

Computational Geometry

Triangulation

A Triangulation of the sets of points $P := p_1, p_2, \dots, p_n$ in the plane is the maximal planar subdivision whose vertex set is P .

A **maximal planar subdivision** is a subdivision S such that no edge between two vertices can be added to S without destroying the planarity (*a graph where no edges cross each other*). This means that any edge that is not in S crosses one of the other edges.

Theorem 9.1 P is the set of n points in the plane. k denotes the number of points in P that lie on the boundary of the convex hull of P . Then any **triangulation** of P has:

$$\text{Triangles: } 2n - 2 - k$$

$$\text{Edges: } 3n - 3 - k$$

Proof

We denote some triangulation of P as T . Let m be the number of triangles of T . The number of faces of T is $n_f = m + 1$ since each triangle represents a bounded face and then we have the unbounded face, hence $m + 1$. The number of edges is therefore: each triangle has 3 edges ($3m$) and the unbounded face has k edges. Since we have counted each edge twice (each edge is shared between two faces) we divide by 2 and get the number of edges n_e :

$$n_e = (3m + k)/2$$

Given Euler's formula:

$$n - n_e + n_f = 2$$

we plug the values of n_e and n_f and get:

$$\begin{aligned}
n - \frac{3m+k}{2} + (m+1) &= 2 \\
2n - (3m+k) + 2(m+1) &= 4 \\
2n - 3m - k + 2m + 2 &= 4 \\
2n - m - k + 2 &= 4 \\
-m &= 4 - 2n + k - 2 \\
m &= 2n - 2 - k
\end{aligned}$$

and we have our number of triangles, m . Plug in the value of m to n_e and we get:

$$\begin{aligned}
n_e &= \frac{3m+k}{2} \\
&= \frac{3(2n-2-k)+k}{2} \\
&= \frac{6n-6-3k+k}{2} \\
&= \frac{6n-6-2k}{2} \\
&= 3n-3-k
\end{aligned}$$

and we have our number of edges.

Illegal edge

An edge is illegal if we can increase the smallest angle by flipping that edge.

$$\min \alpha_i < \min \alpha'_i$$

This means that if we have a triangulation T including in illegal edge e , we can obtain another triangulation T' by flipping e and $A(T') > A(T)$ where $A(T)$ is the angle vector for T . **Angle optimization** means that the size of an angle vector for some triangulation T of P is larger than any other triangulation of P .

Thales's Theorem

Let C be a circle with line l intersecting it in a and b . Let p, q, r and s all lie on the same side of l where p and q lie exactly on C , r is contained within C and s lies outside of C . Then:

$$\angle arb > \angle abp = \angle aqb > \angle asb$$

Lemma 9.4

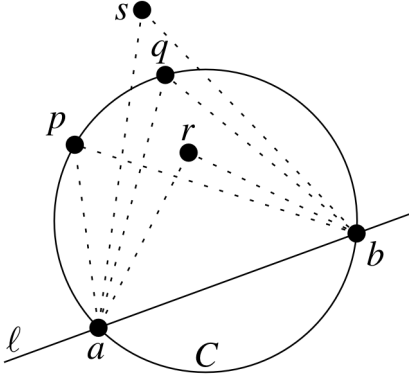


Figure 1: Thales's Theorem

We have edge $\overline{p_i p_j}$ which is incident to triangles $p_i p_j p_k$ and $p_i p_j p_l$. C is a circle through p_i , p_j and p_k . The edge $\overline{p_i p_j}$ is illegal if and only if p_l lies on the interior of C . It's also illegal for the reason that we can find an angle in the two incident triangles which is lower than any after we do the flip.

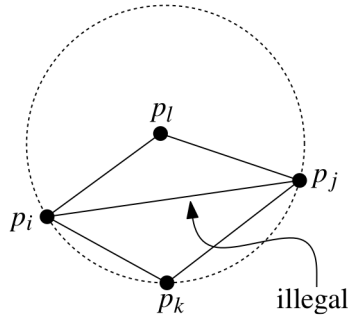


Figure 2: Illegal edge!

The Delaunay Triangulation

Theorem 9.5

The Delaunay graph of P is a plane graph (*no two edges in the embedding cross*).

Proof (by contradiction)

We consider an edge $\overline{p_i p_j}$ that is in the Delaunay graph if and only if there exists a closed disc C_{ij} with p_i and p_j on its boundary and no other point (*site*) contained in it. Note that the center of this disc lies on the common edge of $V(p_i)$ and $V(p_j)$ ($V(p)$ being the Voronoi cell of p) We then define t_{ij} to be the triangle with p_i , p_j and the center of C_{ij} as vertices. As the figure below shows, the edge from the center of the closed disc to p_i (this is one of the edge of t_{ij}) is contained in the $V(p_i)$. The same goes for p_j .

Now we look at another edge in the Delaunay graph: $\overline{p_k p_l}$, having the circle C_{kl} , triangle t_{kl} . For the sake of contradiction, we say that the two edges, $\overline{p_i p_j}$ and $\overline{p_k p_l}$ intersect. Since both p_k and p_l must lay outside C_{ij} (by definition), it implies that the edge $\overline{p_k p_l}$ must therefore intersect one of the edges of t_{ij} incident to the center of C_{ij} (this is because since we said originally that the two edges intersect, then it also intersects one of the edges in the triangle, it cuts the circle in that manner). And by the same argument, $\overline{p_i p_j}$ must therefore intersect one of the edges of t_{kl} incident to the center of C_{kl} . Furthermore, one of the edges of t_{ij} incident to the center must intersect with one of the edges of t_{kl} incident to the center. This would mean that the edges that are incident to the center points are contained in multiple cells, which is a contradiction.

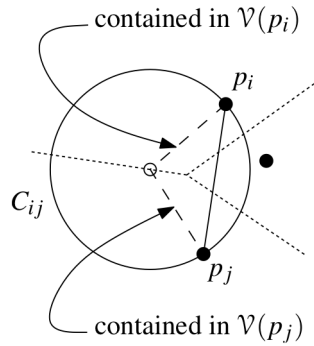


Figure 3: Theorem 9.5