Discussion Section: Week #5
Due: By 11:59pm the day of your Discussion Section

Instructions:

You are allowed to work alone or in small groups. But if you work in a group, the work you turn in should be your own. Each student will submit a physical write-up as a PDF document to CatCourses **by 11:59 pm of your discussion section day**.

Discussion assignments will be graded partially for completeness but a subset of the problems will be graded for correctness. Providing the correct answer, without justification, is not considered complete. For credit you **must** either show steps (if it's a calculation problem) or explain/justify your reasoning (if it's a short answer problem).

Problem Set:

1. Let S be the set of all f in $C^2[a,b]$ such that

$$f''(x) + f(x) = 0$$

for all x in $[a,b]$.
Show that S is a subspace of $C^2[a,b]$.

Solution: The set S is not empty since $f(x) = 0$ is in the set.

(a) Let $a \in \mathbb{R}$ and $f \in S$. Then

$$\begin{aligned}(af)''(x) + (af)(x) &= af''(x) + af(x) \\ &= a(f''(x) + f(x)) \\ &= a(0) \\ &= 0.\end{aligned}$$

So, $af \in S$

(b) Let $f \in S$ and $g \in S$. Then

$$\begin{aligned}(f+g)''(x) + (f+g)'(x) &= f''(x) + g''(x) + f(x) + g(x) \\ &= (f''(x) + f(x)) + (g''(x) + g(x)) \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

So, $f + g \in S$

Since af and $f + G \in S$, S is a subspace of $C^2[a,b]$.

2. (a) Find the special solutions to $Ux = 0$. Reduce U to R and repeat:

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) If the right-hand side is change from $(0, 0, 0)$ to $(a, b, 0)$, what are all solutions?

Solution:

(a)

$$[U|0] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The second and the fourth columns of U are non-pivot columns, so x_2 and x_4 are free variables. Let $x_2 = s$ and $x_4 = t$. Then, solving the second equation gives us

$$x_3 = -2x_4 = -2t;$$

Solving the first equation gives us

$$x_1 = -2x_2 - 3x_3 - 4x_4 = -2s - 3(-2t) - 4t = -2s + 2t.$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + 2t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Now notice that to obtain the reduced system from $Ux = 0$, we only need to subtract thrice of row 2 from row 1.

$$[U|0] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1^* = R_1 - 3R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [R|0]$$

Since the second and the fourth column are non-pivot columns, x_2 and x_4 are free variables. Let $x_2 = s$ and $x_4 = t$. Then, solving the second equation gives us

$$x_3 = -2x_4 = -2t;$$

Solving the first equation gives us

$$x_1 = -2x_2 + 2x_4 = -2s + 2t.$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s+2t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

(b)

$$[U|0] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & a \\ 0 & 0 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The second and the fourth columns of U are non-pivot columns, so x_2 and x_4 are free variables. Let $x_2 = s$ and $x_4 = t$. Then, solving the second equation gives us

$$x_3 = b - 2x_4 = b - 2t;$$

Solving the first equation gives us

$$x_1 = a - 2x_2 - 3x_3 - 4x_4 = a - 2s - 3(b-2t) - 4t = -3b - 2s + 2t.$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (a-3b)-2s+2t \\ s \\ b-2t \\ t \end{bmatrix} = \begin{bmatrix} a-3b \\ 0 \\ b \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Now notice that to obtain the reduced system from $Ux = 0$, we only need to subtract thrice of row 2 from row 1.

$$[U|0] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & a \\ 0 & 0 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1^* = R_1 - 3R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & a-3b \\ 0 & 0 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [R|0]$$

Since the second and the fourth column are non-pivot columns, x_2 and x_4 are free variables. Let $x_2 = s$ and $x_4 = t$. Then, solving the second equation gives us

$$x_3 = b - 2x_4 = b - 2t;$$

Solving the first equation gives us

$$x_1 = (a-3b) - 2x_2 + 2x_4 = (a-3b) - 2s + 2t.$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (a-3b)-2s+2t \\ s \\ b-2t \\ t \end{bmatrix} = \begin{bmatrix} a-3b \\ 0 \\ b \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

3. (a) What is the nullspace of A where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Under what condition on b_1, b_2, b_3 is the is the following system solvable:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Solution:

(a) The nullspace of A is the solution set to $Ax = 0$. Since $A = U$ in Problem 1, we found $Ux = 0$ where the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + 2t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

(b) The systeme $A\vec{x} = \vec{b}$ is solvable is $\vec{b}$ is a linear combination of the columns of A. Since the third row of A is zero, we must require $b_3 = 0$. b_1 and b_2 can be any real number.