Homework Assignment #3

Remember, this Homework Assignment is **not collected or graded**! But you are advised to do it anyway because the problems for Homework Quiz #3 will be heavily based on these problems!

1. Recall that an $n \times n$ matrix A is **symmetric** if $A^T = A$.
   Let R be an $m \times n$ matrix, prove that RR^T and $R^T R$ are both symmetric matrices.

Solution: To prove these properties, we need the property:

$$(AB)^T = B^T A^T.$$

Let's first consider the $m \times m$ matrix RR^T. In order to show it's symmetric, we must show:

$$(RR^T)^T = RR^T.$$

$$(RR^T)^T = R(^T)^T R^T = RR^T.$$

Similarly, for the $n \times n$ matrix $R^T R$. In order to show this is symmetric, we must show:

$$(R^T R)^T = R^T (R^T)^T = R^T R.$$

2. We learned that $n \times n$ matrices A can be factored into LDU where L is a lower triangular matrix, D is a diagonal matrix and U is an upper triangular matrix.
   But if A is symmetric it turns out that this the same LDU factorization process produces an upper triangular matrix U that is actually the transpose of the lower triangular matrix L. As such, we write the factorization as:
   $$A = LDL^T.$$
   Using elementary row matrices, determine the symmetric LDL^T factorization of:
   $$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix}.$$

Solution: Let's start off by performing the first two row operations:

$$E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

We have:

$$E_{13}E_{12}A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix}$$

We will now carry out the last row operation:

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \implies E_{23}E_{13}E_{12}A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Let's first operate on the right hand side. We will factor out a 3 from rows 2 and 3.

$$E_{23}E_{13}E_{12}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's call the diagonal matrix D and the upper triangular matrix U. Not returning to the LDU factorization, we invert the elementary row matrices multiplying A to obtain L

$$A = (E_{23}E_{13}E_{12})^{-1} DU.$$

We can either directly multiply for ourselves, or remember that the lower triangular matrix L has 1's on the diagonal and the multipliers on the off diagonal:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}.$$

We notice that our matrix $U = L^T$. As such, we have:

$$A = LDL^T.$$

3. Invert these matrices A by the Gauss-Jordan method:

(a) $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: We will begin with an augmented matrix of the form: $[A|I]$ and perform row operations:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

We have arrived at the identity on the left hand side which means:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Solution: As before, we will start from an augmented matrix with $[A|I]$ and perform row operations until we make the left hand side equation to the identity matrix

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + (1/2)R_1} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -(5/2) & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - (2/5)R_2}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -5/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 12/5 & -1/5 & -2/5 & 1 \end{array}\right] \xrightarrow{R_3 \to (5/12)R_3} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -5/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & -1/12 & -1/6 & 5/12 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -5/2 & 0 & 5/12 & 5/6 & 5/12 \\ 0 & 0 & 1 & -1/12 & -1/6 & 5/12 \end{array}\right]$$

$$\xrightarrow{R_2 \to (-2/5)R_2} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & -1/3 & -1/6 \\ 0 & 0 & 1 & -1/12 & -1/6 & 5/12 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 5/6 & -1/3 & -1/6 \\ 0 & 1 & 0 & -1/6 & -1/3 & -1/6 \\ 0 & 0 & 1 & -1/12 & -1/6 & 5/12 \end{array}\right]$$

$$\xrightarrow{R_1 \to (1/2)R_1} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 5/12 & -1/6 & -1/12 \\ 0 & 1 & 0 & -1/6 & -1/3 & -1/6 \\ 0 & 0 & 1 & -1/12 & -1/6 & 5/12 \end{array}\right]$$

Since the left hand side is now the identity, we have:

$$A^{-1} = \begin{bmatrix} 5/12 & -1/6 & -1/12 \\ -1/6 & -1/3 & -1/6 \\ -1/12 & -1/6 & 5/12 \end{bmatrix}.$$

(c) $A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution: In this case, we see that row swaps are going to be helpful as we perform the Gauss-Jordan Algorithm.

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Thus, we arrive at

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. The matrix:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

has a special property. When you multiply A by itself you get the matrix of all 0's: $A^2 = 0$.

Is it possible for a non-zero **symmetric** matrix B to have the second property $B^2 = 0$? Prove your answer. (Hint: You might need to remember what $i = \sqrt{-1}$ is.)

Solution: We will start by writing the general form for an 2 by 2 symmetric matrix and squaring it:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & b(a+c) \\ b(a+c) & b^2 + d^2 \end{bmatrix}.$$

Since each of these entries must be 0, we have:

$$a^2 + b^2 = 0, b(a+c) = 0, \text{ and } b^2 + c^2 = 0.$$

From the first equation and last equations we have:

$$a^2 = -b^2 \implies -a^2 + c^2 = 0 \implies a^2 = c^2.$$

Intriguingly, we are left with some interesting behavior. First, notice that if $b = 0$, the middle condition is automatically satisfied but we also have $a^2 = 0$ and $c^2 = 0$ which implies $a = b = c = 0$ and thus we have the zero-matrix.

Let's now restrict $b \neq 0$. We have $a^2 = -b^2$ which can only occur if we allow imaginary numbers. Let's let $b = \beta i$ where β is any non-zero real number. Then, we could satisfy all conditions by allowing:

$$a = \beta, b = \beta i \text{ and } d = -\beta.$$

As such, we end up with matrices of the form:

$$\begin{bmatrix} \beta & i\beta \\ i\beta & -\beta \end{bmatrix}.$$

As such, we have found symmetric matrices which satisfy this intriguing property but we had to remind ourselves about the existence of complex numbers. Indeed, both symmetric matrices and complex numbers will continue to show up throughout the course.

5. A matrix A is said to be **skew symmetric** if $A^T = -A$. Show that if a matrix A is skew-symmetric then all it's diagonal entries must be 0.

Solution: The entries of a matrix A are denoted $a_{i,j}$ where i indicates the row and j indicates the column. If a matrix A is skew-symmetric, then $A^T = -A$. We know that the i, j entry in A^T is the j, i entry in A. As such,

$$A^T = -A \implies a_{j,i} = -a_{i,j}.$$

However, on the diagonal, when $i = j$ this condition implies:

$$a_{ii} = -a_{ii} \implies 2a_{ii} = 0 \implies a_{ii} = 0.$$

As such, all entries on the diagonal of a skew symmetric matrix must be 0.