Optimization Theory and Applications

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Outline of the Course

- Chapter 1: Introduction to Optimization
  - Overview of optimization
  - Examples of optimization problems
  - Mathematical models and classifications of optimization problems
  - Introduction to Matlab optimization toolbox
- Chapter 2: Preliminary Knowledges
  - Notations
  - Vector Space and Matrix
  - Geometry

Outline of the Course

- Chapter 3: Unconstrained Optimization Problem
  - Introduction
  - Conditions for local minimizers
- Chapter 4: One-dimensional Search Methods
  - Introduction
  - Golden section search
  - Bisection method
  - Secant method
  - Newton's method

Outline of the Course

- Chapter 5: Global Search Method
  - Simulated annealing method
  - Genetic algorithm
  - Particle swarm optimization
- Chapter 6: Linear Programming
  - Brief history of linear programming
  - Simple examples of linear programs
  - Standard form linear programs
  - Basic solutions
  - Application examples of linear programming

Outline of the Course

- Chapter 7: Integer Programming
  - Prototype example
  - Some BIP applications
  - Branch and bound method
  - Knapsack problem
  - Application examples of integer programming
- Chapter 8: Equality Constrained Nonlinear Programming
  - Basics of nonlinear programming
  - Equality constraints
  - The theorem of Lagrange
  - Second-order conditions

Outline of the Course

- Chapter 9: Inequality Constrained Nonlinear Programming
  - The theorem of Karush-Kuhn-Tucker
  - Using KKT conditions
  - Application examples
- Chapter 10: Convex Optimization
  - Introduction to optimization
  - Convex functions
  - Convexity and optimization
  - Application examples of convex optimization

Outline of the Course

- Chapter 11: Duality
  - The Lagrange dual function
  - The Lagrange dual problem
- Chapter 12: Reading Session
  - Presentation

Introduction

- Optimization is a discipline for studying how to choose certain actions for achieving the optimal objective under certain constraints
  - E.g., job hunting
- Optimization is part of Operational Research
- Optimization has been widely applied in different areas
  - Computer science, Communications, Manufacturing, Military
  - Transportation, Management
  - Economics, Finance

Course Objectives

- Introduction to optimization theory and methods, with applications in different areas
- Analysis of optimization problems
- Optimization algorithms and their analysis
- Ability to make precise statements about optimization problems
- Understand and apply optimization techniques for your own research

Benefits

- You get 3 credits
- Read papers more easily
- Possibly publish a paper after this course
- Find good jobs

Benefits

岗位要求：

1. 计算机、数据挖掘、应用数学、统计学、运筹学、计量经济学、量化研究或相关专业。

5、计算机，数学，统计学, 运筹学或相关专业，硕士以上学历；

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BASIC QUALIFICATIONS

· Strong analytical, mathematical and statistical skills.

amazon jobs

Operations Research Scientist
US, WA, Seattle | 职位 ID: 337975

Amazon Fulfillment Services is looking for a motivated individual with strong numerical optimization and analysis skills and practical inventory modeling experience to join our Modeling and Optimization...阅读更多内容

Course Prerequisites

- Working knowledge of linear algebra (matrix manipulations, vector spaces, bases, eigenvalues, quadratic forms)
- Working knowledge of calculus of several variables (differentiating functions of n variables, gradients, limits)
- Basic state-space systems in discrete time (desirable but not required)
- An appreciation of rigor
- Time and effort

Grading

- Homework: twice, 20%
- Quiz: once, 20%
- Final exam: 60%
- Course Project ?

Bonus

- Additional up to 5 points: if you can write an acceptable paper using any optimization, operational research, and game theory techniques for you own research problems
- Additional up to 5 points: if you can reproduce the results of a good paper using any optimization, operational research, and game theory techniques
  - List the name, institution of the authors
  - List the source, publication year, and citations
  - Write a brief report with results and submit the source code

Textbook and References

- E. K. P. Chong and S. H. Zak, An Introduction to Optimization, Fourth Edition, New York, 2013.
- S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
- J. Nocedal and S. J. Wright, Numerical Optimization, Springer, 2006.
- F. S. Hiller and G. J. Lieberman, Introduction to Operations Research, Tenth edition, McGraw-Hill Education, 2014.

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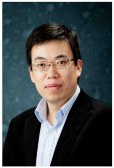

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Homework 1

- Find a famous research colleague whose work are highly related to optimization
- List 3 of her/his representative papers using optimization
- Find the optimization techniques used in these works
- Write a simple report showing the above items

How to Learn This Course

- Read various books and materials
- Read high quality related papers
- Practice
- Try it for your own problems
- Start from the simple one

Overview of Optimization Modeling Approach

- In this course, we focus on the mathematical methods of optimization
- Optimization studies are not just mathematical exercises
- Major phases of a typical optimization problem modeling

Overview of Optimization Modeling Approach

- Define the problem of interest and gather relevant data
- Formulate a mathematical model to represent the problem
- Deriving solutions from the model
- Test the model and refine it as needed
- Prepare to apply the model
- Implementation

Define the problem of interest and gather relevant data

- Practical problems are commonly described in a vague, imprecise way
- Problem definition is crucial. Difficult to extract a ”right” answer from the ”wrong” problem
- Determining appropriate objectives (for who, for what, for how long)
- Any constraints?
- Gather data

Formulating a Mathematical Model

- Reformulate the problem in a form that is convenient for analysis (quantitative analysis)
- Define the decision variables
- Define the objective function
- Define the constraints
- Determining the values of parameters (critical)
  - Uncertainty of the parameter
  - How the solution will change if the parameters changed to other plausible values?
  - Sensitivity analysis
- Precision vs Tractability
- Start from the simple version

Deriving Solutions From the Model

- Develop a computer-based procedure for deriving solutions to the problem from the model
- Usually it is a relatively simple step
- Some standard algorithms exist
- Optimizing vs Satisfying
  - Mathematical model vs Reality
    "Optimizing is the science of the ultimate; satisficing is the art of the feasible"
  - Cost for find optimal solution
  - Heuristics approach and suboptimal solution

Testing the Model

- Find bugs and flaws (program)
- Model validation
- Try as many inputs as possible
- A more systematic approach: Retrospective test
    - Using historical data to reconstruct the past
    - Determine how well the model have performed
    - Any drawback?
    - Possible solution?

Focus of this Course

- Mathematical model formulation
- Deriving the solutions

Mathematical Formulation of Optimization Problems

- Optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

$x = (x_1, \ldots, x_n)$: optimization variables

$f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$: objective function

$f_i : \mathbf{R}^n \rightarrow \mathbf{R}$, $i = 1, \ldots, m$: constraint functions

- Optimal solution: $x*$ has smallest value of f_0 among all vectors that satisfy the constraints

Solving Optimization Problems

- General optimization problem
  - Very difficult to solve
  - Involve some compromise, e.g., very long computation time, or not always finding the solution
- Exceptions: certain problem classes can be solved efficiently and reliably
  - Linear programming problem
  - least-squares problems
  - Convex optimization problems

Examples

- Portfolio optimization
  - Variables: amounts invested in different assets
  - Constraints: budget, max/min investment per asset, minimum return
  - Objective: overall risk or return variance
- Data fitting
  - Variables: model parameters
  - Constraints: prior information, parameter limits
  - Objective: measure of misfit or prediction error

Linear Programming

- Formulation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T X \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- Solving linear programs
  - No analytical formulation for solution
  - Reliable and efficient algorithms
  - Computation time proportional to $n^2 m$
  - A mature technology

An Illustrative Example on Linear Programming Problem

- Problem Setting
  - A company with three plants (Plant 1, Plant 2, and Plant 3) produces windows and doors
  - Plant 1 produces aluminum frames and hardware
  - Plant 2 produces wood frames
  - Plant 3 produces the glass and assembles the products
- Two new products
  - Product 1: A glass door with aluminum framing
  - Product 2: A wood-framed window
  - Product 1 needs Plant 1 and 3
  - Product 2 needs Plant 2 and 3
- Question: which mix of the two products would be most profitable?

Step 1: Define the Problem

- Problem Setting
  - Determine what the production rates should be for the two products in order to maximize their total profit
  - Subject to the restrictions imposed by the limited production capacities available in the three plants

Step 2: Gather the Data

- Data to be gathered
  - Number of hours of production time available per week in each plant
  - Number of hours of production time used in each batch produced of each product
  - Profit per batch produced of each new product

Plant	Production Time per Batch, Hours Product 1	Production Time per Batch, Hours Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

Step 3: Mathematical Formulation

- Decision variables
  - Define x_1, x_2 the number of batches of product 1 and product 2 produced per week, respectively
  - Define Z the total profit per week
- Linear program formulation

$$\text{Maximize} \quad Z = 3x_1 + 5x_2$$

$$\begin{aligned} x_1 \phantom{{}+2x_2} &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \end{aligned}$$

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Integer Linear Programming Problem

- Formulation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T X \leq b_i, \quad i = 1, \ldots, m \\ & x \in Z^n \end{array}$$

- Solving linear programs
  - No analytical formulation for solution
  - ILP is NP hard
  - Computation time depends on the algorithms used

An Illustrative Example on Integer Linear Programming Problem

- Problem Setting
  - A company wants to build a new factory in either Los Angeles or San Francisco, or in both cities
  - Also build at most one warehouse, the location will be restricted to the city where a new factory is being built
- Data gathered: net present value, capital required

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

An Illustrative Example on Integer Linear Programming Problem

- Objective: find the feasible combination of alternatives to maximize the total net present value
- Decision variables:
  $x_j \in \{0, 1\}$, represents decision j is yes or no.
- Let Z denote the total net present value these decisions

An Illustrative Example on Integer Linear Programming Problem

- Formulation

$$\text{Maximize} \qquad Z = 9x_1 + 5x_2 + 6x_3 + 4x_4,$$

subject to

$$\begin{array}{rcrcrcrcl}
6x_1 & + & 3x_2 & + & 5x_3 & + & 2x_4 & \leq & 10 \\
 & & & & x_3 & + & x_4 & \leq & 1 \\
-x_1 & & & + & x_3 & & & \leq & 0 \\
 & - & x_2 & & & + & x_4 & \leq & 0 \\
 & & & & & & x_j & \leq & 1 \\
 & & & & & & x_j & \geq & 0
\end{array}$$

and

$$x_j \text{ is integer,} \qquad \text{for } j = 1, 2, 3, 4.$$

Least squares

- Formulation

  minimize $||Ax - b||_2^2$

- Solving least-squares problems
  - Analytical solution: $x^* = (A^TA)^{-1}A^Tb$
  - Reliable and efficient algorithms
  - Computation time proportional to $n^2k(A \in R^{k\times n})$
  - A mature technology
  - least-squares problems are easy to recognize

Convex Optimization Problem

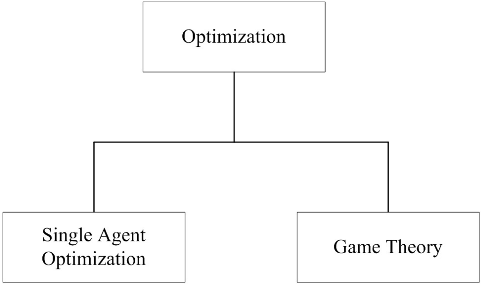
- Formulation

$$\begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{subject to } f_i(x) \le b_i, i = 1, \ldots, m \end{aligned}$$

- Objective and constraint functions are convex
  $f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$, if $\alpha + \beta = 1, \alpha \ge 0, \beta \ge 0$
- least squares problems and linear programs as special cases
- Solving convex optimization problems
  - No analytical solution
  - Reliable and efficient algorithms
  - Computation time proportional to $\max\{n^3, n^2 m, F\}$, where F is cost of evaluating the constraints and their first and second derivatives
  - A mature technology

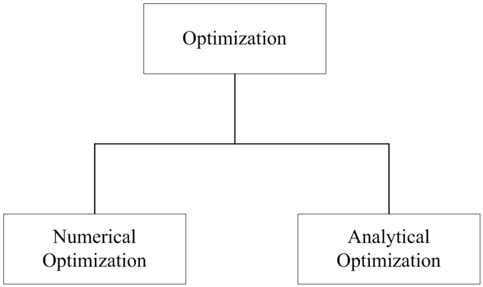
Classification of Optimization Models

- According to agents involved



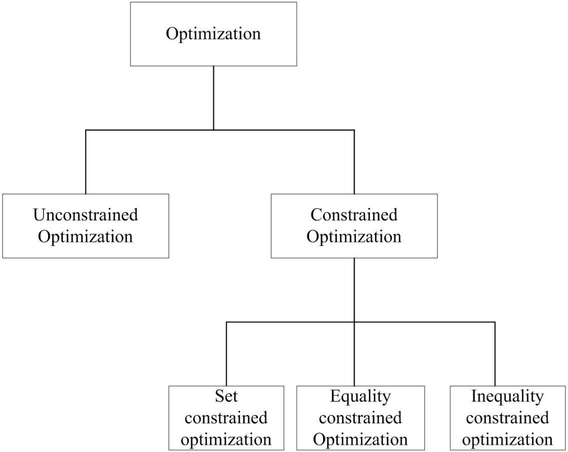
Classification of Optimization Models

- According to solution methods



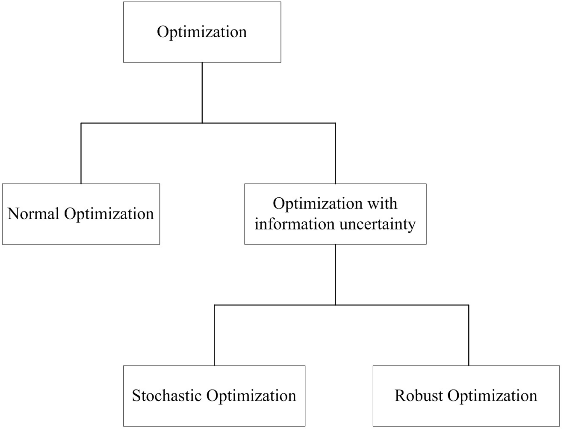
Classification of Optimization Models

- According to constraints involved



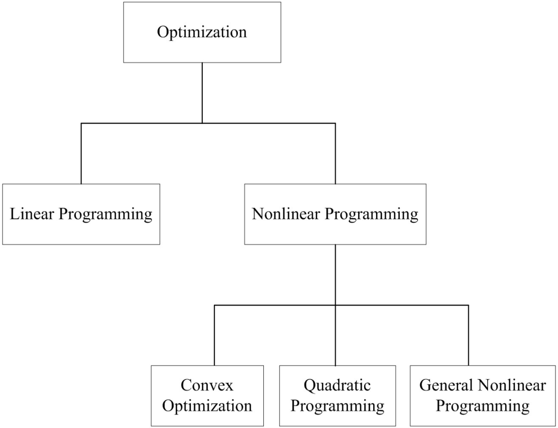
Classification of Optimization Models

- According to information completeness



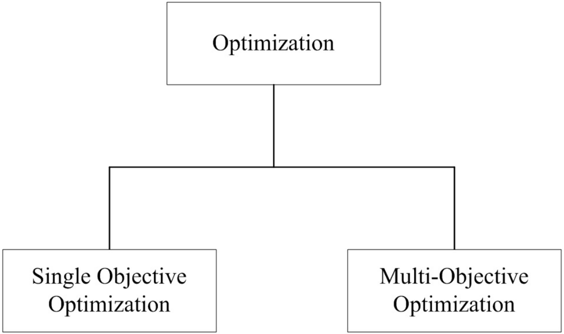
Classification of Optimization Models

- According to property of the problem



Classification of Optimization Models

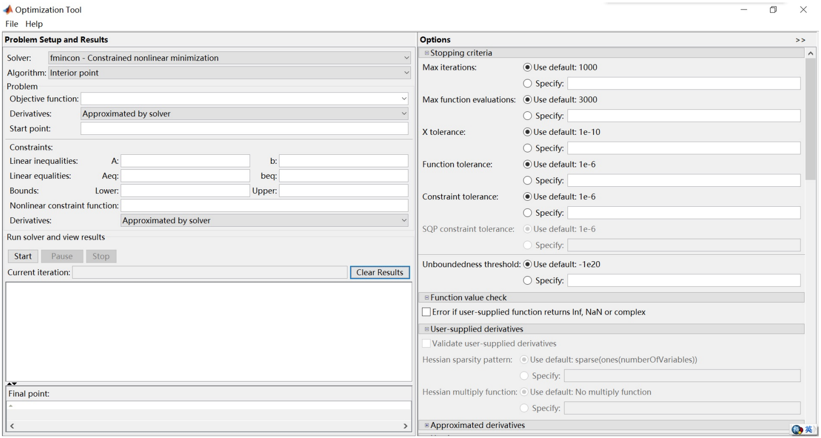
- According to objectives



Related Optimization Softwares

- Excel (with add-ins, e.g., Analytic Solver Platform for Education [ASPE, free for 140 days])
- Matlab Optimization toolbox
- Mathematica
- CVX (free)

Example of Matlab Optimization Toolbox



Example of Matlab Optimization Toolbox

Consider the problem of finding $[x_1, x_2]$ that solves

$$\min_x f(x) = x_1^2 + x_2^2$$

subject to the constraints

$$0.5 \le x_1 \qquad \text{(bound)}$$

$$-x_1 - x_2 + 1 \le 0 \qquad \text{(linear inequality)}$$

$$\left.\begin{aligned} -x_1^2 - x_2^2 + 1 &\le 0 \\ -9x_1^2 - x_2^2 + 9 &\le 0 \\ -x_1^2 + x_2 &\le 0 \\ -x_2^2 + x_1 &\le 0 \end{aligned}\right\} \text{(nonlinear inequality)}$$

Example of Matlab Optimization Toolbox

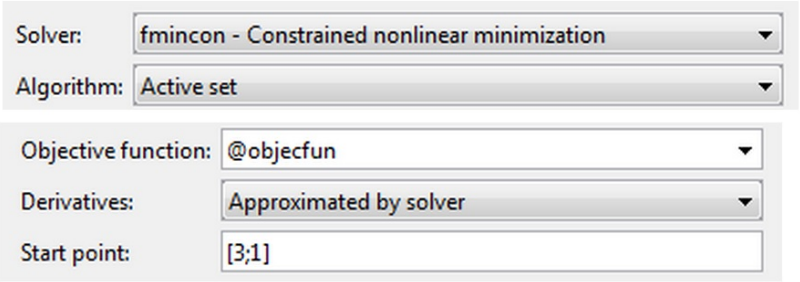
Step 1: Write a file objecfun.m for the objective function.

```
function f = objecfun(x)
f = x(1)^2 + x(2)^2;
```

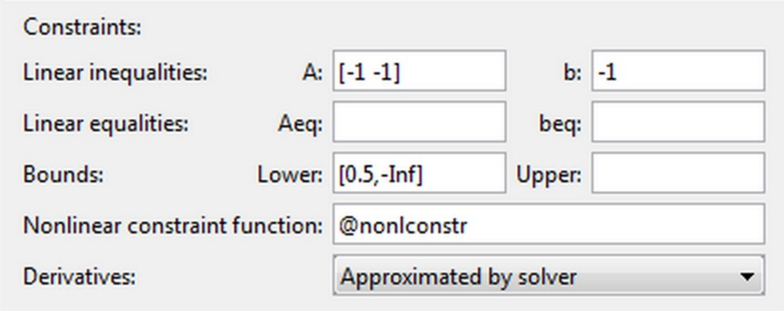
Step 2: Write a file nonlconstr.m for the nonlinear constraints.

```
function [c,ceq] = nonlconstr(x)
c = [-x(1)^2 - x(2)^2 + 1;
     -9*x(1)^2 - x(2)^2 + 9;
     -x(1)^2 + x(2);
     -x(2)^2 + x(1)];
ceq = [];
```

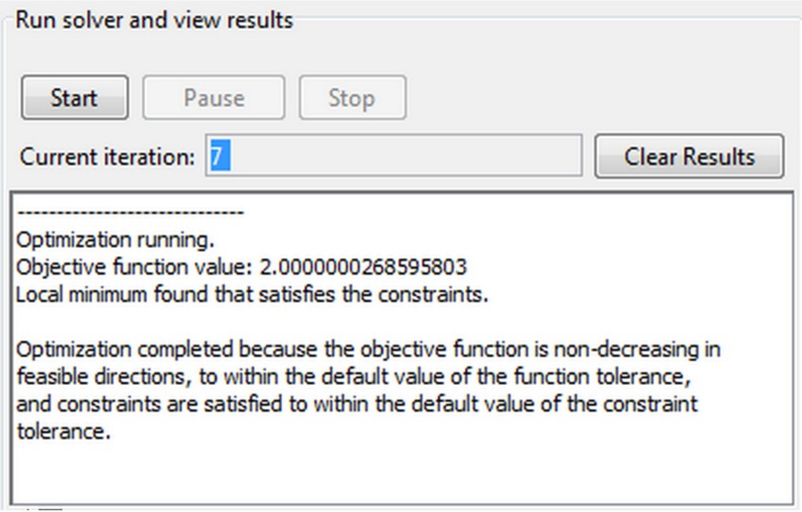
Example of Matlab Optimization Toolbox



Example of Matlab Optimization Toolbox



Example of Matlab Optimization Toolbox



Example of Matlab Optimization Toolbox

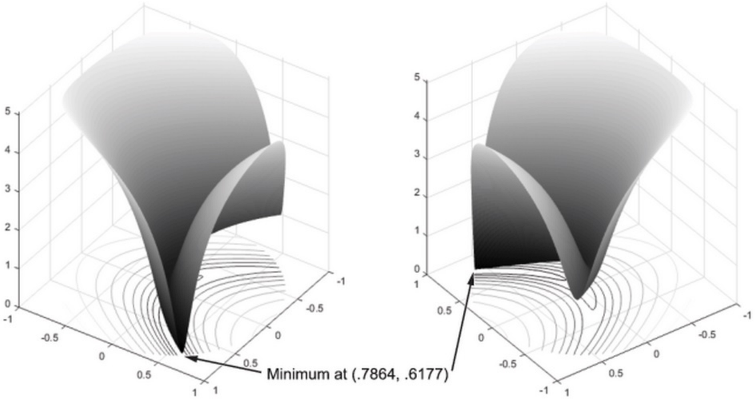
er	F-count	f(x)	Max constraint	Line search steplength	Directional derivative	First-order optimality	Procedure
0	3	10	2				Infeasible start point
1	6	4.84298	-0.1322	1	-5.22	1.74	
2	9	4.0251	-0.01168	1	-4.39	4.08	Hessian modified twice
3	12	2.42704	-0.03214	1	-3.85	1.09	
4	15	2.03615	-0.004728	1	-3.04	0.995	Hessian modified twice
5	18	2.00033	-5.596e-005	1	-2.82	0.0664	Hessian modified twice
6	21	2	-5.327e-009	1	-2.81	0.000522	Hessian modified twice

Example of Matlab Optimization Toolbox

$$f(x) = 100\left(x_2 - x_1^2\right)^2 + (1 - x_1)^2$$

$$x_1^2 + x_2^2 \le 1$$

Example of Matlab Optimization Toolbox



Thanks for attending!