Discussion Section: Week #3
Due: By 11:59pm the day of your Discussion Section

Instructions:

You are allowed to work alone or in small groups. But if you work in a group, the work you turn in should be your own. Each student will submit a physical write-up as a PDF document to CatCourses **by 11:59 pm of your discussion section day**.

Discussion assignments will be graded partially for completeness but a subset of the problems will be graded for correctness. Providing the correct answer, without justification, is not considered complete. For credit you **must** either show steps (if it's a calculation problem) or explain/justify your reasoning (if it's a short answer problem).

Problem Set:

1. Given the matrix
$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$
determine the matrix E_{21} such that $E_{21}A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution:
$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

2. The parabola $y = a + bx + cx^2$ goes through the point $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknown (a, b, c).

Solution: By substituting the $(x, y)-$value pairs into the parabola equation
$$y = a + bx + cx^2,$$

we obtain a system of equations

$$\begin{aligned} a + 1b + 1^2c &= 4 \\ a + 2b + 2^2c &= 8 \\ a + 3b + 3^2c &= 14 \end{aligned}$$

or

$$\begin{aligned} a + 1b + 1c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$$

.

Converting the system in the matrix-vector form $A\vec{x} = \vec{b}$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 14 \end{bmatrix}.$$

To solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we apply Gaussian elimination/elementary row operations to this augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{array}\right]$.

Thus, we have

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{array}\right] &\xrightarrow[R_3^* = R_3 - R_1]{R_2^* = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \\ &\xrightarrow{R_3^* = R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right] \\ &\xrightarrow{R_3^* = \frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]. \end{aligned}$$

From the reduced system $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$, we see that the third rows gives us $1c = 1$, so $c = 1$.

Now the second row of the reduced system is $1b + 3c = 4$, so

$$b = 4 - 3c = 4 - 3(1) = 1.$$

Finally, the first row of the reduced system is $1a + 1b + 1c = 4$, so

$$a = 4 - b - c = 4 - 1 - 1 = 2.$$

Thus, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. You can verify it is the solution to the system of equations. (How?)

3. Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be two (column) vectors in $\mathbb{R}^n$. Then, the dot product of $\vec{x}$ and $\vec{y}$ is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Show that the inner product $\vec{x}^T y$ equals the dot product $\vec{x} \cdot \vec{y}$. Here, $\vec{x}^T$ denotes the transpose of the vector $\vec{x}$. If $\vec{x}$ is a column vector, then $\vec{x}^T$ is a row vector. Since $\vec{x}^T$ is a row vector and $\vec{y}$ is a column vector, $\vec{x}^T \vec{y}$ is just row times column, which is usually how we compute matrix multiplication.

Solution:

$$\begin{aligned} \vec{x}^T y &= \underbrace{\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}}_{\text{row}} \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{\text{column}} \\ &= x_1 y_1 + \cdots + x_n y_n \\ &= \vec{x} \cdot \vec{y}. \end{aligned}$$