CSE 015: Discrete Mathematics
Homework 2

Fall 2021
Provided Solution

1 Quantifiers

a) TRUE. Substituting the value 2 for x in the statement $x < x^3$ gives $2 < 2^3$ which is true.

b) FALSE. Substituting the value -1 for x in the statement $x < x^3$ gives $-1 < -1^3$, i.e., $-1 < -1$ which is false.

c) FALSE. The answer to the previous question provides a counterexample to the statement being true for all integers. Any other counterexample could be equally valid. In fact, any integer smaller than 2 is a counterexample.

d) TRUE. The answer to the first question provides an instance for which the sentence is true. In fact and integer greater or equal than 2 satisfies the formula.

e) FALSE. As just stated above, any integer greater or equal than 2 satisfies the formula so it is not true that there is a unique element in the domain that satisfies the formula.

2 Translating English Sentences into Formulas

a) $\neg\forall x(S(x) \rightarrow M(x))$. Alternative correct answer: $\exists x(S(x) \wedge \neg M(x))$

b) $\forall x(S(x) \oplus M(x))$

c) $\exists x(S(x) \wedge \neg M(x))$ (note that indeed this sentence was the negation of the first one.)

3 Logical Equivalence

Recall that the definition of logical equivalence for statements with predicates states that two statements are equivalent if and only if they have the same truth value no matter which predicates are substituted into the statements, and which domain is used for the variables.

Based on this definition, the two statements are not equivalent. This can be shown in different ways. First approach: the expression on the left states that every element in the domain must make true $A(x) \wedge B(x)$, and for this to be true it means that every element in the domain must make $A(x)$ true. The expression on the right instead states that if an element in the domain makes $A(x)$ true, then it must also make $B(x)$ true. However, this expression does not require that each element in the domain satisfies $A(x)$, and this makes it different from the one on the left. Second approach: write the truth tables for $a \wedge b$ and $a \rightarrow b$ and observe that their truth values are different when a is false. Therefore the two statements have different truth values when $A(x)$ is a predicate not satisfied by every element x in the domain.

4 Nested Quantifiers

a) TRUE. The statement reads, "There exists a real number x such that for each y the product xy is 0". Picking $x = 0$ makes the statement true (in fact, $x = 0$ is the only value that will make it true).

b) TRUE. The statement reads, "There exists two real numbers x and y that added together give 0. Any two opposite real numbers make the statement true.

c) TRUE. The statement reads, "For every real number x, there exists a real number y such that the product xy is 0. No matter the value selected for x, picking $y = 0$ makes the product 0.

d) FALSE . The statement reads, "There exists a real number x such that for each real number y the product xy is 0 and the sum $x+y$ is 0". To satisfy the part $A(x, y)$ (i.,e, $xy = 0$) one must pick $x = 0$ as per the first question above. With $x = 0$ to make the sentence true one should have $\forall y(B(0, y))$ which is obviously false for each $y \neq 0$.

e) TRUE. The statement reads, "There exists two real numbers x and y whose product 0 and whose sum is not 0. Picking $x = 0$ and any value $y \neq 0$ satisfy this statement.

5 Negating Formulas with Nested Quantifiers

In the following we write DM for *De Morgan*

a)

$$\begin{aligned}
\neg\exists x\exists y(P(x) \rightarrow Q(y)) \equiv \quad & \text{(apply DM to the leftmost existential quantifier)} \\
\forall x\neg\exists y(P(x) \rightarrow Q(y)) \equiv \quad & \text{(apply DM to the remaining existential quantifier)} \\
\forall x\forall y\neg(P(x) \rightarrow Q(y)) \equiv \quad & \text{(recall that } \neg(p \rightarrow q) \equiv p \wedge \neg q\text{)} \\
\forall x\forall y(P(x) \wedge \neg Q(y)) \quad &
\end{aligned}$$

Alternative equivalent solution:

$$\begin{aligned}
\neg\exists x\exists y(P(x) \rightarrow Q(y)) \equiv \quad & \text{(recall that } p \rightarrow q \equiv \neg p \vee q\text{)} \\
\neg\exists x\exists y(\neg P(x) \vee Q(y)) \equiv \quad & \text{(apply DM to the leftmost existential quantifier)} \\
\forall x\neg\exists y(\neg P(x) \vee Q(y)) \equiv \quad & \text{(apply DM to the remaining existential quantifier)} \\
\forall x\forall y\neg(\neg P(x) \vee Q(y)) \equiv \quad & \text{(apply DM to the disjunction)} \\
\forall x\forall y(\neg\neg P(x) \wedge \neg Q(x)) \equiv \quad & \text{(remove double negation)} \\
\forall x\forall y(P(x) \wedge \neg Q(x)) \quad &
\end{aligned}$$

b)

$$\begin{aligned}
\neg\exists y(\exists x A(x, y) \vee \forall x B(x, y)) \equiv \quad & \text{(apply DM to the leftmost existential quantifier)} \\
\forall y\neg(\exists x A(x, y) \vee \forall x B(x, y)) \equiv \quad & \text{(apply DM to the disjunction)} \\
\forall y(\neg\exists x A(x, y) \wedge \neg\forall x B(x, y)) \equiv \quad & \text{(apply DM to the exitential quantifier)} \\
\forall y(\forall x\neg A(x, y) \wedge \neg\forall x B(x, y)) \equiv \quad & \text{(apply DM to the rightmost universal quantifier)} \\
\forall y(\forall x\neg A(x, y) \wedge \exists x\neg B(x, y)) \quad &
\end{aligned}$$