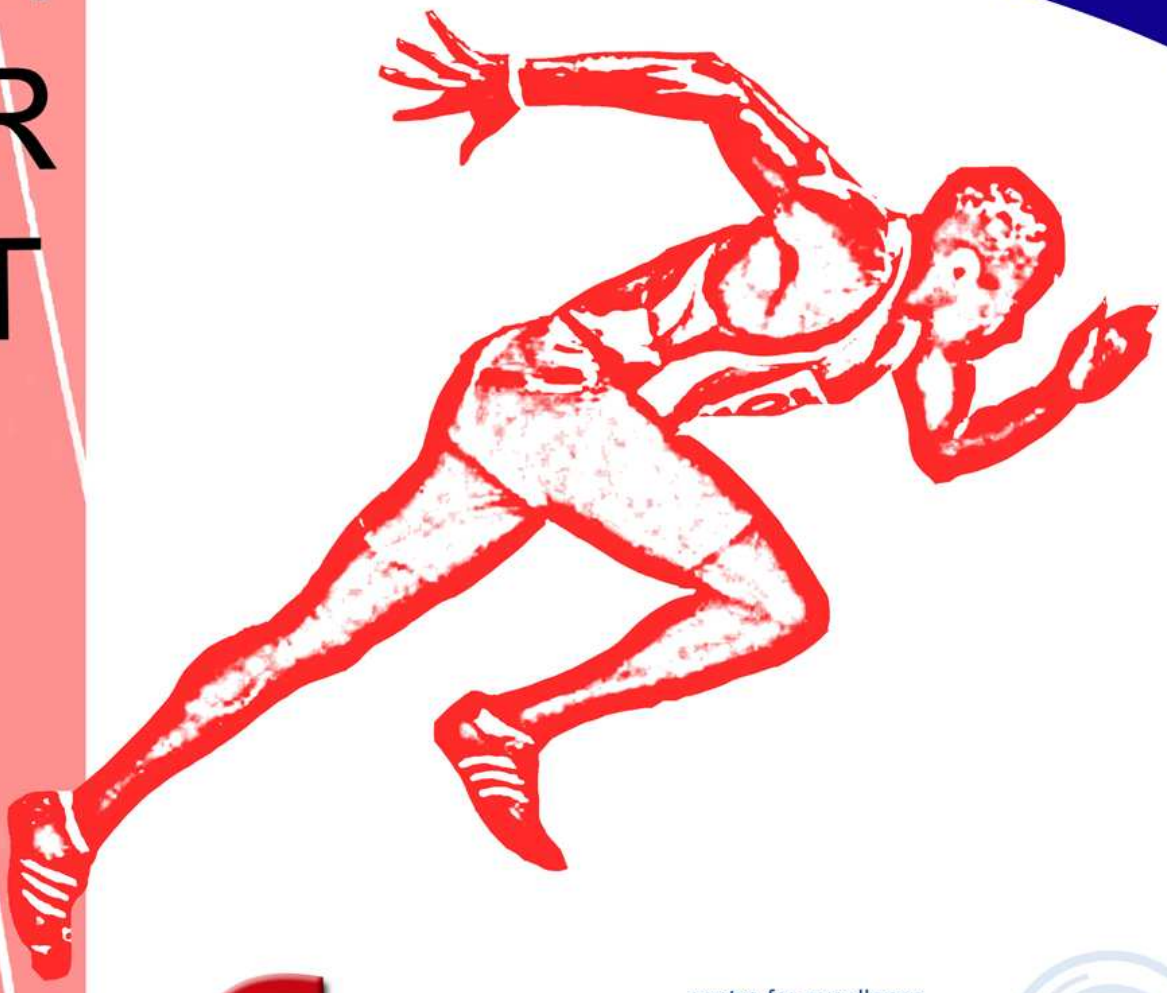
HEAD START MATHS

Rational Numbers/ Fractions/Decimals



CEMTL
Regional Centre for Excellence in Mathematics Teaching and Learning

centre for excellence
sigma Σ
in mathematics & statistics support

Foreword

The Regional Centre for Excellence in Mathematics Teaching and Learning (CEMTL) is collaboration between the Shannon Consortium Partners: University of Limerick, Institute of Technology, Limerick; Institute of Technology, Tralee and Mary Immaculate College of Education, Limerick., and is driven by the Mathematics Learning Centre (MLC) and The Centre for Advancement in Mathematics Education and Technology (CAMET) at the University of Limerick.

CEMTL is committed to providing high quality educational resources for both students and teachers of mathematics. To that end this package has been developed to a high standard and has been peer reviewed by faculty members from the University of Limericks Department of Mathematics and Statistics and sigma, the UK based Centre for Excellence in Teaching and Learning (CETL). Through its secondment programme, sigma provided funding towards the creation of these workbooks.

Please be advised that the material contained in this document is for information purposes only and is correct and accurate at the time of publishing. CEMTL will endeavour to update the information contained in this document on a regular basis.

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2. Rational Numbers/ Fractions/ Decimals

2.1 Introduction to Rational Numbers

In the last workshop we studied Natural numbers and Integers. They were groups of **whole** (full) numbers.

However, if we were to divide 2 by 3 we get the number

$$\frac{2}{3}$$

As you can see, this is not a whole number i.e. it is neither a Natural number nor an Integer. Therefore we now have another set of numbers which are called **Rational numbers**. This set of numbers is represented by the letter $\mathbb{Q}$.

What is a Rational Number?

A Rational number is any number of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Therefore, any numbers that can be written as fractions are Rational numbers e.g.

$$\frac{1}{2}, \quad \frac{3}{4}, \quad \frac{10}{11}, \quad etc.$$

The top number in a fraction is called the **Numerator** and the bottom number is called the **Denominator**.

If the numerator is smaller than the denominator, the fraction is less than 1.

e.g. $\frac{1}{2}, \quad \frac{5}{6}, \quad \frac{3}{10}.$

If the numerator is bigger than the denominator, the fraction is greater than 1.

e.g. $\frac{3}{2} = 1\frac{1}{2}, \quad \frac{7}{3} = 2\frac{1}{3}, \quad \frac{31}{10} = 3\frac{1}{10}.$

Equivalent Fractions

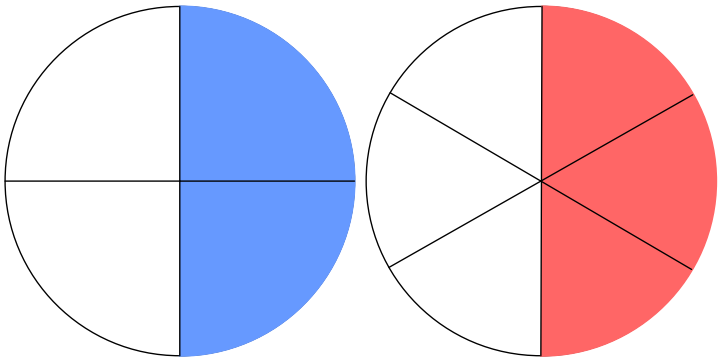


Figure 1: Equivalent Fractions

In the diagrams above, the first pie is divided into quarters and the second into sixths.

It is obvious that $\frac{2}{4} = \frac{3}{6}$ both of which we can see are equal to $\frac{1}{2}$

Fractions like these that are equal to each other are called **Equivalent Fractions**.

Note

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} \quad etc.$$

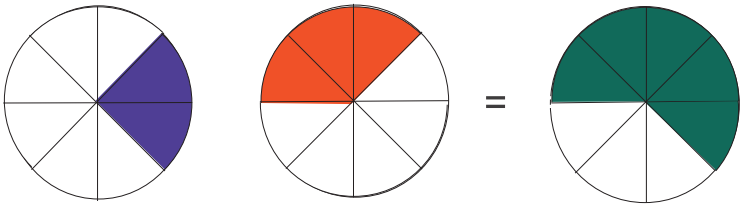
$\frac{1}{2}$ is called the **Simplest Form** because 1 and 2 have no common factors (other than 1) so cannot be broken down or simplified any further.

Exercises 1

Fill in the Missing Numbers to Make Equivalent Fractions

1. $\frac{6}{9} = \frac{2}{\_}$
2. $\frac{\_}{5} = \frac{3}{15}$
3. $\frac{7}{\_} = \frac{1}{4}$
4. $\frac{11}{10} = \frac{121}{\_\_}$
5. $\frac{6}{\_} = \frac{24}{28}$
6. $\frac{\_}{5} = \frac{36}{60}$
7. $\frac{2}{9} = \frac{\_\_}{108}$
8. $\frac{4}{\_} = \frac{16}{12}$
9. $\frac{5}{25} = \frac{\_\_}{60}$
10. $\frac{15}{\_\_} = \frac{21}{28}$

2.2 Addition and Subtraction of Fractions



From the diagram above, we can see that $\frac{2}{8} + \frac{3}{8} = \frac{5}{8}$.

Remember: $\frac{2}{8}$ means 2 lots of $\frac{1}{8}$

So ...

$\frac{2}{8} + \frac{3}{8}$ means 2 lots of $\frac{1}{8}$ plus 3 lots of $\frac{1}{8}$, giving us 5 lots of $\frac{1}{8}$ (or $\frac{5}{8}$ in total).

Therefore when we add fractions with the same denominator, we simply add the numerators (the denominator in your answer remains the same).

Getting a Common Denominator

So what happens if we wish to add or subtract two fractions that have different denominators?

$\Rightarrow$ We must first make the denominators the same.

For example, to add $\frac{1}{2} + \frac{1}{6}$ we write $\frac{1}{2}$ as an equivalent fraction $\frac{3}{6}$

$$\Rightarrow \frac{3}{6} \quad + \quad \frac{1}{6} \quad = \quad \frac{4}{6} \quad = \quad \frac{2}{3}$$

This process is called getting a **Common Denominator**. We do this whenever we wish to add or subtract fractions.

Example:

Evaluate $\frac{1}{3} + \frac{3}{4} + \frac{5}{6}$

Here we look for a denominator that the current denominators; $3, 4$ and 6, all divide exactly into i.e. 12.

We rewrite the problem with each fraction as an equivalent fraction with 12 as denominator:

$$\frac{1}{3} = \frac{4}{12}; \quad \frac{3}{4} = \frac{9}{12}; \quad \frac{5}{6} = \frac{10}{12}$$

then simply add the numerators together.

$$\frac{4}{12} + \frac{9}{12} + \frac{10}{12} = \frac{23}{12} = 1\frac{11}{12}$$

Exercises 2

Evaluate the Following

1. $\frac{3}{4} + \frac{4}{5}$

2. $\frac{2}{3} - \frac{1}{7}$

3. $\frac{1}{2} + \frac{3}{5} + \frac{1}{6}$

4. $\frac{2}{9} + \frac{4}{5}$

5. $\frac{5}{8} - \frac{1}{3}$

Proper, Mixed and Improper Fractions

When the numerator in a fraction is smaller than its denominator, it is referred to as a **Proper Fraction** (a fraction whose value is less than 1).

e.g. $\frac{3}{7}, \quad \frac{1}{5}, \quad \frac{4}{9}.$

When fractions are written in the form $a\frac{b}{c}$, i.e. the sum of a whole number and a proper fraction, we refer to them as **Mixed Fractions**.

e.g. $1\frac{1}{2}, \quad 2\frac{1}{3}, \quad 3\frac{1}{10}.$

When fractions are written with the numerators greater than the denominators we refer to them as **Improper Fractions**.

e.g. $\frac{3}{2}, \quad \frac{7}{3}, \quad \frac{31}{10}.$

Converting Mixed Fractions to Improper Fractions

We can convert a mixed fraction to an improper fraction by writing its components, i.e. the whole number and the proper fraction, as a sum with a common denominator then adding the numerators.

Example

$$1\frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}$$

$$2\frac{1}{3} = \frac{3}{3} + \frac{3}{3} + \frac{1}{3} = \frac{7}{3}$$

$$3\frac{1}{10} = \frac{10}{10} + \frac{10}{10} + \frac{10}{10} + \frac{1}{10} = \frac{31}{10}.$$

Note:

We generally convert mixed fractions to improper fractions before carrying out any operations such as adding, subtracting, multiplying or dividing.

Exercises 3

Write the Following Mixed Fractions as Improper Fractions

1. $2\frac{4}{5}$

2. $13\frac{1}{6}$

3. $7\frac{2}{9}$

4. $8\frac{1}{10}$

5. $10\frac{2}{3}$

Exercises 4

Write the Following Improper Fractions as Mixed Fractions

1. $\frac{33}{10}$

2. $\frac{17}{9}$

3. $\frac{154}{12}$

4. $1\frac{64}{3}$

5. $\frac{15}{2}$

Exercises 5

Evaluate the Following

1. $7\frac{2}{3} + 3\frac{1}{4}$

2. $4\frac{1}{3} + 2\frac{3}{5} - 1\frac{1}{2}$

3. $-\frac{2}{3} - 8\frac{1}{4} + \frac{1}{6}$

4. $-\frac{2}{21} + 2\frac{4}{7}$

5. $-3\frac{3}{4} - \frac{3}{4}$

2.3 Multiplication and Division of Fractions

Multiplying Fractions

When multiplying fractions we multiply top number by top number (numerator by numerator) and bottom number by bottom number (denominator by denominator) then reduce to simplest terms if necessary.

$$\frac{2}{3} \times \frac{6}{7} = \frac{2 \times 6}{3 \times 7} = \frac{12}{21} = \frac{4}{7}.$$

Dividing Fractions

When dividing fractions, we invert the second fraction (i.e. turn it upsidedown) and multiply as normal like this:

Evaluate $\frac{3}{5} \div \frac{4}{7}$

$\frac{3}{5} \div \frac{4}{7}$

$\frac{4}{7}$ turned upside down is $\frac{7}{4}$ so

$\frac{3}{5} \times \frac{7}{4}$

$\frac{3 \times 7}{5 \times 4}$

$\frac{21}{20}$

Mixed Fractions

When multiplying or dividing mixed fractions you should first convert them to improper fractions.

Example

$3\frac{1}{2} \times 4\frac{2}{5} = \frac{7}{2} \times \frac{22}{5} = \frac{77}{5} = 15\frac{2}{5}$

Exercises 6

Simplify Each of the Following

1. $\frac{1}{5} \times \frac{1}{6}$

2. $\frac{2}{3} \times \frac{4}{5}$

3. $\frac{3}{5} \times \frac{5}{9} \times \frac{1}{3}$

4. $1\frac{1}{4} \times 2\frac{1}{7} \times \frac{5}{8}$

5. $\frac{3}{7} \div \frac{7}{3}$

6. $3\frac{1}{3} \div \frac{3}{4}$

7. $10\frac{2}{5} \div 1\frac{1}{10}$

8. $(4\frac{1}{5} + 2\frac{1}{3}) \div (\frac{2}{5} \times \frac{1}{4})$

9. $(3\frac{1}{2} - 1\frac{1}{4}) \div (3\frac{1}{5} \times \frac{15}{4})$

10. $(-\frac{4}{5} + 1\frac{1}{3}) \div (3 \times -\frac{3}{4})$

2.4 Fractions - Problems

Problem 1

Michael won €36,000 in the National Lottery. He wanted to give two thirds of his winnings to charity, with the remainder to be divided equally among his 4 children. How much money did each of his children receive?

Solution

The charity received two thirds of €$36,000$

i.e. $\frac{2}{3} \times \frac{36,000}{1} = \frac{2 \times 36,000}{3 \times 1} = \frac{72,000}{3} =$ €$24,000$

Therefore €$12,000$ remains to be divided between Michael's 4 children'

Each child receives one quarter of €$12,000$

i.e. $\frac{1}{4} \times \frac{12,000}{1} = \frac{1 \times 12,000}{4 \times 1} = \frac{12,000}{4} =$ €$3,000$

Problem 2

When Catherine goes to the gym, she usually spends an hour and a half working out. She spends $\frac{2}{5}$ of her time on the treadmill, $\frac{1}{3}$ of her time on the rowing machine, $\frac{1}{10}$ of her time on free weights and the remainder of the session stretching out. How many minutes does Catherine spend stretching at the end of each gym session?

Solution

Problem 3

Engineering Mathematics has 960 students registered. $\frac{3}{8}$ of the students study mechanical engineering, $\frac{1}{6}$ study civil engineering and the remainder of the students study electrical engineering. How many students study electrical engineering?

Solution

2.5 Decimals

Decimal numbers are numbers that contain a decimal point, for example: 4.158; 5.17897; 123.1 etc.

Each digit in the number has a certain value depending on how close or how far it is from the decimal point.

The digits to the left of the decimal point represent ones, tens, hundreds etc.

The numbers to the right of the decimal point represents tenths, hundredths, thousandths etc.

Example:

In the number 368.429,

the number 3 represents 3 hundreds i.e. 300;

the number 6 represents 6 tens i.e. 60;

the number 8 represents 8 ones/units i.e. 8;

the number 4 represents 4 tenths i.e. $\frac{4}{10}$;

the number 2 represents 2 hundredths i.e. $\frac{2}{100}$;

the number 9 represents 9 thousandths i.e. $\frac{9}{1000}$.

Recurring Decimals

If we try to express $\frac{1}{3}$ as a decimal we get

$$\frac{1}{3} = 1 \div 3 = 0.33333333\ldots$$

The 3's go on indefinitely so we call $0.333333\ldots$ a **Recurring Decimal.**

It can also be written as $0.\dot{3}$

The dot above the 3 tells us that it is recurring.

2.6 Changing Decimals to Fractions

If there is one digit to the right hand side of the decimal point we place the number over 10 and simplify if possible.

Example:

$$0.7 = \frac{7}{10}$$

$$2.3 = \frac{23}{10} = 2\frac{3}{10}$$

If there are two digits to the right hand side of the decimal point we place the number over 100 and simplify if possible.

Example:

$$0.75 = \frac{75}{100} = \frac{3}{4}$$

$$1.45 = \frac{145}{100} = 1\frac{45}{100} = 1\frac{9}{20}$$

If there are three digits to the right hand side of the decimal point we place the number over 1000 and simplify if possible.

Example:

$$0.765 = \frac{765}{1000} = \frac{153}{200}$$

$$2.375 = \frac{2375}{1000} = 2\frac{375}{1000} = 2\frac{3}{8}$$

and so on....

Exercises 7

Write Each of the Following Decimals as a Fraction then Simplify Where Possible

1. 0.5

2. 0.3

3. 0.45

4. 0.67

5. 1.25

6. 3.1

7. 0.258

8. 1.355

9. 21.8

10. 32.455

2.7 Changing Fractions to Decimals

To change a fraction to a decimal we divide the denominator (bottom number) into the numerator (top number).

Example:

$\frac{2}{5} = 2 \div 5 = 0.4$

$4\frac{3}{8} = \frac{35}{8} = 35 \div 8 = 4.375$

Exercises 8

Write Each of the Following Fractions as a Decimal

1. $\frac{4}{10}$

2. $\frac{6}{10}$

3. $\frac{2}{5}$

4. $\frac{1}{8}$

5. $\frac{3}{100}$

6. $\frac{45}{100}$

7. $\frac{25}{50}$

8. $\frac{3}{75}$

9. $\frac{35}{100}$

10. $\frac{40}{65}$

2.8 Decimals - Problems

Problem 1

A certain mobile phone company is offering a deal to its bill paying customers.
150 free text messages to any network within Ireland and the first 200 minutes of calls free to all mobile phone and landline numbers in Ireland for €50.
Additional text messages to any network within Ireland cost 6.5 cent each and phone calls are charged at a rate of 17.3 cent per minute .

Calculate the following bills:

1. 314 minutes of calls and 198 text messages.
2. 623.14 minutes of calls and 214 text messages.
3. 721.1 minutes of calls and 329 text messages.

Problem 2

Gas bills are composed of a standing charge plus a further charge per unit used. If the standing charge is €8.549 and the cost per unit is 4.509 cent, determine the total cost of a bill if

1. 150 units are used,
2. 225 units are used,
3. 530 units are used.

Problem 3

A car achieves 21.7 kilometres to the litre under normal driving conditions. Supposing the petrol tank holds 65.12 litres, what distance should the driver expect to travel on one full tank of petrol?

2.9 Percentages

The word **Percent** means **Per Hundred** and is represented mathematically by the symbol $\%$. For example, if you received 85 marks out of 100 in an assessment, you would get a mark of 85%.

Barack Obama received 53% of the votes in the 2008 US presidential election. This means that out of every 100 people who voted, 53 voted for him.

2.10 Changing Percentages to Fractions

- Write percentage over 100
- Simplify/cancel down to lowest terms

Example:

$$26\% = \frac{26}{100} = \frac{13}{50}$$

$$40\% = \frac{40}{100} = \frac{2}{5}$$

$$10\frac{1}{2}\% = \frac{10.5}{100} = \frac{105}{1000} = \frac{21}{200}$$

Exercises 9

Change the Following Percentages into Fractions

1. 25%

2. 36%

3. 14%

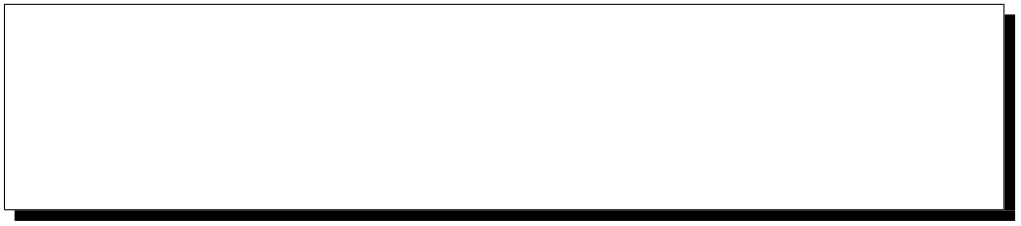
4. $37\frac{1}{2}\%$

5. $66\frac{2}{3}\%$

6. 1%

7. 78%

8. 55%



9. 125%

10. 0.5%

2.11 Changing Fractions to Percentages

- Multiply by $\frac{100}{1}\%$

Example:

$$\frac{1}{2} = \frac{1}{2} \times \frac{100}{1}\% = \frac{100}{2}\% = 50\%$$

$$\frac{3}{5} = \frac{3}{5} \times \frac{100}{1}\% = \frac{300}{5}\% = 60\%$$

$$\frac{1}{3} = \frac{1}{3} \times \frac{100}{1}\% = \frac{100}{3}\% = 33.33\%$$

Exercises 10

Change the Following Fractions into Percentages

1. $\frac{2}{5}$

2. $\frac{3}{7}$

3. $\frac{1}{2}$

4. $\frac{9}{4}$

5. $\frac{15}{2}$

6. $\frac{3}{100}$

7. $\frac{4}{25}$

8. $\frac{55}{3}$

9. $\frac{150}{11}$

10. $\frac{3}{2}$

2.12 Changing Percentages to Decimals

- Write percentage as a fraction with 100 as the denominator
- Write as a decimal

Example:

$$58\% = \frac{58}{100} = 0.58$$

$$10\frac{1}{2}\% = \frac{10.5}{100} = \frac{105}{1000} = 0.105$$

2.13 Changing Decimals to Percentages

- Write decimal as a fraction with 100 as the denominator
- The numerator is the percentage

Example:

$$0.38 = \frac{38}{100} = 38\%$$

$$0.125 = \frac{125}{1000} = \frac{12.5}{100} = 12.5\%$$

Exercises 11

Change the Following Decimals into Percentages

1. 0.9

2. 0.85

3. 0.7

4. 0.23

5. 1.95

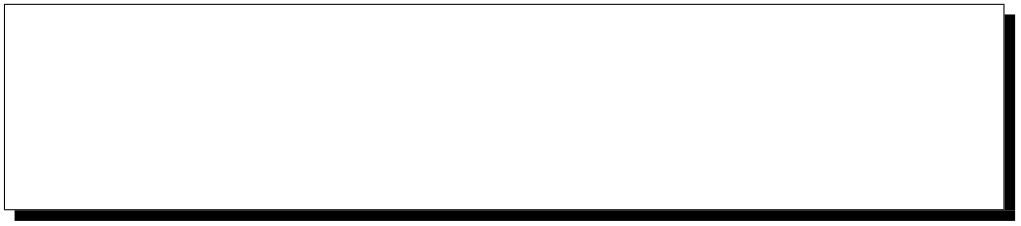
6. 2.54

7. 10.03

8. 0.5

9. 3.33

10. 1.0



2.14 Expressing One Number as a Percentage of Another Number

$$\frac{\text{First Number}}{\text{Second Number}} \times \frac{100}{1}\%$$

Example 1

Express 15 as a percentage of 180.

$$\frac{15}{180} \times \frac{100}{1} = \frac{1500}{180} = 8.33\%$$

Example 2

If the price of petrol has increased from €1.20 per litre to €1.30 per litre, find the percentage increase in price.

Increase = €1.30 - €1.20 = €0.10

$$\% \text{ Increase} = \frac{\text{Increase}}{\text{Original Price}} \times \frac{100}{1}\%$$

$$\% \text{ Increase} = \frac{0.10}{1.20} \times \frac{100}{1}$$

$$\% \text{ Increase} = \frac{1}{12} \times \frac{100}{1}$$

$$\% \text{ Increase} = \frac{100}{12}$$

$$\% \text{ Increase} = 8.33\%$$

Example 3

Find 36% of $4,500$.

The word *of* in mathematics means multiply.

36% of $4,500$

$$\Rightarrow 36\% \times 4,500$$

$$= \frac{36}{100} \times \frac{4,500}{1}$$

$$= \frac{162,000}{100}$$

$$= 1,620.$$

Exercises 12

Find

1. 40% of 800

2. 35% of $1,234$

3. 2% of 90

4. 20% of 1

5. 25% of $1,000$

6. $4\frac{1}{2}\%$ of 300

7. 6.5% of 50

8. 80% of 80

9. 75% of 200

10. 11% of 20

Exercises 13

Express the First Number as a Percentage of the Second Number in Each of the Following

1. $5 : 25$

2. $3 : 60$

3. $10 : 1,000$

4. $15 : 165$

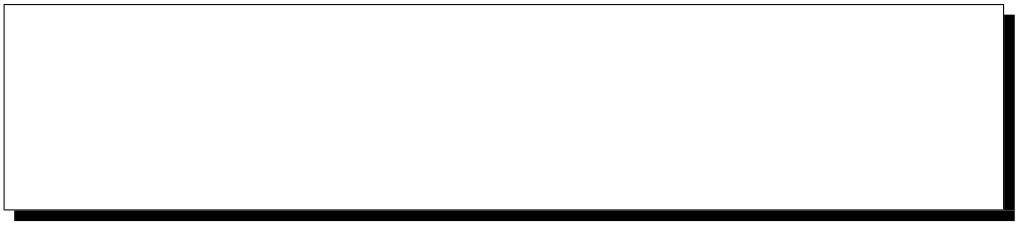
5. $25 : 400$

6. 2 : 150

7. 25 c: €1.75

8. 30 c: €6.60

9. 50 cm: 2m



10. 65 ml: 3.2l

2.15 Percentages - Problems

Problem 1

Students within the Department of Mathematics and Statistics fall into one of three categories: Undergraduate, Postgraduate or Doctoral, as shown in the table below.

Express the number of students in each category as a percentage of the total number of students within the Department.

Category	Number of Students
Undergraduate	188
Postgraduate	42
Doctoral	19

Problem 2

In 1991 Carl Lewis set a world record of 9.92 seconds for the men's 100 metres race. In the 2009 World Championships in Berlin, Usain Bolt set a new record of 9.58 seconds for the same race. Calculate the percentage difference in times.

Problem 3

A survey was carried out to investigate if Irish people would vote yes in the Lisbon Treaty referendum.

In all, 800 people were questioned, 55% of whom were female.

45% of the females questioned said they would vote yes and 40% of the males who were surveyed indicated they would vote yes.

How many of the females questioned stated that they would vote yes in the survey?

How many of the males questioned stated that they would vote no in the survey?

Problem 4

Orla has a number of deductions from her salary each month.

8% goes towards her pension fund. 21% is paid in tax. $7\frac{1}{2}\%$ goes on her car loan.

If her gross monthly wage is €$2,464$, how much does Orla take home at the end of the month?

2.16 Answers

Exercises 1:

1). 3 2). 1 3). 28 4). 110
5). 7 6). 3 7). 24 8). 3
9). 12 10). 20

Exercise 2:

1). $1\frac{11}{20}$ 2). $\frac{11}{21}$ 3). $1\frac{4}{15}$ 4). $1\frac{1}{45}$

5). $\frac{7}{24}$

Exercise 3:

1). $\frac{14}{5}$ 2). $\frac{79}{6}$ 3). $\frac{65}{9}$ 4). $\frac{81}{10}$

5). $\frac{32}{3}$

Exercise 4:

1). $3\frac{3}{10}$ 2). $1\frac{8}{9}$ 3). $12\frac{5}{6}$ 4). $21\frac{1}{3}$

5). $7\frac{1}{2}$

Exercise 5:

1). $10\frac{11}{12}$ 2). $5\frac{13}{30}$ 3). $-8\frac{3}{4}$ 4). $2\frac{10}{21}$

5). $-4\frac{1}{2}$

Exercise 6:

1). $1\frac{1}{30}$ 2). $\frac{8}{15}$ 3). $\frac{1}{9}$ 4). $1\frac{151}{224}$

5). $\frac{9}{49}$ 6). $4\frac{4}{9}$ 7). $9\frac{5}{11}$ 8). $65\frac{1}{3}$

9). $\frac{3}{16}$ 10). $-\frac{32}{135}$

Fractions - Problem 2: 15 minutes

Fractions - Problem 3: 440

Exercise 7:

1). $\frac{1}{2}$ 2). $\frac{3}{10}$ 3). $\frac{9}{20}$ 4). $\frac{67}{100}$

5). $1\frac{1}{4}$ 6). $3\frac{1}{10}$ 7). $\frac{129}{500}$ 8). $\frac{71}{200}$

9). $21\frac{4}{5}$ 10). $32\frac{91}{200}$

Exercises 8:

1). 0.4 2). 0.6 3). 0.4 4). 0.125
5). 0.03 6). 0.45 7). 0.5 8). 0.04
9). 0.35 10). 0.615

Decimals - Problem 1:

1. €72.84
2. €127.36
3. €151.79

Decimals - Problem 2:

1. €15.31
2. €18.69
3. €32.45

Decimals - Problem 3: 1413.104km

Exercises 9:

1). $\frac{1}{4}$ 2). $\frac{9}{25}$ 3). $\frac{7}{50}$ 4). $\frac{3}{8}$
5). $\frac{2}{3}$ 6). $\frac{1}{100}$ 7). $\frac{39}{50}$ 8). $\frac{11}{20}$
9). $1\frac{1}{4}$ 10). $\frac{1}{200}$

Exercises 10:

1). 40% 2). 42.9% 3). 50% 4). 225%
5). 750% 6). 3% 7). 16% 8). 1833.33%
9). 1363.64% 10). 150%

Exercises 11:

1). 90% 2). 85% 3). 70% 4). 23%
5). 195% 6). 254% 7). 1003% 8). 50%
9). 333% 10). 100%

Exercises 12:

1). 320	2). 431.9	3). 1.8	4). 0.2
5). 250	6). 13.5	7). 3.25	8). 64
9). 150	10). 2.2		

Exercises 13:

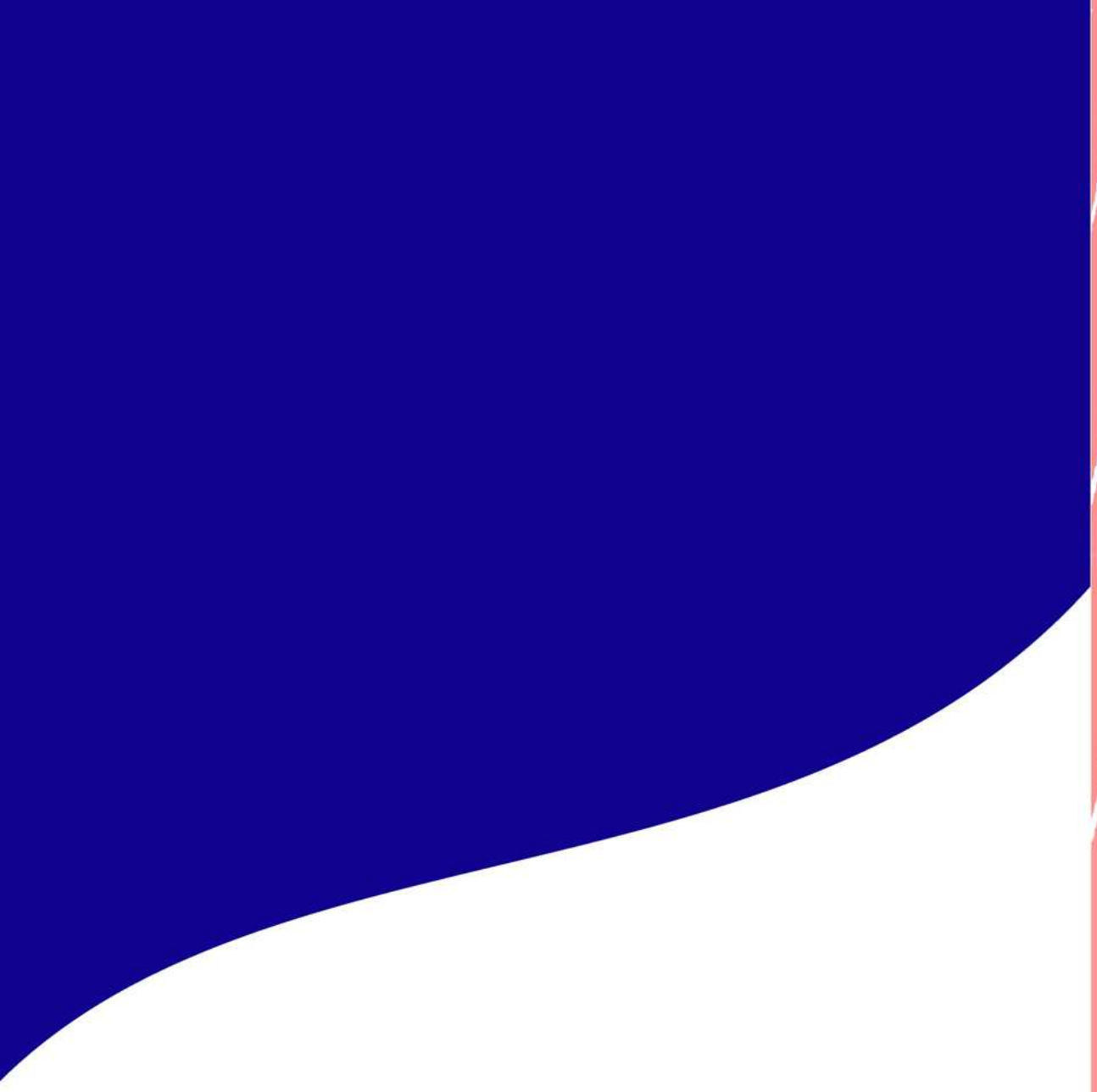
1). 20%	2). 5%	3). 1%	4). 9.09%
5). 6.25%	6). 1.33%	7). 14.29%	8). 4.55%
9). 25%	10). 2.03%		

Percentages - Problem 1: $75.5\%, 16.87\%, 7.63\%$

Percentages - Problem 2: 3.43%

Percentages - Problem 3: $198, 216$

Percentages - Problem 4: €1564.64

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