Homework Assignment #7

Remember, this Homework Assignment is **not collected or graded**! But it is in your best interest to do it as the Homework Quiz will be based on it and it is the best way to ensure you know the material.

Section 3.3: Projections and Least Squares

1. Consider the following vector matrix system:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$$

   (a) Show the system $A\vec{x} = \vec{b}$ has no solution.

   (b) Use Calculus to find $\hat{\vec{x}} = \begin{bmatrix} \hat{x_1} \\ \hat{x_2} \end{bmatrix}$ that minimizes the error between $A\hat{\vec{x}}$ and $\vec{b}$:

$$E(\hat{x_1}, \hat{x_2}) = \|A\hat{\vec{x}} - \vec{b}\|^2 = (\hat{x_1} - \hat{x_2} - 4)^2 + (\hat{x_1} - 5)^2 + (\hat{x_1} + \hat{x_2} - 9)^2.$$

   (c) Solve the normal equations:

$$A^T A\hat{\vec{x}} = A^T\vec{b}$$

   and compare your solution from part (b) and (c).

2. Consider the following matrix A and vector $\vec{b}$:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

   (a) Find the projection, $\vec{p}$ of $\vec{b}$ into the column space of A.

   (b) Let $\vec{e} = \vec{p} - \vec{b}$ be the *error* between $\vec{b}$ and $\vec{p}$ which is in the column space of A. Which of the four fundamental subspaces does $\vec{e}$ belong to?

3. If V is the subspace spanned by $(1, 1, 0, 1)$ and $(0, 0, 1, 0)$ find:

   (a) A basis for the orthogonal complement $V^{\perp}$.

   (b) The projection matrix P onto V. (Recall, the projection matrix takes any $\vec{b}$ and projects it to the column space of V. Explicitly, we know that if the columns of a matrix A are linearly independent: $P = A(A^T A)^{-1}A^T$. See Section 3.3 and Lecture Notes from Week 8 for more details.)

   (c) The vector in V closest to the vector $\vec{b} \in V^{\perp}$ where $\vec{b} = (0, 1, 0, -1)$.

   (d) Is the result in (c) surprising? Why or why not.

4. One of the most significant uses for projections is to solve least-squares problems. Find the best straight-line fit to the measurements:

b	t
4	-2
3	-1
1	0
0	2

More specifically, you are looking for a model of the form: $\alpha + \beta t = b$. Which means you are seeking to project $\vec{b} = (4, 3, 1, 0)$ to the column space of:

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Section 3.4: Orthogonal Bases and Gram-Schmidt

5. Suppose Q_1 and Q_2 are orthogonal matrices. That is they are square matrices with orthonormal columns which we learned means:

$$Q_1^T Q_1 = I = Q_1 Q_1^T \text{ and } Q_2^T Q_2 = I = Q_2 Q_2^T.$$

   Show that $Q = Q_1 Q_2$ is also an orthogonal matrix.

6. Find a the values of $\vec{x} = (x_1, x_2, x_3)$, the third column so that the matrix Q is orthogonal:

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x_1 \\ 1/\sqrt{3} & 2/\sqrt{14} & x_2 \\ 1/\sqrt{3} & -3/\sqrt{14} & x_3 \end{bmatrix}.$$

7. Consider the following two vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

   (a) Explain why (or show) that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

   (b) Use the Gram-Schmidt Process to determine an equivalent set of orthonormal vectors $\vec{q}_1$, $\vec{q}_2$.

   (c) Use your solution to (b) to determine the QR decomposition of the matrix below:

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}.$$