Searching for combinatorial covers using linear programming

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Background

- Builds upon „*Automatic discovery of structural rules of permutation classes*“ (2019) by Christian Bean, Bjarki Guðmundsson and Henning Ulfarsson
- Struct is only written for permutation classes

Our contribution

- CombCov published as a module for Python
- General framework for all kinds of combinatorial objects
- Automated old results of permutation classes avoiding mesh patterns

A combinatorial object: *words*

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Example

abba is a word of length 4 over the alphabet $\Sigma = \{a, b\}$.

A combinatorial object: *words* (continued)

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We say that a word $u = u_1 \cdots u_n$ *contains* another word $v = v_1 \cdots v_k$ as a *subword* if there exists an i such that $u_{i+1} \cdots u_{i+k} = v_1 \cdots v_k$. If u does not contain v, we say that u *avoids* v and define $\mathrm{Av}_n(v)$ as the set of all words of length n avoiding v and write $\mathrm{Av}(v) = \bigcup_{n=0}^{\infty} \mathrm{Av}_n(v)$.

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Example

The word *abba* contains the subword *bb* but avoids *aa*.

Avoidance sets of words

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$\mathrm{Av}(aa) = \{\epsilon, a, b, ab, ba, bb, aba, abb, bab, bba, bbb, \ldots\}$ is an avoidance set of words.

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Research questions

- *Can we find a formula for the number of elements of specific length in the avoidance sets?*
- *Can we describe the avoidance sets such as* $\mathrm{Av}(aa)$ *differently, i.e., in "simpler terms"?*

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  (1) $\bigcup_{i \in I} R_i = R$
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- In the next few slides we go step by step through how the algorithm finds a cover for $\mathrm{Av}(aa)$

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- Next step: Verify *valid* rules and throw away *invalid* rules

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- The rule $a\mathrm{Av}(a)$ generates $R' = \{a, ab\} \subseteq R$ with corresponding bitstring 010100 so it is valid

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- The rule $a\mathrm{Av}(a)$ generates $R' = \{a, ab\} \subseteq R$ with corresponding bitstring 010100 so it is valid
- The rule $a\mathrm{Av}(b)$ generates $R'' = \{a, aa\} \nsubseteq R$ so the rule is invalid

Linear programming

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- Using *Gurobi* (or *COIN CLP/CBC LP*) to solve

$$
\begin{array}{lrl}
\text{Min} & z = x_1 + \cdots + x_9 & \\
\text{s.t.} & x_6 + x_8 + x_9 & = 1 \\
 & x_1 & = 1 \\
 & x_2 + x_7 & = 1 \\
 & x_3 + x_8 + x_9 & = 1 \\
 & x_4 + x_7 & = 1 \\
 & x_5 + x_9 & = 1 \\
\text{with} & x_i \in \{0, 1\} \text{ for } i = 1, \ldots, 9. &
\end{array}
$$

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- By increasing the precision ℓ to $\geqslant 3$ we get the correct solution:

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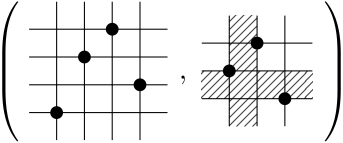
- It is easy to prove that the *enumeration* of this avoidance set of words is the Fibonacci sequence shifted by two, i.e., $|\mathrm{Av}_n(aa)| = F_{n+2}$

Our area of interest

Av



Permutations

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A *permutation of length n* is a bijection from the set of the first n integers, $[\![n]\!] = \{1, \ldots, n\}$ to itself. The set of all permutations of length n is denoted with $\mathfrak{S}_n$ and $\mathfrak{S} = \bigcup_{n=0}^{\infty} \mathfrak{S}_n$ is the set of all permutations. The only permutation of length 0 is called the *empty permutation* and it is denoted by ϵ.

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Example

An example of a permutation of length 5 is $\pi = 35142$ with $\pi(1) = 3$, $\pi(2) = 5$, $\pi(3) = 1$, $\pi(4) = 4$ and $\pi(5) = 2$.

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Example

The permutation $\pi = 35142$ contains the permutation $\sigma = 213$ because $\pi(1)\pi(3)\pi(4) = 314$ is order relative to 213. π avoids 123.

Grid representation of permutations

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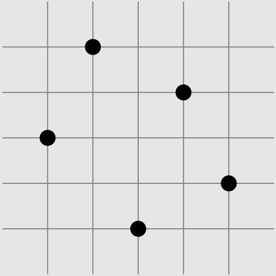

On the left we have plotted $\mathrm{Gr}(\pi)$ with $\pi = 35142$ from previous examples.

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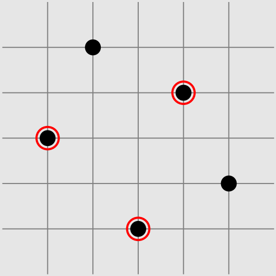

On the left we have plotted $\mathrm{Gr}(\pi)$ with $\pi = 35142$ from previous examples. Highlighted with red circles is an occurrence of $\sigma = 213$ in π.

Mesh patterns

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A *mesh pattern* is a a pair

$$p = (\sigma, R) \text{ with } \sigma \in \mathfrak{S}_k \text{ and } R \subseteq [\![0, k]\!] \times [\![0, k]\!]$$

where $[\![0, k]\!] = \{0, 1, \ldots, k\}$ and R is a set of Cartesian coordinates $\ulcorner i, j \lrcorner$ denoting the lower left corners of the squares in the grid representation of σ which are *shaded*.

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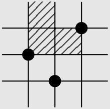

The mesh pattern $p = (\sigma, R)$ where $\sigma = 213$ and $R = \{\ulcorner 1, 2 \lrcorner, \ulcorner 1, 3 \lrcorner, \ulcorner 2, 2 \lrcorner\}$ is shown on the left.

Permutations containing mesh patterns

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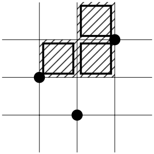

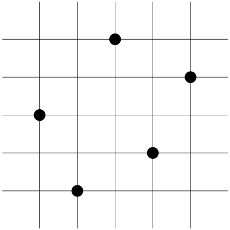

- The mesh pattern $p = (213, \{\boxed{1, 2}, \boxed{2, 2}, \boxed{2, 3}\})$ (left) and the permutation $\pi = 31524$ (middle).

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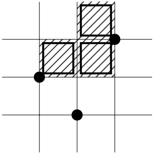

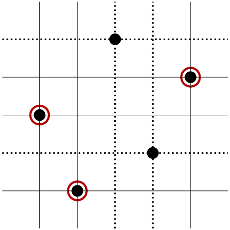

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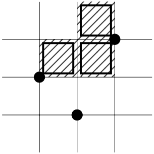
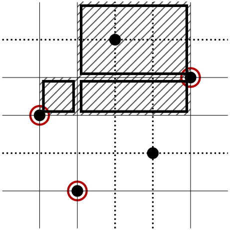
- Even though 314 (highlighted with red circles) is an occurrence of 213 in π

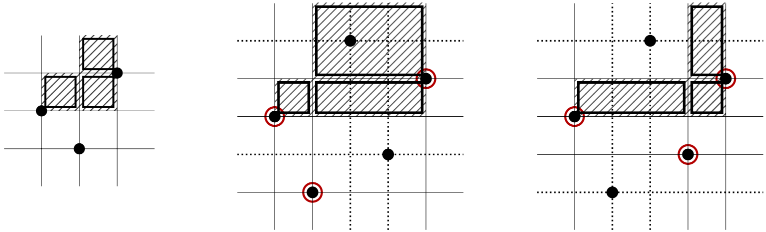
Permutations containing mesh patterns





- The mesh pattern $p = (213, \{\boxed{1,2}, \boxed{2,2}, \boxed{2,3}\})$ (left) and the permutation $\pi = 31524$ (middle).
- Even though 314 (highlighted with red circles) is an occurrence of 213 in π this is not an occurrence of p in π.

Permutations containing mesh patterns



- The mesh pattern $p = (213, \{\ulcorner 1,2 \lrcorner, \ulcorner 2,2 \lrcorner, \ulcorner 2,3 \lrcorner\})$ (left) and the permutation $\pi = 31524$ (middle).
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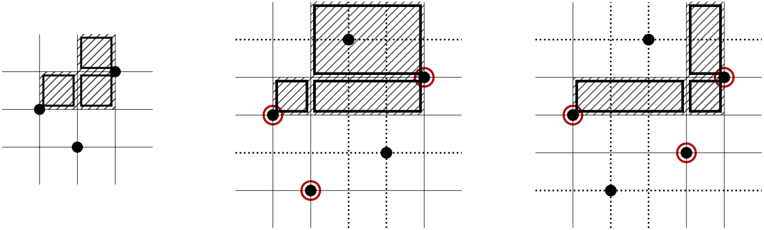
- However, p is contained in π because with the occurrence 324 the shaded areas do not overlap with any point in π.

Permutations containing mesh patterns



- The mesh pattern $p = (213, \{\boxed{1, 2}, \boxed{2, 2}, \boxed{2, 3}\})$ (left) and the permutation $\pi = 31524$ (middle).
- Even though 314 (highlighted with red circles) is an occurrence of 213 in π this is not an occurrence of p in π.
- However, p is contained in π because with the occurrence 324 the shaded areas do not overlap with any point in π.
- The set of all permutations that *avoid* the mesh pattern p is denoted with $\mathrm{Av}(p)$.

Generating functions

Generating functions

Definition

The *generating function (GF)* of the avoidance set $\mathrm{Av}(S)$ is the sum

$$\sum_{n=0}^{+\infty} F_n x^n$$

where $F_n = |\mathrm{Av}_n(S)|$ for all n. The *exponential generating function (EGF)* of $\mathrm{Av}(S)$ is the sum $\sum_{n=0}^{+\infty} \frac{F_n}{n!} x^n$. We say that the avoidance set is *enumerated* by the GF (or EGF) and that the number sequence $(F_n)_{n \geqslant 0}$ is the *enumeration* of $\mathrm{Av}(S)$.

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Example

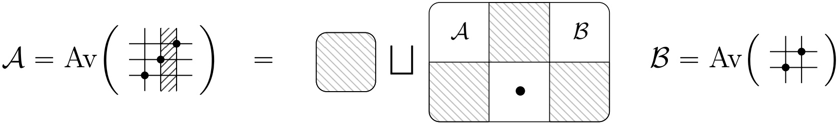
The GF of $\mathrm{Av}(21) = \{\epsilon, 1, 12, \ldots\}$ is $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$ so the enumeration is $(1, 1, 1, \ldots)$. The EGF of $\mathrm{Av}(21)$ is $\sum_{n=0}^{+\infty} \frac{1}{n!} x^n = e^x$.

Interesting results

Now we will look at some interesting results obtained with CombCov.

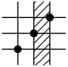
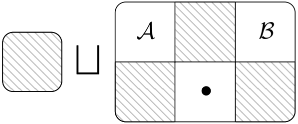
Generalized Pattern Avoidance (2001)

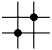
One of the papers that we tried replicating results from was the above one by Anders Claesson where he studies *generalized permutation patterns* that are specific type of mesh patterns. Out of the 6 results in the paper we could replicate 5 of them. Below is one of them.



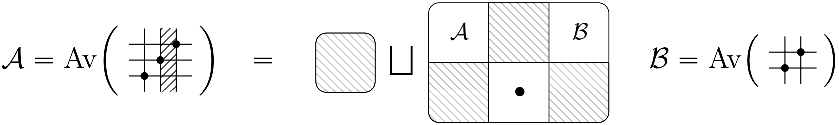
After verifying that the cover is indeed correct it is interesting to derive the *exponential generating function (EGF)* $F = \sum_{n \geqslant 0} \frac{a_n}{n!} x^n$ of the avoidance set and show that $a_n = B_n$, proving that the sequence is enumerated by the Bell numbers.

Deriving the EGF

$\mathcal{A} = \mathrm{Av}\Big($  $\Big)$ $=$ 

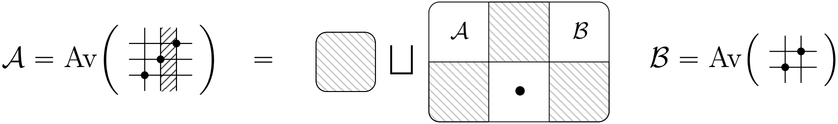
$\mathcal{B} = \mathrm{Av}\Big($  $\Big)$

Deriving the EGF



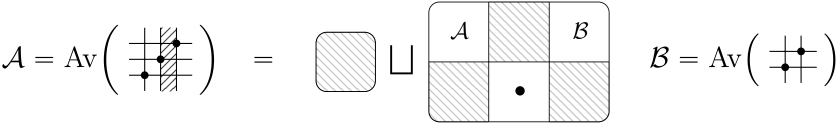
- $\mathcal{A}$ has EGF $F = \sum_{n \geqslant 0} \frac{a_n}{n!} x^n$

Deriving the EGF



- $\mathcal{A}$ has EGF $F = \sum_{n\geqslant 0} \frac{a_n}{n!} x^n$
- $\mathcal{B}$ has EGF $\sum_{n\geqslant 0} \frac{1}{n!} x^n = e^x$

Deriving the EGF



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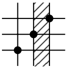
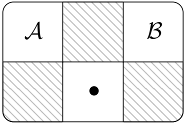
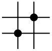
$\mathcal{A} = \mathrm{Av}\Big($  $\Big) =$  $\bigsqcup$ 

$\mathcal{B} = \mathrm{Av}\Big($  $\Big)$

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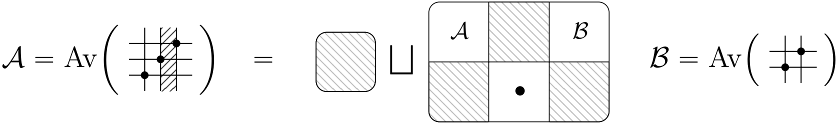
Deriving the EGF



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- This gives us the recurrence relation $F = 1 + \int F e^x dx$
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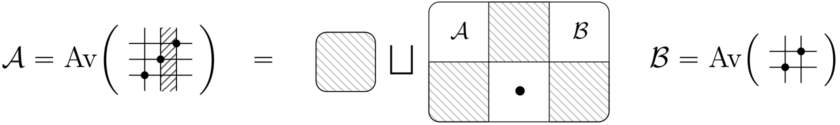
- Knowing that there is only one permutation of length zero in $\mathcal{A}$ we put $a_0 = 1$ into the equation at $x = 0$ and get $A = e^{-1}$

Deriving the EGF



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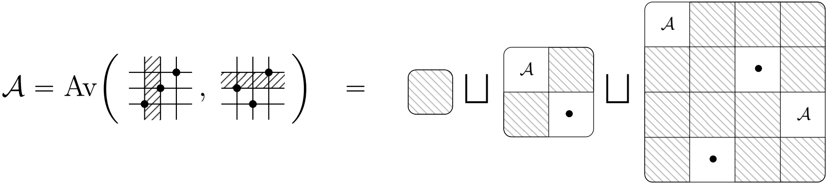
- This gives us the recurrence relation $F = 1 + \int F e^x dx$
- We solve it and get $F = A e^{e^x}$
- Knowing that there is only one permutation of length zero in $\mathcal{A}$ we put $a_0 = 1$ into the equation at $x = 0$ and get $A = e^{-1}$
- We have now shown that $F = e^{e^x - 1}$ which is indeed the EGF for the Bell numbers.

Enumerations of Permutations Simultaneously Avoiding a Vincular and a Covincular Pattern of Length 3 (2017)

Paper by C. Bean, H. Ulfarsson and A. Claesson. Out of 40 results we could replicate 11, one of them shown below.



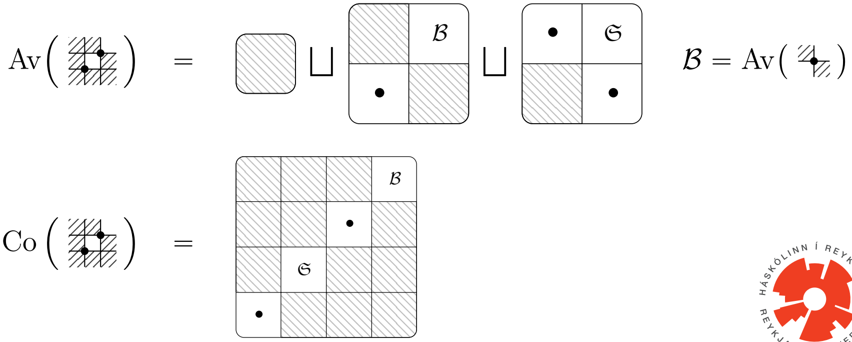
From this cover it is easy to see that the GF satisfies

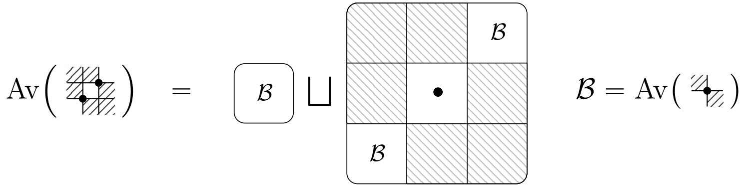
$$F(x) = 1 + xF(x) + x^2F(x)^2$$

which indeed gives us the *Motzkin* numbers M_n to x^n in $F(x)$.

Wilf-Classification of Mesh Patterns of Short Length (2015)

Paper by Í. Hilmarsson, I. Jónsdóttir, S. Sigurðardóttir, L. Viðarsdóttir and H. Ulfarsson. Out of 65 results we managed to replicate 15. Some of them shown here without further comments.

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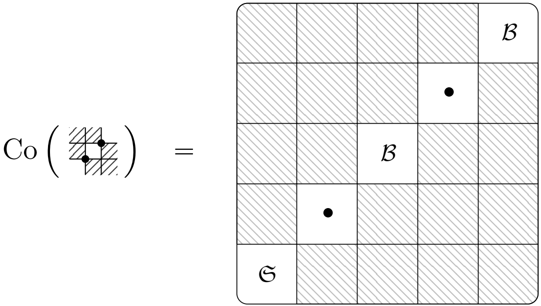
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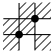
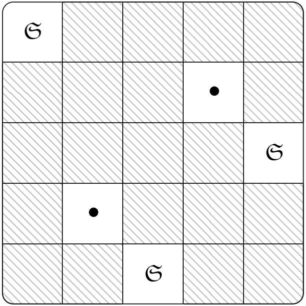
$\mathrm{Co}\Big($  $\Big) =$ 

Wilf classification of bi-vincular permutation patterns (2009)

Preprint by R. Parviainen. Out of 24 results we replicated 7.

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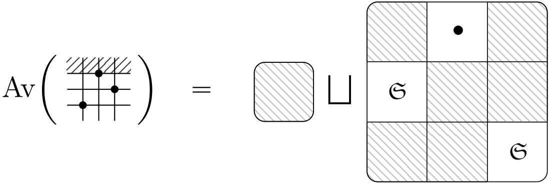
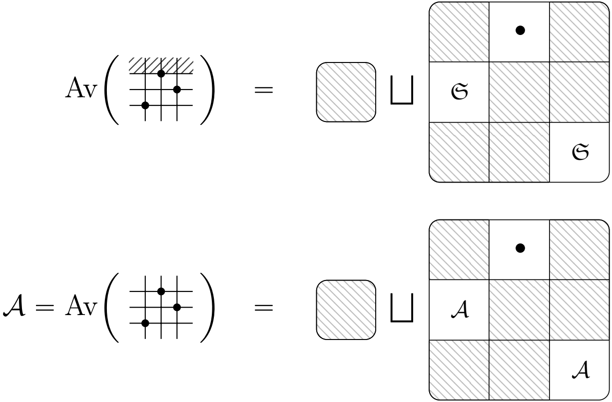


Wilf classification of bi-vincular permutation patterns (2009)

Preprint by R. Parviainen. Out of 24 results we replicated 7. Below is one of them. It is interesting to compare it to the well known $\mathrm{Av}(132)$.



Conclusions

- We showed that CombCov is a powerful tool in guiding humans by coming up with conjectures that would otherwise have required substantial effort to discover manually.
- We were pleasantly surprised in how many published results it found covers for.
- CombCov's shortcomings may be remedied with improved ways of coming up with and generate the rules.

Any questions?

"There is no such thing as a dumb question."

— Carl Sagan