Appendix 2
Coordinate Systems and Vector Relations

The Rectangular, Cylindrical, and Spherical Coordinate Systems

The three systems utilized in this book are rectangular coordinates, circular cylindrical coordinates, and spherical coordinates. These are defined briefly before generalizing.

The intersection of two surfaces is a line; the intersection of three surfaces is a point; thus the coordinates of a point may be given by stating three parameters, each of which defines a coordinate surface. In rectangular coordinates, the three planes $x = x_1$, $y = y_1$, $z = z_1$ intersect at a point designated by the coordinates x_1, y_1, z_1. The elements of length in the three coordinate directions are dx, dy, and dz, the elements of area are $dx\,dy$, $dy\,dz$, and $dz\,dx$, and the element of volume is $dx\,dy\,dz$.

In the circular cylindrical coordinate system, the coordinate surfaces are (1) a set of circular cylinders (r = constant), (2) a set of planes all passing through the axis (ϕ = constant), and (3) a set of planes normal to the axis (z = constant). Coordinates of a particular point may then be given as r_1, ϕ_1, z_1 (Fig. 2*a*). The r, ϕ, and z coordinates are known respectively as the radius, the azimuthal angle, and the distance along the axis. Elements of length are dr, $r\,d\phi$, and dz, and the element of volume is $r\,dr\,d\phi\,dz$. The system shown is a right-hand system in the order of writing r, ϕ, z.

In spherical coordinates the surfaces are (1) a set of spheres (radius r from the origin = constant), (2) a set of cones about the axis (θ = constant), and (3) a set of planes passing through the polar axis (ϕ = constant). The intersection of sphere $r = r_1$, cone $\theta = \theta_1$, and plane $\phi = \phi_1$ gives a point whose coordinates are said to be r_1, θ_1, ϕ_1 (Fig. 2*b*). r is the radius, θ the polar angle or colatitude, and ϕ the azimuthal angle or longitude. Elements of distance are dr, $r d\theta$, and $r \sin\theta\, d\phi$, elements of area are $r\,dr\,d\theta$, $r^2 \sin\theta\, d\theta\, d\phi$, and $r \sin\theta\, d\phi\, dr$; and the element of volume is $r^2 \sin\theta\, dr\, d\theta\, d\phi$.

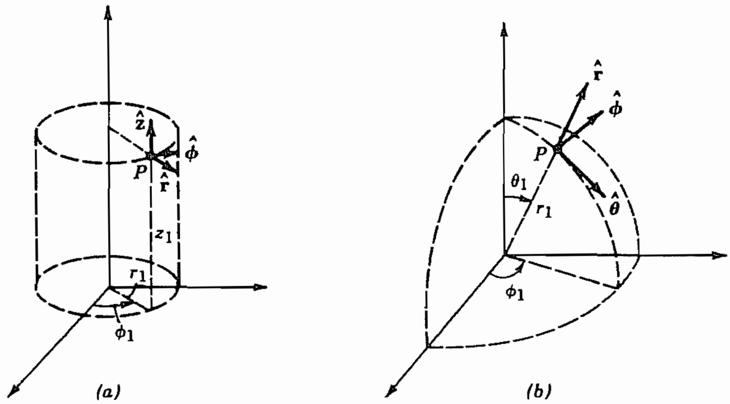

FIG. 2 (*a*) System of circular cylindrical coordinates. (*b*) System of spherical coordinates.

General Curvilinear Coordinates

Each of the preceding three systems and many others utilized in mathematical physics are orthogonal coordinate systems in that the lines of intersection of the coordinate surfaces are at right angles to one another at any given point. It is possible to develop general expressions for divergence, curl, and other vector operations for such systems which make it unnecessary to begin at the beginning each time a new system is met.

Suppose that a point in space is thus defined in any orthogonal system by the coordinate surfaces q_1, q_2, q_3. These then intersect at right angles and a set of three unit vectors, $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$, $\hat{\mathbf{3}}$ may be placed at this point. These should point in the direction of increasing coordinates (Fig. 2*c*). The three coordinates need not necessarily express directly a distance (consider, for example, the angles of spherical coordinates) so that the differential elements of distance must be expressed:

$$dl_1 = h_1\,dq_1, \qquad dl_2 = h_2\,dq_2, \qquad dl_3 = h_3\,dq_3 \tag{1}$$

where h_1, h_2, h_3 in the most general case may each be functions of all three coordinates q_1, q_2, q_3.

Scalar and Vector Products A reference to the fundamental definitions of the two vector multiplications will show that these do not change in form in orthogonal curvilinear coordinates. Thus, for scalar or dot product,

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3, \tag{2}$$

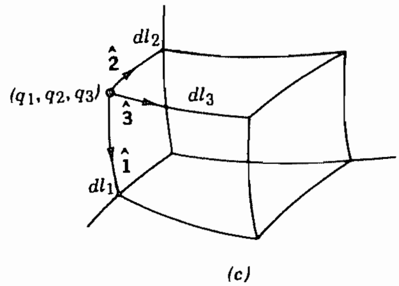

FIG. 2c Element in arbitrary orthogonal curvilinear coordinates.

and, for the vector or cross product,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{1} & \hat{2} & \hat{3} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{3}$$

When one of these vectors is replaced by the operator $\boldsymbol{\nabla}$, the foregoing expressions do not hold, as will be shown below.

Gradient According to previous definitions, the gradient of any scalar Φ will be a vector whose component in any direction is given by the change of Φ for a change in distance along that direction. Thus,

$$\boldsymbol{\nabla}\Phi = \hat{1}\,\frac{\partial\Phi}{h_1\partial q_1} + \hat{2}\,\frac{\partial\Phi}{h_2\partial q_2} + \hat{3}\,\frac{\partial\Phi}{h_3\partial q_3} \tag{4}$$

Divergence In forming the divergence, it is necessary to account for the variations in surface elements as well as the vector components when one changes a coordinate. If the product of surface element by the appropriate component is first formed and then differentiated, both of these changes are taken into account:

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \frac{1}{h_1h_2h_3\,dq_1\,dq_2\,dq_3}\left[dq_1\,\frac{\partial}{\partial q_1}(D_1h_2h_3\,dq_2\,dq_3) \right.$$
$$\left. + \,dq_2\,\frac{\partial}{\partial q_2}(D_2h_1h_3\,dq_1\,dq_3) + dq_3\,\frac{\partial}{\partial q_3}(D_3h_2h_1\,dq_2\,dq_1)\right]$$

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3D_1) + \frac{\partial}{\partial q_2}(h_1h_3D_2) + \frac{\partial}{\partial q_3}(h_2h_1D_3)\right] \tag{5}$$

Note that for the spherical coordinate system $dl_1 = dr$, $dl_2 = r\,d\theta$, and $dl_3 = r\sin\theta\,d\phi$, so that $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin\theta$. A substitution of these in (5) leads directly to Eq. 1.11(9).

Curl In forming the curl, it is necessary to account for the variations in length elements with changes in coordinates as one integrates about an elemental path. This may again be done by forming the product of length element and proper vector component and then differentiating. The result may be written

$$\nabla \times \mathbf{H} = \begin{vmatrix} \dfrac{\hat{\mathbf{1}}}{h_2h_3} & \dfrac{\hat{\mathbf{2}}}{h_3h_1} & \dfrac{\hat{\mathbf{3}}}{h_1h_2} \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1H_1 & h_2H_2 & h_3H_3 \end{vmatrix} \tag{6}$$

Laplacian The Laplacian of a scalar, which is defined as the divergence of the gradient of that scalar, may be found by combining (4) and (5):

$$\begin{aligned} \nabla^2\Phi &= \nabla \cdot \nabla\Phi \\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial q_3}\right)\right] \end{aligned} \tag{7}$$

Laplacian of Vectors For the Laplacian of a vector in a system of coordinates other than rectangular, it is convenient to use the vector identity

$$\nabla^2\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times \nabla \times \mathbf{F} \tag{8}$$

Each of the operations on the right has been defined earlier.

Differentiation of Vectors The derivative of a vector is sometimes required as in Newton's law for the motion of particles.

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\frac{d}{dt}(\hat{\mathbf{1}}v_1 + \hat{\mathbf{2}}v_2 + \hat{\mathbf{3}}v_3) \tag{9}$$

If this is expanded, we have

$$\frac{\mathbf{F}}{m} = \hat{\mathbf{1}}\frac{dv_1}{dt} + \hat{\mathbf{2}}\frac{dv_2}{dt} + \hat{\mathbf{3}}\frac{dv_3}{dt} + v_1\frac{d\hat{\mathbf{1}}}{dt} + v_2\frac{d\hat{\mathbf{2}}}{dt} + v_3\frac{d\hat{\mathbf{3}}}{dt}$$

The last three terms involve changes in the unit vectors, which by definition cannot change magnitude, but may change in direction as one moves along the coordinate system. Consider for example the fourth term:

$$v_1\frac{d\hat{\mathbf{1}}}{dt} = v_1\left(\frac{\partial\hat{\mathbf{1}}}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial\hat{\mathbf{1}}}{\partial q_2}\frac{dq_2}{dt} + \frac{\partial\hat{\mathbf{1}}}{\partial q_3}\frac{dq_3}{dt}\right)$$

Partials of the form $\partial\hat{\mathbf{1}}/\partial q_1$ and so on may be nonzero. As an example, consider the term $\partial\hat{\boldsymbol{\theta}}/\partial\theta$ in spherical coordinates. From Fig. 2*d* the vector $d\theta\, \partial\hat{\boldsymbol{\theta}}/\partial\theta$ is seen to have

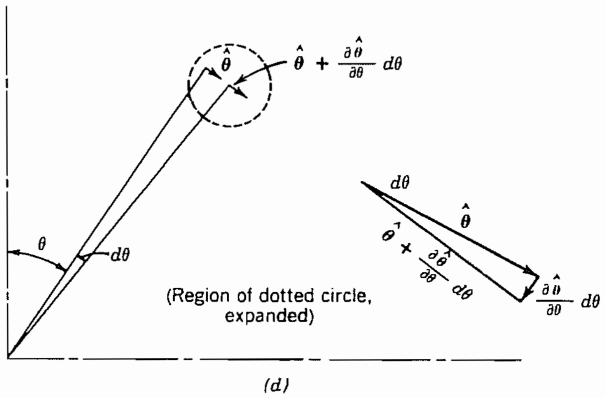

FIG. 2d

magnitude $d\theta$ and has direction given by $-\hat{\mathbf{r}}$. Thus, $\partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\mathbf{r}}$. Other partials of unit vectors in this and the cylindrical coordinate system are listed below.

Coordinates and Derivatives of Unit Vectors for Various Coordinate Systems

Rectangular Coordinates

$$q_1 = x, \qquad q_2 = y, \qquad q_3 = z$$

$$h_1 = 1, \qquad h_2 = 1, \qquad h_3 = 1$$

All partials of unit vectors ($\partial\hat{\mathbf{x}}/\partial x$, $\partial\hat{\mathbf{x}}/\partial y$, etc.) are zero.

Cylindrical Coordinates

$$q_1 = r, \qquad q_2 = \phi, \qquad q_3 = z$$

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = 1$$

All partials of unit vectors are zero except

$$\frac{\partial\hat{\mathbf{r}}}{\partial\phi} = \hat{\boldsymbol{\phi}} \qquad \frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} = -\hat{\mathbf{r}}$$

Spherical Coordinates

$$q_1 = r, \qquad q_2 = \theta, \qquad q_3 = \phi$$

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = r \sin \theta$$

All partials of unit vectors are zero except

$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}, \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\mathbf{r}}$$

$$\frac{\partial \hat{\mathbf{r}}}{\partial \phi} = \hat{\boldsymbol{\phi}} \sin \theta, \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi} = \hat{\boldsymbol{\phi}} \cos \theta, \qquad \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = -(\hat{\mathbf{r}} \sin \theta + \hat{\boldsymbol{\theta}} \cos \theta)$$