Homework Assignment #2

Remember, this Homework Assignment is **not collected or graded**! But you are advised to do it anyway because the problems for Homework Quiz #2 will be heavily based on these problems!

1. Determine all values of the constant k for which the following system has (a) no solution, (b) an infinite number of solutions, (c) a unique solution.

$$\begin{aligned} x_1 + 2x_2 \quad - x_3 &= 3 \\ 2x_1 + 5x_2 \quad + x_3 &= 7 \\ x_1 \; + x_2 - k^2 x_3 &= -k. \end{aligned}$$

Solution:

We will use Gaussian elimination the study how k changes the structure of our solutions:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -k^2 & k \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & 1-k^2 & -3+k \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 4-k^2 & -2+k \end{array}\right].$$

The structure of the system rests on the last row: $\left[\begin{array}{ccc|c} 0 & 0 & (4-k^2) & -2+k \end{array}\right]$.

(a) For no solutions we need $(4 - k^2) = 0$ and $-2 + k \neq 0$. This will occur when $k = -2$.

(b) For infinitely many solutions, we need both: $(4 - k^2) = 0$ and $-2 + k = 0$. This will happen when $k = 2$. In this case, x_3 will be a free variable t. We use back-substitution to construct the full solution set:

$$x_2 + 3x_3 = 1 \implies x_2 + 3t = 1 \implies x_2 = 1 - 3t.$$

$$x_1 + 2x_2 - x_3 = 3 \implies x_1 + 2(1 - 3t) - t = 3 \implies x_1 = 1 + 7t$$

Thus we have:

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}.$$

(c) For a unique solution we require that $(4 - k^2) \neq 0$, in this case we have:

$$x_3 = -\frac{(2-k)}{(4-k^2)} \implies -\frac{1}{2+k}$$

$$x_2 + 3x_3 = 1 \implies x_2 - \frac{3}{2+k} = 1 \implies x_2 = 1 + \frac{3}{2+k}.$$

$$x_1 + 2x_2 - x_3 = 3 \implies x_1 + 2\left(1 + \frac{3}{2+k}\right) + \frac{1}{2+k} = 3 \implies x_1 = 1 - \frac{7}{2+k}$$

2. Consider the following linear system $A\vec{x} = \vec{b}$ where,

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix}$$

and $\vec{b} = \begin{bmatrix} b \\ b \\ b \end{bmatrix}$ for a real number b.

(a) Use elementary matrices to reduce the augmented matrix A to row-echelon form.

Solution: We observe the first operations in row reduction, and their corresponding elementary row matrices, are

$$R_2 \to R_2 - R_1 \implies E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$R_3 \to R_3 - (3/2)R_1 \implies E_{3,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix}.$$

We carry out the multiplication and find:

$$E_{3,1}E_{2,1}A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -4 & 0 \\ 0 & -\frac{7}{2} & 0 \end{bmatrix}.$$

Thus, the last elementary row operation, and corresponding matrix, is given by

$$R_3 = R_2 - \frac{7}{8}R_3 \implies E_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7/8 & 1 \end{bmatrix}$$

Carrying out these operations gives us:

$$E_{3,2}E_{3,1}E_{2,1}A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, this question is written a little awkwardly, and asks about the **augmented matrix** A. But the augmented matrix is: $[A|\vec{b}]$. Notice that the same row matrices are used on this matrix as well:

$$E_{3,2}E_{3,1}E_{2,1}[A|\vec{b}] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & b \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -\frac{b}{2} \end{array}\right]$$

(b) Determine the values of b where so that $A\vec{x} = \vec{b}$ has at least 1 solution and solve the linear system.

Solution: We begin with the REF of the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & b \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -\frac{b}{2} \end{array}\right]$$

We see this system will have a solution if and only if the last row has all 0 entries. (In this case it will correspond to having infinitely many solutions!) This will require $-b/2 = 0 \implies b = 0$.

In that case, we see that our system has a free variable: $x_3 = t$. The second row shows that $-4x_2 = 0 \implies x_2 = 0$.
The top row specifies x_1:

$$2x_1 + x_2 + 4x_3 = b \implies 2x_1 + 4t = 0 \implies x_1 = -2t.$$

Thus the solutions are:

$$\vec{x} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

3. Consider the following system:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 5 \end{bmatrix}$$

(a) Use elementary row operations to write the LU factorization for A (or PA if necessary).

Solution: Let's begin by assuming no row swaps are needed, then we will adjust if we need to. The first row operation we need to perform is to replace R_2 with $R_2 - 2R_1$. This corresponds to multiplication as follows:

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

However, we see that the second row does not have a pivot in the second column. But the third row does. This suggests that we should exchange this second row with the third row so we will have the second pivot occurring in the second column. In order to do this we multiply A by the appropriate permutation matrix P as follows:

$$PA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & -2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Now we will carry out matrix multiplications on this new matrix: PA. In this case, the first operation will to replace our new R_3 by $R_3 - 2R_1$. But now we have a slightly different matrix:

$$E(PA) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & -2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Now the pivots are in the right order! And we only have one more operation to go, we will replace R_4 by $R_4 - 2R_3$

$$F(EPA) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

In this case we have: $FEPA = LU \implies PA = E^{-1}F^{-1}U \implies L = E^{-1}F^{-1}$.
As we discussed in class we have:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

(b) Use the LU factorization to solve the system by solving the corresponding two triangular systems.

Solution: Since we did a row exchange, we need to be a teeny careful with our system:

$$A\vec{x} = \vec{b} \implies PA\vec{x} = P\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 5 \end{bmatrix}$$

Thus, since $PA = LU$ we solve two systems:

$$L\vec{c} = P\vec{b} \text{ and } U\vec{x} = \vec{c}.$$

Using L from the previous section, we write the system in augmented matrix form:

$$[L|P\vec{b}] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 2 & 1 & 5 \end{array}\right].$$

Row 1 gives us: $c_1 = 0 \implies c_1 = 0$. Row 2 of L gives us $2c_1 + c_2 = 0 \implies c_2 = 0$. The third row of L gives us $c_3 = 4$ and finally the last row gives us:

$$2c_3 + c_4 = 5 \implies 8 + c_4 = 5 \implies c_4 = -3.$$

Now we need to solve $U\vec{x} = \vec{c}$. We will start by writing this in the augmented matrix form:

$$[U|\vec{c}] = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right].$$

Now performing elimination from the bottom to the top (backwards elimination). We have: $x_4 = 1$ and $x_3 + 2x_4 = 4 \implies x_3 = 2$. The second row implies

$$x_2 + x_4 = 0 \implies x_2 + 1 = 0 \implies x_2 = -1.$$

Finally, the top row:

$$x_1 - x_2 = 0 \implies x_1 + 1 = 0 \implies x_1 = -1.$$

Thus we have:

$$\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \end{bmatrix}.$$

Finally, notice that we solved the system: $PA\vec{x} = P\vec{b}$ our solution also satisfies $A\vec{x} = \vec{b}$. This is because permutation matrices are invertible.

4. A farmer with 1200 acres is planning to plant 3 kinds of crops: corn, soybeans and oats. The cost of each crop is different per acre, corn seed costs \$20 per acre, while soybean seed costs \$50 per acre and oat costs \$15 per acre. The farmer has \$40,000 to spend on seeds and will spend it all.

   (a) Use the information in the problem to formulate two linear equations with three unknowns. Explain, in words, what each equation means and what the unknowns are.

   Solution: Well, word problems are always the hardest problems in a math class. And we finally arrive at our first word problem for linear algebra.

   In the situation the farmer has to decide on how many acres to plant of the three unknowns: corn (x_1), soybeans (x_2) and oats (x_3). The farmer has a total of 1200 acres and if all are plotted we have:

   $$x_1 + x_2 + x_3 = 1200.$$

   But the farmer also has a total of \$40,000 to spend on this planting. The cost of each crop depends on which plant. But if the farmer wants to spend his entire \$40,000 we know:

   $$20x_1 + 50x_2 + 15x_3 = 40000.$$

   (b) Solve the system you wrote, and explain the solution set.

   Solution: Our system of two equations and two unknowns can be written as:

   $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1200 \\ 20 & 50 & 15 & 40000 \end{array}\right].$$

   We will carry out the row operation $R_2 = R_2 - 20R_1$

   $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1200 \\ 0 & 30 & -5 & 16000. \end{array}\right].$$

   We have two pivots, the third column the number of oat acres is free. (Of course practically speaking, we know that the smallest this variable can be is 0 and the biggest it can be is 1200.) For now we will make it $x_3 = t$. In this case we know:

   $$30x_2 - 5t = 16000 \implies x_2 = 1600/3 + (1/6)t.$$

   Finally, we have:

   $$x_1 + x_2 + x_3 = 1200 \implies x_1 + (1600/3 + (1/6)t) + t = 1200 \implies x_1 = 2000/3 - (7/6)t$$

   Thus we have a rather interesting solution for the farmer. Pick a value of t and then

   - Plant $2000/3 - (7/6)t$ acres of corn
   - Plant $1600/3 + (1/6)t$ acres of soy beans
   - Plant t acres of oats

   Thus, the farmer has an infinite number of solutions to choose from when selecting how many acres of each crop to plant.

   Note that the number of acres for each plant must be non-negative, as such this restricts the value of t within a range: $0 \leq t \leq 4000/7$.

(c) We now introduce a third constraint to the farmer's plan, when he sells his crops he can earn \$100 per acre of corn, \$300 per acre of soybean and \$80 per acre of oats. Suppose the farmer is unusually specific in his goals and wants his crops to bring in **exactly** \$230,000. Will he be able to meet this specific goal?

> **Solution:**
> We now have a third equation that must be simultaneously satisfied by our solutions. We have:
>
> $$100x_1 + 300x_2 + 80x_3 = 230000.$$
>
> We can plug in the solution we found in the previous part to solve for t:
>
> $$100(2000/3 - (7/6)t) + 300(1600/3 + (1/6)t) + 80t = 230000$$
>
> $$\frac{40\,(17000 + t)}{3} = 230000$$
>
> Thus we have, $t = 250$. Now we have a single unique solution to give the farmer:
>
> - Plant 375 acres of corn
> - Plant 575 acres of soy beans
> - Plant 250 acres of oats
>
> We are clearly stretching this problem a bit, where the farmer has an exact figure of the earnings they want to bring in. But you can think about this in the last part of the problem.

(d) Explain both geometrically and algebraically how this additional constraint changes the solution set you found in part b.

> **Solution:** There are a few ways to think about this problem.
> First, geometrically our solution in part (b) was a line (technically a line-segment since it is not a solution for all choices of t.) This last constraint is a plane in 3D which intersects this line in a single point.
> Second, algebraically, we added a constraint and had 3 equations and 3 unknowns. With 2 equations and 3 unknowns, there was always the potential for singularity - where there were infinitely many solutions. With equal numbers of equations and unknowns we have the ability to have a unique solution to the problem.

(e) *(Challenge Problem: Optional)* Suppose the farmer, realizing he might not be achieving the full benefit of his land, instead wants to maximize the amount of money he can earn from selling his crops. Determine the maximum amount the farmer can earn, provided he still meets the two original constraints.

> **Solution:** Alrighty, this farmer is smart. Better than satisfying a condition exactly, he should try to maximize his earnings. Let's notice that the last equation gives us a way to write earnings, E, as a function of t, the number of acres of oats the farmer will plant:
>
> $$E(t) = \frac{40\,(17000 + t)}{3}.$$

We can see that this is a line with positive slope in t. This suggests that if we make t as big as possible, this will make the largest earnings.
Since from part (b) we know that $t \leq 4000/7$, this suggests that to maximize earnings the farmer should plant $4000/7 \approx 571$ acres of oats.
If this were something nonlinear, we'd have to break out our derivatives. But fortunately we can save that excitement for another day.

5. The trace of a matrix A, denoted $tr(A)$, is the sum of its entries along the diagonal. For example, if A is the matrix below:

$$A = \begin{bmatrix} 1 & 7 \\ 2 & 6 \end{bmatrix} \implies tr(A) = 1 + 6 = 7.$$

(a) **Prove** that if A and B are two 2×2 matrices that $tr(AB) = tr(BA)$.

Solution: This will look a lot like the problem from last week about triangular matrix multiplication. We're going to need to take a close look at the problem with a 2 by 2 perspective and then will try to make this work in an n by n case.
In the 2 by 2 case we will write out the multiplications:

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

We have

$$tr(AB) = (a_{11}b_{11} + a_{12}b_{21}) + (a_{21}b_{12} + a_{22}b_{22})$$

$$tr(BA) = (b_{11}a_{11} + b_{12}a_{21}) + (b_{21}a_{12} + b_{22}a_{22}).$$

We notice that the first and last term in the sum of $tr(AB)$ and $tr(BA)$ are identical and the middle terms are in reversed order. As such, these two quantities are the same.
Note this reversal in the middle will also give us an idea about how to approach the proof in the general case below.

(b) *(Challenge Problem: Optional)* **Prove** that if A and B are $n \times n$ matrices that $tr(AB) = tr(BA)$. You might find it helpful to remember the formula for matrix multiplication. If A and B are $n \times n$ matrices and $C = AB$, then

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}.$$

Solution: Just as in our last homework set, we find it helpful to think about matrix multiplication.
In this case it might be helpful to remember the formula for matrix multiplication. If A and B are $n \times n$ matrices and for AB we have

$$(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}.$$

Let's notice that:

$$tr(AB) = \sum_{\ell=1}^{n}(AB)_{\ell\ell} = \sum_{\ell=1}^{n}\left(\sum_{k=1}^{n} a_{\ell,k}b_{k,\ell}\right).$$

Now, let's notice that we can reverse the order of the sums and the terms in the sums: $a_{\ell,k}b_{k,\ell} = b_{k,\ell}a_{\ell}k$.

$$\sum_{\ell=1}^{n}\sum_{k=1}^{n} a_{\ell,k}b_{k,\ell} = \sum_{k=1}^{n}\sum_{\ell=1}^{n} a_{\ell,k}b_{k,\ell} = \sum_{k=1}^{n}\sum_{\ell=1}^{n} b_{k,\ell}a_{\ell,k} = \sum_{k=1}^{n}\left(\sum_{\ell=1}^{n} b_{k,\ell}a_{\ell,k}\right).$$

But the terms in the final sum term is just the entries $(BA)_{k,k}$.

$$\sum_{k=1}^{n}\left(\sum_{\ell=1}^{n} b_{k,\ell}a_{\ell,k}\right) = \sum_{k=1}^{n}(BA)_{k,k} = tr(BA).$$

Thus, we have shown that $tr(AB) = tr(BA)$.