Particle kinematics

Instructor

Ingrid M. Padilla Espinosa, PhD

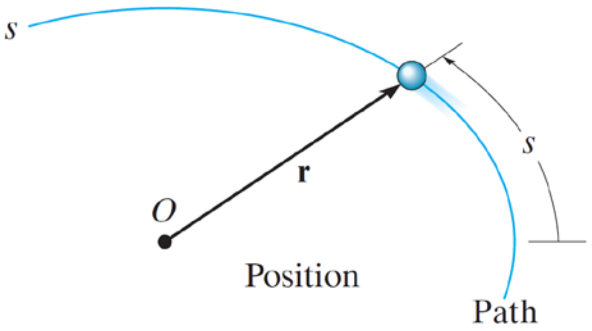
UNIVERSITY OF CALIFORNIA MERCED

Problem Solving. Dynamics is considered to be more involved than statics since both the forces applied to a body and its motion must be taken into account. Also, many applications require using calculus, rather than just algebra and trigonometry. In any case, the most effective way of learning the principles of dynamics is ***to solve problems***. To be successful at this, it is necessary to present the work in a logical and orderly manner as suggested by the following sequence of steps:

1. Read the problem carefully and try to correlate the actual physical situation with the theory you have studied.
2. Draw any necessary diagrams and tabulate the problem data.
3. Establish a coordinate system and apply the relevant principles, generally in mathematical form.
4. Solve the necessary equations algebraically as far as practical; then, use a consistent set of units and complete the solution numerically.
Report the answer with no more significant figures than the accuracy of the given data.
5. Study the answer using technical judgment and common sense to determine whether or not it seems reasonable.
6. Once the solution has been completed, review the problem. Try to think of other ways of obtaining the same solution.

In applying this general procedure, do the work as neatly as possible. Being neat generally stimulates clear and orderly thinking, and vice versa.

Review: Definitions

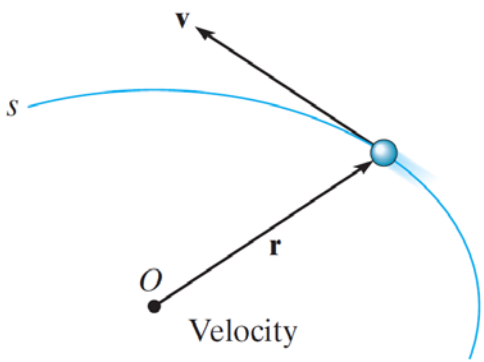


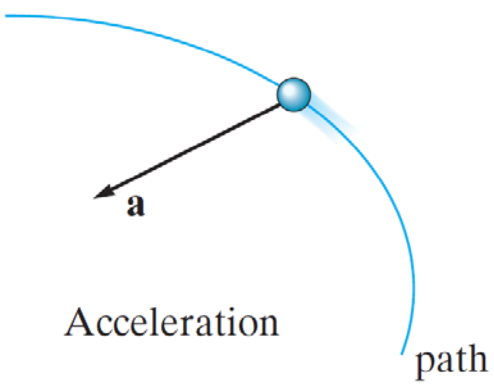
The transition from statics to dynamics can be summarized as: the position vector changes over time

The path described by a particle over time may be rectilinear, curvilinear, circular, etc.

Given $\boldsymbol{r} = \boldsymbol{r}(t)$ we can define the velocity of a particle as the change in position over time. The velocity vector, **v**, is always tangent to the path of motion.

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$$

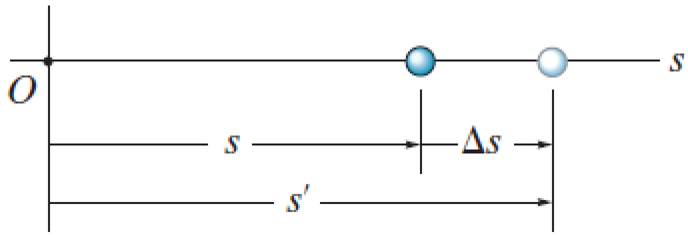




And given $\boldsymbol{v}(t)$ we can define the acceleration of a particle in a similar way

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$$

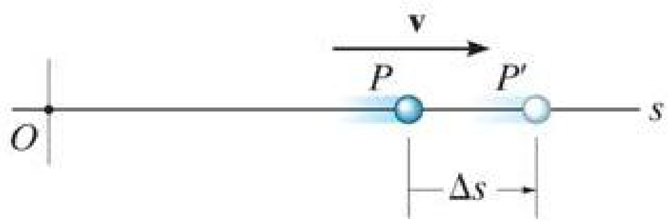
Rectilinear motion



Displacement

A particle travels along a straight-line path defined by the coordinate axis s.

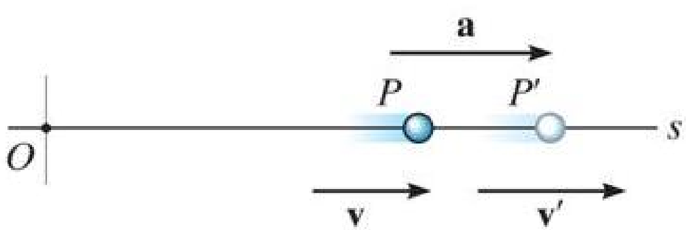
The position of the particle at any instant, relative to the origin, O, is defined by the position vector ***r***, or the scalar s.



Velocity

Velocity is a measure of the rate of change in the position of a particle

Speed is the magnitude of velocity: $v = ds / dt$



Acceleration

Acceleration is the rate of change in the velocity of a particle

Scalar form: $a = dv / dt = d^2s / dt^2$

$a\,ds = v\,dv$

Summary of kinematic relations: Rectilinear motion

Differentiate position to get velocity and acceleration.

$$v = ds/dt \; ; \qquad a = dv/dt \quad \text{or} \quad a = v\, dv/ds$$

Integrate acceleration for velocity and position (constant acceleration).

Velocity:

$$\int_{v_o}^{v} dv = \int_{o}^{t} a\,dt \quad \text{or} \quad \int_{v_o}^{v} v\,dv = \int_{s_o}^{s} a\,ds$$

Position:

$$\int_{s_o}^{s} ds = \int_{o}^{t} v\,dt$$

Note that s_o and v_o represent the initial position and velocity of the particle at $t = 0$.

Constant acceleration

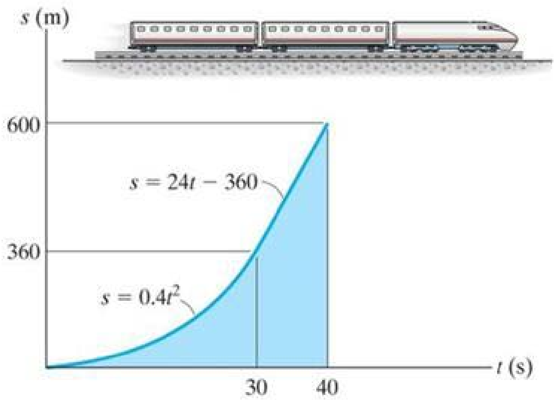
When acceleration is constant ($a = a_c$), example gravity; these equations are:

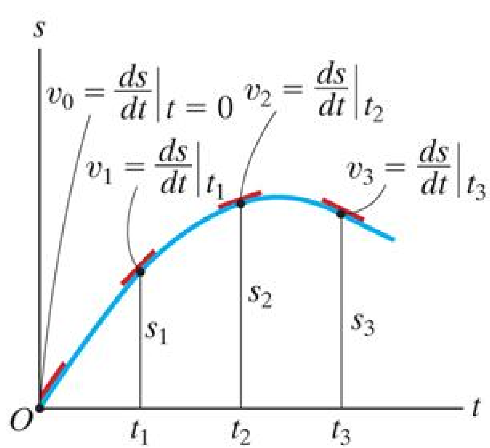
$a_c = dv / dt$ $\Rightarrow$ $\int_{v_0}^{v} dv = \int_0^t a_c \, dt$ $\Rightarrow$ $v = v_0 + a_c t$

$v = ds / dt$ $\Rightarrow$ $\int_{s_0}^{s} ds = \int_0^t v \, dt$ $\Rightarrow$ $s = s_0 + v_0 t + \frac{a_c t^2}{2}$

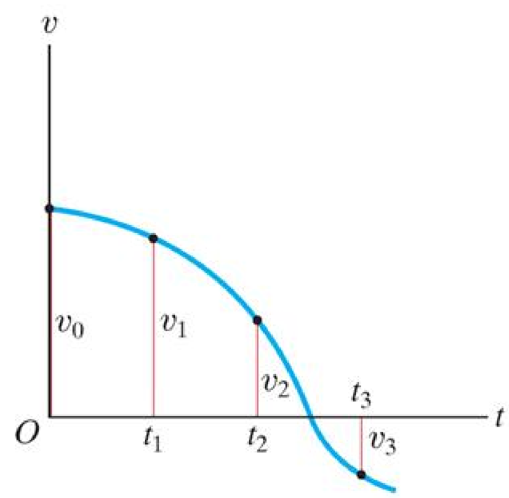
$a \, ds = v \, dv$ $\Rightarrow$ $\int_{v_0}^{v} v \, dv = \int_{s_0}^{s} a_c \, ds$ $\Rightarrow$ $v^2 = (v_0)^2 + 2a_c(s - s_0)$

Erratic motion
Motion graphs: S-t graph



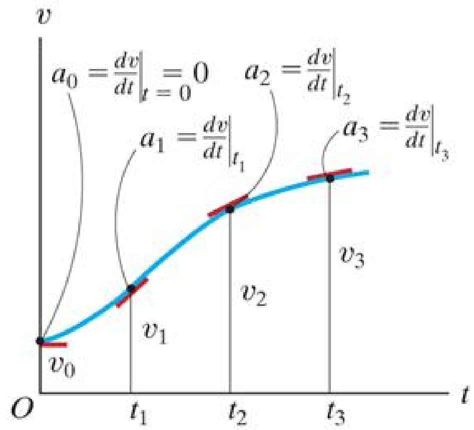


Plots of position vs. time can be used to find velocity vs. time curves. Finding the slope of the line tangent to the motion curve at any point is the velocity at that point (or v = ds/dt).



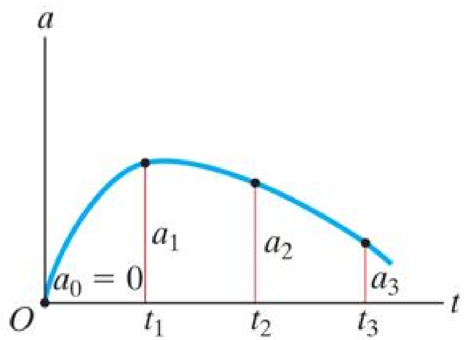
Therefore, the v-t graph can be constructed by finding the slope at various points along the s-t graph.

Motion graphs: **V-t graph**



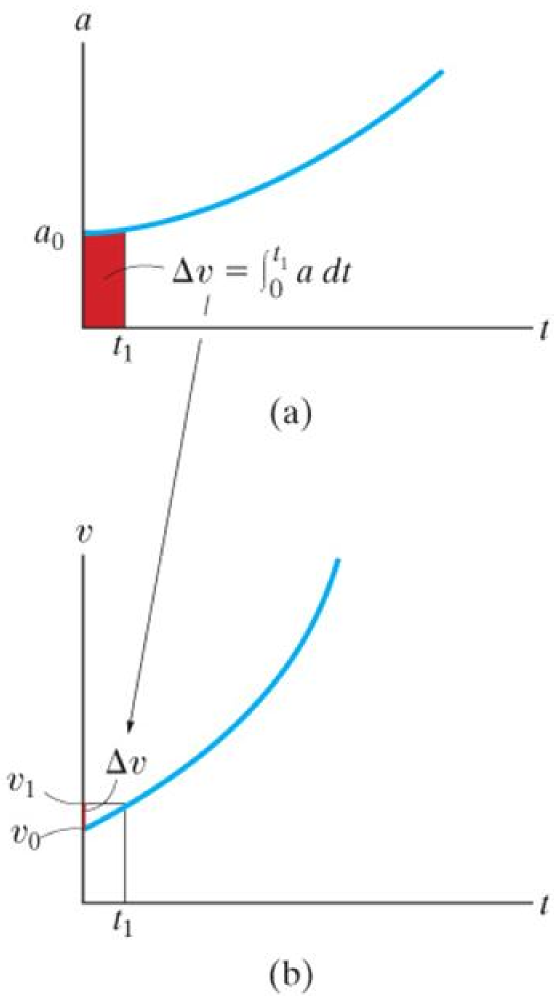
Plots of velocity vs. time can be used to find acceleration vs. time curves. Finding the slope of the line tangent to the velocity curve at any point is the acceleration at that point (or a = dv/dt).

Therefore, the acceleration vs. time (or a-t) graph can be constructed by finding the slope at various points along the v-t graph.



Also, the distance moved (displacement) of the particle is the area under the v-t graph during time Δt.

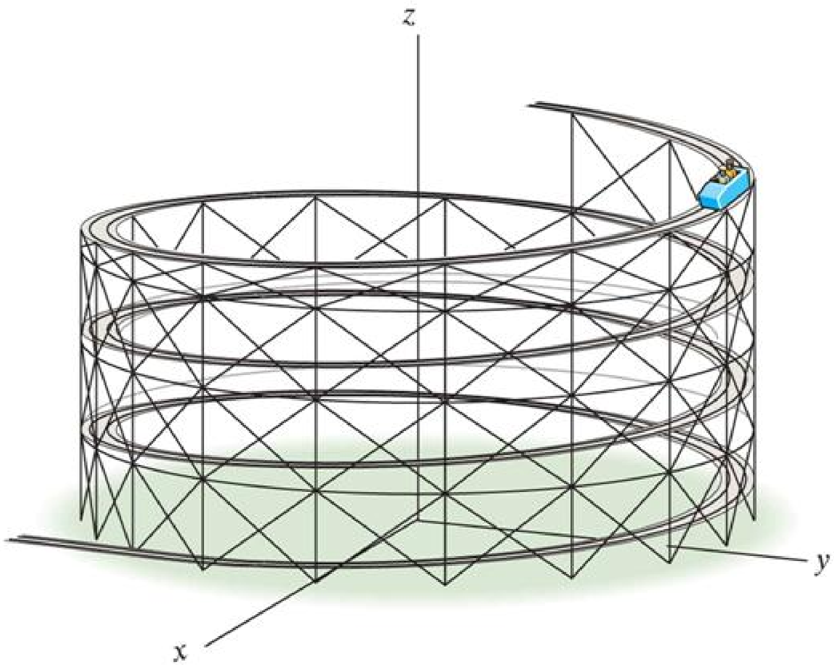
Motion graphs: **A-t graph**



Given the acceleration vs. time or a-t curve, the change in velocity (Δv) during a time period is the area under the a-t curve.

So we can construct a v-t graph from an a-t graph if we know the initial velocity of the particle.

Curvilinear motion



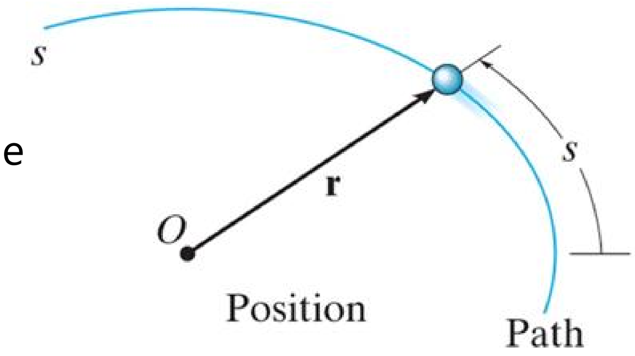
A roller coaster car travels down a fixed, helical path at a constant speed.

The path of motion of a plane can be tracked with radar and its x, y, and z-coordinates (relative to a point on earth) recorded as a function of time.

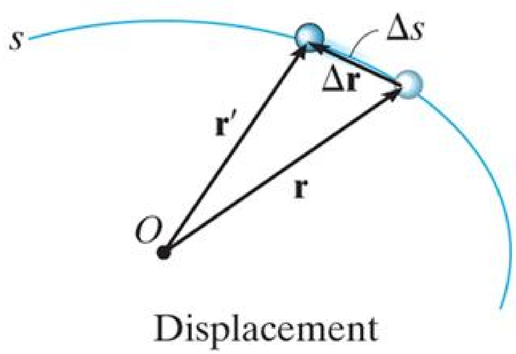
Curvilinear motion

A particle moving along a curved path undergoes curvilinear motion. Since the motion is often three-dimensional, vectors are usually used to describe the motion.

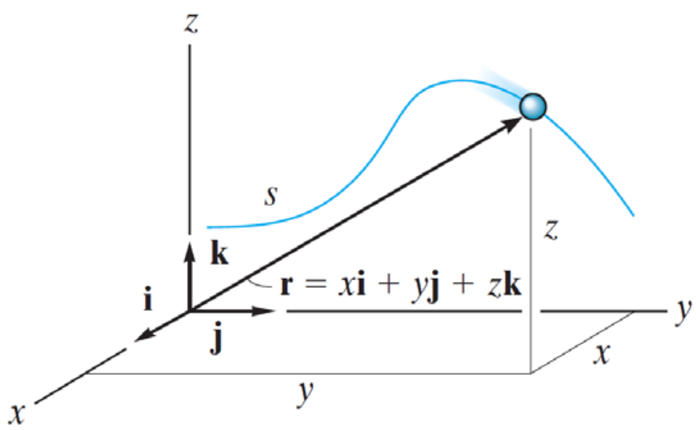
A particle moves along a curve defined by the path function, s.



The position of the particle at any instant is designated by the vector **r** = **r**(t). Both the magnitude and direction of **r** may vary with time.



If the particle moves a distance Δs along the curve during time interval Δt, the displacement is determined by vector subtraction: $\Delta \mathbf{r} = \mathbf{r}' - \mathbf{r}$



Rectangular components

Sometimes, it is convenient to describe the motion of a particle in terms of the cartesian *x,y,z* components (or rectangular components of **r** relative to a fixed frame

In this case, the *x,y,z* components are all functions of time. We can obtain **v** and **a** from those functions

We know that $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ and $r = \sqrt{x^2 + y^2 + z^2}$

So, the first derivative of $\boldsymbol{r}$ gives

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}(x\hat{\boldsymbol{\imath}}) + \frac{d}{dt}(y\hat{\boldsymbol{\jmath}}) + \frac{d}{dt}(z\hat{\boldsymbol{k}})$$

If we assume the vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ <u>do not change over time</u>, we get

$$\boldsymbol{v} = v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}}$$

where $v_x = \dot{x}, v_y = \dot{y}, v_z = \dot{z}$

the dot notation represents the first time-derivative

Similarly, for the acceleration

$$\boldsymbol{a} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}$$

where $a_x = \ddot{x}, a_y = \ddot{y}, a_z = \ddot{z}$

Example (Individual work 10 min)

A particle travels along the path $y = 0.5\, x^2$. When $t = 0$, $x = y = z = 0$.

Find the particle's distance and the magnitude of its acceleration when $t = 1$ s, if $v_x = (5\, t)$ ft/s, where t is in seconds.

Solution

1) Determine x and a_x by integrating and differentiating v_x, respectively, using the initial conditions.

2) Find the y-component of velocity & acceleration by taking a time derivative of the path.

3) Determine the magnitude of the acceleration & position.

1) x-components:

Velocity known as: $v_x = \dot{x} = (5\ t)$ ft/s $\Rightarrow$ $\underline{5 \text{ ft/s at } t=1\text{s}}$

Position: $\int v_x \mathrm{d}t = \int_0^t (5t)\ \mathrm{d}t \Rightarrow x = 2.5\ t^2 \Rightarrow$ $\underline{2.5 \text{ ft at } t=1\text{s}}$

Acceleration: $a_x = \ddot{x} = \mathrm{d}/\mathrm{d}t\ (5\ t) \Rightarrow$ $\underline{5 \text{ ft/s}^2 \text{ at } t=1\text{s}}$

2) y-components:

Position known as: $y = 0.5\ x^2 \Rightarrow$ $\underline{3.125 \text{ ft at t=1s}}$

Velocity: $\dot{y} = 0.5\ (2)\ x\ \dot{x} = x\ \dot{x} \Rightarrow$ $\underline{12.5 \text{ ft/s at t=1s}}$

Acceleration: $a_y = \ddot{y} = \dot{x}\ \dot{x} + x\ \ddot{x} \Rightarrow$ $\underline{37.5 \text{ ft/s}^2 \text{ at t=1s}}$

3) The position vector and the acceleration vector are

Position vector: $\mathbf{r} = [\, x\,\mathbf{i} + y\,\mathbf{j}\,]$ ft

where $x = 2.5$ ft, $y = 3.125$ ft

Magnitude: $r = \sqrt{2.5^2 + 3.125^2} = \underline{4.00 \text{ ft}}$

Acceleration vector: $\mathbf{a} = [\, a_x\,\mathbf{i} + a_y\,\mathbf{j}]$ ft/s^2

where $a_x = 5$ ft/s^2, $a_y = 37.5$ ft/s^2

Magnitude: $a = \sqrt{5^2 + 37.5^2} = \underline{37.8 \text{ ft/s}^2}$

Motion of a projectile

Projectile motion can be treated as two rectilinear motions, one in the horizontal direction experiencing zero acceleration and the other in the vertical direction experiencing constant acceleration (i.e., from gravity).

Kinematic equations: horizontal motion

Since $a_x = 0$, the velocity in the horizontal direction remains constant ($v_x = v_{ox}$) and the position in the x direction can be determined by:

$$x = x_o + (v_{ox})\, t$$

Kinematic equations: vertical motion

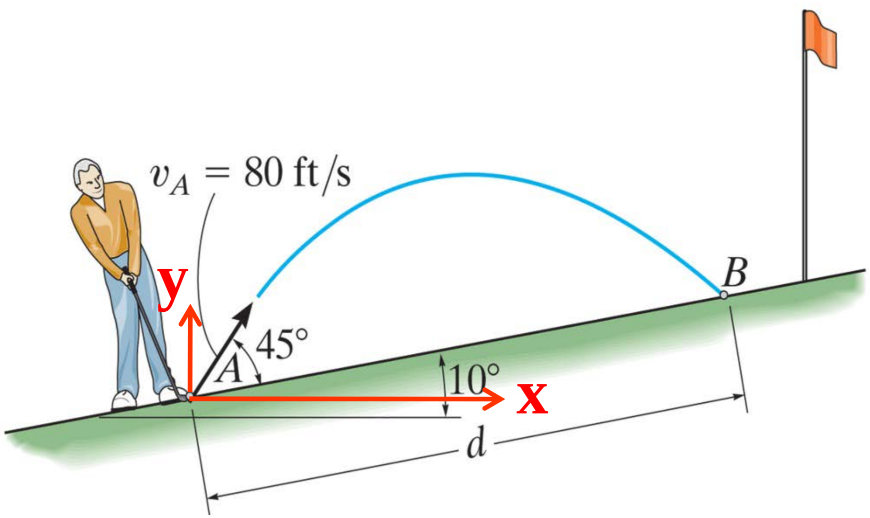
Since the positive y-axis is directed upward, $a_y = -g$. Application of the constant acceleration equations yields:

$$v_y = v_{oy} - g\, t$$

$$y = y_o + (v_{oy})\, t - \tfrac{1}{2}\, g\, t^2$$

$$v_y^2 = v_{oy}^2 - 2\, g\, (y - y_o)$$

Example (Group work 10 min)



Given: The golf ball is struck with a velocity of 80 ft/s as shown.

Find: Distance d to where it will land.

Solution

1) Establish a fixed x, y coordinate system (in this solution, the origin of the coordinate system is placed at A).

2) Apply the kinematic relations in x- and y-directions.

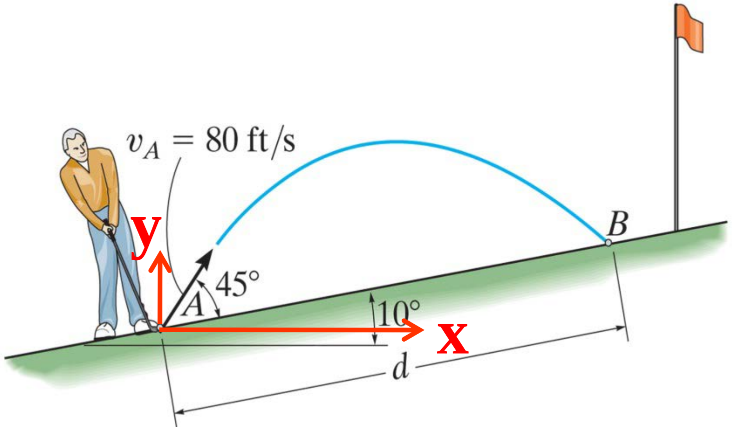
Solution:

Motion in x-direction:

Using $x_B = x_A + v_{ox}(t_{AB})$

$\Rightarrow d \cos 10 = 0 + 80 (\cos 55)\, t_{AB}$

$t_{AB} = 0.02146\, d$



Motion in y-direction:

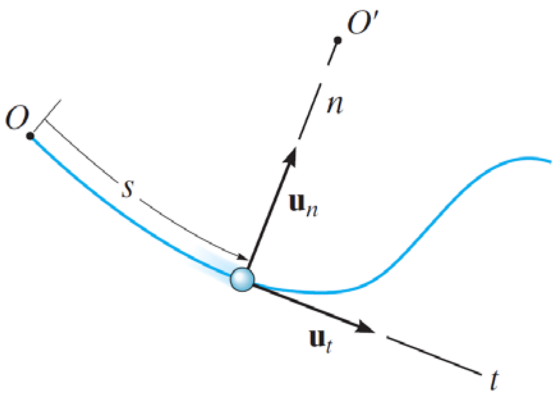
Using $y_B = y_A + v_{oy}(t_{AB}) - ½\, g(t_{AB})^2$

$\Rightarrow d \sin 10 = 0 + 80(\sin 55)(0.02146\, d) - ½\, 32.2\, (0.02146\, d)^2$

$\Rightarrow 0 = 1.233\, d - 0.007415\, d^2$

d = 0, <u>166 ft</u> Only the non-zero answer is meaningful.

Normal and tangential components

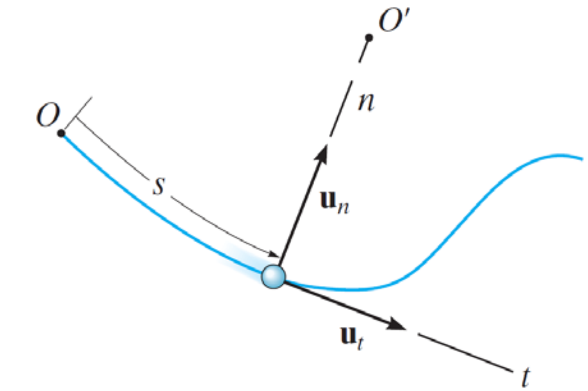


If we know the path a particle will follow, it can be convenient to describe the motion with respect to that path using the *n* (normal) and *t* (tangential) coordinates

The *n* and *t* coordinates are always defined with respect to the particle instead of a fixed point. They change over time

The t-axis is tangent to the path (curve) at the instant considered, positive in the direction of the particle's motion.
The n-axis is perpendicular to the t-axis with the positive direction toward the center of curvature of the curve.

Normal and tangential components



The positive n and t directions are defined by the unit vectors $\mathbf{u_n}$ and $\mathbf{u_t}$, respectively.

The center of curvature, O', always lies on the concave side of the curve. The radius of curvature, ρ, is defined as the perpendicular distance from the curve to the center of curvature at that point.

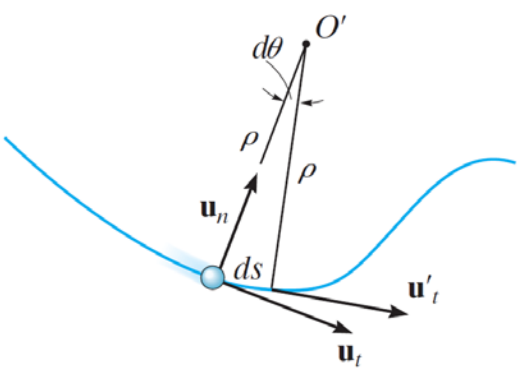
According to our definitions, the position *s* is a function of time, its derivative gives the velocity:

$$\boldsymbol{v} = v\boldsymbol{u}_t$$

Where $\boldsymbol{u}_t$ is the unit vector in the *t* (tangential) direction and The velocity vector is always tangent to the path of motion

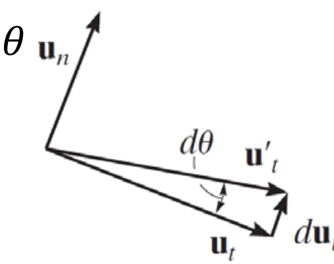
$$v = \dot{s}$$

Differentiating the velocity will get the acceleration $\boldsymbol{a} = \dot{\boldsymbol{v}} = \dot{v}\boldsymbol{u}_t + v\dot{\boldsymbol{u}}_t$



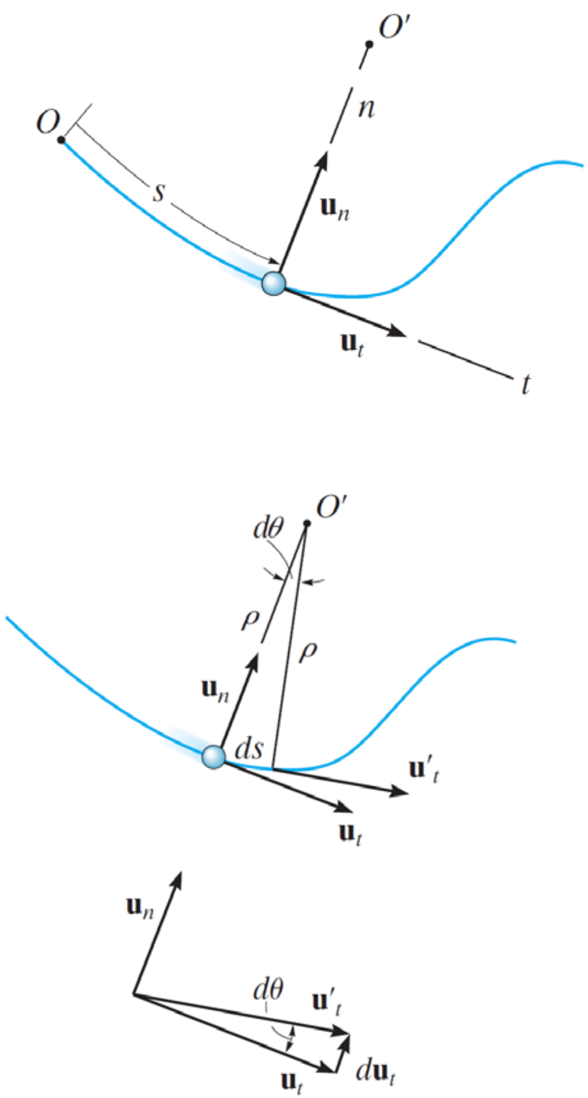
Considering an infinitesimal displacement *ds*

This gives the magnitude of $d\boldsymbol{u}_t$ as $du_t = (1)d\theta$



And the direction of $d\boldsymbol{u}_t$ is given by $\boldsymbol{u}_n$, thus

$$\dot{\boldsymbol{u}}_t = \dot{\theta}\boldsymbol{u}_n$$



Normal and tangential components

We also know that $ds = \rho\, d\theta$ which gives $\dot{\theta} = \dot{s}/\rho$ which means

$$\dot{\boldsymbol{u}}_t = \dot{\theta}\boldsymbol{u}_n = \frac{\dot{s}}{\rho}\boldsymbol{u}_n = \frac{v}{\rho}\boldsymbol{u}_n$$

In summary, the acceleration is given by

$$a = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n$$

With the tangential and normal components of the acceleration:

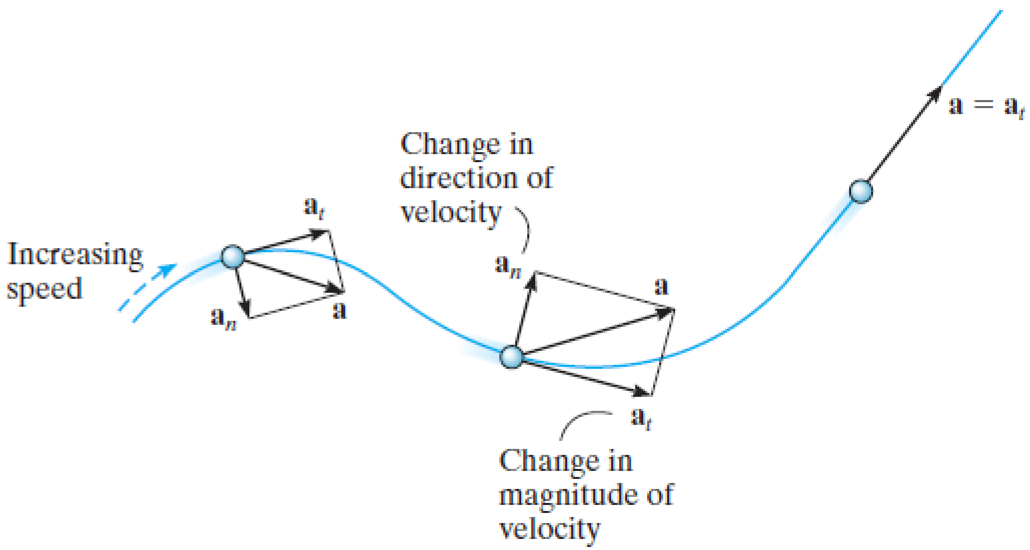
$$a_t = \dot{v} \qquad\qquad a_n = \frac{v^2}{\rho}$$

Two special cases

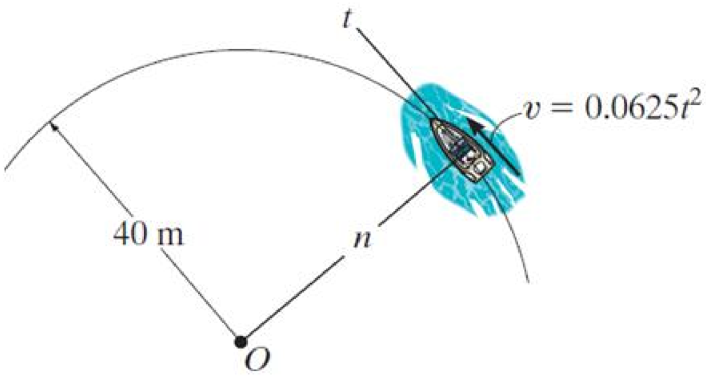
1. If the particle moves along a straight line, then $\rho \to \infty$ and $a_n = 0$. Thus $a = a_t = \dot{v}$, and we can conclude that the *tangential component of acceleration represents the time rate of change in the magnitude of the velocity.*

2. If the particle moves along a curve with a constant speed, then $a_t = \dot{v} = 0$ and $a = a_n = {v^2}/{\rho}$

Therefore, the *normal component of acceleration represents the time rate of change in the direction of the velocity*. Since $\mathbf{a}_n$ *always* acts towards the center of curvature, this component is sometimes referred to as the *centripetal* (or center seeking) *acceleration*.



Example



A boat travels around a circular path, $\rho = 40$ m, at a speed that increases with time, $v = (0.0625\ t^2)$ m/s.

Calculate the magnitudes of the boat's velocity and acceleration at the instant $t = 10$ s.

Solution

1) The boat starts from rest (v = 0 when t = 0).
2) Calculate the velocity at t = 10 s using v(t).
3) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

1) The velocity vector is $\boldsymbol{v} = v\,\boldsymbol{u}_t$, where the magnitude is given by $v = (0.0625t^2)$ m/s. At t = 10s:

$v = 0.0625\,t^2 = 0.0625\,(10)^2 = \underline{6.25 \text{ m/s}}$

2) The acceleration vector is $\mathbf{a} = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n = \dot{v}\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n$.

Tangential component: $a_t = \dot{v} = d(.0625\,t^2)/dt = 0.125\,t \text{ m/s}^2$

At t = 10s: $a_t = 0.125t = 0.125(10) = 1.25 \text{ m/s}^2$

Normal component: $a_n = v^2/\rho \text{ m/s}^2$

At t = 10s: $a_n = (6.25)^2 / (40) = 0.9766 \text{ m/s}^2$

The magnitude of the acceleration is

$a = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{1.25^2 + 0.9766^2} = \underline{1.59 \text{ m/s}^2}$

Example (Group work 15 min)

Starting from rest, a bicyclist travels around a horizontal circular path, $\rho = 10$ m, at a speed of $v = (0.09\ t^2 + 0.1\ t)$ m/s.

Find the magnitudes of her velocity and acceleration when she has traveled 3 m.

Solution

1) The bicyclist starts from rest ($v = 0$ when $t = 0$).
2) Integrate $v(t)$ to find the position $s(t)$.
3) Calculate the time when $s = 3$ m using $s(t)$.
4) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

1) The velocity vector is $v = (0.09\ t^2 + 0.1\ t)$ m/s, where t is in seconds. Integrate the velocity and find the position $s(t)$.

Position: $\int v\,dt = \int (0.09\ t^2 + 0.1\ t)\ dt$

$$s\ (t) = 0.03\ t^3 + 0.05\ t^2$$

2) Calculate the time, t when s = 3 m.

$$3 = 0.03\ t^3 + 0.05\ t^2$$

Solving for t, $t = 4.147$ s

The velocity at $t = 4.147$ s is,

$$v = 0.09\ (4.147\ )^2 + 0.1\ (4.147\ ) = \underline{1.96\ \ m/s}$$

3) The acceleration vector is $\mathbf{a} = a_t \boldsymbol{u}_t + a_n \boldsymbol{u}_n = \dot{v} \boldsymbol{u}_t + (v^2/\rho) \boldsymbol{u}_n$.

Tangential component:

$$a_t = \dot{v} = d(0.09\ t^2 + 0.1\ t) / dt = (0.18\ t + 0.1)\ m/s^2$$

$$\text{At } t = 4.147\ s: \quad a_t = 0.18\ (4.147) + 0.1 = 0.8465\ m/s^2$$

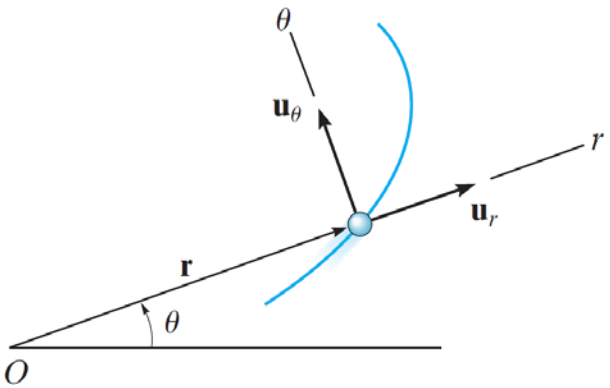
Normal component:

$$a_n = v^2/\rho \quad m/s^2$$

$$\text{At } t = 4.147\ s: \quad a_n = (1.96)^2 / (10) = 0.3852\ m/s^2$$

The magnitude of the acceleration is

$$a = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{0.8465^2 + 0.3852^2} = \underline{0.930\ m/s^2}$$



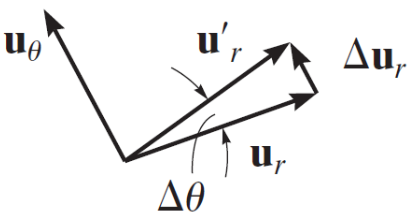
Cylindrical components

Sometimes, motion is most conveniently described using cylindrical coordinates or (in the plane) polar coordinates

In this case, the radial coordinate *r* extends outward from the origin and the transversal coordinate *q* is the <u>counterclockwise</u> angle between a fixed reference line and the *r* axis

At any instant the position is given by $\boldsymbol{r} = r\boldsymbol{u}_r$

The velocity is obtained by differentiation of $\boldsymbol{r}$ thus $\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{u}_r + r\dot{\boldsymbol{u}}_r$

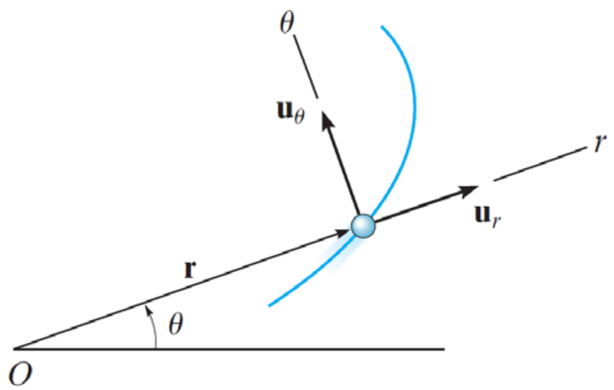


Following a similar rationale as before, the change $\dot{\boldsymbol{u}}_r$ is a change in direction given by the angular velocity $\dot{\theta}$ $\dot{\boldsymbol{u}}_r = \dot{\theta}\boldsymbol{u}_\theta$

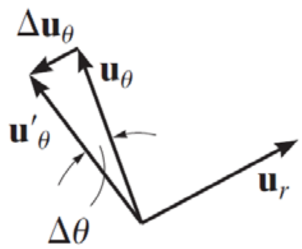
The velocity is then $\boldsymbol{v} = v_r\boldsymbol{u}_r + v_\theta\boldsymbol{u}_\theta$ with $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$

Finally, the acceleration is given by $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{r}\boldsymbol{u}_r + \dot{r}\dot{\boldsymbol{u}}_r + \dot{r}\dot{\theta}\boldsymbol{u}_\theta + r\ddot{\theta}\boldsymbol{u}_\theta + r\dot{\theta}\dot{\boldsymbol{u}}_\theta$

Cylindrical components



To determine $\dot{\boldsymbol{u}}_\theta$ we can follow the same procedure as before. In this case, the infinitesimal change in direction has a magnitude $d\theta$ and is in the direction $-\boldsymbol{u}_r$ thus



$$\dot{\boldsymbol{u}}_\theta = -\dot{\theta}\boldsymbol{u}_r$$

This gives $\boldsymbol{a} = a_r\boldsymbol{u}_r + a_\theta\boldsymbol{u}_\theta$ where

$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

a_r is the radial acceleration, a_θ is the transverse acceleration

The magnitude of acceleration is $a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$

If the particle P moves along a space curve, its position can be written as

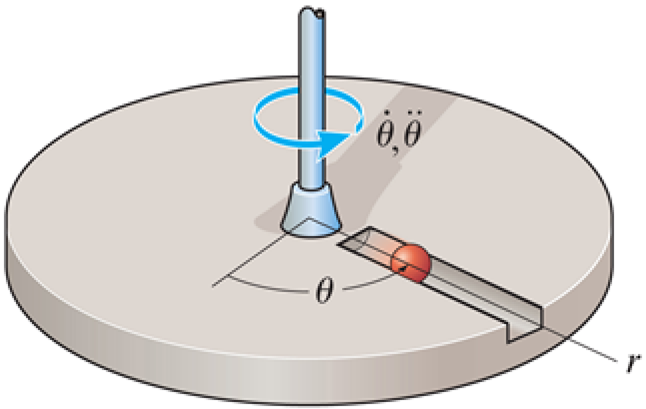
$$\boldsymbol{r_P} = \mathrm{r}\boldsymbol{u_r} + \mathrm{z}\boldsymbol{u_z}$$

Taking time derivatives and using the chain rule:

Velocity: $\boldsymbol{v_P} = \mathrm{r}\boldsymbol{u_r} + \mathrm{r}\theta\boldsymbol{u_\theta} + \mathrm{z}\boldsymbol{u_z}$

Acceleration: $\mathbf{a}_{\boldsymbol{P}} = (\mathrm{r} - \mathrm{r}\theta^2)\boldsymbol{u_r} + (\mathrm{r}\theta + 2\mathrm{r}$

Example



The platform is rotating such that, at any instant, its angular position is $\theta = (4t^{3/2})$ rad, where t is in seconds.
A ball rolls outward so that its position is
$r = (0.1t^3)$ m

Calculate the magnitude of velocity and acceleration of the ball when $t = 1.5$ s.

Use a polar coordinate system and related kinematic equations.

angular position is $\theta = (4t^{3/2})$ rad

$r = (0.1t^3)$ m

Solution

$r = 0.1t^3$, $\dot{r} = 0.3\,t^2$, $\ddot{r} = 0.6\,t$

$\theta = 4\,t^{3/2}$, $\dot{\theta} = 6\,t^{1/2}$, $\ddot{\theta} = 3\,t^{-1/2}$

At t=1.5 s,

$r = 0.3375$ m, $\dot{r} = 0.675$ m/s, $\ddot{r} = 0.9$ m/s^2

$\theta = 7.348$ rad, $\dot{\theta} = 7.348$ rad/s, $\ddot{\theta} = 2.449$ rad/s^2

Substitute into the equation for velocity

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{u_r} + r\dot{\theta}\,\boldsymbol{u_\theta} = 0.675\,\boldsymbol{u_r} + 0.3375\,(7.348)\,\boldsymbol{u_\theta}$$

$$= 0.675\,\boldsymbol{u_r} + 2.480\,\boldsymbol{u_\theta}$$

$$v = \sqrt{(0.675)^2 + (2.480)^2} = \underline{2.57\text{ m/s}}$$

Substitute in the equation for acceleration:

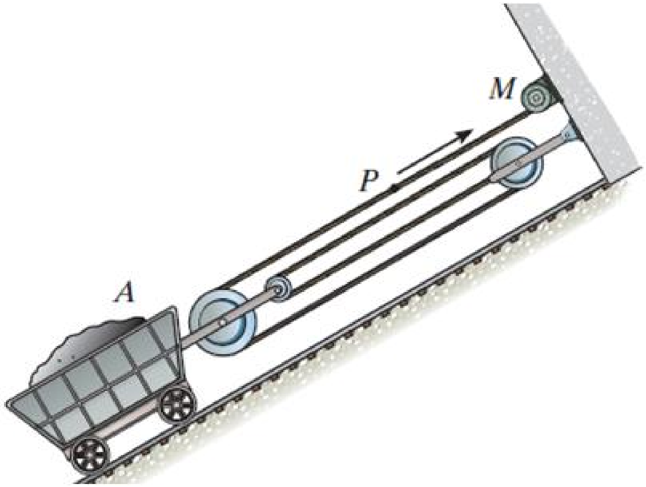
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u_\theta}$$

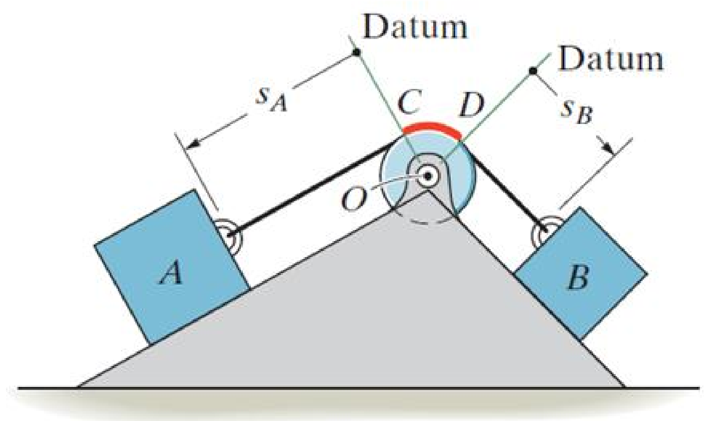
$$\mathbf{a} = [0.9 - 0.3375(7.348)^2]\ \boldsymbol{u_r} + [0.3375(2.449) + 2(0.675)(7.348)]\ \boldsymbol{u_\theta}$$

$$\mathbf{a} = -\,17.33\ \boldsymbol{u_r} + 10.75\ \boldsymbol{u_\theta}\ \text{m/s}^2$$

$$a = \sqrt{(-\,17.33)^2 + (10.75)^2} = \underline{20.4\ \text{m/s}^2}$$

Dependent motion





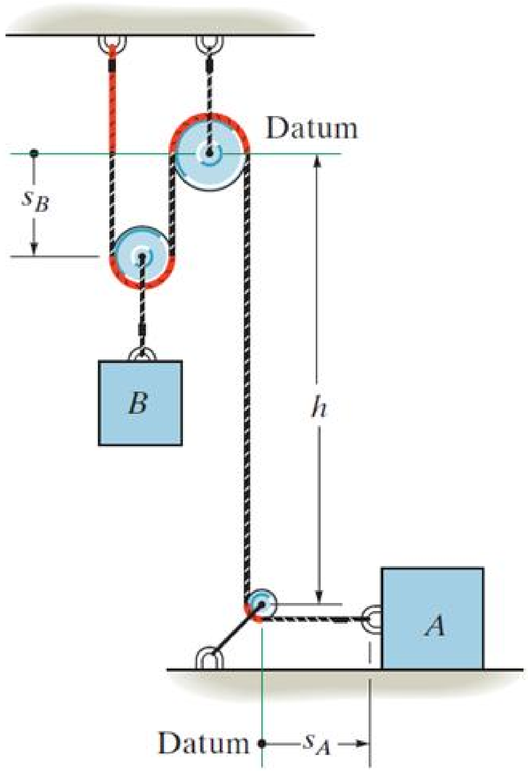
The cable and pulley system shown can be used to modify the speed of the mine car, A, relative to the speed of the motor, M.

It is important to establish the relationships between the various motions in order to determine the power requirements for the motor and the tension in the cable.

The blocks in this figure are connected by an inextensible cord wrapped around a pulley. If block A moves downward along the inclined plane, block B will move up the other incline.

The motion of each block can be related mathematically by defining position coordinates, s_A and s_B. Each coordinate axis is defined from a fixed point or datum line, measured positive along each plane in the direction of motion of each block.

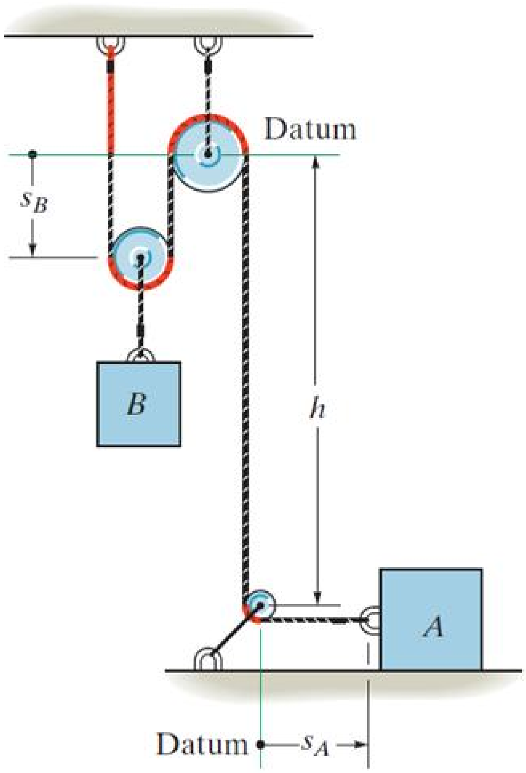
Dependent motion example



Position coordinates (s_A and s_B) are defined from fixed datum lines, measured along the direction of motion of each block.

Note that s_B is only defined to the center of the pulley above block B, since this block moves with the pulley. Also, *h* is a constant.

The red-colored segments of the cord remain constant in length during the motion of the blocks.



The position coordinates are related by the equation

$$2s_B + h + s_A = l_T$$

Where l_T is the total cord length minus the lengths of the red segments.

Since l_T and h remain constant during the motion, the velocities and accelerations can be related by two successive time derivatives:

$$2v_B = -v_A \quad \text{and} \quad 2a_B = -a_A$$

When block B moves downward ($+s_B$), block A moves to the left ($-s_A$). Remember to be consistent with your sign convention!

Dependent motion procedure

These procedures can be used to relate the dependent motion of particles moving along rectilinear paths (only the magnitudes of velocity and acceleration change, not their line of direction).

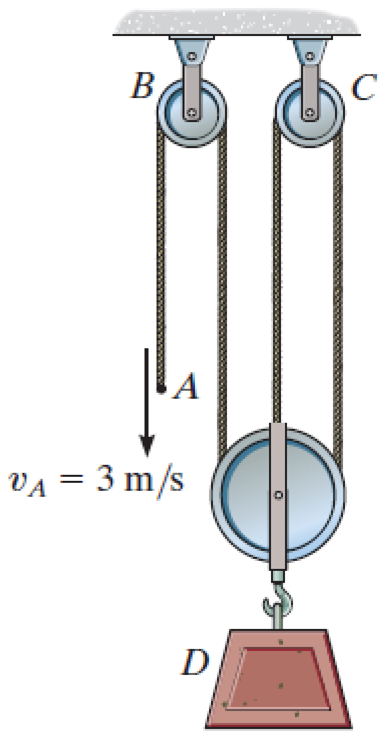
1. Define position coordinates from fixed datum lines, along the path of each particle. Different datum lines can be used for each particle.

2. Relate the position coordinates to the cord length. Segments of cord that do not change in length during the motion may be left out.

3. If a system contains more than one cord, relate the position of a point on one cord to a point on another cord. Separate equations are written for each cord.

4. Differentiate the position coordinate equation(s) to relate velocities and accelerations. Keep track of signs!

Example (Group work 10 min)



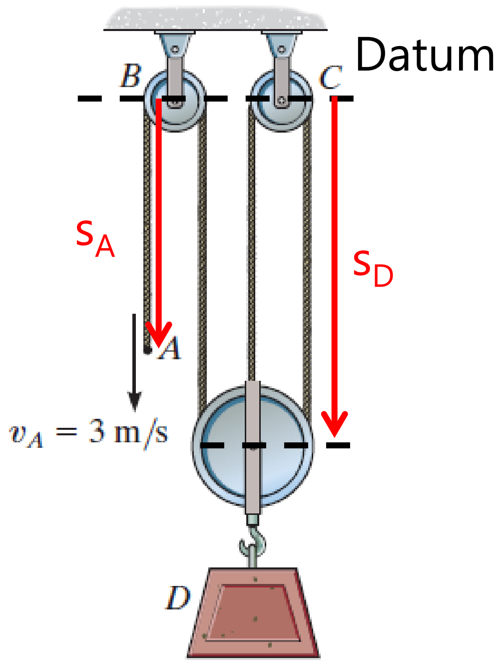
In the figure on the left, the cord at A is pulled down with a speed of 3 m/s.

Find the speed of block *D*

Solution

There is only one cord involved in the motion, so only one position/length equation is required. Define position coordinates for block *D* and cable lengths that change, write the position relation and then differentiate it to find the relationship between the two velocities.

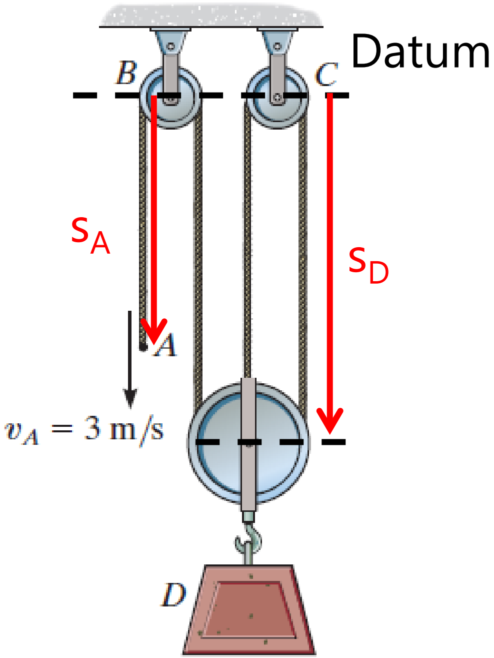
1) A datum line can be drawn through the upper, fixed pulleys. Two coordinates must be defined: one for block *D* (s_D) and one for the changing cable length (s_A).



s_A can be defined to the point *A*.

s_D can be defined to the center of the pulley above D.

All coordinates are defined as positive down and along the direction of motion of each point/object.



2) Write position/length equations for the cord. Define l_T as the length of the cord, minus any segments of constant length.

$$s_A + 3s_D = l_T$$

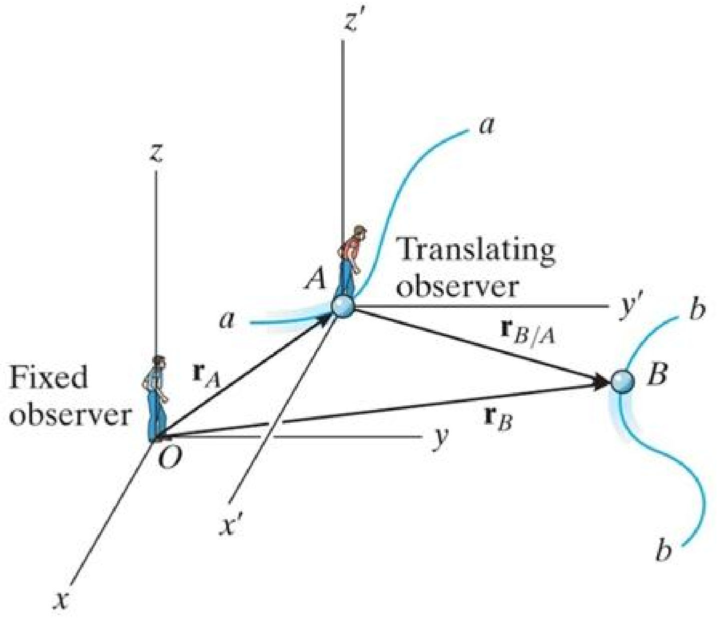
3) Differentiate to find the velocity relationship:

$$v_A + 3v_D = 0$$

Since the cord at A is pulled down with a speed of 3 m/s,

$3 + 3v_D = 0 \Rightarrow v_D = -1$ m/s = 1 m/s $\uparrow$

Relative position



The absolute positions of two particles A and B with respect to the fixed x, y, z-reference frame are given by $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$. The position of B relative to A is represented by

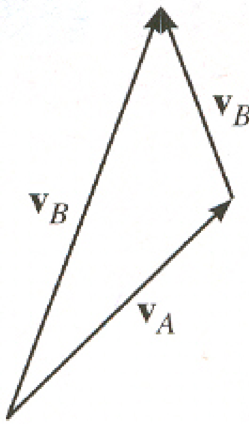
$$\boldsymbol{r}_{B/A} = \boldsymbol{r}_B - \boldsymbol{r}_A$$

Therefore, if $\boldsymbol{r}_B = (10\,\boldsymbol{i} + 2\,\boldsymbol{j})$ m

and $\boldsymbol{r}_A = (4\,\boldsymbol{i} + 5\,\boldsymbol{j})$ m,

then $\boldsymbol{r}_{B/A} = \boldsymbol{r}_B - \boldsymbol{r}_A = (6\,\boldsymbol{i} - 3\,\boldsymbol{j})$ m.

relative velocity



To determine the relative velocity of B with respect to A, the time derivative of the relative position equation is taken.

$$\boldsymbol{v}_{B/A} = \boldsymbol{v}_B - \boldsymbol{v}_A$$

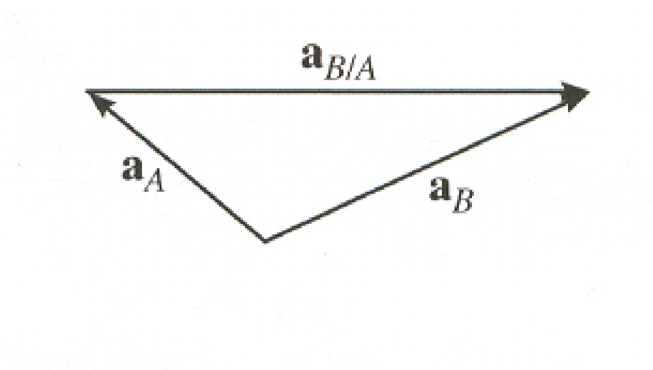
or

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$$

In these equations, $\boldsymbol{v}_B$ and $\boldsymbol{v}_A$ are called absolute velocities and $\boldsymbol{v}_{B/A}$ is the relative velocity of B with respect to A.

Note that $\boldsymbol{v}_{B/A} = - \boldsymbol{v}_{A/B}$.

relative acceleration



The time derivative of the relative velocity equation yields a similar vector relationship between the absolute and relative accelerations of particles A and B.

These derivatives yield: $\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$

or

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$