Homework Assignment #9

Remember, this Homework Assignment is **not collected or graded**! But it is in your best interest to do it as the this material is designed to be a review for Midterm #2.

Chapter 2: Review Questions

1. Let A be an $m \times n$ matrix with rank r. What do you know about $C(A)$ and how r is related to m and n when the number of solutions to $A\vec{x} = \vec{b}$ behaves as follows.
   (a) 0 or 1, depending on $\vec{b}$.
   (b) ∞ independent on $\vec{b}$.
   (c) 0 or ∞ depending on $\vec{b}$
   (d) 1 regardless of $\vec{b}$

2. Consider the following matrix A and $\vec{b}$:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   (a) Under what conditions on $\vec{b}$ does $A\vec{x} = \vec{b}$ have a solution?
   (b) Find the general solution to $A\vec{x} = \vec{b}$ when a solution exists.
   (c) Find a basis for the column space of A.
   (d) What is the rank of A^T?

3. Suppose that the following depicts $PA = LU$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & -3 & 2 \\ 2 & -1 & 4 & 2 & 1 \\ 4 & -2 & 9 & 1 & 4 \\ 2 & -1 & 5 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) What is the rank of A?
   (b) What is a basis for the row space of A?
   (c) True or False: Rows 1, 2, 3 of A are linearly independent.
   (d) What is a basis for the column space of A?
   (e) What is the dimension of the left nullspace of A?
   (f) What is the general solution to $A\vec{x} = \vec{0}$?

Chapter 3: Review Questions

4. Construct the projection matrix P which projects vectors onto the space spanned by $(1, 1, 1)$ and $(0, 1, 3)$.

5. Find all 2 by 2 orthogonal matrices who have entries that are only 0 and 1.

6. What point on the plane $x + y - z = 0$ is the closest to $\vec{b} = (2, 1, 0)^T$.

7. Use Gram-Schmidt to construct an orthonormal pair $\vec{q}_1$ and $\vec{q}_2$ from the vectors:

$$\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Express $\vec{x}$ and $\vec{y}$ as a linear combination of $\vec{q}_1$ and $\vec{q}_2$ and determine the QR factorization for the matrix A, the 4 by 2 matrix whose columns consist of $\vec{x}$ and $\vec{y}$.

8. If Q is an orthogonal matrix, is Q^3 and orthogonal matrix?

9. For any A, $\vec{b}$, $\vec{x}$ and $\vec{y}$ show that:
   (a) If $A\vec{x} = \vec{b}$ and $\vec{y}^T A = \vec{0}$ then show, $\vec{y}^T \vec{b} = 0$.
   (b) If $A\vec{x} = 0$ and $A^T \vec{y} = \vec{b}$ then $\vec{x}^T \vec{b} = 0$.

10. Let $A = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$ and let V be the nullspace of A. Find a basis for V and a basis for $V^\perp$.

Chapter 4 (Section 4.1): Review Questions

11. If $B = M^{-1}AM$ find $\det(B)$ in terms of $\det(A)$. What is $\det(A^{-1}B)$?

12. Use row operations to simplify and compute these determinants:
   (a) Find $\det(A)$ when $A = \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}$.
   (b) Find $\det(A)$ when $A = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}$.
   (c) Consider the following LU factorization of the matrix A.

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

   Find the determinants of L, U, A, $U^{-1}L^{-1}$ and $U^{-1}L^{-1}A$.

Chapter 5 (Sections 5.1 - 5.3): Review Questions

13. Find the eigenvalues and eigenvectors and diagonalize each of the following two matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix}.$$

Use the diagonalization to calculate A^{50} and B^{200}.

14. Find the determinants of A and A^{-1} if:

$$A = S \begin{bmatrix} \lambda_1 & 2 \\ 0 & \lambda_2 \end{bmatrix} S^{-1}.$$

15. If A has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$ that correspond respectively to eigenvectors:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

   (a) Find A.
   (b) Find the eigenvectors and eigenvalues of A^2.