Homework Quiz #3

1. Suppose A is an $n \times m$ matrix. True or False: The matrix AA^T is symmetric.

> **Solution: TRUE.** Let $R = AA^T$, we want to show $R = R^T$. That is,
> $$R^T = \left(AA^T\right)^T = (A^T)^T A^T = AA^T = R.$$

2. Suppose A is an $n \times m$ matrix. True or False: The matrix $A^T A$ is symmetric.

> **Solution: TRUE.** Let $R = A^T A$, we want to show $R = R^T$. That is,
> $$R^T = \left(A^T A\right)^T = A^T (A^T)^T = A^T A = R.$$

3. If A and B are symmetric. (i.e., $A = A^T$ and $B = B^T$). Which of these matrices **might not** be symmetric.

> **Solution:** In what follows below, we will let R be the matrix in question and investigate which ones must be symmetric since $A = A^T$ and $B = B^T$
>
> - $R = ABA$ must be symmetric. We have:
> $$R^T = (ABA)^T = A^T B^T A^T = ABA = R.$$
> - $R = A^2 + B^2$ must be symmetric. Since A and B are symmetric, we know that A^2 and B^2 are symmetric:
> $$(A^2)^T = (AA)^T = A^T A^T = (A^T)^2 = A^2.$$
> We also know that the sum of two symmetric matrices is symmetric.
> - $R = A^2 - B^2$ must be symmetric. (See previous explaination).
> - $R = ABAB$ might not be symmetric:
> $$R^T = (ABAB)^T = B^T A^T B^T A^T = BABA.$$
> Thus, $R^T \neq R$ in general.

4. True or false. The following matrix is **skew-symmetric**
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}.$$

> **Solution: FALSE.** A matrix is skew-symmetric if $A^T = -A$. Since:
> $$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = A,$$
> we have a symmetric matrix.

5. Let
$$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$
What is the entry in the first row and first column of $\vec{x}^T A \vec{y}$.

Solution:
$$\vec{x}^T A \vec{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 5.$$

6. Give an example of a 2×2 matrix A that is **not symmetric** and that satisfies $A^2 = 0$

Solution: Let's look for a generic 2×2 matrix, that is not symmetric (i.e., $b \neq c$ and square it:
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}. = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
We need every entry to be 0 so, for simplicity, let's suppose $a = -d$. This automatically makes the $a_{1,2} = a_{2,1} = 0$.
Thus we have the following:
$$\begin{bmatrix} a^2 + bc & 0 \\ 0 & bc + a^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
We notice that if $b = a$ and $c = -a$ we will satisfy the other conditions:
$$A = \begin{bmatrix} a & a \\ -a & -a \end{bmatrix}.$$
The matrix A is not symmetric and satisfies $A^2 = 0$. Non-zero matrices that have some power $A^k = 0$ are called **nilpotent** matrices, and we will spend time studying them later in the semester.

7. We learned that $n \times n$ matrices A can be factored into LDU where L is a lower triangular matrix, D is a diagonal matrix and U is an upper triangular matrix.
But if A is symmetric it turns out that this the same LDU factorization process produces an upper triangular matrix U that is actually the transpose of the lower triangular matrix L. As such, we write the factorization as:
$$A = LDL^T.$$
Using elementary row matrices, determine the symmetric LDL^T factorization of:
$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix}.$$

Solution: Let's start off by performing the first two row operations:
$$E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

We have:

$$E_{13}E_{12}A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix}$$

We will now carry out the last row operation:

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \implies E_{23}E_{13}E_{12}A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Let's first operate on the right hand side. We will factor out a 3 from rows 2 and 3.

$$E_{23}E_{13}E_{12}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's call the diagonal matrix D and the upper triangular matrix U. Not returning to the LDU factorization, we invert the elementary row matrices multiplying A to obtain L

$$A = (E_{23}E_{13}E_{12})^{-1} DU.$$

We can either directly multiply for ourselves, or remember that the lower triangular matrix L has 1's on the diagonal and the multipliers on the off diagonal:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}.$$

We notice that our matrix $U = L^T$. As such, we have:

$$A = LDL^T.$$