

Homework Assignment #2

Remember, this Homework Assignment is **not collected or graded!** But you are advised to do it anyway because the problems for Homework Quiz #2 will be heavily based on these problems!

1. Determine all values of the constant k for which the following system has (a) no solution, (b) an infinite number of solutions, (c) a unique solution.

$$\begin{aligned}x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 + 5x_2 + x_3 &= 7 \\ x_1 + x_2 - k^2 x_3 &= -k.\end{aligned}$$

2. Consider the following linear system $A\vec{x} = \vec{b}$ where,

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix}$$

and $\vec{b} = \begin{bmatrix} b \\ b \\ b \end{bmatrix}$ for a real number b .

- (a) Use elementary matrices to reduce the augmented matrix A to row-echelon form.
 - (b) Determine the values of b where so that $A\vec{x} = \vec{b}$ has at least 1 solution and solve the linear system.
3. Consider the following system:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 5 \end{bmatrix}$$

- (a) Use elementary row operations to write the LU factorization for A (or PA if necessary).
 - (b) Use the LU factorization to solve the system by solving the corresponding two triangular systems.
4. A farmer with 1200 acres is planning to plant 3 kinds of crops: corn, soybeans and oats. The cost of each crop is different per acre, corn seed costs \$20 per acre, while soybean seed costs \$50 per acre and oat costs \$15 per acre. The farmer has \$40,000 to spend on seeds and will spend it all.
 - (a) Use the information in the problem to formulate two linear equations with three unknowns. Explain, in words, what each equation means and what the unknowns are.
 - (b) Solve the system you wrote, and explain the solution set.
 - (c) We now introduce a third constraint to the farmer's plan, when he sells his crops he can earn \$100 per acre of corn, \$300 per acre of soybean and \$80 per acre of oats. Suppose the farmer is unusually specific in his goals and wants his crops to bring in **exactly** \$230,000. Will he be able to meet this specific goal?
 - (d) Explain both geometrically and algebraically how this additional constraint changes the solution set you found in part b.
 - (e) (*Challenge Problem: Optional*) Suppose the farmer, realizing he might not be achieving the full benefit of his land, instead wants to maximize the amount of money he can earn from selling his crops. Determine the maximum amount the farmer can earn, provided he still meets the two original constraints.

5. The trace of a matrix A , denoted $\text{tr}(A)$, is the sum of its entries along the diagonal. For example, if A is the matrix below:

$$A = \begin{bmatrix} 1 & 7 \\ 2 & 6 \end{bmatrix} \implies \text{tr}(A) = 1 + 6 = 7.$$

- (a) **Prove** that if A and B are two 2×2 matrices that $\text{tr}(AB) = \text{tr}(BA)$.
- (b) (*Challenge Problem: Optional*) **Prove** that if A and B are $n \times n$ matrices that $\text{tr}(AB) = \text{tr}(BA)$. You might find it helpful to remember the formula for matrix multiplication. If A and B are $n \times n$ matrices and $C = AB$, then

$$c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$$