Chapter 5.5-5.6: Review

Symmetric Matrix

A real-vaued $n \times n$ matrix A is called **symmetric** if:

$$A^T = A.$$

Hermitian Matrix

A $n \times n$ matrix A with (possibly) complex values is called **Hermitian** if:

$$A^H = \overline{A}^T = A.$$

Important Properties

- A symmetric (or Hermitian) matrix has **real eigenvalues**.
- Eigenvectors from different eigenvalues for a symmetric (or Hermitian) matrix are **orthogonal**.

Final

- Tuesday May 10 8am – 11am
- Same length ⇒ 75 minute exam
- Stuff since Midterm 2.

⇒ Drop a Midterm. ✓

⇒ Resurrection Final ✓ ~~F~~

⇒

Chapter 5.5-5.6: Review

Orthogonal Matrix

A real-valued $n \times n$ matrix Q is called **orthogonal** if its columns are orthonormal.

$$Q^{-1} = Q^T.$$

Unitary Matrix

A $n \times n$ matrix U with complex values is called **unitary** if its columns are orthonormal.

$$U^{-1} = U^H.$$

Cool Properties

1. Orthogonal and Unitary matrices preserve lengths and inner products.
2. All eigenvalues λ of an Orthogonal or Unitary matrix have $|\lambda| = 1$.
3. Eigenvectors corresponding to different eigenvalues are orthogonal.

For Proofs, see Week 13 Thursday Notes or Chapter 5.5/5.6.

Chapter 5.6: The Spectral Theorem

The Spectral Theorem

Every real symmetric A can be diagonalized by an orthogonal matrix Q. Every Hermitian matrix A can be diagonalized by a unitary matrix U.

$$\text{Real: } Q^{-1}AQ = \Lambda \text{ or } A = Q\Lambda Q^T$$

$$\text{Complex: } U^{-1}AU = \Lambda \text{ or } A = U\Lambda U^H.$$

The columns of Q (or U) contain orthonormal eigenvectors of A.

- If the matrix A is real and symmetric, the eigenvalues and eigenvectors are **real** at every step!

Review: Section 6.1/2 Positive Definite Matrices

Positive Definite Matrix

Each of the following is a necessary and sufficient (i.e. if and only if) condition for a real symmetric matrix A to be **positive definite**.

1. (i) $\vec{x}^T A\vec{x} > 0$ for all non-zero real vectors $\vec{x}$.
2. (ii) All the eigenvalues of A satisfy $\lambda_i > 0$.
3. (iii) All the upper left submatrices A_k have positive determinants.
4. (iv) All the pivots (without row exchanges) satisfy $d_k > 0$.
5. (v) There is a matrix R with independent columns such that $A = R^T R$.

We note that condition (i) is just the definition we have of positive definite matrices.

We will prove (ii) implies (i). (The other conditions still cool, but less relevant for the moment.)

Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

- Suppose that A is a real-symmetric matrix with only positive eigenvalues. We will show that $\vec{x}^T A\vec{x} > 0$ for all choices of $\vec{x}$.
- Since A is symmetric, we know that we have a full set of orthonormal eigenvectors (Spectral Theorem).
- Let $\vec{x}$ be any vector in, we know it can be expressed uniquely as a linear combination of eigenvectors $\vec{x}_i$:
$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n.$$
- Then we have:

$$A\vec{x} = c_1A\vec{x}_1 + c_2A\vec{x}_2 + \cdots + c_nA\vec{x}_n = c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 + \cdots + c_n\lambda_n\vec{x}_n.$$

- To calculate $\vec{x}^T A\vec{x}$ we use $\vec{x}_i^T\vec{x}_j = 0$ if $i \neq j$ and 1 if $i = j$.

$$\vec{x}^T A\vec{x} = \left(c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n\right)\left(c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 + \cdots + c_n\lambda_n\vec{x}_n\right).$$

$$\implies \vec{x}^T A\vec{x} = c_1^2\lambda_1 + c_2^2\lambda_2 + \cdots + c_n^2\lambda_n > 0$$

Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

Examples

2. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. Check that A is positive definite by
   a. finding all its eigenvalues;
   b. calculating the determinants of all upper left submatrices;
   c. finding all pivots;
   d. writing $\boxed{A = R^T R}$ where R is some matrix with independent columns.

(a) Finding all its eigenvalues:

$$0 = \det(A - \lambda I) = \left|\begin{matrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{matrix}\right| \begin{matrix} 2-\lambda & -1 \\ -1 & 2-\lambda \\ 0 & -1 \end{matrix}\right|.$$

$$= (2-\lambda)^3 + 0 + 0 - (2-\lambda) - (2-\lambda) = (2-\lambda)\left((2-\lambda)^2 - 2\right)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 2)$$

Thus the eigenvalues are:

$$\lambda = \{2, 2-\sqrt{2}, 2+\sqrt{2}\}$$

Since all eigenvalues are positive, we know that A is positive definite.

Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

(b) calculating the determinants of all upper left submatrices;

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- $A_1 = \begin{bmatrix} 2 \end{bmatrix}, \det(A_1) = 2 > 0$
- $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \det(A_2) = 4 - 1 = 3 > 0$
- $A_3 = A.$

$$\det(A) = \left| \begin{array}{ccc|cc} 2 & -1 & 0 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 \\ 0 & -1 & 2 & 0 & -1 \end{array} \right| = 2^3 - 2 - 2 = 8 - 4 = 4 > 0.$$

Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

- (b) finding all pivots;

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1/2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + (2/3)R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

Because A is symmetric, $A = LU = LDL^T$

$$A = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

The pivots are the values on the diagonal of U or in the diagonal matrix D and they are:

$$2, 3/2, 4/3$$

- (d) writing $A = R^T R$ where R is some matrix with independent columns. There are actually two choice depending on which decomposition we want to choose. To address this part we need our three matrices from above: $A = LDL^T = L\sqrt{D}\sqrt{D}L^T$

$$\sqrt{D} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3}/\sqrt{2} & 0 \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}.$$

$$R = \sqrt{D}L^T = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3}/\sqrt{2} & 0 \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T = \left(\sqrt{D}L^T\right)^T = L\sqrt{D}^T = L\sqrt{D}.$$

Thus, $\boxed{A = R^T R.}$ This is called the **Cholesky decomposition** and exists only because the diagonal matrix has positive values.

Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

(d) writing $A = R^T R$ where R is some matrix with independent columns. A second choice would be to use the fact that we can orthogonally diagonalize the matrix A:

$$A = Q \Lambda Q^T = Q \sqrt{\Lambda} \sqrt{\Lambda Q^T}$$

Thus as before, $A = R^T R$.

Week 14: Tuesday. 6.1-6.2: Positive Definite Matrices

Positive definite matrices are important for optimization, and lots of other things, but they are not the only interesting case we have to consider.

Positive Semidefinite

Each of the following tests is a necessary and sufficient condition for a symmetric matrix A to be positive semidefinite:

1. (i) $\vec{x}^T A \vec{x} \geq 0$ for all vectors $\vec{x}$ (Defn of positive **semidefinite**)
2. (ii) All the eigenvalues of A satisfy $\lambda_i \geq 0$
3. (iii) No principle submatrices have negative determinants.
4. (iv) No pivots are negative.
5. (v) There is a matrix R, possibly with dependent columns, such that $A = R^T R$.

Note you can also be **negative definite** (all $\lambda_i < 0$), **negative semi definite** (all $\lambda_i \leq 0$) or **indefinite**.

Examples

$\lambda = 0, 3, 3$ pos. semi defin.

3. Decide whether $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is positive definite, negative definite, semidefinite, or indefinite.

Find the eigenvalues

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} \begin{matrix} 2-\lambda & -1 \\ -1 & 2-\lambda \\ -1 & -1 \end{matrix}$$

$= (2-\lambda)^3 - 1 - 1 - (2-\lambda) - (2-\lambda) - (2-\lambda)$

$= (2-\lambda)^3 - 2 - 3(2-\lambda)$

$= (4-4\lambda+\lambda^2)(2-\lambda) - 2 \underbrace{- 6}_{-8} + 3\lambda$

$\cancel{8} - 8\lambda + 2\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - \cancel{8} + 3\lambda$

$= \boxed{-\lambda^3 + 6\lambda^2 - 9\lambda}$

$= \lambda(-\lambda^2 + 6\lambda - 9)$

$= -\lambda(\lambda^2 - 6\lambda + 9)$

$= -\lambda(\lambda - 3)^2$

Week 14: Thursday. 6.3: Singular Value Decomposition

Course Goals

After studying section 6.1-6.2: Positive Definite Matrices, you should

1. *Understand the singular value decomposition (SVD).*
2. *Understand the applicability of the SVD to other fields.*

Course Outcomes

To manifest that you have reached the above course goals, you should be able to

1. *Be able to calculate the SVD of a matrix.*
2. *Understand how to relate the SVD to the pseudoinverse of a matrix.*
3. *Be able to identify bases/dimensions of the 4 fundamental subspaces from the SVD of a matrix.*

Week 14: Thursday. 6.3: Singular Value Decomposition

Singular Value Decomposition

r=rank

Any $m \times n$ matrix A can be factored into:

$$A = \underbrace{U}_{m\times m}\overbrace{\Sigma}^{m\times n}\underbrace{V^T}_{n\times n} = (\text{orthogonal})(\text{diagonal})(\text{orthogonal}).$$

m×n

$\underset{m\times n}{A} \; \underset{n\times n}{A^T}$

- The columns of U ($m \times m$) are the eigenvectors of AA^T.
- The columns of V ($n \times n$) are the eigenvectors of $\cancel{AA^T}$. A^TA
- The r singular values on the diagonal of Σ ($m \times n$) are the square-roots of the non-zero eigenvalues of both AA^T and A^TA.
- We order the singular values $\boxed{\sigma_i \geq \sigma_{i+1}}$.

Week 14: Thursday. 6.3: Singular Value Decomposition

SVD for Positive Definite Matrices

If A is a positive definite matrix, then it is an $n \times n$ matrix and $\Sigma = \Lambda$ and we end up with:

$$A = U\Lambda V^T \text{ and } A^T = (U\Lambda V^T)^T = V\Lambda^T U^T = V\Lambda U^T = A.$$

$U = V$

Thus, this factorization is equivalent to the orthogonal diagonalization of our matrix:

$$A = Q\Lambda Q^T$$

Q orthogonal matrix of eigenvectors

$A \Rightarrow$ SPD $\Rightarrow \lambda_i > 0$

$\Rightarrow$ Symmetric matrices Always have enough eigenvectors.

Week 14: Thursday. 6.3: Singular Value Decomposition

SVD and Fundamental Subspaces

U and V are ~~orthonormal~~ orthogonal matrices and their columns are orthonormal bases for **all four fundamental subspaces**.

- First r columns of U are a basis for $C(A)$
- Last $(m-r)$ columns of U are a basis for $N(A^T)$
- First r columns of V are a basis for $C(A^T)$
- Last $(n-r)$ columns of V are a basis for $N(A)$.

$$A = U\Sigma V^T \implies \boxed{AV = U\Sigma}$$

Thus, when we map a column of V, $\vec{v}_j$ we end up with σ_j times the same column of U:

$$A\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$$

$$\boxed{A\vec{v}_j = \sigma_j \vec{u}_j}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ & & & \ddots \end{bmatrix}$$

Week 14: Thursday. 6.3: Singular Value Decomposition

SVD and Eigenvectors/Eigenvalues of A^TA and AA^T

- If $A = U\Sigma V^T$. Then

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T$$

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma V^T$$

$\Sigma\Sigma^T$ and $\Sigma^T\Sigma$ are both diagonal matrices and let's just call them Σ^2

- But this means:

$$(AA^T)U = U\Sigma^2 \text{ and } (A^TA)V = V\Sigma^2$$

- And this implies:

$$(AA^T)\vec{u}_i = \sigma_i^2\vec{u}_i \text{ and } (A^TA)\vec{v}_i = \sigma_i^2\vec{v}_i.$$

- Thus U and V are the eigenvectors of AA^T and A^TA respectively and the diagonal entries in Σ^2 are the eigenvalues.

Week 14: Thursday. 6.3: Singular Value Decomposition

Example

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$. Compute the singular values and the singular value decomposition of A.

- $AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

  $(AA^T)^T = (A^T)^T A^T = AA^T$ ✓

- $0 = \det(AA^T - \lambda I) = 4\lambda^2 - \lambda^3 = \lambda^2(4 - \lambda) \implies$ Eigenvalues $= 4, 0, 0$
- $A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ ✓
- $0 = \det(A^T A - \lambda I) = \lambda^2 - 4\lambda \implies$ Eigenvals $= 4, 0$
- They have the same non-zero eigenvalues! All non-zero are positive! (AA^T has an additional 0 eigenvalue!)

Week 14: Thursday. 6.3: Singular Value Decomposition

- Let's find the eigenvectors for the 2×2 matrix: $\sigma_1^2 = 4, \sigma_2^2 = 0$

$$\vec{0} = (A^T A - 4I)\vec{x} = \left[\begin{array}{cc|c} 2-4 & 2 & 0 \\ 2 & 2-4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

1 pivot, 1 free variable $\Longrightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Longrightarrow \vec{v}_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$

$$\vec{0} = (A^T A - 0I)\vec{x} = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

1 pivot, 1 free variable $\Longrightarrow \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Longrightarrow \vec{v}_2 = \frac{\vec{x}_2}{\|\vec{x}_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$

- We have our matrix:

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

- Note columns of V are orthogonal as we expect because $A^T A$ is symmetric.

Week 14: Thursday. 6.3: Singular Value Decomposition

- Now to find eigenvectors of AA^T: $\sigma_1^2 = 4, \sigma_2^2 = \sigma_3^2 = 0$
- $\sigma_1^2 = 4$

$$\vec{0} = (AA^T - 4I)\vec{x} = \left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right].$$

2 Pivots, 1 Free variable: $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \implies \vec{u}_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$

- $\sigma_2^2 = \sigma_3^2 = 0$

$$\vec{0} = (AA^T - 0I)\vec{x} = \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1 Pivot, 2 Free Variables:

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \implies \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, U = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}.$$

Let's check $A = U\Sigma V^T$

$$\begin{aligned} U\Sigma V^T &= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \boxed{\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}} \\ &= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = A \checkmark \end{aligned}$$

Week 14: Thursday. 6.3: Singular Value Decomposition

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, U = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}.$$

4 0 4 0 0

Let's check $A^T = V\Sigma^T U^T$.

$$\begin{aligned} V\Sigma^T U^T &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = A^T. \end{aligned}$$

Week 14: Thursday. 6.3: Singular Value Decomposition

- How do we know the singular value decomposition exists?
- How do we actually get the singular value decomposition for a given matrix?
- We will look at a brief **constructive proof** where we show how to build it.
- In practice, we use computers to calculate the SVD - because most of the cases we care about are cases where the matrix in question is quite large.

Proof of the Singular Value Decomposition

Positive Semi-Definite Matrix

Let A be an $m \times n$ matrix, then the matrix $A^T A$ is positive semi-definite.

- We will show that all eigenvalues are positive or 0. Let λ and $\vec{x}$ be an eigenvalue and eigenvector of $A^T A$.

$$A^T A \vec{x} = \lambda \vec{x}.$$

Defn

- We know (because $A^T A$ is symmetric) that λ is real.
- Let's calculate $\|A\vec{x}\|^2$:

$$\|A\vec{x}\|^2 = (A\vec{x})^T (A\vec{x}) = \vec{x}^T \boxed{A^T A \vec{x}} = \lambda \vec{x}^T \vec{x} = \lambda \|\vec{x}\|^2.$$

Thus, $\lambda = \|A\vec{x}\|^2 / \|\vec{x}\|^2 \geq 0 \implies \lambda \geq 0.$

- Thus $A^T A$ is positive semi-definite.

Proof of the Singular Value Decomposition

Rank of $A^T A$

Let A be an $m \times n$ matrix with rank r, then $A^T A$, which is an $n \times n$ matrix also has rank r.

- We know A^T also has rank r.
- We know that the rank of A is equal to $n - \dim(N(A))$. Similarly the rank of $A^T A$ is equal to $n - \dim N(A^T A)$.
-

Proof of the Singular Value Decomposition

- Let A be an $m \times n$ matrix. Then $A^T A$ is a symmetric $n \times n$, matrix.
- Therefore it's eigenvalues are real and it has an orthonormal set of eigenvectors. Let's put these into the columns of an orthogonal matrix V.
- Assume we have ordered the eigenvalues according to:

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n.$$

- Let r be the rank of A. It is also the rank of $A^T A$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\qquad U = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$

$$\boxed{A\vec{v}_j = \sigma_j \vec{u}_j}$$

$A \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} =$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$