Flow networks

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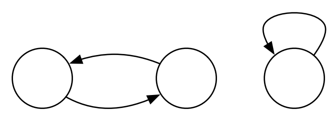
Flow network

A flow network consists of a digraph $G = (V, E)$ with a nonnegative capacity $c(u, v), \forall (u, v) \in E$, and a source $s \in V$, sink $t \in V, s \neq t$. We don't allow anti-parallel edges or self loops:



Constraints

A flow in (G, s, t, c) is a function $f : VxV \to \mathbb{R}$ such that:

1. $\forall (u, v) \in V, 0 \geq f(u, v) \leq c(u, v)$ - **capacity constraint**
2. $\forall u \in V \setminus \{s, t\}, \sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u)$ - **flow conservation**

Flow value

The value $|f|$ of f is defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

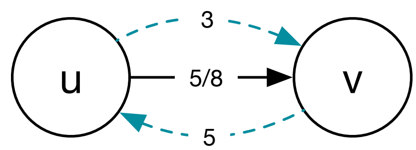
The max flow problem

Finding f that maximizes the flow value $|f|$.

Residual capacity

Given flow f in G, define for any $(u, v) \in VxV$, residual capacity of (u, v) to be

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$



Residual network and augmented flow

Residual network G_f is the flow network (v, E_f), $E_f = \{(u,v) \in VxV : c_f(u,v) > 0\}$.

Given flow f in G and a flow f' in G_f, we define augmented flow $f \uparrow f'$ as:

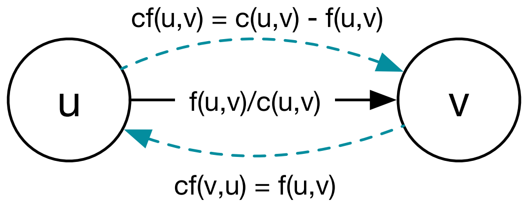
$$(f \uparrow f')(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

Ford-Fulkerson method

Start with $f = 0$.
while there exists a path from s to t in G_f:
send as much flow as possible along p to get f' in G_f and update $f \leftarrow f \uparrow f'$

Lemma

Given flow f in G, flow f' in G_f, then $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$.



Proof

Verify using capacity constraint

Observe that $c_f(v,u) = f(u,v)$ according to the residual capacity explanation here above. Therefore, the flow $f'(v,u) \leq c_f(v,u) = f(u,v)$ and:

$$\begin{aligned}
(|f \uparrow f'|)(u,v) &= f(u,v) + f'(u,v) - f'(v,u) && \text{by definition here above} \\
&\geq f(u,v) + f'(u,v) - f(u,v) && \text{because } f'(u,v) \leq f(u,v) \\
&= f'(u,v) \\
&\geq 0
\end{aligned}$$

and:

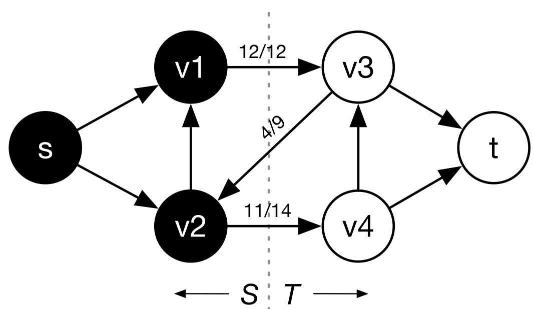
$$\begin{aligned}
(|f \uparrow f'|)(u,v) &= f(u,v) + f'(u,v) - f'(v,u) && \text{by definition here above} \\
&\leq f(u,v) + f'(u,v) && \text{because flows are nonnegative} \\
&\leq f(u,v) + c_f(u,v) && \text{capacity constraints} \\
&= f(u,v) + c(u,v) - f(u,v) && \text{by definition of } c_f \\
&= c(u,v).
\end{aligned}$$

Verify using flow conservation

For $u \in V \setminus \{s, t\}$,

$$\begin{aligned}\sum_{v \in E}(f \uparrow f')(u, v) &= \sum_{v \in V}(f(u, v) + f'(u, v) - f'(v, u)) \\ &= \sum_{v \in V} f(u, v) + \sum_{v \in V} f'(u, v) - \sum_{v \in V} f'(v, u) \\ &= \sum_{v \in V} f(v, u) + \sum_{v \in V} f'(v, u) - \sum_{v \in V} f'(u, v) \\ &= \sum_{v \in V}(f(v, u) + f'(v, u) - f'(u, v)) \\ &= \sum_{v \in V}(f \uparrow f')(v, u)\end{aligned}$$

Cuts



A cut is a partition of V into two subsets (S, T) such that $s \in S$ and $t \in T$.

Capacity of cut

The capacity of (S, T) is:

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$$

The capacity of the cut here above is $12 + 14 = 26$.

Net flow of cut

Given flow f in G, the flow across (S, T) is:

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

The net flow of the cut in the graph here above is $12 + 11 - 4 = 19$.

Lemma

For any cut (S,T), $f(S,T) = |f|$.

Proof

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Corollary

Given any flow f in G and any cut (S,T), $|f| \leq c(S,T)$.

Proof

$$\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u) \\
&\leq \sum_{u \in S} \sum_{v \in T} f(u,v) \\
&\leq \sum_{u \in S} \sum_{v \in T} c(u,v) \\
&= c(S,T)
\end{aligned}$$

Max-flow min-cut theorem

Given flow f in G, the following three statements are equivalent:

1. f is a max flow
2. there is no augmenting path in G_f
3. there is a cut (S,T) such that $|f| = c(S,T)$.

Proof

<u>(1) => (2)</u>

Assume there is a path p from s to t in G_f. Augmenting f with f_p gives flow $f \uparrow f_p$ in G and its value is

$$|f \uparrow f_p| > |f|$$

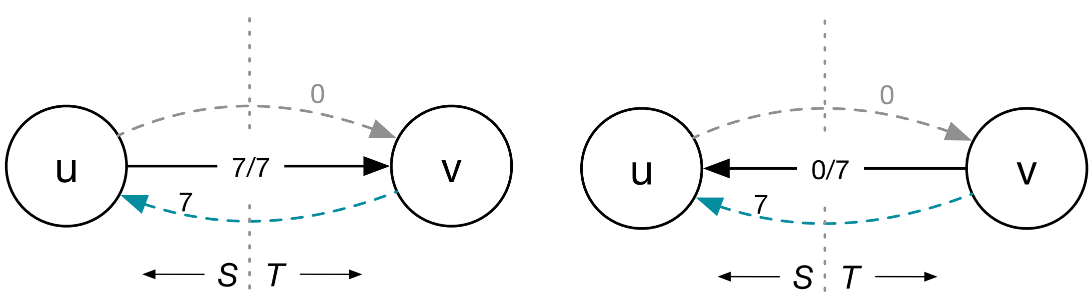
which contradicts the assumption that f is a maximum flow.

<u>(2) => (3)</u>

We assume that G_f has no augmenting path, meaning that there is no path between s and t in G_f. Let's define $S = \{v \in V : \text{there exists a path from } s \text{ to } v \text{ in } G_f\}$ and $T = V \setminus S$. (S, T) is therefore a cut. We now consider a pair of vertices $u \in S$ and $v \in T$. If $(u, v) \in E$, we must have that $f(u, v) = c(u, v)$ because otherwise we would have $(u, v) \in G_f$ which would place v in S (*if there exists an path from s to u, and u to v, then there is a path from s to v*). Similarly, if $(v, u) \in E$, we must have that $f(v, u) = 0$ because otherwise we would have $c_f(u, v) = f(v, u)$ and $(u, v) \in G_f$ which would place v in S.

We thus have:

$$\begin{aligned} f(S,T) &= \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u) \\ &= \sum_{u \in S} \sum_{v \in T} c(u,v) - \sum_{u \in S} \sum_{v \in T} 0 \\ &= c(S,T) \end{aligned}$$



<u>(3) => (1)</u>

By the corollary that $|f| \leq c(S, T)$ for all cuts (S, T), the condition that $|f| = c(S, T)$ thus implies that f is the maximum flow.

Edmonds-Karp

Implementation of Ford-Fulkerson where augmenting paths are shortest paths in G_f.

Lemma

Consider any $v \in V \setminus \{s,t\}$. Then $\delta_f(s,v)$ is monotonically non-decreasing when running Edmond-Karp.

Proof

$\delta_f(s,u)$ is the shortest path distance from s to u in G_f

Assume otherwise and let f be the first flow where distance to some $v \in V \setminus \{s,t\}$ goes down when augmenting flow. Let f' be the next flow $\delta_{f'}(s,v) < \delta_f(s,v)$. Pick v such that $\delta_{f'}(S,v)$ is minimized. Let p be a shortest path in G_f from s to v:

$$p = s \sim u \to v$$

.

$$\delta_{f'}(s,v) = \delta_{f'}(s,u) + 1$$

By choice of v, $\delta_{f'}(s,u) \geq \delta_f(s,u)$