11 Microwave Networks

11.1 INTRODUCTION

We have so far considered individual components of a wave-type system, including transmission lines, waveguides, and cavity resonators. These are used in combination in practical systems, and it is found that many of the ideas from classical network theory, or an extension of these, are useful in handling such interconnections. In this chapter we consider some of these formulations and techniques. For convenience we refer to the subject as that of *microwave networks* since the techniques are most useful in the microwave frequency range, although they apply to any wave-type system from low-frequency transmission-line circuits to dielectric wave-guiding systems in the optical range.

One approach to the analysis of an interconnection of waveguides with other elements might be that of attempting to solve Maxwell's equations subject to boundary conditions for the entire system at once. This would be hopelessly complicated for most practical systems, and it would also give fields everywhere within the system, which is more information than is needed. One usually needs only the characteristics of each part of the system as a transducer or power-transfer element between units or as a coupling element to adjacent units. A finite number of parameters (frequency-dependent, in general) may be defined to give that desired information. These parameters can be obtained by analysis in some cases, found in handbooks[1] for certain standard configurations, or determined from measurement if neither of the first two approaches work.

Specifically, we shall mean by a microwave network a dielectric region of arbitrary shape having certain waveguide or transmission-line inlets and outlets. The waveguides are assumed to support a finite number of noncutoff modes. Examples are the cavity resonator coupled to a single transmission line (Fig. 11.1*a*), the rectangular waveguide with change of height (Fig. 11.1*b*), the microstrip T (Fig. 11.1*c*), and the *magic* T or bridge (Fig. 11.1*d*). These may be said to be microwave networks with, respectively, one, two, three, and four waveguide terminal ports. (This assumes only one noncutoff mode per guide.) In considering the defined arrangements as microwave networks, it

[1] *N. Marcuvitz, Waveguide Handbook, IEEE Press, Piscataway, NJ, 1986.*

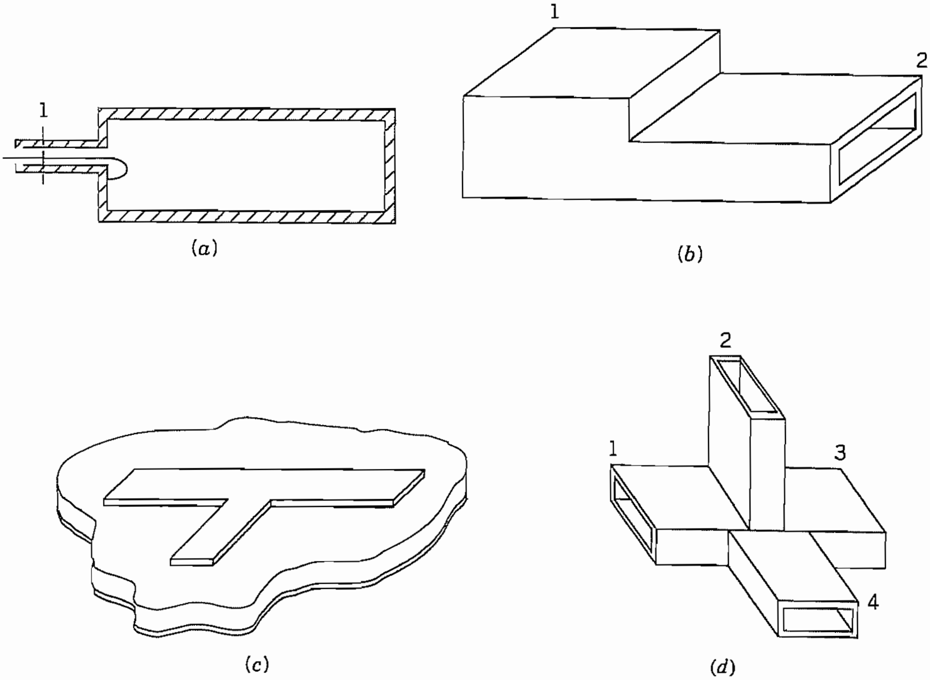


Fig. 11.1 Examples of microwave networks. (*a*) Coupling from a line to a cavity (one port). (*b*) Discontinuity in rectangular guide (two port). (*c*) Microstrip T (three port). (*d*) Magic T or microwave bridge (four port).

will also be assumed that we are interested only in the behavior of the dominant modes in certain of the guides when various load conditions are placed on the remaining guides, and not in the detailed solution of the electromagnetic field in the vicinity of the discontinuities. Dielectric waveguides may also serve as the terminals for the guided modes with fields decaying properly away from the guide.

Although we may wish to excite only the dominant mode in any of the waveguide terminals, it is true that higher-order modes are excited in the vicinity of the junctions, and although these modes may be cut off, they have reactive energy which affects the transmission between the propagating dominant modes of the various guides. But if we are interested only in the manner in which such transmission is affected, it can be expressed in terms of certain coefficients or equivalent circuits, and the details of the higher-mode fields need not be described. Thus the microwave two port of Fig. 11.1*b* may be represented by a T or π network in the same way as a lumped-element two port. It is interesting to note that Carson[2] recognized the validity of this representation as early as 1924, although the thorough development for distributed systems occurred

[2] *J. R. Carson*, Proc. AIEE ***43***, *908 (1924).*

much later.[3–5] Many formulations are possible, and since these are wave-type systems, some of the most useful relate incident and reflected waves in the various guides. Details of both types of formulations will be given in following sections.

Finally, a combination of elements such as those in the foregoing examples is also a microwave network, fitting the definition of the first paragraph. An important part of the study will be concerned with the finding of network parameters for an overall system when they are known for the individual components. The propagating media of this chapter will be considered to be linear and isotropic unless otherwise stated but not necessarily homogeneous.

11.2 The Network Formulation

As noted in the introduction, microwave networks may be described either in terms of parameters relating incident and reflected waves at the terminals or in terms of lumped-element equivalent circuits. Although the former approach may seem more natural, the latter is convenient in many cases and gives a tie to classical network theory, so will be considered first. The equivalent circuit approach does require definition of *voltage* and *current* for the microwave networks.

Voltage and current have been defined in the usual ways for transmission lines propagating the TEM wave (Chapter 5 and Sec. 8.12). For the TE_{10} mode of rectangular guide (used in two of the examples of Fig. 11.1), one might think of a voltage as the line integral of electric field between top and bottom of the guide, but it is not clear if one should take the maximum value at the center or some sort of average. Axial current flows in the top and returns to the bottom much as in a transmission line, but it is not clear if one should worry about the transverse current flow in the sides (Sec. 8.8). For other modes it will become even more confusing if classical ideas of voltage and current are attempted. It is found, however, that a network formulation results if one follows these simple rules:

1. Voltage is defined as proportional to the transverse electric field of the mode and current is defined as proportional to the transverse magnetic field.
2. One condition on the proportionality factors is that average power is given by $\text{Re}[VI^*/2]$ as in a circuit.
3. The second condition on the proportionality factors is that V/I of an incident wave should be a characteristic impedance of the mode of concern, often taken as unity to normalize automatically all impedances.

Concerning the last point, a characteristic impedance is clearly defined for TEM modes, but even there we found normalized values of impedance and admittance useful

[3] *C. G. Montgomery, R. H. Dicke, and E. M. Purcell,* Principles of Microwave Circuits, *MIT Radiation Laboratory Series, Vol. 8, McGraw-Hill, New York, 1948.*

[4] *D. M. Pozar,* Microwave Engineering, *Addison-Wesley, Reading, MA, 1990.*

[5] *R. S. Elliott,* An Introduction to Guided Waves and Microwave Circuits, *Prentice Hall, Englewood Cliffs, NJ, 1993.*

in applying the Smith chart (Chapter 5). For other waves the concept of a characteristic impedance is not so clear. The characteristic wave impedances (Secs. 8.13 and 8.14) are sometimes used, but it is usually better to normalize as suggested. In the following, however, Z_0 will first be retained in the expressions both for generality and for dimensional checks.

For a single traveling wave, using the first rule,

$$\mathbf{E}_t(x, y, z) = V_0 e^{-\gamma z}\mathbf{f}(x, y) \tag{1}$$

$$\mathbf{H}_t(x, y, z) = I_0 e^{-\gamma z}\mathbf{g}(x, y) \tag{2}$$

Applying the second and third rules,

$$\operatorname{Re}(V_0 I_0^*) = 2W_T \quad \text{(3a)} \qquad \frac{V_0}{I_0} = Z_0 \tag{3b}$$

As an example, take the TE_{10} mode in loss-free rectangular guide:

$$E_y = E_0 \sin\frac{\pi x}{a} = V_0 f(x) \tag{4}$$

$$H_x = -\frac{E_0}{Z_z}\sin\frac{\pi x}{a} = I_0 g(x) \tag{5}$$

Utilizing (3a) we have

$$V_0 I_0^* = 2b\int_0^a \frac{E_0^2}{2Z_z}\sin^2\frac{\pi x}{a}\,dx = \frac{abE_0^2}{2Z_z}$$

This result, combined with (3b), gives current and voltage,

$$V_0 = E_0\left(\frac{abZ_0}{2Z_z}\right)^{1/2}, \qquad I_0 = \left(\frac{E_0}{Z_z}\right)\left(\frac{baZ_z}{2Z_0}\right)^{1/2} \tag{6}$$

and, by comparison with (4) and (5), the remaining functions are

$$f(x) = \left(\frac{2Z_z}{abZ_0}\right)^{1/2}\sin\frac{\pi x}{a}, \qquad g(x) = -\left(\frac{2Z_0}{baZ_z}\right)^{1/2}\sin\frac{\pi x}{a} \tag{7}$$

As noted in point 3, Z_0 can be made unity to normalize automatically all subsequent impedances.

The network formulation may now be obtained by making use of the specified linearity and a uniqueness argument. To be specific, consider the region with three waveguide terminals pictured in Fig. 11.2. Each waveguide is assumed to support one propagating mode only, and reference planes are at first chosen far enough from junctions so that all higher-order (cutoff) modes have died out.[6] The forms of the propa-

[6] *The reference planes can actually be chosen for convenience at any place, but transmission-line measurements to determine the network should not be made in the region where local waves are of importance, nor will the calculations from the network give total fields in that region.*

gating or dominant modes are assumed to be known, so that field is completely specified at each reference plane by giving two amplitudes, such as the voltage and current defined above. It is clear that it is not possible to specify independently all voltages and currents of the network and a uniqueness argument tells us how many of these may be specified to determine the problem. As was pointed out in Sec. 3.14 there is one and only one steady-state solution of Maxwell's equations within a region (except for possible undamped, uncoupled modes of no interest to us) if tangential electric field is specified over the closed boundary surrounding that region, or if tangential magnetic field is specified over the closed boundary, or if tangential electric field is specified over some of the boundary and tangential magnetic field over the remainder.

In Fig. 11.2 consider the closed region bounded by the conducting surface S and reference planes 1, 2, and 3. If the conductor is first taken as perfectly conducting, the tangential electric field is known to be zero over the surface S. Then, if voltages are given for each of the reference terminals, tangential electric fields are known there, and, by the statement of uniqueness, one and only one solution of Maxwell's equations is possible. Then **E** and **H** are determinable for any point inside the region, including the reference planes, so that the currents (amplitudes of the tangential magnetic field distributions) may be found there. For linear media, the relations are linear ones and may therefore be written

$$
\begin{aligned}
I_1 &= Y_{11}V_1 + Y_{12}V_2 + Y_{13}V_3 \\
I_2 &= Y_{21}V_1 + Y_{22}V_2 + Y_{23}V_3 \\
I_3 &= Y_{31}V_1 + Y_{32}V_2 + Y_{33}V_3
\end{aligned} \tag{8}
$$

Similarly, if currents are given for all reference planes, tangential magnetic fields are known there, and, with the known zero tangential electric field over S, the uniqueness argument again applies so that tangential electric fields and hence voltages could be found at the reference planes. For linear media,

$$
\begin{aligned}
V_1 &= Z_{11}I_1 + Z_{12}I_2 + Z_{13}I_3 \\
V_2 &= Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3 \\
V_3 &= Z_{31}I_1 + Z_{32}I_2 + Z_{33}I_3
\end{aligned} \tag{9}
$$

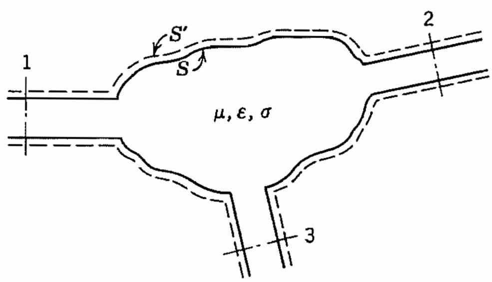


FIG. 11.2 General microwave network with three waveguide terminals.

Forms (8) and (9) are identical with the forms that would be found relating voltages and currents at the terminals of a three-port lumped-element network. Here also the coefficients Y_{ij} and Z_{ij} are functions of frequency and are known as the admittance parameters and impedance parameters, respectively. For an N port, in matrix form,

$$[I] = [Y][V], \qquad [V] = [Z][I] \tag{10}$$

where $[I]$ and $[V]$ are column matrices of order N and $[Y]$ and $[Z]$ are $N \times N$ square matrices.

Although the argument has been given for a perfectly conducting surface S, the foregoing forms also apply to an imperfectly conducting boundary. A reasonably convincing way of seeing this comes from moving the bounding surface several depths of penetration within the conductor to S' (Fig. 11.2). The electric field there is substantially zero, so that an imagined perfect conductor could be introduced along S' without changing the behavior of the system, and the argument would proceed as above. The conducting portion between S and S' will contribute to the parameters Y_{ij} or Z_{ij} since it is now part of the interior, and those coefficients will be complex because of the losses.

11.3 Conditions for Reciprocity

For most systems, the admittance and impedance matrices defined in the preceding article are symmetric. That is,

$$Y_{ij} = Y_{ji}, \qquad Z_{ij} = Z_{ji} \tag{1}$$

This condition follows from a reciprocity theorem due to Lorentz, which states that fields $\mathbf{E}_a$, $\mathbf{H}_a$ and $\mathbf{E}_b$, $\mathbf{H}_b$ from two different sources at the same frequency satisfy the condition

$$\nabla \cdot (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a) = 0 \tag{2}$$

This theorem is easily verified for isotropic (but not necessarily homogeneous) media by substituting Maxwell's equations in complex form and can also be shown to hold for an anisotropic media provided the permittivity and permeability matrices are symmetric. However, it does not hold if the matrices are asymmetric, thus explaining nonreciprocal properties of the gyrotropic media to be met in Chapter 13. If (2) is satisfied, a volume integral of (2), with application of the divergence theorem, gives

$$\oint_S (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a) \cdot \mathbf{dS} = 0 \tag{3}$$

Consider Fig. 11.2 with all reference planes but 1 and 2 closed by perfect conductors (shorted). Fields at 1 and 2 may be written [Eqs. 11.2(1) and 11.2(2)]

$$\mathbf{E}_{t1} = V_1\mathbf{f}_1(x_1, y_1) \qquad \mathbf{H}_{t1} = I_1\mathbf{g}_1(x_1, y_1) \tag{4}$$

$$\mathbf{E}_{t2} = V_2\mathbf{f}_2(x_2, y_2) \qquad \mathbf{H}_{t2} = I_2\mathbf{g}_2(x_2, y_2) \tag{5}$$

By rule 2, voltage and current are defined to have the same relation to power flow in both guides, which requires that

$$\int_{S_1} (\mathbf{f}_1 \times \mathbf{g}_1) \cdot d\mathbf{S} = \int_{S_2} (\mathbf{f}_2 \times \mathbf{g}_2) \cdot d\mathbf{S} \tag{6}$$

The surface integral (3) is zero along the conducting surfaces S of Fig. 11.2 (or S', if imperfectly conducting) and along the shorted planes. For planes 1 and 2, substitution of (4) and (5) gives

$$(V_{1a}I_{1b} - V_{1b}I_{1a}) \int_{S_1} (\mathbf{f}_1 \times \mathbf{g}_1) \cdot d\mathbf{S} + (V_{2a}I_{2b} - V_{2b}I_{2a}) \int_{S_2} (\mathbf{f}_2 \times \mathbf{g}_2) \cdot d\mathbf{S} = 0$$

With (6), this reduces to

$$V_{1a}I_{1b} - V_{1b}I_{1a} + V_{2a}I_{2b} - V_{2b}I_{2a} = 0$$

Relations between current and voltage are introduced from Eq. 11.2(8):

$$\begin{aligned} V_{1a}(Y_{11}V_{1b} + Y_{12}V_{2b}) - V_{1b}(Y_{11}V_{1a} + Y_{12}V_{2a}) \\ + V_{2a}(Y_{21}V_{1b} + Y_{22}V_{2b}) - V_{2b}(Y_{21}V_{1a} + Y_{22}V_{2a}) = 0 \\ (V_{1a}V_{2b} - V_{1b}V_{2a})(Y_{12} - Y_{21}) = 0 \end{aligned} \tag{7}$$

In this argument the sources a and b are arbitrary so that the first factor need not be zero. Hence the second is zero. Then

$$Y_{21} = Y_{12} \tag{8}$$

The argument for the impedance coefficients may be supplied by placing "open circuits" at all but two of the terminals. This is done in the waveguides by placing a perfect short a quarter-wave in front of the reference planes. Moreover, since the numbering system is arbitrary, 1 and 2 may represent any two of the guides and the general relation (1) is valid.

Two-Port Waveguide Junctions

11.4 Equivalent Circuits for a Two Port

The microwave network with two waveguide terminals, as pictured in Fig. 11.1b, is of greatest importance since it includes the cases of discontinuities in a single guide or the coupling between two guides. Most filters, matching sections, phase-correction units, and many other components are of this type. There is a large body of literature on the lumped-element equivalents. The name "two port" is used for these, with a port denoting a single waveguide mode at a specific reference plane for a microwave network and a terminal pair for a lumped-element network.

From Sec. 11.2 the equations for a two port may be written in terms of either impedance or admittance coefficients:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{2}$$

Another convenient form expresses input quantities in terms of output quantities:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \tag{3}$$

Algebraic elimination shows that relations among the above parameters are

$$\begin{aligned} Y_{11} &= \frac{Z_{22}}{\Delta(Z)} = \frac{D}{B} \\ Y_{12} &= \frac{-Z_{12}}{\Delta(Z)} = \frac{-(AD - BC)}{B} \\ Y_{21} &= \frac{-Z_{21}}{\Delta(Z)} = \frac{-1}{B} \\ Y_{22} &= \frac{Z_{11}}{\Delta(Z)} = \frac{A}{B} \end{aligned} \tag{4}$$

where

$$\Delta(Z) = Z_{11}Z_{22} - Z_{12}Z_{21}$$

For a network satisfying reciprocity,

$$Z_{21} = Z_{12}, \qquad Y_{21} = Y_{12}, \qquad AD - BC = 1 \tag{5}$$

We shall assume such reciprocal networks in the remainder of this section.

An infinite number of equivalent circuits may be derived which yield the forms (1) to (5). Two important ones are the well-known T and π forms shown in Figs. 11.4*a* and *b*. They may be shown to be equivalent to (1) and (2), respectively, by setting down the circuit equations. Other interesting ones utilize ideal transformers and sections of transmission lines, two of which are pictured in Figs. 11.4*c* and *d*. These are of greatest importance for lossless microwave networks since the arbitrary reference planes in the input or output guides can be shifted in such a way that only an ideal transformer is left in the representation of Fig. 11.4*c* or an ideal transformer and shunt element in Fig. 11.4*d*. This will be explained in more detail when the measurement problem is discussed

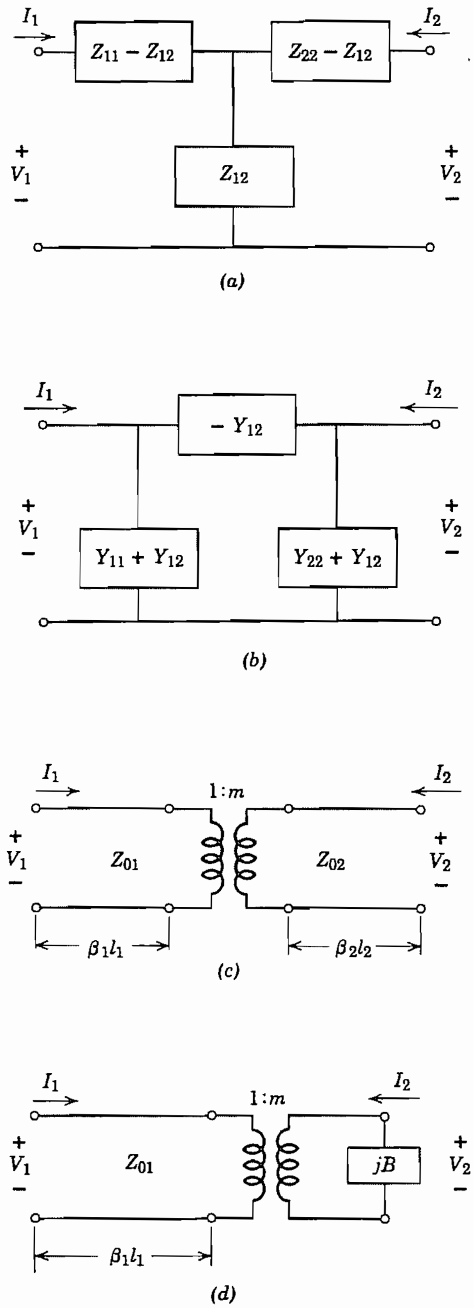


FIG 11.4 (a) T equivalent circuit with (b) π equivalent circuit for a two port satisfying reciprocity. (c) Equivalent circuit for a two port which satisfies reciprocity using sections of transmission line and an ideal transformer. (d) Equivalent circuit using section of transmission line, transformer, and shunt element.

in Sec. 11.6. The quantities of Fig. 11.4c are related to the impedance parameters as follows:

$$\tan \beta_1 l_1 = \left[\frac{1 + c^2 - a^2 - b^2}{2(bc - a)}\right] \pm \sqrt{\left[\frac{1 + c^2 - a^2 - b^2}{2(bc - a)}\right]^2 + 1}$$

$$\tan \beta_2 l_2 = \frac{1 + a \tan \beta_1 l_1}{b \tan \beta_1 l_1 - c} \tag{6}$$

$$\frac{m^2 Z_{01}}{Z_{02}} = \frac{1 + a \tan \beta_1 l_1}{b + c \tan \beta_1 l_1}$$

where

$$a = \frac{-jZ_{11}}{Z_{01}}$$

$$b = \frac{Z_{11}Z_{22} - Z_{12}^2}{Z_{01}Z_{02}} \tag{7}$$

$$c = \frac{-jZ_{22}}{Z_{02}}$$

11.5 Scattering and Transmission Coefficients

The preceding discussions have been given in terms of the voltages, currents, and impedances defined for microwave networks. Definitions of these quantities are somewhat arbitrary. Moreover, the impedances are usually obtained by interpreting measured values of standing wave ratios or reflection coefficients. It is then evident that for some problems it will be more convenient and direct to formulate the transformation properties of the two port in terms of waves. The two independent quantities required for each waveguide terminal are an incident and a reflected wave replacing the voltage and current. This section introduces two of the most useful forms based on wave quantities.

Suppose that incident and reflected voltage waves on the input guide are given in magnitude and phase at the chosen reference plane by V_{1+} and V_{1-} (Fig. 11.5). Similarly, incident and reflected waves looking toward the junction from reference plane 2 are V_{2+} and V_{2-}. It is common to normalize incident and reflected waves as follows:

$$a_n = \frac{V_{n+}}{\sqrt{Z_{0n}}}, \qquad b_n = \frac{V_{n-}}{\sqrt{Z_{0n}}} \tag{1}$$

Thus, voltage and current at reference plane n are related to these wave quantities as follows:

$$V_n = V_{n+} + V_{n-} = \sqrt{Z_{0n}}(a_n + b_n) \tag{2}$$

$$I_n = \frac{1}{Z_{0n}}(V_{n+} - V_{n-}) = \frac{1}{\sqrt{Z_{0n}}}(a_n - b_n)$$

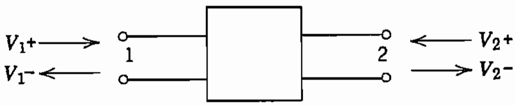


FIG. 11.5 Incident and reflected waves at ports of microwave network.

The average power flowing into terminal n is

$$(W_n)_{av} = \tfrac{1}{2}\,\mathrm{Re}(V_n I_n^*) = \tfrac{1}{2}\,\mathrm{Re}[(a_n a_n^* - b_n b_n^*) + (b_n a_n^* - b_n^* a_n)]$$

The first set of parentheses encloses a purely real quantity, and the second, a purely imaginary quantity. Thus,

$$2(W_n)_{av} = a_n a_n^* - b_n b_n^* \tag{3}$$

That is, $(W_n)_{av}$ is the power carried into terminal n by the incident wave, less that reflected away.

In the first form to be used, we shall relate the two reflected waves to the two incident waves. For a linear medium,

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{4}$$

or

$$[b] = [S][a] \tag{5}$$

with the $[S]$ array known as the *scattering matrix*, and the coefficients S_{11} and so on known as *scattering coefficients*. For a physical interpretation, note that with the source applied to port 1 and the output guide matched so that $a_2 = 0$,

$$b_1 = S_{11} a_1, \qquad b_2 = S_{21} a_1 \tag{6}$$

Thus, S_{11} is just the input reflection coefficient (in magnitude and phase) when the output is matched, and S_{21} is the ratio of waves to the right at output and input under this condition. The energy equation (3) for this matched condition becomes, for the two terminals,

$$\begin{aligned} 2(W_1)_{av} &= (1 - S_{11} S_{11}^*) a_1 a_1^* \\ 2(W_2)_{av} &= -S_{21} S_{21}^* a_1 a_1^* \end{aligned} \tag{7}$$

The negative sign in W_2 arises because power is defined as positive toward each port.

For a passive network with source at 1 as in this example, output power cannot be greater than that supplied at the input. Thus, $(-W_2)_{av} \le (W_1)_{av}$ or

$$S_{21} S_{21}^* \le 1 - S_{11} S_{11}^* \tag{8}$$

The equality holds only when the network is loss free.

By substituting the definitions (2) into Eq. 11.4(1), and utilizing (4), we can relate the scattering coefficients to the impedance coefficients.[7] The results are

$$
\begin{aligned}
FS_{11} &= (Z_{11} - Z_{01})(Z_{22} + Z_{02}) - Z_{12}Z_{21} \\
FS_{12} &= 2\sqrt{Z_{01}Z_{02}}Z_{12} \\
FS_{21} &= 2\sqrt{Z_{02}Z_{01}}Z_{21} \\
FS_{22} &= (Z_{22} - Z_{02})(Z_{11} + Z_{01}) - Z_{21}Z_{12}
\end{aligned}
\tag{9}
$$

where

$$F = (Z_{11} + Z_{01})(Z_{22} + Z_{02}) - Z_{12}Z_{21} \tag{10}$$

From this we see that $S_{21} = S_{12}$ for a network satisfying reciprocity since then $Z_{21} = Z_{12}$. Through the relations of Eq. 11.4(4) the scattering coefficients may also be related to admittance coefficients or the transfer coefficients if necessary. They can be obtained directly from reflection measurements, however, as is shown in a following section.

A second important linear transformation of (4) gives output wave quantities in terms of input quantities:

$$
\begin{aligned}
b_2 &= T_{11}a_1 + T_{12}b_1 \\
a_2 &= T_{21}a_1 + T_{22}b_1
\end{aligned}
\tag{11}
$$

The coefficients T_{ij} are known as the *transmission coefficients* and are related to the scattering coefficients as follows:

$$
\begin{aligned}
T_{11} &= S_{21} - \frac{S_{11}S_{22}}{S_{12}}, &\quad T_{12} &= \frac{S_{22}}{S_{12}} \\
T_{21} &= -\frac{S_{11}}{S_{12}}, &\quad T_{22} &= \frac{1}{S_{12}}
\end{aligned}
\tag{12}
$$

This form is especially useful for cascaded networks, as will be illustrated in Sec. 11.7.

11.6 Measurement of Network Parameters

Measurement of the network parameters of a two-port is straightforward if one can measure relative magnitudes and phases of any two quantities and can apply excitation at either port. Thus, in the impedance form, Eqs. 11.4(1), an open circuit ($I = 0$) may be placed on the output of the two port and Z_{11} is then input impedance. Z_{21} is the ratio of output voltage to input current:

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}, \qquad Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \tag{1}$$

[7] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, Principles of Microwave Circuits, *MIT Radiation Laboratory Series*, pp. 146–148, McGraw-Hill, New York, 1948.

Reversal of the network gives Z_{12} and Z_{22} by similar measurements. In a like fashion, the admittance parameters may be found by applying a short circuit ($V = 0$) at port 2 with excitation at port 1:

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}, \qquad Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \tag{2}$$

Reversal of the procedure gives Y_{12} and Y_{22}. Direct measurement of the scattering parameters is accomplished by successively matching ports 2 and 1 and measuring incident and reflected waves at the two ports:

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0}, \qquad S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}, \qquad S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0}, \qquad S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} \tag{3}$$

Magnitude and phase of input impedance or reflection coefficient can be measured by slotted line techniques as explained in Sec. 5.8, but measurement of phase between input and output quantities, as required for S_{ij}, presents a greater difficulty. The measurement is accomplished in modern network analyzers[8] by measuring incident and reflected waves at each port by directional couplers, to be described in Sec. 11.10. After selection of the pair of signals required for a particular S_{ij} and matching according to (3), the microwave signals for the two quantities to be compared are mixed with local oscillator signals of slightly different frequency to produce lower-frequency signals which retain the information concerning phase and relative magnitudes. Ratios of the complex phasors are calculated digitally to give the desired S_{ij}. Excitation frequency and local oscillator frequency can be swept so that values of the desired parameters may be obtained over a range of frequencies. Display may be in a variety of forms, often in Smith chart format.

Network analyzers which provide both phase and magnitude are called *vector analyzers.* Those which provide magnitudes only are called *scalar analyzers*[9] and are of course simpler since ratios may then be measured by using simple power meters.

If a network analyzer is not available, more laborious methods are possible. Often a two port is used to transform impedances from one side to the other. From Eq. 11.4(1), load impedance $Z_L = -V_2/I_2$ produces input impedance $Z_i = V_1/I_1$ as follows:

$$Z_i = Z_{11} - \frac{Z_{12}^2}{Z_{22} + Z_L} \tag{4}$$

Measurement of three different (Z_L, Z_i) pairs can then determine the three parameters Z_{11}, Z_{22}, and Z_{12}^2. A particularly simple way is to place a good short at three independent positions along the guide to produce the three known load impedances. Algebraic elimination from three equations of the form of (4) shows that, if Z_{L1} produces Z_{i1}, Z_{L2} produces Z_{i2}, and Z_{L3} produces Z_{i3}, the impedance parameters are

8 *R. G. Dildine and J. D. Grace,* Hewlett-Packard J. ***39***, *12 (1988).*

9 *J. H. Egbert et al.,* Hewlett-Packard J. ***37***, *24 (1986).*

$$Z_{11} = \frac{(Z_{i1} - Z_{i3})(Z_{i1}Z_{L1} - Z_{i2}Z_{L2}) - (Z_{i1} - Z_{i2})(Z_{i1}Z_{L1} - Z_{i3}Z_{L3})}{(Z_{i1} - Z_{i3})(Z_{L1} - Z_{L2}) - (Z_{i1} - Z_{i2})(Z_{L1} - Z_{L3})} \tag{5}$$

$$Z_{22} = \frac{(Z_{i1}Z_{L1} - Z_{i2}Z_{L2}) - Z_{11}(Z_{L1} - Z_{L2})}{(Z_{i2} - Z_{i1})} \tag{6}$$

$$Z_{12}^2 = (Z_{11} - Z_{ip})(Z_{22} + Z_{Lp}), \qquad p = 1, 2, 3 \tag{7}$$

If the network is lossless and if purely reactive terminations are used, input impedances will also be pure reactances, and Z's may be replaced by X's everywhere in these equations. The form of (5) to (7) may also be shown to be valid for determination of admittance parameters Y_{11}, Y_{12}, and Y_{22} when pairs of input–output admittances $Y_{L1}Y_{i1}$, $Y_{L2}Y_{i2}$, $Y_{L3}Y_{i3}$ are measured. The Z's are then replaced by Y's in (5) to (7). Note that the sign of Z_{12} cannot be determined from impedance transformation measurements alone, since it does not enter into (4). For the same reason, it is of no interest if results are to be used only for impedance transformations by the network.

The above forms apply also to scattering parameters since input reflection coefficient is related to that at the output by

$$\rho_i = S_{11} - \frac{S_{12}^2}{S_{22} + 1/\rho_L} \tag{8}$$

$$\rho_i = \frac{b_1}{a_1}, \qquad \rho_2 = \frac{a_2}{b_2} \tag{9}$$

Thus, measurement of ρ_i with three independent values of ρ_L may be used to obtain S_{11}, S_{22}, and S_{12} by replacing Z_i with ρ_i, Z_1 with $1/\rho_L$, and Z_{mn} with S_{mn} in (5) to (7).

For regions that may be considered lossless, the representation of Fig. 11.4*c* is especially useful. This follows because a shift of the input reference plane from 1 to 1′ (Fig. 11.6*a*) by a distance $\beta_1 y_0 = \beta_1 l_1 - \pi$ and a shift of output reference plane from 2 to 2′ by $\beta_2 x_0 = \pi - \beta_2 l_2$ gives as the equivalent circuit an ideal transformer with half-wave lines at input and output. But the half-wave lines give unity impedance transformation and so may be ignored, leaving only the ideal transformer representing the region between 2′ and 1′. A load impedance referred to 2′ is multiplied simply by $(1/m)^2$ to give the input impedance referred to 1′.

The parameters of the above representation may be determined as follows. The output guide is perfectly terminated ($Z_L = Z_{02}$); the position of the minimum impedance point on the input guide corresponds to 1′, and the value of this minimum impedance gives m^2:

$$m^2 = \frac{Z_{02}}{Z_{\min}} \tag{10}$$

Similarly, if the network is reversed, the input guide terminated, and like measurements made on the output guide, the location of reference plane 2′ is obtained as well as a check on m^2.

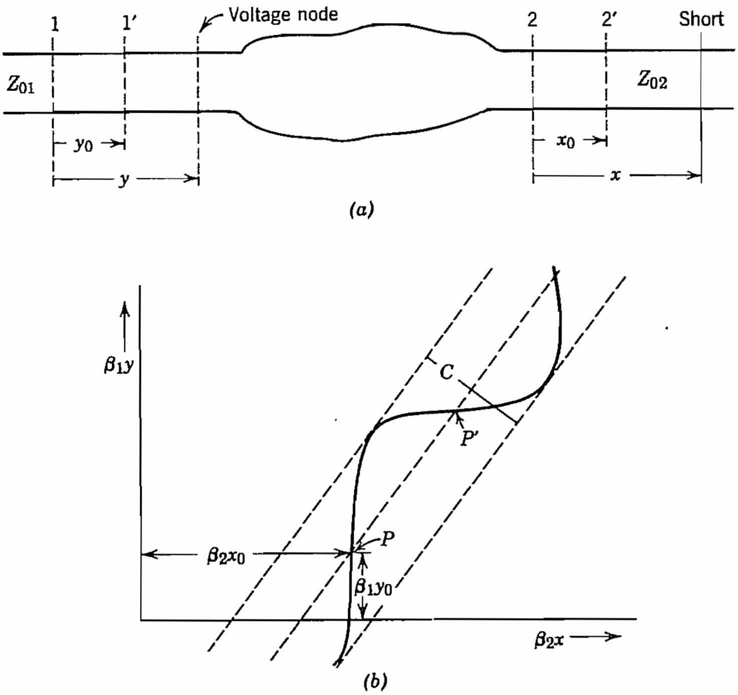


FIG. 11.6 (*a*) General two port. (*b*) Typical *S* curve obtained by measurement on (*a*).

An alternative procedure has advantages in some cases. Weissfloch[10] has shown that for a lossless junction, a plot of position of voltage minimum on the input guide as a function of position of a short on the output guide has the "*S* curve" form shown in Fig. 11.6*b*, where $\beta_1 y$ is the electrical distance of the minimum from the originally selected reference 1, and $\beta_2 x$ is the electrical distance of the short from 2. The form of the equation is easily found to be

$$\tan \beta_1(y - y_0) = \frac{Z_{02}}{m^2 Z_{01}} \tan \beta_2(x - x_0) \tag{11}$$

The new reference planes 1′ and 2′ are given by the positions x_0, y_0 of the maximum slope of the *S* curve, point *P* of Fig. 11.6*b*. The value of this maximum slope is $Z_{02}/m^2 Z_{01}$. The turns ratio may also be determined in terms of the distance *C* between the envelope tangents.

$$\frac{m^2 Z_{01}}{Z_{02}} = \tan^2\left(\frac{\pi}{4} - \frac{\sqrt{2}C}{4}\right) \tag{12}$$

[10] N. Marcuvitz, Waveguide Handbook, p. 122, IEEE Press, Piscataway, NJ, 1986.

For the measurement, many points of input minimum are then measured as a short is moved along the output, and the curve determined. There is the advantage that the consistency of measurement and discrepancies caused by neglected losses may be told more easily than in the methods first described where only a few points are measured. Alternatively, one can use the crossover point P'. Turns ratio is the reciprocal of the above and reference planes are shifted by a quarter-wavelength on each side.

11.7 Cascaded Two Ports

The *ABCD* transfer forms of Sec. 11.4 and the transmission coefficients of Sec. 11.5 are especially useful when two ports are connected in tandem or cascade, because the output quantities of one network become the input quantities of the following one. Thus, referring to Fig. 11.7*a*, we may successively apply the form of Eq. 11.4(3) to networks a and b:

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix}, \qquad \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix}$$

but $V_{2a} = V_{1b}$ and $-I_{2a} = I_{1b}$, so we may combine the two to give

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix} \tag{1}$$

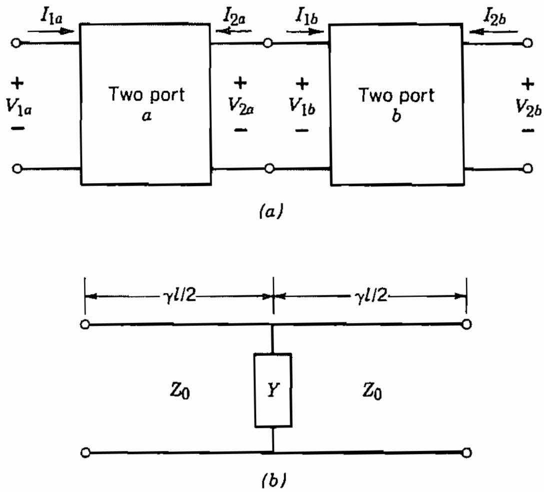
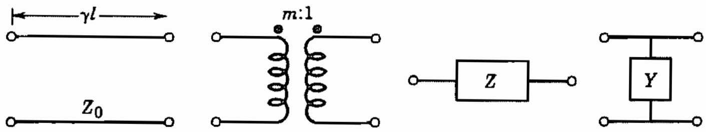


FIG. 11.7 (*a*) Cascade connection of two ports. (*b*) Uniform transmission line with shunt element in center.

Table 11.7



	Transmission Line	Ideal Transformer	Series Impedance	Shunt Admittance
A	$\cosh \gamma l$	m	1	1
B	$Z_0 \sinh \gamma l$	0	Z	0
C	$Y_0 \sinh \gamma l$	0	0	Y
D	$\cosh \gamma l$	$1/m$	1	1

Thus, the transfer matrix for the two networks cascaded is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \tag{2}$$

with obvious extension to more than two cascaded two ports. Similarly for the T matrices of Sec. 11.5, but here we start with the matrix for the last unit since these were defined to give output in terms of input quantities:

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} (T_{11})_b & (T_{12})_b \\ (T_{21})_b & (T_{22})_b \end{bmatrix} \begin{bmatrix} (T_{11})_a & (T_{12})_a \\ (T_{21})_a & (T_{22})_a \end{bmatrix} \tag{3}$$

Table 11.7 gives the $ABCD$ coefficients for some simple units.

Example 11.7a
Transmission Line with Shunt Element

To illustrate the use of Table 11.7, consider the circuit of Fig. 11.7*b* with a shunt admittance at the midpoint of a uniform transmission line. The matrix for the combination is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cosh\left(\frac{\gamma l}{2}\right) & Z_0 \sinh\left(\frac{\gamma l}{2}\right) \\ Y_0 \sinh\left(\frac{\gamma l}{2}\right) & \cosh\left(\frac{\gamma l}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \begin{bmatrix} \cosh\left(\frac{\gamma l}{2}\right) & Z_0 \sinh\left(\frac{\gamma l}{2}\right) \\ Y_0 \sinh\left(\frac{\gamma l}{2}\right) & \cosh\left(\frac{\gamma l}{2}\right) \end{bmatrix} \tag{4}$$

Multiplication of the matrices and use of identities for hyperbolic functions gives

$$A = D = \cosh \gamma l + \left(\frac{Y}{2Y_0}\right) \sinh \gamma l \tag{5}$$

$$B = Z_0\left[\left(\frac{Y}{2Y_0}\right)(-1 + \cosh \gamma l) + \sinh \gamma l\right] \tag{6}$$

$$C = Y_0\left[\left(\frac{Y}{2Y_0}\right)(1 + \cosh \gamma l) + \sinh \gamma l\right] \tag{7}$$

Example 11.7b
PERIODIC SYSTEM AS CASCADE OF *N* TWO PORTS

Periodic circuits, considered in Sec. 9.10 from a field point of view, may also be considered as a cascade of like networks. The overall transfer matrix for N like networks is just the Nth power of that for a single network:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}^N \tag{8}$$

For a network with reciprocity, this results in[11]

$$A = [A_0 \sinh N\Gamma - \sinh(N - 1)\Gamma]/\sinh \Gamma \tag{9}$$

$$B = B_0 \sinh N\Gamma/\sinh \Gamma \tag{10}$$

$$C = C_0 \sinh N\Gamma/\sinh \Gamma \tag{11}$$

$$D = [D_0 \sinh N\Gamma - \sinh(N - 1)\Gamma]/\sinh \Gamma \tag{12}$$

$$\cosh \Gamma = (A_0 + D_0)/2 \tag{13}$$

In matrix terminology, $e^{\pm\Gamma}$ are the characteristic roots of the matrix. Physically, Γ may be considered the propagation constant when the network is properly terminated.

To interpret the above, let us consider that each cell is symmetric ($A_0 = D_0$) in addition to being reciprocal ($A_0D_0 - B_0C_0 = 1$) as already assumed. Let us also terminate the chain in a characteristic impedance Z_c defined so that each cell terminated in Z_c also gives impedance Z_c at its input.[12] That is,

$$Z_c = \frac{V_1}{I_1} = \frac{A_0V_2 - B_0I_2}{C_0V_2 - A_0I_2} = \frac{A_0Z_c + B_0}{C_0Z_c + A_0}$$

11 *See, for example, G. Strang,* Linear Algebra and its Applications, *Academic Press, New York, 1976.*

12 *If the network is asymmetric, image impedances—one for each direction—are used. See, for example, R. S. Elliott,* An Introduction to Guided Waves and Microwave Circuits, *Appendix E, Prentice Hall, Englewood Cliffs, NJ, 1993.*

from which

$$Z_c = \left(\frac{B_0}{C_0}\right)^{1/2} \tag{14}$$

Equations (9)–(12) then reduce to

$$A = D = \cosh N\Gamma \tag{15}$$

$$B = Z_c \sinh N\Gamma \tag{16}$$

$$C = (Z_c)^{-1} \sinh N\Gamma \tag{17}$$

By comparison with Table 11.7, we recognize that each cell now acts as a transmission line of characteristic impedance Z_c and overall propagation constant Γ. The cascaded N sections then just have overall propagation constant $N\Gamma$.

Note that from (13) if $|A_0 + D_0| \leq 2$, Γ is imaginary so that there is phase shift but no attenuation through the network. If $|A_0 + D_0| > 2$, Γ is real and there is attenuation. This filter-type behavior is most important in communication networks and will be illustrated more in the following section.

11.8 Examples of Microwave and Optical Filters

The filtering action described in the preceding section is extremely important for communication systems; filters pass desired frequencies with small attenuation while providing much more attenuation for noise or undesired signals outside the frequency range of interest. We shall give some examples in this section for different types of transmission systems. Many other configurations are useful, and a variety of techniques for design of the different types are discussed in treatises on this subject.[13]

Example 11.8a
Filters with Periodic Shunt Elements in Transmission Lines

The first four forms to be considered, Figs. 11.8*a*–*d*, are specific cases of the symmetrical transmission system with periodic shunt elements analyzed in the preceding section. Figure 11.8*a* shows a coaxial line with capacitive diaphragms at intervals l along the line, and Fig. 11.8*b* is a similar arrangement in microstrip with the loading capacitors made as side taps. If the loading capacitors have value C_d and losses are

[13] *G. L. Matthaei, L. Young, and E. M. T. Jones,* Microwave Filters, Impedance-Matching Networks and Coupling Structures, *Artech House, Norwood, MA, 1980. Also, R. E. Collin,* Foundations of Microwave Engineering, *2nd ed., McGraw-Hill, New York, 1991. See also footnotes 4 and 5.*

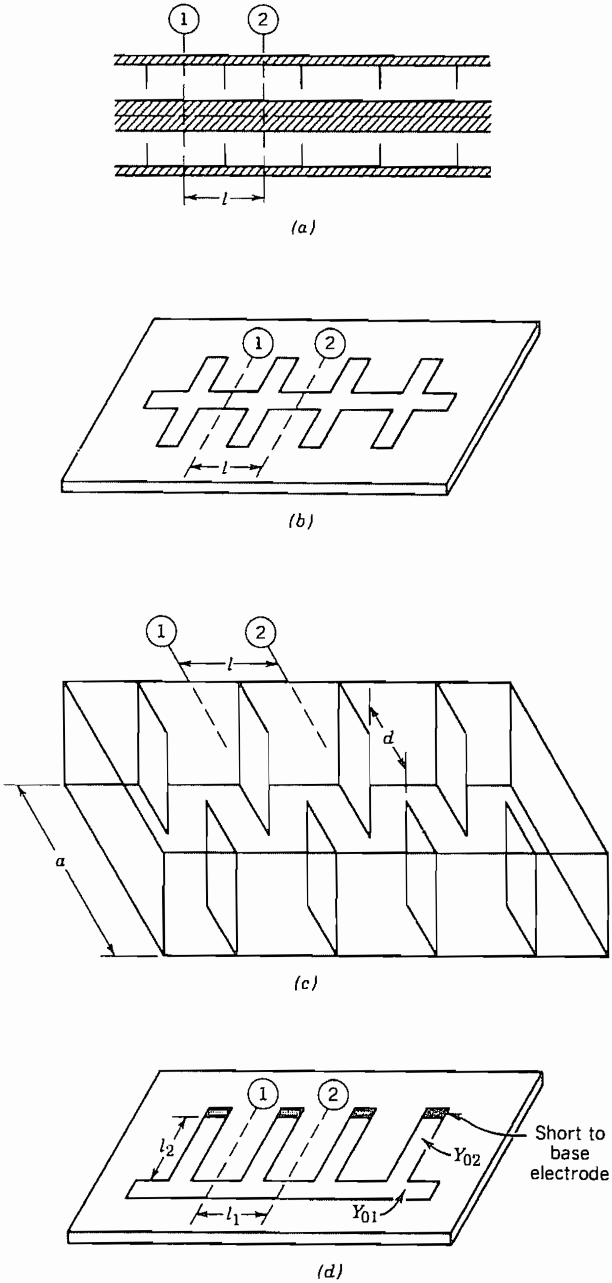


FIG 11.8 (*a*) Coaxial line with capacitive disks at intervals l. (*b*) Microstrip with capacitive tabs at intervals l. (*c*) Rectangular waveguide with symmetric inductive diaphragms introduced from sides at intervals l. (*d*) Microstrip with shorted microstrip stub lines in parallel at intervals l.

neglected in the transmission line so that $\gamma = j\beta$, Eq. 11.7(5) and Eq. 11.7(13) give

$$\cosh \Gamma = \cos \beta l - \frac{\omega C_d}{2Y_0} \sin \beta l \tag{1}$$

These circuits would be expected to pass low frequencies since the periodic capacitors provide small shunting effects in that range. If $\beta l = \omega l \sqrt{LC} \ll 1$, (1) becomes

$$\cosh \Gamma \approx 1 - \frac{\omega^2 l C_d \sqrt{LC}}{2\sqrt{C/L}} \tag{2}$$

where L and C are inductance and capacitance of the microstrip per unit length. The region of imaginary Γ, which we call the *passband* of the filter, is from $-1 \le \cosh \Gamma \le 1$, which is from zero to ω_{c1} in angular frequency, where cutoff is

$$\omega_{c1} = 2\left(\frac{1}{C_d l L}\right)^{1/2} \tag{3}$$

In this approximation the transmission-line sections, short compared with wavelength, act as series inductors and, with the shunt capacitors, produce a classical lumped-element, low-pass filter. However, there are other passbands at higher frequencies. These occur in the vicinity of $\beta l = n\pi$ since $\sin \beta l$ becomes small there. Since $\omega C_d/2Y_0$ increases with frequency, the passbands become narrower as the order n increases (assuming of course that the capacitive representation for the discontinuities holds at these higher frequencies). In any event it is characteristic of the transmission-line circuits that they have multiple passbands, as was found from a different point of view in Sec. 9.10.

Example 11.8b
Filter with Periodic Inductive Diaphragms in Waveguide

Figure 11.8c pictures a rectangular waveguide with diaphragms at intervals l introduced symmetrically from the sides. Such diaphragms are found to act as inductive shunt susceptances to the TE_{10} mode, with admittance given approximately by[14]

$$\frac{Y}{Y_0} = -\frac{j\lambda_g}{a} \cot^2\left(\frac{\pi d}{2a}\right) \tag{4}$$

where a is guide width, d aperture width, and λ_g guide wavelength,

$$\lambda_g = \lambda\left[1 - \left(\frac{\lambda}{2a}\right)^2\right]^{-1/2}$$

[14] *See footnote 1 or 3.*

Equations 11.7(5) and 11.7(13) give for this case

$$\cosh \Gamma = \cos\left(\frac{2\pi l}{\lambda_g}\right) + \frac{\lambda_g}{2a}\cot^2\left(\frac{\pi d}{2a}\right)\sin\left(\frac{2\pi l}{\lambda_g}\right) \tag{5}$$

The waveguide is in itself a high-pass filter, but with typical values of λ_g/a and d/a, the added diaphragms cause it to remain attenuating to higher frequencies than the waveguide cutoff. But at least in the vicinity of $l = \lambda_g/2$ the last term in (5) is small and $|\cosh \Gamma| < 1$ so that there is a passband. Depending upon values of λ_g/a and d/a there may be additional attenuation bands and passbands at still higher frequencies, as in the first example.

Example 11.8c
BANDPASS FILTER IN MICROSTRIP

In Fig. 11.8*d* the shunt elements consist of shorted sections of microstrip of length l_2 and characteristic admittance Y_{02} connected in parallel with the main microstrip at intervals l_1. The shorted stub lines also short the main line at low frequencies (and at frequencies for which $\beta_2 l_2 = n\pi$), but present high impedances in parallel for frequency ranges near those for which the stub lengths are odd multiples of a quarter-wavelength. The shunt admittances of the shorted stubs are

$$Y = -jY_{02}\cot \beta_2 l_2 \tag{6}$$

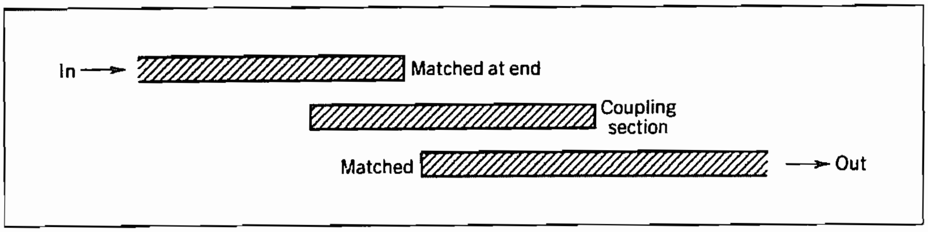
so that Eqs. 11.7(5) and 11.7(13) yield

$$\cosh \Gamma = \cos \beta_1 l_1 + \frac{Y_{02}}{2Y_{01}}\cot \beta_2 l_2 \sin \beta_1 l_1 \tag{7}$$

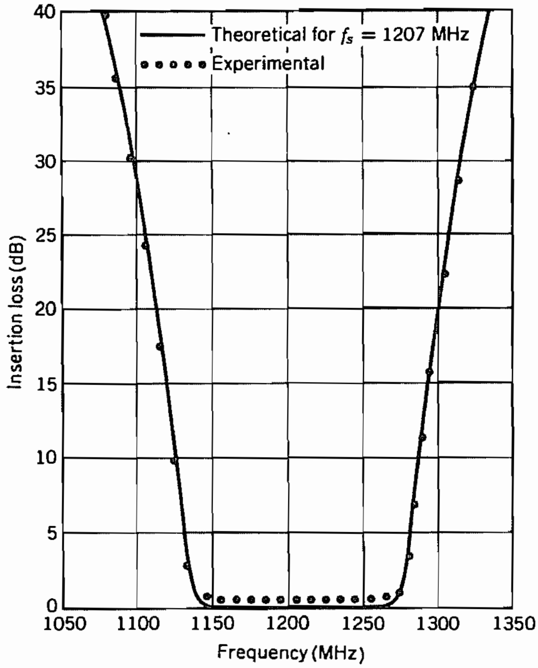
This clearly has passbands, in the sense defined, for $\beta_2 l_2$ near $(2m + 1)\pi/2$, as expected.

Example 11.8d
FILTERING BY COUPLED MICROSTRIP TRANSMISSION LINES

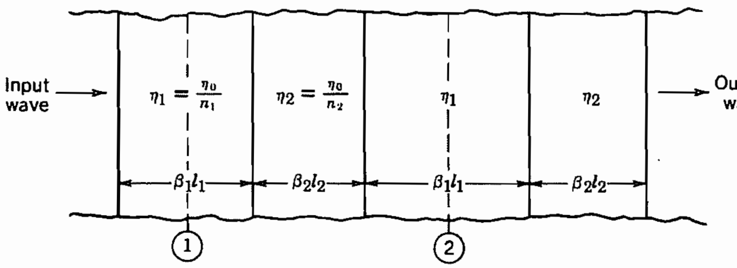
A somewhat different approach to construction of a bandpass filter is illustrated by Fig. 11.8*e*. In this, one microstrip line is coupled to another through an intervening microstrip of finite length. For frequencies near those for which the coupling section is resonant, there is large coupling and nearly perfect transfer. For other frequencies, there is a small transfer of energy. Analysis of this, which requires consideration of the distributed couplings, is not given here, but measured and calculated curves of insertion loss from a six-section filter of this type are shown in Fig. 11.8*f*.



(e)



(f)



(g)

FIG. 11.8 (*e*) Top view of microstrip with coupling section between input and output transmission lines. (*f*) Insertion loss of parallel-coupled-resonator filter of six sections [From S. B. Cohn, *IRE Trans. Microwave Theory Tech.* **MTT-6**, 223 (1958). © 1958 IRE (now IEEE).] (*g*) Optical filter with periodic alternations of refractive index.

Example 11.8e
OPTICAL FILTER

As a final example we consider the optical filter made of alternating layers of dielectric with different refractive indices, as illustrated in Fig. 11.8*g*. (The equivalent of this in transmission-line form is made by alternating sections of transmission line of different characteristic impedance.) As explained in Sec. 11.1, microwave network concepts also apply at optical frequencies. The *ABCD* matrix for the symmetric section between reference planes 1 and 2 is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos \dfrac{\beta_1 l_1}{2} & j\eta_1 \sin \dfrac{\beta_1 l_1}{2} \\ \dfrac{j}{\eta_1} \sin \dfrac{\beta_1 l_1}{2} & \cos \dfrac{\beta_1 l_1}{2} \end{bmatrix} \begin{bmatrix} \cos \beta_2 l_2 & j\eta_2 \sin \beta_2 l_2 \\ \dfrac{j}{\eta_2} \sin \beta_2 l_2 & \cos \beta_2 l_2 \end{bmatrix} \times \begin{bmatrix} \cos \dfrac{\beta_1 l_1}{2} & j\eta_1 \sin \dfrac{\beta_1 l_1}{2} \\ \dfrac{j}{\eta_1} \sin \dfrac{\beta_1 l_1}{2} & \cos \dfrac{\beta_1 l_1}{2} \end{bmatrix} \tag{8}$$

from which we obtain

$$A = D = \cosh \Gamma = \cos \beta_1 l_1 \cos \beta_2 l_2 - \frac{1}{2}\left(\frac{\eta_1}{\eta_2} + \frac{\eta_2}{\eta_1}\right) \sin \beta_1 l_1 \sin \beta_2 l_2 \tag{9}$$

This is propagating (imaginary Γ) for low frequencies, but eventually reaches an attenuating region with $|\cosh \Gamma| > 1$. There are then a series of attenuation bands and passbands as frequency is increased.

In this section we have considered only the propagation through one unit, and have looked for frequency ranges for which Γ is real (attenuation bands) or imaginary (passbands). As shown in the preceding section, if there are N such symmetric units in cascade, and the last is terminated in Z_c defined by Eq. 11.7(14), the overall attenuation is $N\Gamma$ for the attenuation bands and N times the phase shift per unit in the passbands. For a practical filter, the units need be neither symmetric nor all the same. Moreover, Z_c is in general a function of frequency so that it is not usually possible to match its frequency characteristics exactly to that of the terminating impedance. Reflection losses thus also occur and the overall insertion loss—in both the passbands and the attenuation bands—must be considered in the design of the filter. The techniques used for designing a filter which approximates a desired characteristic of insertion loss versus frequency are described elsewhere.[13]

N-Port Waveguide Junctions

11.9 CIRCUIT AND S-PARAMETER REPRESENTATION OF N PORTS

It was shown in Eq. 11.2(10) that the currents and voltages (as defined in Sec. 11.2) are related through impedance parameters,

$$[V] = [Z][I] \tag{1}$$

or through admittance parameters,

$$[I] = [Y][V] \tag{2}$$

where the admittance and impedance matrices are of order $N \times N$. The scattering coefficients, defined in Sec. 11.5 for a two port, may also be extended to the N port:

$$[b] = [S][a] \tag{3}$$

The column matrix $[b]$ represents the N waves leaving the junction, and $[a]$ the N waves incident upon the junction, as defined in Eq. 11.5(1). The representation is pictured in Fig. 11.9a for a four port. Other linear transformations are of course possible, and an indefinite number of equivalent circuits may be drawn, similar to those of Sec. 11.4, but the three formulations given above have proven the most generally useful. We now consider some specializations of the general N port.

Reciprocal Networks For networks satisfying reciprocity, each of the matrices $[Z]$, $[Y]$, and $[S]$ is symmetric. That is,

$$Z_{ji} = Z_{ij}, \qquad Y_{ji} = Y_{ij}, \qquad S_{ji} = S_{ij} \tag{4}$$

The proof is easily accomplished by extending from that of the two port. For example, if all ports except for i and j are shorted at the reference planes, there remains a two port between i and j, for which the proof of Sec. 11.3 shows $Y_{ji} = Y_{ij}$ under the conditions of reciprocity, and similarly for the other relations of (4).

Loss-Free Networks Internal losses may be neglected in many useful microwave junctions, so it is important to know the consequence of negligible loss. The complex Poynting theorem, Eq. 3.13(6), applied to power flow into the N ports gives

$$-\oint (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} = \sum_{m=1}^{N} V_m I_m^* = 2W_L + 4j\omega(U_H - U_E) \tag{5}$$

but

$$V_m = \sum_{n=1}^{N} Z_{mn} I_n \tag{6}$$

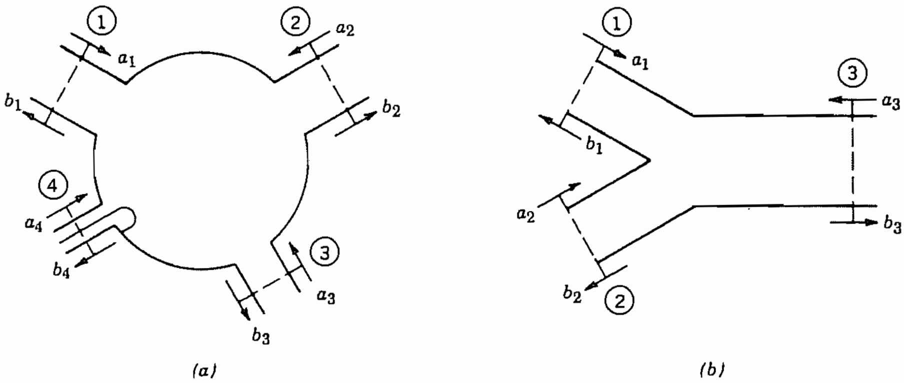


FIG. 11.9 (*a*) General four port showing incident and reflected waves. (*b*) A Y junction (three port) for combining power from sources ① and ②.

so

$$\sum_{n=1}^{N} \sum_{m=1}^{N} Z_{mn} I_n I_m^* = 2W_L + 4j\omega(U_H - U_E) \tag{7}$$

Now let all ports be open circuited except the ith:

$$Z_{ii} I_i I_i^* = 2W_L + 4j\omega(U_H - U_E) \tag{8}$$

So for a loss-free network with $W_L = 0$, Z_{ii} is imaginary since $I_i I_i^*$ is real. To study an off-diagonal term, let all ports but i and j be open-circuited:

$$Z_{ii} I_i I_i^* + Z_{jj} I_j I_j^* + Z_{ij} I_j I_i^* + Z_{ji} I_j^* I_i = 2W_L + 4j\omega(U_H - U_E) \tag{9}$$

The first two terms are imaginary, by the above, so if $W_L = 0$,

$$\mathrm{Re}[Z_{ij} I_j I_i^* + Z_{ji} I_j^* I_i] = 0 \tag{10}$$

For a reciprocal network with $Z_{ji} = Z_{ij}$, Z_{ij} is then also imaginary. Similarly the admittance matrix is imaginary for a loss-free network satisfying reciprocity.

To show the properties of the scattering matrix for loss-free networks,

$$\sum_{m=1}^{N} V_m I_m^* = \sum_{m=1}^{N} (a_m + b_m)(a_m^* - b_m^*) = 2W_L + 4j\omega(U_H - U_E) \tag{11}$$

so for a loss-free network,

$$\sum_{m=1}^{N} b_m b_m^* = \sum_{m=1}^{N} a_m a_m^* \tag{12}$$

This equation can be written in matrix form,

$$[b]_t[b^*] = [a]_t[a^*] \tag{13}$$

where $[b]_t$ denotes the transpose of $[b]$, obtained by interchanging rows and columns. In particular the transpose of a column matrix is a row matrix, and it can be checked that the rule for matrix multiplication applied to (13) does give (12). Now, substituting (3),

$$([S][a])_t([S][a])^* = [a]_t[a^*] \tag{14}$$

The transpose of a product is the product of transposes with order reversed, so

$$[a]_t[S]_t[S^*][a^*] = [a]_t[U][a^*] \tag{15}$$

where $[U]$ is the unit matrix. Thus,

$$[S]_t[S^*] = [U] \tag{16}$$

From this we see that $[S]_t = [S^*]^{-1}$ so (16) may also be written

$$[S^*][S]_t = [U] \tag{17}$$

Matrices for which the transpose is the conjugate of the inverse matrix are called *unitary matrices.* Use of the product rule for matrices shows that they have the following properties:

$$\sum_{n=1}^{N} S_{in}S_{in}^* = 1 \tag{18}$$

$$\sum_{n=1}^{N} S_{in}S_{jn}^* = 0, \qquad i \neq j \tag{19}$$

The above relations may be derived directly from conservation of energy, (18) by applying an incident wave to terminal i with all terminals matched and (19) by applying incident waves to terminals i and j with all terminals matched. Note that reciprocity was not required in the derivation. The relations have important consequences as to what can or cannot be done with loss-free junctions, as will be illustrated with two examples.

Example 11.9a
LIMITATIONS ON LOSS-FREE THREE PORTS

The three-port Y junction pictured in Fig. 11.9*b* is useful as a power divider or power combiner. It is assumed that it satisfies reciprocity. If sources are introduced at terminals 1 and 2 with the combined power obtained at 3, one might wish to have $S_{12} = 0$ to eliminate direct interaction between the two sources. But condition (19), with $i = 1$, $j = 2$, gives

$$S_{11}S_{21}^* + S_{12}S_{22}^* + S_{13}S_{23}^* = 0 \tag{20}$$

Thus if $S_{12} = 0$, either S_{13} or S_{23} is zero also, eliminating one of the two desired couplings. The junction will act as a power combiner, but there is interaction between the two sources, and if the two sources are not identical in magnitude and phase, one source will tend to feed power to the other.

Example 11.9b
LIMITATIONS ON IDEAL ISOLATING NETWORKS

The ideal isolator would be a loss-free, one-way transmission line with $S_{12} = 0$ but $S_{21} \neq 0$. It was noted that the unitary property (17) applies to nonreciprocal as well as reciprocal networks. Equation (18) with $i = 1$ and (19) with $i = 1, j = 2$ give

$$S_{11}S_{11}^* + S_{12}S_{12}^* = 1 \tag{21}$$

$$S_{11}S_{21}^* + S_{12}S_{22}^* = 0 \tag{22}$$

In (22) if $S_{12} = 0$, either $S_{21} = 0$ or $S_{11} = 0$, but $S_{11} \neq 0$ from (21), so this ideal also is impossible. We shall see useful isolators employing nonreciprocal elements in Chapter 13, but because of the limitation shown here, they will have dissipative elements to absorb the reflected wave.

Some consequences of the unitary property of directional couplers and four-port hybrid networks are discussed in the following section.

Shift of Reference Planes If the reference in each port is shifted away from the network by distance l_n, there is additional phase delay in scattering matrix parameters S_{ij} by $\beta_i l_i$ for the ith port and $\beta_j l_j$ for the jth port. Thus, the S' matrix coefficient referred to the new references is related to the original S matrix coefficient by

$$S'_{ij} = S_{ij}e^{-j(\beta_i l_i + \beta_j l_j)} \tag{23}$$

11.10 DIRECTIONAL COUPLERS AND HYBRID NETWORKS

One of the most important four ports is the directional coupler, designed to couple in a separable fashion to the positively and negatively traveling waves in a guide. Figure 11.10*a* gives the simplest conception of this device. Imagine a main waveguide with two small holes placed a quarter-wave apart coupling to an auxiliary guide terminated at each end by a matching resistance and meter as shown. If wave A progresses toward the right, coupled waves from the two holes at terminal 4 follow paths B and C of equal lengths, and the contributions add in that load, its meter indicating the strength of A. The couplings through the two holes cancel at terminal 3, however, since the paths E and D differ in length by a half-wavelength, and the couplings through the two holes are substantially the same in amount if the holes are small. By symmetry, a wave flowing

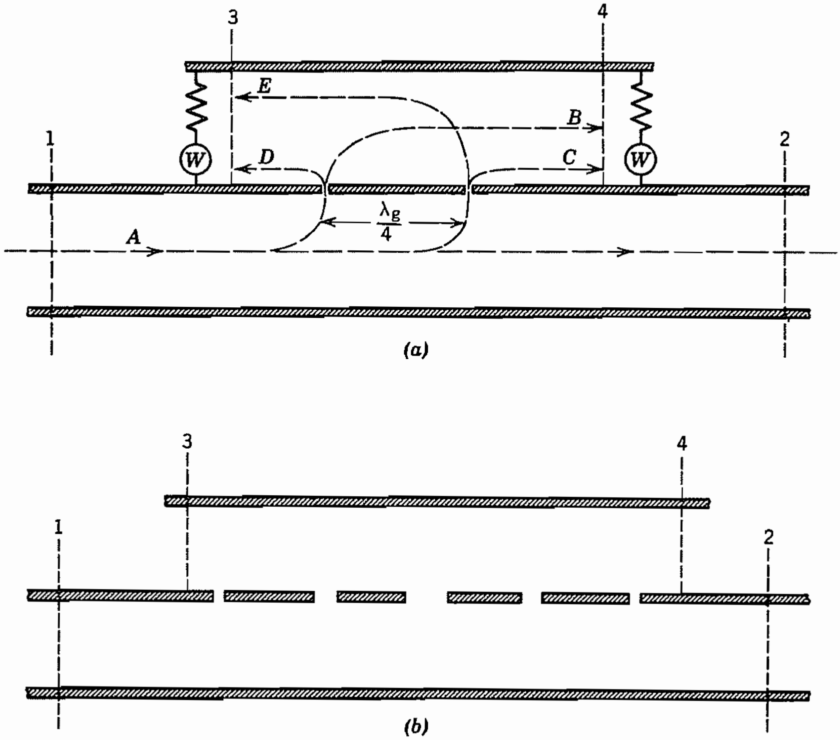


FIG. 11.10 (*a*) Basic directional coupler. (*b*) Broadband directional coupler.

to the left will register at terminal 3 but yield coupled waves which cancel at 4. Thus, meter 4 reads the strength of the wave to the right, and meter 3, that to the left.

This simple coupler is frequency sensitive since it depends on the quarter-wave spacing of holes. A like effect with greater bandwidth may be obtained by supplying several holes with properly graded couplings, as illustrated in Fig. 11.10*b*. Still other embodiments are used.[4] All these couplers may be considered as four ports with the four reference planes as shown in Fig. 11.10*a* or *b*. Losses may normally be neglected. Several important general properties follow.

To study the properties of the coupler, it is most convenient to use the scattering matrix form of Eq. 11.9(3). It is desired not to couple between 1 and 3 with 2 and 4 matched, so $S_{13} = S_{31} = 0$. It is also desired to have no coupling between 2 and 4 with 1 and 3 matched, so $S_{24} = S_{42} = 0$. Moreover, the ideal directional coupler should be matched so that all the power entering at one terminal divides between the other two for which there is coupling, leaving no reflections at the input. Thus S_{11}, S_{22}, and so on are zero. The network is also assumed to satisfy reciprocity so that $S_{12} = S_{21}$, $S_{14} = S_{41}$, and so on. Thus the scattering matrix for an ideal directional coupler has been specialized to

$$[S] = \begin{bmatrix} 0 & S_{12} & 0 & S_{14} \\ S_{12} & 0 & S_{23} & 0 \\ 0 & S_{23} & 0 & S_{34} \\ S_{14} & 0 & S_{34} & 0 \end{bmatrix} \tag{1}$$

For negligible loss within the network, power conservation leads to the unitary property of the scattering matrix as shown in the preceding section. Equation 11.9(18), for different values of the index i, gives

$$i = 1: \qquad S_{12}S_{12}^* + S_{14}S_{14}^* = 1 \tag{2}$$

$$i = 2: \qquad S_{12}S_{12}^* + S_{23}S_{23}^* = 1 \tag{3}$$

$$i = 3: \qquad S_{23}S_{23}^* + S_{34}S_{34}^* = 1 \tag{4}$$

$$i = 4: \qquad S_{14}S_{14}^* + S_{34}S_{34}^* = 1 \tag{5}$$

Comparison of (2) and (3) shows that $|S_{14}| = |S_{23}|$ and comparison of (2) and (5) shows that $|S_{12}| = |S_{34}|$. Moreover, reference plane 2 may be selected with respect to 1 so that S_{12} is real and positive, and similarly 4 with respect to 3 so that S_{34} is real and positive. Then

$$S_{12} = S_{34} \triangleq a \tag{6}$$

There remains the Eqs. 11.9(19), also following from the unitary nature of $[S]$. The specializations already made in (1) satisfy these identically except for

$$i = 1, j = 3: \qquad S_{12}S_{23}^* + S_{14}S_{34}^* = 0 \tag{7}$$

$$i = 2, j = 4: \qquad S_{12}S_{14}^* + S_{23}S_{34}^* = 0 \tag{8}$$

Use of (6) in either of the above requires $S_{14} = -S_{23}^*$. Reference plane 4 may then be selected with respect to 1 so that S_{14} is real and

$$S_{23} = -S_{14} \triangleq b \tag{9}$$

Thus we have reduced the scattering matrix to the very simple form

$$[S] = \begin{bmatrix} 0 & a & 0 & -b \\ a & 0 & b & 0 \\ 0 & b & 0 & a \\ -b & 0 & a & 0 \end{bmatrix} \tag{10}$$

Note that b gives the coupling from the main guide to the auxiliary guide and is known as the coupling factor (often expressed in decibels). The coefficient a may be called the transmission factor and the two are related by any of the energy relations (2) to (5)

$$a^2 + b^2 = 1 \tag{11}$$

Although a and b are the only two parameters of an ideal directional coupler, real units give some coupling to the terminal for which zero coupling is desired. That is, S_{13} and S_{24} will not be exactly zero for a real coupler. The coupling to the desired terminal in the auxiliary guide, as compared with the undesired terminal, is defined as the *front-to-back ratio* or the *directivity*, usually expressed in decibels.

Several important theorems may be proved for loss-free reciprocal four ports.

1. A four port with two pairs of noncoupling elements is completely matched. That is, the setting of S_{11}, S_{22}, and so on equal to zero was not a separate condition but followed from $S_{13} = 0$, $S_{24} = 0$ because of power relations resulting from the complete Eqs. 11.9(18) and 11.9(19).
2. Any completely matched junction of four waveguides is a directional coupler. (Note that this does not mean that an arbitrary four port may be made into a directional coupler by externally introducing matching transformers, since the adjustment of one of these in such a case disturbs matching for the other ports; it must be an internal property giving $S_{11} = S_{22} = S_{33} = S_{44} = 0$.)
3. A four port with two noncoupling terminals matched is a directional coupler. That is, if $S_{13} = 0$, $S_{11} = 0$, and $S_{33} = 0$, the other properties defined earlier follow.

The Magic T and Other Hybrid Networks The special case of a directional coupler with $a^2 = b^2 = \frac{1}{2}$ is of particular interest in that it may be used as a bridge or "hybrid" network. (The latter name is taken from the properties of the classical hybrid coil.[15]) One of the configurations used for this purpose in rectangular guides for the TE_{10} mode is the magic T pictured in Fig. 11.10*c*. A wave introduced into the "E" arm, 2, will, from considerations of symmetry, divide equally between arms 1 and 3 but not couple to the "H" arm, 4. Conversely a wave introduced into 4 divides between arms 1 and 3 with no coupling to 2. Thus, the scattering coefficient S_{24} is zero. By

[15] *C. G. Montgomery, R. H. Dicke, and E. M. Purcell,* Principles of Microwave Circuits, *MIT Radiation Laboratory Series, p. 307, McGraw-Hill, New York, 1948.*

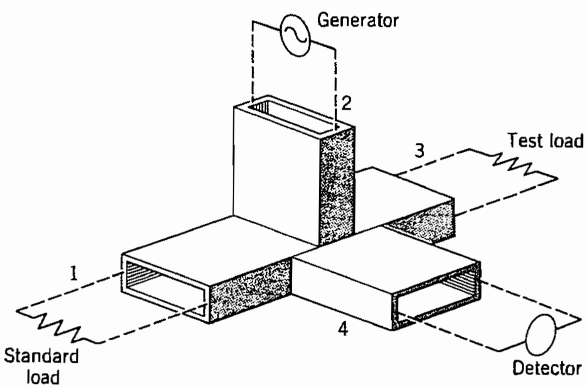


FIG. 11.10c Magic T network as a bridge.

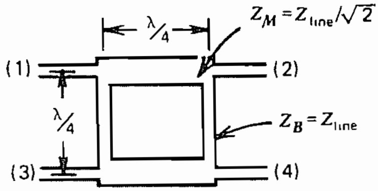


FIG. 11.10*d* Two-branch hybrid for microstrip.

theorem III, above, this becomes a directional coupler if the unit is internally matched so that S_{22} and S_{44} are zero. This matching is normally accomplished by introducing pins or diaphragms or both within the guides near the junction. It then follows from the theorem that S_{11}, S_{33}, and S_{13} are also zero. By symmetry, the transmission and coupling coefficients are equal so that, as stated,

$$a^2 = b^2 = \tfrac{1}{2} \tag{12}$$

A typical use of one of these units is as a bridge. With the generator at 2 and detector at 4 in Fig. 11.10*c*, no output is observed if the loads on 1 and 3 are equal. Thus a standard load may be placed on 1 and test loads on 3 which are nominally the same. Any deviation from the standard will produce a reading in the detector at 4.

Other configurations accomplishing the same goal are known. One general type utilizes cancellation of waves around a "rat race" structure, as pictured for microstrip in Fig. 11.10*d*. The division of power and phase changes by the two paths to each outlet produce the cancellations or additions desired. Variations of this with greater bandwidth are possible.[16] Structures based on the same principles can also be made with coaxial lines.

Frequency Characteristics of Waveguide Networks

11.11 Properties of a One-Port Impedance

Let us consider a closed region with one waveguide terminal or one port, as in the cavity sketched in Fig. 11.1*a*. Reference plane 1 is selected far enough from the junction

[16] *See, for example, J. Frey (Ed.),* Microwave Integrated Circuits, *Sec. IIB, Artech House, Norwood, MA, 1975.*

so that only the dominant mode in the guide is important. Application of the complex Poynting theorem leads to Eq. 11.9(8), which for $i = 1$ is

$$Z = R + jX = \frac{2W_L + 4j\omega(U_H - U_E)}{II^*} \tag{1}$$

where W_L is the average power loss in the region, and U_E and U_H are average stored energies in electric and magnetic fields, respectively. Similarly for input admittance Y,

$$Y = G + jB = \frac{2W_L + 4j\omega(U_E - U_H)}{VV^*} \tag{2}$$

Certain properties of these impedance and admittance functions will be discussed. Note that the comments apply to the function $Z_{ii}(\omega)$ of a general microwave network, since this would be the input impedance of a two port formed by shorting all but the ith terminal. Similarly, the theorems apply to Y_{ii} and to some other combinations of the impedance or admittance functions.

A study of (1) shows several simple results expected from physical reasoning. Impedance is purely imaginary (reactive) if power loss is zero. When power loss is finite, the real (resistance) part of Z is positive for a passive network. If stored electric and magnetic average energies are equal, reactance is zero and the network is said to be resonant. If average magnetic energy is greater than electric, the reactance is positive (inductive), and if electric energy is the greater, reactance is negative (capacitive). Since power and energy would be the same for positive and negative frequencies, (1) shows that $R(\omega)$ is an even function of frequency and $X(\omega)$ an odd function, if the range is extended to negative frequencies. Similar results can be deduced for the admittance function.

Loss-Free One Ports When losses are negligible, (1) and (2) become

$$X = \frac{4\omega(U_H - U_E)}{II^*}, \qquad B = \frac{4\omega(U_E - U_H)}{VV^*} \tag{3}$$

Moreover, the variation of X and B with frequency may be related to stored energy through a variational form of the Poynting theorem (Prob. 11.11c). The results are

$$\frac{dX}{d\omega} = \frac{4(U_E + U_H)}{II^*}, \qquad \frac{dB}{d\omega} = \frac{4(U_E + U_H)}{VV^*} \tag{4}$$

It is evident that the average stored energy $(U_E + U_H)$ is positive, and II^* is positive, so the rate of change of reactance with frequency for the lossless one port will be positive. The reactance must then go through a succession of zeros and poles as sketched in Fig. 11.11. Similarly, the susceptance of the lossless one port has a positive slope for the same reason. It follows that the zeros and poles are all simple (first order) and that a zero must lie between two adjacent poles, and vice versa.

These results were first derived by Foster[17] for lumped-element networks, and so are known as Foster's reactance theorem.

[17] *R. M. Foster,* Bell Syst. Tech. J. **3**, *259 (1924).*

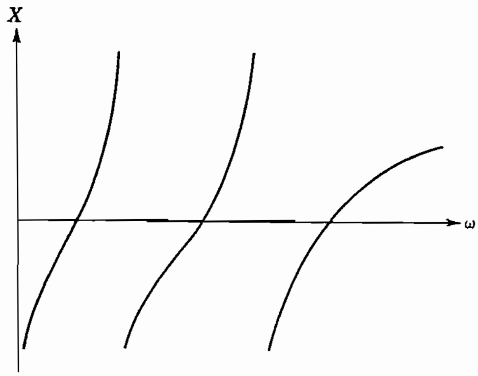


FIG. 11.11 Typical form of reactance versus frequency for a lossless one port.

Relations Between Real and Imaginary Parts of Impedance or Admittance Functions If losses are finite, there remain some constraints between the frequency variation of resistance and of reactance, or of conductance and susceptance. These are of the same form as the Kronig–Kramers relations between real and imaginary parts of permittivity to be discussed in Sec. 13.2. This is because of the location of zeros and poles in the complex frequency plane and the analytic character of the functions in a half-plane.

The complex "frequency" variable commonly utilized is $\alpha + j\omega$. Zeros or poles of Z (or Y) would mean that natural frequencies exist for such values, giving finite solutions without a driving source. For passive networks (those without internal sources of energy), such solutions can only decay from any transient initial state. Thus α must be negative for such natural frequencies of passive networks. In other words the impedance (or admittance) function can have no zeros or poles in the right half $(\alpha + j\omega)$-plane. This analytic property permits the relation of real and imaginary parts.

Insight into the properties of circuits as functions of complex frequency can be obtained by considering an analogy with static electric potential and flux functions as complex variables (Sec. 7.5). In potential function terms, specification of potential everywhere along the $j\omega$ axis determines potential everywhere in the right half-plane, if there are no sources there and potential dies off properly at infinity. Determination of potential at all points permits the finding of electric field, and this in turn permits the construction of a flux function. Conversely, a statement of flux along the $j\omega$ axis defines charge distribution there, and from it the potential is determined everywhere in the source-free region.

If the real part of a function $u + jv$ is defined along the imaginary axis, the imaginary part in a source-free right half-plane[18] (see also Prob. 11.11g) is

$$v(\alpha, \omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\omega' - \omega)u(\omega')\, d\omega'}{\alpha^2 + (\omega' - \omega)^2} \tag{5}$$

[18] *H. Jeffreys and B. S. Jeffreys,* Methods of Mathematical Physics, *3rd ed., Cambridge University Press, London, 1972.*

Thus if $R(\omega)$ is given, this may be used directly to find $X(\omega)$. Several specializations may be used in applying to the network function. Since we wish X for real frequency, α is set equal to zero in (5). $R(\omega')$ is an even function of ω'. Finally we may add the reactance function of any lossless two port in series if we allow idealized elements in the circuit, since such a function is known to have no R. Use of these three points leads to

$$X(\omega) = \frac{1}{\pi}\int_0^\infty R(\omega')\left(\frac{1}{\omega' - \omega} + \frac{1}{-\omega' - \omega}\right) d\omega' + X_0(\omega)$$
$$X(\omega) = \frac{2\omega}{\pi}\int_0^\infty \frac{R(\omega')}{\omega'^2 - \omega^2}\, d\omega' + X_0(\omega) \tag{6}$$

where $X_0(\omega)$ denotes the reactance function for the lossless part.

Since jZ is also analytic in the complex plane, (5) may also be used with u denoting $-X$ and v denoting R. In this application we again set $\alpha = 0$, and utilize the fact that X is an odd function of ω. We can also add in series a constant resistance without changing X, so the result is

$$R(\omega) = \frac{1}{\pi}\int_0^\infty - X(\omega')\left(\frac{1}{\omega' - \omega} - \frac{1}{-\omega' - \omega}\right) d\omega' + R_0$$
$$R(\omega) = -\frac{2}{\pi}\int_0^\infty \frac{\omega' X(\omega')}{\omega'^2 - \omega^2}\, d\omega' + R_0 \tag{7}$$

Similar relations apply to admittance functions and to relations between magnitude and phase. The analogy to potential and flux functions has allowed the interesting use of electrolytic tanks and resistance paper for the study of network functions.[19]

11.12 Equivalent Circuits Showing Frequency Characteristics of One Ports

The functional properties of the impedance (admittance) function developed in the preceding section allow one to develop several equivalent circuits showing the frequency characteristics. We first consider the impedance function, without losses, then the effect of small losses, and finally other forms.

First Foster Form for Loss-Free Case It was shown that the reactance function of a loss-free one port is an odd function of frequency and must have an infinite number of simple poles. It is known from the theory of functions that such a function can be expanded in a series of "partial fractions" about the poles, provided that the following summation is convergent[20]:

$$X(\omega) = \sum_{n=1}^{\infty}\left(\frac{a_n}{\omega - \omega_n} + \frac{a_{-n}}{\omega - \omega_{-n}}\right) + \frac{a_0}{\omega} + f(\omega) \tag{1}$$

[19] W. W. Hansen and O. C. Lundstrom, Proc. IRE **33**, 528 (1945).
[20] See, for example, K. Knopp, Theory of Functions, Part II, Chap. 2, Dover, New York, 1952.

where a_0/ω represents the pole at zero frequency, if any is present, and $f(\omega)$ is an arbitrary *entire function* (one with no singularities in the finite plane). Since the function is odd, $\omega_{-n} = -\omega_n$ and $a_n = a_{-n}$. Moreover, $f(\omega)$ can have only odd powers of ω, and, since it must behave at most like a simple pole at infinity, it is known to be proportional to the first power of ω. With these specializations, (1) becomes

$$X(\omega) = \sum_{n=1}^{\infty} \frac{2\omega a_n}{\omega^2 - \omega_n^2} + \frac{a_0}{\omega} + \omega L_\infty \tag{2}$$

In (2), a_n is known as the *residue* of the pole ω_n. It may be obtained in terms of the slope of the susceptance curve, which can in turn be related to energy storage. For, in the vicinity of ω_n, the nth term of (2) predominates and

$$B(\omega) = -\frac{1}{X(\omega)} \approx -\frac{\omega^2 - \omega_n^2}{2\omega a_n}$$

Differentiation shows that

$$\left.\frac{dB}{d\omega}\right|_{\omega=\omega_n} = -\frac{1}{a_n}$$

Then, utilizing Eq. 11.11(4)

$$a_n = -\frac{1}{(dB/d\omega)_{\omega=\omega_n}} = -\left[\frac{VV^*}{4(U_E + U_H)}\right]_{\omega=\omega_n} \tag{3}$$

The form of (2) suggests an equivalent circuit consisting of antiresonant LC circuits added in series as shown in Fig. 11.12a, since the nth component of this circuit yields a reactance

$$X_n = -\frac{1}{\omega C_n - 1/\omega L_n} = -\frac{\omega/C_n}{\omega^2 - 1/L_n C_n}$$

Comparison with the foregoing equation yields

$$a_n = -\frac{1}{2C_n}, \qquad \omega_n^2 = \frac{1}{L_n C_n}, \qquad a_0 = -\frac{1}{C_0} \tag{4}$$

or

$$C_n = -\frac{1}{2a_n}, \qquad L_n = -\frac{2a_n}{\omega_n^2}, \qquad C_0 = -\frac{1}{a_0} \tag{5}$$

This representation, known as the *first canonical form of Foster*,[17] is then applicable to any lossless one port for which the series in (2) is convergent. To find the circuit, we need to know the antiresonances, with energy storage quantities at those frequencies; both of these quantities were studied for cavity resonators in Chapter 10. The difficult part comes from the fact that the energy must be referred to the voltage in the input guide [see (3)], and this requires some specific knowledge of the coupling network.

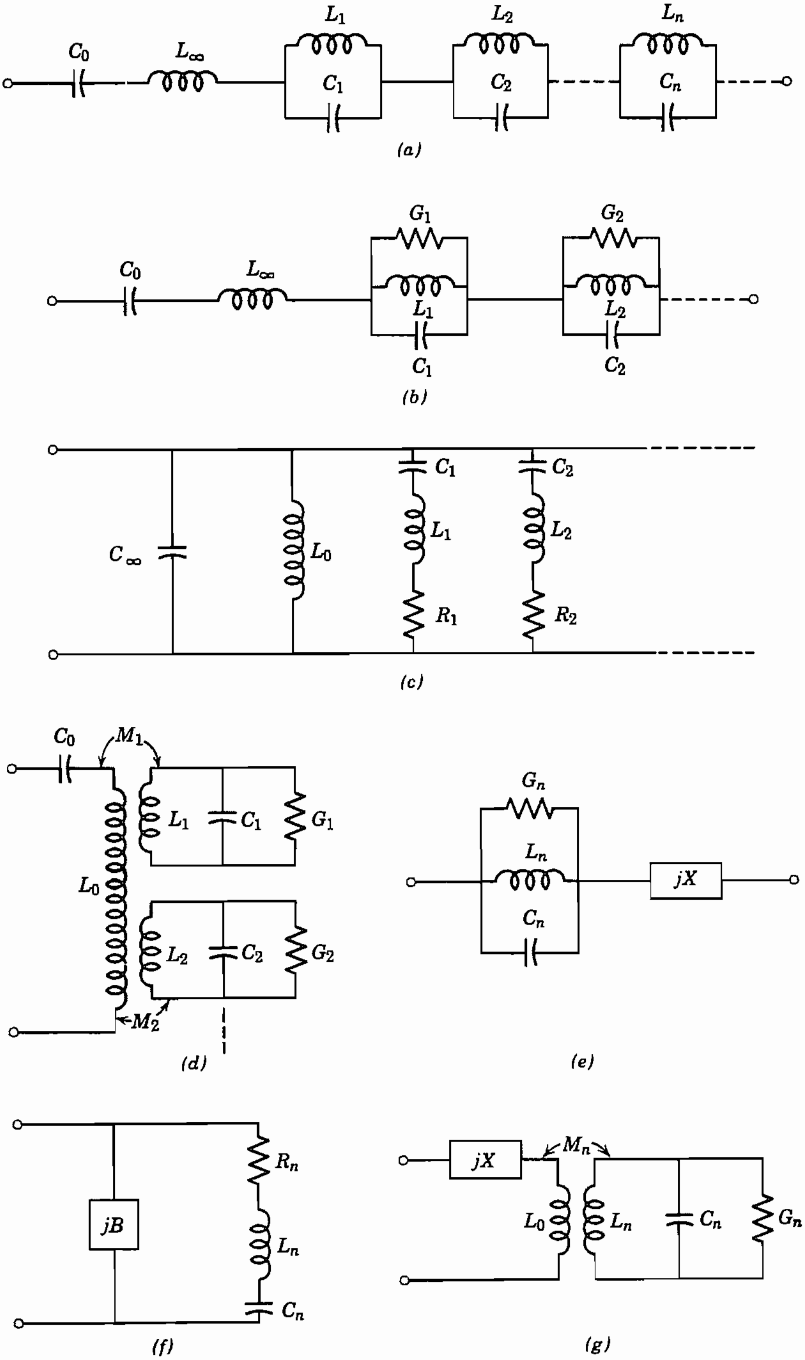


FIG. 11.12a–g Various equivalent circuits for one ports.

The general representation may be useful for interpreting measurements and for forming general conclusions even when this coupling problem cannot be solved.

Effect of Losses The study of losses for practical cavity resonators in the last chapter was concerned with the calculation of a quality factor Q which expressed for a given mode the ratio of energy stored to energy lost per radian. For low-loss cavities, it might be expected that the equivalent circuit of Fig. 11.12*a* would be modified by adding a shunt conductance to each antiresonant element, as shown in Fig. 11.12*b*. The value of a given conductance G_n would be adjusted so that the Q calculated from the nth antiresonant circuit would agree with the known Q_n of the mode which it represents. That is,

$$G_n = \frac{\omega_n C_n}{Q_n} \tag{6}$$

Justification for this procedure can be supplied by the theory of functions by making approximations appropriate to poles which are at a complex frequency near, but not exactly on, the real frequency axis.

If one accepts this modification of the lumped circuit equivalent to account for losses, it is clear that the Q of a cavity, determined from energy calculations, is also useful for interpreting the frequency characteristics in the same manner as for a lumped circuit. This fact was stated without justification in Sec. 10.3.

Second Foster Form An expansion of the susceptance function about its poles yields a form similar to (2):

$$B(\omega) = \sum_{m=1}^{\infty} \frac{2\omega b_m}{\omega^2 - \omega_m^2} + \frac{b_0}{\omega} + \omega C_\infty, \tag{7}$$

where the residues b_m are given by

$$b_m = -\frac{1}{[dX/d\omega]_{\omega=\omega_m}} = -\left[\frac{II^*}{4(U_E + U_H)}\right]_{\omega=\omega_m} \tag{8}$$

When the series is convergent, this equation has the equivalent circuit of Fig. 11.12*c* (known as the *second Foster canonical form*) with

$$L_m = -\frac{1}{2b_m} \qquad C_m = -\frac{2b_m}{\omega_m^2} \qquad L_0 = -\frac{1}{b_0} \tag{9}$$

Figure 11.12*c* also shows series resistance added to each resonant circuit to account for small losses, and as in the foregoing discussion, these are selected to give the known Q for each mode:

$$R_m = \frac{\omega_m L_m}{Q_m} \tag{10}$$

Other Equivalent Circuits Schelkunoff[21] has shown that other equivalent circuits may be derived by adding convergence factors to the series (2) or (7). These factors are necessary if the original series do not converge, the Mittag–Leffler theorem[20] from the theory of functions telling how they may be formed to ensure convergence. They may also be desirable in other cases where the original series converge, but do so slowly. For example, Schelkunoff has shown that the form with one term of the convergence factor is

$$X(\omega) = \sum_{n=1}^{\infty} 2\omega a_n \left(\frac{1}{\omega^2 - \omega_n^2} + \frac{1}{\omega_n^2} \right) + \frac{a_0}{\omega} + \omega L_0 \tag{11}$$

Note that, in addition to the convergence factor added in the series, the series inductance term has been modified, and inspection of (11) shows the L_0 is the entire series inductance of the circuit in the limit of zero frequency. The physical explanation of this procedure is then that this low-frequency inductance has been taken out as a separate term rather than being summed from its contributions from the various modes. It is reasonable to expect that this would often help convergence. A specific example for loop coupling to a cavity will be given in the following section.

The equivalent circuit of Fig. 11.12*d* gives the form of reactance function (11) (loss elements G_n being neglected at first), provided that

$$\frac{M_n^2}{L_n} = -\frac{2a_n}{\omega_n^2}, \qquad \frac{1}{L_n C_n} = \omega_n^2 \tag{12}$$

Here one imagines the input guide coupled to the various natural modes of the resonator through transformers, which gives a very natural way of looking at a problem of loop coupling to a cavity. Note, however, that one cannot determine the elements of the circuit uniquely since there are three elements, L_n, C_n, M_n, to be determined from the two basic quantities a_n and ω_n for each mode. One of the three may be chosen arbitrarily, perhaps by reference to physical intuition, but any choice will give a circuit which properly duplicates the behavior with respect to impedance at input terminals. Small losses are again accounted for by adding conductances G_n, calculated from form (6), to the circuits as shown in Fig. 11.12*d*.

Approximations in the Vicinity of a Single Mode Finally, we note that when we are interested in operation in the vicinity of the natural frequency for one mode, other resonances being well separated, the dominant factor will be the one representing that mode. Other terms will vary only slowly with frequency over this range and may be lumped together as a constant impedance or admittance (predominantly reactive). The equivalent circuits of Figs. 11.12*b*–*d* then reduce to the simplified representations of Figs. 11.12*e*–*g*, respectively. This is an important practical case, enabling one to use simplified lumped-element circuit analysis for the study of cavity resonator coupling problems.

[21] *S. A. Schelkunoff,* Proc. IRE ***32,*** *83 (1944).*

11.13 Examples of Cavity Equivalent Circuits

Two examples will be given to clarify the calculation of element values in the equivalent circuits of Sec. 11.12. It should be stressed again that the difficult part comes in solving enough of the coupling problem to refer the energy quantities within the resonator to defined voltage or current in the guide. In the first example, a uniform line is considered so that energy can be expressed directly in terms of the input current. In the second example, a reasonable approximation to the coupling problem can be made.

Example 11.13a
Open-Circuited Transmission Line

Let us consider a lossless open-circuited line of length l, inductance L per unit length, and capacitance C per unit length (Fig. 11.13a). We shall derive the second Foster form, Fig. 11.12c, starting from Eq. 11.12(7). For this calculation we need the natural modes having infinite susceptance at the input. Current is then a maximum at the input, zero at $z = l$, and length must be an odd multiple of a quarter-wavelength:

$$I_m(z) = I_{0m} \cos \omega_m z\sqrt{LC} \tag{1}$$

$$\omega_m = \frac{2\pi}{\lambda_m\sqrt{LC}} = \frac{2\pi m}{4l\sqrt{LC}} \qquad (m \text{ odd}) \tag{2}$$

The sum of average U_E and U_H is equal to the total energy stored at resonance, which may be computed as maximum energy in magnetic fields:

$$U_E + U_H = (U_H)_{\max} = \int_0^l \frac{LI_{0m}^2}{2} \cos^2 \omega_m z\sqrt{LC}\, dz = \frac{lLI_{0m}^2}{4} \tag{3}$$

Substitution in Eq. 11.12(8) gives the residue for the mth mode:

$$b_m = -\frac{I_{0m}^2}{4(U_E + U_H)} = -\frac{1}{Ll}$$

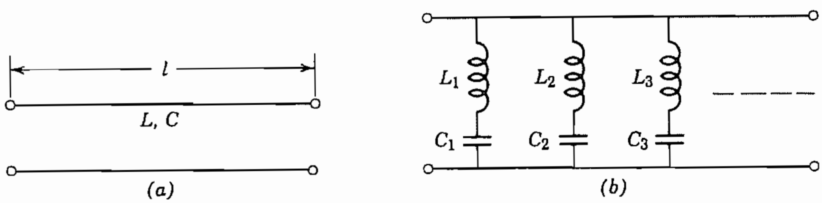


Fig. 11.13 (a) Open-circuited ideal line. (b) Equivalent circuit.

Inductance and capacitance for the mth circuit are found from Eq. 11.12(9):

$$L_m = -\frac{1}{2b_m} = \frac{Ll}{2} \tag{4}$$

$$C_m = -\frac{2b_m}{\omega_m^2} = \frac{8Cl}{\pi^2 m^2} \tag{5}$$

This leads to the equivalent circuit of Fig. 11.13b, which is valid for all frequencies, provided the equation for $B(\omega)$ obtained from Eq. 11.12(7) is convergent. The series is convergent in this case, and in fact can be shown to be equivalent to the closed form

$$B(\omega) = -\sum_{m \text{ odd}} \frac{2\omega}{Ll[\omega^2 - m^2\pi^2/4l^2LC]} = \sqrt{\frac{C}{L}} \tan \omega l\sqrt{LC} \tag{6}$$

The last expression can be recognized as the input susceptance for an open-circuited ideal line obtained from simple transmission-line theory, as it should be.

Example 11.13b
LOOP-COUPLED CAVITY

For a second example, we shall return to the loop-coupled cylindrical cavity discussed in Sec. 10.10 from an energy point of view. In particular, we shall concern ourselves with behavior in the vicinity of resonance for the simple TM_{010} mode, all other resonances being well separated, so that one of the approximate forms of Fig. 11.12e, f, or g is appropriate. The form of Fig. 11.12g, arising from Eq. 11.12(11), is particularly useful because the self-inductance of the loop is separated out, and the remaining series may be thought of as representing more nearly the behavior of the unperturbed cavity. From the physical point of view, it is a natural equivalent circuit, since we picture the input line as being coupled to the cavity mode through a mutual inductance which represents the loop.

The voltage at the loop terminals (computed with no self-inductance drop, as is appropriate for the zero current of antiresonance) is found approximately by taking magnetic field of the unperturbed mode flowing through the small loop of area S, as in Sec. 10.10:

$$V = j\omega\mu HS \tag{7}$$

Energy stored in the mode from Sec. 10.5 may be written

$$(U_E + U_H) = (U_H)_{\max} = \tfrac{1}{2}\pi\mu\, dH^2a^2 \tag{8}$$

Substitution in Eq. 11.12(3) gives the residue for the mode:

$$a_1 = -\frac{VV^*}{4(U_E + U_H)_{\omega=\omega_1}} = -\frac{\omega_1^2\mu^2S^2}{2\pi a^2 d} \tag{9}$$

Resonant frequency and Q are known from the analysis of Sec. 10.5:

$$\omega_1 = \frac{p_{01}}{a\sqrt{\mu\varepsilon}} \tag{10}$$

$$Q_1 = \frac{\eta p_{01} d}{2R_s(d + a)} \tag{11}$$

Here we meet the indeterminancy of the form selected, for we have three quantities, a_1, ω_1, Q_1, to determine four quantities, M_1, L_1, C_1, and G_1. As pointed out before, one of the four may be selected arbitrarily and the same input impedance will result. One choice is to calculate the conductance G_1 from power loss and voltage across the center. This makes sense, for example, when an electron beam is to be shot across the center, in which case a beam admittance, calculated on the same basis, can simply be placed in parallel with G_1 in the equivalent circuit. This can be shown to be

$$G_1 = \frac{R_s}{\eta^2}\frac{2\pi a(d + a)}{d^2} J_1^2(p_{01}) \tag{12}$$

Application of Eqs. 11.12(6) and 11.12(12) then yields

$$C_1 = \frac{Q_1 G_1}{\omega_1} = J_1^2(p_{01})\left(\frac{\pi a^2 \varepsilon}{d}\right) \tag{13}$$

$$L_1 = \frac{1}{\omega_1^2 C_1} = \frac{\mu d}{\pi p_{01}^2 J_1^2(p_{01})} \tag{14}$$

$$M_1 = \left(-\frac{2a_1 L_1}{\omega_1^2}\right)^{1/2} = \frac{\mu S}{\pi a p_{01} J_1(p_{01})} \tag{15}$$

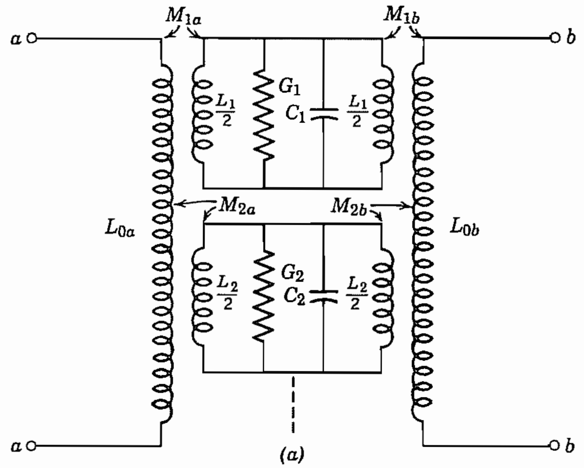
Input impedance, as a function of frequency, is then

$$Z = j\omega L_0 + \frac{\omega^2 M^2}{j\omega L_1 + 1/(G_1 + j\omega C_1)} \tag{16}$$

When evaluated at resonance, $\omega = \omega_1$, and taking $G_1 << \omega C_1$, this yields the same result as was found in Eq. 10.10(3) by energy considerations.

11.14 Circuits Giving Frequency Characteristics of N Ports

In considering frequency characteristics of the type discussed in Sec. 11.11 for a one port, one notes that any coefficient Z_{ii} of an N port must satisfy the conditions for a one port, since this represents the input impedance for a one port formed by shorting all but the ith terminal. Similarly, any Y_{ii} must satisfy conditions for the admittance function for a one port. The transfer coefficients, however, need not satisfy such conditions.



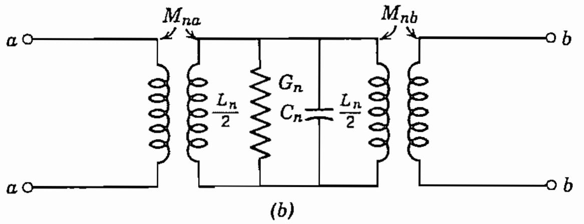


FIG 11.14 (*a*) Equivalent circuit for a cavity coupled to two waveguide terminals. (*b*) Approximation in the vicinity of one resonant mode.

When the N port is a cavity resonator with more than one waveguide coupled to it, representation of frequency characteristics by an equivalent circuit may be desirable, generalizing Sec. 11.12. Thus an extension of the form Fig. 11.13*d* would naturally lead to each waveguide terminal coupled to each of the normal modes by a mutual inductance as illustrated in Fig. 11.14*a* for a cavity with two ports. Justification for this procedure has been supplied by Schelkunoff[21] from complex function theory and by Harrington[22] from normal mode theory. In the vicinity of a resonance, the circuit simplifies to that of Fig. 11.14*b*.

If we wish to determine the characteristics of an N port by measurement, the simplest procedure is usually that of terminating all but two of the ports in known impedances, leaving a two port for which four of the parameters may be obtained by the methods of Sec. 11.6. Repetition of this procedure for different pairs of ports will eventually give all the coefficients.

[22] *R. F. Harrington,* Time-Harmonic Electromagnetic Fields, *McGraw-Hill, New York, 1961.*

Junction Parameters by Analysis

11.15 QUASISTATIC AND OTHER METHODS OF JUNCTION ANALYSIS

It has been noted that we do not require the complete field solution for a network formulation. Nevertheless, solution of the boundary value problem for fields has been useful in obtaining the network representations for certain junctions. The results are largely tabulated in handbooks.[1,23] A brief discussion of the approach to such problems may be helpful for the proper use of the tabulated results.

For junctions small in comparison with wavelength, it may be possible to set down a reasonably good equivalent circuit from quasistatic reasoning. The basis for this follows from the Helmholtz equation:

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} + \left(\frac{2\pi}{\lambda}\right)^2 \mathbf{E} = 0 \tag{1}$$

Thus if variations arise from changes in dimension x, y, or z that are small in comparison with wavelength, the first terms dominate over the last and the equation reduces to Laplace's equation, giving static forms for the field solutions. As an example, consider the step in the parallel-plane line of Fig. 11.15a. The electrostatic solution of this problem is known (Sec. 7.7), and a "fringing" or "excess" capacitance may be found as the excess of total capacitance between electrodes over that which would exist if field lines were straight across. Letting $\alpha = a/b$, one has

$$C_{\mathrm{d}} = \frac{\varepsilon}{\pi}\left[\left(\frac{\alpha^2 + 1}{\alpha}\right)\ln\left(\frac{1+\alpha}{1-\alpha}\right) - 2\ln\left(\frac{4\alpha}{1-\alpha^2}\right)\right] \text{F/m width} \tag{2}$$

The curve for this is plotted in Fig. 11.15b. This capacitance is placed in shunt with the two transmission lines at the junction. As we shall see later, the shunt representation is exact. The use of

$$Y_{\mathrm{d}} \approx j\omega C_{\mathrm{d}} \tag{3}$$

is a good approximation if the transverse dimension b is less than about 0.2λ.

A second example in which quasistatic approximations are useful is the right-angle bend in a parallel-plane line illustrated in Fig. 11.15c. Physical reasoning leads us to include an inductance to account for the Faraday law difference in voltage between planes 1 and 2, and this may be estimated by assuming magnetic field uniform in the corner. There is also an excess capacitance as in the preceding example, and this may be divided equally because of the symmetry. There results the π equivalent circuit as shown in Fig. 11.15d with the element values there indicated and w, width into the page.

Let us return to Fig. 11.15a and argue more carefully the case for the shunt admittance as an exact representation of the junction effect. In an approximate transmission-line

[23] *T. Moreno,* Microwave Transmission Design Data, *Artech House, Norwood, MA, 1989.*

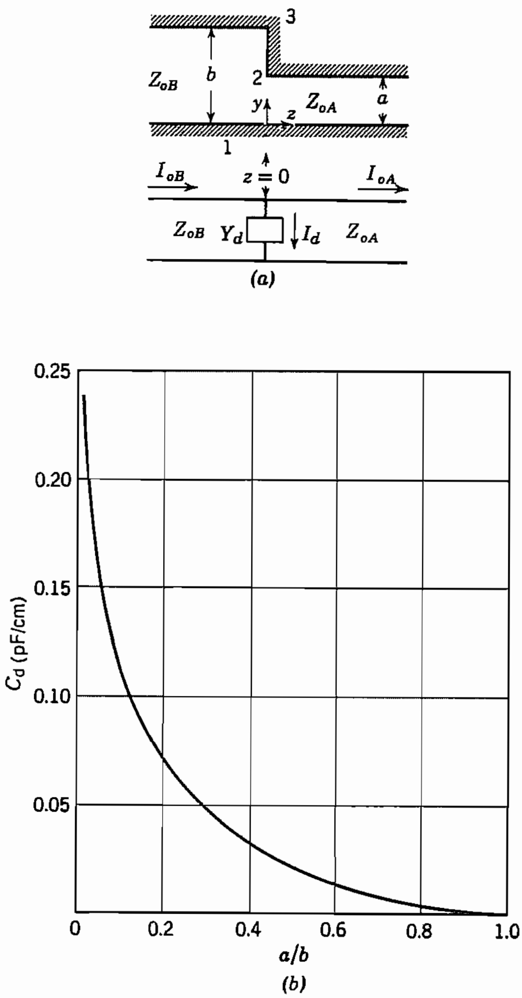


FIG. 11.15 (*a*) Step discontinuity in parallel-plane transmission line and exact equivalent circuit. (*b*) Curve of discontinuity capacitance per unit width for (*a*).

treatment, it is common to consider this as two lines of different characteristic impedance joined at $z = 0$. Such a treatment, however, considers only the TEM or principal transmission-line waves which have E_y and H_x with no variations in y. The perfect conductor portion between points 2 and 3 requires that $E_y = 0$ here. If there were only principal waves, E_y would then have to be zero everywhere at $z = 0$ because of the lack of variations with y in the principal wave. There could then be no energy passing into the second line A regardless of its termination since the Poynting vector would then also be zero across the entire plane, $z = 0$. The difficulty is met by the higher-order waves which are excited at the discontinuity, so that E_y in the principal waves is

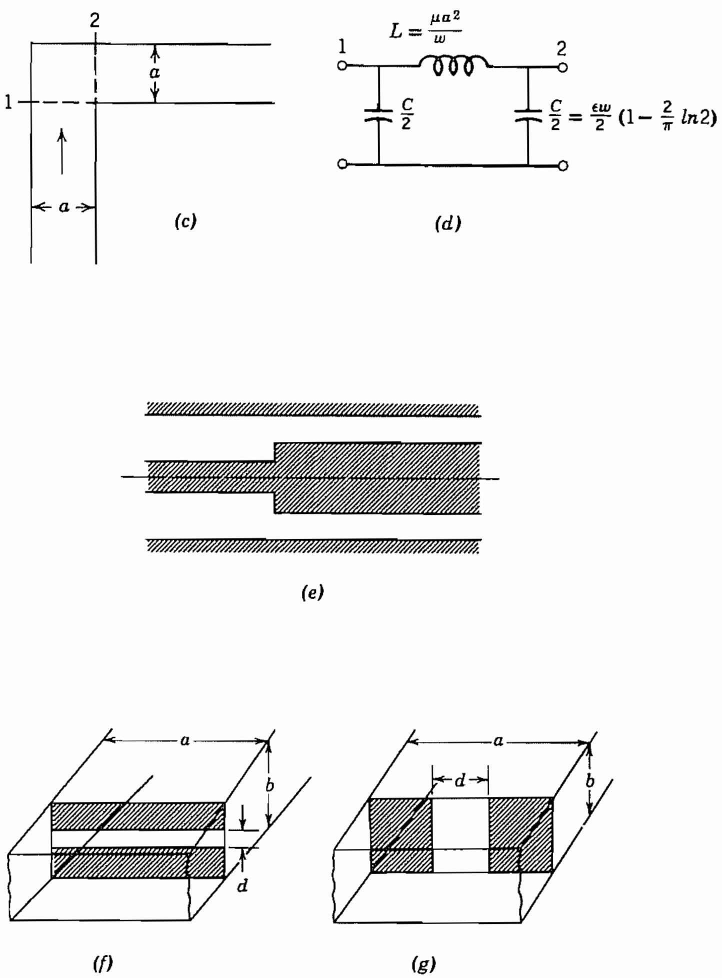


FIG. 11.15 (*c*), (*d*) Right-angle bend in parallel-plane transmission line and equivalent circuit. (*e*) Typical discontinuity in coaxial line. (*f*) Capacitive diaphragm in rectangular guide. (*g*) Inductive diaphragm in rectangular guide.

not generally zero at $z = 0$, but total E_y (sum of principal and higher-order components) is zero from 2 to 3 but not from 1 to 2. For the example of Fig. 11.15*a*, the higher-order waves excited are TM waves, since E_y, E_z, and H_x alone are required in the fringing fields. For spacings between planes small compared with wavelength, these waves are far below cutoff, so that their fields are localized in the region of the discontinuity.

To show that the effect of these local waves on the transmission of the principal waves may be expressed as a lumped admittance placed at $z = 0$ in the transmission-line equivalent circuit, as in Fig. 11.15*a*, consider that current at any value of z may be expressed as one part $I_0(z)$ from the principal wave and a contribution $I'(z)$ from all local waves:

$$I(z) = I_0(z) + I'(z) \tag{4}$$

Now *total* current must be continuous at the discontinuity $z = 0$, but current in the principal wave need not be, since the difference in principal-wave currents may be made up by the local-wave currents.

$$I_{0A}(0) + I'_A(0) = I_{0B}(0) + I'_B(0)$$

or

$$I_{0B}(0) - I_{0A}(0) = I'_A(0) - I'_B(0) \tag{5}$$

Total voltage in the line as defined by $-\int \mathbf{E} \cdot \mathbf{dl}$ between planes, however, is only that in the principal wave, since a study of the local waves shows that their contribution is zero:

$$V(z) = V_0(z)$$

Continuity of total voltage across the discontinuity $z = 0$ then requires continuity of voltage in the principal wave:

$$V_{0A}(0) = V_{0B}(0) = V_0(0)$$

Now, if an equivalent circuit is drawn for the principal wave only, its continuity of voltage but discontinuity of current may be accounted for by a lumped discontinuity admittance at $z = 0$, the current through this admittance being

$$I_{0B}(0) - I_{0A}(0) = I_\text{d} = Y_\text{d} V_0(0)$$

Or, from (5),

$$Y_\text{d} = \frac{I'_A(0) - I'_B(0)}{V_0(0)} \tag{6}$$

The complete analysis[24] reveals that, when local-wave values are substituted in (6), numerical values of Y_d may be calculated which are independent of terminations so long as these are far enough removed from the discontinuity not to couple to the local-wave fields. For Fig. 11.15*a* the shunt admittance acts as a pure capacitance if transverse dimensions are negligible compared with wavelength, and the value is accurately given by Fig. 11.15*b*. Corrections are needed when transverse dimension is comparable with wavelength.[24]

Results are available[25] for several forms of coaxial discontinuity, carried out by an important series method as formulated by Hahn.[26] To a fair approximation, discontinuity capacitance for Fig. 11.15*e* may be found by multiplying values from Fig. 11.15*b*

[24] *J. R. Whinnery and H. W. Jamieson,* Proc. IRE **32**, *98 (1944).*
[25] *J. R. Whinnery, H. W. Jamieson, and T. E. Robbins,* Proc. IRE **32**, *695 (1944).*
[26] *W. C. Hahn,* J. Appl. Phys. **12**, *62 (1941).*

by outer circumference. If the step is in the outer conductor, values from Fig. 11.15*b* are multiplied by inner circumference.

Marcuvitz and Schwinger[27] applied many of the powerful methods for boundary value problems to waveguide discontinuities including integral equation formulations and variational methods—probably the most powerful approximate methods for attacking wave problems of many types. The variational methods are described in several texts.[22,28] Approximate solution of the integral equations leads to approximate but useful forms for the two waveguide discontinuities shown in Figs. 11.15*f* and 11.15*g*. For the diaphragm extending from top and bottom of a rectangular guide propagating the TE_{10} mode (Fig. 11.15*f*), the energy of the higher-order modes is predominantly capacitive. The susceptance for a symmetrical diaphragm of gap d in a guide of width a and height b, is approximately

$$\frac{B}{Y_0} \approx \frac{4b}{\lambda_g} \ln \csc \frac{\pi d}{2b} \tag{7}$$

For the diaphragm extending from the side walls as in Fig. 11.15*g*, higher-order modes give a net stored magnetic energy, and the corresponding inductive susceptance, with d the gap width in this case also, is

$$\frac{B}{Y_0} \approx -\frac{\lambda_g}{a} \cot^2 \frac{\pi d}{2a} \tag{8}$$

The small iris in a conducting thin diaphragm across the rectangular waveguide (Fig. 11.15*h*) is especially useful in coupling between waveguide and a resonant cavity. This also may be represented by a shunt element, and for $d/b << 1$, the susceptance is

$$\frac{B}{Y_0} \approx -\frac{3ab\lambda_g}{2\pi d^3} \tag{9}$$

A variety of numerical methods have also been used to obtain the scattering parameters or network representations of waveguide and transmission-line discontinuities. As an example, consider the symmetric cross junction in microstrip of Fig. 11.15*i*. Its scattering parameters have been obtained for a particular junction by a finite-difference time-domain method.[29] Results are shown as a function of frequency in Fig. 11.15*j*. Note that for low frequencies,

$$|S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 = 4(\tfrac{1}{4}) = 1 \tag{10}$$

as expected from energy conservation. At the higher frequencies the sum is appreciably less than unity, indicating important radiation loss.

Results from a wide variety of microstrip and other planar transmission-line discontinuities are summarized in a handbook[30] and a reprint volume.[31]

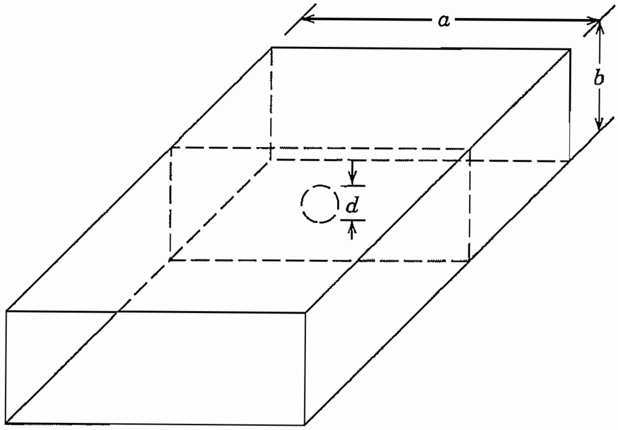
27 *N. Marcuvitz and J. Schwinger,* J. Appl. Phys. ***22****, 806 (1951), and in unpublished work.*

28 *R. E. Collin,* Field Theory of Guided Waves, *2nd ed., IEEE Press, Piscataway, NJ, 1991.*

29 *X. Zhang and K. K. Mei,* IEEE Trans. Microwave Theory Techniques ***36****, 1775 (1988).*

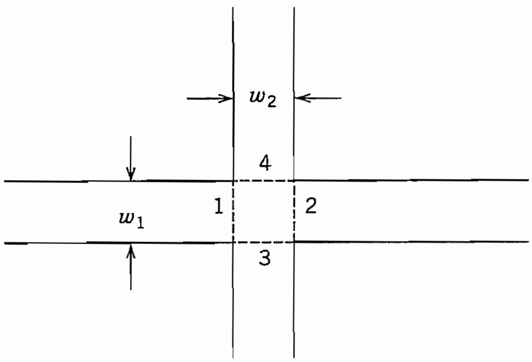
30 *R. K. Hoffmann,* Handbook of Microwave Integrated Circuits, *Artech House, Norwood, MA, 1987.*

31 *T. Itoh (Ed.),* Planar Transmission Line Structures, *IEEE Press, Piscataway, NJ, 1987.*

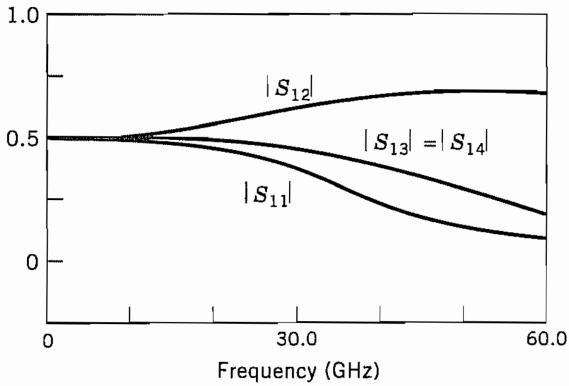


(h)

FIG. 11.15h Iris in thin diaphragm across rectangular waveguide.



(i)



(j)

FIG. 11.15 (i) Top view of cross junction in microstrip. (j) Calculated values from reference of footnote 29 for magnitudes of scattering parameters for junction of (i). $\varepsilon_r = 9.7$, substrate thickness $h = 0.635$ mm, $w_1 = w_2 = 0.56$ mm.

PROBLEMS

11.2a Show that the functions **f** and **g** defined by Eqs. 11.2(1) and (2) are in general related as follows:

$$\mathbf{g}(x, y) = \left(\frac{Z_0}{Z_z}\right)\hat{\mathbf{z}} \times \mathbf{f}(x, y)$$

Show that the values obtained for the rectangular guide satisfy this relation.

11.2b Use the principles of Sec. 11.2 to obtain definitions of voltage and current for the TE_{01} mode in a circular cylindrical guide.

11.2c* Repeat Prob. 11.2b for the TE_{11} mode in a circular guide and for the TM_{11} mode in a rectangular guide.

11.2d With voltage defined as the integral of maximum electric field between top and bottom of the TE_{10} mode in a rectangular guide, and current as the total axial flow in the top surface, compare with the derived quantities of Sec. 11.2. How is the product VI related to power flow in this case?

11.2e Supply the proof of the uniqueness theorem cited in Sec. 3.14 and utilized in Sec. 11.2. To do this, assume that there are two possible solutions, $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$, and apply the Poynting theorem to the difference field $(\mathbf{E}_1 - \mathbf{E}_2, \mathbf{H}_1 - \mathbf{H}_2)$. Note Sec. 1.17 for a typical uniqueness argument.

11.2f Suppose that an N port has a load impedance Z_L connected to the terminals 1, and voltage generators connected to the other $N - 1$ terminals. Show that the following Thévenin equivalent circuits are valid *so far as calculations of effects in the load are concerned*:

(i) A voltage generator V_0 connected to Z_L through a series impedance Z_g. V_0 is the voltage produced at terminals 1 with these terminals open-circuited, and Z_g is the impedance seen looking into 1 with all voltage generators short-circuited (and any current generators open-circuited).

(ii) A current generator I_0 connected across Z_L with internal admittance Y_g in parallel. I_0 is the current that would flow at terminals 1 if these terminals were shorted, and $Y_g = 1/Z_g$.

11.3a Verify Eq. 11.3(2) for isotropic but not necessarily homogeneous media.

11.3b* Show that Eq. 11.3(2) does not apply for anisotropic media with the matrix representing permittivity or permeability asymmetric.

11.3c Complete a proof similar to that of Sec. 11.3 to show that $Z_{21} = Z_{12}$.

11.4a For $Z_{11} - Z_{12} = j2, Z_{22} - Z_{12} = j5, Z_{12} = j$, find the admittance coefficients, the π circuit, and the $ABCD$ constants.

11.4b For the numerical values of Prob. 11.4a, obtain the values in the equivalent circuit of Fig. 11.4*c*.

11.4c For a terminating impedance of 1 Ω, find input impedance using all the forms of Probs. 11.4a and b.

11.4d* Set up the relation between currents and voltages for Fig. 11.4*d*, and from these determine the impedance parameters in terms of Z_{01}, $\beta_1 l_1$, m, and B.

11.5a Relate the scattering coefficients to the admittance coefficients to obtain equations similar to Eqs. 11.5(9).

11.5b Imagine that the source of energy is introduced at reference 2 in Fig. 11.5, with guide 1 perfectly terminated so that $a_1 = 0$. Derive the condition corresponding to Eq. 11.5(8). (Note that although these conditions are derived for special terminations of the network, they relate the basic parameters and must hold, quite apart from the termination and driving conditions.)

11.5c Show that the condition for reciprocity in terms of the transmission coefficients is $T_{11}T_{22} - T_{12}T_{21} = 1$.

11.5d* The matrix equivalents of Eqs. 11.5(9) and Prob. 11.5a are

$$[S] = ([\overline{Z}] - [U])([\overline{Z}] + [U])^{-1} = ([U] - [\overline{Y}])([U] + [\overline{Y}])^{-1}$$

where $\overline{Z}$ is normalized Z, that is, $\overline{Z}_{nn} = Z_{nn}/Z_{0n}$, $\overline{Z}_{nm} = Z_{nm}/\sqrt{Z_{0n}Z_{0m}}$, and similarly for $\overline{Y}$. $[U]$ is the unit matrix. Derive these from the matrix statements of the several formulations.

11.5e Find scattering and transmission coefficients for the network with numerical values given in Prob. 11.4a. Utilize the results to find input reflection coefficient if the output is matched.

11.6a As noted in the text, network analyzers typically heterodyne the microwave signal $E_s \cos(\omega_s t + \phi_s)$ with a local oscillator signal $E_L \cos \omega_L t$ by forming a product in a nonlinear device. Show that phase is preserved in the lower-frequency signal of frequency $\omega_L - \omega_S$.

11.6b If one selects the point of minimum slope P' of Fig. 11.6*b* to determine x_0, y_0 and equates this slope to Z_{02}/m^2Z_{01}, a second correct representation results. Show that transformations calculated by the latter are equivalent to those from the representation described.

11.6c* For the numerical values of Prob. 11.4a, plot an "S" curve as in Fig. 11.6*b*. Show that the values of m^2, x_0, and y_0 agree with those calculated in Prob. 11.4b.

11.7a Derive the transmission parameters, T_{11} and so on, for each of the elements of Table 11.7.

11.7b In a transmission line with shunt element, as in Ex. 11.7a, let the line be lossless so that $\gamma = j\beta$ and the admittance be a pure reactance, $Y = jB$. Write the *ABCD* parameters for one section and for N sections of such a line.

11.7c Repeat Prob. 11.7b for a lossless transmission line with series reactances replacing the shunt susceptances.

11.7d* A dielectric window is formed by two contacting dielectric slabs, ε_1 of length l_1 and ε_2 of length l_2, placed with air (ε_0) on each side. Set up the matrix product for the overall *ABCD* matrix and also for the $[T]$ matrix and compare. For only an outward propagating wave on the right, find reflections on the left for $k_1l_1 = k_2l_2 = \pi$ and $k_1l_1 = k_2l_2 = \pi/2$. Explain results physically.

11.7e Equation 11.7(13) for propagation constant may also be derived by a difference equation approach. Assuming that voltage and current at the nth output are $\exp(-n\Gamma)$ times the values at the beginning, relate quantities at the nth, $(n + 1)$st, and $(n - 1)$st and derive the desired relation.

11.8a For the filter of Ex. 11.8a, take $C_d = 1$ pF, $Y_0 = 0.02$ S, $l = 1$ cm, and find the upper cutoff frequency of the first passband and upper and lower cutoff frequencies of the next higher passband.

11.8b A coaxial transmission line has periodic gaps, distance l apart, in the inner conductor.

These act as series capacitors C. Find the expression for Γ and comment on the nature of the passbands and stop bands.

11.8c Derive Eq. 11.8(9) from 11.8(8). Note that for $\beta_1 l_1$ and $\beta_2 l_2$ each a multiple of π, one has real solutions for Γ. Explain in terms of the wave impedance transformation for such electrical lengths. For $\eta_2 = \eta_1(1 + \delta)$ show that to first order in δ, $\Gamma = j(\beta_1 l_1 + \beta_2 l_2)$. Explain.

11.9a Show that Eqs. 11.9(18) and (19) follow from 11.9(16) or (17). Verify the statement that reciprocity is not required for these relations.

11.9b Derive Eqs. 11.9(18) and (19) from conservation of energy as suggested in the paragraph following 11.9(19).

11.9c For the Y junction of Fig. 11.9b, show that for symmetric sources ($a_1 = a_2$) and port 3 matched so that $a_3 = 0$, the junction does act as a power combiner with all power incident on 1 and 2 exiting at 3. Assume symmetry ($S_{11} = S_{22}$, $S_{13} = S_{23}$) and that $S_{33} = 0$.

11.10a Prove theorem 1 [following Eq. 11.10(11)] for directional couplers.

11.10b Prove theorem 2 [following Eq. 11.10(11)] for directional couplers.

11.10c Prove theorem 3 [following Eq. 11.10(11)] for directional couplers.

11.10d For the "rat race" hybrid of Fig. 11.10d, give physical arguments to explain why a signal introduced at port 2 gives no response at 4 when ports 1 and 3 are equally loaded.

11.11a It was noted in Sec 11.11 that diagonal terms of the impedance matrix for an N port have the properties of one-port impedance. To show that off-diagonal terms do not have all these properties, demonstrate that Z_{12} for length l of a loss-free transmission line does not satisfy Foster's reactance theorem.

11.11b Illustrate by example that $dX/d\omega$ is not always positive for lossy networks.

11.11c Suppose that a small variation $\delta\omega$ in frequency produces variations $\delta\mathbf{E}$ and $\delta\mathbf{H}$ in fields. Starting from Maxwell's equations, show that

$$\oint_S (\mathbf{E} \times \delta\mathbf{H} - \delta\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = j\delta\omega \int_V (\mu H^2 - \varepsilon E^2)\, dV$$

11.11d Apply the result of Prob. 11.11c to derive Eq. 11.11(4) for a lossless network with one waveguide input.

11.11e For a simple circuit with resistance R and capacitance C in parallel, show that $R(\omega)$ and $X(\omega)$ satisfy Eqs. 11.11(6) and (7).

11.11f If Z is analytic in the right half-plane, ln Z is also. Utilize this fact and the form of Eq. 11.11(5) to relate magnitude and phase of a passive network, obtaining equations analogous to Eqs. 11.11(6) and (7).

11.11g* Although Eq. 11.11(5) may be derived in a variety of ways, one interesting point of view is through potential theory. Let $u(\omega)$ correspond to a flux function given along the $j\omega$ axis. By the Cauchy–Riemann equations, $du/d\omega$ determines $dv/d\alpha$ for $\alpha = 0$, which is the normal electric field entering that plane. This may be interpreted to arise from a surface charge distribution $\rho_s = -2\, dv/d\alpha$ along the plane $\alpha = 0$. (Question: Why the factor of 2 when it is absent if the plane is a conductor?) An element of charge $\rho_s d\omega'$ at ω' may be thought of as a line charge perpendicular to the plane, for which the potential at point α, ω can be found. Write an expression for the potential and integrate by parts to obtain Eq. 11.11(5). What conditions did you have to assume at infinity for the result to be valid?

11.11h To what does scattering matrix reduce for a one port? Give the properties of this similar to those discussed in Sec. 11.11 for impedance or admittance.

11.12* Modify the form of $B(\omega)$, Eq. 11.12(7), to correspond to the form 11.12(11) with a convergence factor added, and find an equivalent circuit representation. *Hint:* Change the form of Fig. 11.12*d* by using T networks to replace the transformers, and look for the dual of this circuit.

11.13a Derive the equivalent circuit of Fig. 11.12*a* for a shorted length l of ideal line. Show that the series form for $X(\omega)$ converges to the usual expression for reactance of a shorted ideal line.

11.13b Derive the first Foster form for the open line and the second Foster form for the shorted line. Show that the series forms converge to proper expressions for reactance and susceptance, given that

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 + n^2\pi^2}$$

11.13c Show that the capacitance C_1 derived for the loop-coupled cavity of Ex. 11.13b is that which would be obtained by referring energy stored in electric fields to voltage at the center of the cavity. Compare the mutual M_1 to that which would be derived by referring induced voltage in the loop to total vertical current in the cavity wall.

11.13d Consider the coupling from coaxial transmission line to the TM_{010} mode of the cylindrical resonator by a small probe of length s extending along the axis from the top. What assumptions concerning the coupling need be made to derive the equivalent circuit of form Fig. 11.12*e*? Similarly for Fig. 11.12*f*?

11.14a Suppose that two well-separated loops, each of area 0.5 cm^2, are coupled to the TM_{010} cylindrical mode as for one loop in Sec. 11.13. Draw the equivalent circuit, and calculate element values (except L_0) for an air-filled cavity resonant at 4 GHz with $d = 1$ cm, $R_s = 0.02$ ohm.

11.14b The Q of a cavity mode is sometimes measured by finding the curves of transmission versus frequency between two guides coupled to the cavity as in Prob. 11.14a. Under what condition will the Q be well approximated by $f_0/\Delta f$, where f_0 is resonant frequency and Δf the frequency difference between points of amplitude response $1/\sqrt{2}$ the maximum? Is this condition satisfied by the values of Prob. 11.14a, assuming a 50-Ω output load?

11.15a Determine the form of the proper local waves in the example of Fig. 11.15*a*. Show that voltage between planes, $-\int \mathbf{E} \cdot \mathbf{dl}$, is zero for each of these.

11.15b Imagine a parallel-plane transmission line with two steps such as the one in Fig. 11.15*a*. The first is from spacing b to spacing a; the second is removed from the first by a half-wavelength and is from spacing a back to b. The line to the right of b is perfectly terminated by its characteristic impedance, Z_{0B}. If it were not for the discontinuity capacitances, the line to the left of the first discontinuity would also be perfectly terminated. Calculate reflection coefficient in this line, taking into account the discontinuity capacitances from Fig. 11.15*b*. Take $a = 1$ cm, $b = 2$ cm, $\lambda = 12$ cm. Assume air dielectric.

11.15c Using Fig. 11.15*b*, calculate an approximate discontinuity capacitance for the coaxial line of Fig. 11.15*e*. Take $r_1 = 0.5$ cm, $r_2 = 1$ cm, $r_3 = 1.2$ cm. Assume air dielectric.

11.15d A rectangular waveguide of dimensions 0.900×0.400 in. propagating the TE_{10} mode at 9 GHz feeds a horn. Standing wave ratio in the guide is measured as 2.5 with a voltage minimum 0.55 cm in front of the horn entrance. Find the dimensions and placing of a capacitive diaphragm to produce a match for waves approaching from the left.

11.15e Repeat Prob. 11.15d, using an inductive diaphragm.

11.15f* Reason physically as to the field components required in higher-order modes for the two waveguide discontinuities of Figs. 11.15*f* and 11.15*g*, and the type of energy storage predominant in each.

11.15g For $a \gg \lambda/2$, the capacitive waveguide discontinuity reduces to the corresponding capacitive diaphragm problem in a parallel-plane transmission line. Express Eq. 11.15(7) as a capacitance per unit width in this limit, and plot as a function of d/b. Compare with the step capacitance of Fig. 11.15*b*.

11.15h* A resonant cavity is made by closing one end of a rectangular waveguide with a conductor, and placing another conductor, with diaphragm, distance l in front of it. Wavelength $\lambda = 4$ cm, $a = 3$ cm, $b = 1.5$ cm, $d = 0.2$ cm, and the resonant mode of concern is the TE_{101} mode. Estimate the Q resulting from loading by the waveguide in front of the diaphragm. (See Fig. 11.15*h*.)

11.15i For the microstrip cross junction of Fig. 11.15*i*, use results of Fig. 11.15*j* to estimate the fraction of incident power radiated if a 30 GHz wave is incident at terminal 1 with all other terminals matched.