Homework Assignment #5

Remember, this Homework Assignment is **not collected or graded**! But you are advised to do it anyway because this is a Review for Exam #1. In addition, Homework Quiz #5 will be heavily based on these problems!

1. For which values of a will the following vector matrix system fail to have 3 pivots?

$$\begin{aligned} ax + 2y + 3z &= b_1 \\ ax + ay + 4z &= b_2 \\ ax + ay + az &= b_3. \end{aligned}$$

Solution: Let's consider the system:

$$\left[\begin{array}{ccc|c} a & 2 & 3 & b_1 \\ a & a & 4 & b_2 \\ a & a & a & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & a-2 & a-3 & b_3 - b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & 0 & a-4 & b_3 - b_2 - 2b_1 \end{array}\right]$$

We see that the values of the pivots are $a, (a-2)$ and $(a-4)$. As such, if $a = 0, 2,$ or 4 the system will fail to have 3 pivots.

2. Write down the 3 by 3 elementary row matrices that perform the following elementary row operations:

(a) Subtracts 5 times row 1 from row 2.

Solution: $E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

(b) Subtracts -7 times row 2 from row 3.

Solution: $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}.$

(c) P exchanges rows 1 and 2.

Solution: $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

3. Consider the following:

(a) If A is invertible, what is the inverse of A^T

Solution: $(A^T)^{-1} = (A^{-1})^T$ if A^{-1} exists.

(b) If A is invertible and symmetric, what is the transpose of A^{-1}?

Solution: If A is symmetric then, the inverse of A is symmetric. Let's take the transpose of AA^{-1} and $A^{-1}A$. This will show that $(A^{-1})^T$ is also the inverse of A.

$$AA^{-1} = I \implies (AA^{-1})^T = I^T \implies (A^{-1})^T A^T = I \implies (A^{-1})^T A = I.$$

$$A^{-1}A = I \implies (A^{-1}A)^T = I^T \implies A^T(A^{-1})^T = I \implies A(A^{-1})^T = I.$$

The last equality on each line shows that $(A^{-1})^T$ is also A^{-1}. But since the inverse of a matrix is unique this must mean that:

$$(A^{-1})^T = A^{-1}$$

and thus A^{-1} is also symmetric.

(c) Illustrate both formulas for the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution: Right now the only formula we have for building an inverse is doing the Gauss-Jordan Algorithm. So, we will do that:

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to 1/2R_1} \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1/2 & -1/2 & 1 \end{array}\right] \xrightarrow{R_2 \to 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right].$$

Thus $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

We can see that A^{-1} is also symmetric and thus, $(A^{-1})^T = (A^T)^{-1}$.

4. Suppose A is the following matrix:

$$A = \begin{bmatrix} 1 & v_1 & 0 & 0 \\ 0 & v_2 & 0 & 0 \\ 0 & v_3 & 1 & 0 \\ 0 & v_4 & 0 & 1 \end{bmatrix}$$

(a) Factor $A = LU$ assuming $v_2 \neq 0$.

Solution: The matrix A is almost already in REF, we just need to perform two last row operations. These row operations will be possible if $v_2 \neq 0$.

We want to perform: (1) $R_4 \to R_4 - (v_4/v_2)R_2$ and (2) $R_3 \to R_3 - (v_3/v_2)R_2)$. We can do this by the following multiplications:

$$E_{24}E_{23}A = U$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-v_3/v2&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-v_4/v2&0&1\end{bmatrix}\begin{bmatrix}1&v_1&0&0\\0&v_2&0&0\\0&v_3&1&0\\0&v_4&0&1\end{bmatrix}=\begin{bmatrix}1&v_1&0&0\\0&v_2&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Thus we have: $A = LU$ where:

$$L=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&v3/v2&1&0\\0&v4/v2&0&1\end{bmatrix} \text{ and } U=\begin{bmatrix}1&v_1&0&0\\0&v_2&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

(b) Find A^{-1}. You'll see is has the same form as A.

Solution: We can see that we have almost arrived at the inverse. Just two more operations. We will need to: (1) $R_2 \to (1/v_2)R_2$, (2) $R_1 \to R_1 - v_1R_2$.
We will employ these to see that we have:

$$E_{12}F_2E_{24}E_{23}A = I \implies A^{-1} = (E_{12}F_2E_{24}E_{23})$$

$$E_{12}=\begin{bmatrix}1&-v_1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \text{ and } F_2=\begin{bmatrix}1&0&0&0\\0&1/v_2&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Multiplying we have:

$$A^{-1}=E_{12}F_2E_{24}E_{23}=\begin{bmatrix}1&-v1/v2&0&0\\0&1/v_2&0&0\\0&-v_3/v_2&1&0\\0&-v_4/v_2&0&1\end{bmatrix}.$$

We see this is a matrix of a similar form to A and exists/is well defined as long as $v_2 \neq 0$.

5. Use Gauss Jordan Elimination to calculate the inverse of:

$$A=\begin{bmatrix}2&1&0\\1&2&1\\0&1&2\end{bmatrix}.$$

Solution: Gauss Jordan Elimination requires us begin from an augmented matrix of the form: $[A|I]$ and proceed to an augmented matrix of the form $[I|A^{-1}]$.

$$\left[\begin{array}{ccc|ccc}2&1&0&1&0&0\\1&2&1&0&1&0\\0&1&2&0&0&1\end{array}\right]\xrightarrow{R_2\to R_2-(1/2)R_1}\left[\begin{array}{ccc|ccc}2&1&0&1&0&0\\0&3/2&1&-1/2&1&0\\0&1&2&0&0&1\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-(2/3)R_2}\left[\begin{array}{ccc|ccc}2&1&0&1&0&0\\0&3/2&1&-1/2&1&0\\0&0&4/3&1/3&-2/3&1\end{array}\right]\xrightarrow{R_3\to 3/4R_3}\left[\begin{array}{ccc|ccc}2&1&0&1&0&0\\0&3/2&1&-1/2&1&0\\0&0&1&1/4&-1/2&3/4\end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right] \xrightarrow{R_2 \to 2/3 R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right] \xrightarrow{R_1 \to 1/2 R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right].$$

Thus we arrive at:

$$A^{-1} = \begin{bmatrix} 3/4 & -1/2 & 1/4 \\ -1/2 & 1 & -1/2 \\ 1/4 & -1/2 & 3/4 \end{bmatrix}.$$

6. The trace of a matrix tr(A) is defined as the sum of entries along the diagonal. So, for a 4×4 matrix A we have:

$$\text{tr}(A) = a_{11} + a_{22} + a_{33} + a_{44}.$$

In the following, you will consider S_4, the set of all 4×4 matrices with tr$(A) = 0$.

(a) Show that S_4 is a subspace.

Solution: First, we notice that the 0-matrix (the 4 by 4 matrix of all 0's) is in S_4 because it's trace is 0. We next want to show that S_4 is closed under addition and scalar multiplication.

- Closure Under Addition:

  Suppose that A and B both belong to S_4. Then we know, by definition, tr$(A) =$ tr$(B) = 0$. We want to show that $C = A + B$ also belongs to S_4. Thus, we have to show tr$(C) = 0$. We see that:

$$\begin{aligned} \text{tr}(C) &= c_{11} + c_{22} + c_{33} + c_{44} \\ &= (a_{11} + b_{11}) + (a_{22} + b_{22}) + (a_{33} + b_{33}) + (a_{44} + b_{44}) \\ &= (a_{11} + a_{22} + a_{33} + a_{44}) + (b_{11} + b_{22} + b_{33} + b_{44}) \\ &= \text{tr}(A) + \text{tr}(B) \\ &= 0. \end{aligned}$$

  Thus, we have shown that $C \in S_4$ and that the set is closed under addition.

- Closure Under Scalar Multiplication:

  Suppose that A belongs to S_4, we want to show that for any real constant $\alpha \in \mathbb{R}$ we must have $C = (\alpha A)$ in S_4. That is, tr$(C) = 0$. We repeat as above:

$$\begin{aligned} \text{tr}(C) &= c_{11} + c_{22} + c_{33} + c_{44} \\ &= \alpha a_{11} + \alpha a_{22} + \alpha a_{33} + \alpha a_{44} \\ &= \alpha(a_{11} + a_{22} + a_{33} + a_{44}) \\ &= \alpha \text{tr}(A) \\ &= 0. \end{aligned}$$

  Thus we have shown $C = \alpha A \in S_4$ for any choice of α. Thus, S_4 is closed under scalar multiplication.

Because S_4 is closed under both addition and scalar multiplication, it is a subspace of these vector space of 4×4 matrices.

(b) Find a basis for S_4 and determine its dimension.

Solution: We first look at what a typical 4 by 4 matrix looks like. We will look at a basis for this bigger vector space and then consider S_4. A general 4 by 4 matrix has 15 entries:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

For simplicity, let's define $E_{i,j}$ to be the 4 by 4 matrix with 0 entries everywhere except $e_{i,j} = 1$. For example, $E_{1,3}$ is defined as follows:

$$E_{1,3} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can see that the vector space of 4×4 matrices has dimension 16 (each entry in the matrix can be chosen independently) and thus a basis for this set of matrices is:

$$B = \{E_{11}, E_{12}, E_{13}, E_{14}, E_{21}, E_{22}, E_{23}, E_{24}, E_{31}, E_{32}, E_{33}, E_{34}, E_{41}, E_{42}, E_{43}, E_{44}\}.$$

Let's notice that this is a linearly independent spanning set of the set of 4 by 4 matrices. Now, let's start by keeping the ones in this basis that are also in S_4.
Each one of these matrices belongs to S_4 except for: E_{11}, E_{22}, E_{33} and E_{44}. Each of these have tr$(E_{ii}) = 1$.
Now for an element of S_4 we don't have quite the same freedom, because we need to restrict four of the entries as follows:

$$a_{11} + a_{22} + a_{33} + a_{44} = 0 \implies a_{44} = -(a_{11} + a_{22} + a_{33}).$$

This means that a general entry $S \in S_4$ looks like this:

$$S = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & -(a_{11} + a_{22} + a_{33}) \end{bmatrix}.$$

This lets us see that S_4 really has dimension 15, one of the entries in the matrix is fully determined by the other values. Thus a basis for S_4 has to have 3 special entries:

$$F_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, F_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, F_{33} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Thus, the basis we have for S_4 is the following:

$$B = \{F_{ii} \text{ for } i = 1, 2, 3 \text{ and } E_{ij} \text{ for } i, j = 1, 2, 3, 4 \text{ and } i \neq j\}.$$

7. Find a basis for the following subspaces of $\mathbb{R}^4$:

(a) The vectors satisfying $x_1 = 2x_4$.

Solution: Let's start by forcing the equation above into the matrix vector format. (This will let us see that we really have 1 pivot and 3 free variables:

$$\begin{bmatrix} 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

In this case we see that $x_4 = t$, $x_3 = s$, and $x_2 = r$. The equation itself implies $x_1 = 2t$. Thus our three special solutions (which form a basis) are the following:

$$B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(b) The vectors for which:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_3 + x_4 &= 0. \end{aligned}$$

Solution: Let's start by writing things once again as a vector and matrix system:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We see that this system has 2 pivots and 2 free variables. The free variables are $x_4 = t$ and $x_2 = s$. The second equation gives us: $x_3 = -t$ and the first equation:

$$x_1 + x_2 + x_3 = 0 \implies x_1 = -x_2 - x_3 \implies x_1 = -s + t.$$

These two special solutions are the basis for the nullspace of this matrix (which is the basis for the vectors which solve this sytsem.)

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(c) The subspace spanned by the vectors:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\}.$$

Solution: We know that the span of any number of vectors is itself a vector space. However, the set above is a basis only if the set of vectors are linearly independent.

In order to determine if the vectors are linearly independent, we will look for solutions $\vec{\alpha}$ where:

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{0}.$$

But this is asking us about the nullspace of this matrix:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - R_1, R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 3R_2]{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let's notice that we have 2 pivots and 1 free variable in this system, this means that we have non-trivial solutions and (as such) the vectors are linearly dependent. Because columns 1 and 2 have pivots, we know that together $\{\vec{v}_1\vec{v}_2\}$ form our basis (a linearly independent spanning set). But let's finish solving for α_i so we can see the dependency.

$$\alpha_3 = t, \alpha_2 + \alpha_3 = 0 \implies \alpha_2 = -t$$

$$\text{and } \alpha_1 + \alpha_2 + \alpha_3 = 0 \implies \alpha_1 = -\alpha_2 - 2\alpha_3 = t - 2t = -t.$$

Thus,

$$\vec{\alpha} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

But this means that:

$$t\left(-\vec{v}_1 - \vec{v}_2 + \vec{v}_3\right) = 0 \text{ for all } t \in \mathbb{R}.$$

(That is $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$ which we can also see from how they were written.)
The basis for this set is thus:

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

8. Suppose A is an $m \times n$ matrix with rank r. Suppose that there **some** choices for $\vec{b}$ where:

$$A\vec{x} = \vec{b}$$

has **no solution**.

(a) What inequalities ($<$ or $\leq$) must be true between m, n and r?

Solution: Let's first notice that A is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. This means $\vec{b} \in \mathbb{R}^m$ and $\vec{x} \in \mathbb{R}^n$. If there are some $\vec{b}$ where there are no solutions to $A\vec{x} = \vec{b}$ this means that $C(A) \neq \mathbb{R}^m$. We know that $\dim(C(A)) = r$, so this requires that $r < m$. But we also know that $r \leq n$ because the rank (the number of pivots) can never be larger than the number of columns or rows.

(b) How do we know that $A^T\vec{y} = 0$ has a non-zero solution?

Solution: The Fundamental Theorem of Linear Algebra tells us that:

- $\dim(C(A)) = r$,

- dim$(N(A)) = n - r$,
- dim$(C(A^T)) = r$,
- dim$N(A^T)) = m - r$.

Since we know that $r < m$ (from part (a)) this means that $m - r > 0$. As such the dimension of the left nullspace will be larger than 0 and there will be non-trivial (i.e., non-zero) solutions to:

$$A^T\vec{y} = \vec{0} \implies \vec{y}^T A = \vec{0}.$$

9. Find dimensions and bases for the four fundamental subspaces for:

(a) $A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 5 & 5 \end{bmatrix}$.

Hint: You should be able to figure these out without extensive calculations.

Solution:

(a) $A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

We will start by reducing the matrix to REF (although you do not need to).

$$\begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{R_3 \to R_3 - (1/3)R_1} \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that the rank of this matrix is 2 and the columns 2 and 3 (which have the pivots) correspond to the linearly independent columns of A. We also see that we have 2 free variables which gives us:

$$x_4 = t, x_1 = s.$$

The second row gives us $x_3 = 0$ and the top row:

$$3x_2 + 3x_3 + 3x_4 = 0 \implies x_2 + x_4 = 0 \implies x_2 = -t.$$

We learned that the row space of A is equal to the row space of U, it's REF form. And so we can see that rows 1 and 2 in U correspond to the basis for $C(A^T)$.

- $C(A)$ has dimension 2 and the following basis:

$$B_{C(A)} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

- $N(A)$ has dimension 2 and the following basis corresponding to the special solutions to $A\vec{x} = \vec{0}$.

$$B_{N(A)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

- $C(A^T)$ has dimension 2 and the following basis:

$$B_{C(A^T)} = \left\{ \begin{bmatrix} 0 & 3 & 3 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix} \right\}.$$

- $N(A^T)$ has dimension 1 (number of rows minus rank). To get it's basis we will go back to A and consider finding the nullspace of A^T.

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This system has 2 pivots and 1 free variable. The pivots correspond to $x_1 = x_3 = 0$ and $x_2 = t$ is the pivot. As such, the basis for $N(A^T)$ is given by:

$$B_{N(A^T)} = \left\{ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \right\}.$$

(b) $B = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 5 & 5 \end{bmatrix}$.

For this system, let's go ahead and use some shortcuts. We know the column space is equal to the span of the columns of B. Since the two columns are identical, this means:

$$\text{dim}(C(B)) = 1 \text{ and } B_{C(B))} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}.$$

The rowspace of B is the span of the rows of B. We can see that each row is a multiple of the first. As such:

$$\text{dim}(C(B^T)) = 1 \text{ and } B_{C(B^T)} = \left\{ \begin{bmatrix} 1 & 1 \end{bmatrix} \right\}.$$

A little more work is needed to find the nullspace of B and the left nullspace of B. To do those we will go through the reduction with row operations.

$$N(B) \implies \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 5 & 5 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \text{dim}(N(B)) = 1 \text{ and } B_{N(B)} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

$$N(B^T) \implies \begin{bmatrix} 1 & 4 & 5 \\ 1 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{dim}(N(B^T)) = 2 \text{ \& } B_{N(B^T)} = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

10. Consider the following matrix A and vector $\vec{b}$:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(a) Find a basis for the nullspace of A.

Solution: We will perform row operations on the matrix A to reduce it to REF. For simplicity, we will switch the second and third rows immediately.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This system has 2 pivots and 2 free variables. We see that $x_4 = 0, x_3 = t, x_2 = s$ and $x_1 = -2s$.

This means that the dimension of the nullspace of A is 2 and the basis for the nullspace of A is found by the two special solutions:

$$B_{N(A)} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(b) Find the general solution to $A\vec{x} = \vec{b}$ when a solution exists.

Solution: We already know that the general solution to $A\vec{x} = \vec{b}$ is the sum of a particular solution plus the nullspace. Since in A we learned that the nullspace has dimension 2, we know there will be infinitely many solutions to this system.

But, we will proceed as if we had not previously found the nullspace.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 2 & 4 & 0 & 1 & b_3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 2 & 4 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 0 & b_2 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & -5 & b_3 - 2b_1 \\ 0 & 0 & 0 & 0 & b_2 \end{array}\right].$$

This system will have no solutions with $b_2 \neq 0$ but otherwise will have the following.

$$x_4 = (b_3 - 2b_1)/(-5), x_3 = t, x_2 = s \text{ and}$$

$$x_1 + 2x_2 + 3x_4 = b_1 \implies x_1 = b_1 - 3x_4 - 2x_2 \implies x_1 = b_1 + \frac{3}{5}(b_3 - 2b_1) - 2s.$$

$$\vec{x} = \begin{bmatrix} b_1 + \frac{3}{5}(b_3 - 2b_1) \\ 0 \\ 0 \\ (b_3 - 2b_1)/(-5) \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(c) Find a basis for the column space of A.

Solution: A basis for the column space of A consists of the linearly independent columns of A. We can find those columns by selecting the columns in the corresponding pivot positions in the REF of A. (In this case it's columns 1 and 4 of A.)

$$B_{C(A)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Alternatively, we could just look at the columns in the original matrix A and notice that the first and second columns are multiples of one another (so we only need to keep one), the third column is the 0 vector (which never belongs in a basis) and the last vector is not a multiple (or linear combination) of the others.

(d) Find the rank of A and A^T.

Solution: The rank of a matrix (and it's transpose) are the same. As such the rank of both is 2 since we have already obtained the rank of the matrix A in part (a). However, if we did not know that the rnak of A and A^T are the same, we could simply perform row operations on A^T and see how many pivots we obtain.

$$A^T = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

This matrix has 2 pivots and 1 free variable which means that the rank of the matrix is 2.