Homework Assignment #11

Remember, this Homework Assignment is **not collected or graded**! But it is in your best interest to do it as the this material will be covered on your Homework Quiz.

This Homework will go through properties of matrices, eigenvalues and eigenvectors that, while important, have not yet come up in our course.

Section 5.5: Symmetric and Hermitian Matrices

1. If $x = 2 + i$ and $y = 1 + 3i$ find: $\overline{x}, x\overline{x}, xy, 1/x, x/y$.
   Check that the absolute value $|xy|$ equals $|x|$ times $|y|$ and that the absolute value $|1/x| = 1/|x|$.

Solution:

- $\overline{x} = \overline{2+i} = 2 - i$
- $x\overline{x} = (2+i)(\overline{2+i}) = (2+i)(2-i) = 4 - 2i + 2i - i^2 = 4 - (-1) = 5.$
- $xy = (2+i)(1+3i) = 2 + i + 6i + 3i^2 = 2 - 3 + 4i = -1 + 7i$
- $1/x$:
$$\frac{1}{x} = \frac{1}{x}\left(\frac{\overline{x}}{\overline{x}}\right) = \frac{1}{(2+i)}\frac{(2-i)}{(2-i)} = \frac{(2-i)}{(2+i)(2-i)} = \frac{(2-i)}{5} = (2/5) - i(1/5).$$
- x/y:
$$\frac{x}{y} = \frac{x}{y}\left(\frac{\overline{y}}{\overline{y}}\right) = \frac{x\overline{y}}{|y|^2} = \frac{(2+i)(1-3i)}{1+3^2} = \frac{2+i-6i-3i^2}{10} = \frac{5-5i}{10} = \frac{1}{2} - i\frac{1}{2}.$$
- Verifying $|xy| = |x||y|$
$$|xy| = |-1+7i| = \left((-1+7i)(\overline{-1+7i})\right)^{1/2} = \left((-1)^2 + 7^2\right)^{1/2} = \sqrt{50}$$
$$|x| = \sqrt{x\overline{x}} = \sqrt{5} \text{ and } |y| = \sqrt{y\overline{y}} = \sqrt{1^2+3^2} = \sqrt{10}$$
$$|xy| = \sqrt{50} = \sqrt{5}\sqrt{10} = |x||y|$$
- Verifying $|1/x| = 1/|x|$
$$|1/x| = |2/5 - i(1/5)| = \left((2/5)^2 + (1/5)^2\right)^2 = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{x\overline{x}}} = \frac{1}{|x|}.$$

2. Write out the matrix A^H and compute $C = A^H A$ if
$$A = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}.$$
   What special property do C and C^H hold?

Solution:

$$C = A^H A = \overline{\begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}}^T \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$$

While there might be different properties of C that we would look at, because we're familiar with real matrices, we might be inclined to think that just as $A^T A$ and AA^T are symmetric for real matrices, $A^H A$ might lead to a Hermitian matrix.

$$C^H = (A^H A)^H = A^H (A^H)^H = A^H A = C.$$

Thus, we see that C is Hermitian. We also can verify this directly by taking its conjugate transpose:

$$C^H = \overline{\begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix}}^T = \begin{bmatrix} 2 & -i & i \\ i & 1 & 0 \\ -i & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix} = C$$

3. Here we will consider determinants of matrices with complex entries.

   (a) How does the determinant of A^H relate to the determinant of A? (Hint: Try thinking about the LU decomposition of A.)

Solution: This question might seem a bit challenging, but we can use two different properties about matrices and determinants to prove this.

First, let's notice that even matrices with complex values can be decomposed into the LU decomposition using Gaussian Elimination. Thus there exists an invertible lower triangular matrix L with 1's on the diagonal and an upper triangular matrix U that represents the row-echelon form of A so that $A = LU$.

Then, we remember that the determinant of a product is the product of the determinants:

$$A = LU \implies \det(A) = \det(LU) = \det(L)\det(U) = 1\det(U) = \prod_{i=1}^{n} u_{ii}.$$

We have used the that the determinant of a lower (or upper) triangular matrix is the product of the diagonal entries. For L these are all 1's and for U these are the product of the diagonal entries.

Second, we know that $A^H = (LU)^H = U^H L^H$. Since L^H and U^H are still triangular matrices (although L^H is now upper triangular and U^H is now lower triangular) their determinants are the product of their values on the diagonal:

$$\det(A^H) = \det(U^H L^H) = \det(U^H)\det(L^H) = \prod_{i=1}^{n} \overline{u_{ii}}$$

because $\det(L^H) = 1$ because all the values on the diagonal of L^H and the diagonal entries of U^H are the conjugate of the diagonal entries of U.

Thus, this means that:

$$\det(A^H) = \overline{\det(A)}.$$

   (b) Prove that the determinant of any Hermitian matrix is real.

Solution: From the previous part of our problem, we have:

$$\det(A^H) = \overline{\det(A)}.$$

But if A is Hermitian, we have $A^H = A$, thus:

$$\det(A^H) = \det(A) \implies \det(A) = \overline{\det(A)}.$$

The only numbers that are equal to their own conjugates are real numbers.

4. Find a third column so that U is unitary.

$$U = \begin{bmatrix} 1/\sqrt{3} & i/\sqrt{2} & u_1 \\ 1/\sqrt{3} & 0 & u_2 \\ i/\sqrt{3} & 1/\sqrt{2} & u_3 \end{bmatrix}.$$

Solution: Here we will use the fact that a unitary matrix U satisfies: $U^H U = I$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U^H U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -i/\sqrt{3} \\ -i/\sqrt{2} & 0 & 1/\sqrt{2} \\ \overline{u_1} & \overline{u_2} & \overline{u_3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & i/\sqrt{2} & u_1 \\ 1/\sqrt{3} & 0 & u_2 \\ i/\sqrt{3} & 1/\sqrt{2} & u_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{3}}(u_1 + u_2 - iu_3) \\ 0 & 1 & \frac{1}{\sqrt{2}}(-iu_1 + u_3) \\ \frac{1}{\sqrt{3}}(\overline{u_1} + \overline{u_2} + i\overline{u_3}) & \frac{1}{\sqrt{2}}(i\overline{u_1} + \overline{u_3}) & \overline{u_1}u_1 + \overline{u_2}u_2 + \overline{u_3}u_3 \end{bmatrix}$$

Notice that the off diagonal entries are themselves conjugates of one another. And from the 2×3 entry we are left with

$$\frac{1}{\sqrt{2}}(-iu_1 + u_3) = 0 \implies u_1 = -iu_3.$$

From the 1×3 entry we have:

$$\frac{1}{\sqrt{3}}(u_1 + u_2 - iu_3) = 0 \implies (u_1 + u_2 - iu_3) = 0 \implies -2iu_3 + u_2 = 0 \implies u_2 = 2iu_3.$$

The last entry gives us:

$$1 = \overline{u_1}u_1 + \overline{u_2}u_2 + \overline{u_3}u_3 = \overline{-iu_3}(-iu_3) + \overline{2iu_3}(2iu_3) + \overline{u_3}u_3 = i(-i)\overline{u_3}u_3 + (-2i)(2i)\overline{u_3}u_3 + \overline{u_3}u_3$$

$$= \overline{u_3}u_3(1 + 4 + 1) \implies 1/6 = \overline{u_3}u_3 = \|u_3\|^2.$$

Thus, we select a complex value that lies on the circle of radius $1/6$. We could go ahead and pick $u_3 = 1/\sqrt{6}$. Then we have:

$$u_1 = -i(1/\sqrt{6}), u_2 = 2i(1/\sqrt{6}), u_3 = 1/6.$$

Note that there are infinitely many choices for this vector.

Section 5.6: Similarity Transformations

5. Consider similar matrices A and B. Show that if λ is an eigenvalue for A that it is also an eigenvalue for B.

(Hint: Remember that two matrices are similar if there exists an M such that: $B = M^{-1}AM$.)

Solution: By the definition of similarity we have: $BM^{-1} = M^{-1}A$. If λ is an eigenvalue of A then we have:

$$A\vec{x} = \lambda\vec{x}.$$

But then we have:

$$BM^{-1}\vec{x} = M^{-1}A\vec{x} = M^{-1}\lambda\vec{x} = \lambda M^{-1}\vec{x}.$$

Thus $M^{-1}\vec{x}$ is an eigenvector of B with the same eigenvalue λ.

6. Can A ever be similar to a matrix of the form $A+I$? (Hint: What are the eigenvalues of $A+I$ in terms of the eigenvalues of A?)

Solution: We know (and were reminded) in our previous problem similar matrices have the same eigenvalues. However, if λ is an eigenvalue of A with eigenvector $\vec{x}$ then:

$$(A+I)\vec{x} = A\vec{x} + I\vec{x} = \lambda\vec{x} + 1\vec{x} = (\lambda+1)\vec{x}.$$

Thus $A+I$ will always have eigenvalues that are shifted by 1 from A and therefore can never be similar to A.

7. If A has eigenvalues $0, 1$ and 2 find the eigenvalues of $A(A-I)(A-2I)$.

Solution: Let λ be an eigenvalue of A with eigenvector $\vec{x}$ then we have: $A\vec{x} = \lambda\vec{x}$.

We already showed before, but can be reminded again, that A^k has an eigenvalue λ^k for the same eigenvector. Here we will prove this by induction:

- Base Case:
$$A^2\vec{x} = A(A\vec{x}) = A\lambda\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x}.$$
- Inductive Hypothesis: Assume
$$A^k\vec{x} = \lambda^k\vec{x}$$
- Inductive Step:
$$A^{k+1}\vec{x} = AA^k\vec{x} = A\lambda^k\vec{x} = \lambda^k A\vec{x} = \lambda^{k+1}.$$

Thus we have:

$$\begin{aligned} A(A-I)(A-2I)\vec{x} &= (A^2-A)(A-2I)\vec{x} \\ &= (A^3-2A^2-A^2+2A)\vec{x} \\ &= A^3\vec{x} - 3A^2\vec{x} + 2A\vec{x} \\ &= \left(\lambda^3 - 3\lambda^2 + 2\lambda\right)\vec{x}. \end{aligned}$$

Since we have 3 choices for λ we have:

- $\lambda = 0 \implies 0^3 - 2(0^2) + 2(0) = 0$
- $\lambda = 1 \implies 1^3 - 3(1^2) + 2(1) = 1 - 3 + 2 = 0$
- $\lambda = 2 \implies 2^3 - 3(2^2) + 2(2) = 8 - 12 + 4 = 0$

Thus, all eigenvalues of the matrix A are equal to 0.

8. Find an orthogonal matrix Q that diagonalizes the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution: We see that A is symmetric, which we know means that we can always diagonalize A and that there is a set of orthogonal eigenvectors. Let's first find the eigenvectors and eigenvalues:

$$\begin{aligned} 0 = \det(A - \lambda I) &= \left|\begin{matrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{matrix}\right| \begin{matrix} 1-\lambda & 1 \\ 1 & 1-\lambda \\ 1 & 1 \end{matrix}\right| \\ &= (1-\lambda)^3 + 1 + 1 - 3(1-\lambda) \\ &= (1-\lambda)(1 - 2\lambda + \lambda^2) - 1 + 3\lambda \\ &= 1 - 2\lambda + \lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 1 + 3\lambda \\ &= 3\lambda^2 - \lambda^3 \\ &= \lambda^2(3 - \lambda). \end{aligned}$$

Thus, we have two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 3$. Let's now find the eigenvectors for each and enforce (if needed) their orthonormality.

- $\lambda_1 = 0$:

$$\vec{0} = A\vec{x} \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We have 1 pivot and 2 free variables: $x_3 = t$, $x_2 = s$ and $x_1 = -t - s$. This gives us:

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

We notice that we have two linearly independent vectors, but they are not orthogonal! Thus we will use the Gram-Schmidt Process to enforce orthogonality:

$$\vec{q}_1 = \frac{\vec{x}_1}{\|\vec{x}\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\vec{Q}_2 = \vec{x}_2 - (\vec{x}_2^T \vec{q}_1)\vec{q}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - (1/\sqrt{2}) \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{Q}_2}{\|\vec{Q}_2\|} = \begin{bmatrix} -1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \\ -1/\sqrt{6} \end{bmatrix}.$$

- $\lambda_2 = 3$.

$$\vec{0} = (A - 3I)\vec{x} \implies \left[\begin{array}{ccc|c} 1-3 & 1 & 1 & 0 \\ 1 & 1-3 & 1 & 0 \\ 1 & 1 & 1-3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + (1/2)R_1]{R_2 \to R_2 + (1/2)R_1}$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Here we have 2 pivots and 1 free variable. $x_3 = t, x_2 = t$ and $x_1 = t$

$$\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies \vec{q}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Thus we have the following:

$$Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Section 6.1-6.2: Positive Definite Matrices

9. Test if A^TA is positive definite for these 3 choices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Solution: We will determine the positive definite status of each matrix by computing their eigenvalues. Because A^TA is symmetric for any choice of A, we know the eigenvalues will be real. So the question we have is are the eigenvalues positive, negative or zero.

Matrices with only positive eigenvalues are positive definite.

(a)

$$A^TA = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}.$$

$$0 = \det(A^TA - \lambda I) = \lambda^2 - \text{tr}(A^TA)\lambda + \det(A^TA) = \lambda^2 - 14\lambda + 9$$

$$\lambda_{1,2} = \frac{14 \pm \sqrt{14^2 - 4(9)}}{2} = \frac{14 - \sqrt{160}}{2} = 7 \pm 2\sqrt{10}.$$

Since $2\sqrt{10} < 7$ we have both eigenvalues as larger than 0, thus the matrix A^TA is positive definite.

(b)

$$A^TA = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}.$$

$$0 = \det(A^TA - \lambda I) = \lambda^2 - \text{tr}(A^TA)\lambda + \det(A^TA) = \lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1).$$

Since the eigenvalues are 11 and 1 and both are positive, A^TA is positive definite.

(c)

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

$$\begin{aligned} 0 = \det(A^TA - \lambda I) &= \left| \begin{matrix} 2-\lambda & 3 & 3 \\ 3 & 5-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{matrix} \right| \begin{matrix} 2-\lambda & 3 \\ 3 & 5-\lambda \\ 3 & 4 \end{matrix} \right| \\ &= (2-\lambda)(5-\lambda)^2 + 36 + 36 - 9(5-\lambda) - 16(2-\lambda) - 9(5-\lambda) \\ &= (2-\lambda)(25 - 10\lambda + \lambda^2) + 72 - 122 + 34\lambda \\ &= \lambda^3 + 12\lambda^2 - 45\lambda + 50 - 50 + 34\lambda \\ &= \lambda^3 + 12\lambda^2 - 11\lambda \\ &= \lambda(\lambda^2 + 12\lambda - 11) \\ &= \lambda(\lambda - 11)(\lambda - 1). \end{aligned}$$

Thus the system has 3 eigenvalues: $11, 1, 0$. Since 2 are positive and 1 is 0, the matrix A^TA is positive semi-definite.

10. Are the following matrices positive definite, positive semi-definite, negative definite, negative semi-definite or indefinite.

$$A = \begin{bmatrix} 1 & (1+i) \\ (1-i) & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & (4+i) \\ (4-i) & 6 \end{bmatrix}.$$

Note: Both matrices are Hermitian. We only care about the positive definite status of matrices that are either real and symmetric or Hermitian.

Solution: First, we notice that both matrices are Hermitian. That is:

$$A^H = \begin{bmatrix} \overline{1} & \overline{1-i} \\ \overline{1+i} & \overline{2} \end{bmatrix} = \begin{bmatrix} 1 & (1+i) \\ (1-i) & 2 \end{bmatrix} = A$$

$$B^H = \begin{bmatrix} \overline{3} & \overline{(4-i)} \\ \overline{(4+i)} & \overline{6} \end{bmatrix} = \begin{bmatrix} 3 & (4+i) \\ (4-i) & 6 \end{bmatrix} = B$$

In order to determine their positive definite status, we need to find their eigenvalues.

(a) A:

$$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = \lambda^2 - 3\lambda + (2 - (1+i)(1-i)) = \lambda^2 - 3\lambda + (2-2) = \lambda(\lambda - 3)$$

Since the eigenvalues of this matrix are 0 and 3 this matrix is positive semi-definite.

(b) B:

$$0 = \det(B - \lambda I) = \lambda^2 - \mathrm{tr}(B)\lambda + \det(B) = \lambda^2 - 9\lambda + (18 - (4^2 + 1^2)) = \lambda^2 - 9\lambda + 1$$

$$\lambda_{1,2} = \frac{9 \pm \sqrt{81 - 4(1)}}{2}.$$

Since $\sqrt{81-4} < \sqrt{81} = 9$ we know that both roots are positive. Thus this matrix is positive definite.

11. For which values of b is the following matrix positive definite:

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}.$$

Solution: We first notice that A is symmetric, these are the only matrices whose positive definite status is something we are concerned with.

We then look for the characteristic equation:

$$0 = \det(A - \lambda I) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 - 10\lambda + (9 - b^2).$$

The eigenvalues for this system are:

$$\lambda_{1,2} = \frac{10 \pm \sqrt{100 - 4(9 - b^2)}}{2} = 5 \pm \sqrt{16 + b^2}.$$

One of these eigenvalues will always be positive: $\lambda_1 = 5 + \sqrt{16 + b^2}$. But the other one we have to look at the sign of:

$$\lambda_2 = 5 - \sqrt{16 + b^2}$$

$$\lambda_2 > 0 \implies 5 - \sqrt{16 + b^2} > 0 \implies 5 > \sqrt{16 + b^2} \implies 25 > 16 + b^2 \implies 9 > b^2.$$

Thus, when

- $-3 < b < 3$ the matrix is positive definite (2 positive eigenvalues)
- $-3 = b$ or $b = 3$ the matrix is positive semi-definite (1 positive and 1 zero eigenvalue)
- $b < -3$ or $b > 3$ the matrix is indefinite (1 positive and 1 negative eigenvalue)

Section 6.3: Singular Value Decomposition

12. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Compute the singular value decomposition of A. What special structure is true about this singular value decomposition. (Hint: A satisfies a very important property.)

Solution: We will use the method we discussed in class.

- We will first determine the eigenvalues and orthonormal eigenvectors of $A^T A$.
- Then (if 0 is an eigenvalue) we will determine the orthonormal eigenvectors of AA^T corresponding to 0 as an eigenvalue.
- Then we will enforce:

$$A\vec{v}_j = \sigma_j \vec{u}_j$$

where $\lambda_j = \sigma_j^2$ is any non-zero eigenvalue of $A^T A$.

- Determine the eigenvalues and eigenvectors of A^TA:

$$A^TA = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 0 & 2\\0 & 1 & 0\\2 & 0 & 2\end{bmatrix}.$$

$$\begin{aligned}0 = \det(A^TA - \lambda I) &= \left|\begin{matrix}2-\lambda & 0 & 2\\0 & 1-\lambda & 0\\2 & 0 & 2-\lambda\end{matrix}\,\middle|\,\begin{matrix}2-\lambda & 0\\0 & -\lambda\\2 & 0\end{matrix}\right|\\ &= (2-\lambda)^2(1-\lambda) + 0 + 0 - 4(1-\lambda) - 0 - 0\\ &= (1-\lambda)\left((2-\lambda)^2 - 4\right)\\ &= (1-\lambda)(\lambda^2 - 4\lambda + 4 - 4)\\ &= (1-\lambda)(\lambda)(\lambda - 4).\end{aligned}$$

This we have $\lambda = 4, 1, 0$ which correspond to $\sigma_1 = 2, \sigma_2 = 1$ and $\sigma_3 = 0$ as singular values. We next determine the eigenvectors:

  - $\lambda = 4$:

$$\vec{0} = (A^TA - 4I)\vec{x} \implies$$

$$\left[\begin{matrix}2-4 & 0 & 2\\0 & 1-4 & 0\\2 & 0 & 2-4\end{matrix}\,\middle|\,\begin{matrix}0\\0\\0\end{matrix}\right] = \left[\begin{matrix}-2 & 0 & 2\\0 & -3 & 0\\2 & 0 & -2\end{matrix}\,\middle|\,\begin{matrix}0\\0\\0\end{matrix}\right] = \left[\begin{matrix}-2 & 0 & 2\\0 & -3 & 0\\0 & 0 & 0\end{matrix}\,\middle|\,\begin{matrix}0\\0\\0\end{matrix}\right].$$

    We have 2 pivots and 1 free variable. Our eigenvector $\vec{x}_1$ and normal eigenvector $\vec{v}_1$ are:

$$\vec{x}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} \implies \vec{v}_1 = \begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}.$$

  - $\lambda = 1$

$$\vec{0} = (A^TA - I)\vec{x} \implies$$

$$\left[\begin{matrix}2-1 & 0 & 2\\0 & 1-1 & 0\\2 & 0 & 2-1\end{matrix}\,\middle|\,\begin{matrix}0\\0\\0\end{matrix}\right] = \left[\begin{matrix}1 & 0 & 2\\0 & 0 & 0\\2 & 0 & 1\end{matrix}\,\middle|\,\begin{matrix}0\\0\\0\end{matrix}\right] = \left[\begin{matrix}1 & 0 & 2\\0 & 0 & 0\\0 & 0 & -3\end{matrix}\,\middle|\,\begin{matrix}0\\0\\0\end{matrix}\right].$$

    We have 2 pivots and 1 free variable. Thus our eigenvector $\vec{x}_2 = \vec{v}_2$ is the following:

$$\vec{x}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} = \vec{v}_2.$$

    Notice that the eigenvectors we have found are orthogonal thus far. This is to be expected because we are looking at the eigenvectors from different eigenvalues of a symmetric matrix.

  - $\lambda = 0$

$$\vec{0} = A^TA\vec{x} \implies$$

$$\left[\begin{matrix}2 & 0 & 2\\0 & 1 & 0\\2 & 0 & 2\end{matrix}\,\middle|\,\begin{matrix}0\\0\\0\end{matrix}\right] = \left[\begin{matrix}2 & 0 & 2\\0 & 1 & 0\\0 & 0 & 0\end{matrix}\,\middle|\,\begin{matrix}0\\0\\0\end{matrix}\right].$$

    We have 2 pivots and 1 free variable. The final $\vec{x}_3$ and $\vec{v}_3$ are given by:

$$\vec{x}_3 = \begin{bmatrix}-1\\0\\1\end{bmatrix} \implies \vec{v}_3 = \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}.$$

Thus in total our V matrix is given by:

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}.$$

- Now we need to find the eigenvector that corresponds to the eigenvalue 0 for AA^T. But let's notice that because A is symmetric we have $A^TA = AA^T$.

  $\lambda = 0$

$$\vec{0} = AA^T\vec{x} \implies$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

  We have 2 pivots and 1 free variable and indeed see that this is the same eigenvector we found in the last part. $\vec{u}_3 = \vec{v}_3$.

- We next need to find the other two columns of U. We (again) will enforce our condition:

$$\vec{u}_j = \frac{1}{\sigma_j} A\vec{v}_j$$

  - $\vec{u}_1$:
    We enforce:

$$\vec{u}_1 = \frac{1}{2}A\vec{v}_1 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \vec{v}_1$$

  - $\vec{u}_2$:
    We enforce:

$$\vec{u}_2 = A\vec{v}_1 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{v}_2$$

  Finally, we note that $U = V$ and that our SVD is simply a eigenvector diagonalization. This is because A is symmetric:

$$A = U\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}U^T$$

13. Suppose that the singular value decomposition of A, a 5 by 4 matrix, is the following:

$$A = U\Sigma V^T = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 & \vec{u}_5 \end{bmatrix}\begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \\ \vec{v}_4^T \end{bmatrix}.$$

    (a) Are the columns of U eigenvectors of A^TA or AA^T? Are the columns of V eigenvectors of A^TA or AA^T?

Solution:
$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T$$
Thus the columns of U are the eigenvectors of AA^T.

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V.$$
Thus the columns of V are the eigenvectors of $A^T A$.

(b) What is the rank of A?

Solution: The rank of A is equal to the number of non-zero singular values (i.e., non-zero values of the diagonal of Σ.) In this case it's 3.

(c) What is the dimension of $N(A)$? What is a basis for $N(A)$? (In terms of the columns of U or V.)

Solution: The dimension of $N(A)$ is the number of columns of A (i.e., dimension of V) minus the rank. Since A is a 5 by 4 matrix the dimension of $N(A)$ is 4 - 3 = 1. A basis for $N(A)$ is given by the fourth column in V which is $\vec{v}_4$.

(d) What is the dimension of $N(A^T)$? What is a basis for $N(A^T)$? (In terms of the columns of U or V.)

Solution: The dimension of $N(A^T)$ is the number of columns of A^T (i.e., dimension of U) minus the rank. Since A is a 5 by 4 matrix, and A^T a 4 by 5 dimensional matrix, the dimension of $N(A^T)$ is 5 - 3 = 2. A basis for $N(A^T)$ is given by the last two columns in U which are $\{\vec{u}_4, \vec{u}_5\}$.

(e) What is a basis for $C(A)$? (In terms of the columns of U or V.)

Solution: A basis for $C(A)$ is given by the the first 3 columns of U: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

(f) What is a basis for $C(A^T)$? (In terms of the columns of U or V.)

Solution: A basis for $C(A^T)$ is given by the first 3 columns of V which correspond to $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.