Appendix A

Set notation and concepts

A.1 Basic concepts and notation

A set is a collection of items. You can write a set by listing its elements (the items it contains) inside curly braces. For example, the set that contains the numbers 1, 2 and 3 can be written as $\{1, 2, 3\}$. The order of elements do not matter in a set, so the same set can be written as $\{2, 1, 3\}$, $\{2, 3, 1\}$ or using any permutation of the elements. The number of occurrences also does not matter, so we could also write the set as $\{2, 1, 2, 3, 1, 1\}$ or an infinity of other ways. All of these describe the same set. We will normally write sets without repetition, but the fact that repetitions do not matter is important to understand the operations on sets.

We will typically use uppercase letters to denote sets and lowercase letters to denote elements in a set, so we could write $M = \{2, 1, 3\}$ and $x = 2$ as an element of M. The empty set can be written either as an empty list of elements ($\{\}$) or using the special symbol $\emptyset$. The latter is more common in mathematical texts.

A.1.1 Operations and predicates

We will often need to check if an element belongs to a set or select an element from a set. We use the same notation for both of these: $x \in M$ is read as "x is an element of M" or "x is a member of M". The negation is written as $x \notin M$, which is read as "x is not an element of M" or "x is not a member of M".

We can use these in conditional statements like "if $3 \in M$ then ...", for asserting a fact "since $x \notin M$, we can conclude that ..." or for selecting an element from a set: "select $x \in M$", which will select an arbitrary element from M and let x be equal to this element.

We can combine two sets M and N into a single set that contains all elements from both sets. We write this as $M \cup N$, which is read as "M union N" or "the union of M and N". For example, $\{1, 2\} \cup \{5, 1\} = \{1, 2, 5, 1\} = \{1, 2, 5\}$. The following statement holds for membership and union:

$$x \in (M \cup N) \Leftrightarrow x \in M \vee x \in N$$

where $\Leftrightarrow$ is bi-implication ("if and only if") and $\vee$ is logical disjunction ("or").

We can also combine two sets M and N into a set that contains only the elements that occur in both sets. We write this as $M \cap N$, which is read as "M intersect N" or "the intersection of M and N". For example, $\{1, 2\} \cap \{5, 1\} = \{1\}$. The following statement holds for membership and intersection:

$$x \in (M \cap N) \Leftrightarrow x \in M \wedge x \in N$$

where $\wedge$ is logical conjunction ("and").

We can also talk about set difference (or set subtraction), which is written as $M \setminus N$, which is read as "M minus N" or "M except N'. $M \setminus N$ contains all the elments that are members or M but not members of N. For example, $\{1, 2\} \setminus \{5, 1\} = \{2\}$. The following statement holds for membership and set difference:

$$x \in (M \setminus N) \Leftrightarrow x \in M \wedge x \notin N$$

Just like arithmetic operators, set operators have precedence rules: $\cap$ binds more tightly than $\cup$ (just like multiplication binds tighter than addition). So writing $A \cup B \cap C$ is the same as writing $A \cup (B \cap C)$. Set difference has the same precedence as union (just like subtraction has the same precedence as addition).

If all the elements of a set M are also elements of a set N, we call M a subset of N, which is written as $M \subseteq N$. This can be defined by

$$M \subseteq N \Leftrightarrow (x \in M \Rightarrow x \in N)$$

where $\Rightarrow$ is logical implication ("only if").

The converse of subset is superset: $M \supseteq N \Leftrightarrow N \subseteq M$.

A.1.2 Properties of set operations

Just like we have mathematical laws saying that, for example $x + y = y + x$, there are also similar laws for set operations. Here is a selection of the most commonly used laws:

$A \cup A = A$	union is idempotent
$A \cap A = A$	intersection is idempotent
$A \cup B = B \cup A$	union is commutative
$A \cap B = B \cap A$	intersection is commutative
$A \cup (B \cup C) = (A \cup B) \cup C$	union is associative
$A \cap (B \cap C) = (A \cap B) \cap C$	intersection is associative
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	union distributes over intersection
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	intersection distributes over union
$A \cup \emptyset = A$	the empty set is a unit element of union
$A \cap \emptyset = \emptyset$	the empty set is a zero element of intersection
$A \subseteq B \Leftrightarrow A \cup B = B$	subset related to union
$A \subseteq B \Leftrightarrow A \cap B = A$	subset related to intersection
$A \subseteq B \Leftrightarrow A \setminus B = \emptyset$	subset related to set difference
$A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B$	subset is antisymmetric
$A \subseteq B \wedge B \subseteq C \Leftrightarrow A \subseteq C$	subset is transitive
$A \setminus (B \cup C) = (A \setminus B) \setminus C$	corresponds to $x-(y+z)=(x-y)-z$

Since $\cup$ and $\cap$ are associative, we will often omit parentheses and write, *e.g.*, $A \cup B \cup C$ or $A \cap B \cap C$.

A.2 Set-builder notation

We will often build a new set by selecting elements from other sets and doing operations on these elements. We use the very flexible set-builder notation for this. A set builder has the form $\{e \mid p\}$, where e is an expression and p is a list of predicates separated by commas. Typically, p will contain predicates of the form $x \in M$, which defines x to be any element of M. The set builder will evaluate the expression e for all elements x of M that fulfills the other predicates in p and build a set of the results. We read $\{e \mid p\}$ as "the set of all elements of the form e where p holds", or just "e where p". Some mathematical texts use a colon instead of a bar, *i.e.*, writing $\{e : p\}$ instead of $\{e \mid p\}$.

A simple example is

$$\{x^3 \mid x \in \{1, 2, 3, 4\},\ x < 3\}$$

which builds the set $\{1^3, 2^3\} = \{1, 8\}$, as only the elements 1 and 2 from the set $\{1, 2, 3, 4\}$ fulfill the predicate $x < 3$.

We can take elements from more than one set, for example

$$\{x+y \mid x \in \{1, 2, 3\},\ y \in \{1, 2, 3\},\ x < y\}$$

which builds the set $\{1+2, 1+3, 2+3\} = \{3, 4, 5\}$. All combinations of elements from the two sets that fulfill the predicate are used.

We can separate the predicates in a set builder by $\wedge$ or "and" instead of commas. So the example above can, equivalently, be written as

$$\{x+y \mid x \in \{1,2,3\},\ y \in \{1,2,3\} \text{ and } x < y\}$$

A.3 Sets of sets

The elements of a set can be other sets, so we can, for example, have the set $\{\{1,2\},\{2,3\}\}$ which is a set that has the two sets $\{1,2\}$ and $\{2,3\}$ as elements. We can "flatten" a set of sets to a single set which is the union of the element sets using the "big union" operator:

$$\bigcup\{\{1,2\},\{2,3\}\} = \{1,2,3\}$$

Similarly, we can take the intersection of the element sets using the "big intersection" operator:

$$\bigcap\{\{1,2\},\{2,3\}\} = \{2\}$$

We can use these "big" operators together with set builders, for example

$$\bigcap\{\{x^n \mid n \in \{0,1,2\}\} \mid x \in \{1,2,3\}\}$$

which evaluates to $\bigcap\{\{1\},\{1,2,4\},\{1,3,9\}\} = \{1\}$.

When a big operator is used in combination with a set builder, a special abbreviated notation can be used: $\bigcup\{e \mid p\}$ and $\bigcap\{e \mid p\}$can be written, respectively, as

$$\bigcup_p e \quad \text{and} \quad \bigcap_p e$$

For example,

$$\bigcap\{\{x^n \mid n \in \{0,1,2\}\} \mid x \in \{1,2,3\}\}$$

can be written as

$$\bigcap_{x \in \{1,2,3\}} \{x^n \mid n \in \{0,1,2\}\}$$

A.4 Set equations

Just like we can have equations where the variables represent numbers, we can have equations where the variables represent sets. For example, we can write the equation

$$X = \{x^2 \mid x \in X\}$$

This particular equation has several solutions, including $X = \{0\}$, $X = \emptyset$ and $X = \{0, 1\}$ or even $X = [0, 1]$, where $[0, 1]$ represents the interval of real numbers between 0 and 1. Usually, we have an implied universe of elements that the sets can draw from. For example, we might only want sets of integers as solutions, so we don't consider intervals of real numbers as valid solutions.

When there are more solutions, we are often interested in a solution that has the minimum or maximum possible number of elements. In the above example (assuming we want sets of integers), there is a unique minimal (in terms of number of elements) solution, which is $X = \emptyset$ and a unique maximal solution $X = \{0, 1\}$. Not all equations have unique minimal or maximal solutions. For example, the equation

$$X = \{1, 2, 3\} \setminus X$$

has no solution at all, and the equation

$$X = \{1, 2, 3\} \setminus \{6/x \mid x \in X\})$$

has exactly two solutions: $X = \{1, 2\}$ and $X = \{1, 3\}$, so there are no unique minimal or maximal solutions.

A.4.1 Monotonic set functions

The set equations we have seen so far are of the form $X = F(X)$, where F is a function from sets to sets. A solution to such an equation is called a *fixed-point* for F.

As we have seen, not all such equations have solutions, and when they do, there are not always unique minimal or maximal solutions. We can, however, define a property of the function F that guarantees a unique minimal and a unique maximal solution to the equation $X = F(X)$.

We say that a set function F is *monotonic* if $X \subset Y \Rightarrow F(X) \subseteq F(Y)$.

Theorem A.1 *If we draw elements from a finite universe U and F is a monotonic function over sets of elements from U, then there exist natural numbers m and n, so the unique minimal solution to the equation $X = F(X)$ is equal to $F^m(\emptyset)$ and the unique maximal solution to the equation $X = F(X)$ is equal to $F^n(U)$.*

where $F^i(A)$ is F applied i times to A. For example $F^3(A) = F(F(F(A)))$.

Proof: It is trivially true that $\emptyset \subseteq F(\emptyset)$. Since F is monotonic, this implies $F(\emptyset) \subseteq F(F(\emptyset))$. This again implies $F(F(\emptyset)) \subseteq F(F(F(\emptyset)))$ and, by induction, $F^i(\emptyset) \subseteq F^{i+1}(\emptyset)$. So we have a chain

$$\emptyset \subseteq F(\emptyset) \subseteq F(F(\emptyset)) \subseteq F(F(F(\emptyset))) \subseteq \cdots$$

Since the universe U is finite, the sets $F^i(\emptyset)$ can not all be different. Hence, there exist an m such that $F^m(\emptyset) = F^{m+1}(\emptyset)$, which means $X = F^m(\emptyset)$ is a solution to the equation $X = F(X)$. To prove that it is the unique minimal solution, assume that another solution A exist. Since $A = F(A)$, we have $A = F^m(A)$. Since $\emptyset \subseteq A$ and F is monotonic, we have $F^m(\emptyset) \subseteq F^m(A) = A$. This implies that $F^m(\emptyset)$ is a subset of all solutions to the equation $X = F(X)$, so there can not be a minimal solution different from $F^m(\emptyset)$.

The proof for the maximal solution is left as an exercise.

fixed-point iteration

The proof provides an algorithm for finding minimal solutions to set equations of the form $X = F(X)$, where F is monotonic and the universe is finite: Simply compute $F(\emptyset)$, $F^2(\emptyset)$, $F^3(\emptyset)$ and so on until $F^{m+1}(\emptyset) = F^m(\emptyset)$. This is easy to implement on a computer:

```
X := ∅;
repeat
   Y := X;
   X := F(X)
until X = Y;
return X
```

A.4.2 Distributive functions

A function can have a stronger property than being monotonic: A function F is *distributive* if $F(X \cup Y) = F(X) \cup F(Y)$ for all sets X and Y. This clearly implies monotonicity, as $Y \supseteq X \Leftrightarrow Y = X \cup Y \Rightarrow F(Y) = F(X \cup Y) = F(X) \cup F(Y) \supseteq F(X)$.

We also solve set equations over distributive functions with fixed-point iteration, but we exploit the distributivity to reduce the amount of computation we must do: If we need to compute $F(A \cup B)$ and we have already computed $F(A)$, then we need only compute $F(B)$ and add the elements from this to $F(A)$. We can implement an algorithm for finding the minimal solution that exploits this:

```
X := ∅;
W := F(∅);
while W ≠ ∅ do
   pick x ∈ W;
   W := W\{x};
   X := X ∪ {x};
   W := W ∪ (F({x})\X);
return X
```

We keep a work set W that by invariant is equal to $F(\mathtt{X}) \setminus \mathtt{X}$. A solution must include any $x \in \mathtt{W}$, so we move this from W to X while keeping the invariant by adding $F(x) \setminus \mathtt{X}$ to W. When W becomes empty, we have $F(\mathtt{X}) = \mathtt{X}$ and, hence, a solution. While the algorithm is more complex than the simple fixed-point algorithm, we can compute F one element at a time and we avoid computing F twice for the same element.

A.4.3 Simultaneous equations

We sometimes need to solve several simultaneous set equations:

$$\begin{array}{rcl} X_1 & = & F_1(X_1,\ldots,X_n) \\ & \vdots & \\ X_n & = & F_n(X_1,\ldots,X_n) \end{array}$$

If all the F_i are monotonic in all arguments, we can solve these equations using fixed-point iteration. To find the unique minimal solution, start with $X_i = \emptyset$ and then iterate applying all F_i until a fixed-point is reached. The order in which we do this doesn't change the solution we find (it will always be the unique minimal solution), but it might affect how fast we find the solution. Generally, we need only recompute X_i if a variable used by F_i changes.

If all F_i are distributive in all arguments, we can use a work-set algorithm similar to the algorithm for a single distributive function.

Exercises

Exercise A.1

What set is built by the set builder

$$\{x^2 + y^2 \mid x \in \{1, 2, 3, 4\},\ y \in \{1, 2, 3, 4\},\ x < y^2\}?$$

Exercise A.2

What set is built by the set expression

$$\bigcup_{x \in \{1,2,3\}} \{x^n \mid n \in \{0, 1, 2\}\}?$$

Exercise A.3

Find all solutions to the equation

$$X = \{1, 2, 3\} \setminus \{x+1 \mid x \in X\})$$

Hint: Any solution must be a subset of $\{1, 2, 3\}$.

Exercise A.4

Prove that if elements are drawn from a finite universe U and F is a monotonic function over sets of elements from U, then there exists an n such that $X = F^n(U)$ is the unique maximal solution to the set equation $X = F(X)$.