1 Preliminaries

1.1 Set and Collections

Definition 1.1.1 *Let $\mathcal{D}$ be a collection of sets. The union of $\mathcal{C}$, denoted by $\bigcup \mathcal{C}$, is the set defined by*

$$\bigcup \mathcal{C} = \{x | x \in S \text{ for some } S \in \mathcal{C}\}.$$

If $\mathcal{C}$ is a non-empty collection, its intersection is the set $\bigcap \mathcal{C}$ given by

$$\bigcap \mathcal{C} = \{x | x \in S \text{ for every } S \in \mathcal{C}\}.$$

Here, the generalized union and generalized intersection of sets simply involve performing the union or intersection operation on each element within the set.

Note that if $\mathcal{C}$ and $\mathcal{D}$ are two collections such that $\mathcal{C} \subseteq \mathcal{D}$, then

$$\bigcup \mathcal{C} \subseteq \bigcup \mathcal{D} \text{ and } \bigcap \mathcal{D} \subseteq \bigcap \mathcal{C}$$

Within the framework of collections of subsets of a given set $\mathcal{S}$, we extend the previous definition by taking $\bigcap \varnothing = S$ for the empty collection of subsets of $\mathcal{S}$.

This is consistent with the fact that $\varnothing \subseteq \mathcal{C}$ implies $\bigcap \mathcal{C} \subseteq \mathcal{S}$.

Definition 1.1.2 *The symmetric difference of sets denoted by $\oplus$ is defined by $U \oplus V = (U - V) \cup (V - U)$ for all sets U, V.*

The symmetric difference operation is easily shown to satisfy symmetry and associativity, and also $U \oplus U = \varnothing$. Next, we will prove the associativity.

Proof 1.1.1

$$\begin{aligned} \mathit{LEFT} &= ((U - V) \cup (V - U)) \oplus T \\ &= (((U - V) \cup (V - U)) - T) \cup (T - ((U - V) \cup (V - U))) \\ &= ((U\bar{V} \cup V\bar{U}) \cap \bar{T}) \cup (T \cap (\bar{U}\bar{V} \cup UV)) \\ &= \sum_{i=1,2,4,7} m_i \end{aligned}$$

$$\mathit{RIGHT} = \mathit{LEFT}$$

However, the above method is rather cumbersome. We can instead adopt the characteristic function approach for the proof. Here, we first supplement the relevant concepts of characteristic functions.

Definition 1.1.3 *For any set A, the function*

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A \end{cases}$$

is called the characteristic function of the set A.

The characteristic function has the following properties:

1. The necessary and sufficient condition for $A = X$ is $\chi_A(x) \equiv 1$, and the necessary and sufficient condition for $A = \varnothing$ is $\chi_A(x) \equiv 0$;
2. The necessary and sufficient condition for $A \subset B$ is
$$\chi_A(x) \leqslant \chi_B(x), (\forall x \in X);$$
3. $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$.
4. $\chi_{A\cap B}(x) = \chi_A(x) \cdot \chi_B(x)$;
5.
$$\chi_{\bigcup_{\alpha\in\Lambda} A_\alpha}(x) = \max_{\alpha\in\Lambda} \chi_{A_\alpha}(x)$$
$$\chi_{\bigcap_{\alpha\in\Lambda} A_\alpha}(x) = \min_{\alpha\in\Lambda} \chi_{A_\alpha}(x)$$
6. Let $\{A_k\}$ be an arbitrary sequence of sets, then
$$\chi_{\overline{\lim}_{k\to\infty} A_k}(x) = \overline{\lim_{k\to\infty}}\, \chi_{A_k}(x)$$
$$\chi_{\underline{\lim}_{k\to\infty} A_k}(x) = \underline{\lim_{k\to\infty}}\, \chi_{A_k}(x);$$
7. The necessary and sufficient condition for $\lim_{k\to\infty} A_k$ to exist is that $\lim_{k\to\infty} \chi_{A_k}(x)$ exists $(\forall x \in X)$, and when the limit exists, we have
$$\chi_{\lim_{k\to\infty} A_k}(x) = \lim_{k\to\infty} \chi_{A_k}(x) \quad (x \in X).$$

$$\overline{\lim_{k\to\infty}}\, A_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

The limit superior of a sequence of sets reflects the collection of elements that ”repeatedly appear” in the sequence of sets.

$$\overline{\lim_{k\to\infty}}\, A_k = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

The limit inferior of a sequence of sets reflects the collection of elements that exhibit ”stable membership” within the sequence of sets.

$$\overline{\lim_{k\to\infty}}\, \chi_{A_k}(x) = \lim_{n\to\infty} \big(\sup_{k\geqslant n} \chi_{A_k(x)}\big)$$
$$\underline{\lim_{k\to\infty}}\, \chi_{A_k}(x) = \lim_{n\to\infty} \big(\inf_{k\geqslant n} \chi_{A_k(x)}\big)$$

Next, we will employ the method of characteristic functions to complete the proof of the associativity of the symmetric difference operation.

Proof 1.1.2

$$\begin{aligned}
\chi_{A\oplus B}(x) &= \chi_{(A-B)\cup(B-A)}(x) \\
&= \chi_{A-B}(x) + \chi_{B-A}(x) \\
&= \chi_A(x)\cdot\chi_{\bar{B}}(x) + \chi_{\bar{A}}(x)\cdot\chi_B(x) \\
&= \chi_A(x) + \chi_B(x) - 2\chi_{A\cap B}(x)
\end{aligned}$$

$$\begin{aligned}
\mathit{LEFT} &= \chi_{(U\oplus V)\oplus T}(x) \\
&= \chi_U(x) + \chi_V(x) + \chi_T(x) - 2\chi_{U\cap V}(x) - 2\chi_{V\cap T}(x) - 2\chi_{U\cap T}(x) + 4\chi_{U\cap V\cap T}(x) \\
&= \mathit{RIGHT}
\end{aligned}$$

Definition 1.1.4 *An ordered pair is a collection of sets $\{\{x,y\},\{x\}\}$. It can be readily verified that x and y are determined uniquely.*

Let $\{\{x,y\},\{x\}\}$ be an ordered pair. Then x is the first component of p and y is the second component of p.

Let X, Y be two sets. Their product is the set $X \times Y$ that consists of all pairs of the form (x,y) as an ordered pair on the set S.