Appendix B

Linear Algebra

This chapter provides a review of basic concepts in linear algebra.

B.1 Vector spaces

You are no doubt familiar with **vectors** in $\mathbb{R}^2$ or $\mathbb{R}^3$, i.e.

$$\vec{x} = \begin{bmatrix} 2.2 \\ 3 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}. \tag{B.1}$$

From the point of view of algebra, vectors are much more general objects. They are elements of sets called **vector spaces** that satisfy the following definition.

Definition B.1.1 (Vector space)**.** *A vector space consists of a set $\mathcal{V}$ and two operations $+$ and $\cdot$ satisfying the following conditions.*

1. *For any pair of elements $\vec{x}, \vec{y} \in \mathcal{V}$ the **vector sum** $\vec{x} + \vec{y}$ belongs to $\mathcal{V}$.*
2. *For any $\vec{x} \in \mathcal{V}$ and any scalar $\alpha \in \mathbb{R}$ the **scalar multiple** $\alpha \cdot \vec{x} \in \mathcal{V}$.*
3. *There exists a **zero vector** or **origin** $\vec{0}$ such that $\vec{x} + \vec{0} = \vec{x}$ for any $\vec{x} \in \mathcal{V}$.*
4. *For any $\vec{x} \in \mathcal{V}$ there exists an additive inverse $\vec{y}$ such that $\vec{x} + \vec{y} = \vec{0}$, usually denoted as $-\vec{x}$.*
5. *The vector sum is commutative and associative, i.e. for all $\vec{x}, \vec{y} \in \mathcal{V}$*
$$\vec{x} + \vec{y} = \vec{y} + \vec{x}, \quad (\vec{x} + \vec{y}) + z = \vec{x} + (\vec{y} + z). \tag{B.2}$$
6. *Scalar multiplication is associative, for any $\alpha, \beta \in \mathbb{R}$ and $\vec{x} \in \mathcal{V}$*
$$\alpha\,(\beta \cdot \vec{x}) = (\alpha\,\beta) \cdot \vec{x}. \tag{B.3}$$
7. *Scalar and vector sums are both distributive, i.e. for all $\alpha, \beta \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathcal{V}$*
$$(\alpha + \beta) \cdot \vec{x} = \alpha \cdot \vec{x} + \beta \cdot \vec{x}, \quad \alpha \cdot (\vec{x} + \vec{y}) = \alpha \cdot \vec{x} + \alpha \cdot \vec{y}. \tag{B.4}$$