ENGR 65 Electric Circuits

Lecture 14: Inverse Laplace Transform and RC, RL, and RLC Circuits

Today's Topics

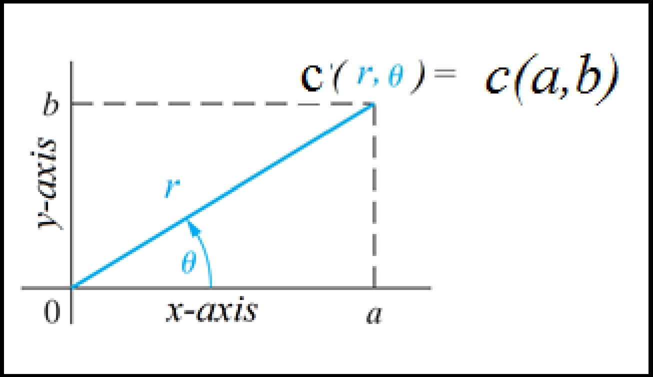
- The review of complex numbers and their operations
- The inverse Laplace transform
  1. Rational functions
     a. Proper rational functions
     b. Improper rational functions
  2. The partial fraction expansion
- The applications of the Laplace transform in first-order RC/RL circuits and second-order RLC circuits

- The topics covered in Sections 12.6 and 12.7

Complex Number Notation

There are two ways to describe a complex number

- Rectangular form: $c = a + jb$
- Polar form: $c = re^{j\theta} = r\angle\theta^0$



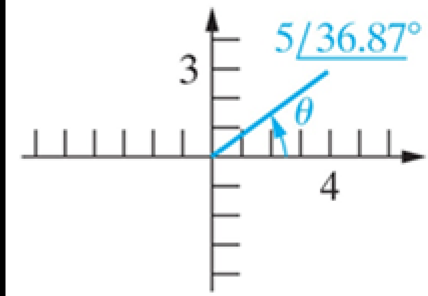
$$a = r cos\,\theta$$
$$b = r\sin\theta$$

$$\begin{cases} a^2 + b^2 = r^2 \\ \tan\theta = \dfrac{b}{a} \end{cases} \Rightarrow \begin{cases} r = \sqrt{a^2 + b^2} \\ \theta = tan^{-1}(\dfrac{b}{a}) \end{cases}$$

Some Examples

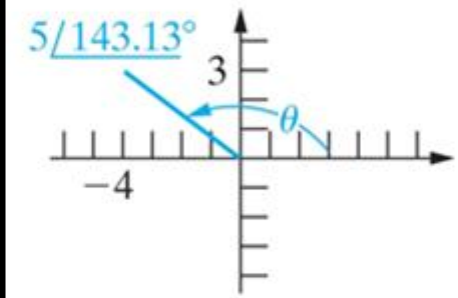
$$c = a + jb = 4 + j3$$

$$\begin{cases} r = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = 5 \\ \theta = tan^{-1}\left(\dfrac{b}{a}\right) = tan^{-1}\dfrac{3}{4} = 36.87^0 \end{cases}$$



$4 + j3 = 5\angle 36.87°$

(a)



$-4 + j3 = 5\angle 143.13°$

(b)

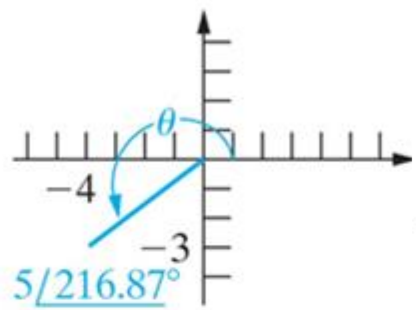
$143.13^0 = 180^0 - 36.87^0$

a) $c = 4 + j3 = 5e^{j36.87^0}$

b) $c = -4 + j3 = 5e^{j143.13^0}$

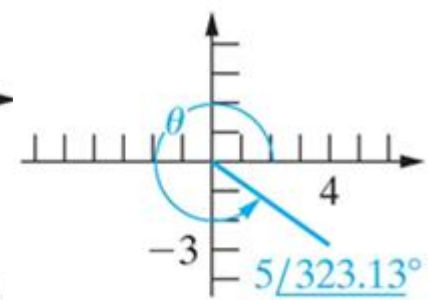
c) $c = -4 - j3 = 5e^{-j143.13^0}$

d) $c = 4 - j3 = 5e^{-j36.87^0}$



$-4 - j3 = 5\angle 216.87° = 5\angle -143.13°$

(c)



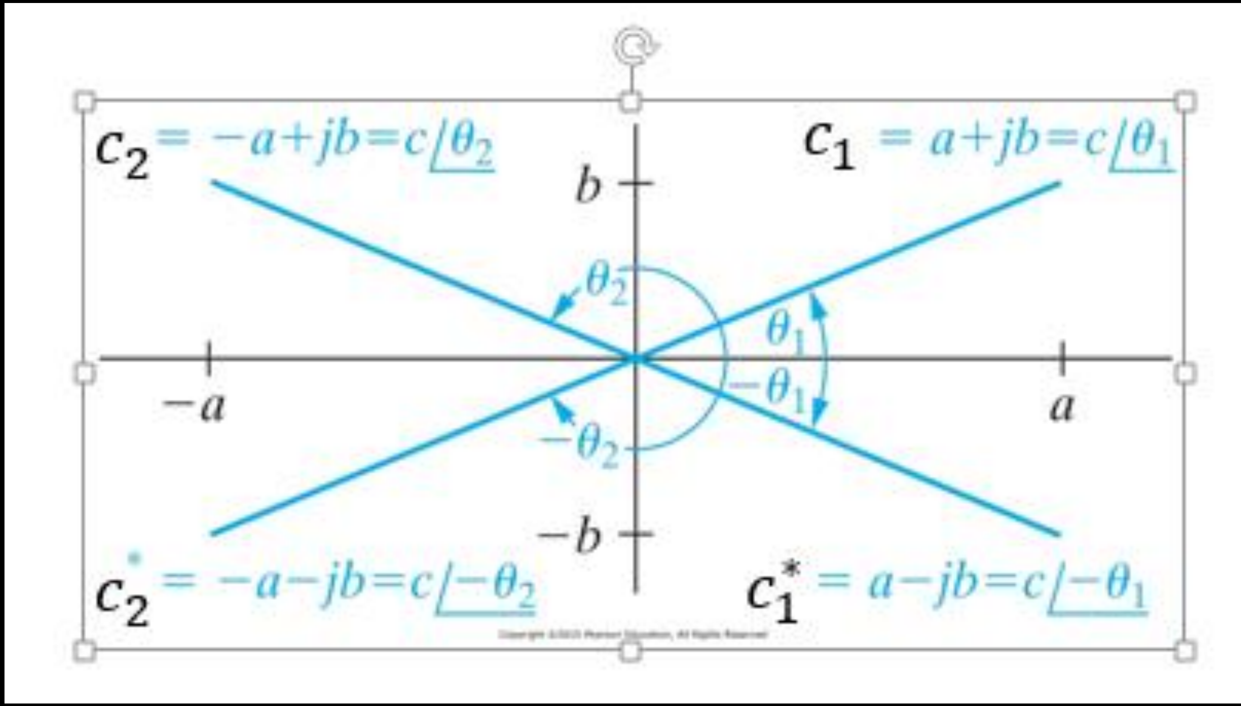
$4 - j3 = 5\angle 323.13° = 5\angle -36.87°$

(d)

$-143.13^0 = -180^0 + 36.87^0$

Conjugate Complex Numbers

$If\ c_1 = a + jb, c = a - jb$, c_1 and c are called conjugate numbers, denoted by $c = c_1^*$



Complex Numbers Operations

Addition/Subtraction

If we have two complex numbers

$$c_1 = a_1 + jb_1$$
$$c_2 = a_2 + jb_2$$

Then $c = c_1 \pm c_2 = (a_1 + jb_1) \pm (a_2 + jb_2)$
$= (a_1 \pm a_2) + j(b_1 \pm b_2)$

For example: $c_1 = 8 + j16, c_2 = 12 - j3$

$$c_1 + c_2 = (8 + 12) + j(16 - 3) = 20 + j13$$
$$c_1 - c_2 = (8 - 12) + j(16 + 3) = -4 + j19$$

Complex Number Operations

Multiplication/Division

If we have two complex numbers

$$c_1 = a_1 + jb_1 = r_1 e^{j\theta_1}$$
$$c_2 = a_2 + jb_2 = r_2 e^{j\theta_2}$$

$$c_1 c_2 = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

$$\frac{c_1}{c_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$$

For example: $c_1 = 4 + j3 = 5e^{j36.87^0}, \;\; c_2 = 4 - j3 = 5e^{-j36.87^0}$

$$c_1 c_2 = 5 \times 5e^{j[36.87^0+(-36.87^0)]} = 25$$

$$\frac{c_1}{c_2} = \frac{5}{5} e^{j36.87^0-(-36.87^0)} = e^{j73.74^0}$$

Useful Identities

$$j = \sqrt{-1}$$
$$j^2 = -1$$
$$(-j)j = 1$$
$$j = \frac{1}{-j}$$
$$e^{\pm j\pi} = \cos(\pi) \pm jsin(\pi) = -1$$
$$e^{\pm j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) \pm jsin\left(\frac{\pi}{2}\right) = \pm j$$
$$a^2 - b^2 = (a+b)(a-b)$$
$$a^2 + b^2 = a^2 - (jb)^2 = (a+jb)(a-jb)$$

The Inverse Laplace Transform

An integral inverse Laplace transform is defined as:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{\gamma - jT}^{\gamma + jT} F(s) e^{st} ds$$

where the integration is performed along the vertical line $\mathrm{Re}(s) = \gamma$ in the complex plane such that γ is greater than the real part of all roots of $F(s)$.

Rational Functions

The general form of Laplace transform of a function is:

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}$$

where the coefficient *a* and *b* are real constant, and the exponents *m* and *n* are natural numbers.

If $m > n$, $\frac{N(s)}{D(s)}$ is called a proper rational function;

If $m \leq n$, $\frac{N(s)}{D(s)}$ is called an improper rational function.

Partial Fraction Expansion

A *proper rational function* can be expanded into a sum of partial fraction by writing a term or a series terms for each root of D(s).

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}$$

$$= \frac{K(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0)}{(s+p_1)(s+p_2)\ldots(s+p_m)} \qquad m > n$$

The roots of D(s) might be

1. distinct and real
2. distinct and complex
3. repeated and real
4. repeated and complex

Improper Rational Functions

- *Improper rational functions* cannot be expanded into partial fractions. However, you can convert an improper rational function into a polynomial + a proper rational function by using a long division method.

For example $F(s) = \frac{s^2+s+1}{s+1} = s + \frac{1}{s+1}$

Polynomial function of s

Proper rational function of s

And we know: $\mathcal{L}\{\delta'(t)] = s$, (Problem 12.12) $\mathcal{L}\{\delta^{(n)}(t)] = s^n$ (Problem 12.11)

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \delta'(t) + e^{-t}, \qquad t > 0$$

The Inverse Laplace Transform

1. *D(s) has d*istinct real roots, *and* $l < q$

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_l s^l + a_{l-1} s^{l-1} + \cdots + a_1 s + a_0}{b_q s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0}$$

$$= \frac{K(a_l s^l + a_{l-1} s^{l-1} + \cdots + a_1 s + a_0)}{(s+p_1)(s+p_2)\ldots(s+p_q)}$$

$$\equiv \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \frac{k_3}{s+p_3} + \cdots + \frac{k_q}{s+p_q}$$

where $-p_i, i = 1,2,3,\ldots,q$, are real and distinct.

$$k_i = F(s)(s+p_i)|_{s=-p_i}$$

$$f(t) = \mathcal{L}^{-1}\{\frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \frac{k_3}{s+p_3} + \cdots + \frac{k_q}{s+p_q}\}$$

$$= (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + k_3 e^{-p_3 t} + \cdots + k_q e^{-p_q t})u(t)$$

The Inverse Transform-Example

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{k_1}{s} + \frac{k_2}{s+8} + \frac{k_3}{s+6}$$

Both sides are multiplied by s and then evaluated at s=0:

$$\left.\frac{96(s+5)(s+12)s}{s(s+8)(s+6)}\right|_{s=0} \equiv k_1 + \left.\frac{k_2 s}{s+8}\right|_{s=0} + \left.\frac{k_3 s}{s+6}\right|_{s=0}$$

$$k_1 = \left.\frac{96(s+5)(s+12)s}{s(s+8)(s+6)}\right|_{s=0} = F(s)s\Big|_{s=0} = \frac{96(5)(12)}{(8)(6)} = 120$$

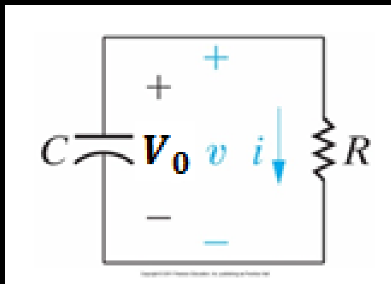
$$Similarly, \qquad k_2 = \left.\frac{96(s+5)(s+12)}{s(s+6)}\right|_{s=-8} = F(s)(s+8)\Big|_{s=-8} = \frac{96(-3)(4)}{(-8)(-2)} = -72$$

$$k_3 = \left.\frac{96(s+5)(s+12)}{s(s+8)}\right|_{s=-6} = F(s)(s+6)\Big|_{s=-6} = \frac{96(-1)(6)}{(-6)(2)} = 48$$

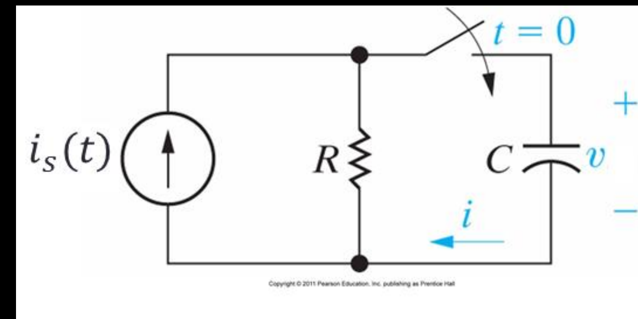
$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{120}{s} + \frac{-72}{s+8} + \frac{48}{s+6}$$

$$f(t) = (120 - 72e^{-8t} + 48e^{-6t})u(t)$$

Applications of the Laplace Transform – The First Order RC Circuit



Apply Laplace transform
to the both sides of the equation

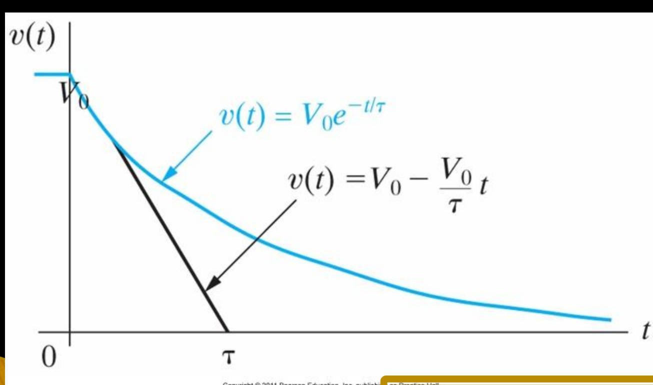


$$C\frac{dv(t)}{dt} + \frac{v(t)}{R} = 0 \Rightarrow V(s) = \frac{V_0}{s + \frac{1}{RC}}$$

$$C\frac{dv(t)}{dt} + \frac{v(t)}{R} = i_s(t) \text{ and } \quad i_s(t) = I_s$$

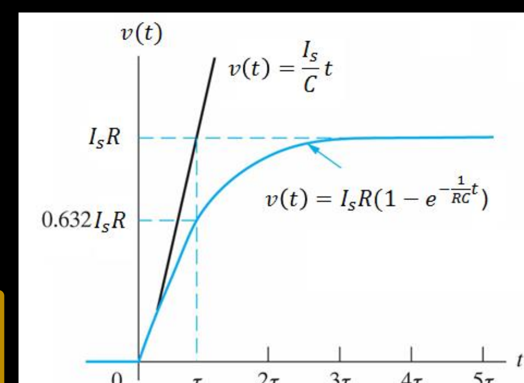
The natural response is: $v(t) = \mathcal{L}^{-1}\{V(s)\} = V_0 e^{-\frac{1}{RC}t}$

$$V(s) = \frac{\frac{1}{C} I_s}{s(s + \frac{1}{RC})}$$



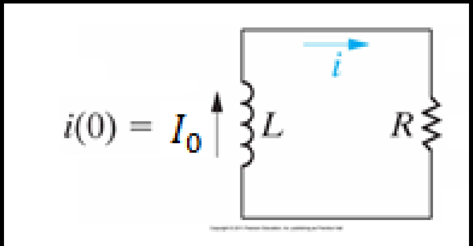
The step response is:

$$v(t) = I_s R(1 - e^{-\frac{1}{RC}t})$$

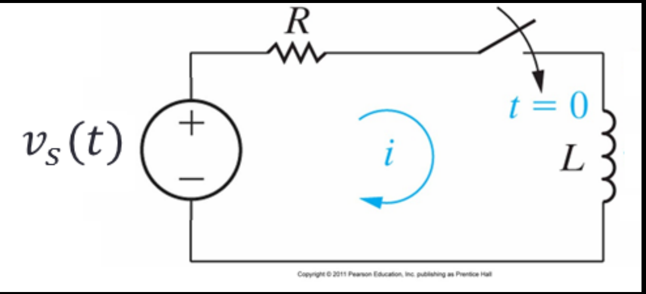


$$\boxed{\text{The time constant of RC circuits is: } \tau = RC}$$

Applications of the Laplace Transform– The First Order RL Circuit



Apply Laplace transform
to the both sides of the equation



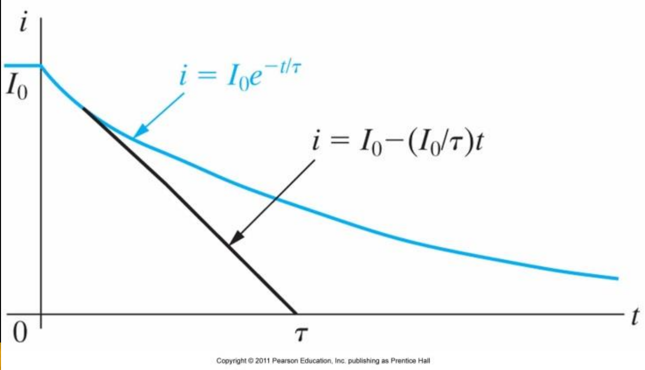
$$L\frac{di(t)}{dt} + Ri(t) = 0 \Rightarrow \mathrm{I}(s) = \frac{I_0}{s + R/L}$$

$$L\frac{di(t)}{dt} + Ri(t) = v_s(t) \text{ and } v_s(t) = V_s$$

The natural response is:

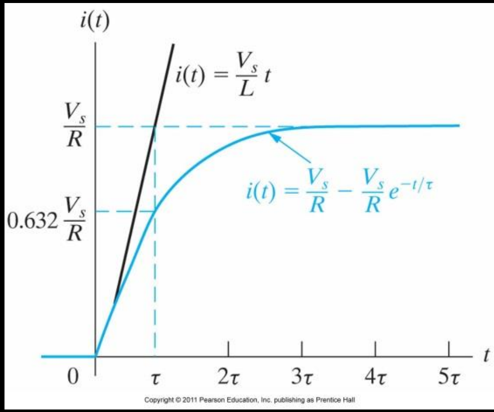
$$i(t) = \mathcal{L}^{-1}\{I(s)\} = I_0\, e^{-\frac{R}{L}t}$$

$$\mathrm{I}(s) = \frac{V_s/L}{s(s + \frac{R}{L})}$$



The step response is:

$$i(t) = \frac{V_s}{R}(1 - e^{-\frac{R}{L}t})$$



The time constant of RL circuits is $\tau = \frac{L}{R}$

The Inverse Transform

2. *D(s) has d*istinct complex roots *and* $l < q$

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_l s^l + a_{l-1} s^{l-1} + \cdots + a_1 s + a_0}{b_q s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0}$$

$$= \frac{K(a_l s^l + a_{l-1} s^{l-1} + \cdots + a_1 s + a_0)}{(s + p_1)(s + p_2) \ldots (s + p_q)} \equiv \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \frac{k_3}{s + p_3} + \cdots + \frac{k_q}{s + p_q}$$

where $-p_i, i = 1,2,3, \ldots, q$, are complex and distinct.

$$k_i = F(s)(s + p_i)|_{s=-p_i}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + k_3 e^{-p_3 t} + \cdots + k_q e^{-p_q t})u(t)$$

- Note: In realizable circuits, complex roots always appear in conjugate pair and their corresponding coefficients are themselves complex conjugate.

The Inverse Transform–Example

$$F(s)=\frac{100(s+3)}{(s+6)(s^2+6s+25)}=\frac{100(s+3)}{(s+6)(s+3-j4)(s+3+j4)}$$

$$\equiv\frac{k_1}{s+6}+\frac{k_2}{s+3-j4}+\frac{k_3}{s+3+j4}$$

$$k_1=F(s)(s+6)\Big|_{s=-6}=\frac{100(s+3)}{(s^2+6s+25)}\Bigg|_{s=-6}=\frac{100(-3)}{36+6(-6)+25}=-12$$

$$k_2=F(s)(s+3-j4)\Big|_{s=-3+j4}=\frac{100(s+3)}{(s+6)(s+3+j4)}\Bigg|_{s=-3+j4}=\frac{100(j4)}{(3+j4)(j8)}$$

$$=6-\mathrm{j}8=10e^{-j53.13^0}$$

$$\boldsymbol{k_3}=F(s)(s+3+j4)\Big|_{s=-3-j4}=\frac{100(s+3)}{(s+6)(s+3-j4)}\Bigg|_{s=-3-j4}=\frac{100(-j4)}{(3-j4)(-j8)}$$

$$=6+j8=10e^{j53.13^0}$$

k_2 and k_3 are complex conjugate. So $k_3=k_2^*$

The Inverse Transform-Example

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)} \equiv \frac{-12}{s+6} + \frac{10e^{-j53.13^0}}{s+3-j4} + \frac{10e^{j53.13^0}}{s+3+j4}$$

$$f(t) = \left[-12e^{-6t} + 10e^{-j53.13^0}e^{-(3-j4)t} + 10e^{j53.13^0}e^{-(3+j4)t}\right]u(t)$$

$$= \left[-12e^{-6t} + 10 \times e^{-3t}\left(2 \times \frac{e^{j(4t-53.13^0)} + e^{-j(4t-53.13^0)}}{2}\right)\right]\mathrm{u(t)}$$

$$= [-12e^{-6t} + 20e^{-3t} cos(4t - 53.13°)]u(t)$$

In general:

$$\mathcal{L}^{-1}\left\{\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}\right\} = 2|K|e^{-\alpha t}\cos(\beta t+\theta)$$

Where $K = |K|e^{j\theta}$ and $K^* = |K|e^{-j\theta}$.

The Inverse Transform

3. *D(s) has r*epeated real roots *and* $l < q$

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_l s^l + a_{l-1} s^{l-1} + \cdots + a_1 s + a_0}{b_q s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0}$$

$$= \frac{K(a_l s^l + a_{l-1} s^{l-1} + \cdots + a_1 s + a_0)}{(s+p)^q}$$

$$\equiv \frac{k_1}{(s+p)^q} + \frac{k_2}{(s+p)^{q-1}} + \frac{k_3}{(s+p)^{q-2}} + \cdots + \frac{k_q}{s+p}$$

where $-p$ is a real root of multiplicity q.

$$k_i = \frac{1}{(i-1)!} \frac{d^{(i-1)}}{ds^{(i-1)}} [F(s)(s+p)^q]|_{s=-p}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= [k_1 \frac{t^{q-1}}{(q-1)!} e^{-pt} + k_2 \frac{t^{q-2}}{(q-2)!} e^{-pt} + k_3 \frac{t^{q-3}}{(q-3)!} e^{-pt} + \cdots + k_q e^{-pt}] u(t)$$

The Inverse Transform–Example

$$F(s)=\frac{100(s+25)}{s(s+5)^3}\equiv\frac{k_4}{s}+\frac{k_1}{(s+5)^3}+\frac{k_2}{(s+5)^2}+\frac{k_3}{s+5}$$

$$k_4=F(s)s\Big|_{s=0}=\frac{100(s+25)}{(s+5)^3}\Bigg|_{s=0}=\frac{100(25)}{125}=20$$

$$k_1=F(s)(s+5)^3\Big|_{s=-5}=\frac{100(s+25)}{s}\Bigg|_{s=-5}=\frac{100(20)}{-5}=-400$$

$$k_2=\frac{d[F(s)(s+5)^3]}{ds}\Bigg|_{s=-5}=\frac{d}{ds}\left[\frac{100(s+25)}{s}\right]_{s=-5}=100[\frac{s-(s+25)}{s^2}]\Bigg|_{s=-5}=-100$$

$$k_3=\frac{1}{2}\frac{d^2[F(s)(s+5)^3]}{ds^2}\Bigg|_{s=-5}=\frac{1}{2}\frac{d^2}{ds^2}\left[\frac{100(s+25)}{s}\right]_{s=-5}=50(\frac{50}{s^3})\Bigg|_{s=-5}=-20$$

$$F(s)=\frac{20}{s}-\frac{400}{(s+5)^3}-\frac{100}{(s+5)^2}-\frac{20}{s+5}$$

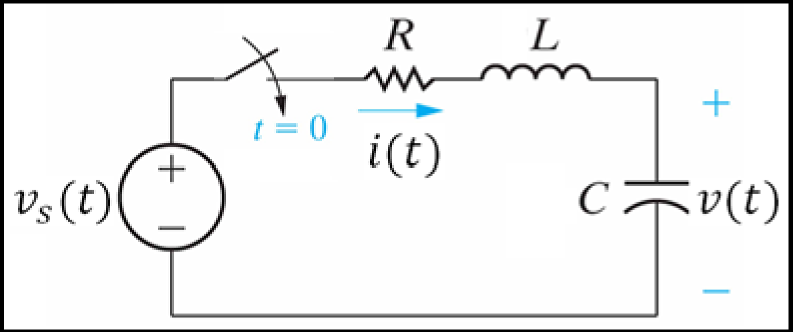
$$f(t)=(20-200t^2e^{-5t}-100te^{-5t}-20e^{-5t})\mathrm{u(t)}$$

Applications in Series-connected RLC Circuits

At t = 0, the switch is closed and $i(0^-) = 0$, $v(0^-) = 0$.

$$Ri(t) + L\frac{d}{dt}i(t) + v(t) = v_s(t)$$

$$i(t) = C\frac{d}{dt}v(t)$$



Applying Laplace transform to the both sides of the above equations and assuming $V(s) = \mathcal{L}\{v(t)\}$, $V_s(s) = \mathcal{L}\{v_s(t)\}$, we have:

$$RI(s) + sLI(s) + V(s) = V_s(s)$$

$$I(s) = sCV(s)$$

$$V(s) = \frac{V_s(s)/LC}{s^2 + \left(\frac{R}{L}\right)s + (1/LC)}$$

From $D(s) = s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right) = 0$, we have

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2}$$

where $\alpha = \frac{R}{2L}$: neper frequency

$\omega_0 = \frac{1}{\sqrt{LC}}$: resonant radian frequency

Applications in Series-connected RLC Circuits

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}\ .$$

Let's assume $v_s(t) = V_f$; a DC source

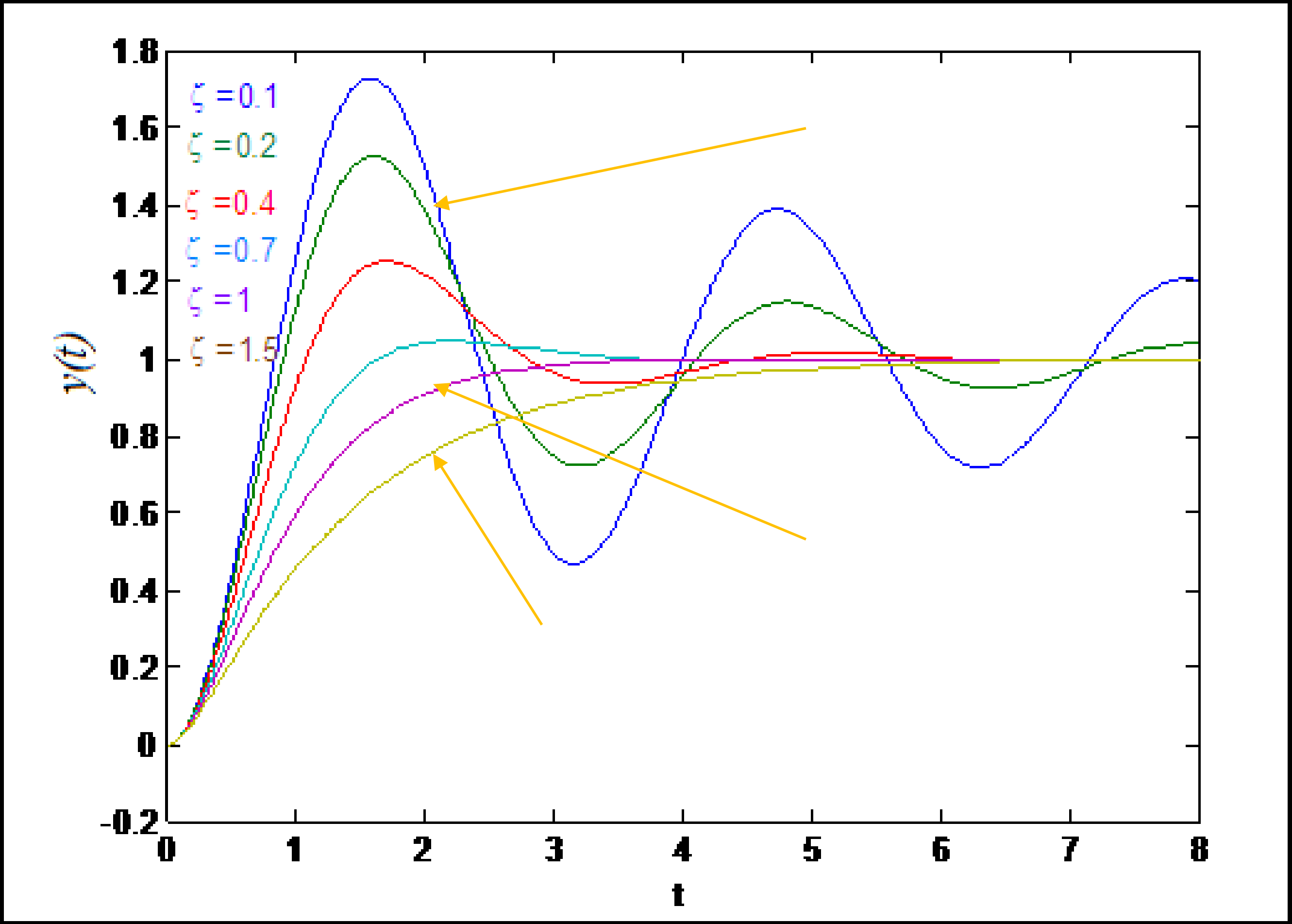
1. If $\alpha > \omega_0$, s_1 *and* s_2 are real and distinct, the voltage response is called overdamped. $v(t) = V_f + k_1 e^{s_1 t} + k_2 e^{s_2 t}$
2. If $\alpha < \omega_0$, s_1 *and* s_2 are distinct and complex, the voltage response is called underdamped.

$$v(t) = V_f + k_1 e^{-\alpha t}\cos(\omega_d t) + k_2 e^{-\alpha t}\sin(\omega_d t) = V_f + k e^{-\alpha t}\cos(\omega_d t + \theta)$$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called damped radian frequency.

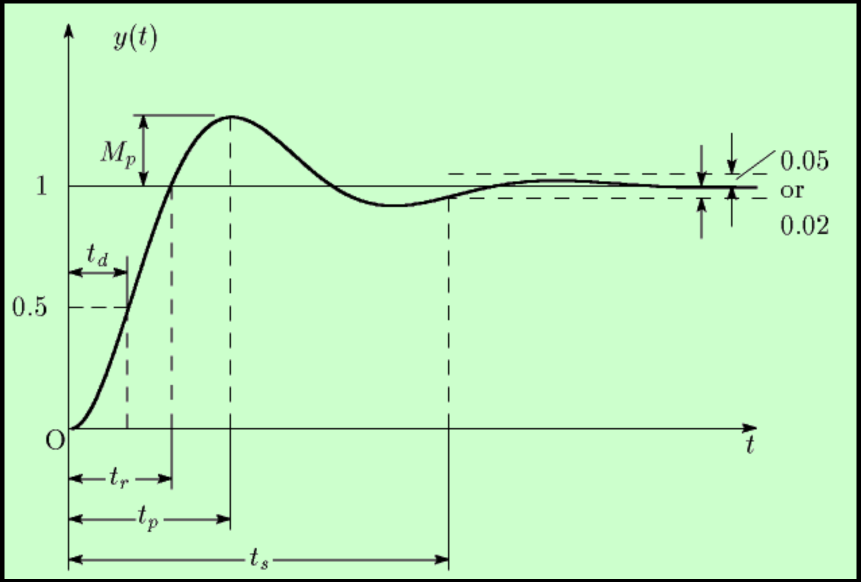
3. If $\alpha = \omega_0$, s_1 *and* s_2 are repeated real roots, the response is called critically damped. $v(t) = V_f + k_1 t e^{-\alpha t} + k_2 e^{-\alpha t}$

Applications in Series-connected RLC Circuits



$\zeta = \frac{\alpha}{\omega_0}$ is called damping ratio.

Applications in Series-connected RLC Circuits



t_p: Peak time(required for the response to reach the first peak value).

t_s: Settling time(required for the response to reach and stay with 2% or 5% final value).

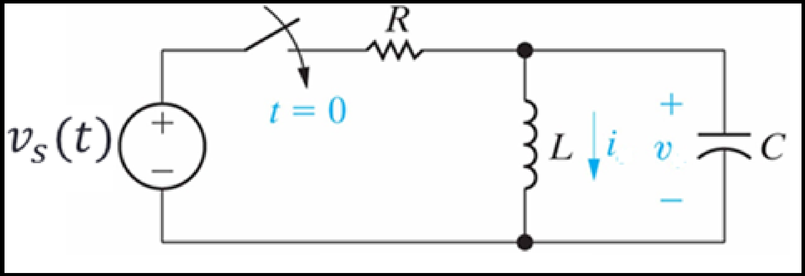
M_p: Maximum percent overshot

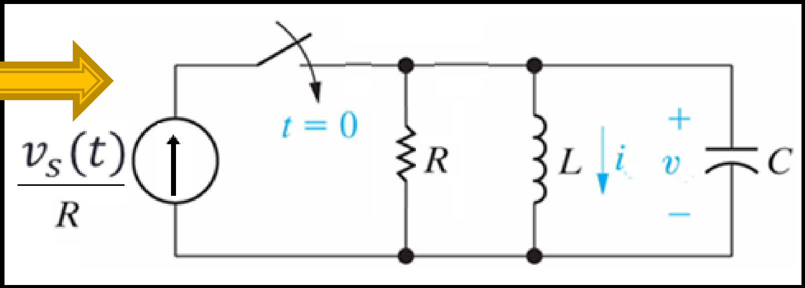
$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)}$$

t_d: Delay time(required for the response to reach half of the final the very first time).

t_r: Rise time(required for the response to rise from 0 to100% of the final value. (For the overdamped or critically damped circuit, the rise time from 10% to 90% is commonly used).

Applications in Parallel-connected RLC Circuits





At t = 0, the switch is closed and $i(0^-) = 0$, $v(0^-) = 0$. Applying KCL to the above circuit, we have:

$$\frac{v(t)}{R} + \frac{1}{L}\int_{0^-}^{t} v(x)dx + C\frac{d}{dt}v(t) = \frac{v_s(t)}{R}$$

Applying Laplace transform to both sides of the above equations and assuming $V(s) = \mathcal{L}\{v(t)\}$, $V_s(s) = \mathcal{L}\{v_s(t)\}$, we have:

$$V(s) = \frac{sV_s(s)/RC}{s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)}.$$

$$I(s) = \frac{V_s(s)/RLC}{s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)}$$

From $D(s) = s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right) = 0$, we have:

$$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2}$$

where $\alpha = \frac{1}{2RC}$: neper frequency

$\omega_0 = \frac{1}{\sqrt{LC}}$: resonant radian frequency

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$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$.   Let's assume $v_s(t) = V_f$; $A\ DC\ source$

1. If $\alpha > \omega_0$, $s_1\ and\ s_2$ are real and distinct, the voltage response is called overdamped. $v(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$
2. If $\alpha < \omega_0$, $s_1 and\ s_2$ are distinct and complex, the voltage response is called underdamped.
   $v(t) = k_1 e^{-\alpha t}\cos(\omega_d t) + k_2 e^{-\alpha t}\sin(\omega_d t) = ke^{-\alpha t}\cos(\omega_d t + \theta)$
   where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called damped radian frequency.
3. If $\alpha = \omega_0$, $s_1 and\ s_2$ are repeated real roots, the response is called critically damped. $v(t) = k_1 te^{-\alpha t} + k_2 e^{-\alpha t}$

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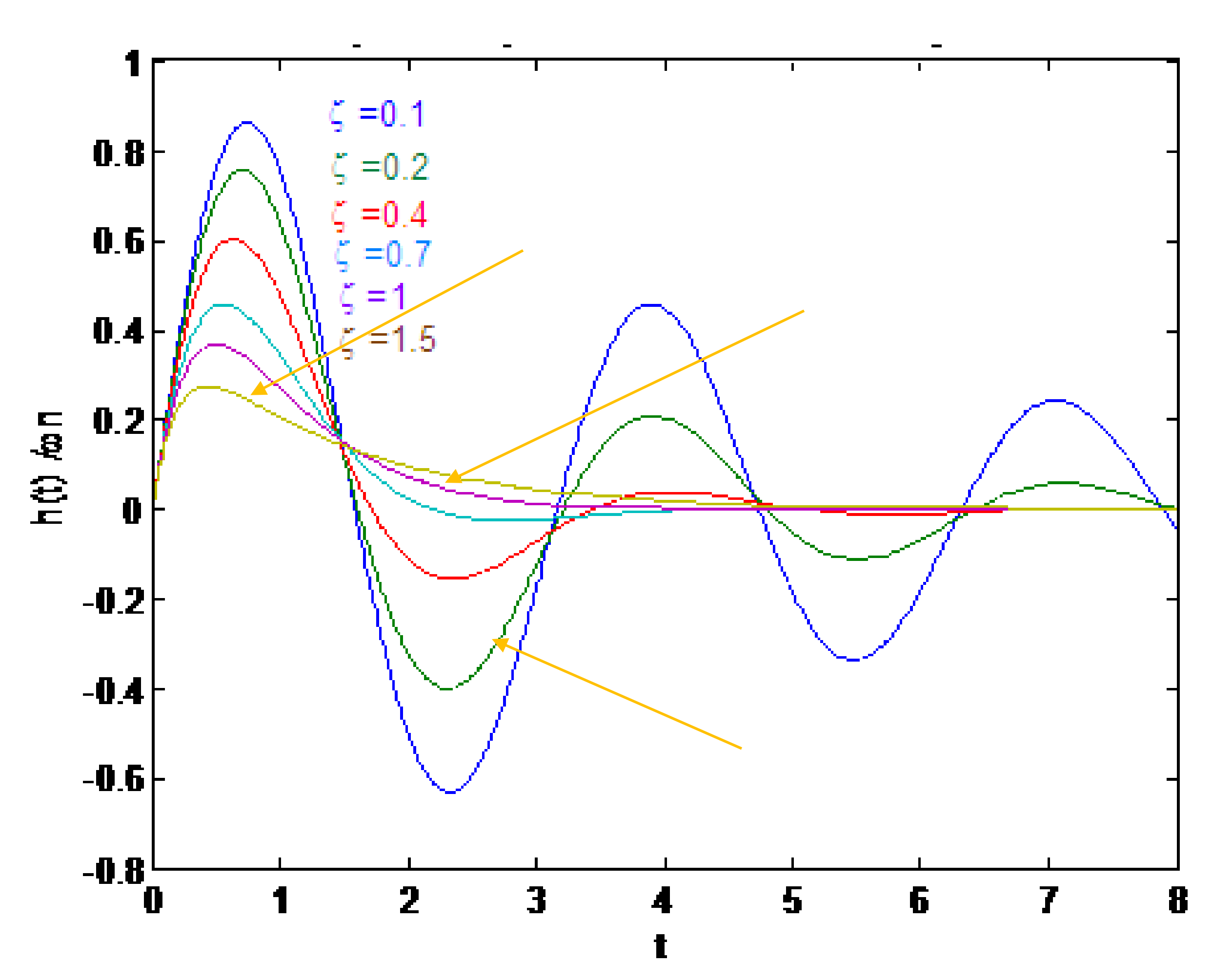

The Inverse Transform

4. D(s) has Repeated complex roots $and\ l < 2q$

$$F(s) = \frac{N(s)}{D(s)} = \frac{K(a_l s^l + a_{l-1}s^{l-1} + \cdots + a_1 s + a_0)}{(s+p)^q (s+\bar{p})^q} \equiv \frac{k_1}{(s+p)^q} + \frac{k_1{}'}{(s+\bar{p})^q} + \cdots + \frac{k_q}{s+p} + \frac{k_q{}'}{s+\bar{p}}$$

$$k_i = \frac{1}{(i-1)!}\frac{d^{(i-1)}}{ds^{(i-1)}}[F(s)(s+p)^q]|_{s=-p} \quad , \qquad k_i{}' = \frac{1}{(i-1)!}\frac{d^{(i-1)}}{ds^{(i-1)}}[F(s)(s+\bar{p})^q]|_{s=-\bar{p}}$$

where $-p\ and\ -\bar{p}$ are a complex root of multiplicity q. They are conjugate.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = [k_1\frac{t^{q-1}}{(q-1)!}e^{-pt} + k_1{}'\frac{t^{q-1}}{(q-1)!}e^{-\bar{p}t} + \cdots + k_q\, e^{-pt} + k_q{}' e^{-\bar{p}t}]u(t)$$

Note: in realizable circuit, complex roots always appear in conjugate pair and their corresponding coefficients k_i and $k_i{}'$ are themselves conjugate.

The Inverse Transform

$$F(s) = \frac{768}{(s^2+6s+25)^2} = \frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$\equiv \frac{k_1}{(s+3-j4)^2} + \frac{k_2}{s+3-j4} + \frac{k_1{}^*}{(s+3+j4)^2} + \frac{k_2{}^*}{s+3+j4}$$

$$k_1 = F(s)(s+3-j4)^2\Big|_{s=-3+j4} = \frac{768}{(s+3+j4)^2}\Bigg|_{s=-3+j4} = \frac{768}{(j8)^2} = -12$$

$$k_2 = \frac{d}{ds}F(s)(s+3-j4)^2\Big|_{s=-3+j4} = -\frac{2(768)}{(s+3+j4)^3}\Bigg|_{s=-3+j4} = -\frac{1536}{(j8)^3}$$

$$= -j3 = 3e^{-j90^0}$$

$$F(s) \equiv \frac{-12}{(s+3-j4)^2} + \frac{3e^{-j90^0}}{s+3-j4} + \frac{-12}{(s+3+j4)^2} + \frac{3e^{j90^0}}{s+3+j4}$$

$$f(t) = \left[-24te^{-3t}cos4t + 6e^{-3t}\cos\left(4t-90^0\right)\right]u(t).$$

Summary

- If F(s) is a proper rational function, the inverse Laplace transform can be found by using a partial fraction expansion.
- If F(s) is an improper rational function, it can be expanded as a polynomial function plus a proper rational function.
- Four situations in the inverse Laplace transform were discussed based on the different types of roots of D(s).

- In next class, we will discuss
- Poles and zeros
- Initial- value and final-value theorems