Discussion Section: Week #1
Due: By 11:59pm the day of your Discussion Section

Instructions:

You are allowed to work alone or in small groups. But if you work in a group, the work you turn in should be your own. Each student will submit a physical write-up as a PDF document to CatCourses **by the end of your Discussion section**.

Discussion assignments will be graded partially for completeness but a subset of the problems will be graded for correctness. Providing the correct answer, without justification, is not considered complete. For credit you **must** either show you steps (if it's a calculation problem) or explain/justify your reasoning (if it's a short answer problem).

Problem Set:

1. Explain why the system

$$\begin{cases} u+v+w=2 \\ u+2v+3w=1 \\ v+2w=0 \end{cases}$$

is singular by finding a combination of the three equations that adds up to 0=1. What value should replace the last zero on the right side to allow the equations to have solutions and what is one of the solutions?

Solution: Adding $-R_1$ to R_2 first, and then $-R_2$ to R_3 yields a bottom row that says $0u + 0v + 0w = 1$, hence showing that the system is singular. To make the system consistent, the last zero should be -1, that way the bottom row yields $0u + 0v + 0w = 0$, which is consistent since R_2 will be exactly R_3. One such solution then would be to take $w = 1$ since it is a free variable, and then by the equations in the system we get $v = -3$ and $u = 4$.

2. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{cases} ax+2y=0 \\ 2x+ay=0 \end{cases}$$

Solution: By inspection, we see that $a = 2$ creates a line of solutions since the system will be redundant as both equations will be exactly the same. The resulting system then says y is a free variable, hence we get that by equation one, $x = -y$, thus forming our lines of solutions. Similarly, if $a = -2$, we get that the system row reduces to have a bottom row of all 0's, hence by equation 1, we get our solution set is given by the line $x = y$.

3. It is impossible for a system of linear equations to have exactly two solutions. Explain why.

Solution: For the sake of contradiction, suppose that there are two solutions. This means that in the system $Ax = b$, we have that x_1 and x_2 solve the system and we claim that these are the only solutions. Define a new vector $x_3 = \frac{1}{2}x_1 + \frac{1}{2}x_2$. Then observe:

$$Ax_3 = A(\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2) = \tfrac{1}{2}Ax_1 + \tfrac{1}{2}Ax_2 = \tfrac{1}{2}b + \tfrac{1}{2}b = b.$$

Hence we see that there is another solution to this system, therefore there can not be exactly two solutions.

Another approach is to think about ways a system of equations has a solution, no solution, or infinitely many solutions. For simplicity, suppose we have a system of two equations and two unknowns. Then, if the system has two solutions, the two lines intersect twice. This, however, is possible if these two lines are on top of each other, which implies there are more than two intersections (solutions). In fact, there are infinitely many solutions.