Equilibrium of a rigid body and statical determinacy

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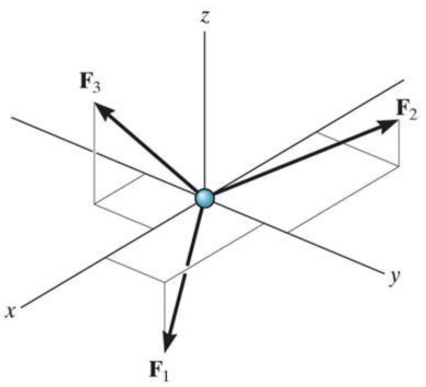
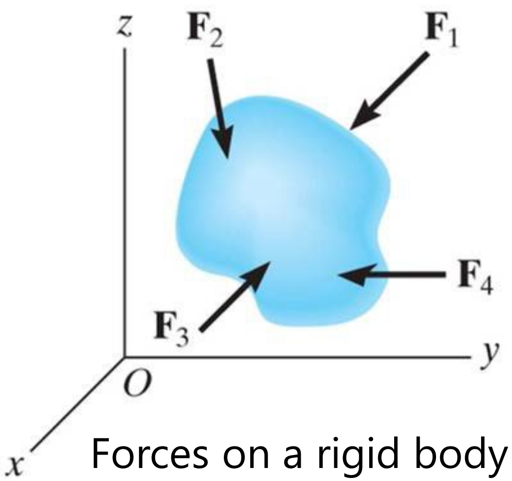
Mechanical constraints



Forces on a particle

- Up to now, we have not taken into consideration reaction forces.
- We know that for every action there must be an equal or opposite reaction. This must be included in the FBD.
- In real situations, support reactions are distributed loads. We will replace them with point loads for simplicity.

Conditions for rigid-body equilibrium



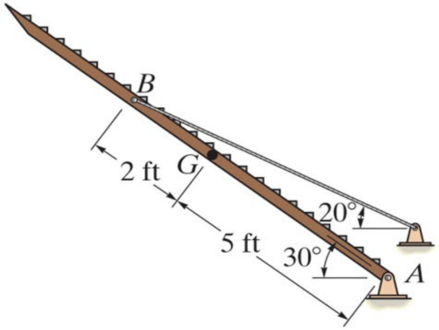
Forces on a rigid body

- In contrast to the forces on a particle, the forces on a rigid-body are not usually concurrent and may cause rotation of the body (due to the moments created by the forces).
- For a rigid body to be in equilibrium, the net force as well as the net moment about any arbitrary point O must be equal to zero.

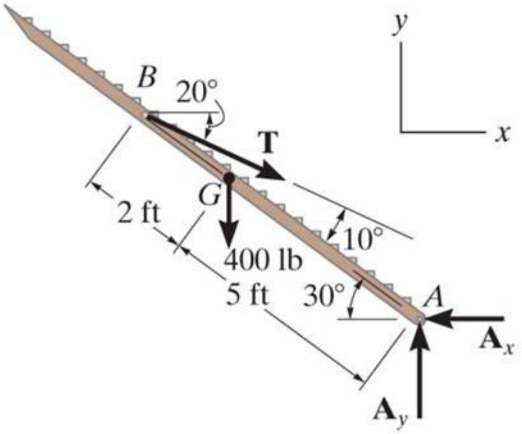
$\sum F = 0$ (no translation)

and $\sum M_O = 0$ (no rotation)

The process of solving rigid body equilibrium problems



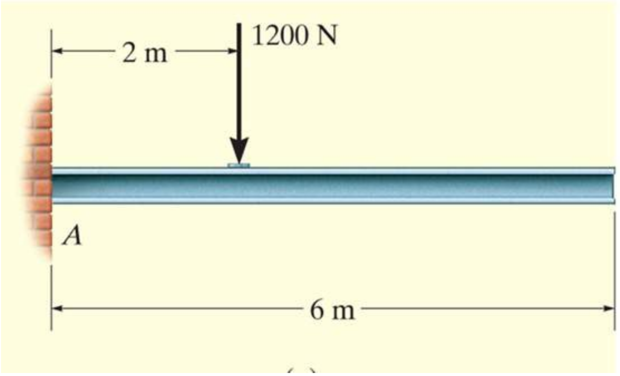
For analyzing an actual physical system, first we need to create an <u>idealized model</u> (above right).



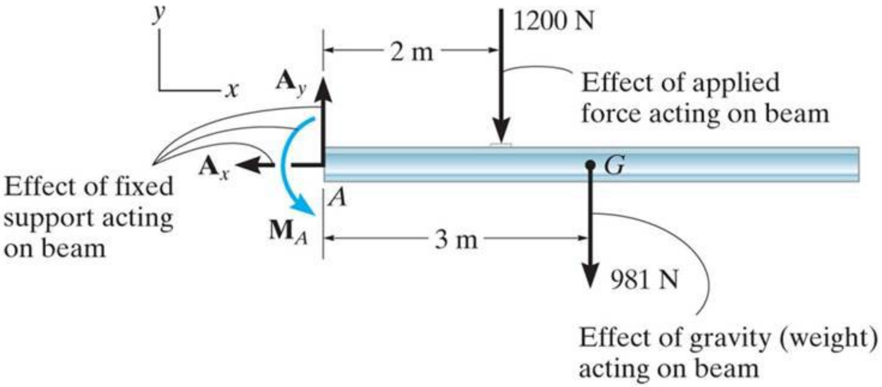
Then we need to draw a <u>free-body diagram (FBD) showing all the external (active and reactive) forces</u>.

Finally, we need to <u>apply the equations of equilibrium</u> to solve for any unknowns.

Free-body diagrams



Idealized model



Free-body diagram (FBD)

1. <u>Draw an outlined shape</u>. Imagine the body to be isolated or cut "free" from its constraints and draw its outlined shape.

2. <u>Show all the external forces and couple moments.</u> These <u>typically</u> include:

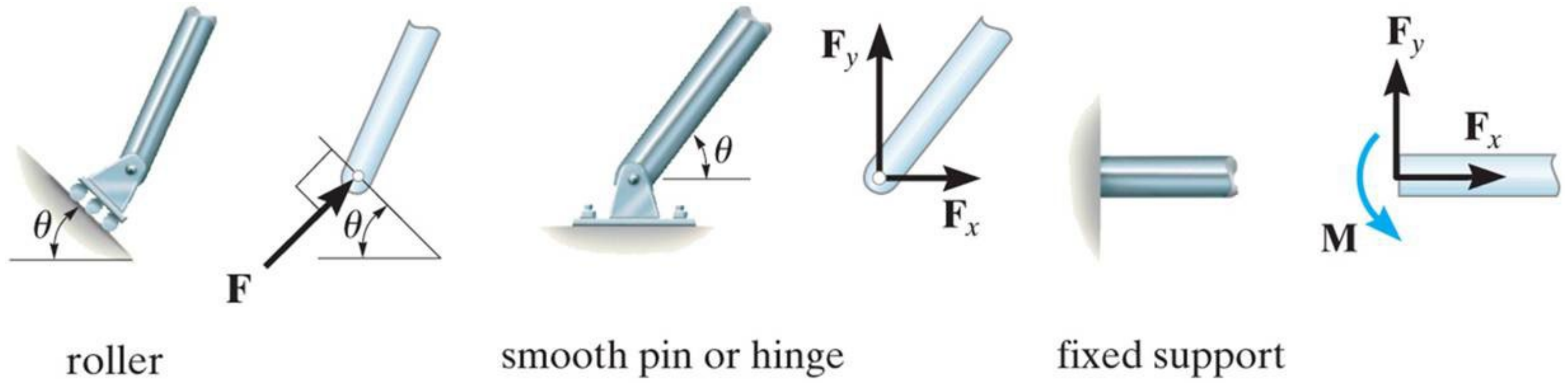
   a) applied loads

   b) <u>support reactions</u>

   c) the weight of the body.

<u>3. Label loads and dimensions on the FBD</u>: All known forces and couple moments should be labeled with their magnitudes and directions. For the unknown forces and couple moments, use letters like A_x, A_y, M_A, etc.. Indicate any necessary dimensions.

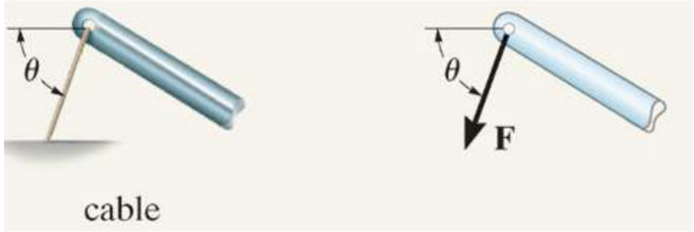
Support reactions

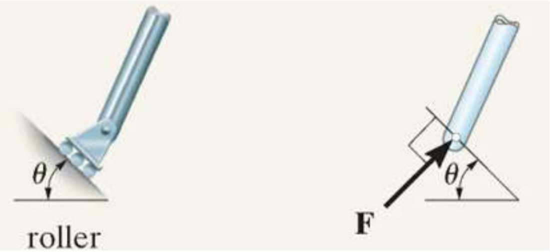


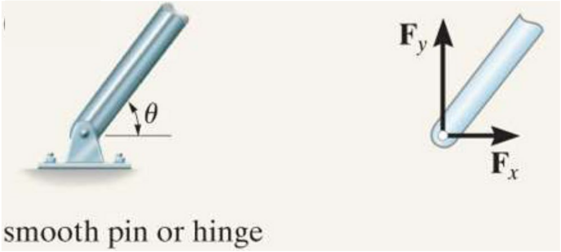
As a general rule, if a support prevents translation of a body in a given direction, then a force is developed on the body in the opposite direction.

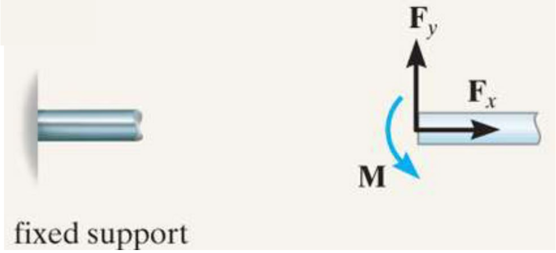
Similarly, if rotation is prevented, a couple moment is exerted on the body in the opposite direction.

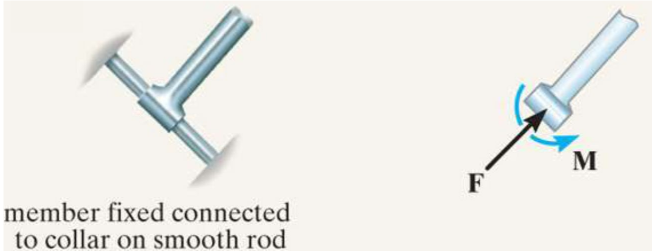
Some 2D examples

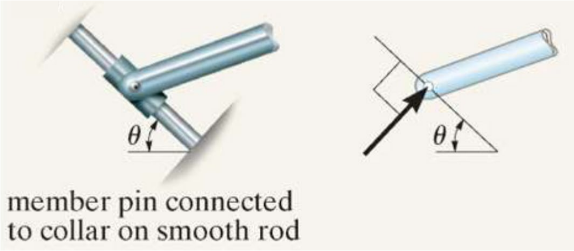




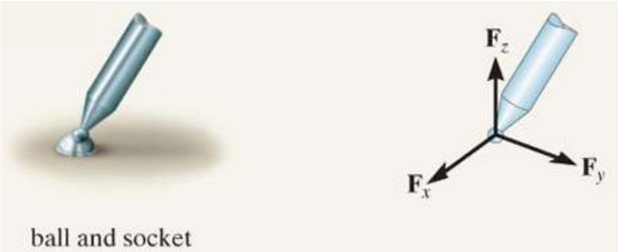


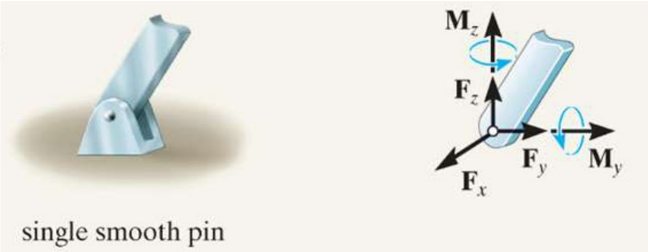


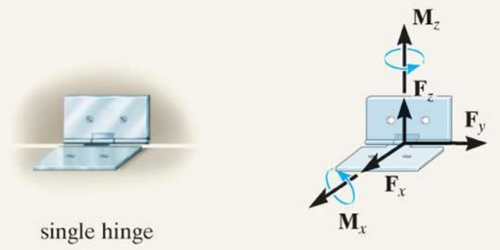


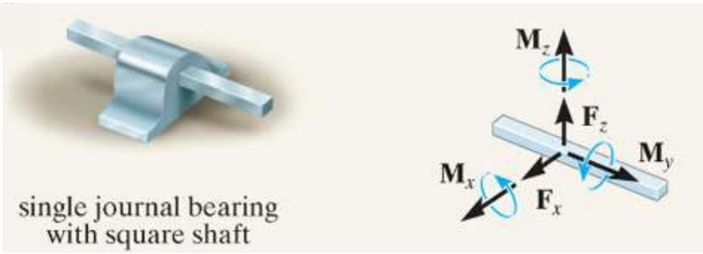


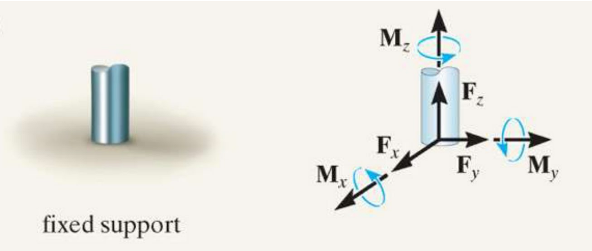
Some 3D examples







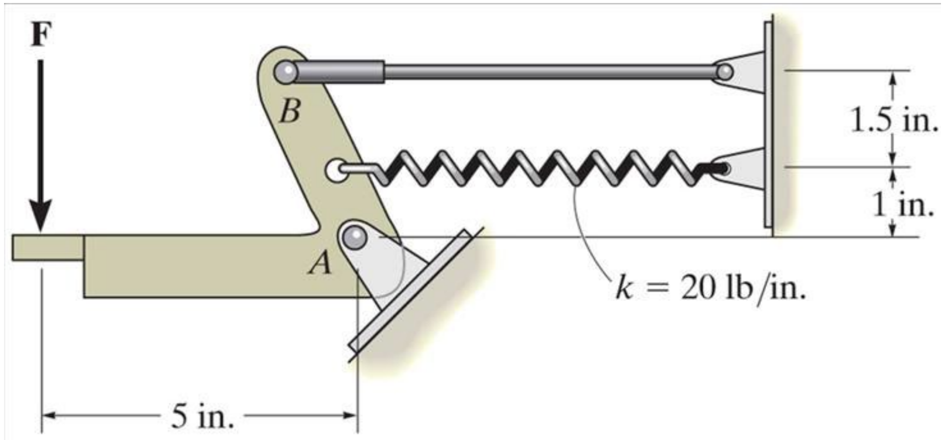




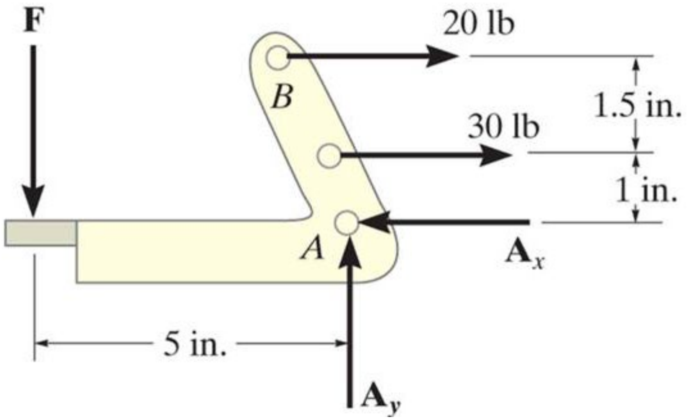
Example

Given: The operator applies a vertical force to the pedal so that the spring is stretched 1.5 in. and the force in the short link at *B is 20 lb*.

Draw: A an idealized model and free-body diagram of the foot pedal.

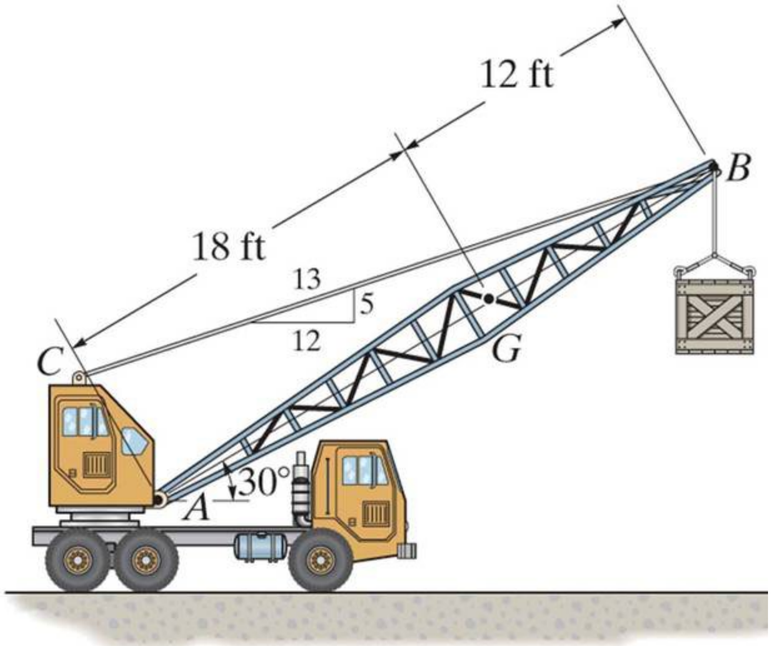


Idealized model

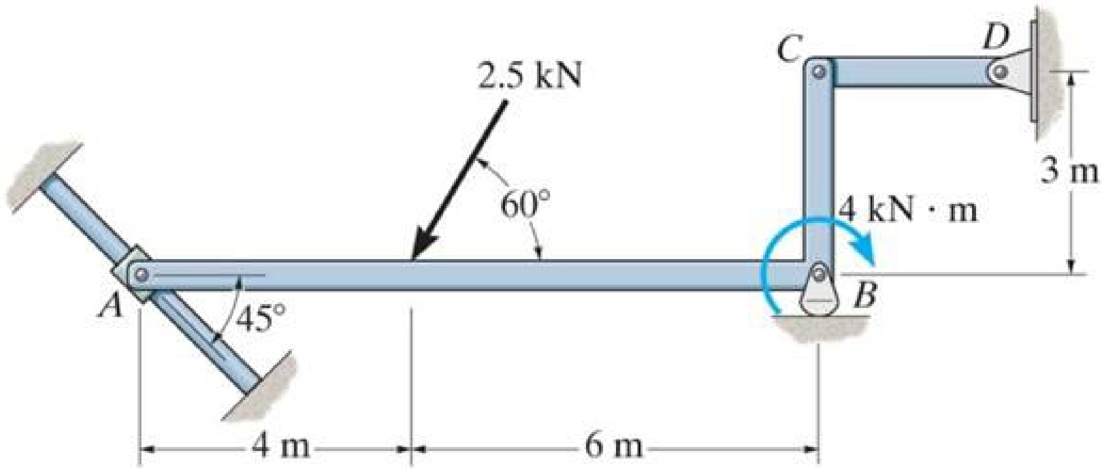


Free-body diagram (FBD)

Individual work (15 min)

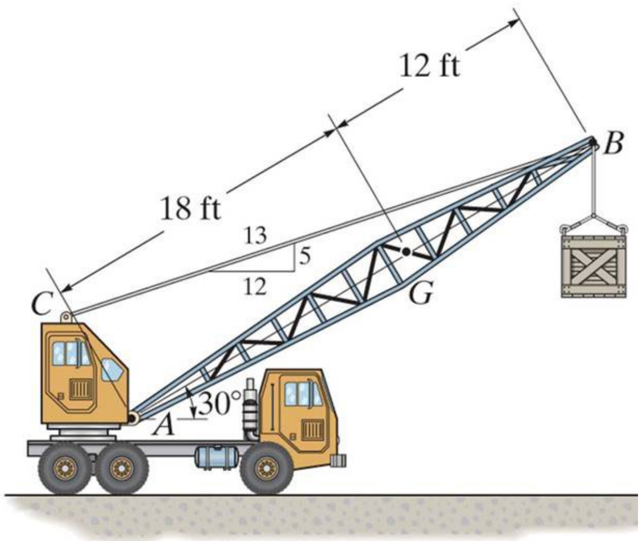


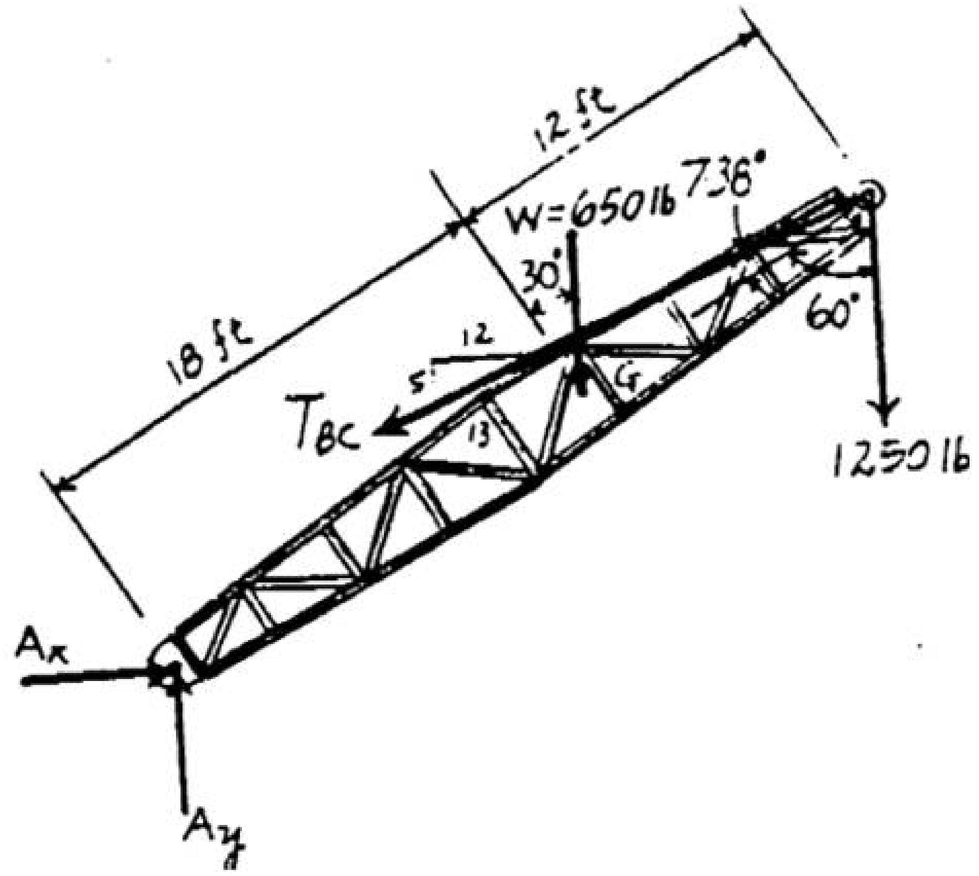
Draw a FBD of the crane boom, which is supported by a pin at A and cable BC. The load of 1250 lb is suspended at B and the boom weighs 650 lb.



Draw a FBD of member ABC, which is supported by a smooth collar at A, roller at B, and link CD.

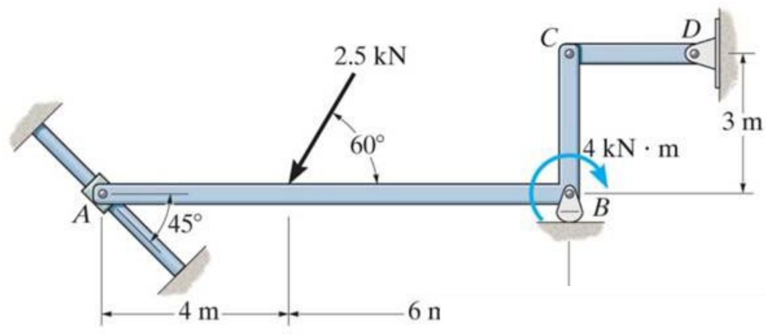
Solution

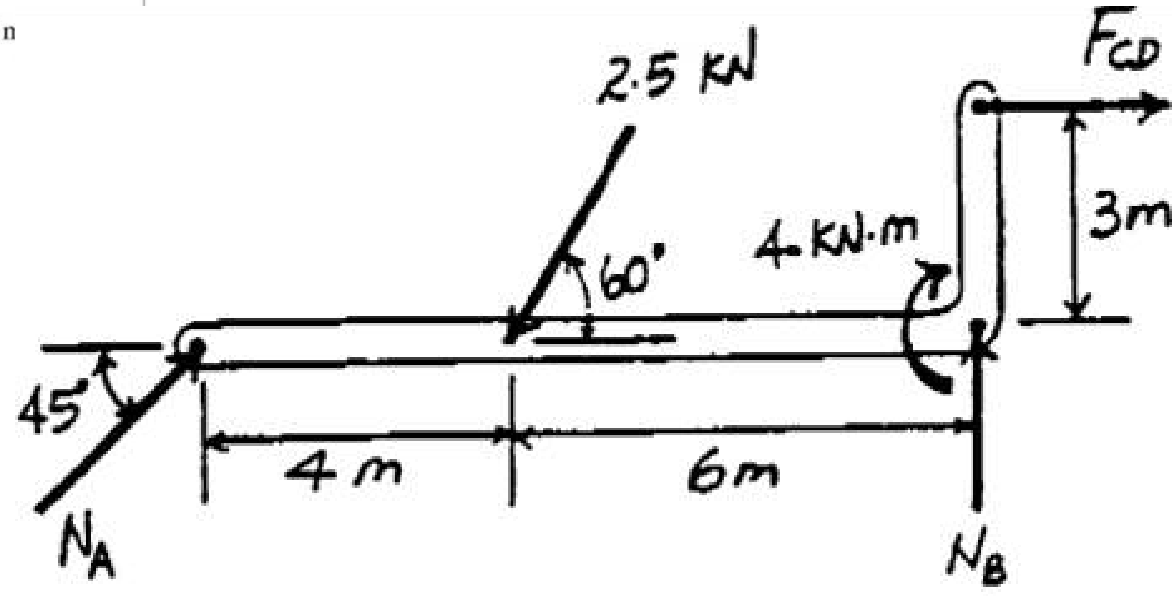




FBD

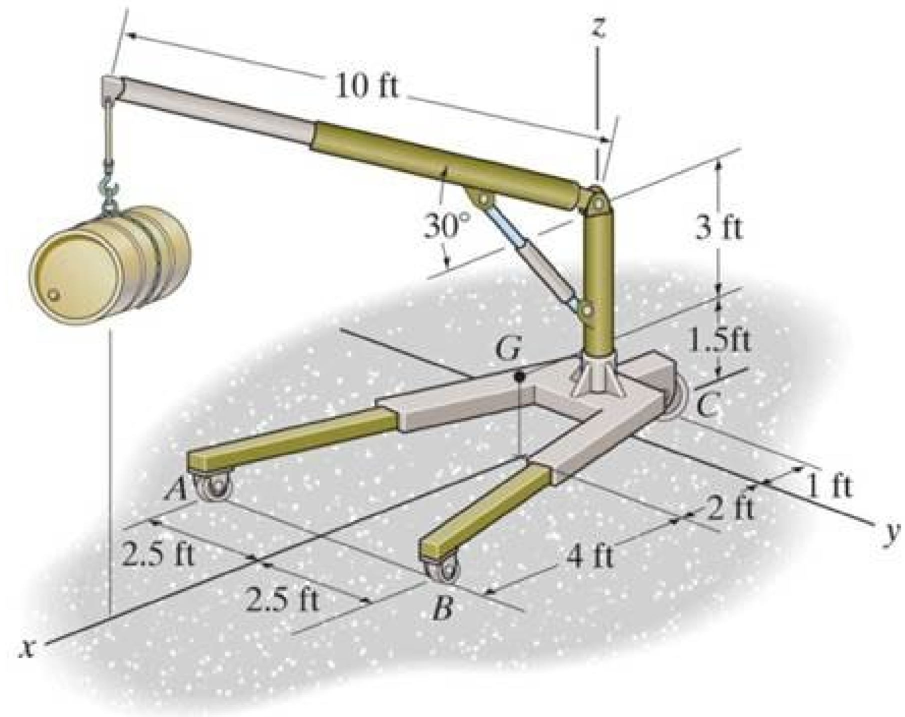
Solution





FBD

Application example

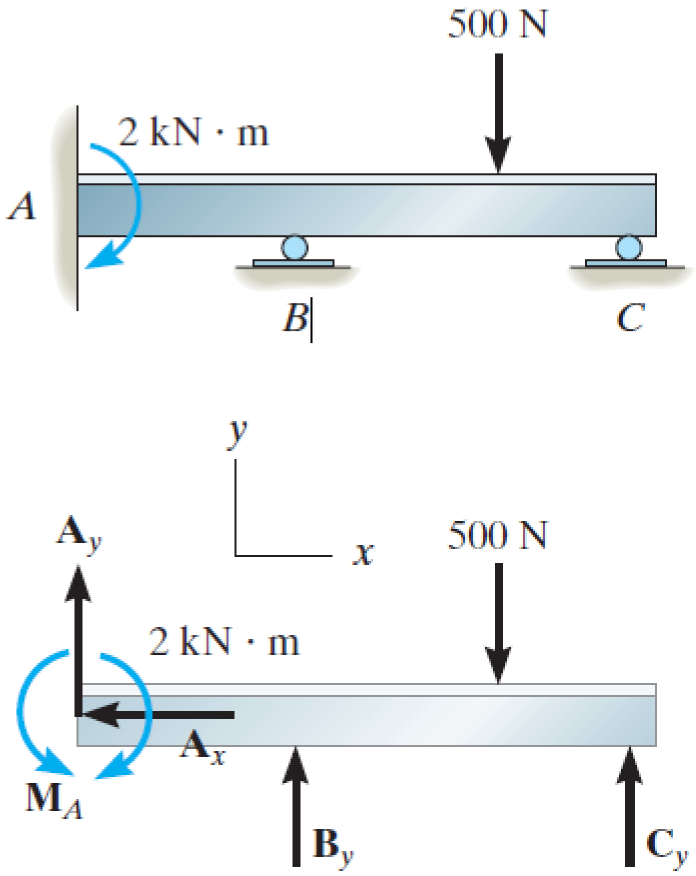


The crane, which weighs 200 lb, is supporting an oil drum.

What is the largest oil drum weight that the crane can support without overturning ?

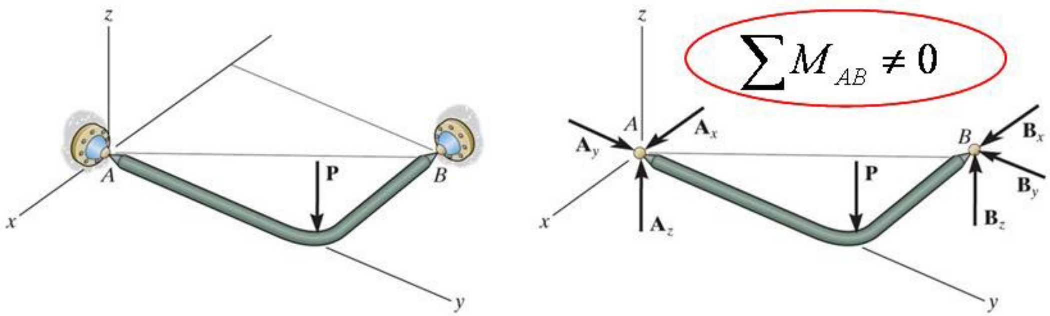
Constraints and statical determinacy

- Some bodies may have more supports than are necessary for equilibrium
- A problem is *statically determinate* if we can find all of the support reactions using only equilibrium equations
- If there are more unknowns than equations, we need additional information such as compatibility conditions
- *Statically indeterminate* means that there will be more unknown loadings on the body than equations of equilibrium available for their solution.

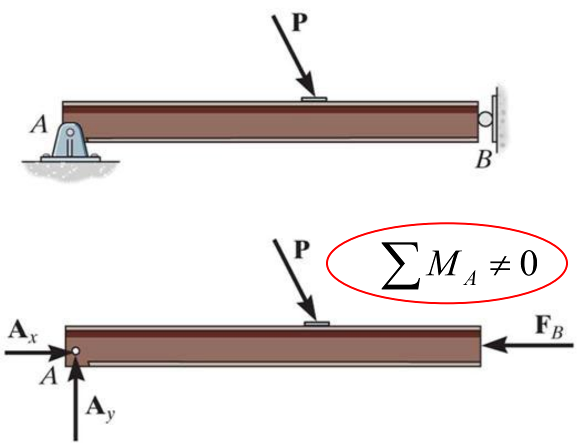


(a)

Improper constraints



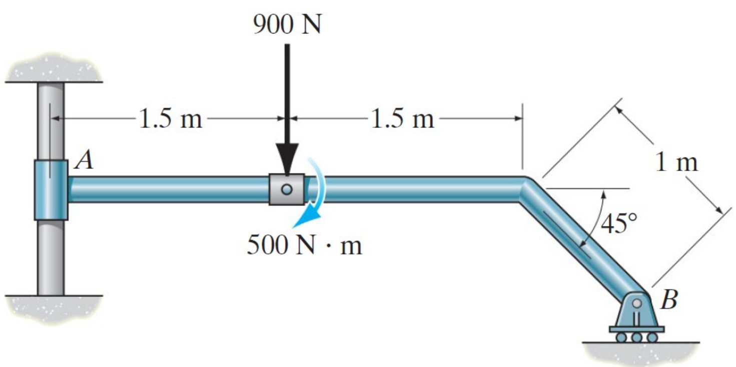
Here, we have 6 unknowns but there is nothing restricting rotation about the AB axis.



In some cases, there may be as many unknown reactions as there are equations of equilibrium.

However, if the supports are not properly constrained, the body may become unstable for some loading cases.

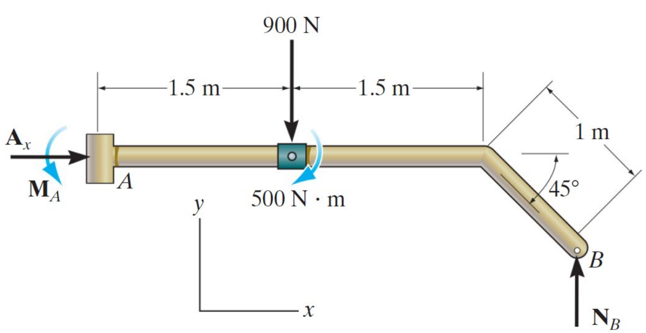
Example



Determine the support reactions on the pipe. The collar at A is fixed to the member and can slide vertically along the vertical shaft

Since we have three unknowns, the problem is statically determinate... Why?

First, let's draw the FBD



(b)

We can use the equilibrium equations

$$\sum F_x = 0 \Rightarrow A_x = 0$$

$$\sum F_y = 0 \Rightarrow N_B - 900\text{ N} = 0 \Rightarrow N_B = 900\text{ N}$$

$$\sum M_A = 0 \Rightarrow M_A - (1.5)(900) - 500 + (3 + \cos(45°))(N_B) = 0 \Rightarrow M_A = 1486\text{ N} \cdot \text{m}$$

Equations of equilibrium

As stated earlier, when a body is in equilibrium, the net force and the net moment equal zero

2D

$$\boxed{\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M_O &= 0 \end{aligned}}$$

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma M_A &= 0 \\ \Sigma M_B &= 0 \end{aligned}$$

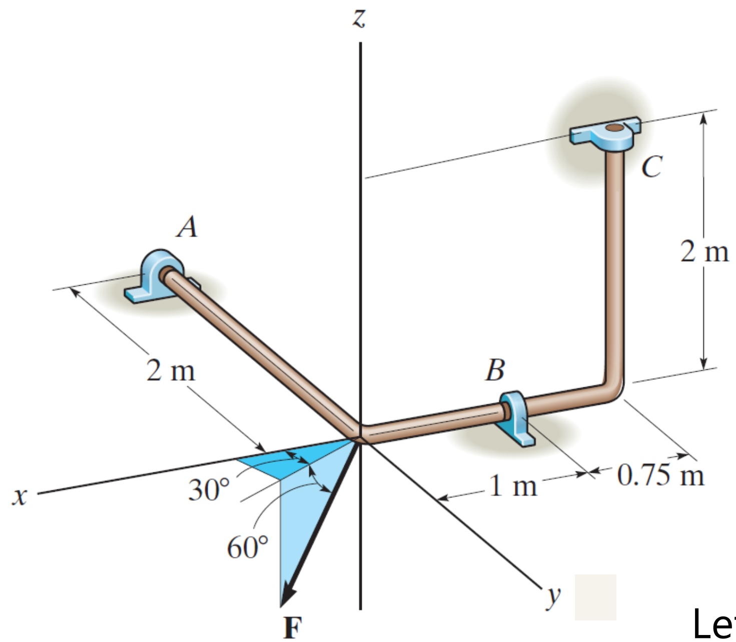
the line passing through points *A* and *B* is *not parallel* to the *y* axis.

3D

$$\boxed{\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} \Sigma M_x &= 0 \\ \Sigma M_y &= 0 \\ \Sigma M_z &= 0 \end{aligned}}$$

The moment equations can be determined about any point. Usually, choosing the point where the maximum number of unknown forces are present simplifies the solution. Any forces occurring at the point where moments are taken do not appear in the moment equation since they pass through the point.



Example

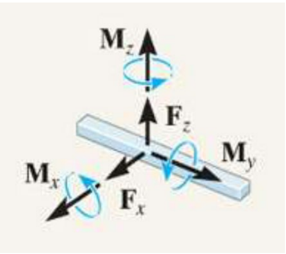
The bent rod is supported at *A, B, C* by smooth journal bearings. Determine the components of reaction at the bearings if the rod is subjected to the force *F* = 800 N. The bearings are in proper alignment and exert only force reactions on the rod.

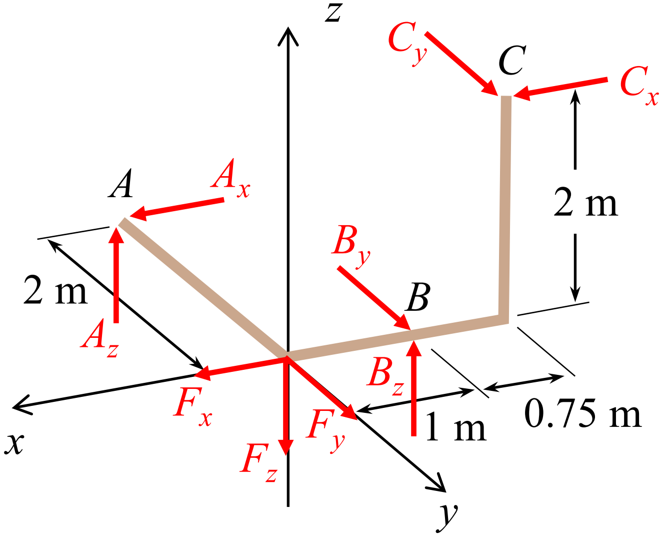
Let's begin by calculating the components of **F**

$$F_x = 800\cos(60°)\cos(30°) = 346.41\text{ N}$$

$$F_y = 800\cos(60°)\sin(30°) = 200\text{ N}$$

$$F_z = 800\sin(60°) = 692.82\text{ N}$$





We can now sum forces

$$F_x + A_x + C_x = 0$$

$$F_y + B_y + C_y = 0$$

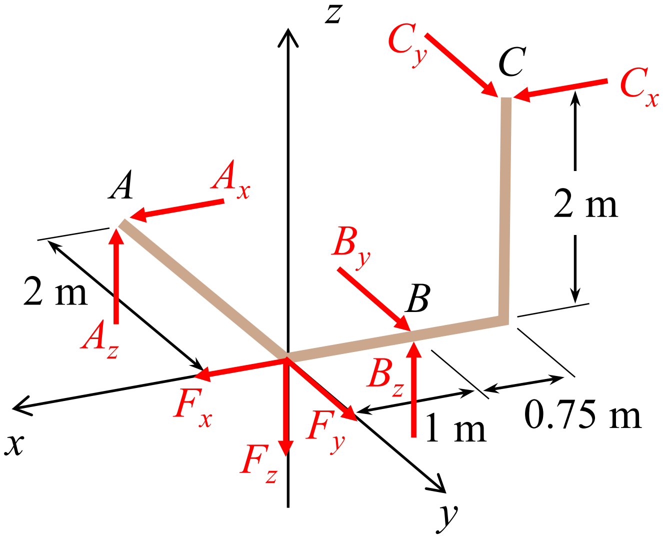
$$-F_z + A_z + B_z = 0$$

And moments on *A* in *x, y, z*

$$-2F_z + 2B_z - 2C_y = 0$$

$$B_z + 2C_x = 0$$

$$-2F_x - 1B_y - 1.75C_y - 2C_x = 0$$



- We now have 6 equations with 6 unknowns
- One way to solve this problem is to write it in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 & -2 & -1.75 \end{bmatrix} \begin{bmatrix} A_x \\ A_z \\ B_y \\ B_z \\ C_x \\ C_y \end{bmatrix} = \begin{bmatrix} -F_x \\ -F_y \\ F_z \\ F_z \\ 0 \\ 2F_x \end{bmatrix}$$

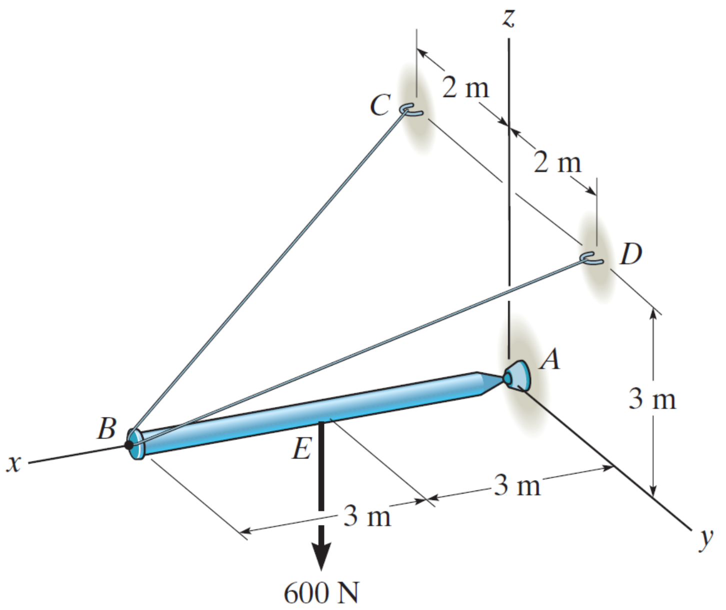
Any system of linear equations of the form $Ax = b$ can be solved by $x = A^{-1}b$

Which gives

$$\begin{bmatrix} A_x \\ A_z \\ B_y \\ B_z \\ C_x \\ C_y \end{bmatrix} = \begin{bmatrix} -400\ \mathrm{N} \\ 800\ \mathrm{N} \\ 600\ \mathrm{N} \\ -107.18\ \mathrm{N} \\ 53.59\ \mathrm{N} \\ -800\ \mathrm{N} \end{bmatrix}$$

Notice some signs are negative, which means our assumption on direction was incorrect

Try other methods at home, such as substitution or reduction. The result is the same

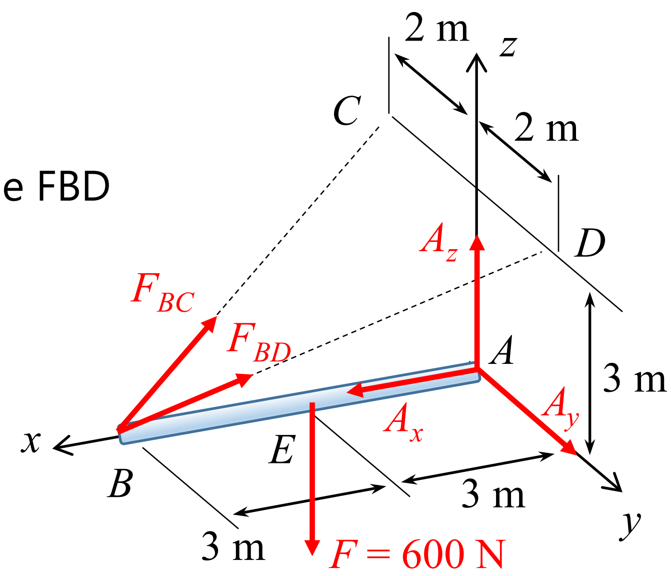


Individual work (15 min)

Determine the reactions at the ball-and-socket joint A and the tension in each cable necessary for equilibrium of the rod

Solution

Let's begin with the FBD



In order to sum forces, we need the components of $\mathbf{F_{BC}}$ and $\mathbf{F_{BD}}$

We can use the unit vector so that $\mathbf{F} = F\,\hat{\mathbf{u}}_F$

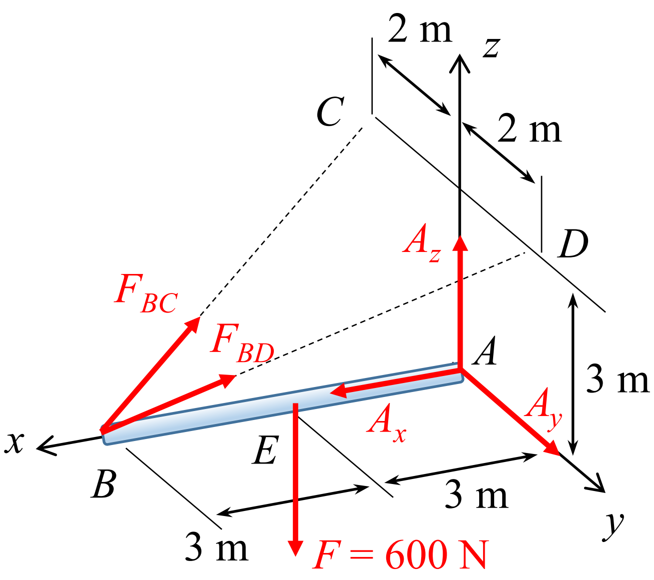
In order to find the unit vectors of $\mathbf{F}_{BC}$ and $\mathbf{F}_{BD}$ we can use the position vectors $\mathbf{r}_{BC}$ and $\mathbf{r}_{BD}$

$$\mathbf{r}_{BC} = (0-6)\hat{\mathbf{i}} + (-2-0)\hat{\mathbf{j}} + (3-0)\hat{\mathbf{k}}$$

$$\mathbf{r}_{BC} = -6\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

$$r_{BC} = \sqrt{(-6)^2 + (-2)^2 + (3)^2} = 7$$

$$\hat{\mathbf{u}}_{\mathrm{BC}} = \frac{\mathbf{r}_{\mathrm{BC}}}{r_{BC}} = -\frac{6}{7}\hat{\mathbf{i}} - \frac{2}{7}\hat{\mathbf{j}} + \frac{3}{7}\hat{\mathbf{k}}$$



In a similar way, we can obtain the unit $\hat{\mathbf{u}}_{BD}$ vector

$$\mathbf{r}_{\mathrm{BD}} = -6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

$$r_{BD} = \sqrt{(-6)^2 + (2)^2 + (3)^2} = 7$$

$$\hat{\mathbf{u}}_{BD} = \frac{\mathbf{r}_{\mathrm{BD}}}{r_{BD}} = -\frac{6}{7}\hat{\mathbf{i}} + \frac{2}{7}\hat{\mathbf{j}} + \frac{3}{7}\hat{\mathbf{k}}$$

This means the components of $\mathbf{F}_{BC}$ and $\mathbf{F}_{BD}$ are

$$\mathbf{F}_{BC} = -\frac{6}{7}F_{BC}\,\hat{\mathbf{i}} - \frac{2}{7}F_{BC}\,\hat{\mathbf{j}} + \frac{3}{7}F_{BC}\,\hat{\mathbf{k}}$$

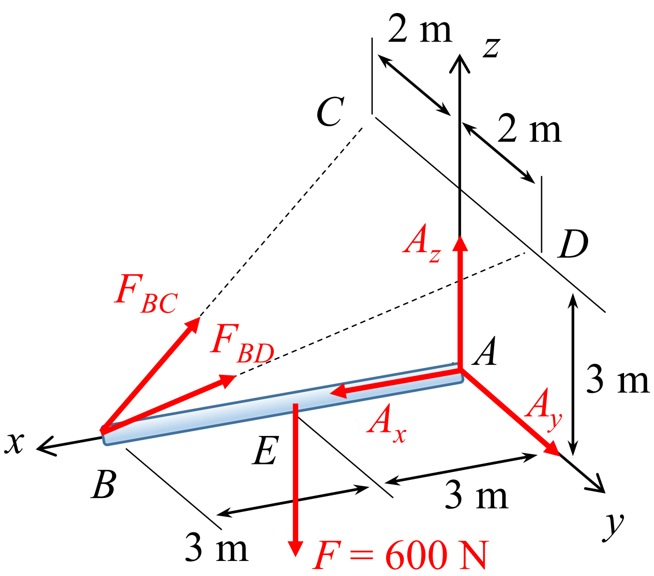
$$\mathbf{F}_{\mathrm{BD}} = -\frac{6}{7}F_{BD}\,\hat{\mathbf{i}} + \frac{2}{7}F_{BD}\,\hat{\mathbf{j}} + \frac{3}{7}F_{BD}\,\hat{\mathbf{k}}$$

We can now use equilibrium of forces in *x, y, z*

$$A_x - \frac{6}{7}F_{BC} - \frac{6}{7}F_{BD} = 0$$

$$A_y - \frac{2}{7}F_{BC} + \frac{2}{7}F_{BD} = 0$$

$$A_z - 600 + \frac{3}{7}F_{BC} + \frac{3}{7}F_{BD} = 0$$



We still need three more equations. We can sum moments around A to obtain the remaining information

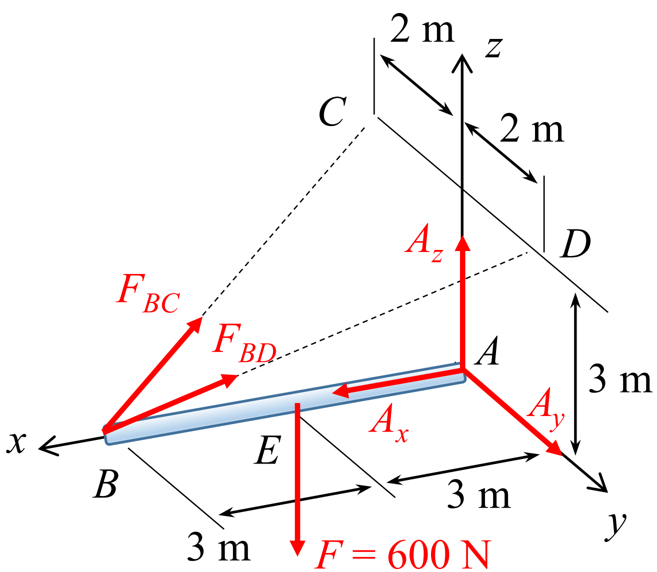
$$\sum \mathbf{M}_{\mathrm{A}} = \mathbf{r}_{\mathrm{E}} \times \mathbf{F} + \mathbf{r}_{\mathrm{B}} \times \mathbf{F}_{\mathrm{BC}} + \mathbf{r}_{\mathrm{B}} \times \mathbf{F}_{\mathrm{BD}} = \mathbf{0}$$

Notice this represents <u>three</u> scalar equations!

$$\sum \mathbf{M}_{\mathrm{A}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 0 & 0 \\ 0 & 0 & -600 \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 0 & 0 \\ -\frac{6}{7}F_{BC} & -\frac{2}{7}F_{BC} & \frac{3}{7}F_{BC} \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 0 & 0 \\ -\frac{6}{7}F_{BD} & \frac{2}{7}F_{BD} & \frac{3}{7}F_{BD} \end{vmatrix} = \mathbf{0}$$

$$\sum \mathbf{M}_{\mathrm{A}} = 1800\,\hat{\mathbf{j}} - \frac{18}{7}F_{BC}\,\hat{\mathbf{j}} - \frac{12}{7}F_{BC}\,\hat{\mathbf{k}} - \frac{18}{7}F_{BD}\,\hat{\mathbf{j}} + \frac{12}{7}F_{BD}\,\hat{\mathbf{k}} = \mathbf{0}$$

$$\left[1800 - \frac{18}{7}(F_{BC} + F_{BD})\right]\hat{\mathbf{j}} + \frac{12}{7}(F_{BD} - F_{BC})\hat{\mathbf{k}} = \mathbf{0}$$



Which means our remaining equations are:

$$1800 - \frac{18}{7}(F_{BC} + F_{BD}) = 0$$

$$\frac{12}{7}(F_{BD} - F_{BC}) = 0$$

In other words, $M_y = M_z = 0$

Immediately, we have $F_{BD} - F_{BC} = 0 \Rightarrow F_{BD} = F_{BC}$

So that $1800 - \frac{18}{7}2(F_{BC}) = 0 \Rightarrow \boxed{F_{BC} = 350\text{ N}}$ $\boxed{F_{BD} = 350\text{ N}}$

Which gives $A_z - 600 + \frac{3}{7}(350) + \frac{3}{7}(350) = 0 \Rightarrow \boxed{A_z = 300\text{ N}}$

Also, $A_y - \frac{2}{7}(350) + \frac{2}{7}(350) = 0 \Rightarrow \boxed{A_y = 0\text{ N}}$

Finally, $A_x - \frac{6}{7}(350) - \frac{6}{7}(350) = 0 \Rightarrow \boxed{A_x = 600\text{ N}}$