Homework Assignment #9

Remember, this Homework Assignment is **not collected or graded**! But it is in your best interest to do it as the this material is designed to be a review for Midterm #2.

Chapter 2: Review Questions

1. Let A be an $m \times n$ matrix with rank r. What do you know about $C(A)$ and how r is related to m and n when the number of solutions to $A\vec{x} = \vec{b}$ behaves as follows.

   (a) 0 or 1, depending on $\vec{b}$.

   (b) ∞ independent on $\vec{b}$.

   (c) 0 or ∞ depending on $\vec{b}$

   (d) 1 regardless of $\vec{b}$

Solution:

(a) In order to get 0 or 1 solution we need to have a pivot in every column but more rows than columns so that it is possible we will have a row of all 0's in the REF for of A.

This means the rank $r = n$ and that $m > n$.

(b) In order to have infinity solutions we have to have a pivot in every row (so we always have a solution) but more columns than rows (so that we will have some free variables).

This means the rank $r = m$ and $n > m$.

(c) In order to have 0 we need to have fewer pivots than rows, in order to have ∞ solutions we need fewer pivots than columns.

This means $r < m$ and $r < n$. There is no relationship between m and n that we need to be aware of. That is we can find the same solution behavior in a 2×3 matrix and a 4×3 matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(d) In order to have a unique solution for every $\vec{b}$ we need $N(A^T) = 0$ and $N(A) = 0$ which means that $m = n = r$. That is, we have a square invertible matrix.

2. Consider the following matrix A and $\vec{b}$:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   (a) Under what conditions on $\vec{b}$ does $A\vec{x} = \vec{b}$ have a solution?

Solution: Let's consider the augmented matrix:

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 2 & 4 & 0 & 1 & b_3 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & -5 & b_3 - 2b_1 \end{array}\right].$$

Solutions exist only when $b_2 = 0$.

(b) Find the general solution to $A\vec{x} = \vec{b}$ when a solution exists.

Solution: We first note that $b_2 = 0$. Then we pick up our augmented matrix calculation from above:

$$\left[A \mid \vec{b}\right] \implies \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & b_3 - 2b_1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & -5 & b_3 - 2b_1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This system has 2 pivots and 2 free variables. We have $x_3 = t$ and $x_2 = s$. We have $x_4 = (-1/5)(b_3 - 2b_1)$ and:

$$x_1 = b_1 - 2x_2 - 3x_4 = b_1 - 2s + (3/5)(b_3 - 2b_1) = (-1/5)b_1 + (3/5)b_3 - 2s.$$

Thus the solution we have is:

$$\vec{x} = (-1/5)\begin{bmatrix} b_1 - 3b_3 \\ 0 \\ 0 \\ b_3 - 2b_1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Find a basis for the column space of A.

Solution: The basis for $C(A)$ is given by the columns in A that correspond to pivots in the row-echelon form of A. In this case, this is columns 1 and 4. Thus we have:

$$C(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Note that:

- We note that these two vectors are linear independent.
- The second column of A is twice the first column of A, so it need not be included in a basis for $C(A)$.
- The third column of A is the zero vector, which is never part of a basis.

(d) What is the rank of A^T?

Solution: We know that the rank of A^T is the same as the rank of A and is the number of pivots. In this case rank A^T is equal to 2.

3. Suppose that the following depicts $PA = LU$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & -3 & 2 \\ 2 & -1 & 4 & 2 & 1 \\ 4 & -2 & 9 & 1 & 4 \\ 2 & -1 & 5 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the rank of A?

> **Solution:** The rank of A is the number of pivots in U, the row-echelon form of A. Here we see that U has 3 pivots, so the rank of A is 3.

(b) What is a basis for the row space of A?

> **Solution:** A basis for the row space of A is either the non-zero rows in U, the row-echelon form of PA. Or the corresponding rows in the matrix PA. In this case, both are the same:
>
> $$C(A^T) = \left\{ \begin{bmatrix} 2 & -1 & 4 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -3 & 2 \end{bmatrix} \right\}.$$

(c) True or False: Rows 1, 2, 3 of A are linearly independent.

> **Solution:** False. As we can see from the $PA = LU$ factorization: $R_3 = 2R_2 + R_1$. As such, these rows are linearly dependent.

(d) What is a basis for the column space of A?

> **Solution:** A basis for the column space of A are the columns in A that correspond to the pivots of U, the row-echelon form of A. That is,
>
> $$C(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 5 \end{bmatrix} \right\}.$$
>
> If we denote the i-th column of A by C_i, then we notice that: $C_2 = (-1/2)C_1$ and $C_4 = -3C_3 - 14C_2$. Thus, these are linearly dependent.

(e) What is the dimension of the left nullspace of A?

> **Solution:** The dimension of the left nullspace is the number of the rows minus the rank of A. We have 4 rows and the rank of A is 3, thus the dimension of the left nullspace is 1.
> If we want to see this directly, we can solve: $A^T\vec{x} = \vec{0}$:
>
> $$\left[\begin{array}{cccc|c} 0 & 2 & 4 & 2 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 1 & 4 & 9 & 5 & 0 \\ -3 & 2 & 1 & -1 & 0 \\ 2 & 1 & 4 & 5 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{cccc|c} 1 & 4 & 9 & 5 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 2 & 4 & 2 & 0 \\ -3 & 2 & 1 & -1 & 0 \\ 2 & 1 & 4 & 5 & 0 \end{array}\right] \xrightarrow[R_5 \to R_5 - 2R_1]{R_4 \to R_4 + 3R_1}$$
>
> $$\left[\begin{array}{cccc|c} 1 & 4 & 9 & 5 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 2 & 4 & 2 & 0 \\ 0 & 14 & 28 & 14 & 0 \\ 0 & -7 & -14 & -5 & 0 \end{array}\right] \xrightarrow[R_4 \to R_4 + 14R_2,\, R_5 \to R_5 - 7R_2]{R_3 \to R_3 + 2R_2} \left[\begin{array}{cccc|c} 1 & 4 & 9 & 5 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_5 \leftrightarrow R_3}$$
>
> $$\left[\begin{array}{cccc|c} 1 & 4 & 9 & 5 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We have 3 pivots and 1 free variable. We have $x_4 = 0, x_3 = t$ and thus, $x_2 = -2t$ and:

$$x_1 = -4x_2 - 9x_3 - 5x_4 = -4(-2t) - 9(t) - 5(0) = -t.$$

Thus our solution to $A^T\vec{x} = \vec{0}$ is of the form:

$$\vec{x} = t\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

And thus the left nullspace of A has dimension 1.

(f) What is the general solution to $A\vec{x} = \vec{0}$?

Solution: We first note that:

$$A\vec{x} = \vec{0} \implies PA\vec{x} = P\vec{0} = \vec{0}.$$

But then we have:

$$PA\vec{x} = \vec{0} \implies LU\vec{x} = \vec{0} \implies U\vec{x} = L^{-1}\vec{0} = \vec{0}.$$

Thus we have to solve:

$$\left[\begin{array}{ccccc|c} 2 & -1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This system has 3 pivots and 2 free variables. We have $x_5 = 0, x_4 = t, x_2 = s$. Then row 2 gives us: $x_3 = 3x_4 - 2x_5 = 3t$. And row 1 gives us:

$$2x_1 = x_2 - 4x_3 - 2x_4 - x_5 \implies 2x_1 = s - 4(3t) - 2(t) - 0 \implies x_1 = (s/2) - 7t.$$

Thus we have:

$$\vec{x} = \left\{ s\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Chapter 3: Review Questions

4. Construct the projection matrix P which projects vectors onto the space spanned by $(1, 1, 1)$ and $(0, 1, 3)$.

Solution: We know that if we want to project onto the column space of A we need to consider the matrix:

$$P = A(A^TA)^{-1}A^T.$$

Let's let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

This means:

$$A^TA = \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix} \text{ and } (A^TA)^{-1} = \frac{1}{14}\begin{bmatrix} 10 & -4 \\ -4 & 3 \end{bmatrix}.$$

Then the matrix P is defined:

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \left(\frac{1}{14}\right) \begin{bmatrix} 10 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{bmatrix}.$$

5. Find all 2 by 2 orthogonal matrices who have entries that are only 0 and 1.

Solution: We need matrices of the form:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

that are also orthogonal matrices with only entries 0 and 1.

That is we need:

$$a^2 + c^2 = 1^2, b^2 + d^2 = 1^2 \text{ and } ac + bd = 0.$$

There are only two choices for this. Either:

- $a = 0, c = 1$ and $b = 1, d = 0$
- $a = 1, c = 0$ and $b = 0, d = 1$

This gives us only two choices:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We notice that $A_1^TA_1 = A_2^TA_2 = I$ as required. Note that these matrices involve permutations of the identity matrix.

6. What point on the plane $x + y - z = 0$ is the closest to $\vec{b} = (2, 1, 0)^T$.

Solution: We first note that the point $\vec{b}$ is not on the plane $x + y - z = 0$ because it does not satisfy the equation. This means that we now need to project the point $\vec{b}$ onto the space spanned by the two vectors defining the plane.

That is, we first need to solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \end{array}\right].$$

We can see that we have 1 pivot and 2 free variables: $x_3 = t, x_2 = s$ and $x_1 = -x_2 + x_3 = -s + t$. Thus we have two special solutions which correspond to a basis for the plane:

$$\left\{ s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus, we define a matrix A with these two vectors as columns so $C(A)$ will be the plane:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, we have to find the projection $\vec{p}$ of $\vec{b}$ defined as follows:

$$\vec{p} = P\vec{b} = A(A^TA)^{-1}A^T\vec{b}.$$

We have:

$$A^T = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, A^TA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \text{ and } (A^TA)^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then we build:

$$P = A(A^TA)^{-1}A^T = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Finally we have:

$$\vec{p} = P\vec{b} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

7. Use Gram-Schmidt to construct an orthonormal pair $\vec{q}_1$ and $\vec{q}_2$ from the vectors:

$$\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Express $\vec{x}$ and $\vec{y}$ as a linear combination of $\vec{q}_1$ and $\vec{q}_2$ and determine the QR factorization for the matrix A, the 4 by 2 matrix whose columns consist of $\vec{x}$ and $\vec{y}$.

Solution: We know that the algorithm for Gram-Schmidt is as follows:

- $\vec{q}_1 = \frac{\vec{x}}{||\vec{x}||}$.
- $\vec{Y} = \vec{y} - (\vec{y}^T\vec{q}_1)\vec{q}_1$ and then $\vec{q}_2 = \frac{\vec{Y}}{||\vec{Y}||}$

Thus we have:

$$\vec{q}_1 = \frac{\vec{x}}{\sqrt{4^2+5^2+2^2+2^2}} = \frac{1}{7}\begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}.$$

We then build:

$$\vec{Y} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{7}\left((1)(4)+(2)(5)+(0)(2)+(0)(2)\right)\begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 8/7 \\ 10/7 \\ 4/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix}.$$

Then we normalize to get:

$$\vec{q}_2 = \frac{\vec{Y}}{\|\vec{Y}\|} = \frac{\vec{Y}}{\sqrt{(1+(3)4^2)/(49)}} = \vec{Y}.$$

We then are looking for the following:

$$Q^T A = R \implies \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \end{bmatrix} \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{q}_1^T \vec{x} & \vec{q}_1^T \vec{y} \\ \vec{q}_2^T \vec{x} & \vec{q}_2^T \vec{y} \end{bmatrix}$$

- $\vec{q}_1^T \vec{x} = \|\vec{x}\| = 7$
- $\vec{q}_1^T \vec{y} = (1)(4/7) + (2)(5/7) + (0)(2/7) + (0)(2/7) = 2$
- $\vec{q}_2^T \vec{x} = (-1/7)(4) + (4/7)(5) + (-4/7)(2) + (-4/7)(2) = 0$ (as expected)
- $\vec{q}_2^T \vec{y} = (-1/7)(1) + (4/7)(2) + (-4/7)(0) + (-4/7)(0) = 7/7 = 1$

Thus we have:

$$A = QR \implies \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 0 & 1 \end{bmatrix}.$$

8. If Q is an orthogonal matrix, is Q^3 and orthogonal matrix?

Solution: The answer is yes, if Q is an orthogonal matrix so is Q^3 and indeed any power of Q is again an orthogonal matrix. We will prove this by mathematical induction. That is, we will show that if Q is an orthogonal matrix then Q^n is also orthogonal.

Remember if Q is an orthogonal matrix, it is a square matrix with orthonormal columns that satisfies:

$$Q^T Q = Q Q^T = I.$$

- **Base Case:** Let's show that Q^2 is an orthogonal matrix. (This was a previous homework problem.) That is, we need to show that : $(Q^2)^T Q^2 = I$.

  $$(Q^2)^T Q^2 = Q^T Q^T Q Q = Q^T I Q = Q^T Q = I.$$

  Thus, we have proven the Base Case.

- **Inductive Hypothesis:** We will assume that for k we have Q^k is an orthogonal matrix. That is,

  $$(Q^k)^T Q^k = I.$$

- **Inductive Step:** Now let's show that if Q^k is orthogonal this implies that $Q^{(k+1)}$ is orthogonal:

  $$(Q^{(k+1)})^T Q^{(k+1)} = (QQ^k)^T Q^{(k+1)} = (Q^k)^T Q^T Q Q^k = (Q^k)^T I Q^k = (Q^k)^T Q^k = I.$$

  The last equality uses our inductive hypothesis. And thus we have shown that Q^k being orthogonal implies $Q^{(k+1)}$ is orthogonal.

Thus Q^k is an orthogonal matrix if Q is for any power of k.

9. For any A, $\vec{b}$, $\vec{x}$ and $\vec{y}$ show that:

   (a) If $A\vec{x} = \vec{b}$ and $\vec{y}^T A = \vec{0}$ then show, $\vec{y}^T\vec{b} = 0$.

   Solution: Notice that this is asking us to show that $N(A^T)$ is orthogonal to $C(A)$.

   $$\vec{y}^T\vec{b} = \vec{y}^T(A\vec{x}) = (\vec{y}^T A)\vec{x} = \vec{0}^T\vec{x} = 0.$$

   (b) If $A\vec{x} = 0$ and $A^T\vec{y} = \vec{b}$ then $\vec{x}^T\vec{b} = 0$.

   Solution: Notice this is asking us to show that the row space $C(A^T)$ is orthogonal to the nullspace $N(A)$.

   $$\vec{x}^T\vec{b} = \vec{x}^T(A^T\vec{y}) = (\vec{x}^T A^T)\vec{y} = (A\vec{x})^T\vec{y} = \vec{0}^T\vec{y} = 0.$$

10. Let $A = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$ and let V be the nullspace of A. Find a basis for V and a basis for $V^\perp$.

Solution: To find the nullspace, we need to solve $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \end{array}\right].$$

The matrix has 1 pivot and 2 free variables. We have $x_3 = t$, $x_2 = s$ and we have:

$$3x_1 = -x_2 + x_3 \implies x_1 = \frac{1}{3}(-s + t).$$

Thus, each of these special solutions corresponds to a basis vector for $N(A)$:

$$V = N(A) = \text{ span } \left\{ \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

In order to find $V^\perp$ we have two different ways to operate. First, we could appeal to the fact that we know $C(A^T)$ is the orthogonal complement to $N(A)$. Then we look for a basis for the row-space which is (since we have only one row) that row.
Second, we could try to seek a generic vector of the form: $\vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ that satisfies:

$$\vec{x}^T \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} = 0 \text{ and } \vec{x}^T \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\implies -\frac{1}{3}x_1 + x_2 + 0x_3 = 0 \text{ and } \frac{1}{3}x_1 + 0x_2 + x_3 = 0$$

$$\implies \left[\begin{array}{ccc|c} -1/3 & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} -1/3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

We see that this system has 2 pivots and 1 free variable. We have: $x_3 = t$, $x_2 = -t$ and $x_1 = -3t$. Thus we have:

$$V^\perp = \text{span} \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

We notice that this vector is just the negative of the single row matrix as predicted.

Chapter 4 (Section 4.1): Review Questions

11. If $B = M^{-1}AM$ find $\det(B)$ in terms of $\det(A)$. What is $\det(A^{-1}B)$?

Solution: We know from our properties of determinants that the determinant of a product is the product of the determinants. We also know that $\det(M^{-1}) = 1/\det(M)$. Together this lets us determine:

$$\det(B) = \det(M^{-1}AM) = \det(M^{-1})\det(A)\det(M) = \frac{1}{\det(M)}\det(A)\det(M) = \det(A).$$

We also know that:

$$\det(A^{-1}B) = \det(A^{-1}B) = \det(A^{-1})\det(B) = \frac{1}{\det(A)}\det(A) = 1.$$

12. Use row operations to simplify and compute these determinants:

(a) Find $\det(A)$ when $A = \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}$.

Solution: While we could use the basket weaving method to calculate a determinant, row operations actually make this one pretty reasonable. Remember, any row-operation does not change the determinant calculation.

$$\begin{vmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{vmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{vmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{vmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

The last equality is because we know any matrix with two identical rows has determinant 0.

(b) Find $\det(A)$ when $A = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}$.

Solution: We could use the basket weaving method, but row operations similarly make this calculation reasonable.

$$\begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{vmatrix} \xrightarrow[R_3 \to R_3 - t^2 R_1]{R_2 \to R_2 - tR_1} \begin{vmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & t-t^3 & 1-t^4 \end{vmatrix} \xrightarrow{R_3 \to R_3 - tR_2} \begin{vmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & 0 & 1-t^2 \end{vmatrix} = (1)(1-t^2)^2.$$

Thus, $\det(A) = (1-t^2)^2$.

(c) Consider the following LU factorization of the matrix A.

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of L, U, A, $U^{-1}L^{-1}$ and $U^{-1}L^{-1}A$.

Solution:

- $\det(L) = 1$ because the determinant of an upper/lower triangular of diagonal matrix is the product of diagonal entries.
- $\det(U) = 3(2)(-1) = -6$ because the determinant of an upper/lower triangular of diagonal matrix is the product of diagonal entries.
- $\det(A) = \det(LU) = \det(L)\det(U) = 1(-6) = -6$ because the determinant of a product is the product of the determinants.
- Since $A = LU$ we know $A^{-1} = U^{-1}L^{-1}$. And we know, $\det(A^{-1}) = 1/\det(A)$. Thus,
$$\det(U^{-1}L^{-1}) = \det(A^{-1}) = 1/\det(A) = -1/6.$$
- Note that $U^{-1}L^{-1}A = A^{-1}A = I$ and $\det(I) = 1$. Thus,
$$\det(U^{-1}L^{-1}A) = \det(A^{-1}A) = \det(I) = 1.$$

Chapter 5 (Sections 5.1 - 5.3): Review Questions

13. Find the eigenvalues and eigenvectors and diagonalize each of the following two matrices:
$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix}.$$
Use the diagonalization to calculate A^{50} and B^{200}.

Solution: Let's do these one at a time.

- A:
First we need to solve for the eigenvalues
$$0 = \det(A - \lambda I) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$
This gives us $\lambda_1 = 3$ and $\lambda_2 = 1$. We then need to find eigenvectors:
  - $\lambda_1 = 3$:
$$(A - 3I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 1-3 & 0 & 0 \\ 2 & 3-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$
We have 1 pivot and 1 free variable. Our eigenvector is:
$$\vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
  - $\lambda_2 = 1$
$$(A - I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 1-1 & 0 & 0 \\ 2 & 3-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$
We have 1 pivot and 1 free variable. Let $x_2 = t$ then we have $x_1 = -t$. Thus our eigenvector is:
$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

In order to diagonalize, we need a matrix of eigenvalues Λ, the matrix of eigenvectors S and its inverse S^{-1}.

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \text{ and } S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus we have:

$$A = S\Lambda S^{-1}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

In order to calculate A^{50} we use our diagonalization to obtain:

$$A^{50} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{50} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1+3^{50} & 3^{50} \end{bmatrix}.$$

- B

  We recall,

$$B = \begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix}$$

  We first find the eigenvalues by solving:

$$0 = \det(B - \lambda I) = \lambda^2 - \text{tr}(B)\lambda + \det(B) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1).$$

  This gives us $\lambda_1 = 1$ and $\lambda_2 = 2$.

  Now we need to find the eigenvectors:

  - $\lambda_1 = 1$:

$$(B-I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 7-1 & 2 & 0 \\ -15 & -4-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 6 & 2 & 0 \\ -15 & -5 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + (5/2)R_1} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

    We have 1 pivot and 1 free variable. This gives us: $x_2 = t$ and $x_1 = -t/3$. For simplicity we will let $t = 3$ and consider the eigenvector:

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

  - $\lambda_2 = 2$:

$$(B - 2I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 7-2 & 2 & 0 \\ -15 & -4-2 & 0 \end{array}\right] \left[\begin{array}{cc|c} 5 & 2 & 0 \\ -15 & -6 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + 3R_1} \left[\begin{array}{cc|c} 5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

    We have 1 pivot and 1 free variable. This gives us $x_2 = t$ and $x_1 = -2t/5$. For simplicity we will let $t = 5$ and end up with the eigenvector:

$$\vec{x}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

In order to diagonalize, we need a matrix of eigenvalues Λ, the matrix of eigenvectors S and its inverse S^{-1}.

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, S = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix}, \text{ and } S^{-1} = \begin{bmatrix} 5 & 2 \\ -3 & -1 \end{bmatrix}.$$

Then we have:

$$B = S\Lambda S^{-1} \implies \begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -3 & -1 \end{bmatrix}.$$

In order to calculate B^{200} we will use the diagonalization:

$$B^{200} = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{200} \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -5 + (6)2^{200} & 2(-1 + 2^{200}) \\ 15(1 - 2^{200}) & 6 - (5)2^{200} \end{bmatrix}.$$

14. Find the determinants of A and A^{-1} if:

$$A = S \begin{bmatrix} \lambda_1 & 2 \\ 0 & \lambda_2 \end{bmatrix} S^{-1}.$$

Solution: We know that the determinant of a product is the produce of the determinants, we also know that $\det(A^{-1}) = 1/\det(A)$. Thus,

$$\det(A) = \det(S) \begin{vmatrix} \lambda_1 & 2 \\ 0 & \lambda_2 \end{vmatrix} (1/\det(S)) = \lambda_1 \lambda_2.$$

Thus, $\det(A^{-1}) = 1/\det(A) = 1/(\lambda_1 \lambda_2)$.

15. If A has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$ that correspond respectively to eigenvectors:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(a) Find A.

Solution: Here we will use the fact that we can diagonalize the matrix A by its eigenvalues and eigenvectors. That is, we need to determine S, the matrix of eigenvectors, S^{-1} and Λ the matrix of eigenvalues and we will have:

$$A = S\Lambda S^{-1}.$$

We have:

$$S = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\det(S)} \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Thus we have:

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \left(\frac{1}{5}\right) \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \left(\frac{1}{5}\right) \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

(b) Find the eigenvectors and eigenvalues of A^2.

Solution: The next question is to find the eigenvalues and eigenvectors of A^2. We could find those directly, or we could use the fact that we know the eigenvectors and eigenvalues of A:

$$A\vec{x}_1 = (0)\vec{x}_1 \implies A^2\vec{x}_1 = A(A\vec{x}_1) = A(0\vec{x}_1) = 0\vec{x}_1$$

$$A\vec{x}_2 = \vec{x}_2 \implies A^2\vec{x}_2 = A(A\vec{x}_2) = A\vec{x}_2 = \vec{x}_2.$$

Thus, $\lambda_1 = 0$ and $\lambda_2 = 1$ are still eigenvectors of A^2 and the eigenvalues are the same. Indeed, if λ is an eigenvalue and $\vec{x}$ an eigenvector of A then:

$$A\vec{x} = \lambda\vec{x} \implies A^2\vec{x} = A(A\vec{x}) = A\lambda\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x}.$$