Appendix A

Set theory

This chapter provides a review of basic concepts in set theory.

A.1 Basic definitions

A **set** is a collection of objects. The set containing every possible object that we consider in a certain situation is called the **universe** and is usually denoted by Ω. If an object x in Ω belongs to set S, we say that x is an **element** of S and write $x \in S$. If x is not an element of S then we write $x \notin S$. The **empty set**, usually denoted by $\emptyset$, is a set such that $x \notin$ for all $x \in \Omega$ (i.e. it has no elements). If all the elements in a set B also belong to a set A then B is a **subset** of A, which we denote by $B \subseteq A$. If in addition there is at least one element of A that does not belong to B then B is a proper subset of A, denoted by $B \subset A$.

The elements of a set can be arbitrary objects and in particular they can be sets themselves. This is the case for the power set of a set, defined in the next section.

A useful way of defining a set is through a statement concerning its elements. Let S be the set of elements such that a certain statement $s(x)$ holds, to define S we write

$$S := \{x \mid s(x)\} . \tag{A.1}$$

For example, $A := \{x \mid 1 < x < 3\}$ is the set of all elements greater than 1 and smaller than 3. Let us define some important sets and set operations using this notation.

A.2 Basic operations

Definition A.2.1 (Set operations)**.**

- *The **complement** S^c of a set S contains all elements that are not in S.*

$$S^c := \{x \mid x \notin S\} . \tag{A.2}$$

- *The **union** of two sets A and B contains the objects that belong to A or B.*

$$A \cup B := \{x \mid x \in A \text{ or } x \in B\} . \tag{A.3}$$

This can be generalized to a sequence of sets A_1, A_2, ...

$$\bigcup_n A_n := \{x \mid x \in A_n \text{ for some } n\}, \tag{A.4}$$

where the sequence may be infinite.

- *The **intersection** of two sets A and B contains the objects that belong to A and B.*

$$A \cap B := \{x \mid x \in A \text{ and } x \in B\}. \tag{A.5}$$

Again, this can be generalized to a sequence,

$$\bigcap_n A_n := \{x \mid x \in A_n \text{ for all } n\}. \tag{A.6}$$

- *The **difference** of two sets A and B contains the elements in A that are not in B.*

$$A/B := \{x \mid x \in A \text{ and } x \notin B\}. \tag{A.7}$$

- *The **power set** 2^S of a set S is the set of all possible subsets of S, including $\emptyset$ and S.*

$$2^S := \left\{S' \mid S' \subseteq S\right\}. \tag{A.8}$$

- *The **cartesian product** of two sets S_1 and S_2 is the set of all ordered pairs of elements in the sets*

$$S_1 \times S_2 := \{(x_1, x_2) \mid x_1 \in S_1, x_2 \in S_2\}. \tag{A.9}$$

An example is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the set of all possible pairs of real numbers.

Two sets are equal if they have the same elements, i.e. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. It is easy to verify for instance that $(A^c)^c = A$, $S \cup \Omega = \Omega$, $S \cap \Omega = S$ or the following identities which are known as *De Morgan's laws.*

Theorem A.2.2 (De Morgan's laws)**.** *For any two sets A and B*

$$(A \cup B)^c = A^c \cap B^c, \tag{A.10}$$
$$(A \cap B)^c = A^c \cup B^c. \tag{A.11}$$

Proof. Let us prove the first identity; the proof of the second is almost identical.

First we prove that $(A \cup B)^c \subseteq A^c \cap B^c$. A standard way to prove the inclusion of a set in another set is to show that if an element belongs to the first set then it must also belong to the second. Any element x in $(A \cup B)^c$ (if the set is empty then the inclusion holds trivially, since $\emptyset \subseteq S$ for any set S) is in A^c; otherwise it would belong to A and consequently to $A \cup B$. Similarly, x also belongs to B^c. We conclude that x belongs to $A^c \cap B^c$, which proves the inclusion.

To complete the proof we establish $A^c \cap B^c \subseteq (A \cup B)^c$. If $x \in A^c \cap B^c$, then $x \notin A$ and $x \notin B$, so $x \notin A \cup B$ and consequently $x \in (A \cup B)^c$. □