CSE 015: Discrete Mathematics
Homework 6

Fall 2021
Provided Solution

1 Recursively defined functions

Throughout this exercise you have to recall that $f(0) = 3$.

a) $f(1) = -2f(0) = -6$
$f(2) = -2f(1) = 12$
$f(3) = -2f(2) = -24$
$f(4) = -2f(3) = 48$
$f(5) = -2f(4) = -96$

b) $f(1) = 3f(0) + 7 = 16$
$f(2) = 3f(1) + 7 = 55$
$f(3) = 3f(2) + 7 = 172$
$f(4) = 3f(3) + 7 = 523$
$f(5) = 3f(4) + 7 = 1576$

c) $f(1) = f(0)^2 - 2f(0) - 2 = 1$
$f(2) = f(1)^2 - 2f(1) - 2 = -3$
$f(3) = f(2)^2 - 2f(2) - 2 = 13$
$f(4) = f(3)^2 - 2f(3) - 2 = 141$
$f(5) = f(4)^2 - 2f(4) - 2 = 19597$

d) $f(1) = 3^{f(0)/3} = 3$
$f(2) = 3^{f(1)/3} = 3$
$f(3) = 3^{f(2)/3} = 3$
$f(4) = 3^{f(3)/3} = 3$
$f(5) = 3^{f(4)/3} = 3$

2 Recursively defined sequences

a) Let us first compute the values of a_n for $n = 1, 2, 3, 4$ by applying the given formula.

$$a_1 = 4 \cdot 1 - 2 = 2 \qquad a_2 = 4 \cdot 2 - 2 = 6 \qquad a_3 = 4 \cdot 3 - 2 = 10 \qquad a_4 = 4 \cdot 4 - 2 = 14$$

The recursive definition for this sequence is therefore the following:
Induction Basis: $a_1 = 2$
Inductive Step: $a_n = a_{n-1} + 4$ for $n \geq 2$

b) Let us first compute the values of a_n for $n = 1, 2, 3, 4$ by applying the given formula.

$$a_1 = 1 + (-1)^1 = 0 \qquad a_2 = 1 + (-1)^2 = 2 \qquad a_3 = 1 + (-1)^3 = 0 \qquad a_4 = 1 + (-1)^4 = 2$$

The recursive definition for this sequence is therefore the following:
Induction Basis: $a_1 = 0$
Inductive Step: $a_n = 2 - a_{n-1}$ for $n \geq 2$

c) Let us first compute the values of a_n for $n = 1, 2, 3, 4$ by applying the given formula.

$$a_1 = 1 \cdot 0 = 0 \qquad a_2 = 2 \cdot 1 = 2 \qquad a_3 = 3 \cdot 2 = 6 \qquad a_4 = 4 \cdot 3 = 12$$

Induction Basis: $a_1 = 0$
Inductive Step: $a_n = a_{n-1} + 2(n-1)$ for $n \geq 2$

d) Let us first compute the values of a_n for $n = 1, 2, 3, 4$ by applying the given formula.

$$a_1 = 1^2 = 1 \qquad a_2 = 2^2 = 4 \qquad a_3 = 3^2 = 9 \qquad a_4 = 4^2 = 16$$

Induction Basis: $a_1 = 1$
Inductive Step: $a_n = a_{n-1} + 2n - 1$ for $n \geq 2$
Note: you could directly obtain the result observing that $n = n-1+1$ and therefore $n^2 = [(n-1)+1]^2$, i.e., $(n-1)^2 + 2(n-1) + 1 = (n-1)^2 + 2n - 1$ (and then recall that $(n-1)^2 = a_{n-1}$).

3 Mathematical Induction 3

As suggested, it is convenient to first determine a few strings in S.
Basis of induction: the empty string $\varepsilon \in S$.
Inductive step: pick a string $x \in S$ and build a new one using the given formula. So far the only string in S is the empty string ε, so picking $x = \varepsilon$ we can build the string $0\varepsilon 1 = 01$.
Let us apply the inductive step again. Now we have two strings in S, namely ε and 01. Let us select $x = 01$ (picking $x = \varepsilon$ is now useless, because if would produce a string already in S, i.e., 01.) Then, using the rule we get the string $0x1 = 0011$.
Let us apply the inductive step again. Now we have three strings in S, namely ε, 01, and 0011. Let us select $x = 0011$ (picking any other string is useless, because if would produce a string already in S.) Then, using the rule we get the string $0x1 = 000111$.
At this point it is clear what the pattern is, as you keep producing new strings. The set S can be described as follows:

S is the set of strings made of 0s and 1s, that are composed by a sequence of k consecutive 0s followed by a sequence of k consecutive 1s, for $k \geq 0$.

Of course the same set could be also described using a different sentence or set of sentences. Any such description would be fine as long as it correctly characterizes all and only the strings in S.