Homework Assignment #4

Remember, this Homework Assignment is **not collected or graded**! But you are advised to do it anyway because the problems for Homework Quiz #4 will be heavily based on these problems!

1. Let V be a vector space with subspaces S_1 and S_2. Define $S = S_1 \cap S_2$ as the set of all vectors that belong to both S_1 and S_2.
   (a) Explain how we know that S will be non-empty.
   (b) Prove that S will be a subspace of V.

2. The complete solution to $A\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is:
$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
   What is the matrix A?

3. Are the following statements true or false? Give a reason if true and a counter example if false.
   (a) A square matrix (i.e., $n \times n$) has no free variables.
   (b) An invertible matrix has no free variables.
   (c) An $m \times n$ matrix has no more than n pivots.
   (d) An $m \times n$ matrix has no more than m pivots.

4. What are the **special solutions** (i.e., the non-zero solutions defined by free variables) to the system
$$A\vec{x} = \vec{0}$$
   for the following choices of A:
   (a) $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
   (b) $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$.

5. Let P_3 be the vector space of polynomials up to degree 3. That is, an element of P_3 is of the form $p(x) = a_0 + a_1 x + a_2 x^2$.
   (a) Explain why this is a vector space of dimension 3.
   (b) Let $p_1(x) = 1 + x$, $p_2(x) = x(x-1)$, $p_3(x) = 1 + 2x^2$. Verify that $\{p_1, p_2, p_3\}$ is a basis for P_3.
   (c) Determine the coefficients of linear combination representing a generic polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ relative to this basis. That is, find coefficients α_i so that:
$$p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x).$$

6. Find a basis for the each of the following subspaces of $\mathbb{R}^3$. (Hint: It MIGHT help to write these as a matrix-vector system, $A\vec{x} = \vec{0}$ as these subspaces are the nullspace for a matrix A.)
   (a) The plane $2x - 3y + z = 0$
   (b) The intersection of the plane $2x - 3y + z = 0$ with the xy plane.