Optimization Theory and Applications

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Introduction

- So far, we have considered unconstrained (or only set constrainted) optimization problems:

$$\text{minimize} \;\; f(\mathbf{x})$$

where $f : \mathbb{R}^n \to \mathbb{R}$

- We now turn to problems with *constraints*
- In constrained problems, the solution must lie inside some prespecified set, called the *constraint set* or *feasible set*

Introduction

- General constrained optimization problem:

$$\text{minimize} \quad f(\mathbf{x})$$

$$\text{subject to} \quad \mathbf{x} \in \Omega$$

  where $\Omega \subset \mathbb{R}^n$ is the constraint set

- Points in Ω are called *feasible points*
- Example: $\Omega = \{\mathbf{x} : x_i \geq 0, i = 1, \ldots, n\}$
- The desired solution $\mathbf{x}^*$ must lie inside Ω

Introduction

- To begin, we first consider a special class of constrained optimization problems
- Consider the case where

$$f(\mathbf{x}) = \mathbf{c}^T \mathbf{x}$$

  where $\mathbf{c} \in \mathbb{R}^n$ is a given vector
- Note that f is a linear function
- Example: $f(\mathbf{x}) = 3x_1 - 4x_2 + x_3$, here

$$\mathbf{c}^T = [3, -4, 1]$$

Introduction

- Next, consider

$$\Omega = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \text{ and } \mathbf{x} \geq 0\}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$

- The notation $\mathbf{x} \geq 0$ means that every component of $\mathbf{x}$ must be ≥ 0
- Example: $\Omega = \{\mathbf{x} : 4x_1 + x_2 = 5, x_1, x_2 \geq 0\}$, here

$$\mathbf{A} = [4, 1], \qquad \mathbf{b} = [5]$$

Introduction

- We can write the problem in a standard form as

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

- Name: ***Linear programming*** (LP) problem, or ***linear program***
- Note that the objective function and constraint set are expressed using linear functions
- The constraint set involves equations and inequalities

Brief History of Linear Programming

- Problems exist for thousands of years
- Efficient methods for solving LP problems appear in 1930s
- In 1939, Kantorovich presented a number of solutions to some production and transportation planning problems
- During WWII, Koopmans contributed significantly to the solution of transportation problems
- Kantorovich and Koopmans were awarded Nobel Prize in Economics in 1975 for contributions on the theory of optimal allocation of resources

Brief History of Linear Programming

- In 1947, Dantzig developed the "***simplex method***" which is efficient and elegant, and has been declared one of the 10 algorithms with the greatest influence on the development and practice of science and engineering in the twentieth century

- In 1984, Karmarkar developed an algorithm with polynomial complexity which leads "***Interior point methods***" which is a class of algorithms that solves linear and nonlinear convex optimization problems efficiently

Examples of Linear Programming Problems

- *Example 1*: Produces four products: X_1, X_2, X_3 and X_4
- Three inputs to this production process; labor in person-weeks, kilograms of raw material A, and boxes of raw material B
- The manufacturer cannot use more than the available amounts of labor and the two raw materials
- Every production decision must satisfy the restrictions on the availability of inputs

	Product				Input
Inputs	X_1	X_2	X_3	X_4	Availabilities
Person-weeks	1	2	1	2	20
Kilograms of material A	6	5	3	2	100
Boxes of material B	3	4	9	12	75
Production levels	x_1	x_2	x_3	x_4	

Examples of Linear Programming Problems

- Constraints:

$$\begin{aligned} & x_1 + 2x_2 + x_3 + 2x_4 \leq 20 \\ & 6x_1 + 5x_2 + 3x_3 + 2x_4 \leq 100 \\ & 3x_1 + 4x_2 + 9x_3 + 12x_4 \leq 75 \end{aligned}$$

  Also,

$$x_i \geq 0, \qquad i = 1, 2, 3, 4$$

- Suppose unit product revenue of X_1, X_2, X_3, X_4 are 6,4,7,and 5, the total revenue for any production decision (x_1, x_2, x_3, x_4) is

$$f(x_1, x_2, x_3, x_4) = 6x_1 + 4x_2 + 7x_3 + 5x_4$$

- The problem is then to maximize f subject to given constraints

Examples of Linear Programming Problems

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \mathbf{0}, \end{aligned}$$

where

$$\mathbf{c}^\top = [6, 4, 7, 5],$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 6 & 5 & 3 & 2 \\ 3 & 4 & 9 & 12 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 20 \\ 100 \\ 75 \end{bmatrix}$$

Examples of Linear Programming Problems

- *Example 2*: Optimal diet
- Nutrition table

Vitamin	Food type: Milk	Food type: Eggs	Daily Requirements
V	2	4	40
W	3	2	50
Intake	x_1	x_2	
Unit cost	3	5/2	

- Total cost: $3x_1 + 5x_2/2$
- Dietary constraints:

  Vitamin V: $2x_1 + 4x_2 \geq 40$

  Vitamin W: $3x_1 + 2x_2 \geq 50$
- Physical constraint: $x_1, x_2 \geq 0$

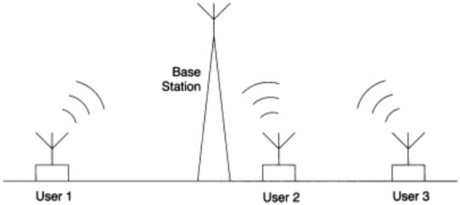
Examples of Linear Programming Problems

- Optimization problem

$$
\begin{array}{rl}
\text{minimize} & 3x_1 + \dfrac{5}{2}x_2 \\
\text{subject to} & 2x_1 + 4x_2 \geq 40 \\
& 3x_1 + 2x_2 \geq 50 \\
& x_1, x_2 \geq 0.
\end{array}
$$

Examples of Linear Programming Problems

- Consider the following wireless communication system



- There are n mobile users each transmits a signal to the base station with power p_i and an attenuation factor of h_i (i.e., the actual signal power received at the base station from user i is $h_i p_i$)

Examples of Linear Programming Problems

- *Concept of interference*: when the base station is receiving from user i, the total power received from all other users is considered *interference* (i.e., the interference for user i is $\sum_{j \neq i} h_j p_j$)
- For user i, the Signal-to-interference ratio must exceed a threshold γ_i
- Question: how to minimize the total power transmitted by all users subject to having reliable communications for all users?

Examples of Linear Programming Problems

- The total power transmitted is $p_1 + \cdots + p_n$. The signal-to-interference ratio for user i is

$$\text{SIR } = \frac{h_i p_i}{\sum_{j \neq i} h_j p_j}$$

- Hence, the problem can be written as

$$\begin{array}{ll} \text{minimize} & p_1 + \cdots + p_n \\ \text{subject to} & \frac{h_i p_i}{\sum_{j \neq i} h_j p_j} \geq \gamma_i, \quad i- = 1, \ldots, n \\ & p_1, \ldots, p_n \geq 0 \end{array}$$

Examples of Linear Programming Problems

- We can write the above as the linear programming problem

$$\begin{array}{rl} \text{minimize} & p_1 + \cdots + p_n \\ \text{subject to} & h_i p_i - \gamma_i \sum_{j \neq i} h_j p_j \geq 0, \ \ i = 1, \ldots, n \\ & p_1, \ldots, p_n \geq 0. \end{array}$$

In matrix form, we have

$$\boldsymbol{c} = [1, \ldots, 1]^\top$$

$$\boldsymbol{A} = \begin{bmatrix} h_1 & -\gamma_1 h_2 & \cdots & -\gamma_1 h_n \\ -\gamma_2 h_1 & h_2 & \cdots & -\gamma_2 h_n \\ \vdots & & \ddots & \vdots \\ -\gamma_n h_1 & -\gamma_n h_2 & \cdots & h_n \end{bmatrix}, \quad \boldsymbol{b} = \boldsymbol{0}.$$

Two-dimensional Linear Programs

- Many fundamental concepts of linear programming are easily illustrated in two-dimensional space
- We consider linear problems in $\mathbb{R}^2$ first
- Consider the following LP

$$
\begin{aligned}
\text{maximize} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\
& \boldsymbol{x} \ge \mathbf{0},
\end{aligned}
$$

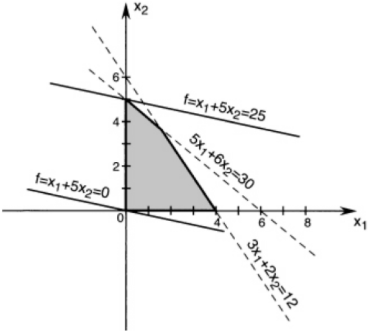
where $\boldsymbol{x} = [x_1, x_2]^\top$ and

$$\boldsymbol{c}^\top = [1, 5],$$

$$\boldsymbol{A} = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 30 \\ 12 \end{bmatrix}.$$

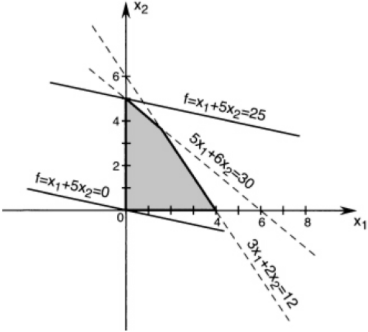
Two-dimensional Linear Programs

- The set of equations $\{f = \mathbf{c}^T\mathbf{x} = x_1 + 5x_2\}$ specifies a family of straight lines in $\mathbf{R}^2$
- Solve the problem using geometric arguments
  - We first draw the constraints in $\mathbb{R}^2$, which is a polyhedron
  - We then draw level sets of the objective function on the same diagram



Two-dimensional Linear Programs

- Maximizing $\mathbf{c}^T\mathbf{x} = x_1 + 5x_2$ subject to the constraints equals finding the straight line $f = x_1 + 5x_2$ that intersects the shaded region and has the largest f



Linear Programs

- Observation:
  - The solution lies on the boundary of the constraint set
  - In fact, unless the level sets happen to be parallel to one of the edges of the constraint set, the solution will lie on a corner point (or vertex)
  - Even if the level sets happen to be parallel to one of the edges of the constraint set, a corner point will be an optimal solution
- It turns out that the solution of an LP problem (if it exists) always lies on a vertex of the constraint set
- Therefore, instead of looking for candidate solutions everywhere in the constraint set, we need only focus on the vertices

Standard Form LP Problems

- For the purpose of analyzing and solving LP problems, we will consider only problems in a particular *standard form*
- LP problem in *standard form*:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

where $\mathbf{A}$ is $m \times n$, $m < n$, $\text{rank}(\mathbf{A}) = m$, and $\mathbf{b} \geq 0$

Standard Form LP Problems

- Examples:
- In standard form:

$$
\begin{aligned}
\text{minimize} \quad & 3x_1 + 5x_2 - x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + 4x_3 = 4 \\
& -5x_1 - 3x_2 + x_3 = 15 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

- Not in standard form:

$$
\begin{aligned}
\text{maximize} \quad & 3x_1 + 5x_2 - x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + 4x_3 \leq -4 \\
& -5x_1 - 3x_2 + x_3 \geq 15 \\
& x_2 \leq 0.
\end{aligned}
$$

Standard Form LP Problems

- All our analyses and algorithms will apply only to standard form LP problems
- What about other variations of LP problems?
  - Any LP problem can be converted into an equivalent standard form LP problem
  - Equivalent: solution to one problem gives the solution to the other

Standard Form LP Problems

- How to convert from given LP problem to an equivalent problem in standard form
  - If problem is a maximization, simply multiply the objective function by -1 to get minimization
  - If $\mathbf{A}$ is not of full rank, can remove one or more rows
  - If a component of $\mathbf{b}$ is negative, say the ith component, multiply the ith constraint by -1 to obtain a positive right-hand side
  - What about inequality constraints?

Converting to Standard Form: Slack Variables

- Suppose we have a constraint of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

- We can convert the above inequality constraint into the standard equality constraint by introducing a slack variable
- Specifically, the above constraint is equivalent to

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + x_{n+1} = b$$
$$x_{n+1} \geq 0$$

Converting to Standard Form: Surplus Variables

- Suppose we have a constraint of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$$

- We can convert the above inequality constraint into the standard equality constraint by introducing a surplus variable

- Specifically, the above constraint is equivalent to

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n - x_{n+1} = b$$
$$x_{n+1} \geq 0$$

Converting to Standard Form: Nonpositive variable

- Suppose one of the variables (say x_1) has the constraint

$$x_1 \leq 0$$

- We can convert the variable into the usual nonnegative variable by changing every occurrence of x_1 by its negative $x_1' = -x_1$
- Example: Consider the constraint

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

$$x_1 \leq 0$$

- By introducing $x_1' = -x_1$, we obtain the following equivalent constraint

$$-a_1 x_1' + a_2 x_2 + \cdots + a_n x_n = b$$

$$x_1' \geq 0$$

Converting to Standard Form: Free Variables

- Suppose one of the variables (say x_1) does not have a nonnegativity constraint (e.g., there is no $x_1 \geq 0$ constraint)
- We can introduce variables $u_1 \geq 0$ and $v_1 \geq 0$ and replace x_1 by $u_1 - v_1$
- Example: the constraint

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is equivalent to

$$a_1(u_1 - v_1) + a_2x_2 + \cdots + a_nx_n = b$$

$$u_1, v_1 \geq 0$$

Converting to Standard Form: An Example

- Consider the LP problem (not in standard form)

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 5x_2 - x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 4x_3 \le -4 \\ & -5x_1 - 3x_2 + x_3 \ge 15 \\ & x_2 \le 0, x_3 \ge 0 \end{aligned}$$

- Convert it to a standard form
- How?

Converting to Standard Form: An Example

- We first convert the problem into a minimization problem by multiplying the objective function by -1
- Next, we introduce a slack variable $x_4 \geq 0$ to convert the first inequality constraint into

$$x_1 + 2x_2 + 4x_3 + x_4 = -4$$

- We then multiply the equation by -1 to make the right hand side positive

$$-x_1 - 2x_2 - 4x_3 - x_4 = 4$$

- Next, we introduce a surplus variable $x_5 \geq 0$ to convert the second inequality constraint into

$$-5x_1 - 3x_2 + x_3 - x_5 = 15$$

Converting to Standard Form: An Example

- Next, we substitute $x_1 = u_1 - v_1$, with $u_1, v_1 \geq 0$
- Finally, we replace $x_2 = -x_2'$ in the equations
- Resulting equivalent LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & -3(u_1 - v_1) + 5x_2' + x_3 \\ \text{subject to} \quad & -(u_1 - v_1) + 2x_2' - 4x_3 - x_4 = 4 \\ & -5(u_1 - v_1) + 3x_2' + x_3 - x_5 = 15 \\ & u_1, v_1, x_2', x_3, x_4, x_5 \geq 0 \end{aligned}$$

Basic Solutions

- Consider LP problem in standard form:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

  where $\mathbf{c} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m, m < n$, rank $\mathbf{A} = m$, and $\mathbf{b} \geq 0$

- The *feasible points* are those $\mathbf{x}$ that satisfy $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$

Basic Solutions

- Recall the conditions for having solutions of a linear equations system
  - The system has solution if $\text{rank}[\mathbf{A}] = \text{rank}[\mathbf{A}, \mathbf{b}]$
  - The solution is unique if $\text{rank}[\mathbf{A}] = n$
  - There are infinite solutions if $\text{rank}[\mathbf{A}] < n$
- Since $m < n$, there are infinitely many points $\mathbf{x}$ satisfying $\mathbf{Ax} = \mathbf{b}$

Basic Solutions

- How to find points $\mathbf{x}$ satisfying $\mathbf{A}\mathbf{x} = \mathbf{b}$?
- Simple way of solving linear equations: use elementary row operations
- Recall three kinds of elementary row operations that we perform on a given matrix
  - Interchanging two rows of the matrix
  - Multiplying one row of the matrix by a (nonzero) constant
  - Adding a constant times one row to another row
- We can find solutions to $\mathbf{A}\mathbf{x} = \mathbf{b}$ using elementary row operations

Basic Solutions

- First, we form the "augmented matrix" $[\mathbf{A}, \mathbf{b}]$
- Assume that the first m columns of $\mathbf{A}$ are linearly independent
- Using elementary row operations, we can reduce the augmented matrix $[\mathbf{A}, \mathbf{b}]$ into the form

$$\begin{bmatrix} 1 & & & 0 & \cdots & y_1 \\ & 1 & & & \cdots & y_2 \\ & & \ddots & & \cdots & \vdots \\ 0 & & & 1 & & y_m \end{bmatrix}$$

- One specific solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{bmatrix} \mathbf{y}_0 \\ 0 \end{bmatrix}$$

where $\mathbf{y}_0 = [y_1, \ldots, y_m]^T$

Basic Solutions

- Example: use elementary operations to find a specific solution to

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

Basic Solutions

- Equivalently, we reorder the matrix $\mathbf{A}$, and let $\mathbf{B}$ be a square matrix whose columns are m linearly independent columns of $\mathbf{A}$
- Now $\mathbf{A} = [\mathbf{B}, \mathbf{D}]$, where $\mathbf{D}$ is an $m \times (n-m)$ matrix
- The matrix $\mathbf{B}$ is nonsingular, and we can solve the equation

  $$\mathbf{B}\mathbf{x}_B = \mathbf{b}$$

  and the solution is $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$
- Let $\mathbf{x} = [\mathbf{x}_B^T, \mathbf{0}^T]^T$. Then, $\mathbf{x}$ is a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$
- Note that $\mathbf{x}_B^T = \mathbf{y}_0$

Basic Solutions

- Definition: We call $[\mathbf{x}_B^T, \mathbf{0}^T]^T$ a ***basic solution*** to $\mathbf{A}\mathbf{x} = \mathbf{b}$ with respect to the basis $\mathbf{B}$. We refer to the components of the vector $\mathbf{x}_B$ as ***basic variables*** and the columns of $\mathbf{B}$ as ***basic columns***

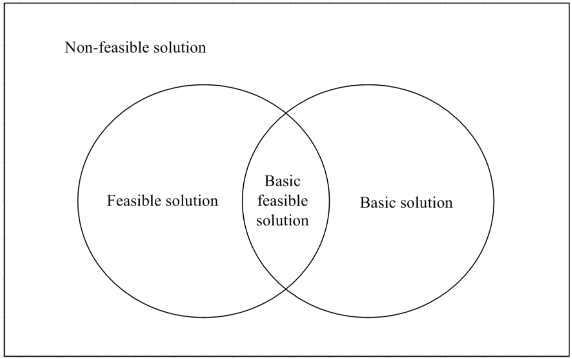
- E.g., consider the basic solution
$$\mathbf{x} = [x_1, x_2, \ldots, x_m, 0, \ldots, 0]^T$$
- The variables $x_1, \ldots, x_m$ are basic variables, while $x_{m+1}, \ldots, x_n$ are nonbasic variables

- Note that the value of nonbasic variables is always 0

Basic Solutions

- A vector $\mathbf{x}$ satisfying $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq 0$ is a ***feasible solution***
- A feasible solution that is also basic is called a ***basic feasible solution*** (BFS)
- Note that a basic solution $\mathbf{x}$ is feasible if every basic variable is ≥ 0
- If some of the basic variables of a basic solution are 0, then the basic solution is a ***degenerate basic solution***
- If the basic feasible solution is a degenerate basic solution, then it is called a ***degenerate basic feasible solution***

Basic Solutions



Basic Solutions

- Example: Consider the equation $\mathbf{Ax} = \mathbf{b}$ with
$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$
  - For the basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2]$, the basic solution is $\mathbf{x} = [6, 2, 0, 0]^T$ (basic feasible solution)
  - For the basis $\mathbf{B} = [\mathbf{a}_2, \mathbf{a}_3]$, the basic solution is $\mathbf{x} = [0, 2, -6, 0]^T$ (how about this one?)
  - For the basis $\mathbf{B} = [\mathbf{a}_3, \mathbf{a}_4]$, the basic solution is $\mathbf{x} = [0, 0, 0, 2]^T$ (how about this one?)
  - For the basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_4]$, the basic solution is $\mathbf{x} = [0, 0, 0, 2]^T$
  - For the basis $\mathbf{B} = [\mathbf{a}_2, \mathbf{a}_4]$, the basic solution is $\mathbf{x} = [0, 0, 0, 2]^T$

Basic Solutions

- Note that $\mathbf{x} = [6, 2, 0, 0]^T$ is a basic feasible solution
- $\mathbf{x} = [0, 0, 0, 2]^T$ is a degenerate basic feasible solution
- $\mathbf{x} = [3, 1, 0, 1]^T$ is a solution, but it is not basic
- $\mathbf{x} = [0, 2, -6, 0]^T$ is not a feasible solution

Basic Solutions

- *Example*: Consider the equation $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & 1 & 1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

- We form the augmented matrix $[\mathbf{A}, \mathbf{b}]$ of the system

$$[\mathbf{A}, \mathbf{b}] = \begin{bmatrix} 2 & 3 & -1 & -1 & -1 \\ 4 & 1 & 1 & -2 & 9 \end{bmatrix}$$

- Using elementary row operations, we transform this matrix into the form

$$\begin{bmatrix} 1 & 0 & \frac{2}{5} & -\frac{1}{2} & \frac{14}{5} \\ 0 & 1 & -\frac{3}{5} & 0 & -\frac{11}{5} \end{bmatrix}$$

- The total number of possible basic solutions is at most

$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!} = \frac{4!}{2!(4-2)!} = 6$$

Basic Solutions

- Check the feasibility of each of the basic solutions
  - For the basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2]$, set $x_3 = x_4 = 0$ and solve $\mathbf{B}\mathbf{x}_B = \mathbf{b}$, we have $\mathbf{x}_B = [14/5, -11/5]^T$, and hence $\mathbf{x} = [14/5, -11/5, 0, 0]^T$ is a basic solution, but is not feasible
  - For the basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_3]$, $\mathbf{x} = [4/3, 0, 11/3, 0]^T$ is a basic feasible solution
  - For the basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_4]$
  $$\mathbf{B} = [\mathbf{a_1}, \mathbf{a}_4] = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$
  is singular. Therefore, $\mathbf{B}$ cannot be a basis
  - For the basis $\mathbf{B} = [\mathbf{a}_2, \mathbf{a}_3]$, $\mathbf{x} = [0, 2, 7, 0]^T$ is a basic feasible solution
  - For the basis $\mathbf{B} = [\mathbf{a}_2, \mathbf{a}_4]$, $\mathbf{x} = [0, -11/5, 0, -28/5]^T$ is a basic solution, but is not feasible
  - For the basis $\mathbf{B} = [\mathbf{a}_3, \mathbf{a}_4]$, $\mathbf{x} = [0, 0, 11/3, -8/3]^T$ is a basic solution, but is not feasible

Properties of Basic Solutions

- We will show that when solving LP problem, we need only consider *basic feasible solutions*, since the optimal value (if exists) is always achieved at a basic feasible solution
- ***Definition:*** Any vector $\mathbf{x}$ that yields the minimum value of the objective function $\mathbf{c}^T\mathbf{x}$ over the set of vectors satisfying the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$ is an ***optimal feasible solution***
- An optimal feasible solution that is basic is an ***optimal basic feasible solution***

Properties of Basic Solutions

- ***Fundamental Theorem of LP:*** Consider a linear program in standard form
  - If there exists a feasible solution, then there exists a basic feasible solution
  - If there exists an optimal feasible solution, then there exists an optimal basic feasible solution

Consequences of FTLP

- The fundamental theorem of LP reduces the task of solving a LP problem to that of searching over the basic feasible solutions
- We only need to check basic feasible solutions for optimality
- There are only a finite number of BFSs, in fact, at most
$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$
- Note that though the number is finite, it could be very large
- For example $m = 5$ and $n = 50$, then
$$\binom{50}{5} = 2118760$$
- The brute force approach of exhaustively comparing all the BFSs is impractical
- Therefore, a more efficient method of solving LP is needed

Geometric View of Linear Programs

- We analyze a geometric interpretation of the above fundamental theorem of LP, which leads to the simplex method for solving LP problems
- Recall a set $\Theta \subset \mathbb{R}^n$ is convex if for every $\mathbf{x}, \mathbf{y} \in \Theta$ and every real number α, $0 < \alpha < 1$, the point

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in \Theta$$

Geometric View of Linear Programs

- ***Theorem***: if the feasible set

  $$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

  exists, then it is convex

- Proof:
  - Let $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the constraints
  - Then for all $\alpha \in (0, 1)$,
    $\mathbf{A}(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) = \alpha\mathbf{A}\mathbf{x}_1 + (1-\alpha)\mathbf{A}\mathbf{x}_2 = \mathbf{b}$
  - Also, for $\alpha \in (0, 1)$, we have $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \geq 0$

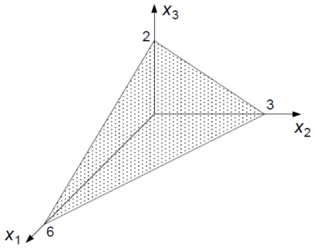
Geometric View of Linear Programs

- Recall that a point $\mathbf{x}$ in a convex set Θ is an ***extreme point*** of Θ if there are no two distinct points $\mathbf{x}_1$ and $\mathbf{x}_2$ in Θ such that $\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$ for some $\alpha \in (0,1)$
- An extreme point does not lie strictly within the line segment between two points
- Therefore, if $\mathbf{x}$ is an extreme point, and $\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$, then $\mathbf{x}_1 = \mathbf{x}_2$

Geometric View of Linear Programs

- Extreme points of the constraint set are equivalent to basic feasible solutions (BFS)
- Geometrically, a BFS corresponds to a ”corner” point (vertex) of the constraint set
- Example: consider the constraint $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$, where

$$\mathbf{A} = [1 \ \ 2 \ \ 3], \qquad b = [6]$$



- The BFSs are $[6, 0, 0]^T, [0, 3, 0]^T, [0, 0, 2]^T$

Geometric View of Linear Programs

- Formal FTLP
  - If there exists a feasible solution, then there exists a basic feasible solution
  - If there exists an optimal feasible solution, then there exists an optimal basic feasible solution
- Informal interpretation of the fundamental theorem of LP
  - If the feasible set is nonempty, then it has a vertex
  - If the problem has a minimizer (optimal), then one of the vertices is a minimizer

Geometric View of Linear Programs

- Implication:
  - The set of extreme points equals to the the set of basic feasible solutions
  - Together with fundamental theorem of LP, we can see that solving LP problems we need only examine the extreme points of the constraint set

Geometric View of Linear Programs

- *Example*: Consider the following LP problem:

$$\begin{array}{ll} \text{maximize} & 3x_1 + 5x_2 \\ \text{subject to} & x_1 + 5x_2 \leq 40 \\ & 2x_1 + x_2 \leq 20 \\ & x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{array}$$

- We introduce slack variables x_3, x_4, x_5 to convert this LP into standard form

$$\begin{array}{ll} \text{minimize} & -3x_1 - 5x_2 \\ \text{subject to} & x_1 + 5x_2 + x - 3 = 40 \\ & 2x_1 + x_2 + x_4 = 20 \\ & x_1 + x_2 + x_5 = 12 \\ & x_1, x_2, \ldots, x_5 \geq 0 \end{array}$$

Geometric View of Linear Programs

- We can represent the constraints as

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 40 \\ 20 \\ 12 \end{bmatrix}$$

  That is, $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 + x_5\mathbf{a}_5 = \mathbf{b}, \mathbf{x} \geq 0$

- Note that $\mathbf{x} = [0, 0, 40, 20, 12]^T$ is a feasible basic solution, and an extreme point of the feasible set, but is not an optimal point
- We need to search among other extreme points
- We move from one extreme point to an adjacent extreme point such that the value of the objective function decreases
- Definition: two extreme points are adjacent if the corresponding basic columns differ by only one vector

Geometric View of Linear Programs

- Begin with basis $[\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5]$, with $\mathbf{x} = [0, 0, 40, 20, 12]^T$
- Next, we move to the basis $[\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5]$, with $\mathbf{x} = [10, 0, 30, 0, 2]^T$, the objective function value is -30
- Next, we move to the basis $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, with $\mathbf{x} = [8, 4, 12, 0, 0]^T$, the objective function value is -40
- Next, we move to the basis $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, with $\mathbf{x} = [5, 7, 0, 3, 0]^T$, the objective function value is -50