Appendix 5
Solution for Retarded Potentials: Green's Functions

The inhomogeneous wave equation for potential function Φ, Eq. 3.19(7), is

$$\nabla^2\Phi(\mathbf{r}, t) - \mu\varepsilon \frac{\partial^2\Phi(\mathbf{r}, t)}{\partial t^2} = -\frac{\rho(\mathbf{r}, t)}{\varepsilon} \tag{1}$$

Let us take the Fourier transform to obtain

$$\nabla^2\overline{\Phi}(\mathbf{r}, \omega) + \omega^2\mu\varepsilon\overline{\Phi}(\mathbf{r}, \omega) = -\frac{\overline{\rho}(\mathbf{r}, \omega)}{\varepsilon} \tag{2}$$

where

$$\overline{\Phi}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \Phi(\mathbf{r}, t)e^{-j\omega t}dt \tag{3}$$

$$\overline{\rho}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \rho(\mathbf{r}, t)e^{-j\omega t}dt \tag{4}$$

A solution to (2) may be written

$$\overline{\Phi}(\mathbf{r}, \omega) = \int_V \frac{\overline{\rho}(\mathbf{r}, \omega)}{\varepsilon} G(\mathbf{r}, \mathbf{r}')dV' \tag{5}$$

where $G(\mathbf{r}, \mathbf{r}')$, called a Green's function, is the contribution from a unit source at $\mathbf{r}'$ to potential $\overline{\Phi}$ at $\mathbf{r}$. (This is analogous to the impulse response function in circuit theory.) Thus G satisfies the equation

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') + k^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{6}$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac delta function and $k^2 = \omega^2\mu\varepsilon$. Let us next shift the origin to $\mathbf{r}'$ and define $R = |\mathbf{r} - \mathbf{r}'|$. The response is now symmetric about the point source, so (6) in spherical coordinates may be written

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dG}{dR}\right) + k^2 G = -\frac{\delta(R)}{4\pi R^2} \tag{7}$$

(The factor $1/4\pi R^2$ is introduced so that integration over a volume in spherical coordinates gives a unit source.) Multiplication of (7) by R^2 and integration yield

$$R^2 \frac{dG}{dR} + k^2 \int_0^R R^2 G \, dR = -\frac{1}{4\pi} \tag{8}$$

It can be checked that a solution is

$$G = \frac{e^{\pm jkR}}{4\pi R} = \frac{e^{\pm jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{9}$$

A theorem due to Green states that for two functions of space, f and g,

$$\int_V [g\nabla^2 f - f\nabla^2 g] dV = \oint_S \left[g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right] dS \tag{10}$$

where $\partial/\partial n$ represents the outward normal derivative. In (10), let $g = G$ and $f = \overline{\Phi}$.

$$\int_V [G\nabla^2 \overline{\Phi} - \overline{\Phi}\nabla^2 G] dV = \oint_S \left[G \frac{\partial \overline{\Phi}}{\partial n} - \overline{\Phi} \frac{\partial G}{\partial n} \right] dS \tag{11}$$

Surface S now goes to infinity to include all sources. From (9), G decreases as $1/R$ and we anticipate the result that $\overline{\Phi}$ does also. Derivatives $\partial\overline{\Phi}/\partial n$ and $\partial G/\partial n$ then die off as $1/R^2$ and dS increases only as R^2. Thus, the surface integral approaches zero as $S \to \infty$. Use of (2) and (6) in the left side of (11) then gives

$$\int_V \left\{ G\left[-\frac{\overline{\rho}}{\varepsilon} - k^2\overline{\Phi} \right] - \overline{\Phi}[-k^2 G - \delta(\mathbf{r} - \mathbf{r}')] \right\} dV' = 0 \tag{12}$$

but

$$\int \overline{\Phi}\delta(\mathbf{r} - \mathbf{r}') dV' = \overline{\Phi}(\mathbf{r}) \tag{13}$$

so

$$\overline{\Phi}(\mathbf{r}, \omega) = \int_V G(\mathbf{r}, \mathbf{r}') \frac{\overline{\rho}(\mathbf{r}', \omega)}{\varepsilon} dV' = \int_V \frac{\overline{\rho}(\mathbf{r}', \omega) e^{\pm jkR}}{4\pi\varepsilon R} dV' \tag{14}$$

We next take the inverse Fourier transform,

$$\Phi(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\Phi}(\mathbf{r}, \omega) e^{j\omega t} d\omega \tag{15}$$

$$\Phi(\mathbf{r}, t) = \int_V \frac{dV'}{4\pi\varepsilon R} \int_{-\infty}^{\infty} \frac{\overline{\rho}(\mathbf{r}', \omega) e^{j(\omega t \pm kR)}}{2\pi} d\omega \tag{16}$$

But the integral in ω is recognized as $\rho(\mathbf{r}', t \pm kR/\omega)$. So

$$\Phi(\mathbf{r}, t) = \int_V \frac{\rho(\mathbf{r}', t \pm R/v) dV'}{4\pi\varepsilon R}$$

where $v = \omega/k = 1/\sqrt{\mu\varepsilon}$. Only the minus sign is retained under the argument of causality. The plus sign would give a response anticipating the change in the source, which is allowed only in science fiction. This is the result in Eq. 3.20(3).

An exactly parallel development applies to the rectangular components of vector potential **A**, thus leading to the result in Eq. 3.20(4).

The Green's function introduced in (5) measured the response at r arising from an elemental source at r'. It was a free-space Green's function for the wave equation, applicable to a region extending to infinity. The concept applies also to closed regions such as waveguides or cavity resonators. For such applications one would solve the inhomogeneous wave equation (6) subject to the appropriate boundary conditions of the problem. The resulting functions have extensive use in the solution of advanced boundary-value problems.[1–3]

The concept of the Green's function applies to differential equations other than the wave equation, in each case giving an effect at one point arising from an elemental source at another point (or, as in the impulse response of circuit theory, an effect at an instant of time arising from an impulse at an earlier time). The Green's functions for Laplace's equation have had especially extensive study.[4]

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2 *R. E. Collin,* Field Theory of Guided Waves, *2nd ed., IEEE Press, Piscataway, NJ, 1991.*
3 *C. A. Balanis,* Advanced Engineering Electromagnetics, *Chap. 14, Wiley, New York, 1989.*
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