Discussion Section: Week #12
Due: By 11:59pm the day of your Discussion Section

Instructions:

You are allowed to work alone or in small groups. But if you work in a group, the work you turn in should be your own. Each student will submit a physical write-up as a PDF document to CatCourses **by 11:59 pm of your discussion section day**.

Discussion assignments will be graded partially for completeness but a subset of the problems will be graded for correctness. Providing the correct answer, without justification, is not considered complete. For credit you **must** either show steps (if it's a calculation problem) or explain/justify your reasoning (if it's a short answer problem).

Problem Set:

1. Find the determinant of

   (a) a rank one matrix

$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}.$$

   (b) the upper triangular matrix

$$U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Solution:

(a) Since each row of the matrix A is a scalar multiple of $\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$, the determinant of A, i.e., $|A| = 0$.

(b) The determinant of an upper triangular matrix is the product of the diagonal entries, i.e., $|U| = 4(1)(2)(2) = 16$.

2. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

   Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1 \lambda_2 \lambda_3$ equals the determinant.

Solution: The characteristic equation is

$$(3-\lambda)(1-\lambda)(0-\lambda)=0$$

So, $\lambda = 3,\ 1,\ 0.$
For $\lambda = 0,$

$$(A-0I)v_1 = 0$$
$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the first two columns of $A-0I$ are the pivot columns, v_{13} is a free variable. Let $v_{13}=t$. Then, we have

$$\begin{aligned} v_{12}+2v_{13}=0 &\implies v_{12}=-2t \\ 3v_{11}+4v_{12}+2v_{13}=0 &\implies v_{11}=\frac{1}{3}(-4(-2t)-2t) \\ &\implies v_{11}=2t \end{aligned}$$

Thus,

$$\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix},$$

so the eigenvector associated with the eigenvalue $\lambda = 0$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

For $\lambda = 1,$

$$(A-1I)v_2 = 0$$
$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the first and the third column of $A-1I$ are the pivot columns, v_{22} is a free variable. Let $v_{22}=t$. Then, we have

$$\begin{aligned} -v_{23}=0 &\implies v_{23}=0 \\ 2v_{21}+4v_{22}+2v_{23}=0 &\implies v_{21}=\frac{1}{2}(-4t-2(0)) \\ &\implies v_{21}=-2t \end{aligned}$$

Thus,

$$\begin{bmatrix} v_{21} \\ v_{12} \\ v_{23} \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},$$

so the eigenvector associated with the eigenvalue $\lambda = 1$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = 3$,

$$\begin{gathered} (A - 3I)v_3 = 0 \\ \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{gathered}$$

Since the second and the third column of $A - 3I$ are the pivot columns, v_{31} is a free variable. Let $v_{31} = t$. Then, we have

$$\begin{aligned} -3v_{33} = 0 &\implies v_{33} = 0 \\ -2v_{32} + 2v_{33} = 0 &\implies v_{32} = 0 \end{aligned}$$

Thus,

$$\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

so the eigenvector associated with the eigenvalue $\lambda = 3$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Finally, we confirm that

$$\begin{aligned} tr(A) = 3 + 1 + 0 &= \lambda_3 + \lambda_2 + \lambda_1 \\ &= \lambda_1 + \lambda_2 + \lambda_3 \end{aligned}$$

and

$$\begin{aligned} |A| = 3(1)(0) &= \lambda_3 \lambda_2 \lambda_1 \\ &= \lambda_1 \lambda_2 \lambda_3. \end{aligned}$$