ENGR 065 Electric Circuits

Lecture 13: The Laplace Transform and the Functional/Operational Transform

Today's Topics

- What is the Laplace transform?
- The functional transform
  - The Laplace transform of specified functions of t
- The operational transform
  - Mathematical operations of how f(t) is transformed to F(s) or vice versa
    1. Multiplication by a constant
    2. Addition/subtraction
    3. Differentiation
    4. Integration
    5. Translation in the time domain
    6. Translation in the frequency domain
    7. Scale changing

- Covered in Sections 12.1, 12.2, 12.3, 12.4 and 12.5

Laplace Transform

Why Laplace transform in this course?

Transform a set of differential equations in the time domain to a set of algebraic equations in the frequency domain.

How does it work?

a. Transform a problem from the time domain to the frequency domain
b. Obtain the solutions for the problem in the frequency domain
c. Inversely transform the solutions back to the time domain

Pierre-Simon Laplace (1749–1827).

A French scientist who made important contributions in engineering, mathematics, statistics, physics, astronomy, and philosophy.

Laplace Transform

- Definition of Laplace transform

The Laplace transform (one-sided, unilateral) of a function $f(t)$ is defined by:

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

It is also denoted as $F(s) = \mathcal{L}\{f(t)\}$, where $f(t) \leq ce^{kt}, c > 0, k > 0$.

The Functional Transform

- The functional transform is the Laplace transform of specified functions of t.
- Some of these specified functions are:
    1. Impulse function: $\delta(t)$
    2. Step function: $u(t)$
    3. Ramp function: t
    4. Exponential function: e^{-at}
    5. Sinusoidal function: $sin(\omega t)$
    6. Sinusoidal function: $cos(\omega t)$

Some Useful Results

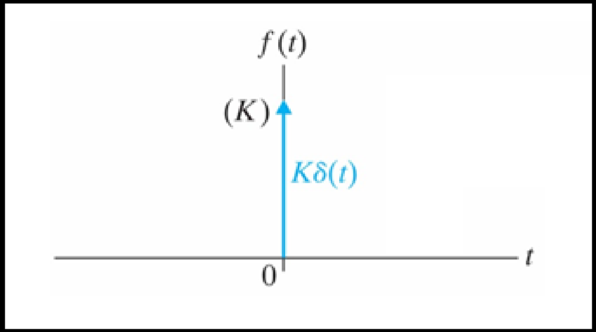
- $\lim_{t\to\infty} e^{\alpha t} = +\infty \ (\alpha > 0)$
- $\lim_{t\to\infty} e^{-\alpha t} = 0 \ \ (\alpha > 0)$
- $\lim_{t\to 0} e^{\alpha t} = \lim_{t\to 0} e^{-\alpha t} = 1 \ \ (\alpha > 0)$
- $\frac{d}{dt}(e^{at}) = ae^{at}$
- $\int e^{at} dt = \frac{1}{a} e^{at} + c$
- $\frac{d}{dt}(uv) = \frac{du}{dt} v + u\frac{dv}{dt} = u'v + uv'$ (product rule)
- $\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{\frac{du}{dt}v - u\frac{dv}{dt}}{v^2} = \frac{u'v - uv'}{v^2}$ (quotient rule)
- If a function $f(t)$ is continuous on the interval [a, b], for every t in the interval [a, b], $\frac{d}{dt}\int_a^t f(x)dx = f(t)$

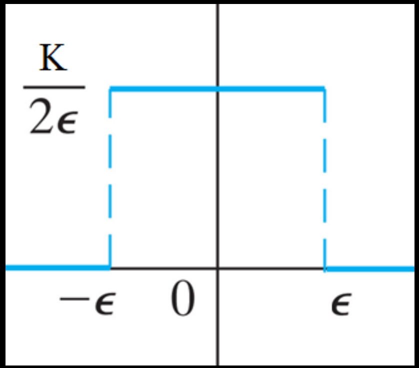
The Impulse Function

An impulse function is a function that is zero everywhere except at one single point, and when integrated over all reals gives a nonzero value. Mathematically, it is defined as

$$f(t) = \begin{cases} K\delta(t), & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} K\delta(t)dt = K$$

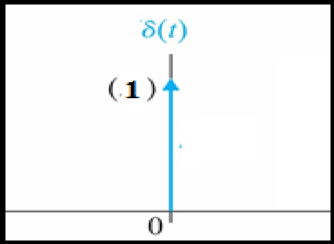




$\varepsilon \to 0$

If $K = 1$, the function is called the unit impulse function, and denoted as

$$\delta(t) = 0 \; for \; t \neq 0, and \int_{-\infty}^{+\infty} \delta(t)dt = 1$$



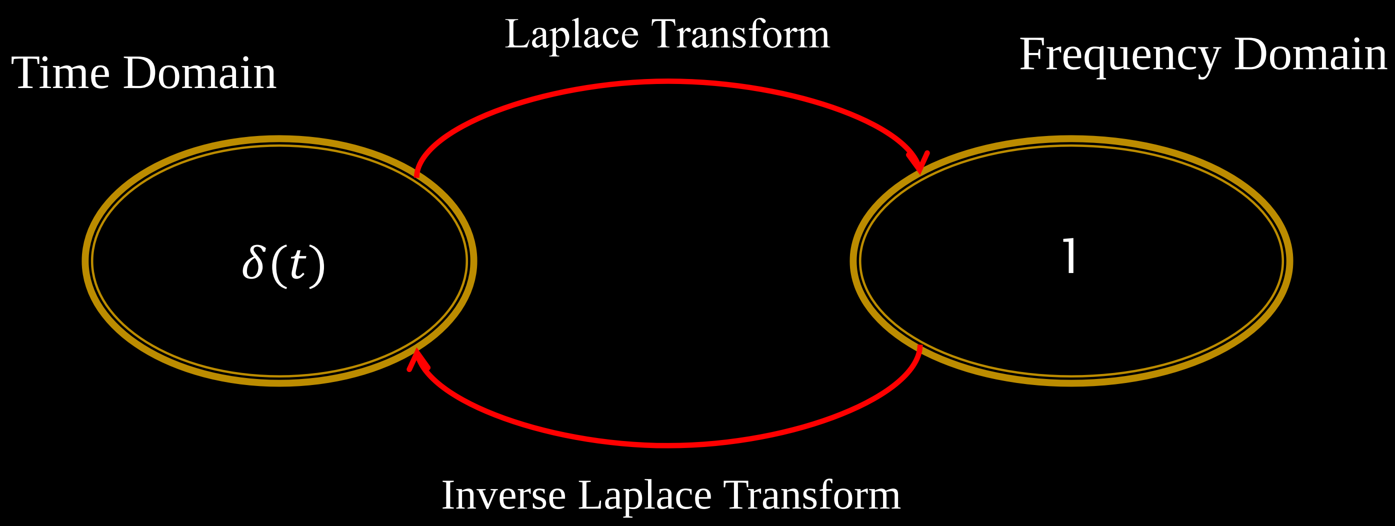
In fact, $\int_{-\infty}^{+\infty} \delta(t)dt = \int_{-\infty}^{0^-} \delta(t)dt + \int_{0^-}^{0^+} \delta(t)dt$

$$+\int_{0^+}^{\infty} \delta(t)dt = 0 + \int_{0^-}^{0^+} \delta(t)dt + 0 = \int_{0^-}^{0^+} \delta(t)dt = 1$$

The Laplace Transform of $\delta(t)$

- The Laplace transform of $\delta(t)$

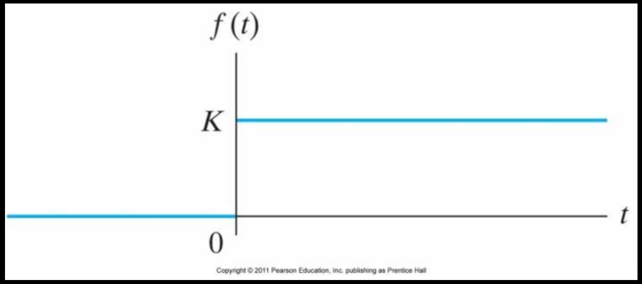
$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-s\times 0} = 1$$



The Step Function

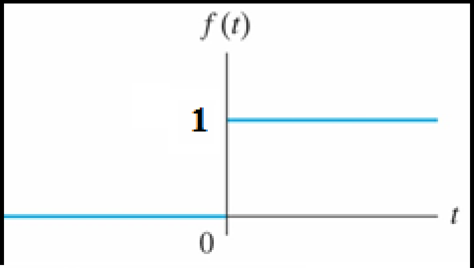
- The step function is a mathematical function of a single real variable that remains constant within each of a series of adjacent intervals but changes in value from one interval to the next.
- In this course, the step function is defined as:

$$f(t) = ku(t) = \begin{cases} 0, & t < 0 \\ K, & t > 0 \end{cases}$$



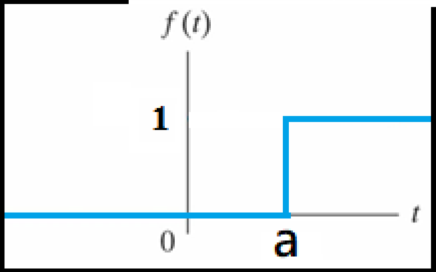
where

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$u(t)$ is called the unit step function.

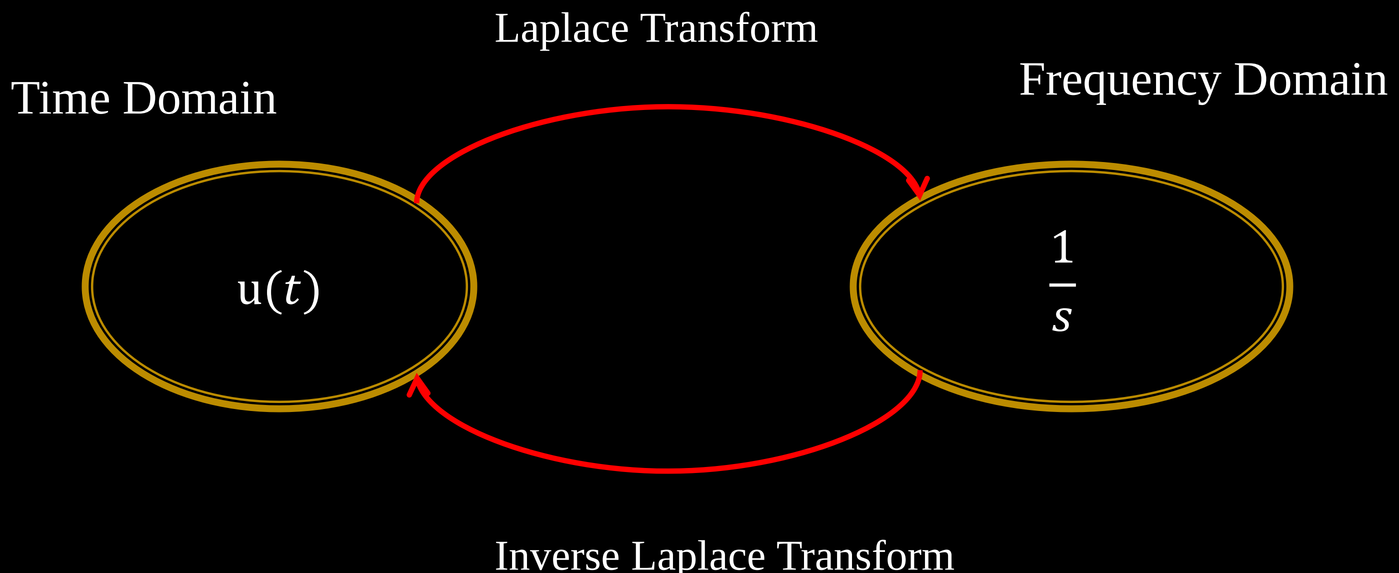
- $u(t-a) = \begin{cases} 0, & t-a < 0 \text{ (t<a)} \\ 1, & t-a > 0 \text{ (t>a)} \end{cases}$



The Laplace Transform of u(t)

- The Laplace transform of u(t)

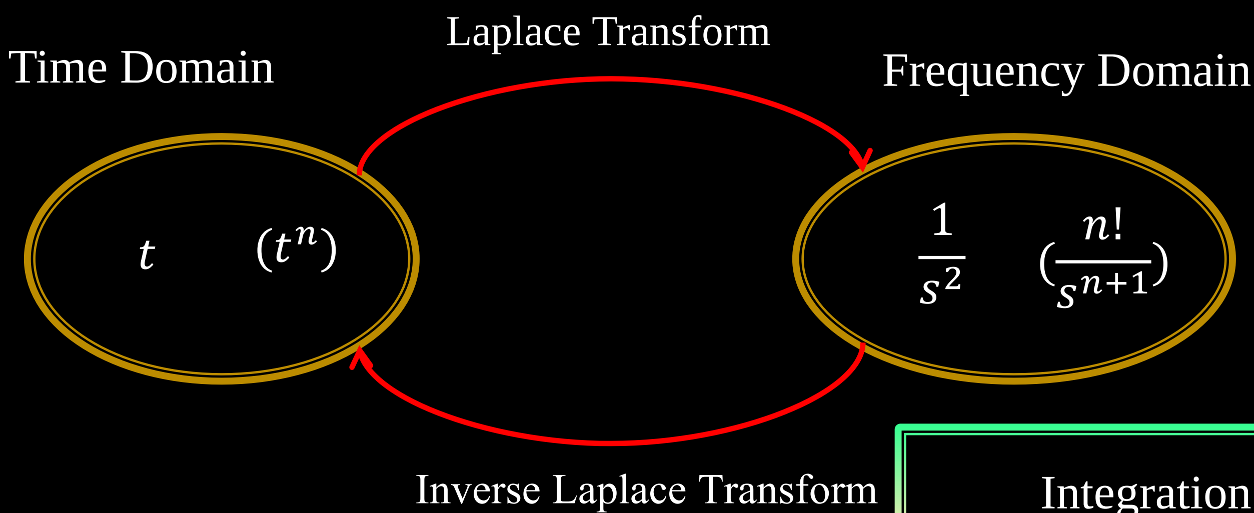
$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} u(t)e^{-st}dt = \int_{0^+}^{\infty} e^{-st}dt = \frac{1}{s}$$



The Laplace Transform of *t*

- The Laplace transform of t

$$\mathcal{L}\{t\} = \int_{0^-}^{\infty} t e^{-st} dt = \frac{1}{s^2}\,, \qquad s > 0$$



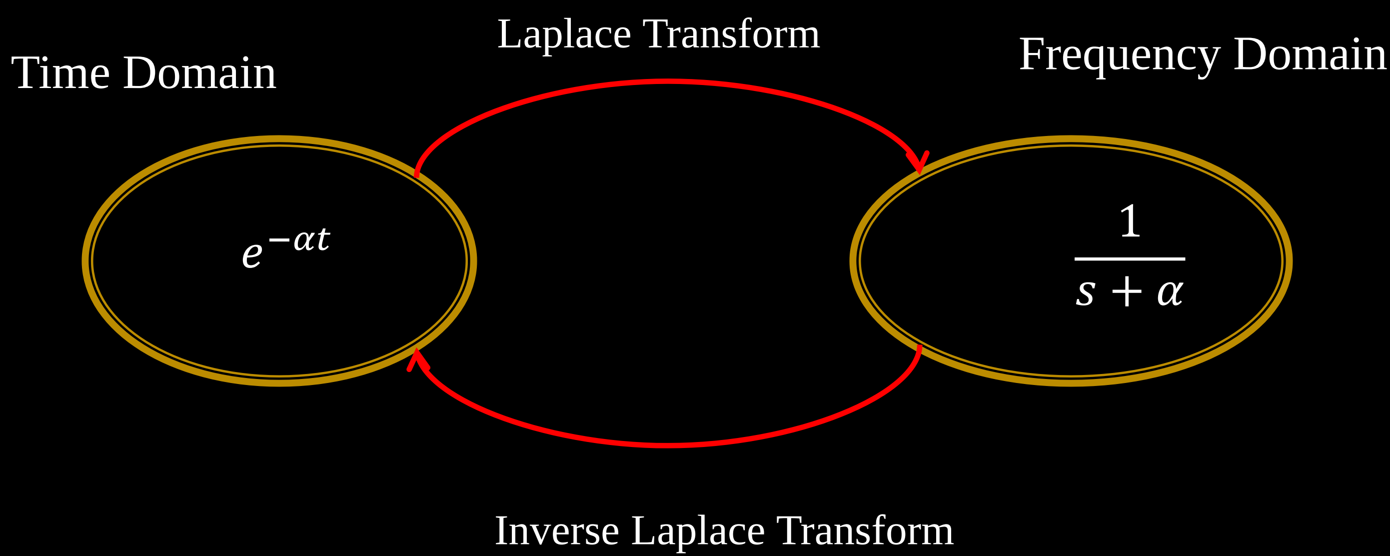
Integration by parts

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)dx$$

The Laplace Transform of $e^{-\alpha t}$

- The Laplace transform of $e^{-\alpha t}$

$$\mathcal{L}\{e^{-\alpha t}\} = \int_{0^-}^{\infty} e^{-\alpha t} e^{-st} dt = \int_{0^-}^{\infty} e^{-(s+\alpha)t} dt = \frac{1}{s+\alpha}\,, s + a > 0$$



The Operational Transform

Mathematical operations of how f(t) is transformed to F(s) or vice versa. Assume $\mathcal{L}\{f_i(t)\} = \int_{0^-}^{\infty} f_i(t) e^{-st} dt = F_i(s)$

1. Multiplication by a constant

$$\mathcal{L}\{Kf(t)\} = KF(s)$$

Proof:

$$\mathcal{L}\{Kf(t)\} = \int_{0^-}^{\infty} Kf(t) e^{-st} dt = K\int_{0^-}^{\infty} f(t) e^{-st} dt = KF(s)$$

The Operational Transform

2. Addition/Subtraction

$$\mathcal{L}\{f_1(t) \pm f_2(t) \pm f_3(t)\} = F_1(s) \pm F_2(s) \pm F_3(s)$$

Proof:

$$\mathcal{L}\{f_1(t) \pm f_2(t) \pm f_3(t)\} = \int_{0^-}^{\infty} \{f_1(t) \pm f_2(t) \pm f_3(t)\} e^{-st} dt$$

$$= \int_{0^-}^{\infty} f_1(t) e^{-st} dt \pm \int_{0^-}^{\infty} f_2(t) e^{-st} dt \pm \int_{0^-}^{\infty} f_3(t) e^{-st} dt$$

$$= F_1(s) \pm F_2(s) \pm F_3(s)$$

The Laplace Transform of sin(ωt)

- The Laplace transform of $\sin(\omega t)$

$$\mathcal{L}\{sin(\omega t)\} = \int_{0^-}^{\infty} sin(\omega t)e^{-st}dt = \frac{\omega}{s^2+\omega^2}$$

Laplace Transform

Time Domain

Frequency Domain

$\sin(\omega t)$

$\frac{\omega}{s^2+\omega^2}$

Inverse Laplace Transform

Euler's identity:
For any real number θ:

$$e^{j\theta} = cos\theta + jsin\theta$$

$$e^{-j\theta} = \cos\theta - jsin\theta$$

$$cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

The Operational Transform

3. Differentiation

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\frac{d^2}{dt^2}f(t)\right\} = s^2F(s) - sf(0^-) - f'(0^-)$$

$$\mathcal{L}\left\{\frac{d^n}{dt^n}f(t)\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$$

It transforms differential operations into algebraic operations.

Proof:

Using the integration by parts $\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)dx$

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = \int_{0^-}^{\infty} f'(t)e^{-st}dt = f(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\,(-se^{-st})dt$$

$$= -f(0^-) + s\int_{0^-}^{\infty} f(t)\,e^{-st}dt = sF(s) - f(0^-)$$

$note: u(t) = e^{-st}, v' = f'(t)$, so $u'(t) = -se^{-st}, v(t) = f(t)$

The Operational Transform

3. Differentiation – cont'd.

$$\mathcal{L}\left\{\frac{d^2}{dt^2}f(t)\right\} = s^2F(s) - sf(0^-) - f'(0^-)$$

Proof: Let $g(t) = \frac{df(t)}{dt}$, so $G(s) = sF(s) - f(0^-)$ and $g(0^-) = f'(0^-)$

Because $\frac{dg(t)}{dt} = \frac{d^2f(t)}{dt^2}$,

$$\mathcal{L}\left\{\frac{d^2}{dt^2}f(t)\right\} = \mathcal{L}\left\{\frac{d}{dt}g(t)\right\} = sG(s) - g(0^-) = s[sF(s) - f(0^-)] - f'(0^-)$$
$$= s^2F(s) - sf(0^-) - f'(0^-)$$

Repeating the same process, we can prove:

$$\mathcal{L}\left\{\frac{d^n}{dt^n}f(t)\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$$

Example #1

$$\mathcal{L}\{cos\ (\omega t)\}$$

$$= \frac{1}{\omega}\, \mathcal{L}\left\{\frac{d}{dt} sin\ (\omega t)\right\}$$

$$= \frac{1}{\omega}\left[s\, \frac{\omega}{s^2 + \omega^2} - \sin(0^-)\right]$$

$$= \frac{1}{\omega}\left[s\, \frac{\omega}{s^2 + \omega^2} - 0\right]$$

$$= \frac{s}{s^2 + \omega^2}$$

The Operational Transform

4. Integration

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s}$$

Proof:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \int_{0^-}^{\infty}[\int_{0^-}^{t} f(x)dx]e^{-st}dt = \int_{0^-}^{\infty} uv'dt$$

$$u = \int_{0^-}^{t} f(x)dx, \qquad u' = f(t), \qquad v' = e^{-st}, \qquad v = -\frac{e^{-st}}{s} \ ,$$

$$\int_{0^-}^{\infty}\left[\int_{0^-}^{t} f(x)dx\right]e^{-st}dt = -\frac{e^{-st}}{s}\int_{0^-}^{t} f(x)dx\Big|_{0^-}^{\infty} + \int_{0^-}^{\infty}\frac{e^{-st}}{s}f(t)\,dt$$

$$= 0 + \frac{1}{s}\int_{0^-}^{\infty} f(t)e^{-st}dt = \frac{F(s)}{s}$$

The Operational Transform

5. Translation in the time domain

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \qquad a > 0$$

Proof: $\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0^-}^{\infty} f(t-a)u(t-a)e^{-st}dt$

$$= \int_{a^-}^{\infty} f(t-a)e^{-st}dt$$

Let $x = t - a$.

$So\ dx = dt.\ When\ t = a^-, x = 0^-; when\ t = \infty, x = \infty; and\ t = x + a.$

$$\int_{a^-}^{\infty} f(t-a)e^{-st}dt = \int_{0^-}^{\infty} f(x)e^{-s(x+a)}dx = e^{-sa}\int_{0^-}^{\infty} f(x)e^{-sx}dx$$

$$= e^{-as}F(s) \qquad a>0$$

The Operational Transform

6. Translation in the frequency domain

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

7. Scale changing (Assignment)

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \qquad a > 0$$

Laplace Transform Table-1

Type	$f(t)\ (t > 0^-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Laplace Transform Table-2

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

The Laplace transform of $-2\delta(t)$ is

A. 1
B. -1
C. 2
D. -2

The Laplace transform of e^{10t} is

A. $\frac{1}{s+10}$

B. $\frac{1}{s-10}$

C. $\frac{1}{s}$

D. $-\frac{1}{s}$

The Laplace transform of $\frac{3}{4}e^{-3t}$ is

A. $\dfrac{1}{s+3}$

B. $\dfrac{1}{s-3}$

C. $\dfrac{3}{4(s+3)}$

D. $\dfrac{-3}{4(s+3)}$

The Laplace transform of $\frac{3}{4}e^{-3t} + \cos(5t)$ is

A. $\dfrac{3}{4(s+3)} + \dfrac{5}{s^2+25}$

B. $\dfrac{3}{4(s+3)} + \dfrac{s}{s^2+25}$

C. $\dfrac{1}{s+3} + \dfrac{5}{s^2+25}$

D. $\dfrac{1}{s+3} + \dfrac{s}{s^2+25}$

The Laplace transform of $4\int_{0^-}^{t} xe^{-2x}\,dx$ is

A. $\dfrac{4}{s(s+2)^2}$

B. $\dfrac{4}{(s+2)^2}$

C. $\dfrac{4}{s(s+2)}$

D. $\dfrac{4}{s^3}$

Example #2 – Find $f(t)$

If $f(t)$ satisfies the following equation, find $f(t)$.

$$f''(t) + 3f'(t) + 2f(t) = sin2t$$

Step 1: Applying Laplace transform to the both sides of the above equation and assuming $F(s) = \mathcal{L}\{f(t)\}$, we have:

$$s^2F(s) - sf(0^-) - f'(0^-) + 3sF(s) - 3f(0^-) + 2\mathrm{F(s)} = \frac{2}{s^2+4}$$

Step 2: Find F(s). Rearranging the above equation, we have:

$$F(s) = \frac{\frac{2}{s^2+4} + (s+3)f(0^-) + f'(0^-)}{s^2+3s+2}$$
$$= \frac{\left(s^2+4\right)[(s+3)f(0^-) + f'(0^-)] + 2}{(s^2+4)(s^2+3s+2)}$$

Find $f(t) - cont'd$

For example, if $f(0^-) = 1, f'(0^-) = 2,$

$$F(s) = \frac{(s^2+4)[(s+3)f(0^-) + f'(0^-)] + 2}{(s^2+4)(s^2+3s+2)}$$

$$= \frac{s^3 + 5s^2 + 4s + 22}{(s^2+4)(s^2+3s+2)}$$

- Step 3: f(t) can be found by applying the inverse Laplace transform.

Summary

- The definition of Laplace transform
- Two types of transforms were introduced in this lecture
  - The functional transform
  - The operational transform

- In next lecture, we will discuss
- The inverse Laplace transform
- How to apply the Laplace transform to simple circuit analysis.