WH #10

1) a) Let $P(X=0, Y=-1) = P(X=1, Y-1) = P(X=2, Y=-1) = a$

$\sum_x \sum_y P(X=x, Y=y) = 1$

$3a = 1 - .5$

$a = 1/6$

	$x=0$	$x=1$	$x=2$	$P(Y=y)$
$y=-1$	$1/6$	$1/6$	$1/6$	$1/2$
$y=1$	0	$1/2$	0	$1/2$
$P(X=x)$	$1/6$	$4/6$	$1/6$	1

b) 2 discrete random variables X and Y are independent if $P(X=x, Y=y) = P(X=x)\cdot P(Y=y)$ for all x and y.

$\sum_y P(X=0, Y=y) = P(X=0, Y=-1) = 1/6$

$P(X=0, Y=1) = 0$

$1/6 = P(X=0)$

$\sum_x P(X=x, Y=-1) = P(X=0, Y=-1) = 1/6$

$P(X=1, Y=-1) = 1/6$

$P(X=2, Y=-1) = 1/6$

$1/2 = P(Y=-1)$

$\boxed{1/6 = P(X=0, Y=-1) \neq P(X=0)\cdot P(Y=-1) = 1/12}$

$\therefore$ X and Y are not independent

c) $E(XY) = \sum_x \sum_y xy P(X=x, Y=y)$

$E(X) = \sum_x x \cdot P(X=x) = 0\times(1/6) + 1\times(4/6) + 2\cdot(1/6)$
$= 1$

$E(Y) = \sum_y y \cdot P(Y=y) = -1\times(1/2) + 1(1/2) = 0$

$E(X,Y) = \sum_x \sum_y xy P(X=x, Y=y) = \cancel{0(-1)\times(1/6)} + 1\times(-1)\times\frac{1}{6}$
$+ 2\times(-1)\times\frac{1}{6} + \cancel{0\times1\times0}$
$+ 1\times(1)\times\frac{1}{2} + \cancel{2\times1\times0}$
$= -1/6 - \frac{2}{6} + 1/2 = 0$

$\therefore Cov(X,Y)$
$= E(XY) - E(X)E(Y)$
$0 - 1\cdot 0 = 0$

$\therefore \rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}} = \boxed{0 = \text{The correlation of X and Y.}}$

d) $\rho_{XY} = 0$ = correlation
X and Y are not independent

∴ zero correlation does not guarantee independence

2) a) $E[XY] = \int_x \int_y xy f(x,y) = \int_x \int_y xy f(x) \cdot f(y)$

$= \int_x x f(x) \cdot \int_y f(y) = \boxed{E(X)\cdot E(Y) = E(XY)}$

b) $cov(X,Y) = E(XY) - E(X)E(Y)$
if X and Y are independent $E(X,Y) = 0$
$E(X,Y) = E(X)E(Y)$
$cov(X,Y) = E(X)E(Y) - E(X)E(Y) = 0$

c) correlation $= \rho = \frac{cov(x,y)}{\sqrt{var(x)\,var(y)}} = 0$

independence implies 0 correlation

3) $\iint f(x)\,dy\,dx = 1$

a) $\int_{-2}^{2}\int_{-1}^{1} k(4x^2+y^2)\,dy\,dx = 1$

$k\int_{-2}^{2}\left[4x^2y + y^3/3\right]_{-1}^{1} dx = 1$

$k\int_{-2}^{2} (4x^2 + 1/3 - (-4x^2 - 1/3))\,dx = 1$

$k\int_{-2}^{2} 8x^2 + 2/3\,dx = 1$

$k\left[\frac{8x^3}{3} + 2/3\,x\right]_{-2}^{2} = 1$

$k\left[\frac{8(2^3)}{3} + \frac{2}{3}(2) - \left(\frac{8(-2)^3}{3} + \frac{2}{3}(-2)\right)\right] = 1$

$k\cdot\left[8/3(16+1)\right] = 1$

$\frac{k\,136}{3} = 1$

$k = 3/136$ $f(x) = \frac{3}{136}(4x^2+y^2)$; $-2 \le x \le 2$, $-1 \le y \le 1$ next page

Marginal distribution of X

$g(x) = \int f(x,y)\,dy$

$g(x) = \int_{-1}^{1} 3/136\,(4x^2 + y^2)\,dy$

$= 3/136\,[4x^2 y + y^3/3]_{-1}^{1}$

$= 3/136\,(4x^2(2) + 2/3)$

$= 3/68\,(4x^2 + 1/3)$

$E(x) = \int_{-2}^{2} x\,g(x)\,dx$

$E(x) = 3/68 \int_{-2}^{2} (4x^3 + \frac{x}{3})\,dx$

$E(x) = 3/68\,[x^4 + \frac{x^2}{6}]_{-2}^{2}$

* $E(x) = 3/68\,[16 + 4/6 - (16 + 4/6)] = 0$

$E(x^2) = \int_{-2}^{2} x^2 g(x)\,dx$

$E(x^2) = 3/68 \int_{-2}^{2} (4x^4 + \frac{x^2}{3})\,dx$

$E(x^2) = 3/68\,[4x^5/5 + \frac{x^3}{9}]_{-2}^{2}$

$E(x^2) = 3/68\,[\frac{4(2^5)}{5} + \frac{2^3}{9} - (\frac{4(-2^5)}{5} + \frac{(-2)^3}{9})]$

$E(x^2) = 3/68\,[\frac{128}{5} + \frac{8}{9} - -\frac{128}{5} - \frac{8}{9}]$

$E(x^2) = 3/68\,[256/5 + 16/9]$

$3/68\,[2304/45 + 80/45] = 3/68\,[\frac{2384}{45}] = \frac{7152}{3060}$

$Var(x) = \frac{7152}{3060} - \frac{596}{255} = 2.3373$

b) marginal distribution of Y

$h(y) = \int_{-2}^{2} f(x,y)\,dx$

$= \int_{-2}^{2} 3/136\,(4x^2 + y^2)\,dx$

$= 3/136\,[4x^3/3 + y^2 x]_{-2}^{2}$

$= 3/136\,[4/3 \cdot 16 + y^2(2) - (-4/3 \cdot 16 + y^2(-2))]$

$= 3/136\,[4/3 \cdot 16 + 4y^2]$

$= 3/34\,(y^2 + 16/3); \; -1 \le y \le 1$

$E(y) = \int_{-1}^{1} y\,h(y)\,dy$

$E(y) = \int_{-1}^{1} 3/34\,(y^3 + 16/3\,y)\,dy$

$E(y) = 3/34\,[y^4/4 + 8/3\,y^2]_{-1}^{1} = 0$ next page

$E(y^2) = \int_{-1}^{1} y^2 h(y)\,dy$

$E(y^2) = \int_{-1}^{1} 3/34\,(y^4 + 16/3\,y^2)\,dy$

$E(y^2) = \frac{3}{34}\left[\frac{y^5}{5} + \frac{16y^3}{9}\right]_{-1}^{1}$

$E(y^2) = \frac{3}{34}\,(2/5 + \frac{16}{9} \cdot 2)$

$E(y^2) = 89/255 \approx .3490 =$

$\boxed{Var(y) = E(y^2) - E^2(y) = 89/255}$

c) $E(x,y) = \int_{-2}^{2}\int_{-1}^{1} xy\,f(x,y)\,dy\,dx$

$E(x,y) = \int_{-2}^{2}\int_{-1}^{1} 3/136\,(4x^3 y + xy^3)\,dy\,dx$

$= \int_{-2}^{2} 3/136\,[2x^3 y^2 + \frac{xy^4}{3}]_{-1}^{1}\,dx$

$E(x,y) = 3/136 \int_{-2}^{2} 0\,dx = 0$

714.06 $Cov(x,y) = E(x,y) - E(x)E(y) = 0 = cov(x,y)$

The random variables X and Y are uncorrelated

4) $f(x,y) = \frac{k \ln x}{\sqrt{y}}$ $\quad 50 < x < 200$, $\quad 20 < y < 100$

$\int_{20}^{100}\int_{50}^{200} f(x,y)\,dx\,dy = 1$

$\int_{20}^{100} \frac{k}{\sqrt{y}} \int_{50}^{200} \ln x\,dx\,dy = 1$

$\int_{20}^{100} \frac{k}{\sqrt{y}} [x \ln x - x]_{50}^{200}\,dy = 1$

$\int_{20}^{100} k/\sqrt{y}\,[200 \ln 200 - 200 - 50 \ln 50 + 50]\,dy = 1$

$k[200 \ln 4 + 150 \ln 50 - 150] \times 2[\sqrt{y}]_{20}^{100} = 1$

$k \times c_1 \times c_2 = 1$

$k = \frac{1}{c_1 \times c_2}$ $\quad c_1 = 200 \ln 4 + 150 \ln 50 - 150$, $\quad c_2 = 2(10 - \sqrt{20})$

next page

marginal of x

$f(x) = \int_{20}^{100} f(x,y)\,dy = \frac{\ln x}{c_1 c_2} \int_{20}^{100} \frac{1}{\sqrt{y}}\,dy$

$= \frac{\ln x}{c_1 c_2} 2[\sqrt{y}]_{20}^{100}$

$= \frac{\ln x}{c_1 \times c_2} \times 2[10 - \sqrt{20}]$

$f(x) = \frac{1}{c_1} \ln x \quad 50 < x < 200$

marginal of y

$f(y) = \int_{50}^{200} f(x,y)\,dx$

$= \frac{1}{c_1 c_2 \sqrt{y}} \int_{50}^{200} \ln x\,dx$

$= \frac{1}{c_1 c_2 \sqrt{y}} [x \ln x - x]_{50}^{200}$

$\frac{1}{c_1 c_2 \sqrt{y}} [200 \ln 4 + 150 \ln 50 - 150]$

$f(y) = \frac{1}{c_2 \sqrt{y}} \quad 20 < y < 100$

$f(x,y) = \frac{k \ln x}{\sqrt{y}} = \frac{1}{c_1 c_2} \frac{\ln x}{\sqrt{y}} = \frac{\ln x}{c_1} \times \frac{1}{c_2 \sqrt{y}}$

$f(x,y) = f(x) f(y)$

x and y are independent random variable

$\mathrm{Cov}(x,y) = 0$

$\boxed{r \approx 2.3912 \times 10^{-5}}$

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5) a) $F_X(x) = P(X \le x) = \int_0^x f_X(x) = \int_0^x \frac{3}{4} x(2-x)$

$\int_0^x \frac{3}{4} x(2-x) = \frac{3}{4} \int_0^x 2x - x^2 = \frac{3}{4}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^x$

$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]$

$\boxed{F_X(x) = \frac{3}{4} x^2 (1 - x/3) \quad 0 \le x \le 2}$
$\quad 1 \quad x > 2$

b) $F_Y(y) = F_X(y^2)$

$= \frac{3}{4}(y^2)^2\left(1 - \frac{y^2}{3}\right)$

$= \frac{3}{4} y^4 - \frac{1}{4} y^6$

$= \boxed{\frac{1}{4} y^4 (1 - y^2) \quad 0 \le y \le \sqrt{2}}$

$= 0 \quad y < 0$

c) $F_Y(y) = \frac{d}{dy} F_Y(y)$

$\frac{d}{dy}\left[\frac{3y^4}{4} - \frac{y^6}{4}\right] = \left[\frac{3}{4} \times 4y^3 - \frac{1}{4} \times 6y^5\right] = \left[3y^3 - \frac{3}{2} y^5\right]$

$= 3y^3\left[1 - \frac{y^2}{2}\right]$

$= F_Y(y) = 3y^3\left[1 - \frac{y^2}{2}\right] = \boxed{\frac{3}{2} y^3 (2 - y^2) \quad 0 \le y \le \sqrt{2}}$

$f(x) = \lambda e^{-\lambda x} \quad , x > 0 = P(x \le x) = \frac{1}{2} e^{-x/2} \quad , x > 0$

$F(x) = 1 - e^{-x/2} \quad , x > 0$

6) a) $y = \frac{1}{2} x \qquad x > 0 \quad , y > 0$

CDF $\quad G(y) = P(Y \le y) = P(\frac{1}{2} x \le y)$

$= P(x \le 2y)$

$= F(2y)$

$= 1 - e^{-2y/2} \quad , y > 0$

$= 1 - e^{-y} \quad , y > 0$

b) $g(y) = \frac{d}{dy} G(y)$

$= \frac{d}{dy}(1 - e^{-y})$

$\boxed{g(x) = e^{-y} \; , y > 0}$

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c) Y has an exponential distribution

$\boxed{Y \sim \mathrm{Exp}(1)}$