U\vec{x}\|^2 = (U\vec{x})^H(U\vec{x}) = \vec{x}^H U^H U\vec{x} = \vec{x}^H\vec{x} = \|\vec{x}\|^2.$$

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- To show that eigenvalues have absolute value 1, let's note that if $U\vec{x} = \lambda\vec{x}$ this gives us:

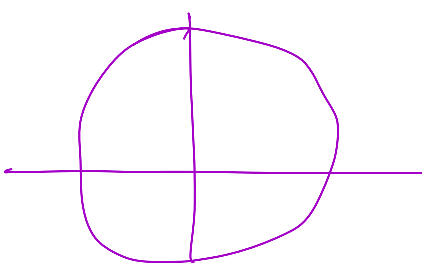
$$\|\vec{x}\|^2 = \|U\vec{x}\|^2 = (U\vec{x})^H(U\vec{x}) = (\lambda\vec{x})^H(\lambda\vec{x}) = \overline{\lambda}\vec{x}^H\vec{x}\lambda = |\lambda|^2\|\vec{x}\|.$$

This gives us:

$$\|\vec{x}\|^2 = |\lambda|^2\|\vec{x}\|^2 \implies |\lambda|^2 = 1 \implies |\lambda| = 1.$$

Note that λ might be complex! This is the absolute value in the sense of a complex number.

$\sqrt{a^2+b^2}$



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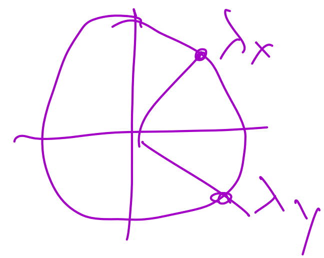
- To show that for a unitary matrix U eigenvectors from distinct eigenvalues are orthogonal, we will use the fact that the a U preserves inner products.
  Let $\vec{x}$ and $\vec{y}$ be eigenvectors from distinct eigenvalues λ_x and λ_y.

$$\vec{x}^H\vec{y} = (U\vec{x})^H(U\vec{y}) = (\lambda_x\vec{x})^H(\lambda_y\vec{y}) = \overline{\lambda_x}\vec{x}^H\lambda_y\vec{y} = \overline{\lambda_x}\lambda_y\vec{x}^H\vec{y}.$$

Looking at both sides of the equal sign we have:

$$\boxed{(1-\overline{\lambda_x}\lambda_y)\vec{x}^H\vec{y} = 0.}$$



Thus, this either means: $\vec{x}^H\vec{y} = 0$ or $1 = \overline{\lambda_x}\lambda_y$.
Since $\overline{\lambda_x}\lambda_x = 1$ and $\overline{\lambda_y}\lambda_y = 1$ it is not possible that $\lambda_x \neq \lambda_y$ and $\overline{\lambda_x}\lambda_y = 1$.
Thus it must be that $\boxed{\vec{x}^H\vec{y} = 0.}$

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Similar Matrices

Matrices A and B are **similar matrices** if there exists an invertible matrix M such that:

$$\boxed{B = M^{-1}AM.} \Rightarrow \quad MBM^{-1} = A$$

Examples

1. Suppose that $B = M^{-1}AM$.
   a. How is $\det A$ related to $\det B$ and why?
   b. Why do A and B have the same eigenvalues?
   c. How are their eigenvectors related?
   d. Explain why similar matrices are diagonalizable or not at the same time.

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(a) How is $\det A$ related to $\det B$ and why?

$$\begin{aligned}
\det(B) &= \det(M^{-1}AM) \\
&= \det(M^{-1})\det(A)\det(M) \\
&= (1/\det(M))\det(A)\det(M) \\
&= \det(A).
\end{aligned}$$

(b) Why do A and B have the same eigenvalues?

$$\begin{aligned}
0 = \det(B - \lambda I) &= \det(M^{-1}AM - \lambda I) \\
&= \det(M^{-1}AM - \lambda M^{-1}IM) \\
&= \det\left(M^{-1}(A - \lambda I)M\right) \\
&= \det(\cancel{M^{-1}})\det(A - \lambda I)\det(\cancel{M}) \\
&= \det(A - \lambda I).
\end{aligned}$$

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- How are the eigenvectors of A and B related if $B = M^{-1}AM$? Let's suppose that $B\vec{x} = \lambda\vec{x}$.

$$B\vec{x} = \lambda\vec{x} \implies M^{-1}AM\vec{x} = \lambda\vec{x} \implies M(M^{-1}AM\vec{x}) = M(\lambda\vec{x})$$

$$\implies AM\vec{x} = \lambda M\vec{x}.$$

  Thus the matrix M converts between eigenvectors of B and A.

- Explain why similar matrices are diagonalizable or not at the same time.
  - A and B have the same number of linearly independent eigenvectors.
  - As such, two similar matrices are either both diagonalizable or neither of them is diagonalizable.

Note that when A is diagonalizable there is a matrix S of eigenvectors where $A = S\Lambda S^{-1}$ or $\Lambda = S^{-1}AS$. A matrix that is diagonalizable is similar to a diagonal matrix.

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Examples

2. If the vectors $\vec{x}_1$ and $\vec{x}_2$ are in the columns of S, what are the eigenvalues and eigenvectors of

$$A = S\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} \quad \text{and} \quad B = S\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} S^{-1}?$$

Although these matrices look quite similar, the problems are subtle.

A is similar to $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

B is similar to $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

what are the eigenvalues/eigenvectors of A & B

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- (a) $A = S \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} S^{-1}$

  Because $AS = S\Lambda$ we know S's columns $(\vec{x}_1, \vec{x}_2)$ are the eigenvectors of A and the values on the diagonal of Λ are the eigenvalues.

$$AS = S\Lambda \implies A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \vec{x}_1 & \lambda_2 \vec{x}_2 \end{bmatrix}.$$

$$\lambda_1 = 2, \vec{x}_1 \text{ and } \lambda_2 = 1, \vec{x}_2.$$

  This is just the usual way we do diagonalization.

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(b) $B = S \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} S^{-1}$

B , $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = R$

B shares its eigenvalues with the upper triangular matrix in the middle: $\lambda_1 = 2, \lambda_2 = 1$. To find eigenvectors of B multiply by S on the left we have:

$$S \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = BS.$$

$\left[\begin{array}{cc|c} 2-2 & 3 & 6 \\ 0 & 1-2 & 0 \end{array}\right]$ $\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -1 & 0 \end{array}\right]$

Now, let's notice that $\vec{e}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\vec{e}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ are eigenvectors of the matrix in the middle:

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 2

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \vec{e}_1 = 2\vec{e}_1 \text{ and } \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \vec{e}_2 = \vec{e}_2.$$

$$\lambda_i (S\vec{e}_i) = S(\lambda_i \vec{e}_i) = S \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \vec{e}_i = (BS)\vec{e}_i = B(S\vec{e}_i).$$

Thus we see that B has two eigenvectors: $S\vec{e}_1$ and $S\vec{e}_2$.

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We have learned several different matrix factorization methods to far:

$$A = LU, A = LDU, A = QR.$$

Now, we are going to learn one more.

Schur Lemma

An $n \times n$ matrix A is similar to a triangular matrix T. That is, there is a unitary matrix U such that:

$$U^{-1}AU = T.$$

$U^{-1} = U^H$

Why does Schur's Lemma work? Let's take a little bit of a look.

$A = \underline{M}\,\underline{B}\,\underline{M^{-1}}$

* A & B have the same eigenvalues

* M converts eigenvectors from A to B (B to A)

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For simplicity - assume A is a 4 by 4 matrix. We will show there is unitary matrix U such that: $U^{-1}AU = T$ where T is an upper triangular matrix.

- A has at least 1 eigenvalue λ_1 and at least 1 eigenvector $\vec{u}_1$.
- Let's assume $\vec{u}_1$ has length 1, and use it to create a unitary matrix U_1 with $\vec{u}_1$ as it's first column and any other vectors as the latter columns in a way that makes it unitary.
- To do this, we just make $\vec{u}_1$ can do this by finding an orthogonal basis for $V^{\perp}$ if $V = \text{span}\{\vec{u}_1\}$. Let's call these vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Then we have:
$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{a} & \vec{b} & \vec{c} \end{bmatrix}.$$
- Let's notice that:
$$AU_1 = A\begin{bmatrix} \vec{u}_1 & \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \lambda\vec{u}_1 & A\vec{a} & A\vec{b} & A\vec{c} \end{bmatrix} = U_1 \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & \boxed{\begin{matrix} * & * & * \\ * & * & * \\ * & * & * \end{matrix}} \\ 0 & & & \\ 0 & & & \end{bmatrix}$$
- Notice that we use $*$'s because we don't really need to know what these values are.

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- We will re-write this slightly as

$$U_1^{-1}AU_1 = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

- Consider the red 3×3 submatrix of $*$'s. This sub matrix has at least 1 eigenvalue λ_2 and 1 eigenvector $\vec{u}_2 = \begin{bmatrix} x & y & z \end{bmatrix}^T$. Let's find two other orthonormal vectors to complete $\mathbb{R}^3$ and define a new unitary matrix U_2:

$$U_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & * & * \\ 0 & y & * & * \\ 0 & z & * & * \end{bmatrix} \implies U_2^{-1}(U_1^{-1}AU_1)U_2 = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

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- Now we work with the matrix of blue $*$'s. This 2 by 2 submatrix has at least 1 eigenvalue λ_3 and 1 eigenvector. Let's call is $\vec{u}_3 = \begin{bmatrix} x & y \end{bmatrix}^T$ and let's use it to create a unitary matrix U_3:

$$U_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & * \\ 0 & 0 & y & * \end{bmatrix}.$$

- Then combining together we have:

$$U_3^{-1} \left(U_2^{-1} U_1^{-1} A U_1 U_2 \right) U_3 = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & \lambda_3 & * \\ 0 & 0 & 0 & * \end{bmatrix} = T$$

- Since $U = U_1 U_2 U_3$ is itself a unitary matrix, we have: $U^{-1} A U = T$.

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Examples

3. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. Find a unitary matrix U such that $U^{-1}AU = T$ where T is an upper triangular matrix with eigenvalues of A on the diagonal.

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- We will first find an eigenvector and eigenvalue for the matrix A. Then we will use the idea from the proof of the Schur Lemma to figure this out.

$$0 = \det(A - \lambda I) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

$$(A - I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 2-1 & -1 & 0 \\ 1-1 & 0 & 0 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

- We have 1 pivot and 1 free variable. (Only one eigenvector!!)

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- We make this unitary and require:

$$\vec{u}_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

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- Now, we need to find the second orthonormal vector that will complete our unitary matrix! Let's seek another vector, $\vec{x}_2$ such that $\vec{x}_2^T\vec{u}_1 = 0$.

$$0 = \vec{x}_2^T\vec{u}_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}(x_1 + x_2).$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \end{array}\right]$$

- This matrix has 1 pivot and 1 free variable and we have the eigenvector $\vec{x}_2$ (and normalized eigenvector $\vec{u}_2$):

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \implies \vec{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

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- Thus we have:

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

- We remember that $U^{-1} = U^T$.
- We then find:

$$\begin{aligned} U^{-1}AU &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -3/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

- Thus, we have shown that A is similar to an upper triangular matrix with the eigenvalues on the diagonal.

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The Spectral Theorem

Every real symmetric A can be diagonalized by an orthogonal matrix Q.
Every Hermitian matrix A can be diagonalized by a unitary matrix U.

$$\text{Real: } Q^{-1}AQ = \Lambda \text{ or } A = Q\Lambda Q^T$$

$$\text{Complex: } U^{-1}AU = \Lambda \text{ or } A = U\Lambda U^H.$$

The columns of Q (or U) contain orthonormal eigenvectors of A.

- If the matrix A is real and symmetric, the eigenvalues and eigenvectors are **real** at every step!

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Proof

Use Schur's lemma to prove the Spectral Theorem: *Every Hermitian matrix A can be diagonalized by a unitary U:*

$$U^H A U = U^{-1} A U = \Lambda \quad \text{or} \quad A = U \Lambda U^H.$$

- Let A be a Hermitian matrix. Then $A^H = A$.
- Schur's Lemma, we have a unitary U such that: $U^{-1}AU = T$.
- Because U is unitary we have: $U^{-1} = U^H$.
- Let's take the conjugate transpose of this expression:

$$T^H = (U^H A U)^H = U^H A^H U = U^H A U = T \implies T^H = T$$

- Because T an upper triangular matrix, it's conjugate transpose T^H is lower triangular. But the only way that an upper and lower triangular matrix can be equal is if it is a diagonal matrix.

1. What is the Spectral theorem for symmetric matrices?
2. Suppose that $A = U\Lambda U^{-1} = U\Lambda U^H$, where Λ is diagonal and U is unitary. Show that A is normal. That is $AA^H = A^H A$.
3. What about the diagonalizability of a normal matrix?
4. Construct a diagonalizable matrix A which is NOT normal.
5. The eigenvalues of $A_{7\times 7}$ are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 2$, and $\lambda_5 = \lambda_6 = \lambda_7 = 3$. There are two linearly independent eigenvectors corresponding to the eigenvalue 2 and only one corresponding to 3. What is the Jordan form of A?