8 Waveguides with Cylindrical Conducting Boundaries

8.1 INTRODUCTION

A waveguide is a structure, or part of a structure, that causes a wave to propagate in a chosen direction with some measure of confinement in the planes transverse to the direction of propagation. If the waveguide boundaries change direction, within reasonable limits, the wave is constrained to follow it. For example, in a transmission line used to transfer energy from a transmitter to an antenna, the energy follows the path of the line, at least for paths with only small discontinuities. The guiding of the waves in all such systems is accomplished by an intimate connection between the fields of the wave and the currents and charges on the boundaries or by some condition of reflection at the boundary.

In this chapter we concentrate on cylindrical structures with conducting boundaries. Multiconductor lines can be used for frequencies from dc up to the millimeter-wave range. At the highest frequencies, they are often in the form of metallic films on insulating substrates. Hollow conducting cylinders of various cross-sectional shapes are used in the microwave and millimeter-wave frequency ranges (approximately 1–100 GHz).

Generally, in waveguide analyses we are interested in the distribution of the electromagnetic fields; but of greatest importance is the dependence of the propagation constant upon frequency. From the propagation constant one finds wave velocities, phase variation, and attenuation along the guide and the pulse dispersion properties of the guide.

Several different types of guides are analyzed in this chapter, including the simple parallel-plate structure (and some more practical, related forms) and hollow-tube guides of rectangular and circular cross section. An introduction to means for exciting waves in waveguides is presented. The chapter concludes with a study of the general properties of waves in cylindrical waveguides with conducting boundaries.

General Formulation for Guided Waves

8.2 Basic Equations and Wave Types for Uniform Systems

We consider here cylindrical systems with axes taken along the z axis. We also consider time-harmonic waves with time and distance variations described by $e^{(j\omega t-\gamma z)}$, as in the study of transmission-line waves. The character of the propagation constant γ tells much about the properties of the wave, such as the degree of attenuation and the phase and group velocities. The fields in the wave must satisfy the wave equation and the boundary conditions. We will assume that there is no net charge density in the dielectric and that any conduction currents are included by allowing permittivity and therefore $k^2 = \omega^2\mu\varepsilon$ to be complex. The wave equations, which reduce to the Helmholtz equations for phasor fields (Sec. 3.11), are

$$\nabla^2\mathbf{E} = -k^2\mathbf{E}, \qquad \nabla^2\mathbf{H} = -k^2\mathbf{H}$$

The three-dimensional ∇^2 may be broken into two parts:

$$\nabla^2\mathbf{E} = \nabla_t^2\mathbf{E} + \frac{\partial^2\mathbf{E}}{\partial z^2}$$

The last term is the contribution to ∇^2 from derivatives in the axial direction. The first term is the two-dimensional Laplacian in the transverse plane, representing contributions to ∇^2 from derivatives in this plane. With the assumed propagation function $e^{-\gamma z}$ in the axial direction,

$$\frac{\partial^2\mathbf{E}}{\partial z^2} = \gamma^2\mathbf{E}$$

The foregoing wave equations may then be written

$$\nabla_t^2\mathbf{E} = -(\gamma^2 + k^2)\mathbf{E} \tag{1}$$

$$\nabla_t^2\mathbf{H} = -(\gamma^2 + k^2)\mathbf{H} \tag{2}$$

Equations (1) and (2) are the differential equations that must be satisfied in the dielectric regions of the transmission lines or guides. The boundary conditions imposed on fields follow from the configuration and the electrical properties of the boundaries.

The usual procedure is to find two components of the fields, usually the z components of $\mathbf{E}$ and $\mathbf{H}$, that satisfy the wave equations (1) and (2) and the boundary conditions; then the other field components can be found from these by using Maxwell's equations. To facilitate finding the other components, it usually is most convenient to have them explicitly in terms of the z components of $\mathbf{E}$ and $\mathbf{H}$.

The curl equations with the assumed functions $e^{(j\omega t-\gamma z)}$ are written below for fields in the dielectric system, assumed here to be linear, homogeneous, and isotropic:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \qquad\qquad \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\frac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x \quad (3) \qquad\qquad \frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x \quad (6)$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \quad (4) \qquad\qquad -\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad (7)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad (5) \qquad\qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \quad (8)$$

It must be remembered in all analysis to follow that these coefficients, E_x, H_x, E_y, and so on, are functions of x and y only, by our agreement to take care of the z and time functions in the assumed $e^{(j\omega t - \gamma z)}$.

From the foregoing equations, it is possible to solve for E_x, E_y, H_x, or H_y in terms of E_z and H_z. For example, H_x is found by eliminating E_y from (3) and (7), and a similar procedure gives the other components.

$$E_x = -\frac{1}{\gamma^2 + k^2}\left(\gamma \frac{\partial E_z}{\partial x} + j\omega\mu \frac{\partial H_z}{\partial y}\right) \quad (9)$$

$$E_y = \frac{1}{\gamma^2 + k^2}\left(-\gamma \frac{\partial E_z}{\partial y} + j\omega\mu \frac{\partial H_z}{\partial x}\right) \quad (10)$$

$$H_x = \frac{1}{\gamma^2 + k^2}\left(j\omega\varepsilon \frac{\partial E_z}{\partial y} - \gamma \frac{\partial H_z}{\partial x}\right) \quad (11)$$

$$H_y = -\frac{1}{\gamma^2 + k^2}\left(j\omega\varepsilon \frac{\partial E_z}{\partial x} + \gamma \frac{\partial H_z}{\partial y}\right) \quad (12)$$

For propagating waves, it is convenient to use the substitution $\gamma = j\beta$ where β is real if there is no attenuation. Rewriting the above with this substitution,

$$E_x = -\frac{j}{k_c^2}\left(\beta \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y}\right) \quad (13)$$

$$E_y = \frac{j}{k_c^2}\left(-\beta \frac{\partial E_z}{\partial y} + \omega\mu \frac{\partial H_z}{\partial x}\right) \quad (14)$$

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon \frac{\partial E_z}{\partial y} - \beta \frac{\partial H_z}{\partial x}\right) \quad (15)$$

$$H_y = -\frac{j}{k_c^2}\left(\omega\varepsilon \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y}\right) \quad (16)$$

$$\nabla_t^2 E_z = -k_c^2 E_z \quad (17)$$

$$\nabla_t^2 H_z = -k_c^2 H_z \quad (18)$$

where

$$k_c^2 \triangleq \gamma^2 + k^2 = k^2 - \beta^2 \tag{19}$$

In studying guided waves along uniform systems, it is common to classify the wave solutions into the following types:

1. Waves that contain neither electric nor magnetic field in the direction of propagation. Since electric and magnetic field lines both lie entirely in the transverse plane, these may be called *transverse electromagnetic (TEM) waves*. They are the usual *transmission-line waves* along a multiconductor guide.
2. Waves that contain electric field but no magnetic field in the direction of propagation. Since the magnetic field lies entirely in transverse planes, they are known as *transverse magnetic (TM)* waves. They have also been referred to in the literature as *E waves*, or waves of electric type.
3. Waves that contain magnetic field but no electric field in the direction of propagation. These are known as *transverse electric (TE)* waves, and have also been referred to as *H waves*, or waves of magnetic type.
4. *Hybrid waves* for which boundary conditions require all field components. These may often be considered as a coupling of TE and TM modes by the boundary.

The preceding is not the only way in which the possible wave solutions may be divided, but is a useful way in that any general field distribution excited in an ideal guide may be divided into a number (possibly an infinite number) of the above types with suitable amplitudes and phases. The propagation constants of these tell how the individual waves change phase and amplitude as they travel down the guide, so that they may be superposed at any later position and time to give the total resultant field there. Since it is disadvantageous to have a signal carried by several waves traveling at different velocities because of the resultant distortion, waveguides are normally designed so that only one wave can propagate even if many are excited at the entrance to the guide. As we shall see in Chapter 14, multimode guides are sometimes used in optical communications; they have also been used in "overmoded" millimeter-wave systems, but the single-mode guide is the norm.

Cylindrical Waveguides of Various Cross Sections

8.3 Waves Guided by Perfectly Conducting Parallel Plates

One of the simplest wave-guiding systems for analysis is that formed by a slab of dielectric with parallel-plane conductors on top and bottom. The fields are assumed to be the same as if the plates were of infinite width, which means that any edge effects or other variations along one transverse coordinate are neglected for a first-order analysis of this model. We saw in Chapter 5 that such a system may be considered a two-conductor transmission line, with upper and lower plates acting as the two conductors of the line. But we shall see that the system also guides waves of other types. Analysis of this system helps in the understanding of all the wave types before going on to more complicated boundaries for the guiding system. We consider the three classes of waves defined in the preceding section.

TEM Waves The transverse electromagnetic waves have neither E_z nor H_z. From Eqs. 8.2(9)–(12) we see that all transverse components must be zero also *unless* $\gamma^2 + k^2 = 0$. The propagation constant for a TEM wave must then be

$$\gamma_{\text{TEM}} = \pm jk \tag{1}$$

That is, propagation is with the velocity of light in the dielectric medium. Since this argument does not make use of the specific configuration of parallel planes, it applies to TEM waves in any shape of guide, as will be discussed more later. Moreover, if $\gamma^2 + k^2$ is zero, we see from Eqs. 8.2(1) and 8.2(2) that both electric and magnetic fields satisfy Laplace's equation so that both have the spatial distribution of two-dimensional *static* fields. It is known that the static electric field between parallel-plane conductors is uniform and normal to the planes so that we may write

$$E_x = E_0 \tag{2}$$

and magnetic field, from Eq. 8.2(4) with $E_z = 0$, is

$$H_y = \frac{\gamma}{j\omega\mu} E_x = \pm \frac{j\omega\sqrt{\mu\varepsilon}}{j\omega\mu} E_x = \pm \sqrt{\frac{\varepsilon}{\mu}} E_x \tag{3}$$

where the upper sign is for positively traveling waves and the lower sign for negatively traveling waves. When interpreted in terms of voltage and current, we find the same results as those obtained from the transmission-line analysis.

TM Waves Transverse magnetic waves have finite E_z but no H_z so we may use Eq. 8.2(17). The transverse Laplacian is taken as d^2/dx^2 because of the neglect of the y derivatives:

$$\frac{d^2E_z}{dx^2} = -k_c^2 E_z \tag{4}$$

$$k_c^2 = \gamma^2 + k^2 \tag{5}$$

Solution of (4) is in sinusoids:

$$E_z = A \sin k_c x + B \cos k_c x \tag{6}$$

Boundary conditions are next applied at the conducting planes at $x = 0$ and a. Since these are taken as perfectly conducting, $E_z = 0$ there. Placing $E_z = 0$ at $x = 0$ in (6) requires $B = 0$. The condition $E_z = 0$ at $x = a$ then requires either $A = 0$, in which case we have nothing left, or $\sin k_c a = 0$, which is satisfied by

$$k_c a = m\pi, \qquad m = 1, 2, 3, \ldots \tag{7}$$

So a solution for E_z which satisfies boundary conditions is

$$E_z = A \sin \frac{m\pi x}{a} \tag{8}$$

The transverse field components are now obtained from E_z by means of Eqs. 8.2(9) to 8.2(12), letting $H_z = 0$ and $\partial/\partial y = 0$:

$$E_x = -\frac{\gamma}{k_c^2}\frac{dE_z}{dx} = -\frac{\gamma a}{m\pi} A \cos \frac{m\pi x}{a} \tag{9}$$

$$H_y = -\frac{j\omega\varepsilon}{k_c^2}\frac{dE_z}{dx} = -\frac{j\omega\varepsilon a}{m\pi} A \cos \frac{m\pi x}{a} \tag{10}$$

$$H_x = 0, \qquad E_y = 0 \tag{11}$$

It is seen that there is an infinite number of solutions for the various integral values of m, each with a different field distribution. These solutions are called the *modes* of the guide (in this case, TM modes).

Let us now examine the properties of the propagation constant. Solving (5) for γ we have

$$\gamma = \sqrt{k_c^2 - k^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\varepsilon} \tag{12}$$

At sufficiently high frequencies, the second term in the radicand is larger than the first and we can rewrite (12) as

$$\gamma = j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{(m\pi/a)^2}{\omega^2\mu\varepsilon}} \tag{13}$$

We see that the phase constant approaches that for a plane wave as frequency approaches infinity. Lowering the frequency reduces β and it goes to zero at the frequency

$$\omega_c = \frac{m\pi}{a\sqrt{\mu\varepsilon}} = \frac{m\pi v}{a} \tag{14}$$

where v is the velocity of light $(\mu\varepsilon)^{-1/2}$ in the given material. We call ω_c the cutoff frequency since propagation takes place only where β is real. The variation of β with ω is often plotted as in Fig. 8.3*a* for the reasons discussed in Sec. 5.12. It is conveniently expressed in terms of the cutoff frequency

$$\gamma = j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}, \qquad \omega \geq \omega_c \tag{15}$$

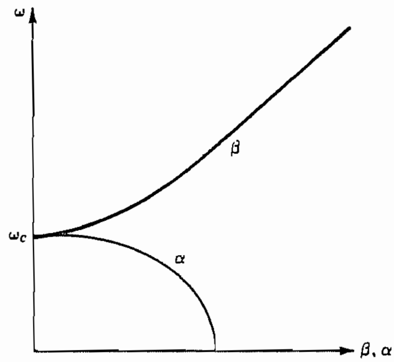


FIG. 8.3a Phase constant and attenuation as functions of frequency for a TM or TE waveguide mode.

The cutoff point may usefully be expressed in terms of the wavelength at the cutoff frequency

$$\lambda_c = \frac{2\pi v}{\omega_c} = \frac{2a}{m} \tag{16}$$

That is, cutoff occurs when the spacing between plates is m half-wavelengths, measured at the velocity of light for the dielectric material. We see the important feature that TM waves can propagate above a cutoff frequency (or, equivalently, at wavelengths shorter than the cutoff wavelength); at lower frequencies γ is real and is given by

$$\gamma = \alpha = \frac{m\pi}{a}\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2}, \qquad \omega \le \omega_c \tag{17}$$

As seen in Fig. 8.3*a* there is attenuation without phase shift for frequencies below the cutoff frequency of a given mode, phase shift without attenuation for frequencies above cutoff, and neither attenuation nor phase shift exactly at cutoff. Each of these modes then acts as a high-pass filter; the attenuation below cutoff, like that for a loss-free filter, is a reactive attenuation representing reflection but no dissipation for this nondissipative system.

For the propagating regime, $\omega > \omega_c$, we can define a phase velocity in the usual way:

$$v_p = \frac{\omega}{\beta} = \frac{v}{\sqrt{1 - (\omega_c/\omega)^2}} \tag{18}$$

Group velocity can be derived from this as in Sec. 5.15:

$$v_g = \frac{d\omega}{d\beta} = v\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \tag{19}$$

Phase velocity is always greater than the velocity of light in the medium and group velocity is always less, both approaching each other and v at frequencies far above cutoff. A wavelength along the guide, λ_g, may also be defined as the distance for which phase shift increases by 2π,

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - (\omega_c/\omega)^2}} \tag{20}$$

where λ is wavelength of a plane wave in the dielectric medium,

$$\lambda = \frac{2\pi v}{\omega} \tag{21}$$

The ratio of transverse electric field to transverse magnetic field of a single propagating wave may be defined as a characteristic wave impedance and is useful for certain types of reflection problems, much as the field impedance for plane waves was found to be in Chapter 6. For the TM wave, this impedance, from (9) and (10) with $\eta = (\mu/\varepsilon)^{1/2}$, is

$$Z_{\mathrm{TM}} = \frac{E_x}{H_y} = \frac{\beta}{\omega\varepsilon} = \eta\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \tag{22}$$

Note that this ratio is not a function of x and y. It is imaginary for frequencies below cutoff, so that a Poynting calculation will show no average power flow in that regime. It is real for frequencies above cutoff so that a Poynting calculation in that regime shows finite average power carried by the wave.

A plot of electric field lines in a TM_1 mode from the equations for E_x and E_z is shown in Fig. 8.3*b* (Prob. 8.3c). Note that induced charges on top and bottom plates are of the same sign in a given z plane for this wave, in contrast to the sign relations found for the TEM wave. Electric field lines starting on these charges turn and go axially down the guide, ending on charges of opposite sign a distance a half guide wavelength down the guide. The entire pattern translates along the guide at the phase velocity for a traveling wave in one direction.

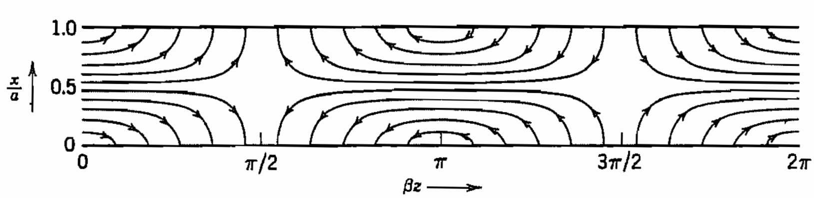


FIG. 8.3*b* Electric field lines of TM_1 wave between plane conductors.

TE Waves The transverse electric class of waves has nonzero H_z but no E_z. Equation 8.2(18) is then utilized:

$$\nabla_t^2 H_z = \frac{d^2 H_z}{dx^2} = -k_c^2 H_z \tag{23}$$

$$k_c^2 = \gamma^2 + k^2 = k^2 - \beta^2 \tag{24}$$

The solution will again be written in terms of sinusoids, but this time only the cosine term is retained since E_y, proportional to the derivative of H_z with x, must become zero at the perfectly conducting plane $x = 0$:

$$H_z = B \cos k_c x \tag{25}$$

From Eqs. 8.2(13) to 8.2(16), remembering that E_z is zero,

$$H_x = -\frac{j\beta}{k_c^2}\frac{dH_z}{dx} = \frac{j\beta}{k_c} B \sin k_c x \tag{26}$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{dH_z}{dx} = -\frac{j\omega\mu}{k_c} B \sin k_c x \tag{27}$$

$$E_x = 0, \qquad H_y = 0 \tag{28}$$

Also, E_y must be zero at the conducting plane $x = a$, so k_c is determined from (27) as some multiple of π/a. As with the TM wave, this is identified from (24) as the value of k at cutoff:

$$k_c = 2\pi f_c \sqrt{\mu\varepsilon} = \frac{m\pi}{a}, \qquad m = 1, 2, 3, \ldots \tag{29}$$

The propagation constant from (24) may then be written

$$\gamma = \alpha = \left(\frac{m\pi}{a}\right)\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2}, \qquad \omega < \omega_c \tag{30}$$

$$\gamma = j\beta = jk\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}, \qquad \omega > \omega_c \tag{31}$$

The forms for attenuation constant in the cutoff range and phase constant in the propagation range are thus exactly the same as for the TM waves (Fig. 8.3*a*), and by (14) and (29) the conditions for cutoff are the same for TE modes as for TM modes of the same order. The expressions for phase velocity, group velocity, and guide wavelength in the propagation range follow from (31) and are exactly the same as (18) to (20). Wave or field impedance for the TE wave is

$$Z_{TE} = -\frac{E_y}{H_x} = \frac{j\omega\mu}{\gamma} = \frac{\eta}{\sqrt{1 - (\omega_c/\omega)^2}} \tag{32}$$

For frequencies below cutoff this wave impedance is imaginary, but for frequencies

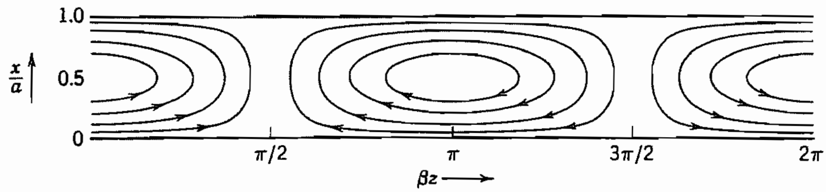


FIG. 8.3c Magnetic field lines of TE_1 wave between plane conductors.

above cutoff it is real and always greater than η, as contrasted with the wave impedance for TM waves, which is always less than η.

The form of the field lines for the first-order TE mode is indicated in Fig. 8.3*c*. Here the magnetic field lines form closed curves surrounding the y-direction displacement current. There is no charge induced on the conducting plates and only a y component of current corresponding to the finite H_z tangential to the plates.

Example 8.3
INTERPRETATION OF GROUP VELOCITY AS AN ENERGY VELOCITY

Let us define a velocity of energy flow in terms of energy stored per unit length and average power flow:

$$v_{\mathrm{E}} = \frac{W_T}{u} \tag{33}$$

Take the TM wave as example. Average power flow is found from the complex Poynting theorem for a width w:

$$\begin{aligned} W_T &= w \int_0^a \frac{1}{2}\,\mathrm{Re}(E_x H_y^*)dx \\ &= \frac{w}{2} A^2 \beta\omega\varepsilon\left(\frac{a}{m\pi}\right)^2 \int_0^a \cos^2\frac{m\pi x}{a}\,dx = \frac{waA^2}{4}\frac{a^2}{m^2\pi^2}\beta\omega\varepsilon \end{aligned} \tag{34}$$

Time-average energy storage per unit length, including both electric and magnetic parts, is

$$u = w\int_0^a \left\{\frac{\varepsilon}{4}\,[|E_x|^2 + |E_z|^2] + \frac{\mu}{4}\,|H_y|^2\right\} dx \tag{35}$$

Using the fields from (8)–(10) this is

$$\begin{aligned} u &= \frac{A^2 w}{4}\int_0^a \left\{\varepsilon\left[\sin^2\frac{m\pi x}{a} + \frac{\beta^2 a^2}{m^2\pi^2}\cos^2\frac{m\pi x}{a}\right] + \frac{\mu\omega^2\varepsilon^2 a^2}{m^2\pi^2}\cos^2\frac{m\pi x}{a}\right\} dx \\ &= \frac{A^2 wa\varepsilon}{8}\left[1 + \frac{\beta^2 a^2}{m^2\pi^2} + \frac{k^2 a^2}{m^2\pi^2}\right] \end{aligned}$$

But from (13), $m^2\pi^2/a^2 + \beta^2$ is just k^2, so this reduces to

$$u = \frac{A^2 wa\varepsilon k^2 a^2}{4m^2\pi^2} \tag{36}$$

and energy velocity from (33) is

$$v_E = \frac{\beta\omega}{k^2} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \tag{37}$$

which is exactly the same as expression (19) for group velocity. The same equivalence is found for TE waves. We will say more about the use of these velocities for signal propagation at the end of the chapter.

8.4 Guided Waves Between Parallel Planes as Superposition of Plane Waves

The modes of the parallel-plane guide, studied in the preceding section from the wave equation, can also be found to be superpositions of plane waves propagating at various angles. This picture is very helpful in developing a physical feeling for the different types of modes. The TEM wave is clearly just a portion of a uniform plane wave polarized with electric field in the x direction and propagating in the z direction:

$$E_x(x, z) = E_0 e^{-jkz} \tag{1}$$

The magnetic field for such a wave, from Sec. 6.2, is

$$H_y(x, z) = \frac{E_0}{\eta} e^{-jkz} \tag{2}$$

where $k = \omega\sqrt{\mu\varepsilon}$. These are of the same form as Eqs. 8.3(2) and 8.3(3).

For TM and TE waves, we superpose uniform plane waves propagating at an angle θ from the normal, as pictured in Fig. 8.4a. It was found in Sec. 6.9 that when plane waves strike a conducting plane at an angle, tangential electric field must be zero not only at that plane, but also at other planes which are multiples of a half-wavelength

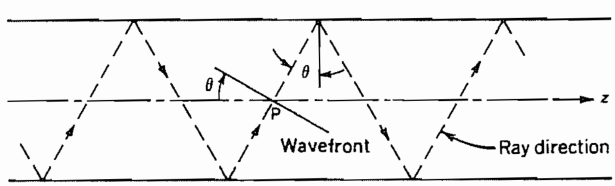


Fig. 8.4a Diagram showing ray directions and wavefront direction of a uniform plane-wave component of TM or TE waves between parallel planes.

measured at phase velocity in the direction normal to the plane. That is, boundary conditions are satisfied if spacing a between plates is

$$a = \frac{m\lambda}{2\cos\theta} \tag{3}$$

or

$$\cos\theta = \frac{m\lambda}{2a} = \frac{\lambda}{\lambda_c} = \frac{\omega_c}{\omega} \tag{4}$$

where

$$\lambda_c = \frac{2\pi v}{\omega_c} = 2a/m \tag{5}$$

The phase constant in the z direction is

$$\beta = \mathrm{k}\sin\theta = k\sqrt{1 - \cos^2\theta} \tag{6}$$

or, using (4),

$$\beta = k\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \tag{7}$$

This is exactly the same as found from the detailed analysis in Sec. 8.3. Since β is the same, phase and group velocity will be also. We see from Fig. 8.4a that phase velocity is the velocity of the imaginary point P of intersection of the plane-wave fronts with the z axis, and is greater than velocity v of the plane wave in the medium, as found in Sec. 8.3. Group velocity is the component of v in the z direction and is always less than v. At cutoff (Fig. 8.4b), the waves simply bounce laterally back and forth between planes without any forward progression so that group velocity is zero and phase velocity infinite. For frequencies much above cutoff, the angle is very flat (Fig. 8.4c) and the wavefronts are nearly normal to z, so that both v_p and v_g approach v.

Both TM and TE wave types show the behavior described above. TM waves correspond to the plane waves polarized with electric field in the plane of incidence, and TE waves, to those polarized with magnetic field in the plane of incidence.

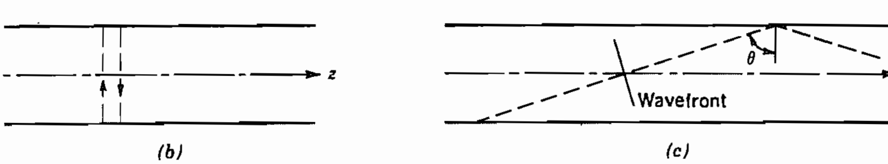


FIG. 8.4 (b) Special case of (a) when wave is at cutoff. (c) Special case for frequency far above cutoff so that $\theta \to \pi/2$ and wavefront is nearly normal to guide axis.

Example 8.4

DETAILED FIELD EXPRESSIONS FOR TM_m WAVE FROM PLANE-WAVE COMPONENTS

To show that this point of view is exactly equivalent to that of Sec. 8.3, consider the field expressions given by Eqs. 6.9(11)–(13) for a wave polarized with electric field in the plane of incidence striking a conductor at an angle. We rewrite these with the coordinates x and z primed, since we take different coordinate systems from those in Sec. 6.9:

$$E_{x'}(x', z') = -2jE_+ \cos\theta \sin(kz' \cos\theta) e^{-jkx' \sin\theta} \tag{8}$$

$$E_{z'}(x', z') = -2E_+ \sin\theta \cos(kz' \cos\theta) e^{-jkx' \sin\theta} \tag{9}$$

$$\eta H_y(x', z') = 2E_+ \cos(kz' \cos\theta) e^{-jkx' \sin\theta} \tag{10}$$

To change to our present coordinate system, let $x' = z$ and $z' = -x$. Also let $k \sin \theta = \beta$ by (6), and $k \cos\theta = m\pi/a$ from (3). Equations (8)–(10) then become

$$E_z(z, x) = +\frac{2jE_+ m\pi}{ka} \sin\left(\frac{m\pi x}{a}\right) e^{-j\beta z} \tag{11}$$

$$E_x(z, x) = \frac{2E_+ \beta}{k} \cos\left(\frac{m\pi x}{a}\right) e^{-j\beta z} \tag{12}$$

$$\eta H_y(z, x) = 2E_+ \cos\left(\frac{m\pi x}{a}\right) e^{-j\beta z} \tag{13}$$

If we next set the multiplier $(2jE_+ m\pi/ka)$ equal to a constant A and remember that the propagation factor $e^{-j\beta z}$ has been suppressed in Eqs. 8.3(8)–(10), we find that (11)–(13) are identical to Eqs. 8.3(8)–(10).

8.5 PARALLEL-PLANE GUIDING SYSTEM WITH LOSSES

When the dielectric filling the region between conductors of the parallel-plane guide is imperfect, we may replace permittivity by its complex forms $\varepsilon' - j\varepsilon''$ (Sec. 6.4) in the expressions of Sec. 8.3. Here we are concerned primarily with the effect on the propagation constant. For the TEM wave we have

$$\gamma_{\text{TEM}} = j\omega\sqrt{\mu(\varepsilon' - j\varepsilon'')} \tag{1}$$

If $\varepsilon''/\varepsilon'$ is small in comparison with unity, a binomial expansion shows that the real part of this, which is the attenuation constant, is to first order

$$(\alpha_d)_{\text{TEM}} \cong \frac{\omega\sqrt{\mu\varepsilon'}}{2} \frac{\varepsilon''}{\varepsilon'} \tag{2}$$

For TM and TE waves from Eq. 8.3(12)

$$\begin{aligned}\gamma_{\mathrm{TM,TE}} &= \left[\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu(\varepsilon' - j\varepsilon'')\right]^{1/2} \\ &\approx \left[\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\varepsilon'\right]^{1/2}\left\{1 + \frac{j\omega^2\mu\varepsilon''}{2}\left[\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\varepsilon'\right]^{-1}\right\}\end{aligned} \tag{3}$$

The second expression is obtained by making a binomial expansion of the first. The attenuation constant (real part) for $\omega > \omega_c$ is then

$$(\alpha_d)_{\mathrm{TM,TE}} = \frac{\omega\sqrt{\mu\varepsilon'}(\varepsilon''/\varepsilon')}{2\sqrt{1 - (\omega_c/\omega)^2}} \tag{4}$$

Retention of second-order terms gives a correction to phase constant, which may be important in considering the dispersive properties of the guided wave.

An exact solution is considerably more difficult to obtain when the finite conductivity of the conducting boundaries must be considered. The approach is to obtain field solutions in both dielectric and conductor regions, with proper continuity conditions applied at the boundary between them. This approach cannot even be carried out for some geometrical configurations, although it can for this simple system and shows that the approximations to be used in the following analysis are justified so long as the plane boundaries are made of much better conducting material than the intervening dielectric region.[1] The expression to be used is that giving attenuation in terms of power loss per unit length and average power transferred by the mode, Eq. 5.11(19). This is an exact expression, but the approximation comes in by calculating power transfer as that of the ideal guide, and loss per unit length as that from currents of the ideal guide flowing in the real conductors. For the TEM wave, the power transfer for a width w is

$$(W_T)_{\mathrm{TEM}} = \frac{awE_0^2}{2\eta} \tag{5}$$

The average power loss per unit area in the plates, if plates are thick compared with skin depth, is $\frac{1}{2}R_s|J_{sz}|^2$, so for a unit length and width w, counting both plates,

$$(w_L)_{\mathrm{TEM}} = 2\,\frac{wR_s}{2}\,|H_y|^2 = wR_s\left(\frac{E_0}{\eta}\right)^2 \tag{6}$$

Attenuation constant is then calculated as

$$(\alpha_c)_{\mathrm{TEM}} = \frac{w_L}{2W_T} = \frac{R_s}{\eta a} \tag{7}$$

This is the same as that which would be obtained from the transmission-line formula, Eq. 5.11(22).

[1] *More precisely, it is required that displacement current in the conductor be negligible in comparison with displacement current in the dielectric, and conduction current in the dielectric be small in comparison with conduction current in the conductor.*

For the TM wave of order m, average power transfer from the Poynting theorem and the results of Sec. 8.3 for $\omega > \omega_c$ is

$$
\begin{aligned}
(W_T)_{\mathrm{TM}} &= w \int_0^a \frac{1}{2} (E_x H_y^*)\, dx \\
&= \frac{w}{2} \int_0^a \left(-\frac{j\beta a A}{m\pi} \cos \frac{m\pi x}{a} e^{-j\beta z} \right) \left(\frac{j\omega\varepsilon a}{m\pi} A \cos \frac{m\pi x}{a} e^{j\beta z} \right) dx \\
&= \frac{w}{2} \frac{\omega\varepsilon\beta a^2 A^2}{m^2\pi^2} \int_0^a \cos^2 \frac{m\pi x}{a}\, dx = \left(\frac{w\omega\varepsilon\beta a^2 A^2}{2m^2\pi^2} \right) \frac{a}{2}
\end{aligned} \tag{8}
$$

The current flow in both upper and lower plates is

$$
|J_{sz}| = |H_y|_{x=0} = |H_y|_{x=a} = \frac{\omega\varepsilon a A}{m\pi} \tag{9}
$$

So average power loss per unit length for a width w is

$$
(w_L)_{\mathrm{TM}} = \frac{2wR_s}{2} |J_{sz}|^2 = wR_s \left(\frac{\omega\varepsilon a A}{m\pi} \right)^2 \tag{10}
$$

The attenuation constant from conductor losses is then approximately

$$
(\alpha_c)_{\mathrm{TM}} = \frac{w_L}{2W_T} = \frac{2R_s\omega\varepsilon}{\beta a} = \frac{2R_s}{\eta a \sqrt{1 - (\omega_c/\omega)^2}} \tag{11}
$$

By a similar calculation we find attenuation for a TE mode to be

$$
(\alpha_c)_{\mathrm{TE}} = \frac{2R_s(\omega_c/\omega)^2}{\eta a \sqrt{1 - (\omega_c/\omega)^2}} \tag{12}
$$

Curves of normalized attenuation versus frequency are shown in Fig. 8.5. There are several interesting features of these curves. Note first that expressions (11) and (12)

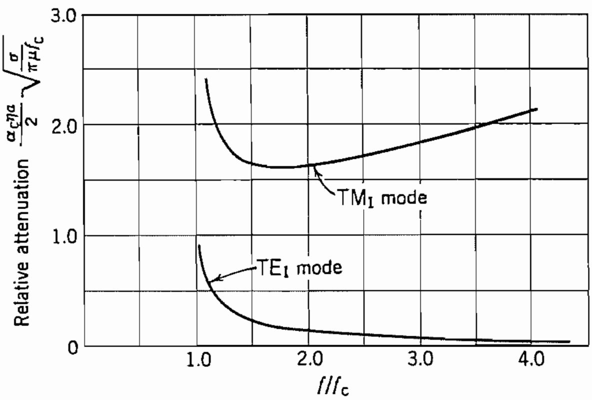


FIG. 8.5 Attenuation curves of waves between imperfectly conducting planes.

approach infinity as $\omega \to \omega_c$, but the approximations break down at cutoff so that attenuation is actually finite although high there. The curve for the TM wave starts to decrease with increasing frequency above cutoff, but reaches a minimum (at $\omega = \sqrt{3}\omega_c$) and thereafter increases with increasing frequency because of the increase of R_s with frequency. The curve for the TE wave is always lower than that for the TM and, moreover, continues to decrease with increasing frequency. This behavior arises because the currents in the conductors are the y-directed currents related to H_z, and this component of field approaches zero at high frequencies as the wavefront becomes substantially normal to the axis, as explained in Sec. 8.4.

8.6 Planar Transmission Lines

Several different forms of wave-guiding structures made from parallel metal strips on a dielectric substrate have found use in microwave and millimeter-wave circuits as well as in high-speed digital circuits.[2] In this section we will examine in some detail three types, called *stripline*, *microstrip*, and *coplanar waveguide*. Emphasis will be on the lowest-order mode, which is a TEM wave in the stripline and a quasi-TEM wave in microstrip and coplanar waveguide.

Stripline The stripline consists of a conducting strip lying between, and parallel to, two wide conducting planes, as shown in Fig. 8.6*a*. The region between the strip and the planes is filled with a uniform dielectric. Such a structure, with a uniform dielectric and more than one conductor, can support a TEM wave. If the strip width w is much greater than the spacing d and the two planes are at a common potential, the structure is roughly approximated by two parallel-plane lines connected in parallel. More precise results are found from the capacitance per unit length. For a TEM wave, the phase velocity is $v_p = (\mu\varepsilon)^{-1/2}$, and from the transmission-line formalism it is also given by $v_p = (LC)^{-1/2}$. Then the characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{\sqrt{LC}}{C} = \frac{\sqrt{\mu\varepsilon}}{C} \tag{1}$$

Thus, for systems with uniform ε and μ the characteristic impedance can be found from the capacitance, which can be determined in a number of ways, as we saw in Chapters 1 and 7. An approximate expression for the characteristic impedance of the stripline, assuming a zero-thickness strip, has been found by conformal transformation[3] to be

$$Z_0 \approx \frac{\eta}{4} \frac{K(k)}{K(\sqrt{1 - k^2})} \tag{2}$$

[2] T. Itoh (Ed.), Planar Transmission Line Structures, IEEE Press, Piscataway, NJ, 1987.

[3] R. E. Collin, Field Theory of Guided Waves, 2nd ed., Sec. 4.3, IEEE Press, Piscataway, NJ, 1991.

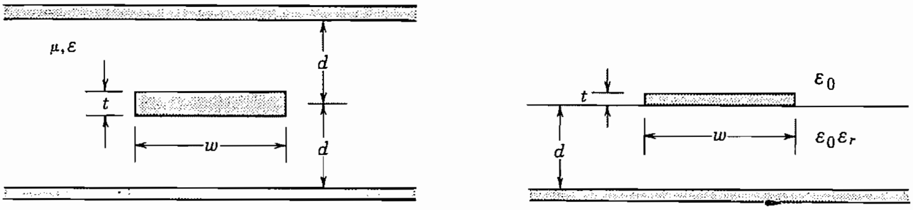


FIG. 8.6 (*a*) Stripline. (*b*) Microstrip.

where $\eta = \sqrt{\mu/\varepsilon}$, k is given by

$$k = \left[\cosh\left(\frac{\pi w}{4d}\right)\right]^{-1} \tag{3}$$

and $K(k)$ is the complete elliptic integral of the first kind (see Ex. 4.7a). A convenient approximate expression for Z_0, accurate over the range $w/2d > 0.56$, is[4]

$$Z_0 \approx \frac{\eta\pi}{8 \ln[2 \exp(\pi w/4d)]} \tag{4}$$

Approximate techniques used to obtain corrections to Z_0 for $t > 0$ are discussed by Hoffmann.[4] The velocity of propagation does not depend on thickness and is $(\mu\varepsilon)^{-1/2}$ as for all TEM waves. Both velocity of propagation and characteristic impedance, neglecting loss, are independent of frequency and may be used up to the cutoff frequency of modes between the ground plates, Eq. 8.3(14).

An approximate expression for attenuation resulting from conductor surface resistivity R_s is[3]

$$\alpha_c = \frac{R_s}{2\eta d}\left[\frac{\pi w/2d + \ln(8d/\pi t)}{\ln 2 + \pi w/4d}\right] \text{ nepers/m} \tag{5}$$

which is valid if $w > 4d$ and $t < d/5$. Approximations for other dimensions are given in footnote 4. Attenuation from lossy dielectrics is exactly as in Eq. 8.5(2).

Microstrip Probably the most widely used thin-strip line is one formed with the strip lying on top of an insulator with a conductive backing. A different dielectric (usually air) is above the insulator and strip (Fig. 8.6*b*). In such an arrangement, there cannot be a true TEM wave. The reason is that such a wave, as shown in connection with Eq. 8.3(1), requires that $\gamma^2 + k^2 = 0$. The propagation constant γ is a single quantity for the wave, so $\gamma^2 + k_1^2$ and $\gamma^2 + k_2^2$ cannot both be zero if $k_1 \neq k_2$. Exact solution of this problem is complicated by the finite width of the strip and the two different dielectrics. As mentioned above for the stripline, the lowest-order approximation for the microstrip is a section of parallel-plane guide neglecting fringing. A much better, though

[4] *R. K. Hoffmann,* Handbook of Microwave Integrated Circuits, *Artech House, Norwood, MA, 1987.*

still approximate, approach is to consider that the lowest-order wave is approximately a TEM wave so the distribution of fields in the transverse plane is nearly the same as that for static fields. This so-called *quasistatic* approach employs calculations made with static fields to determine the transmission-line parameters for propagation of the lowest-order mode, even though it is not a pure TEM wave. The approximation is very useful in practical applications but has some limitations that we will mention below.

A common approach to obtaining simple, accurate expressions is to first find the characteristic impedance Z_{00} of an electrode structure identical to the one of interest but with the strip electrode having zero thickness and the dielectric being free space everywhere. This problem can be solved exactly by the method of conformal mapping, but the expressions are very complex and more useful results are the various approximations to the exact expressions. A particularly useful expression is[4]

$$Z_{00} = 377\left[\frac{w}{d} + 1.98\left(\frac{w}{d}\right)^{0.172}\right]^{-1} \tag{6}$$

which is accurate to $<0.3\%$ for all $(w/d) > 0.06$. Then to get the characteristic impedance of the actual line, it is corrected by the so-called *effective dielectric constant* ε_{eff}, which, if filling the entire space, would give the same capacitance as that of the actual structure. Since the inductance is unaffected by the presence of a dielectric, correction for the capacitance gives the correction for the characteristic impedance.

The static approximation to the effective relative dielectric constant is also found by conformal mapping methods; one of several useful approximations is[5]

$$\varepsilon_{\text{eff}} = 1 + \frac{(\varepsilon_r - 1)}{2}\left[1 + \frac{1}{\sqrt{1 + 10d/w}}\right] \tag{7}$$

which is always between unity and ε_r. Applying (6) and (7) in

$$Z_0 = Z_{00}/\sqrt{\varepsilon_{\text{eff}}} \tag{8}$$

we obtain the static approximation for the actual characteristic impedance. The results for a variety of insulator dielectric constants are shown in Fig. 8.6*c*. Corrections for nonzero strip thickness have been published.[4]

Since the phase velocity in a TEM wave is $v_p = (\mu\varepsilon)^{-1/2}$, the ratio $c/v_p = \sqrt{\varepsilon_{\text{eff}}}$ in the quasistatic case. As frequency increases, the longitudinal field components become increasingly important; this can be represented by a frequency-dependent effective dielectric constant $\varepsilon_{\text{eff}}(f)$ to express the variation of phase velocity or phase constant. Approximate expressions for $\varepsilon_{\text{eff}}(f)$ and for attenuation from conductor and dielectric losses have been obtained from numerical calculations or empirically.[4] Some results follow:

$$\sqrt{\varepsilon_{\text{eff}}(f)} = \frac{\beta}{k_0} = \frac{\sqrt{\varepsilon_r} - \sqrt{\varepsilon_{\text{eff}}(0)}}{1 + 4F^{-1.5}} + \sqrt{\varepsilon_{\text{eff}}(0)} \tag{9}$$

[5] *H. A. Wheeler*, IEEE Trans. Microwave Theory Tech. ***MTT-25***, *631 (1977).*

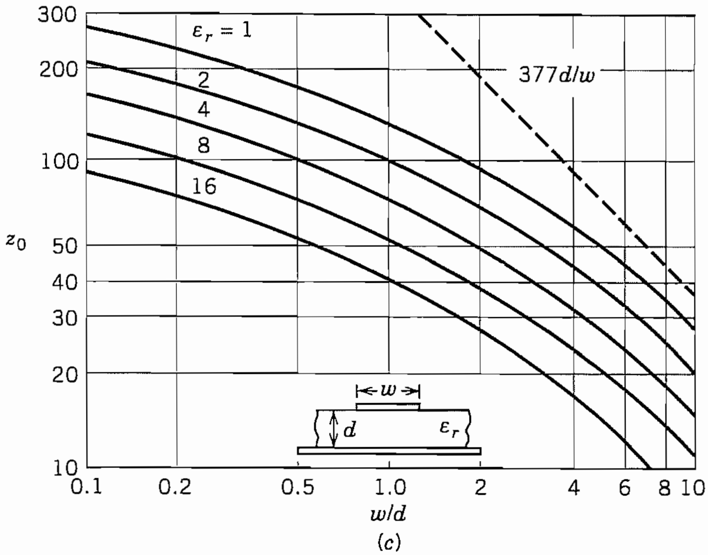


FIG. 8.6c Approximate characteristic impedance of stripline. The dashed line is the parallel-plane approximation for air dielectric. Adapted from H. A. Wheeler, *IEEE Trans. Microwave Theory Tech.* **MTT-25**, 631 (1977). © 1977 IEEE.

where $\varepsilon_{\text{eff}}(0)$ is given by (7) and

$$F = \frac{4fd\sqrt{\varepsilon_r - 1}}{c}\left\{0.5 + \left[1 + 2\ln\left(1 + \frac{w}{d}\right)\right]^2\right\} \tag{10}$$

The frequency above which the frequency-dependent effective dielectric constant in (9) and (10) is needed is given empirically by[4]

$$f_{\max} = \frac{21 \times 10^6}{(w + 2d)\sqrt{\varepsilon_r + 1}} \tag{11}$$

The quasi-TEM mode can be used up to near the frequency where a higher-order mode can propagate, which is $(f_c)_{\text{HE1}} = cZ_0/2\eta_0 d$. The relationship of the higher-order modes to the quasi-TEM mode is analogous to the relationship of the TM and TE modes to the TEM mode in the parallel-plate guide (Sec. 8.3), but here the higher-order modes are hybrids of TE and TM components.

Attenuation from losses in conductors with surface resistance R_s is

$$\alpha_c = R_s\sqrt{\varepsilon_{\text{eff}}(0)}A/d \tag{12}$$

where the parameter A is given for various strip widths and thicknesses in Fig. 8.6*d*. Attenuation from a dielectric having a loss tangent $\tan\delta_\varepsilon$ is

$$\alpha_\varepsilon = \frac{\pi f \tan\delta_\varepsilon}{c}\sqrt{\frac{\varepsilon_r(1 + F_1)}{2}}\left[1 + \frac{1 - F_1}{\varepsilon_r(1 + F_1)}\right]^{-1} \quad \text{nepers/m} \tag{13}$$

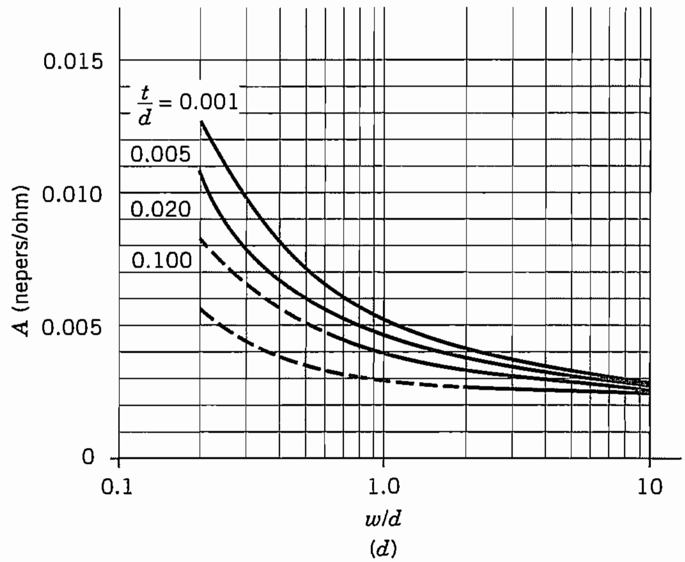


FIG. 8.6d Factor for calculation of conductor losses in microstrip using the loss formula, Eq. 8.6(12). Both conductors are assumed to have the same R_s and thicknesses satisfying $t > 3\delta$. Broken parts of the curves may require adjustment of w to account for nonzero electrode thickness t. See footnote 4 from which these data are taken.

where $F_1 = [1 + (10d/w)]^{-1/2}$. The total attenuation can be taken to be the sum of (12) and (13), although there are also radiation and scattering losses.

Coplanar Waveguide Of the several different forms of stripline wave-guiding systems in which all conductors are on one surface of a dielectric substrate, the most widely used is the *coplanar waveguide* shown in Fig. 8.6*e*, in which the signal voltage is applied between the center strip and the grounded outer strips.

As with the microstrip, the fundamental mode of propagation in the coplanar waveguide is a quasi-TEM mode. Because the dielectric is not homogeneous in the transverse plane, the wave cannot be a pure TEM mode. The distribution of electric fields in the space above the line is the same as in the substrate (assuming negligible thickness of the strips and infinite substrate thickness). We assume here that air is above the strips so $\varepsilon_r = 1$. If the capacitance per unit length for a line with the same conductors but $\varepsilon_r = 1$ everywhere is C_0, in the actual line the capacitance contributions above and below the substrate surface will be $C_0/2$ and $C_0\varepsilon_r/2$. Then an effective dielectric constant can be defined for the quasistatic limit as

$$\varepsilon_{\text{eff}} = \frac{C}{C_0} = \frac{\varepsilon_r + 1}{2} \tag{14}$$

and the phase velocity is $v_p = (\mu\varepsilon_0\varepsilon_{\text{eff}})^{-1/2}$.

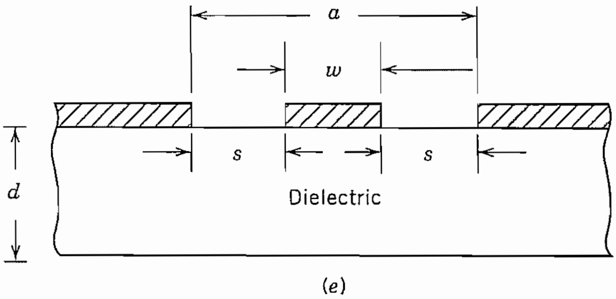


FIG. 8.6e Coplanar waveguide.

The characteristic impedance for zero-thickness conductors, infinitely wide ground conductors, and infinite-thickness substrate can be expressed in terms of complete elliptic integrals of first-order $K(k)$ and $K(k')$:

$$Z_0 = \frac{Z_{00}}{\sqrt{\varepsilon_{\text{eff}}}} = \eta_0 \frac{K(k')}{4\sqrt{\varepsilon_{\text{eff}}}K(k)} \tag{15}$$

where $k = w/a$ (see Fig. 8.6e) and $k' = (1 - k^2)^{1/2}$. Very good approximations to (15) (<0.24% error) that are much more convenient in practice are

$$Z_0 = \frac{\eta_0}{\pi\sqrt{\varepsilon_{\text{eff}}}} \ln\left(2\sqrt{\frac{a}{w}}\right) \qquad \text{for } 0 < w/a < 0.173 \tag{16}$$

and

$$Z_0 = \frac{\pi\eta_0}{4\sqrt{\varepsilon_{\text{eff}}}} \left[\ln\left(2\,\frac{1 + \sqrt{w/a}}{1 - \sqrt{w/a}}\right)\right]^{-1} \qquad \text{for } 0.173 < w/a < 1 \tag{17}$$

These expressions for effective dielectric constant and characteristic impedance are also accurate to within several percent if the substrate thickness d is finite but greater than the total gap width a. More complex relations are required for thinner substrates.[4]

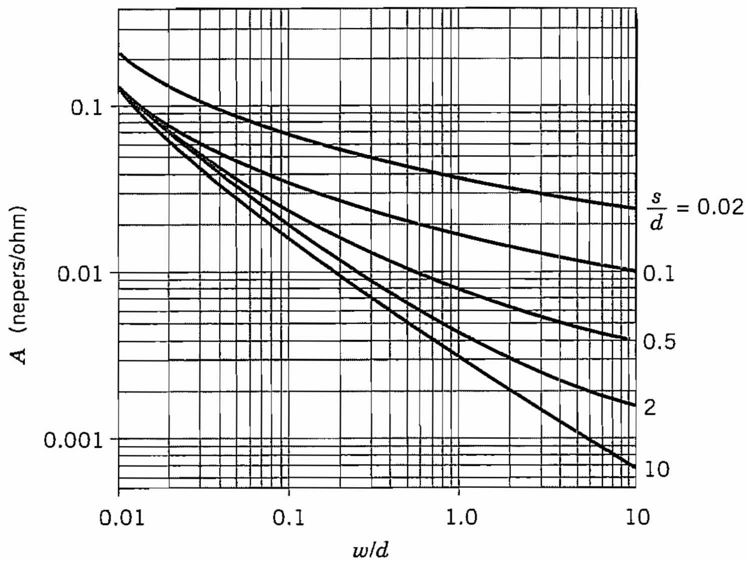
One reason for interest in the coplanar waveguide is that dispersion is typically less than in the microstrip for microwave and lower frequencies. A form nearly identical to (9) has been shown to give a good fit to numerical calculations of the dispersion in the coplanar waveguide over a very wide range of parameters.[6] Thus,

$$\sqrt{\varepsilon_{\text{eff}}(f)} = \frac{\beta}{k_0} = \frac{\sqrt{\varepsilon_r} - \sqrt{\varepsilon_{\text{eff}}(0)}}{1 + bF_2^{-1.8}} + \sqrt{\varepsilon_{\text{eff}}(0)} \tag{18}$$

where $F_2 = 2fd\sqrt{\varepsilon_r - 1}/c$ and $\varepsilon_{\text{eff}}(0)$ is the value in (14). Also,

$$b = \exp[u \ln(w/s) + r] \tag{19}$$

[6] *G. Hasnian, A. Dienes, and J. R. Whinnery,* IEEE Trans. Microwave Theory Tech. ***MTT-34,*** *738 (1986).*



(*f*)

FIG. 8.6*f* Factor for calculation of conductor losses in coplanar waveguide using the general loss formula, Eq. 8.6(12). Conductors all have the same R_s and have thicknesses satisfying $t > 3\delta$. Data taken from footnote 4.

where parameters u and r depend on substrate thickness according to

$$u = 0.54 - 0.64q + 0.015q^2$$

and

$$r = 0.43 - 0.86q + 0.54q^2$$

in which $q = \ln(w/d)$.

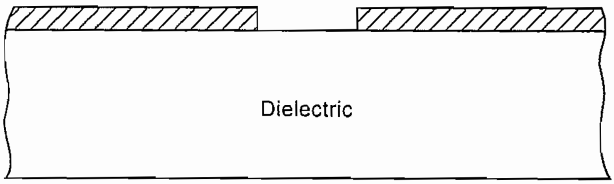
Attenuation from conductor losses in the coplanar waveguide can be found using the general form (12) with parameter A given by the data in Fig. 8.6*f*.

Attenuation resulting from losses in the dielectric is found using[4]

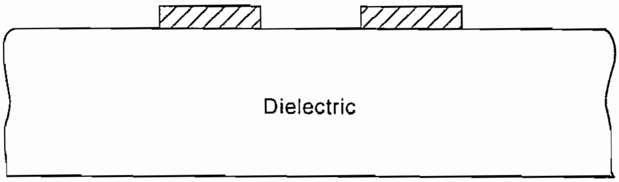
$$\alpha_d = \frac{\pi f\sqrt{\varepsilon_{\text{eff}}(0)}}{c}\left[\frac{1 - 1/\varepsilon_{\text{eff}}(0)}{1 - 1/\varepsilon_r}\right]\tan\delta_\varepsilon \quad \text{nepers/m} \tag{20}$$

with $\varepsilon_{\text{eff}}(0)$ given by (14), which is accurate only if $d/a > 1$. More accurate expressions exist for thinner substrates.[4] The quasi-TEM solution used here applies up to $F_2 = 1$.

There are several other varieties of strip-type lines. Two prominent types are the *slot-line waveguide* and *coplanar strips* shown, respectively, in Figs. 8.6*g* and *h*. These are both versions of two-conductor transmission lines. As with the microstrip and coplanar waveguide, the lowest mode is not TEM but rather quasi-TEM, because of the different dielectrics above and below the conductors.



(g)



(h)

FIG. 8.6 (g) Slot-line waveguide. (h) Coplanar-strip waveguide.

8.7 RECTANGULAR WAVEGUIDES

The most important of the hollow-pipe guides is that of rectangular cross section. As in Fig. 8.7*a*, a dielectric region of width a and height b extends indefinitely in the axial (z) direction and is closed by conducting boundaries on the four sides. In the ideal guide, both conductor and dielectric are loss-free. There can be no transverse electromagnetic (TEM) wave inside the hollow pipe since, as was shown in Sec. 8.3, TEM waves have transverse variations like static fields, and no static fields can exist inside a region bounded by a single conductor. Transverse magnetic (TM) and transverse electric (TE) waves can exist and will be analyzed below.

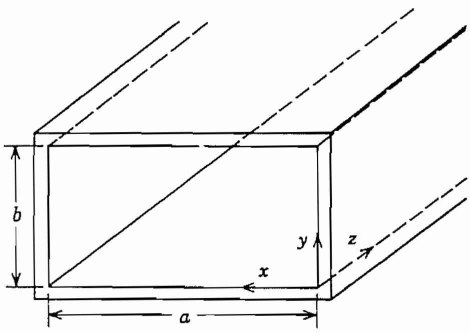


FIG. 8.7a Coordinate system for rectangular guide.

TM Waves Transverse magnetic waves have zero H_z but nonzero E_z. The differential equation governing E_z is Eq. 8.2(17), here expressed in rectangular coordinates:

$$\nabla_t^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -k_c^2 E_z \tag{1}$$

This equation was solved in Sec. 7.19 by separation of variables procedures and found to have solutions of the form

$$E_z = (A' \sin k_x x + B' \cos k_x x)(C' \sin k_y y + D' \cos k_y y) \tag{2}$$

where

$$k_x^2 + k_y^2 = k_c^2 \tag{3}$$

The perfectly conducting boundary at $x = 0$ requires $B' = 0$ to produce $E_z = 0$ there. Similarly the ideal boundary at $y = 0$ requires $D' = 0$. We let $A'C'$ be a new constant A and have

$$E_z = A \sin k_x x \sin k_y y \tag{4}$$

Axial electric field E_z must also be zero at $x = a$ and $y = b$. This can only be so (except for the trivial solution $A = 0$) if $k_x a$ is an integral multiple of π:

$$k_x a = m\pi, \qquad m = 1, 2, 3, \ldots \tag{5}$$

Similarly, to make E_z zero at $y = b$, $k_y b$ must also be a multiple of π:

$$k_y b = n\pi, \qquad n = 1, 2, 3, \ldots \tag{6}$$

So the cutoff condition of the transverse magnetic wave with m variations in x and n in y (designated TM_{mn}) is found from (3):

$$\omega_{c_{m,n}} = \frac{k_{c_{m,n}}}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2} \tag{7}$$

Since k_c^2 is $k^2 - \beta^2$ as in Eq. 8.2(19), attenuation for frequencies below the cutoff frequency of a given mode and phase constant for frequencies above the cutoff frequency have the same forms as for the parallel-plane guiding system:

$$\alpha = k_{c_{m,n}}\sqrt{1 - \left(\frac{\omega}{\omega_{c_{m,n}}}\right)^2}, \qquad \omega < \omega_{c_{m,n}} \tag{8}$$

$$\beta = k\sqrt{1 - \left(\frac{\omega_{c_{m,n}}}{\omega}\right)^2}, \qquad \omega > \omega_{c_{m,n}} \tag{9}$$

Phase and group velocities then also have the same forms as before [Eqs. 8.3(18) and 8.3(19)].

The remaining field components of the TM_{mn} wave are found from Eqs. 8.2(13)–(16) with $H_z = 0$ and E_z from (4):

$$E_x = -\frac{j\beta k_x}{k^2_{c_{m,n}}} A \cos k_x x \sin k_y y \tag{10}$$

$$E_y = -\frac{j\beta k_y}{k^2_{c_{m,n}}} A \sin k_x x \cos k_y y \tag{11}$$

$$H_x = \frac{j\omega\varepsilon k_y}{k^2_{c_{m,n}}} A \sin k_x x \cos k_y y \tag{12}$$

$$H_y = -\frac{j\omega\varepsilon k_x}{k^2_{c_{m,n}}} A \cos k_x x \sin k_y y \tag{13}$$

where k_x, k_y, $k_{c_{m,n}}$ and β are defined by (5), (6), (7), and (9), respectively, and all fields are multiplied by the propagation terms, $e^{-j\beta z}$. Plots of electric and magnetic field lines in the TM_{11} and TM_{21} modes are shown in Table 8.7. Note that electric field lines (shown solid) begin on charges on the guide walls at some fixed z plane, turn and go axially down the guide, and end on charges of opposite sign a half-guide wavelength down the guide. Magnetic field lines (shown dashed) surround the displacement currents represented by the changing electric fields as the pattern moves down the guide with velocity v_p. The pattern for the TM_{21} mode is that of two TM_{11} modes side by side and of opposite sense.

The attenuation resulting from losses in the conducting walls can be calculated for the TM_{mn} wave following the procedure used to find Eq. 8.5(11):

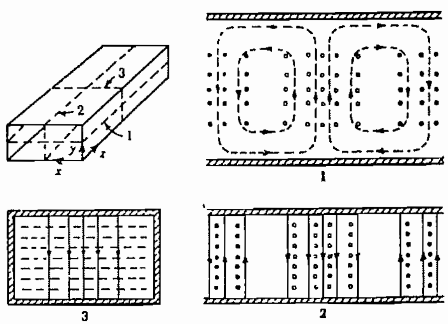
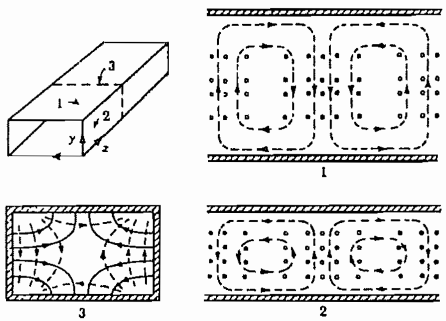
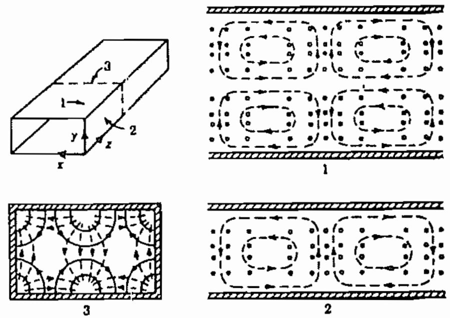
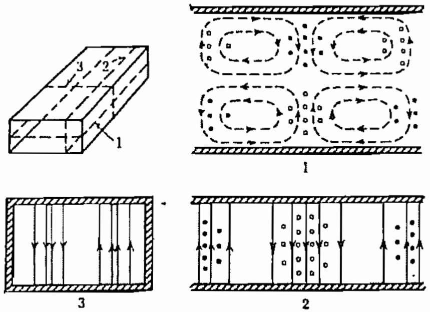
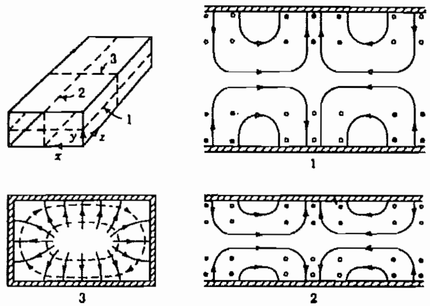
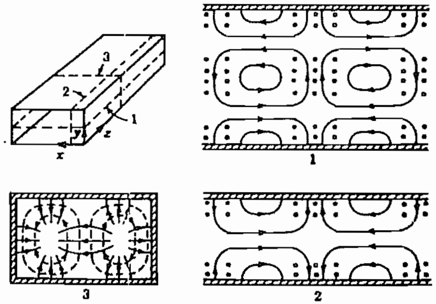
$$(\alpha_c)_{TM_{mn}} = \frac{2R_s}{b\eta\sqrt{1 - (f_c/f)^2}} \frac{[m^2(b/a)^3 + n^2]}{[m^2(b/a)^2 + n^2]} \tag{14}$$

We make two points before leaving this class of waves. Note from (4) and the definitions of k_x and k_y given by (5) and (6) that neither m nor n can be zero for the TM wave without its disappearing entirely. The second point is that we have required as boundary condition that E_z be zero along the perfectly conducting boundary, but should also be sure that other tangential components of electric field are zero there. From (10) and the definitions (5) and (6) we can see that E_x is zero as required at $y = 0$ and $y = b$; from (11) we see that E_y is zero at $x = 0$ and $x = a$. Thus all tangential components do satisfy the boundary conditions at the conductors. It can be shown (Prob. 8.7h) that imposition of the boundary condition on E_z necessarily causes the other tangential components of **E** to be zero on the boundaries because of the form of the relations 8.2(13)–(16).

TE Waves Transverse electric waves have zero E_z and nonzero H_z so that the start is from Eq. 8.2(18), again expressed in rectangular coordinates:

$$\nabla_t^2 H_z = \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = -k_c^2 H_z \tag{15}$$

Table 8.7
Summary of Wave Types for Rectangular Guides[a]

TE_{10}	TE_{11}	TE_{21}
		
TE_{20}	TM_{11}	TM_{21}
		

[a] Electric field lines are shown solid and magnetic field lines are dashed.

Solution by the separation of variables techniques of Sec. 7.19 gives

$$H_z = (A'' \sin k_x x + B'' \cos k_x x)(C'' \sin k_y y + D'' \cos k_y y) \tag{16}$$

where

$$k_c^2 = k_x^2 + k_y^2 \tag{17}$$

Imposition of boundary conditions in this case is a little less direct, but from Eqs. 8.2(13) and 8.2(14) we find electric field components as

$$\begin{aligned} E_x &= -\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} \\ &= -\frac{j\omega\mu k_y}{k_c^2}(A'' \sin k_x x + B'' \cos k_x x)(C'' \cos k_y y - D'' \sin k_y y) \end{aligned} \tag{18}$$

$$\begin{aligned} E_y &= \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} \\ &= \frac{j\omega\mu k_x}{k_c^2}(A'' \cos k_x x - B'' \sin k_x x)(C'' \sin k_y y + D'' \cos k_y y) \end{aligned} \tag{19}$$

For E_x to be zero at $y = 0$ for all x, $C'' = 0$, and for $E_y = 0$ at $x = 0$ for all y, $A'' = 0$. Defining $B''D'' = B$, we have then

$$H_z = B \cos k_x x \cos k_y y \tag{20}$$

We also require E_x to be zero at $y = b$ so that $k_y b$ must be a multiple of π. E_y is zero at $x = a$ so that $k_x a$ is also a multiple of π:

$$k_x a = m\pi, \qquad k_y b = n\pi \tag{21}$$

In contrast to the TM waves, one but not both of m and n may be zero without the wave's vanishing. Although we found the boundary conditions by first calculating electric field, we can see from the way in which $\mathbf{E}$ is related to H_z that the derivative of H_z normal to the conducting boundary must be zero for the tangential electric field to be zero there, so boundary conditions can be imposed directly on the form (16) without requiring the explicit forms for E_x and E_y.

The forms of transverse electric field with the derived simplifications to (18) and (19) are

$$E_x = \frac{j\omega\mu k_y}{k_{c_{m,n}}^2} B \cos k_x x \sin k_y y \tag{22}$$

$$E_y = -\frac{j\omega\mu k_x}{k_{c_{m,n}}^2} B \sin k_x x \cos k_y y \tag{23}$$

Corresponding transverse magnetic field components from Eqs. 8.2(15) and 8.2(16) are

$$H_x = \frac{j\beta k_x}{k^2_{c_{m,n}}} B \sin k_x x \cos k_y y \tag{24}$$

$$H_y = \frac{j\beta k_y}{k^2_{c_{m,n}}} B \cos k_x x \sin k_y y \tag{25}$$

Since comparison of (21) with (5) and (6) shows that k_x and k_y have the same forms for TM and TE waves, cutoff frequency and propagation characteristics for a TE_{mn} mode found from (17) are exactly the same as for the same order TM_{mn} mode. That is, the expressions (7), (8), and (9) apply here without change. Modes that have different field distributions but the same cutoff frequencies are said to be *degenerate modes.*

Table 8.7 gives the field distribution for several different TE modes. Since electric field is confined to the transverse plane, we find that for each one of the TE modes shown, electric fields begin on charges for a portion of the boundary and end on charges of opposite sign on another portion, in the same x–y plane. Magnetic field lines surround the displacement currents represented by the changing transverse electric fields. For the TE_{10} mode having no variations in the vertical direction, electric fields go between top and bottom of the guide in straight lines, and magnetic fields lie entirely in planes parallel to top and bottom. The TE_{10} mode is so important that it will be discussed separately in the following section.

Figure 8.7*b* shows a line diagram indicating the cutoff frequencies of several of the lowest-order modes referred to that of the so-called *dominant* TE_{10} mode for a guide with a side ratio $b/a = \frac{1}{2}$, which is close to the value used in most practical guides. Normally, such a guide is designed so that its cutoff frequency for the TE_{10} mode is somewhat (say, 30%) below the operating frequency. In this way only one mode can propagate so signal distortion caused by multimode propagation is avoided. Also, by not being too close to the cutoff frequency, dispersion caused by having different group velocities for different frequency components of the signal is minimized for the one propagating mode. Higher-order modes may be excited at the entrance to the guide but they are below their cutoff frequencies and die away in a short distance from the source.

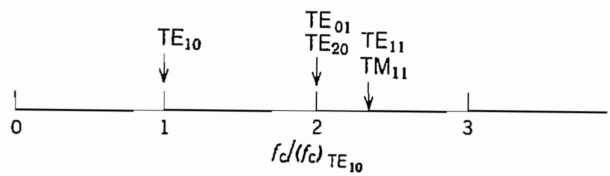


FIG. 8.7b Relative cutoff frequencies of waves in a rectangular guide ($b/a = \frac{1}{2}$).

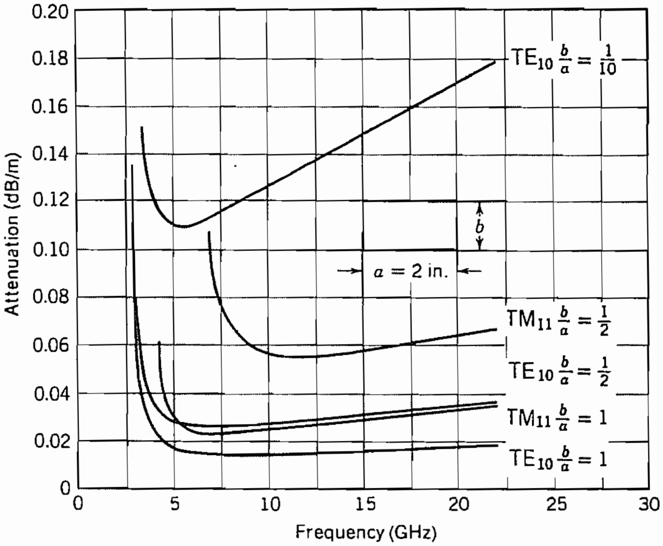


FIG. 8.7c Attenuation due to copper losses in rectangular waveguides of fixed width.

The attenuation constant for TE_{mn} $(n \neq 0)$ modes is found using power transfer and power loss per unit length as in Eq. 8.5(11).

$$(\alpha_c)_{TE_{mn}} = \frac{2R_s}{b\eta\sqrt{1 - (f_c/f)^2}} \left\{ \left(1 + \frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2 + \left[1 - \left(\frac{f_c}{f}\right)^2\right]\left[\frac{(b/a)((b/a)m^2 + n^2)}{(b^2m^2/a^2) + n^2}\right] \right\} \tag{26}$$

And for TE_{m0} modes

$$(\alpha_c)_{TE_{m0}} = \frac{R_s}{b\eta\sqrt{1 - (f_c/f)^2}} \left[1 + \frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right] \tag{27}$$

Figure 8.7c shows attenuation versus frequency for TM_{11} and TE_{10} modes in rectangular copper waveguides with various side ratios b/a found using (14) and (27), respectively. It is seen that small b/a ratios give large attenuations because of the high ratio of surface to cross-sectional area.

8.8 The TE_{10} Wave in a Rectangular Guide

One of the simplest of all the waves which may exist inside hollow-pipe waveguides is the dominant TE_{10} wave in the rectangular guide, which is one of the TE modes studied in the preceding section. This mode is of great engineering importance, partly for the following reasons:

1. Cutoff frequency is independent of one of the dimensions of the cross section. Consequently, for a given frequency this dimension may be made small enough so that the TE_{10} wave is the only wave which will propagate, and there is no difficulty with higher-order waves that end effects or discontinuities may cause to be excited.
2. The polarization of the field is definitely fixed, electric field passing from top to bottom of the guide. This fixed polarization may be required for certain applications.
3. For a given frequency the attenuation due to copper losses is not excessive compared with other wave types in guides of comparable size.

Let us now rewrite the expressions from the previous section for general TE waves in rectangular guides, Eqs. 8.7(20)–(24), setting $m = 1$, $n = 0$, in which case $k_y = 0$ and $k_c = k_x = \pi/a$.

$$H_z = B \cos k_x x \tag{1}$$

$$E_y = -\frac{j\omega\mu B}{k_x} \sin k_x x \tag{2}$$

$$H_x = \frac{j\beta B}{k_x} \sin k_x x \tag{3}$$

All other components are zero. This set may be rewritten in a useful alternate form,

$$E_y = -Z_{TE} H_x = E_0 \sin\left(\frac{\pi x}{a}\right) \tag{4}$$

$$H_z = \frac{jE_0}{\eta}\left(\frac{\lambda}{2a}\right)\cos\left(\frac{\pi x}{a}\right) \tag{5}$$

where

$$E_0 = -\frac{j\omega\mu B}{k_x} = -\frac{j2\eta a B}{\lambda} \tag{6}$$

$$Z_{TE} = \eta\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-1/2} = \eta\left[1 - \left(\frac{\lambda}{2a}\right)^2\right]^{-1/2} \tag{7}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}, \qquad \lambda = \frac{v}{f} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} \tag{8}$$

Cutoff frequency, wavelength, and wavenumber are

$$f_c = \frac{1}{2a\sqrt{\mu\varepsilon}}, \qquad \lambda_c = 2a, \qquad k_c = \frac{\pi}{a} \tag{9}$$

Phase and group velocities and wavelength measured along the guide are

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(\lambda/2a)^2}}, \qquad v_g = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1-\left(\frac{\lambda}{2a}\right)^2} \tag{10}$$

$$\lambda_g = \frac{v_p}{f} = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}} \tag{11}$$

The attenuation arising from an imperfect dielectric is obtained by replacing ε with $\varepsilon' - j\varepsilon''$ in the equation for γ. Since k_c in the TE$_{m0}$ modes is of the same form as in the parallel-plane modes, Eq. 8.5(3) applies here and leads to a result equivalent to Eq. 8.5(4):

$$\alpha_d = \frac{k\varepsilon''/\varepsilon'}{2\sqrt{1-(\lambda/2a)^2}} \tag{12}$$

To find attenuation if the conductor is imperfect, we first calculate the power transferred by the wave from the Poynting theorem:

$$W_T = \frac{1}{2}\,\mathrm{Re}\int_0^a\int_0^b (-E_yH_x^*)\,dx\,dy \tag{13}$$

Utilizing the forms (4), we have

$$W_T = \frac{E_0^2 b}{2Z_{TE}}\int_0^a \sin^2\frac{\pi x}{a}\,dx = \frac{E_0^2 ba}{4Z_{TE}} \tag{14}$$

Next we find approximate losses in the walls by using currents of the ideal mode in material of surface resistivity R_s. Current in a conductor is related to the tangential magnetic field H_z at the side walls $x = 0$ and $x = a$ so there is current per unit width $|J_{sy}| = |H_z|$ there. Both components H_x and H_z are tangential at top and bottom surfaces giving rise to surface current densities $|J_{sz}| = |H_x|$ and $|J_{sx}| = |H_z|$. Thus, power loss per unit length is

$$(w_L)_{\text{SIDES}} = 2\left(\frac{bR_s}{2}|H_z|^2_{x=0}\right) = \frac{bR_sE_0^2\lambda^2}{4\eta^2a^2} \tag{15}$$

$$\begin{aligned}(w_L)_{\text{TOP AND BOTTOM}} &= 2\,\frac{R_s}{2}\int_0^a (|H_x|^2 + |H_z|^2)\,dx \\ &= R_s\int_0^a\left[\frac{E_0^2}{Z_{TE}^2}\sin^2\frac{\pi x}{a} + \frac{E_0^2\lambda^2}{4\eta^2a^2}\cos^2\frac{\pi x}{a}\right]dx \\ &= \frac{a}{2}R_s\left(\frac{E_0^2}{Z_{TE}^2} + \frac{E_0^2\lambda^2}{4\eta^2a^2}\right)\end{aligned} \tag{16}$$

Adding the two contributions (15) and (16) and substituting Z_{TE} from (7),

$$w_L = \frac{R_sE_0^2}{\eta^2}\left[\frac{b\lambda^2}{4a^2} + \frac{a}{2}\left(1 - \frac{\lambda^2}{4a^2} + \frac{\lambda^2}{4a^2}\right)\right] = \frac{R_sE_0^2}{2\eta^2}\left(a + \frac{b\lambda^2}{2a^2}\right)$$

The attenuation from conductor losses, Eq.5.9(4), is then

$$\alpha_c = \frac{w_L}{2W_T} = \frac{R_s Z_{TE}}{\eta^2 ba}\left(a + \frac{b\lambda^2}{2a^2}\right) \tag{17}$$

or

$$\alpha_c = \frac{R_s}{b\eta\sqrt{1 - (\lambda/2a)^2}}\left[1 + \frac{2b}{a}\left(\frac{\lambda}{2a}\right)^2\right] \tag{18}$$

A study of the field distributions (1) to (3) or (4) and (5) shows the field patterns for this wave sketched in Table 8.7. First it is noted that no field components vary in the vertical or y direction. The only electric field component is the vertical one E_y passing between top and bottom of the guide. This is a maximum at the center and zero at the conducting walls, varying as a half-sine curve. The corresponding charges induced by the electric field lines ending on conductors are (1) charges zero on side walls and (2) a charge distribution on top and bottom with $\rho_s = \varepsilon E_y$ on the bottom and $-\varepsilon E_y$ on the top. The magnetic field forms closed paths surrounding the vertical electric displacement currents arising from E_y, so that there are components H_x and H_z. Component H_x is zero at the two side walls and a maximum in the center, following the distribution of E_y. Component H_z is a maximum at the side walls and zero at the center. Component H_x corresponds to a longitudinal current flow down the guide in the top, and opposite in the bottom; H_z corresponds to transverse currents in the top and bottom and vertical currents on the side walls. These current distributions are sketched in Fig. 8.8*a*.

This simple wave type is a convenient one to study to strengthen some of our physical pictures of wave propagation. Electric field is confined to the transverse plane and so passes between opposite charges of equal density on the top and bottom. The electric field E_y and the transverse magnetic field H_x are maximum at planes a half guide wavelength apart. Halfway between those planes is the maximum rate of change of E_y for the traveling wave and therefore the location of maximum displacement current. The conduction currents in the metal walls, related to the tangential magnetic field, vary with position. Displacement currents provide the continuity of total current. The magnetic fields surround the electric displacement currents inside the guide and so must have an axial as well as a transverse component.

As a fairly crude way of looking at the problem, one might also think of this mode being formed by starting with a parallel-plate transmission line A of width w to carry the longitudinal current in the center of the guide, and then adding shorted troughs B of depth l on the two sides to close the region, as pictured in Fig. 8.8*b*. Since one would expect the lengths l to be around a quarter-wavelength to provide a high impedance at the center, the overall width should be something over a half-wavelength, which we know to be true for propagation. The picture is only a rough one because the fields in the two regions are not separated, and propagation is not purely longitudinal in the center portion or transverse in the side portions.

A third viewpoint follows from that used in studying the higher-order waves between parallel planes. There it was pointed out that one could visualize the TM and TE waves

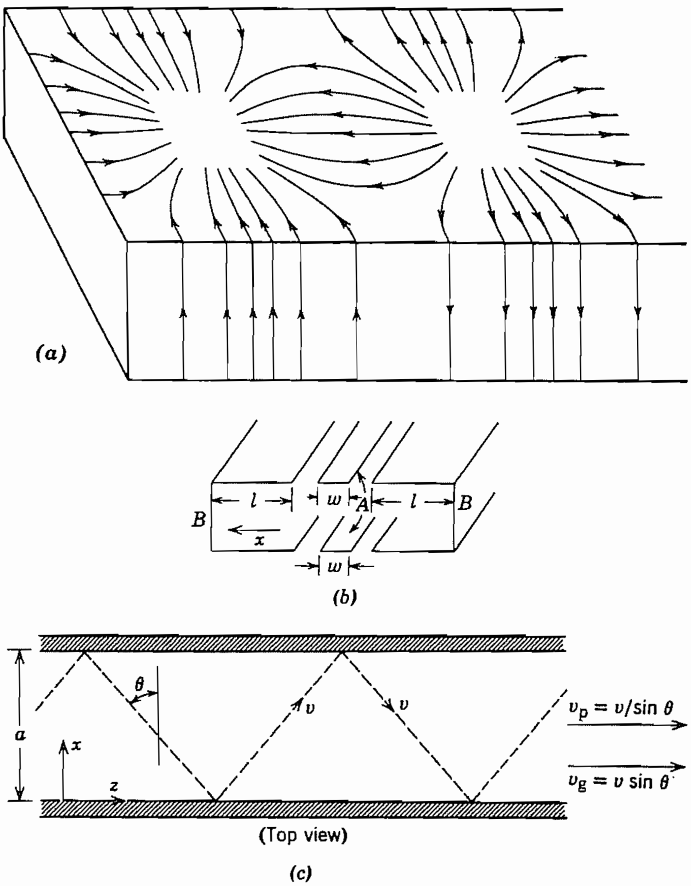


FIG. 8.8 (*a*) Current flow in walls of rectangular guide with TE_{10} mode. (*b*) Guide roughly divided into axial- and transverse-current regions. (*c*) Path of uniform plane-wave component of TE_{10} wave in rectangular guide.

in terms of plane waves bouncing between the two planes at such an angle that the interference pattern maintains a zero of electric field tangential to the two planes. Similarly, the TE_{10} wave in the rectangular guide may be thought of as arising from the interference between incident and reflected plane waves, polarized so that the electric vector is vertical, and bouncing between the two sides of the guide at such an angle with the sides that the zero electric field is maintained at the two sides. One such component uniform plane wave is indicated in Fig. 8.8*c*. As in the result of Sec. 8.4, when the width a is exactly $\lambda/2$, the waves travel exactly back and forth across the guide with no component of propagation in the axial direction. At slightly higher frequencies there is a small angle θ such that $a = \lambda/2 \cos\theta$, and there is a small propagation in the axial direction, a very small group velocity in the axial direction $v \sin\theta$, and a very large phase velocity $v/\sin\theta$. At frequencies approaching infinity, θ approaches 90 degrees, so that the wave travels down the guide practically as a plane wave in space propagating in the axial direction.

All the foregoing points of view explain why the dimension b should not enter into the determination of cutoff frequency. Since the electric field is always normal to top and bottom, the placing of these planes plays no part in the boundary condition. However, the dimension b does affect other characteristics of the guide. Small b gives a larger separation between cutoff frequencies of the TE_{10} and TE_{01} modes. But it increases attenuation as shown in Fig. 8.7c and limits power-handling capabilities because of breakdown-field limits.

8.9 Circular Waveguides

Hollow-pipe waveguides of circular cross section are used in a number of instances, for example, when circular polarization is to be transmitted to certain classes of antennas. Also, as will be shown, the class of TE_{0n} modes (called *circular electric*) is interesting because of the low attenuation in this class at high frequencies. As before, we start with ideal dielectric and conducting boundary and make approximate modifications to these solutions when the materials have small losses. Before treating separately the TM and TE classes of waves, it is desirable to have the set of equations 8.2(13)–(16) transformed to circular cylindrical coordinates. A straightforward transformation gives

$$E_r = -\frac{j}{k_c^2}\left[\beta\frac{\partial E_z}{\partial r} + \frac{\omega\mu}{r}\frac{\partial H_z}{\partial\phi}\right] \tag{1}$$

$$E_\phi = \frac{j}{k_c^2}\left[-\frac{\beta}{r}\frac{\partial E_z}{\partial\phi} + \omega\mu\frac{\partial H_z}{\partial r}\right] \tag{2}$$

$$H_r = \frac{j}{k_c^2}\left[\frac{\omega\varepsilon}{r}\frac{\partial E_z}{\partial\phi} - \beta\frac{\partial H_z}{\partial r}\right] \tag{3}$$

$$H_\phi = -\frac{j}{k_c^2}\left[\omega\varepsilon\frac{\partial E_z}{\partial r} + \frac{\beta}{r}\frac{\partial H_z}{\partial\phi}\right] \tag{4}$$

where

$$k_c^2 = \gamma^2 + k^2 = k^2 - \beta^2 \tag{5}$$

TM Waves The transverse part of the Laplacian in Eq. 8.2(17) for E_z is expressed in circular cylindrical coordinates for this configuration, with the coordinate system as shown in Fig. 8.9a.

$$\nabla_t^2 E_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial\phi^2} = -k_c^2 E_z \tag{6}$$

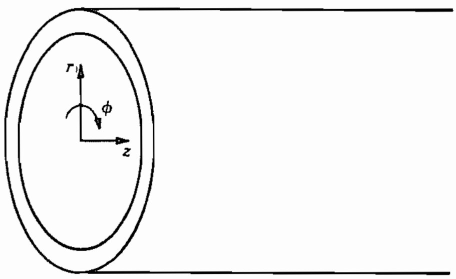


FIG. 8.9a Hollow-pipe circular cylindrical guide showing coordinate system.

Separation of variables techniques in Sec. 7.20 led to the solution

$$E_z(r, \phi) = [A'J_n(k_c r) + B'N_n(k_c r)][C' \cos n\phi + D' \sin n\phi] \tag{7}$$

where J_n and N_n are nth-order Bessel functions of first and second kind, respectively. The second kind, $N_n(k_c r)$, is infinite at $r = 0$ for any n and so cannot be included in the interior solution which includes the axis. Also, for simplicity, we choose the origin of ϕ so that we have just the cos $n\phi$ variation. Letting $A'C' = A$,

$$E_z = AJ_n(k_c r) \cos n\phi \tag{8}$$

From (1) to (4) with $H_z = 0$, the remaining field components are

$$E_r = Z_{\text{TM}} H_\phi = -\frac{j\beta}{k_c} AJ'_n(k_c r) \cos n\phi \tag{9}$$

$$E_\phi = -Z_{\text{TM}} H_r = \frac{j\beta n}{k_c^2 r} AJ_n(k_c r) \sin n\phi \tag{10}$$

where the prime denotes the derivative with respect to the argument and

$$Z_{\text{TM}} = \frac{\beta}{\omega\varepsilon} \tag{11}$$

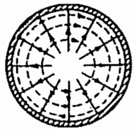
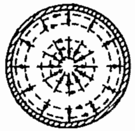
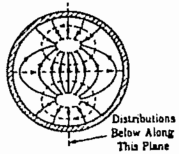
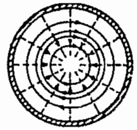
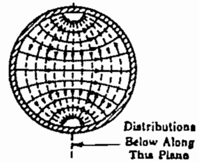
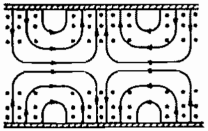
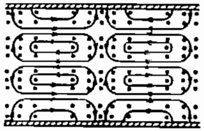
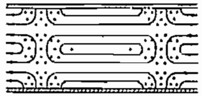
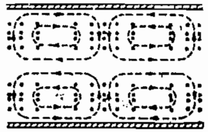
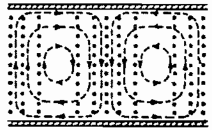
The boundary condition imposed by the perfect conductor at $r = a$ requires E_z and E_ϕ to be zero there. We see from (8) that E_z is zero at the boundary if $k_c a$ is one of the zeros of the Bessel function,

$$k_c a = \omega_c \sqrt{\mu\varepsilon} a = \frac{2\pi a}{\lambda_c} = p_{nl} \tag{12}$$

where $J_n(p_{nl}) = 0$. We see from (10) that this makes E_ϕ zero there also.

Equation (12) allows one to calculate cutoff frequency or wavelength for any mode order. For any n there is an infinite number of zeros of $J_n(k_c r)$ so there is a doubly infinite set of modes, denoted TM_{nl}. Note that the first subscript denotes angular variations and the second radial variations, differing from the usual cyclic order of coordinates. Field distributions of the TM_{01}, TM_{02}, and TM_{11} modes are given in Table 8.9

Table 8.9
Summary of Wave Types for Circular Guides[a]

Wave Type	TM_{01}	TM_{02}	TM_{11}	TE_{01}	TE_{11}
Field distributions in cross-sectional plane, at plane of maximum transverse fields					
Field distributions along guide					
Field components present	E_z, E_r, H_ϕ	E_z, E_r, H_ϕ	$E_z, E_r, E_\phi, H_r, H_\phi$	H_z, H_r, E_ϕ	$H_z, H_r, H_\phi, E_r, E_\phi$
p_{nl} or p'_{nl}	2.405	5.52	3.83	3.83	1.84
$(k_c)_{nl}$	$\frac{2.405}{a}$	$\frac{5.52}{a}$	$\frac{3.83}{a}$	$\frac{3.83}{a}$	$\frac{1.84}{a}$
$(\lambda_c)_{nl}$	$2.61a$	$1.14a$	$1.64a$	$1.64a$	$3.41a$
$(f_c)_{nl}$	$\frac{0.383}{a\sqrt{\mu\varepsilon}}$	$\frac{0.877}{a\sqrt{\mu\varepsilon}}$	$\frac{0.600}{a\sqrt{\mu\varepsilon}}$	$\frac{0.609}{a\sqrt{\mu\varepsilon}}$	$\frac{0.293}{a\sqrt{\mu\varepsilon}}$
Attenuation due to imperfect conductors	$\frac{R_S}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_S}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_S}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_S}{a\eta}\frac{(f_c/f)^2}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_S}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}\left[\left(\frac{f_c}{f}\right)^2+0.420\right]$

[a] Electric field lines are shown solid and magnetic field lines are dashed.

together with expressions for cutoff frequency and wavelength, and approximate attenuation from conductor losses if the cylindrical wall has surface resistivity R_s. Phase velocity, group velocity, guide wavelength, and attenuation for frequencies below cutoff are of the same forms in terms of cutoff frequency as we have seen for the parallel-plane and rectangular guides. Wave impedance, when β in terms of cutoff frequency is substituted in (11), also has the form found for TM modes in the guides studied earlier.

TE Waves The differential equation for the nonzero H_z of TE waves, Eq. 8.2(18), expressed in cylindrical coordinates, is

$$\nabla_t^2 H_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \phi^2} = -k_c^2 H_z \tag{13}$$

The solution for this from Sec. 7.20 is

$$H_z(r, \phi) = BJ_n(k_c r)\cos n\phi \tag{14}$$

Here we have left out the second solution and chosen the cosine variation with ϕ with the same justification as for the TM waves. The remaining field components, from (1) to (4) with $E_z = 0$, are

$$E_r = Z_{\text{TE}}H_\phi = \frac{j\omega\mu n}{k_c^2 r} BJ_n(k_c r)\sin n\phi \tag{15}$$

$$E_\phi = -Z_{\text{TE}}H_r = \frac{j\omega\mu}{k_c} BJ_n'(k_c r)\cos n\phi \tag{16}$$

where

$$Z_{\text{TE}} = \frac{\omega\mu}{\beta} \tag{17}$$

In this case the boundary condition imposed by the perfect conductor at $r = a$ requires that $E_\phi = 0$ there, or

$$k_c a = \omega_c\sqrt{\mu\varepsilon}a = \frac{2\pi a}{\lambda_c} = p_{nl}' \tag{18}$$

where $J_n'(p_{nl}') = 0$. Again, expressions for phase and group velocity and wavelength along the guide are as before.

The field distributions of the TE_{01} and TE_{11} modes and some data for these are given in Table 8.9. Note that the field distribution of the TE_{11} mode is quite a bit like that of the TE_{10} mode in the rectangular guide, with electric field going from top to bottom of the guide, so this is the one that would be primarily excited if a TE_{10}-mode rectangular guide were properly tapered and connected to the circular guide. It also has the lowest cutoff frequency of any mode in a given size circular pipe, as shown by Fig. 8.9*b*. In the TE_{01} mode, the electric field lines do not end on the guide walls, but form closed

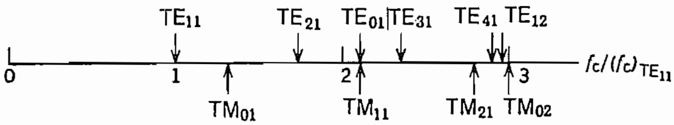


FIG. 8.9b Relative cutoff frequencies of waves in a circular guide.

circles surrounding the axial time-varying magnetic field. This latter wave is especially interesting as a potential low-loss transmission system at high frequencies and will be considered more in the example.

Example 8.9

CIRCULAR ELECTRIC TE_{01} MODE IN OVERSIZE GUIDE

The field expressions for the TE_{01} mode from the general forms (14)–(16) are

$$H_z = BJ_0(k_c r) \tag{19}$$

and

$$E_\phi = -Z_{TE}H_r = -\frac{j\omega\mu}{k_c} BJ_1(k_c r) \tag{20}$$

$$k_c a = p'_{01} = p_{11} = 3.83 \ldots \tag{21}$$

The average power transfer by the mode, from the Poynting theorem is

$$W_T = \int_0^a \frac{2\pi r}{2} (-E_\phi H_r^*)\, dr = \frac{\omega^2\mu^2B^2\pi}{k_c^2 Z_{TE}} \int_0^a rJ_1^2(k_c r)\, dr \tag{22}$$

The Bessel integral is evaluated by Eq. 7.15(22):

$$W_T = \frac{\omega^2\mu^2B^2\pi}{k_c^2 Z_{TE}} \left[\frac{a^2}{2} J_0^2(p_{11}) \right] \tag{23}$$

Conduction current in the guide walls is purely circumferential, related to the tangential H_z, so that wall losses per unit length are

$$w_L = 2\pi a \frac{R_s}{2} |H_z|^2_{r=a} = \pi a R_s B^2 J_0^2(p_{11}) \tag{24}$$

Attenuation per unit length, in terms of power transfer and loss, is then

$$\alpha_c = \frac{w_L}{2W_T} = \frac{k_c^2 Z_{TE} R_s}{\omega^2\mu^2 a} \tag{25}$$

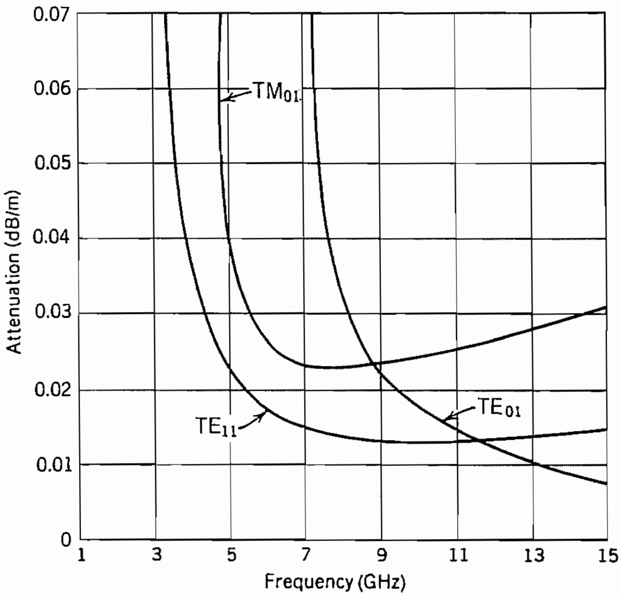


FIG. 8.9c Attenuation due to copper losses in circular waveguides; diameter = 2 in.

When Z_{TE} is substituted from (17), this may be put in the form

$$\alpha_c = \frac{R_s(\omega_c/\omega)^2}{a\eta\sqrt{1 - (\omega_c/\omega)^2}} \tag{26}$$

R_s is proportional to the square root of frequency, but the overall expression decreases with frequency. Thus we have the unusual result that attenuation in this mode, for a given size guide, decreases with increasing frequency, as shown in Fig. 8.9*c* which gives a comparison with TE_{11} and TM_{01} modes for a 2-in.-diameter guide. The low attenuation is because the mode fields are very little coupled to the guide walls at high frequencies. However, other modes may propagate, as shown by Fig. 8.9*b*, so that there are problems with mode conversion when such guides bend to go around corners. Practical ways of solving such problems were developed,[7] and this system was demonstrated as a low-loss guiding system for millimeter waves, but has been replaced by optical fiber.

8.10 Higher Order Modes on Coaxial Lines

The lowest-order mode on a coaxial line is a TEM wave; this was assumed implicitly in the transmission-line treatment of Ex. 5.2 where we used the capacitance and in-

[7] *See, for example, S. E. Miller,* Bell System Tech. J. ***33**, 1209 (1954).*

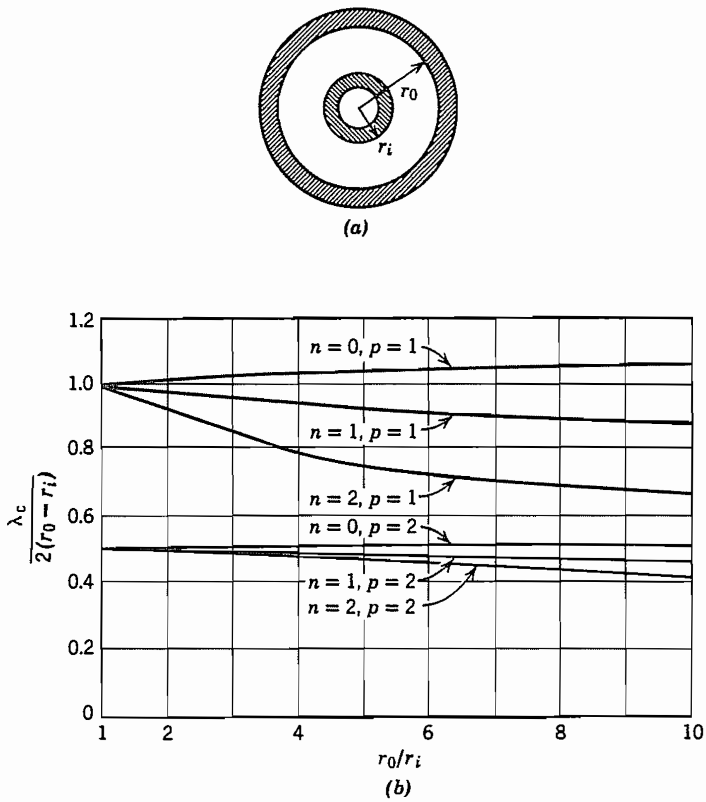


FIG. 8.10 (*a*) Cross section of a coaxial line. (*b*) Cutoff wavelength for some higher-order TM waves in coaxial lines.

ductance found from static fields. As in the parallel-plane guide, higher-order (TM and TE) modes can also exist. Normally, the line is designed in such a way that the cutoff frequencies of the higher-order moues are well above the operating frequency. Even in that case, these modes can be of importance near discontinuities.

The general forms useful for the TM and TE modes in circular cylindrical coordinates are listed in Sec. 7.20. The boundary conditions require that E_z for TM waves be zero at r_0 and r_i (Fig. 8.10*a*).

For TM waves,

$$A_n J_n(k_c r_i) + B_n N_n(k_c r_i) = 0$$
$$A_n J_n(k_c r_0) + B_n N_n(k_c r_0) = 0$$

or

$$\frac{N_n(k_c r_i)}{J_n(k_c r_i)} = \frac{N_n(k_c r_0)}{J_n(k_c r_0)} \tag{1}$$

For TE waves, the derivative of H_z normal to the two conductors must be zero at the inner and outer radii. [See discussion following Eq. 8.7(21).] Then, in place of (1),

$$\frac{N'_n(k_c r_i)}{J'_n(k_c r_i)} = \frac{N'_n(k_c r_0)}{J'_n(k_c r_0)} \tag{2}$$

Solutions to the transcendental equations (1) and (2) determine the values of k_c and hence cutoff frequency for any wave type and any particular values of r_i and r_0. By analogy with the parallel-plane guide, we would expect to find certain modes with a cutoff such that the spacing between conductors is of the order of p half-wavelengths.

$$\lambda_c \approx \frac{2}{p}(r_0 - r_i), \qquad p = 1, 2, 3, \ldots \tag{3}$$

This is verified by Fig. 8.10*b* for values of r_0/r_i near unity.

Probably more important is the lowest-order TE wave with circumferential variations. This is analogous to the TE_{10} wave of a rectangular waveguide, and physical reasoning from the analogy leads one to expect cutoff for this wave type when the average circumference is about equal to wavelength. The field picture of the TE_{10} mode given in Sec. 8.8 should make this reasonable. Solution of (2) reveals this simple rule to be within about 4% accuracy for r_0/r_i up to 5. In general, for the nth-order TE wave with circumferential variations,

$$\lambda_c \approx \frac{2\pi}{n}\left(\frac{r_0 + r_i}{2}\right), \qquad n = 1, 2, 3, \ldots \tag{4}$$

There are, of course, other TE waves with further radial variations, and the lowest order of these has a cutoff about the same as the lowest-order TM wave.

8.11 Excitation and Reception of Waves in Guides

The problems of exciting or receiving waves in a waveguide are not simple field problems. In this section we give only a qualitative introduction to the manners of excitation of fields in various kinds of guides. Approaches to analysis and measurement of these junctions are given in Chapter 11. Reception of the energy of a wave uses the same kind of structure as excitation and is just the reverse process. To excite any particular desired wave, one should study the field pattern and use one of the following concepts.

1. Introduce the excitation in a probe or antenna oriented in the direction of electric field. The probe is most often placed near a maximum of the electric field of the mode pattern, but exact placing is a matter of impedance matching. Examples are shown in Figs. 8.11*a* and *b*.
2. Introduce the excitation through a loop oriented in a plane normal to the magnetic field of the mode pattern (Fig. 8.11*c*).
3. Couple to the desired mode from another guiding system, by means of a hole or iris, the two guiding systems having some common field component over the

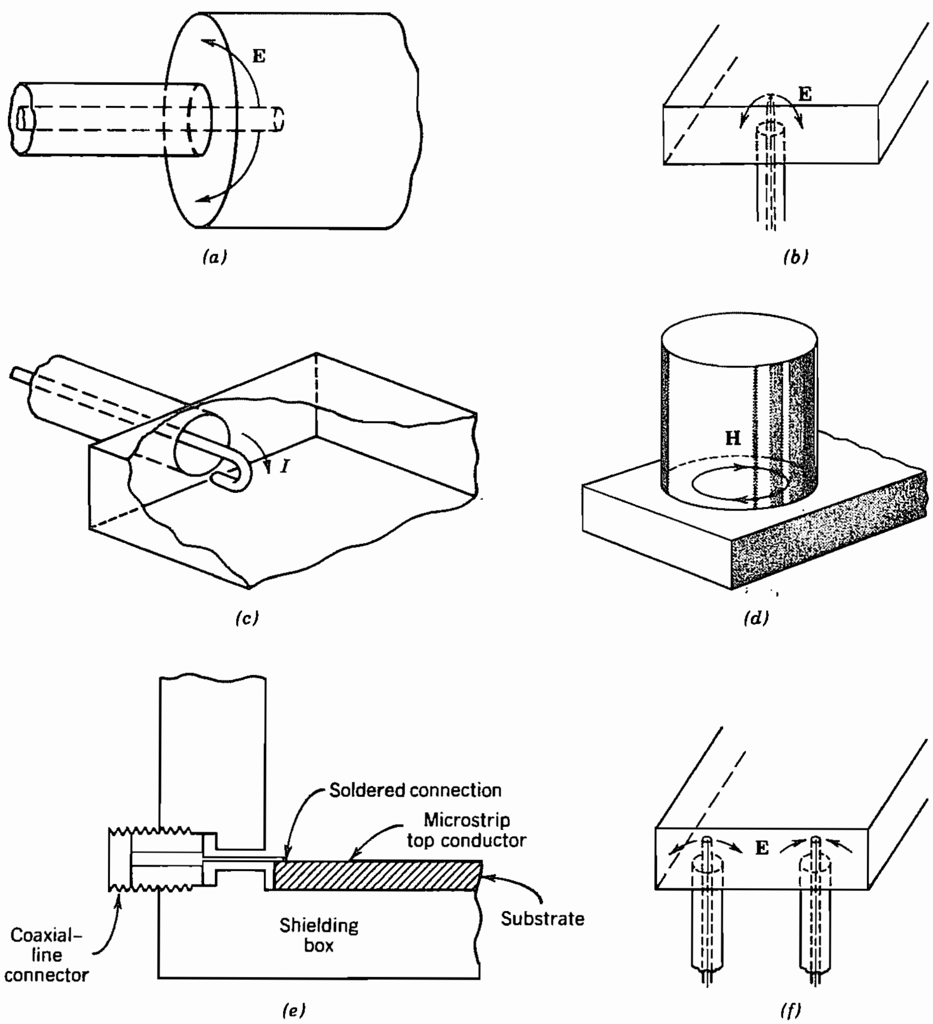


Fig. 8.11 (*a*) Antenna in end of circular guide for excitation of TM_{01} wave. (*b*) Antenna in bottom of rectangular guide for excitation of the TE_{10} wave. (*c*) Loop in end of rectangular guide for excitation of TE_{10} wave. (*d*) Junction between circular guide (TM_{01} wave) and rectangular guide (TE_{10} wave); large-aperture coupling. (*e*) Coaxial line coupling to microstrip. (*f*) Excitation of the TE_{20} wave in rectangular guide by two oppositely phased antennas.

extent of the hole. An example of coupling between waveguides using a large iris is shown in Fig. 8.11*d*. The coupling is sometimes done with a small hole as for coupling to resonant cavities (Sec. 10.10).

4. Introduce currents from one kind of transmission line into another, as in coupling from a coaxial line to microstrip shown in Fig. 8.11*e*.
5. For higher-order waves combine as many of the exciting sources as are required, with proper phasings (Fig. 8.11*f*).

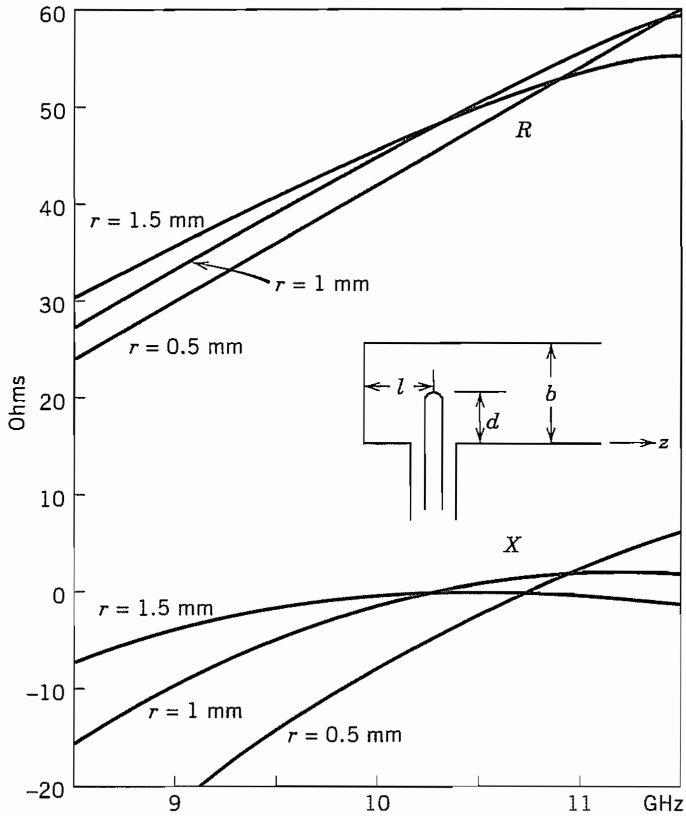
6. Gradually taper a transition between two types of guides, as for a TE_{10} wave in a rectangular guide to a TE_{11} in a circular guide.

Since most of these exciting methods are in the nature of concentrated sources, they will not in general excite purely one wave, but all waves that have field components in a favorable direction for the particular exciting source. That is, we see that one wave alone will not suffice to satisfy the boundary conditions of the guide complicated by the exciting source, so that many higher-order waves must be added for this purpose. If the guide is large enough, several of these waves will then proceed to propagate. Most often, however, only one of the excited waves is above cutoff. This will propagate down the guide and (if absorbed somewhere) will represent a resistive load on the source, comparable to the radiation resistance of antennas which we shall encounter further in Chapter 12. The higher-order waves that are excited, if all below cutoff, will be localized in the neighborhood of the source and will represent purely reactive loads on the source. For practical application, it is then necessary to add, in the line that feeds the probe or loop or other exciting means, an arrangement for matching to the load that has a real part representing the propagating wave and an imaginary part representing the localized reactive waves. In a practical design, it is important to be concerned that the match is good over the frequency band of interest.

Example 8.11
EXCITATION OF A WAVEGUIDE BY A COAXIAL LINE

Let us look in more depth at the structure in Fig. 8.11*b* where a coaxial line is inserted in the center of the broad side of a waveguide of rectangular cross section to excite a TE_{10} mode. The waveguide is short-circuited at a distance l from the probe to aid in matching the coaxial line to the waveguide. The fields associated with the probe excite both the desired TE_{10} mode and other higher-order modes. The latter are cutoff and do not propagate, but they store reactive energy and therefore constitute a reactive component of the load on the coaxial line. Proper choice of the size and location of the probe for a given frequency and guide dimensions makes the standing wave between the probe and the shorted end contain reactive energy of opposite sign and equal magnitude so that the net reactive component of the input impedance is zero. These adjustments are used to make the real part of the load impedance on the coaxial line equal to its characteristic impedance so that perfect matching is achieved and all the power is coupled into the guide. Figure 8.11*g* shows the calculated results for probes of various radii in a guide of dimensions appropriate for use at about 10 GHz (X band).[8] Similar graphs can be calculated using the methods in the reference of footnote 8 for other guide sizes and probe radii.

[8] R. E. Collin, *Field Theory of Guided Waves*, 2nd ed., Sec. 7.1, IEEE Press, Piscataway, NJ, 1991.



(g)

FIG. 8.11g Probe input resistance and reactance as a function of frequency for $d = 0.62$ cm, $l = 0.495$ cm, guide width $a = 2.286$ cm, guide height $b = 1.016$ cm. For the thin probe of radius $r = 0.5$ mm, $l = 0.505$ cm. Reproduced by permission from R. E. Collin, *Field Theory of Guided Waves*, 2nd ed., Sec. 7.1, IEEE Press, Piscataway, NJ, 1991.

General Properties of Guided Waves

8.12 General Properties of TEM Waves on Multiconductor Lines

The classical two-conductor transmission system was studied extensively in Chapter 5, starting from a distributed circuits point of view. We used wave solutions to verify the results for the special case of TEM waves between parallel planes in Sec. 8.3. At this point we can show that TEM waves in any two-conductor cylindrical system with isotropic, homogeneous dielectric, and loss-free conductors are exactly those predicted by the transmission-line equations.

The general relations between wave components as expressed by Eqs. 8.2(9)–(12) show that, with E_z and H_z zero, all other components must of necessity also be zero, unless $\gamma^2 + k^2$ is at the same time zero. Thus, a transverse electromagnetic wave must satisfy the condition

$$\gamma = \pm jk = \pm\frac{j\omega}{v} = \pm j\omega\sqrt{\mu\varepsilon} \tag{1}$$

For a perfect dielectric, the propagation constant γ is thus a purely imaginary quantity, signifying that any completely transverse electromagnetic wave must propagate unattenuated and with velocity v, the velocity of light in the dielectric bounded by the guide.

With (1) satisfied, the wave equations, as written in the form of Eqs. 8.2(1) and 8.2(2), reduce to

$$\nabla_{xy}^2\mathbf{E} = 0, \qquad \nabla_{xy}^2\mathbf{H} = 0 \tag{2}$$

These are exactly the form of the two-dimensional Laplace equation written for **E** and **H** in the transverse plane. Since E_z and H_z are zero, E and H lie entirely in the transverse plane. Since electric and magnetic fields both satisfy Laplace's equation under static conditions, the field distribution in the transverse plane is exactly a static distribution if boundary conditions to be applied to the fields in (2) are the same as those for a static field distribution. The boundary condition for the TEM wave on a perfect conducting guide is that electric field at the surface of the conductor can have a normal component only, which is the same as the condition at a conducting boundary in statics. The line integral of the electric field between conductors is the same for all paths lying in a given transverse plane, and may be thought of as corresponding to a potential difference between the conductors for that value of z.

To study the character of the magnetic field, note Eqs. 8.2(3) and 8.2(6) with zero E_z and H_z:

$$H_y = \frac{j\omega\varepsilon}{\gamma}E_x = \frac{E_x}{\eta} \tag{3}$$

and

$$H_x = -\frac{\gamma}{j\omega\mu}E_y = -\frac{E_y}{\eta} \tag{4}$$

[The signs of (3) and (4) are for a positively traveling wave; for a negatively traveling wave they are opposite.] Study shows that (3) and (4) are conditions that require that electric and magnetic fields be everywhere normal to each other. In particular, magnetic field must be tangential to the conducting surfaces since electric field is normal to them. The magnetic field pattern in the transverse plane then corresponds exactly to that arising from dc currents flowing entirely on the surfaces of the perfect conductors.

These characteristics show that a transverse electromagnetic wave may be guided by two or more conductors, or outside a single conductor, but not inside a closed con-

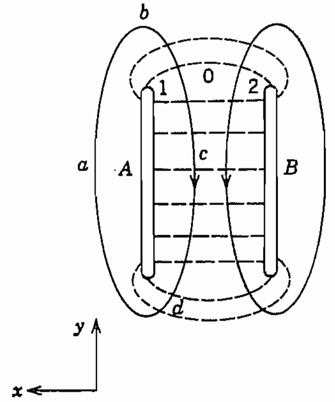


FIG. 8.12 Two-conductor transmission line with integration paths.

ducting region, since it can have only the distribution of the corresponding two-dimensional static problem, and no electrostatic field can exist inside a source-free region completely closed by a conductor (see Prob. 8.12d).

We next may show an exact identity with the ordinary transmission-line equations for TEM waves on the systems that support them. Consider a transmission line consisting of two conductors A and B of any general shape (Fig. 8.12). The voltage between the two conductors may be found by integrating electric field over any path between conductors, such as 1–0–2 of the figure. It will have the same value no matter which path is chosen, since $\mathbf{E}$ satisfies Laplace's equation in the transverse plane and so may be considered the gradient of a scalar potential insofar as variations in the transverse plane are concerned.

$$V = -\int_1^2 \mathbf{E} \cdot \mathbf{dl} = -\int_1^2 (E_x\, dx + E_y\, dy) \tag{5}$$

Differentiating this equation with respect to z

$$\frac{\partial V}{\partial z} = -\int_1^2 \left(\frac{\partial E_x}{\partial z} dx + \frac{\partial E_y}{\partial z} dy\right) \tag{6}$$

But the curl relation,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

shows that, if E_z is zero,

$$\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t} \quad \text{and} \quad \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \tag{7}$$

By substituting (7) in (6), we have

$$\frac{\partial V}{\partial z} = -\frac{\partial}{\partial t}\int_1^2 (-B_y\,dx + B_x\,dy) \tag{8}$$

A study of Fig. 8.12 reveals that the quantity inside the integral is the magnetic flux flowing across the path 1–0–2 per unit length in the z direction. According to the usual definition of inductance, this may be written as the product of inductance L per unit length and current I, so (8) becomes

$$\frac{\partial V}{\partial z} = -\frac{\partial}{\partial t}(LI) = -L\frac{\partial I}{\partial t} \tag{9}$$

Equation (9) is one of the differential equations used as a starting point for conventional transmission-line analysis [Eq. 5.2(3)]. The other may be developed by starting with current in line A as the integral of magnetic field about a path a–b–c–d–a. (There is no contribution from displacement current since there is no E_z.)

$$I = \oint \mathbf{H}\cdot d\mathbf{l} = \oint (H_x\,dx + H_y\,dy) \tag{10}$$

Differentiating with respect to z,

$$\frac{\partial I}{\partial z} = \oint\left(\frac{\partial H_x}{\partial z}\,dx + \frac{\partial H_y}{\partial z}\,dy\right) \tag{11}$$

From the curl equation,

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

it follows that, if $H_z = 0$,

$$\frac{\partial H_y}{\partial z} = -\frac{\partial D_x}{\partial t} \quad\text{and}\quad \frac{\partial H_x}{\partial z} = \frac{\partial D_y}{\partial t} \tag{12}$$

Substituting (12) in (11), we have

$$\frac{\partial I}{\partial z} = -\frac{\partial}{\partial t}\oint (D_x\,dy - D_y\,dx) \tag{13}$$

Inspection of Fig. 8.12 shows that this must be the electric displacement flux per unit length of line crossing from one conductor to the other. Since it corresponds to the charge per unit length on the conductors, it may be written as the product of capacitance per unit length and the voltage between lines and (13) becomes

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t} \tag{14}$$

Equations (9) and (14) are exactly the equations used as a beginning for transmission-line analysis, if losses are neglected (Sec. 5.2). It is seen that these equations may be derived exactly from Maxwell's equations provided the conductors are perfect, and since fields in the transverse plane satisfy Laplace's equation, the inductance and capacitance appearing in the equations are the same as those calculated in statics. This is of course not the case for the TM and TE waves met in the preceding sections. Waves on the ideal transmission line have been shown to be TEM waves with phase velocity $(\mu\varepsilon)^{-1/2}$. Transmission-line phase velocity is $(LC)^{-1/2}$ so it follows that inductance and capacitance per unit length of an ideal transmission line are related by $LC = \mu\varepsilon$.

Transmission Lines with Losses If the conductor of the transmission line has finite losses, the above argument does not apply exactly. There must be finite E_z at the conductor to force the axial currents through the imperfect conductors. In that case $\gamma^2 + k^2$ of Eqs. 8.2(9)–(12) cannot be zero, and Eqs. 8.2(1) and 8.2(2) do not reduce to Laplace's equation. But so long as the conductors are reasonably good, the axial component of electric field is small compared with the transverse component and corrections are small. The usual way of handling the losses through a series resistance in the transmission-line equations can then be shown by perturbation arguments to be an excellent approximation.[9] Losses in the dielectric, however, do not in themselves disturb the TEM nature of the wave since these cause conduction currents to flow only in transverse directions. In this case treatment by inclusion of shunt conductance computed from static concepts and by the wave method with ε replaced by $\varepsilon' - j\varepsilon''$ can be shown to be the same (Prob. 8.12c).

In addition to the principal TEM or *transmission-line* mode on the two-conductor system, there may propagate higher-order modes as well. The higher-order modes may be excited at discontinuities in the transmission line and may cause dispersion effects or radiation. It is difficult to work out the forms of the higher-order modes in open structures such as the two-wire line, but is straightforward to derive them for the coaxial line as was done in Sec. 8.10.

8.13 General Properties of TM Waves in Cylindrical Conducting Guides of Arbitrary Cross Section

In the earlier sections we have seen several specific examples of TM waves; it is the purpose of the present section to generalize the formulation for any cylindrical structure. The analysis can be done in a generalized coordinate system[10] and might appear more general, but for simplicity we will use rectangular coordinates with the understanding that boundaries may be of arbitrary shape.

The Differential Equation With the assumed propagation constant $e^{(j\omega t - \gamma z)}$, the finite axial component of electric field for the TM waves must satisfy the wave equation

[9] R. E. Collin, Field Theory of Guided Waves, *2nd ed., Sec. 4.1, IEEE Press, Piscataway, NJ, 1991.*

[10] R. E. Collin, Field Theory of Guided Waves, *2nd ed., Sec. 5.1, IEEE Press, Piscataway, NJ, 1991.*

in the form of Eq. 8.2(17):

$$\nabla_{xy}^2 E_z = -k_c^2 E_z \tag{1}$$

$$k_c^2 = (\gamma^2 + k^2) = \gamma^2 + \omega^2\mu\varepsilon \tag{2}$$

The value of k_c, which is a constant for a particular mode, is determined by the boundary condition to be applied to (1).

Boundary Condition for a Perfectly Conducting Guide As in the examples, the first step in the solution of a practical waveguide problem is to assume that the waveguide boundaries are perfectly conducting. The appropriate boundary condition is $E_z = 0$. It is easily shown from the general relations for the transverse field components in Sec. 8.2 that

$$E_x = \mp\frac{\gamma}{k_c^2}\frac{\partial E_z}{\partial x} \qquad E_y = \mp\frac{\gamma}{k_c^2}\frac{\partial E_z}{\partial y} \tag{3}$$

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y} \qquad H_y = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x} \tag{4}$$

Relations (3) may be written in the vector form

$$\mathbf{E}_t = \mp\frac{\gamma}{k_c^2}\nabla_t E_z \tag{5}$$

where $\mathbf{E}_t$ is the transverse part of the electric field vector, and ∇_t represents the transverse part of the gradient. By the nature of the gradient, the transverse electric vector $\mathbf{E}_t$ is normal to any line of constant E_z. It is then normal to the conducting boundary, as required, once the boundary is made a curve of constant $E_z = 0$. Thus $E_z = 0$ is the only required boundary condition for solutions of (1).

Cutoff Properties of TM Waves Solution of the homogeneous differential equation (1) subject to the given boundary condition is possible only for discrete values of the constant k_c. These are the *characteristic values*, *allowed values*, or *eigenvalues* of the problem, any one of which determines a particular TM mode for the given guide. It can be shown (Prob. 8.13e) that, for any lossless dielectric region which is completely closed by perfect conductors, the allowed values of k_c must be real. Hence the propagation constant from (2),

$$\gamma = \sqrt{k_c^2 - k^2} \tag{6}$$

always exhibits cutoff properties. That is, for a particular mode in a perfect dielectric, γ is real for the range of frequencies such that $k < k_c$, γ is zero for $k = k_c$, and γ is imaginary for $k > k_c$. The cutoff frequency of a given mode is then given by

$$2\pi f_c\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda_c} = k_c \tag{7}$$

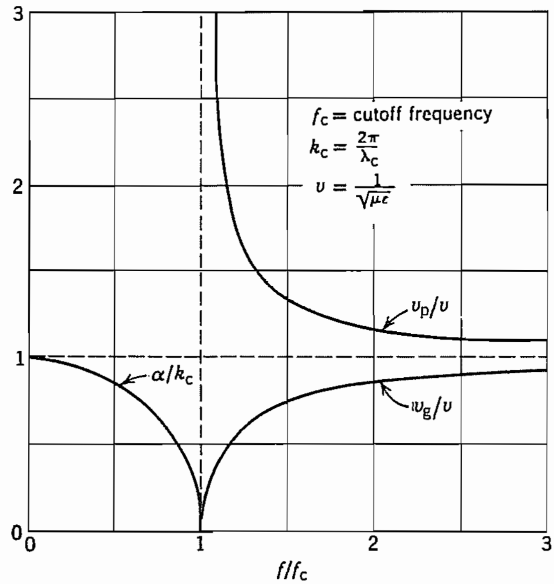


FIG. 8.13 Frequency characteristics of all TE and TM wave types.

and (6) may be written in terms of frequency f and cutoff frequency f_c:

$$\gamma = \alpha = k_c\sqrt{1 - \left(\frac{f}{f_c}\right)^2}, \qquad f < f_c \tag{8}$$

$$\gamma = j\beta = jk\sqrt{1 - \left(\frac{f_c}{f}\right)^2}, \qquad f > f_c \tag{9}$$

The phase velocity for all TM modes in an ideal guide then has the form

$$v_p = \frac{\omega}{\beta} = v\left[1 - \left(\frac{f_c}{f}\right)^2\right]^{-1/2} \tag{10}$$

The group velocity is

$$v_g = \frac{d\omega}{d\beta} = v\left[1 - \left(\frac{f_c}{f}\right)^2\right]^{1/2} \tag{11}$$

Universal curves for attenuation constant, phase velocity, and group velocity as functions of f/f_c are shown in Fig. 8.13. Phase velocity is infinite at cutoff frequency and is always greater than the velocity of light in the dielectric; group velocity is zero at cutoff and is always less than the velocity of light in the dielectric. As the frequency increases far beyond cutoff, phase and group velocities both approach the velocity of light in the dielectric.

Magnetic Fields of the Waves Once the distribution of E_z is found by solution of the differential equation (1) subject to the boundary condition $E_z = 0$, the transverse electric field of a given mode may be found from relation (3) or (5). The transverse magnetic field may be found from relations (4). By comparing (3) and (4), we see that

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \pm\frac{\gamma}{j\omega\varepsilon} \tag{12}$$

These relations show that transverse electric and magnetic fields are at right angles and that their magnitudes are related by the quantity $\gamma/j\omega\varepsilon$, which may be thought of as the wave impedance or field impedance of the mode:

$$Z_{\mathrm{TM}} = \frac{\gamma}{j\omega\varepsilon} = \eta\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{13}$$

The wave impedance is imaginary (reactive) for frequencies less than the cutoff frequency and purely real for frequencies above cutoff, approaching the intrinsic impedance of the dielectric at infinite frequency. This type of behavior is also found in the study of lumped-element filters, and it emphasizes that the wave can produce no average power transfer for frequencies below cutoff, where the impedance is imaginary.

The relations between electric and magnetic fields may also be given in the following vector form, which expresses the properties described above:

$$\mathbf{H} = \pm\frac{\hat{\mathbf{z}} \times \mathbf{E}_t}{Z_{\mathrm{TM}}} \tag{14}$$

where $\hat{\mathbf{z}}$ is the unit vector in the z direction. The upper sign is for positively traveling waves, the lower sign for negatively traveling waves.

Power Transfer in the Waves The power transfer down the guide is zero below cutoff if the conductor of the guide is perfect. Above cutoff it may be obtained in terms of the field components by integrating the axial component of the Poynting vector over the cross-sectional area. Since it has been shown that transverse components of electric and magnetic fields are in phase and normal to each other, the axial component of the average Poynting vector is one-half the product of the transverse field magnitudes. For a positively traveling wave,

$$W_T = \int_{\mathrm{cs}} \frac{1}{2}\,\mathrm{Re}[\mathbf{E} \times \mathbf{H}^*]_z \cdot \mathbf{dS} = \frac{1}{2}\int_{\mathrm{cs}} |E_t||H_t|\,dS = \frac{Z_{\mathrm{TM}}}{2}\int_{\mathrm{cs}} |H_t|^2\,dS \tag{15}$$

By use of (4), this may be written

$$W_T = \frac{Z_{\mathrm{TM}}\omega^2\varepsilon^2}{2k_c^4}\int_{\mathrm{cs}} |\nabla_t E_z|^2\,dS \tag{16}$$

Making use of the relation

$$\int_{\text{cs}} |\nabla_t E_z|^2 \, dS = k_c^2 \int_{\text{cs}} E_z^2 \, dS \tag{17}$$

(Prob. 8.13e), we obtain

$$W_T = \frac{Z_{\text{TM}}\omega^2\varepsilon^2}{2k_c^2} \int_{\text{cs}} E_z^2 \, dS = \frac{Z_{\text{TM}}}{2\eta^2} \left(\frac{f}{f_c}\right)^2 \int_{\text{cs}} E_z^2 \, dS \tag{18}$$

Attenuation Due to Imperfectly Conducting Boundaries When the conducting boundaries are imperfect, an exact solution would require solution of Maxwell's equations in both the dielectric and conducting regions. Because this procedure is impractical for most geometrical configurations, we take advantage of the fact that most practical conductors are good enough to cause only a slight modification of the ideal solution, and the expression $w_L/2W_T$ in formula 5.11(19) may be used. To compute the average power loss per unit length, we require the current flow in the guide walls, which is taken to be the same as that in the ideal guide. By the $\hat{\mathbf{n}} \times \mathbf{H}$ rule, the current per unit width in the boundary is equal to the transverse magnetic field at the boundary and flows in the axial direction since magnetic field is entirely transverse:

$$w_L = \oint_{\text{bound}} \frac{R_s}{2} |J_{sz}|^2 \, dl = \frac{R_s}{2} \oint_{\text{bound}} |H_t|^2 \, dl \tag{19}$$

The attenuation constant is then approximately

$$\alpha_c = \frac{w_L}{2W_T} = \frac{R_s \oint_{\text{bound}} |H_t|^2 \, dl}{2Z_{\text{TM}} \int_{\text{cs}} |H_t|^2 \, dS} \quad \text{nepers/m} \tag{20}$$

If desired, the power loss and hence the attenuation constant may be written in terms of the distribution of E_z only. By use of (4)

$$W_L = \frac{R_s}{2} \frac{\omega^2\varepsilon^2}{k_c^4} \oint_{\text{bound}} |\nabla E_z|^2 \, dl \tag{21}$$

Since E_z is zero at all points along the boundary, there is no tangential derivative of E_z there; E_z has only the derivative normal to the conductor:

$$W_L = \frac{R_s\omega^2\varepsilon^2}{2k_c^4} \oint_{\text{bound}} \left[\frac{\partial E_z}{\partial n}\right]^2 dl = \frac{R_s}{2\eta^2 k_c^2} \left(\frac{f}{f_c}\right)^2 \oint \left[\frac{\partial E_z}{\partial n}\right]^2 dl \tag{22}$$

An alternative form for the attenuation constant is then

$$\alpha_c = \frac{R_s}{2k_c^2 Z_{\text{TM}}} \left[\oint \left(\frac{\partial E_z}{\partial n}\right)^2 dl \Big/ \int_{\text{cs}} E_z^2 \, dS\right] \tag{23}$$

Attenuation Due to Imperfect Dielectric It is noted that the general form for propagation constant (9) is exactly the same as that for the special case of the parallel-plane guide, Eq. 8.3(15). Hence, the modification caused by an imperfect dielectric, taken into account by replacing $j\omega\varepsilon$ by $j\omega(\varepsilon' - j\varepsilon'')$ or $\sigma + j\omega\varepsilon$, yields the same form for attenuation as Eq. 8.5(4):

$$\alpha_{\mathrm{d}} = \frac{k\varepsilon''/\varepsilon'}{2\sqrt{1 - (f_c/f)^2}} = \frac{\sigma\eta}{2\sqrt{1 - (f_c/f)^2}} \quad \text{nepers/m} \tag{24}$$

It is especially interesting to note that the form of the attenuation produced by an imperfect dielectric is the same for all modes and all shapes of guides, though of course the amount of attenuation is a function of the cutoff frequency, which does depend on the guide and the mode.

8.14 General Properties of TE Waves in Cylindrical Conducting Guides of Arbitrary Cross Section

Finally, we consider waves that have magnetic field but no electric field in the axial direction. Because of the treatment is similar to that of TM waves in the preceding section, it will be given more briefly.

The Differential Equation The finite H_z of the waves must satisfy the wave equation in the form of Eq. 8.2(18):

$$\nabla_t^2 H_z = -k_c^2 H_z \tag{1}$$

$$k_c^2 = \gamma^2 + k^2 \tag{2}$$

Boundary Conditions for a Perfectly Conducting Guide Allowable solutions to (1) are determined by the single boundary condition that at perfect conductors the normal derivative of H_z must be zero:

$$\frac{\partial H_z}{\partial n} = 0 \qquad \text{at boundary} \tag{3}$$

To show that (3) is the required boundary condition, the transverse fields of the wave from Eqs. 8.2(9)–(12) are written:

$$E_x = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} \qquad E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} \tag{4}$$

$$H_x = \mp\frac{\gamma}{k_c^2}\frac{\partial H_z}{\partial x} \qquad H_y = \mp\frac{\gamma}{k_c^2}\frac{\partial H_z}{\partial y} \tag{5}$$

Relation (5) may be written in the vector form

$$\mathbf{H}_t = \mp\frac{\gamma}{k_c^2}\nabla_t H_z \tag{6}$$

If H_z has no normal derivative at the boundary, its transverse gradient has only a component tangential to the boundary, so by (6), $\mathbf{H}_t$ does also. Comparison of (4) and (5) shows that transverse electric and magnetic field components are normal to one aother, so electric field is normal to the conducting boundary as required.

Cutoff Properties of TE Waves It was mentioned in the preceding section on TM waves that k_c is always real for dielectric regions completely closed by perfect conductors; the same can be shown for TE waves. By (2), γ then shows cutoff properties exactly the same as for TM waves:

$$\gamma = \sqrt{k_c^2 - k^2} \tag{7}$$

Formulas for attenuation constant below cutoff, phase constant, and phase and group velocities above cutoff then follow exactly as in Eqs. 8.13(8)–(11) and the universal curves of Fig. 8.13 apply.

Electric Field of the Wave The electric field is everywhere transverse and everywhere normal to the transverse magnetic field components. Transverse components of electric and magnetic field may again be related through a field or wave impedance

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = Z_{TE} \tag{8}$$

where, from (4) and (5),

$$Z_{TE} = \frac{j\omega\mu}{\gamma} = \eta\left[1 - \left(\frac{f_c}{f}\right)^2\right]^{-1/2} \tag{9}$$

This impedance is imaginary for frequencies below cutoff, infinite at cutoff, and purely real for frequencies above cutoff, approaching the intrinsic impedance η as f/f_c becomes large.

Electric field may also be written in the vector form

$$\mathbf{E} = \mp Z_{TE}(\hat{\mathbf{z}} \times \mathbf{H}_t) \tag{10}$$

where $\hat{\mathbf{z}}$ is the unit vector in the z direction, and the upper and lower signs apply respectively to positively and negatively traveling waves.

Power Transfer in TE Waves Average power transfer in the propagating range is, as usual, obtained from the Poynting vector:

$$\begin{aligned} W_T &= \frac{1}{2}\int_{cs} \mathrm{Re}[\mathbf{E} \times \mathbf{H}^*] \cdot d\mathbf{S} = \frac{1}{2}\int_{cs} |E_t||H_t|\, dS \\ &= \frac{Z_{TE}}{2}\int_{cs} |H_t|^2\, dS \end{aligned} \tag{11}$$

Or, using (6) and

$$\int_{\mathrm{cs}} |\nabla_t H_z|^2 \, dS = k_c^2 \int_{\mathrm{cs}} H_z^2 \, dS \tag{12}$$

(see Prob. 8.13e), we obtain

$$W_T = \frac{\eta^2 (f/f_c)^2}{2Z_{\mathrm{TE}}} \int_{\mathrm{cs}} H_z^2 \, dS \tag{13}$$

Attenuation Due to Imperfectly Conducting Boundaries As with the TEM mode, there cannot be a true transverse electric wave in most guides with imperfect conductors, since most (but not all) of the TE modes have axial currents that require a certain finite axial electric field when conductivity is finite. This axial field is very small compared with the transverse field, however, so the waves are not renamed.

The axial component of current arises from the transverse component of magnetic field at the boundary:

$$|J_{sz}| = |H_t| = \frac{\beta}{k_c^2} |\nabla_t H_z| = \frac{\beta}{k_c^2} \frac{\partial H_z}{\partial l} \tag{14}$$

The last form follows since it has been shown that the transverse gradient of H_z has only a tangential component $\partial/\partial l$ at the boundary. There is in addition a transverse current arising from the axial magnetic field:

$$|J_{st}| = |H_z| \tag{15}$$

The power loss per unit length is then

$$w_L = \frac{R_s}{2} \oint [|H_z|^2 + |H_t|^2] \, dl \tag{16}$$

The attenuation caused by the conductor losses is

$$\alpha_c = \frac{R_s \oint [|H_z|^2 + |H_t|^2] \, dl}{2Z_{\mathrm{TE}} \int |H_t|^2 \, dS} \quad \text{nepers/m} \tag{17}$$

Attenuation Due to Imperfect Dielectric Since the propagation constant of the TE waves has the same form as for the TM waves, it follows that the form for attenuation due to an imperfect dielectric does also. For a reasonably good dielectric, the approximate form, Eq. 8.13(24), may be used.

8.15 Waves Below and Near Cutoff

The higher-order waves that may exist in transmission lines and all waves that may exist in hollow-pipe waveguides are characterized by cutoff frequencies. If the waves are to be used for propagating energy, we are of course interested only in the behavior above cutoff. However, the behavior of these reactive or evanescent waves below cutoff is important in at least two practical cases:

1. Application to waveguide attenuators
2. Effects of discontinuities in transmission systems

The attenuation properties of these waves below cutoff have been developed in the previous analyses. It has been found that below the cutoff frequency there is an attenuation only and no phase shift in an ideal guide. The characteristic wave impedance is a purely imaginary quantity, reemphasizing the fact that no energy can propagate down the guide. This is not a dissipative attenuation, as is that due to resistance and conductance in transmission systems with propagating waves. It is a purely reactive attenuation, analogous to that in a filter section made of reactive elements, when this is in the cutoff region. The energy is not lost but is reflected back to the source so that the guide acts as a pure reactance to the source.

The expression for attenuation below cutoff in an ideal guide, Eq. 8.13(8), may be written as

$$\gamma = \alpha = k_c\sqrt{1 - \left(\frac{f}{f_c}\right)^2} = \frac{2\pi}{\lambda_c}\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \qquad (1)$$

As f is decreased below f_c, α increases from zero toward the constant value

$$\alpha = \frac{2\pi}{\lambda_c} \qquad (2)$$

when $(f/f_c)^2 \ll 1$. This is an important point in the use of waveguide attenuators, since it shows that the amount of this attenuation is substantially independent of frequency if the operating frequency is far below the cutoff frequency.

Now let us look for a moment at the relations among the fields of both transverse magnetic and transverse electric waves below cutoff. If $\gamma = \alpha$ as given by (1) is substituted in the expressions for field components of transverse magnetic waves, Eqs. 8.13(3) and 8.13(4),

$$\begin{aligned} H_x &= \frac{j}{\eta}\left(\frac{f}{f_c}\right)\frac{1}{k_c}\frac{\partial E_z}{\partial y} \qquad & E_x &= -\sqrt{1 - \left(\frac{f}{f_c}\right)^2}\,\frac{1}{k_c}\frac{\partial E_z}{\partial x} \\ H_y &= -\frac{j}{\eta}\left(\frac{f}{f_c}\right)\frac{1}{k_c}\frac{\partial E_z}{\partial x} \qquad & E_y &= -\sqrt{1 - \left(\frac{f}{f_c}\right)^2}\,\frac{1}{k_c}\frac{\partial E_z}{\partial y} \end{aligned} \qquad (3)$$

For a given distribution of E_z across the guide section, which is determined once the guide shape and size and the wave type are specified, it is evident from relations (3) that, as frequency decreases, $f/f_c \to 0$, the components of magnetic field approach zero whereas the transverse components of electric field approach a constant value. We draw the conclusion that electric fields are dominant in transverse magnetic or E waves far below cutoff. Similarly, magnetic fields are dominant in transverse electric or H waves far below cutoff. If the waves are far below cutoff, the dimensions of the guide are small compared with wavelength. For any such region small compared with wavelength, the wave equation will reduce to Laplace's equation so that low-frequency analyses neglecting any tendency toward wave propagation are applicable.

The presence of losses in the guide below cutoff causes the phase constant to change from the zero value for an ideal guide to a small but finite value, and modifies slightly the formula for attenuation. These modifications are most important in the immediate vicinity of cutoff, for with losses there is no longer a sharp transition but a more gradual change from one region to another. It should be emphasized again that the approximate formulas developed in previous sections may become extremely inaccurate in this region. For example, the approximate formulas for attenuation caused by conductor or dielectric losses would yield an infinite value at $f = f_c$. The actual value is large compared with the minimum attenuation in the pass range since it is approaching the relatively larger magnitude of attenuation in the cutoff regime, but it is nevertheless finite. Previous formulas have also shown an infinite value of phase velocity at cutoff, and with losses it too will be finite.

8.16 Dispersion of Signals Along Transmission Lines and Waveguides

We have in several instances noted the dispersive properties of transmission systems when phase velocity, group velocity, or both vary with frequency. In Chapter 5 we considered a simple two-frequency group in a dispersive system, but we now wish to be more general, using the Fourier integral of Sec. 7.11. There are two classes of problems of concern. One is that of a *base-band* signal, in which the detailed signal is of concern. Examples are audio or video signals, or electrical pulses from a computer, before being placed on other *carrier* frequencies. The other is that of modulated signals in which the base-band signal is placed on a high-frequency carrier. For the latter case we shall consider amplitude modulation and examine the distortion of the envelope.

Base-Band Signals Given an audio signal, series of pulses, or similar electrical waveform, we can express it as a Fourier integral as in Eq. 7.11(15). For a time function $f(t)$, the transform pair may be written

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{j\omega t}\,d\omega \tag{1}$$

$$g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{2}$$

If each frequency component is delayed in phase by βz in propagating distance z along the transmission system, (1) gives the delayed function at z as

$$f(t, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{j(\omega t-\beta z)}\,d\omega \tag{3}$$

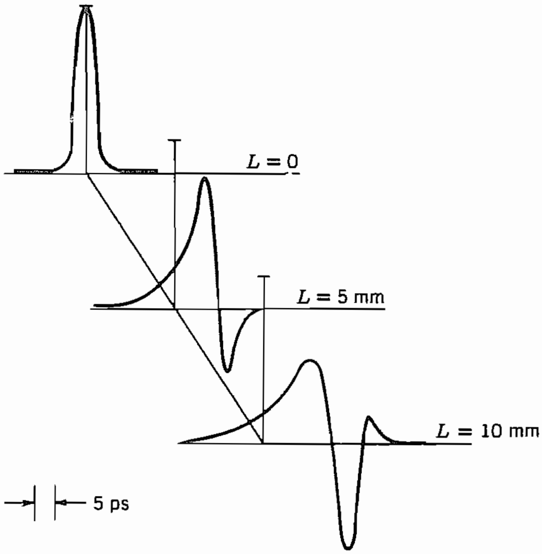


FIG. 8.16a Propagation of a 5-ps gaussian pulse along a microstrip line. Strip width = 0.32 mm, dielectric thickness = 0.4 mm, and $\varepsilon_r = 6.9$. Reproduced by permission from K. K. Li, G. Arjavalingam, A. Dienes, and J. R. Whinnery, *IEEE Trans.* **MTT-30**, 1270 (1982). © 1982 IEEE.

Now if

$$\beta = \frac{\omega}{v_p} \tag{4}$$

with v_p independent of ω, (3) is

$$f(t, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{j\omega(t-z/v_p)}\,d\omega = f\left(t - \frac{z}{v_p}\right) \tag{5}$$

Thus the original function maintains its shape and propagates at the phase velocity, as we have assumed in many wave problems. But any dispersion in v_p modifies the function, at least to some degree.

Transmission lines are often used for base-band signals and have some dispersion through loss terms and internal inductance as affected by skin effect. Some lines, as the microstrip line of Sec. 8.6, have additional dispersion from the presence of multiple dielectrics. Figure 8.16*a* shows the result of a numerical calculation from (3), using the dispersion relation of Eq. 8.6(18), for the change in shape of a 5-ps gaussian pulse in propagating along a typical microstrip used with short electrical pulses.[11]

[11] *K. K. Li, G. Arjavalingam, A. Dienes, and J. R. Whinnery,* IEEE Trans. ***MTT-30**, 1270 (1982).*

Modulated Signals If the signal (1) is used to amplitude modulate a carrier of amplitude V_c and angular frequency ω_c, the resulting modulated wave may be written

$$v_m(t) = \mathrm{Re}\{V_c e^{j\omega_c t}[1 + mf(t)]\} \tag{6}$$

where m is a modulation coefficient. In substituting (1) in (6), we use ω_m for the frequency of the modulating (base-band) signal, and assume that its significant frequency components extend only over a band $-\omega_B \le \omega \le \omega_B$:

$$v_m(t, 0) = \mathrm{Re}\left\{V_c e^{j\omega_c t}\left[1 + \frac{m}{2\pi}\int_{-\omega_B}^{\omega_B} g(\omega_m)e^{j\omega_m t}\, d\omega_m\right]\right\} \tag{7}$$

Or letting $\omega = \omega_c + \omega_m$,

$$v_m(t, 0) = \mathrm{Re}\left\{V_c e^{j\omega_c t} + \frac{mV_c}{2\pi}\int_{\omega_c-\omega_B}^{\omega_c+\omega_B} g(\omega - \omega_c)e^{j\omega t}\, d\omega\right\} \tag{8}$$

Frequencies above ω_c in the integral in (8) correspond to upper sideband terms and those below ω_c to lower sideband terms. Each frequency component propagates according to its appropriate phase constant β. Let us expand β as a Taylor series about ω_c:

$$\beta(\omega) = \beta(\omega_c) + (\omega - \omega_c)\left.\frac{d\beta}{d\omega}\right|_{\omega_c} + \frac{(\omega - \omega_c)^2}{2}\left.\frac{d^2\beta}{d\omega^2}\right|_{\omega_c} + \cdots \tag{9}$$

So the modulated signal, after propagating a distance z, is

$$\begin{aligned} v_m(t, z) = \mathrm{Re}\Bigg\{ & V_c e^{j[\omega_c t - \beta(\omega_c)z]} \\ & \times \left[1 + \frac{m}{2\pi}\int_{-\omega_B}^{\omega_B} g(\omega_m)e^{j[\omega_m(t - z/v_g) - (\omega_m^2/2)z(d^2\beta/d\omega^2) + \cdots]}\, d\omega_m\right]\Bigg\} \end{aligned} \tag{10}$$

where

$$\frac{1}{v_g} = \left.\frac{d\beta}{d\omega}\right|_{\omega_c} \tag{11}$$

Now if $d^2\beta/d\omega^2$ and higher terms are negligible, (10) is interpreted as

$$v_m(t, z) = \mathrm{Re}\left\{V_c e^{j[\omega_c t - \beta(\omega_c)z]}\left[1 + mf\left(t - \frac{z}{v_g}\right)\right]\right\} \tag{12}$$

so the envelope propagates without distortion at group velocity v_g (though the carrier inside moves at a generally different phase velocity). But if the higher-order terms are not negligible, the envelope is distorted and there is said to be *group dispersion.* For a gaussian envelope,

$$f(t) = Ce^{-(2t/\tau)^2} \tag{13}$$

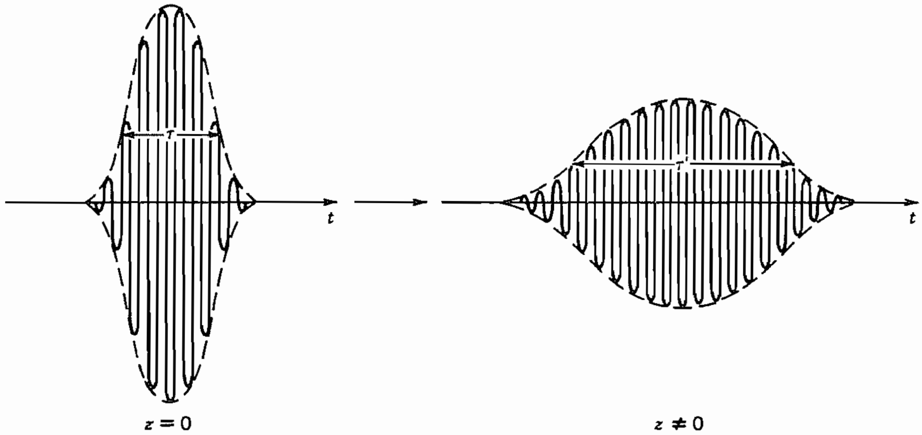


FIG. 8.16b Illustration of the spread of the modulated envelope of a pulse as it travels down a system with group dispersion.

It can be shown (Prob. 8.16c) that the term $d^2\beta/d\omega^2$ causes the envelope to spread to a width τ' after propagating distance z, with τ' given by

$$\tau' = \tau\left[1 + \left(\frac{8z}{\tau^2}\frac{d^2\beta}{d\omega^2}\right)^2\right]^{1/2} \tag{14}$$

The spread of a gaussian envelope, illustrated in Fig. 8.16*b*, clearly limits data rates as pulses begin to overlap their neighbors. Although a factor in some waveguide problems (Prob. 8.16a) the limitation is most important for optical fibers and will be met again in Chapter 14.

PROBLEMS

8.2a As we will see later, one mode of a rectangular waveguide is a TM wave with $H_z = 0$ and $E_z = A \sin(\pi x/a) \sin(\pi y/b)$ with z and t dependence assumed to be $e^{j(\omega t - \beta z)}$. Find expressions for the transverse field components. At a given plane what are the phase relations among the transverse components and between them and E_z.

8.2b The division into TM and TE classes is not the only way of classifying guided waves, as noted in Sec. 8.2. Another frequently useful division employs longitudinal-section electric (LSE) with $E_x = 0$ but all other components present and longitudinal-section magnetic (LSM) with $H_x = 0$ but all other components present. Find the relations between E_z and H_z for each of these classes.

8.3a Add induced charges and current flows, with attention to sign, to the pictures of Figs. 8.3*b* and *c* for the positively traveling TM_1 and TE_1 waves. Repeat for negatively traveling waves.

8.3b Calculate cutoff frequency for TE_1, TE_2, TE_3, TM_1, TM_2, TM_3 waves between planes 1.5 cm apart with air dielectric. Repeat for a glass dielectric with $\varepsilon'/\varepsilon_0 = 4$. Suppose excitation at 8 GHz is provided at a cross section of the air-filled line and all waves are excited. Which wave(s) will propagate without attenuation? At what distance from the excitation plane will each of the nonpropagating waves be attenuated to $1/e$ of its value at the excitation plane?

8.3c The slope of an electric field line in the x–z plane is $dx/dz = E_x/E_z$. Show that the curve for an electric field line of a TM_1 wave, obtained from the expressions for E_x and E_z of the wave, is defined by

$$\cos \beta z = [\cos \pi x_0/a][\cos(\pi x/a)]^{-1}$$

where x_0 is the value of x for a given curve at $z = 0$. Plot one or two lines to verify the form shown in Fig. 8.3*b*. [*Hint:* First express fields as real functions of z.)

8.3d Similarly to Prob. 8.3c, derive the expression defining magnetic field lines for a TE_1 wave and plot one or two lines to verify the form shown in Fig. 8.3*c*.

8.3e* Find the expression for electric field lines for a TM_2 wave, plot one or two lines, and sketch the remainder to give a plot similar to Fig. 8.3*b*. Similarly, plot and sketch magnetic field lines for a TE_2 wave.

8.3f Show that the expression for energy velocity as derived for TM_m waves [Eq. 8.3(37)] also applies to TE_m waves.

8.4a Calculate the angle θ as defined in Fig. 8.4*a* for ray directions of a TM_1 mode between planes 1.5 cm apart with glass dielectric, $\varepsilon'/\varepsilon_0 = 4$, for frequencies of 5, 6, 10, and 30 GHz.

8.4b Obtain the expressions for wave impedance of TM and TE waves, using the picture of uniform plane waves reflecting at an angle.

8.4c* By suitably changing coordinates as in Ex. 8.4, show that the expressions 6.09(18)–(20) for a wave polarized with electric field normal to the plane of incidence striking a conductor at an angle correspond exactly to the field expressions for a TE_m wave.

8.5a Find average power transfer and conductor loss for a TE mode between parallel planes to verify the expression for attenuation, Eq. 8.5(12).

8.5b Calculate attenuation in decibels per meter for a TM_1 wave between copper planes 1.5 cm apart with air dielectric. Frequency is 12 GHz. For the same frequency and spacing, a glass dielectric with $\varepsilon'/\varepsilon_0 = 4$, $\varepsilon''/\varepsilon' = 2 \times 10^{-3}$ is introduced. Calculate attenuation from both dielectric and conductor losses.

8.5c Prove that the frequency of minimum attenuation for a TM_m mode, from conductor losses, is $\sqrt{3}f_c$, where f_c is cutoff frequency. Give the expression for the minimum attenuation and calculate for silver conductors 2 cm apart and air dielectric for the $m = 1, 2$, and 3 modes.

8.5d Show that the transmission-line formula for attenuation constant, Eq. 5.9(7), gives precisely the same result as the approximate wave analysis of Sec. 8.5 for the TEM wave.

8.5e Derive the approximate formula for attenuation constant due to dielectric losses by using $\alpha = w_L/2W_T$.

8.5f* Since E_z is equal and opposite at top and bottom conductors for TEM wave in the parallel-plane line, it is reasonable to assume a linear variation between the two values:

$$E_z = (1 + j)\frac{R_s E_0}{\eta}\left(1 - \frac{2x}{a}\right)$$

Find the modification in the distribution for E_x to satisfy the divergence equation for **E**. Find the corresponding modification in H_y from Maxwell's equations. Describe qualitatively the average Poynting vector as a function of position in the guide.

8.6a For a symmetric stripline as in Fig. 8.6*a* with $w = 1$ mm, $d = 2$ mm, $\varepsilon_r = 2.7$, and thickness t negligible (but larger than several penetration depths), calculate Z_0 and phase velocity of the TEM mode and the cutoff frequency of the next higher mode. [Note that tables of elliptic integrals are required.]

8.6b For $\varepsilon_r = 1$, the lossless microstrip of Fig. 8.6*b* can propagate a true TEM wave at the velocity of light. Find inductance and capacitance per unit length for a 50-Ω line with such a dielectric and the required w/d for this from Fig. 8.6*c*. Calculate the difference due to fringing fields between the capacitance per unit length found above and that given by the parallel-plane approximation, and express this as an equivalent extra width, $\Delta w/d$. Now maintaining w/d constant, assuming inductance is independent of ε_r, and transmission-line equations applicable, repeat for other values of ε_r and plot the extra equivalent $\Delta w/d$ due to fringing as a function of ε_r.

8.6c Calculate the characteristic impedance for a copper microstrip line with an alumina (Al_2O_3 ceramic) dielectric and air above the line. The dimensions should be $w/d = 8$ and $d = 0.2$ mm. Compare the results obtained using the formulas with the graphical data in Sec. 8.6. Find the fractional change of Z_0 between $f = 0$ and $f = 3$ GHz. Calculate the maximum frequency at which the static approximation should be used.

8.6d Design a stripline with the same materials and substrate thickness d and having the characteristic impedance found in Prob. 8.6c for the microstrip line. Calculate and compare the attenuations in the microstrip and stripline at 3 GHz assuming conductor thicknesses of 0.01 mm. Neglect dielectric losses.

8.6e It is desired to make a 15-Ω stripline with the maximum possible delay achievable with no more than 3 dB attenuation at 10 GHz. Consider two possible lines. One is to be made with copper conductors with $w = 100$ μm and alumina (Al_2O_3 ceramic) dielectric and is to be used at room temperature. The other is made with superconducting niobium conductors with $w = 100$ μm and undoped silicon dielectric, having $\varepsilon_r = 11.7$ and loss tangent $\tan \delta_\varepsilon = 10^{-5}$ at 4.2 K, at which temperature the line is to be used. Take $R_s = 10^{-5}$ Ω for niobium at 4.2 K and the strip thickness to be 5 μm for copper and 1 μm for niobium. Find the maximum delay achievable with each of the lines.

8.6f Consider the coplanar waveguide strip transmission line shown in Fig. 8.6*f*. Assuming the line is on an infinitely thick dielectric substrate, the electric fields are distributed symmetrically above and below the line.

(i) Argue that this leads to an effective dielectric constant $\varepsilon_{\text{eff}} = (\varepsilon_r + 1)/2$.
(ii) Find the dimensions to give a line with $Z_0 = 50$ Ω using $\varepsilon_r = 3.78$ and the following design formula[4]

$$\frac{w}{a} = \tanh^2\left(\frac{\pi\eta_0}{8Z_{00}} - \frac{\ln 2}{2}\right)$$

where Z_{00} is the characteristic impedance when the dielectric constant is $\varepsilon_r = 1$

everywhere, and w is the width of the strip located in the center of a gap of width a.

8.6g The various frequency components in a signal (e.g., a pulse) propagate at phase velocities determined by the effective dielectric constants at those frequencies. As will be discussed in Sec. 8.16, this variation of velocity leads to dispersion of signals. The fractional variation of phase velocity with frequency in a coplanar waveguide is lower at low frequencies than it is in microstrip. Consider the following 50-Ω lines with copper conductors and 0.635-mm-thick alumina (Al_2O_3 ceramic) substrates. The coplanar line has a strip width w of 0.266 mm and gaps s of 0.117 mm each. The strip width in the microstrip line is 0.598 mm.

(i) Plot the fractional change of phase velocity of the quasi-TEM mode as a function of frequency in the range $0 < f < 50$ GHz for the coplanar guide and $0 < f < 35$ GHz for the microstrip. For the microstrip, mark $f_{\max}$, the limit of applicability of the static formulation, and also the cutoff frequency of the next higher mode, $(f_c)_{\text{HE1}} = cZ_0/2\eta_0 d$. Also mark the cutoff frequency $f_{\text{TE}} = c/4d\sqrt{\varepsilon_r - 1}$ of the next higher mode for the coplanar waveguide.

(ii) Find the fractional change of the phase velocity at the cutoff frequency of the next higher mode for the coplanar waveguide.

8.6h Compare the total attenuation at 3 GHz in nepers/meter for the two lines described in Prob. 8.6g and explain the physical reason why the higher one is higher.

8.7a For a rectangular waveguide with inner dimensions 3 $\times$ 1.5 cm and air dielectric, calculate the cutoff frequencies of the TE_{10}, TE_{20}, TE_{11}, TE_{12}, TE_{21}, TE_{22}, TM_{11}, TM_{22} modes. Repeat for a glass dielectric with $\varepsilon'/\varepsilon_0 = 4$. Find lengths to the $1/e$ distances for the nonpropagating modes excited at 10 GHz.

8.7b Derive the expression for magnetic lines in the transverse plane of a TM_{11} wave and plot one or two such lines, comparing with Table 8.7. (See approach in Prob. 8.3c.)

8.7c Derive the expression for electric field lines in the transverse plane of a TE_{11} wave and plot one or two such lines, comparing with Table 8.7. (See approach in Prob. 8.3c.)

8.7d Show that the expression for attenuation because of conductor loss for a TM_{mn} mode in the rectangular guide is as given by Eq. 8.7(14).

8.7e Show that the expression for attenuation because of conductor loss for a TE_{mn} mode (neither m nor n zero) in the rectangular guide is as given by Eq. 8.7(26). Explain why this does not apply to $m = 0$ or $n = 0$ case.

8.7f Recalling that surface resistivity R_s is a function of frequency, find the frequency of minimum attenuation for a TM_{mn} mode. Show that the expression for attenuation of a TE_{mn} mode must also have a minimum.

8.7g* Of the wave types studied so far, those transverse magnetic to the axial direction were obtained by setting $H_z = 0$; those transverse electric to the axial direction were obtained by setting $E_z = 0$. For the rectangular waveguide, obtain the lowest-order mode with $H_x = 0$ but all other components present. This may be called a wave transverse magnetic to the x direction. Show that it may also be obtained by superposing the TM and TE waves given previously of just sufficient amounts so that H_x from the two waves exactly cancel. This is a longitudinal-section wave as discussed in Prob. 8.2.

8.7h* Repeat Prob. 8.7g for a wave transverse electric to the x direction.

8.7i From the form of Eqs. 8.2(9)–(12), show that for a TM wave, imposition of the condition $E_z = 0$ on a perfectly conducting boundary of a cylindrical guide causes the other tangential component of **E** also to be zero along that boundary.

8.8a For $f = 3$ GHz, design a rectangular waveguide with copper conductor and air dielectric so that the TE_{10} wave will propagate with a 30% safety factor ($f = 1.30 f_c$) but also so that the wave type with next higher cutoff will be 20% below its cutoff frequency. Calculate the attenuation due to copper losses in decibels per meter.

8.8b For Prob. 8.8a, calculate the attenuation in decibels per meter of the three modes with cutoff frequencies closest to that of the TE_{10} mode, neglecting losses.

8.8c Design a guide for use at 3 GHz with the same requirements as in Prob. 8.8a except that the guide is to be filled with a dielectric having a permittivity four times that of air. Calculate the increase in attenuation due to copper losses alone, assuming that the dielectric is perfect. Calculate the additional attenuation due to the dielectric, if $\varepsilon''/\varepsilon' = 0.01$.

8.8d Find the maximum power that can be carried by a 6-GHz TE_{10} wave in an air-filled guide 4 cm wide and 2 cm high, taking the breakdown field in air at that frequency as 2×10^6 V/m.

8.8e The transmission-line analogy can be applied to the transverse field components, the ratios of which are constants over guide cross sections and are given by wave impedances, just as in the case of plane waves in Chapter 6. A rectangular waveguide of inside dimensions 4×2 cm is to propagate a TE_{10} mode of frequency 5 GHz. A dielectric of constant $\varepsilon_r = 3$ fills the guide for $z > 0$, with an air dielectric for $z < 0$. Assuming the dielectric-filled part to be matched, find the reflection coefficient at $z = 0$ and the standing wave ratio in the air-filled part.

8.8f Find the length and dielectric constant of a quarter-wave matching section to be placed between the air and given dielectric of Prob. 8.8e.

8.9a Derive the set of Eqs. 8.9(1)–(4) by utilizing Maxwell's equations in circular cylindrical coordinates and assuming propagation as $e^{-j\beta z}$.

8.9b What inner radius do you need for an air-filled round pipe to propagate the TE_{11} wave at 6 GHz with operating frequency 20% above the cutoff frequency? What is the guide wavelength for this mode? Find the attenuation in decibels per meter of the TM_{01} mode at this frequency, neglecting losses for that calculation.

8.9c Show that the expression for attenuation from conductor losses of a TM_{nl} mode is

$$\alpha_c = \frac{R_s}{a\eta\sqrt{1 - (\omega_c/\omega)^2}}$$

At what value of ω/ω_c is this a minimum?

8.9d* Show that the expression for attenuation from conductor losses of a TE_{nl} mode is

$$\alpha_c = \frac{R_s}{a\eta\sqrt{1 - (\omega_c/\omega)^2}}\left[\left(\frac{\omega_c}{\omega}\right)^2 + \frac{n^2}{p_{nl}'^2 - n^2}\right]$$

8.9e For a circular air-filled guide with copper conductor, select a radius so that the TE_{01} mode has attenuation of 0.3 dB/km for a frequency of 4 GHz. Estimate the number of modes (counting only the symmetric ones with $n = 0$) that have cutoff frequencies below the operating frequency.

8.10 Use the asymptotic forms of Bessel functions in Eqs. 8.10(1) and (2) for TM and TE waves, respectively, to show that for large $k_c r_i$ and r_0/r_i near unity, the cutoff wavelength of the $n = 0$, $p = 1$ modes is approximately twice the spacing between conductors.

8.11a Sketch examples of mode couplings by each of the six methods described in Sec. 8.11 using for each a system different from the one utilized in Fig. 8.11 to illustrate it.

8.11b Plot fraction of power coupled from a coaxial line into a waveguide (Fig. 8.11*g*) as a function of frequency from 10 to 11 GHz if probe radius is 1.5 mm and other dimensions are as stated in the figure caption.

8.12a Demonstrate that, although in a TEM wave **E** does satisfy Laplace's equation in the transverse plane and so may be considered a gradient of a scalar insofar as variations in the transverse plane are concerned, **E** is not the gradient of a scalar when variations in all directions (x, y, and z) are included.

8.12b Two perfectly conducting cylinders of arbitrary cross-sectional shapes are parallel and separated by a dielectric of conductivity σ and permittivity ε. Show that the ratio of electrostatic capacitance per unit length to dc conductance per unit length is ε/σ.

8.12c If the conductors are perfect but the dielectric has conductivity σ as well as permittivity ε, show that γ must have the following value for a TEM wave to exist ($E_z = 0$, $H_z = 0$):

$$\gamma = \pm[j\omega\mu(\sigma + j\omega\varepsilon)]^{1/2}$$

Explain why the distribution of fields may be a static distribution as in the loss-free line, unlike the case for a lossy conducting boundary.

8.12d How many linearly independent TEM waves may exist on a three-conductor transmission line? Describe current relations for a basic set. Complete the proof that there can be no static field, and hence no TEM wave, inside a single infinite cylindrical conductor.

8.13a Show that the circuit of Fig. P8.13*a* may be used to represent the propagation characteristics of the transverse magnetic wave, if the characteristic wave impedance and propagation constant are written by analogy with transmission-line results in terms of an impedance Z_1 and an admittance Y_1 per unit length, and the medium is μ_1, ε_1.

$$Z_{\text{TM}} = \sqrt{\frac{Z_1}{Y_1}}, \qquad \gamma = \sqrt{Z_1 Y_1}$$

Note the similarity between this and the circuits of conventional filter sections, remembering of course that all constants in this circuit are in reality distributed constants.

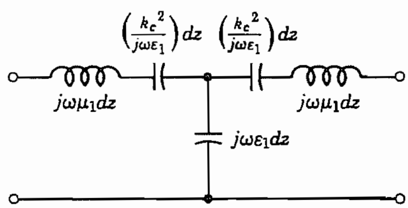


FIG. P8.13*a*

8.13b Show that all field components for a TM wave may be derived from the axial component of the vector potential **A**. Obtain the expressions relating E_x, H_x, and so on to A_z, the differential equation for A_z, and the boundary conditions to be applied at a perfect conductor. Repeat using the axial component of the Hertz potential defined in Prob. 3.19b.

8.13c Show for a TM wave that the magnetic field distribution in the transverse plane can be derived from a scalar flux function, and relate this to E_z. With transverse electric field derivable from a scalar potential function and transverse magnetic field derivable from a scalar flux function, does it follow that both are static-type distributions as in the TEM wave? Explain.

8.13d Show that energy velocity equals group velocity for the TM modes in a lossless waveguide of general cross section.

8.13e* Show that $E_z(x, y)$ for a general TM wave in a perfectly conducting guide satisfies the equation

$$k_c^2 = \left[\int_S (\nabla_t E_z)^2 \, dS\right]\left[\int_S E_z^2 \, dS\right]^{-1}$$

where ∇_t represents the transverse gradient and the integral is over the cross section of the guide. From this argue that k_c^2 is real and positive for waves in which phase is constant over the transverse plane.

8.13f* Numerical methods can be used to find the propagation constants for waveguides of arbitrary cross section. Following the procedures used in solving the Laplace or Poisson equations in Sec. 1.21 to get a difference equation solution for the scalar Helmholtz equation $\nabla^2\psi + k_c^2\psi = 0$, one finds the residual at the kth step to be $R^{(k)}(x, y) = \psi^{(i)}(x, y + h) + \psi^{(i)}(x, y - h) + \psi^{(i)}(x + h, y) + \psi^{(i)}(x - h, y) - (4 - k_c^2h^2)/\psi^{(k-1)}(x, y)$. The change of variable from one iteration step to the next in the successive overrelaxation method is governed by $\psi^{(k)} = \psi^{(k-1)} + \Omega R^{(k)}/(4 - k_c^2h^2)$. Apply the equations with Ω set to 1.0 for convenience to make anumerical evaluation of k_c^2 for a TM_{11} mode in a rectangular waveguide. Assume a rectangular guide with side ratio 1:2. The Helmholtz equation to be solved is Eq. 8.13(1). Divide the waveguide into a grid of 18 squares and number the interior points 1–10 left to right, top to bottom. A reasonable initial guess for the product $k_c^2h^2 = u^2h^2$ can be formed assuming a one-dimensional variation in the smallest dimension; here take $k_c^2h^2 = 1.1$. Start with E_z having the following values at the grid points as a first guess: for points 1, 5, 6, and 10, $E_z = 30$; for points 2, 4, 7, and 9, $E_z = 50$; for points 3 and 8, $E_z = 70$. Use simple relaxation twice to improve the values of E_z for the given $k_c^2h^2$. Then calculate an improved value of $k_c^2h^2$ using the relation

$$k_c^2h^2 = \frac{\sum E_z(x, y)[E_{zN} + E_{zE} + E_{zS} + E_{zW} - 4E_z(x, y)]}{\sum E_z^2(x, y)}$$

where N, E, S, W indicate the points surrounding the grid point at (x, y) and the summations are over all grid points. Next make two more steps of relaxation to adjust the fields to the new $k_c^2h^2$. Then use the above formula to get a second correction to $k_c^2h^2$. Compare the result with the value of $k_c^2h^2$ found using differential equations in Sec. 8.7.

8.14a Derive the equivalent circuit for a TE wave analogous to that of a TM wave given in Prob. 8.13a.

8.14b Show that fields satisfying Maxwell's equations in a homogeneous charge-free, current-free dielectric may be derived from a vector potential **F**:

$$\mathbf{E} = -\frac{1}{\varepsilon}\nabla \times \mathbf{F}$$

$$\mathbf{H} = \frac{1}{j\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{F}) - j\omega\mathbf{F}$$

$$(\nabla^2 + k^2)\mathbf{F} = 0$$

Obtain expressions for all field components of a TE wave from the axial component F_z of the above potential function, and give the differential equation and boundary conditions for F.

8.14c Show that if one utilizes the potential function **A** instead of the **F** of Prob. 8.14b for derivation of a TE wave, more than one component is required.

8.14d Show for a TE mode that transverse distribution of electric field can be derived from a scalar flux function. How is this related to H_z?

8.14e Show that the energy velocity equals the group velocity for the TE modes in a lossless waveguide of general cross section.

8.14f Show for a TM wave in any shape of guide passing from one dielectric material to another, that at one frequency the change in cutoff factor may cancel the change in η, and the wave may pass between the two media without reflection. Identify this condition with the case of incidence at polarizing angle in Sec. 6.13. Determine the requirement for a similar situation with TE waves, and show why it is not practical to obtain this.

8.15 A particular waveguide attenuator is circular in cross section with radius 1 cm. Plot attenuation in decibels per meter for the TE_{11} mode over the frequency range 1–4 GHz. Also plot attenuation of the mode with next nearest cutoff frequency.

8.16a For a hollow-pipe waveguide, with β given by Eq. 8.13(9), find the group dispersion term $d^2\beta/d\omega^2$. Find the length of waveguide for which the width of a gaussian pulse with $\tau = 1$ ns is doubled if frequency is 10 GHz and $\omega_c/\omega = 0.85$.

8.16b Find $d^2\beta/d\omega^2$ for a transmission line with series resistance R and shunt conductance G independent of frequency, where $R/\omega L$ and $G/\omega C$ are small compared with unity. Repeat for a coaxial line with $G = 0$ and R governed by skin effect. Is the resulting group dispersion likely to be significant in usual applications?

8.16c* Start with a gaussian function $f(t)$ given by Eq. 8.16(13) and find its $g(\omega)$. Using this in Eq. 8.16(10), show that the envelope broadens with z as given by Eq. 8.16(14).

8.16d* From the solution of Prob. 8.16c find phase ϕ at z for the high-frequency pulse with gaussian envelope and find the frequency "chirp," defined as $d\phi/dt$.