CSE100: Design and Analysis of Algorithms
Lecture 07 – Selection and Median (cont.)

Feb 8th 2022

More Recursion, Beyond the Master Theorem

Solving Recurrence Relations (review)

- A **recurrence relation** expresses $T(n)$ in terms of $T(\text{less than } n)$
- For example, $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$
- Two methods of solution:
    1. Master Theorem (aka, generalized “tree method”)
    2. Substitution method (aka, guess and check)

What have we learned? (review)

- The substitution method can work when the master theorem doesn't.
  - For example, with different-sized sub-problems.

- Step 1: generate a guess
  - Throw the kitchen sink at it.
- Step 2: try to prove that your guess is correct
  - You may have to leave some constants unspecified till the end – then see what they need to be for the proof to work!!
- Step 3: profit
  - Pretend you didn't do Steps 1 and 2 and write down a nice proof.

The k-SELECT problem (review)

A is an array of size n, k is in {1,…,n}

- SELECT(A, k):
  - Return the k'th smallest element of A.

For today, assume all arrays have distinct elements.

7	4	3	8	1	5	9	14

- SELECT(A, 1) = 1
- SELECT(A, 2) = 3
- SELECT(A, 3) = 4
- SELECT(A, 8) = 14

- SELECT(A, 1) = MIN(A)
- SELECT(A, n/2) = MEDIAN(A)
- SELECT(A, n) = MAX(A)

Being sloppy about floors and ceilings!

Note that the definition of Select is 1-indexed…

The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
4. Return of the Substitution Method.

Idea: divide and conquer!

Say we want to find SELECT(A, k)

9	8	3	6	1	4	2

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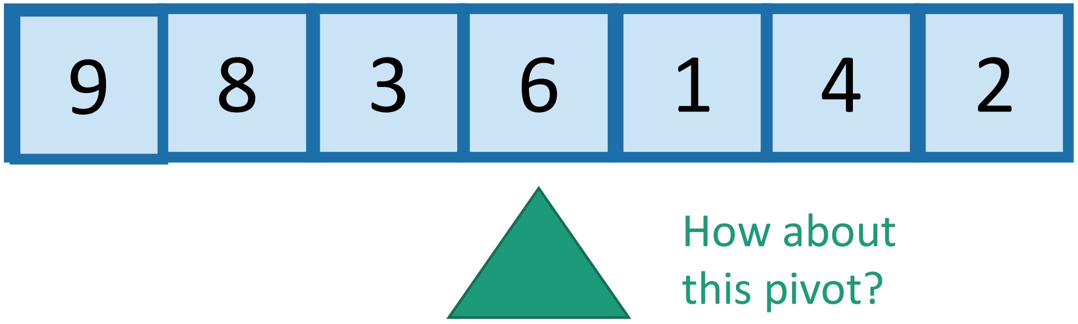

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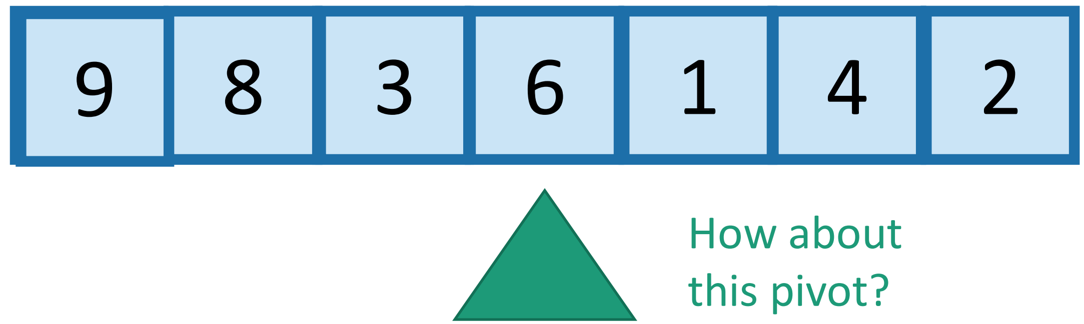

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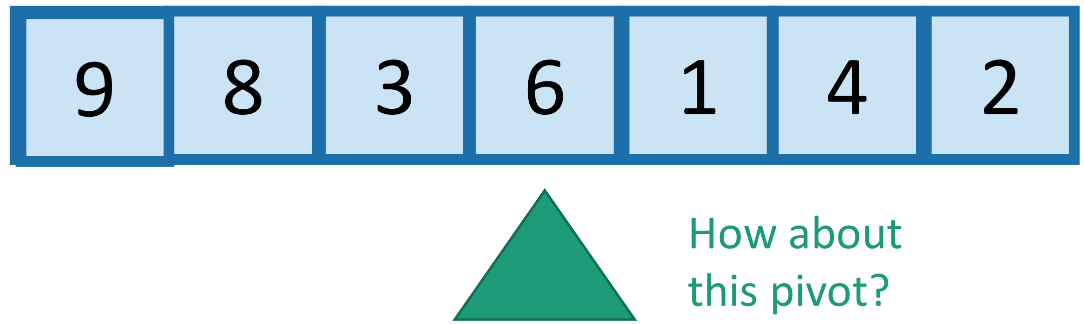

How about this pivot?

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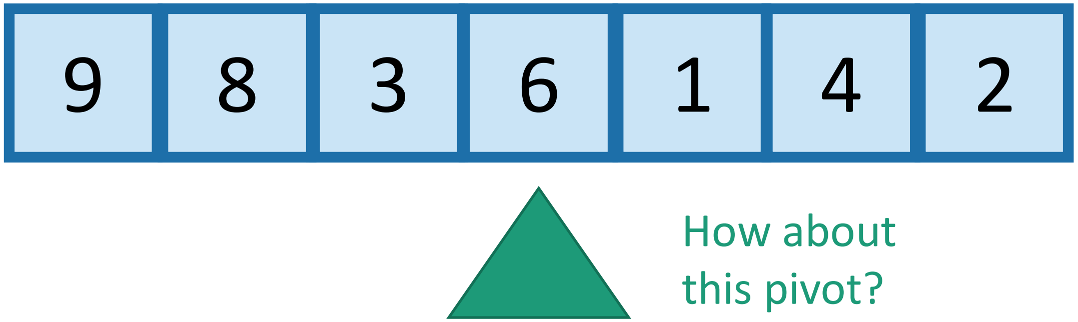

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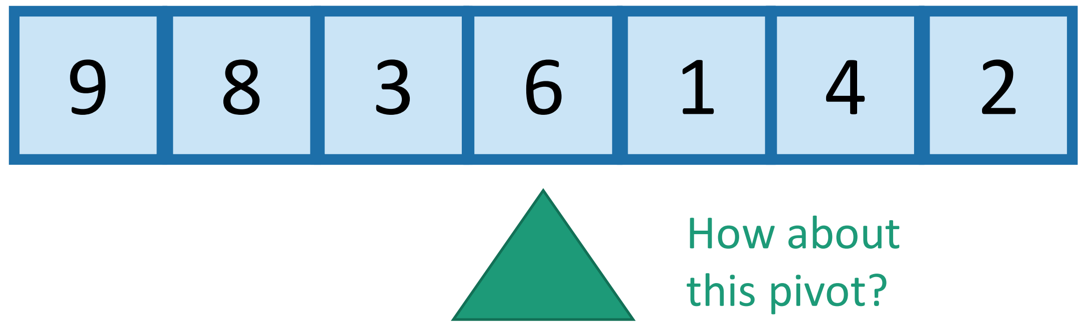

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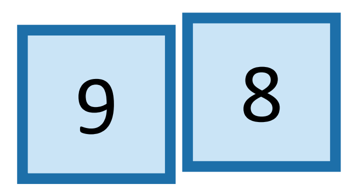

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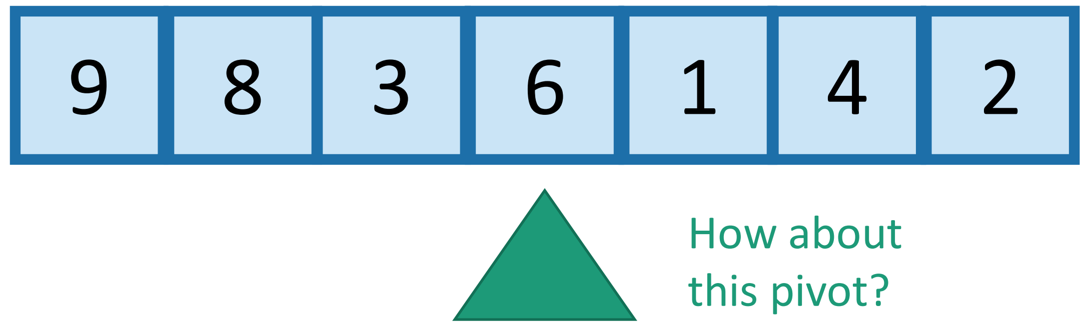

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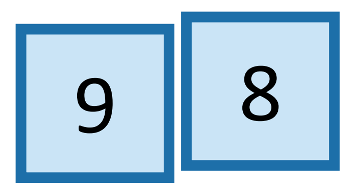

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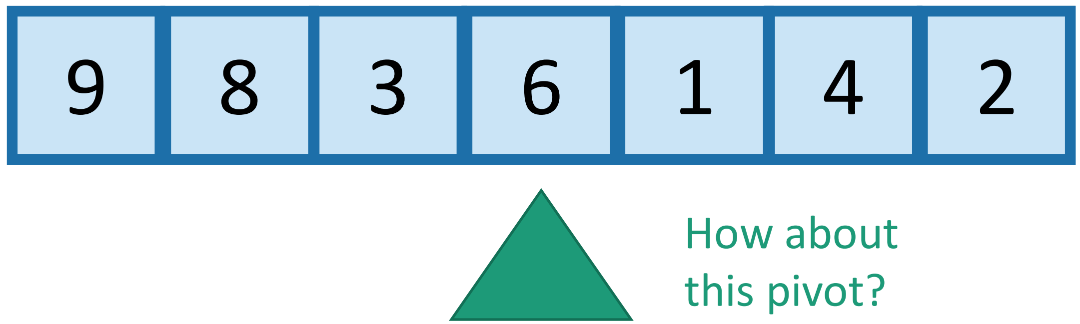

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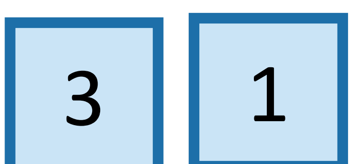

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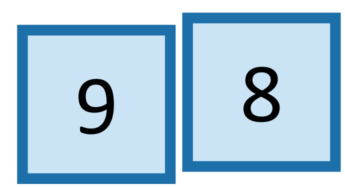

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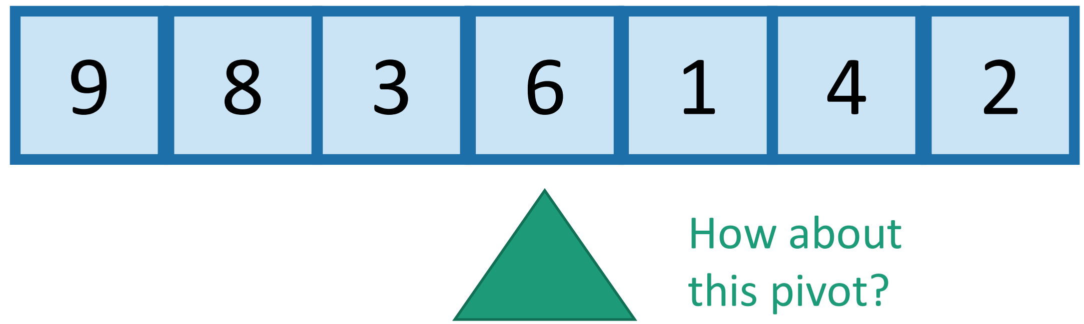

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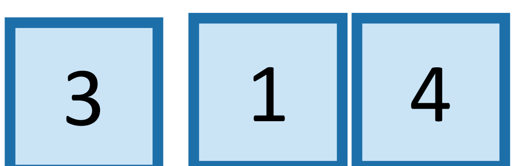

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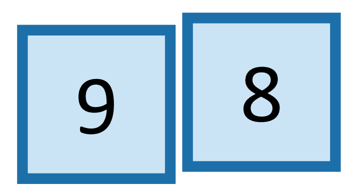

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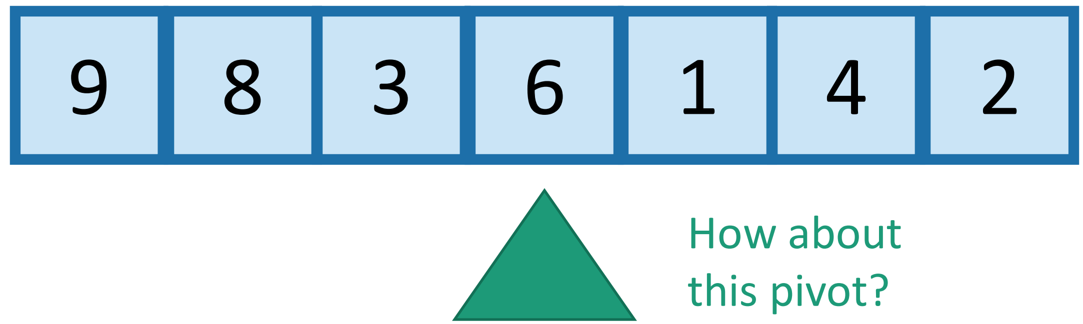

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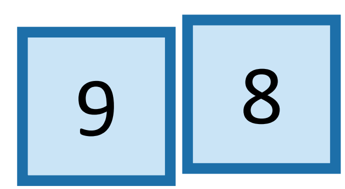

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This PARTITION step takes time O(n). (Notice that we don't sort each half).

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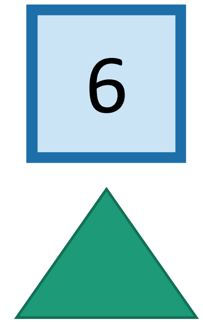

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Idea continued…

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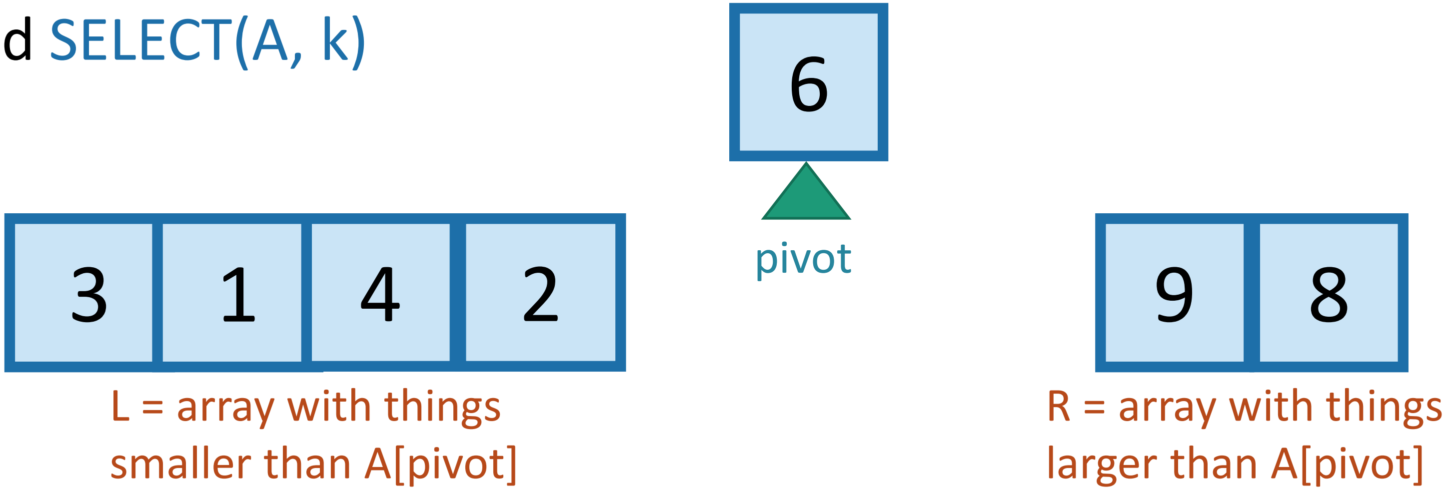
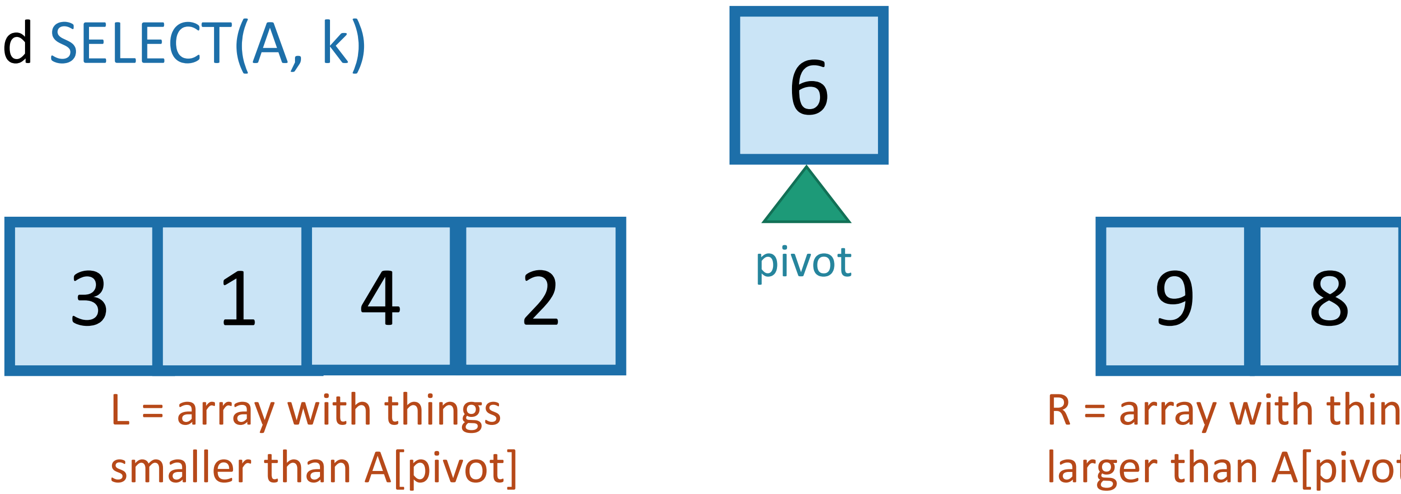


Idea continued…

Say we want to find SELECT(A, k)



- If k = 5 = len(L) + 1:
  - We should return A[pivot]
- If k < 5:
  - We should return SELECT(L, k)
- If k > 5:
  - We should return SELECT(R, k – 5)

This suggests a recursive algorithm

(still need to figure out how to pick the pivot…)

Let’s make that a bit more formal

- PARTITION(A, p):
  - L = new array
  - R = new array
  - **For** i=1,…,n:
    - **If** i==p:
      - continue
    - **Else if** A[i] <= A[p]:
      - L.append(A[i])
    - **Else if** A[i] > A[p]:
      - R.append(A[i])
  - **Return** L, A[p], R

initialize L and R

go through elts one at a time…put small ones in L, big ones in R.

- This is the O(n) PARTITION algorithm that we saw before.
- For clarity, I’m just going to initialize two new arrays, L and R. (Assume they are dynamically sized, and that we can append stuff in time O(1) and access any index in time O(1)).
- However, you can implement this (and everything else we will do in this lecture) in-place, without any of these considerations. (Fun exercise! Or see CLRS.)

More formal part II

- SELECT(A, k):
  - **If** len(A) <= 50:
    - A = MergeSort(A)
    - Return A[k]
  - p = CHOOSEPIVOT(A)
  - L, A[p], R = PARTITION(A, p)
  - **If** len(L) = k - 1:
    - **Return** A[p]
  - **Else If** len(L) > k - 1:
    - **Return** SELECT(L, k)
  - **Else if** len(L) < k - 1:
    - return SELECT(R, k – (len(L) + 1))

We'll see why I chose 50 later. It's pretty arbitrary.

- PARTITION(A, p):
  - L = new array
  - R = new array
  - **For** i=1,…,n:
    - **If** i==p:
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  - **Return** L, A[p], R

Base Case: If the len(A) = O(1), then any sorting algorithm runs in time O(1).

Case 1: We got lucky and found exactly the k'th smallest value!

Case 2: The k'th smallest value is in the first part of the list

Case 3: The k'th smallest value is in the second part of the list

Correctness

There seems to be some whitespace not yet used.

That's better.

- Recursion invariant:

At the return of each recursive call of size < n, SELECT(A, k) returns the k^{th} smallest element of A.

- Base case ("Initialization"):
- If len(A) <= 50, then the MergeSort approach is "clearly" correct.
- Inductive step: ("Maintenance")
  - Suppose that the recursion invariant holds for n.
  - Want to show that it holds for n + 1.
  - Three cases:
    - **if len(L) = k-1**, then A[p] is the correct thing to return.
    - **If len(L) > k-1**, then the k^{th} smallest element of L is the correct thing to return
      - And by induction, this is indeed what we return.
    - **If len(L) < k-1**, then the (k – (len(L)+1))st smallest elt of R is the correct thing to return.
      - And by induction, this is indeed what we return.

- SELECT(A, p=k):
  - **If** len(A) <= 50:
    - A =MergeSort(A)
    - Return A[k]
  - p = CHOOSEPIVOT(A)
  - L, A[p], R = PARTITION(A,p)
  - **If** len(L) = k - 1:
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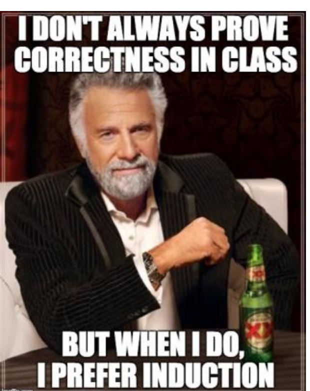
    - **Return** SELECT(L, k)
  - **Else if** len(L) < k - 1:
    - return SELECT(R, k – len(L) – 1)

Note: something like this is totally acceptable on your exams (maybe with one more sentence, or an example, saying why those are the right things to return.)



Note: Soon I'm going to stop proving correctness in class – eventually all these arguments will start to look the same.

- Conclusion ("Termination")
- By induction, the recursion invariant holds for n + 1, which means that SELECT(A,k) is correct.

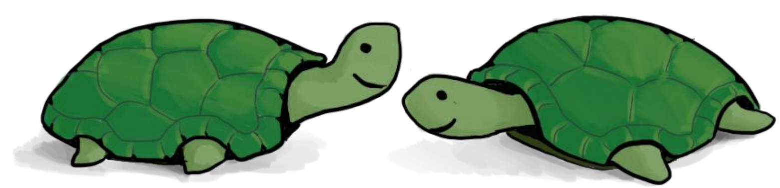
What is the running time?

Assuming we pick the pivot in time O(n)…

- $T(n) = \begin{cases} T(\mathbf{len(L)}) + O(n) & \mathbf{len(L)} > k - 1 \\ T(\mathbf{len(R)}) + O(n) & \mathbf{len(L)} < k - 1 \\ O(n) & \mathbf{len(L)} = k - 1 \end{cases}$

- What are **len(L)** and **len(R)**?
- That depends on how we pick the pivot…

What would be a "good" pivot?
What would be a "bad" pivot?



Think-Pair-Share Terrapins

The best way would be to always pick the pivot so that len(L) = k-1. But say we don't have control over k, just over how we pick the pivot.

The ideal pivot

- We split the input exactly in half:
  - len(L) = len(R) = (n-1)/2

Apply here, the Master Theorem does NOT. Making unsubstantiated assumptions about problem sizes, we are.

Jedi master Yoda

- Let's pretend that's the case and use the **Master Theorem**!
  - $T(n) \leq T\left(\frac{n}{2}\right) + O(n)$
  - So, a = 1, b = 2, d = 1
  - $T(n) \leq O\left(n^d\right) = O(n)$

<u>Note</u>: This is a rhetorical point for intuition in lecture. It is **NOT OKAY** as a final solution on your exam.

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

How about runtime?

- Let's try a recurrence relation…
- $T(n) = \begin{cases} T\big(len(L)\big) + O(n) & len(L) < k - 1 \\ T\big(len(R)\big) + O(n) & len(L) > k - 1 \\ O(n) & len(L) = k - 1 \end{cases}$
- What is len(L), len(R)?
- Let's **pretend** that len(L) is about ~~n/2~~. 7n/10
- $T(n) \leq T\left(\cancel{\frac{n}{2}}\right) + O(n)$
- $T(n) = O(n)$
- That would be great!

(we can even assume something a little weaker)

- SELECT(A, k):
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Recall the Master Theorem

which totally doesn't apply here, we are cheating by pretending we know the problem size.

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In our case:

- $T(n) \leq T\left(\frac{7n}{10}\right) + O(n)$
- So a = 1, b = 10/7, d = 1
- $T(n) \leq O\left(n^d\right) = O(n)$

Lucky the Lackadaisical Lemur

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Can we get away with n-1?

- $T(n) \leq T\left(\cancel{\frac{n}{2}}\right) + O(n)$

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In our case:

- $T(n) \leq T(n-1) + O(n)$
- So a = 1, b = 1/(1 -1/n) , d = 1
- $T(n) \leq O(n)$ still?
- NO!!! b needs to be independent of n for the master thm to work. Actual running time is O(n^2).

Lucky the Lackadaisical Lemur

How about runtime?

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- $T(n) \leq T\left(\cancel{\frac{n}{2}}\right) + O(n)$
- $T(n) = O\left(n^2\right)$
- Not good enough ☹!

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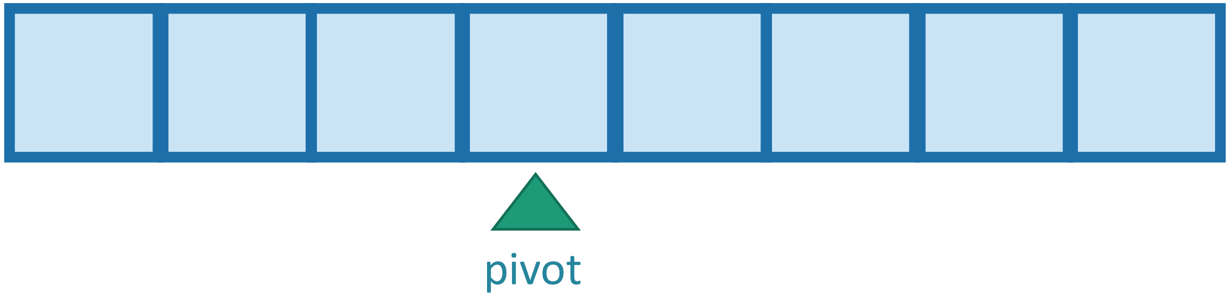
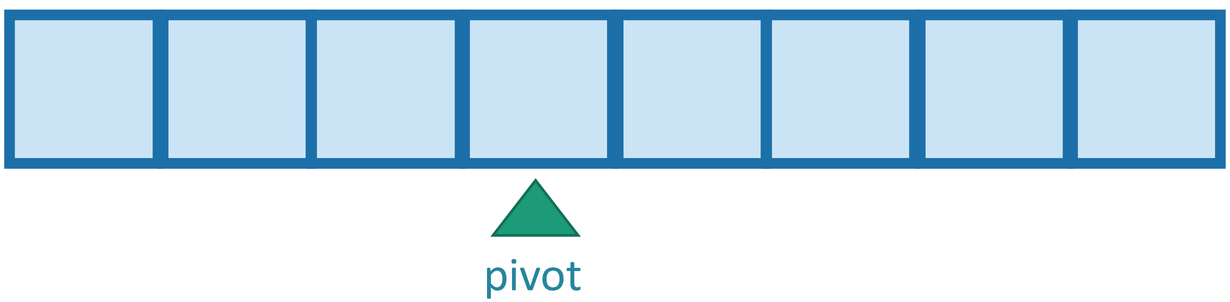
The worst pivot



The worst pivot

- Say our choice of pivot doesn't depend on A.



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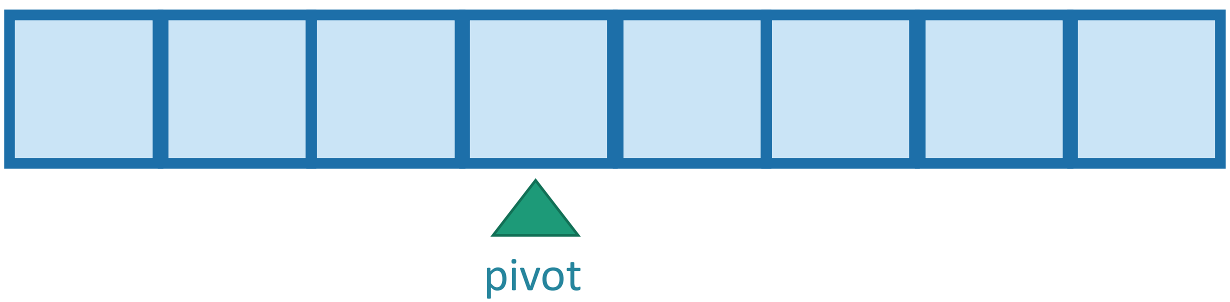

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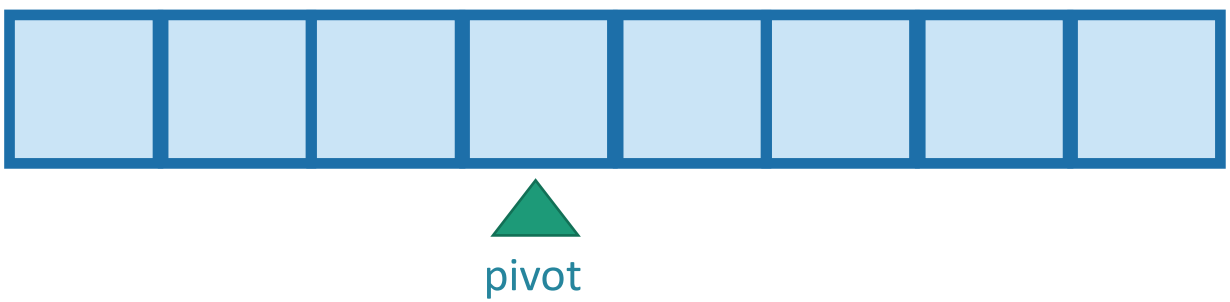

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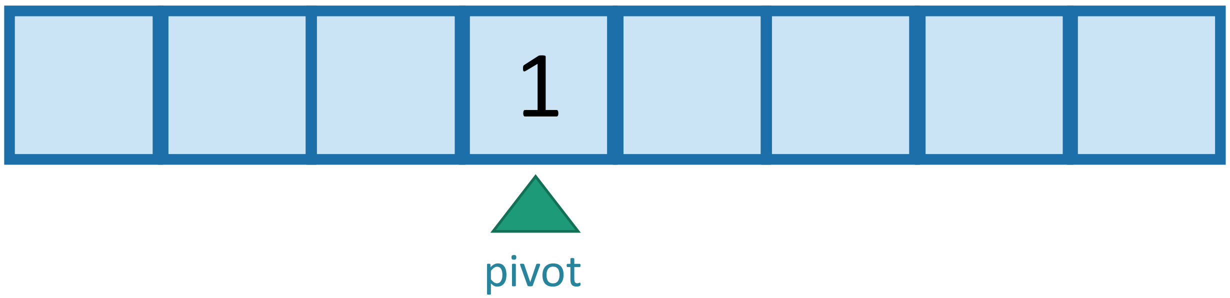
- A bad guy who **knows what pivots we will choose** gets to come up with A.



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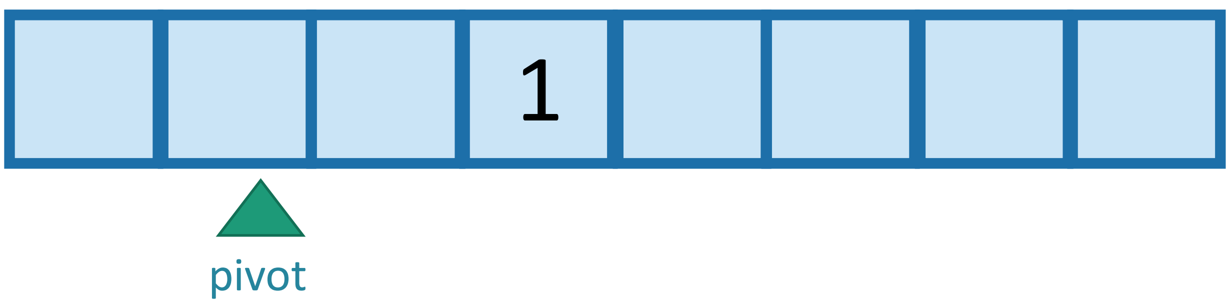

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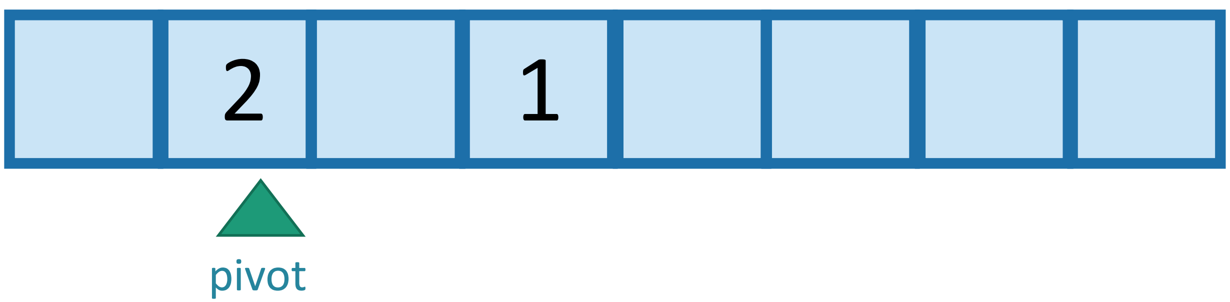

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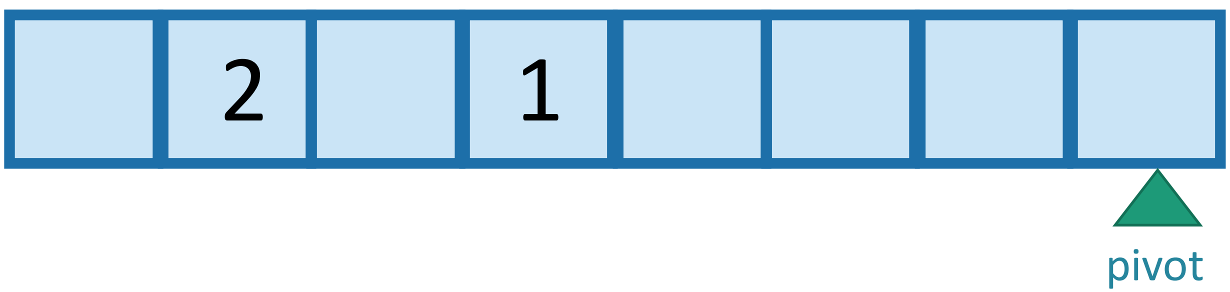

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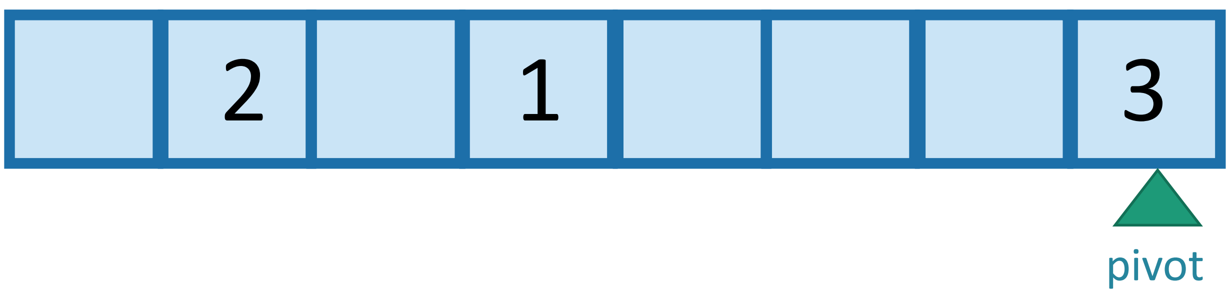
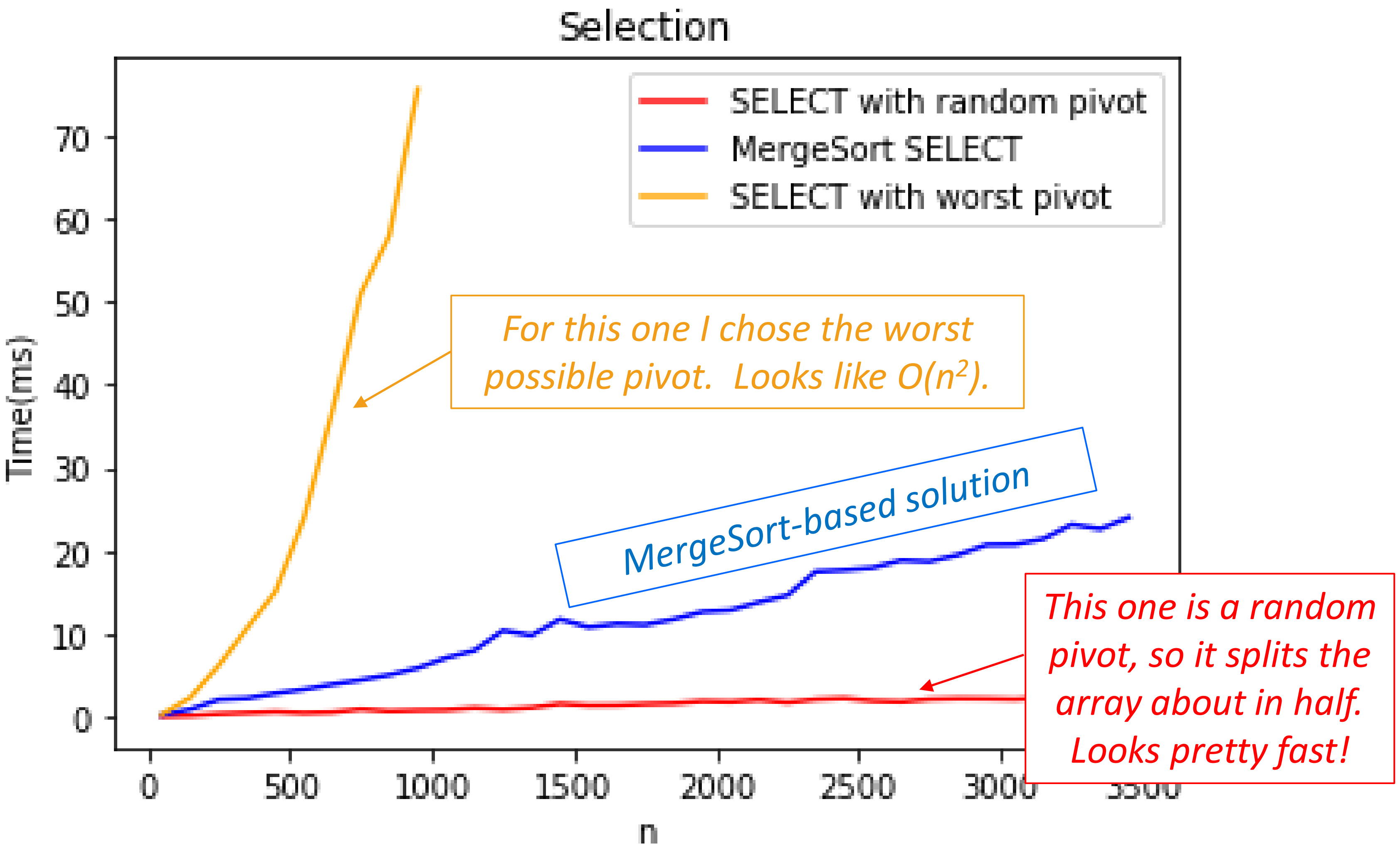


The distinction matters!



How do we pick a good pivot?

- Randomly?
  - That works well if there's no bad guy.
  - But if there is a bad guy who gets to see our pivot choices, that's just as bad as the worst-case pivot.

Aside:

- In practice, there is often no bad guy. In that case, just pick a random pivot and it works really well!
- (More on this later with randomized algos)

How do we pick a good pivot?

- For today, let's assume there's this bad guy.
- Reasons:
  - This gives us a very strong guarantee
  - We'll get to see a really clever algorithm.
    - Necessarily it will look at A to pick the pivot.
  - We'll get to use the substitution method.

The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
   a) The outline of the algorithm.
   b) How to pick the pivot.
4. Return of the Substitution Method.

Approach

- First, we'll figure out what the ideal pivot would be.
  - But we won't be able to get it.
- Then, we'll figure out what a **pretty good** pivot would be.
  - But we still won't know how to get it.
- Finally, we will see how to get our pretty good pivot!
  - And then we will celebrate.

How do we pick our ideal pivot?

- We’d like to live in the ideal world.

- Pick the pivot to divide the input in half.
- Aka, pick the median!
- Aka, pick SELECT(A, n/2)!

How about a good enough pivot?

- We'd like to approximate the ideal world.

- Pick the pivot to divide the input about in half!
- Maybe this is easier!

Moral of this extremely shady logic

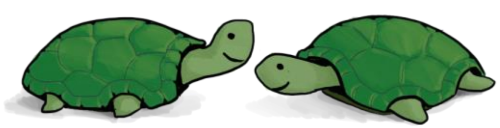
- If we can pick a pivot so that L and R **somewhat** balanced (even like 7n/10), then we're doing great. Otherwise, no good.
- **Try 1: Let's pick the pivot to be the median!**
  - Then L and R are always n/2. (or $\left\lfloor\frac{n}{2}\right\rfloor$ or $\left\lceil\frac{n}{2}\right\rceil$).
- **Problem:** That's exactly the problem we're trying to solve to begin with.
- **Solution:**
  - We can't find the median of n things (yet), but we can *recursively* find the median of n/5 things…
  - that will give us something "close enough" to the median that we can (rigorously) apply the previous analysis.

A good enough pivot

We still don't know that we can get such a pivot, but at least it gives us a goal and a direction to pursue!

Lucky the lackadaisical lemur

- We split the input not quite in half:
  - 3n/10 < len(L) < 7n/10
  - 3n/10 < len(R) < 7n/10
- If we could do that (let's say, in time O(n)), the **Master Theorem** would say:
  - $T(n) \leq T\left(\frac{7n}{10}\right) + O(n)$



Think-Pair-Share Terrapins!

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$. Then

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

A good enough pivot

We still don't know that we can get such a pivot, but at least it gives us a goal!

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  - 3n/10 < len(L) < 7n/10
  - 3n/10 < len(R) < 7n/10
- If we could do that (let's say, in time O(n)), the **Master Theorem** would say:
  - $T(n) \leq T\left(\frac{7n}{10}\right) + O(n)$
  - So a = 1, b = 10/7, d = 1
  - $T(n) \leq O\left(n^d\right) = O(n)$

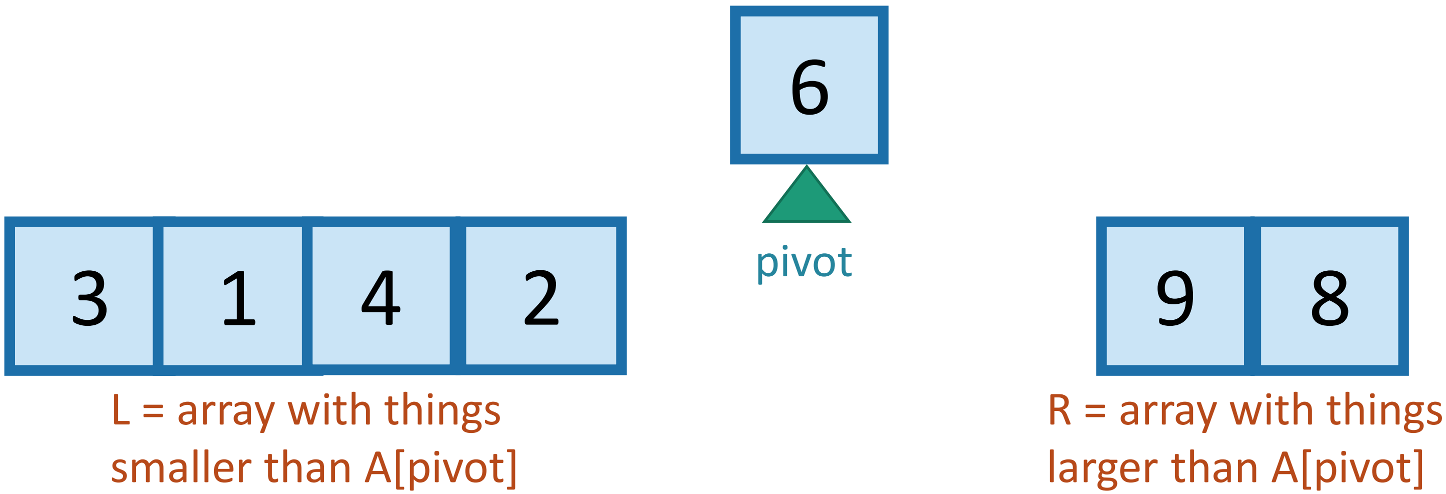
STILL GOOD!

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

Goal

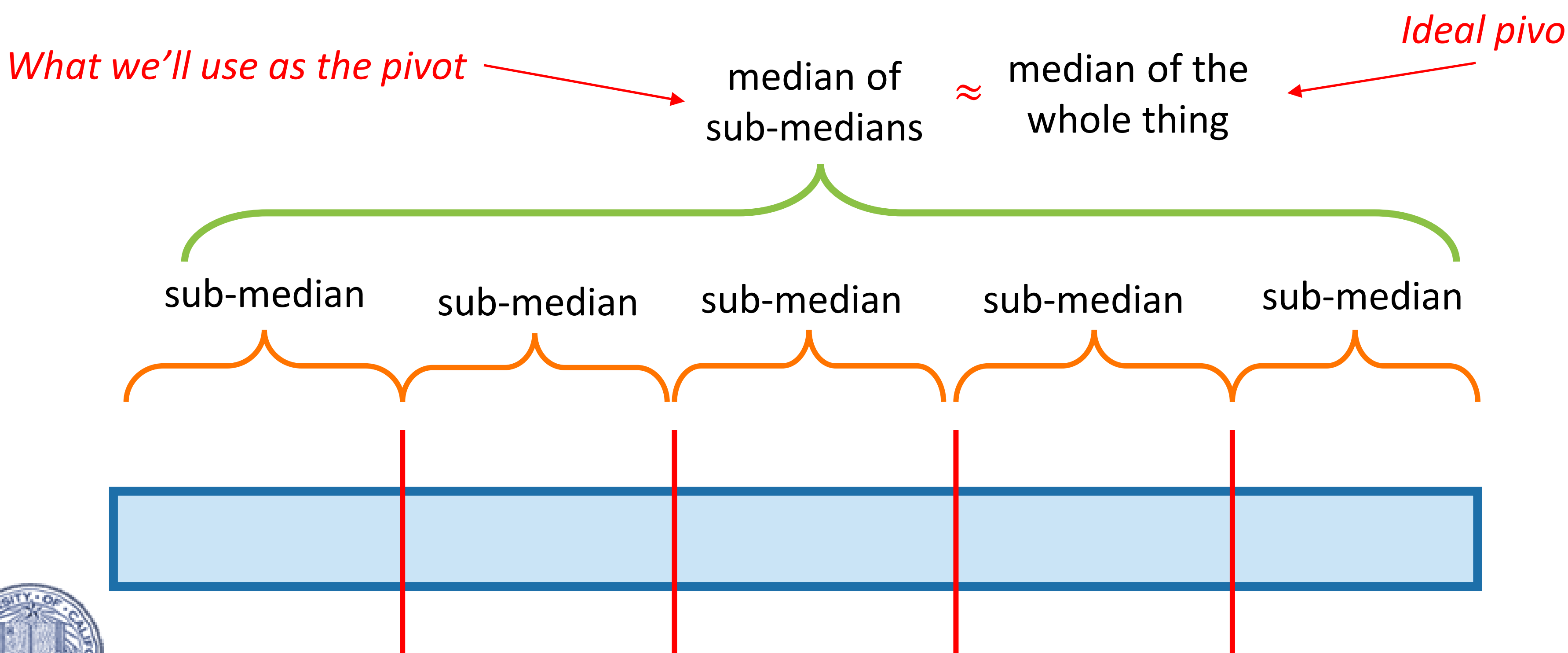
- In time O(n), pick the pivot so that



$$\frac{3n}{10} < \mathrm{len}(L) < \frac{7n}{10} \qquad \frac{3n}{10} < \mathrm{len}(R) < \frac{7n}{10}$$

Another divide-and-conquer alg!

- We can't solve SELECT(A,n/2) (yet)
- But we can divide and conquer and solve SELECT(B,m/2) for smaller values of m (where len(B) = m).
- **Lemma*:** The median of sub-medians is close to the median.



*we will make this a bit more precise.

How to pick the pivot

- CHOOSEPIVOT(A):
  - Split A into m = $\left\lceil \frac{n}{5} \right\rceil$ groups, of size <=5 each.
  - **For** i=1, .., m:
    - Find the median within the i^{th} group, call it p_i
  - p = SELECT([p_1, p_2, p_3, …, p_m] , m/2)
  - **return** p

- SELECT(A, p=k):
  - **If** len(A) <= 50:
    - A = MergeSort(A)
    - Return A[k]
  - p = CHOOSEPIVOT(A)
  - L, A[p], R = PARTITION(A,p)
  - **If** len(L) = k - 1:
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  - **Else if** len(L) > k - 1:
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1	8	9	3	15	5	9	1	3	4	12	2	1	5	20	15	13	2	4	6	12	1	15	22	3

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Pivot is SELECT([8 | 4 | 5 | 6 | 12] , 3) = 6:

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6

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PARTITION around that 6:

1	3	5	1	3	4	2	1	2	4	1	3	5	6	8	9	15	9	12	20	15	13	12	15	22

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PARTITION around that 5:

1	3	5	1	3	4	2	1	2	4	1	3	5	6	8	9	15	9	12	20	15	13	12	15	22

This part is L

This part is R: it's almost the same size as L.

So this gives the whole algorithm

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  - **Else if** len(L) < k - 1:
    - return SELECT(R, k – len(L) – 1)

- PARTITION(A, p):
  - L = new array
  - R = new array
  - **For** i=1,…,n:
    - **If** i==p, continue
    - **Else If** A[i] <= A[p]:
      - L.append(A[i])
    - **Else if** A[i] > A[p]:
      - R.append(A[i])
  - **Return** L, A[p], R

Note: We use recursion in two ways! Both in SELECT itself, and in CHOOSEPIVOT.

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- Does it work?
- Yes, our proof before worked for any pivoting strategy.