Computational Geometry

Triangulation

A Triangulation of the sets of points $P := p_1, p_2, \ldots, p_n$ in the plain is the maximal planar subdivision whose vertex set is P.

A **maximal planar subdivision** is a subdivision S such that no edge between two vertices can be added to S without destroying the planarity (*a graph where no edges cross each other*). This means that any edge that is not in S crosses one of the other edges.

Theorem 9.1 P is the set of n points in the plane. k denotes the number of points in P that lie on the boundary of the convex hull of P. Then any **triangulation** of P has:

$$\text{Triangles: } 2n - 2 - k$$

$$\text{Edges: } 3n - 3 - k$$

Proof

We denote some triangulation of P as T. Let m be the number of triangles of T. The number of faces of T is $n_f = m + 1$ since each triangle represents a bounded face and then we have the unbounded face, hence $m + 1$. The number of edges is therefore: each triangle has 3 edges ($3m$) and the unbounded face has k edges. Since we have counted each edge twice (each edge is shared between two faces) we divide by 2 and get the number of edges n_e:

$$n_e = (3m + k)/2$$

Given Euler's formula:

$$n - n_e + n_f = 2$$

we plug the values of n_e and n_f and get:

$$
\begin{aligned}
n - \frac{3m+k}{2} + (m+1) &= 2 \\
2n - (3m+k) + 2(m+1) &= 4 \\
2n - 3m - k + 2m + 2 &= 4 \\
2n - m - k + 2 &= 4 \\
-m &= 4 - 2n + k - 2 \\
m &= 2n - 2 - k
\end{aligned}
$$

and we have our number of triangles, m. Plug in the value of m to n_e and we get:

$$
\begin{aligned}
n_e &= \frac{3m+k}{2} \\
&= \frac{3(2n-2-k)+k}{2} \\
&= \frac{6n-6-3k+k}{2} \\
&= \frac{6n-6-2k}{2} \\
&= 3n-3-k
\end{aligned}
$$

and we have our number of edges.

Illegal edge

An edge is illegal if we can increase the smallest angle by flipping that edge.

$$\min\ \alpha_i < \min\ \alpha_i'$$

This means that if we have a triangulation T including in illegal edge e, we can obtain another triangulation T' by flipping e and $A(T') > A(T)$ where $A(T)$ is the angle vector for T. **Angle optimization** means that the size of an angle vector for some triangulation T of P is larger than any other triangulation of P.

Thales's Theorem

Let C be a circle with line l intersecting it in a and b. Let p, q, r and s all lie on the same side of l where p and q lie exactly on C, r is contained within C and s lies outside of C. Then:

$$\measuredangle arb > \measuredangle abp = \measuredangle aqb > \measuredangle asb$$

Lemma 9.4

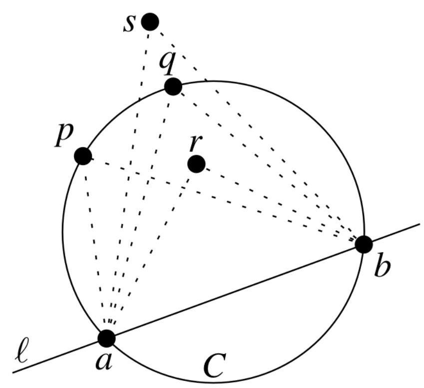

Figure 1: Thales's Theorem

We have edge $p_i\bar{p}_j$ which is incedent to triangles $p_ip_jp_k$ and $p_ip_jp_l$. C is a circle through p_i, p_j and p_k. The edge $p_i\bar{p}_j$ is illegal if and only if p_l lies on the interior of C.

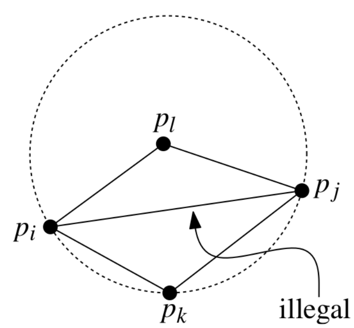

Figure 2: Illegal edge!