Homework #2
SOLUTION

PROBLEM 1:

Consider the two image subsets, S_1 and S_2, shown in the following figure. For $V = \{1\}$, determine whether these two subsets are (a) 4-adjacent, (b) 8-adjacent, or (c) m-adjacent.

	S_1				S_2				
0	0	0	0	0	0	0	1	1	0
1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	1	1	0	0	0
0	0	1	1	1	0	0	0	0	0
0	0	1	1	1	0	0	1	1	1

SOLUTION:

Let p and q be as shown in Fig. P2.11. Then, (a) S_1 and S_2 are not 4-connected because q is not in the set $N_4(p)$; (b) S_1 and S_2 are 8-connected because q is in the set $N_8(p)$; (c) S_1 and S_2 are m-connected because (i) q is in $N_D(p)$, and (ii) the set $N_4(p) \cap N_4(q)$ is empty.

	S_1				S_2				
0	0	0	0	0	0	0	1	1	0
1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	1 q	1	0	0	0
0	0	1	1	p 1	0	0	0	0	0
0	0	1	1	1	0	0	1	1	1

Figure P2.11

PROBLEM 2:

Consider the image segment shown.

★**(a)** Let $V = \{0, 1\}$ and compute the lengths of the shortest 4-, 8-, and m-path between p and q. If a particular path does not exist between these two points, explain why.

(b) Repeat for $V = \{1, 2\}$.

	3	1	2	1 (q)
	2	2	0	2
	1	2	1	1
(p)	1	0	1	2

SOLUTION:

(a) When $V=\{0,1\}$, 4-path does not exist between p and q because it is impossible to get from p to q by traveling along points that are both 4-adjacent and also have values from V. Figure P2.15(a) shows this condition; it is not possible to get to q. The shortest 8-path is shown in Fig. P2.15(b); its length is 4. The length of the shortest m- path (shown dashed) is 5. Both of these shortest paths are unique in this case.

(b) One possibility for the shortest 4-path when $V=\{1,2\}$ is shown in Fig. P2.15(c); its length is 6. It is easily verified that another 4-path of the same length exists between p and q. One possibility for the shortest 8-path (it is not unique) is shown in Fig. P2.15(d); its length is 4. The length of a shortest m-path (shown dashed) is 6. This path is not unique.

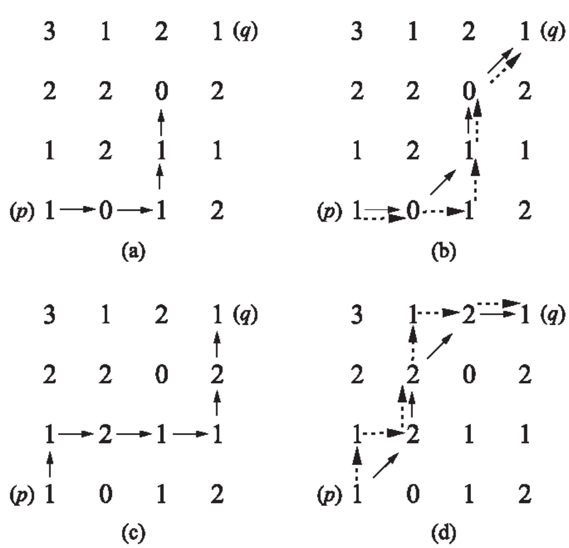

Figure P.2.15

PROBLEM 3:

Let *H[·]* be the operator that determines the minimum pixel value in an image. That is, if the image *f(x,y)* has the following pixel values

f(x,y)=

30	23	6
110	128	234
12	4	175

then

$$H[f] = 4.$$

Prove that *H[·]* is non-linear.

(Remember that to prove non-linearity, you just need to come up with a counter-example; i.e., images *f1* and *f2* and constants *a* and *b* such that *H[a*f1+b*f2]* ≠ *a*H[f1]+b*H[f2]*.)

SOLUTION:

We just need to find a specific pair of images *f1* and *f2* and pair of constants *a* and *b* such that *H[a*f1+b*f2]* ≠ *a*H[f1]+b*H[f2]*. There are of course endless options but the following will work. Let

f1=

1	2	3
4	5	6
7	8	9

f2=

18	17	16
15	14	13
12	11	10

a= 1 and *b*=1

Then, *a*f1+b*f2*=

19	19	19
19	19	19
19	19	19

so that *H[a*f1+b*f2]*=19.

But, *a*H[f1]*=1 and *b*H[f2]*=10 so that *a*H[f1]+b*H[f2]*=11.

We're done since we've found a specific pair of images *f1* and *f2* and pair of constants *a* and *b* such that $H[a*f1+b*f2] \neq a*H[f1]+b*H[f2]$.

PROBLEM 4:

In general, do affine transformations commute?

That is, given two affine transformations T_1 and T_2, does the transformation T_1T_2 give the same result as T_2T_1? (T_1 and T_2 are the matrix representations of the transformations. You can interpret T_1T_2 as first applying T_1 and then applying T_2 or you can interpret it as the matrix multiplication of T_1 and T_2.)

If not, provide a counter example. That is, provide affine transformation matrices T_1 and T_2 such that a point mapped by T_1T_2 is different from the same point mapped by T_2T_1.

SOLUTION:

Affine transformations, in general, do not commute. We just need to find a counterexample to prove this.

Consider a rotation transformation and a shear transformation and their respective matrices:

Rotation matrix = $T_R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Shear matrix = $T_S = \begin{bmatrix} 1 & 0 & 0 \\ s_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Case 1</u>: First rotate and then shear. That is, find where the pixel at (v, w) is mapped to by the combination of these transformations.

First, the rotation:

$$[x' \quad y' \quad 1] = [v \quad w \quad 1] \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v\cos\theta - w\sin\theta \quad v\sin\theta + w\cos\theta \quad 1]$$

Now, the shear:

$$[x \quad y \quad 1] = [x' \quad y' \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ s_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v\cos\theta - w\sin\theta \quad v\sin\theta + w\cos\theta \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ s_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v\cos\theta - w\sin\theta + s_v(v\sin\theta + w\cos\theta) \quad v\sin\theta + w\cos\theta \quad 1]$$

So, $(x, y) = (v\cos\theta - w\sin\theta + s_v(v\sin\theta + w\cos\theta),\ v\sin\theta + w\cos\theta)$

<u>Case 2</u>: First shear and then rotate. That is, find where the pixel at (v, w) is mapped to by the combination of these transformations.

First, the shear:

$$[x' \quad y' \quad 1] = [v \quad w \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ s_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v + s_v w \quad w \quad 1]$$

Now, the rotation:

$$[x \quad y \quad 1] = [x' \quad y' \quad 1]\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [v + s_v w \quad w \quad 1]\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [(v + s_v w)\cos\theta - w\sin\theta \quad (v + s_v w)\cos\theta - w\sin\theta \quad 1]$$

So, $(x, y) = \big((v + s_v w)\cos\theta - w\sin\theta\,,\ (v + s_v w)\cos\theta - w\sin\theta\big)$

This is not equal to (x, y) in case 1 above.

So, these two transformations do not commute and we have found our counterexample.

We can then conclude that affine transformations do not, in general, commute.

(Note that there are of course an endless number of possible counterexamples in addition to the one above.)

PROBLEM 5:

Suppose that only pixels with values 5, 10, 30, and 150 occur in a grayscale image. And suppose that these pixels occur with the following probabilities in the image:

$$p(5) = 0.10$$
$$p(10) = 0.55$$
$$p(30) = 0.05$$
$$p(150) = 0.30$$

(You can assume the usual case where grayscale images have pixels with values 0 through 255.)

(a) Compute the mean of the pixel values in the image.

(b) Compute the variance (σ^2) of the pixel values in the image.

SOLUTION:

(a) We can use the following equation to compute the mean:

$$mean = \sum_{k=0}^{255} z_k p(z_k)$$

where $z_k$is the pixel intensity and $p(z_k)$ is the probability that a pixel with intensity z_k occurs.

Since $p(z_k) = 0$ for $z_k \neq \{5, 10,30,150\}$ this reduces to

$$mean = 5(0.10) + 10(0.55) + 30(0.05) + 150(0.30) = 52.5$$

(b) We can use the following equation to compute the variance:

$$variance = \sigma^2 = \sum_{k=0}^{255} (z_k - mean)^2 p(z_k)$$

This reduces to

$variance = \sigma^2 = (5 - 52.5)^2(0.10) + (10 - 52.5)^2(0.55) + (30 - 52.5)^2(0.05) +$
$(150 - 52.5)^2(0.30)$ =4096.2