# Chapter 5.5-5.6: Review

## Symmetric Matrix

A real-vaued $n \times n$ matrix $A$ is called **symmetric** if:

$$A^T = A.$$

## Hermitian Matrix

A $n \times n$ matrix $A$ with (possibly) complex values is called **Hermitian** if:

$$A^H = \overline{A}^T = A.$$

## Important Properties

- A symmetric (or Hermitian) matrix has **real eigenvalues**.
- Eigenvectors from different eigenvalues for a symmetric (or Hermitian) matrix are **orthogonal**.

# Chapter 5.5-5.6: Review

## Orthogonal Matrix

A real-valued $n \times n$ matrix $Q$ is called **orthogonal** if its columns are orthonormal.

$$Q^{-1} = Q^T.$$

## Unitary Matrix

A $n \times n$ matrix $U$ with complex values is called **unitary** if its columns are orthonormal.

$$U^{-1} = U^H.$$

## Cool Properties

1. Orthogonal and Unitary matrices preserve lengths and inner products.
2. All eigenvalues $\lambda$ of an Orthogonal or Unitary matrix have $|\lambda| = 1$.
3. Eigenvectors corresponding to different eigenvalues are orthogonal.

*For Proofs, see Week 13 Thursday Notes or Chapter 5.5/5.6.*

# Chapter 5.5-5.6: Review

## Similar Matrices

Matrices $A$ and $B$ are **similar matrices** if there exists an invertible matrix $M$ such that:

$$B = M^{-1}AM.$$

## Properties

1. Similar matrices have the same eigenvalues.
2. The eigenvectors of similar matrices are related by the matrix $M$. For example, if $\vec{x}$ is an eigenvalue of $B$ then $M\vec{x}$ is an eigenvector of $A$ with the same eigenvalue:
$$B = M^{-1}AM \text{ and } B\vec{x} = \lambda\vec{x} \implies A(M\vec{x}) = \lambda(M\vec{x}).$$
3. A diagonalizable matrix is one that is **similar** to a diagonal matrix: $A = S\Lambda S^{-1}$.
4. 2 similar matrices are either BOTH diagonalizable (i.e., full set of linearly independent eigenvectors) or NEITHER are diagonalizable.

# Chapter 5.5-5.6: Schur Lemma

We have learned several different matrix factorization methods to far:

$$A = LU, A = LDU, A = QR.$$

Now, we are going to learn one more.

**Schur Lemma**

An $n \times n$ matrix $A$ is similar to a triangular matrix $T$. That is, there is a unitary matrix $U$ such that:

$$U^{-1}AU = T.$$

Why does Schur's Lemma work? Let's take a little bit of a look.

# Chapter 5.5-5.6: Schur Lemma Proof

For simplicity - assume $A$ is a 4 by 4 matrix. We will show there is unitary matrix $U$ such that: $U^{-1}AU = T$ where $T$ is an upper triangular matrix.

- $A$ has at least 1 eigenvalue $\lambda_1$ and at least 1 eigenvector $\vec{u}_1$.
- Let's assume $\vec{u}_1$ has length 1, and use it to create a unitary matrix $U_1$ with $\vec{u}_1$ as it's first column and any other vectors as the latter columns in a way that makes it unitary.
- To do this, we just make $\vec{u}_1$ can do this by finding an orthogonal basis for $V^{\perp}$ if $V = \text{span}\{\vec{u}_1\}$. Let's call these vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Then we have:
$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{a} & \vec{b} & \vec{c} \end{bmatrix}.$$
- Let's notice that:
$$AU_1 = A\begin{bmatrix} \vec{u}_1 & \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \lambda\vec{u}_1 & A\vec{a} & A\vec{b} & A\vec{c} \end{bmatrix} = U_1 \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$
- Notice that we use $*$'s because we don't really need to know what these values are.

# Chapter 5.5-5.6: Schur Lemma Proof

- We will re-write this slightly as

T

$$U_1^{-1}AU_1 = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

- Consider the red $3 \times 3$ submatrix of $*$'s. This sub matrix has at least 1 eigenvalue $\lambda_2$ and 1 eigenvector $\vec{u}_2 = \begin{bmatrix} x & y & z \end{bmatrix}^T$. Let's find two other orthonormal vectors to complete $\mathbb{R}^3$ and define a new unitary matrix $U_2$:

$$U_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & * & * \\ 0 & y & * & * \\ 0 & z & * & * \end{bmatrix} \Longrightarrow U_2^{-1}(U_1^{-1}AU_1)U_2 = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

$\vec{u}_2$ unitary

# Chapter 5.5-5.6: Schur Lemma Proof

- Now we work with the matrix of blue $*$'s. This 2 by 2 submatrix has at least 1 eigenvalue $\lambda_3$ and 1 eigenvector. Let's call is $\vec{u}_3 = \begin{bmatrix} x & y \end{bmatrix}^T$ and let's use it to create a unitary matrix $U_3$:

$$U_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & * \\ 0 & 0 & y & * \end{bmatrix}.$$

2x2 unitary matrix

- Then combining together we have:

$$U_3^{-1} \left( U_2^{-1} U_1^{-1} A U_1 U_2 \right) U_3 = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & \lambda_3 & * \\ 0 & 0 & 0 & * \end{bmatrix} = T$$

- Since $U = U_1 U_2 U_3$ is itself a unitary matrix, we have: $\boxed{U^{-1} A U = T}$.

## Examples

3. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. Find a unitary matrix $U$ such that $U^{-1}AU = T$ where $T$ is an upper triangular matrix.

Similar

$0 = \det(A - \lambda I)$

$= \lambda^2 - tr(A)\lambda + \det(A)$

$tr(A) = 2 \rightarrow$ sum of diagonal

$\det(A) = 1 \rightarrow$

# Chapter 5.5-5.6: Schur Lemma Application

- We will first find an eigenvector and eigenvalue for the matrix $A$. Then we will use the idea from the proof of the Schur Lemma to figure this out.

$$0 = \det(A - \lambda I) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

$$(A - I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 2-1 & -1 & 0 \\ 1-1 & 0 & 0 \end{array}\right] \implies \boxed{\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]}.$$

- We have 1 pivot and 1 free variable. (Only one eigenvector!!)

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$x_2 = t$

$x_1 = t$

- We make this unitary and require:

$\vec{a}, \vec{b}$

$\vec{q}_1, \vec{q}_2$

$$\boxed{\vec{u}_1 = \frac{\vec{x}_1}{||\vec{x}_1||} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.}$$

$U = [\vec{u}_1 \ \vec{u}_2]$

- Now, we need to find the second orthonormal vector that will complete our unitary matrix! Let's seek another vector, $\vec{x}_2$ such that $\vec{x}_2^T \vec{u}_1 = 0$.

$\vec{x}_2 = \begin{bmatrix} x \\ y \end{bmatrix}$

Find $\vec{x}_2$ so that $\vec{x}_2^T \vec{u}_1 = 0$

$$0 = \vec{x}_2^T \vec{u}_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}(x_1 + x_2). = 0, \sqrt{2}$$

$$\begin{bmatrix} 1 & 1 & | & 0 \end{bmatrix}$$

- This matrix has 1 pivot and 1 free variable and we have the eigenvector $\vec{x}_2$ (and normalized eigenvector $\vec{u}_2$):

the other vector

$$\boxed{\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}} \implies \boxed{\vec{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}.$$

# Chapter 5.5-5.6: Schur Lemma Application

- Thus we have:

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

- We remember that $U^{-1} = U^T$.
- We then find:

$$\begin{aligned} U^{-1}AU &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \overbrace{\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}^{U} \\ &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -3/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

A is similar to a upper triangular matrix

T upper triangular matrix

- Thus, we have shown that $A$ is similar to an upper triangular matrix with the eigenvalues on the diagonal.

# Chapter 5.6: The Spectral Theorem

### The Spectral Theorem

Every real symmetric $A$ can be diagonalized by an orthogonal matrix $Q$.
Every Hermitian matrix $A$ can be diagonalized by a unitary matrix $U$.

$$\text{Real: } Q^{-1}AQ = \Lambda \text{ or } A = Q\Lambda Q^T$$

$$\text{Complex: } U^{-1}AU = \Lambda \text{ or } A = U\Lambda U^H.$$

The columns of $Q$ (or $U$) contain orthonormal eigenvectors of $A$.

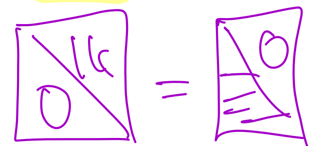
- If the matrix $A$ is real and symmetric, the eigenvalues and eigenvectors are **real** at every step!

## Proof

Use Schur's lemma to prove the Spectral Theorem: *Every Hermitian matrix $A$ can be diagonalized by a unitary $U$:*

$$U^H AU = U^{-1}AU = \Lambda \quad \text{or} \quad A = U\Lambda U^H.$$

- Let $A$ be a Hermitian matrix. Then $A^H = A$.
- Schur's Lemma, we have a unitary $U$ such that: $U^{-1}AU = T$.
  $U^H$
- Because $U$ is unitary we have: $U^{-1} = U^H$.
- Let's take the conjugate transpose of this expression:



$$T^H = (U^H AU)^H = U^H A^H U = U^H AU = T \Longrightarrow T^H = T$$

- Because $T$ an upper triangular matrix, it's conjugate transpose $T^H$ is lower triangular. But the only way that an upper and lower triangular matrix can be equal is if it is a diagonal matrix.

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

## Course Goals

*After studying section 6.1-6.2: Positive Definite Matrices, you should*

1. *Understand the concept of positive definite matrices and some of their applications.*

## Course Outcomes

*To manifest that you have reached the above course goals, you should be able to*

1. *Determine whether a given matrix is positive/negative definite or semidefinite,by using any of the five equivalent necessary and sufficient conditions for (semi)definiteness.*
2. *Use definite matrices to determine whether a critical point of a multi–variable function is a minimum or maximum.*

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

- We will start thinking about the positivity (or negativity) of the eigenvalues..
- We will learn a test that will guarantee all eigenvalues are positive that brings together: *pivots*, *determinants* and *eigenvalues*.
- Signs of eigenvalues are important!!

$$\frac{d\vec{u}}{dt} = A\vec{u} \implies \vec{u}(t) = e^{At}\vec{u}(0) = Se^{\Lambda t}S^{-1}\vec{u}(0).$$

For example, if $A$ is a 2 by 2 matrix with eigenvalues $\lambda_1, \lambda_2$ we have:

$$\vec{u}(t) = S\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} S^{-1}\vec{u}(0)$$

  - If $\lambda_i < 0$ for $i = 1, 2$ then $\vec{u}(t) \to 0$ as $t \to \infty$.
  - If $\lambda_i > 0$ for $i = 1, 2$ then $\vec{u}(t) \to \infty$ as $t \to \infty$.
  - If $\lambda_1 < 0$ and $\lambda_2 > 0$ then, the system behavior becomes dependent on the initial condition.

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

## Examples

1. (Calculus review) Consider the two-variable function

$$f(x, y) = -1 + 4(e^x - x) - 5x \sin y + 6y^2.$$

   a. Check that $(0, 0)$ is a critical point.
   b. What are the terms, of order 0, 1, and 2, in the Taylor expansion of $f$ at $(0, 0)$?
   c. Determine whether $(0, 0)$ is a maximum, minimum or neither.

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

(a) Check that $(0,0)$ is a critical point.

$$f(x,y) = -1 + 4(e^x - x) - 5x\sin y + 6y^2$$

$f_x(x,y) = 4(e^x - 1) - 5\sin y$ and $f_y(x,y) = -5x\cos(y) + 12y$

$f_x(0,0) = 4(e^0 - 1) - 5\sin(0) = 0$ and $f_y(0,0) = -5(0)\cos(0) + 12(0) = 0$

Thus $(0,0)$ is a critical point. It could be a maximum, a minimum or a saddle point.

(b) What are the terms, of order 0, 1, and 2, in the Taylor expansion of $f$ at $(0,0)$? Recall the Taylor Series in 2D centered at $(a,b)$ is:

$$\begin{aligned} T(x,y) &= f(a,b) \\ &+ (x-a)f_x(a,b) + (y-b)f_y(a,b) \\ &+ \frac{1}{2!}\left((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)\right. \\ &+ \left.(y-b)^2 f_{yy}(a,b)\right) + \text{higher order terms} \end{aligned}$$

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

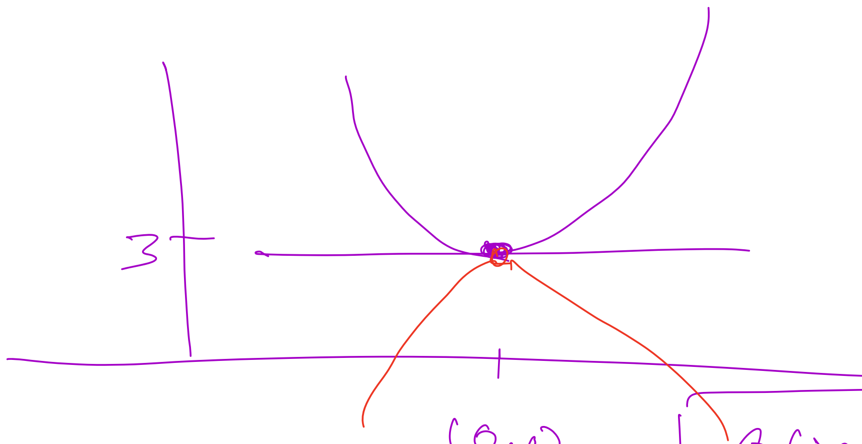
$$f(x, y) = -1 + 4(e^x - x) - 5x \sin y + 6y^2$$

- Order 0: $f(0,0) = -1 + 4(e^0 - 0) - 5(0)\sin(0) + 6(0)^2 = -1 + 4 = 3.$
- Order 1: The terms are 0 because we showed $(0,0)$ is a critical point. $f_x(0,0) = 0$ and $f_y(0,0) = 0$
- Order 2:
  - $f_{xx}(x, y) = \frac{\partial}{\partial x}(4(e^x - 1) - 5\sin y) = 4e^x$. Then $f_{xx}(0,0) = 4$.
  - $f_{yx}(x, y) = \frac{\partial}{\partial x}(-5x\cos(y) + 12y) = -5\cos(y)$. Then $f_{yx}(0,0) = -5$.
  - $f_{yy}(y, y) = \frac{\partial}{\partial y}(-5x\cos(y) + 12y) = 5x\sin(y) + 12$. Then $f_{yy}(0,0) = 12$.

Thus the Taylor Series Expansion up to the quadratic term is:

$$T(x, y) = f(0,0) + \frac{1}{2!}(x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)).$$

$$\boxed{T(x, y) = 3 + \frac{1}{2}\left(4x^2 - 10xy + 12y^2\right) = 3 + 2x^2 - 5xy + 6y^2.}$$

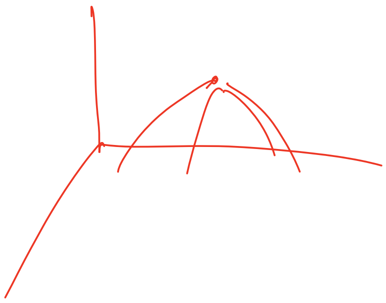



(0,0)

$T(x,y) = 3 + \boxed{Q(x,y)}$

$Q(x,y) > 0$
the point
$f(0,0) = 3$
minimum

($Q(x,y) < 0$
maximum)



# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

- Determine whether $(0,0)$ is a maximum, minimum or neither.
  Near $(0,0)$ the function $f(x,y)$ behaves according to its Taylor series:

$$T(x,y) = 3 + \frac{1}{2}(4x^2 - 10xy + 12y^2).$$

So, if the quadratic part is

- always positive, then $f(x,y)$ has a minima at $T(0,0) = f(0,0) = 3$
- always negative, then $f(x,y)$ has a maxima at $T(0,0) = f(0,0) = 3$
- neither, then $f(x,y)$ has a saddle point at $T(0,0) = f(0,0) = 3$.

From Calculus, we remember the second derivative test for a minima:

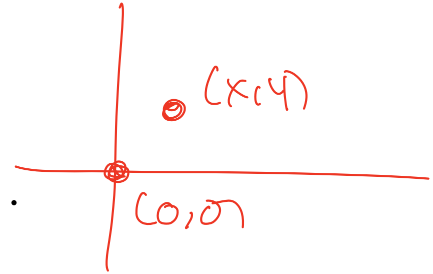
$$f_{xx}(0,0) > 0 \text{ and } f_{xx}(0,0)f_{yy}(0,0) > f_{xy}(0,0)^2$$

$$f_{xx}(0,0) = 4 > 0, f_{xx}(0,0)f_{yy}(0,0) = 4(12) > (-5)^2 = f_{xy}(0,0)^2.$$

Thus, the critical point $(0,0)$ is a true minima of the function $f(x,y)$.
Here we will make this test better by linking it to linear algebra.

For the last problem, write the quadratic part as:

$$\vec{x}^T A \vec{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$




This let's us write the Taylor Series centered at (for simplicity 0) as follows:

$f_{xy} = f_{yx}$

$$\begin{matrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{matrix}$$

$$\begin{aligned} T(0,0) &= f(0,0) \\ &+ x f_x(0,0) + y f_y(0,0) \\ &+ \frac{1}{2!} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \text{higher order terms} \end{aligned}$$

This expansion with a matrix also holds for higher dimensions and this matrix of second derivatives is called the Hessian. Here is our general expression for the second order approximation for the Taylor Series:

$$T(\vec{x}) = f(\vec{0}) + \vec{x}^T \nabla f + \boxed{\vec{x}^T H \vec{x}} + (H.O.T.)$$

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

## Positive Definite

An $n \times n$ matrix $A$ is positive definite if for any vector $\vec{x}$ we have:

$$\boxed{\vec{x}^T A \vec{x} > 0.}$$

Recall,

$$T(\vec{x}) = f(\vec{0}) + \overset{0}{\cancel{\vec{x}^T \nabla f}} + \boxed{\vec{x}^T H \vec{x}}$$

where $H$ is the Hessian, the symmetric matrix of second partial derivatives for a scalar valued function taking inputs from $\vec{x} \in \mathbb{R}^n$.

## Positive Definite Hessian Implies Minima

The second-derivative test involves determining if the Hessian matrix, $H$ is positive definite.

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

## Positive Definite Matrix

Each of the following is a necessary and sufficient (i.e. if and only if) condition for a real symmetric matrix $A$ to be **positive definite**.

1. (i) $\vec{x}^T A\vec{x} > 0$ for all non-zero real vectors $\vec{x}$.
2. (ii) All the eigenvalues of $A$ satisfy $\lambda_i > 0$.
3. (iii) All the upper left submatrices $A_k$ have positive determinants.
4. (iv) All the pivots (without row exchanges) satisfy $d_k > 0$.
5. (v) There is a matrix $R$ with independent columns such that $A = R^T R$.

We note that condition (i) is just the definition we have of positive definite matrices.
We will prove (ii) implies (i). (The other conditions still cool, but less relevant for the moment.)

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

- Suppose that $A$ is a real-symmetric matrix with only positive eigenvalues. We will show that $\vec{x}^T A\vec{x} > 0$ for all choices of $\vec{x}$.
- Since $A$ is symmetric, we know that we have a full set of orthonormal eigenvectors (Spectral Theorem).
- Let $\vec{x}$ be any vector in, we know it can be expressed uniquely as a linear combination of eigenvectors $\vec{x}_i$:
$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n.$$
- Then we have:

$$A\vec{x} = c_1 A\vec{x}_1 + c_2 A\vec{x}_2 + \cdots + c_n A\vec{x}_n = c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 + \cdots + c_n\lambda_n\vec{x}_n.$$

- To calculate $\vec{x}^T A\vec{x}$ we use $\vec{x}_i^T\vec{x}_j = 0$ if $i \neq j$ and $1$ if $i = j$.

$$\vec{x}^T A\vec{x} = \left(c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n\right)\left(c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 + \cdots + c_n\lambda_n\vec{x}_n\right).$$

$$\implies \vec{x}^T A\vec{x} = c_1^2\lambda_1 + c_2^2\lambda_2 + \cdots + c_n^2\lambda_n > 0$$

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

## Examples

2. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. Check that $A$ is positive definite by
   - (a) finding all its eigenvalues;
   - (b) calculating the determinants of all upper left submatrices;
   - (c) finding all pivots;
   - (d) writing $A = R^T R$ where $R$ is some matrix with independent columns.

(a) Finding all its eigenvalues:

$$0 = \det(A - \lambda I) = \left|\begin{matrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{matrix}\right| \begin{matrix} 2-\lambda & -1 \\ -1 & 2-\lambda \\ 0 & -1 \end{matrix}\right|.$$

$$= (2-\lambda)^3 + 0 + 0 - (2-\lambda) - (2-\lambda) = (2-\lambda)\left((2-\lambda)^2 - 2\right)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 2)$$

Thus the eigenvalues are:

$$\lambda = \{2, 2-\sqrt{2}, 2+\sqrt{2}\}$$

Since all eigenvalues are positive, we know that $A$ is positive definite.

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

(b) calculating the determinants of all upper left submatrices;

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- $A_1 = \begin{bmatrix} 2 \end{bmatrix}, \det(A_1) = 2 > 0$
- $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \det(A_2) = 4 - 1 = 3 > 0$
- $A_3 = A.$

$$\det(A) = \left| \begin{array}{ccc|cc} 2 & -1 & 0 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 \\ 0 & -1 & 2 & 0 & -1 \end{array} \right| = 2^3 - 2 - 2 = 8 - 4 = 4 > 0.$$

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

(b) finding all pivots;

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1/2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + (2/3)R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

Because $A$ is symmetric, $A = LU = LDL^T$

$$A = LDL^T = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}}_{L^T}$$

The pivots are the values on the diagonal of $U$ or in the diagonal matrix $D$ and they are:

$$2, 3/2, 4/3$$

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

- (d) writing $A = R^T R$ where $R$ is some matrix with independent columns. There are actually two choice depending on which decomposition we want to choose. To address this part we need our three matrices from above: $\boxed{A = LDL^T} = L\sqrt{D}\sqrt{D}L^T$

$$\sqrt{D} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3}/\sqrt{2} & 0 \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}.$$

$$\boxed{R = \sqrt{D}L^T} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3}/\sqrt{2} & 0 \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T = \left(\sqrt{D}L^T\right)^T = L\sqrt{D}^T = L\sqrt{D}.$$

Thus, $A = R^T R$. This is called the **Cholesky decomposition** and exists only because the diagonal matrix has positive values.

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

(d) writing $A = R^T R$ where $R$ is some matrix with independent columns. A second choice would be to use the fact that we can orthogonally diagonalize the matrix $A$:

$$A = Q\Lambda Q^T = Q\sqrt{\Lambda}\sqrt{\Lambda Q^T}$$

Thus as before, $A = R^T R$.

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

Positive definite matrices are important for optimization, and lots of other things, but they are not the only interesting case we have to consider.

## Positive Semidefinite

Each of the following tests is a necessary and sufficient condition for a symmetric matrix $A$ to be positive semidefinite:

1. $\vec{x}^T A\vec{x} \geq 0$ for all vectors $\vec{x}$ (Defn of positive **semidefinite**)
2. All the eigenvalues of $A$ satisfy $\lambda_i \geq 0$
3. No principle submatrices have negative determinants.
4. No pivots are negative.
5. There is a matrix $R$, possibly with dependent columns, such that $A = R^T R$.

Note you can also be **negative definite** (all $\lambda_i < 0$), **negative semi definite** (all $\lambda_i \leq 0$) or **indefinite**.

# Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

## Examples

3. Decide whether $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is positive definite, negative definite, semidefinite, or indefinite.