The topological classification of surfaces

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Abstract

In this project we examine the existence of the classification theorem for surfaces, and give a concise method to manually classify a surface with or without boundary. We introduce topological space, and understand the reasons behind its uniqueness. We discuss the relevant topological properties needed to classify surfaces, including explanations of their importance in topological space. We also examine methods of constructing, altering and performing surgery on surfaces, whilst drawing attention to importance of the central topological property, homeomophism.

Contents

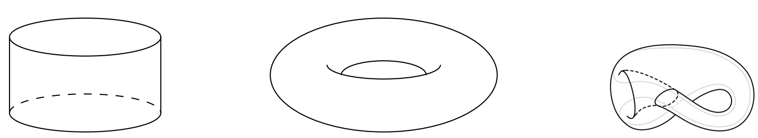
1 Introduction

We begin by first discussing the relevance of topology as a mathematical study. Whilst relatively new, the subject has a rich history and builds on many mathematical concepts from areas such as real analysis, linear algebra and group theory. One of the earliest known studies of topology was from the early 18th century, when Euler published a paper called *Solutio problematis ad geometriam situs pertinentis*, discussing the infamous seven bridges of Königsberg problem. By 1750, Euler's theorem for polyhedra $v - e + f = 2$ was documented; a theorem that we will later discuss in significant detail. Since then, many notable mathematicians have contributed to what is now known as general topology; an abstract branch of geometry. The *Möbius strip* was introduced almost a century later, with Möbius' work birthing two crucially important topological terms; *triangulation* and *orientability*. The University of St. Andrews [2] provides a brilliantly comprehensive history of topology, from Euler's contributions to geometry, to 20th century topology.

Often described as *rubber sheet* geometry, we wish to discuss the characteristics of shapes and surfaces in abstract space, where conventional laws do not apply. Once the importance of measures such as distance and size are disregarded, we look for unique ways to form an equivalence between two or more objects. In an abstract space, can we visualise a way to deform one shape into another? In the most simple cases, this is true. However, there are two criteria that must be followed in order to successfully visualise an equivalence relation between two or more surfaces:

1. Points on an object you wish to deform are never identified.

2. We never tear an object.



(a) Cylinder. **(b)** Torus. **(c)** Klein Bottle.

Figure 1

There are three surfaces shown in figure 1; a cylinder, torus and Klein bottle. Some of these surfaces are topologically equivalent. To construct a torus, first imagine the cylinder is made of rubber. We are then able to take each circular end and join them together, creating a torus. Constructing a Klein bottle is slightly harder to visualise. In topological space, we are able to assume that a surface can pass through itself, so by joining one circular end of the cylinder and joining it with the *inside* of the other circular end creates a Klein bottle. It is important to note that whilst we assume surfaces can pass through themselves, we have not torn the surface at any point, and we have made no disfigurations which alter the integrity of any surface. We simply carry out a series of manipulations to show equivalence. A detailed diagram of this including other examples can be found in [1, section 1.3].

1.1 Euclidean space

"A surface is a topological space with the same local properties as the familiar plane of Euclidean geometry." - Massey[3]

The *Euclidean space*, denoted $\mathbb{R}^n$, is the space of all co-ordinates

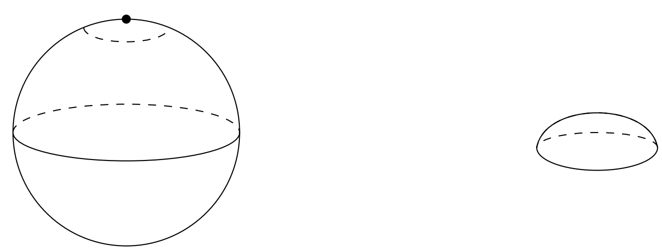
$$\{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in \mathbb{R}\}.$$

This space is bounded by a fundamental set of axioms[4] which defines the conventional laws of geometry. Euclid's revolutionary work on geometry is how we today understand the size, shape and direction of a collection of co-ordinates. For example, in $\mathbb{R}^3$, take the unit sphere $x^2 + y^2 + z^2 = 1$. This Euclidean surface is easy to understand as the sphere is bounded by the centre point $(0, 0, 0)$ and radius $r = 1$; just as outlined by one of the axioms (*"A circle can be constructed when a point for its centre and a distance for its radius are given"*[4]).

Example 1.1. The unit n sphere can be denoted as

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\},$$

where $\mathbb{S}^0$ denotes a line interval, $\mathbb{S}^1$ denotes a unit circle, and $\mathbb{S}^2$ denotes a unit sphere.



(a) Sphere with cross section. **(b)** Cap-like cross section.

Figure 2

When taking a small section about a point of a 3-dimensional surface we aim to be left with a 2-dimensional image, known as a *neighbourhood*. This presents a clear problem when working in Euclidean space. For any point on the unit sphere, when taking a disc-like neighbourhood of this point we are still left with a 3-dimensional cap-like image shown in figure 2b. It is tempting to look at the image as if we were looking down on it, however in Euclidean space this does not negate the fact that it is still 3-dimensional; it merely visually represents a 2-dimensional image. Thus, to work in topological space we need to ignore some of the conventions set in Euclidean space. The centre point and radius of a sphere become irrelevant, along with the physical size of the sphere. We seek to construct a space in which is sphere is just a 3-dimensional surface.

1.2 Topological space

Definition 1.1. Let X be any set. A *topology*, τ, on X is a collection of subsets of X such that:

- $\emptyset, X \in \tau$.
- The union of elements of τ is also an element of τ.
- The intersection of *finitely* many elements of τ is also an element of τ.

The elements of τ are open sets, and the set (X, τ) is a *topological space.*

In essence, the topology τ is a collection of open sets that are closed when we take both the union of any number of elements, and a finite intersection of elements of τ. From this definition, we get an intuitive sense that a topological space is vast, and encompasses the previous spaces discussed.

Example 1.2. Let $X := \{x, y\}$. There are four possible topologies on X, namely $\{\emptyset, \{x, y\}\}$, $\{\emptyset, \{x\}, \{x, y\}\}$, $\{\emptyset, \{y\}, \{x, y\}\}$, and $\{\emptyset, \{x\}, \{y\}, \{x, y\}\}$.

Example 1.3. Take the Euclidean space $\mathbb{R}$. An open subset $U \subset \mathbb{R}$ is a collection of points $x \in U$ that lie in the open interval $(a, b) \in \mathbb{R}$. This also describes the metric space $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$. By definition 1.1, this collection of open sets forms a topology, therefore $(\mathbb{R}, d)$ is a topological space.

Definition 1.2. The function $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous at $x \in \mathbb{R}^m$ when for $\epsilon > 0$, there exist some $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Definition 1.3. The function $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous at a point $a \in \mathbb{R}^m$ if for any given neighbourhood, $N_1(f(a))$, there exists a neighbourhood $N_2(a)$ that satistfies $f(x) \in N_1(f(a))$ whenever $x \in N_2(a)$.

Therefore, we can say that for a function to be continuous, we need all the values of $f(x)$ to lie in a neighbourhood of $f(x_0)$. If we choose small enough neighbourhoods about x and x_0, then it would follow that f is continuous.

By merging definitions 1.1 and 1.3 together, we can now define a more succinct definition of topological space. Here, we can keep the notion of a neighbourhood, but remove any reliance on distance. These definitions have been adapted from Armstrong's notes on abstract spaces[1, section 1.4].

Definition 1.4. The set X is a topological space if for each point $x \in X$, there exists a subset, N, comprised of neighbourhoods of x such that:

- x lies entirely in N.
- The intersection of two neighbourhoods of x is also a neighbourhood of x.
- A subset $U \subseteq X$ that contains N is also a neighbourhood of x.
- The interior set $\dot{N}$ is also a neighbourhood of x, where $\dot{N}$ is the largest open set containing N.

2 Topological properties

2.1 Homeomorphism

"Two surfaces are homeomorphic if there exists some transformation, f, such that the surfaces are indistinguishable in topological space"

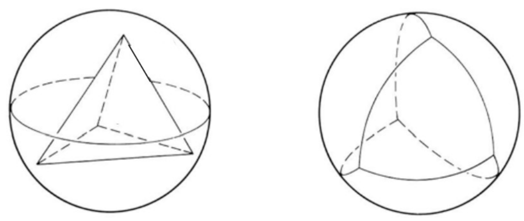


Figure 3: Homeomorphism between the tetrahedron and the sphere.

Previously, we discussed a process that outlines how surfaces can equivalent in topological space. This equivalence relation is known as *homeomorphism.* For a complete proof that this is an equivalence relation, Kinsey[5, chapter 2] discusses this in significant detail. Using the informal definition, we can visually see that in figure 3, we can 'blow up' a tetrahedron to become a sphere, and so this transformation makes the two surfaces homeomorphic.

Homeomorphism is simply an alternative term for *topological isomorphism.* We know from linear algebra that two vector spaces V and W are *isomorphic* when there exists a function $f : V \to W$ such that f is bijective and structure-preserving. We observe that in the case of the tetrahedron and the sphere, we do not identify points on either surface, so the structure-preserving property is not necessary in topological space. This disregard for distance is what differentiates homeomorphism and isomorphism. However, we know that the transformation must be some map that connects two topological spaces.

Definition 2.1. Let X and Y be topological spaces. A function $f : X \to Y$ is *homeomorphic* if it is bijective, continuous, and has a continuous inverse.

Example 2.1. *Any open interval (a, b), where $a < b$, is homeomorphic to $(0, 1)$.* For $f : (a, b) \to (0, 1)$, let

$$f(x) = \frac{x-a}{b-a}, f^{-1}(x) = x(b-a) + a.$$

We observe that these functions are one-to-one and onto, hence we have a bijection. We also see that both functions are linear functions of the variable x, so by defintion these we have a continuous function with a continuous inverse, thus we have a homeomorphism.

Example 2.2. $[0, 1)$ *is not homeomorphic to the unit circle S^1.* Define the function $f : [0, 1) \to S^1$, where

$$f(x) = e^{2\pi i x}.$$

Although we know that this function is bijective and continuous, the function fails to be homeomorphic. This is because the inverse function

$$f^{-1}(x) = \frac{\ln x}{2\pi i},$$

is discontinuous at $x = 0$.

2.2 Compactness

"Compactness is an important property and is the topologists' form of finiteness, in the sense that a compact set does not go on forever"[5]

The wealth of properties that compactness yields is what makes it so fundamental. It is the way that we understand that $(-1, 1)$ is homeomorphic to $(-\infty, \infty)$, but $[-1, 1]$ is not.

We saw in example 2.2 that the notion of closed and bounded sets can influence topological properties. However, due to the absence of distance in topological space, boundedness is not a topological property. Instead, we formulate some new terminology. Let X be a topological space. Take a collection of neighbourhoods, N, of X, where the union of all these neighbourhoods equals the whole set X. This collection is called an *open cover* of X. If N' is a subset of N, and the union of the neighbourhoods of N' is also equal to X, then N' is a *subcover* of X.

Example 2.3. Let $X := \{a, b, c, d, e\}$ be a topological space. The collection of subsets $N := \{\{a\}, \{bc\}, \{ad\}, \{bce\}\}$ is an open cover of X, and the set $N := \{\{ad\}, \{bce\}\}$ is a subcover of X.

Conversely, not all open covers have subcovers. If we take the collection of open balls of radius 1 with integer coordinates, we form an open cover of the plane. However, if we take just one element away from this collection, the union of this subset fails to cover the entire plane, as it does not contain the centre of the ball. Thus we find that there is no finite subcover for this collection.

Definition 2.2. A topological space X is *compact* if every open cover of X has a finite subcover.

This is an essential definition for topologists, as it yields many interesting properties. First, the definition itself sets compactness as a fundamental topological property. This means that if X is a compact space and X is homeomorphic to Y, then Y must be compact.

Theorem 2.1. *(Heine-Borel Theorem). A set $A \in \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

This theorem is closely related to, and can be easily proved using the Bolzano-Weierstrass theorem. For a complete proof, see Armstrong[1, section 3.2]. This is a critical theorem, and explains why example 2.2 is not homeomorphism, the interval $[0, 1)$ is not closed and bounded, so cannot be compact, and thus cannot form a homeomorphism. Additionally, we see that as a result of the Heine-Borel theorem, compactness preserves more topological properties.

Theorem 2.2. *The continuous image of a compact space is compact.*

Proof. Let X be compact, and define the function $f : X \to Y$ to be a surjective continuous function. Let N be an open cover of Y. The inverse $f^{-1}(n)$ will be an open subset of X by the continuity of f, so the collection $\gamma := \{f^{-1}(n) : n \in F\}$ is an open cover of X. We know that X is compact, so γ has a finite subcover $X = f^{-1}(n_1) \cup ... \cup f^{-1}(n_k)$. The function f is surjective, so we have that $f(f^{-1}(n_i)) = n_i$ for $1 \leq i \leq k$. Therefore we can show that $Y = n_1 \cup ... \cup n_k$, which forms a finite subcover of F. $\square$

2.3 Connectedness

Mathematicians have an intuitive sense of whether a space is connected. It is easy to understand that surfaces such as the ones shown in figure 1 are connected, as they appear to be in one piece. Therefore, defining connectedness for topological spaces is relatively simple.

Definition 2.3. A space X is *connected* if whenever it is split into two nonempty subsets A and B, such that $X = A \cup B$, then either $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$.

Theorem 2.3. $\mathbb{R}$ *is a connected space.*

Proof. We shall prove this by contradiction. Assume $\mathbb{R}$ is not connected. Then by definition 2.3, $\mathbb{R} = A \cup B$ such that $A \cap B = \emptyset$. Let $a \in A$ and $b \in B$. We define the two nonempty subsets $\bar{A} = A \cap [a, b]$ and $\bar{B} = B \cap [a, b]$, so that $\bar{A} \cup \bar{B} = [a, b]$.

$\bar{A}$ will have a least upper bound in $[a, b]$, $\sup \bar{A} = x$, so $a \leq x \leq b$. Now, if $x \in \bar{B}$ then we can take a neighbourhood of x, $B_\epsilon(x) \subseteq \bar{B}$. This implies that $x - \frac{\epsilon}{2} \in \bar{B}$, giving us a better value for the supremum of $\bar{A}$, so $x \notin \bar{B}$. Similarly, if $x \in \bar{A}$, then $\bar{A}$ is open and $x + \frac{\epsilon}{2} \in \bar{A}$ for some ϵ, so $x \notin \bar{A}$. Therefore, we have that $\bar{A} \cup \bar{B} \neq [a, b]$, thus a contradiction. $\square$

We have shown previously that homeomorphism and connectedness are topological properties by proving that the property is upheld by the continuous inverse. This is true for connectedness, that is that connectedness is in fact a topological property.

Theorem 2.4. *The continuous image of a connected space is connected.*

Proof. Let $f : X \to Y$ be a surjective continuous function, and assume that X is connected. Take a subset $A \subseteq Y$ and assume that it is *clopen*, meaning A is both closed and open. Then $f^{-1}(A)$ is also clopen (surjectivity). Since X is connected, and by definition 2.3, $f^{-1}(A)$ must either be X or the empty set. Conversely, A must then either be Y or the empty set, thus Y is connected. $\square$

Corollary 2.1. If $f : X \to Y$ is a homeomorphism, then X is connected if and only if Y is connected.

We will see in section 3 that just showing if a space is connected is not always sufficient. In order to classify certain surfaces, we will need to divide the surface into sections. How do we show that by splitting a surface, we still have connectedness? We do this by introducing *paths*. In a topological space X, a path is some function $\gamma : [0, 1] \to X$ which maps the points $\gamma(0)$ and $\gamma(1)$ continuously.

Definition 2.4. A space is *path-connected* if any two points in the space can be joined by a path.

Corollary 2.2. If a space is path-connected, then it is connected.

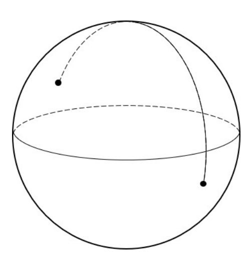


Figure 4: Path-connectedness on the 2-sphere.

Example 2.4. *The n-sphere is path-connected for $n > 0$.* We define the n-sphere as $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. If we take the set $X = \mathbb{R}^{n+1} \backslash \{0\}$, this is clearly path-connected. There exists a linear function between any two points in X, and can be redirected through a third point if the function passes through 0.

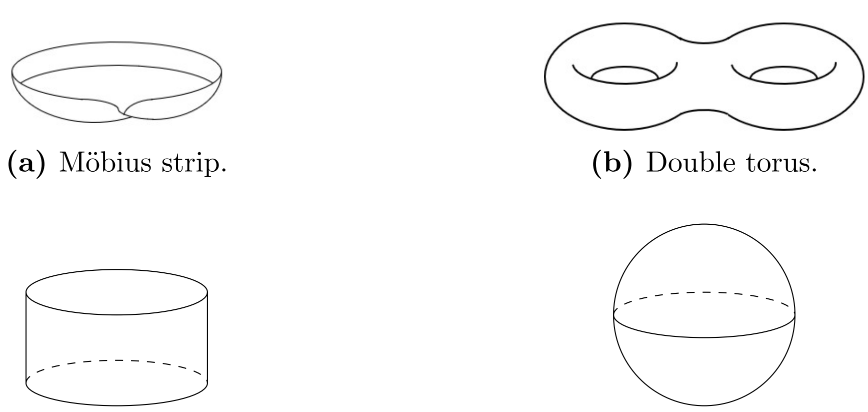
We take an arbitrary path $\gamma : [0,1] \to X$ and a function $f : X \to \mathbb{S}^n$, where $f(x) = \frac{x}{\|x\|}$. $f(x)$ is a continuous function for all $x \neq 0$ (for proof, see [1, section 2.2]). Thus, we have that

$$[0,1] \xrightarrow{\gamma} X \xrightarrow{f} \mathbb{S}^n,$$

defines a path in $\mathbb{S}^n$.

2.4 Boundary

The topological properties in this section are what make up the classification theorem that we will discuss in section 4, and we finish this section with the last property. The boundary is one of three characteristics that we can use to classify surfaces.



(a) Möbius strip. **(b)** Double torus. **(c)** Cylinder. **(d)** 2-sphere.

Figure 5

Definition 2.5. Let S be a surface equipped with topology τ. A *boundary point* of S is a point in τ such that every neighbourhood is half disc-like. The set of all boundary points is the *boundary number*, β, of S.

If we examine the surfaces in figure 5, we see that no two of these surfaces are homeomorphic. Therefore we could assume that each of these surfaces have nothing in common. However, we see that for the double torus and the 2-sphere, there is no point on the surface with a half disc-like neighbourhood. We can therefore say that the boundary number for these two surfaces $\beta = \emptyset$.

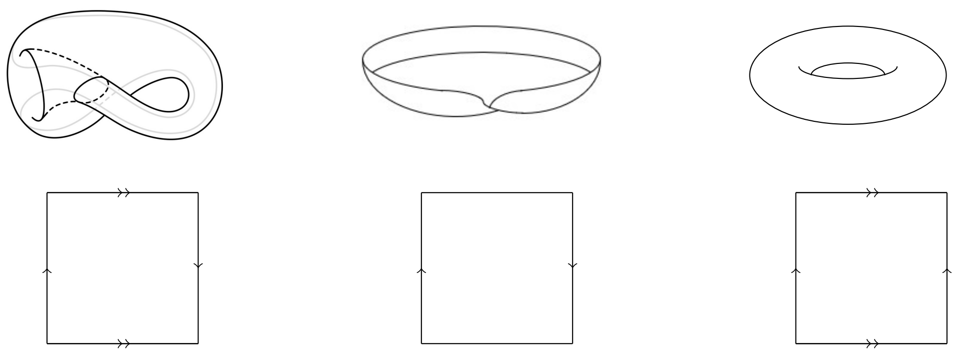
Conversely, we see that this is not also true for the cylinder and Möbius strip. If we take any point on the edge of either circular end of the cylinder (we assume the cylinder is hollow), we see that each of these points contain half disc-like neighbourhoods, thus for the cylinder, $\beta = 2$. Due to the fact that the Möbius strip has a half-twist, we observe that by following the edge of the surface from any point traverses the whole Möbius strip. Equallly, we see that this edge is a boundary point, so $\beta = 1$.

Corollary 2.3. A surface is said to be *without boundary* if $\beta = \emptyset$, otherwise the surface is said to be *with boundary*.

3 Surfaces

3.1 Construction

Surfaces are a complex part of topology. In many texts we find that figures used for reference are often surfaces that are either easy to work with, or instantly recognisable. However, it is often the case that a surface is not immediately easy to work with. Topologists have therefore developed various methods of representing surfaces in different formats. One method we will examine is known as the *"combinatorial approach to topology"* [5].



(a) Klein bottle. **(b)** Möbius strip. **(c)** Torus.

Figure 6: Representation of surfaces as planar diagrams.

This 2-dimensional representation of surfaces is widely used, and extremely useful. By using planar diagrams, we can understand the formation of a surface by the *glueing* of edges. Massey [3, section 1.3] provides a clear and concise explanation of this process for a Möbius strip.

Lemma 3.1. *Glueing lemma.* Let A and B be subsets of a topological space X. If there exist functions $f : A \to C$ and $g : B \to C$, then we can define

$$f \cup g : A \cup B \to C,$$

where $f \cup g$ is formed by the *glueing* together of the functions f and g.

Proof. We prove this by showing that $f \cup g$ is continuous. Let A and B be compact and connected in $A \cup B$, and assume that f and g are continuous functions. Take Z to be a compact and connected subset of C. We observe that the inverse $f^{-1}(Z)$ is compact and connected in A, as f is continuous. So $f^{-1}(Z)$ is compact and connected in $A \cup B$. Similarly, we have that $g^{-1}(Z)$ is compact and connected in $A \cup B$.

We know that $(f \cup g)^{-1}(Z) = f^{-1}(Z) \cup g^{-1}(Z)$, so $(f \cup g)^{-1}(Z)$ must also be compact and connected in $A \cup B$, thus we conclude that $f \cup g$ is continuous. $\square$

There exists an alternative way to construct spaces. We observe that in Euclidean space, we can use the Cartesian product $\mathbb{R} \times \mathbb{R}$ to be equivalent to $\mathbb{R}^2$. This can also be applied to spaces.

Definition 3.1. Let X and Y be any space. The *product space* $X \times Y$ is the set of all ordered pairs (x, y) such that

$$X \times Y := \{(x, y) : x \in X, y \in Y\}.$$

Therefore, we see that we can now construct the surfaces in figure 6. Clearly, the torus is the product space $\mathbb{S}^1 \times \mathbb{S}^1$. The Klein bottle and Möbius strip are more complex. We know that the product space $[0, 1] \times [0, 1]$ forms the unit square. Using the planar diagrams, we see that the Klein bottle can be expressed as $[0, 1] \times [0, 1]$ with the equivalence relations $\{(x, 0) \sim (1 - x, 1) : 0 \leq x \leq 1\}$ and $\{(0, y) \sim (1, y) : 0 \leq y \leq 1\}$. The Möbius strip is $[0, 1] \times [0, 1]$ with the equivalence relation $\{(x, 0) \sim (1 - x, 1) : 0 \leq x \leq 1\}$. These are examples of *quotient spaces.* For further reading, refer to [1, section 3.4] or [5, section 3.4].

Corollary 3.1. The Klein bottle is homeomorphic to the glueing of two Möbius strips.

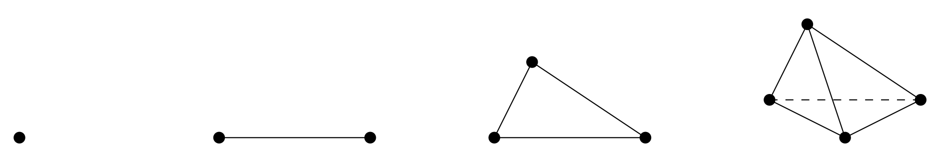
It is useful to understand here that although the Klein bottle and Möbius strip themselves are not homeomorphic, the quotient space shows that a Möbius strip is contained within the Klein bottle, and so we can form a homeomorphism. By glueing together two Möbius strips along their boundaries, we form a surface homeomorphic to the Klein bottle.

3.2 Triangulation

"Problems may be dealt with effectively by working with spaces that can be broken up into pieces which we can recognise, ... the so called triangulable spaces" - Armstrong[1]

Arguably the most crucial element for classification, triangulation is a process that allows one to split a seemingly impossible surface into smaller triangular segments, making calculations much easier. In order to find a homeomorphism between surfaces, we may need to break up a surface using a unique rule.

Definition 3.2. An *n-simplex* is the n-dimensional representation of $n+1$ points on a surface. A finite collection of simplices is called a *simplicial complex* if whenever an simplex lies in the collection then so does each of its faces, and whenever two simplices of the collection intersect, they do so in a common face.



(a) 0-simplex. **(b)** 1-simplex. **(c)** 2-simplex. **(d)** 3-simplex.

Figure 7

Definition 3.3. A *triangulation* of a topological space X consists of a simplicial complex K, and a homeomorphism $h : |K| \to X$.

Figure 7 gives a representation of an n-simplex for $0 \leq n \leq 3$. In order to build a simplicial complex, we must make sure there is no overlap or disjointness between simplices. This means we form a perfect tessellation of simplices that covers the whole surface. Where this is possible, we say that a surface is *triangulated* with respect to the polyhedron $|K|$.

Lemma 3.2. Let K be a simplicial complex.

1. $|K|$ is a compact space.
2. If $|K|$ is a connected space, then it is path connected.

Proof. For (1), we see that clearly, any n-simplex is closed and bounded. Therefore, by the Heine-Borel theorem, $|K|$ is a compact space. For (2), assume that $|K|$ is connected. Then each simplex in $|K|$ is joined to at least one other simplex at a common edge. Thus there must exist a path from any one simplex to another. For any two points $x, y \in |K|$, there exists a path $\gamma : [0,1] \to |K|$ such that $\gamma(0) = x$ and $\gamma(1) = y$. This is defined by an arbitrary function $f : |K| \to |K|$, giving a path connected space

$$[0,1] \xrightarrow{\gamma} |K| \xrightarrow{f} |K|.$$

$\square$

3.3 Barycentric subdivision

Now that we have all the conditions necessary to triangulate a surface, we can propose an algorithm to outline the triangulation process.

Theorem 3.1. *(Barycentric subdivison algorithm). Let K be a simplicial complex.*

1. *Take a 2-simplex $\Delta V_1V_2V_3 \in K$.*
2. *Insert all vertices $V'_{ij} = \frac{V_i+V_j}{2}$, where $i \neq j$.*
3. *Connect all V and V'.*
4. *Insert vertices V'' at the intersection of any two connected V and V'.*
5. *Repeat as many times as necessary.*

The result produces $\Delta V_1V_2V_3$ as an accumulation of smaller 2-simplices with respect to V, V' and V''.

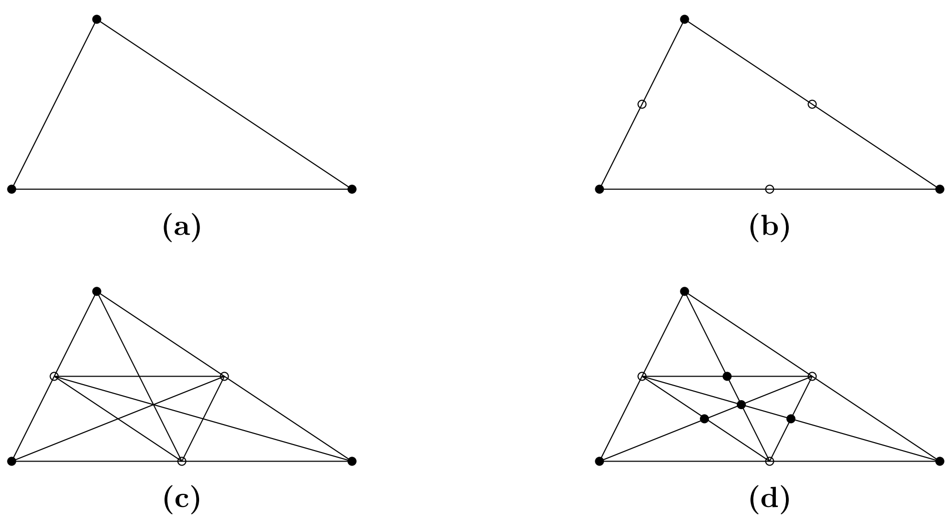


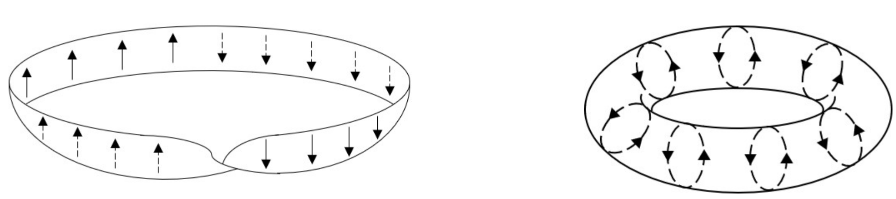
Figure 8: Process of barycentric subdivision on a 2-simplex.

3.4 Orientation

We finish this section by discussing the last property of surfaces necessary for classification. For simplicity of explanation, we begin by assuming all of the surfaces in question are connected and can be triangulated. We then have two ways to explain the idea of *orientable* surfaces, adapted from both Massey [3] and Armstrong [1].

Definition 3.4. A surface is *orientable* if it does not contain a Möbius strip. A surface with a Möbius strip is *non-orientable*.

Corollary 3.2. An orientable surface has an orientation number $\omega = 0$, and a non-orientable surface has orientation number $\omega = 1$.

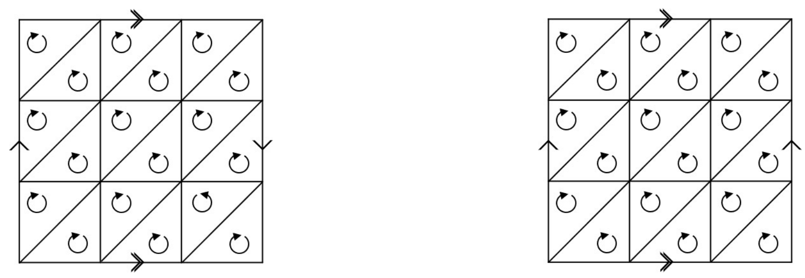


(a) Orientation path on the Möbius strip. **(b)** Orientation path on the torus.

Figure 9

Figure 9 gives orientation paths for two connected surfaces. Imagine we were to place an arrow at any point on a Möbius strip, with the arrowhead facing north.

Regardless of the direction of travel, as the arrow traverses the surface of a Möbius strip and returns to the initial position, we notice the arrowhead now faces south. We can therefore say that on a Möbius strip, the arrow travels on an *orientation reversing* path. Conversely, we see that on the torus, regardless of the direction of travel, the orientation of the arrow remains unchanged. In this case, we say the arrow travels on an *orientation preserving* path.



(a) **(b)**

Figure 10: Triangulated planar diagrams for (a) Klein bottle and (b) torus.

The other way in which to approach orientation is to look at the *compatibility* of each simplex on a triangulated surface. A triangulated surface is orientable when each adjacent simplex has the same orientation. We see that each simplex of a triangulated torus, shown in figure 10, has the same orientation. We therefore deduce that the triangulation of a torus is wholly compatible, thus is an *orientable* surface. However, a Klein bottle will always have one set of adjacent simplices that are incompatible, thus the Klein bottle is a *non-orientable* surface.

The most efficient way to determine the orientability of a surface is to examine its planar diagram. If the opposite sides of the diagram have opposing directions, then we know that the surface contains a twist, or Möbius strip, and therefore is non-orientable.

Theorem 3.2. *Homeomorphism preserves orientation.*

Knowing that homeomorphic surfaces have the same orientation is a useful result that we will use in classification. We will not cover the proof of this theorem, however further reading in order to understand the proof can be found in [3, chapter XIV], or researching the *Poincaré duality* theorem.

4 Classification of surfaces

"Results which allow one to classify completely a collection of objects are among the most important and aesthetically pleasing in mathematics" - Armstrong[1]

4.1 Euler characteristic

We begin the process of classification by discussing some of Euler's works on geometry. In 1750, a formula appeared in a letter outlining a characteristic for polyhedra, and in 1752, Euler published 2 papers explaining this discovery [2].

Definition 4.1. Let S be a countable surface with v number of vertices, e edges and f faces. The *Euler characteristic* of S is

$$\chi(S) = v - e + f.$$

By countable, we mean that the number of vertices, edges and faces can be easily counted using a visualisation of the surface in $\mathbb{R}^3$. When Euler spoke of the formula in his papers, his main point of reference were convex polyhedra, such as the platonic solids. He then stated the following theorem.

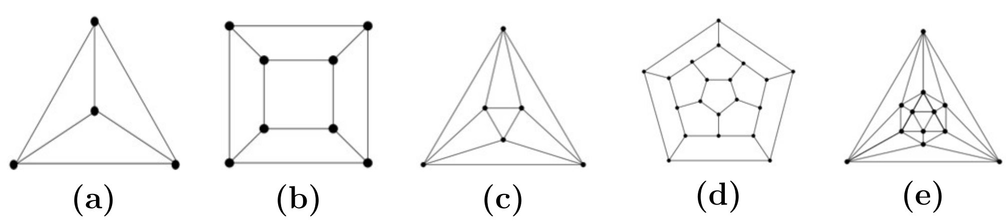


Figure 11: Planar projections of the platonic solids **(a)** tetrahedron, **(b)** cube, **(c)** octahedron, **(d)** dodecahedron and **(e)** icosahedron.

Theorem 4.1. *Let P be a polyhedron which satisfies:*

1. *Any two vertices of P can be connected by a chain of edges.*
2. *Any loop on P made of straight line segments separates P into two pieces.*

Then $v - e + f = 2$ for all P.

Proof. Let a planar graph denote the connected set of all vertices and edges of P. If this graph contains no loops, then we say it is a *tree*, T. By elementary graph theory, the number of vertices minus the number of edges is always equal to 1. Therefore we have that $v(T) - e(T) = 1$. We choose a tree, T, such that it contains all vertices of P, but not necessarily all edges.

Now we construct a second graph, Γ. Let each face A of P be represented by a vertex $\hat{A}$. Two vertices $\hat{A}$ and $\hat{B}$ of Γ are joined by an edge if and only if the corresponding faces in P are adjacent, and and separated by an edge that is not in T. By this condition, we deduce that Γ has no loops, therefore it must be connected, and thus a tree. (The proof no longer works here for polyhedra with a Euler characteristic that is not 2, as the graph Γ will contain a loop.) We now know that $v(\Gamma) - e(\Gamma) = 1$. Since $v(T) = v$, $e(T) + e(\Gamma) = e$ and $v(\Gamma) = f$, we get that

$$v(T) - [e(T) + e(\Gamma)] + v(\Gamma) = 2.$$

□

This theorem proves that for any convex polyhedron, we know they all have a Euler characteristic of 2. However, we have seen in section 3.2 that counting the vertices, edges and faces of a surface is not always as simple. This explains the importance of triangulation. Armstrong [1, section 7.3] outlines an alternative formula to calculate the Euler characteristic of a surface

Definition 4.2. Let S be a compact connected surface, and take K to be the triangulation of S. If L is a finite simplicial complex of dimension n, in $|K|$, then the Euler characteristic is

$$\chi(L) = \sum_{i=0}^{n} (-1)^i \alpha_i,$$

where α_i denotes the number of i-simplices in L.

Example 4.1. We know that the cube has a Euler characteristic of 2, and can be easily calculated using definition 4.1. However, we can also use definition 4.2 to achieve the same result.

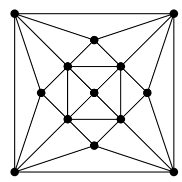


Figure 12: Simplical complex of the cube.

Let L be the simplicial complex in figure 12, consisting of a collection of 2-simplices. We have a 2-dimenstional space, therefore $i = 2$. We can then count the number of 0-simplices, $\alpha_0 = 14$, 1-simplices, $\alpha_1 = 32$, and 2-simplices, $\alpha_2 = 20$. Therefore

$$\chi(L) = (1)(14) + (-1)(32) + (1)(20) = 2.$$

Lemma 4.1. The Euler characteristic remains unchanged regardless of barycentric subdivision.

Proof. Let K and K' be two different triangulations of a surface S. We know from definition 3.3 that there exists two homeomorphisms $h_1 : |K| \to S$ and $h_2 : |K'| \to S$. We know that homeomorphism is a topological property, therefore as $|K|$ and $|K'|$ are homeomorphic to S, they themselves must be homeomorphic. Now, let K be denoted by an arbitrary polygon with v number of vertices, e edges and f faces. Take a second polygon L with v' number of vertices, e' edges and f' faces, where L is formed from adding one element from K.

If we add a single edge to K, then we obtain $v' = v$, $e' = e + 1$,and $f' = f + 1$. Therefore

$$\chi(L) = (f + 1) - (e + 1) + v = \chi(K).$$

□

Corollary 4.1. Let X and Y be compact connected surfaces. The Euler characteristic of their connected sum, $X \oplus Y$, is

$$\chi(X \oplus Y) = \chi(X) + \chi(Y) - \chi(\mathbb{S}^2) = \chi(X) + \chi(Y) - 2.$$

Corollary 4.2. The Euler characteristic is preserved under homeomorphism.

Corllary 4.2 describes the relation that if two surfaces are homeomorphic, then they have the same Euler characteristic. The proof of this mimics the proof of lemma 4.1.

4.2 Classification theorem

We approach this theorem cautiously. When put into practise, the act of classification is a multi-step process that requires one to accumulate all of the subject knowledge stated above. We begin by stating the theorem, and then gathering information on each of the surfaces in question.

Theorem 4.2. *(The classification theorem.) Any compact surface S is homeomorphic to either the sphere, the sphere with n number of handles, or the sphere with n-discs removed and replaced with Möbius strips.*

Sphere.

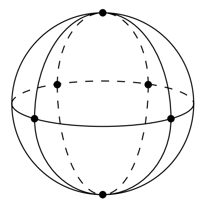


Figure 13: Triangulation of sphere.

We begin by noting that the sphere is an *orientable* surface, as it does not contain a Möbius strip, therefore $\omega = 0$. As shown is section 2.4, we know that it has no boundary, so $\beta = \emptyset$. Lastly, as shown in figure 13, we can easily triangulate the sphere to calculate its Euler characteristic, $\chi = 2$.

Sphere with n-handles.

(a) Sphere with 2-handles. **(b)** Double torus.

Figure 14

The sphere with n-handles is homeomorphic to the connected sum of n tori. If we use figure 14 for reference, for $n = 2$ we can see the 2-sphere is homeomorphic to a double torus by holding the handles of a 2-sphere, and stretching to form a double torus. Similar to the sphere, we can intuitively see that $\beta = \emptyset$ and $\omega = 0$.

Lemma 4.2. The sphere with n-handles has a Euler characteristic of the form $2-2n$.

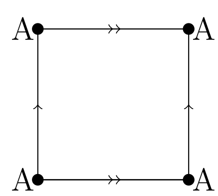


Figure 15: Planar projection of a torus.

Proof. We choose to prove by induction that this holds for the connected sum of n-tori. A single torus is a connected surface, therefore we can represent it as the planar projection shown in figure 15, giving $\chi = 0$. Now assume that for n, $\chi(n\text{-tori}) = 2-2n$. Then for $n+1$, we have that

$$\chi((n+1)\text{-tori}) = \chi(n\text{-tori} \oplus \text{torus}) = (2-2n) + 0 - 2 = -2n,$$

therefore the Euler characteristic of n-tori is $2-2n$ for all n. $\square$

Sphere with n-discs removed and replaced with Möbius strips.

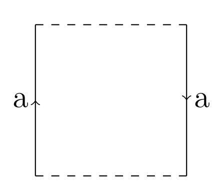


Figure 16: The projective plane.

Figure 16 shows the projective plane. Topologically, the projective plane is formed by the glueing of one single opposing edge of the Möbius strip. Therefore, we can say that the sphere with n-discs removed and replaced with Möbius strips is homeomorphic to the connected sum of n projected planes. We can see that this results in a surface without boundary, $\beta = 0$. Furthermore, even though the projective plane can be formed using a Möbius strip, it is non-orientable, so $\omega = 0$.

Lemma 4.3. The sphere with n-discs removed and replaced with Möbius strips has a Euler characteristic of the form $2 - n$.

Proof. We prove by induction that this is true for the connected sum of n projective planes. As shown in figure 16, we can deduce that for the projective plane, $\mathbb{P}^2$, $\chi = 1$. Now assume that $\chi(n\mathbb{P}^2) = 2 - n$. Then for $n + 1$ projective planes, we have that

$$\chi((n+1)\mathbb{P}^2) = \chi(n\mathbb{P}^2 \oplus \mathbb{P}^2) = (2 - n) + 1 - 2 = 1 - n,$$

therefore the Euler characteristic of n projective planes is $2 - n$ for all n. □

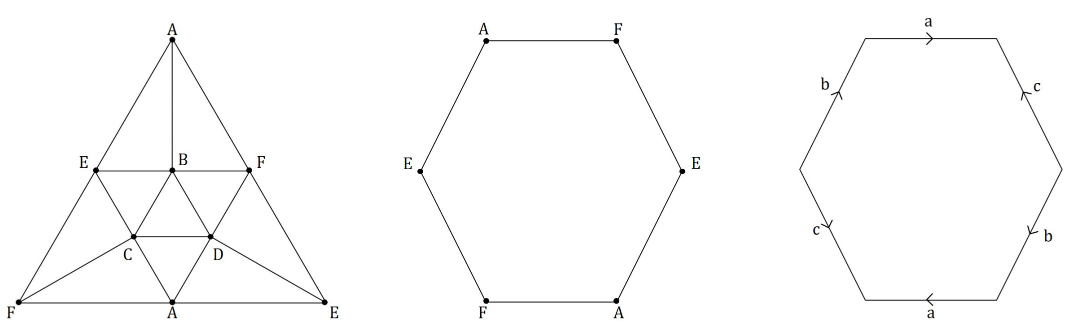
Proposition 4.1. Surfaces that are homeomorphic have the same boundary number, orientation number and Euler characteristic.

4.3 Proof of 4.2

There are a selection of ways in order to prove theorem 4.2, however it is often the case that the most simplistic approach makes the most sense. Chapter 1 of Massey [3] provides a rigorous algebraic proof of the theorem, however we will focus on the approach proposed by Kinsey [5], which is a proof by construction. This is beneficial as not only does it provide a concrete proof, but we also obtain a step by step method for classifying surfaces.

Proof. Let S be a compact surface. By lemma 3.2, we know that S has a finite simplicial complex K, and as such is also connected. We can therefore show K as any homeomorphic simplicial complex.

Step 1: Simplify simplicial complex into simple planar diagram.



(a) **(b)** **(c)**

Figure 17: Process for the first step.

We can see from figure 17 that as K is a connected simplicial complex, we can order the simplices in many different ways. The only criterion we need to follow is that the ordering must ensure that any two neighbouring simplices are connected via a single edge. For example, we could order (a) in the following way:

$$\Delta(ABE, ABF, BDF, DEF, ADE, ACD, ACF, CEF, BCE, BCD).$$

This process reduces K to the planar diagram shown in (c), where $a = AF$, $b = EA$ and $c = EF$.

Step 2: Simplify recurring combinations of edges.

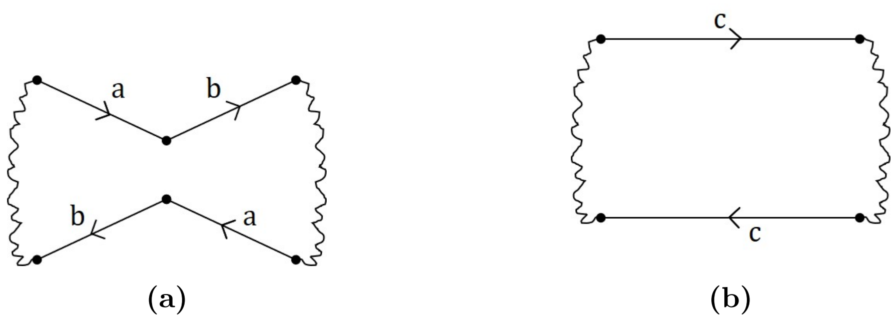


Figure 18: Process for the second step.

Throughout the proof we will see figures featuring jagged edges. This simply denotes other edges in the planar diagram that we are not currently working with. A sequence of edges that appears twice in the same order can be shortened, for example, in figure 18 we label $ab^{-1} = c$. It is useful here to acknowledge the direction of each pair. Figure 18 shows a *twisting pair*, meaning that if we were to traverse the perimeter of the planar diagram, the direction of the pair remains unchanged. Contrarily, if we get an *opposing pair*, then the direction is opposite.

Step 3: Remove adjacent opposing pairs.

Figure 19: Process for the third step.

We can glue opposing pairs together to eliminate them, as shown in figure 19. If all of the edges of the planar diagram are opposing pairs, then we stop, concluding that the surface is homeomorphic to the sphere. This is because as shown by Massey: *"We summarize our results by writing the symbols corresponding to each of the surfaces... sphere: aa^{-1}"* [3].

Step 4: Reduce planar diagram to just one vertex.

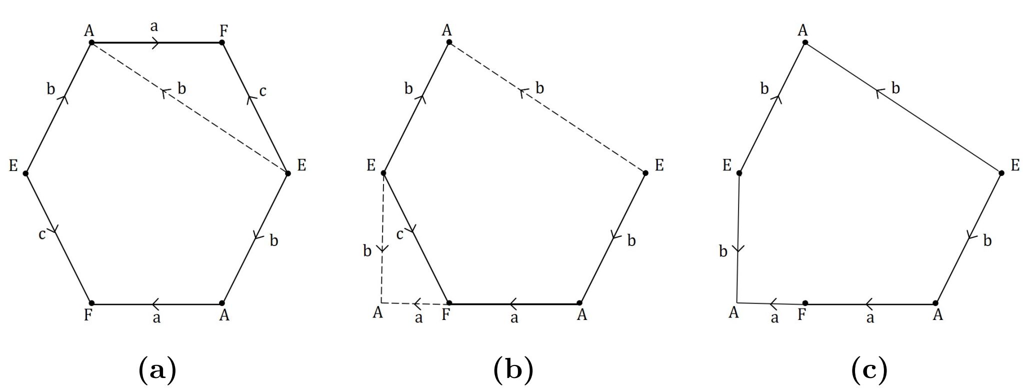


Figure 20: Process for the fourth step.

For most diagrams, this step will require multiple iterations in order to reduce to one vertex. The process involved simply choosing a vertex to dominate, and then using a *cut and paste* operation on the diagram. If we refer back to the original diagram in figure 17, we choose to dominate the vertex A, as shown in figure 20. We see here that we perform a cutting operation on ΔAEF, and then glue to a different vertex in order to eliminate the vertex F. In the example shown in figure 20, we perform a cut and paste operation that actually elimates both F vertices.

Step 5: Gather twisted pairs.

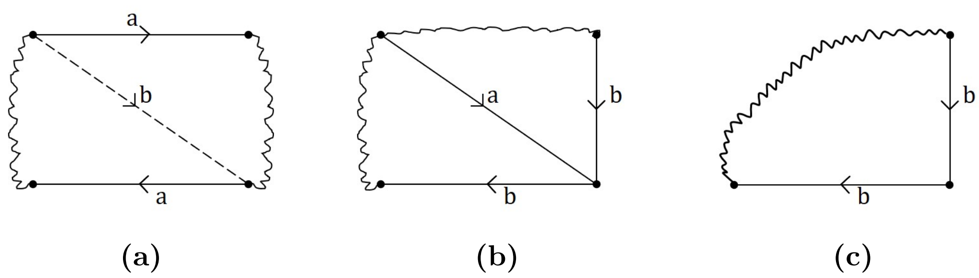


Figure 21: Process for the fifth step.

Here, we apply another iterative process of cut and paste operations on twisted pairs of edges, as shown in figure 21. If we cut along the edge b, we obtain two separate pieces that can be rejoined by glueing at the edge a. If after this step we have a diagram consisting of only twisted edges, then we can conclude that the surface is a connected sum of n projective planes. This is because as shown by Massey: *"The connected sum of n projective planes: $a_1a_1a_2a_2...a_na_n$"* [3].

Step 6: Gather sets of opposing pairs.

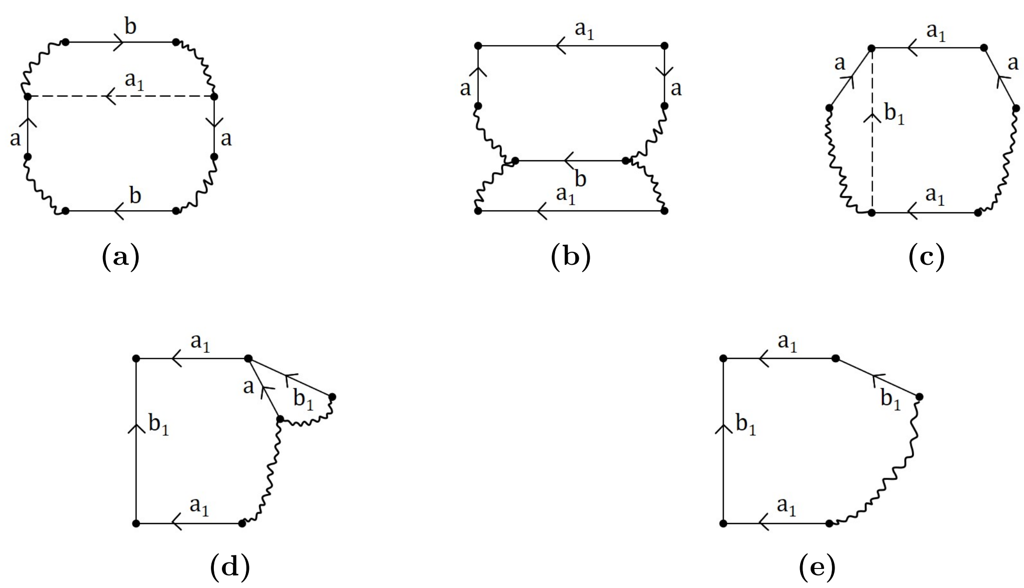


Figure 22: Process for the sixth step.

Clearly, if we have performed steps 1 to 5 already, then all that remains will be sets of opposing pairs. Therefore, we perform the last cut and paste operation, shown in figure 22, to collect pairs of opposing pairs together. If we a finish this step with a collection of edges only of the form shown in (e), then we can conclude that the surface is a connected sum of n tori. This is because as shown by Massey: *"The connected sum of n tori: $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}...a_nb_na_n^{-1}b_n^{-1}$"* [3]. □

If the surface has not been classified by this point, then we know that for a surface *without boundary*, it must be a connected sum of tori and projective planes. We can classify this surface with the following proposition.

Proposition 4.2. The connected sum of tori and projective planes is equal to the connected sum of projective planes.

For proof of proposition 4.2, refer to the diagram on page 88 of Kinsey [5], which provides a clear visualisation of its existence.

As shown in section 4.2, we showed that theorem 4.2 works for all compact surfaces *without boundary*. For surfaces without boundary we modify the classification theorem slightly.

Theorem 4.3. *(The classification theorem.) Any compact surface S with boundary is homeomorphic to either the sphere, the connected sum of n tori, or the connected sum of n projective planes, with a finite number of discs removed.*

Proof. To begin, we take a simplicial complex K for the surface S, where K adheres to a certain criterion. Any edge not on the boundary of the surface *must not* have both vertices on the boundary. As a result of this, we obtain a simplicial complex K such that the boundary of the surface is shown in the center of K. We then follow the same process outlined in the proof of 4.2 to classify the surface. $\square$

5 Conclusion

We have shown how the classification theorem can be used to classify surfaces up to homeomorphism. The 6 step method introduced in section 4.2 as a proof for the classification theorem is a robust method for classification. We rely heavily on the ability to triangulate a surface, however we see from the method that the ability to classify the surface is extremely simple once the surface is represented by a simplicial complex.

We begin with our surface S, and then triangulate S to obtain a simplicial complex K. We can then begin the method of classification. If the process stops at step 3, then we conclude that the surface is homeomorphic to the sphere. Examples of this include regular polyhedra and combinatorial surfaces.

If the process stops at step 5, then we have two scenarios. If the surface is compact, then it is homeomorphic to the connected sum of projective planes. An example of this would be the Klein bottle. If the surface is not compact, then it is homeomorphic to the connected sum of projective planes with a finite number of discs removed. An example of this would be a sphere with two discs removed, and one replaced with a Möbius strip.

If the process ends at the final step, then the surface is homeomorphic to the connected sum of tori. In the event of the process not leading to classification, we deduce that the surface must be a connected sum of tori and projective planes, and we have shown that this type of surface is homeomorphic to the connected sum of projective planes.

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