Homework Assignment #1

This course requires you to be very familiar with matrix operations including vector operations and familiarity making mathematical arguments. As such, this first Homework Assignment will be an opportunity for you to review problems in these areas. Remember, this Homework Assignment is **not collected or graded**! But you are advised to do it anyway because the problems for Homework Quiz #1 will be heavily based on these problems!

1. Consider the following system of 3 equations and 3 unknowns.

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 - x_2 - 2x_3 &= -7 \\ 5x_1 + x_2 - x_3 &= 4. \end{aligned}$$

   (a) Write the system as a matrix-vector equation: $A\vec{x} = \vec{b}$.
   (b) Show that for all values of t, $x_1 = 1 - t, x_2 = 2 + 3t$ and $x_3 = 3 - 2t$ is a solution to the system.
   (c) Recall that each equation above represents a plane in 3D. Explain in words the geometric interpretation of the infinitely many solutions you demonstrated in part (b).

2. **Prove** that if a, b and c are all positive integers and $a^2 + b^2 = c^2$ then it is not possible for all three numbers to be odd. (Hint: Remember that an odd integer can be written as $2n + 1$ for some value of n whereas an even number can be written as $2n$ for some value of n. The phrasing of the problem suggests a proof by contradiction.)

3. Consider the two following systems:

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & -2 \\ & & x_2 & & & = & 3 \\ & & & & 2x_3 & = & 4. \end{array} \tag{1}$$

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & -2 \\ -3x_1 & - & x_2 & + & x_3 & = & 5 \\ 3x_1 & + & 2x_2 & + & x_3 & = & 2. \end{array} \tag{2}$$

   (a) Explain geometrically what these sets of equations represent according to the **row perspective**.
   (b) Write these equations in matrix vector form and explain what they represent according to the **column perspective**
   (c) Find the solution to equations (1) and explain why equations (1) are easier to solve than equations (2).
   (d) Demonstrate that the second set of equations can be made to look like to the first set of equations by performing the following operations:
      - Add the first and second equations of (2) together and replace the second equation in (2) with the resulting equation.
      - Subtract the first equation from the third equation given in (2) and replace the third equation in (2) with the resulting equation.
   (e) Are the solutions to equations (1) and (2) the same? Explain why or why not.

4. Consider the following matrix and vector:

$$A = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Carry out the multiplication to verify that $A\vec{x} = 0$.

(b) Let $\alpha \neq 0$ be any real number. **Prove** that for any vector $\vec{y} = \alpha\vec{x}$ we also have $A\vec{y} = 0$.

(c) Investigate if there are any other vectors $\vec{z}$ that have the property $A\vec{z} = 0$. (There are many ways to answer this problem. The goal is that you explore.)

5. A **upper triangular** matrix is one that has only 0's **below** the diagonal. See for example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

Mathematically, we can express the entries of a matrix A in terms of $a_{i,j}$ where i is the row and j is the column. (For example, in the example above $a_{2,3} = 4$.) As such, we can succinctly say that: A matrix A is **upper triangular** if $a_{i,j} = 0$ for $i > j$.

(a) **Prove** that the product of two 2×2 upper triangular matrices A and B is again a upper triangular matrix.

(b) **Prove** that the product of two $n \times n$ upper triangular matrices A and B is again a upper triangular matrix.

In this case it might be helpful to remember the formula for matrix multiplication. If A and B are $n \times n$ matrices and $C = AB$, then

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$