Homework Assignment #6

Remember, this Homework Assignment is **not collected or graded**! But it is in your best interest to do it as the Homework Quiz will be based on it and it is the best way to ensure you know the material.

Section 2.7: Linear Transformations

1. For the vector space P_3 of polynomials with degree at most 3. What matrix represents taking the second derivative: $\frac{d^2}{dt^2}$. Remember in this context we are representing a vector in P_3 in terms of it's standard basis which makes polynomials in this vector space appear like vectors in $\mathbb{R}^4$.

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \implies \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Solution: First, let's take the second derivative of a generic vector in P_3.

$$\frac{d^2}{dt^2}p(x) = \frac{d}{dt}\left(\frac{d}{dt}\left(a_0 + a_1x + a_2x^2 + a_3x^3\right)\right) = \frac{d}{dt}\left(a_1 + 2a_2x + 3a_3x^2\right) = 2a_2 + 6a_3x.$$

We see that we are left with a vector in P_1. Since we can treat P_3 like $\mathbb{R}^4$ we can treat P_1 as a vector in $\mathbb{R}^2$. Our second derivative matrix, A, thus will have dimension 2×4 and satisfy the following:

$$A\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2a_2 \\ 6a_3 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2a_2 \\ 6a_3 \end{bmatrix}.$$

To determine the nullspace we consider: $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \end{array}\right] \implies x_1 = s, x_2 = t, x_3 = x_4 = 0.$$

Thus,

$$N(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\} = \{a_0 + a_1x \text{ for } a_0, a_1 \in \mathbb{R}\}.$$

As expected, the nullspace of the matrix that performs the second derivative on an element of P_3 has as its nullspace the subspace of P_3 consisting of polynomials with largest power 1 (i.e., linear functions.).

2. P_3 is the vector space of polynomials with degree at most 3. We again represent a "vector" in this space as follows:

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$$

Let S to be the set of all polynomials in P_3 with $\int_0^1 p(x)dx = 0$.

(a) Prove that S is a subspace by showing it is closed under addition and scalar multiplication.

Solution: We need to verify that S is a subspace by verifying that it is closed under addition of vectors from and the multiplication of a vector by a scalar.
First, assume $p_1(x)$ and $p_2(x)$ belong to S. This means that:

$$\int_0^1 p_1(x)dx = \int_0^1 p_2(x)dx = 0.$$

To show S is closed under addition we need to verify that if $p(x) = p_1(x) + p_2(x)$ then we have $\int_0^1 p(x)dx = 0.$
Fortunately, we already know (from Calculus) that for integrable functions $f(x)$ and $g(x)$ we have:

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Since polynomials are integrable functions we have:

$$\int_0^1 p(x)dx = \int_0^1 (p_1(x) + p_2(x))\,dx = \int_0^1 p_1(x)dx + \int_0^1 p_2(x)dx = 0 + 0 = 0.$$

Thus $p(x) \in S$ and S is closed under addition.
To show that S is closed under scalar multiplication. We need to show that if $p(x) \in S$ then $g(x) = \alpha p(x)$ also belongs to S for any $\alpha \in \mathbb{R}$.
Again, we appeal to what we learned in Calculus where for an integrable function $f(x)$:

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx.$$

Thus we have:

$$\int_0^1 g(x)dx = \int_0^1 \alpha p(x)dx = \alpha \int_0^1 p(x)dx = \alpha(0) = 0.$$

Thus, $g(x) \in S$ and S is closed under scalar multiplication.
Thus S is a subspace of P_3.

(b) Find a basis for S. (Hint: Enforce the conditions that have to hold on the coefficients of $p(x)$ for the integral to be 0.)

Solution: We will take the integral of a generic element of P_3 and determine which conditions need to hold in order for its integral from 0 to 1 to be 0.

$$\begin{aligned} 0 = \int_0^1 p(x)dx &= \int_0^1 \left(a_0 + a_1x + a_2x^2 + a_3x^3\right) dx \\ &= \left(a_0x + (a_1/2)x^2 + (a_2/3)x^3 + (a_3/4)x^4\right)\Big|_0^1 \\ &= (a_0 + (a_1/2) + (a_2/3) + (a_3/4)) - 0. \end{aligned}$$

Taken together we have:

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 = 0.$$

We have 1 equation and 4 unknowns which means that we have 3 free variables:

$a_3 = t, a_2 = s, a_1 = r$ and

$$a_0 = -\frac{1}{2}r - \frac{1}{3}s - \frac{1}{4}t$$

$$\text{Basis} = \left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Or to write in a way that expresses these vectors as polynomials in P_3:

$$\text{Basis} = \left\{-1/2 + x, -1/3 + x^2, -1/4 + x^3\right\}.$$

Section 3.1: Orthogonal Vectors and Subspaces

3. Suppose B is an invertible $n \times n$ matrix. How do we know that the i-th row of an invertible matrix B is orthogonal to the j-th column of B^{-1} if $i \neq j$.

Solution: First, let's figure out some notation. Let's suppose A and B are $n \times n$ matrices and $C = AB$. As usual we will denote the entry in the i-th row and j-th column of A, B and C respectively by lower case letters, $a_{i,j}, b_{i,j}$ and $c_{i,j}$. Then we know that:

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{j,k} = \vec{a}_i^T \vec{b}_j.$$

Where $\vec{a}_i$ is the i-th row of A and $\vec{b}_j$ is the j-th column of B.

Notice that if we were looking at $BB^{-1} = I$

$$i_{i,j} = \vec{b}_i^T \vec{b}_j^{-1} = i\text{-th row of } B \times j\text{-th column of } B^{-1}.$$

Since if $i \neq j$ the identity matrix has a 0, we know that this product will be 0.

4. Find the orthogonal complement in $\mathbb{R}^3$ of the vector space V consisting of a plane spanned by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Solution: We know that the orthogonal complement in $\mathbb{R}^3$ is the set of vectors that are orthogonal to every vector in V. Since we know that any vector in V is a linear combination of $\vec{v}_1$ and $\vec{v}_2$ where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

we can just look for all vectors $\vec{v} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ where:

$$\vec{v}^T \vec{v}_1 = 0 \text{ and } \vec{v}^T \vec{v}_2 = 0.$$

$$\vec{v}_T\vec{v}_1 = x_1 + x_2 + 2x_3 = 0 \text{ and } \vec{v}^T\vec{v}_2 = x_1 + 2x_2 + 2x_3 = 0.$$

This gives us:

$$A\vec{v} = \vec{0} \implies \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

This system has 2 pivots and only one free variable ($x_3 = t$). The second row gives $x_2 = 0$ and the top row gives

$$x_1 + x_2 + 2x_3 = 0 \implies x_1 + 0 + 2t = 0 \implies x_1 = -2t.$$

This gives us:

$$\vec{v} = \left\{ t \begin{bmatrix} -2 & 0 & 1 \end{bmatrix}^T | t \in \mathbb{R} \right\}.$$

Thus we can see that a basis for $V^\perp$ is given by:

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note that V has dimension 2 and $V^\perp$ has dimension 1 (as expected) since we are in $\mathbb{R}^3$.

5. Find a basis for the orthogonal complement of the rowspace of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

Solution: There are two conceptual ways to approach this problem. We will see that both these conceptual ways lead towards the exact same mathematical formulation.

In the first we can remember that the rowspace of A: $C(A^T)$ has orthogonal complement $N(A)$. Thus, we are looking for a basis for $N(A)$. This means we are looking for solutions to:

$$A\vec{x} = \vec{0} \implies \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]. \tag{1}$$

In the second way, we simply notice the rows of the matrix A and realize that we are looking at the vector space defined by the span of the two rows: $\vec{a}_1, \vec{a}_2$:

$$V = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

We can simply look for vectors $\vec{x}$ so that:

$$\vec{x}^T\vec{a}_1 = \vec{x}^T\vec{a}_2 = 0.$$

This gives us:

$$x_1 + 2x_3 = 0 \text{ and } x_1 + x_2 + 4x_3 = 0 \implies \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]. \tag{2}$$

But notice that equation (1) and equation (2) are the same! So these two conceptual ways of solving the problem lead us to the same mathematical equation!

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

The system has 2 pivots and 2 free variable. The free variable gives us $x_3 = t$ and the rows give us: $x_2 = -2t$ and $x_1 = -2t$. Thus, the solution to this system is a 1-dimensional space:

$$V = \left\{ t \begin{bmatrix} -2 & -2 & 1 \end{bmatrix}^T | t \in \mathbb{R} \right\}.$$

Thus the basis for this vector space is:

$$\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

6. Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. Show that $(\vec{x} - \vec{y})$ is orthogonal to $(\vec{x} + \vec{y})$ if and only if $\|\vec{x}\| = \|\vec{y}\|$. Note that: $\|x\|^2 = \vec{x}^T \vec{x}$. (Hint: This problem is easier than it looks, just write out the inner product.)

Solution: Let's suppose $\vec{x}, \vec{y} \in \mathbb{R}^n$. We know that:

$$\overrightarrow{(x-y)} = \{(x_1 - y_1), (x_2 - y_2), \ldots, (x_n - y_n)\} \text{ and}$$

$$\overrightarrow{(x+y)} = \{(x_1 + y_1), (x_2 + y_2), \ldots, (x_n + y_n)\}.$$

Let's suppose that $\overrightarrow{(x-y)}$ is orthogonal to $\overrightarrow{(x+y)}$. Then we have the following:

$$0 = \overrightarrow{(x-y)}^T \overrightarrow{(x+y)} = \sum_{i=1}^n (x_i - y_i)(x_i + y_i) = \sum_{i=1}^n (x_i^2 - y_i^2) = \sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2 = \|\vec{x}\|^2 - \|\vec{y}\|^2.$$

The beginning and end of the equation give us:

$$0 = \|\vec{x}\|^2 - \|\vec{y}\|^2 \implies \|\vec{x}\|^2 = \|\vec{y}\|^2 \implies \|\vec{x}\| = \|\vec{y}\|.$$

As such, we have that $\overrightarrow{(x-y)}$ is orthogonal to $\overrightarrow{(x+y)}$ if and only if $\|\vec{x}\| = \|\vec{y}\|$.

7. Suppose that A is a symmetric matrix. That is, $A^T = A$.

(a) Why is its column space perpendicular to its nullspace?

Solution: For a symmetric matrix we know that $C(A) = C(A^T)$. But for any matrix, we know that its row space is the orthogonal complement to its nullspace. That is, $C(A^T)$ is perpendicular to $N(A)$.
But since $C(A^T) = C(A)$ we know that for a symmetric matrix $C(A)$ is perpendicular to $N(A)$.

(b) If $A\vec{x} = \vec{0}$ and $A\vec{z} = 5\vec{z}$, which fundamental subspaces contain $\vec{x}$ and $\vec{z}$?

Solution: We know that $\vec{x} \in N(A)$ because $A\vec{x} = \vec{0}$. But since $A = A^T$ we know that $\vec{x} \in N(A^T)$.
Similarly, since $A\vec{z} = 5\vec{z} \implies A(1/5)\vec{z} = \vec{z}$ we know that $\vec{z} \in C(A)$. But since $A = A^T$ we know that $\vec{z} = C(A^T)$.
Thus, we know that $\vec{z}$ is perpendicular to $\vec{x}$.

Section 3.2: Projections onto Lines

8. Find the matrix that projects any point in $\mathbb{R}^2$ onto the line $x + y = 0$.

Solution: We first need to determine the direction the line is in. Our equation can be written as the following vector matrix system:

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}.$$

We have 1 pivot and 1 free variable. Let's let $y = t$ then $x = -t$ and our line can be written as $t\vec{a}$ for $t \in \mathbb{R}$ where:

$$\vec{a} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We know that:

$$P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

9. Project the vector $\vec{b}$ onto the line through $\vec{a}$. Denote this projected vector $\vec{p}$ and verify that $\vec{e} = \vec{b} - \vec{p}$ is perpendicular to $\vec{a}$.

(a) $\vec{b} = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}^T$ and $\vec{a} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

Solution: We will begin by constructing $P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}$ and then calculating $P\vec{b} = \vec{p}$.

$$P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$P\vec{b} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}.$$

Now we need to find $\vec{e} = \vec{b} - \vec{p}$ and verify that $\vec{e}$ is perpendicular to $\vec{a}$.

$$\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

We thus have $\vec{e}^T\vec{a}$:

$$\vec{e}^T\vec{a} = \begin{bmatrix} -2/3 & 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -2/3 + 1/3 + 1/3 = 0.$$

(b) $\vec{b} = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}^T$ and $\vec{a} = \begin{bmatrix} -1 & -3 & -1 \end{bmatrix}$.

> **Solution:** We will begin by constructing $P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}$ and then calculating $P\vec{b} = \vec{p}$.
>
> $$P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{11}\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}\begin{bmatrix} -1 & -3 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}.$$
>
> $$P\vec{b} = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ 33 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$
>
> Notice that in this case $\vec{b} = \vec{p}$. This is because we actually already have $\vec{b}$ in the direction of $\vec{a}$.
> As such, $\vec{e} = \vec{b} - \vec{p} = \vec{0}$. Notice then $\vec{e}^T\vec{a} = 0$ by default and thus $\vec{e}$ is orthogonal to $\vec{a}$.

10. Find the projection matrix P that projects every point in $\mathbb{R}^3$ onto the line of intersection between the planes:

$$\begin{aligned} x + y + t &= 0 \\ x - t &= 0. \end{aligned}$$

> **Solution:** First, we have to find the line that satisfies both planes. We will solve the following vector matrix equation:
>
> $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right].$$
>
> This system has 2 pivots and 1 free variable. Even though we started off with x, y and t as our coordinates. For simplicity, we will stick with dimensions x_1, x_2 and x_3. As such, we will set $x_3 = t$. The last row specifies $x_2 = -2t$ and the top row gives:
>
> $$x_1 + x_2 + x_3 = 0 \implies x_1 - 2t + t = 0 \implies x_1 - t = 0 \implies x_1 = t.$$
>
> As such our line is of the form:
>
> $$t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$
>
> We next construct our projection matrix, P with our usual formula:
>
> $$P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$