ENGR 065 Electric Circuits

Lecture 18: The Steady-State Sinusoidal Responses

Today's Topics

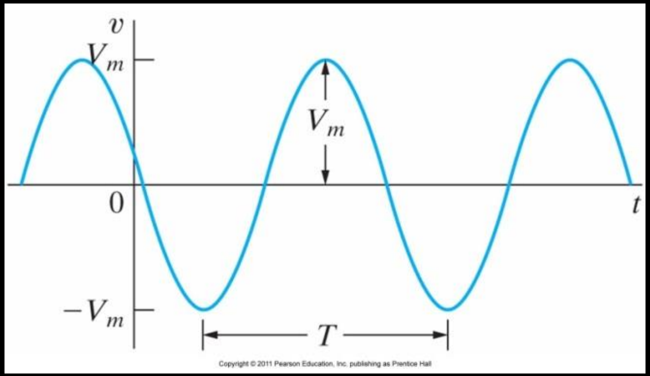
- Sinusoidal sources
- The total responses of RC and RL circuits
- The steady-state sinusoidal response of circuits

- Covered in Section 13.7

Sinusoidal Sources

- A sinusoidal voltage/current source (independent or dependent) produces a voltage/current that varies sinusoidally with time.

Period, T, is the length of time required for the sinusoidal function to pass through all its possible values. Its unit is seconds.



$$v = V_m \cos(\omega t + \phi)$$

The reciprocal of T is frequency, f. It gives the number of cycles per second. It has a unit of Herz, abbreviated Hz.

$$f = \frac{1}{T}.$$

ω represents the angular frequency of the sinusoidal function, or

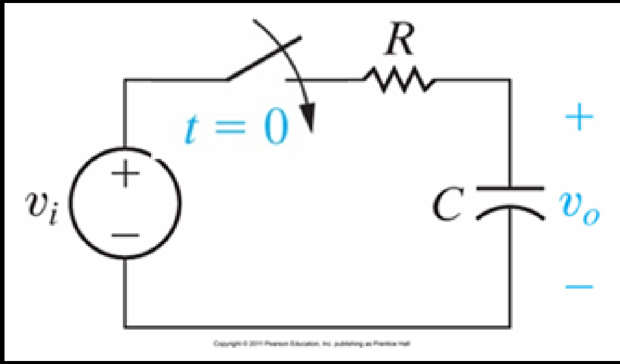
$$\omega = 2\pi f = \frac{2\pi}{T} \quad \text{(radians/second)}$$

The coefficient V_m gives the maximum amplitude of the sinusoidal voltage.

The angle ϕ is called the phase angle of the sinusoidal voltage. It determines the value of the sinusoidal function at $t = 0$;

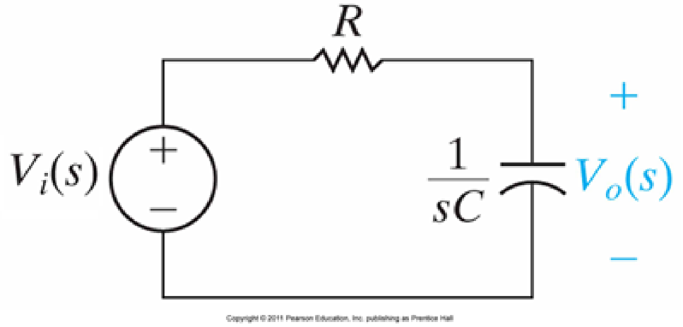
The Total Response of RC Circuit

Suppose the voltage source is $v_i = V_m \cos(\omega t + \phi)$ and the initial voltage $v_0(t)$ in the circuit is zero. What is the response of $v_o(t)$? Applying KVL to the circuit on the right:



$$RC\frac{dv_o(t)}{dt} + v_o(t) = V_m \cos(\omega t + \phi)$$

Applying Laplace transform to the both sides of the above equation, we have:



$$RCsV_o(s) + V_o(s) = \frac{V_m(s\cos\phi - \omega\sin\phi)}{s^2 + \omega^2},$$

$$V_o(s) = \frac{\frac{V_m}{RC}(s\cos\phi - \omega\sin\phi)}{(s + \frac{1}{RC})(s^2 + \omega^2)}$$

The Total Response of RC Circuit

$$V_o(s) = \frac{\frac{V_m}{RC}(s\cos\phi - \omega\sin\phi)}{\left(s+\frac{1}{RC}\right)(s^2+\omega^2)} = \frac{k_1}{s+\frac{1}{RC}} + \frac{k_2}{s-j\omega} + \frac{k_2^*}{s+j\omega}$$

$$k_1 = V_o(s)\left(s+\frac{1}{RC}\right)\Big|_{s=-\frac{1}{RC}} = \frac{-V_m}{\sqrt{(\omega RC)^2+1}}\cos(\phi+\theta)$$

where $\theta = -\arctan(\omega RC)$.

$$k_2 = V_o(s)(s-j\omega)|_{s=j\omega} = \frac{V_m}{2\sqrt{(\omega RC)^2+1}}e^{j(\phi+\theta)}$$

where $\theta = -\arctan(\omega RC)$.

The Total Response of RC Circuit

The solution for $v_0(t)$ is

$$v_o(t) = \frac{-V_m}{\sqrt{(\omega RC)^2+1}}\cos(\phi+\theta)\,e^{-\left(\frac{1}{RC}\right)t} + \frac{V_m}{\sqrt{(\omega RC)^2+1}}\cos(\omega t+\phi+\theta)$$

Transient component steady-state component

where $\theta = -\arctan(\omega RC)$.

The total response = transient response + steady-state response.

As $t \to \infty$, $\frac{-V_m}{\sqrt{(\omega RC)^2+1}}\cos(\phi+\theta)\,e^{-\left(\frac{1}{RC}\right)t} = 0$, which means

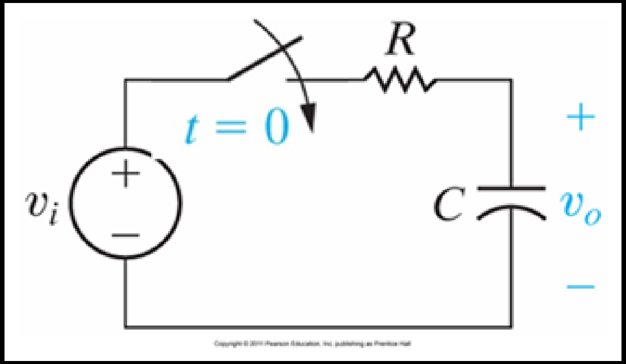
the transient response approaches to zero as $t \to \infty$.

The Steady-State Sinusoidal Response

We denote the second part as v_{oss}. It is

$$v_{oss} = \frac{V_m}{\sqrt{(\omega RC)^2+1}} \cos(\omega t + \phi + \theta)$$

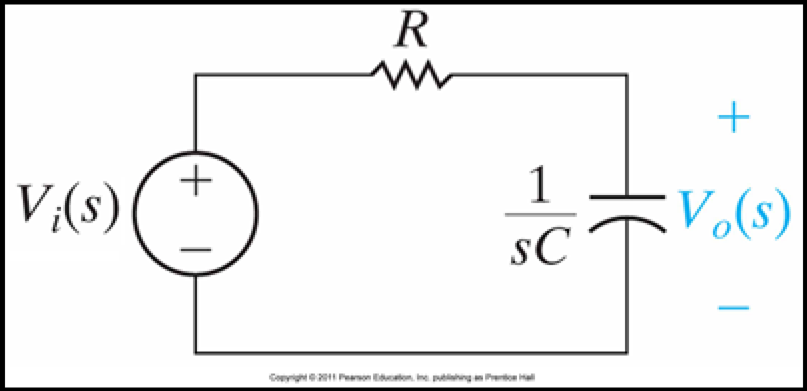
where $\theta = -\arctan(\omega RC)$. It is the voltage across the capacitor as $t \to \infty$.



Four notes:

1. The steady-state response/output (the voltage across the capacitor) has the same frequency as the source/input.
2. The magnitude of the steady-state response is reduced by the factor of $\sqrt{(\omega RC)^2+1}$.
3. The phase angle of the steady-state response is lagged by a degree of $\arctan(\omega RC)$.
4. The magnitude and the phase angle of the response vary with frequency.

The Steady-State Sinusoidal Response



Let's take a look at the transfer function of the circuit.
Here, the input is $V_i(s)$, and the output is $V_0(s)$, and the transfer function is:

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{1/RC}{s + 1/RC}$$

Replacing s with jω (s= jω), we have

$$H(j\omega) = \frac{1/RC}{j\omega + 1/RC} = \frac{1}{\sqrt{(\omega RC)^2+1}} e^{j[-\arctan(\omega RC)]} = |H(j\omega)| e^{j\angle H(j\omega)},$$

where $|H(j\omega)| = \frac{1}{\sqrt{(\omega RC)^2+1}}$, and $\angle H(j\omega) = -\arctan(\omega RC)$

The Steady-State Sinusoidal Response

Comparing the steady-state output $v_{oss} = \frac{V_m}{\sqrt{(\omega RC)^2+1}}\cos(\omega t + \phi + \theta)$ to

$|H(j\omega)| = \frac{1}{\sqrt{(\omega RC)^2+1}}$, and $\angle H(j\omega) = -\arctan(\omega RC)$

we know that the steady-state output is equal to

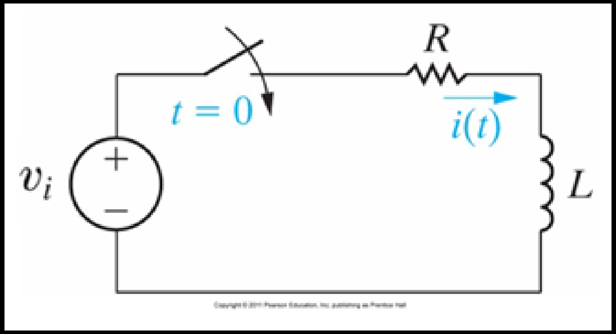
$$v_{oss} = V_m \frac{1}{\sqrt{(\omega RC)^2+1}} \cos(\omega t + \phi + \theta)$$
$$= V_m |H(j\omega)| \cos(\omega t + \phi + \angle H(j\omega))$$

The Total Response of RL Circuit

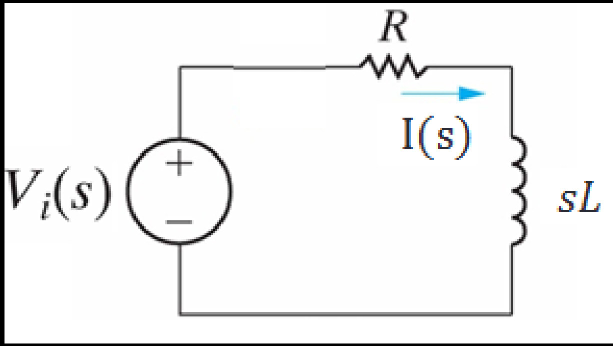
Suppose the voltage source is $v_s = V_m \cos(\omega t + \phi)$ and the initial current in the circuit is zero. What is the total response of $i(t)$?

Applying KVL:

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$



Applying Laplace transform to the both sides of The above formula, we have:



$$LsI(s) + RI(s) = \frac{V_m(s\cos\phi - \omega\sin\phi)}{s^2 + \omega^2},$$

$$I(s) = \frac{\frac{V_m}{L}(s\cos\phi - \omega\sin\phi)}{(s + \frac{R}{L})(s^2 + \omega^2)}$$

The Total Response of RL Circuit

By using the partial fraction expansion, we can find the solution for $i(t)$ is

$$i(t) = \frac{-V_m}{\sqrt{R^2 + (\omega L)^2}} \cos(\phi + \theta)\, e^{-\left(\frac{R}{L}\right)t} + \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cos(\omega t + \phi + \theta)$$

Transient component

steady-state component

where $\theta = -tan^{-1}(\frac{\omega L}{R})$

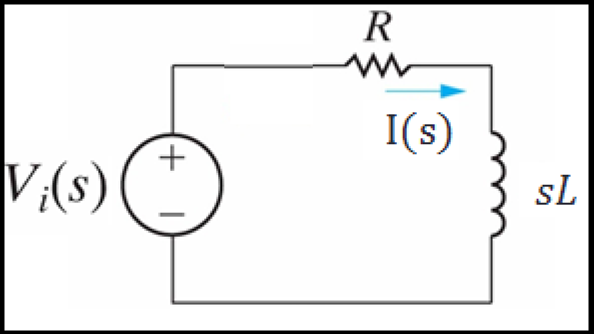
The total response = transient response + steady-state response.

The transient response is approaching to zero as $t \rightarrow \infty$, which is

$$\lim_{t\to\infty} \frac{-V_m}{\sqrt{R^2+(\omega L)^2}} \cos(\phi + \theta)\, e^{-\left(\frac{R}{L}\right)t} = 0.$$

The Steady-State Sinusoidal Response

The transfer function of the circuit is



$$H(s) = \frac{1/L}{s+R/L},$$

$$H(j\omega) = \frac{1/L}{j\omega+R/L} = \frac{1}{j\omega L+R} = \frac{1}{\sqrt{R^2+(\omega L)^2}} \angle - tan^{-1}\left(\frac{\omega L}{R}\right),$$

$$|\ H(j\omega)| = \frac{1}{\sqrt{R^2+(\omega L)^2}},\ \angle H(j\omega) = -\,tan^{-1}\left(\frac{\omega L}{R}\right)$$

From the second term of $i(t)$, we know that

$$i_{ss}(t) = V_m|H(j\omega)|\cos(\omega t + \phi + \angle H(j\omega))$$

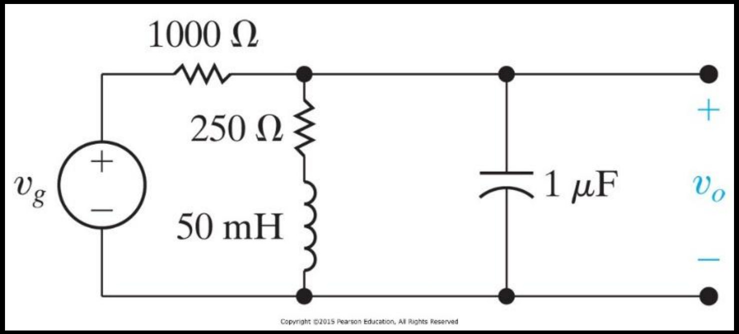
The Steady-State Sinusoidal Response

We can extend the results from the RC and RL circuits to any circuits and conclude that, in general, if the input of a circuit is a sinusoidal source of $\mathrm{x(t)} = \mathrm{A}\cos(\omega t + \phi)$, the steady-state response (output) of the circuit is:

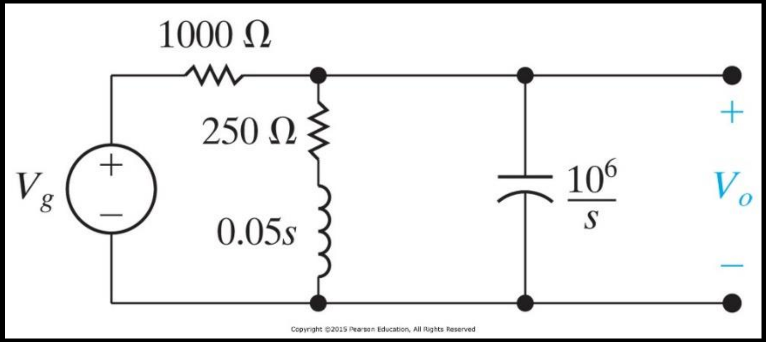
$$y_{ss}(t) = A|H(j\omega)|\cos\left[\omega t + \phi + \theta(\omega)\right]$$

where $H(j\omega)$ is the transfer function, $H(s) = \frac{Y(s)}{X(s)}$, of the circuit when $s = j\omega$. $H(j\omega) = |H(j\omega)|e^{j\theta(\omega)}$, where $|H(j\omega)|$ and $\theta(\omega) = \angle H(j\omega)$ is the magnitude and phase anlge of $\mathrm{H(j\omega)}$, respectively.

Example (p. 501)



No initial stored energy. The sinusoidal voltage source is 120 cos(5000t+30^0) V. Find the steady-state expression for $v_0(t)$.



Step 1: No initial stored energy
Step 2: $v_g = 120$ cos(5000t+30^0) V.
Step 3. Draw the circuit model in the s domain
Step 4 Use the node-voltage method to
find $V_o(s)$

$$\frac{V_0 - V_g}{1000} + \frac{V_0}{250 + 0.05s} + \frac{V_0 s}{10^6} = 0$$

Solving for $V_0\ yields$

Example (p. 501) – cont'd

$$V_0(s) = \frac{1000(s + 5000)V_g(s)}{s^2 + 6000s + 25 \times 10^6}$$

Step 5: The transfer function is

$$H(s) = \frac{V_0(s)}{V_g(s)} = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}$$

Step 6: From $v_g = 120 \cos(5000t + 30^0)\ V$, we know

$$V_m = 120\ V, \omega = 5000\ rad/s,\ \varphi = 30^0$$

Step 7: Find H(jω)

$$\mathrm{H(j5000)} = \frac{1000(j5000 + 5000)}{(j5000)^2 + 6000(j5000) + 25 \times 10^6} = \frac{\sqrt{2}}{6} e^{-j45^0}$$

$$|H(j\omega)| = \frac{\sqrt{2}}{6} \qquad \theta = \angle H(j\omega) = -45^0$$

Example (p. 501) – cont'd

Step 8: Find the steady-state expression for $v_0(t)$.

$$v_{0ss}(t)=V_m|H(j\omega)|\cos(\omega t + \phi + \angle H(j\omega))$$

$$v_{0ss}(t)=120 \times \frac{\sqrt{2}}{6}\cos(5000\mathrm{t} + 30^0 - 45^0)$$

$$= 20\sqrt{2}\cos(5000\mathrm{t} - 15^0)\ \mathrm{V}$$

Summary

The transfer function of a circuit can be used to find the circuit's steady-state response to a sinusoidal source.

If the source of the circuit is $x(t) = A\cos(\omega t + \phi)$, the steady-state response to the source is $y_{ss}(t)$, the transfer function is $H(s) = \frac{Y_{ss}(s)}{X(s)}$, then

$$y_{ss}(t) = A|H(j\omega)|\cos[\omega t + \phi + \theta(\omega)]$$