Discussion Section: Week #9
Due: By 11:59pm the day of your Discussion Section

Instructions:

You are allowed to work alone or in small groups. But if you work in a group, the work you turn in should be your own. Each student will submit a physical write-up as a PDF document to CatCourses **by 11:59 pm of your discussion section day**.

Discussion assignments will be graded partially for completeness but a subset of the problems will be graded for correctness. Providing the correct answer, without justification, is not considered complete. For credit you **must** either show steps (if it's a calculation problem) or explain/justify your reasoning (if it's a short answer problem).

Problem Set:

1. Write out $E^2 = \|Ax - b\|^2$ and set to zero its derivatives with respect to u and v, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Compare the resulting equations with $A^T A\hat{x} = A^T b$, confirming that calculus gives the normal equations. Find the solution $\hat{x}$ and the projection $p = A\hat{x}$. Why is $p = b$?

Solution:

$$E^2 = \|Ax - b\|^2 = (u-1)^2 + (v-3)^2 + (u+v-4)^2$$

$$\begin{aligned} \frac{\partial E^2}{\partial u} &= \frac{\partial}{\partial u}\left[(u-1)^2 + (v-3)^2 + (u+v-4)^2\right] \\ &= 2(u-1) + 2(u+v-4) \\ \frac{\partial E^2}{\partial v} &= \frac{\partial}{\partial v}\left[(u-1)^2 + (v-3)^2 + (u+v-4)^2\right] \\ &= 2(v-3) + 2(u+v-4) \end{aligned}$$

Setting the partial derivatives equal 0, we obtain a system of equations

$$\begin{aligned} 2(u-1) + 2(u+v-4) = 0 &\longrightarrow 2u + v = 5 \\ 2(v-3) + 2(u+v-4) = 0 &\longrightarrow u + 2v = 7 \end{aligned}$$

To solve

$$\begin{aligned} 2u + v &= 5 \\ u + 2v &= 7, \end{aligned}$$

we write it in matrix-vector form and performed Gaussian elimination.

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 2 & 7 \end{array}\right] \xrightarrow{R_1^* = R_1 - 2R_2} \left[\begin{array}{cc|c} 0 & -3 & -9 \\ 1 & 2 & 7 \end{array}\right]$$
$$\xrightarrow{R_1^* = -\frac{1}{3}R_1} \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 7 \end{array}\right]$$
$$\xrightarrow{R_2^* = R_2 - 2R_1} \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 0 & 1 \end{array}\right]$$

Thus, $\hat{x} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Now we compute $\hat{x}$ by $(A^T A)^{-1} A^T b$. First, however, notice that

$$\begin{aligned} A^T A \hat{x} &= A^T b \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \hat{x} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \hat{x} &= \begin{bmatrix} 5 \\ 7 \end{bmatrix}. \end{aligned}$$

This is the same system of equations we solved when we minimized the least squares above.

$$
\begin{aligned}
\hat{x} = (A^TA)^{-1}A^Tb &= \left(\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\3\\4\end{bmatrix}\right)\\
&= \left(\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\right)^{-1}\begin{bmatrix}5\\7\end{bmatrix}\\
&= \frac{1}{2(2)-1(1)}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix}\\
&= \frac{1}{3}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix}\\
&= \frac{1}{3}\begin{bmatrix}3\\9\end{bmatrix}\\
\hat{x} &= \begin{bmatrix}1\\3\end{bmatrix}.
\end{aligned}
$$

The projection $p = A\hat{x}$ is

$$
\begin{aligned}
p = A\hat{x} &= \begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}\\
&= \begin{bmatrix}1\\3\\4\end{bmatrix}
\end{aligned}
$$

The reason $p = b$ is because b is in the column space of A; that is, b can be written as a linear combination of the vectors of A.