Homework Assignment #7

Remember, this Homework Assignment is **not collected or graded**! But it is in your best interest to do it as the Homework Quiz will be based on it and it is the best way to ensure you know the material.

Section 3.3: Projections and Least Squares

1. Consider the following vector matrix system:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$$

(a) Show the system $A\vec{x} = \vec{b}$ has no solution.

Solution: Let's go to the augmented matrix $\left[A \mid \vec{b}\right]$ and see what we find.

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 1 & 0 & 5 \\ 1 & 1 & 9 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 1 & 1 \\ 0 & 2 & 5 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{array}\right].$$

We can see the last row implies $0x_1 + 0x_2 = 3$ which is impossible. Thus our $A\vec{x} = \vec{b}$ has no solution.

(b) Use Calculus to find $\hat{\vec{x}} = \begin{bmatrix} \hat{x_1} \\ \hat{x_2} \end{bmatrix}$ that minimizes the error between $A\hat{\vec{x}}$ and $\vec{b}$:

$$E(\hat{x_1}, \hat{x_2}) = \|A\hat{\vec{x}} - \vec{b}\|^2 = (\hat{x_1} - \hat{x_2} - 4)^2 + (\hat{x_1} - 5)^2 + (\hat{x_1} + \hat{x_2} - 9)^2.$$

Solution: Here we will need to take partial derivatives:

$$\frac{d}{d\hat{x_1}} E(\hat{x_1}, \hat{x_2}) = 2\left((\hat{x_1} - \hat{x_2} - 4) + (\hat{x_1} - 5)^2 + (\hat{x_1} + \hat{x_2} - 9)\right) = 6(\hat{x_1} - 6)$$

$$\frac{d}{d\hat{x_2}} E(\hat{x_1}, \hat{x_2}) = 2\left((\hat{x_1} - \hat{x_2} - 4)(-1) + (\hat{x_1} + \hat{x_2} - 9)\right) = 4\hat{x_2} - 10.$$

We note that in order for both equations to be 0 we require: $\hat{x_1} = 6$ and $\hat{x_2} = 5/2$. We need to check that this is truly a minima and so we do our second derivative test.

- $\frac{d^2}{d\hat{x_1}^2} E(\hat{x_1}, \hat{x_2}) = 6.$
- $\frac{d^2}{d\hat{x_2}^2} E(\hat{x_1}, \hat{x_2}) = 4.$
- $\frac{d^2}{d\hat{x_1}\hat{x_2}} E(\hat{x_1}, \hat{x_2}) = 0.$

This tells us that $D(6, 5/2) = 6(4) - 0^2 > 0$. Since $\frac{d^2}{d\hat{x_1}^2} E(\hat{x_1}, \hat{x_2}) > 0$ we know our function has a minima.

(c) Solve the normal equations:

$$A^T A\hat{\vec{x}} = A^T \vec{b}$$

and compare your solution from part (b) and (c).

Solution: We now will use the normal equations to try and solve our system:

$$A^T A\hat{\vec{x}} = A^T\vec{b} \implies \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 5 \end{bmatrix}.$$

This system works out really nicely as the normal equations are equivalent to:

- $3x_1 = 18$ which implies $x_1 = 6$.
- $2x_2 = 5$ which implies $x_2 = 5/2$.

As expected, this system produces the same solution that we found with calculus. This demonstrates that looking at the orthogonal projection of $\vec{b}$ onto the column space of A will indeed produce the vector $\vec{x}$ which minimizes the distance from $A\vec{x}$ to $\vec{b}$.

2. Consider the following matrix A and vector $\vec{b}$:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

(a) Find the projection, $\vec{p}$ of $\vec{b}$ into the column space of A.

Solution: In order to project $\vec{b}$ onto the column space of A we will need to solve the normal equations:

$$A^T A\vec{x} = A^T\vec{b} \implies \vec{x} = (A^T A)^{-1} A^T\vec{b} \implies A\vec{x} = A(A^T A)^{-1} A^T\vec{b}.$$

Alternatively, we could determine seek $\vec{p} = P\vec{b}$ where the projection matrix P is defined as follows; $P = A(A^T A)^{-1} A^T$.
Just for fun, we will do this BOTH ways! First, let's solve the normal equations:

$$A^T A\vec{x} = A^T\vec{b}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}.$$

We will move to the augmented matrix:

$$\left[\begin{array}{cc|c} 6 & -8 & -11 \\ -8 & 18 & 27 \end{array}\right] \xrightarrow{R_2 \to R_2 + (8/6)R_1} \left[\begin{array}{cc|c} 6 & -8 & -11 \\ 0 & 22/3 & 37/3 \end{array}\right].$$

The last row gives us $x_2 = 37/22$ and the top row gives us:

$$6x_1 - 8x_2 = -11 \implies 6x_1 - 8(37/22) = -11 \implies x_1 = 9/22.$$

In which case the projection $\vec{p}$ of $\vec{b}$ is simply:

$$\vec{p} = A\begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}.$$

Now, let's find the projection matrix P. In this case we first want to invert $A^T A$

$$(A^T A)^{-1} = \frac{1}{44}\begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}.$$

In which case:

$$P = A(A^T A)^{-1}A^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}\left(\frac{1}{44}\begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}\right)\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} = (1/44)\begin{bmatrix} 40 & 12 & 4 \\ 12 & 8 & -12 \\ 4 & -12 & 40 \end{bmatrix}.$$

This implies:

$$P = \begin{bmatrix} \frac{10}{11} & \frac{3}{11} & \frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} & -\frac{3}{11} \\ \frac{1}{11} & -\frac{3}{11} & \frac{10}{11} \end{bmatrix}.$$

We note that

$$\vec{p} = P\vec{b} = \begin{bmatrix} \frac{10}{11} & \frac{3}{11} & \frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} & -\frac{3}{11} \\ \frac{1}{11} & -\frac{3}{11} & \frac{10}{11} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}.$$

(b) Let $\vec{e} = \vec{p} - \vec{b}$ be the *error* between $\vec{b}$ and $\vec{p}$ which is in the column space of A. Which of the four fundamental subspaces does $\vec{e}$ belong to?

Solution: We know that $\vec{p}$ is, by construction, in $C(A)$ and that $\vec{e}$ is in $N(A^T)$. So let's find $\vec{e}$ and then verify it is in $N(A^T)$.

$$\vec{e} = \vec{p} - \vec{b} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 12/11 \\ -36/11 \\ -12/11 \end{bmatrix}.$$

If $\vec{e}$ is in the left nullspace, we can either show: $\vec{e}^T A = \vec{0}$ or $A^T\vec{e} = \vec{0}$. Let's do the former:

$$\begin{bmatrix} 12/11 & -36/11 & -12/11 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 12/11 - 36/11 + 24/11 \\ 12/11 + 36/11 - 48/11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

3. If V is the subspace spanned by $(1,1,0,1)$ and $(0,0,1,0)$ find:

(a) A basis for the orthogonal complement $V^{\perp}$.

Solution: The orthogonal complement of V consists of all the vectors that are orthogonal to every vector in V. Because we are given a basis for V (note the two vectors given are linearly independent) we can just look for all vectors $\vec{x} \in \mathbb{R}^4$ that are orthogonal to both basis vectors.

That is:

$$\vec{x}^T \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0 \implies x_1 + x_2 + x_4 = 0$$

and

$$\vec{v}^T \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0 \implies x_3 = 0.$$

We can combine these into the following matrix and vector system:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix is already in REF so we can see that there are 2 pivots and 2 free variables. The second row requires $x_3 = 0$. The first row designates our free variables Let $x_4 = s$ and $x_2 = t$ then we have:

$$x_1 + x_2 + x_4 = 0 \implies x_1 + t + s = 0 \implies x_1 = -t - s.$$

This gives us:

$$V^\perp = \left\{ \begin{bmatrix} -t-s \\ t \\ 0 \\ s \end{bmatrix}, \text{ where } t, s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

These two vectors are the basis. (Notice: The rowspace of this matrix is equal to V:, $C(A^T) = V$. Notice that the nullspace of this matrix is equal to $V^\perp$.)

(b) The projection matrix P onto V. (Recall, the projection matrix takes any $\vec{b}$ and projects it to the column space of V. Explicitly, we know that if the columns of a matrix A are linearly independent: $P = A(A^TA)^{-1}A^T$. See Section 3.3 and Lecture Notes from Week 8 for more details.)

Solution: Let's define A to be the matrix whose columns are the basis for V. Then $C(A) = V$ and formula $P = A(A^TA)^{-1}A^T$ will project any vector $\vec{b} \in \mathbb{R}^4$ to the column space of A.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \implies (A^TA) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that:

$$(A^TA)^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we have:

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 1/3 & 0 \\ 0 & 1 \\ 1/3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\implies P = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix}$$

(c) The vector in V closest to the vector $\vec{b} \in V^{\perp}$ where $\vec{b} = (0, 1, 0, -1)$.

Solution: By definition we know that the vector in V that is closest to $\vec{b} = (0, 1, 0, -1)$ is found by $P\vec{b}$.
This gives us:

$$P\vec{b} = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(d) Is the result in (c) surprising? Why or why not.

Solution: The result we obtained, that $\vec{0}$ is the vector in V closest to $\vec{b}$ is not surprising.
First, we note that $\vec{b}$ definitively belongs to $V^{\perp}$ because we can write $\vec{b}$ as a linear combination of the basis vectors we found in (a):

$$\vec{b} = (1) \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then, we know that $\vec{b}$ by definition is orthogonal to every vector in V. This means that the orthogonal projection of $\vec{b}$ must be the 0-vector. This is equivalent to projecting the vector $[0, 1]$ onto the x-axis in $\mathbb{R}^2$.

4. One of the most significant uses for projections is to solve least-squares problems. Find the best straight-line fit to the measurements:

b	t
4	-2
3	-1
1	0
0	2

More specifically, you are looking for a model of the form: $\alpha + \beta t = b$. Which means you are seeking to project $\vec{b} = (4, 3, 1, 0)$ to the column space of:

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Solution: We are looking for values α, β that satisfy the following:

$$A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \vec{b} \implies \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

Let's first demonstrate the system has no solution by performing row reduction on the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 1 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - R_1, R_3 \to R_3 - R_1} \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 0 & 2 & -3 \\ 0 & 4 & -4 \end{array}\right] \xrightarrow[R_4 \to R_4 - 4R_2]{R_3 \to R_3 - 2R_2} \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right].$$

Because row 3 implies $0\alpha + 0\beta = -1$ the system has no solution.

Thus, we need to consider least-square solutions by solving the normal equations:

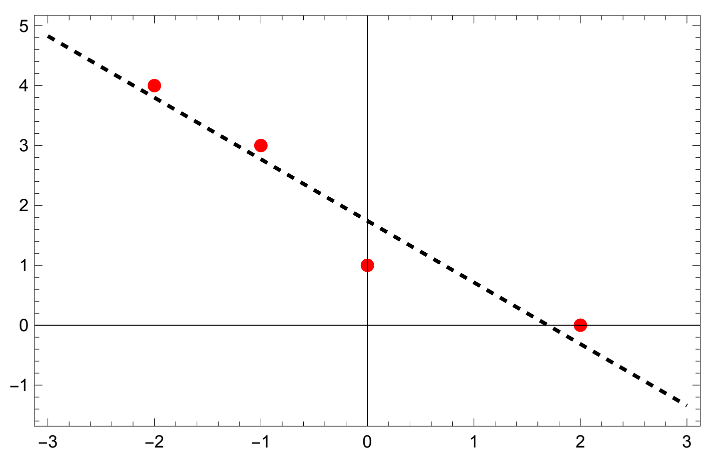
$$A^T A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = A^T \vec{b} \implies \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\implies \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\implies \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 61 \\ -36 \end{bmatrix}.$$

We can verify that this is indeed a reasonable linear approximation by plotting the points (red dots) against the linear equation (dashed line).



Section 3.4: Orthogonal Bases and Gram-Schmidt

5. Suppose Q_1 and Q_2 are orthogonal matrices. That is they are square matrices with orthonormal columns which we learned means:

$$Q_1^T Q_1 = I = Q_1 Q_1^T \text{ and } Q_2^T Q_2 = I = Q_2 Q_2^T.$$

Show that $Q = Q_1 Q_2$ is also an orthogonal matrix.

Solution: As we learned in class, matrices with orthonormal columns have really cool properties. Here, we will prove one more.

Suppose Q_1 and Q_2 are orthogonal matrices (i.e., square matrices with orthonormal columns). We want to show that the matrix $Q = Q_1 Q_2$ is also an orthonormal matrix.

That is, we want to show that $Q^T Q = QQ^T = I$.

$$\begin{aligned}
Q^T Q &= (Q_1Q_2)^T(Q_1Q_2) && \text{by definition of } Q \\
&= (Q_2^T Q_1^T)(Q_1Q_2) && \text{by definition of transpose} \\
&= Q_2^T(Q_1^T Q_1)Q_2 && \text{by definition of matrix multiplication} \\
&= Q_2^T I Q_2 && \text{by definition orthogonal matrix} \\
&= I && \text{by definition of orthogonal matrix.}
\end{aligned}$$

Similarly, $QQ^T = I$ and thus we have shown that the product of two orthogonal matrices is again an orthogonal matrix.

6. Find a the values of $\vec{x} = (x_1, x_2, x_3)$, the third column so that the matrix Q is orthogonal:

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x_1 \\ 1/\sqrt{3} & 2/\sqrt{14} & x_2 \\ 1/\sqrt{3} & -3/\sqrt{14} & x_3 \end{bmatrix}.$$

Solution: Let's simply carry out the multiplication and see what we find:

$$Q^T Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{14} & 2/\sqrt{14} & -3/\sqrt{14} \\ x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x_1 \\ 1/\sqrt{3} & 2/\sqrt{14} & x_2 \\ 1/\sqrt{3} & -3/\sqrt{14} & x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{x_1+x_2+x_3}{\sqrt{3}} \\ 0 & 1 & \frac{x_1+2x_2-3x_3}{\sqrt{14}} \\ \frac{x_1+x_2+x_3}{\sqrt{3}} & \frac{x_1+2x_2-3x_3}{\sqrt{14}} & x_1^2+x_2^2+x_3^2 \end{bmatrix}.$$

Because we want this matrix to be equal to the identity, we need to require:

$$\frac{x_1+x_2+x_3}{\sqrt{3}} = 0 \implies x_1 + x_2 + x_3 = 0$$

$$\frac{x_1+2x_2-3x_3}{\sqrt{14}} = 0 \implies x_1 + 2x_2 - 3x_3 = 0,$$

and $x_1^2 + x_2^2 + x_3^2 = 1$.

The first two conditions correspond to a system of linear equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right].$$

This system has 2 pivots and 1 free variable leaving: $x_3 = t$, $x_2 = 4t$ and

$$x_1 + x_2 + x_3 = 0 \implies x_1 + 4t + t = 0 \implies x_1 = -5t.$$

As such our vector of possible solutions is simply:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}.$$

We can now impose the last condition, which will guarantee that we have a unit vector:

$$x_1^2 + x_2^2 + x_3^2 = 1 \implies t^2 25 + t^2 16 + t^2 = 1 \implies t^2(42) = 1 \implies t = \pm 1/\sqrt{42}.$$

Thus we have 2 possible solutions to the last column of Q:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \pm(1/\sqrt{42}) \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}.$$

7. Consider the following two vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

(a) Explain why (or show) that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

Solution: We know that vectors $\vec{v}_1$ and $\vec{v}_2$ are linearly independent because they are not multiples of one another.
In addition, we could also seek to find α_1 and α_2 such that:

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = \vec{0} \implies \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Because the matrix has 2 pivots the only solution to this system is $\alpha_1 = \alpha_2 = 0$.
Thus $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

(b) Use the Gram-Schmidt Process to determine an equivalent set of orthonormal vectors $\vec{q}_1$, $\vec{q}_2$.

Solution: The first vector $\vec{q}_1$ is simply a unit vector in the direction of $\vec{v}_1$.

$$\vec{q}_1 = \vec{v}_1 / \|\vec{v}_1\| = \frac{1}{\sqrt{2}} \vec{v}_1 = (1/\sqrt{2}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To determine the second vector $\vec{q}_2$ we first determine an intermediate vector $\vec{b}$ by subtracting the orthogonal projection of $\vec{v}_2$ in the direction of $\vec{q}_1$ from $\vec{v}_2$.

$$\begin{aligned}
\vec{b} &= \vec{v}_2 - \vec{q}_1^T \vec{v}_2 \vec{q}_1 \\
&= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - (4/\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -2 \end{bmatrix}.
\end{aligned}$$

Then we have:

$$\vec{q}_2 = \vec{b}/\|\vec{b}\| = 1/\sqrt{8} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(c) Use your solution to (b) to determine the QR decomposition of the matrix below:

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}.$$

Solution: We know the QR decomposition of the matrix A is defined as follows:

$$A = QR \implies \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \begin{bmatrix} \vec{q}_1^T \vec{v}_1 & \vec{q}_1^T \vec{v}_2 \\ \vec{q}_2^T \vec{v}_1 & \vec{q}_2^T \vec{v}_2 \end{bmatrix}.$$

We have:

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1\sqrt{2} \end{bmatrix}$$

$$R = \begin{bmatrix} \vec{q}_1^T \vec{v}_1 & \vec{q}_1^T \vec{v}_2 \\ \vec{q}_2^T \vec{v}_1 & \vec{q}_2^T \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1/\sqrt{2} & 1\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1/\sqrt{2} & -1\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1/\sqrt{2} & -1\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 4/\sqrt{2} \\ 0 & 4/\sqrt{2} \end{bmatrix}.$$