Øving 1

1) $P_Y(y) = P(Y \leq y) = P(P_X(x) \leq y)$

$= P(P_X^{-1}(P_X(x)) \leq P_X^{-1}(y)) = P(x \leq P_X^{-1}(y))$

$= P(P^{-1}(y)) = y$

$p(y) \quad \frac{\partial P_Y(y)}{\partial y} = \underline{1} \quad , \underline{y \in [0, 1]}$

2a) $p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x \in (0, 1, 2 \dots)$

$G(t) = E_x(t^x) = \sum_0^\infty \frac{\lambda^x e^{-\lambda}}{x!} \cdot t^x =$

$e^{-\lambda} \sum_0^\infty \frac{(\lambda t)^x}{x!} = e^{-\lambda} e^{\lambda t} = \underline{\underline{e^{\lambda(t-1)}}}$

b) $p(x) = \binom{n}{x} r^x (1-r)^{n-x}$

$G(t) = E_x(t^x) = \sum_0^\infty \binom{n}{x} r^x t^x (1-r)^{n-x}$

$\binom{n}{x} = \frac{n!}{x!(n-x)!}$

$G(t) = \sum_0^\infty \frac{n!}{x!(n-x)!} (rt)^x (1-r)^{n-x} = \underline{\underline{(1-r+rt)^n}}$

Siste kommer av Binomial teoremet.

c) Binomial moment genererende funksjon

$G(t) = \left(1 + \frac{n(rt-r)}{n}\right)^n = \left(1 + \frac{\lambda(t-1)}{n}\right)^n$

$\lim_{n \to \infty} G(t) = \underline{\underline{e^{\lambda(t-1)}}}$

Ser at når $n \to \infty$ så blir bearnoulli en poisson fordeling. Betyr at poisson er et spesialtilfelle av Bearnoulli for $n \to \infty$

2d) Vet at sum av to tilfeldige variable tilsvarer produktet av deres moment genererende funksjoner

$$G(t) = G_{N_1} \cdot G_{N_2} = e^{\lambda_1(t_1-1)} e^{\lambda_2(t_2-1)} = \underline{\underline{e^{(\lambda_1+\lambda_2)(t-1)}}}$$

Ser at dette også er poisson fordelt.

3 $$p(t_1 \mid t_1 \geq t_0) = \frac{p(t_1 \geq t_0 \mid T_1)\, p(t_1)}{p(t_0 \leq t_1)}$$

$$p(t_1 \geq t_0 \mid t_1) = \begin{cases} 1 & t_1 \geq t_0 \\ 0 & T_1 < t_0 \end{cases}$$

$$p(T_1) = \lambda e^{-\lambda t_1}$$

$$p(t_0 \leq t_1) = \int_{t_0}^{\infty} \lambda e^{-\lambda T_1}\, dT_1 = \left[-e^{-\lambda T_1}\right]_0^{\infty} = e^{-\lambda t_0}$$

$$p(t_1 \mid t_1 \geq t_0) = \begin{cases} \lambda e^{-\lambda(t_1 - t_0)} & t_1 \geq t_0 \\ 0 & t_1 \leq t_0 \end{cases}$$

b) Fordelingen fa $T_n - T_0 = \sum_{i=1}^{n} \Delta T_i$ blir det samme som å se på produktet av momentfunksjonen.

$$G_{\Delta T}(t) = \frac{1}{\left(1 - \frac{\Delta t}{\lambda}\right)}$$

$$G_{T_n - T_0}(t) = \prod^{n} \frac{1}{\left(1 - \frac{\Delta t}{\lambda}\right)} = \frac{1}{\left(1 - \frac{\Delta t}{\lambda}\right)^n}$$

Dette er en gamma fordeling ~~med~~ $\Gamma(n, \lambda)$

3c) $P(\Delta T_{n+1} > t \mid T_n \in [0,t]) = P(\Delta T_{n+1} > t - t_n \mid T_n = t_n \in [0,t])$

$$= 1 - \int_0^{t-t_n} \lambda e^{-\lambda \Delta t} \, d\Delta t = 1 - [-e^{-\lambda(t-t_n)} + 1] = \underline{\underline{e^{-\lambda(t-t_n)}}}$$

d) $p_N(n) = P(T_{n+1} > t \wedge T_n \leq t \mid T_1 > t_0) =$

$P(T_{n+1} > t, T_n \leq t \mid T_1 \geq t_0) =$

$P(T_{n+1} > t \mid T_n \leq t, T_1 \geq t_0) \cdot P(T_n \leq t \mid T_1 \geq t_0)$

$$P(T_{n+1} > t \mid T_n \leq t, T_1 \geq t_0) = e^{-\lambda(t-t_n)}$$

$$P(T_n \leq t \mid T_1 \geq t_0) = P(\underbrace{T_n - t_0}_{\Delta T} \leq t - t_0 \mid T_1 \geq t_0) =$$

$$P(\Delta T \leq \Delta t), \quad \Delta t = t_n - t_0$$

$$= \int_0^{t-t_0} \frac{\lambda^n \Delta t^{n-1}}{(n-1)!} e^{-\lambda \Delta t} \, d\Delta t$$

$$p_N(n) = \lambda^n e^{-\lambda(t-t_0)} \int_{t_0}^{t} \frac{(t_n - t_0)^{n-1}}{(n-1)!} \, dt_n = \frac{(t_n - t_0)^n \cdot \lambda^n e^{-\lambda(t-t_0)}}{n!}$$

$$= \frac{(\lambda(t_n - t_0))^n}{n!} e^{-\lambda(t-t_0)}$$

Dette er da en poisson fordeling med parameter $-\lambda(t-t_0)$.

4a) $P(n_D, n)$

$P(n_d | n) = B(n_D; P_D, n)$

$P(n_D, n_u | n) = P(n_D | n)$

$$P(n_D, n) = P(n_d | n) \cdot P(n)$$

$$= \binom{n}{n_D} P_D^{n_D} (1-P_D)^{n-n_D} \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!}$$

$$= \frac{\cancel{n!}}{n_D!(n-n_D)!} P_D^{n_D} (1-P_D)^{n-n_D} \cdot \frac{\lambda^n e^{-\lambda}}{\cancel{n!}} =$$

$$\frac{P_D^{n_D}}{n_D!} \frac{(1-P_D)^{n_u}}{n_u!} \cdot \lambda^{n_u + n_D} \cdot e^{-\lambda} =$$

$$\frac{(\lambda P_D)^{n_D}}{n_D!} e^{-\lambda P_D} \cdot \frac{(\lambda(1-P_D))^{n_u}}{n_u} e^{-\lambda(1-P_D)} = \underline{\underline{P(n_D) \cdot P(n_u)}}$$

4(b) $P(n_D|m) = \dfrac{P(m|n_d)\cdot P(n_d)}{P(m)}$

$$P(n_D) = \frac{(\lambda P_D)^{n_D}}{n_D!} e^{-\lambda P_D}$$

$$m = n_D + m_{fa}$$

$$P(m, n_d) = P(m|n_D)\, P(n_D)$$

$$P(m_{fa}+n_D, n_D) = P(m_{fa}+n_D | n_D)\cdot P(n_D)$$

$$P(m_{fa}, n_D) = P(m_{fa}|n_D)\cdot P(n_D)$$

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$$P(n_D|m) = \frac{P(m_{fa}, n_d)}{P(m)} = \frac{P(m_{fa})\, P(n_D)}{P(m)}$$

$P(m) = P(n_D + m_{fa}) \sim$ Poisson fordelt.

$$P(n_D|m) = \frac{\dfrac{\Lambda^{m_{fa}} e^{-\Lambda}}{m_{fa}!}\cdot\dfrac{(\lambda P_D)^{n_D} e^{-\lambda P_D}}{n_D!}}{\dfrac{(\Lambda+\lambda P_D)^m e^{-(\Lambda+P_D)}}{m!}} = \frac{m!}{m_{fa}!\,n_D!}\,\frac{\Lambda^{m_{fa}}(\lambda P_D)^{n_D}}{(\Lambda+\lambda P_D)^m}$$

$$= \frac{m!}{(m-n_D)\,n_D}\,\frac{\Lambda^{m-n_D}}{(\Lambda-\lambda P_D)^{m-n_D}}\cdot\frac{(\lambda P_D)^{n_D}}{(\Lambda+\lambda P_D)^{n_D}}$$