高等数学笔记

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目录

1 极限的计算

问题 1.1 计算

$$\lim_{x\to 0}\frac{\sqrt[m]{1+\alpha x}\,\sqrt[n]{1+\beta x}-1}{x}$$

解答:

$$\begin{aligned}
\text{原式} &= \lim_{x\to 0}\sqrt[m]{1+\alpha x}\frac{\sqrt[n]{1+\beta x}-1}{x}+\lim_{x\to 0}\sqrt[n]{1+\beta x}\frac{\sqrt[m]{1+\alpha x}-1}{x}\\
&= \frac{\alpha}{m}+\frac{\beta}{n}
\end{aligned}$$

问题 1.2 计算

$$\lim_{x\to 0}\frac{1-\cos x\cos x\ldots\cos x}{x^2}$$

解答:

$$\text{原式}=\lim_{x\to 0}\frac{(1-\cos x)+(\cos x-\cos x\cos x\ldots\cos x)}{x^2}$$

$$
\begin{aligned}
&= \frac{1}{2} + \lim_{x\to0}\frac{1-\cos 2x\ldots\cos x}{x^2} \\
&= \cdots \quad (\text{反复使用上面的拆项方法}) \\
&= \frac{1}{2} + \frac{1}{2}\cdot 2^2 + \frac{1}{2}\cdot 3^2 + \cdots + \frac{1}{2}\cdot n^2 \\
&= \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(2n+1)}{12}
\end{aligned}
$$

注 前面两个都是添项减项法的运用。其实还有另一种做法。

问题 1.3 计算

$$
\lim_{x\to0}\frac{\sqrt[m]{1+\alpha x}\sqrt[n]{1+\beta x}-1}{x}
$$

解答:

$$
\begin{aligned}
\text{原式} &= \lim_{x\to0}\frac{\ln\left[(1+\alpha x)^{\frac{1}{m}}\cdot(1+\beta x)^{\frac{1}{n}}\right]}{x} \\
&= \lim_{x\to0}\frac{\frac{1}{m}\ln(1+\alpha x)}{x} + \lim_{x\to0}\frac{\frac{1}{n}\ln(1+\beta x)}{x} \\
&= \frac{\alpha}{m} + \frac{\beta}{n}
\end{aligned}
$$

问题 1.4 计算

$$
\lim_{x\to0}\frac{1-\cos x\cos x\ldots\cos x}{x^2}
$$

解答:

$$
\begin{aligned}
\text{原式} &= -\lim_{x\to0}\frac{\cos x\cos 2x\cdots\cos nx-1}{x^2} \\
&= -\lim_{x\to0}\frac{\ln\cos x\cdots\cos nx}{x^2} \\
&= -\sum_{k=1}^{n}\lim_{x\to0}\frac{\ln\cos kx}{x^2} \\
&= -\sum_{k=1}^{n}\lim_{x\to0}\frac{\cos kx-1}{x^2}
\end{aligned}
$$

$$= -\sum_{k=1}^{n}(-\frac{1}{2}k^2) = \frac{n(n+1)(2n+1)}{12}$$

注 先反用 $\ln x \sim x-1(x \to 1)$，再正用 $\ln x \sim x-1(x \to 1)$，尤其适用于含有"$A_1A_2\dots A_n - 1$" 结构的极限。

问题 1.5 计算

$$\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}\cdots\sqrt[n]{\cos nx}}{x^2}$$

解答:

$$\begin{aligned}
\text{原式} &= \lim_{x\to 0}\frac{1-\cos x(\cos 2x)^{\frac{1}{2}}\cdots(\cos nx)^{\frac{1}{n}}}{x^2}\\
&= \lim_{x\to 0}\frac{-\ln\left(\cos x(\cos 2x)^{\frac{1}{2}}\cdots(\cos nx)^{\frac{1}{n}}\right)}{x^2}\\
&= -\lim_{x\to 0}\frac{\sum_{k=1}^{n}\frac{1}{k}\ln(\cos kx)}{x^2}\\
&= -\sum_{k=1}^{n}\lim_{x\to 0}\frac{\frac{1}{k}\ln(\cos kx)}{x^2}\\
&= -\sum_{k=1}^{n}\frac{1}{k}\lim_{x\to 0}\frac{-\frac{(kx)^2}{2}}{x^2} \quad \text{（利用等价无穷小} \ln(\cos u) \sim -\frac{u^2}{2}\text{）}\\
&= \frac{n(n+1)}{4}
\end{aligned}$$

问题 1.6 计算

$$\lim_{x\to 0}\frac{(1-\cos x)\left[x-\ln(1+\tan x)\right]}{x^4}$$

解答:

$$\begin{aligned}
\text{原式} &= \lim_{x\to 0}\frac{\frac{x^2}{2}\left[x-\ln(1+\tan x)\right]}{x^4}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{x-\ln(1+\tan x)}{x^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\lim_{x\to 0}\frac{x-\tan x}{x^2}+\lim_{x\to 0}\frac{\tan x-\ln(1+\tan x)}{x^2}\right] \\
&= \frac{1}{2}\left[0+\frac{1}{2}\right] \\
&= \frac{1}{4}
\end{aligned}
$$

注 $x-\ln(1+x)\sim\frac{x^2}{2}\quad(x\to 0)$

问题 1.7 计算

$$
\lim_{x\to 0}\frac{(3+2\tan x)^x-3^x}{3\sin^2 x+x^3\cdot\cos\frac{1}{x}}
$$

解答:

$$
\begin{aligned}
\text{原式} &= \lim_{x\to 0}\frac{(3+2\tan x)^x-3^x}{3\sin^2 x}\quad(\text{无穷小的吸收律}) \\
&= \lim_{x\to 0}\frac{3^x\left[(1+\frac{2}{3}\tan x)^x-1\right]}{3x^2} \\
&= \frac{1}{3}\lim_{x\to 0}\frac{x\ln(1+\frac{2}{3}\tan x)}{x^2} \\
&= \frac{1}{3}\lim_{x\to 0}\frac{\frac{2}{3}\tan x}{x} \\
&= \frac{2}{9}
\end{aligned}
$$

问题 1.8 设 $f(x),g(x)$ 在 $x=0$ 的邻域 U 内有定义，且对 $\forall x\in U$，均有 $f(x)\neq g(x)$，且 $\lim_{x\to 0}f(x)=\lim_{x\to 0}g(x)=a>0$，计算

$$
\lim_{x\to 0}\frac{[f(x)]^{g(x)}-[g(x)]^{g(x)}}{f(x)-g(x)}
$$

解答:

$$
\begin{aligned}
\text{原式} &= \lim_{x\to 0}[g(x)]^{g(x)}\cdot\frac{\left[\frac{f(x)}{g(x)}\right]^{g(x)}-1}{f(x)-g(x)} \\
&= a^a\lim_{x\to 0}\frac{g(x)\cdot\ln\frac{f(x)}{g(x)}}{f(x)-g(x)}
\end{aligned}
$$

$$
= a^{a+1} \lim_{x\to 0} \frac{\frac{f(x)-g(x)}{g(x)}}{f(x)-g(x)}
$$

$$
= a^a
$$

问题 1.9 计算

$$
\lim_{x\to 0^+} (e^x - 1 - x)^{\frac{1}{\ln x}}
$$