Homework Assignment #1

This course requires you to be very familiar with matrix operations including vector operations and familiarity making mathematical arguments. As such, this first Homework Assignment will be an opportunity for you to review problems in these areas. Remember, this Homework Assignment is **not collected or graded**! But you are advised to do it anyway because the problems for Homework Quiz #1 will be heavily based on these problems!

1. Consider the following system of 3 equations and 3 unknowns.

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 - x_2 - 2x_3 &= -7 \\ 5x_1 + x_2 - x_3 &= 4. \end{aligned}$$

(a) Write the system as a matrix-vector equation: $A\vec{x} = \vec{b}$.

Solution: We simply separate the coefficients $a_{i,j}$ and the unknowns x_i.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 4 \end{bmatrix}$$

(b) Show that for all values of t, $x_1 = 1 - t, x_2 = 2 + 3t$ and $x_3 = 3 - 2t$ is a solution to the system.

Solution: There are several ways to solve this problem, but the most direct is to simply substitute $\vec{x} = [(1-t), (2+3t), (3-2t)]^T$ and carry out the vector matrix multiplication.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} (1-t) \\ (2+3t) \\ (3-2t) \end{bmatrix} = \begin{bmatrix} 1(1-t) + 1(2+3t) + 1(3-2t) \\ 1(1-t) - 1(2+3t) - 2(3-2t) \\ 5(1-t) + 1(2+3t) - 1(3-2t) \end{bmatrix}$$

$$= \begin{bmatrix} (1+2+3) + t(-1+3-2) \\ (1-2-6) + t(-1-3+4) \\ (5+2-3) + t(-5+3+2) \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 4 \end{bmatrix}$$

(c) Recall that each equation above represents a plane in 3D. Explain in words the geometric interpretation of the infinitely many solutions you demonstrated in part (b).

Solution: The solution to this system of equations is the intersection of all 3 planes. We know that the intersection of 3 planes can be empty, have a unique solution or have infinitely many solutions.

In this case, we can see we have infinitely many solutions and that those solutions depend on a single parameter t. Because it depends on a single parameter (rather than two different parameters) we are inclined to think about the intersection as a line. (As opposed to a plane which would depend on two different parameters.) Indeed when we write the solutions $\vec{x}$ in a slightly different way, the geometric interpretation of a line becomes more clear:

$$\vec{x} = \begin{bmatrix} (1-t) \\ (2+3t) \\ (3-2t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$

2. **Prove** that if a, b and c are all positive integers and $a^2 + b^2 = c^2$ then it is not possible for all three numbers to be odd. (Hint: Remember that an odd integer can be written as $2n + 1$ for some value of n whereas an even number can be written as $2n$ for some value of n. The phrasing of the problem suggests a proof by contradiction.)

Solution: We will assume that a, b and c are each odd numbers and that $a^2 + b^2 = c^2$ and we will try to find a contradiction. This contradiction will mean that our assumption is wrong and at least one of the 3 numbers must be even.

If a, b and c are odd, then each number can be written as:

$$a = 2n_a + 1, b = 2n_b + 1 \text{ and } c = 2n_c + 1$$

for positive numbers n_a, n_b and n_c.

Then let's square a and b and add them together:

$$a^2 + b^2 = (2n_a + 1)^2 + (2n_b + 1)^2 = (4n_a^2 + 4n_a + 1) + (4n_b^2 + 4n_b + 1) = 4n^a + 4n^b + 4n_a + 4n_b + 2.$$

Notice we can factor a 2 from each term in the expression:

$$a^2 + b^2 = 2(2n_a^2 + 2n_b^2 + 2n_a + 2n_b + 1).$$

As such $(a^2 + b^2)$ is an even number.

However, we note that c^2 is an odd number because:

$$c^2 = (2n_c + 1) = 4n_c^2 + 4n_c + 1 = 2(n_c^2 + 2n_c) + 1.$$

Since every number is either even or odd, it is not possible for $a^2 + b^2$ (which we showed was even) to be equal to c^2 (which we have shown must be odd). Thus we have a contradiction and at least one of the numbers a, b or c must be odd.

3. Consider the two following systems:

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & -2 \\ & & x_2 & & & = & 3 \\ & & & & 2x_3 & = & 4. \end{array} \tag{1}$$

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & -2 \\ -3x_1 & - & x_2 & + & x_3 & = & 5 \\ 3x_1 & + & 2x_2 & + & x_3 & = & 2. \end{array} \tag{2}$$

(a) Explain geometrically what these sets of equations represent according to the **row perspective**.

Solution: Each of these equations specifies a plane in three dimensions. A solution represents the geometric intersection of these 3 planes. As we have discussed, the intersection of planes can be either empty (no solution), a single point (one solution) or a line or a plane (infinitely many solutions).

Looking more closely at each case, we can investigate which of these options makes the most sense. Figure 1.5 in your textbook shows some of different ways our system can be singular (i.e., have no solution or infinitely many solutions.) Without doing calculations, it's hard to rule out these cases. But we can make a little progress.Two of the four scenarios involve having either two or all three planes parallel. We know that the normal vector to a plane in 3D is given by the coefficients of the variables. For example, the normal vector to the first plane is $(3, 2, -1)$ in both cases. In order to have 0 solutions we would need two of the planes to be parallel, i.e., their normal vectors would be a multiple of one another. That doesn't hold for either (1) or (2).

We could use an approach from vector calculus to find the line in common between a pair of planes, and then see if it either lies in the third plane (which would be 0 or infinite solutions) or intersects the third plane in a point. But as we will see later with Gaussian Elimination, we have better tools to identify the intersection.

(b) Write these equations in matrix vector form and explain what they represent according to the **column perspective**

Solution: Each of these systems can be written as a vector matrix equation and, with the laws of matrix multiplication, be expressed as a sum of columns.

First we look at (1):

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} \implies \begin{bmatrix} 3x_1 + 2x_2 - 1x_3 \\ 0x_1 + 1x_2 + 0x_3 \\ 0x_1 + 0x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

$$\implies x_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}.$$

Writing the system like this makes it clear that the matrix vector form shows that $A\vec{x}$ simply is a linear combination of the different columns of A. That is, the vector $\vec{b}$ corresponds to a sum of the columns of A with the coefficients in the sum given by the unknown values x_i. It's again easy to "pick out" the solution to x_3 and x_2.

For (2) we have a similar formulation:

$$\begin{bmatrix} 3 & 2 & -1 \\ -3 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} \implies \begin{bmatrix} 3x_1 + 2x_2 - 1x_3 \\ -3x_1 - 1x_2 + 1x_3 \\ 3x_1 + 2x_2 + 1x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

$$\implies x_1 \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

Here we observe something interesting. Since x_1, x_2 and x_3 can be any constants, we could do something like define a new $\hat{x_1} = 3x_1$ and instead solve the related system:

$$\hat{x_1} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}.$$

(c) Find the solution to equations (1) and explain why equations (1) are easier to solve than equations (2).

Solution: Since the 2nd and 3rd rows of A involve only one of the unknown variables, we will start there. Row 2 gives us: $1x_2 = 3 \implies x_2 = 3$. Similarly, row 3 gives us: $2x_3 = 4 \implies x_3 = 2$

We can now substitute these values in for the first row:

$$3x_1 + 2x_2 - x_3 = -2 \implies 3x_1 + 2(3) - 1(2) = -2 \implies 3x_1 = -6 \implies x_1 = -2.$$

Thus we have a unique solution: $\vec{x} = [-2, 3, 2]^T$.
This "back-substitution" that we did for (1) was only possible because we had two rows which depended on only one unknown variable. This is not true and would not have been possible with (2).

(d) Demonstrate that the second set of equations can be made to look like to the first set of equations by performing the following operations:

- Add the first and second equations of (2) together and replace the second equation in (2) with the resulting equation.
- Subtract the first equation from the third equation given in (2) and replace the third equation in (2) with the resulting equation.

Solution: In order to do this, we're going to write (2) as an augmented matrix and carry out the row operations:

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ -3 & -1 & 1 & 5 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ 0 & 1 & 0 & 3 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right].$$

Thus we arrive at the same system as (1).

(e) Are the solutions to equations (1) and (2) the same? Explain why or why not.

Solution: We will soon learn that the row operations that we carried out are in-fact equivalent to multiplying by invertible matrices. That is, there exist invertible matrices $E_{1,2}$ and $E_{1,3}$ that define the specific operations that we carried out above. Then, we can consider row operations on our system $A\vec{x} = \vec{b}$ as multiplying both sides of the equation on the left as follows:

$$E_{1,3}E_{1,2}A\vec{x} = E_{1,3}E_{1,2}\vec{b}.$$

It is also possible to prove that a solution (x_1, x_2, x_3) of our original system, would also solve a system consisting of a linear combination (what we have done above) will be preserved. Suppose we have x, y, z that solve our original system. That is we know that,

$$\begin{bmatrix} a_{1,1}x + a_{1,2}y + a_{1,3}z \\ a_{2,1}x + a_{2,2}y + a_{2,3}z \\ a_{3,1}x + a_{3,2}y + a_{3,3}z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Suppose we add a multiple, α, of row 1 to row 2 as we did above, then we have the following:

$$\begin{bmatrix} a_{1,1}x + a_{1,2}y + a_{1,3}z \\ \alpha(a_{1,1}x + a_{1,2}y + a_{1,3}z) + (a_{2,1}x + a_{2,2}y + a_{2,3}z) \\ a_{3,1}x + a_{3,2}y + a_{3,3}z \end{bmatrix} = \begin{bmatrix} b_1 \\ \alpha b_1 + b_2 \\ b_3 \end{bmatrix}.$$

Notice that row 1 and row 3 of the equality are still satisfied, they never changed, but the second row is also satisfied because we know that:

$$\alpha(a_{1,1}x + a_{1,2}y + a_{1,3}z) = \alpha b_1 \text{ and } (a_{2,1}x + a_{2,2}y + a_{2,3}z) = b_2$$

because (x, y, z) are solutions.

4. Consider the following matrix and vector:

$$A = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Carry out the multiplication to verify that $A\vec{x} = 0$.

Solution: We can see that:

$$\begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(2) - 6(1) + 0(1) \\ 0(2) + 2(1) - 2(1) \\ 1(2) - 1(1) - 1(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) Let $\alpha \neq 0$ be any real number. **Prove** that for any vector $\vec{y} = \alpha\vec{x}$ we also have $A\vec{y} = 0$.

Solution: There are two ways to solve this problem, one is quick and uses the fact that multiplication by a scalar α is commutative with matrices and vectors we have the following:

$$A\vec{y} = A\left(\alpha\vec{x}\right) = \alpha A\vec{x} = \alpha\vec{0} = \vec{0}.$$

The second way is to simply carry out the same multiplication we did directly:

$$\begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2\alpha \\ 1\alpha \\ 1\alpha \end{bmatrix} = \begin{bmatrix} \alpha\left(3(2) - 6(1) + 0(1)\right) \\ \alpha\left(0(2) + 2(1) - 2(1)\right) \\ \alpha\left(1(2) - 1(1) - 1(1)\right) \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Investigate if there are any other vectors $\vec{z}$ that have the property $A\vec{z} = 0$. (There are many ways to answer this problem. The goal is that you explore.)

Solution: There are several ways one can think about this problem. First, we could directly look at the second row of the system $A\vec{x} = \vec{0}$ and observe:

$$2x_2 - 2x_3 = 0 \implies x_2 = x_3.$$

Then from the first row we have

$$3x_1 - 6x_2 = 0 \implies x_1 = 2x_2.$$

This lets see that we can write $x_1 = 2x_3$ and $x_2 = x_3$. Let's check that this satisfies the third row:

$$x_1 - x_2 - x_3 = 2x_3 - x_3 - x_3 = 0.$$

Let's go ahead and let $x_3 = t$ be a free variable, and we have just shown that any solution to $A\vec{x} = \vec{0}$ is of the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

We notice this is exactly the form of $\vec{y}$ in the previous part of the question.

5. A **upper triangular** matrix is one that has only 0's **below** the diagonal. See for example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

Mathematically, we can express the entries of a matrix A in terms of $a_{i,j}$ where i is the row and j is the column. (For example, in the example above $a_{2,3} = 4$.) As such, we can succinctly say that: A matrix A is **upper triangular** if $a_{i,j} = 0$ for $i > j$.

(a) **Prove** that the product of two 2×2 upper triangular matrices A and B is again a upper triangular matrix.

Solution: We first define A and B to be 2×2 upper triangular matrices which means that, $a_{1,2} = b_{1,2} = 0$.
We will prove the matrix $C = AB$ is also an upper triangular matrix.by showing $c_{1,2} = 0$.
By the definition of matrix multiplication we know that:

$$c_{1,2} = \sum_{k=1}^{2} a_{1,k} b_{k,2} = a_{1,1} b_{1,2} + a_{1,2} b_{2,2} = a_{1,1}(0) + (0) b_{2,2} = 0.$$

Since all other terms for C can be non-zero, we have shown that C must be an upper triangular matrix.

(b) **Prove** that the product of two $n \times n$ upper triangular matrices A and B is again a upper triangular matrix.
In this case it might be helpful to remember the formula for matrix multiplication. If A and B are $n \times n$ matrices and $C = AB$, then

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Solution: We first A and B to be $n \times n$ upper triangular matrices which means that, $a_{i,j} = b_{i,j} = 0$ when $i > j$.
We want to show the product $C = AB$ must also be an upper triangular matrix by showing that $c_{i,j} = 0$ when $i > j$.
Let $i > j$, then by the definition of matrix multiplication we have:

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

We know that $a_{i,k} = 0$ when $i > k$. It is more convenient to reverse the inequality and think about this as $a_{i,k} = 0$ when $k < i$. This means the terms $k = 1, 2, \ldots, (i-1)$ in the sum have $a_{i,k} = 0$.
However, we know that $b_{k,j} = 0$ when $k > j$. This means the terms in the sum for $k = (j+1), \ldots n$ have $b_{k,j} = 0$.
The only non-zero terms in the sum are from $k = i$ to $k = j$:

$$c_{i,j} = \sum_{k=i}^{j} a_{i,k} b_{k,j}.$$

However, because $i > j$, this sum is empty and we have: $c_{i,j} = 0$ whenever $i > j$. Thus, C is an upper triangular matrix.