$$U = \begin{bmatrix} 1/\sqrt{3} & i/\sqrt{2} & u_1 \\ 1/\sqrt{3} & 0 & u_2 \\ i/\sqrt{3} & 1/\sqrt{2} & u_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U^H U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -i/\sqrt{3} \\ -i/\sqrt{2} & 0 & 1/\sqrt{2} \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & i/\sqrt{2} & u_1 \\ 1/\sqrt{3} & 0 & u_2 \\ i/\sqrt{3} & 1/\sqrt{2} & u_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{i}{\sqrt{6}} - \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{3}}(u_1+u_2-iu_3) \\ 0 & 1 & \frac{1}{\sqrt{2}}(-iu_1+u_3) \\ * & * & \bar{u}_1 u_1 + \bar{u}_2 u_2 + \bar{u}_3 u_3 \end{bmatrix}$$

$$\bar{u}_1 u_1 + \bar{u}_2 u_2 + \bar{u}_3 u_3 = \vec{u}^T \vec{u} = 1$$

(1) $= 0$

(2) $= 0$

(3) $= 1$

$$(U^H U)^H = U^H (U^H)^H = U^H U$$

Hermitian

$$U = [\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3]$$

$$U^H = \begin{bmatrix} \vec{u}_1^H \\ \vec{u}_2^H \\ \vec{u}_3^H \end{bmatrix} \Rightarrow U^H U = \begin{bmatrix} \vec{u}_1^H \vec{u}_1 & \vec{u}_1^H \vec{u}_2 & \\ & & \\ & & \end{bmatrix}$$

$\|\vec{u}_1\|^2$

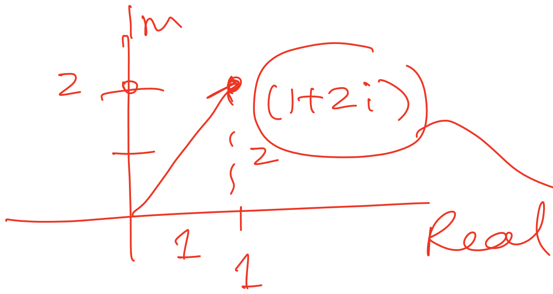
$\|$

$$\|\vec{u}_1\|^2 = \vec{u}_1^H \vec{u}_1$$

$1+2i$

$$\vec{u}_1^H \vec{u}_2 = 0$$

inner product



$1^2+2^2=5$

$(1+2i)(\overline{1+2i})$

$=(1+2i)(1-2i)$

$=1+\cancel{2i}-\cancel{2i}-4i^2$

$=5$

Similar Matrices

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Can A be similar to $A+I = B$

~~Hint~~: They can never be symmetric

$\lambda, \vec{x}$ are an eigenvalue & eigen. vect.

(A is similar to B.)

$M \Leftarrow$

$$\boxed{B = M^{-1}AM}$$

Suppose $\vec{x}$, λ for A

$BM^{-1} = M^{-1}AMM^{-1}$

$= M^{-1}A$

$BM^{-1}\vec{x} = M^{-1}A\vec{x}$

$BM^{-1}\vec{x} = M^{-1}(\lambda\vec{x})$

$\det(A+\lambda I)=0$

$\det(A-\lambda I)=\det(B-\lambda I)$

$BM^{-1}\vec{x}=\lambda M^{-1}\vec{x}$

$B\vec{y}=\lambda\vec{y}$

$\Rightarrow \det((A+I)-\lambda I)=0$

$=\det(A-(\lambda-1)I)$

$A+I$

$\Rightarrow \vec{x},\ \lambda\ ;\ A\vec{x}=\lambda\vec{x}$

$(A+I)\vec{x}=A\vec{x}+I\vec{x}$

$=\lambda\vec{x}+\vec{x}$

$=(\lambda+1)\vec{x}$

(If $(\vec{x},\lambda)$ are an eigenpair for A then $(\vec{x},(\lambda+1))$ are an eigenpair for $A+I$)

A ; $\lambda=2$ is an eigenvalue of A

$(A+2I) \Rightarrow \lambda+2$

$A\vec{x} = 2\vec{x}$

$(A+2I)\vec{x} = \underline{A\vec{x}} + 2I\vec{x}$

$= \lambda\vec{x} + 2\vec{x}$

$= (\lambda+2)\vec{x}$

$(A+3I) \Longrightarrow \lambda+3$

A has 3 eigenvalues 0, 1, 2

$(A+3I)$ what are its eigenvalues

$A\vec{x} = \vec{x} \Leftarrow$ def of eigenval 1

$(A+3I)\vec{x} = A\vec{x} + 3I\vec{x}$

$= \vec{x} + 3\vec{x}$

$= (1+3)\vec{x}$

$= 4\vec{x}$

7. A has eigenvalues 0, 1, 2

find the eigenvalues of

$$B = \boxed{A(A-I)(A-2I)} = A^3 - 3A^2 + 2A$$

$$\Rightarrow A\vec{x} = \lambda\vec{x}$$

$A^2 \Rightarrow$

$$A^2\vec{x} = A(A\vec{x})$$
$$= A\lambda\vec{x}$$
$$= \lambda A\vec{x}$$
$$= \lambda^2\vec{x}$$

$A^k \Rightarrow \lambda^k$

$A + 5I$

$$(A^3 - 3A^2 + 2A)\vec{x}$$
$$= A^3\vec{x} - 3A^2\vec{x} + 2A\vec{x}$$
$$= \lambda^3\vec{x} - 3\lambda^2\vec{x} + 2\lambda\vec{x}$$
$$= (\lambda^3 - 3\lambda^2 + 2\lambda)\vec{x}$$

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \lambda_1 = 0, \quad \lambda_2 = 3$$

$\boxed{\text{Diagonalize } A \text{ w/ an orthogonal matrix}}$

① Symmetric matrices are always diagonalizable orthogonal ✓

⇐ ②

don't know from symmetric

A; λ_1 $\vec{x}_1$ λ_2 $\vec{x}_2$ $\lambda_1 \neq \lambda_2$

linearly independent

$$(A - \cancel{\lambda I})\vec{x} = A\vec{x} = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cc} \uparrow & \uparrow \\ x_2 & x_3 \\ \| & \downarrow \\ s & t \end{array}$$

$x_1 = -t - s$

$$\Rightarrow \vec{x} = \begin{bmatrix} -t-s \\ s \\ t \end{bmatrix} \Rightarrow s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \; t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = s\begin{bmatrix}-1\\1\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$A = S^{-1}\Lambda S$$

$$\text{span}\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$$

$$\vec{q}_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|} \qquad \vec{Q} = \vec{x}_2 - \vec{x}_2^T \vec{q}_1 \, \vec{q}_1$$

$$\vec{q}_2 = \frac{\vec{Q}}{\|\vec{Q}\|}$$

Why do we only care about P.D. of Symmetric matrice.

$\vec{x}^T A \vec{x}$ ← Real ← Symmetric matric

$\vec{x}^T A \vec{x} > 0$ ← Positive Def

only positive eigenvalues

$$\vec{X} = c_1 \vec{X_1} + c_2 \vec{X_2} + \cdots + c_n \vec{X_n}$$

orthogonal

$$\vec{X}^T (A\vec{X}) = \lambda_1 c_1^2 + \cancel{\lambda_2} c_2^2 + \cdots$$

$$= \lambda_1 c_1^2 + \lambda_2 c_2^2 \cdots + \lambda_n c_n^2$$

$$(c_1\vec{X_1} + c_2\vec{X_2} + c_3\vec{X_3})^T A (c_1\vec{X_1} + c_2\vec{X_2} + c_3\vec{X_3})$$

$$c_i \vec{X_i}^T A c_j \vec{X_j}$$

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$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

SVD

$$A = U \Sigma V^T$$

- V = Eigenvectors of A^TA

U = " " of AA^T

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

① Determine eigen values & eigenvectors of A^TA

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \lambda_r > 0 = $$

$$\boxed{A\vec{v}_j = \sigma_j \vec{u}_j}$$