Homework Assignment #8

Remember, this Homework Assignment is **not collected or graded**! But it is in your best interest to do it as the Homework Quiz will be based on it and it is the best way to ensure you know the material.

Section 4.2: Properties of the Determinant

1. If a 4×4 matrix A has $\det(A) = 1/2$. What is the value of:
   (a) $\det(2A)$
   (b) $\det(-A)$
   (c) $\det(A^2)$
   (d) $\det(A^{-1})$

   Hint: It might be helpful to remember the Properties of the Determinant we discussed in class (Week 10, Tuesday) and that are given in your Textbook in Section 4.1.

Solution:

(a) $\det(2A)$: Property (3) of the Determinant tells us that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\begin{vmatrix} \alpha a & \alpha b \\ c & d \end{vmatrix} = \alpha \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \alpha \det(A).$$

Since we can factor a constant 2 from each of the 4 rows of the matrix $2A$ we have:

$$\det(2A) = 2^4 \det(A).$$

(b) $\det(-A)$: By the same reasoning as part(a) we have,

$$\det(-A) = (-1)^4 \det(A) = \det(A).$$

(c) $\det(A^2)$: Property (9) of the Determinant tells us that: $\det(AB) = \det(A)\det(B)$. This means that:

$$\det(A^2) = \det(AA) = \det(A)\det(A) = \det(A)^2.$$

(d) $\det(A^{-1})$: We know that $A^{-1}A = I$ and we know that $\det(I) = 1$. Thus we have:

$$AA^{-1} = I \implies \det(A)\det(A^{-1}) = \det(I) = 1 \implies \det(A^{-1}) = 1/\det(A).$$

2. Recall that if Q is an orthogonal matrix, that is an $n \times n$ matrix with orthonormal columns, then Q is invertible and we have: $Q^{-1} = Q^T$. This means $Q^T Q = QQ^T = I$.

   Use properties of the Determinant to show that $\det(Q) = \pm 1$. (Hint: In particular you'll need Properties 9 and 10).

Solution: Let's remind ourselves of Property 9 and 10:

- Property 9: $\det(AB) = \det(A)\det(B)$
- Property 10: $\det(A) = \det(A^T)$.

$$\begin{aligned} I &= QQ^T \\ \det(I) &= \det(QQ^T) \\ 1 &= \det(Q)\det(Q^T) \\ 1 &= \det(Q)\det(Q) \\ 1 &= \det(Q)^2. \end{aligned}$$

Thus we have $\det(Q) = \sqrt{1} = \pm 1$.

3. Use row operations to show that the 3×3 matrix given has the following determinant.

$$A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

$$\det(A) = (b-a)(c-a)(c-b)$$

Solution: Since row operations applied to a matrix do not change the determinant, we will attempt to perform row reduction to transform this matrix an upper triangular matrix.

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & a & a^2 \\ 0 & (b-a) & b^2 - a^2 \\ 0 & (c-a) & c^2 - a^2 \end{bmatrix}$$

Here we will assume that $(b-a)$ is different from 0, otherwise, the determinant of the matrix is 0, which satisfies the formula given.

$$\begin{bmatrix} 1 & a & a^2 \\ 0 & (b-a) & b^2 - a^2 \\ 0 & (c-a) & c^2 - a^2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{(c-a)}{(b-a)} R_2} \begin{bmatrix} 1 & a & a^2 \\ 0 & (b-a) & b^2 - a^2 \\ 0 & 0 & (c^2 - a^2) - \frac{(c-a)}{(b-a)}(b^2 - a^2) \end{bmatrix}.$$

This matrix is now upper triangular, which means the determinant is the product of the diagonal entries leaving us:

$$\det(A) = 1(b-a)\left((c^2 - a^2) - \frac{(c-a)}{(b-a)} b^2 - a^2\right).$$

We need to rearrange the last terms:

$$\begin{aligned} (c^2 - a^2) - \frac{(c-a)}{(b-a)} b^2 - a^2 &= c^2 - a^2 - \frac{(c-a)}{(b-a)}(b-a)(b+a) \\ &= (c^2 - a^2) - (c-a)(b+a) \\ &= (c-a)(c+a) - (c-a)(b+a) \\ &= (c-a)((c+a) - (b+a)) \\ &= (c-a)(c-b). \end{aligned}$$

Thus we have: $\det(A) = (b-a)(c-a)(c-b)$.

Section 5.1: Introduction to Eigenvalues and Eigenvectors

4. Find the eigenvalues and eigenvectors of A:

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace of A and $(\lambda_1\lambda_2\lambda_3)$ equals the determinant of A.

Hint: To calculate the 3×3 determinant, you might find it helpful to remember your Properties of Determinants, the basket-weaving method which I've now learned should more likely be credited to French mathematician Pierre Sarrus.

Solution: Let's start by calculating the trace and determinant of A so we will have those for later. First, let's note that:

$$\text{tr}(A) = 0 + 2 + 0 = 2.$$

Then, to calculate the determinant of A we will employ a row-swap between R_1 and R_3 which changes the sign of the determinant:

$$\det(A) = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -2^3 = -8.$$

Next, we will calculate the eigenvalues and their corresponding eigenvectors. That is, we will find the values of λ which satisfy: $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix}$$

Here we will employ the method of Sarrus to calculate the 3×3 determinant:

$$\det(A - \lambda I) = (-\lambda)(2-\lambda)(-\lambda) + 0(0)(2) + 2(0)(0) - (-\lambda)(0)(0) - (0)(0)(-\lambda) - 2(2-\lambda)(2)$$

$$\implies \det(A - \lambda I) = \lambda^2(2-\lambda) - (2-\lambda) = (2-\lambda)(\lambda^2 - 4) = (2-\lambda)(\lambda+2)(\lambda-2).$$

Thus we have a repeated root in our characteristic polynomial and only 2 distinct eigenvalues: $\lambda_{1,2} = 2, \lambda_3 = -2$.

Now, we are left with finding eigenvectors for each of the eigenvalues by solving $(A - \lambda I)\vec{x} = \vec{0}$:

- $\lambda_{1,2} = 2$

$$(A - 2I)\vec{x} = \vec{0} \implies \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} \vec{x} = \vec{0}.$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_1 + R_3} \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We have 1 pivot and 2 free variables, which means that $x_3 = t$ and $x_2 = s$. And Row 1 gives us $x_1 = t$. Thus we have two eigenvectors that correspond to each of these special solutions:

$$\vec{x}_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

- $\lambda_3 = -2$.

$$(A-(-2)I)\vec{x} = \vec{0} \implies \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \vec{x} = \vec{0}.$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_1 - R_3} \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We have 2 pivots and 1 free variable, $x_3 = t$ which implies that $x_1 = -t$. Then we have only a single eigenvector:

$$\vec{x}_3 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}.$$

Finally, let's notice that we satisfy the condition on the trace and the determinant:

- Trace of A is equal to the sum of the eigenvalues:

$$\text{tr}(A) = 2 \iff \lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 - 2 = 2.$$

- Determinant of A is equal to the product of the eigenvalues:

$$\det(A) = -8 \iff \lambda_1\lambda_2\lambda_3 = (2)(2)(-2) = -8.$$

The Method of Sarrus

Pierre Frédéric Sarrus (10 March 1798, Saint-Affrique - 20 November 1861) was a French mathematician. Sarrus was professor at the University of Strasbourg, France (1826-1856) and member of the Academy of Sciences in Paris (1842). He discovered a mnemonic rule for solving the determinant of a 3-by-3 matrix, named Sarrus' scheme, which provides an easy-to-remember method of working out the determinant of a 3-by-3 matrix (as illustrated below)

det(M)= $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \left(a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{33}\right) - \left(a_{13}a_{22}a_{31} + a_{23}a_{32}a_{11} + a_{33}a_{12}a_{31}\right)$