CSE 185

Histogram Equalization

Let variable r denote the continuous intensity values an image to be processed can take

$r \in [0, L-1]$ $\quad r=0 \rightarrow$ black
$r=L-1 \rightarrow$ white

Consider transformations (intensity mappings) of form

$$s = T(r) \qquad 0 \leq r \leq L-1$$

that produce an output intensity level s for every pixel in the input image having intensity r.

Assume that

(a) $T(r)$ is a monotonically increasing function in the interval $0 \leq r \leq L-1$; and
(b) $0 \leq T(r) \leq L-1$ for $0 \leq r \leq L-1$

Condition (a) guarantees that relative ordering of output values matches relative ordering of input values (prevent reversals of intensity).

Condition (b) guarantees the range of output intensities is the same as the input.

Note that under these constraints it is OK for more than one input to be mapped to the same output (many-to-one).

The intensity levels in an image may be viewed as random variables in the interval $[0, L-1]$ (continuous in this case). A fundamental descriptor of a random variable is its probability density function (PDF).

$p_r(r')$ is probability that $r=r'$

Let $p_r(r)$ and $p_s(s)$ be PDFs of r and s.

A fundamental result from basic probability theory is that if $p_r(r)$ and $T(r)$ are known, and $s=T(r)$ is continuous and differentiable over the range of values of interest then

$$p_s(s) = p_r(r) \left| \frac{dr}{ds} \right| \qquad (1)$$

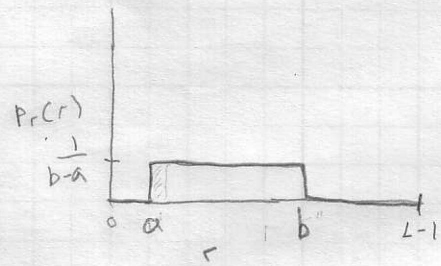
A transformation function of particular importance in histogram processing has the form

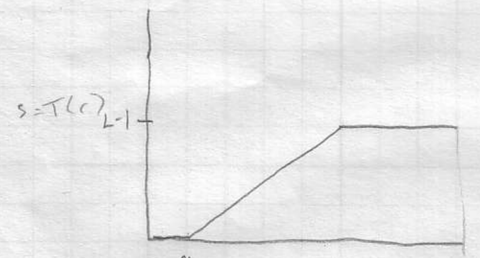
$$s = T(r) = (L-1) \int_0^r p_r(w)\,dw \qquad (2)$$

where w is a dummy variable of integration.

The RHS of this equation can be recognized as the cumulative distribution function (CDF) of the r.v. r.

Example:





Because PDFs are always positive and because the integral of a function is the area under the function, it follows that the transformation function (2) satisfies condition (a).

When the upper limit in (2) is $r=(L-1)$, the integral evaluates to 1 so the maximum value of s is $(L-1)$ and condition (b) is also satisfied.

To find $p_s(s)$, we use (1). From Leibniz's rule in basic calculus, the derivative of a definite integral with respect to its upper limit is the integrand evaluated at the limit:

$$\frac{ds}{dr} = \frac{dT(r)}{d(r)} = (L-1)\frac{d}{dr}\left[\int_0^r p_r(w)\,dw\right] = (L-1)\,p_r(r).$$

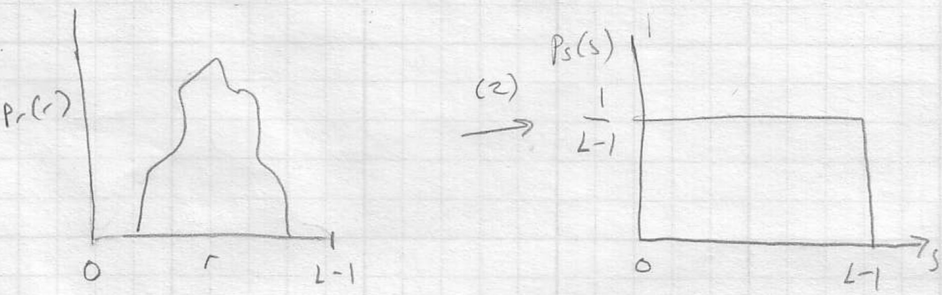
Substituting for $\frac{dr}{ds}$ in (2) we get

$$p_s(s) = p_r(r)\left|\frac{ds}{dr}\right| = p_r(r)\left|\frac{1}{(L-1)\,p_r(r)}\right| = \frac{1}{L-1} \qquad 0 \le s \le L-1 \qquad (3)$$

$p_s(s)$ is here recognized as a uniform PDF.

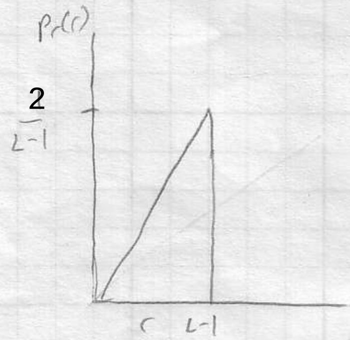
We have demonstrated that performing the intensity transformation in (2) yields a r.v. s characterized by a uniform PDF.

Not that $p_s(s)$ is always uniform, independent of the form of $p_r(r)$.



Example

Suppose $p_r(r) = \begin{cases} \frac{2r}{(L-1)^2} & \text{for } 0 \le r \le L-1 \\ 0 & \text{otherwise} \end{cases}$



$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw = \frac{2}{L-1}\int_0^r r\,dw = \frac{r^2}{L-1}$$

Suppose we form a new image with intensities s obtained using this transformation; that is, the s values are formed by squaring the corresponding intensity values and dividing by $(L-1)$.

We can verify the PDF of the intensities in the new image is uniform simply by substituting $p_r(r)$ into (1) and using the fact that $s = \frac{r^2}{(L-1)}$

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right| = \frac{2r}{(L-1)^2}\left|\left[\frac{ds}{dr}\right]^{-1}\right| = \frac{2r}{(L-1)^2}\left|\left[\frac{d}{dr}\frac{r^2}{L-1}\right]^{-1}\right|$$

$$= \frac{2r}{(L-1)^2}\left|\frac{(L-1)}{2r}\right| = \frac{1}{L-1}$$

For discrete values, we deal with probabilities (histogram values) and summations instead of PDFs and integrals.

$$p_r(r_k) = \frac{n_k}{MN} \qquad k = 0, 1, 2, \ldots, L-1$$

The discrete form of the transformation in (2) is

$$s_k = T(r_k) = (L-1) \sum_{j=0}^{k} p_r(r_j)$$

$$= \frac{(L-1)}{MN} \sum_{j=0}^{k} n_j \qquad k = 0, 1, 2, \ldots, L-1$$

Example 3.5 from book.