WH #7

1) $F(a) = P(X > 6.75) = 1 - F(6.75) = \cancel{1} - [\cancel{1} - e^{-6.75/4.5}]$

$4.5 = \frac{1}{\lambda} \Rightarrow \lambda = 1/4.5$

$= e^{-6.75/4.5} = \boxed{.2231}$

2) $45 = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{45}$

$F(a) = P(X > 45\,\text{min}) = 1 - F(60) = \cancel{1} - [\cancel{1} - e^{-60/45}] = \boxed{.2636}$

3) a) $F(x) = P(X \geq 1) = \int_1^x f(t)\,dt = \int_1^x \frac{\alpha}{x^{\alpha+1}}\,dx = \alpha \int_1^x x^{-(\alpha+1)}$

$= \cancel{\alpha}\left[\frac{x^{-\alpha-1+1}}{-\cancel{\alpha}}\right]_1^x$

$= -x^{-\alpha}$

$= -x^{-x} - -1^{-\alpha}$

$= -x^{-\alpha} + 1$

$= 1 - \frac{1}{x^\alpha};\ 1 \leq x$

b) $\mu = E[X] = \int_{-\infty}^{\infty} x f(x) = \int_1^\infty x \frac{\alpha}{x^{\alpha+1}}\,dx$

$= \int_1^\infty \frac{\alpha x}{x^\alpha \cdot x}\,dx = \alpha \int_1^\infty x^{-\alpha}\,dx = \frac{\alpha x^{-\alpha+1}}{-\alpha+1}\Big|_1^\infty$

$= \frac{\alpha}{-\alpha+1}\left(\frac{1}{x^{\alpha-1}}\right)\Big|_1^\infty = \frac{\alpha}{-\alpha+1}\left[\lim_{x\to\infty}\left(\frac{1}{x^{\alpha-1}}\right) - \frac{1}{1^{\alpha-1}}\right]$

$= \frac{\alpha}{-\alpha+1}[0-1] = \frac{\alpha}{\alpha-1};\ \alpha > 1 = \mu$

c)

$EX^2 = \int_1^\alpha x^2 f(x)\,dx = \alpha \int_1^\alpha x^{-\alpha+1}\,dx = \alpha \frac{x^{-\alpha+2}}{-\alpha+2}\Big]_1^\alpha = \alpha\left[0 - \frac{1}{-\alpha+2}\right] = \frac{\alpha}{\alpha-2},\ \alpha > 2$

$V(X) = EX^2 - (EX)^2 = \frac{\alpha}{\alpha-2} - \left[\frac{\alpha}{\alpha-1}\right]^2 = \frac{\alpha(\alpha-1)^2 - \alpha^2(\alpha-2)}{(\alpha-2)(\alpha-1)^2}$

$= \frac{\alpha(\alpha^2-2\alpha+1) - \alpha^3 + 2\alpha^2}{(\alpha-2)(\alpha-1)^2}$

$= \frac{\alpha^3 - 2\alpha^2 + \alpha - \alpha^3 + 2\alpha^2}{(\alpha-2)(\alpha-1)^2}$

$= \frac{\alpha}{(\alpha-2)(\alpha-1)^2},\ \alpha > 2$

$\sigma = \sqrt{\frac{\alpha}{(\alpha-2)(\alpha-1)^2}},\ \alpha > 2$

4) $\int_{x_m}^m f(x)\,dx = 1/2$ median of a Pareto Distribution

$\int_{x_m}^m \frac{\alpha x_m^\alpha}{x^{\alpha+1}}\,dx = 1/2$

$\alpha x_m^\alpha \left[\int_{x_m}^m x^{-\alpha-1}\,dx\right] = 1/2$

$\alpha x_m^\alpha \left[\frac{x^{-\alpha-1+1}}{-\alpha-1+1}\right]_{x_m}^m = 1/2$

$\alpha x_m^\alpha \left[\frac{x^{-\alpha}}{-\alpha}\right]_{x_m}^m = 1/2$

$-x_m^\alpha\left[m^{-\alpha} - x_m^{-\alpha}\right] = 1/2$

$-x_m^\alpha m^{-\alpha} + x_m^\alpha x_m^{-\alpha} = 1/2$

$-x_m^\alpha m^{-\alpha} + 1 = 1/2$

$1/2 = x_m^\alpha m^{-\alpha}$ log both sides

$\log(1/2) = \log(x_m^\alpha m^{-\alpha})$

$\log(1/2) = \log(x_m^\alpha) + \log(m^{-\alpha})$

$\log(1/2) = \alpha \log(x_m) - \alpha \log(m)$

$\alpha \log(m) = \alpha \log(x_m) - \log(1/2)$

$\log(m) = \log(x_m) - \frac{1}{\alpha}\log(1/2)$

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$\log m = \log X_m - \log\left(\frac{1}{2}\right)^{1/\alpha}$

$\log m = \log\left[\frac{X_m}{(1/2)^{1/\alpha}}\right]$

$m = \frac{X_m}{(1/2)^{1/\alpha}} = \boxed{X_m(2)^{1/\alpha}}$

5) a) $f(x) = \frac{4}{x^5} = \frac{\alpha}{x^{\alpha+1}}$ $\quad f(x) = \begin{cases} \frac{4}{x^5} & ; x \geq 1 \\ 0 & ; \text{otherwise} \end{cases}$

$\boxed{\alpha = 4}$

b) $\mu = E(x) = \int_1^\infty x \cdot \frac{4}{x^5}\,dx = 4\int_1^\infty x^{-4}\,dx$

$4\left[\frac{x^{-3}}{-3}\right]_1^\infty = \frac{4}{3}\left[\frac{-1}{x^3}\right]_1^\infty = \frac{4}{3}[0-1] = \boxed{4/3 = \mu}$

$\sigma = E[x^2] = \int_1^\infty x^2 \frac{4}{x^5}dx = 4\int_1^\infty x^{-3}dx = 4\left[\frac{x^{-2}}{-2}\right]_1^\infty = 2\left[-\frac{1}{x^2}\right]_1^\infty$

$= 2[0-1] = 2$

$\sigma = \sqrt{2-(4/3)^2} = 2 - \frac{16}{9} = \frac{18-16}{9} = 2/9$

$\boxed{\sigma = \frac{\sqrt{2}}{3} = \frac{2}{3}\sqrt{1/2}}$

c) $(\mu - 2\sigma, \mu + 2\sigma) = \left(\frac{4}{3} - 2\frac{\sqrt{2}}{3}, \frac{4}{3} + 2\frac{\sqrt{2}}{3}\right) = (.39, 2.28)$

contains 95% of the total data

consider $\int_{.39}^{2.28} f(x)\,dx = \int_1^{2.28} \frac{4}{x^5}\,dx = 4\left[\frac{x^{-4}}{-4}\right]_1^{2.28} = 1\left[\frac{-1}{x^4}\right]_1^{2.28}$

$= 1\left[-\frac{1}{2.28^4} + \frac{1}{1}\right]$

$= 1 - \frac{1}{2.28^4}$

$= .963 \approx .95$

d) CDF $F(x) = 1 - \left(\frac{1}{x}\right)^\alpha = 1 - \left(\frac{1}{x}\right)^4$

$F(3) = 1 - \left(\frac{1}{3}\right)^4 = 80/81$

$F(2) = 1 - \left(\frac{1}{2}\right)^4 = 15/16$

$\int_2^3 f(x)\,dx = F(3) - F(2) = .05$

Yes, 3 cell phones is unusually high compared to the range rule of thumb.