Homework Assignment #4

Remember, this Homework Assignment is **not collected or graded**! But you are advised to do it anyway because the problems for Homework Quiz #4 will be heavily based on these problems!

1. Let V be a vector space with subspaces S_1 and S_2. Define $S = S_1 \cap S_2$ as the set of all vectors that belong to both S_1 and S_2.

   (a) Explain how we know that S will be non-empty.

   Solution: We know that $\vec{0}$ will be part of any subspace, so at minimum S contains $\vec{0}$.

   (b) Prove that S will be a subspace of V.

   Solution: To show this, we will need to verify S is itself a vector space. This means we will have to show that S is closed under vector addition and scalar multiplication.

   Let $\vec{v}_1$ and $\vec{v}_2$ be in S, we want to show that $\vec{v} = \vec{v}_1 + \vec{v}_2$ also is in S.

   Note that $\vec{v}_1$ and $\vec{v}_2$ are in S_1 and S_2 (every vector in S is also in S_1 **and** S_2). Since we were told that S_1 and S_2 are subspaces, this means they are closed under vector addition. That is, $\vec{v} \in S_1$ and $\vec{v} \in S_2$. But this means $\vec{v} \in S_1 \cap S_2 = S$.

   A similar argument holds for scalar multiplication. If $\vec{v} \in S$ then $\vec{v} \in S_1 \cap S_2$. Thus $\alpha\vec{v} \in S_1$ and $\alpha\vec{v} \in S_2$ for any scalar α. Thus, $\alpha\vec{v} \in S$

2. The complete solution to $A\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is:

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   What is the matrix A?

Solution: First - let's figure out the dimensions of A. Since $\vec{x}$ and $\vec{b} = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$ are both in $\mathbb{R}^2$ we know that A is a 2 by 2 matrix.

We also know - from the general solution given above that the nullspace of A is dimension 1. So there is 1 pivot and 1 free variable the REF. Since A is 2 by 2 this means that we have a matrix of the form $R_2 = \alpha R_1$ for rows R_2, R_1.

Let's start by making $N(A) = \text{span}\{\begin{bmatrix} 0 & 1 \end{bmatrix}^T\}$

$$\begin{bmatrix} a_1 & a_2 \\ \alpha a_1 & \alpha a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \alpha R_1} \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We see that $x_2 = t$ because it's a free variable. But in order for $x_1 = 0$, we need $a_2 = 0$.

Thus, the matrix A must be of the form:

$$\begin{bmatrix} a_1 & 0 \\ \alpha a_1 & 0 \end{bmatrix}.$$

We will now try to determine a_1 such that $A \begin{bmatrix} 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$.

$$\begin{bmatrix} a_1 & 0 \\ \alpha a_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \implies a_1(1) = 1 \text{ and } \alpha a_1(1) = 3 \implies a_1 = 1, \alpha = 3.$$

We thus arrive at the following matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

3. Are the following statements true or false? Give a reason if true and a counter example if false.

(a) A square matrix (i.e., $n \times n$) has no free variables.

Solution: This is **FALSE**. Many square matrices have free variables. Indeed any square matrix ($n \times n$) with less than n pivots will have free variables.
Here's one example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) An invertible matrix has no free variables.

Solution: This is **TRUE**. If a matrix is invertible it will have no free variables. The nullspace of an invertible matrix is only the zero vector. This means that the solution to any $A\vec{x} = \vec{b}$ is unique and is equal to: $\vec{x} = A^{-1}\vec{b}$.

(c) An $m \times n$ matrix has no more than n pivots.

Solution: This is **TRUE** a matrix can never have more pivots than it has columns. Indeed:

$$\# \text{ pivots } \leq \min\{n, m\} \leq n.$$

(d) An $m \times n$ matrix has no more than m pivots.

Solution: This is **TRUE** a matrix can never have more pivots than it has rows. Indeed:

$$\# \text{ pivots } \leq \min\{n, m\} \leq m.$$

4. What are the **special solutions** (i.e., the non-zero solutions defined by free variables) to the system

$$A\vec{x} = \vec{0}$$

for the following choices of A:

(a) $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies x_3 = t \text{ and } x_4 = s.$$

The second row implies:

$$x_2 + 4x_3 + 5x_4 = 0 \implies x_2 + 4t + 5s = 0 \implies x_2 = -4t - 5s.$$

The first row implies:

$$x_1 + 2x_3 + 3x_4 = 0 \implies x_1 + 2t + 3s = 0 \implies x_1 = -2t - 3s.$$

Thus, the full solution for the null space, the linear combination of our two special solutions:

$$\left\{ t \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix} \text{ for } s, t \in \mathbb{R} \right\}.$$

(b) $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right] \implies x_3 = t, x_2 = s \text{ and } x_1 = 0.$$

Thus, the full solution for the null space, the linear combination of our two special solutions:

$$\left\{ t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ for } s, t \in \mathbb{R} \right\}.$$

5. Let P_3 be the vector space of polynomials up to degree 3. That is, an element of P_3 is of the form $p(x) = a_0 + a_1 x + a_2 x^2$.

(a) Explain why this is a vector space of dimension 3.

Solution: One way to see this to observe the the following set of vectors:

$$v_0(x) = 1, v_1(x) = x, \text{ and } v_2(x) = x^2$$

constitute a **linearly independent spanning set** and therefore are a basis.
Since this is 3 vectors, we have a vector space of dimension 3.
In this case, we might have also observed that we have, because of the coefficients, essentially a *copy* of $\mathbb{R}^3$.

(b) Let $p_1(x) = 1 + x$, $p_2(x) = x(x-1)$, $p_3(x) = 1 + 2x^2$. Verify that $\{p_1, p_2, p_3\}$ is a basis for P_3.

Solution: From the previous question, we know the vector space $P_3(x)$ is a vector space of dimension 3. As such, we need only show that $p_i(x)$ are linearly independent.
That is, we need to study which choices of α_i solve the following equation:

$$\alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) = 0.$$

$$\implies \alpha_1(1+x) + \alpha_2(x(x-1)) + \alpha_3(1+2x^2) = 0$$
$$\implies \alpha_1 + \alpha_1 x + \alpha_2 x^2 - \alpha_2 x + \alpha_3 + \alpha_3(2x^2) = 0$$
$$\implies 1(\alpha_1+\alpha_3) + x(\alpha_1 - \alpha_2) + x^2(\alpha_2 + 2\alpha_3) = 0.$$

If this statement has to be true for **every** x value, we need all of the coefficients for each term to be equal to 0:

$$\begin{aligned} \alpha_1 + \alpha_3 &= 0 \\ \alpha_1 - \alpha_2 &= 0 \\ \alpha_2 + 2\alpha_3 &= 0. \end{aligned}$$

We write this as the following vector matrix system:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We carry out normal row operations:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

There are 3 pivots, and so **the only** solution to this system is the trivial solution: $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus the original set of vectors are linearly independent.

(c) Determine the coefficients of linear combination representing a generic polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ relative to this basis. That is, find coefficients α_i so that:

$$p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x).$$

Solution: We have shown that the set of vectors must be a basis, but now we will formally show they are a spanning set by finding a linear combination (α_i) that will let us create any arbitrary vector from P_3.
That is, for a given $p(x) = a_0 + a_1 x + a_2 x^2$ we will find α_i such ghat:

$$p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x)$$

$$\implies a_0 + a_1 x + a_2 x^2 = \alpha_1(1+x) + \alpha_2(x(x-1)) + \alpha_3(1+2x^2)$$
$$\implies a_0 + a_1 x + a_2 x^2 = 1(\alpha_1+\alpha_3) + x(\alpha_1 - \alpha_2) + x^2(\alpha_2 + 2\alpha_3).$$

If this has to be true for **every** value of x, we require:

$$a_0 = \alpha_1 + \alpha_3, a_1 = \alpha_1 - \alpha_2, \text{ and } a_3 = \alpha_2 + 2\alpha_3.$$

We can write this as a matrix vector equation and perform our usual row operations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 1 & -1 & 0 & a_1 \\ 0 & 1 & 2 & a_2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & -1 & -1 & a_1 - a_0 \\ 0 & 1 & 2 & a_2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & -1 & -1 & a_1 - a_0 \\ 0 & 0 & 1 & a_2 + (a_1 - a_0) \end{array}\right]$$

We could perform back-substitution, but let's use the idea from the Gauss-Jordan algorithm to perform row operations until the LHS becomes the identity matrix.

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & -1 & 0 & a_2 + 2(a_1 - a_0) \\ 0 & 0 & 1 & a_2 + (a_1 - a_0) \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & 1 & 0 & 2(a_0 - a_1) - a_2 \\ 0 & 0 & 1 & a_2 + (a_1 - a_0) \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_0 - a_2 - (a_1 - a_0) \\ 0 & 1 & 0 & 2(a_0 - a_1) - a_2 \\ 0 & 0 & 1 & a_2 + (a_1 - a_0) \end{array}\right].$$

Thus we have:

$$\alpha_1 = 2a_0 - a_1 - a_2, \alpha_2 = 2a_0 - 2a_1 - a_2, \text{ and } \alpha_3 = -a_0 + a_1 + a_2.$$

We have found the **unique** representation of a generic vector from P_3 in terms of our basis.

6. Find a basis for the each of the following subspaces of $\mathbb{R}^3$. (Hint: It MIGHT help to write these as a matrix-vector system, $A\vec{x} = \vec{0}$ as these subspaces are the nullspace for a matrix A.)

(a) The plane $2x - 3y + z = 0$

Solution: Both of these questions are easier than they look. Let's write them as a vector matrix equation:

$$\begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [0].$$

The matrix A is a 1 by 3. We have 1 pivot and we have 2 free variables: $z = t, y = s$.
Thus:

$$2x - 3y + z = 0 \implies 2x - 3s + t = 0 \implies x = \frac{1}{2}(3s - t).$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} 3/2 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} -1/2 & 0 & 1 \end{bmatrix}^T \right\}.$$

(b) The intersection of the plane $2x - 3y + z = 0$ with the xy plane.

Solution: We now have two equations to simultaneously satisfy. Our original equation $(2x - 3y + z = 0)$ and the xy plane $(z = 0)$.

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This system now has 2 pivots, and 1 free variable: $y = s$. The second row requires $z = 0$. And back substitution to the first equation gives us:

$$2x - 3y + z = 0 \implies 2x - 3s + 0 = 0 \implies x = \frac{3}{2}s.$$

Then we have:

$$N(A) = \left\{ \begin{bmatrix} 3/2 & 1 & 0 \end{bmatrix}^T \right\}.$$