Week 13: Tuesday. 5.5: Complex Matrices

Course Goals

After studying section 5.5: Complex Matrices, you should

1. *Understand how to deal with complex vectors and complex matrices.*
2. *Understand special properties of Hermitian and unitary matrices.*

Course Outcomes

To manifest that you have reached the above course goals, you should be able to

1. *Find the inner products and lengths of complex vectors.*
2. *Find the conjugate transpose (Hermitian transpose or adjoint) of a matrix.*
3. *Test whether a given matrix is Hermitian or unitary.*
4. *Discuss what special properties the eigenvalues and eigenvectors of each kind of special matrices have.*

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Hermitian Matrices

A "symmetric" matrix that has *complex* values is called **Hermitian**.

Symmetric Matrices have Amazing Eigen-structure

1. Every symmetric (and Hermitian) matrix has **real eigenvalues**.
2. Every symmetric (and Hermitian) matrix is diagonalizable with an **orthonormal** set of eigenvectors.

$A^H \rightarrow$ if A is real

$A^T = A^H$

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absolute value

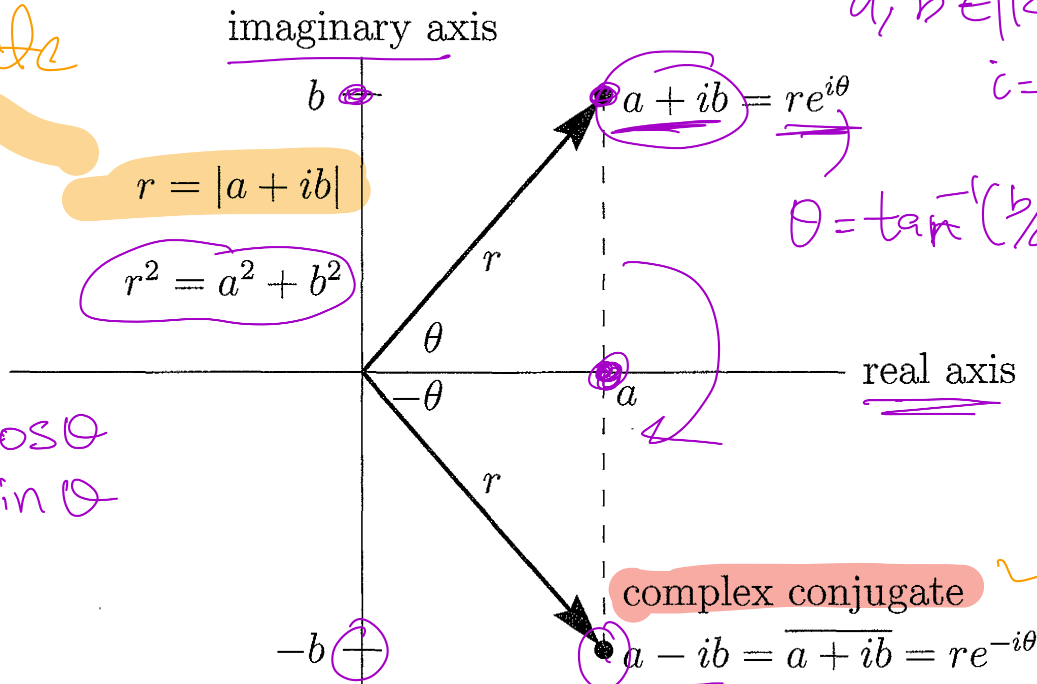
$a, b \in \mathbb{R}$

$i = \sqrt{-1}$

$\theta = \tan^{-1}(b/a)$

$\mathbb{R}^2 \cong \mathbb{C}$

Math 122



$a = r\cos\theta$

$b = r\sin\theta$

Figure 5.4 The complex plane, with $a + ib = re^{i\theta}$ and its conjugate $a - ib = re^{-i\theta}$.

$\overline{a+ib} = a - ib$ $\qquad$ $\bar{a} = a$

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Complex Number Arithmetic

$\sqrt{-1}$

1. Addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$.
2. Multiplication: $(\sqrt{-1})(\sqrt{-1}) = -1$
   $(a+ib)(c+id) = ac + ibc + iad + i^2bd = (ac - bd) + i(bc+ad)$.
3. Complex Conjugate: $\overline{a+ib} = a - ib$ ← flipping about the real axis
4. Absolute Value: $|a+ib| = \sqrt{a^2+b^2}$.
5. Polar Form: $a + ib = \boxed{r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}}$.
   $r = |a+ib|, \theta = \arctan(b/a)$

- The absolute value is also the square root of $z\bar{z}$ $\quad |z| = (z\bar{z})^{1/2}$

$$\sqrt{(a+ib)\overline{(a-ib)}} = \sqrt{a^2 - \cancel{abi} + \cancel{abi} - i^2b^2} = \sqrt{a^2+b^2} = |a+ib|.$$

- To divide by a complex number, multiply by the complex conjugate.

$$\frac{2+i}{3+4i} = \frac{2+i}{3+4i}\frac{3-4i}{3-4i} = \frac{6+3i-7i-4i^2}{3^2+4^2} = \frac{10-5i}{25} = a+ib$$

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Examples

1. (§5.5: 1) For the complex numbers $1 + \sqrt{3}i$ and $1 - i$
   a. find their positions in the complex plane;
   b. find their sum and product;
   c. find their conjugates and their absolute values.
   d. Write the original numbers and their product into $re^{i\theta}$ form.

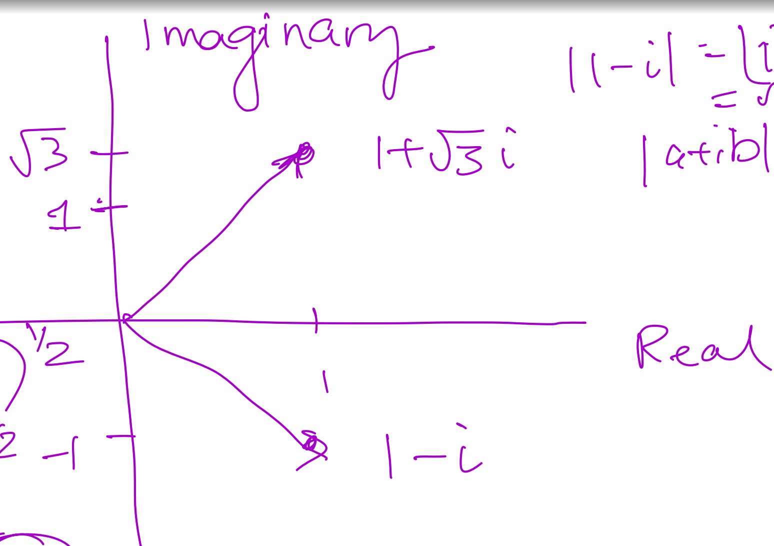
special Δ's

(a)

$\overline{1+\sqrt{3}i} = 1-\sqrt{3}i$

$\overline{1-i} = 1+i$

$|1+\sqrt{3}i| = \left(\overline{(1+\sqrt{3}i)}(1+\sqrt{3}i)\right)^{1/2}$
$= \left((1-\sqrt{3}i)(1+\sqrt{3}i)\right)^{1/2}$
$= (1+3)^{1/2}$
$= 2$



$|1-i| = (1^2+1^2)^{1/2}$
$= \sqrt{2}$

$|a+ib| = (a^2+b^2)^{1/2}$

(b) Find their sum and product and (d) Write the original numbers and product into $re^{i\theta}$ form.

- $(1+\sqrt{3}i)+(1-i)=2+i(\sqrt{3}-1)$.
- $(1+\sqrt{3}i)(1-i)=1+\sqrt{3}i-i-i^2\sqrt{3}=(1+\sqrt{3})+i(\sqrt{3}-1)$

  We could also do:

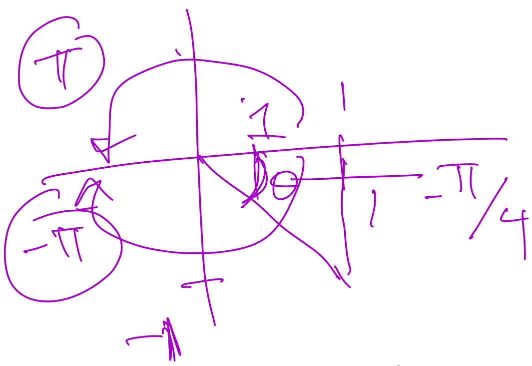
$$(1+\sqrt{3}i)=2e^{i\pi/3} \text{ and } (1-i)=\sqrt{2}e^{-i\pi/4}$$

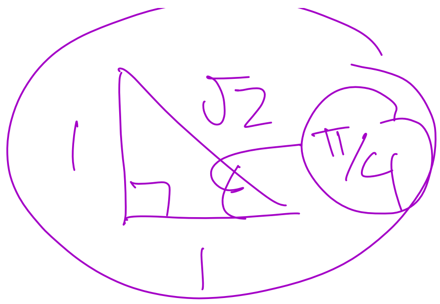
$$(1+\sqrt{3}i)(1-i)=2e^{i\pi/3}\left(\sqrt{2}e^{-i\pi/4}\right)=2\sqrt{2}e^{i(\pi/3-\pi/4)}=2\sqrt{2}e^{i\pi/12}$$

$$x=r\cos(\theta) \implies 2\sqrt{2}\cos(\pi/12)=1+\sqrt{3}$$

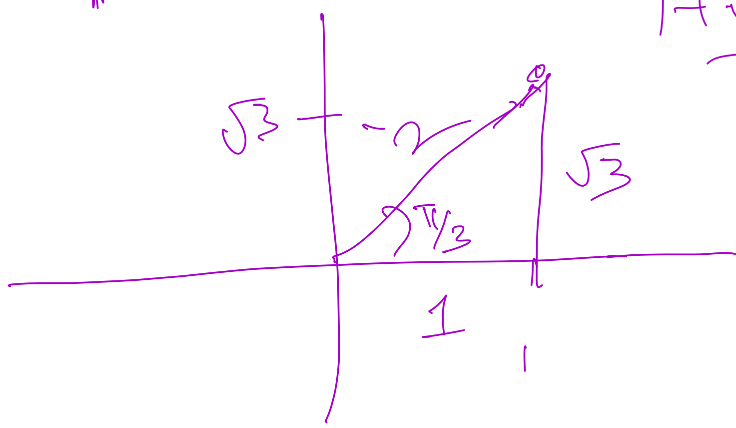
$$y=r\sin(\theta) \implies 2\sqrt{2}\sin(\pi/12)=-1+\sqrt{3}$$

$re^{i\theta} = r\cos\theta + i r\sin\theta$





$1 + \sqrt{3}\, i = 2e^{i\pi/3}$



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(c) Find their conjugates and absolute values.

- Conjugates:
  - $\overline{(1+\sqrt{3}i)} = 1 - i\sqrt{3}$.
  - $\overline{(1-i)} = 1 + i$
- Absolute Value:
  - $|1+\sqrt{3}|^2 = (1+\sqrt{3}i)\overline{(1+\sqrt{3}i)} = (1+\sqrt{3}i)(1-\sqrt{3}i) = 1^2 + 3 = 4$
  - $|(1-i)|^2 = (1-i)\overline{(1-i)} = (1-i)(1+i) = 1^2 + 1^2 = 2.$

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Examples

2. (§5.5: 5(a)) If $z = re^{i\theta}$, what are z^2, z^{-1}, and $\bar{z}$ in polar coordinates?

- $z^2 = (re^{i\theta})(re^{i\theta}) = r^2 e^{i2\theta}$.
- $z^{-1} = (re^{i\theta})^{-1} = (1/r)e^{-i\theta}$.
- $\bar{z} = \overline{re^{i\theta}} = re^{-i\theta}$.

  To see why here's a little tip:

$$
\begin{aligned}
\bar{z} &= \overline{r\cos(\theta) + ir\sin(\theta)} \\
&= r\cos(\theta) - ir\sin(\theta) \\
&= r\cos(-\theta) + ir\sin(-\theta) \\
&= re^{-i\theta}
\end{aligned}
$$

sin odd

cos(θ) even

$\cos(\theta) = \cos(-\theta)$

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Examples

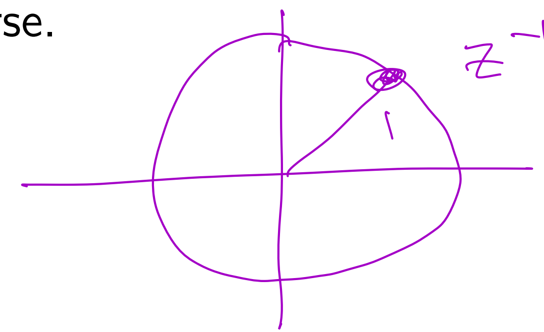
3. (§5.5: 5(a)) Which complex numbers satisfy $z^{-1} = \bar{z}$?

Here we might as well use the polar coordinate forms we had in the previous slide:

- $z^{-1} = (re^{i\theta})^{-1} = (1/r)e^{-i\theta}$.
- $\bar{z} = \overline{re^{i\theta}} = re^{-i\theta}$.

$$(1/r)e^{-i\theta} = re^{-i\theta} \Longrightarrow 1/r = r \Longrightarrow r = 1.$$

Thus, when we have complex number that lies on the unit circle, its inverse is equal to its converse.



$z^{-1} = \bar{z}$

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Examples

4. (§5.5: 2) What can you say about
   a. the sum of a complex number and its conjugate? what about difference?
   b. the conjugate of a number on the unit circle? $(\bar{z} = z^{-1})$
   c. the product of two numbers on the unit circle?

$\mathrm{Re}(a+ib) = a$

- Sum: $(a+ib) + \overline{a+ib} = (a+ib) + (a-ib) = 2a = 2\mathrm{Real}(a+ib)$
- Difference:
  $(a+ib) - \overline{a+ib} = (a+ib) - (a-ib) = 2bi = i2\cancel{\text{Complex}}(a+ib)$.
- Product of 2 Numbers on the Unit Circle (also on the unit circle)

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

$\mathbb{C}$

$r=1$

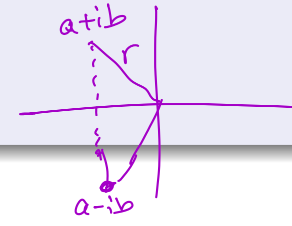
$\mathrm{Im}(a+ib) = b$

Examples

5. (§5.5: 6) Find the lengths and the inner product of

$\mathbb{C}^2$

$$\vec{x} = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} 2+4i \\ 4i \end{bmatrix}.$$



Conjugate Transpose: $\vec{x}^H = \overline{\vec{x}^T}$.

- Lengths:

$$\|\vec{x}\|^2 = \vec{x}^H\vec{x} = \begin{bmatrix} \overline{2-4i} & \overline{4i} \end{bmatrix} \begin{bmatrix} 2-4i \\ 4i \end{bmatrix} = \begin{bmatrix} 2+4i & -4i \end{bmatrix} \begin{bmatrix} 2-4i \\ 4i \end{bmatrix}$$
$$= (2+4i)(2-4i) + (-4i)(4i) = 4 + 8i - 8i + 16 + 16 = 36$$

$$\|\vec{y}\|^2 = \vec{y}^H\vec{y} = \begin{bmatrix} \overline{2+4i} & \overline{4i} \end{bmatrix} \begin{bmatrix} 2+4i \\ 4i \end{bmatrix} = \begin{bmatrix} 2-4i & -4i \end{bmatrix} \begin{bmatrix} 2+4i \\ 4i \end{bmatrix}$$
$$= (2-4i)(2+4i) + (-4i)(4i) = 4 + 8i - 8i + 16 + 16 = 36$$

$$\|\vec{x}\| = (\vec{x}^H\vec{x})^{1/2} = 6 \text{ and } \|\vec{y}\| = (\vec{y}^H\vec{y})^{1/2} = 6.$$

Examples

5. (§5.5: 6) Find the lengths and the inner product of

$$\vec{x} = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 2+4i \\ 4i \end{bmatrix}.$$

Conjugate Transpose: $\vec{x}^H = \overline{\vec{x}^T}$.

Real $\boxed{\vec{x}^T\vec{y} = \vec{y}^T\vec{x}}$

- Inner Product:

$$\vec{x}^H\vec{y} = \begin{bmatrix} \overline{2-4i} & \overline{4i} \end{bmatrix}\begin{bmatrix} 2+4i \\ 4i \end{bmatrix} = \begin{bmatrix} 2+4i & -4i \end{bmatrix}\begin{bmatrix} 2+4i \\ 4i \end{bmatrix}$$

$$(2+4i)^2 + (-4i)(4i) = 4 + 16i - 16 + 16 = 4 + 16i$$

$$\boxed{(\vec{x}^H\vec{y})^H} = \vec{y}^H(\vec{x}^H)^H = \boxed{\vec{y}^H\vec{x}}$$

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Examples

6. What is the proper equivalence of the transpose A^T of a real matrix in the complex case?

A **Conjugate Transpose:** $A^H = A^* = \overline{A}^T$.

- Note that if A is a real matrix, $A^H = A^T$.
- The generalization of a real **symmetric** matrix is a **Hermitian** matrix.
- Similarly to $(AB)^T = B^T A^T$ we have $(AB)^H = B^H A^H$.

$A = A^H$
Hermitian

Example

Take the conjugate transpose of

$$A = \begin{bmatrix} 2+i & 3i \\ 4-i & 5 \\ 0 & 0 \end{bmatrix} \Longrightarrow A^H = \begin{bmatrix} 2-i & 4+i & 0 \\ -3i & 5 & 0 \end{bmatrix}$$

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Examples

7. $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$.
   a. Check that A is a Hermitian matrix.
   b. Diagonalize A.
   c. Have you noticed anything special about its eigenvalues and eigenvectors?

Let's check: $A^H = \overline{A}^T = A$.

$$A^H = \overline{\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}}^T = \begin{bmatrix} 2 & 3+3i \\ 3-3i & 5 \end{bmatrix}^T = \boxed{\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} = A}$$

Thus A is a Hermitian matrix.

YAY

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$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$$

To diagonalize the matrix A, we find eigenvectors and eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) \\ &= \lambda^2 - 7\lambda + 10 - (3+3i)(3-3i) \\ &= \lambda^2 - 7\lambda + (10 - 18) \\ &= \lambda^2 - 7\lambda - 8 \\ &= (\lambda - 8)(\lambda + 1). \end{aligned}$$

$$\implies \lambda_1 = -1, \lambda_2 = 8.$$

Real Eigenvalues

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- $\lambda_1 = -1$

$$(A-\lambda_1 I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 2-(-1) & 3-3i & 0 \\ 3+3i & 5-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 3-3i & 0 \\ 3+3i & 6 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2-(1+i)R_1} \left[\begin{array}{cc|c} 3 & 3-3i & 0 \\ 0 & 0 & 0 \end{array}\right] \implies x_2 = t, x_1 = (-1+i)t$$

$$\implies \vec{x}_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \quad \times$$

- $\lambda_2 = 8$

$$(A-\lambda_2 I)\vec{x} = \vec{0} \implies \left[\begin{array}{cc|c} 2-8 & 3-3i & 0 \\ 3+3i & 5-8 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -6 & 3-3i & 0 \\ 3+3i & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2+(1/2)(1+i)R_1} \left[\begin{array}{cc|c} -6 & 3-3i & 0 \\ 0 & 0 & 0 \end{array}\right] \implies x_2 = 2t, x_1 = (1-i)t$$

$$\vec{x}_2 = \begin{bmatrix} \cancel{(1-i)} \\ \cancel{1} \end{bmatrix} \qquad \begin{bmatrix} (1-i) \\ 2 \end{bmatrix}$$

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To diagonalize we have:

$$\Lambda = \begin{bmatrix} \overset{\lambda_1}{-1} & 0 \\ 0 & \underset{\lambda_2}{8} \end{bmatrix} \text{ and } S = \begin{bmatrix} \overset{\vec{x}_1}{-1+i} & \overset{\vec{x}_2}{1-i} \\ 1 & 2 \end{bmatrix}.$$

Let's first take a look at the eigenvectors a little more closely:

$$\vec{x}_1^H \vec{x}_2 = \overline{\left[(-1+i) \quad 1\right]} \begin{bmatrix} (1-i) \\ 2 \end{bmatrix} = (-1-i)(1-i)+2 = -1-\cancel{i}+\cancel{i}-1+2 = 0.$$

$\Rightarrow$ orthogonal

- The eigenvectors are orthogonal! Which means that we could make S into an orthonormal matrix by making the columns have length 1 in which case $S^{-1} = S^H$.
- We call a complex matrix with orthonormal columns a **unitary** matrix. (orthogonal)
- Also, are we a little surprised that we have real valued eigenvalues even though our matrix had complex numbers in it?

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We will finish diagonalizing by taking unit vectors:

$$S = [\vec{x}_1/\|\vec{x}_1\| \quad \vec{x}_2/\|\vec{x}_2\|].$$

$$\|\vec{x}_1\|^2 = \vec{x}_1^H\vec{x}_1 = \overline{\begin{bmatrix}(-1+i) & 1\end{bmatrix}}\begin{bmatrix}-1+i \\ 1\end{bmatrix} = (-1-i)(-1+i)+1 = 3.$$

$$\|\vec{x}_2\|^2 = \vec{x}_2^H\vec{x}_2 = \overline{\begin{bmatrix}(1-i) & 2\end{bmatrix}}\begin{bmatrix}(1-i) \\ 2\end{bmatrix} = (1+i)(1-i)+4 = 6$$

$$S = \begin{bmatrix}(1/\sqrt{3})(-1-i) & (1/\sqrt{6})(1-i) \\ 1/\sqrt{3} & 2/\sqrt{6}\end{bmatrix}$$

Then $S^{-1} = S^H$. Thus we have:

$$A = S\Lambda S^H.$$

unitary	
Complex	Real
unitary	orthogonal
Hermitian	Symmetri

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$n \times n$, $\vec{x} \in \mathbb{C}^n$

Examples

8. A is any Hermitian matrix. Show that A has the following properties.

   a. $\vec{x}^H A \vec{x}$ is a real number for any complex vector $\vec{x}$.
   b. All eigenvalues of A are real.
   c. Eigenvectors corresponding to different eigenvalues are orthogonal.

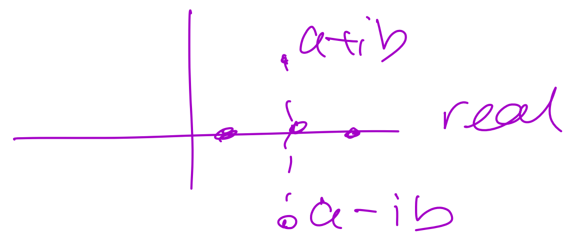
a. Suppose that $A = A^H$. We will show $r = \vec{x}^H A \vec{x}$ must be a real number by showing that $r^H = r$. 1×1

r 1×1 $\quad$ $1 \times n$ $\; n \times n$ $\; n \times 1$

$$r^H = \overline{\vec{x}^H A \vec{x}} = \left(\vec{x}^H A \vec{x}\right)^H = (A\vec{x})^H \vec{x} = \boxed{\vec{x}^H A^H \vec{x}} = \vec{x}^H A \vec{x} = r.$$

$a + ib = a - ib$

$b = 0$



$(\vec{x}^H)^H = \vec{x}$

- (b) Suppose that $A = A^H$, show that all eigenvalues are real. That is, if

$$\boxed{A\vec{x} = \lambda\vec{x}}$$

λ has to be real

λ must be real.

Let's multiply both sides of the equation by $\vec{x}^H$:

$$\boxed{\vec{x}^H A\vec{x}} = \vec{x}^H \lambda\vec{x} = \lambda \boxed{\vec{x}^H \vec{x}} = \lambda \|\vec{x}\|^2.$$

$\|\vec{x}\|^2 \in \mathbb{R}$, $\neq 0$

But this gives us:

$$\boxed{\lambda = \frac{\vec{x}^H A\vec{x} \leftarrow \mathbb{R}}{\|\vec{x}\|^2 \leftarrow \mathbb{R}}.}$$

From our previous exercise we know that $\vec{x}^H A\vec{x}$ is real. But we also know that the length of an eigenvector is a positive real number. (We also know that the length of an eigenvector is larger than 0.)

Thus, λ is the ratio of two real numbers which is necessarily real.

Note this means real symmetric matrices have real eigenvalues. ✱

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- Let A be a Hermitian matrix, $A^H = A$. Show that eigenvectors from distinct eigenvalues are orthogonal.

  Suppose that λ_x and λ_y are two distinct eigenvalues (i.e., $\lambda_x \neq \lambda_y$) corresponding to distinct eigenvectors $\vec{x}$ and $\vec{y}$.
  We know that λ_x and λ_y are real and that.

$$A\vec{x} = \lambda_x \vec{x} \text{ and } A\vec{y} = \lambda_y \vec{y}.$$

  Let's use the fact that $\vec{x}$ and $\vec{y}$ are eigenvectors to look closely at this calculation: $\vec{x}^H \vec{y} = 0$

$$\lambda_x \vec{x}_H \vec{y} = (\lambda_x \vec{x})^H \vec{y} = (A\vec{x})^H \vec{y} = \vec{x}^H A^H \vec{y} = \vec{x}^H A \vec{y} = \vec{x}^H (\lambda_y \vec{y}) = \lambda_y \vec{x}^H \vec{y}.$$

  Both ends of the equation given us:

$$\rightarrow \boxed{\lambda_x \boxed{\vec{x}_H \vec{y}} = \lambda_y \boxed{\vec{x}_H \vec{y}}.} \Rightarrow \lambda_x = \lambda_y$$

  If $\vec{x}_H \vec{y} \neq 0$ we could divide both sides by this (complex) number and be left with $\lambda_x = \lambda_y$. But this is a contradiction because $\lambda_x \neq \lambda_y$.
  Thus, $\vec{x}^H \vec{y} = 0$, eigenvectors from distinct eigenvalues are orthogonal.

This same proof tells us that eigenvectors from distinct eigenvalues of a real symmetric matrix are also orthogonal.

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Examples

9. What can you say about the eigenvalues and eigenvectors of a symmetric matrix?

- The same proof to (b) we have above shows that eigenvalues from a symmetric matrix are real.
- The same proof to (c) shows that eigenvectors from distinct eigenvalues of a real symmetric matrix are orthogonal.
- We will **soon see** that symmetric matrices (and Hermitian matrices) are **ALWAYS** diagonalizable. In that there is always a full set of orthogonal eigenvectors.

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Examples

10. What is a natural generalization of notion of orthogonal matrices to the complex case?

We say that a complex valued matrix that has orthonormal columns is a **unitary matrix**. We tend to use the notation U for a unitary matrix and know that:

$$U^{-1} = U^H.$$

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Examples

11. U is a unitary matrix. Show that U has the following properties.
    a. U preserves inner products, and as a consequence lengths.
    b. All eigenvalues have absolute value 1.
    c. Eigenvectors corresponding to different eigenvalues are orthogonal.

a. To show U preserves inner products, we need to show that $\vec{x}^H\vec{y} = (U\vec{x})^H(U\vec{y})$.

$$(U\vec{x})^H(U\vec{y}) = \vec{x}^H U^H U\vec{y} = \vec{x}^H\vec{y}.$$

This is because $U^H U = I$. This also says that U preserves lengths because:

$$\|U\vec{x}\|^2 = (U\vec{x})^H(U\vec{x}) = \vec{x}^H U^H U\vec{x} = \vec{x}^H\vec{x} = \|\vec{x}\|^2.$$

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- (b) To show that eigenvalues have absolute value 1, let's note that if $U\vec{x} = \lambda\vec{x}$ this gives us:

$$\|\vec{x}\|^2 = \|U\vec{x}\|^2 = (U\vec{x})^H(U\vec{x}) = (\lambda\vec{x})^H(\lambda\vec{x}) = \overline{\lambda}\vec{x}^H\vec{x}\lambda = |\lambda|^2\|\vec{x}\|.$$

This gives us:

$$\|\vec{x}\|^2 = |\lambda|^2\|\vec{x}\|^2 \implies |\lambda|^2 = 1 \implies |\lambda| = 1.$$

Note that λ might be complex! This is the absolute value in the sense of a complex number.

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- To show that for a unitary matrix U eigenvectors from distinct eigenvalues are orthogonal, we will use the fact that the a U preserves inner products.
  Let $\vec{x}$ and $\vec{y}$ be eigenvectors from distinct eigenvalues λ_x and λ_y.

  $$\vec{x}^H\vec{y} = (U\vec{x})^H(U\vec{y}) = (\lambda_x\vec{x})^H(\lambda_y\vec{y}) = \overline{\lambda_x}\vec{x}^H\lambda_y\vec{y} = \overline{\lambda_x}\lambda y\vec{x}^H\vec{y}.$$

  Looking at both sides of the equal sign we have:

  $$(1 - \overline{\lambda_x}\lambda_y)\vec{x}^H\vec{y} = 0.$$

  Thus, this either means: $\vec{x}^H\vec{y} = 0$ or $1 = \overline{\lambda_x}\lambda_y$.
  Since $\overline{\lambda_x}\lambda_x = 1$ and $\overline{\lambda_y}\lambda_y = 1$ it is not possible that $\lambda_x \neq \lambda_y$ and $\overline{\lambda_x}\lambda_y = 1$.
  Thus it must be that $\vec{x}^H\vec{y} = 0$.