Homework Quiz #4

1. Let A be a 3 by 3 matrix. Which of the following statements can never be true.

   - $C(A)$ has dimension 2, and $N(A)$ has dimension 2.
   - $C(A)$ has dimension 3, and $N(A)$ has dimension 0.
   - $C(A)$ has dimension 0, and $N(A)$ has dimension 3.

   Solution: We know that the number of columns 3, must be the sum of the dimension of the column space $C(A)$ and the nullspace $N(A)$. Thus, first option is not ever true.

2. Decide if the following statement is true or false:

   If the columns of A are linearly independent, then $A\vec{x} = \vec{b}$ has exactly 1 solution for any choice of $\vec{b}$.

   Solution: **FALSE.** We know that the only choices of $\vec{b}$ where there will be a solution to $A\vec{x} = \vec{b}$ will be if $\vec{b} \in C(A)$. If the columns of A are linearly independent it will mean that there will be either 0 or 1 solution.

3. Consider the matrix A:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   What is the dimension of $C(A)$?

   Solution: The dimension of $C(A)$ are the number of pivots in the row echelon form of the matrix. Since A is already in the row echelon form, we see that there are 2 pivots. This means the dimension of $C(A)$ is 2.

   We could also see this by noting that the third column and fourth column are linear combinations of the first and second columns.

4. Consider the matrix A:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   What is the dimension of $N(A)$?

   Solution: The dimension of $N(A)$ are the number of columns minus the number pivots in the row echelon form of the matrix. Since A is already in the row echelon form, we see that there are 2 pivots. This means the dimension of $N(A)$ is 4 - 2 = 2..

5. Consider the following set of vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Are the vectors linearly independent or linearly dependent?

Solution: We will try to determine the set of coefficients c_1, c_2 and c_3 that satisfy:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_2\vec{v}_3 = \vec{0}$$

are $c_1 = c_2 = c_3 = 0$. We can write this system as:

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 3c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + 3c_2 + c_3 &= 0. \end{aligned}$$

$$\implies \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 2 & 3 & 1 & 0. \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + (-1/5)R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 0 & -18/5 & 0 \end{array}\right].$$

We can see that we have 3 pivots which would imply $c_1 = c_2 = c_3 = 0$. Thus the vectors are linearly independent.

6. What is the dimension of the vector space consisting of all 2 by 2 matrices?

Solution: We know that a generic 2 by 2 matrix is given by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

This leads us to conclude that the dimension is 4 because it requires specifying 4 independent real numbers. Indeed we can also write down a canonical basis for this vector space:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

To show they are linearly independent, let's try finding a linear combination of the matrices that leads to the 0 matrix. That is, we want to find coefficients c_1, c_2, c_4, c_4 so that:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \implies c_1 = c_2 = c_3 = c_4 = 0.$$

Thus any generic matrix can be written as the following linear combination

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aM_1 + bM_2 + cM_3 + dM_4.$$

7. A set of 4 vectors in $\mathbb{R}^5$ is (always, sometimes or never) linearly dependent.

Solution: Because the dimension of $\mathbb{R}^5$ is 5, larger than 4, we know that a set of 4 vectors is **sometimes** linearly dependent.

8. A set of 7 vectors in $\mathbb{R}^5$ is (always, sometimes or never) linearly dependent.

Solution: Because the dimension of $\mathbb{R}^5$ is 5, which is smaller than 7, we know that a set of 7 vectors is **always** linearly dependent.

9. Let P_3 be the vector space of polynomials up to degree 3. This means that a vector in P_3 will look like:

$$p(x) = a_0 + a_1 x + a_2 x^2$$

where the values of a_i can be any real number.
Consider the set of vectors:

$$p_1(x) = 1 + x, p_2(x) = x(x-1), \text{ and } p_3(x) = 1 + 2x^2.$$

Prove this set of vectors is a **basis** for P_3 by showing two properties:

(a) The set is linearly independent.
(b) The set of vectors spans P_3.

Solution: We will take this in two parts.

- **Linear Independence:**
  To show the set $p_1(x), p_2(x)$ and $p_3(x)$ are linearly independent, we need to show that the only solution to the following linear equation is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

$$\alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) = 0.$$

$$\begin{aligned}
&\implies \alpha_1(1+x) + \alpha_2(x(x-1)) + \alpha_3(1+2x^2) = 0 \\
&\implies \alpha_1 + \alpha_1 x + \alpha_2 x^2 - \alpha_2 x + \alpha_3 + \alpha_3(2x^2) = 0 \\
&\implies 1(\alpha_1 + \alpha_3) + x(\alpha_1 - \alpha_2) + x^2(\alpha_2 + 2\alpha_3) = 0.
\end{aligned}$$

If this statement has to be true for **every** x value, we need all of the coefficients for each term to be equal to 0:

$$\begin{aligned}
\alpha_1 + \alpha_3 &= 0 \\
\alpha_1 - \alpha_2 &= 0 \\
\alpha_2 + 2\alpha_3 &= 0.
\end{aligned}$$

We write this as the following vector matrix system:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We carry out normal row operations:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

There are 3 pivots, and so **the only** solution to this system is the trivial solution: $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus the original set of vectors are linearly independent.

- **Spanning Set:**

  To show the set of vectors is a spanning set, we need to a linear combination (α_i) that will let us create any arbitrary vector from P_3.

  That is, for a given $p(x) = a_0 + a_1x + a_2x^2$ we will find α_i such ghat:

$$p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x)$$

$$\implies a_0 + a_1x + a_2x^2 = \alpha_1(1+x) + \alpha_2(x(x-1)) + \alpha_3(1+2x^2)$$

$$\implies a_0 + a_1x + a_2x^2 = 1(\alpha_1 + \alpha_3) + x(\alpha_1 - \alpha_2) + x^2(\alpha_2 + 2\alpha_3).$$

  If this has to be true for **every** value of x, we require:

$$a_0 = \alpha_1 + \alpha_3, a_1 = \alpha_1 - \alpha_2, \text{ and } a_3 = \alpha_2 + 2\alpha_3.$$

  We can write this as a matrix vector equation and perform our usual row operations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 1 & -1 & 0 & a_1 \\ 0 & 1 & 2 & a_2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & -1 & -1 & a_1 - a_0 \\ 0 & 1 & 2 & a_2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & -1 & -1 & a_1 - a_0 \\ 0 & 0 & 1 & a_2 + (a_1 - a_0) \end{array}\right]$$

  We could perform back-substitution, but let's use the idea from the Gauss-Jordan algorithm to perform row operations until the LHS becomes the identity matrix.

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & -1 & 0 & a_2 + 2(a_1 - a_0) \\ 0 & 0 & 1 & a_2 + (a_1 - a_0) \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & 1 & 0 & 2(a_0 - a_1) - a_2 \\ 0 & 0 & 1 & a_2 + (a_1 - a_0) \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_0 - a_2 - (a_1 - a_0) \\ 0 & 1 & 0 & 2(a_0 - a_1) - a_2 \\ 0 & 0 & 1 & a_2 + (a_1 - a_0) \end{array}\right].$$

  Thus we have:

$$\alpha_1 = 2a_0 - a_1 - a_2, \alpha_2 = 2a_0 - 2a_1 - a_2, \text{ and } \alpha_3 = -a_0 + a_1 + a_2.$$

  We have found the **unique** representation of a generic vector from P_3 in terms of our basis.

Since we have shown $p_1(x), p_2(x)$ and $p_3(x)$ are a linearly independent spanning set for P_3 we have shown they are a basis.