Appendix 3 Sketch of the Derivation of Magnetic Field Laws

In presenting various forms for the laws of static magnetic fields, many of the less obvious steps were left out, and the order was chosen as that most convenient for presentation of the laws rather than that of the logical development. This appendix sketches the omitted steps.

We will start from Ampère's law, Eq. 2.3(2), which we may write

$$\mathbf{H}(\mathbf{r}) = \int \frac{I'(\mathbf{r}')\,\mathbf{dl}' \times \mathbf{R}}{4\pi R^3} \tag{1}$$

by assuming the origin of the coordinate system to be at the current element at each point through the integration. $I'(\mathbf{r}')$ is the current in a contributing element $\mathbf{dl}'$ at point (x', y', z'), and $\mathbf{R}$ is the vector running from $\mathbf{dl}'$ to point (x, y, z) at which $\mathbf{H}(\mathbf{r})$ is to be computed:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \hat{\mathbf{x}}(x - x') + \hat{\mathbf{y}}(y - y') + \hat{\mathbf{z}}(z - z')$$

We wish to find $\mathbf{B} = \mu\mathbf{H}$ in terms of the derivatives at the point of observation (x, y, z) of the vector potential $\mathbf{A}$. It may be shown that

$$\frac{\mathbf{dl}' \times \mathbf{R}}{R^3} = \nabla\left(\frac{1}{R}\right) \times \mathbf{dl}' \tag{2}$$

where ∇ denotes derivatives with respect to x, y, and z. Also, using the vector identity of Prob. 2.6b,

$$\nabla\left(\frac{1}{R}\right) \times \mathbf{dl}' = \nabla \times \left(\frac{\mathbf{dl}'}{R}\right) - \frac{1}{R}\nabla \times \mathbf{dl}' \tag{3}$$

Since ∇ represents derivatives with respect to x, y, and z, which are not involved with $\mathbf{dl}'$, the last term is zero. Therefore

$$\mathbf{B}(\mathbf{r}) = \mu \int \frac{I'(\mathbf{r}')}{4\pi} \nabla \times \left(\frac{\mathbf{dl}'}{R}\right) = \nabla \times \mathbf{A}(\mathbf{r}) \tag{4}$$

where

$$\mathbf{A}(\mathbf{r}) = \mu \int \frac{I'(\mathbf{r}')\mathbf{dl}'}{4\pi R} \tag{5}$$

The curl operation in (4) could be taken outside of the integral since it is with respect to $\mathbf{r}$, and the integration is with respect to $\mathbf{r}'$. Thus, the vector potential forms of Sec. 2.9 have been derived from Ampère's law for which the arbitrary coordinate origin was chosen at the current element for simplicity.

For the next step, let us note the x component of (5):

$$A_x(\mathbf{r}) = \mu \int \frac{I'_x(\mathbf{r}')\,dx'}{4\pi R} = \mu \int_{V'} \frac{J'_x(\mathbf{r}')\,dV'}{4\pi R} \tag{6}$$

This may be compared with Poisson's equation and the integral expression for electrostatic potential:

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon}, \qquad \Phi(\mathbf{r}) = \int_{V'} \frac{\rho(\mathbf{r}')\,dV'}{4\pi\varepsilon R} \tag{7}$$

Although these equations were obtained from a consideration of the properties of electrostatic fields, the second of the two equations, (7), may be considered a solution in integral form for the first, for any continuous scalar functions Φ and ρ/ε. Consequently, by direct analogy between (6) and (7), we write

$$\nabla^2 A_x = -\mu J_x \tag{8}$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{9}$$

This was the differential equation relating $\mathbf{A}$ to current density discussed in Sec. 2.12. By reversing the steps of that section, the differential equation for magnetic field may be derived from it:

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{10}$$

And as was shown in Sec. 2.8, the integral form may be derived from this by use of Stokes's theorem:

$$\oint \mathbf{H} \cdot \mathbf{dl} = I \tag{11}$$

There remains the argument for $\nabla \cdot \mathbf{A} = 0$ used in Sec. 2.12. In this it is necessary to make use of the del operator with respect to both $\mathbf{r}$ and $\mathbf{r}'$. The former will be denoted ∇, and the latter ∇'. Recall that the integration is with respect to the primed variables. Then,

$$\frac{1}{\mu}\nabla \cdot \mathbf{A} = \nabla \cdot \int_{V'} \frac{\mathbf{J}'(\mathbf{r}')\,dV'}{4\pi R} = \int_{V'} \nabla \cdot \left(\frac{\mathbf{J}'(\mathbf{r}')}{R}\right)\frac{dV'}{4\pi} \tag{12}$$

Using a vector equivalence of Prob. 1.11a,

$$\frac{1}{\mu}\nabla \cdot \mathbf{A} = -\int_{V'} \left[\frac{\nabla \cdot \mathbf{J}'(\mathbf{r}')}{R} + \mathbf{J}'(\mathbf{r}') \cdot \nabla\left(\frac{1}{R}\right)\right] \frac{dV'}{4\pi}$$

The first term is zero since $\mathbf{J}'$ is not a function of $\mathbf{r}$. In the latter term, from the definition of R, we can write

$$\nabla\left(\frac{1}{R}\right) = -\nabla'\left(\frac{1}{R}\right)$$

Then,

$$\begin{aligned} \frac{1}{\mu}\nabla \cdot \mathbf{A} &= -\int_{V'} \mathbf{J}'(\mathbf{r}') \cdot \nabla'\left(\frac{1}{R}\right) \frac{dV'}{4\pi} \\ &= \int_{V'} \left[\frac{1}{R}\nabla' \cdot \mathbf{J}'(\mathbf{r}') - \nabla' \cdot \frac{\mathbf{J}'(\mathbf{r}')}{R}\right] \frac{dV'}{4\pi} \end{aligned}$$

In the last step we have again used a vector equivalence of Prob. 1.11a. The first term is zero because we are concerned with direct currents, which, by continuity, give $\nabla' \cdot \mathbf{J}' = 0$. The second term is transformable to a surface integral by the divergence theorem. Thus,

$$\frac{1}{\mu}\nabla \cdot \mathbf{A} = \oint_{S'} \frac{\mathbf{J}'(\mathbf{r}')}{4\pi R} \cdot d\mathbf{S}' \tag{13}$$

But, if the surface encloses all the current, as it must, there can be no current flow through the surface and the result in (13) is seen to be zero. Thus all the major laws given have been shown to follow from the original experimental law of Ampère. It should be noted that the argument given is for a homogeneous medium (permeability not a function of position). For an inhomogeneous or anisotropic medium, the equations forming the fundamental starting point are

$$\nabla \times \mathbf{H} = \mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0 \tag{14}$$

and for an anisotropic medium, $\mathbf{B}$ and $\mathbf{H}$ are not related by a simple constant.