Homework Assignment #6

Remember, this Homework Assignment is **not collected or graded**! But it is in your best interest to do it as the Homework Quiz will be based on it and it is the best way to ensure you know the material.

Section 2.7: Linear Transformations

1. For the vector space P_3 of polynomials with degree at most 3. What matrix represents taking the second derivative: $\frac{d^2}{dt^2}$. Remember in this context we are representing a vector in P_3 in terms of it's standard basis which makes polynomials in this vector space appear like vectors in $\mathbb{R}^4$.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \implies \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

2. P_3 is the vector space of polynomials with degree at most 3. We again represent a "vector" in this space as follows:

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

   Let S to be the set of all polynomials in P_3 with $\int_0^1 p(x)dx = 0$.

   (a) Prove that S is a subspace by showing it is closed under addition and scalar multiplication.

   (b) Find a basis for S. (Hint: Enforce the conditions that have to hold on the coefficients of $p(x)$ for the integral to be 0.)

Section 3.1: Orthogonal Vectors and Subspaces

3. Suppose B is an invertible $n \times n$ matrix. How do we know that the i-th row of an invertible matrix B is orthogonal to the j-th column of B^{-1} if $i \neq j$.

4. Find the orthogonal complement in $\mathbb{R}^3$ of the vector space V consisting of a plane spanned by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

5. Find a basis for the orthogonal complement of the rowspace of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

6. Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. Show that $(x - y)$ is orthogonal to $(x + y)$ if and only if $\|x\| = \|y\|$.
   Note that: $\|x\|^2 = \vec{x}^T \vec{x}$. (Hint: This problem is easier than it looks, just write out the inner product.)

7. Suppose that A is a symmetric matrix. That is, $A^T = A$.

   (a) Why is its column space perpendicular to its nullspace?

   (b) If $A\vec{x} = \vec{0}$ and $Az = 5z$, which fundamental subspaces contain $\vec{x}$ and $\vec{z}$?

Section 3.2: Projections onto Lines

8. Find the matrix that projects any point in $\mathbb{R}^2$ onto the line $x + y = 0$.

9. Project the vector $\vec{b}$ onto the line through $\vec{a}$. Denote this projected vector $\vec{p}$ and verify that $\vec{e} = \vec{b} - \vec{p}$ is perpendicular to $\vec{a}$.

   (a) $\vec{b} = \begin{bmatrix}1 & 2 & 2\end{bmatrix}^T$ and $\vec{a} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$.

   (b) $\vec{b} = \begin{bmatrix}1 & 3 & 1\end{bmatrix}^T$ and $\vec{a} = \begin{bmatrix}-1 & -3 & -1\end{bmatrix}$.

10. Find the projection matrix P that projects every point in $\mathbb{R}^3$ onto the line of intersection between the planes:

$$\begin{aligned} x + y + t &= 0 \\ x - t &= 0. \end{aligned}$$