Name:	Section:

75 Minutes. Answer all questions, without the use of notes, books or calculators. Partial credit will be awarded for correct work, unless otherwise specified. When you are asked to explain yourself, please write clearly and use complete sentences. Solve each problem directly onto the exam paper and write your name at the top of each page. Good luck!

1. (2 Points Each) True or False. For each of the following, state whether the statement is True or False (1 Point) and provide a short (1 - 2 sentence) justification (1 Point) . Remember a true statement must ALWAYS be true).

   (a) (2 Points) If A is an 2×2 matrix, the set of vectors $\vec{x}$ which solve $A\vec{x} = \vec{0}$ is a subspace.

   Solution: TRUE. This is just the definition of $N(A)$.

   (b) (2 Points) If A is an 3×3 matrix, the set of vectors $\vec{x}$ which solve $A\vec{x} = [1, 2, 0]^T$ is a subspace.

   Solution: FALSE. This does not contain $\vec{0}$.

   (c) (2 Points) If A is an 3×5 matrix with rank equal to 3, then there will always be at least one solution to the matrix-vector equation $A\vec{x} = \vec{b}$ for every possible $\vec{b}$.

   Solution: TRUE. This says that $C(A)$ has dimension 3.

   (d) (2 Points) A 5×6 matrix never has linearly independent columns.

   Solution: TRUE. The columns are 6 vectors from $\mathbb{R}^5$. As such, the set will always be linearly dependent.

Name:	Section:

2. (15 Points) Given the following system of linear equations:

$$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -x_1 + x_2 - 2x_3 + x_4 &= b_2 \\ 2x_1 - 2x_2 + 7x_3 + 7x_4 &= b_3 \end{aligned}$$

(a) (5 Points) Find all possible values of b_1, b_2 and b_3 for which this system has solutions.

(b) (5 Points) Relate your solution in (a) to $C(A)$ and a basis for $C(A)$.

(c) (5 Points) Find all possible solutions when $b_1 = 1, b_2 = -2, b_3 = 1$.

Solution:

(a) We consider the following system:

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ -1 & 1 & -2 & 1 & b_2 \\ 2 & -2 & 7 & 7 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 0 & 1 & 3 & b_1 + b_2 \\ 0 & 0 & 1 & 3 & -2b_1 + b_3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 0 & 1 & 3 & b_1 + b_2 \\ 0 & 0 & 0 & 0 & -3b_1 - b_2 + b_3 \end{array}\right]$$

In order for the system to have solutions we need $-3b_1 - b_2 + b_3 = 0$.

(b) The equation we found above specifies the column space of A, $C(A)$, because this specifies the vectors $\vec{b}$ so that $A\vec{x} = \vec{b}$ has at least one solution.

The matrix A has rank 2, because it has 2 pivots. And we can also see that a single equation for the b_i has two free variables. Let's allow: $b_3 = t$, $b_2 = s$ and

$$-3b_1 - b_2 + b_3 = 0 \implies 3b_1 = -b_2 + b_3 \implies b_1 = \frac{1}{3}(-s + t).$$

$$C(A) = \text{span}\left\{ \begin{bmatrix} -1/3 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1/3 & 0 & 1 \end{bmatrix}^T \right\}.$$

(c) For the specific choice we had above $b_1 = 1, b_2 = -2, b_3 = 1$.

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 0 & 1 & 3 & b_1 + b_2 \\ 0 & 0 & 0 & 0 & -3b_1 - b_2 + b_3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We have 2 pivots and 2 free variables. We will let $x_4 = t$ and $x_2 = s$. Thus we have:

$$x_3 + 3x_4 = -1 \implies x_3 = -1 - 3x_4 = -1 - 3t.$$

$$x_1 - x_2 + 3x_3 + 2x_4 = 1 \implies x_1 = 1 + x_2 - 3x_3 - 2x_4 = 1 + s - 3(-1 - 3t) - 2t = 4 + s + 7t$$

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} 4 & 0 & -1 & 0 \end{bmatrix}^T + s\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T + t\begin{bmatrix} 7 & 0 & -3 & 1 \end{bmatrix}^T$$

Name:	Section:

3. (20 Points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

(a) (5 Points) Using any method you like, determine the LU factorization of A.

(b) (4 Points, 1 Point Each) What is the dimension of $N(A)$, $C(A)$, $N(A^T)$ and $C(A^T)$

(c) (8 Points, 2 Points Each) Determine a basis for: $N(A)$, $C(A)$, $N(A^T)$ and $C(A^T)$.

Solution:

(a) We begin by noticing the first operation we would perform is $R_3 \to R_3 + R_1$ performing row operations and noticing:

$$E_{13}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

Thus we have $L^{-1} = E_{13}$ and

$$A = LU \implies \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) The rank of A is equal to 2 since we have two pivots. This also means the dimension of $C(A)$ and $C(A^T)$ are both 2. The dimension of the nullspace is equal to the number of columns minus the rank, dim$N(A) = 4 - 2 = 2$. And the dimension of the left nullspace is similarly equal to the number of rows (3) minus the rank which is 2. So dim$N(A^T) = 1$.

(c) The basis for $C(A)$ are the columns in A in positions with pivots in U (1 and 2):

$$\text{Basis for } C(A) = \left\{ \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}^T \right\}.$$

The basis for $C(A^T)$ are simply rows 1 and 2:

$$\text{Basis for } C(A^T) = \left\{ \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T \right\}.$$

The basis for $N(A)$ is found by identifying the special solutions to $A\vec{x} = \vec{0}$. We have two pivots and two free variables. $x_4 = t, x_3 = s$. From row 2 in U we have $x_2 = -s$ and $x_1 = 2s - t$.

$$\text{Basis for } N(A) = \left\{ \begin{bmatrix} 2 & -1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}^T \right\}.$$

We know that $N(A^T)$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 & 2x_1 + x_2 + 2x_3 & x_2 & x_1 + x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\implies x_1 + x_3 = 0, 2x_1 + x_2 + 2x_2 = 0, x_2 = 0 \text{ and } x_1 + x_3 = 0. \implies x_2 = 0, x_1 + x_3 = 0.$$

$$\text{Basis for } N(A^T) = \left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T \right\}.$$

Name:	Section:

4. (10 Points) Give the dimension and a basis for each of the following subspaces of $\mathbb{R}^3$.
   (a) (5 Points) The plane $x + y + z = 0$.
   (b) (5 Points) The intersection of the plane $x + y + z = 0$ with the xy plane.

Solution:

(a) We can frame this as a matrix-vector equation:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

There is 1 pivot and 2 free variables. $z = t, y = s$ and $x = -y - z = -t - s$.

$$\text{Basis } = \left\{ \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T \right\}.$$

(b) We have one more equation to add because the xy plane is $z = 0$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now our matrix has 2 pivots and 1 free variable, $y = t$. This implies $x = -y - z = -t$.

$$\text{Basis } = \left\{ \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T \right\}.$$

Name:	Section:

5. (12 Points) Let V be the set of all 2×2 matrices such that $[1\ 5]^T$ is in the nullspace. That is, the set of all matrices A such that:

$$A \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Show that this set of matrices is a subspace by verifying:

(a) (2 Points) The zero matrix belongs to V.

(b) (5 Points) The set V is closed under vector addition.

(c) (5 Points) The set V is closed under scalar multiplication.

Solution:

(a) We first notice that the 2×2 0 matrix belongs to V:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, the 2×2 zero matrix belongs to V.

(b) To show the system is closed under vector addition, let's suppose A and B are in V. We want to show that for $C = (A + B)$ we have $C \in V$.

The criteria for being in V is that $C \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \vec{0}$.

$$C \begin{bmatrix} 1 \\ 5 \end{bmatrix} = (A + B) \begin{bmatrix} 1 \\ 5 \end{bmatrix} = A \begin{bmatrix} 1 \\ 5 \end{bmatrix} + B \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \vec{0} + \vec{0} = \vec{0}.$$

Thus $C \in V$ and we have shown V is closed under vector addition.

(c) To show the system is closed under scalar multiplication, let $C = kA$ for $A \in V$. We want to show that $C \in V$.

$$C \begin{bmatrix} 1 \\ 5 \end{bmatrix} = (kA) \begin{bmatrix} 1 \\ 5 \end{bmatrix} = k \left(A \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right) = k\vec{0} = \vec{0}.$$

Thus, $C \in V$ and V is closed under scalar multiplication.

Grading Rubric

- Problem 1 (a - d):
  - 2 Points: Correct answer (True/False) and valid clear reasoning.
  - 1.5 Points: Correct answer (True/False) and incomplete reasoning.
  - 1 Point: Either only correct answer (True/False) alone OR correct reasoning.
  - 0.5 Point: Incorrect answer (True/False) AND incomplete reasoning.
  - 0 Points: Blank, Incorrect answer (True/False) and/or incorrect reasoning.
- Problem 2:
  - Part (a): +3 for correct reduced matrix U and +2 for correct right-hand size vector $\vec{b}$ after reduction.
  - Part (b): +3 if using C(A) as basis rather than from $-3b_1 - b_2 + b_3 = 0$. -1 if a wrong vector is added to the basis. +2 if using the pivot columns of U instead.
  - Part (c): -1 Correct work but from incorrect part (a). 0 point for irrelevant answer. If students re-do the part with Gaussian elimination, the part is graded based on algebraic work and correct solution.
- Problem 3:
  - (a) +3 for correct L or U. +5 for correct L and U. -1 for algebraic or operation mistake. +2 for anything reasonable
  - (b) No partial credit. +1 for each dimension.
  - (c) -1 for algebraic error or incorrect vector. +2 per space.
- Problem 4:
  - 5 Points: Correct work supported by appropriate work. At most one trivial arithmetic or notation error.
  - 4 points: Mostly correct work with at most one meaningful mistake
  - 3 points: Relevant work towards the solution but has errors
  - 2 points: Some work but off track from the solution
  - 1 point: Progress towards a solution exists but minimal
  - 0 Point: Blank, or work irrelevant to a correct solution
- Problem 5:
  - (a) +2 for showing 0 matrix is in the space. +1 for anything reasonable.
  - (b) +2 for stating 2 matrices living in the subspace and needing to check addition. +3 for proof. +2 for anything reasonable.
  - (c) +2 for stating a matrix in the space as well as a scalar $\in \mathbb{R}$. +3 for proof. +2 for anything reasonable.