6

Plane-Wave Propagation and Reflection

6.1 Introduction

The first example of the application of Maxwell's equations in Chapter 3 was that of electromagnetic wave propagation in a simple dielectric medium. We now return to the plane wave example and extend it in this chapter, before considering the more general guided, resonant, and radiating waves.

Plane waves are good approximations to real waves in many practical situations. Radio waves at large distances from the transmitter, or from diffracting objects, have negligible curvature and are well represented by plane waves. Much of optics utilizes the plane-wave approximation. More complicated electromagnetic wave patterns can be considered as a superposition of plane waves, so in this sense the plane waves are basic building blocks for all wave problems. Even when that approach is not followed, the basic ideas of propagation, reflection, and refraction, which are met simply here, help the understanding of other wave problems. The methods developed in the preceding chapter on transmission lines will be very valuable for such problems. A large part of this chapter is concerned with the reflection and refraction phenomena when waves pass from one medium to another, with examples for both radio waves and light.

Plane-Wave Propagation

6.2 Uniform Plane Waves in a Perfect Dielectric

The uniform plane wave was given in Chapter 3 as the first example of the use of Maxwell's equations. We now discuss its properties in more detail, restricting attention to media for which μ and ε are constants. For a uniform plane wave, variations in two directions, say x and y, are assumed to be zero, with the remaining (z) direction taken as the direction of propagation. As in Sec. 3.9, Maxwell's equations in rectangular coordinates then reduce to

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

In component form these are

$$\frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t} \qquad (1) \qquad\qquad \frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t} \qquad (4)$$

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \qquad (2) \qquad\qquad \frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t} \qquad (5)$$

$$0 = \mu \frac{\partial H_z}{\partial t} \qquad (3) \qquad\qquad 0 = \varepsilon \frac{\partial E_z}{\partial t} \qquad (6)$$

As noted in Sec. 3.9, the above equations (3) and (6) show that both E_z and H_z are zero, except possibly for constant (static) parts which are not of interest in the wave solution. That is, electric and magnetic fields of this simple wave are transverse to the direction of propagation.

In Sec. 3.9 we showed that combination of the above equations (2) and (4) leads to the one-dimensional wave equation in E_x,

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \qquad (7)$$

which has a general solution

$$E_x = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) \qquad (8)$$

where

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \qquad (9)$$

which is the velocity of light for the medium. The first term of (8) can be interpreted as a wave propagating with velocity v in the positive z direction, and the second as a wave propagating with the same velocity in the negative z direction. That is,

$$E_{x+} = f_1\left(t - \frac{z}{v}\right), \qquad E_{x-} = f_2\left(t + \frac{z}{v}\right) \tag{10}$$

By use of either (2) or (4), magnetic field H_y was found to be

$$H_y = H_{y+} + H_{y-} = \frac{E_{x+}}{\eta} - \frac{E_{x-}}{\eta} \tag{11}$$

where

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{12}$$

The quantity η is thus seen to be the ratio of E_x to H_y in a single traveling wave of this simple type, and as defined by (12) it may also be considered a constant of the medium, and will be a useful parameter in the analysis of more complicated waves. It has dimensions of ohms and is known as the *intrinsic impedance* of the medium. For free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.73 \approx 120\pi\ \Omega \tag{13}$$

Now looking at the remaining two components, E_y and H_x, combination of (1) and (5) leads to the wave equation in E_y

$$\frac{\partial^2 E_y}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_y}{\partial t^2} \tag{14}$$

which also has solutions in the form of positively and negatively traveling waves as in (8). We write this

$$E_y = f_3\left(t - \frac{z}{v}\right) + f_4\left(t + \frac{z}{v}\right) = E_{y+} + E_{y-} \tag{15}$$

Either (1) or (5) then shows that magnetic field is

$$H_x = -\frac{E_{y+}}{\eta} + \frac{E_{y-}}{\eta} \tag{16}$$

To stress the relationship of electric and magnetic fields for the waves we write the results of (11) and (16) as

$$\frac{E_{x+}}{H_{y+}} = -\frac{E_{y+}}{H_{x+}} = \eta, \qquad \frac{E_{x-}}{H_{y-}} = -\frac{E_{y-}}{H_{x-}} = -\eta \tag{17}$$

These results show a number of things. First, relations (17) are sufficient to require that **E** and **H** shall be perpendicular to one another in each of the traveling waves. They also require that the value of E at any instant must be η times the value of H at that instant, for each wave. Finally we note that, if $\mathbf{E} \times \mathbf{H}$ is formed, it points in the positive

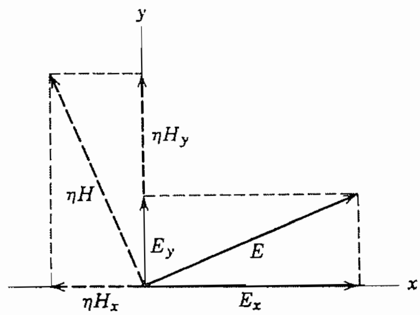


FIG. 6.2a Relations between **E** and **H** for a wave propagating in positive z direction (out of page).

z direction for the positively traveling parts of (17) and in the negative z direction for the negatively traveling part, as expected. These relations are indicated for a positively traveling wave in Fig. 6.2a.

The energy relations are also of interest. The stored energy in electric fields per unit volume is

$$u_E = \frac{\varepsilon E^2}{2} = \frac{\varepsilon}{2}(E_x^2 + E_y^2) \tag{18}$$

and that in magnetic fields is

$$u_H = \frac{\mu H^2}{2} = \frac{\mu}{2}(H_x^2 + H_y^2) \tag{19}$$

By (17), u_E and u_H are equal for a single propagating wave, so the energy density at each point at each instant is equally divided between electric and magnetic energy. The Poynting vector for the positive traveling wave is

$$P_{z+} = E_{x+}H_{y+} - E_{y+}H_{x+} = \frac{1}{\eta}(E_{x+}^2 + E_{y+}^2) \tag{20}$$

and is always in the positive z direction except at particular planes where it may be zero for a given instant. Similarly, the Poynting vector for the negatively traveling wave is always in the negative z direction except where it is zero. The time-average value of the Poynting vector must be the same for all planes along the wave since no energy can be dissipated in the perfect dielectric, but the instantaneous values may be different at two different planes, depending on whether there is a net instantaneous rate of increase or decrease of stored energy between those planes.

In Sec. 3.10 we also studied the important phasor forms for a plane wave with E_x and H_y. Extending that analysis to include the remaining components, we have

$$E_x(z) = E_1 e^{-jkz} + E_2 e^{jkz} \tag{21}$$

$$\eta H_y(z) = E_1 e^{-jkz} - E_2 e^{jkz} \tag{22}$$

$$E_y(z) = E_3 e^{-jkz} + E_4 e^{jkz} \tag{23}$$

$$\eta H_x(z) = -E_3 e^{-jkz} + E_4 e^{jkz} \tag{24}$$

where

$$k = \frac{\omega}{v} = \omega\sqrt{\mu\varepsilon} \quad \mathrm{m}^{-1} \tag{25}$$

This constant is the phase constant for the uniform plane wave, since it gives the change in phase per unit length for each wave component. It may also be considered a constant of the medium at a particular frequency defined by (25), known as the *wave number*, and will be found useful in the analysis of all waves, as will be seen.

The *wavelength* is defined as the distance the wave propagates in one period. It is then the value of z which causes the phase factor to change by 2π:

$$k\lambda = 2\pi \quad \text{or} \quad k = \frac{2\pi}{\lambda} \tag{26}$$

or

$$\lambda = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{v}{f} \tag{27}$$

This is the common relation between wavelength, phase velocity, and frequency. The *free-space wavelength* is obtained by using the velocity of light in free space in (27) and is frequently used at the higher frequencies as an alternative to giving the frequency. It is also common in the optical range of frequencies to utilize *a refractive index n* given by

$$n = \frac{c}{v} = \sqrt{\frac{\mu}{\mu_0}\frac{\varepsilon}{\varepsilon_0}} \tag{28}$$

For most materials in the optical range $\mu = \mu_0$, so that n is just the square root of the relative permittivity for that frequency.

To summarize the properties for a single wave of this simple type, which may be described as a *uniform plane wave*:

1. Velocity of propagation is $v = 1/\sqrt{\mu\varepsilon}$.
2. There is no electric or magnetic field in direction of propagation.
3. The electric field is normal to the magnetic field.
4. The value of the electric field is η times that of the magnetic field at each instant.
5. The direction of propagation is given by the direction of $\mathbf{E} \times \mathbf{H}$.
6. Energy stored in the electric field per unit volume at any instant and any point is equal to energy stored in the magnetic field.
7. The instantaneous value of the Poynting vector is given by $E^2/\eta = \eta H^2$, where E and H are the instantaneous values of total electric and magnetic field strengths.

Example 6.2
PROPAGATION OF A MODULATED WAVE IN A NONDISPERSIVE MEDIUM

If a radio wave of angular frequency ω_0 is amplitude modulated by a sine wave of angular frequency ω_m, the resulting function may be written

$$E(t) = A[1 + m \cos \omega_m t]\cos \omega_0 t \tag{29}$$

Suppose this function at $z = 0$ excites a uniform plane wave propagating in the positive z direction. To obtain the form of the propagating wave it is straightforward to replace t by $t - z/v$ to obtain

$$E(z, t) = A\left[1 + m \cos \omega_m\left(t - \frac{z}{v}\right)\right]\cos \omega_0\left(t - \frac{z}{v}\right) \tag{30}$$

The interpretation of this expression is that the entire function propagates in the z direction with velocity v as illustrated in Fig. 6.2*b*. All this is correct provided that the medium is nondispersive (i.e., that v is independent of frequency). If there is dispersion, an expansion of (29) shows that different frequency components are present and that each component then propagates at its appropriate v. The result then is that the envelope moves with a different velocity than the modulated wave as shown in the discussion of group velocity (Sec. 5.15).

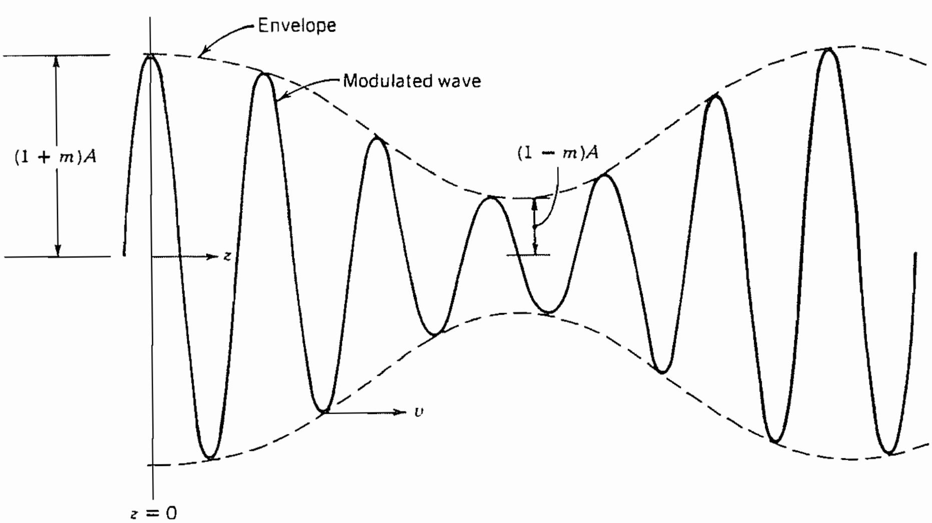


FIG. 6.2*b* Simple modulated wave having form of Eq. (30). The wave is plotted versus z at time $t = 0$. In a nondispersive medium, envelope and modulated portion move with velocity v in z direction.

6.3 Polarization of Plane Waves

If several plane waves have the same direction of propagation, it is straightforward to superpose these for a linear medium. The orientations of the field vectors in the individual waves and the resultant are described by the *polarization*[1] of the waves. In this discussion we are concerned primarily with sinusoidal waves of the same frequency.

Let us take a positively traveling wave only, use phasor representation, and assume there are both x and y components of electric field. The general expression for such a wave is then

$$\mathbf{E} = (\hat{\mathbf{x}}E_1 + \hat{\mathbf{y}}E_2 e^{j\psi})e^{-jkz} \tag{1}$$

where E_1 and E_2 are taken as real and ψ is the phase angle between x and y components. The corresponding magnetic field is

$$\mathbf{H} = \frac{1}{\eta}(-\hat{\mathbf{x}}E_2 e^{j\psi} + \hat{\mathbf{y}}E_1)e^{-jkz} \tag{2}$$

The several classes of polarization then depend upon the phase and relative amplitudes E_1 and E_2.

Linear or Plane Polarization If the two components are in phase, $\psi = 0$, they add at every plane z to give an electric vector in some fixed direction defined by angle α with respect to the x axis, as pictured in Fig. 6.3*a*:

$$\alpha = \tan^{-1}\frac{E_y}{E_x} = \tan^{-1}\frac{E_2}{E_1} \tag{3}$$

This angle is real and hence the same for all values of z and t. Since $\mathbf{E}$ maintains its direction in space, this polarization is called *linear*. It is also called *plane polarization* since the electric vector defines a plane as it propagates in the z direction. In commu-

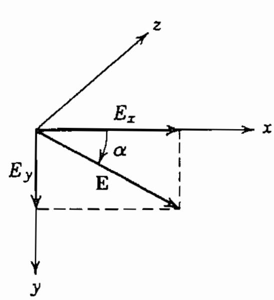


Fig. 6.3*a* Components of a linearly (plane) polarized wave.

[1] *The term* polarization *is used in electromagnetics both for this purpose and for the unrelated concept of the contribution of atoms and molecules to dielectric properties as described in Secs. 1.3 and 13.2. Usually, the intended usage is clear from the context.*

nications engineering it is common to describe polarization by the plane of the electric vector, so that the term *vertical polarization* implies that **E** is vertical. In older optics texts the magnetic field defined the plane of polarization, but it is now also common in optics to use electric field.[2] To avoid ambiguity in either case it is best to specify explicitly, "polarized with electric field in the vertical plane."

Circular Polarization A second important special case arises when amplitudes E_1 and E_2 are equal and phase angle is $\psi = \pm \pi/2$. Equation (1) then becomes

$$\mathbf{E} = (\hat{\mathbf{x}} \pm j\hat{\mathbf{y}})E_1 e^{-jkz} \tag{4}$$

The magnitude of **E** is seen to be $\sqrt{2}E_1$ from the above, and it may be inferred that it rotates in circular manner, but to see this clearly let us go to the instantaneous forms:

$$\begin{aligned} \mathbf{E}(z, t) &= \mathrm{Re}[(\hat{\mathbf{x}} \pm j\hat{\mathbf{y}})E_1 e^{j\omega t} e^{-jkz}] \\ &= E_1[\hat{\mathbf{x}} \cos(\omega t - kz) \mp \hat{\mathbf{y}} \sin(\omega t - kz)] \end{aligned} \tag{5}$$

The sum of the squares of instantaneous E_x and E_y,

$$E_x^2(z, t) + E_y^2(z, t) = E_1^2[\cos^2(\omega t - kz) + \sin^2(\omega t - kz)] = E_1^2 \tag{6}$$

does define the equation of a circle. The instantaneous angle α with respect to the x axis is

$$\alpha = \tan^{-1} \frac{E_y(z, t)}{E_x(z, t)} = \tan^{-1}\left(\mp\frac{\sin(\omega t - kz)}{\cos(\omega t - kz)}\right) = \mp(\omega t - kz) \tag{7}$$

In a given z plane, the vector thus rotates with constant angular velocity with $\alpha = \mp\omega t$. For a fixed time, the vector traces out a spiral in z as pictured in Fig. 6.3*b*. The propagation may be pictured as the movement of this "corkscrew" in the z direction with velocity v.

Note that $\psi = +\pi/2$ leads to $\alpha = -\omega t$ (for $z = 0$) and $\psi = -\pi/2$ leads to rotation in the opposite direction. The first case is called the left-hand or counterclockwise sense of circular polarization (looking in the direction of propagation) and the latter, the right-hand or clockwise. The magnetic field for the circularly polarized wave, using $E_1 = E_2$ and $\psi = \pm\pi/2$, is

$$\mathbf{H} = \frac{E_1}{\eta}(\mp j\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{-jkz} \tag{8}$$

Elliptic Polarization For the general case, with $E_1 \neq E_2$, or $E_1 = E_2$ but ψ other than 0 or $\pm\pi/2$, the terminus of the electric field traces out an ellipse in a given z plane so that the condition of polarization is called *elliptical*. To see this, again let us take instantaneous forms of (1),

$$\begin{aligned} \mathrm{E}(z, t) &= \mathrm{Re}[(\hat{\mathbf{x}}E_1 + \hat{\mathbf{y}}E_2 e^{j\psi})e^{j\omega t}e^{-jkz}] \\ &= \hat{\mathbf{x}}E_1 \cos(\omega t - kz) + \hat{\mathbf{y}}E_2 \cos(\omega t - kz + \psi) \end{aligned} \tag{9}$$

[2] *M. Born and E. Wolf,* Principles of Optics, *6th ed., p. 28, Macmillan, New York, 1980.*

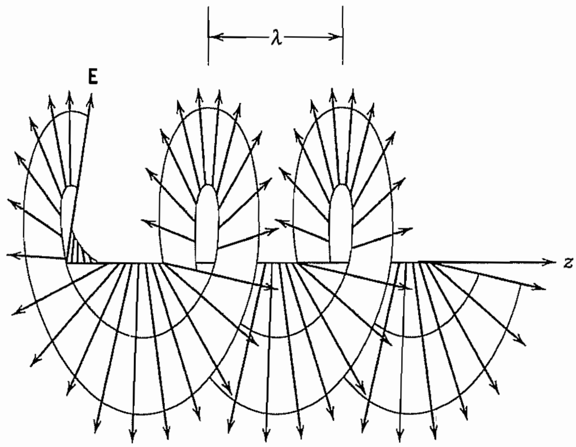


FIG. 6.3b Circularly polarized wave. Terminus of the electric field vectors forms a spiral of period equal to the wavelength at any instant of time. This spiral moves in the z direction with velocity v, so that the vector in a given z plane traces out a circle as time progresses.

or in a given z plane, say $z = 0$,

$$\begin{aligned} E_x(z, t) &= E_1 \cos \omega t \\ E_y(z, t) &= E_2 \cos(\omega t + \psi) \end{aligned} \tag{10}$$

These are the parametric equations of an ellipse. If $\psi = \pm\pi/2$ the major and minor axes of the ellipse are aligned with the x and y axes, but for general ψ the ellipse is tilted as illustrated in Fig. 6.3c.

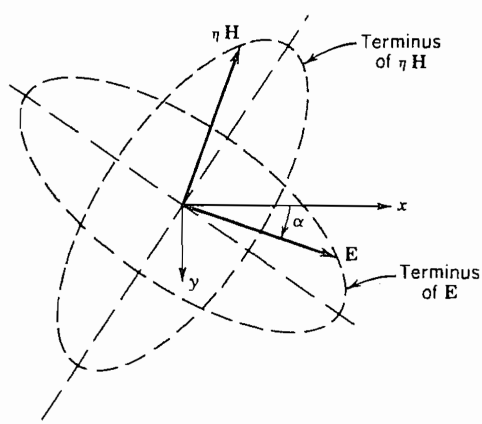


FIG. 6.3c Elliptically polarized wave. The locus of the terminus of electric and magnetic field vectors is in each case an ellipse for a given z plane as time progresses.

Unpolarized Wave We sometimes also speak of an unpolarized wave in which there is a component in any arbitrary direction for each instant of time. Note that this concept applies only to the superposition of waves of different frequency, or of random phases, since superposition of any number of components of the same frequency and defined phase reduces to one of the three cases described above. We may meet unpolarized waves when we have a frequency spectrum (as in sunlight) or a random variation of phase between components, as in the propagation of a radio wave through the ionosophere.

Example 6.3
LINEARLY POLARIZED WAVE AS SUPERPOSITION OF TWO CIRCULARLY POLARIZED WAVES

Just as the circularly polarized wave may be looked at as the superposition of two linearly polarized waves, so may a linearly polarized wave be considered a superposition of two oppositely circulating circular polarized waves. To show this, let us add right-hand and left-hand circularly polarized waves of the same amplitude. Using (4),

$$\mathbf{E} = (\hat{\mathbf{x}} + j\hat{\mathbf{y}})E_1 e^{-jkz} + (\hat{\mathbf{x}} - j\hat{\mathbf{y}})E_1 e^{-jkz} = 2\hat{\mathbf{x}}E_1 e^{-jkz}$$

and magnetic field, using (8), is

$$\mathbf{H} = \frac{E_1}{\eta}(-j\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{-jkz} + \frac{E_1}{\eta}(j\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{-jkz} = \frac{2E_1}{\eta}\hat{\mathbf{y}}e^{-jkz}$$

The results are the expressions for **E** and **H** in a wave polarized with electric field in the x direction. To obtain **E** in the y direction, we have only to subtract the two circularly polarized parts.

6.4 WAVES IN IMPERFECT DIELECTRICS AND CONDUCTORS

Materials properties and their effect on wave propagation will be dealt with extensively in Chapter 13. Here we consider only isotropic, linear materials and bring in the effect of losses on the response of applied fields. The effect of losses can enter through a response to either the electric or magnetic field, or both. An example of a material in which the response to the magnetic field leads to losses is the so-called *ferrite* (or *ferrimagnetic* materials), and in this case the permeability must be a complex quantity $\mu = \mu' - j\mu''$. For most materials of interest in wave studies, magnetic response is very weak and the permeability is a real constant that differs very little from the permeability of free space; this will be assumed throughout this text unless otherwise specified.

A study of the response of bound electrons in atoms and ions in molecules (Sec. 13.2) shows that the total current density resulting from their motion is

$$\mathbf{J} = j\omega\varepsilon\mathbf{E} = j\omega(\varepsilon' - j\varepsilon_b'')\mathbf{E}$$

If, in addition, the material contains free electrons or holes, there is a conduction current density

$$\mathbf{J} = \sigma\mathbf{E}$$

At sufficiently high frequencies, σ can be complex (Sec. 13.3), but we will assume that frequency is low enough to consider it real (satisfactory through the microwave range). Then the total current density is

$$\mathbf{J} = j\omega\left(\varepsilon' - j\varepsilon_b'' - j\frac{\sigma}{\omega}\right)\mathbf{E} \tag{1}$$

In materials called *dielectrics*, there are usually few free electrons and any free-electron current component is included in ε'' and σ is taken as zero. On the other hand, in materials considered conductors such as normal metals and semiconductors, the conduction current dominates and the effect of the bound electrons ε_b'' is subsumed in the conductivity. Although different physically, the two loss terms enter into equations in the same way through the relation $\sigma = \omega\varepsilon''$.

For dielectric materials with real permeability and complex permittivity,

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} = j\omega(\varepsilon' - j\varepsilon'')\mathbf{E} \tag{2}$$

where conduction currents are included in the loss factor ε''.

The wave number, Eq. 6.2(25), is complex in this case and is

$$k = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')} \tag{3}$$

The wave number k may be separated into real and imaginary parts:

$$jk = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon'\left[1 - j\left(\frac{\varepsilon''}{\varepsilon'}\right)\right]} \tag{4}$$

where

$$\alpha = \omega\sqrt{\left(\frac{\mu\varepsilon'}{2}\right)\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]} \tag{5}$$

and

$$\beta = \omega\sqrt{\left(\frac{\mu\varepsilon'}{2}\right)\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]} \tag{6}$$

According to Sec. 6.2, the exponential propagation factor for phasor waves is e^{-jkz} which becomes, when k is complex,

$$e^{-jkz} = e^{-\alpha z}e^{-j\beta z}$$

Thus the wave attenuates as it propagates through the material and the attenuation depends upon the dielectric losses and the conduction losses, as would be expected.

The intrinsic impedance, or ratio of electric to magnetic field for a uniform plane wave, becomes

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon'[1 - j(\varepsilon''/\varepsilon')]}} \tag{7}$$

An important parameter appearing in (4) through (7) is the ratio $\varepsilon''/\varepsilon'$. For low-loss materials such as the examples given in Table 6.4a, this ratio is much less than unity. We may refer to such a material as an *imperfect dielectric.* Under these conditions, the attenuation constant (5) and the phase constant (6) may be approximated by expanding both as binomial series:

$$\alpha \approx \frac{k\varepsilon''}{2\varepsilon'} \tag{8}$$

$$\beta \approx k\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \tag{9}$$

where $k = \omega\sqrt{\mu\varepsilon'}$. Dielectric losses are usually described by the "loss tangent" $\tan\delta = \varepsilon''/\varepsilon'$. It is seen that there is a small increase of the phase constant and,

Table 6.4a
Properties of Common Dielectrics at 25°C

	$\varepsilon'/\varepsilon_0$			$10^4 \tan\delta^a$		
Material[b]	$f = 10^6$	$f = 10^8$	$f = 10^{10}$	$f = 10^6$	$f = 10^8$	$f = 10^{10}$
Fused quartz (SiO_2)	3.8	3.8	3.8	1	2	2
Alumina (96%)	8.8	8.8	8.8	3.3	3.0	14
Alsimag ceramic	5.7	5.6	5.2	30	16	20
MgO	9.6	9.6	—	<3	<3	—
$SrTiO_3$	230	230	—	2	1	—
Polyethylene	2.3	2.3	2.3	<2	2	5
Polystyrene	2.6	2.6	2.5	0.7	<1	10
Teflon	2.1	2.1	2.1	<2	<2	5

[a] $\tan\delta = \varepsilon''/\varepsilon'$.
[b] A microwave-absorbing material, Eccosorb, is available with $1.5 < (\varepsilon'/\varepsilon_0) < 50$ and $0.08 < (\varepsilon''/\varepsilon_0) < 1$.

therefore, decrease of the phase velocity when losses are present in a dielectric. The intrinsic impedance in this case is, from (7),

$$\eta \approx \eta' \left\{ \left[1 - \frac{3}{8} \left(\frac{\varepsilon''}{\varepsilon'} \right)^2 \right] + j \frac{\varepsilon''}{2\varepsilon'} \right\} \tag{10}$$

where $\eta' = \sqrt{\mu/\varepsilon'}$.

In a material such as a semiconductor where the losses are predominantly conductive,

$$\nabla \times \mathbf{H} = (j\omega\varepsilon' + \sigma)\mathbf{E} = j\omega\varepsilon' \left[1 - j\frac{\sigma}{\omega\varepsilon'} \right] \mathbf{E} \tag{11}$$

For radian frequencies much larger than σ/ε' the loss term is small and the approximations (8) to (10) may be used with $\varepsilon''/\varepsilon'$ replaced by $\sigma/\omega\varepsilon'$. For frequencies much below σ/ε' the material may be considered a *good conductor*. If we make use of the fact that

$$\frac{\sigma}{\omega\varepsilon'} >> 1$$

in (4), we find

$$jk = j\omega\sqrt{\frac{\mu\sigma}{j\omega}} = (1 + j)\sqrt{\pi f \mu \sigma} = \frac{1 + j}{\delta} \tag{12}$$

where δ is the depth of penetration defined by Eq. 3.16(11) and used extensively in Chapter 3. The propagation factor for the wave shows that the wave decreases in magnitude exponentially, and has decreased to $1/e$ of its original value after propagating a distance equal to depth of penetration of the material. The phase factor corresponds to a very small phase velocity

$$v_p = \frac{\omega}{\beta} = \omega\delta = c\frac{2\pi\delta}{\lambda_0} \tag{13}$$

where c is the velocity of light in free space and λ_0 is free-space wavelength. Since δ/λ_0 is usually very small, this phase velocity is usually much less than the velocity of light.

Equation (7) gives, for a good conductor,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = (1 + j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1 + j)R_s \tag{14}$$

where R_s is the surface resistivity or high-frequency skin effect resistance per square of a plane conductor of great depth. Equation (14) shows that electric and magnetic fields are 45 degrees out of time phase for the wave propagating in a good conductor. Also, since R_s is very small (0.014 Ω for copper at 3 GHz), the ratio of electric field to magnetic field in the wave is small.

Table 6.4b
Material Parameters for Microwave Frequencies and Below

Material	Conductivity (S/m)	$\frac{\varepsilon'}{\varepsilon_0}$	Frequency at which $\sigma = \omega\varepsilon'$ (Hz)
Copper	5.80×10^7	—	(Optical)
Platinum	0.94×10^7	—	(Optical)
Germanium (lightly doped)	10^2	16	1.1×10^{11}
Seawater	4	81	8.9×10^8
Fresh water	10^{-3}	81	2.2×10^5
Silicon (lightly doped)	10	12	1.5×10^{10}
Representative wet earth	10^{-2}	30	6.0×10^6
Representative dry earth	10^{-3}	7	2.6×10^6

The above-mentioned results for the good conductor agree with those found in Chapter 3 by using what appeared to be a different analysis. In Sec. 3.16 the assumption that the conduction current greatly exceeds the displacement current is made at the outset, with the result that the differential equation is not the wave equation but the so-called *diffusion equation*. The assumptions in both approaches are identical, however, so it is to be expected that the results would be the same.

Table 6.4b lists several materials for which the dominant losses are those resulting from finite conductivity. The values listed for conductivity and real part of permittivity are those applicable up through the microwave frequency range. The frequencies at which displacement currents equal conduction currents are listed and serve to indicate the range of frequencies in which the two above-mentioned approximations are valid.

Plane Waves Normally Incident on Discontinuities

6.5 Reflection of Normally Incident Plane Waves from Perfect Conductors

If a uniform plane wave is normally incident on a plane perfect conductor located at $z = 0$, we know that there must be some reflected wave in addition to the incident wave. The boundary conditions cannot be satisfied by a single one of the traveling wave solutions, but will require just enough of the two so that the resultant electric field at the conductor surface is zero for all time. From another point of view, we know from the Poynting theorem that energy cannot pass the perfect conductor, so all energy brought by the incident wave must be returned in a reflected wave. In this simple case the incident and reflected waves are of equal amplitudes and together form a standing wave pattern whose properties will now be studied.

Let us consider a single-frequency, uniform plane wave and select the orientation of axes so that total electric field lies in the x direction. The phasor electric field, including waves traveling in both the positive and negative z directions (Fig. 6.5), is

$$E_x = E_+ e^{-jkz} + E_- e^{jkz}$$

If $E_x = 0$ at $z = 0$ as required by the perfect conductor, $E_- = -E_+$:

$$E_x = E_+(e^{-jkz} - e^{jkz}) = -2jE_+ \sin kz \tag{1}$$

The relation of the magnetic field to the electric field for the incident and reflected waves is given by Eq. 6.2(11). Hence

$$\begin{aligned} H_y &= \left(\frac{E_+}{\eta} e^{-jkz} - \frac{E_-}{\eta} e^{jkz}\right) \\ &= \frac{E_+}{\eta}(e^{-jkz} + e^{jkz}) = \frac{2E_+}{\eta} \cos kz \end{aligned} \tag{2}$$

Equations (1) and (2) state that, although total electric and magnetic fields for the combination of incident and reflected waves are still mutually perpendicular in space and related in magnitude by η, they are now in time quadrature. The pattern is a standing wave since a zero of electric field is always at the conductor surface, and also always at $kz = -n\pi$ or $z = -n\lambda/2$. Magnetic field has a maximum at the conductor surface, and there are other maxima each time there are zeros of electric field. Similarly, zeros of magnetic field and maxima of electric field are at $kz = -(2n + 1)\pi/2$, or $z = -(2n + 1)\lambda/4$. This situation is sketched in Fig. 6.5, which shows a typical standing wave pattern such as was found for the shorted transmission line in Sec. 5.13. At an instant in time, occurring twice each cycle, all the energy of the line is in the magnetic field; 90 degrees later the energy is stored entirely in the electric field. The *average*

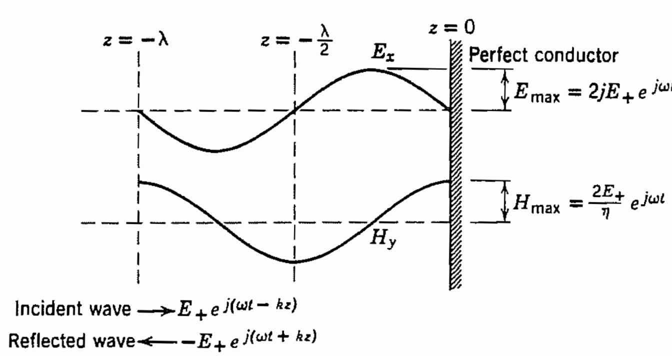


FIG. 6.5 Standing wave patterns of electric and magnetic fields when a plane wave is reflected from a perfect conductor, each shown at instants of time differing by one-quarter of a cycle.

value of the Poynting vector is zero at every cross-sectional plane; this emphasizes the fact that on the average as much energy is carried away by the reflected wave as is brought by the incident wave.

These points are also shown by the instantaneous forms for fields:

$$E_x(z, t) = \mathrm{Re}[-2jE_+ \sin kz e^{j\omega t}] = 2E_+ \sin kz \sin \omega t \tag{3}$$

$$H_y(z, t) = \mathrm{Re}\left[\frac{2E_+}{\eta} \cos kz e^{j\omega t}\right] = \frac{2E_+}{\eta} \cos kz \cos \omega t \tag{4}$$

6.6 Transmission-Line Analogy of Wave Propagation: The Impedance Concept

In the problem of wave reflections from a perfect conductor, we found all the properties previously studied for standing waves on an ideal transmission line. The analogy between the plane-wave solutions and the waves along an ideal line is in fact an exact and complete one. It is desirable to make use of this whether we start with a study of classical transmission-line theory and then undertake the solution of wave problems or proceed in the reverse order. In either case the algebraic steps worked out for the solution of one system need not be repeated in analyzing the other; any aids (such as the Smith chart or computer programs) developed for one may be used for the other; any experimental techniques applicable to one system will in general have their counterparts in the other system. We now show the basis for this analogy.

Let us write side by side the equations for the field components in positively and negatively traveling uniform plane waves and the corresponding expressions found in Chapter 5 for an ideal transmission line. For simplicity we orient the axes so that the wave has E_x and H_y components only:

$$E_x(z) = E_+e^{-jkz} + E_-e^{jkz} \quad (1) \qquad V(z) = V_+e^{-j\beta z} + V_-e^{j\beta z} \quad (5)$$

$$H_y(z) = \frac{1}{\eta}[E_+e^{-jkz} - E_-e^{jkz}] \quad (2) \qquad I(z) = \frac{1}{Z_0}[V_+e^{-j\beta z} - V_-E^{j\beta z}] \quad (6)$$

$$k = \omega\sqrt{\mu\varepsilon} \quad (3) \qquad \beta = \omega\sqrt{LC} \quad (7)$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (4) \qquad Z_0 = \sqrt{\frac{L}{C}} \quad (8)$$

We see that if in the field equations we replace E_x by voltage V, H_y by current I, permeability μ by inductance per unit length L, and dielectric constant ε by capacitance per unit length C, we get exactly the transmission-line equations (5) to (8). To complete the analogy, we must consider the continuity conditions at a discontinuity between two regions. For the boundary between two dielectrics, we know that total tangential electric and magnetic field components must be continuous across this boundary. For the case of normal incidence (other cases will be considered separately later), E_x and H_y are the tangential components, so these continuity conditions are in direct correspondence to those of transmission lines which require that total voltage and current be continuous at the junction between two transmission lines.

To exploit this analogy fully, it is desirable to consider the ratio of electric to magnetic fields in the wave analysis, analogous to the ratio of voltage to current which is called impedance and used so extensively in the transmission-line analysis. It is a good idea to use such ratios in the analysis, quite apart from the transmission-line analogy or the name given these ratios, but in this case it will be especially useful to make the identification with impedance because of the large body of technique existing under the heading of "impedance matching" in transmission lines, most of which may be applied to problems in plane-wave reflections. Credit for properly evaluating the importance of the wave impedance concept to engineers and making its use clear belongs to Schelkunoff.[3]

At any plane z, we shall define the field or wave impedance as the ratio of total electric field to total magnetic field at that plane:

$$Z(z) = \frac{E_x(z)}{H_y(z)} \tag{9}$$

For a single positively traveling wave this ratio is η at all planes, so that η, which has been called the intrinsic impedance of the medium, might also be thought of as a *characteristic wave impedance* for uniform plane waves. For a single negatively traveling wave the ratio (9) is $-\eta$ for all z. For combinations of positively and negatively traveling waves, it varies with z. The input value Z_i distance l in front of a plane at which the "load" value of this ratio is given as Z_L may be found from the corresponding transmission-line formula, Eq. 5.7(13), taking advantage of the exact analogy. The intervening dielectric has intrinsic impedance η:

$$Z_i = \eta\left[\frac{Z_L \cos kl + j\eta \sin kl}{\eta \cos kl + jZ_L \sin kl}\right] \tag{10}$$

It may be argued that in wave problems the primary concern is with reflections and not with impedances directly. This is true, but as in the transmission-line case there is a one-to-one correspondence between reflection coefficient and impedance mismatch ratio. The analogy may again be invoked to adapt Eqs. 5.7(8) and 5.7(9) to give the reflection and transmission coefficients for a dielectric medium of intrinsic impedance η when it is terminated with some known load value of field impedance Z_L:

$$\rho = \frac{E_-}{E_+} = \frac{Z_L - \eta}{Z_L + \eta} \tag{11}$$

$$\tau = \frac{E_2}{E_{1+}} = \frac{2Z_L}{Z_L + \eta} \tag{12}$$

[3] *S. A. Schelkunoff,* Bell Syst. Tech. J. ***17****, 17 (1938).*

We see from this that there is no reflection when $Z_L = \eta$ (i.e., when impedances are matched). There is complete reflection $|\rho| = 1$ when Z_L is zero, infinity, or purely imaginary (reactive). Other important uses of formulas (10) to (12) will follow in succeeding sections.

Example 6.6a
FIELD IMPEDANCE IN FRONT OF CONDUCTING PLANE

If we calculate $Z(z)$ for the plane wave normally incident on a plane conductor, as studied in Sec. 6.5, we recognize that the plane acts as a short circuit since it constrains E_x to be zero there, corresponding to zero voltage in the transmission-line analogy. If we take $Z_L = 0$ in (10), we obtain

$$Z_i = j\eta \tan kl \tag{13}$$

The same result is obtained by taking the ratio of phasor electric and magnetic fields at $z = -l$ from Eqs. 6.5(1) and 6.5(2). Like the impedance of a shorted transmission line, this is always imaginary (reactive). It is zero at $kl = n\pi$ and infinite at $kl = (2n + 1)\pi/2$.

Example 6.6b
ELIMINATION OF WAVE REFLECTIONS FROM CONDUCTING SURFACES

To eliminate wave reflections from a plane perfectly conducting surface, it is clear that coating the conductor with a thin film having conductivity σ does not help since the perfect conductor merely shorts this out. As in Fig. 6.6, one can place a sheet with resistivity η Ω/square a quarter-wavelength in front of the conductor surface. As seen from (13), this is at a point of infinite impedance because of wave reflections so that there is no shunting effect. The wave impinging on the front (if the sheet is thin com-

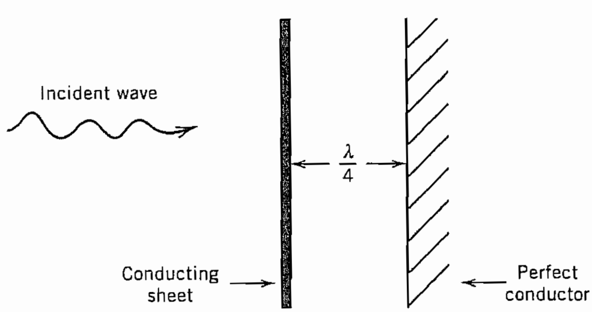


FIG. 6.6 Elimination of reflection from perfect conductor by placing a conducting sheet $\lambda/4$ in front of it.

pared with its depth of penetration) then sees wave impedance η and is perfectly matched. The needed conductivity of the sheet is then

$$\frac{1}{\sigma d} = \eta, \qquad d << \delta \tag{14}$$

The arrangement just described is not very convenient to make, and is sensitive to frequency of operation and to angle of incidence. For this reason coatings of "anechoic chambers" more often use a nonuniform-line approach to matching, with a porous, lossy material, with pyramidal taperings on the surface facing the incident wave.

6.7 Normal Incidence on a Dielectric

If a uniform plane wave is normally incident on a single dielectric boundary from a medium with $\sqrt{\mu_1/\varepsilon_1} = \eta_1$ to one with $\sqrt{\mu_2/\varepsilon_2} = \eta_2$, the wave reflection and transmission may be found from the concepts and equations of Sec. 6.6. Select the direction of the electric field as the x direction, and the direction of propagation of the incident wave as the positive z direction, with the boundary at $z = 0$ (Fig. 6.7a). The medium to the right is assumed to be effectively infinite in extent, so that there is no reflected wave in that region. The field impedance there is then just the intrinsic impedance η_2 for all planes, and in particular this becomes the known load impedance at the plane $z = 0$. Applying Eq. 6.6(11) to give the reflection coefficient for medium 1 referred to $z = 0$,

$$\rho = \frac{E_{1-}}{E_{1+}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{1}$$

The transmission coefficient giving the amplitude of electric field transmitted into the second dielectric, from Eq. 6.6(12), is

$$\tau = \frac{E_2}{E_{1+}} = \frac{2\eta_2}{\eta_2 + \eta_1} \tag{2}$$

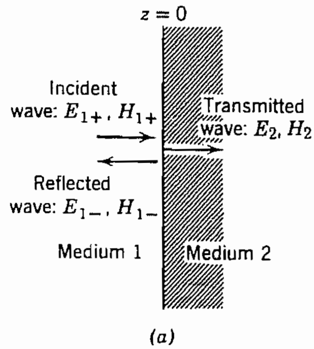
The fraction of incident power density reflected is

$$\frac{P_{1-}}{P_{1+}} = \left(\frac{E_{1-}^2}{2\eta_1}\right)\left(\frac{E_{1+}^2}{2\eta_1}\right)^{-1} = |\rho|^2 \tag{3}$$

And the fraction of the incident power density transmitted into the second medium is

$$\frac{P_2}{P_{1+}} = 1 - |\rho|^2 \tag{4}$$

From (1), we see that there is no reflection if there is a match of impedances, $\eta_1 = \eta_2$. This would of course occur for the trivial case of identical dielectrics, but also for the



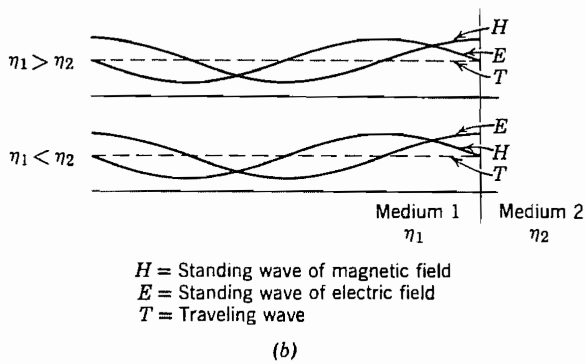


Fig. 6.7 Reflection and transmission of a plane wave from a plane boundary between two dielectric media.

case of different dielectrics if they could be made with the same *ratio* of μ to ε. This latter case is not usually of importance since we do not commonly find high-frequency dielectric materials with permeability different from that of free space, but it is interesting since we might not intuitively expect a reflectionless transmission in going from free space to a dielectric with both permittivity and permeability increased by, say, 10 times.

In the general case there will be a finite value of reflection in the first region, and from (1) we can show that the magnitude of ρ is always less than unity. (It approaches unity as η_2/η_1 approaches zero or infinity.) The reflected wave can then be combined with a part of the incident wave of equal amplitude to form a standing wave pattern as in the case of complete reflection studied in Sec. 6.5. The remaining part of the incident wave can be thought of as a traveling wave carrying the energy that passes on into the second medium. The combination of the traveling and standing wave parts then produces a space pattern with maxima and minima, but with the minima not zero in general. As for corresponding transmission lines, it is convenient to express the ratio of ac amplitude at the electric field maximum to the minimum ac amplitude (occurring a quarter-wavelength away) as a standing wave ratio S:

$$S = \frac{|E_x(z)|_{\max}}{|E_x(z)|_{\min}} = \frac{1 + |\rho|}{1 - |\rho|} \tag{5}$$

By utilizing (1), it may be shown for real η that

$$S = \begin{cases} \eta_2/\eta_1 & \text{if} \quad \eta_2 > \eta_1 \\ \eta_1/\eta_2 & \text{if} \quad \eta_1 > \eta_2 \end{cases} \tag{6}$$

Since η_1 and η_2 are both real for perfect dielectrics, ρ is real and the plane $z = 0$ must be a position of a maximum or minimum. It is a maximum of electric field if ρ is positive, since reflected and incident waves then add, so it is a maximum of electric field and minimum of magnetic field if $\eta_2 > \eta_1$. The plane $z = 0$ is a minimum of electric field and a maximum of magnetic field if $\eta_1 > \eta_2$. These two cases are sketched in Fig. 6.7*b*.

Example 6.7a
Reflection from Quartz and Germanium at Infrared Wavelengths

Let us find reflection of plane waves normally incident from air onto quartz, and also onto germanium, at a wavelength of 2 μm, neglecting losses. Equation (1) may be written in terms of refractive indices, Eq. 6.2(28). Since $\mu_1 = \mu_2$ and $n_1 = \sqrt{\varepsilon_1/\varepsilon_0}$, $n_2 = \sqrt{\varepsilon_2/\varepsilon_0}$, (1) becomes

$$\rho = \frac{n_1 - n_2}{n_1 + n_2} \tag{7}$$

Using data in Fig. 13.2*b*, n_2 for quartz is about 1.5 at 2 μm and n_1 may be taken as unity. Reflection coefficient ρ is then -0.2 and fraction of incident power reflected ρ^2 is 0.04 or 4%. For germanium, n_2 is 4 and ρ is found to be -0.6, so that ρ^2 is 0.36 or 36%.

Example 6.7b
Reflection from a Good Conductor

From Eq. 6.4(14) the characteristic wave impedance of a conductor is seen to be $(1 + j)R_s$, with R_s the surface resistivity defined in Sec. 3.17. Thus for a plane wave normally incident from a dielectric of intrinsic impedance η onto such a conductor, (1) yields

$$\rho = \frac{(1 + j)R_s - \eta}{(1 + j)R_s + \eta} = -\frac{1 - (1 + j)R_s/\eta}{1 + (1 + j)R_s/\eta} \tag{8}$$

Examination of values for typical conductors from Table 3.17a shows that R_s/η is normally very small. Thus a binomial expansion of (8) with first-order terms only retained gives

$$\rho \approx -1 + \frac{2(1+j)R_s}{\eta} \tag{9}$$

Then to first order,

$$|\rho|^2 \approx 1 - \frac{4R_s}{\eta} \tag{10}$$

so the fraction of power transmitted into the conductor is approximately $4R_s/\eta$. For air into copper at 100 MHz, this is $4 \times (2.61 \times 10^{-3})/377 = 2.76 \times 10^{-5}$, or only 0.0028%.

6.8 Reflection Problems with Several Dielectrics

We are next interested in considering the case of several parallel dielectric discontinuities with a uniform wave incident in some material to the left, as pictured in the case for three dielectric materials in Fig. 6.8*a*. We might at first be tempted to treat the problem by considering a series of wave reflections, the incident wave breaking into one part reflected and one part transmitted at the first plane; of the part transmitted into region 2 some is transmitted at the second plane and some is reflected back toward the first plane; of the latter part some is transmitted and some reflected; and so on through an infinite series of wave reflections. This lengthy procedure can be avoided by considering total quantities at each stage of the discussion, and again the impedance formulation is useful in the solution.

If the region to the right has only a single outwardly propagating wave, the wave or field impedance at any plane in this medium is η_3, which then becomes the load imped-

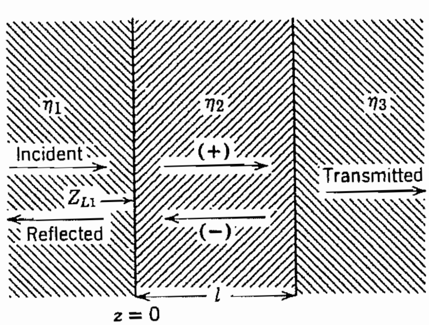


Fig. 6.8*a* Wave reflections from a system with a dielectric medium interfaced between two other media.

ance to place at $z = l$. The input impedance for region 2 is then given at once by Eq. 6.6(10), and since this is the impedance at $z = 0$, it may also be considered the load impedance for region 1:

$$Z_{L1} = Z_{i2} = \eta_2\left(\frac{\eta_3 \cos k_2 l + j\eta_2 \sin k_2 l}{\eta_2 \cos k_2 l + j\eta_3 \sin k_2 l}\right) \tag{1}$$

The reflection coefficient in region 1, referred to $z = 0$, is given by Eq. 6.6(11):

$$\rho = \frac{Z_{L1} - \eta_1}{Z_{L1} + \eta_1} \tag{2}$$

The fraction of power reflected and transmitted is given by Eqs. 6.7(3) and 6.7(4), respectively.

If there are more than the two parallel dielectric boundaries, the process is simply repeated, the input impedance for one region becoming the load value for the next, until one arrives at the region in which reflection is to be computed. It is of course desirable in many cases to utilize the Smith chart described in Sec. 5.9 in place of (1) to transform load to input impedances and to compute reflection coefficient or standing wave ratio once the impedance mismatch ratio is known, just as the chart is used in transmission-line calculations.

We now wish to consider several special cases which are of importance.

Example 6.8a
HALF-WAVE DIELECTRIC WINDOW

If the input and output dielectrics are the same in Fig. 6.8*a* ($\eta_1 = \eta_3$), and the intervening dielectric window is some multiple of a half-wavelength referred to medium 2 so that $k_2 l = m\pi$, then (1) gives

$$Z_{L1} = \eta_3 = \eta_1 \tag{3}$$

so that by (2) reflection at the input face is zero.

A window such as the above will of course give reflections for frequencies other than that for which its thickness is a multiple of a half-wavelength. For example, a Corning 707 glass window with $\varepsilon_r = 4$ and thickness 0.025 m corresponds to $k_2 l = \pi$ at a frequency of 3 GHz. Let us calculate the reflection at 4 GHz using the Smith chart. Normalized load impedance for region 2 is $377/188.5 = 2$, so we enter the chart at point A of Fig. 6.8*b*. We then move on a circle of constant radius $4/3 \times \lambda/2$ or 0.667λ toward the generator. This is one complete circuit of the chart plus 0.167λ, ending at point B where we read normalized input impedance $Z_{i2}/Z_{02} = 0.62 - j0.38$. Renormalizing to the characteristic impedance of the input region, we multiply by $188.5/377$ or $\frac{1}{2}$ and find $0.31 - j0.19$. This is entered at point C, for which we read

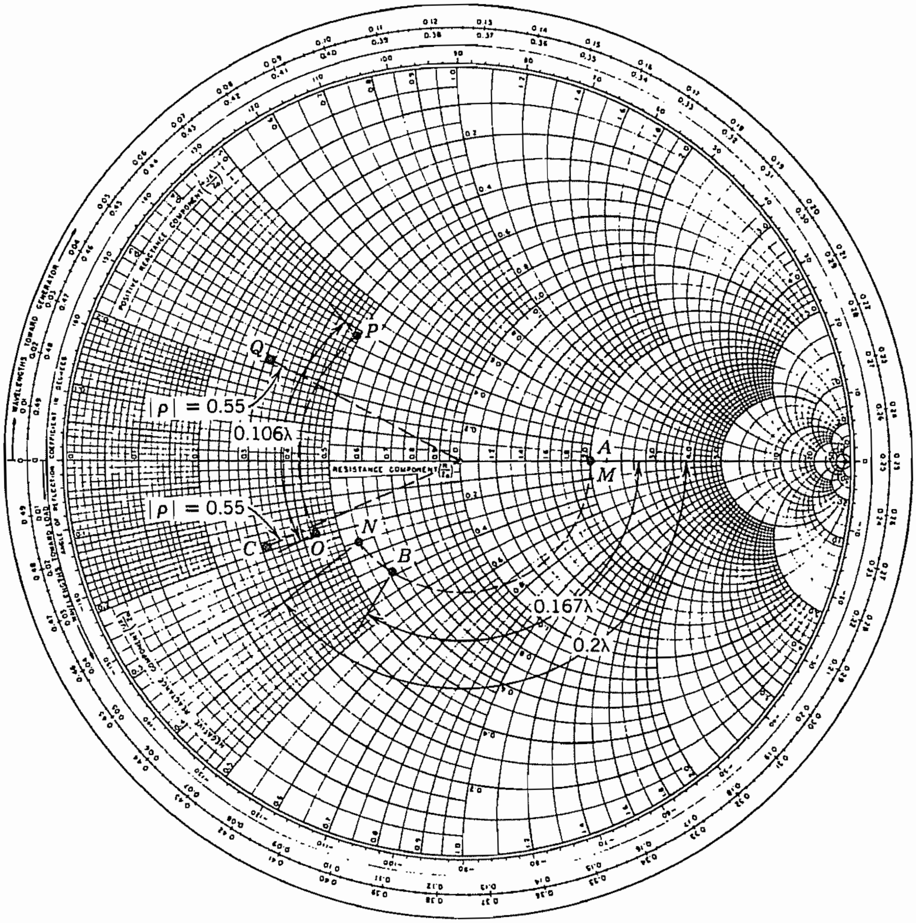


FIG. 6.8b Smith Chart constructions for Exs. 6.8a and 6.8d.

from the radial scale a reflection coefficient $|\rho| = 0.55$, so that $|\rho|^2$ is 0.30 or 30%. This value can be checked by numerical calculation using (1) and (2).

Example 6.8b
ELECTRICALLY THIN WINDOW

If $\eta_1 = \eta_3$ and $k_2 l$ is so small compared with unity that $\tan k_2 l \approx k_2 l$, (1) becomes

$$Z_{L1} \approx \eta_2 \left(\frac{\eta_1 + j\eta_2 k_2 l}{\eta_2 + j\eta_1 k_2 l} \right) \approx \eta_1 \left[1 + jk_2 l \left(\frac{\eta_2}{\eta_1} - \frac{\eta_1}{\eta_2} \right) \right] \tag{4}$$

Substituting in (2), we see that

$$\rho \approx j\frac{k_2 l}{2}\left(\frac{\eta_2}{\eta_1} - \frac{\eta_1}{\eta_2}\right) \tag{5}$$

The magnitude of reflection coefficient is thus proportional to the electrical length of the dielectric window for small values of $k_2 l$; and the fraction of incident power reflected is proportional to the square of this length.

For example, a polystyrene window ($\varepsilon_r \approx 2.54$) 3 mm thick, with a normally incident plane wave at 3 GHz from air, would give $\rho = -j0.145$ so that 2% of incident power is reflected.

Example 6.8c

QUARTER-WAVE COATING FOR ELIMINATING REFLECTIONS

Another important case is that of a quarter-wave coating placed between two different dielectrics. If its intrinsic impedance is the geometric mean of those on the two sides, it will eliminate all wave reflections for energy passing from the first medium into the third. To show this, let

$$k_2 l = \frac{\pi}{2} \qquad \eta_2 = \sqrt{\eta_1 \eta_3} \tag{6}$$

From (1),

$$Z_{L1} = \frac{\eta_2^2}{\eta_3} = \frac{\eta_1 \eta_3}{\eta_3} = \eta_1$$

This is a perfect match to dielectric 1, so that $\rho = 0$.

This technique is used, for example, in coating optical lenses to decrease the amount of reflected light, and is exactly analogous to the technique of matching transmission lines of different characteristic impedances by introducing a quarter-wave section having characteristic impedance the geometric mean of those on the two sides. In all cases the matching is perfect only at specific frequencies for which the length is an odd multiple of a quarter-wavelength, but is approximately correct for bands of frequencies about these values. Multiple coatings are used to increase the frequency band obtainable with a specified permissible reflection.[4]

As a numerical example of this technique, consider the coating needed to eliminate reflections from a 488-nm-wavelength argon laser beam (taken as a plane wave) in going from air to fused silica with refractive index 1.46. It follows from (6) that refractive index of the coating should be the geometric mean of that of the air and window, or about 1.21. Note from Fig. 13.2*b* that there are few materials with this low index, but if one is found its thickness should be a quarter-wavelength measured in the coating, which is about 0.1 μm.

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[4] *C. A. Balanis,* Advanced Engineering Electromagnetics, *Sec. 5.5, Wiley, New York, 1989.*

Example 6.8d
REFLECTION FROM A TWO-PLY DIELECTRIC

Now take the two-ply dielectric of Fig. 6.8*c* in which $d_2 = 2$ mm, $\varepsilon_2/\varepsilon_0 = 2.54$ (polystyrene) and $d_3 = 3.0$ mm, $\varepsilon_3/\varepsilon_0 = 4$ (Corning 707 glass). The normally incident wave is at frequency 10 GHz ($\lambda_0 = 3$ cm). We will do this just on the Smith chart and enter points on Fig. 6.8*b*. The steps are clear extensions of the above.

Starting at the 3–4 interface,

$$\frac{Z_{L3}}{Z_{03}} = \frac{377}{188.5} = 2.00 \qquad \text{(point } M\text{)}$$

Moving toward generator,

$$\frac{d_3}{\lambda_3} = \frac{0.30}{3.0/\sqrt{4}} = 0.2 \text{ wavelength} \qquad \text{(point } N\text{)}$$

We read $Z_{i3}/Z_{03} = 0.55 - j0.235$ and renormalize to get load on region 2:

$$\frac{Z_{L2}}{Z_{02}} = (0.55 - j0.235)\sqrt{\frac{2.54}{4}} = 0.438 - j0.187 \qquad \text{(point } O\text{)}$$

Moving toward generator,

$$\frac{d_2}{\lambda_2} = \frac{0.20}{3.0/\sqrt{2.54}} = 0.106 \text{ wavelength} \qquad \text{(point } P\text{)}$$

Finally we read $Z_{i2}/Z_{02} = 0.49 + j0.38$ and renormalize to air to get load on region 1:

$$\frac{Z_{L1}}{Z_{01}} = (0.49 + j0.38)\sqrt{\frac{1}{2.54}} = 0.31 + j0.24 \qquad \text{(point } Q\text{)}$$

Radius to Q (as fraction of radius of chart) gives $|\rho| = 0.55$ so that over 30% of incident power is reflected.

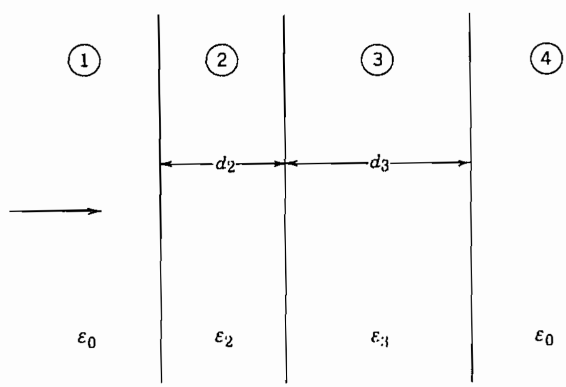


FIG. 6.8c A composite window with two slab dielectrics and free space on either side.

Plane Waves Obliquely Incident on Discontinuities

6.9 Incidence at Any Angle on Perfect Conductors

We next remove the restriction to normal incidence which has been assumed in all the preceding examples. It is possible and desirable to extend the impedance concept to apply to this case also, but before doing this we shall consider the reflection of uniform plane waves incident at an arbitrary angle on a perfect conductor to develop certain ideas of the behavior at oblique incidence. It is also convenient to separate the discussion into two cases, polarization with electric field in the plane of incidence and normal to the plane of incidence. Other cases may be considered a superposition of these two. The plane of incidence is defined by a normal to the surface on which the wave impinges and a ray following the direction of propagation of the incident wave. That is, it is the plane of the paper as we have drawn sketches in this chapter. We consider here loss-free media.[5] Polarization with **E** in the plane of incidence may also be referred to as *transverse magnetic* (TM) since magnetic field is then transverse. When **E** is normal to the plane of incidence it may be called *transverse electric* (TE).[6]

Polarization with Electric Field in the Plane of Incidence (TM) In Fig. 6.9*a* the ray drawn normal to the incident wavefront makes an angle θ with the normal to the conductor. We know that since energy cannot pass into the perfect conductor, there must be a reflected wave, and we draw its direction of propagation at some unknown angle θ'. The electric and magnetic fields of both incident and reflected waves must lie perpendicular to their respective directions of propagation by the properties of uniform plane waves (Sec. 6.2), so the electric fields may be drawn as shown by $\mathbf{E}_+$ and $\mathbf{E}_-$. The corresponding magnetic fields $\mathbf{H}_+$ and $\mathbf{H}_-$ are then both normally out of the paper, so that $\mathbf{E} \times \mathbf{H}$ gives the direction of propagation for each wave. Moreover, with the senses as shown,

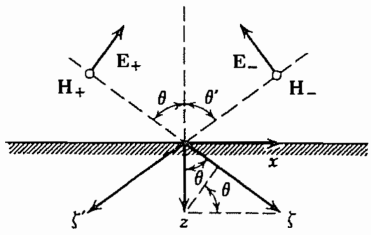
$$\frac{E_+}{H_+} = \frac{E_-}{H_-} = \eta \tag{1}$$

If we draw a ζ direction in the actual direction of propagation for the incident wave as shown, and a ζ' direction so that the reflected wave is traveling in the negative ζ' direction, we know that the phase factors for the two waves may be written as $e^{-jk\zeta}$ and $e^{jk\zeta'}$, respectively. The sum of incident and reflected waves at any point x, z $(z < 0)$ can be written

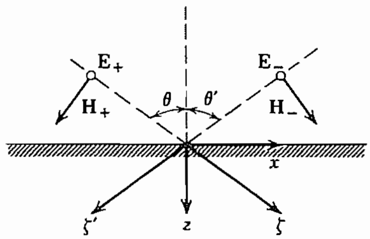
$$\mathbf{E}(x, z) = \mathbf{E}_+ e^{-jk\zeta} + \mathbf{E}_- e^{jk\zeta'} \tag{2}$$

[5] *For a treatment of waves in free space obliquely incident on the plane surface of a lossy medium, see C. A. Balanis,* Advanced Engineering Electromagnetics, *Sec. 5.4, Wiley, New York, 1989.*

[6] *An alternate designation, common in the scientific literature, employs* P *(for German "parallel") and* S *(for German "senkrecht," i.e., perpendicular), respectively, for the orientations of* **E** *in the two cases.*



(a)



(b)

FIG. 6.9 Wave incident at an angle on a dielectric interface. (*a*) Polarization with electric field in plane of incidence. (*b*) Polarization with electric field perpendicular to plane of incidence.

where $\mathbf{E}_+$ and $\mathbf{E}_-$ are reference values at the origin. We now express all coordinates in terms of the rectangular system aligned with the conductor surface. The conversion of ζ and ζ' from the diagram is

$$\zeta = x \sin \theta + z \cos \theta \tag{3}$$

$$\zeta' = -x \sin \theta' + z \cos \theta' \tag{4}$$

so that, if these are substituted in the phase factors of (2), and the two waves broken into their x and z components, we have

$$E_x(x, z) = E_+ \cos \theta e^{-jk(x\sin\theta + z\cos\theta)} - E_- \cos \theta' e^{jk(-x\sin\theta' + z\cos\theta')} \tag{5}$$

$$E_z(x, z) = -E_+ \sin \theta e^{-jk(x\sin\theta + z\cos\theta)} - E_- \sin \theta' e^{jk(-x\sin\theta' + z\cos\theta')} \tag{6}$$

The magnetic field in the two waves is

$$H_y(x, z) = H_+ e^{-jk(x\sin\theta + z\cos\theta)} + H_- e^{jk(-x\sin\theta' + z\cos\theta')} \tag{7}$$

The next step is the application of the boundary condition of the perfect conductor, which is that, at $z = 0$, E_x must be zero for all x. From (5),

$$E_x(x, 0) = E_+ \cos \theta e^{-jkx\sin\theta} - E_- \cos \theta' e^{-jkx\sin\theta'} = 0 \tag{8}$$

This equation can be satisfied for all x only if the phase factors in the two terms are equal, and this in turn requires that

$$\theta = \theta' \tag{9}$$

That is, *the angle of reflection is equal to the angle of incidence.* With this result in (8), it follows that the two amplitudes must be equal:

$$E_+ = E_- \tag{10}$$

If the results (9) and (10) are substituted in (5), (6), and (7), we have the final expressions for field components at any point $z < 0$:

$$E_x(x, z) = -2jE_+ \cos\theta \sin(kz\cos\theta)e^{-jkx\sin\theta} \tag{11}$$

$$E_z(x, z) = -2E_+ \sin\theta \cos(kz\cos\theta)e^{-jkx\sin\theta} \tag{12}$$

$$\eta H_y(x, z) = 2E_+ \cos(kz\cos\theta)e^{-jkx\sin\theta} \tag{13}$$

The foregoing field has the character of a traveling wave with respect to the x direction, but that of a standing wave with respect to the z direction. That is, E_x is zero for all time at the conducting plane, and also in parallel planes distance nd in front of the conductor, where

$$d = \frac{\lambda}{2\cos\theta} = \frac{1}{2f\sqrt{\mu\varepsilon}\cos\theta} \tag{14}$$

The ac amplitude of E_x is a maximum in planes an odd multiple of $d/2$ in front of the conductor. H_y and E_z are maximum where E_x is zero, are zero where E_x is maximum, and are everywhere 90 degrees out of time phase with respect to E_x. Perhaps the most interesting result from this analysis is that the distance between successive maxima and minima, measured normal to the plane, becomes *greater* as the incidence becomes more oblique. A superficial survey of the situation might lead one to believe that they would be at projections of the wavelength in this direction, which would become smaller with increasing θ. This point will be pursued more in the following section.

Polarization with Electric Field Normal to the Plane of Incidence (TE) In this polarization (Fig. 6.9*b*), $\mathbf{E}_+$ and $\mathbf{E}_-$ are normal to the plane of the paper, and $\mathbf{H}_+$ and $\mathbf{H}_-$ are then as shown. Proceeding exactly as before, we can write the components of the two waves in the x, z system of coordinates as

$$E_y(x, z) = E_+e^{-jk(x\sin\theta + z\cos\theta)} + E_-e^{jk(-x\sin\theta' + z\cos\theta')} \tag{15}$$

$$\eta H_x(x, z) = -E_+ \cos\theta e^{-jk(x\sin\theta + z\cos\theta)} + E_- \cos\theta' e^{jk(-x\sin\theta' + z\cos\theta')} \tag{16}$$

$$\eta H_z(x, z) = E_+ \sin\theta e^{-jk(x\sin\theta + z\cos\theta)} + E_- \sin\theta' e^{jk(-x\sin\theta' + z\cos\theta')} \tag{17}$$

The boundary condition at the perfectly conducting plane is that E_y is zero at $z = 0$ for all x, which by the same reasoning as before leads to the conclusion that $\theta = \theta'$ and $E_+ = -E_-$. The field components, (15) to (17), then become

$$E_y = -2jE_+ \sin(kz\cos\theta)e^{-jkx\sin\theta} \tag{18}$$

$$\eta H_x = -2E_+ \cos\theta \cos(kz\cos\theta)e^{-jkx\sin\theta} \tag{19}$$

$$\eta H_z = -2jE_+ \sin\theta \sin(kz\cos\theta)e^{-jkx\sin\theta} \tag{20}$$

This set again shows the behavior of a traveling wave in the x direction and a standing wave pattern in the z direction, with zeros of E_y and H_z and maxima of H_x at the conducting plane and at parallel planes distance nd away, with d given by (14).

6.10 Phase Velocity and Impedance for Waves at Oblique Incidence

Phase Velocity Let us consider an incident wave, such as that of Sec. 6.9, traveling with velocity $v = 1/\sqrt{\mu\varepsilon}$ in a positive direction, which makes angle θ with a desired z direction aligned normally to some reflecting surface. We saw that it is possible to express the phase factor in terms of the x and z coordinates:

$$\mathbf{E}(x, z) = \mathbf{E}_+ e^{-jk\zeta} = \mathbf{E}_+ e^{-jk(x\sin\theta + z\cos\theta)} \tag{1}$$

For many purposes it is desirable to concentrate on the change in phase as one moves in the x direction, or in the z direction. We may then define the two phase constants for these directions:

$$\beta_x = k \sin \theta \tag{2}$$

$$\beta_z = k \cos \theta \tag{3}$$

Wave (1) in instantaneous form is then

$$\mathbf{E}(x, z, t) = \text{Re}[\mathbf{E}_+ e^{j(\omega t - \beta_x x - \beta_z z)}] \tag{4}$$

If we wish to keep the instantaneous phase constant as we move in the x direction, we keep $\omega t - \beta_x x$ constant (the last term does not change if we move only in the x direction), and the velocity required for this is defined as the phase velocity referred to the x direction:

$$v_{px} = \left.\frac{\partial x}{\partial t}\right|_{(\omega t - \beta_x x) = \text{const}} = \frac{\omega}{\beta_x}$$

or

$$v_{px} = \frac{\omega}{k \sin \theta} = \frac{1}{\sqrt{\mu\varepsilon}\sin\theta} = \frac{v}{\sin\theta} \tag{5}$$

Similarly for the z direction,

$$v_{pz} = \frac{\omega}{\beta_z} = \frac{v}{\cos\theta} \tag{6}$$

where v is the velocity normal to its wave front, $1/\sqrt{\mu\varepsilon}$.

We see that in both cases the phase velocity is *greater* than the velocity measured normal to the wavefront and will in fact be so for any oblique direction. There is no violation of relativistic principles by this result, since no material object moves at this velocity. It is the velocity of a fictitious point of intersection of the wavefront and a

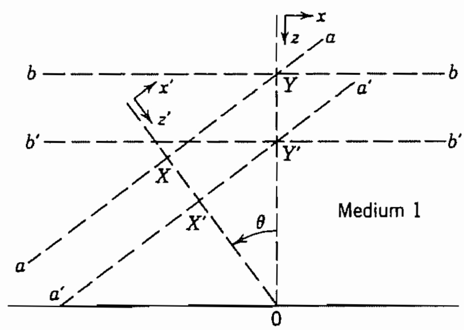


FIG. 6.10a Uniform plane wave moving at angle θ toward a plane.

line drawn in the selected direction. Thus in Fig. 6.10*a*, if a plane of constant phase *aa* moves to $a'a'$ in a given interval of time, the distance moved normal to the wavefront is XX', but the distance moved by this constant phase reference along the *z* direction is the greater distance YY'. Since

$$YY' = XX' \sec \theta$$

this picture would again lead to the result (6) for phase velocity in the *z* direction.

Thus it is the phase constant β in a particular direction that is reduced by cosine or sine of the angle between normal to the boundary and normal to the wavefront, whereas phase velocity is increased by the same factor. The concept of a phase velocity, and the understanding of why it may be greater than the velocity of light, is essential to the discussion of guided waves in later chapters, as well as to the remainder of this chapter.

Wave Impedance In the problems of oblique incidence on a plane boundary between different media, it is also useful to define the wave or field impedance as the ratio of electric to magnetic field components in planes parallel to the boundary. The reason for this is the continuity of the *tangential* components of electric and magnetic fields at a boundary and the consequent equality of the above-defined ratio on the two sides of the boundary. That is, if the value of this ratio is computed as an input impedance for a region to the right in some manner, it is also the value of load impedance at that plane for the region to the left, just as in the examples of normal incidence.

Thus, for incident and reflected waves making angle θ with the normal as in Sec. 6.9, we may define a characteristic wave impedance referred to the *z* direction in terms of the components in planes transverse to that direction. From Eqs. 6.9(5) and 6.9(7) *for waves polarized with electric field in the plane of incidence,*

$$(Z_z)_{TM} = \frac{E_{x+}}{H_{y+}} = -\frac{E_{x-}}{H_{y-}} = \eta \cos \theta \qquad (7)$$

$+$ and $-$ refer, respectively, to incident and reflected wave; the sign of the ratio is chosen for each wave to yield a positive impedance. From Eqs. 6.9(15) and 6.9(16) *for waves polarized with electric field normal to the plane of incidence,*

$$(Z_s)_{TE} = -\frac{E_{y+}}{H_{x+}} = \frac{E_{y-}}{H_{x-}} = \eta \sec \theta \tag{8}$$

we see that, for the first type of polarization, the characteristic wave impedance is always less than η, as we would expect, since only a component of total electric field lies in the transverse x–y plane, whereas the total magnetic field lies in that plane. In the latter polarization, the reverse is true and Z_z is always greater than η.

The interpretation of the example of the last section from the foregoing point of view is then that the perfect conductor amounts to a zero impedance or short to the transverse field component E_x. We would then expect a standing wave pattern in the z direction with other zeros at multiples of a half-wavelength away, this wavelength being computed from phase velocity in the z direction. This is consistent with the interpretation of Eq. 6.9(14).

Example 6.10
DIFFRACTION ORDERS FROM A BRAGG GRATING

It is known that a periodic grating of wires, slots, or similar perturbations reradiates, or diffracts, a plane wave incident upon it into various directions described as the *diffraction orders* for the grating. The directions are defined as those for which phase constants along the grating match on the two sides, except that multiples of 2π difference may exist between grating elements and the contributions still add constructively. Consider a grating as in Fig. 6.10*b*, with a plane wave incident from the bottom at angle θ_1 from the normal. Phase constant in the x direction is $k \sin \theta_1$ so that the phase difference between induced effects in adjacent grating elements is $kd \sin \theta_1$. For the reradiated

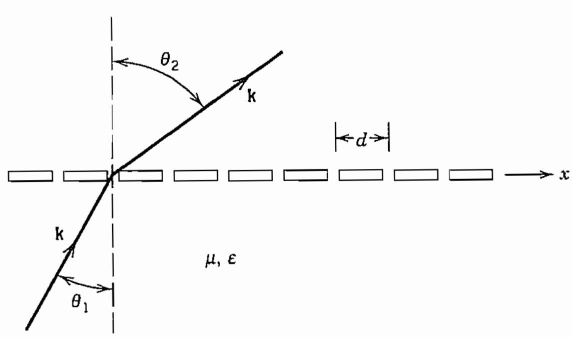


FIG. 6.10*b* Diffraction grating with plane wave incident at an angle.

wave in the top, at angle θ_2 from the normal, there will then be constructive interference if

$$kd \sin \theta_1 = kd \sin \theta_2 \pm 2m\pi, \qquad m = 0, 1, 2, \ldots$$

or

$$\sin \theta_2 = \sin \theta_1 \pm \frac{m\lambda}{d}$$

Angle $\theta_2 = \theta_1$ for $m = 0$ (the *principal order*) but there can be other diffraction orders or *lobes* provided that $|\sin \theta_1 \pm m\lambda/d| \leq 1$, which requires $d > \lambda/2$.

6.11 Incidence at Any Angle on Dielectrics

Law of Reflection For a uniform plane wave incident at angle θ_1 from the normal to the plane boundary between two dielectrics ε_1 and ε_2 (Fig. 6.11), there is a reflected wave at some angle θ_1' with the normal and a transmitted (*refracted*) wave into the second medium at some angle θ_2 with the normal. For either type of polarization, the continuity condition on tangential components of electric and magnetic field at the boundary $z = 0$ must be satisfied for all values of x. As in the argument applied to the problem of reflection from the perfect conductor, this is possible for all x only if inci-

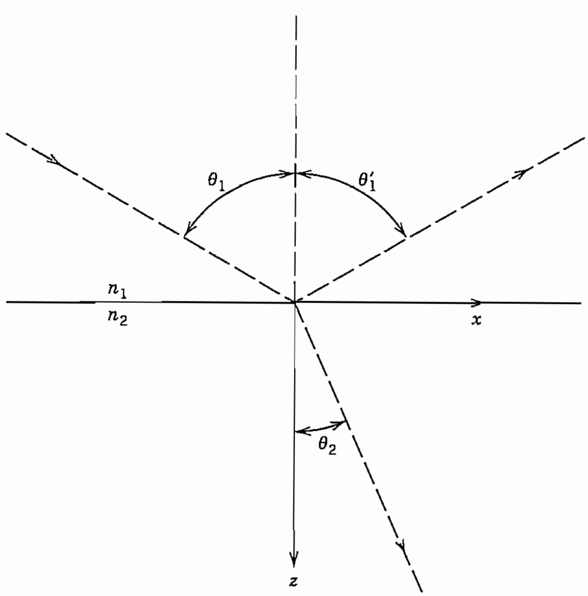


Fig. 6.11 Oblique incidence on boundary between two isotropic dielectrics.

dent, reflected, and refracted waves all have the same phase factor with respect to the x direction:

$$k_1 \sin \theta_1 = k_1 \sin \theta_1' = k_2 \sin \theta_2 \tag{1}$$

The first pair in (1) gives the result

$$\theta_1' = \theta_1 \tag{2}$$

or the angle of reflection is equal to the angle of incidence.

Snell's Law of Refraction From the last pair of (1) we find a relation between the angle of refraction θ_2 and the angle of incidence θ_1:

$$\frac{\sin \theta_2}{\sin \theta_1} = \frac{k_1}{k_2} = \frac{v_2}{v_1} = \frac{n_1}{n_2} \tag{3}$$

This relation is a familiar one in optics and is known as Snell's law. The *refractive index n* is defined to be unity for free space so its value for any other dielectric is a measure of the phase velocity of electromagnetic waves in the medium, relative to free space. It is common to use the refractive index to characterize properties of dielectrics in the infrared and optical frequency ranges as explained in Sec. 6.2. At microwave and lower frequencies it is more common to express the velocities in (3) in terms of permittivity and permeability. For most dielectrics n_1/n_2 may be replaced by $(\varepsilon_1/\varepsilon_2)^{1/2}$ since $\mu_1 \approx \mu_2 \approx \mu_0$.

Reflection and Transmission for Polarization with E in Plane of Incidence To compute the amount of the wave reflected and the amount transmitted, we may use the impedance concept as extended for oblique incidence in the last section. To show the validity of this procedure, we write the continuity conditions for total E_x and H_y, including both incident and reflected components in region 1:

$$E_{x+} + E_{x-} = E_{x2} \tag{4}$$

$$H_{y+} + H_{y-} = H_{y2} \tag{5}$$

Following Sec. 6.10, if we define wave impedances in terms of the tangential components for this TM polarization,

$$Z_{z1} = \frac{E_{x+}}{H_{y+}} = -\frac{E_{x-}}{H_{y-}} \tag{6}$$

$$Z_L = \frac{E_{x2}}{H_{y2}} \tag{7}$$

Equation (5) may be written

$$\frac{E_{x+}}{Z_{z1}} - \frac{E_{x-}}{Z_{z1}} = \frac{E_{x2}}{Z_L} \tag{8}$$

An elimination between (4) and (8) results in equations for reflection and transmission coefficients defined in terms of tangential electric field:

$$\rho = \frac{E_{x-}}{E_{x+}} = \frac{Z_L - Z_{z1}}{Z_L + Z_{z1}} \tag{9}$$

$$\tau = \frac{E_{x2}}{E_{x+}} = \frac{2Z_L}{Z_L + Z_{z1}} \tag{10}$$

For the present case, in which we assume there is no returning wave in medium 2, the load impedance Z_L is just the characteristic wave impedance for the refracted wave referred to the z direction, obtainable from (3) and Eq. 6.10(7):

$$Z_L = \eta_2 \cos\theta_2 = \eta_2\sqrt{1 - \left(\frac{v_2}{v_1}\right)^2 \sin^2\theta_1} \tag{11}$$

And the characteristic wave impedance for medium 1 referred to the z direction is

$$Z_{z1} = \eta_1 \cos\theta_1 \tag{12}$$

The second form of (11) is applicable even when the result for Z_L is complex. Note that, for dielectrics with $\mu_1 = \mu_2$,

$$\frac{\eta_2}{\eta_1} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \frac{n_1}{n_2} \tag{13}$$

The total fields in region 1 may then be written as the sum of incident and reflected waves, utilizing (9) and the basic properties of uniform plane waves. We shall use H_{y+} (denoted H_+) of the incident wave as the reference component since it is parallel to the boundary:

$$E_x = \eta_1 H_+ \cos\theta_1 e^{-j\beta_x x}[e^{-j\beta_z z} + \rho e^{j\beta_z z}] \tag{14}$$

$$H_y = H_+ e^{-j\beta_x x}[e^{-j\beta_z z} - \rho e^{j\beta_z z}] \tag{15}$$

$$E_z = \eta_1 H_+ \sin\theta_1 e^{-j\beta_x x}[-e^{-j\beta_z z} + \rho e^{j\beta_z z}] \tag{16}$$

$$\beta_x = k_1 \sin\theta_1 \qquad \beta_z = k_1 \cos\theta_1 \tag{17}$$

This field again has the character of a traveling wave field in the x direction and a standing wave field in the z direction, but here the minima in the z direction do not in general reach zero. The ratio of maxima to minima could be expressed as a standing wave ratio and would be related to the magnitude of reflection coefficient by the usual expression, Eq. 6.7(5).

Reflection and Transmission for Polarization with *E* Normal to Plane of Incidence For this (TE) polarization, the basic relations (9) and (10) between impedances and reflection or transmission may also be shown to apply. Note that they were first introduced in connection with transmission-line waves in Chapter 5 but have now found usefulness for many wave problems through the impedance concept applied to wave phenomena. Again we define in terms of tangential electric field:

$$\rho = \frac{E_{y-}}{E_{y+}} = \frac{Z_L - Z_{z1}}{Z_L + Z_{z1}} \tag{18}$$

$$\tau = \frac{E_{y2}}{E_{y+}} = \frac{2Z_L}{Z_L + Z_{z1}} \tag{19}$$

For this polarization, the proper wave impedances are obtained from Eq. 6.10(8):

$$Z_L = \eta_2 \sec \theta_2 = \eta_2 \left[1 - \left(\frac{v_2}{v_1} \right)^2 \sin^2 \theta_1 \right]^{-1/2} \tag{20}$$

$$Z_{z1} = \eta_1 \sec \theta_1 \tag{21}$$

The total fields in region 1 are (E_+ denotes the value of E_{y+} in the incident wave)

$$E_y = E_+ e^{-j\beta_x x}[e^{-j\beta_z z} + \rho e^{j\beta_z z}] \tag{22}$$

$$\eta_1 H_x = -E_+ \cos \theta_1 e^{-j\beta_x x}[e^{-j\beta_z z} - \rho e^{j\beta_z z}] \tag{23}$$

$$\eta_1 H_z = E_+ \sin \theta_1 e^{-j\beta_x x}[e^{-j\beta_z z} + \rho e^{j\beta_z z}] \tag{24}$$

$$\beta_x = k_1 \sin \theta_1, \qquad \beta_z = k_1 \cos \theta_1 \tag{25}$$

Example 6.11
Reflection and Transmission of a Circularly Polarized Wave at Oblique Incidence

A circularly polarized wave, incident at an angle on a dielectric discontinuity as in Fig. 6.11, can be resolved into the two linearly polarized parts (Sec. 6.3) and relations of this section used. To be specific, let us assume such a wave incident at angle $\theta_1 = 60$ degrees from air onto fused quartz with $\varepsilon_r = 3.78$. By Snell's law,

$$\theta_2 = \sin^{-1}\left[\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \sin \theta_1\right] = \sin^{-1}\frac{0.866}{\sqrt{3.78}} = \sin^{-1}(0.445) = 26.4^\circ$$

For the TM component, wave impedances are

$$Z_{z1} = \eta_1 \cos \theta_1 = 377 \cos 60^\circ = 188.5\ \Omega$$

$$Z_{z2} = \eta_2 \cos \theta_2 = \frac{377}{\sqrt{3.78}} \cos 26.4^\circ = 174\ \Omega$$

from which we calculate ρ and τ from (9) and (10) as

$$\rho = \frac{174 - 188.5}{174 + 188.5} = -0.04$$

$$\tau = \frac{2 \times 174}{174 + 188.5} = 0.96$$

For the TE component,

$$Z_{z1} = \eta_1 \sec \theta_1 = 756\ \Omega$$

$$Z_{z2} = \eta_2 \sec \theta_2 = 217\ \Omega$$

$$\rho = -0.554$$

$$\tau = 0.446$$

So the second component is highly reflected, the first part is weakly reflected, and both reflected and transmitted waves will be elliptically polarized.

6.12 Total Reflection

A study of the general results from Sec. 6.11 shows that there are several particular conditions of incidence of special interest. The first is one that leads to a condition of total reflection. From the basic formula for reflection coefficient, Eq. 6.11(9) or 6.11(18), we know that there is complete reflection ($|\rho| = 1$) if the load impedance Z_L is zero, infinity, or purely imaginary. To show the last condition, let $Z_L = jX_L$ and note that Z_{z1} is real:

$$|\rho| = \left|\frac{jX_L - Z_{z1}}{jX_L + Z_{z1}}\right| = \frac{\sqrt{X_L^2 + Z_{z1}^2}}{\sqrt{X_L^2 + Z_{z1}^2}} = 1 \qquad (1)$$

The value of Z_L for TM polarization, given by Eq. 6.11(11), is seen to become zero for some critical angle $\theta = \theta_c$ such that

$$\sin \theta_c = \frac{v_1}{v_2} \qquad (2)$$

The value of Z_L for TE polarization, given by Eq. 6.11(20), becomes infinite for this same condition. For both polarizations, Z_L is imaginary for angles of incidence greater than θ_c, so there is total reflection for such angles of incidence.

For loss-free dielectrics having $\mu_1 = \mu_2$, (2) reduces to

$$\sin \theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad (3)$$

It is seen that there are real solutions for the critical angle in this case only when $\varepsilon_1 > \varepsilon_2$, or when the wave passes from an optically dense to an optically rarer medium. From Snell's law, Eq. 6.11(3), we find that the angle of refraction is $\pi/2$ for $\theta = \theta_c$ and is imaginary for greater angles of incidence. So from this point of view also we expect no transfer of energy into the second medium. Although there is no energy transfer, there are finite values of field in the second region as required by the continuity conditions at the boundary. Fields die off exponentially with distance from the boundary as the phase constant β_z becomes imaginary.

Although the reflected wave has the same amplitude as the incident wave for angles of incidence greater than the critical, it does not in general have the same phase. The phase relation between E_{x-} and E_{x+} for the first type of polarization is also different from that between E_{y-} and E_{y+} for the second type of polarization incident at the same angle. Thus, if the incident wave has both types of polarization components, the reflected wave under these conditions is elliptically polarized (Sec. 6.3).

The phenomenon of total reflection is very important at optical frequencies, as it provides reflection with less loss than from conducting mirrors. The use in total reflecting prisms is a well-known example, and its importance to dielectric waveguides will be shown in later chapters.

Example 6.12
EVANESCENT DECAY IN SECOND MEDIUM

It was noted above that fields are nonzero in the second medium under conditions of total reflection, but die off exponentially from the boundary (i.e., are evanescent). Let us show this more specifically using expressions already developed. We have so far considered the propagation factor in the z direction as

$$e^{-j\beta_z z} \tag{4}$$

where, for medium 2,

$$\beta_z = k_2 \cos\theta_2 \tag{5}$$

and using Snell's law,

$$\beta_z = k_2\sqrt{1 - \left(\frac{v_2}{v_1}\right)^2 \sin^2\theta_1} \tag{6}$$

This becomes imaginary, which by (4) represents an attenuation factor $e^{-\alpha z}$, for $(v_2/v_1)\sin\theta_1 > 1$, with

$$\alpha = k_2\sqrt{\left(\frac{v_2}{v_1}\right)^2 \sin^2\theta_1 - 1} \quad \text{Np/m} \tag{7}$$

Since $k_2 = 2\pi\sqrt{\varepsilon_{2r}}/\lambda_0$, α will be of order of magnitude 2π Np (or 54.5 dB) per wavelength for angles well beyond the critical.

6.13 POLARIZING OR BREWSTER ANGLE

Let us next ask under what conditions there might be no reflected wave when the uniform plane wave is incident at angle θ on the dielectric boundary. We know that this occurs for a matching of impedances between the two media, $Z_L = Z_{z1}$. For the wave with TM polarization and for a medium with $\mu_1 = \mu_2$, Eqs. 6.11(11) and 6.11(12) become

$$Z_L = \sqrt{\frac{\mu_1}{\varepsilon_2}}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1} \tag{1}$$

$$Z_{z1} = \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_1 \tag{2}$$

These two quantities may be made equal for a particular angle $\theta_1 = \theta_{\mathrm{p}}$ such that

$$\cos\theta_{\mathrm{p}} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_{\mathrm{p}}} \tag{3}$$

This equation has a solution

$$\theta_{\mathrm{p}} = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}} = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \tan^{-1}\left(\frac{n_2}{n_1}\right) \tag{4}$$

Note that (4) yields real values of θ_{p} for either $\varepsilon_1 > \varepsilon_2$ or $\varepsilon_2 > \varepsilon_1$, and so for TM polarization there is always some angle for which there is no reflection; all energy incident at this angle passes into the second medium.

For TE polarization a study of Eqs. 6.11(20) and 6.11(21) shows that there is no angle yielding an equality of impedances for materials with different dielectric constants but like permeabilities. Hence, a wave incident at angle θ_{p} with both polarization components present has some of the second polarization component but none of the first reflected. The reflected wave at this angle is thus plane polarized with electric field normal to the plane of incidence, and the angle θ_{p} is correspondingly known as the *polarizing angle*. It is also alternatively known as the *Brewster angle*. Early gas lasers made use of end windows placed at the Brewster angle to provide for oscillation for only one of the two polarizations. For the TM polarization, there is no reflection from the ends of the tube, so an external optical resonator governs the oscillation behavior.

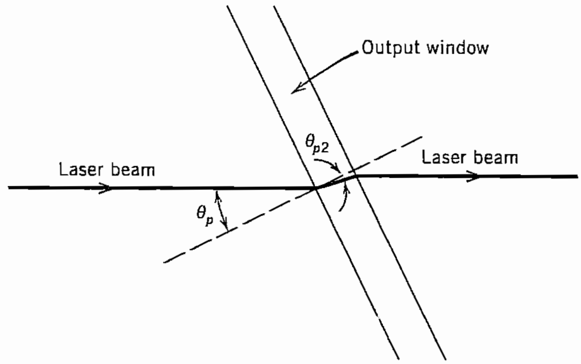


FIG. 6.13 Window set at the Brewster angle to eliminate reflection for a laser beam with TM polarization.

Example 6.13

LASER LIGHT THROUGH A WINDOW SET AT THE BREWSTER ANGLE

Let us calculate the angle needed to pass without reflection TM polarized light from a helium–neon laser ($\lambda = 0.633\ \mu m$) using fused quartz with refractive index $n_2 = 1.46$. From (4),

$$\theta_p = \tan^{-1}\frac{1.46}{1} = 55.6°$$

Note that this is the angle between the beam axis and the normal to the window (Fig. 6.13). We find the angle in the glass by Snell's law,

$$\theta_2 = \sin^{-1}\left[\frac{1}{1.46}\sin 55.6°\right] = 34.4°$$

Note that this is the proper Brewster angle for going from glass to air:

$$\theta_{p2} = \tan^{-1}\frac{1}{1.46} = 34.4°$$

So the exit from the window is without reflection also.

6.14 MULTIPLE DIELECTRIC BOUNDARIES WITH OBLIQUE INCIDENCE

If there are several dielectric regions with parallel boundaries, the problem may be solved by successively transforming impedances through the several regions, using the standard transmission-line formula, Eq. 6.6(10), or a graphical aid such as the Smith chart. For each region the phase constant and characteristic wave impedance must include the function of angle from the normal as well as the properties of the dielectric material. Thus for the ith region, from the concepts of Sec. 6.11, the phase constant is

$$\beta_{zi} = k_i \cos \theta_i \tag{1}$$

and the characteristic wave impedance is

$$Z_{zi} = \eta_i \cos \theta_i \qquad \text{for TM polarization} \tag{2}$$

$$Z_{zi} = \eta_i \sec \theta_i \qquad \text{for TE polarization} \tag{3}$$

When the impedance is finally transformed to the surface at which it is desired to find reflection, the reflection coefficient is calculated from the basic reflection formula, Eq. 6.11(9), and the fraction of the incident power reflected is just the square of its magnitude. The angles in the several regions are found by successively applying Snell's law, starting from the first given angle of incidence.

Example 6.14
Oblique Incidence on a Two-Ply Dielectric Window

We return to the two-ply dielectric of Ex. 6.8d with $n_2 = 2.54$, $n_3 = 4.0$ and consider incidence of a plane wave at angle $\theta_1 = 40°$ from the normal (Fig. 6.14). From Snell's law, angles in the dielectric are $\theta_2 = 23.8°$, $\theta_3 = 18.7°$, and $\theta_4 = 40.0°$. Note that the exit angle is equal to the entrance angle as expected (Prob. 6.14d).

Wave impedance for the ith region using TE polarization is given by

$$Z_{zi} = \eta_i / \cos \theta_i$$

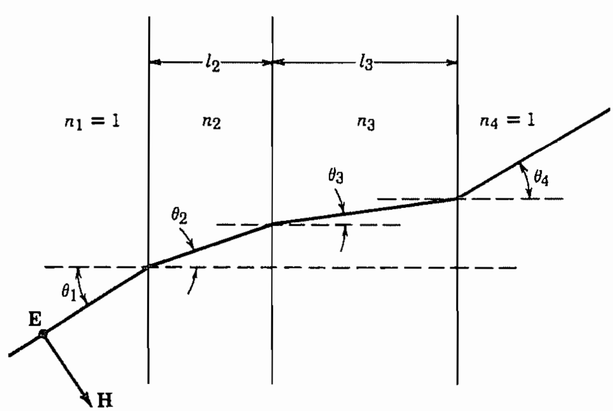


Fig. 6.14 Composite window with wave incident at an angle.

yielding $Z_{z1} = Z_{z4} = 492\ \Omega$, $Z_{z2} = 258.5\ \Omega$, and $Z_{z3} = 199\ \Omega$. The length in wavelengths for each region is given by

$$\ell_i/\lambda_i = \ell_i \cos\theta_i \sqrt{\varepsilon_{ri}}/\lambda_0$$

yielding $\ell_2/\lambda_2 = 0.097$ and $\ell_3/\lambda_3 = 0.189$. Use of the above values on a Smith chart leads to the normalized impedances

$$Z_{L3}/Z_{03} = 2.47, \qquad Z_{i3}/Z_{03} = 0.48 - j0.325$$

$$Z_{L2}/Z_{02} = 0.37 - j0.25, \qquad Z_{i2}/Z_{02} = 0.38 + j0.30$$

$$Z_{L1}/Z_{01} = 0.20 + j0.158$$

This last value corresponds to $|\rho| = 0.66$ or $|\rho|^2 = 0.435$. This is somewhat greater than the value for normal incidence in Ex. 6.8d.

PROBLEMS

6.2a For an inhomogeneous dielectric, find the differential equation for E_x (replacing Eq. 6.2(7)) if ε is a function of z only. Repeat for ε a function of x only ($\mu = \mu_0$ in both cases).

6.2b A step-function uniform plane wave is generated by suddenly impressing a constant electric field $E_x = C$ at $z = 0$ at time $t = 0$ and maintaining it thereafter. A perfectly conducting plane is placed normal to the z direction at $z = 600$ m. Sketch total E_x and ηH_y versus z at $t = 1\ \mu$s and at $t = 3\ \mu$s.

6.2c Write the instantaneous forms corresponding to phasor fields given by Eqs. 6.02(21)–(24). Find the instantaneous Poynting vector and show that the average part is equal to the average Poynting vector of the positively traveling wave minus that for the negatively traveling wave.

6.2d For the modulated wave of Ex. 6.2, suppose that the medium is dispersive with wave number k varying linearly with frequency over the frequency band of interest:

$$k \approx \frac{\omega}{v_0} + \frac{\Delta k(\omega - \omega_0)}{\omega_0}$$

Describe propagation of the modulated wave in this case.

6.3a Check to show that Eqs. 6.3(1) and (2) satisfy Maxwell's equations under the specializations appropriate to plane waves.

6.3b Sketch the locus of **E** for the following special cases of Eq. 6.3(1), identifying the type of polarization for each: (i) $E_1 = 1, E_2 = 2, \psi = 0$; (ii) $E_1 = 1, E_2 = 2, \psi = \pi$; (iii) $E_1 = 1, E_2 = 1, \psi = \pi/2$; (iv) $E_1 = 1, E_2 = 2, \psi = \pi/2$; (v) $E_1 = 1, E_2 = 1, \psi = \pi/4$.

6.3c A circularly polarized wave of strength E_1 travels in the positive z direction with the reflected circularly polarized wave of strength E_1' returning:

$$\mathbf{E} = (\hat{\mathbf{x}} + j\hat{\mathbf{y}})E_1e^{-jkz} + (\hat{\mathbf{x}} + j\hat{\mathbf{y}})E_1'e^{jkz}$$

Find the average Poynting vector. Repeat for reflected wave $(\hat{\mathbf{x}} - j\hat{\mathbf{y}})E_1'e^{jkz}$ and discuss types of terminations at $z = 0$ to give the two forms of reflected waves.

6.3d Show that any arbitrary elliptically polarized wave may be broken up into two oppositely rotating circularly polarized components instead of the two plane-polarized components.

6.4a For a uniform plane wave of frequency 10 GHz propagating in polystyrene, calculate the attenuation constant, phase velocity, and intrinsic impedance. What are the most important differences over air dielectric?

6.4b Plot a curve showing attenuation constant in sea water from 10^4 to 10^9 Hz, assuming that the constants given do not vary over this range. Comment on the implications of the results to the problem of communicating by radio waves through seawater.

6.4c A horn antenna excited at 1 GHz is buried in dry earth. How thick could the earth covering be if half the power is to reach the surface? Neglect reflections at the surface in your first calculation and then estimate additional fraction lost by the reflection.

6.4d Plot attenuation in nepers per micrometer versus wavelength for nickel and silver over the wavelength range of Fig. 13.3*b*, using data of that figure.

6.4e Derive the expression for group velocity of a uniform plane wave propagating in a good conductor and compare with phase velocity in the conductor.

6.4f Determine the group velocity for uniform plane waves in a lossy dielectric, assuming (i) σ and ε' independent of frequency and (ii) assuming $\varepsilon''/\varepsilon'$ independent of frequency. Use the approximate form for β assuming $\varepsilon''/\varepsilon'$ small and compare in each case with phase velocity.

6.4g* For a plane wave in loss-free dielectric ε_1, impinging upon a second loss-free dielectric ε_2 at normal incidence, the average Poynting vector in region 2 is just the difference between the average Poynting vectors of incident and reflected waves in region 1. Show that this is not so if ε_1 is lossy (even if ε_2 is taken to be loss free). Explain.

6.5a Find the instantaneous Poynting vector for plane z for the standing wave of Sec. 6.5. Note the planes for which it is zero for all values of t and comment on the significance of these planes. Show that the average Poynting vector is zero as stated in Sec. 6.5.

6.5b Evaluate instantaneous values of stored energy in electric field and in magnetic field between the conductor and plane z in the standing wave of Sec. 6.5. Note planes for which the two forms of energy have the same maximum values (occurring at different times) and verify the statements concerning stored energy made in Sec. 6.5.

6.6a Write the formulas for a uniform plane wave with E_y and H_x only, and give the correspondence to voltage and current in the transmission-line equations.

6.6b Consider a lossy ferrite with both μ and ε complex, $\mu' - j\mu''$ and $\varepsilon' - j\varepsilon''$, respectively. Show that the transmission-line analogy for a plane wave in this material has both series resistance and shunt conductance. Determine expressions for these elements.

6.6c Reflection and transmission coefficients are given in Eqs. 6.6(11) and (12) in terms of electric field. Give corresponding expressions in terms of magnetic field and in terms of power ratios.

6.6d A conducting film of impedance 377 Ω/square is placed a quarter-wave in air from a plane conductor to eliminate wave reflections at 9 GHz. Assume negligible displacement currents in the film. Plot a curve showing the fraction of incident power reflected versus frequency for frequencies from 6 to 18 GHz.

6.7a A 10-GHz radar produces a substantially plane wave which is normally incident upon a still ocean. Find the magnitude and phase of reflection coefficient and percent of incident energy reflected and percent transmitted into the body of the sea.

6.7b For a certain dielectric material of effectively infinite depth, reflections of an incident plane wave from free space are observed to produce a standing wave ratio of 2.7 in the free space. The face is an electric field minimum. Find the dielectric constant.

6.7c Check the expression for power transmitted into a good conductor, given at the end of Ex. 6.7b, by assuming that magnetic field at the surface is the same as for reflection from a perfect conductor and computing the conductor losses due to the currents associated with this magnetic field.

6.8a Calculate the reflection coefficient and percent of incident energy reflected when a uniform plane wave is normally incident on a plexiglas radome (dielectric window) of thickness $\frac{3}{8}$ in., relative permittivity $\varepsilon_r = 2.8$, with free space on both sides. Frequency corresponds to free-space wavelength of 20 cm. Repeat for $\lambda_0 = 10$ cm and 3 cm.

6.8b* For a sandwich-type radome consisting of two identical thin sheets (thickness 1.5 mm, relative permittivity $\varepsilon_r = 4$) on either side of a thicker foam-type dielectric (thickness 1.81 cm, relative permittivity $\varepsilon_r = 1.1$), calculate the reflection coefficient for waves striking at normal incidence. Take frequency 3×10^9 Hz; repeat for 6×10^9 Hz. *Suggestion:* Use the Smith chart.

6.8c What refractive index and what thickness do you need to make a quarter-wave antireflection coating between air and silicon at 10 GHz? At $\lambda_0 = 10\ \mu$m? Assume undoped silicon with losses negligible.

6.8d For the 10-μm design of Prob. 6.8c, plot fraction of incident energy reflected versus wavelength from $\lambda_0 = 15\ \mu$m to $\lambda_0 = 5\ \mu$m.

6.8e* Imagine two quarter-wave layers of intrinsic impedance η_2 and η_3 between dielectrics of intrinsic impedances η_1 and η_4. Show that perfect matching occurs if $\eta_2/\eta_3 = (\eta_1/\eta_4)^{1/2}$. For $\eta_4 = 4$, $\eta_3 = 3$, $\eta_2 = 1.5$, $\eta_1 = 1$, calculate reflection coefficient at a frequency 10% below that for perfect matching. Compare with the result for a single quarter-wave matching coating with $\eta = 2$.

6.8f** A dielectric window of polystyrene (see Table 6.4a) is made a half-wavelength thick (referred to the dielectric) at 10^8 Hz, so that there would be no reflections for normally incident uniform plane waves from space, neglecting losses in the dielectric. Considering the finite losses, compute the reflection coefficient and fraction of incident energy reflected from the front face. Also determine the fraction of the incident energy lost in the dielectric window.

6.8g A slab of dielectric of length l, constants ε' and ε'', is backed by a conducting plane at $z = l$ which may be considered perfect. Determine the expression for field impedance at the front face, $z = 0$. Find value for $\varepsilon'/\varepsilon_0 = 4$, $\varepsilon''/\varepsilon' = 0.01$, $f = 3 \times 10^9$ Hz, $l = 1.25$ cm. Compare with the value obtained with losses neglected.

6.9a Write instantaneous forms for the field components of Eqs. 6.9(5)–(7) for TM polarization. Find the average and instantaneous components of the Poynting vector for both the x and z directions.

6.9b Repeat Prob. 6.9a for the TE polarization, defined by Eqs. 6.9(18)–(20).

6.9c For the TM polarization with E_x, E_z, and H_y, find stored energy in electric fields and in magnetic fields, for unit area, between planes $z = 0$ and $z = \pi/(k \cos \theta)$ and interpret.

6.10a Show that a generalization of Eq. 6.10(1) for a wave oblique to all three axes can be written in the convenient form

$$\mathbf{E}(x, y, z) = \mathbf{E}_+ e^{-j\mathbf{k}\cdot\mathbf{r}}$$

where

$$\mathbf{k} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z$$
$$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$$

6.10b A diffraction grating as in Ex. 6.10 has grating spacing 2 μm. Find the number and angle of the diffracted orders for an argon laser beam with $\lambda_0 = 0.488$ μm at (i) normal incidence, $\theta_1 = 0$, and (ii) incidence with $\theta_1 = 60$ degrees.

6.10c Find the modified relation for diffracted angle in Ex. 6.10 if there is dielectric ε_1 on the incident side of the grating and a different ε_2 on the exit side ($\mu = \mu_0$ for both).

6.11a Plot versus incident angle the phase and magnitude of ρ for a wave incident from air onto a ceramic with $\varepsilon_r = 4.95$ for both TM and TE polarizations.

6.11b Repeat Prob. 6.11a with the wave passing from the ceramic into air. (Note that reflection is total beyond some angle, as will be explained in Sec. 6.12.)

6.11c Obtain the special forms of ρ and τ for grazing incidence ($\theta = \pi/2 - \delta$ with $\delta << 1$) for both polarizations and $\varepsilon_2 > \varepsilon_1$.

6.11d For both polarizations, give the conditions for which the standing wave pattern in z shows a minimum of tangential electric field at the boundary surface; repeat for a maximum of tangential E at the surface.

6.11e Write expressions for E_x, H_y, and E_z in region 2 as functions of x and z for polarization with **E** in the plane of incidence. Obtain the Poynting vector for each region and demonstrate the power balance. (Take $\varepsilon_2 > \varepsilon_1$ to avoid the special case to be treated in Sec. 6.12.)

6.11f A wave passes from air to a medium with $\varepsilon_r = 10$, $\mu_r = 10$. For normal incidence there is no reflection since $\eta_1 = \eta_2 = \eta_0$. There is a nonzero reflection for incident angles other than zero. Plot reflection coefficient versus incident angle for both TM and TE polarizations.

6.12a A microwave transmitter is placed below the surface of a freshwater lake. Neglecting absorption, find the cone over which you could expect radiation to pass into the air.

6.12b Using data of Fig. 13.2*b*, find the critical angle for a wave of wavelength 3 μm passing from silica into air and also for silicon into air. For both cases plot phase of reflection coefficient versus incident angle over the range $\theta_c \le \theta \le \pi/2$ for a wave polarized with **E** in the plane of incidence.

6.12c In a GaAs laser, the generated radiation ($\lambda_0 = 0.85$ μm) is "trapped" or guided along the thin junction region by total reflection from the adjacent layers. (The mechanism of dielectric guiding will be explored more in Chapters 8 and 14.) If the junction layer has refractive index $n = 3.60$, the upper layer $n = 3.45$, and the lower layer $n = 3.50$, find critical angle at upper and lower surfaces.

6.12d* Defining ψ as the phase E_{x-}/E_{x+} and ψ' as the phase of E_{y-}/E_{y+}, find expressions for ψ and ψ' under conditions of total reflection. Show that the phase difference between these two polarization components, $\delta = \psi - \psi'$, is given by

$$\tan\left(\frac{\delta}{2}\right) = \frac{(\eta_2/\eta_1)[(v_2/v_1)^2 - 1] \sin^2 \theta}{[(\eta_2/\eta_1)^2 - 1] \cos \theta \sqrt{(v_2/v_1)^2 \sin^2 \theta - 1}}$$

6.12e Optical fibers, now of importance in optical communications, guide light by the phenomenon of total reflection (as will be discussed more in Chapter 14). The evanescent fields outside of the guiding core could cause coupling or "crosstalk" between adjacent fibers. To estimate the size of this coupling, take a plane model with silica core having refractive index $n = 1.535$ and external cladding of a borosilicate glass having $n = 1.525$. The optical signal has free-space wavelength of 0.85 μm. How far away can the next core be placed if field is to be 10^{-3} the surface value at that plane, if (i) incident angle of waves within the core is 85 degrees, and (ii) if incident angle is 89 degrees.

6.12f For several applications of total reflection (note Probs. 6.12c and e), refractive indices on the two sides of the boundary are not very different. If $n_2 = n_1(1 - \delta)$ where $\delta << 1$, show that $\theta_c \approx \pi/2 - \Delta$ where $\Delta = \sqrt{2\delta}$.

6.13a In Ex. 6.13, it was found that polarizing angle for entrance to the window is also correct for exit from the window. Prove this for general ratios of n_2/n_1.

6.13b Examine Fig. 13.2*b* to determine materials which might be suitable as Brewster windows for a CO_2 laser operating at 10.6 μm. Calculate the appropriate window angle for each such material.

6.13c A green ion laser beam, operating at $\lambda_0 = 0.545$ μm, is generated in vacuum, and then passes through a glass window of refractive index 1.5 into water with $n = 1.34$. Design a window to give zero reflection at the two surfaces for a wave polarized with **E** in the plane of incidence.

6.13d For $\varepsilon_1 = \varepsilon_2$ but $\mu_1 \neq \mu_2$, show that the TE polarization will have an incident angle with zero reflection like the Brewster angle for TM polarization. Also, what conditions must be satisfied for such an angle for TE polarization if both $\varepsilon_1 \neq \varepsilon_2$ and $\mu_1 \neq \mu_2$?

6.14a An incident wave in medium 1 of permittivity ε_1 makes angle θ_1 with the normal. Find the proper length and permittivity of a medium 2 to form a "quarter-wave matching section" to a medium of permittivity ε_3. Consider both polarizations.

6.14b* A uniform plane wave of free-space wavelength 3 cm is incident from space on a window of permittivity 3 and thickness equal to a half-wavelength referred to the dielectric material so that it gives no reflections for normal incidence. For general angles of incidence, plot the fraction of incident energy reflected versus θ for polarization with **E** in the plane of incidence and also for polarization normal to the plane of incidence.

6.14c By use of the transmission-line analogies, determine the spacing between a thin film and a parallel perfect conductor, and the conductivity properties of that film if reflections are to be perfectly eliminated for a wave incident at an angle θ from the normal for the two types of polarization. (See Ex. 6.6b.)

6.14d For a series of parallel boundaries between different dielectrics, as in Fig. 6.14, show that the relation between exit angle and angle of incidence upon the first boundary is given by Snell's law utilizing indices of refraction of entrance and exit materials only, provided there is no intermediate surface at which total reflection occurs.

6.14e An optical instrument called an *ellipsometer* employs a beam of monochromatic light with elliptical polarization incident at an oblique angle on a surface whose properties are to be determined. Measurements on the incident and reflected beams yield a value of the complex ratio of reflection coefficients for components in the plane of incidence and normal to it. Find the expression for the ratio of reflection coefficients that will be measured by the ellipsometer in terms of incident angle, refractive index of the substrate supporting the film, and the unknown film thickness and refractive index. Assuming angle and substrate index known, and neglecting losses, explain how the two unknowns could be determined.