Homework Assignment #10

Remember, this Homework Assignment is **not collected or graded**! But it is in your best interest to do it as the this material will be covered on your Homework Quiz.

This Homework will go through properties of matrices, eigenvalues and eigenvectors that, while important, have not yet come up in our course.

1. Suppose that λ is an eigenvalue of a matrix A. Let V be the following:

$$V = \{\vec{x} \text{ such that } A\vec{x} = \lambda\vec{x}\}.$$

Show that V is a vector space by showing that:

- V **is Closed Under Addition**: If $\vec{x}, \vec{y} \in V$ then $(\vec{x} + \vec{y}) \in V$.
- V **is Closed Under Scalar Multiplication**: If $\vec{x} \in V$ then $(\alpha\vec{x}) \in V$ for all $\alpha \in \mathbb{R}$.

Solution: Let A, λ and V be as specified in the problem statement.

- Closure Under Addition:
  We know that if $\vec{x} \in V$ and $\vec{y} \in V$ we have:

$$A\vec{x} = \lambda\vec{x} \text{ and } A\vec{y} = \lambda\vec{y}.$$

  To show that $(\vec{x} + \vec{y}) \in V$ we need to show:

$$A(\vec{x} + \vec{y}) = \lambda(\vec{x} + \vec{y}).$$

  But because matrix-vector operations satisfy the distributive property, this statement holds:

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \lambda\vec{x} + \lambda\vec{y} = \lambda(\vec{x} + \vec{y}).$$

  Thus, V is closed under addition.

- Closure Under Scalar Multiplication:
  Let $\vec{x} \in V$, we want to show that if $\alpha \in \mathbb{R}$ we have $(\alpha\vec{x}) \in V$. But this holds directly:

$$A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha\lambda\vec{x} = \lambda(\alpha\vec{x}).$$

  Thus, V is closed under scalar multiplication.

2. Square matrices A and B are called **similar** if there exists an invertible matrix S such that:

$$B = S^{-1}AS.$$

(a) True or False: Matrices that are diagonalizable similar to a diagonal matrix?

Solution: TRUE. We know that if an $n \times n$ matrix A is diagonalizable it has n linearly independent eigenvectors. When we put these into a matrix S we have:

$$AS = S\Lambda \implies A = S\Lambda S^{-1} \implies \Lambda = S^{-1}AS$$

where Λ is a diagonal matrix whose entries are the eigenvalues of A in the same order as the eigenvectors in S.

But this means by the definition we just learned that, A and Λ are similar matrices

(b) Show similar matrices A and B will have the same eigenvalues by showing that their characteristic polynomials are the same. Recall that the characteristic polynomial is of the form:

$$p_A(\lambda) = \det(A - \lambda I) \text{ and } p_B(\lambda) = \det(B - \lambda I).$$

(Hint: You'll have to remember your properties of determinants of products of matrices.)
Note that the eigenvalues of A and B (respectively) are the values of λ that satisfy:

$$p_A(\lambda) = 0 \text{ and } p_B(\lambda) = 0.$$

If we show that $p_A(\lambda) = p_B(\lambda)$ this means the eigenvalues of both matrices will be the same.

Solution: We will assume that $B = S^{-1}AS$ and show that $p_B(\lambda) = p_A(\lambda)$.

$$\begin{aligned}
p_B(\lambda) &= \det(B - \lambda I) \\
&= \det\left(S^{-1}AS - \lambda I\right) \\
&= \det\left(S^{-1}AS - \lambda S^{-1}IS\right) \\
&= \det\left(S^{-1}(A - \lambda I)S\right) \\
&= \det(S^{-1})\det(A - \lambda I)\det(S) \\
&= (1/\det(S))\det(A - \lambda I)\det(S) \\
&= \det(A - \lambda I) \\
&= p_A(\lambda).
\end{aligned}$$

3. Let A be an $n \times n$ matrix and let $B = I - 2A + A^2$.

(a) Show that if $\vec{x}$ is an eigenvector of A belonging to an eigenvalue λ of A, then $\vec{x}$ is also an eigenvector of B with eigenvalues μ.
How are λ and μ related?

Solution: Let's let $\vec{x}$ be an eigenvector of A with eigenvalue λ and let $B = I - 2A + A^2$. We will find the value μ such that $B\vec{x} = \mu\vec{x}$.

$$\begin{aligned}
B\vec{x} &= (I - 2A + A^2)\vec{x} \\
&= I\vec{x} - 2A\vec{x} + A^2\vec{x} \\
&= \vec{x} - 2(\lambda\vec{x}) + A(A\vec{x}) \\
&= \vec{x} - (2\lambda)\vec{x} + A(\lambda\vec{x}) \\
&= \vec{x} - (2\lambda)\vec{x} + \lambda(A\vec{x}) \\
&= \vec{x} - (2\lambda)\vec{x} + \lambda^2\vec{x} \\
&= (1 - 2\lambda + \lambda^2)\vec{x}.
\end{aligned}$$

Thus we have $\mu = (1 - 2\lambda + \lambda^2)$.

(b) Show that if $\lambda = 1$ then B will not be invertible.

Solution: If we have $\lambda = 1$ then we have:

$$\mu = (1 - 2\lambda + \lambda^2) = (1 - 2 + 1) = 0.$$

Since we know that the determinant of a matrix is equal to the product of the eigenvalues this means that B has a 0 eigenvalue. This means that $B\vec{x} = 0\vec{x}$ for some non-zero vectors $\vec{x}$ and thus B is not invertible.

4. Difference equations are often used to model population dynamics for species that reproduce only at particular time intervals. A scientist studied a beetle which in the laboratory live for a maximum of 3 years.

   - Only half the beetles born will survive their first year.
   - Only one third of the beetles that survive the first year, survive their second year.
   - **Every** beetle that enters the third year produces exactly 6 beetles as they die (at the end of the third year).

   Consider the following matrix:
$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}.$$

   (a) Explain why the matrix A is a reasonable model for the beetle growth described in this problem. That is, if $\vec{x}_k$ is a vector with three components representing the number of newly born beetles ($x_1^{(k)}$), 1 year old beetles ($x_2^{(k)}$), and 2 year old beetles($x_3^{(k)}$), in year k then:
$$\vec{x}_{k+1} = A\vec{x}_k.$$

Solution: Let's carry out the multiplication and see what we end up with:
$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{bmatrix} = \vec{x}_{k+1} = A\vec{x}_k = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} 6x_3^{(k)} \\ (1/2)x_1^{(k)} \\ (1/3)x_2^{(k)} \end{bmatrix}.$$

This system correctly represents the facts we were given about beetle reproduction.

- The newly born beetles in year $(k+1)$, $x_1^{(k+1)}$ will be 6 times the number of 2 year old beetles in year k, $x_3^{(k)}$.
- The number of 1 year old beetles in year $(k+1)$, $x_2^{(k+1)}$ will be half of the number of newly born beetles in year k, $x_1^{(k)}$.
- The number of 2 year old beetles in year $(k+1)$, $x_3^{(k+1)}$ will be one third times the number of 1 year old beetles in year k, $x_2^{(k)}$.

   (b) If we start off with 6 newborn beetles, how many beetles will be in each of the next 3 years? (Do you see a pattern, can you predict how many there will be for all future years?)

   Hint: Diagonalization, while possible, is not the best way to solve this problem. Just go ahead and plug in solutions. If you do feel strongly committed to diagonalization, you might find it helpful to remember that:
$$\lambda^3 = 1 \implies \lambda_1 = 1, \lambda_{2,3} = (1/2)(-1 \pm \sqrt{-3}).$$

Solution: Let's first try this the direct way:

$$\vec{x}_0 = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}.$$

$$\vec{x}_1 = A\vec{x}_0 = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}.$$

$$\vec{x}_2 = A\vec{x}_1 = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}_3 = A\vec{x}_2 = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \vec{x}_0.$$

Thus, the system will cycle and repeat these same states again and again forever:

$$\vec{x}_k = \begin{cases} \vec{x}_0 = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} & \text{if } k \text{ divided by 3 equals } 0 \\ \vec{x}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} & \text{if } k \text{ divided by 3 equals } 1 \\ \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \text{if } k \text{ divided by 3 equals } 2 \end{cases}.$$