5 Transmission Lines

5.1 Introduction

In Chapter 3, we saw that the interchange of electric and magnetic energy gives rise to the propagation of electromagnetic waves in space. More specifically, the magnetic fields that change with time induce electric fields as explained by Faraday's law, and the time-varying electric fields induce magnetic fields, as explained by the generalized Ampère's law. This interrelationship also occurs along conducting or dielectric boundaries, and can give rise to waves that are guided by such boundaries. These waves are of paramount importance in guiding electromagnetic energy from a source to a device or system in which it is to be used. Dielectric guides, hollow-pipe waveguides, and surface guides are all important for such purposes, but one of the simplest systems to understand—and one very important in its own right—is the two-conductor transmission line. This system may be considered a distributed circuit and so is useful in establishing a relation between circuit theory and the more general electromagnetic theory expressed in Maxwell's equations. The concepts of energy propagation, reflections at discontinuities, standing versus traveling waves and the resonance properties of standing waves, phase and group velocity, and the effects of losses upon wave properties are easily extended from these transmission-line results to the more general classes of guiding structures.

A parallel two-wire system is a typical and important example of the transmission lines to be studied in this chapter. In any transverse plane, electric field lines pass from one conductor to the other, defining a voltage between conductors for that plane. Magnetic field lines surround the conductors, corresponding to current flow in one conductor and an equal but oppositely directed current flow in the other. Both voltage and current (and, of course, the fields from which they are derived) are functions of distance along the line. In the two following sections we set down the transmission-line equations from distributed-circuit theory, but then discuss its relation to field theory.

Transmission-line effects are not always desirable ones. A cable interconnecting two high-speed computers may be intended as a direct connection, but will at the very least introduce a time delay (around 5 ns/m in typical dielectric-filled cable). Moreover, if the interconnections are not impedance matched at the two ends there will be reflections of the waves (as we shall see later in the chapter). These "echoes" of pulses representing the digits could introduce serious errors. A still further complication is *dispersion*. In a real transmission line the velocity of propagation varies to some degree with frequency, so the frequency components which represent the pulse (by Fourier analysis) travel at different velocities and the pulse distorts as it travels. If dispersion is excessive, the pulses may be blurred enough so that individual digits cannot be clearly distinguished. All of these effects occur also in the interconnections of elements in printed circuits and even in semiconductor integrated circuits, but the close spacings in the last case limit performance only for extremely short pulses.

Transmission-line analysis is useful, by analogy, in studying a variety of wave phenomena, such as the propagation of acoustic waves and their reflection from materials with different acoustic properties. An especially interesting analog is that of the propagation of signals along a nerve of the human body.

Time and Space Dependence of Signals on Ideal Transmission Lines

5.2 Voltage and Current Variations Along an Ideal Transmission Line

We begin by considering the transmission line as a distributed circuit. In Sec. 2.5 we identified an inductance per unit length associated with the flux produced by the oppositely directed currents in a pair of parallel conductors. When the currents vary with time, there is a voltage change along the line. Likewise, the distributed capacitance produces displacement current between the conductors when the voltage is time-varying and leads to change in the current flowing along the conductors. The interrelationship leads to the wave equation for voltage and current along an ideal lossless transmission line.

Figure 5.2 shows a representative two-conductor line and the circuit model for a differential length. It should be kept in mind that the external inductance per unit length of a parallel-conductor line is not associated with one conductor or the other. Also, the circuit model is simply a representation of a differential length of line; there is not a one-for-one identification of the two sides of the circuit with the two conductors of the modeled line.

Consider a differential length of line dz, having the distributed inductance, L per unit length, and the distributed capacitance, C per unit length.[1] The length dz then has

[1] *Note that, as in Chapter 4, the same symbols are used here for inductance and capacitance per unit length as for total inductance and capacitance. Some texts use l and c for the distributed quantities, but there is then confusion with length and light velocity, respectively.*

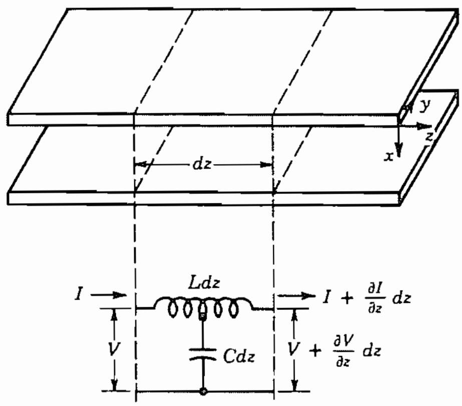


FIG. 5.2 Section of a representative transmission line and the equivalent circuit for a differential length.

inductance $L\,dz$ and capacitance $C\,dz$. The change in voltage across this length is then equal to the product of this inductance and the time rate of change of current. For such a differential length, the voltage change along it at any instant may be written as the length multiplied by the rate of change of voltage with respect to length. Then

$$\text{voltage change} = \frac{\partial V}{\partial z}\,dz = -(L\,dz)\,\frac{\partial I}{\partial t} \tag{1}$$

Note that time and space derivatives are written as partial derivatives, since the reference point may be changed in space or time in independent fashion.

Similarly, the change in current along the line at any instant is merely the current that is shunted across the distributed capacitance. The rate of decrease of current with distance is given by the capacitance multiplied by time rate of change of voltage. Partial derivatives are again called for:

$$\text{current change} = \frac{\partial I}{\partial z}\,dz = -(C\,dz)\,\frac{\partial V}{\partial t} \tag{2}$$

The length dz may be canceled in (1) and (2):

$$\frac{\partial V}{\partial z} = -L\,\frac{\partial I}{\partial t} \tag{3}$$

$$\frac{\partial I}{\partial z} = -C\,\frac{\partial V}{\partial t} \tag{4}$$

Equations (3) and (4) are the fundamental differential equations for the analysis of the ideal transmission line. Note that they are identical in form with the pairs, Eqs. 3.9(5) and 3.9(9) or Eqs. 3.9(6) and 3.9(8), found from Maxwell's equations for plane electromagnetic waves. As was done there, (3) and (4) can be combined to form a wave equation for either of the variables. For example, one can differentiate (3) partially with respect to distance and (4) with respect to time:

$$\frac{\partial^2 V}{\partial z^2} = -L \frac{\partial^2 I}{\partial z\, \partial t} \tag{5}$$

$$\frac{\partial^2 I}{\partial t\, \partial z} = -C \frac{\partial^2 V}{\partial t^2} \tag{6}$$

Partial derivatives are the same taken in either order (assuming continuous functions) so (6) may be substituted directly in (5):

$$\frac{\partial^2 V}{\partial z^2} = LC \frac{\partial^2 V}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2} \tag{7}$$

where

$$v = (LC)^{-1/2} \tag{8}$$

All real signals are continuous functions, as required for (7) to apply. The discontinuous step waves used later as examples are to be understood as approximations to real signals. Equations (3) and (4) are known as the *telegraphist's equations*, and the differential equation (7) is the one-dimensional wave equation. A similar equation may be obtained in terms of current by differentiating (4) with respect to z and (3) with respect to t, and combining the results:

$$\frac{\partial^2 I}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 I}{\partial t^2} \tag{9}$$

We saw in Sec. 3.9 that an equation of the form (7) has a solution

$$V(z, t) = F_1\left(t - \frac{z}{v}\right) + F_2\left(t + \frac{z}{v}\right) \tag{10}$$

where F_1 and F_2 are arbitrary functions. A constant value of $F_1(t - z/v)$ would be seen by an observer moving in the $+z$ direction with a velocity v, so $F_1(t - z/v)$ represents a wave traveling in the $+z$ direction with velocity v. Similarly, $F_2(t + z/v)$ represents a wave moving in the $-z$ direction with velocity v.

To find the current on the line in terms of the functions F_1 and F_2, substitute the expression for voltage given by (10) in the transmission-line equation (3):

$$-L \frac{\partial I}{\partial t} = -\frac{1}{v} F_1'\left(t - \frac{z}{v}\right) + \frac{1}{v} F_2'\left(t + \frac{z}{v}\right) \tag{11}$$

This expression may be integrated partially with respect to t:

$$I = \frac{1}{Lv}\left[F_1\left(t - \frac{z}{v}\right) - F_2\left(t + \frac{z}{v}\right)\right] + f(z) \tag{12}$$

If this result were substituted in the other transmission-line equation (4), it would be found that the function of integration, $f(z)$, could only be a constant. This is a possible superposed dc solution not of interest in studying the wave solution, so the constant will be ignored. Equation (12) may then be written

$$I = \frac{1}{Z_0}\left[F_1\left(t - \frac{z}{v}\right) - F_2\left(t + \frac{z}{v}\right)\right] \tag{13}$$

where

$$Z_0 = Lv = \sqrt{\frac{L}{C}}\ \Omega \tag{14}$$

The constant Z_0 as defined by (14) is called the *characteristic impedance* of the line, and is seen from (10) and (12) to be the ratio of voltage to current for a single one of the traveling waves at any given point and given instant. The negative sign for the negatively traveling wave is expected since the wave propagates to the left, and by our convention current is positive if flowing to the right.[2]

Example 5.2

CHARACTERISTIC IMPEDANCE AND WAVE VELOCITY FOR A COAXIAL LINE

Let us find expressions for the characteristic impedance and wave velocity for an ideal coaxial line and examine some typical values. We will assume the conductor spacing is large enough to neglect internal inductance. Using C from Eq. 1.9(4) and L from Eq. 2.5(6) in (14) we find

$$Z_0 = \frac{\ln b/a}{2\pi}\sqrt{\frac{\mu}{\varepsilon}} \tag{15}$$

where a and b are the radii of the inner and outer conductors at the dielectric surfaces, respectively. A common commercial coaxial cable is designated RG58/U. It has a dielectric of relative permittivity 2.26 and the radii are $a = 0.406$ mm and $b = 1.48$ mm. Substituting these values in (15) and taking $\mu \cong \mu_0$ one finds that $Z_0 = 51.6\ \Omega$. This is slightly below the published normal value $Z_0 = 53.5\ \Omega$ in part because of the neglect of the frequency-dependent internal inductance of the conductors. There is not much variation of the relative permittivity among the various materials used as the dielectrics in coaxial lines and since the radius ratio comes in only in a logarithm, one finds that most commercial coaxial lines have characteristic impedance in a limited

[2] *Since Z_0 as defined here is real, it is more logical to call it a "characteristic resistance," especially since the concept of impedance implies use with the phasor forms appropriate to steady-state sinusoidal excitation. That is an important special case to be considered later, but even for transmission lines used with pulses or other general signals, it is common to refer to the defined Z_0 as* characteristic impedance.

range, usually $50\ \Omega \lesssim Z_0 \lesssim 80\ \Omega$. The wave velocity (8) becomes

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{16}$$

which is the same as the velocity of plane waves in the same dielectric (Sec. 3.9). This result obtains for all two-conductor transmission lines when the internal inductance and losses can be neglected.[3] The wave velocity is usually between about 0.5 and 0.7 of the velocity of light in vacuo, 3×10^8 m/s, for lines with plastic dielectrics.

5.3 Relation of Field and Circuit Analysis for Transmission Lines

Although we largely utilize the distributed-circuit model for transmission-line analysis in this chapter, let us relate the equations obtained in Sec. 5.2 to field concepts of Chapter 3. First, let us take the special case of a parallel-plane transmission line, as indicated in Fig. 5.2, with the conducting planes assumed wide enough in the y direction so that fringing at the edges is not important. If the planes are also assumed perfectly conducting, it is clear that a portion of a uniform plane wave with E_x and H_y, as studied in Chapter 3, can be placed in the dielectric region between the planes and will satisfy the boundary condition that electric field enter normally to the perfectly conducting planes. The Maxwell equations for such a wave [Eqs. 3.9(6) and 3.9(8)] are

$$\frac{\partial E_x(z, t)}{\partial z} = -\mu \frac{\partial H_y(z, t)}{\partial t} \tag{1}$$

$$\frac{\partial H_y(z, t)}{\partial z} = -\varepsilon \frac{\partial E_x(z, t)}{\partial t} \tag{2}$$

If we define voltage as the line integral of $-\mathbf{E}$ between planes at a given z,

$$V(z, t) = -\int_1^2 \mathbf{E} \cdot \mathbf{dl} = -\int_0^a E_x\, dx = -aE_x(z, t) \tag{3}$$

Current for a width b, with positive sense defined for the upper plane, is related to the tangential magnetic field by

$$I(z, t) = -bH_y(z, t) \tag{4}$$

With these substituted in Eqs. 5.2(3) and 5.2(4), we find the result identical to (1) and (2) if

$$C = \frac{\varepsilon b}{a}\ \text{F/m}, \qquad L = \frac{\mu a}{b}\ \text{H/m} \tag{5}$$

These are, respectively, the capacitance per unit length and inductance per unit length

3 *It follows that knowledge of either L or C for such ideal lines determines the other.*

for such a system of parallel-plane conductors, calculated from static concepts. The field and circuit concepts are thus identical in this simple case.

We would also find field and distributed-circuit approaches identical if we applied them to the coaxial transmission line with ideal conductors or to two-conductor systems of other shape so long as conductors are perfect. This is because such systems can be shown to propagate *transverse electromagnetic* (TEM) waves, for which both electric and magnetic fields have only transverse components. The absence of an axial magnetic field means that there are no induced transverse electric fields and no corresponding contributions to the line integral $\int \mathbf{E} \cdot \mathbf{dl}$ taken between the two conductors, so long as the integration paths remain in the transverse plane; thus the voltage between conductors can be taken as *uniquely defined for that plane*. Similarly the absence of an axial electric field means that there is no displacement current contribution to $\oint \mathbf{H} \cdot \mathbf{dl}$ for paths in a given transverse plane, and if such a closed path surrounds one conductor, the integral will be just the conduction current flow in that conductor for that plane at that instant of time. Moreover, the transverse **E** and **H** fields can be shown to satisfy Laplace's equation *in the transverse plane* (Prob. 5.3), thus explaining the appropriateness of using Laplace solutions for the calculation of the L and C of the transmission line.

When the finite resistances of conductors are taken into account, the identity of circuit and field analysis is no longer an exact one, but has been shown to be a good approximation, for practical transmission lines. This field basis for TEM waves will be developed more in Chapter 8.

5.4 Reflection and Transmission at a Resistive Discontinuity

Most transmission-line problems are concerned with junctions between a given line and another of different characteristic impedance, a load resistance, or some other element that introduces a discontinuity. By Kirchhoff's laws, total voltage and current must be continuous across the discontinuity. The total voltage in the line may be regarded as the sum of voltage in a positively traveling wave, equal to V_+ at the point of discontinuity, and voltage in a reflected or negatively traveling wave, equal to V_- at the discontinuity. The sum of V_+ and V_- must be V_L, the voltage appearing across the junction:

$$V_+ + V_- = V_L \tag{1}$$

Similarly, the sum of current in the positively and negatively traveling waves of the line, at the point of discontinuity, must be equal to the current flowing into the junction or load:

$$I_+ + I_- = I_L \tag{2}$$

The simplest form of discontinuity is one in which a load resistance R_L is connected to the transmission line at the junction, as shown in Fig. 5.4*a*. Another case that is equivalent is that of Fig. 5.4*b* in which the first ideal line is connected to a second ideal line of infinite length and characteristic impedance Z_{0L}; here $R_L = Z_{0L}$. Still other forms

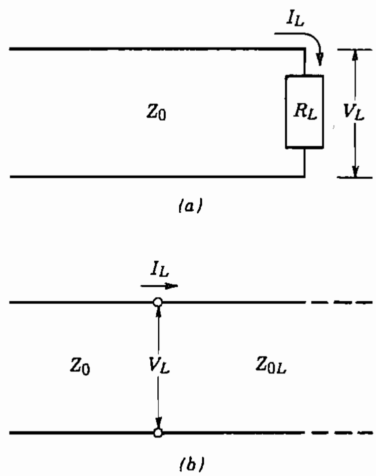


FIG. 5.4 (*a*) Ideal transmission line with a resistive load. (*b*) Ideal transmission line of characteristic impedance Z_0 with a second ideal line of infinite length and characteristic impedance Z_{0L} as a load.

of load circuits can produce an effective resistance R_L at the junction. In all these cases $V_L = R_L I_L$. By utilizing the relations between voltage and current for the two traveling waves as found in Sec. 5.2, Eq. (2) becomes

$$\frac{V_+}{Z_0} - \frac{V_-}{Z_0} = \frac{V_L}{R_L} \tag{3}$$

By eliminating between (1) and (3), the ratio of voltage in the reflected wave to that in the incident wave (*reflection coefficient*) and the ratio of the voltage on the load to that in the incident wave (*transmission coefficient*) may be found:

$$\rho \triangleq \frac{V_-}{V_+} = \frac{R_L - Z_0}{R_L + Z_0} \tag{4}$$

$$\tau \triangleq \frac{V_L}{V_+} = \frac{2R_L}{R_L + Z_0} \tag{5}$$

The most interesting, and perhaps the most obvious, conclusion from the foregoing relations is this: there is no reflected wave if the terminating resistance is exactly equal to the characteristic impedance of the line. All energy of the incident wave is then transferred to the load and τ of (5) is unity.

In Sec. 5.7 the definitions of reflection and transmission coefficients will be given for the case of sinusoidal signals and will include other than purely resistive loads.

The instantaneous incident power at the load is $W_T^+ = I_+ V_+ = V_+^2/Z_0$. The fractional power reflected is, therefore, the constant

$$\frac{W_T^-}{W_T^+} = \rho^2 \tag{6}$$

The remainder of the power goes into the load resistor or the second line, so

$$\frac{W_{TL}}{W_T^+} = 1 - \rho^2 \tag{7}$$

5.5 Pulse Excitation on Transmission Lines

Transmission lines are increasingly used for digital or pulse-coded information. We consider some simple examples utilizing the boundary and continuity conditions given above.

Example 5.5a
Pulse on Short-Circuited Line

Let us consider a signal in the form of a pulse having a constant value V_0 between $t = 0$ and $t = t_1/5$ and zero otherwise. The pulse is fed into an ideal transmission line of length l such that $l = vt_1$. We will analyze what happens when the pulse reaches the end of the line, which will be taken as short-circuited.

Drawing (i) in Fig. 5.5*a* shows the pulse moving along the line at $t \approx 0.3t_1$. The arrows connecting the charges on the conductors are electric field vectors. The integral of the electric field is the voltage between conductors. The current flows in the conductors only where there is voltage, and current continuity is accounted for by displacement currents $\varepsilon\, \partial \mathbf{E}/\partial t$ at the leading and trailing edges of the pulse.

At time t_1 the leading edge of the pulse reaches the end of the line. Drawing (ii) in Fig. 5.5*a* shows the pulse shortly after $t = t_1$. The short circuit requires that the voltage be zero. To maintain the voltage at zero during the time that the incident pulse is at the termination $t_1 < t < t_2$, a negative-z-traveling (reflected) wave having opposite voltage polarity and equal amplitude is generated as shown in drawing (iii). Note that this result is predicted by (4) for $R_L = 0$. Also, the zero voltage on the load agrees in (5) with $R_L = 0$. Note that the polarity of the current in the reflected wave is the same as in the incident wave, as could be argued from Eqs. 5.2(10) and 5.2(12) with the fact that $V_- = -V_+$. The total voltage on the line at the time used for drawings (ii) and (iii) is their superposition; this is shown in drawing (iv), where it is seen that the voltages are almost completely canceled. At a still later time, the reflected pulse is seen on its way to the generator [drawing (v)]. At $t = 2t_1$, the pulse will reach the pulse generator. What happens there depends on the impedance seen looking into the generator. Typically, the characteristic impedance of the transmission line equals the output impedance

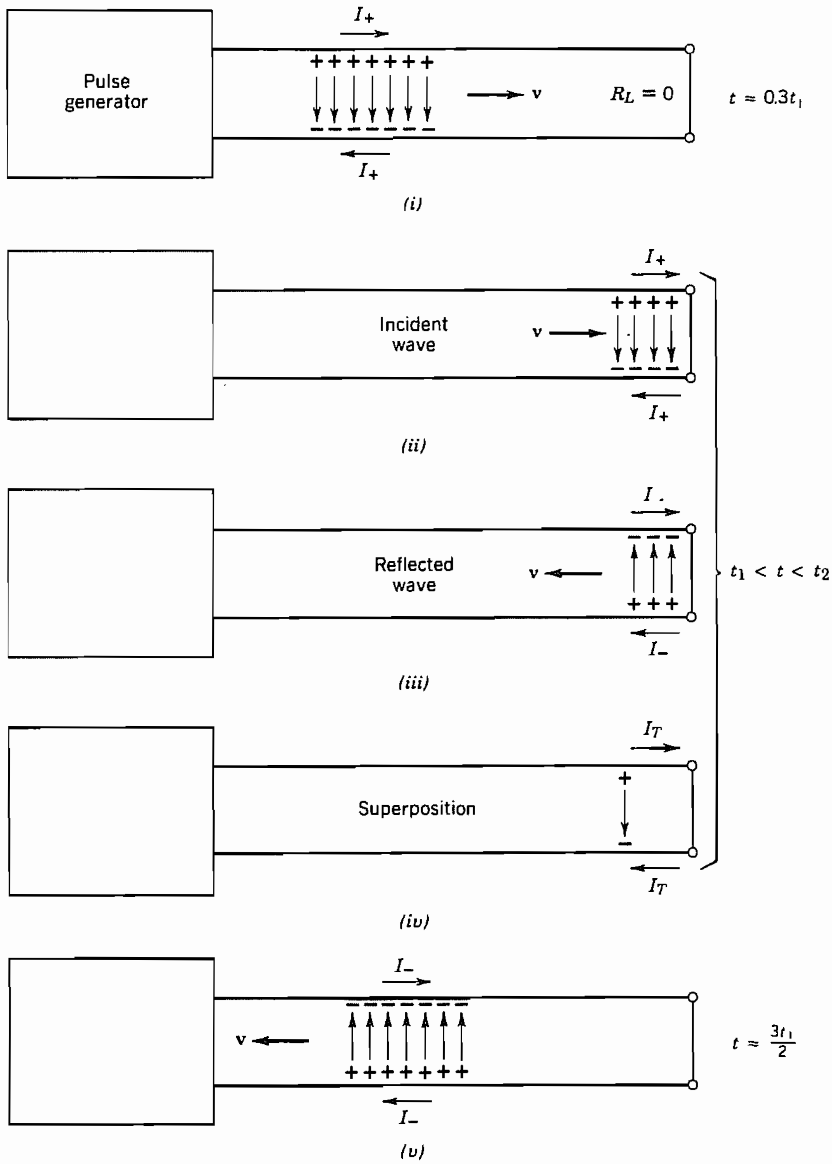


FIG. 5.5a Reflection of a pulse at the end of a shorted line. The times t_1 and t_2 are those for which the leading edges and trailing edges, respectively, reach the end of the line. Drawings (*ii*), (*iii*), and (*iv*) are for various instants during the period in which the reflected wave is generated to maintain the voltage at zero across the short circuit.

of the generator (it is *matched*). In that case, the reflected pulse is absorbed in the generator. Otherwise, another reflection takes place.

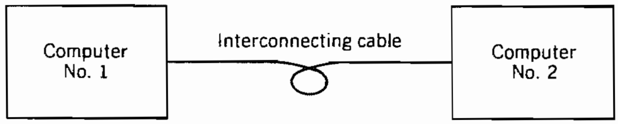


FIG. 5.5b Transmission line cable for transmitting digital signals between two computers.

Example 5.5b
PULSE REFLECTIONS ON A TRANSMISSION LINE INTERCONNECTING TWO COMPUTERS

The aim of this example is to show the importance of transmission-line matching in controlling reflections for a transmission line used to interconnect two computers. Consider the two computers shown in Fig. 5.5*b* interconnected by a coaxial cable 100 m long and with a velocity of propagation 2×10^8 m/s, so that there is a time delay of 500 ns for a pulse to propagate from input of the cable to its output. Consider a portion of the digitally coded signal, made up of 10-ns pulses with basic spacing 20 ns as sketched in Fig. 5.5*c*. This is sketched versus distance in Fig. 5.5*d* at 5 ns before the

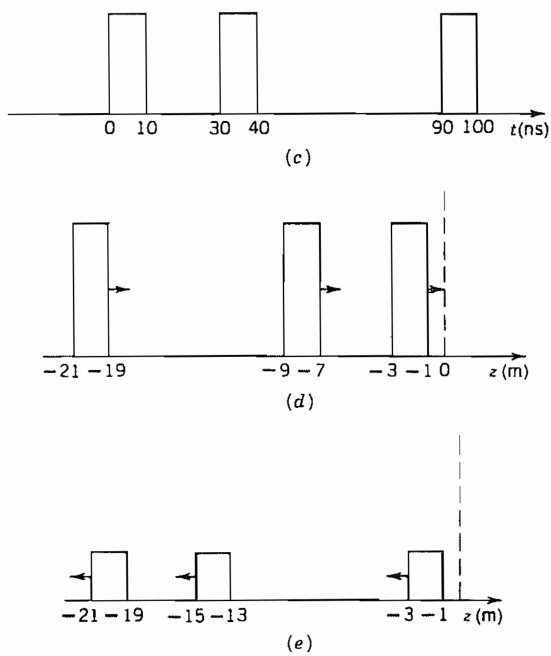


FIG. 5.5 (*c*) A portion of a pulse-coded signal versus time for the computer interconnection of Fig. 5.5*b*. (*d*) Voltage versus distance for a time 5 ns before leading edge of first pulse reaches input of computer No. 2 (defined as $z = 0$). (*e*) Voltage versus distance for reflected signal 110 ns after instant of sketch (*d*).

first pulse edge reaches computer No. 2. If input impedance of the second computer matches the characteristic impedance of the line, there is no reflection and the entire signal is accepted. But suppose its input impedance is 100 Ω and the characteristic impedance of the cable is 50 Ω. By (4), the reflection coefficient is

$$\rho = \frac{R_L - Z_0}{R_L + Z_0} = \frac{100 - 50}{100 + 50} = \frac{1}{3}$$

So the train of pulses is reflected, at $\frac{1}{3}$ amplitude, toward computer No. 1 as sketched versus distance in Fig. 5.5*e* at 110 ns after time of Fig. 5.5*d*. If there is impedance mismatch at the terminals of computer No. 1 additional reflection will take place when the signal returns there and a spurious signal will be superposed on whatever desired code is being sent between the computers at that time. Since the reflected "echo" is of lower amplitude than the original signal, differentiation is possible on the basis of amplitude level, but there is an obvious advantage in matching impedances well enough that reflected signals are small.

Example 5.5c
SQUARE WAVE IN z APPLIED TO INFINITE LINE

As a third example, consider an infinite line suddenly charged at $t = 0$ with a square wave in distance from $z = -1$ m to $z = +1$ m. Voltage distribution at $t = 0$ is then as in Fig. 5.5*f*. This may be considered a superposition of two such square waves, each of amplitude $V_0/2$. One of these moves to the right and the other to the left, each with velocity v. Thus at $t = 1.667$ ns (taking $v = 3 \times 10^8$ m/s) the two partial waves and

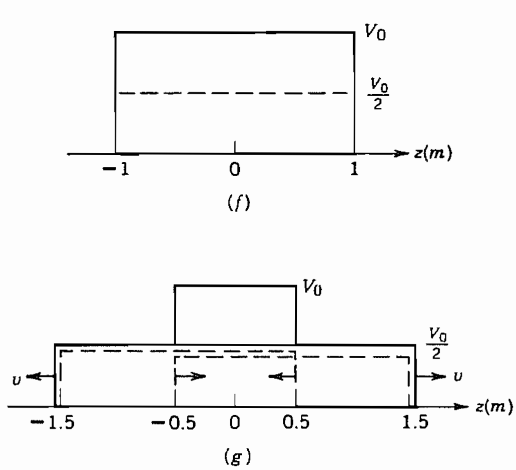


FIG. 5.5 Voltage and current distributions for Ex. 5.5c. (*f*) Voltage distribution at $t = 0$. (*g*) Traveling waves (dashed) and total voltage versus z (solid) at $t = 1.667$ ns.

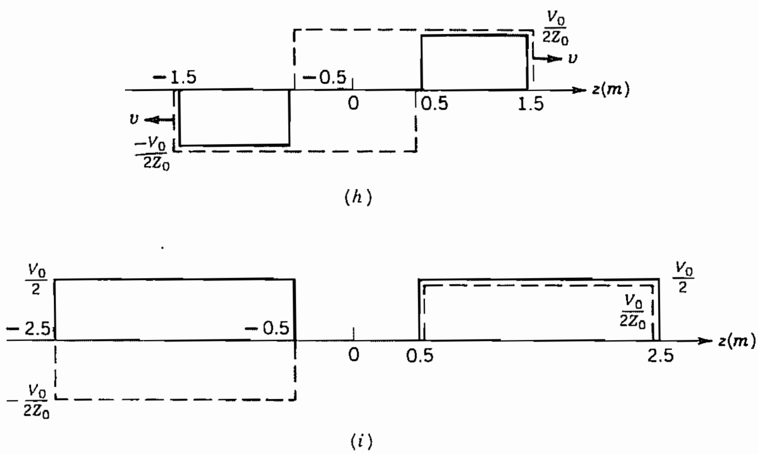


FIG. 5.5 (*h*) Traveling waves of current (dashed) and total current (solid) versus z at t = 1.667 ns. (*i*) Voltage (solid) and current (dashed) versus z at $t = 5$ ns.

their sum are shown in Fig. 5.5*g*. Figure 5.5*h* shows the corresponding current distribution, taking into account the different sign relations between current and voltage for positively and negatively traveling waves. Figure 5.5*i* shows voltage and current distributions at $t = 5$ ns.

Example 5.5d
PULSE REFLECTIONS WHEN PULSE IS LONGER THAN TRAVEL TIME DOWN THE LINE

Example 5.5b considered reflections for pulses much shorter than the travel time down the transmission line. For many interconnections, especially those within integrated circuits, the delay time is shorter than pulse width. Reflections because of mismatch may still cause a problem, causing structure within the pulse. To illustrate the point, consider the first part of the pulse as a step function of voltage V_0 applied to an ideal transmission line at $z = -l$, with terminating impedances at end $z = 0$ so high that it may be considered an open circuit. The front of the wave travels along the line in the positive z direction and reaches the end at $t = t_1$. Since current must be zero at the open-circuited end $z = 0$, a reflected step wave must be generated with current opposite to that of the positively traveling wave, starting at the instant of arrival of the latter at $z = 0$. From Eq. 5.2(13), we know that current in the negatively traveling wave is $-V/Z_0$, so it follows that $V_- = V_+$.

As time goes on, the conditions to be met are that the total voltage at the input of the line ($z = -l$) must be the value V_0 of the pulse applied there and the current at the open end must be zero. These conditions can be satisfied by the sum of two square waves, one traveling in the positive z direction and one in the negative z direction. Figure 5.5*j* shows the voltages of the two individual waves and the superposed, or total,

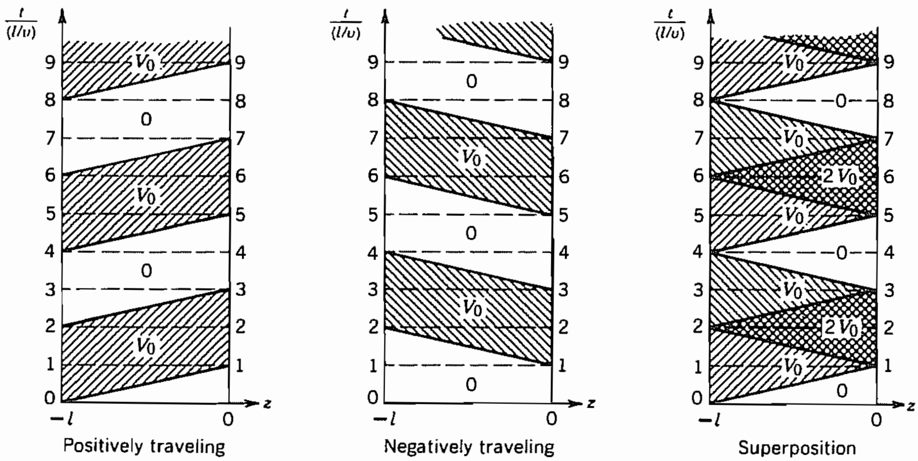


FIG. 5.5j Positively and negatively traveling wave components and their superposition to match boundary conditions on a transmission line with a pulse of voltage V_0 applied to the line at $z = -l$ and the line open-circuited at $z = 0$. The applied pulse length is much longer than the travel time along the line.

voltage. Note that the total voltage is always equal to V_0 at $z = -l$, as required by the boundary condition. As discussed in the preceding paragraph, the condition where the negatively traveling wave has the same polarity of voltage as the positively traveling wave gives a zero total current. Thus, for either the situation with both waves having zero value or both with V_0, the boundary condition at the open end is satisfied. It is seen in Fig. 5.5*j* that the sum of the two waves meets that requirement. Since the two waves satisfy the transmission-line equations and their sum satisfies the boundary conditions, they constitute the unique solution.

Note that a square wave of voltage is produced at $z = 0$ with voltage zero for intervals of $2L/v$, interspersed with intervals of the same value with $V = 2V_0$.

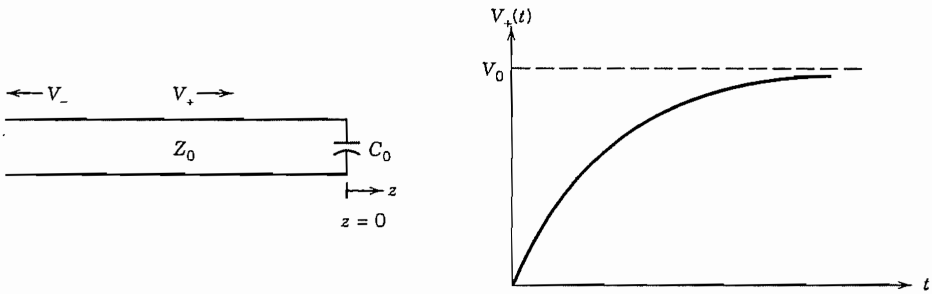
Example 5.5e
TRANSMISSION LINE WITH CAPACITIVE TERMINATION

As a somewhat different example, consider the ideal transmission line of Fig. 5.5*k* terminated in an uncharged capacitor. We take the incident wave as the exponential buildup of Fig. 5.5*l*,

$$V_+(t) = V_0[1 - e^{-t/\tau_0}] \qquad (1)$$

where, for convenience, time zero is taken as the instant the forward wave arrives at the capacitor. By continuity of voltage and current,

$$V_+(t) + V_-(t) = V_c(t) \qquad (2)$$



(k) (l)

FIG. 5.5 (*k*) Ideal transmission line terminated with a capacitor; (*l*) form of forward-traveling voltage wave incident on C_0.

$$\frac{V_+(t)}{Z_0} - \frac{V_-(t)}{Z_0} = I_c(t) = C_0 \frac{dV_c(t)}{dt} \tag{3}$$

By combining these equations and using (1) we obtain

$$\frac{dV_-(t)}{dt} + \frac{V_-(t)}{\tau_1} = \frac{V_0}{\tau_1}(1 - e^{-t/\tau_0}) - \frac{V_0}{\tau_0} e^{-t/\tau_0} \tag{4}$$

where $\tau_1 = C_0 Z_0$ is a time constant set by the capacitor and transmission-line impedance. A solution of this first-order differential equation is

$$V_-(t) = V_0\left[1 + A_1 e^{-t/\tau_1} - \left(\frac{\tau_0 + \tau_1}{\tau_0 - \tau_1}\right) e^{-t/\tau_0}\right] \tag{5}$$

The arriving wave has zero voltage at the instant of arrival and the capacitor is uncharted, so $V_-(0) = 0$ and

$$A_1 = \frac{2\tau_1}{\tau_0 - \tau_1} \tag{6}$$

Using (2), the voltage is found to build up in the capacitor as

$$V_c(t) = V_0\left[2 + \frac{2\tau_1}{\tau_0 - \tau_1} e^{-t/\tau_1} - \frac{2\tau_0}{\tau_0 - \tau_1} e^{-t/\tau_0}\right] \tag{7}$$

It can be checked that (3) is satisfied. The z variation of the reflected wave is obtained by substituting $t + z/v$ for t in (5).

5.6 PULSE FORMING LINE

One way of forming pulses of a desired length is by charging a transmission line of length l to a dc voltage V_0 and then connecting to a resistor as shown in Fig. 5.6*a*. (In

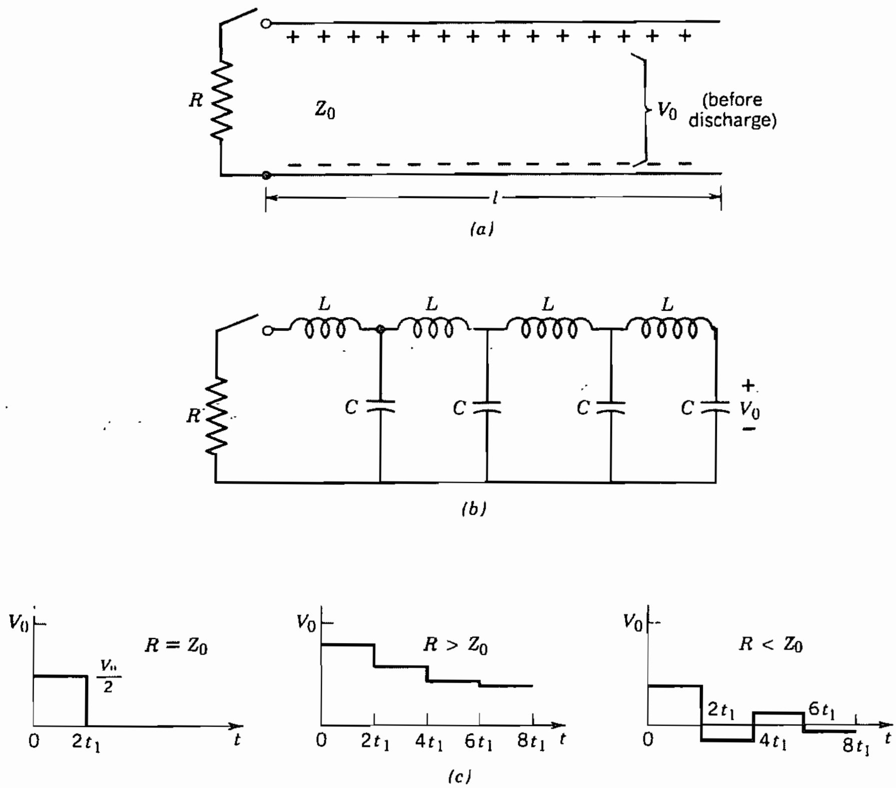


FIG. 5.6 (*a*) Transmission line of length l (time delay one way $= t_1$) charged to potential V_0 and connected to resistance R at $t = 0$. (*b*) Lumped element approximation to (*a*). (*c*) Wave shapes of resistor voltage for different relations of R to Z_0.

practice, the line is often approximated by lumped elements, so the lumped-circuit approximation is shown in Fig. 5.6*b*.) If the resistance R is matched to the characteristic impedance, a pulse of height $V_0/2$ is formed across R for a time $2t_1$, where t_1 is one-way propagation time down the line and the line completely discharged. It may then be recharged and the process repeated. It may at first seem puzzling that voltage across the resistor is not just V_0 when the switch is closed, but this is because a traveling wave to the right is excited by the connection. Voltage across the resistor just after closing the switch is then the sum of dc voltage and the voltage of the positive wave, V_+:

$$V_R = V_0 + V_+ \tag{1}$$

The current flowing in the resistor is just the negative of current in the positively traveling wave.

$$I_R = -I_+ = -\frac{V_+}{Z_0} \tag{2}$$

Since $V_R = RI_R$, combination of (1) and (2) gives

$$V_R = \left(\frac{R}{R + Z_0}\right)V_0 = -\frac{R}{Z_0}V_+ \tag{3}$$

Thus if $R = Z_0$, $V_R = V_0/2$ and $V_+ = -V_0/2$. Current in the wave started to the right; $I_+ = -V_0/2Z_0$ by (2). When this current reaches the open end, $z = l$, there must then be a reflected wave with current $I_- = V_0/2Z_0$ so that the net current is zero at the open end as required. This requires a voltage in the reflected wave $V_- = -Z_0 I_- = -V_0/2$, which travels to the left, canceling the remaining voltage on the line and bringing zero voltage and current to the resistance at $t = 2t_1$. From then on all is still. Thus in the case of $R = Z_0$, the wave to the right discharges half the voltage initially on the line, and the wave to the left the other half, yielding a rectangular pulse as shown in Fig. 5.6*c*.

If $R \neq Z_0$, the wave started to the right upon closing of the switch is other than $V_0/2$, so cancellation of the voltage on the charged line in one round trip does not occur, and there are further reflections when the wave returns to the input. Figure 5.6*c* sketches the form of resistor current for $R > Z_0$ and for $R < Z_0$. (See also Prob. 5.6b.)

Sinusoidal Waves on Ideal Transmission Lines

5.7 Reflection and Transmission Coefficients and Impedance and Admittance Transformations for Sinusoidal Voltages

The preceding discussion has involved little restriction on the type of variation with time of the voltages applied to the transmission lines. Many practical problems are concerned with sinusoidal time variations. If a sinusoidal voltage is supplied to a line, it can be represented at $z = 0$ by

$$V(0, t) = V \cos \omega t \tag{1}$$

The corresponding wave traveling in the positive z direction is

$$V_+(z, t) = |V_+| \cos \omega\left(t - \frac{z}{v_p}\right)$$

and that traveling in the negative z direction is

$$V_-(z, t) = |V_-| \cos\left[\omega\left(t + \frac{z}{v_p}\right) + \theta_p\right]$$

The total voltage is the sum of the two traveling waves:

$$V(z, t) = |V_+| \cos \omega\left(t - \frac{z}{v_p}\right) + |V_-| \cos\left[\omega\left(t + \frac{z}{v_p}\right) + \theta_p\right] \tag{2}$$

The corresponding current, from Eq. 5.2(13), is

$$I(z, t) = \frac{|V_+|}{Z_0} \cos \omega\left(t - \frac{z}{v_p}\right) - \frac{|V_-|}{Z_0} \cos\left[\omega\left(t + \frac{z}{v_p}\right) + \theta_\rho\right] \tag{3}$$

In Sec. 5.2 we saw that a constant point on a wave described by $F(t - z/v)$ is seen by an observer moving with a velocity v in the positive z direction. The argument of a sinusoid is called its *phase*, so the velocity for which phase is constant is called the *phase velocity* v_p.

For sinusoidal time variations, it is useful to rewrite (2) and (3) in phasor form:

$$V = V_+ e^{-j\beta z} + V_- e^{j\beta z} \tag{4}$$

$$I = \frac{1}{Z_0} [V_+ e^{-j\beta z} - V_- e^{j\beta z}] \tag{5}$$

where

$$\beta = \frac{\omega}{v_p} = \omega\sqrt{LC} \tag{6}$$

We may take V_+ as the reference for zero phase so that it is real. Then V_- is in general complex and equal to $|V_-|e^{j\theta_\rho}$, with θ_ρ being the phase angle between reflected and incident waves at $z = 0$ as in the instantaneous form (2).

The quantity β is called the *phase constant* of the line since βz measures the instantaneous phase at a point z with respect to $z = 0$. Moreover, voltage (or current) is observed to be the same at any two points along the line that are separated in z such that βz differs by multiples of 2π. The shortest distance between points of like current or voltage is called a *wavelength* λ. By the foregoing reasoning,

$$\beta\lambda = 2\pi$$

or

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC} \tag{7}$$

The expressions for reflection and transmission coefficients in Eqs. 5.4(4) and 5.4(5) can now be written in a special form for sinusoidal waves. It is convenient to choose the origin of the z coordinate at the discontinuity to be analyzed, as shown in Fig. 5.7 for three representative discontinuities. It is assumed that a previous analysis gave the equivalent value of impedance looking in the $+z$ direction at $z = 0$ in the two lower lines. We will see below how this is done. The ratio of total phasor voltage to total phasor current at any point is the definition of impedance. We set the impedance at $z = 0$, the load impedance Z_L, equal to the ratio (4) to (5). Solving for the ratio V_-/V_+, the reflection coefficient for sinusoidal waves is obtained:

$$\rho = \frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{8}$$

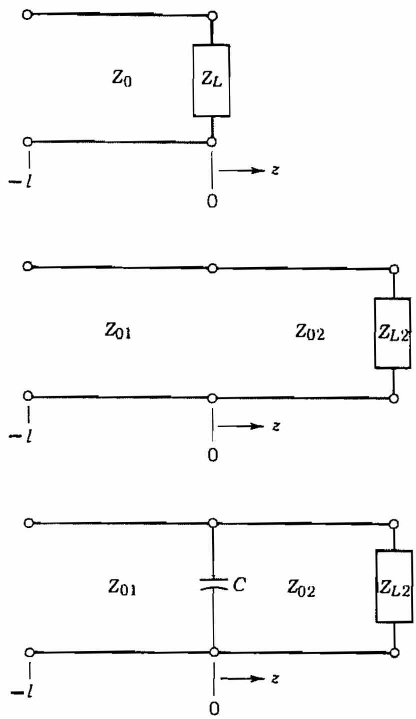


FIG. 5.7 Representative situations where a line of length l with a discontinuity at $z = 0$ is the subject of the analysis.

Also, (4) and (5) can be combined to give the transmission coefficient:

$$\tau = \frac{V_L}{V_+} = \frac{2Z_L}{Z_L + Z_0} \tag{9}$$

The load voltage V_L is the total voltage at $z = 0$.

The expressions for power reflected and transmitted at a discontinuity, given for real functions of time in Eqs. 5.4(6) and 5.4(7), can be adapted to sinusoidal signals of the complex exponential type. Since power in this case is $VI^*/2$, and for a single wave in a loss-free line this is $VV^*/2Z_0 = |V|^2/2Z_0$, the fractional reflected power is

$$\frac{W_T^-}{W_T^+} = \frac{|V_-|^2}{|V_+|^2} = |\rho|^2 \tag{10}$$

and the remainder goes into the load:

$$\frac{W_{TL}}{W_T^+} = 1 - |\rho|^2 \tag{11}$$

Now let us find expressions for the input impedance and admittance at $-l$. We find impedance by dividing (4) by (5) for $z = -l$:

$$Z_i = Z_0\left[\frac{e^{j\beta l} + \rho e^{-j\beta l}}{e^{j\beta l} - \rho e^{-j\beta l}}\right] \tag{12}$$

Or, substituting (8),

$$Z_i = Z_0\left[\frac{Z_L \cos \beta l + jZ_0 \sin \beta l}{Z_0 \cos \beta l + jZ_L \sin \beta l}\right] \tag{13}$$

By defining admittances $Y_i = 1/Z_i$, $Y_L = 1/Z_L$, and $Y_0 = 1/Z_0$, we can find an expression of the same form for Y_i:

$$Y_i = Y_0\left[\frac{Y_L \cos \beta l + jY_0 \sin \beta l}{Y_0 \cos \beta l + jY_L \sin \beta l}\right] \tag{14}$$

Example 5.7
CASCADED THIN-FILM LINES

Suppose the second diagram of Fig. 5.7 represents two thin-film transmission lines in a microwave integrated circuit. The load Z_{L2} represents a device having a real impedance of 20 Ω at the signal frequency, 18 GHz. Line 2, of characteristic impedance $Z_{02} = 30\ \Omega$, has a length $l_2 = 2$ mm. Line 1, of characteristic impedance $Z_{01} = 20\ \Omega$, has a length $l_1 = 1.5$ mm. The phase velocities for both lines are the same, 2×10^8 m/s. Let us find the impedance at the input to line 1.

First we must solve for the way the load impedance Z_{L2} transforms along the line attached to it. To do this we apply formula (13) to that section of line. For both lines (6) gives

$$\beta = \frac{\omega}{v_p} = \frac{(2\pi)(18 \times 10^9) \text{ rad/s}}{2 \times 10^8 \text{ m/s}} = 566 \text{ rad/m}$$

For variety we will take angles in degrees here, as both degrees and radians are commonly used in transmission-line calculations. Thus $\beta\ell_2 = 64.9$ degrees and $\beta\ell_1 = 48.6$ degrees:

$$Z_{i2} = 30\left[\frac{20 \cos 64.9° + j30 \sin 64.9°}{30 \cos 64.9° + j20 \sin 64.9°}\right]$$
$$= 36.7 + j11.8\ \Omega$$

Now we can find Z_{i1} at the input to line 1 using Z_{i2} as the load Z_{L1}:

$$Z_{i1} = 20\left[\frac{(36.7 + j11.8) \cos 48.6° + j20 \sin 48.6°}{20 \cos 48.6° + j(36.7 + j11.8) \sin 48.6°}\right]$$
$$= 18.9 - j14.6\ \Omega$$

Systems with several lines of different characteristic impedances in cascade can be analyzed as we have in this example. In each case the analysis starts with the point farthest from the signal source, transforming the impedance back successively to the next discontinuity until the input is reached. In general, Z_0, β, and l will be different for each section.

5.8 STANDING WAVE RATIO

Let us examine the phases of the voltages in Eq. 5.7(4). One coefficient, say V_+, can be chosen to be real by choice of origin of time. The reflection coefficient, Eq. 5.7(8), which is a complex number, can be written in the form $|\rho|e^{j\theta_\rho}$ so V_- in Eq. 5.7(4) can be replaced with $V_+|\rho|e^{j\theta_\rho}$, giving

$$V = V_+e^{-j\beta z} + V_+|\rho|e^{j(\theta_\rho + \beta z)} \tag{1}$$

Let us write this as a real function of time with $-z$ replaced by l, the distance from the end of the line:

$$V(t, -l) = V_+ \cos(\omega t + \beta l) + V_+|\rho| \cos(\omega t - \beta l + \theta_\rho)$$

Considering any particular instant, say $t = 0$, we can readily see that the argument of the cosine in the incident wave (first term), which is its phase, increases with distance from $z = 0$ and that the phase of the reflected wave (second term) decreases. These phases are shown in the top drawings of Fig. 5.8 where it is clear that at some distance z_0, the phases are the same. At z_π they differ by π rad, one having decreased by $\pi/2$ and the other having increased by $\pi/2$. At $z_{2\pi}$ they differ by 2π rad, and so on. So there are a series of locations where the two sinusoids are in phase and another series of locations where they are π rad out of phase. Where they are in phase, they add directly at each instant, and where π rad out of phase, they subtract. At the former locations the total voltage has its maximum amplitude, and at the latter, its minimum. Analysis of the sum of the incident and reflected waves given in (1) (see Prob. 5.8f) shows that the total voltage can also be represented as the sum of a standing wave and a traveling wave. The total voltage is shown in the lower drawing of Fig. 5.8 for several particular times in a cycle selected to show the voltage when it has its maximum and minimum peak values. The broken line shows the voltage *amplitude* along the transmission line.

The maximum voltage is

$$V_{\max} = |V_+| + |V_-| \tag{2}$$

and the minimum, found a quarter-wavelength from the maximum, is

$$V_{\min} = |V_+| - |V_-| \tag{3}$$

The standing wave ratio is then defined as the ratio of the maximum voltage amplitude

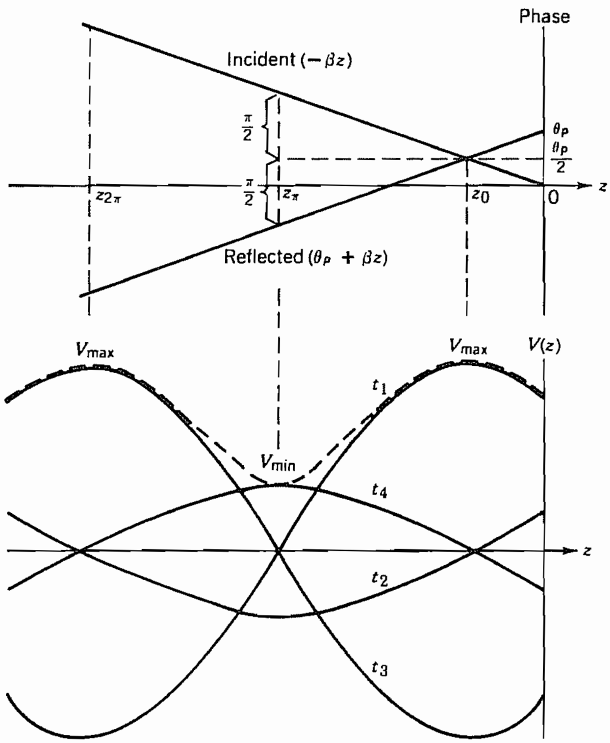


FIG. 5.8 The upper graph shows the phases of the incident and reflected waves at $t = 0$ on a line with a reflection coefficient $\rho = |\rho|e^{j\theta_\rho}$. The total voltage $V(z)$ is shown for selected times in the lower graph, $\omega t_1 = -\theta_\rho/2$, $\omega t_2 = -\theta_\rho/2 + \pi/2$, $\omega t_3 = -\theta_\rho/2 + \pi$, $\omega t_4 = -\theta_\rho/2 + 3\pi/2$. The broken line gives the voltage amplitude along the line.

to the minimum voltage amplitude:

$$S = \frac{V_{\text{max}}}{V_{\text{min}}} \tag{4}$$

By substituting (2) and (3) and the definition of reflection coefficient, Eq. 5.7(8), we find

$$S = \frac{|V_+| + |V_-|}{|V_+| - |V_-|} = \frac{1 + |\rho|}{1 - |\rho|} \tag{5}$$

It is seen that standing wave ratio is directly related to the magnitude of reflection coefficient ρ, giving the same information as this quantity. The inverse relation is

$$|\rho| = \frac{S - 1}{S + 1} \tag{6}$$

Figure 5.8 is plotted for $S = 3$, corresponding to $|\rho| = \frac{1}{2}$.

Because of the negative sign appearing in the current equation, Eq. 5.7(5), it is evident that at the position where the two traveling wave terms add in the voltage relations, they subtract in the current relation, and vice versa. The maximum voltage position is then a minimum current position. The value of the minimum current is

$$I_{\min} = \frac{|V_+| - |V_-|}{Z_0} \tag{7}$$

At this position impedance is purely resistive and has the maximum value it will have at any point along the line:

$$Z_{\max} = Z_0\left[\frac{|V_+| + |V_-|}{|V_+| - |V_-|}\right] = Z_0 S \tag{8}$$

At the position of the voltage minimum, current is a maximum, and impedance is a minimum and real:

$$I_{\max} = \frac{|V_+| + |V_-|}{Z_0} \tag{9}$$

$$Z_{\min} = Z_0\left[\frac{|V_+| - |V_-|}{|V_+| + |V_-|}\right] = \frac{Z_0}{S} \tag{10}$$

Example 5.8
SLOTTED-LINE IMPEDANCE MEASUREMENT

A *slotted line* is an instrument that can be used to measure impedances. It is a transmission line containing a movable probe with which the standing wave ratio and the position of a voltage minimum or maximum can be found. The unknown impedance is connected to the end of the slotted line.

Suppose a measurement on a slotted line of characteristic impedance $Z_0 = 50\ \Omega$ reveals a standing wave ratio $S = 3$ and the closest voltage minimum is 0.33λ from the unknown load impedance. Let us see how Z_L is deduced. In Fig. 5.8, we see that at z_π where the minimum is found, the phase of the incident wave, $-\beta z$ attains the value $(\theta_\rho + \pi)/2$, so

$$l_{\min} = -z_{\min} = \frac{\theta_\rho + \pi}{2\beta}$$

and

$$\theta_\rho = 2\beta(0.33\lambda) - \pi = 2\left(\frac{2\pi}{\lambda}\right)(0.33\lambda) - \pi = 1.0 \text{ rad}$$

where we have used Eq. 5.7(7). Using the given S and (6) we find $|\rho|$ to be 0.5. Then

$$\rho = 0.5e^{j1.0}$$

From Eq. 5.7(8) we can get Z_L in term of ρ and Z_0. Thus

$$Z_L = Z_0\left[\frac{1 + \rho}{1 - \rho}\right] = 50\left[\frac{1 + 0.5e^{j1.0}}{1 - 0.5e^{j1.0}}\right] = 52.8 + j59.3\ \Omega$$

5.9 THE SMITH TRANSMISSION-LINE CHART

Many graphical aids for transmission-line computations have been devised. Of these, the most generally useful has been one presented by Smith,[4] which consists of loci of constant resistance and reactance plotted on a polar diagram in which radius corresponds to magnitude of reflection coefficient. The chart enables one to find simply how impedances are transformed along the line or to relate impedance to reflection coefficient or to standing wave ratio and position of a voltage minimum. By combinations of operations, it enables one to understand the behavior of complex impedance-matching techniques and to devise new ones. It is much used in displaying the locus of impedance of many useful devices as frequency is varied. Although computer programs are available for transmission-line calculations, its role in displaying and understanding matching mechanisms remains useful. This chart utilizes the reflection coefficient plane. Impedance for any point along a transmission line with a passive load then lies within the unit circle. Loci of constant resistance are circles and loci of constant reactance are circles orthogonal to those of constant resistance. We will first show the basis for this, and in the following section give some examples of the chart's use.

The discussion of the chart will begin with Eq. 5.7(12), which gives impedance in terms of reflection coefficient. We define a normalized impedance

$$\zeta(l) = (r + jx) \triangleq \frac{Z_i}{Z_0} \tag{1}$$

and a complex variable w equal to the reflection coefficient at the end of the line, shifted in phase to correspond to the input position l:

$$w = u + jv \triangleq \rho e^{-2j\beta l} \tag{2}$$

Equation 5.7(12) may then be written

$$\zeta(l) = \frac{1 + w}{1 - w} \tag{3}$$

or

$$r + jx = \frac{1 + (u + jv)}{1 - (u + jv)} \tag{4}$$

4 P. H. Smith, Electronics **12**, 29–31 (1939); **17**, 130 (1944).

This equation may be separated into real and imaginary parts as follows:

$$r = \frac{1 - (u^2 + v^2)}{(1 - u)^2 + v^2} \tag{5}$$

$$x = \frac{2v}{(1 - u)^2 + v^2} \tag{6}$$

or

$$\left(u - \frac{r}{1 + r}\right)^2 + v^2 = \frac{1}{(1 + r)^2} \tag{7}$$

$$(u - 1)^2 + \left(v - \frac{1}{x}\right)^2 = \frac{1}{x^2} \tag{8}$$

If we then wish to plot the loci of constant resistance r on the w plane (u and v serving as rectangular coordinates), (7) shows that they are circles with centers on the u axis at $[r/(1+r), 0]$ and with radii $1/(1 + r)$. The curves for $r = 0, \frac{1}{2}, 1, 2, \infty$ are sketched in Fig. 5.9*a*. From (8), the curves of constant x plotted on the w plane are also circles, with centers at $(1, 1/x)$ and with radii $1/|x|$. Circles for $x = 0, \pm\frac{1}{2}, \pm 1, \pm 2, \infty$ are sketched in Fig. 5.9*a*. Any point on a given transmission line will have some impedance with positive resistance part, and so will correspond to a particular point on the inside of the unit circle of the w plane. Several uses of the chart will follow. Many

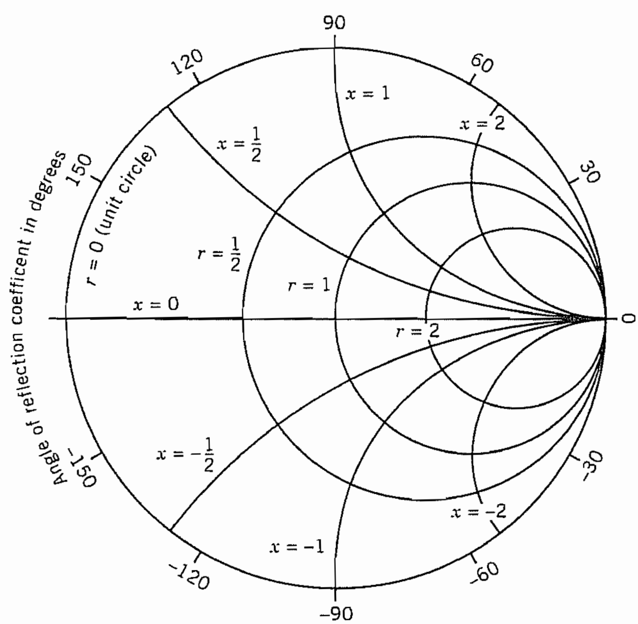


FIG. 5.9a Basic features of the Smith Chart.

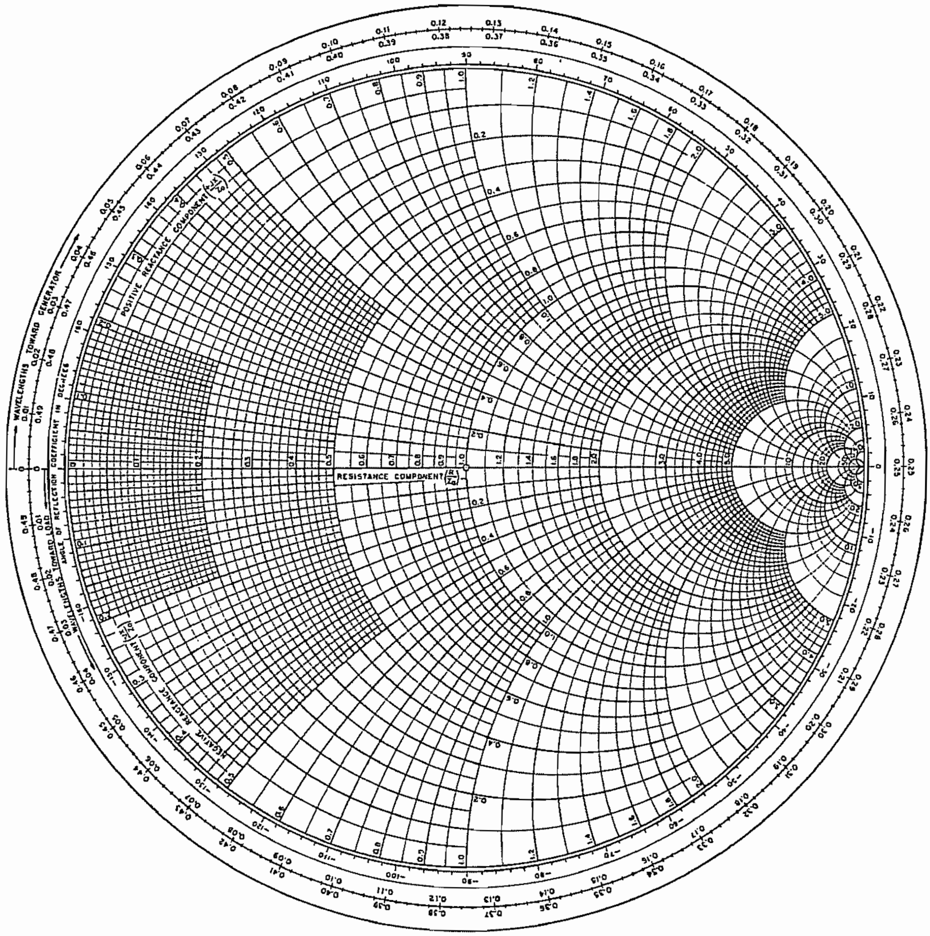


FIG. 5.9*b* Smith Chart.

extensions and combinations of the ones to be cited will be obvious to the reader. A chart with more divisions is given in Fig. 5.9*b*.

5.10 Some Uses of the Smith Chart

In this section we show the use of the Smith chart in displaying the relation between impedance and reflection coefficient, in transferring impedances or admittances along the line, and in impedance matching. Other uses are illustrated by the problems and still other extensions or combinations will be evident to the reader.

To Find Reflection Coefficient Given Load Impedance, and Conversely The point within the unit circle of the Smith chart corresponding to a particular position on a transmission line may of course be located at once if the normalized impedance corresponding to that position is known. This is done with a reasonable degree of accuracy by utilizing the orthogonal families of circles giving resistance and reactance as described above. Thus, point A of Fig. 5.10a is the intersection of the circles $r = 1$ and $x = 1$ and corresponds to a position with normalized impedance $1 + j1$. It is clear from Eq. 5.9(2) that $|w| = |\rho|$ and from Eq. 5.9(7) that $|w| = (u^2 + v^2)^{1/2} = 1$ on the $r = 0$ circle, which is the outer edge of the graph. A measure of the radius to some point on the graph as a fraction of the radius to the $r = 0$ circle thus gives $|\rho|$ directly. If the point on the Smith chart is the normalized load impedance, $l = 0$, and $\angle w = \angle\rho$, so the phase angle of the reflection coefficient can be read directly. One can, of course, reverse the process to find Z_L if ρ is given.

Example 5.10a

REFLECTION COEFFICIENT FROM LOAD IMPEDANCE

Suppose a transmission line of characteristic impedance $Z_0 = 70\ \Omega$ is terminated with a load $Z_L = 70 + j70\ \Omega$. The normalized load impedance is $\zeta(0) = 1 + j1$, shown as point A in Fig. 5.10a. The magnitude of ρ is 0.45 and $\angle\rho = \angle w = 1.11$ rad so $\rho = 0.45e^{j1.11}$. The angle may be found by reading the outside wavelength scales, recognizing that a quarter-wave is π radians on the chart.

To Transform Impedance Along the Line As position l along a loss-free line, measured relative to the load, is changed, only the phase angle of w changes, as can be seen from Eq. 5.9(2) wherein ρ is a complex number, the reflection coefficient at the load. Thus, change of position along an ideal line is represented on the chart by movement along circles centered at the origin of the w plane. The angle through which w changes is proportional to the length of the line and, by Eq. 5.9(2), is just twice the electrical length of line βl. (Most charts have a scale around the outside calibrated in fractions of a wavelength, so that the angle need not be computed explicitly. See Fig. 5.10a.) Finally, the direction in which one moves is also defined by Eq. 5.9(2). If one moves toward the generator (increasing l), the angle of w becomes increasingly negative, which corresponds to clockwise motion about the chart. Motion toward the load corresponds to decreasing l and thus corresponds to counterclockwise motion about the chart.

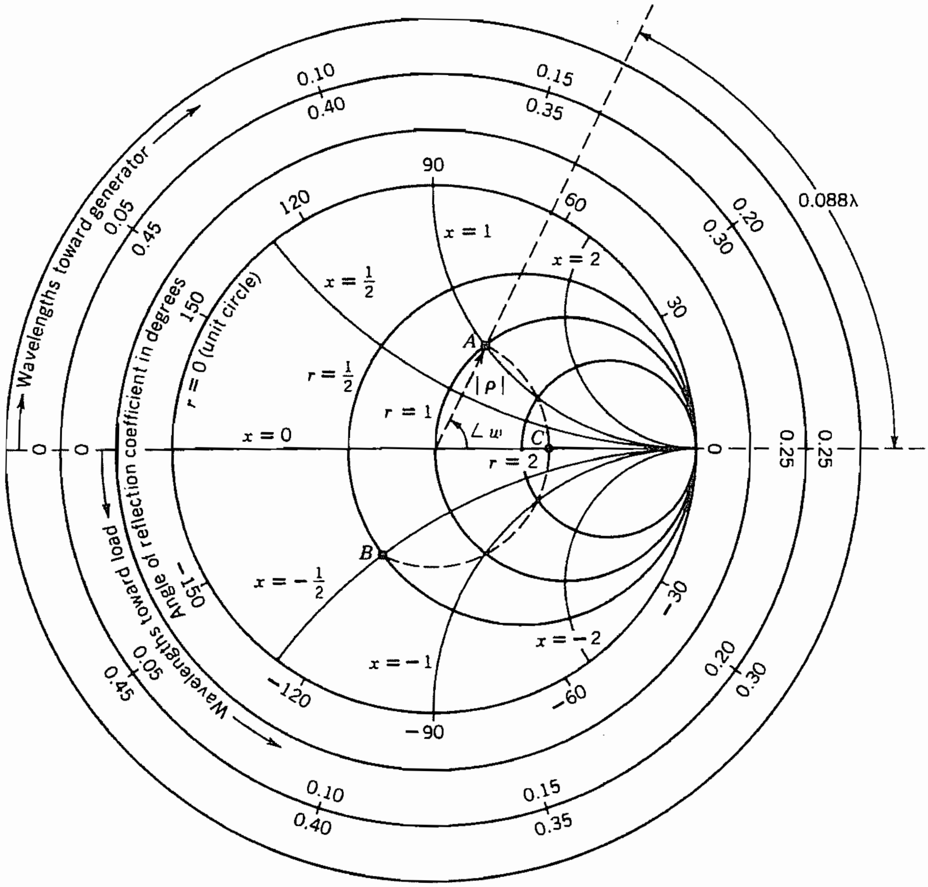


FIG. 5.10*a* Smith chart for impedances. Points A, B, and C and associated broken lines relate to Exs. 5.10a, 5.10b, and 5.10c.

Example 5.10b
IMPEDANCE TRANSFORMATION

Consider the line and load of Ex. 5.10a for which the normalized load impedance is $1 + j1$ and is shown at point A in Fig. 5.10a. If the line is a quarter-wave long (90 electrical degrees), we move through an angle of 180 degrees at constant radius on the chart toward the generator (clockwise) to point B. The normalized input impedance is then read as $0.5 - j0.5$ for point B. If input impedance is given and load impedance desired, the reverse of this procedure can obviously be used.

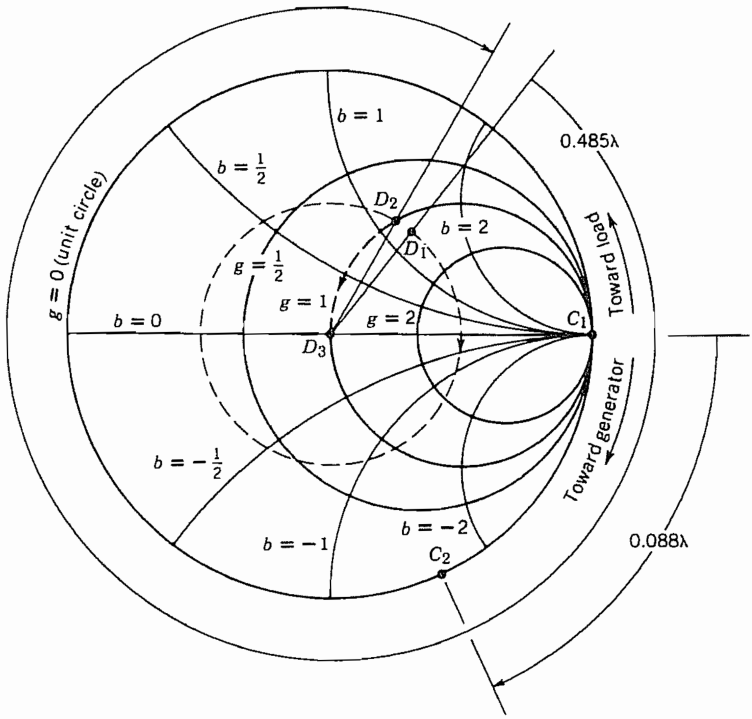


FIG.5.10*b* Polar transmission-line chart for admittances. The constructions involving points C_1, C_2, and D_1–D_3 relate to Ex. 5.10d.

To Find Standing Wave Ratio and Position of Voltage Maximum from a Given Impedance, and Conversely If we wish the standing wave ratio of an ideal transmission line terminated in a known load impedance, we make use of the information found in the preceding section. Equation 5.8(8) shows that the location of maximum impedance is also the location of maximum voltage and the impedance there is real. We see that

$$S = \frac{Z_{\max}}{Z_0} = \zeta_{\max} \tag{1}$$

and can see from Fig. 5.10*a* that the point where impedance is real and maximum along any ideal transmission line (represented by a $|\rho| =$ constant circle) lies on the right side along the horizontal axis (u axis). Thus, in following about the circle on the chart determined by the given load impedance, we note its crossing of the right-hand u axis of the w plane. The value of the normalized resistance of this point is then the standing wave ratio; the angle moved through to this position from the load impedance fixes the position of the voltage maximum.

The reversal of this procedure to determine the load impedance, if standing wave ratio and position of a voltage maximum are given, is straightforward, as is the extension to finding position of voltage minimum or finding input impedance in place of load impedance.

Example 5.10c
DETERMINATION OF STANDING WAVE RATIO AND LOCATION OF VOLTAGE MAXIMUM

Let us continue our analysis of the ideal transmission line and load discussed in Exs. 5.10a and 5.10b. The normalized load impedance is $1 + j1$ plotted at point A in Fig. 5.10*a* Moving along the line away from the load (clockwise), one arrives at the pure-resistance point C by going 0.088 wavelength. The value of maximum normalized resistance, which equals the standing wave ratio S, by (1) is read as 2.6.

Use as an Admittance Diagram Since admittance transforms along the ideal line in exactly the same manner as impedance, Eq. 5.7(14), it is evident that exactly the same chart may be used for transformation of admittances with the same procedure as for impedances described in the above. Admittance is read for impedance, conductance for resistance, and susceptance for reactance as seen in Fig. 5.10*b*. There are differences to remember: the right-hand u axis now represents an admittance maximum and, therefore, a current maximum instead of a voltage maximum; the phase of the reflection coefficient read as described above and corresponding to a given normalized load admittance is that for current in the reflected wave compared with current in the incident wave and is therefore different by π from that based on voltages. (See Prob. 5.7d.)

Example 5.10d
ADMITTANCE ANALYSIS OF VARIABLE SHORTED-STUB TUNER

In this example we will use the Smith chart for admittances to analyze a mismatched line that employs a variable shorted-stub tuner to produce a unity standing wave ratio in the line leading up to the stub. Figure 5.10*c* shows the arrangement. Suppose $Z_0 = 50\ \Omega$ in the main line and $Z_{0s} = 70\ \Omega$ in the stub. Assume $Z_L = 20 - j20\ \Omega$. We will find the appropriate stub location $z = -l_m$ and the stub length l_s. The admittance chart rather than the impedance form is used because of the convenience in handling shunt circuits in the admittance formulation.

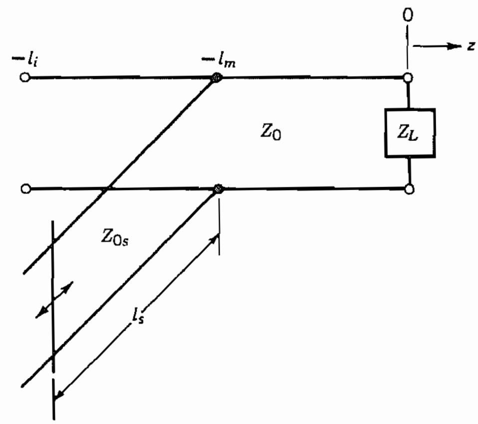


FIG. 5.10c Variable shorted-stub tuner connected in shunt to provide matching at $-l_m$ and therefore $S = 1$ for $z < -l_m$.

The load admittance is $Y_L = 1/Z_L = 0.025 + j0.025$ S and the characteristic admittance is $Y_0 = 1/Z_0 = 0.020$ S. The normalized load admittance is $1.25 + j1.25$; this is at point D_1 in Fig. 5.10*b*.

The input admittance of the shorted stub is seen from Eq. 5.7(14) to be purely imaginary since $Y_i = -jY_0 \cot \beta l$. This suggests that it should be placed at a point along the main line $-z = l_m$, where the admittance has a normalized real part $g = 1$ and an imaginary part b that can be canceled by Y_{is}. Following a path of constant $|\rho| = |w|$ toward the generator, we come to the $g = 1$ curve where $b = 1.13$ (point D_2) corresponding to an unnormalized susceptance of $B = (0.020)(1.13) = 0.0226$ S. That is seen to be 0.485λ from the load. A stub with an input susceptance of -0.0226 S is connected in shunt to cancel out the imaginary part of the admittance. This moves us on the admittance chart from D_2 to D_3 where $g = 1$ and the line is perfectly matched for waves approaching the stub location from the generator.

Now let us find the length l_s, of the stub. Since $Z_{0s} = 70\ \Omega$, $Y_{0s} = 0.014$ S. The normalized input susceptance of the stub must be $b_{is} = -0.0226/0.014 = -1.61$. The shorted end of the stub has an infinite admittance. That is at C_1 on the Smith admittance chart in Fig. 5.10*b* at the right end of the real axis. To transform this admittance to the normalized susceptance -1.61 we move clockwise as shown to point C_2. The length of the stub must be 0.088λ.

The line to the right of the stub appears as a conductance in parallel with a capacitance; addition of the shunt stub provides an inductance which makes the combination appear as a parallel tuned circuit. It should be clear that this matching technique leaves a standing wave in the line between the load and the stub as well as in the stub and provides exact matching only at the one frequency.

To Transform Impedance Along Cascaded Lines It is often useful to find the input impedance of cascaded lines of differing characteristics as in Ex. 5.7. The Smith chart involves impedances normalized to the characteristic impedance, so a study of impedance transformation in a cascade of lines requires renormalization for each line. One starts at the load and transforms, line by line, back toward the generator.

Example 5.10e
IMPEDANCE TRANSFORMATION ALONG CASCADED LINES

For the cascaded ideal lines in Fig. 5.10*d*, let us find the fraction of the power incident in line 1 in a 10-GHz signal that is absorbed in the load. This is given by a knowledge of $|\rho|$ in line 1 using Eq. 5.7(11) and recognizing that the power passing the junction at the end of line 1 must be absorbed in the load since we are assuming ideal lossless lines.

Using the parameters for line 3 given in Fig. 5.10*d* we find the normalized load impedance $\zeta_{L3} = Z_L/Z_{03} = 2$. The wavelength $\lambda_3 = v_3/f = 2$ cm, so $l_3 = 0.1\lambda_3$. The load impedance ζ_{L3} is marked as point E_1 on the Smith chart in Fig. 5.10*e*. We move along the constant $|w|$ circle toward the generator by 0.1λ. The point E_2 is at the normalized input impedance $\zeta_{i3} = 0.98 - j0.70$.

To transform the impedance along line 2, we must first denormalize ζ_{i3} and then normalize it to line 2 to get the load impedance ζ_{L2}; thus, $\zeta_{L2} = Z_{03}\zeta_{i3}/Z_{02} = 0.70 - j0.50$. This point is marked E_3. The wavelength in line 2 is $\lambda_2 = v_2/f = 1.5$ cm, so $l_2 = 0.2\lambda_2$. We move along a constant $|w|$ circle clockwise from E_3 by 0.2λ to the input of line 2 (marked E_4) where $\zeta_{i2} = 0.65 + j0.46$.

To find the reflection coefficient at the load point for line 1, we renormalize ζ_{i2}, so $\zeta_{L1} = Z_{02}\zeta_{i2}/Z_{01} = 0.91 + j0.64$. This is the point E_5. Measuring the distance from E_5 to the origin and dividing by the radius of the $r = 0$ line, we obtain $|\rho_1| = 0.32$. Then

$$\frac{W_{TL}}{W_{T1}^{+}} = 1 - |\rho_1|^2 = 0.90$$

is the fraction of the incident power in line 1 that is dissipated in the load.

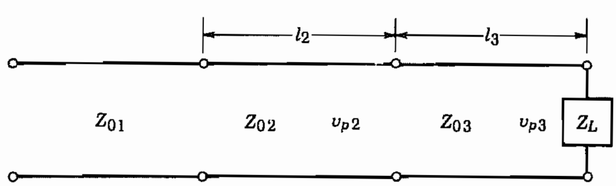


FIG 5.10*d* Cascaded transmission lines with parameters for Ex. 5.10e. $Z_{01} = 50\ \Omega$, $Z_{02} = 70\ \Omega$, $Z_{03} = 50\ \Omega$, $Z_L = 100\ \Omega$, $l_2 = 3$ mm, $l_3 = 2$ mm, $v_{p2} = 1.5 \times 10^8$ m/s, $v_{p3} = 2 \times 10^8$ m/s.

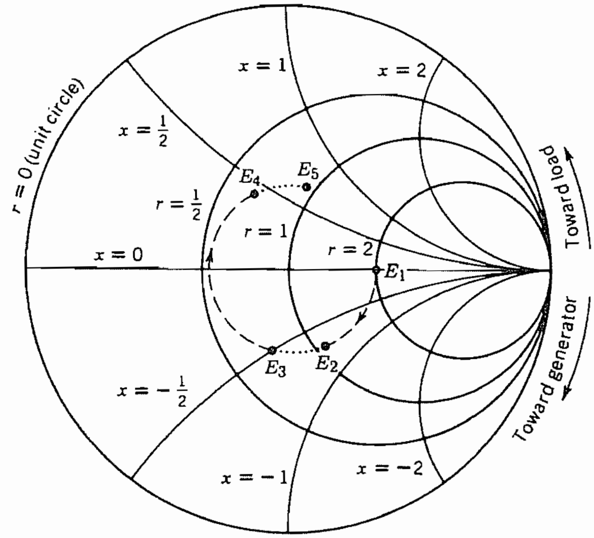


FIG. 5.10e Determination of $|\rho|$ for Ex. 5.10e.

Nonideal Transmission Lines

5.11 Transmission Lines with General Forms of Distributed Impedances: Lossy Lines

For lines with losses or for filter-type transmission circuits, we may generalize the distributed series element in the circuit model to an impedance Z per unit length, and the distributed shunt element to a general admittance Y per unit length, as shown in Fig. 5.11*a*. For steady-state sinusoids, using complex notation, the differential equations for voltage and current variations with distance are then

$$\frac{dV}{dz} = -ZI \tag{1}$$

$$\frac{dI}{dz} = -YV \tag{2}$$

Differentiation of (1) and substitution in (2) then yield

$$\frac{d^2V}{dz^2} = \gamma^2 V \tag{3}$$

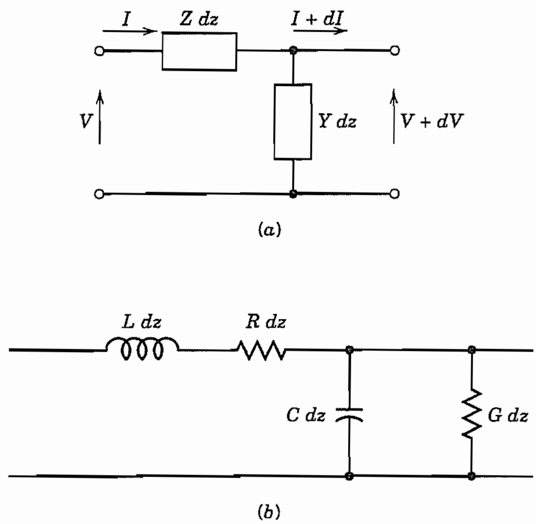


FIG. 5.11 (*a*) Differential length of general transmission line. (*b*) Lossy line with series resistance and shunt conductance.

where

$$\gamma = \sqrt{ZY} \tag{4}$$

The solution to (3) may be written in terms of exponentials, as can be verified by substitution of the expression

$$V = V_+ e^{-\gamma z} + V_- e^{\gamma z} \tag{5}$$

From (1), the corresponding solution for current is

$$I = \frac{1}{Z_0} [V_+ e^{-\gamma z} - V_- e^{\gamma z}] \tag{6}$$

where

$$Z_0 = \frac{Z}{\gamma} = \sqrt{\frac{Z}{Y}} \tag{7}$$

The characteristic impedance Z_0 is in general complex, indicating that the voltage and current for a single traveling wave are not in phase. The quantity γ is called the *propagation constant* and is also generally complex,

$$\gamma = \alpha + j\beta = \sqrt{ZY} \tag{8}$$

so that if (5) is written in terms of α and β,

$$V = V_+ e^{-\alpha z} e^{-j\beta z} + V_- e^{\alpha z} e^{j\beta z} \tag{9}$$

Thus α tells the rate of exponential attenuation of each wave and is correspondingly called the attenuation constant. The constant β tells the amount of phase shift per unit length for each wave and is called the phase constant, as in the loss-free case.

The formula for reflection coefficient derived in Eq. 5.7(8) applies to this case also, remembering that Z_0 is complex. To find the input impedance at $z = -l$ in terms of a given reflection coefficient $\rho = V_-/V_+$ at $z = 0$, division of (5) by (6) yields

$$Z_i = Z_0\left[\frac{V_+e^{\gamma l} + V_-e^{-\gamma l}}{V_+e^{\gamma l} - V_-e^{-\gamma l}}\right] = Z_0\left[\frac{1 + \rho e^{-2\gamma l}}{1 - \rho e^{-2\gamma l}}\right] \tag{10}$$

This may be put in terms of load impedance by substituting Eq. 5.7(8):

$$Z_i = Z_0\left[\frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l}\right] \tag{11}$$

Transmission Line with Series and Shunt Losses A very important case in practice is one in which losses must be considered in the transmission line. In general there may be distributed series resistance in the conductors of the line, and distributed shunt conductance because of leakage through the dielectric of the line. Distributed impedance and admittance are then (Fig. 5.11*b*)

$$Z = R + j\omega L, \qquad Y = G + j\omega C \tag{12}$$

where L includes both external and internal inductance. These may be used as the values of Z and Y in (4) and (7) to determine propagation constant and characteristic impedance. The formula (10) applies to impedance transformations, and the Smith transmission-line chart may be utilized with a modification which recognizes that γ is complex. The procedure is as in Sec. 5.10 except that, in moving along the line toward the generator, one moves not along a circle but along a spiral of radius decreasing according to the exponential $e^{-2\alpha l}$.

For many important problems, losses are finite but relatively small. If $R/\omega L << 1$ and $G/\omega C << 1$, the following approximations are obtained by retaining up to second-order terms in the binomial expansions of (4) and (7), with (12) substituted.

$$\alpha \approx \frac{R}{2\sqrt{L/C}} + \frac{G\sqrt{L/C}}{2} \tag{13}$$

$$\beta \approx \omega\sqrt{LC}\left[1 - \frac{RG}{4\omega^2 LC} + \frac{G^2}{8\omega^2 C^2} + \frac{R^2}{8\omega^2 L^2}\right] \tag{14}$$

$$Z_0 \approx \sqrt{\frac{L}{C}}\left[\left(1 + \frac{R^2}{8\omega^2 L^2} - \frac{3G^2}{8\omega^2 C^2} + \frac{RG}{4\omega^2 LC}\right) + j\left(\frac{G}{2\omega C} - \frac{R}{2\omega L}\right)\right] \tag{15}$$

In using the foregoing approximate formulas, it is often sufficient to retain only first-order correction terms, in which case β reduces to its ideal value of $2\pi/\lambda$, α is computed

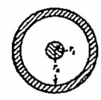
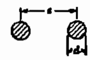
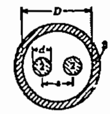
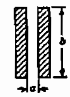
Table 5.11a
General Formulas for Transmission Lines

Quantity	General Line	Ideal Line	Approximate Results for Low-Loss Lines
Propagation constant $\gamma = \alpha + j\beta$	$\sqrt{(R + j\omega L)(G + j\omega C)}$	$j\omega\sqrt{LC}$	(See α and β below)
Phase constant β	$\mathrm{Im}(\gamma)$	$\omega\sqrt{LC} = \dfrac{\omega}{v} = \dfrac{2\pi}{\lambda}$	$\omega\sqrt{LC}\left[1 - \dfrac{RG}{4\omega^2 LC} + \dfrac{G^2}{8\omega^2 C^2} + \dfrac{R^2}{8\omega^2 L^2}\right]$
Attenuation constant α	$\mathrm{Re}(\gamma)$	0	$\dfrac{R}{2Z_0} + \dfrac{GZ_0}{2}$
Characteristic impedance Z_0	$\sqrt{\dfrac{R + j\omega L}{G + j\omega C}}$	$\sqrt{\dfrac{L}{C}}$	$\sqrt{\dfrac{L}{C}}\left[1 + j\left(\dfrac{G}{2\omega C} - \dfrac{R}{2\omega L}\right)\right]$
Input impedance Z_i	$Z_0\left[\dfrac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l}\right]$	$Z_0\left[\dfrac{Z_L \cos \beta l + jZ_0 \sin \beta l}{Z_0 \cos \beta l + jZ_L \sin \beta l}\right]$	
Impedance of shorted line	$Z_0 \tanh \gamma l$	$jZ_0 \tan \beta l$	$Z_0\left[\dfrac{\alpha l \cos \beta l + j \sin \beta l}{\cos \beta l + j\alpha l \sin \beta l}\right]$
Impedance of open line	$Z_0 \coth \gamma l$	$-jZ_0 \cot \beta l$	$Z_0\left[\dfrac{\cos \beta l + j\alpha l \sin \beta l}{\alpha l \cos \beta l + j \sin \beta l}\right]$

Impedance of quarter-wave line	$Z_0\left[\dfrac{Z_L \sinh \alpha l + Z_0 \cosh \alpha l}{Z_0 \sinh \alpha l + Z_L \cosh \alpha l}\right]$	$\dfrac{Z_0^2}{Z_L}$	$Z_0\left[\dfrac{Z_0 + Z_L \alpha l}{Z_L + Z_0 \alpha l}\right]$
Impedance of half-wave line	$Z_0\left[\dfrac{Z_L \cosh \alpha l + Z_0 \sinh \alpha l}{Z_0 \cosh \alpha l + Z_L \sinh \alpha l}\right]$	Z_L	$Z_0\left[\dfrac{Z_L + Z_0 \alpha l}{Z_0 + Z_L \alpha l}\right]$
Voltage along line $V(z)$	$V_i \cosh \gamma z - I_i Z_0 \sinh \gamma z$	$V_i \cos \beta z - j I_i Z_0 \sin \beta z$	
Current along line $I(z)$	$I_i \cosh \gamma z - \dfrac{V_i}{Z_0} \sinh \gamma z$	$I_i \cos \beta z - j \dfrac{V_i}{Z_0} \sin \beta z$	
Reflection coefficient ρ	$\dfrac{Z_L - Z_0}{Z_L + Z_0}$	$\dfrac{Z_L - Z_0}{Z_L + Z_0}$	
Standing-wave ratio	$\dfrac{1 + \lvert\rho\rvert}{1 - \lvert\rho\rvert}$	$\dfrac{1 + \lvert\rho\rvert}{1 - \lvert\rho\rvert}$	

R, L, G, C	Distributed resistance, inductance, conductance, capacitance per unit length
l	Length of line
Subscript i	Denotes input end quantities
Subscript L	Denotes load end quantities
z	Distance along line from input end
λ	Wavelength measured along line
v	Phase velocity of line; equals velocity of light in dielectric of line for an ideal line

Table 5.11b
Formulas for Specific Transmission Line Configurations

			$p = \frac{s}{d}$ $q = \frac{s}{D}$ 	 Formulas for $a \ll b$
Capacitance C, farads/meter	$\frac{2\pi\epsilon}{\ln\left(\frac{r_0}{r_i}\right)}$	$\frac{\pi\epsilon}{\cosh^{-1}\left(\frac{s}{d}\right)}$	----	$\frac{\epsilon b}{a}$
External inductance L, henrys/meter	$\frac{\mu}{2\pi}\ln\left(\frac{r_0}{r_i}\right)$	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{s}{d}\right)$	----	$\mu\frac{a}{b}$
Conductance G, siemens/meter	$\frac{2\pi\sigma}{\ln\left(\frac{r_0}{r_i}\right)} = \frac{2\pi\omega\epsilon''}{\ln\left(\frac{r_0}{r_i}\right)}$	$\frac{\pi\sigma}{\cosh^{-1}\left(\frac{s}{d}\right)} = \frac{\pi\omega\epsilon''}{\cosh^{-1}\left(\frac{s}{d}\right)}$	----	$\frac{\sigma b}{a} = \frac{\omega\epsilon'' b}{a}$
Resistance R, ohms/meter	$\frac{R_s}{2\pi}\left(\frac{1}{r_0} + \frac{1}{r_i}\right)$	$\frac{2R_s}{\pi d}\left[\frac{s/d}{\sqrt{(s/d)^2 - 1}}\right]$	$\frac{2R_{s2}}{\pi d}\left[1 + \frac{1+2p^2}{4p^4}(1-4q^2)\right] + \frac{8R_{s3}}{\pi D}q^2\left[1 + q^2 - \frac{1+4p^2}{8p^4}\right]$	$\frac{2R_s}{b}$
Internal inductance L_i, henrys/meter (for high frequency)	← $\frac{R}{\omega}$ →			
Characteristic impedance at high frequency Z_0, ohms	$\frac{\eta}{2\pi}\ln\left(\frac{r_0}{r_i}\right)$	$\frac{\eta}{\pi}\cosh^{-1}\left(\frac{s}{d}\right)$	$\frac{\eta}{\pi}\left\{\ln\left[2p\left(\frac{1-q^2}{1+q^2}\right)\right] - \frac{1+4p^2}{16p^4}(1-4q^2)\right\}$	$\eta\frac{a}{b}$
Z_0 for air dielectric	$60\ln\left(\frac{r_0}{r_i}\right)$	$120\cosh^{-1}\left(\frac{s}{d}\right) \cong 120\ln\left(\frac{2s}{d}\right)$ if $s/d \gg 1$	$120\left\{\ln\left[2p\frac{(1-q^2)}{(1+q^2)}\right] - \frac{1+4p^2}{16p^4}(1-4q^2)\right\}$	$120\pi\frac{a}{b}$
Attenuation due to conductor α_c	← $\frac{R}{2Z_0}$ →			
Attenuation due to dielectric α_d	← $\frac{GZ_0}{2} = \frac{\sigma\eta}{2} = \frac{\pi}{\lambda}\left(\frac{\epsilon''}{\epsilon'}\right)$ →			
Total attenuation dB/meter	← $8.686(\alpha_c + \alpha_d)$ →			
Phase constant for low-loss lines β	← $\omega\sqrt{\mu\epsilon'} = \frac{2\pi}{\lambda}$ →			

All units above are mks.

$\epsilon = \epsilon' - j\epsilon''$ = permittivity, farads/meter
μ = permeability, henrys/meter } for the dielectric
$\eta = \sqrt{\mu/\epsilon}$ ohms

ϵ'' = loss factor of dielectric = σ/ω
R_s = skin effect surface resistivity of conductor, ohms
λ = wavelength in dielectric

Formulas for shielded pair obtained from Green, Leibe, and Curtis, *Bell System Tech. Journ.*, 15, pp. 248–284 (April 1936).

from (13), and Z_0 includes a first-order reactive part, given by the last part of (15). Note that the first-order effects of the two loss factors tend to cancel in Z_0 whereas they add in α.

Several of the important formulas for loss-free, low-loss, and general lines are summarized in Table 5.11a. Formulas for properties of lines of several different cross-sectional forms are listed in Table 5.11b.

Physical Approximations for Low-Loss Lines The approximate form (13) for attenuation in transmission lines with small losses may be derived from physical ideas. This approach will be especially useful in estimating attenuation of more general guiding systems to be considered later in the text. Let us consider the positively traveling voltage wave of (9) and its corresponding current:

$$V = V_+ e^{-\alpha z} e^{-j\beta z} \tag{16}$$

$$I = I_+ e^{-\alpha z} e^{-j\beta z} \tag{17}$$

The average power transfer at any position is then given by $W_T = \frac{1}{2}\,\mathrm{Re}(VI^*)$:

$$W_T = \tfrac{1}{2} V_+ I_+ e^{-2\alpha z} \tag{18}$$

It is assumed here that the imaginary part of Z_0 is negligible so that I_+ is in phase with V_+.

The rate of decrease of the average power (18) with distance along the line must equal the average power loss w_L in the line, per unit length:

$$\frac{\partial W_T}{\partial z} = -w_L = -2\alpha\left(\frac{1}{2} V_+ I_+ e^{-2\alpha z}\right) = -2\alpha W_T$$

or

$$\alpha = \frac{w_L}{2W_T} \tag{19}$$

This is an important formula relating attenuation constant to power loss per unit length and average power transfer. By the nature of the development, it applies to the attenuation of a traveling wave along any uniform system.

To apply (19) to a transmission line with series resistance R and shunt conductance G, we first calculate the average power loss per unit length, part of which comes from the current flow through the resistance and another part from voltage appearing across the shunt conductance. For convenience we calculate w_L and W_T at $z = 0$:

$$w_L = \frac{I_+^2 R}{2} + \frac{V_+^2 G}{2} = \frac{V_+^2}{2}\left[G + \frac{R}{Z_0^2}\right] \tag{20}$$

The average power transferred by the wave at $z = 0$ is

$$W_T = \frac{1}{2} V_+ I_+ = \frac{1}{2}\frac{V_+^2}{Z_0} \tag{21}$$

So (19) gives attenuation in agreement with (13):

$$\alpha = \frac{1}{2}\left[GZ_0 + \frac{R}{Z_0}\right] \quad \mathrm{Np/m} \tag{22}$$

The *neper* (Np) is a unit-free name for attenuation that measures the decay of voltage amplitude. One neper per meter indicates that the *amplitude* has decayed to $1/e$ of its incoming value in 1 m. The *decibel* (dB) is an alternative measure describing the rate of *power* decrease by the formula $10 \log_{10} W_{T2}/W_{T1}$.[5] Attenuation in decibels per meter is 8.686 times the attenuation in nepers per meter.

[5] *Decibels are often used for ratios of voltage amplitudes using the formula $20 \log_{10} V_2/V_1$. This is only correct if the voltages are across identical impedances, as in the case of voltages at points along a transmission line.*

Example 5.11
ATTENUATION IN A THIN-FILM TRANSMISSION LINE

Let us find the attenuation in an aluminum thin-film parallel-plane transmission line for a signal of 18 GHz. The structure has the form in Fig. 2.5c and fringing fields will be neglected. The metal thickness h is 2.0 μm with a dielectric thickness d also of 2.0 μm. The width of the conductors is typical of photolithographically defined lines, $w = 10$ μm. The relative permittivity of the dielectric is 3.8 and it is assumed to be lossless. From Eq. 1.9(3) the capacitance per unit length is $C = \varepsilon w/d = 1.67 \times 10^{-10}$ F/m. From Eq. 2.5(3) the external inductance per unit length is $L = \mu_0 d/w = 2.51 \times 10^{-7}$ H/m.To see how to treat the internal inductance and resistance of the conductors, the depth of penetration must be compared with the film thickness. From Table 3.17 the depth of penetration for aluminum is $\delta = 0.0826/\sqrt{f}$, so for 18 GHz, $\delta = 0.616$ μm. Thus the aluminum films are 3.2 times the depth of penetration so they are well approximated by arbitrarily thick layers. Then the internal inductance and resistance are given by the surface impedance. From Table 3.17 the surface resistivity is $R_s = 3.26 \times 10^{-7}\sqrt{f}$, so from Eqs. 3.17(4) and 3.17(6) the surface impedance is

$$Z_s = 3.26 \times 10^{-7}\sqrt{f}(1 + j) \tag{23}$$

and the internal impedance per unit length for both electrodes is

$$Z_i = \frac{2Z_s}{w} = 8.75 \times 10^3(1 + j) \quad \Omega \tag{24}$$

The characteristic impedance is found from $Z_0 = \sqrt{L/C}$, where L includes both external and internal inductances. The latter is found from (24) by dividing by ω so $L_i = 7.74 \times 10^{-8}$ H. Adding this to the external inductance and substituting the sum and the capacitance into Z_0, we find $Z_0 = 44.3$ Ω. Note that if we had neglected internal inductance, Z_0 would have been calculated as 38.4 Ω. The resistance per unit length R is the real part of (24). Substituting Z_0 and R in (22) we get the attenuation constant:

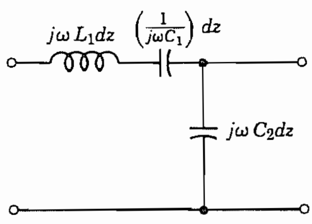
$$\alpha = \frac{R}{2Z_0} = 0.988 \text{ Np/cm} \tag{25}$$

From this we see that the wave attenuates by about a factor of e in a distance of 1 cm.

5.12 FILTER-TYPE DISTRIBUTED CIRCUITS: THE ω–β DIAGRAM

Suppose the distributed series impedance of the general transmission line is formed by inductance and capacitance in series, as shown in Fig. 5.12a. The propagation constant γ is then

$$\gamma = \sqrt{j\omega C_2\left(j\omega L_1 + \frac{1}{j\omega C_1}\right)} = j\omega\sqrt{L_1C_2\left(1 - \frac{\omega_c^2}{\omega^2}\right)} \tag{1}$$



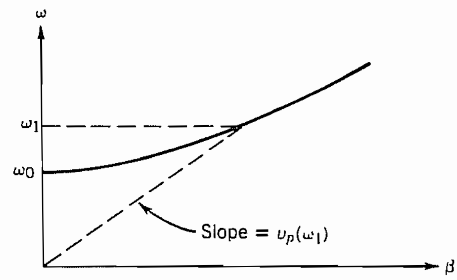


Fig. 5.12 (*a*) Filter-type distributed circuit. (b) Its ω–β diagram, showing phase velocity.

where

$$\omega_c = (L_1 C_1)^{-1/2} \tag{2}$$

The interesting characteristic of this system is that for the lower range of frequencies, $\omega < \omega_c$, γ is purely real, representing an attenuation without losses in the system

$$\gamma = \alpha = \omega\sqrt{L_1 C_2\left(\frac{\omega_c^2}{\omega^2} - 1\right)}, \qquad \omega < \omega_c \tag{3}$$

The attenuation in this circuit occurs below the cutoff frequency defined by (2), so that the system is a distributed high-pass filter. The reactive attenuation which occurs arises essentially because of continuous reflections in the system, and is of the same nature as the attenuation in a loss-free, lumped-element filter in the attenuating band.

For frequencies above ω_c, the propagation constant γ is purely imaginary so $\gamma = j\beta$ and is given by (1). It is found useful to plot relations between β and ω with β on the abscissa and ω on the ordinate. These are called *ω–β diagrams.* Figure 5.12*b* shows the ω–β relation (1) for the line discussed here. Note that β goes to zero for $\omega = \omega_c$ and does not exist for $\omega < \omega_c$. We saw in (3) that, for this line, there is only attenuation for $\omega < \omega_c$. An important reason for choosing the coordinates of the ω–β diagram as done is that the phase velocity at any frequency, which from Eq. 5.7(6) is $v_p = \omega/\beta$, can be seen immediately as the slope of a line to the origin from the curve, as illustrated in Fig. 5.12*b*. We see that the phase velocity for the line under consideration is

$$v_p = \frac{1}{\sqrt{L_1 C_2}}\left[1 - \frac{\omega_c^2}{\omega^2}\right]^{-1/2} \tag{4}$$

which is seen to vary strongly for frequencies just above cutoff. Signals with several frequency components propagating in this range will thus have large *dispersion*, as will be discussed more in Sec. 5.15.

Resonant Transmission Lines

5.13 Purely Standing Wave on an Ideal Line

An important special case of standing waves on a transmission line, introduced in general in Sec. 5.8, is one in which all the incident energy is reflected. The reflected wave has the same amplitude as the incident wave so $S = \infty$. It is clear from Eq. 5.7(8) that $|\rho| = 1$ so that $|V_-| = |V_+|$ if any of the following conditions exist: (1) short-circuit load, $Z_L = 0$; (2) open-circuit load, $Z_L = \infty$; (3) purely reactive load. The last is less obvious than the others but is easily shown (Prob. 5.13b). In each case $|V_-| = |V_+|$ because the load cannot dissipate power and it must, therefore, be fully reflected.

Suppose that a transmission line, shorted at one end, is excited by a sinusoidal voltage at the other. Let us select the position of the short as the reference, $z = 0$. The short imposes the condition that, at $z = 0$, voltage must always be zero. From Eq. 5.7(4),

$$V(0) = V_+ + V_- = 0$$

If $V_- = -V_+$ is substituted in Eqs. 5.7(4) and 5.7(5),

$$V = V_+[e^{-j\beta z} - e^{j\beta z}] = -2jV_+ \sin \beta z \tag{1}$$

$$I = \frac{V_+}{Z_0}[e^{-j\beta z} + e^{j\beta z}] = 2\frac{V_+}{Z_0} \cos \beta z \tag{2}$$

These results, typical for standing waves, show the following.

1. Voltage is always zero not only at the short, but also at multiples of $\lambda/2$ to the left; that is,

$$V = 0 \quad \text{at} \quad -\beta z = n\pi \quad \text{or} \quad z = -n\frac{\lambda}{2}$$

2. Voltage is a maximum at all points for which βz is an odd multiple of $\pi/2$. These are at distances odd multiples of a quarter-wavelength from the short circuit. Figure 5.13 shows this and also the time evolution of the voltage found by multiplying (1) and (2) by $e^{j\omega t}$ and taking the real part. Time origin is chosen so that V_+ is real.
3. Current is a maximum at the short circuit and at all points where voltage is zero; it is zero at all points where voltage is a maximum. Figure 5.13 shows the time variation of current along the line.
4. Current and voltage are not only displaced in their space patterns, but also are 90 degrees out of time phase, as indicated by the j appearing in (1) and as seen in Fig. 5.13.
5. The ratio between the maximum current on the line and the maximum voltage is Z_0, the characteristic impedance of the line.
6. The total energy in any length of line a multiple of a quarter-wavelength long is

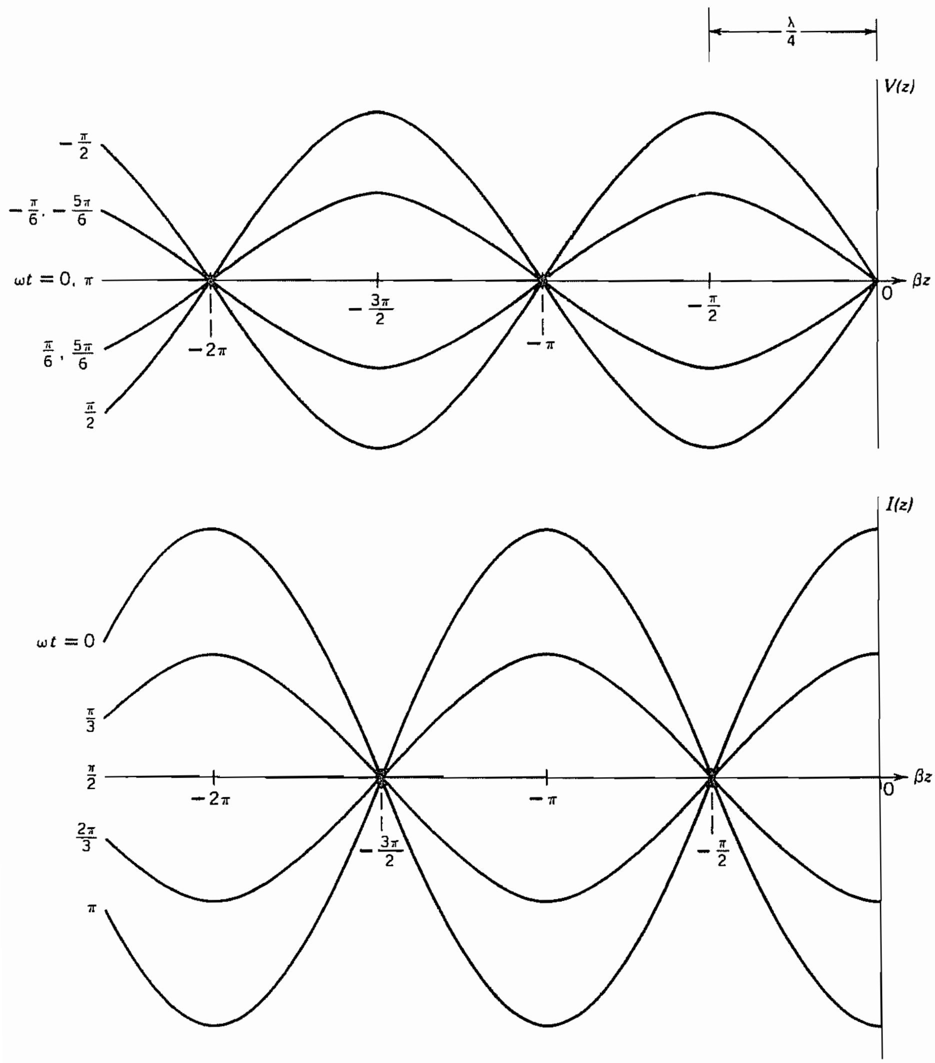


FIG. 5.13 Time evolution of voltage and current on a shorted transmission line. The zeros and extrema remain at the same locations.

constant, merely interchanging between energy in the electric field of the voltages and energy in the magnetic field of the currents.

To check the energy relation just stated, let us calculate the magnetic energy of the currents at a time when the current pattern is a maximum and voltage is zero everywhere along the line. Current is given by (2). The energy is calculated for a quarter-wavelength of the line, assuming V_+ to be real.

$$U_M = \frac{L}{2}\int_{-\lambda/4}^{0} |I|^2\,dz = \frac{L}{2}\int_{-\lambda/4}^{0} \frac{4V_+^2}{Z_0^2}\cos^2 \beta z\,dz$$

$$= \frac{2V_+^2 L}{Z_0^2}\left[\frac{z}{2} + \frac{1}{4\beta}\sin 2\beta z\right]_{-\lambda/4}^{0}$$

Since $\beta = 2\pi/\lambda$ by Eq. 5.7(7), the foregoing is

$$U_M = \frac{V_+^2 L\lambda}{4Z_0^2} \tag{3}$$

The maximum energy stored in the distributed capacitance effect of the line is calculated for the quarter-wavelength when the voltage pattern is a maximum and current is everywhere zero. Voltage is given by (1).

$$U_E = \frac{C}{2}\int_{-\lambda/4}^{0} |V|^2\,dz = \frac{C}{2}\int_{-\lambda/4}^{0} 4V_+^2 \sin^2 \beta z\,dz$$

$$= 2CV_+^2\left[\frac{z}{2} - \frac{1}{4\beta}\sin 2\beta z\right]_{-\lambda/4}^{0} = \frac{CV_+^2\lambda}{4} \tag{4}$$

By the definition of Z_0, (3) may also be written

$$U_M = \frac{V_+^2 L\lambda}{4L/C} = \frac{V_+^2 C\lambda}{4} = U_E \tag{5}$$

Thus, the maximum energy stored in magnetic fields is exactly equal to that stored in electric fields 90 degrees later in phase. It can also be shown that the sum of electric and magnetic energy at any other part of the cycle is equal to this same value.

Expressions (1) and (2) are also valid for a transmission line with short circuits both at $z = 0$ and another point where $z = n(-\pi/\beta)$, for any integer n. With some way to couple energy into a section of line short-circuited at both ends, at a frequency such that the above criterion on z is satisfied (recall that $\beta = \omega/v$), there will be voltages and currents satisfying (1) and (2). At each such frequency, the line is said to have a *resonance*. This idea will be developed further in Sec. 5.14.

5.14 Input Impedance and Quality Factor for Resonant Transmission Lines

Resonant systems play a very significant role in communication systems for impedance matching and filtering and we have already seen some aspects of this in Ex. 5.10d, where resonant sections of lines were used for matching impedances. In Sec. 5.13 we analyzed standing waves on short-circuited ideal lines. In the present section we consider a low-loss line shorted at either one or both ends so that standing waves similar to those discussed in Sec. 5.13 occur. The line is supplied by a voltage source connected

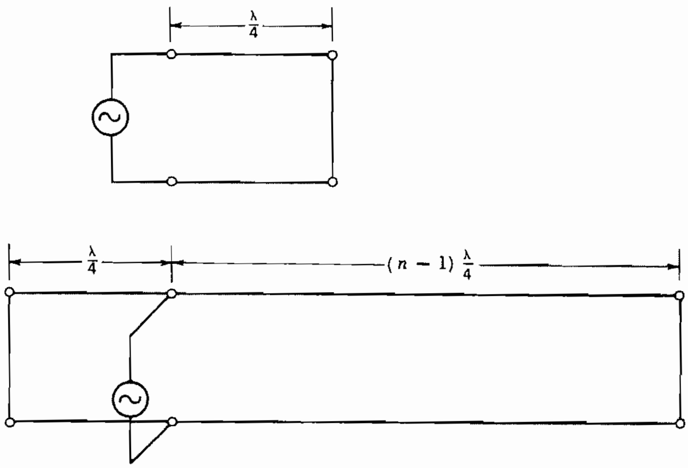


FIG. 5.14a Resonant transmission lines driven by voltage sources at positions of maximum voltage.

at a voltage maximum in either of the ways shown in Fig. 5.14a. We will find approximate expressions for the resistance seen by the source and for the quality factor Q of the line, considered as a resonant circuit.

For an ideal line, there are points of voltage maximum and zero current at odd multiples of a quarter-wavelength from the short-circuited ends so the impedance is infinite there. When losses are present, however, the impedance at these positions is high but finite, representing the energy dissipated in the losses of the line. Let us find these losses approximately for a line of n quarter-wavelengths using the expressions for voltage and current derived for an ideal line, Eqs. 5.13(1) and 5.13(2), assuming that they are not greatly changed by the small losses. The average power dissipated in the shunt conductance is then

$$W_G = \int_0^{n\lambda/4} (2V_+ \sin \beta z)^2 \frac{G}{2}\, dz = \left(\frac{4V_+^2 G}{4}\right)\left(\frac{n\lambda}{4}\right) \tag{1}$$

and the average power dissipated in the series resistance is

$$W_R = \int_0^{n\lambda/4} \left(\frac{2V_+ \cos \beta z}{Z_0}\right)^2 \frac{R}{2}\, dz = \left(\frac{4V_+^2 R}{4Z_0^2}\right)\left(\frac{n\lambda}{4}\right) \tag{2}$$

The input resistance (at a voltage maximum) must be such that the voltage appearing across this resistance will produce losses equal to the sum of (1) and (2). The magnitude of voltage there is $2V_+$. Thus

$$\frac{1}{2}\frac{(2V_+)^2}{R_i} = \frac{nV_+^2\lambda}{4}\left(G + \frac{R}{Z_0^2}\right)$$

or

$$R_i = \frac{8Z_0}{n\lambda[GZ_0 + (R/Z_0)]} \tag{3}$$

A general expression for the quality factor Q of any resonant system is

$$Q \triangleq \frac{\omega_0 \text{ (energy stored)}}{\text{average power loss}} = \frac{\omega_0 U}{W_L} \tag{4}$$

For a resonant transmission line of n quarter-wavelengths, the stored energy for each quarter-wavelength is taken as that for the ideal line, Eq. 5.13(5), and the power loss is given by the sum of (1) and (2). The result is

$$Q = \frac{\omega_0 U}{W_L} = \frac{4\omega_0 C V_+^2 n\lambda}{4V_+^2 n\lambda[G + (R/Z_0^2)]} = \frac{\omega_0 C Z_0}{GZ_0 + (R/Z_0)} \tag{5}$$

We see that Q is independent of n; this results from the fact that both the stored energy and the power dissipated are proportional to the length. Thus, Q is a property of the line, independent of the number of resonant quarter-wavelengths.

The input resistance for a shorted quarter-wavelength section or at the maximum voltage point of a line $n\lambda/4$ (n even) long shorted at both ends can be rewritten using (3) and (5) with Eqs. 5.2(8) and 5.2(14) and 5.7(6) and 5.7(7):

$$R_i = \frac{8Q}{n\lambda\omega_0 C} = \frac{4QZ_0}{n\pi} \tag{6}$$

The input resistance measures the power supplied to maintain a given voltage level; Q increases as the losses decrease, leading to a higher input resistance.

If the frequency and, therefore, λ are changed so the distance from the input point to the short circuit differs from $\lambda/4$, the input impedance acquires a reactive component of first-order importance. With the same amount of frequency change, the voltage and current patterns do not change much, so the resistive part (6) does not change much.

It will be convenient to complete the analysis in terms of admittance. The susceptance that arises is in parallel with the conductance equivalent of (6). Let us calculate it for the lower circuit in Fig. 5.14*a*. It consists of two susceptances in parallel, that for the $\lambda/4$ section on the left and that of the remaining $(n - 1)\lambda/4$ portion on the right. For each section, the load admittance is infinite so Eq. 5.7(14) gives, in the lossless approximation,

$$jB_i = -jY_0(\cot \beta l_L + \cot \beta l_R) \tag{7}$$

where $Y_0 = 1/Z_0$ and l_L and l_R are the lengths of the left and right sides of the line. Letting $\beta = \omega/v_p = \omega_0(1 + \delta)/v_p = \beta_0 + \beta_0\delta$ and taking $\beta_0 l_L = \pi/2$ and $\beta_0 l_R = (n - 1)\pi/2$, the cotangents in (7) can be approximated for small δ to give

$$B_i = Y_0\left[\frac{\pi}{2}\,\delta + \frac{(n - 1)\pi}{2}\,\delta\right] = \frac{n\pi}{2}\,\delta Y_0 \tag{8}$$

Then using (6) and (8), we have the admittance at the feed point in the $n\lambda/4$ line in the lower circuit of Fig. 5.14*a*:

$$Y_i = \frac{n\pi}{2} Y_0\left(\frac{1}{2Q} + j\delta\right) \tag{9}$$

For the upper circuit in Fig. 5.14*a* the cot βl_R terms in (7) and (8) are missing so (9) describes that circuit with $n = 1$. From this we see that the fractional frequency shift for which the susceptance becomes equal to the conductance, a common measure of circuit sharpness, is

$$\delta_1 = \frac{1}{2Q} \tag{10}$$

or

$$Q = \frac{\omega_0}{2\ \Delta\omega_1} = \frac{f_0}{2\ \Delta f_1} \tag{11}$$

where 2 Δf_1 is the frequency width between points where the admittance magnitude reaches $\sqrt{2}$ times its value at resonance ($\omega = \omega_0$). Thus Q, as defined by (4), is useful as a measure of sharpness of frequency response, as it is for lumped-element circuits. Resonant low-loss transmission lines can have Q's of thousands in the UHF range of frequencies.

Example 5.14

OPEN-ENDED PARALLEL-PLANE TRANSMISSION LINE

Consider standing waves in an open-circuited section of transmission line and the contribution to Q from radiation at the ends. Radiation loss may be expressed in terms of load conductances G_L at each end, as pictured in Fig. 5.14*b*, and when radiation is

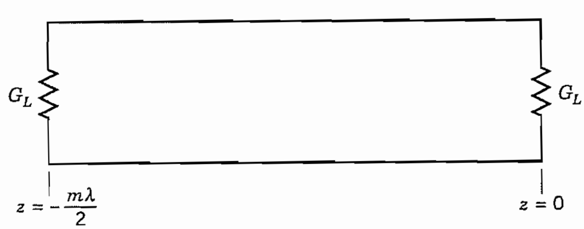


FIG. 5.14*b* Model for open-ended transmission line.

small, $G_L << 1/Z_0$, fields in the line are essentially those of a completely open-circuited line,

$$V = 2V_+ \cos \beta z \tag{12}$$

$$I = -j \frac{2V_+}{Z_0} \sin \beta z \tag{13}$$

Power loss from the two end conductances is then

$$W_L = 2 \frac{(2V_+)^2}{2} G_L \tag{14}$$

Energy storage for a length some multiple of a half-wavelength is

$$U = \int_{-m\lambda/2}^{0} \frac{C}{2} (2V_+)^2 \cos^2\beta z \, dz = \frac{CV_+^2 m\lambda}{2} \tag{15}$$

Using (4), the Q from the radiation component is found to be

$$Q = \frac{\omega_0 C m\lambda}{8G_L} = \frac{m\pi}{4Z_0 G_L} \tag{16}$$

There may also be contributions to Q from conductor and dielectric losses, as in (5). Losses, when small, add, so reciprocals of Q's add also:

$$\frac{1}{Q_{\text{total}}} = \frac{1}{Q_1} + \frac{1}{Q_2} + \cdots \tag{17}$$

Special Topics

5.15 Group and Energy Velocities

A function of time with arbitrary wave shape may be expressed as a sum of sinusoidal waves by Fourier analysis. If it happens that v_p is the same for each frequency component and there is no attenuation, the component waves will add in proper phase at each point along the line to reproduce the original wave shape exactly, but delayed by the time of propagation z/v_p. The velocity v_p in this case describes the rate at which the wave moves down the line and could be said to be the velocity of propagation. This case occurs, for example, in the ideal loss-free transmission line already studied for which v_p is a constant equal to $(LC)^{-1/2}$. If v_p changes with frequency, there is said to be *dispersion* and a signal may change shape as it travels. This causes distortion of analog signals and limits data rates because of the spreading of pulses in digital signals.

In the nondispersive situation described above, a wave of a given shape propagates along a line without distortion. The velocity of the group of frequency components is the same as the phase velocity of any one. In many transmission systems the velocity of the envelope of the wave such as that in Fig. 5.15*a* can be different from the phase velocities of the frequency components, and it is useful to introduce a so-called *group velocity* to describe the motion of the group, or envelope of the wave. This is the typical case when a high-frequency carrier is modulated by a digital or analog signal.

Let us consider the simplest possible group, a wave having two equal-amplitude sinusoidal components of slightly different frequency. The voltage at $z = 0$ with unity-amplitude components is

$$V(t) = \sin(\omega_0 - d\omega)t + \sin(\omega_0 + d\omega)t \tag{1}$$

Then for a lossless line, the voltage at any point is

$$V(t, z) = \sin[(\omega_0 - d\omega)t - (\beta_0 - d\beta)z] + \sin[(\omega_0 + d\omega)t - (\beta_0 + d\beta)z] \tag{2}$$

in which β is to be regarded as a function of ω; $d\beta$ corresponds to $d\omega$. Expression (2) can be put in the form

$$V(t, z) = 2\cos[(d\omega)t - (d\beta)z]\,\sin(\omega_0 t - \beta_0 z) \tag{3}$$

From (3) we see that the voltage in this wave group has the form shown in Fig. 5.15*b* for one instant of time. The sinusoid of center frequency moves at the phase velocity $v_p = \omega_0/\beta_0$ whereas the envelope, described by $\cos[(d\omega)t - (d\beta)z]$, has the form of a wave but it moves at a different velocity. This is found by keeping the argument of the cosine term a constant:

$$v_g = \frac{d\omega}{d\beta} \tag{4}$$

Velocity v_g is called the group velocity and is shown in Fig. 5.15*b*. Note that v_g is the slope of ω–β curve at the center frequency of the group. Forming the derivative $dv_p/d\omega$ using $v_p = \omega/\beta$, another useful form for group velocity can be derived:

$$v_g = \frac{v_p}{1 - (\omega/v_p)(dv_p/d\omega)} \tag{5}$$

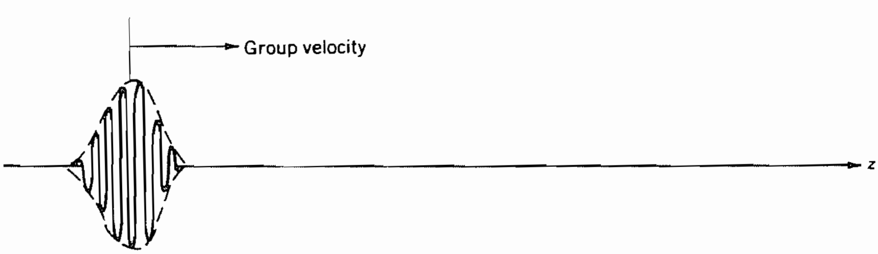


FIG. 5.15*a* Envelope or group velocity.

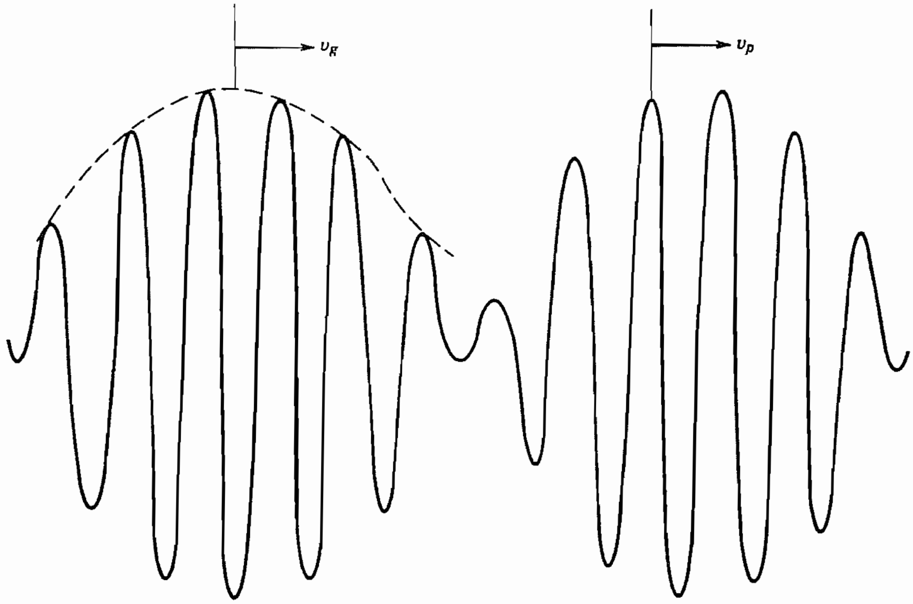


FIG. 5.15*b* Phase and group velocities for a group of two sinusoids of slightly different frequencies.

For groups more complicated than the two-frequency one considered above, we will show later by Fourier analysis that the envelope of a modulated wave retains its shape so long as v_g is constant (i.e., the ω–β curve is linear) over the range of frequencies required to represent the wave. In that case, a pulse such as that illustrated in Fig. 5.15*a* would retain its *envelope* shape and propagate with delay time τ_d over distance l:

$$\tau_d = \frac{l}{v_g} = l\,\frac{d\beta}{d\omega} \tag{6}$$

If $d\beta/d\omega$ is not constant over the frequency band of the signal, there is a broadening or distortion of the envelope. This is known as *group dispersion* and will be seen in several later examples.

Group velocity is often referred to as the "velocity of energy travel." This concept has validity for many important cases, but is not universally true.[6,7] To illustrate the basis for the concept, let us define here a separate velocity v_E based on energy flow so that power transfer is stored energy multiplied by this velocity. That is,

$$v_E = \frac{W_T}{u_{av}} \tag{7}$$

6 J. A. Stratton, Electromagnetic Theory, pp. 330–340, McGraw-Hill, New York, 1941.
7 L. Brillouin, Wave Propagation and Group Velocity, Academic Press, New York, 1960.

where W_T is average power flow in a single wave and u_{av} is average energy storage per unit length. If this definition is applied to the ideal transmission line of Sec. 5.7, we find that $v_E = v_g = v_p$. More interesting is the case of the filter-type circuit of Fig. 5.12*a*, which is a case of *normal dispersion* ($dv_p/d\omega < 0$). Here

$$W_T = \frac{1}{2} Z_0 II^* = \frac{II^*}{2}\left[\frac{L_1}{C_2}\left(1 - \frac{\omega_c^2}{\omega^2}\right)\right]^{1/2}$$

$$u_{av} = \frac{1}{2}\left(\frac{C_2 VV^*}{2} + \frac{L_1 II^*}{2} + \frac{C_1}{2}\frac{II^*}{\omega^2 C_1^2}\right)$$

$$= \frac{L_1}{4}\left(\frac{C_2 Z_0^2}{L_1} + 1 + \frac{\omega_c^2}{\omega^2}\right)II^* = \frac{L_1 II^*}{2}$$

so

$$v_E = (L_1 C_2)^{-1/2}\left[1 - \frac{\omega_c^2}{\omega^2}\right]^{1/2} \tag{8}$$

This is equal to group velocity $d\omega/d\beta$, as can be found by differentiating Eq. 5.12(1), and is different from phase velocity.

The identity of group velocity and energy velocity can also be shown to be true for simple waveguides, and it also applies to many other cases of normal dispersion. It does not usually apply to systems with *anomalous dispersion* ($dv_p/d\omega > 0$), including the simple transmission line with losses. In any event the concept of an energy velocity is useful only when there is limited dispersion so that the input signal can be recognized at the output.

5.16 Backward Waves

A wave in which phase velocity and group velocity have opposite signs is known as a *backward wave*. Conditions for these may seem unexpected or rare, but they are not. Consider for instance the distributed system of Fig. 5.16*a* in which there are series capacitances and shunt inductances—the dual of the simple transmission line of Sec. 5.2. From Sec. 5.11,

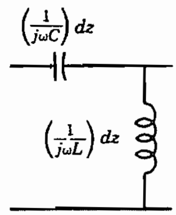
$$\gamma = j\beta = \sqrt{ZY} = \sqrt{\left(\frac{1}{j\omega C}\right)\left(\frac{1}{j\omega L}\right)} = -\frac{j}{\omega\sqrt{LC}} \tag{1}$$

This ω–β relation is shown in Fig. 5.16*b*. The phase and group velocities are

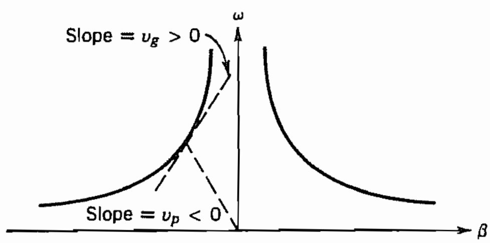
$$v_p = \frac{\omega}{\beta} = -\omega^2\sqrt{LC} \tag{2}$$

and

$$v_g = \frac{d\omega}{d\beta} = \omega^2\sqrt{LC} \tag{3}$$



(a)



(b)

FIG. 5.16 (*a*) Equivalent circuit for a transmission line which propagates backward waves. (*b*) ω–β relation for line of (*a*) showing phase and group velocity directions discussed in the text.

So it is seen that this very simple transmission system satisfies the conditions for backward waves. If energy is made to flow in the positive z direction, group velocity will be in this direction, as this is a case where v_g does represent energy flow (see Prob. 5.16c). However, the phase becomes increasingly negative or "lagging" in the direction of propagation because of the C–L configuration. Thus there is a negative phase velocity.

There are many other filter-type circuits having other combinations of series and shunt inductances and capacitances on which can exist waves with increasingly lagging phase in the direction of energy propagation. Also, all periodic circuits (Sec. 9.10) have equal numbers of forward and backward "space harmonics."

5.17 NONUNIFORM TRANSMISSION LINES

For a transmission line with varying spacing or size of conductors, as illustrated in Fig. 5.17, a natural extension of the transmission-line analysis would lead one to consider impedance and admittance as varying with distance in the transmission-line equations. Actually, fields may be distorted so that the formulation is not this simple, but it is a good approximation in a number of important cases, and the methods discussed apply

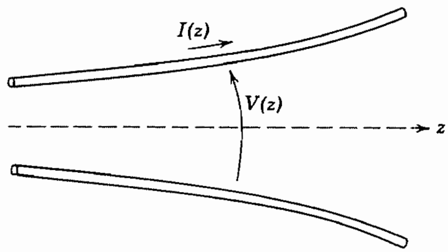


FIG. 5.17 Nonuniform transmission line.

to some wave problems with spatial variations of the medium (i.e., inhomogeneous materials). The remainder of this section will consider cases where such nonuniform transmission-line theory yields a good approximation.

If impedance and admittance per unit length vary with distance, the transmission-line equations corresponding to Eqs. 5.11(1) and 5.11(2) are

$$\frac{dV(z)}{dz} = -Z(z)I(z) \tag{1}$$

$$\frac{dI(z)}{dz} = -Y(z)V(z) \tag{2}$$

Differentiate (1) with respect to z, denoting z differentiation by primes:

$$V'' = -[ZI' + Z'I] \tag{3}$$

To obtain a differential equation in voltage alone, I may be substituted from (1) and I' from (2). The result is

$$V'' - \left(\frac{Z'}{Z}\right)V' - (ZY)V = 0 \tag{4}$$

A similar procedure, starting with differentiation of (2), yields a second-order differential equation in I:

$$I'' - \left(\frac{Y'}{Y}\right)I' - (ZY)I = 0 \tag{5}$$

If Z' and Y' are zero, (4) and (5) reduce, as they should, to the equations for a uniform line (Sec. 5.11). When these derivatives are nonzero, representing the nonuniform line discussed, the equations may be solved numerically for arbitrary variations of Z and Y with distance. A few forms of the variation permit analytic solutions, including the "radial transmission line," where either Z or Y is proportional to z, and their product is constant. Another important case is the "exponential line," which is taken as the example for this article.

Example 5.17

LINE WITH EXPONENTIALLY VARYING PROPERTIES

Let us consider a loss-free exponential line with Z and Y varying as follows:

$$Z = j\omega L_0 e^{qz}, \qquad Y = j\omega C_0 e^{-qz} \tag{6}$$

These variations yield constant values of ZY, Z'/Z, and Y'/Y so that (4) and (5) become equations with constant coefficients,

$$V'' - qV' + \omega^2 L_0 C_0 V = 0 \tag{7}$$

$$I'' + qI' + \omega^2 L_0 C_0 I = 0 \tag{8}$$

These have solutions of the exponential propagating form,

$$V = V_0 e^{-\gamma_1 z}, \qquad I = I_0 e^{-\gamma_2 z} \tag{9}$$

where

$$\gamma_1 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 - \omega^2 L_0 C_0} \tag{10}$$

$$\gamma_2 = +\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 - \omega^2 L_0 C_0} \tag{11}$$

We see the interesting property of "cutoff" again, for γ_1 and γ_2 are purely real for low frequencies $\omega < \omega_c$ where

$$\omega_c^2 L_0 C_0 = \left(\frac{q}{2}\right)^2 \tag{12}$$

The attenuation represented by these real values, like that for the loss-free filter-type lines, is reactive. This represents no power dissipation but only a continuous reflection of the wave. For $\omega > \omega_c$, however, the values of γ have both real and imaginary parts, which is a behavior different from that of the loss-free filters. Again the real parts represent no power dissipation (see Prob. 5.17b). The values of γ approach purely imaginary values representing phase change only for $\omega >> \omega_c$.

The greatest use of this type of line is in matching between lines of different characteristic impedance. Unlike the resonant matching sections (Prob. 5.7c), this type of matching is insensitive to frequency. Note the variation of characteristic impedance:

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_0 e^{-\gamma_1 z}}{I_0 e^{-\gamma_2 z}} = \frac{V_0}{I_0} e^{-(\gamma_1 - \gamma_2)z} = Z_0(0) e^{qz} \tag{13}$$

Thus Z_0 can be changed by an appreciable factor if qz is large enough. The transmission-line approximation will become poor, however, if there is too large a change of Z and Y in a wavelength or in a distance comparable to conductor spacing.

The design of nonuniform matching sections is explored in detail by Elliot.[8]

PROBLEMS

5.2a Sketch the function

$$V(z, t) = \cos \omega\left(t + \frac{z}{v}\right) + \frac{1}{2}\cos 2\omega\left(t + \frac{z}{v}\right)$$

versus $\omega z/v$ for values of $\omega t = 0, \pi/2, \pi, 3\pi/2$ and explain how this shows traveling-wave behavior.

5.2b (i) Derive an expression for the characteristic impedance of the parallel-plate line in Fig. 5.2 having a width w and spacing a neglecting the internal inductance of the conductors. Thin-film transmission lines in some computer circuits can be modeled approximately by the parallel-plane line. The line width is usually about 5 μm and the spacing is by means of dielectric of 1-μm thickness and relative permittivity 2.5 (as is usually true for dielectrics, the relative permeability can be taken as ≈ 1.0).

(ii) Calculate the characteristic impedance Z_0 and wave velocity v.

(iii) Suppose the dielectric thickness is halved and find the new values of Z_0 and v.

A better model for such lines is given in Chapter 8.

5.2c The capacitance per unit length of a parallel-wire line having radii R with distance $2d$ between axes is $C = \pi\varepsilon/\cosh^{-1}(d/R)$. Find characteristic impedance of a line with air dielectric and spacing between axes 1 cm if (i) wire radius is 2 mm and (ii) wire radius is 0.5 mm.

5.2d Calculate propagation time along the following transmission lines interconnecting computer elements:

(i) A thin-film line on GaAs ($\varepsilon_r = 11$) between circuit elements 100 μm apart

(ii) Transmission line interconnecting two devices on a silicon computer chip 1 mm apart, $\varepsilon_r = 12$

(iii) Coaxial cable 100 m long with $\varepsilon_r = 2.4$, used to interconnect computer terminal and central processor

5.2e A second type of solution to the wave equation, to be studied later in the chapter, is the *standing wave* solution. Find under what conditions the following such solution satisfies Eq. 5.2(7):

$$V(z, t) = V_m \cos \omega t \sin \beta z$$

Find the current $I(z, t)$ corresponding to this voltage distribution.

[8] *R. S. Elliott,* An Introduction to Guided Waves and Microwave Circuits, *Chap. 8, Prentice Hall, Englewood Cliffs, NJ, 1993.*

5.2f Expand the cosine and sine in the expression of Prob. 5.2e in terms of complex exponentials, $\cos x = \frac{1}{2}(e^{jx} + e^{-jx})$, and so on, and after multiplying out, show that the products can be interpreted as traveling waves of the form of Eq. 5.2(10).

5.2g Examine the expression for characteristic impedance of a coaxial transmission line, Eq. 5.2(15),and explain why it is difficult to obtain high characteristic impedances for such a line without having unreasonable dimensions or very high losses. Plot dc resistance per unit length for such a line versus Z_0 if the outer conductor is a tubular copper conductor of inner radius 1 cm and wall thickness 1 mm, and the inner conductor a solid copper cylinder.

5.2h Repeat Prob. 5.2g but plot ac resistance at 100 MHz using the approximation of Ex. 3.17.

5.2i Use Eqs. 5.2(3) and (4) to show that the spatial rate of change of power flow on an ideal line is equal to the negative of the time rate of change of the stored energy per unit length.

5.3 To show some simple properties of transverse electromagnetic (TEM) waves, utilize Maxwell's equations in rectangular coordinates (though the boundaries need not be rectangular). Take the dielectric as source-free and without losses. Show that if $H_z = 0$ and $E_z = 0$,

(i) Propagation must be at the velocity of light in the dielectric.

(ii) Both **E** and **H** (which are transverse) satisfy the Laplace equation in x and y.

5.4a Derive Eq. 5.4(7) directly from voltage and current for the load.

5.4b Plot ρ^2 and $1 - \rho^2$ as functions of R_L/Z_0 and note region of reasonable power transfer to the load.

5.5a Analyze, as in Ex.5.5a and with drawings like those in Fig. 5.5*a*, the case of a pulse of length $t_1/5$ reaching a termination at $l = vt_1$ with $R_L = 2Z_0$. Find an expression for the energy dissipated in the load in terms of the voltage of the incident pulse.

5.5b A transmission line of characteristic impedance $Z_{01} = 50\ \Omega$ and length $l = 200$ m is connected to a second line of characteristic impedance $Z_{02} = 100\ \Omega$ and infinite length. Velocity of propagation in both lines is 2×10^8 m/s. Voltage $V_0 = 100$ V is suddenly applied at the input to line 1 at $t = 0$. Sketch current versus distance z at $t = 1.3\ \mu$s. Calculate power in the incident wave, the reflected wave, and the wave transmitted into line 2, showing that there is a power balance.

5.5c Plot the reflected wave from the terminal of computer No. 2, as in Fig. 5.4g, if $Z_0 = 50\ \Omega$ and $R_L = 10\ \Omega$.

5.5d At $t = 0$ a charge distribution is suddenly placed in the central portion of an infinite line as in Ex. 5.5c except that the voltage distribution in z is triangular, with maximum voltage V_0 at $z = 0$, falling to zero at $z = \pm 1$ m. Find voltage and current distributions at $t = 1.667$ ns and at $t \doteq 5$ ns, as in Ex. 5.5c.

5.5e Repeat Ex. 5.5e but with the transmission line terminated with inductor L.

5.5f The problem is as in Ex. 5.5e except that the transmission line continues beyond the capacitor, where it is terminated by its characteristic impedance. Find $V_-(t)$, $V_c(t)$, and $V_2(t)$ in this case, where $V_2(t)$ is the voltage at the input to the continuation transmission line.

5.6a An ideal open ended line of length l is charged to dc voltage V and shorted at its input at time $t = 0$. Sketch the current wave shape through the short as a function of time.

5.6b A charged cable is connected suddenly to a load resistor equal to its 50-Ω characteristic impedance. If its length is 3 m and its phase velocity is one-half the velocity of light, how long is the pulse in the load? Sketch the waveform at the midpoint of the line assuming the cable is initially charged to 100 V and the load is 25 Ω instead of 50.

5.6c The circuit shown in Fig. P5.6*c* is a so-called Blumlein pulse generator (A. T. Starr, *Radio and Radar Technique*, Pitman & Sons, London, 1953) and has the property that it produces a voltage pulse equal to the voltage to which the lines are initially charged by the source V_c. The resistor R_c can be considered essentially infinite. At a time τ after the switch is closed, a voltage V_c appears across the output line terminals 0–0′ and that voltage remains across the terminals 0–0′ for a time 2τ. The initially charged lines are of equal length. Analyze the behavior of the circuit to show the above-described behavior, treating the output line as a lumped resistor $R_L = 2Z_0$.

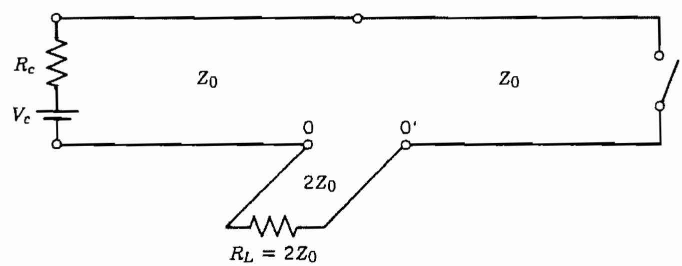


FIG. P5.6*c*

5.7a The alternative approach to derivation of the phasor forms for voltage and current along a transmission line is to replace $\partial/\partial t$ by $j\omega$ in Eqs. 5.2(3) and (4). Write such equations and show that Eqs. 5.7(4) and (5) satisfy them.

5.7b Find the special cases of Eq. 5.7(13) for a shorted line, an open line, a half-wave line with load impedance Z_L, and a quarter-wave line with load impedance Z_L.

5.7c When two transmission lines are to be connected in cascade, a reflection of the wave to be transmitted from one to the other will occur if they do not have the same characteristic impedances. Show that a quarter-wavelength line inserted between the cascaded lines will cause the first line to see its characteristic impedance Z_{01} as a termination and thus eliminate reflection in transfer if $\beta_2 l_2 = \pi/2$ and $Z_{02} = \sqrt{Z_{01}Z_{03}}$, where Z_{02} and Z_{03} are the characteristic impedances of the quarter-wave section and the final line, respectively.

5.7d Derive an expression for a reflection coefficient for current $\rho_I = I_-/I_+$ and show that it differs in phase from the voltage reflection coefficient by π rad.

5.7e A television receiving line of negligible loss is one-third of a wavelength long and has characteristic impedance of 100 Ω. The detuned receiver acts as a load of 100 + j100 Ω. Find the input impedance. Sketch a phasor diagram showing the values of V_+, V_-, I_+, and I_- at both the load and input, and check the calculated results for impedance from this diagram.

5.7f A strip transmission line of characteristic impedance 20 Ω is used at a frequency of 10 GHz with a load that is a microwave diode with conductance 0.05 S in parallel with a 1-pF capacitor. The line is one-eighth wavelength long at the design frequency. Find reflection coefficient at the load and the input admittance.

5.7g If $Z_L << Z_0$ and the line not near a multiple of quarter-wavelength in length, show that the first-order terms of a binomial expansion for Eq. 5.7(13) give the following approximate expression for input impedance:

$$Z_i \approx jZ_0 \tan \beta l + Z_L \sec^2 \beta l$$

5.7h If $Z_L >> Z_0$ and the line not near a multiple of quarter-wavelength in length, show that the first-order terms of a binomial expansion for Eq. 5.7(13) give the following approximate expression for input impedance:

$$Z_i \approx -jZ_0 \cot \beta l + \frac{Z_0^2}{Z_L} \csc^2 \beta l$$

5.8a An impedance of $100 + j100\ \Omega$ is placed as a load on a transmission line of characteristic impedance 50 Ω. Find the reflection coefficient in magnitude and phase and the standing wave ratio of the line.

5.8b Suppose that reflection coefficient is given in magnitude and phase as $|\rho|e^{j\phi}$ at the load at $z = 0$. Find the value of (negative) z for which voltage is a maximum. Show that current is in phase with voltage at this position, so that impedance there is real, as stated. Calculate the position of maximum voltage for the numerical values of Prob. 5.8a.

5.8c A slotted line measurement shows a standing wave ratio of 1.5 with voltage minimum 0.1λ in front of the load. Find magnitude and phase of reflection coefficient at the load and the input impedance for a length 0.2λ of the line.

5.8d An ideal transmission line is terminated by a resistance with value half the characteristic impedance, $R_L = Z_0/2$. What resistance can you put in parallel with the line $\lambda/4$ in front of the load to eliminate reflections on the generator side of that resistance? Can you find a value for such a parallel resistance if the load resistance is $2Z_0$?

5.8e* Give two designs for a power splitter consisting of one 50-Ω input line T-connected to two 50-Ω lines with matched terminations, using quarter-wave transformers (see Prob. 5.7c) as necessary to ensure unity standing wave ratio at the input at the design frequency and an equal power split. Plot power reflected in the input line as a function of frequency.

5.8f Show that Eq. 5.8(1) can be written in phasor notation in the form of a standing wave plus a traveling wave. Rewrite as a real function of time and, for the example in Fig. 5.8, calculate $V(z)$ at a value of ωt shifted in phase by $\pi/4$ rad from ωt_1.

5.9 The Smith chart uses loci of constant resistance and reactance on the reflection coefficient plane. Other charts have used loci of reflection coefficient magnitude and phase plotted on the impedance plane. Show that curves of constant $|\rho|^2$ are circles on the impedance plane, and give radii and position of the centers as functions of $|\rho|^2$. Similarly define the circles corresponding to constant phase of ρ. Explain the advantages of the Smith chart.

5.10a A 50-Ω line is terminated in a load impedance of $75 - j69\ \Omega$. The line is 3.5 m long and is excited by a source of energy at 50 MHz. Velocity of propagation along the line is 3×10^8 m/s. Find the input impedance, the reflection coefficient in magnitude and phase, the value of the standing wave ratio, and the position of a voltage minimum, using the Smith chart.

5.10b The standing wave ratio on an ideal 70-Ω line is measured as 3.2, and a voltage minimum is observed 0.23 wavelength in front of the load. Find the load impedance using the Smith chart.

5.10c Show that the Smith chart for admittance has the form in Fig. 5.10*b*, developing equations corresponding to Eqs. 5.9(1)–(8).

5.10d Repeat Prob. 5.10b using the chart to determine load admittance. Check to see if the result is consistent with the impedance found in Prob. 5.10b.

5.10e A 70-Ω line is terminated in an impedance of $50 + j10\ \Omega$. Find the position and value of a reactance that might be added in series with the line at some point to eliminate reflections of waves incident from the source end. Use the Smith chart.

5.10f Repeat Prob. 5.10e to determine the position and value of shunt susceptance to be placed on the line for matching.

5.10g* A 50-Ω transmission line is terminated with a load of $Z_L = 20 + j30$. A *double-stub tuner* consisting of a pair of shorted 50-Ω transmission lines connected in shunt to the main line at points spaced by 0.25λ is located with one stub at 0.2λ from the load. (See Fig. P5.10*g*.) Find the lengths of the stubs to give unity standing wave ratio for $-z < 0.45\ \lambda$.

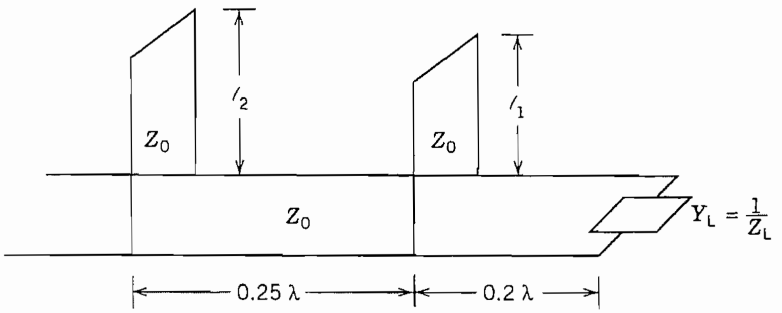


FIG. P5.10*g*

5.10h* Two antennas have impedances of $100 + j100\ \Omega$ for a particular frequency and are fed by transmission lines of characteristic impedance 300 Ω. By inspection of the Smith chart, show that it is possible to choose different lengths for the two feed lines so that when combined in series at the input, the series combination perfectly matches the 300-Ω line to which they are connected. Give the lengths of the two lines in fractions of a wavelength. By study of the procedure you have used, state whether or not this compensation approach will work if the two antennas have arbitrary but equal impedances.

5.10i* The problem is as in Prob. 5.10h except that the two transmission lines are connected in parallel.

5.10j A certain coaxial line has an alternating dielectric of vacuum and a material with $\varepsilon = 4\varepsilon_0$ and $\mu = \mu_0$ and is terminated at the end of a vacuum section by an impedance equal to the characteristic impedance of the vacuum regions. At frequency f_0 the dielectric and vacuum regions are each $\lambda/2$ long (λ appropriate to each region). Show on a Smith chart the path of impedance variation along the line for operating frequencies f_0 and $f_0/2$. Also plot the standing wave ratio as a function of distance from the load.

5.10k* Show on the Smith chart regions of admittance which cannot be matched by the double-stub arrangement of Prob. 5.10g. Repeat for a spacing of $\lambda/8$ between stubs. Repeat for a three-stub arrangement with spacing $\lambda/8$ between stubs.

5.11a Use the formula for input impedance of a transmission line with losses to check Eq. 5.11(22), making approximations consistent with $R/\omega L << 1$ and $G/\omega C << 1$.

5.11b Defining α_c from conductor losses and α_d from dielectric losses by identifying the appropriate parts of Eq. 5.11(13), write the approximate expressions, Eqs. 5.11(14) and (15) for β and Z_0 in terms of α_c and α_d.

5.11c If the transmission line of Ex. 5.11 is made with thinner films, the currents in the metal films can be almost uniform. With that approximation for films 0.2 μm thick, calculate the characteristic impedance and attenuation. Comment on the effects of using thinner films.

5.11d* Show that the variation of complex power along a lossy transmission line carrying sinusoidal waves can be written as $(d/dz)(\frac{1}{2}VI^*) + \frac{1}{2}(RII^* + GVV^*) + (j/2)(XII^* - BVV^*) = 0$, where X and B are the imaginary parts of Z and Y, respectively. Also, show that a direct term-by-term identification with the complex Poynting theorem can be made by multiplying the above expression by dz and considering the volume for Eq. 3.13(6) to be of dz thickness and infinite width.

5.11e Calculate for frequencies 1 MHz and 1 GHz the attenuation in decibels per meter for an air-filled coaxial transmission line with copper conductors using the skin-effect approximations for high-frequency resistance of Ex. 3.17. The inner conductor is a solid cylinder of radius 2 mm and the outer conductor is a thick tubular cylinder of inner radius 1 cm.

5.11f Repeat Prob. 5.11e if the transmission line is now filled with a polystyrene dielectric. The equivalent conductances of polystyrene at 1 Mz and 1 GHz are, respectively, $\sigma_d = 10^{-8}$ S/m and 2.8×10^{-5} S/m.

5.12a For the filter-type circuit studied in Sec. 5.12, find expressions for characteristic impedance Z_0. Show that this is real in the propagating region and imaginary in the attenuating region. What does this signify with respect to power flow in a single traveling wave?

5.12b A certain continuous transmission line has an equivalent circuit consisting of series inductance L_1 H/m and a shunt element consisting of capacitance C F/m and inductance L_2 H · m in parallel. Let $\omega_1^2 = 1/L_2C$ and

(i) Obtain expressions for γ, α,and β in terms of L_1, L_2, ω_1, and ω.
(ii) Plot γ^2 versus ω.
(iii) Replot with ω versus α, where α is real, and versus β, where β is real.

5.13a Write the instantaneous expressions for voltage and current represented by the complex values in Eqs. 5.13(1) and (2). Make an integration of total energy, electric plus magnetic, for a quarter-wavelength of the line and show that it is independent of time.

5.13b Show that $|\rho| = 1$ for a purely reactive load on an ideal transmission line. Find and sketch as in Fig. 5.13 suitably normalized $V(z, t)$ and $I(z, t)$ for a line terminated in a pure inductive reactance equal in magnitude to the characteristic impedance.

5.13c Calculate the instantaneous power flow for a short-circuited line $W_T(t, z) = V(t, z)I(t, z)$ and plot the results in a diagram like Fig. 5.13. Discuss this $W_T(t, z)$ in connection with conclusion 6 reached from (1) and (2) in Sec. 5.13.

5.13d One of the limitations of energy-storage systems for large energies is the breakdown of air, unless the system can be evacuated. For air with breakdown strength 3×10^6 V/m, estimate the maximum energy storage in a resonant air-filled half-wave parallel-plate line. Spacing between plates is 2 cm and characteristic impedance is 50 Ω. Is there any advantage with respect to breakdown over the use of a parallel-plate capacitor? Discuss an inductor as an energy-storage system from the same point of view.

5.14a Plot Eq. 5.14(7) normalized to Y_0 as a function of frequency near where $n = 1$ and see over what range (8) is a reasonable approximation.

5.14b A half-wavelength coaxial transmission line with air dielectric has copper conductors with the dimensions of Prob. 5.11e, and is designed for resonance at 6 GHz. Find the bandwidth $2\Delta f_1$.

5.15a Is phase or group velocity the larger for normal dispersion $(dv_p/d\omega < 0)$? For anomalous dispersion $(dv_p/d\omega > 0)$?

5.15b Find the phase and group velocities for a transmission line with small losses.

5.15c Consider a transmission line with very high leakage conductance G per unit length so that series resistance R and shunt capacitance C are negligible. Find phase and group velocities.

5.15d For Probs. 5.15b and 5.15c, show that an energy velocity as defined by Eq. 5.15(7) is not equal to group velocity.

5.15e Certain water waves of large amplitude have phase velocities given by $v_p = g/\omega$, where g is acceleration due to gravity and ω is angular frequency. Determine the ratio of group velocity to phase velocity for such a wave.

5.15f For Prob. 5.12b, plot v_p and v_g versus ω and show that $v_p v_g = 1/L_1 C$ for all frequencies.

5.16a Plot the ω–β diagram for a distributed transmission system with series inductance L_1 per unit length, and a shunt admittance made up of L_2 in parallel with C_2, for a unit length of the system. Is this a backward or forward wave system? Show cutoff frequency and illustrate phase and group velocity on the plot.

5.16b Repeat Prob. 5.16a for a system made up of a series impedance per unit length of L_1 and C_1 in parallel and the shunt admittance resulting from inductance L_2.

5.16c Calculate the velocity of energy propagation, as defined in Sec. 5.15 for the backward wave line of Fig. 5.16*a*. Show that it does correspond to group velocity for this line and discuss the concepts of normal and anomalous dispersion for backward waves.

5.17a Show that Eqs. 5.17(9) with definitions (10) and (11) do give the solutions of (7) and (8) with the variations (6). Describe ways in which the exponential variation of L and C might be achieved, at least approximately, (i) for a parallel-plane transmission line and (ii) for a coaxial line.

5.17b Show that average power transfer is independent of z in the loss-free exponential line considered here for frequncies above cutoff, $\omega > \omega_c$.

5.17c There are two values of Eqs. 5.17(10) and 5.17(11) representing positively and negatively traveling waves, as expected. Write the complete solutions for $V(z)$ and $I(z)$, showing both waves, using as constants the voltage amplitudes in positively and negatively traveling waves at $z = 0$. Note the interchange of behavior of positive and negative waves if the sign of q is changed, and explain physically.

5.17d Modify the analysis for the exponential line to include losses, retaining constancy of ZY, Z'/Z, and Y'/Y, and interpret the effect of attenuation constant. Assume R_0/X_0 and G_0/B_0 small.