Classifying platonic solids using the Euler characteristic

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In 1736, Leonhard Euler released a paper outlining the infamous *Seven bridges of Könisberg* problem [2]. This brought forward many new concepts in geometry, which would later grow into the study of topology. Euler discussed and then proved whether it was possible for a person to traverse each of the seven bridges between the four regions of Könisberg only once. To show this, he introduced the idea of a *graph*, denoting each of the bridges as *edges* and every region as *vertices*, shown in figure 1.

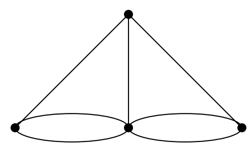


Figure 1: The seven bridges of Könisberg problem as a graph.

Euler stated that for a graph to have a walk in which each edge is used only once, it must have at most two vertices of odd degree. Since the graph above has four vertices of odd degree, he concluded that the problem is unsolvable. This study of graph theory is what lead Euler to discover that it was possible to solve problems using an absence of distance. He therefore turned his attention to polyhedra, and discussed the idea of a potential relation between the vertices, edges and faces of regular polyhedra. In 1752, he proved that this relation was always equal to 2, and thus the Euler characteristic was born.

Definition 1. Let A be a polyhedron with v number of vertices, e edges and f faces. The *Euler characteristic* of A is

$$\chi(A) = v + e - f.$$

Historically there are five known regular polyhedra, namely, the regular tetrahedron, cube, octahedron, dodecahedron, and icosahedron. This group of geometrical shapes is known as the *platonic solids*.

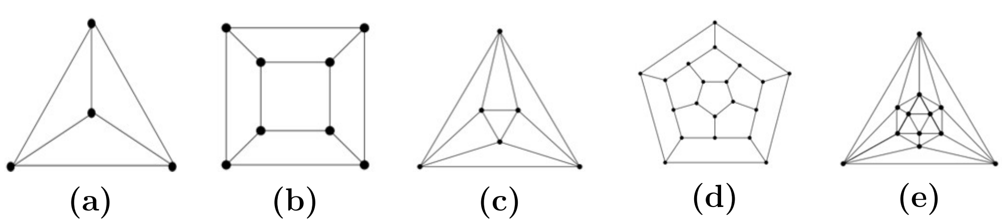


Figure 2: The platonic solids **(a)** tetrahedron, **(b)** cube, **(c)** octahedron, **(d)** dodecahedron and **(e)** icosahedron.

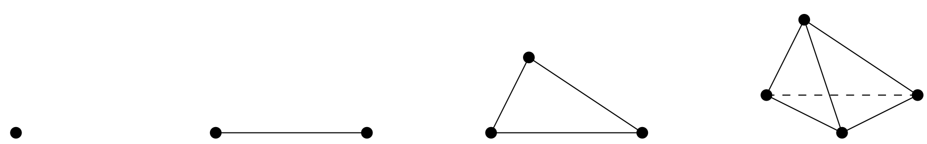
In fact, the Euler characteristic is equal to 2 for any *connected planar graph*. We say a graph is *connected* if there exists a path between any two vertices, and is *planar* if the graph can be drawn on the surface of a sphere without any edges crossing each other. The platonic solids can be drawn as connected planar graphs, as shown in figure 2, and the characteristic $v + e - f = 2$ can be proved multiple ways (see [3] for a complete list of proofs). This common property means that we can strive to find any potential relationship between the platonic solids and other shapes in topological space. To do this, we must now consider them as *surfaces*.

Informally, a topological space is a collection of sets in which laws such as continuity, connectedness and convergence are not described by distance, but rather by the relationship between the sets themselves [4]. A *surface* in topological space is any geometrical shape that can be represented on a plane. In the case of the platonic solids, we have shown them as planar graphs, which can be embedded in a plane.

In topological space, the act of classification comes from equivalence relations between surfaces. The idea of topological equivalence is similar to the notion of isomorphism. Originating from the Greek word *iso*, meaning equal, an isomorphism is a map that upholds relations between elements. Let X and Y be topological spaces. If a function $h : X \to Y$ exists that is bijective, continuous, and has a continuous inverse, then we can say that X and Y are topologically equivalent, or *homeomorphic.*[1]

According to Definition 1, we can find the Euler characteristic of any polyhedron by counting the number of vertices, edges and faces of the shape. But in topological space, there exist an infinite amount of surfaces on the plane, many of which cannot be counted as easily as the platonic solids. We therefore devise an all-encompassing technique for finding the Euler characteristic of a surface.

Definition 2. An *n-simplex* is the planar graph formed between $n + 1$ points in n-dimensional space. A finite collection of simplexes is called a *simplicial complex* if whenever a simplex lies in the collection then so does each of its faces, and whenever two simplexes of the collection intersect they do so in a common face.[1]



(a) 0-simplex **(b)** 1-simplex **(c)** 2-simplex **(d)** 3-simplex

Figure 3

A visual representation of Defintion 2 can be seen in figure 3 above. A 0-simplex consists of a single point, a 1-simplex is a line segment, a 2-simplex forms a triangle, and 3-simplex a tetrahedron.

Definition 3. A *triangulation* of a topological space X consists of a simplicial complex K and a homeomorphism $h : |K| \to X$.

In order to find the Euler characteristic of any surface, we must triangulate it, creating a simplicial complex that allows us to count the number of edges, faces and vertices. We denote the polyhedron associated with the triangulation $|K|$. Although it is not necessary, we can triangulate each of the platonic solids. If we imagine that they could not be counted by their visual diagram, we can take the planar graphs of each polyhedron in figure 2 and form five simplicial complexes.

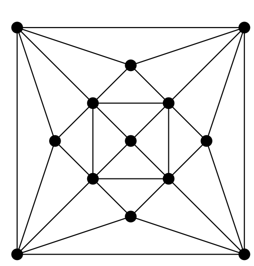


Figure 4: Simplical complex of the cube.

For example, in figure 4 we see that we can form a simplicial complex of the cube, with 14 vertices, 36 edges and 24 faces. We see from this process that we do not need to triangulate the tetrahedron, octahedron or icosahedron, as these planar graphs are already simplicial complexes. It is also very clear that visually the simplicial complex of the cube is a homeomorphism to the cube itself, there exists a continuous and bijective function that maps all points of the simplicial complex in figure 4 to the planar graph in figure 2, therefore we conclude that we have a triangulation of the cube.

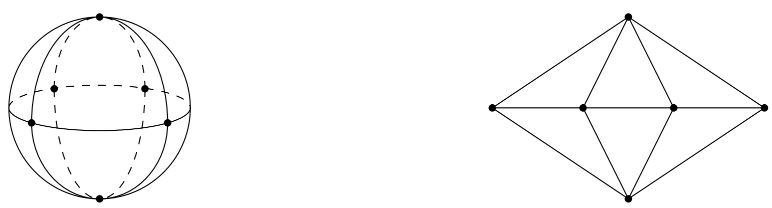
[1] Adapted from [1, section 1.4].

Theorem 1. *Surfaces with the same Euler characteristic are topologically equivalent.*

This is an interesting theorem that was founded and proved well after Euler's death. It leads into the formal classification theorem for any topological surface, which states *"any surface is homeomorphic to the sphere, the sphere with n handles, or the sphere with n Mobius strips attached"*. We will not prove this, however a rigorous proof of this theorem can be found in [1, section 7]. Given this information, we now know that the platonic solids must be homeomorphic to one of the aforementioned options.

Lemma 1. *The platonic solids are homeomorphic to the sphere.*

Proof. We know that the platonic solids have a Euler characteristic of 2, therefore we must show that the Euler characteristic of a sphere is the same. We do this by triangulating the sphere.



(a) Triangulation of sphere. **(b)** Associated simplicial complex.

Figure 5

There are an infinite amount of potential triangulations of the sphere, an example of a triangulation with 6 vertices is given in figure 5. If we count the associated simplicial complex, we see it has 6 vertices, 8 edges and 6 faces. Therefore the Euler characteristic of the sphere is 2, and by Theorem 1, the platonic solids are homeomorphic to the sphere. □

To conclude, the birthplace of topology begins with Euler's contribution to geometry. Subsequent research in this field has proven that the Euler characteristic is the foundation for classifying surfaces. This is critical for mathematicians, as we always strive to find generalisations. For over 2000 years, there have only been five known regular polyhedra, namely the platonic solids, but research has shown that there are in fact nine. Furthermore, we now know that the breadth of different surfaces in topological space is infinite, but by breaking down surfaces into different categories, we make infinity more manageable. By using the techniques shown above, we can classify surfaces into one of three categories. The Euler characteristic has shown that in the case of the platonic solids, they are homeomorphic to the sphere, and are part of a long list containing all irregular polyhedra topologically indistinguishable to the sphere.

References

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[3] Eppstein D. *Twenty Proofs of Euler's Formula: V-E+F=2.* Accessed online at `https://www.ics.uci.edu/~eppstein/junkyard/euler/` on Jan 3, 2021.

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