Homework Quiz #2

1. Consider the following system of equations, you will determine how the solutions depend on k

$$\begin{aligned} x_1 + 1x_2 &= 3 \\ x_1 + k^2 x_2 &= -k. \end{aligned}$$

   (a) For what values of k does the system have 0 solutions?
   (b) For what values of k does the system have infinitely many solutions?
   (c) For $k = 2$ how many solutions does the system have?

Solution:
We will use Gaussian elimination the study how k changes the structure of our solutions:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & k^2 & -k \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & k^2 - 1 & (-k-3) \end{array}\right]$$

The structure of the system rests on the last row: $\left[\begin{array}{cc|c} 0 & k^2 - 1 & -(k+3) \end{array}\right]$.

   (a) For no solutions we need $(k^2 - 1) = 0$ and $-(k+3) \neq 0$. When $k = 1$ or -1 we will have no solutions because the last equations will be: $0 = -2$ or $0 = -4$ respectively.

   (b) For infinitely many solutions, we need both: $(k^2 - 1) = 0$ and $-(k+3) = 0$. These two conditions can not simultaneously be satisfied for all values of k.

   (c) When $k = 0$ we have the following:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right]$$

   The last row gives, $x_2 = 3$ plugging in to the top equation we have:

$$x_1 + x_2 = 3 \implies x_1 + 3 = 3 \implies x_1 = 0.$$

2. Consider the following linear system A where,

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix}$$

   (a) Which matrix is the correct elementary row matrix necessary to remove 0 from the second row, first column. (i.e., the normal first step of Gaussian elimination).
   (b) What is the value of the pivot in the second row of the row echelon form of A.
   (c) What is the L matrix in the LU decomposition of A?

Solution: The answer to all these questions can be found by carrying out Gaussian Elimination and then computing the LU decomposition.
We observe the first operations in row reduction, and their corresponding elementary row matrices, are

$$R_2 \to R_2 - R_1 \implies E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$R_3 \to R_3 - (3/2)R_1 \implies E_{3,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix}.$$

Notice the matrix $E_{2,1}$ corresponds to the solution to **question 4 on Homework Quiz # 2.** We carry out the multiplication and find:

$$E_{3,1}E_{2,1}A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -4 & 0 \\ 0 & -\frac{7}{2} & 0 \end{bmatrix}.$$

Thus, the last elementary row operation, and corresponding matrix, is given by

$$R_3 = R_2 - \frac{7}{8}R_3 \implies E_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7/8 & 1 \end{bmatrix}$$

Carrying out these operations gives us U, the row-echelon form of A:

$$U = E_{3,2}E_{3,1}E_{2,1}A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here we see that the pivot in the second row of the row-echelon form of A is -4, the answer to **question 5 on Homework Quiz # 2.**

The LU decomposition is found by either inverting all of the elementary row matrices, or using the properties we discussed in class (Week 2, Thursday).

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3/2 & 7/8 & 1 \end{bmatrix}$$

This is the same matrix given in **question 6 on Homework Quiz # 2**. Thus the answer to that question is true.

3. A farmer with 1200 acres is planning to plant 3 kinds of crops: corn, soybeans and oats. The cost of each crop is different per acre, corn seed costs \$20 per acre, while soybean seed costs \$50 per acre and oat costs \$15 per acre. The farmer has \$40,000 to spend on seeds and will spend it all.

   (a) Use the information in the problem to formulate two linear equations with three unknowns.

Solution: Well, word problems are always the hardest problems in a math class. And we finally arrive at our first word problem for linear algebra.

In the situation the farmer has to decide on how many acres to plant of the three unknowns: corn (x_1), soybeans (x_2) and oats (x_3). The farmer has a total of 1200 acres and if all are plotted we have:

$$x_1 + x_2 + x_3 = 1200.$$

But the farmer also has a total of \$40,000 to spend on this planting. The cost of each crop depends on which plant. But if the farmer wants to spend his entire \$40,000 we know:

$$20x_1 + 50x_2 + 15x_3 = 40000.$$

(b) Solve the system you wrote, and explain the solution set.

Solution: Our system of two equations and two unknowns can be written as:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1200 \\ 20 & 50 & 15 & 40000 \end{array}\right].$$

We will carry out the row operation $R_2 = R_2 - 20R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1200 \\ 0 & 30 & -5 & 16000. \end{array}\right].$$

We have two pivots, the third column the number of oat acres is free. (Of course practically speaking, we know that the smallest this variable can be is 0 and the biggest it can be is 1200.) For now we will make it $x_3 = t$. In this case we know:

$$30x_2 - 5t = 16000 \implies x_2 = 1600/3 + (1/6)t.$$

Finally, we have:

$$x_1 + x_2 + x_3 = 1200 \implies x_1 + (1600/3 + (1/6)t) + t = 1200 \implies x_1 = 2000/3 - (7/6)t$$

Thus we have a rather interesting solution for the farmer. Pick a value of t and then

- Plant $2000/3 - (7/6)t$ acres of corn
- Plant $1600/3 + (1/6)t$ acres of soy beans
- Plant t acres of oats

Thus, the farmer has an infinite number of solutions to choose from when selecting how many acres of each crop to plant.
Note that the number of acres for each plant must be non-negative, as such this restricts the value of t within a range: $0 \leq t \leq 4000/7$.