You're expected to work on the discussion problems before coming to the lab. Discussion session is not meant to be a lecture. TA will guide the discussion and correct your solutions if needed. We will not release 'official' solutions. If you're better prepared for discussion, you will learn more. TAs will record names of the students who actively engage in discussion and report them to the instructor; they are also allowed to give some extra points to those students at their discretion. The instructor will factor in participation in final grade.

1. (Advanced) Let f be a flow in a network, and let α be a real number. The scalar flow product, denoted αf, is a function from VxV to $\mathbb{R}$ defined by

   $$(\alpha f)(u, v) = \alpha \cdot f(u, v).$$

   Prove that the flows in a network form a *convex set*. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1-\alpha) f_2$ for all α in the range $0 \leq \alpha \leq 1$.
   Sol.
   To see that the flows form a convex set, we show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1-\alpha) f_2$ for all α in the range $0 \leq \alpha \leq 1$.

   For capacity contraints, first observe that $\alpha \leq 1$ implies that 1 - $\alpha \geq 0$. Thus, for any $u, v \in V$, we have

   $$\begin{aligned} \alpha f_1(u, v) + (1-\alpha) f_2(u, v) &\geq 0 \cdot f_1(u, v) + 0 \cdot (1-\alpha) f_2(u, v) \\ &= 0. \end{aligned}$$

   Since $f_1(u, v) \leq c(u, v)$ and $f_2(u, v) \leq c(u, v)$, we also have

   $$\begin{aligned} \alpha f_1(u, v) + (1-\alpha) f_2(u, v) &\leq \alpha c(u, v) + (1-\alpha) c(u, v) \\ &= (\alpha + (1-\alpha)) c(u, v) \\ &= c(u, v). \end{aligned}$$

   For flow conservation, observe that since f_1 and f_2 obey flow conservation, we have $\sum_{v \in V} f_1(v, u) = \sum_{v \in V} f_1(u, v)$ and $\sum_{v \in V} f_2(v, u) = \sum_{v \in V} f_2(u, v)$ for any $u \in V - \{s, t\}$. We need to show that

   $$\sum_{v \in V} (\alpha f_1(v, u) + (1-\alpha) f_2(v, u)) = \sum_{v \in V} (\alpha f_1(u, v) + (1-\alpha) f_2(u, v))$$

   for any $u \in V - \{s, t\}$. We multiply both sides of the equality for f_1 by α, multiply both sides ofteh equality for f_2 by 1 - α, and add the left-hand and right-hand sides ot the resulting equalities to get

   $$\alpha \sum_{v \in V} f_1(v, u) + (1-\alpha) \sum_{v \in V} f_2(v, u) = \alpha \sum_{v \in V} f_1(u, v) + (1-\alpha) \sum_{v \in V} f_2(u, v)).$$

   Observing that

$$
\begin{aligned}
\alpha \sum_{v \in V} f_1(v,u) + (1-\alpha) \sum_{v \in V} f_2(v,u) &= \sum_{v \in V} \alpha f_1(v,u) + \sum_{v \in V} (1-\alpha) f_2(v,u) \\
&= \sum_{v \in V} (\alpha f_1(v,u) + (1-\alpha) f_2(v,u))
\end{aligned}
$$

and, likewise, that

$$
\alpha \sum_{v \in V} f_1(u,v) + (1-\alpha) \sum_{v \in V} f_2(u,v) = \sum_{v \in V} (\alpha f_1(u,v) + (1-\alpha) f_2(u,v))
$$

completes the proof that flow conservation, and thus that flows form a convex set.

2. (Basic) Professor Adam has two children who, unfortunately, dislike each other. The problem is so severe that not only do they refuse to walk to school together, but in fact each one refuses to walk on any block that the other child has stepped on that day. The children have no problem with their paths crossing at a corner. Fortunately both the professors house and the school are on corners, but beyond that he is not sure if it is going to be possible to send both of his children to the same school. The professor has a map of his town. Show how to formulate the problem of determining whether both his children can go to the same school as a maximum-flow problem.
Sol.
Create a vertex for each corner, and if there is a street between corners u and v, create directed edges (u, v), (v, v_u) and (v_u, u), where v_u is a unique vertex created for only this street between corners u and v. (We need vertex v_u to avoid antiparallel edges. Note that if there is a street between corners u and v and between corners x and v, then the vertices v_u and v_x are distinct.) Set the capacity of each edge to 1. Let the source be the corner on which the professors house sits, and let the sink be the corner on which the school is located. We wish to find a flow of value 2 that also has the property that $f(u, v)$ is an integer for all vertices u and v. Such a flow represents two edge-disjoint paths from the house to the school.

3. (Basic) In Figure 26.1(b) (see CLRS page 710), what is the flow across the cut $(\{s, v_2, v_4\}, \{v_1, v_3, t\})$? What is the capacity of this cut?
Sol.
The flow across the cut $(\{s, v_2, v_4\}, \{v_1, v_3, t\})$ is 19 (23 units going from $(\{s, v_2, v_4\}$ to $\{v_1, v_3, t\}$ and 4 units going back), and the capacity of the cut is 31.

4. (Intermediate) Suppose that we redefine the residual network to disallow edges into s. Argue that the procedure FORD-FULKERSON still correctly computes a maximum flow.

Sol.
Let G_f be the residual network just before an iteration of the **while** loop of FORD-FULKERSON, and let E_s be the set of residual edges of G_f into s. Well show that the augmenting path p chosen by FORD-FULKERSON does not include an edge in E_s. Thus, even if we redefine G_f to disallow edges in E_s, the path p still remains an augmenting path in the redefined network. Since p remains unchanged, an iteration of the **while** loop of FORD-FULKERSON updates the flow in the same way as before the redefinition. Furthermore, by disallowing some edges, we do not introduce any new augmenting paths. Thus, FORD-FULKERSON still correctly computes a maximum flow.

Now, we prove that FORD-FULKERSON never chooses an augmenting path p that includes an edge $(v, s) \in E_s$. Why? The path p always starts from s, and if p included an edge (v, s), the vertex s would be repeated twice in the path. Thus, p would no longer be a simple path. Since FORD-FULKERSON chooses only simple paths, p cannot include (v, s).

5. (Advanced) Suppose that you wish to find, among all minimum cuts in a flow network G with integral capacities, one that contains the smallest number of edges. Show how to modify the capacities of G to create a new flow network G' in which any minimum cut in G' is a minimum cut with the smallest number of edges in G.
Sol.
Let (S, T) and (X, Y) be two cuts in G (and G'). Let c' be the capacity function of G'. One way to define c' is to add a small amount δ to the capacity of each edge in G. That is, if u and v are two vertices, we set

$$c'(u, v) = c(u, v) + \delta \ .$$

Thus, if $c(S, T) = c(X, Y)$ and (S, T) has fewer edges than (X, Y), then we would have c'(S,T) ¡ c'(X,Y). We have to be careful and choose a small δ, lest we change the relative ordering of two unequal capacities. That is, if $c(S, T) < c(X, Y)$, then no matter many more edges (S, T) has than (X, Y), we still need to have $c'(S, T) < c'(X, Y)$. With this definition of c', a minimum cut in G' will be a minimum cut in G that has the minimum number of edges.

How should we choose the value of δ? Let m be the minimum difference between capacities of two unequal-capacity cuts in G. Choose $\delta = m/(2|E|)$. For any cut (S, T), since the cut can have at most —E— edges, we can bound $c'(S, T)$ by

$$c(S, T) \leq c'(S, T) \leq c(S, T) + |E| \cdot \delta \ .$$

Let $c(S, T) < c(X, Y)$. We need to prove that $c'(S, T) < c'(X, Y)$. We have

$$
\begin{aligned}
c'(S,T) &\leq c(S,T) + |E| \cdot \delta \\
&= c(S,T) + m/2 \\
&< c(X,Y) \quad (since c(X,Y) - c(S,T) \geq m) \\
&\leq c'(X,Y).
\end{aligned}
$$

Because all capacities are integral, we can choose $m = 1$, obtaining $\delta = 1/2|E|$. To avoid dealing with fractional values, we can scale all capacities by $2|E|$ to obtain

$$c'(u,v) = 2|E| \cdot c(u,v) + 1 \ .$$

6. (Basic) Let $G = (V, E)$ be a bipartite graph with vertex partition $V = L \cup R$, and let G' be its corresponding flow network. Give a good upper bound on the length of any augmenting path found in G' during the execution of FORD-FULKERSON.
Sol.
By definition, an augmenting path is a simple path $s \to t$ in the residual network G'_f. Since G has no edges between vertices in L and no edges between vertices in R, neither does the flow network G' and hence neither does G'_f. Also, the only edges involving s or t connect s to L and R to t. Note that although edges in G' can go only from L to R, edges in G'_f can also go from R to L.

Thus any augmenting path must go

$$s \to L \to R \to \cdots \to L \to R \to t \ ,$$

crossing back and forth between L and R at most as many times as it can do so without using a vertex twice. It contains s, t, and equal numbers of distinct vertices from L and Rat most $2 + 2{\cdot}min(|L|, |R|)$ vertices in all. The length of an augmenting path (i.e., its number of edges) is thus bounded above by $2 \cdot min(|L|, |R|) + 1$.