Thomas J. Pfaff

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To the students who patiently worked with drafts of this book.

Preface

I want to begin by acknowledging the calculus textbooks that influenced this book. The first is *Calculus in Context* [12]. The use of programming was in many ways ahead of its time and I wonder what it would be like if it were written today with programming and computation easily accessible with R. The use of R throughout this text was inspired by *Calculus in Context*. The second influential text is *Calculus Concepts* [26]. The use of curve fitting as a way to add real data and context to calculus is wonderful. The function gallery in this text is due to *Calculus Concepts*.

The overarching goals of this calculus textbook are to integrate scientific programming with the use of R and to use it both as a tool for applied problems and to aid in learning calculus ideas. As a secondary goal, the combination of meaningful functions with units and an emphasis on reading graphs, all but a few made with R, makes this textbook well suited for various quantitative literacy requirements. Further, while I consider this an applied calculus text, a student that decides to go on in mathematics should develop sufficient algebraic skills so that they can be successful in a more traditional second semester calculus course. Student have used this book in their first semester and been successful in further math courses and have gone on to major or minor in math. Integrating R is meant as more of an enhancement of calculus and not so much as a replacement for basic skills. Hopefully, the applications provide some motivation to learn techniques and theory.

I have chosen R over more traditional math languages such as MatLab or Mathematica for a number of reasons. R is free and used more widely outside academia. These two reasons make R potentially more useful to more students. Data analysis has grown and we can make an argument that students in calculus I are more likely to do something with data than with calculus. In particular, the graphing capabilities of R can be a real asset to students in other courses and in a future job. For these students learning R is more valuable. A similar argument Python, although R might be easier for students to learn. There is still the question of whether or not the time invested in learning R, or any program, is worth it at all. Obviously, I think it is. My experience is that the attention to detail in coding builds student's skills that are transferable to other software environments. I do think incorporating R adds to students learning calculus. My favorite quote from students that I hear regularly is that I understand the math but I can't make R work. In almost all cases the issue here was with the math. Finally, R allows us to do more interesting stuff in calculus and for students that have taken some form of calculus before college, R gives them something new to learn.

The book contains chapters in the appendix for algebra review. Algebra skills can always be improved and these chapters are included to solve algebraic issues before they occur in calculus. I have used these chapters by assigning some algebra problems regularly from day one of the course. I have found this helps student focus on the calculus and less on algebra issues later in the course. Some students have commented that the algebra problems serve as a nice warm up before doing calculus problems.

R is introduced as the first chapter of the text as it is a key component of the book. There are no expectation that a student has any coding experience in order to use this text. R code is set off in boxes so they are easy to identify and when code is specifically referred to in the exposition it has **this font**. Further, there is an appendix of R code used to create the figures in the text as well an R glossary with a list of R functions used in this book. Deciphering code used to create figures is a helpful way to learn the language but note that the figure code will sometimes require understanding more than is used throughout the text. Problems are sometimes designated as those that require R and others where R should not be used, but other times it is up to the user to decide if R would help or not.

A subset of student will be reluctant or anxious about learning R. There is a bit of learning curve, but most of the issues are basic syntax such as parenthesis and commas. There is no partial credit with coding as it must be correct to run. As students work through this initial learning process I tell them to copy and paste the code and error message from the console window in an email. I try to respond as quickly as I can to keep them moving. This goes on for a couple of weeks but then most students start to get the basic syntax fixed and with patience and encouragement the resistance will fade. In reality, for most of the book R is used as more of a fancy calculator. You will even find that some students will figure out that R can be used to make their lives easier in other courses.

The R Code boxes highlight the code used in problem-solving. I have chosen to show the code as it is in the console window and lines are preceded by a >. Now because of the space on a page lines of code that need more than one line will not have a > on lines other than the first. This makes it easier to determine lines of code. In some cases lines of code will be preceded by a + sign. These are typically lines inside a for loop. The + signs do not get typed as they are placed there by R when the code is run. Finally, one of the most common errors is to have a + sign in the console instead of a >. In this case R expects that there is more code to come and is typically because there is a missing) to end a line. Press the Esc key to get back to > and then look for the error in the editor.

I have used versions of this text for a few years for a four credit first semester calculus course. Chapter 1 introduces students to using R. Terminology for describing a graph, maximum, minimum, concavity, etc. is informally introduced in

Chap. 2. The function gallery in Chap. 3 provides examples of curve fitting and the exercises are designed to get students comfortable working with functions in context and with units, including responding to questions in sentences. Chapter 4 introduces the idea of trying to quantify how fast a function is changing, while Chap. 5 introduces the notation of the derivative and what it represents. We summarize formulas that quantify change in Chap. 6 and I often use Project 6.2 in class as it is open ended and generates great discussions. Chapter 7 introduces local linearization as the microscope equation. Estimating derivatives is covered in Chap. 8. I do not typically do Projects 8.2, 8.3, and 8.4, although I often discuss 8.2, while 8.3 and 8.4 are included for those with a little more time or inclination to do some more programming. All the projects in the book are optional and do not need to be covered to move forward. Chapters 9 through 14, The Derivative Graphically, The Formal Derivative as a Limit, Basic Derivative Rules, Product Rule, Quotient Rule, and Chain Rule cover expected topics. I do use project 10.2, while I do not typically have time for Project 12.2. Derivatives with R, Chap. 15 also includes material about nth derivatives. I use both of the projects, 15.2 and 15.3, although not immediately. It is worth noting to students that they are very similar projects and well done code on the first project makes the second much easier. Chapters 1 through 15 should be covered in order.

Chapter 16 is optional, although I cover it because I think it is important to stress that not every increasing concave up function is exponential. I typically do not have time for the project. Chapters 17 and 18 cover finding local and global maximums and minimums as well as inflection point and need to be covered. Chapters 19 through 22 are optional and I cover all but the related rates chapter. I have found the chapter on partial derivatives, Chap. 20, to be really helpful to students in learning to deal with different variables when finding derivatives. The interpretations are less important to me at this point. The surge function chapter, 22, is one of my favorites but it is challenging with the model building and code to go with it. To me it really brings together model building, coding, and uses of calculus.

The differential equations chapters, 23–26, are optional but note that one needs to cover Chaps. 23 and 24 to cover Chaps. 25 and 26. There is also a lot of code in the differential equation chapters but the real work is in the population growth models, Chap. 24, as the code in the following chapters, 25 and 26, is nearly identical. I do not expect students to create their own differential equation code, but I do expect that can work with the code presented. I do cover all of these chapters, although not necessarily as sufficiently as I would like. Chapter 27 works well as an in class project to motivate area under the curve as a measurement.

Chapters 28 though 32 develop the idea of integration and cover the basic techniques as well as substitution and parts. I do cover these chapters and coverage will vary depending on the expectations of the first semester calculus course you are teaching. In terms of testing, during in class exams (or as I like to call them, opportunities for student to demonstrate their learning) I will have code for them to answer question about while on the final I have a couple of problems where they need R to solve. In this case, I allow student to use code they have on their computer from the course.

Two other features worth pointing out. There are times when life is a lot easier when basic facts are memorized. Formal memorization is done with forced recall of information. Flash cards really help get information stored in our memory. The M-Boxes in this book, where M stands for both math and memorize, make it clear what facts should be memorized. It is really hard to solve a problem, quickly or otherwise, if the tools to solve the problem are not readily available.

Finally, in the writing I tended towards informal rather than formal and I have tried to keep the exposition as short as possible. The risk of being more informal is that at times details are left out or hidden. I do not see this as a problem in a first semester calculus course. Those that go on and take more math will fill in details as necessary while those that do not, and I hope this group is a close to zero as possible, will be fine.

Two students need to be thanked for catching numerous typos. Amanda Birenbach and Owen Pfaff (yes, related and he enjoyed finding my mistakes). I also want to thank David Brown for his willingness to be the first faculty member, other than me, to use this text. Thanks also to Peter Maceli as the second faculty member to use this text. Kudos to the contributors to R that keep R going. They are listed here https://www.r-project.org/contributors.html. Kudos also to those that create packages for R. In particular, this book requires the Deriv and rootSolve packages and I thank the authors of these packages [14] [33]. The reviewers provided valuable feedback that improved this book and I thank them for their time and thoughtful comments.

There is companion website for this textbook with hopefully useful materials at https://sustainabilitymath.org/acr/ and your thoughts and comments are welcome. I also welcome feedback and there is an email on the website. I am particularly interested in ideas for calculus problems that require R to solve. These can be applied or theoretical. I am confident that others can come up with better problems than I have. Please, send them to me. Finally, if you have chosen to use this textbook, I am humbled and grateful as I never thought I would write textbook. My younger self would be shocked.

Ithaca, USA Thomas J. Pfaff

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Chapter 1
A Brief Introduction to R

Before we begin learning R we should address two questions. What is R? From the R Project: “R is a language and environments for statistical computing and graphics,” and “One of R’s strengths is the ease with which well-designed publication-quality plots can be produced, including mathematical symbols and formulae where needed.” [37] In short R is a programming language that is particularly useful for computations and graphics.

Why R? Traditionally, examples and exercises in a calculus class are rigged to be done by hand. We still have many of those problems in this text as there is value in hand calculations, but once we add R we can do much more. We can analyze more complicated functions and models. We can create graphics to help us understand calculus as well as real world phenomena. At the same time R is a widely used, popular, and powerful scientific programming language. R is used by major corporations and so this isn’t a program used just in a math course. R is also free. R enhances what we can do and learn in calculus and is worth learning. Think of learning R with calculus as value added.

The are no assumptions that you have programming experience. First, relax as it is not that bad. Second, we really only scratch the surface of R, and most of the time we use it as an advanced calculator. Still, there is a learning curve, and it will be frustrating at first. Programming requires precision and it will take some time to get accustomed to making sure your syntax is perfect. This will be frustrating, but experience suggests that most of the early problems are with making sure commas and parentheses are correct. This will be frustrating and make you feel as if you are lost. You are not. Keep at it and after a few weeks of regular use it will get better. Also keep in mind that it is a tool that can be used in other classes. Any class with computations or a need for graphics is a class that R can help. Let’s get started. One last reminder, relax, be patient, and keep at it.

Download R to your Mac or PC from `www.r-project.org`. There should be a download R link in the first paragraph, which will take you to a page to select a CRAN (Comprehensive R Archive Network) site. This will be the location from which the software will be downloaded. Pick a location close to your location, although it may not matter much. On the next page choose your operating system and go from there.

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Once installed open R and notice that the window you will see is the *console window*. This is the window where the code is run. You can type directly in this window on the lines that begin with $>$. We will generally not type directly into the R console window but for the moment follow the simple examples in R boxes 1.1 and 1.2.

R Code 1.1: Basic Arithmetic

```
> 2+2
```

```
[1] 4
```

R Code 1.2: Basic Arithmetic

```
> 3^2
```

```
[1] 9
```

Now go to File and open a new script (PC) or document (Mac) file. Note that this window is called the *R Editor*. This is where you will generally type code and these are the files you will save so you can open them and reuse them in the same way you work with docx (word documents) or xlxs (excel files). The file extension for R scripts or document files is. R. Now in the R editor type $2 + 2$. To execute this line in the R Console, make sure your cursor is on the same line and use Ctrl + R (PC) or Command + Enter (Mac). The line will be sent to the R Console and it will look as if the line was typed directly there. Note if you want to run only one line of code in the R Editor you don't need to highlight the entire line; all you need to do is have the cursor on the line before using Ctrl + R (PC) or Command + Enter (Mac). If you want to run multiple lines, which we will do soon enough, then highlight all the lines you want to send to the R Console and use Ctrl + R (PC) or Command + Enter (Mac). At this point you are strongly encouraged to create a folder on your computer as a location to save your R files. Really, do this. You will want to reuse code and to do that you will need to find your code. Also, use a sensible file naming convention. Your file name for R scripts should help you know both what is in the file and when you created the file. For example, you might name the script you have open Brief-R-Fall-2023, if this is the fall 2023 semester. To save the file in the folder you just created go to file then save as.

Basic R Tips

- NOTE: In this book the R Code boxes show what you will see in the R Console window. The > should not be typed in the R Editor window.

- Create a folder for all your R files for calculus. You might call this folder Calculus-R.
- Always type your code in an R Editor window. If you make a mistake it can be edited and the file can be saved. Reusing code through copy and paste will save time.
- When you open a new R Editor window save it immediately and give it a useful name that will help you identify the content. Also include some type of date in the title. Do not use . in file names.
- If you want to run only one line of code in the R Editor just make sure your cursor is on that line somewhere and use Ctrl + R (PC) or Command + Enter (Mac).
- If you want to run multiple lines of code in the R Editor highlight all of those lines and use Ctrl + R (PC) or Command + Enter (Mac).
- You should get in the habit of adding comments to your code in the R Editor. Comment lines should begin with a #. If you run a line that begins with # it will be ignored in the R Console. Also a string of # is a good way to make sections in the R Editor.
- Start new script or document files for new content. Don't keep using the same file as it will get really long and hard to find code.
- When you close R it will ask: Save workspace image? You should say no to this unless you fully understand why you might want to say yes. Be careful because it will also ask you if you want to save any open script (PC) or document (Mac) files in the R Editor. You likely want to say yes to this.
- Warning: Like all computer languages, R is extremely picky. Your code must be perfect for it to run. This means that even syntax such as parenthesis and commas must be perfect or your code won't work. Be patient it takes time to get used to this.

We will now go through a lengthy example where we will define a function, graph it, find its roots, mark them on the graph, define another function, add it to the graph, and find where the functions intersect not are interested. The advantage of using R here is that we can work with more complicated functions and create richer more detailed graphs. We begin with R Code box 1.3. This code is available on the companion web site sustainabilitymath.org/acr.

Note that in these R Code boxes we show the code in the R Console. You should be typing in an R Editor window and not typing > in the R Editor window. The point of showing the console view is each line of code starts with > and if a line is too long to fit in the box it will continue onto the next line and not have > in the line.

In R Code box 1.3 we have defined the function f. The syntax here is the R function **function** defines a function. The next piece **(x)** denotes x as the variable of the function. R functions are not limited to one variable, then can have multiple variables of multiple types. Next we have $\{\mathbf{4*x^5+15*x^4-140*x^3-430*x^2+1200*x+1000}\}$

which defines the function. Note that it is wrapped in braces, { and }. Each multiplication must be defined with a $*$, for example, $4x$ will not work as it must be $4 * x$, and powers are defined with a caret, ˆ. When the line **f <– function(x)**$\{\mathbf{4 * x^5 + 15 * x^4 - 140 * x^3 - 430 * x^2 + 1200 * x + 1000}\}$ is run all that will happen is the line is repeated in the R Console. The object **f** has been defined as $4x^5 + 15x^4 - 140x^3 - 430x^2 + 1200x + 1000$.

Note that $< -$ is the assignment operator. Whatever is on the right-hand side is assigned to the object on the left-hand side. We should say here that $4x^5 + 15x^4 - 140x^3 - 430x^2 + 1200x + 1000$ has been assigned to **f**. Also note that **f** is a variable name we defined. Pay attention to variables that we as users have chosen to use as opposed to functions or names that already exist in R. In the next line we have **f(-4.2)**, in other words, plug -4.2 into the function and we get -1812.985. We would write this as $f(-4.2) = -1812.985$.

R Code 1.3: Define and Evaluate a Function

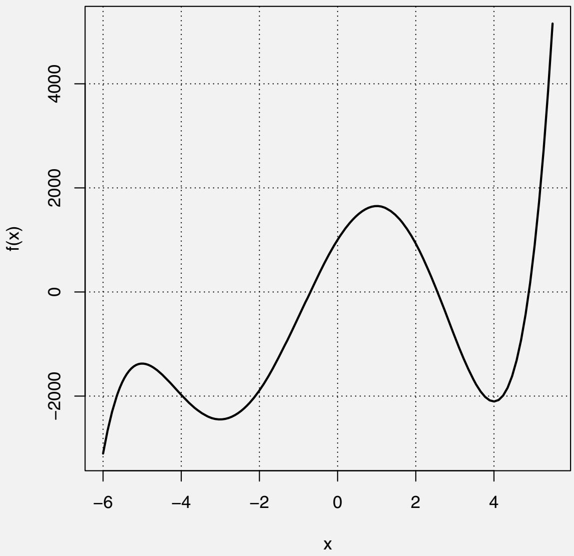
```
> f<-function(x){4*x^5+15*x^4-140*x^3-430*x^2+
1200*x+1000}
> f(-4.2)
```

```
[1] -1812.985
```

We now move to graph the function **f**. R Code box 1.4 provides the code. The R function **curve** will graph functions. The first three arguments of **curve** must be in order and are the function to graph, the left side of the x-axis, and the right side of the x-axis. Here we add **lwd=2** which sets the line width to 2 times the default. Take a moment to notice that functions in R use the same notion in math in that we use parenthesis for argument that goes into the function. The R function **curve** can accept many other arguments, which we will explore later, but if you are inclined to play then here are a few: **ylim=c(a,b)** defines the range for the y-axis, **col="purple"** will color the function purple, **xlab=" text "** labels the x-axis and **ylab=" text "** does the same for the y-axis, and **main=" text "** adds a title to the graph. The second line in R Code box 1.4 adds grid lines with the function **grid**. The use of **NULL** in the first two arguments tells R to put grid lines where it put axis ticks when in created the graph with **curve**. We set the color of the grid lines to black with **col="black"**. The default color is a light gray which is often a little faint. A new window will open with the graph. Take a moment to recognize that the x-axis, when $y = 0$, is not along the bottom, although the numbering of the x-axis is at the bottom.

R Code 1.4: Graph a Function

```
> curve(f,-6,5.5,lwd=2)
> grid(NULL,NULL,col="black")
```



Based on the graph it looks like the function **f** intersects the x-axis in three locations. These are roots of **f**. Our next goal is to find, or at least estimate, these values. Note that the function **f** is a fifth-degree polynomial and we aren't finding roots by hand. In order to find the roots of **f** we will need to install a package.

Installing a Package in R: We need the rootSolve package. To install this on your computer run the code **install.packages("rootSolve")** as seen in R Code box 1.5. You will again be asked to select a CRAN location. You only need to install a package on your computer once. This is essentially installing software that R will use on your computer. **Don't** keep running this line of code so put a # in front of it to comment out this line as in R Code box 1.6. It is a good idea to create an R script that has nothing but a list of **install.packages** commands with different packages. This keeps your list of packages separate from your other code and when you update R you can open this script and reinstall your packages easily.

R Code 1.5: Install the rootSolve Package

```
> install.packages("rootSolve")
```

R Code 1.6: Comment Out the Install of the rootSolve Package

```
> #intall.packages("rootSolve")
```

Using a Package in R: The fact that a package is installed on your computer does not mean R has access to it automatically. The R function **library** opens a package to be used during an R session. See R Code box 1.7. Once you open a package to be used during a session you don't need to keep running this line of code. So, again put a # in front of the line to comment out the line as in R Code box 1.8. Note that if you close R and then start it again you will need to open the package if you want to use it. The reason R doesn't open all of your packages when you open R is that it would take up too much computer space. Packages are add-ons to R that you only open when you need them during an R session.

R Code 1.7: Load the rootSolve Package to Use in an R Session

```
> library(rootSolve)
```

R Code 1.8: Comment out Loading of the rootSolve Package

```
> #library(rootSolve)
```

Now we can find the roots of **f**. The R function we want to use is **uniroot.all**. We need two arguments for this function. The first is the function **f** that we would like to find roots of and an interval to search as **uniroot.all** will not search for roots from $-\infty$ to ∞. To input the interval we have to use the **c** command, which stands for concatenate. In the case, **c(-10,10)**, defines the interval from -10 to 10. Based on our graph this is a larger interval than we really need. In review, **uniroot.all(f,c(-10,10))** will estimate all the roots of the function f between $x = -10$ and $x = 10$. The three roots in the output shown in R Code box 1.9 match what we would expect based on the graph.

R Code 1.9: Find Roots of a Function

```
> uniroot.all(f,c(-10,10))
----------------------------------------
[1] -0.7001584  2.5542663  4.8943181
```

We might consider checking to make sure we have roots by evaluating them in the function to see if the output is 0. We do this in R Code box 1.10. We are a bit lazy and evaluate **f** at -0.7. Notice that the output isn't 0 as it should be. The first possible issue is that we rounded the root to one decimal place and that was not accurate enough. We could copy and paste the output into the function but that isn't efficient nor could we easily reuse that code. R Code box 1.11 demonstrated how to name output.

R Code 1.10: Evaluate a Function

```
> f(-0.7)
```

```
[1] 0.24922
```

In the first line of R Code box 1.11 the output is stored in **f_roots**, which is a variable name we created. When using our own variable names we look to comprise between short and meaningful. In this case **f_roots** is the variable that stores the roots of the function f. If we run **f_roots** we will get the same output as in R Code box 1.9. If we want only the first root in the list we add square brackets with a 1 as shown with **f_roots[1]**. If we want the second or third root we would use **f_roots[2]** and **f_roots[3]**, respectively. In general, square brackets are used to denote a location in a vector, whereas parenthesis indicates an input for function. Note here that **f_roots** is a variable name we chose, as opposed **uniroot.all** which is a function defined in R. The goal is to choose variable names that have meaning so they are easy to understand without them being too long.

R Code 1.11: Store Roots as a Variable and Display the First Root

```
> f_roots<-uniroot.all(f,c(-10,10))
> f_roots[1]
```

```
[1] -0.7001584
```

Now we can directly input the first root into the function as seen in R Code box 1.12. There are two key benefits to this. First, all decimal places are kept for the value of the root that we then use as the input to the function **f**. Second, the code here is not dependent on the function **f**. In other words, if we change the definition of **f** back in R Code box 1.3 the code here still works properly. Now, does **f(f_roots[1])** return 0? This depends on the desired accuracy. For the purposes of this text we will use the default accuracy of **uniroot.all**. To improve the accuracy define the tolerance in **uniroot.all** using **tol**, such as **uniroot.all(f,c(-10,10), tol=0.1^10)**.

R Code 1.12: Evaluate a Function with a Variable

```
> f(f_roots[1])
```

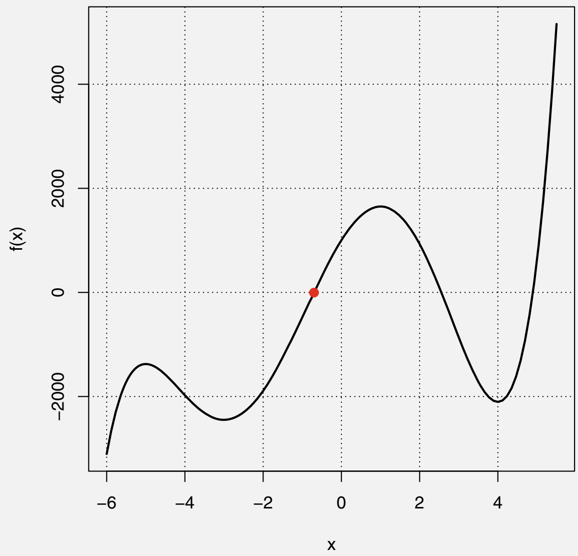
```
[1] -0.001181759
```

We move to adding dots on our graph at the roots of **f**. The R function that adds points to a graph is the function **points**. The first two arguments must be the x-value followed by the y-value of the point. R Code box 1.13 provides code to add a point at the first root. We include three optional arguments: pch=16 defines the point character as solid (try different numbers from 1-25 to see what happens), col="red" to color the point read, and cex=1.25 to increase the size of the point by 25% (cex for character expansion). In order to add the other two points we can use the same line of code but replace the 1 with a 2 and then a 3, but there is a faster way.

Note that the **points** function does not create a graph; it can only add elements to a currently open graph. On the other hand **curve** does initiate a new graph.

R Code 1.13: Add a Point to a Graph

```
> points(f_roots[1],f(f_roots[1]),pch=16,col="red", cex=1.25)
```



Consider R Code box 1.14. We used the **c** function, concatenate, to create a vector with values 1, 2, 3, and 4, and then used that as input into **f**. R will evaluate **f** at each of the values 1, 2, 3, and 4, and return the vector of the four results. Similarly, the **points** function will accept a vector of x and y-values, make sure each has the same number of values, and plot all pairs of points. We do this in R Code box 1.15 where **f_roots** is the vector of x-values of roots of **f** and **f(f_roots)** is the corresponding vector of y-values. Again note that at this point we can go back to R Code box 1.3 and change the function **f** to another function and the code will work. Try it!

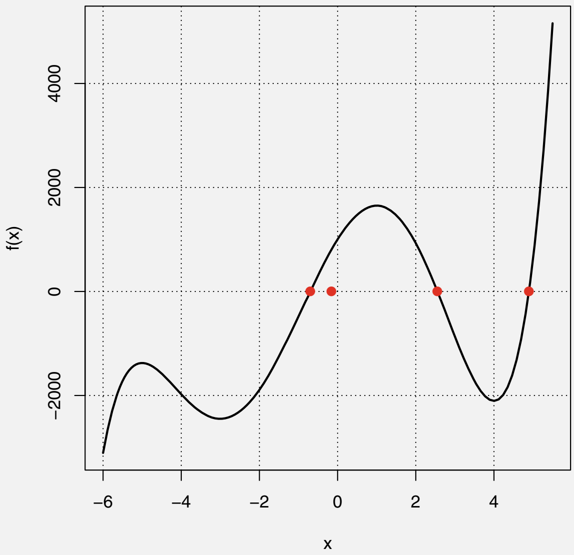
R Code 1.14: Evaluate a Function at Multiple Values

```
> f(c(1,2,3,4))
```

```
[1]  1649   928  -863 -2104
```

R Code 1.15: Add Multiple Points to a Graph

```
> points(f_roots,f(f_roots),pch=16,col="red",cex=1.25)
```



When we look at the function **f**, it is clear it crosses the value 2000. What is the corresponding y-value? In other words, we would like to solve the equation $f(x) = 2000$. The **uniroot.all** function only finds roots of a function, but with a little algebra $f(x) = 2000$ can be written as $f(x) - 2000 = 0$. In other words, solving $f(x) = 2000$ is equivalent to finding the roots of $f(x) - 2000$. To do this in R, we first define a new function, **f_2000**, which is **f(x)-2000**, in the first line of R Code box 1.16 and then use **uniroot.all** to find the roots of **f_2000**. The output of 5.187652 matches up with the graph.

R Code 1.16: Define a Function from a Function

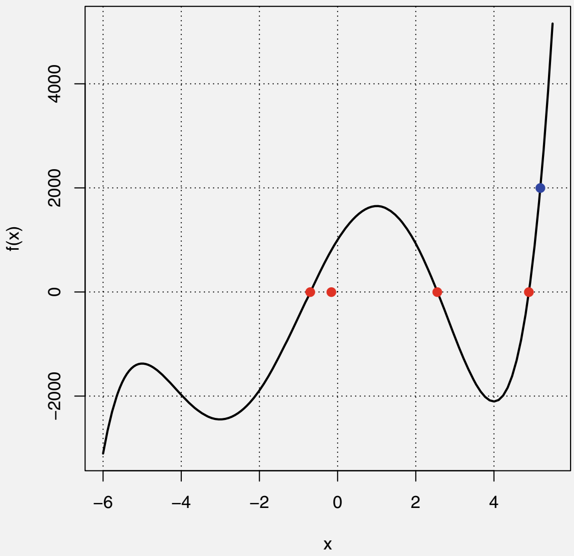
```
> f_2000<-function(x){f(x)-2000}
> uniroot.all(f_2000,c(2,6))
```

```
[1] 5.187652
```

To add the point on the graph where **f** crosses 2000, we will set the output of **uniroot.all** to **f_2000_roots** and then use **points** to plot the point. The only change to **points** is we color this point blue.

R Code 1.17: Identify when f(x) is 2000 and Add Points to the Graph

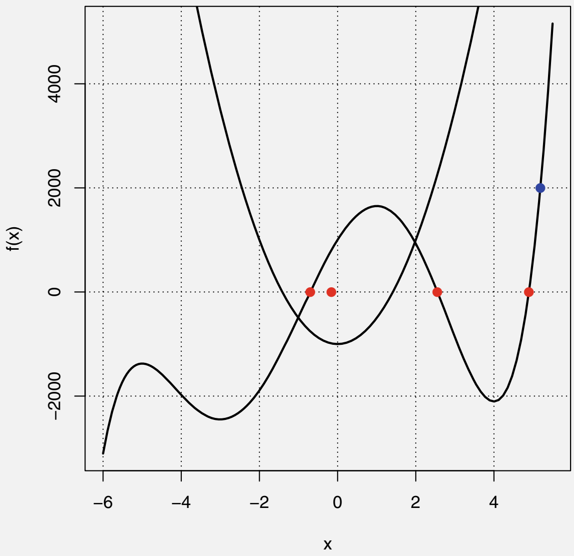
```
> f_2000_roots<-uniroot.all(f_2000,c(2,6))
> points(f_2000_roots,f(f_2000_roots),pch=16,
col="blue",cex=1.25)
```



In continuing with this example, we will add the function $g(x) = 500x^2 - 1000$ to the current graph, identify the points where $g(x)$ and $f(x)$ intersect, plot those points on the graph, and for fun connect those points with a line segment. We begin with R Code box 1.18 where we first define the function **g** and then use **curve** to add it to the graph. If we want to use **curve** to add to a graph instead of creating a new graph we need to include the argument **add=TRUE** where TRUE is in all caps. Recall that the first three arguments to **curve** must be the function, left x-value, and right x-value. Here we used the same x-value as the original graph. If we only wanted part of **g** to appear on the graph we could use a smaller range for the x-values. We use **lwd=2** to double the default line width.

R Code 1.18: Define a Function and Graph It

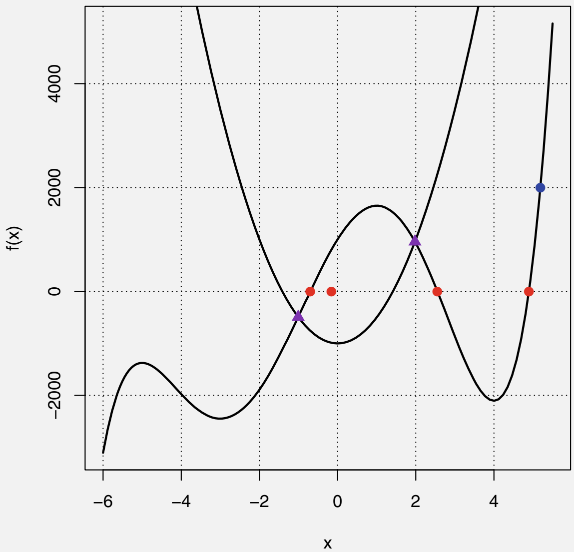
```
> g<-function(x){500*x^2-1000}
> curve(g,-6,5.5,lwd=2,add=TRUE)
```



To find the intersection points of two functions we need to solve $f(x) = g(x)$ or equivalently solve $f(x) - g(x) = 0$, which is the same as finding the roots of $f(x) - g(x)$. Since **uniroot.all** only locates roots we define a new function **f_g_diff** in R Code box 1.19. Note the syntax in **function(x){f(x)-g(x)}** as function(x){f-g} will fail. In the second line of code we find the roots of **f_g_diff** and set the output to the variable **f_g_diff_roots**. Finally we plot the points using **points** where the point character is set to 17 for a triangle, pch=17, and the color is purple. If we investigate the output of **f_g_diff_roots**, we will find there are actually three intersection points, but the third is off the graph.

R Code 1.19: Identify Intersection Points and Add Them to the Graph

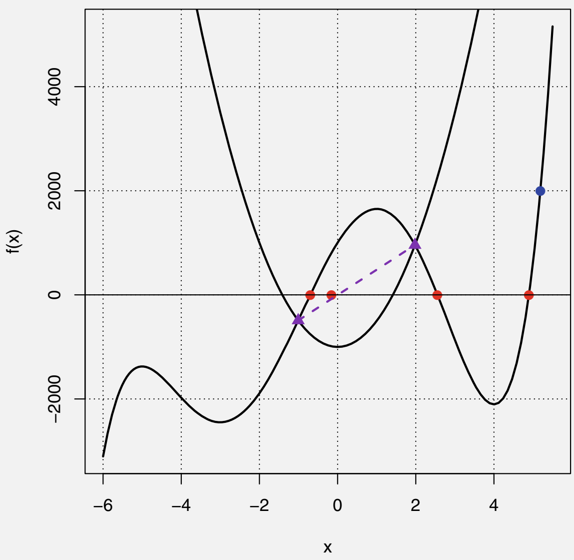
```
> f_g_diff<-function(x){f(x)-g(x)}
> f_g_diff_roots<-uniroot.all(f_g_diff,c(-10,10))
> points(f_g_diff_roots,f(f_g_diff.roots),pch=17,
col="purple",cex=1.25)
```



We add two more elements to this graph. First we connect the two intersection points with a dashed line. There isn't really a good reason to do this other than to illustrate the use of the **segments** function which draws a line segment between two points. The first four arguments of **segments** must be the x and y coordinates of the first point followed by the x and y coordinates of the second point. Here we use the points **f_g_diff_roots[1],f(f_g_diff_roots[1])** for the first point and **f_g_diff_roots[2], f(f_g_diff_roots[2])** for the second. We set the argument **lty=2**, line type, to get the dashed line. We also color it purple. Lastly, to call attention to the x-axis we use the function **abline** with argument **h=0** for a horizontal line with y-value equal to 0. Can you guess how to add a vertical line? Note how this line is drawn over the red dots. In R graph layers are added on top of each other. If we wanted this horizontal line underneath the red points, we would run this line of code before we added the red points.

R Code 1.20: Add a Line Segment to a Graph

```
> segments(f_g_diff_roots[1],f(f_g_diff_roots[1]),
f_g_diff_roots[2],f(f_g_diff_roots[2]),lty=2,
lwd=2,col="purple")
> abline(h=0)
```



In creating this graph we never set the range for the y-axis as we let R do that for us. In graphing the function $f(x)$ this was fine, but when we added the function $g(x)$ the graph went outside the axis range. We can set the y-axis range with the argument **ylim** in the **curve** function. For example, in R Code box 1.4 when we first created the graph we could have done **curve(f,-6,5.5,ylim=c(-3000,10000),lwd=2)**, which would set the y-axis from -3000 to 10000. Note that we wouldn't know to do this until after we added **g** to the graph. Since we work in the R Editor window it isn't a big deal to go back, edit the code, highlight it, and run it again (also because we never copied and pasted output).

Common R Errors

1. Pay attention to the output in the R Console. Learning to understand error messages can help you find problems with you code, although sometimes error messages are hard to interpret.
2. If you see a + in the R Console window instead of a > this is a problem. It means that previous code didn't end and it is waiting for more lines of code. Use Esc to get back to >. Look in your code to see if your parenthesis aren't matching. There is a good chance you are missing a `)`, although this isn't the only possible problem.
3. Coping and pasting code from a pdf file into R often doesn't work, especially in Macs. It is better to type directly into the R editor.

Throughout this text you should refer back to this box of tips for how to complete exercises in R.

More Basic R Tips and Functions

- The assignment operator <– should be viewed as an arrow pointing to the left. The idea is that what is written on the rights is being assigned, by the arrow, to the object on the left. In mathematics we might say let $a = 5$. In R we assign the value of 5 to the name a with **a<– 5**.
- Parenthesis (and) are used for argument for a function in R, where as square brackets [and] are used to define a location in an object such as a vector or matrix.
- If you find yourself copying and pasting (or worse retyping) an output result to be used in your code, stop. You should be defining a variable.
- The R function **curve()** initiates a new graph unless the argument **add=TRUE** is included.
- The R functions **points()** and **segments()** add elements to a current graph, they will not initiate a new graph.
- When you see, for example, $e+02$ or $e-12$ with a number it is scientific notion, in other words $e+02$ and $e-12$ is equivalent to $\times 10^2$ and $\times 10^{-12}$. For example, $3.2e+02 = 320$ and $1.36e-12 = 0.00000000000136$.
- Common functions and how to code them in R:
  - $\sqrt{x}$ is **sqrt(x)**
  - $\sqrt[n]{x}$ is **x^(1/n)**
  - e^x is **exp(x)**
  - $\sin(x)$ and $\cos(x)$ are **sin(x)** and **cos(x)**
  - π is **pi**
  - The natural log, $\ln(x)$ is **log(x)**

- Saving a graph. Using print screen is not how one should save a graph made in R to include in a report. Here is how it is done.

PC Make sure the window with the graph is active (click on it). Go to File -> Save as. Choose the file type and go from there.

Mac Make sure the window with the graph is active (click on it). Go to File -> Save in the menu bar, and choose a location to save the file. It will save as a PDF file, which we can double-click to open in Preview, and then use the File -> Save as menu choice to convert it to another format.

- The concatenate or combine command **c()** creates a vector of the objects in the command.
- The colon command **a:b** creates a vector of numbers starting at **a** and increasing by unit until **b** is exceeded. For example **1:5** is the vector $(1, 2, 3, 4, 5)$.
- **paste()** allows us to concatenate a variable and characters. For example, if $a = 3.1415$ then **paste("(",a,")", sep="")** will output **(3.1415)**. **paste()** is particularly useful for math in labeling and adding information to graphs.
- **round()** rounds a value. For example if $a = 3.1415$ then **round(a,2)** will output **3.14**. **round()** is often useful within **paste()**.
- **text()** will place text on a graph. Using 1, **text(1,3, paste("(",a,")", sep=""))** will place **(3.1415)** at the point $(1, 3)$ on the graph.
- **pos** is an option that can be used within **text()** to place the next below, to the left, above, or to the right of the point with pos set to 1, 2, 3, or 4, respectively.
- **expression()** is used to output math expressions. For example, **expression(f(x)==x^2*sin(x))** will output $f(x) = x^2 \sin(x)$. Search R expression to see tables of the syntax. Two quick tips are that == outputs one equal sign and you need * between parts of the expression.
- **par(mar=c(a,b,c,d))** sets the margins around a graph where the values of a, b, c, and d represent the number of lines below, to the left, above, and to the right of the graph (clockwise starting at the bottom). Use this before the use of **curve()**.

1.1 Exercises

Tip: For each problem save your code as you may be able to reuse parts of it for other problems. Think copy, paste, and edit.

1. Calculate $2^7 - e^{7.5} + \sin(3\pi) - \ln(150) + \sqrt{92}$.
2. Calculate $3^7 + e^{5.26} - \sin(5\pi) + \ln(250) + \sqrt{111}$.
3. Calculate $5^{-4} + e^{-9.2} + \cos(-6\pi/4) - \ln(625) + \sqrt[4]{218}$.
4. Calculate $7^{-8} - e^{-15.8} - \cos(-7\pi/3) + \ln(456) - \sqrt[5]{568}$.
5. Evaluate $f(3.2\pi)$ and $f(5.6)$ where $f(x) = 2^x - e^x + \sin(2\pi x) - \ln(x) + \sqrt{x}$.
6. Evaluate $g(6.7\pi)$ and $f(9.3)$ where $g(x) = 3^x + e^x - \sin(5\pi x) + \ln(x) + \sqrt{x}$.
7. Evaluate $f(1)$, $f(3)$, $f(8)$, and $f(11)$ in one step (see R Box 1.14) where $f(x) = 5^x + e^x + \cos(6\pi x) - \ln(x + 10) + \sqrt[3]{x}$.
8. Evaluate $g(2)$, $g(4)$, $g(9)$, and $g(13)$ in one step (see R Box 1.14) where $g(x) = 7^x + e^{-x} - \sin(2\pi x) + \ln(x + 10) + \sqrt[3]{x}$.
9. Graph $\sin(x)$ and $\cos(x)$ from 0 to 2π. Color each function, add a grid to your graph, and label the axis.
10. Graph $\sqrt{x}$ and $\sqrt[5]{x}$ from 0 to 1000. Color each function, add a grid to your graph, and label the axis.
11. Graph $\sqrt{x}$ and $\ln(x)$ from 0 to 500. Color each function, add a grid to your graph, and label the axis.
12. Graph e^x and x^2 from 0 to 100. Color each function, add a grid to your graph, and label the axis.
13. Find the roots of $f(x) = x^2 + 8x + 15$ by both factoring and using R. Graph the function and place points on the graph to identify the roots. Choose a window so that the roots are clearly identified.
14. Find the roots of $f(x) = x^2 - 5x - 36$ by both factoring and using R. Graph the function and place points on the graph to identify the roots. Choose a window so that the roots are clearly identified.
15. Find the roots of $f(x) = -x^3 - 13x^2 + 19x + 5$. Graph the function and place points on the graph to identify the roots. Choose a window so that the roots are clearly identified.
16. Find the roots of $f(x) = 237173 + 40930.4x - 7783.38x^2 - 29.85x^3 + x^4$. Graph the function and place points on the graph to identify the roots. Choose a window so that the roots are clearly identified.
17. Find the roots of $f(x) = e^x - 42$ by both algebra and using R. Graph the function and place points on the graph to identify the roots. Choose a window so that the roots are clearly identified. Tip: In R the function e^x is codded as **exp(x)**.
18. Find the roots of $f(x) = \ln(x) - 100$ by both algebra and using R. Graph the function and place points on the graph to identify the roots. Choose a window so that the roots are clearly identified. Tip: In R the function $\ln(x)$ is codded as **log(x)**.
19. At what point(s) does $f(x) = x^2 - 5x - 36$ equal 10? Graph the function and place points on the graph to identify when it equals 10.
20. At what point(s) does $f(x) = x^2 + 8x + 15$ equal 50? Graph the function and place points on the graph to identify when it equals 50.

21. At what point(s) does $f(x) = -x^3 - 13x^2 + 19x + 5$ equal -100000? Graph the function and place points on the graph to identify when it equals -100000.
22. At what point(s) does $f(x) = 237173 + 40930.4x - 7783.38x^2 - 29.85x^3 + x^4$ equal -11000000? Graph the function and place points on the graph to identify when it equals -11000000.
23. When do $\sin(x)$ and $\cos(x)$ intersect on the interval $[0, 4\pi]$? Graph the functions and place a point whenever they intersect.
24. Where do the functions $x^2 + 2$ and x^6 intersect? Graph the functions and place a point whenever they intersect.
25. Where do the functions $\sqrt{x}$ and $1 + \ln(x)$ intersect? Graph the functions and place a point whenever they intersect.
26. Where do the functions $e^{x/4} - 0.75$ and $\sin(x)$ intersect? Graph the functions and place a point whenever they intersect. Tip: Graph $\sin(x)$ first (do you know why?).
27. Graph $f(x) = x^2 + 3$ on $[-5, 5]$ and add a line segment connecting the points $(1, f(1))$ and $(4, f(4))$. Based on the graph, does this line segment have a positive or negative slope? What is the slope of the line segment?
28. Graph $g(x) = x^2 - 4$ on $[-6, 6]$ and add a line segment connecting the points $(-3, g(-3))$ and $(2, g(2))$. Based on the graph, does this line segment have a positive or negative slope? What is the slope of the line segment?
29. Graph $f(x) = x\sin(x)$ on $[0, 6]$ and add a line segment connecting the points $(1.5, f(1.5))$ and $(5.5, f(5.5))$. Based on the graph, does this line segment have a positive or negative slope? What is the slope of the line segment?
30. Graph $g(x) = x\cos(x)$ on $[-5, 2]$ and add a line segment connecting the points $(-4, g(-4))$ and $(1, g(1))$. Based on the graph, does this line segment have a positive or negative slope? What is the slope of the line segment?
31. Graph $f(x) = 4x^5 + 15x^4 - 140x^3 - 430x^2 + 1200x + 1000$ (same function from R Box 1.3) from -10 to 10. Why does this look so different from the graph here from -6 to 5.5. How important is the window?
32. The graphs $f(x)$ and $g(x)$ from R Boxes 1.3 and 1.18 intersect at a third point. What is that point?

33. Again consider the intersection of the two graphs $f(x)$ and $g(x)$ from R Boxes 1.3 and 1.18. What happens to the intersection points if we consider $g(x) = ax^2 - 1000$ with values of $a = 500$, $a = 1000$, and $a = 2000$. Make a conjecture about the intersection points as the value of a gets larger.

34. Consider the functions $x^2 + 2$ and x^n where n is a positive integer. Make a conjecture about what happens to the intersection points as the value of n gets larger. Provide examples to justify your conjecture.

35. Consider $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$.

    a. Graph $f(x)$ from 1 to 4.
    b. Find and mark all roots of $f(x)$ with a purple point.
    c. Mark where $f(x) = -5$ on the graph.
    d. Mark intersection points with $g(x) = -9x^4 + 108x^3 - 486x^2 + 972x - 709$, again from $x = 1$ to $x = 4$. Add a secant line from first to last intersection point.

Chapter 2
Describing a Graph

We take a moment here to informally provide common terminology used to describe a graph. This will allow us to learn some of language of a graph now. Eventually we will use calculus to identify parts of the graph with these characteristics. Figure 2.1 has these definitions which are informally given in M-Box 2.1.

M-Box 2.1: Describing a Graph

Informally and referencing figure 2.1:

- A graph is **increasing** when it is going up or rising.
- A graph is **decreasing** when it is going down or falling.
- A graph is **concave up** when it is curved upward.
- A graph is **concave down** when it is curved downward.
- An **inflection point** is where the concavity changes.
- A **local max** (or maximum) is a local high point of the graph. Graphs can have more than one local max.
- A **local min** (or minimum) is a local low point of the graph. Graphs can have more than one local min.
- A **global max** (or maximum) is the absolute highest point on the graph on a fixed interval. A **global max** may or may not be the same as a local max.
- A **global min** (or minimum) is the absolute lowest point on the graph on a fixed interval. A global min may or may not be the same as a local min.

Example 2.1. *Describe the* $\sin(x)$ *graph in figure 2.2.*

Solution. Based on the figure:

- The graph is increasing from $x = 0$ to $x = \pi/2$ and from $x = 3\pi/2$ to 2π.
- The graph is decreasing from $x = \pi/2$ to $x = 3\pi/2$.
- The graph is concave up from $x = \pi$ to $x = 2\pi$.
- The graph is concave down from $x = 0$ to $x = \pi$.

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_2

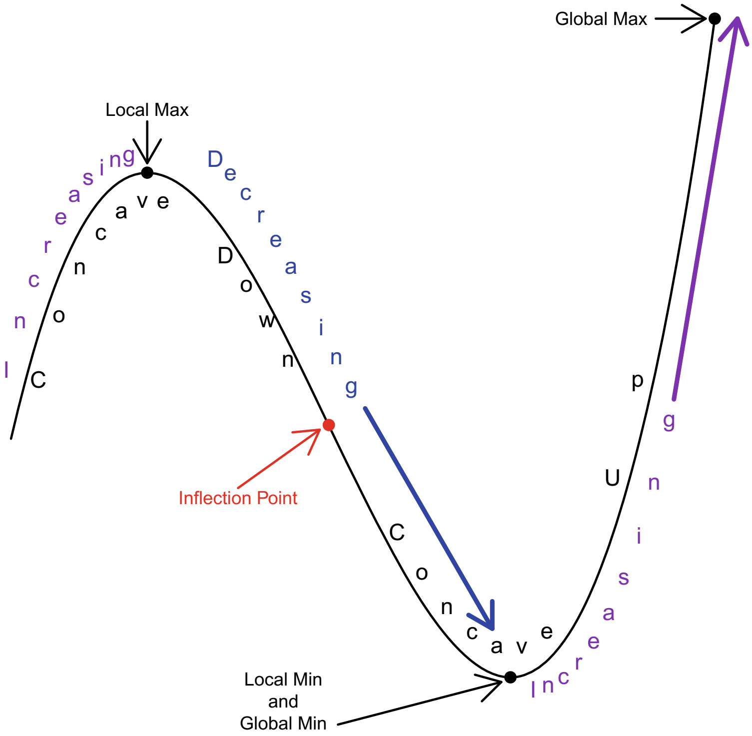


Fig. 2.1 Terminology used to describe characteristics of a graph.

- There is an inflection point at $(\pi, 0)$.
- There is a local max at $(\pi/2, 1)$. This is also the global max.
- There is a local min at $(3\pi/2, -1)$. This is also the global min.

Notice that in describing when the graph is increasing, decreasing, concave up, and concave down we provided a range based on the x-values of when it starts and then stops the behavior. On the other hand, for inflection point, local max, and local min, a point (both x and y value) was provided. □

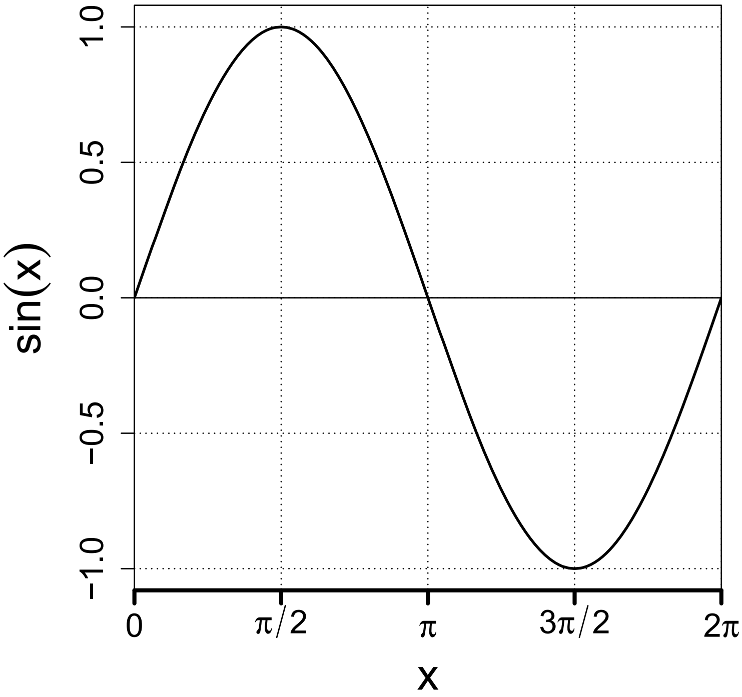
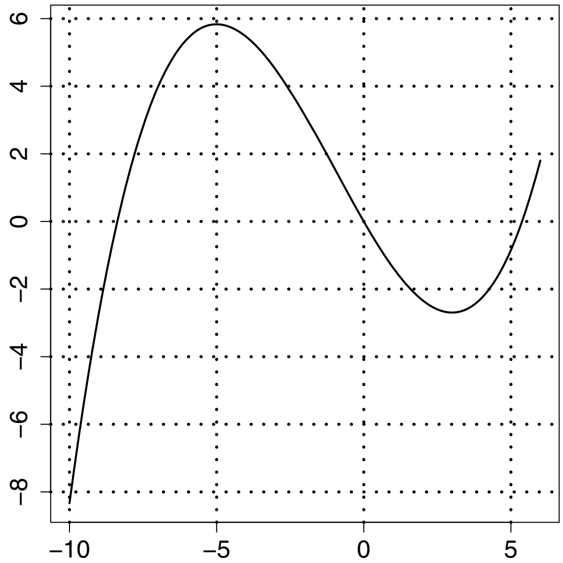


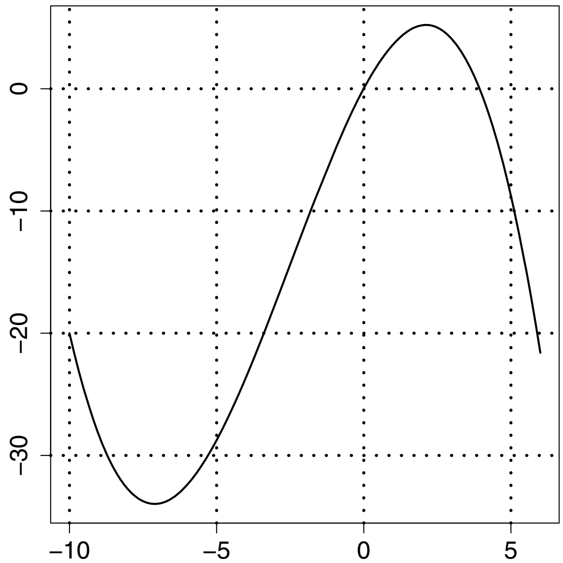
Fig. 2.2 Graph of $\sin(x)$.

2.1 Exercises

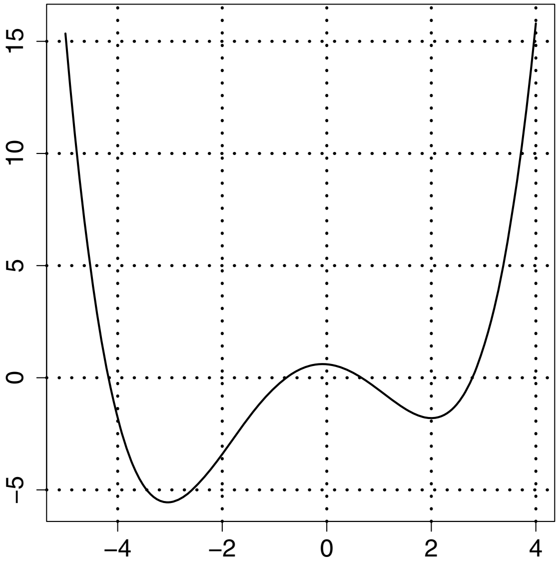
1. Describe the graph as done in example 2.1.



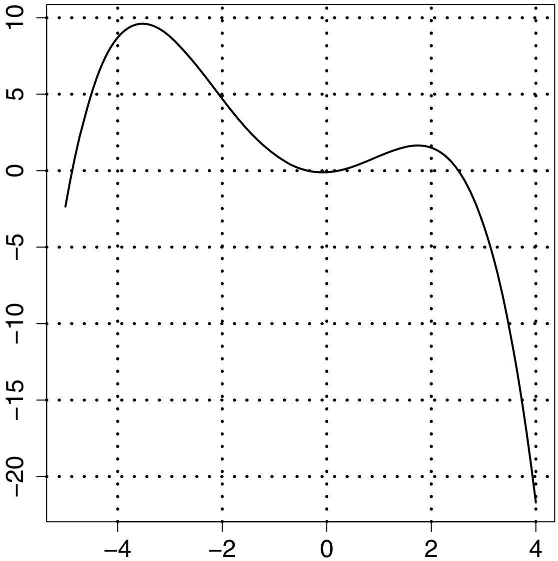
2. Describe the graph as done in example 2.1.



3. Describe the graph as done in example 2.1.



4. Describe the graph as done in example 2.1.



5. Provide a function that is always concave up and increasing. If you can't give an explicit function, then sketch a graph of such a function. Does the graph have any inflection points or local extrema (local max or local min)?
6. Provide a function that is always concave down and increasing. If you can't give an explicit function, then sketch a graph of such a function. Does the graph have any inflection points or local extrema (local max or local min)?
7. Provide a function that is always concave down and decreasing. If you can't give an explicit function, then sketch a graph of such a function. Does the graph have any inflection points or local extrema (local max or local min)?
8. Provide a function that is always concave up and decreasing. If you can't give an explicit function, then sketch a graph of such a function. Does the graph have any inflection points or local extrema (local max or local min)?
9. Provide an example of a graph that has exactly one inflection point but does not have any local extrema (local max or local min)? A sketch of the graph of the function is acceptable.
10. Provide an example of a graph that has exactly two inflection points and only one local extrema (local max or local min)? A sketch of the graph of the function is acceptable.
11. Use R to graph $x^3 - 10x^2 + 3x + 2$ on the interval $[-1, 10]$. Describe the graph as done in example 2.1. Your values will be estimated from reading the graph. Learning Tip: Try sketching the graph yourself before using R.
12. Use R to graph $2 - 3x + 12x^2 - x^3$ on the interval $[-4, 9]$. Describe the graph as done in example 2.1. Your values will be estimated from reading the graph. Learning Tip: Try sketching the graph yourself before using R.
13. Use R to graph $x^4 - 4x^3 - 2x^2 + 10x + 2$ on the interval $[-2, 4]$. Describe the graph as done in example 2.1. Your values will be estimated from reading the graph. Learning Tip: Try sketching the graph yourself before using R.
14. Use R to graph $\sin(\sqrt{x})$ on the interval $[0, 30]$. Describe the graph as done in example 2.1. Your values will be estimated from reading the graph. Learning Tip: Try sketching the graph yourself before using R.

15. Use R to graph $\sin(x^2)$ on the interval [0, 7.1]. How many local maximums and inflection points does the graph have on the interval? R tip: Add n=10000 to the **curve** function for a smoother graph. Learning Tip: Try sketching the graph yourself before using R.

16. Use R to graph $\sin(x^3)$ on the interval [0, 7.1]. How many local maximums and inflection points does the graph have on the interval? Compare this to the $\sin(x^2)$ function in the previous problem. R tip: Add n=10000 to the **curve** function for a smoother graph. Learning Tip: Try sketching the graph yourself before using R. If you are curious take a look at $\sin(e^x)$.

17. Use R to graph $x^2 + 25\sin(x)$ on the interval [0, 9]. How many local maximums, local minimums, and inflection points does the graph have on the interval? How many more of each do we get if the interval is expanded to [0, 20]? How would this change if the 25 is replaced with a 50 in front of $\sin(x)$? Explain why this happens?

18. Use R to graph $\sin(1/x)$ on the interval [0.05, 1]. How many local maximums and inflection points does the graph have on the interval? How many more of each do we get if the interval is expanded to [0.03, 1]? What happens if the interval is expanded to [0.01, 1]? R tip: Add n=10000 to the **curve** function for a smoother graph. Learning Tip: Try sketching the graph yourself before using R. If you are curious try other values on the left side of the interval, say 0.001, etc. Can you explain what is happening?

Chapter 3
The Function Gallery

Each graph below is a real world data set with a fitted curve. The fitted curves will be used throughout the text. Note that each graph caption provides the function of the fitted curve, as well as, the beginning and end of the data set where appropriate. These functions are here to generally demonstrate the idea of curve fitting as a way functions are created and specifically to see how well the curves fit the particular data. Examples that are current will be updated and posted on the companion web site https://sustainabilitymath.org/acr/ along with the data and R code for the curve fitting. The exercises at the end of this chapter will help you get used to using functions with units and context. **TIP:** Once you type these functions into an R script you should save the code so you don't have to type the function into R repeatedly. **WARNING:** If you copy and paste the functions in the captions from a pdf into R it may not work, especially if you are using a mac. The most common problems are the negative sign is incorrect, but it can be deleted and retyped.

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_3

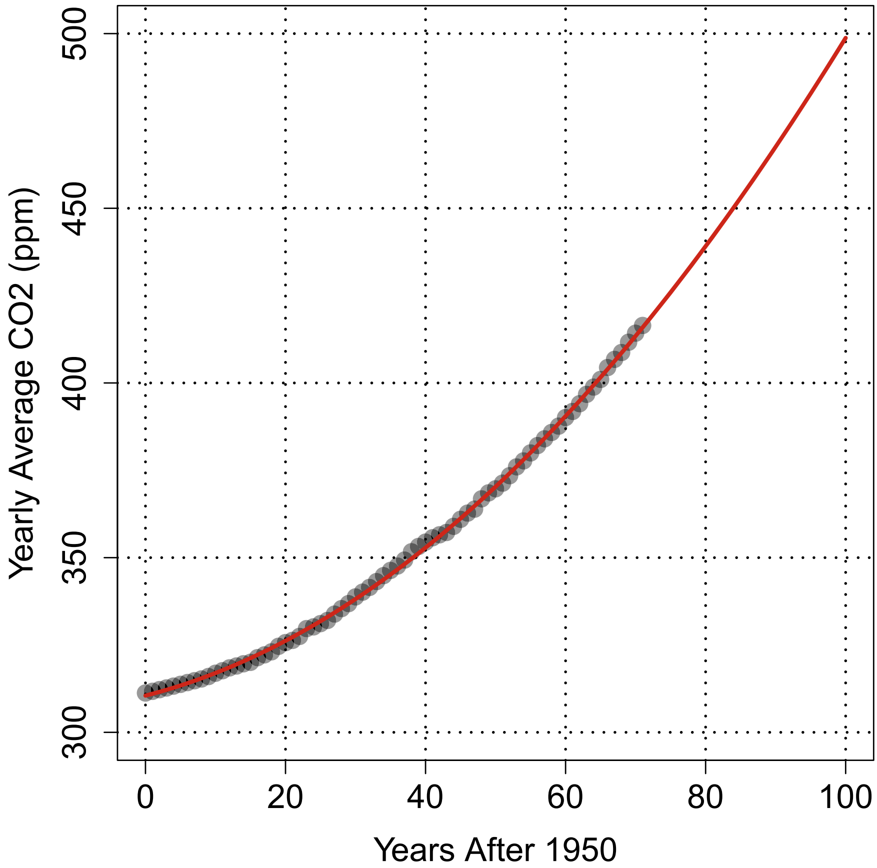


Fig. 3.1 $CO2(t) = 0.0137331303009963t^2 + 0.509146893268788t + 310.512512186844$ average yearly atmospheric CO2 at Mauna Loa in ppm t years after 1950. The data is from 1950 through 2021. [34]

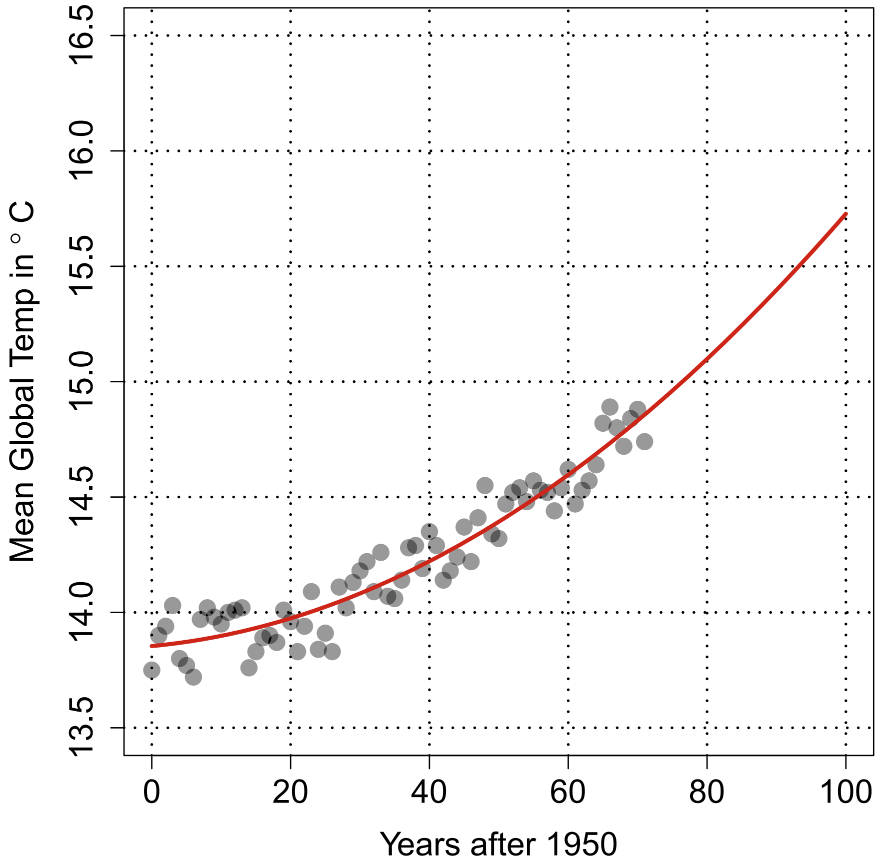


Fig. 3.2 $GTemp(t) = 0.000159118973994531t^2 + 0.00282082411785264t + 13.8542160310996$ global average temperature in degrees Celsius t years after 1950. The data is from 1950 through 2021. [15]

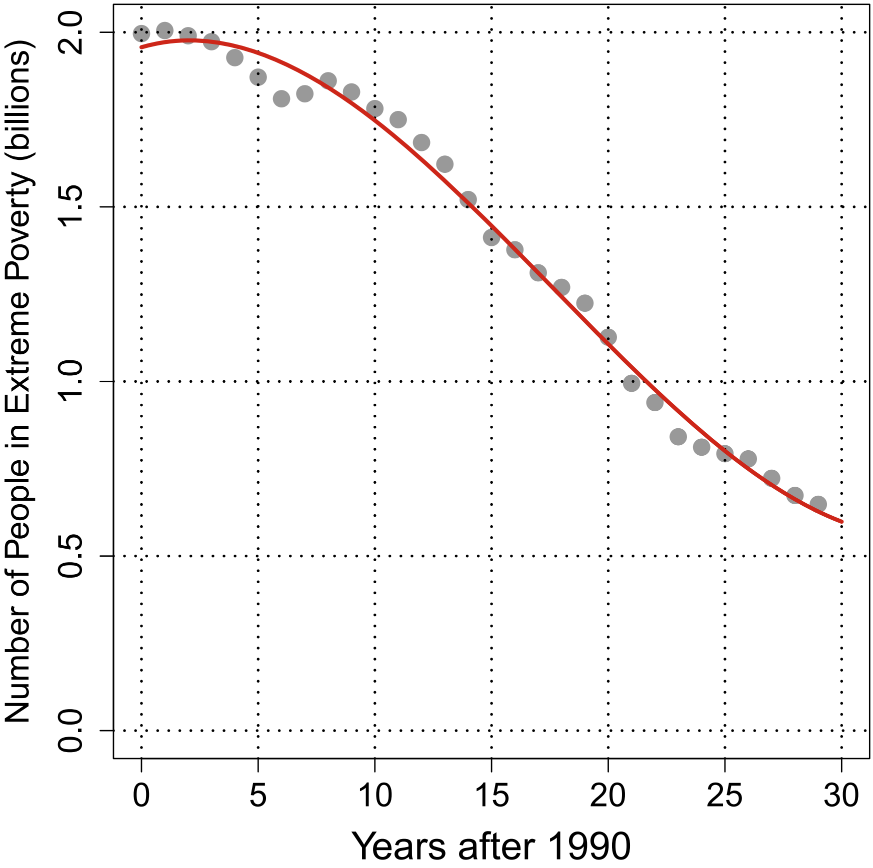


Fig. 3.3 $P(t) = 1.95718412183052 + 0.0192626524183241t + -0.00496217402961009t^2 + 0.000093676815174674 1t^3$ number of people in extreme poverty in billions, t years after 1990. Extreme poverty is defined as living below the International Poverty Line of $2.15 per day. This data is adjusted for inflation and for differences in the cost of living between countries. The data is from 1990 through 2019. Note that overall world population has been growing. [21]

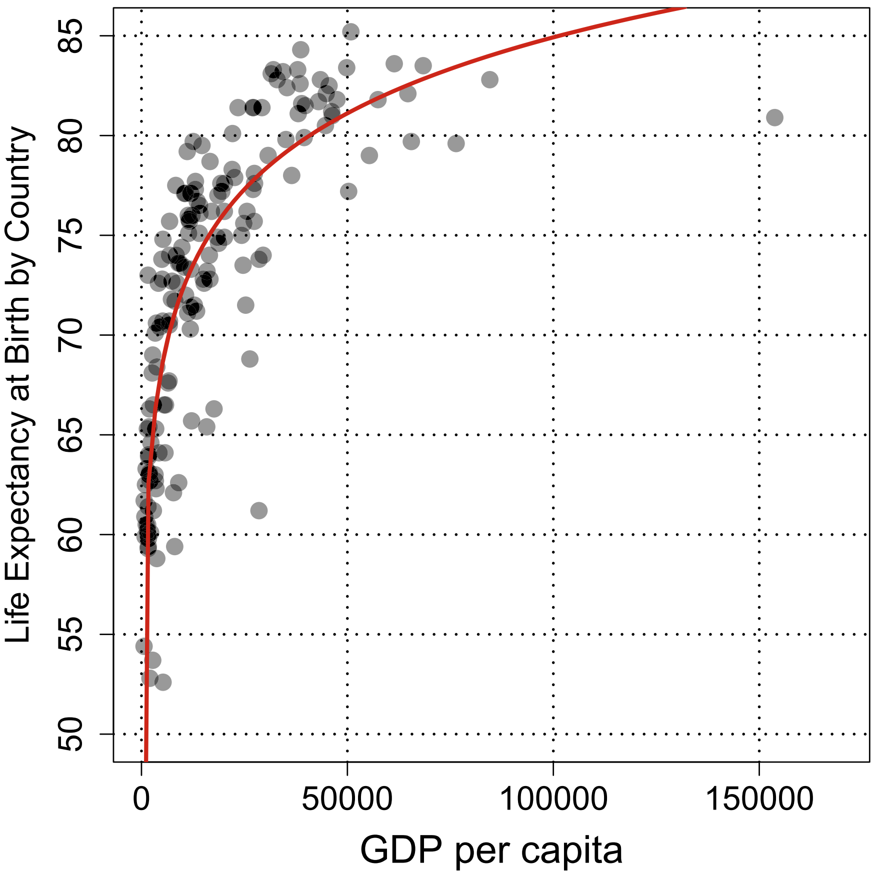


Fig. 3.4 $LeGdp(x) = 5.49220288353388 log(x + 1) + 21.6908132339644$ life expectancy at birth for 166 countries in 2018, where x is GDP per capita in 2011 international dollars, which corrects for inflation and cross-country price differences. The outlier is Qatar. Note $log(x + 1)$ is the natural log. [30]

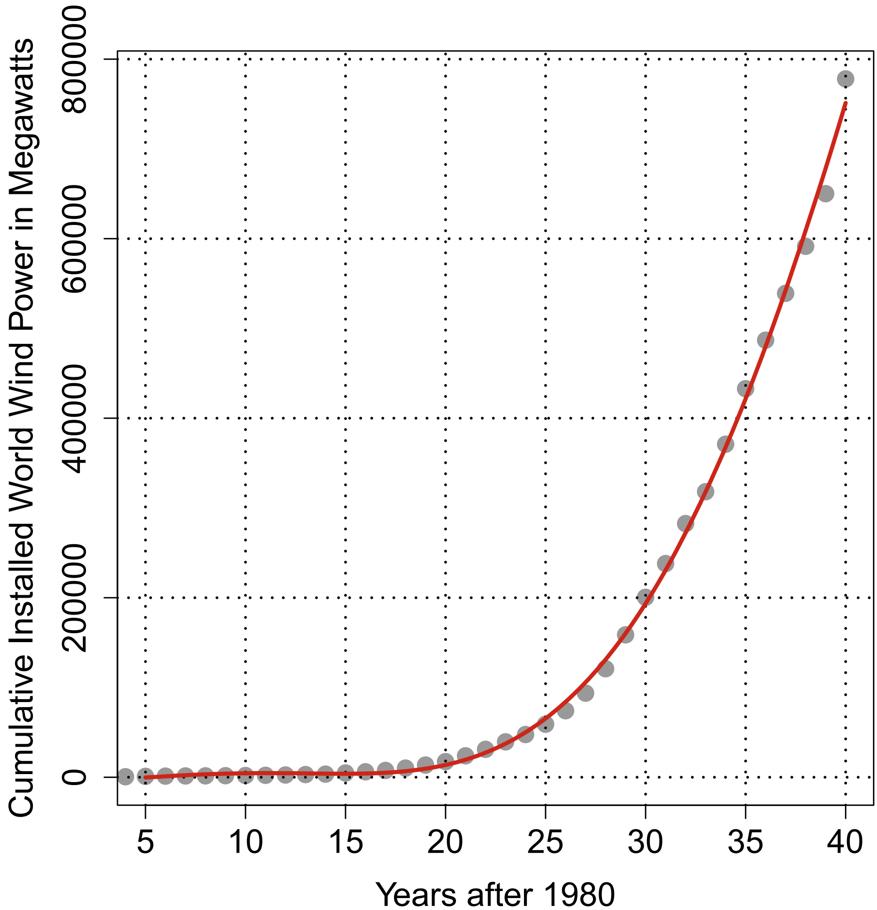


Fig. 3.5 $Wwind(t) = -0.0348424026738882t^5 + 3.66966512288051t^4 - 105.918177599909t^3 + 1169.39739731949t^2 - 4260.57451226362t + 2859.79180744146$ cumulative installed world wind power in megawatts t years after 1980. The data is from 1980 through 2020. [24]

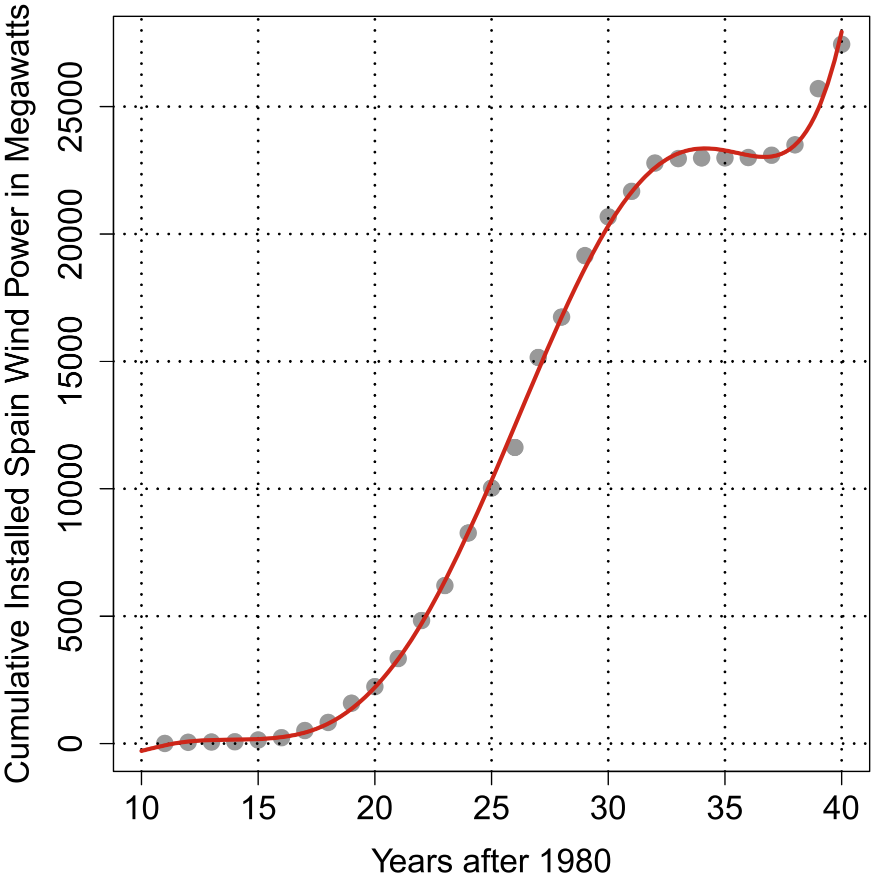


Fig. 3.6 $Swind(t) = 0.00161675544440024t^6 + -0.207341202671209t^5 + 10.2471436839421t^4 + -247.214861506442t^3 + 3087.43087123947t^2 + -18940.4936525017t + 44234.0826986463$ cumulative installed Spain wind power in megawatts t years after 1980. The data is from 1992 through 2020. [24]

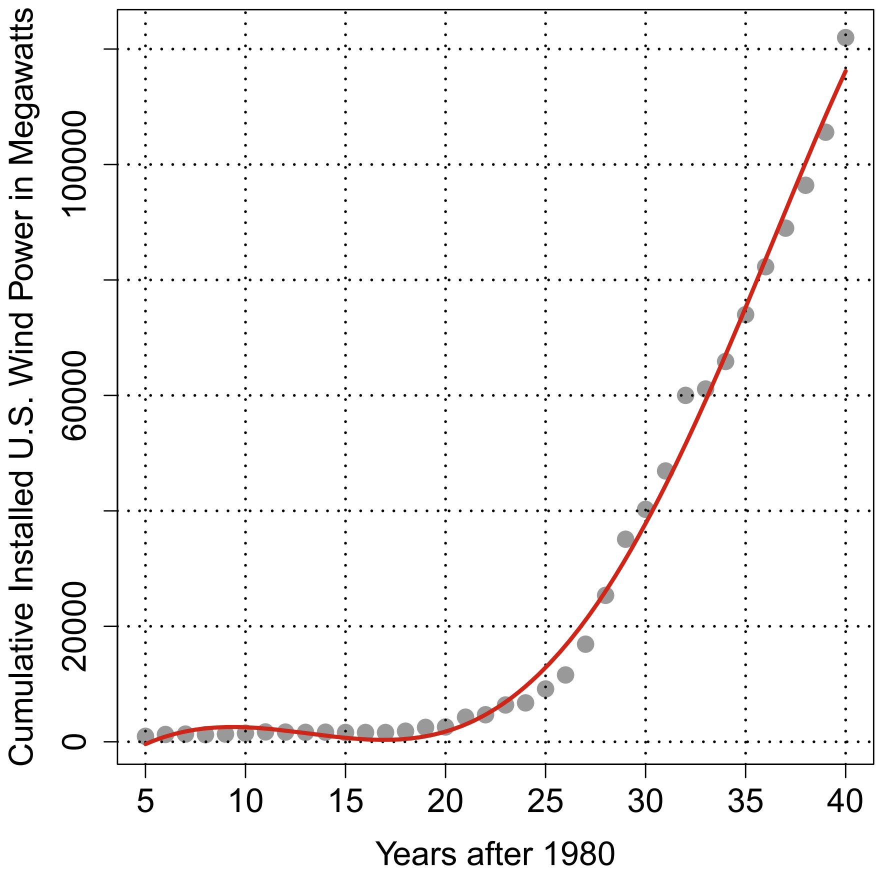


Fig. 3.7 $USwind(t) = -0.006067298467655 57t^5 + 0.4037964459704 04t^4 - 279.748222478446t^2 + 4142.93534737984t - 14361.298551044$ cumulative installed U.S. wind power in megawatts t years after 1980. The data is from 1985 through 2020. [24]

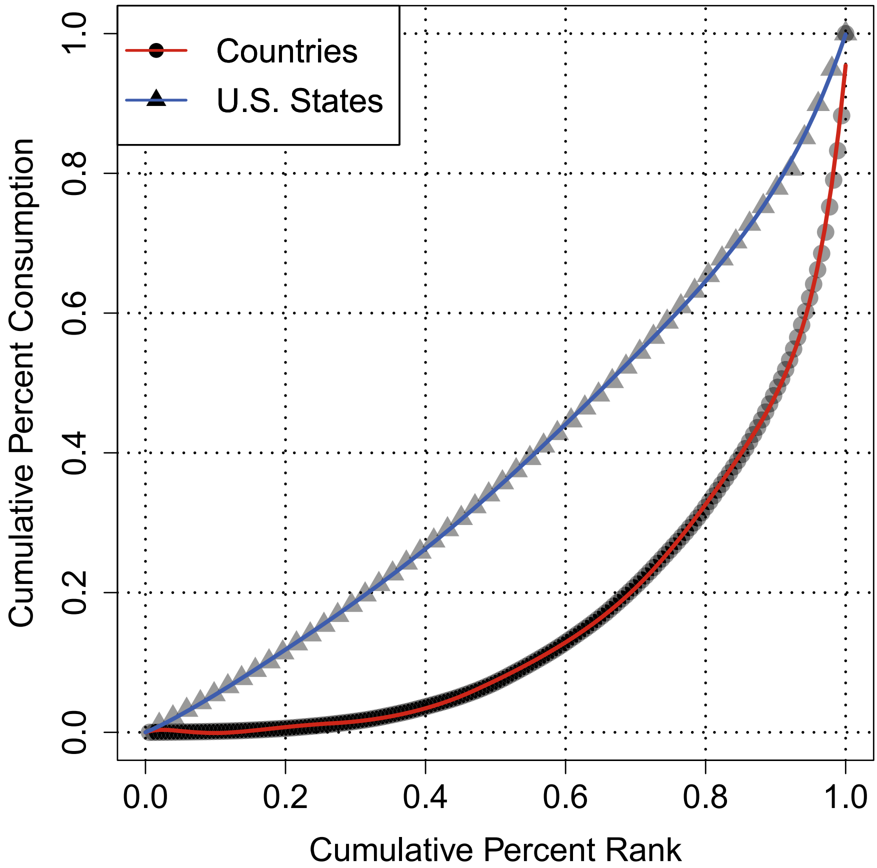


Fig. 3.8 The distribution of energy consumption in the U.S. (2014 data) and World (2011 data) can be modeled by $ECus(x) = 7.2038917391x^6 - 17.8551679663x^5 + 16.5816140612x^4 - 7.0654275059x^3 + 1.7077246274x^2 + 0.4260396828x$ and $ECw(x) = 678.0352163746x^9 - 2796.2519054480x^8 + 4802.0852334478x^7 - 4441.8091503689x^6 + 2389.4054597788x^5 - 751.8800491391x^4 + 132.3874503758x^3 - 11.3747211453x^2 + 0.3569478992x$. To interpret these functions consider the example: $ECus(0.63) = 0.47$ means that the bottom 63% of states in the U.S. have per capita energy use in the bottom 48% of all states. The function ECw(x) is similar and replaces countries for states. [13]

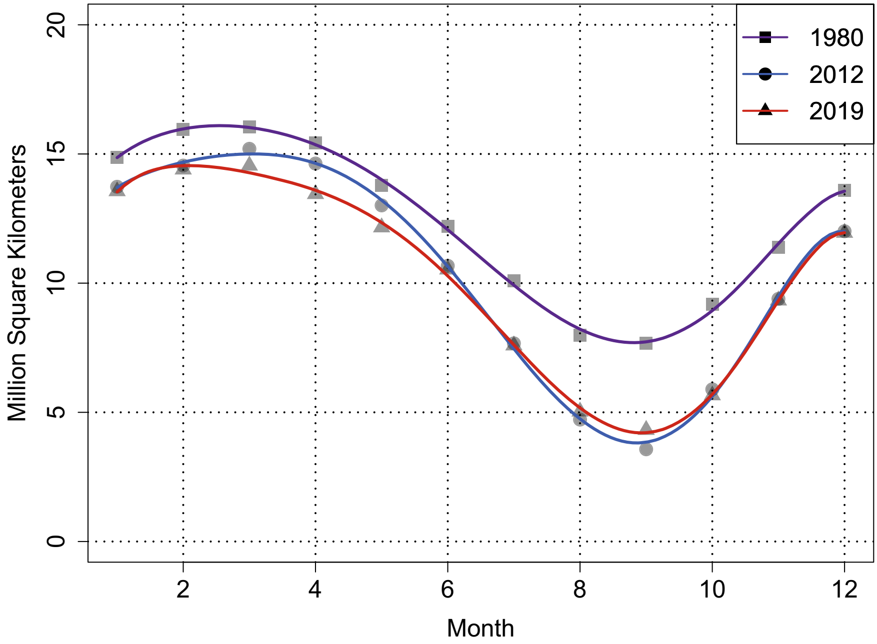


Fig. 3.9 Average monthly Arctic ice extent in million square kilometers with curve fits for 1980, 2012 (current record low year), and 2019. We interpret 1 to mean the middle of January, etc.

$AI\_1980(x) = 11.4612878787871 + 5.3194875879835x - 2.4766110734985x^2 + 0.6539677816055x^3 - 0.1073611634825x^4 + 0.0087432283811x^5 - 0.0002627314815x^6$

$AI\_2012(x) = 10.7484090909081 + 4.9595950483327x - 2.7846102155594x^2 + 0.9400642525363x^3 - 0.1762030228758x^4 + 0.0150792326546x^5 - 0.0004622140523x^6$

$AI\_2019(x) = 7.900757575757 + 9.987580105349x - 5.796657844468x^2 + 1.682711461332x^3 - 0.262366871962x^4 + 0.019775106838x^5 - 0.000558959695x^6$. [32]

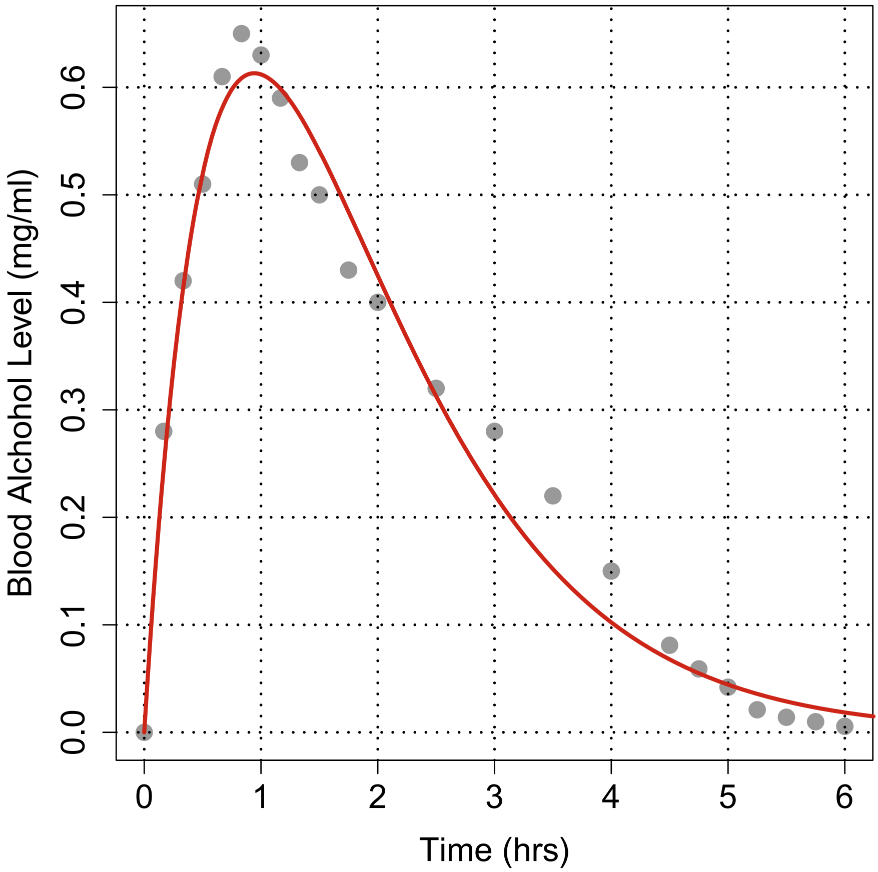


Fig. 3.10 Blood alcohol concentration of eight fasting adult males after consuming a 95% ethanol oral dose of 45ml. [38] The curve is known as a surge function and given by $S(x) = 1.76393642046205xe^{-1.05841662684339x}$. Note that driving impairment begins around 0.5 mg/ml and the legal limit in most states is 0.8mg/ml. The dosage here is about 2.5 standard shots of vodka and about one and a half 16oz of 6% beer.

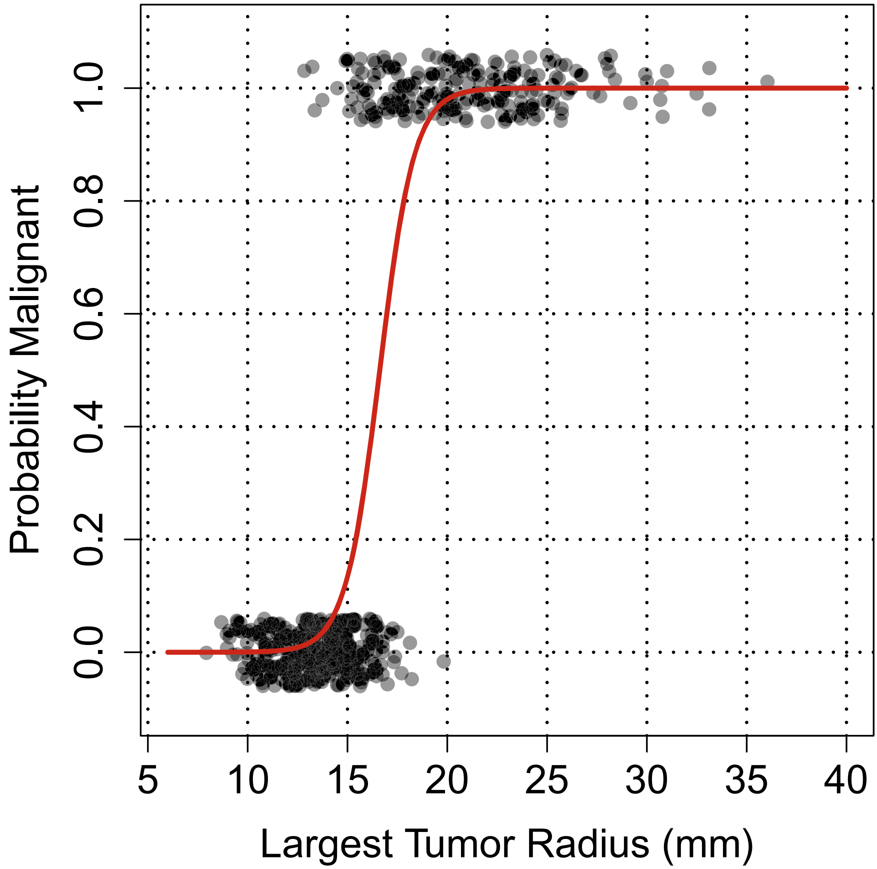


Fig. 3.11 The data is from the Diagnostic Wisconsin Breast Cancer Data where "features are computed from a digitized image of a fine needle aspirate (FNA) of a breast mass. They describe the characteristics of the cell nuclei present in the image." [40] The x-axis is the size of the largest radius of nuclei and the y-axis is the probability the breast mass is malignant. The curve is known as logistic regression and given by $L(x) = \dfrac{e^{-19.1744645062916+1.15383554750228x}}{1+e^{-19.1744645062916+1.15383554750228x}}$.

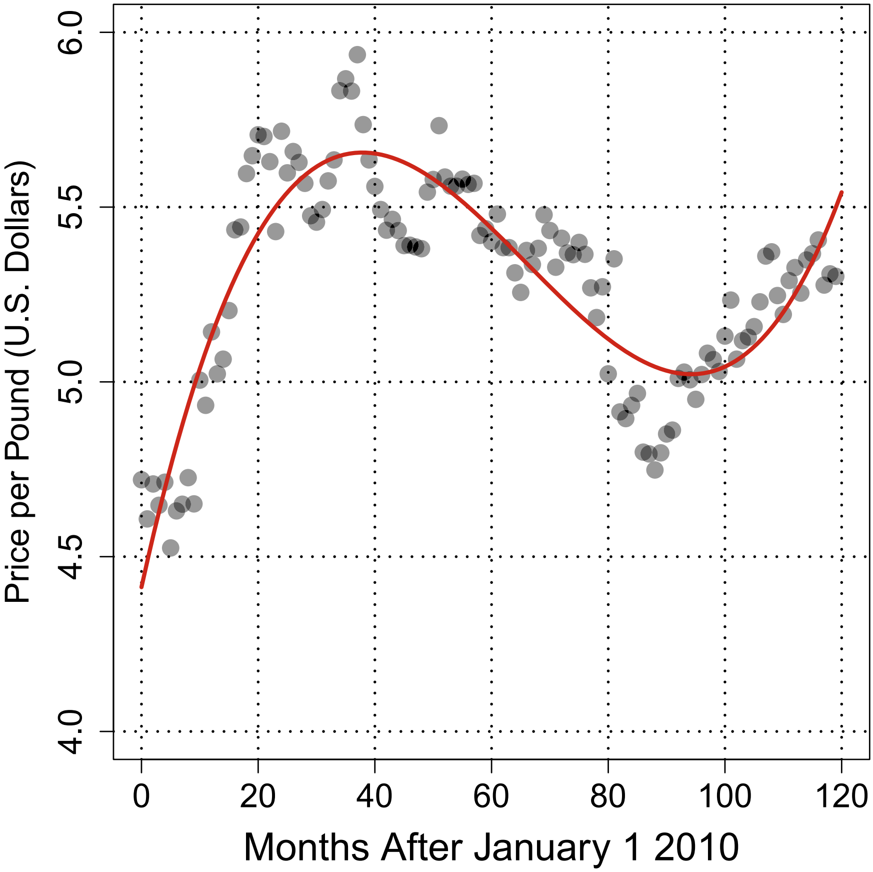


Fig. 3.12 The average price of a pound of cheddar cheese in U.S. cities can be modeled by $Ch(t) = 4.41281093468948 + 0.0758375702747065t - 0.0014041565369063t^2 + 0.00000708849730870583t^3$ dollars, where t is the number of months since January 1, 2010 ($t = 0$). The data is from January 1, 2010 to December 1, 2019. [3]

3.1 Exercises

1. Use the $CO2(t)$ function and graph in figure 3.1 to answer the following questions.
   a. What are the input and output units?
   b. The data the model is based on begins and ends in what years?
   c. Describe the graph using the definitions from Chapter 2.
   d. What is $CO2(63)$? Use your result in a sentence with proper context and units.
   e. What is the yearly average CO2 in 1995? Use your result in a sentence with proper context and units.
   f. According to the model when will CO2 reach 450 ppm? Use your result in a sentence with proper context and units.
2. Use the $Gtemp(t)$ function and graph in figure 3.2 to answer the following questions.
   a. What are the input and output units?
   b. The data the model is based on begins and ends in what years?
   c. Describe the graph using the definitions from Chapter 2.
   d. What is $Gtemp(48)$? Use your result in a sentence with proper context and units.
   e. What is the mean global temperature in 2020? Use your result in a sentence with proper context and units.
   f. According to the model when will the mean global temperature reach 15.25 °C? Use your result in a sentence with proper context and units.
3. Use the $P(t)$ function and graph in figure 3.3 to answer the following questions.
   a. What are the input and output units?
   b. Describe the graph using the definitions from Chapter 2.
   c. What meaning does the inflection point have in the context of the data?
   d. What was $P(11)$? Use your result in a sentence with proper context and units.
   e. What was the number of people in extreme poverty in 2007? Use your result in a sentence with proper context and units.
   f. According to the model in what year was the number of people in extreme poverty 900,000,000? Use your result in a sentence with proper context and units.
4. Use the $LeGdp(x)$ function and graph in figure 3.4 to answer the following questions.
   a. What are the input and output units?
   b. Should the model be used to extrapolate?
   c. Describe the graph using the definitions from Chapter 2.
   d. What is $LeGdp(48)$? Use your result in a sentence with proper context and units.

e. What is the expected life expectancy for a country with a GDP per capita of \$10,000? Use your result in a sentence with proper context and units.
f. According to the model what would be the GDP per capta of a country with a life expectancy of 85? Use your result in a sentence with proper context and units.

5. Use the $Wwind(t)$ function and graph in figure 3.5 to answer the following questions.

   a. What are the input and output units?
   b. The data the model is based on begins and ends in what years?
   c. Describe the graph using the definitions from Chapter 2.
   d. What is $Wwind(37)$? Use your result in a sentence with proper context and units.
   e. What is the cumulative installed world wind power in 2016? Use your result in a sentence with proper context and units.
   f. According to the model when will the cumulative installed world wind power reach 1,000,000 MW? Use your result in a sentence with proper context and units.

6. Use the $Swind(t)$ function and graph in figure 3.6 to answer the following questions.

   a. What are the input and output units?
   b. The data the model is based on begins and ends in what years?
   c. Describe the graph using the definitions from Chapter 2.
   d. What is $Swind(24)$? Use your result in a sentence with proper context and units.
   e. What is the cumulative installed Spain wind power in 2019? Use your result in a sentence with proper context and units.
   f. According to the model when will the cumulative installed Spain wind power reach 30,000 MW? Use your result in a sentence with proper context and units.

7. Use the $USwind(t)$ function and graph in figure 3.7 to answer the following questions.

   a. What are the input and output units?
   b. The data the model is based on begins and ends in what years?
   c. Describe the graph using the definitions from Chapter 2.
   d. What is $USwind(33)$? Use your result in a sentence with proper context and units.
   e. What is the cumulative installed U.S. wind power in 2017? Use your result in a sentence with proper context and units.
   f. According to the model when did the cumulative installed U.S. wind power reach 85,000 MW? Use your result in a sentence with proper context and units.

8. Use the $ECus(x)$ and $ECw(x)$ functions and graph in figure 3.8 to answer the following questions.
   a. What are the input and output units of both functions?
   b. Describe both graphs using the definitions from Chapter 2.
   c. Which of the two graphs has more concavity? Why?
   d. What is $ECus(0.9)$ and $ECw(0.9)$? Use your results in a sentence or two with proper context and units. In your response compare the results and their meaning relative to each other.
   e. What is the cumulative percent consumption for the U.S. and the world with a cumulative percent rank of 80%? Use your results in a sentence with proper context and units.
   f. According to each model when does the cumulative percent consumption reach 75%? Use your results in a sentence with proper context and units.
9. Use the three Arctic sea ice extent curves in 3.9 to answer the following questions.
   a. What are the input and output units of the functions?
   b. Describe all three graphs using the definitions from Chapter 2.
   c. Approximately when does the inflection point occur for each of the three years? Use your results in a sentence with proper context and units.
   d. What is $AI\_1980(9)$ and $AI\_2012(9)$? Use your results in a sentence with proper context and units.
   e. What is the Arctic sea ice extent for 2019 in July? Use your result in a sentence with proper context and units.
   f. In 2012 when did Arctic sea ice extent reach 8 million square kilometers? Use your result in a sentence with proper context and units.
10. Use the $S(x)$ function and graph in 3.10 to answer the following questions.
    a. What are the input and output units of the function?
    b. Describe the graph using the definitions from Chapter 2.
    c. Approximately when does the inflection point occur? Use your result in a sentence with proper context and units.
    d. What is $S(1.5)$? Use your result in a sentence with proper context and units.
    e. What is the blood alcohol level two and half hours after the 95% oral dose of ethanol is consumed? Use your result in a sentence with proper context and units.
    f. When is the blood alcohol level 0.35 mg/ml? Use your result in a sentence with proper context and units.
11. Use the $L(x)$ function and graph in 3.11 to answer the following questions.
    a. What are the input and output units of the function?
    b. Describe the graph using the definitions from Chapter 2.
    c. Approximately when does the inflection point occur? Use your result in a sentence with proper context and units.
    d. What is $L(17)$? Use your result in a sentence with proper context and units.

e. What is the probability of a malignant tumor when the largest tumor radius is 18 mm? Use your result in a sentence with proper context and units.
f. If the probability a tumor is malignant is 70% then what is the size of the largest tumor? Use your result in a sentence with proper context and units.

12. Use the $Ch(t)$ function and graph in figure 3.12 to answer the following questions.

a. What are the input and output units?
b. The data the model is based on begins and ends at what dates?
c. Describe the graph using the definitions from Chapter 2.
d. What is $Ch(42)$? Use your result in a sentence with proper context and units.
e. What is the average price of a pound of cheddar cheese in U.S. cities on December 1, 2017? Use your result in a sentence with proper context and units.
f. According to the model when did the average price of a pound of cheddar cheese in U.S. cities $5.00 per pound? Use your result in a sentence with proper context and units.

Part I
Change and the Derivative

Chapter 4
How Fast is CO2 Increasing?

In the Function Gallery, chapter 3, we were introduced to functions that modeled data. The exercises were focused on the units and finding x-values and y-values, but there is much more information in these graphs. In this chapter we begin to develop quantifying how fast a function changes. We will start with how fast a function changes over an (input) interval by using a secant line which effectively averages the rate of change over an input range. We then begin to develop the notion of how fast a function changes at a specific input value with the goal of quantifying this speed. The analogy here is the speed of a car at moment in time. Once we can quantify change at a point we will be able to explicitly calculate key values of a function that were introduced in Describing a Graph, chapter 2. We will use the model in figure 4.1, which measures the average amount of carbon dioxide, CO_2, in the atmosphere each year with the model constructed with data from 1950 through 2017. The function gallery, chapter 3, has an updated CO2 function with more recent data.

In order to understand how CO_2 levels have changed and are changing in figure 4.1 we ask two questions. First, on average how fast was CO2 increasing from 1950 to 2017 (last year of the data in figure 4.1)? Second, how fast are CO2 levels increasing in 2017? The first question is addressed by calculating the slope of the secant line from 1950 through 2017.

M-Box 4.1: Slope of the Secant Line I

The slope of the secant line from $(a, f(a))$ to $(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a} \tag{4.1}$$

This is also the average rate of change of the function from $x = a$ to $x = b$. The units are the output units of $f(x)$ divided by the input units of $f(x)$.

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_4

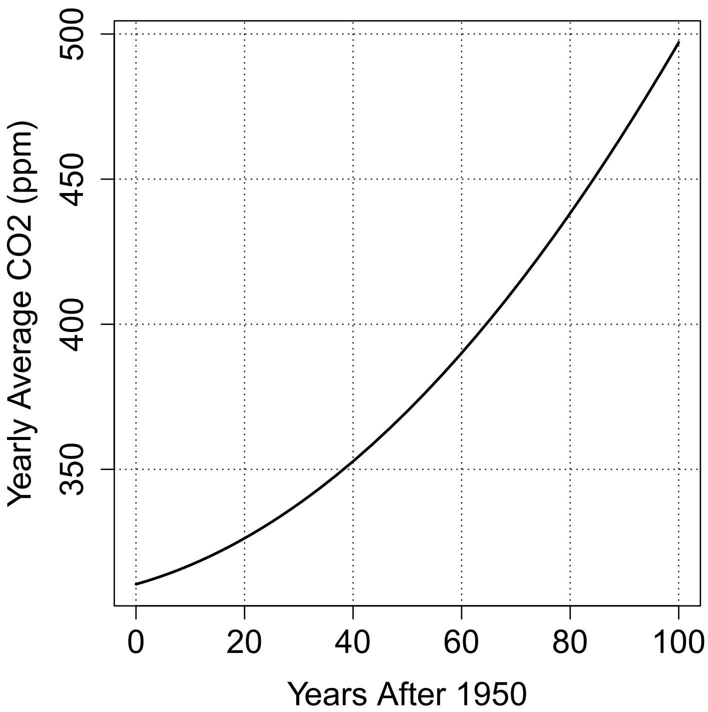


Fig. 4.1 $CO2(t) = 0.0134594696825t^2 + 0.520632601929t + 310.423363171$ average yearly atmospheric CO_2, carbon dioxide, at Mauna Loa in ppm t years after 1950. The data used for this function is from 1950 through 2017. An updated function with the data can be found in the function gallery.

Example 4.1. *Suppose $R(x) = 100x - 4x^2$ is the revenue in dollars from selling x widgets. Calculate and interpret the slope of the secant line from $x = 5$ to $x = 10$. NOTE: Throughout this text recognize when problems are made up so that they are easier to calculate to illustrate or practice a concept. In the case, we have a coefficient that is easy to work with as opposed to the coefficients in the CO2 function in figure 4.1.*

Solution. We first calculate the slope of the secant line noting that $R(10)=100(10) - 4(10^2)=\$600$ and $R(5) = 100(5) - 4(5^2) = \$400$:

$$\begin{aligned}\frac{R(10)-R(5)}{10\text{ widgets}-5\text{ widgets}} &= \frac{\$600-\$400}{10\text{ widgets}-5\text{ widgets}}\\ &= \frac{\$200}{5\text{ widgets}}\\ &= 40\text{ dollars/widget.}\end{aligned}$$

In a sentence this result tells us that on average revenue increased by 40 dollars per widget when the number of widgets sold increased from 5 to 10 widgets. Note that in the calculations we listed the units throughout to see how we ended up with dollars per widget. We will typically not do this as it clutters the calculation. □

To calculate the slope of the secant line from 1950 to 2017 of the CO2 function we turn to R. It is not that we cannot do this by hand, but it would be real easy to make a mistake given the coefficients of the function. As another bonus, once the CO2 function is typed into R you can save it so you do not have to type it again.

Recall that the input variable is years after 1950 so that 67 is 2017. Note that we define the variables **a** and **b** instead of entering

(CO2(67)-CO2(0))/(67-0).

This allows us to change the values **a** and **b** in one place, which may not be that important at the moment, but it will be later and so we should get into this habit now. In putting the value 1.422417 into context we can say that *average yearly CO2 levels at the Mauna Loa site increased on average by 1.4224 ppm per year from 1950 to 2017*. Geometrically, the 1.4224 ppm per year from 1950 to 2017 is represented in figure 4.2 as the slope of the secant line from 1950 through 2107, which is the dashed line. In other words, if CO2 levels increased by the same amount from 1950 to 2017 it would look like the secant line.

Note that there is a real question of how many digits should be reported in a result. In this case, the data that created the CO2 function has five significant digits and so when reporting results we should round to five significant digits, but you cannot get that information from the function gallery. The issue of significant digits is important, but for the sake of this text the number of digits reported will often be enough so that answers can be compared and checked.

R Code 4.1: Calculating the Slope of a Secant Line

```
> CO2<-function(t){0.0134594696825104*t^2+
0.520632601928747*t+310.423363171355}
> a<-0
> b<-67
> (CO2(b)-CO2(a))/(b-a)
```

```
[1] 1.422417
```

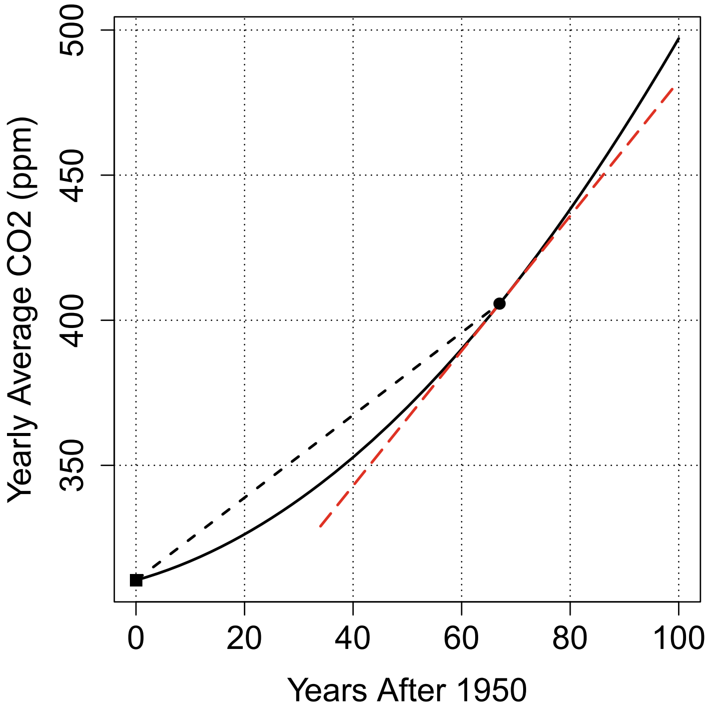


Fig. 4.2 The graph from figure 4.1 with the secant line from 1950, $x = 0$, to 2017, $x = 67$, (dashed black line) and tangent line at 2017, $x = 67$, (dashed red line).

The second question is more challenging as what do we mean by "How fast are CO2 levels increasing in 2017?" This question is addressing an instantaneous change or a change at a given input value. As an analogy, if our graph was the location of a car at time t the instantaneous change would be the car's velocity. How do we get this information from the graph? In viewing the graph, it appears that it is increasing faster as time increases. To capture how fast it is increasing at a particular point we turn to the tangent line. Loosely, a tangent line to a curve at a point, touches the curve at the point and follows the shape of the curve at that point. In figure 4.2 the long dashed line is tangent to the point $(67, CO2(67))$ on the curve. We will learn how to add tangent lines to the graph later. For now, the question is how to estimate the slope of the tangent line.

In figure 4.3 we "zoom" in on the **CO2** function so that the x-axis ranges from 66.99 to 67.01. With an x-axis range this small, $\Delta x = 0.02$, it appears that we are looking at a line and we cannot distinguish between the curve and the tangent line, or at least really close to a line as we cannot be certain. An estimate of the slope of tangent line can be found by simply calculating the slope of the secant line between the two endpoints of our "zoomed in" graph. We think of straddling the location where we want to estimate the slope of the tangent line. In this case, the endpoints can be represented as $(a-h, f(a-h))$ and $(a+h, f(a+h))$ where $a = 67$, the x-value of the point where the tangent line is drawn, and where $h = 0.01$ is the distance from a to each endpoint. For figure 4.3 we can estimate the slope of the tangent line a $a = 67$ by using the slope of the secant line calculation in R Code box 4.2 or

(CO2(a+h)-CO2(a-h))/(a+h-(a-h))

to get 2.324202 (Tip: Parenthesis are important in R. Pay close attention to them.) What does 2.324202 mean? *In 2017 average annual CO2 levels at the Mauna Loa site were increasing at an estimated rate of 2.3242 ppm per year.* Since the function is increasing faster each year it is not surprising that rate of increase in 2017 is larger than the average change, 1.4224 ppm per year, from 1950 to 2017.

R Code 4.2: Calculating the Slope of a Secant Line with a and h

```
> CO2<-function(t){0.0134594696825104*t^2+
0.520632601928747*t+310.423363171355}
> a<-67
> h<-0.01
> (CO2(a+h)-CO2(a-h))/(a+h-(a-h))
```

```
[1] 2.324202
```

We should pause for a moment and emphasize that 2.324202 ppm per year is an estimate of the instantaneous change of CO2 in 2017 or the slope of the tangent line at $(67, CO2(67))$. We have no sense of the accuracy of our estimate and we will deal with this issue shortly.

In summary, what have we learned about atmospheric CO2? Based on the model, average yearly CO2 levels at the Mauna Loa site increased on average by 1.4224 ppm per year from 1950 to 2017 and we estimate that in 2017 CO2 was increasing at a rate of 2.3242 ppm per year.

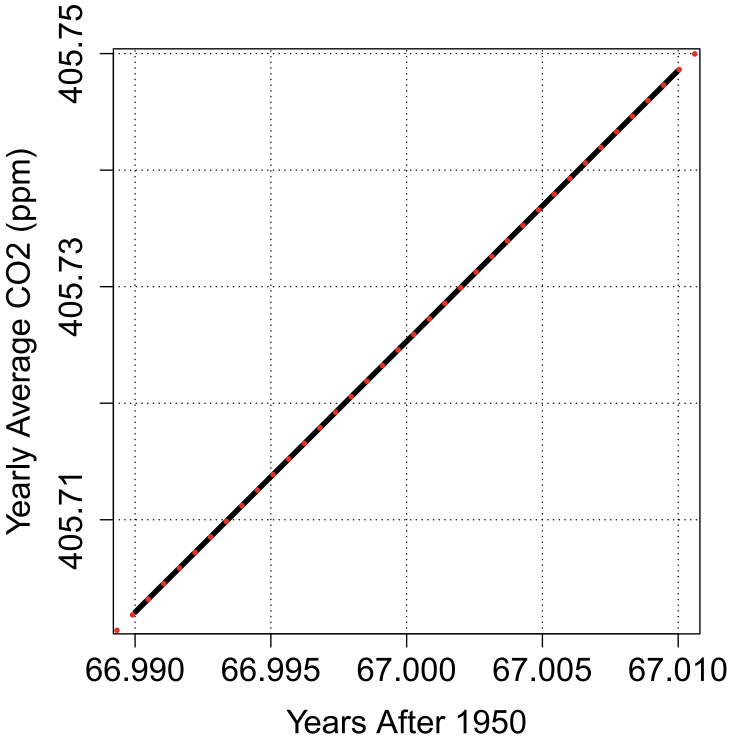


Fig. 4.3 The graph from figure 4.1 with an x-axis from 66.99 to 67.01 and tangent line at 2017, $a = 67$, (dashed red line).

4.1 Exercises

Note: For problems that refer to the function gallery, Chapter 3, the function and the beginning and end of the data set are given in the captions of the graphs.

1. Suppose $R(x) = 50x - 2x^2$ is the revenue in dollars from selling x widgets. Calculate and interpret the slope of the secant line from $x = 2$ to $x = 8$. Report your result in a sentence with proper context and units.
2. Suppose $R(x) = 40x - x^2$ is the revenue in dollars from selling x widgets. Calculate and interpret the slope of the secant line from $x = 5$ to $x = 20$. Report your result in a sentence with proper context and units.
3. Suppose $C(x) = x^2 - 250x + 20000$ is the cost in dollars from producing x widgets. Calculate and interpret the slope of the secant line from $x = 5$ to $x = 10$. Report your result in a sentence with proper context and units.
4. Suppose $C(x) = 2x^2 - 200x + 10000$ is the cost in dollars from producing x widgets. Calculate and interpret the slope of the secant line from $x = 10$ to $x = 20$. Report your result in a sentence with proper context and units.
5. Explain each line of the R code below.

```
> f<-function(x){x^2+5*x-10}
> curve(f,-10,10,lwd=2)
> grid(NULL,NULL,col="black")
> f(5)
[1] 40
```

6. Explain each line of the R code below.

```
> f<-function(x){sin(x) - cos(x)}
> curve(f,0,2*pi,lwd=2)
> grid(NULL,NULL,col="black")
> f(pi)
[1] 1
```

7. Explain each line of the R code below.

```
> f<-function(x){sin(x)}
> a<- 0
> b<- pi/2
> (f(b)-f(a))/(b-a)
[1] 0.6366197724
```

8. Explain each line of the R code below.

```
> f<-function(x){x^2-x+1}
> a<- 2
```

```
> b<- 6
> (f(b)-f(a))/(b-a)
[1] 7
```

9. Find the average rate of change of $f(x) = 4x^5 + 15x^4 - 140x^3 - 430x^2 + 1200x + 1000$ from $x = -6$ to $x = 3$. Does the average rate of change here give a good representation of how the function changed throughout the interval from $x = -6$ to $x = 3$?
10. Find the average rate of change of $f(x) = 4x^5 + 15x^4 - 140x^3 - 430x^2 + 1200x + 1000$ from $x = 3$ to $x = 6$. Does the average rate of change here give a good representation of how the function changed throughout the interval from $x = 3$ to $x = 6$?
11. Find the average rate of change of $g(x) = e^x + (2.5)^x \sin(2\pi x) - 10$ from $x = 1$ to $x = 4$. Recall the function is input in R as `exp(x)+(2.5)^x*sin(2*pi*x)-10`.
12. Using the CO2 function from the function gallery (not from this chapter), find the average rate of change from the beginning to the end of the data set. Report your result in a sentence in context with units. How does your result here compare to the result in this chapter? Conjecture as to why the results differ.
13. Using the global temperature function from the function gallery, find the average rate of change from the beginning to the end of the data set. Report your result in a sentence in context with units.
14. Using the cumulative installed world wind power function from the function gallery, find the average rate of change from the beginning to the end of the data set. Report your result in a sentence in context with units.
15. Using the cumulative installed U.S. wind power function from the function gallery, find the average rate of change from the beginning to the end of the data set. Report your result in a sentence in context with units.
16. Using the cumulative installed Spain wind power function from the function gallery, find the average rate of change from the beginning to the end of the data set. Report your result in a sentence in context with units.
17. The data below is from the Bermuda Atlantic Time-series Study and is the depth and temperature of ocean water 80 kilometers southeast of Bermuda taken in July 1996. [8] What is the average rate of change of the water temperature from a depth of 3.8 meters to 161.2 meters? What is the average rate of change of the water temperature from a depth of 161.2 meters to 500.5 meters? Report your result in a sentence or two in context with units.

Depth (m)	3.8	58.8	101.0	161.2	250.6	399.8	500.5	700.1	1398.0	3000.3	4001.2
Temp (F)	79.1	68.7	66.0	65.1	64.7	63.8	61.5	54.8	40.6	37.0	36.0

18. The data below is the end of month elevation of Lake Mead. Lake Mead is the reservoir created by Hoover Dam and supplies water to about 20 million people and farmland. [28] What is the average rate of change of the height of Lake

Mead from Jan 2022 to June 2022? Report your result in a sentence in context with units.

Month/Year	Nov 21	Dec 21	Jan 22	Feb 22	Mar 22	Apr 22	May 22	Jun 22
Height (ft)	1064.97	1066.39	1067.09	1066.78	1061.49	1054.69	1047.69	1043.02

19. The data below is the percent of California in drought at the beginning of October from the U.S. Drought Monitor. [16] What is the average rate of change of the percent of California in drought from 2019 to 2021? Report your result in a sentence in context with units.

Year	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Percent in Drought	88.05	97.37	100	99.86	100	22.12	87.82	4.71	84.65	100

20. The table below converts the speed at which someone is running in mph to the time it takes to complete a mile in minutes. What is the average rate of change of the time to complete a mile when the running speed increases from 4 mph to 6 mph? Report your result in a sentence in context with units.

Running Speed (mph)	3	4	5	6	7	8	9	10
Time to Complete a mile (min)	20.00	15.00	12.00	10.00	8.57	7.5	6.67	6.00

21. Using the table of data in problem 17 estimate the slope of the tangent line at a depth of 58.8 feet and 700.1 feet. Report your results in a sentence in context with units. Do the estimates seem reasonable?
22. Using the table of data in problem 18 estimate the slope of the tangent line in Dec 21 and Apr 22. Report your results in a sentence in context with units. Do the estimates seem reasonable?
23. Using the table of data in problem 19 estimate the slope of the tangent line in 2013 and 2018. Report your results in a sentence in context with units. Do the estimates seem reasonable?
24. Using the table of data in problem 20 estimate the slope of the tangent line at 4mph and 8mph. Report your results in a sentence in context with units. Do the estimates seem reasonable?
25. Estimate the instantaneous rate of change of $f(x) = 4x^5 + 15x^4 - 140x^3 - 430x^2 + 1200x + 1000$ at $a = 3$ with $h = 0.1$ and $h = 0.0001$.
26. Estimate the instantaneous rate of change of $f(x) = 4x^5 + 15x^4 - 140x^3 - 430x^2 + 1200x + 1000$ at $a = 0.75$ with $h = 0.1$ and $h = 0.0001$. Do you think you have a good estimate? Why or why not?
27. Estimate the instantaneous rate of change $g(x) = e^x + (2.5)^x \sin(2\pi x) - 10$ at $a = 4.3$ with $h = 0.1$ and $h = 0.001$. Do you think you have a good estimate? Why or why not?
28. Using the CO2 function from the function gallery (not from this chapter), estimate the instantaneous rate of change for the last year in the data set with h=0.01. Report your result in a sentence in context with units. How does your result here

compare to the result in this chapter? Conjecture as to why the results differ. Hint: R box 4.2.

29. Using the global temperature function from the function gallery, estimate the instantaneous rate of change for the last year in the data set with h=0.001 and h=0.0001. Report your result in a sentence in context with units.
30. Using the cumulative installed world wind power function from the function gallery, estimate the instantaneous rate of change for the last year in the data set with h=0.01 and h=0.001. Report your result in a sentence in context with units.
31. Using the cumulative installed U.S. wind power function from the function gallery, estimate the instantaneous rate of change for the last year in the data set with h=0.1 and h=0.01. Report your result in a sentence in context with units.
32. Using the cumulative installed Spain wind power function from the function gallery, estimate the instantaneous rate of change for the last year in the data set with h=0.001 and h=0.0001. Report your result in a sentence in context with units.

Chapter 5
The Idea of the Derivative

In Chapter 4 we estimated that in 2017 CO_2 levels were increasing at a rate of 2.32 ppm per year. This is an instantaneous measure of change or in a sense the speed of the function at 2017. At this point the 2.32 ppm is an estimate of the slope of the tangent line. Before we turn to calculating tangent line slopes explicitly we want to introduce terminology and notation for the slope of the tangent line and then develop an understanding of the information provided by the slope of the tangent line.

In M-Box 5.1 we introduced notation for the derivative or the instantaneous rate of change at a point. In the context of CO2 we write $CO2'(67) \approx 2.32$ ppm per year. The prime notation is used to represent the instantaneous rate of change of a specific original function, in this case $CO2(t)$. Note the relationship between the function and the derivative of the function. We use $f(a)$ for the value of the function at $x = a$ and $f'(a)$ for the derivative or the instantaneous rate of change of the function $f(x)$ at $x = a$.

M-Box 5.1: Slope of the Tangent Line Notation - The Derivative

We use $f'(a)$ to represent the **slope of the tangent line** of the function $f(x)$ **at** $x = a$. We also refer to $f'(a)$ as the **derivative** or the **instantaneous rate of change** of the function $f(x)$ at $x = a$. **UNITS**: Output units of $f(x)$ divided by input units of $f(x)$.

Example 5.1. *A celeriac (a root vegetable - look it up) is dropped from the top of a building. Let $f(t)$ be the distance in feet of the celeriac from the top of the building at time t seconds. Suppose that $f'(3) = 96.52$. Report this result in a sentence in context with units.*

Solution. Three seconds after being dropped the celeriac is falling at a rate (or speed) of 96.52 ft/sec. □

Example 5.2. *Boiling water is poured into a cup to make green tea. Let $G(t)$ represent the temperature in $^\circ F$ of the green tea at time t minutes since the hot water was poured.*

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_5

1. *Is $G(3)$ or $G(6)$ bigger?*
2. *What does $G'(3)$ represent and should it be positive or negative.*
3. *Should $G'(6)$ be larger or smaller than $G'(3)$?*
4. *Sketch a graph of $G(t)$ from $t = 0$ to $t = 10$ and represent $G'(3)$ and $G'(6)$ on the graph.*

Solution. We will assume the water was poured at 212°F and that the room temperature is 70°F. The cup of tea will cool off as time passes and so $G(3)$ should be larger, a higher temperature, than $G(6)$. Here $G'(3)$ has units °F/minute and is the rate at which the green tea is cooling at 3 minutes after the water was poured. Note that $G'(3)$ should be negative since the green tea is cooling. The tea should be cooling faster at $t = 3$ minutes than at $t = 6$ minutes since the difference between the room temperature and green tea temperature is greater. This means that $G'(3)$ is a larger negative number than $G'(6)$ so that $G'(3) < G'(6)$ (i.e., $-10 < -3$). A possible graph is given in figure 5.1 with key feature that the graph is concave up and intersects $(0, 212)$ based on our stated assumptions.

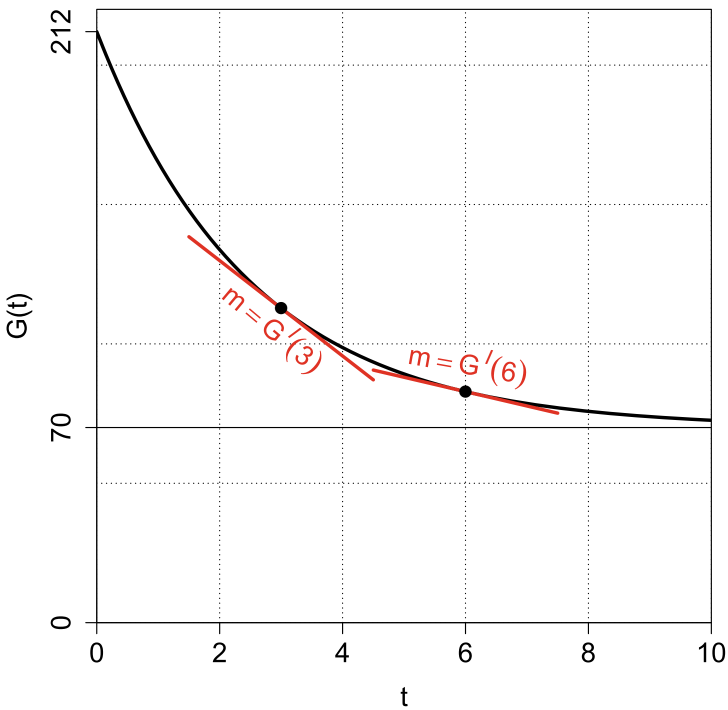


Fig. 5.1 Possible graph of a cup of green tea that starts at boiling and placed in a 70° F room.

We use a variety of synonyms to refer to the derivative based on context and list these in M-Box 5.2. This is so important that it is repeated as M-Box 10.3 later with additional notation. Note that the words we use are typically contextual. For example, derivative is more an algebraic context while slope of the tangent line is geometric.

M-Box 5.2: Derivative Synonyms

The following are synonyms: Slope of the Tangent Line, Derivative, Rate of Change, and Slope of a Function.

5.1 Exercises

1. Boiling water is poured into a cup to make tea. Let $f(t)$ represent the temperature of the water in °F t minutes after being poured. Suppose $f'(10) = -3.5$. Report this result in a sentence in context with units.
2. A pot of water is put on a hot stove to be boiled to cook pasta. Let $f(t)$ represent the temperature of the water in °F t minutes after the pot has been placed on the hot stove. Suppose $f'(4) = 5.5$. Report this result in a sentence in context with units.
3. A 1kg cylinder is rolled down a 30-degree angle ramp. Let $s(t)$ represent the distance traveled in meters after t seconds. Suppose $s'(2) = 6.538$. Report this result in a sentence in context with units.
4. A ball is thrown up in the air at a speed of 17.88 meters per second. Let $h(t)$ be the height of the ball after t seconds. Suppose $h'(1.5) = 3.18$. Report this result in a sentence in context with units.
5. Let $C(x)$ be the cost in thousands of dollars to produce x items for a business. Suppose $C'(58) = 8.25$. Report this result in a sentence in context with units.
6. Let $d(p)$ be the demand in millions of people for a steaming service at a monthly price of p dollars. Suppose $d'(15) = -1.6$. Report this result in a sentence in context with units.
7. A river starts at the top of a mountain. Let $D(x)$ be the flow rate of the river in cubic feet per second a distance x miles from the start of the river. Suppose $D'(10) = 1000$. Report this result in a sentence in context with units.
8. A chemical is leaking from a pipe into soil. Let $C(x)$ be the amount of this chemical in the soil, in ppm, x feet from the leaking pipe. Suppose $C'(1.5) = -0.5$. Report this result in a sentence in context with units.
9. Let $T(e)$ be the temperature, in °F, at a fixed time of the day based on the elevation e in feet above sea level. Suppose $T'(3000) = -0.0035$. Report this result in a sentence in context with units.
10. Create an exercise similar to the ones above and answer it.

11. A rock is dropped from 2000 ft. Let $s(t)$ represent the distance the rock has traveled at time t seconds. What does $s(2)$ and $s'(2)$ represent (include units). Which should be larger $s(2)$ or $s(4)$? Which should be larger $s'(2)$ or $s'(4)$? Assume the rock hits the ground in 11 sec. Sketch a graph of $s(t)$ from 0 to 11 seconds. Represent $s'(2)$ and $s'(4)$ on the graph.
12. A ripe watermelon is launched straight up from the ground (Why a ripe watermelon? Because it will make a big splat when it hits the ground.). Let $s(t)$ represent the height in feet of the watermelon at time t seconds. What does $s(1)$ and $s'(1)$ represent (include units). Which should be larger $s(1)$ or $s(3)$? Which should be larger $s'(1)$ or $s'(3)$? Assume the watermelon reaches its maximum height at 4 seconds and hits the ground at 8 seconds. Sketch a graph of $s(t)$ from 0 to 8 seconds. Represent $s'(1)$ and $s'(3)$ on the graph.
13. The average car can go from 0 to 60 mph in about 8 seconds. An average car starts from stop and accelerates as fast as possible. Let $s(t)$ be the distance the car has traveled since it started accelerating where time t is in seconds. What does $s(3)$ and $s'(3)$ represent (include units). Which should be larger $s(3)$ or $s(5)$? Which should be larger $s'(3)$ or $s'(5)$? What should $s'(8)$ equal? Sketch a graph of $s(t)$ from 0 to 8 seconds. Represent $s'(3)$ and $s'(5)$ and $s'(8)$ on the graph.
14. In dry conditions the average car can go from 60 mph to 0 in about 7 seconds and will travel about 130 ft. An average car traveling at 60 mph hits the breaks. Let $s(t)$ be the distance the car has traveled since it started decelerating where time t is in seconds. What does $s(1)$ and $s'(1)$ represent (include units). Which should be larger $s(1)$ or $s(4)$? Which should be larger $s'(1)$ or $s'(4)$? Sketch a graph of $s(t)$ from 0 to 7 seconds. Represent $s'(1)$ and $s'(4)$ on the graph.
15. A backpacker is hiking up a mountain that continues to get steeper until they reach the top. Let $B(t)$ represent the distance, in miles, the backpacker has traveled at time t in hours. What does $B(2)$ and $B'(2)$ represent (include units)? Which should be larger $B(2)$ or $B(5)$? Which should be larger $B'(2)$ or $B'(5)$? Sketch a graph of $B(t)$ from 0 to 6 hours. Assume the backpacker reaches the summit at 6 hours. Represent $B'(2)$ and $B'(5)$ on your graph.
16. A backpacker is hiking along a level trail and does not get tired during the hike. Let $B(t)$ represent the distance, in miles, the backpacker has traveled at time t in hours. What does $B(1)$ and $B'(1)$ represent (include units)? Which should be larger $B(1)$ or $B(3)$? Which should be larger $B'(1)$ or $B'(3)$? Sketch a graph of $B(t)$ from 0 to 4 hours. Assume the backpacker reaches the end of the trail at 4 hours. Represent $B'(1)$ and $B'(3)$ on your graph.
17. Consider a square of side length s that is expanding. Let $A(s)$ be the area of the square with side length s feet. What does $A(3)$ and $A'(3)$ represent (include units). Which should be larger $A(3)$ or $A(6)$? Which should be larger $A'(3)$ or $A'(6)$? Sketch a graph of $A(s)$ from 0 to 10 feet. Represent $A'(3)$ and $A'(6)$ on the graph.
18. Consider a cube of side length s that is expanding. Let $v(s)$ be the volume of the cube with side length s inches. What does $v(4)$ and $v'(4)$ represent (include units). Which should be larger $v(4)$ or $v(8)$? Which should be larger $v'(4)$ or

$v'(8)$? Sketch a graph of $v(s)$ from 0 to 10 inches. Represent $v'(4)$ and $v'(8)$ on the graph.

19. On a cold winter day a wood stove is heating a home. The stove is currently at 200 °F. Let $T(x)$ be the temperature x feet from the stove. What does $T(4)$ and $T'(4)$ represent (include units). Which should be larger $T(4)$ or $T(8)$? Which should be larger $T'(4)$ or $T'(8)$? Sketch a graph of $T(x)$ from 0 to 10 feet. Represent $T'(4)$ and $T'(8)$ on the graph.
20. The resistance due to drag on an eight-person rowing shell (a boat) increasing relative to the square of the velocity. Let $R(v)$ be the drag resistance in Newtons with a given velocity v in m/s. What does $R(4)$ and $R'(4)$ represent? Which should larger $R(4)$ or $R(6)$? Which should be larger $R'(4)$ or $R'(6)$? Sketch a graph of $R(v)$ from $v = 0$ m/s to $v = 6$ m/s ($v = 6.2$ m/s is about as fast as an eight will travel).
21. Create your own problem using the format of the last few problems above and provide a solution to your problem.

For the following True or False exercises, provide an explanation or example to justify your response.

22. (T/F) It is possible to have $f(4) > 0$ and $f'(4) < 0$ or $f'(4) > 0$ or $f'(4) = 0$.
23. (T/F) It is possible to have $f(6) < 0$ and $f'(6) < 0$ or $f'(6) > 0$ or $f'(6) = 0$.
24. (T/F) It is possible to have $f(7) = 0$ and $f'(7) > 0$ or $f'(7) < 0$ or $f'(7) = 0$.
25. (T/F) It would generally makes sense if the units for $g'(16)$ are feet.
26. (T/F) It would generally makes sense to add or subtract $h'(42)$ and $h(42)$.
27. (T/F) It would generally makes sense to multiply $f'(9)$ and $f(9)$.
28. (T/F) It would generally makes sense to divide $g'(5)$ by $g(5)$.
29. (T/F) The following sentence makes sense: The rock was falling at a rate of 15 meters.

Chapter 6
Formulas Quantifying Change

We continue with the CO2 function example from chapter 4 chapter and provide some context about changing atmospheric CO2 levels by using different formulas to quantify change. We will focus on the period of the data, which is 1950 to 2017. The total *change*, M-Box 6.1, in CO2 levels from 1950 to 2017 is

$$CO2(67)\,\text{ppm} - CO2(0)\,\text{ppm} = 95.3\,\text{ppm}.$$

In other words, from 1950 through 2017 atmospheric CO2 levels increased by 95.3 ppm. A related calculation is the *average rate of change* over a time period, M-Box 6.2. In this case

$$(CO2(67)\,\text{ppm} - CO2(0)\,\text{ppm})/(67\,\text{year} - 0\,\text{year}) = 1.42\text{ppm per year.} \tag{6.1}$$

In other words, from 1950 through 2017 CO2 levels increased on average by 1.42 ppm per year. Two details to note. First, the input for the $CO2(t)$ function is years after 1950, but when we report the results we don't say 67 years after 1950; we say 2017. Similarly, we use year as the unit in the denominator in the calculation in 6.1. The average rate of change is also the slope of the secant line, as seen in Chapter 4, and averages the change over the time period. This does not mean CO2 necessarily increased by 1.42 ppm per year as this is the yearly average over that time period.

M-Box 6.1: Change

The quantity $f(b) - f(a)$ represents how much the function $f(x)$ changes **from** $x = a$ to $x = b$. **UNITS**: Output units of $f(x)$.

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_6

M-Box 6.2: Average Rate of Change

The quantity

$$\frac{f(b) - f(a)}{b - a}$$

is the average rate of change of the function $f(x)$ **from** $x = a$ **to** $x = b$. It is also the slope of the secant line of the function $f(x)$ **from** $x = a$ **to** $x = b$. **UNITS**: Output units of $f(x)$ divided by input units of $f(x)$ (think "rise/run").

The change in CO2 over from 1950 to 2017 is an absolute quantity but it doesn't provide a sense of if this change is a lot or a little. For example, if CO2 levels went from 1,000,000 ppm to 1,000,095.3 ppm then the change would be relatively small. This is similar for average rate of change. To understand the relative change over the time period we calculate the *percent change* from 1950 through 2017, M-Box 6.3. We have

$$\frac{CO2(67) \text{ ppm} - CO2(0) \text{ ppm}}{CO2(0) \text{ ppm}} \times 100\% = 23.49\%.$$

In other words, from 1950 through 2017 atmospheric CO2 levels increased by 23.49%.

M-Box 6.3: Percent Change

The quantity

$$\frac{f(b) - f(a)}{f(a)} \times 100\%$$

is the percentage change of the function $f(x)$ **from** $x = a$ **to** $x = b$. **UNITS**: Unit free, it is a percent.

So far, M-Box 6.1 through M-Box 6.3 are measures of change that are calculated over a fixed time frame. On the other hand the derivative, M-Box 5.1, represents a measure of change at a point or instantaneously. Just as we have Change and Percent Change, we have *rate of change (derivative)* and *percentage rate of change*, M-Box 6.4. For the year 2017, the percentage rate of change of CO2 is

$$\frac{CO2'(67) \text{ ppm/year}}{CO2(67) \text{ ppm}} \times 100\% = \frac{2.324202 \text{ ppm/year}}{405.72530690537 \text{ ppm}} \times 100\% \approx 0.57\% \text{ per year}.$$

In other words, in 2017 atmospheric CO2 was increasing at a rate of 0.57% per year. Note that at this point this is an approximation because we only have an approximation for $CO2'(67)$. We'll learn how to calculate derivatives soon enough. Is an increase of a little more than half a percent a lot or a little? From a mathematical perspective we do not really know. We should ask an ecologist or climate scientist, but we are able to clearly articulate how atmospheric CO2 was changing in 2017.

M-Box 6.4: Percentage Rate of Change

The quantity

$$\frac{f'(a)}{f(a)} \times 100\%$$

is the percentage rate of change of the function $f(x)$ **at** $x = a$.
UNITS: Percent per input units of $f(x)$.

Selected R code for the calculations are given in R Code box 6.1. Notice that a and b are defined in the first two lines because they are used repeatedly and so only have to be changed once if we want to evaluate these quantities of change at different values. Each line has a comment explaining the code. Lastly, as a convention we will use **CO2_p** to represent $CO2'(x)$ in R. At this point we have to input the value of **CO2_p(b)** by hand, but in a later chapter we will learn how to calculate derivatives in R.

R Code 6.1: Calculating Versions of Change

```
> a<-0  #1950
> b<-67 #2017
> CO2(b)-CO2(a) #change
> (CO2(b)-CO2(a))/(b-a) #Ave Rate Change
> (CO2(b)-CO2(a))/CO2(a)*100 # percent change
> CO2_p(b)/CO2(b)*100 # percentage rate of change
```

```
[1] 95.30194
[1] 1.422417
[1] 30.700635
[1] 0.572851
```

6.1 Exercises

1. In the R code below identify the type of change for the function f calculated and its value. Assume the output units are meters and the input units seconds. You should report your result in a sentence with proper context.

```
> f<-function(x){exp(x)}
> a<- 1
> b<- 10
> f(b)-f(a)
[1] 22023.74751
```

2. In the R code below identify the type of change for the function f calculated and its value. Assume the output units are meters and the input units seconds. You should report your result in a sentence with proper context.

```
> f<-function(x){sqrt(x)}
> b <- 16
> f_p(b) <-0.125
> f_p(b)/f(b)*100
[1] 3.125
```

3. In the R code below identify the type of change for the function f calculated and its value. Assume the output units are meters and the input units seconds. You should report your result in a sentence with proper context.

```
> f<-function(x){x^2}
> a<- 5
> b<- 12
> (f(b)-f(a))/f(a)*100
[1] 476
```

4. In the R code below identify the type of change for the function f calculated and its value. Assume the output units are meters and the input units seconds. You should report your result in a sentence with proper context.

```
> f<-function(x){sqrt(x)}
> a<- 10
> b<- 15
> (f(b)-f(a))/(b-a)
[1] 0.1421411372
```

5. In the R code below identify the type of change for the function f calculated and its value. Assume the output units are meters and the input units seconds. You should report your result in a sentence with proper context.

```
> f<-function(x){x^3}
> b <- 2
> f_p(b) <- 12
> f_p(b)/f(b)*100
[1] 150
```

6. In the R code below identify the type of change for the function f calculated and its value. Assume the output units are meters and the input units seconds. You should report your result in a sentence with proper context.

```
> f<-function(x){exp(x)}
> a<- 0
> b<- 2
> (f(b)-f(a))/f(a)*100
[1] 638.9056099
```

7. A formula for the dropped rock from problem 11 in section 5.1 is given by $s(t) = 16t^2$ feet traveled, t seconds after being dropped. What is the change, average rate of change, and percentage change from 2 to 4 seconds? Report your result in a sentence in context with units.
8. A formula for the watermelon from problem 12 in section 5.1 is given by $s(t) = 128t - 16t^2$ feet from the ground, t seconds after being launched. What is the change, average rate of change, and percentage change from 1 to 3 seconds? Report your result in a sentence in context with units.
9. A formula for the accelerating car given in problem 13 in section 5.1 is given by $s(t) = 5.5t^2$ feet where t is in seconds. What is the change, average rate of change, and percentage change from 0 to 4 and from 4 to 8 seconds? Which values are larger? Explain why this makes sense. Note: We assume a constant acceleration of 11 ft/sec^2. Report your result in a sentence in context with units.
10. A formula for the square problem 17 in section 5.1 is given by $A(s) = s^2$ feet squared. What is the change, average rate of change, and percentage change from 3 to 6 feet? Report your result in a sentence in context with units.
11. A formula for the cube problem 18 in section 5.1 is given by $V(s) = s^3$ inches cubed. What is the change, average rate of change, and percentage change from 4 to 8 inches? Report your result in a sentence in context with units.
12. A formula for the derivative of the dropped rock from problem 11 and 7 in section 5.1 is given by $s'(t) = 32t$. What is the rate of change and percentage rate change at 4 seconds? Report your result in a sentence in context with units. Calculate $s'(2)$. Is it larger or smaller than $s'(4)$? Why does this make sense? Compare to your answer from problem 11.
13. A formula for the derivative of the watermelon from problem 12 (Section 5.1) and 8 (section 6.1) is given by $s'(t) = 128 - 32t$. What is the rate of change and percentage rate change at 3 seconds? Report your result in a sentence in context with units. Calculate $s'(1)$. Is it larger or smaller than $s'(3)$? Why does this make sense? Compare to your answer from problem 12 in section 5.1.
14. A formula for the derivative of the accelerating car given in problem 13 (Section 5.1) and 9 (section 6.1) is given by $s'(t) = 11t$ feet, where t is in seconds. What is the rate of change and percentage rate change at 3 seconds? Report your result in a sentence in context with units. Calculate $s'(5)$. Is it larger or smaller than $s'(3)$? Why does this make sense? Compare to your answer from problem 13 in section 5.1.

15. A formula for the derivative of the square problem 17 (Section 5.1) and 10 (section 6.1) is given by $A'(s) = 2s$. What is the rate of change and percentage rate change at 6 feet? Report your result in a sentence in context with units. Calculate $A'(3)$. Is it larger or smaller than $A'(6)$? Why does this make sense? Compare to your answer from problem 17 in section 5.1.
16. A formula for derivative of the cube problem 18 (Section 5.1) and 11 (section 6.1) is given by $V'(s) = 3s^2$. What is the rate of change and percentage rate change at 4 inches? Report your result in a sentence in context with units. Calculate $V'(8)$. Is it larger or smaller than $V'(4)$? Why does this make sense? Compare to your answer from problem 18 in section 5.1.
17. The table below is the elevation from the near the beginning of the hike of Cliff Mt, an Adirondack high peak, to the peak. [20] What is the average elevation change from the 2 miles to the 7.5 miles mark? Over the same interval what is the change and percentage change in elevation? Is change or percentage change a better measure of how much work went into reaching the peak? Report your results in a sentence in context with units.

Mile from Start	2	3	4	5	6	7	7.5
Elevation (ft)	2398	2520	3238	3808	3349	3353	3913

18. The table below is the elevation from the start of the hike of Mt Marcy, an Adirondack high peak and highest peak in NYS, to the peak. [20] What is the average elevation change from the beginning to the end? Over the same interval what is the change and percentage change in elevation? Is change or percentage change a better measure of how much work went into reaching the peak? Report your results in a sentence in context with units.

Mile from Start	0	1	2	3	4	5	6	7
Elevation (ft)	2205	2279	2393	2762	3504	3899	4468	5275

19. The table below is the average Arctic ice extent for the month of March, which is the month of maximum ice. [18] On average how much did ice change from 1997 to 2020? Over the same time period what is the change and percentage change in ice? Is change or percentage change a better measure of how much work went into reaching the peak? Report your results in a sentence in context with units.

Year	1997	2000	2003	2006	2009	2011	2014	2017	2020
March Extent (MSK)	15.47	15.22	15.48	14.42	14.98	14.55	14.76	14.29	14.73

20. The table below is the average Arctic ice extent for the month of September, which is the month of minimum ice. [18] On average how much did ice change from 1997 to 2020? Over the same time period what is the change and percentage change in ice? Is change or percentage change a better measure of how much

work went into reaching the peak? Report your results in a sentence in context with units. Compare your results to the previous problem.

Year	1997	2000	2003	2006	2009	2011	2014	2017	2020
September Extent (MSK)	6.69	6.25	6.12	5.86	5.26	4.56	5.22	4.82	4.00

21. The table below is hourly wages in 2021 dollars for the bottom 10th percentile of earners. [36] On average how much did wages change from 1975 to 2020? Over the same time period what is the change and percentage change in wages? Is change or percentage change a better measure of how much work went into reaching the peak? Report your results in a sentence in context with units.

Year	2020	2010	2005	2000	1995	1990	1985	1980	1975
10th	\$11.53	\$10.00	\$9.78	\$9.64	\$8.81	\$8.63	\$8.53	\$9.64	\$9.30

22. The table below is hourly wages in 2021 dollars for the top 90th percentile of earners. [36] On average how much did wages change from 1975 to 2020? Over the same time period what is the change and percentage change in wages? Is change or percentage change a better measure of how much work went into reaching the peak? Report your results in a sentence in context with units. Compare your results to the previous problem.

Year	2020	2010	2005	2000	1995	1990	1985	1980	1975
90th	\$55.34	\$47.41	\$44.65	\$42.57	\$39.11	\$38.15	\$36.35	\$35.26	\$34.96

23. If a curve is increasing and concave up from $x = a$ to $x = b$, will the average rate of change from $x = a$ to $x = b$ be larger or smaller than the rate of change at $x = b$. Explain.
24. If a curve is increasing and concave down from $x = a$ to $x = b$, will the average rate of change from $x = a$ to $x = b$ be larger or smaller than the rate of change at $x = b$. Explain.
25. If a curve is increasing and neither concave down nor concave up from $x = a$ to $x = b$, will the average rate of change from $x = a$ to $x = b$ be larger or smaller than the rate of change at $x = b$. Explain.
26. The average power (in calories per second) to move an eight-person rowing shell (a boat) at an average speed of v m/s is given by $P(v) = 9.6v^3$. For each of the problems below include units and use your calculation in a sentence in context properly.

    a. Find the average rate of change from $v = 3$ to $v = 6$.
    b. Find the percent change from $v = 3$ to $v = 6$.
    c. Find the percentage rate of change at $v = 3$ given that $P'(3) = 259.2$.

27. Figure 6.1 is a profile graph of a hiking trail for Mount Skylight. Estimate the following using this graph (When appropriate include units and use your calculation in a sentence in context properly.):

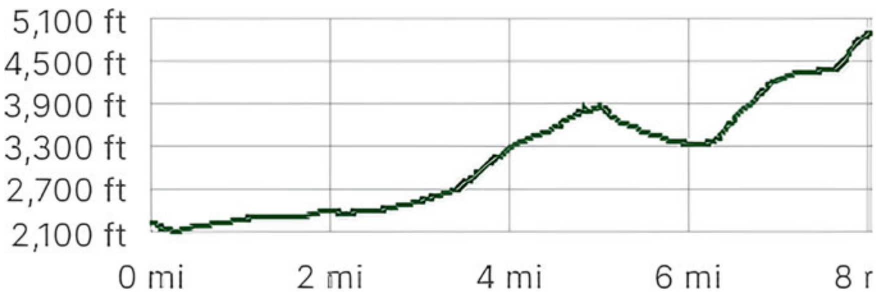


Fig. 6.1 Profile graph of a hiking trail for Mount Skylight, which is one of the Adirondack 46. Graph made with GAIA GPS (gaiagps.com).

a. What is the total change from the beginning of the hike to the peak? Is this the same as the total amount of elevation a hiker has to climb (or the total amount of "going up" that is hiked) during this hike?
b. What is the average rate of change from the beginning of the hike to the peak?
c. What is the percent change from the beginning of the hike to the peak?
d. At what mile is the derivative the greatest?
e. Over what mile of the hike is the average rate of change the greatest? Estimate this value.

28. Figure 6.2 is a profile graph of a hiking trail for Rocky Peak Ridge. Estimate the following using this graph (When appropriate include units and use your calculation in a sentence in context properly.):

a. What is the total change from the beginning of the hike to the peak? Is this the same as the total amount of elevation a hiker has to climb (or the total amount of "going up" that is hiked) during this hike?
b. What is the average rate of change from the beginning of the hike to the peak?
c. What is the percent change from the beginning of the hike to the peak?
d. At what mile is the derivative the greatest?

Fig. 6.2 Profile graph of a hiking trail for Rocky Peak Ridge, which is one of the Adirondack 46. Graph made with GAIA GPS (gaiagps.com).

e. Over what mile of the hike is the average rate of change the greatest? Estimate this value.

29. Use the extreme poverty function from the Function Gallery, Chapter 3, for the following problems.

a. Use $P'(29) = -0.032$ in a sentence with proper context and units.
b. Find the average rate of change of extreme poverty from 1990 to 2019. Use your result in a sentence that explains the context of the calculation.
c. Find the percentage change of extreme poverty from 1990 to 2019. Use your result in a sentence that explains the context of the calculation.
d. Find the percentage rate of change, given $P'(29) = -0.032$, of extreme poverty in 2019. Use your result in a sentence that explains the context of the calculation.

30. Use the life expectancy by GDP function from the Function Gallery, Chapter 3, for the following problems.

a. Use $LeGdp'(21000) = 0.000261521$ in a sentence with proper context and units. Tip: It might make sense to convert to units of per \$1000 then use per \$1.
b. Find the average rate of change of life expectance from a GDP per capita of \$1000 to \$21000. Use your result in a sentence that explains the context of the calculation.
c. Find the percentage change of extreme of life expectance from a GDP per capita of \$1000 to \$21000. Use your result in a sentence that explains the context of the calculation.
d. Find the percentage rate of change, given $LeGdp'(21000) = 0.000261521$, of life expectance given a GDP per capita of \$21000. Use your result in a sentence that explains the context of the calculation.

31. Use the global temperature function from the Function Gallery, Chapter 3, for the following problems.

a. Use $GTemp'(60) = 0.022$ in a sentence with proper context and units.
b. Find the average rate of change of global temperature from 1950 to 2010. Use your result in a sentence that explains the context of the calculation.
c. Find the percentage change of global temperature from 1950 to 2010. Use your result in a sentence that explains the context of the calculation.
d. Find the percentage rate of change, given $GTemp'(60) = 0.022$, of global temperature in 2010. Use your result in a sentence that explains the context of the calculation.

32. Use the CO2 function from the Function Gallery, Chapter 3, for the following problems.

a. Use $CO2'(65) = 2.29$ in a sentence with proper context and units.
b. Find the average rate of CO2 from 1950 to 2015. Use your result in a sentence that explains the context of the calculation.

c. Find the percentage change of CO2 from 1950 to 2015. Use your result in a sentence that explains the context of the calculation.
d. Find the percentage rate of change, given $CO2'(65) = 2.29$, of CO2 in 2015. Use your result in a sentence that explains the context of the calculation.

33. Use the cumulative installed world wind power function from the Function Gallery, Chapter 3, for the following problems.

a. Use $Wwind'(30) = 35136.29$ in a sentence with proper context and units.
b. Find the average rate of change of cumulative installed world wind power from 1980 to 2010. Use your result in a sentence that explains the context of the calculation.
c. Find the percentage change of cumulative installed world wind power from 1980 to 2010. Use your result in a sentence that explains the context of the calculation.
d. Find the percentage rate of change, given $Wwind'(30) = 35136.29$, of cumulative installed world wind power in 2010. Use your result in a sentence that explains the context of the calculation.

34. Use the cumulative installed Spain wind power function from the Function Gallery, Chapter 3, for the following problems.

a. Use $Swind'(25) = 2118.14$ in a sentence with proper context and units.
b. Find the average rate of change of cumulative installed Spain wind power from 1992 to 2005. Use your result in a sentence that explains the context of the calculation.
c. Find the percentage change of cumulative installed Spain wind power from 1992 to 2005. Use your result in a sentence that explains the context of the calculation.
d. Find the percentage rate of change, given $Swind'(25) = 2118.14$, of cumulative installed Spain wind power in 2005. Use your result in a sentence that explains the context of the calculation.

35. Use the cumulative installed U.S. wind power function from the Function Gallery, Chapter 3, for the following problems.

a. Use $USwind'(35) = 8287.95$ in a sentence with proper context and units.
b. Find the average rate of change of cumulative installed U.S. wind power from 1990 to 2015. Use your result in a sentence that explains the context of the calculation.
c. Find the percentage change of cumulative installed U.S. wind power from 1990 to 2015. Use your result in a sentence that explains the context of the calculation.
d. Find the percentage rate of change of cumulative installed U.S. wind power in 2015. Use your result in a sentence that explains the context of the calculation.

36. The following is quoted from the January 2019 article "EIA Forecasts Renewable Will be Fastest Growing Source of Electricity Generation" by Tyler Hodge. [22]

What type of change is represented by “9GW” and “44%?” Include any relevant time period in your response.

> In addition to utility-scale solar in the electric power sector, some residences and businesses have installed small-scale solar photovoltaic systems to supply some of the electricity they consume. EIA forecasts that small-scale solar generating capacity will grow by almost 9 GW during the next two years, an increase of 44%.

37. The following is quoted from the article “Redrawing the Map: How the World’s Climate Zones are Shifting” by Nicola Jones. [25] What type of change is represented by “10 percent?” Include any relevant time period in your response.

> When Natalie Thomas and Sumant Nigam, ocean and atmospheric scientists at the University of Maryland, looked at records stretching from 2013 back to 1920, they found that these boundaries for the Sahara had crept both northward and southward, making the entire region about 10 percent larger.

38. The following is quoted from the article “Redrawing the Map: How the World’s Climate Zones are Shifting” by Nicola Jones. [25] What type of change is represented by “13.3” Include any relevant time period in your response.

> Lauren Parker and John Abatzoglou of the University of Idaho tracked what would happen to hardiness zones from 2041 to 2070 under future global warming scenarios, and found the lines will continue to march northward at a “climate velocity” of 13.3 miles per decade.

39. The following is quoted from the article “If Carbon Dioxide Hits a New High Every Year, Why isn’t Every Year Hotter than the Last?” by Rebecca Lindsey. [27] What type of change is represented by “20 parts per million” and “0.04°C?” Include any relevant time period in your response.

> Atmospheric carbon dioxide levels rose by around 20 parts per million over the 7 decades from 1880-1950, while the temperature increased by an average of 0.04°C per decade.

40. The following is quoted from the article “If Carbon Dioxide Hits a New High Every Year, Why isn’t Every Year Hotter than the Last?” by Rebecca Lindsey. [27] What type of change is represented by “100 ppm” and “0.14°C?” Include any relevant time period in your response. Hint: 7 decades follow the previous problem.

> Over the next 7 decades, however, carbon dioxide climbed nearly 100 ppm—5 times as fast! To put those changes in some historical context, the amount of rise in carbon dioxide levels since the late 1950s would naturally, in the context of past ice ages, have taken somewhere in the range of 5,000 to 20,000 years; we’ve managed to do it in about 60. At the same time, the rate of warming averaged 0.14°C per decade.

41. The following is quoted from the article “Keeping Score on Earth’s Rising Seas” by Pat Brennan. [10] What type of change is represented by “3.1”? Include any relevant time period in your response.

> Since 1993, global sea level has been rising by an average 3.1 millimeters per year, with the rise accelerating by 0.1 millimeter per year, according to the study published Aug. 28 in the journal, “Earth System Science Data.”

42. The following is quoted from the article "Global Trade in Liquefied Natural Gas grew by 4.5% in 2021" by Victoria Zaretskaya. [41] What type of change is represented by "29%" and "14.0?" Include any relevant time period in your response.

> Global LNG export capacity has increased by 29%, or 14.0 Bcf/d, over the past five years (2017–21).

6.2 Project: Which Mountain to Climb?

The goal of this project is to make a case in a short report based on quantitative information for why you would rather hike Mount Redfield or Mount Haystack (see the graphs in figures 6.3 and 6.4), under the assumption that you would prefer to hike the easier mountain. Your argument must include at least three different measures of change from this chapter.

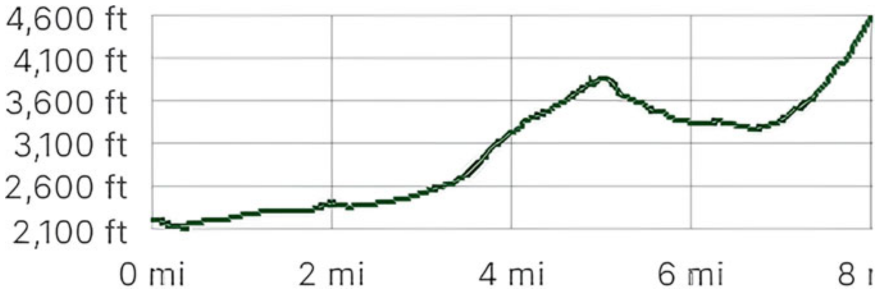


Fig. 6.3 Profile graph of a hiking trail for Mount Redfield, which is one of the Adirondack 46. Graph made with GAIA GPS (gaiagps.com).

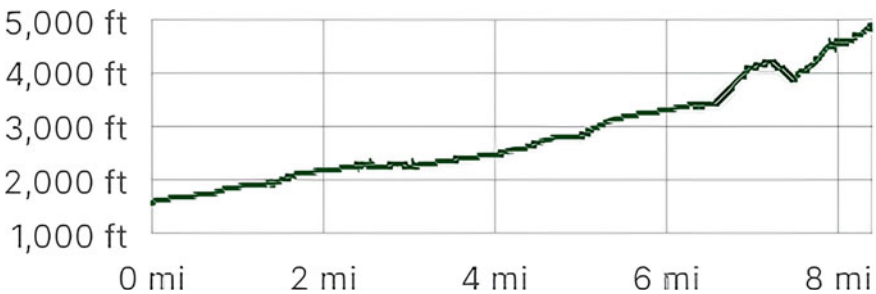


Fig. 6.4 Profile graph of a hiking trail for Mount Haystack, which is one of the Adirondack 46. Graph made with GAIA GPS (gaiagps.com).

Chapter 7
The Microscope Equation

If we are in a car and we look at the speedometer and it says we are traveling 60 mph, then about how far will we travel in the next minute? 30 seconds? Two minutes? Traveling at 60 mph is the same as a rate of 1 mile per minute and we would answer one mile, a half a mile, and two miles, but we should recognize that in our responses we are assuming the speed is constant. These answers are approximations as we might speed up or slow down, but unless our change in speed is drastic, say slamming on the breaks, the results are likely reasonably close or, in other words, a good estimate. On the other hand, if we wanted to estimate how far we travel over the next hour we would say 60 miles, but we would recognize that this response may not be great as our speed may well change over the course of an hour. Our reasoning here is formalized as the microscope equation in M-Box 7.1.

M-Box 7.1: The Microscope Equation

We can estimate how much a function $f(x)$ changes from $x = a$ to $x = b$ with

$$f(b) - f(a) \approx f'(a)(b - a)$$

which is written compactly as

$$\triangle y \approx f'(a)\triangle x.$$

The equation is also useful in this form for estimating a value at $f(b)$ instead of a change.:

$$f(b) \approx f'(a)(b - a) + f(a),$$

note this is similar to the point slope form of a line.

The microscope equation is essentially a specialized version of the point slope form of a line. We call this the microscope equation because if we look under a microscope, in other words really zoomed in, what we will see is a line. Another way to say this is that the curve is locally linear. It is critical to recognize that we obtain

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_7

estimates here, hence the use of $\approx$ and not $=$ in M-Box 7.1. In general the farther away we move from point $x = a$ to point $x = b$ our approximation will likely get worse. In other words, are estimations are likely only good "under the microscope." One more note about M-Box 7.1 is that we introduce the notation $\triangle x$, which is simply short hand for change in x or $b - a$. Similarly, $f(b) - f(a)$ is the change in y and we can write $\triangle y$.

Consider figure 7.1. The line has point $(a, f(a))$. The slope is $m = f'(a)$ since $f'(a)$ is by definition the slope of the tangent line at $x = a$ of the function $f(x)$. Using the point slope form of the line, the equation of the tangent line at $x = a$ of the function $f(x)$ is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - f(a) &= f'(a)(x - a), \end{aligned}$$

or

$$y = f'(a)(x - a) + f(a).$$

If we evaluate at $x = b$ we get

$$y(b) = f'(a)(b - a) + f(a).$$

The value of $f(b)$ is close to the y-value at b on the line, $y(b)$ and so we write

$$f(b) \approx f'(a)(b - a) + f(a).$$

In other words, $f(b)$ is approximately the y-value at b on the line. This is the last equation in M-Box 7.1. If we move $f(a)$ to the left side then we get the first equation in M-Box 7.1. We call this the microscope equation because under a microscope the tangent line at $x = a$ is the same at the function near $x = a$. Note though that "near" is not defined here and is not the same for all functions. Note that in figure 7.1 the farther we move from $x = a$ the bigger the gap between the function and the line.

Now, let us go back to the opening example and use the microscope equation to answer the questions. We need a little notation first. So, let $f(t)$ be the distance in miles the car has traveled at time t in hours. Given this, $f'(t)$ is the rate, mph, at which the car is traveling at time t. Assume we look at speedometer at time $t = 0$, which give $f'(0) = 60$ mph or $f'(0) = 1$ mile per minute (our questions are in minutes so we will convert our time units to minutes) and $f(0) = 0$. In the microscope equation the value of a is the x-value where the tangent line is constructed. In this example, $a = 0$. The value of b is the x-value at which we want an estimation. The question, How far will the car travel in one minute?, leads us to $b = 1$. Using the microscope equations we get

$$\begin{aligned} f(b) &\approx f'(a)(b - a) + f(a) \\ f(1) &\approx f'(0)(1 - 0) + f(0) \\ f(1) &\approx 1(1 - 0) + 0 = 1. \end{aligned}$$

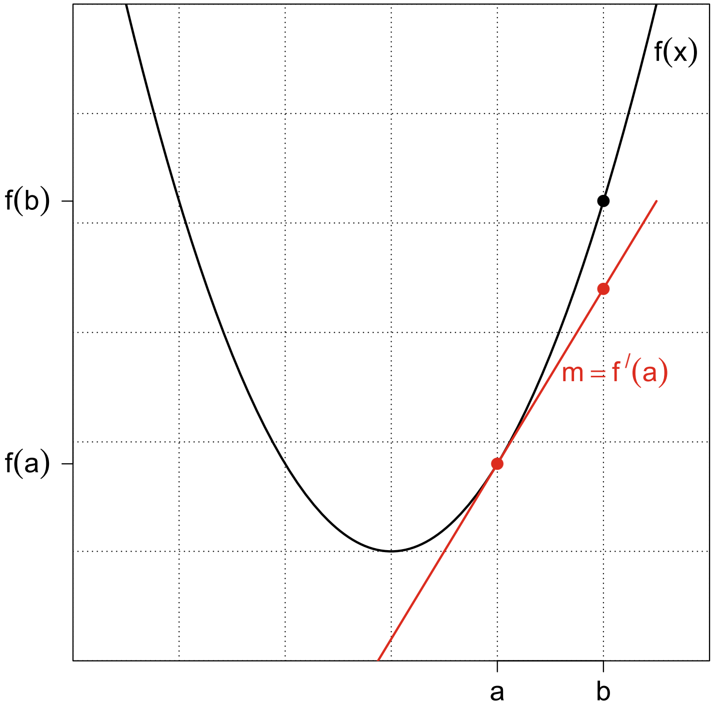


Fig. 7.1 Generic graph of a function with a tangent line at $x = a$.

We get $f(1) \approx 1$ mile and so we estimate that we will travel about one mile in the next minute. Will the care, in fact, travel one mile in the next minute? This would only happen under certain circumstances. For example, if the car's speed doesn't change for the next minute or, more generally, if the average rate of change of the car over the next minute is one mile per minute. How far will be traveling in 30 seconds? The only thing that changes is we now have $b = 0.5$. Here is the computation:

$$f(0.5) \approx f'(a)(b - a) + f(a)$$
$$f(0.5) \approx f'(0)(0.5 - 0) + f(0)$$
$$f(0.5) \approx 1(0.5 - 0) + 0 = 0.5$$

We get $f(0.5) \approx 0.5$ mile and so we estimate that we will travel about one mile in the next minute. Which is likely to be a better approximation, $f(0.5) \approx 0.5$ mile or $f(1) \approx 1$ mile? We would say $f(0.5) \approx 0.5$ mile because the farther from $x = 0$ the worse our estimation is likely to be.

One more example. In the previous chapter we estimated that $CO2'(67) \approx 2.32$ ppm per year. About how much will CO2 increase over the next two years? We use the microscope equation with $a = 67$ and $b = 69$ to get

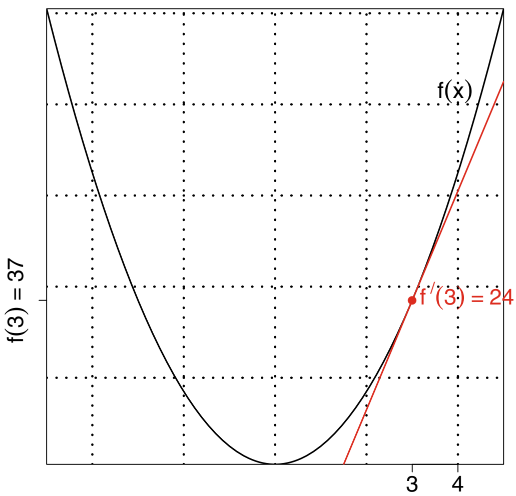
$$\begin{aligned} CO2(b) - CO2(a) &\approx CO2'(a)(b-a) \\ CO2(69) - CO2(67) &\approx CO2'(67)(69-67) \\ CO2(69) - CO2(67) &\approx 2.32(69-67) \\ CO2(69) - CO2(67) &\approx 4.64. \end{aligned}$$

In two years, we would estimate that CO2 levels would increase about 4.64 ppm. Another way to think about the equation $CO2'(67) \approx 2.32$ ppm per year is as a rise over run (slope of a line) and so if we run two years we should rise $2 \times 2.32 = 4.64$ pppm. In the examples here we really did not need a formal microscope equation, but in the exercises and later chapters being comfortable with the microscope equation will be necessary.

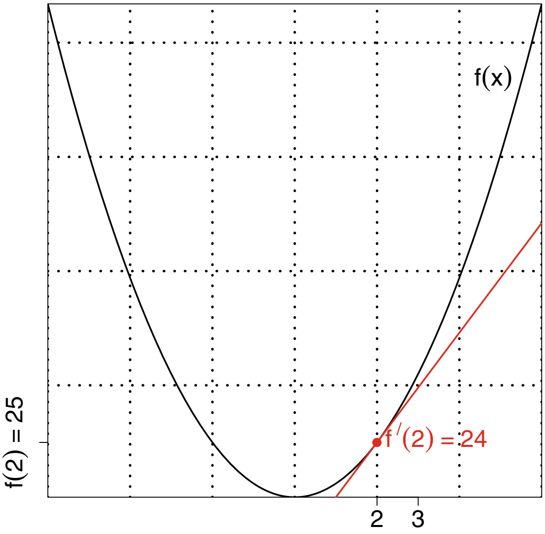
The microscope equation also formalizes the idea of using a line to estimate values on a function, using the notion of local linearity. For example, in physics when modeling a simple pendulum $\sin(x)$ will be substituted with x as $f(x) = x$ is the equation of the tangent line of $\sin(x)$ at $x = 0$.

7.1 Exercises

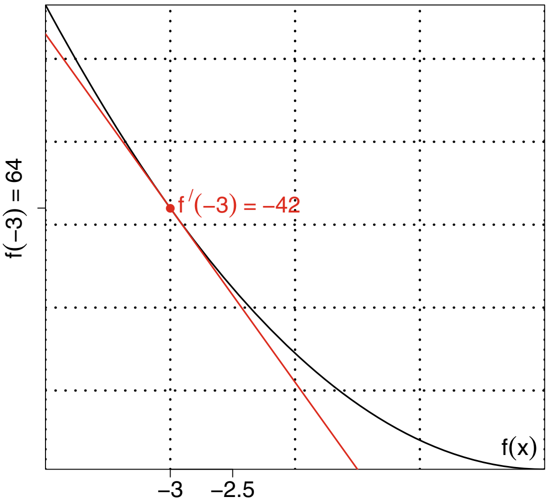
1. Given the information in the graph estimate $f(4)$.



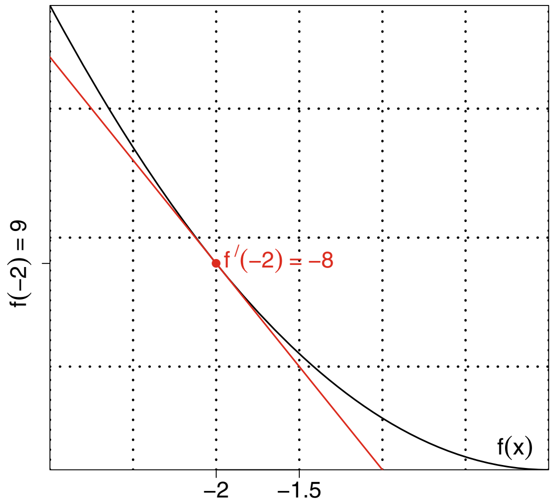
2. Given the information in the graph estimate $f(3)$.



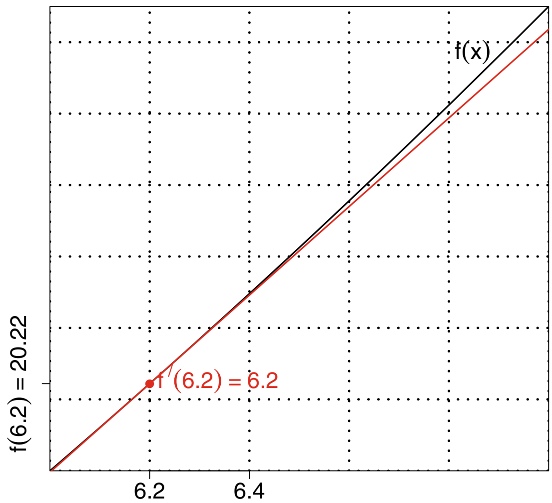
3. Given the information in the graph estimate $f(-0.25)$.



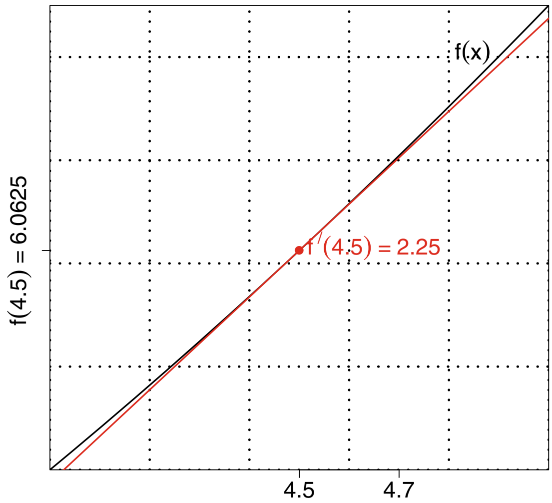
4. Given the information in the graph estimate $f(-1.5)$.



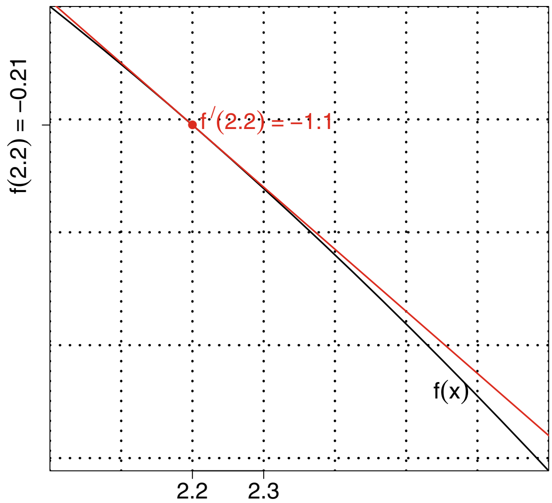
5. Given the information in the graph estimate $f(6.4)$.



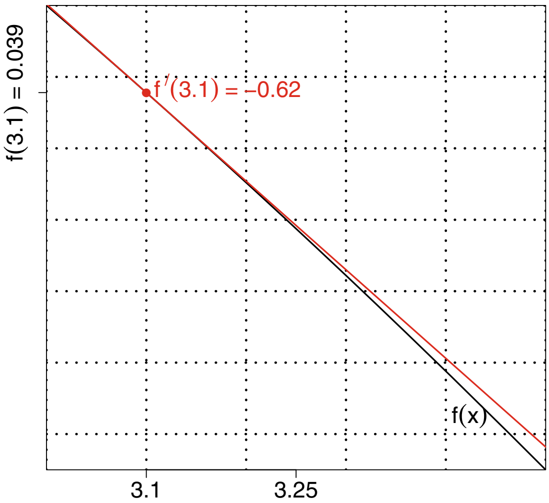
6. Given the information in the graph estimate $f(4.7)$.



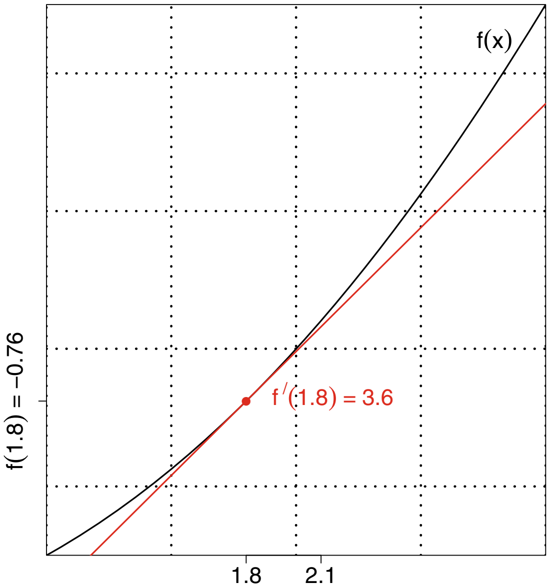
7. Given the information in the graph estimate $f(2.3)$.



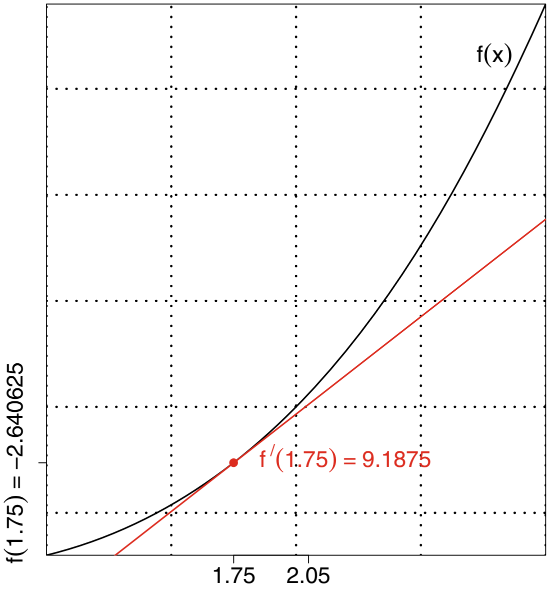
8. Given the information in the graph estimate $f(3.25)$.



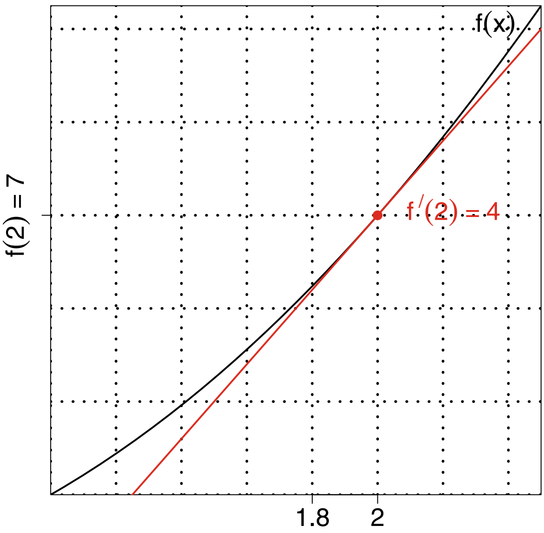
9. Given the information in the graph estimate $f(2.1)$.



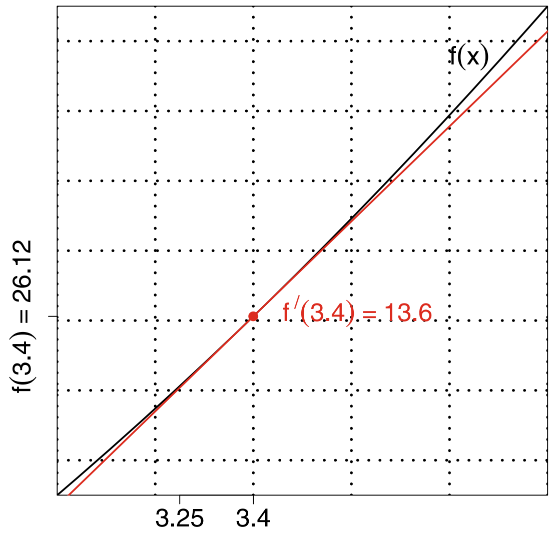
10. Given the information in the graph estimate $f(2.05)$.



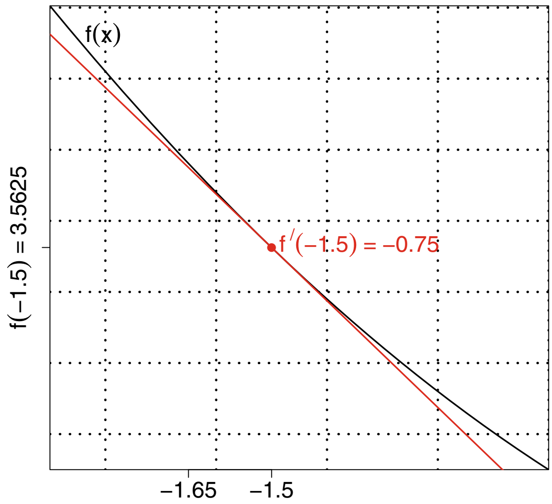
11. Given the information in the graph estimate $f(1.8)$.



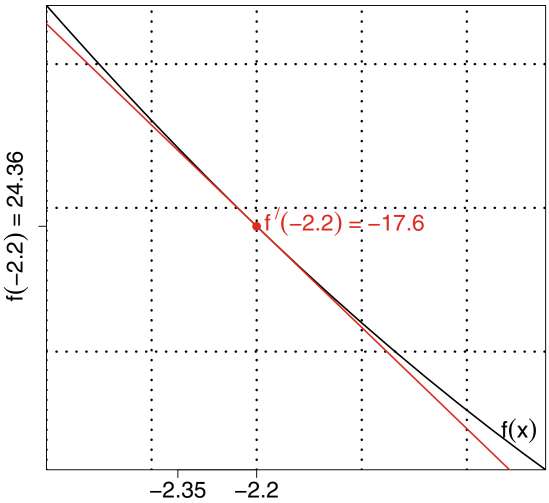
12. Given the information in the graph estimate $f(3.25)$.



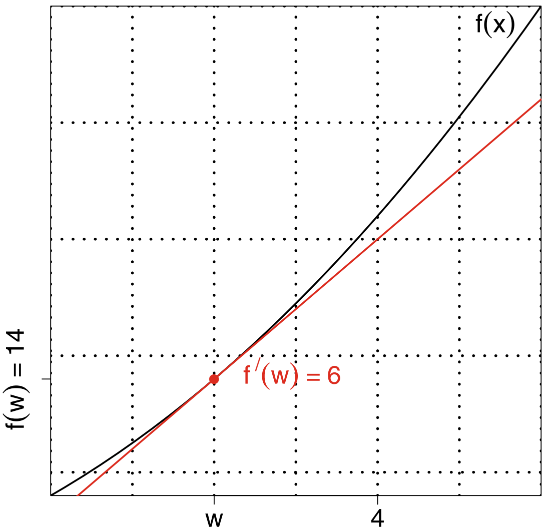
13. Given the information in the graph estimate $f(-1.65)$.



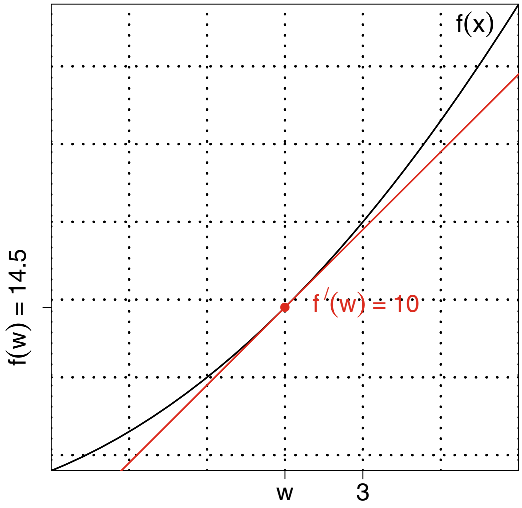
14. Given the information in the graph estimate $f(-2.35)$.



15. Given the information in the graph find w. Assume the value of the line at $x = 4$ is 20.



16. Given the information in the graph find w. Assume the value of the line at $x = 3$ is 19.5.



17. Using the global temperature function from the Function Gallery, Chapter 3, given that $GTemp'(60) = 0.022$ estimate how much the global temperature would have been in 2011 and it 2012. Use your results in a sentence that explains the context of the calculation.
18. Using the CO2 function from the Function Gallery, Chapter 3, given that $CO2'(65) = 2.29$ estimate how much CO2 would have been in 2016 and 2017. Use your results in a sentence that explains the context of the calculation.
19. Using the cumulative installed world wind power function from the Function Gallery, Chapter 3, given that $Wwind'(30) = 35136.29$ estimate how much cumulative installed world wind power would have changed from 2010 to 2011, and 2010 to 2012. Use your results in a sentence that explains the context of the calculation.
20. Using the cumulative installed Spain wind power function from the Function Gallery, Chapter 3, given that $Swind'(25) = 2118.14$ estimate how much cumulative installed Spain wind power will change from 2005 to 2006, and 2005 to 2007. Use your results in a sentence that explains the context of the calculation.
21. Using the cumulative installed U.S. wind power function from the Function Gallery, Chapter 3, given that $USwind'(35) = 8287.95$ about how much will cumulative installed U.S. wind power will change from 2015 to 2016, and 2015 to 2017. Use your results in a sentence that explains the context of the calculation.

22. If $f(8) = 5$ and $f'(8) = 6$, then we would estimate that $f(9) =$ ________.
23. If $f(4) = 7$ and $f'(4) = 3$, then we would estimate that $f(5) =$ ________.
24. If $g(8) = 5$ and $g'(8) = 2$, then we would estimate that $g(8.5) =$ ________.
25. If $g(10) = 9$ and $g'(10) = 5$, then we would estimate that $g(10.5) =$ ________.
26. If $h(6) = 4$ and $h'(6) = 4$, then we would estimate that $h(6.2) =$ ________.
27. If $h(7) = 2$ and $h'(7) = 4$, then we would estimate that $h(7.4) =$ ________.
28. If $g(9) = 3$ and $g'(9) = -6$, then we would estimate that $g(9.8) =$ ________.
29. If $g(8) = 4$ and $g'(8) = -9$, then we would estimate that $g(8.4) =$ ________.
30. If $f(3) = 6$ and $f'(3) = 5$, then we would estimate that $f($________$) = 8.5$ and $f($________$) = 5$.
31. If $f(8) = 12$ and $f'(8) = 6$, then we would estimate that $f($________$) = 13.8$ and $f($________$) = 9.6$.
32. If $g(15) = 20$ and $g'(15) = -7$, then we would estimate that $g($________$) = 20.7$ and $g($________$) = 17.9$.
33. If $g(48) = 35$ and $g'(48) = -2$, then we would estimate that $g($________$) = 36.6$ and $g($________$) = 33.8$.
34. If $h(31) = 40$ and $h'(31) =$ ________, then we would estimate that $h(31.5) = 48$.
35. If $h(44) = 23$ and $h'(44) =$ ________, then we would estimate that $h(44.25) = 24$
36. If $f(21) = 35$ and $f'(21) =$ ________, then we would estimate that $f(21.4) = 34.2$.
37. If $f(28) = 9$ and $f'(28) =$ ________, then we would estimate that $f(28.6) = 8.7$.
38. Assume $f(5) = 8$, $f'(5) = 3$, and $f'(5.5) = 4.2$. Use the microscope equation to first estimate $f(5.5)$ and then $f(6)$.
39. Assume $f(7) = 10$, $f'(7) = 2$, and $f'(7.5) = 3.5$. Use the microscope equation to first estimate $f(7.5)$ and then $f(8)$.
40. Assume $g(1) = -6$, $g'(1) = 4$, and $g'(1.25) = 2$. Use the microscope equation to first estimate $g(1.25)$ and then $g(1.5)$.
41. Assume $h(8) = 4$, $h'(8) = -3$, and $h'(8.25) = 1$. Use the microscope equation to first estimate $h(8.25)$ and then $h(8.5)$.

Chapter 8
Successive Approximations to Estimate Derivatives

In the chapter How Fast is CO2 Increasing?, chapter 4 we estimated the slope of the tangent line at $x = 67$ (or 2017) of the CO2 function in figure 4.1 using

(CO2(a+h)-CO2(a-h))/(a+h-(a-h))

$a = 67$ and $h = 0.01$ to get 2.324202 ppm per year. The calculation is presented R Code 4.2. The question is, how accurate is our estimate of the slope of the tangent line of CO2 at $x = 67$ (or 2017)? Is $h = 0.01$ small enough so that the secant line that straddles the tangent line provides an accurate slope estimation? If we look at figure 8.1 which is the CO2 function from figure 4.1 zoomed in with a tangent line at 2017, $x = 67$, (dashed red line). Two square points are added at $(67 - 0.01, CO2(67 - 0.01))$ and $(67 + 0.01, CO2(67 + 0.01))$. Note that it appears that the tangent line connects the two points and hence is the same as the second line connecting the two points. Based on this graph it would seem that our secant line slope approximation of the tangent line slope at $x = 67$ is good (whatever good means?).

The graph provides us with evidence that our approximation is good, but we should also check this algebraically. We can evaluate the slope of the secant line with smaller values of h and examine how our estimate of the slope changes. For example, we let h range over 0.1, 0.01, 0.001, and 0.0001, and calculate the corresponding slope of the secant lines from $67 - h$ to $67 + h$. We do this in R Code 8.1. The code here is almost identical to R Code 4.2, with the only difference being that h is defined to be a vector of values using the concatenate or combine function **c**. The expression for the secant line slope will be evaluated for each value of h, as R interprets a vector in a formula to mean evaluate the formula for each element of the vector. This is a powerful tool that allows us easily to do multiple calculations.

R Code 8.1: Secant Line Slopes with Different Values of h

```
> CO2<-function(t){0.0134594696825104*t^2+0.520632601928747*t+310.423363171355}
> a<-67
> h<-c(0.1,0.01,0.001,0.0001)
```


T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_8

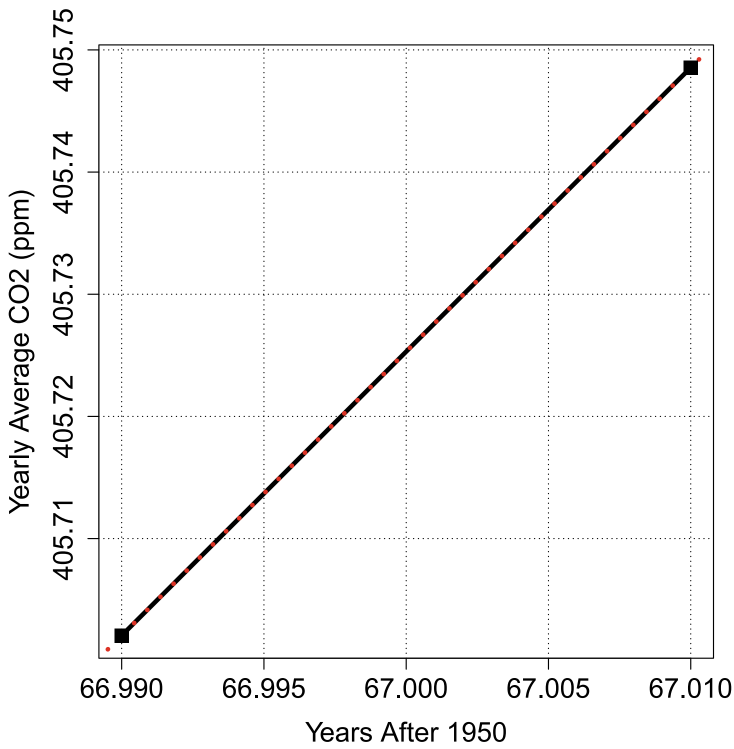


Fig. 8.1 Graph of the CO2 function from figure 4.1 with a tangent line at 2017, $x = 67$, (dashed red line) and square points on the function at $(67 - 0.01, CO2(67 - 0.01))$ and $(67 + 0.01, CO2(67 + 0.01))$. Note that it appears that the tangent line connects the two points and hence is the same as the second line connecting the two points.

```
> (CO2(a+h)-CO2(a-h))/(a+h-(a-h))
```

```
[1] 2.324202 2.324202 2.324202 2.324202
```

For each value of h over 0.1, 0.01, 0.001, and 0.0001 we get the same slope of the secant line, namely 2.324202. In other words, the slope has not changed in the first six decimal places as h got smaller and so it seems we have a good approximation. If we tried $h = 0.00001$ we would expect to get the same result.

Now, is the CO2 function special or do secant line slopes approximate the tangent line slope easily? The answer is not always and so we now create an example. Since we are using R we can create such examples with function that would be difficult, nearly impossible?, to do by hand. Consider the function $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$. We would like to estimate the slope of the tangent line at $a = 4.30931$. We pause for a moment as you should be suspicious of the point a here. It is carefully chosen to create this example.

R Code 8.2: Secant Line Slopes with Different Values of h

```
> f<- function(x){exp(x)+(2.5)^x*sin(2*pi*x)-10}
> h<- c(0.1,0.01,0.001,0.0001)
> a<- 4.30931
> (f(a+h)-f(a-h))/(a+h-(a-h))
```

```
[1] -1.08443172 -0.00571720  0.00656511  0.00668808
```

The slopes of the secant line from $x = a - h$ to $x = a + h$ for $a = 4.30931$ and h over $0.1, 0.01, 0.001$ and 0.0001 for this function is in R Code 8.2. In this case, our secant line slopes not only do not converge in any sense, the first two values are negative while the last two are positive. It is not even clear if we should conclude that the slope of the tangent line at 4.30931 is positive or negative. What is going on? Figure 8.2, which has the secant line in blue for $h = 0.01$, gives us a hint. It appears that at $a = 4.30931$ we may have a horizontal tangent line and as h gets smaller the secant lines are shifted from negative to positive. An even better example of why the "straddle" method for a tangent line approximation has issues is given in section 8.2, which is a short project. We now move to a slightly different approach to overcome these issues.

Instead of "straddling" the tangent line with a secant line we will calculate secant line slopes to the left and to the right of the tangent line. This formula is given in M-Box 8.1. This second definition for the slope of a secant line may not at first look like a secant line slope. This is a good example of how information is lost due to simplification. To arrive at the formula, given the points $(a, f(a))$ and $(a + h, f(a + h))$ the slope of the secant line is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

M-Box 8.1: Slope of the Secant II

The slope of the secant line from $(a, f(a))$ to $(a + h, f(a + h))$ is given by

$$\frac{f(a+h) - f(a)}{h}$$

Note that if $h < 0$ the secant line is drawn from $(a, f(a))$ to the left to $(a + h, f(a + h))$ since $a + h < a$.

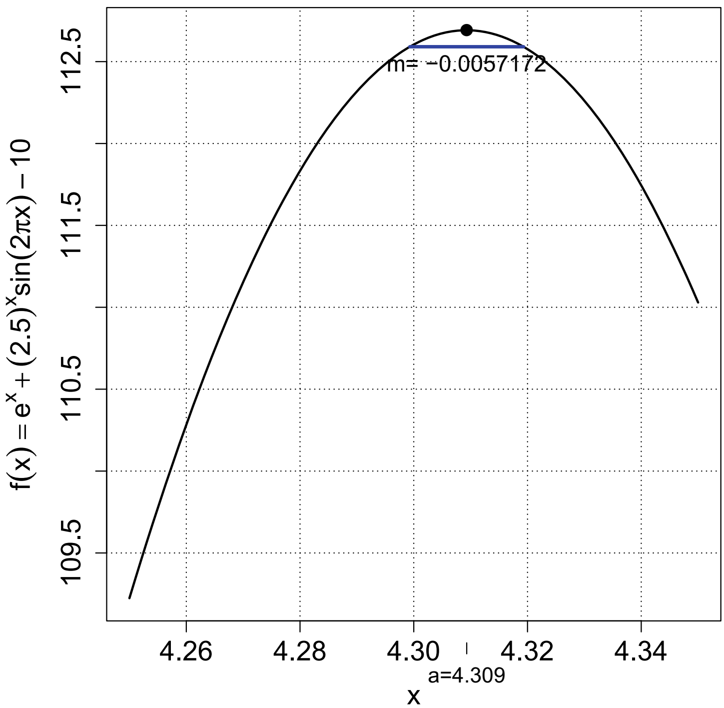


Fig. 8.2 Graph of the function $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$ with a second line from $(4.30931 - 0.01, f(4.30931 - 0.01))$ to $(4.30931 + 0.01, f(4.30931 + 0.01))$.

In this case, we are calculating the slope of the secant line from $(a, f(a))$ to either the right or left of the point depending on whether or not h is positive or negative. Before moving back to the function $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$, we will do a simpler example. In general, we will not do this by hand in this chapter and once we move to R it no longer matters how difficult the function $f(x)$ is to work with.

Example 8.1. *Consider the function $f(x) = x^2$ and the point $a = 1$. Estimate the slope of the tangent line at $a = 1$ to the left with $h = -0.1$ and to the right with $h = 0.01$.*

Solution. In both cases we use the same formula. For $h = -0.1$ we have

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{f(1-0.1)-f(1)}{-0.1} \\
&= \frac{f(0.9)-f(1)}{-0.1} \\
&= \frac{(0.9)^2-(1)^2}{-0.1} \\
&= \frac{0.81-1}{-0.1} \\
&= \frac{-0.19}{-0.1} \\
&= 1.9.
\end{aligned}$$

For $h = 0.01$ we have

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{f(1+0.01)-f(1)}{0.01} \\
&= \frac{f(1.01)-f(1)}{0.01} \\
&= \frac{(1.01)^2-(1)^2}{0.01} \\
&= \frac{1.0201-1}{0.01} \\
&= \frac{0.0201}{0.01} \\
&= 2.01.
\end{aligned}$$

Generally, although it is not a guarantee, the slope of the tangent line will be between the slopes of the secant lines on the left and the right. In the case, we have evidence that the slope of the tangent line of $f(x) = x^2$ at $a = 1$ is between 1.9 and 2.01. $\square$

Back to the function $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$. R Code box 8.3 calculates secant line slopes from $(4.30931, f(4.30931))$ to points to the right $(4.30931 + h, f(4.30931 + h))$, where the distance on the x-axis between the points, h, decreases from 0.1 to 0.00001. R Code box 8.4 is the same except the secant line slopes from $(4.30931, f(4.30931))$ to $(4.30931 + h, f(4.30931 + h))$ are to the left, the values of h are negative, but the distances between the points along the x-axis still decrease from 0.1 to 0.00001.

R Code 8.3: Secant Lines Slopes to the "Right"

```
> f<-function(x){(exp(x)+(2.5)^x*sin(2*pi*x))-10}
> h<-c(0.1,0.01,0.001,0.0001,0.00001)
> a<-4.30931
> (f(a+h)-f(a))/h
```

```
[1] -98.15304952 -10.04911578  -0.99811537
[4] -0.09378031  -0.00335753
```

R Code 8.4: Secant Lines Slopes to the "Left"

```
> f<-function(x){(exp(x)+(2.5)^x*sin(2*pi*x))-10}
> h<-c(-0.1,-0.01,-0.001,-0.0001,-0.00001)
> a<-4.30931
> (f(a+h)- f(a))/h
```

```
[1] 95.9841861 10.0376814  1.0112456
[4] 0.1071565  0.0167362
```

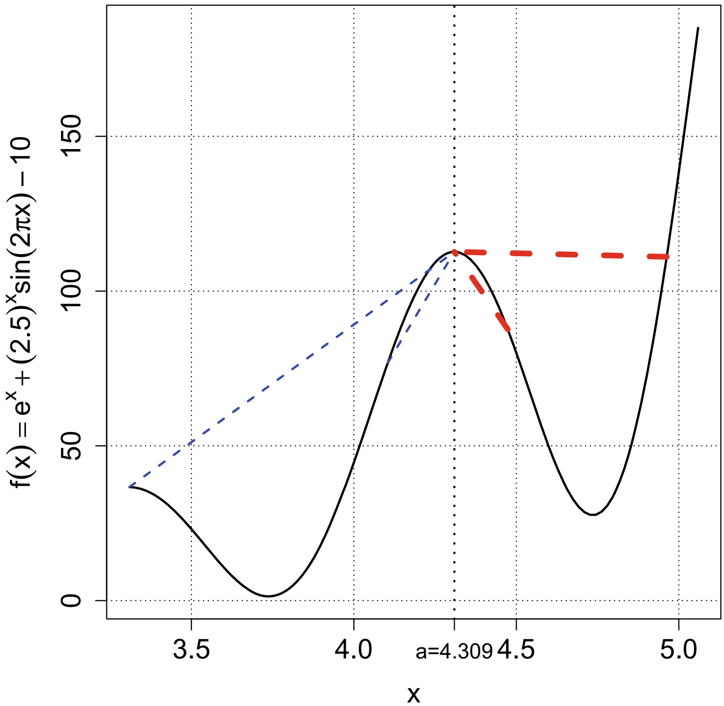


Fig. 8.3 Graph of the function $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$ with four secant lines centered on $(4.30931, f(4.30931))$.

The results in R Code boxes 8.3 and 8.4 suggest that secant line slopes to the right of $(4.30931, f(4.30931))$ are negative and getting closer to 0, while secant line slopes to the left are positive and also getting closer to 0. For example, figure 8.3 has four secant lines from $(4.30931, f(4.30931))$; two to the left and two to the right. Notice how the lines to the left have positive slopes while the lines to the right have negative slopes. Also notice how as we use successively smaller values of h the change from one estimate of slope to the next gets smaller. For example, in R Code box 8.3 as we go from $h = 0.1$ to $h = 0.01$ the slope changed from -98.15304952 to -10.04911578, but when we go from $h = 0.0001$ to $h = 0.00001$ the slope changed from -0.09378031 to -0.00335753 with no change in the first decimal place. At this point we have some confidence that the slope from the right is about -0.0. In R Code box 8.4 we have yet to repeat the first decimal place. But, for example, if we use $h = -0.0000001$ and $h = -0.00000001$ then we get slopes of 0.006790088 and 0.006701839. In this case, we have evidence that the slope is about 0.0067, although this is not a guarantee. We can and will do better.

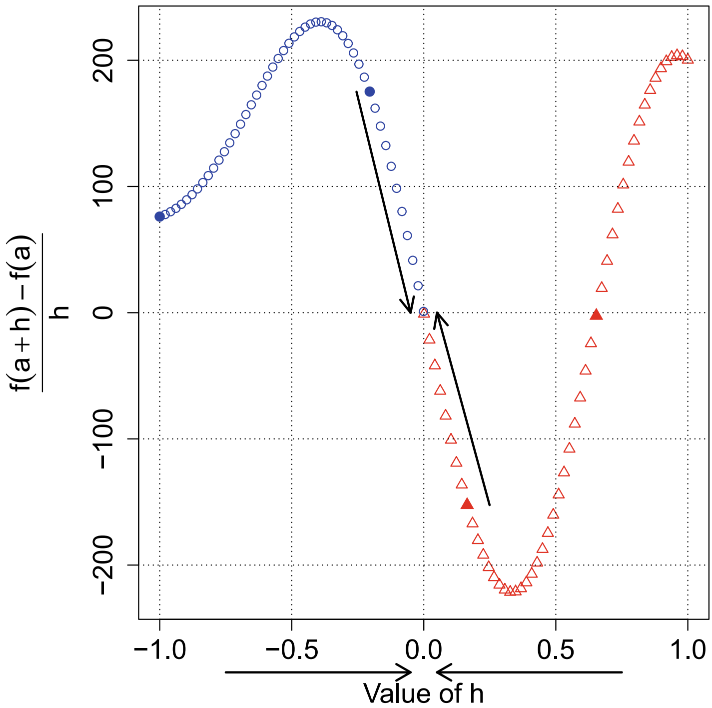


Fig. 8.4 Values of the slopes of the secant lines of $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$ from $(4.30931, f(4.30931))$ to $(4.30931 + h, f(4.30931 + h))$.

One more graph to illustrate how the slopes in the secant lines change are given in figure 8.4. Each point on the graph is the slope from $(4.30931, f(4.30931))$ to $(4.30931 + h, f(4.30931 + h))$ of the function $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$. The blue circles are secant line slopes with a negative value of h, while the red triangles are secant line slopes with a positive value of h. The four darkened points coincide with the four secant lines in figure 8.3. What we learn from this graph is that since from both sides the secant line slopes end up at a value of 0, we are confident that the slope of the tangent line of $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$ at $(4.30931, f(4.30931))$ is 0 or at least really close to 0. An even better example of why the "straddle" method for a tangent line approximation has issues is given in section 8.2, which is a short project.

Example 8.2. *Using the function $f(x) = x\sin(x)$ estimate $f'(a)$ with $a = \pi$ to six decimal places. Justify your response with data.*

Solution. We begin by setting up the necessary code, which starts in R Code box 8.5. The function **f** is defined and we use **curve** for a quick graph, which is given in figure 8.5. The graph includes secant lines with $h = -1$ and $h = 1$ for context. It is not absolutely necessary to create a graph, but minimally it can give us a sense of the magnitude of $f'(\pi)$ as well as the sign. In this case, $f'(\pi)$ is clearly negative. It also appears that secant lines to the left should be less negative than the secant lines to the right. Our computational results should reflect this and if not we know something is incorrect.

We include **options(digits=12)** in the R code to increase the digits in the output as the default would not provide enough digits, or decimal places, for the accuracy required of the problem. We define **a=pi** and then values of h as vectors for both positive values **h_p** and negative values **h_n**. Note that this is the final answer but ahead of time we did not know how small h the values of h were necessary to get the slopes to stabilize to six decimal places. There is some trial and error involved and at the same time we do not need to provide more computations than necessary. The output for the slope of the secant lines using **h_p** are in R Code box 8.6. The results with **h_n** are in R Code box 8.5.

R Code 8.5: Secant Line Slopes "Left" and "Right"

```
> f<-function(x){x*sin(x)}
> curve(f,0,6,lwd=2)
> options(digits=12)
> a <- pi
> h_p <- c(0.1,0.01,0.001,0.0001,0.00001,
0.000001,0.0000001)
> h_n <- c(-0.1,-0.01,-0.001,-0.0001,-0.00001,
-0.000001,-0.0000001)
> (f(a+h_p)-f(a))/h_pr
```

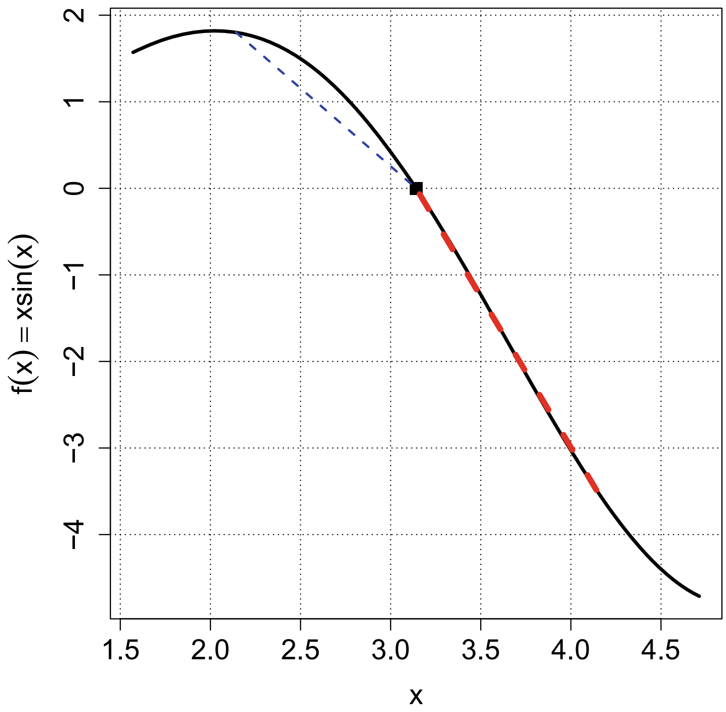


Fig. 8.5 Graph of $f(x) = x\sin(x)$ with a square point at $x = \pi$ and secant lines drawn to $x = \pi - 1$ and $x = \pi + 1$.

```
[1] -3.23619269985 -3.15154012731 -3.14259212982
[4] -3.14169264836 -3.14160265356 -3.14159365403
[7] -3.14159274845
```

R Code 8.6: Continuation of the Previous R Code

```
> (f(a+h_n)-f(a))/h_n
```

```
[1] -3.03652586656 -3.13154046064 -3.14059213016
[4] -3.14149264836 -3.14158265356 -3.14159165403
[7] -3.14159254845
```

We organize the data in a table to make it easier to interpret. Note that the results in the tables agree with our observations about the graph above. The values of h get closer to 0 as we go down the table and we also see that the decimal places begin to stabilize. From the last line of the table we can conclude that the slope of the tangent line is between -3.14159274845 and -3.14159254845. In particular, we are confident that the $f'(\pi) = -3.141593$, rounded to six decimal places.

h	secant slope	h	secant slope
0.1	-3.23619269985	-0.1	-3.03652586656
0.01	-3.15154012731	-0.01	-3.13154046064
0.001	-3.14259212982	-0.001	-3.14059213016
0.0001	-3.14169264836	-0.0001	-3.14149264836
0.00001	-3.14160265356	-0.00001	-3.14158265356
0.000001	-3.14159365403	-0.000001	-3.14159165403
0.0000001	-3.14159274845	-0.0000001	-3.14159254845

□

The computational approach to approximating the slope of tangent lines has advantages and disadvantages. The complexity of the function does not matter much since all computations are done by the computer and we could create code to generalize this procedure for any function. On the other hand, we are estimating the slope of a function at one point, and it would be more useful to have a function that provides slopes of the tangent line at any point, which we will work through in the next chapter.

8.1 Exercises

1. Estimate (by hand) the slope of the tangent line at $a = 2$ for the function $f(x) = x^2$ to the left with $h = -0.01$
2. Estimate (by hand) the slope of the tangent line at $a = 2$ for the function $f(x) = x^2$ to the right with $h = 0.001$
3. Estimate (by hand) the slope of the tangent line at $a = 1$ for the function $f(x) = x^2 + 4$ to the right with $h = 0.1$
4. Estimate (by hand) the slope of the tangent line at $a = 3$ for the function $f(x) = x^2 + 4$ to the left with $h = -0.01$
5. Estimate (by hand) the slope of the tangent line at $a = 1$ for the function $f(x) = x^2 + x - 5$ to the right with $h = 0.01$
6. Estimate (by hand) the slope of the tangent line at $a = 2$ for the function $f(x) = x^2 + x - 5$ to the left with $h = -0.01$
7. Estimate (by hand) the slope of the tangent line at $a = -1$ for the function $f(x) = 2x^2 - 3x + 10$ to the right with $h = 0.1$
8. Estimate (by hand) the slope of the tangent line at $a = -4$ for the function $f(x) = 2x^2 - 3x + 10$ to the left with $h = -0.01$

9. Explain each line of the R code below.

```
> f<-function(x){sin(x)}
> x_values<- c(0,pi/4,pi/2,pi)
> round(f(x_values),2)
[1] 0.00 0.71 1.00 0.00
```

10. Explain each line of the R code below.

```
> f<-function(x){x^2}
> x_values<- c(1,2,3,4)
> f(x_values)
[1]  1  4  9 16
```

11. Explain each line of the R code below.

```
> f<-function(x){x+4}
> a<- 1
> h<- c(1,0.5,0.1,0.01)
> f(a+h)
[1] 6.00 5.50 5.10 5.01
```

12. Explain each line of the R code below.

```
> f<-function(x){x^2}
> a<- 2
> h<- c(1,0.5,0.1,0.01)
> f(a+h)
[1] 9.0000 6.2500 4.4100 4.0401
```

13. Explain each line of the R code below.

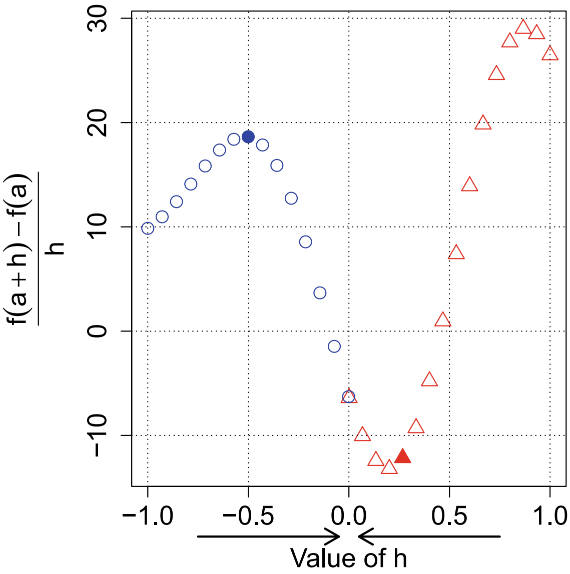
```
> f<-function(x){x^2}
> a<- 2
> h<- c(0.1,0.01,0.001,0.0001)
> (f(a+h)-f(a))/h
[1] 4.1000 4.0100 4.0010 4.0001
```

14. Explain each line of the R code below.

```
> f<-function(x){2*x^2 -3*x + 10}
> a<- 1
> h<- c(0.1,0.01,0.001,0.0001)
> (f(a+h)-f(a))/h
[1] 1.2000 1.0200 1.0020 1.0002
```

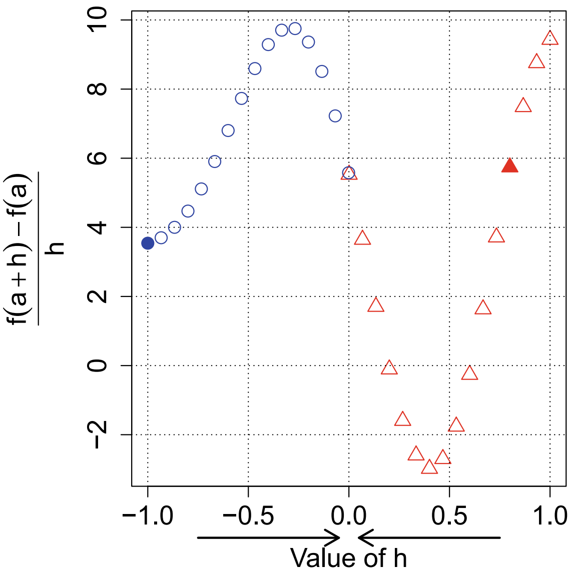
15. The graph here is similar to figure 8.4 but with a different function $f(x)$. The graph represents results in estimating $f'(8)$.

    a. Fill in the value of a, h, and m for the solid blue dot in the formula $\frac{f(a+h)-f(a)}{h} = m$.
    b. Fill in the value of a, h, and m for the solid red triangle in the formula $\frac{f(a+h)-f(a)}{h} = m$.
    c. Given this graph, what is your estimate for $f'(8)$?



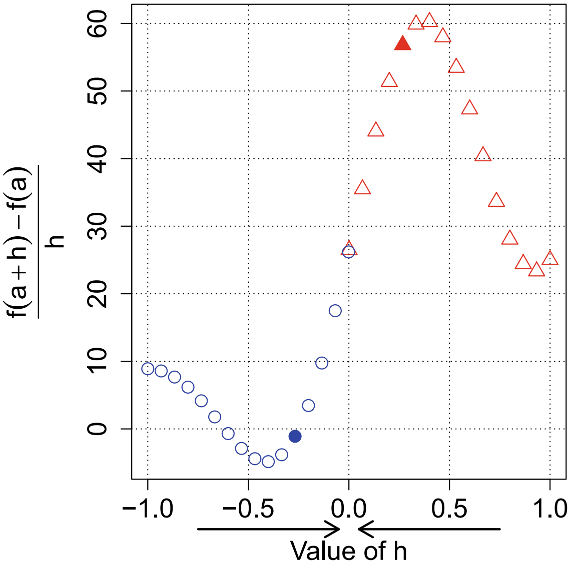
16. The graph here is similar to figure 8.4 but with a different function $f(x)$. The graph represents results in estimating $f'(3)$.

    a. Fill in the value of a, h, and m for the solid blue dot in the formula $\frac{f(a+h)-f(a)}{h} = m$.
    b. Fill in the value of a, h, and m for the solid red triangle in the formula $\frac{f(a+h)-f(a)}{h} = m$.
    c. Given this graph, what is your estimate for $f'(3)$?



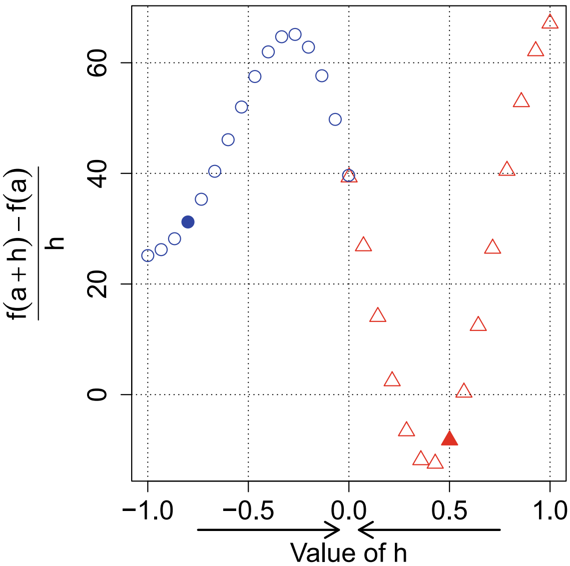
17. The graph here is similar to figure 8.4 but with a different function $f(x)$. The graph represents results in estimating $f'(-2)$.

a. Fill in the value of a, h, and m for the solid blue dot in the formula $\frac{f(a+h)-f(a)}{h} = m$.
b. Fill in the value of a, h, and m for the solid red triangle in the formula $\frac{f(a+h)-f(a)}{h} = m$.
c. Given this graph, what is your estimate for $f'(-2)$?



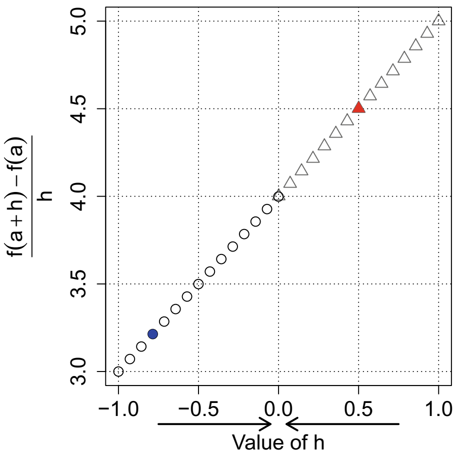
18. The graph here is similar to figure 8.4 but with a different function $f(x)$. The graph represents results in estimating $f'(-5)$.

a. Fill in the value of a, h, and m for the solid blue dot in the formula $\frac{f(a+h)-f(a)}{h} = m$.
b. Fill in the value of a, h, and m for the solid red triangle in the formula $\frac{f(a+h)-f(a)}{h} = m$.
c. Given this graph, what is your estimate for $f'(-5)$?



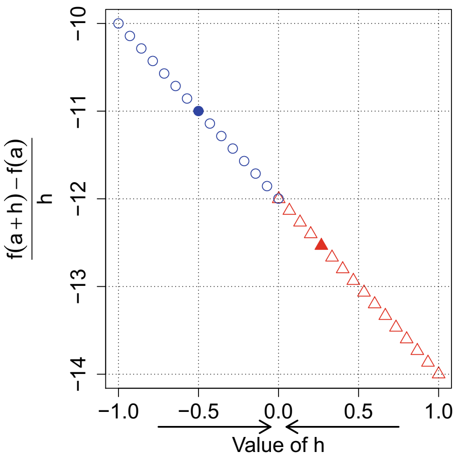
19. The graph here is similar to figure 8.4 but with a different function $f(x)$. The graph represents results in estimating $f'(2)$.

a. Fill in the value of a, h, and m for the solid blue dot in the formula $\frac{f(a+h)-f(a)}{h} = m$.
b. Fill in the value of a, h, and m for the solid red triangle in the formula $\frac{f(a+h)-f(a)}{h} = m$.
c. The function $f(x) = x^2$, given this evaluate $\frac{f(a+h)-f(a)}{h}$ for the two equations above. How close were you in reading off the graph for the value of m in each case?
d. Given this graph, what is your estimate for $f'(2)$?



20. The graph here is similar to figure 8.4 but with a different function $f(x)$. The graph represents results in estimating $f'(3)$.

a. Fill in the value of a, h, and m for the solid blue dot in the formula $\frac{f(a+h)-f(a)}{h} = m$.
b. Fill in the value of a, h, and m for the solid red triangle in the formula $\frac{f(a+h)-f(a)}{h} = m$.
c. The function $f(x) = -2x^2$, given this evaluate $\frac{f(a+h)-f(a)}{h}$ for the two equations above. How close were you in reading off the graph for the value of m in each case?
d. Given this graph, what is your estimate for $f'(3)$?



21. Using the function $f(x) = x^2$ create a table of the values of the slope of the secant line from $a = 2$ to $a + h$ with $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$ and $h = -0.1, -0.01, -0.001, -0.0001, -0.00001$. Make a conjecture for the value of $f'(a)$. Explain your answer.

22. Repeat exercise 21 with $a = 3$, $a = 4$, and $a = 5$. Make a conjecture for the value of $f'(a)$ for each value of a. Explain your reasoning.
23. Repeat exercises 21 with $a = -3$, $a = -4$, and $a = -5$. What is your estimate of $f'(a)$ for each value of a? Explain your reasoning.
24. Based on exercises 21 to 23, make a conjecture for a formula for $f'(x)$. Explain your reasoning.
25. Repeat exercise 21 for $f(x) = x^3$ at $a = 4$.
26. Repeat exercise 21 for $f(x) = x^3$ at $a = -2$.
27. Repeat exercise 21 for $f(x) = \sin(x)$ at $a = 0$. Tip: Use $\sin(x)$ to define the function.
28. Repeat exercise 21 for $f(x) = \sin(x)$ at $a = \pi/2$. Tip: Use pi/2 to define a in R.
29. Repeat exercise 21 for $f(x) = \cos(x)$ at $a = 0$.
30. Repeat exercise 21 for $f(x) = \cos(x)$ at $a = \pi/2$.
31. Repeat exercise 21 for $f(x) = \ln(x)$ at $a = 1$. Note: Use log(x) for the natural log, ln, in R.
32. Repeat exercise 21 for $f(x) = \ln(x)$ at $a = 2$.
33. Repeat exercise 21 for $f(x) = \sqrt{x}$ at $a = 1$. Tip: Use sqrt(x) to define the function.
34. Repeat exercise 21 for $f(x) = \sqrt{x}$ at $a = 4$.

For the next set of exercises add options(digits=12) to your R code to increase the number of digits in the output.

35. Estimate $f'(a)$ for $f(x) = \sqrt{x}$ and $a = 2$ to seven decimal places. Your answer must include your estimate of $f'(2)$, the values of h (positive and negative) used for your estimate, and the complete output of the slope of the secant line associated with the values of h.
36. Repeat exercise 35 with $f(x) = \sqrt{x}$, $a = 12$, and an accuracy of eight decimal places.
37. Repeat exercise 35 with $f(x) = e^x$, $a = 1$, and an accuracy of six decimal places.
38. Repeat exercise 35 with $f(x) = e^x$, $a = 2$, and an accuracy of five decimal places.
39. Repeat exercise 35 with $f(x) = \sin(x)$, $a = 1$, and an accuracy of five decimal places.
40. Repeat exercise 35 with $f(x) = \sin(x)$, $a = 3$, and an accuracy of five decimal places.
41. Repeat exercise 35 with $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$, $a = 4$, and an accuracy of three decimal places.
42. Repeat exercise 35 with $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$, $a = 5$, and an accuracy of three decimal places.
43. Repeat exercise 35 with the global temperature function from the function gallery, use the last year in the data set for a, and an accuracy of four decimal places. Use your result in a sentence that explains the context of the calculation.

44. Repeat exercise 35 with the world wind function from the function gallery, use the last year in the data set for a, and an accuracy of four decimal places. Use your result in a sentence that explains the context of the calculation.
45. Repeat exercise 35 with the U.S. wind function from the function gallery, use the last year in the data set for a, and an accuracy of four decimal places. Use your result in a sentence that explains the context of the calculation.
46. Repeat exercise 35 with the Spain wind function from the function gallery, use the last year in the data set for a, and an accuracy of four decimal places. Use your result in a sentence that explains the context of the calculation.

8.2 Project: Which Secant Line Approximation is Better?

This project provides an example to demonstrate what can go wrong with using a secant line slope that approximates a tangent line slope by the "straddle" method in M-Box 4.1, while using the "left" and "right" method from M-Box 8.1. We will attempt to estimate the slope of the tangent line at $a = 0$ of the function $f(x) = 1 + x - \sqrt[3]{(x^2)(1 - x^2)^2}$. Create a graph of the function in R and add three secant lines from $x = 0 - h$ to $x = 0 + h$ for $h = 0.1, 0.2$, and 0.4. Also, add three secant lines to the left and three to the right of $a = 0$ using the same values of h. You now have nine secant lines on your graph.

1. What are the slopes of each of the nine secant lines?
2. Based on the three straddle secant lines, what is your estimate of the slope of the tangent line at $a = 0$?
3. Based on the three secant lines to the left, what is your estimate of the slope of the tangent line at $a = 0$?
4. Based on the three secant lines to the right, what is your estimate of the slope of the tangent line at $a = 0$?
5. Given the information you have what is your final estimate of the slope of the tangent line at $a = 0$? If you have any concerns about your estimate or you think there is some issue, then please articulate your thoughts in a few sentences.

8.3 Project: Estimating e

One way to estimate the value of e is based on the following fact. Draw values from a uniform random variable from 0 to 1. Sum the values and record the number of values it takes to exceed 1. The value of e is approximated by the average number of draws it takes to exceed 1. Use the idea of successive approximation to estimate e to as many decimal places as you can. R-Code box 8.1 has code to get you started, without much explanation. The key is the **while** loop. The code within a while loop will continue as long as the statement of the while loop remains true. Figuring out the code is the key challenge of this project. Explain each line of code? Provide your estimate of e? Do you think this way of estimating e is effective from a computational power perspective? [39]

R Code 8.1:

```
> SumUnif<-function() {
+   valueSum <- runif(1,0,1)
+   draws <- 1
+   while (valueSum < 1) {
+     valueSum <- valueSum+runif(1,0,1)
+     draws <- draws+1
+   }
+   return(draws)
+ }
> trials<-100
> output<-replicate(trials,SumUnif())
> mean(output)
```

8.4 Project: Estimating π

Consider a square with side length 1 and an inscribed circle with radius 1, both centered at the origin. If we randomly selected points inside the square then the proportion of those inside the square will equal the ratio of the area of the circle to the square, which in this case is $\pi/1 = \pi$. To simplify this, we will only select points in the first quadrant and then multiply by 4. Use the idea of successive approximation to estimate π to as many decimal places as you can. R-Code box 8.1 has code to get you started, without explanation. Figuring out the code is the key challenge of this project. Explain each line of code? Provide your estimate of π? Do you think this way of estimating π is effective from a computational power perspective? [17]

R Code 8.1:

```
> ApproxPi <- function(n) {
+     x <- runif(n)
+     y <- runif(n)
+     dist_x_y <- sqrt(x^2 + y^2)
+     return(4 * sum(dist_x_y < 1.0) / n)
+ }
> ApproxPi(100)
```

Chapter 9
The Derivative Graphically

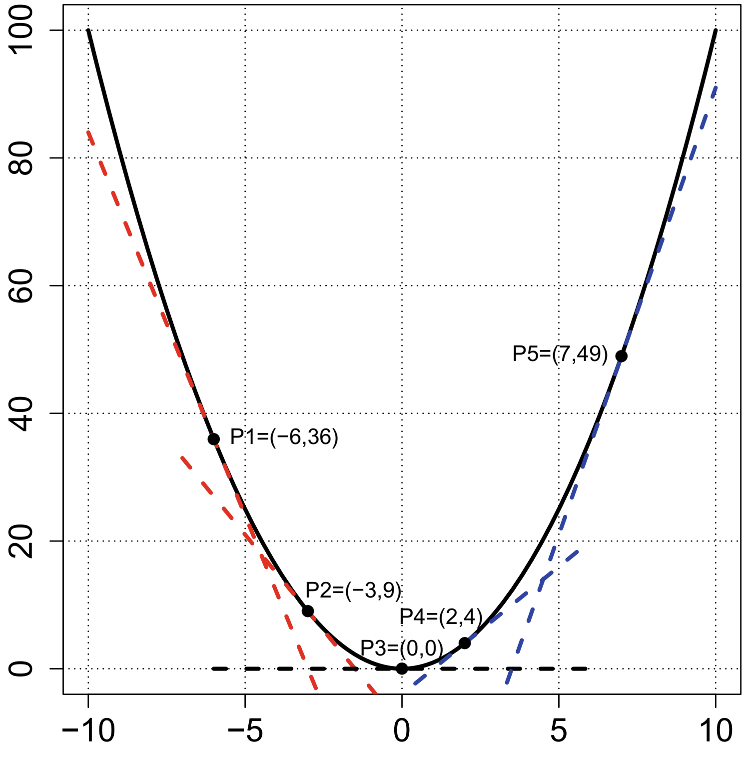


Fig. 9.1 Graph of the function $f(x) = x^2$ with five tangent lines.

In the previous chapter we estimated the slope of the tangent line at one specific point. We now take a qualitative look at the slopes of tangent lines and sketch the graph of the derivative of a function. Consider figure 9.1 which is the graph of $f(x) = x^2$ with five tangent lines at the points labeled P1 through P5. The slopes of the tangent lines at P1 and P2 are negative, while the slopes at P4 and P5 are

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_9

positive. If we were to order the slopes at the points we would have P1 $<$ P2 $<$ P3 $<$ P4 $<$ P5. Let $|$P1$|$, etc. represent the absolute value of the slope of the tangent line. How would we order the absolute value of the tangent line slopes? The answer is $|$P3$|$ $<$ $|$P4$|$ $<$ $|$P2$|$ $<$ $|$P1$|$ $<$ $|$P5$|$.

In general for figure 9.1, the slope of all tangent lines with $x < 0$ are negative while the slope of all tangent lines with $x > 0$ are positive. We can say more, since from $x = -10$ to $x = 0$ the slopes are increasing from larger negative to less negative and eventually 0. Similarly, from $x = 0$ to $x = 10$ the slope of the tangent lines are increasing. A sketch of the graph of the derivative function would start at a negative y-value at $x = -10$, increase to 0 at $x = 0$, and then become a positive y-value and continue to increase until $x = 10$.

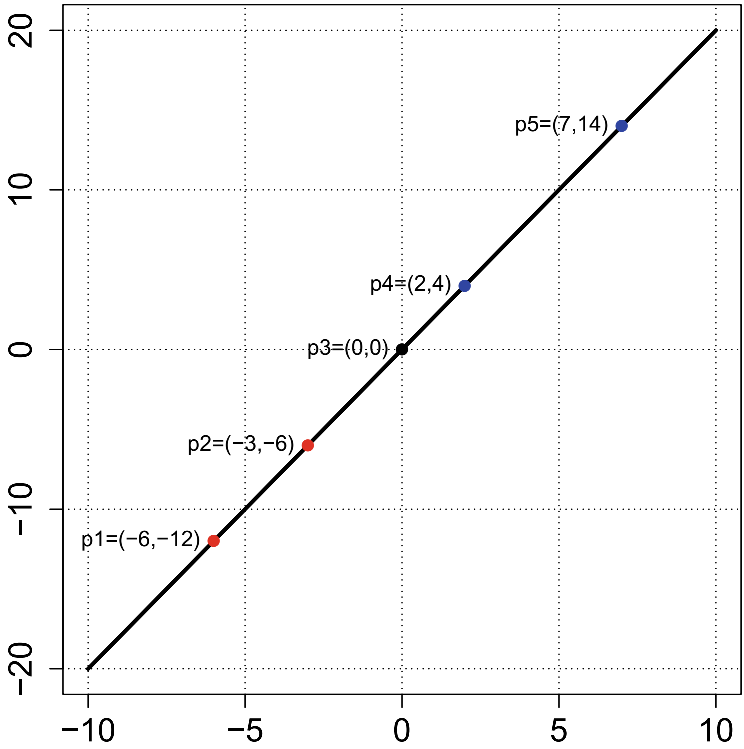


Fig. 9.2 Graph of the derivative of the function $f(x) = x^2$ with five points labeled.

The graph of the derivative of $f(x) = x^2$ is given in figure 9.2 (Do not worry about where this came from, you will know soon enough). Note that the points are labeled with a lowercase p to correspond with the uppercase P in the original graph. So, for example, p2= $(-3, -6)$ in the derivative graph has the same x value as the point P2= $(-3, 9)$ in the first graph, figure 9.1. What is the connection? The y-value of p2 is the slope of the tangent line at the point P2. In other words, the slope of the tangent line at P2 is -12. We'll worry later about how we arrived at the graph of the derivative of $f(x) = x^2$; for now focus on the relationship between the two graphs. Let us check the statement we made above and pay close attention to when we are talking about, for example, P1 (capital P points) or p1 (lowercase p points):

1. *The slopes of the tangent lines at P1 and P2 are negative,* The y-value of the points p1 and p2 are negative and both points p1 and p2 are below the x-axis.
2. *The slopes at P4 and P5 are positive.* Similarly, the y-value of the points p1 and p2 are positive and both points p4 and p5 are above the x-axis.
3. If we were to order the slopes at the points we would have P1 $<$ P2 $<$ P3 $<$ P4 $<$ P5. Notice that the points p1 through p5 are ordered vertically meaning the y-values, the slopes at the corresponding capital P points, are increasing.
4. *How would we order the absolute value of the tangent line slopes? The answer is* |P3| $<$ |P4| $<$ |P2| $<$ |P1| $<$ |P5|. Check this by considering the absolute value of the y-value of each of the lowercase p points.
5. *In general, the slope of all tangent lines with $x < 0$ are negative* In figure 9.2 all points to the left of $x = 0$ are below the x-axis, which means the slope of tangent lines of figure 9.1 to the left of $x = 0$ are negative.
6. *The slope of all tangent lines with $x > 0$ are positive.* Similarly, in figure 9.2 all points to the right of $x = 0$ are above the x-axis, which means the slope of tangent lines of figure 9.1 to the right of $x = 0$ are positive.

In figure 9.3 we stacked the two graphs and added vertical lines over both graphs at the labeled points. The key in interpreting the two graphs is to recognize that the slope of the tangent line at capital P points in the top graphs is given by the y-value of the lower case p points in the bottom graph. The corresponding capital P and lower case p have the same x-value.

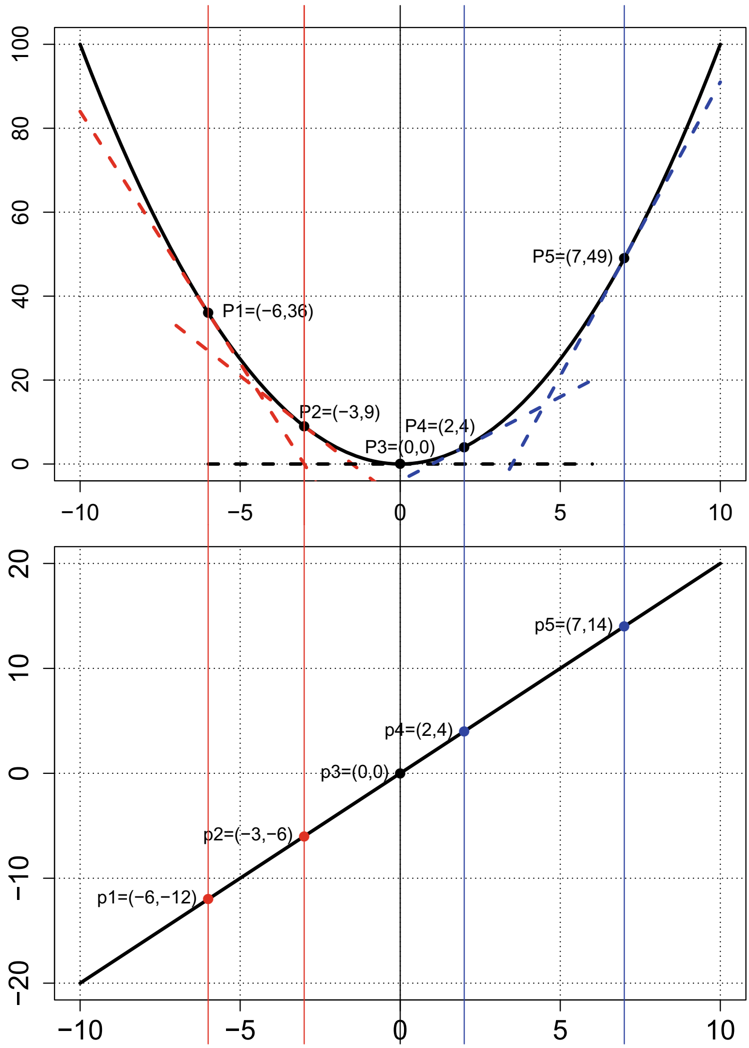


Fig. 9.3 The graph of $f(x) = x^2$ and its derivative below.

M-Box 9.1: Relationship Between a Graph and its Derivative Graph

The slope of the tangent line at point (a, b) of $f(x)$ is the y–value of the point at $x = a$ of the derivative graph, $f'(x)$, of $f(x)$.

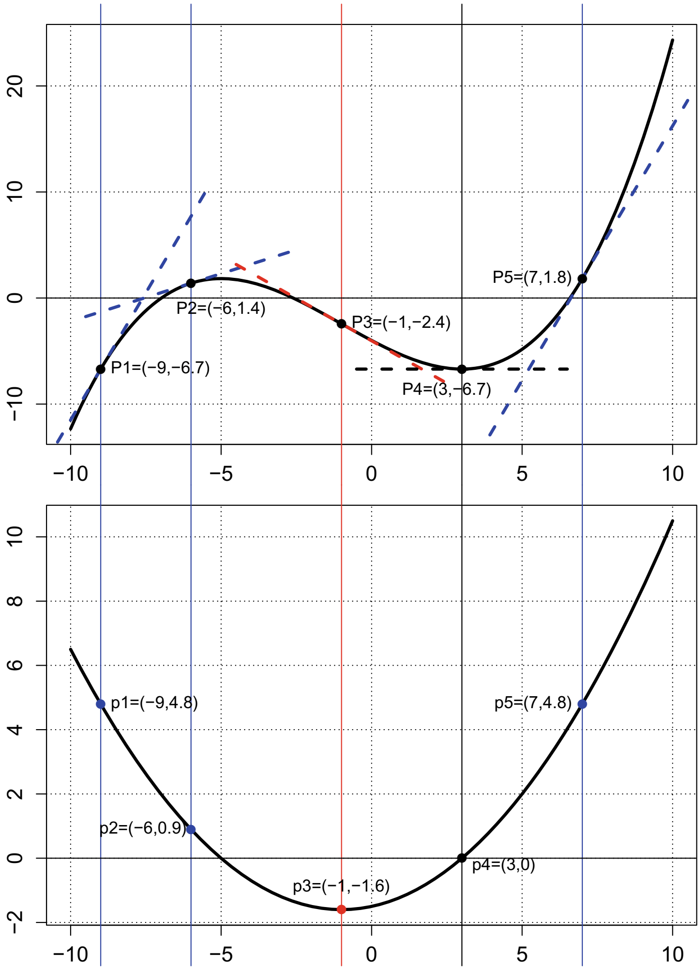


Fig. 9.4 The graph of $f(x) = .1 * (x^3/3 + x^2 - 15 * x) - 5$ and its derivative below.

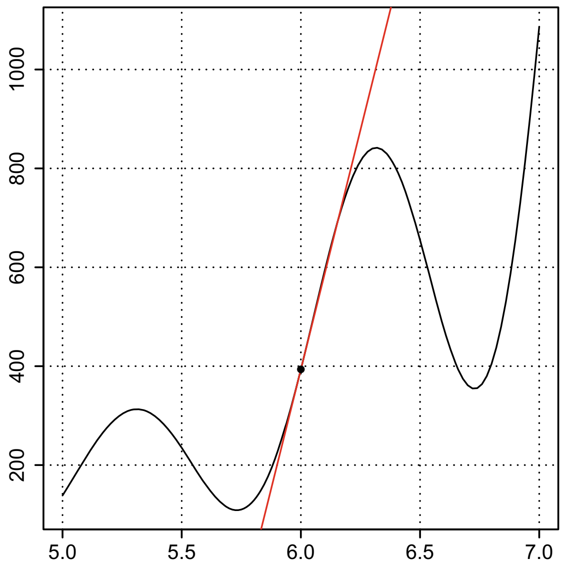
Another example is given in figure 9.4 where the top graph is $f(x) = 0.1(x^3/3 + x^2 - 15x) - 4$ and the bottom graph is its derivative. It is worth repeating the connection between the two graphs. The connection between the graphs is the slope of the tangent line at capital P points in the top graphs is given by the y-value of the lower case p points in the bottom graph. Here are some, not all, observations about the relationship between these two graphs:

1. At points P1, P2, and P5 the slope of the tangent lines are positive, hence points p1, p2, and p5 are above the x-axis with positive y-values.

2. At point P3 the slope of the tangent line is negative, hence point p3 lies below the x-axis with a negative y-value. In fact, at point P3 on the top graph is decreasing the fastest and hence point p3 is the lowest point on the bottom graph.
3. The slope of the tangent line at point P4 is 0, hence point p4 is on the x-axis.
4. From $x = -5$ to $x = 3$ the top graph is decreasing and hence the bottom graph is below the x-axis on this domain.
5. The slope of the tangent line at points P1 and P5 is the same, 4.8, even though point P1 is below the x-axis while point P5 is above.

At this point we have only a couple of ways to calculate, in fact only estimate, the slope of the tangent line. First, Chapter 8 provides a method of successive approximations to estimate the tangent line slope if we are given the function explicitly. If all we are given is a graph, then we could sketch the tangent line, choose two points, and estimate a slope. For example,

Example 9.1. *Estimate the slope of the tangent line of the graph at* $x = 6$.

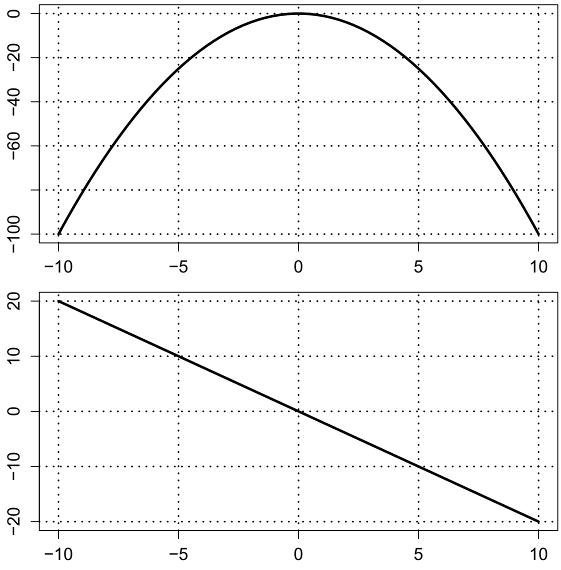


Solution. Note that if the tangent line was not drawn in then that should be done first. The key here is to pick the "best" two points on the tangent line that we can. The tangent line was placed at $x = 6$ and that is usually a good place to start. The point (6, 400) is our first point. Note that this point is on both the tangent line and the function. For a second point it often helps to choose a place where the tangent line crosses a grid line (if there are grid lines). Reasonable choices would be (6.3, 1000), (6.2, 800), (6.1, 600), or (5.9, 200). Using (6, 400) and (6.3, 1000) gives a slope of $(1000 - 400)/(6.3 - 6) = 600/0.3 = 2000$. In this example, any of the second points gives the same slope. This won't always be the case. Also, two people won't always agree on the values of a point, but they should be relatively close. □

9.1 Exercises

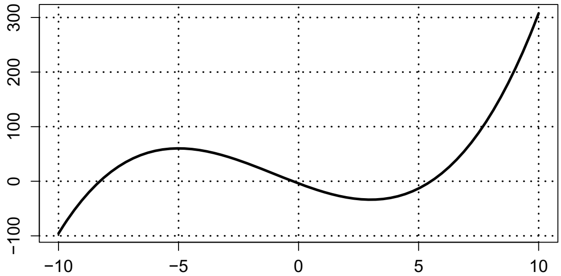
1. In the graphs below the top graph is a function $f(x)$ and the one underneath its derivative $f'(x)$. Answer the following questions by estimating values from reading the graphs.

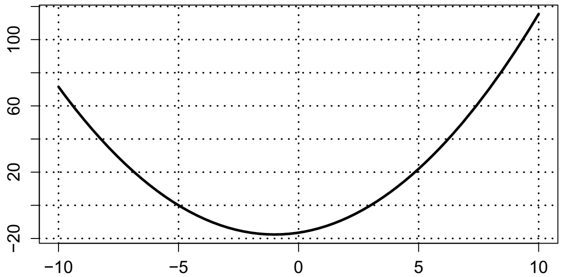
   a. What is $f(5)$?
   b. What is $f'(5)$?
   c. What is the slope of the tangent line of $f(x)$ at $x = 0$?
   d. What is the rate of change of $f(x)$ at $x = -5$?
   e. What is $f(-5)$?
   f. What is the value of x at which the slope of the tangent line of $f(x)$ is -10?



2. In the graphs below the top graph is a function $f(x)$ and the one underneath its derivative $f'(x)$. Answer the following questions by estimating values from reading the graphs.

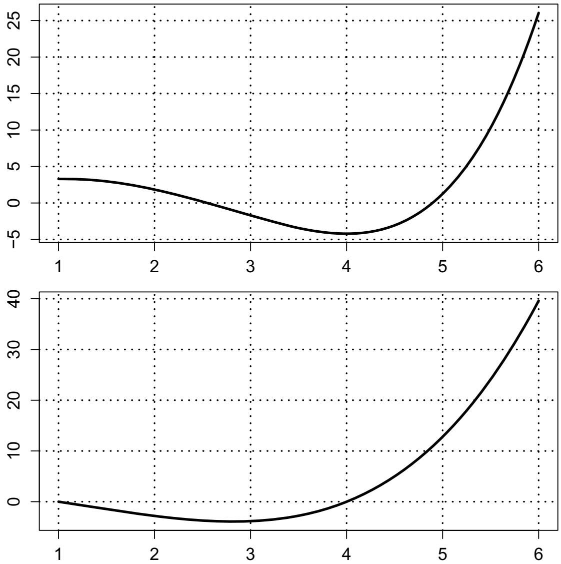
   a. What is $f(0)$?
   b. What is $f(-10)$?
   c. What is the slope of the tangent line of $f(x)$ at $x = -5$?
   d. What is the value of x at which the slope of the tangent line of $f(x)$ is 80?
   e. What is $f'(0)$?
   f. What is the rate of change of $f(x)$ at $x = 10$?





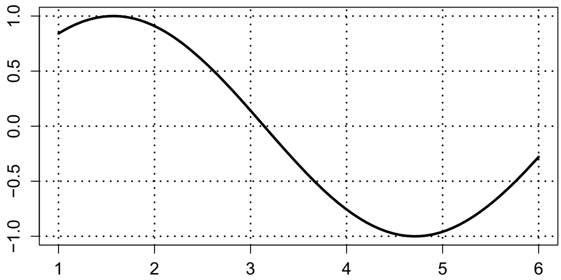
3. In the graphs below the top graph is a function $f(x)$ and the one underneath its derivative $f'(x)$. Answer the following questions by estimating values from reading the graphs.

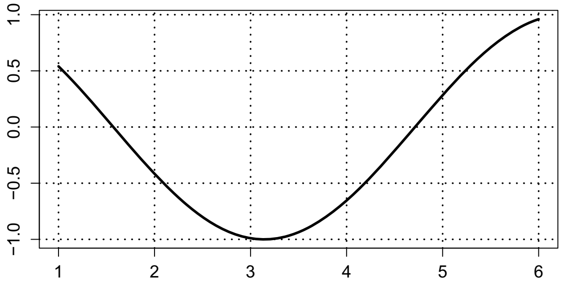
   a. What is the slope of the tangent line of $f(x)$ at $x = 6$?
   b. What is $f'(4)$?
   c. What is $f(4)$?
   d. What is the rate of change of $f(x)$ at $x = 2$?
   e. What is the value of x at which the slope of the tangent line of $f(x)$ is 20?
   f. What is $f(6)$?



4. In the graphs below the top graph is a function $f(x)$ and the one underneath its derivative $f'(x)$. Answer the following questions by estimating values from reading the graphs.

   a. What is the value of x at which the slope of the tangent line of $f(x)$ is 1?
   b. What is $f(2)$?
   c. What is the slope of the tangent line of $f(x)$ at $x = 5$?
   d. What is $f'(2)$?
   e. What is $f(3)$?
   f. What is the rate of change of $f(x)$ at $x = 1$?





5. If $T(t)$ is temperate at time t during a 24 hour day, would we expect $T'(13)$ to be positive or negative? Explain. In words what does $T'(13)$ represent?

6. If $W(t)$ is the temperature of a bathtub of water where t is the time in minutes since the tub was filled, would we expect $W'(12)$ to be positive or negative? Explain. In words what does $W'(12)$ represent?

7. A constant volume of water in flowing through a canal. Let $D(w)$ be the depth in inches of the water given the width of the canal in feet. Would we expect $D'(20)$ to be positive or negative. Explain. In words what does $D'(20)$ represent?

8. Let $O(a)$ be the effective percentage of oxygen at an altitude of a feet above sea level. Would we expect $O'(5000)$ to be positive or negative. Explain. In words what does $O'(5000)$ represent?

9. A ball is thrown up in the air and $s(t)$ is the distance above the ground t minutes after it is thrown. Would we expect $s'(3)$ to be positive or negative. Explain. In words what does $s'(3)$ represent?

10. A roller coaster reaches a high point and begins to go down. Let $D(t)$ be the distance traveled from the peak where t is in seconds. Would we expect $D'(3)$ to be positive or negative? Explain. In words what does $D'(3)$ represent?

11. A spherical balloon is being blown up. Let $D(x)$ be the diameter of the balloon in centimeters given x liters of air in the balloon. Would we expect $D'(2)$ to be positive or negative? Explain. In words, what does $D'(2)$ represent?

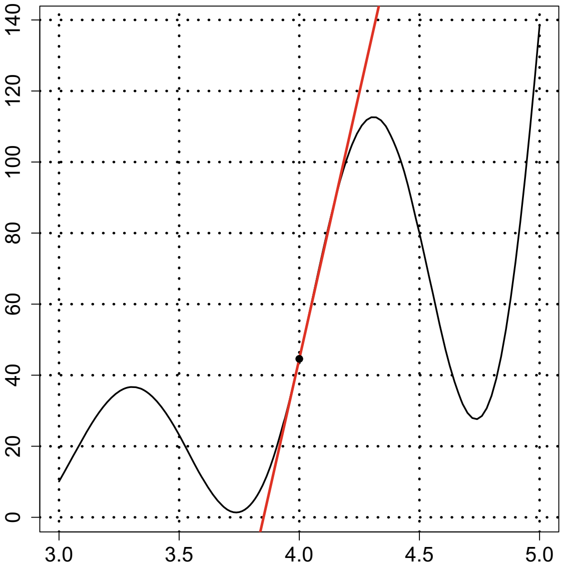
12. A rubber band maintains the shape of a circle as it is being stretched. Let $A(N)$ be the area in cm squared of the circle made by the rubber band given a force of n Newtons stretching the rubber band. Would we expect $A'(4)$ to be positive or negative? Explain. In words, what does $A'(4)$ represent?

13. A fully electric car begins a trip. Let $W(t)$ be a measurement of the number of watts in the battery where t is the time in minutes since the beginning of the trip. Would we expect $W'(28)$ to be positive or negative? Explain. In words what does $W'(28)$ represent?

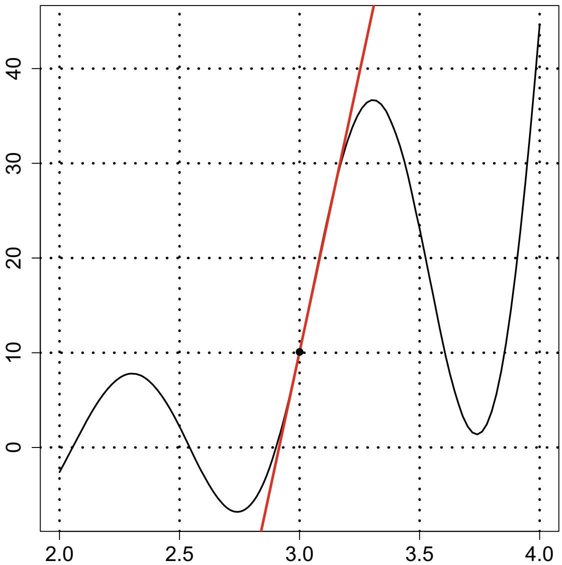
14. Create a scenario similar to the previous few problems. In other words, define a function and the derivative at a specific point. Then explain what the derivative means and if it is likely positive or negative.

For the next eight problems, estimate the slope of the tangent line based on the graph.

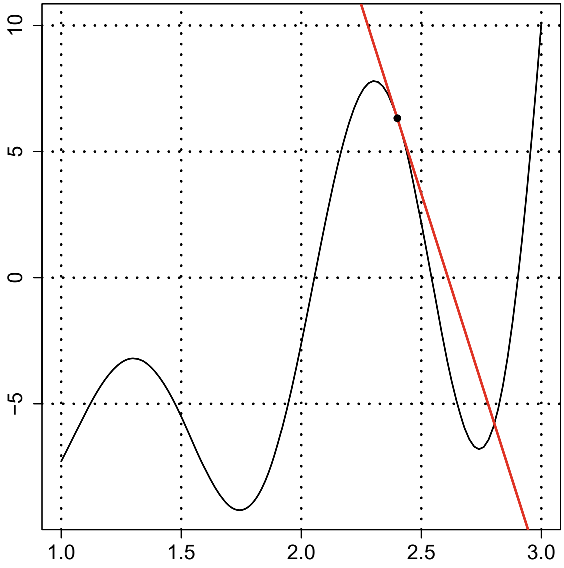
15. Estimate $f'(4)$.



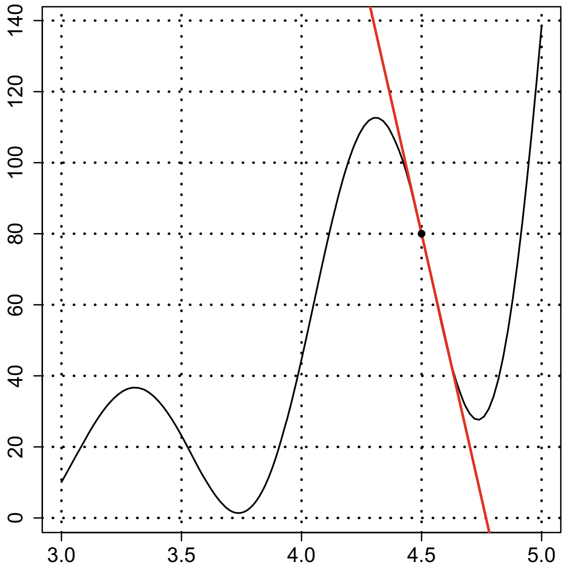
16. Estimate $f'(3)$.



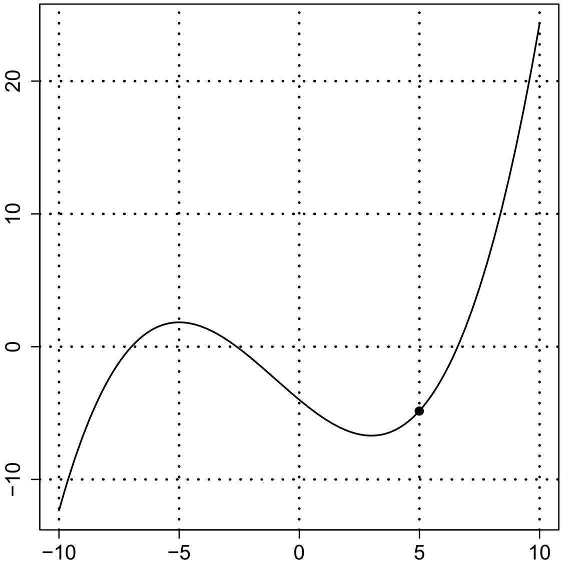
17. Estimate $f'(2.4)$.



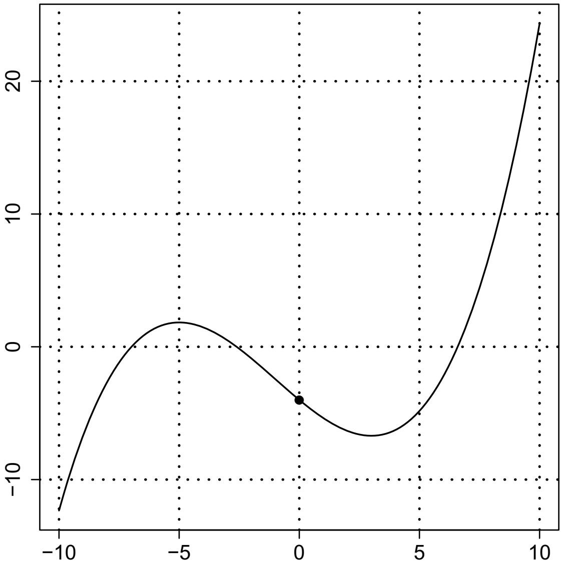
18. Estimate $f'(4.5)$.



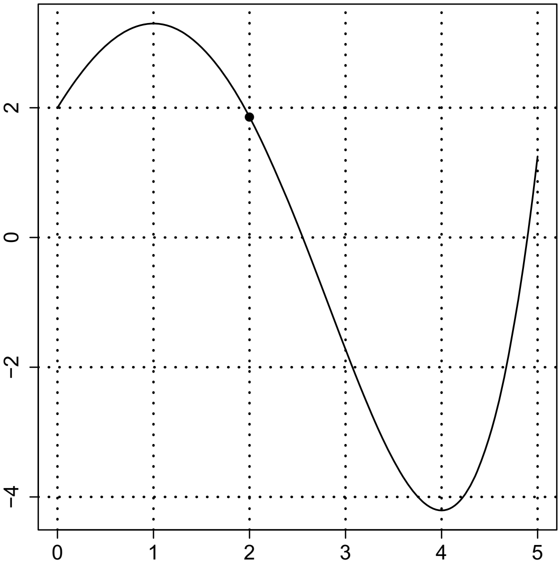
19. Estimate $f'(5)$.



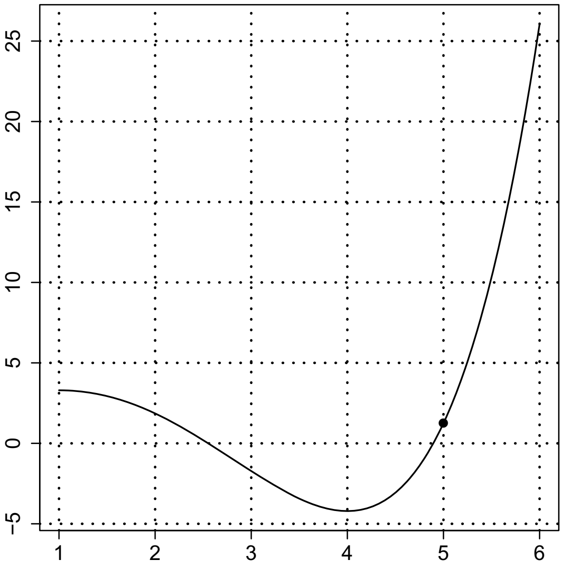
20. Estimate $f'(0)$.



21. Estimate $f'(2)$.

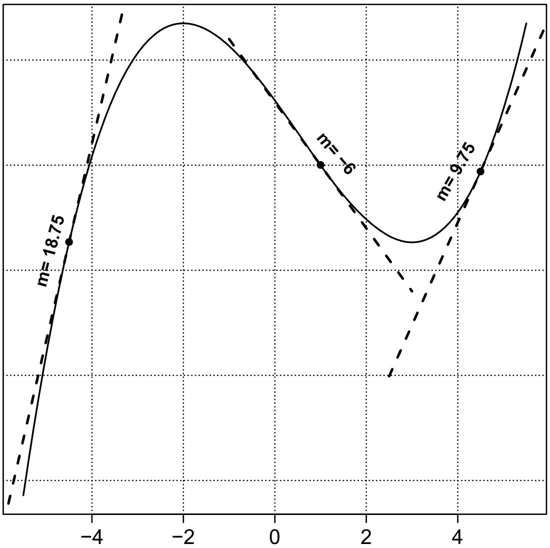


22. Estimate $f'(5)$.



For the next eight problems, sketch the graph of the derivative and include both x-axis and y-axis values using the information given.

23.



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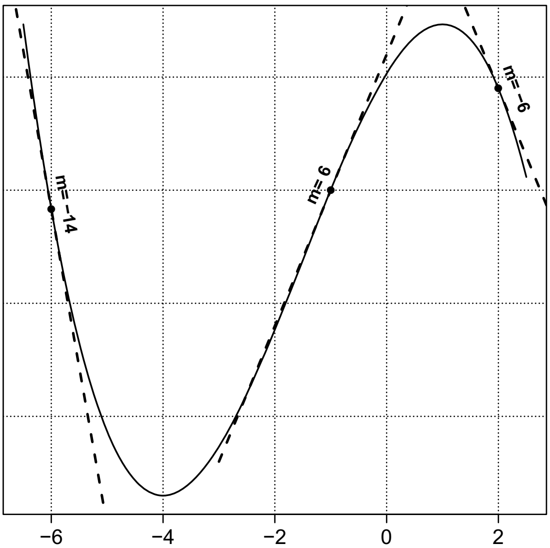

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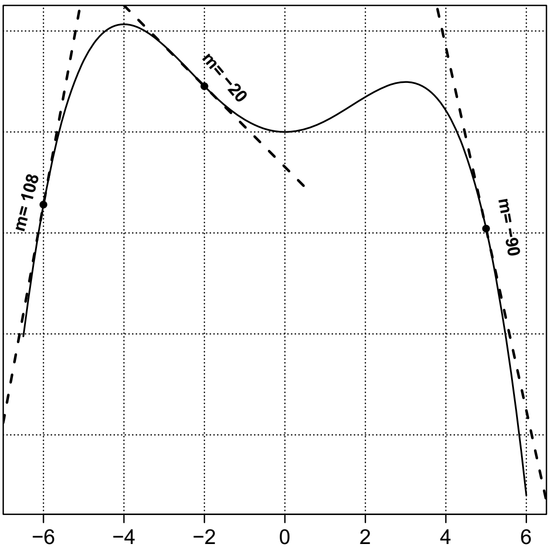

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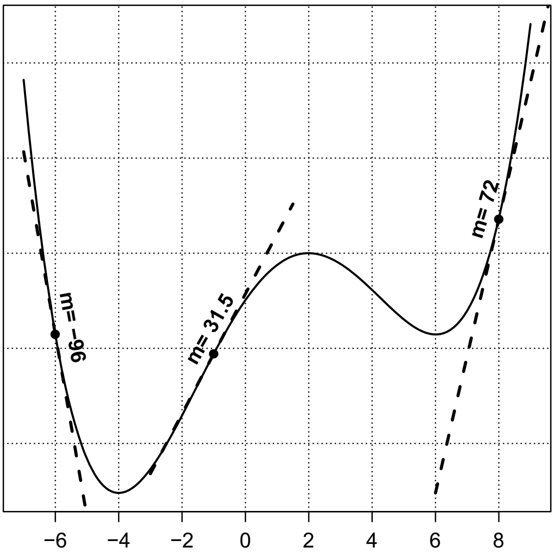

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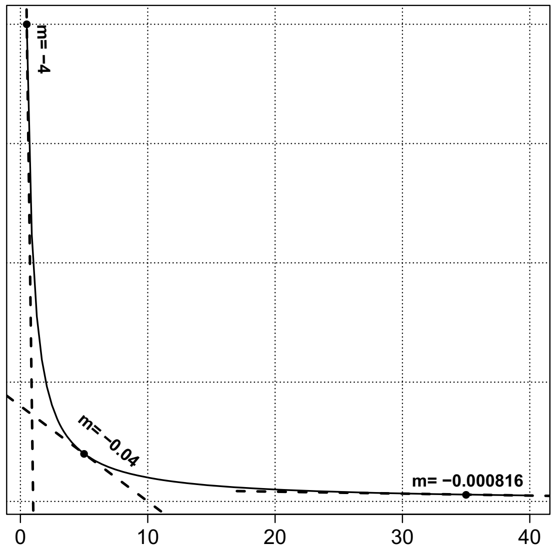

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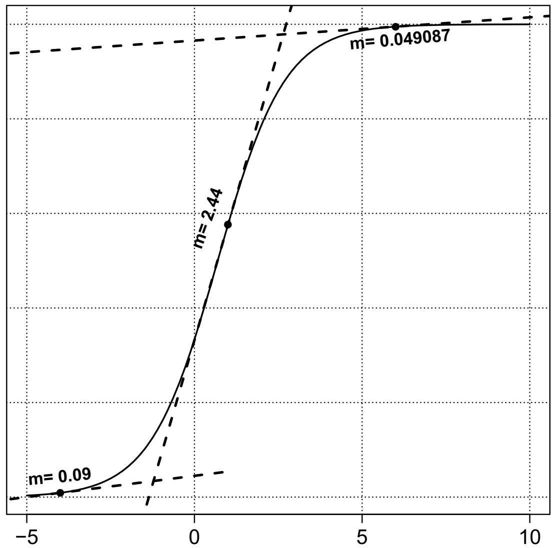

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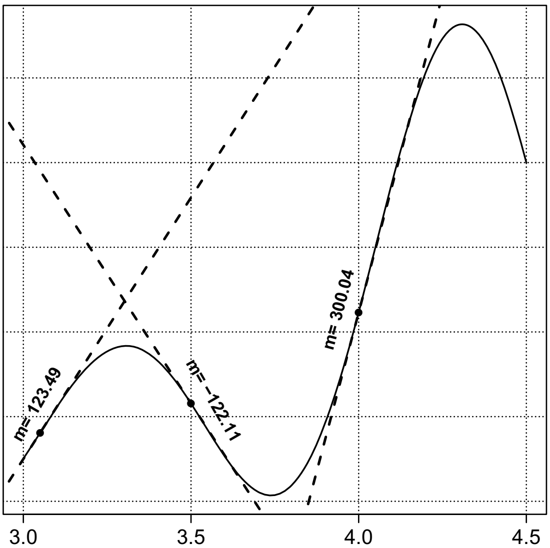

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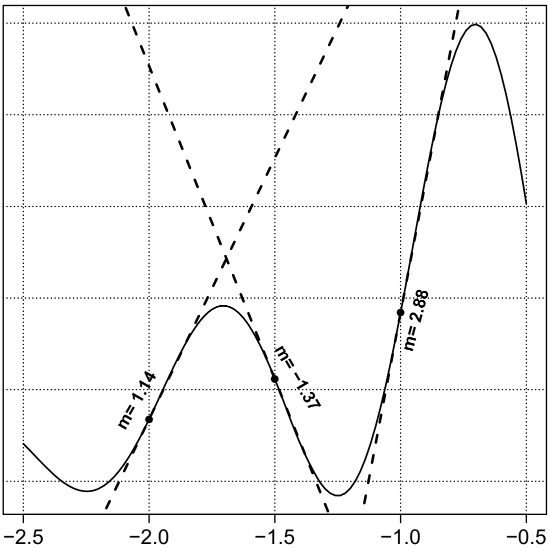
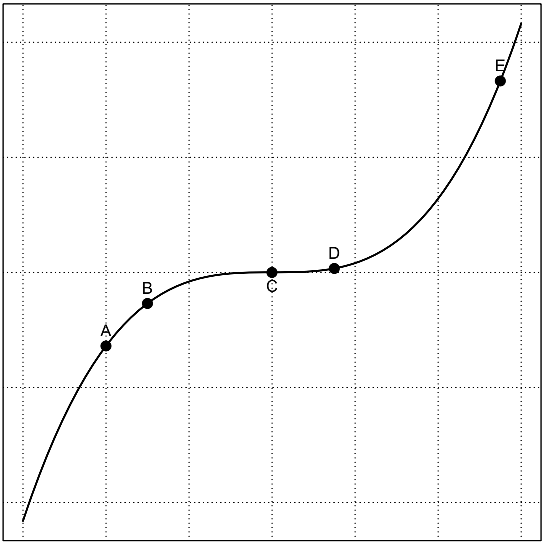


For the next ten problems, order the points from smallest to largest according to the derivative at the point.

31.



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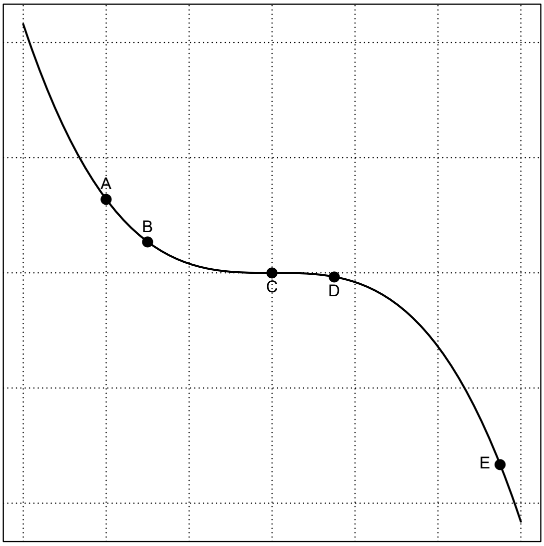

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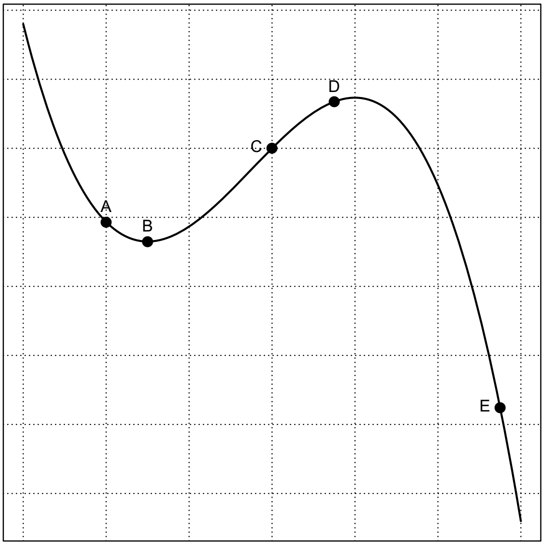

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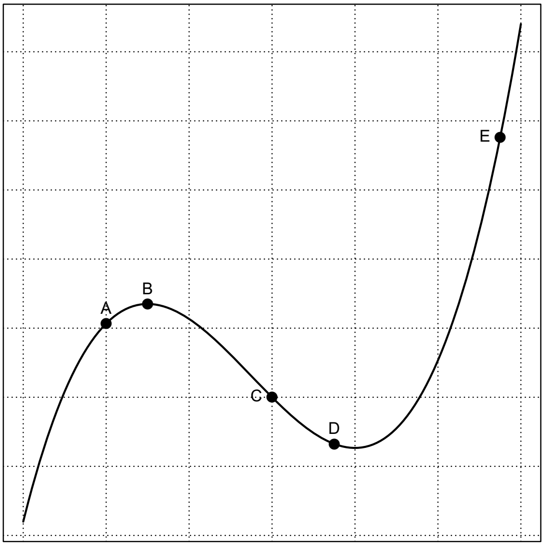

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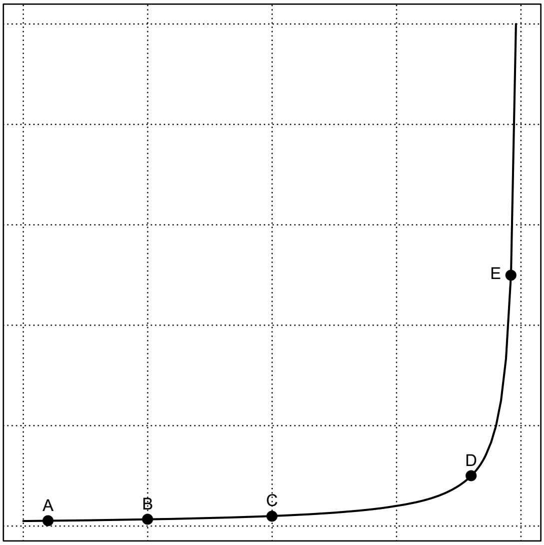

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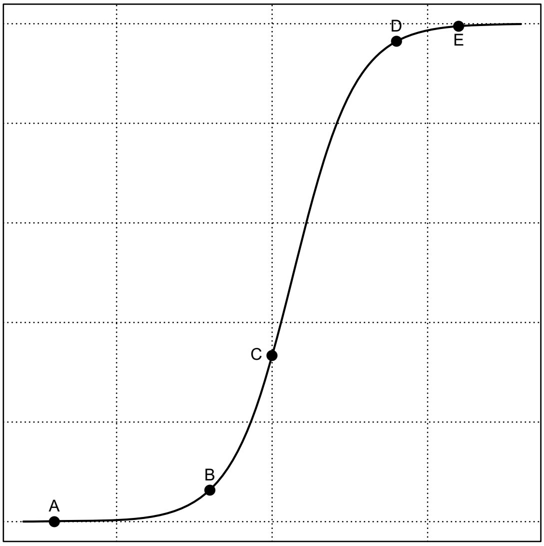

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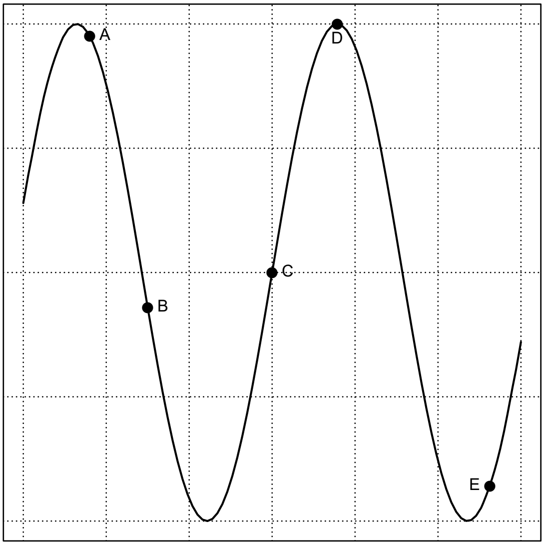

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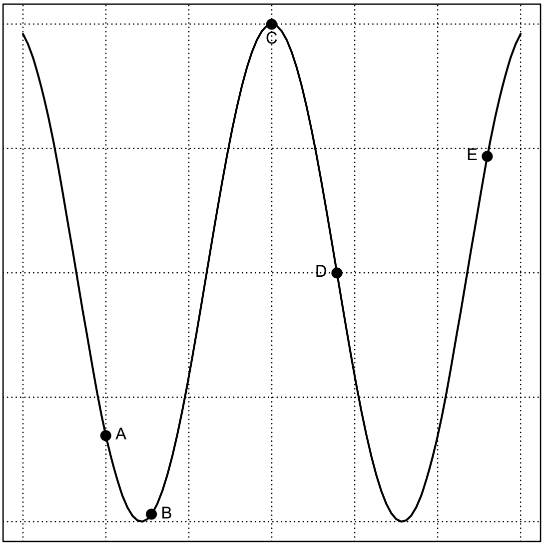

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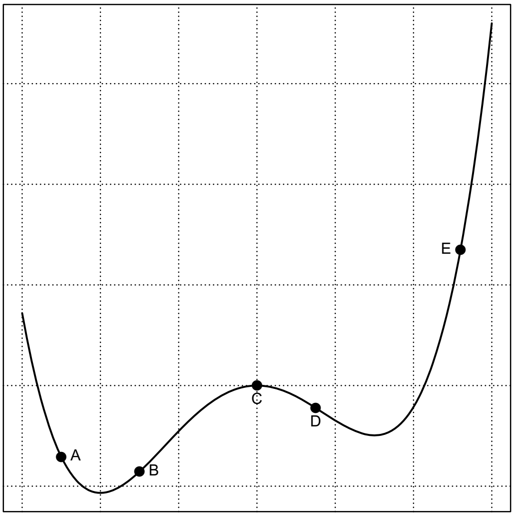

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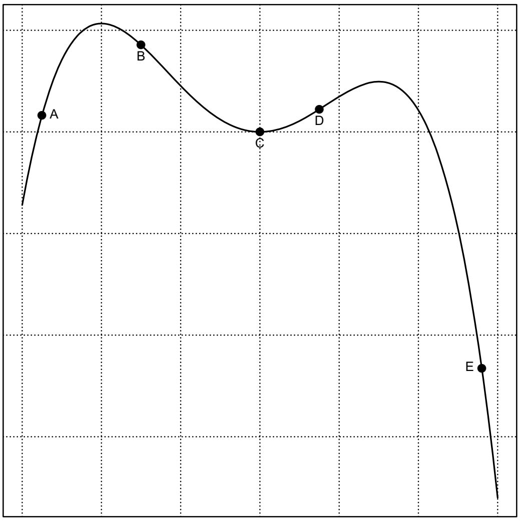

41. Sketch the graph of the derivative of the function in problem 27. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.

42. Sketch the graph of the derivative of the function in problem 28. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.

43. Sketch the graph of the derivative of the function in problem 29. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.
44. Sketch the graph of the derivative of the function in problem 30. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.
45. Sketch the graph of the derivative of the function in problem 31. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.
46. Sketch the graph of the derivative of the function in problem 32. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.
47. Sketch the graph of the derivative of the function in problem 33. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.
48. Sketch the graph of the derivative of the function in problem 34. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.
49. Sketch the graph of the derivative of the function in problem 35. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.
50. Sketch the graph of the derivative of the function in problem 36. Label the x-axis with A, B, etc. and place an A′, B′, etc. on the graph that corresponds to the points A, B, etc. of the original graph. The points A′, B′, etc. must be relatively correct (location vertically) in their placement.

Chapter 10
The Formal Derivative as a Limit

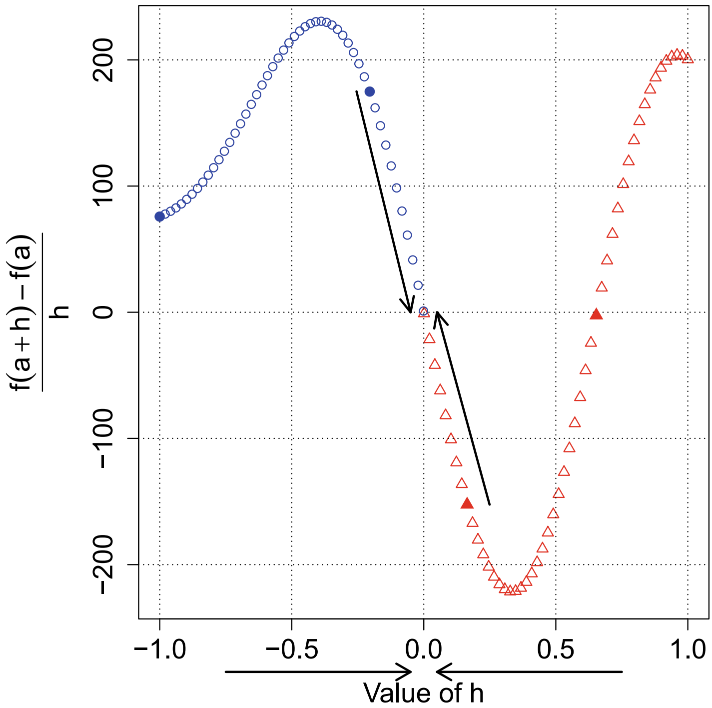

Fig. 10.1 Values of the slopes of the secant lines of $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$ from $(4.30931, f(4.30931))$ to $(4.30931 + h, f(4.30931 + h))$.

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_10

As we move from successively approximating the slope of tangent line to a formal algebraic definition, we will capture the idea of successive approximation with that of a limit. Let's first review figure 10.1, which is a repeat of figure 8.4. Each point on this graph represents

$$\frac{f(a+h)-f(a)}{h}$$

where $a = 4.30931$ and the value of h is the x-axis. In other words, each point is the slope of a secant line from $a = 4.30931$ to $a + h = 4.30931 + h$. We should view this graph from the outside towards $h = 0$ following the arrows. Notice that as h gets closer to 0, from both sides, the slope of the secant line converges to, in this case, 0.

These ideas are captured symbolically in M-Box 10.1. There are three parts of this definition that we need to explain. First, the symbol $f'(a)$ is the notation for the derivative of the function $f(x)$ at the specific point $x = a$ (this is the same as in M-Box 5.1). We add the prime after $f(x)$ so that it is clear that $f'(x)$ is the derivative of the function $f(x)$. Second,

$$\lim_{h\to 0}$$

in words is the limit as h approaches 0. This captures the idea of successive approximation in a formal definition (there are really formal ways to compute limits, but for now we will deal with the idea of a limit informally). When you see $\lim_{h\to 0}$ think about the values of the slope of the secant line as h gets small or as we follow the arrows in figure 10.1. Finally,

$$\frac{f(a+h)-f(a)}{h}$$

is the slope of the secant line from $(a, f(a))$ to $(a + h, f(a + h))$ just as it is in M-Box 8.1. We illustrate this definition in example 10.1

M-Box 10.1: The Definition of the Derivative at a Point

The derivative of the function $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Example 10.1 *Find the slope of the tangent line of $f(x) = x^2$ at $a = 2$.*

Using the definition we have

$$f'(2) = \lim_{h\to 0} \boxed{\frac{f(2+h)-f(2)}{h}} \tag{10.1}$$
$$= \lim_{h\to 0} \boxed{\frac{(2+h)^2-2^2}{h}} \tag{10.2}$$
$$= \lim_{h\to 0} \boxed{\frac{4+4h+h^2-4}{h}} \tag{10.3}$$
$$= \lim_{h\to 0} \boxed{\frac{4h+h^2}{h}} \tag{10.4}$$
$$= \lim_{h\to 0} \boxed{\frac{h(4+h)}{h}} \tag{10.5}$$
$$= \lim_{h\to 0} \boxed{4+h} \tag{10.6}$$
$$= 4 \tag{10.7}$$

We will explain each line in example 10.1. In line (10.1) we substituted 2 for a in the definition. If we try to plug $h = 0$ in the boxed expression in line (10.1) we get a fraction of the form $\frac{0}{0}$. This is not helpful and so in the boxed expressions from lines (10.1) through (10.6) we perform algebra to simplify the expression to get away from the $\frac{0}{0}$ form. In each case the item in the box is still the slope of the secant line from $(2, f(2)$ to $(2+h, f(2+h))$ or, in this case with $f(x) = x^2$ from $(2, 4)$ to $(2+h, (2+h)^2)$. In line (10.2) we have evaluated the function at $2+h$ and 2, using that

$$f(2+h) = (2+h)^2$$

and

$$f(2) = 2^2.$$

In line (10.3) we squared both terms calculating that

$$(2+h)^2 = 4 + 4h + h^2$$

and $2^2 = 4$. In line (10.4) we simplified by adding 4 and -4 to get 0. In line (10.5) we have factored out an h in the numerator. In line (10.6) we have canceled the h in the numerator with the one in the denominator. In line (10.6) the expression in the box, $4+h$, is a formula for the slope of the secant line of the function $f(x) = x^2$ from $(2, f(2))$ to $(2+h, f(2+h))$. At this point, using the idea of successive approximation we could evaluate $4+h$ at $h = 0.1, 0.01, 0.001, \ldots$ to get $4.1, 4.01, 4.001, \ldots$. On the other side, we could evaluate $4+h$ at $h = -0.1, -0.01, -0.001, \ldots$ to get $3.9, 3.09, 3.009, \ldots$. Either way, as h gets smaller and smaller the value of $4+h$ approaches 4. In essence, we can plug in 0 for h, although that is not the idea of the limit and recall we already noted that we cannot plug in 0 for h in line (10.1) because that would leave us with an expression of the form $\frac{0}{0}$. What does 4 represent here? In words, the slope of the tangent line (or the derivative) of $f(x) = x^2$ at $a = 2$ is 4. In notation we write $f'(2) = 4$.

If we wanted to know the slope of the tangent line of $f(x) = x^2$ at, say, $a = 5$, we would have to perform a similar calculation as the one in example 10.1. What would be nice is to derive a derivative formula, in other words, a function that we can use to find the derivative at any point of a function. M-Box 10.2 is the definition for the derivative function.

M-Box 10.2: The Definition of the Derivative Function

The derivative of the function $f(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The formulas in M-Box 10.1 and 10.2 look the same, but in mathematics an a and an x are interpreted differently. The use of a refers to a specific point, such as $a = 2$ in the example above. On the other hand, x is a variable representing any value. We will work through an example and then discuss the differences between M-Box 10.1 and 10.2.

Example 10.2 *Find the derivative of $f(x) = x^2$.*

Using the definition we have

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \boxed{\frac{f(x+h) - f(x)}{h}} && (10.8)\\
&= \lim_{h \to 0} \boxed{\frac{(x+h)^2 - x^2}{h}} && (10.9)\\
&= \lim_{h \to 0} \boxed{\frac{x^2 + 2xh + h^2 - x^2}{h}} && (10.10)\\
&= \lim_{h \to 0} \boxed{\frac{2xh + h^2}{h}} && (10.11)\\
&= \lim_{h \to 0} \boxed{\frac{h(2x+h)}{h}} && (10.12)\\
&= \lim_{h \to 0} \boxed{2x + h} && (10.13)\\
&= 2x && (10.14)
\end{aligned}$$

We begin the calculation in line (10.8) with the definition. Note that

$$\frac{f(x+h) - f(x)}{h}$$

is still a slope of a secant line; the secant line from an unknown $(x, f(x))$ to $(x + h, f(x + h))$. Recognize that the boxed expressions from line (10.8) through line (10.13) are still the slope of a secant line and we are simplifying the expression so that we can understand the limit as h goes to 0. As in example 10.1 the expression in the box in line (10.8) would give us $\frac{0}{0}$ if we try to plug in $h = 0$. In line (10.9) we evaluate the function as

$$f(x + h) = (x + h)^2$$

and

$$f(x) = x^2.$$

In line (10.10) we square $(x + h)^2$ to get $x^2 + 2xh + h^2$. In line (10.11) we cancel the x^2 and the $-x^2$. In line (10.12) we factor out an h as $2xh + h^2 = h(2x + h)$. In line (10.13) we cancel the h in the numerator with the one in the denominator.

The expression $2x + h$ is still the slope of the secant line from $(x, f(x))$ to $(x + h, f(x + h))$ for any x. For example, the slope of the secant line from $x = 2$ to $x = 2 + 0.1$ is $2(2) + 0.1 = 4.1$ or from $x = 9$ to $x = 9 + 0.01$ is $2(9) + 0.01 = 18.01$. What we can see in this form is that as $h \to 0$, $2x + h \to 2x$. What does $2x$ represent? The derivative of the function

$$f(x) = x^2$$

is

$$f'(x) = 2x.$$

In other words, for any value of x the slope of the tangent line of $f(x) = x^2$ is given by $2x$.

The formulas in M-Box 10.1 and 10.2 are similar as are the calculations in examples 10.1 and 10.2. The differences are contextual and often the difficulty of the calculation. For M-Box 10.1 and example 10.1 we calculate the slope of the tangent line at a specific point. In the example it was at 2. It is often easier to work through the calculation at a specific value. It is also easier to explain. For M-Box 10.2 and example 10.2 the calculation was generalized to derive a function. This allows us to easily find the slope of the tangent line at any point, but the computation can be more difficult. One last point, $f'(a)$ in M-Box 10.1 represents a number while $f'(x)$ in M-Box 10.2 represents a function.

At this point, we have been using the terms derivative and slope of the tangent line interchangeably and so we note M-Box 10.3, which is partly repeat of M-Box 5.2. The additional information is the list of common notation for the derivative. In this text we will generally use $f'(x)$ and on occasion $\frac{dy}{dx}$. Finally, the words we use to refer to the derivative should reflect the context. We typically use slope of the tangent line and slope of a function when we are thinking of the graph or geometrically. We typically use derivative in algebraic contexts and rate of change in the context of applications.

M-Box 10.3: Derivative Synonyms

The following are synonyms: Slope of the Tangent Line, Derivative, Rate of Change, and Slope of a Function. In terms of notation $f'(x)$, $\frac{dy}{dx}$, and $\dot{y}(x)$ all represent the derivative of the function $y = f(x)$, with the last, the dot notation, common in physics.

10.1 Exercises

Use the definition of the derivative at a point, M-Box 10.1, **for the following problems.**

1. Find the derivative of $f(x) = x^2 + 5$ at $a = 3$.
2. Find the derivative of $f(x) = x^2 + 5$ at $a = 7$.
3. Find the derivative of $f(x) = x^2 + x$ at $a = -4$.
4. Find the derivative of $f(x) = x^2 + x$ at $a = 9$.
5. Find the derivative of $f(x) = x^2 - 2x + 8$ at $a = 2$.
6. Find the derivative of $f(x) = x^2 - 2x + 8$ at $a = -5$.
7. Find the derivative of $f(x) = 3x^2 + 5x - 11$ at $a = 7$.
8. Find the derivative of $f(x) = 3x^2 + 5x - 11$ at $a = 6$.
9. Find the derivative of $f(x) = 4x^2 + 3x + 10$ at $a = -5$.
10. Find the derivative of $f(x) = 4x^2 + 3x + 10$ at $a = -3$.
11. Find the derivative of $f(x) = x^3 + 7$ at $a = 2$.
12. Find the derivative of $f(x) = x^3 + 7$ at $a = 4$.
13. Find the derivative of $f(x) = x^3 + 3x - 7$ at $a = -8$.
14. Find the derivative of $f(x) = x^3 + 3x - 7$ at $a = -7$.
15. Find the derivative of $f(x) = x^3 + 5x^2 - x + 9$ at $a = 2$.
16. Find the derivative of $f(x) = x^3 + 5x^2 - x + 9$ at $a = 3$.
17. Find the derivative of $f(x) = x^4 - 5x + 11$ at $a = 5$.
18. Find the derivative of $f(x) = x^4 - 5x + 11$ at $a = 6$.

Use the definition of the derivative, M-Box 10.2, **for the following problems.**

19. Find the derivative of $f(x) = x^2 + 7$.
20. Find the derivative of $f(x) = x^2 + 1$.
21. Find the derivative of $f(x) = x^2 + 4x$.
22. Find the derivative of $f(x) = x^2 + 3x$.
23. Find the derivative of $f(x) = x^2 - 7x + 6$.
24. Find the derivative of $f(x) = x^2 - 5x + 9$.
25. Find the derivative of $f(x) = 6x^2 + 4x - 8$.
26. Find the derivative of $f(x) = 12x^2 + 7x + 10$.
27. Find the derivative of $f(x) = 9x^2 - 6x + 6$.
28. Find the derivative of $f(x) = 5x^2 - 9x + 6$.

29. Find the derivative of $f(x) = x^3 - 8$.
30. Find the derivative of $f(x) = x^3 - 9$.
31. Find the derivative of $f(x) = x^3 + 6x - 9$.
32. Find the derivative of $f(x) = x^3 - 5x + 8$.
33. Find the derivative of $f(x) = x^3 - 4x^2 - x - 10$.
34. Find the derivative of $f(x) = x^3 + x^2 - x + 12$.
35. Find the derivative of $f(x) = x^4 + 2x + 1$.
36. Find the derivative of $f(x) = x^4 - 11x + 3$.

Model the scenario with a function and then use the definition of the derivative at a point, M-Box 10.1, for the following problems.

37. How fast is the area of a square with side length s inches growing when $s = 5$ inches? What is the percentage rate of change when $s = 5$ inches? Summarize your results in a sentence or two with proper context.
38. A rectangle has one side that is 10 inches longer than the first side. How fast is the area of the square growing when the first side is 7 inches long? What is the percentage rate of change? Summarize your results in a sentence or two with proper context.
39. How fast is the volume of a cube with side length s feet growing when $s = 10$ feet? What is the percentage rate of change when $s = 10$ feet? Summarize your results in a sentence or two with proper context.
40. Consider a rectangular box with a square side with side length s feet and a length of 5 feet. How fast is the volume growing if the square side is of length 7 feet? What is the percentage rate of change? Summarize your results in a sentence or two with proper context.

10.2 Project: An Origin Story of the number e

The goal of this project is to provide an example of where the number e comes from while at the same time exploring functions of the type $f(x) = b^x$ for $b > 0$ (**don't miss** this fact about b being positive) and their derivatives. Problems with an (R) are to be done in R.

1. **(R) Graph** Choose three or more values of b that are all greater than 1 and graph $f(x) = b^x$ in R on the same graph. Use different colors. How does changing b change the shape of the function. Consider both $x > 0$ and $x < 0$, when describing these graphs. Repeat this for three values of b that are between 0 and 1. **Tip:** Set the range of the x-axis to be symmetric around the origin.
2. **Using the definition of the derivative at a point** provide an expression for $f'(0)$ with $f(x) = b^x$. Your answer will still have $\lim_{h\to 0}$ involved. Hint: There really isn't anything that can be simplified. Still, pay attention to what you have as it will be used later.
3. **Justify** each step in the calculation of $f'(x)$ with $f(x) = b^x$. In other words, explain why each of the equal signs with a number above it is true. Hint: For number 7 consider your result in 2.

$$
\begin{aligned}
f'(x) &\overset{(1)}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&\overset{(2)}{=} \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} \\
&\overset{(3)}{=} \lim_{h\to 0} \frac{b^x b^h - b^x}{h} \\
&\overset{(4)}{=} \lim_{h\to 0} \frac{b^x (b^h - 1)}{h} \\
&\overset{(5)}{=} \lim_{h\to 0} \frac{b^x (b^h - b^0)}{h} \\
&\overset{(6)}{=} b^x \lim_{h\to 0} \frac{b^h - b^0}{h} \\
&\overset{(7)}{=} b^x f'(0)
\end{aligned}
$$

4. **Based on the calculation** in 3 explain why this statement is true: The derivative of a function of the form $f(x) = b^x$ is equal to b^x times the slope of its tangent line at $a = 0$.
5. **(R) Estimate** to four decimal places $f'(0)$ with $f(x) = 2^x$ and $g'(0)$ with $g(x) = 3^x$. Hint: This is the successive approximation game in R.
6. **Should there be a number**, call it e, such that $f'(0) = 1$ with $f(x) = e^x$? Hint: You don't have to calculate anything here as you need to reason this from 5 and your graphs from 1.

7. **If we can find such a number**, call it e, such that $f'(0) = 1$ with $f(x) = e^x$, then what will be the derivative of $f(x) = e^x$? Hint: You don't need to calculate anything just explain why this follows from 3.
8. **(R) Estimate e** using the idea of successive approximation. Hint: This is done in R but you need to estimate and adjust the value of b in the function $f(x) = b^x$ based on your result of the estimate of the derivative. So, for example, try a number between 2 and 3 and see what you get for an estimate of $f'(0)$. Continue to adjust and add decimal places. The goal here is to see how well you can estimate e and not for you to look up the number and plug it in nor should you be using the function $exp(x)$ in any way in your code. Your answer for this should include your estimate for e, how accurate you think you are, and your code.

Chapter 11
Basic Derivative Rules

The definition of the derivative function given in M-Box 10.2 is time-consuming to apply to every function for which we want a derivative. For example, the derivative of $f(x) = 3x^2$, $f(x) = 5x^2$, and $f(x) = 7x^2$ would each be a similar yet separate calculation. It turns out we can use the definition of the derivative in M-Box 10.2 to derive general rules so that we do not need to use the limit formula each time. M-Box 11.1 lists basic rules of derivatives. Note that all of these rules are derived by definition of the derivative in M-Box 10.2. They are not definitions but the result of applying a definition. We prove rule 11.2 and note that the case $n = 2$ for rule 11.3 was done in example 10.2.

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M-Box 11.1: Basic Derivative Rules

Function	Derivative	
$f(x) = c$	$f'(x) = 0$	(11.1)
$f(x) = mx + b$	$f'(x) = m$	(11.2)
$f(x) = x^n$	$f'(x) = nx^{n-1}$	(11.3)
$f(x) = \sin(x)$	$f'(x) = \cos(x)$	(11.4)
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$	(11.5)
$f(x) = e^x$	$f'(x) = e^x$	(11.6)
$f(x) = a^x$	$f'(x) = \ln(a)a^x$	(11.7)
$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$	(11.8)

Example 11.1. *Proof of rule 11.2.*

Solution. If $f(x) = mx + b$ then from the limit definition of the derivative we have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \boxed{\frac{f(x+h) - f(x)}{h}} && (11.9)\\
&= \lim_{h\to 0} \boxed{\frac{m(x+h) + b - (mx+b)}{h}} && (11.10)\\
&= \lim_{h\to 0} \boxed{\frac{mx + mh + b - mx - b}{h}} && (11.11)\\
&= \lim_{h\to 0} \boxed{\frac{mh}{h}} && (11.12)\\
&= \lim_{h\to 0} \boxed{m} && (11.13)\\
&= m && (11.14)
\end{aligned}
$$

□

Two concepts to emphasize about the proof. First the equations that are boxed are still formulas for the slope of a secant line. In this case, they are formulas for the slope of the secant line from x to $x + h$ of the function $f(x) = mx + b$. The algebra performed to simplify the expression from equation (11.9) through (11.13) doesn't change the fact that the expressions are the same formula for the secant line slope, but in a different or simplified form. The m on line (11.13) is the slope of the secant line from x to $x + h$ of the function $f(x) = mx + b$. This makes sense because any secant line on a line is the same as the line and has the same slope.

Second, the result in equation (11.14) comes from evaluating the limit in (11.13). Notice that the expression in the limit in (11.13) does not involve h. In other words, regardless of h we always get m in the limit and hence m in (11.14). Intuitively, this makes since because any tangent line at any point on a line is the line itself and so it has the same slope.

Before moving to the next result we pause to consider a graph. First, what happens to a function $f(x)$ if it is multiplied by a scaler c (assume $c > 0$ for now)? In other words, how does $f(x)$ compare to $cf(x)$? Also, is there a relationship between the tangent lines of $f(x)$ and $cf(x)$? We explore these questions in the graph in figure 11.1. The graph includes x^2 and $5x^2$ with tangent lines at $x = -4$ and $x = 6$ for each curve. Which graph is x^2? It is the "flatter" graph. Multiplying x^2 by the scaler 5 makes it "steeper." How much "steeper" is $5x^2$? In other words, what appears to be the relationship between the associated tangent lines? M-Box 11.2 answers the question. In words, the value of the derivative is scaled by the same amount as the original function. We prove this result in example 11.2.

Example 11.2. *Proof of the Constant Multiple Rule Theorem, M-Box 11.2.*

Solution. Let $g(x) = cf(x)$ (we do this to make it easier to write otherwise we would have to write $(cf(x))'$). From the limit definition of the derivative we have

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} && (11.15) \\ &= \lim_{h \to 0} \frac{cf(x+h) - cf(x))}{h} && (11.16) \\ &= \lim_{h \to 0} \frac{c(f(x+h) - f(x))}{h} && (11.17) \\ &= c \boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} && (11.18) \\ &= cf'(x) && (11.19) \end{aligned}$$

We explain each step in the proof. Line (11.15) is the definition of the derivative. In line (11.16) we replaced $g(x)$ with $cf(x)$ as defined. We factor out c in line (11.17). In line (11.18) we factor c outside the limit. Intuitively this makes sense as the limit is based on h. There are formal ways to prove this step but they are beyond the scope of this book. Notice that in line (11.18) the boxed equation is the definition of the derivative of the function $f(x)$, which gets us line (11.19). □

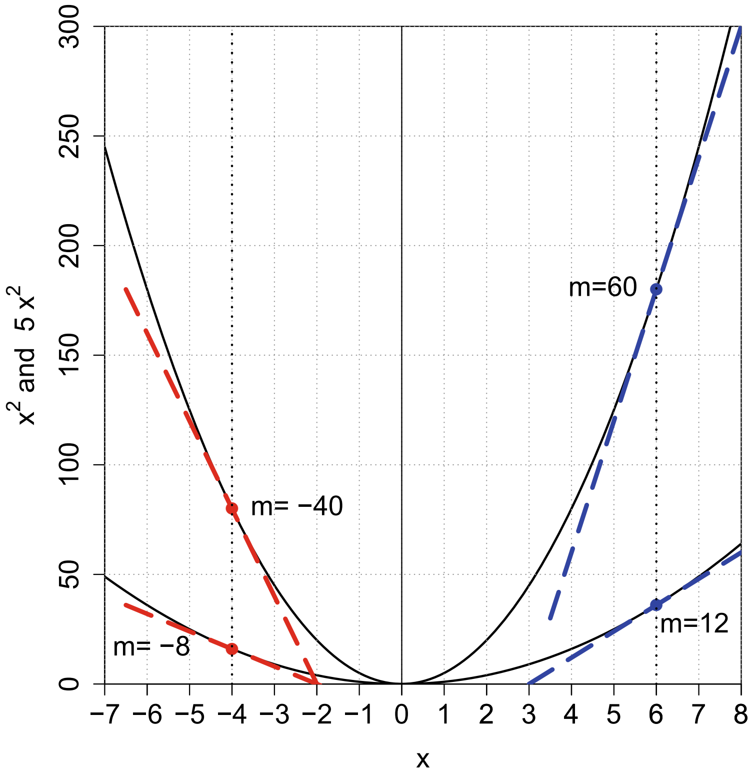

Fig. 11.1 Graph of x^2 and $5x^2$ with tangent lines at $x = -4$ and $x = 6$ for both curves.

M-Box 11.2: Theorem - The Constant Multiple Rule

The derivative of $g(x) = cf(x)$ is $g'(x) = cf'(x)$.

In figure 11.1 we multiplied the function x^2 by 5 and the slopes of the tangent lines at $x = -4$ and $x = 6$ were also multiplied by 5. Graphically, the Constant Multiple Rule, M-Box 11.2, proves that figure 11.1 is not a special example in that if we multiply a function by a constant c the graph will be “stretched” by a factor of c so that corresponding tangent line slopes will be multiplied by the factor c.

Algebraically, the ramification of M-Box 11.2 is that we can find the derivative of functions such as $f(x) = 10x^4$ or $g(x) = 42\sin(x)$ by using a combination of basic rules, M-Box 11.1, and M-Box 11.2 instead of using the definition of the derivative, which are the next two examples.

Example 11.3. *What is the derivative of $f(x) = 10x^4$?*

Solution. From M-Box 11.2, the derivative of $f(x) = 10x^4$ will be 10 times the derivative of x^4. Using line (11.3) with $n = 4$ of M-Box 11.1, the derivative of x^4 is $4x^3$. Hence, $f'(x) = 10(4x^3) = 40x^3$. □

Example 11.4. *What is the derivative of* $g(x) = 42\sin(x)$*?*

Solution. From M-Box 11.2, the derivative of $g(x) = 42\sin(x)$ will be 42 times the derivative of $\sin(x)$. Using line (11.4) of M-Box 11.1, the derivative of $\sin(x)$ is $\cos(x)$. Hence, $g'(x) = 42\cos(x)$. □

We consider one more graph, figure 11.2. This is a graph of $f(x) = x^2$, $g(x) = 12x$, and $h(x) = f(x) + g(x) = x^2 + 12x$ with tangent lines at $x = 2$ and $x = 6$. The key question here is what happens to slopes of tangent line when we add two functions? Can you see the pattern in the graph? The result is given in M-Box 11.3.

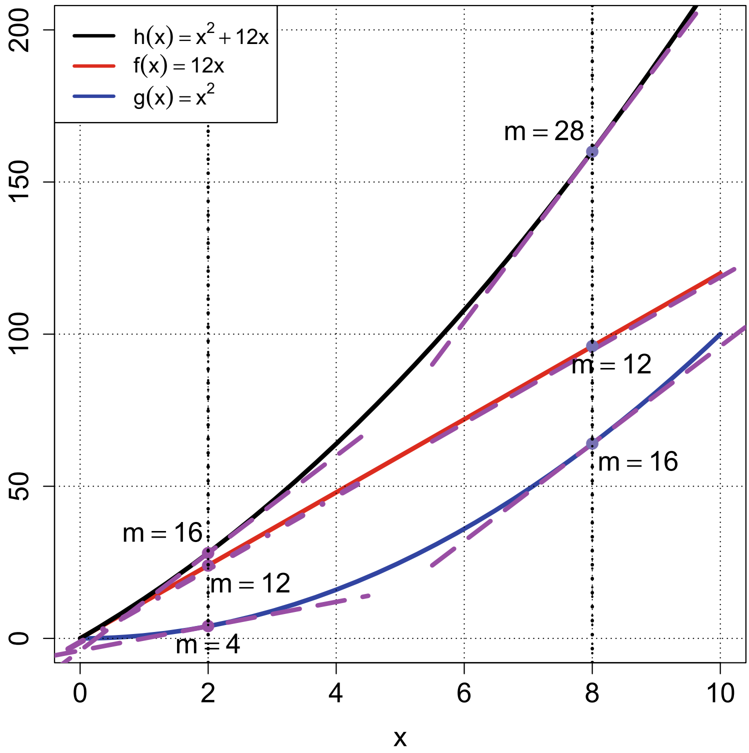

Fig. 11.2 Graph of x^2, $12x$, and $x^2 + 12x$ with tangent lines at $x = 2$ and $x = 6$ for all curves.

M-Box 11.3: Theorem - The Sum and Difference Rule

The derivative of $h(x) = f(x) + g(x)$ is $h'(x) = f'(x) + g'(x)$ and the derivative of $k(x) = f(x) - g(x)$ is $k'(x) = f'(x) - g'(x)$.

Graphically, The Sum and Difference Rule, M-Box 11.3, whose proof is left as an exercise, says that if we add two functions (or subtract) then the slopes of the tangent lines of the new function $h(x)$ (or $k(x)$) are the addition (or subtraction) of the slopes of the tangent lines of the original two functions $f(x)$ and $g(x)$. Algebraically, we can now find the derivative of functions of the form $f(x) = x^8 + \cos(x)$ or $g(x) = x^{13} + 5^x + 89$ without resorting to the definition of the derivative as the example demonstrates.

Example 11.5. *What is the derivative of $f(x) = x^8 + \cos(x)$?*

Solution. From M-Box 11.3, the derivative of $f(x) = x^8 + \cos(x)$ will be the derivative of x^8 plus the derivative of $\cos(x)$, which are $8x^7$ by line (11.3) of M-Box 11.1 and $-\sin(x)$ by line (11.5) of M-Box 11.1, respectively. Hence $f'(x) = 8x^7 - \sin(x)$. □

M-Box 11.3 generalizes to the sum and difference of more than two functions. For example, the derivative of $k(x) = f(x) + g(x) - h(x)$ can be viewed as the difference of two functions $f(x) + g(x)$ and $h(x)$. We can then reduce the derivative of $f(x) + g(x)$ to the derivative of $f(x)$ and $g(x)$ so that

$$k'(x) = f'(x) + g'(x) - h'(x).$$

In other words, to find the derivative of the sums and differences of any number of functions we can apply derivative rules to each function as in the next example.

Example 11.6. *What is the derivative of $g(x) = x^{13} - 5^x - e^x + 89$?*

Solution. Applying M-Box 11.3 as noted in the comment above to $g(x)$ we need to find the derivative of x^{13}, 5^x, e^x, and 89. Using the rules in M-Box 11.1 (you should identify each of these) the derivatives of x^{13}, 5^x, e^x, and 89 are $13x^{12}$, $\ln(5)5^x$, e^x, and 0, respectively. Putting this together we get

$$g'(x) = 13x^{12} - \ln(5)5^x - e^x,$$

while noting that we do not include the +0 at the end. □

One final example that combines the Constant Multiple Rule and the Sum and Difference Rule.

Example 11.7. *What is the derivative of $h(x) = 15x^5 - 8\cos(x) + 7\ln(x) - 11$?*

Solution. The Sum and Difference Rule, M-Box 11.3, allows us to take the derivative of each component of this function. Applying the Constant Multiple Rule, M-Box

11.2 and our basic derivative rule, M-Box 11.1, the derivatives of $15x^5$, $8\sin(x)$, and $7\ln(x)$ are $15(5x^4) = 75x^4$, $-8\sin(x)$, and

$$7\left(\frac{1}{x}\right) = \frac{7}{x},$$

respectively. The derivative of 11 is 0 and so

$$h'(x) = 75x^4 - (-8\sin(x)) + \frac{7}{x} - 0 = 75x^4 + 8\sin(x) + \frac{7}{x}.$$

□

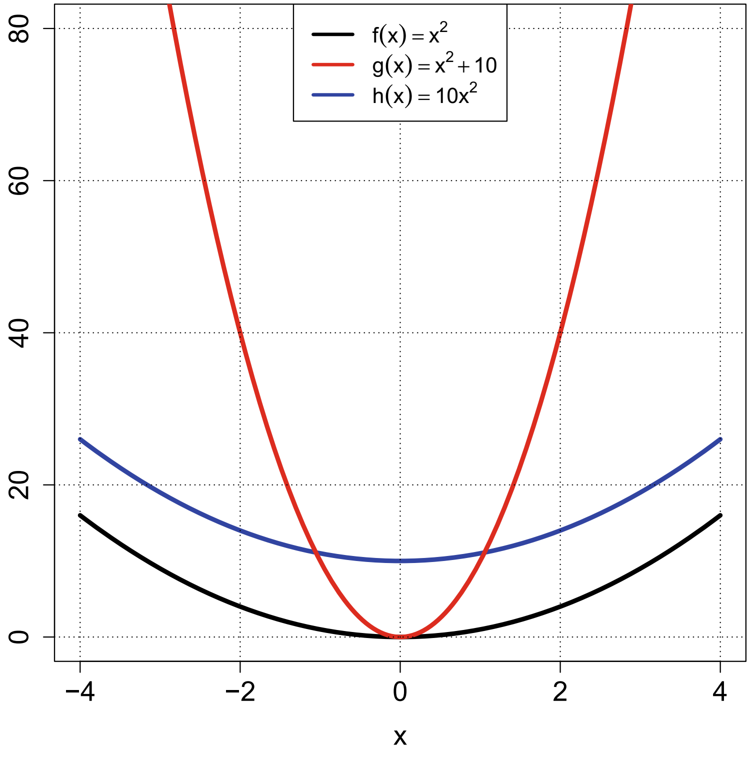

Fig. 11.3 Graph of $f(x) = x^2$, $g(x) = x^2 + 10$, and $h(x) = 10x^2$.

Some points we need to mention. The functions $g(x) = cf(x)$ and $h(x) = f(x) + c$ are not the same. For example, in figure 11.3 we see the relationship between the functions $f(x) = x^2$, $g(x) = x^2 + 10$, and $h(x) = 10x^2$. The function $g(x) = x^2 + 10$ shifts $f(x) = x^2$ vertically by 10, while $h(x) = 10x^2$ scales the function $f(x) = x^2$ by 10 which stretches the function vertically. The slopes of the tangent lines of $f(x) = x^2$ and $g(x) = x^2 + 10$ are the same because shifting a function vertically does not change the slopes of the tangent lines. Hence $f'(x) = 2x$ and $g'(x) = 2x$. On the other hand, the tangent lines of $f(x) = x^2$ and $h(x) = 10x^2$ are clearly not the same. By the constant multiple rule, $h'(x) = 10(2x)$ which is 10 times the derivative of $f(x)$.

In general, if we have the functions $g(x) = cf(x)$ and $h(x) = f(x) + c$, the function $g(x)$ vertically scales the function $f(x)$ by the constant c. The Constant Multiple Rule allows us to take the derivative to get $g'(x) = cf'(x)$. The function $h(x)$ vertically shifts the function $f(x)$ up or down by the constant c. The shape of the function $h(x)$ is the same as the shape of the function $f(x)$ and hence we would expect the derivative of $h(x)$ to be the derivative of $f(x)$ if we are thinking graphically. Algebraically, we apply the Sum and Difference Rule to get that $h'(x) = f'(x) + 0 = f'(x)$ as we expect. In summary, multiplying by a constant scales a function and by the Constant Multiple Rule the derivative, or slopes of the tangent line, are also scaled by that constant. On the other hand, adding or subtracting, which is just adding a negative number, a constant to a function does not change the shape of the function and hence does not change the derivative or slopes of the tangent. Note that in this paragraph we fluently moved between thinking graphically and algebraic computations. Calculus ideas begin with graphs and are formally captured in algebra.

It is often asked why calculus does not just start with this chapter while skipping the Definition of the Derivative, M-Box 10.2. It needs to be emphasized that all the rules in this chapter are derived, or proved, starting from the Definition of the Derivative, M-Box 10.2, they do not exist without the definition. The Definition of the Derivative algebraically captures the idea of using secant lines to approximate the slope of a tangent line. In particular, it connects the graph and the algebra and this is particularly important in understanding the information the derivative of a function provides about the original function. We will take advantage of this connection when we get to applications of the derivative. For example,

Example 11.8. *Using the CO2 model in figure 4.1, how fast was CO2 increasing in 2017?*

Solution. By the rules in this chapter the derivative of

$$CO2(t) = 0.0134594696825t^2 + 0.520632601929t + 310.423363171$$

is

$$CO2'(t) = 0.026918939365t + 0.520632601929.$$

The variable t represents years after 1950 and so we evaluate the derivative at $t = 67$ to get the slope the tangent line in 2017 or how fast CO2 was increasing in 2017,

$$CO2'(67) = 0.026918939365(67) + 0.520632601929 = 2.324201539384.$$

Based on this model, Mauna Loa average yearly atmospheric CO2 was increasing at a rate of 2.324201539384 ppm/yr. Note: This value should be rounded to the number of significant digits in the data collected but we were not given that so we leave it with the number of significant digits in the model. □

This is basically the result we obtain in the successive approximation chapter. So again, it is fair to ask why we went through the successive approximation idea instead of getting to this chapter. There are at least three reasons. First, we will use the successive approximation idea in a later chapter. In an era of computational power, with software such as R, we can use ideas such as successive approximation to obtain results in cases where algebraic or theoretical results do not exist or we do not know them. The idea of successive approximation is connected to the idea of the limit used in the Definition of the Derivative. While the results here are valuable in and of themselves, there are bigger ideas at play that are transferable to other problems and we want to be conscious of that.

11.1 Exercises

Find the derivative and state the line of the Basic Derivative Rules, M-Box 11.1, used to find the derivative.

1. $f(x) = x^{78}$
2. $g(x) = \cos(x)$
3. $g(x) = \sin(x)$
4. $f(x) = x^{54}$
5. $h(x) = 12$
6. $g(x) = 7^x$
7. $h(x) = 8^x$
8. $f(x) = 18$
9. $g(x) = \ln(x)$
10. $h(x) = e^x$
11. $k(x) = e^x$
12. $f(x) = \ln(x)$

Use the derivative rules to find the derivative of the following 50 functions.

13. $f(x) = 10x - 3$
14. $f(x) = 19x - 4$
15. $g(x) = 7 - 25x$
16. $h(x) = 21 - 15x$
17. $g(x) = 3x^{24}$
18. $f(x) = 6x^8$
19. $h(x) = 13x^7 + 24$
20. $f(x) = 11x^4 - 6$
21. $f(x) = 4x^{4/14}$
22. $h(x) = 5x^{3/7}$
23. $h(x) = 3x^{-5/8}$
24. $f(x) = 3x^{-6/5}$
25. $g(x) = 6x^{22/14}$
26. $h(x) = 11x^{14/3}$
27. $g(x) = \frac{2}{x^4}$
28. $f(x) = \frac{6}{x^{11}}$
29. $f(x) = \frac{10}{16x^5}$
30. $h(x) = \frac{9}{4x^5}$
31. $g(x) = \frac{9}{19x^7}$
32. $h(x) = \frac{12}{15x^9}$
33. $g(x) = \sqrt{x}$
34. $f(x) = \sqrt[5]{x}$
35. $h(x) = \frac{2}{21\sqrt{x}}$
36. $h(x) = \frac{-4}{2\sqrt[3]{x}}$
37. $g(x) = 5\sin(x)$
38. $g(x) = 6\cos(x)$
39. $f(x) = -4e^x + 18$
40. $h(x) = -2e^x + 23$
41. $f(x) = 9^x$
42. $g(x) = 12^x$
43. $h(x) = 5(6^x)$
44. $f(x) = 3(24^x)$
45. $g(x) = 5\ln(x)$
46. $g(x) = 19\ln(x)$
47. $f(x) = \pi^3$
48. $f(x) = 8\pi^4$
49. $f(x) = -7\ln(x) + 5x$
50. $h(x) = -15\ln(x) - 6x$
51. $f(x) = -8x^{11} - 6x^{-9} - 11x$
52. $h(x) = 8x^{-1} - 12x^{-2} + x$

53. $f(t) = 5t^{-14} - 2t^{-13/2} + 12e^t$
54. $f(t) = 13t^{-12} - 5t^{-12/5} + 8e^t$
55. $h(t) = 11t^7 - \cos(t) - 9e^t - 8$
56. $g(t) = 14t^8 + 12\sin t - 3e^t + 9$
57. $h(w) = 3^w - 15w^{5/6} - 8\sqrt{w}$
58. $f(w) = 11^w + 12w^{2/3} + 6\sqrt{w}$
59. $f(x) = \frac{-12}{9x^{10/7}} - 15\ln(x) + 3x^{-10}$
60. $h(x) = \frac{6}{15x^{7/3}} - 12\ln(x) + 9x^{-13}$
61. $f(x) = \frac{5}{10^{-x}} - \frac{4}{7\sqrt{x}}$
62. $f(x) = \frac{3}{7^{-x}} - \frac{5}{9\sqrt[3]{x}}$
63. Use the limit definition of the derivative to prove that the derivative of $f(x) + g(x)$ is $f'(x) + g'(x)$ as given in M-Box 11.3.
64. Use the limit definition of the derivative to prove that the derivative of $f(x) - g(x)$ is $f'(x) + g'(x)$ as given in M-Box 11.3.
65. Use the definition of the derivative to prove that the derivative of $f(x) + c$ for some constant c is $f'(x)$. Explain why this makes sense graphically.
66. Use the Sum and Difference Rule, M-Box 11.3 to prove that the derivative of $f(x) + c$ for some constant c is $f'(x)$.
67. If the point (3, 8) is on the graph of $f(x)$ then what point is on the graph of $5f(x)$? Similarly, if (3, −10) is on the graph of $f'(x)$ then what point is on the graph of the derivative of $5f(x)$?
68. If the point (7, −9) is on the graph of $g(x)$ then what point is on the graph of $-3g(x)$? Similarly, if (7, 4) is on the graph of $g'(x)$ then what point is on the graph of the derivative of $-3g(x)$?
69. If the point (2, 9) is on the graph of $f(x)$ then what point is on the graph of $f(x) + 42$? Similarly, if (2, 14) is on the graph of $f'(x)$ then what point is on the graph of the derivative of $f(x) + 42$?
70. If the point (−5, 0) is on the graph of $h(x)$ then what point is on the graph of $h(x) + 11$? Similarly, if (−5, −10) is on the graph of $h'(x)$ then what point is on the graph of the derivative of $h(x) + 11$?
71. How do slopes of tangent lines of $f(x)$ compare to those of $f(x) + 10$? Your answer should include a graph to illustrate your response.

When applying the derivative recall that if we want to know how fast something is changing at a particular point, then that is the derivative evaluated at that point.

72. The area of a hexagon with side length s inches is given approximately by $A(s) = 2.598s^2$. How fast is the hexagon growing when the side is of length 9 inches? Write a sentence using your results in context properly.
73. The area of a pentagon with side length s inches is given approximately by $A(s) = 2.378s^2$. How fast is the pentagon growing when the side is of length 6 inches? Write a sentence using your results in context properly.
74. Create a function that outputs the surface area of a cube given its volume. How fast is the surface area growing if the volume is 132 cubic feet? What is the percentage rate of change? Summarize your results in a sentence or two with proper context.
75. Using the life expectancy function in the function gallery, how fast was life expectancy changing when GDP was \$15,000 and \$60,000? Write a sentence using and comparing your results in context properly.

76. Using the CO2 function in the function gallery, how fast was CO2 increasing in 1980, 2000, and the last year in the data set? How much faster, as a percent, was CO2 increasing in the last year of the data set as compared to 1980? Write a sentence using your results in context properly.
77. Using the Global Temperature function in the function gallery, how fast was Global Temperature increasing in 1980, 2000, and the last year in the data set? How much faster, as a percent, was Global Temperature increasing in the last year of the data set as compared to 1980? Write a sentence using your results in context properly.
78. Using the World Wind function in the function gallery, how fast was cumulative installed wind power increasing in 2000 and the last year in the data set? How much faster, as a percent, wind power increasing in the last year of the data set as compared to 2000? Write a sentence using your results in context properly.
79. Using the U.S. Wind function in the function gallery, how fast was cumulative installed wind power increasing in 2000 and the last year in the data set? How much faster, as a percent, wind power increasing in the last year of the data set as compared to 2000? Write a sentence using your results in context properly.
80. Find the equation of the tangent line of $f(x) = x^2$ at $x = 5$. What is the y-value of the tangent line given $x = 5.1$? How does this compare to 5.1^2? Why are these two numbers close (or not)? Sketch a graph representing your calculations.
81. Find the equation of the tangent line of $f(x) = x^4$ at $x = 7$. What is the y-value of the tangent line given $x = 7.1$? How does this compare to 7.1^4? Why are these two numbers close (or not)? Sketch a graph representing your calculations.
82. Find the equation of the tangent line of $f(x) = \sqrt{x}$ at $x = 9$. What is the y-value of the tangent line given $x = 9.1$? How does this compare to $\sqrt{9.1}$? Why are these two numbers close (or not)? Sketch a graph representing your calculations.
83. Find the equation of the tangent line of $f(x) = \sqrt{x}$ at $x = 16$. What is the y-value of the tangent line given $x = 16.1$? How does this compare to $\sqrt{16.1}$? Why are these two numbers close (or not)? Sketch a graph representing your calculations.

Chapter 12
Product Rule

The CO2 emissions of a country, in metric tons, can be thought of as made up of two parts. The first is the amount of CO2 emitted per person, metric tons (MT) per person, and the second is the population, in persons:

$$\text{Total CO2 MT} = \text{CO2 MT per person} \times \text{Population}.$$

These quantities change over time so we can view this as a function of t in years

$$\text{Total CO2 MT}(t) = \text{CO2 MT per person}(t) \times \text{Population}(t).$$

How is Total CO2 in MT changing at any point in time? The rate of change of Total CO2 in MT would certainly be positive if the rate of change of CO2 MT per person is positive *and* the rate of change of population is positive. But, could the rate of change of Total CO2 in MT be decreasing if the rate of change of CO2 MT per person is positive? Yes, if the population was decreasing fast enough. We immediately see that the derivative of Total CO2 in MT is a bit complicated.

It would be nice if the derivative of a product of two function was the product of the derivatives. So, could the following be true? **HINT: The following is not true!**

$$\text{Total CO2 MT}'(t) = \text{CO2 MT per person}'(t) \times Population'(t).$$

Why is this incorrect? First, the units on the left are metric tons/year and on the right they are metric tons/person/year × persons/year = metric tons. The units on the left and right are not equal and so the equation cannot be correct. Checking units like this to see if an equation or model makes sense, known as dimensional analysis, is a useful method. Second, if CO2 MT per person$'(t)$ is negative (decreasing CO2 per person) and $Population'(t)$ were positive (increasing population) then by multiplying a positive and a negative we get a negative and hence Total CO2 MT$'(t)$ would be negative, but that doesn't have to be the case because population could be growing so fast that even with decreasing CO2 MT per person the total could be going up.

Let us consider a simple (mostly made up) example to help understand how to calculate Total CO2 MT$'(t)$ or really how we would estimate total CO2 change over

T. J. Pfaff, *Applied Calculus with R*,
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1 year. Suppose the CO2 MT per person is 16.1 MT per person (U.S. per capita emission in 2018) and decreasing at a rate of 0.2 MT per person per year. In other words, next year each person would emit approximately

$$16.1 - 0.2 = 15.9 \text{ MT}$$

next year. Suppose our population has 100 people and growing at a rate of 8 people per year. In other words, our population would have approximately 108 people next year.

Approximately how much will Total CO2 in MT change next year? The 8 people are each going to add 15.9 MT of CO2 next year to the total or

$$15.9 \text{ MT} \times 8 \text{ people} = 127.2 \text{ MT}.$$

At the same time the 100 people in the population are decreasing their emissions by 0.2 MT per person. This gives a total of

$$100 \text{ people} \times (-0.2) \text{ MT per person} = -20 \text{ MT}.$$

Hence a decrease of CO2 next year by 20 MT. We see that the additional people to the population adds more CO2 then the decrease in per capita emission for a total approximate change of

$$128.8 - 20 = 108.8 \text{ MT}$$

next year.

The change has two components that are added (or add a negative number). The 128.8 is the result of a change (increase in population) times an amount (individual CO2 emissions). The 20 comes from a change (new emission level per person) times an amount (how many people). This derivative is called the product rule and given in M-Box 12.1. We move to an example to graphically illustrate the rate of change of a function that arises from the product of two functions, which will first require us to consider the implications of multiplying two functions.

Consider the function $f(x) = x^2 \sin(x)$ in figure 12.1, which includes the function $g(x) = x^2$ and $h(x) = -x^2$ for reference. The product of two functions, in this case x^2 and $\sin(x)$ can make for an interesting new function. We'd like to understand how each part of $f(x) = x^2 \sin(x)$ influences the graph. Think about these questions:

1. Which part of $f(x) = x^2 \sin(x)$ (x^2 or $\sin(x)$) determines whether or not the function is positive or negative?
2. Which part of $f(x) = x^2 \sin(x)$ determines the height at local peaks or maximums as well as the local minimums?
3. Which part of $f(x) = x^2 \sin(x)$ determines when (an x-value) the local maximums and local minimums occur?
4. Which part of $f(x) = x^2 \sin(x)$ determines when the $x - axis$ is crossed?
5. Which part of $f(x) = x^2 \sin(x)$ determines when the function is increasing or decreasing?

Take a moment to write down an answer to each of these questions before we move on. Now examine figure 12.2 which is $f(x) = x^2 \sin(x)$ on the interval from 0 to 2π with $g(x) = 4\sin(x)$ as a reference graph (the 4 is used to scale the function in the graph; it does not change the characteristics of $\sin(x)$ other than the amplitude). Based on this graph, should any of your answers to the above questions be changed?

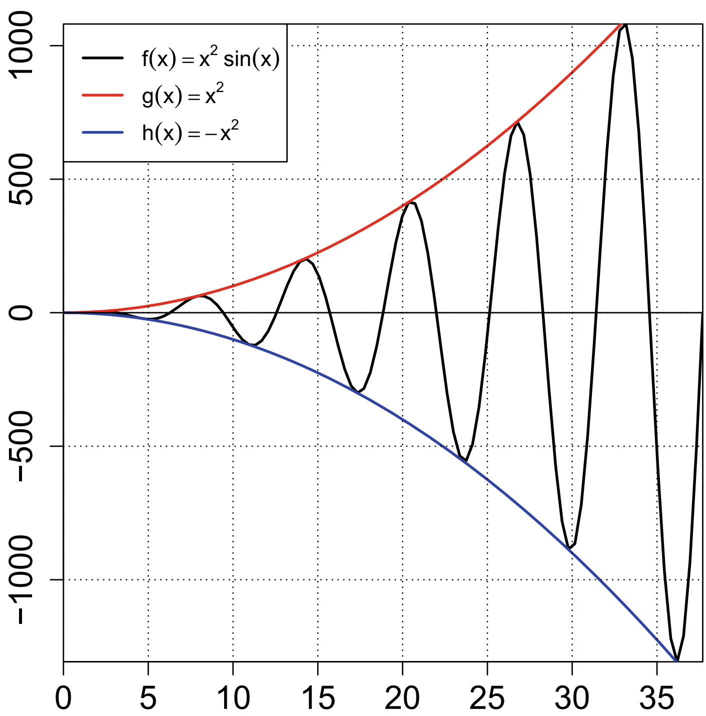


Fig. 12.1 A graph of $f(x) = x^2 \sin(x)$, $g(x) = x^2$, and $h(x) = -x^2$.

Again, considering figure 12.2 answer these questions:

1. Why are the local minimums and maximums of $f(x) = x^2 \sin(x)$ not aligned with that of $4\sin(x)$?
2. Is $f(x) = x^2 \sin(x)$ negative (or positive) at the same values of x as $4\sin(x)$? If so why?
3. Could $2x\cos(x)$ be the derivative of $f(x) = x^2 \sin(x)$? Hint: Evaluate $2x\cos(x)$ at $x = \pi/2$ and $x = 3\pi/2$. Does this math up with how $f(x) = x^2 \sin(x)$ should be changing at these x-values?

The key here is to recognize that multiplying $\sin(x)$ by x^2 changes $\sin(x)$ in more than just the amplitude. We now have two competing functions. The function x^2 is

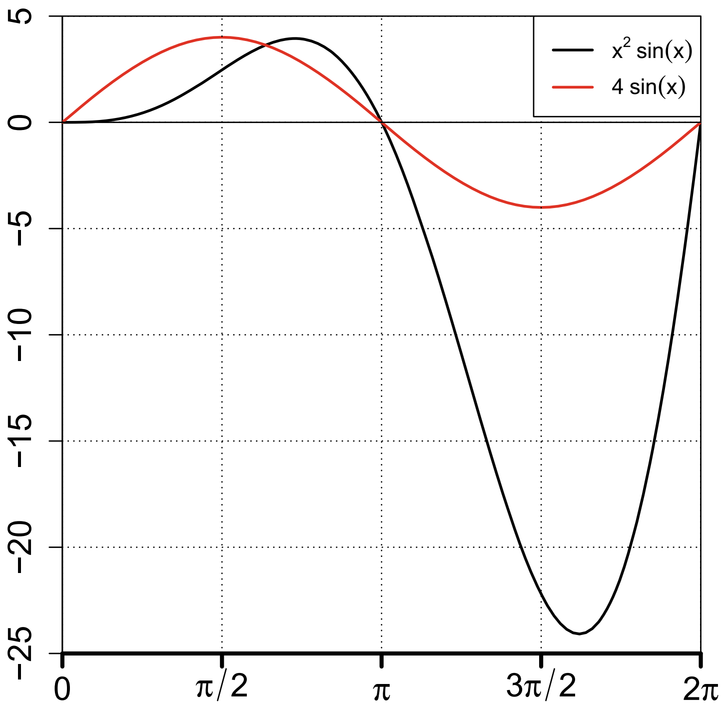


Fig. 12.2 A graph of $f(x) = x^2 \sin(x)$ with $g(x) = 4\sin(x)$ as a reference function. Notice how multiplying $\sin(x)$ by x^2 changes $\sin(x)$ in more than just the amplitude.

always increasing and the rate of increase in increasing as its derivative is $2x$. So, for example, while $\sin(x)$ has a local maximum at $x = \pi/2$ the function $x^2 \sin(x)$ has a local max a little after $x = \pi/2$ (see figure 12.2). The reason for this is that the growth of x^2 as we pass $x = \pi/2$ is more than the rate of decrease of $\sin(x)$. Eventually though $\sin(x)$ gets small enough, and eventually 0, and x^2 cannot compensate. The product of two functions can get complicated and is interesting, and the derivative will need to account for these competing processes.

Fig 12.3 is a graph of $f(x) = x^2 \sin(x)$ and the "derivative" we would like $g(x) = 2x\cos(x)$. There are a numerous reasons that $g(x)$ cannot be the derivative of $f(x)$ that can be articulated from interpreting the graph. Here are a few: At $x = 2.29$ on $f(x) = x^2 \sin(x)$ we have a local maximum and the derivative must be 0 but $g(x) = 2x\cos(x)$ is not 0 as it does not cross the x-axis. Similarly at $x = 5.09$ the derivative is 0 but $g(x) = 2x\cos(x)$ is not 0. From $x = \pi/2$ to $x = 2.29$ the function $f(x) = x^2 \sin(x)$ is increasing and so the derivative is positive but $g(x) = 2x\cos(x)$ is below the x-axis and so negative. The same statement can be made from $x = 3\pi/2$ to $x = 2\pi$. In other words, $g(x) = 2x\cos(x)$ cannot be the derivative of

$f(x) = x^2 \sin(x)$ and one might say it really is not even close. At this point, we have hopefully driven home the point that the derivative of a product of two functions is not as simple as the product of the derivatives. The correct Product Rule is M-Box 12.1 and it is a theorem which is derived from the definition of the derivative, although we will not prove this theorem here.

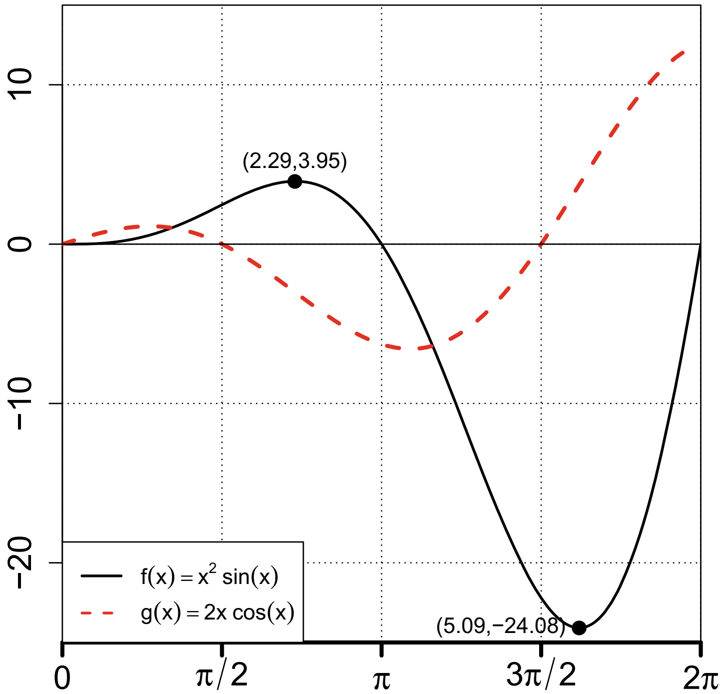


Fig. 12.3 A graph of $f(x) = x^2 \sin(x)$ and the function $g(x) = 2x \cos(x)$ to illustrate why $g(x)$ can't be the derivative of $f(x)$.

M-Box 12.1: Theorem - The Product Rule

If $h(x) = f(x)g(x)$ then

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

or in different notation, if $h(x) = u(x)v(x)$ then

$$h'(x) = udv + vdu.$$

Example 12.1. *What is the derivative of* $f(x) = x^2 \sin(x)$

Solution. To help us keep track of information we set up the following table as a bookkeeping method and to demonstrate correct reasoning to anyone reading our work:

$$\begin{tikzcd} u = x^2 \arrow[dr, "1"] & v = \sin(x) \arrow[dl, bend right, "2"'] \\ du = 2x & dv = \cos(x) \end{tikzcd}$$

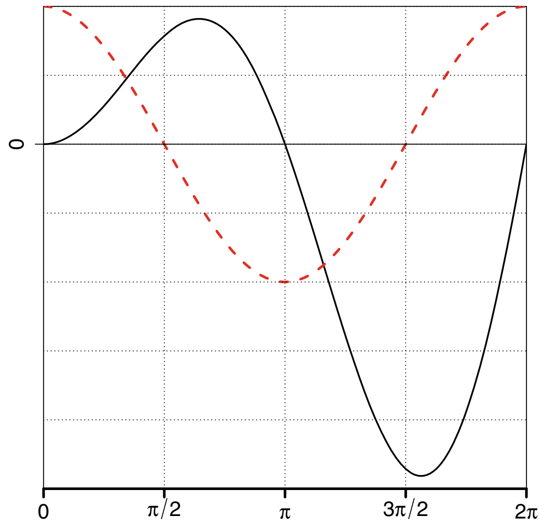
We start by listing the two functions that are being multiplied, $u = x^2$ and $v = \sin(x)$. Below each of these write their derivatives as $du = 2x$ and $dv = \cos(x)$. The product rule then says to multiply udv, the first arrow, and add it to multiplying vdu, the second arrow. We get $f'(x) = x^2 \cos(x) + 2x \sin(x)$. Note that we reorder the last term to make it easier to read as $\sin(x)2x$ could be interpreted as $\sin(x2x)$. □

We should note that the notation used here u, v, du, and dv are used as a bookkeeping tool. But, we should point out that du and dv are known as differentials, really small changes in u and v, respectively. Similarly, dx and dy are differentials and if we consider $\frac{dy}{dx} = f'(x)$ we can write this as $dy = f'(x)dx$, which says that a really small change in x results in a $f'(x)dx$ change in y or dy. You should see this as the microscope equation with dx and dy instead of δx and δy, respectively. If, for example, δx is small then $\delta x \approx dx$. We stop this tangent for now and just say that this is worth noting.

12.1 Exercises

In the following graphs the solid black curve is a function and the dashed red curve is a potential derivative. Provide a rationale as to why the red curve cannot be the derivative. Problems may have multiple responses and while you only need to provide one you should try to identify as many as possible.

1.



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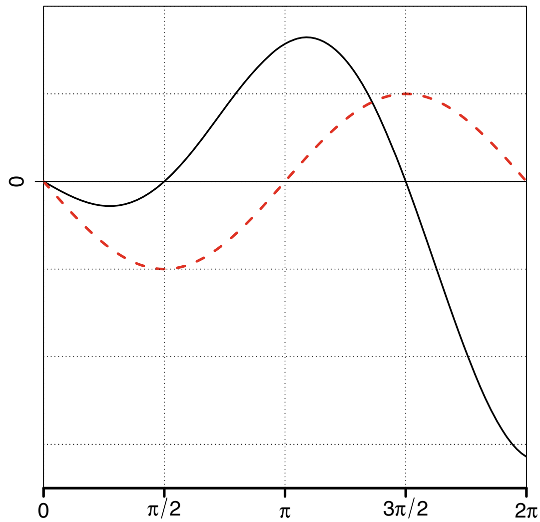

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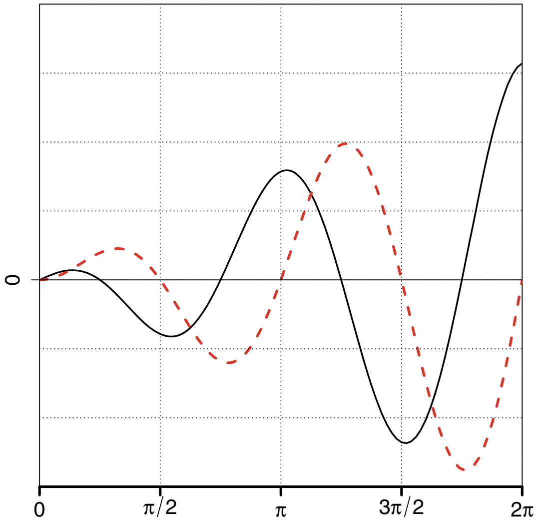

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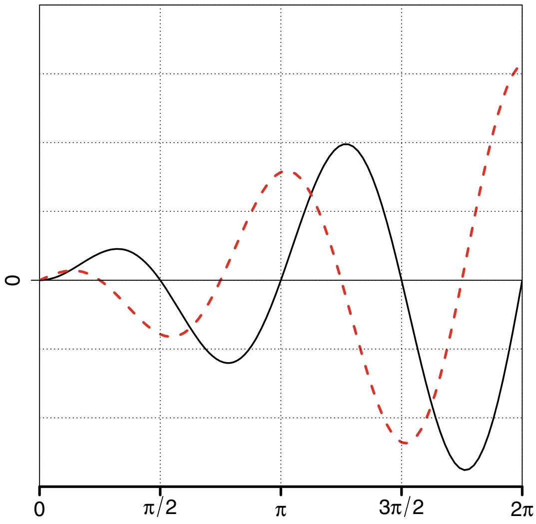

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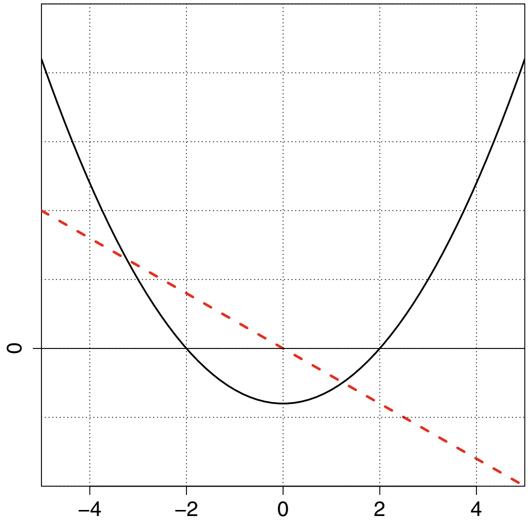

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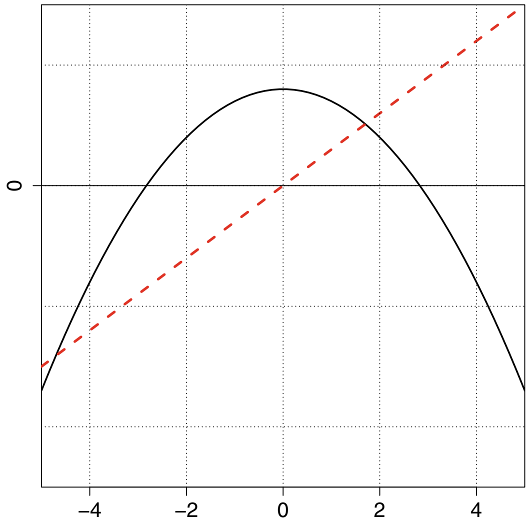

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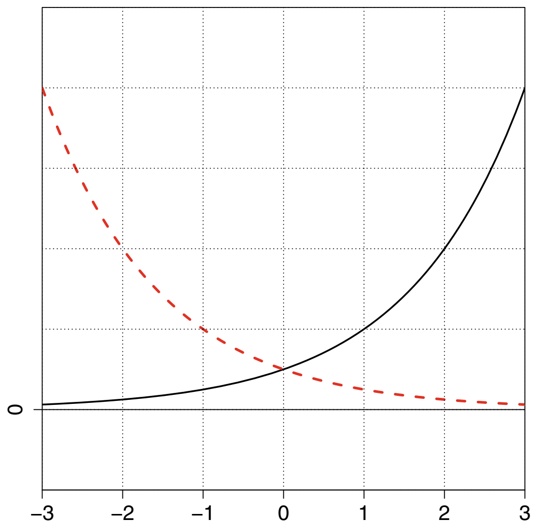

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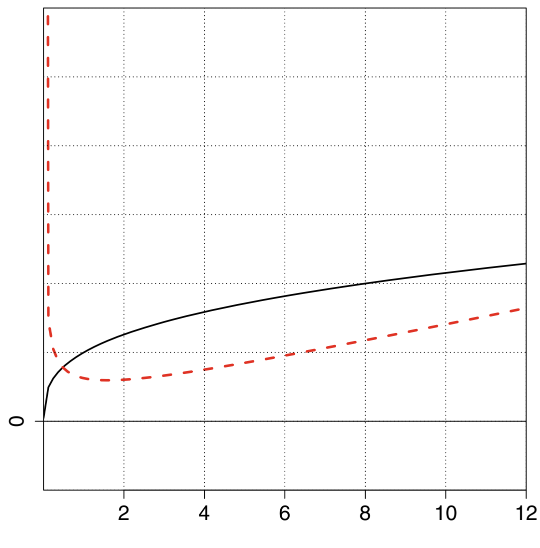


Find the derivative of the following 18 functions.

9. $g(x) = (x^3 + x^2 + 5x + 11)(3x^2 - 4x - 42)$
10. $h(x) = (3x^4 - 6x^2 + 7)(2x^5 + 7x^3 - 8x + 5)$
11. $f(x) = x^2 \cos(x)$
12. $f(x) = x^{11} \sin(x)$
13. $g(t) = (12t^2 - 2t + 9)\sqrt{t}$
14. $h(t) = (7t^4 + 4t^2 - 13)\sqrt{t}$
15. $h(x) = (9x^2 + 3x - 2)\sin(x)$.
16. $g(x) = (4x^7 - 4x^5 + x)\cos(x)$
17. $f(x) = \sqrt[3]{x}\cos(x)$
18. $h(x) = \sqrt[4]{x}\sin(x)$

19. $g(x) = 6x^{-4}e^x$
20. $h(x) = 4x^{-9}e^x$
21. $f(t) = 12\cos(t)\ln(t)$
22. $f(t) = -3\ln(t)\sin(t)$
23. $g(x) = (2x^{15} - 11x^{13} + 3)5^x$
24. $g(x) = (5x^{-10} + 14x^{-3} + 12)8^x$
25. $f(x) = 5x^3e^x\sin(x)$
26. $g(x) = 6x^7e^x\cos(x)$
27. Find the equation of the tangent line to the function $f(x) = x^2\cos(x)$ at $a = \pi/2$
28. Find the equation of the tangent line to the function $f(x) = x^2\cos(x)$ at $a = 3\pi/2$
29. Find the equation of the tangent line to the function $f(x) = xe^x$ at $a = 0$
30. Find the equation of the tangent line to the function $f(x) = x^2e^x$ at $a = 0$

12.2 Project: Conceptual Product Rule Graphic

Create a graph of $f(x) = e^x\sin(x)$ similar to the one in figure 12.1 and/or 12.2. Identify two points on the graph, place identifiable points at those locations, and explain why they show that we cannot have $f'(x) = e^x\cos(x)$. Find the correct derivative of $f(x)$, add it to the graph, and explain why the issues you had with these two points are not a problem with the correct derivative. The main goal here is to make a meaningful graphic. TIP: There is an appendix with R code for all figures.

Chapter 13
Quotient Rule

From the proceeding section we recognized that multiplying two functions creates interesting dynamics and so it should not be surprising that dividing functions also creates interesting dynamics. Let us start with an applied example.

If we would like to know what percent of cumulative installed world wind power is installed by the U.S. and how that is changing we would create the function

$$USwindPercent(t) = \frac{USwind(t)}{Wwind(t)} \times 100\%$$

using the function $USwind(t)$ and $Wwind(t)$ from the function gallery. Reading off the graphs for 2010 (chosen because it is easy to read off values) this is about

$$\frac{40,000}{200,000} \times 100\% = 20\%.$$

In other words, in 2010 the U.S. had 20% of the installed world wind power. The question becomes how do we find $USwindPercent'(t)$? Note that the units of $USwindPercent'(t)$ should be percentage points per year. We would like the derivative to be

$$\frac{USwind'(t)}{Wwind'(t)} \times 100\%$$

but the units of this are just percent and missing per year. Hence, this cannot be the derivative.

Another, just as important, explanation as to why

$$\frac{USwind'(t)}{Wwind'(t)} \times 100\%$$

is not the derivative of $USwindPercent(t)$ is as follows: If we presume that both the U.S. and the World are increasing their wind power then both the numerator and denominator of

$$\frac{USwind'(t)}{Wwind'(t)}$$

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_13

would be positive and that would imply that $USwindPercent'(t)$ would be positive (positive divided by a positive is positive). But, if everyone in the world was increasing their wind power by a lot and the U.S. only by a little then the share of U.S. wind power in the world would be decreasing and $USwindPercent'(t)$ would have to be negative.

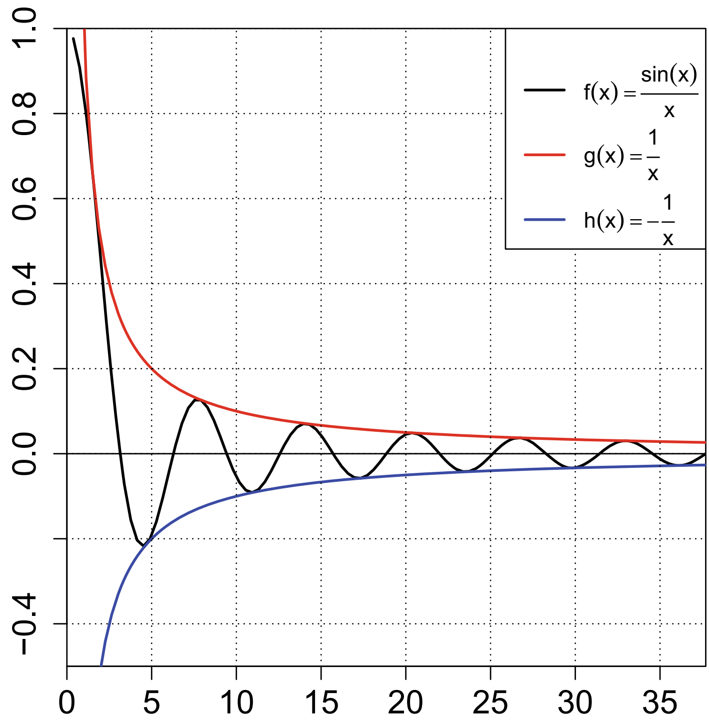


Fig. 13.1 The graph of $f(x) = \frac{\sin(x)}{x}$ along with $g(x) = \frac{1}{x}$ and $h(x) = \frac{-1}{x}$ for reference.

Before we get to the Quotient rule, M-Box 13.1, here is a graphical example similar to what we did with the product rule to illustrate the rate of change of a function that arises from dividing two functions. Consider the function

$$f(x) = \frac{\sin(x)}{x}$$

in figure 13.1, which includes the function $g(x) = \frac{1}{x}$, and $h(x) = \frac{-1}{x}$ for reference. The quotient of two functions, in this case $\sin(x)$ over x can make for an interesting new function. We would like to understand how each part of

$$f(x) = \frac{\sin(x)}{x}$$

influences the graph. Consider these questions:

1. Which part of $f(x) = \frac{\sin(x)}{x}$ ($\sin(x)$ in the numerator or x in the denominator) determines whether the function is positive or negative?
2. Which part of $f(x) = \frac{\sin(x)}{x}$ determines the height of local peaks or maximums as well as the local minimums?
3. Which part of $f(x) = \frac{\sin(x)}{x}$ determines when (an x-value) the local maximums and local minimums occur?
4. Which part of $f(x) = \frac{\sin(x)}{x}$ determines when the x-axis is crossed?
5. Which part of $f(x) = \frac{\sin(x)}{x}$ determines when the function is increasing or decreasing?

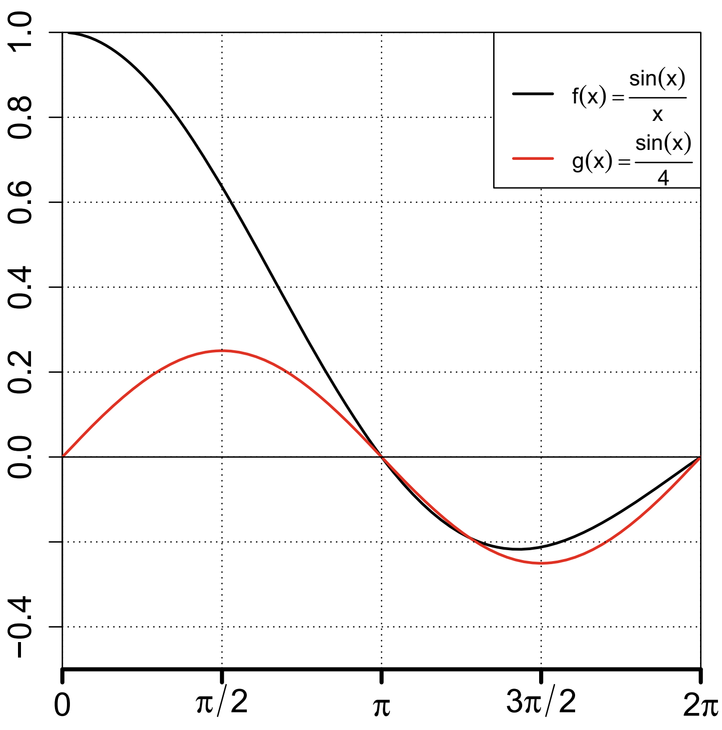


Fig. 13.2 The graph of $f(x) = \frac{\sin(x)}{x}$ along with $g(x) = \frac{\sin(x)}{4}$ for reference.

Take a moment to write down an answer to each of these questions before moving on. Now examine figure 13.2 which is $f(x) = \frac{\sin(x)}{x}$ on the interval from 0 to 2π

with $g(x) = \frac{\sin(x)}{4}$ as a reference graph. Based on this graph, should any of your answers to the above questions be changed? Again considering figure 13.2 answer these questions:

1. Why are the local minimums and maximums of $f(x) = \frac{\sin(x)}{x}$ not aligned with that of $g(x) = \frac{\sin(x)}{4}$?
2. Is $f(x) = \frac{\sin(x)}{x}$ negative (or positive) at the same value of x as $\sin(x)$? If so why?
3. Could $\frac{\cos(x)}{1}$ be the derivative of $f(x) = \frac{\sin(x)}{x}$? Hint: Is $\frac{\cos(x)}{1}$ positive or negative from 0 to $\pi/2$ and does this match how $f(x) = \frac{\sin(x)}{x}$ is changing on that interval? Also, evaluate $\frac{\cos(x)}{1}$ at $\pi/2$ and $3\pi/2$. Does this match up with $f(x) = \frac{\sin(x)}{x}$?

The quotient of two functions is even more complicated than the product of two functions. It should not be surprising then that the rule for the derivative of a quotient, M-Box 13.1, is more involved.

M-Box 13.1: Theorem - The Quotient Rule

If $h(x) = \frac{f(x)}{g(x)}$ then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

or in different notation, if $h(x) = \frac{u(x)}{v(x)}$ then

$$h'(x) = \frac{vdu - udv}{v^2}$$

Example 13.1. *Find the derivative of* $f(x) = \frac{\sin(x)}{x}$.

Solution. To help us keep track of information and make it easier for a reader to follow our reasoning, we set up the following table:

$$\begin{tikzcd} u = \sin(x) \arrow[dr, bend left, "2"] & v = x \arrow[dl, "1"'] \\ du = \cos(x) & dv = 1 \end{tikzcd}$$

We start by listing the two functions that make up the quotient with u the top function, $u = \sin(x)$ and $v = x$. Below each of these write their derivatives as $du = \cos(x)$ and $dv = 1$. The quotient rule then says to multiply vdu (the first arrow), subtract from it udv (the second arrow), and then dived by $v^2 = x^2$. We get

$$f'(x) = \frac{x\cos(x) - 1\sin(x)}{x^2}.$$

Note that the order of the arrows is reversed from the product rule which helps to remind us to subtract in the numerator. □

The graph of $f(x) = \frac{\sin(x)}{x}$ and its derivative is given in figure 13.3. Take a moment to examine the relationships between $f(x)$ and $f'(x)$.

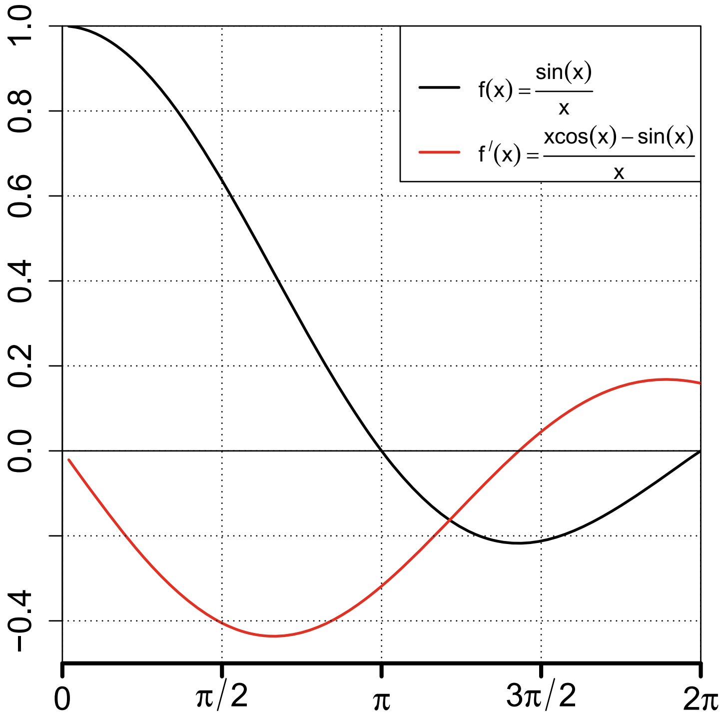


Fig. 13.3 The graph of $f(x) = \frac{\sin(x)}{x}$ and its derivative.

13.1 Exercises

Find the derivative of the following 24 functions.

1. $g(x) = \dfrac{x^2}{5x^2 + 6x + 2}$
2. $f(x) = \dfrac{x^3}{7x^2 + 4x + 8}$
3. $f(x) = \dfrac{x^2 + 5x + 2}{x^3 - x}$
4. $g(t) = \dfrac{t^4 - 3t - 5}{t^2 + 7}$

5. $g(x) = \dfrac{5x^2}{5x^2 + 6x + 2}$

6. $f(x) = \dfrac{9x^3}{7x^2 + 4x + 8}$

7. $f(x) = \dfrac{4\sqrt{x}}{5 + x^2}$

8. $g(x) = \dfrac{4\sqrt[3]{x}}{8 + x^2}$

9. $f(x) = \dfrac{8x^5}{x + \cos(x)}$

10. $h(t) = \dfrac{3t^4}{t - \sin(t)}$

11. $h(x) = \dfrac{\sin(x)}{x^2 + 5x + 3}$

12. $h(x) = \dfrac{\cos(x)}{5x^4 + x^2 - 14}$

13. $f(t) = \dfrac{e^t}{4 + \sqrt{t}}$

14. $f(t) = \dfrac{9^t}{5 - \sqrt[3]{t}}$

15. $h(x) = \dfrac{3x^9}{5x + 6e^x}$

16. $g(x) = \dfrac{5x^7}{4x^2 + 3e^x}$

17. $f(x) = \dfrac{\cos(x)}{5 + 4e^x}$

18. $f(x) = \dfrac{\sin(x)}{5 + \ln(x)}$

19. $g(z) = \dfrac{\ln(z)}{z^2 + 5}$

20. $h(z) = \dfrac{\ln(z)}{z^3 + 9}$

21. $f(x) = \dfrac{5x^2 e^x}{\sin(x)}$

22. $f(x) = \dfrac{7x^4 5^x}{\cos(x)}$

23. $f(x) = \dfrac{9x^2 \cos(x)}{e^x}$

24. $f(x) = \dfrac{3x^3 \sin(x)}{e^x}$

25. Note that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$. Use this fact to find a formula for the derivative of $\tan(x)$.

26. Note that $\sec(x) = \dfrac{1}{\cos(x)}$. Use this fact to find a formula for the derivative of $\sec(x)$.

Chapter 14
Chain Rule

Before moving to our last derivative result, the chain rule, you might consider reviewing the Appendix on function composition, F. The chain rule is considered the most challenging of the three results, the produce rule, quotient rule, and chain rule, partly due to function composition itself being confusing. We are going to use $\sin(x^2)$ as a main example, which is a composition of $\sin(x)$ and x^2. In other words, if $f(x) = \sin(x)$ and $g(x) = x^2$ then $h(x) = f(g(x)) = \sin(x^2)$. In this example, $g(x) = x^2$ is the inside function while $f(x) = \sin(x)$ is the outside function, because $g(x)$ is inside $f(x)$ in $h(x) = f(g(x))$. Now, if we are given $h(x) == \sin(x^2)$ how do we know this is a composition and how do we know which is the inside function. What may help is considered what we would do if we were to evaluate $h(x)$ at some value of x, say $x = 5$. We would first do 5^2 to get 25 after that we would then evaluate $\sin(25)$. Here x^2 is the inside function because we did that first, squared 5, and then took that output and used it to find $\sin(25)$, making $\sin(x)$ the outside function. We will consider both how function composition changes functions along with the impact on the derivative.

From the previous section we considered the graphs of $f(x) = x^2 \sin(x)$ and $f(x) = \frac{\sin(x)}{x}$ to gain an intuitive understanding of the product rule and quotient rule. Multiplying and dividing functions are two ways to build new more complicated functions from simple functions. One more way to build a new function is through function composition. In a similar manner to the product and quotient rule sections we will consider the function $f(x) = \sin(x^2)$, which is a composition of $\sin(x)$ and x^2, in figure 14.1. Our goal is to understand how each part, $\sin(x)$ and x^2, contribute to the graph of $f(x) = \sin(x^2)$. Consider these questions:

1. Which part of $f(x) = \sin(x^2)$, $\sin(x)$, or x^2 contributes to the height of local peaks or maximums as well as the local minimums?
2. Which part of $f(x) = \sin(x^2)$ determines when (an x-value) the local maximums and local minimums occur?
3. Which part of $f(x) = \sin(x^2)$ determines when the x-axis is crossed?
4. Which part of $f(x) = \sin(x^2)$ determines how often the function crosses the x-axis?

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_14

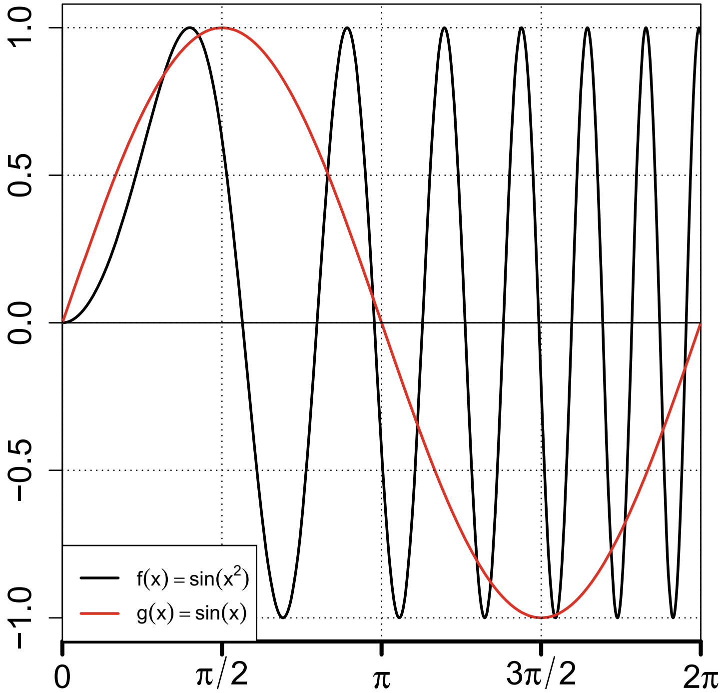


Fig. 14.1 The graph of $f(x) = \sin(x^2)$ with $g(x) = \sin(x)$ as a reference.

5. Why are neither $\cos(x^2)$ or $\cos(2x)$ the derivative of $f(x) = \sin(x^2)$?
6. What attributes must the derivative of $f(x) = \sin(x^2)$ have?
7. How would $g(x) = \sin(x^3)$ compare to $f(x) = \sin(x^2)$?

Now consider the graphs in figure 14.2 and answer these questions:

1. Why does the derivative graph oscillate as much as $f(x) = \sin(x^2)$?
2. What part of $f(x) = \sin(x^2)$ is creating the oscillation of the derivative?
3. Why does the derivative graph increase in magnitude?
4. What part of $f(x) = \sin(x^2)$ is forcing the increase in magnitude of the derivative?

The derivative of a function composition is given in M-Box 14.1.

M-Box 14.1: Theorem - The Chain Rule

If $h(x) = f(g(x))$ then

$$h'(x) = f'(g(x))g'(x)$$

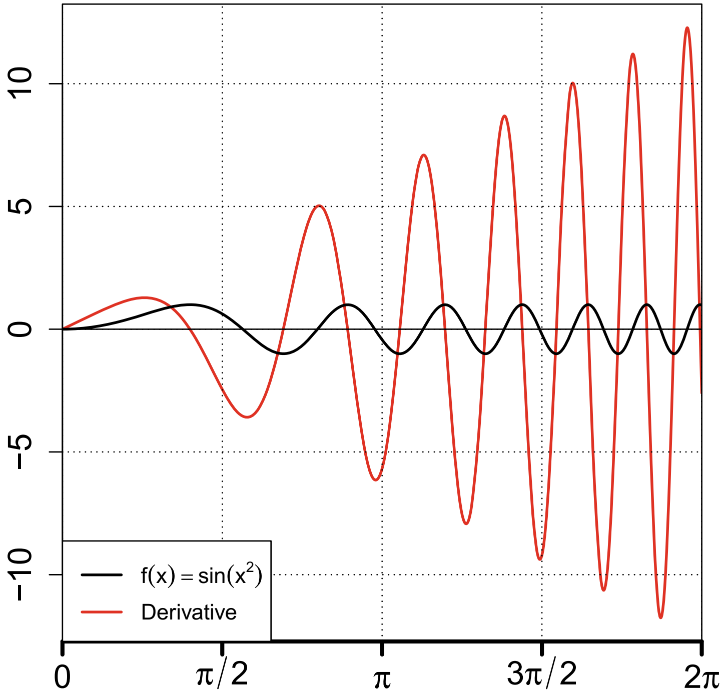


Fig. 14.2 The graph of $f(x) = \sin(x^2)$ and the graph of its derivative.

or, to avoid confusion of $f(x)$ and $g(x)$ in problems: If $f(x) = u(v(x))$ then

$$f'(x) = u'(v(x))v'(x)$$

Example 14.1. *Find* $f'(x)$ *with* $f(x) = \sin(x^2)$.

Solution. We identify $u(x) = \sin(x)$, the outside function, and $v(x) = x^2$, the inside function. Now $u'(x) = \cos(x)$ and $v'(x) = 2x$ so that

$$\begin{aligned} f'(x) &= u'(v(x))v'(x) \\ &= u'(x^2)v'(x) \\ &= \cos(x^2)2x \\ &= 2x\cos(x^2). \end{aligned}$$

□

We note that as you become comfortable using the chain rule you will not need to define $u(x)$ and $v(x)$ as we did in the previous example. You should aim to recognize the inside and outside function and recognize that the chain rule is the derivative of the outside function, leaving the inside function alone, and then multiplying by the inside function. Here are two more examples but without defining $u(x)$ and $v(x)$.

Example 14.2. *Find $f'(x)$ with $f(x) = e^{\sin(x)}$.*

Solution. If we were to evaluate $f(x)$ at say 10 (chosen at random) we realize we would first find $\sin(10)$ and then that becomes the power of e^x. This makes $\sin(x)$ the inside function and e^x the outside function. The derivative of e^x is e^x and the derivative of $\sin(x)$ is $\cos(x)$. Now, by the chain rule the derivative of $f(x)$ is the derivative of the outside function leaving the inside function alone, giving us $e^{\sin(x)}$ and then multiplying by the derivative of the inside function, $\cos(x)$. Hence,

$$f'(x) = e^{\sin(x)}\cos(x),$$

although we should write this as

$$f'(x) = \cos(x)e^{\sin(x)}.$$

□

Example 14.3. *Find $f'(x)$ with $f(x) = (x^2 + 5x - 2)^5$.*

Solution. If we were to evaluate $f(x)$ at say 7 (chosen at random) we realize we would first find $^2 + 5(7) - 2$ and then we raise that to the 5th power. This makes $x^2 + 5x - 2$ the inside function and x^5 the outside function. The derivative of x^5 is $5x^4$ and the derivative of $x^2 + 5x - 2$ is $2x + 5$. Now, by the chain rule the derivative of $f(x)$ is the derivative of the outside function leaving the inside function alone, giving us $5(x^2 + 5x - 2)^4$ and then multiplying by the derivative of the inside function, $2x + 5$. Hence,

$$f'(x) = 5(x^2 + 5x - 2)^4(2x + 5)$$

although we should write this as

$$f'(x) = (10x + 25)(x^2 + 5x - 2)^4,$$

where the $2x + 5$ was moved to the front and then the 5 was distributed through it. □

Example 14.4. *Find $f'(x)$ with $f(x) = e^{\sin(x)}$.*

Solution. If we were to evaluate $f(x)$ at say 10 (chosen at random) we realize we would first find $\sin(10)$ and then that becomes the power of e^x. This makes $\sin(x)$ the inside function and e^x the outside function. The derivative of e^x is e^x and the derivative of $\sin(x)$ is $\cos(x)$. Now, by the chain rule the derivative of $f(x)$ is the derivative of the outside function leaving the inside function alone, giving us $e^{\sin(x)}$ and then multiplying by the derivative of the inside function, $\cos(x)$. Hence,

$$f'(x) = e^{\sin(x)} \cos(x),$$

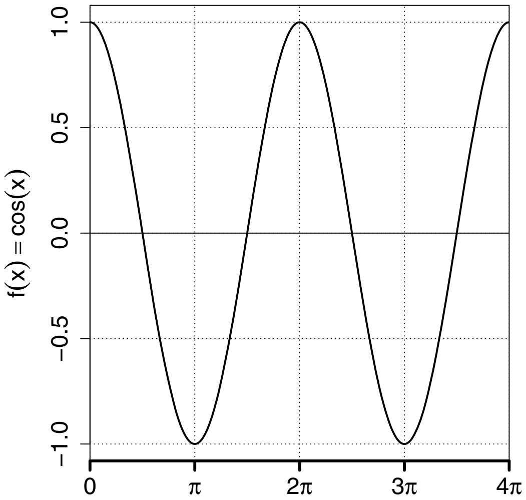
although we should write this as

$$f'(x) = \cos(x)e^{\sin(x)}.$$

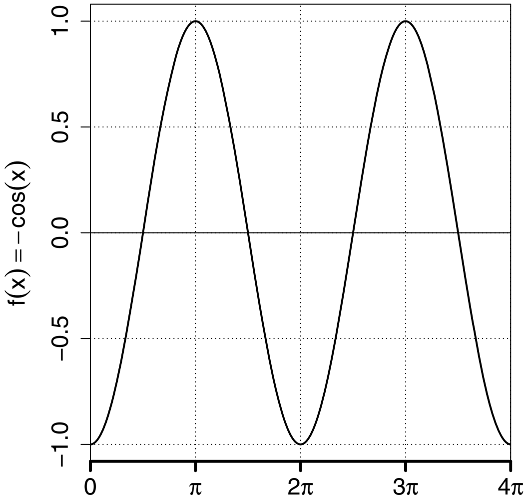
Note that this example is a little extra confusing because the derivative of e^x is e^x. □

14.1 Exercises

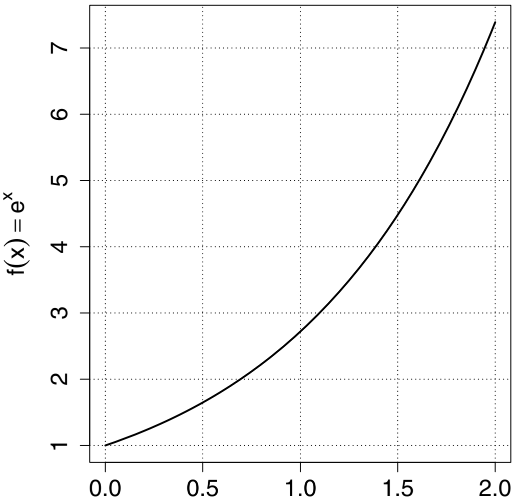
1. Sketch the graph of $\cos(\sqrt{x})$ on the same graph as $f(x) = \cos(x)$ given here. The important aspect is that your sketch is roughly correct relative to the given graph. Now check your sketch by graphing the two functions in R. Use two different colors.



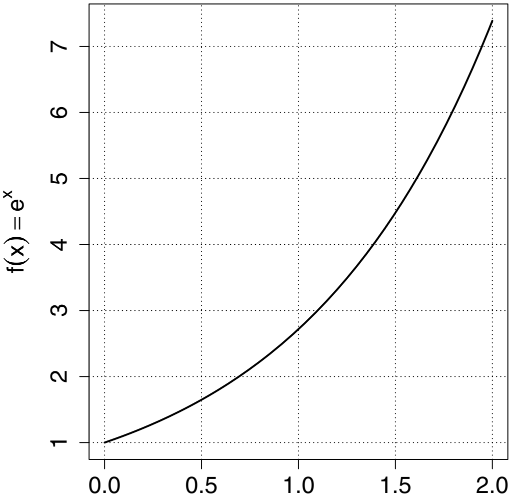
2. Sketch the graph of $-\cos(\sqrt{x})$ on the same graph as $f(x) = -\cos(x)$ given here. The important aspect is that your sketch is roughly correct relative to the given graph. Now check your sketch by graphing the two functions in R. Use two different colors.



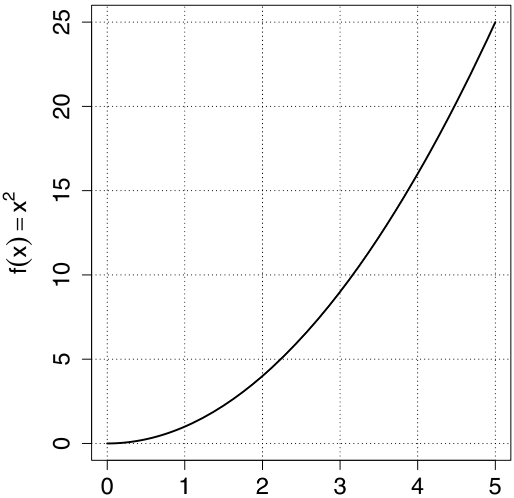
3. Sketch the graph of $e^{\sqrt{x}}$ on the same graph as $f(x) = e^x$ given here. The important aspect is that your sketch is roughly correct relative to the given graph. Now check your sketch by graphing the two functions in R. Use two different colors.



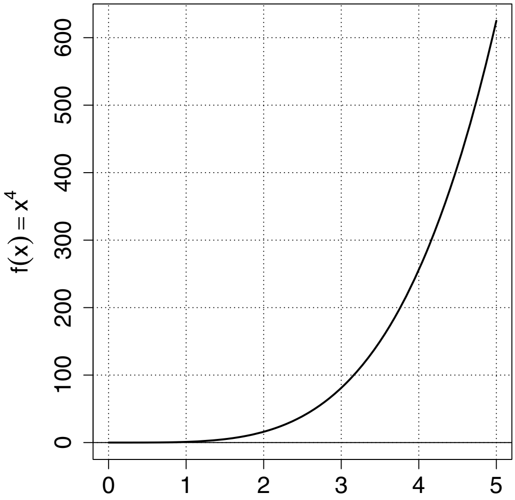
4. Sketch the graph of e^{x^2} on the same graph as $f(x) = e^x$ given here. The important aspect is that your sketch is roughly correct relative to the given graph. Now check your sketch by graphing the two functions in R. Use two different colors.



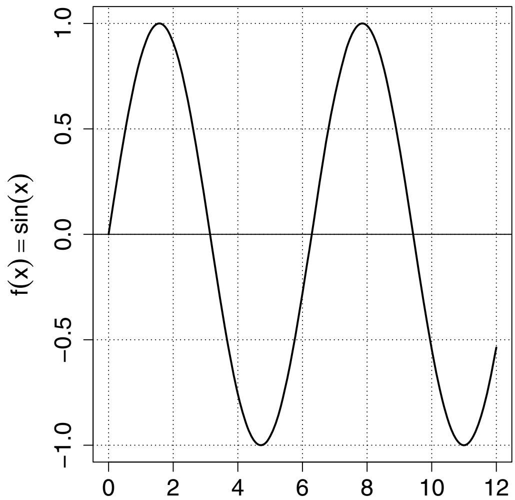
5. Sketch the graph of $(3\sin(x))^2$ on the same graph as $f(x) = x^2$ given here. The important aspect is that your sketch is roughly correct relative to the given graph. Now check your sketch by graphing the two functions in R. Use two different colors.



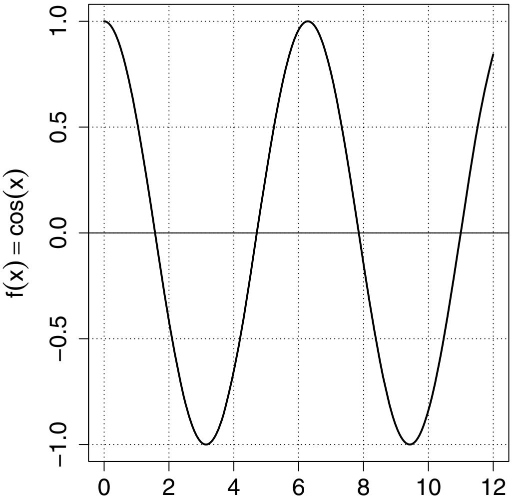
6. Sketch the graph of $(5\cos(x))^4$ on the same graph as $f(x) = x^4$ given here. The important aspect is that your sketch is roughly correct relative to the given graph. Now check your sketch by graphing the two functions in R. Use two different colors.



7. Sketch the graph of $\sin(\cos(x))$ on the same graph as $f(x) = \sin(x)$ given here. The important aspect is that your sketch is roughly correct relative to the given graph. Now check your sketch by graphing the two functions in R. Use two different colors.



8. Sketch the graph of $\cos(\sin(x))$ on the same graph as $f(x) = \cos(x)$ given here. The important aspect is that your sketch is roughly correct relative to the given graph. Now check your sketch by graphing the two functions in R. Use two different colors.



Find the derivative of the following functions.

9. $g(x) = \cos(\sqrt{x})$
10. $g(x) = -\cos(\sqrt{x})$
11. $g(x) = e^{\sqrt{x}}$
12. $g(x) = e^{x^2}$
13. $g(x) = (3\sin(x))^2$
14. $g(x) = (5\cos(x))^4$
15. $g(x) = \sin(\cos(x))$
16. $g(x) = \cos(\sin(x))$
17. $f(x) = (12x^2 - 5x + 12)^{10}$
18. $g(x) = (3x^{-7} - 11x^9 + 13)^{12}$
19. $f(t) = (2t^4 - 9t^3 - 12)^6$
20. $h(x) = (14x^{-13} + 4x^{-9} + 3)^{11}$
21. $g(x) = (-2x^4 - 13x^2 + 6\sin(x))^{-3}$
22. $g(t) = (6t^9 - 6t^{-13} - 5\cos t)^{-7}$
23. $f(x) = \sqrt{13x^5 - 3x^4 + 9}$
24. $h(x) = \sqrt{4x^{-15} - 13x^6 + 8}$
25. $h(t) = \sqrt[3]{2x^{-8} - 12x^3 - 10}$
26. $f(x) = \sqrt[3]{5x^{-8} + 11x^9 - 13}$
27. $h(x) = \dfrac{15}{8(4x^3 - 9x^4 - 13)^4}$
28. $g(x) = \dfrac{10}{6(x^{13} - 5x^3 + 7)^6}$
29. $f(t) = \dfrac{13}{\sqrt{2t^7 - 8t^3 + 14}}$
30. $g(t) = \dfrac{7}{\sqrt{8t^7 - 2t^{12} + 15}}$
31. $f(x) = e^{3x^6+5}$
32. $h(x) = e^{2x^8-11}$
33. $f(x) = e^{5x^6-8x+\cos(x)}$
34. $f(t) = e^{-t^{-12}+13t-\sin(x)}$
35. $g(x) = 7^{3x^2-8x+7}$
36. $h(t) = 13^{7t^5-3t+9}$
37. $f(t) = 9^{t^{15}-5\sin(t)}$
38. $g(t) = 6^{t^5-14\cos(t)}$

39. $f(x) = \cos(3x^{11} - 15x^3 + 12)$
40. $f(x) = \cos(14x^{10} - 7x^4 + 2)$
41. $h(t) = \sin(5t^3 - 12t^4 - 11)$
42. $f(t) = \sin(8t^{12} - 7t^9 - 13)$
43. $f(x) = \ln(15x^8 - 15x^3 - 8)$
44. $g(x) = \ln(12x^3 - 13x^2 - 14)$
45. $f(t) = \ln(8t^{-15} + 4\sin(t))$
46. $h(x) = \ln(14x^{-11} - 5\cos(x))$
47. $f(x) = \sin(e^{5x^2+6x+9})$
48. $f(x) = \ln(\sqrt{5x^4 + \cos(4x^7 + 9)})$
49. $g(x) = (9^{12x^2} - e^{15x^2+\sin(x^7)})^8$
50. $h(t) = e^{(t^6+\cos t^2)^7}$

Derivatives may now be a combination of chain, product, and quotient rules.

51. $f(x) = 2x^2e^{x^4}$
52. $f(x) = 7x^3e^{x^9}$
53. $h(x) = \dfrac{(x^2 + 5x - 4)^3}{\cos(x^2)}$
54. $h(x) = \dfrac{(x^2 - 3x + 14)^3}{\sin(x^3)}$
55. $g(x) = \cos(x^3 - 5)\ln(x^7 + 2)$
56. $h(t) = \sin(t^5 + 7)\ln(x^9 - 3)$
57. $f(t) = 3t^2(6t^2 + 3t - 9)^8$
58. $g(t) = 5t^7(6t^2 - 5t + 11)^{12}$
59. $h(x) = \dfrac{7x^2}{5 + e^{x^2+4}}$
60. $h(x) = \dfrac{8x^5}{9 - e^{x^2-3}}$
61. $g(x) = \sin(x^2)e^{2x^3-5x+7}$
62. $g(x) = \cos(x^4)e^{3x^5+9x-42}$
63. $f(x) = \dfrac{12x^6}{10 - \ln(x^3 + 9)}$
64. $g(x) = \dfrac{9x^4}{14 + \ln(x^5 - 1)}$
65. $f(x) = 5^{3-x^4}\sqrt{x^2 + 5x - 2}$
66. $g(x) = 7^{8+x^2}\sqrt{x^3 - 6x + 11}$
67. $g(x) = \dfrac{5^{4x^2+10}}{\sqrt{15x^3 - 5}}$
68. $h(t) = \dfrac{8^{2t^{11}+13}}{\sqrt{5t^4 - 8}}$
69. $f(t) = (7t^{13} + 42t - 11)^9e^{6t^5+19}$
70. $f(x) =$
$(4x^{14} + 7x^8 - 14)^4e^{12x^8-10}$
71. $f(x) = \dfrac{e^{8x^6-7x+2}}{\cos(4x^5 - 10)}$
72. $f(t) = \dfrac{e^{10t^3-2t+9}}{\sin(8t^7 - 12)}$
73. $h(x) = \sin(3x^{14} - 10x^7 + 1)\ln(5x^9 - 8)$
74. $g(x) = \cos(15x^{14} - 6x^8 + 5)\ln(12x^{15} - 7x^9 + 13)$

Chapter 15
Derivatives with R

The Deriv package in R has a function that performs symbolic differentiation. The first line in R Code box 15.1 loads the Deriv package with the **library** command (this assumes the packages has been installed on your computer with **install.packages(“Deriv")** as noted in Chapter 1). Recall that a package has to be loaded only once per session and so we won’t have **library(Deriv)** in the examples below after the first example. The second line defines the $CO2$ function. The third line sets **CO2_p** to the derivative of $CO2$ using the **Deriv** function. We use the convention of adding **_p** to denote the derivative of a function in R since we don’t have the option of using a prime. In other words, we use **f_p** for the derivative of **f** in R because we cannot use $f'(x)$. The last line outputs the derivative function. Note that **Deriv(CO2)** by itself will produce this output but by assigning **CO2_p** to **Deriv(CO2)** we can use the **CO2_p** function. For example, R Code box 15.2 outputs the derivative of CO2 for 2017 with **CO2_p(67)**, which we estimated in Chapter 4 and calculated in example 11.8.

R Code 15.1: The Derivative of the CO2 Function

```
> library(Deriv)
> CO2<-function(x){0.0134594696825*x^2+
0.520632601929*x+310.423363171}
> CO2_p<-Deriv(CO2)
> CO2_p
```

```
function (x)
0.026918939365 * x + 0.520632601929
```

R Code 15.2: Evaluate a Derivative

```
> CO2_p(67)
```


T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_15

```
[1] 2.324201539384
```

In Chapter 8 we estimated the derivative of

$$f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$$

at $x = 4.30931$. R Code box 15.3 outputs $f'(4.30931)$. Note that $f'(4.30931)$ is not 0 all though it is "close." R code box 15.4 provides some insight in that $x = 4.30931$ is not actually a root since we rounded to the fifth decimal place. For R code box 15.4 we loaded the rootSolve package and used the **uniroot.all** function to find the roots of **f_p**. Recall that the second argument in **uniroot.all** provides a domain in which to find any roots. In summary, **uniroot.all** has two inputs. The first input is a function and the second is a range in which to locate any roots.

R Code 15.3: Evaluate a Derivative

```
> f<-function(x){exp(x)+(2.5)^x*sin(2*pi*x)-10}
> f_p<-Deriv(f)
> f_p(4.30931)
```

```
[1] 0.00668932509134379
```

R Code 15.4: Find the Roots of a Derivative

```
> library(rootSolve)
> uniroot.all(f_p, c(4,4.5))
```

```
[1] 4.30931270608666
```

Combining the **Deriv** and **uniroot.all** functions will allow us to find local maximum and minimums of functions. In general, local maximums or minimums occur when the derivative is zero and so roots of the derivative provide key information about the original function. We will discuss this in more detail in Chapter 17, but for now note the example in R Code box 15.5. We define **f** to be the function $\sin(x)$ and set its derivative to **f_p**. We then find the roots of **f_p** with **uniroot.all** on the interval from 0 to 2π. You are likely familiar with the output as they are $\pi/2 = 1.570796$ and $3\pi/2 = 4.712389$.

R Code 15.5: Find the Roots of a Derivative

```
> f<-function(x){sin(x)}
> f_p<-Deriv(f)
```

```
> uniroot.all(f_p,c(0,2*pi))
[1] 1.570796 4.712389
```

In Chapter 9 we considered the function

$$f(x) = 0.1(x^3/3 + x^2 - 15x) - 4.$$

R Code box 15.6 demonstrates how we graphed the function and added one of the tangent lines to the graph. As we go through the code, focus on **f_tan** as this is a function of two variables that we created from two previous functions, **f** and **f_p**. The first line defines the function **f** and the second is its derivative **f_p**.

Recall the point slope form of a line

$$y - y_1 = m(x - x_1).$$

For tangent lines to the function $f(x)$ at $x = a$, we have $x_1 = a$, $y_1 = f(a)$, and $m = f'(x)$. Solving for y with these variables gives

$$y = f'(a)(x - a) + f(a).$$

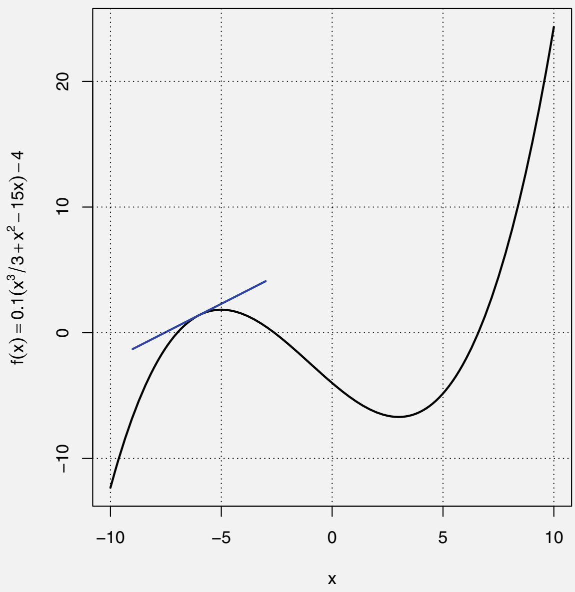
We define the function **f_tan** (tangents to f) as a function of a and x as $f'(a)(x - a) + f(a)$ or in R language f_p(a)*(x-a)+f(a). Here the value of a is the location of the tangent line to $f(x)$ and x is an input value on the line. So, for example, **f_tan(3,10)** would return the y-value corresponding to the x-value of 10 on the tangent line at $x = 3$ of the function f. Try it.

The **par(mar=c(4,5,1,1))** sets the margins around the graph at 4, 5, 1, and 1 line starting at the bottom and proceeding clockwise (bottom, left, top, and right). We plot the function **f** with **curve** from -10 to 10 with a line width of 2 (lwd=2). We label the y-axis using ylab. The function **expression** is for mathematical expression. Note we use double = signs to print out one equal sign on the graph. The next use of **curve** adds the tangent line. Here we plot **f_tan(-6,x)** which is the equation of the tangent line at $x = -6$ of the function **f**. The tangent line is plotted from -9 to -3 (x-values) with a line width of 2 and colored blue. We need to include add=TRUE to add this to the graph started in the previous line, otherwise a new graph would be generated. The last line adds a grid with the first two arguments NULL adding grid lines a tick marks of the current graph.

R Code 15.6: Placing a Tangent Line on a Graph

```
> f<-function(x){.1*(x^3/3+x^2-15*x)-4}
> f_p<-Deriv(f)
```

```
> f_tan<-function(a,x){f_p(a)*(x-a)+f(a)}
> par(mar=c(4,5,1,1))
> curve(f,-10,10,lwd=2,
ylab=expression(f(x)==.1*(x^3/3+x^2-15*x)-4))
> curve(f_tan(-6,x),-9,-3,lwd=2,col="blue",add=TRUE)
> grid(NULL,NULL,col="black")
```



M-Box 15.1: The Second and nth Derivatives

Given a function $f(x)$ the second derivative $f''(x) = \frac{d}{dx}(f'(x))$ or simply the derivative of the derivative. The second derivative provides information about concavity. More generally $f^{(n)}(x)$ is the nth derivative of a function, in other words continue taking derivatives of a function n times.

We will have need of the second derivative and we should note that we can continue to take derivatives of a function. M-Box 15.1 provides a definition of the second derivative as well as nth derivatives. There are no new formulas here only a little notation as we see in the next example.

Example 15.1. *Find* $f^{(3)}(x)$ *given* $f(x) = x^5 - 3x^4 + 6x^2$.

Solution. We are asked to find the third derivative of the function $f(x)$. We have

$$
\begin{aligned}
f(x) &= x^5 - 3x^4 + 6x^2 \\
f'(x) &= 5x^4 - 12x^3 + 12x \\
f''(x) &= 20x^3 - 36x^2 + 12 \\
f^{(3)}(x) &= 60x^2 - 72x
\end{aligned}
$$

□

The last example is how to get derivatives of derivatives such as the second derivative in R. R Code box 15.7 is the second derivative of $f(x) = 0.1(x^3/3 + x^2 - 15x) - 4$. Note that we just need to add n=2 to the **Deriv** function for the second derivative. The last example, R Code box 15.8, is the 48th derivative of $\sin(x)$. Why did we get $\sin(x)$ as a result? Did we break R?

R Code 15.7: The 2nd Derivative of a Function

```
> f_pp<-Deriv(f,n=2)
> f_pp
```

```
function (x)
0.1 * (2 + 2 * x)
```

R Code 15.8: The 48th Derivative of a Function

```
> g<-function(x){sin(x)}
> Deriv(g,n=48)
```

```
function (x)
sin(x)
```

Note that the output of the derivative of function in R can be difficult to read due to the algorithm used to compute derivatives. For example, R Code box 15.9 calculates the derivative of $f(x) = \cos(x)\sin(x^2)$. In the output the variable `.e1<-x^2` means each **.e1** in the next line should be replaced with `x^2` if we want the derivative function. In most cases this is not an issue as we will simply use **f_p** for any results we desire.

R Code 15.9: Derivative with an Output Substitution

```
> f<-function(x){cos(x)*sin(x^2)}
> f_p<-Deriv(f)
> f_p
```

```
function (x)
{
    .e1 <- x^2
    2 * (x * cos(x) * cos(.e1)) - sin(x) * sin(.e1)
}
```

R Derivative Tips

- If you are using R to take the derivative of more than one function then do not keep defining the function as **f**. This is a good way to make a mistake if you have multiple functions **f** in your code. The easiest thing to do in this case is use the convention of naming your functions **f1**, **f2**, etc.
- You cannot define the constant function **f<–function(x){c}** in R. If the variable of the function is **x** then there must be an **x** in the function definition. On the other hand this will work: **f<–function(x){c + 0*x}**, although it has questionable value. You can also use **abline(h=c)** to graph a horizontal line.
- The **Deriv** command will by default take the derivative of the function with respect to the variable defined when the function was created. For example, if **f<–function(t){t^n**} then **Deriv(f)** will output what we expect. We could take the derivative with respect to n, which would be done with **Deriv(f,"n")**.

15.1 Exercises

All exercises, except the first eight, should be done in R unless otherwise noted.

1. Find $f''(x)$ given $f(x) = 3x^5 - 6x^3 + x - 10$.
2. Find $f''(x)$ given $f(x) = 5x^6 + 7x^4 - 2x + 22$.
3. Find $f''(x)$ given $f(x) = \ln(x)$.
4. Find $f''(x)$ given $f(x) = \sqrt{x}$.
5. Find $f^{(4)}(x)$ given $f(x) = e^{2x}$.
6. Find $f^{(4)}(x)$ given $f(x) = \sin(3x)$.
7. Find $f^{(6)}(x)$ and $f^{(7)}(x)$ given $f(x) = x^6$.
8. Make a conjecture and explain your reasoning for $f^{(n-1)}(x)$ and $f^{(n)}(x)$ given $f(x) = x^n$.
9. Check all the rules in M-Box 11.1 with R. **R tip:** In R the $\ln(x)$ function is log(x). Note: Just a reminder that the rules are derived from the limit definition and someone programmed the rules into R.
10. Find the derivative of $f(x) = x^3 \sin(x)$ by hand and then check it with R.
11. Find the derivative of $f(x) = x^2 e^x$ by hand and then check it with R.

12. Find the derivative of $f(x) = \frac{x^2+4}{\cos(x)}$ by hand and then check it with R.
13. What is the 100th derivative of $\cos(x)$? Make a conjecture and then check it with R.
14. What is the 100th derivative of e^{2x}? Make a conjecture and then check it with R.
15. What is the 10th derivative of $\frac{1}{x}$? Make a conjecture and then check it with R. Repeat this question for the 12th derivative.
16. What is the 20th derivative of $(2x + 9)^{100}$? Make a conjecture and then check it with R. Repeat this question for the 101st derivative.
17. Using the global temperature function in the function gallery, how fast was global temperature increasing in the last year of the data set? Using the microscope equation, approximately what will global temperature be one and two years past the last year in the data? Use your results in a sentence that explains the context of the calculation.
18. Using the world wind function in the function gallery, how fast was world wind capacity increasing in the last year of the data set? Using the microscope equation, approximately what will world wind capacity be one and two years past the last year in the data? Use your results in a sentence that explains the context of the calculation.
19. Using the U.S. wind function in the function gallery, how fast was U.S. wind capacity increasing in the last year of the data set? Using the microscope equation, approximately what will U.S. wind capacity be one and two years past the last year in the data? Use your results in a sentence that explains the context of the calculation.
20. Using the Spain wind function in the function gallery, how fast was Spain wind capacity increasing in the last year of the data set? Using the microscope equation, approximately what will Spain's wind capacity be one and two years past the last year in the data? Use your results in a sentence that explains the context of the calculation.
21. What is the x and y-value of the first local **max** on the interval $(0, \infty)$ of the function $x^2 \sin(x)$? Graph $x^2 \sin(x)$ and place a point at this value. Label the point on your graph using the text() function.
22. What is the x and y-value of the first local **min** on the interval $(0, \infty)$ of the function $x^2 \sin(x)$? Graph $x^2 \sin(x)$ and place a point at this value. Label the point on your graph using the text() function.
23. What is the x and y-value of the first local **max** on the interval $(0, \infty)$ of the function $\frac{\sin(x)}{x}$? Graph $\frac{\sin(x)}{x}$ and place a point at this value. Label the point on your graph using the text() function.
24. What is the x and y-value of the first local **min** on the interval $(0, \infty)$ of the function $\frac{\sin(x)}{x}$? Graph $\frac{\sin(x)}{x}$ and place a point at this value. Label the point on your graph using the text() function.
25. What is the x and y-value of the first local **max** on the interval $(0, \infty)$ of the function $\sin(x^2)$? Graph $\sin(x^2)$ and place a point at this value. Label the point on your graph using the text() function.

26. What is the x and y-value of the first local **min** on the interval $(0, \infty)$ of the function $\sin(x^2)$? Graph $\sin(x^2)$ and place a point at this value. Label the point on your graph using the text() function.
27. What is the x and y-value of the first local **max** on the interval $(0, \infty)$ of the function $\sin(\sqrt{x})$? Graph $\sin(\sqrt{x})$ and place a point at this value. Label the point on your graph using the text() function.
28. What is the x and y-value of the first local **min** on the interval $(0, \infty)$ of the function $\sin(\sqrt{x})$? Graph $\sin(\sqrt{x})$ and place a point at this value. Label the point on your graph using the text() function.
29. The logistic model in the function gallery (figure 3.11)

$$L(x) = \frac{e^{-19.1744645062916+1.15383554750228x}}{1 + e^{-19.1744645062916+1.15383554750228x}}$$

gives the probably cells are malignant based on the maximum radius size in mm. Find and interpret $L(17)$ and $L'(17)$. Using the microscope equation, estimate $L(18)$. Write a sentence using your results in context properly.
30. The probability a house in Windsor Canada in the summer of 1987 has a driveway based on the price of the house in thousands of dollars is given by

$$D(x) = \frac{e^{-1.71810950755824+0.0612745196451642x}}{1 + e^{-1.71810950755824+0.0612745196451642x}}$$

Data from [1]. Find and interpret $D(50)$ and $D'(50)$. Using the microscope equation estimate $D(51)$. Write a sentence using your results in context properly.
31. Based on a sample of house cats the probability a cat is male based on its weight in kilograms is given by

$$C(x) = \frac{e^{-8.67943082163772+3.635139933871x}}{1 + e^{-8.67943082163772+3.635139933871x}}$$

Data from [1]. Find and interpret $C(2)$ and $D'(2)$. Using the microscope equation estimate $C(2.25)$. Write a sentence using your results in context properly.
32. Based on data from the 1995 issue of US News and World Report the probability a college is private based on the room and board cost in thousands of dollars is given by

$$P(x) = \frac{e^{-2.65749434686338+0.879952201297837x}}{1 + e^{-2.65749434686338+0.879952201297837x}}$$

Data from [1]. Find and interpret $P(4)$ and $P'(4)$. Using the microscope equation estimate $P(4.5)$. Write a sentence using your results in context properly.

For the next set of problems R will be used to create a graph to illustrate using a tangent line to approximate a point on a curve. For the given function, point, and value of h, do the following: Graph the function, add a tangent

line to the graph at the point, place dots on the graph and tangent line at the point a + h, and a point (use triangle) where the tangent line touches the curve. Evaluate the function and tangent line at the point a + h. What is the difference between the value at the tangent line and at the point? Make sure you select an appropriate window for your graph. Note that the computation to these problems can be done by hand, which makes them valuable as first-order approximations of a function (take Calc II to learn more).

33. $f(x) = \sqrt{x}$, $a = 4$, $h = 0.5$.
34. $f(x) = \sqrt{x}$, $a = 9$, $h = 0.6$.
35. $f(x) = \sqrt[3]{x}$, $a = 8$, $h = 0.7$. Tip: Use $x^{1/3}$ in R.
36. $f(x) = \sqrt[3]{x}$, $a = 27$, $h = 0.9$. Tip: Use $x^{1/3}$ in R.
37. $f(x) = \sin(x)$, $a = \pi$, $h = 0.8$.
38. $f(x) = \cos(x)$, $a = 4$, $h = 0.6$.
39. $f(x) = \ln(x)$, $a = 1$, $h = 0.4$. Tip: log(x) in the natural log in R.
40. $f(x) = e^x$, $a = 0$, $h = 0.3$. Tip: Recall e^x is exp(x) in R.

15.2 Project: Mauna Loa CO2 Projections

Answer the following questions using the model of yearly average CO2 measurement at Mauna Loa from the function gallery, Chapter 3. All responses to the questions must be typed and work should be done in R. Copy your R code at the end of the document as an appendix. In general, all typed answers should be in sentence form. Part of your grade will be based on your use of the English language. Some R code is provided in R Code box CO2 at the bottom to plot the observed data and find the quadratic model to fit the data. Note: The code pulls data directly from the text's web site and may have more recent data than the function gallery.

1. Create a graph that contains a scatter plot of the CO2 data (see R Coded Box CO2 to get the scatter plot), the fitted curve (from the function gallery), a tangent line at the last year of data, and dots on both the curve and the line at 2050 (use different colors). Include this graph in your report as you will be asked to reference it in some of the problems below.
2. According to the model what will CO2 levels be in 2050? Along with your answer, explain how your response is represented on the graph.
3. What is the rate of change of CO2 and the percentage rate of change in the last year of the data set? Along with your answer, explain how your first response is represented on the graph.
4. Assuming that CO2 levels continue to grow constantly at the last year in the data set rates, what will CO2 levels reach in 2050? Along with your answer, explain how your response is represented on the graph.
5. Atmospheric CO2 levels of 450ppm yield a likely chance that global average temperature increases will be at least 2° Celsius.* According to the model, in what year do we reach a CO2 level of 450ppm? If we assume CO2 levels continue to grow constantly at the last year in the data set rates, in what year do we reach a CO2 level of 450ppm?
   NOTE: According to Warren**, at 1° Celsius, in addition to the trends we are already observing, oceans will further acidify, natural ecosystems will start to collapse, and as many as 18–60 million people in the developing world will go hungry. At 1.5° Celsius the Greenland ice sheet will melt, eventually causing a 7m rise in sea level, inundating coastal areas. At 2° Celsius agricultural yields in the rich nations will start to fall and 1–3 billion people will experience water scarcity. At 3° Celsius the Amazon rainforest is expected to collapse and at 4° Celsius most of Africa and Australia will lose all agricultural production.
6. Fill in the blank: In order to avoid reaching 450ppm of atmospheric CO2 the trend in the data would have to become (Calculus Term/Phrase).
7. Provide a (general or real world related) question that you would like answered based on your work here. This should not be something that you could answer yourself with a little work.
8. Summarize your work on questions 1–5 in a short paragraph as if it were a short scientific news article (don't editorialize).

R Code: CO2

```
> dataURL<-
"https://sustainabilitymath.org/excel/Mauna-Loa-CO2-R.csv"
> CO2.data <- read.csv(url(dataURL),header=TRUE)
> x<-CO2.data$Years.After.1950
> y<-CO2.data$CO2.parts.per.million
> plot(x,y,type="p",cex=1.25,pch=16,xlim=c(0,100),
ylim=c(300,500), xlab="Years after 1950", ylab="CO2 ppm")
> title(main="Mauna Loa Average Yearly CO2")
> grid (NULL,NULL,lty=6,col="black")
> Quad_fit<-lm(y~x+I(x^2))
> summary(Quad.fit)
> Coef<-Quad_fit$coefficients
> CO2<-function(x){Coef[[1]] + Coef[[2]]*x + Coef[[3]]*x^2}
```

* According to IPCC Fifth Assessment Report (AR5) page 22: https://www.ipcc.ch/pdf/assessment-report/ar5/syr/AR5_SYR_FINAL_SPM.pdf

** Warren, R. 2006. Impacts of global climate change at different annual mean global temperature increases, in H.J. Schellnhuber et al. (eds.) Avoiding Dangerous Climate Change. Cambridge University Press, Cambridge.

15.3 Project: Climate Change Projections

Answer the following questions using the model of global average temperature from the function gallery, Chapter 3. All responses to the questions must be typed and work should be done in R. Copy your R code at the end of the document as an appendix. In general, all typed answers should be in sentence form. Part of your grade will be based on your use of the English language. Some R code is provided in R Code box CO2 at the bottom to plot the observed data and find the quadratic model to fit the data. Note: The code pulls data directly from the text's web site and may have more recent data than the function gallery.

1. Create a graph that contains a scatter plot of the temperature data in degrees Fahrenheit (see R Coded Box Global Temperature to get the scatter plot), the fitted curve (Use function composition to convert the GTemp(t) function so that the output is Fahrenheit, recall that $F = 9C/5 + 32$), a tangent line at the last year of the data set, and dots on both the curve and the line at 2050 and 2100 (use different colors). Label your x-axis and y-axis properly. Include this graph in your report as you will be asked to reference it in some of the problems below.
2. Based on the model what is the predicted global average temperature for 2050? 2100? How much of a change is that since 2000 for each of those years? Along with your answers, explain how your responses are represented on the graph.
3. What is the rate of change of average global temperature in the last year of the data set? If we assume the average global temperature continues to rise at the last year of the data set rates then what is the predicted average global temperature for 2050 and 2100. How much of a change in temperature will there be compared to 2000 for each of those years? Along with your answers, explain how your responses are represented on the graph.
4. Which information is more useful: the temperature prediction or the change in temperature prediction? Why?
5. Use the information above to fill in the blanks (use Fahrenheit).
   According to the model, if current temperature trends continue, in 2100 the average global temperature will be ______________________________ which is an increase of ______________________________ above the 2000 average temperature. On the other hand, if we assume that the rate of temperature increase remains constant at the (insert last year of the data set here) rates of ______________________________, then the average global temperature will be ______________________________ in 2100, which is an increase of ______________________________ above the 2000 average temperature.
6. The projections of global temperature here are based on time series, but time isn't driving climate change. Sophisticated models of future climate are based on scenarios of human behavior. Consider Figure SPM.8a from the IPCC sixth assessment report (2021) (https://www.ipcc.ch/report/ar6/wg1/figures/summary-for-policymakers/figure-spm-8/). Which two projections for 2100 match closest to your two projections? Note the graph is in °C, multiply by $9/5$ to get a change in °F. What are the underlying assumptions of each of these two scenarios?

7. Provide at least one, general or real world related, question that you would like answered based on your work. This should not be something that you couldn't answer yourself with a little work.

R Code: Global Temperature

```
> dataURL<-
"https://sustainabilitymath.org/excel/Global-Temperature-R.csv"
> Temp.data <-read.csv(url(dataURL),header=TRUE)
> x<-Temp.data$Years.After.1950
> y<-9*Temp.data$Ave.Global.Temp/5  +32
> plot(x,y,cex=1.25,pch=16,xlim=c(0,150),ylim=c(55,70),
xlab="Years after 1950",ylab="Degrees Fahrenheit")
> title(main="Global Annual Mean Land and Ocean Temperature")
> grid (NULL,NULL,lty=6,col="black")
> Quad_fit<-lm(y~x+I(x^2))
> summary(Quad.fit)
> Coef<-Quad_fit$coefficients
> Gtemp<-function(x){Coef[[1]] + Coef[[2]]*x + Coef[[3]]*x^2}
```

Chapter 16
End Behavior of a Function - L'Hospital's Rule

One of our motivations for this chapter is that it is common to hear someone refer to exponential growth any time they see a graph that is concave up and increasing. This is not true. In the function gallery, for example, both the global temperature and CO2 models are increasing and concave up but the functions are not exponential functions as they are quadratic polynomials. As we will see in this chapter, there is a big difference in the growth of an exponential function as compared to a quadratic polynomial. We start with an example.

Consider the functions

$$f(x) = x^2$$

and

$$g(x) = \frac{1}{x^2 + 1}$$

in figure 16.1. Informally, for $f(x) = x^2$ it appears that as x gets larger, read the graph from left to right along the x-axis, the y-values continually increase. In other words, as we increase our x-values, for example, $x = 10, 100, 1000, \ldots$ the y-values which are $y = 10^2, 100^2, 1000^2, \cdots = 100, 10000, 1000000, \ldots$ keep getting larger. It appears that $f(x) = x^2$ tends to infinity as x go to infinity.

For

$$g(x) = \frac{1}{x^2 + 1}$$

it appears that as x gets larger, again read the graph from left to right along the x-axis, the y-values shrink towards zero. In other words, as we increase our x-values, for example, $x = 10, 100, 1000, \ldots$ the y-values which are

$$\begin{aligned} y &= \frac{1}{10^2+1}, \frac{1}{100^2+1}, \frac{1}{1000^2+1}, \ldots \\ &= 0.009900990099, 0.000099990001, 0.000000999999, \ldots . \end{aligned}$$

It appears that

$$g(x) = \frac{1}{x^2 + 1}$$

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_16

tends to 0 as x go to infinity. We express these ideas with the following notation:

$$\lim_{x\to\infty} x^2 = \infty,$$
$$\lim_{x\to\infty} \frac{1}{x^2+1} = 0$$

We need to be careful in that reading a graph provides intuition but is not formal reasoning. Logically, for $f(x) = x^2$ as $x \to \infty$ then the square of x will also tend to infinity. In the case of

$$g(x) = \frac{1}{x^2+1},$$

as $x \to \infty$ the denominator tends to infinity, $x^2 + 1$ will tend to infinity as x goes to infinity, and since the numerator is fixed value the fraction tends to 0. In other words, as we divide by a larger and larger value the fraction will get smaller. Both of these are informal arguments and there are formal ways to prove these statements, but our goal here is to develop the ideas informally. We refer to the *end behavior* of a function as the result of the

$$\lim_{x\to\infty} f(x)$$

given in M-Box 16.1.

M-Box 16.1: Definition of End Behaviour

The end behaviour of a function $f(x)$ is the result of

$$\lim_{x\to\infty} f(x).$$

Another example is given in figure 16.2 with the function

$$f(x) = \frac{10}{1+e^{-.5*(x-20)}}.$$

Informally, the graph is suggesting that

$$\lim_{x\to\infty} \frac{10}{1+e^{-0.5(x-20)}} = 10.$$

The informal argument is that as $x \to \infty$ the part of the function $e^{-0.5(x-20)}$ tends to 0 leaving us with the fraction $10/(1+0) = 10$. How do we know $e^{-0.5(x-20)}$ tends to 0 as $x \to \infty$? To answer this question we start with some algebra to get that

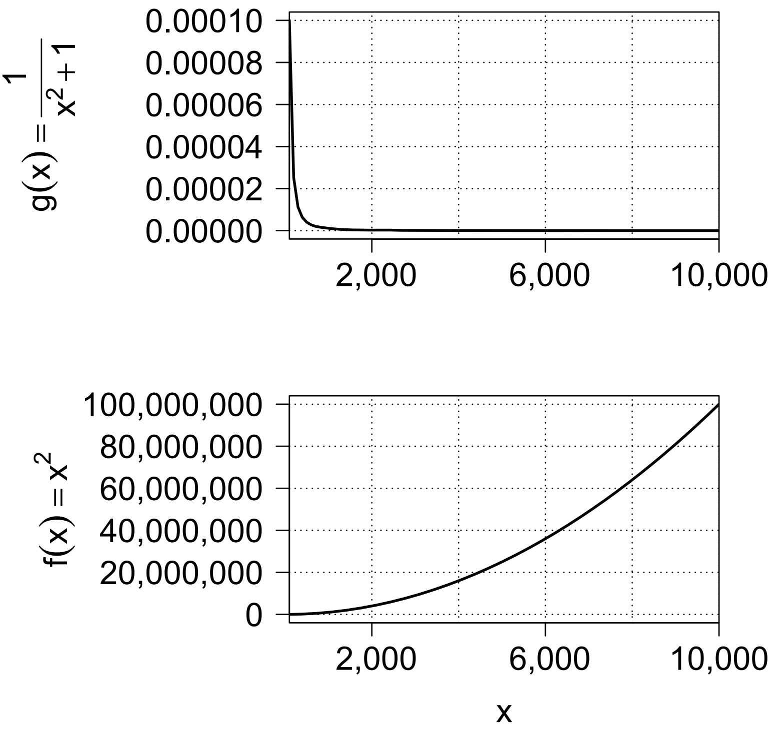


Fig. 16.1 The graphs of the functions $f(x) = x^2$ and $g(x) = \frac{1}{x^2+1}$.

$$\begin{aligned} e^{-0.5*(x-20)} &= e^{-0.5x+10} \\ &= e^{-0.5x}e^{10} \\ &= \frac{e^{10}}{e^{0.5x}} \end{aligned}$$

As $x \to \infty$ the denominator $e^{0.5x}$ will tend to infinity and as the numerator is the constant e^{10}, the fraction will tend to 0.

We should note that not all functions have a defined end behavior that is stable. For example,

$$\lim_{x \to \infty} \sin(x)$$

is undefined because as the value of x increases the y-values of $\sin(x)$ oscillate between -1 and 1.

The question of the end behavior is important not just for individual functions but also when comparing functions. As we noted in the introduction to this chapter concave up increasing functions are often referred to as exponential, but an increasing concave up functions does not have to be exponential. Consider figure 16.3, which

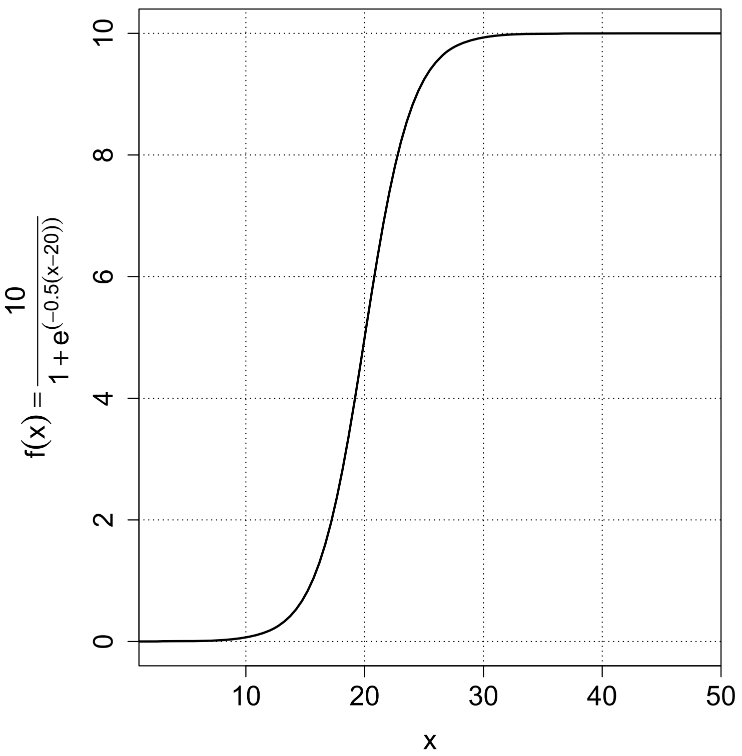


Fig. 16.2 Graph of the function the function $f(x) = \frac{10}{1+e^{-0.5*(x-20)}}$.

has two such functions. One of the functions is e^x while the other is $6x^2 + 1$. The end behavior of both of these functions is infinity, but they are very different. We will use L'Hospital's (loh-pee-TAHL) Rule, M-Box 16.2, to compare the end behavior of these two functions in the next example. L'Hospital's Rule allows us to compare two competing processes. The assumptions of L'Hospital's Rule are that both the function in the numerator and the denominator tend to infinity. The question becomes does one of them get there faster?

M-Box 16.2: Theorem - L'Hospital's Rule for End Behavior

If $f(x)$ and $g(x)$ are two differentiable functions such that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

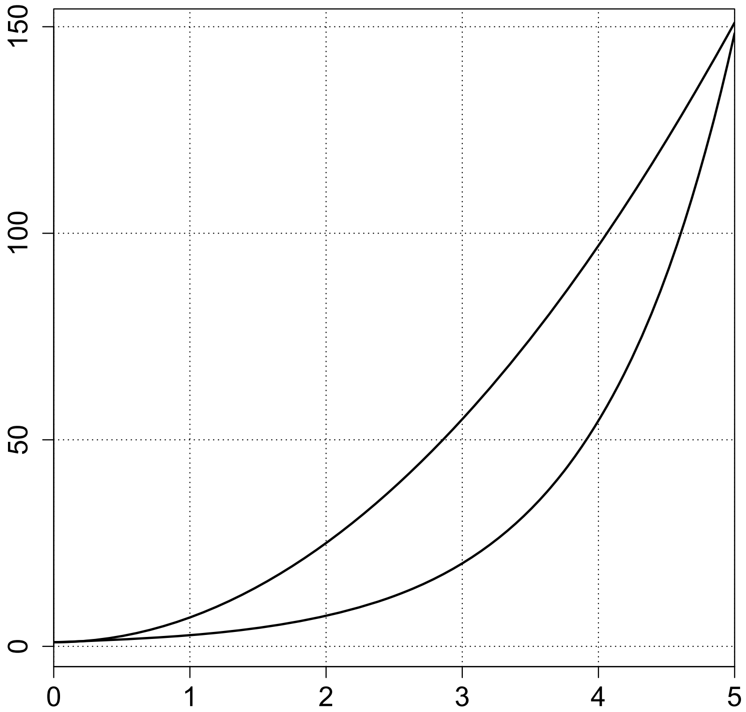


Fig. 16.3 Graph of two concave up functions.

- If this limit is 0 the function in the denominator grows faster than the function in the numerator.
- If this limit is ∞ the function in the numerator grows faster than the function in numerator.
- If this limit is neither ∞ nor 0 then the functions grow at the same rate or are proportional.

Warning This is not a quotient rule calculation.

Example 16.1. *Given* $f(x) = e^x$ *and* $g(x) = 6x^2 + 1$ *what is* $\lim_{x\to\infty} \frac{f(x)}{g(x)}$*?*

Solution. The solution will require two uses of L'Hospital's Rule. First note that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ and the assumptions of L'Hospital's Rule are satisfied. Using L'Hospital's Rule we get

$$\lim_{x\to\infty} \frac{e^x}{6x^2+1} \stackrel{L.H.}{=} \lim_{x\to\infty} \frac{e^x}{12x}.$$

Now $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} 12x = \infty$ so that we can apply the rule again to get

$$\lim_{x\to\infty} \frac{e^x}{12x} \stackrel{L.H.}{=} \lim_{x\to\infty} \frac{e^x}{12} = \infty.$$

Note that each time we applied L'Hospital's Rule we placed an $L.H.$ above the equals sign. This helps the reader follow our work. In the last limit, $\lim_{x\to\infty} \frac{e^x}{12}$, is ∞ because the denominator is a fixed value while the numerator tends to ∞. □

What does exercise 16.1 tell us about the end behavior of $f(x) = e^x$ as compared to $g(x) = 6x^2 + 1$? While both function tend to infinity, $f(x) = e^x$ does so at a rate much faster than $g(x) = 6x^2 + 1$. In fact, the ratio of the two functions goes to infinity. Note that we could have evaluated the

$$\lim_{x\to\infty} \frac{6x^2+1}{e^x},$$

in which case we would get 0 and conclude that the denominator grows faster. In other words, we get the same result that $f(x) = e^x$ goes to infinity faster than $g(x) = 6x^2 + 1$. A key point here is that the end behavior of $f(x) = e^x$ is very different than that of $g(x) = 6x^2 + 1$ (or any quadratic function as the calculation is almost the same). When using a function to model data, as we did in the function gallery, our choice of function matters not just in how it fits the data but what it might say when we extrapolate. Future predictions with an exponential function will be very different than a quadratic (see the project in section 16.2 as an example).

Informally, why does L'Hospital's Rule work. If both

$$\lim_{x\to\infty} f(x) = \infty$$

and

$$\lim_{x\to\infty} g(x) = \infty$$

then instead of comparing the functions with

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

we compare their rates of growth with

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

If one function is growing faster, derivative information, then it will get to infinity faster.

Example 16.2. *Compare the end behavior of the functions* $f(x) = 3x^2 + 2x - 5$ *and* $g(x) = 10x^2 + 500x - 10$.

Solution. We will use L'Hospital's Rule twice since $\lim_{x\to\infty} 3x^2 + 2x - 5 = \infty$, $\lim_{x\to\infty} 10x^2 + 500x - 10 = \infty$, $\lim_{x\to\infty} 6x + 2 = \infty$, and $\lim_{x\to\infty} 20x + 500 = \infty$. Now,

$$\lim_{x\to\infty} \frac{3x^2 + 2x - 5}{10x^2 + 500x - 10} \overset{L.H.}{=} \lim_{x\to\infty} \frac{6x + 2}{20x + 500}$$
$$\overset{L.H.}{=} \lim_{x\to\infty} \frac{6}{20} = \frac{6}{20} = \frac{3}{10}.$$

In this case the limit of the ratios is a constant and the end behavior is proportional. Informally, $f(x) \approx (3/10)g(x)$ as $x \to \infty$, and neither function grows faster. □

16.1 Exercises

In all cases when you give a result for a limit you need to (informally) explain your reasoning.

1. What is $\lim\limits_{x\to\infty} 2x^3$?
2. What is $\lim\limits_{x\to\infty} 4x^2$?
3. What is $\lim\limits_{x\to\infty} 3^x$?
4. What is $\lim\limits_{x\to\infty} 5^x$?
5. What is $\lim\limits_{x\to\infty} e^{-x}$?
6. What is $\lim\limits_{x\to\infty} 4^{-x}$?
7. What is the end behavior of $f(x) = \frac{2}{x}$?
8. What is the end behavior of $g(x) = \frac{4}{x^2}$?
9. What is the end behavior of $f(x) = x^2 - 2x$?
10. What is the end behavior of $g(x) = x^3 - 2x^2$?
11. What is the end behavior of $g(x) = \frac{8}{2-e^{-x}}$?
12. What is the end behavior of $g(x) = \frac{7}{4-2^{-x}}$?
13. Compare the end behaviors of the functions $f(x) = x$ and $g(x) = e^x$. Is the end behavior proportional or does one grow faster than the other?
14. Compare the end behaviors of the functions $f(x) = x^2$ and $g(x) = e^x$. Is the end behavior proportional or does one grow faster than the other?
15. Compare the end behaviors of the functions $f(x) = 3x^2 + 5x + 7$ and $g(x) = 5x^2 + 8x - 10$. Is the end behavior proportional or does one grow faster than the other?
16. Compare the end behaviors of the functions $f(x) = 6x^2 + 3x - 1$ and $g(x) = 8x^2 + 2x + 5$. Is the end behavior proportional or does one grow faster than the other?
17. Compare the end behaviors of the functions $f(x) = 10^x$ and $g(x) = 5x^2$. Is the end behavior proportional or does one grow faster than the other?
18. Compare the end behaviors of the functions $f(x) = 8^x$ and $g(x) = 3x^2$. Is the end behavior proportional or does one grow faster than the other?

19. Compare the end behaviors of the functions $f(x) = \sqrt{x}$ and $g(x) = \ln(x)$. Is the end behavior proportional or does one grow faster than the other?
20. Compare the end behaviors of the functions $f(x) = \sqrt[3]{x}$ and $g(x) = \ln(x)$. Is the end behavior proportional or does one grow faster than the other?
21. Compare the end behaviors of the functions $f(x) = x^2 + 9$ and $g(x) = 4x + e^x$. Is the end behavior proportional or does one grow faster than the other?
22. Compare the end behaviors of the functions $f(x) = x^2 - 4x + 2$ and $g(x) = 7x + e^x$. Is the end behavior proportional or does one grow faster than the other?
23. Why can't we use L'Hospital's Rule to compare the end behavior of $f(x) = x^2$ and $\cos(x)$?
24. Why can't we use L'Hospital's Rule to compare the end behavior of $f(x) = \ln(x)$ and $\sin(x)$?
25. L'Hospital's rule is not helpful in comparing the end behavior of say $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$. Explain why. Use some algebra to compare the end behavior of these two functions.
26. L'Hospital's rule is not helpful in comparing the end behavior of say $f(x) = 5^x$ and $g(x) = e^x$. Explain why. Use some algebra to compare the end behavior of these two functions.
27. The function $f(x) = e^x$ grows faster than $g(x) = x^n$ for any positive integer n. In comparing the end behavior of these functions using L'Hospital's rule what is the final expression before the limit can be executed? Hint $n! = n(n-1)(n-2)\ldots(3)(2)(1)$.
28. The function $f(x) = \ln(x)$ grows slower than $g(x) = \sqrt[n]{x}$ for any positive integer n. In comparing the end behavior of these functions using L'Hospital's rule what is the final expression before the limit can be executed?

16.2 Project: Comparing Exponential and Quadratic Models in Population Predictions

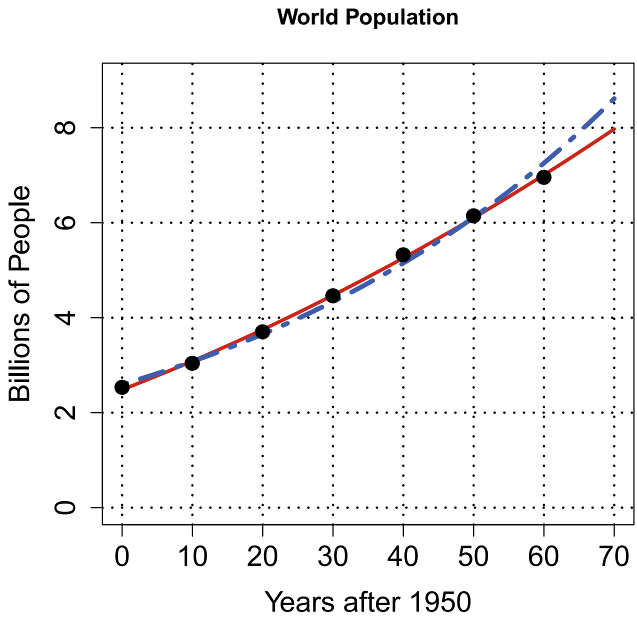


Fig. 16.4 U.N world population with two fitted curves. The solid red curve is a quadratic model and the dashed blue curve is an exponential model.

The goal of this project is to see the differences in extrapolating with an exponential model as compared to a quadratic model. Figure 16.4 has U.N. Population data with two fitted curves. It is natural to think of population growth as exponential and the blue dashed curve is visually a good fit to the data (Note for those with a statistics background: the adj R square is 0.993, although a residual plot will have a pattern). The quadratic model, the solid red curve, is also a candidate to fit the data (adj R square is 0.998). The questions below will demonstrate how using these models for future predictions (extrapolation) of world population provide very different results and, in fact, neither match U.N. projections (June 2020).

1. For each model in figure 16.4, the red solid line is $WPq(x) = 2.4845139485476 + 0.0571519644893x + 0.0003037208235x^2$ and the blue dashed line is $WPe(x) = 2.593403628e^{0.0171488328x}$, what is the predicted population size for 2025 and 2050?

2. What is the 2020 growth rate (derivative) for each model?
3. For each model find the equation of the tangent line at 2020.
4. For each model what does the tangent line predict for population size for 2025 and 2050?
5. Summarize your information for each model in a few sentences. What is the difference between the predictions of your two models and what does this say about the differences between an exponential model and a quadratic model? We tend to think that populations always grow exponential, do you believe that the exponential model is the better fit or not?
6. The U.N. is predicting a world population of 8.18 billion in 2025 and 9.74 billion in 2050 (June 2020). How do your projections compare to the predictions given by the U.N. and what does this say about the type of model the U.N. is using?

Part II
Applications of the Derivative

Chapter 17
How Do We Know the Shape of a Function?

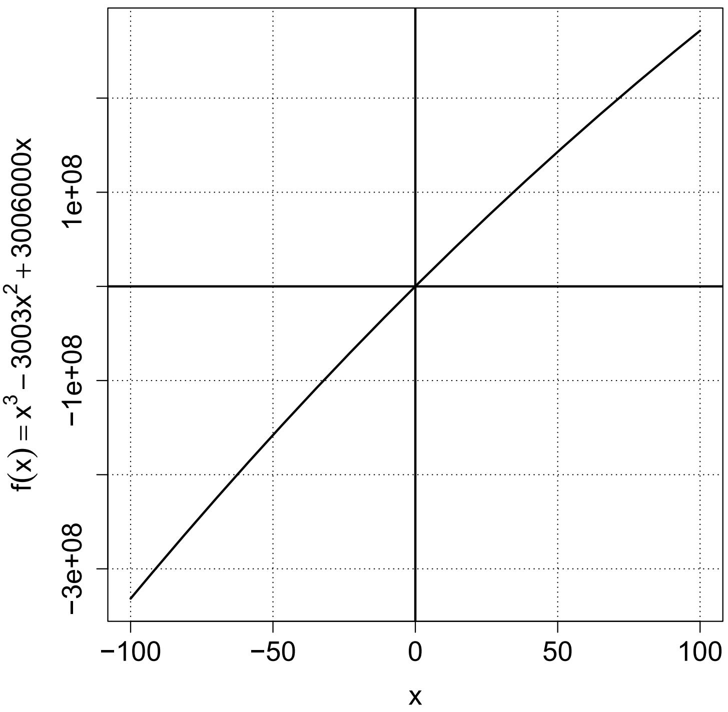


Fig. 17.1 Graph of $f(x) = x^3 - 3003x^2 + 3006000x$.

Consider the graph of $f(x) = x^3 - 3003x^2 + 3006000x$ in figure 17.1. Can we conclude the function does not have any local maximums, local minimums, or

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inflection points? If we think we do not have the correct window (domain and range of the graph) how do we decide what window to use? In this chapter we will use the derivative, both the first and second, to completely understand the shape of a graph. In order to see how we use the first and second derivatives in understanding key aspects of the shape of a graph, consider figure 17.2 which has the graph of

$$f(x) = x^3/3 - 3x^2 + 8x + 2$$
$$f'(x) = x^2 - 6x + 8$$
$$f''(x) = 2x - 6$$

Here are the key relationships (**Tip**: Take a moment to review the definitions in M-Box 2.1 from Chapter 2):

1. The function $f(x)$ is increasing on the x-axis intervals $(-\infty, 2)$ and $(4, \infty)$, which is also when $f'(x) > 0$.
2. The function $f(x)$ is decreasing on the x-axis interval $(2, 4)$, which also is when $f'(x) < 0$.
3. The function $f(x)$ has a local max at $x = 2$ and a local min at $x = 4$, which is also when $f'(x) = 0$ ($f'(2) = 0$ and $f'(4) = 0$).
4. The function $f(x)$ has an inflection point at $x = 3$, which is when $f'(x)$ has a local min (in this case) and $f''(x) = 0$ ($f''(3) = 0$).
5. The function $f(x)$ is concave down on the x-axis interval $(-\infty, 3)$, which is when $f'(x)$ is decreasing and $f''(x) < 0$.
6. The function $f(x)$ is concave up on the x-axis interval $(3, \infty)$, which is when $f'(x)$ is increasing and $f''(x) > 0$.

The relationships between $f(x)$, $f'(x)$ and $f''(x)$ are formalized in M-Boxes 17.1 through 17.6. The first box M-Box 17.1 gives a name to important. We all any point $x = c$ were $f'(c) = 0$ or $f'(c)$ fails to exist. We will not worry much about places where $f'(c)$ fails to exist, but we should be aware, for example, that a corner on a function such as at $x = 0$ of $f(x) = |x|$ is a place where the derivative fails to exist. The issue here is that we get secant line slopes of -1 to the left of $x = 0$ and 1 to the right of zero.

M-Box 17.1: Critical Point

A critical point of a function $f(x)$ is a point $x = c$ in the domain of the function such that either $f'(c) = 0$ or $f'(c)$ fails to exist.

A key result is given in M-Box 17.2, which states that the only place where a local maximum or minimum can occur is at a critical point. Be careful here as the reverse is not true in that a critical point does not have to be a local maximum or minimum. The question then becomes is the critical point a maximum or a minimum or neither? We have two ways to decide.

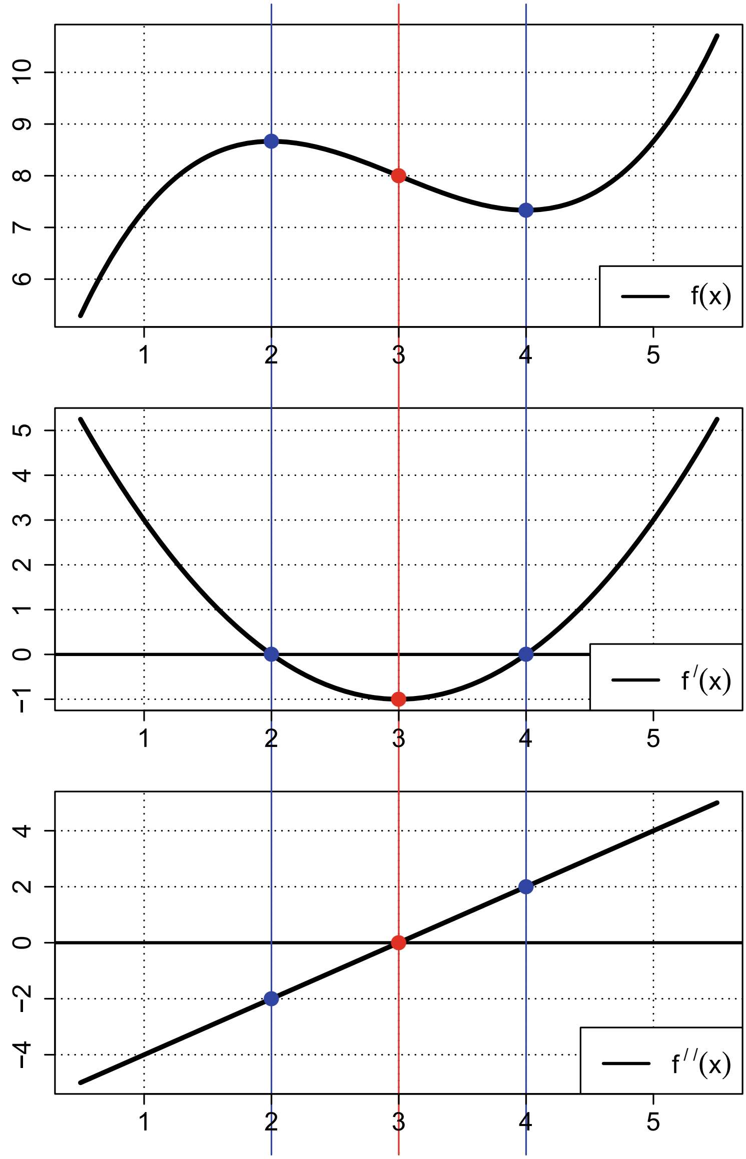

Fig. 17.2 Graph of $f(x) = x^3/3 - 3x^2 + 8x + 2$, $f'(x) = x^2 - 6x + 8$, and $f''(x) = 2x - 6$.

M-Box 17.2: Theorem - Local Max and Min Points of a Function

Let $f(x)$ be a continuous function. If $(c, f(c))$ is a local maximum or local minimum then $x = c$ must be a critical point. In other words, local maximum or minimum can only occur at critical points. Note that the revere is not necessarily true. If $x = c$ is a critical point then it does not have to be a local maximum or minimum. This means that the roots of $f'(x)$ do not have to be a local maximum or minimum.

The first derivative test, M-Box 17.3, is one way to decide the status of a critical point. The idea is that given a critical point $x = c$, if a point to the left of the critical point say $x = a$ has $f'(a) > 0$ and a point to the right say $x = b$ has $f'(b) < 0$ then we have a local maximum at $x = c$. The reasoning is that to the left of $x = c$ the function is going up and after $x = c$ the function is going down, hence $x = c$ is a local maximum. The reverse is that if $f'(a) < 0$ and $f'(b) > 0$ then $x = c$ is a local minimum. If the sign does not change then $x = c$ is neither a local maximum nor local minimum.

M-Box 17.3: Theorem - First Derivative Test

Let $x = c$ be a critical point of a continuous function $f(x)$ and that$f'(x)$ exists in some neighborhood around c, say $[c - h, c + h]$. If $x \in [c - h, c + h]$ then

1. if $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then $(c, f(c))$ is a local maximum,
2. if $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then $(c, f(c))$ is a local minimum,
3. if $f'(x)$ has the same sign on the interval $[c - h, c + h]$ then $(c, f(c))$ is neither a maximum or minimum.

Next, M-Box 17.4 provides a way to determine the concavity of a function. In short, if the second derivative, the derivative of the derivative, is positive then the curve is concave up whereas if it is negative, it is concave down. The intuition here, for example, is that if the second derivative is positive then the first derivative $f'(x)$ is increasing which means that the slopes of the tangent lines of $f(x)$ are getting steeper forcing a concave up shape.

M-Box 17.4: Theorem - Concavity of a Function

If a function $f(x)$ has $f''(x) > 0$ on an interval or at $x = a$ then the function is concave up on the interval or at $x = a$. Alternatively, if the function has $f''(x) < 0$ on an interval or at $x = a$, then the function is concave down on the interval or at $x = a$.

We now have our second method of deciding the status of a critical point, which is the second derivative test given in M-Box 17.5. If $x = c$ is a critical point and $f''(c) > 0$ then the function is concave up and the critical point is a local minimum. Similarly, if $f''(c) < 0$ then the function is concave down and the critical point is a local maximum. Which test should we use? The first derivative test does not require finding a second derivative, but we do have to evaluate the first derivative at two points that we have to choose. If finding the second derivative is challenging then it may be the way to go. In this text, we will use the second derivative test because finding the second derivative is easy enough with R.

M-Box 17.5: Theorem - Second Derivative Test

If $f'(c) = 0$ and $f''(c) > 0$ then the curve is concave up at $x = c$ and $(c, f(c)$ is a local minimum. Alternatively, if $f'(c) = 0$ and $f''(c) < 0$ then the curve is concave down at $x = c$ and $(c, f(c)$ is a local maximum. If $f''(c) = 0$ then no conclusions can be drawn.

Our last result in this chapter summarizes briefly identifying inflection points in M-Box 17.6. This is really nothing more than applying the results above starting with $f'(x)$ instead of starting with $f(x)$ as an inflection point is a local maximum or minimum of $f'(x)$.

M-Box 17.6: Identifying Inflection Points

An inflection point of a function $f(x)$ has $f''(a) = 0$ or the second derivative is undefined at $x = a$. In other words, find the roots of $f''(x)$ (and any value where $f''(x)$ is undefined) and they are the only places $f(x)$ can have an inflection point, but the roots of $f''(x)$ do not have to be inflection points. NOTE: This is nothing more than M-Box 17.2 applied to $f'(x)$ as inflection points of $f(x)$ are places where $f'(x)$ has a local max or min.

We put all these M-Boxes in the next example. We will use the function in figure 17.2 and do the calculation "by hand" and the related R code is then in R code box 17.1.

Example 17.1. *Identify any local maximums, local minimums, and inflection points of the function* $f(x) = \dfrac{x^3}{3} - 3x^2 + 8x + 2.$

Solution. To find local maximums or minimums we need the critical points. The derivative of $f(x)$ is $f'(x) = x^2 - 6x + 8$. The roots of $f'(x)$ are found by

$$\begin{aligned} f'(x) &= 0 \\ x^2 - 6x + 8 &= 0 \\ (x-4)(x-2) &= 0 \end{aligned}$$

and the roots are $x = 4$ and $x = 2$. Here $f'(x)$ does not have any values of x where it is undefined. Hence, the two roots are the only possible location of maximums and minimums of the graph. To check to see if they are a maximum or minimum we need the second derivative, which is $f''(x) = 2x - 6$. Now $f''(2) = -2 < 0$ and so the curve is concave down at $x = 2$, which makes $(2, f(2)) = (2, 8.67)$ a local maximum. Similarly, $f''(4) = 2 > 0$ and so the curve is concave up at $x = 4$, which makes $(4, f(4)) = (4, 7.33)$ a local minimum.

To find any inflection points we need the roots of $f''(x)$, which are found by

$$\begin{aligned} f''(x) &= 0 \\ 2x - 6 &= 0 \\ 2x &= 6 \\ x &= 3. \end{aligned}$$

Hence, there is one possible inflection point. We know from early that the curve is concave down at $x = 2$ and concave up at $x = 4$ and so $(3, f(3)) = (3, 8)$ is, in fact, an inflection point. Based on this work we know the graph in figure 17.2 has no other local maximums, local minimums, or inflection points. In fact, for all $x > 4$ the function will be increasing and concave up, otherwise there would be another local maximum or inflection point. Similarly, as $x \to -\infty$ the curve will continue as decreasing and concave down. In other words, we know the shape of the entire function even though we can only graph a portion of the function (we can only graph on some finite interval for the x-axis). □

In example 17.1 all the calculations are done "by hand." R Code box 17.1 has the corresponding code which can be used for examples that cannot be done by hand. The code starts by loading the Deriv and rootSolve packages. We define the function **f** and then find the first and second derivative with **f_p<–Deriv(f)** and **f_pp<–Deriv(f,n=2)**. We use **uniroot.all** to find the roots of the first and second derivative storing them using the names **root_deriv** and **root_deriv2**. Note that the second input in **uniroot.all** is **c(0,5)** and informs R to look for roots between $x = 0$ and $x = 5$. Deciding on the proper interval to use is not always simple.

We see that **root_deriv** stores two roots or critical points, $x = 2$ and $x = 4$ which we found "by hand" through factoring. The next line evaluates the roots in the function **f** to find their y-values of 8.66667 and 7.33333. We then evaluate the two critical points, $x = 2$ and $x = 4$, in the second derivative, **f_pp(root_deriv)** to check concavity. The last two lines first outputs what is stored in **root_deriv2**, $x = 3$, and evaluates the values (only one in this case) in the function.

R Code 17.1: Finding Key Points of a Function

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){x^3/3-3*x^2+8*x +2}
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(0,5))
```

```
> root_deriv2<-uniroot.all(f_pp,c(0,5))
> root_deriv
[1] 2 4
> f(root_deriv)
[1] 8.666667 7.333333
> f_pp(root_deriv)
[1] -2  2
> root_deriv2
[1] 3
> f(root_deriv2)
[1] 8
```

Two observations about example 17.1 and the related R code in R code box 17.1. First, the example was chosen so that it could be done "by hand." In other words, the roots of the first and second derivatives can be found algebraically. If we restrict examples to functions whose first and second derivative roots can be calculated this way, then the variety of functions we can explore are very limited with third degree polynomials as one of the main examples. On the other hand, while R has algorithms to find roots of functions the **uniroot.all** command requires a range of x-value to search for the roots. This implies we have some sense of where the roots might be found. This is not always so simple.

Now you may say, why not give **uniroot.all** a really big interval. The **uniroot.all** finds roots by using a selection of test points and testing to see when the function values change signs. If the interval is too big and the roots too close, then the test points may skip over the roots. Consider R Code box 17.2. We can find the roots of

$$f(x) = (x-1)^2 - 0.4$$

by hand to get $x = 0.98$ and $x = 1.02$. When we use R and give **uniroot.all** an interval from -100 to 100 it returns numeric(0), which means no roots were found. Now we can add an option argument to **uniroot.all**, n=1000, we says use 1000 test points and it will find the roots. Still, this is just an example to illustrate the issue. Computers do not solve all of our problems and mathematical theory is valuable. Also, finding roots is not easy.

R Code 17.2: Finding Key Points of a Function

```
> f<-function(x){(x-1)^2-0.04}
> uniroot.all(f,c(-100,100))
```

```
numeric(0)
```

Let us return to figure 17.1 and the function

$$f(x) = x^3 - 3003x^2 + 3006000x.$$

What was wrong with our graph? With a little effort we find that the critical points of $f(x)$ are $x = 1000$ and $x = 1002$ and our problem is we made a bad choice of our x-axis. Figure 17.3 is the same function as in figure 17.1 but with an x-axis from 995 to 1005.

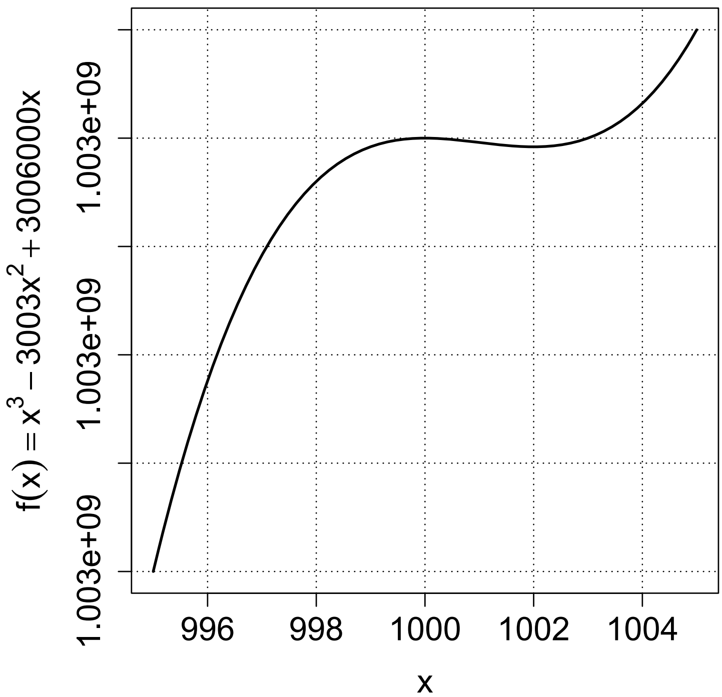

Fig. 17.3 Graph of $f(x) = x^3 - 3003x^2 + 3006000x$ with a different x-axis than in figure 17.1.

17.1 Exercises

1. Answer the following questions from the R code below.

   a. What are the x and y-values of the critical points and on what interval were they located?
   b. Use the second derivative test to decide if the critical points are a local maximum, local minimum, or neither.
   c. What are the x and y-values of possible inflection points?
   d. What is the rate of change at the inflection points?

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){**hidden**}
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> roots_p <- uniroot.all(f_p,c(-10,10))
> roots_pp <- uniroot.all(f_pp,c(-10,10))
> roots_p
[1] -2.774845  1.441520
> roots_pp
[1] -0.6666667
> f(roots_p)
[1]  193.3202 -181.4683
> f(roots_pp)
[1] 5.925926
> f_p(roots_p)
[1] 0.00  0.000
> f_p(roots_pp)
[1] -133.3333
> f_pp(roots_p)
[1] -126.4907  126.4912
> f_pp(roots_pp)
[1] 0
```

2. Answer the following questions from the R code below.

   a. What are the x and y-values of the critical points and on what interval were they located?
   b. Use the second derivative test to decide if the critical points are a local maximum, local minimum, or neither.
   c. What are the x and y-values of possible inflection points?
   d. What is the rate of change at the inflection points?

```
> library(Deriv)
> library(rootSolve)
```

```
> f<-function(x){**hidden**}
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> roots_p <- uniroot.all(f_p,c(0,20))
> roots_pp <- uniroot.all(f_pp,c(0,20))
> roots_p
[1]  2.779365 10.553962
> roots_pp
[1] 6.666667
> f(roots_p)
[1]  263.1139 -206.8176
> f(roots_pp)
[1] 28.14815
> f_p(roots_p)
[1] 0.00 0.00
> f_p(roots_pp)
[1] -90.66667
> f_pp(roots_p)
[1] -46.64762  46.64755
> f_pp(roots_pp)
[1] 0
```

3. Answer the following questions from the R code below.
   a. What are the x and y-values of the critical points and on what interval were they located?
   b. Use the second derivative test to decide if the critical points are a local maximum, local minimum, or neither.
   c. What are the x and y-values of possible inflection points?
   d. What is the rate of change at the inflection points?

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ **hidden**}
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> roots_p <- uniroot.all(f_p,c(-5,5))
> roots_pp <- uniroot.all(f_pp,c(-5,5))
> roots_p
[1] -1.2629429  0.4228395  2.3400769
> roots_pp
[1] -0.5408343  1.5408343
> f(roots_p)
[1] -11.716884  -2.899003 -15.321613
> f(roots_pp)
[1] -7.764733 -9.846401
```

```
> f_p(roots_p)
[1] 0.00 0.00 0.00
> f_p(roots_pp)
[1]   8.020553 -10.020553
> f_pp(roots_p)
[1]  24.29561 -12.92856  27.63060
> f_pp(roots_pp)
[1] 0.00 0.00
```

4. Answer the following questions from the R code below.
    a. What are the x and y-values of the critical points and on what interval were they located?
    b. Use the second derivative test to decide if the critical points are a local maximum, local minimum, or neither.
    c. What are the x and y-values of possible inflection points?
    d. What is the rate of change at the inflection points?

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){**hidden**}
> f(-3)
[1] -36730.62
> curve(f,-30,-10,lwd=2)
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> roots_p <- uniroot.all(f_p,c(-30,-10))
> roots_pp <- uniroot.all(f_pp,c(-30,-10))
> roots_p
[1] -22.52589 -19.15395 -15.31985
> roots_pp
[1] -21.08167 -16.91833
> f(roots_p)
[1]  77.16884 -11.00997 113.21613
> f(roots_pp)
[1] 37.64733 58.46401
> f_p(roots_p)
[1] 0.00  0.00 0.00
> f_p(roots_pp)
[1] -40.10276  50.10276
> f_pp(roots_p)
[1] -60.73903  32.32224 -69.07649
> f_pp(roots_pp)
[1] 0.00 0.00
```

5. A line $f(x) = mx + b$ cannot have a local maximum or minimum. Use the results of this section to prove this.
6. The function $f(x) = 1/x$ cannot have a local max or minimum. Use the results of this section to prove this.
7. In a previous course you may have been told that if the parabola $f(x) = ax^2 + bx + c$ has $a > 0$ then it is concave up. Alternatively, if $a < 0$ then it is concave down. Use the result of this section to justify these two results.
8. A cubic function $f(x) = ax^3 + bx^2 + cx + d$ can have only one inflection point. Use the results of this section to prove this.
9. The function $f(x) = \sqrt{x}$ on the interval $(0, \infty)$ cannot have a local maximum or minimum, and is always concave down. Use the results of this section to prove this.
10. The function $f(x) = e^x$ on the interval $(0, \infty)$ cannot have a local maximum or minimum and is always concave up. Use the results of this section to prove this.
11. An exponential function of the form $f(x) = a^x$ is increasing if $a > 1$ and decreasing if $0 < a < 1$. Use the results of this section to prove this.
12. A 4th degree polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ can have at most two inflection points. Use the results of this section to identify the x-values of these potential inflection points.

For the following exercises, first find the local maximums, local minimums, and inflection points of the given function by hand. This includes using the second derivative test to identify a critical point as a max or min. The local maximums, local minimums, and inflection points should be given with the x-value and y-value. Then sketch a graph of the function by hand. Once this is done by hand, repeat the process with all the calculation in R and include a graph of the function placing points at the local maximums, local minimums, and inflection points. TIP: A well-written R code makes reuse easy.

13. $f(x) = 3x^2 + 5x - 3$
14. $f(x) = -4x^2 - 9x + 4$
15. $f(x) = \dfrac{x^3}{3} + 2x^2 - 12x - 8$
16. $f(x) = \dfrac{x^3}{3} + 6x^2 + 32x + 10$
17. $g(x) = x^3 + 3x^2 - 24x + 5$
18. $g(x) = x^3 + 6x^2 - 36x - 8$
19. $g(x) = \dfrac{x^4}{4} + \dfrac{2x^3}{3} - 4x^2 + 9$
20. $g(x) = \dfrac{x^4}{4} + 3x^3 + 9x^2 - 5$
21. $f(x) = xe^x$
22. $f(x) = xe^{-x}$
23. $f(x) = \dfrac{e^{-x}}{7x}$
24. $f(x) = \dfrac{e^x}{4x}$

25. The function

$$E(t) = 1.56587994153652 + 0.0436841781992499t - 0.00108765220881672t^2 + 0.0000064185051554861 2t^3$$

models the average price of a dozen grade A large eggs in U.S. cities in dollars, where t is the number of months after January 1, 2012. Find the local maximum, local minimum, and inflection point. Report your result in a few sentence with proper context. This should include the corresponding rate at the inflection point. Data from [4]

26. The function

$$G(t) = 2.76899429346421 + 0.0534408607430349t - 0.00116759790367557t^2 + 0.00000580880799149856t^3$$

models the average price of a gallon of gas in U.S. cities in dollars, where t is the number of months after February 1, 2010. Find the local maximum, local minimum, and inflection point. Report your result in a few sentence with proper context. This should include the corresponding rate at the inflection point. Data from [5]

27. The average monthly Arctic sea ice extent, in million square kilometers (msk), for 2018 can be modeled by $I(t) = -0.0005842184t^6 + 0.0205973950t^5 - 0.2720494488t^4 + 1.7289649946t^3 - 5.8584004508t^2 + 10.0743685090t + 7.3435605857$ where $t = 1$ is the middle of Jan, etc. Find the maximum and minimum ice extent and the fastest rate of melting for t in [1, 12]. Write a sentence using your results in context properly.

28. Using the breast cancer logistic function from the function gallery

$$L(x) = \frac{e^{-19.1744645062916+1.15383554750228x}}{1 + e^{-19.1744645062916+1.15383554750228x}}$$

figure 3.11, at what cell radius is the chance of malignancy increasing the fastest? What is the corresponding rate of increase? Write a sentence using your results in context properly.

29. Based on a sample of house cats the probability a cat is male based on its weight in kilograms is given by

$$C(x) = \frac{e^{-8.67943082163772+3.635139933871x}}{1 + e^{-8.67943082163772+3.635139933871x}}$$

Data from [1]. At what weight is the chance of being male increasing the fastest? What is the corresponding rate of increase? Write a sentence using your results in context properly.

30. "A population of women who were at least 21 years old, of Pima Indian heritage and living near Phoenix, Arizona, was tested for diabetes according to World Health Organization criteria. The data were collected by the US National Institute of Diabetes and Digestive and Kidney Diseases." [[2],[1]] Using logistic regression the probability of diabetes given results of an oral glucose tolerance test (mg/dL) is given by

$$G(x) = \frac{e^{-5.94680800161305+0.0424209811372968x}}{1 + e^{-5.94680800161305+0.0424209811372968x}}$$

At what glucose level of having diabetes increasing the fastest? What is the corresponding rate of increase? Write a sentence using your results in context properly.
31. Use the extreme poverty function from the function gallery and find and interpret the inflection point. Write a sentence using your results in context properly. You should include the rate.
32. (Challenge:) Use the life expectancy function, $LeGdp(x)$ from the function gallery for this problem. If we calculate the average rate of change from $x = 0$ to $x = 100000$ we get 0.0006323138 increase in life expectancy per dollar of GDP per capita. Find and interpret the location on the function where the rate of change is 0.0006323138.

17.2 Project: Arctic Sea Ice Analysis

Answer the following questions using the 1980 and 2012 Arctic sea ice extent functions from the function gallery. The fitted curves are to the Arctic ice data for the years 1980 (second full year of data) and 2012 (record low year). The R Code box Arctic Sea Ice below will get you started. You must submit a typed report with the graph, your responses to the questions, and your R code. In general, all typed answers should be in sentence form. Part of your grade will be based on your use of the English language, which includes the use of units with numbers.

1. Create one graph that contains a scatter plot of both years of data, the fitted curves from the function gallery, tangent lines and points at the location where ice is melting the fastest, and dots representing the max and min. Make appropriate use of colors. Include this graph in your report as you will be asked to reference it in some of the problems below.
2. What are the maximum and minimum values and locations for each year? Summarize your results in a few sentences for each model and compare any similarities and/or difference between the two years. Along with your answers, explain how your responses are represented on the graph.
3. Use your models to find the month of fastest melting of sea ice. Summarize your results in a few sentences for each model, including how fast the ice was melting at these points, and compare any similarities and difference between the two years. Your summary should address the question: Is the speed of ice melting meaningfully different between the two years (Hint: Units are particularly important here.)? Along with your answers, explain how your responses are represented on the graph.
4. Read these two short articles: What is the Cryosphere? Why it Matters: https://nsidc.org/learn/what-cryosphere/why-cryosphere-matters and The Ice-Albedo feedback: http://www.windows2universe.org/earth/polar/ice_albedo_feedback.html Explain how the analysis here along with global warming provides an example of this positive feedback loop.

R Code: Arctic Sea Ice

```
> dataURL<-
"http://sustainabilitymath.org/excel/Arctic-Ice-Calc-R.csv"
> Ice <-read.csv(url(dataURL),header=TRUE)
> ## Define variables, mostly to save typing
> x<-Ice$Month
> y1<-Ice$X1980.extent.in.million.square.km
> y2<-Ice$X2012.extent.in.million.square.km
> plot(x,y1,cex=1.25,pch=16,xlim=c(1,12),ylim=c(0,20),
col="red2",xlab="Month",ylab="Million Square Kilometers")
> points(x,y2,cex=1.25,pch=16,xlim=c(1,12),ylim=c(0,20),
```

```
col="royalblue")
> title(main="Arctic Sea Ice Extent")
> grid (NULL,NULL, col = "black")
> legend("topright",c("1980", "2012"),pch=c(16,16),
col=c("red2","royalblue"),y.intersp=1.25,bg="white")
```

Chapter 18
Finding Extremes

A person throws a ball vertically into the air at a speed of 26.8 m/s (about 60mph) and leaving their hand 1.8 meters (about 6ft) above the ground. The height of the ball is modeled by

$$s(t) = -4.9t^2 + 26.8t + 1.8288$$

meters t seconds after the ball leaves the persons hand. How high does the ball go? What is the fastest the ball travels? Both of these questions are about extremes of the function. The first question asks about the global maximum of the function, while the second asks about the global maximum of the derivative. Note that global maximums or minimums may occur at local maximums or minimums or at endpoints, which is stated in M-Box 18.1.

M-Box 18.1: Extreme Value Theorem

A continuous function $f(x)$ on a closed interval $[a, b]$ has both a global maximum and global minimum. The global maximum and global minimum may occur at a local maximum or minimum or at one of the endpoints $x = a$ or $x = b$.

Example 18.1. *What is the maximum height and speed of a ball thrown vertically if the height is modeled by* $s(t) = -4.9t^2 + 26.8t + 1.8288$ *meters* t *seconds after the ball leaves the persons hand.*

Solution. We first need to decide on the appropriate interval for the problem. We know $t \geq 0$, and the largest value of t is when the ball hits the ground or when $s(t) = 0$. The roots of $s(t)$ are -0.067 and 5.537 seconds (found with R, but you could use the quadratic formula). The first root can be ignored and so the interval for the values of t is [0, 5.537].

The maximum height of the ball can happen at either a critical point or endpoint of the interval. We have

$$s'(t) = -9.8t + 26.8.$$

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_18

The root of $s'(t)$ is 2.735. The possible maximum of the function can happen at $x = 0$, $x = 2.735$, or $x = 5.537$. Evaluating these values in $s(t)$ we get $s(0) = 1.8288$m, $s(2.735) = 38.4737$m, and $s(5.537) = 0$m. Hence the maximum height of the ball is 38.4737 meters and it happens at 2.735 seconds. In this case the maximum occurred at a critical point.

What about the maximum speed of the ball? This can happen at an inflection point or an endpoint. In this case the second derivative is $s''(t) = -9.8m/s^2$. The second derivative does not have any roots and so $s(t)$ does not have any inflection points. The maximum speed must occur at an endpoint. Now $s'(0) = 26.8m/s$ and $s'(5.537) = -27.4626m/s$. The fastest the ball travels is $-27.4626m/s$, which is when the ball hits the ground. □

A few notes on R Code box 18.1 which is for example 18.1. The variable **endPoints** was defined for the end points of the problem and we rounded the roots to match the example. Later in the code we define the variable **keyPoints** which concatenated into one vector the **endPoints** and **root_deriv** vectors. This in turn made it easier to evaluate all these points in the function in the line with **f(keyPoints)**. In the line above we just had **keyPoints** which printed out the points. This just made it easier to match the inputs and outputs of the function. This was similarly done with **keyPoints2**. The output of **root_deriv2** is **numeric(0)** which means that the vector is empty, in other words, there are no roots of the second derivative. In the last line there we identify the velocity of the ball, which is $s'(t)$, when it is traveling the fastest.

R Code 18.1: Finding Extremes I

```
> library(Deriv)
> library(rootSolve)
> s<-function(t){-4.9*t^2 +26.8*t+1.8288}
> uniroot.all(s,c(0,10))
[1] 5.536796
> endPoints<-c(0,5.537)
> s_p<-Deriv(s)
> s_pp<-Deriv(s,n=2)
> root_deriv<-uniroot.all(s_p,c(0,5))
> root_deriv
[1] 2.734694
> keyPoints<-c(endPoints,root_deriv)
> keyPoints
[1] 0.000000 5.537000 2.734694
> s(keyPoints)
[1]  1.8288000 -0.0056081 38.4736980
> root_deriv2<-uniroot.all(s_pp,c(0,5.537))
> root_deriv2
numeric(0)
> keyPoints2<-c(endPoints, root_deriv2)
> keyPoints2
```

```
[1] 0.000 5.537
> s_p(keyPoints2)
[1]  26.8000 -27.4626
```

Example 18.2. *A model of the average price is dollars of a fresh whole fortified gallon of milk in U.S. cities is given by*

$$m(t) = 3.3796980630769 + 0.0225309091316052t \\ - 0.000605546478028601t^2 + 0.00000348503830283816t^3$$

$*$ *where* t *is months after January 1, 2011 (*$t = 0$*) to November 1, 2022 (*$t = 142$*). During this time period what was the maximum price of a gallon of milk, what was the minimum price of a gallon of milk, what was the maximum rate of decrease of the price of a gallon of milk, and what was the maximum rate of increase of the price of a gallon of milk? Data from [6]*

Solution. We are given a time frame from January 1, 2011 to November 1, 2022, which provides us with the endpoints $t = 0$ and $= 142$. We identify the endpoints by solving

$$m'(t) = 0.0225309091316052 - 0.001211093t + 0.00001045511t^2 = 0.$$

Using the quadratic formula or R will yield two critical points $t = 23.28406$ and $t = 92.55336$. Evaluating $m(t)$ at the endpoints and two critical points we get

$$\begin{aligned} m(0) =& 3.379698 \\ m(23.28406) =& 3.040844 \\ m(92.55336) =& 3.620007 \\ m(142) =& 4.347516 \end{aligned}$$

The maximum price was \$4.35 per gallon in November 2022, while the minimum price was \$3.04 in December 2012. Recall that $t = 0$ is January 1, 2011, making $t = 11$ is Dec 1, 2011 and $t = 23$ Dec 1, 2012. In this example the global minimum was at a local minimum was the global maximum was at an endpoint.

To find the maximum rate of increase and decrease, we follow the same procedure but "one derivative down" as we are looking for a global maximum and minimum of $m'(t)$. We need to solve (note we switched the order of terms so that the first term is not negative)

$$m''(t) = 0.00002091022t - 0.001211093 = 0$$

and find $t = 57.91868$. We not evaluate $m'(t)$ at the endpoints and the possible inflection point to get

$$m'(0) = 0.02253091$$
$$m'(57.91868) = -0.01254154$$
$$m'(142) = 0.06137265$$

Milk prices were increasing the fastest in November 2022 at a rate of \$0.06 per gallon and decreasing the fastest in November 2015 (rounding to $t = 59$) at a rate of -\$0.01 per gallon. In this example the maximum of $m'(t)$ occurred at an endpoint while the minimum occurred at an inflection point. R Code box 18.2 has the code for this problem and uses similar conventions discussed above, for example 18.1. □

R Code 18.2: Finding Extremes II

```
> library(Deriv)
> library(rootSolve)
> m<-function(t){3.3796980630769+0.0225309091316052*t
-0.000605546478028601*t^2+0.00000348503830283816*t^3}
> endPoints<-c(0,142)
> m_p<-Deriv(m)
> m_pp<-Deriv(m,n=2)
> root_deriv<-uniroot.all(m_p,c(0,142))
> root_deriv
[1] 23.28406 92.55336
> keyPoints<-c(endPoints,root_deriv)
> keyPoints
[1]   0.00000 142.00000  23.28406  92.55336
> m(keyPoints)
[1] 3.379698 4.347516 3.620007 3.040844
> root_deriv2<-uniroot.all(m_pp,c(0,142))
> root_deriv2
[1] 57.91868
> keyPoints2<-c(endPoints, root_deriv2)
> keyPoints2
[1]   0.00000 142.00000  57.91868
> m_p(keyPoints2)
[1]  0.02253091  0.06137265 -0.01254154
```

18.1 Exercises

1. Answer the following questions from the R code below. Consider this a problem of finding extremes.

   a. What are the endpoints and critical points for this problem?
   b. What is the global maximum point and is it an endpoint or a critical point?
   c. What is the global minimum point and is it an endpoint or a critical point?
   d. What is the maximum rate of increase and at what point does it occur? Is this an endpoint or an inflection point?
   e. What is the minimum rate of increase and at what point does it occur? Is this an endpoint or an inflection point?

```
> library(Deriv)
> library(rootSolve)
> g<-function(x){**Hidden**}
> endPoints<-c(-12,12)
> g_p<-Deriv(g)
> g_pp<-Deriv(g,n=2)
> root_deriv<-uniroot.all(g_p,c(-12,12))
> root_deriv
[1] -7.291503  3.291495
> keyPoints<-c(endPoints,root_deriv)
> keyPoints
[1] -12.000000  12.000000  -7.291503   3.291495
> g(keyPoints)
[1] -40.00000 176.00000  17.04052 -57.04052
> root_deriv2<-uniroot.all(g_pp,c(-12,12))
> root_deriv2
[1] -2
> keyPoints2<-c(endPoints,root_deriv2)
> keyPoints2
[1] -12  12  -2
> g_p(keyPoints2)
[1]  27.0  63.0 -10.5
```

2. Answer the following questions from the R code below.

   a. What are the endpoints and critical points for this problem?
   b. What is the global maximum point and is it an endpoint or a critical point?
   c. What is the global minimum point and is it an endpoint or a critical point?
   d. What is the maximum rate of increase and at what point does it occur? Is this an endpoint or an inflection point?
   e. What is the minimum rate of increase and at what point does it occur? Is this an endpoint or an inflection point?

```
> library(Deriv)
> library(rootSolve)
> g<-function(x){**Hidden**}
> endPoints<-c(-15,20)
> g_p<-Deriv(g)
> g_pp<-Deriv(g,n=2)
> root_deriv<-uniroot.all(g_p,c(-15,20))
> root_deriv
[1] -10.082915   6.082904
> keyPoints<-c(endPoints,root_deriv)
> keyPoints
[1] -15.000000  20.000000 -10.082915   6.082904
> g(keyPoints)
[1]  -43.8750 -792.0000 -132.0208  132.0208
> root_deriv2<-uniroot.all(g_pp,c(-15,20))
> root_deriv2
[1] -2
> keyPoints2<-c(endPoints, root_deriv2)
> keyPoints2
[1] -15  20  -2
> g_p(keyPoints2)
[1]  -38.875 -157.000   24.500
```

3. Answer the following questions from the R code below.
   a. What are the endpoints and critical points for this problem?
   b. What is the global maximum point and is it an endpoint or a critical point?
   c. What is the global minimum point and is it an endpoint or a critical point?
   d. What is the maximum rate of increase and at what point does it occur? Is this an endpoint or an inflection point?
   e. What is the minimum rate of increase and at what point does it occur? Is this an endpoint or an inflection point?

```
> library(Deriv)
> library(rootSolve)
> h<-function(x){**Hidden**}
> endPoints<-c(7,16.5)
> h_p<-Deriv(g)
> h_pp<-Deriv(g,n=2)
> root_deriv<-uniroot.all(h_p,c(7,16.5))
> root_deriv
[1]  8.096158 11.172584 14.276415
> keyPoints<-c(endPoints,root_deriv)
> keyPoints
[1]  7.000000 16.500000  8.096158 11.172584 14.276415
> h(keyPoints)
[1]   32.19234 -193.78356   63.63498 -122.87617  201.84322
```

```
> root_deriv2<-uniroot.all(h_pp,c(7,16.5))
> root_deriv2
[1]  9.819001 12.871282 15.955634
> keyPoints2<-c(endPoints, root_deriv2)
> keyPoints2
[1]  7.000000 16.500000  9.819001 12.871282 15.955634
> h_p(keyPoints2)
[1]   46.13902 -214.71652  -96.56024  165.75628 -254.63688
```

4. Answer the following questions from the R code below.

   a. What are the endpoints and critical points for this problem?
   b. What is the global maximum point and is it an endpoint or a critical point?
   c. What is the global minimum point and is it an endpoint or a critical point?
   d. What is the maximum rate of increase and at what point does it occur? Is this an endpoint or an inflection point?
   e. What is the minimum rate of increase and at what point does it occur? Is this an endpoint or an inflection point?

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){**Hidden**}
> endPoints<-c(-20,-10)
> f_p<-Deriv(g)
> f_pp<-Deriv(g,n=2)
> root_deriv<-uniroot.all(f_p,c(-20,-10))
> root_deriv
[1] -17.39316 -14.27646 -11.17271
> keyPoints<-c(endPoints,root_deriv)
> keyPoints
[1] -20.00000 -10.00000 -17.39316 -14.27646 -11.17271
> f(keyPoints)
[1] 1000.0000  999.9998 1000.0000 1000.0000 1000.0000
> root_deriv2<-uniroot.all(f_pp,c(-20,-10))
> root_deriv2
[1] -19.05767 -15.95553 -12.87119
> keyPoints2<-c(endPoints, root_deriv2)
> keyPoints2
[1] -20.00000 -10.00000 -19.05767 -15.95553 -12.87119
> f_p(keyPoints2)
[1]  199.75063  -94.78758  363.23309 -254.63689  165.75628
```

For the following problems find the local maximums, local minimums, global maximums, global maximums, and inflection points of the given function on the given interval with R. This includes using the second derivative test with R to identify a critical point as a max or min. The local maximums, local minimums, and inflection points should be given with the x and y-value as well as the

associated rate at the inflection points. Also identify the location and rate where the curve is increasing the fastest and increasing the slowest (or decreasing the fastest). Finally, include a graph of the function and place points at the local maximums, local minimums, and inflection points.

5. $f(x) = e^x + (2.5)^x \sin(2\pi x) - 10$, $[6, 8]$
6. $f(x) = e^x + (2.5)^x \cos(\pi x) - 1000$, $[4, 7.5]$
7. $g(x) = 10x^6 - 36x^5 - 345x^4 + 1020x^3 + 2820x^2 - 7200x - 10000$, $[-5, 6]$
8. $g(x) = -10x^6 + 60x^5 + 150x^4 - 1000x^3 - 270x^2 + 2700x + 10000$, $[-4, 6]$
9. $f(x) = x\sin(x)$, $[-6, 6]$
10. $f(x) = x^2 \sin(x)$, $[0, 10]$
11. $g(x) = \sin(x^2)$, $[0, \pi]$
12. $g(x) = \sin(\sqrt{x})$,$[0, 10\pi]$
13. $f(x) = x\cos(x)\sin(\sqrt{x})$, $[2, 14]$ (for fun graph this on a larger interval)
14. $f(x) = \sin(x)\cos(e^x)$,$[0, \pi]$ (for fun graph this on a larger interval)

Use R as desired for these problems.

15. The function

$$E(t) = 1.56587994153652 + 0.0436841781992499t - 0.00108765220881672t^2 + 0.00000641850515548612t^3$$

models the average price of a dozen grade A large eggs in U.S. cities in dollars, where t is the number of months after January 1, 2012. Find the global maximum, global minimum, and the rates of fastest increases and decrease, between January 1, 2012 ($t = 0$) and November 1, 2022 ($t = 142$). Report your result in a few sentence with proper context. Data from [4]

16. The function

$$Ch(t) = 4.41281093468948 + 0.0758375702747065t - 0.0014041565369063t^2 + 0.00000708849730870583t^3$$

models the average price of a pound of cheddar cheese in U.S. cities in dollars, where t is the number of months after January 1, 2010. Find the global maximum, global minimum, and the rates of fastest increases and decrease, between January 1, 2010 ($t = 0$) and December 1, 2019 ($t = 119$). Report your result in a few sentence with proper context. Data from [3]

17. A watermelon is launched vertically starting at 3 meters. The distance of the watermelon from the ground in meters is given by $s(t) = 3 + 75t - 4.9t^2$ where t is in seconds. What is the maximum height of the watermelon? What is the maximum speed of the watermelon? Write a sentence using your results in context properly.

18. A few professional baseball players can throw a ball 100mph. If such a person throws a ball vertically releasing the ball 1.8288 meters above the ground, then the height is modeled by $s(t) = -4.9t^2 + 44.704t + 1.8288$ meters t seconds after the ball leaves the persons hand. What is the maximum height of the ball? What is the maximum speed of the ball? Write a sentence using your results in context properly.
19. Find the x and y-value of the global maximum and global minimum and the x-value and rate where the fastest increase, and fastest decrease occurs of the function $f(x) = \sin(x^2)$ on the interval $[0, 3]$
20. Find the x and y-value of the global maximum and global minimum and the x-value and rate where the fastest increase, and fastest decrease occurs of the function $f(x) = x^2 \sin(x)$ on the interval $[0, 14]$
21. The profit for a company is modeled by $p(x) = -5x^2 + 100x - 200$ dollars when x items are sold. How many items sold maximizes profit? How many items sold maximizes average profit (the function $p(x)/x$)? Write a sentence using your results in context properly.
22. The distribution of energy consumption in the U.S. (2014 data) and World (2011 data) are shown in figure 3.8 in the Function Gallery chapter. To interpret these functions consider the example: ECus(0.63)=0.47 means that the bottom 63% of states in the U.S. have per capita energy use in the bottom 48% of all states. The function ECw(x) is similar and replaces countries for states. Find the value $x = c$ such that ECus'(c) = 1 Answer the same question for ECw(x). The values of c have a significant real world interpretation. What is it? Note: The existence of c is guaranteed by the Mean Value Theorem, which is not covered in this text.
23. A particle is moving around the origin. At time t the particle is at $(x, y) = (16(sin(t))^3, 13cos(t) - 5cos(2t) - 2cos(3t) - cos(4t))$ for $t \in [0, 2\pi]$. Let $h(t)$ be the distance from the origin to the particle. Find the maximum and minimum distance between the particle and the origin and the location at which it is getting closets to the origin the fastest. Part of the challenge here is defining the function $h(t)$. Hint: The distance between two points in the plane is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. To graph the location of the particle, which will help you see if your function $h(t)$ might be correct, here is R code that will do the trick:

```
> values<-seq(0,2*pi,by=0.01)
> x_t<-16*sin(values)^3
> y_t<-13*cos(values) -5*cos(2*values)-2*cos(3*values) - cos(4*values)
> plot(x_t,y_t,pch=16)
> abline(h=0,v=0,lwd=2)
```

Chapter 19
Optimization

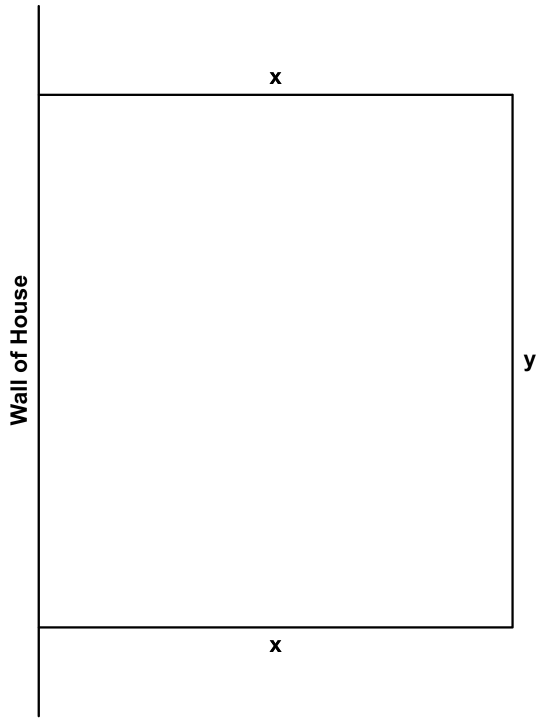

Fig. 19.1 Sketch of situation for fence problem.

Suppose we want to build a fenced in area adjacent to a house and we are limited to 100 feet of fencing based on how fencing is sold. What is the largest area that can

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be enclosed? This type of problem, one where we want to maximize or minimize something, in this case the area of the rectangle, given some constraint, in this case a limit on the perimeter, is an optimization problem. In order to solve this problem we first sketch a picture and define variables, which is done in figure 19.1. Our **objective function**, what we want maximized in this case, is

$$A(x, y) = xy$$

which is the area of the enclosed space. Note that this is a function of two variables x and y. We can only find the maximum or minimum of a function of one variable. Our **constraint equation**, is $2x + y = 100$, in other words we have only 100 feet of fencing for the two x-sides and one y-side. We can solve this equation for y to get $y = 100 - 2x$. We can now substitute this equation into $A(x, y)$ to get

$$A(x) = x(100 - 2x) = 100x - 2x^2.$$

Our area function is now a function of one variable. To find a maximum of $A(x)$ we take the derivative,

$$A'(x) = 100 - 4x,$$

and solve $100 - 4x = 0$ to get the critical point at $x = 25$. Is $x = 25$ a maximum or a minimum?

First, $A''(x) = -4$ which means the curve is always concave down so that $x = 25$ is a local max as well as a global max. On the other hand, the range for x is $[0, 50]$ and checking the endpoints gives $A(0) = 0$ and $A(50) = 50$, while $A(25) = 1250$. Again $x = 25$ provides a maximum. Going back to our constraint equation we find that $y = 50$, the dimension of the maximum area is 25×50 for a total area of 1250ft^2. The general steps of optimization problems are summarized in M-Box 19.1.

M-Box 19.1: General Steps for Optimization Problems

1. Draw a picture when appropriate and include variables.
2. Identify the function that requires a maximum or minimum, the objective function.
3. Identify the constraint equation. This equation captures any limitations for the problem.
4. Solve the constraint equation for one variable, choose wisely, and substitute into the objective function.
5. With the objective equation in one variable, find the maximum or minimum using derivatives.
6. Verify that you actually have a maximum or minimum.
7. Provide the final answer the problem desires with appropriate units.

19.1 Exercises

1. Let x and y be two positive numbers. Minimize $x^2 + y^2$ under the constraint that the sum of x and y is 20.
2. Let x and y be two positive numbers. Maximize xy^2 under the constraint that the sum of x and y is 42.
3. A rectangular pen with a partition is to be built with 1000 feet of fencing. What are the dimensions and maximum possible area of the pen?
4. A rectangular pen is to be built along the side of a house (so no fencing needed on one side) with a partition perpendicular to the house with 1000 feet of fencing. What are the dimensions and the maximum possible area of the pen.
5. A rectangular fenced in yard is to be built. The side facing the road will be nicer and will cost \$20 per foot. The other three sides will only cost \$10 per foot. If \$1500 is to be spent on the yard, what are the dimensions and largest possible area of the yard?
6. A rectangular fenced in yard is to be built with four internal pens built like a checkerboard. The exterior fencing will cost \$25 per foot and the interior fencing will cost \$10 per foot. If \$2000 dollars is available to build the yard, what are the dimensions and maximum area of the entire yard?
7. A rectangular box with square base and no top is to be constructed using 50ft^2 of material. What are the dimensions of the box with maximum volume?
8. A metal can is to hold 23.75 cubic inches of volume. What are the dimensions of a can that minimizes the total material used to construct the can?
9. A 4ft by 5ft sheet of cardboard is to be made into a box by cutting equal sized squares out of each corner and folding up the edges. What is the dimension of the box with maximum volume?
10. A Norman window (look it up) will be built with a perimeter of 42 feet. What are the dimensions and total area of the window with maximum area?
11. A 40-inch piece of wire will be cut into two pieces. One piece will make a square and the other a circle. What is the length of the two pieces of cut wire that maximize the total area of the two shapes? What are the corresponding dimensions of the shapes?
12. A rectangle will be inscribed in a 3, 4, and 5 foot right triangle. What are the dimensions of the largest such rectangle?
13. A pen in the shape of a right triangle is to be built. Due to the space for the pen the hypotenuse of the triangle can be at most 11 yards. Find the pen with maximum area.
14. A pen in the shape of a square is to be built with the limitation that the diagonal of square is at most 25 feet. Find the pen with maximum area.
15. A home owner is going to build an in ground pool with an area of 800 square feet. A deck will be build around the pool that is 10 feet wide on the ends and 4 feet wide on the sides. What is the smallest piece of property needed for the pool and deck?
16. A home owner is going to build an in ground pool with an area of 800 square feet. A deck will be build around the pool that is 10 feet wide on the ends and 4

feet wide on the sides. The sides with the 10 foot deck will cost \$30 per square foot (better materials for setting up furniture on) and the sides with the 4 foot deck will cost \$20 per square foot. What are the dimensions of the pool and deck that will cost the least amount of money to build the deck?

17. A shed with flat roof will be built adjacent to a house and so the side against the house does not require material. The base of the shed will have one side 5 feet longer than the other. The material that can be used to enclose the shed has a total area of 1500 square feet. What are the dimensions of the shed with the largest volume?
18. A shed with flat roof will be built adjacent to a house and so the side against the house does not require material. The bottom of the shed will also be left open. The base of the shed will have one side 5 feet longer than the other. The material that can be used to enclose the shed has a total area of 1500 square feet. What are the dimensions of the shed with the largest volume?
19. Minimize the quantity $x^4 + 2y^2$ under the condition that $x + y = 100$, $x \geq 0$ and $y \geq 0$.
20. Minimize the quantity $10x^3 + 2y^5$ under the condition that $x + y = 50$, $x \geq 0$ and $y \geq 0$.
21. The idea behind this problem is to calculate the path through two regions where the time to cross between the regions is different. We want to travel from (0, 0) to (4, 4). The region defined by the four points (0, 0), (0, 4), (4, 0), and (4, 4) is divided by the line $y = 3 + x/2$. The units of distance in the region or in miles. It takes 10 minutes per mile to travel through the lower half of the region and 25 minutes per mile to travel through the upper half of the region. The path taken will be a line from (0, 0) to any point, call it (a, b), on the line $y = 3 + x/2$ and then a line from there to the point (4, 4). What point (a, b) minimizes the travel time from (0, 0) to (4, 4) and what it the time it takes to get from (0, 0) to (4, 4)?
22. The idea behind this problem is to calculate the path through two regions where the time to cross between the regions is different. We want to travel from (0, 0) to (4, 2). The region defined by the four points (0, 0), (0, 4), (2, 0), and (4, 2) is divided by the curve $y = 2 - x^2/9$. The units of distance in the region or in miles. It takes 10 minutes per mile to travel through the lower half of the region and 20 minutes per mile to travel through the upper half of the region. The path taken will be a line from (0, 0) to any point, call it (a, b), on the curve $y = 2 - x^2/9$ and then a line from there to the point (4, 2). What point (a, b) minimizes the travel time from (0, 0) to (4, 2) and what it the time it takes to get from (0, 0) to (4, 2)?
23. The top half of a sphere of radius 10 feet is constructed and placed on a flat surface. The flat surface has an x-axis and y-axis so the sphere is centered on $(0, 0)$. You can enter the sphere anywhere (the sphere is constructed of magic material so you can enter anywhere). Since there is a grid on the ground you take a path through the sphere by following the function $y = -x^2 + 2x + 5$. You are also carrying a measuring device so you know the height of the sphere above you along your path. What is the highest point of the sphere along your path and where on the grid did it occur? Tip: The formula for the sphere is $f(x, y) = \sqrt{100 - x^2 - y^2}$.

24. Follow the same scenario as in the previous problem but instead of walking through a sphere the "dome" shape is given by $f(x, y) = \sqrt{100 - 2x^2 - y^2 - y}$ (sort of egg shaped). You will follow the same path of $y = -x^2 + 2x + 5$. What is the highest point of the sphere along your path and where on the grid did it occur?

Chapter 20
Derivatives of Functions of Two Variables

The idea of the derivative generalizes to functions of more than one variable where can capture the rate of change relative to one of the variables. Consider the function $V(r, h) = \pi r^2 h$ which is the volume of a cylinder as seen in figure 20.1. This is the function of the two variables r and h. If, for example, the value of h was fixed at, say 10 inches, then the formula becomes $V(r) = 10\pi r^2$, which is the volume of a cylinder of height 10 inches and radius r. The derivative here will provide a formula for the growth of volume of this specific cylinder. In this case $\frac{dV}{dr} = 20\pi r$. Note that we are using the $\frac{dV}{dr}$ instead of $V'(r)$ as given in M-Box 10.3. We read $\frac{dV}{dr}$ as the derivative of the function V with respect to the variable r. As we generalize the derivative idea to multivariable functions this becomes important. The formula $\frac{dV}{dr} = 20\pi r$ provides us with how the volume of a cylinder of height 10 inches grows with respect to the radius.

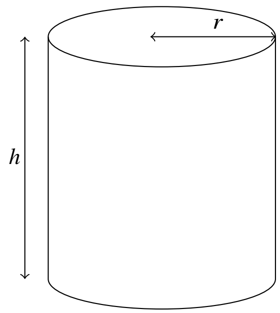

Fig. 20.1 A cylinder of height h and radius r.

We would like a formula that can be used for any height h. We can do this by simply taking the (partial) derivative of $V(r, h) = \pi r^2 h$ with respect to r treating h as if it were a constant. Here is what we get

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https://doi.org/10.1007/978-3-031-28571-4_20

$$\frac{\partial V}{\partial r} = 2\pi r h. \tag{20.1}$$

This is read as the partial derivative of v with respect to the variable r. The use of ∂ instead of d indicates that we are dealing with a function of more than one variable. We can also do this for the variable h to get

$$\frac{\partial V}{\partial h} = \pi r^2. \tag{20.2}$$

This is the partial derivative of v with respect to the variable h. What does this all mean? Suppose we have a cylinder of radius 2 feet and height 3 feet. How fast is the volume growing with respect to the radius? This is

$$\frac{\partial V}{\partial r}(2, 3) = 2\pi 2(3) = 12\pi \text{ feet cubed per foot of radius.} \tag{20.3}$$

In other words, the volume would increase about 12π cubic feet if the radius was increased from 2 feet to 3 feet. Similarly, how fast is the volume growing with respect to the height?

$$\frac{\partial V}{\partial h}(2, 3) = \pi 2^2 = 4\pi \text{ feet cubed per foot of height.} \tag{20.4}$$

In other words, the volume would increase about 4π cubic feet if the height was increased from 3 feet to 4 feet. Overall, an increase of the radius by a foot will add more volume than adding a foot of height.

There is a graphical interpretation to this. Figure 20.2 is the three-dimensional graph of the function $V(r, h) = \pi r^2 h$ from two different perspectives. (Note: Appendix I R Code for Figures has the code for this graph and if you run it you will be able to rotate the graph.) In reading this graph, the x-axis and y-axis are for the variables r and h, respectively. Given a value of r and h we then plot on the z-axis. For example, the yellow dot has $r = 0$ and $h = 0$ to get $V(0, 0) = \pi 0^2(0) = 0$ or the point $(0, 0, 0)$, while the red dot has $r = 2$ and $h = 3$ to get $V(2, 3) = \pi 2^2(4) = 12\pi$ or the point $(2, 3, 12\pi)$.

The graph in figure 20.2 has tangent lines at the point $(2, 3, 12\pi)$. One of the tangent lines is parallel to the x-axis and the other to the y-axis. The slope of the tangent line parallel to the x-axis is given by equation 20.3 or $\frac{\partial V}{\partial r}(2, 3) = 2\pi 2(3) = 12\pi$ feet cubed per foot of radius. In other words, the partial derivative with respect to r provides a formula for the slope of the tangent lines parallel to the x-axis or really the r-axis. Similarly, the slope of the tangent line parallel to the y-axis is given by equation 20.4 or $\frac{\partial V}{\partial h}(2, 3) = \pi 2^2 = 4\pi$ feet cubed per foot of height. In other words, the partial derivative with respect to h provides a formula for the slope of the tangent lines parallel to the y-axis or really the h-axis.

This idea extends to second derivatives. For example,

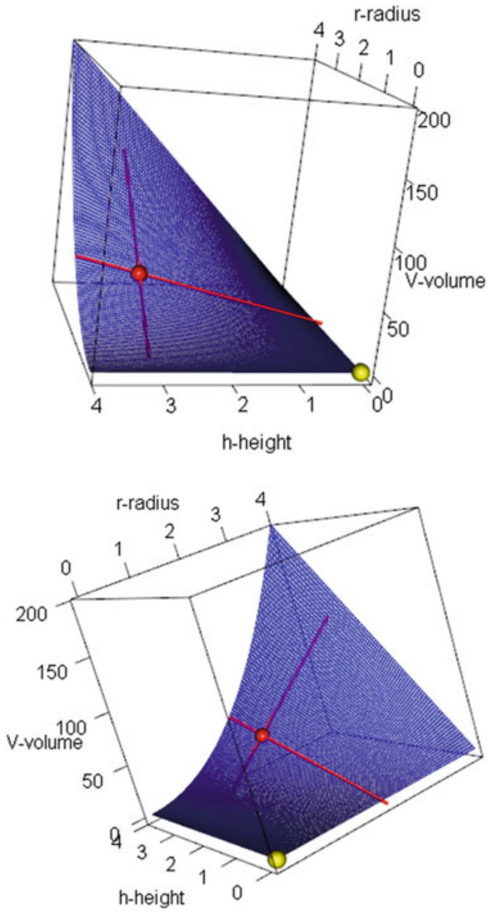

Fig. 20.2 Graph of the function for the volume of a cylinder, $V(r,h) = \pi r^2 h$. The yellow dot is at (0, 0, 0) and the red dot is (2, 3, 12π).

$$\frac{\partial^2 V}{\partial r^2} = 2\pi h \tag{20.5}$$

and

$$\frac{\partial^2 V}{\partial h^2} = 0. \tag{20.6}$$

In reading the notion $\partial^2 V$ indicates we are taking a second derivative of the function V and ∂r^2 says the derivatives are with respect to the variable r both times. The notation ∂r^2 does not mean square r. The interpretation is concavity parallel to the respective axis. For example, $\frac{\partial^2 V}{\partial h^2} = 0$ says that if you follow any path parallel to the y-axis (the h variable) there isn't any concavity or curvature. This should not be surprising since the volume formula of a cylinder is linear in h. In figure 20.2 the tangent line parallel to the y-axis sits on the surface. On the other hand, parallel to the x-axis (r variable) the graph is concave up.

There is one last idea here and it is that we can take two derivatives, once with respect to one of the variables and then the other. For example,

$$\frac{\partial^2 V}{\partial h \partial r} = \frac{\partial V}{\partial h}\left(\frac{\partial V}{\partial r}\right) = \frac{\partial V}{\partial h}(2\pi r h) = 2\pi r \tag{20.7}$$

where we first took a derivative with respect to r to get $2\pi r h$ and then did the derivative of this with respect to h. The other direction goes like this

$$\frac{\partial^2 V}{\partial r \partial h} = \frac{\partial V}{\partial r}\left(\frac{\partial V}{\partial h}\right) = \frac{\partial V}{\partial r}\left(\pi r^2\right) = 2\pi r \tag{20.8}$$

In this example, both paths resulted in the same derivative and this will happen if the surface is continuous. Geometrically, the interpretation of equation 20.7 is that since it is positive as we walk parallel to the x-axis (r) but look parallel to the y-axis the slopes parallel to the y-axis will increase. It is reversed for equation 20.8.

M-Box 20.1: Partial Derivative Summary

Let $f(x, y)$ be a function of two variables.

1. $\frac{\partial f}{\partial x}$ The first derivative of $f(x, y)$ with respect to the variable x. This is the rate of change of the function moving parallel to the x–axis.
2. $\frac{\partial f}{\partial y}$ The first derivative of $f(x, y)$ with respect to the variable y. This is the rate of change of the function moving parallel to the y–axis.
3. $\frac{\partial^2 f}{\partial x^2}$ The second derivative of $f(x, y)$ with respect to the variable x in both derivatives. This is the concavity of the function moving parallel to the x–axis.

4. $\dfrac{\partial^2 f}{\partial y^2}$ The second derivative of $f(x, y)$ with respect to the variable y in both derivatives. This is the concavity of the function moving parallel to the y–axis.
5. $\dfrac{\partial^2 f}{\partial y \partial x}$ The second derivative of $f(x, y)$ with respect to the variable x for the first derivative and then with respect to y for the second derivative. This is the rate of change of the slope looking toward the x–axis while moving parallel to the y–axis.
6. $\dfrac{\partial^2 f}{\partial x \partial y}$ The second derivative of $f(x, y)$ with respect to the variable y for the first derivative and then with respect to x for the second derivative. This is the rate of change of the slope looking toward the y–axis while moving parallel to the x–axis.

Algebraically calculating partial derivatives there are two things to keep in mind. Every symbol in the equation other than the variable with respect to which we are differentiating should be treated as if it were a constant. Second, all the usual derivative rules can be used. Finally, R Code box 20.1 demonstrates partial derivatives with the Deriv package. The only addition to the code is to add an option in **Deriv** function specifying the variable. For example, **Deriv(V,"r")** is simply the derivative of the function V with respect to the variable r.

R Code 20.1: Partial Derivatives

```
> library(Deriv)
> V<-function(r,h){pi*r^2*h}
> Deriv(V,"r")
function (r, h)
2 * (h * pi * r)
> Deriv(V,"h")
function (r, h)
pi * r^2
> Deriv(V,"r",n=2)
function (r, h)
2 * (h * pi)
> Deriv(V,"h",n=2)
function (r, h)
0
> result<-Deriv(V,"r")
> Deriv(result,"h")
function (r, h)
2 * (pi * r)
```

20.1 Exercises

In problems 1-10 find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial y \partial x}$ **and** $\frac{\partial^2 f}{\partial x \partial y}$. **These should be done by hand but you should check your work in R.**

1. $f(x, y) = 5x^3y^2$
2. $f(x, y) = 7x^4y^5$
3. $f(x, y) = 8x^2\cos(y)$
4. $f(x, y) = 9\sin(x)y^3$
5. $f(x, y) = 3x^6e^{y^2}$
6. $f(x, y) = 7y^4e^{x^2}$
7. $f(x, y) = 8e^{x^2y^2}$
8. $f(x, y) = 11\sin(x^3y^5)$
9. $f(x, y) = 4x\cos(6xy^2)$
10. $f(x, y) = 7ye^{9x^2y}$
11. A rectangular box with a square base has volume given by $V(b, h) = b^2h$. How fast is a box with $b = 4$ feet and $h = 5$ feet growing with respect to the base? With respect to the height? Report your results in a sentence or two with appropriate context and units.
12. The volume of a right circular cone is given by $V(b, h) = \pi r^2h/3$. How fast is a right circular cone with $r = 10$ inches and $h = 12$ inches growing with respect to the radius? With respect to the height? Report your results in a sentence or two with appropriate context and units.
13. The formula for the top half of a sphere centered at the origin with radius 10 feet is given by $S(x, y) = \sqrt{100 - x^2 - y^2}$. A lazy ant is standing on the sphere at $(5, -6, S(5, -6))$. Would you advise the ant to proceed parallel to the x-axis or parallel to the y-axis? Your answer must include clear quantitative information with units.
14. The formula for the top half of a sphere centered at the origin with radius of 25 feet is given by $S(x, y) = \sqrt{25 - x^2 - y^2}$. A lazy ant is standing on the sphere at $(-1, -2, S(-1, -2))$. Would you advise the ant to proceed parallel to the x-axis or parallel to the y-axis? Your answer must include clear quantitative information with units.
15. The logistic model in the function gallery (figure 3.11) is an example of using one variable to predict a binary outcome. Models such as these can be developed with more than one variable. For example, the probability a house in Windsor Canada in the summer of 1987 has a driveway based on the price, p, of the house in thousands of dollars and the lot size, l, in hundreds of square feet is given by

$$D(x) = \frac{e^{-2.6344455121961+0.0432104251080119p+0.0457419277727325l}}{1 + e^{-2.6344455121961+0.0432104251080119p+0.0457419277727325l}}$$

Data from [1]. Find and interpret $D(35, 25)$, $\frac{\partial D}{\partial p}(35, 25)$, $\frac{\partial D}{\partial l}(35, 25)$. Based on these calculations, is the probability a house has air conditioning likely to increase more if the price of \$35000 is increased by \$1000 or if the lot size of 2500 square feet is increased by 100 square feet? Report your results in a sentence or two with appropriate context and units.

16. "A population of women who were at least 21 years old, of Pima Indian heritage and living near Phoenix, Arizona, was tested for diabetes according to World Health Organization criteria. The data were collected by the US National Institute of Diabetes and Digestive and Kidney Diseases." [[2],[1]] The logistic model in the function gallery (figure 3.11) is an example of using one variable to predict a binary outcome. Models such as these can be developed with more than one variable. For example, Using logistic regression the probability of diabetes given results of an oral glucose tolerance test (mg/dL), g, and bmi (body mass index in kg of weight / (height in meters)^2), b, is given by

$$D(x) = \frac{e^{-8.02530241788477+0.039311076164066g+0.0726453273490191b}}{1 + e^{-8.02530241788477+0.039311076164066g+0.0726453273490191b}}$$

Find and interpret $D(130, 45)$, $\frac{\partial D}{\partial g}(130, 45)$, $\frac{\partial D}{\partial b}(130, 45)$. Based on these calculations, is the probability of diabetes likely to increase more if the glucose test goes up 1 mg/dL, from 130 to 131, or if the bmi goes up 1 kg/m^2, from 45 to 46? Report your results in a sentence or two with appropriate context and units.

Chapter 21
Related Rates

Consider the equation for the area of a circle

$$A = \pi r^2. \tag{21.1}$$

We tend to think of this as the function $A(r)$ where r is the input and area is the output. We also get a function for the rate of area with $A'(r)$. Of course, we could view this relationship as

$$r = \sqrt{\frac{A}{\pi}}$$

and in this case we may think of this as the function $r(A)$. Similarly, we can calculate $r'(A)$. The point here is that the area and radius of a circle are related. Let us view equation 21.1 in a different light by writing

$$A(t) = \pi(r(t))^2. \tag{21.2}$$

Pause for a moment and think about what is being implied by adding the "of t" to the area and radius and how it relates to figure 21.1.

Consider a rubber band in the shape of a perfect circle that stays as a perfect circle as it is stretched. We start stretching the rubber band at time $t = 0$ seconds. Now, $A(t) = \pi(r(t))^2$ is an equation for the area of the circle at time t. For example, if we are told that $r(3) = 5$ inches, the radius after stretching the rubber band for 3 seconds, then

$$A(3) = \pi(r(3))^2 = \pi(5)^2 = 25\pi \text{ square inches.}$$

Further, as we stretch the rubber band the radius grows and the area grows. In other words, the rates of growth of the radius and area are related. Hence the title of the chapter; related rates. We can take the derivative of both sides of 21.2 with respect to the variable t to get

$$A'(t) = \pi 2(r(t))r'(t). \tag{21.3}$$

First, take note of the use of the chain rule in finding the derivative of $(r(t))^2$ as this is a function composition with $r(t)$ as the inside function and the square as the

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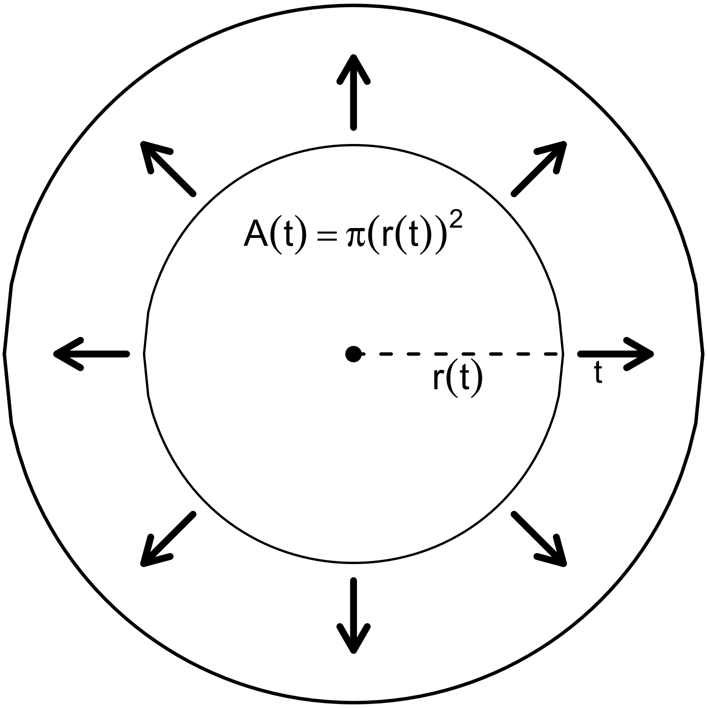

Fig. 21.1 A representation of an expanding circle and the equation $A(t) = \pi(r(t))^2$.

outside function. Second, realize that we do not have an explicit formula for the rate of change of the area, $A'(t)$. What we do know, is that the rate of change of the area, $A'(t)$, is related to the radius, $r(t)$, and the rate of change of the radius, $r'(t)$. If we now, for example, that $r(3) = 4$ inches and $r'(3) = 0.5$ inches per second which means that at time $t = 3$ the radius is 2 inches and growing at a rate of 0.5 inches per second, then we know from equation 21.3 that

$$A'(3) = \pi 2(r(3))r'(3) = \pi 2(4)(0.5) = 4\pi \text{ square inches per second.}$$

We conclude that at time $t = 3$ that the area is growing at a rate of 4π square inches per second. We now provide two examples of related rate problems with steps in M-Box 21.1 on how to approach these problems between the examples.

Example 21.1. *A perfect sphere shaped balloon is being blown up. After two second we know the radius is 3 inches and growing at a rate of 0.25 cubic inches per second. How fast is the volume changing at two seconds?*

Solution. It might help to draw a picture of a sphere, with radius r (go ahead and do it). We are asked to find $V'(2)$ where $v(t)$ is the volume of a spherical balloon, where t is the seconds after inflation begin. In notation, the information given is the $r(2) = 3$ inches and $r'(2) = 0.25$ cubic inches per second, where $r(t)$ is the radius of the balloon at time t. The problem is about the volume and radius of a sphere and so we need a formula relating the two. The equation for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

and adding the time component we get

$$V(t) = \frac{4}{3}\pi(r(t))^3. \tag{21.4}$$

We find the derivative of equation 21.4 with respect to t, noting we need a chain rule to calculate the derivative of $(r(t))^3$ to get

$$V'(t) = \frac{4}{3}\pi 3(r(t))^2 r'(t). \tag{21.5}$$

When finding the derivative of $(r(t))^3$, we recognize $r(t)$ as the inside function and to the power of 3 as the outside function. In the derivative expression $3(r(t))^2 r'(t)$ the three in front comes from bringing the third power down in front and reducing it by one in the power, which also explains why we now have $(r(t))^2$ in equation 21.5. We now multiply by the derivative of the inside function, $r(t)$, which we express as $r'(t)$ since we do not have an explicit expression for $r(t)$. Note that the $\frac{4}{3}\pi$ is a constant that stays with the derivative formula. Evaluating at time $t = 2$ gives

$$V'(2) = \frac{4}{3}\pi 3(r(2))^2 r'(2) = \frac{4}{3}\pi 3(3)^2(0.25) = 9\pi \text{ cubic inches per second.} \tag{21.6}$$

In summary, given at time 2 seconds a balloon being inflated has $r(2) = 3$ inches and $r'(2) = 0.25$ inches per second then the volume of the balloon is growing at a rate of 9π cubic inches per second. □

M-Box 21.1: Steps for Related Rates Problems

1. Identify the rate that needs to be found.
2. If appropriate draw a picture of the scenario and label it using the variables given in the problem.
3. Construct the relevant equation.
4. Find the derivative, often with respect to time, paying particular attention to any possible chain rule.
5. Substance the known values into the derivative equation.

6. Solve the equation for the desired rate. Note: We could solve for the rate in question before substituting in the known values first, but substituting in values first tends to be easier.
7. Answer the question with an appropriate sentence.

Example 21.2. *An extension ladder is extended to 25 feet and placed against the side of a house. The ladder was put at an angle of 60 degrees instead of the proper angle of 75 degrees. Due to this and a slick surface the ladder starts to slide away from the house. When the ladder is 24 feet from the house it is sliding away from the house at a rate of one foot per second. How fast is the ladders sliding down the house at this point?*

Solution. We will Follow the steps in M-Box 21.1. (1) We want to know the rate at which the ladder is sliding down the house. (2) A well labeled picture, figure 21.2, will help us here. We let $h(t)$ be the height of the ladder against the house and $d(t)$ the distance from the house to the base of the ladder. The goal is to find information about $h'(t)$. (4) We need to relate $h(t)$ and $d(t)$. Since the ladder is 25 feet long we use the Pythagorean theorem to get

$$(h(t))^2 + (d(t))^2 = 25^2. \tag{21.7}$$

Taking the derivative of equation 21.7, noting that we need to use the chain rule twice, we get

$$2(h(t))h'(t) + 2(d(t))d'(t) = 0. \tag{21.8}$$

(5) Although we were not given a specific time, we are told at some time t, $d(t) = 24$ feet, and $d'(t) = 1$ foot per second. At this time we now have

$$2(h(t))h'(t) + 2(24)(1) = 0. \tag{21.9}$$

To find $h'(t)$ we still need to know $h(t)$. Given the ladder is 25 feet, the hypotenuses, and the distance from the house is 24 feet, one of the legs, we use the Pythagorean theorem to solve

$$24^2 + (h(t))^2 = 25^2$$

to get $h(t) = 7$. We substitute this into equation 21.9 to get.

$$2(7)h'(t) + 2(24)(1) = 0. \tag{21.10}$$

(6) We solve equation 21.10 to get $h'(t) = -48/14 \approx= 3.43$ feet per second. Note the negative here is because the $h(t)$ is getting smaller as the ladder slides down the house to the ground. (7) When the ladder is 24 feet from the house and sliding away from the house at a rate of one foot per second, the ladder is sliding down the house at a rate of 3.43 feet per second. □

In solving related rates problems there are two steps to emphasize. In step (3) in M-Box 21.1 the relevant equation can really come from anywhere, but we note

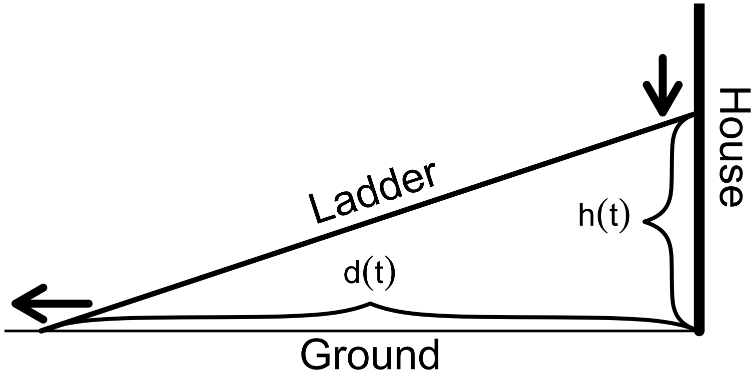

Fig. 21.2 A representation of a ladder sliding down a wall.

that if there is a good picture from step (2) this will help. We finding derivatives in step (5) the chain rule is highly likely. To help see the chain rule note that these two examples

$$\frac{d}{dx}(\sin(x))^9 = 9(\sin(x))^8 \cos(x)$$

and

$$\frac{d}{dx}(f(x))^9 = 9(f(x))^8 f'(x)$$

the role of $f(x)$ is the same as that of $\sin(x)$ with the only difference being that we have an explicit expression for the derivative of $\sin(x)$ and we can only express the derivative of $f(x)$ as $f'(x)$.

21.1 Exercises

1. Find the derivative with respect to t of the equation $f(t) = 5g(t) + 22$.
2. Find the derivative with respect to t of the equation $f(t) - 7g(t) = 62$.
3. Find the derivative with respect to t of the equation $f(t) = 6(h(t))^2 + 4h(t) - 15$.
4. Find the derivative with respect to t of the equation $f(t) = 7(h(t))^3 - 5h(t) + 94$.
5. Find the derivative with respect to t of the equation $g(t) = \sin(7h(t))$.
6. Find the derivative with respect to t of the equation $g(t) = \cos(4h(t))$.
7. Find the derivative with respect to t of the equation $6(h(t))^4 - 3(f(t))^3 = 8t$.
8. Find the derivative with respect to t of the equation $7(h(t))^7 + 4(f(t))^5 = 9t^2$.
9. Find the derivative with respect to t of the equation $\ln(h(t)) + e^{f(t)} = \sin(t)$.
10. Find the derivative with respect to t of the equation $f(t) + \sqrt{h(t)} = e^t$.

11. Find the derivative with respect to t of the equation $g(t)f(t) = 5t^9$.
12. Find the derivative with respect to t of the equation $g(t)/f(t) = 8t$.
13. A square has elastic sides and is expanding. If at time $t = 3$ the square is 16 square feet and the sides are increasing at a rate of 0.75 feet per second, then how fast is the area increasing?
14. An equilateral triangle elastic sides and is expanding. If at time $t = 2$ the triangle is 10 square feet and the sides are increasing at a rate of 0.5 feet per second, then how fast is the area increasing?
15. A balloon is in the shape of a cube and is expanding. At time $t = 5$ the balloon has a volume of 64 cubic inches and the sides are increasing at a rate of 3 inches per second. How fast is the volume increasing?
16. A balloon is in the shape of a sphere and is expanding. At time $t = 6$ the balloon has a radius of 6 inches and volume is increasing at a rate of 5 cubic inches per second. How fast is the radius increasing?
17. An extension ladder is extended to 17 feet and placed against the side of a house. The ladder starts to slide away from the house. When the ladder is 15 feet from the house it is sliding away from the house at a rate of six inches per second. How fast is the ladders sliding down the house at this point?
18. An extension ladder is extended to 29 feet and placed against the side of a house. The ladder starts to slide away from the house. When the ladder is 21 feet from the ground it is sliding down the house at a rate of 1.5 feet per second. How fast is the ladders sliding away from the house at this point?
19. A person 1.75 meters tall is walking away from a lamp post that 6 meters high at a speed of 1.2 meters per second. How fast is the person's shadow changing?
20. A person 1.5 meters tall is walking towards a lamp post that 8 meters high at a speed of 0.75 meters per second. How fast is the person's shadow changing?
21. The equation $q(t) = 20(p(t))^{-2}$ is the relationship between the quantity of widgets, $q(t)$, demanded given a price $p(t)$ in thousands of dollars. Supposed our widgets cost \$1000 and the price is increasing at a rate of \$500 per week. How fast is demand changing?
22. The equation $q(t) = \sqrt{200 - (p(t))^2}$ is the relationship between the quantity of widgets, $q(t)$ in millions of widgets, demanded given a price $p(t)$ in hundreds of dollars. Supposed our widgets cost \$500 and the price is increasing at a rate of \$100 per month. How fast is demand changing?

23. A rock is tossed into a still pond causing ripples to move out from where the rock landed in concentric circles. At what rate is the area of disturbed rate increasing when the radius of the outermost circle is 10 feet and increasing at a rate of 3 feet per second?

24. A spherical snowball is melting and the radius is increasing at a rate of 0.1 inch per second. How fast is the volume decreasing when the radius is 3 inches?

25. The formula for kinetic energy is $E_k = \frac{1}{2}mv^2$. Suppose a 1400 kg car is traveling at 18 m/s and accelerating at 4 m/s^2. How fast is the kinetic energy changing? Hint: The units of a joule, J, are $\text{kg} \cdot \text{m}^2/\text{s}^2$.

26. A car is traveling west towards a fixed point, P, at a speed of 80 km/hr. Another car is traveling north away from the fixed point, P, at a speed of 105 km/hr. If the car traveling west is 28 km from the fixed point and the car traveling north is 45 km from the fixed point, then how fast is the distance between the cars changing?

27. How fast is the area of a rectangle changing when the length is 20 in and decreasing at 2 in/s and the width is 8 in and increasing at 3 in/s?

28. How fast is the area of a triangle changing when the height is 10 in and increasing at 4 in/s and the base is 30 in and decreasing at 5 in/s?

Chapter 22
Surge Function

Figure 22.1 is from the Function Gallery chapter, but we repeat it here as it is central to this chapter. In the chapter our goal is to understand the surge function in general, a function of the form

$$f(t) = ate^{-bt},$$

and then study the impact of repeated doses of a chemical over time. For example, in figure 22.1 we see the data and model for one dose of ethanol, but what happens when another dose is taken, say, an hour later, and then again in another hour? We begin more generally or abstractly and then study a few specific examples in the exercises.

A model for the concentration of a drug in the blood stream after taking the drug at time $t = 0$ is given by

$$f(t) = ate^{-bt}.$$

Three examples of this function, or really family of functions based on a and b, with different values of a and b are graphed in figure 22.2. Notice how the function reaches its maximum fairly quickly, the surge, but then decreases more slowly. R Code box 22.1 defines the basic surge functions with parameters a and b. In the second line we evaluate the function with $a = 1$ and $b = 1$, which is the solid black curve in figure 22.2, at time $t = 2$. Notice that in figure 22.2 the solid black curve is about 0.27 when $t = 2$. We think of a and b as fixed parameters associated with, for example, a particular drug and t is the variable. Note the fitted curve in figure 22.1 gives us $a = 1.76393642046205$ and $b = -1.05841662684339$. If we want to graph the surge function from $t = 0$ to $t = 10$ with $a = 1$ and $b = 1$ we would graph **Surge(1,1,t)** within curve as **curve(Surge(1,1,x),0,10,lwd=2)**.

R Code 22.1: Basic Surge Function

```
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge(1,1,2)
[1] 0.2706706
```


T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_22

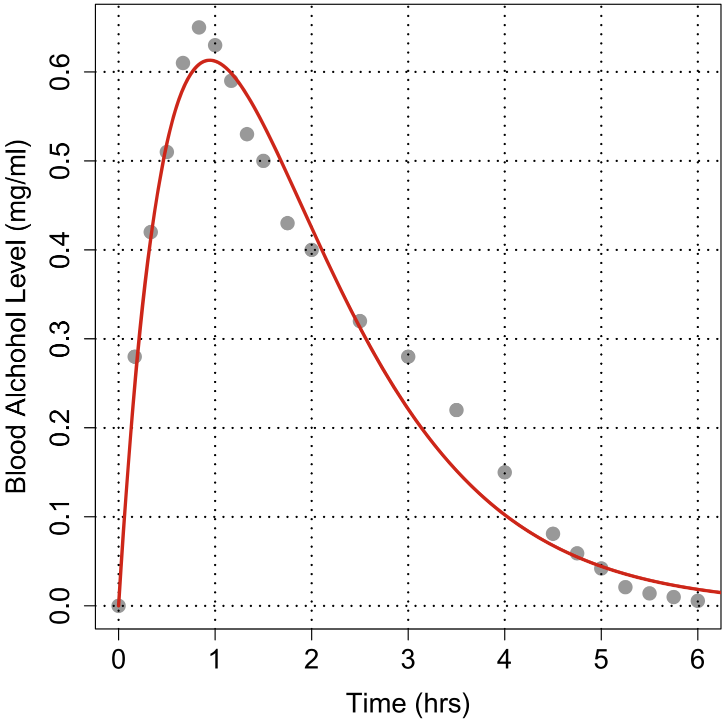

Fig. 22.1 Blood alcohol concentration of eight fasting adult males after consuming a 95% ethanol oral dose of 45ml. [38] The curve is known as a surge function and given by $S(x) = 1.76393642046205xe^{-1.05841662684339x}$. Note that driving impairment begins around 0.5 and the legal limit in most states is 0.08. The dosage here is equivalent to about 2 pints of 5% beer or 2.5 shots of vodka.

M-Box 22.1: The Surge Function

The surge function $f(t) = ate^{-bt}$ has input unit of time (hours is typical), with $t = 0$ the time the drug is ingested, and output units of mg/ml (milligram per milliliter of blood is typical). The maximum of the functions is at $(1/b, a/(be^1))$.

A single surge function models the amount of a drug in the blood when the drug is taken once, but medicine is often taken multiple times a day. We wish to model and analyze the amount of the drug in the blood when the drug is ingested multiple times a day. The modeling process for this is challenging and it is easy to lose site of the calculus. The calculus comes into play once we have the model and we wish

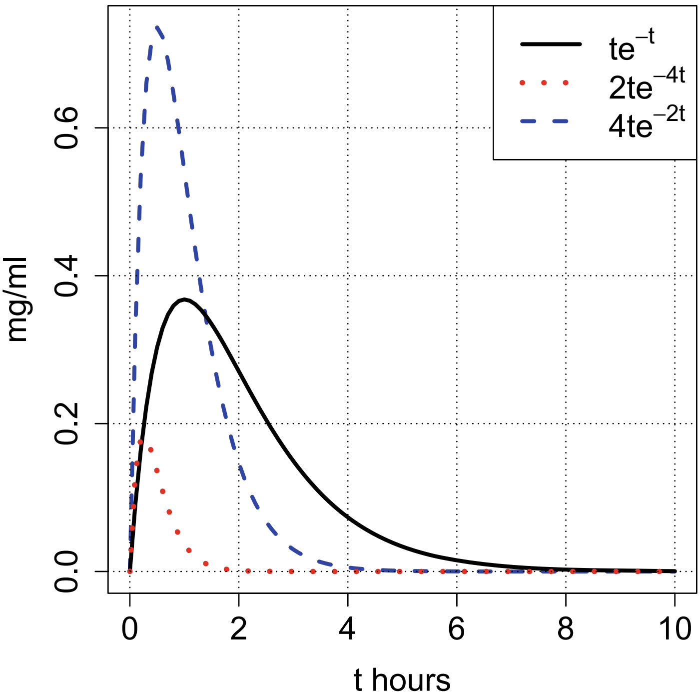

Fig. 22.2 Three examples of a surge function.

to know the maximum amount of drug in the blood stream. Moreover, we will need R to find these maximums. We now move to building the model.

To build the model we need to shift the surge function by t_s time units. For example, if a pill is taken at time 6 hours we need to adjust the function

$$f(t) = ate^{-bt}$$

as it assumes the drug is ingested at time $t = 0$. To correct for this we compose $f(t)$ with $g(t) = t - t_s$, where, for example, if the drug is taken 6 hours since time 0 then $g(t) = t - t_s = t - 6$. See figure 22.3 for an example with a shift of 6 hours and notice that we use the function **Surge(4,2,t-6)** with t replaced by $t - 6$ for the composition. Note that the function is 0 at time 6 hours as it should be and the shape of the function after 6 hours it appears correct. The problem is that before 6 hours the function is negative and that makes no sense. One last fix is to create a piecewise function that is 0 for values less than 6. For example,

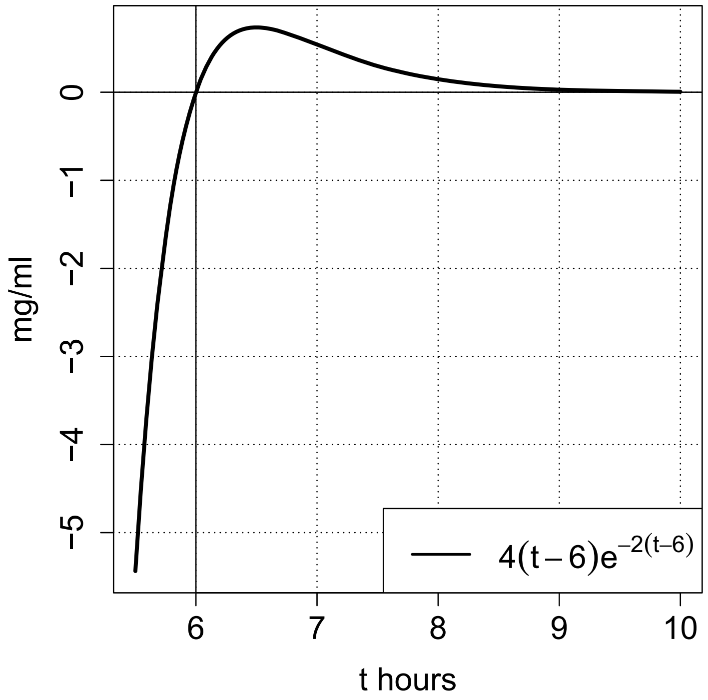

Fig. 22.3 A surge function shifted by 6 hours.

$$f(t) = \begin{cases} 0 & t \leq 6 \\ 4(t-6)e^{-2(t-6)} & t > 6 \end{cases} \tag{22.1}$$

To create a piecewise function in R we use an **ifelse(A,B,C)** function. In **ifelse(A,B,C)**, A is a logical statement that is either true or false. If it is true than do B, otherwise do C. An example is give in R Code box 22.2, where the Surge_Piece function has an **ifelse()** function. In this case if $x < s$ (we use s as the shift to the next drug dosage or spacing between doses) the functions returns 0 otherwise it returns the value of the shifted Surge function. In Surge_Piece(4,2,6,5), we have $a = 4$, $b = 2$, with a shift of $s = 6$ and evaluate the function at time 5. The output is 0 since $5 < 6$. In the second, Surge_Piece(4,2,6,7), we get a non-zero value since $7 > 6$. A graph of this function is given in figure 22.4.

R Code 22.2: Piecewise Surge Function

```
> Surge_Piece<-function(a,b,s,t)
{ifelse(x<s,0,Surge(a,b,t-s) )}
```

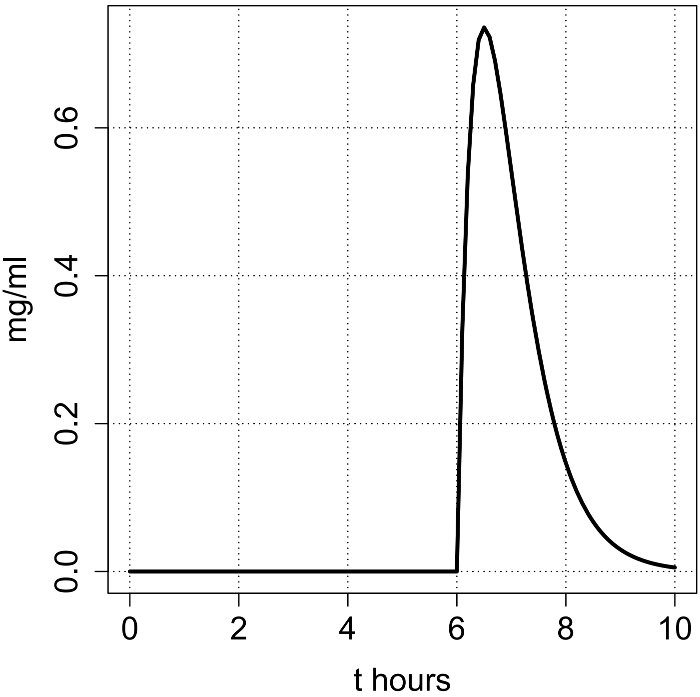

Fig. 22.4 Graph of the function given in equation 22.1.

```
> Surge_Piece(4,2,6,5)
[1] 0
> Surge_Piece(4,2,6,7)
[1] 0.5413411
```

Our final goal is to add these functions to model the drug levels in the blood over the course of a day. Suppose the drug has parameters $a = 1$ and $b = 0.5$. We let time 0 represent 8am and the drug is taken at 8am, 2pm, and 8pm. A graph of this scenario is given in figure 22.5. R Code box 22.3 answers the question of what is the maximum amount of drug in the blood during this 24 hour period. It appears that the drug maximum is 0.905 mg/ml at time 13.69 hours (about 9:30pm).

Some comments on R Code box 22.3: The function **h** depends on both **Surge** and **Surge_Piece** being defined and run. Those two functions are the first two lines to stress this point. For the sake of clarity we define **a**, **b**, and **s** before we define the function **h** so that **h** now has only the one variable t when it is defined. When **h** is defined we use **Surge(a,b,t)** but note that would be the same as **Surge_Piece(a,b,0*s,t)**, which would follow the pattern of using **1*s** (we do not type the 1) and then **2*s**. The idea here is, for example, the **2*s** represents taking the third dose two time periods (shift or spacing) since the first dose. Lastly, if we want to model only taking two doses then **h** would only have **Surge(a,b,t)** and **Surge_Piece(a,b,s,t)**. Similarly, if we wanted to model four doses we would add in a **Surge_Piece(a,b,3*s,t)**, etc.

One technical note is that in figure 22.5 the local minimums are at "corners" and the derivative is undefined. The **uniroot.all** function is not solving for when the derivative is 0 algebraically. It is using an algorithm to identify points where the value of the derivative is negative on one side and positive on the other. In the case of these "corners" this works well as the derivative is negative to the left and positive to the right at these locations. On the other hand, the derivative does not exist at these points as seen by the fact that the secant line slopes to the left and to the right will not converge to the same value. Yet the second lines on the left will converge to a value and the ones on the right will converge to a value; they just differ. This could lead to a discussion of one sided limits, but that is beyond the scope of this text yet worth mentioning for completeness.

R Code 22.3: Adding Surge Functions

```
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge_Piece<-function(a,b,s,t)
{ifelse(t<s,0,Surge(a,b,t-s) )}
> a<-1
> b<-0.5
> s<-6
> h<-function(t){Surge(a,b,t)+Surge_Piece(a,b,s,t)+
  Surge_Piece(a,b,2*s,t)}
> h_p<-Deriv(h)
> roots_p<-uniroot.all(h_p,c(0,24))
> roots_p
[1]  1.999886  5.999580  7.715511 11.999587 13.687851
> h(roots_p)
[1] 0.7357589 0.2987642 0.8904860 0.3285137 0.9049905
```

22.1 Exercises

Note: Most of the exercises are designed to get comfortable working with the surge function. In these case a and b are not applied to any particular scenario. The last few exercises use value of a and b derived from modeling data.

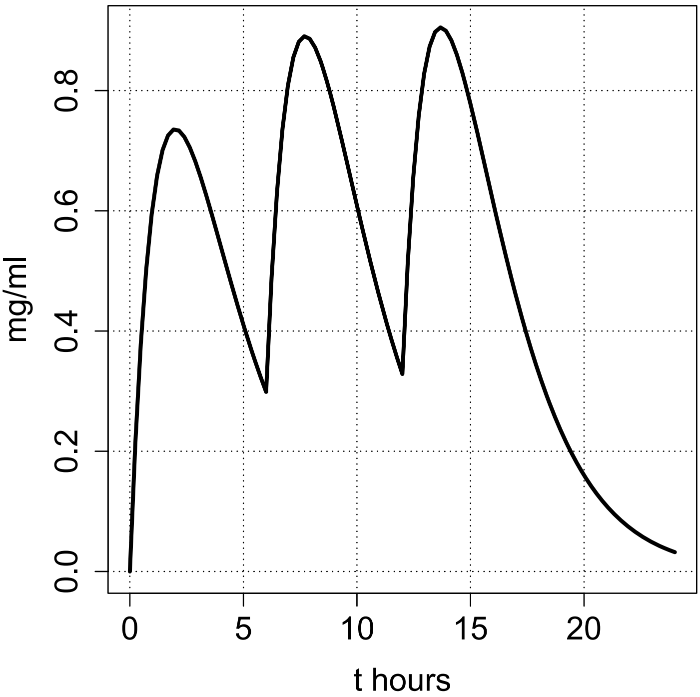

Fig. 22.5 Graph of the sum of three surge function with $a = 1$ and $b = 0.5$ and time 0 representing 8am; modeling taking the drug at 8am, 2pm, and 8pm.

1. Find $f(-2)$, $f(2)$, and $f(10)$.

$$f(t) = \begin{cases} 5 & t \leq 0 \\ x^2 & t > 0 \end{cases}$$

2. Find $f(-3)$, $f(1)$, and $f(8)$.

$$f(t) = \begin{cases} 3x - 5 & t \leq 3 \\ 4 & t > 3 \end{cases}$$

3. Find $f(-1)$, $f(2)$, and $f(5)$.

$$f(x) = \begin{cases} 3x^3 & x < 4 \\ 2^x & x \geq 4 \end{cases}$$

4. Find $f(-5)$, $f(2)$, and $f(7)$.

$$f(x) = \begin{cases} 5x - 2 & x < -2 \\ 2x - 8 & x \geq -2 \end{cases}$$

5. Find $g(-8)$, $g(9)$, and $g(0)$.

$$g(x) = \begin{cases} x^3 - 2 & x < -5 \\ 6 & -5 \leq x < 9 \\ \sqrt{x} & x \geq 9 \end{cases}$$

6. Find $g(-2)$, $g(5)$, and $g(4)$.

$$g(x) = \begin{cases} 8 & x \leq 0 \\ \sqrt{x} & 0 < x \leq 4 \\ x^2 - 5 & x > 4 \end{cases}$$

7. Find $f(-5)$, $f(8)$, and $f(3)$ if f<–function(x){ifelse($x < 3, 0, x^2$)}
8. Find $f(6)$, $f(-2)$, and $f(-4)$ if f<– function(x){ifelse($x < -2, 7, 3x - 4$)}
9. Find $f(-2)$, $f(2)$, and $f(0)$ if f<– function(x){ifelse($x > 0, 2x, x^3$)}
10. Find $f(7)$, $f(-6)$, and $f(5)$ if f<– function(x){ifelse($x > 5, x^2, 4x - 5$)}
11. Find $g(3)$, $g(2)$, and $g(-1)$ if g<– function(x){ifelse($x \geq 2, x, x^2 + 1$)}
12. Find $g(4)$, $g(-2)$, and $g(-1)$ if g<– function(x){ifelse($x \leq -1, x^3, \sqrt{x}$)}
13. Prove the maximum of the surge function $f(t) = ate^{-bt}$ is at $(1/b, a/(be^1))$.
14. Find the inflection point of $f(t) = ate^{-bt}$.
15. A surge function models the drug in the blood stream in mg/ml with parameters $a = 6$ and $b = 0.5$. If the drug is taken at $10am$, how much of the drug is in the blood system at 2pm? When does the maximum amount in the blood occur and how much will be in the blood stream at that time?
16. A surge function models the drug in the blood stream in mg/ml with parameters $a = 3$ and $b = 0.4$. If the drug is taken at $3pm$, how much of the drug is in the blood system at 4pm? When does the maximum amount in the blood occur and how much will be in the blood stream at that time?
17. A surge function models the drug in the blood stream in mg/ml with parameters $a = 2$ and $b = 0.75$. If the drug is taken at $8am$, how much of the drug is in the blood system at 11am? When does the maximum amount in the blood occur and how much will be in the blood stream at that time?
18. A surge function models the drug in the blood stream in mg/ml with parameters $a = 10$ and $b = 0.3$. If the drug is taken at noon, how much of the drug is in the blood system at 8pm? When does the maximum amount in the blood occur and how much will be in the blood stream at that time?
19. A surge function models the drug in the blood stream in mg/ml with parameters $a = 10$ and $b = 0.5$. If the drug is taken at $6am$, how much of the drug is in the blood system at 11:30am? When does the maximum amount in the blood occur and how much will be in the blood stream at that time?
20. A surge function models the drug in the blood stream in mg/ml with parameters $a = 15$ and $b = 0.4$. If the drug is taken at 2:00pm, how much of the drug is in the blood system at 4:15pm? When does the maximum amount in the blood occur and how much will be in the blood stream at that time?

21. A surge function models the drug in the blood stream in mg/ml with parameters $a = 20$ and $b = 0.8$. If the drug is taken at $10:00pm$, how much of the drug is in the blood system at 1:45am? When does the maximum amount in the blood occur and how much will be in the blood stream at that time?
22. A surge function models the drug in the blood stream in mg/ml with parameters $a = 5$ and $b = 0.2$. If the drug is taken at 6:00pm, how much of the drug is in the blood system at 9:45pm? When does the maximum amount in the blood occur and how much will be in the blood stream at that time?

The remaining problems require R.

23. How does the parameter a change the surge function? With $b = 0.5$ graph a few surge functions with different values of a and explain how a changes the surge function.
24. How does the parameter b change the surge function? With $a = 10$ graph a few surge functions with different values of b and explain how b changes the surge function.
25. A drug with surge parameters $a = 8$ and $b = 0.5$ is taken at 8am and 11am. How much of the drug is in the blood stream at 9am and 4pm? What is the maximum amount of drug in the blood stream and when does it happen?
26. A drug with surge parameters $a = 6$ and $b = 0.75$ is taken at 10am and four hours later. How much of the drug is in the blood stream at 2pm and 8pm? What is the maximum amount of drug in the blood stream and when does it happen?
27. A drug with surge parameters $a = 5$ and $b = 0.2$ is taken at 6am and six hours later. How much of the drug is in the blood stream at 9pm? What is the maximum amount of drug in the blood stream and when does it happen?
28. A drug with surge parameters $a = 10$ and $b = 0.4$ is taken at 10am and seven hours later. How much of the drug is in the blood stream at 10pm? What is the maximum amount of drug in the blood stream and when does it happen?
29. A drug with surge parameters $a = 8$ and $b = 0.5$ is expected to be taken a second time seven hours after the first dose. Suppose the drug is taken a second time in five hours instead of seven. How much higher is the maximum amount in the blood stream? Report your result as a difference and a percent change. Should there be a concern in taking the drug two hours earlier?
30. A drug with surge parameters $a = 10$ and $b = 0.75$ is expected to be taken a second time six hours after the first dose. Suppose the drug is taken a second time in five hours instead of six. How much higher is the maximum amount in the blood stream? Report your result as a difference and a percent change. Should there be a concern in taking the drug one hour earlier?

31. A drug with surge parameters $a = 20$ and $b = 0.5$ is first taken at 8am and then six and 12 hours later. How much of the drug is in the blood stream at midnight? What are the locations and amounts of drug in the blood at the local maximums?
32. A drug with surge parameters $a = 12$ and $b = 0.4$ is first taken at 10am and then five and ten hours later. How much of the drug is in the blood stream at 10pm? What are the locations and amounts of drug in the blood at the local maximums?
33. A drug with surge parameters $a = 10$ and $b = 0.4$ is taken three times at 2-hour intervals. How much higher is the global maximum compared to the first maximum? Does this suggest that 3-hour intervals should be extended? Explain.
34. A drug with surge parameters $a = 12$ and $b = 0.35$ is taken three times at 4-hour intervals. How much higher is the global maximum compared to the first maximum? Does this suggest that 4-hour intervals should be extended? Explain.
35. A drug with surge parameters $a = 15$ and $b = 0.5$ is taken three times at 9-hour intervals. The minimum dosage to be effective in the blood stream is 2.25mg/ml. Is this protocol effective? If not, suggest a change in the time between doses. Your response to this should be due to exact values not simply reading a graph.
36. A drug with surge parameters $a = 10$ and $b = 0.4$ is taken three times at 8-hour intervals. The minimum dosage to be effective in the blood stream is 4 mg/ml. Is this protocol effective? If not, suggest a change in the time between doses. Your response to this should be due to exact values not simply reading a graph.
37. It is suggested that a drug with surge parameters $a = 15$ and $b = 0.25$ be taken every six hours. Based on three doses does the six-hour recommendation make sense? Base your answer on the values of the local maximums and minimums.
38. It is suggested that a drug with surge parameters $a = 10$ and $b = 0.4$ be taken every three hours. Based on three doses does the three-hour recommendation make sense? Base your answer on the values of the local maximums and minimums.
39. A drug is going to be taken with an initial loading dose so that the parameter $a = 20$ for the first dose but $a = 10$ for the next two doses. Assume that $b = 0.5$ in all cases and the drug is taken at 8-hour intervals. Explain the result of this model including comparing the local maximums and minimums values and times.
40. A drug is going to be taken with an initial loading dose so that the parameter $a = 15$ for the first dose but $a = 5$ for the next two doses. Assume that $b = 0.4$ is all cases and the drug is taken at 8-hour intervals. Explain the result of this model including comparing the local maximums and minimums values and times.

The data in figure 22.1 is from the paper *Pharmacokinetics of Ethanol After Oral Administration in the Fasting States* and in the paper there is data for three other initial consumption scenarios; 95% ethanol oral dose of 15ml, 30ml, and 60ml. [38] **The values of a and b in the following problems are from curve fitting this data. Note that the data was collected on adult males and there is considerable variation in how alcohol impacts different people. The problems below are meant to provide general information on alcohol consumption.**

41. In the 95% ethanol oral dose of 15ml the parameters for the surge function are $a = 1.10493682724572$ and $b = 2.50572657235535$. The amount of alcohol in a 95% ethanol oral dose of 15ml is about 0.8 of a standard shot of vodka and about 8oz of 6% beer. Suppose someone consumes this amount of alcohol every 15 minutes. What are the fewest number of doses necessary to reach an impaired state, 0.5mg/ml? What is the maximum alcohol level in this case and when does it occur?

42. In the 95% ethanol oral dose of 30ml the parameters for the surge function are $a = 1.47206184750394$ and $b = 1.4203887662126$. The amount of alcohol in a 95% ethanol oral dose of 30ml is about 1.6 standard shots of vodka and about 16oz of 6% beer. A person confidently states they can have three doses (drinks) of this amount spaced out every 45 minutes and be safe to drive two hours after the first dose (the limit is 0.8mg/ml). Does the model say this is a true statement? If not, what is the time reaches 0.8mg/ml and when does it drop below 0.8mg/ml? Also, find the maximum blood concentration and the time of the maximum.

43. In the 95% ethanol oral dose of 45ml the parameters for the surge function are $a = 1.76393642046205$ and $b = 1.05841662684339$. The amount of alcohol in a 95% ethanol oral dose of 45ml is about 2.5 standard shots of vodka and about one and a half 16oz of 6% beer. A person has three doses two hours apart. What is the maximum amount of alcohol in the body and when does it occur? At what point in the night does the concentration in the blood stream fall below 0.8 permanently? At what time is the alcohol being removed from the blood stream the fastest and what is that rate?

44. In the 95% ethanol oral dose of 60ml the parameters for the surge function are $a = 1.63386311838393$ and $b = 0.784910359900638$. The amount of alcohol in a 95% ethanol oral dose of 60ml is about 3.25 standard shots of vodka and about two 16oz of 6% beer. A person consumes three doses with 1.5 hours between doses. Blackouts are likely when the blood alcohol level reaches 1.5 mg/ml. [9] At what time is this person first likely to blackout? According to the model when does the blood alcohol level fall below 1.5 mg/ml and at that time how fast is blood alcohol level decreasing? Also, find the maximum blood concentration and the time of the maximum.

Chapter 23
Differential Equations - Preliminaries

A differential equation is an equation involving a function or functions and their derivatives. Differential equations are used to model real world phenomenon. Before analyzing various differential equation models we have some preliminary work to do. In building differential equation we will use the language of variables beginning proportional. In analyzing differential equation we will use a for loop in R. We cover both of these concepts in this chapter. First is M-Box 23.1 which translates the statement x is proportional to y into the equation $x = ky$ for some constant k. This will be helpful in translating statements about real world situation into differential equations to analyze. Note that there is not anything particularly special about using k as the constant. Other letters can be used especially if they make more sense in context, such as say r if the context is related to a growth rate.

M-Box 23.1: x is proportional to y

If x is proportional to y then we get the equation $x = ky$ for some constant k. Similarly, if $x = ky$ for some constant k the we say x is proportional to y.

The main idea we will use to analyze our differential equations will be to use the microscope equation repeatedly over very small time increments. To do this efficiently we will take advantage of a for loop, which is common to almost all programming languages. Our first example of a for loop in R is in R Code box 23.1.

The code begins by setting **A** to the value 1 and **B** to the value 2. The third line starts the for loop with **for (i in 1:5)**, which states that the variable **i** will range through the values 1 through 5 or the vector $(1, 2, 3, 4, 5)$. Recall that the colon command **a:b** in R creates a vector of numbers starting at **a** and increasing by unit until **b** is exceeded. What this means is that anything between the beginning brace { and ending brace } will be run first for $i = 1$, then $i = 2$, then $i = 3$, then $i = 4$, and finally for $i = 5$.

As we enter the loop the first time with $i = 1$, **A<–A+1** will set the value of **A** to what the value of **A** is currently plus 1. Hence **A** is now equal to 2. The next line

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is **B<−A+B** and note that the value of **A** is now 2 from the line above while the current value of **B** is also 2. Adding the two sets **B** to 4. We reach the end brace } which loops us back to the beginning and moves **i** to the next value which is 2.

We proceed to run the two lines of code with $i = 2$, $A = 2$, and $B = 2$. The value of **A** will become 3 and the value of **B** will become 7. We reach the end brace } which loops us back to the beginning and moves **i** to the next value which is now 3.

We proceed to run the two lines of code with $i = 3$, $A = 3$, and $B = 7$. The value of **A** will become 4 and the value of **B** will become 11. We reach the end brace } which loops us back to the beginning and moves **i** to the next value which is now 4.

We proceed to run the two lines of code with $i = 4$, $A = 4$, and $B = 11$. The value of **A** will become 5 and the value of **B** will become 16. We reach the end brace } which loops us back to the beginning and moves **i** to the next value which is now 5.

We proceed to run the two lines of code with $i = 5$, $A = 5$, and $B = 16$. The value of **A** will become 6 and the value of **B** will become 22. We reach the end brace } which loops us back to the top but we have exhausted all values for **i** and we go back to the end brace } and continue with the code. The last lines simply display the final values of **A** and **B**.

R Code 23.1: A Basic For Loop I

```
> A<-1
> B<-2
> for (i in 1:5){
+ A<-A+1
+ B<-A+B }
> A
[1] 6
> B
[1] 22
```

We have one more example of a for loop in R Code box 23.2. In this case the variable of the for loop **i** is part of the calculation within the loop. The code begins by setting **A** to the value 1. The second line starts the for loop with **for (i in 1:4)**, which states that the variable **i** will range through the values 1 through 4 or the vector $(1, 2, 3, 4)$.

As we enter the loop the first time with $i = 1$, **A<−A+i** will set the value of **A** to what the value of **A** is currently plus the current value of i which is 1. Hence **A** is now equal to 2. We reach the end brace } which loops us back to the beginning and moves **i** to the next value which is 2.

We proceed to run the one line of code with $i = 2$, $A = 2$. The value of **A** will become 4; the current value of **A** plus the current value of **i** or $2 + 2 = 4$. We reach the end brace } which loops us back to the beginning and moves **i** to the next value which is now 3.

We proceed to run the one line of code with $i = 3$, $A = 4$. The value of **A** will become 7; the current value of **A** plus the current value of **i** or $3 + 4 = 7$. We reach the end brace } which loops us back to the beginning and moves **i** to the next value which is now 4.

We proceed to run the one line of code with $i = 4$, $A = 7$. The value of **A** will become 11; the current value of **A** plus the current value of **i** or $4 + 7 = 11$. We reach the end brace } which loops us back to the beginning but we have exhausted all values for **i** and we go back to the end brace } and continue with the code. The last lines simply display the final value of **A**.

R Code 23.2: A Basic For Loop II

```
> A<-1
> for (i in 1:4){
+ A<-A+i }
> A
[1] 11
```

One tip that can help in understanding for loops and possibly allow you to debug code. You can add, for example, with R code 23.2, **print(A)** within the for loop to see how **A** changes at each step. In fact, you can add **print(A)** before and after the line **A<–A+i** if that helps. Try it once so you see how it works.

23.1 Exercises

Translate the statement into an equation.

1. A is proportional to B.
2. W is proportional to R.
3. x is proportional to the inverse of y.
4. The inverse of x is proportional to y.
5. x is proportional to the square of y.
6. x is proportional to the cube of y.
7. A is proportional to the product of B and C.
8. A is proportional to the quotient of B and C.
9. x is proportional to the difference of y and z.
10. x is proportional to ratio of y and z minus 1.
11. $f(t)$ is proportional to t.
12. $f(t)$ is proportional to the square of t.
13. The derivative of $f(t)$ is proportional to $f(t)$.
14. The derivative of $f(t)$ is proportional to the square of $f(t)$.
15. The derivative of $f(t)$ is proportional to the product of t and $f(t)$.
16. The derivative of $f(t)$ is proportional to the quotient of t and $f(t)$.
17. The derivative of $f(t)$ is proportional to the difference of $f(t)$ and A.
18. The derivative of $f(t)$ is proportional to the square of the difference of $f(t)$ and B.

19. The second derivative of a function is proportional to the function.
20. The sum of the first and second derivative of a function is proportional to the function.
21. The cube of the difference between the second derivative of a function and the constant B is proportional to the function.
22. The square of the second derivative of a function is proportional to the first derivative of the function.
23. The second derivative of a function is proportional to the square of the third derivative.
24. The square of the sum of the first three derivatives of a function is proportional to the function.

First find the final value of the variable in the question then run the code in R and check your answer.

25. Find the final value of A.
```
A<- 1
for (i in 1:5){
A<- A+3 }
```
26. Find the final value of A.
```
A<- 3
for (i in 1:5){
A<- A+2 }
```
27. Find the final value of A.
```
A<- 5
for (i in 1:6){
A<- A+5 }
```
28. Find the final value of A.
```
A<- 5
for (i in 1:4){
A<- A*2 }
```
29. Find the final value of A and B.
```
A<- 5
B<- 2
for (i in 1:5){
A<- A+2
B<- A+B }
```
30. Find the final value of A and B.
```
A<- 4
B<- 0
for (i in 1:3){
A<- A+5
B<- A+B }
```
31. Find the final value of A.
```
A<- 3
for (i in 1:5){
A<- A+i }
```
32. Find the final value of A.
```
A<- 1
for (i in 1:4){
A<- A+2*i }
```
33. Find the final value of A.
```
A<- 5
for (i in 1:3){
A<- A+5*i }
```
34. Find the final value of A.
```
A<- 48
for (i in 1:4){
A<- A/i }
```
35. Find the final value of A and B.
```
A<- 4
B<- 0
for (i in 1:3){
A<- A+2*i
B<- A*B }
```
36. Find the final value of A and B.
```
A<- 1
B<- 0
for (i in 1:4){
A<- A+3*i
B<- A*B }
```

37. Find the final value of A and B.
```
A<– 10
B<– 0
for (i in 2:5){
A<– (i-1)*A
B<– (i+1)*B+A }
```

38. Find the final value of A and B.
```
A<– 5
B<– 1
for (i in 3:5){
A<– (i-1)*A
B<–(i+1)*A }
```

39. Find the final value of A, B, and C.
```
A<– 1
B<– 2
C<– 3
for (i in 2:4){
A<– A*B+i
B<– A*B-i
C <– A+B+C+2*i }
```

40. Find the final value of A, B, and C.
```
A<– 0
B<– 2
C<– 4
for (i in 3:5){
A<– A*(B-1)+i
B<– (A+1)*B-i
C <– A+B+i*C }
```

41. Create a for loop that sums the first five integers $(1 + 2 + 3 + 4 + 5)$. Check your work in R and then create a for loop that sums the first 100 integers. Use R to find the sum.

42. Create a for loop that sums the square of the first five integers $(1^2 + 2^2 + 3^2 + 4^2 + 5^2)$. Check your work in R and then create a for loop that sums the square of the first 100 integers. Use R to find the sum.

43. Create a for loop that sums the reciprocals of the first five integers $\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)$. Check your work in R and then create a for loop that sums the reciprocals of the first 100 integers. Use R to find the sum. Now do the sum for 1000, 10000, etc. Conjecture as to whether or not you think the sum will go to infinity or not as you increase the length of the sum.

44. Create a for loop that sums the reciprocals of the square of the first five integers $\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}\right)$. Check your work in R and then create a for loop that sums the square of the reciprocals of the first 100 integers. Use R to find the sum. Now do the sum for 1000, 10000, etc. Conjecture as to whether or not you think the sum will go to infinity or not as you increase the length of the sum.

Chapter 24
Differential Equations - Population Growth Models

One of the simplest differential equation models starts with the observation that a population will grow at a rate proportional to its size, assuming no resource limitations. In other words, the larger the population the faster it grows. In an equation, if $P(t)$ is the population size at time t then "a population will grow," $P'(t)$, "proportional to its size," $P(t)$, gives

$$P'(t) = rP(t),$$

where r is the per capita (per person) growth rate. For example, if $P(t)$ is the number of people in the population with t in years then if, say, $r = 0.1$ each person in the population increases the population size by 0.1 per year. In essence, r is really the difference of the per person growth/birth rate, b, minus the per person death rate, d, where the rates are constant and do not depend on the population size (this is where the assumption of no resource limitations is used). If we now check units, $P'(t)$ is people per year and $rP(t)$ is per year times people, which is people per year. Our goal now is to determine what we can say about the function $P(t)$ given this differential equation. This model is given in M-Box 24.1.

As an example we will assume $r = 0.1$ and at time $t = 0$ our population size is 500 people or $P(0) = 500$. We will estimate the population size one year later or when $t = 1$, in other words estimate $P(1)$. Recall the microscope equation $\Delta y \approx f'(a)\Delta x$ or for this example $\Delta y \approx P'(0)\Delta t$. Now

$$P'(0) = rP(0) = 0.1(500) = 50,$$

which we get from the differential equation, and $\Delta t = 1 - 0 = 1$. Hence

$$\Delta y \approx P'(0)\Delta t = 50(1) = 50$$

and our population increases by approximately 50 people from time $t = 0$ to $t = 1$ and so

$$P(1) \approx P(0) + P'(0)\Delta y = 500 + 50 = 550.$$

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In one line, we can take the general microscope equation

$$P(t_1) - P(t_0) \approx P'(t_0)(t_1 - t_0)$$

and write it as

$$P(t_1) \approx P'(t_0)(t_1 - t_0) + P(t_0),$$

which is simply using a tangent line of $P(t)$ at t_0 and evaluating it at t_1. We will use this idea repeatedly.

How good is this estimate and how can we improve the estimate? Recall that with the tangent line problems the larger the step from the location of the tangent line the worse the estimate. So instead of taking a step size of one year at once let us take two half year steps.

$$P(0.5) \approx P'(0)\Delta t + P(0) = rP(0)\Delta t + P(0) = 0.1(500)(0.5) + 500 = 525 \tag{24.1}$$

$$P(1) \approx P'(0.5)\Delta t + P(0.5) = rP(0.5)\Delta t + P(0.5) = 0.1(525)(0.5) + 525 = 551.25 \tag{24.2}$$

Note that equation 24.2 uses the value of $P(0.5)$ estimated in equation 24.1 to estimate $P(1)$ stepping from $t = 0.5$ to $t = 1$. Other than this the two equations 24.1 and 24.2 are the same use of the microscope equation but over different time periods.

M-Box 24.1: Uninhibited Growth Model

Let $P(t)$ be the size of a population at time t. If the population grows proportional to its size, then the **initial value equations** for this model are

$$P'(t) = rP(t) \qquad P(0) = P_0.$$

In this case, by taking two half steps instead of one step our estimate for $P(1)$ increased by 1.25 people, due to the fact that people added to the population at time $t = 0.5$ increase the growth of the population. Can we do better? Yes, we can try four-quarter steps or ten one-tenth steps. Once our estimation stabilizes to the accuracy we can accept then we can stop. In other words, this is a successive approximation situation which is best done in R. This process of estimation is referred to as Euler's Method. In R we first generate the data in R Code box 24.1, which will use the scenario above but with step sizes of 0.01 for the example above. We will then look at R code to find the final value and graph the result. Note that you will find this code on the companion web site sustainabilitymath.org/acr.

R Code 24.1: Uninhibited Growth Data

```
> ## Define starting values and variables.
>
> t_initial <- 0
> t_final <- 1
```

```
> y_0 <- 500
> r <- 0.1
> delta_t <- 0.01
>
> ## Set up value for the For Loop.
>
> t <- t_initial
> y <- y_0
> # Initializing the time vector with starting value
> t_data <- t
> # Initializing the y vector with starting value
> y_data <- y
> number_steps <- (t_final-t_initial)/delta_t
>
> ## Run the For Loop.
>
> for (i in 1:number_steps){
+   y_prime <- r*y # The differential equation
+   y <- y_prime*delta_t+y # Microscope equation
+   t <- t+delta_t # Increments t to its next value
+   t_data[i+1] <- t # Add new value to time vector
+   y_data[i+1] <- y # Add new value to pop. vector
+ }
```

In R Code box 24.1 the first five lines of code define the set up of the model. We set the starting time **t_initial** to 0, the ending time **t_final** to 1, the initial population **y_0** to 500, the growth rate **r** to 0.1, and the step size **delta_t** to 0.01. These values need to be changed for different models, but the rest of the code stays fixed. Before entering the for loop we set **t** to the value of **t_initial** and **y** to the value of **y_0**. We do this because **t** and **y** will change within the for loop. We also initialize two new variables, **t_data** and **y_data**, with the starting values of **t** and **y**. The variables **t_data** and **y_data** will store the results at each step of the process within the for loop.

The variable **number_steps** is how many iterations of the microscope equations will be needed with the given step size and interval length. For example, in equations 24.1 and 24.2 we used the microscope equation twice, two steps, to go from time 0 to time 1 with steps of size 0.5 or

$$\frac{1-0}{0.5} = 2.$$

In general, given a time interval, **t_final** - **t_initial**, and a step size, **delta_t**, the number of steps is

$$\frac{\textbf{t\_final} - \textbf{t\_initial}}{\textbf{delta\_t}},$$

which is set to **number_steps**.

The for loop uses **i** to range from 1 to the **number_steps**. There are five lines within the for loop which are started with { and ended with }. The first line, **y_prime<– r*y** is the differential equation model that computes the value of the derivative given the current value of **y**. For example, the first time we enter the loop, $i = 1$, the value of **y** is 500 and so **y_prime** = 0.1(500)=50. The next line is the microscope equation. In words and equations, we take the current slope, **y_prime**, multiply it by the step size or how far forward we are predicting, **y_prime*delta_t**, and add it to our current value, **y**, and this is the new value of **y**. In comparing to equation 24.1 as we enter the loop the first time, **y** starts out as $P(0)$ and then become $P(0.01)$ since our step size is 0.01. The next time through the loop will move **y** from $P(0.01)$ to $P(0.02)$, similar to equation 24.2.

We then increase the value of **t** with by one step size with **t<–t+delta_t** increases the value of **t** by one step size. The last two lines append the new values of **t** and **y** to the vectors **t_data** and **y_data** at the $i + 1$ location of the vectors by assigning **t** to **t_data[i+1]** and **y** to **y_data[i+1]**. Note that, for example, **y_data** is the entire vector of y data while **y_data[i+1]** is the value in the $i + 1$ location of the vector **y_data**.

The code to create a graph of an approximation of the function $P(t)$ for this scenario is given in R Code box 24.2. The first line **par(mar=c(4,5,5,2))** sets the number of lines for the margins around the graph starting at the bottom and moving clockwise. All of our data is stored in the vectors **t_data** and **y_data** and we use **plot()** to plot these values. Using **type="l"** graphs a line instead of the default of points. We set **lwd=2**, line width, for a slightly thicker line than the default. We label the x-axis and y-axis with **xlab** and **ylab**. Note that the text we want on the graph is in quotes for **xlab** and we use the **expression()** function for mathematical typeset of $P(t)$. The last two options in **plot()**, **cex.lab=1.5** and **cex.axis=1.5**, scale the axis labels and axis numbers to 1.5 times the default values.

The next line adds a title to the graph with the **title()** function. Within **title()** we use the **paste()** function. Within **paste()** we separate text within quotes and variables without quotes by commas, which will put on the graph the value of the variables and not the name of the variable. Within the text \ **n** is used to start a new line. The last part of **paste()** is **sep=""** which tells R to separate the text and variables without a space. Notice that there is no space between the quotes. This allows us to put space in the output where we would like it and so spaces within quotes matters, for example, between the, and r. The last line, **grid()** add grid lines and using the options NULL and NULL places the grid lines where the tick marks are placed in the graph. We color the grid lines black as the default is gray and is often not dark enough.

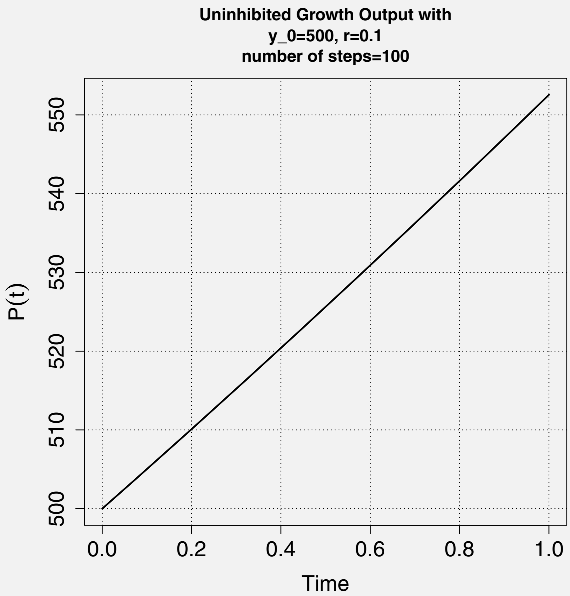
The graph in R Code box 24.2 appears linear but note that we are only graphing from 0 to 1. You should experiment with the code and change **t_final** to, for example, a 10 and then 100 to see how the graph changes. You will notice that the graph of $P(t)$ is concave up and increasing. This can be reasoned from the original differential equation

$$P'(t) = rP(t).$$

If $r > 0$ the growth of $P(t)$ increases the value of $P'(t)$ and hence the slope of the tangent lines increase. This creates a concave up shape. Of course, we are also assuming $P(t) > 0$ as it represents a population. What is the shape of $P(t)$ if $r < 0$? One more note here is to consider the impact of different step sizes on the graph.

R Code 24.2: Uninhibited Growth Graphical Solution

```
> par(mar=c(4,5,5,2))
> plot(t_data,y_data,type="l",lwd=2,xlab="Time",
ylab=expression(P(t)),cex.lab=1.5, cex.axis=1.5)
> title(paste("Uninhibited Growth Output with\ny_0=",
y_0,", r=",r,"\nnumber of steps=",number_steps,sep=""))
> grid(NULL,NULL,col="black")
```



We may want to know the final value or $P(1)$ in our example. The R Boxes 24.3 and 24.4 provide that output. In R Box 24.3 we first provide code to set the number of digits of output in R with the **options()** function. We want the last value in the **y_data** vector. We use square brackets to output a particular value (or values really). How many values are in the **y_data** vector? The answer is number_steps + 1, we have an initial value plus all the steps, and we could use that but instead we used the

length() function. In this case **length(y_data)** is the length of the **y_data**, which is also the location of the last value and hence we use **y_data[length(y_data)]**. Two notes here. First is that R indexes starting with 1 and not 0. Second, note that we use parenthesis, (and), with a function and its input but brackets, [and], when we want the value at a location in a vector.

In R Box 24.4 we provide a nicer output using **paste()**, where text is in quotes and variables are not. Note the use of **round(t,1)**. Here **t** is the final value (we could have used that instead of **length(y_data)** but two methods gives us a check on our work), but due to rounding in the code we might get, for example, in this case, a string of 0.99999 instead of 1. We simply round the output to ones place. Note that the final value with step sizes of 0.01 gives 552.56 which is slightly larger than the result of using step sizes of 0.5, 551.25, in 24.2.

R Code 24.3: Uninhibited Growth Final Value I

```
> options(digits=8) # Sets number of digits of output
> y_data[length(y_data)]
```

```
[1] 552.55785
```

R Code 24.4: Uninhibited Growth Final Value II

```
> paste("The value of y at time t=",round(t,1)," is ",
y_data[length(y_data)],sep="")
```

```
[1] "The value of y at time t=1 is 552.557848860384"
```

One last bit of code to provide information about our results is in R Code box 24.5 which provides a table of the data. We set a value for **rows** for the number of rows of data. We then use **cbind()** to combine our two vectors of data as columns. We then use the **head()** function which will output the beginning of the data with the number of rows the second option in **head()**. Choose the value of rows wisely as it doesn't help to output hundreds, thousands, or millions of lines of data. Note that if **head()** is changed to **tail()** the output will be the end of the data. The code here is useful in checking your work when practicing with hand calculations.

R Code 24.5: Uninhibited Growth Table

```
> rows <- 5
> head(cbind(t_data,y_data), rows)
```

```
     t_data   y_data
[1,]   0.00 500.0000
[2,]   0.01 500.5000
[3,]   0.02 501.0005
```

```
[4,]   0.03 501.5015
[5,]   0.04 502.0030
```

One goal of building models is to arrive at a model that captures the important characteristics of the real world phenomenon, while being as simple as possible so that results can be obtained. For $r > 0$ the uninhibited growth model has a solution, $P(t)$, that is concave up and increasing which we might expect of uninhibited growth and reasoned above from the model. In this case there are methods to find the function $P(t)$ explicitly and it is exponential. The value of Euler's method here is that it can be applied to differential equations that are either difficult to solve (you should take a differential equations course) or not solvable algebraically. We'll see other examples in this chapter and following chapters as we increase the complexity of our models.

The uninhibited growth model in M-Box 24.1 was constructed under the assumption that the birth rate b and death rate d were constant so that $r = b - d$ is a constant. In many ecological situations this assumption is not true. To address this we construct a new model under the assumption that the birth and death rates are linear functions of the population. [29] As we will see this adds complexity to our model and we will arrive at a new differential equation.

Let $P(t)$ be a population size at time t. If we assume the birth and death rates are linear we get

$$B(t) = b_0 - b_1 P(t)$$

and

$$D(t) = d_0 + d_1 P(t).$$

The idea here is that as the population increases resource pressures will decrease the birth rate linearly and increase the death rate linearly. Now the growth of the population will equal $(B - D)P(t)$, to get the differential equation

$$P'(t) = (B - D)P(t).$$

Note that we do not write $B(t)$ and $D(t)$ as we just use B and D, but both are, in fact, functions of t and we need to deal with that. To simplify the equation $P'(t) = (B - D)P(t)$ we first simplify $B - D$ as such

$$\begin{aligned}
B - D &= (b_0 - b_1 P(t)) - (d_0 + d_1 P(t)) \\
&= (b_0 - d_0) - (b_1 + d_1) P(t) \\
&= r - r_1 P(t) \\
&= r \left(1 - \frac{r_1}{r} P(t)\right) \\
&= r \left(1 - \frac{1}{\frac{r}{r_1}} P(t)\right) \\
&= r \left(1 - \frac{1}{K} P(t)\right)
\end{aligned}$$

where $r = b_0 - d_0$, $r_1 = b_1 + d_1$, and $K = \frac{r}{r_1}$. Our differential equation now becomes

$$\begin{aligned}
P'(t) &= (B - D) P(t) \\
&= r \left(1 - \frac{1}{K} P(t)\right) P(t) \\
&= r \left(1 - \frac{P(t)}{K}\right) P(t) \\
&= r P(t) \left(1 - \frac{P(t)}{K}\right)
\end{aligned}$$

This result, the logistic differential equation, is summarized in M-Box 24.2.

M-Box 24.2: The Logistic Differential Equation

Let $P(t)$ be the size of a population at time t. If we assume that the birth and death rates are linear function of the population, then the **initial value equations** for this model are

$$P'(t) = r P(t) \left(1 - \frac{P(t)}{K}\right) \qquad P(0) = P_0,$$

where K is the carrying capacity and r is the intrinsic growth rate.

The related code for the logistic model is almost identical to the uninhibited growth model code. R Code box 24.6 generates the data for the logistic model in the same way R Code box 24.1 does for the uninhibited growth model. There are only three differences. The first two are the addition of the variables r and k in the first sections as required by the logistic model. Note that in this example we have a starting time of 0 and a final time of 20 with an initial population of $y\_0 = 10$. The value of r and k are 0.5 and 1000, respectively. The other difference is that in the for loop the differential equation model is different because this is a different model.

R Code 24.6: Logistic Growth Data

```
> ## Define starting values and variables.
>
> t_initial <- 0
> t_final <- 20
> y_0 <- 10
> r <- 0.5
> k <- 1000
> delta_t <- 0.01
>
> ## Set up values for the For Loop.
>
> t <- t_initial
> y <- y_0
> number_steps <- (t_final-t_initial)/delta_t
> # Initializing the time vector with starting value
> t_data <- t
> # Initializing the y vector with starting value
> y_data <- y
>
> ## Run the For Loop.
>
> for (i in 1:number_steps){
+ y_prime <- r*(1-y/k)*y  # The Differential equation
+ y <- y_prime*delta_t + y # Microscope equation
+ t <- t+delta_t # Increments t to its next value
+   t_data[i+1] <- t # Add new value to time vector
+   y_data[i+1] <- y # Add new value to the pop. vector
+ }
```

Once we have the data we can graph the results. The code and the resulting graph for the logistic model is in R Code box 24.7 and again nearly identical to the uninhibited growth code in R Code box 24.2. On the other hand, the graph is very different and we should try to explain how the differential equation and graph relate to each other. Recall the logistic differential equation where $r > 0$ and $K > 0$

$$P'(t) = r\,P(t)\left(1 - \frac{P(t)}{K}\right).$$

If $P(t)$ is much smaller than K then the value of

$$\frac{P(t)}{K}$$

will be small making

$$1 - \frac{P(t)}{K}$$

close to 1. This leaves us with an equation of close to $P'(t) = rP(t)$ which is uninhibited growth. The graph of the logistic solution starts off as concave up and increasing like the uninhibited growth model. But as $P(t)$ gets closer to K the value of

$$\frac{P(t)}{K}$$

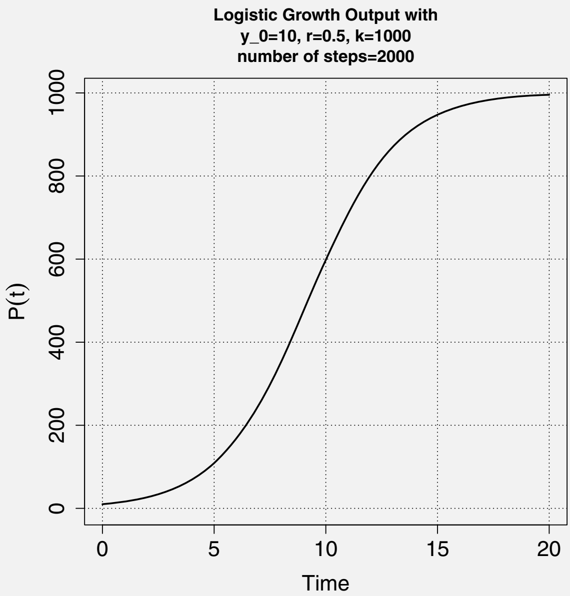
gets close to 1 making

$$1 - \frac{P(t)}{K}$$

close to 0. In other words as $P(t)$ gets closer to K the value of $P'(t)$ gets smaller and tends towards 0. This is what the graph does. Starting around half way the slopes of the tangent lines start to decrease and tend towards 0. Finally we note the horizontal asymptote at the value of K. We call K the carrying capacity as the model levels off at K. Note how the change of birth and death rates from constant in the uninhibited growth model to changing linearly in the logistic model generates a qualitatively different solution.

R Code 24.7: Logistic Growth Graphical Solution

```
> par(mar=c(4,5,5,2))
> plot(t_data,y_data,ylim=c(0,max(y_data)),type="l",
lwd=2,xlab="Time",ylab=expression(P(t)),cex.lab=1.5,
cex.axis=1.5)
> title(paste("Logistic Growth Output with\ny_0=",y_0,
", r=",r,", k=",k,"\nnumber of steps=",number_steps,
sep=""))
> grid(NULL,NULL,col="black")
```



We finish this section with three final examples of code to follow what was done in the uninhibited growth example; all three of which are identical to the uninhibited growth example. For the final value in the code we can use R Code boxes 24.8 and 24.9. For a table of values we have R Code boxes 24.10.

R Code 24.8: Logistic Growth Final Value I

```
> options(digits=8) # Sets number of digits of output
> y_data[length(y_data)]
```

```
[1] 995.53446
```

R Code 24.9: Logistic Growth Final Value II

```
> paste("The value of y at time t=",round(t,1)," is "
,y_data[length(y_data)],sep="")
```

```
[1] "The value of y at time t=20 is 995.534462814629"
```

R Code 24.10: Logistic Growth Table

```
> rows <- 5
> head(cbind(t_data,y_data), rows)
```

```
     t_data    y_data
[1,]   0.00 10.000000
[2,]   0.01 10.049500
[3,]   0.02 10.099243
[4,]   0.03 10.149229
[5,]   0.04 10.199460
```

24.1 Exercises

First do the problem by hand and then check your work in R where appropriate.

1. If we are estimating a population size from $t = 0$ to $t = 2$ hours and use 1000 steps, what is the step size?
2. If we are estimating a population size from $t = 0$ to $t = 3$ hours and use 1500 steps, what is the step size?
3. If we are estimating a population size from $t = 1$ to $t = 4$ hours and use 2000 steps, what is the step size?
4. If we are estimating a population size from $t = 2$ to $t = 6$ hours and use 3000 steps, what is the step size?
5. If we are estimating a population size from $t = 0$ to $t = 2$ hours and want a step size of 0.1, how many steps do we need?
6. If we are estimating a population size from $t = 0$ to $t = 3$ hours and want a step size of 0.2, how many steps do we need?
7. If we are estimating a population size from $t = 1$ to $t = 4$ hours and want a step size of 0.01, how many steps do we need?
8. If we are estimating a population size from $t = 2$ to $t = 6$ hours and want a step size of 0.05, how many steps do we need?

9. Assume E.coli grows proportional to its population size. E.coli divides in every 20 minutes so $r = 3$ per hour. If we have 100 E.coli cells in a dish, approximately how many will there be in 2 hours? Use 2 steps and 4 steps.

10. Assume Staphylococcus aureus grows proportional to its population size. Staphylococcus aureus divides in every 30 minutes so $r = 2$ per hour. If we have 50 Staphylococcus aureus cells in a dish, approximately how many will there be in 3 hours? Use 1 steps and 3 steps.

11. Assume E.coli grows proportional to its population size. E.coli divides in every 20 minutes so $r = 3$ per hour. If we have 20 E.coli cells in a dish, approximately how many will there be in 4 hours? Use step sizes of 4/3 and then step sizes of 4/5.

12. Assume Staphylococcus aureus grows proportional to its population size. Staphylococcus aureus divides in every 30 minutes so $r = 2$ per hour. If we have 75 Staphylococcus aureus cells in a dish, approximately how many will there be in 4 hours? Use step sizes of 2 and then step sizes of 1.

13. Assume E.coli is placed in a petri dish which will limit its population size to 10,000,000 (so we need to use the logistic model). E.coli divides every 20 minutes and so $r = 3$ per hour. If we have 100 E.coli cells in a dish, approximately how many will there be in 1 hour? Use 2 steps and 4 steps.

14. Assume Staphylococcus aureus is placed in a petri dish which will limit its population size to 10,000,000 (so we need to use the logistic model). Staphylococcus aureus divides in every 30 minutes and so $r = 2$ per hour. If we have 50 Staphylococcus aureus cells in a dish, approximately how many will there be in 3 hours? Use 1 step and 3 steps.

15. Yeast used to rise bread has a carrying capacity based on the amount of sugar available. Three grams of yeast are used in the recipe. Assume yeast has a growth rate of 0.5 per hour (depends on the yeast, temperature, and other conditions) and for a particular recipe a carrying capacity of 30 grams. Use the logistic model and estimate the amount of yeast after 2 hours using step sizes of 2 and then step sizes of 1.

16. Yeast used to rise bread has a carrying capacity based on the amount of sugar available. Three grams of yeast are used in the recipe. Assume yeast has a growth rate of 0.6 per hour (depends on the yeast, temperature, and other conditions) and for a particular recipe a carrying capacity of 40 grams. Use the logistic model and estimate the amount of yeast after 4 hours using step sizes of 2 and then step sizes of 1.

These problems should be done in R.

17. Assume E.coli satisfies the initial value problem $E'(t) = 3E(t)$ with $E(0) = 1$, where t is in hours. Estimate to an accuracy of 10,000 the number of bacterial at 5 hours. What step sizes were required? Include an appropriate graph of your solution.
18. Assume E.coli satisfies the initial value problem $E'(t) = 3E(t)$ with $E(0) = 1$, where t is in hours. Estimate to an accuracy of 100,000 the number of bacterial at 7 hours. What step sizes were required? Include an appropriate graph of your solution.
19. Assume Staphylococcus aureus satisfies the initial value problem $S'(t) = 2S(t)$ with $S(0) = 1$, where t is in hours. Estimate to a whole number accuracy the number of bacterial at 4 hours. What step sizes were required? Include an appropriate graph of your solution.
20. Assume Staphylococcus aureus satisfies the initial value problem $S'(t) = 2S(t)$ with $S(0) = 1$, where t is in hours. Estimate to a whole number accuracy the number of bacterial at 7 hours. What step sizes were required? Include an appropriate graph of your solution.
21. In the E.coli problem above, how long will it take an initial population of 100 to reach 1000? Take an estimate and check approach to t_final and use successive approximation. Creating a table is a good idea here.
22. In the Staphylococcus aureus problem above, how long will it take an initial population of 50 to reach 1000? Take an estimate and check approach to t_final and use successive approximation. Creating a table is a good idea here.
23. In 1859, an Australian farmer imported two dozen rabbits and set them free. Rabbits are not native to Australia and have no real limits to their growth (this is for real, google it). Assume rabbit populations grow proportional to their size with an $r = 0.19$ per month. If $R(t)$ is the number of rabbits in month t, what is the initial value equation? How many rabbits were there 6 years later (estimate to within 1000 rabbits)?
24. In the Australian rabbit problem, how long does it take the rabbits to reach 100,000 rabbits? Take an estimate and check approach to t_final and use successive approximation. Creating a table is a good idea here.

The Logistic Model

25. With $k = 1000$, $r = 0.25$, and an initial population of 10, create a graph of the logistic model from $t = 0$ to $t = 50$. Describe the shape of the graph using calculus terms. If there is an inflection point, about where does it occur?

26. With $k = 1000$, $r = 0.20$, and an initial population of 2000, create a graph of the logistic model from $t = 0$ to $t = 50$. Describe the shape of the graph using calculus terms. If there is an inflection point, about where does it occur? Based on the initial value equations why did you get this shape?

27. With $r = 0.25$ and an initial population of 10, choose different values of K and create graphs of the logistic model from $t = 0$ to $t = 50$. Explain how changing K changes the graph and how this connects to the differential equation.

28. With $k = 1000$ and an initial population of 10, choose different values of r and create graphs of the logistic model from $t = 0$ to $t = 50$. Explain how changing r changes the graph and how this connects to the differential equation.

29. Yeast used to rise bread has a carrying capacity based on the amount of sugar available. A full packet of yeast is 7 grams and used in the recipe. Assume yeast has a growth rate of 0.6 (depends on the yeast, temperature, and other conditions) and for a particular recipe a carrying capacity of 25 grams. What is the differential equation model for this scenario? How much yeast is in the bread after 5 hours (accurate to two decimal places)? How long does it take the yeast to get within 10% of the carrying capacity?

30. Yeast used to rise bread has a carrying capacity based on the amount of sugar available. Three grams of yeast are used in the recipe. Assume yeast has a growth rate of 0.5 (depends on the yeast, temperature, and other conditions) and for a particular recipe a carrying capacity of 30 grams. What is the differential equation model for this scenario? How much yeast is in the bread after 8 hours (accurate to two decimal places)? How long does it take the yeast to get within 10% of the carrying capacity?

31. Assume E.coli is placed in a petri dish which will limit its population size to 10,000,000. E.coli divides every 20 minutes and so $r = 3$ per hour. If we have 100 E.coli cells in a dish, approximately how many will there be in 4 hours? Estimate to within 1,000 cells and include the number of steps needed.

32. Assume Staphylococcus aureus is placed in a petri dish which will limit its population size to 5,000,000. Staphylococcus aureus divides in every 30 minutes and so $r = 2$ per hour. If we have 50 Staphylococcus aureus cells in a dish, approximately how many will there be in 6 hours? Estimate to within 1,000 cells and include the number of steps needed.

Related Differential Equations

33. Assume a population grows proportional to the square of the population. What differential equation represents this scenario? If $r = 0.01$, how big will a population of size 10 reach in 5 years? Estimate to a whole number. How long will it take the population to reach 1000?

34. Assume a population grows proportional to the square root of the population. What differential equation represents this scenario? If $r = 0.01$, how big will a population of size 10 reach in 5 years? Estimate to a whole number. How long will it take the population to reach 1000? Compare your results to the previous exercise.

35. **Decay** Carbon 11 decays proportional to its amount with $r = -0.034$ per minute. What is the differential equation for this scenario? If we start with 1000 grams of carbon 11, how much will we have in 10 minutes (estimate this within two decimal places)? What is the half-life of carbon 11, in other words, if we start with 1000 grams how long until we have 500 grams (estimate this within two decimal places)?

36. **Decay** Hydrogen-3 (tritium) decays proportional to its amount with $r = -0.05626$. What is the differential equation for this scenario? If we start with 10,000 grams of tritium, how much will we have in 7 years (estimate this to the nearest whole number)? What is the half-life of tritium, in other words, if we start with 10,000 grams how long until we have 5000 grams (estimate to the nearest tenth of a year)?

37. **Newton's Law of Cooling** Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference between the current temperature and the surrounding temperature. Suppose a just poured cup of tea, green tea brewed properly at 180 degrees, is in a room that is 70 degrees. Assume the proportionality constant is 0.05 per minute. What is the initial value problem for this scenario? Create a graph of the differential equation you found. Within a tenth of a degree how long does it take the tea to reach 90 degrees?

38. **Falling Object with Air Resentence** Let $s(t)$ be the distance traveled by a dropped object or, say a raindrop. The rate of change of velocity $v(t)$ has two factors. Gravity, $g = 9.8m/s^2$ increases the velocity while air resistance which is proportional to the square of the velocity reduces velocity. What is the differential equation for this scenario? Suppose the proportionality constant for a typical raindrop is 0.25. What are the units on the 0.25 and what is the terminal velocity of the raindrop?

Chapter 25
Differential Equations - Predator Prey

The previous differential equation models involved only one species. We now consider a differential equation model with two species that interact. One classic example explores the interaction between, say, foxes (predators) and rabbits (prey). Let $R(t)$ and $F(t)$ represent the number of rabbits and foxes time t. We assume that rabbits will grow proportional to their population with growth constant b_1.

The rabbits die based on a death rate per encounter with foxes, d_1. If there are $R(t)$ rabbits and $F(t)$ foxes then there are $R(t)F(t)$ pairs of foxes and rabbits that may interact. For example, if there are 2 foxes and 10 rabbits, the each fox could interact with 10 rabbits for a total of $2 \times 10 = 20$ possible interactions. Each possible interaction happens with some probability and when it does, the rabbit has some other probability of surviving or not. We can simplify this a bit with the death rate d_1 which includes the rate at which interactions occur in which the rabbit does not survive. For the rate of change of the rabbit population we would then get

$$R'(t) = b_1 R(t) - d_1 R(t) F(t).$$

Informally, we can think of the $d_1 R(t) F(t)$ term with a few cases. If there are lots of foxes and lots of rabbits, then the rabbits are easy to find and with lots of foxes around we get more encounters and a high death rate from $d_1 R(t) F(t)$. If there are lots of foxes and few rabbits or if there are few foxes and lots of rabbits, then there are fewer encounters making $d_1 R(t) F(t)$ smaller. If there are few foxes and few rabbits, then encounters are even lower and $d_1 R(t) F(t)$ is again smaller. In all cases the d_1 represents the death rate per encounter and the $R(t)F(t)$ quantifies the number of possible encounters.

Foxes on the other hand grow based on their ability to turn predated rabbits into offspring. We again use $R(t)F(t)$ to quantify encounters between foxes and rabbits and b_2 the foxes efficiency of turning encounters into offspring or population growth. The rate b_2 captures the chance of encounters, the chance a fox eats, and how many rabbits they would need to consume to support offspring and population growth. This gives $b_2 R(t) F(t)$ as the growth term for foxes. Note we use b_2 but this isn't a birth rate. Still, we can informally think of the parts of the differential equations as "births"

T. J. Pfaff, *Applied Calculus with R*,
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or "growth" and "deaths." Consider the four cases above and how they impact the growth of the fox population. Foxes decline proportional to their population size with a natural death rate of d_2. The rate of change of the fox population is then given by

$$F'(t) = b_2 R(t) F(t) - d_2 F(t).$$

It might seem that rabbits would be involved in the death rate of foxes since they will die off if they don't eat. The goal of modeling is to start off simple and we assume the population will decline if the death rate is larger than the growth rate and the growth rate includes the rabbits. The resulting set of differential equations, known as the Lotka–Volterra predator–prey model, are given in M-Box 25.1. This is a simplified model, as beginning models always are, and assumes that the foxes only eat rabbits, that the rabbits have no limitation on growth, and that the only predator for the rabbits are the foxes.

M-Box 25.1: Lotka-Volterra Predator-Prey Model

Let $N_1(t)$ be the population size of prey that grow proportional to their population size and $N_2(t)$ the population size of predators. The Lotka-Volterra equations are

$$\begin{aligned} N_1'(t) &= b_1 N_1 - d_1 N_1 N_2 \\ N_2'(t) &= b_2 N_1 N_2 - d_2 N_2, \end{aligned}$$

with initial conditions $N_1(0)$ and $N_2(0)$. The model assumes that the predators eat only this prey, that the prey have no limitation on growth, and that the only predator for the prey are this one predator. This is a general way to express this model, but you can replace the $N_1(t)$ with $R(t)$ and $N_2(t)$ with $F(t)$ if that helps and we will use $R(t)$ and $F(t)$ in the code.

Some of the key questions related to this model are: Can the two populations coexist? If so, under what circumstances? Under what circumstances will the populations die out? How sensitive is the system to the parameters b_1, b_2, d_1, and d_2, and the initial population sizes $N_1(0)$ and $N_2(0)$? Note that these types of models are used more to understand these qualitative questions than to try and predict the number of, say, foxes and rabbits at some given time. Some of these questions will be explored in the exercises and we now look at the related code to explore these questions.

As an example, set $b_1 = 0.05$, $d_1 = 0.005$, $b_2 = 0.0004$, and $d_2 = 0.04$ while starting with $R(0) = 200$ rabbits (prey) and $F(t) = 10$ foxes (predator). R Code 25.1 uses Euler's method to estimate the number of rabbits and foxes at a given time. The code is very similar to the code in Chapter 24. The only real difference is that there are now two sets of everything; parameters, differential equations, and microscope equations. The first nine lines of code set the values for all of our parameters including **delta_t** for the step sizes. Before entering the for loop we set the values of **t**, **R**, **F**, and **number_steps**. We also initialize the vectors **t_data**,**R_data**, and

F_data to store the time, rabbit, and fox data as we repeatedly use the microscope equation.

Again, the for loop is similar to that in Chapter 24 but now we have two differential equations and two microscope equations. There is one new line of code **if (round(R)==0)**{**R<–0**}, which if we round the current value of R and it is 0 we set the value of R to 0. In other words, if the number of rabbits falls below 0.5 we say the rabbits are extinct and set $R = 0$. This makes sense if we are using population counts and not, say, population in thousands or millions or as density. When you are exploring the code consider removing this line to see how it may change outcomes. The last three lines within the for loop add the new values to the entering the for loop we set the values of **t**, **R**, **F**, and **number_steps**. We also initialize the vectors **t_data**,**R_data**, and **F_data** vectors.

R Code 25.1: Lotka-Volterra Predator-Prey Data

```
> ## Define starting values and variables.
>
> t_initial <- 0
> t_final <- 400
> R_0 <- 200
> b_1 <- 0.05
> d_1 <- 0.005
> F_0 <- 10
> b_2 <- 0.0004
> d_2 <- 0.04
> delta_t <- 0.01
>
> ## Set up values for the For Loop.
>
> t <- t_initial
> R <- R_0
> F <- F_0
> number_steps <- (t_final-t_initial)/delta_t
> # Initializing the time vector with starting value
> t_data <- t
> # Initializing the Rabbit vector with starting value
> R_data <- R
> # Initializing the Fox vector with starting value
> F_data <- F
>
> ## Run the For Loop.
>
> for (i in 1:number_steps){
```

```
+ R_prime <- b_1*R-d_1*R*F # The rabbit DE
+ F_prime <- b_2*R*F-d_2*F # The fox DE
+
+ R <- R_prime*delta_t + R # Rabbit microscope equation
+ F <- F_prime*delta_t + F # Fox microscope equation
+ t <- t+delta_t # Increments t to its next value
+
+ if (round(R)==0){R<-0}  # Round prey down to 0
+ t_data[i+1]<- t  # Add new value to the time vector
+ R_data[i+1] <- R  # Add new value to Rabbit vector
+ F_data[i+1] <- F  # Add new value to Fox vector
+ }
```

Now that we have the data lets make a graph. This is a little more complicated as we want to graph both the foxes and the rabbits in the same graph, but there is often a scale difference between the rabbits and foxes, and if we plotted them with the same y-axis it would typically be difficult to see the fox graph. The solution here is to create a graph with a left and right y-axis with different scales. This is done in R Code box 25.2.

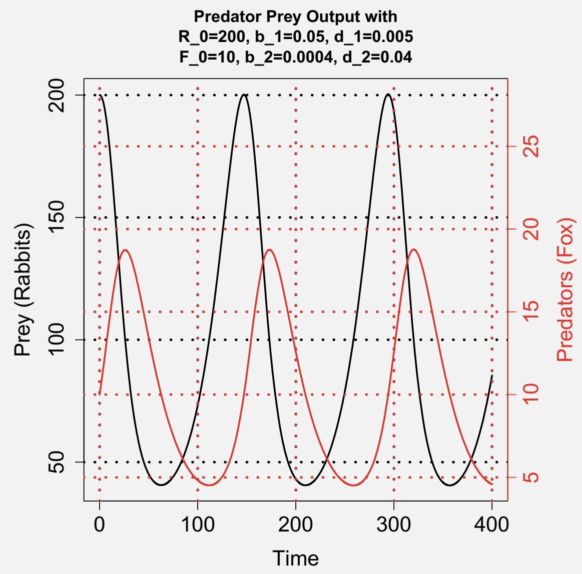
The first four lines of code in R Code box 25.2 are similar to the graph code in Chapter 24. We set the margins with **par(mar=c(5,5,5,5))** and then graph the rabbit data with **plot()**. We then add a title with the **title()** function and use **paste()** to concatenate text and the values of variables. Finally a grid is added with **grid()**. We added lwd=3 to make the grid lines thicker. This helps for printing in a text but is too big when you are running your code. Consider setting this to 1 or 2 when running the code yourself. What we have done so far is all the black parts of the graph, which is a typical graph with the y-axis on the left.

In order to get the foxes on a right y-axis we begin with **par(new=TRUE)**, which says to start a new graph but on top of the current graph. The next line plots the fox data similar to plotting the rabbits with two key differences. First note that **axes=FALSE**, which turns off adding axis for the fox graph. We do not want a new set of axis, but we will need to create a right y-axis. We also set the range of the right y-axis with ylim. The default is to create a graph that uses the full y-axis. If we allowed this the foxes graph would go to the same height as the rabbits and it would look like they are the same. This might be deceiving. We set the range to have a min the smallest of the fox data, **min(F_data)** and the max to **1.5*max(F_data)**. Setting the max to **1.5*max(F_data)** makes the y-axis bigger than necessary resulting in the fox graph not reaching the top of the axis. Play with the code and take out the 1.5 to see what happens and change it to a different value as the 1.5 is not particularly special.

The **axis()** function adds an axis to a graph. Side 4 is the right side. The **at** says where to consider putting tick marks and we use the same range we set in **plot()** but put it inside the **pretty()** function. The **pretty()** function tells R to choose “nice” values. Take it out and see what happens. We color the axis and scale the values as we have been doing. The **mtext()** function adds text to the margin. The key here is **side=4** is the right side and **line=3** is the third line from the axis. Again we color and scale the text. Finally we add a grid with the red2 color and use lwd=3 again.

R Code 25.2: Lotka-Volterra Predator-Prey Graphical Solution

```
> par(mar=c(5,5,5,5))
> plot(t_data,R_data,type="l",lwd=2,xlab="Time",
ylab="Prey (Rabbits)",cex.lab=1.5,cex.axis=1.5)
> title(paste("Predator Prey Output with\nR_0=", R_0,
 ", b_1=",b_1, ", d_1=", d_1, "\nF_0=", F_0, ",
  b_2=", b_2, ",d_2=", d_2,sep="") )
> grid(NULL,NULL,col="black",lwd=3)
> par(new=TRUE)
> plot(t_data,F_data,type="l",ylim=c(min(F_data),
1.5*max(F_data)),lwd=2,axes=FALSE,xlab="",
ylab="",col="red2")
> axis(side=4,at=pretty(c(min(F_data),
1.5*max(F_data))),col="red2",cex.axis=1.5,
col.axis="red2")
> mtext("Predators (Fox)",side=4,line=3,col="red2",
cex=1.5)
> grid(NULL,NULL,col="red2",lwd=3)
```



The resulting graph here is interesting and make sure to read the left and right y-axis correctly. In this case the two populations coexist but in an oscillating fashion where the number of foxes lag behind the number of rabbits. This makes sense because as the rabbit population grows we can get more foxes, but eventually the fox population gets large enough to decrease the rabbit population. This then leads to a decrease in the fox population and a rebound in the rabbit population.

If we would like the final values for the number of rabbits and foxes we can use the code in R Code boxes 25.3 and 25.4. On the other hand we may be interested in the maximum values (or minimum), which is in R Code box 25.5 for the rabbits. The fox code is similar and provided on the supplement page at sustainabilitymath.org/acr. The maximum number of rabbits is 200.35362 found with **max(R_data)**. What is the t-values or time this occurred? We first find the entry value in the **R_data** vector where the maximum occurred, **which.max(R_data)**, and then input that value into the **t_data** vector, **t_data[which.max(R_data)]** to get a time value of 294.13 (day?, years? we haven't specified the units yet). The minimums can be found by changing max to min in the code. Lastly we can generate a table of data as shown in R Code box 25.6.

R Code 25.3: Predator Prey Final Value I

```
> options(digits=8) # Sets number of digits of output
> R_data[length(R_data)]
> F_data[length(F_data)]
```

```
[1] 85.395452
[1] 4.5879773
```

R Code 25.4: Predator Prey Final Value II

```
> paste("The value of R at time t=",round(t,1)," is ",
R_data[length(R_data)], sep="")
> paste("The value of F at time t=",round(t,1)," is ",
F_data[length(F_data)], sep="")
```

```
[1] "The value of R at time t=400 is 85.395451675426"
[1] "The value of F at time t=400 is 4.58797728729927"
```

R Code 25.5: Predator Prey Maximum Value

```
> max(R_data) # Rabbit max
> which.max(R_data) # Location of first rabbit max
> t_data[which.max(R_data)] # Time value 1st rabbit max
```

```
[1] 200.35362
[1] 29414
[1] 294.13
```

R Code 25.6: Predator Prey Table

```
> rows <- 5
> head(cbind(t_data,R_data,F_data),rows)
```

```
     t_data    R_data    F_data
[1,]   0.00 200.00000 10.000000
[2,]   0.01 200.00000 10.004000
[3,]   0.02 199.99996 10.008002
[4,]   0.03 199.99988 10.012005
[5,]   0.04 199.99976 10.016010
```

We should pause for a moment here and reflect on what we have done and the knowledge used. The ideas of calculus and the derivative were used to set up the differential equations. The microscope equation provided a way to estimate solutions.

We then used some coding to do all the calculations set up by calculus, providing us with an interesting solution graph. We will see in the exercises that there are relationships between maximums, minimums, and inflection points in the predator–prey graph thereby using our calculus knowledge to help analyze and understand the relationships between the foxes and rabbits. Our calculus and coding skills have allowed us to develop, find a solution, and analyze two species interacting model.

25.1 Exercises

1. Consider the Lotka–Volterra model in M-Box 25.1. What are the units for b_1 and b_2?
2. Consider the Lotka–Volterra model in M-Box 25.1. What are the units for d_1 and d_2?
3. Based on the graph in R Code 25.2, what is the maximum and minimum population size of the prey? How much time is there between occurrences of the maximums?
4. Based on the graph in R Code 25.2, what is the maximum and minimum population size of the predators? How much time is there between occurrences of the maximums?
5. Based on the graph in R Code 25.2, the inflection points with positive slopes of the prey graph occur at the same time as what on the predator graph? Explain why this happens in terms of the dynamics of the predators and prey.
6. Based on the graph in R Code 25.2, the inflection points with positive slopes of the predator graph occur at the same time as what on the prey graph? Explain why this happens in terms of the dynamics of the predators and prey.
7. Based on the graph in R Code 25.2, the inflection points with negative slopes of the prey graph occur at the same time as what on the predator graph? Explain why this happens in terms of the dynamics of the predators and prey.
8. Based on the graph in R Code 25.2, the inflection points with negative slopes of the predator graph occur at the same time as what on the prey graph? Explain why this happens in terms of the dynamics of the predators and prey.
9. (This should be done by hand.) Based on the graph in R Code 25.1, what is the value of R and F after the completion of the for loop the second time?
10. (This should be done by hand.) Based on the graph in R Code 25.1, what is the value of R and F after the completion of the for loop the third time?

For the next 14 problems we will use $b_1 = 0.05$, $d_1 = 0.005$, $b_2 = 0.0004$, **and** $d_2 = 0.04$ **while starting with 200 rabbits (prey) and 10 foxes (predator) each time you begin the problem. Assume for discussion that time is in days. Let time range from 0 to 500. Tip: Include dev.new() in the code if you want to create a new plot window each time to compare changes where useful.**

11. Increase b_1 to 0.07, 0.09, and 0.1. Compare and contrast what happens to the foxes and rabbits in each case. Explain why this makes sense or not relative to the fox and rabbit populations of the differential equations.
12. Change d_1 to 0.004, 0.006, and 0.008. Compare and contrast what happens to the foxes and rabbits in each case. Explain why this makes sense or not relative to the fox and rabbit populations of the differential equations.
13. Change b_2 to 0.0003, 0.0006, and 0.0008. Compare and contrast what happens to the foxes and rabbits in each case. Explain why this makes sense or not relative to the fox and rabbit populations of the differential equations.
14. Change d_2 to 0.03, 0.05, and 0.07. Compare and contrast what happens to the foxes and rabbits in each case. Explain why this makes sense or not relative to the fox and rabbit populations of the differential equations.
15. Presumably more rabbits is good for the foxes (more food to eat). Increase the starting number of rabbits to 400, 700, and 1000. Compare and contrast what happens to the foxes and rabbits in each case. Explain why this makes sense or not relative to the fox and rabbit populations of the differential equations. Is increasing the number of rabbits good for the foxes?
16. Presumably more foxes are bad for rabbits (more hungry foxes). Increase the starting number of foxes to 20, 30, and 40. Compare and contrast what happens to the foxes and rabbits in each case. Explain why this makes sense or not relative to the fox and rabbit populations of the differential equations. Is increasing the number of foxes bad for the rabbits?
17. Change the starting number of rabbits to 1000, time period to 0 to 20, and use 80 steps. What are the values of R_prime, F_prime, R, and F the first two times through the for loop?
18. Change the starting number of rabbits to 500, time period to 0 to 10, and use 50 steps. What are the values of R_prime, F_prime, R, and F the first two times through the for loop?
19. Using the same setup from the header above but change b_1 to 0.1, calculate $R(1)$ and $F(1)$ using two steps. Does this first by hand and then check with R.
20. Using the same setup from the header above but change b_1 to 0.1, calculate $R(2)$ and $F(2)$ using two steps. Doe this first by hand and then check with R.
21. Using the setup in the header above, what is the maximum number of foxes and rabbits reached?
22. Using the setup in the header above, what is the minimum number of foxes and rabbits reached? Use code, do not just read the graph and so provide at least two decimal places.
23. Approximately, what is the fastest rate of decline of the rabbits?
24. Approximately, what is the fastest rate of increase of the rabbits?

25. **Model Variant** Consider the Lotka–Volterra model in M-Box 25.1. Suppose the prey N_1 grow logistically. In this scenario what differential equations do we get?

26. **Model Variant** Consider the Lotka–Volterra model in M-Box 25.1. Suppose there is a limit on the size of the predator population or a carrying capacity K for the predators. In this scenario what differential equations do we get?

27. **Model Variant** Consider the Lotka–Volterra model in M-Box 25.1. Suppose there is another predator that continuously takes 10% of the prey from the population, but doesn't interact in any other way with the system. In this scenario what differential equations do we get?

28. **Model Variant** Consider the Lotka–Volterra model in M-Box 25.1. Species often have a minimum viable population (MVP) to survive. There are many reasons for this such as inbreeding, too difficult to find a mate, or risk of extinction due to an environmental change or natural catastrophes. In essence, this is the same as a carrying capacity but a minimum instead of a maximum. Suppose the fox population needs an MVP of M to survive. In this scenario what differential equations do we get?

Chapter 26
Differential Equations - SIR Model

Epidemiology is the study of the incidence, distribution, and possible control of diseases. In this section we consider a model to understand how a disease moves through a population. We let $S(t)$ be the number of people susceptible to the disease, $I(t)$ be the number of people infected with the disease, and $R(t)$ be the number of people recovered from the disease at time t, which is often days. For now we assume a person can only get the disease once and is then immune to the disease once recovered. We will build a set of differential equations with these three functions to model the spread of a disease through a population.

We begin by considering the rate of change of $R(t)$ or $R'(t)$. As an example, suppose we have 120 people with the disease and that disease lasts for 10 days. On average, 1/10 of the 120 people, or 12 people, will recover each day so that at this moment $R'(t) = 12$. In general,

$$R'(t) = I(t)/d$$

where d is the number of days the disease lasts.

Next, we consider $S'(t)$ which is the rate at which people move from susceptible to infected. How does an individual get sick? They must first come into contact with an infected person and the disease must then be transmitted, which is not a guarantee. Similar to the predator–prey model from Chapter 25 there are $S(t)I(t)$ possible pairs of interactions between a susceptible person and an infected person. For example, if there are 50 susceptible people and 10 infected people then there are 500 possible pairings of a susceptible person and an infected person that can happen at any moment. Now suppose that, say, 2% of these pairings happen. Then we have $0.02(50)(10) = 10$ chances for an infection to move from an infected person to a susceptible person. If we let c be the contact rate between susceptible and infected we have $cI(t)S(t)$ as the number of opportunities for the disease to spread. Hence, if we let m be the transmission rate (m because we cannot use t) then the number of new sick people is $mcI(t)S(t)$. In our example, if 50% of the time the disease spreads from a susceptible to an infected then we have $0.5(0.02)(50)(10) = 5$ new infected people. Thus,

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_26

$$S'(t) = -mcI(t)S(t),$$

where the negative is due to the number of susceptible people decreasing.

Another way to view the $cI(t)S(t)$ piece of $S'(t)$ is to assume, for example, that on average an individual interacts with say 2% (the value of c) of the population. One individual would then come in contact with $0.02I(t)$ infected people. We then multiply this by the total number of susceptible people to get $0.02I(t)S(t)$ or, in general,$cI(t)S(t)$. This they gets multiplied by the transmission rate m to get $mcI(t)S(t)$.

Finally, $I'(t)$ is made up of the newly infected people minus those that recovered. In other words,

$$I'(t) = mcI(t)S(t) - I(t)/d.$$

Notice that

$$S'(t) + I'(t) + R'(t) = 0$$

as it must be since there are no deaths in the model as people only move from one category to another. The general model is given in M-Box 26.1.

M-Box 26.1: SIR Model

The SIR model (Susceptible, Infected, Recovered) with functions $S(t)$ the number of people susceptible to a disease, $I(t)$ the number of people infected with the disease, and $R(t)$ the number of people recovered from the disease at time t (often in days), yields the differential equations

$$\begin{aligned} S'(t) &= -mcI(t)S(t) \\ I'(t) &= mcI(t)S(t) - \frac{I(t)}{d} \\ R'(t) &= \frac{I(t)}{d}, \end{aligned}$$

where m is the transmission rate of the disease when a susceptible comes into contact with an infected, c is the rate at which a susceptible comes into contact with an infected, and d is the number of days an infected has the disease.

R Code box 26.1 is set up with the following example:

$$\begin{aligned} S'(t) &= -0.00001I(t)S(t) \\ I'(t) &= 0.00001I(t)S(t) - \frac{I(t)}{14} \\ R'(t) &= \frac{I(t)}{14} \end{aligned}$$

with $S(0) = 50000$, $I(0) = 2000$, and $R(0) = 500$. Note that while in this case we wrote $I(t)/14$ and we will do this in the code it is cumbersome. Alternatively,

$1/14 = 0.0714$ and we can write

$$R'(t) = \frac{1}{14} I(t) = 0.0714 I(t).$$

The code in R Code box 26.1 follows the same format as the growth and predator–prey code. We start by defining all of the starting values. Follow that up with the usual setup for the values for the for loop including defining **number_steps** and initializing the **t_data**, **S_data**, **I_data**, and **R_data** vectors. In the for loop we evaluate the three differential equations, calculate the three microscope equations, and add the new values to the three vectors of the model plus time.

R Code 26.1: SIR Data

```
> ## Define starting values and variables.
>
> t_initial <- 0
> t_final <- 30
> S_0 <- 50000
> I_0 <- 2000
> R_0 <- 500
> mc <- 0.00001
> d <- 14
> delta_t <- 0.01
>
> ## Set up values for the For Loop.
>
> t <- t_initial
> S <- S_0
> I <- I_0
> R <- R_0
> number_steps <- (t_final-t_initial)/delta_t
> # Initializing time vector with starting value
> t_data <- t
> # Initializing Susceptible vector with starting value
> S_data <- S
> # Initializing Infected vector with starting value
> I_data <- I
> # Initializing Recovered vector with starting value
> R_data <- R
>
> ## Run the For Loop.
>
> for(i in 1:number_steps){
+ S_prime <- -mc*S*I    # The differential equations
+ I_prime <- mc*S*I-I/d
+ R_prime <- I/d
+
+ S <- S_prime*delta_t+S # Microscope equations
```

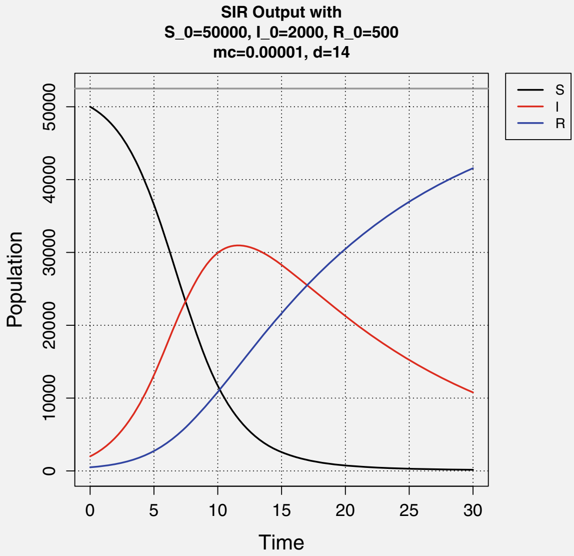
```
+ I <- I_prime*delta_t+I
+ R <- R_prime*delta_t+R
+ t <- t+delta_t # Increments t to its next value
+
+ t_data[i+1] <- t  # Add new value to the time vector
+ S_data[i+1] <- S  # Add new value to the Sus. vector
+ I_data[i+1] <- I  # Add new value to the Inf. vector
+ R_data[i+1] <- R  # Add new value to the Rec. vector
+ }
```

The graph of the SIR model has only a few things different that the graphs for the growth model and R Code box 26.2 has the details. The first line, **options(scipen = 999)** will keep R from putting the axis values into scientific notation. The par command sets the lines of the margin of the graph. We start by plotting the **S_data** first. We set the y-axis range from zero to the sum of the initial values of susceptible, infected, and recovered as this represents the total population for this model. We use **title()** and **paste()** to add a title with useful information to the graph. Something new here is the **abline()** function which adds a horizontal line, the h in **h=(S_0+I_0+R_0)**, at the sum of initial values of susceptible, infected, and recovered. This just emphasizes the part of the graph that represents the total population. We then add grid lines with **grid()**. We still need to add the infected and recovered data to the graph and this is done with the **lines()** function. The **lines()** takes x and y data and adds a curve to the existing graph by connecting the dots of the data. We will next add a legend to keep track of which curve is which data. The **par(xpd=TRUE)** allows us to add elements outside the plot area. We then use the **legend()** function. The first element is where to put the legend and then **c("S","I","R")** provides the text for the legend. The next two elements, **lwd=c(2,2,2)** and **col=c("black","red2","blue2")**, say to put lines of line width two next to our text elements and then color those lines appropriately. The last elements **inset=c(-0.2,0)** shift the legend to the right, -0.2, and leave the y orientation fixed with 0. Why -0.2? Technically the shift is a percentage of the plot region. In practice, estimate, check, and adjust is what is done in many cases. The last line, **par(xpd=FALSE)**, keeps any future graph elements from being drawn outside the graph frame. For example, if we left this out then the next time the graphing code is run the grid lines will extend outside the plot frame.

R Code 26.2: SIR Graphical Solution

```
> options(scipen = 999) # Disable scientific notation
> par(mar=c(5,4,5,6))
> plot(t_data,S_data,type="l", ylim=c(0,S_0+I_0+R_0),
```

```
lwd=2,xlab="Time", ylab="Population",cex.lab=1.5,
cex.axis=1.25)
> title(paste("SIR Output with\nS_0=", S_0, ", I_0=",
 I_0, ", R_0=",R_0,"\nmc=", mc, ", d=", d,sep="") )
> abline(h=(S_0+I_0+R_0),lwd=2,col="gray50")
> grid(NULL,NULL,col="black")
> lines(t_data,I_data,lwd=2,col="red2")
> lines(t_data,R_data,lwd=2,col="blue2")
> par(xpd=TRUE)
> legend("topright",c("S","I","R"),lwd=c(2,2,2),
col=c("black","red2","blue2"),inset=c(-0.2,0))
> par(xpd=FALSE)
```



As with the predator–prey model we have a number of data outputs. R Code boxes 26.3 and 26.4 provide final values for the three groups. The infected population has a maximum and this is found with the use of R Code box 26.5. This is a particularly good example of how the maximum value and the time at which it occurs has value. In this example, there are a maximum of 30,974 people and it occurs around day 12. This informs us of how many infected people that have to be managed and when that will happen. Both are valuable pieces of information. The final box, R Code box 26.6 provides a table of data.

R Code 26.3: SIR Final Value I

```
> options(digits=8) # Sets number of digits of output
> S_data[length(S_data)]
> I_data[length(I_data)]
> R_data[length(R_data)]
```

```
[1] 158.26282
[1] 10777.775
[1] 41563.962
```

R Code 26.4: SIR Final Value II

```
> paste("The value of S at time t=",round(t,1)," is ",
S_data[length(S_data)], sep="")
> paste("The value of I at time t=",round(t,1)," is ",
I_data[length(I_data)], sep="")
> paste("The value of R at time t=",round(t,1)," is ",
R_data[length(R_data)], sep="")
```

```
[1] "The value of S at time t=30 is 158.262817509613"
[1] "The value of I at time t=30 is 10777.7754046645"
[1] "The value of R at time t=30 is 41563.9617778259"
```

R Code 26.5: SIR Maximum Value

```
> max(I_data) # Infected max
> which.max(I_data) # Location of infected max
> t_data[which.max(I_data)] # Time value of inf. max
```

```
[1] 30973.797
[1] 1163
[1] 11.62
```

R Code 26.6: SIR Table

```
> rows <- 5
> head(cbind(t_data,S_data,I_data,R_data),rows)
```

```
     t_data    S_data    I_data    R_data
[1,]   0.00 50000.000 2000.0000 500.00000
[2,]   0.01 49990.000 2008.5714 501.42857
```

```
[3,]   0.02 49979.959 2017.1776 502.86327
[4,]   0.03 49969.877 2025.8186 504.30411
[5,]   0.04 49959.754 2034.4946 505.75112
```

26.1 Exercises

1. In the SIR model in M-Box 26.1, what are the units for $S(t)$, $I(t)$, $R(t)$, $S'(t)$, $I'(t)$ and $R'(t)$? Assume t is days.
2. In the SIR model in M-Box 26.1, what are the units for m, c, and d? Assume t is days.
3. In R Code box 26.2, the maximum of $I(t)$ is related to what point on the $S(t)$ curve? Explain why this makes sense in the context of the SIR model.
4. In R Code box 26.2, the maximum of $I(t)$ is related to what point on the $R(t)$ curve? Explain why this makes sense in the context of the SIR model.

For the next set of problems (5-26) use the SIR model below (problems with an (R) in front require R, the others are to be done by hand):

$$\begin{aligned} S'(t) &= -0.00005I(t)S(t) \\ I'(t) &= 0.00005I(t)S(t) - 0.0625I(t) \\ R'(t) &= 0.0625I(t) \end{aligned}$$

with $S(0) = 20000$, $I(0) = 1000$, **and** $R(0) = 50$.

5. How long does someone have the disease?
6. Suppose the rate at which a susceptible comes in contact with an infected is 0.001. What is the transmission rate of the disease?
7. Suppose the transmission rate is 2%. What is the rate at which a susceptible comes in contact with an infected?
8. Why must $S(t) + I(t) + R(t) = 21050$ for all t?
9. Why must $S'(t) + I'(t) + R'(t) = 0$ for all t?
10. Is there a value of t such that $S'(t) = -R'(t)$? Is so when does it occur relative to either $S(t)$, $I(t)$ and/or $R(t)$?
11. In terms of $S(t)$, when does $I(t)$ have a maximum?
12. Why must $S'(t) \leq 0$ and $R'(t) \geq 0$? Answer in terms of the idea of the model and the equations.

13. Using steps of 1 day calculate $S(1)$, $I(1)$, and $R(1)$.
14. Using steps of 1 day calculate $S(2)$, $I(2)$, and $R(2)$.
15. Using steps of 1/2 day calculate $S(1)$, $I(1)$, and $R(1)$.
16. Using steps of 1/2 day calculate $S(2)$, $I(2)$, and $R(2)$.
17. (R) Estimate $S(5)$, $I(5)$, and $R(5)$ to a whole number. How many steps and what was the step sizes you used?
18. (R) Estimate $S(20)$, $I(20)$, and $R(20)$ to a whole number. How many steps and what was the step sizes you used?
19. (R) Estimate the time (two decimal places) and value of the maximum of $I(t)$.
20. (R) Estimate the first time (two decimal places) $S(t) < 5$.
21. (R) Suppose due to a quarantine effort we are able to cut the mc coefficient in half. How does this impact the system? Answer in terms of what you see in the graph. In particular, how does that change the local max of $I(t)$ in terms of both the value and the time?
22. (R) Suppose due to better drugs we are able to cut the number of days someone is ill in half. How does this impact the system? Answer in terms of what you see in the graph.
23. What is the end behavior of $S(t)$? What does this mean in this context?
24. What is the end behavior of $R(t)$? What does this mean in this context?
25. Estimate the maximum rate of increase of the recovered population.
26. Estimate the fastest rate of decrease of the susceptible population.

For the next set of problems (23-40) use the SIR model below (problems with an (R) in front require R, the others are to be done by hand):

$$\begin{aligned} S'(t) &= -0.00003I(t)S(t) \\ I'(t) &= 0.00003I(t)S(t) - 0.0833I(t) \\ R'(t) &= 0.0833I(t) \end{aligned}$$

with $S(0) = 30600$, $I(0) = 500$, **and** $R(0) = 125$.

27. Suppose the rate at which a susceptible comes in contact with an infected is 0.002. What is the transmission rate of the disease?
28. How long does someone have the disease?
29. Why must $S(t) + I(t) + R(t) = 31225$ for all t?
30. Suppose the transmission rate is 1.5%. What is the rate at which a susceptible comes in contact with and infected?
31. Is there a value of t such that $S'(t) = -R'(t)$? Is so when does it occur relative to either $S(t)$, $I(t)$ and/or $R(t)$?
32. Why must $S'(t) + I'(t) + R'(t) = 0$ for all t?

33. Why must $S'(t) \leq 0$ and $R'(t) \geq 0$? Answer in terms of the idea of the model and the equations.
34. In terms of $S(t)$, when does $I(t)$ have a maximum?
35. Using steps of 1 day calculate $S(2)$, $I(2)$, and $R(2)$.
36. Using steps of 1 day calculate $S(1)$, $I(1)$, and $R(1)$.
37. Using steps of 1/2 day calculate $S(2)$, $I(2)$, and $R(2)$.
38. Using steps of 1/2 day calculate $S(1)$, $I(1)$, and $R(1)$.
39. (R) Estimate $S(20)$, $I(20)$, and $R(20)$ to a whole number. How many steps and what was the step sizes you used?
40. (R) Estimate $S(5)$, $I(5)$, and $R(5)$ to a whole number. How many steps and what was the step sizes you used?
41. (R) Estimate the first time (two decimal places) $S(t) < 5$.
42. (R) Estimate the time (two decimal places) and value of the maximum of $I(t)$.
43. (R) Suppose due to better drugs we are able to cut the number of days someone is ill in half. How does this impact the system? Answer in terms of what you see in the graph.
44. (R) Suppose due to a quarantine effort we are able to cut the mc coefficient in a third. How does this impact the system? Answer in terms of what you see in the graph. In particular, how does that change the local max of $I(t)$ in terms of both the value and the time?
45. What is the end behavior of $S(t)$? What does this mean in this context?
46. What is the end behavior of $R(t)$? What does this mean in this context?
47. Estimate the maximum rate of increase of the recovered population.
48. Estimate the fastest rate of decrease of the susceptible population.
49. **SIS Model** Suppose a disease that spreads through populations is one where it can be contracted again immediately after recovering. In other words there isn't a recovered group only a susceptible and an infected group (maybe this should be called an SI model). Build a model similar to the SIR model in M-Box 26.1 for this scenario.
50. **SIR Model Variant** Suppose a disease that spreads through populations is one where it can be contracted again but only after a fixed number of days, say d_r, of being recovered. In this case there is a recovered group but the population can move from recovered to susceptible. Build a model similar to the SIR model in M-Box 26.1 for this scenario.

51. **SEIR Model** Suppose a disease that spreads through populations is one where a person is exposed before becoming infected. Assume that after d_e days there is a fifty-fifty chance of moving from exposed to infected or exposed to recovered. This disease can only be contracted once. Build a model similar to the SIR model in M-Box 26.1 for this scenario.
52. **SIRD Model** Suppose a disease that spreads through populations is one where an infected person has a p percent chance of recovering and a $1 - p$ percent chance of dying. The disease can only be contracted once. Build a model similar to the SIR model in M-Box 26.1 for this scenario.
53. **SIR Model Variant** Suppose in our SIR model that our population is growing at a rate proportional to the total population with rate r. All new people in the population are susceptible. Build a model similar to the SIR model in M-Box 26.1 for this scenario.
54. **SIR Model Variant** Suppose in our SIR model that our population of recovered people is decreasing proportional to the number of recovered people, with a death rate of d unrelated to this disease. Build a model similar to the SIR model in M-Box 26.1 for this scenario.

Chapter 27
Project: The Gini Coefficient—Prelude to Section III

The distribution of energy consumption in the U.S. (2014 data) and World (2011 data) can be modeled by $ECus(x) = 7.2038917391x^6 - 17.8551679663x^5 + 16.5816140612x^4 - 7.0654275059x^3 + 1.7077246274x^2 + 0.4260396828x$ and $ECw(x) = 678.0352163746x^9 - 2796.2519054480x^8 + 4802.0852334478x^7 - 4441.8091503689x^6 + 2389.4054597788x^5 - 751.8800491391x^4 + 132.3874503758x^3 - 11.3747211453x^2 + 0.3569478992x$. Both functions and the related data are shown in figure 3.8 in the Function Gallery chapter.

To interpret these functions consider the example: ECus(0.63)=0.47 means that the bottom 63% of states in the U.S. have per capita energy use in the bottom 48% of all states. The function ECw(x) is similar and replaces countries for states. These functions are Lorenz curves and are commonly used in the context of income.

Answer the following questions using these models. All responses to the questions must be typed and relevant work should be done in R. Copy your R code at the end of the document as an appendix. In general, all typed answers should be in sentence form. Part of your grade will be based on your use of the English language.

1. Find out how much energy per person the bottom 75% and the bottom 95% of U.S. States consume. Also, find out how much energy per person the bottom 75% and the bottom 95% of countries of the world consume. Write sentences to explain the meaning of your answers in both parts. Based on your answers, does it appear as if there is more energy consumption inequality in the U.S. or countries in the world? Explain why your answer makes sense in economic terms. Would you expect there to be more inequality within states than by state?
2. Should ECus(0)=0, ECus(1)=1, ECw(0)=0, and ECw(1)=1? Explain why or why not? How accurate are our models?
3. What function for ECus(x) would represent perfect equality in U.S. per capita energy consumption by state? Would this be the same for ECw(x)? Explain your response in both cases.
4. Find the value $x = c$ such that ECus'(c) = 1 Answer the same question for ECw(x). The values of c have a significant real world interpretation. What is it? Note: The

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existence of c is guaranteed by the Mean Value Theorem, which is not covered in this text.

5. Sketch a graph of ECus(x) and the function representing perfect equality from 3. Sketch a similar graph but with ECw(x). Based on these graphs, make conjectures about how one might measure how much ECw(x) deviates from equality and eventually decide which function ECus(x) or ECw(x) represent greater inequality. Calculate this if you can.

Part III
Accumulation and the Integral

Chapter 28
Area Under Curves

Up to this point we have developed techniques to extra information about curves related to their rate of change. We can now quantify how fast a curve is increasing or decrease, identify maximum and minimum points, and identify inflection points. There is still more valuable information in graphs that we would like to quantify. For example, in the Function Gallery figure 3.8 has data and models to represent distribution of energy consumption in the U.S. and World. The line $y = x$ would represent perfect equality of the distribution of energy. The area between the curve and $y = x$ is used to quantify how much the given resource, in this case energy, deviates from equality; known as the Gini coefficient (technically the Gini coefficient is this area divided by 2). The problem now is how do we calculate this area? Here is another example.

Figure 28.1 is a graph of total water flow at the USGS station on the Cayuga Inlet in Ithaca NY for the month of August in 2019. [35] Note the y-axis is a log scale and in units of cubic feet of water per second, while the x-axis is a date and time recorded every 15 minutes. At this point you recognize the derivative information in the graph. Steep positive slopes represent rain events increasing the water flow. After rain events the slopes are negative and the shape is concave up. The water is draining out of the 86.7 square mile watershed. There is more information in this graph that how fast the water is flowing. The area under the curve represents the total amount of water that passed through the gauge station over a given period of time. The question becomes how do we calculate area under a curve? As with derivatives we will start by estimating the area under a curve and work to improve that estimate.

Figure 28.2 represents three different ways to estimate the area under

$$f(x) = x^2 + 5$$

from $x = 0$ to $x = 8$. We are using rectangles or boxes to estimate area and in these graphs we have chosen four boxes, but note that more boxes would provide a better estimate. Once we decide on the number of boxes will partition the interval from $x = 0$ to $x = 8$ into four intervals of equal size. The size of each interval will be

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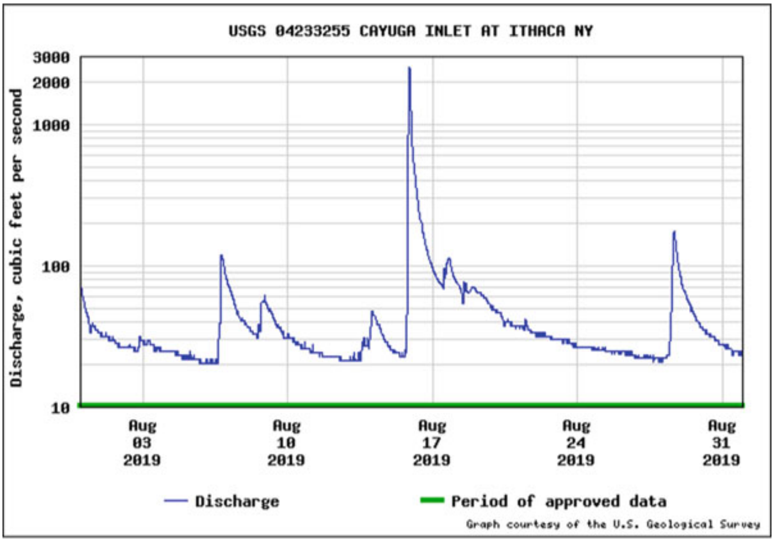


Fig. 28.1 Graph of discharge of Cayuga Lake inlet in cubic feet per second for Aug 2019.

$$\Delta x = \frac{8-0}{4} = 2.$$

In this case, the intervals are [0, 2], [2, 4], [4, 6], and [6, 8]. These intervals will be the base of our rectangles. We have three choices on the height of the boxes. In the first graph, the height of the box is given by the function value at the middle of each interval which are $f(1)$, $f(3)$, $f(5)$, and $f(7)$. In the middle graph, we used the left endpoint of each interval to determine the height with $f(0)$, $f(2)$, $f(4)$, and $f(6)$. In the last graph, the right endpoint of each interval to determine the height with $f(2)$, $f(4)$, $f(6)$, and $f(8)$. Example 28.1 provides some details of the calculations.

Example 28.1. *Estimate the area under $f(x) = x^2 + 5$ from $x = 0$ to $x = 8$ using four left boxes. See the middle graph in figure 28.2.*

Solution. Using four boxes from $x = 0$ to $x = 8$ we get $\Delta x = \frac{8-0}{4} = 2$ for the width of each box. The interval is partitioned by $x = 0, 2, 4, 6$ and 8. Since we are using left boxes the height of each box is given by

$$\begin{aligned} f(0) =& 0^2 + 5 = 5 \\ f(2) =& 2^2 + 5 = 9 \\ f(4) =& 4^2 + 5 = 21 \\ f(6) =& 6^2 + 5 = 41 \end{aligned}$$

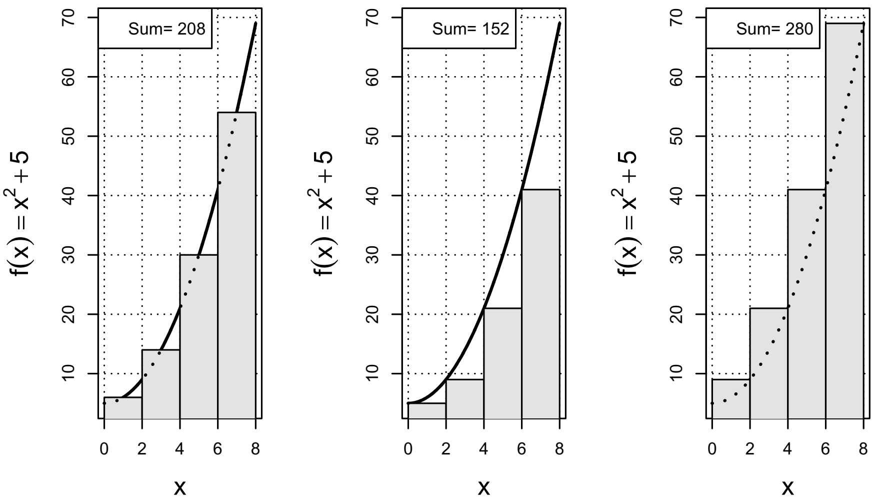


Fig. 28.2 Four Riemann sum boxes (midpoint, left, and right boxes) for $f(x) = x^2 + 5$ from $x = 0$ to $x = 8$.

The area of each box is then

$$\begin{aligned}\Delta x f(0) =& 2(5) = 10\\ \Delta x f(2) =& 2(9) = 18\\ \Delta x f(4) =& 2(21) = 42\\ \Delta x f(4) =& 2(41) = 82\end{aligned}$$

Hence, the estimated area is the sum of the area of the four boxes which is $10 + 18 + 42 + 82 = 52$. □

In order to increase the accuracy of our estimate we will increase the number of boxes which will decrease the base width, Δx, of each box. Figure 28.3 demonstrates how as the boxes increase and Δx decreases the accuracy of our estimate improves. The idea is the same as decreasing the value of h in our secant lines approximating our tangent lines. In this case we will let the number of boxes go to infinity

$$\lim_{n\to\infty}$$

where as in derivatives we had

$$\lim_{h\to 0}$$

Before moving forward we need summation notation, which is officially defined in M-Box 28.1 and is demonstrated in the next three examples. Note the similarities

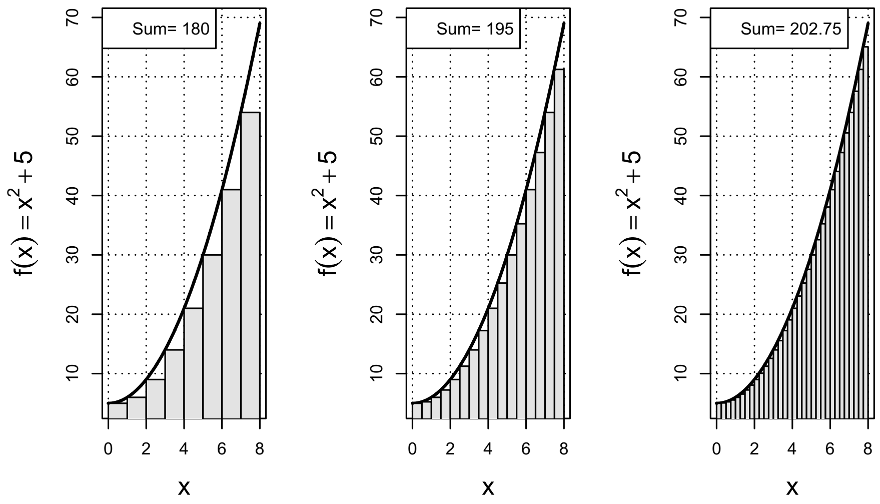


Fig. 28.3 Left boxes with 8, 16, and 32 boxes for $f(x) = x^2 + 5$ from $x = 0$ to $x = 8$.

between the summation notation and for loops. This notation will allow us to write out a formula for the sum of our boxes.

Example 28.2. *Evaluate* $\sum_{i=1}^{5} i$

Solution. By definition,

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15.$$

□

Example 28.3. *If* $f(x) = x^2 + 3$, *evaluate* $\sum_{i=0}^{6} f(i)$

Solution.

$$\sum_{i=0}^{6} f(i) = f(0) + f(1) + f(2) + f(3) + f(4) + f(5) + f(6)$$
$$= 3+(1+3)+(4+3)+(9+3) + (16+3)+(25+3) + (36+3) = 112$$

□

Example 28.4. *If* $f(x) = x^3 - 10$, $\Delta x = 0.5$, $(x_1, x_2, x_3, x_4) = (2, 2.5, 3, 3.5)$, *evaluate* $\sum_{i=1}^{4} f(x_i)\Delta x$.

Solution. Note that the sum here represents four left boxes of the function $f(x) = x^3 - 10$ from $x = 2$ to $x = 4$. We have

$$\begin{aligned}\sum_{i=1}^{4} f(x_i)\Delta x &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= f(2)0.5 + f(2.5)0.5 + f(3)0.5 + f(3.5)0.5 \\ &= (-2)0.5 + (5.625)0.5 + (17)0.5 + (32.875)0.5 = 53.5\end{aligned}$$

□

M-Box 28.1: Summation Notation

With $c \leq n$

$$\sum_{i=c}^{n} a_i = a_c + a_{c+1} + \cdots + a_{n-1} + a_n$$

The definition of a Riemann sum is given in M-Box 28.2. This definition algebraically captures the idea of letting the number of boxes go off to infinity. The idea of

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

is exactly the same as idea of

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The following definition, M-Box 28.3 defines the notation

$$\int_a^b f(x)dx$$

in the same way we defined $f'(x)$. Can we make sense of

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

as we were able to calculate

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}?$$

Well, it is in the book so the answer is probably yes.

One final note about M-Box 28.2 as

$$\int_a^b f(x)dx$$

is referred to as signed area. Notice that in

$$\sum_{i=0}^{n-1} f(x_i)\Delta x$$

the values of $f(x_i)$ can be negative meaning the function is below the x-axis. In this case the quantity $f(x_i)\Delta x$ will be negative. In other words, boxes below the x-axis contribute a negative quantity to the sum while boxes above the x-axis contribute a positive quantity. We can have

$$\int_a^b f(x)dx = 0$$

with "area" above the x-axis canceling with "area" below the x-axis. This will provide a useful interpretation where

$$\int_a^b f(x)dx$$

will represent a total change. For example, if we drive a car to the store and back home we can let $v(t)$ be the velocity of the car at time t. A negative velocity represents a direction, likely driving home. In this case

$$\int_a^b v(t)dt = 0$$

where a is the start time and b is the end time. The interpretation is that the total change of location is 0, we are back where we started.

M-Box 28.2: The Riemann Sum

If $f(x)$ is a continuous function on the interval $[a, b]$. Let $[a, b]$ be partitioned by $\{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}$ where $x_i - x_{i-1} = \frac{b-a}{n} = \Delta x$. In other words, the $\{x_i\}$ values split up the interval into n equal parts of length $\frac{b-a}{n}$, which is represented by Δx. Now, n left boxes is represented by

$$\sum_{i=0}^{n-1} f(x_i)\Delta x$$

and n right boxes is represented by

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

Note that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \tag{28.1}$$

if this limit exists.

M-Box 28.3: The Definite Integral

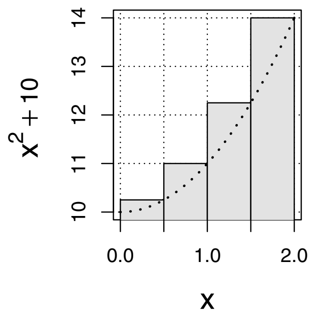
The signed area of between the continuous function $f(x)$ and the x-axis from $x = a$ to $x = b$ is represented by $\int_a^b f(x)dx$, the definite integral. In fact,

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x$$

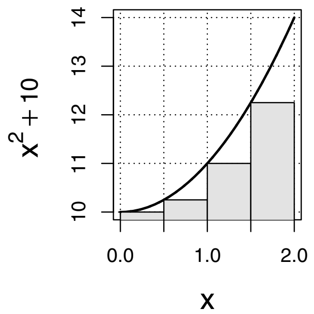
where $\sum_{i=0}^{n-1} f(x_i)\Delta x$ is the Riemann sum given in M Box 28.2. Note left or right boxes can be used here due to equation 28.1.

28.1 Exercises

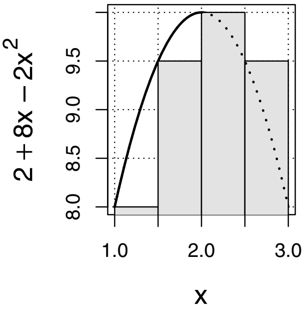
1. The graph shows four right boxes from $x = 0$ to $x = 2$. What is Δx, the area of each box, and the estimated area under the curve?



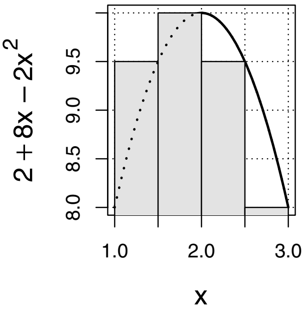
2. The graph shows four left boxes from $x = 0$ to $x = 2$. What is Δx, the area of each box, and the estimated area under the curve?



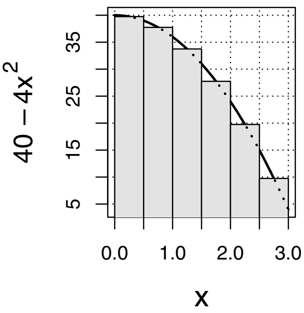
3. The graph shows four left boxes from $x = 1$ to $x = 3$. What is Δx, the area of each box, and the estimated area under the curve?



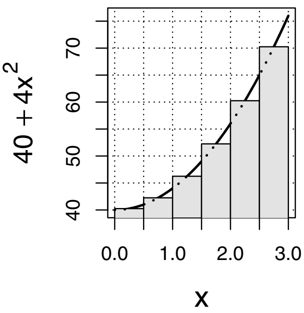
4. The graph shows four right boxes from $x = 1$ to $x = 3$. What is Δx, the area of each box, and the estimated area under the curve?



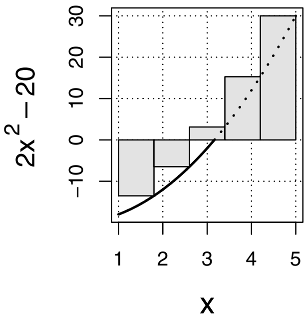
5. The graph shows six midpoint boxes from $x = 0$ to $x = 3$. What is Δx, the area of each box, and the estimated area under the curve?



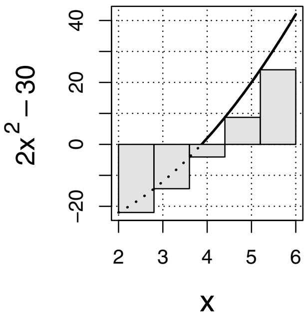
6. The graph shows six midpoint boxes from $x = 0$ to $x = 3$. What is Δx, the area of each box, and the estimated area under the curve?



7. The graph shows five right boxes from $x = 1$ to $x = 5$. What is Δx, the area of each box, and the estimated area under the curve?



8. The graph shows five left boxes from $x = 2$ to $x = 6$. What is Δx, the area of each box, and the estimated area under the curve?



9. Estimate the area under the function $f(x) = e^x$ from $x = 0$ to $x = 2$ using four right boxes. Include a sketch of the graph and boxes.

10. Estimate the area under the function $f(x) = x^3 + 2x^2 = 3$ from $x = -1$ to $x = 3$ using four right boxes. Include a sketch of the graph and the boxes.

11. Estimate the (signed) area under the function $f(x) = 2 - x^3$ from $x = 0$ to $x = 2$ using four left boxes. Include a sketch of the graph and boxes.
12. Estimate the (signed) area under the function $f(x) = 2x + 2$ from $x = -2$ to $x = 4$ using four left boxes. Include a sketch of the graph and boxes.
13. Estimate $\int_0^\pi \sin(x)dx$ using four left boxes. Include a sketch of the graph and boxes.
14. Estimate $\int_0^\pi \cos(x)dx$ using four left boxes. Include a sketch of the graph and boxes.
15. Estimate $\int_1^5 (x^2 + 3)dx$ using six right boxes. Include a sketch of the graph and boxes.
16. Estimate $\int_2^6 (x^3 - 4)dx$ using eight right boxes. Include a sketch of the graph and boxes.
17. Estimate $\int_1^3 (3x^2 - 10)dx$ using four left boxes. Include a sketch of the graph and boxes.
18. Estimate $\int_2^5 (15 - x^2)dx$ using six right boxes. Include a sketch of the graph and boxes.
19. Which is larger, $\int_0^1 x\,dx$ or $\int_0^1 2x\,dx$? Explain why.
20. Which is larger, $\int_0^1 x^2\,dx$ or $\int_0^1 2x^2\,dx$? Explain why.
21. Which is larger, $\int_0^5 \sqrt{x}\,dx$ or $\int_0^{10} \sqrt{x}\,dx$? Explain why.
22. Which is larger, $\int_0^2 e^x\,dx$ or $\int_0^4 e^x\,dx$? Explain why.
23. Which is larger, $\int_0^{2\pi} \sin x\,dx$ or $\int_0^{2\pi} \cos x\,dx$? Explain why.
24. Which is larger, $\int_0^\pi \sin x\,dx$ or $\int_0^\pi \cos x\,dx$? Explain why.
25. Which is larger, $\int_0^1 x^2\,dx$ or $\int_0^1 \sqrt{x}\,dx$? Explain why.
26. Which is larger, $\int_1^2 x^2\,dx$ or $\int_1^2 \sqrt{x}\,dx$? Explain why.
27. Which is larger, $\int_{-2}^0 x\,dx$ or $\int_{-3}^0 x\,dx$? Explain why.
28. Which is larger, $\int_{-3}^0 x\,dx$ or $\int_{-3}^0 2x\,dx$? Explain why.
29. Which is larger, $\int_{-5}^0 x^2\,dx$ or $\int_{-5}^0 x^3\,dx$? Explain why.
30. Which is larger, $\int_{-1}^0 x^2dx$ or $\int_{-1}^0 x^4\,dx$? Explain why.
31. The data below is from the graph in figure 28.1 and represents estimated hourly discharge on Aug 6, 2019 from 5pm to 11pm. Use left boxes to estimate the total discharge from 5pm to 11pm. Write a sentence using your results in context properly.

Hour (pm)	5	6	7	8	9	10	11
Discharge (ft^3/hr)	125280	139680	403200	414000	396000	363600	326880

32. The data below is from the graph in figure 28.1 and represents estimated hourly discharge on Aug 6, 2019 from 5pm to 11pm. Use right boxes to estimate the total discharge from 5pm to 11pm. Write a sentence using your results in context properly.

Hour (pm)	5	6	7	8	9	10	11
Discharge (ft^3/hr)	125280	139680	403200	414000	396000	363600	326880

33. The data below if from the graph in figure 28.1 and represents estimated hourly discharge on Aug 15, 2019 from 7am to 8:30am. Use right boxes to estimate the total discharge from 7am to 8:30am. Write a sentence using your results in context properly.

Hour (am)	7:00	7:15	7:30	7:45	8:00	8:15	8:30
Discharge (ft^3/hr)	247680	341640	756000	1749600	2523600	3672000	5112000

34. The data below is from the graph in figure 28.1 and represents estimated hourly discharge on Aug 15, 2019 from 7am to 8:30am. Use left boxes to estimate the total discharge from 7am to 8:30am. Write a sentence using your results in context properly.

Hour (am)	7:00	7:15	7:30	7:45	8:00	8:15	8:30
Discharge (ft^3/hr)	247680	341640	756000	1749600	2523600	3672000	5112000

35. Calculate $\sum_{i=1}^{5}(i^2-2)$.
36. Calculate $\sum_{i=2}^{7}(i^2+1)$.
37. Calculate $\sum_{i=0}^{5}(2i+5)$.
38. Calculate $\sum_{i=3}^{7}(2i^2-3)$.
39. Let $f(x)=5x^2-7$. Calculate $\sum_{i=1}^{4} f(i)$.
40. Let $f(x)=8-2x^2$. Calculate $\sum_{i=2}^{5} f(i)$.
41. Let $g(x)=x^3-1$. Calculate $\sum_{i=0}^{4} g(i)$.
42. Let $g(x)=1-x^3$. Calculate $\sum_{i=0}^{4} g(i)$.
43. Let $h(x)=x^2$. Calculate $\sum_{i=0}^{3} h(i^2)$.
44. Let $h(x)=10-x^2$. Calculate $\sum_{i=1}^{4} h(i^2)$.
45. (R Challenge) Use a for loop to calculate $\sum_{i=1}^{100} i$, $\sum_{i=1}^{1000} i$, and $\sum_{i=1}^{10000} i$. Do you think $\lim_{n\to\infty}\sum_{i=1}^{n} i$ equals ∞ or some number?
46. (R Challenge) Use a for loop to calculate $\sum_{i=1}^{100}\frac{1}{i}$, $\sum_{i=1}^{1000}\frac{1}{i}$, and $\sum_{i=1}^{10000}\frac{1}{i}$. Do you think $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i}$ equals ∞ or some number?
47. (R Challenge) Use a for loop to calculate $\sum_{i=1}^{100}\frac{1}{i^2}$, $\sum_{i=1}^{1000}\frac{1}{i^2}$, and $\sum_{i=1}^{10000}\frac{1}{i^2}$. Do you think $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i^2}$ equals ∞ or some number?
48. (R Challenge) Use a for loop to calculate $\sum_{i=1}^{100}\frac{1}{\sqrt{i}}$, $\sum_{i=1}^{1000}\frac{1}{\sqrt{i}}$, and $\sum_{i=1}^{10000}\frac{1}{\sqrt{i}}$. Do you think $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{i}}$ equals ∞ or some number?

Chapter 29
The Accumulation Function

In Chapter 9 we sketched graphs of the derivative of a function before we had a formal algebraic way to computer derivatives. We realized that we can view the derivative as a function related to the original function. Here we will proceed in a similar way and qualitatively graph how much (signed) area is accumulated between a function and the x-axis (note: we will often just say the area under the curve) as we move along the x-axis.

Consider figure 29.1, which has five graphs of $f(x) = 2x$ and a sixth graph which is a graph of the accumulation function $A(x)$ of $f(x) = 2x$. Unlike graphing the slopes of the tangent lines of a function we need a starting point. Here we start at $x = 0$. In the first graph the amount of (signed) area from $x = 0$ to $x = 0$ is 0. Hence $A(0) = 0$ and we plot $(0, 0)$ on the graph of $A(x)$. In the second graph we evaluate $A(1)$ by first shading in the area under $f(x) = 2x$ from $x = 0$ to $x = 1$. This shaded area is a triangle with base 1, height 2, and area 1. Hence $A(1) = 1$ and we plot $(1, 1)$ on the graph of $A(x)$.

In the third graph we evaluate $A(2)$ by first shading in the area under $f(x) = 2x$ from $x = 0$ to $x = 2$. This shaded area is a triangle with base 2, height 4, and area 4. Hence $A(2) = 4$ and we plot $(2, 4)$ on the graph of $A(x)$. In the fourth graph we evaluate $A(3)$ by first shading in the area under $f(x) = 2x$ from $x = 0$ to $x = 3$. This shaded area is a triangle with base 3, height 6, and area 9. Hence $A(3) = 9$ and we plot $(3, 9)$ on the graph of $A(x)$. In the fifth graph we evaluate $A(4)$ by first shading in the area under $f(x) = 2x$ from $x = 0$ to $x = 4$. This shaded area is a triangle with base 4, height 8, and area 16. Hence $A(4) = 16$ and we plot $(4, 16)$ on the graph of $A(x)$.

In the graph of $A(x)$, a point on the graph (a, b) means that area under $f(x) = 2x$ from $x = 0$ to $x = a$ is b, but note that the starting point $x = 0$ isn't represented on the graph of $A(x)$. The graph of $A(x)$ is increasing and concave up. As we look at the shaded areas in the other graphs as x gets larger we shade in more area and hence $A(x)$ is increasing. As we move to the right, we shade in more area than before. For example, the area from $x = 2$ to $x = 3$ is more than $x = 1$ to $x = 2$. Hence $A(x)$ is concave up.

T. J. Pfaff, *Applied Calculus with R*,
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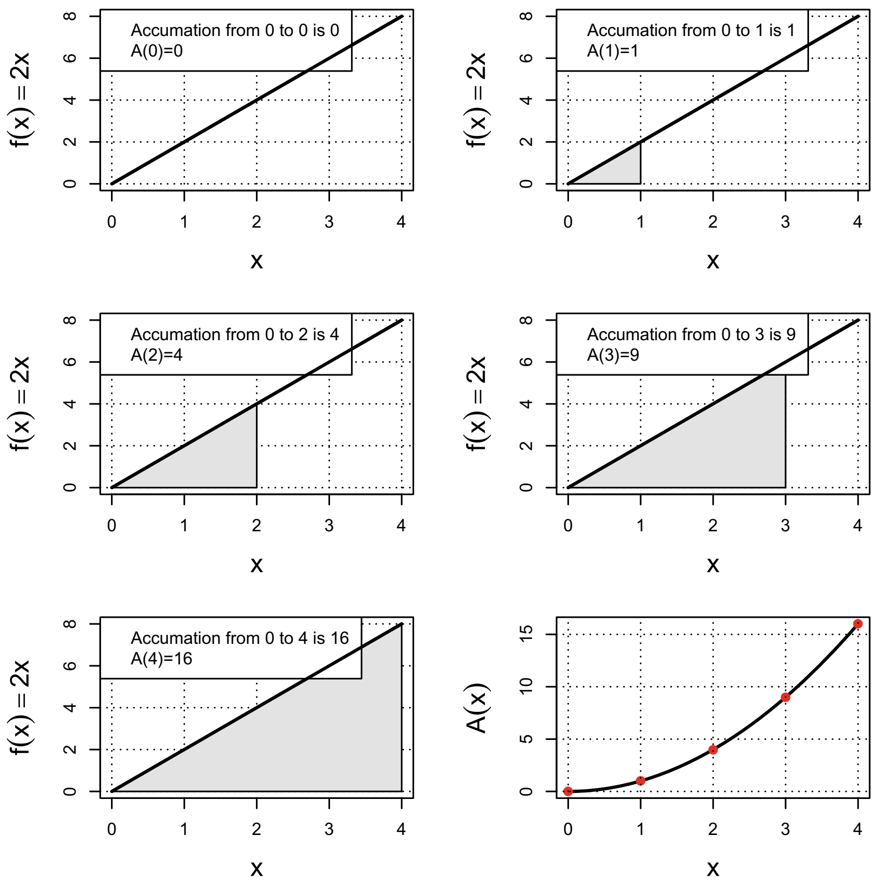


Fig. 29.1 Example of the accumulation function $A(x)$ with the function $f(x) = 2x$.

A formal definition of the accumulation function is given in M-Box 29.1 using our notation of the integral. In figure 29.1 $f(t) = 2t$ and $a = 0$ so that

$$A(x) = \int_a^x f(t)dt = \int_0^x 2t dt$$

In our five graphs with the shaded area we calculated

$$A(0) = \int_0^0 2t dt = 0$$

$$A(1) = \int_0^1 2t dt = 1$$

$$A(2) = \int_0^2 2t dt = 4$$

$$A(3) = \int_0^3 2t dt = 9$$

$$A(4) = \int_0^4 2t dt = 16$$

Notice that we changed the variable of our function f from x to t since in our expression we needed to differentiate between x the variable of $A(x)$ and t the variable of the function $f(t)$, otherwise we would have $A(x) = \int_a^x f(x)dx$ which we be (even more?) confusing. Let's do one more example.

M-Box 29.1: The Accumulation Function

Let $A(x) = \int_a^x f(t)dt$. The function $A(x)$ is the accumulation function of the function $f(t)$ with starting value a.

Consider figure 29.2, which has five graphs of $f(x) = \sin(x)$ and a sixth graph which is a graph of the accumulation function $A(x)$ of

$$f(x) = \sin(x)$$

Note that in the graphs we revert back to x as the variable for both functions since the x-axis is really the same for both $f(x)$ and $A(x)$. Here again we start at $x = 0$. In the first graph the amount of (signed) area from $x = 0$ to $x = 0$ is 0. Hence $A(0) = 0$ and we plot $(0, 0)$ on the graph of $A(x)$. In the second graph we evaluate $A(1.57)$ by first shading the area under

$$f(x) = \sin(x)$$

from $x = 0$ to $x = 1.57 = \pi/2$. This shaded area is 1 and we'll learn how to compute that later, but now that we know that we should understand the values of the shaded region for the next three graphs. Hence $A(1.57) = 1$ and we plot $(1.57, 1)$ on the graph of $A(x)$.

In the third graph we evaluate $A(3.14)$ by first shading in the area under $f(x) = \sin(x)$ from $x = 0$ to $x = 3.14 = \pi$. This shaded area is 2. Hence $A(3.14) = 2$ and we plot $(3.14, 2)$ on the graph of $A(x)$. In the fourth graph we evaluate $A(4.71)$ by first shading the area under

$$f(x) = \sin(x)$$

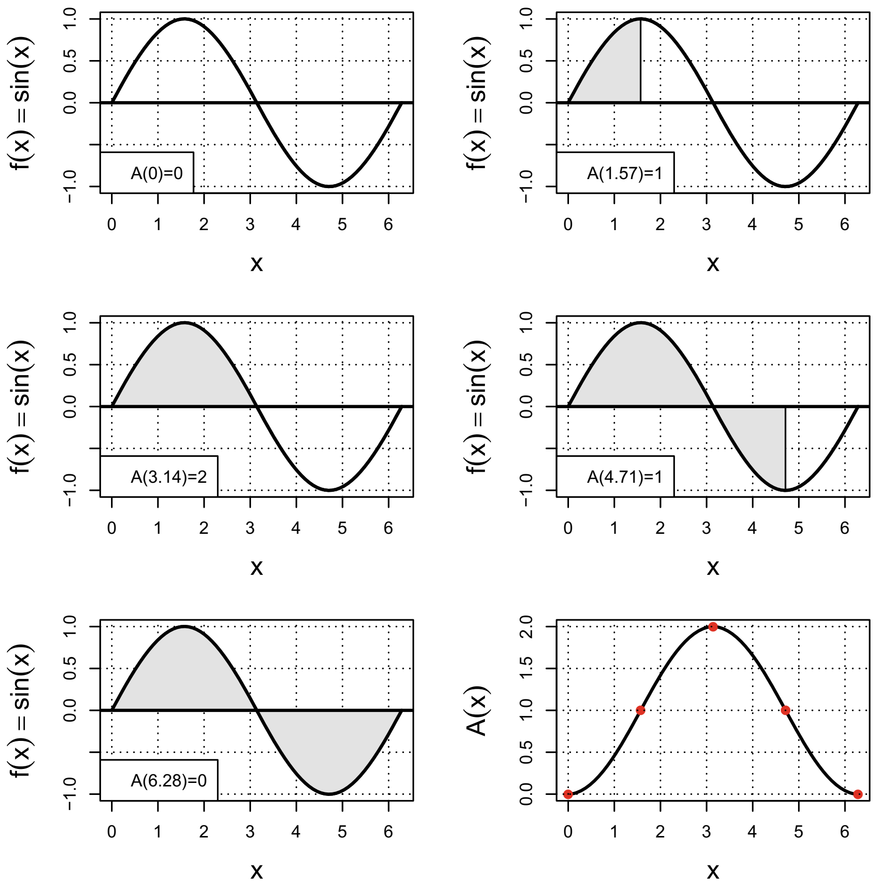


Fig. 29.2 Example of the accumulation $A(x)$ function with the function $f(x) = \sin(x)$.

from $x = 0$ to $x = 4.71 = 3\pi/2$. This shaded area is 1 because while the area from $x = 3.14$ to $x = 4.71$ is 1 is it below the axis and contributes a -1 to our sum. Hence $A(4.71) = 1$ and we plot $(4.71, 1)$ on the graph of $A(x)$. In the fifth graph we evaluate $A(6.28)$ by first shading in the area under $f(x) = \sin(x)$ from $x = 0$ to $x = 6.28 = 2\pi$. This shaded area is 0 because while the area from $x = 3.14$ to $x = 6.28$ is 2 is it below the axis and contributes a -2 to our sum and cancels with the area from $x = 0$ to $x = 3.14$ which is 2. Hence $A(6.28) = 0$ and we plot $(6.28, 0)$ on the graph of $A(x)$. In terms of the notation of M-Box 29.1 the five values of the shaded region give

$$A(0) = \int_0^0 \sin(t)dt = 0$$

$$A(1) = \int_0^{1.57} \sin(t)dt = 1$$

$$A(3.14) = \int_0^{3.14} \sin(t)dt = 2$$

$$A(4.71) = \int_0^{4.71} \sin(t)dt = 1$$

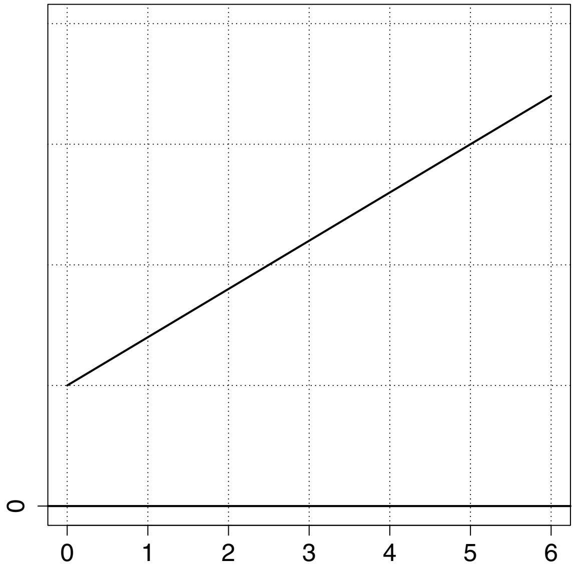
$$A(6.28) = \int_0^{6.28} \sin(t)dt = 0$$

A few observation about the graph of $A(x)$ that we must recognize. The point $(1.57, 1)$ on $A(x)$ is an inflection point. This is connected to the fact that $\sin(x)$ has a local maximum at $x = 1.57$. Once we pass $x = 1.57$ on $\sin(x)$ we are still accumulating area above the x-axis but that amount is going down. Hence, while $A(x)$ is still increasing after $x = 1.57$ the rate is decreasing or slowing down. This is the exact relationship between a graph and its derivative. If we start with $A(x)$ and sketch its derivative we would have a local maximum at $x = 1.57$.

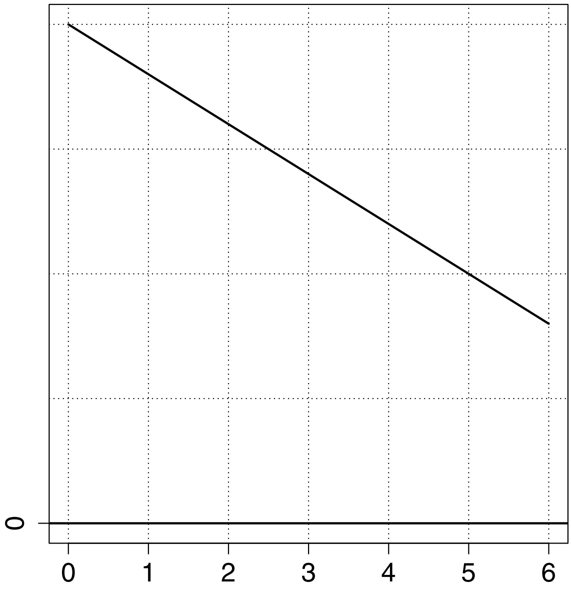
Another observation is that the local maximum of $A(x)$ at $(3.14, 2)$ is connected to the fact that $\sin(x)$ crosses the x-axis at $x = 3.14$. Once $\sin(x)$ crosses the x-axis the area we accumulate is below the axis and hence subtracted from our current total. This means that $A(x)$ must start to decrease. In the reverse direction, since $A(x)$ has a local maximum at $(3.14, 2)$ means the slope of the tangent line is 0. Further, $A(x)$ is decreasing after $(3.14, 2)$ and so the slope of the tangent lines are negative and its slope or derivative graph must be below the x-axis, which $\sin(x)$ is. Finally, $(4.71, 1)$ on $A(x)$ is an inflection point corresponding to the local maximum of $\sin(x)$ at $x = 4.71$. At this point we should recognize that the accumulation function is in some way an operation that reverses the derivative. We will formalize this relationship in the next chapter and it will allow us to calculate the (signed) area under a function.

29.1 Exercises

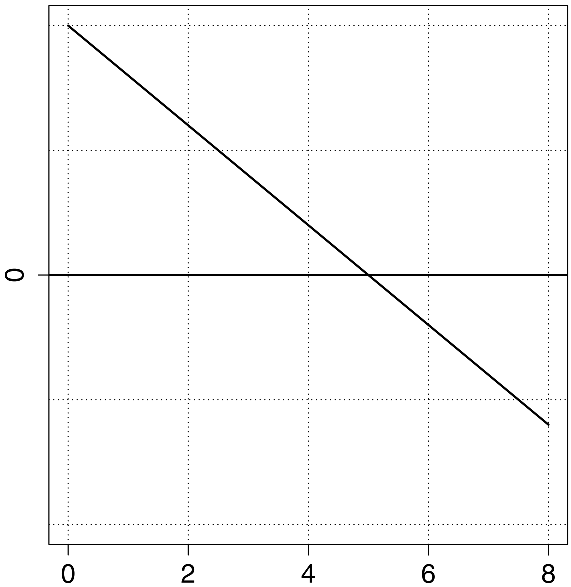
1. Graph the accumulation function, including concavity, of the given function in the graph. Use the facts that $\int_0^1 f(t)dt = 6$, $\int_0^3 f(t)dt = 24$, and $\int_0^6 f(t)dt = 66$.



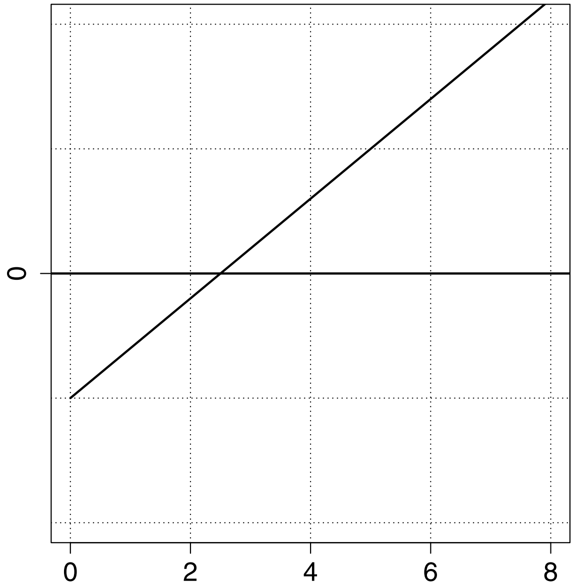
2. Graph the accumulation function, including concavity, of the given function in the graph. Use the facts that $\int_0^1 f(t)dt = 19$, $\int_0^3 f(t)dt = 51$, and $\int_0^6 f(t)dt = 84$.



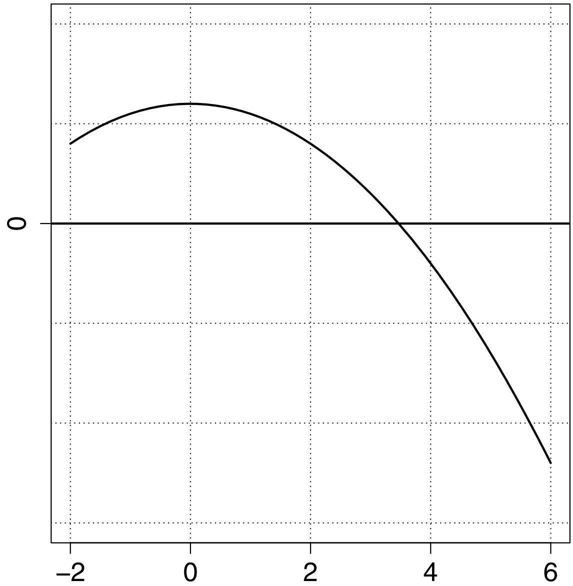
3. Graph the accumulation function, including concavity, of the given function in the graph. Use the facts that $\int_0^2 f(t)dt = 16$, $\int_0^5 f(t)dt = 25$, and $\int_0^8 f(t)dt = 16$.



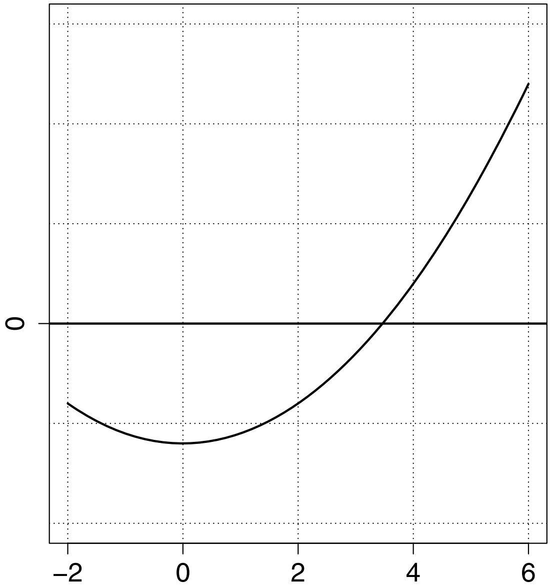
4. Graph the accumulation function, including concavity, of the given function in the graph. Use the facts that $\int_0^2 f(t)dt = -6$, $\int_0^5 f(t)dt = 0$, and $\int_0^8 f(t)dt = 24$.



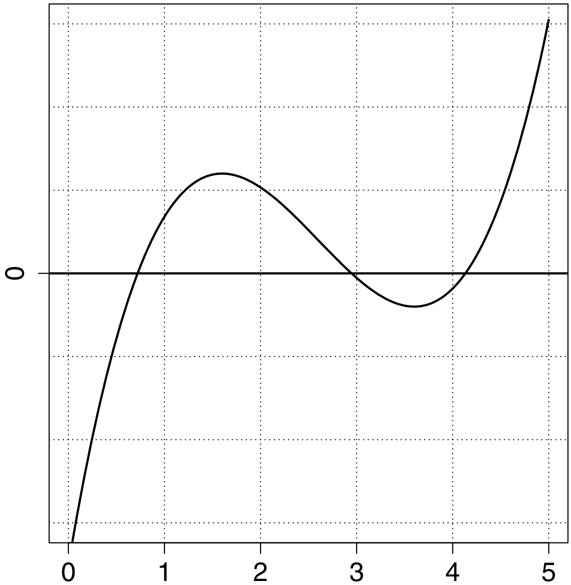
5. Graph the accumulation function, including concavity, of the given function in the graph. Use the facts that $\int_{-2}^{-1} f(t)dt = 5$, $\int_{-2}^{4} f(t)dt = 24$, and $\int_{-2}^{6} f(t)dt = 11$.



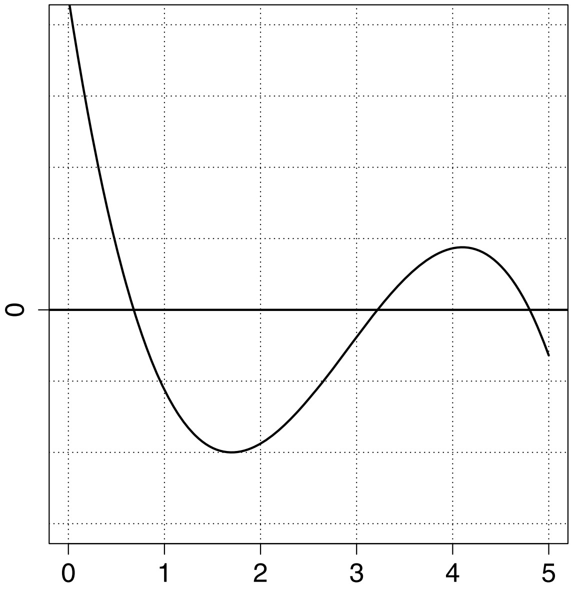
6. Graph the accumulation function, including concavity, of the given function in the graph. Use the facts that $\int_{-2}^{-1} f(t)dt = -5$, $\int_{-2}^{2} f(t)dt = -21$, and $\int_{-2}^{6} f(t)dt = -11$.



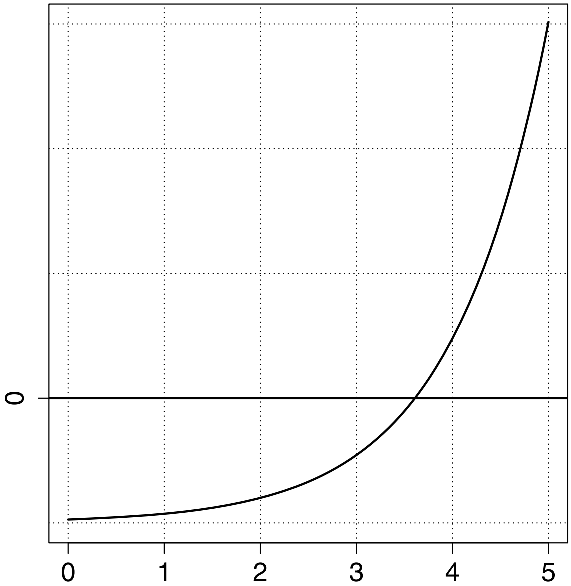
7. Graph the accumulation function, including concavity, of the given function in the graph. Use the facts that $\int_{0}^{1} f(t)dt = -5$, $\int_{0}^{3} f(t)dt = 3$, and $\int_{0}^{5} f(t)dt = 7$.



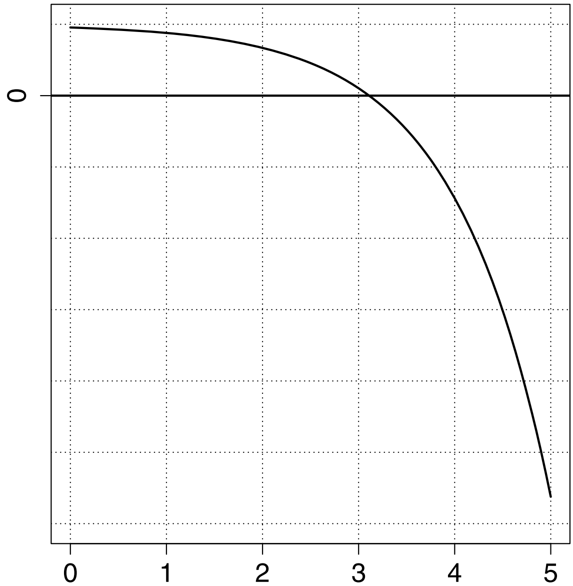
8. Graph the accumulation function, including concavity, of the given function in the graph. Use the facts that $\int_0^1 f(t)dt = 6$, $\int_0^3 f(t)dt = -9$, and $\int_0^5 f(t)dt = -5$.



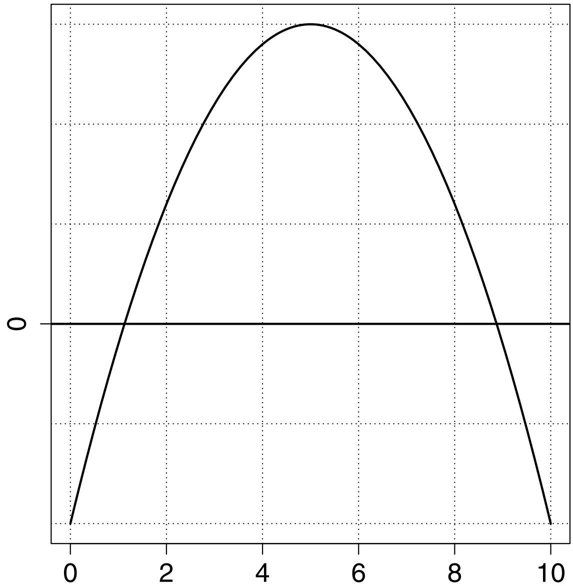
9. Graph the accumulation function, including concavity, of the given function in the graph.



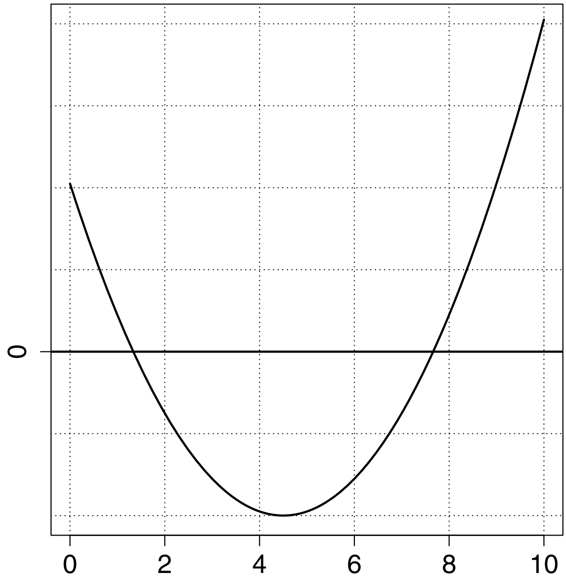
10. Graph the accumulation function, including concavity, of the given function in the graph.



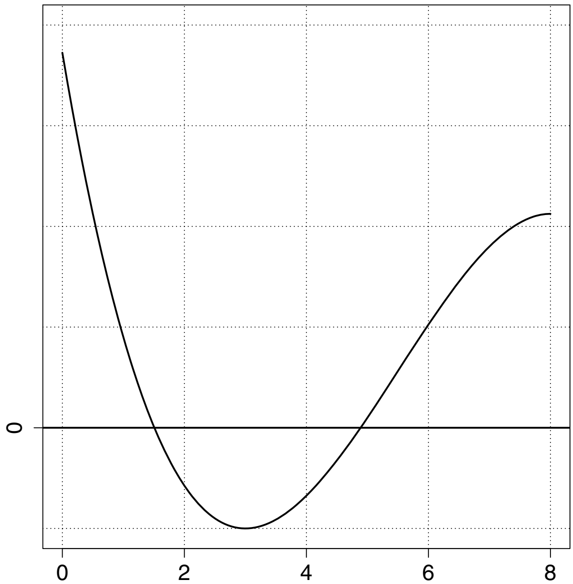
11. Graph the accumulation function, including concavity, of the given function in the graph.



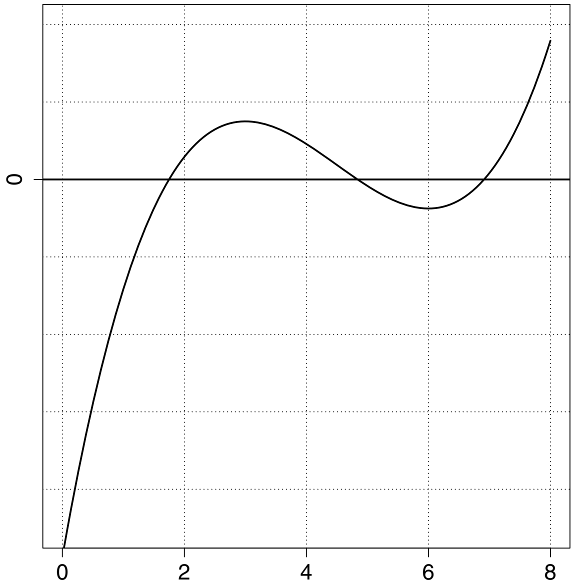
12. Graph the accumulation function, including concavity, of the given function in the graph.



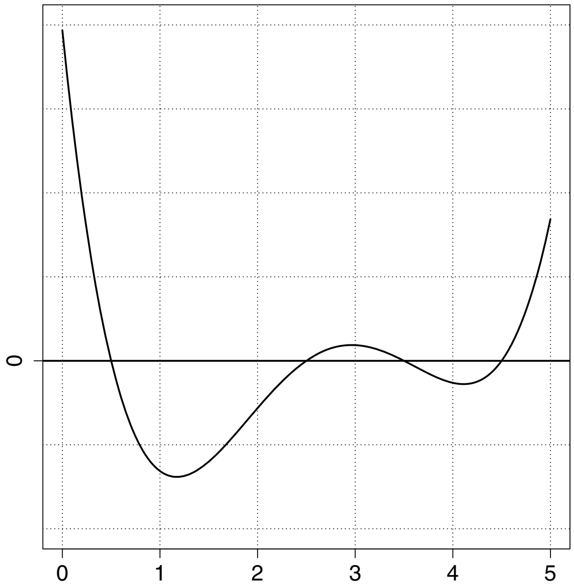
13. Graph the accumulation function, including concavity, of the given function in the graph.



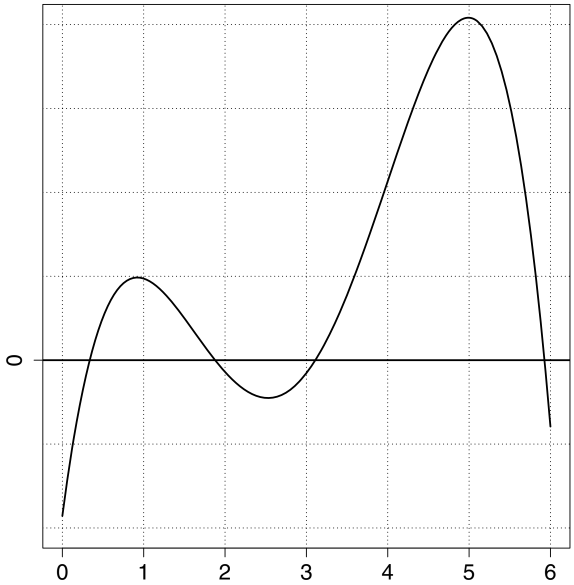
14. Graph the accumulation function, including concavity, of the given function in the graph.



15. Graph the accumulation function, including concavity, of the given function in the graph.



16. Graph the accumulation function, including concavity, of the given function in the graph.



29.2 Project: Hubbard Brook - The Importance of a Watershed

Definitions: Watershed: The area that drains into a stream, river, or lake. Normalize: To make conform to or reduce to a norm or standard.

Exercise: Below you will find scatter plots in figure 29.3 and data in table 29.1 for the mean daily stream flow from watersheds 2 and 6 of the Hubbard Brook Ecosystem Study for July 1966. [23] Data is collected from stream gauges at the head of each watershed. Each watershed has similar characteristics and they are near each other. The size of the watershed and rainfall for the month of July for watershed 2 and 6 is 11.8 hectares and 98mm, and 13.2 hectares and 99.6 mm, respectively. The one difference is that watershed 2 had all trees and shrubs cut and left in place the previous winter while watershed 6 was left as is to serve as a control. Answer the following questions in the form of a typed short report. The table of data is available at the companion web site https://sustainabilitymath.org/acr/.

1. Based on the graphs in figure 29.3 does it appear that the runoff during July 1966 from both watersheds are similar? If so why, if not why not? Are there any notable differences between the two graphs?
2. Use a Riemann sum to calculate the total water flow for each watershed for July 1966. Explain what type of Riemann sum you used (left, right, midpoint) and why you used it for this data.
3. Normalize your total flow from both watersheds so you can compare the total flow fairly (total rainfall and size are not exactly the same for both).
4. Comment on the ecosystem service provided by the trees and shrubs and how clear cutting and development may impact flooding.

Day in July 1966	Watershed 2 Streamflow (ft^3/day)	Watershed 6 Streamflow (ft^3/day)
1	6857.32608	598.30272
2	6239.94624	308.49984
3	5832.03456	182.29536
4	5374.51200	168.27264
5	4817.76768	88.81056
6	64488.63168	12031.49376
7	55845.31392	10820.86560
8	24110.88768	2743.77888
9	16206.22080	1205.95392
10	15831.38304	916.15104
11	32936.11200	4197.46752
12	20373.53472	1729.46880
13	15958.16640	1233.99936
14	13185.46944	747.87840
15	11515.23648	406.65888
16	10704.92544	238.38624
17	9690.65856	158.92416
18	8527.55904	116.85600
19	11790.85248	434.70432
20	14161.15008	1089.09792
21	10919.90592	369.26496
22	9222.11136	252.40896
23	8185.79520	172.94688
24	7270.75008	102.83328
25	6438.38976	56.09088
26	5997.40416	32.71968
27	5346.95040	23.37120
28	6559.66080	112.18176
29	6146.23680	294.47712
30	5247.72864	102.83328
31	4872.89088	46.74240

Table 29.1 Mean daily stream flow data of watershed 2 and watershed 6 for July 1966 from the Hubbard Brook Ecosystem study

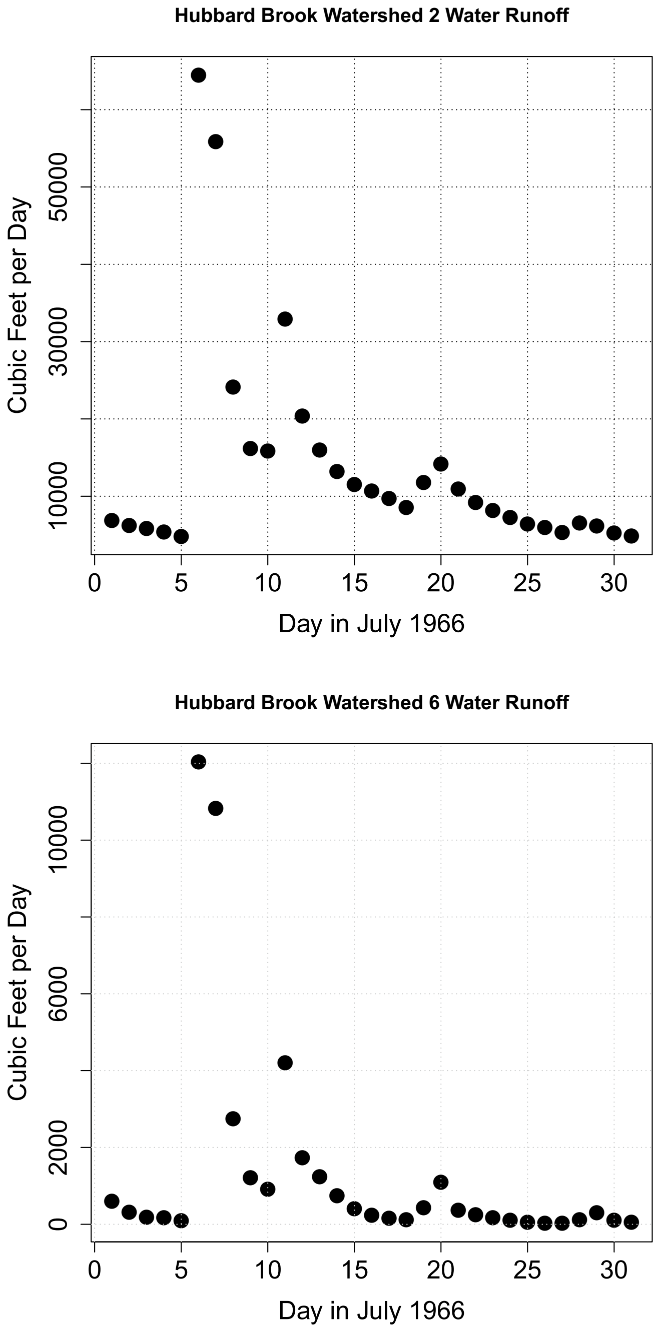


Fig. 29.3 Mean daily steam flow for July 1966 from watershed 2 and 6 of the Hubbard Brook Ecosystem study. Watershed 2 was clear cut the previous winter. Watershed 6 was left as is to serve as a control watershed.

Chapter 30
The Fundamental Theorem of Calculus

From Chapter 29 we recognize that there is a relationship between the accumulation function and the derivative. Specifically, if we start with the graph of $f(x)$ and sketch its accumulation function $A(x)$ then the slope or derivative graph of $A(x)$ is just $f(x)$ again. This relationship is formalized with The Fundamental Theorem of Calculus given in M-Box 30.1. Note that the starting point, given by a in the formula, is not specified. In fact, any a will do as the effect of a is to simply shift $A(x)$ up or down. Shifting a function up or down does not change its derivative.

M-Box 30.1: The Fundamental Theorem of Calculus

If $A(x)$ is an accumulation function for a continuous function $f(t)$ then

$$\frac{d}{dx}A(x) = f(x) or A'(x) = f(x)$$

or

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x).$$

At the moment, The Fundamental Theorem of Calculus given in M-Box 30.1 does not appear to help us find the area under a curve. To get there we first need the definition on an antiderivative in M-Box 30.2, which says that $F(x)$ is an antiderivative of the function $f(x)$ if $F'(x) = f(x)$. For example, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ because the derivative of x^2 is $2x$. We should immediately recognize that an antiderivative is not unique as $F(x) = x^2 + 42$ or $F(x) = x^2 - 99$ are also antiderivatives of $f(x)$. We should also recognize that we can find antiderivatives with a little guess and check as the next examples show.

M-Box 30.2: The Antiderivative

The function $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$. In other words, the derivative of $F(x)$ is $f(x)$. We use

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_30

$$\int f(x)dx$$

to represent the general antiderivative of $f(x)$.

Example 30.1. *What is an antiderivative of $f(x) = x^2$?*

Solution. Based on our experience with derivatives we know that when we take a derivative of a polynomial the power is reduced by one. It would make sense than that our first guess for an antiderivative $F(x)$ would be x^3. We now check by taking the derivative of x^3 to get $3x^2$. We are close but we have an extra 3 that we don't want. Recall that when we multiply a function by a constant that constant also multiplies the derivative. To cancel out the 3, we can multiply by $1/3$. Our adjusted guess is now

$$F(x) = \frac{x^3}{3}$$

which is a correct antiderivative of $f(x) = x^2$. We can be more general and go with

$$F(x) = \frac{x^3}{3} + c$$

where c is any constant (or number). Checking our guess to be certain we have it correct:

$$F'(x) = \frac{3x^2}{3} + 0 = x^2 = f(x)$$

□

Example 30.2. *What is an antiderivative of $f(x) = \sin(x)$?*

Solution. We know that $\sin(x)$ and $\cos(x)$ are derivatives of each other so our first guess would be $\cos(x)$. When we check we recognize that the derivative of $\cos(x)$ is $-\sin(x)$ and so we are off by a -1. Our adjusted guess is then $F(x) = -\cos(x)$ or for a general antiderivative

$$F(x) = -\cos(x) + c$$

Checking our guess to be certain we have it correct:

$$F'(x) = -(-\sin(x)) + 0 = \sin(x) = f(x)$$

□

M-Box 30.3: The Definite Integral

If $f(x)$ is continuous function with an antiderivative $F(x)$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

is the definite integral of $f(x)$ from $x = a$ to $x = b$. The values a and b are the limits of integration with a the lower limit and b the upper limit. **Units**: In the last expression the units on $F(b) - F(a)$ are the input units of $f(x)$ times the output units of $f(x)$. Also, $F(b) - F(a)$ is the formula for change and represents the total change (in terms input units of $f(x)$ times the output units of $f(x)$) over the interval $x = a$ to $x = b$. The definite integral also represents the (signed) area between the function and the x–axis.

With all the algebraic tools in hand we can now state a formula for The Definite Integral in M-Box 30.3. If we want to compute the (signed) area under the curve $f(x)$ from $x = a$ to $x = b$ then we need to find any antiderivative $F(x)$ and find the difference $F(b) - F(a)$. An example:

Example 30.3. *Find* $\int_1^4 x^2 dx$*?*

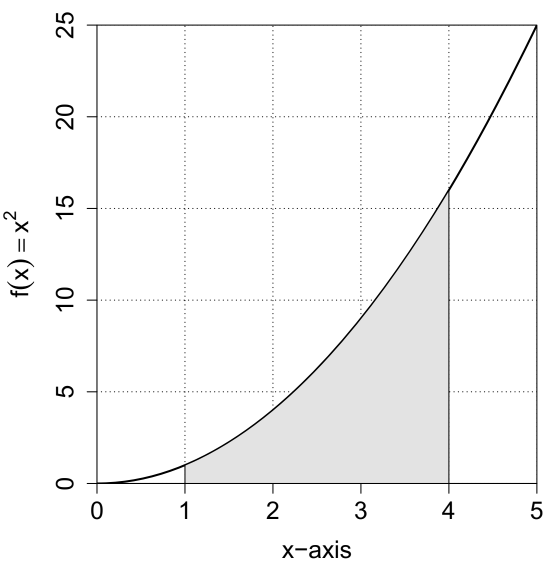


Fig. 30.1 The region represented by $\int_1^4 x^2 dx$ is shaded.

Solution. Figure 30.1 has the area shaded that is represented by

$$\int_1^4 x^2 dx$$

To calculate this definite integral we first need an antiderivative. In example 30.1 we found that

$$F(x) = \frac{x^3}{3}$$

is an antiderivative. Note we can use any antiderivative and so it is easiest to use one where $c = 0$. Now

$$F(4) = \frac{4^3}{3} = 64/3$$

and

$$F(1) = \frac{1^3}{3} = 1/3$$

and so

$$\int_1^4 f(x)dx = F(4) - F(1) = 64/3 - 1/3 = 63/3 = 21$$

The shaded area in figure 30.1 is 21 $f(x)$ input units times $f(x)$ output units. $\square$

The real challenge in finding the area under the curve is our ability to compute antiderivatives. It turns out that, in general, finding antiderivatives is more challenging than computing derivatives. Still, there is a lot we can do and we start with reversing the derivatives in Basic Derivative Rules M-Box 11.1. This list is given in Basic Antiderivative Rules M-Box 30.4.

M-Box 30.4: Basic Antiderivative Rules

Function	Antiderivative	
$f(x) = k$	$F(x) = kx + C$	(30.1)
$f(x) = mx + b$	$F(x) = mx^2/2 + bx + C$	(30.2)
$f(x) = x^n$	$F(x) = \frac{1}{n+1}x^{n+1} + C$	(30.3)
$f(x) = \sin(x)$	$F(x) = -\cos(x) + C$	(30.4)
$f(x) = \cos(x)$	$F(x) = \sin(x) + C$	(30.5)
$f(x) = e^x$	$F(x) = e^x + C$	(30.6)
$f(x) = a^x$	$F(x) = \frac{1}{\ln(a)}a^x + C$	(30.7)
$f(x) = \frac{1}{x}$	$F(x) = \ln(x) + C$	(30.8)

Antiderivatives inherit two other rules from derivatives. We get a constant multiple rule for antiderivatives, M-Box 30.5, which is the equivalent of the constant multiple rule M-Box 11.2. We also have a sum and difference rule for antiderivatives, M-Box 30.6, which is the equivalent of the sum and difference rule M-Box 11.3.

M-Box 30.5: Theorem-Constant Multiple Rule for Antiderivatives

The antiderivative of $cf(x)$ is $cF(x)$. In other words, $\int cf(x)dx = c\int f(x)dx$.

M-Box 30.6: Theorem-Sum and Difference Rule for Antiderivatives

The antiderivative of $f(x) + g(x)$ is $F(x) + G(x)$ and the antiderivative of $f(x) - g(x)$ is $F(x) - G(x)$. In other words, $\int(f(x) + g(x))dx = F(x) + G(x)$ and similarly for the difference.

Here is an example using some of our new antiderivative rules:

Example 30.4. *Find* $\int \left(12x^5 + 4^x - 5\sqrt{x}\right) dx$.

Solution. The sum and difference rule allow us to treat each piece of the integrand individually. In other words, we need to find the antiderivative of $12x^5$, 4^x, and $5\sqrt{x}$. Using the constant multiple rule and rule 30.3 we have

$$\int 12x^5 dx = 12\int x^5 dx = 12\frac{x^6}{6} = \frac{12x^6}{6} = 2x^6$$

Using rule 30.7 we have

$$\int 4^x dx = \frac{1}{\ln(4)}4^x = \frac{4^x}{\ln(4)}$$

For the last term we will convert $\sqrt{x} = x^{1/2}$ just as we did with derivatives. Using the constant multiple rule 30.3 we have

$$\int 5\sqrt{x}dx = 5\int x^{1/2}dx = 5\frac{x^{(1/2)+1}}{3/2} = 5\frac{2x^{3/2}}{3} = \frac{10x^{3/2}}{3}$$

Note how we simplified the $\frac{1}{3/2}$ to $2/3$. Putting it all together we get

$$\int \left(12x^5 + 4^x - 5\sqrt{x}\right) dx = 2x^6 + \frac{4^x}{\ln(4)} - \frac{10x^{3/2}}{3} + c$$

Notice that we add the c at the end as opposed to adding a constant for each of the three pieces. This is the same since we would simply add the three constants from each piece and call it c. □

What about integration in R? There is an algebraic package that does symbolic integration called Ryacas, but it is more than we need and it would take some time to learn. Base R does have numerical integration and we provide an example in R Code box 30.1. Simply define the function and use **integrate(f,a,b)** where a and b are the limits of integration. R is using an numeric algorithm, such as lots of Riemann sums, to find the value and it provides an estimation of the error. In this example the error is less than 0.00000000000023. Of course, we calculated this value exactly as 21 in example 30.3 but the **integrate** function can only estimate the value, although with a very small possible error. There is a place for exact values calculated with antiderivatives and numerical estimates with R. It is not the case that one way is necessarily better than the other.

R Code 30.1: Numerical Integration

```
> f<-function(x){x^2}
> integrate(f,1,4)
```

```
21 with absolute error < 2.3e-13
```

30.1 Exercises

Evaluate the Expression

1. $\int x^4 dx$
2. $\int 10 dx$
3. $\int x dx$
4. $\int x^3 dx$
5. $\int 9\sin(x) dx$
6. $\int 11\cos(x) dx$
7. $\int 5^x dx$
8. $\int 8^x dx$
9. $\int 5x^3 dx$
10. $\int -3x^6 dx$
11. $\int \frac{2}{x^7} dx$
12. $\int \frac{-8}{x^{14}} dx$
13. $\int 5\sqrt{x} dx$
14. $\int -2\sqrt[3]{x} dx$
15. $\int \frac{6}{5\sqrt{x}} dx$
16. $\int \frac{-7}{9\sqrt[3]{x}} dx$
17. $\int (3x^2 + 5x + 8) dx$
18. $\int (5x^3 - 4x + 12) dx$
19. $\int (2x^4 + 7x^2 - 15) dx$
20. $\int (8x^4 - 9x^2 + 18) dx$
21. $\int (4e^x + x^{-5} - 2\sqrt{x}) dx$
22. $\int (2e^x + x^{-11} - 7\sqrt[3]{x}) dx$
23. $\int (4(9^x) + 3x^{-8} - 56\cos(x)) dx$
24. $\int (-3(7^x) - 4x^{-9} + 15\sin(x)) dx$
25. $\int \left(\frac{5}{x} + 3\sin(x) + 7\sqrt[4]{x}\right) dx$
26. $\int \left(\frac{-6}{x} + 9\cos(x) - 11\sqrt{x}\right) dx$
27. $\int \left(\frac{7}{x^8} - 9(5^x) - 82x^5\right) dx$
28. $\int \left(\frac{-4}{x^2} - 8(4^x) + 42x^{11}\right) dx$

Evaluate the expression leaving your result in exact form and then check your work using R.

29. $\int_0^4 3x^2 dx$
30. $\int_1^7 6x^2 dx$
31. $\int_0^{\pi} \sin(x) dx$
32. $\int_0^{2\pi} \cos(x) dx$
33. $\int_8^{20} 3e^x dx$
34. $\int_{-4}^{4} 7e^x dx$
35. $\int_{-\pi}^{\pi} 7\sin(x) dx$
36. $\int_{-2\pi}^{2\pi} -2\cos(x) dx$
37. $\int_{-5}^{0} 2x^3 dx$
38. $\int_{-7}^{-2} 7x^5 dx$
39. $\int_1^9 \frac{2}{x} dx$
40. $\int_2^8 \frac{7}{x} dx$
41. $\int_{-3}^{1} 3(7^x) dx$
42. $\int_{-5}^{-2} 8(9^x) dx$
43. $\int_{-10}^{-1} \frac{2}{3x^5} dx$
44. $\int_2^9 \frac{-7}{8x^4} dx$
45. $\int_0^4 3\sqrt{x} dx$
46. $\int_0^8 5\sqrt[3]{x} dx$
47. $\int_{-2}^{0} 4(3^x) dx$
48. $\int_2^5 -4(7^x) dx$
49. $\int_1^9 \left(4\sqrt{x} - \frac{7}{3\sqrt{x}}\right) dx$
50. $\int_8^{27} \left(7\sqrt[3]{x} - \frac{5}{4\sqrt[3]{x}}\right) dx$

51. $\int_{2}^{10} \left(2x^3 - 27x^2 + 84x + 36\right) dx$

52. $\int_{-5}^{6} \left(-2x^3 + 10x^2 + 40x - 30\right) dx$

In the following problems you will first set up an appropriate integral and then calculate it. Use R as desired.

53. An empty balloon (a perfect sphere for this problem) is being filled with air at a rate of 0.5 cubic inches per second. How much air was put in the balloon after 5 seconds?

54. An empty balloon that is in the shape of a cube is being filled with air at the rate of 0.6 cubic cm per second. How much air was put in the balloon from 1 second to 6 seconds?

55. A watermelon is launched up into the air. The velocity of the watermelon is given by $v(t) = 128 - 32t$ feet per second. How far from the ground is the watermelon at time $t = 4$ seconds?

56. The average car can go from 0 to 60 mph in about 8 seconds. The velocity of a car is given by $v(t) = 11t$ feet per second. How far does the car travel in 6 seconds?

57. If you have taken a statistics course (and you should) you will be familiar with the 68-95-99.7 rule which says that 68% of values are within one standard deviation, 95% of values are within one standard deviation, and 99.7% of values are within one standard deviation. This is calculated as the area under the normal density function. The standard normal density, mean 0, and standard deviation 1, is given by $f(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$ Verify the 68% rule by integrating from -1 to 1.

58. If you have taken a statistics course (and you should) you will be familiar with the 68-95-99.7 rule which says that 68% of values are within one standard deviation, 95% of values are within one standard deviation, and 99.7% of values are within one standard deviation. This is calculated as the area under the normal density function. The standard normal density, mean 0 and standard deviation 1, is given by $f(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$ Verify the 95% rule.

59. In figure 28.1 there is a surge of water flow that starts around Aug 6 at 6:00pm (time 0) and ends around Aug 7 at 11:30 pm (time 30.75). We modeled the data with a surge function similar to figure 3.10 with $a = 152659.795954899$ and $b = -0.225525145773889$. Two adjustments were made to the data to be able to fit a surge function to the data. Specifically, the data had to be shifted to start at $(0, 0)$. The first is that Aug 6 at 6:00pm is time $t = 0$ and hence Aug 7 at 11:30 pm is time $t = 30.75$. The second is that the water flow was shifted down by 125280 cubic feet per hour (do not forget this when answering the question). Find the total volume of water that flowed into the Cayuga inlet from Aug 6 at 6:00pm to Aug 7 at 11:30 pm. Write a sentence using your results in context properly.
60. In figure 28.1 there is a surge of water flow that starts around Aug 28 at 2:00pm (time 0) and ends around Aug 29 at 6:45am (time 16). We modeled the data with a surge function similar to figure 3.10 with $a = 397406.090506807$ and $b = -0.39484448003015$. Two adjustments were made to the data to be able to fit a surge function to the data. Specifically, the data had to be shifted to start at $(0, 0)$. The first is that Aug 28 at 2:00pm is time $t = 0$ and hence Aug 29 at 6:45am is time $t = 16$. The second is that the water flow was shifted down by 210240 cubic feet per hour (do not forget this when answering the question). Find the total volume of water that flowed into the Cayuga inlet from Aug 28 at 2:00pm to Aug 29 at 6:45am. Write a sentence using your results in context properly.
61. The Gini coefficient is the ratio of the area between a resource distribution curve, or Lorenz curve, as seen in figure 3.8 (also see Chapter 27) and the line $y = x$ representing equal distribution. Specifically the formula is Gini Coef $= 1 - 2\int_0^1 L(x)dx$. Find the Gini coefficient for U.S. energy consumption in 2014. Use the function from the function gallery.
62. The Gini coefficient is the ratio of the area between a resource distribution curve, or Lorenz curve, as seen in figure 3.8 (also see Chapter 27) and the line $y = x$ representing equal distribution. Specifically the formula is Gini Coef $= 1 - 2\int_0^1 L(x)dx$. Find the Gini coefficient for world energy consumption in 2011. Use the function from the function gallery. Compare the result to the previous problem. Is there more inequality in energy use within the U.S. or by county in the world?

63. Consider a rectangle of length l and width w. We know the area is lw, but we will derive this formula using integration. Set up an appropriate integral to find the area of this rectangle and solve it. Hint: Graph the function you need to integrate and recognize that you should treat l and w as if they are numbers in the integration. Note: This is circular reasoning since the area of a rectangle is used to develop the integral in the first place.

64. Consider a right triangle with base b feet and height h feet. We know the area is $bh/2$, but we will derive this formula with integration. Set up an appropriate integral to find the area of this triangle and solve it. Hint: Graph the function you need to integrate and recognize that you should treat b and h as if they are numbers in the integration.

65. In [11] it says "Thanks to satellite and tide gauge data, we know that sea level is rising about 3.3 millimeters (0.13 inches) a year, a rate that grows by another 1 millimeter (0.04 inches) per year every decade or so." Given this we can create a function to model the rate of sea level change as $s(t) = 3.3 + 0.1t$ millimeters per year where t is years after 2020. Set up an integral that represents the total increase in sea level from 2020 to 2050. Report your result in a sentence and use inches. Do the same from 2020 to 2100. Note that these increases in sea level are in addition to what has occurred up to 2020.

66. In [31] it says "Increasing rates of global warming have accelerated Greenland's ice mass loss from 25 billion tons per year in the 1990s to a current average of 234 billion tons per year." Assume that 25 billion tons per year is for 1990 and 234 billion tons per year is 2019. Create a linear model that represents the rate of Greenland's ice mass, where t is years since 1990. Set up an integral and calculate how many tons of Greenland ice mass will be lost (melted really) from 2020 to 2050 and from 2020 to 2100. Report your results in a sentence or two. As a challenge how much will this increase sea level? Note that this is additional ice melt on top of what has happened up to 2020.

67. Suppose that the increase of Greeland's ice sheet melting isn't linear but quadratic of the form $at^2 + c$. Use the two points in the previous problem and find a quadratic model quadratic of the form $at^2 + c$ to fit the two points. Tip: Let t be years since 1990. Set up an integral and calculate how many tons of Greenland ice mass will be lost (melted really) from 2020 to 2050 and from 2020 to 2100. Report your results in a sentence or two. Note this isn't necessarily realistic but it does demonstrate the difference between linear and quadratic models, although the rate of change of ice melting doesn't have to increase linearly.

Chapter 31
Techniques of Integration - The u Substitution

This chapter and the next add to our techniques for finding antiderivatives. The basic integration techniques given in M-Box 30.4 basically take the basic derivative rules from M-Box 30.4 and reverses them. We follow this same train of thought and look to reverse the chain rule and then the product rule. The integration technique called the u substitution is used to help undo the chain rule. Recall that the chain rule allows us to find the derivative of a function that is the composition of functions. The main idea is given in M-Box 31.1 with a couple of examples to follow.

M-Box 31.1: The u Substitution

In order to find

$$\int f'(g(x))g'(x)dx$$

set $u = g(x)$ so that $\frac{du}{dx} = g'(x)$ or $du = g'(x)dx$. Make the substitution to get

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(u) + c = f(g(x)) + c,$$

where in the last step we use $u = g(x)$ in reverse.

Example 31.1. *Find* $\int 2xe^{x^2}dx$.

Solution. Set $u = x^2$ so that $du = 2xdx$. Now

$$\int 2xe^{x^2}dx = \int e^{x^2}2xdx \tag{31.1}$$

$$= \int e^u du \tag{31.2}$$

$$= e^u + c \tag{31.3}$$

$$= e^{x^2} + c \tag{31.4}$$

T. J. Pfaff, *Applied Calculus with R*,
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Explanations:

- (set $u = x^2$) This was chosen because x^2 is "inside" the function e^x. To help identify the "inside" of a function composition consider if we were to evaluate e^{x^2} at a value, say 3, we would do 3^2 first. This identifies x^2 as the "inside" of the function composition. We also want to set u to something whose derivative is also part of the integrand. Here the derivative of x^2, $2x$, is part of the integral.
- 31.1 The integrand was rearranged simply to make the substitution easier to follow.
- 31.2 The substitution was executed by replacing u with x^2 and $2xdx$ with du. In this step we are renaming the variable x with the variable u based on the relationship between u and x, namely $u = x^2$. It must be the case that the substitution replaces all occurrences of x.
- 31.3 We find the antiderivative of e^u.
- 31.4 We replace u with x^2 so that the solution is in the original variable.

Finally, we should check our answer by starting with $y = e^{x^2} + c$ and taking the derivative to get $y' = 2xe^{x^2}$. □

Example 31.2. *Find* $\int 6x(x^2+5)^{11}dx$.

Solution. Set $u = x^2 + 5$ so that $du = 2xdx$. Note that $2xdx$ does not match the $6xdx$ in the integrand. Notice how we fix that in the steps below. Now

$$\int 6x(x^2+5)^{11}dx = 3\int 2x(x^2+5)^{11}dx \tag{31.5}$$

$$= 3\int u^{11}du \tag{31.6}$$

$$= 3\frac{u^{12}}{12} + c \tag{31.7}$$

$$= \frac{(x^2+5)^{12}}{4} + c \tag{31.8}$$

Explanations:

- (set $u = x^2 + 5$) This was chosen because $x^2 + 5$ is "inside" the function $(x^2 + 5)^{11}$ and because the derivative of $x^2 + 5$ is $2x$ and this is in the integrand, although the constant is off.
- 31.5 The constant multiple rule of integrals allows us to move constants outside the integration. In this case we factored 6 as 2×3 and pulled the 3 out of the integral. We did this so the $du = 2xdx$ matches our substitution.
- 31.6 We make the substitution replacing $(x^2 + 5)^{11}$ with u^{11} and $2xdx$ with du.
- 31.7 We find the antiderivative of u^{11}.
- 31.8 Substitute $x^2 + 5$ back in for u.

□

Example 31.3. *Find* $\int_0^1 xe^{x^2+3}dx$.

Solution. Given this is a definite integral we have two paths for evaluations. When we do the u substitution the limits of integration should be in terms of u. We can avoid this by removing the limits of integration, find the antiderivative back in terms of x and then evaluate. We demonstrate both methods.
Change the Limits - Faster Let $u = x^2 + 3$ so that $du = 2x, u(0) = 3$ and $u(1) = 4$. Note that we will adjust the integral for the 2.

$$\begin{aligned}\int_0^1 xe^{x^2+3}dx &= \frac{1}{2}\int_0^1 2xe^{x^2+3}dx \\ &= \frac{1}{2}\int_3^4 e^u du \\ &= \frac{1}{2}e^u\Big|_3^4 \\ &= \frac{1}{2}\left(e^4 - e^3\right)\end{aligned}$$

Note the use of $|_3^4$ as notation that says we still need to evaluate at the limits.
Find Antiderivative First - Slower Again let $u = x^2 + 3$ so that $du = 2x$. Note that we will adjust the integral for the 2. We now find the antiderivative:

$$\begin{aligned}\int xe^{x^2+3}dx &= \frac{1}{2}\int 2xe^{x^2+3}dx \\ &= \frac{1}{2}\int e^u du \\ &= \frac{1}{2}e^u \\ &= \frac{1}{2}e^{x^2+3}\end{aligned}$$

With $F(x) = \frac{1}{2}e^{x^2+3}$ we get

$$\int_0^1 xe^{x^2+3}dx = F(1) - F(0) = \frac{1}{2}\left(e^4 - e^3\right)$$

□

31.1 Exercises

1. $\int 2x\cos(x^2)dx$
2. $\int 2x\sin(x^2)dx$

3. $\int 2xe^{x^2}dx$

4. $\int 3x^2e^{x^3}dx$

5. $\int (2x+5)(x^2+5x-9)^{32}dx$

6. $\int (6x+7)(3x^2+7x-13)^{19}dx$

7. $\int 3x\sin(9x^2)dx$

8. $\int x\cos(4x^2)dx$

9. $\int \cos(x)\sqrt{\sin(x)}dx$

10. $\int e^x\sqrt{e^x+3}dx$

11. $\int \frac{9x^2-21}{x^3-7x+11}dx$

12. $\int \frac{4x^3+9x}{2x^4+9x^2-15}dx$

13. $\int \frac{\cos(x)}{\sqrt{8+\sin(x)}}dx$

14. $\int \frac{\sin(x)}{\sqrt{11+2\cos(x)}}dx$

15. $\int \frac{\ln(5x)}{x}dx$

16. $\int \frac{\ln(3x)}{9x}dx$

17. $\int \frac{x}{(x^2-4)^9}dx$

18. $\int \frac{x^2}{\sqrt{x^3+8}}dx$

These problems should be done by hand and left as an exact answer not a decimal, but feel free to check your work in R.

19. $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} x\cos(x^2)dx$

20. $\int_{-\sqrt{2\pi}}^{\sqrt{2\pi}} 5x\sin(x^2)dx$

21. $\int_0^1 7xe^{3x^2}dx$

22. $\int_{-1}^1 8x^2e^{2x^3}dx$

23. $\int_{-2}^0 (1-x)^4dx$

24. $\int_1^3 (1-2x)^5dx$

25. $\int_{-1}^1 (x^2+2)(x^3+6x-7)^{17}dx$

26. $\int_0^2 (2x^3+4x)(x^4+4x^2-33)^{12}dx$

27. $\int_{-\pi}^{\pi} \cos^8(x)\sin(x)dx$

28. $\int_{-\pi}^{\pi} \sin^7(x)\cos(x)dx$

29. $\int_1^4 \frac{2\ln(x^2)}{x}dx$

30. $\int_2^9 \frac{\ln(x^3)}{9x}dx$

Chapter 32
Techniques of Integration - Integration by Parts

In deriving a formula for reversing the product rule we start with the product rule

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

and then integrate both sides with respect to x to get

$$\int (f(x)g(x))'dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx.$$

Now $\int (f(x)g(x))'dx = f(x)g(x)$ and so

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

or

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

In short, integration by parts converts the problem of finding $\int f(x)g'(x)dx$ to the problem of finding $\int f'(x)g(x)dx$. A good choice of $f(x)$ and $g(x)$ in problems and this conversion works, while a poor choice makes matters worse. Example 32.1 will illustrate this point.

M-Box 32.1: Integration by Parts

Given continuous differentiable function $f(x)$ and $g(x)$ we have

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

A version that is easier to remember is

$$\int udv = uv - \int vdu.$$

T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4_32

Example 32.1. *Find* $\int x\cos(x)dx$.

Solution. Using integration by parts set $u = x$ and $dv = \cos(x)dx$. Now $\frac{du}{dx} = 1$ or $du = 1dx$ and $v = \int dv = \int \cos(x)dx = \sin(x)$. A good way to organize this is with the following diagram:

$$\begin{array}{cc} u = x & dv = \cos(x)dx \\ \\ du = 1dx & v = \sin(x) \end{array}$$

We now add arrows to the diagram to complete the integration by parts formula.

$$\begin{tikzcd} u = x \arrow[dr, "1"] & dv = \cos(x)dx \\ du = 1dx \arrow[r, "2"] & v = \sin(x) \end{tikzcd}$$

We now have

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx.$$

The calculation is finished by noting that

$$\int \sin(x)dx = -\cos(x) + C.$$

Putting it all together we get

$$\begin{aligned} \int x\cos(x)dx &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) - (-\cos(x) + C) \\ &= x\sin(x) + \cos(x) + C \end{aligned}$$

□

In example 32.1 if we set $dv = x$ then $v = x^2$ and we would have to find

$$\int \frac{x^2\sin(x)}{2}$$

which is more difficult than the integral we started with. When trying to choose the u and dv look to choose a u that gets simpler when we take the derivative to get du. Polynomials are a good choice because their derivative is one power lower. For dv make a choice that first can be integrated and second does not get more complicated. Functions such as $\cos(x)$, $\sin(x)$, or e^x are good choices. Of course, there are always cases that break these rules and if the first go around does not work switch the roles of u and dv and try again. Note that with the “new” integral any technique is fair game including using integration by parts again. At the same time, any integration technique is fair game on dv. One more example to demonstrate a technique keeping in mind that we can always multiply by 1.

Example 32.2. *Find* $\int \ln(x)dx$.

Solution. This doesn't immediately look like an integration by parts example. At the same time we don't have an antiderivative for $\ln(x)$. We choose $u = \ln(x)$ and $dv = 1$.

$$\begin{array}{cc} u = \ln(x) & dv = 1dx \\ \\ du = \frac{1}{x}dx & v = x \end{array}$$

We now add arrows to the diagram to complete the integration by parts formula.

$$\begin{tikzcd} u = \ln(x) \arrow[dr, "1"] & dv = 1dx \\ du = \frac{1}{x}dx \arrow[r, "2"] & v = x \end{tikzcd}$$

We now have

$$\int \ln(x)dx = x\ln(x) - \int 1dx.$$

Putting it all together we get

$$\begin{aligned} \int \ln(x)dx &= x\ln(x) - \int 1dx \\ &= x\ln(x) - x + c \end{aligned}$$

□

32.1 Exercises

Evaluate the Expression

1. $\displaystyle\int xe^x dx$
2. $\displaystyle\int xe^{3x} dx$
3. $\displaystyle\int x\cos(5x)dx$
4. $\displaystyle\int x\sin(7x)dx$

5. $\int 4x(2x+8)^{11}dx$

6. $\int 5x(7x+8)^{15}dx$

7. $\int 8x\ln(x)dx$

8. $\int 3x^2\ln(x)dx$

9. $\int (5x+9)e^{2x}dx$

10. $\int (11x+7)\sin(4x)dx$

11. $\int x^2e^x dx$

12. $\int x^2\sin(4x)dx$

13. $\int \frac{\ln(x)}{x^2}dx$

14. $\int \frac{\ln(x)}{x^3}dx$

15. $\int 4x^3e^{x^2}dx$

16. $\int 4x^7\sin(x^4)dx$

17. $\int \cos(x)\sin(x)dx$

18. $\int \sin(x)e^x dx$

These problems should be done by hand and left as an exact answer not a decimal, but feel free to check your work in R.

19. $\int_1^4 2xe^{4x}dx$

20. $\int_2^7 5xe^{5x}dx$

21. $\int_0^{\pi/2} 4x\cos(2x)dx$

22. $\int_0^{\pi/2} 3x\sin(2x)dx$

23. $\int_{-2}^3 6x(3x+7)^9dx$

24. $\int_{-1}^1 8x(3x-11)^{12}dx$

25. $\int_1^e 7x\ln(x)dx$

26. $\int_1^{e^2} 4x^3\ln(x)dx$

27. $\int_0^5 (9x-8)e^{-4x}dx$

28. $\int_{-\pi}^{\pi} (7x-3)\sin(2x)dx$

29. $\int_0^{\sqrt{\pi}} 4x^3\cos(x^2)dx$

30. $\int_0^1 6x^7e^{x^4}dx$

Appendix A
Algebra Review - Functions and Graphs

In the most general sense a function is any rule that maps one or more elements from one set to one or more elements of another set. As you start using R in this text you will see many examples of functions such as **curve()**, **uniroot.all()**, and **Deriv()**. For the purposes of this chapter we will focus on functions that take as an input a real number and return a real number. For example, the formula for the area of a square with side length s inches is the function $A(s) = s^2$. The input is s the side length of the square in inches. In words, $A(s)$ is spoken as A of s. The output is represented as $A(s)$ and calculated by s^2 square inches. In words, the definition of the function, $A(s) = s^2$, would be A of s is (or is equal to) s squared. We use $A(s)$ here instead of $f(s)$ because the function returns area so using $A(s)$ provides some context. If our square has a side length of 4 inches, then $A(4) = 4^2 = 16$. Here $A(4)$, which has units square inches because it represents an output, is spoken as A of 4. In summary, 4 inches is an input value, $A(4)$ square inches represents the output, and $4^2 = 16$ square inches is the value of the output. When functions are applied to something and have units they are often easier to understand, but in calculus we consider functions in general. The next example will consider a general function (Figure A.1).

Let $f(x) = x^2 + 3$ graphed in figure A.1. This function takes any real number as an input, the x inside the f, and returns the square of x plus 3. As above $f(x)$ is spoken as f of x. The use of f as the function name is commonly used in mathematics to represent a general function, while g and h are also used. The variable x is the typical input variable to represent a real number. The second most common input variable is t when the input is time. This same function may also be written as $y = x^2 + 3$ or even $y(x) = x^2 + 3$. The choice is often based on context. Where $y = x^2 + 3$ is often used when we are thinking in terms of a graph. Now let's turn our attention to the graph of $f(x) = x^2 + 3$.

T. J. Pfaff, *Applied Calculus with R*,
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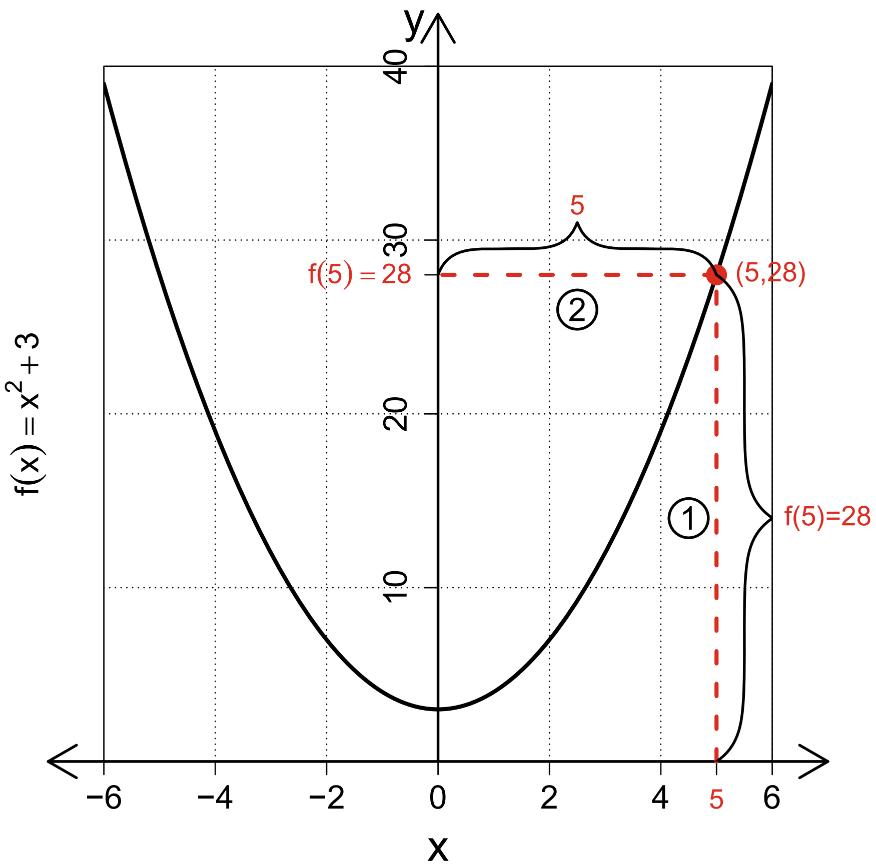


Fig. A.1 The graph of $f(x) = x^2 + 3$ with the point (5, 28) labeled.

The graph of $f(x) = x^2 + 3$ is given in figure A.1 or really just part of the graph from $x = -6$ to $x = 6$. There is nothing special about using -6 to 6 here. When we think of the Cartesian coordinate system we have an x-axis, typically the horizontal axis, and the y-axis, typically the vertical axis. When first looking at a graph it is a good idea to clearly identify the positive and negative sides of both the x-axis and y-axis as well as the location of (0, 0). In figure A.1 the y-axis ranges from $y - 0$ to $y = 40$ and we have already noted that the x-axis is from $x = -6$ to $x = 6$. The value of (0, 0) is at the middle of the bottom part of the graph.

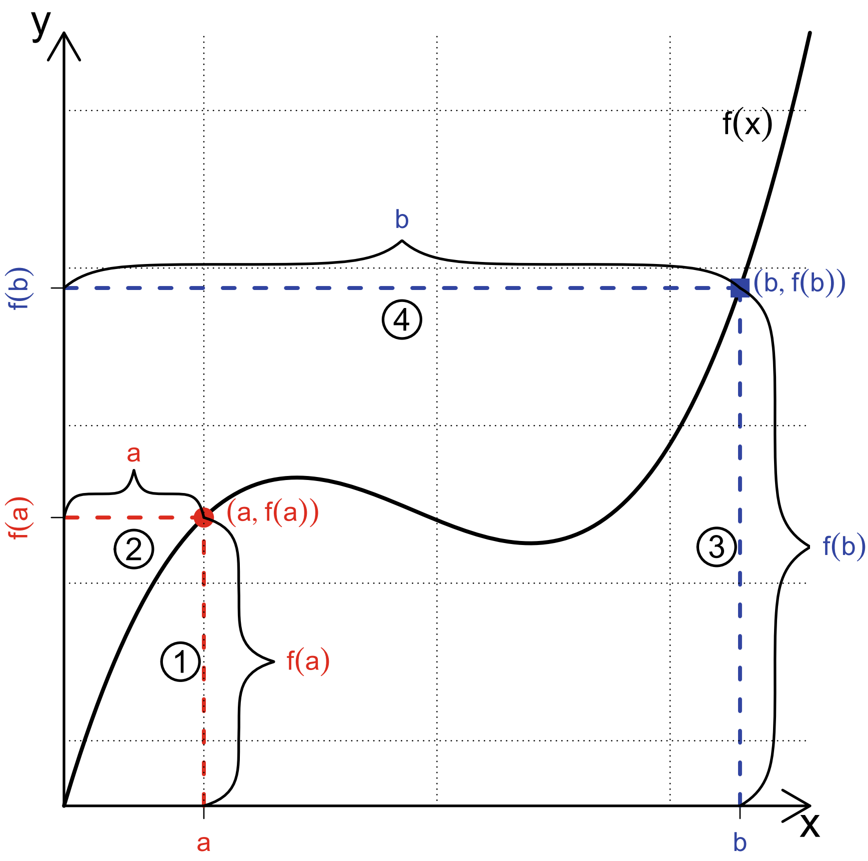


Fig. A.2 The graph of a function with key parts labeled.

Calculators or any computer graphing is done by just plotting "lots" (left undefined) of points and connecting the dots. One such dot at (5, 28) is noted in the figure. Note that points on a graph have an x-value, input values of the function, and a y-value, output value of a function. In this case, we take $x = 5$ and evaluate $f(5)$ to get $f(5) = 5^2 + 3 = 28$. Similar to the above, 5 is an input value, and $f(5)$ is an output value. Algebraically, $f(5) = 28$ but when we move to the graph we think of 28 as the y-value paired with the x-value of 5. Hence the point on the graph (5, 28). The point on the graph (5, 28) also represents two distances or lengths of line segments. The length of the (vertical) line segment from $x = 5$ or really the point on the graph (5, 0) to (5, 28) is 28 units. This line segment is denoted by the circled 1 in the graph. The length of the (horizontal) line segment from $y = 28$ or really the point on the graph (0, 28) to the point (5, 28) is 5 units. This line segment is denoted by the circled 2 in the graph. Note that how we can talk about the value 5 on the x-axis or the point on the graph (5, 0). Similarly for $y = 28$ or the point (0, 28).

Now let's add a layer of abstraction and consider figure A.2. This is part of a graph of some function $f(x)$. The x-axis and y-axis are labeled and we can assume the origin, (0, 0), is at the bottom left corner and we are viewing only positive value for both x and y. The value $x = a$, an input, is labeled on the x-axis and it is also the point $(a, 0)$. If evaluate the function at a it is represented on the y-axis as $f(a)$ (f of a) and it is also the point $(0, f(a))$. The value $f(a)$ is a y-value and an output. The red circle on the graph in the point $(a, f(a))$ where a is the x-value and $f(a)$ is the y-value. The distance from $(a, f(a))$ to the x-axis at $(a, 0)$, the line segment represented by the circled 1, is represented by $f(a)$. Similarly, the distance from $(a, f(a))$ to the y-axis at $(0, f(a))$, the line segment represented by the circled 2, is represented by a.

We can repeat the same analysis for the blue square point. The value $x = b$, an input, is labeled on the x-axis and it is also the point $(b, 0)$. If evaluate the function at b it is represented on the y-axis as $f(b)$ (f of b) and it is also the point $(0, f(b))$. The value $f(b)$ is a y-value and an output. The blue square on the graph in the point $(b, f(b))$ where b is the x-value and $f(b)$ is the y-value. The distance from $(b, f(b))$ to the x-axis at $(b, 0)$, the line segment represented by the circled 3, is represented by $f(b)$. Similarly, the distance from $(b, f(b))$ to the y-axis at $(0, f(b))$, the line segment represented by the circled 2, is represented by b.

So, what's the point? One way to think about this is that algebraic representations of a function are compact way to store information. We started this chapter with the function $A(s) = s^2$; the formula for the area of a square. We have a way to find the area of square given any side length we have. Another way to do something similar is to make a really big table and lookup values of side lengths and the corresponding area. The function $A(s) = s^2$ is really compact compared to such a table. Ok, so why graph functions? One way to think of this is it helps us understand the relationships between the input and output. The graph in figure A.1 is the same shape for that of $A(s) = s^2$; it is just shifted 3 units up (think about it). In this case the negative values of x have no physical meaning. If we focus on the positive value of x we see that the graph increases faster and faster (this will be quantified in the calculus course). In other words, the bigger the square the more area we gain if the side is increased by an inch (more on this in other chapters).

A.1 Exercises

For the next four exercises consider the function $A(s) = s^2/2$ which is the area of an Isosceles right triangle with side length s feet.

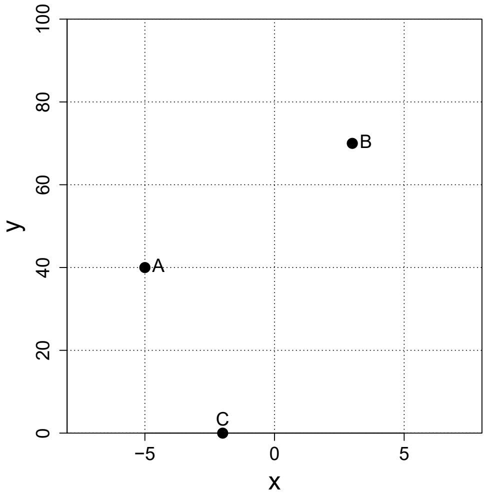
1. What are the units of $A(4)$? What is $A(4)$? What are the units for the 4?
2. What are the units of $A(10)$? What is $A(10)$? What are the units for the 10?
3. What are the coordinates on the graph of $A(s)$ with $s = 6$?
4. What are the coordinates on the graph of $A(2)$ with $s = 2$?

For the next ten exercise consider an arbitrary function $f(x)$ with input units feet and ouput units cubic feet.

5. What is the y-value associated with an x-value of 8?
6. What is the y-value associated with an x-value of 42?
7. What are the coordinates on the graph of $f(x)$ with $x = 9$?
8. What are the coordinates on the graph of $f(x)$ with $x = 15$?
9. What are the units for $f(7)$?
10. What are the units for the 4 in $f(4)$?
11. Is $f(99)$ a y-value or x-value?
12. Is $f(12)$ an output or an input?
13. The value of $f(1)$ is the output associate with an input of what?
14. The value of $f(a)$ is the output associate with an input of what?

For the next four exercises explain the location of the origin, give an approximate value for the range of the x-axis and y-axis, and find the coordinates of the points A, B, and C.

15.



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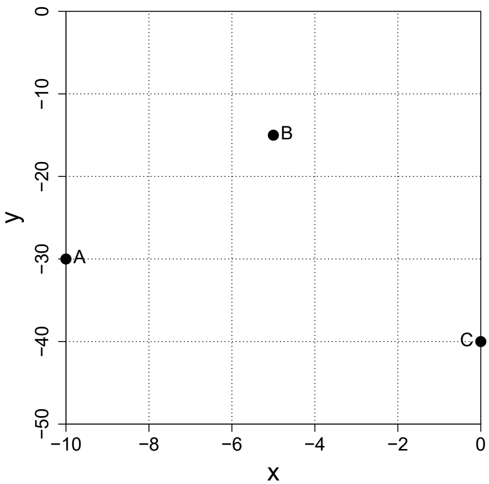

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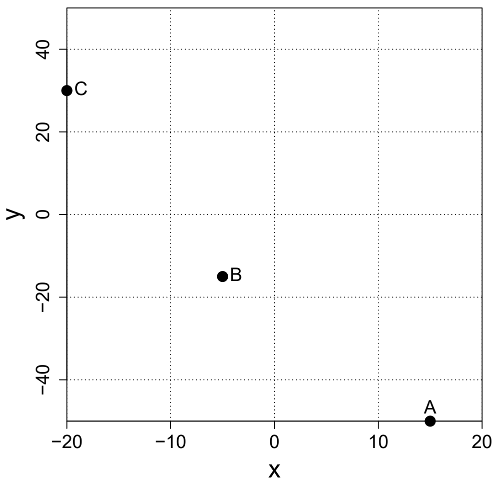

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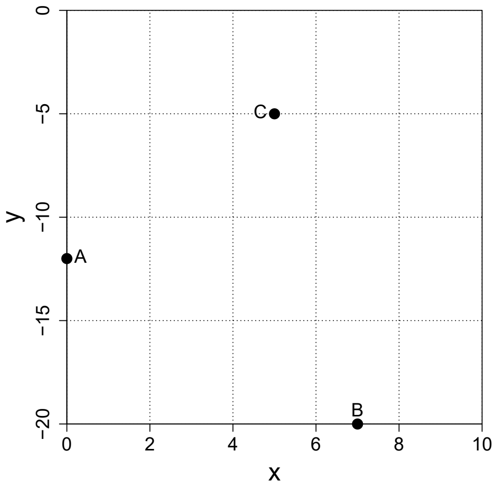
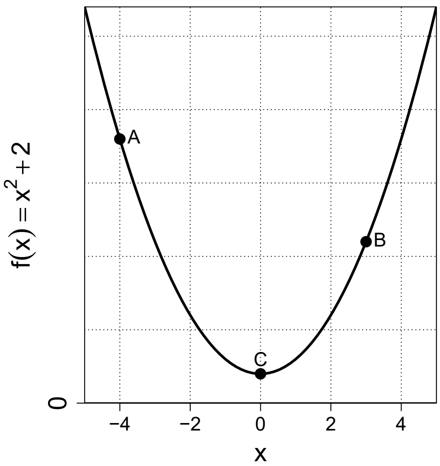


For the next eight exercises explain the location of the origin, give an approximate value for the range of the x-axis, and find the coordinates of the points A, B, and C.

19.



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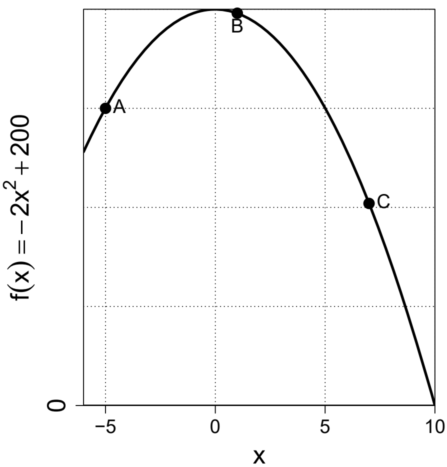

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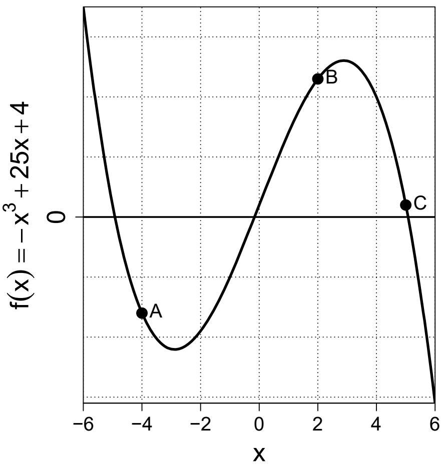

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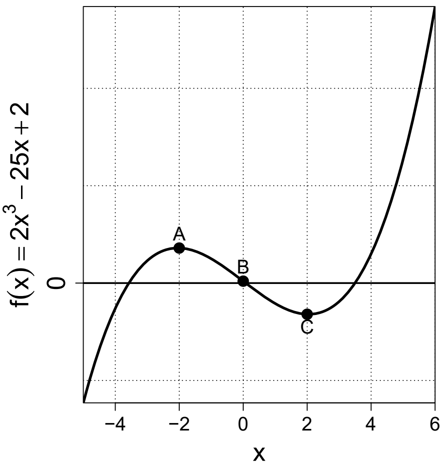

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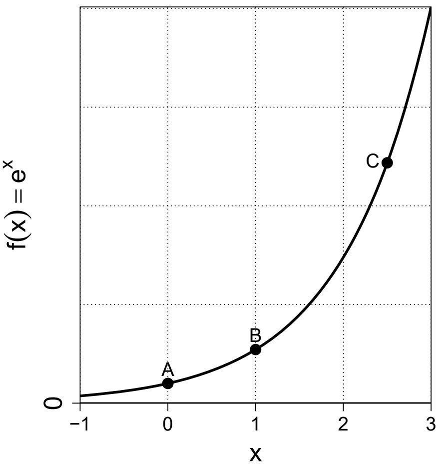

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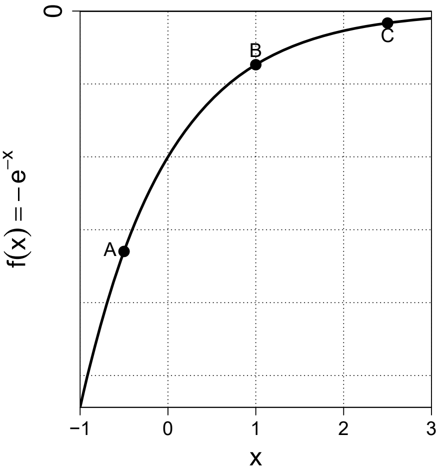

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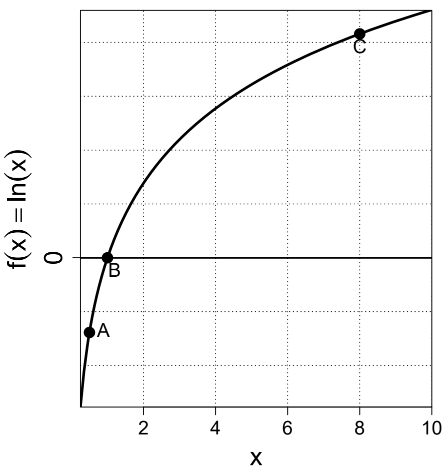

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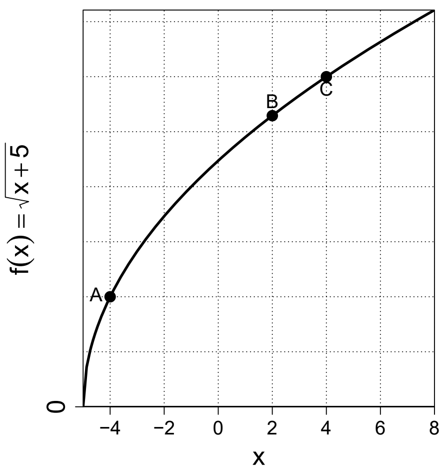
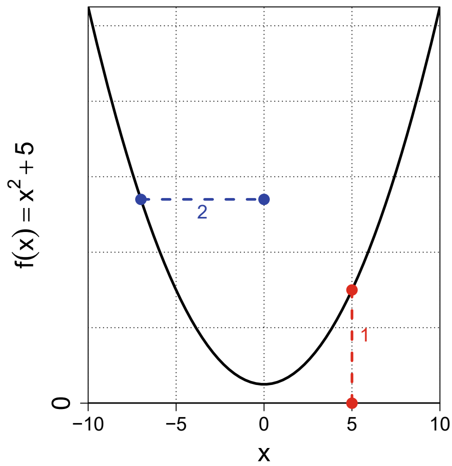
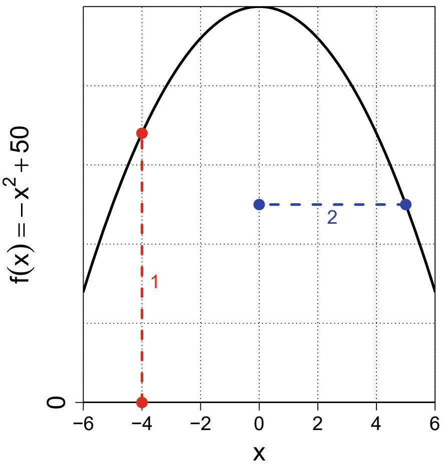


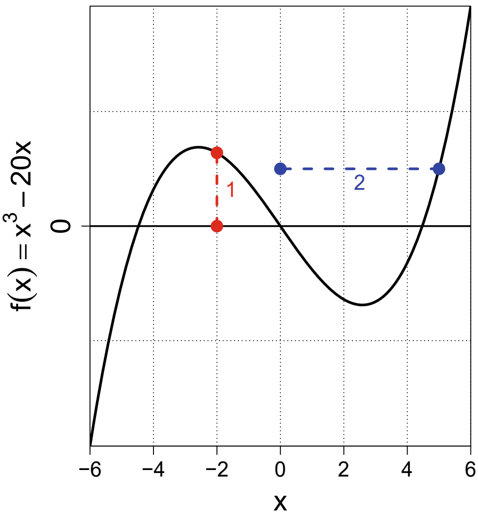
27. Find the length of line segments 1 and 2.



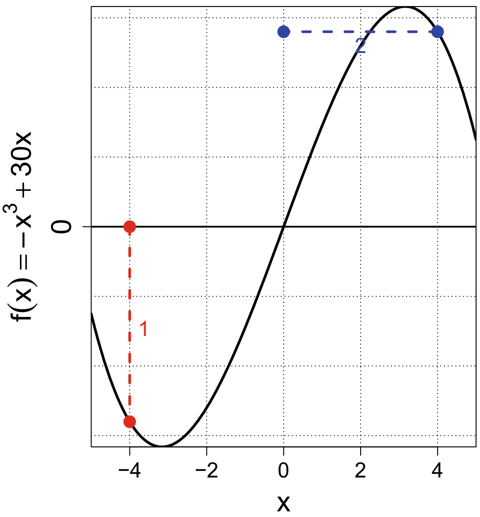
28. Find the length of line segments 1 and 2.



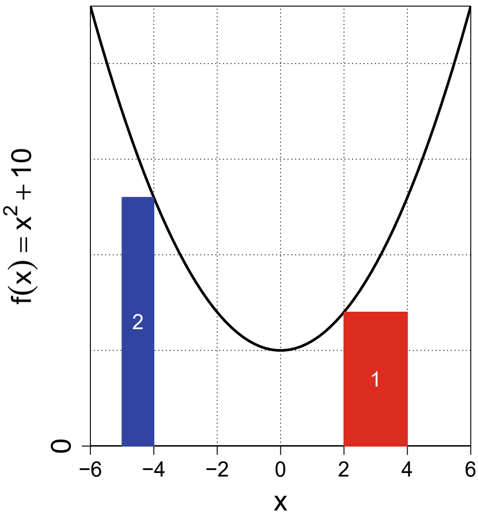
29. Find the length of line segments 1 and 2.



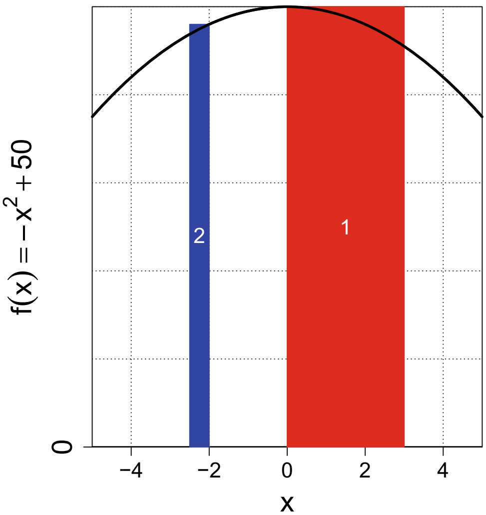
30. Find the length of line segments 1 and 2.



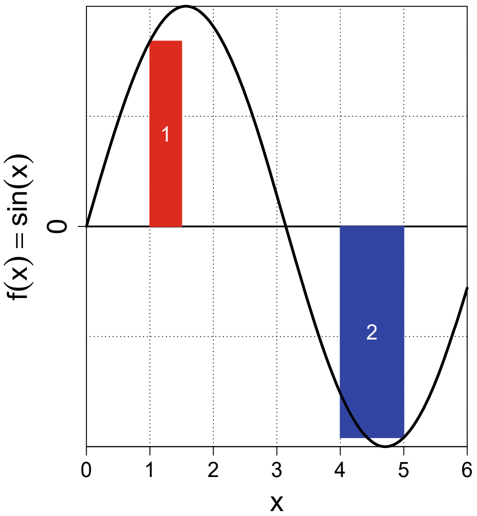
31. Find the area of rectangle 1 and 2.



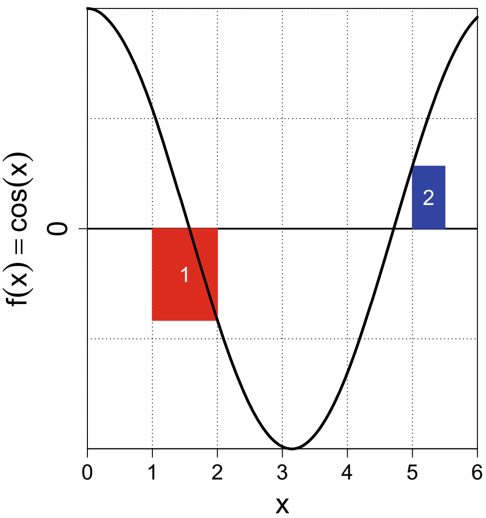
32. Find the area of rectangle 1 and 2.



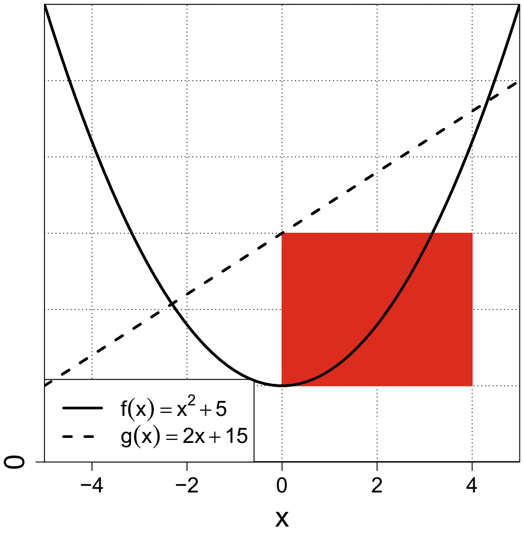
33. Find the area of rectangle 1 and 2.



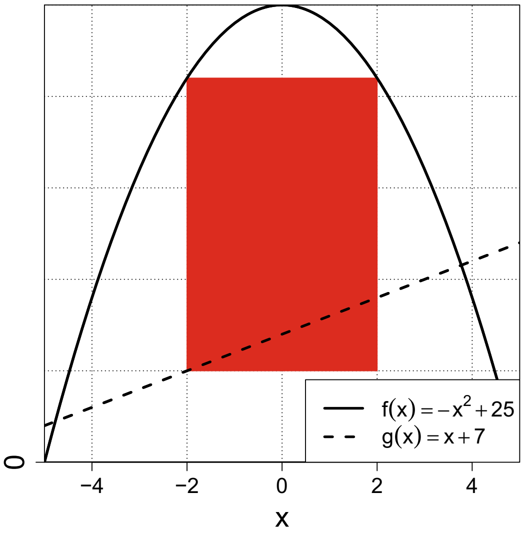
34. Find the area of rectangle 1 and 2.



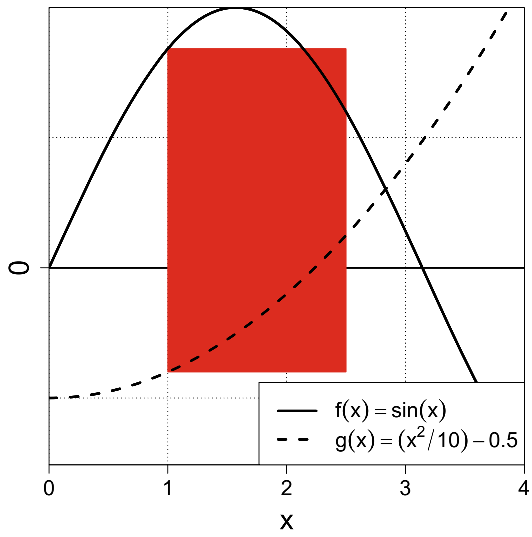
35. Find the area of the rectangle.



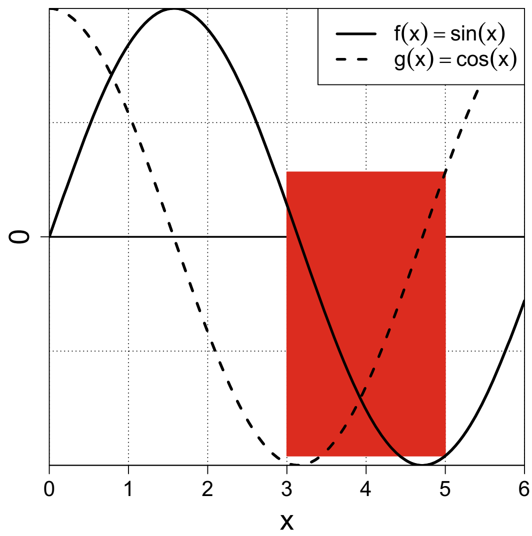
36. Find the area of the rectangle.



37. Find the area of the rectangle.



38. Find the area of the rectangle.



Appendix B
Algebra Review - Adding and Multiplying Fractions

The basic rules for adding and multiplying fractions are given in M-Box B.1. The following examples will illustrate using these rules.

M-Box B.1: Fractions Rules

$$a = \frac{a}{1} \tag{B.1}$$

$$a\,\frac{c}{d} = \frac{a}{1}\,\frac{c}{d} = \frac{ac}{d} \tag{B.2}$$

$$\frac{a}{b}\,\frac{c}{d} = \frac{ac}{bd} \tag{B.3}$$

$$\frac{a}{\frac{b}{c}} = a\,\frac{c}{b} = \frac{ac}{b} \tag{B.4}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\,\frac{d}{c} = \frac{ad}{bc} \tag{B.5}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{d}{d}\,\frac{a}{b} + \frac{b}{b}\,\frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd} \tag{B.6}$$

Example B.1. *Simplify* $\frac{5}{\frac{x}{y}}$.

Solution.. Using B.4 and then B.2 we have

$$\frac{5}{\frac{x}{y}} = 5\,\frac{y}{x} = \frac{5}{1}\,\frac{y}{x} = \frac{5y}{x}$$

□

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https://doi.org/10.1007/978-3-031-28571-4

Example B.2. *Simplify* $\frac{\frac{18z}{a}}{\frac{9w}{y}}$

Solution.. Using B.5 we have

$$\frac{\frac{18z}{a}}{\frac{9w}{y}} = \frac{18z}{a}\frac{y}{9w} = \frac{18zy}{a9w} = \frac{18yz}{9aw} = \frac{2yz}{aw}$$

In the last step $\frac{18}{9} = 2$. □

Example B.3. *Simplify* $\frac{12}{2x} + \frac{z}{8y}$

Solution.. The common denominator of the fractions is $8xy$. We will multiply the first fraction by $\frac{4y}{4y}$ and the second by $\frac{x}{x}$. Since both these fractions are equivalent to 1 and multiplying by 1 doesn't change a value we haven't changed the value of each fraction in the sum. Now following B.6 we have

$$\frac{12}{2x} + \frac{z}{8y} = \frac{12}{2x}\frac{4y}{4y} + \frac{z}{8y}\frac{x}{x} = \frac{12(4y)}{2x4y} + \frac{zx}{8yx} = \frac{48y}{8xy} + \frac{xz}{8xy} = \frac{48y + zx}{8xy}$$

□

Example B.4. *Represent the fraction* $\frac{5xb}{7yz}$ *as multiplying fractions that have only one of the terms and a 1.*

Solution.. The fraction can be represented as

$$\frac{5}{1}\frac{x}{1}\frac{b}{1}\frac{1}{7}\frac{1}{y}\frac{1}{z}$$

□

B.1 Exercises

For each of the equal signs with a number identify the rule used from M-Box B.1 and how it was used.

1. $\frac{a}{\frac{b}{c}} \overset{1}{=} \frac{\frac{a}{1}}{\frac{b}{c}} \overset{2}{=} \frac{a}{1}\frac{c}{b} \overset{3}{=} \frac{ac}{b}$

2. $\frac{x}{\frac{z}{y}} \overset{1}{=} \frac{\frac{x}{z}}{\frac{y}{1}} \overset{2}{=} \frac{x}{z}\frac{1}{y} \overset{3}{=} \frac{z}{yz}$

3. $\frac{x}{\frac{y}{z}} + 8\frac{x}{3} \overset{1}{=} \frac{xz}{y} + 8\frac{x}{3} \overset{2}{=} \frac{xz}{y} + \frac{8x}{3} \overset{3}{=} \frac{3xz}{3y} + \frac{8xy}{3y} \overset{4}{=} \frac{3xz + 8xy}{3y}$

4. $\frac{w}{\frac{x}{z}} + 4\frac{w}{5} \overset{1}{=} \frac{wz}{x} + 4\frac{w}{5} \overset{2}{=} \frac{wz}{x} + \frac{4w}{5} \overset{3}{=} \frac{5wz}{5x} + \frac{4wx}{5x} \overset{4}{=} \frac{5wz + 4wx}{5x}$

5. $\dfrac{30x+3y}{15xy} \overset{1}{=} \dfrac{30x}{15xy} + \dfrac{3y}{15xy} \overset{2}{=} \dfrac{2}{y} + \dfrac{1}{5x} \overset{3}{=} 2\dfrac{1}{y} + \dfrac{1}{5x}$

6. $\dfrac{30w+2z}{10wz} \overset{1}{=} \dfrac{30w}{10wz} + \dfrac{2z}{10wz} \overset{2}{=} \dfrac{3}{z} + \dfrac{1}{5w} \overset{3}{=} 3\dfrac{1}{z} + \dfrac{1}{5w}$

Simplify to a single faction.

7. $\dfrac{1}{\frac{1}{5}}$

8. $\dfrac{1}{\frac{1}{9}}$

9. $\dfrac{1}{\frac{1}{x}}$

10. $\dfrac{1}{\frac{1}{w}}$

11. $\frac{5}{x}\frac{w}{3}$

12. $\frac{3z}{x}\frac{w}{5y}$

13. $\frac{10x}{6w}\frac{3w}{2y}$

14. $\frac{8w}{3z}\frac{5z}{2x}$

15. $8\dfrac{x}{\frac{2}{y}}$

16. $12\dfrac{w}{\frac{3}{z}}$

17. $15\dfrac{3}{\frac{x}{z}}$

18. $21\dfrac{3}{\frac{w}{y}}$

19. $\dfrac{\frac{6z}{5w}}{\frac{z}{3x}}$

20. $\dfrac{\frac{4w}{3x}}{\frac{6z}{x}}$

21. $\dfrac{\frac{7z}{2w}}{\frac{3x}{14w}}$

22. $\dfrac{\frac{x}{2y}}{\frac{6x}{2z}}$

23. $6\dfrac{x}{3w} + 5\dfrac{2z}{y}$

24. $5\dfrac{w}{2z} + 3\dfrac{4x}{y}$

25. $3\dfrac{z}{4w} + x\dfrac{2y}{z}$

26. $8\dfrac{x}{3z} + w\dfrac{3y}{x}$

27. $4\dfrac{xy}{2} + 3\dfrac{z}{\frac{x}{w}}$

28. $6\dfrac{wz}{3} + 4\dfrac{w}{\frac{z}{y}}$

29. $12x\dfrac{2y}{3z} - 5\dfrac{2z}{\frac{w}{xy}}$

30. $9w\dfrac{3z}{2x} - 4\dfrac{3w}{\frac{z}{wy}}$

Appendix C
Algebra Review - Exponents

M-Box C.1: Exponent Rules

$$x^0 = 1 \quad \text{or} \quad 1 = x^0 \tag{C.1}$$

$$x^a x^b = x^{a+b} \quad \text{or} \quad x^{a+b} = x^a x^b \tag{C.2}$$

$$(x^a)^b = x^{ab} \quad \text{or} \quad x^{ab} = (x^a)^b \tag{C.3}$$

$$x^a y^a = (xy)^a \quad \text{or} \quad (xy)^a = x^a y^a \tag{C.4}$$

$$\sqrt[n]{x} = x^{1/n} \quad \text{or} \quad x^{1/n} = \sqrt[n]{x} \tag{C.5}$$

$$\frac{1}{x^a} = x^{-a} \quad \text{or} \quad x^{-a} = \frac{1}{x^a} \tag{C.6}$$

$$x^a = \frac{1}{x^{-a}} \quad \text{or} \quad \frac{1}{x^{-a}} = x^a \tag{C.7}$$

$$\frac{x^a}{x^b} = x^{a-b} \quad \text{or} \quad x^{a-b} = \frac{x^a}{x^b} \tag{C.8}$$

In M-Box C.1 we have explicitly listed each rule in both orders to make it clear that a rule can be used in either direction. For example, we may want to replace $x^a x^b$ with x^{a+b} or we may want to replace x^{a+b} with $x^a x^b$, but we tend to think of the equal sign as one directional. Note that rule C.8, $\frac{x^a}{x^b} = x^{a-b}$, is really just a combination of rules C.7 followed by rule C.2. There are typically multiple paths to simplify expressions with exponents and sooner or later a lot will be done in your head. Here is a path we will typically take: change roots to powers, distribute (rules C.4 and C.5), bring all variables to the numerator, combine like terms, and move terms so that they have a positive exponent.

Example C.1. *Simplify* $\frac{2^3 5^{-3} 3^0}{3^{-2} 2 \sqrt{25}}$

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Solution.. In this case we will simplify the exponents and evaluate them at the end and so, for example, we will leave 2^3 and not convert it to 8 at first. But, we will compute $\sqrt{25}$ and write is as 5, instead of $25^{1/2}$.

$$\frac{2^3 5^{-3} 3^0}{3^{-2} 2\sqrt{25}} = \frac{2^3 5^{-3} 1}{3^{-2} 2(5)} \tag{C.9}$$
$$= 2^3 5^{-3} 3^2 2^{-1} 5^{-1} \tag{C.10}$$
$$= 2^{3-1} 5^{-3-1} 3^2 \tag{C.11}$$
$$= 2^2 5^{-4} 3^2 \tag{C.12}$$
$$= \frac{2^2 3^2}{5^4} \tag{C.13}$$
$$= \frac{4(9)}{625} \tag{C.14}$$
$$= \frac{36}{625} \tag{C.15}$$

□

Example C.2. *Simplify* $\frac{4^2 2^{-4} (9^2)^{1/4} \sqrt{3}}{9^{-3} 3^2 (2\cdot 3)^{-3} \sqrt[3]{8}}$.

Solution..

$$\frac{4^2 2^{-4} (9^2)^{1/4} \sqrt{3}}{9^{-3} 3^2 (2 \cdot 3)^{-3} \sqrt[3]{8}} = \frac{4^2 2^{-4} (9^2)^{1/4} 3^{1/2}}{9^{-3} 3^2 (2 \cdot 3)^{-3} 2} \tag{C.16}$$
$$= \frac{(2^2)^2 2^{-4} 9^{2(1/4)} 3^{1/2}}{(3^2)^{-3} 3^2 2^{-3} 3^{-3} 2} \tag{C.17}$$
$$= \frac{2^{2(2)} 2^{-4} 9^{1/2} 3^{1/2}}{3^{2(-3)} 3^2 2^{-3} 3^{-3} 2} \tag{C.18}$$
$$= \frac{2^4 2^{-4} (3^2)^{1/2} 3^{1/2}}{3^{-6} 3^2 2^{-3} 3^{-3} 2} \tag{C.19}$$
$$= 2^4 2^{-4} 3^{2(1/2)} 3^{1/2} 3^6 3^{-2} 2^3 3^3 2^{-1} \tag{C.20}$$
$$= 2^{4-4+3-1} 3^{1+1/2+6-2+3} \tag{C.21}$$
$$= 2^2 3^{8+1/2} \tag{C.22}$$
$$= 4 3^8 3^{1/2} \tag{C.23}$$
$$= 4(6561) 3^{1/2} \tag{C.24}$$
$$= 26244\sqrt{3} \tag{C.25}$$

□

Example C.3. *Simplify* $\frac{x^5 x^{-3} y^{-7}}{x^4 y^{-7}}$.

Solution..

$$\frac{x^5x^{-3}y^{-7}}{x^4y^{-7}} = x^5x^{-3}x^{-4}y^{-7}y^7 \tag{C.26}$$

$$= x^{5-3-4}y^{-7+7} = x^{-2}y^0 \tag{C.27}$$

$$= \frac{1}{x^2} \tag{C.28}$$

□

Example C.4. *Simplify* $\frac{(x^4)^3y^{-3}}{x^{-7}y^9}$.

Solution..

$$\frac{(x^4)^3y^{-3}}{x^{-7}y^9} = x^{12}y^{-3}x^7y^{-9} \tag{C.29}$$

$$= x^{12+7}y^{-3-9} = x^{19}y^{-12} \tag{C.30}$$

$$= \frac{x^{19}}{y^{12}} \tag{C.31}$$

or

$$\frac{(x^4)^3y^{-3}}{x^{-7}y^9} = \frac{x^{12}x^7}{y^9y^3} \tag{C.32}$$

$$= \frac{x^{12+7}}{y^{9+3}} \tag{C.33}$$

$$= \frac{x^{19}}{y^{12}} \tag{C.34}$$

□

Example C.5. *Simplify* $\frac{2x^8zy^6z^2}{4x^{10}y^5x^2z^7}$.

Solution.. Note $z = z^1$.

$$\frac{6x^8zy^6z^2}{4x^{10}y^5x^2z^7} = \frac{6x^8z^1y^6z^2x^{-10}y^{-5}x^{-2}z^{-7}}{4} \tag{C.35}$$

$$= \frac{6x^{8-10-2}y^{6-5}z^{1+2-7}}{4} \tag{C.36}$$

$$= \frac{3x^{-4}y^1z^{-5}}{2} \tag{C.37}$$

$$= \frac{3y}{2x^4z^5} \tag{C.38}$$

□

Example C.6. *Simplify* $\frac{12(xy)^{-3}y^4\sqrt[8]{x}}{15\sqrt[4]{x}y^{-6}\sqrt{y}}$

Solution..

$$\frac{12(xy)^{-3}y^4\sqrt[8]{x}}{15\sqrt[4]{x}y^{-6}\sqrt{y}} = \frac{12x^{-3}y^{-3}y^4x^{1/8}}{15x^{1/4}y^{-6}y^{1/2}} \tag{C.39}$$

$$= \frac{12x^{-3}y^{-3}y^4x^{1/8}x^{-1/4}y^6y^{-1/2}}{15} \tag{C.40}$$

$$= \frac{12x^{-3+1/8-1/4}y^{-3+4+6-1/2}}{15} \tag{C.41}$$

$$= \frac{12x^{-3-1/8}y^{7-1/2}}{15} \tag{C.42}$$

$$= \frac{12x^{-24/8-1/8}y^{14/2-1/2}}{15} \tag{C.43}$$

$$= \frac{12x^{-25/8}y^{13/2}}{15} \tag{C.44}$$

$$= \frac{4y^{13/2}}{5x^{25/8}} \tag{C.45}$$

□

How do we interpret the last line? Note that $y^{13/2} = (y^13)^{1/2} = \sqrt{y^{13}}$ or $y^{13/2} = (y^{1/2})^{13} = (\sqrt{y})^{13}$. The best form is the one that is easiest to calculate. For example $8^{2/3}$ may be best viewed as $(8^{1/3})^2 = 2^2 = 4$ whereas $(8^2)^{1/3} = 64^{1/3} = 4$, but the cube root of 64 is "harder" then the cube root of 8.

C.1 Exercises

The purpose of the next set of exercises is to help you follow the steps of a calculation carefully, along with assessing your knowledge of the rules of exponents.

1. Explain each step, from C.9 to C.15, in example C.1.
2. Explain each step, from C.16 to C.25, in example C.2.
3. Explain each step, from C.26 to C.28, in example C.3.
4. Explain each step, from C.29 to C.34, in example C.4.
5. Explain each step, from C.35 to C.38, in example C.5.
6. Explain each step, from C.39 to C.45, in example C.6.

Simplify to the form a^b or $\frac{1}{a^b}$ where $b > 0$.

7. $\frac{2^2}{2^3}$
8. $\frac{5^4}{5^2}$
9. $\frac{3^2 3^8}{3^5}$
10. $\frac{9^8 9^2}{9^{12}}$
11. $\frac{4^2 4^{-3}}{4^{-5} 4^4}$
12. $\frac{7^{-3} 7^6}{7^8 7^{-6}}$
13. $\frac{2^3 2^{-11} 2}{4^2 2^7 2^0}$
14. $\frac{25^3 5^{-4} 5^2}{5^{-3} 5^7}$
15. $\frac{8^3 64^{1/2} 8^{-2}}{64^4 8^{-9} 8^0}$
16. $\frac{3^4 9^{1/2} 3^{-5} 3^0}{9^5 9^{-8} 3^2}$
17. $\frac{5^{-3} 25^{3/2} 5^6 5}{25^3 5^{-11}}$
18. $\frac{7^4 49^5 7^{-8}}{49^{5/2} 7^{-5} 7}$

Simplify so that each base appears once with a positive exponent.

19. $\frac{2^3 5^{-6} 2^{-5}}{5^{-10} 5^3 \sqrt{25}}$ 20. $\frac{3^4 3^{-8} \sqrt{49}}{7^5 7^{-2}}$ 21. $\frac{3^{-8} 5^{11} \sqrt{25}}{3^6 3^{-4} 5^{-1}}$ 22. $\frac{7^5 10^3 \sqrt{100}}{7^4 10^5 7^{-3}}$

23. $\frac{(5^2)^3 8^{-2} (8^5)^{-2}}{5^{-3} 8^3 \sqrt[3]{125}}$ 24. $\frac{3^{-5} (2^2)^3 \sqrt[3]{8}}{(3^{-2})^{-3} 2^5 3^{-4}}$ 25. $\frac{(5^2)^{-3} 7^{-4} \sqrt[3]{343}}{5^5 (7^3)^{-4} 5^{-11}}$ 26. $\frac{(2^3)^4 (7^3)^5 \sqrt[3]{343}}{(2^{-3})^3 (7^2)^8 2^{21}}$

27. $\frac{(2^4)^4 (2\cdot 5)^6}{(5^2)^6 2^{-3} \sqrt{25}}$ 28. $\frac{(5^{-2})^{-3} (3\cdot 5)^4}{(3^4)^{-2} 5^4 \sqrt{9}}$ 29. $\frac{(3\cdot 7)^{-5} (7^3)^{-2}}{7^{-11} (3^2)^5 \sqrt[3]{27}}$ 30. $\frac{(2\cdot 7)^{-4} (2^4)^{-5}}{(7^3)^5 \sqrt[3]{8}}$

31. $\frac{(2^4)^5 3^{5/2}}{(2\cdot 3)^5 \sqrt{3}}$ 32. $\frac{5^{3/2} (7^{-2})^3}{(5\cdot 7)^3 \sqrt{5}}$ 33. $\frac{5^{-7/2} (3^4)^{-3}}{(3\cdot 5)^{-7} \sqrt{5}}$ 34. $\frac{3^{-5/2} (7^2)^{-6}}{(3\cdot 7)^{-4} \sqrt{3}}$

35. $\frac{5^{3/4} (3^{-2})^7 (7\cdot 3)^{-4}}{5^{-7/4} 7^{12} 3^{-11} \sqrt{5}}$ 36. $\frac{(2\cdot 5)^{-3} 7^{5/4} (2^3)^{-2}}{2^{-13} 5^7 7^{-9/4} \sqrt{7}}$ 37. $\frac{(3^2)^7 2^{5/6} 21^{-3} \sqrt[3]{2}}{2^{1/6} (3\cdot 7)^{-5} 3^8}$ 38. $\frac{3^{7/6} (2^3)^9 10^{-5} \sqrt[3]{3}}{3^{-1/2} (5\cdot 2)^{-4} 5^7}$

Simplify to the form expression$\sqrt[n]{b}$, where expression is a product of a base with positive exponents, and b, and n are integers.

39. $\frac{(5^2)^4 (3^6)^{1/2} 3^7 7^{2/3}}{25^3 9^2 (3\cdot 5)^{-2} \sqrt[3]{7}}$ 40. $\frac{(2^1 6)^{1/4} 49^3 4^3 5^{1/2}}{2^{-3} 7^{-5} (2\cdot 7)^3 \sqrt[4]{5}}$

41. $\frac{(7\cdot 3)^9 (3^3)^{-2} 7^{-2} \sqrt[5]{2}}{21^3 2^{-2/5} 3^{-2} (7^{1/3})^6}$ 42. $\frac{(2\cdot 5)^7 2^{-5} (5^2)^{-4} \sqrt[4]{3}}{10^{-3} 3^{-1/2} (5^{1/4})^{12} 2^3}$

43. $\frac{(2^{3/2})^8 (11^{-1/2})^3 5^{10} 11^{5/2}}{((2\cdot 5)^2)^3 5^{-4} \sqrt{11}}$ 44. $\frac{5^{12} (7^{-1/2})^7 (11^{4/5})^{10} 7^{9/2}}{((5\cdot 11)^4)^2 11^{-3} \sqrt{7}}$

45. $\frac{(2\cdot 3)^8 (7^4)^5 8^2 2^{11} \sqrt[3]{8}}{2^{3^2} 3^{-5} 7^7 8^{5/3} 3^5}$ 46. $\frac{(7\cdot 8)^5 (11^7)^3 5^2 8^3 11 \sqrt[3]{5}}{7^{2^4} 8^{-3} 11^{12} 5^{8/3} 8^3}$

Convert the expression so that all variables appear in the numerator even if they have a negative exponent. Numbers may be left in the numerator and the denominator, but must be simplified.

47. $\frac{1}{x^5}$ 48. $\frac{1}{x^7}$ 49. $\frac{3}{y^8}$ 50. $\frac{4}{y^3}$

51. $\frac{1}{3x^8}$ 52. $\frac{1}{7x^9}$ 53. $\frac{4}{6z^5}$ 54. $\frac{8}{9z^4}$

55. $\frac{4y^4}{5y^{-3} 6y^8}$ 56. $\frac{11y^8}{6y^{-7} 4y^{10}}$ 57. $\frac{8x^3}{2x^5 8x^{-2}}$ 58. $\frac{5w^9}{9w^5 9w^{-3}}$

59. $\frac{2x^3 5y^3}{8y^{-7} 9x^5}$ 60. $\frac{11w^3 8y^2}{6y^{-4} 5w^{10}}$ 61. $\frac{2z^6 8x^4}{9x^{-3} 5z^{12}}$ 62. $\frac{3w^2 10z^5}{6z^{-7} 8w^{12}}$

63. $\frac{12w^8 6x^9}{11x^{-5} 3w^7}$ 64. $\frac{4y^8 10z^5}{11z^{-9} 7y^3}$ 65. $\frac{5x^3 2y^{-11}}{7x^{10} 6y^{-9}}$ 66. $\frac{12w^2 9y^{-11}}{6w^7 3y^{-5}}$

Simplify the expression so that all variables appear only once, all exponents are positive, and fractions are reduced.

67. $\frac{9x^7}{6x^9}$ 68. $\frac{8x^2}{2x^{11}}$ 69. $\frac{9y^{10}}{6y^{11}}$ 70. $\frac{11y^6}{7y^9}$

71. $\frac{5z^{11} 7z^{12}}{4z^9 4z^{-5}}$ 72. $\frac{3z^{11} 5z^4}{7z^{-2} 9z^7}$ 73. $\frac{2x^6 4x^4}{5x^{10} 8x^{-9}}$ 74. $\frac{9x^2 5x^9}{8x^{-7} 2x^{10}}$

75. $\frac{6w^{11} \sqrt[6]{w}}{7w^4 \sqrt[3]{w}}$ 76. $\frac{9w^8 \sqrt[10]{w}}{11w^7 \sqrt[5]{w}}$ 77. $\frac{9y^6 \sqrt{y}}{12y^{-7} \sqrt[4]{y}}$ 78. $\frac{8y^5 \sqrt[7]{y}}{12w^{-3} \sqrt[14]{y}}$

79. $\frac{4(z^3)^8}{8z^{-6} \sqrt[5]{z}}$ 80. $\frac{6(z^7)^4}{5z^{-12} \sqrt[4]{z}}$ 81. $\frac{4z^8 \sqrt[9]{z}}{12(z^7)^{11}}$ 82. $\frac{11z^7 \sqrt[8]{z}}{22(z^6)^4}$

83. $\frac{x^{-3} y^9 x^2}{x^9 y^{-6}}$ 84. $\frac{x^{-8} z^9 x^6}{x^{10} z^6}$ 85. $\frac{w^{12} z^{-3}}{z^5 w^{-3} z^{-4}}$ 86. $\frac{w^6 x^{-5}}{x^{-6} w^{-4} x^9}$

87. $\frac{12(yz)^4 y^{10}}{2y^{11} z^{-6}}$ 88. $\frac{10(wy)^{12} w^3}{4w^7 y^{-11}}$ 89. $\frac{2(xy)^4 x^{-9}}{10x^{12} y^3}$ 90. $\frac{6(xz)^5 z^{-2}}{9x^7 z^{10}}$

91. $\dfrac{(wz)^4 w^{12}}{(w^8)^5 z^{2/9} \sqrt[9]{z}}$

92. $\dfrac{(wx)^8 x^{10}}{(w^6)^4 x^{3/5} \sqrt[5]{x}}$

93. $\dfrac{(yz)^{11} \sqrt[8]{z}}{(y^{-4})^9 z^{5/8} y^7}$

94. $\dfrac{(wy)^4 \sqrt[7]{y}}{(w^9)^{-11} y^{3/7} w^8}$

95. $\dfrac{5x^{10} y^5 z^3 x^2 y^{12} z^{-4}}{15x^{-4} y^7 z^2 x^8 y^{-7} z^3}$

96. $\dfrac{8w^2 y^2 z^{-6} w^8 y^3 z^8}{12w^{-2} 2y^{10} z^{11} w^6 y^9 z^{-11}}$

97. $\dfrac{10(wxz)^3 w^{11} (x^{-8})^{10} z^6}{5w^7 (xz)^{-9} w^5 x^{-2} z^4}$

98. $\dfrac{12(wxy)^{11} w^{-7} x^8 (y^5)^{-7}}{4w^5 x^7 (yw)^{-4} x^{12} y^8}$

99. $\dfrac{10(wx)^{-9} (z^{11})^7 x^{3/5} \sqrt{z}}{30(wxz)^3 w^{-6} x^{10} z^{3/4}}$

100. $\dfrac{3w^{4/5} (xy)^{-8} (w^7)^4 \sqrt{y}}{15(wxy)^7 w^9 x^{2/7} y^{-6}}$

101. $\dfrac{12((wyz)^{-2})^3 w^{-3/7} (y^5)^{10} z^9}{9(wy)^6 z^{4/5} w^6 \sqrt[5]{z}}$

102. $\dfrac{25((xyz)^{11})^{-4} x^{-3/5} y^8 (z^3)^5}{5(xy)^9 x^8 y^{4/11} \sqrt[3]{z}}$

103. $\dfrac{w^4 x^3 y^{10} z^{-6} w^{12} x^3 y^9 z^4}{w^4 x^8 y^2 z^9 w^3 x^{-7} y^2 z^5}$

104. $\dfrac{w^8 x^{-12} y^6 z^3 w^9 x^4 y^{11} z^7}{w^8 x^8 y^2 z^9 w^4 x^3 y^{12} z^{-4}}$

105. $\dfrac{4(wx)^{11} (zw)^{-2/3} x^7 y^{-9} z^2}{16w^6 x^{11} (y^3)^6 (z^4)^{3/5} w^{-7} y^{11} \sqrt[4]{z}}$

106. $\dfrac{8(wx)^{12} y^{10} z^3 (xy)^{-5/2} z^{-9}}{2w^{-4} (x^7)^{8/3} (y^5)^3 z^7 w^{12} x^5 \sqrt[3]{y}}$

Appendix D
Algebra Review - Lines

While there is more than one form of the line, we use the point slope form as the one formula to memorize. In the formula y is considered a variable while y_1 is an unknown value, similarly for x and x_1. Still, if we solve for m we get $m = \frac{y-y_1}{x-x_1}$, which reminds us of the formula for slope $m = \frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$.

M-Box D.1: Point Slope Form of a Line

$$y - y_1 = m(x - x_1) \tag{D.1}$$

Example D.1. *What is the equation of the line created by the two points* $(-3, 8)$ *and* $(2, -2)$*?*

Solution.. We first find the slope of the line. Note that the points can be ordered in either way as long as we are consistent. Here we take $(x_1, y_1) = (2, -2)$ and use parenthesis to help keep track of negative signs in the numerator.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{8 - (-2)}{-3 - 2} \\ &= \frac{10}{-5} = -2 \end{aligned}$$

We now use the point slope form of the line. We can use either of our two points for (x_1, y_1) and so choose the point that may be easier to work with. Here will use $(x_1, y_1) = (2, -2)$ and although this is the same point we used for (x_1, y_1) in calculating the slope, it doesn't have to be. Now

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$$\begin{aligned}
y - y_1 &= m(x - x_1)\\
y - (-2) &= -2(x - 2)\\
y + 2 &= -2x + 4\\
y &= -2x + 4 - 2\\
y &= -2x + 2
\end{aligned}$$

□

Example D.2. *Given the function $f(x) = x^2 + 3x + 9$, what is the equation of the line with slope 7 that goes through the point on the curve with $x = 2$? What is the point on the line with $x = 3$? What is the point on the line with $y = 40$?*

Solution.. In order to create the equation of a line we need a point and a slope. The slope is given as $m = 7$. Since the point goes through the curve with $x = 2$ the associated y-value is $f(2) = 2^2 + 3(2) + 9 = 19$ and so the point is $(2, 19)$. We now have

$$\begin{aligned}
y - y_1 &= m(x - x_1)\\
y - 19 &= 7(x - 2)\\
y - 19 &= 7x - 14\\
y &= 7x - 14 + 19\\
y &= 7x + 5
\end{aligned}$$

What is the point on the line with $x = 3$? We plug $x = 3$ into the equation of the line to get $y = 7(3) + 5 = 26$ and so the point is $(3, 26)$.

What is the point on the line with $y = 40$? Here we solve $40 = 7x + 5$ to get $x = (40 - 5)/7 = 35/7 = 5$ and so the point is $(5, 40)$. □

Example D.3. *What is the equation of the line through the point $(a, f(a))$ with slope s?*

Solution.. In this example we are given a point and a slope and so we use the point slope form of the line to get

$$\begin{aligned}
y - y_1 &= m(x - x_1)\\
y - f(a) &= s(x - a)\\
y - f(a) &= sx - sa\\
y &= sx - sa + f(a)
\end{aligned}$$

□

Example D.4. *If $f'(a)$ represents that slope of the tangent line to the function $f(x)$ at $x = a$, what is the equation of the line?*

Solution.. For the desired line we are given that the slope is $m = f'(a)$. Since the line is tangent to the function $f(x)$ at $x = a$, then the point $(a, f(a))$ must be on the line. Using the point slope form of the line we get

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - f(a) &= f'(a)(x - a) \\ y &= f'(a)(x - a) + f(a) \end{aligned}$$

□

D.1 Exercises

Practice working with lines because we all need to.

1. In example D.1 use the other point in the point slope calculation to find the equation of the line. Did you get the same result? Should you?
2. In example D.3 if we evaluate the line y at $x = a$, what is the result. Explain why this makes sense given how the line was created.
3. In the equation $y = sx - sa + f(a)$ from example D.3, which expressions represent unknown constants and which are variables? Does $y(1)$ make sense? If yes, then find the value.
4. In example D.4 if we evaluate the line y at $x = a$, what is the result. Explain why this makes sense given how the line was created.
5. In the equation $y = f'(a)(x - a) + f(a)$ from example D.4, which expressions represent unknown constants and which are variables?
6. In the equation $y = f'(a)(x - a) + f(a)$ from example D.4, evaluate the line at $x = a + 1$ and explain the meaning of the result in the context of the example.

Find the equation of the line from the given slope and point. Find the corresponding y-value from the given x-value and find the corresponding x-value from the given y-value.

7. $m = -12$, $(-2, 9)$; $x = 4$; $y = -1$
8. $m = 1$, $(13, -13)$; $x = 20$; $y = 3$
9. $m = 7$, $(13, -10)$; $x = 2$; $y = 0$
10. $m = 5$, $(17, -12)$; $x = 17$; $y = 9$
11. $m = 18$, $(-12, 6)$; $x = -6$; $y = 18$
12. $m = -4$, $(-2, -2)$; $x = 7$; $y = 16$
13. $m = -5$, $(10, 16)$; $x = 16$; $y = 17$
14. $m = -2$, $(-8, 13)$; $x = -12$; $y = 8$

Find the equation of the line that goes through the two given points. Find the corresponding y-value from the given x-value and find the corresponding x-value from the given y-value.

15. $(-14, 12), (6, -3); x = 9; y = 3$
16. $(-9, 17), (-15, -1); x = 3; y = 8$
17. $(-19, -8), (16, -14); x = -6;$ $y = -12$
18. $(-12, -8), (-3, 8); x = -16;$ $y = -1$
19. $(-14, -19), (2, -12); x = -11;$ $y = -7$
20. $(-5, 2), (-5, 10); x = -1;$ $y = -10$
21. $(7, 4), (4, 16); x = -5; y = 17$
22. $(-10, -2), (-3, 3); x = 6;$ $y = -18$

The next set of problems are similar to the ones above except that instead of numbers the points and slope are letters. The letters were randomly generated for these problems. Due to the non-standard use of notation these problems can be extra challenging. The point is to get comfortable working with algebraic symbols in the context of lines.

Find the equation of the line that goes through the two given points.

23. $(-A, e), (-s, N)$
24. $(p, x), (-N, g)$
25. $(m, O), (-t, z)$
26. $(O, -h), (-M, l)$
27. $(o, Y), (U, T)$
28. $(q, -m), (x, -I)$
29. $(d, Q), (-Y, -H)$
30. $(-E, N), (-R, Z)$

Find the equation of the line from the given slope and point. Find the corresponding y-value from the given x-value and find the corresponding x-value from the given y-value.

31. $m = o, (j, t); x = v; y = H$
32. $m = e, (M, p); x = U; y = k$
33. $m = z, (p, b); x = n; y = w$
34. $m = q, (u, p); x = D; y = S$
35. $m = f'(D), (D, f(D)); x = r;$ $y = M$
36. $m = f'(B), (B, f(B)); x = a;$ $y = s$
37. $m = g'(c), (c, g(c)); x = J;$ $y = U$
38. $m = g'(k), (k, g(k)); x = j;$ $y = Z$

Appendix E
Algebra Review - Expanding, Factoring, and Roots

In this section we highlight a few important tools for multiplying polynomial terms, factoring, and finding roots. We begin with the pneumonic FOIL.

M-Box E.1: FOIL

FOIL stands for First, Outside, Inside, and Last as a way to remember how to multiply $(x_1 + y_1)(x_2 + y_2)$, in other words (use the subscripts to keep track of what is being multiplied),

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$$

FOIL is a nice way to remember how to multiplying two binomial terms, but it obscures the general idea. What we are really doing is this:

$$\begin{aligned}(x_1 + y_1)(x_2 + y_2) &= x_1(x_2 + y_2) + y_1(x_2 + y_2)\\ &= x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2\end{aligned}$$

This then allows us to generalize what we are doing to more terms. For instance

$$\begin{aligned}(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) &= (x_1 + y_1)(x_2(x_3 + y_3) + y_2(x_3 + y_3))\\ &= x_1(x_2(x_3 + y_3) + y_2(x_3 + y_3)) + y_1(x_2(x_3 + y_3) + y_2(x_3 + y_3))\\ &= x_1x_2(x_3 + y_3) + x_1y_2(x_3 + y_3) + y_1x_2(x_3 + y_3) + y_1y_2(x_3 + y_3)\\ &= x_1x_2x_3 + x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + y_1x_2x_3 + y_1x_2y_3 + y_1y_2x_3 + y_1y_2y_3\\ &= x_1x_2x_3 + x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + y_1x_2x_3 + y_1x_2y_3 + y_1y_2x_3 + y_1y_2y_3\end{aligned}$$

Can you see the pattern with the subscripts and the variables x and y? Here is an example with fewer variables.

Example E.1. *Expand* $(x + 2)^3$.

We begin by writing out $(x + 2)^3$ as $(x + 2)(x + 2)(x + 2)$. Now,

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$$\begin{aligned}(x+2)(x+2)(x+2) &= (x+2)(x^2+2x+2x+4)\\ &= (x+2)(x^2+4x+4)\\ &= x(x^2+4x+4)+2(x^2+4x+4)\\ &= x^3+4x^2+4x+2x^2+8x+8\\ &= x^3+6x^2+12x+8\end{aligned}$$

Note that through the calculation we simplified by combining like terms.
Solution. □

M-Box E.2: Synonyms

> If x is a value such that $f(x)=0$ then x is a root, zero, and x-intercept of the function $f(x)$. The point $(x,0)$ is on the graph of $f(x)$.

Particularly important inputs of a function $f(x)$ are values of x such that $f(x)=0$. These values of x are known as zeroes, roots, or x-intercepts. In general, solving the equation $f(x)=0$ to find roots is a difficult problem. Chapter 1 demonstrates how to use R to find roots. There are few simple cases that can be done that we will review quickly.

Example E.2. *Find the roots of* $f(x)=x^2+2x-8$.

Solution.. The function x^2+2x-8 factors as $(x-2)(x+4)$. Now solving $x^2+2x-8=0$ is the same as solving $(x-2)(x+4)=0$. A product can equal 0 only if one or both of the terms are 0. Hence $(x-2)(x+4)=0$ if and only if $x-2=0$ or $x+4=0$, in other words, when $x=2$ or $x=-4$. The roots of $f(x)=x^2+2x-8$ are 2 and -4. □

M-Box E.3: Factoring a Monic Polynomial of Degree Two

> We can factor x^2+bx+c into $(x+c_1)(x+c_2)$ where $c_1c_2=c$ and $c_1+c_2=b$. The values of b, c, c_1, and c_2 may be positive, negative or 0.

The general idea of factoring a monic (the coefficient of the highest power is 1) polynomial of degree 2 (the highest power of x) is given in M-Box E.3. The quadratic formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ is a general formula for finding the roots of ax^2+bx+c. There are also formulas for factoring polynomial in specific instances such as

$$x^3+d^3=(x+d)(x^2-dx+d^2) \tag{E.1}$$

$$x^3-d^3=(x-d)(x^2+dx+d^2) \tag{E.2}$$

We note that factoring out common terms can help in finding roots or solving for zero. For example,

Example E.3. *Find the zeroes of* $f(x) = 2x^3 + 10x^2 + 12x$*. Removing the common factor* $2x$ *and factoring yields*

$$\begin{aligned} 2x^3 + 10x^2 + 12x &= 2x(x^2 + 5x + 6x) \\ &= 2x(x+2)(x+3) \end{aligned}$$

Hence, the zeroes of $f(x) = 2x^3 + 10x^2 + 12x$ *are* 0, −2, *and* −3.

Solution. □

Lastly, we point out that only a few examples are provided here and only with small degree polynomials. The problem of finding roots is one that applies to all functions, such as $f(x) = \sin x$, $f(x) = e^x - 10$, and $f(x) = x^2 - \ln x$. If algebraic or "by hand" methods allude to us, then use technology.

E.1 Exercises

1. Verify that the roots in example E.2 are correct. Hint: Plug the values into $f(x)$ and show that the output is 0.
2. From M-Box E.3 what are the values of b, c, c_1, and c_2 that correspond to example E.2.
3. Verify equation E.1 by expanding the right-hand side to show that it equals the left-hand side.
4. Verify equation E.2 by expanding the right-hand side to show that it equals the left-hand side.

Expand the expression.

5. $(x+8)(x-9)$
6. $(x-6)(x+10)$
7. $(y+7)(y+8)$
8. $(y-1)(y+7)$
9. $(4x+6)(x+5)$
10. $(6x-2)(x-10)$
11. $(7z+5)(z-3)$
12. $(5z-6)(z-2)$
13. $(x-8)^2$
14. $(x-3)^2$
15. $(y+10)^2$
16. $(y+7)^2$
17. $(x+4)^3$
18. $(x+9)^3$
19. $(x-10)^3$
20. $(x-2)^3$
21. $(2x+6)^2$
22. $(3x+5)^2$
23. $(x+h)^2$
24. $(x+h)^3$
25. $(x+7)(x^2-4x+1)$
26. $(x+8)(x^2-6x+4)$
27. $(5x-6)(x^2+2x+6)$
28. $(4x-8)(x^2+7x+3)$
29. $(9x-2)(x^2-3x+9)$
30. $(5x-10)(x^2+6x+8)$

Find the zeroes or roots of the function and check your answer.

31. $f(x) = (x+2)(x+4)$
32. $f(x) = (x+6)(x+1)$
33. $g(x) = (x+9)(x-1)$
34. $g(x) = (x+4)(x-7)$
35. $f(x) = x^2 + 5x - 14$
36. $f(x) = x^2 + 7x - 30$

37. $f(x) = x^2 - 12x + 35$
38. $f(x) = x^2 - 10x + 24$
39. $g(x) = x^2 - 4x - 5$
40. $g(x) = x^2 - 4x - 12$
41. $h(x) = x^2 + 8x - 20$
42. $h(x) = x^2 + 9x + 14$
43. $f(x) = x^2 + 10x + 24$
44. $f(x) = x^2 + 2x - 80$
45. $f(x) = x^2 - 3x - 4$
46. $f(x) = x^2 + 6x + 5$
47. $f(x) = x^2 + 14x + 33$
48. $f(x) = x^2 + 16x + 48$
49. $f(x) = x^2 - 18x + 72$
50. $f(x) = x^2 - 20x + 96$
51. $f(x) = x^2 + x - 110$
52. $f(x) = x^2 + 15x + 36$

Solve the equations.

53. Solve $f(x) = -31$, where $f(x) = x^2 + 14x + 9$.
54. Solve $f(x) = -21$, where $f(x) = x^2 - 11x + 9$.
55. Solve $f(x) = 1$, where $f(x) = x^2 - 24$.
56. Solve $f(x) = 16$, where $f(x) = x^2 + 4x - 61$.
57. Solve $f(x) = -36$, where $f(x) = x^2 - 3x - 40$.
58. Solve $f(x) = -23$, where $f(x) = x^2 - 13x + 19$.
59. Solve $f(x) = 5$, where $f(x) = x^2 - x - 85$.
60. Solve $f(x) = 36$, where $f(x) = x^2 + 5x - 30$.
61. Solve $f(x) = 8$, where $f(x) = x^2 + 9x - 44$.
62. Solve $f(x) = 2$, where $f(x) = x^2 - 17x + 32$.
63. Solve $f(x) = -10$, where $f(x) = x^2 + x - 66$.
64. Solve $f(x) = -26$, where $f(x) = x^2 - 28x + 134$.
65. Solve $f(x) = -41$, where $f(x) = x^2 + 12x - 9$.
66. Solve $f(x) = 47$, where $f(x) = x^2 - 4x - 145$.
67. Solve $f(x) = 16$, where $f(x) = x^2 + 4x - 29$.
68. Solve $f(x) = -20$, where $f(x) = x^2 - 8x - 29$.
69. Solve $f(x) = -25$, where $f(x) = x^2 - 4x - 70$.
70. Solve $f(x) = 3$, where $f(x) = x^2 - 12x - 25$.
71. Solve $f(x) = -15$, where $f(x) = x^2 + 16x + 24$.
72. Solve $f(x) = 10$, where $f(x) = x^2 + 22x + 130$.
73. Solve $f(x) = 11$, where $f(x) = x^2 - 23x + 101$.
74. Solve $f(x) = 33$, where $f(x) = x^2 + 8x - 95$.

Appendix F
Algebra Review - Function Composition

One way to build new more complicated functions from simpler function is through function composition. The idea here is to substitute an entire function $f(x)$ for the variable x in another function. This is given formally in M-Box F.1.

M-Box F.1: Composition of Functions

Given functions $f(x)$ and $g(x)$, the function $f(g(x))$ replaces the input variable x of $f(x)$ with $g(x)$. Similarly $g(f(x))$ replaces the input variable x of $g(x)$ with $f(x)$. Note that, for example, for $f(g(x))$ to make sense in context the output units of $g(x)$ must be the same as the input units of $f(x)$.

Example F.1. *If $f(x) = 2x^3 + 4x^2 - 5x + 2$, then find $f(x+h)$.*

In this example we can think of $g(x) = x + h$ so that the function composition is $f(g(x))$.

$$f(x+h) = 2(x+h)^3 + 4(x+h)^2 - 5(x+h) + 2 \tag{F.1}$$

$$= 2(x^3 + 3x^2h + 3xh^2 + h^3) + 4(x^2 + 2xh + h^2) + 5(x+h) + 2 \tag{F.2}$$

$$= 2x^3 + 6x^2h + 6xh^2 + 2h^3 + 4x^2 + 8xh + 4h^2 + 5x + 5h + 2 \tag{F.3}$$

Example F.2. *Convert the Global Temperature function from the function gallery so that the output is degrees Fahrenheit instead of Celsius. The Global Temperature function is $GTemp(t) = 0.0001954645347t^2 + 0.00476546338t + 13.87628243$. We convert from Celsius to Fahrenheit with the function $F(C) = 9/5C + 32$. Notice that the input units of $F(C)$ are degrees Celsius and the output units of $GTemp(t)$ are also degrees Celsius. This checks that at least are units match up. The desired function is $F(GTemp(t)) = 9/5(0.0001954645347t^2 + 0.00476546338t + 13.87628243) + 32 = 0.0003518361625t^2 + 0.008577834083t + 56.97730837$.*

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F.1 Exercises

Find $f(g(x))$ and $g(f(x))$. Simplify whenever appropriate.

1. $f(x) = x + 3$; $g(x) = x^2$
2. $f(x) = x - 5$; $g(x) = x^2$
3. $f(x) = x - 6$; $g(x) = x^2 + 8$
4. $f(x) = x - 4$; $g(x) = x^2 + 9$
5. $f(x) = x + 7$; $g(x) = 6x^2 - 4x + 5$
6. $f(x) = x + 6$; $g(x) = 2x^2 - 7x + 10$
7. $f(x) = x - 3$; $g(x) = 5x^2 - 2x - 14$
8. $f(x) = x - 7$; $g(x) = 2x^2 - 7x - 9$
9. $f(x) = x^2 + 4x + 12$; $g(x) = x^2 - 7x + 6$
10. $f(x) = x^2 + 6x + 7$; $g(x) = x^2 - 3x + 10$
11. $f(x) = e^x$; $g(x) = x^2 + 2x - 4$
12. $f(x) = e^x$; $g(x) = x^2 + 4x - 8$
13. $f(x) = e^{2x}$; $g(x) = x^3 + 5x + 8$
14. $f(x) = e^{3x}$; $g(x) = x^3 + 7x - 1$
15. $f(x) = e^x$; $g(x) = \ln x$
16. $f(x) = e^x$; $g(x) = \sqrt{x}$
17. $f(x) = \sin x$; $g(x) = x^2 + 3x - 12$
18. $f(x) = \sin x$; $g(x) = x^2 + 6x - 14$
19. $f(x) = \sin x$; $g(x) = e^{2x^2}$
20. $f(x) = \sin x$; $g(x) = e^{3x^2}$
21. $f(x) = \cos x$; $g(x) = \sin x$
22. $f(x) = \cos x$; $g(x) = \tan x$
23. $f(x) = \tan x$; $g(x) = \ln x$
24. $f(x) = \tan x$; $g(x) = \sqrt{x}$
25. $f(x) = \sqrt{x}$; $g(x) = e^{3x^2}$
26. $f(x) = \sqrt{x}$; $g(x) = e^{5x^2}$
27. $f(x) = \sqrt{x}$; $g(x) = 2x^4 + 5x^2 + 3$
28. $f(x) = \sqrt{x}$; $g(x) = 3x^4 - 6x^2 - 9$
29. $f(x) = \sqrt{x}$; $g(x) = \sin x + \cos x$
30. $f(x) = \sqrt{x}$; $g(x) = \sin x + \ln x$

Find $g(f(x))$. Simplify whenever appropriate.

31. $f(x) = x + h$; $g(x) = x^2 - 2$
32. $f(x) = x + h$; $g(x) = x^2 - 5$
33. $f(x) = x + h$; $g(x) = 3x^2 - 5x + 9$
34. $f(x) = x + h$; $g(x) = 2x^2 - 7x + 4$
35. $f(x) = x + h$; $g(x) = 4x^3$
36. $f(x) = x + h$; $g(x) = 7x^3$
37. $f(x) = x + h$; $g(x) = 2x^3 - 3x^2 + x - 7$
38. $f(x) = x + h$; $g(x) = 5x^3 - 4x^2 + x - 8$

The following are function compositions of the form $f(g(x))$, identify the functions $f(x)$ and $g(x)$.

39. $(x + 7)^2 + 3(x + 7) + 8$
40. $(x - 5)^2 + 8(x - 5) + 10$
41. $5(x^2 + 3x)^2 - 2(x^2 + 3x) + 3$
42. $4(x^2 + 5x)^2 - 8(x^2 + 5x) + 12$
43. e^{5x^2+2x}
44. e^{8x^3-4x}
45. $(x^2 + \sin x)^{11}$
46. $(3x^4 + \cos x)^{13}$
47. $\sin(5x^2 + 3x - 2)$
48. $\cos(3x^2 - 2x + 8)$
49. $\sqrt{-3x^3 + 5x - 9}$
50. $\sqrt{-4x^5 + 6x^3 - 11}$
51. $\ln(5x + \sin x)$
52. $\ln(7x^2 - \cos x)$

Find $h(g(f(x)))$. Simplify whenever appropriate.

53. $f(x) = x^2$, $g(x) = (x-5)^3$, $h(x) = x^2 - x + 3$
54. $f(x) = x^3$, $g(x) = (x+8)^2$, $h(x) = 3x^2 + x - 9$
55. $f(x) = x^2 - x + 2$, $g(x) = e^x + 1$, $h(x) = x^7$
56. $f(x) = x^2 + 2x - 3$, $g(x) = \sin(x) + 3$, $h(x) = \sqrt{x}$
57. $f(x) = x + h$, $g(x) = 5x^2$, $h(x) = e^x$
58. $f(x) = x + h$, $g(x) = 6x^3$, $h(x) = \sin(x)$
59. $f(x) = \cos(x)$, $g(x) = 3x^2 - 7x + 9$, $h(x) = e^x$
60. $f(x) = \ln(x)$, $g(x) = 7x^2 + 5x - 8$, $h(x) = \sin(x)$

The following are function compositions of the form $f(g(h(x)))$, identify the functions $f(x)$, $g(x)$, and $h(x)$.

61. $4(x^4-8)^2 + 6(x^4-8) - 9$
62. $5(x^3+1)^2 + 7(x^3+1) + 12$
63. $e^{(\sin(x))^2+42}$
64. $e^{\sqrt{5x^2-x+3}}$
65. $(\sin(x^2) - \cos(x^2))^5$
66. $(e^{x^2} + x^2 + 1)^7$
67. $\ln(\sin(x^2))$
68. $5^{\cos(x^2)}$

Appendix G
R Glossary

This is a list of the R functions used in the text with brief comments on usage. This is not a complete description of these functions as most have more options and uses. If you type ?functionname in R, for example ?abline, a browser window will open with full details about the function. For options within functions, such as pch for point character, a web search of R pch will usually do the trick. Following the list is all the R tips given in the text.

abline To place horizontal at $y = y1$ or a vertical line at $x = x1$ on a graph use **abline(h=y1, v=x1)**. The value $x1$ and $y1$ may be vectors for multiple lines. For example, **abline(h= c(1,2,3,4))** will place horizontal lines at $y = 1$, $y = 2$, $y = 3$ and $y = 4$. For other lines, use **abline(a=a1, b=b1)** where a1 is the intercept and b1 is the slope of the line. Graphing parameters such as **lwd** and **col** are available within **abline**.

axis For a user define axis labeling use **axis(side=a,at=, label=)**. The value of a is the axis where 1, 2, 3, and 4, are the bottom, left, top, and right axis, respectively. The value of at is where to place the label. Labels are what to place at the defined locations. Key options are **col** to color the axis line, **cex.axis=** for scaling the axis labels, and **col.axis=** to color the axis labels. In the function that generated the graph use **xaxt="n"**, **yaxt="n"**, or **axis=FALSE** to remove the x-axis, y-axis, or both, respectively.

basic arithmetic Multiplication us * such as 3*x. You must use a multiplication sign in all cases of multiplying. Division us / such a 4/2. For exponents us ˆ such as xˆ2.

cbind Column bind combines multiple vectors in columns to create a matrix. For example, try **cbind(1:4,5:8)**.

colon command - a:b The colon command creates a vector of numbers starting at **a** and increasing by 1 unit until **b** is exceeded. For example **1:5** is the vector $(1, 2, 3, 4, 5)$. The values of **a** and **b** do note have to be integers.

concatenate - c The concatenate or combine command **c()** creates a vector of the objects in the command. For example, **c(1,3,5,7,9)** or **c("a","b","c","d")**.

comment For comments in the R editor precede the line with a hashtag, #.

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curve The code **curve(f,a,b)** graphs a function, f, from $x = a$ to $x = b$. If $add = TRUE$ is used then the function will be added to the current graph otherwise a graph frame is initiated. Optional arguments $lwd =$ sets the line width, $xlab =$ "$text$" and $ylab =$ "" adds axis text, $col =$ "$somecolor$" colors the function, $ylim = c(y_1, y_2)$ sets the y-axis range from a min of y_1 to a max of y_2.

Defined Functions and Values in R

- $\mathbf{cos(x)}$ is input as cos(x).
- $\boldsymbol{e^x}$ is input as exp(x).
- $\mathbf{ln(x)}$ is input as log(x).
- $\boldsymbol{\pi}$ is input as pi.
- $\mathbf{sin(x)}$ is input as sin(x).
- $\boldsymbol{\sqrt{x}}$ is input as sqrt(x), whereas $\sqrt[n]{x}$ is input as x^(1/n).

Deriv The **Deriv** command will by default take the derivative of the function with respect to the variable defined when the function was created. For example, if **f<–function(t){t^n**} then **Deriv(f)** will output what we expect; nt^{n-1}. We could take the derivative with respect to n, which would be done with **Deriv(f,"n")**. To use the **Deriv** function the Deriv package has to be loaded with **library**.

expression is used to output math expressions. For example, **expression(f(x)==x^2*sin(x))** will output $f(x) = x^2 \sin(x)$. Search R expression to see tables of the syntax. Two quick tips are that == outputs one equal sign and you need * between parts of the expression.

for loop A for loop will repeat until all values in the vector of loop have been exhausted. The basic syntax for a for loop is: **for (i in vector){ code }**. The vector object can be any vector of numbers or characters. The variable does not have to be i. The code between the braces will be executed for each element in the vector in order.

function The basic example of defining a one variable function is **f<–function(x){ equation }**. What is input for equation is the formula for the function. We note that functions in R can have more than one variable and the variable may be of any type.

grid We add an axis grid to a graph with **grid(NULL,NULL, col="black")**. The use of NULL here places the grid lines at the axis tick marks in the graph. Other options are available. This text colors the grid lines black because the default gray is often too faint.

head To view the first few lines of a vector, matrix, or data frame use **head**. For example **head(1:1000)** will output the numbers 1 to 6. Use the optional argument n=a, where a is the number of lines to output.

ifelse The function **ifelse** outputs one of two options based on a logical statement. For example, **ifelse(x>3,1,2)** will output 1 if x>3 is true and 2 if x>3 is false.

install.packages Use **install.packages** to install a package for R on a computer. For example, **install.packages("Deriv")** installs the Deriv package on a computer. The package name must be in quotes. R will ask for a CRAN (Comprehensive R Archive Network) location. This is the location of the server that the packages will be downloaded from. Choosing a site close to your current location makes sense. Packages only need to be installed once on a computer.

integrate To obtain a numerical approximation of a definite integral use **integrate(f,a,b)** to find $\int_a^b f(x)dx$.

legend The **legend(location, legend)** function adds a legend at a specified location, which can be a keyword in quotes such as topright or x and y coordinates, with the information in the vector legend. There are other key options such as **lwd**, **pch**, and **col** which should be vectors of the same length as legend. For example,**legend("topright", c("a","b","c"), pch=c(15,16,17),col=c("red", "blue","black"))** will add a legend at the top right of the graph with tree lines a, b, and c. Next to a, b, and c will be a square, circle, and triangle, respectively, given by pch and they will be colored red, blue, and black, respectively.

library In order to use a package in a current R session it needs to be loaded with the **library** function. For example, **library(Deriv)** loads the Deriv package to be used. There are no quotes around Deriv, unlike with **install.packages**. Packages should only be loaded once per session.

lines Given a vector of x and y coordinates, the function **lines(x,y)** will connected the coordinates with lines and add the result to a current graph. In essence, the points are plotted and the points are connected. Graphical options such as lwd and col may be used in **lines**.

max Use **max** to find the maximum value in a vector of numbers.

min Use **min** to find the minimum value in a vector of numbers.

mtext In order to add text to the margin of a graph use **mtext(text,side=,line=)** where the text to be added in quotes, the side in the side of the graph with 1, 2, 3, and 4 adding text to the bottom, left, top, and right margins. Graphical options such as col and cex, for scaling the size of the text may be used.

options(digits=a) Set the value of a to the number of desired digits of the output of numbers.

par(mar=c(a,b,c,d) Set the values of a, b, c, and d for the number of lines below, to the left, above, and to the right of the graph (clockwise starting at the bottom). Use this before the use of, say, **curve()**.

par(new=TRUE) This command will allow another graph to be placed on the same currently opened graph.

par(xpd=TRUE) If the desire is to allow elements placed on a graph to extend beyond the margins of the current graph us **par(xpd=TRUE)**. Replace TRUE with FALSE to turn this off.

paste This function allows us to concatenate a variable and characters. For example, if $a = 3.1415$ then **paste("(",a,")", sep="")** will output **(3.1415)**. The function **paste()** is particularly useful for math in labeling and adding information to graphs.

plot The **plot** function is a robust plotting command. In this text we used it to plot points. For example, **plot(x,y,xlim=c(a,b),ylim=c(c,d))** will plot the collection of points given by the vector x, for the x-coordinates, and the vector y, for the y-coordinates. The xlim and ylim options set the range for the x and y-axis, respectively. If these are left out the ranges will be set based on the given points.

Key options: pch= where 16 is a solid dot and the default is a hollow dot. type= “l” or “b” to connect the points as a curve or for both the points and lines connecting the points. xlab=“text” and ylab=“text” will add axis labels. main=“text” will add a title. Other graphing options such as col and lwd are all available.

points To add points to a current graph use **points(x,y)** where x and y may be vectors of coordinates for point or single points. Graphing options cex, col, and pch are useful here.

pos To place text on a graph with **text(x,y,pos=** below, to the left, above, or to the right of the point (x, y) set pos to 1, 2, 3, or 4, respectively.

round If, for example, $a = 3.1415$ then **round(a,2)** will output **3.14**. **round()** is often useful within **paste()**.

segments To connect two points $(x1, y1)$ and $(x2, y2)$ on a graph with a line use **segments(x1,y1,x2,y2)**. Graphing options such as lwd, lty, and col are useful here.

seq To generate a sequence of points the **seq** function is valuable and has many uses. **seq(a,b,by=d)** will create a sequence from a to b by step of d, whereas **seq(a,b,length=d)** will create a sequence from a to b with d points evenly spaced.

tail To view the last few lines of a vector, matrix, or data frame use **tail**. For example **tail(1:1000)** will output the numbers 995 to 1000. Use the optional argument n=a, where a is the number of lines to output.

text Use **text(a,b, content)** to place text in content, the text in content should be in quotes, on a graph at the point (a, b). The option pos and the function paste are useful with text.

title To add a tile or a subtitle to a graph use **title(main=,sub=)** where text set to main or sub should be in quotes. Within quoted text use \ n for a line break. The function paste is useful here.

uniroot.all To find the roots of a function between $x = a$ and $x = b$ use **uniroot.all(f,c(a,b))**. The rootSolve package must be loaded to use **uniroot.all**.

which.max If we have the vector example<–(2, 5, 1, 6, 1, 2, 6) then **which.max(example)** will return 4, the location in the vector of the first, if there are ties, maximum value.

which.min If we have the vector example<–(2, 5, 1, 6, 1, 2, 6) then **which.min(example)** will return 3, the location in the vector of the first, if there are ties, minimum value.

while loop A while loop will continue until the condition in the loop condition of the loop returns false. The basic syntax for a for loop is: **while (condition){ code }**. The condition must be a logical statement such as $x > 2$. The code is repeated until, for example, $x > 2$ returns false. Caution is warranted here because within the code of a while loop the value, in this example, of x should change and eventually cause the condition to return false, otherwise the loop will not stop.

Basic R Tips

- NOTE: In this book the R Code boxes show what you will see in the R Console window. The `>` should not be typed in the R Editor window.
- Create a folder for all your R files for calculus. You might call this folder Calculus-R.
- Always type your code in an R Editor window. If you make a mistake it can be edited and the file can be saved. Reusing code through copy and paste will save time.
- When you open a new R Editor window save it immediately and give it a useful name that will help you identify the content. Also include some type of date in the title. Do not use . in file names.
- If you want to run only one line of code in the R Editor just make sure your cursor is on that line somewhere and use Ctrl + R (PC) or Command + Enter (Mac).
- If you want to run multiple lines of code in the R Editor highlight all of those lines and use Ctrl + R (PC) or Command + Enter (Mac).
- You should get in the habit of adding comments to your code in the R Editor. Comment lines should begin with a `#`. If you run a line that begins with `#` it will be ignored in the R Console. Also a string of `#` is a good way to make sections in the R Editor.
- Start new script or document files for new content. Don't keep using the same file as it will get really long and hard to find code.
- When you close R it will ask: Save workspace image? You should say no to this unless you fully understand why you might want to say yes. Be careful because it will also ask you if you want to save any open script (PC) or document (Mac) files in the R Editor. You likely want to say yes to this.
- Warning: Like all computer languages, R is extremely picky. Your code must be perfect for it to run. This means that even syntax such as parenthesis and commas must be perfect or your code won't work. Be patient it takes time to get used to this.
- Pay attention to the output in the R Console. Learning to understand error messages can help you find problems with you code, although sometimes error messages are hard to interpret.
- If you see a `+` in the R Console window instead of a `>` this is a problem. It means that previous code didn't end and it is waiting for more lines of code. Use Esc to get back to `>`. Look in your code to see if your parenthesis aren't matching. There is a good chance you are missing a `)`, although this isn't the only possible problem.
- Coping and pasting code from a pdf file into R often doesn't work, especially in Macs. It is better to type directly into the R editor.

- The assignment operator <– should be viewed as an arrow pointing to the left. The idea is that what is written on the rights is being assinged, by the arrow, to the object on the left. In mathematics we might say let $a = 5$. In R we assign the value of 5 to the name a with **a<– 5**.
- Parenthesis (and) are used for argument for a function in R, where as square brackets [and] are used to define a location in an object such as a vector or matrix.
- If you find yourself copying and pasting (or worse retyping) an output result to be used in your code, stop. You should be defining a variable.
- When you see, for example, $e{+}02$ or $e{-}12$ with a number it is scientific notion, in other words $e{+}02$ and $e{-}12$ is equivalent to $\times 10^2$ and $\times 10^{-12}$. For example, $3.2e{+}02 = 320$ and $1.36e - 12 = 0.00000000000136$.
- Saving a graph. Using print screen is not how one should save a graph made in R to include in a report. Here is how it is done.

PC Make sure the window with the graph is active (click on it). Go to File -> Save as. Choose the file type and go from there.

Mac Make sure the window with the graph is active (click on it). Go to File -> Save in the menu bar, and choose a location to save the file. It will save as a PDF file, which we can double-click to open in Preview, and then use the File -> Save as menu choice to convert it to another format.

- If you are using R to take the derivative of more than one function then do not keep defining the function as **f**. This is a good way to make a mistake if you have multiple functions **f** in your code. The easiest thing to do in this case is use the convention of naming your functions **f1**, **f2**, etc.
- You cannot define the constant function **f<–function(x){c}** in R. If the variable of the function is **x** then there must be an **x** in the function definition. On the other hand this will work: **f<–function(x){c + 0*x}**, although it has questionable value. You can also use **abline(h=c)** to graph a horizontal line.

Appendix H
Answers to Odd Problems

Section 1.1

1.
```
> 2^7-exp(7.5)+sin(3*pi)-log(150)+sqrt(92)
[1] -1675.461
```

3.
```
> 5^(-4)-exp(-9.2)+cos(-6*pi/4)-log(625)+ 218^(1/4)
[1] -2.593751
```

5.
```
> f<-function(x){2^x -exp(x) + sin(2*pi*x) - log(x) + sqrt(x)}
> f(3.2*pi)
[1] -22164.02
> f(5.6)
[1] -221.8676
```

7.
```
> f(c(1,3,8,11))
[1] 7.320387e+00 1.449628e+02 3.936061e+05 4.888800e+07
```

9.
```
> f<-function(x){sin(x)}
> g<-function(x){cos(x)}
> curve(f,0,2*pi,lwd=2,xlab="x-axis", ylab="sin(x) and cos(x)",
col="red")
> curve(g,0,2*pi,lwd=2,col="blue",add=TRUE)
> grid(NULL,NULL,col="black")
```

11.
```
> f<-function(x){sqrt(x)}
> g<-function(x){log(x)}
> curve(f,0,500,lwd=2,xlab="x-axis", ylab="ln(x) and sqrt(x)",
col="red")
```


T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4

```
> curve(g,0,500,lwd=2,col="blue",add=TRUE)
> grid(NULL,NULL,col="black")
```

13.

```
> library(rootSolve) #Only run once per R session
> f<-function(x){x^2+8*x+15}
> roots<-uniroot.all(f,c(-10,10))
> roots
[1] -5 -3
> curve(f,-10,5,lwd=2,xlab="x-axis", ylab="x^2+8x+15",col="purple")
> points(roots,f(roots),pch=16,cex=1.5)
> grid(NULL,NULL,col="black")
```

15.

```
> library(rootSolve) #Only run once per R session
> f<-function(x){-x^3-13*x^2+19*x+5}
> roots<-uniroot.all(f,c(-20,10))
> roots
[1] -14.3037242  -0.2281752   1.5320378
> curve(f,-20,5,lwd=2,xlab="x-axis", ylab="-x^3-13*x^2+19*x+5",
col="purple")
> points(roots,f(roots),pch=16,cex=1.5)
> grid(NULL,NULL,col="black")
```

17.

```
> library(rootSolve) #Only run once per R session
> f<-function(x){exp(x)-42}
> roots<-uniroot.all(f,c(-5,5))
> roots
[1] 3.737671
> curve(f,-5,5,lwd=2,xlab="x-axis", ylab="exp(x)-42",col="purple")
> points(roots,f(roots),pch=16,cex=1.5)
> grid(NULL,NULL,col="black")
```

19.

```
> library(rootSolve) #Only run once per R session
> f<-function(x){x^2-5*x-36}
> fminus10<-function(x){f(x)-10}
> roots<-uniroot.all(fminus10,c(-5,10))
> roots
[1] -4.728549  9.728549
> curve(f,-6,11,lwd=2,xlab="x-axis", ylab="x^2-5*x-36",
col="purple")
> points(roots,f(roots),pch=16,cex=1.5)
> grid(NULL,NULL,col="black")
```

21.

```
> library(rootSolve) #Only run once per R session
> f<-function(x){-x^3-13*x^2+19*x+56}
> fplus100000<-function(x){f(x)+100000}
> roots<-uniroot.all(fplus100000,c(30,50))
> roots
```

```
[1] 42.59458
> curve(f,-30,50,lwd=2,xlab="x-axis", ylab="-x^3-13*x^2+19*x+56",
col="purple")
> points(roots,f(roots),pch=16,cex=1.5)
> grid(NULL,NULL,col="black")
```

23.

```
> library(rootSolve) #Only run once per R session
> f<-function(x){sin(x)}
> g<-function(x){cos(x)}
> curve(f,0,4*pi,lwd=2,xlab="x-axis", ylab="sin(x) and cos(x)",
col="red")
> curve(g,0,4*pi,lwd=2,col="blue",add=TRUE)
> grid(NULL,NULL,col="black")
> fminusg<-function(x){f(x)-g(x)}
> roots<-uniroot.all(fminusg,c(0,4*pi))
> roots
[1]  0.7854292  3.9270218  7.0686145 10.2102072
> points(roots,f(roots),pch=16,cex=1.5)
```

25.

```
> library(rootSolve) #Only run once per R session
> f<-function(x){sqrt(x)}
> g<-function(x){log(x)+1}
> curve(f,0,2,ylim=c(-2,2),lwd=2,xlab="x-axis",
ylab="sqrt(x) and 1+ln(x)",col="red")
> #Note the y-axis was set with ylim.
> curve(g,0,2,lwd=2,col="blue",add=TRUE)
> grid(NULL,NULL,col="black")
> fminusg<-function(x){f(x)-g(x)}
> roots<-uniroot.all(fminusg,c(0,3))
> roots
[1] 1.00015
> points(roots,f(roots),pch=16,cex=1.5)
```

27.

```
> f<-function(x){x^2+2}
> curve(f,-5,5,lwd=2,xlab="x-axis", ylab="x^2+2",col="red")
> grid(NULL,NULL,col="black")
> segments(1,f(1), 4,f(4),lwd=2,col="blue")
> #Positive slope
> slope=(f(4)-f(1))/(4-1)
> slope
[1] 5
```

29.

```
f<-function(x){x*sin(x)}
curve(f,0,6,lwd=2,xlab="x-axis", ylab="x*sin(x)",col="red")
grid(NULL,NULL,col="black")
segments(1.5,f(1.5), 5.5,f(5.5),lwd=2,col="blue")
```

```
#negative slope
slope=(f(5.5)-f(1.5))/(5.5-1.5)
slope
```

31.
```
> f<-function(x){4*x^5+15*x^4-140*x^3-430*x^2+1200*x+1000}
> curve(f,-10,10,lwd=2,xlab="x-axis",
ylab="4*x^5+15*x^4-140*x^3-430*x^2+1200*x+1000",col="purple")
> grid(NULL,NULL,col="black")
> # y-axis scale changed so we can't see the detials
> # of the changes around 0.
> # The window really matters.
```

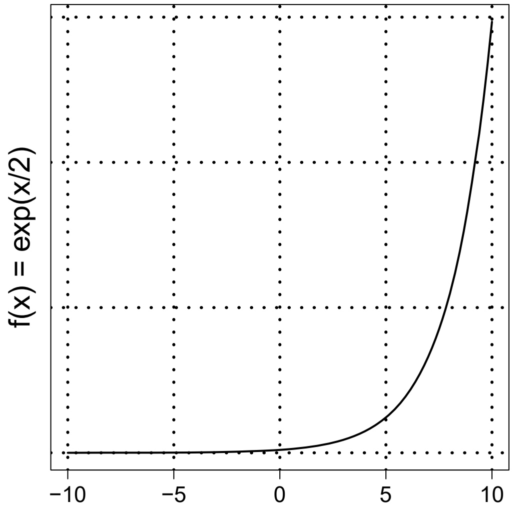
33.
```
> library(rootSolve) #Only run once per R session
> f<-function(x){4*x^5+15*x^4-140*x^3-430*x^2+1200*x+1000}
> g500<-function(x){500*x^2-1000}
> g1000<-function(x){1000*x^2-1000}
> g2000<-function(x){2000*x^2-1000}
> curve(f,-2,8.5,lwd=2,xlab="x-axis", ylab="f, g500, g1000, and g2000")
> curve(g500,-6,10,lwd=2,col="red2",add=TRUE)
> curve(g1000,-6,10,lwd=2,col="purple",add=TRUE)
> curve(g2000,-6,10,lwd=2,col="blue2",add=TRUE)
> grid(NULL,NULL,col="black")
>
> fminusg500<-function(x){f(x)-g500(x)}
> roots1<-uniroot.all(fminusg500,c(-2,8.5))
> roots1
[1] -1.008062  1.978687  6.248472
> points(roots1,f(roots1),pch=16,cex=1.5,col="red2")
>
> fminusg1000<-function(x){f(x)-g1000(x)}
> roots2<-uniroot.all(fminusg1000,c(-2,8.5))
> roots2
[1] -0.8617282  1.5564982  7.1425554
> points(roots2,f(roots2),pch=16,cex=1.5,col="purple")
>
> fminusg2000<-function(x){f(x)-g2000(x)}
> roots3<-uniroot.all(fminusg2000,c(-2,8.5))
> roots3
[1] -0.7045925  1.1472819  8.4059667
> points(roots3,f(roots3),pch=16,cex=1.5,col="blue2")
```

Section 2.1

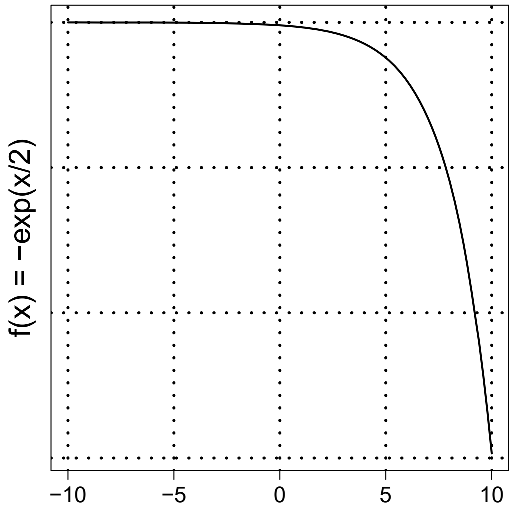
1. Answers are approximate. Increasing: $x = -10$ to $x = -5$ and $x = 3$ to $x = 6$. Decreasing: $x = -5$ to $x = 3$. Concave down: $x = -10$ to $x = -1$. Concave up: $x = -1$ to $x = 6$. Inflection point at $(-1, 1)$. Local max and global max at $(-5, 6)$. Local min at $(3, -3)$. Global min at $(-10, -8)$.

3. Answers are approximate. Increasing: $x = -3$ to $x = 0$ and $x = 2$ to $x = 4$. Decreasing: $x = -5$ to $x = -3$ and $x = 0$ to $x = 2$. Concave down: $x = -1.5$ to $x = 1$. Concave up: $x = -5$ to $x = -1.5$ and $x = 1$ to $x = 4$. Inflection points at $(-1.5, -2)$ and $(1, -1)$. Local max at $(0, 0.5)$. Global max at $(4, 15.5)$. Local min at $(-3, -5.5)$, also global min, and $(2, -2)$.

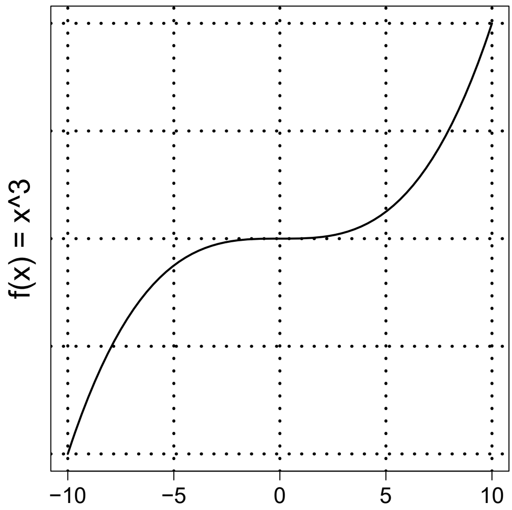
5.



7.



9.



11. Answers are approximate. Increasing: $x = -1$ to $x = 0$ and $x = 6.5$ to $x = 10$. Decreasing: $x = 0$ to $x = 6.5$. Concave down: $x = -1$ to $x = 3$. Concave up: $x = 3$ to $x = 10$. Inflection point at $(3, -60)$. Local max at $(0, 0.1)$. Global max at $(10, 40)$. Local and global min at $(6.5, -140)$.

13. Answers are approximate. Increasing: $x = -1$ to $x = 1$ and $x = 3$ to $x = 4$. Decreasing: $x = -2$ to $x = -1$ and $x = 1$ to $x = 3$. Concave down: $x = 0$ to $x = 2$. Concave up: $x = -2$ to $x = 0$ and $x = 2$ to $x = 4$. Inflection points at $(0, 2)$ and $(2, -2)$. Local max at $(1, 7)$. Global max at $(-2, 23)$. Local min at $(-1, -5)$ and $(3, -13)$, also the global min.

15. Eight local maximums. Sixteen inflection points and likely another one just after $x = 0$.

17. On $[0, 9]$: Two local maximums. One local minimum. Two Inflection points. On $[0, 20]$: Two local maximums. Two local minimums. Five Inflection points. On $[0, 20]$ with $x^2 + 50\sin(x)$: Three local maximums. Three local minimums. Five Inflection points.

Section 3.1

1. $CO2(t)$ function:

   a. The input t is years after 1950 and the output is yearly average CO2 in ppm at Mauna Loa.
   b. The data is yearly from 1950 through 2021.
   c. CO2 measured in ppm is increasing and concave up at Mauna Loa.
   d. In 2013 the model estimates that the yearly average CO2 was 397.09556 ppm.
   e. In 1995 the model estimates that the yearly average CO2 was 361.23371 ppm.
   f. The model predicts that 450 ppm will occur in 2033.94. R Hint:

```
> f<-function(t){CO2(t)-450}
> uniroot.all(f,c(60,100))+1950
[1] 2033.93531473
```

3. $P(t)$ function:

   a. The input t is years after 1990 and the output is the number of people in extreme poverty in billions.
   b. The graph is decreasing and concave down until around $x = 18$ where it changes to concave up.
   c. The number of people in extreme poverty is still decreasing but slower than before.
   d. In 2011 the model estimates that 1.7 billion people were in extreme poverty.
   e. In 2017 the model estimates that 1.3 billion people were in extreme poverty.

f. The model predicts that 900,000,000 people will be in extreme poverty in 2013. R Hint:

```
> f<-function(x){P(x) - 0.9}
> uniroot.all(f,c(0,30))+1990
[1] 2013.25
```

5. $Wwind(t)$ function:

a. The input t is years after 1980 and the output is cumulative installed world wind power in megawatts
b. The data is yearly from 1980 through 2020.
c. Cumulative installed world wind power is increasing and concave up.
d. In 2017 the model estimates that cumulative installed world wind power was 542,483 megawatts.
e. In 2016 the model estimates that cumulative installed world wind power was 480,141 megawatts.
f. The model predicts that 1,000,000 megawatts will occur in 2023.19. Note that the function does not extrapolate well and if your range in uniroot.all is large enough you will get another around 2043, which makes no sense. R Hint:

```
> f<-function(x){cubic.World(x) - 1000000}
> uniroot.all(f,c(40,50))+1980
[1] 2023.19465946
```

7. $USwind(t)$ function:

a. The input t is years after 1980 and the output is cumulative installed U.S. wind power in megawatts.
b. The data is yearly from 1985 through 2020.
c. Cumulative installed U.S. wind power is increasing and concave up, although the fitted curve has a chance in concavity around $t = 14$ which is not reflected in the data.
d. In 2013 the model estimates that cumulative installed U.S. wind power was 59,134 megawatts.
e. In 2017 the model estimates that cumulative installed U.S. wind power was 92,001 megawatts.
f. The model predicts that 85,000 megawatts will occur in 2016.17. R Hint:

```
> f<-function(x){USwind(x) - 85000}
> uniroot.all(f,c(30,50))+1980
[1] 2016.16796576
```

9. Arctic Ice functions:

a. For all functions, the input x is the month of the year with Jan$=1$, etc. and the output is million square kilometers of Arctic sea ice.

b. All three functions have a local maximum around $x = 3$, a local minimum around $x = 9$, and an inflection point between $x = 6$ to $x = 7$ where the functions change from concave down to concave up.
c. For AI_1980 the inflection point occurs in June, while the other two years, 2012 and 2019, the inflection point is in July.
d. In the middle of September the models for Arctic sea ice extent for 1980 and 2021 estimate 7.7 msk and 3.8 msk of ice, respectively.
e. In the middle of July the model estimates 7.6 msk of ice in 2019.
f. For 2012 the model estimates that 8 msk was reached around Jul 10 (0.856*30=25.69 days past the middle of June) and again around Nov 5. R Hint:

```
> f<-function(x){p.2012(x) - 8}
> uniroot.all(f,c(1,12))
[1]  6.856084471 10.668971998
```

11. $L(x)$ function:

a. the input x is the radius of the largest tumor in millimeters and the output is the probability of a malignant tumor.
b. The graph has horizontal asymptotes at $y = 0$ and $y = 1$ with an inflection point around $x = 17$ as the curve changes from concave up to concave down.
c. There is an inflection point at a tumor radius of approximately 17mm.
d. If the radius of the largest tumor is 17mm then the model predicts the probability a tumor is malignant is 61%.
e. If the radius of the largest tumor is 18mm then the model predicts the probability a tumor is malignant is 83%.
f. If the probability a tumor is malignant is 70% then the model predicts that the radius of the largest tumor is 17.35mm. R Hint:

```
> f<-function(x){Lfun(x) - 0.7}
> uniroot.all(f,c(15,20))
[1] 17.35238
```

Section 4.1

1. On average revenue increased by 30 dollars per widget when the number of widgets sold increased from 2 to 8 widgets.
3. On average cost decreased by 235 dollars per widget when the number of widgets produced increased from 5 to 10 widgets.

5. The function f is defined as $x^2 + 5x - 10$ and then graphed from $x = -10$ to $x = 10$ with line width 2. A grid is added to the graph. The function is then evaluated at 5 with an output of 40 or $f(5) = 40$.

7. The function f is defined as $sin(x)$. The value of a and b are set to 0 and $\pi/2$, respectively. The slope of the secant line from a to b is calculated to be 0.6366197724. In other words, the slope of the secant line of $sin(x)$ from $x = 0$ to $x = \pi/2$ is 0.6366197724.

9. ARC: 249 units/unit. This is not a particularly good representation because the function fluctuates greatly between the endpoints.

11. ARC: 17.29329. The curve is increasing and concave up and the ARC is a decent representation over the interval, although the function does fluctuate a little over the interval.

13. On average global temperate rose 0.014 °C per year from 1950 to 2021.

15. U.S. cumulative installed wind power increases on average by 3331.24 MW per year from 1985 to 2020.

17. On average the water temperature decreases at a rate of 0.09 °F from a depth of 3.8 meters to a 161.2 meters. On the other hand, the water temperature only decreases on average at a rate of 0.01 °F from a depth of 161.2 meters to a 500.5 meters.

19. On average the percent of CA in drought increases 47.6 percent per year from 2019 to 2021.

21. At 58.8 feet the water temperature is estimated to be decreasing at a rate of 0.13 °F per meter, while at 700.1 feet the water temperature is estimated to be decreasing at a rate of 0.02 °F per meter. The trends in the data seem consistent so the estimates may be ok, but the intervals seem large and that is a concern. Calculations: $(66 - 79.1)/(101 - 3.8)$ and $(40.6 - 61.5)/(1398 - 500.5)$

23. In 2013 the percent of CA in drought is estimated to be increasing at 0.12 percentage points per year, while in 2018 it is decreasing at 8.7 percentage points per year. The 2018 estimate may be way off given the up down trend in the data. Calculations: $(100 - 88.05)/(2104 - 2012)$ and $(4.71 - 22.12)/(2019 - 2017)$

25. For $h = 0.1$: -3831.999. For $h = 0.0001$: -3840. Decreasing h from 0.1 to 0.0001 led to about a 0.2% change in the estimation. The value of -3840 is likely close to the actual instantaneous rate of change.

27. For $h = 0.1$: 4.24106. For $h = 0.001$: 4.234001. Decreasing h from 0.1 to 0.001 led to about a 0.16% change in the estimation. The value of 4.234001 is likely close to the actual instantaneous rate of change.

29. Global average temperature is estimated to be increasing at a rate of 0.025 °C per year in 2021 with $h = 0.001$ and 0.025 °C per year in 2021 with $h = 0.0001$.

31. U.S. cumulative installed wind power is estimated to be increasing at a rate of 7473.2 MW per year in 2020 with $h = 0.1$ and a rate of 7473.5 MW per year in 2020 with $h = 0.01$

Section 5.1

1. Ten minutes after being poured, the temperature of the tea is decreasing at a rate of 3.5°F / minute.

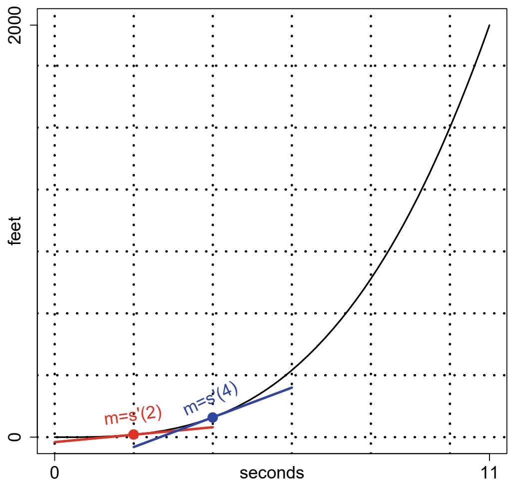
3. At 2 seconds the cylinder is rolling down the ramp at a rate of 6.538 meters/second. Or, at 2 seconds the cylinder is rolling down the ramp at a velocity of 6.538 meters/second.

5. When producing 58 items the cost is increasing at a rate of 8.25 thousand dollars per item produced.

7. Ten miles from the start of the river the flow rate is increasing at a rate of 1000 cubic feet/second per mile.

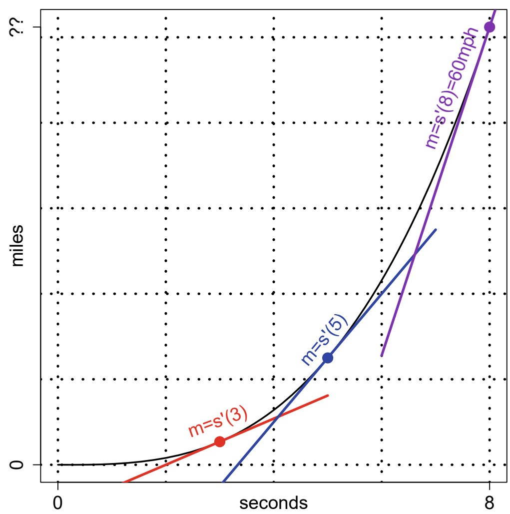
9. At an elevation of 3000 feet, the temperature is decreasing at a rate of 0.0035°F / foot of elevation.

11. $s(2)$ is the distance in feet the rock has traveled after 2 seconds. $s'(2)$ represents the rate of change in feet/second at 2 seconds, in this context this is the speed of the rock at 2 seconds. The rock will travel farther after 4 seconds than 2 seconds and so $s(4)$ is larger than $s(2)$. The rock will pick up speed, rate of change will increase, as it falls and so $s'(4)$ is larger than $s'(2)$.

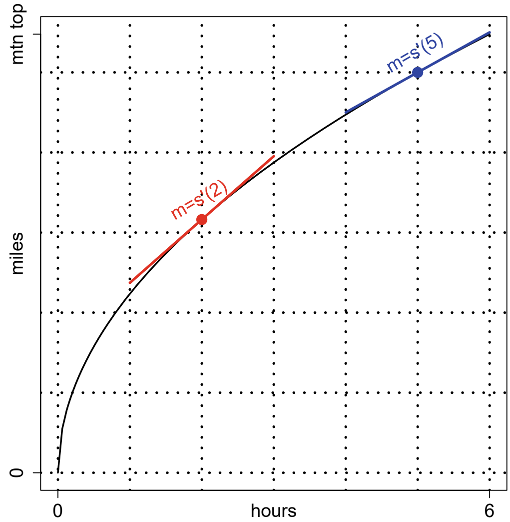


13. $s(3)$ is the distance the car has traveled after 3 seconds. $s'(3)$ is the rate of change in miles per second of the car at 3 seconds, in this case it is the speed of the car. The car is moving forward so it will have traveled farther after 5 seconds, $s(5)$ is greater than $s(3)$. We are told the car is accelerating and so it will be traveling faster at 5 seconds, hence $s'(5)$ is greater than $s'(3)$. The problem states that the car accelerates to 60mph in 8 seconds and so $s'(8) = 60\text{mph} = 1/60$miles per second.

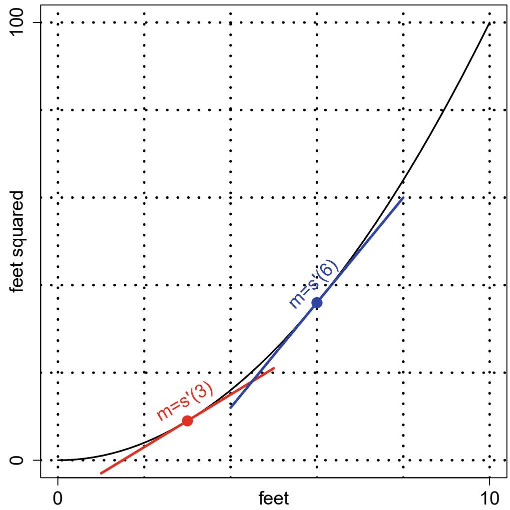
15.



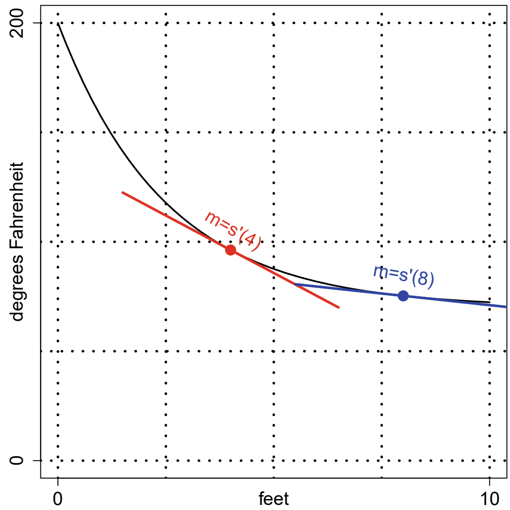
17. $B(2)$ is the distance traveled up the mountain, in miles, after 2 hours. $B'(2)$ is the rate of change of the backpacker, in miles per hour, at 2 hours or, in this case, the hiking speed of the backpacker. The backpacker will have traveled farther after 5 hours than 2 and so $B(5)$ is larger than $B(2)$. We'll assume the backpacker gets tired and travels slower the steeper the trail and so the hiking speed at 5 hours will be slower than at 2 hours. Hence $B'(2)$ is larger than $B'(5)$.



19. $A(3)$ is the area of the square with a side length of 3 feet. $A'(3)$ is the rate of change of the area, in feet squared per foot, with a side of length 3 feet. The area of a square with a side length of 6 feet is larger than one with a side length of 3 feet and so $A(6)$ is larger than $A(3)$. A small increase in the side length will add more area when the side is 6 feet than when it is 3 feet. Hence $A'(6)$ is larger than $A'(3)$.



21. $T(4)$ is the temperature 4 feet from the wood stove. $T'(4)$, which is negative, is the rate of change, in °F per foot of the temperature 4 feet from the stove. In other words, how much the temperature is decreasing as we move away from the stove. The temperature will be cooler the farther from the wood stove and so $T(4)$ will be larger than $T(8)$. As we move away from the stove the temperature will decrease faster the closer we are to the stove. Hence $|T'(8)|$ is larger than $|T'(4)|$ which means that $T'(4)$ is larger (or less negative) than $T'(8)$.



23. True. The function can be going up, down, or be constant at any value.
25. False. The units should be some form of rate.
27. False. Generally the units won't make sense.
29. False. 15 meters isn't a rate.

Section 6.1

1. The change from 1 second to 10 seconds is 22023.74751 meters.
3. The percentage change from 5 seconds to 12 seconds is 476%.
5. The percentage rage of change at 2 seconds is 12 percent per second.
7. Change: 192 feet. Average rate of change: 96 ft/sec. Percentage change: 300%. During 2 to 4 seconds after being dropped the rock fell 192 feet, traveled on average at 96 ft/sec and the total distance traveled increased 300%.

9. Change 0 to 4: 88 ft. Average rate of change 0 to 4: 22 ft/sec. Percentage change from 0 to 4: NA - can't divide by zero. During 0 to 4 seconds the car traveled 88 feet at an average rate of 22 ft/sec. Change 4 to 8: 264 ft. Average rate of change 4 to 8: 66 ft/sec. Percentage change from 4 to 8: 300%. During 4 to 8 seconds the car traveled 264 feet at an average rate of 66 ft/sec and the total distance traveled increased 300%. Since the car is accelerating it will cover more distance during the four second interval from 4 to 8 second than from 0 to 4 seconds.
11. Change 4 to 8: 448 cubic inches. Average rate of change 4 to 8: 112 cubic feet of volume per inch of side length. Percentage change from 4 to 8: 700%. As the cube increases its side length from 4 to 8 inches the volume increases 448 cubic inches at an average rate of 112 cubic feet of volume per inch of side length and a 700% increase in volume
13. Rate of change at 3 seconds: $s'(1) = 32$ ft/sec. Percent rate of change at 3 seconds: 13.33 % per second.$s'(1) = 96$ ft/sec. At 3 seconds after being launched the watermelon is rising at a rate of 32 ft/sec and increasing its height at a rate of 13.33 % per second. Gravity will cause the watermelon to slow down as it rises and so $s'(1)$ is larger than $s'(3)$.
15. Rate of change at 6 feet: $A'(6) = 12$ square feet of area / foot of side length. Percent rate of change at 6 feet: 33.33 % per foot of side length. $A'(3) = 6$ square feet of area/foot of side length. At a side length of 6 feet the square is increasing at a rate of square feet of area/foot of side length and increasing its area at a rate of 33.33 % per foot of side length. A larger square gains more area with an increase of the side length than a smaller square, hence $A'(6)$ is larger than $A'(3)$.
17. From the 2 mile to the 7.5 mile mark the elevation increases on average by 275 feet per mile. Over the same interval the total elevation change is 1515 feet or a 63% increase. Change is more relevant than percent change. If we started at 1000 feet and climbed to 2515 the change is the same but the parentage change is larger.
19. From 1997 to 2020 March Arctic ice extent decreased on average by 0.03 msk per year. This is a total decrease of 0.74 msk or a decrease of 4.7%. Both change and percentage change are relevant. The 0.74 msk needs the context of 4.7%.
21. Wages for the 10th percentile increase on average at 0.04 dollars per year from 1975 to 2020. This is an increase of \$2.23 or 23.97%. Both change and percent change are meaningful here.
23. An increasing concave up curve would also have its rate of change increasing, hence the rate of change at $x = b$ would be the largest and also larger than the average.
25. A curve that is neither concave up or down is a line. The average rate of change between any two points is the slope of the line. Similarly, the rate of change, or slope of the tangent line, is the slope of the line at all points. Hence the two values would be equal.

27. Answers are approximated based on reading values off the graph. (a) $4900 - 2200 = 2700$ ft. The total elevation climbed is greater due to going down and back up during the hike. (b) $(4900 - 2200)/(8 - 0) = 337.5$ feet/mile. During the 8 mile trial the average elevation change is 337.5 feet/mile. (c) $((4900 - 2200)/2200)100\% = 122.73\%$. The hike has a 122.73% increase in elevation. (d) Either around mile 6.5 or right near the 8 mile mark. (e) From mile 6 to mile 7 the elevation changes from 3300 feet to 4300 feet, for an average rate of change of 1000 feet/mile.
29. (a) In 2019 the number of people in extreme poverty was decreasing at a rate of 32,000,000 people per year. (b) The number of people in extreme poverty decreased on average by $46,000,000$ people per year from 1900 to 2019. (c) The number of people in extreme poverty decreased 67.9% from 1990 to 2019. (d) In 2019 the number of people in extreme poverty was decreasing increasing at a rate of 5.13 % per year.
31. (a) Average global temperature was increasing at a rate of 0.022 degrees Celsius per year in 2010. (b) Average global temperature increased on average 0.012 degrees Celsius per year from 1950 to 2010. (c) Average global temperature increased 5.4% from 1950 to 2010. (d) In 2010 average global temperature was increasing at a rate of 0.15 % per year.
33. (a) Cumulative installed world wind power was increasing at a rate of 35136.29 megawatts per year in 2010. (b) From 1980 to 2010 cumulative installed world wind power increases on average by 6353.60 megawatts per year. (c) Cumulative installed world wind power increased by 665.1 % from 1980 to 2010. (d) In 2010, cumulative installed world wind power was increasing at a rate of 18.2% per year.
35. (a) Cumulative installed U.S. wind power was increasing at a rate of 8287.95 megawatts per year in 2015. (b) From 1985 to 2015 cumulative installed U.S wind power increases on average by 2908.26 megawatts per year. (c) Cumulative installed U.S. wind power increased by 2880.1 % from 1985 to 2015. (d) In 2015, cumulative installed world wind power was increasing at a rate of 11% per year.
37. The 10 percent is a percentage change of the area of the Sahara from 1920 to 2013.
39. Atmospheric CO2 increases by 20 ppm (a change) from 1880-1959, while the average rate of change of temperature was 0.04°C per decade.
41. The average rate of change of sea level since 1993 is 3.1 mm per year.

Section 7.1

1. 61	3. 43	5. 21.46	7. -0.32
9. 0.32	11. 6.2	13. 3.675	15. 3

17. We would estimate that average global temperature would have been 14.62 degrees Celsius in 2011 and 14.64 degrees Celsius in 2012.
19. We estimate that cumulative installed world wind power would have increased 3516 megawatts from 2010 to 2011 and 70272 megawatts from 2010 to 2012.

21. We estimate that U.S. cumulative installed wind power increased 8288 megawatts from 2015 to 2016 and 16576 megawatts from 2015 to 2017.
23. 10
25. 12.5
27. 3.6
29. -2
31. 8.3; 7.6
33. 47.2; 48.6
35. 4
37. 0.4
39. 11; 12.75
41. 3.25; 3.5

Section 8.1

1. 3.99
3. 2.1
5. 3.01
7. -6.8
9. In the first line the function **f** is defined to be $\sin(x)$. In the second line the variable **x_values** is defined to be the vector of four values 0, $\pi/4$, $\pi/2$, π. In the third line **f(x_values)** evaluates the four values in the vector **x_values** and since this is inside **round(,2)** the output values are rounded to 2 decimal places. The last line are the four output values rounded to two decimal places. For example $\sin(pi/4) = 0.71$ rounded to two decimal places.
11. The first line defines the function **f** to be $x + 4$. The second line defines the variable **a** to be equal to 1. The third line defines **h** to be the vector of four values 1, 0.5, 0.1, 0.01. The fourth line evaluates the function **f** at the four input values $a + h = 1 + 1, 1 + 0.5, 1 + 0.1, 1 + 0.01$. The last line are the four output values.
13. The first line defines the function **f** to be x^2. The second line defines the variable **a** to be equal to 2. The third line defines **h** to be the vector of four values 0.1, 0.01, 0.001, 0.0001. The fourth lines evaluates the expression $(f(a+h) - f(a))/h$ at the input values $a + h = 2 + 0.1, 2 + 0.01, 2 + 0.001, 2 + 0.0001$. The last line are the four output values of the expression. In this case the output values are the slope of the secant line from $a = 2$ to the value $a + h$ of the function $f(x) = x^2$.
15. (a) $\dfrac{f(8-0.5) - f(8)}{-0.5} = 19$ (b) $\dfrac{f(8+0.25) - f(8)}{0.25} = -12$ (c) $f'(8) \approx -6$
17. (a) $\dfrac{f(-2-0.25) - f(-2)}{-0.25} = -1$ (b) $\dfrac{f(-2+0.25) - f(-2)}{0.25} = 57$ (c) $f'(-2) \approx 27$
19. (a) $\dfrac{f(2-0.75) - f(2)}{-0.75} = 3.25$ (b) $\dfrac{f(2+0.5) - f(2)}{0.5} = 4.5$ (c) $\dfrac{f(2-0.75) - f(2)}{-0.75} = \dfrac{f(1.25) - f(2)}{-0.75} = \dfrac{1.25^2 - 2^2}{-0.75} = \dfrac{-2.4375}{-0.75} = 3.25$ and $\dfrac{f(2+0.5) - f(2)}{0.5} = \dfrac{f(2.5) - f(2)}{0.5} = \dfrac{2.5^2 - 2^2}{0.5} = \dfrac{2.25}{0.5} = 4.5$ (d) $f'(2) \approx 4$
21. Based on the table we conjecture that $f'(2) = 4$.

h	secant slope	h	secant slope
0.1	4.10000	-0.1	3.90000
0.01	4.01000	-0.01	3.99000
0.001	4.00100	-0.001	3.99900
0.0001	4.00010	-0.0001	3.99990
0.00001	4.00001	-0.00001	3.99999

23. Based on the three tables below we conjecture that $f'(-3) = -6$, $f'(-4) = -8$, and $f'(-3) = -10$.

h	secant slope	h	secant slope
0.1	-5.90000	-0.1	-6.10000
0.01	-5.99000	-0.01	-6.01000
0.001	-5.99900	-0.001	-6.00100
0.0001	-5.99990	-0.0001	-6.00010
0.00001	-5.99999	-0.00001	-6.00001

h	secant slope	h	secant slope
0.1	-7.90000	-0.1	-8.10000
0.01	-7.99000	-0.01	-8.01000
0.001	-7.99900	-0.001	-8.00100
0.0001	-7.99990	-0.0001	-8.00010
0.00001	-7.99999	-0.00001	-8.00001

h	secant slope	h	secant slope
0.1	-9.90000	-0.1	-10.10000
0.01	-9.99000	-0.01	-10.01000
0.001	-9.99900	-0.001	-10.00100
0.0001	-9.99990	-0.0001	-10.00010
0.00001	-9.99999	-0.00001	-10.00001

25. Based on the table we conjecture that $f'(4) = 48$.

h	secant slope	h	secant slope
0.1	49.21000	-0.1	46.81000
0.01	48.12010	-0.01	47.88010
0.001	48.01200	-0.001	47.98800
0.0001	48.00120	-0.0001	47.99880
0.00001	48.00012	-0.00001	47.99988

27. Based on the table we conjecture that $f'(0) = 1$.

h	secant slope	h	secant slope
0.1	0.998334	-0.1	0.998334
0.01	0.999983	-0.01	0.999983
0.001	1.000000	-0.001	1.000000
0.0001	1.000000	-0.0001	1.000000
0.00001	1.000000	-0.00001	1.000000

29. Based on the table we conjecture that $f'(0) = 0$.

h	secant slope	h	secant slope
0.1	-0.049958	-0.1	0.049958
0.01	-0.005000	-0.01	0.005000
0.001	-0.000500	-0.001	0.000500
0.0001	-0.000050	-0.0001	0.000050
0.00001	-0.000005	-0.00001	0.000005

31. Based on the table we conjecture that $f'(1) = 1$.

h	secant slope	h	secant slope
0.1	0.953102	-0.1	1.053605
0.01	0.995033	-0.01	1.005034
0.001	0.999500	-0.001	1.000500
0.0001	0.999950	-0.0001	1.000050
0.00001	0.999995	-0.00001	1.000005

33. Based on the table we conjecture that $f'(1) = 0.5$.

h	secant slope	h	secant slope
0.1	0.488088	-0.1	0.513167
0.01	0.498756	-0.01	0.501256
0.001	0.499875	-0.001	0.500125
0.0001	0.499988	-0.0001	0.500013
0.00001	0.499999	-0.00001	0.500001

35. Based on the table, both the left and the right secant line slopes agree to (more than) seven decimal places and so we estimate that $f'(2) = 0.3535534$. Note that in the second to last line there was not agreement to seven decimal places. Also note that, for example, for the positive values of h we had successive values that agreed to 7 decimal places but then it changed at $h = 0.000000001$.

h	secant slope	h	secant slope
0.1	0.349241122458	-0.1	0.358086871641
0.01	0.353112550269	-0.01	0.353996440651
0.001	0.353509207464	-0.001	0.353597595819
0.0001	0.353548971286	-0.0001	0.353557810122
0.00001	0.353552948651	-0.00001	0.353553832544
0.000001	0.353553346377	-0.000001	0.353553434751
0.0000001	0.353553384347	-0.0000001	0.353553395449
0.00000001	0.353553386567	-0.00000001	0.353553408772
0.000000001	0.353553408772	-0.000000001	0.353553408772
0.0000000001	0.353552742638	-0.0000000001	0.353554963084

37. Based on the table, both the left and the right secant line slopes agree to six decimal places and so we estimate that $f'(1) = 2.718281$.

h	secant slope	h	secant slope
0.1	2.858841954874	-0.1	2.586787173021
0.01	2.731918655787	-0.01	2.704735610978
0.001	2.719641422533	-0.001	2.716923140478
0.0001	2.718417747083	-0.0001	2.718145918896
0.00001	2.718295419957	-0.00001	2.718268237079
0.000001	2.718283187431	-0.000001	2.718280469161
0.0000001	2.718281968406	-0.0000001	2.718281688630

39. Based on the table, both the left and the right secant line slopes agree to five decimal places and so we estimate that $f'(1) = 0.54030$.

h	secant slope	h	secant slope
0.1	0.497363752535	-0.1	0.581440751804
0.01	0.536085981012	-0.01	0.544500620738
0.001	0.539881480360	-0.001	0.540722951275
0.0001	0.540260231419	-0.0001	0.540344378517
0.00001	0.540298098506	-0.00001	0.540306513208
0.000001	0.540301885121	-0.000001	0.540302726670

41. Based on the table, both the left and the right secant line slopes agree to three (actually 4) decimal places and so we estimate that $f'(1) = -0.653$.

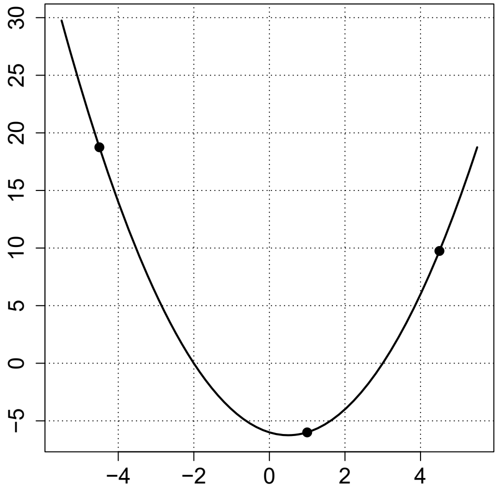
h	secant slope	h	secant slope
0.1	-0.614746157565	-0.1	-0.690363361240
0.01	-0.649848745915	-0.01	-0.657416707801
0.001	-0.653265110707	-0.001	-0.654021913139
0.0001	-0.653605779648	-0.0001	-0.653681459900

43. Answer changes each year the function is updated. No answer provided.
45. Answer changes each year the function is updated. No answer provided.

Section 9

1. [a.] -25 [b.] -10 [c.] 0 [d.] 10 [e.] -25 [f.] 5
3. [a.] 39.5 [b.] 0 [c.] -4.2 [d.] -2.8 [e.] 5.34 [f.] 26
5. Generally the temperature is increasing around 1:00pm and so we would expect $T'(13) > 0$, which is the rate of change of temperature at 1:00pm in degrees per hour.
7. As the canal gets wider the depth of the water should decrease and so $D'(20) < 0$, which represent the rate of change of the depth of the water when the canal is 20 feet wide in inches per feet.
9. Unless the person has an amazing arm after 3 minutes the ball is sitting on the ground and not moving and so $s'(3) = 0$, in other words, the ball isn't moving.
11. If air is being blown into the balloon then it is increasing in size and so $D'(2) > 0$, which is the rate of increase of the diameter of the balloon with 2 liters of air in centimeters per liter.
13. As the car is driven the energy in the battery will decrease and so $W'(28) < 0$, which is the rate, in watts per minute, at which the battery loses watts at 28 minutes of driving.
15. Using the points $(4, 45)$ and $(4.2, 100)$ we get a slope of 275.
17. Using the points $(2.3, 10)$ and $(2.6, 0)$ we get a slope of -33.33.
19. Using the points $(5, -5)$ and $(2.5, -10)$ we get a slope of 6.
21. Using the points $(2, 1.9)$ and $(3, -1)$ we get a slope of -2.9.

23.



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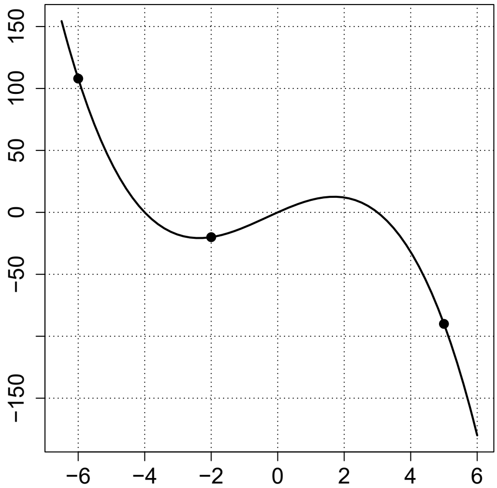

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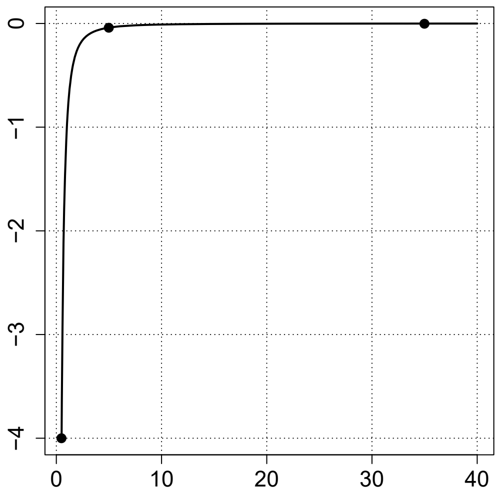

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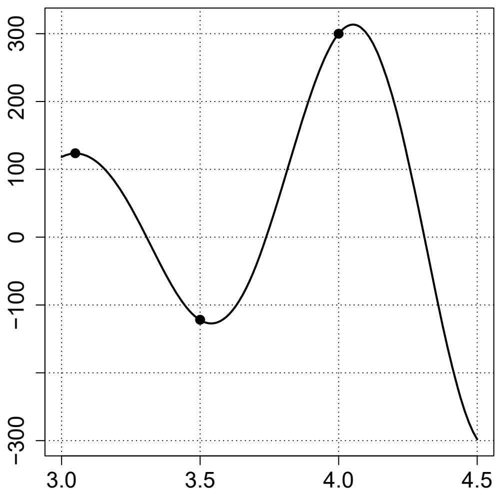


31. C,D,B,A,E

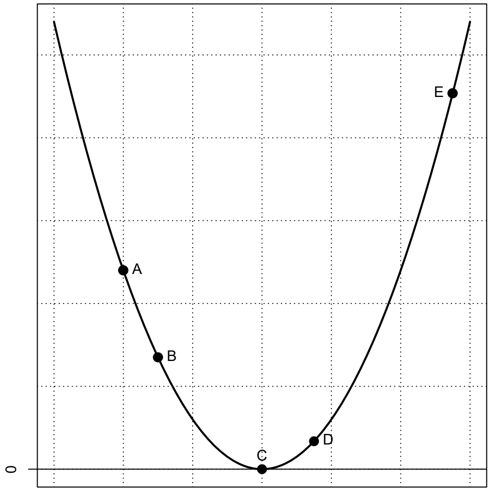
33. E,A,B,D,C

35. A,B,C,D,E

37. B,A,D,E,C

39. A,D,C,B,E

41.



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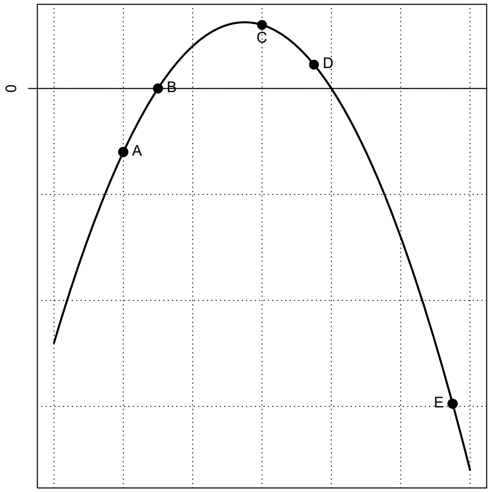

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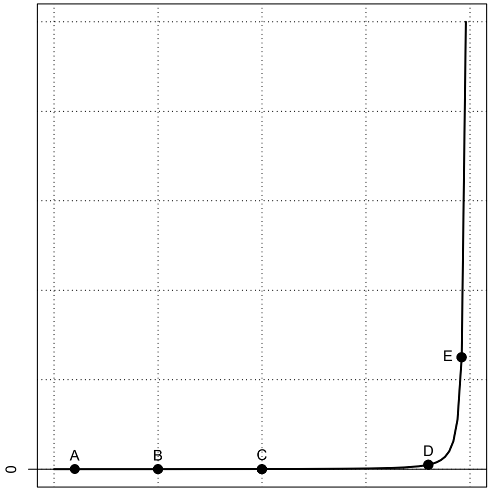

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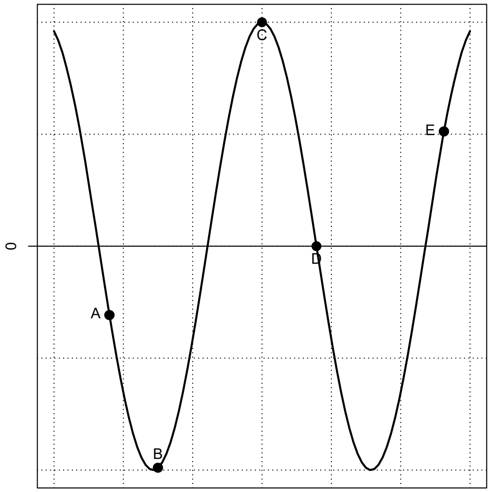
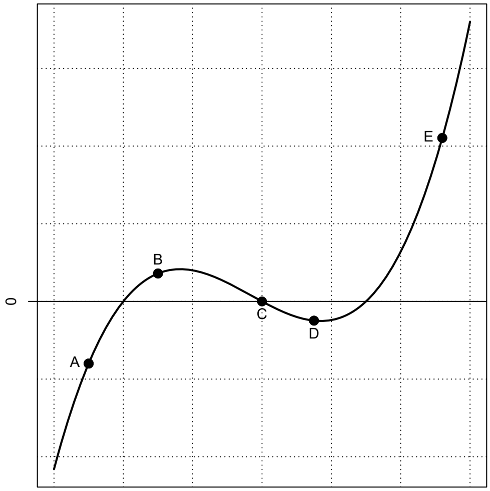


49.



Section 10.1

1. $f'(3) = \lim_{h\to 0} \dfrac{f(3+h) - f(3)}{h} = \cdots = \lim_{h\to 0}(6+h) = 6$
3. $f'(-4) = \lim_{h\to 0} \dfrac{f(-4+h) - f(-4)}{h} = \cdots = \lim_{h\to 0}(-7+h) = -7$
5. $f'(2) = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h} = \cdots = \lim_{h\to 0}(h+2) = 2$
7. $f'(7) = \lim_{h\to 0} \dfrac{f(7+h) - f(7)}{h} = \cdots = \lim_{h\to 0}(3h+47) = 47$
9. $f'(-5) = \lim_{h\to 0} \dfrac{f(-5+h) - f(-5)}{h} = \cdots = \lim_{h\to 0}(4h-37) = -37$
11. $f'(2) = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h} = \cdots = \lim_{h\to 0}(h^2+6h+12) = 12$
13. $f'(-8) = \lim_{h\to 0} \dfrac{f(-8+h) - f(-8)}{h} = \cdots = \lim_{h\to 0}(h^2-24h+195) = 195$
15. $f'(2) = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h} = \cdots = \lim_{h\to 0}(h^2+11h+31) = 31$
17. $f'(5) = \lim_{h\to 0} \dfrac{f(5+h) - f(5)}{h} = \cdots = \lim_{h\to 0}(h^3+20h^2+150h+495) = 495$
19. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \cdots = \lim_{h\to 0}(2x+h) = 2x$
21. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \cdots = \lim_{h\to 0}(2x+4+h) = 2x+4$
23. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \cdots = \lim_{h\to 0}(2x-7+h) = 2x-7$
25. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \cdots = \lim_{h\to 0}(12x+4+6h) = 12x+4$

27. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \cdots = \lim_{h\to 0}(18x - 6 + 9h) = 18x + 6$
29. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \cdots = \lim_{h\to 0}(3x^2 + 3h + h^2) = 3x^2$
31. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \cdots = \lim_{h\to 0}(3x^2 + 3h + h^2 + 6) = 3x^2 + 6$
33. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \cdots = \lim_{h\to 0}(h^2 + 3xh - 4h + 3x^2 - 8x - 1) = 3x^2 - 8x - 1$
35. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \cdots = \lim_{h\to 0}(h^3 + 4h^2x + 6hx^2 + 4x^3 + 2) = 4x^3 + 2$
37. $A(s) = s^2$. Calculate $A'(s) = 2s$. $A'(5) = 10$. Percentage rate of change: 40% per inch. With a side length of 5 inches the area of the square is growing at a rate of 10 inches squared per inch or 40% per inch.
39. $V(s) = s^3$. Calculate $V'(s) = 3s^2$. $V'(10) = 300$. Percentage rate of change: 30%per foot. With a side length of 10 feet the volume of the cube is growing at a rate of 300 feet cubed per foot or 30% per foot.

Section 11.1

1. $f'(x) = 78x^{77}$. Line (11.3).
3. $g'(x) = \cos(x)$. Line (11.4).
5. $h'(x) = 0$. Line (11.1).
7. $g'(x) = \ln(7)7^x$. Line (11.7).
9. $g'(x) = 1/x$. Line (11.8).
11. $k'(x) = e^x$. Line (11.6).
13. $f'(x) = 10$
15. $g'(x) = -25$
17. $g'(x) = 72x^{23}$
19. $h'(x) = 91x^6$
21. $f'(x) = \frac{16}{14}x^{-5/7}$
23. $h'(x) = \frac{-15}{8}x^{-13/8}$
25. $g'(x) = \frac{66}{7}x^{8/14}$
27. $g'(x) = \dfrac{-8}{x^5}$
29. $f'(x) = \dfrac{-25}{8x^6}$
31. $g'(x) = \dfrac{-63}{19x^8}$
33. $g'(x) = \dfrac{1}{2\sqrt{x}}$
35. $h'(x) = \dfrac{-1}{21\sqrt{x^3}}$
37. $g'(x) = 5\cos(x)$
39. $f'(x) = -4e^x$
41. $f'(x) = (\ln(9))9^x$
43. $h'(x) = 5(\ln(6))6^x$
45. $g'(x) = \dfrac{5}{x}$
47. $f'(x) = 0$
49. $f'(x) = \dfrac{-7}{x} + 5$
51. $f'(x) = -88x^{10} + \dfrac{54}{x^{10}} - 11$
53. $f'(t) = \dfrac{-70}{t^{15}} + \dfrac{13}{t^{15/2}} + 12e^t$
55. $h'(t) = 77t^6 + \sin(t) - 9e^t$
57. $h'(w) =$ $(\ln(3))3^w - \dfrac{25}{2w^{1/6}} - \dfrac{4}{\sqrt{w}}$
59. $f'(x) = \dfrac{40}{21x^{17/7}} - \dfrac{15}{x} - \dfrac{30}{x^{11}}$
61. $f'(x) = 5(\ln(10))10^x + \dfrac{2}{7x^{3/2}}$

63. Let $k(x) = f(x) + g(x)$. Start with $k'(x) = \lim_{h\to 0} \dfrac{k(x+h) - k(x)}{h}$, substitute in $f(x) + g(x)$, use algebra to get to $\lim_{h\to 0} \left[\dfrac{f(x+h) - f(x)}{h} + \dfrac{g(x+h) - g(x)}{h} \right]$, and then finish the argument.
65. Let $k(x) = f(x) + c$. Apply the definition of the derivative to $k(x)$ and show that you get $f'(x)$.
67. We get $f(3) = 8$ so that $5f(3) = 40$. Hence (3, 40) is on the graph of $5f(x)$. Since the derivative of $5f(x)$ is $5f'(x)$, similar reasoning gives us that $(3, -50)$ is on the graph of the derivative of $5f(x)$.
69. We get that $f(2) = 9$ so that $f(2) + 42 = 51$ which gives that the point (2, 51) is on the graph of $f(x) + 42$. The derivative of $f(x) + 42$ is $f'(x)$. Hence, $f'(2) = 14$ gives that (2, 14) is on the graph of the derivative of $f(x) + 42$
71. The slopes of the tangent lines of $f(x)$ and $f(x) + 10$ are exactly the same because the derivatives of $f(x)$ and $f(x) + 10$ are both $f'(x)$. A graph of the two functions should show that $f(x) + 10$ is the same as $f(x)$ just shifted ten units up. Hence they have the same shape and the same derivative.
73. $A'(s) = 4.756s$, $A'(6) = 28.536$, A pentagon with side length of 6 will grow at a rate of 28.536 square inches per inch of side length.
75. When GDP per capita is \$15,000 life expectancy is increasing at a rate of 3.66 year per \$10,000. When GDP per capita is \$60,000 life expectancy is increasing at a rate of 0.92 year per \$10,000. This suggests, if it is a causal relationship, that a country with a GDP of \$15,000 gains more life expectancy for the same GDP per capita increase than a country with a GDP per capita of \$60,000
77. Global average temperature in 1980, 2000, and 2021 was increasing at a rate of 0.012, 0.019, and 0.025 degrees Celsius per year, respectively. Global average temperature was rising 105% faster in 2021 as compared to 1980.
79. U.S. cumulative installed wind energy in 2000 and 2020 was increasing at a rate of 1021 and 7474 megawatts per year, respectively. U.S. cumulative installed wind energy was increasing 632% faster in 2020 as compared to 2020.
81. Tangent line is $y = 10x - 25$. $y(5.1) = 26$, $5.1^2 = 26.01$. The values are close since the tangent line is close to the curve at $x = 5.1$.
83. Tangent line is $y = \dfrac{x}{8} + 2$. $y(16.1) = 4.0125$. A calculator will give $\sqrt{16.1} = 4.01248$. The values are close since the tangent line is close to the curve at $x = 16.1$.

Section 12.1

1. The dashed red curve crosses the x-axis at $\pi/2$, but the solid black curve does not have a horizontal tangent line at $\pi/2$.
3. The dashed red curve crosses the x-axis at $\pi/2$, but the solid black curve does not have a horizontal tangent line at $\pi/2$.

5. From $x = -4$ to $x = 0$, the solid black curve is decreasing and hence all tangent lines have a negative slope, but the dashed red curve is positive $x = -4$ to $x = 0$.
7. The slopes of the tangent lines are increasing on the solid black curve while the dashed red curve is decreasing.
9. $g'(x) = (6x - 4)(x^3 + x^2 + 5x + 11) + (3x^2 - 4x - 42)(3x^2 + 2x + 5)$
11. $f'(x) = x^2 \sin(x) + 2x \cos(x)$
13. $g'(t) = \dfrac{12t^2 - 2t + 9}{2\sqrt{t}} + (24x - 2)\sqrt{x}$
15. $h'(x) = (9x^2 + 3x - 2)\cos(x) + (18x + 3)\sin(x)$
17. $f'(x) = \dfrac{\cos(x)}{3x^{2/3}} - \sqrt[3]{x}\sin(x)$
19. $g'(x) = \dfrac{6e^x}{x^4} - \dfrac{24e^x}{x^5}$
21. $f'(t) = \dfrac{12\cos(t)}{t} - 12\sin(t)\ln(t)$
23. $g'(x) = (2x^{15} - 11x^{13} + 3)5^x \ln(5) + (30x^14 - 143x^{12})5^x$
25. $f'(x) = 5x^3 e^x(\sin(x) + \cos(x)) + 15x^2 e^x e^x \sin(x)$
27. $y = \dfrac{-\pi^2 x}{4} + \dfrac{\pi^3}{8}$
29. $y = x$

Section 13.1

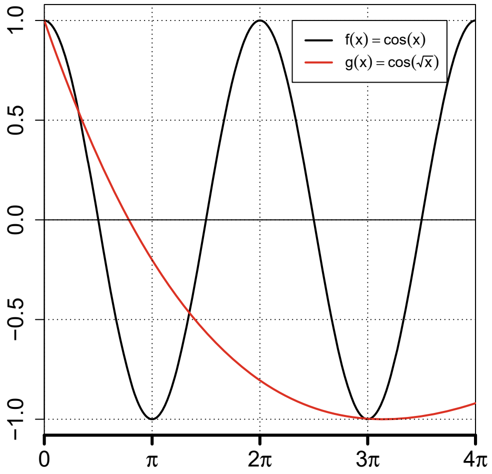
1. $g'(x) = \dfrac{2x(9x + 2)}{(5x^2 + 6x + 2)^2}$
3. $f'(x) = \dfrac{-x^4 - 10x^3 - 7x^2 + 2}{(x^3 - x)^2}$
5. $g'(x) = \dfrac{10x(3x + 2)}{(5x^2 + 6x + 2)^2}$
7. $f'(x) = \dfrac{10 - 6x^2}{\sqrt{x}(5 + x^2)^2}$
9. $f'(x) = \dfrac{8x^4(4x + x\sin(x) + 5\cos(x)}{(x + \cos(x))^2}$
11. $h'(x) = \dfrac{((x^2 + 5x + 3)\cos(x) - (2x + 5)\sin(x))}{(x^2 + 5x + 3)^2}$
13. $f'(t) = \dfrac{e^t(2t + 8\sqrt{t} - 1)}{2\sqrt{t}(\sqrt{t} + 4)^2}$
15. $h'(x) = \dfrac{6x^8(20x - 3e^x(x - 9))}{(5x + 6e^x)^2}$
17. $f'(x) = \dfrac{-5\sin(x) - 4e^x(\sin(x) + \cos(x))}{(4e^x + 5)^2}$
19. $g'(z) = \dfrac{z^2 - 2z^2\ln(z) + 5}{z(z^2 + 5)^2}$
21. $f'(x) = \dfrac{5xe^x(x + 2)\sin(x) - 5x^2 e^x\cos(x)}{\sin^2(x)}$

23. $f'(x) = \dfrac{18x\cos(x) - 9x^2\sin(x) - 9x^2\cos(x)}{e^x}$

25. $\dfrac{1}{\cos^2(x)} = \sec^2(x)$

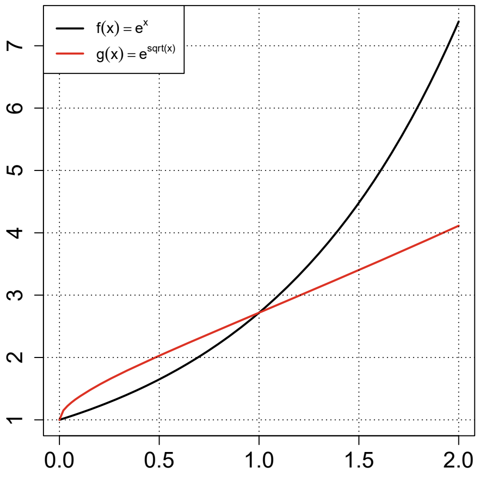
Section 14.1

1.
```
> f<-function(x){cos(x)}
> g<-function(x){cos(sqrt(x))}
> a<-0
> b<-4*pi
> par(mar=c(3,3,2,2))
> curve(f,a,b,n=10000,xaxt="n",xaxs="i",ylab="",xlab="",lwd=2,
cex.axis=1.5,cex.lab=1.5)
> curve(g,a,b,lwd=2,xaxt="n",xaxs="i",col="red",add=TRUE)
> grid(NA,NULL,col="black")
> v1=c(0,pi,2*pi, 3*pi,4*pi)
> v2= c("0", expression(pi),expression(2*pi),expression(3*pi),
expression(4*pi))> axis(side=1,lwd=3,at=v1,labels=v2,
cex.axis=1.5, cex.lab=1.5)
> abline(v=v1,lty=3)
> abline(h=0)
> v4<-c(expression(f(x)==cos(x)),expression(g(x)==cos(sqrt(x))))
> legend(2.3*pi,1,v4,lty=c(1,1),lwd=c(2,2),bg="white",
col=c("black","red"),y.intersp=1.25)
```



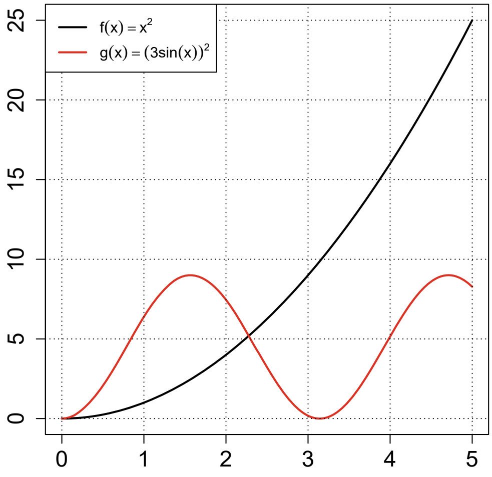
3.
```
> f<-function(x){exp(x)}
> g<-function(x){exp(sqrt(x))}
> a<-0
> b<-2
> par(mar=c(3,5,2,2))
> curve(f,a,b,n=10000,ylab="",xlab="",lwd=2,cex.axis=1.5,cex.lab=1.5)
> curve(g,a,b,col="red",add=TRUE,lwd=2)
> grid(NULL,NULL,col="black")
> v4<-c(expression(f(x)==e^x),expression(g(x)==e^"sqrt(x)"))
> legend("topleft",v4,lty=c(1,1),lwd=c(2,2),bg="white",
col=c("black","red"),y.intersp=1.25)
```

Did you get the shape near $x = 0$ correct?



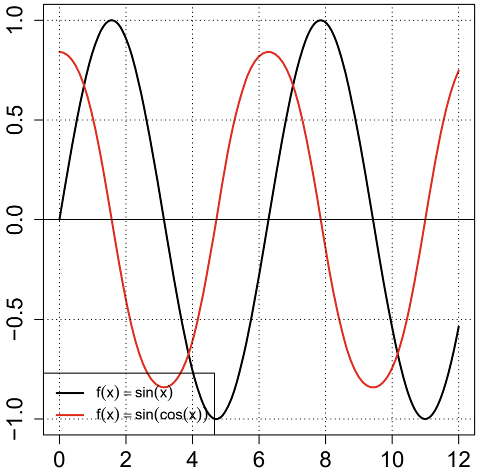
5.
```
> f<-function(x){x^2}
> g<-function(x){(3*sin(x))^2}
> a<-0
> b<-5
> par(mar=c(3,3,2,2))
> curve(f,a,b,n=10000,ylab="",xlab="",lwd=2,cex.axis=1.5,cex.lab=1.5)
> curve(g,a,b,col="red",add=TRUE,lwd=2)
> grid(NULL,NULL,col="black")
> v4<-c(expression(f(x)==x^2),expression(g(x)==(3*sin(x))^2))
> legend("topleft",v4,lty=c(1,1),lwd=c(2,2),
bg="white",col=c("black","red"),y.intersp=1.25)
```

Did you get the amplitude of $g(x) = (3\sin(x))^2$ correct?



7.
```
> f<-function(x){sin(x)}
> g<-function(x){sin(cos(x))}
> a<-0
> b<-12
> par(mar=c(3,3,2,2))
> curve(f,a,b,n=10000,ylab="",xlab="",lwd=2,cex.axis=1.5,cex.lab=1.5)
> curve(g,a,b,col="red",add=TRUE,lwd=2)
> grid(NULL,NULL,col="black")
> abline(h=0)
> v4<-c(expression(f(x)==sin(x)),expression(g(x)==sin(cos(x))) )
> legend("bottomleft",v4,lty=c(1,1),lwd=c(2,2),col=c("black","red"),
 y.intersp=1.25)
```

Did you get the amplitude of $g(x) = \sin(cos(x)))$ correct?



9. $g'(x) = \dfrac{-\sin(\sqrt{x})}{2\sqrt(x)}$
11. $g'(x) = \dfrac{e^{\sqrt{x}}}{2\sqrt(x)}$
13. $g'(x) = 6\cos(x)\sin(x)$
15. $g'(x) = -\sin(x)\cos(\cos(x))$
17. $f'(x) = 10(24x - 5)(12x^2 - 5x + 12)^9$
19. $f'(t) = 6t^2(8t - 27)(2t^4 - 9t^3 - 12)^5$
21. $g'(x) = \dfrac{-3(-8x^3 - 26x + 6\cos(x)}{(-2x^4 - 13x^2 + 6\sin(x))^4}$
23. $f'(x) = \dfrac{x^3(65x - 12)}{2\sqrt{13x^5 - 3x^4 + 9}}$
25. $h'(x) = \dfrac{-16x^{-9} - 36x^2}{3(2x^{-8} - 12x^3 - 10)^{2/3}}$
27. $h'(x) = \dfrac{-15(12x^2 - 36x^3)}{2(4x^3 - 9x^4 - 13)^5}$
29. $f'(t) = \dfrac{312t^2 - 182t^6}{(2t^7 - 8t^3 + 14)^{3/2}}$
31. $f'(x) = 18x^5 e^{3x^6+5}$
33. $f'(x) = (30x^5 - 8 - \sin(x))e^{5x^6 - 8x + \cos(x)}$
35. $g'(x) = (6x - 8)\ln(7)7^{3x^2 - 8x + 7}$
37. $f'(t) = (15t^{14} - 5\cos(t))\ln(9)9^{t^{15} - 5\sin(t)}$
39. $f'(x) = (45x^2 - 33x^{10})\sin(3x^{11} - 15x^3 + 12)$
41. $h'(t) = (15t^2 - 48t^3)\cos(5t^3 - 12t^4 - 11)$
43. $f'(x) = \dfrac{120x^7 - 45x^2}{15x^8 - 15x^3 - 8}$

45. $f'(t) = \dfrac{4\cos(t) - 120t^{-16}}{8t^{-15} + 4\sin(t)}$

47. $f'(x) = (10x + 6)\cos(e^{5x^2+6x+9})e^{5x^2+6x+9}$

49. $g'(x) = 8(9^{12x^2} - e^{15x^2+\sin(x^7)})^7(24x\ln(9)9^{12x^2} - (30x + 7x^6\cos(x^7))e^{15x^2+\sin(x^7)})$

51. $f'(x) = 4e^(x^4)(2x^5 + x)$

53. $h'(x) = \dfrac{(6x + 15)(x^2 + 5x - 4)^2\cos(x^2) + 2x(x^2 + 5x - 4)^3\sin(x^2)}{(\cos(x^2))^2}$

55. $g'(x) = \dfrac{7x^6\cos(x^3 - 5)}{x^7 + 2} - 3x^2\ln(x^7 + 2)\sin(x^3 - 5)$

57. $f'(t) = 24t^2(12t + 3)(6t^2 + 3t - 9)^7 + 6t(6t^2 + 3t - 9)^8$

59. $h'(x) = \dfrac{14x(5 + e^{x^2+4}) - 14x^3e^{x^2+4}}{(5 + e^{x^2+4})^2}$

61. $g'(x) = e^{2x^3-5x+7}((6x^2 - 5)\sin(x^2) + 2x\cos(x^2))$

63. $f'(x) = \dfrac{72x^5(10 - \ln(x^3 + 9)) - \frac{36x^8}{x^3+9}}{(10 - \ln(x^3 + 9))^2}$

65. $f'(x) = 5^{3-x^4}\left(\dfrac{2x + 5}{\sqrt{x^2 + 5x - 2}} - 4x^3\ln(5)\sqrt{x^2 + 5x - 2}\right)$

67. $g'(x) = \dfrac{5^{4x^2+10}\left(16x\ln(5)(15x^3 - 5) - 45x^2\right)}{2(15x^3 - 5)^{3/2}}$. Tip: Simplify by multiplying top and bottom by $2\sqrt{15x^3 - 5}$

69. $f'(t) = 9(91t^{12} + 42)(7t^{13} + 42t - 11)^8e^{6t^5+19} + 30t^4(7t^{13} + 42t - 11)^9e^{6t^5+19}$

71. $f'(x) = \dfrac{e^{8x^6-7x+2}\left((48x^5 - 7)\cos(4x^5 - 10) + 20x^4\sin(4x^5 - 10)\right)}{(\cos(4x^5 - 10))^2}$

73. $h'(x) = \dfrac{45x^8\sin(3x^{14} - 10x^7 + 1)}{5x^9 - 8} + (42x^{13} - 70x^6)\ln(5x^9 - 8)\cos(3x^{14} - 10x^7 + 1)$

Section 15.1

1. $f''(x) = 60x^3 - 36x$
3. $f''(x) = -1/x^2$
5. $f^{(4)}(x) = 16e^{2x}$
7. $f^{(6)}(x) = 6!$ (Note: $n! = n(n-1)(n-2)\cdots 3(2)(1)$.) and $f^{(7)}(x) = 0$
9. Note that to define a constant function we used $c + 0 * x$ because R needs a variable in the function definition.

```
> library(Deriv)

> f1<-function(x){c+0*x}
> Deriv(f1)
function (x)
0
> f2<-function(x){m*x+b}
> Deriv(f2)
function (x)
m
> f3<-function(x){x^n}
```

```
> Deriv(f3)
function (x)
n * x^(n - 1)
> f4<-function(x){sin(x)}
> Deriv(f4)
function (x)
cos(x)
> f5<-function(x){cos(x)}
> Deriv(f5)
function (x)
-sin(x)
> f6<-function(x){exp(x)}
> Deriv(f6)
function (x)
exp(x)
> f7<-function(x){a^x}
> Deriv(f7)
function (x)
a^x * log(a)
> f8<-function(x){log(x)}
> Deriv(f8)
function (x)
1/x
```

11. By hand we might simplify to $f'(x) = e^x(x^2 + 2x)$.

```
> f<-function(x){x^2*exp(x)}
> Deriv(f)
function (x)
x * (2 + x) * exp(x)
```

13. Any multiple of 4 for derivatives will result in $\cos(x)$.

```
> Deriv(f,n=100)
function (x)
cos(x)
```

15. The pattern here is the nth derivative is $f^n(x) = (-1)^n n!/x^{n+1}$, where $n! = n(n-1)(n-2)\ldots 2 * 1$. The R code includes the check of 10! and 12!.

```
> f<-function(x){1/x}
> Deriv(f,n=10)
function (x)
3628800/x^11
> factorial(10)
[1] 3628800
> Deriv(f,n=12)
function (x)
479001600/x^13
> factorial(12)
[1] 479001600
```

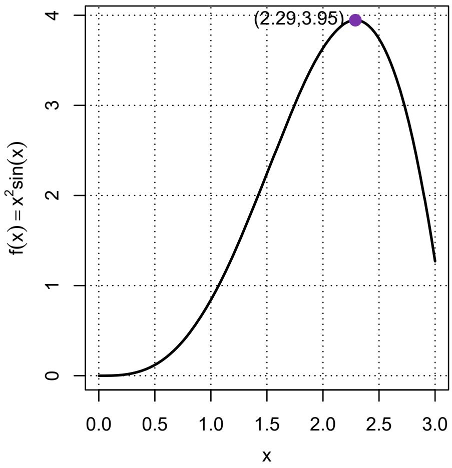
17. From the code below. In 2021 the temperature was increasing at a rate of 0.02541572 degrees Celsius per year. We estimate that in 2022 and 2023 the global temperature will be 14.88203 and 14.90744 degrees Celsius, respectively.

```
> library(Deriv)
> GTemp <-function(t){0.000159118973994531*t^2+
0.00282082411785264*t+ 13.8542160310996}
> GTemp_p<-Deriv(GTemp)
> GTemp_p(71)
[1] 0.02541572
> GTemp_p(71)*1+GTemp(71)
[1] 14.88203
> GTemp_p(71)*2+GTemp(71)
[1] 14.90744
```

19. From the code below. In 2020 the U.S. wind capacity was increasing at a rate of 7473.547 MW per year. We estimate that in 2021 and 2022 the U.S. wind capacity will be 123660 and 131133.6 MW, respectively.

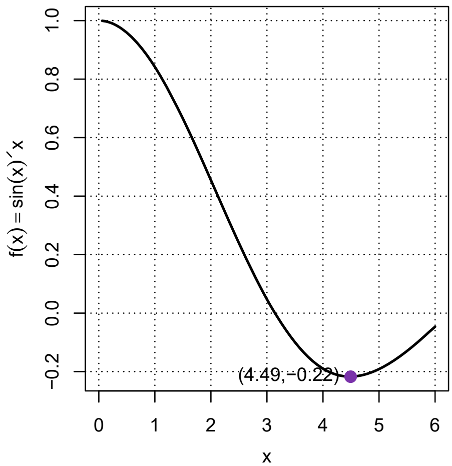
```
> library(Deriv)
> USwind<-function(t){- 0.00606729846765557*t^5+0.403796445970404*t^4
- 279.748222478446*t^2+4142.93534737984*t -14361.298551044}
> USwind_p<-Deriv(USwind)
> USwind_p(40)
[1] 7473.547
> USwind_p(40)*1+USwind(40)
[1] 123660
> USwind_p(40)*2+USwind(40)
[1] 131133.6
```

21. This code is one example to answer the problem and yours doesn't have to match this exactly. The domain used in uniroot.all() and curve() were found by first graphing the function over various intervals.



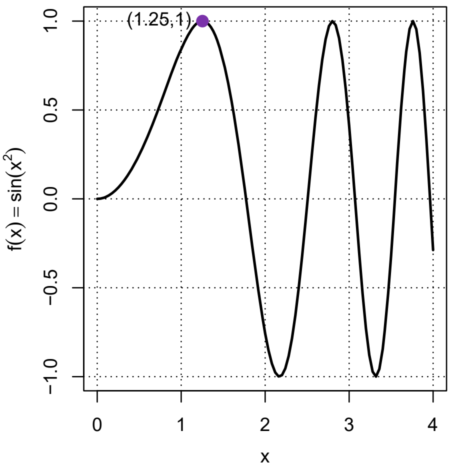

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){x^2*sin(x)}
> f_p<-Deriv(f)
> root<-uniroot.all(f_p,c(0.1,3))
> par(mar=c(4,5,2,2))
> curve(f,0,3,lwd=2,ylab=expression(f(x)==x^2*sin(x)))
> grid(NULL,NULL,col="black")
> points(root,f(root),pch=16,cex=1.5,col="purple")
> text(root,f(root),paste("(",round(root,2),",",round(f(root),2),")",
sep=""),pos=2)
```

23. Note how little changed from the code above. This code is one example to answer the problem and yours doesn't have to match this exactly. The domain used in uniroot.all() and curve() were found by first graphing the function over various intervals.



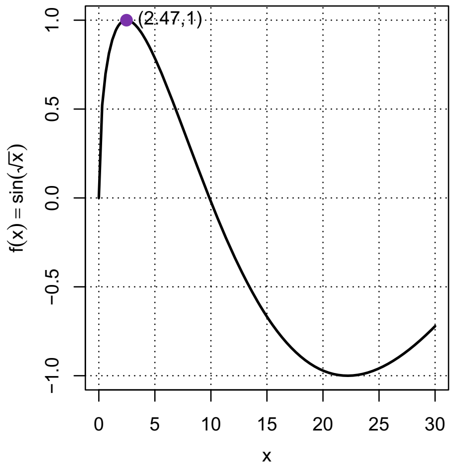
```
> library(Deriv)
> library(rootSolve)
> f<-function(x){sin(x)/x}
> f_p<-Deriv(f)
> root<-uniroot.all(f_p,c(0.1,6))
> par(mar=c(4,5,2,2))
> curve(f,0,6,lwd=2,ylab=expression(f(x)==sin(x)/x))
> grid(NULL,NULL,col="black")
> points(root,f(root),pch=16,cex=1.5,col="purple")
> text(root,f(root),paste("(",round(root,2),",",round(f(root),2),")",
sep=""),pos=2)
```

25. Note how little changed from the code above. This code is one example to answer the problem and yours doesn't have to match this exactly. The domain used in uniroot.all() and curve() were found by first graphing the function over various intervals.




```
> library(Deriv)
> library(rootSolve)
> f<-function(x){sin(x^2)}
> f_p<-Deriv(f)
> root<-uniroot.all(f_p,c(0.1,2))
> par(mar=c(4,5,2,2))
> curve(f,0,4,lwd=2,ylab=expression(f(x)==sin(x^2)))
> grid(NULL,NULL,col="black")
> points(root,f(root),pch=16,cex=1.5,col="purple")
> text(root,f(root),paste("(",round(root,2),",",round(f(root),2),")",
sep=""),pos=2)
```

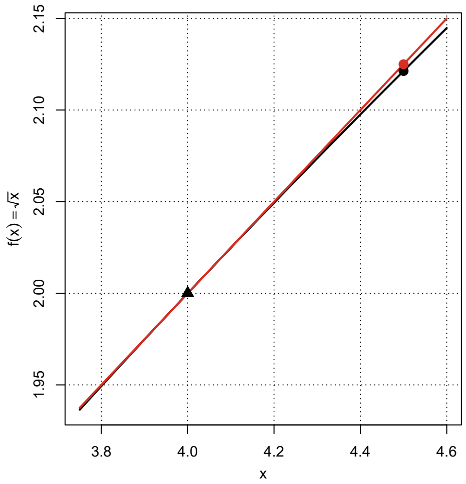
27. Note how little changed from the code above. This code is one example to answer the problem and yours doesn't have to match this exactly. The domain used in uniroot.all() and curve() were found by first graphing the function over various intervals.




```
> library(Deriv)
> library(rootSolve)
> f<-function(x){sin(sqrt(x))}
> f_p<-Deriv(f)
> root<-uniroot.all(f_p,c(0.1,3))
> par(mar=c(4,5,2,2))
> curve(f,0,30,lwd=2,ylab=expression(f(x)==sin(sqrt(x))))
> grid(NULL,NULL,col="black")
> points(root,f(root),pch=16,cex=1.5,col="purple")
> text(root,f(root),paste("(",round(root,2),",",round(f(root),2),")",
sep=""),pos=4)
```

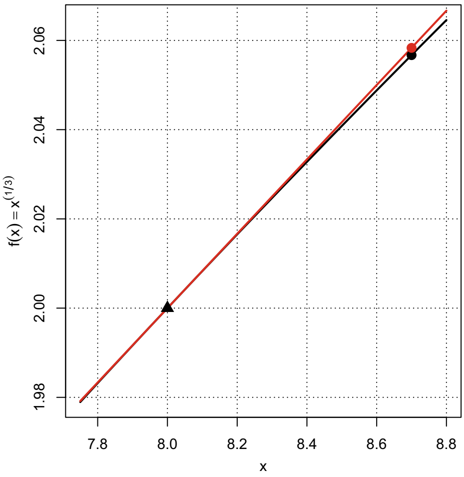
29. $L(17) = 0.61$: The probability the cells are malignant given a maximum nuclei radius of 17 mm is 61%. $L'(17) = 0.27$: Given a maximum nuclei radius of 17 mm, the chance the cells are malignant is increasing at a rate of 27 percentage points per mm of nuclei radius. In other words, by the microscope equations the odds of the malignancy with an 18mm radius is approximately 89%.
31. $C(2) = 0.19$. The probability a 2kg cat is male is 20%. $C'(2) = 0.57$: The chances a 2kg cat is male is increasing at a rate of 57 percentage points per kg of weight. Note: $C'(2)$ seems large and is due to the units in kg. According to the microscope equation $C(2.25) = 0.19 + (0.25)0.57 = 0.3325$.

33. Tangent line are often good approximations to functions close to where the tangent line is on the curve. In this example, $f(4+0.5) = \sqrt{4.5} = 2.12132$ while the value on the tangent line is 2.125. In the code note the use of `x_left` and `x_right`, which are used to make it easier to adjust the window of the graph. Similarly, defining `a` and `h` makes the code easily adaptable.



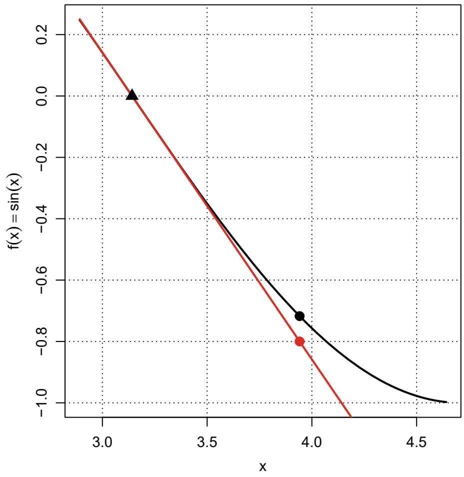

```
> f<-function(x){sqrt(x)}
> a<-4
> h<-0.5
> x_left<-0.25
> x_right<-0.6
> f_p<-Deriv(f)
> f_tan<-function(a,x){f_p(a)*(x-a)+f(a)}
> par(mar=c(4,5,2,2))
> curve(f,a-x_left,a+x_right,lwd=2,ylab=expression(f(x)==sqrt(x)))
> grid(NULL,NULL,col="black")
> curve(f_tan(a,x),col="red2",lwd=2,add=TRUE)
> points(a,f(a),pch=17,cex=1.5)
> points(a+h,f(a+h),pch=16,cex=1.5)
> points(a+h,f_tan(a,a+h),pch=16,col="red2",cex=1.5)
> f(a+h)
[1] 2.12132
> f_tan(a,a+h)
[1] 2.125
```

35. Tangent line are often good approximations to functions close to where the tangent line is on the curve. In this example, $f(8+0.7) = \sqrt[3]{8.7} = 2.05671$ while the value on the tangent line is 2.058333. In the code note the use of `x_left` and `x_right`, which are used to make it easier to adjust the window of the graph. Similarly, defining `a` and `h` makes the code easily adaptable. Note how little the code for this problem had to be changed from the previous problem.



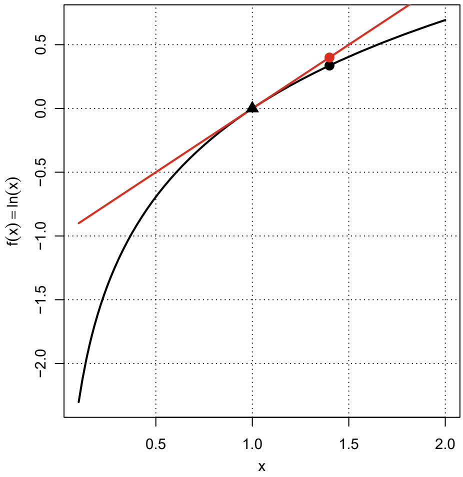

```
> f<-function(x){x^(1/3)}
> a<-8
> h<-0.7
> x_left<-0.25
> x_right<-0.8
> f_p<-Deriv(f)
> f_tan<-function(a,x){f_p(a)*(x-a)+f(a)}
> par(mar=c(4,5,2,2))
> curve(f,a-x_left,a+x_right,lwd=2,ylab=expression(f(x)==x^(1/3)))
> grid(NULL,NULL,col="black")
> curve(f_tan(a,x),col="red2",lwd=2,add=TRUE)
> points(a,f(a),pch=17,cex=1.5)
> points(a+h,f(a+h),pch=16,cex=1.5)
> points(a+h,f_tan(a,a+h),pch=16,col="red2",cex=1.5)
> f(a+h)
[1] 2.05671
> f_tan(a,a+h)
[1] 2.058333
```

37. Tangent line are often good approximations to functions close to where the tangent line is on the curve. In this example, $f(\pi + 0.8) = \sin(\pi + 0.8) = -0.7173561$ while the value on the tangent line is -0.8. In the code note the use of `x_left` and `x_right`, which are used to make it easier to adjust the window of the graph. Similarly, defining `a` and `h` makes the code easily adaptable. Note how little the code for this problem had to be changed from the previous problem.




```
> f<-function(x){sin(x)}
> a<-pi
> h<-0.8
> x_left<-0.25
> x_right<-1.5
> f_p<-Deriv(f)
> f_tan<-function(a,x){f_p(a)*(x-a)+f(a)}
> par(mar=c(4,5,2,2))
> curve(f,a-x_left,a+x_right,lwd=2,ylab=expression(f(x)==sin(x)))
> grid(NULL,NULL,col="black")
> curve(f_tan(a,x),col="red2",lwd=2,add=TRUE)
> points(a,f(a),pch=17,cex=1.5)
> points(a+h,f(a+h),pch=16,cex=1.5)
> points(a+h,f_tan(a,a+h),pch=16,col="red2",cex=1.5)
> f(a+h)
[1] -0.7173561
> f_tan(a,a+h)
[1] -0.8
```

39. Tangent line are often good approximations to functions close to where the tangent line is on the curve. In this example, $f(1+.4) = \ln(1.4) = 0.3364722$ while the value on the tangent line is 0.4. In the code note the use of `x_left` and `x_right`, which are used to make it easier to adjust the window of the graph. Similarly, defining `a` and `h` makes the code easily adaptable. Note how little the code for this problem had to be changed from the previous problem.




```
> f<-function(x){log(x)}
> a<- 1
> h<-0.4
> x_left<- .9
> x_right<-1
> f_p<-Deriv(f)
> f_tan<-function(a,x){f_p(a)*(x-a)+f(a)}
> par(mar=c(4,5,2,2))
> curve(f,a-x_left,a+x_right,lwd=2,ylab=expression(f(x)==ln(x)))
> grid(NULL,NULL,col="black")
> curve(f_tan(a,x),col="red2",lwd=2,add=TRUE)
> points(a,f(a),pch=17,cex=1.5)
> points(a+h,f(a+h),pch=16,cex=1.5)
> points(a+h,f_tan(a,a+h),pch=16,col="red2",cex=1.5)
> f(a+h)
[1] 0.3364722
> f_tan(a,a+h)
[1] 0.4
```

Section 16.1

1. We have $\lim\limits_{x\to\infty} 2x^3 = \infty$ because as x increases $2x^3$ increases.

3. We have $\lim\limits_{x\to\infty} 3^x = \infty$ because power of 3 get larger as we raise the power.

5. We have $\lim\limits_{x\to\infty} e^{-x} = 0$ since $e^{-x} = \frac{1}{e^x}$ and $\lim_{x\to\infty} e^x = \infty$ gives $\lim_{x\to\infty} \frac{1}{e^x} = 0$.

7. We find that $\lim\limits_{x\to\infty} \dfrac{2}{x} = 0$ since the denominator goes to infinity and the numerator is a fixed value.

9. We find that $\lim_{x\to\infty} x^2 - 2x = \infty$ since $x^2 - 2x = x(x-2)$ and once x is larger than 2 (x-2) is positive and growing and multiplied by x which is positive and growing.

11. We find that $\lim_{x\to\infty} \frac{8}{2-e^{-x}} = 4$ since $\lim_{x\to\infty} e^{-x} = 0$ the $\lim_{x\to\infty} \frac{8}{2-e^{-x}}$ will tend to $8/2 = 4$.

13. $\lim_{x\to\infty} \frac{x}{e^x} = 0$. Hence e^x grows faster.

15. $\lim_{x\to\infty} \frac{3x^2+5x+7}{5x^2+8x-10} = 3/5$. Hence the functions grow proportionally.

17. $\lim_{x\to\infty} \frac{10^x}{5x^2} = \infty$. Hence 10^x grows faster.

19. $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)} = \infty$. Hence $\sqrt{x}$ grown faster.

21. $\lim_{x\to\infty} \frac{x^2+9}{4x+e^x} = 0$. Hence $4x + e^x$ grows faster.

23. $\lim_{x\to\infty} \cos(x)$ is not ∞ and so L'Hospital's Rule can't be applied here.

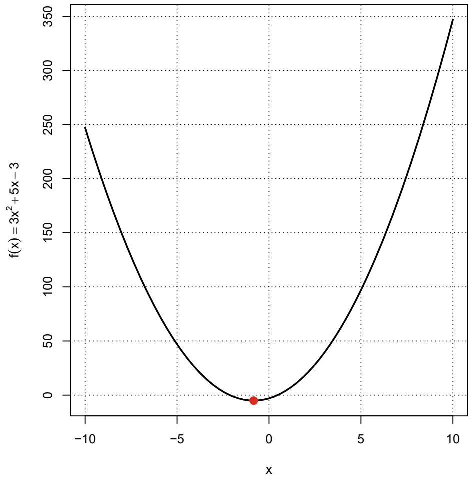
25. If we try using L'Hospital's Rule $\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{3x^{2/3}}{2x^{1/2}}$, which is a limit that isn't any easier to evaluate. Here we can algebraically simplify the first limit and solve the problem without L'Hospital's Rule.

27. If we use L'Hospital's Rule n times we get $\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$. Hence e^x grows faster than any x^n for a positive integer n.

Section 17.1

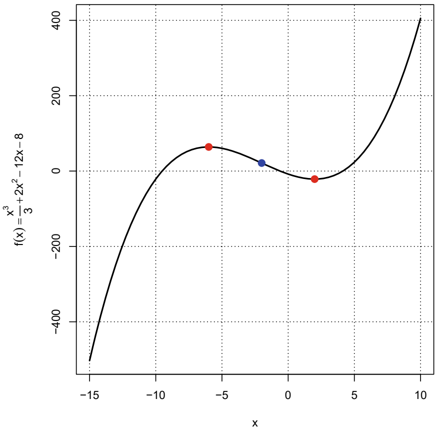
1. (a.) $(-2.774845, 193.3202)$, $(1.441520, -181.4683)$ on the interval from $x = -10$ to $x = 10$ (b.) $f''(-2.774845) = -126.4907 < 0$ and $(-2.774845, 193.3202)$ is a local maximum. $f''(-1.441520) = 126.4912 > 0$ and $(1.441520, -181.4683)$ is a local minimum. (c.) $(-0.6666667, 5.925926)$ (d.) -133.3333
3. (a.) $(-1.2629429, -11.716884)$, $(0.4228395, -2.899003)$, $(2.3400769, -15.321613)$ on the interval from $x = -5$ to $x =$ (b.) $f''(-1.2629429) = 24.29561 > 0$ and $(-1.2629429, -11.716884)$ is a local minimum. $f''(0.4228395) = -12.92856 < 0$ and $(0.4228395, -2.899003)$ is a local maximum. $f''(2.3400769) = 27.63060 > 0$ and $(2.3400769, -15.321613)$ is a local minimum. (c.) $(-0.5408343, -7.764733)$, $(1.5408343, -9.846401)$ (d.) 8.020553 and -10.020553, resp.
5. $f'(x) = m$, hence the first derivative is only ever 0 if $m = 0$, in which case we have a horizontal line.
7. $f''(x) = 2a$, hence if $a > 0$ then $f''(x) > 0$ and the curve is concave up. Similarly, if $a < 0$ then $f''(x) < 0$ and the curve is concave up.
9. $f'(x) = \frac{1}{2\sqrt{x}}$, hence the first derivative can never equal 0 on $(0, \infty)$.
11. $f'(x) = \ln(a)a^x$, hence if $a > 1$ then $\ln(a) > 0$, $f'(x) > 0$ and the function in increasing. Similarly, if $0 < a < 1$ then $\ln(a) < 0$, $f'(x) < 0$, and the function in increasing.

13. Solving $f'(x) = 6x + 5 = 0$ yields a critical point at $x = 5/6$. $f''(x) = 6 > 0$ so the curve is always concave up making $(-5/6, f(-5/6) = -0.83333)$ a local min. $f''(x) = 6$ hence there are no inflection points.



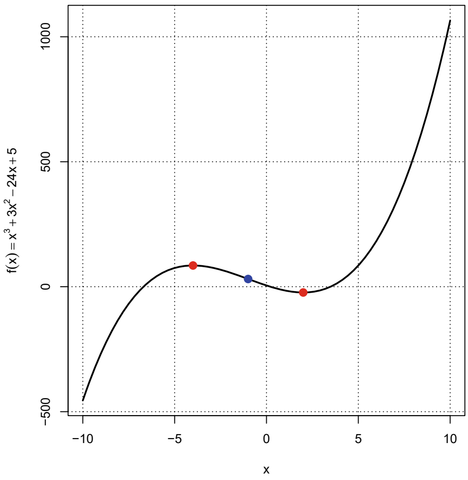

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ 3*x^2+5*x-3}
> a<- -10
> b<- 10
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> root_deriv
[1] -0.8333333
> f(root_deriv)
[1] -5.083333
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> root_deriv2
numeric(0)
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=expression(f(x)==3*x^2+5*x-3))
> grid(NULL,NULL,col="black")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
```

15. Solving $f'(x) = x^2 + 4x - 12 = (x + 6)(x - 2) = 0$ yields two critical points at $x = -6$ and $x = 2$. $f''(x) = 2x + 4$ so $f''(-6) = -8 < 0$ and the curve is concave down making $(-6, f(-6) = 64)$ a local maximum. Similarly, $f''(2) = 8 > 0$ and the curve is concave up making $(2, f(2) = -21.33)$ a local minimum. Solving $f''(x) = 2x + 4 = 0$ yields an inflection point at $(-2, f(-2) = 21.33)$. Note this must be an inflection point since we already know the curve must change concavity between $(-6, f(-6) = 64)$ and $(2, f(2) = -21.33)$.



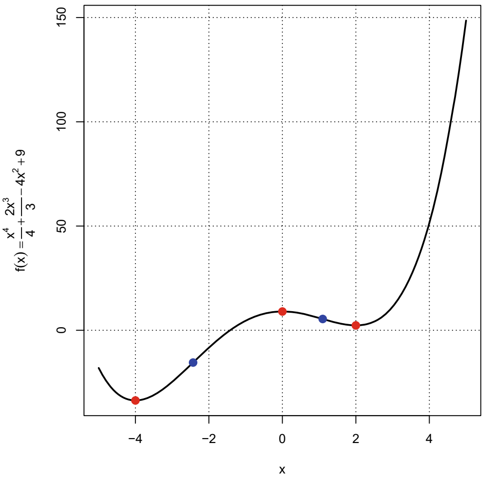

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ x^3/3+2*x^2-12*x-8}
> a<- -15
> b<- 10
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> root_deriv
[1] -6  2
> f(root_deriv)
[1]  64.00000 -21.33333
> f_pp(root_deriv)
[1] -8  8
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> root_deriv2
[1] -2
> f(root_deriv2)
[1] 21.33333
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=expression(f(x)==frac(x^3,3)+2*x^2-12*x-8  ))
> grid(NULL,NULL,col="black")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

17. Solving $f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 4) = 3(x+4)(x-2) = 0$ yields two critical points at $x = -4$ and $x = 2$. $f''(x) = 6x + 6$ so $f''(-4) = -18 < 0$ and the curve is concave down making $(-4, f(-4) = 85)$ a local maximum. Similarly, $f''(2) = 18 > 0$ and the curve is concave up making $(2, f(2) = -23)$ a local minimum. Solving $f''(x) = 6x + 6 = 0$ yields an inflection point at $(-1, f(-1) = 31)$. Note this must be an inflection point since we already know the curve must change concavity between $(-4, f(-4) = 85)$ and $(2, f(2) = -23)$.



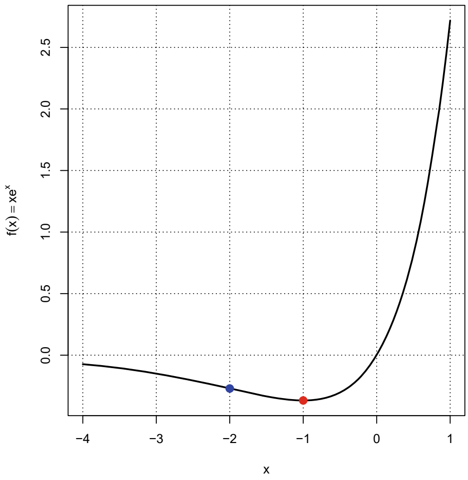

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ x^3+3*x^2-24*x+5}
> a<- -10
> b<- 10
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> root_deriv
[1] -4  2
> f(root_deriv)
[1]  85 -23
> f_pp(root_deriv)
[1] -18  18
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> root_deriv2
[1] -1
> f(root_deriv2)
[1] 31
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=expression(f(x)==x^3+3*x^2-24*x+5))
> grid(NULL,NULL,col="black")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

19. Solving $f'(x) = x^3 + 2x^2 - 8x = x(x+4)(x-2) = 0$ yields three critical points at $x = 0$, $x = -4$ and $x = 2$. $f''(x) = 3x^2 + 4x - 8$ so $f''(0) = -8 < 0$ and the curve is concave down making $(0, f(0) = 9)$ a local maximum. $f''(-4) = 24 > 0$ and the curve is concave up making $(-4, f(-4) = -33.67)$ a local minimum. $f''(2) = 12 > 0$ and the curve is concave up making $(2, f(2) = 2.33)$ a local minimum. Solving $f''(x) = 3x^2 + 4x - 8 = 0$ (quadratic formula) yields an inflection points at $(-2.43, f(-2.43) = -15.477)$ and $(1.097, f(1.097) = 5.428)$. Note these must be an inflection point since they are between local maximums and minimums meaning the concavity changed.



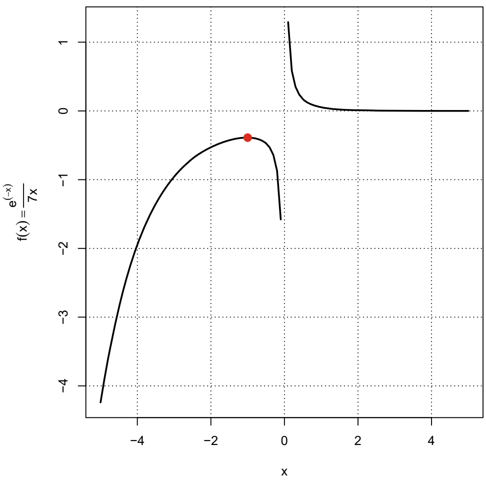

```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ x^4/4+2*x^3/3-4*x^2+9}
> a<- -5
> b<- 5
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> root_deriv
[1]  0 -4  2
> f(root_deriv)
[1]   9.000000 -33.666667   2.333333
> f_pp(root_deriv)
[1] -8 24 12
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> root_deriv2
[1] -2.430501  1.097087
> f(root_deriv2)
[1] -15.477046   5.428064
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=
expression(f(x)==frac(x^4,4)+frac(2*x^3,3)-4*x^2+9 ))
> grid(NULL,NULL,col="black")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

21. Solving $f'(x) = xe^x + e^x = e^x(x+1)$ yields the critical points at $x = -1$. $f''(x) = e^x + e^x(x+1) = e^x(x+2)$ so $f''(-1) = e^{-1} > 0$ and the curve is concave down making $(-1, f(-1) = -e^{-1})$ a local minimum. Solving $f''(x) = e^x + e^x(x+1) = e^x(x+2) = 0$ yields an inflection point at $(-2, f(-2) = -2e^{-2})$. We already know that the curve is concave down at $(-1, f(-1) = -e^{-1})$. Testing a point to the left of $x = -2$, say $x = -3$, we find $f''(-3) = -e^{-3} < 0$ and confirming that concavity changed and $(-2, f(-2) = -2e^{-2})$ is an inflection point.




```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ x*exp(x)}
> a<- -4
> b<- 1
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> root_deriv
[1] -1
> f(root_deriv)
[1] -0.3678794
> f_pp(root_deriv)
[1] 0.3678794
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> root_deriv2
[1] -2
> f(root_deriv2)
[1] -0.2706706
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=expression(f(x)==x*e^x ))
> grid(NULL,NULL,col="black")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

23. Solving $f'(x) = \dfrac{-e^{-x}(x+1)}{7x^2} = 0$ yields the critical points at $x = -1$. $f''(x) = \dfrac{e^{-x}(x^2+2x+2)}{7x^3}$ so $f''(-1) = \dfrac{-e^{x}}{7} < 0$ and the curve is concave down making $(-1, f(-1) = -\frac{-e^x}{7})$ a local maximum. There are no solutions to $f''(x) = \dfrac{e^{-x}(x^2+2x+2)}{7x^3} = 0$ and so there are no inflection point. The function is undefined at $x = 0$ and, in fact, the concavity is down to the left of $x = 0$ and up to the right of $x = 0$, for example, $f''(1) = \frac{5e^{-x}}{7} > 0$. Hence the concavity changes at the undefined point $x = 0$.




```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ exp(-x)/(7*x)}
> a<- -5
> b<- 5
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> root_deriv
[1] -1
> f(root_deriv)
[1] -0.388326
> f_pp(root_deriv)
[1] -0.388326
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> root_deriv2
numeric(0)
> f(root_deriv2)
numeric(0)
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=expression(f(x)==frac(e^(-x),7*x) ))
> grid(NULL,NULL,col="black")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

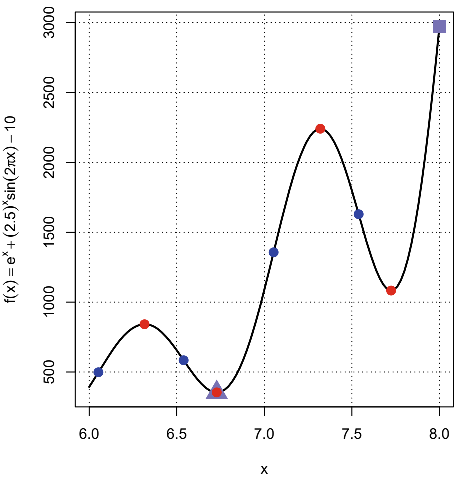
25. The local maximum price is \$2.08 per dozen in March 2014, $t = 26.12$. The local minimum price is \$1.36 per dozen in March 2014, $t = 86.85$. There is an

inflection point in September 2016, $t = 4.71$ where the price was decreasing at a rate of \$0.02 per dozen.

27. The maximum ice extent is 14.1863 msk occurring at $t = 2.269$ or roughly the third week in February. The minimum ice extent is 4.7532 msk occurring at $t = 8.93$ or roughly the end of August. The ice is melting the fastest at $t = 6.868$, or roughly the end of June at a melting rate of 2.69 msk/month.
29. The probability is growing fastest at a weight of 2.387624 kg with a rate of 91 percentage points per kg.
31. The inflection point occurs at $x = 17.65707$ with a rate of -0.0683548 billion people per year. In 2007 the number of people in extreme poverty was decreasing the fastest at -0.0683548 billion people per year, although it was still decreasing after 2007 the rate is slowing down.

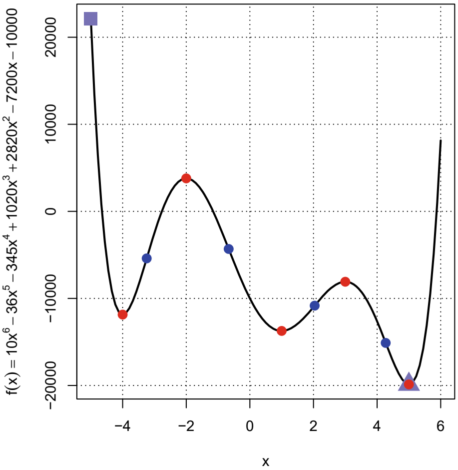
Section 18.1

1. (a.) The endpoint are $x = -12$ and $x = 12$. The critical points $x = -7.291503$ and $x = 3.291495$ (b.) (12, 176.00000), endpoint. (c.) (3.291495, −57.04052), critical point. (d.) Rate 27.0 at (−12.00000, −40.00000), endpoint. (e.) Rate −10.5 at $x = -2$, y-value not given, inflection point.
3. (a.) The endpoint are $x = 7$ and $x = 16.5$. The critical points are at the following x-values: 7.000000, 16.500000, 8.096158, 11.172584, and 14.276415. (b.) (14.276415, 201.84322), critical point. (c.) (16.500000, −193.78356), endpoint. (d.) Rate 165.75628 at $x = 12.871282$, y-value not given, inflection point. (e.) Rate −254.63688, at $x = 15.955634$, y-value not given, inflection point.
5. From the code. Local maximums at (6.316092, 841.8417) and (7.320000, 2240.6381). Local minimums at (6.728268, 354.3028) and (7.724320, 1082.8425). Global maximum at (8, 2970.9580). Global minimum at (6.728268, 354.3028). Inflection points at (6.052572, 498.2990), (6.539252, 584.0769), (7.053148, 1356.5301) and (7.538643, 1628.8955) with rates, in units per units, 2024.005, −1835.925, 5152.861, and −4438.855. The function is increasing the fastest at (8, 2970.9580) with a rate of 12568.338 units per units, and decreasing the fastest at (7.538643, 1628.8955) with a rate of −4438.855 units per units.



```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ exp(x)+(2.5)^x*sin(2*pi*x)-10}
> a<- 6
> b<- 8
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> keyPoints<-c(a,b,root_deriv)
> keyPoints
[1] 6.000000 8.000000 6.316092 6.728268 7.320000 7.724320
> f(keyPoints)
[1]  393.4288 2970.9580  841.8417  354.3028 2240.6381 1082.8425
> f_pp(keyPoints)
[1]   3214.574  20550.613 -12492.902  18304.092 -31111.190  45272.593
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> keyPoints2<-c(a,b,root_deriv2)
> keyPoints2
[1] 6.000000 8.000000 6.052572 6.539252 7.053148 7.538643
> f(keyPoints2)
[1]  393.4288 2970.9580  498.2990  584.0769 1356.5301 1628.8955
> f_p(keyPoints2)
[1]  1937.410 12568.338  2024.005 -1835.925  5152.861 -4438.855
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=expression(f(x)==e^x+(2.5)^x*sin(2*pi*x)-10 ))
> grid(NULL,NULL,col="black")
> gMax<-c(keyPoints[2])
> gMin<-c(keyPoints[4] )
> points(gMax,f(gMax),pch=15,cex=2,col="#7570b3")
> points(gMin,f(gMin),pch=15,cex=2,col="#7570b3")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

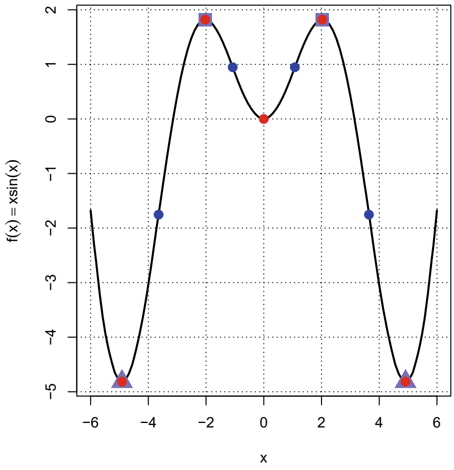
7. From the code. Local maximums at $(-1.9999977, 3792)$ and $(2.9999914, -8083)$. Local minimums at $(-3.9999904, -11856)$, $(0.9999994, -13731)$ and $(4.9999893, -19875)$. Global maximum at $(-5, 22125)$. Global minimum at $(4.9999893, -19875)$. Inflection points at $(-3.2434066, -5400.099)$, $(-0.6659449, -4318.084)$, $(2.0377631, -10828.692)$ and $(4.2714318, -15109.748)$ with rates, in units per units, 12327.356, -9234.571, 4326.813, and -9431.913. The function is increasing the fastest at $(6, 8144)$ with a rate of 72000 units per units, and decreasing the fastest at $(6, 8144)$ with a rate of -86400 units per units.




```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ 10*x^6-36*x^5-345*x^4+1020*x^3+2820*x^2-7200*x-10000}
> a<- -5
> b<- 6
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> keyPoints<-c(a,b,root_deriv)
> keyPoints
[1] -5.0000000  6.0000000 -3.9999904 -1.9999977  0.9999994
[6]  2.9999914  4.9999893
> f(keyPoints)
[1]  22125   8144 -11856   3792 -13731  -8083 -19875
> f_pp(keyPoints)
[1] 149040.000 126600.000  37799.308 -12599.990   7200.002
[6]  -8399.950  30239.353
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> keyPoints2<-c(a,b,root_deriv2)
> keyPoints2
[1] -5.0000000  6.0000000 -3.2434066 -0.6659449  2.0377631  4.2714318
> f(keyPoints2)
[1]  22125.000   8144.000  -5400.099  -4318.084 -10828.692 -15109.748
> f_p(keyPoints2)
[1] -86400.000  72000.000  12327.356  -9234.571   4326.813  -9431.913
> # For Graph
> par(mar=c(5,6,2,2))
```

```
> curve(f,a,b,lwd=2,ylab=expression(f(x)==
10*x^6-36*x^5-345*x^4+1020*x^3+2820*x^2-7200*x-10000 ))
> grid(NULL,NULL,col="black")
> gMax<-c(keyPoints[1])
> gMin<-c(keyPoints[7] )
> points(gMax,f(gMax),pch=15,cex=2,col="#7570b3")
> points(gMin,f(gMin),pch=17,cex=2.5,col="#7570b3")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

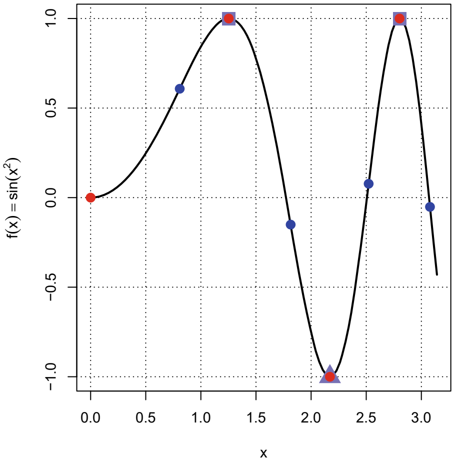
9. From the code. Local maximums at $(-2.028756, 1.819706)$ and $(2.028756, 1.819706)$. Local minimums at $(0, 0)$, $(-4.913017, -4.81447)$ and $(4.913017, -4.81447)$. Global maximum at $(-2.028756, 1.819706)$ and $(2.028756, 1.819706)$. Global minimum at $(-4.913017, -4.81447)$ and $(4.913017, -4.81447)$. Inflection points at $(-3.643598, -1.753242)$, $(-1.076807, 0.948073)$, $(1.076807, 0.948073)$ and $(3.643598, -1.753242)$ with rates, in units per units, 3.675233, -1.391008, 1.391008, and -3.675233. The function is increasing the fastest at $(6, -1.676493)$ with a rate of 5.481606 units per units, and decreasing the fastest at $(-6, -1.676493)$ with a rate of -5.481606 units per units.




```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ x*sin(x)}
> a<- -6
> b<- 6
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> keyPoints<-c(a,b,root_deriv)
> keyPoints
[1] -6.000000 6.000000 0.000000 -4.913017 -2.028756 2.028756 4.913017
```

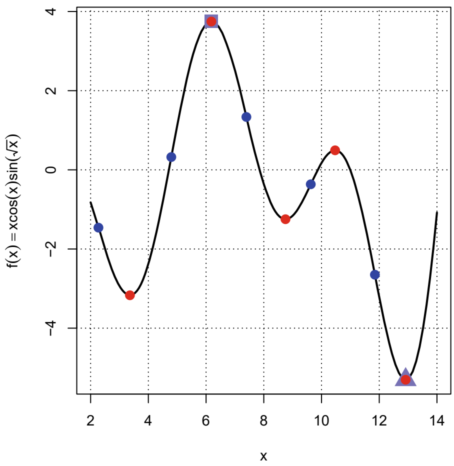
```
> f(keyPoints)
[1] -1.676493 -1.676493 0.000000 -4.814470 1.819706 1.819706 -4.814470
> f_pp(keyPoints)
[1]  3.596834 3.596834 2.000000 5.213040 -2.703944 -2.703944 5.213040
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> keyPoints2<-c(a,b,root_deriv2)
> keyPoints2
[1] -6.000000  6.000000 -3.643598 -1.076807  1.076807  3.643598
> f(keyPoints2)
[1] -1.676493 -1.676493 -1.753242  0.948073  0.948073 -1.753242
> f_p(keyPoints2)
[1] -5.481606  5.481606  3.675233 -1.391008  1.391008 -3.675233
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=expression(f(x)==x*sin(x)))
> grid(NULL,NULL,col="black")
> gMax<-c(keyPoints[c(5,6)])
> gMin<-c(keyPoints[c(4,7)] )
> points(gMax,f(gMax),pch=15,cex=2,col="#7570b3")
> points(gMin,f(gMin),pch=17,cex=2.5,col="#7570b3")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

11. From the code. Local maximums at $(1.253197, 1)$ and $(2.802425, 0.9999999)$. Local minimums at $(0, 0)$ and $(2.170751, -1)$. Global maximum at $(1.253197, 1)$ and $(2.802425, 0.9999999)$. Global minimum at $(2.170751, -1)$. Inflection points at $(0.80832, 0.607874)$, $(1.814594, -0.150585)$, $(2.522042, 0.077435)$ and $(3.078391, -0.051689)$ with rates, in units per units, 1.283665, -3.587805, 5.028939, and -6.148551. The function is increasing the fastest at $(2.522042, 0.077435)$ with a rate of 5.028939 units per units, and decreasing the fastest at $(3.078391, -0.051689)$ with a rate of -6.148551 units per units.



```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ sin(x^2)}
> a<- 0
> b<- pi
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> keyPoints<-c(a,b,root_deriv)
> keyPoints
[1] 0.000000 3.141593 0.000000 1.253197 2.170751 2.802425
> f(keyPoints)
[1]  0.0000000 -0.4303012  0.0000000  1.0000000 -1.0000000  0.9999999
> f_pp(keyPoints)
[1]   2.000000  15.182240   2.000000  -6.281429  18.848175 -31.413564
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> keyPoints2<-c(a,b,root_deriv2)
> keyPoints2
[1] 0.0000000 3.1415927 0.8083198 1.8145945 2.5220424 3.0783908
> f(keyPoints2)
[1]  0.00000000 -0.43030122  0.60787441 -0.15058544  0.07743501
[6] -0.05168862
> f_p(keyPoints2)
[1]  0.000000 -5.671739  1.283665 -3.587805  5.028939 -6.148551
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=expression(f(x)==sin(x^2)))
> grid(NULL,NULL,col="black")
> gMax<-c(keyPoints[c(4,6)])
> gMin<-c(keyPoints[c(5)] )
> points(gMax,f(gMax),pch=15,cex=2,col="#7570b3")
> points(gMin,f(gMin),pch=17,cex=2.5,col="#7570b3")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

13. From the code. Local maximums at $(6.183531, 3.747686)$ and $(10.471137, 0.493808)$. Local minimums at $(3.362034, -3.168048)$, $(8.751563, -1.247079)$ and $(12.91514, -5.303144)$. Global maximum at $(6.183531, 3.747686)$. Global minimum at $(12.91514, -5.303144)$. Inflection points at $(2.270792, -1.459892)$, $(4.795329, 0.323552)$, $(7.397172, 1.335856)$, $(9.628267, -0.364342)$ and $(11.850322, -2.649082)$ with rates, in units per units, -2.407148, 3.906937, -3.084678, 1.555666 and $--3.768691$. The function is increasing the fastest at $(14, -1.081009)$ with a rate of 7.54315 units per units, and decreasing the fastest at $(11.850322, -2.649082)$ with a rate of -3.768691 units per units.




```
> library(Deriv)
> library(rootSolve)
> f<-function(x){ x*cos(x)*sin(sqrt(x))}
> a<- 2
> b<- 14
> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
> root_deriv<-uniroot.all(f_p,c(a,b))
> keyPoints<-c(a,b,root_deriv)
> keyPoints
[1]  2.000000 14.000000 3.362034 6.183531 8.751563 10.471137 12.915140
> f(keyPoints)
[1] -0.8221113 -1.0810092 -3.1680484  3.7476861 -1.2470793  0.4938080
[7] -5.3031441
> f_pp(keyPoints)
[1] -1.106421 5.255459 3.825465 -4.212234 3.063925 -3.341682 6.632717
> root_deriv2<-uniroot.all(f_pp,c(a,b))
> keyPoints2<-c(a,b,root_deriv2)
> keyPoints2
[1]  2.000000 14.000000 2.270792 4.795329 7.397172 9.628267 11.850322
> f(keyPoints2)
```

```
[1] -0.8221113 -1.0810092 -1.4598923  0.3235522  1.3358562 -0.3643424
[7] -2.6490825
> f_p(keyPoints2)
[1] -2.253290 7.543150 -2.407148 3.906937 -3.084678 1.555666 -3.768691
> # For Graph
> par(mar=c(5,6,2,2))
> curve(f,a,b,lwd=2,ylab=expression(f(x)==x*cos(x)*sin(sqrt(x))))
> grid(NULL,NULL,col="black")
> gMax<-c(keyPoints[c(4)])
> gMin<-c(keyPoints[c(7)] )
> points(gMax,f(gMax),pch=15,cex=2,col="#7570b3")
> points(gMin,f(gMin),pch=17,cex=2.5,col="#7570b3")
> points(root_deriv,f(root_deriv),pch=16,cex=1.5,col="red2")
> points(root_deriv2,f(root_deriv2),pch=16,cex=1.5,col="blue2")
```

15. On the interval [0, 142] the maximum price of eggs occurred in November 2022 at \$4.22 per dozen, while the minimum price occurred in March 2019 at \$1.36 per dozen. Egg prices were increasing the fastest in November 2022 at a rate of \$.12 per dozen per month, while the price was decreasing the fastest in September 2016 at a rate of \$0.02 per dozen per month.
17. The maximum height is 289.9898 meters and happens at 7.65 seconds. The maximum speed is the maximum of the derivative over the interval [0, 15.346] where 15.346 seconds is a root of $s(t)$ representing when the watermelon hits the ground. The maximum speed is 75.39 meters per second occurring at time 15.346 seconds.
19. The global min is (0, 0). The global max is (2.28881, 3.945302). The curve is increasing the fastest at (1.519738, 2.306593) with a rate of 3.153388 units/units. The curve is decreasing the fastest at (3, 1.27008) with a rate of −8.063212 units/units.
21. Profit is maximized by selling 10 items with a profit of \$300. Average profit is maximized by selling at 6.32, or 6 items with a profit of \$36.75 per item.
23. Sorry, yes it is an odd problem, but this one could be used as a project so no answer.

Section 19.1

1. Objective: $g(x, y) = x^2 + y^2$. Constraint: $x + y = 20$. Minimize $f(x) = x^2 + (20 - x)^2$ when $x \in [0, 20]$ to get global minimum (10, 200).

3. Objective: $A(x, y) = xy$. Constraint: $2x + 3y = 1000$. Maximize/Minimize $f(y) = \dfrac{1000 - 3y}{2}x$ when $y \in [0, 1000/3]$ to get global maximum (1666.6667, 416666.67). The pen will have a total area of 416666.67 square feet with dimensions 166.6667 feet by 250 feet.

5. Let x be the length of the fence along the road. Objective: $A(x, y) = xy$. Constraint: $20x + 10x + 10y + 10y = 30x + 20y = 1500$. Maximize $f(x) = x\dfrac{1500 - 30x}{20}$ when $x \in [0, 50]$ to get global maximum $(25, 937.5)$. The maximum area is 937.5 feet squared with dimensions 25 feet by 37.5 feet.

7. Let x be the length of the side of the square bottom and y the height of the box. Objective: $V(x, y) = x^2y$. Constraint: $x^2 + 4y^2 = 50$. Maximize $f(y) = (\sqrt{50 - 4y^2}^2 y = 50y - 4y^3$ when $y \in [0, \sqrt{50}/2]$ to get global maximum $(2.04, 68.04)$. The maximum volume is 68.04 feet cubed with dimensions $x = 5.78$ feet and $y = 2.04$ feet.

9. Let x be the side length of the square being cut out. Objective: $f(x) = (5 - 2x)(4 - 2x)$. Constraint: This is built into the fact that we start with a 4ft by 5ft piece of cardboard. Maximize $f(x) = (5 - 2x)(4 - 2x)x$ with $x \in [0, 2]$ to get $x = 0.736$ feet for a total volume of 6.563 cubic feet. The box is $0.736 \times 2.528 \times 2.528$.

11. Let x be the length of a piece of the wire, making the other piece $40 - x$. We'll use the x piece to make the circle and the $40 - x$ to make the square. Objective: $f(x) = \left(\dfrac{40 - x}{4}\right)^2 + \pi\left(\dfrac{x}{2\pi}\right)^2$. Constraint: Built into starting with the 40inch piece of wire. Maximize $f(x) = \left(\dfrac{40 - x}{4}\right)^2 + \pi\left(\dfrac{x}{2\pi}\right)^2$ with $x \in [0, 40]$ to get $x = 40$, in other words use all the material to make a circle with 40 inch circumference and area 127.324 square inches. Note: The minimum area is achieved with $x = 17.6$.

13. Let h be the length of the hypotenuse and x and y the lengths of the other two sides of the triangle. Objective: $A(x, y) = xy/2$. Constraint: $x^2 + y^2 = h^2 = 11^2$. Maximize $f(x) = \dfrac{x\sqrt{121 - x^2}}{2}$ with $x \in [0, 11]$ to get $x = 7.78$ for a total area of 30.25 square yards.

15. Let x and y be the lengths of the sides of the pool. Objective: $A(x, y) = (x + 8)(y + 20)$. Constraint $xy = 800$. Minimize $f(x) = (x + 8)(\frac{800}{x} + 20)$ with $x > 0$ to get $x = 17.89$ feet for a total area for the property 1675.54 square feet with dimensions $25.89\text{feet} \times 64.72\text{feet}$.

17. Let x be the length of the shorter shed based and y the height. Objective: $V(x, y) = yx(x+5)$. Constraint $2x(x+5) + y(x+5) + 2xy = 1500$. Maximize $f(x) = \dfrac{(1500 - 2x(x+5))x(x+5)}{3x+5}$ with $x \in [0, 25]$ (larger than 25 and the numerator of $f(x)$ is negative) to get $x = 13.25$ for total volume of 5492.47 square feet and dimensions $13.35\text{feet} \times 18.35\text{feet} \times 22.42\text{feet}$

19. Objective: $x^4 + 2y^2$. Constraint $x + y = 100$. Minimize $f(y) = (y - 100)^4 + 2y^2$ with $y \in [0, 100]$ to get $y = 95.43$ for minimum of 18649.95.

21. Objective: $f(a, b) = 10\sqrt{a^2 + b^2} + 25\sqrt{(4-a)^2 + (4-b)^2}$. Constraint: $b = 3 + a/2$. Minimize: $g(a) = 10\sqrt{a^2 + (3 + a/2)^2} + 25\sqrt{(4-a)^2 + (4 - (3 + a/2))^2}$ with $a \in [0, 4]$ to get $a = 2.37$ and a time of 98 minutes.

23. Objective: $f(x, y) = \sqrt{100 - x^2 - y^2}$. Constraint $y = -x^2 + 2x + 5$. Maximize $g(x) = \sqrt{100 - x^2 - (-x^2 + 2x + 5)^2}$ with $x \in [-2.94, 4.84]$ to get a maximum height of 9.899 feet at $(-1.3894, 0.2908)$.

Section 20.1

1. $$\frac{\partial f}{\partial x} = 15x^2y^2$$
$$\frac{\partial f}{\partial y} = 10x^3y$$
$$\frac{\partial^2 f}{\partial x^2} = 30xy^2$$
$$\frac{\partial^2 f}{\partial y^2} = 10x^3$$
$$\frac{\partial^2 f}{\partial y \partial x} = 30x^2y$$
$$\frac{\partial^2 f}{\partial x \partial y} = 30x^2y$$

3. $$\frac{\partial f}{\partial x} = 16x\cos(y)$$
$$\frac{\partial f}{\partial y} = -8x^2\sin(y)$$
$$\frac{\partial^2 f}{\partial x^2} = 16\cos(y)$$
$$\frac{\partial^2 f}{\partial y^2} = -8x^2\cos(y)$$
$$\frac{\partial^2 f}{\partial y \partial x} = -16x\sin(y)$$
$$\frac{\partial^2 f}{\partial x \partial y} = -16x\sin(y)$$

5. $\frac{\partial f}{\partial x} = 18x^5 e^{y^2}$
$\frac{\partial f}{\partial y} = 6x^6 y e^{y^2}$
$\frac{\partial^2 f}{\partial x^2} = 90x^4 e^{y^2}$
$\frac{\partial^2 f}{\partial y^2} = 6x^6(2y^2 + 1)e^{y^2}$
$\frac{\partial^2 f}{\partial y \partial x} = 36x^5 y e^{y^2}$
$\frac{\partial^2 f}{\partial x \partial y} = 36x^5 y e^{y^2}$

7. $\frac{\partial f}{\partial x} = 16xy^2 e^{x^2y^2}$
$\frac{\partial f}{\partial y} = 16x^2 y e^{x^2y^2}$
$\frac{\partial^2 f}{\partial x^2} = 16y^2(2x^2y^2 + 1)e^{x^2y^2}$
$\frac{\partial^2 f}{\partial y^2} = 16x^2(2x^2y^2 + 1)e^{x^2y^2}$
$\frac{\partial^2 f}{\partial y \partial x} = 32xy(x^2y^2 + 1)e^{x^2y^2}$
$\frac{\partial^2 f}{\partial x \partial y} = 32xy(x^2y^2 + 1)e^{x^2y^2}$

9. $\frac{\partial f}{\partial x} = 4(\cos(6xy^2) - 6xy^2 \sin(6xy^2))$
$\frac{\partial f}{\partial y} = -48x^2 y \sin(6xy^2)$
$\frac{\partial^2 f}{\partial x^2} = -48y^2(\sin(6xy^2) + 3xy^2 \cos(6xy^2))$
$\frac{\partial^2 f}{\partial y^2} = -48x^2(\sin(6xy^2) + 12xy^2 \cos(6xy^2))$
$\frac{\partial^2 f}{\partial y \partial x} = 96xy(\sin(6xy^2) + 3xy^2 \cos(6xy^2))$
$\frac{\partial^2 f}{\partial x \partial y} = 96xy(\sin(6xy^2) + 3xy^2 \cos(6xy^2))$

11. A rectangular box with a square base of side 4 feet and height 5 feet is growing at a rate of 40 cubic feet per foot of base and 25 cubic feet per foot of height.

13. The ant should move parallel to the x-axis since it is down at a rate of 0.8 feet per foot where as parallel to the y-axis it is an increase of 0.96 feet per foot.

15. $D(35, 25) = 0.505$, $\frac{\partial D}{\partial p}(35, 25) = 0.108$ probability points per thousand dollars of price, $\frac{\partial D}{\partial l}(35, 25) = 0.114$ probability points per 100 square feet of lot size. A house with a price of \$35000 and a lot size of 2500 square feet has just over a 50% chance of having a driveway. An increase of cost of \$1000 increases this probability by about 0.108 while an increase of 100 square feet to the lot size is about 0.114, which is greater.

Section 21.1

1. $f'(t) = 5g'(t)$

3. $f'(t) = 12h(t)h'(t) + 4h'(t)$

5. $g'(t) = 7h'(t)\cos(7h(t))$

7. $24(h(t))^3 h'(t) - 9(f(t))^2 f'(t) = 8$

9. $\frac{h'(t)}{h(t)} + f'(t)e^{f(t)} = \cos(t)$

11. $g(t)f'(t) + g'(t)f(t) = 45t^8$.

13. The area of the square is increasing at a rate of 6 square feet per second.

15. The volume of the cube is increasing at rate of 144 cubic inches per second.

17. The ladders is sliding down the house at a rate of $15/16 \approx 0.94$ feet per second.

19. The person's shadow is increasing at a rate of 0.47 feet per second.

21. The demand for the widgets is decreasing at a rate of 20 per week.

23. The area of disturbed water is increasing at a rate of 60π feet per second.

25. The kinetic energy of the car is increasing at a rate of 100,800 joules (J) per second.

27. The area of the rectangle is increasing at rate of 33 square inches per second.

Section 22.1

1. $f(-2) = 5$, $f(2) = 4$, $f(10) = 100$

3. $f(-1) = -3$, $f(2) = 24$, $f(5) = 32$

5. $g(-8) = -514$, $g(9) = 3$, $g(0) = 6$

7. $f(-5) = 0$, $f(8) = 64$, $f(3) = 9$

9. $f(-2) = -8$, $f(2) = 4$, $f(0) = 0$

11. $g(3) = 3$, $g(2) = 2$, $g(-1) = 2$

13. Solve $f'(t) = ae^{-bt} - abte^{-bt} = 0$ to get $t = 1/b$ and $f(1/b) = a/be$.

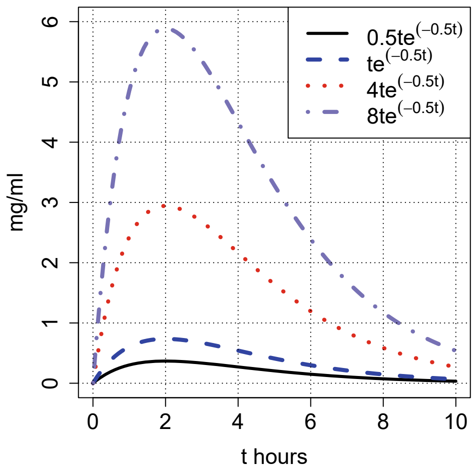
15. Use the model $f(t) = 6te^{-0.5t}$, $f(4) = 3.248$ mg/ml. The maximum occurs at $1/b = 1/0.5 = 2$ or noon with an amount of $f(2) = 4.415$ mg/ml or $a/(be) = 4.415$ mg/ml.

17. Use the model $f(t) = 2te^{-0.75t}$, $f(3) = 0.632$ mg/ml. The maximum occurs at $1/b = 1/0.75 = 1.33$ or about 9:20am with an amount of $f(1/0.75) = 0.981$ mg/ml or $a/(be) = 0.981$ mg/ml.

19. Use the model $f(t) = 10te^{-0.5t}$, $f(5.5) = 3.52$ mg/ml. The maximum occurs at $1/b = 1/0.5 = 2$ or about 8:00am with an amount of $f(2) = 7.36$ mg/ml or $a/(be) = 7.36$ mg/ml.

21. Use the model $f(t) = 20te^{-0.8t}$, $f(3.75) = 3.73$ mg/ml. The maximum occurs at $1/b = 1/0.8 = 1.25$ or about 11:15pm with an amount of $f(1.25) = 9.20$ mg/ml or $a/(be) = 9.20$ mg/ml.

23. Since $b = 0.5$ is fixed the location of the maximum will remain at $1/0.5 = 2$ for all values of a. As a increases the value of the maximum increases since the value of the maximum is $a/(be)$, which makes the curve steeper as it reaches the maximum value.



```
> par(mar=c(5,5,2,2))
> Surge=function(a,b,x){a*x*exp(-b*x)}
> curve(Surge(0.5,0.5,x),0,10,ylim=c(0,6),xlab="t hours",ylab="mg/ml",
lwd=3,cex.axis=1.5,cex.lab=1.5)
> curve(Surge(1,0.5,x),0,10,xlab="t hours",ylab="mg/ml",lwd=4,
col="blue2",lty=2,add=TRUE)
> curve(Surge(4,0.5,x),0,10,xlab="t hours",ylab="mg/ml",lwd=4,
add=TRUE,lty=3,col="red2")
> curve(Surge(8,0.5,x),0,10,xlab="t hours",ylab="mg/ml",lwd=4,
add=TRUE,lty=4,col="#7570b3")
> grid(NULL,NULL,col="black")
> legend("topright", c( expression(0.5*t*e^(-0.5*t)),
expression(t*e^(-0.5*t)),expression(4*t*e^(-0.5*t)),
expression(8*t*e^(-0.5*t))),lwd=c(3,4,4,4),lty=c(1,2,3,4),
col=c("black","blue2","red2","#7570b3"),bg="white",cex=1.5)
```

25. Results from the code below. Note that the code for a graph is included. It is a good idea to graph the function as a check of your results. There was 4.85 mg/ml and 4.46 mg/ml in the blood stream at 9:00am and 4:00pm, respectively. The maximum amount in the blood was 9.466 mg/ml occurring at $t = 4.45$ or roughly 12:30pm.

```
> library(Deriv)
> library(rootSolve)
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s))}
> a<-8
> b<-0.5
> s<-3
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)}
> h(1)
[1] 4.852245
> h(8)
[1] 4.455601
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1] 1.999886 2.999957 4.452686
> h(roots)
[1] 5.886071 5.355162 9.465374
> curve(h,0,24,ylim=c(0,12),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
```

27. Results from the code below. Note that the code for a graph is included. It is a good idea to graph the function as a check of your results. There was 11.17 mg/ml in the blood stream at 9:00pm. The maximum amount in the blood was 15.80 mg/ml occurring at $t = 9.6$ or roughly 3:40pm.

```
> library(Deriv)
> library(rootSolve)
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s))}
> a<-5
> b<-0.2
> s<-6
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)}
> h(15)
[1] 11.17248
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1] 4.999999 5.999554 9.611150
> h(roots)
[1]  9.196986  9.035961 15.798714
> curve(h,0,24,ylim=c(0,20),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
```

29. Results from the code below. Note that the code for a graph is included. It is a good idea to graph the function as a check of your results. When the drug is taken a second time seven hours after the first dose the maximum amount in the blood was 6.72 mg/ml occurring at $t = 8.79$ after the initial dose. When the drug is taken a second time five hours after the first dose the maximum amount in the blood was 7.70 mg/ml occurring at $t = 6.62$ after the initial dose. When the drug is taken a second time 2 hours early the maximum is 0.98 mg/ml or 13% greater. The maximum amount also occurs over two hours earlier.

```
> library(Deriv)
> library(rootSolve)
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s))}
> a<-8
> b<-0.5
> ## With s at 7
> s<-7
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)}
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1] 1.999886 6.999597 8.794783
> h(roots)
[1] 5.886071 1.691297 6.718950
> curve(h,0,24,ylim=c(0,10),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> ## With s at 5
> s<-5
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)}
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
```

```
[1] 1.999886 4.999521 6.620673
> h(roots)
[1] 5.886071 3.283872 7.699258
> curve(h,0,24,ylim=c(0,10),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5,col="red2",add=TRUE)
> grid(NULL,NULL,col="black")
```

31. Results from the code below. Note that the code for a graph is included. It is a good idea to graph the function as a check of your results. There was 12.28 mg/ml in the blood stream at midnight. The local maximums occur at $t = 2$ (10:00am), $t = 7.72$ (approximately 3:45pm) and $t = 13.69$ (approximately 9:45pm), with amounts 14.72 mg/ml, 17.81 mg/ml and 18.10 mg/ml, respectively.

```
> library(Deriv)
> library(rootSolve)
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s) )}
> a<-20
> b<-0.5
> s<-6
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)
+Surge_Piece(a,b,2*s,x)}
> h(16)
[1] 12.28176
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1]  1.999886  5.999580  7.715511 11.999587 13.687851
> h(roots)
[1] 14.715178  5.975284 17.809721  6.570274 18.099811
> curve(h,0,24,ylim=c(0,20),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
```

33. Results from the code below. Note that the code for a graph is included. It is a good idea to graph the function as a check of your results. When the drug is taken at 2 hour intervals there is only one local maximum of 22.96 mg/ml at time $t = 5.47$. With three does we would expect three local maximums. The following doses are occurring before the previous dose's peak occurs. Unless this is on purpose it seems questionable. At three hour intervals we get three clear local maximums, with the largest occurring at $t = 7.46$ with 19.41 mg/ml of drug in the blood stream. In the case the maximum is 3.55 mg/ml or 15.5% lower.

```
> library(Deriv)
> library(rootSolve)
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s))}
> a<-10
> b<-0.4
> # With two hour intervals
> s<-2
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)
+Surge_Piece(a,b,2*s,x)}
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1] 5.466671
> h(roots)
[1] 22.95926
> curve(h,0,24,ylim=c(0,25),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> # With three hour intervals
> s<-3
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)
+Surge_Piece(a,b,2*s,x)}
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1] 2.499851 2.999783 4.805576 5.999610 7.459798
> h(roots)
[1]  9.196986  9.035957 15.798714 14.479633 19.407139
> curve(h,0,24,ylim=c(0,25),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5,col="red2",add=TRUE)
> grid(NULL,NULL,col="black")
```

35. Results from the code below. Note that the code for a graph is included. It is a good idea to graph the function as a check of your results. When the drug is taken at 9 hour intervals both local minimums, 1.50 mg/ml and 1.53 mg/ml, fall below the effective level of 2.25 mg/ml. Reducing the time between doses to 7.5 hours results in local minimums of 2.65 mg/ml and 3.77 mg/ml. Of course 7.5 hour spacing is unusual so 7 hours would only raise the local minimums.

```
> library(Deriv)
> library(rootSolve)
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s))}
> a<-15
> b<-0.5
> # With nine hour intervals
> s<-9
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)
+Surge_Piece(a,b,2*s,x)}
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
```

```
[1]  1.999886  8.999934 10.901097 17.999552 19.898929
> h(roots)
[1] 11.036383  1.499753 11.724567  1.533303 11.738813
> curve(h,0,24,ylim=c(0,15),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> # With seven and a half hour intervals
> s<-7.5
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)
+Surge_Piece(a,b,2*s,x)}
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1]  1.999886  7.499802  9.327662 14.999509 16.819684
> h(roots)
[1] 11.036383  2.645939 12.312414  2.770694 12.368492
> curve(h,0,24,ylim=c(0,15),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5,col="red2",add=TRUE)
> grid(NULL,NULL,col="black")
```

37. Results from the code below. Note that the code for a graph is included. It is a good idea to graph the function as a check of your results. If we compare the first local maximum, 22.07 mg/ml, with the third local maximum 41.48 mg/ml (also the global maximum) we see a near doubling of the drug in the blood stream. Unless this is on purpose it seems like the time between doses is too short. It is also worth noting that the local minimum rises from 20.08 mg/ml to 29.04 mg/ml; a nearly 50% increase.

```
> library(Deriv)
> library(rootSolve)
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s))}
> a<-15
> b<-0.25
> s<-6
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)
+Surge_Piece(a,b,2*s,x)}
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1]  3.999996  5.999580  8.905456 11.999629 14.478969
> h(roots)
[1] 22.07277 20.08242 35.49515 29.04456 41.09676
> curve(h,0,24,ylim=c(0,45),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
```

39. Results from the code below. Note that the code for a graph is included. It is a good idea to graph the function as a check of your results. In the loading dose case the value of 20 is in the first Surge function in the definition of the function $h(t)$. In the normal case this is left at a. In the loading dose case the first local maximum is notably higher, 29.43 mg/ml compared to 22.07 mg/ml. Similarly, the first local minimum is notably higher at 21.65 mg/ml compared to 16.24 mg/ml. The local minimums of the loading dose case are close to the same where as in the normal case the first local minimum is lower, 16.34 mg/ml, as compared to the second local minimum, 20.64 mg/ml. Overall the loading dose case maintains more consistent minimums and maximums. Note that the times of the maximums and minimums are the same for both cases (do you know why?).

```
> library(Deriv)
> library(rootSolve)
> Surge<-function(a,b,t){a*t*exp(-b*t)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s) )}
> a<-15
> b<-0.25
> s<-8
> # Loading Dose Case
> h<-function(x){Surge(20,b,x)+Surge_Piece(a,b,s,x)
+Surge_Piece(a,b,2*s,x)}
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1]  3.999996  7.999738 10.777093 15.999659 18.729548
> h(roots)
[1] 29.43036 21.65435 35.37355 22.10231 35.16909
> curve(h,0,24,ylim=c(0,45),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> # Normal Dosing Case
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)
+Surge_Piece(a,b,2*s,x)}
> h_p<-Deriv(h)
> roots<-uniroot.all(h_p,c(0,24))
> roots
[1]  3.999996  7.999907 11.046377 15.999820 18.807570
> h(roots)
[1] 22.07277 16.24042 31.80675 20.63650 34.30880
> curve(h,0,24,ylim=c(0,45),xlab="Hours Since First Dose",
ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5,col="red2",add=TRUE)
> grid(NULL,NULL,col="black")
```

41. Results from the code below. We first added extra doses one at a time to the model function $h(x)$ until graphically $h(x)$ exceeded 0.5. This occurred after four doses. With four doses the maximum amount is 0.53 ml/mg and occurs at 57 minutes from time 0.95.

```
> library(Deriv)
> library(rootSolve)
> s<-0.25
> a<-1.10493682724572
> b<-2.50572657235535
> s<-0.25
> Surge<-function(a,b,x){a*x*exp(-b*x)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s) )}
> h<-function(x){Surge(a,b,x)+ Surge_Piece(a,b,s,x)
+Surge_Piece(a,b,2*s,x)
+Surge_Piece(a,b,3*s,x)}
> curve(h,0,24,xlab="t hours", ylab="mg/ml",lwd=3,cex.axis=1.5,
cex.lab=1.5)
> grid(NULL,NULL,col="black")
> h_p<-Deriv(h)
> roots_p<-uniroot.all(h_p,c(0,24))
> roots_p
[1] 0.9508981
> h(roots_p)
[1] 0.5258677
```

43. Results from the code below. The maximum is 0.92mg/ml and occurs at time 4.68 or 4 hours and 41 minutes after the initial dose. The concentration in the blood falls below 0.8 at time 5.27 or 5 hours and 16 minutes after the initial dose. The alcohol is being removed from the blood fastest at time 5.62 or 5 hours and 38 minutes after the initial dose with a rate of 0.35 mg/ml per hour.

```
> library(Deriv)
> library(rootSolve)
> a<-1.76393642046205
> b<-1.05841662684339
> s<-2
> Surge<-function(a,b,x){a*x*exp(-b*x)}
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s) )}
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)+
 Surge_Piece(a,b,2*s,x)}
> h_p<-Deriv(h)
> roots_p<-uniroot.all(h_p,c(0,24))
> roots_p
[1] 0.9447918 1.9997345 2.7298618 3.9999847 4.6815144
> h(roots_p)
[1] 0.6131007 0.4248624 0.8624051 0.5271065 0.9194393
> h_0.8<-function(x){h(x)-0.8}
> roots_p8<-uniroot.all(h_0.8,c(0,24))
> roots_p8
[1] 2.409498 3.144843 4.266612 5.271175
> h(roots_p8)
[1] 0.8000559 0.7999956 0.8000106 0.8000009
> h_pp<-Deriv(h,n=2)
> roots_pp<-uniroot.all(h_pp,c(0,24))
> roots_pp
[1] 1.889558 1.999648 3.674670 3.999875 5.626322
> h_p(roots_pp)
[1] -0.2387228 -0.2372243 -0.3357944 -0.3199287 -0.3580018
```

Section 23.1

1. $A = kB$
3. $x = \frac{k}{y}$
5. $x = ky^2$
7. $A = kBC$
9. $x = k(y - z)$
11. $f(t) = kt$
13. $f'(t) = kf(t)$
15. $f'(t) = ktf(t)$
17. $f'(t) = k(f(t) - A)$
19. $f''(t) = kf(t)$
21. $f(t) = k(f''(t) - B)^3$
23. $f''(t) = k(f'''(t))^2$
25. $A = 16$
27. $A = 35$
29. $A = 15$, $B = 57$
31. $A = 18$
33. $A = 35$
35. $A = 16$, $B = 0$
37. $A = 240$, $B = 2400$
39. $A = 4297$, $B = 683219$, $C = 687733$

41. To sum the first 100 integers:

```
> A<-1
> for (i in 2:100){
+ A<-A+i
+ }
> A
[1] 5050
```

43. The code below is for the first 1000 reciprocals. If you increase the number or reciprocals you add the sum does increase but slowly (100000000 reciprocals sums to 18.9979). This does go off to infinity, but this isn't a proof (why not? and How would you prove this?)

```
> A<-1
> for (i in 2:1000){
+ A<-A+1/i
+ }
> A
[1] 7.485471
```

Section 24.1

1. 0.002
3. 0.0015
5. 20
7. 300

9. 1600, 3906.25

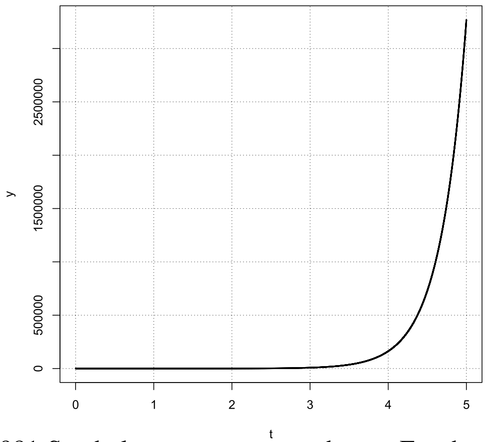
11. 12500, 45435.424

13. 40190.08, 40190.08

15. 5.7, 6.2

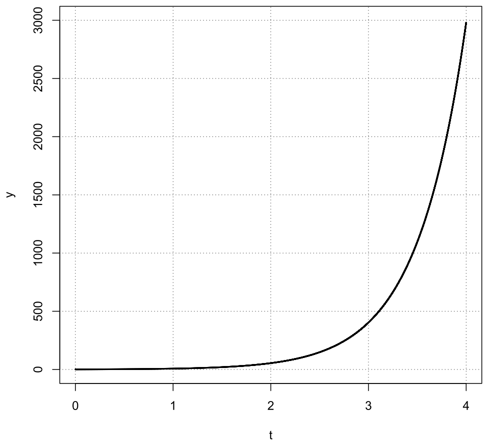
17. Based on the table we estimate 3,270,000 E.Coli at 5 hours. For the graph we used step sizes of 0.00001 to speed up making the graph and the final estimate is sufficiently accurate.

Step Size	Estimate
0.1	497,929
0.01	2,621,877
0.001	3,196,429
0.0001	3,261,672
0.00001	3,268,282
0.000001	3,268,944
0.0000001	3,269,010



19. Based on the table we estimate 2,981 Staphylococcus aureus at hours. For the graph we used step sizes of 0.00001 making the graph and the final estimate is sufficiently accurate.

Step Size	Estimate
0.1	1,469.77
0.01	2,754.66
0.001	2,957.24
0.0001	2,978.57
0.00001	2,980.72
0.000001	2,980.93

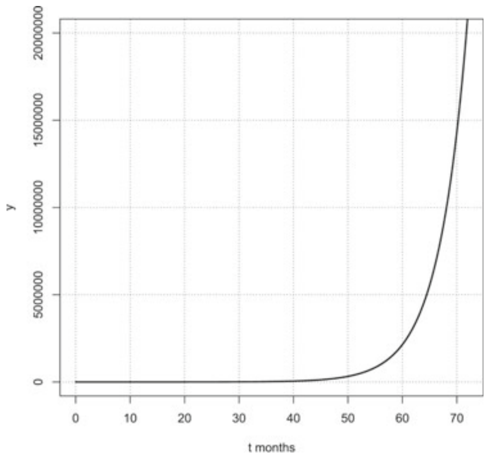


21. Based on previous problems we used a step size of 0.0000001. From the table, it will take about 0.7675 hours to reach 1000 E.coli.

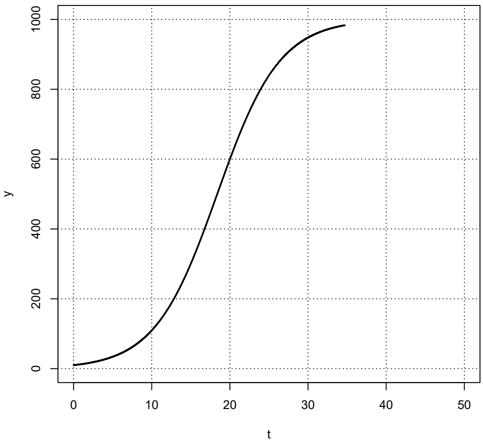
Time	Estimate
0.760	977.67
0.761	980.61
0.762	983.55
0.763	986.51
0.764	989.47
0.765	992.44
0.766	995.43
0.767	998.42
0.768	1,001.42
0.769	1,004.42
0.770	1,007.44

23. $R'(t) = 0.19R(t)$, $R(0) = 24$, with t in months. We estimate 20,958,500 rabbits after 72 months. For the graph we used step sizes of 0.0001 making the graph and the final estimate is sufficiently accurate.

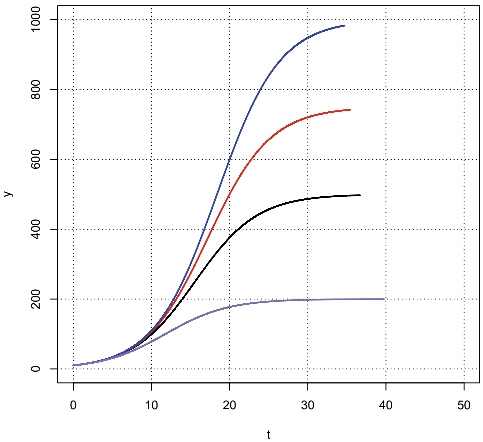
Step Size	Estimate
0.1	18,434,176
0.01	20,688,205
0.001	20,931,262
0.0001	20,955,755
0.00001	20,958,206
0.000001	20,958,451



25. We used step sizes of 0.001 based on running the numerical result code first. The graph is increasing concave up from $t = 0$ to about $t = 18$ and then increasing concave down. There is an inflection point around $t = 18$ and a horizontal asymptote at $y = 1000$.



27. We used step sizes of 0.001. The value of k is the value of the horizontal asymptote. Since all graphs had the same initial value $y_0 = 10$, the larger the value of k the faster the curve will increase in the middle. Overall, the shapes of all of the graphs are the same.



29. $P'(t) = 0.6(1 - P(t)/25)P(t)$ with $P(0) = 7$. Based on the table there will be 22.16 grams of yeast at 5 hours. Using estimate and check with step size 0.0001 it will take 5.2362 hours to reach about 22.5 (90% of 25) grams of yeast.

Step Size	Estimate
0.1	22.2154
0.01	22.1679
0.001	22.1632
0.0001	22.1627

31. Based on the table there will be 6, 194, 000 E.coli after 4 hours.

Step Size	Estimate
0.1	2,806,727
0.01	5,832,730
0.001	6,158,531
0.0001	6,190,633
0.00001	6,193,838
0.000001	6,194,158
0.0000001	6,194,190

33. $y' = ry^2$ with $y(0) = 10$. After 5 years the population is 20. It will take approximately 9.9005 to reach 1000, using step size of 0.0001.

Step Size	Estimate
0.1	19.7336
0.01	19.9724
0.001	19.9972
0.0001	19.9997
0.00001	20.0000

35. $y' = ry^2$ with $y(0) = 1000$, although here r is negative and we typically write this as $y' = -ry^2$, where $r > 0$. After 10 minutes the amount is 711.77 grams. It will take approximately 20.4 minutes to reach 500 grams, using step size of 0.00001.

Step Size	Estimate
0.1	711.3581
0.01	711.7292
0.001	711.7662
0.0001	711.7699
0.00001	711.7703
0.000001	711.7703

37. Sorry, yes it is an odd problem, but this one could be used as a project so no answer.

Section 25.1

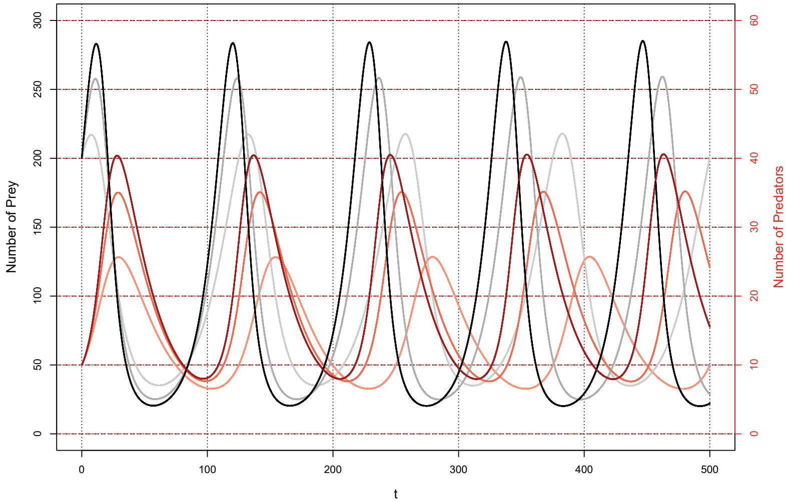
1. Note the units on $N_1'(t)$ are prey per time unit, $N'2(t)$ are predators per time unit, N_1 is number of prey, and N_2 is number of predators. So the units on b_1 must be per time unit. The units on b_2 must be per prey time unit.

3. The maximum prey is 200 and the minimum is about 45. The time between time units is about 150 time units.

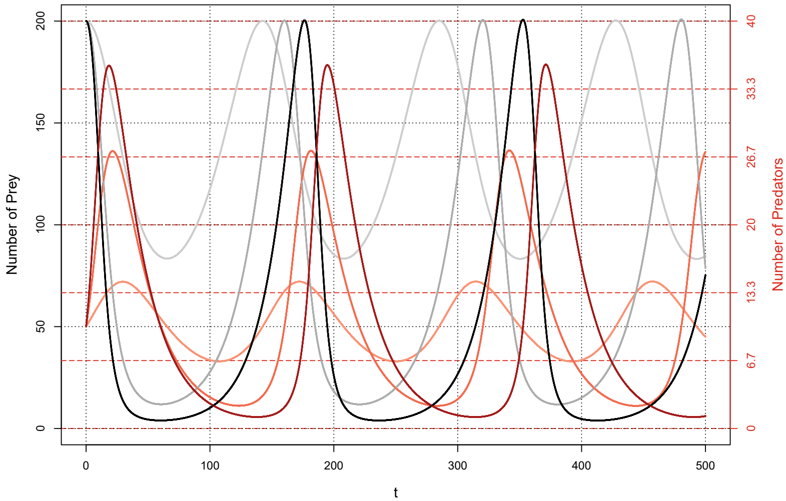
5. On the prey graph, the positive slope inflection points, when prey are growing the fastest, occur when the predators are at a minimum. Prey will grow the fastest when the fewest predators are around. Once predator populations start to increase the rate of growth of prey will begin to decrease.

7. On the prey graph, the negative slope inflection points, when prey are decreasing the fastest, occur when the predators are at a maximum. Prey will decrease the fastest when the most predators are around. Once predator populations start to decrease the rate of growth of prey will begin to increase.

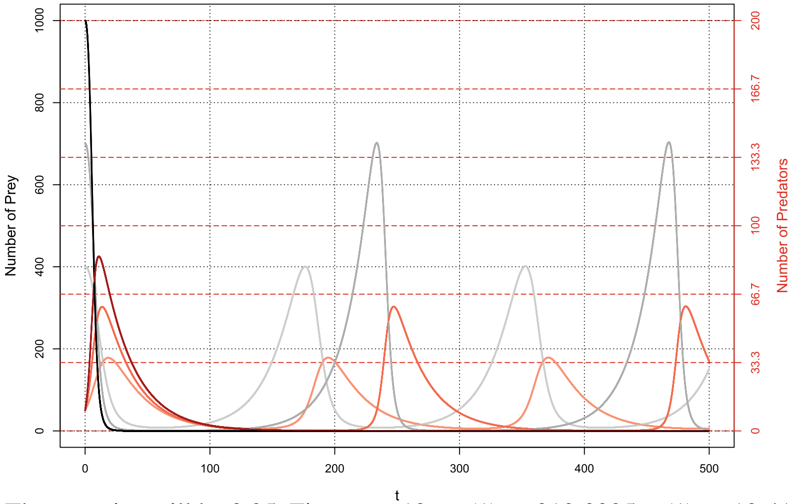
9. At time $t = 0.02$ there are 10.008 foxes and 199.99996 rabbits.
11. In the graph the curves get darker as b_1 increases. As b_1 increases the maximums of both the rabbits and foxes increase and the minimums decrease. The time between the maximums decreases. If this trend continues, a large enough value of b_1 may eventually lead to extinction of both prey and predator. Increasing the growth rate of the rabbits allows the rabbits to grow faster which allows for greater fox population growth. But, with more foxes around we also can decrease the rabbit population more.



13. In the graph the curves get darker as b_2 increases. The maximum rabbit population stays at 200 while the fox population maximum grows as b_2 increases. The larger b_2 the greater the ability of foxes to convert rabbit population to fox growth. On the other hand the minimum of the rabbit population gets smaller and stays low for a longer period. It would seem that a large enough b_2 will collapse both populations. In essence a large enough b_2 allows the foxes to consume all of the rabbits.



15. In the graph the curves get darker as the initial number of rabbits increases. Even at 400 initial rabbits we spike the number of foxes so that the populations almost die off with a longer period of small populations of both. Similarly, but worse for 700 initial rabbits. With 1000 initial rabbits the populations go extinct. The larger starting population of rabbits allows for a spike or overshoot of the fox population that nearly (400, 700 cases) or does kill off the rabbits. Too many rabbits are not necessarily good for the foxes.



17. The step size will be 0.25. First time through: $R\_prime = 0$, $F\_prime = 3.6$, $R = 1000$, $F = 10.9$. Second time through: $R\_prime = -4.5$, $F\_prime = 3.924$, $R = 998.875$, $F = 11.881$.

19. $R(1) = 210.0225$ $F(1) = 10.4142$

21. Using the code for maximum: max(F_data)= 18.7802 and max(R_data)= 200.531

23. The fastest rate of decrease of rabbits, an inflection point, occurs when foxes are at their maximum. We estimate the slope of the tangent line by straddling around this location to get a decrease of about 4.4 rabbits per day. Example code:
(R_data[which.max(F_data)+1]-R_data[which.max(F_data)-1])/(t_data[which.max(F_data)+1] - t_data[which.max(F_data)-1])

25. $$N_1'(t) = b_1(1 - N_1/K)N_1 - d_1 N_1 N_2$$
$$N_2'(t) = b_2 N_1 N_2 - d_2 N_2$$

27. $$N_1'(t) = b_1 N_1 - d_1 N_1 N_2 - 0.1 N_1$$
$$N_2'(t) = b_2 N_1 N_2 - d_2 N_2$$

Section 26.1

1. $S(t)$, $I(t)$, and $R(t)$ are all people. $S'(t)$, $I'(t)$, and $R'(t)$ are all people per day.

3. The maximum of $I(t)$ appears to line up with the inflection point on $S(t)$. At the inflection point on $S(t)$ the rate of change begins to increase (less negative) meaning fewer people are moving from susceptible to infected. If fewer people are moving from susceptible to infected we would expect the number of infected to begin to decrease.

5. $\frac{1}{0.0625} = 16$ days

7. $mc = 0.00005$. If $m = 0.02$ then $c = 0.0025$.

9. If one category gains people then another must lose them at the same rate. Note adding the three differential equations does yield 0.

11. Set $I'(t) = 0$ and solve for $S(t)$ to get $S(t) = \frac{0.0625}{0.00005} = 1250$ people.

13. $S(1) = 19000$, $I(1) = 1937.5$, $R(t) = 112.5$

15. $S(1) = 18783.984$, $I(1) = 2138.867$, $R(t) = 127.148$

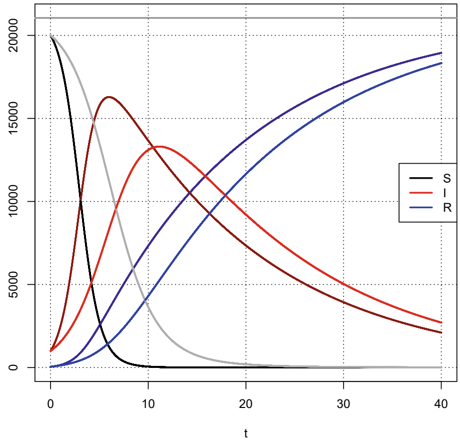
17. Based on the table we estimate $S(5) = 18429$, $I(5) = 2468$, and $R(5) = 152$ with step sizes of 0.00001 and number of steps 500000.

Step Size	Estimate S	Estimate I	Estimate R
0.1	18,517.24	2,386.87	145.90
0.01	18,438.71	2,459.73	151.56
0.001	18,430.38	2,467.46	152.16
0.0001	18,429.54	2,468.24	152.22
0.00001	18,429.45	2,468.32	152.23

19. Based on the table we estimate $I(5.9815) = 16284$ as the maximum. Step sizes of 0.0001 were used.

Time	Estimate I
5.975	16,284.259
5.976	16,284.263
5.977	16,284.267
5.978	16,284.271
5.979	16,284.273
5.980	16,284.274
5.981	16,284.275
5.982	16,284.275
5.983	16,284.274
5.984	16,284.272
5.985	16,284.269

21. The darker colors are the original scenario. The lighter colors are with mc cut in half. The maximum of $I(t)$ is cut by about 3000 people and delayed by about 5 days. Step sizes of 0.001 were used. TIP: To get these on the same graph simply comment out the plot set up when running the code the second time and change the colors in segments.



23. $\lim_{t\to\infty S(t)=0}$ In this model everyone will eventually get the disease and hence the number of susceptible will go to 0.

25. The recovered population increases by an estimated 1018.444 people per day at its fastest. To find this note that the maximum of the infected is where susceptible has an infection point. Code: (R_data[which.max(I_data) +1]-R_data[which.max(I_data) -1])/(t_data[which.max(I_data) +1]-t_data[which.max(I_data) -1])

27. $mc = 0.00003$. If $c = 0.002$ then $m = 0.015$.

29. The sum represents the total population.

31. Yes, solve the equation to get the value of t when $S(t) = 2776.67$.

33. The number of susceptible people only goes down while recovered only goes up.

35. $S(2) = 29311.50$, $I(2) = 1670.43$, $R(2) = 243.065$.

37. $S(2) = 28969.25$, $I(2) = 1979.87$, $R(2) = 275.88$.

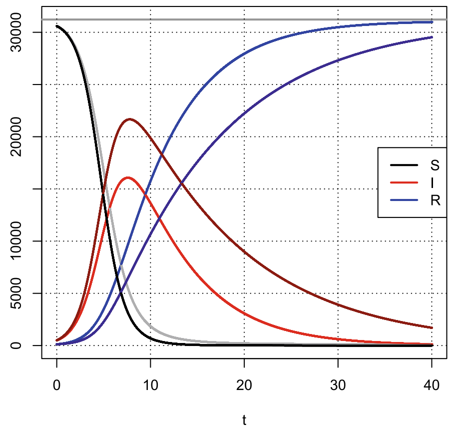
39. Based on the table we estimate $S(20) = 10.67$, $I(20) = 8983.81$, and $R(20) = 22230.52$ with step sizes of 0.0000001 and number of steps 200000000.

Step Size	Estimate S	Estimate I	Estimate R
0.1	18,517.24	2,386.87	145.90
0.1	8.91	9,043.49	22,172.60
0.01	10.49	8,989.78	22,224.73
0.001	10.65	8,984.41	22,229.94
0.0001	10.67	8,983.87	22,230.46
0.00001	10.67	8,983.82	22,230.51
0.000001	10.67	8,983.81	22,230.52
0.0000001	10.67	8,983.81	22,230.52

41. Based on the table we estimate $S(23.21) < 5$ for the first time. Step sizes of 0.00001 were used.

Time	Estimate I
23.16	5.045
23.17	5.035
23.18	5.025
23.19	5.014
23.20	5.004
23.21	4.994
23.22	4.983

43. The darker colors are the original scenario. The lighter colors are with d cut in half. The maximum of $I(t)$ is cut by about 6000 people but it occurs at the same time. Step sizes of 0.001 were used. TIP: To get these on the same graph simply comment out the plot set up when running the code the second time and change the colors in segments.



45. $\lim_{t\to\infty S(t)=0}$ In this model everyone will eventually get the disease and hence the number of susceptible will go to 0.

47. The recovered population increases by an estimated 1805.625 people per day at its fastest. To find this note that the maximum of the infected is where susceptible has an infection point. Code: (R_data[which.max(I_data) +1]-R_data[which.max(I_data) -1])/(t_data[which.max(I_data) +1]-t_data[which.max(I_data) -1])

49. $$S'(t) = -mcI(t)S(t) + \frac{I(t)}{d}$$
$$I'(t) = mcI(t)S(t) - \frac{I(t)}{d}$$

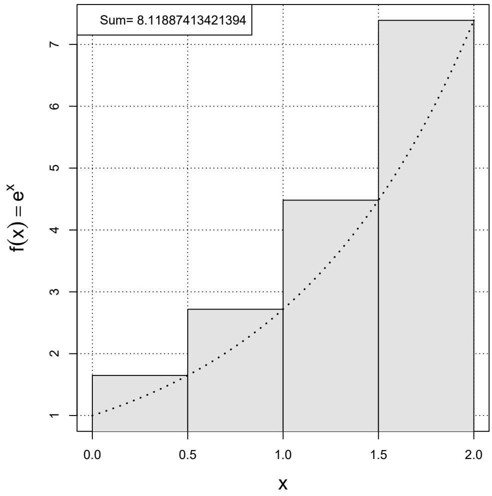
51. $$S'(t) = -mcI(t)S(t)$$
$$E'(t) = mcI(t)S(t) - \frac{E(t)}{d_e}$$
$$I'(t) = 0.5\frac{E(t)}{d_e} - \frac{I(t)}{d}$$
$$R'(t) = 0.5\frac{E(t)}{d_e} + \frac{I(t)}{d}$$

53. $$S'(t) = -mcI(t)S(t) + r(S(t) + I(t) + R(t))$$
$$I'(t) = mcI(t)S(t) - \frac{I(t)}{d}$$
$$R'(t) = \frac{I(t)}{d}$$

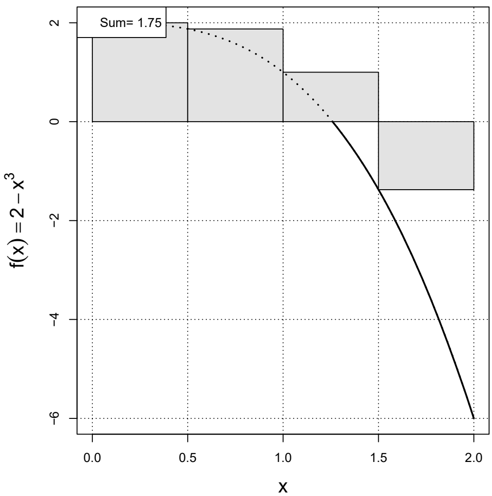
Section 28.1

1. $\Delta x = 0.5$; 5.125, 5.5, 6.125, 7; 23.75
3. $\Delta x = 0.5$, 4, 4.75, 5, 4.75; 18.5
5. $\Delta x = 0.5$; 3.4375, 4.4375, 4.9375, 4.9375, 4.4375; 22.1875
7. $\Delta x = 0.8$, -10.816, -5.184, 2.496, 12.224, 24; 22.72

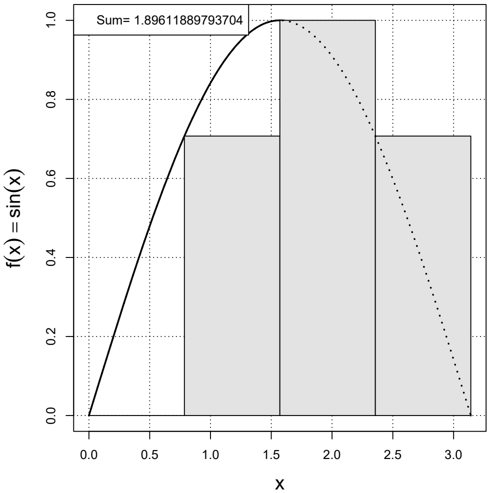
9.


Sum= 8.11887413421394
f(x) = e^x
x
0.0
0.5
1.0
1.5
2.0
1
2
3
4
5
6
7

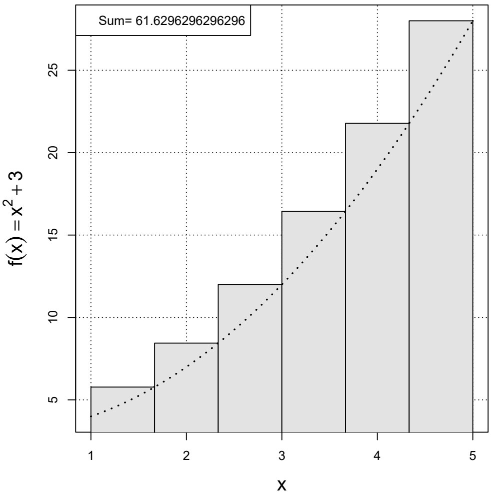
11.


Sum= 1.75
f(x) = 2 − x^3
x
0.0
0.5
1.0
1.5
2.0
2
0
−2
−4
−6

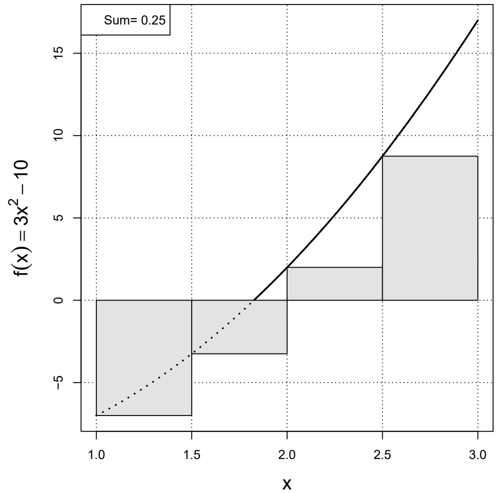
13.


Sum= 1.89611889793704
f(x) = sin(x)
0.0
0.2
0.4
0.6
0.8
1.0
0.0 0.5 1.0 1.5 2.0 2.5 3.0
x

15.


Sum= 61.6296296296296
f(x) = x^2 + 3
5
10
15
20
25
1 2 3 4 5
x

17.



$\int_0^1 2xds$ is larger since $2x > x$ on [0, 1].

19. $\int_0^{10} \sqrt{x}$ is larger since $\sqrt{x} > 0$ when $x > 0$ and this integral goes to 10 instead of just 5.

21. They are both equal to 0.

23. $\int_0^1 \sqrt{x}dx$ is larger since $\sqrt{x} > x^2$ on [0, 1].

25. $\int_{-2}^{0} xdx$ is larger or less negative since it is over a shorter interval and $x < 0$ on the intervals.

27. $\int_{-5}^{0} x^2dx$ is larger since it is positive and $\int_{-5}^{0} x^2dx$ is negative.

29. We estimate that from 5pm to 11pm 1,841,760 cubic feet of water flowed into the Cayuga Inlet on Aug 6, 2019. Note that with using left boxes we do not use the value at 11pm.

31. We estimate that from 7am to 8:30am 3,538,710 cubic feet of water flowed into the Cayuga Inlet on Aug 15, 2019. Note that with using right boxes we do not use the value at 7am.

33. 45

35. 60

37. 122

39. 95

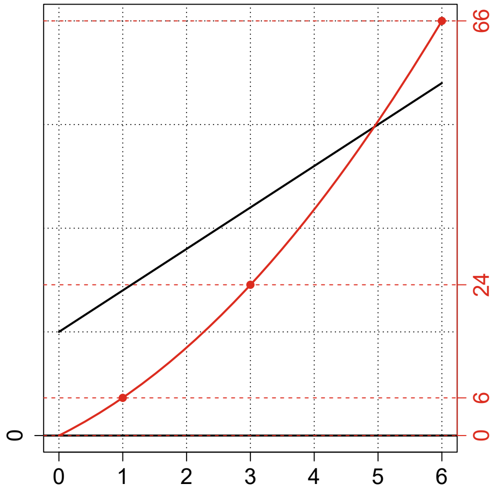
41. 794

43. 5050,500500,50005000, seems to go to infinity.

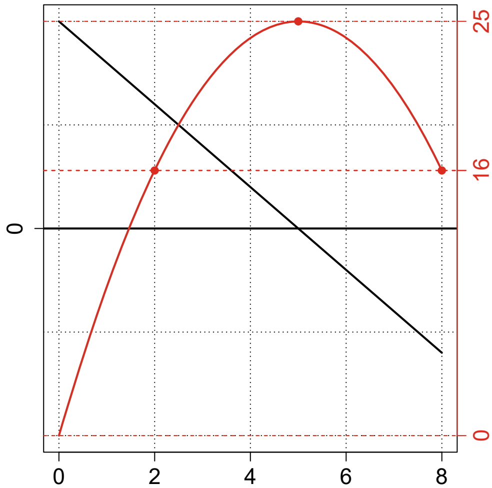
45. 5.187378, 7.485471, 9.787606, hard to tell what the limit will be but it does converge to $\pi^2/6$.

Section 29.1

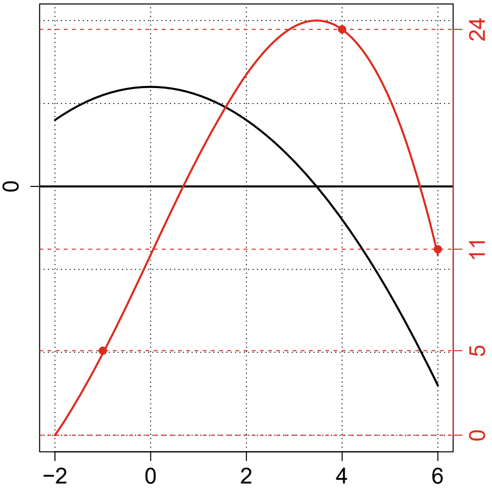
1.



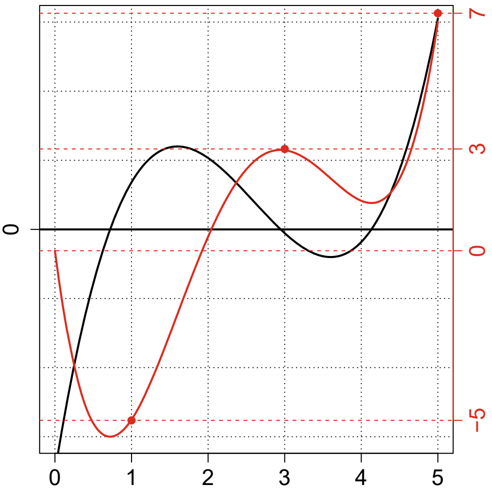
3.



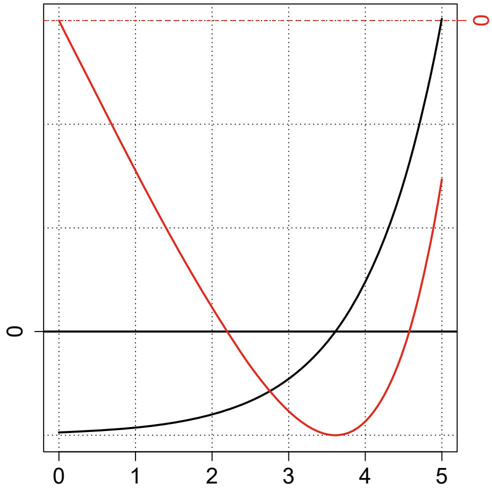
5.



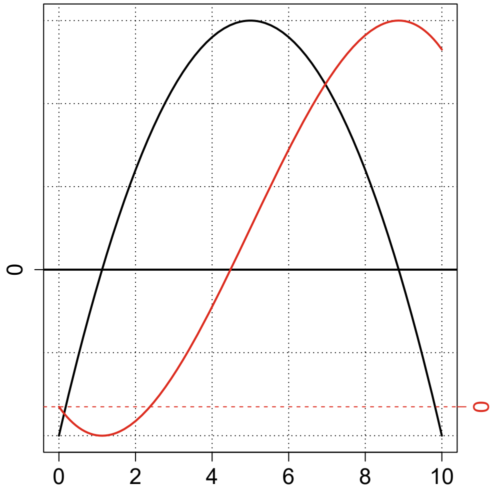
7.



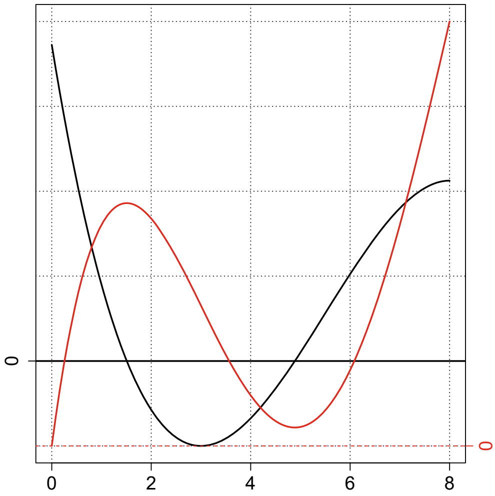
9.



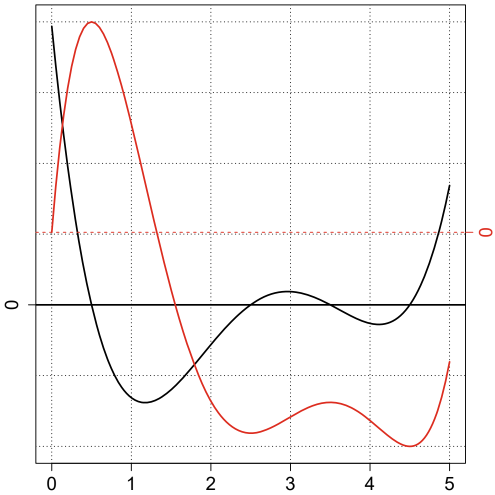
11.



13.



15.



Section 30.1

1. $\dfrac{x^5}{5} + c$

3. $\dfrac{x^2}{2} + c$

5. $-9\cos(x) + c$

7. $\dfrac{5^x}{\ln(5)} + c$

9. $\dfrac{5x^4}{4} + c$

11. $\dfrac{-1}{4x^6} + c$

13. $\dfrac{10x^{3/2}}{3} + c$

15. $\dfrac{12\sqrt{x}}{5} + c$

17. $x^3 + \dfrac{5x^2}{2} + 8x + c$

19. $\dfrac{2x^5}{5} + \dfrac{7x^3}{3} - 15x + c$

21. $4e^x - \frac{1}{4x^4} - \frac{4x^{3/2}}{3} + c$

23. $\frac{4(9^x)}{\ln(9)} - \frac{3}{7x^7} - 56\sin(x) + c$

25. $5\ln(x) - 3\cos(x) + \frac{28x^{5/4}}{5} + c$

27. $\frac{-1}{x^7} - \frac{9(5^x)}{\ln(5)} - \frac{41x^6}{3} + c$

29. $F(x) = x^3$, $F(4) - F(0) = 64$

31. $F(x) = -\cos(x)$, $F(\pi) - F(0) = 2$

33. $F(x) = 3e^x$, $F(20) - F(8) = 3(e^{20} - e^8) \approx 1.46 \times 10^9$

35. $F(x) = -7\cos(x)$, $F(\pi) - F(-\pi) = 0$

37. $F(x) = x^4/2$, $F(0) - F(-5) = -312.5$

39. $F(x) = 2\ln(x)$, $F(9) - F(1) = 2\ln(9) = \ln(81)$

41. $F(x) = \frac{3(7^x)}{\ln(7)}$, $F(1) - F(-3) = \frac{7200}{343\ln(7)}$

43. $F(x) = \frac{-1}{6x^4}$, $F(-1) - F(-10) = -3333/20000 \approx -0.167$

45. $F(x) = 2x^{3/2}$, $F(4) - F(0) = 16$

47. $F(x) = \frac{4(3^x)}{\ln(3)}$, $F(0) - F(-2) = \frac{32}{9\ln(3)} \approx 3.24$

49. $F(x) = \frac{8x^{3/2}}{3} - \frac{14\sqrt{x}}{3}$, $F(9) - F(1) = 60$

51. $F(x) = x^4/2 - 9x^3 + 42x^3 + 36x$, $F(10) - F(2) = 384$

53. $\int_0^5 0.5dt = 2.5$ cubic inches.

55. $\int_0^4 (128 - 32t)dt = 256$ feet.

57. Use R: $\int_{-1}^{1} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = 0.6826895$

59. 2978296 +125280 * 30.75 = 6830656. From Aug 6 at 6:00pm to Aug 7 at 11:30 pm approximately 6,830,656 cubic feet of water flowed into the Cayuga inlet.

61. 0.6652967676

63. $\int_0^l w dx = lw$

65. Sorry, yes it is an odd problem, but this one could be used as a project so no answer.

67. Sorry, yes it is an odd problem, but this one could be used as a project so no answer.

Section 31.1

1. $u = x^2$, $\sin(x^2) + c$

3. $u = x^2$, $e^{x^2} + c$

5. $u = x^2 + 5x - 9$, $\frac{(x^2 + 5x - 9)^{33}}{33} + c$

7. $u = 9x^2$, $\frac{-\cos(9x^2)}{6} + c$

9. $u = \sin(x)$, $\frac{2\sin^{3/2}(x)}{3} + c$

11. $u = x^3 - 7x + 11$, $3\ln(x^3 - 7x + 11) + c$

13. $u = 8 + \sin(x)$, $2\sqrt{\sin(x) + 8} + c$

15. $u = \ln(5x)$, $\frac{\ln^2(5x)}{2} + c$

17. $u = x^2 + 4$, $\frac{-1}{16(x^2 - 4)^8} + c$

19. $u = x^2$, 0

21. $u = 3x^2$, $\frac{7(e^3 - 1)}{6}$

23. $u = 1 - x$, 242/5

25. $u = x^3 + 6x - 7$, $\dfrac{-(14)^{18}}{54}$

27. $u = \cos(x)$, 0

29. $u = \ln(x^2)$, $2\ln^2(4)$

Section 32.1

1. $u = x$, $dv = e^x$, $e^x(x-1) + c$

3. $u = x$, $dv = \cos(5x)$, $\dfrac{5x\sin(5x) + \cos(5x)}{25} + c$

5. $u = 4x$, $dv = (2x+8)^{11}$, $\dfrac{(9-x)(2x+8)^{12}}{78} + c$

7. $u = \ln(x)$, $dv = 8x$, $2x^2(2\ln(x) - 1) + c$

9. $u = 5x + 9$, $dv = e^{2x}$, $\dfrac{(10x+13)e^{2x}}{4} + c$

11. $u = x^2$, $dv = e^x$, $e^x(x^2 - 2x + 2) + c$

13. $u = \ln(x)$, $dv = 1/x^2$, $\dfrac{1+\ln(x)}{x} + c$

15. $u = 2x^2$, $dv = 2xe^{x^2}$, $2e^(x^2)(x^2 - 1) + c$

17. $u = \cos(x)$, $dv = \sin(x)$, $-1/2cos^2(x) + c$

19. $u = 2x$, $dv = e^{4x}$, $\dfrac{3(5e^{48} - e^4)}{8}$

21. $u = 4x$, $dv = \cos(2x)$, -2

23. $u = 6x$, $dv = (3x+7)^9$, $\dfrac{630(16^{10}) + 67}{165}$

25. $u = \ln(x)$, $dv = 7x$, $\dfrac{7(1+e^2)}{4}$

27. $u = 9x - 8$, $dv = e^{-4x}$, $-\dfrac{23}{16} - \dfrac{157}{16e^{20}}$

29. $u = 2x^2$, $dv = 2x\cos(x^2)$, -4

Section A.1

1. $A(4)$ is in square feet. $A(4) = 8.4$ feet.

3. (6, 18)

5. $f(8)$

7. $(9, f(9))$

9. cubic feet

11. y-value

13. $x = 1$

15. Origin: bottom middle. x-axis: -8 to 8. y-axis: 0 to 100. $A = (-5, 40)$, $B = (3, 70)$, $C = (-2, 0)$.

17. Origin: center. x-axis: -20 to 20. y-axis:-50 to 50. $A = (15, -50)$, $B = (-5, -18)$, $C = (-20, 30)$.

19. Origin: bottom middle. x-axis: -5 to 5. $A = (-4, 18)$, $B = (3, 11)$, $C = (0, 2)$

21. Origin: center. x-axis: -6 to 6. $A = (-4, -160)$, $B = (2, 62)$, $C = (5, 254)$.

23. Origin: . x-axis: bottom, left of middle. $A = (0, 1)$, $B = (1, e \approx 2.718)$, $C = (2.5, e^{2.5} \approx 12.182)$.

25. Origin: left side, just below middle. x-axis: 0 to 10. $A = (0.5, \ln(0.5) \approx -0.693)$, $B = (1, 0)$, $C = (8, \ln(8) \approx 2.079)$
27. 1: 30, 2: 7
29. 1: 32, 2: 5
31. 1: 28, 2: 13
33. 1: ≈ 0.42, 2: ≈ 0.959
35. 40
37. ≈ 1.862

Section B.1

1. 1: Rule B.1, 2: Rule B.5, 3: Rule B.3
3. 1: Rule B.4, 2: Rule B.2, 3 and 4: Rule B.6
5. 1: Rule B.6 right to left, 2: Rule B.3 right to left where $a = b$, 3: Rule B.2 right to left
7. 5
9. x
11. $\dfrac{5w}{3x}$
13. $\dfrac{5x}{2y}$
15. 4xy
17. $\dfrac{45z}{x}$
19. $\dfrac{18x}{5w}$
21. $\dfrac{7z}{2x}$
23. $\dfrac{2xy + 10wz}{wy}$
25. $\dfrac{3z^2 + 8wxy}{4wz}$
27. $\dfrac{2x^2y + 3wz}{x}$
29. $\dfrac{27wz^2 + 24w^2xy}{2xz}$

Section C.1

1. Example C.1. Step C.9: $3^0 = 1$ and $\sqrt{25} = 5$. Step C.10: Moved $3^{-2}2(5)$ to the numerator by changing the sign of the exponents to $3^2 2^{-1} 5^{-1}$. Step C.11: Added exponents with the same base, $2^3 2^{-1} = 2^{3-1}$, $5^{-3}5^{-1} = 5^{-3-1}$. Step C.12: Computed the addition in the exponents, $2^{3-1} = 2^2$, $5^{-3-1} = 5^{-4}$ Step C.13: Moved 5^{-4} to the denominator by changing the sign of the exponent to 5^4. Step C.14: Computed the powers, $2^2 = 4$, $3^2 = 9$, $5^4 = 625$. Step C.15: Multiplied $4(9) = 36$.
3. Example C.3. Step C.26: Moved x^4y^{-7} to the numerator by changing the sign of the exponents to $x^{-4}y^7$. Step C.27: Combined exponents with the same base and then did the addition, $x^5x^{-3}x^{-4} = x^{5-3-4} = x^{-2}$, $y^{-7}y^7 = y^{7-7} = y^0$. Step C.28: Moved x^{-2} to the denominator by changing the sing to x^2 and $y^0 = 1$.
5. Example C.5. Step C.35: Moved $x^{10}y^5x^2z^7$ to the numerator by changing the sign of the exponents to $x^{-10}y^{-5}x^{-2}z^{-7}$. Changed z to z^1 to keep track of the exponent. Note that the 4 stays in the denominator. Step C.36: Combined exponents with the same base, $x^8x^{-10}x^{-2} = x^{8-10-2}$, $y^6y^{-5} = y^{6-5}$, $z^1z^2z^{-7} = z^{1+2-7}$. Simplified $\frac{6}{4} = \frac{3}{2}$. Step C.37: Added the exponents, $x^{8-10-2} = x^{-4}$, $y^{6-5} = y^1$, $z^{1+2-7} = z^{-5}$. Step C.38: Moved terms in the numerator with negative exponents, x^{-4} and z^{-5} to the denominator by changing the sing of the exponent to x^4 and z^5. Wrote y^1 as y by convention.

7. $\frac{1}{2}$
9. 3^5
11. $4^0 = 1$
13. $\frac{1}{2^{18}}$
15. 8^3
17. 5^{12}
19. $\frac{1}{2^2}$
21. $\frac{5^{13}}{3^{10}}$
23. $\frac{5^8}{8^{13}}$
25. 7^9
27. $\frac{2^{25}}{5^7}$
29. $\frac{1}{3^{16}}$
31. $\frac{2^{15}}{3^3}$
33. $\frac{5^4}{3^5}$
35. $\frac{5^2}{3^7 7^{16}}$
37. $(2)3^8 7^2$
39. $3^8 5^3 \sqrt[3]{7}$
41. $3^2 7^2 \sqrt[5]{8}$
43. $2^6 5^8 \sqrt{11}$
45. $2^{10} 3^8 7^{13} \sqrt[3]{64}$
47. x^{-5}
49. $3y^{-8}$
51. $\frac{x^{-8}}{3}$
53. $\frac{2z^{-5}}{3}$
55. $\frac{2y^{-1}}{15}$
57. $\frac{1}{2}$
59. $\frac{5x^{-2}y^{10}}{36}$
61. $\frac{16x^7z^{-6}}{45}$
63. $\frac{24x^{14}w}{11}$
65. $\frac{5x^{-7}y^{-2}}{21}$
67. $\frac{3}{2x^2}$
69. $\frac{3}{2y}$
71. $\frac{35z^{19}}{16}$
73. $\frac{x^9}{5}$
75. $\frac{6w^{41/6}}{7}$
77. $\frac{3y^{53/4}}{4}$
79. $\frac{z^{149/5}}{2}$
81. $\frac{1}{3z^{620/9}}$
83. $\frac{y^{15}}{x^{10}}$
85. $\frac{w^{15}}{z^4}$
87. $6y^3z^{10}$
89. $\frac{y}{5x^{17}}$
91. $\frac{z^{11/3}}{w^{24}}$
93. $y^{40}z^{21/2}$
95. $\frac{x^8y^{17}}{3z^6}$
97. $\frac{2w^2z^{14}}{x^{66}}$
99. $\frac{z^{295/4}}{3w^6z^{107/5}}$
101. $\frac{4y^{38}z^2}{w^{129/7}}$
103. $\frac{w^9x^5y^{15}}{z^{16}}$
105. $\frac{w^{434/3}x^7}{4y^{38}z^{79/60}}$

Section D.1

1. $y = -2x + 2$ Yes, this is the same as it should be.
3. a and s represent unknown constants whereas x is a variable. $y(1)$ does make sense and $y(1) = s - sa + f(a)$.
5. a represents and unknown constant whereas x is a variable.
7. $y = -12x - 15$, $(4, -63)$, $(-7/6, -1)$
9. $y = 7x - 101$, $(4, -87)$, $(101/7, 0)$
11. $y = 18x + 222$, $(16, 114)$, $(-205/18, 17)$
13. $y = -5x + 66$, $(16, -14)$, $(49/5, 17)$
15. $y = \frac{-3}{4}x + \frac{3}{2}$, $(9, -57/4)$, $(-42/3, 3)$
17. $y = \frac{22}{3}x - \frac{394}{3}$, $(-6, -526/3)$, $(358/22, -12)$
19. $y = \frac{7}{16}x - \frac{206}{16}$, $(-11, -283/16)$, $(94/7, -7)$
21. $y = -4x + 32$, $(-5, 52)$, $(15/4, 17)$
23. $y = \frac{N-e}{A-s}x + \frac{AN-Ae}{A-s} + e$
25. $y = \frac{z-O}{-t-m}x - \frac{mz-mO}{-t-m} + O$
27. $y = \frac{T-Y}{U-o}x - \frac{oT-oY}{U-o} + Y$

29. $y = \frac{-H-Q}{-Y-d}x - \frac{-dH-dQ}{-Y-d} + Q$

31. $y = ox - oj + t$, $(v, ov - oj + t)$, $(\frac{H-t+oj}{o}, H)$

33. $y = zx - zp + b$, $(n, zn - zp + b)$, $(\frac{w-b+zp}{z}, w)$

35. $y = f'(D)x - f'(D)D + f(D)$, $(r, f'(D)r - f'(D)D + f(D))$, $(\frac{M-f(D)+f'(D)D}{f'(D)}, M)$

37. $y = g'(c)x - g'(c)c + g(c)$, $(J, g'(c)J - g'(c)c + g(c))$, $(\frac{U-g(c)+g'(c)c}{g'(c)}, U)$

Section E.1

1. $f(2) = 2^2 + 2(2) - 8 = 0$ and $f(-4) = (-4)^2 + 2(-4) - 8 = 0$
3. Expanding $(x + d)(x^2 - dx + d^2)$ will yield $x^3 + d^3$
5. $x^2 - x - 72$
7. $y^2 + 15y + 56$
9. $4x^2 + 26x + 30$
11. $7z^2 - 16z - 21$
13. $x^2 - 16x + 64$
15. $y^2 + 20y + 100$
17. $x^3 + 12x^2 + 48x + 64$
19. $x^3 - 30x^2 + 300x - 1000$
21. $4x^2 + 24x + 36$
23. $h^2 + 2hx + x^2$
25. $x^3 + 3x^2 - 27x + 7$
27. $5x^3 + 4x^2 + 18x - 36$
29. $9x^3 - 29x^2 + 87x - 18$
31. $x = -2$, $x = -4$
33. $x = -9$, $x = 1$
35. $x = -7$, $x = 2$
37. $x = 5$, $x = 7$
39. $x = -1$, $x = 5$
41. $x = -10$, $x = 2$
43. $x = -6$, $x = -4$
45. $x = -1$, $x = 4$
47. $x = -11$, $x = -3$
49. $x = 6$, $x = 12$
51. $x = -11$, $x = 10$
53. $x = -10$, $x = -4$
55. $x = 5$, $x = -5$
57. $x = -1$, $x = 4$
59. $x = -9$, $x = 10$
61. $x = -13$, $x = 4$
63. $x = -8$, $x = 7$
65. $x = -8$, $x = -4$
67. $x = -9$, $x = 5$
69. $x = -5$, $x = 9$
71. $x = -3$, $x = -13$
73. $x = 5$, $x = 18$

Section F.1

1. $f(g(x)) = x^2 + 3$
   $g(f(x)) = (x + 3)^2$
3. $f(g(x)) = x^2 + 2$
   $g(f(x)) = (x - 6)^2 + 8$
5. $f(g(x)) = 6x^2 - 4x + 12$
   $g(f(x)) = 6(x + 7)^2 - 4x - 23$
7. $f(g(x)) = 5x^2 - 2x - 17$
   $g(f(x)) = 5(x - 3)^2 - 2x - 20$
9. $f(g(x)) = (x^2 - 7x + 6)^2 + 4x^2 - 28x + 36$
   $g(f(x)) = (x^2 + 4x + 12)^2 - 7x^2 - 28x - 78$
11. $f(g(x)) = e^{x^2+2x-4}$
    $g(f(x)) = e^{2x} + 2e^x - 4$
13. $f(g(x)) = e^{2x^3+10x+16}$
    $g(f(x)) = e^{6x} + 5e^{2x} + 8$
15. $f(g(x)) = x$
    $g(f(x)) = x$
17. $f(g(x)) = \sin(x^2 + 3x - 12)$
    $g(f(x)) = \sin^2(x) + 3\sin(x) - 12$
19. $f(g(x)) = \sin(e^{2x^2})$
    $g(f(x)) = e^{2\sin^2(x)}$

21. $f(g(x)) = \cos(\sin(x))$
$g(f(x)) = \sin(\cos(x))$

23. $f(g(x)) = \tan(\ln(x))$
$g(f(x)) = \ln(\tan(x))$

25. $f(g(x)) = \sqrt{e^{3x^2}}$
$g(f(x)) = e^{3x}$ for $x > 0$

27. $f(g(x)) = \sqrt{2x^4 + 5x^2 + 3}$
$g(f(x)) = 2x^2 + 5x + 3$ for $x > 0$

29. $f(g(x)) = \sqrt{\sin(x) + \cos(x)}$
$g(f(x)) = \sin(\sqrt{x}) + \cos(\sqrt{x})$

31. $g(f(x)) = (x + h)^2 - 2$

33. $g(f(x)) =$
$3(x + h)^2 - 5(x + h) + 9$

35. $g(f(x)) = 4(x + h)^3$

37. $g(f(x)) = 2(x + h)^3 - 3(x + h)^2 + x + h - 7$

39. $f(x) = x^2 + 3x + 8$
$g(x) = x + 7$

41. $f(x) = 5x^2 - 2x + 3$
$g(x) = x^2 + 3x$

43. $f(x) = e^x$
$g(x) = 5x^2 + 2x$

45. $f(x) = x^{11}$
$g(x) = x^2 + \sin(x)$

47. $f(x) = \sin(x)$
$g(x) = 5x^2 + 3x - 2$

49. $f(x) = \sqrt{x}$
$g(x) = -3x^3 + 5x - 9$

51. $f(x) = \ln(x)$
$g(x) = 5x + \sin(x)$

53. $h(g(f(x))) =$
$(x^2 - 5)^6 - (x^2 - 5)^3 + 3$

55. $h(g(f(x))) = e^{7x^2 - 7x + 14}$

57. $h(g(f(x))) = e^{5(x+h)^2}$

59. $h(g(f(x))) = e^{3\cos^2(x) - 7\cos(x) + 9}$

61. $f(x) = 4x^2 + 6x - 9$
$g(x) = x - 8$
$h(x) = x^2$

63. $f(x) = e^x$
$g(x) = x^2 + 42$
$h(x) = \sin(x)$

65. $f(x) = x^5$
$g(x) = \sin(x) - \cos(x)$
$h(x) = x^2$

67. $f(x) = \ln(x)$
$g(x) = \sin(x)$
$h(x) = x^2$

Appendix I
R Code for Figures

R Code: Figure 2.2

```
> par(mar=c(5,5,2,2))
> f<-function(x){sin(x)}
> curve(f,0,2*pi,lwd=2,xaxt="n",xlab="x",xaxs="i",ylab=expression(sin(x)),
cex.axis=1.5,cex.lab=2)
> grid(NA,NULL,col="black")
> v1<-c(0,pi/2,pi,3*pi/2,2*pi)
> v2<-c("0",expression(pi/2),expression(pi),expression(3*pi/2),
expression(2*pi))
> axis(side=1,lwd=3,at=v1,labels=v2,cex.axis=1.5,cex.lab=1.5)
> abline(v=v1,lty=3)
> abline(h=0)
```

R Code: Figure 4.1

```
> par(mar=c(5,5,2,2))
> CO2<-function(t){0.0134594696825104*t^2+0.520632601928747*t+
310.423363171355}
> curve(CO2,0,100,lwd=2,xlab="Years After 1950",ylab="Yearly Average
CO2 (ppm)",cex.axis=1.5,cex.lab=1.5 )
> grid(NULL,NULL,col="black")
```

R Code: Figure 4.2

```
> par(mar=c(5,5,2,2))
> CO2<-function(t){0.0134594696825104*t^2+0.520632601928747*t+
310.423363171355}
> curve(CO2,0,100,lwd=2,xlab="Years After 1950", ylab="Yearly Average
CO2 (ppm)",cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> a<-0
> b<-67
```


T. J. Pfaff, *Applied Calculus with R*,
https://doi.org/10.1007/978-3-031-28571-4

```
> segments(a,CO2(a),b,CO2(b),lty=2,lwd=2)
> CO2.p=Deriv(CO2)
> CO2.tan.67=function(x){CO2.p(67)*(x-67)+CO2(67)}
> curve(CO2.tan.67,34,100, lwd=2,lty=5, col="red",add=TRUE)
> points(b,CO2(b),cex=1.25,pch=16)
> points(a,CO2(a),cex=1.25,pch=15)
```

R Code: Figure 4.3

```
> par(mar=c(5,5,2,2))
> a<-67
> h<-0.01
> curve(CO2,a-h,a+h, lwd=4,xlab="Years After 1950", ylab="Yearly Average
 CO2 (ppm)",cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> curve(CO2.tan.67,34,100, lwd=4,lty=3,col="red",add=TRUE)
```

R Code: Figure 5.1

```
> par(mar=c(5,5,2,2))
> G<-function(t){142*exp(-t/2.5)+70}
> plot(G,xlim=c(0,10),ylim=c(0,220),lwd=3,ylab="G(t)",xlab="t",xaxs="i",
yaxs="i",yaxt="n",cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> axis(2,label=c(0,70,212), at=c(0,70,212),cex.axis=1.5)
> abline(h=70)
>
> G_p=Deriv(G)
> f=function(x){G_p(3)*(x-3)+G(3)}
> curve(f,1.5,4.5,lwd=3,col="red",add=TRUE)
> text(3,G(3),expression(m==G^{~bold("/")}*(3)),col="red",srt=-40,pos=1,
cex=1.5)
> points(3,G(3),pch=16,cex=1.5)
>
> g<-function(x){G_p(6)*(x-6)+G(6)}
> curve(g,4.5,7.5,lwd=3,col="red",add=TRUE)
> text(6,G(6),expression(m==G^{~bold("/")}*(6)),col="red",srt=-10,pos=3,
cex=1.5)
> points(6,G(6),pch=16,cex=1.5)
```

R Code: Figure 7.1

```
> par(mar=c(3,4,2,2))
> f<-function(x){x^2+5}
> curve(f,-6,6,lwd=2,ylim=c(0,30),xlab="",ylab="",xaxs="i",yaxs="i",
yaxt="n",xaxt="n",cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> g<-function(x){4*x+1} #at 2
> curve(g,-5,5,lwd=2,col="red2",add=TRUE)
> axis(1,at=c(2,4),label=c("a","b"),cex.axis=1.5)
> points(c(2,4),c(f(2),g(4)),cex=1.5,col="red2",pch=16)
> points(c(4),c(f(4)),cex=1.5,pch=16)
> axis(2,at=c(f(2),f(4)),label=c(expression(f(a)),expression(f(b))),
cex.axis=1.5,las=1)
> text(4.75,f(4.75),expression(f(x)),cex=1.5,pos=4)
> text(3,g(3),expression( m==f^{" /"}*(a)),pos=4,col="red2",cex=1.5)
```

R Code: Figure 8.1

```
> par(mar=c(5,5,2,2))
> a<-67
> h<-0.01
> CO2_p<-Deriv(CO2)
> CO2_tan_67<-function(x){CO2.p(67)*(x-67)+CO2(67)}
> curve(CO2,a-h,a+h, lwd=4,xlab="Years After 1950",
ylab="Yearly Average CO2 (ppm)",cex.axis=1.5,cex.lab=1.5 )
> grid(NULL,NULL,col="black")
> curve(CO2.tan.67,34,100, lwd=4,lty=3, col="red",add=TRUE)
> points(c(a+h,a-h),c(CO2(a+h),CO2(a-h)),pch=15,cex=1.75)
```

R Code: Figure 8.2

```
> par(mar=c(5,5,2,2))
> f<- function(x){(exp(x)+(2.5)^x*sin(2*pi*x))-10}
> name<-expression(f(x)==e^x+(2.5)^x*sin(2*pi*x)-10)
> a<-4.30931
> curve(f, 4.25,4.35,lwd=2, ylab=name,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> points(a,f(a),pch=16,cex=1.5)
> axis(1,line=1,at=a,label=paste("a=",round(a,3),sep=""),
cex.axis=1.15)
> segments(a-0.01,f(a-0.01),a+0.01,f(a+0.01),col="blue2",lwd=3)
> slope<-(f(a+0.01)-f(a-0.01))/(0.02)
> text(a,f(a+0.01),paste("m=",round(slope,8)),pos=1,cex=1.25)
```

R Code: Figure 8.3

```
> name<-expression(f(x)==e^x+(2.5)^x*sin(2*pi*x)-10)
> a<-4.30931
> par(mar=c(5,5,2,2))
> curve(f, a-1,a+.75,lwd=2, ylab=name,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> abline(v=a,lwd=2,lty=3)
> axis(1,at=a,label=paste("a=",round(a,3),sep=""),cex.axis=1.15)
> hp=seq(1,1/1000,length=50)
> hm=seq(-1,-1/1000,length=50)
> point1=hm[1]
> point2=hm[40]
> point3=hp[18]
> point4=hp[42]
> segments(a+point1,f(a+point1),a,f(a),lty=2,col="blue",lwd=2)
> segments(a+point2,f(a+point2),a,f(a),lty=2,col="blue",lwd=2)
> segments(a+point3,f(a+point3),a,f(a),lty=2,col="red",lwd=5)
> segments(a+point4,f(a+point4),a,f(a),lty=2,col="red",lwd=5)
```

R Code: Figure 8.4

```
> f<-function(x){(exp(x)+(2.5)^x*sin(2*pi*x))-10}
> a<-4.30931
> hp<-seq(1,1/1000,length=50)
> hm<-seq(-1,-1/1000,length=50)
> par(mar=c(4,6,3,3))
> plot(hp,(f(a+hp)-f(a))/hp,col="red",pch=2,xlim=c(-1,1),ylim=c(-225,225),
xlab="Value of h",ylab=expression(frac(f(a+h)-f(a),h)),cex.axis=1.5,
cex.lab=1.5)
> grid(NULL,NULL,col="black")
> points(hm,(f(a+hm)- f(a))/hm,col="blue",pch=1)
> arrows(-.75, -285, -0.05, -285, length=.15, lwd=2, xpd = TRUE)
> arrows(.75, -285, 0.05, -285,code=2, length=.15, lwd=2, xpd = TRUE)
> point1=hm[1]
> point2=hm[40]
> point3=hp[18]
> point4=hp[42]
> arrows(point2-0.05,(f(a+point2)-f(a))/point2,-0.05,0,angle=25,lwd=2,
length=.15)
> arrows(point4+0.085,(f(a+point4)-f(a))/point4,0.05,0,code=2,angle=25,
lwd=2,length=.15)
> points(point1,(f(a+point1)- f(a))/point1, col="blue",pch=16,cex=1.25)
> points(point2,(f(a+point2)- f(a))/point2, col="blue",pch=16,cex=1.25)
> points(point3,(f(a+point3)- f(a))/point3, col="red",pch=17,cex=1.25)
> points(point4,(f(a+point4)- f(a))/point4, col="red",pch=17,cex=1.25)
```

R Code: Figure 8.5

```
> par(mar=c(5,5,2,2))
> f<-function(x){x*sin(x)}
> name<-expression(f(x)==x*sin(x))
> a<- pi
> curve(f, a-pi/2,a+pi/2,lwd=3, ylab=name,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
> points(a,f(a),pch=15,cex=1.5)
> segments(a-1,f(a-1),a,f(a),lty=2,col="blue",lwd=2)
> segments(a+1,f(a+1),a,f(a),lty=2,col="red",lwd=5)
```

R Code: Figure 9

```
> f<-function(x){x^2}
> f_p<-Deriv(f)
> f_tan<-function(a,x){f.p(a)*(x-a)+f(a)}
> par(mar=c(2,2.5,1,1))
> curve(f, -10, 10,lwd=3,cex.axis=1.5,xlab="",ylab="")
> grid(NULL,NULL,col="black")
> curve(f_tan(-6,x),-10,-2,add=TRUE,lwd=3,lty=2,col="red")
> text(-5.9,36,"P1=(-6,36)",pos=4)
> points(-6,f(-6),pch=16,cex=1.25)
> curve(f_tan(-3,x),-7,1, add=TRUE,lwd=3,lty=2, col="red")
> text(-3.5,12,"P2=(-3,9)",pos=4)
> points(-3,f(-3),pch=16,cex=1.25)
> curve(f_tan(0,x),-6,6,add=TRUE,lwd=3,lty=2)
> text(0,0,"P3=(0,0)",pos=3)
> points(0,f(0),pch=16,cex=1.25)
> curve(f_tan(2,x),-2,6,add=TRUE,lwd=3,lty=2,col="blue")
> text(1.25,5,"P4=(2,4)",pos=3)
> points(2,f(2),pch=16,cex=1.25)
> curve(f_tan(7,x),3,10,add=TRUE,lwd=3,lty=2, col="blue")
> text(7,49,"P5=(7,49)",pos=2)
> points(7,f(7),pch=16,cex=1.25)
```

R Code: Figure 9.2

```
> a=-6
> b=-3
> c=0
> d=2
> e=7
> par(mar=c(2,2.5,1,1))
> curve(f_p,-10, 10,lwd=3,cex.axis=1.5,xlab="",ylab="")
> grid(NULL,NULL,col="black")
> text(a,f_p(a),paste("p1=(",a,",",f_p(a),")",sep=""),pos=2)
> points(a,f_p(a),pch=16,cex=1.25,col="red")
> text(b,f_p(b),paste("p2=(",b,",",f_p(b),")",sep=""),pos=2)
> points(b,f_p(b),pch=16,cex=1.25,col="red")
> text(c,f_p(c),paste("p3=(",c,",",f_p(c),")",sep=""),pos=2)
> points(c,f_p(c),pch=16,cex=1.25,col="black")
> text(d,f_p(d),paste("p4=(",d,",",f_p(d),")",sep=""),pos=2)
```

```
> points(d,f_p(d),pch=16,cex=1.25,col="blue")
> text(e,f_p(e),paste("p5=(",e,",",f_p(e),")",sep=""),pos=2)
> points(e,f_p(e),pch=16,cex=1.25,col="blue")
```

R Code: Figure 9.3

```
> par(mfrow=c(2,1))
> par(mar=c(2,2.5,1,1))
> curve(f, -10, 10,lwd=3,cex.axis=1.25,xlab="",ylab="")
> grid(NULL,NULL,col="black")
> curve(f_tan(-6,x),-10,-2,add=TRUE,lwd=3,lty=2,col="red")
> text(-5.9,36,"P1=(-6,36)",pos=4)
> points(-6,f(-6),pch=16,cex=1.25)
> curve(f_tan(-3,x),-7,1, add=TRUE,lwd=3,lty=2, col="red")
> text(-3.5,12,"P2=(-3,9)",pos=4)
> points(-3,f(-3),pch=16,cex=1.25)
> curve(f_tan(0,x),-6,6,add=TRUE,lwd=3,lty=2)
> text(0,0,"P3=(0,0)",pos=3)
> points(0,f(0),pch=16,cex=1.25)
> curve(f_tan(2,x),-2,6,add=TRUE,lwd=3,lty=2,col="blue")
> text(1.25,5,"P4=(2,4)",pos=3)
> points(2,f(2),pch=16,cex=1.25)
> curve(f_tan(7,x),3,10,add=TRUE,lwd=3,lty=2, col="blue")
> text(7,49,"P5=(7,49)",pos=2)
> points(7,f(7),pch=16,cex=1.25)
>
> abline(v=a,col="red",xpd=TRUE)
> abline(v=b,col="red",xpd=TRUE)
> abline(v=c,col="black",xpd=TRUE)
> abline(v=d,col="blue",xpd=TRUE)
> abline(v=e,col="blue",xpd=TRUE)
>
> par(mar=c(2,2.5,1,1))
> curve(f_p,-10, 10,lwd=3,cex.axis=1.5,xlab="",ylab="")
> grid(NULL,NULL,col="black")
> text(a,f_p(a),paste("p1=(",a,",",f_p(a),")",sep=""),pos=2)
> points(a,f_p(a),pch=16,cex=1.25,col="red")
> text(b,f_p(b),paste("p2=(",b,",",f_p(b),")",sep=""),pos=2)
> points(b,f_p(b),pch=16,cex=1.25,col="red")
> text(c,f_p(c),paste("p3=(",c,",",f_p(c),")",sep=""),pos=2)
> points(c,f_p(c),pch=16,cex=1.25,col="black")
> text(d,f_p(d),paste("p4=(",d,",",f_p(d),")",sep=""),pos=2)
> points(d,f_p(d),pch=16,cex=1.25,col="blue")
> text(e,f_p(e),paste("p5=(",e,",",f_p(e),")",sep=""),pos=2)
> points(e,f_p(e),pch=16,cex=1.25,col="blue")
>
> abline(v=a,col="red",xpd=TRUE)
> abline(v=b,col="red",xpd=TRUE)
> abline(v=c,col="black",xpd=TRUE)
> abline(v=d,col="blue",xpd=TRUE)
> abline(v=e,col="blue",xpd=TRUE)
```

R Code: Figure 9.4

```
> f<-function(x){.1*(x^3/3+1*x^2-15*x)-4}
> l=<- -10
> r<-10
> a<- -9
> b<- -6
> c<- -1
> d<-3
> e<-7
> g<-3.5
>
> f_p<-Deriv(f)
> f_tan<-function(w,x){f_p(w)*(x-w)+f(w)}
> par(mfrow=c(2,1))
> par(mar=c(2,2,1,1))
> curve(f, l, r,lwd=3,cex.axis=1.25)
> grid(NULL,NULL,col="black")
>
> abline(v=a,col="blue",xpd=TRUE)
> abline(v=b,col="blue",xpd=TRUE)
> abline(v=c,col="red",xpd=TRUE)
> abline(v=d,col="black",xpd=TRUE)
> abline(v=e,col="blue",xpd=TRUE)
> abline(h=0,col="black")
>
> curve(f_tan(a,x),a-g,a+g,add=TRUE,lwd=3,lty=2,col="blue")
> text(a,f(a),paste("P1=(",a,",",round(f(a),1),")",sep=""),pos=4)
> points(a,f(a),pch=16,cex=1.25)
> curve(f_tan(b,x),b-g,b+g,add=TRUE,lwd=3,lty=2,col="blue")
> text(b+1,f(b),paste("P2=(",b,",",round(f(b),1),")",sep=""),pos=1)
> points(b,f(b),pch=16,cex=1.25)
> curve(f_tan(c,x),c-g,c+g,add=TRUE,lwd=3,lty=2,col="red")
> text(c,f(c),paste("P3=(",c,",",round(f(c),1),")",sep=""),pos=4)
> points(c,f(c),pch=16,cex=1.25)
> curve(f_tan(d,x),d-g,d+g,add=TRUE,lwd=3,lty=2,col="black")
> text(d-0.5,f(d)-0.5,paste("P4=(",d,",",round(f(d),1),")",sep=""),pos=1)
> points(d,f(d),pch=16,cex=1.25)
> curve(f_tan(e,x),e-g,e+g,add=TRUE,lwd=3,lty=2,col="blue")
> text(e,f(e),paste("P5=(",e,",",round(f(e),1),")",sep=""),pos=2)
> points(e,f(e),pch=16,cex=1.25)
> curve(f_p,-10,10,lwd=3,cex.axis=1.25)
> grid(NULL,NULL,col="black")
>
> abline(v=a,col="blue",xpd=TRUE)
> abline(v=b,col="blue",xpd=TRUE)
> abline(v=c,col="red",xpd=TRUE)
> abline(v=d,col="black",xpd=TRUE)
> abline(v=e,col="blue",xpd=TRUE)
> abline(h=0,col="black")
>
> text(a,f_p(a),paste("p1=(",a,",",f_p(a),")",sep=""),pos=4)
> points(a,f_p(a),pch=16,cex=1.25,col="blue")
> text(b+.2,f_p(b),paste("p2=(",b,",",f_p(b),")",sep=""),pos=2)
> points(b,f_p(b),pch=16,cex=1.25,col="blue")
> text(c,f_p(c),paste("p3=(",c,",",f_p(c),")",sep=""),pos=3)
> points(c,f_p(c),pch=16,cex=1.25,col="red")
> text(d,f_p(d)-0.25,paste("p4=(",d,",",f_p(d),")",sep=""),pos=4)
```

```
> points(d,f_p(d),pch=16,cex=1.25,col="black")
> text(e,f_p(e),paste("p5=(",e,",",f_p(e),")",sep=""),pos=2)
> points(e,f_p(e),pch=16,cex=1.25,col="blue")
```

R Code: Example 9.1

```
> f<-function(x){exp(x)+(2.5)^x*sin(2*pi*x)-10}
> f_p<-Deriv(f)
> f_tan<-function(w,x){f_p(w)*(x-w)+f(w)}
> par(mar=c(2,2.5,1,1))
> curve(f,5,7,lwd=2,cex.axis=1.5)
> grid(NULL, NULL,col="black",lwd=2)
> curve(f_tan(6,x),lwd=2,col="red",add=TRUE)
> points(6,f(6), cex=1.25,pch=16)
```

R Code: Figure 11.1

```
> par(mar=c(4,5,1,1))
> f<-function(a,x){2*a*(x-a)+a^2 }
> g<-function(a,x){5*2*a*(x-a)+5*a^2}
> curve(x^2,-7,8,ylim=c(0,300),lwd=2, xaxs="i", yaxs="i",
ylab=expression(paste(x^2, " and  ", 5~x^2)),cex.axis=1.5,cex.lab=1.5 )
> curve(5*x^2,-7,8,lwd=2,add=TRUE)
> abline(h=seq(0,300,by=50),v=-7:8,lty=3,col="grey50")
> axis(1,at=c(-7:8), label=c(-7:8),cex.axis=1.5)
> b=2.5
> curve(f(-4,x), -4-b, -4+b,lty=5, lwd=4, add=TRUE,col="red2")
> curve(f(6,x), 6-b-.5, 6+b,lty=5, lwd=4, add=TRUE,col="blue2")
> curve(g(-4,x), -4-b, -4+b,lty=5, lwd=4, add=TRUE, col="red2")
> curve(g(6,x), 6-b, 6+b,lty=5, lwd=4, add=TRUE, col="blue2")
> points(c(-4,-4),c(f(-4,-4),g(-4,-4)), pch=16,cex=1.5,col="red2")
> points(c(6,6),c(f(6,6),g(6,6)), pch=16,cex=1.5,col="blue2")
> abline(v=c(-4,6),lty=3,lwd=2)
> abline(v=0)
> text(6,f(6,6), "          m=12",pos=1,cex=1.5)
> text(6,g(6,6), "m=60 ",pos=2, cex=1.5)
> text(-4,f(-4,-4), "m= -8    ",pos=2,cex=1.5)
> text(-4,g(-4,-4), " m= -40 ",pos=4, cex=1.5)
```

R Code: Figure 11.2

```
> library(Deriv)
> f<-function(x){12*x}
> g<-function(x){x^2}
> h<-function(x){f(x)+g(x)}
> fp<-Deriv(f)
> gp<-Deriv(g)
> hp<-Deriv(h)
> f_tan<-function(a,x){fp(a)*(x-a)+f(a) }
> g_tan<-function(a,x){gp(a)*(x-a)+g(a) }
```

```
> h_tan<-function(a,x){hp(a)*(x-a)+h(a) }
> min_x<-0
> max_x<-10
> min_y<-0
> max_y<-200
> par(mar=c(4,2.5,1,1))
> curve(f,min_x,max_x,ylim=c(min_y,max_y),col="red2",lwd=4,ylab="",
cex.axis=1.5,cex.lab=1.5 )
> curve(g,min_x,max_x,lwd=4,add=TRUE,col="blue2")
> curve(h,min_x,max_x,lwd=4,add=TRUE)
> grid(NULL,NULL, col="black")
> names=c(expression(h(x)==x^2+12*x),expression(f(x)==12*x),
expression(g(x)==x^2))
> x1<-2
> b<-2.5
> curve(f_tan(x1,x)-1.25, x1-b, x1+b,lty=4, lwd=5, add=TRUE,col="#984ea3")
> curve(g_tan(x1,x), x1-b, x1+b,lty=5, lwd=4, add=TRUE,col="#984ea3")
> curve(h_tan(x1,x), x1-b, x1+b,lty=5, lwd=4, add=TRUE,col="#984ea3")
> points(c(x1,x1,x1),c(f(x1),g(x1),h(x1)),cex=1.5,pch=16,col="#984ea3")
> m1<-fp(x1)
> m2<-gp(x1)
> m3<-hp(x1)
> text(x1+.3,f(x1), bquote( "      m"==.(m1)),pos=1,cex=1.5)
> text(x1,g(x1), bquote( "m"==.(m2) ),pos=1,cex=1.5)
> text(x1-.3,h(x1), bquote( "m"==.(m3)~"      " ),pos=3,cex=1.5)
> x2<-8
> b<-2.5
> curve(f_tan(x2,x)-1.25, x2-b, x2+b,lty=5, lwd=4, add=TRUE,col="#984ea3")
> curve(g_tan(x2,x), x2-b, x2+b,lty=5, lwd=4, add=TRUE,col="#984ea3")
> curve(h_tan(x2,x), x2-b, x2+b,lty=5, lwd=4, add=TRUE,col="#984ea3")
> points(c(x2,x2,x2),c(f(x2),g(x2),h(x2)),cex=1.5,pch=16,col="#7570b3")
> m1<-fp(x2)
> m2<-gp(x2)
> m3<-hp(x2)
> text(x2+.3,f(x2), bquote( "m"==.(m1) ),pos=1,cex=1.5)
> text(x2+.3,g(x2), bquote( "       m"==.(m2) ),pos=1,cex=1.5)
> text(x2-.4,h(x2), bquote( "m"==.(m3)~"     " ),pos=3,cex=1.5)
> abline(v=c(x1,x2),lty=3,lwd=3)
> legend("topleft",names, lty=c(1,1,1), lwd=c(3,3,3),
col=c("black","red2","blue2"),bg="white",cex=1.2)
```

R Code: Figure 11.3

```
> f<-function(x){x^2}
> g<-function(x){x^2+10}
> h<-function(x){10*x^2}
> par(mar=c(4,2.5,1,1))
> curve(f,-4,4,ylim=c(0,80),lwd=4,ylab="",cex.axis=1.5,
cex.lab=1.5 )
> curve(g,-4,4,lwd=4,add=TRUE,col="blue2")
> curve(h,-4,4,lwd=4,col="red2",add=TRUE)
> grid(NULL,NULL, col="black")
> names=c(expression(f(x)==x^2),expression(g(x)==x^2+10),
expression(h(x)==10*x^2))
> legend("top",names, lty=c(1,1,1), lwd=c(3,3,3),
```

```
col=c("black","red2","blue2"),bg="white",cex=1.2)
```

R Code: Figure 12.1

```
> f<-function(x){x^2*sin(x)}
> g<-function(x){x^2}
> a<-0
> b<-12*pi
> par(mar=c(3,3,2,2))
> curve(f,a,b,lwd=2,xaxs="i",yaxs="i",ylab="",xlab="",cex.axis=1.5,
cex.lab=1.5)
> curve(g,a,b,add=TRUE,lwd=2,col="red")
> curve(-g(x),a,b,add=TRUE,lwd=2,col="blue")
> grid(NULL,NULL,col="black")
> abline(h=0)
> v4<-c(expression(f(x)==x^2*~sin(x)),expression(g(x)==x^2),
expression(h(x)==-x^2))
> legend("topleft",v4,lty=c(1,1,1),lwd=c(2,2,2),bg="white",
col=c("black","red","blue"),y.intersp=1.25)
```

R Code: Figure 12.2

```
> f<-function(x){x^2*sin(x)}
> a<-0
> b<-2*pi
> par(mar=c(3,3,2,2))
> curve(f,a,b,ylim=c(-25,5),lwd=2,xaxt="n",xaxs="i",yaxs="i",ylab="",
xlab="",cex.axis=1.5,cex.lab=1.5)
> curve(4*sin(x),a,b,col="red",lwd=2,add=TRUE)
> grid(NA,NULL,col="black")
> v1<-c(0,pi/2,pi,3*pi/2,2*pi)
> v2<-c("0",expression(pi/2),expression(pi),expression(3*pi/2),
expression(2*pi))
> axis(side=1,lwd=3,at=v1,labels=v2,cex.axis=1.5, cex.lab=1.5)
> abline(v=v1,lty=3)
> abline(h=0)
> v3<-c(expression(x^2*~sin(x)),expression(4*~sin(x)))
> legend("topright",v3,lty=c(1,1),lwd=c(2,2),col=c("black","red"),
bg="white",y.intersp=1.25)
```

R Code: Figure 12.3

```
> f<-function(x){x^2*sin(x)}
> f_p<-Deriv(f)
> g<-function(x){2*x*cos(x)}
> a<-0
> b<-2*pi
> rootsf_p<-uniroot.all(f_p,c(a+.1,b))
> par(mar=c(3,3,2,2))
> curve(f,a,b,ylim=c(-25,15),lwd=2,xaxt="n",xaxs="i",yaxs="i",ylab="",
```

```
xlab="",cex.axis=1.5,cex.lab=1.5 )
> curve(g,a,b,col="red",lty=2,lwd=3,add=TRUE)
> grid(NA,NULL,col="black")
> v1<-c(0,pi/2,pi,3*pi/2,2*pi)
> v2<-c("0", expression(pi/2),expression(pi),expression(3*pi/2),
expression(2*pi))
> axis(side=1,lwd=3,at=v1,labels =v2 , cex.axis=1.5, cex.lab=1.5)
> abline(v=v1,lty=3)
> abline(h=0)
> v3<-c(expression(f(x)==x^2*~sin(x)), expression(g(x)==2*x*~cos(x)))
> legend("bottomleft", v3, lty=c(1,2), lwd=c(2,2), col=c("black","red"),
bg="white", y.intersp=1.25)
> points(rootsf_p,f(rootsf_p),pch=16,cex=1.5)
> text(rootsf_p[1],f(rootsf_p)[1],paste("(",round(rootsf_p[1],2), ",",
round(f(rootsf_p[1]),2),")",sep=""),pos=3)
> text(rootsf_p[2],f(rootsf_p)[2],paste("(",round(rootsf_p[2],2), ",",
round(f(rootsf_p[2]),2),")",sep=""),pos=2)
```

R Code: Figure 13.1

```
> f<-function(x){sin(x)/x}
> g<-function(x){1/x}
> a<-0
> b<-12*pi
> par(mar=c(3,3,2,2))
> curve(f,a,b,ylim=c(-0.5,1),lwd=2,xaxs="i",yaxs="i",ylab="",
xlab="",cex.axis=1.5,cex.lab=1.5)
> curve(g,a,b,add=TRUE,lwd=2,col="red")
> curve(-g(x),a,b,add=TRUE,lwd=2,col="blue")
> grid(NULL,NULL,col="black")
> abline(h=0)
> v4<-c(expression(f(x)==frac(sin(x),x)),expression(g(x)==frac(1,x)),
expression(h(x)==-frac(1,x)))
> legend("topright",v4,lty=c(1,1,1),lwd=c(2,2,2),bg="white",
col=c("black","red","blue"), y.intersp=1.25)
```

R Code: Figure 13.2

```
> f=function(x){sin(x)/x}
> a=0
> b=2*pi
> par(mar=c(3,3,2,2))
> curve(f,a,b,ylim=c(-.5,1),lwd=2,xaxt="n",xaxs="i",yaxs="i",ylab="",
xlab="",cex.axis=1.5,cex.lab=1.5)
> curve(sin(x)/4,a,b,col="red",lwd=2,add=TRUE)
> grid(NA,NULL,col="black")
> v1<-c(0,pi/2,pi,3*pi/2,2*pi)
> v2<-c("0", expression(pi/2),expression(pi),expression(3*pi/2),
expression(2*pi))
> axis(side=1,lwd=3,at=v1,labels =v2 , cex.axis=1.5, cex.lab=1.5)
> abline(v=v1,lty=3)
> abline(h=0)
```

```
> v3<-c(expression(f(x)==frac(sin(x),x)),expression(g(x)==frac(sin(x),4)))
> legend("topright", v3, lty=c(1,1), lwd=c(2,2), col=c("black","red"),
bg="white", y.intersp=1.25)
```

R Code: Figure 13.3

```
> f<-function(x){sin(x)/x}
> h<-function(x){(x*cos(x) -sin(x))/x^2}
> a<-0
> b<-2*pi
> par(mar=c(3,3,2,2))
> curve(f,a,b,ylim=c(-.5,1),lwd=2,xaxt="n",xaxs="i",yaxs="i",ylab="",
xlab="",cex.axis=1.5,cex.lab=1.5)
> curve(h,a,b,col="red",lwd=2,add=TRUE)
> grid(NA,NULL,col="black")
> v1<-c(0,pi/2,pi,3*pi/2,2*pi)
> v2<-c("0", expression(pi/2),expression(pi),expression(3*pi/2),
expression(2*pi))
> axis(side=1,lwd=3,at=v1,labels =v2 , cex.axis=1.5, cex.lab=1.5)
> abline(v=v1,lty=3)
> abline(h=0)
> v3<-c(expression(f(x)==frac(sin(x),x)),
expression(f^" /"*(x)==frac(x*cos(x)-sin(x),x)))
> legend("topright", v3, lty=c(1,1), lwd=c(2,2), col=c("black","red"),
bg="white",y.intersp=1.25)
```

R Code: Figure 14.1

```
> f<-function(x){sin(x^2)}
> a<-0
> b<-2*pi
> par(mar=c(3,3,2,2))
> curve(f,a,b, n=10000,lwd=2,xaxt="n",xaxs="i",ylab="",xlab="",
cex.axis=1.5,cex.lab=1.5)
> curve(sin(x),a,b,col="red",lwd=2,add=TRUE)
> grid(NA,NULL,col="black")
> v1<-c(0,pi/2,pi,3*pi/2,2*pi)
> v2<-c("0", expression(pi/2),expression(pi),expression(3*pi/2),
expression(2*pi))
> axis(side=1,lwd=3,at=v1,labels =v2,cex.axis=1.5,cex.lab=1.5)
> abline(v=v1,lty=3)
> abline(h=0)
> v4=c(expression(f(x)==sin(x^2)),expression(g(x)==sin(x)))
> legend("bottomleft",v4, lty=c(1,1),lwd=c(2,2),bg="white",
col=c("black","red"),cex=0.95,y.intersp=1.25)
```

R Code: Figure 14.2

```
> f<-function(x){sin(x^2)}
> a<-0
> b<-2*pi
> par(mar=c(3,3,2,2))
> g<-function(x){cos(x^2)*2*x}
> curve(g,a,b,n=10000,col="red",xaxt="n",xaxs="i",ylab="",xlab="",lwd=2,
cex.axis=1.5,cex.lab=1.5)
> curve(f,a,b, n=10000,lwd=2,add=TRUE)
> grid(NA,NULL,col="black")
> v1<-c(0,pi/2,pi,3*pi/2,2*pi)
> v2<-c("0", expression(pi/2),expression(pi),expression(3*pi/2),
expression(2*pi))
> axis(side=1,lwd=3,at=v1,labels =v2 , cex.axis=1.5, cex.lab=1.5)
> abline(v=v1,lty=3)
> abline(h=0)
> v4=c(expression(f(x)==sin(x^2)),"Derivative")
> legend("bottomleft",v4,lty=c(1,1),lwd=c(2,2),bg="white",
col=c("black","red"),y.intersp=1.25)
```

R Code: Figure 16.1

```
> g<-function(x){1/(x^2+1)}
> f<-function(x){x^2}
> options(scipen = 999)
> par(mfrow=c(2,1))
> par(mar=c(5,11,1,2))
> curve(g,100,10000,lwd=2,las=1,xlab="",xaxs="i",cex.axis=1.5,cex.lab=1.5,
ylab="",xaxt="n")
> axis(1,at=c(2000,6000,10000),
labels=format(c(2000,6000,10000),big.mark=","),cex.axis=1.5)
> title(ylab=expression(g(x)==frac(1,x^2+1)), line=8, cex.lab=1.5)
> grid(NULL,NULL,col="black")
> par(mar=c(5,11,1,2))
> curve(f,100,10000,lwd=2,las=1,xaxs="i",cex.axis=1.5,cex.lab=1.5,ylab="",
yaxt="n",xaxt="n")
> axis(1,at=c(2000,6000,10000),labels=format(c(2000,6000,10000),
big.mark=","),cex.axis=1.5)
> axis(2,at=seq(0,100000000,by=20000000),
labels=format(seq(0,100000000,by=20000000),big.mark=","),
cex.axis=1.5,las=1)
> title(ylab=expression(f(x)==x^2),line=8,cex.lab=1.5)
> grid(NULL,NULL,col="black")
```

R Code: Figure 16.2

```
> f<-function(x){10/(1+exp(-.5*(x-20)))}
> par(mar=c(4,6.5,1,2))
> curve(f,1,50,lwd=2,xaxs="i",cex.axis=1.5,cex.lab=1.5,
ylab=expression(f(x)==frac(10,1+e^(-.5*(x-20)))))
> grid(NULL,NULL,col="black")
```

R Code: Figure 16.3

```
> f<-function(x){exp(x)}
> g<-function(x){6*x^2+1}
> par(mar=c(4,3,1,1))
> curve(f,0,5,lwd=2,xlab="",xaxs="i",ylab="",cex.axis=1.5,cex.lab=1.5)
> curve(g,0,5,lwd=2,xaxs="i",add=TRUE)
> grid(NULL,NULL,col="black")
```

R Code: Figure 17.1

```
> par(mar=c(5,5,2,2))
> f<-function(x){x^3-3003*x^2+3006000*x}
> curve(f,-100,100,lwd=2,xlab="x", cex.axis=1.5,cex.lab=1.5,
ylab=expression(f(x)==x^3-3003*x^2+3006000*x))
> grid(NULL,NULL,col="black")
> abline(h=0,v=0,lwd=2)
```

R Code: Figure 17.2

```
> library(Deriv)
>
> f<-function(x){x^3/3-3*x^2+8*x +2}
>
> l<-0.5
> r<- 5.5
> a<-2
> b<-4
> c<-3

> f_p<-Deriv(f)
> f_pp<-Deriv(f,n=2)
>
> dev.new(width=5, height=7.5, unit="in")
>
> par(mfrow=c(3,1))
> par(mar=c(3,3,1,1))
>
> curve(f, l, r,lwd=3,xaxt="n",yaxt="n")
> grid(NULL,NULL,col="black")
> abline(v=a,col="blue",xpd=TRUE)
> abline(v=b,col="blue",xpd=TRUE)
> abline(v=c,col="red",xpd=TRUE)
> points(c(2,4),c(f(2),f(4)),col="blue", pch=16,cex=2)
> points(3,f(3),col="red", pch=16,cex=2)
> legend("bottomright", expression(f(x)), lwd=2,cex=1.5,bg="white")
> axis(1,cex.axis=1.5)
> axis(2,cex.axis=1.5)
>
> curve(f.p,l,r,lwd=3,xaxt="n",yaxt="n")
> grid(NULL,NULL,col="black")
```

```
> abline(h=0,lwd=2)
> abline(v=a,col="blue",xpd=TRUE)
> abline(v=b,col="blue",xpd=TRUE)
> abline(v=c,col="red",xpd=TRUE)
> points(c(2,4),c(f.p(2),f.p(4)),col="blue", pch=16,cex=2)
> points(3,f.p(3),col="red", pch=16,cex=2)
> legend("bottomright", expression(f^" /"*(x)), lwd=2,cex=1.5,bg="white")
> axis(1,cex.axis=1.5)
> axis(2,cex.axis=1.5)
>
> curve(f.pp,l,r,lwd=3,xaxt="n",yaxt="n")
> grid(NULL,NULL,col="black")
> abline(h=0,lwd=2)
> abline(v=a,col="blue",xpd=TRUE)
> abline(v=b,col="blue",xpd=TRUE)
> abline(v=c,col="red",xpd=TRUE)
> points(c(2,4),c(f.pp(2),f.pp(4)),col="blue", pch=16,cex=2)
> points(3,f.pp(3),col="red", pch=16,cex=2)
> legend("bottomright", expression(f^" / /"*(x)), lwd=2,cex=1.5,bg="white")
> axis(1,cex.axis=1.5)
> axis(2,cex.axis=1.5)
```

R Code: Figure 17.3

```
> par(mar=c(5,5,2,2))
> f=function(x){x^3-3003*x^2+3006000*x}
> curve(f,995,1005,lwd=2,xlab="x",cex.axis=1.5,cex.lab=1.5,
ylab=expression(f(x)==x^3-3003*x^2+3006000*x))
> grid(NULL,NULL,col="black")
```

R Code: Figure 19.1

```
> par(mar=c(0,0,0,0))
> plot(0,0,type="n", xlim=c(1,7), ylim=c(1,9),axes=FALSE)
> segments(2,1,2,9,lwd=2)
> text(1.9,5,"Wall of House",srt=90,font=2,pos=3,cex=1.25)
> segments(2,8,6,8,lwd=2)
> segments(2,2,6,2,lwd=2)
> segments(6,2,6,8,lwd=2)
> text(4,8,"x",font=2,pos=3,cex=1.25)
> text(6,5,"y",font=2,pos=4,cex=1.25)
> text(4,2,"x",font=2,pos=1,cex=1.25)
```

R Code: Figure 20.2

```
> library(rgl)
> open3d()
> cylinder<-function(r,h){pi*r^2*h}
> x<-y<-seq(0,4,length=100)
> z<-outer(x,y,cylinder)
```

```
> persp3d(x,y,z,col="blue",zlim=c(0,max(z)),xlab="r-radius",
 ylab="h-height",zlab="V-volume",front = "lines", back = "lines")
> rgl.spheres(0, 0, 0, r = 4, color = "yellow")
> rgl.spheres(2, 3, cylinder(2,3), r = 4, color = "red")
> abclines3d(2, 3, 12*pi, a=1, b = 0, c = 12*pi, color="red",lwd=3)
> abclines3d(2, 3, 12*pi, a=0, b = 1, c = 4*pi, color="red",lwd=3)
> filename <- paste("surface3", formatC(1, digits = 1, flag = "0"), ".png", sep = "")
> snapshot3d(filename)
```

R Code: Figure 21.1

```
> f1<-function(x){sqrt(9-x^2)}
> f2<-function(x){-sqrt(9-x^2)}
> g1<-function(x){sqrt(25-x^2)}
> g2<-function(x){-sqrt(25-x^2)}
> par(mar=c(0,0,0,0))
> plot(0,0,xlim=c(-5.5,5.5),ylim=c(-5.5,5.5),type="n",xlab="", ylab="",
xaxs="i",yaxs="i", xaxt="n", yaxt="n",frame=FALSE)
> curve(f1,-3,3,lwd=2,add=TRUE)
> curve(f2,-3,3,lwd=2,add=TRUE)
> curve(g1,-5,5,lwd=3,add=TRUE)
> curve(g2,-5,5,lwd=3,add=TRUE)
> arrows(0,-3.25,0,-4.25,lwd=6)
> arrows(0,3.25,0,4.25,lwd=6)
> text(3.5,0,expression(t),cex=1.75,pos=1)
> arrows(3.25,0,4.25,0,lwd=6)
> arrows(-3.25,0,-4.25,0,lwd=6)
> arrows(3*sqrt(2)/2+0.177,3*sqrt(2)/2+0.177,3*sqrt(2)/2+.707+0.177,
3*sqrt(2)/2+.707+0.177,lwd=6)
> arrows(-(3*sqrt(2)/2+0.177),3*sqrt(2)/2+0.177,
-(3*sqrt(2)/2+.707+0.177),3*sqrt(2)/2+.707+0.177,lwd=6)
> arrows(3*sqrt(2)/2+0.177,-(3*sqrt(2)/2+0.177),3*sqrt(2)/2+.707+0.177,
-(3*sqrt(2)/2+.707+0.177),lwd=6)
> arrows(-(3*sqrt(2)/2+0.177),-(3*sqrt(2)/2+0.177),
-(3*sqrt(2)/2+.707+0.177),-(3*sqrt(2)/2+.707+0.177),lwd=6)
> text(0,1.75,expression(A(t)==pi*(r(t))^2),cex=2)
> segments(0,0,3,0,lwd=3,lty=2)
> points(0,0,pch=16,cex=2)
> text(1.5,0,expression(r(t)),cex=2,pos=1)
```

R Code: Figure 21.2

```
> windows.options(width = 6, height = 3, reset = FALSE)
> par(mar=c(0,0,0,0))
> plot(0,0,xlim=c(0.5,11),ylim=c(0,7),type="n",xlab="",ylab="",
xaxs="i",yaxs="i", xaxt="n", yaxt="n",frame=FALSE)
> segments(1,1,10,5,lwd=4)
> segments(10,1,10,10,lwd=8)
> segments(0,1,10,1,lwd=2)
> text(5.5,1,"Ground",pos=1,cex=2)
> text(10.25,5.5,"House",srt=-90,pos=4,cex=2)
> text(5.5,4,"Ladder",srt=18,pos=1,cex=2)
```

```
> arrows(1.65,1.5, 0.65,1.5,lwd=6)
> arrows(9.5,6, 9.5,5,lwd=6)
> brackets(10,1,10,5,lwd=3,h=0.75,curvature = 0.85)
> text(9.25,3,expression(h(t)),pos=2,cex=1.5)
> brackets(1,1,10,1,lwd=3,h=0.5,curvature = 1)
> text(5.5,1.5,expression(d(t)),pos=3,cex=1.5)
```

R Code: Figure 22.2

```
> par(mar=c(5,5,2,2))
> Surge<-function(a,b,x){a*x*exp(-b*x)}
> curve(Surge(4,2,x),0,10,xlab="t hours", ylab="mg/ml",lwd=3,col="blue",
lty=2,cex.axis=1.5,cex.lab=1.5)
> curve(Surge(1,1,x),0,10,xlab="t hours", ylab="mg/ml",lwd=3,add=TRUE)
> curve(Surge(2,4,x),0,10,xlab="t hours", ylab="mg/ml",lwd=4,add=TRUE,
lty=3,col="red")
> grid(NULL,NULL,col="black")
> legend("topright", c(expression(t*e^-t),expression(2*t*e^{-4*t}),
expression(4*t*e^{-2*t})), lwd=c(3,4,3),lty=c(1,3,2),
col=c("black","red","blue"),bg="white",cex=1.5)
```

R Code: Figure 22.3

```
> par(mar=c(5,5,2,2))
> Surge<-function(a,b,x){a*x*exp(-b*x)}
> curve(Surge(4,2,x-6),5.5,10,xlab="t hours", ylab="mg/ml",lwd=3,
cex.axis=1.5,cex.lab=1.5)
> abline(h=0,v=6)
> grid(NULL,NULL,col="black")
> legend("bottomright", expression(4*(t-6)*e^{-2*(t-6)}),lwd=2,bg="white",
cex=1.5)
```

R Code: Figure 22.4

```
> par(mar=c(5,5,2,2))
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s) )}
> curve(Surge_Piece(4,2,6,x),0,10,xlab="t hours",ylab="mg/ml",lwd=3,
cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
```

R Code: Figure 22.5

```
> Surge_Piece<-function(a,b,s,x){ifelse(x<s,0,Surge(a,b,x-s) )}
> Surge<-function(a,b,x){a*x*exp(-b*x)}
> a<-1
> b<-0.5
> s<-6
> h<-function(x){Surge(a,b,x)+Surge_Piece(a,b,s,x)+Surge_Piece(a,b,2*s,x)}
> curve(h,0,24,xlab="t hours", ylab="mg/ml",lwd=3,cex.axis=1.5,cex.lab=1.5)
> grid(NULL,NULL,col="black")
```

R Code: Figure 28.2

```
> MidBox<-function(a,b,n,f){
+ dx<-(b-a)/n
+ mid<-seq(a+dx/2,b,dx)
+ par(mar=c(4,5,2,2))
+ curve(f,a,b,lwd=2,ylab=name,cex.lab=1.5)
+ grid(NULL,NULL,col="black")
+ for(i in 1:n){
+ polygon(c(mid[i]-dx/2,mid[i]-dx/2, mid[i]+dx/2,mid[i]+dx/2),
c(0,f(mid[i]),f(mid[i]),0),col="gray85", border="black")
+ }
+ curve(f,a,b,lty=3,lwd=2,add=TRUE)
+ legend("topleft",paste("Sum=",sum(f(mid)*dx)),bg="white")
+ }
> LeftBox<-function(a,b,n,f){
+ dx<-(b-a)/n
+ mid<-seq(a,b-dx,dx)
+ par(mar=c(4,5,2,2))
+ curve(f,a,b,lwd=2,ylab=name,cex.lab=1.5)
+ grid(NULL,NULL,col="black")
+ for(i in 1:n){
+ polygon(c(mid[i],mid[i], mid[i]+dx,mid[i]+dx),c(0,f(mid[i]),f(mid[i]),0),
col="gray85",border="black")
+ }
+ curve(f,a,b,lty=3,lwd=2,add=TRUE)
+ legend("topleft",paste("Sum=",sum(f(mid)*dx)),bg="white")
+ }
> RightBox<-function(a,b,n,f){
+ dx<-(b-a)/n
+ mid<-seq(a+dx,b,dx)
+ par(mar=c(4,5,2,2))
+ curve(f,a,b,lwd=2,ylab=name,cex.lab=1.5)
+ grid(NULL,NULL,col="black")
+ for(i in 1:n){
+ polygon(c(mid[i]-dx,mid[i]-dx, mid[i],mid[i]),c(0,f(mid[i]),f(mid[i]),0),
col="gray85",border="black")
+ }
+ curve(f,a,b,lty=3,lwd=2,add=TRUE)
+ legend("topleft",paste("Sum=",sum(f(mid)*dx)),bg="white")
+ }
>
> name<-expression(f(x)==x^2+5)
```

```
> a<-0
> b<-8
> dev.new(width=6,height=3.5,unit="in")
> par(mfrow=c(1,3))
> h<-function(x){x^2+5}
> MidBox(a,b,4,h)
> LeftBox(a,b,4,h)
> RightBox(a,b,4,h)
```

R Code: Figure 28.3

```
> LeftBox<-function(a,b,n,f){
+ dx<-(b-a)/n
+ mid<-seq(a,b-dx,dx)
+ par(mar=c(4,5,2,2))
+ curve(f,a,b,lwd=2,ylab=name,cex.lab=1.5)
+ grid(NULL,NULL,col="black")
+ for(i in 1:n){
+ polygon(c(mid[i],mid[i], mid[i]+dx,mid[i]+dx),c(0,f(mid[i]),f(mid[i]),0),
col="gray85",border="black")
+ }
> name<-expression(f(x)==x^2+5)
> a<-0
> b<-8
> dev.new(width=6, height=3.5, unit="in")
> par(mfrow=c(1,3))
> h<-function(x){x^2+5}
> LeftBox(a,b,8,h)
> LeftBox(a,b,16,h)
> LeftBox(a,b,32,h)
```

R Code: Figure 29.3

```
> ## URL for the data from sustainabilitymath.org
> dataURL<-"http://sustainabilitymath.org/excel/HubbardBrook-R.csv"
> ## Read data from online csv file
> Hubbard.data <-read.csv(url(dataURL),header=TRUE)
> ## Define x and y, mostly to save typing
> x=Hubbard.data$Day.in.July.1966
> Watershed.2=Hubbard.data$Watershed2.Streamflow.ft3.day
> Watershed.6=Hubbard.data$Watershed6.Streamflow.ft3.day
> ## Create Plot for Watershed 2
> plot(x, Watershed.2, cex=2, pch=16, xlim=c(1,31),xlab="Day in July 1966",
ylab="Cubic Feet per Day",cex.axis=1.5,cex.lab=1.5)
> title(main="Hubbard Brook Watershed 2 Water Runoff")
> grid(NULL,NULL,col="black")
> ## Create Plot for Watershed 6
> plot(x, Watershed.6, cex=2, pch=16, xlim=c(1,31),xlab="Day in July 1966",
ylab="Cubic Feet per Day",cex.axis=1.5,cex.lab=1.5)
> title(main="Hubbard Brook Watershed 6 Water Runoff")
> grid (NULL ,NULL, col = "gray")
```

R Code: Figure 30.1

```
> f<-function(x){x^2}
> par(mar=c(4,5,2,2))
> curve(f,0,5,lwd=2,cex.axis=1.5,cex.lab=1.5,xaxs="i",yaxs="i",
xlab="x-axis",ylab=expression(f(x)==x^2))
> grid(NULL,NULL,col="black")
> a<-1
> b<-4
> xlist<-seq(a,b,by=0.1)
> polygon(c(xlist,rev(xlist)),c(f(xlist),0*rev(xlist)),col="gray85",
border="black")
```

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