1 Stationary Electric Fields

1.1 INTRODUCTION

Electric fields have their sources in electric charges—electrons and ions. Nearly all real electric fields vary to some extent with time, but for many problems the time variation is slow and the field may be considered stationary in time (static) over the interval of interest. For still other important cases (called *quasistatic*) the spatial distribution is nearly the same as for static fields even though the actual fields may vary rapidly with time. Because of the number of important cases in each of these classes, and because static field concepts are simple and thus good for reviewing the vector operations needed to describe fields quantitatively, we start with static electric fields in this chapter and static magnetic fields in the next. The student approaching the problem in this way must remember that these are special cases, and that the interactions between time-varying fields can give rise to other phenomena, most notably the wave phenomena to be studied later in the text.

Before beginning the quantitative development of this chapter, let us comment briefly on a few applications to illustrate the kinds of problems that arise in electrostatics or quasistatics. Electron and ion guns are good examples of electrostatic problems where the distribution of fields is of great importance in the design. Electrode shapes are designed to accelerate particles from a source and focus them into a beam of desired size and velocity. Electron guns are used in cathode-ray oscilloscopes, in television tubes, in the microwave traveling wave tubes of radar and satellite communication systems, in electron microscopes, and for electron-beam lithography used for precision definition of integrated-circuit device features.

Many electronic circuit elements may have quite rapidly varying currents and voltages and yet at any instant have fields that are well represented by those calculated from static field equations. This is generally true when the elements are small in comparison with wavelength. The passive capacitive, inductive, and resistive elements are thus commonly analyzed by such *quasistatic* laws, up to very high frequencies; so also are the semiconductor diodes and transistors which constitute the active elements of electronic circuits.

Transmission lines, including the strip line used in microwave and millimeter-wave integrated circuits even for frequencies well above 10 GHz, have properties that can be calculated using the laws for static fields. This is far from being a static problem, but we will see later in the text that for systems having no structural variations along one axis (along the transmission line), the fields in the transverse plane satisfy, exactly or nearly exactly, static field laws.

There are many other examples of application of knowledge of static field laws. The electrostatic precipitators used to remove dust and other solid particles from air, xerography, and power switches and transmission systems (which must be designed to avoid dielectric breakdown) all use static field concepts. Electric fields generated by the human body are especially interesting examples. Thus the fields that are detected by electroencephalography (fields of the brain) and electrocardiography (fields of the heart) are of sufficiently low frequency to be distributed in the body in the same way that static fields would be.

In all the examples mentioned, the general problem is that of finding the distribution of fields produced by given sources in a specified medium with defined boundaries on the region of interest. Our approach will be to start with a simple experimental law (Coulomb's law) and then transform it into other forms which may be more general or more useful for certain classes of problems.

Most readers will have met this material before in physics courses or introductory electromagnetics courses, so the approach will be that of review with the purposes of deepening physical understanding and improving familiarity with the needed vector algebra before turning to the more difficult time-varying problems.

Basic Laws and Concepts of Electrostatics

1.2 Force Between Electric Charges: The Concept of Electric Field

It was known from ancient times that electrified bodies exert forces upon one another. The effect was quantified by Charles A. Coulomb through brilliant experiments using a torsion balance.[1] His experiments showed that like charges repel one another whereas opposite charges attract; that force is proportional to the product of charge magnitudes;

[1] *An excellent description of Coulomb's experiments and the groundwork of earlier researchers is given in R. S. Elliott,* Electromagnetics, *McGraw-Hill, New York, 1966. For a detailed account of the history of this and other aspects of electromagnetics, see E. T. Whittaker,* A History of the Theories of Aether and Electricity, *Am. Inst. Physics, New York, 1987, or P. F. Mottelay,* Biographical History of Electricity and Magnetism, *Ayer Co. Publishers, Salem, NH, 1975.*

that force is inversely proportional to the square of the distance between charges; and that force acts along the line joining charges. Coulomb's experiments were done in air, but later generalizations show that force also depends upon the medium in which charges are placed. Thus force magnitude may be written

$$f = K\frac{q_1 q_2}{\varepsilon r^2} \tag{1}$$

where q_1 and q_2 are charge strengths, r is the distance between charges, ε is a constant representing the effect of the medium, and K is a constant depending upon units. Direction information is included by writing force as a vector $\mathbf{f}$ (denoted here as boldface) and defining a vector $\hat{\mathbf{r}}$ of unit length pointing from one charge directly away from the other:

$$\mathbf{f} = K\frac{q_1 q_2}{\varepsilon r^2}\hat{\mathbf{r}} \tag{2}$$

Various systems of units have been used, but that to be used in this text is the International System (SI for the equivalent in French) introduced by Giorgi in 1901. This is a meter–kilogram–second (mks) system, but the great advantage is that electric quantities are in the units actually measured: coulombs, volts, amperes, etc. Conversion factors to the classical systems still used in many references are given in Appendix 1. Thus in the SI system, force in (2) is in newtons (kg-m/s^2), q in coulombs, r in meters, and ε in farads/meter. The constant K is chosen as $1/4\pi$ and the value of ε for vacuum found from experiment is

$$\varepsilon_0 = 8.854 \times 10^{-12} \approx \frac{1}{36\pi} \times 10^{-9}\,\frac{F}{m} \tag{3}$$

For other materials,

$$\varepsilon = \varepsilon_r \varepsilon_0 \tag{4}$$

where ε_r is the *relative permittivity* or *dielectric constant* of the material and is the value usually tabulated in handbooks. Here we are considering materials for which ε is a scalar independent of strength and direction of the force and of position. More general media are discussed in Sec. 1.3 and considered in more detail in Chapter 13. Thus in SI units Coulomb's law is written

$$\mathbf{f} = \frac{q_1 q_2}{4\pi\varepsilon r^2}\hat{\mathbf{r}} \tag{5}$$

Generalizing from the example of two charges, we deduce that a charge placed in the vicinity of a system of charges will also experience a force. This might be found by adding vectorially the component forces from the individual charges of the system, but it is convenient at this time to introduce the concept of an *electric field* as the force per unit charge for each point of the region influenced by charges. We may define this by introducing a test charge Δq at the point of definition small enough not to disturb

the charge distribution to be studied. Electric field $\mathbf{E}$ is then

$$\mathbf{E} = \frac{\mathbf{f}}{\Delta q} \tag{6}$$

where $\mathbf{f}$ is the force acting on the infinitesimal test charge Δq.

The electric field arising from a point charge q in a homogeneous dielectric is then given by the force law (5):

$$\mathbf{E} = \frac{q}{4\pi\varepsilon r^2}\,\hat{\mathbf{r}} \tag{7}$$

Since $\hat{\mathbf{r}}$ is the unit vector directed from the point in a direction away from the charge, the electric field vector is seen to point away from positive charges and toward negative charges as seen in the lower half of Fig. 1.2*a*. The units of electric field magnitude in the SI system are in volts per meter, as may be found by substituting units in (7):

$$E = \frac{\text{coulombs meter}}{\text{farads (meter)}^2} = \frac{\text{volts (V)}}{\text{meter (m)}}$$

We can see from the form of (7) that the total electric field for a system of point charges may be found by adding vectorially the fields from the individual charges, as is illustrated at point P of Fig. 1.2*a* for the charges q and $-q$. In this manner the electric field vector could be found for any point in the vicinity of the two charges. An *electric*

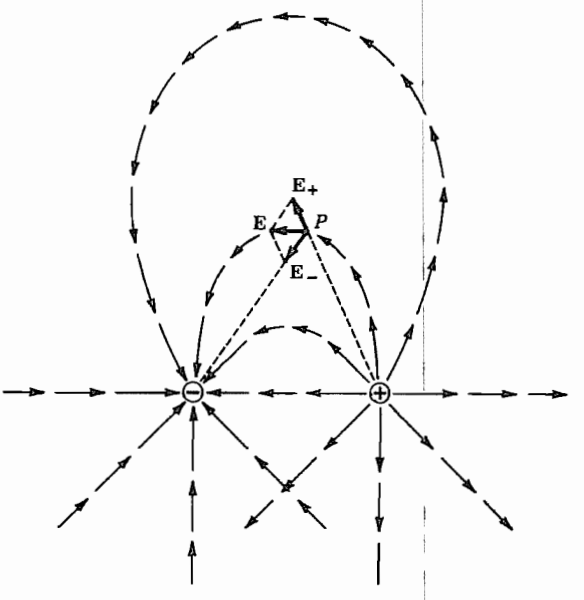

FIG. 1.2*a* Electric fields around two opposite charges. Lower half of figure shows the separate fields $\mathbf{E}_+$ and $\mathbf{E}_-$ of the two charges. Upper half shows the vector sum of $\mathbf{E}_+$ and $\mathbf{E}_-$. Construction of $\mathbf{E}$ is shown at one point P.

field line is defined as a line drawn tangent to the electric field vector at each point in space. If the vector is constructed for enough points of the region, the electric field lines can be drawn in roughly by following the direction of the vectors as illustrated in the top half of Fig. 1.2*a*. Easier methods of constructing the electric field will be studied in later sections, but the present method, although laborious, demonstrates clearly the meaning of the electric field lines.

For fields produced by continuous distributions of charges, the superposition is by integration of field contributions from the differential elements of charge. For volume distributions, the elemental charge dq is $\rho\, dV$ where ρ is charge per unit volume (C/m^3) and dV is the element of volume. For surface distributions, a surface density ρ_s (C/m^2) is used with elemental surface dS. For filamentary distributions, a linear density ρ_l (C/m) is used with elemental length dl. An example of a continuous distribution follows.

Example 1.2
FIELD OF A RING OF CHARGE

Let us calculate the electric field at points on the z axis for a ring of positive charge of radius a located in free space concentric with and perpendicular to the z axis, as shown in Fig. 1.2*b*. The charge ρ_l along the ring is specified in units of coulombs per meter so the charge in a differential length is $\rho_l\, dl$. The electric field of $\rho_l\, dl$ is designated by **dE** in Fig. 1.2*b* and is given by (7) with $r^2 = a^2 + z^2$. The component along the z axis is $dE \cos\theta$ where $\cos\theta = z/(a^2 + z^2)^{1/2}$. Note that, by symmetry, the component perpendicular to the axis is canceled by that of the charge element on the opposite side of the ring. The total field at points on the axis is thus directed along the axis and is the integral of the differential axial components. Taking $dl = a\, d\phi$ we have

$$E = \int_0^{2\pi} \frac{\rho_l a z\, d\phi}{4\pi\varepsilon(a^2 + z^2)^{3/2}} = \frac{\rho_l a z}{2\varepsilon(a^2 + z^2)^{3/2}} \tag{8}$$

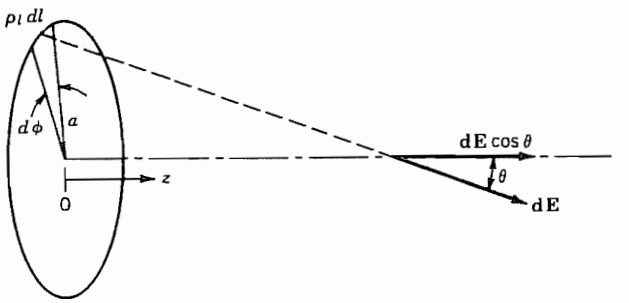

FIG. 1.2*b* Electric field of a ring of charge.

1.3 The Concept of Electric Flux and Flux Density: Gauss's Law

It is convenient in handling electric field problems to introduce another vector more directly related to charges than is electric field **E**. If we define

$$\mathbf{D} = \varepsilon\mathbf{E} \tag{1}$$

we notice from Eq. 1.2(7) that **D** about a point charge is radial and independent of the material. Moreover, if we multiply the radial component D_r by the area of a sphere of radius r, we obtain

$$4\pi r^2 D_r = q \tag{2}$$

We thus have a quantity exactly equal to the charge (in coulombs) so that it may be thought of as the *flux* arising from that charge. **D** may then be thought of as the *electric flux density* (C/m^2). For historical reasons it is also known as the *displacement vector* or *electric induction.*

It is easy to show (Prob. 1.3*b*) that for an arbitrarily shaped closed surface as in Fig. 1.3*a*, the normal component of **D** integrated over the surface surrounding a point charge also gives q. The analogy is that of fluid flow in which the fluid passing surface S of Fig. 1.3*b* in a given time is the same as that passing plane $S_\perp$ perpendicular to the flow. So fluid flow rate out of a region does not depend upon the shape of the surface used to monitor it, so long as all surfaces enclose the same source. By superposition, the result can be extended to a system of point charges or a continuous distribution of charges, leading to the conclusion

$$\textit{electric flux flowing out of a closed surface} = \textit{charge enclosed} \tag{3}$$

This is Gauss's law and, although argued here from Coulomb's law for simple media, is found to apply to more general media. It is thus a most general and important law. Before illustrating its usefulness, let us look more carefully at the role of the medium.

A simplified picture showing why the force between charges depends upon the presence of matter is illustrated in Fig. 1.3*c*. The electron clouds and the nuclei of the atoms experience oppositely directed forces as a result of the presence of the isolated charges.

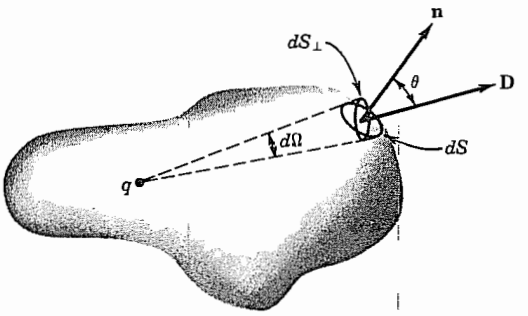

Fig. 1.3*a* Charge q and arbitrary surrounding surface.

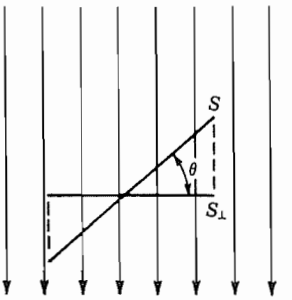

FIG. 1.3*b* Flow of a fluid through surfaces S and $S_\perp$.

Thus the atoms are distorted or *polarized.* There is a shift of the center of symmetry of the electron cloud with respect to the nucleus in each atom as indicated schematically in Fig. 1.3*c*. Similar distortions can occur in molecules, and an equivalent situation arises in some materials where naturally polarized molecules have a tendency to be aligned in the presence of free charges. The directions of the polarization are such for most materials that the equivalent charge pairs in the atoms or molecules tend to counteract the forces between the two isolated charges. The magnitude and the direction of the polarization depend upon the nature of the material.

The above qualitative picture of polarization introduced by a dielectric may be quantified by giving a more fundamental definition than (1) between **D** and **E**:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{4}$$

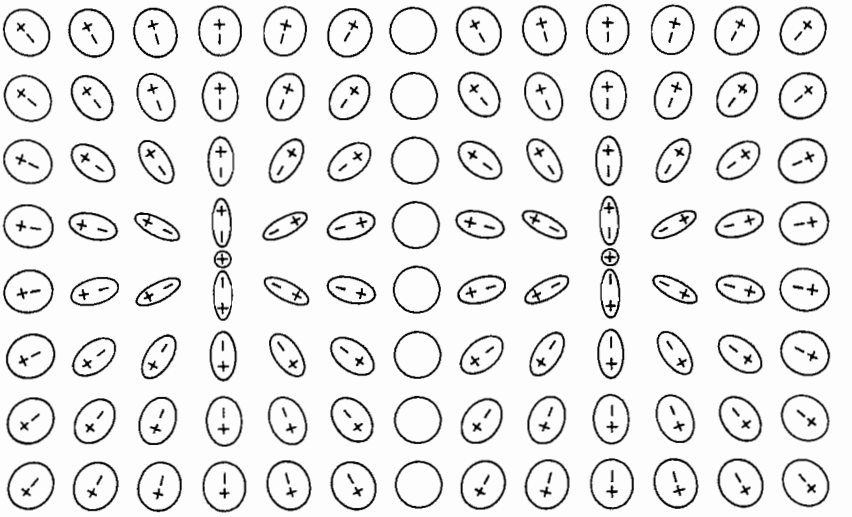

FIG. 1.3*c* Polarization of the atoms of a dielectric by a pair of equal positive charges.

The first term gives the contribution that would exist if the electric field at that point were in free space and the second measures the effect of the polarization of the material (as in Fig. 1.3*c*) and is called the *electric polarization.*

The polarization produced by the electric field in a material depends upon material properties. If the properties do not depend upon position, the material is said to be *homogeneous.* Most field problems are solved assuming homogeneity; inhomogeneous media, exemplified by the earth's atmosphere, are more difficult to analyze. If the response of the material is the same for all directions of the electric field vector, it is called *isotropic.* Special techniques are required for handling a field problem in an anisotropic medium such as an ionized gas with an applied steady magnetic field. A material is called *linear* if the ratio of the response **P** to the field **E** is independent of amplitude. Nonlinearities are generally not present except for high-amplitude fields. It is also possible that the character of a material may be *time variable*, imposed, for example, by passing a sound wave through it. Throughout most of this text, the media will be considered to be homogeneous, isotropic, linear, and time invariant. Exceptions will be studied in the final chapters.

For isotropic, linear material the polarization is proportional to the field intensity and we can write the linear relation

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \tag{5}$$

where the constant χ_e is called the *electric susceptibility.* Then (4) becomes the same as (1),

$$\mathbf{D} = \varepsilon_0(1 + \chi_e)\mathbf{E} = \varepsilon \mathbf{E} \tag{6}$$

and the relative permittivity, defined in Eq. 1.2(4) is $\varepsilon_r = \varepsilon/\varepsilon_0 = 1 + \chi_e$

Although we will describe a dielectric material largely by its permittivity, the concepts of polarization and susceptibility are in a sense more fundamental and are considered in more detail in Chapter 13.

1.4 Examples of the Use of Gauss's Law

The simple but important examples to be discussed in this section show that Gauss's law can be used to find field strength in a very easy way for problems with certain kinds of symmetry and given charges. The symmetry gives the direction of the electric field directly and ensures that the flux is uniformly distributed. Knowledge of the charge gives the total flux. Symmetry is then used to get flux density **D** and hence $\mathbf{E} = \mathbf{D}/\varepsilon$.

Example 1.4a
Field in a Planar Semiconductor Depletion Layer

For the first example, we consider a one-dimensional situation where a metal is in intimate (atomic) contact with a semiconductor. We assume that some of the typically valence 4 (e.g., silicon) atoms have been replaced by "dopant" atoms of valence 5

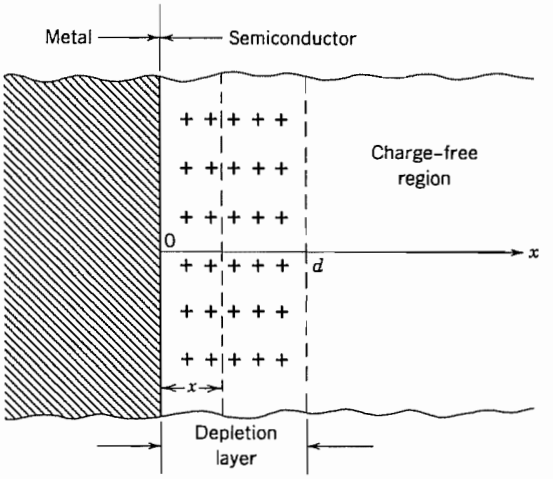

FIG. 1.4*a* Model of a metal–semiconductor contact. The region to which Gauss's law is applied is the region between parallel planes shown dashed.

(e.g., phosphorus). The one extra electron in each atom is not needed for atomic bonding and becomes free to move about in the semiconductor. Upon making the metal contact, it is found that the free electrons are forced away from the surface for a distance d. The region $0 \leq x \leq d$ is called a *depletion region* because it is depleted of the free electrons. Since the dopant atoms were neutral before losing their extra electrons, they are positively charged when the region is depleted. This can be modeled as in Fig. 1.4*a*. In the region $x > d$ the donors are assumed to be completely compensated by free electrons and it is therefore charge-free. (The abrupt change from compensated to uncompensated behavior at $x = d$ is a commonly used idealization.) By symmetry, the flux is $-x$-directed only. The surface used in application of Gauss's law consists of two infinite parallel planes, one at $x < d$ and one at $x = d$. This approximation is made because the transverse dimensions of the contact are assumed to be much larger than d. With no applied fields, there is no average movement of the electrons in the compensated region $x \geq d$ so $\mathbf{E}$ must be zero there. Thus $\mathbf{D}$ is also zero in that region and all the flux from the charged dopant atoms must terminate on negative charges at the metal contact. Therefore, Gauss's law gives, for a unit area,

$$-D_x(x) = N_D e(d - x) \tag{1}$$

where N_D and e are the volume density of donor ions and the charge per donor (magnitude of electronic charge), respectively. Then the x component of the electric field is

$$E_x = \frac{D_x}{\varepsilon} = \frac{N_D e(x - d)}{\varepsilon} \tag{2}$$

It should be clear that the simplicity of solution depended upon the symmetry of the system; that is, that there were no variations in y and z.

Example 1.4b
FIELD ABOUT A LINE CHARGE OR BETWEEN COAXIAL CYLINDERS

We introduced the line charge in ring form in Ex. 1.2. Let us now find the field **E** produced by a straight, infinitely long line of uniformly distributed charge. The radius of the line is negligibly small and can be thought of as the two-dimensional equivalent of a point charge. Practically, a long thin charged wire is a good approximation. The symmetry of this problem reveals that the force on a test charge, and hence the electric field, can only be radial. Moreover, this electric field will not vary with angle about the line charge, nor with distance along it. If the strength of the radial electric field is desired at distance r from the line charge, Gauss's law may be applied to an imaginary cylindrical surface of radius r and any length l. Since the electric field (and hence the electric flux density **D**) is radial, there is no normal component at the ends of the cylinder and hence no flux flow through them. However, **D** is exactly normal to the cylindrical part of the surface, and does not vary with either angle or distance along the axis, so that the flux out is the surface area $2\pi rl$ multiplied by the electric flux density D_r. The charge enclosed is the length l multiplied by the charge per unit length q_l. By Gauss's law, flux out equals the charge enclosed:

$$2\pi rlD_r = lq_l$$

If the dielectric surrounding the wire has constant ε,

$$E_r = \frac{D_r}{\varepsilon} = \frac{q_l}{2\pi\varepsilon r} \tag{3}$$

Hence, the electric field about the line charge has been obtained by the use of Gauss's law and the special symmetry of the problem.

The same symmetry applies to the coaxial transmission line formed of two coaxial conducting cylinders of radii a and b with dielectric ε between them (Fig. 1.4*b*). Hence the result (3) applies for radius r between a and b. We use this result to find the capacitance in Sec. 1.9.

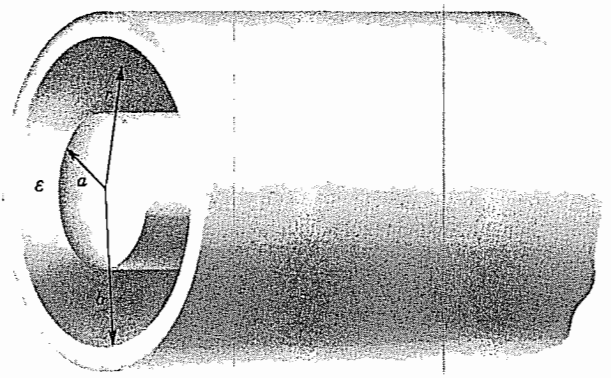

FIG. 1.4*b* Coaxial line.

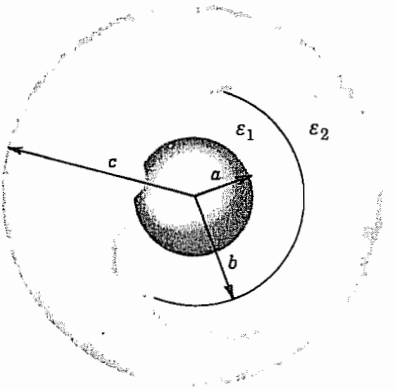

FIG. 1.4c Spherical electrodes separated by two layers of dielectric materials.

Example 1.4c
FIELD BETWEEN CONCENTRIC SPHERICAL ELECTRODES WITH TWO DIELECTRICS

Figure 1.4*c* shows a structure formed of two conducting spheres of radii a and c, with one dielectric ε_1 extending from $r = a$ to $r = b$, in spherical coordinates,[2] and a second, ε_2, from $r = b$ to $r = c$. This problem has spherical symmetry about the center, which implies that the electric field will be radial, and independent of the angular direction about the sphere. If the charge on the inner sphere is Q and that on the outer sphere is $-Q$, the charge enclosed by an imaginary spherical surface of radius r selected anywhere between the two conductors is only that charge Q on the inner sphere. The flux passing through it is the surface $4\pi r^2$ multiplied by the radial component of the flux density D_r. Hence, using Gauss's law,

$$D_r = \frac{Q}{4\pi r^2} \tag{4}$$

The equation for the flux density is the same for either dielectric, since the flux passes from the positive charge on the center conductor continuously to the negative charge on the outer conductor. The electric field has a different value in the two regions, however, since in each dielectric, D and E are related by the corresponding permittivity:

$$E_r = \frac{Q}{4\pi\varepsilon_1 r^2} \qquad a < r < b \tag{5}$$

$$E_r = \frac{Q}{4\pi\varepsilon_2 r^2} \qquad b < r < c \tag{6}$$

[2] *Note that* r *is used both for radius from the axis in the circular cylindrical coordinate system and for radius from the origin in the spherical coordinate system.* ρ *is frequently used for the former but may be confused with charge density, and* R *is used for the latter, but is here reserved for distance between source and field points.*

The radial flux density is continuous at the dielectric discontinuity at $r = b$, but the radial electric field is discontinuous there.

Example 1.4d
FIELDS OF A SPHERICAL REGION OF UNIFORM CHARGE DENSITY

Consider a region of uniform charge density ρ extending from $r = 0$ to $r = a$. As in the preceding example, Gauss's law can be written as (4) where, in this case, $Q = \frac{4}{3}\pi r^3\rho$ for $r \leq a$ and $Q = \frac{4}{3}\pi a^3\rho$ for $r \geq a$. Then the flux densities for the two regions are

$$D_r = \frac{r}{3}\rho \qquad r \leq a \tag{7}$$

$$D_r = \frac{a^3}{3r^2}\rho \qquad r \geq a \tag{8}$$

1.5 SURFACE AND VOLUME INTEGRALS: GAUSS'S LAW IN VECTOR FORM

Gauss's law, given in words by Eq. 1.3(3), may be written

$$\oint_S D \cos\theta \, dS = q \tag{1}$$

The symbol $\oint_S$ denotes the integral over a surface and of course cannot be performed until the actual surface is specified. It is in general a double integral. The circle on the integral sign is used if the surface is closed.

The surface integral can also be written in a still more compact form if vector notation is employed. Define the unit vector normal to the surface under consideration, for any given point on the surface, as $\hat{\mathbf{n}}$. Then replace $D\cos\theta$ by $\mathbf{D} \cdot \hat{\mathbf{n}}$. This particular product of the two vectors $\mathbf{D}$ and $\hat{\mathbf{n}}$ denoted by the dot between the two is known as the *dot product* of two vectors, or the *scalar product*, since it results by definition in a scalar quantity equal to the product of the two vector magnitudes and the cosine of the angle between them. Also, the combination $\hat{\mathbf{n}}\,dS$ is frequently abbreviated further by writing it $\mathbf{dS}$. Thus the elemental vector $\mathbf{dS}$, representing the element of surface in magnitude and orientation, has a magnitude equal to the magnitude of the element dS under consideration and the direction of the outward normal to the surface at that point. The surface integral in (1) may then be written in any of the equivalent forms:

$$\oint_S D \cos\theta \, dS = \oint_S \mathbf{D} \cdot \hat{\mathbf{n}} \, dS = \oint_S \mathbf{D} \cdot \mathbf{dS} \tag{2}$$

All of these say that the normal component of the vector $\mathbf{D}$ is to be integrated over the general closed surface S.

If the charge inside the region is given as a density of charge per unit volume in coulombs per cubic meter for each point of the region, the total charge inside the region must be obtained by integrating this density over the volume of the region. This is analogous to the process of finding the total mass inside a region when the variable mass density is given for each point of a region. This process may also be denoted by a general integral. The symbol $\int_V$ is used to denote this, and, as with the surface integral, the particular volume and the variation of density over that volume must be specified before the integration can be performed. In the general case, it is performed as a triple integral.

Gauss's law may then be written in this notation:

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho \, dV \tag{3}$$

Although the above may at first appear cryptic, familiarity with the notation will immediately reveal that the left side is the net electric flux out of the region and the right side is the charge within the region.

Example 1.5
ROUND BEAM OF UNIFORM CHARGE DENSITY

Consider the circular cylinder of uniform charge density ρ and infinite length shown in Fig. 1.5. A region of integration is taken in the form of a prism of square cross section. We will demonstrate the validity of (3), utilizing the fact that $\mathbf{D}$ has only a radial component. The right side of (3) is

$$\int_V \rho \, dV = \int_0^l dz \int_{-b/2}^{b/2} dy \int_{-b/2}^{b/2} \rho \, dx = lb^2\rho \tag{4}$$

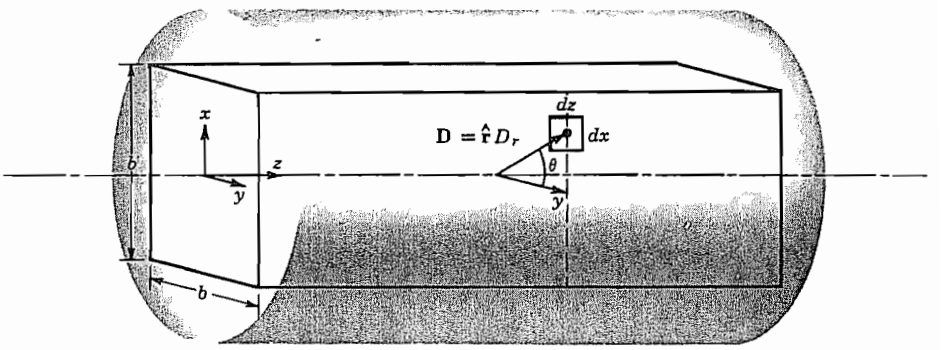

FIG. 1.5 Square cylindrical region of integration in a circular cylinder of free charge.

At any radius, D_r can be found as in Sec. 1.4 to be

$$D_r = \frac{(\pi r^2)\rho}{2\pi r} = \frac{r\rho}{2} \tag{5}$$

To do the surface integration on the left side of (3), we note that $\mathbf{D} \cdot \mathbf{dS} = D_r\, dx\, dz \cos\theta$, that $\cos\theta = b/2r$, and that the four sides make equal contributions to the integral. Thus

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = 4 \int_0^l dz \int_{-b/2}^{b/2} D_r \cos\theta\, dx = lb^2\rho \tag{6}$$

and from (6) and (4) we see that (3) is satisfied. Problem 1.5b contains a similar situation, but is somewhat complicated by having the charge density dependent upon radius.

1.6 Tubes of Flux: Plotting of Field Lines

For isotropic media, the electric field **E** is in the same direction as flux density **D**. A charge-free region bounded by **E** or **D** lines must then have the same flux flowing through it for all selected cross sections, since no flux can flow through the sides parallel with **D**, and Gauss's law will show the conservation of flux for this source-free region. Such a region, called a *flux tube*, is illustrated in Fig. 1.6*a*. Surface S_3 follows the direction of **D**, so there is no flow through S_3. That flowing in S_1 must then come out S_2 if there are no internal charges. These tubes are analogous to the flow tubes in the fluid analogy used in Sec. 1.3.

The concept of flux tubes is especially useful in making maps of the fields, and will be utilized later (Sec. 1.19) in a useful graphical field-mapping technique. We show in the following example how it may be used to obtain field lines in the vicinity of parallel line charges.

Example 1.6
Flux Tubes and Field Lines About Parallel Lines of Opposite Charge

To show how field lines may be found by constructing flux tubes, we use, as an example, two infinitely long, parallel lines of opposite charge. It is obvious from symmetry that the plane in which the two charge lines lie will contain **D** lines and hence can be a boundary of a flux tube. We introduce the *flux function*

$$\psi = \int_S \mathbf{D} \cdot \mathbf{dS} \tag{1}$$

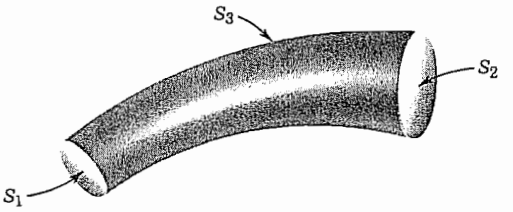

FIG. 1.6*a* Tube of flux.

which measures the flux crossing some chosen surface. For example, suppose S is the cross-hatched surface in Fig. 1.6*b*. First, let the angles α_1 and α_2 shrink to zero so S also vanishes. Then the flux function will be zero along the plane of the lines of charge. (The surface S also could be chosen in some other way, making the flux function different from zero on that plane. The resulting additive constant is arbitrary, so we take it to be zero.) We get other flux tube boundaries by taking nonzero values of the angles α_1 and α_2. First we derive an expression for the flux passing between the $\psi = 0$ plane and line L in Fig. 1.6*b*. Then paths will be formed along which L may be moved while keeping the same flux between it and the $\psi = 0$ surface. Moving L along such a path therefore generates a surface which is the boundary of a flux tube of infinite length parallel to L. The flux may be divided into the part from the positive line charge and the part from the negative line charge, since the effects are superposable. The flux from the positive line goes out radially so that the amount (per unit length) crossing S is $q_l(\alpha_1/2\pi)$. The flux passing radially inward toward the negative line charge through S adds directly to that of the positive line charge and has the magnitude $q_l(\alpha_2/2\pi)$. The total flux per unit length crossing S is

$$\psi = \frac{q_l}{2\pi}(\alpha_1 + \alpha_2) \tag{2}$$

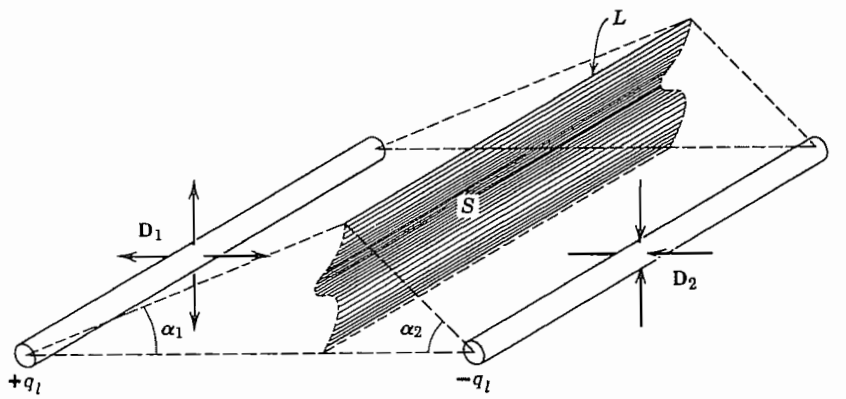

FIG. 1.6*b* Construction of flux tubes about line charges.

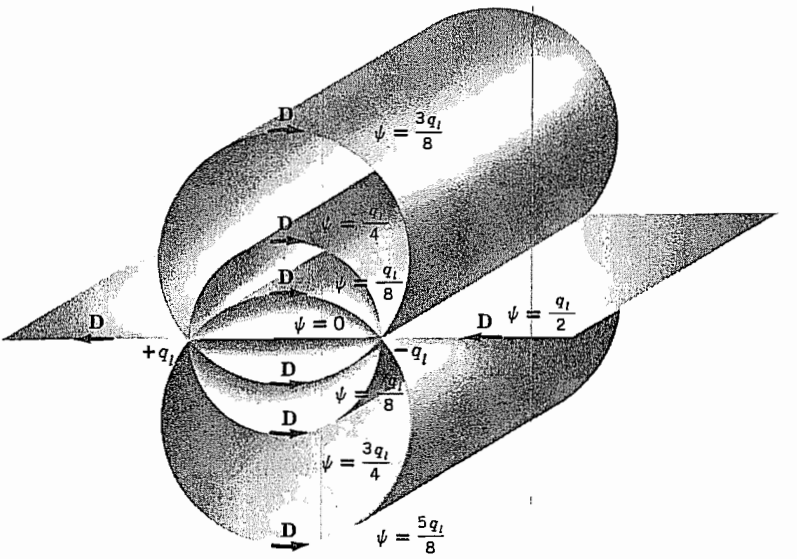

FIG. 1.6c Tubes of flux between line charges.

The surface generated by moving L in such a way as to keep ψ constant is a circular cylinder that passes through the charge lines with its axis in the plane normal to the $\psi = 0$ line midway between the line charges. Figure 1.6*c* shows several flux tubes, with the values of ψ indicating the amount of flux between the $\psi = 0$ surface and the one being considered, as α_2 is increased from zero to 2π. Note that as a path is taken around one of the lines, the flux function goes from 0 to q_l; the total flux per unit length coming from a line charge is q_l. It is clear from this example that the flux function ψ is not single-valued since it continues to increase as α_1 or α_2 increases; more flux lines are crossed as motion about the line charge continues. We must therefore limit α_1 and α_2 to the range 0 to 2π to ensure unique values for ψ.

Since the boundaries of the flux tubes lie along $\mathbf{D}$ vectors and $\mathbf{D} = \varepsilon\mathbf{E}$, they also lie along $\mathbf{E}$ vectors. Thus, by plotting flux tubes, we find the directions of the electric field vectors surrounding the charges. There is a given amount of flux in each tube so the flux density $\mathbf{D}$, and therefore also $\mathbf{E}$, become large where the cross section of the tube becomes small.

1.7 Energy Considerations: Conservative Property of Electrostatic Fields

Since a charge placed in the vicinity of other charges experiences a force, movement of the charge represents energy exchange. Calculation of this requires integration of force components over the path (line integrals). It will be found that the electrostatic

system is conservative in that no net energy is exchanged if a test charge is moved about a closed path, returning to its initial position.

Consider the force on a small positive charge Δq moved from infinity to a point P in the vicinity of a system of positive charges: q_1 at Q_1, q_2 at Q_2, q_3 at Q_3, and so on (Fig. 1.7*a*). The force at any point along its path would cause the particle to accelerate and move out of the region if unconstrained. A force equal to the negative of that from surrounding charges must then be applied to bring Δq from infinity to its final position. The differential work done on Δq from q_1 in the system is the negative of the force component in the direction of the path, multiplied by differential path length:

$$dU_1 = -\mathbf{F}_1 \cdot \mathbf{dl}$$

Or, using the definition of the scalar product, the angle θ as defined in Fig. 1.7*a*, and the force as stated above, we write the line integral for total work related to q_1 as

$$U_1 = -\int_{\infty}^{PQ_1} \mathbf{F}_1 \cdot \mathbf{dl} = -\int_{\infty}^{PQ_1} \frac{\Delta q q_1 \cos\theta \, dl}{4\pi\varepsilon r^2} \tag{1}$$

where r is the distance from q_1 to the differential path element **dl** at each point in the integration. Since $dl \cos\theta$ is dr, the integral is simply

$$U_1 = -\int_{\infty}^{PQ_1} \frac{\Delta q q_1 \, dr}{4\pi\varepsilon r^2}$$

and similarly for contributions from other charges, so that the total work integral is

$$U = -\int_{\infty}^{PQ_1} \frac{\Delta q q_1}{4\pi\varepsilon r^2} dr - \int_{\infty}^{PQ_2} \frac{\Delta q q_2}{4\pi\varepsilon r^2} dr - \int_{\infty}^{PQ_3} \frac{\Delta q q_3}{4\pi\varepsilon r^2} dr - \cdots$$

Note that there is no component of the work arising from the test charge acting upon itself. Integrating,

$$U = \frac{\Delta q q_1}{4\pi\varepsilon PQ_1} + \frac{\Delta q q_2}{4\pi\varepsilon PQ_2} + \frac{\Delta q q_3}{4\pi\varepsilon PQ_3} + \cdots \tag{2}$$

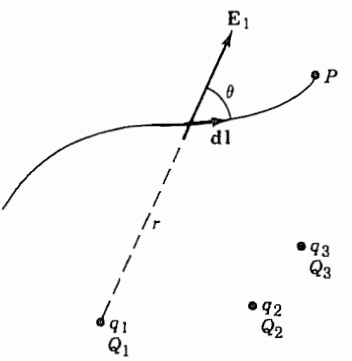

FIG. 1.7*a* Integration path for force on test charge.

Equation (2) shows that the work done is a function only of final positions and not of the path of the charge. This conclusion leads to another: if a charge is taken around any closed path, no net work is done. Mathematically this is written as the closed line integral

$$\oint \mathbf{E} \cdot \mathbf{dl} = 0 \tag{3}$$

This general integral signifies that the component of electric field in the direction of the path is to be multiplied by the element of distance along the path and the sum taken by integration as one moves about the path. The circle through the integral sign signifies that a closed path is to be considered. As with the designation for a general surface or volume integral, the actual line integration cannot be performed until there is a specification of a particular path and the variation of E about that path.

In the study of magnetic fields and time-varying electric fields, we shall find corresponding line integrals which are not zero.

Example 1.7
DEMONSTRATION OF CONSERVATIVE PROPERTY

To illustrate the conservative property and the use of line integrals, let us take the line integral (3) around the somewhat arbitrary path through a uniform sphere of charge density ρ shown in Fig. 1.7*b*. The path is chosen, for simplicity, to lie in the $x = 0$

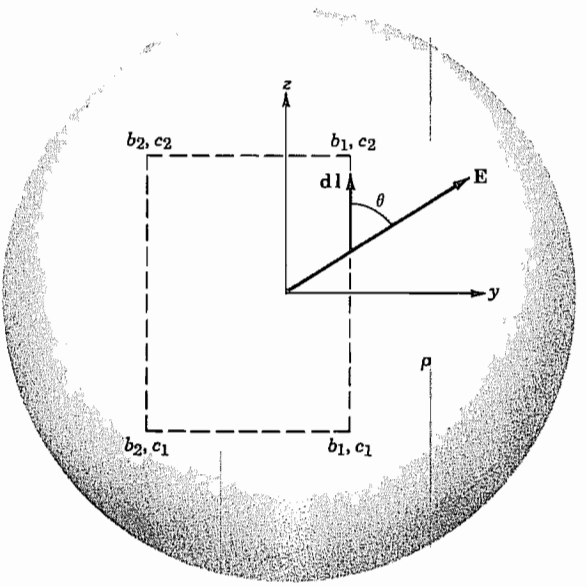

FIG. 1.7*b* Path of integration (broken line) through electric field of sphere of uniform charge to show conservative property.

plane. The integrand $\mathbf{E} \cdot \mathbf{dl}$ involves electric field components E_y and E_z. The radial electric field $E_r = \rho r/3\varepsilon_0$ is found from Eq. 1.4(7). The components are $E_y = E_r(y/r)$ and $E_z = E_r(z/r)$ so $E_y = \rho y/3\varepsilon_0$ and $E_z = \rho z/3\varepsilon_0$. The integral (3) becomes

$$\oint \mathbf{E} \cdot \mathbf{dl} = \int_{c_1}^{c_2} E_z\, dz + \int_{b_1}^{b_2} E_y\, dy + \int_{c_2}^{c_1} E_z\, dz + \int_{b_2}^{b_1} E_y\, dy$$

$$= \frac{\rho}{3\varepsilon_0}\left\{\frac{z^2}{2}\bigg|_{c_1}^{c_2} + \frac{y^2}{2}\bigg|_{b_1}^{b_2} + \frac{z^2}{2}\bigg|_{c_2}^{c_1} + \frac{y^2}{2}\bigg|_{b_2}^{b_1}\right\} = 0$$

The general conservative property of electrostatic fields is thus illustrated in this example.

1.8 Electrostatic Potential: Equipotentials

The energy considerations of the preceding section lead directly to an extremely useful concept for electrostatics—that of potential. The electrostatic potential function is defined as the work done per unit charge. Here we start generally and define a potential difference between points 1 and 2 as the work done on a unit test charge in moving from P_1 to P_2.

$$\Phi_{P_2} - \Phi_{P_1} = -\int_{P_1}^{P_2} \mathbf{E} \cdot \mathbf{dl} \tag{1}$$

The conclusion of the preceding section that the work in moving around any closed path is zero shows that this potential function is single-valued; that is, corresponding to each point of the field there is only one value of potential.

Only a difference of potential has been defined. The potential of any point can be arbitrarily fixed, and then the potentials of all other points in the field can be found by application of the definition to give potential differences between all points and the reference. This reference is quite arbitrary. For example, in certain cases it may be convenient to define the potential at infinity as zero and then find the corresponding potentials of all points in the field; for determination of the field between two conductors, it is more convenient to select the potential of one of these as zero.

If the potential at infinity is taken as zero, it is evident that the potential at the point P in the system of charges is given by U of Eq. 1.7(2) divided by Δq, so

$$\Phi = \frac{q_1}{4\pi\varepsilon PQ_1} + \frac{q_2}{4\pi\varepsilon PQ_2} + \frac{q_3}{4\pi\varepsilon PQ_3} + \cdots \tag{2}$$

This may be written in a more versatile form as

$$\Phi(\mathbf{r}) = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon R_i} \tag{3}$$

where R_i is the distance of the ith charge at $\mathbf{r}_i'$ from the point of observation at $\mathbf{r}$, as seen in Fig. 1.8*a*.

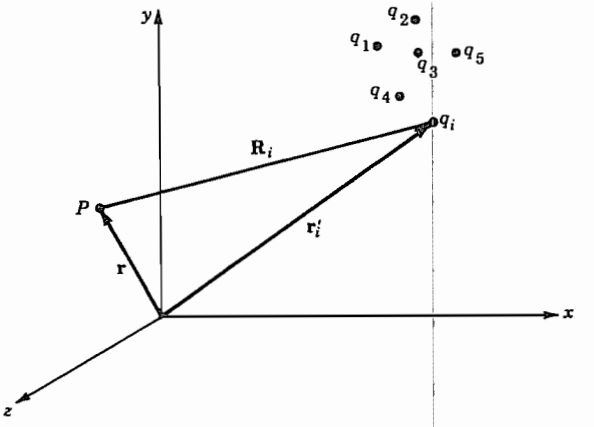

FIG. 1.8a Potential of $q_1, q_2, q_3, \ldots, q_i$ is found at point P.

$$R_i = |\mathbf{r} - \mathbf{r}_i'| = [(x - x_i')^2 + (y - y_i')^2 + (z - z_i')^2]^{1/2} \tag{4}$$

Here x, y, and z are the rectangular coordinates of the point of observation and x_i', y_i', and z_i' are the rectangular coordinates of the ith charge.[3] Generalizing to the case of continuously varying charge density,

$$\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')\, dV'}{4\pi\varepsilon R} \tag{5}$$

The $\rho(\mathbf{r}')$ is charge density at point (x', y', z'), and the integral signifies that a summation should be made similar to that of (2) but continuous over all space. If the reference for potential zero is not at infinity, a constant must be added of such value that potential is zero at the desired reference position.

$$\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')\, dV'}{4\pi\varepsilon R} + C \tag{6}$$

It should be kept in mind that (2)–(6) were derived assuming that the charges are located in an infinite, homogeneous, isotropic medium. If conductors or dielectric discontinuities are present, differential equations for the potential (to be given shortly) are used for each region.

We will see in Sec. 1.10 how the electric field $\mathbf{E}$ can be found simply from $\Phi(\mathbf{r})$. It is usually easier to find the potential by the scalar operations in (3) and (5) and, from it, the field $\mathbf{E}$ than to do the vector summations discussed in Sec. 1.2. Such convenience in electrostatic calculations is one reason for introducing this potential.

In any electrostatic field, there exist surfaces on which the potential is a constant, so-called equipotential surfaces. Since the potential is single-valued, surfaces for different

[3] *Throughout the text we use primed coordinates to designate the location of sources and unprimed coordinates for the point at which their fields are to be calculated.*

values of potential do not intersect. The pictorial representation of more than one such surface in a three-dimensional field distribution is quite difficult. For fields that have no variation in one dimension and are therefore called two-dimensional fields, the third dimension can be used to represent the potential. Figure 1.8*b* shows such a representation for the potential around a pair of infinitely long, parallel wires at potentials Φ_1 and $-\Phi_1$. The height of the surface at any point is the value of the potential. Note that lines of constant height or constant potential can be drawn. These equipotentials can be projected onto the x–y plane as in Fig. 1.8*c*. In such a representation, the equipo-

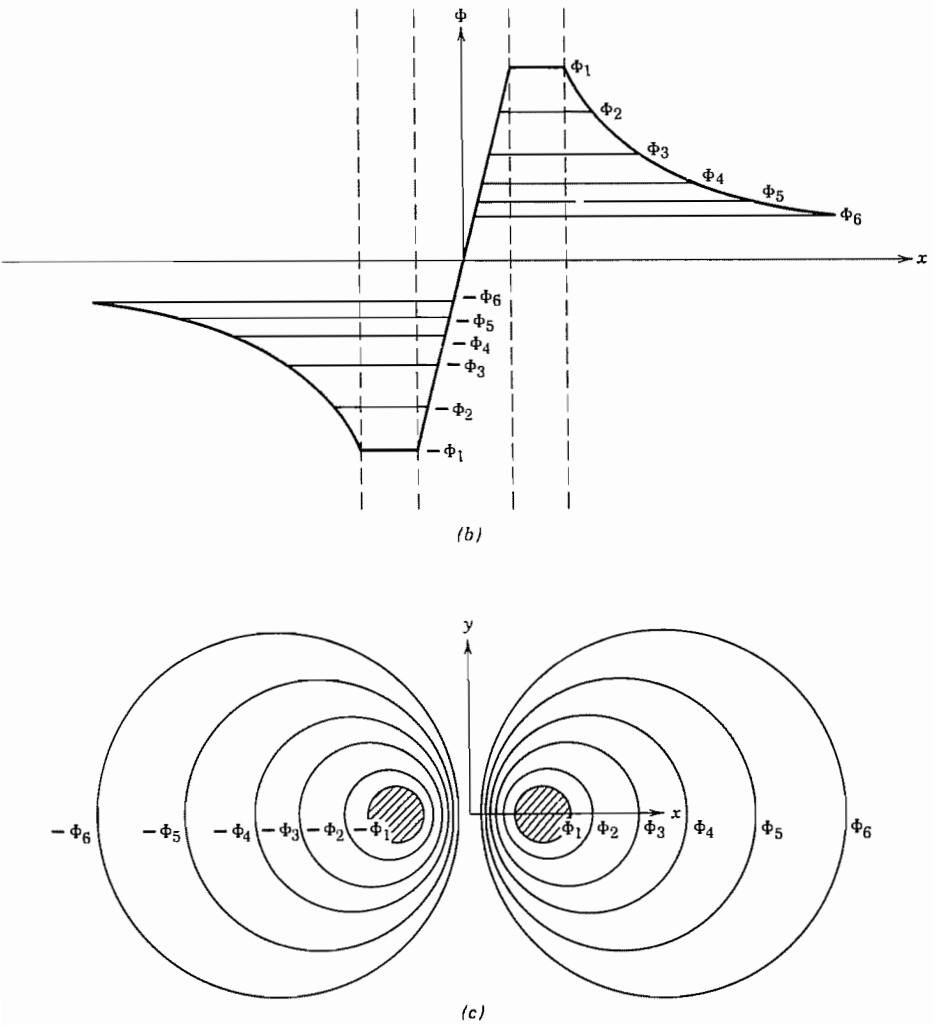

FIG. 1.8 (*b*) Plot of a two-dimensional potential distribution using the third dimension to show the potential. (*c*) Equipotentials for the same potential system as in (*b*) plotted onto the x–y plane.

tentials look like the contour lines of a topographic map and, in fact, measure potential energy of a unit charge relative to a selected zero-potential point just as contours measure potential energy relative to some reference altitude, often sea level. It should be kept in mind that these lines are actually traces in the x–y plane of three-dimensional cylindrical equipotential surfaces.

We will discuss the boundary conditions on conductors in some detail in Sec. 1.14. At this point it is sufficient to say that the electric fields inside of a metallic conductor can be considered to be zero in electrostatic systems. Therefore, (1) shows that the conductor is an equipotential region.

Example 1.8a
POTENTIALS AROUND A LINE CHARGE AND BETWEEN COAXIAL CYLINDERS

As an example of the relations between potential and electric field, consider first the problem of the line charge used as an example in Sec. 1.4, with electric field given by Eq. 1.4(3). By (1) we integrate this from some radius r_0 chosen as the reference of zero potential to radius r:

$$\Phi = -\int_{r_0}^{r} E_r \, \mathrm{dr} = -\int_{r_0}^{r} \frac{q_l \, dr}{2\pi\varepsilon r} = -\frac{q_l}{2\pi\varepsilon} \ln\left(\frac{r}{r_0}\right) \tag{7}$$

Or this expression for potential about a line charge may be written

$$\Phi = -\frac{q_l}{2\pi\varepsilon} \ln r + C \tag{8}$$

Note that it is not desirable to select infinity as the reference of zero potential for the line charge, for then by (7) the potential at any finite point would be infinite. As in (6) the constant is added to shift the position of the zero potential.

In a similar manner, the potential difference between the coaxial cylinders of Fig. 1.4*b* may be found:

$$\Phi_a - \Phi_b = -\int_{b}^{a} \frac{q_l \, dr}{2\pi\varepsilon r} = \frac{q_l}{2\pi\varepsilon} \ln\left(\frac{b}{a}\right) \tag{9}$$

Example 1.8b
POTENTIAL OUTSIDE A SPHERICALLY SYMMETRIC CHARGE

We saw in Eq. 1.4(4) that the flux density outside a spherically symmetric charge Q is $D_r = Q/4\pi r^2$. Using $\mathbf{E} = \mathbf{D}/\varepsilon_0$ and taking the reference potential to be zero at infinity, we see that the potential outside the charge Q is the negative of the integral of

$\hat{\mathbf{r}}E_r \cdot \hat{\mathbf{r}}\, dr$ from infinity to radius r:

$$\Phi(r) = -\int_{\infty}^{r} \frac{Q\, dr_1}{4\pi\varepsilon_0 r_1^2} = \frac{Q}{4\pi\varepsilon_0 r} \tag{10}$$

Example 1.8c
POTENTIAL OF A UNIFORM DISTRIBUTION OF CHARGE HAVING SPHERICAL SYMMETRY

Consider a volume of charge density ρ that extends from $r = 0$ to $r = a$. Taking $\Phi = 0$ at $r = \infty$, the potential outside a is given by (10) with $Q = \frac{4}{3}\pi\varepsilon a^3\rho$, so

$$\Phi(r) = \frac{a^3\rho}{3\varepsilon_0 r} \qquad r \geq a \tag{11}$$

In particular, at $r = a$

$$\Phi(a) = \frac{a^2\rho}{3\varepsilon_0} \tag{12}$$

Then to get the potential at a point where $r \leq a$ we must add to (12) the integral of the electric field from a to r. The electric field is given as $E_r = \rho r/3\varepsilon_0$ (Ex. 1.7) and the integral is

$$\Phi(r) - \Phi(a) = -\int_{a}^{r} \frac{\rho r_1}{3\varepsilon_0}\, dr_1 = \frac{\rho}{6\varepsilon_0}(a^2 - r^2) \tag{13}$$

So the potential at a radius r inside the charge region is

$$\Phi(r) = \frac{\rho}{6\varepsilon_0}(3a^2 - r^2) \qquad r \leq a \tag{14}$$

Example 1.8d
ELECTRIC DIPOLE

A particularly important set of charges is that of two closely spaced point charges of opposite sign, called an *electric dipole.*

Assume two charges, having opposite signs to be spaced by a distance 2δ as shown in Fig. 1.8*d*. The potential at some point a distance r from the origin displaced by an angle θ from the line passing from the negative to positive charge can be written as the sum of the potentials of the individual charges:

$$\Phi = \frac{q}{4\pi\varepsilon}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{15}$$

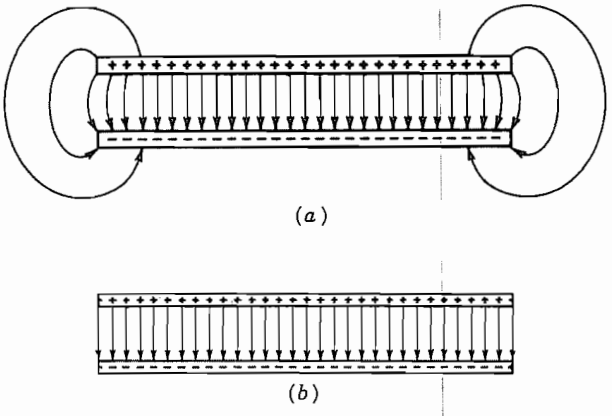

FIG. 1.9 (*a*) Parallel-plane capacitor with fringing fields. (*b*) Idealization.

the surface charge density ρ_s on each plate. Since the field $\mathbf{E} = \mathbf{D}/\varepsilon$ is assumed uniform, the potential difference $\Phi_a - \Phi_b$ is, by Eq. 1.8(1),

$$\Phi_a - \Phi_b = \frac{\rho_s d}{\varepsilon} \tag{2}$$

The total charge on each plate of area A is $\rho_s A$ so (1) and (2) give the familiar expression

$$C = \frac{\varepsilon A}{d} \quad \text{F} \tag{3}$$

In practice, (3) is modified by the fringing fields, which are increasingly important as the ratio of plate spacing to area is increased.

Next consider a capacitor made of coaxial, circular cylindrical electrodes. We assume that fields are only radial and neglect any fringing at the ends if it is of finite length. The charge on each conductor is distributed uniformly in this idealization, as required by symmetry, with the total charge per unit length being q_l. The potential difference found from the field produced by this charge is given by Eq. 1.8(9). The capacitance per unit length is thus

$$C = \frac{2\pi\varepsilon}{\ln(b/a)} \quad \text{F/m} \tag{4}$$

where b and a are the radii of larger and smaller conductors, respectively.

Finally, consider two concentric spherical conductors of radii a and b, with $b > a$, separated by a dielectric ε. Using symmetry, Gauss's law, and $\mathbf{E} = \mathbf{D}/\varepsilon$, it is clear that at any radius

$$E_r = \frac{Q}{4\pi\varepsilon r^2} \tag{5}$$

where Q is the charge on the inner conductor (equal in magnitude and opposite in sign

to the charge on the inside of the outer sphere). Integration of (5) between spheres gives $\Phi_a - \Phi_b$ and this, substituted into (1), yields

$$C = \frac{4\pi\varepsilon}{(1/a) - (1/b)} = \frac{4\pi\varepsilon ab}{b - a} \quad \text{F} \tag{6}$$

The flux tubes in these three highly symmetric structures are very simple, being bounded by parallel surfaces in the first example and by cylindrically or spherically radial surfaces in the last two. In Sec. 1.21 we will see a way of finding capacitance graphically for two-dimensional structures of arbitrary shape in which the flux tubes have more complex shapes.

Capacitance of an isolated electrode is sometimes calculated; in that case, the flux from the charge on the electrode terminates at infinity and the potential on the electrode is taken with respect to an assumed zero at infinity. More extensive considerations of capacitance are found in Sec. 4.9.

Differential Forms of Electrostatic Laws

1.10 Gradient

We have looked at several laws of electrostatics in macroscopic forms. It is also useful to have their equivalents in differential forms. Let us start with the relation between electric field and potential. If the definition of potential difference is applied to two points a distance **dl** apart,

$$d\Phi = -\mathbf{E} \cdot \mathbf{dl} \tag{1}$$

where **dl** may be written in terms of its components and the defined unit vectors:

$$\mathbf{dl} = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz \tag{2}$$

We expand the dot product:

$$d\Phi = -(E_x\, dx + E_y\, dy + E_z\, dz)$$

Since Φ is a function of x, y, and z, the total differential may also be written

$$d\Phi = \frac{\partial\Phi}{\partial x}\, dx + \frac{\partial\Phi}{\partial y}\, dy + \frac{\partial\Phi}{\partial z}\, dz$$

From a comparison of the two expressions,

$$E_x = -\frac{\partial\Phi}{\partial x}, \qquad E_y = -\frac{\partial\Phi}{\partial y}, \qquad E_z = -\frac{\partial\Phi}{\partial z} \tag{3}$$

so

$$\mathbf{E} = -\left(\hat{\mathbf{x}}\frac{\partial\Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}\right) \tag{4}$$

or

$$\mathbf{E} = -\text{grad}\,\Phi \tag{5}$$

where grad Φ, an abbreviation of the *gradient* of Φ, is a vector showing the direction and magnitude of the maximum spatial variation of the scalar function Φ, at a point in space.

Substituting back in (1), we have

$$d\Phi = (\text{grad}\,\Phi)\cdot\mathbf{dl} \tag{6}$$

Thus the change in Φ is given by the scalar product of the gradient and the vector **dl**, so that, for a given element of length **dl**, the maximum value of $d\Phi$ is obtained when that element is oriented to coincide with the direction of the gradient vector. From (6) it is also clear that grad Φ is perpendicular to the equipotentials because $d\Phi = 0$ for **dl** along an equipotential.

The analogy between electrostatic potential and gravitational potential energy discussed in Sec. 1.8 is useful for understanding the gradient. It is easy to see in Fig. 1.8*b* that the direction of maximum rate of change of potential is perpendicular to the equipotentials (which are at constant heights on the potential hill).

If we define a vector operator ∇ (pronounced del)

$$\nabla \triangleq \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{7}$$

then grad Φ may be written as $\nabla\Phi$ if the operation is interpreted as

$$\nabla\Phi = \hat{\mathbf{x}}\frac{\partial\Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z} \tag{8}$$

and

$$\mathbf{E} = -\text{grad}\,\Phi \triangleq -\nabla\Phi \tag{9}$$

The gradient operator in circular cylindrical and spherical coordinates is given on the inside front cover.

Example 1.10
ELECTRIC FIELD OF A DIPOLE

As an example of the use of the gradient operator, we will find an expression for the field around an electric dipole. The potential for a dipole is given in Sec. 1.8 in spherical coordinates so the spherical form of the gradient operator is selected from the inside

front cover.

$$\nabla\Phi = \hat{\mathbf{r}}\frac{\partial\Phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

Substituting Eq. 1.8(16) and noting the independence of ϕ, we get

$$\nabla\Phi = -\hat{\mathbf{r}}\frac{\delta q\cos\theta}{\pi\varepsilon r^3} - \hat{\boldsymbol{\theta}}\frac{\delta q\sin\theta}{2\pi\varepsilon r^3}$$

Then from (9) the electric field is

$$\mathbf{E} = \frac{\delta q}{\pi\varepsilon r^3}\left(\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\frac{\sin\theta}{2}\right) \tag{10}$$

1.11 The Divergence of an Electrostatic Field

The second differential form we shall consider is that of Gauss's law. Equation 1.5(3) may be divided by the volume element ΔV and the limit taken:

$$\lim_{\Delta V\to 0}\frac{\oint_s \mathbf{D}\cdot\mathbf{dS}}{\Delta V} = \lim_{\Delta V\to 0}\frac{\int_V \rho\, dV}{\Delta V} \tag{1}$$

The right side is, by inspection, merely p. The left side is the outward electric flux per unit volume. This will be defined as the *divergence of flux density*, abbreviated div **D**. Then

$$\text{div}\,\mathbf{D} = \rho \tag{2}$$

This is a good place to comment on the size scale implicit in our treatment of fields and their sources; the comments also apply to the central set of relations, Maxwell's equations, toward which we are building. In reality, charge is not infinitely divisible—the smallest unit is the electron. Thus, the limit of ΔV in (1) must actually be some small volume which is still large enough to contain many electrons to average out the granularity. For our relations to be useful, the limit volume must also be much smaller than important dimensions in the system. For example, to neglect charge granularity, the thickness of the depletion layer in the semiconductor in Ex. 1.4a should be much greater than the linear dimensions of the limit volume, which, in turn, should be much greater than the average spacing of the dopant atoms. Similarly the permittivity ε is an average representation of atomic or molecular polarization effects such as that shown in Fig. 1.3*c*. Therefore, when we refer to the field at a point, we mean that the field is an average over a volume small compared with the system being analyzed but large enough to contain many atoms. Analyses can also be made of the fields on a smaller scale, such as inside an atom, but in that case an average permittivity cannot be used. In this book, we concentrate on situations where ρ, ε_r, and other quantities are averages

over small volumes. There are thin films of current practical interest which are only a few atoms thick and semiconductor devices such as that of Ex. 1.4a where these conditions are not well satisfied, so that results from calculations using ρ and ε as defined must be used with caution.

Now let us return to seeking an understanding of (2). Consider the infinitesimal volume as a rectangular parallelepiped of dimensions Δx, Δy, Δz as shown in Fig. 1.11*a*. To compute the amount of flux leaving such a volume element as compared with that entering it, note that the flux passing through any face of the parallelepiped can differ from that which passes through the opposite face only if the flux density perpendicular to those faces varies from one face to the other. If the distance between the two faces is small, then to a first approximation the difference in any vector function on the two faces will simply be the rate of change of the function with distance times the distance between faces. According to the basis of calculus, this is exactly correct when we pass to the limit, since the higher-order differentials are then zero.

If the vector $\mathbf{D}$ at the center x, y, z has a component $D_x(x)$, then

$$\begin{aligned} D_x\left(x + \frac{\Delta x}{2}\right) &\cong D_x(x) + \frac{\Delta x}{2}\frac{\partial D_x(x)}{\partial x} \\ D_x\left(x - \frac{\Delta x}{2}\right) &\cong D_x(x) - \frac{\Delta x}{2}\frac{\partial D_x(x)}{\partial x} \end{aligned} \tag{3}$$

In this functional notation, the arguments in parentheses show the points for evaluating the function D_x. When not included, the point (x, y, z) will be understood. The flux flowing out the right face is $\Delta y\, \Delta z\, D_x(x + \Delta x/2)$, and that flowing in the left face is $\Delta y\, \Delta z\, D_x(x - \Delta x/2)$, leaving a net flow out of $\Delta x\, \Delta y\, \Delta z(\partial D_x/\partial x)$, and similarly for the y and z directions. Thus the net flux flow out of the parallelepiped is

$$\Delta x\, \Delta y\, \Delta z \frac{\partial D_x}{\partial x} + \Delta x\, \Delta y\, \Delta z \frac{\partial D_y}{\partial y} + \Delta x\, \Delta y\, \Delta z \frac{\partial D_z}{\partial z}$$

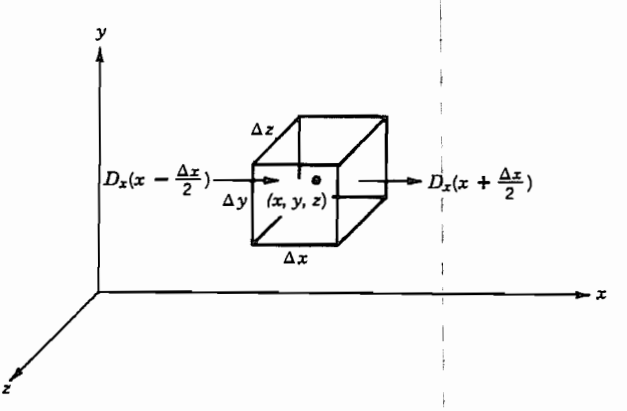

FIG. 1.11*a* Volume element used in div $\mathbf{D}$ derivation.

By Gauss's law, this must equal $\rho\,\Delta x\,\Delta y\,\Delta z$. So, in the limit,

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \tag{4}$$

An expression for div **D** in rectangular coordinates is obtained by comparing (2) and (4):

$$\text{div } \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \tag{5}$$

If we make use of the vector operator ∇ defined by Eq. 1.10(7) in (4), then (5) indicates that div **D** can conveniently be written as $\nabla \cdot \mathbf{D}$. It should be remembered that ∇ is not a true vector but rather a vector operator. It has meaning only when it is operating on another quantity in a defined manner. Summarizing,

$$\nabla \cdot \mathbf{D} \triangleq \text{div } \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \tag{6}$$

The divergence is made up of spatial derivatives of the field, so (6) is a partial differential equation expressing Gauss's law for a region of infinitesimal size. The physical significance of the divergence must be clear. It is, as defined, a description of the manner in which a field varies at a point. It is the amount of flux per unit volume emerging from an infinitesimal volume at a point. With this picture in mind, (6) seems a logical extension of Gauss's law. In fact (6) can be converted back to the large-scale form of Gauss's law through the *divergence theorem*, which states that the volume integral of the divergence of any vector **F** throughout a volume is equal to the surface integral of that vector flowing out of the surrounding surface,

$$\int_V \nabla \cdot \mathbf{F}\, dV = \oint_S \mathbf{F} \cdot \mathbf{dS} \tag{7}$$

Although not a proof, this is made plausible by considering Fig. 1.11*b*. The divergence multiplied by volume element for each elemental cell is the net surface integral out of

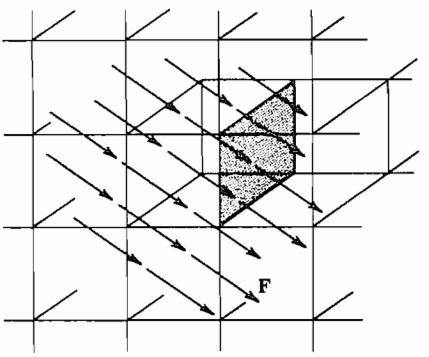

FIG. 1.11*b* Solid divided into subvolumes to illustrate the divergence theorem.

that cell. When summed by integration, all internal contributions cancel since flow out of one cell goes into another, and only the external surface contribution remains. Applications of (7) to (6) with $\mathbf{F} = \mathbf{D}$ gives

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \nabla \cdot \mathbf{D}\, dV = \int_V \rho\, dV \tag{8}$$

which is the original Gauss's law.

It will be useful to have expressions for the divergence and other operations involving ∇ in other coordinate systems for simpler treatment of problems having corresponding symmetries. Let us, as an example, develop here the divergence of $\mathbf{D}$ in spherical coordinates.[4] We use the left side of (1) as the definition of div $\mathbf{D}$ and apply it to the differential volume shown in Fig. 1.11*c*. We will find first the net radial outward flux from the volume. Both the radial component D_r and the element of area $r^2\, d\theta \sin\theta\, d\phi$ change as we move from r to $r + dr$. Thus the net flux flow out the top over that in at the bottom is

$$d\psi_r = (r + dr)^2 \sin\theta\, d\theta\, d\phi \left(D_r + \frac{\partial D_r}{\partial r} dr\right) - r^2 \sin\theta\, d\theta\, d\phi D_r$$

To first-order differentials, this leaves

$$d\psi_r = r^2 \sin\theta\, d\theta\, d\phi \frac{\partial D_r}{\partial r} dr + 2r\, dr \sin\theta\, d\theta\, d\phi\, D_r$$

$$= \sin\theta\, dr\, d\theta\, d\phi \frac{\partial}{\partial r}(r^2 D_r)$$

Similarly for the θ and ϕ directions,

$$d\psi_\theta = d\theta \frac{\partial}{\partial \theta}(D_\theta\, r \sin\theta\, d\phi\, dr) = r\, dr\, d\theta\, d\phi \frac{\partial}{\partial \theta}(\sin\theta\, D_\theta)$$

$$d\psi_\phi = d\phi \frac{\partial}{\partial \phi}(D_\phi\, r\, d\theta\, dr) = r\, dr\, d\theta\, d\phi \frac{\partial D_\phi}{\partial \phi}$$

The divergence is then the total $d\psi$ divided by the element of volume

$$\nabla \cdot \mathbf{D} = \frac{d\psi_r + d\psi_\theta + d\psi_\phi}{r^2 \sin\theta\, dr\, d\theta\, d\phi} \tag{9}$$

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r \sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

For the corresponding expression in circular cylindrical coordinates, see inside front cover and Prob. 1.11*c*.

[4] *Note that here, as with other curvilinear coordinate systems, it is not the scalar product of the gradient operator and the vector in spherical coordinates, but must be obtained from the basic definition given by (1) and (2).*

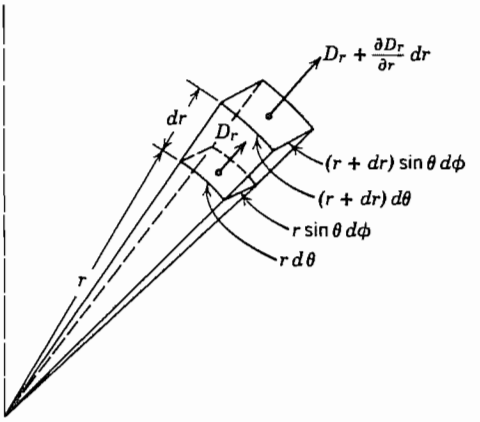

FIG. 1.11c Element of volume in spherical coordinates.

Example 1.11
UNIFORM SPHERE OF CHARGE

We will show an example of the application of (6) using a sphere of charge of uniform density ρ having a radius a with divergence written in spherical coordinates. Since the spherical symmetry of the charge region leads to $D_\theta = D_\phi = 0$, the last two terms of (9) vanish. We use the value of $D_r = r\rho/3$ from Eq. 1.4(7) for the region inside the charge sphere and obtain

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[(r^2)\left(\frac{\rho r}{3}\right)\right] = \rho \tag{10}$$

For the region outside the sphere we use $D_r(r) = a^3\,\rho/3r^2$ from Eq. 1.4(8) to show that

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[(r^2)\left(\frac{a^3\rho}{3r^2}\right)\right] = 0 \tag{11}$$

This example shows that divergence is zero outside the charge region but equal to charge density within it. The same result is obtained if one uses divergence expressed in rectangular coordinates or other coordinates not so natural to the symmetry (Prob. 1.11d).

1.12 LAPLACE'S AND POISSON'S EQUATIONS

The differential relations of the two preceding sections allow us to derive an important differential equation for potential. Differential equations can be applied to problems more general than those solved by symmetry in the first part of the chapter and it is

often convenient to work with potential as the dependent variable. This is because potential is a scalar and because the specified boundary conditions are often given in terms of potentials.

If the permittivity ε is constant throughout the region, the substitution of $\mathbf{E}$ from Eq. 1.10(5) in Eq. 1.11(6) with $\mathbf{D} = \varepsilon\mathbf{E}$ yields

$$\text{div}(\text{grad}\ \Phi) = \nabla \cdot \nabla\Phi = -\frac{\rho}{\varepsilon}$$

But, from the equations for gradient and divergence in rectangular coordinates [Eqs. 1.10(7) and 1.11(6)],

$$\nabla \cdot \nabla\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \tag{1}$$

so that

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = -\frac{\rho}{\varepsilon} \tag{2}$$

This is a differential equation relating potential variation at any point to the charge density at that point and is known as *Poisson's equation*. It is often written

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon} \tag{3}$$

where $\nabla^2\Phi$ (del squared of Φ) is known as the *Laplacian* of Φ.

$$\nabla^2\Phi \triangleq \nabla \cdot \nabla\Phi = \text{div}(\text{grad}\ \Phi) \tag{4}$$

In the special case of a charge-free region, Poisson's equation reduces to

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

or

$$\nabla^2\Phi = 0 \tag{5}$$

which is known as *Laplace's equation*. Although illustrated in its rectangular coordinate form, ∇^2 can be expressed in cylindrical or spherical coordinates through the relations given on the inside front cover.

Any number of possible configurations of potential surfaces will satisfy the requirements of (3) and (5). All are called solutions to these equations. It is necessary to know the conditions existing around the boundary of the region to select the particular solution which applies to a given problem. We will see in Sec. 1.17 a proof of the uniqueness of a function that satisfies both the differential equation and the boundary conditions.

Quantities other than potential can also be shown to satisfy Laplace's and Poisson's equations, both in other branches of physics and in other parts of electromagnetic field theory. For example, the magnitudes of the rectangular components of $\mathbf{E}$ and the component E_z in cylindrical coordinates also satisfy Laplace's equation.

A number of methods exist for solution of two- and three-dimensional problems with Laplace's or Poisson's equation. The separation of variables technique is a very general method for solving two- and three-dimensional problems for a large variety of partial differential equations including the two of interest here. Conformal transformations of complex variables yield many useful two-dimensional solutions of Laplace's equation. Increasingly important are numerical methods using digital computers. All these methods are elaborated in Chapter 7. Examples for this chapter, after discussion of boundary conditions, will be limited to one-dimensional examples for which the differential equations may be directly integrated. These show clearly the role of boundary and continuity conditions in the solutions.

1.13 Static Fields Arising from Steady Currents

Stationary currents arising from dc potentials applied to conductors are not static because the charges producing the currents are in motion, but the resulting steady-state fields are independent of time. Quite apart from the question of designation, there is a close relationship to laws and techniques for the fields arising from purely static charges.

We consider ohmic conductors for which current density is proportional to electric field $\mathbf{E}$ through conductivity[5] σ siemens per meter (S/m):

$$\mathbf{J} = \sigma\mathbf{E} \tag{1}$$

Such a relationship comes from internal "collisons" and is discussed more in Chapter 13. Since the electric field is independent of time, it is derivable from a scalar potential as in Sec. 1.10, so

$$\mathbf{J} = -\sigma\nabla\Phi \tag{2}$$

For a stationary current, continuity requires that the net flow out of any closed region be zero since there cannot be a buildup or decay of charge within the region in the steady state,

$$\oint_S \mathbf{J} \cdot \mathbf{dS} = 0 \tag{3}$$

or in differential form,

$$\nabla \cdot \mathbf{J} = 0 \tag{4}$$

Substitution of (2) in (4), with σ taken as constant, yields

$$\nabla \cdot \nabla\Phi \triangleq \nabla^2\Phi = 0 \tag{5}$$

Thus potential satisfies Laplace's equation as in other static field problems (Sec. 1.12). In addition to the boundary condition corresponding to the applied potentials, there is

[5] *The SI unit for conductivity is siemens per meter (S/m), but older literature sometimes uses mho as the conductivity unit.*

a constraint at the boundaries between conductors and insulators since there can be no current flow across such boundaries. Referring to (2), this requires that for such boundaries, on the conductor side,

$$\frac{\partial \Phi}{\partial n} = 0 \tag{6}$$

where n denotes the normal to such boundaries. Continuity relations between different conductors are considered in the following section.

1.14 Boundary Conditions in Electrostatics

Most practical field problems involve systems containing more than one kind of material. We have seen some examples of boundaries between various regions in the examples of earlier sections, but now we need to develop these systematically to utilize the differential equations of the preceding sections.

Let us consider the relations between normal flux density components across an arbitrary boundary by using the integral form of Gauss's law. Consider an imaginary pillbox bisected as shown in Fig. 1.14*a* by the interface between regions 1 and 2. The thickness of the pillbox is considered to be small enough that the net flux out the sides vanishes in comparison with that out the flat faces. If we assume the existence of net surface charge ρ_s on the boundary, the total flux out of the box must equal $\rho_s \, \Delta S$. By Gauss's law,

$$D_{n1} \, \Delta S - D_{n2} \, \Delta S = \rho_s \, \Delta S$$

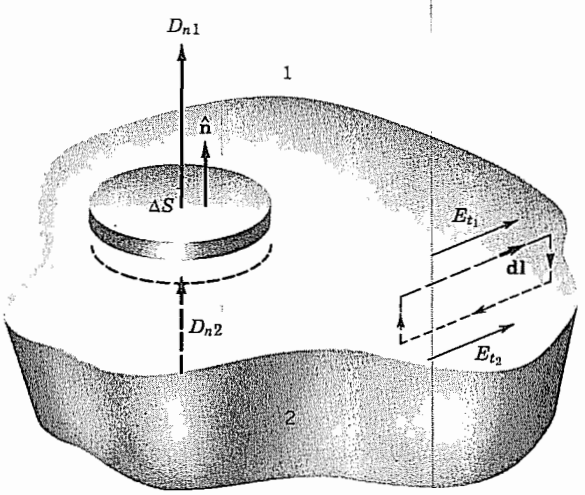

Fig. 1.14*a* Boundary between two different media.

or

$$D_{n1} - D_{n2} = \rho_s \tag{1}$$

where ΔS is small enough to consider D_n and ρ_s to be uniform.

A second relation may be found by taking a line integral about a closed path of length Δl on one side of the boundary and returning on the other side as indicated in Fig. 1.14*a*. The sides normal to the boundary are considered to be small enough that their net contributions to the integral vanish in comparison with those of the sides parallel to the surface. By Eq. 1.7(3) any closed line integral of electrostatic field must be zero:

$$\oint \mathbf{E} \cdot \mathbf{dl} = E_{t1}\,\Delta l - E_{t2}\,\Delta l = 0$$

or

$$E_{t1} = E_{t2} \tag{2}$$

The subscript t denotes components tangential to the surface. The length of the tangential sides of the loop is small enough to take E_t as a constant over the length. Since the integral of $\mathbf{E}$ across the boundary is negligibly small,

$$\Phi_1 = \Phi_2 \tag{3}$$

across the boundary. Equations (1) and (2), or (1) and (3), form a complete set of boundary conditions for the solution of electrostatic field problems.

Consider an interface between two dielectrics with no charge on the surface. From (1) and Eq. 1.3(1).

$$\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2} \tag{4}$$

It is clear that the normal component of $\mathbf{E}$ changes across the boundary, whereas the tangential component, according to (2), is unmodified. Therefore the direction of the resultant $\mathbf{E}$ must change across such a boundary except where either E_n or E_t are zero. Suppose that at some point at a boundary between two dielectrics the electric field in region 1 makes an angle θ_1 with the normal to the boundary. Thus, as seen in Fig. 1.14*b*,

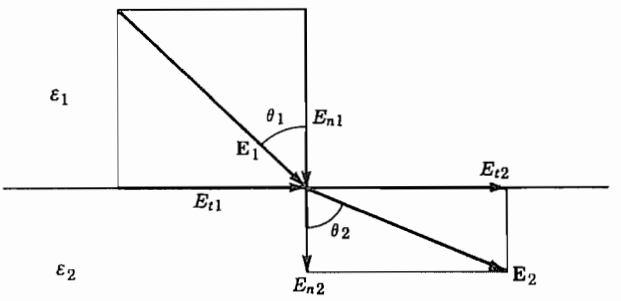

FIG. 1.14*b* Vector relations among electric field components at a point on a boundary between dielectrics ($\varepsilon_2 > \varepsilon_1$).

$$\theta_1 = \tan^{-1} \frac{E_{t1}}{E_{n1}} \tag{5}$$

and

$$\theta_2 = \tan^{-1} \frac{E_{t2}}{E_{n2}} \tag{6}$$

Then using (2) and (4) we see that

$$\theta_2 = \tan^{-1} \left(\frac{\varepsilon_2}{\varepsilon_1} \tan \theta_1 \right) \tag{7}$$

Let us now consider the boundary properties of conductors, as exemplified by a piece of semiconductor with metallic contacts. The currents flowing in both the semiconductor and the metallic contacts are related to the fields by the respective conductivities so potential drops occur in them, as developed in Sec. 1.13.

At the interface of two contiguous conductors, the normal component of current density is continuous across the boundary, because otherwise there would be a continual buildup of charges there. The normal component of electric field is therefore discontinuous and given by

$$E_{n1} = \frac{\sigma_2}{\sigma_1} E_{n2} \tag{8}$$

The argument used in dielectric problems for tangential components of electric field also applies in the case of a discontinuity of conductivity so that tangential electric field is continuous across the boundary.

In problems with metallic electrodes on a less conductive material such as a semiconductor, one normally assumes that the conductivity of the metallic electrode is so high compared with that of the other material that negligible potential drops occur in the metal. The electrodes are assumed to be perfect conductors with equipotential surfaces. In this text we normally assume for dc problems that the electrodes in either electrostatic or dc conduction problems are equipotentials and therefore the tangential electric fields at their surfaces are zero.

Example 1.14
BOUNDARY CONDITIONS IN A DC CONDUCTION PROBLEM

The structure in Fig. 1.14*c* illustrates some of the points we have made about boundary conditions. We assume that a potential difference is provided between the two metallic electrodes by an outside source. The electrodes are considered to be perfect conductors and, therefore, equipotentials. The space directly between them is filled with a layer of conductive material having conductivity σ and permittivity ε. The space surrounding the conductive system is filled with a dielectric having permittivity ε_1 and $\sigma = 0$.

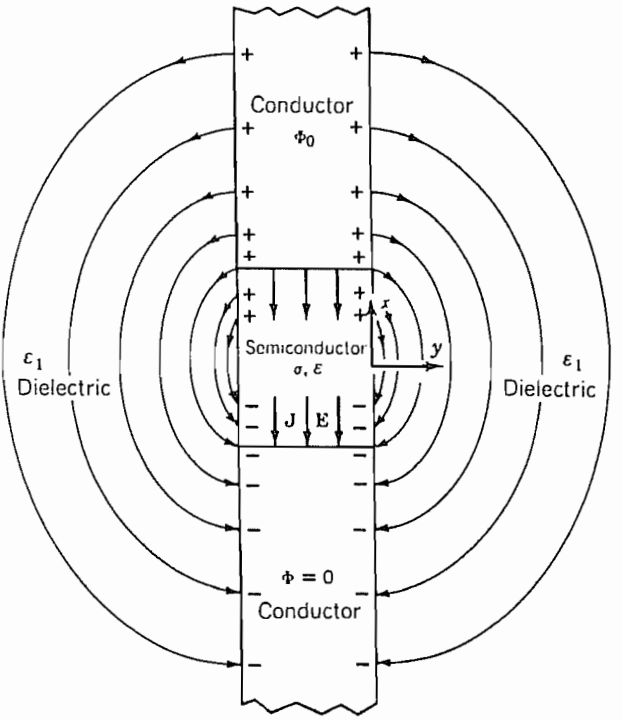

FIG. 1.14c Structure involving both dc conduction and static fields in a dielectric to illustrate boundary conditions.

The normal component of the current density at the conductor–dielectric interface must be zero since the current in the dielectric is zero. This follows formally from (8) using zero conductivity for the dielectric. Therefore, since the electric field is $\mathbf{J}/\sigma$ the normal component of electric field inside the conductor must be zero. Then the electric fields inside the conductor at the boundaries with the dielectric are wholly tangential. Since the electrodes are assumed to be equipotentials, the tangential field there is zero and the field lines are perpendicular to the electrode surfaces.

It is clear that since there is an electric field in the conductive medium and it has a permittivity, there is also a flux density following the same paths as the current density. The flux must terminate on charges so there will be charges on the electrode-to-conductor boundary.

The electric field in the dielectric region terminates on charges both on the perfectly conducting electrodes and on the sides of the other conductive medium. On the boundary of the imperfect conductor there is a tangential component and also surface charges, so the field makes an oblique angle with the conductor surface.

1.15 Direct Integration of Laplace's Equation: Field Between Coaxial Cylinders with Two Dielectrics

The first simple example of Laplace's equation we will take is that of finding the potential distribution between two coaxial conducting cylinders of radii a and c (Fig. 1.15), with a dielectric of constant ε_1 filling the region between a and b, and a second dielectric of constant ε_2 filling the region between b and c. The inner conductor is at potential zero, and the outer at potential V_0. Because of the symmetry of the problem, the solution could be readily obtained by using Gauss's law as in Example 1.4b, but the primary purpose here is to demonstrate several processes in the solution by differential equations.

The geometrical form suggests that the Laplacian $\nabla^2\Phi$ be expressed in cylindrical coordinates (see inside front cover), giving for Laplace's equation

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{1}$$

It will be assumed that there is no variation in the axial (z) direction, and the cylindrical symmetry eliminates variations with angle ϕ. Equation (1) then reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = 0 \tag{2}$$

Note that in (2) the derivative is written as a total derivative, since there is now only one independent variable in the problem. Equation (2) may be integrated directly:

$$r\frac{d\Phi}{dr} = C_1 \tag{3}$$

Integrating again, we have

$$\Phi_1 = C_1 \ln r + C_2 \tag{4}$$

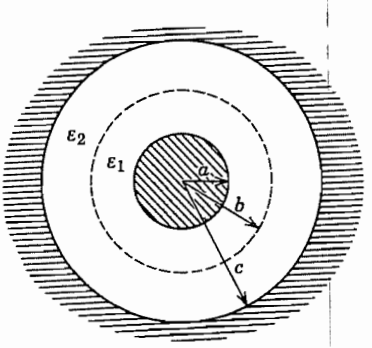

Fig. 1.15 Coaxial cylinders with two dielectrics.

This has been labeled Φ_1 because we will consider that the result of (4) is applicable to the first dielectric region ($a < r < b$). The same differential equation with the same symmetry applies to the second dielectric region, so the same form of solution applies there also, but the arbitrary constants may be different. So, for the potential in region 2 ($b < r < c$), let us write

$$\Phi_2 = C_3 \ln r + C_4 \tag{5}$$

The boundary conditions at the two conductors are:

(a) $\Phi_1 = 0$ at $r = a$

(b) $\Phi_2 = V_0$ at $r = c$

In addition, there are continuity conditions at the boundary between the two dielectric media. The potential and the normal component of electric flux density must be continuous across this charge-free boundary (Sec. 1.14):

(c) $\Phi_1 = \Phi_2$ at $r = b$

(d) $D_{r1} = D_{r2}$ at $r = b$, or $\varepsilon_1 (d\Phi_1/dr) = \varepsilon_2 (d\Phi_2/dr)$ there.

The application of condition (a) to (4) yields

$$C_2 = -C_1 \ln a \tag{6}$$

The application of (b) to (5) yields

$$C_4 = V_0 - C_3 \ln c \tag{7}$$

Condition (c) applied to (4) and (5) gives

$$C_1 \ln b + C_2 = C_3 \ln b + C_4 \tag{8}$$

And condition (d) applied to (4) and (5) gives

$$\varepsilon_1 C_1 = \varepsilon_2 C_3 \tag{9}$$

Any one of the constants, as C_1, may be obtained by eliminating between the four equations, (6) to (9):

$$C_1 = \frac{V_0}{\ln(b/a) - (\varepsilon_1/\varepsilon_2) \ln(b/c)} \tag{10}$$

The remaining constants, C_2, C_3, and C_4, may be obtained from (6), (9), and (7), respectively. The results are substituted in (4) and (5) to give the potential distribution in the two dielectric regions:

$$\Phi_1 = \frac{V_0 \ln(r/a)}{\ln(b/a) + (\varepsilon_1/\varepsilon_2) \ln(c/b)} \qquad a < r < b \tag{11}$$

$$\Phi_2 = V_0 \left[1 - \frac{(\varepsilon_1/\varepsilon_2) \ln(c/r)}{\ln(b/a) + (\varepsilon_1/\varepsilon_2) \ln(c/b)} \right] \qquad b < r < c \tag{12}$$

It can be checked that these distributions do satisfy Laplace's equation and the boundary and continuity conditions of the problem. Only in such simple problems as this will it be possible to obtain solutions of the differential equation by direct integration, but the method of applying boundary and continuity conditions to the solutions, however obtained, is well demonstrated by the example.

1.16 Direct Integration of Poisson's Equation: The pn Semiconductor Junction

The *pn* semiconductor junction is an important practical example in which properties can be found by direct integration of Poisson's equation. Figure 1.16*a* shows a simplified *pn* junction. The basic semiconductor is typically a valence 4 material such as silicon (or a compound semiconductor such as gallium arsenide that behaves much the same). The *n* region of the figure has been "doped" with valence 5 impurity atoms such as phosphorus (donors), which although electrically neutral in themselves, have more electrons than needed for bonding with adjacent silicon atoms and so contribute electrons which can move relatively freely about the material. The *p* region of the figure has valence 3 impurities such as boron (acceptors) which have fewer electrons than needed for bonding with adjacent silicon atoms. These too are electrically neutral in themselves, but leave *holes* that move from atom to atom with electric fields or other forces much like small positive charges. Although the transition between *p* and *n* regions must be over some finite region, we assume an idealized model in which it is abrupt—a step discontinuity.

When the junction is formed, the excess electrons in the *n*-type region at first diffuse into the *p*-type side. The holes diffuse to the *n*-type side. The electrons flowing into the *p*-type side fill the vacancies in the acceptor bonds, causing them to become negatively charged. (Remember that they were originally electrically neutral). Likewise, the holes moving into the *n*-type side are filled by the excess electrons there. The result is a zone near the junction in which there is a net negative charge density in a region on the *p*-type side and a net positive charge density on the *n*-type side called a *depletion region*, as in the metal–semiconductor junction of Sec. 1.4. The density on the *n*-type side is eN_D since each of the donor atoms has been stripped of one electron. The density on the *p*-type side is $-eN_A$ since each acceptor atom has one additional electron. The widths of the zones stabilize when the potential arising in the charge regions is sufficient to prevent further diffusion. Outside the charged regions, the semiconductors are neutral. No fields exist there in the equilibrium situation that we shall examine; if a field did exist it would cause motion of the charges and violate the assumption of equilibrium. The regions of charge are shown (not to scale) in Fig. 1.16*b*.

One can deduce the form of the gradient of potential $d\Phi/dx = -E_x = -D_x/\varepsilon$ from Gauss's law as was done in Ex. 1.4a for the metal–semiconductor contact. We have just argued that E_x is zero outside the charge regions ($x < -d_p$ and $x > d_n$). All the flux from the positive charges in $0 < x < d_n$ must therefore end on the negative charges in $-d_p < x < 0$. This determines the relation between d_n and d_p in terms of

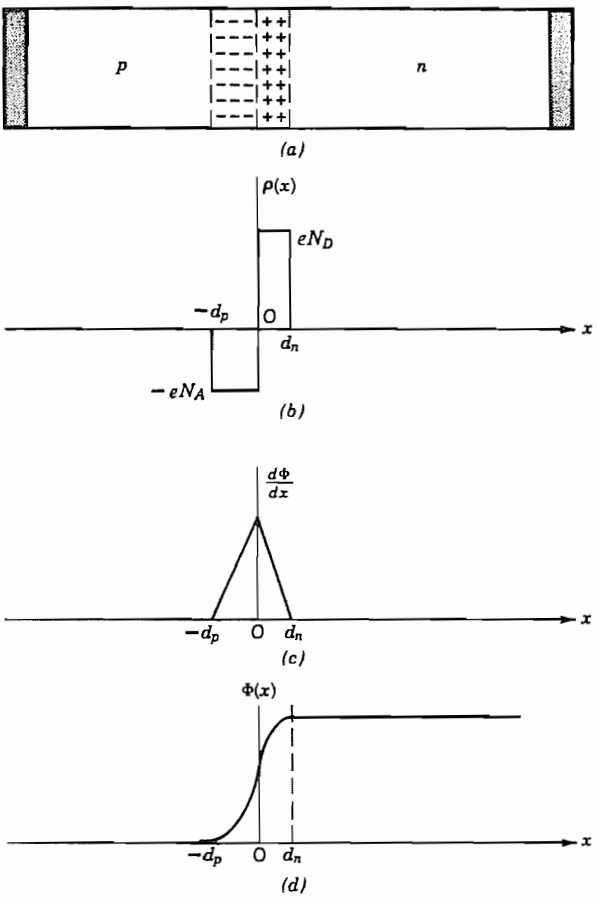

FIG. 1.16 (*a*) *pn* diode showing regions of uncompensated charge. (*b*) Charge density in the diode. (*c*) Potential gradient. (*d*) Potential.

the given values of N_A and N_D. The flux density D_x is negative and its magnitude increases linearly from $x = -d_p$ to $x = 0$ since the charge density is taken to be constant. It then falls linearly to zero between $x = 0$ and $x = d_n$. The gradient therefore takes the form shown in Fig. 1.16*c*. The potential is the integral of the linearly varying gradient so it has the square-law form in Fig. 1.16*d*.

Now let us directly integrate Poisson's equation to get the complete analytic forms. The boundary conditions on the integration are that the gradient is zero at $x = -d_p$ and at d_n. The potential may be taken arbitrarily to have its zero at $x = -d_p$. Specializing Poisson's equation 1.12(3) to one dimension and substituting for charge density the value $-eN_A$ for the region $-d_p < x < 0$, we have

$$\frac{d^2\Phi}{dx^2} = \frac{eN_A}{\varepsilon} \tag{1}$$

The permittivity ε is not appreciably affected by the dopant, at least for low-frequency considerations. Integrating (1) from $x = -d_p$ to an arbitrary $x = 0$ making use of the zero boundary condition on the gradient at $x = -d_p$ gives

$$\left.\frac{d\Phi}{dx}\right|_x = \frac{eN_A}{\varepsilon}(d_p + x) \tag{2}$$

Integrating a second time taking $\Phi(-d_p) = 0$ gives the potential for $-d_p \leq x \leq 0$ as

$$\Phi(x) = \frac{eN_A}{2\varepsilon}(x + d_p)^2 \tag{3}$$

The gradient and potential evaluated at $x = 0$ are

$$\left.\frac{d\Phi}{dx}\right|_0 = \frac{eN_A}{\varepsilon} d_p \tag{4}$$

$$\Phi(0) = \frac{eN_A}{2\varepsilon} d_p^2 \tag{5}$$

These constitute the boundary conditions for the integration in the region $0 \leq x \leq d_n$. Poisson's equation for this region differs from (1) in the choice of charge density:

$$\frac{d^2\Phi}{dx^2} = -\frac{eN_D}{\varepsilon} \tag{6}$$

Integrating (6) with the boundary condition (4) gives

$$\left.\frac{d\Phi}{dx}\right|_x = -\frac{eN_D}{\varepsilon} x + \frac{eN_A}{\varepsilon} d_p \tag{7}$$

Then using the condition that the total positive charge must equal the total negative charge and therefore that $N_D = N_A(d_p/d_n)$, (7) can be put in the form

$$\left.\frac{d\Phi}{dx}\right|_x = \frac{eN_A d_p}{\varepsilon}\left(1 - \frac{x}{d_n}\right) \tag{8}$$

which is seen to be zero at $x = d_n$, as expected from Gauss's law as discussed above. Integrating (8) with the boundary condition (5), we find

$$\Phi(x) = \frac{eN_A d_p^2}{2\varepsilon}\left(1 + 2\frac{x}{d_p} - \frac{x^2}{d_n d_p}\right) \tag{9}$$

The maximum value of the potential is reached at $x = d_n$:

$$\Delta\Phi = \Phi(d_n) = \frac{eN_A d_p^2}{2\varepsilon}\left(1 + \frac{N_A}{N_D}\right) \tag{10}$$

Note that distance d_p in (10) is not immediately known. Since the potential barrier arises

to stop the diffusion of the charge carriers, it is expected that diffusion must also be considered. Diffusion theory reveals that the height of the potential barrier (10) is

$$\Delta\Phi = \frac{k_B T}{e} \ln\left[\frac{N_A N_D}{n_i^2}\right] \tag{11}$$

where k_B is Boltzmann's constant, N_A and N_D are acceptor and donor doping densities, respectively, and n_i is the electron density of intrinsic (undoped) silicon, which is about 1.5×10^{10} electrons/cm^3 at $T = 300$ K. Once $\Delta\Phi$ is calculated, d_p can be found from (10) and all quantities in the field and potential expressions are then known.

1.17 Uniqueness of Solutions

It can be shown that the potentials governed either by the Laplace or Poisson equation in regions with given potentials on the boundaries are unique. With normal derivatives of potential (or, equivalently, charges) specified on the boundaries, the potential is unique to within an additive constant. Here we will prove the theorem for a charge-free region with potential specified on the boundary. The proofs of the other parts of the theorem are left as problems.

The usual way to demonstrate uniqueness of a quantity is first to assume the contrary and then show this assumption to be false. Imagine two possible solutions, Φ_1 and Φ_2. Since they must both reduce to the given potential along the boundary,

$$\Phi_1 - \Phi_2 = 0 \tag{1}$$

along the boundary surface. Since they are both solutions to Laplace's equation,

$$\nabla^2\Phi_1 = 0 \text{ and } \nabla^2\Phi_2 = 0$$

or

$$\nabla^2(\Phi_1 - \Phi_2) = 0 \tag{2}$$

throughout the entire region.

In the divergence theorem, Eq. 1.11(7), **F** may be any continuous vector quantity. In particular, let it be the quantity

$$(\Phi_1 - \Phi_2)\nabla(\Phi_1 - \Phi_2)$$

Then

$$\int_V \nabla \cdot [(\Phi_1 - \Phi_2)\nabla(\Phi_1 - \Phi_2)]\, dV = \oint_S [(\Phi_1 - \Phi_2)\nabla(\Phi_1 - \Phi_2)] \cdot \mathbf{dS}$$

From the vector identity

$$\text{div}(\psi\mathbf{A}) = \psi \text{ div } \mathbf{A} + \mathbf{A} \cdot \text{grad } \psi$$

the equation may be expanded to

$$\int_V (\Phi_1 - \Phi_2)\nabla^2(\Phi_1 - \Phi_2)\, dV + \int_V [\nabla(\Phi_1 - \Phi_2)]^2\, dV = \oint_S (\Phi_1 - \Phi_2)\nabla(\Phi_1 - \Phi_2) \cdot \mathbf{dS}$$

The first integral must be zero by (2); the last integral must be zero, since (1) holds over the boundary surface. There remains

$$\int_V [\nabla(\Phi_1 - \Phi_2)]^2\, dV = 0 \tag{3}$$

The gradient of a real scalar is real. Thus its square can only be positive or zero. If its integral is to be zero, the gradient itself must be zero:

$$\nabla(\Phi_1 - \Phi_2) = 0 \tag{4}$$

or

$$(\Phi_1 - \Phi_2) = \text{constant} \tag{5}$$

This constant must apply even to the boundary, where we know that (1) is true. The constant is then zero, and $\Phi_1 - \Phi_2$ is everywhere zero, which means that Φ_1 and Φ_2 are identical potential distributions. Hence the proof of uniqueness: Laplace's equation can have only one solution which satisfies the boundary conditions of the given region. If by any method we find a solution to a field problem that fits all boundary conditions and satisfies Laplace's equation, we may be sure it is the only one.

Special Techniques for Electrostatic Problems

1.18 The Use of Images

The so-called method of images is a way of finding the fields produced by charges in the presence of dielectric[6] or conducting boundaries with certain symmetries. Here we concentrate on the more common situations, those with conducting boundaries. (But see Prob. 1.18d.)

[6] *W. R. Smythe,* Static and Dynamic Electricity, *Hemisphere Publishing Co., Washington, DC, 1989.*

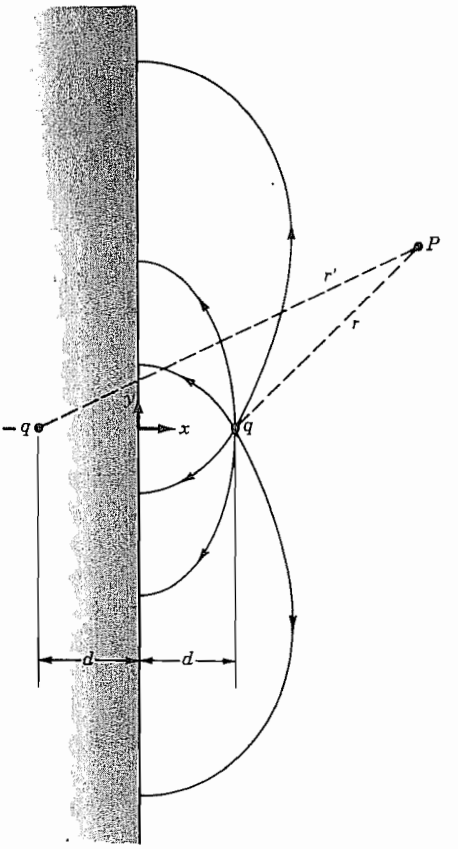

FIG. 1.18*a* Image of a point charge in a conducting plane. The field lines shown are for the charge q with the conductor.

Example 1.18a
POINT IMAGE IN A PLANE

The simplest case is that of a point charge near a grounded[7] conducting plane (Fig. 1.18*a*). Boundary conditions require that the potential along the plane be zero. The requirement is met if in place of the conducting plane an equal and opposite image charge is placed at $x = -d$. Potential at any point P is then given by

$$\Phi = \frac{1}{4\pi\varepsilon}\left(\frac{q}{r} - \frac{q}{r'}\right) \tag{1}$$
$$= \frac{q}{4\pi\varepsilon}\{[(x-d)^2 + y^2 + z^2]^{-1/2} - [(x+d)^2 + y^2 + z^2]^{-1/2}\}$$

[7] *Use of the term* grounded *implies a source for the charge that builds up on the plane.*

This reduces to the required zero potential along the plane $x = 0$, so that (1) gives the potential for any point to the right of the plane. The expression of course does not apply for $x < 0$, for inside the conductor the potential must be everywhere zero.

If the plane is at a potential other than zero, the value of this constant potential is simply added to (1) to give the expression for potential at any point for $x > 0$.

The charge density on the surface of the conducting plane must equal the normal flux density at that point. This is easily found by using

$$\rho_s = D_x = \varepsilon E_x = -\varepsilon \frac{\partial \Phi}{\partial x}\bigg|_{x=0} \tag{2}$$

Substituting (1) in (2) and performing the indicated differentiation gives

$$\rho_s = -\frac{qd}{2\pi}(d^2 + y^2 + z^2)^{-3/2} \tag{3}$$

Analysis of (3) shows that the surface charge density has its peak value at $y = z = 0$, with circular contours of equal charge density centered about that point. The density decreases monotonically to zero as y and/or z go to infinity. One application of the image method is in studying the extraction of electrons from a metallic surface as in the metal–semiconductor surface shown in Fig. 1.4*a*.

Example 1.18b

IMAGE OF A LINE CHARGE IN A PLANE

If there is a line charge of strength q_l C/m parallel to a conducting plane and distance d from it, we proceed as above, placing an image line charge of strength $-q_l$ at $x = -d$. The potential at any point $x > 0$ is then

$$\Phi = -\frac{q_l}{2\pi\varepsilon} \ln\left(\frac{r}{r'}\right) = \frac{q_l}{4\pi\varepsilon} \ln\left[\frac{(x + d)^2 + y^2}{(x - d)^2 + y^2}\right] \tag{4}$$

Example 1.18c

IMAGE OF A LINE CHARGE IN A CYLINDER

For a line charge of strength q_l parallel to the axis of a conducting circular cylinder, and at radius r_q from the axis, the image line charge of strength $-q_l$ is placed at radius $r' = a^2/r_q$, where a is the radius of the cylinder (Fig. 1.18*b*). The combination of the two line charges can be shown to produce a constant potential along the given cylinder of radius a. Potential outside the cylinder may be computed from the original line charge and its image. (Add q_l on axis if cylinder is uncharged.) If the original line charge is within a hollow cylinder a, the rule for finding the image is the same, and potential inside may be computed from the line charges.

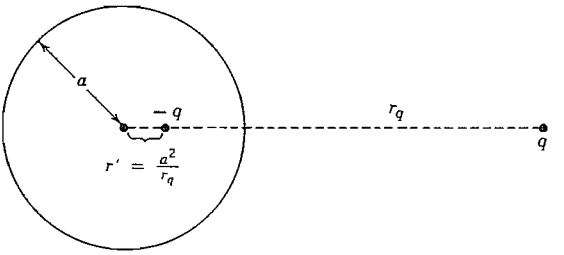

FIG. 1.18b Image of line charge q_l in a parallel conducting cylinder.

Example 1.18d
IMAGE OF A POINT CHARGE IN A SPHERE

For a point charge q placed distance r from the center of a conducting sphere of radius a, the image is a point charge of value $(-qa/r_q)$ placed at a distance (a^2/r_q) from the center (Fig. 1.18*c*). This combination is found to give zero potential along the spherical surface of radius a, and may be used to compute potential at any point P outside of radius a. (Or, if the original charge is inside, the image is outside, and the pair may be used to compute potential inside.)

Example 1.18e
MULTIPLE IMAGINGS

For a charge in the vicinity of the intersection of two conducting planes, such as q in the region of AOB of Fig. 1.18*d*, there might be a temptation to use only one image in each plane, as 1 and 2 of Fig. 1.18*d*. Although $+q$ at Q and $-q$ at 1 alone would give constant potential as required along OA, and $+q$ at Q and $-q$ at 2 alone would give constant potential along OB, the three charges together would give constant potential

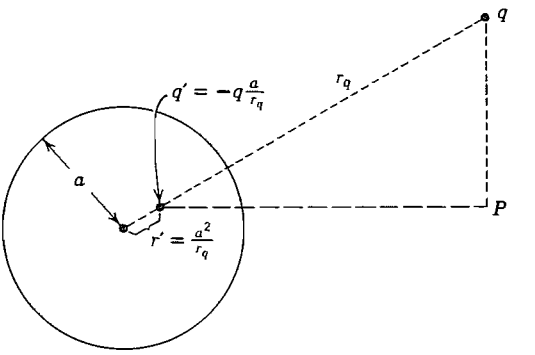

FIG. 1.18c Image of a point charge in a sphere.

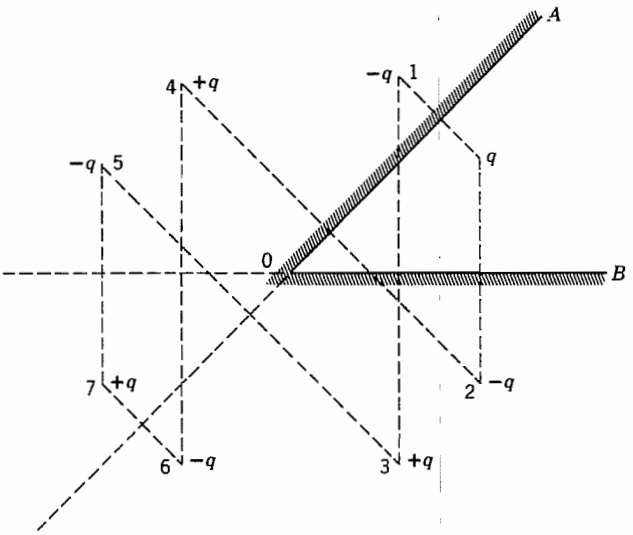

Fig. 1.18*d* Multiple images of a point or line charge between intersecting planes.

along neither *OA* nor *OB*. It is necessary to image these images in turn, repeating until further images coincide or until all further images are too far distant from the region to influence the potential. It is possible to satisfy exactly the required conditions with a finite number of images only if the angle *AOB* is an exact submultiple of 180 degrees, as in the 45-degree case illustrated by Fig. 1.18*d*.

1.19 Properties of Two-Dimensional Fields: Graphical Field Mapping

Many important electrostatic problems may be considered as two-dimensional, as in the pair of parallel wires of Fig. 1.8*b* or the coaxial system of Fig. 1.15. In these the field distribution is the same in all cross-sectional planes, and although real systems are never infinitely long, the idealization is often a useful one. In the examples cited above, the field distributions could be found analytically, but for cylindrical systems with more complicated boundaries, numerical techniques may be called for and will be introduced in the next section. We wish to give first some properties of two-dimensional fields that can be used to judge the correctness of field maps and can even be used to make useful pictures of the fields and to obtain approximate values of such things as capacitance, conductance, and breakdown voltage. Perhaps the greatest value in making a few such maps is the feel they give for field behavior.

It has already been established that equipotentials and electric field lines intersect at right angles, as in the coaxial system of Fig. 1.19*a*, where field lines are radial and equipotentials are circles in any given cross-sectional plane. It has also been shown that the region between two field lines may be considered a flux tube, and if the amount of

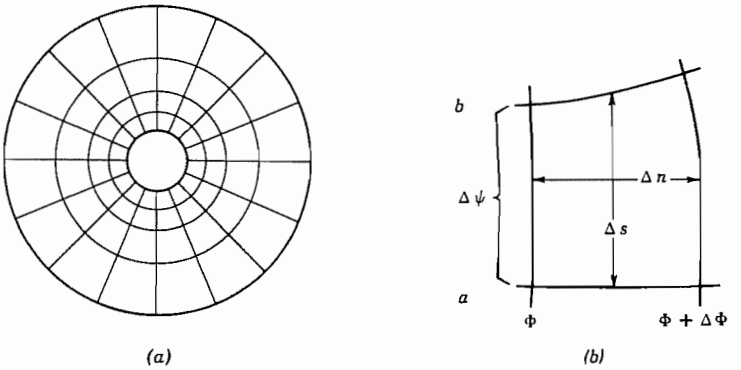

FIG. 1.19 (*a*) Map of field between coaxial conducting cylinders. (*b*) Curvilinear rectangle for graphical field mapping.

flux is properly chosen, the map is made up of small curvilinear figures with equal side ratios, that is, "curvilinear squares." This also is illustrated in Fig. 1.19*a*. To show this more generally, consider one of the curvilinear rectangles from a general plot, as in Fig. 1.19*b*. If Δn is the distance between two adjacent equipotentials, and Δs the distance between two adjacent field lines, the magnitude of electric field, assuming a small square, is approximately $\Delta\Phi/\Delta n$. The electric flux flowing along a flux tube bounded by the two adjacent field lines for a unit length perpendicular to the page is then

$$\Delta\psi = D\ \Delta s = \varepsilon E\ \Delta s = \frac{\varepsilon\ \Delta\Phi\ \Delta s}{\Delta n}$$

or

$$\frac{\Delta s}{\Delta n} = \frac{\Delta\psi}{\varepsilon\ \Delta\Phi} \tag{1}$$

So, if the flux per tube $\Delta\psi$, the potential difference per division $\Delta\Phi$, and the permittivity ε are constant throughout the plot, the side ratio $\Delta s/\Delta n$ must also be constant, as stated above.

We saw in Sec. 1.14 that conducting surfaces are equipotentials in an electrostatic field. Thus, the electric field lines meet the electrodes at right angles.

In applying the principles to the sketching of fields, some schedule such as the following will be helpful.

1. Plan on making a number of rough sketches, taking only a minute or so apiece, before starting any plot to be made with care. The use of transparent paper over the basic boundary will speed up this preliminary sketching.
2. Divide the known potential difference between electrodes into an equal number of divisions, say four or eight to begin with.
3. Begin the sketch of equipotentials in the region where the field is known best, as for example in some region where it approaches a uniform field. Extend the equipotentials according to your best guess throughout the plot. Note that they should

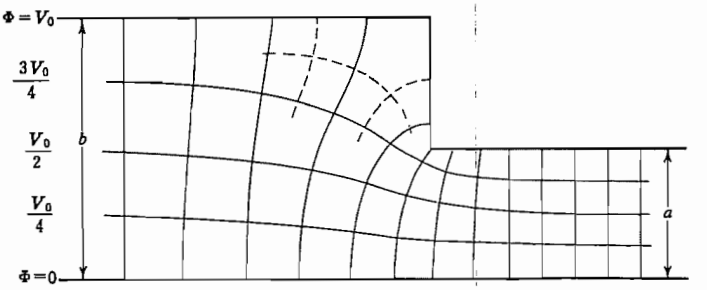

FIG. 1.19c Map of fields between a plane and stepped conductor.

tend to hug acute angles of the conducting boundary, and be spread out in the vicinity of obtuse angles of the boundary.

4. Draw in the orthogonal set of field lines. As these are started, they should form curvilinear squares, but as they are extended, the condition of orthogonality should be kept paramount, even though this will result in some rectangles with ratios other than unity.
5. Look at the regions with poor side ratios and try to see what was wrong with the first guess of equipotentials. Correct them and repeat the procedure until reasonable curvilinear squares exist throughout the plot.
6. In regions of low field intensity, there will be large figures, often of five or six sides. To judge the correctness of the plot in this region, these large units should be subdivided. The subdivisions should be started back away from the region needing subdivision, and each time a flux tube is divided in half, the potential divisions in this region must be divided by the same factor. As an example, Fig. 1.19*c* shows a map made to describe the field between a plane conductor at potential zero and a stepped plane at potential V_0 with a step ratio of $\frac{1}{2}$.

1.20 Numerical Solution of the Laplace and Poisson Equations

Numerical methods are becoming increasingly attractive as digital computer speed and memory capacity continue to increase. Among the powerful methods are those using finite differences,[8] finite elements,[8] Fourier transformations,[9] or method of moments (Sec. 7.3). Still others will undoubtedly be developed as computing capabilities continue to increase. Here we illustrate the idea through the elemental difference equation approach and some of its extensions.

[8] *L. Collatz,* The Numerical Treatment of Differential Equations, *Springer-Verlag, New York, 1966. D. Potter,* Computational Physics, *Wiley, New York, 1973. L. J. Segerlind,* Applied Finite Element Analysis, *2nd ed., Wiley, New York, 1984. R. Sorrentino (Ed.),* Numerical Methods for Passive Microwave and Millimeter Wave Structures, *IEEE Press, New York, 1989.*

[9] *R. W. Hockney and J. W. Eastwood,* Computer Simulation Using Particles, *Am. Inst. Physics, New York, 1988.*

We consider first the Poisson equation with potential specified on the boundary. For simplicity we take a two-dimensional problem (no variations in z). The internal region is divided by a grid of mutually orthogonal lines with potential eventually to be determined at each of the grid points. Rectangular coordinates are used and potential at a point (x, y) is expanded in a Taylor series:

$$\Phi(x + h, y) \approx \Phi(x, y) + h\frac{\partial\Phi(x, y)}{\partial x} + \frac{h^2}{2}\frac{\partial^2\Phi(x, y)}{\partial x^2} \tag{1}$$

$$\Phi(x - h, y) \approx \Phi(x, y) - h\frac{\partial\Phi(x, y)}{\partial x} + \frac{h^2}{2}\frac{\partial^2\Phi(x, y)}{\partial x^2} \tag{2}$$

By adding (1) and (2) and rearranging, we have the approximation

$$\frac{\partial^2\Phi(x, y)}{\partial x^2} = \frac{\Phi(x + h, y) - 2\Phi(x, y) + \Phi(x - h, y)}{h^2} \tag{3}$$

The second partial derivative with respect to y can be obtained in the same way. Then Poisson's equation in two dimensions

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = -\frac{\rho}{\varepsilon}$$

can be expressed in the approximate form

$$\Phi(x + h, y) + \Phi(x - h, y) + \Phi(x, y + h) + \Phi(x, y - h) - 4\Phi(x, y) = -\frac{\rho h^2}{\varepsilon} \tag{4}$$

where the distance increment h is taken, for simplicity, to be equal in the two directions. It is of interest to note that, if space charge is zero, the potential at a given point is the average of the potentials at the surrounding points.

Note that potential is known on boundary points so that a straightforward approach is to solve a set of equations such as (4) for the unknown potentials at the grid points in terms of the known values on boundary points. This is sometimes done by a matrix inversion technique, but if memory capacity is a problem, it may be better to use a method for direct iterative adjustment of grid potentials. This starts from an initial guess and corrects by bringing in the given values on the boundary through successive passes through the grid. We illustrate this first by a simple averaging technique.

Example 1.20
NUMERICAL SOLUTION OF LAPLACE EQUATION BY SIMPLE AVERAGING

As an illustration of iteration with simple averaging, let us find the potentials for the grid of points in the structure in Fig. 1.20a. This is an infinite cylinder of square cross section with the potentials specified on the entire boundary. The space charge will be assumed to be zero so we will be solving Laplace's equation. The broken lines represent

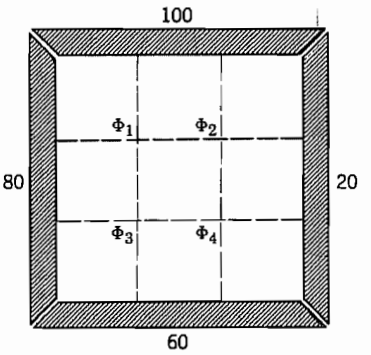

FIG. 1.20*a* Cylinder of square cross section and grid for difference equation solution.

the grid to be used to approximate the region for the finite-difference solution. The coarse grid was chosen to simplify the example; a finer grid would be used in most practical problems. The four unknown potentials designated Φ_1 to Φ_4 are assumed initially to be the average of the boundary potentials, 65 V. The first calculation is to find Φ_1 as the average of the four surrounding potentials (80, 100, $\Phi_2 = 65$, $\Phi_3 = 65$). Therefore, in the column labeled Step 1 in Table 1.20, Φ_1 is given the value 77.50. Then Φ_2 is found as the average of 100, 20, 77.50, and 65, and this value is put in the table in Step 1. The procedure is repeated for Φ_3 and Φ_4. The Step 2 proceeds in the same way. It is seen that after several steps the potentials converge to definite values. Since (4) is approximate, the potentials have converged to approximate answers. They would differ less from the correct solution if the grid were made finer. The correct potentials for the four points are also listed in the table.

Mesh Relaxation The above method, in which successive averaging of the potentials leads to a final result not far from the correct potentials, is convenient for small problems and, in many cases, has a satisfactory rate of convergence. A more generally useful method of calculation is based on a defined *residual* for each grid point that measures the amount by which the potential there differs from the value dictated by

Table 1.20
Iterative Calculation for Example 1.20

	Step					Correct
	1	2	3	4	5	Potentials
Φ_1	77.50	79.07	77.89	77.60	77.51	75.2
Φ_2	65.63	63.28	62.70	62.55	62.51	60.5
Φ_3	70.63	68.28	67.70	67.52	67.52	65.4
Φ_4	54.06	52.89	52.60	52.51	52.51	50.7

the potentials on the neighboring grid points. The residual for the kth pass through the grid is defined by

$$R^{(k)} = \Phi^{(i)}(x, y + h) + \Phi^{(i)}(x, y - h) + \Phi^{(i)}(x + h, y) + \Phi^{(i)}(x - h, y) - 4\Phi^{(k-1)}(x, y) + \frac{h^2\rho}{\varepsilon} \tag{5}$$

where $i = k$ or $k - 1$ since, at any grid point, the potentials at some of the neighboring points will generally have already been adjusted on that pass through the grid, as was seen in the above illustration. On each pass (corresponding to Steps 1–5 in Table 1.20) and at each grid intersection, one calculates the residual $R^{(k)}$ and then the new potential according to

$$\Phi^{(k)} = \Phi^{(k-1)} + \Omega \frac{R^{(k)}}{4} \tag{6}$$

where Ω is called the *relaxation factor* because it determines the rate at which the potentials relax toward the correct solution. It is taken in the range $1 \leq \Omega \leq 2$. Selection of $\Omega = 1$, called *simple relaxation*, corresponds to the method of averaging illustrated above. When $\Omega > 1$, the procedure is called the method of *successive overrelaxation* (SOR). If Ω is fixed, it is usually taken near 2 for problems with many mesh points, but this can cause an initial increase in error, so it is often better to start with $\Omega = 1$ and increase it gradually with each iteration step. One procedure that is found useful for large grids is the *cyclic Chebyshev method* in which the following program of varying Ω is used:

$$\Omega^{(1)} = 1 \tag{7}$$

$$\Omega^{(2)} = \frac{1}{1 - \frac{1}{2}\eta_{it}} \tag{8}$$

where

$$\eta_{it} = \frac{1}{4}\left(\cos\frac{\pi}{n} + \cos\frac{\pi}{m}\right)^2 \tag{9}$$

for a grid with n mesh points in one direction and m points in the other.

$$\Omega^{(k+1)} = \frac{1}{1 - \frac{1}{4}\eta_{it}\Omega^{(k)}} \tag{10}$$

$$\Omega^{(\infty)} = \Omega_{opt} = \frac{2}{1 + \sqrt{1 - \eta_{it}}} \tag{11}$$

When this method is used, the mesh is swept like a checkerboard, with all the red squares being treated on the first pass with $\Omega^{(1)}$, all the black squares being treated on the second pass using $\Omega^{(2)}$, the reds on the third with $\Omega^{(3)}$, and so on. This method requires an additional memory cell in the computer for each mesh point but the speeded convergence usually makes it worthwhile. The convergence can be improved appreci-

ably by having a good initial guess for the potentials, say by using results from a similar problem.

Boundary Conditions In setting up the boundary conditions on a grid, the easiest situation occurs when potentials are specified on grid points. A more difficult problem is where the normal derivatives are specified. The derivatives are usually zero, as would be the case at an insulating surface in a conduction problem (Ex. 1.14) or along a line of symmetry used as an artificial boundary to reduce the required grid size. Examples of such boundaries are shown in Figs. 1.20*b* and 1.20*c*. In one case the structure has obvious symmetry in the x–y plane so that a solution need be found for only one-fourth of the structure. In the other example, the boundary may be taken along the axis of a cylindrically symmetric system. In the latter case, the difference equations can include the symmetry (Prob. 1.20*e*). In the former case, to make the normal derivative of potential zero at a boundary, an imaginary grid point is set up outside the boundary and its potential is kept the same as at the point symmetrically located just inside the boundary. Sometimes the boundary points do not lie on mesh points; in such cases, linear interpolation is used to set the potentials at mesh points nearest to the boundary.

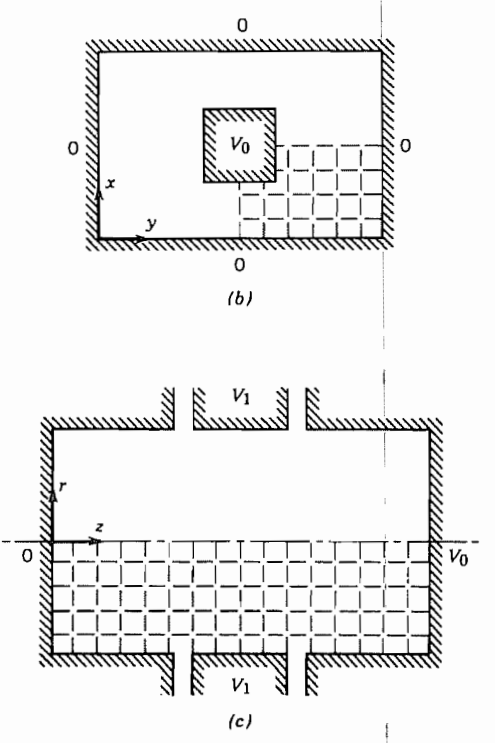

FIG. 1.20*b,c* Examples in rectangular and cylindrical coordinates where symmetry reduces the required size of the grid for finite-difference solutions.

1.21 Examples of Information Obtained from Field Maps

Field maps of the two-dimensional regions, made by either numerical or graphical techniques, may be used to find field strength within the dielectric region or integral properties of the systems, such as capacitance per unit length and conductance per unit length. Field strengths are simply $\Delta\Phi/\Delta n$ in the notation of Sec. 1.19, provided the divisions are fine enough. Danger of breakdown is obviously greatest near acute angles where spacing of equipotentials is smallest, as in the right-angle corner of Fig. 1.19*c*.

For capacitance, we need to know electric flux density at the conductors, which corresponds to surface charge density. Values of potential, and not field, are typically obtained from numerical solutions, but the approximation to a normal derivative at the boundary is readily calculated. Equipotentials and field lines may be drawn in and then calculation of capacitance becomes particularly simple. By Gauss's law, the charge induced on a conductor is equal to the flux ending there. This is the number of flux tubes N_f multiplied by the flux per tube. The potential difference between conductors is the number of potential divisions N_p multiplied by the potential difference per division. So, for a two-conductor system, the capacitance per unit length is

$$C = \frac{Q}{\Phi_2 - \Phi_1} = \frac{N_f \Delta\psi}{N_p \Delta\Phi}$$

The ratio $\Delta\psi/\Delta\Phi$ can be obtained from Eq. 1.19(1):

$$C = \frac{N_f}{N_p}\left(\frac{\varepsilon \Delta s}{\Delta n}\right) \tag{1}$$

And, for a small-square plot with $\Delta s/\Delta n$ equal to unity,

$$C = \varepsilon \frac{N_f}{N_p} \quad \text{F/m} \tag{2}$$

For example, in the coaxial line plot of Fig. 1.19*a*, there are 4 potential divisions and 16 flux tubes, so the capacitance, assuming air dielectric, is

$$C \approx \frac{10^{-9}}{36\pi} \times \frac{16}{4} = 35.3 \times 10^{-12} \text{ F/m} \tag{3}$$

Calculation from Eq. 1.9(4), with $b/a = 5.2$, gives 33.6×10^{-12} F/m, indicating that the map is not perfect.

This same technique can be used to find the conductance between two electrodes placed in a homogeneous, isotropic, conductive material. The conductivity of the electrode materials must be much greater than that of the surrounding region to ensure that, when current flows, there is negligible voltage drop in the electrodes and they can be considered to be equipotential regions. The potential and electric field are related in the same way as for the case in which there is no conductivity (Sec. 1.13). There is a current density $\mathbf{J} = \sigma \mathbf{E}$, where σ is conductivity, and current tubes replace the flux tubes of the dielectric problem. The current in a tube is

$$\Delta I = J\,\Delta s = \sigma E\,\Delta s = \frac{\sigma\,\Delta\Phi\,\Delta s}{\Delta n} \tag{4}$$

The conductance per unit length between two electrodes is defined as

$$G = \frac{I}{\Phi_2 - \Phi_1} = \frac{N_f\,\Delta I}{N_p\,\Delta\Phi} \quad \text{S/m} \tag{5}$$

Using (4) and taking $\Delta s/\Delta n = 1$,

$$G = \sigma\frac{N_f}{N_p} \quad \text{S/m} \tag{6}$$

From (2) and (6) we see the useful conclusion that the conductance per unit length of electrodes is related to the capacitance per unit length between the same electrodes by the ratio σ/ε. This can be of use, for example, in transmission-line problems in giving the conductance per unit length between conductors when the capacitance is known.

Before digital computer methods for solving field problems became so readily available, the analogy seen above between the field distributions in conducting and dielectric media formed the basis for an important means of determining fields in dielectric systems. Electrodes corresponding to those of the dielectric problem are set up in an electrolytic tank or on conduction paper and equipotentials measured in the conducting system (see Prob. 1.21c).

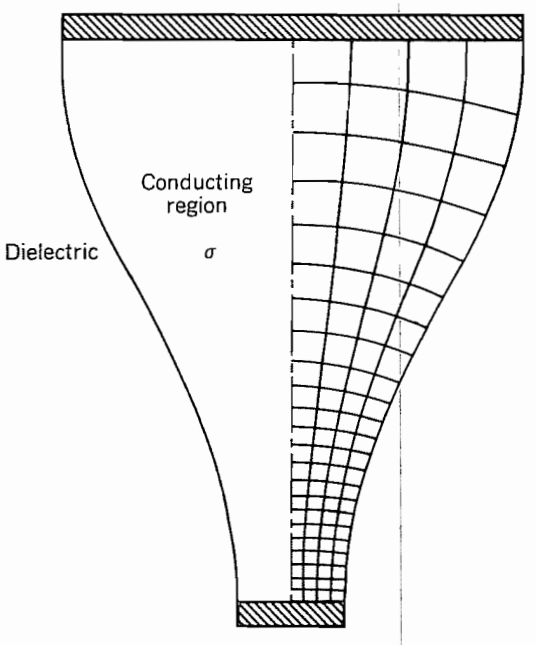

FIG. 1.21 Field map for a conductive medium partially filling the space between two highly conductive electrodes.

If a part of the boundary of the conducting region is a nonconducting dielectric, as in Fig. 1.21, current does not flow in the nonconductive region. As pointed out in Sec. 1.14, the normal component of **E** inside the conductive region must then vanish at the boundary with the dielectric region. By use of this condition and the fact that **E** is perpendicular to the equipotential electrode surfaces, the field in the conductor can be mapped. Suppose, for example, the conductive material between the electrodes in Fig. 1.21 is silicon with the common value of conductivity $\sigma = 100$ S/m. Then $G = 100 \times (8/23) = 35$ S/m.

Energy in Fields

1.22 Energy of an Electrostatic System

The aim of this section is to derive an expression for electrostatic energy in terms of field quantities. The result we will obtain can be shown to be true in general; for simplicity, however, it is shown here for charges in an unbounded region.

The work required to move a charge in the vicinity of a system of charges was discussed in Sec. 1.7. The work done must appear as energy stored in the system, and consequently the potential energy of a system of charges may be computed from the magnitudes and positions of the charges. To do this, let us consider bringing the charges from infinity to their positions in space. No force is required to bring the first charge in since no electric field acts on the charge. When the second charge q_2 is brought to a position separated from the location of q_1 by a distance R_{12}, an energy

$$U_{12} = \frac{q_1 q_2}{4\pi\varepsilon R_{12}} \tag{1}$$

is expended, as was shown in Sec. 1.7. When the third charge is brought from infinity, it experiences the fields of q_1 and q_2 and an energy of

$$U_{13} + U_{23} = \frac{q_1 q_3}{4\pi\varepsilon R_{13}} + \frac{q_2 q_3}{4\pi\varepsilon R_{23}} \tag{2}$$

is expended. The total energy expended to assemble these three charges is the sum of (1) and (2).

In summing over the three charges, we may write

$$U = \frac{1}{2}\sum_{i=1}^{3} q_i \sum_{j=1}^{3} \frac{q_j}{4\pi\varepsilon R_{ij}} \qquad i \neq j$$

With the factor $\frac{1}{2}$, this yields the sum of (1) and (2), since by convention i and j are summed over all the particles, and each contribution to energy enters twice. In physical terms, the factor of 2 would result from assuming all other charges in position when finding the energy of the ith charge. The term $i = j$ has been excluded since the self-energy of the point charge (i.e., the electron, ion, etc.) does not affect the energy of the field. Its contribution to the total system energy does not depend upon the relative positions of the charges. For n charges, the direct extension gives

$$U_E = \frac{1}{2} \sum_{i=1}^{n} q_i \sum_{j=1}^{n} \frac{q_j}{4\pi\varepsilon R_{ij}} \qquad i \neq j \tag{3}$$

where the subscript E indicates energy stored in electric charges and fields. By use of Eq. 1.8(3) for potential, this becomes

$$U_E = \frac{1}{2} \sum_{i=1}^{n} q_i \Phi_i \tag{4}$$

Extending (4) to a system with continuously varying charge density ρ per unit volume, we have

$$U_E = \frac{1}{2} \int_V \rho \Phi \, dV \tag{5}$$

The charge density ρ may be replaced by the divergence of $\mathbf{D}$ by Eq. 1.11(2):

$$U_E = \frac{1}{2} \int_V (\nabla \cdot \mathbf{D}) \Phi \, dV$$

Using the vector equivalence of Prob. 1.11a,

$$U_E = \frac{1}{2} \int_V \nabla \cdot (\Phi \mathbf{D}) \, dV - \frac{1}{2} \int_V \mathbf{D} \cdot (\nabla \Phi) \, dV$$

The first volume integral may be replaced by the surface integral of $\Phi\mathbf{D}$ over the closed surface surrounding the region, by the divergence theorem [Eq. 1.11(7)]. But, if the region is to contain all fields, the surface should be taken at infinity. Since Φ dies off at least as fast as $1/r$ at infinity, D dies off at least as fast as $1/r^2$, and area only increases as r^2, this surface integral approaches zero as the surface approaches infinity.

$$\int_V \nabla \cdot (\Phi \mathbf{D}) \, dV = \oint_{S_\infty} \Phi \mathbf{D} \cdot \mathbf{dS} = 0$$

Then there remains

$$U_E = -\frac{1}{2} \int_V \mathbf{D} \cdot (\nabla \Phi) \, dV = \frac{1}{2} \int_V \mathbf{D} \cdot \mathbf{E} \, dV \tag{6}$$

This result seems to say that the energy is actually in the electric field, each element of volume dV appearing to contain the amount of energy

$$dU_E = \tfrac{1}{2}\mathbf{D} \cdot \mathbf{E}\, dV \tag{7}$$

The right answer is obtained if this *energy density* picture is used. Actually, we know only that the total energy stored in the system will be correctly computed by the total integral in (6).

The derivation of (6) was based on a system of charges in an unbounded, linear, homogeneous region. The same result can be shown if there are surface charges on conductors in the region. With both volume and surface charges present, (5) becomes

$$U_E = \frac{1}{2}\int_V \rho\Phi\, dV + \frac{1}{2}\int_S \rho_s\Phi\, dS \tag{8}$$

The proof that this leads to (6) is left to Prob. 1.22e. Note that in this equation, and the special case of (5), Φ is defined with its reference at infinity, since the last term of (3) is identified as Φ. The advantage of (6) is that it is independent of the reference for potential.

For a nonlinear medium, the incremental energy when fields are changed (Prob. 1.22f) is

$$dU_E = \int \mathbf{E} \cdot \mathbf{dD}\, dV \tag{9}$$

Example 1.22
ENERGY STORED IN A CAPACITOR

It is interesting to check these results against a familiar case. Consider a parallel-plate capacitor of capacitance C and a voltage V between the plates. The energy is known from circuit theory to be $\frac{1}{2}CV^2$, which is commonly obtained by integrating the product of instantaneous current and instantaneous voltage over the time of charging. The result may also be obtained by integrating the energy distribution in the field throughout the volume between plates according to (6). For plates of area A closely spaced so that the end effects may be neglected, the magnitude of field at every point in the dielectric is $E = V/d$ (d = distance between plates) and $D = \varepsilon V/d$. Stored energy U_E given by (6) becomes simply

$$U_E = \frac{1}{2}(\text{volume})(DE) = \frac{1}{2}(Ad)\left(\frac{\varepsilon V}{d}\right)\left(\frac{V}{d}\right)$$

This can be put in terms of capacitance using Eq. 1.9(3):

$$U_E = \frac{1}{2}\left(\frac{\varepsilon A}{d}\right)V^2 = \frac{1}{2}CV^2 \tag{10}$$

PROBLEMS

1.2a (i) Compute the force between two charges of 1 C each, placed 1 m apart in vacuum. (ii) The esu unit of charge (statcoulomb) is defined as one that gives a force of 1 dyne when placed 1 cm from a like charge in vacuum. Use this fact to check the conversion between statcoulombs and coulombs given in Appendix 1.

1.2b Calculate the ratio of the electrostatic force of repulsion between two electrons to the gravitational force of attraction, assuming that Newton's law of gravitation holds. The electron's charge is 1.602×10^{-19} C, its mass is 9.11×10^{-31} kg, and the gravitational constant K is 6.67×10^{-11} N-m^2/kg^2.

1.2c In his experiment performed in 1785, Coulomb suspended a horizontal rod from its center by a filament with which he could apply a torque to the rod. On one end of the rod was a charged pith ball. In the plane in which the rod could rotate was placed another, similarly charged, pith ball at the same radius. By turning the top of the filament he applied successively larger torques to the rod with the amount of torque proportional to the angle turned at the top. With the angle at the top set to 36 degrees, the angle between the two pith balls was also 36 degrees. Raising the angle at the top to 144 degrees decreased the angular separation of the pith balls to 18 degrees. A further increase of the angle at the top to 575.5 degrees decreased the angular separation of the balls to 8.5 degrees. Determine the maximum difference between his measurements and the inverse-square law. (For more details, see R. S. Elliott.[1])

1.2d Construct the electric field vector for several points in the x–y plane for like charges q at $(d/2, 0, 0)$ and $(-d/2, 0, 0)$, and draw in roughly a few electric field lines.

1.2e* Repeat Prob. 1.2d for charges of $2q$ and $-q$ at $(d/2, 0, 0)$ and $(-d/2, 0, 0)$, respectively. Find a point where the field is zero.

1.2f Calculate the electric field at points along the axis perpendicular to the center of a disk of charge of radius a located in free space. The charge on the disk is a *surface charge* ρ_s C/m^2 uniform over the disk.

1.3a Show by symmetry arguments and the results of Sec. 1.3 that there is no electric field at any point inside a spherical shell of uniform surface charge.

1.3b Show that the integral of normal flux density over a general closed surface as in Fig. 1.3*a* with charge q inside gives q. *Hint:* Relate surface element to element of solid angle.

1.3c Calculate that electric flux emanating from a point charge q and passing through a mathematical plane disk of radius a located a distance d from the charge. The charge lies on the axis of the disk. Show that in the limit where $a/d \to \infty$, the flux through the disk becomes $q/2$.

1.4a A coaxial transmission line has an inner conducting cylinder of radius a and an outer conducting cylinder of radius c. Charge q_l per unit length is uniformly distributed over the inner conductor and $-q_l$ over the outer. If dielectric ε_1 extends from $r = a$ to $r = b$ and dielectric ε_2 from $r = b$ to $r = c$, find the electric field for $r < a$, for $a < r < b$, for $b < r < c$, and for $r > c$. Take the conducting cylinders as infinitesimally thin. Sketch the variation of D and E with radius.

1.4b A long cylindrical beam of electrons of radius a moving with velocity $v_z = v_0[1 + \delta_1(r/a)^2]$ has a charge-density radial variation $\rho = \rho_0[1 - \delta_2(r/a)^2]$. Find the radial electric field in terms of the axial velocity v_0 and the total beam current I_0 and sketch its variation with radius.

1.4c Derive the expression for the field about a line charge, Eq. 1.4(3), from the field of a point charge.

1.4d* A sphere of charge of radius a has uniform density ρ_0 except for a spherical cavity of zero charge with radius b, centered at $x = d$, $y = 0$, $z = 0$, where $d < a$ and $b < a - d$. Find electric field along the x axis from $-\infty < x < \infty$. *Hint:* Use superposition.

1.4e* As in Prob. 1.4d, but now find an expression for electric field for a general point inside the cavity, showing that the field in the cavity is constant.

1.5a A point charge q is located at the origin of coordinates. Express the electric field vector in its rectangular coordinate components, and evaluate the surface integral for S chosen as the face perpendicular to the x axis of a cube of side lengths $2a$ centered on the charge. Use symmetry to show that Gauss's law is satisfied.

1.5b Perform the integrations in Eq. 1.5(3) for an infinitely long circular cylindrical ion beam with $\rho = \rho_0[1 + (r/a)^2]$ using the square prism shown in Fig. 1.5 and plane ends at $z = 0$ and l orthogonal to the axis.

1.5c If **A**, **B**, and **C** are vectors, show that

$$\mathbf{B} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{B}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

1.5d Vector **A** makes angles α_1, β_1, γ_1 with the x, y, and z axes respectively, and **B** makes angles α_2, β_2, γ_2 with the axes. If θ is the angle between the vectors, make use of the scalar product $\mathbf{A} \cdot \mathbf{B}$ to show that

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2$$

1.6a Show how the flux function may be used to plot the field from *point* charges q and $-q$ distance d apart. *Hint:* Make use of solid angles and relate these to angle θ from the axis joining charges.

1.6b Plot the field from like charges q distance d apart (Prob. 1.2d) by making use of the flux function.

1.6c Plot the field of charges $2q$ and $-q$ distance d apart (Prob. 1.2e) by use of flux. Note that not all flux lines terminate at both ends on charges.

1.7a Evaluate $\oint \mathbf{F} \cdot \mathbf{dl}$ for vectors $\mathbf{F} = \hat{\mathbf{x}}\, zxy + \hat{\mathbf{y}}\, x^2$ and $\mathbf{F} = \hat{\mathbf{x}}\, y - \hat{\mathbf{y}}\, x$ about a rectangular path from (0, 1) to (1, 1) to (1, 2) to (0, 2) and back to (0, 1). Repeat for a triangular path from (0, 0) to (0, 1) to (1, 1) back to (0, 0). Are either or both nonconservative?

1.7b A point charge q is located at the origin of a system of rectangular coordinates. Evaluate $\int \mathbf{E} \cdot \mathbf{dl}$ in the x–y plane first along the x axis from $x = 1$ to $x = 2$, and next along a rectangular path as follows: along a straight line from the point (1, 0) on the x axis to the point $(1, \frac{1}{2})$; along a straight line from $(1, \frac{1}{2})$ to $(2, \frac{1}{2})$; along a straight line from $(2, \frac{1}{2})$ to (2, 0).

1.8a A circular insulating disk of radius a is charged with a uniform surface density of charge ρ_s C/m^2. Find an expression for electrostatic potential Φ at a point on the axis distance z from the disk.

1.8b* A charge of surface density ρ_s is spread uniformly over a spherical surface of radius a. Find the potential for $r < a$ and for $r > a$ by integrating contributions from the differ-

ential elements of charge. Check the results by making use of Gauss's law and the symmetry of the problem.

1.8c Check the result Eq. 1.8(8) for the potential about a line charge by integrating contributions from the differential elements of charge. Note that the problem is one of handling properly the infinite limits.

1.8d A flat layer of charge of density ρ_0 lies perpendicular to the z axis and is infinitely broad in the x and y directions. Using Gauss's law and Eq. 1.8(1), find the dependence of the potential difference across the layer on its thickness d.

1.8e Consider two parallel sheets of charge having equal surface charge densities but with opposite sign. The sheets are both of infinite transverse dimension and are spaced by a distance d. Using Gauss's law and Eq. 1.8(1), find the electric fields between and outside the sheets and find the dependence of potential difference between the pair of sheets on the spacing. (This is called a *dipole layer*.)

1.8f In a system of infinite transverse dimension, a sheet of charge of ρ_s C/m^2 lies between, and parallel to, two conducting electrodes at zero potential spaced by distance d. Find the distribution of electric field and potential between the electrodes for arbitrary location of the charge sheet. Sketch the results for the cases where the sheet is (i) in the center and (ii) at position $d/4$.

1.8g Show that all the equipotential surfaces for two parallel line charges of opposite sign are cylinders whose traces in the perpendicular plane are circles as shown in Fig. 1.8c.

1.8h A linear quadrupole is formed by two pairs of equal and opposite charges located along a line such that $+q$ lies at $+\delta$, $-2q$ at the origin, and $+q$ at $-\delta$. Find an approximate expression for the potential at large distances from the origin. Plot an equipotential line.

1.8i Show that the magnitude of the torque on a dipole in an electric field is the product of the magnitude of the dipole moment and the magnitude of the field component perpendicular to the dipole.

1.9 Find the capacitance of the system of two concentric spherical electrodes containing two different dielectrics used as Ex. 1.4c.

1.10a Find the gradient of the scalar function $M = e^{\alpha x} \cos \beta y \cosh \alpha z$.

1.10b For two point charges q and $-q$ at $(d/2, 0, 0)$ and $(-d/2, 0, 0)$, respectively, find the potential for any point (x, y, z) and from this derive the electric field. Check the result by adding vectorially the electric field from the individual charges.

1.10c Three positive charges of equal magnitude q are located at the corners of an equilateral triangle. Find the potential at the center of the triangle and the force on one of the charges.

1.10d For two line charges q_l and $-q_l$ at $(d/2, 0)$ and $(-d/2, 0)$, respectively, find the potential for any point (x, y) and from this derive the electric field.

1.10e* Find the expression for potential outside the large sphere of Prob. 1.4d. Also find the electric field for that region as well as for the region outside the small sphere but inside the large sphere.

1.10f Find E_x and E_y in the void in the sphere of charge in Prob. 1.4d by first finding the potential. The zeros for the potentials of both large and small spheres should be at infinity.

1.11a Utilize the rectangular coordinate form to prove the vector equivalences

$$\nabla(\psi\Phi) = \psi\nabla\Phi + \Phi\nabla\psi$$

$$\nabla \cdot (\psi\mathbf{A}) = \psi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\psi$$

where ψ and Φ are any scalar functions and $\mathbf{A}$ is any vector function of space.

1.11b Show that the vector identity $\nabla \cdot \psi\mathbf{A} = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A}$ (inside back cover) is satisfied for $\psi = xyz$ and $\mathbf{F} = \hat{\mathbf{x}}x^3 + \hat{\mathbf{y}}xyz + \hat{\mathbf{z}}yz^2$.

1.11c Derive the expression for divergence in the circular cylindrical coordinate system.

1.11d Evaluate the divergence of $\mathbf{D}$ in Exs. 1.4a, 1.4b, and 1.4c and compare with the known charge densities. Evaluate $\nabla \cdot \mathbf{D}$ for Ex.1.11 using rectangular coordinates.

1.11e Given a vector $\mathbf{F} = \hat{\mathbf{x}}x^2$, evaluate $\oint_S \mathbf{F} \cdot \mathbf{dS}$ for S taken as the surface of a cube of sides $2a$ centered about the origin. Then evaluate the volume integral of $\nabla \cdot \mathbf{F}$ for this cube and show that the two results are equivalent, as they should be by the divergence theorem.

1.11f The width of the depletion region at a metal–semiconductor contact (Ex. 1.4a) can be calculated using the relation $d^2 = (2\varepsilon\Phi_B/eN)$, where Φ_B is the barrier potential, e is electron charge, and N is the density of dopant ions. Calculate d for ion densities of 10^{16}, 10^{18}, and 10^{20} cm^{-3} assuming a barrier potential of 0.6 V. Comment on the applicability in this calculation of the concept of smoothed-out charge as assumed in using Poisson's equation.Take $\varepsilon_r = 11.7$.

1.12a Find the gradient and Laplacian of a scalar field varying as $1/r$ in two dimensions and in three dimensions. Use the operators in rectangular form and also in a more appropriate coordinate system in each case.

1.12b Find the electric field and charge density as functions of x, y, and z if potential is expressed as

$$\Phi = C \sin \alpha x \sin \beta y \, e^{\gamma z} \quad \text{where} \quad \gamma = \sqrt{\alpha^2 + \beta^2}$$

1.12c Find the electric field and charge density as functions of x for a space-charge-limited, parallel-plane diode with potential variation given by $\Phi = V_0(x/d)^{4/3}$. Find the convection current density $\mathbf{J} = \rho\mathbf{v}$ and note that it is independent of x.

1.12d Argue from Laplace's equation that relative extrema of the electrostatic potential cannot exist and hence that a charge placed in an electrostatic field cannot be in stable equilibrium (Earnshaw's theorem).

1.12e The potential around a perpendicular intersection of the straight edges of two large perfectly conducting planes, where the line of intersection is taken to be the axis of a cylindrical coordinate system, can be shown to be expressible as

$$\Phi = Ar^{2/3} \sin \tfrac{2}{3}\phi$$

Show that this function satisfies Laplace's equation in cylindrical coordinates and satisfies a zero-potential boundary condition on the planes. Find a generalization to an arbitrary angle α between planes and verify that it satisfies Laplace's equation.

1.13a Which of the following may represent steady currents: $\mathbf{J} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ or $\mathbf{J} = (\hat{\mathbf{x}}x + \hat{\mathbf{y}}y)(x^2 + y^2)^{-1}$? Sketch the form of the two vector fields.

1.13b Conducting coaxial cylinders of radii a and b have a conducting dielectric with permittivity ε_1 and conductivity σ_1 for the sector $0 < \phi < \alpha$, and loss-free dielectric ε_2 for

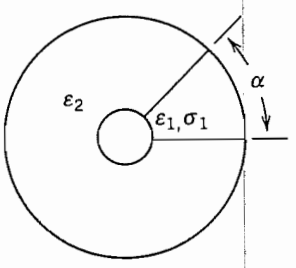

FIG. P1.13*b*

the remainder of the dielectric regions (Fig. P1.13*b*). Find capacitance and conductance per unit length.

1.14a Sketch the field and current lines for a structure of the form in Fig. 1.14*c* but with the dielectric and perfect conductor regions exchanged and a potential difference applied between the two perfect conductors. Check all continuity conditions at boundaries.

1.14b A solution to the problem of Fig. 1.14*c* can be shown to be

$$\Phi(x, y) = \frac{\Phi_0}{2} + \frac{\Phi_0}{2\pi}\left\{\left(1 + \frac{2x}{a}\right)\tan^{-1}\left(\frac{x + a/2}{y}\right) + \left(1 - \frac{2x}{a}\right)\tan^{-1}\left(\frac{x - a/2}{y}\right) + \frac{y}{a}\ln\left[\frac{y^2 + (x - a/2)^2}{y^2 + (x + a/2)^2}\right]\right\}$$

where a is the length of the conductive region. Show that this satisfies the boundary conditions on the surface $y = 0$ and find induced surface charge density along this boundary.

1.15a Obtain by means of Laplace's equation the potential distribution between two concentric spherical conductors separated by a single dielectric. The inner conductor of radius a is at potential V_0, and the outer conductor of radius b is at potential zero.

1.15b Obtain by means of Laplace's equation the potential distribution between two concentric spherical conductors with two dielectrics as in Ex. 1.4c.

1.15c Two coaxial cylindrical conductors of radii a and b are at potentials zero and V_0, respectively. There are two dielectrics between the conductors, with the plane through the axis being the dividing surface. That is, dielectric ε_1 extends from $\phi = 0$ to $\phi = \pi$, and ε_2 extends from $\phi = \pi$ to $\phi = 2\pi$. Obtain the potential distribution from Laplace's equation.

1.15d Obtain the electrostatic capacitances for the two conductor systems described in Sec. 1.15 and in Probs. 1.15a, b, and c.

1.16a Assume the charge-density profile shown in Fig. 1.16*b* with $N_A = 10^{16}\ \text{cm}^{-3}$ and $N_D = 10^{19}\ \text{cm}^{-3}$, $T = 300$ K, and $\varepsilon_r = 12$. Find the height of the potential barrier and the width of the space-charge region $d_n + d_p$. Determine maximum value of electric field.

1.16b Calculate $\Phi(x)$ for the metal–semiconductor junction in Ex. 1.4a by integrating Poisson's equation. Call total barrier height Φ_i in this case. Find the width of the space-charge region d, assuming N_D constant.

1.16c* To illustrate the effect of a continuous charge profile in the *pn* junction example of Sec. 1.16, consider a charge density in the depletion region of the form $\rho =$

$(eN_c x/a)\exp[-|x/a|]$. Find the electric field and potential as a function of x, and sketch ρ, E, and Φ versus x. Find the potential difference between $x = -\infty$ and $x = \infty$ and compare with Eq. 1.16(10), taking $N_A = N_D$.

1.17a Prove that, if charge density ρ is given throughout a volume, any solution of Poisson's equation 1.12(3) must be the only possible solution provided potential is specified on a surface surrounding the region.

1.17b Show that the potential in a charge-free region is uniquely determined, except for an arbitrary additive constant, by specification of the normal derivatives of potential on the bounding surfaces.

1.18a Prove that the line charge and its image as described for a conducting cylinder in Ex. 1.18c will give constant potential along a cylindrical surface at radius a in the absence of the conducting cylinder.

1.18b Prove that the point charge and its image as described for the spherical conductor in Ex. 1.18d gives zero potential along a spherical surface at radius a in the absence of the conducting sphere.

1.18c A circularly cylindrical electron beam of radius a and uniform charge density ρ passes near a conducting plane that is parallel to the axis of the beam and distance s from the axis. Find the electric field acting to disperse the beam for the edge near the plane and for the edge farthest from the plane.

1.18d* For a point charge q lying in a dielectric ε_1 distance $x = d$ from the plane boundary between ε_1 and a second dielectric ε_2, the given charge plus an image charge $q(\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$ placed at $x = -d$ with all space filled by a dielectric ε_1 may be used to compute the potential for any point $x > 0$. To find the potential for a point $x < 0$, a single charge of value $2q\varepsilon_2/(\varepsilon_1 + \varepsilon_2)$ is placed at the position of q with all space filled by dielectric ε_2. Show that these images satisfy the required continuity relations at a dielectric boundary.

1.18e Find and plot the surface charge density induced on the conducting plane as a function of y, when a line charge q_l lying parallel to the z axis is at $x = d$ above the plane.

1.18f Discuss the applicability of the image concept for the case of a line charge parallel to and in the vicinity of the intersection of two conducting planes with an angle $AOB = 270$ degrees. (See Fig. 1.18*d*.)

1.18g Find the potential at all points outside a conducting sphere of radius a held at potential Φ_0 when a point charge q is located a distance d from the center of the sphere $(a < d)$.

1.19a Map fields between an infinite plane conductor at potential zero and a second conductor at potential V_0, as in Fig. 1.19*c*, but for step ratios a/b of $\frac{1}{4}$ and $\frac{3}{4}$.

1.19b Map fields between an infinite flat plane and a cylindrical conductor parallel to the plane. The conductor has diameter d, and its axis is at height h above the plane. Take $d/h = 1, \frac{1}{4}$.

1.19c The outer conductor of a two-conductor transmission line is a rectangular tube of sides $3a$ and $5a$. The inner conductor is a circular cylinder of radius a, with axis coincident with the central axis of the rectangular cylinder. Sketch equipotentials and field lines for the region between conductors, assuming a potential difference V_0 between conductors.

1.19d Two infinite parallel conducting planes defined by $y = a$ and $y = -a$ are at potential zero. A semi-infinite conducting plane of negligible thickness at $y = 0$ and extending

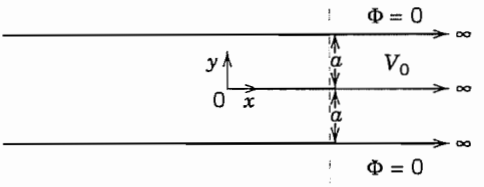

FIG. P1.19*d*

from $x = 0$ to $x = \infty$ is at potential V_0. (See Fig. P1.19*d*.) Sketch a graphical field map for the region between conductors.

1.19e In Prob.1.12e take A to be 35 and r to be measured in centimeters and make a plot of the 100-V equipotential. Construct a graphical field map between the zero and 100-V equipotentials showing the 25-, 50-, and 75-V equipotentials. Then sketch in the same equipotentials found from the formula given in Prob. 1.12e to evaluate your field map. Find the radial distance from the corner where the gradient exceeds the breakdown field in air, 30 kV/cm. What does this suggest about the shape the corner should have to avoid breakdown?

1.20a Subdivide the region in Fig. 1.19*c* into a mesh of squares of sides $a/2$. Terminate the region on the right a distance a from the corner and on the left a distance $3a/2$ from the corner. Take $b = 2a$ and $V_0 = 100$ V. Consider the potentials at the left and right edges of the above-defined grid to be fixed at the values found in linear variation from top to bottom. Start with all interior grid points at 50 V. Find the potentials at the mesh points assuming zero space charge and applying the simple averaging method.

1.20b Repeat Prob. 1.20a using the cyclic Chebyshev method.

1.20c Solve for the potentials at the grid points in the problem in Fig. 1.20a by direct inversion of the set of difference equations expressing Laplace's equation for all grid points. Compare the results with those in Table 1.20 and discuss differences. Does direct inversion give the exact values of potentials at the grid points? Explain your answer.

1.20d Set up the difference equation for a three-dimensional potential distribution in rectangular coordinates. Consider a cubical box with the following potentials on the various sides: top, 80 V; right side, 60 V; bottom, 0 V; left side, 100 V; front, 40 V; back, 100 V. Define a grid of the same coarseness as in Fig. 1.20*a* and assume initial potentials for all interior grid points to be the average of the boundary potentials. Calculate the first set of corrected potentials by the three-dimensional equivalent of the simple scheme used for Table 1.20.

1.20e Derive the difference equation for potential in cylindrical coordinates with axial symmetry assumed ($\partial\Phi/\partial\phi = 0$).

1.20f* An electron beam accelerated from zero potential passes normally through a pair of parallel-wire grids. Model the beam as infinitely broad and without transverse variation. Set up a one-dimensional difference equation for the potential between the grids. Divide the 5-mm space between grids into five segments. Take both grid potentials to be 1000 V and the beam current to be 10^4 A/m^2. Assume 1000 V as a first guess for all difference-equation grid points. Take three steps of potential adjustment with space charge based on the first guess. Recalculate space charge based on the new potentials and again iterate the potential three times. Repeat recalculations of space charge and

potentials until the latter differ by no more than 3% between recalculations of space charge. Use the simple iterative form with $\Omega = 1$.

1.21a Assume that Fig. 1.19*c* is full scale, and that V_0 is 1000 V. Find the approximate direction of the minimum and maximum electric fields in the figure. Plot a curve of electric field magnitude along the bottom plane as a function of distance along this plane, and a curve showing surface charge density induced on this plane as a function of distance.

1.21b Calculate the capacitance per unit length from your plots for Probs. 1.19b and c.

1.21c Describe the simplest way to use resistance paper to determine the capacitance per wire between a grid of parallel round wires and an electrode lying parallel to the grid. (See Fig. P1.21*c*.) Assume the grid to be infinitely long and wide. Defend all decisions made in the design of the analog.

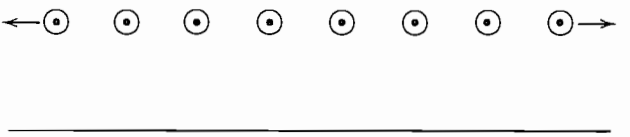

FIG. P1.21c

1.22a For a given potential difference V_0 between conductors of a coaxial capacitor, evaluate the stored energy in the electrostatic field per unit length. By equating this to $\frac{1}{2}CV^2$, evaluate the capacitance per unit length.

1.22b The energy required to increase the separation of a parallel-plate capacitor by a distance dx is equal to the increase of energy stored. Find the force acting between the plates per unit cross-sectional area assuming constant charge on the plates.

1.22c Discuss in more detail the exclusion of the self-energy term in Eq. 1.22(3), and explain why the problem disappeared in going to continuous distributions, as in Eq. 1.22(5).

1.22d Show the equality of the energies found using Eqs. 1.22(5) and (6) for a spherical volume of charge of radius a and charge density ρ C/m^3.

1.22e Consider an arbitrarily shaped, charged finite conductor embedded in a homogeneous-dielectric region of infinite extent that also contains a volume-charge density distribution. Starting from Eq. 1.22(8) show that (6) results. Make use of the identity in Prob. 1.11a and consider the dielectric to be bounded by the surface of the conductor and that at infinity.

1.22f If an incremental charge distribution is brought into a field, the incremental energy may be written

$$\delta U_E = \int_V \Phi \delta \rho dV$$

Use this to develop Eq. 1.22(9) for an unbounded region with a medium which may be nonlinear.