Name:	Section:

75 Minutes. Answer all questions, without the use of notes, books or calculators. Partial credit will be awarded for correct work, unless otherwise specified. When you are asked to explain yourself, please write clearly and use complete sentences. Solve each problem directly onto the exam paper and write your name at the top of each page. Good luck!

1. (2 Points Each) True or False. For each of the following, state whether the statement is True or False (1 Point) and provide a short (1 - 2 sentence) justification (1 Point) . Remember a true statement must ALWAYS be true).

   (a) (2 Points). If Q is a square matrix with orthonormal columns and $\vec{x}$ any vector, then we know that: $\|\vec{x}\|^2 = \|Q\vec{x}\|^2$. (Recall: $\|\vec{x}\|^2 = \vec{x}^T\vec{x}$.)

   Solution: TRUE. The easiest way to see this is to remember that for a matrix with orthonormal columns we have $Q^TQ = I$. Then we have:

   $$\|Q\vec{x}\|^2 = (Q\vec{x})^T(Q\vec{x}) = \vec{x}^TQ^TQ\vec{x} = \vec{x}^TI\vec{x} = \vec{x}^T\vec{x} = \|\vec{x}\|^2.$$

   Thus $\|Q\vec{x}\|^2 = \|\vec{x}\|^2$ for all choices of $\vec{x}$.

   (b) (2 Points) If A is a 2×2 matrix and $\det(A) = 2$, then $\det(kA) = 2k$.

   Solution: FALSE. There are several ways to show this. Since we have a 2×2 matrix we can show this directly:

   $$\det(kA) = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2ac - k^2bd = k^2(ab - cd) = k^2\det(A) = k^2 2.$$

   (c) (2 Points) A 3×3 matrix A with eigenvalues $1, 1, 2$ is always invertible.

   Solution: TRUE. Again there are several ways to see that this is true. One direct way is to see that:

   $$\det(A) = \lambda_1\lambda_2\lambda_3 = 1(1)(2) \neq 0$$

   which implies that the matrix A is invertible.

   (d) (2 Points) A 3×3 matrix A with eigenvalues $1, 1, 2$ is always diagonalizable.

   Solution: FALSE. We know that a matrix is diagonalizable if and only if we have a full set of linearly independent eigenvectors. Since 1 is an eigenvalue of multiplicity 2, we may not have enough eigenvectors.
   For example, the matrix:

   $$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

   $$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)^2.$$

This implies: $\lambda_1 = 2, \lambda_{2,3} = 1$. However we can see that we do not get two linearly independent eigenvectors when $\lambda = 1$:

$$(A - I)\vec{x} = \vec{0} \implies \left[\begin{array}{ccc|c} 2-1 & 0 & 0 & 0 \\ 0 & 1-1 & 1 & 0 \\ 0 & 0 & 1-1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We have 2 pivots and only 1 free variable. This means that we have $x_1 = x_3 = 0$ and $x_2 = t$, so we have only 1 eigenvector:

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We, of course have one more eigenvector, for $\lambda_1 = 2$:

$$(A - 2I)\vec{x} = \vec{0} \implies \left[\begin{array}{ccc|c} 2-2 & 0 & 0 & 0 \\ 0 & 1-2 & 1 & 0 \\ 0 & 0 & 1-2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right].$$

Again, we have 2 pivots and 1 free variable. Together these give us $x_3 = x_2 = 0$ and $x_1 = t$. Thus, our eigenvector is:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since we do not have 3 eigenvectors we can not diagonalize this matrix.

Name:	Section:

2. (15 Points) In this problem we will consider orthogonal complements to subspaces S of $\mathbb{R}^3$. To receive full credit you need to explain your solutions.

   (a) (5 Points) If S is the zero subspace of $\mathbb{R}^3$ find a basis for $S^\perp$?

   Solution: For any vector space S, we know that $S^\perp = \{\vec{x}$ such that $\vec{x}^T\vec{y} = 0$ for all $\vec{y} \in S\}$. This means we are seeking:

$$\vec{x}^T\vec{0} = 0 \implies 0x_1 + 0x_2 + 0x_3 = 0 \implies \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right].$$

   This system has 0 pivots and 3 free variables and thus we have:

$$S^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

   We note that in fact, this is just the canonical basis for $\mathbb{R}^3$ and the space $S^\perp = \mathbb{R}^3$.

   (b) (10 Points) If $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ find a basis for $S^\perp$

   Solution: We are looking for all vectors $\vec{x}$ which satisfy:

$$\vec{x}^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \implies x_1 + x_2 + x_3 = 0 \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \end{array}\right].$$

   This system has 1 pivot and 2 free variables. Let's define $x_3 = t$ and $x_2 = s$ then we have: $x_1 = -t - s$. As such we have two special solutions, each of which corresponds to a basis vector for $S^\perp$.

$$S^\perp = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Name:	Section:

3. (20 Points) Consider the following matrix

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}.$$

(a) (5 Points) Find the eigenvalues of A.

Solution: We will determine the eigenvalues of A by solving the equation $0 = \det(A - \lambda I)$

$$0 = \det(A - \lambda I) \implies 0 = \begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda) + 10 = -12 + 3\lambda - 4\lambda + \lambda^2 + 10$$

$$\implies 0 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \implies \lambda_1 = 2, \lambda_2 = -1.$$

(b) (10 Points) Find the eigenvectors of A.

Solution: We need to find eigenvectors for both eigenvalues.

- $\lambda_1 = 2$:

  $$(A - 2I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 4-2 & -5 & 0 \\ 2 & -3-2 & 0 \end{array}\right] \implies \left[\begin{array}{cc|c} 2 & -5 & 0 \\ 2 & -5 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 2 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

  We have 1 pivot and 1 free variable. Let $x_2 = t$ and then $x_1 = (5/2)t$. Thus we have eigenvector:

  $$\vec{x}_1 = \begin{bmatrix} 5/2 \\ 1 \end{bmatrix}.$$

- $\lambda_2 = -1$:

  $$(A + I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 4+1 & -5 & 0 \\ 2 & -3+1 & 0 \end{array}\right] \implies \left[\begin{array}{cc|c} 5 & -5 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - (2/5)R_1} \left[\begin{array}{cc|c} 5 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

  Again, we have 1 pivot and 1 free variable. Let $x_2 = t$ and $x_1 = t$. Thus we have eigenvector:

  $$\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(c) (5 Points) Compute A^{105}. (Note: You can leave this in expanded form.)

Solution: We will use the fact that this matrix can be diagonalized. That is, if S is the matrix of eigenvectors and Λ is a diagonal matrix with eigenvalues on the diagonal, then we have:

$$A = S \Lambda S^{-1}.$$

We note that:

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, S = \begin{bmatrix} 5/2 & 1 \\ 1 & 1 \end{bmatrix}.$$

We need to find S^{-1}, this requires us to determine $\det(S)$: $\det(S) = (5/2)(1) - (1)(1) = 5/2 - 2/2 = 3/2$. Then we have:

$$S^{-1} = \frac{2}{3}\begin{bmatrix} 1 & -1 \\ -1 & 5/2 \end{bmatrix}.$$

Then we have:

$$A^{105} = (S\Lambda S^{-1})^{105} = S\Lambda^{105}S^{-1} = \begin{bmatrix} 5/2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^{105} & 0 \\ 0 & -1^{105} \end{bmatrix}\left(\frac{2}{3}\right)\begin{bmatrix} 1 & -1 \\ -1 & 5/2 \end{bmatrix}$$

$$= \left(\frac{2}{3}\right)\begin{bmatrix} 5/2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^{105} & -2^{105} \\ 1 & -5/2 \end{bmatrix} = \left(\frac{2}{3}\right)\begin{bmatrix} (5)2^{104} + 1 & -(5)2^{104} - 5/2 \\ 2^{105} + 1 & -2^{105} - 5/2 \end{bmatrix}.$$

Name:	Section:

4. (12 Points). Consider the following three vectors:

$$\vec{a} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{b} = \begin{bmatrix}0\\1\\1\end{bmatrix}, \text{ and } \vec{c} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

You will use the Gram-Schmidt Process to construct an equivalent set of orthonormal vectors: $\vec{q}_i$. Recall the procedure we put reviewed in the textbook was as follows:

- $\vec{q}_1 = \frac{\vec{a}}{\|\vec{a}\|}$
- $\vec{B} = \vec{b} - (\vec{b}^T\vec{q}_1)\vec{q}_1$ and then $\vec{q}_2 = \frac{\vec{B}}{\|\vec{B}\|}$
- $\vec{C} = \vec{c} - (\vec{c}^T\vec{q}_1)\vec{q}_1 - (\vec{c}^T\vec{q}_2)\vec{q}_2$ and then $\vec{q}_3 = \frac{\vec{C}}{\|\vec{C}\|}$

Notice that in this case $\vec{q}_1 = \vec{a}$ because $\vec{a}$ already has unit length: $\|\vec{a}\| = \sqrt{1^2+0^2+0^2} = \sqrt{1} = 1$. Then we have:

$$\vec{q}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} = \vec{a}$$

(a) (5 points) Calculate $\vec{q}_2$:

Solution: Now we have to calculate $\vec{q}_2$ in this two step system:

$$\vec{B} = \vec{b} - (\vec{b}^T\vec{q}_1)\vec{q}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} - ((0)(1)+(1)(0)+(1)(0))\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

(Notice, that $\vec{B} = \vec{b}$ because $\vec{a}$ and $\vec{b}$ are orthogonal!) But then this means we have:

$$\vec{q}_2 = \frac{\vec{B}}{\|\vec{B}\|} = \frac{\vec{B}}{\sqrt{0^2+1^2+1^2}} = \begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}.$$

(b) (5 Points) Calculate $\vec{q}_3$:

Solution: We now follow the last step:

$$\vec{C} = \vec{c} - (\vec{c}^T\vec{q}_1)\vec{q}_1 - (\vec{c}^T\vec{q}_1)\vec{q}_2 = \vec{c} - (1(1)+1(0)+1(0))\,\vec{q}_1 - \left(1(0)+1/\sqrt{2}+1/\sqrt{2}\right)\vec{q}_2$$

$$= \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{2}{\sqrt{2}}\begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Thus, we end up with $\vec{C} = \vec{0}$. Thus we can not calculate a third orthonormal vector.

(c) (2 Points). Did you find 3 non-zero orthonormal vectors? If yes, use them to create a square orthogonal matrix Q. If no, explain why the algorithm failed.

Solution: The algorithm failed to produce 3 non-zero orthonormal vectors because $\vec{c}$ was in the span of $\vec{a}$ and $\vec{b}$. In fact we can see that $\vec{c} = \vec{a} + \vec{b}$.
This is why the Gram Schmidt process failed to produce a non-zero $\vec{q}_3$.

Name:	Section:

5. (10 Points). Let A be an $n \times n$ matrix and $\vec{x}$ be an eigenvector with eigenvalue λ.

   (a) (5 Points). Show $\vec{x}$ is an eigenvector of the matrix $B = A^2$ and find its eigenvalue.

   Solution:
   We know that $\vec{x}$ is an eigenvector of A which means there exists some real number λ such that: $A\vec{x} = \lambda x$. Let's multiply both sides of the equation by A:

$$\begin{aligned} A\vec{x} &= \lambda x \\ A\left(A\vec{x}\right) &= A\left(\lambda\vec{x}\right) \\ A^2\vec{x} &= \lambda(A\vec{x}) \\ A^2\vec{x} &= \lambda\left(\lambda\vec{x}\right) \\ A^2\vec{x} &= \lambda^2\vec{x}. \end{aligned}$$

   Thus we see that $\vec{x}$ is an eigenvector of A^2 with eigenvalue λ^2.

   (b) (5 Points). Suppose that $A^2 = A$, Find all possible values the eigenvalue λ can be.

   Solution:
   From the first part, we know that:

$$A\vec{x} = \lambda\vec{x} \text{ and } A^2\vec{x} = \lambda^2\vec{x}.$$

   However, if $A^2 = A$ then this implies:

$$\lambda^2\vec{x} = A^2\vec{x} = A\vec{x} = \lambda\vec{x} \implies \lambda^2\vec{x} = \lambda\vec{x}.$$

   To isolate λ we can multiply both sides of this equation by $\vec{x}^T$ and obtain:

$$\lambda^2\vec{x}^T\vec{x} = \lambda\vec{x}^T\vec{x} \implies \lambda^2\|\vec{x}\|^2 = \lambda\|\vec{x}\|^2.$$

   Since $\vec{x} \neq \vec{0}$ we know that $\|\vec{x}\|^2 \neq 0$ and thus, $\lambda^2 = \lambda$.
   The only real numbers for which this is true are $\lambda = 0$ or 1.

Name:	Section:

6. **(Extra Credit: 5 Points)** For an $n \times m$ matrix A, prove that every $\vec{y} \in N\left(A^T\right)$ is orthogonal to every vector $\vec{b}$ in the column space of A.

Solution: Suppose that $\vec{y} \in N\left(A^T\right)$, then we know that:

$$A^T \vec{y} = \vec{0} \implies \vec{y}^T A = \vec{0}^T.$$

If $\vec{b} \in C(A)$ this means that there exists some $\vec{x}$ such that:

$$A\vec{x} = \vec{b}.$$

Let's take the inner product of $\vec{y}$ with $\vec{b}$ and apply these properties:

$$\vec{y}^T \vec{b} = \vec{y}^T \left(A\vec{x}\right) = \left(\vec{y}^T A\right) \vec{x} = \vec{0}^T \vec{x} = 0.$$

Thus, for every $\vec{y} \in N\left(A^T\right)$ and $\vec{b} \in C(A)$ we have: $\vec{y}^T \vec{b} = 0$.

Grading Rubric

- Problem 1 (a - d): 2 Points Each, 8 Points Total:
  - 2 Points: Correct answer (True/False) and valid clear reasoning.
  - 1.5 Points: Correct answer (True/False) and incomplete reasoning.
  - 1 Point: Either only correct answer (True/False) alone OR correct reasoning.
  - 0 Points: Blank, Incorrect answer (True/False) AND incorrect reasoning.
- Problem 2 (15 Points)
  - Part (a), 5 Points: +2 for mentioning orthogonal subspaces. +1 for $S^\perp = \vec{0}$.
  - Part (b), 10 Points: +1 for any random basis vector that is not orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$. +4 for any incorrect basis vector(s) that are orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$. +6 points for one of the correct basis vectors.
- Problem 3:
  - (a) 5 Points: +5 if computed correctly, +3 if RREF and then computed eigenvalues
  - (b) 10 points: +5 on each from computing $N(A - \lambda I)$, if error in previous part and done correctly, full or mostly full points awarded
  - (c) 5 points: +3 for diagonalization formula, +2 for computation
- Problem 4:
  - (a) -1 for a small algebraic error.
  - (b) -1 for a small algebraic error.
  - (c) +2 based on results from (a) and (b), even if (a) or (b) is or are incorrect. If the matrix Q does not contain exact vectors from (a) and (b), -1 point. If the reason is a bit off, -1 point. If the reason is irrelevant, -2 points.
- Problem 5:
  - (a) 5 Points: +3 for showing eigenvector, +2 for eigenvalue. +2 for anything reasonable. -1 for assuming diagonalizible.
  - (b) 5 Points: +3 for setting up eigenvalue equation, +2 for solving. +2 for anything reasonable. -1 for assuming diagonalizible.
- Problem 6 (Extra Credit):
  - 1 Point: For citing the fundamental theorem of orthogonality, general property of fundamental subspaces or for correctly setting up/defining what $\vec{b}$ and $\vec{y}$ in terms of the matrix.
  - 2.5 Points: Beginning of correct, but incorrect work.
  - 5 Points: Correct (or mostly correct) linear algebra argument.