Complete the following tasks. To show work for the double-integrals, demonstrate that you can set up the integrals. From there, you may use software such as *Wolfram Alpha* to compute the values of the double integrals. Some answers have been provided. Assemble your work into one PDF document and upload the PDF back into our CatCourses page.

1. A joint probability density function of $X$ and $Y$ is defined overall all real numbers and has the form
$$f(x, y) = \frac{C}{(1 + x^2)(1 + y^2)}.$$
   (a) Find the constant $C$.
   (b) Find the probability that $(X, Y)$ falls inside the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$.
   (c) Are $X$ and $Y$ independent?

2. Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a random variable—$X$ for the right tire and $Y$ for the left tire, with joint probability density function
$$f(x, y) = \begin{cases} K(x^2 + y^2) & 20 \leq x \leq 30, 20 \leq y \leq 30 \\ 0 & \text{otherwise} \end{cases}$$
   (a) What is the value of $K$?
   (b) What is the probability that both tires are underfilled?
   (c) What is the probability that the difference in air pressure between the two tires is at most 2 psi?
   (d) Determine the marginal distribution function of air pressure in the right tire alone.
   (e) Are $X$ and $Y$ independent and why?

3. Consider the variables $X$ and $Y$ from the example in Section 9.2. They have joint probability density:
$$f(x, y) = \frac{2}{75}\left(2x^2y + xy^2\right) \text{ for } 0 \leq x \leq 3 \text{ and } 1 \leq y \leq 2.$$
   (a) Find the marginal densities for $X$ and $Y$.
   (b) Compute $\mathrm{E}[X]$, $\mathrm{E}[Y]$ and $\mathrm{E}[X + Y]$.
   (c) Compute $\mathrm{E}[X^2]$, $\mathrm{E}[Y^2]$, $\mathrm{E}[XY]$ and $\mathrm{E}\left[(X + Y)^2\right]$.
   (d) Compute $\mathrm{Var}(X + Y)$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$.
   (e) Does $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$? Explain what you observe.

4. Let us try using the following bivariate function to describe the presence of sneeze particles from a person's mouth (distance units are in feet).[1]

$$f(x, y) = kye^{-xy}, \quad x > 0, \quad 0 < y < 5$$

   (a) Find the value of $k$ so that $f$ is a probability density function.

   (b) Mindful of social distancing, compute the proportion of sneeze particles that exceed $x = 6$ feet in horizontal distance.

   (c) Compute the standard deviation of $Y$.

5. A hobbyist family is making PPE (personal protective equipment) to donate to local health care workers.[2] Let

   - $T_1 \sim U(1, 4)$ be the amount of time (in hours) to 3D print a face mask and
   - $T_2$ be an exponentially distributed random variable with an average of 3 hours to represent the time (in hours) to cut out and sew a suit.

   Describe the distribution of time to complete construction of one suit-and-mask outfit by computing the mean and standard deviation of the sum $T_1+T_2$ assuming independence between $T_1$ and $T_2$.

[1] Optional reading: In case anyone is interested in the physics setup I assumed, the $x$ axis is positive away from the person, and the $y$ axis is positive downward toward the ground. This was an exam question during the Spring 2020 semester.

[2] Hint: there is only one input variable, time, so there is no need for double integrals. This was an exam question during the Spring 2020 semester.

Here are some of answers. Note that numbers may slightly vary depending on when and where the rounding took place.

1. (a) $C = \dfrac{1}{\pi^2}$
   (b) $\dfrac{1}{16}$
   (c) Yes (but how do you know?)
2. (a) $K = \dfrac{3}{380000}$
   (b) 0.3024
   (c) 0.3593
   (d) $f_Y(y) = \dfrac{3}{380000}\left(10y^2 + \dfrac{19000}{3}\right), \quad 20 \leq y \leq 30$
   (e) No (but how do you know?)
3. (a) $f_X(x) = \dfrac{2x}{225}(9x+7), \quad 0 \leq x \leq 3$ and $f_Y(y) = \dfrac{3y}{25}(y+4), \quad 1 \leq y \leq 2$
   (b) $\mathrm{E}[X+Y] = 3.75$
   (c) $\mathrm{E}[(X+Y)^2] = 14.532$
   (d) $\mathrm{Var}(X+Y) = 0.4695$
   (e)
4. (a) 0.2
   (b) 0.0333
   (c) 1.4434 feet
5. - mean 5.5 hours
   - standard deviation 3.1225 hours