# Discussion Section: Week #8
# Due: By 11:59pm the day of your Discussion Section

## Instructions:

You are allowed to work alone or in small groups. But if you work in a group, the work you turn in should be your own. Each student will submit a physical write-up as a PDF document to CatCourses **by 11:59 pm of your discussion section day**.

Discussion assignments will be graded partially for completeness but a subset of the problems will be graded for correctness. Providing the correct answer, without justification, is not considered complete. For credit you **must** either show steps (if it's a calculation problem) or explain/justify your reasoning (if it's a short answer problem).

## Problem Set:

1. Find a vector $x$ orthogonal to the row space of $A$, and a vector $y$ orthogonal to the column space, and a vector $z$ orthogonal to the nullspace. (All nonzero)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$$

**Solution:** We see that a basis for $C(A)$ can be found by finding the pivots of $A$. This yields $C(A) =$

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}$$

$R(A) =$

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \right\}$$

and $N(A) =$

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Using the definition of orthogonality, we want the dot product with respect to each basis vector to be 0. The easiest to start with is $N(A)$ because there is only a single vector in there. We see that $z =$

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

works as $z \cdot x \in N(A) = 0$. We see that by the Fundamental Theorem of Orthogonality that this is an element in $R(A)$. Similarly, we see that $x =$

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is orthogonal to both elements in $R(A)$. Finally for $C(A)$, we can find a basis for $N(A^T)$ and these will be orthogonal to any element in $C(A)$. Computing $N(A^T)$, we get $N(A^T) =$

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

which has dot product of 0 with both elements in $C(A)$.

2. Show that $x - y$ is orthogonal to $x + y$ if and only if $\|x\| = \|y\|$.

**Solution:** By definition we get that

$$\langle x+y, x-y\rangle = 0 \Leftrightarrow \langle x, x-y\rangle + \langle y, x-y\rangle = 0 \Leftrightarrow \langle x,x\rangle - \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle = 0 \Leftrightarrow$$

$$\langle x,x\rangle = \langle y,y\rangle \Leftrightarrow \|x\|^2 = \|y\|^2 \Leftrightarrow \|x\| = \|y\|$$

Using the properties of the linearity and symmetry of the inner product operator and the definition of a norm.

Here is another way to arrive the same conclusion:

$$\begin{aligned} (x-y)^T(x+y) &= (x^T - y^T)(x+y) \\ &= x^Tx + x^Ty - y^Tx - y^Ty. \end{aligned}$$

Since $x^Ty = y^Tx$, $x^Tx = \|x\|^2$ and $y^Ty = \|y\|^2$, $(x-y)^T(x+y)$ simplifies to

$$\begin{aligned} (x-y)^T(x+y) &= (x^T - y^T)(x+y) \\ &= x^Tx + x^Ty - y^Tx - y^Ty \\ &= x^Tx - y^Ty \\ &= \|x\|^2 - \|y\|^2. \end{aligned}$$

Since we assume $x - y$ and $x + y$ are orthogonal, we have

$$
\begin{aligned}
0 = (x-y)^T(x+y) &= (x^T - y^T)(x+y) \\
&= x^Tx + x^Ty - y^Tx - y^Ty \\
&= x^Tx - y^Ty \\
&= \|x\|^2 - \|y\|^2.
\end{aligned}
$$

This implies

$$
\begin{aligned}
0 &= \|x\|^2 - \|y\|^2 \\
\|x\|^2 &= \|y\|^2 \\
\|x\| &= \|y\|.
\end{aligned}
$$

If we start from $\|x\| = \|y\|$, we can reverse the steps from the work above to show that $x - y$ and $x + y$ are orthogonal.