CSE 015: Discrete Mathematics
Homework 5

Fall 2021
Provided Solution

1 Mathematical Induction 1

a) The statement $P(1)$ is the statement obtained for $n = 1$, i.e., the statement

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2$$

b) The statement $P(1)$ can be shown to be true by substitution. Since $1^3 = 1$, and $1^2 = 1$ the left and right side are equal. Note that on the right side of the equality we can divide by 2 both numerator and denominator.

c) The inductive hypothesis is that the predicate is true for a generic k, i.e., we consider the statement obtained for $n = k$:

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

d) the inductive step asks to show that $P(k) \to P(k+1)$. This can be done in three steps. First, write the sum for $n = k+1$

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3$$

Next, use the inductive hypothesis, i.e., use the fact that the sum of the first k terms is equal to $(\frac{k(k+1)}{2})^2$. This leads to the following expression:

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

At this point just add the two terms and apply basic algebraic rules to rewrite the numerator

$$\frac{k^2(k+1)^2 + 2^2(k+1)^3}{2^2} = \frac{(k+1)^2(k^2 + 4(k+1))}{2^2} = \frac{(k+1)^2(k+2)^2}{2^2} = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

This concludes the proof by observing that the last term is the right end side of $P(n)$ for $n = k+1$. In other words, we have shown that for every $k > 1$, if $P(k)$ is true, then $P(k+1)$ is true, i.e., we completed the inductive step.

2 Mathematical Induction 2

a) Based on the given definitions, the first even natural number is 0 and the corresponding sum is 0. The first two even numbers are 0 and 2, and the sum is 2. The first three are 0,2 and 4, and the sum is 6. The first four even numbers are 0,2,4, and 6, and the sum is 12. We can organize this data in a table, where on the left column we have k and on the right column we have the result of the sum of the first k even numbers (shown in the middles column). In the table we also add a few more rows to ease the task of guessing the forumula.

k	Sum	Result
1	0	0
2	$0+2$	2
3	$0+2+4$	6
4	$0+2+4+6$	12
5	$0+2+4+6+8$	20
6	$0+2+4+6+8+10$	30

One can observe that for each row of the table the result associated with k is $k(k-1)$. For example, for $k=3$ the result is $6 = 3 \cdot 2 = k(k-1)$. The same is true for every row of the table. So $k(k-1)$ is a guess that works for values of k up to 6. In the homework you could stop for $k=4$ and the same guess works, too. However, this is just a guess and before we can say that it holds for all possible values, we have to prove it using mathematical induction.

b) We can follow exactly the same steps we followed in the previous exercise. The statement $P(n)$ gives the formula for the sum of the first n even natural numbers:

$$0+2+4+\cdots+(2n-2) = n(n-1)$$

Note that $2n-2$ is the n-th even number when we assume that the first even number is 0. Indeed, the first even natural number and $0 = 2 \cdot 1 - 2$. Next, 2 is the second even natural number and $2 = 2 \cdot 2 - 2$. The third even natural number is $4 = 2 \cdot 3 - 1$. And so on.

Basis of induction: for $n=1$ the predicate is $0 = n(n-1) = 1 \cdot (1-1)$ and it is verified.

Inductive step: in this case the the inductive hypothesis is that the formula is valid for k, i.e.,

$$0+2+4+\cdots+(2k-2) = k(k-1)$$

and we want to show it is valid for $k+1$. The sum of the first $k+1$ even natural numbers is

$$0+2+4+\cdots+(2k-2)+2k$$

Next, we use the inductive hypothesis for the sum of the first k terms and the sum therefore is

$$k(k-1)+2k$$

and finally we rewrite it as

$$k^2-k+2k = k^2+k = k(k+1)$$

We observe that the last term is the formula given by $P(n)$ for $n=k+1$. This completes the inductive step and the proof, thus showing that our guess is valid for all values of n.

3 Mathematical Induction 3

a) The statement for $P(2)$ is the statement for $n = 2$, i.e., $2! < 2^2$.

b) $P(2)$ is true because $2! = 2 \cdot 1 = 2$ and $2^2 = 4$.

c) The inductive hypothesis is for a generic $k > 1$, $k! < k^k$.

Note that the above steps do not conclude the proof. The next step (not asked in the question), would be to show that if $k! < k^k$, then $(k + 1)! < (k + 1)^{(k+1)}$.