Homework Quiz #5

1. P_3 is the vector space of polynomials with degree at most 3. We again represent a "vector" in this space as follows:

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Let S to be the set of all polynomials in P_3 with $\int_0^1 p(x)dx = 0$. True or False, the set S is a subspace.

Solution: TRUE. We will show that S is a subspace by demonstrating S is closed under addition of vectors from and the multiplication of a vector by a scalar.

First, assume $p_1(x)$ and $p_2(x)$ belong to S. This means that:

$$\int_0^1 p_1(x)dx = \int_0^1 p_2(x)dx = 0.$$

To show S is closed under addition we need to verify that if $p(x) = p_1(x) + p_2(x)$ then we have $\int_0^1 p(x)dx = 0$.

Fortunately, we already know (from Calculus) that for integrable functions $f(x)$ and $g(x)$ we have:

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Since polynomials are integrable functions we have:

$$\int_0^1 p(x)dx = \int_0^1 (p_1(x) + p_2(x))\, dx = \int_0^1 p_1(x)dx + \int_0^1 p_2(x)dx = 0 + 0 = 0.$$

Thus $p(x) \in S$ and S is closed under addition.

To show that S is closed under scalar multiplication. We need to show that if $p(x) \in S$ then $g(x) = \alpha p(x)$ also belongs to S for any $\alpha \in \mathbb{R}$.

Again, we appeal to what we learned in Calculus where for an integrable function $f(x)$:

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx.$$

Thus we have:

$$\int_0^1 g(x)dx = \int_0^1 \alpha p(x)dx = \alpha \int_0^1 p(x)dx = \alpha(0) = 0.$$

Thus, $g(x) \in S$ and S is closed under scalar multiplication.

Thus S is a subspace of P_3.

2. Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. Show that $(\vec{x} - \vec{y})$ is orthogonal to $(\vec{x} + \vec{y})$ if and only if $\|\vec{x}\| = \|\vec{y}\|$.

Solution: Let's suppose $\vec{x}, \vec{y} \in \mathbb{R}^n$. We know that:

$$(\vec{x} - \vec{y}) = \{(x_1 - y_1), (x_2 - y_2), \ldots, (x_n - y_n)\} \text{ and}$$

$$(\vec{x} + \vec{y}) = \{(x_1 + y_1), (x_2 + y_2), \ldots, (x_n + y_n)\}.$$

Let's suppose that $(\vec{x}-\vec{y})$ is orthogonal to $(\vec{x}+\vec{y})$. Then we have the following:

$$0 = (\vec{x}-\vec{y})^T(\vec{x}+\vec{y}) = \sum_{i=1}^{n}(x_i - y_i)(x_i + y_i) = \sum_{i=1}^{n}(x_i^2 - y_i^2) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 = \|\vec{x}\|^2 - \|\vec{y}\|^2.$$

The beginning and end of the equation give us:

$$0 = \|\vec{x}\|^2 - \|\vec{y}\|^2 \implies \|\vec{x}\|^2 = \|\vec{y}\|^2 \implies \|\vec{x}\| = \|\vec{y}\|.$$

As such, we have that $(\vec{x}-\vec{y})$ is orthogonal to $(\vec{x}+\vec{y})$ if and only if $\|\vec{x}\| = \|\vec{y}\|$.

3. Project the vector $\vec{b}$ onto the line through $\vec{a}$.

$$\vec{b} = \begin{bmatrix}1\\2\\2\end{bmatrix} \text{ and } \vec{a} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Remember, this projection should have the form $\vec{p} = \alpha\vec{a}$.

Solution: We will begin by constructing $P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}$ and then calculating $P\vec{b} = \vec{p}$.

$$P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}.$$

$$P\vec{b} = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}5\\5\\5\end{bmatrix}.$$

Thus we notice that:

$$\vec{p} = \frac{5}{3}\vec{a}.$$

4. Find a basis for the orthogonal complement of the rowspace of A:

$$A = \begin{bmatrix}1 & 0 & 2\\1 & 1 & 4\end{bmatrix}.$$

Solution: There are two conceptual ways to approach this problem. We will see that both these conceptual ways lead towards the exact same mathematical formulation.

In the first we can remember that the rowspace of A: $C(A^T)$ has orthogonal complement $N(A)$. Thus, we are looking for a basis for $N(A)$. This means we are looking for solutions to:

$$A\vec{x} = \vec{0} \implies \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\1 & 1 & 4 & 0\end{array}\right].$$

In the second way, we simply notice the rows of the matrix A and realize that we are looking at the vector space defined by the span of the two rows: $\vec{a}_1, \vec{a}_2$:

$$V = \text{span}\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}\right\}.$$

We can simply look for vectors $\vec{x}$ so that:

$$\vec{x}^T\vec{a}_1 = \vec{x}^T\vec{a}_2 = 0.$$

This gives us:

$$x_1 + 2x_3 = 0 \text{ and } x_1 + x_2 + 4x_3 = 0 \implies \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]. \tag{1}$$

But notice that equation (1) and equation (2) are the same! So these two conceptual ways of solving the problem lead us to the same mathematical equation!

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

The system has 2 pivots and 2 free variable. The free variable gives us $x_3 = t$ and the rows give us: $x_2 = -2t$ and $x_1 = -2t$. Thus, the solution to this system is a 1-dimensional space:

$$V = \left\{ t \begin{bmatrix} -2 & -2 & 1 \end{bmatrix}^T | t \in \mathbb{R} \right\}.$$

Thus the basis for this vector space is:

$$\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}.$$