UNIVERSITY OF CALIFORNIA, MERCED

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CSE 015: Discrete Mathematics
Homework 5
Fall 2021

Preliminary Notes

- **This homework must be solved individually.** You can discuss your ideas with others, but when you prepare your solution you must work individually. Your submission must be yours and yours only. No exceptions, and be reminded of the CSE academic honesty policy discussed in class.
- Your solution must be exclusively submitted via CatCourses. Pay attention to the posted deadline because **the system automatically stops accepting submissions when the deadline passes. Late submissions will receive a 0**. You only need to submit the PDF and you have to use the template file provided in CatCourses. Please note that the system does not allow to submit any other file format. Do not submit the LaTeXsource of your solution.
- By now you should have become somewhat familiar with LaTeX. You still will not be penalized for poor typesetting, but it is in your own interest to prepare your submission in a way that is easy to understand, so try using the appropriate LaTeXsymbols. If you do not know how to type a certain math symbol, search on the Internet and you will quickly find the answer.[1] **If in your LaTeXsubmission you embed screenshots or scans of your handwritten solution those will not be graded.** You are encouraged to collaborate with other students to determine how to best format your submission or improve your LaTeXskills.
- Start early.

1 Mathematical Induction 1

Let $P(n)$ be the statement that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

for the positive natural number n. The validity of this statement can be proved by mathematical induction. The following questions guide you through the process.

a) What is the statement $P(1)$?

b) Basis of induction: show that the statement $P(1)$ is true.

c) What is the inductive hypothesis of a proof that $P(n)$ is true for all positive natural numbers n? Note that the question is about the specific $P(n)$ considered in this exercise.

d) Complete the inductive step, i.e., show that $\forall k > 1(P(k) \rightarrow P(k+1))$

[1] see `https://www.caam.rice.edu/~heinken/latex/symbols.pdf` for example.

2 Mathematical Induction 2

In class we have determined the formula for the sum of the first k odd natural numbers. That process involved two parts. First, we computed by hand the result for a few values of k, and made a guess about the formula. Then, we proved that the formula was correct by using mathematical induction. Following the same approach, determine the formula to compute the sum of the first k even natural numbers. Recall that a number is even if it can be written as $2p$, where p is an integer number. When defining the first k even natural numbers, we include 0. That is, the first even natural number is 0, the first two even natural numbers are 0 and 2, the first three even natural numbers are 0,2, and 4, the first four even natural numbers are 0,2,4 and 6, and so on.

a) Compute the sum of the first k even natural numbers and make a guess about the formula. You are free to pick any $k \geq 4$ (i.e., you can stop at $k = 4$, but if you need a few more examples, to make your guess, feel free to continue).

b) Use the principle of mathematical induction to show your formula is correct. To this end, you have to first show the basis step, and then the induction step.

Observation #1: some of you may find strange to consider 0 as the first even natural number. This is a minor detail, and if we decided to exclude it from the count, the formula would only be slightly different. However, note that based on our definitions, 0 is indeed the first even natural number because 0 is a natural number and $0 = 2p$ for the integer $p = 0$.
Observation #2: the formula you need to prove is the formula that gives the sum of the first k even natural number. For example, if we call it $P(n)$, $P(3) = 0 + 2 + 4$, $P(4) = 0 + 2 + 4 + 6$, and so on.

3 Mathematical Induction 3

Mathematical induction can be used not only to prove equalities, but also to prove inequalities. The predicate $P(n)$ is the statement $n! < n^n$ where n is an integer greater than 1.

a) What is the statement $P(2)$?

b) Show that $P(2)$ is true, i.e., complete the basis step of the proof by induction.

c) What is the inductive hypothesis of a proof by mathematical induction that $P(n)$ is true for all natural numbers n greater than 1?

Recall that $n!$ is the factorial of n, i.e., $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.