CSE100: Design and Analysis of Algorithms
Lecture 18 – Strongly Connected Components

Mar 29th 2022

Finding strongly connected components

Last Lecture

- Breadth-first and depth-first search
- Plus, applications!
  - Topological sorting
    - Clarification: Does it work if you don't start at a source?
    - Answer: It does!! Try it ☺
  - In-order traversal of BSTs
  - Shortest path in unweighted graphs
  - Testing bipartite-ness

- The key was paying attention to the structure of the tree that these search algorithms implicitly build.

Today

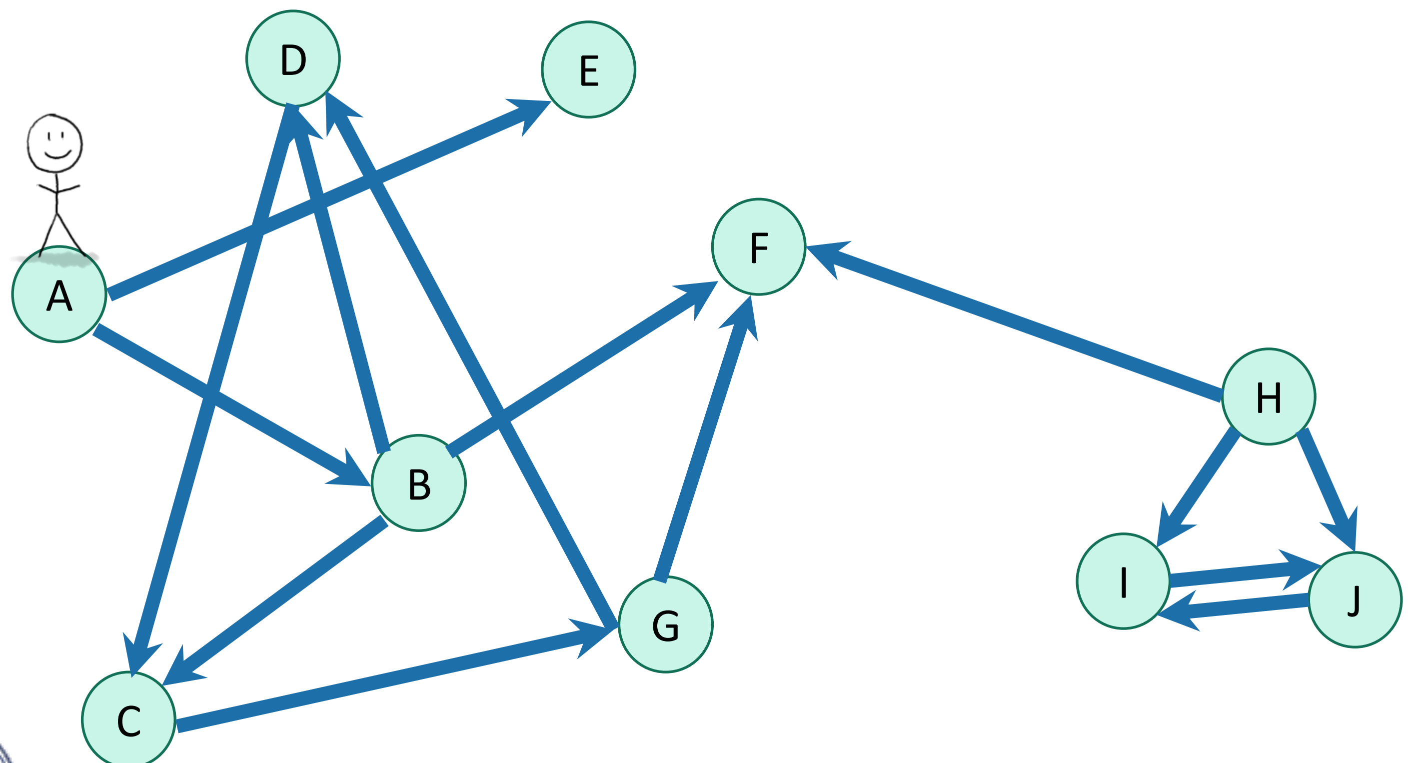
- One more application:

Finding

Strongly Connected Components

When you can't reach everything

- Run DFS repeatedly to get a depth-first forest



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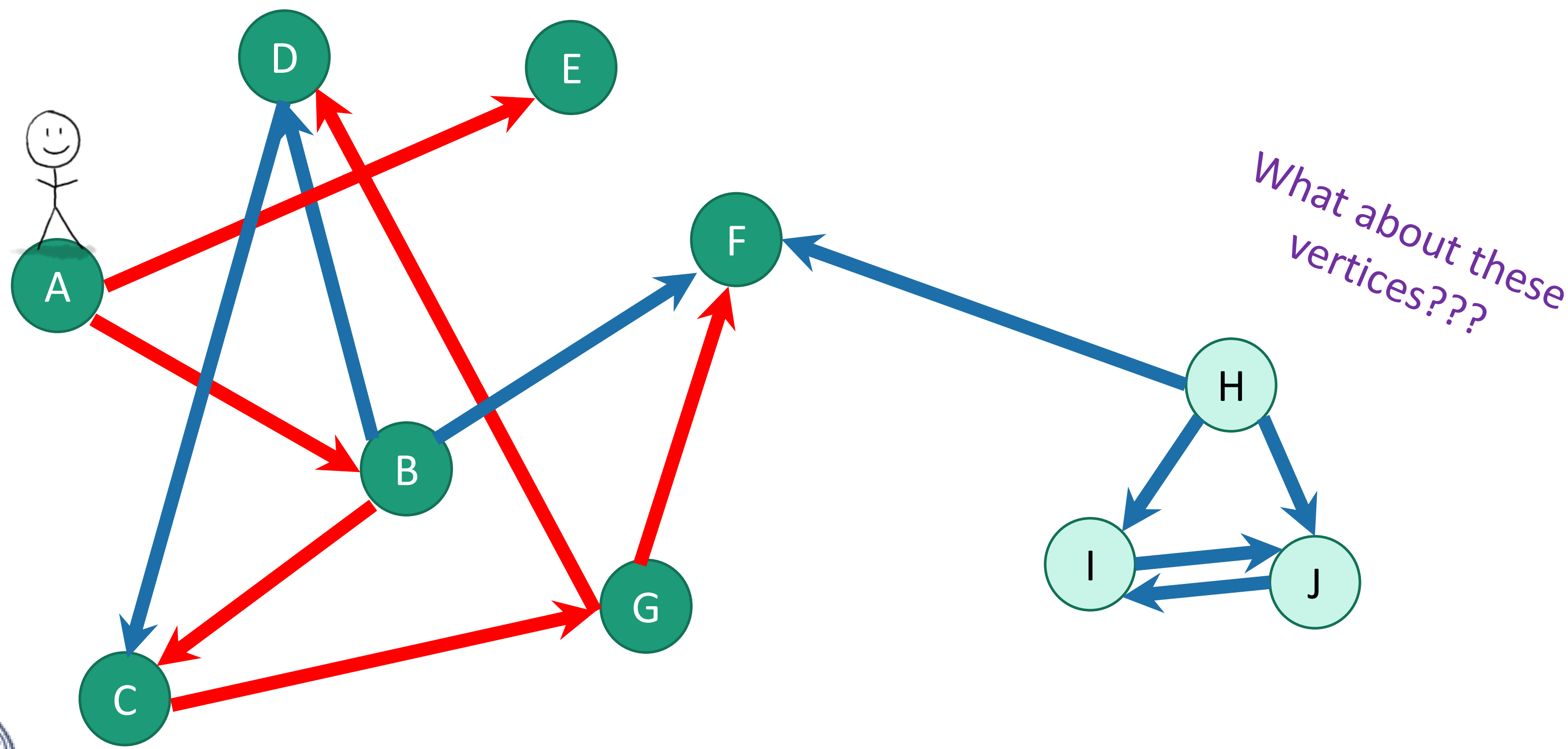

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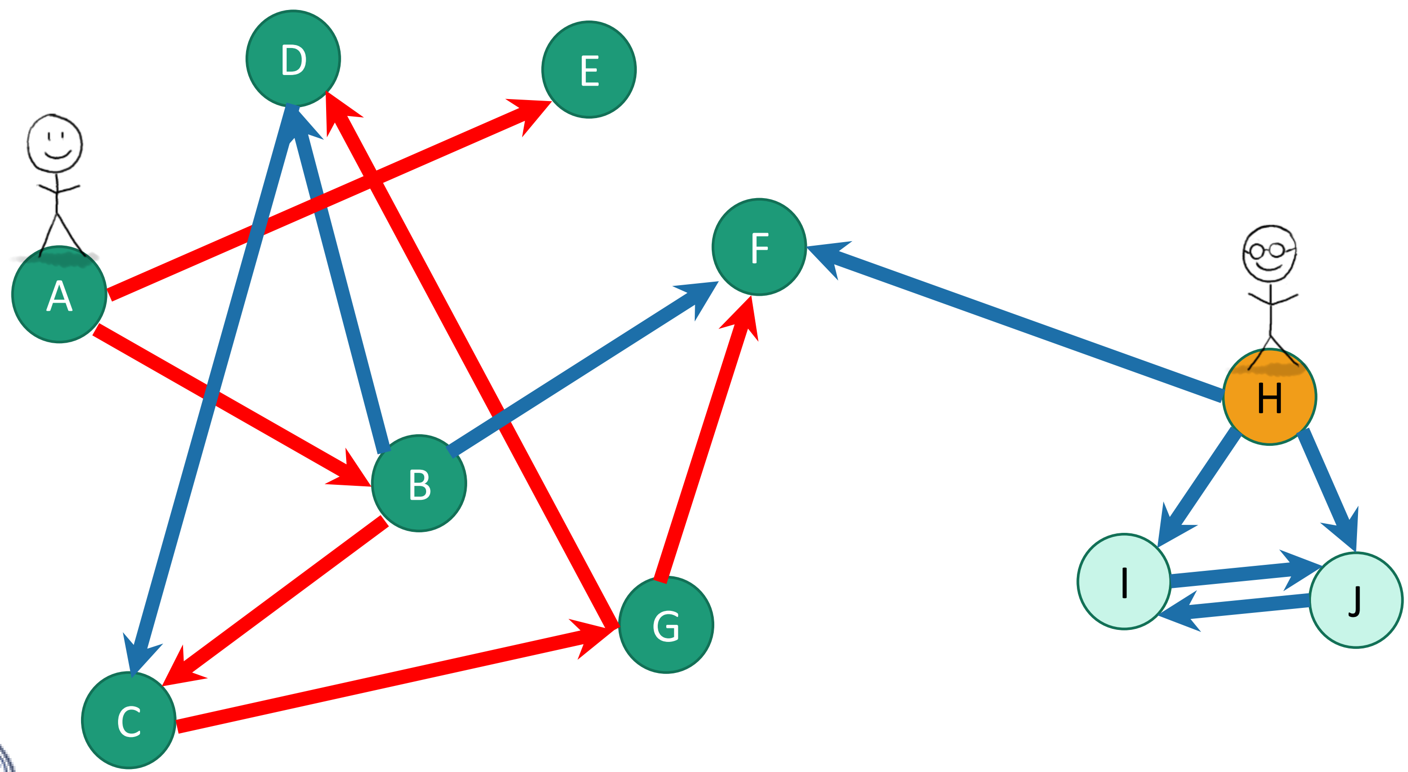

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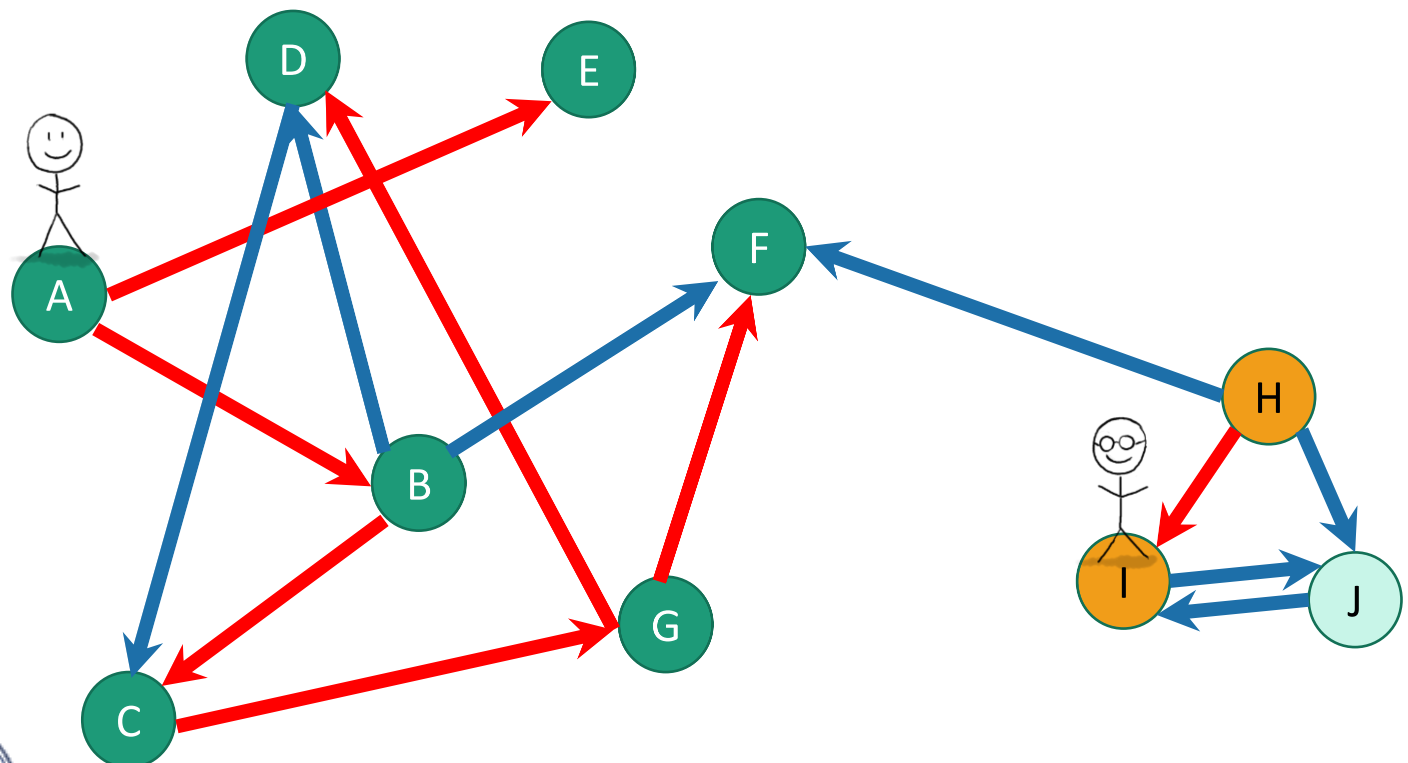

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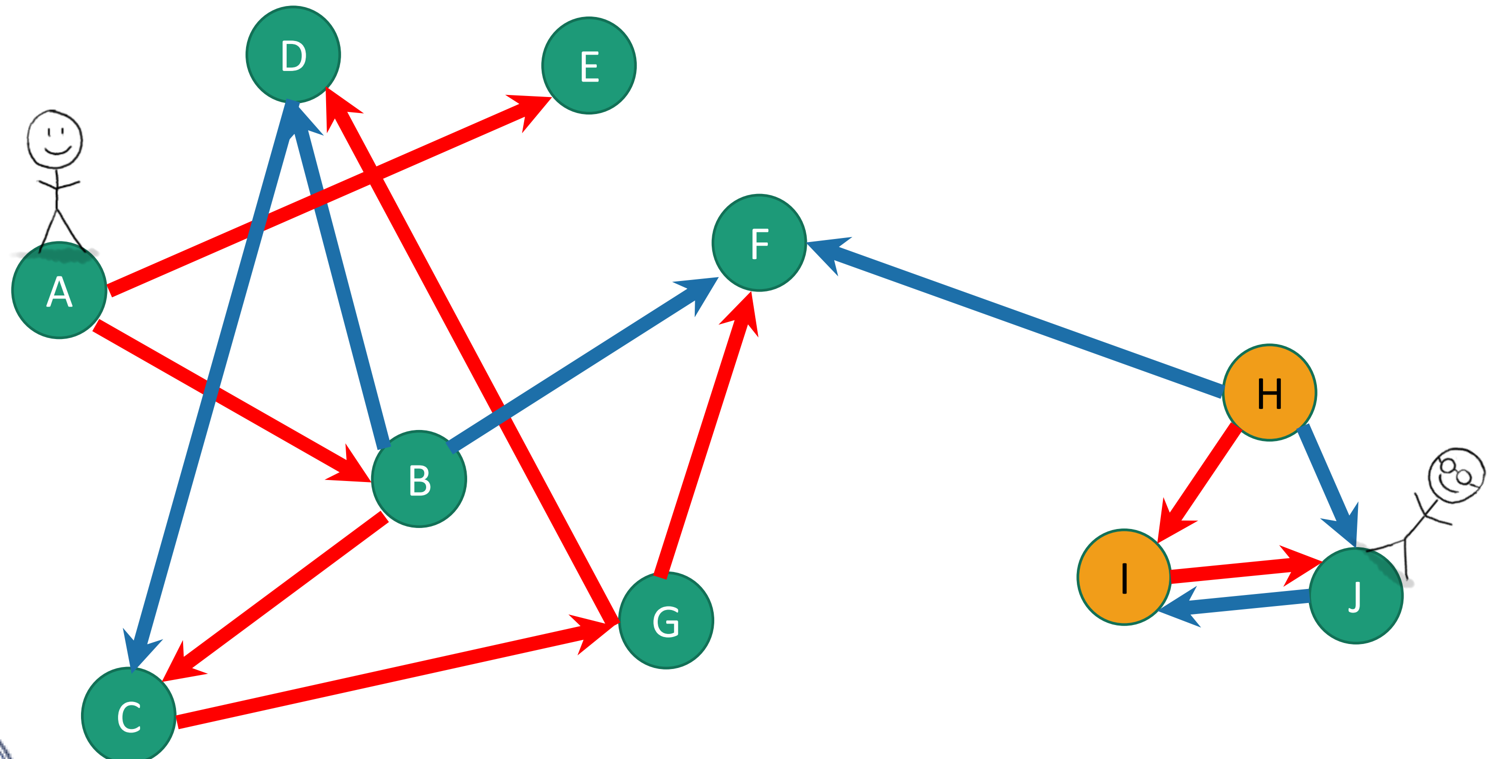

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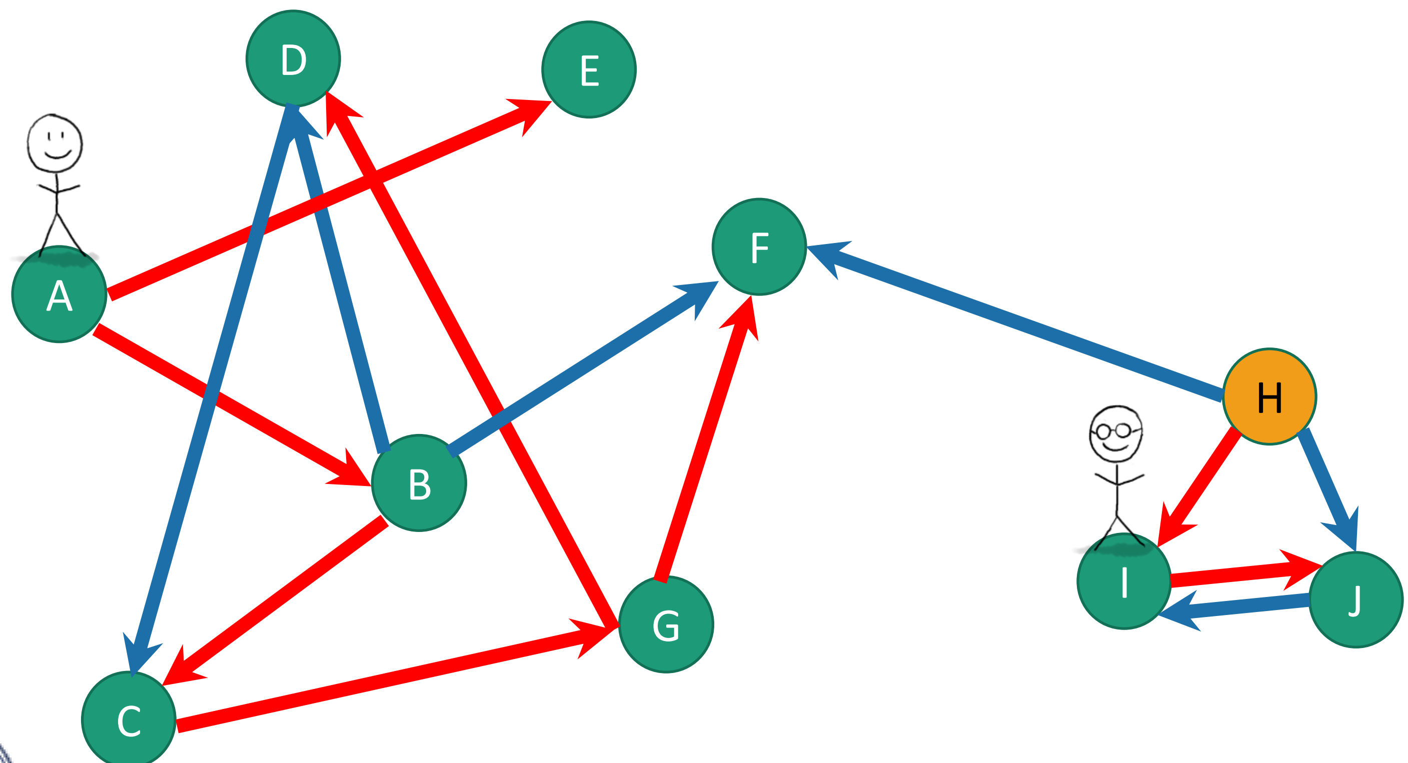

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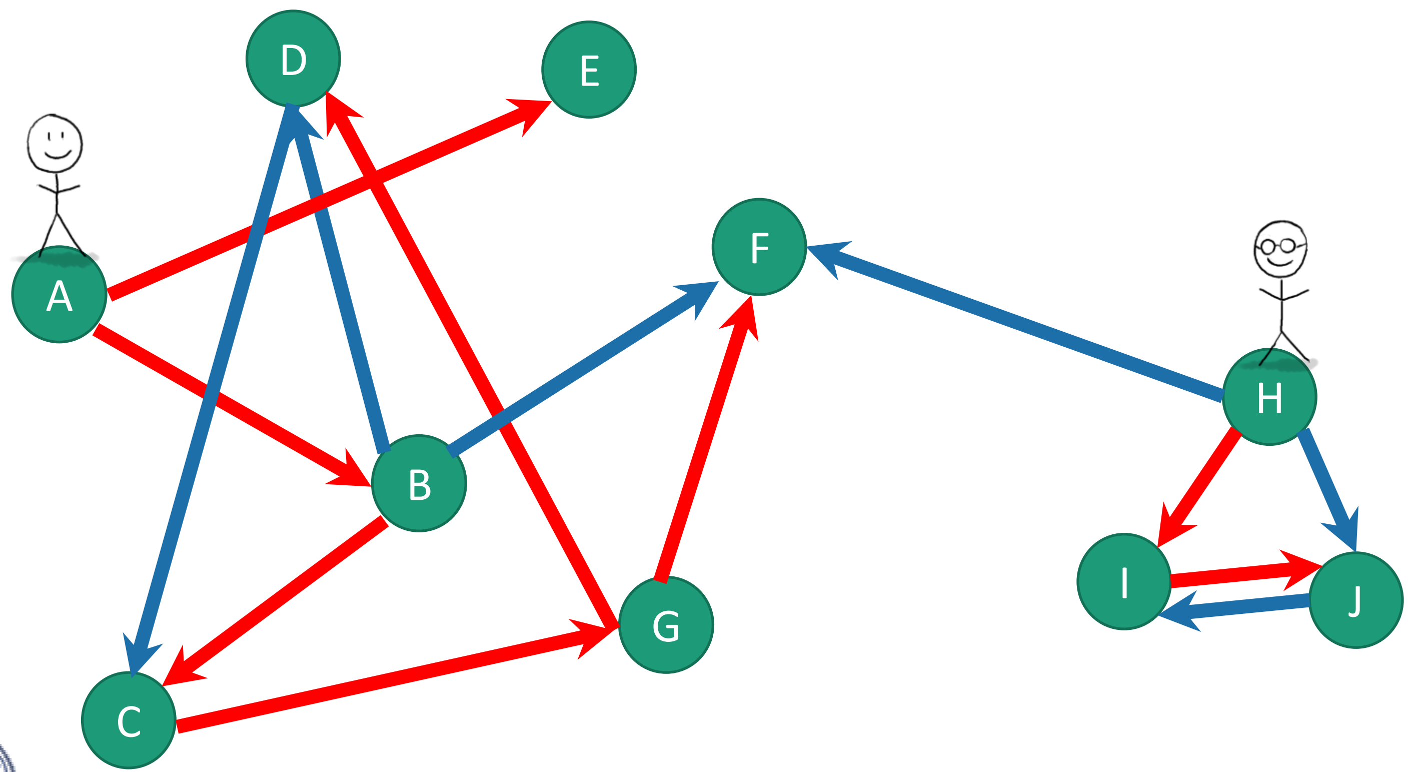

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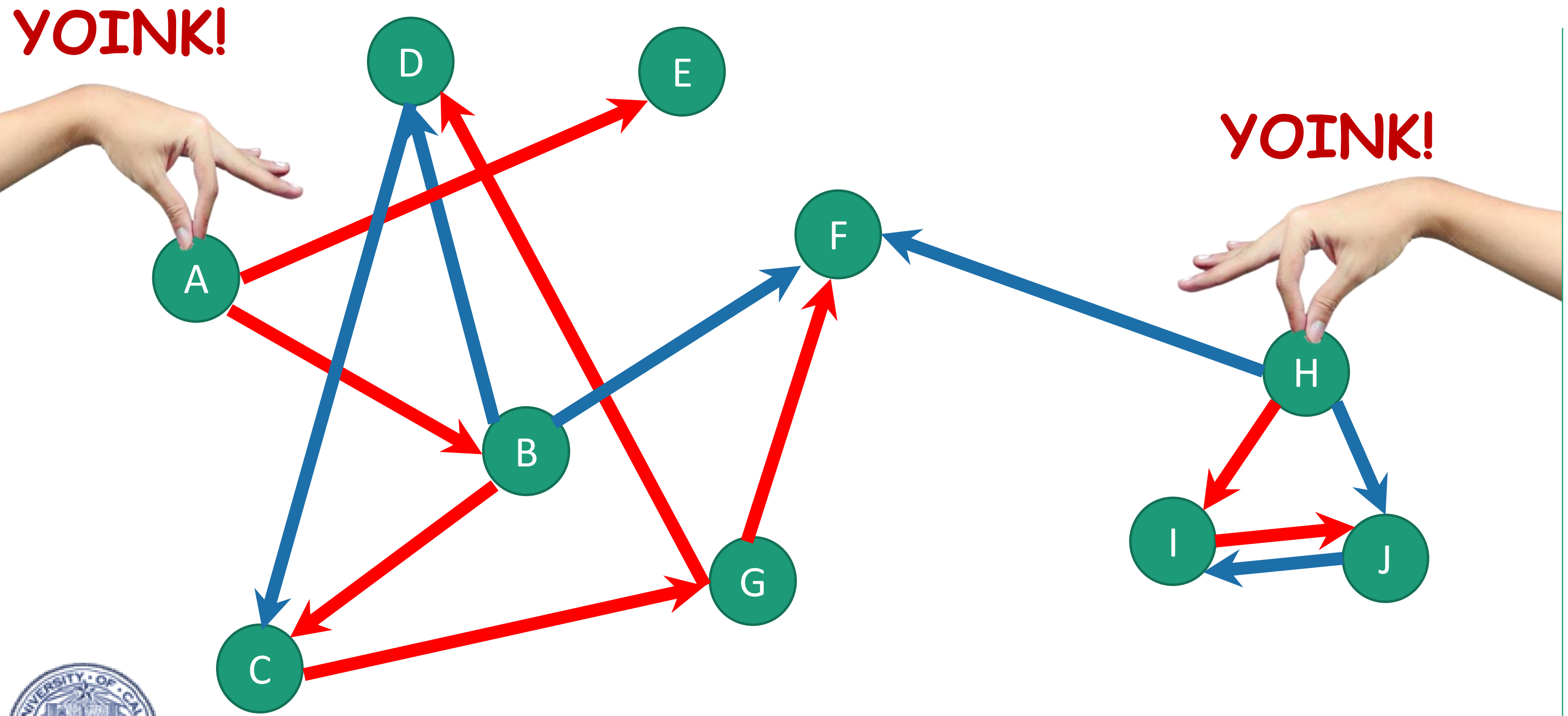

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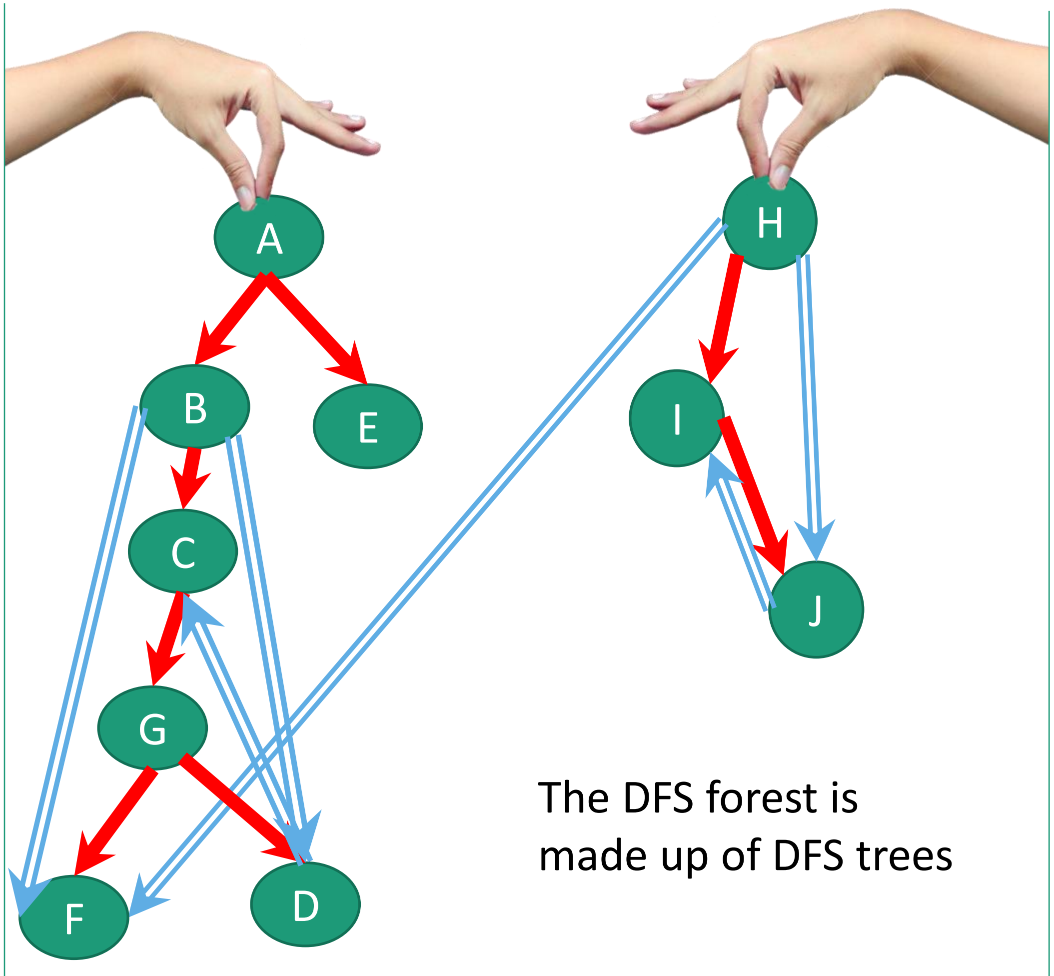
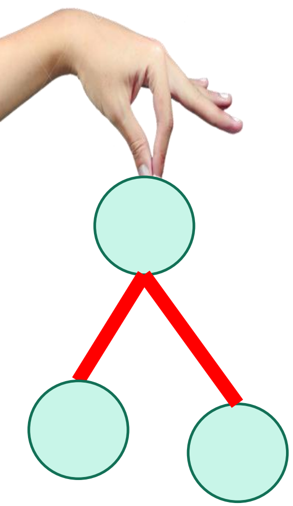


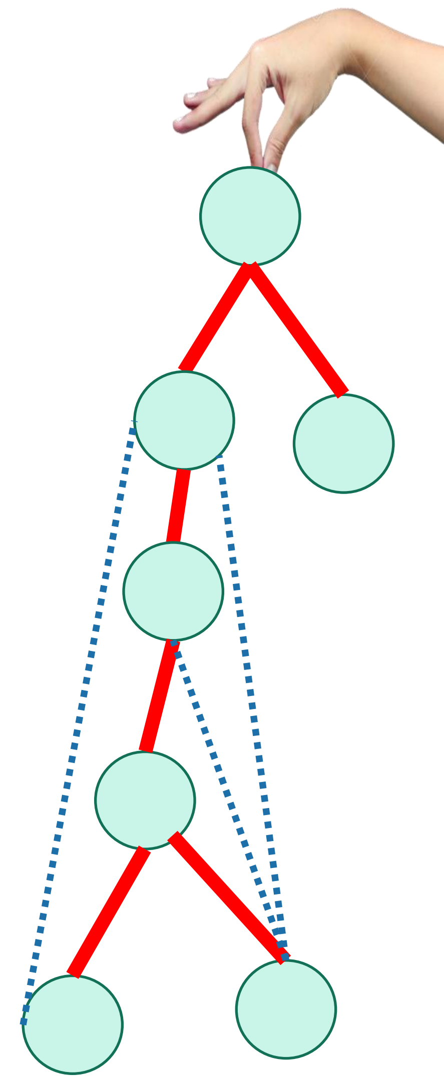
The DFS forest is made up of DFS trees

Recall: the parentheses theorem

(Works the same with DFS forests)



DFS tree



- If v is a descendent of w in this tree:

timeline

w.start v.start v.finish w.finish

- If w is a descendent of v in this tree:

v.start w.start w.finish v.finish

- If neither are descendants of each other:

v.start v.finish w.start w.finish

(or the other way around)

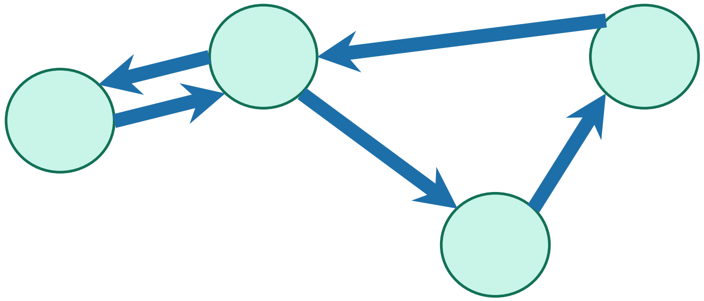
If v and w are in different trees, it's always this last one.

Enough of review

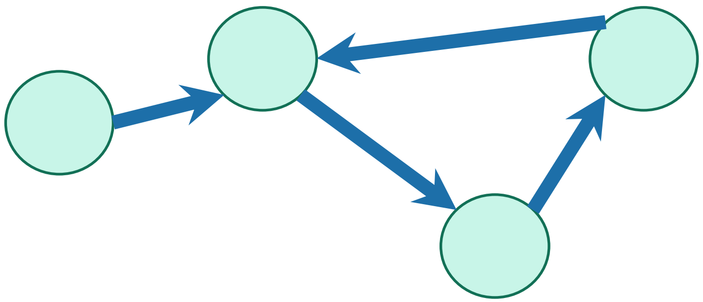
Strongly connected components

Strongly connected components

- A directed graph G = (V,E) is **strongly connected** if:
- for all v,w in V:
  - there is a path from v to w and
  - there is a path from w to v.



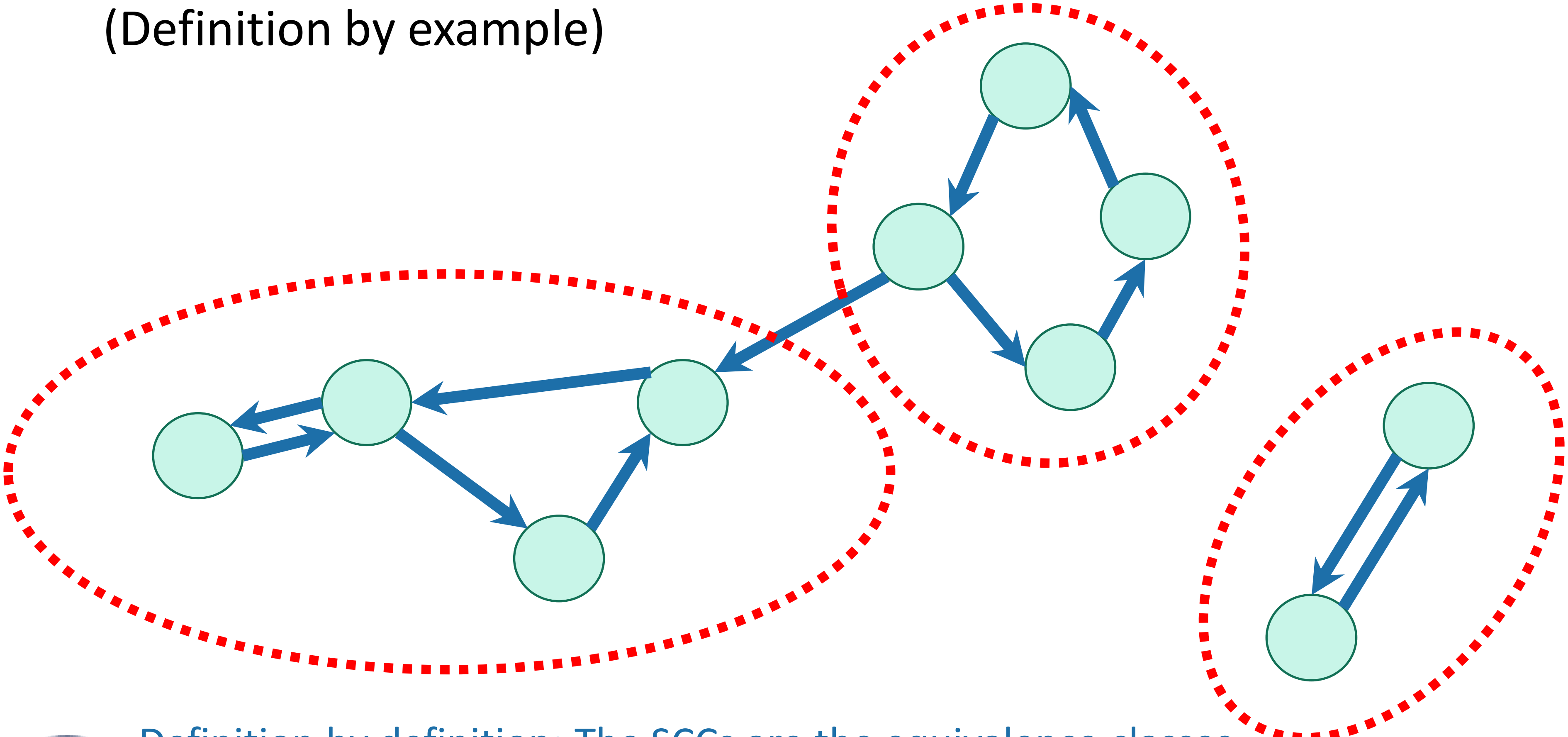
strongly connected



not strongly connected

We can decompose a graph into **strongly connected components** (SCCs)

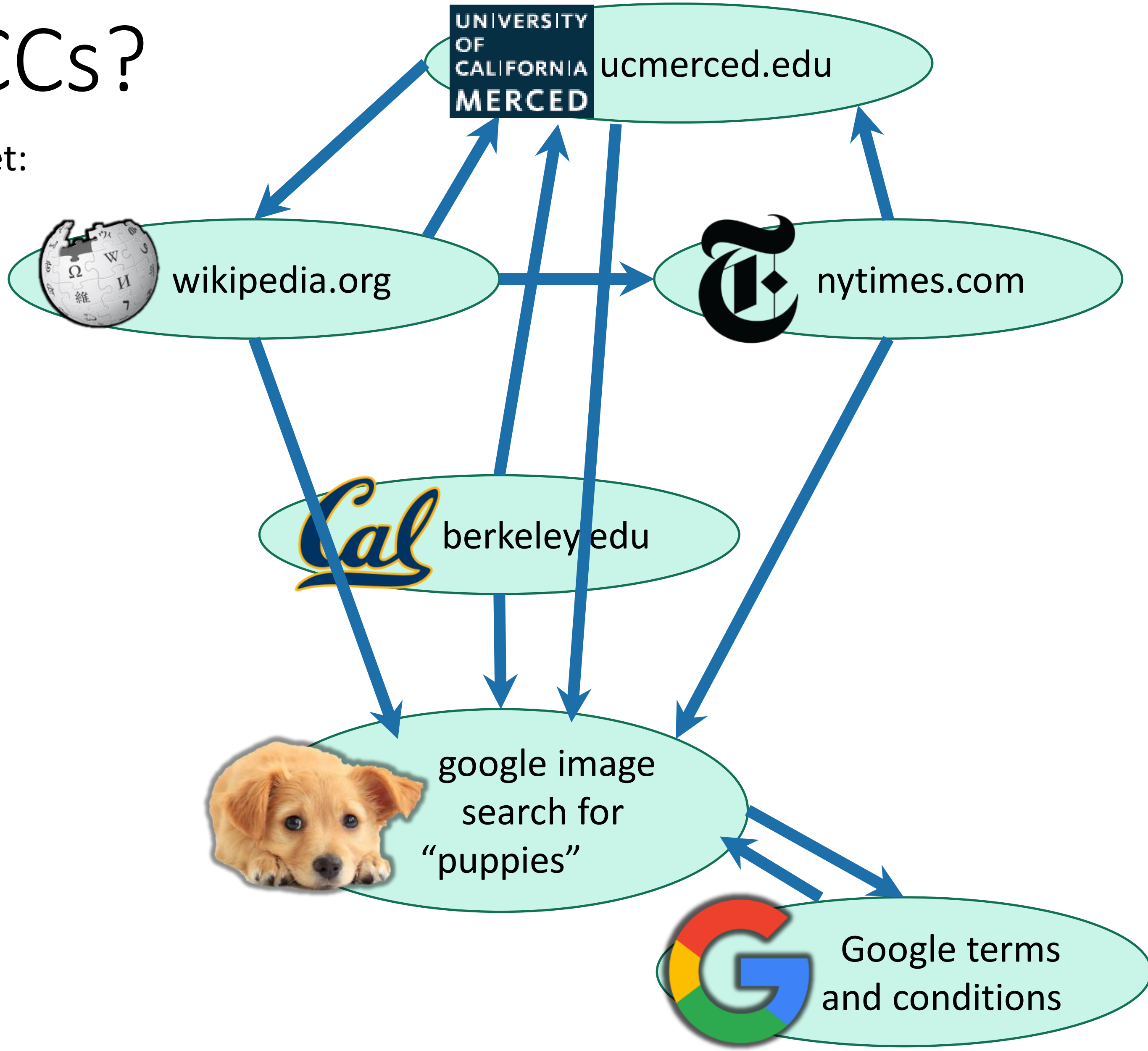
(Definition by example)

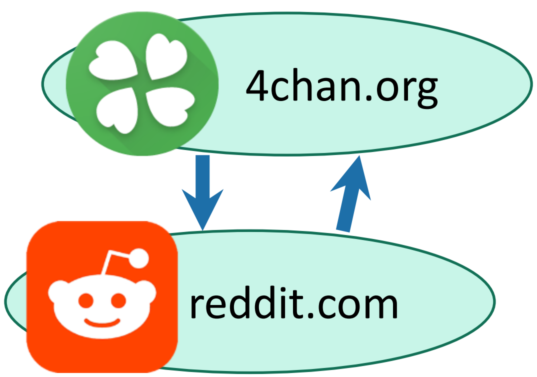


Definition by definition: The SCCs are the equivalence classes under the “are mutually reachable” equivalence relation.

Why do we care about SCCs?

Consider the internet:

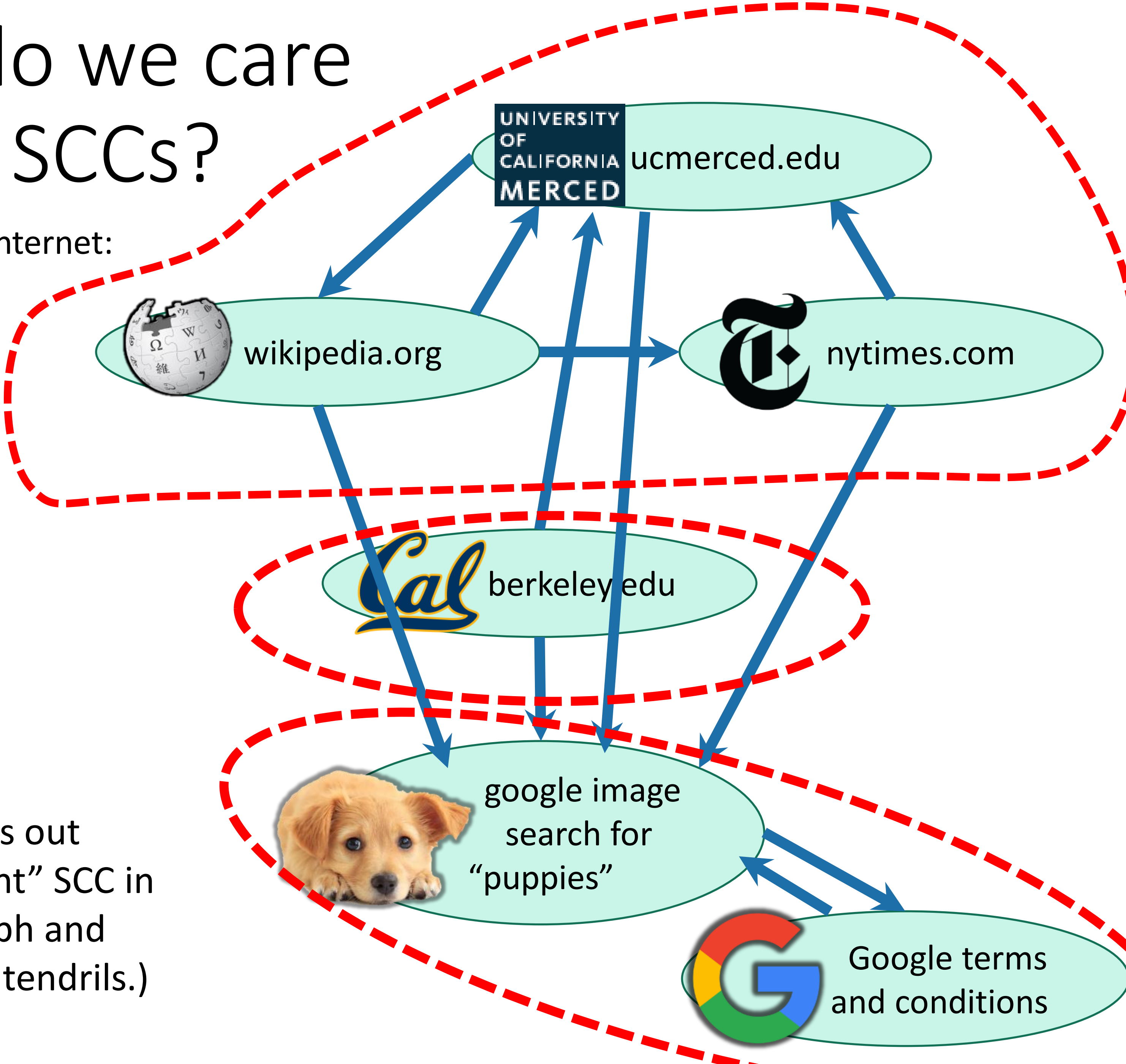




Let's ignore this corner of the internet for now…but everything today works fine if the graph is disconnected.

Why do we care about SCCs?

Consider the internet:



(In real life, turns out there's one "giant" SCC in the internet graph and then a bunch of tendrils.)

Why do we care about SCCs?

- Strongly connected components tell you about **communities.**
- Lots of graph algorithms only make sense on SCCs.
  - (So, sometimes we want to find the SCCs as a first step)

How to find SCCs?

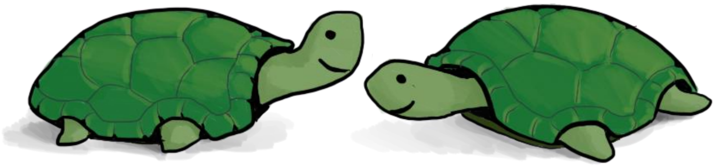
Try 1:

- Consider all possible decompositions and check.

Try 2:

- Something like…
  - Run DFS a bunch to find out which u’s and v’s belong in the same SCC
  - Aggregate that information to figure out the SCCs

Come up with a straightforward way to use DFS to find SCCs. What’s the running time? More than n^2 or less than n^2?



One straightforward solution

This will not be our final solution so don't worry too much about it…

- SCCs = [ ]
- For each u:
  - Run DFS from u
  - For each vertex v that u can reach:
    - If v is in an SCC we've already found:
      - Run DFS from v to see if you can reach u
      - If so, add u to v's SCC
      - Break
  - If we didn't break, create a new SCC which just contains u.

Running time AT LEAST $\Omega(n^2)$, no matter how smart you are about implementing the rest of it…

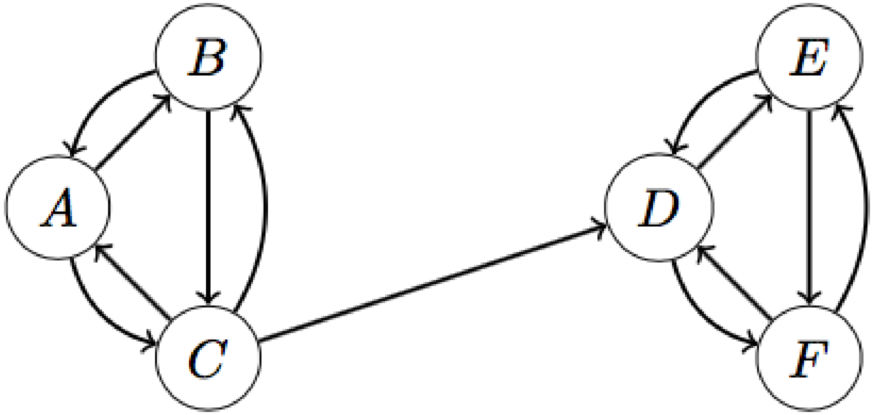
Today

- We will see how to find strongly connected components in time O(n+m)

- !!!!!

Example exercise

- Run DFS starting at D:



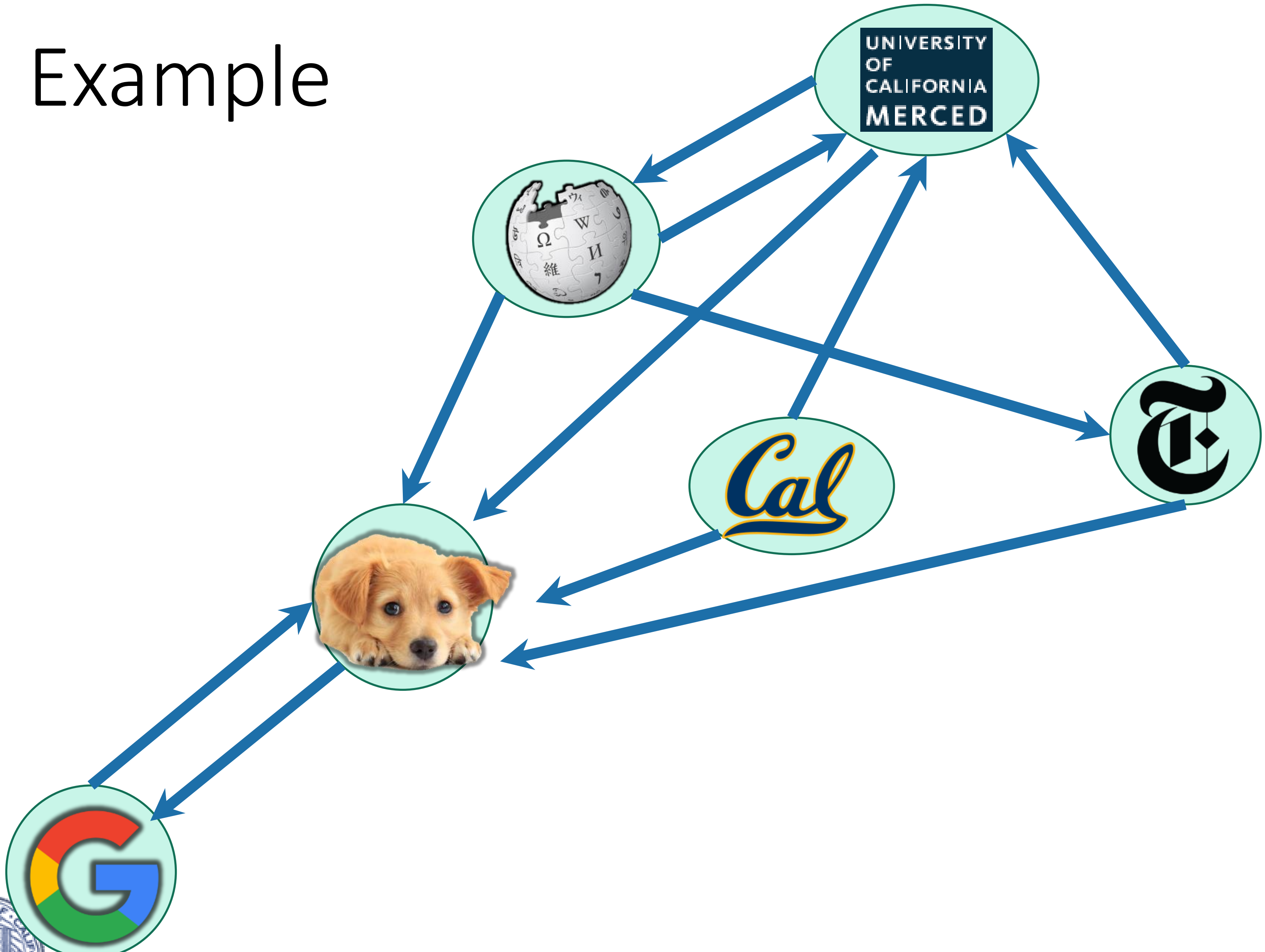
- That will identify SCCs…
- Issues:
  - How do we know where to start DFS?
  - It wouldn't have found the SCCs if we started from A.

Algorithm

Running time: O(n + m)

- Do DFS to create a DFS forest.
  - Choose starting vertices in any order.
  - Keep track of finishing times.
- Reverse all the edges in the graph.
- Do DFS again to create **another DFS forest**.
  - This time, order the nodes in the reverse order of the finishing times that they had from the **first** DFS run.
- The SCCs are the different trees in the **second DFS forest.**

Example



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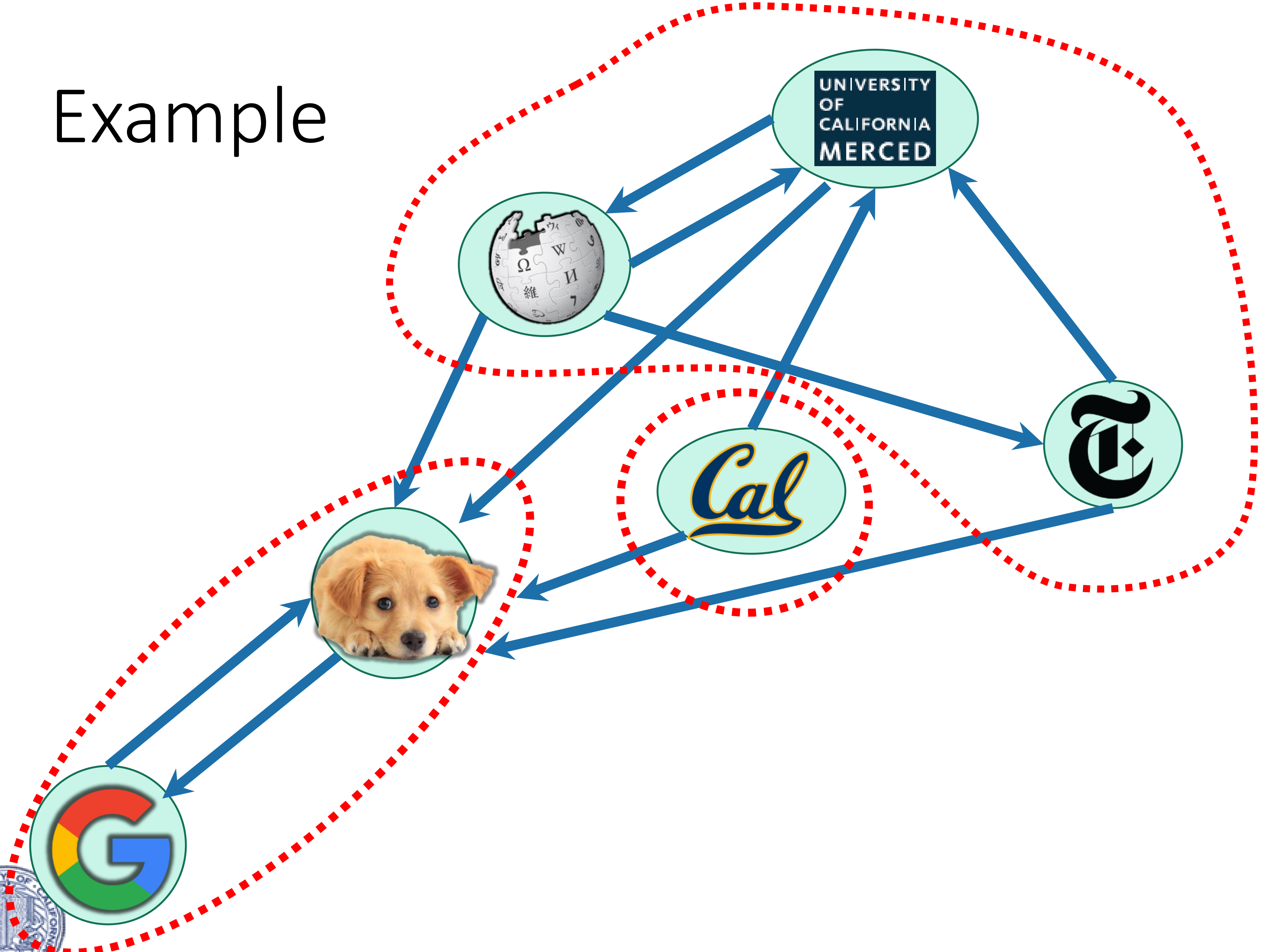

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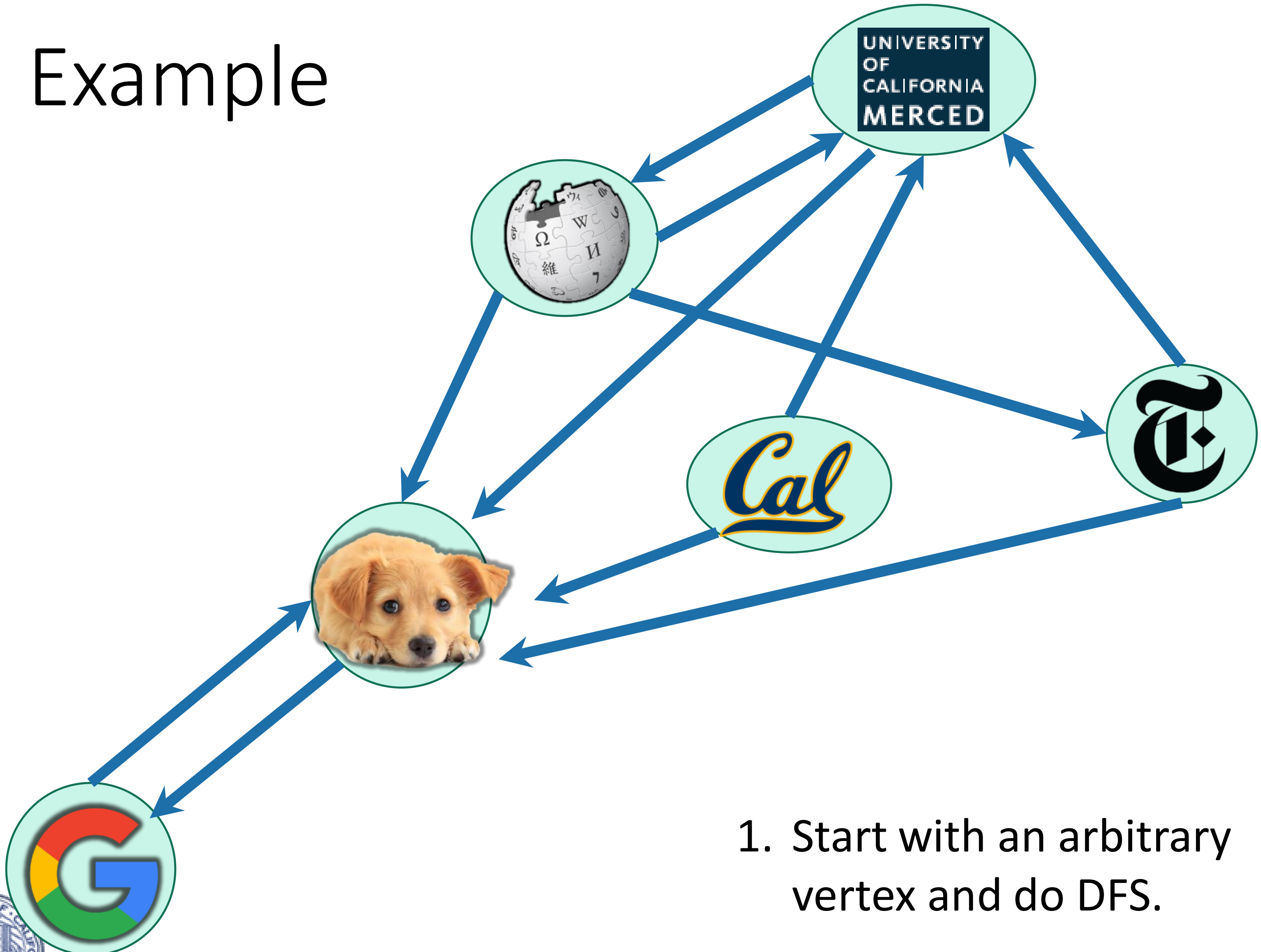
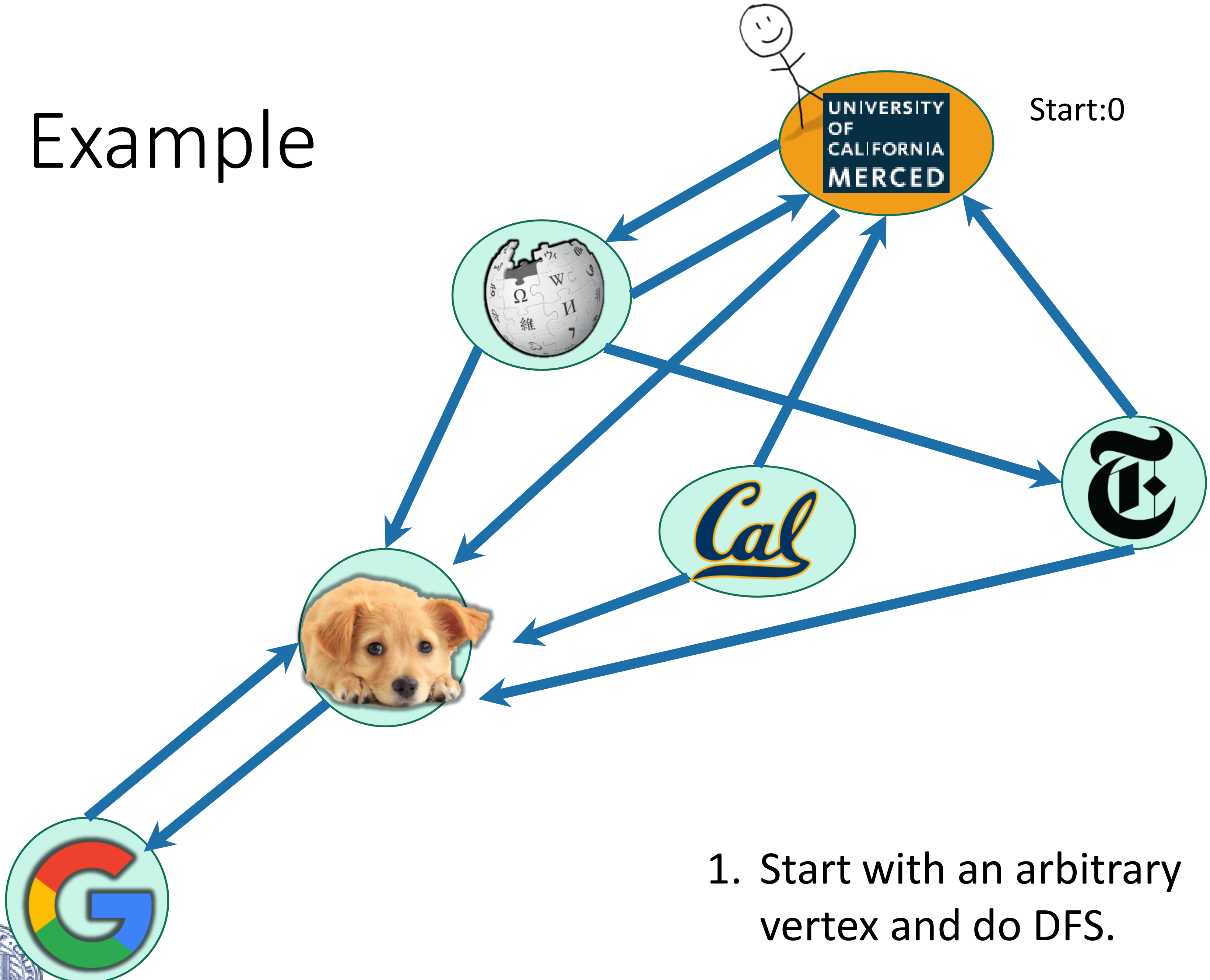


1. Start with an arbitrary vertex and do DFS.

Example



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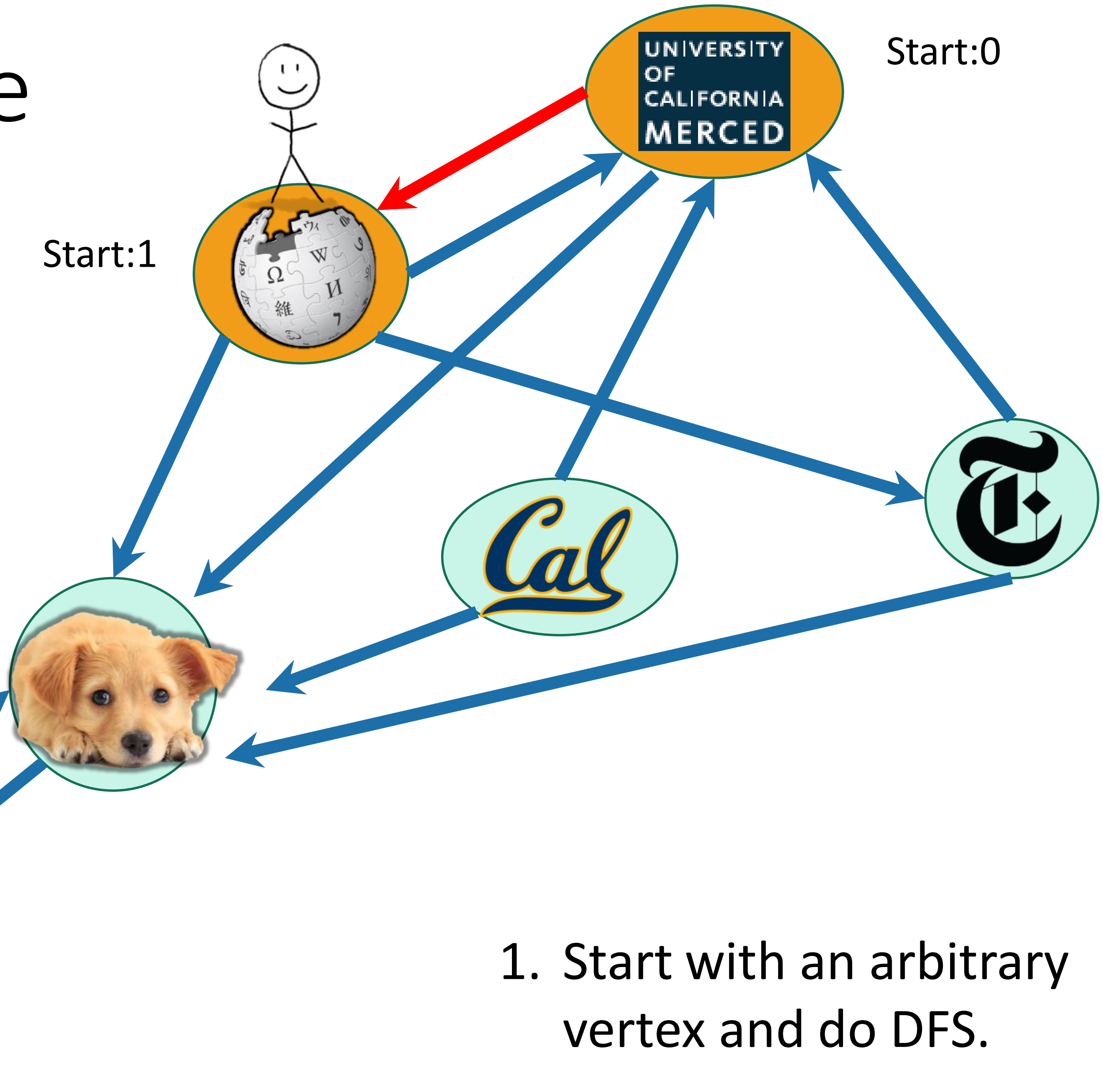

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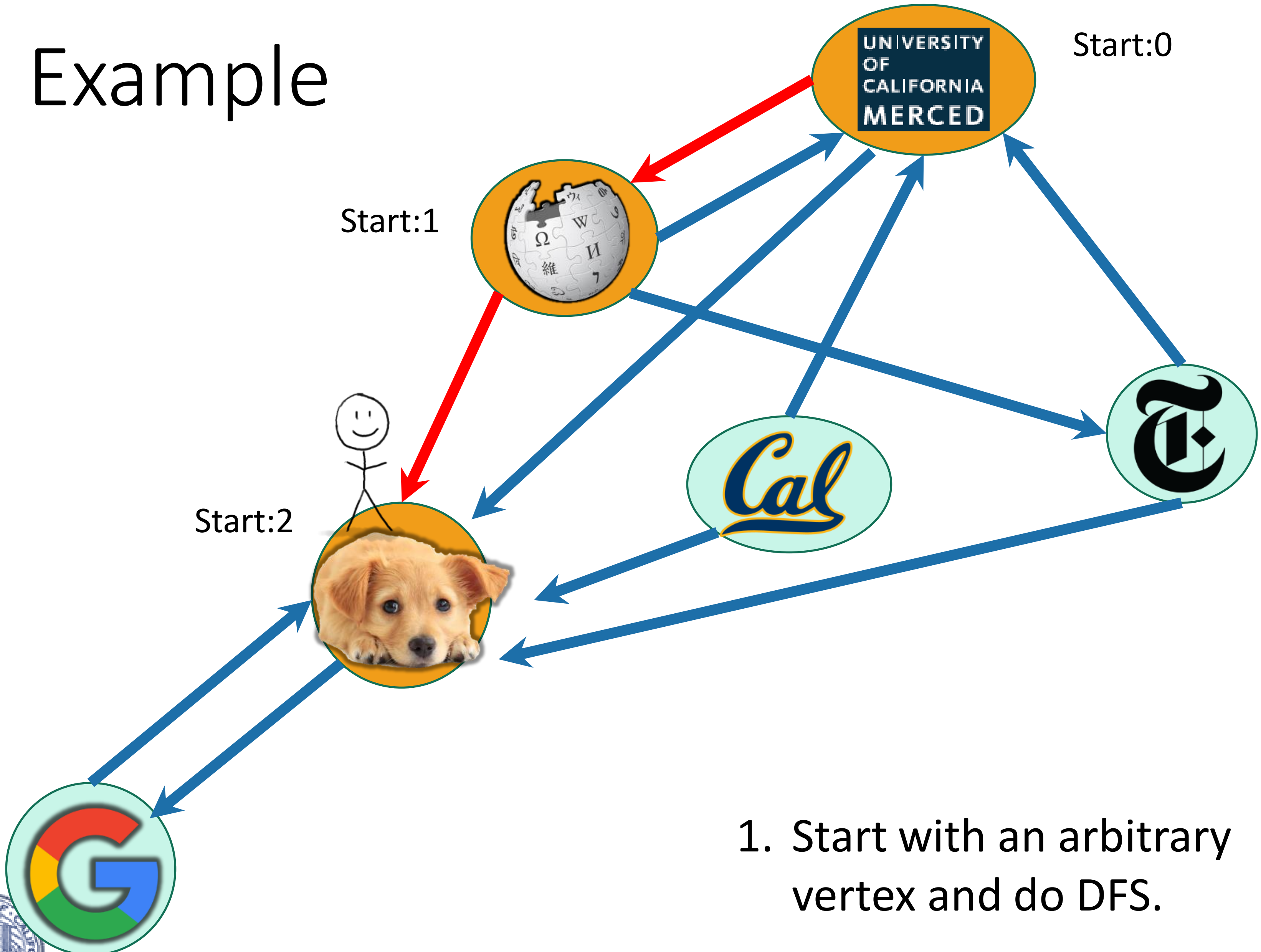

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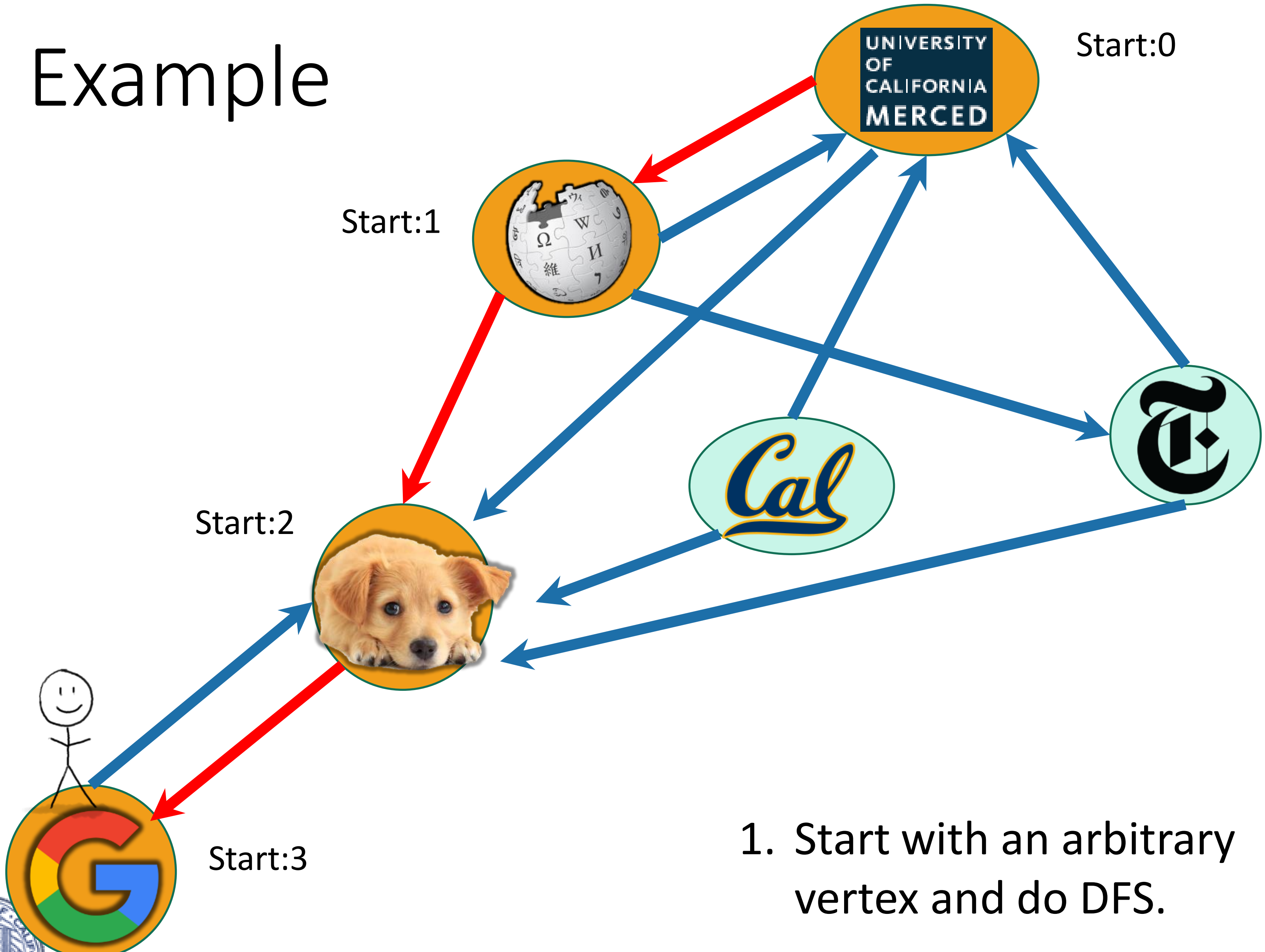

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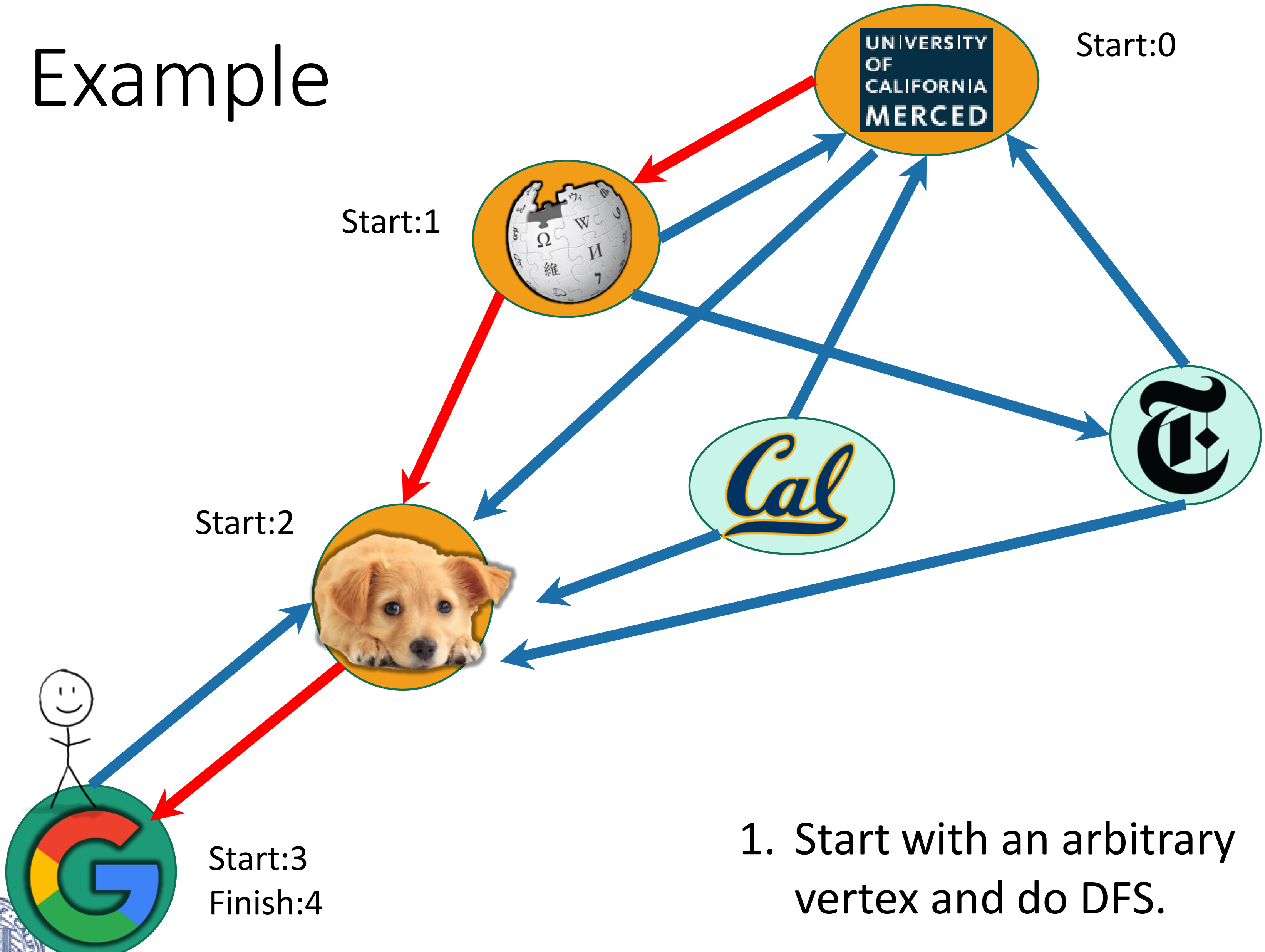

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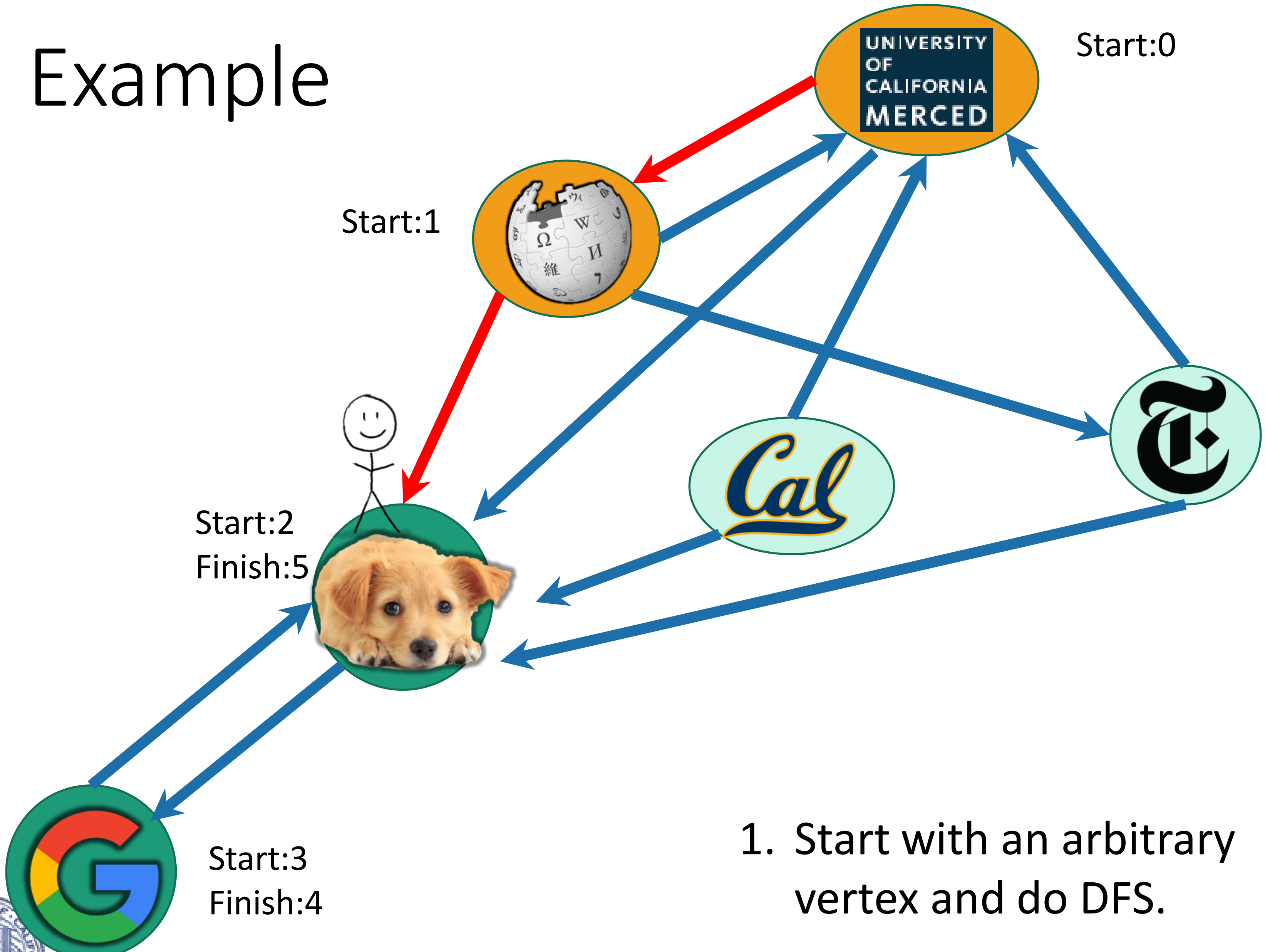

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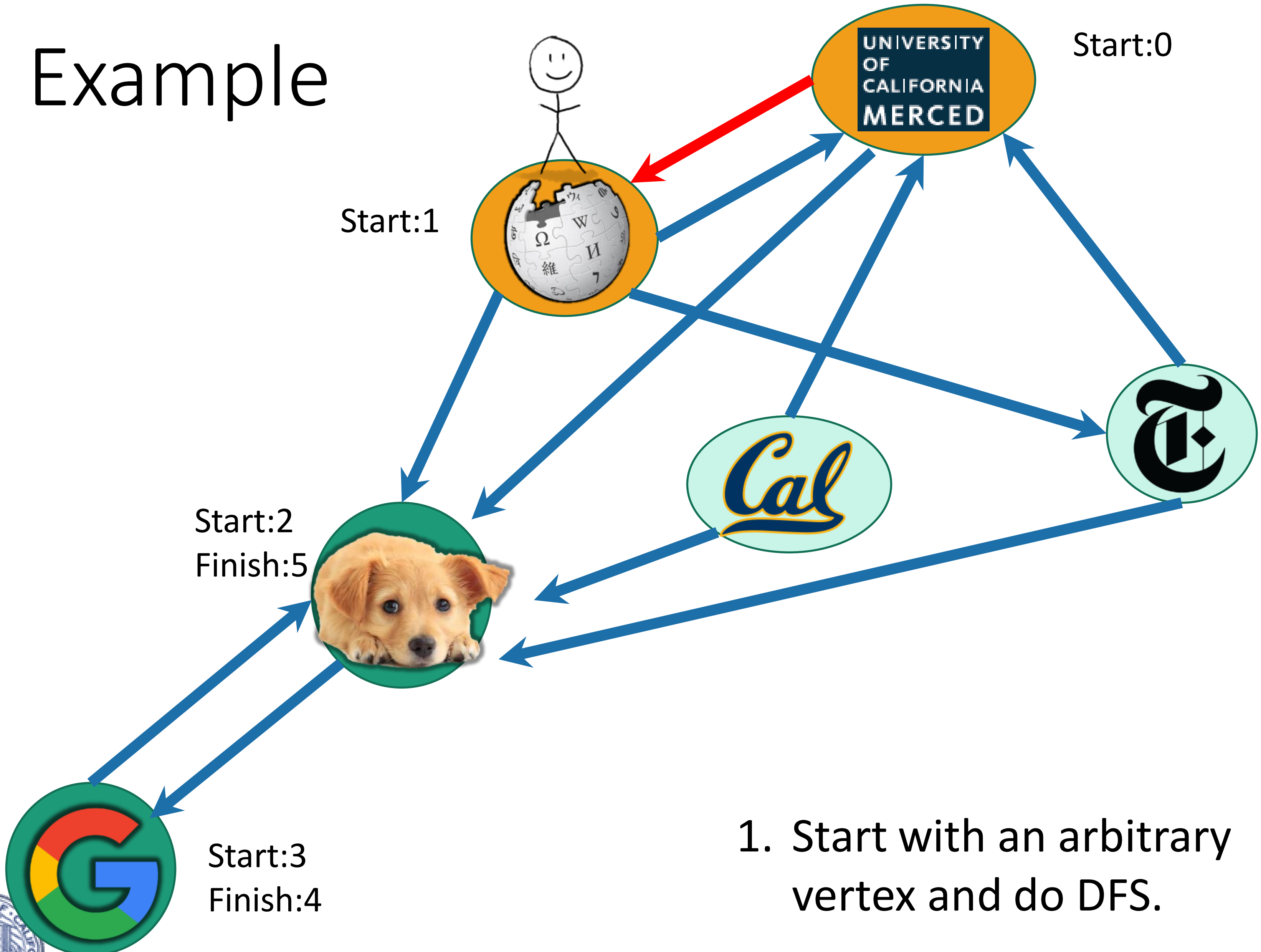

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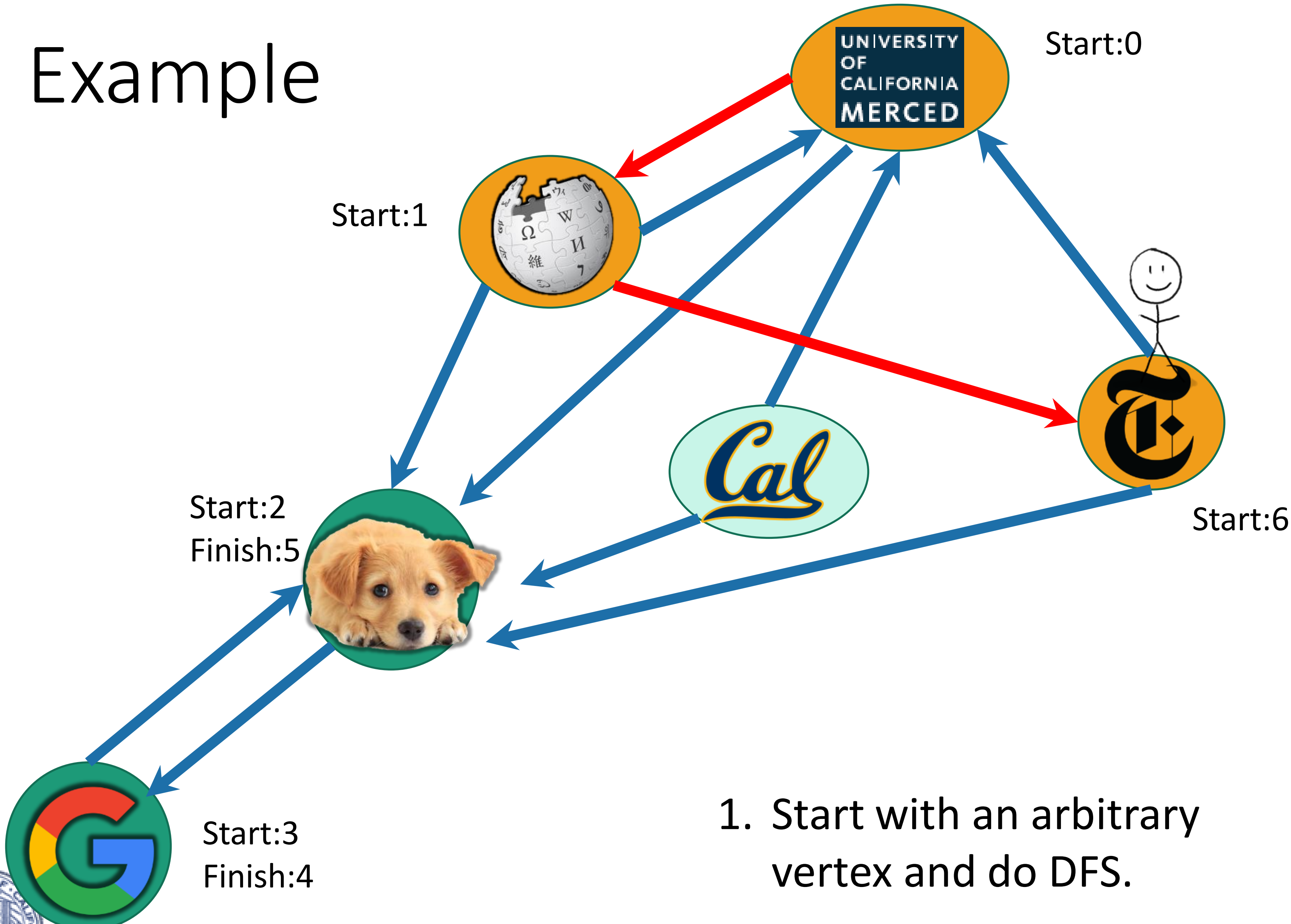

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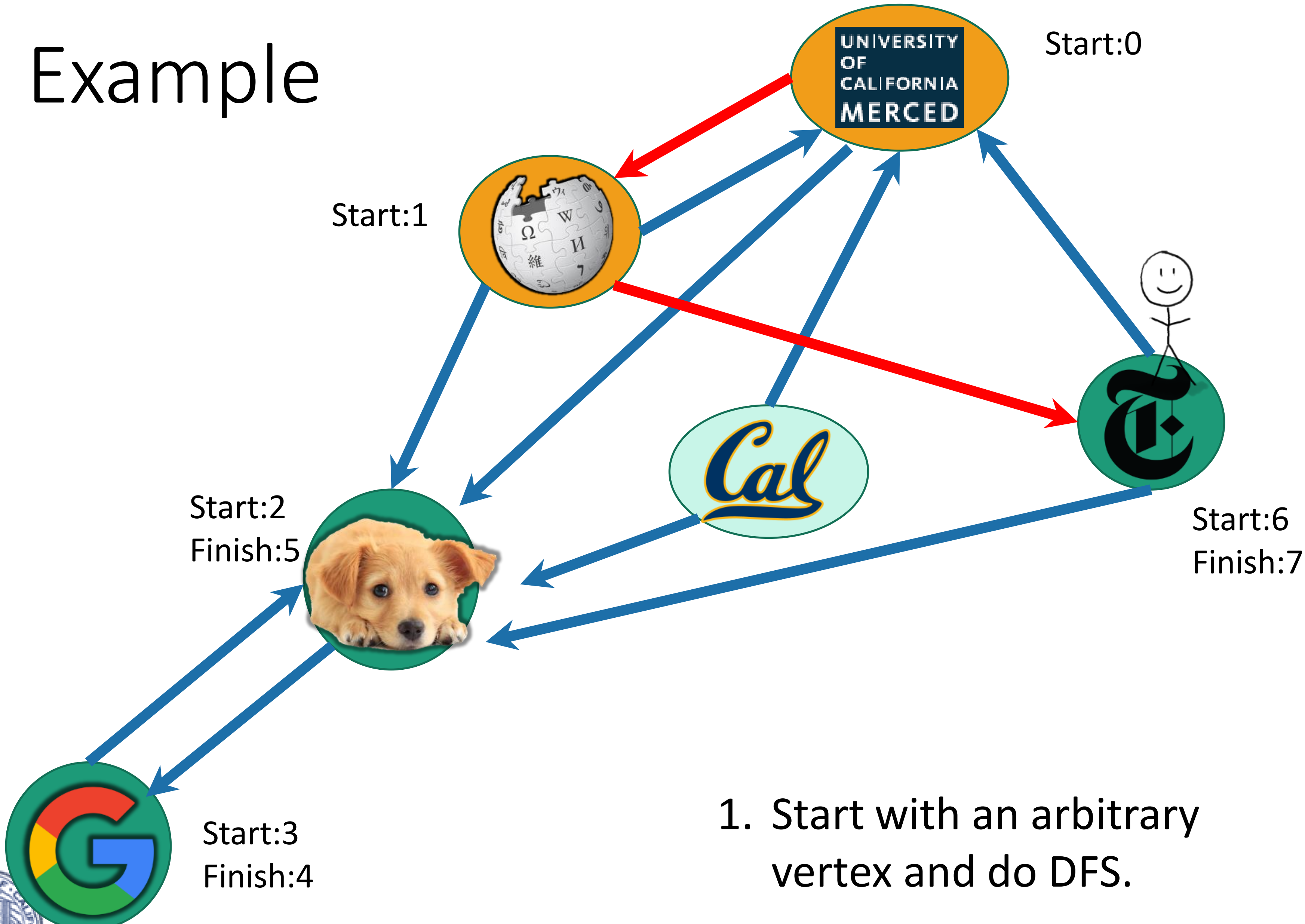

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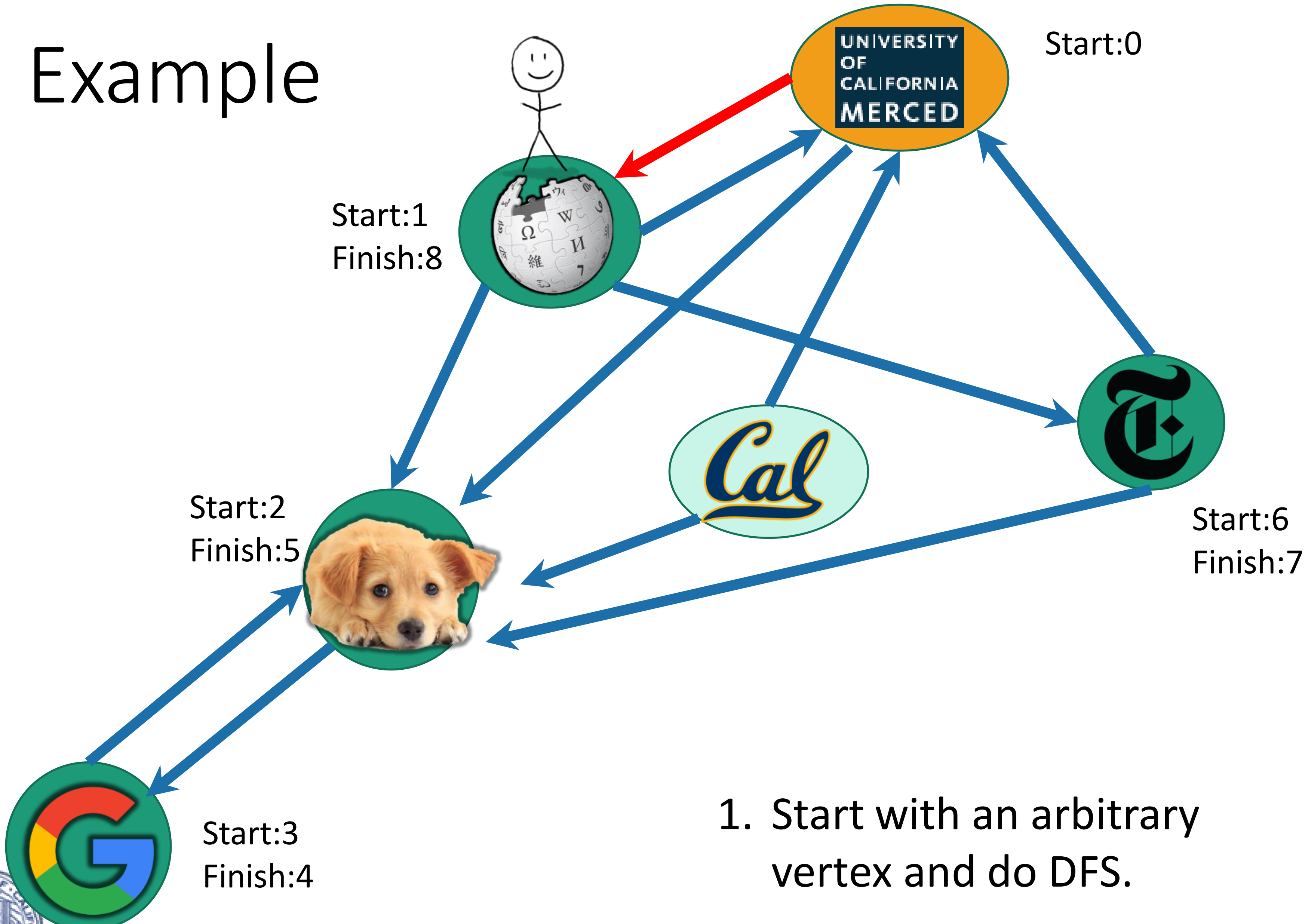

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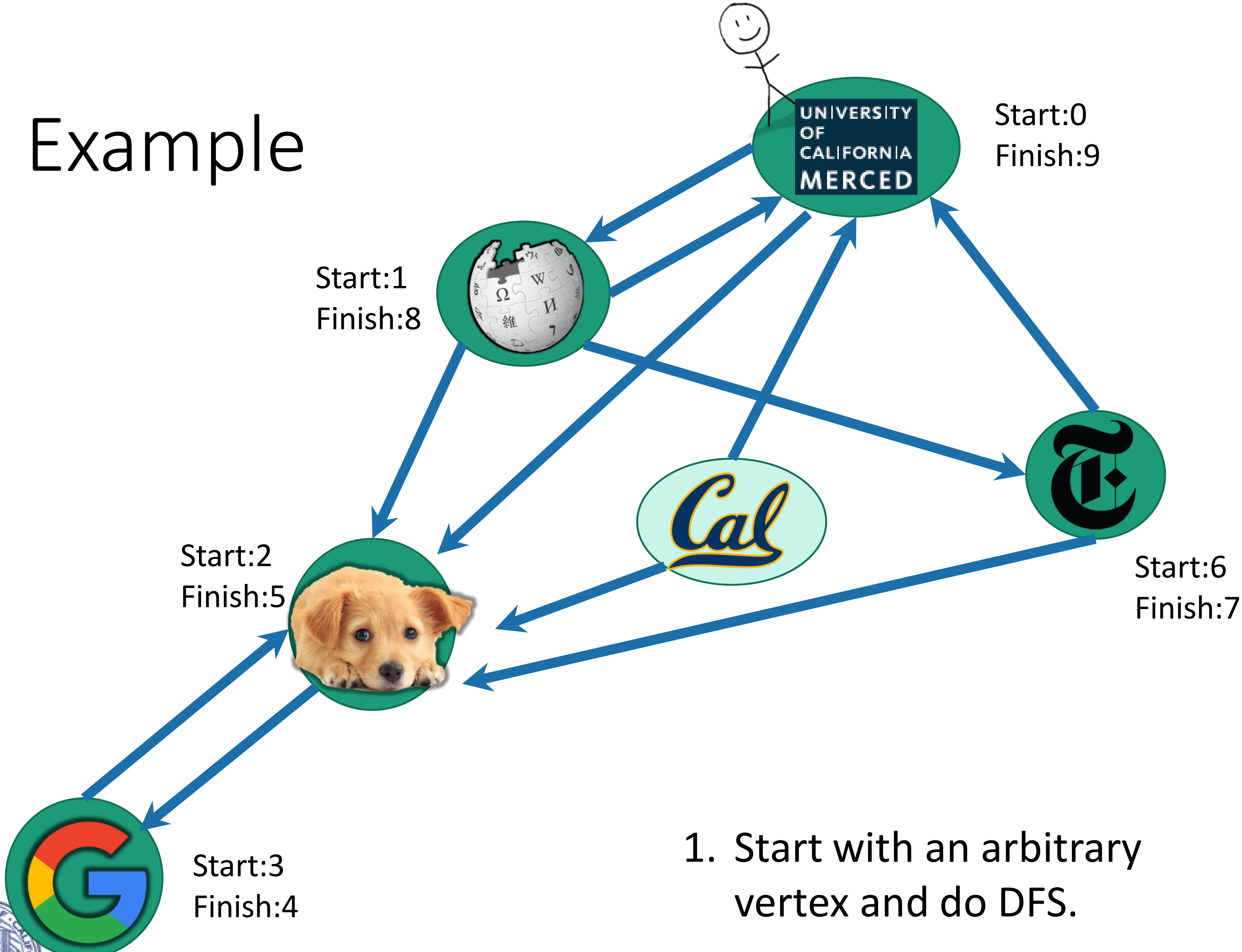

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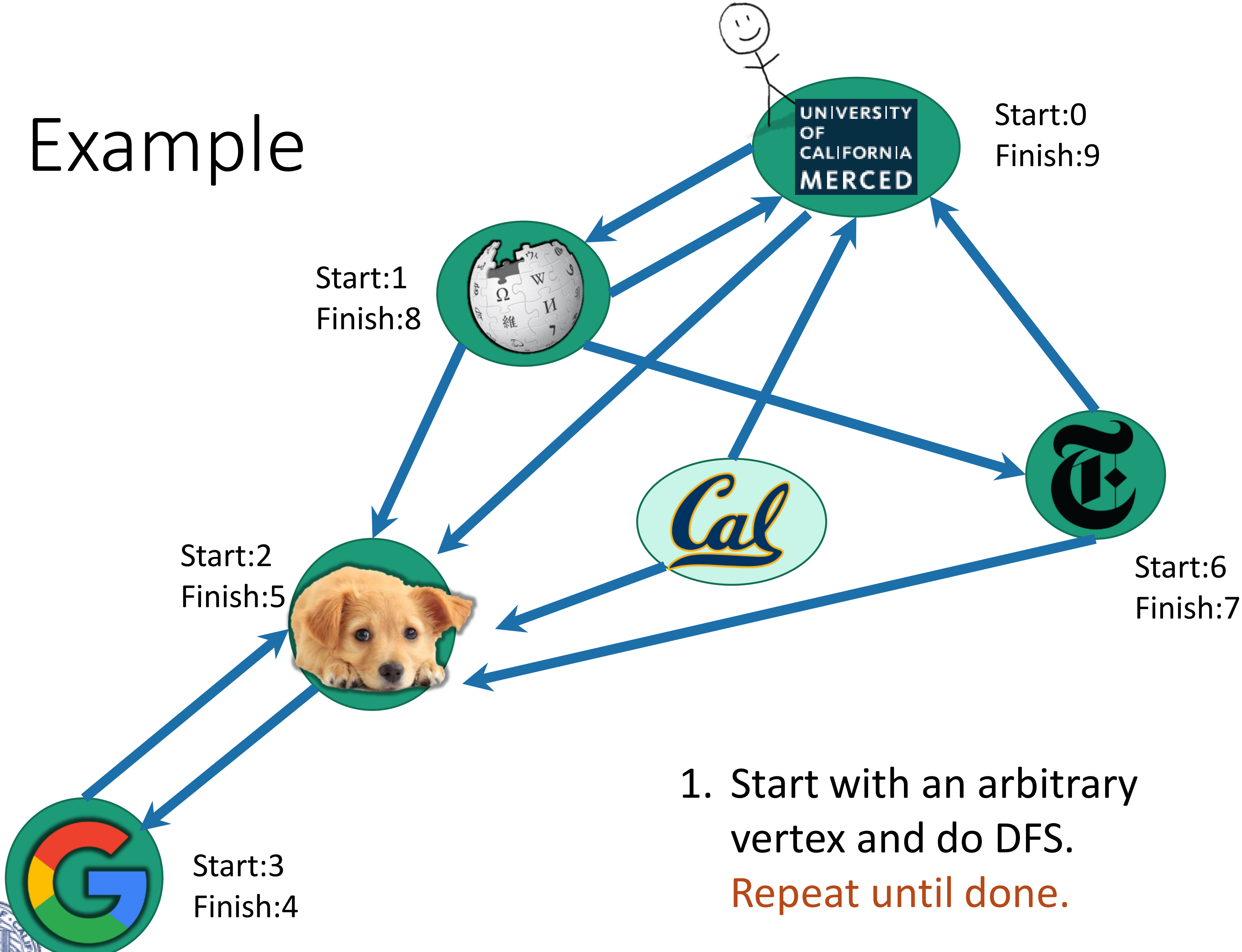

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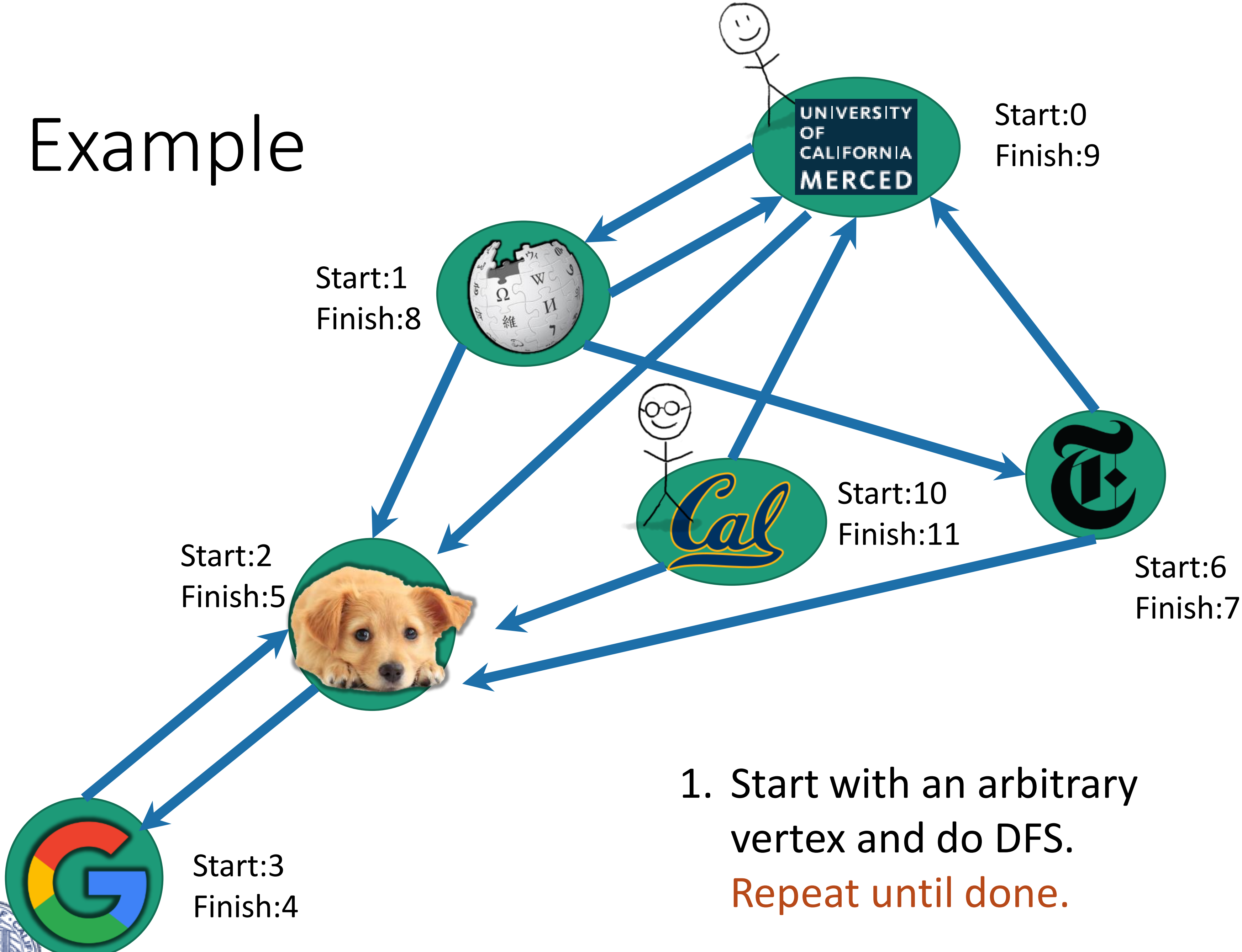

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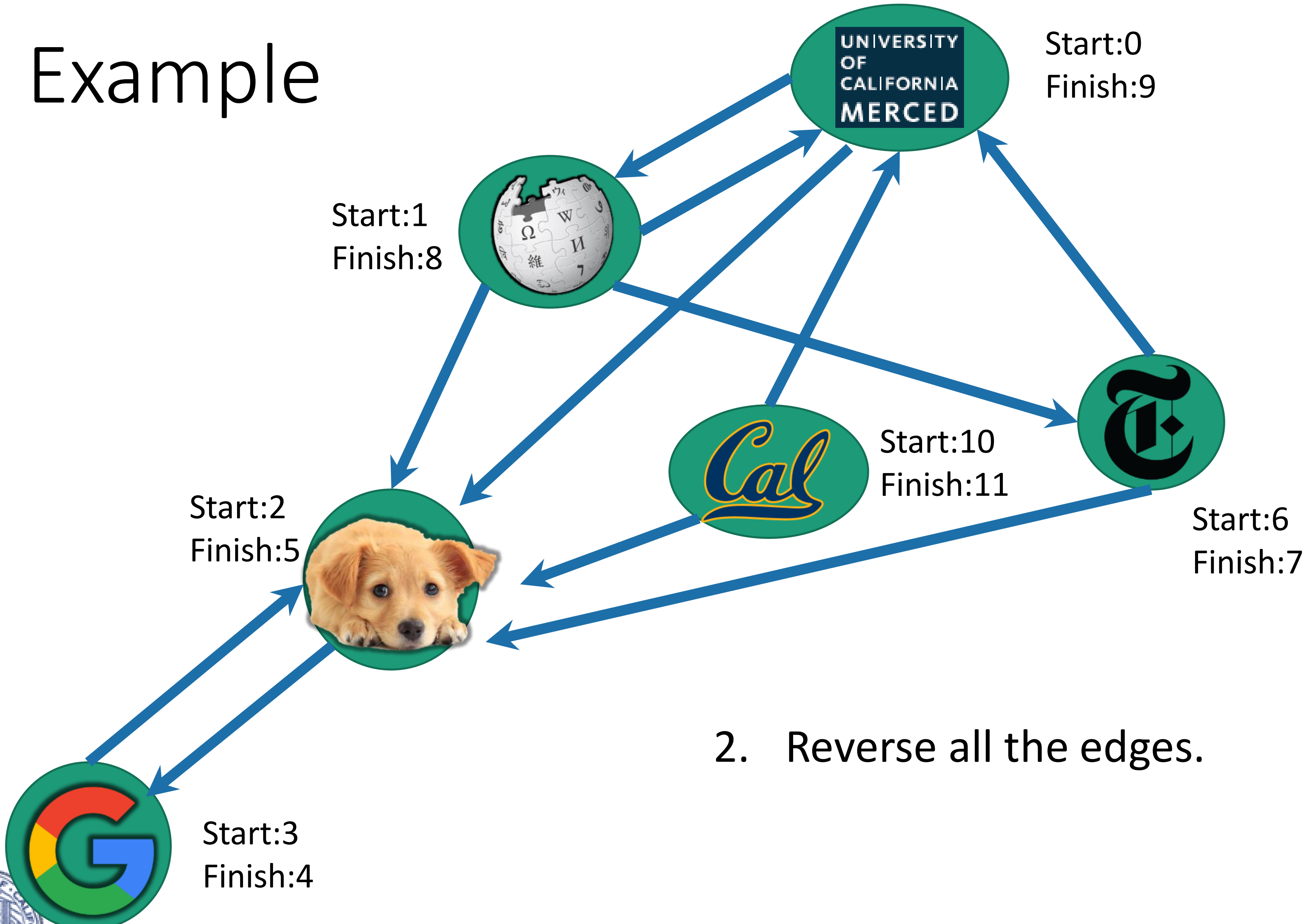
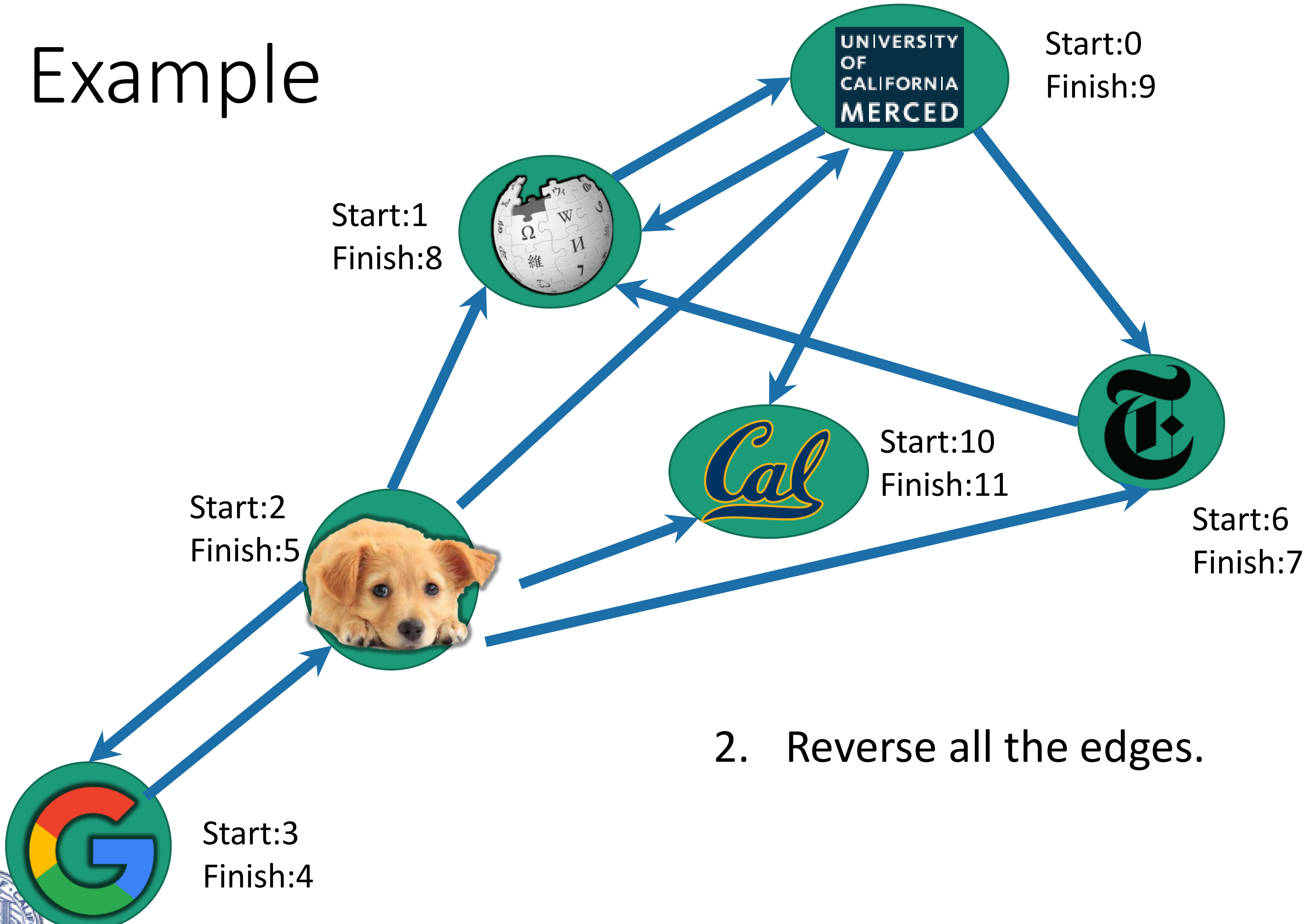


2. Reverse all the edges.

Example



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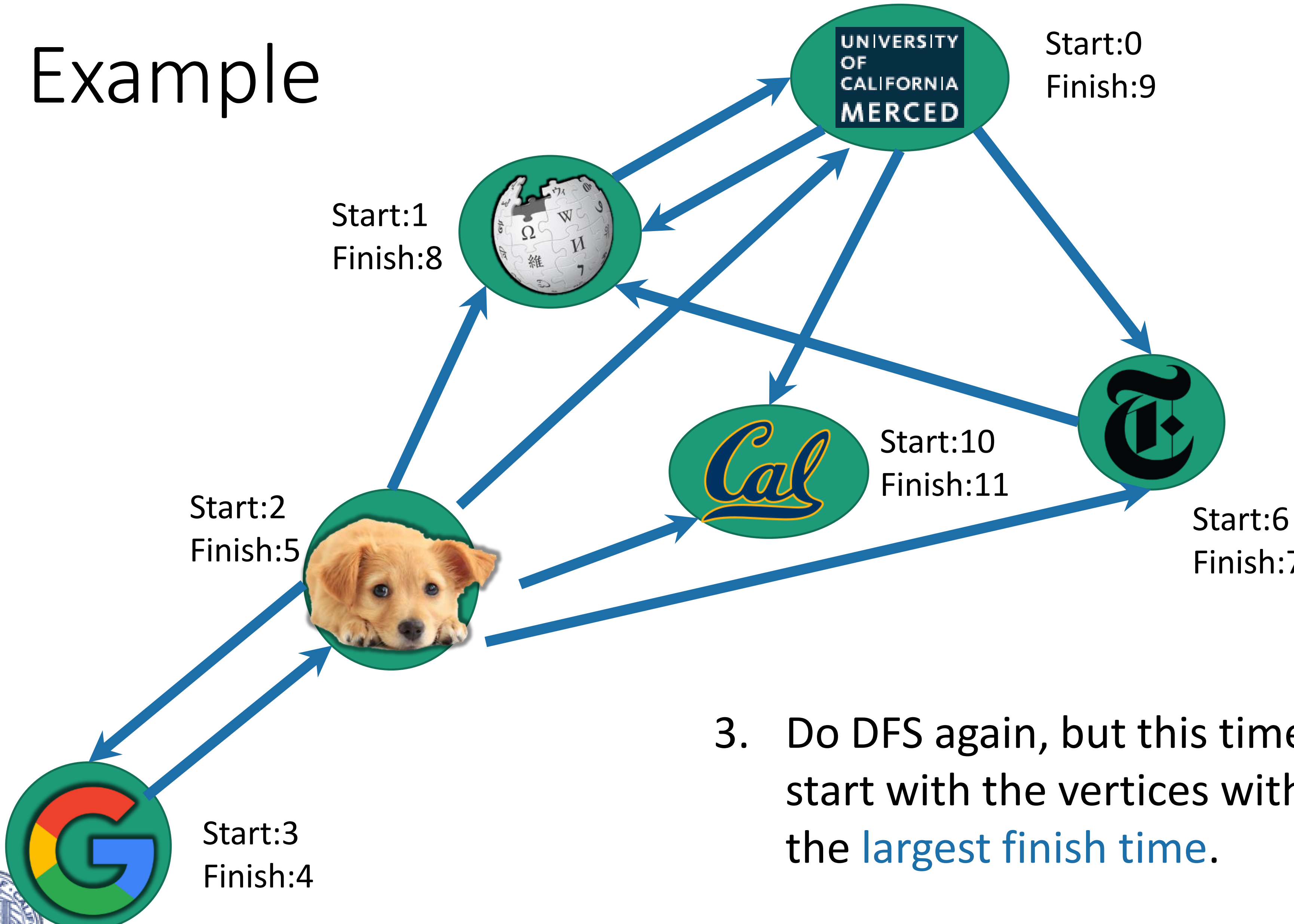

3. Do DFS again, but this time, start with the vertices with the largest finish time.

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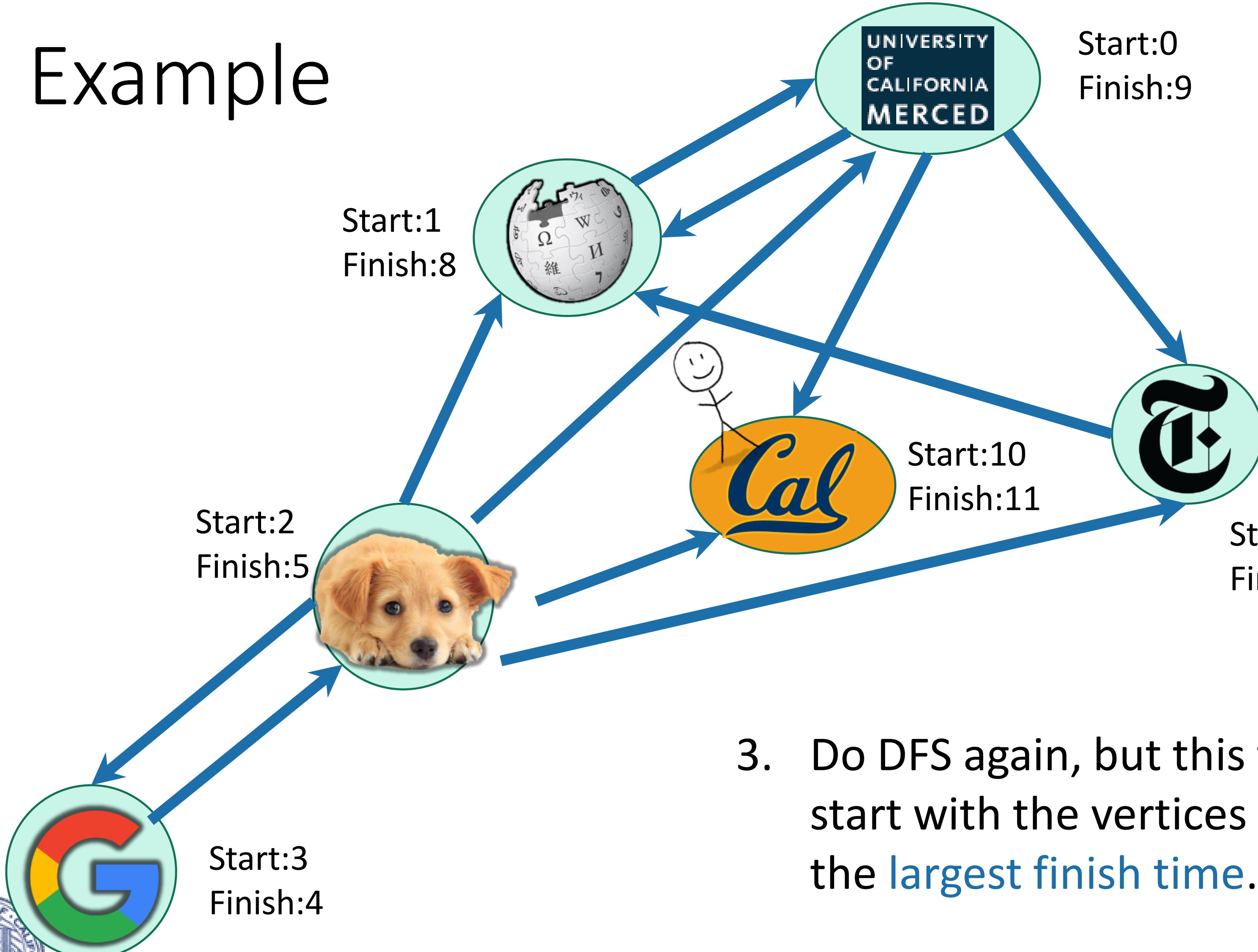

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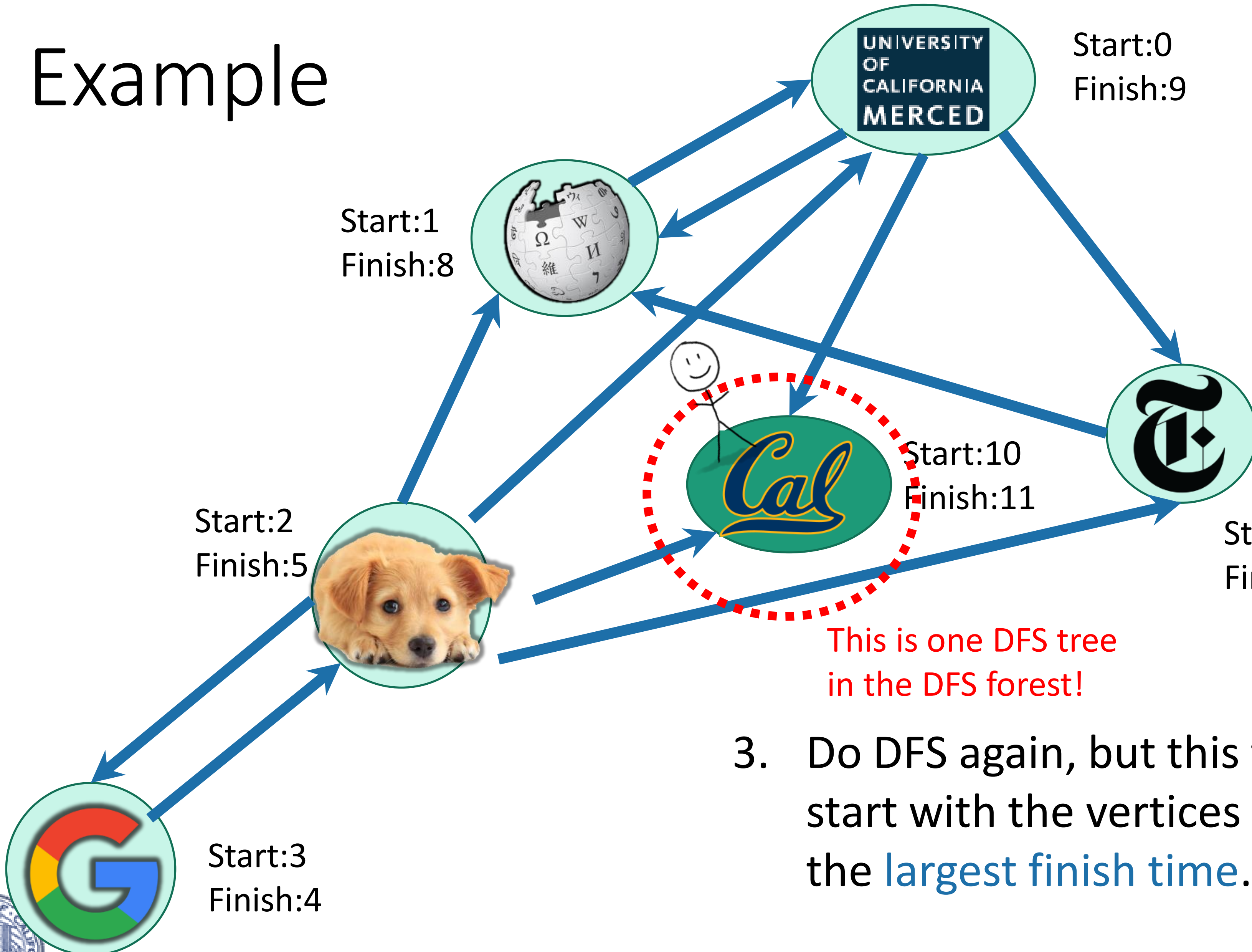

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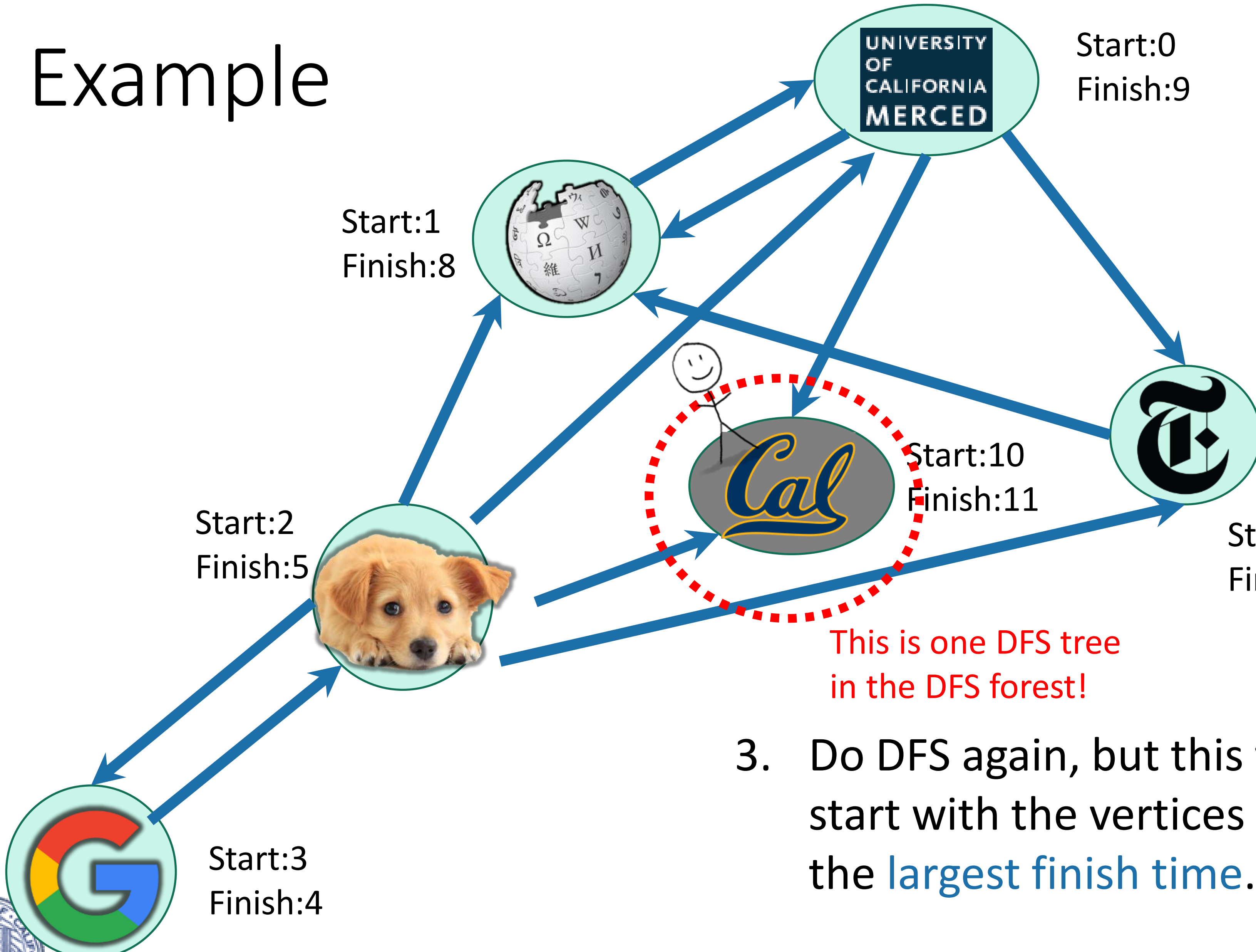

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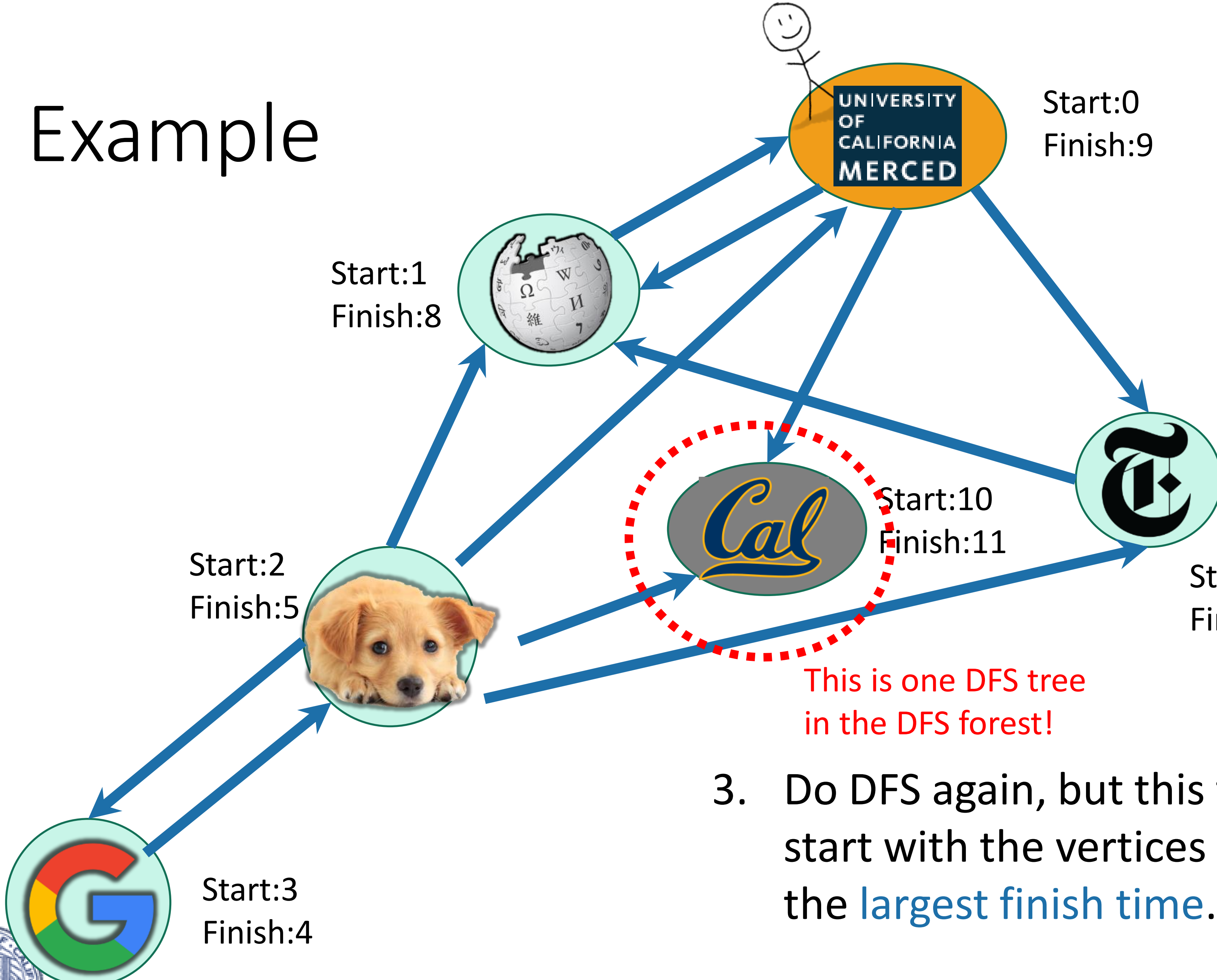

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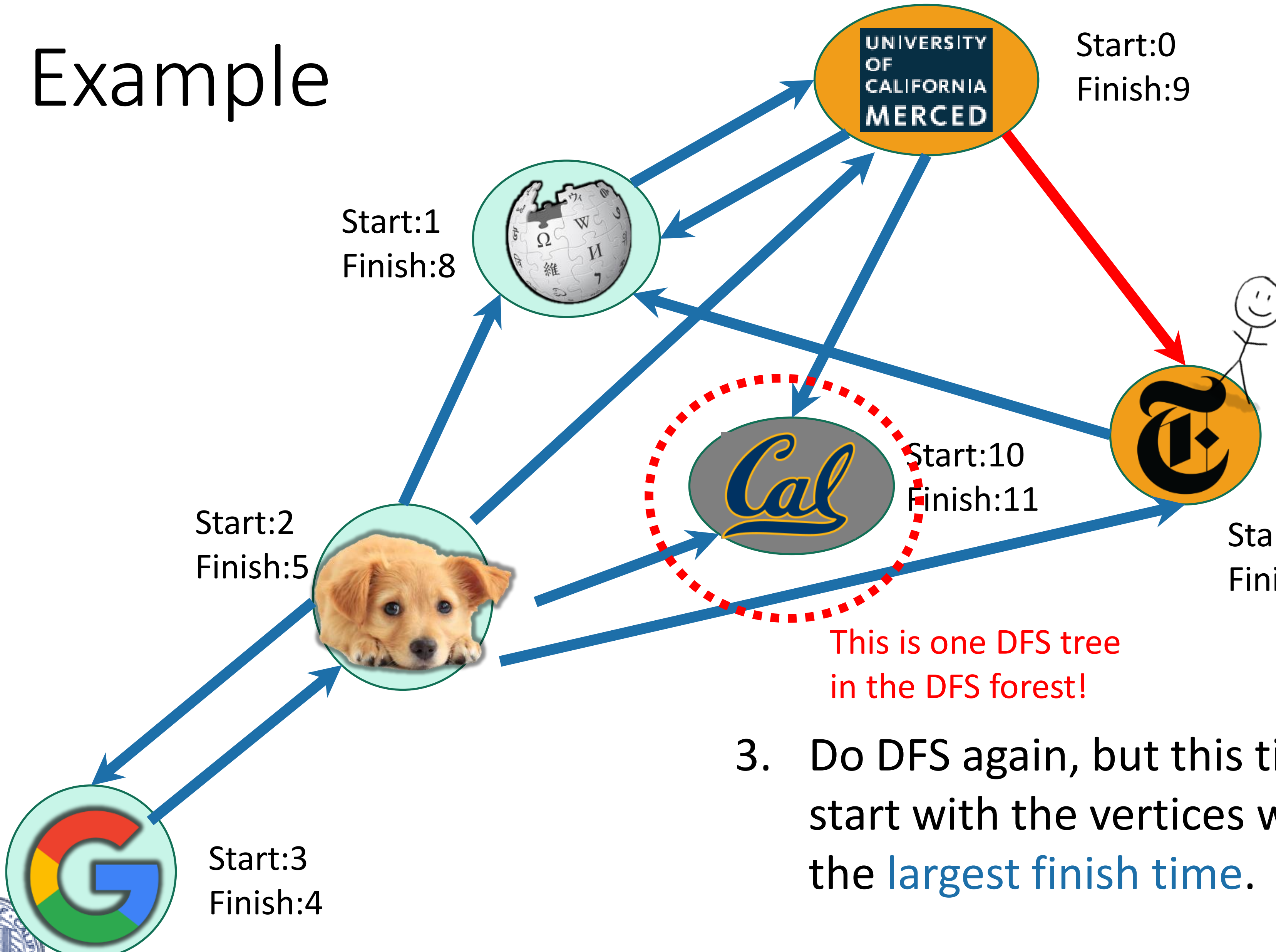

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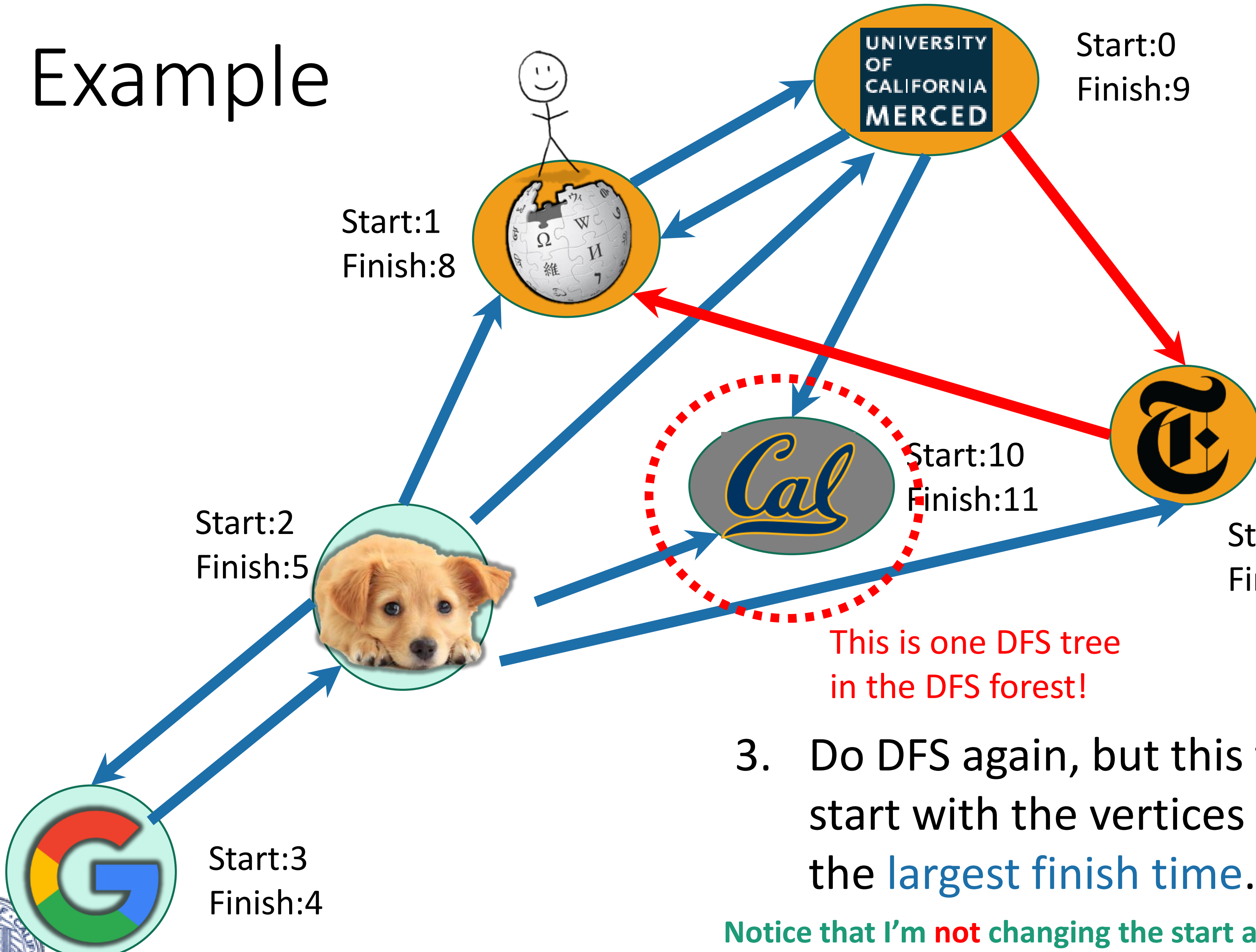

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Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

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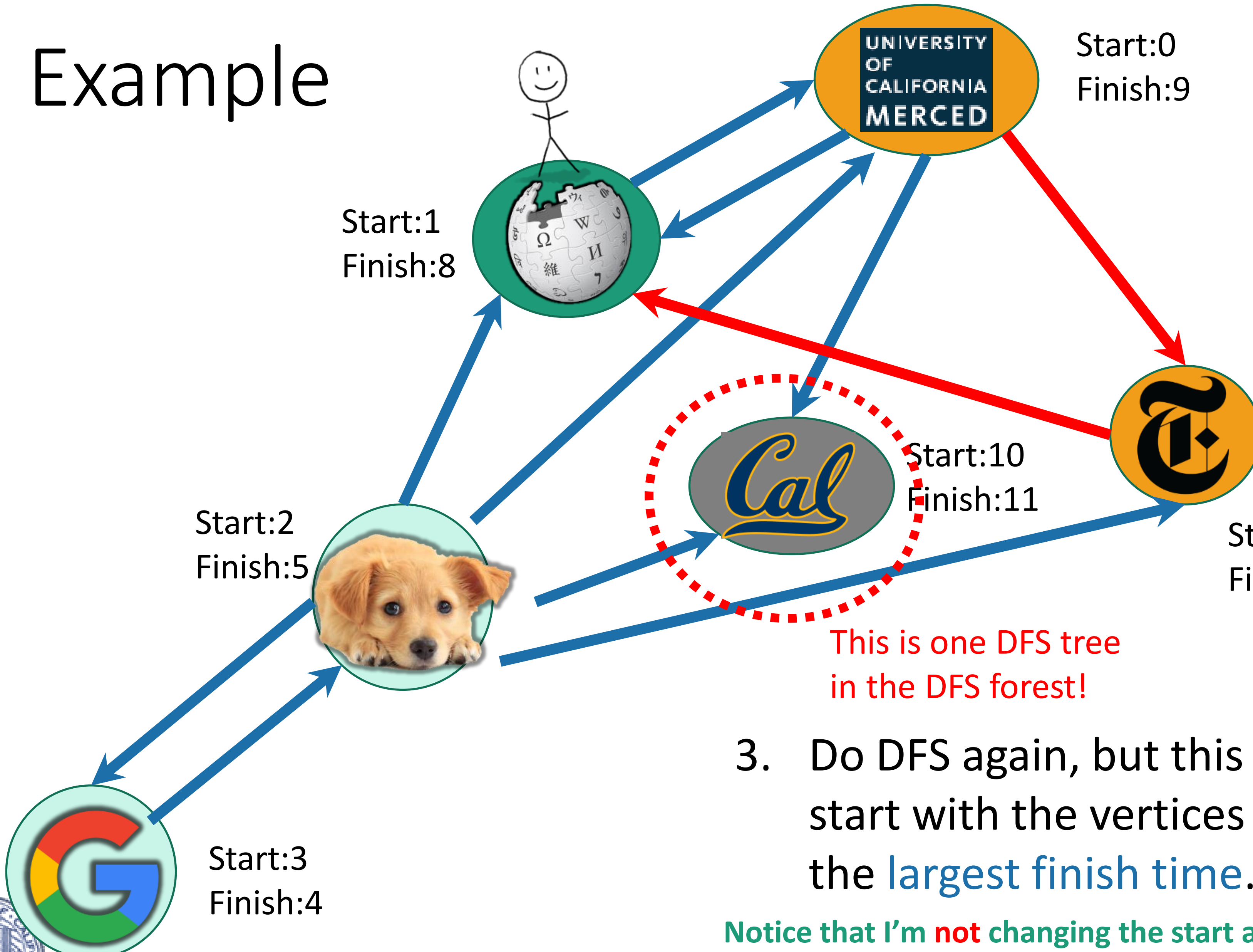

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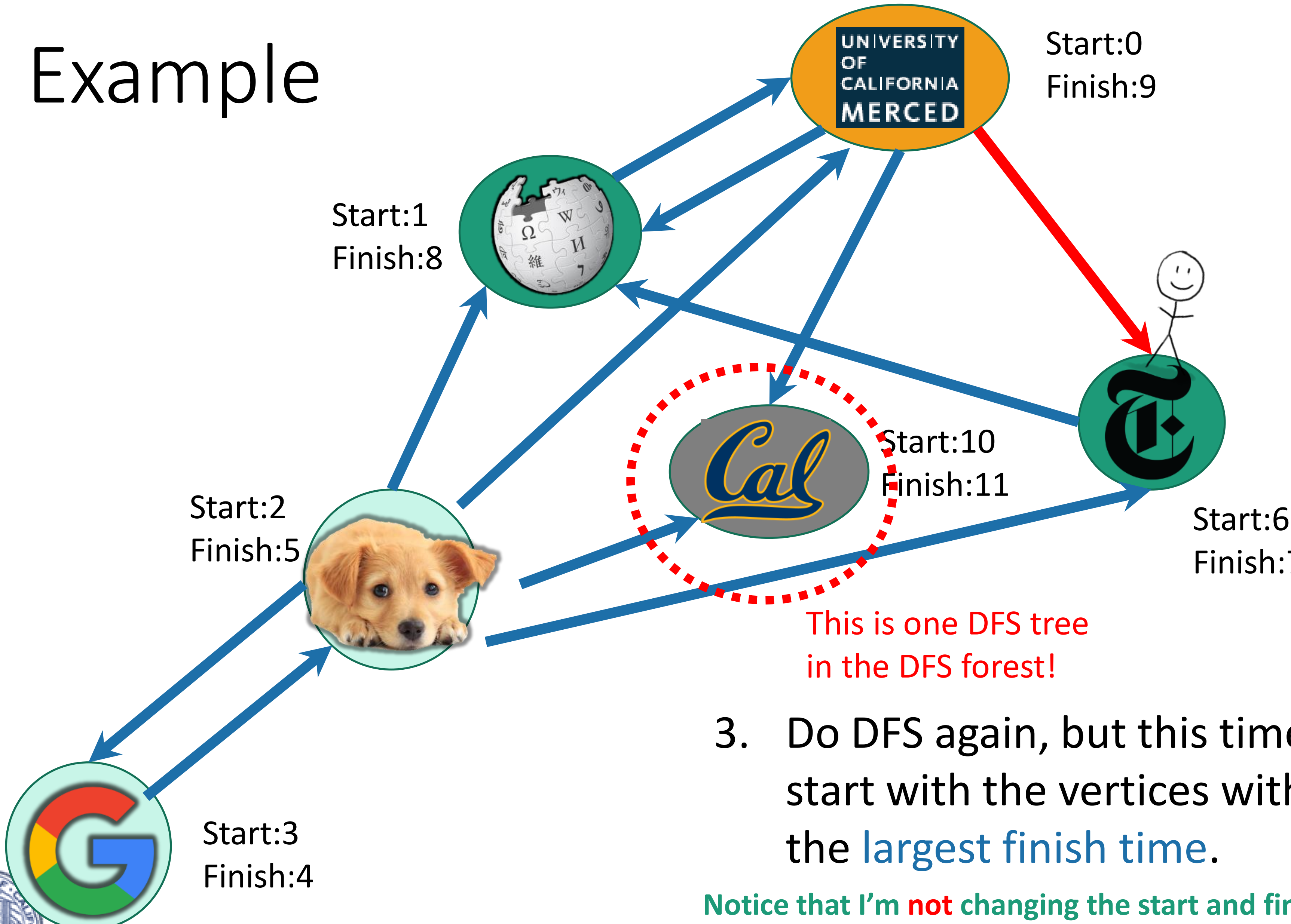

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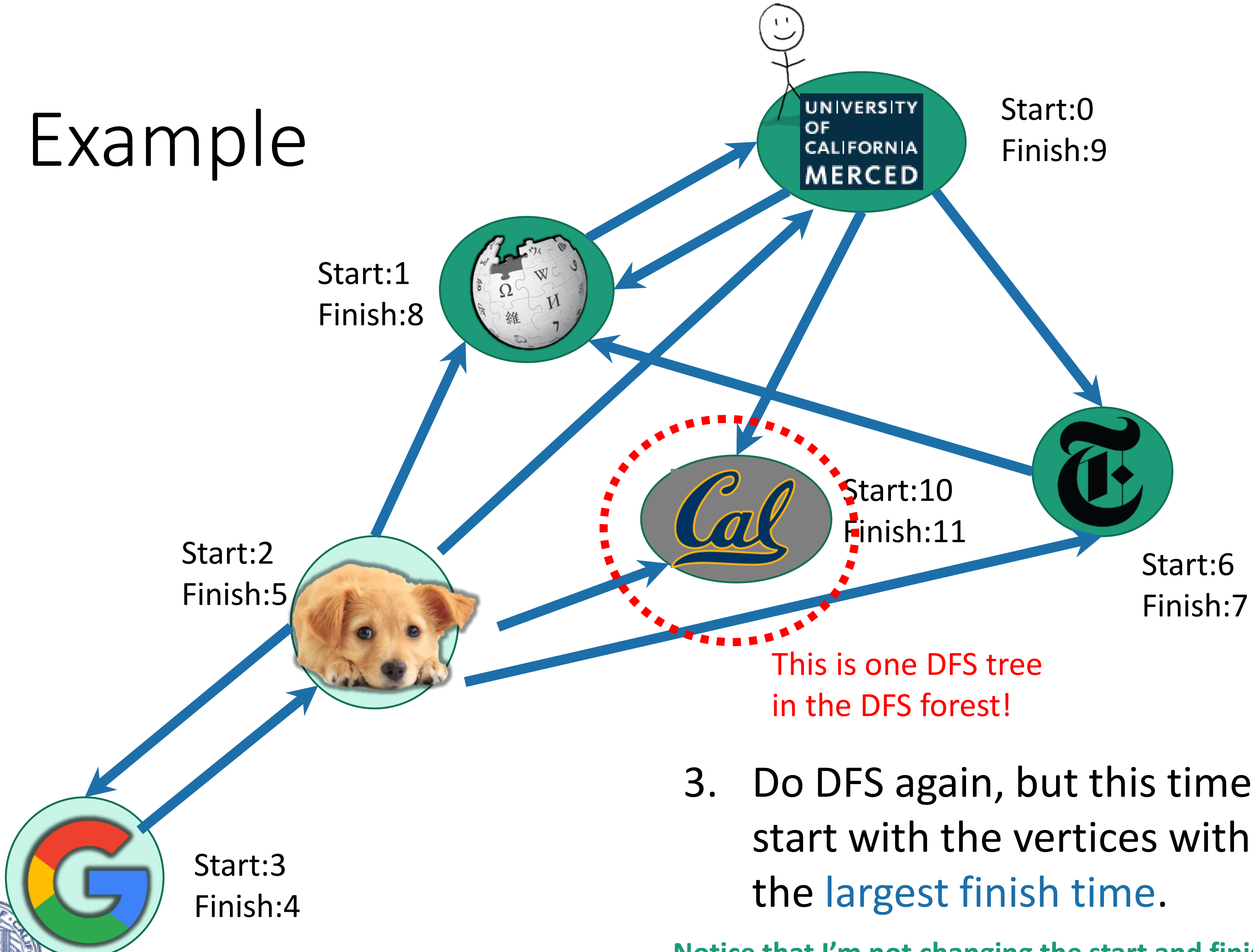

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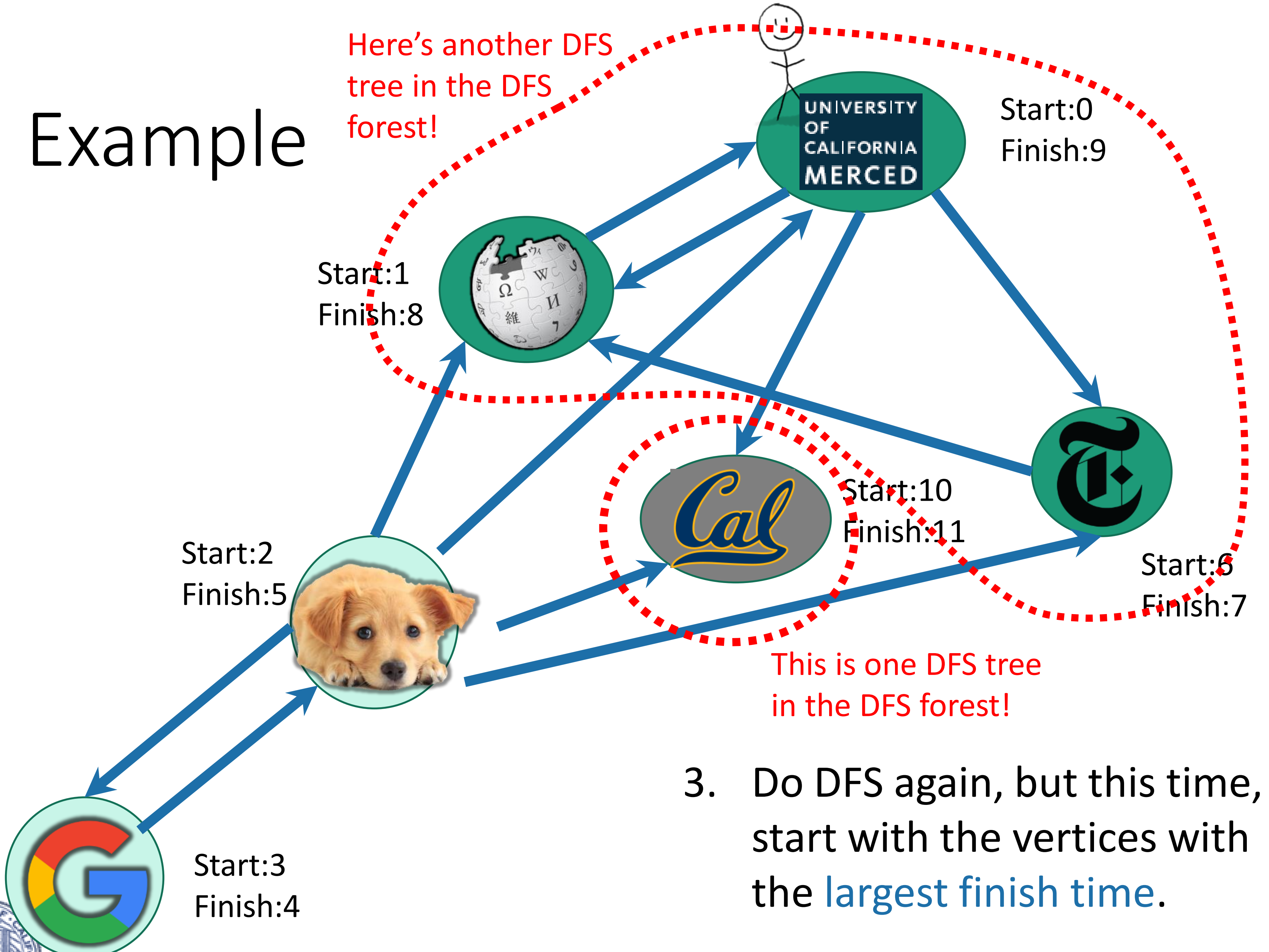

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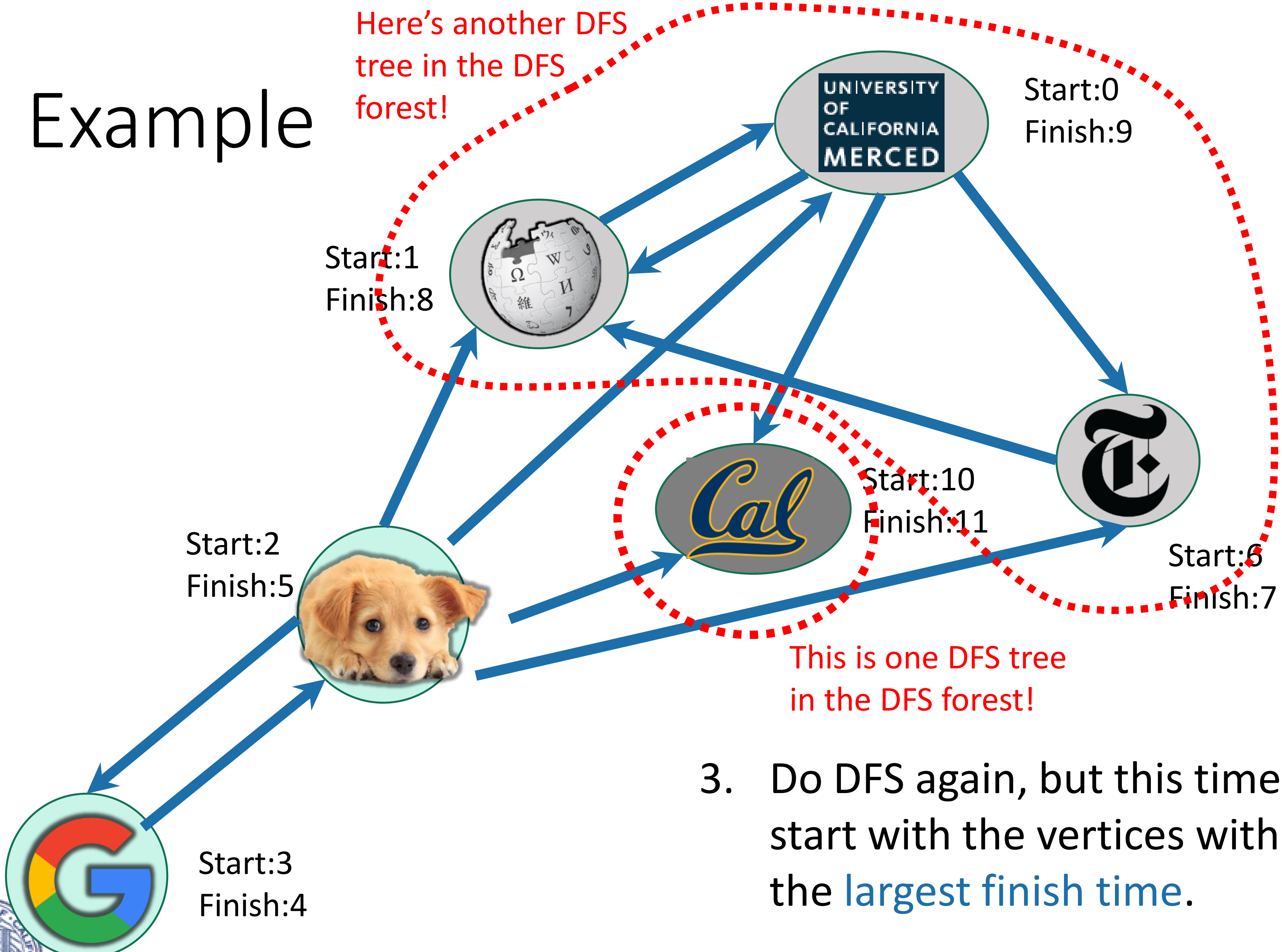

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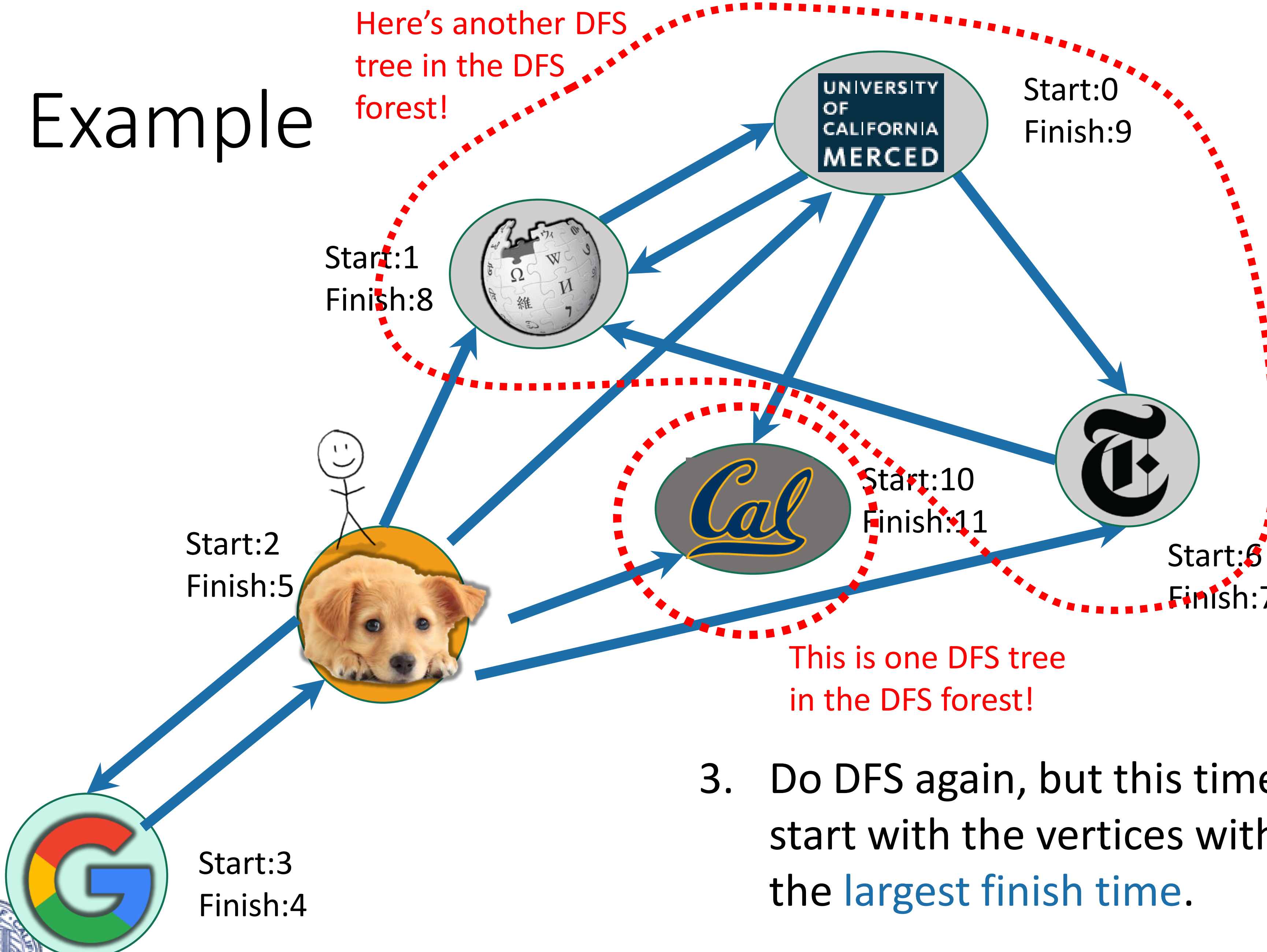

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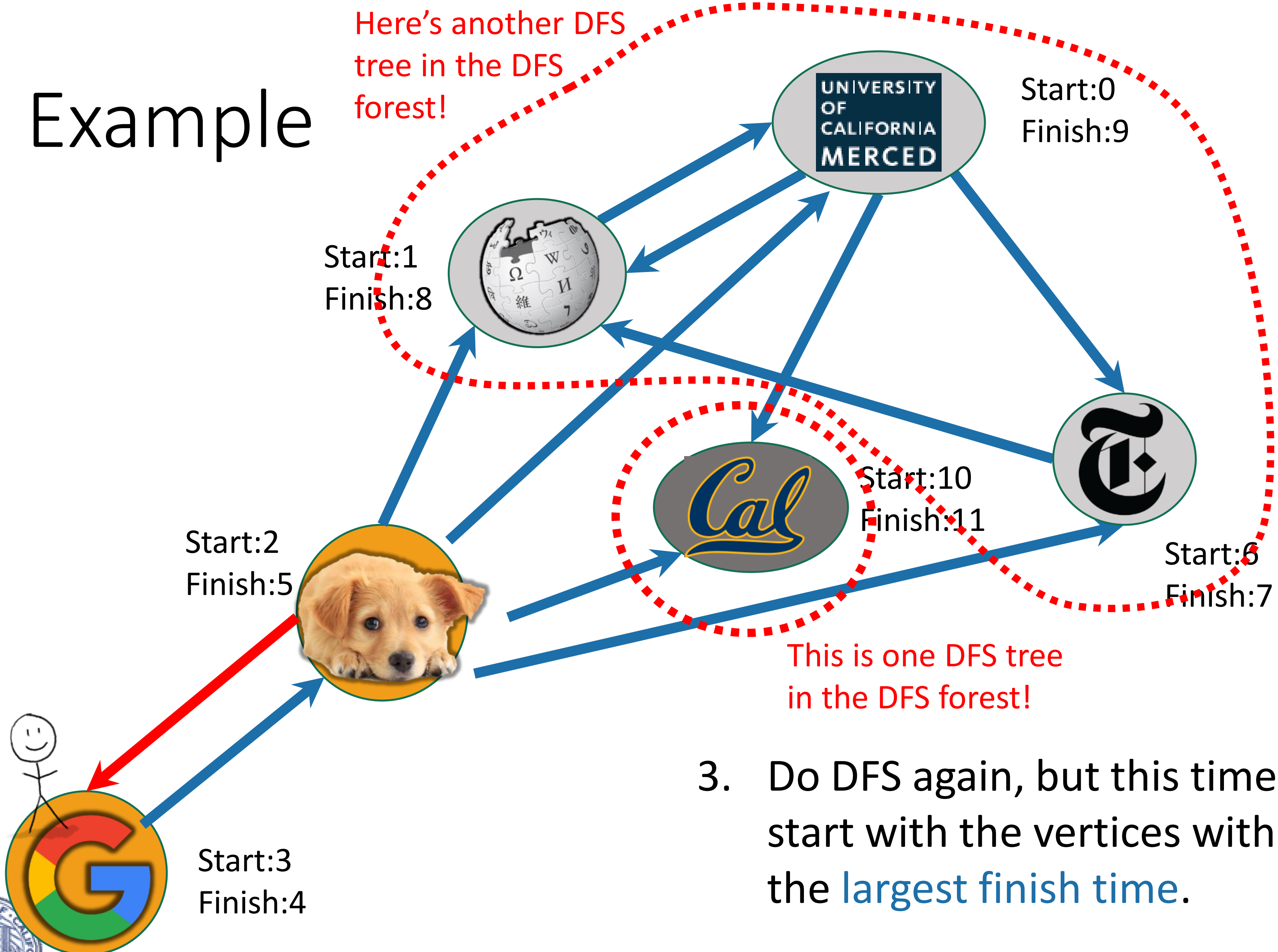

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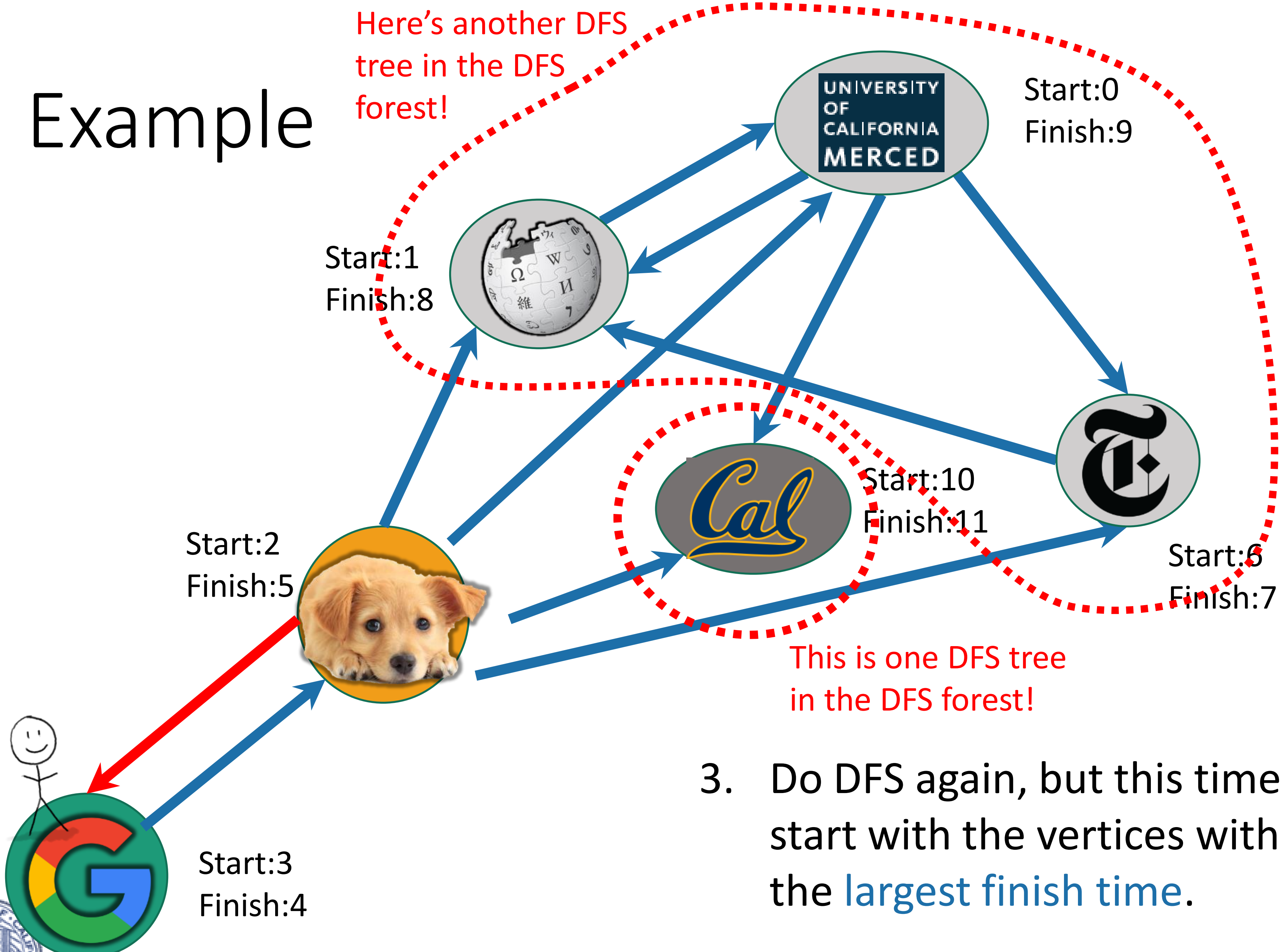

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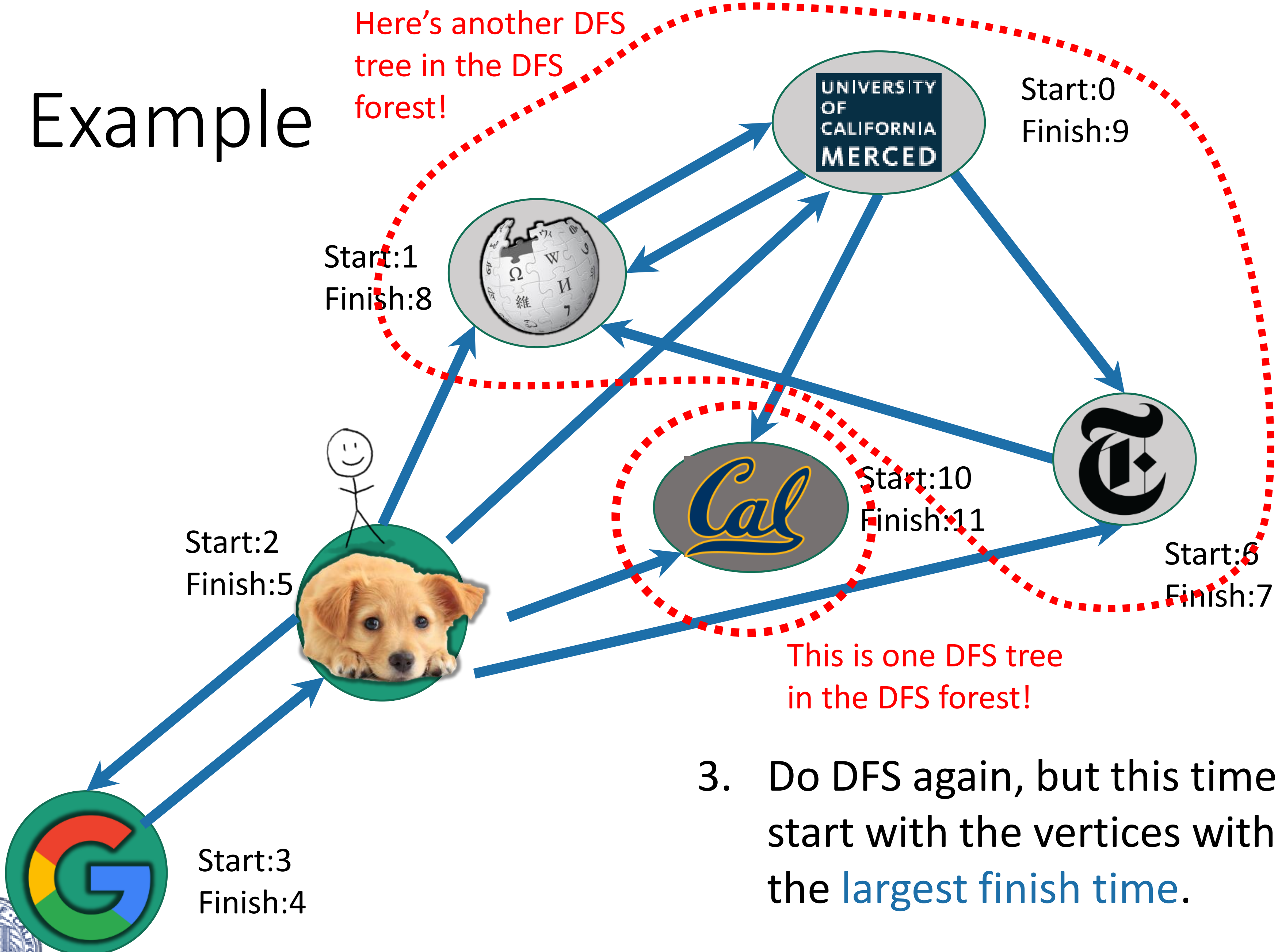

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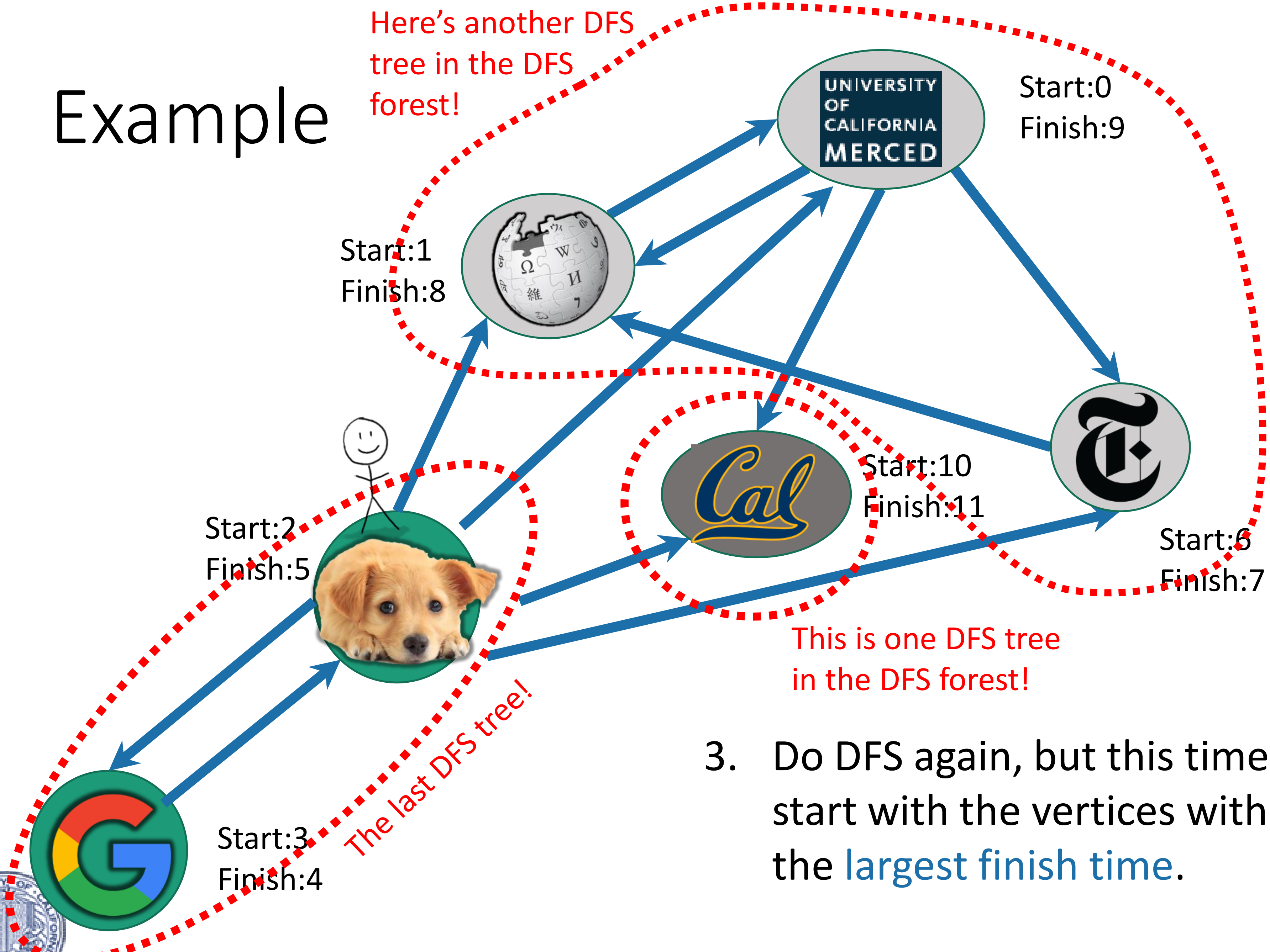

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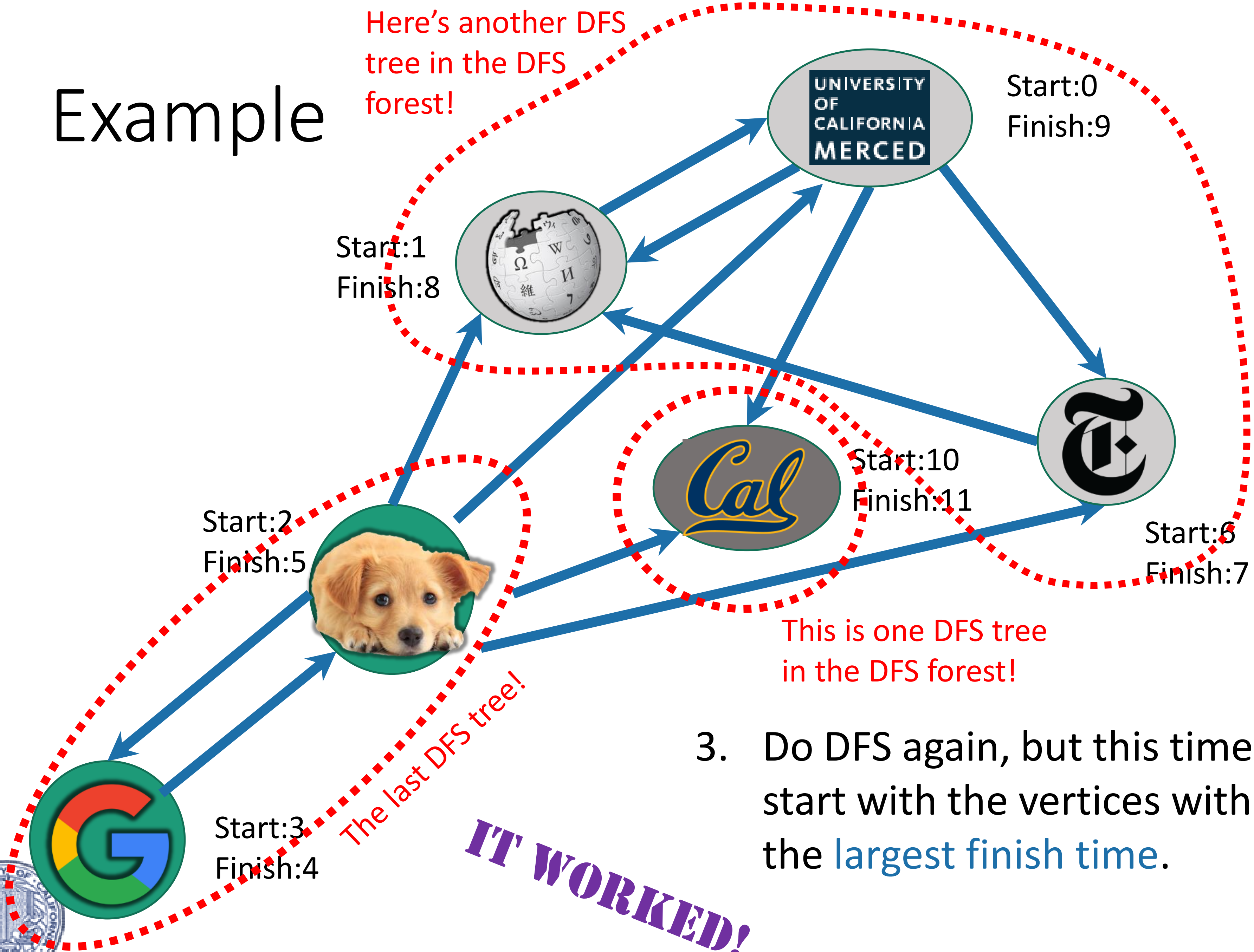

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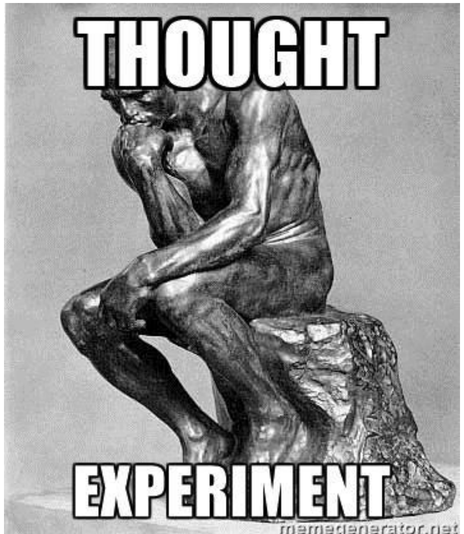
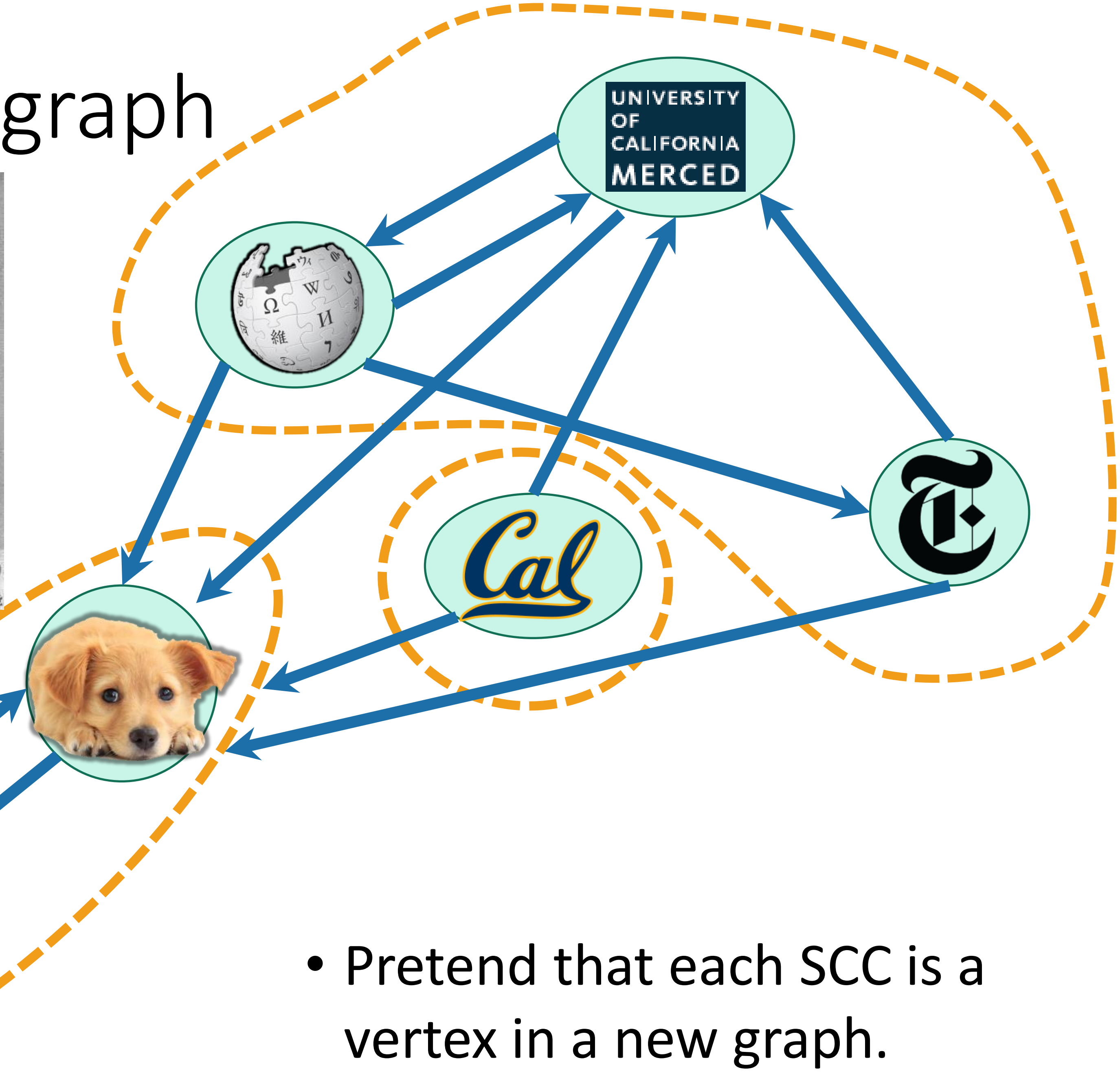
One question

The SCC graph



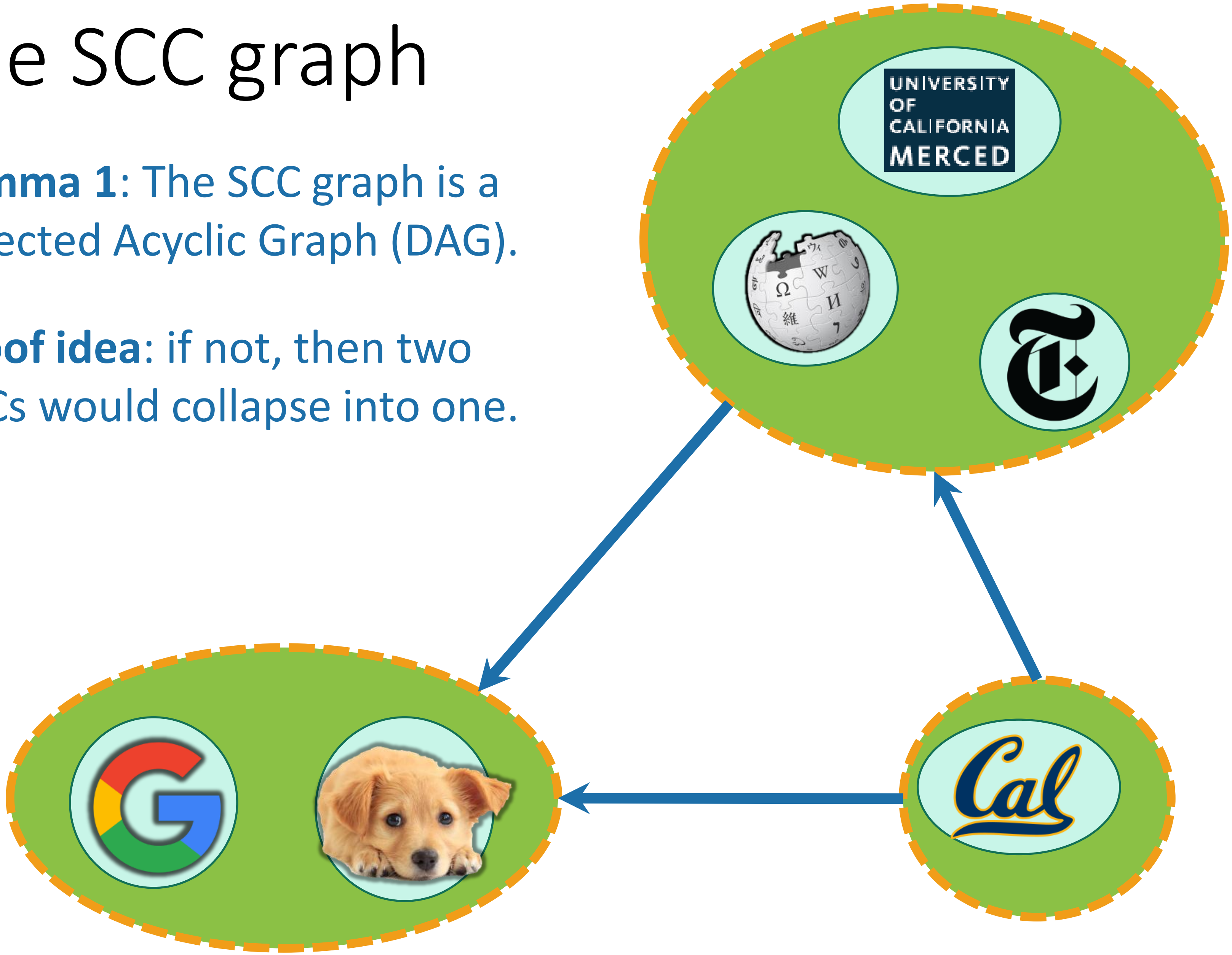


- Pretend that each SCC is a vertex in a new graph.

The SCC graph

Lemma 1: The SCC graph is a Directed Acyclic Graph (DAG).

Proof idea: if not, then two SCCs would collapse into one.

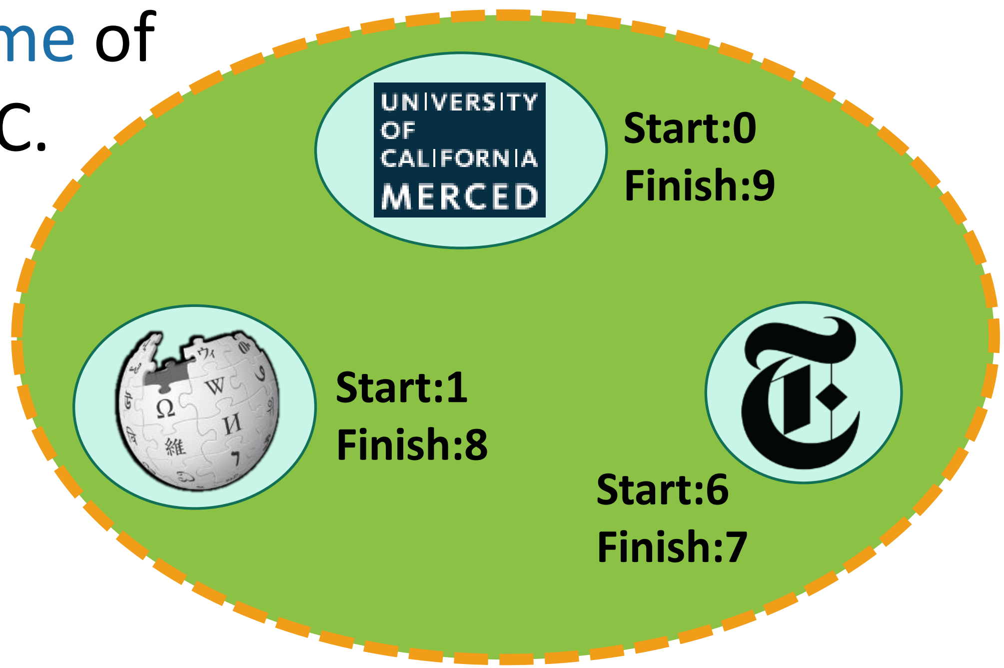


all times are with respect
to the **first** DFS run

Starting and finishing times in a SCC

Definitions:

- The **finishing time** of a SCC is the largest finishing time of any element of that SCC.
- The **starting time** of a SCC is the smallest starting time of any element of that SCC.



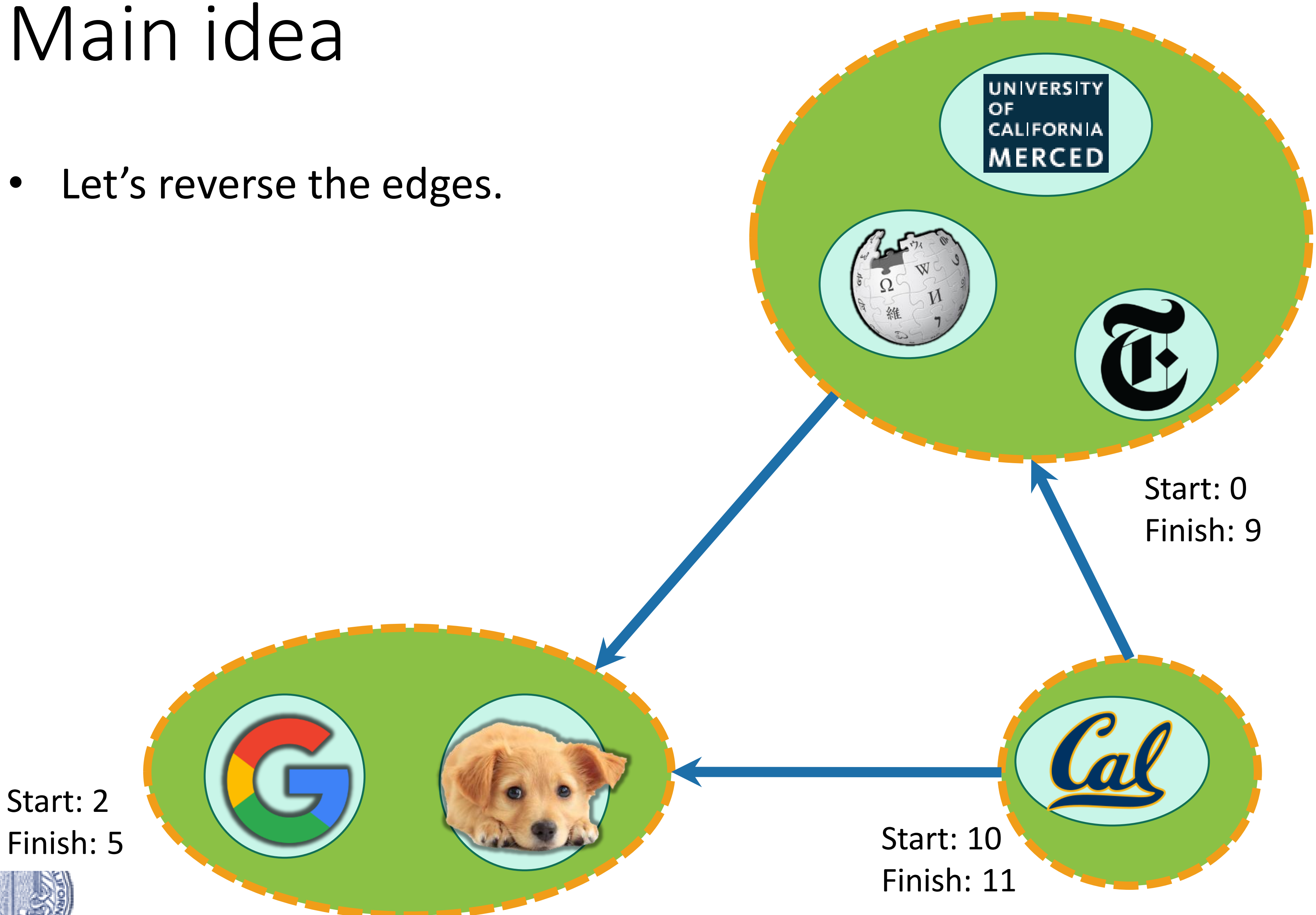
Start: 0
Finish: 9

Our SCC DAG
with start and finish times

- Last time we saw that Finishing times allowed us to **topologically sort** the vertices.
- Notice that works in this example too…

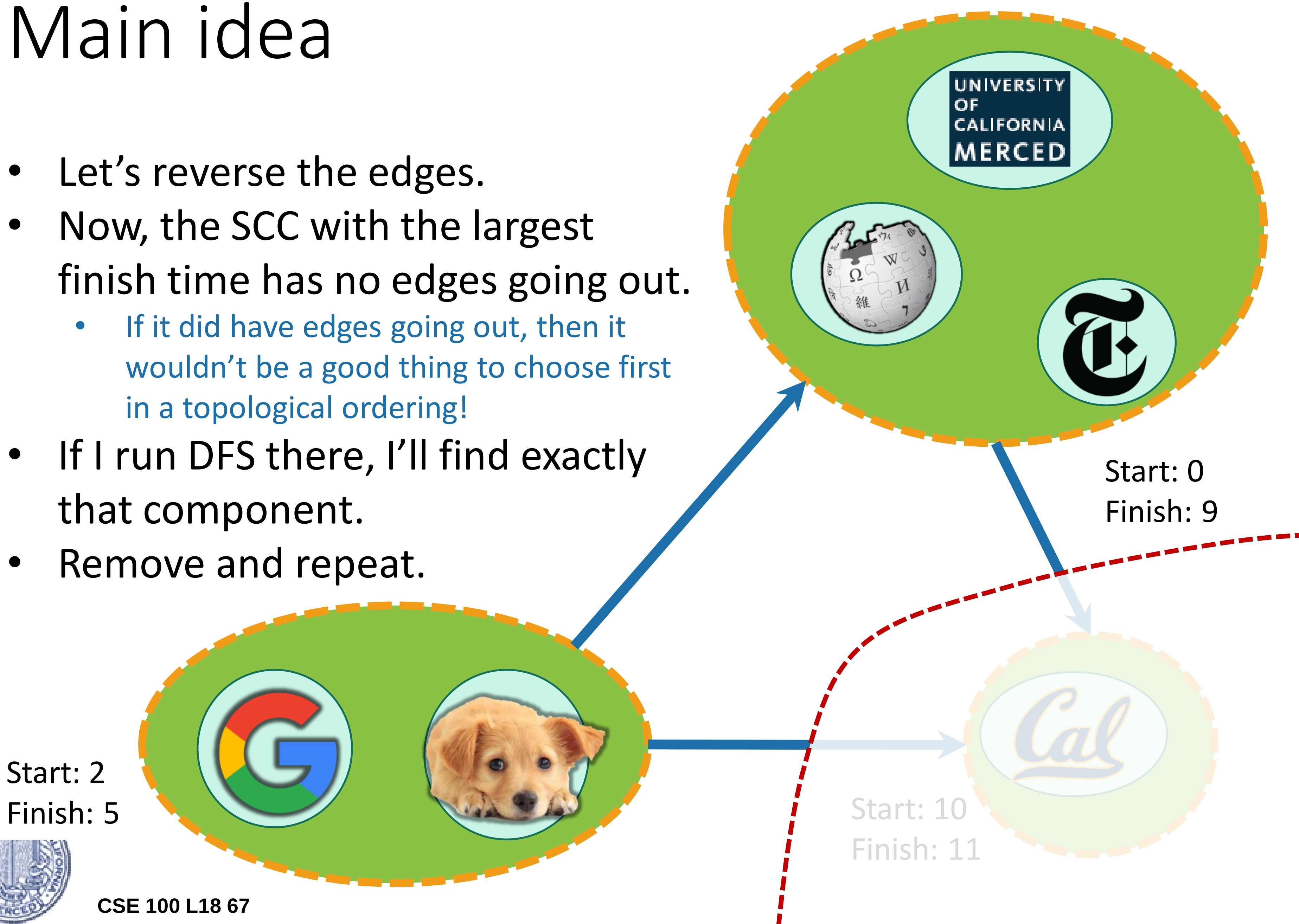
Main idea

- Let’s reverse the edges.



Main idea

- Let's reverse the edges.
- Now, the SCC with the largest finish time has no edges going out.
  - If it did have edges going out, then it wouldn't be a good thing to choose first in a topological ordering!
- If I run DFS there, I'll find exactly that component.
- Remove and repeat.



Let’s make this idea formal.

Back the the parentheses theorem

- If v is a descendent of w in this tree:

timeline

w.start v.start v.finish w.finish

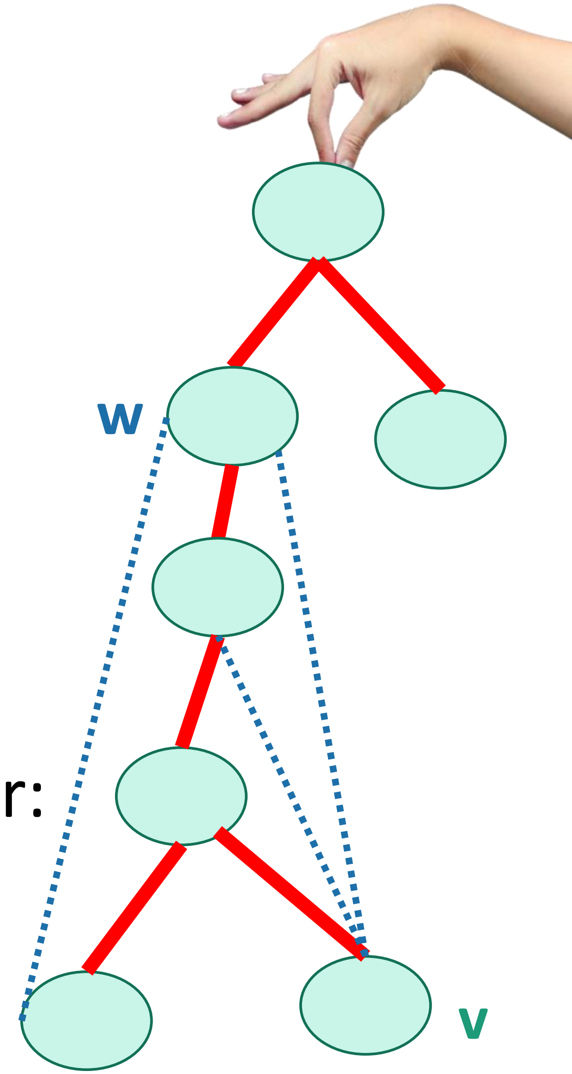
- If w is a descendent of v in this tree:

v.start w.start w.finish v.finish

- If neither are descendents of each other:

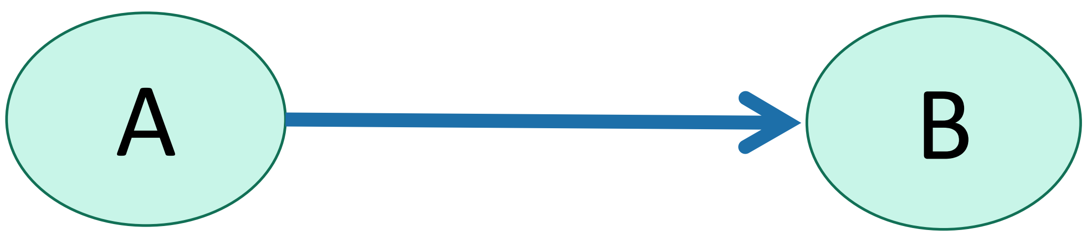
v.start v.finish w.start w.finish

(or the other way around)



As we saw last time…

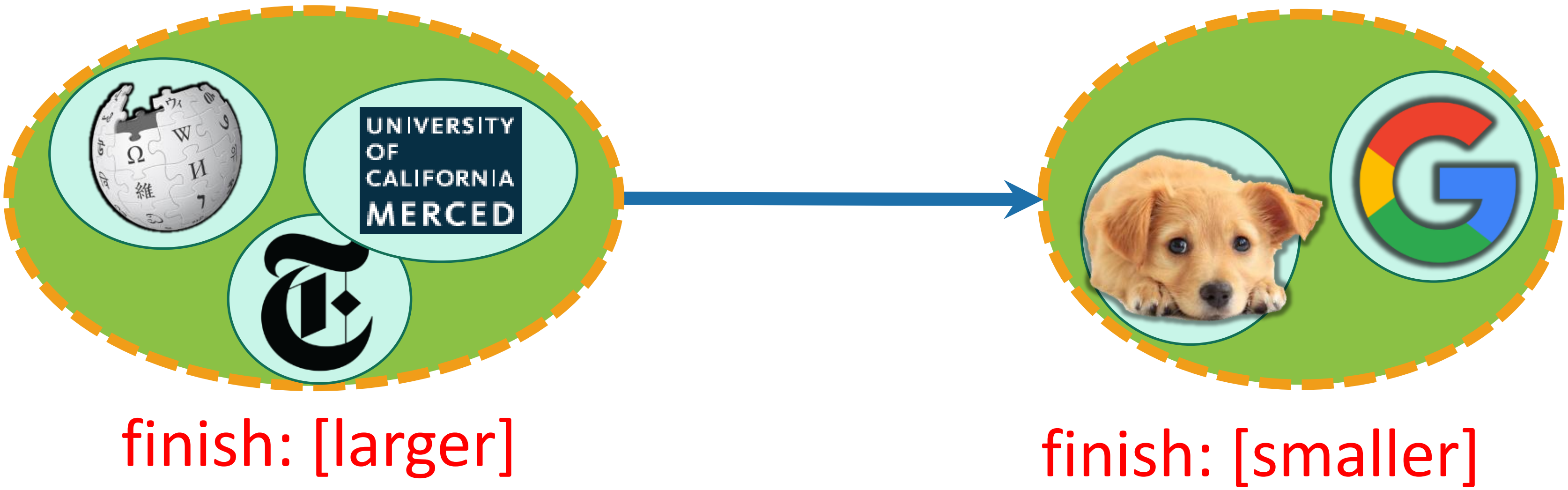
Claim: In a DAG, we'll always have:



finish: [larger] finish: [smaller]

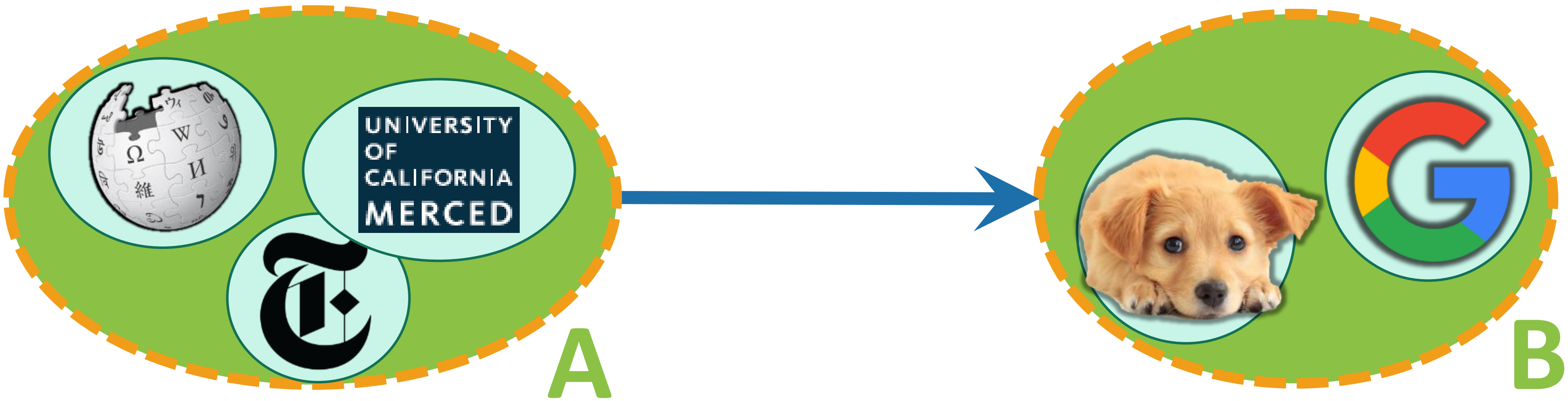
Same thing, in the SCC DAG.

- Claim: we'll always have



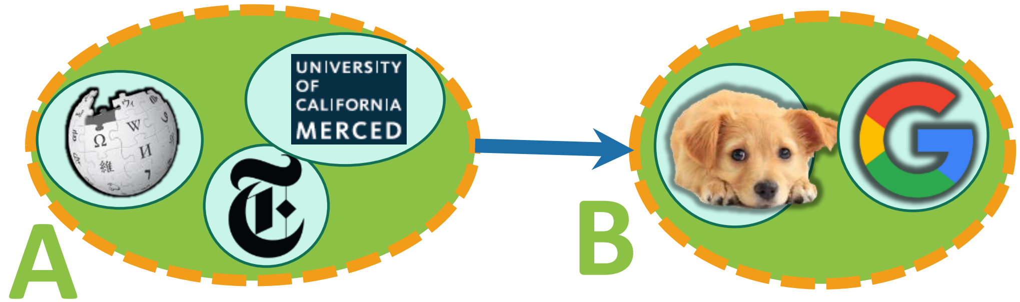
Let's call it Lemma 2

- If there is an edge like this:



- Then A.finish > B.finish.

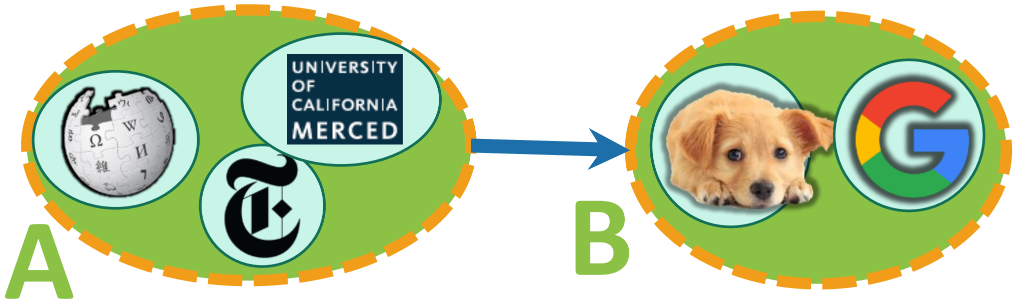
Proof idea



Want to show A.finish > B.finish.

- **Two cases**:
  - We reached A before B in our first DFS.
  - We reached B before A in our first DFS.

Proof idea



Want to show A.finish > B.finish.

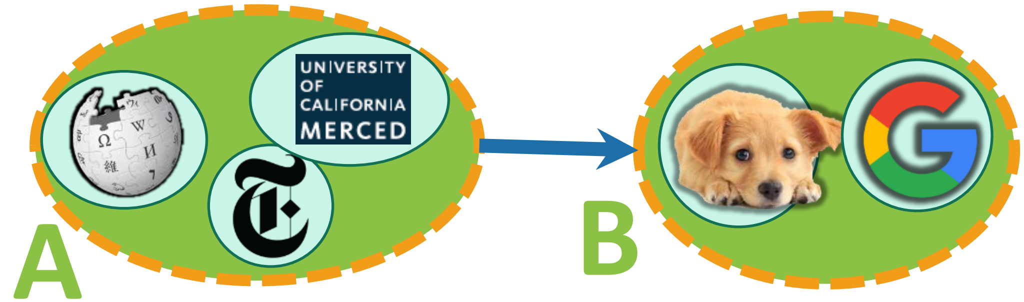
- **Case 1**: We reached A before B in our first DFS.
- Say that:
  - x has the largest finish time in A;
  - y has the largest finish in B;
  - z was discovered first in A;
- Then:
  - Reach A before B
  - => we will discover y via z
  - => y is a descendant of z in the DFS forest.

So A.finish = x.finish
B.finish = y.finish
x.finish >= z.finish

aka,
A.finish > B.finish

- Then

Proof idea



Want to show A.finish > B.finish.

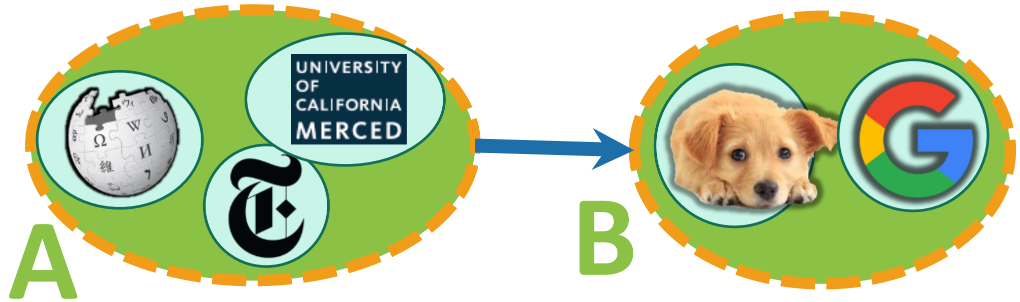
- **Case 2**: We reached B before A in our first DFS.

- There are no paths from B to A
  - because the SCC graph has no cycles

- So we completely finish exploring B and never reach A.
- A is explored later after we restart DFS.

aka,
A.finish > B.finish

Proof idea



Want to show A.finish > B.finish.

- **Two cases**:
  - We reached A before B in our first DFS.
  - We reached B before A in our first DFS.

- In either case:

A.finish > B.finish

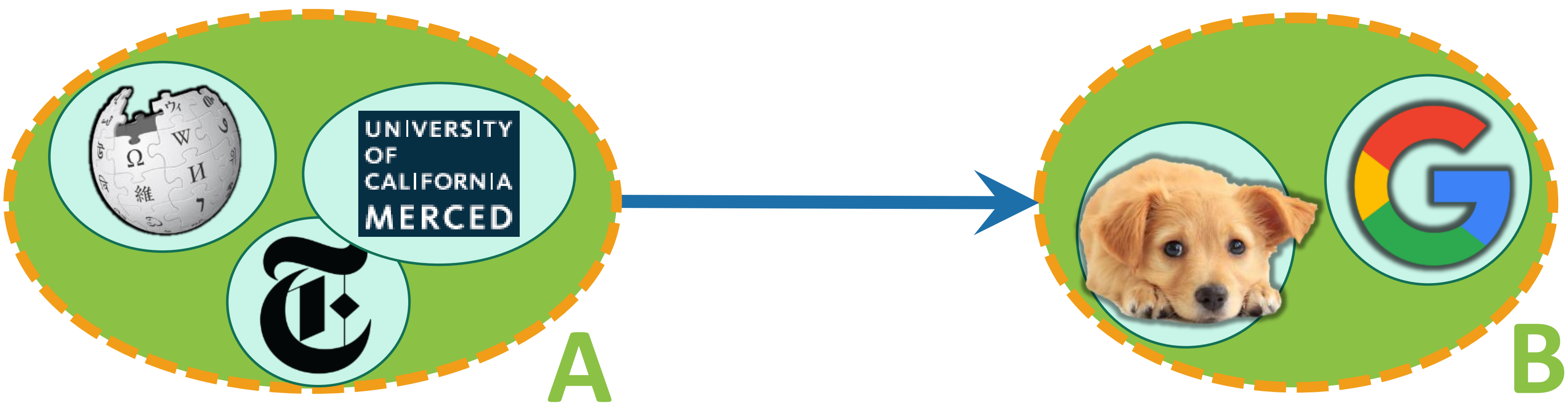
which is what we wanted to show.

Notice: this is exactly the same two-case argument that we did last time for topological sorting, just with the SCC DAG!

This establishes:

Lemma 2

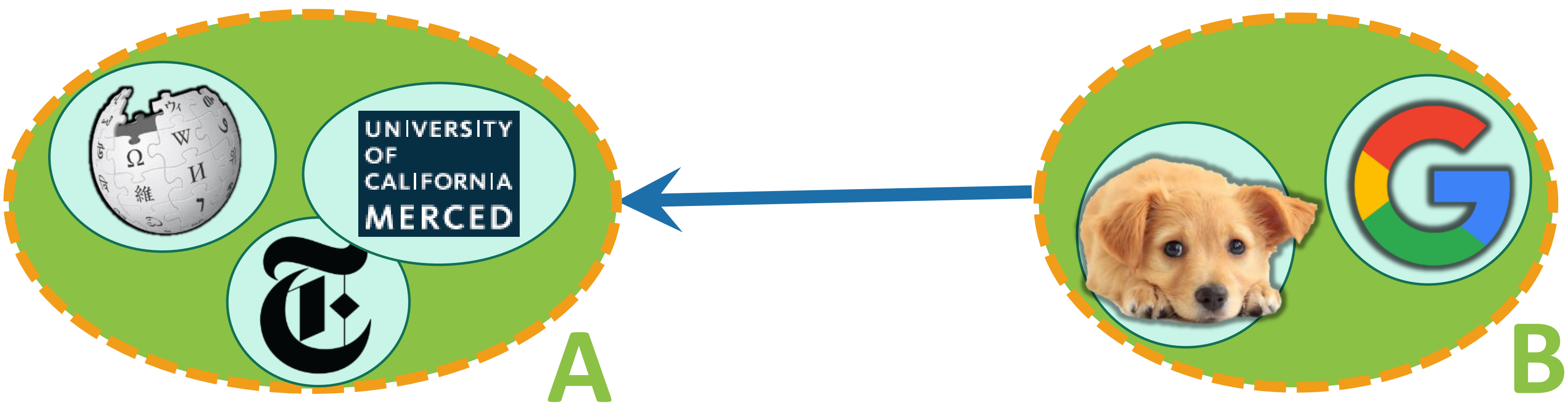
- If there is an edge like this:



- Then A.finish > B.finish.

This establishes:
Corollary 1

- If there there is an edge like this in the **reversed graph**:

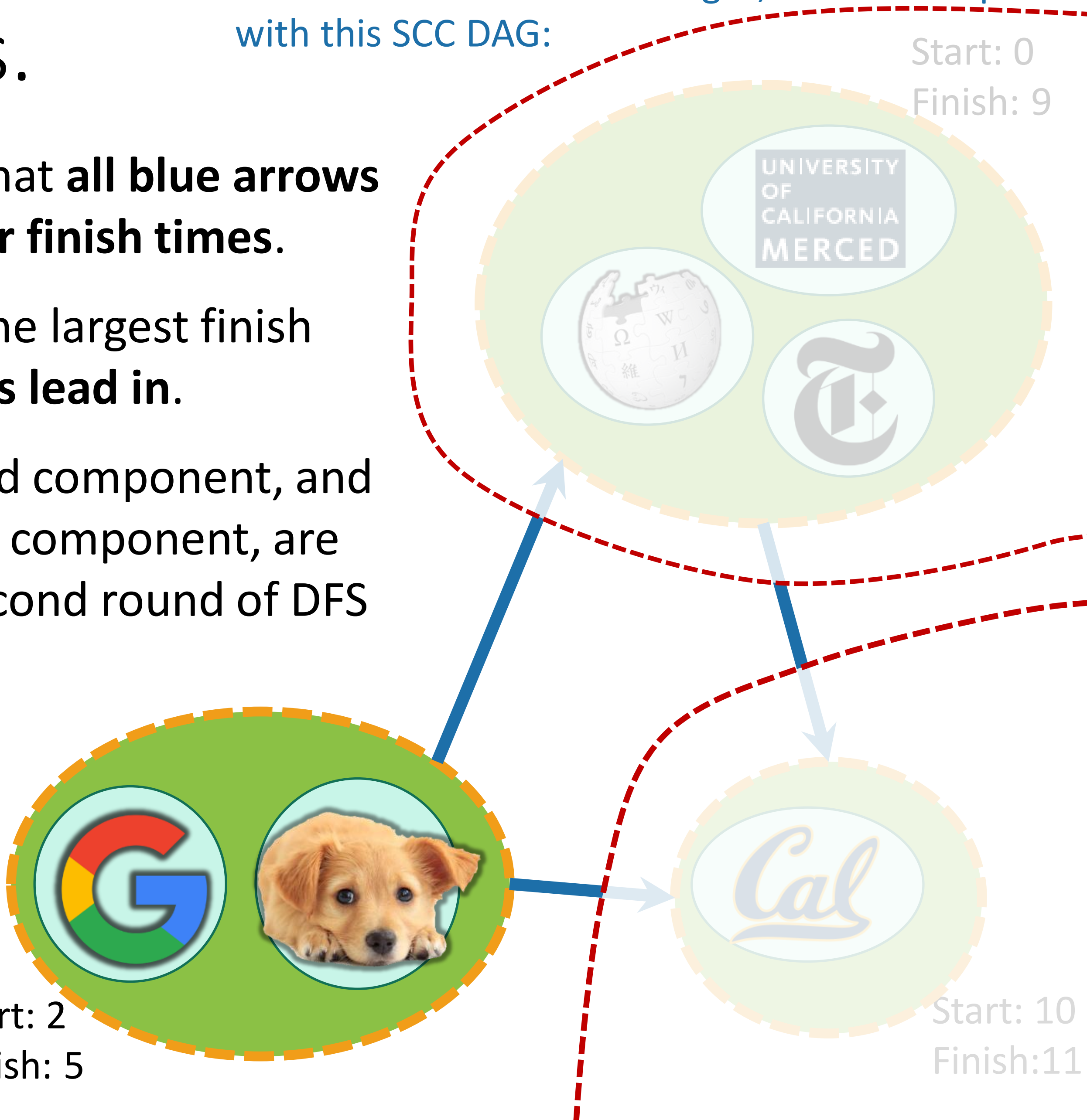


- Then A.finish > B.finish.

Now we see why this finds SCCs.

Remember that after the first round of DFS, and after we reversed all the edges, we ended up with this SCC DAG:

- The Corollary says that **all blue arrows point towards larger finish times**.
- So if we start with the largest finish time, **all blue arrows lead in**.
- Thus, that connected component, and only that connected component, are reachable by the second round of DFS



- Now, we've deleted that first component.
- The next one has the **next biggest finishing time.**
- So **all remaining blue arrows lead in.**
- Repeat.

Formally, we prove it by induction

- **Theorem**: The algorithm we saw before will correctly identify strongly connected components.

- **Inductive hypothesis**:
  - The first t trees found in the second (reversed) DFS forest are the t SCCs with the largest finish times.
  - Moreover, what's left unvisited after these t trees have been explored is a DAG on the un-found SCCs.
- **Base case**: (t=0)
  - The first 0 trees found in the reversed DFS forest are the 0 SCCs with the largest finish times. **(TRUE)**
  - Moreover, what's left unvisited after 0 trees have been explored is a DAG on all the SCCs. **(TRUE by Lemma 1.)**

Inductive step

- **Assume by induction that the first t trees are the last-finishing SCCs, and the remaining SCCs form a DAG.**
- Consider the $(t+1)^{st}$ tree produced, suppose the root is **x**.
- Suppose that **x** lives in the SCC **A**.
- Then **A**.finish > **B**.finish for all remaining SCCs **B**.
  - This is because we chose **x** to have the largest finish time.
- Then there are no edges leaving **A** in the remaining SCC DAG.
  - This follows from the Corollary.
- Then DFS started at **x** recovers exactly **A**.
  - It doesn't recover any more since nothing else is reachable.
  - It doesn't recover any less since A is strongly connected.
  - (Notice that we are using that A is still strongly connected when we reverse all the edges).
- **So the $(t+1)^{st}$ tree is the SCC with the $(t+1)^{st}$ biggest finish time.** (Also the remaining SCCs still form a DAG, since removing a vertex won't create cycles).

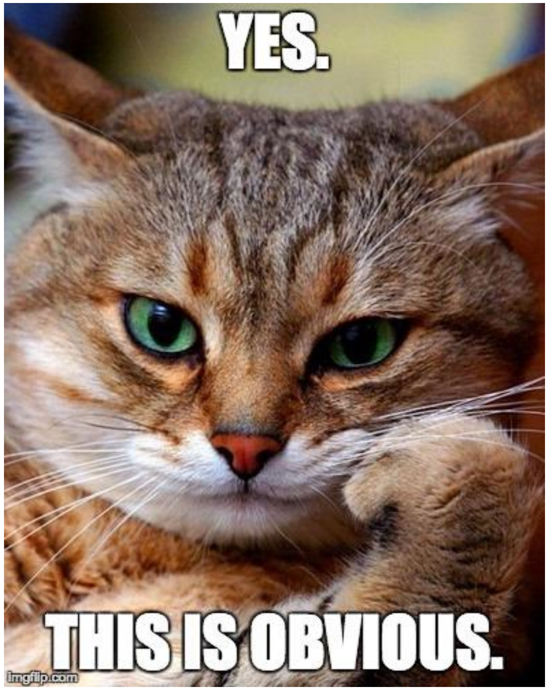
Formally, we prove it by induction

- **Theorem**: The algorithm we saw before will correctly identify strongly connected components.
- **Inductive hypothesis**:
  - The first t trees found in the second (reversed) DFS forest are the t SCCs with the largest finish times.
  - Moreover, what's left unvisited after these t trees have been explored is a DAG on the un-found SCCs.
- **Base case**: *[done]*
- **Inductive step**: *[done]*
- **Conclusion:** The second (reversed) DFS forest contains all the SCCs as its trees!
  - (This is the first bullet of **IH** when t = #SCCs)

Punchline:
we can find SCCs in time O(n + m)

`Algorithm:`

- Do DFS to create a **DFS forest**.
  - Choose starting vertices in any order.
  - Keep track of finishing times.
- Reverse all the edges in the graph.
- Do DFS again to create **another DFS forest**.
  - This time, order the nodes in the reverse order of the finishing times that they had from the first DFS run.
- The SCCs are the different trees in the **second DFS forest.**



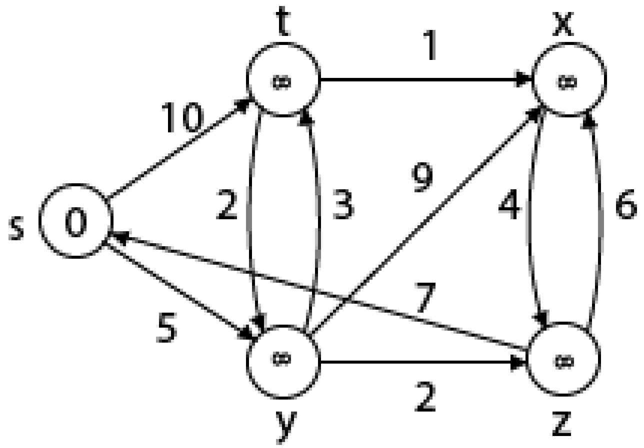
(Clearly it wasn't obvious since it took all class to do! But hopefully it is less mysterious now.)

Recap

- Depth First Search reveals a very useful structure!
  - We saw in the previous class that this structure can be used to do **Topological Sorting** in time O(n+m)
  - Today we saw that it can also find **Strongly Connected Components** in time O(n + m)
  - This was pretty non-trivial.

Next time

- Dijkstra's algorithm!



Before Next Time

- Midterm 2 discussion