Cubic Spline Interpolation by Recurrence Equations

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Motivation

- Cubic spline interpolation is the most frequently used interpolation method for one variable functions.
- Cubic spline interpolation is usually solved using a tridiagonal matrix-vector equation.
- The r cubic spline interpolation is a method based on a simple recurrence equation.

Cubic Spline Interpolation (Definition)

n implies
n+1 data points

Let $f(t)$ be a function from $\mathcal{R}$ to $\mathcal{R}$. Suppose we know about f only its value at locations $t_0 < \ldots < t_n$. Let $f(t_i) = a_i$. Piecewise cubic spline interpolation of $f(t)$ is the problem of finding the b_i, c_i and d_i coefficients of the cubic polynomials S_i for $0 \leq i \leq n-1$ written in the form:

$$S_i(t) = a_i + b_i(t - t_i) + c_i(t - t_i)^2 + d_i(t - t_i)^3 \quad (1)$$

where each piece S_i interpolates the interval $[t_i, t_{i+1}]$.

For a smooth fit between the adjacent pieces, the cubic spline interpolation requires that the following conditions hold for $0 \leq i \leq n-2$:

$$S_i(t_{i+1}) = S_{i+1}(t_{i+1}) = a_{i+1}, \quad (4)$$

$$S_i'(t_{i+1}) = S_{i+1}'(t_{i+1}) = b_{i+1} \quad (5)$$

$$S_i''(t_{i+1}) = S_{i+1}''(t_{i+1}) = 2c_{i+1} \quad (6)$$

Cubic Spline Interpolation (Solution)

Taking once and twice the derivative of Equation (1) yields, respectively, the equations:

$$S_i'(t) = b_i + 2c_i(t - t_i) + 3d_i(t - t_i)^2 \tag{2}$$

$$S_i''(t) = 2c_i + 6d_i(t - t_i) \tag{3}$$

Equations (1-3) imply that

$$S_i(t_i) = a_i$$

$$S_i'(t_i) = b_i$$

$$S_i''(t_i) = 2c_i$$

Cubic Spline Interpolation (Solution)

Let $h_i = t_{i+1} - t_i$. Substituting Equations (1-3) into Equations (4-6), respectively, yields:

$$a_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = a_{i+1} \tag{7}$$

$$b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1} \tag{8}$$

$$c_i + 3d_i h_i = c_{i+1} \tag{9}$$

Equation (9) yields a value for d_i, which we can substitute into Equations (7-8). Hence Equations (7-9) can be rewritten as:

$$a_{i+1} - a_i = b_i h_i + \frac{2c_i + c_{i+1}}{3} h_i^2 \tag{10}$$

$$b_{i+1} - b_i = (c_i + c_{i+1}) h_i \tag{11}$$

$$d_i = \frac{1}{3h_i}(c_{i+1} - c_i). \tag{12}$$

Solving Equation (10) for b_i yields:

$$b_i = (a_{i+1} - a_i)\frac{1}{h_i} - \frac{2c_i + c_{i+1}}{3}h_i \tag{13}$$

which implies for $j \leq n - 3$ the condition:

$$b_{i+1} = (a_{i+2} - a_{i+1})\frac{1}{h_{i+1}} - \frac{2c_{i+1} + c_{i+2}}{3}h_{i+1} \tag{14}$$

Substituting into Equation (11) the values for b_i and b_{i+1} from Equations (13-14) yields:

Cubic Spline Interpolation (Solution)

$$(a_{i+1} - a_i)\frac{1}{h_i} - (2c_i + c_{i+1})\frac{h_i}{3} + (c_i + c_{i+1})h_i =$$
$$(a_{i+2} - a_{i+1})\frac{1}{h_{i+1}} - (2c_{i+1} + c_{i+2})\frac{h_{i+1}}{3}$$

The above can be rewritten as:

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} =$$
$$\frac{3}{h_{i-1}}a_{i-1} - \left(\frac{3}{h_{i-1}} + \frac{3}{h_i}\right)a_i + \frac{3}{h_i}a_{i+1} \tag{15}$$

Equal spacing assumption

Let h = $h_0 = \ldots = h_{n-1}$

Dividing Equation (15) by h yields:

$$c_{i-1} + 4c_i + c_{i+1} = \frac{3}{h^2}(a_{i-1} - 2a_i + a_{i+1}) \tag{16}$$

Usual Cubic Spline Interpolation

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & \ddots & & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & & h_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 & & 1 \end{bmatrix}$$

and $\mathbf{b}$ and $\mathbf{x}$ are the vectors

$$\mathbf{b} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Theorem 1: Ax = B has solution:

$$A = \begin{bmatrix} r & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \\ e_n \end{bmatrix}$$

where $r = 2 + \sqrt{3} \approx 3.732.$

Define α_i as:

$$\alpha_0 = \frac{e_0}{r}$$

$$\alpha_i = \frac{e_i - \alpha_{i-1}}{r} \quad \text{for} \quad 1 \leq i \leq n-1$$

$$\alpha_n = e_n$$

The solution is given by the recurrence equation:

$$c_n = \alpha_n$$

$$c_i = \alpha_i - \frac{c_{i+1}}{r} \quad \text{for} \quad 0 \leq i \leq n-1$$

Proof: Theorem 1 follows from Lemma 1 given in reference [1] and in the Appendix.

[1] Revesz, P. Z. "A recurrence equation-based solution for the cubic spline interpolation problem," *International Journal of Mathematical Models and Methods in Applied Sciences*, vol. 9, p. 446-452, 2015. http://www.naun.org/main/NAUN/ijmmas/2015/b042001-396.pdf

r Cubic Spline

Equation (16) requires a solution to the following:

$$A = \begin{bmatrix} r & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{3r}{2h}\left(\frac{(a_1 - a_0)}{h}\right) \\ \frac{3}{h^2}(a_0 - 2a_1 + a_2) \\ \vdots \\ \frac{3}{h^2}(a_{n-2} - 2a_{n-1} + a_n) \\ 0 \end{bmatrix}.$$

The boundary conditions in the first and last entry could vary because the smoothness conditions leave them underspecified.

This can be solved using Theorem 1.

Example:
r Cubic Spline Interpolation of the Temperature Data for a Weather Station

***t* days**
***a* Fahrenheit degrees**

t	*a*
6	75
13	78
20	72
27	68

Solution: We have to find the cubic polynomials:

$$\begin{cases} S_0(t) = a_0 + b_0(t-t_0) + c_0(t-t_0)^2 + d_0(t-t_0)^3 & \text{for } t \in [t_0, t_1] \\ S_1(t) = a_1 + b_1(t-t_1) + c_1(t-t_1)^2 + d_1(t-t_1)^3 & \text{for } t \in [t_1, t_2] \\ S_2(t) = a_2 + b_2(t-t_2) + c_2(t-t_2)^2 + d_2(t-t_2)^3 & \text{for } t \in [t_2, t_3] \end{cases}$$

given $t_0 = 6,\ t_1 = 13,\ t_2 = 20,\ t_3 = 27,\ a_0 = 75,\ a_1 = 78,\ a_2 = 72,\ a_3 = 68.$

Here $n = 3,\ h = 7$ (the index can be omitted because there is even spacing).

$$B = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \frac{3r}{2h}\left(\frac{(a_1-a_0)}{h}\right) \\ \frac{3}{h^2}(a_0 - 2a_1 + a_2) \\ \frac{3}{h^2}(a_1 - 2a_2 + a_3) \\ 0 \end{bmatrix} = \begin{bmatrix} 0.3427 \\ -0.5510 \\ 0.1224 \\ 0 \end{bmatrix}$$

$$\alpha_0 = \frac{e_0}{r} = 0.0918$$

$$\alpha_1 = \frac{e_1 - \alpha_0}{r} = -0.1723$$

$$\alpha_2 = \frac{e_2 - \alpha_1}{r} = 0.0790$$

$$\alpha_3 = e_3 = 0$$

$$c_3 = \alpha_3 = 0$$

$$c_2 = \alpha_2 - \frac{c_3}{r} = 0.0790$$

$$c_1 = \alpha_1 - \frac{c_2}{r} = -0.1934$$

$$c_0 = \alpha_0 - \frac{c_1}{r} = 0.1437$$

By Equation (13) we get:

$$b_0 = (a_1 - a_0)\frac{1}{h} - \frac{2c_0 + c_1}{3}h = 0.2094$$

$$b_1 = (a_2 - a_1)\frac{1}{h} - \frac{2c_1 + c_2}{3}h = -0.1388$$

$$b_2 = (a_3 - a_2)\frac{1}{h} - \frac{2c_2 + c_3}{3}h = -0.9399$$

By Equation (12) we get:

$$d_0 = \frac{1}{3h}(c_1 - c_0) = -0.0161$$

$$d_1 = \frac{1}{3h}(c_2 - c_1) = 0.0130$$

$$d_2 = \frac{1}{3h}(c_3 - c_2) = -0.0038$$

$$\begin{cases} S_0(t) = 75 + 0.2094(t-6) + 0.1437(t-6)^2 - 0.0161(t-6)^3 & \text{for } t \in [6, 13] \\ S_1(t) = 78 - 0.1388(t-13) - 0.1934(t-13)^2 + 0.0130(t-13)^3 & \text{for } t \in [13, 20] \\ S_2(t) = 72 - 0.9399(t-20) + 0.0790(t-20)^2 - 0.0038(t-20)^3 & \text{for } t \in [20, 27] \end{cases}$$

Appendix
(optional)

Lemma 1: Ax = B has solution:

$$A = \begin{bmatrix} r & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n-1} \\ e_n \end{bmatrix}$$

where $r = 2 + \sqrt{3} \approx 3.732.$

Define α_i as:

$$\begin{aligned} \alpha_0 &= 0 \\ \alpha_i &= \frac{e_i - \alpha_{i-1}}{r} \\ \alpha_n &= e_n \end{aligned}$$

Then we can express the solution by the recurrence equation:

$$x_n = \alpha_n$$
$$x_i = \alpha_i - \frac{x_{i+1}}{r} \quad \text{for} \quad 1 \le i \le n-1$$

Proof of Lemma 1

Let $r = 2 + \sqrt{3} \approx 3.732$. Then the first three equations are:

$$
\begin{aligned}
rx_1 + x_2 &= e_1 \\
x_1 + 4x_2 + x_3 &= e_2 \\
x_2 + 4x_3 + x_4 &= e_3
\end{aligned}
$$

Proof of Lemma 1

Dividing the first row by r and subtracting it from the second row gives:

$$
\begin{aligned}
rx_1 + \ x_2 \qquad\qquad\qquad & = e_1 \\
rx_2 + \ x_3 \qquad\qquad & = e_2 - \frac{e_1}{r} \\
x_2 + 4x_3 + x_4 \qquad & = e_3
\end{aligned}
$$

Recurrence Equation

Dividing the second row by r and subtracting it from the third row gives:

$$
\begin{aligned}
rx_1 + x_2 &= e_1 \\
rx_2 + x_3 &= e_2 - \frac{e_1}{r} \\
rx_3 + x_4 &= e_3 - \frac{e_2}{r} + \frac{e_1}{r^2}
\end{aligned}
$$

Note that each row $1 \le i \le n-1$ will be the following:

$$x_i + \frac{x_{i+1}}{r} = \sum_{0 \le k \le (i-1)} (-1)^k \frac{e_{i-k}}{r^{k+1}}$$

Solution of the Recurrence Equation

The solution to the linear equation system can be described in terms of the α constants as follows:

$$\vdots$$

$$x_{n-3} = \alpha_{n-3} - \frac{\alpha_{n-2}}{r} + \frac{\alpha_{n-1}}{r^2} - \frac{\alpha_n}{r^3}$$

$$x_{n-2} = \alpha_{n-2} - \frac{\alpha_{n-1}}{r} + \frac{\alpha_n}{r^2}$$

$$x_{n-1} = \alpha_{n-1} - \frac{\alpha_n}{r}$$

$$x_n = \alpha_n$$

Therefore, x_i for each row $1 \leq i \leq n$ will be:

$$x_i = \sum_{0 \leq k \leq (n-i)} \left(\frac{-1}{r} \right)^k \alpha_{i+k}$$

Natural Cubic Spline

Since the values of a_i are known, the values of c_i can be found by solving the tridiagonal matrix-vector equation $Ax = B$. Under the natural cubic spline interpolation, we have:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{3}{h^2}(a_0 - 2a_1 + a_2) \\ \vdots \\ \frac{3}{h^2}(a_{n-2} - 2a_{n-1} + a_n) \\ 0 \end{bmatrix}.$$

Clamped Cubic Spline

Under the clamped spline interpolation we have:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2 \end{bmatrix} \qquad x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{3}{h^2}(a_1 - a_0) - \frac{3}{h} f'(x_0) \\ \frac{3}{h^2}(a_0 - 2a_1 + a_2) \\ \vdots \\ \frac{3}{h^2}(a_{n-2} - 2a_{n-1} + a_n) \\ \frac{3}{h} f'(x_n) - \frac{3}{h^2}(a_n - a_{n-1}) \end{bmatrix}.$$

Boundary Conditions

The initial condition is equivalent to the clamped spline interpolation when

$$e_1 = \frac{3r}{2h}\left(\frac{(a_1 - a_0)}{h} - f'(x_0)\right) + \left(1 - \frac{r}{2}\right)\tilde{c_1}.$$

where $\tilde{c_1}$ is an estimate for the value of c_1.

The ending condition is equivalent to the natural spline interpolation when

$$e_n = 0$$

Note: When the values of $f'(x_0)$ and ĉ are unknown, then try zeros.

Other boundary conditions may be chosen when warranted.

Example of a Falling Object

An object is released from a height of 400 feet with zero initial velocity. The object's position is 384, 336 and 256 feet from earth at one, two and three seconds after release. It is in free fall with a gravitational acceleration of $32 ft/sec^2$ at one second after release and at four seconds after release. Find cubic polynomials that interpolate the flight path of the object for the intervals $[0,1]$, $[1,2]$ and $[2,3]$:

$$\begin{cases} S_0(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3 \\ S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3 \\ S_2(x) = a_2 + b_2(x-2) + c_2(x-2)^2 + d_2(x-2)^3 \end{cases}$$

$$A = \begin{bmatrix} r & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -16r - 16 \\ -96 \\ -96 \\ -16 \end{bmatrix}$$

$$\alpha_1 = \frac{e_1}{r} = -16 - \frac{16}{r}$$

$$\alpha_2 = \frac{e_2 - \alpha_1}{r} = -16 - \frac{16}{r}$$

$$\alpha_3 = \frac{e_3 - \alpha_2}{r} = -16 - \frac{16}{r}$$

$$\alpha_4 = e_4 = -16$$

$$c_3 = \alpha_4 = -16$$

$$c_2 = \alpha_3 - \frac{c_3}{r} = -16$$

$$c_1 = \alpha_2 - \frac{c_2}{r} = -16$$

$$c_0 = \alpha_1 - \frac{c_1}{r} = -16$$

$$b_0 = \tfrac{1}{1}(384 - 400) - \tfrac{1}{3}(-16 - 32) = 0$$

$$b_1 = \tfrac{1}{1}(336 - 384) - \tfrac{1}{3}(-16 - 32) = -32$$

$$b_2 = \tfrac{1}{1}(256 - 336) - \tfrac{1}{3}(-16 - 32) = -64$$

$$d_0 = \frac{1}{3}(-16 - (-16)) = 0$$

$$d_1 = \frac{1}{3}(-16 - (-16)) = 0$$

$$d_2 = \frac{1}{3}(-16 - (-16)) = 0$$

$$\begin{cases} S_0(x) = 400 - 16x^2 \\ S_1(x) = 384 - 32(x-1) - 16(x-1)^2 = 400 - 16x^2 \\ S_2(x) = 336 - 64(x-2) - 16(x-2)^2 = 400 - 16x^2 \end{cases}$$