WH#9

1) a) $\iint f(x,y)\,dx\,dy = 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{C}{(1+x^2)(1+y^2)}\,dx\,dy = 1$$

$$C\int_{-\infty}^{\infty} \frac{1}{(1+x^2)}\,dx \int_{-\infty}^{\infty} \frac{1}{1+y^2}\,dy = 1$$

$$\tan^{-1}(x)\Big]_{-\infty}^{\infty} \quad \pi C\int_{-\infty}^{\infty}\frac{1}{1+y^2}\,dy = 1$$

$$\left(\frac{\pi}{2} - -\frac{\pi}{2}\right) \quad \pi C\left(\tan^{-1}(y)\right)\Big]_{-\infty}^{\infty} = 1$$

$$\pi C\left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = 1 \Rightarrow \pi^2 C = 1$$

$$\boxed{C = 1/\pi^2}$$

b) $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$

$$P((x,y)\in R) = 1/\pi^2 \int_0^1\int_0^1 \frac{1}{(1+x^2)(1+y^2)}\,dx\,dy$$

$$\frac{1}{\pi^2}\int_0^1 \frac{1}{(1+x^2)}\,dx \int_0^1 \frac{1}{(1+y^2)}\,dy$$

$$\frac{1}{\pi^2}\left[\tan^{-1}(x)\Big]_0^1 \cdot \tan^{-1}(y)\Big]_0^1\right] = \frac{1}{\pi^2}\left[\frac{\pi}{4}\cdot\frac{\pi}{4}\right] = \boxed{1/16}$$

c) $\frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{1}{(1+x^2)(1+y^2)}\,dy$ Marginal PDF of X

$$\frac{1}{\pi^2}\frac{1}{(1+x^2)}\int_{-\infty}^{\infty}\frac{1}{(1+y^2)}\,dy$$

$$\frac{1}{\pi^2}\frac{1}{(1+x^2)}\left[\tan^{-1}(y)\right]_{-\infty}^{\infty}$$

$$\frac{1}{\pi^2}\frac{1}{(1+x^2)}\left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = \frac{1}{\pi}\frac{1}{(1+x^2)} = f(x)$$

Marginal PDF of Y, $f(y) = \frac{1}{\pi}\frac{1}{(1+y^2)}$

$$f(x)f(y) = \frac{1}{\pi^2}\cdot\frac{1}{(1+x^2)(1+y^2)} = f(x,y)$$

$\boxed{\text{X and Y are independent}}$

2) a) $\int_{y=20}^{y=30} \int_{x=20}^{x=30} K(x^2+y)^2 \, dx\, dy = 1$

$K \int_{20}^{30} \left[\frac{x^3}{3} + x y^2 \right]_{20}^{30} dy = 1$

$K \int_{20}^{30} \left[\frac{1}{3}(30^3 - 20^3) + 10y^2 \right] dy = 1$

$K \int_{20}^{30} \left[\frac{19000}{3} + 10y^2 \right] dy = 1$

$K \cdot \left. \frac{19000y}{3} \right]_{20}^{30} + 10 \left. \frac{y^3}{3} \right]_{20}^{30} = 1$

$K \cdot \left[\frac{190{,}000}{3} + \left(\frac{190{,}000}{3} \right) \right] = 1$

$K \cdot \frac{380{,}000}{3} = 1$ $\boxed{K = 3/380{,}000}$

b) $P(X < 26 \text{ and } Y < 26) = \int_{x=20}^{x=26} \int_{y=20}^{y=26} K(x^2+y^2)\, dy\, dx$

$= K \int_{x=20}^{x=26} \left. x^2 y + \frac{y^3}{3} \right]_{y=20}^{y=26}$

$K \int_{x=20}^{x=26} 6x^2 + 3192$

$K \left[2x^3 + 3192x \right]_{x=20}^{x=26}$

$\frac{3}{380{,}000} \left[38{,}304 \right] = \boxed{.3024}$

c) $P(|X - Y| \le 2)$

$= 1 - \int_{x=20}^{x=28} \int_{y=22}^{y=30} K(x^2+y)^2 \, dy\, dx$

$1 - K \int_{x=20}^{x=28} \left. x^2 y + \frac{y^3}{3} \right]^{y=30}$

$1 - K \int_{x=20}^{x=28} x^2 8 + \frac{16352}{3}$

$1 - K \cdot \left[\frac{8x^3}{3} + \frac{16352x}{3} \right]_{20}^{28}$

$1 - K \cdot [211{,}157.3333 - 130{,}346.6667] = .3593$

d) $f_Y(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x=20}^{x=30} K\,(x^2+y^2)\,dy$

$K\left[\frac{x^3}{3} + xy^2\right]\Big|_{x=20}^{x=30}$

$K\left[\frac{19,000}{3} + 10y^2\right]$

$f_Y(y) = \frac{3}{380,000}\left(10y^2 + \frac{19000}{3}\right), \quad 20 \le y \le 30$

e) $f(x,y) \neq f(x) \cdot f(y)$

Not independent

3) a) $f_X(x) = \int_1^2 f(x,y)\,dy$

$= \int_1^2 \frac{2}{75}(2x^2y + xy^2)\,dy$

$= \frac{2}{75}\left[\frac{\cancel{2}x^2y^2}{\cancel{2}} + \frac{xy^3}{3}\right]_1^2$

$\frac{2}{75}\left[4x^2 + \frac{8x}{3} - x^2 - \frac{x}{3}\right] = \frac{2}{75}\left[3x^2 + \frac{7x}{3}\right]$

$= \frac{2}{75}\left[\frac{9x^2}{3} + \frac{7x}{3}\right]$

$= \frac{2x}{225}(9x+7), \quad 0 \le x \le 3$

$f_Y(y) = \int_0^3 f(x,y)\,dx$

$= \frac{2}{75}\int_0^3 2x^2y + xy^2\,dx$

$= \frac{2}{75}\left[\frac{2x^3}{3}y + \frac{x^2y^2}{2}\right]_0^3$

$\frac{2}{75}\left[\frac{54y}{3} + \frac{9y^2}{2}\right]$

$= \frac{2}{75}\left[\frac{108y}{6} + \frac{27y^2}{6}\right] = \frac{216y}{450} + \frac{54y^2}{450} = \frac{3y}{25}(y+4), \quad 1 \le y \le 2$

b) $E(x) = \int_0^3 x \cdot f(x)\,dx$

$= \int_0^3 x \cdot \frac{2}{75}\left[3x^2 + \frac{7x}{3}\right] dx$

$\frac{2}{75}\int_0^3 \left(3x^3 + \frac{7x^2}{3}\right) dx$

$\frac{2}{75}\left[\frac{3x^4}{4} + \frac{7x^3}{9}\right]_0^3 = \frac{2}{75}\left[60.75 + 21\right] = \boxed{2.18}$

$$E[Y] = \int_1^2 y \cdot \frac{2}{75}\left(18y + \frac{9}{2}y^2\right) dy$$
$$= \frac{2}{75}\int_1^2 \left(18y^2 + \frac{9}{2}y^3\right) dy$$
$$= \frac{2}{75}\left[\frac{18y^3}{3} + \frac{9y^4}{2\cdot 4}\right]_1^2$$
$$= \frac{2}{75}\left[\frac{18\cdot 2^3}{3} + \frac{9\cdot 2^4}{8} - \frac{18}{3} - \frac{9}{8}\right]$$
$$\doteq \boxed{1.57}$$

$$E(x+y) = E(x) + E(y) = 2.18 + 1.57 = \boxed{3.75}$$

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c) $$E[X^2] = \int_0^3 x^2 \cdot \frac{2}{75}\left(3x^2 + \frac{7x}{3}\right) dx$$
$$= \frac{2}{75}\int_0^3 \left(3x^4 + \frac{7x^3}{3}\right) dx = \frac{2}{75}\left[\frac{3x^5}{5} + \frac{7x^4}{12}\right]_0^3$$
$$= \frac{2}{75}\left[\frac{729}{5} + \frac{567}{12}\right]$$
$$\boxed{E(x^2) = 5.148}$$

$$E[Y^2] = \int_1^2 y^2 \cdot \frac{2}{75}\left(18y + \frac{9}{2}y^2\right) dy$$
$$= \frac{2}{75}\int_1^2 \left(18y^3 + \frac{9y^4}{2}\right) dy$$
$$= \frac{2}{75}\left[\frac{18y^4}{4} + \frac{9y^5}{2\cdot 5}\right]_1^2$$
$$\frac{2}{75}\left[\frac{288}{4} + \frac{288}{10} - \frac{18}{4} - \frac{9}{10}\right] = \boxed{2.544 = E[Y^2]}$$

$$E[XY] = \int_1^2\int_0^3 xy \cdot \frac{2}{75}\left(2x^2y + xy^2\right) dx\, dy$$
$$\frac{2}{75}\int_1^2 y\left[\frac{2x^4}{4}y + \frac{x^3}{3}y^2\right]_0^3 dy$$
$$\frac{2}{75}\int_1^2 y\left[\frac{162y}{4} + \frac{27}{3}y^2\right] dy$$
$$= \frac{2}{75}\int_1^2 \left(\frac{162}{4}y^2 + 9y^3\right) dy$$
$$= \frac{2}{75}\left[\frac{162}{4}\cdot\frac{y^3}{3} + \frac{9}{4}y^4\right]_1^2 = \frac{2}{75}\left[108 + 36 - 13.5 - \frac{9}{4}\right]$$
$$E(XY) = 3.42$$

$E((x+y)^2) = E[x^2] + E[y^2] + 2E[x,y]$

$= 5.148 + 2.544 + (2 \cdot 3.42)$

$\boxed{= 14.532}$

d) $Var(x) = E(x^2) - (E(x))^2$

$= 5.148 - (2.18)^2 = \boxed{.3956}$

$Var(Y) = E[y^2] - E((y))^2$

$= 2.544 - (1.57)^2 = \boxed{.0791}$

$Var(x+Y) = Var(x) + Var(Y) + cov(x,y)$

$cov(x,y) = E(xy) - E(x)E(y)$

$= 3.42 - 2.18 \cdot 1.57 = -.0026$

$Var(x+y) = .3956 + .0791 - .0026$

$= \boxed{.4721}$

$Var(x+y) \neq Var(x) + Var(Y)$

4) a) $k\int_0^\infty \int_0^5 y e^{-xy} = k\int_0^\infty y \left[\int_0^5 e^{-xy} dx\right] dy$

$= k\int_0^5 y \left[\frac{e^{-xy}}{-y}\right]_0^\infty dy = k\int_0^5 e^{-xy}\Big]_0^\infty dy$

$= k\int_0^5 (1-0)\, dy = k \cdot x\Big]_0^5 = 5k = 1$

$\boxed{k = 1/5 = .2}$

b) $P(x>6) = \frac{1}{5}\int_0^5 y\left[\int_6^\infty e^{-xy} dx\right] dy$

$= 1/5 \int_0^5 y \left[\frac{e^{-xy}}{-y}\right]_6^\infty dy = 1/5 \int_0^5 (e^{-xy})_\infty^6 dy$

$= 1/5 \int e^{-6y} dy = \frac{1}{5}\left[\frac{e^{-6y}}{-6}\right]_0^5$

$= \frac{1}{30}\left[e^{-6y}\right]_5^0 = \frac{1}{30}(1 - e^{-30})$

$\boxed{= .033}$

c) $F_Y(y) = \frac{1}{5} y \int_0^\infty e^{-xy} dx$

$= \frac{1}{5} y \left[\frac{e^{-xy}}{-y}\right]_0^\infty = \frac{1}{5} \quad 0 < y < 5$

$E[y] = 1/5 \int_0^5 y\, dy = \frac{1}{5}\left[\frac{y^2}{2}\right]_0^5 = 2.5$

$E(y^2) = \frac{1}{5}\int_0^5 y^2 dy$

$= 1/5 \left[\frac{y^3}{3}\right]_0^5 = \frac{25}{3}$

$\sigma = \sqrt{V(Y)} = \sqrt{E(y^2) - E^2(y)} = \sqrt{\frac{25}{3} - (2.5^2)} = \boxed{1.4434}$

5) $E(T_1) = \int_1^4 t \frac{dt}{3} = \int_1^4 \frac{t^2}{6} = \frac{1}{6}(4^2 - 1^2) = 2.5$

$E(T_2) = \int_1^4 t^2 \frac{dt}{3} = \frac{1}{9}(4^3 - 1) = 7$

$Var(T_1) = 7 - (5/2)^2 = .75$

$E(T_1 + T_2) = 2.5 + 3 = \boxed{5.5 \text{ hours} = \text{mean}}$

$V(T_2) = 18 - 3^2 = 9$

$$\begin{aligned} Var(T_1 + T_2) &= Var(T_1) + Var(T_2) \\ &= .75 + 9 \\ &= 9.75 \end{aligned}$$

$\sigma = \sqrt{9.75} = \boxed{3.1225}$