3 Maxwell's Equations

3.1 Introduction

The laws of static electric and magnetic fields have been studied in Chapters 1 and 2. It has been noted that these are useful in predicting effects for many time-varying problems, but there are important dynamic effects not described by the static relations, so other time-varying problems require a more complete formulation. One important dynamic effect is the generation of electric fields by time-varying magnetic fields as expressed through Faraday's law. A second is the complementary effect whereby time-varying electric fields produce magnetic fields. This latter effect is expressed through the concept of *displacement current*, introduced by Maxwell.

Faraday's law is well known to us through its importance in transformers, motors, generators, induction heaters, and similar devices. The effect can be simply demonstrated by moving a coil of wire through the field of a strong permanent magnet and noting the trace on an oscilloscope connected to the coil (Fig. 3.1*a*). With readily available magnets and practical numbers of turns in the coil, movement by hand will generate millivolts, and such voltages are readily observed on the oscilloscope. An alternative demonstration utilizes an electromagnet with its flux threading a fixed coil. A switch to turn on and off the current in the electromagnet causes buildup and decay of the magnetic field and generates the voltage to be observed.

The above-described demonstrations and useful devices utilize induced effects in conductors. An interesting example showing that changing magnetic fields induce electric fields in space is that of the betatron. This useful particle accelerator, illustrated in Fig. 3.1*b*, accelerates electrons or other charged particles by means of a circumferential electric field induced by a changing magnetic field between poles N and S of an electromagnet. The charges are in an evacuated chamber, clearly illustrating that Faraday's law applies in space as well as along conductors.

The second dynamic effect, referred to above as a displacement current effect, is probably best known to us through the concept of a capacitance current. We may, however, think of this only as current in the conductors to capacitor terminals, supplying the time rate of change of charge on capacitor plates. We shall see that the changing electric flux in the dielectric between plates contributes to magnetic fields, just as does conduction current, and acts to complete the current path. Displacement currents also

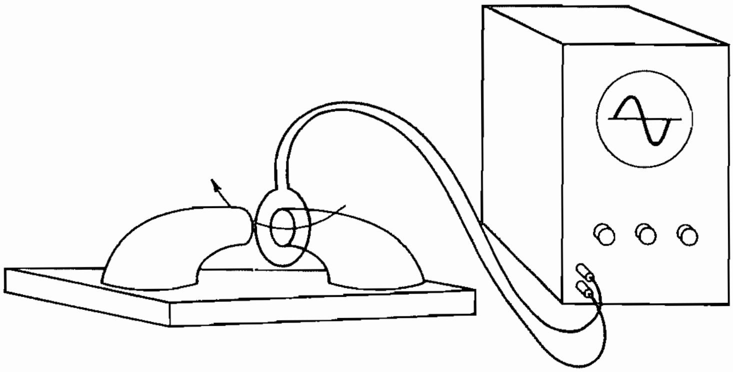

FIG. 3.1*a* Experimental arrangement to demonstrate the induced voltage predicted by Faraday's law. Coil can be moved with sufficient speed by hand to display the induced voltage on a simple oscilloscope.

exist in the vicinity of moving charges, and so are important in vacuum tubes or solid-state electron devices. For example, time-varying effects in the Schottky barrier of Ex. 1.4a or the *pn* junction of Sec. 1.16 produce displacement currents in the respective depletion regions. Effects of these displacement currents must be understood in the analysis of devices using such junctions.

There is a far-reaching consequence of the fact that changing magnetic flux density produces a change in electric field and vice versa: it leads to propagation of electro-

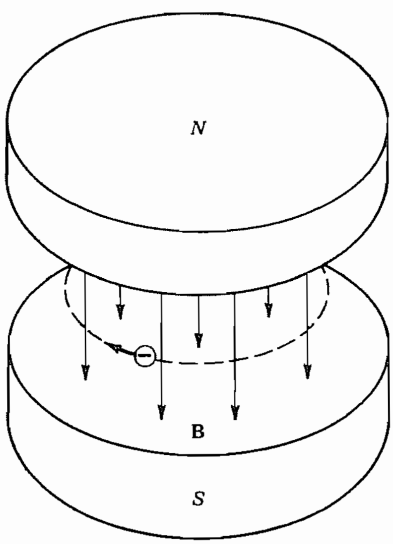

FIG. 3.1*b* Schematic illustration of a betatron, which is used to accelerate electrons by means of an electric field induced by a changing magnetic field.

magnetic waves. In general, wave phenomena result when there are two forms of energy, and the presence of a time rate of change of one leads to a change of the other. In a sound wave, for example, an initial pressure variation in air (potential energy) in one location causes a motion of the air molecules (kinetic energy) that varies both in time and in space. This builds up excess pressure at another position, and the effect continues. Similarly, changing the magnetic field (or flux density) at one position generates a change of electric field in both time and space, by Faraday's law. The subsequent change of electric field produces a change of magnetic field through the displacement current, and so on. In energy terms, the energy interchanges between electric and magnetic types as the wave progresses.

Electromagnetic waves exist in nature in the radiation that takes place when atoms or molecules change from one energy state to a lower one, with frequencies from the microwave through visible into x-ray regions of the spectrum. (Still lower frequencies are generated by lightning and other natural fluctuations.) These natural radiations are utilized in astronomy and radio astronomy. Telecommunications, navigational guidance, radar, and power transmission depend upon our ability to generate, guide, store, radiate, receive, and detect electromagnetic waves. This involves many kinds of structures whose properties the designer must be able to predict. The complete set of laws for time-varying electromagnetic phenomena is known as Maxwell's equations and is central to such predictions.

Large-Scale and Differential Forms of Maxwell's Equations

3.2 Voltages Induced by Changing Magnetic Fields

Faraday discovered experimentally that a voltage is induced in a conducting circuit when the magnetic field linking that circuit is altered. The voltage is proportional to the time rate of change of magnetic flux linking the circuit. For a circuit of n turns, the induced voltage V can be written

$$V = n\frac{d\psi_{\mathrm{m}}}{dt} \tag{1}$$

where ψ_{m} is the magnetic flux linking each turn of the coil. This equation may be used directly to find the voltage induced in the secondary coil of a transformer, for example, or to find the voltage induced in a single circuit because of a time-varying current interacting with the self-inductance of that circuit. For an electric machine, such as a generator or a motor, the change in flux linkages to be used in (1) may be found from the movement of the coil of the machine through a spatially variable magnetic field.

Faraday's experiments included both stationary and moving systems. The question of moving systems may be approached in several ways and will be discussed more in the following section.

One very important generalization of (1) is to a path in space or other nonconducting medium. Such an extension is plausible since the resistance of the path does not appear in the equation. Nevertheless the extension should have experimental verification and it does. Much of the experimental support comes from the wave behavior to be studied in the remainder of the book. As described in Sec. 3.1, the betatron[1] accelerates charged particles in a vacuum by means of an electric field induced by a changing magnetic field, as predicted by Faraday's law. (See Prob. 3.2c.)

Before defining Faraday's law more precisely, we should be clear about several definitions. By *voltage* between two points along a specified path, we mean the negative line integral of electric field between the points along that path. For static fields, we saw that the line integral is independent of the path and equal to the *potential difference* between the two points, but this is not true when there are contributions from Faraday's law. When there is a contribution from changing magnetic flux, the voltage about a closed path is frequently called the *electromotive force (emf)* of that path.

$$\text{emf} \equiv \text{voltage about closed path} \equiv -\oint \mathbf{E} \cdot d\mathbf{l} \tag{2}$$

It is equal, by Faraday's law, to the time rate of change of magnetic flux through the path. For a circuit which is not moving,

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \psi_m}{\partial t} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{3}$$

where the flux ψ_m is found by evaluating the normal component of flux density $\mathbf{B}$ over any surface which has the desired path as its boundary, as indicated by the last term in (3). The negative sign is introduced in the law to agree with the sense relations revealed by experiment, using the usual right-hand convention in the integrals of (3). Thus, as in Fig. 3.2*a*, if the *rate of change* of flux is positive in the directions shown by the vertical arrow, the line integral will be positive in the direction shown—opposite to the conventional right-hand positive direction. If there are several turns, the line integral of (3) is taken about all of them, and if flux through each is the same, we have the form first stated in (1).

[1] *D. W. Kerst and R. Serber,* Phys. Rev. ***60**, 53 (1941).*

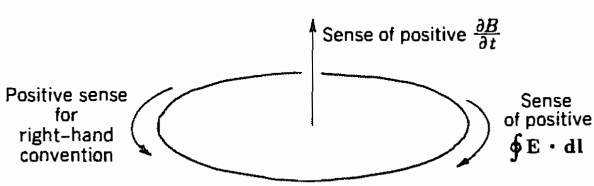

FIG. 3.2*a* Sense relations for Faraday's law.

To transform (3) to differential equation form, we can apply Stokes's theorem (Sec. 2.8) to the left side of (3) and move the time differentiation inside the integral:

$$\int_S (\nabla \times \mathbf{E}) \cdot \mathbf{dS} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} \tag{4}$$

For this equation to be valid for an arbitrary surface, the integrands must be equal so that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

Faraday's law (4) of course reduces to the static case when time derivatives are zero and, as we saw in Sec. 1.7, the line integral of electric field about a closed path is then zero. For the time-varying field it is not in general zero, showing that work can be done in taking a charge about a closed path in such a field. This work comes from the changing stored energy of the magnetic field.

Example 3.2
AIR BREAKDOWN FROM INDUCED ELECTROMOTIVE FORCE

Consider the possibility of an ionizing breakdown in air because of electric fields generated by changing magnetic fields. An axially symmetric electromagnet (Fig. 3.2*b*) has

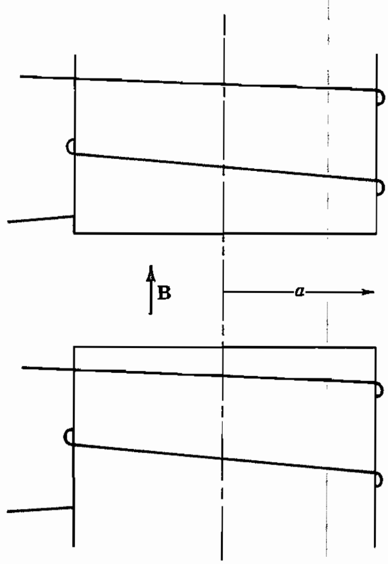

FIG. 3.2*b* Electromagnet with time-varying current producing a time-varying magnetic field.

radius 0.20 m and has essentially uniform field up to this radius and negligible field beyond it. Suppose that it is desired to raise magnetic field from zero to 10 T (tesla) linearly with time in as short a time τ as possible without such breakdown. Because of the axial symmetry we can write Faraday's law for a loop of radius r as

$$2\pi r|E_\phi| = \pi r^2 \frac{\partial B_z}{\partial t} = \pi r^2 \frac{B_{\max}}{\tau} \tag{6}$$

Electric field is thus maximum at the outer radius of 0.20 m. If we take breakdown strength of air as 3×10^6 V/m, then

$$\tau = \frac{rB_{\max}}{2|E_\phi|} = \frac{0.2 \times 10}{2 \times 3 \times 10^6} = \frac{1}{3} \quad \mu\text{s} \tag{7}$$

3.3 Faraday's Law for a Moving System

For the use of Eq. 3.2(1) for a moving system, one must find the change of magnetic flux threading a circuit as it moves through the field. A simple and classical example is that of an elemental ac generator as pictured in Fig. 3.3*a*. This indicates a single rectangular loop rotating at constant angular frequency Ω in the uniform field B_0 between the two pole pieces. When the plane of the loop is at angle ϕ with respect to the horizontal axis, the magnetic flux passing through it is

$$\psi_m = 2B_0 al \sin\phi \tag{1}$$

But angle ϕ changes with time and may be written Ωt. Thus

$$\psi_m = 2B_0 al \sin \Omega t \tag{2}$$

And if voltage is the rate of change of this flux (neglecting signs),

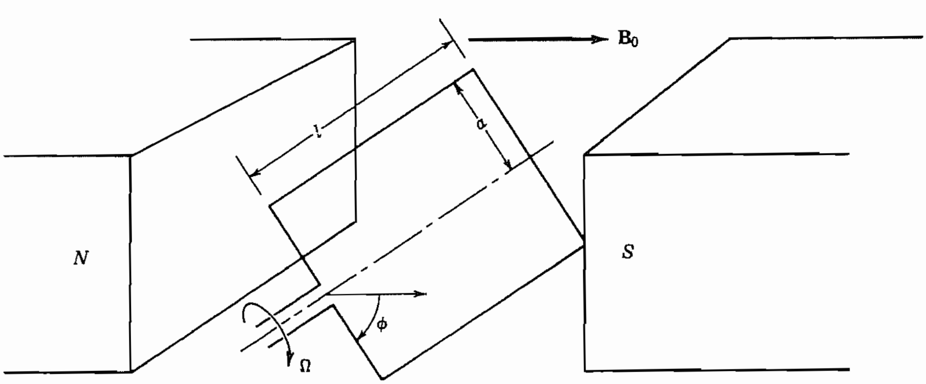

Fig. 3.3a Elemental generator with rotating loop between permanent magnets.

$$V = \frac{d\psi_m}{dt} = 2\Omega B_0 al \cos \Omega t \tag{3}$$

Thus we see the sinusoidal ac voltage produced by this basic generator. Now let us introduce a slightly different point of view to get the same answer. A point of view used effectively by Faraday is that the electric field of the moving conductor is generated by its motion through, and hence "cutting" of, the lines of force. Faraday gave much physical sigificance to the flux tubes and lines of force. This point of view can be developed rigorously by writing the time derivative on the right side of Eq. 3.2(3) as a total derivative instead of a partial derivative:

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS} \tag{4}$$

For a closed path moving in space with velocity $\mathbf{v}$, this may be transformed by a vector transformation developed by Helmholtz,[2] which, with $\nabla \cdot \mathbf{B} = 0$, is

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\int_S \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B})\right] \cdot \mathbf{dS} \tag{5}$$

The first term on the right is the one we have seen before. The second term gives an added contribution to emf and, by use of Stokes's theorem, may be written as the line integral of $\mathbf{v} \times \mathbf{B}$ about the closed path. The result may be interpreted as a *motional electric field* given at each point of the circuit path by

$$\mathbf{E}_m = \mathbf{v} \times \mathbf{B} \tag{6}$$

In the example of Fig. 3.3*a*, the motional field in the upper conductor, by (6), is

$$E_{mz} = (a\Omega)B_0 \cos \phi \tag{7}$$

That in the lower conductor is the negative of this. There is no motional field along the side elements since $\mathbf{v} \times \mathbf{B}$ gives a contribution normal to the wires for these sides. Thus the line integral about the loop yields

$$\oint \mathbf{E} \cdot \mathbf{dl} = la\Omega B_0 \cos \Omega t - (-la\Omega B_0 \cos \Omega t) = 2la\Omega B_0 \cos \Omega t \tag{8}$$

which is identical to (3).

The differential form of Faraday's law, Eq. 3.2(5), may be transformed to a set of moving coordinates with the same result. This may be done by a Galilean transformation for low velocities and by a Lorentz transformation for relativistic velocities.[2] Although relativity is beyond the scope of this text, it is important to know that Maxwell's equations are consistent with the theory of relativity, although Einstein developed that theory later. Their invariance to Lorentz transformations, in fact, had much to do with the development of the theory of special relativity.

[2] See, for example, C. T. Tai, Proc. IEEE **60**, 936 (1972).

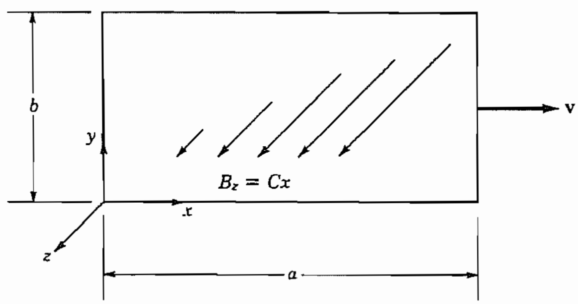

FIG. 3.3*b* Rectangular loop of wire moving through magnetic field which varies with distance.

Example 3.3

RECTANGULAR LOOP MOVING THROUGH INHOMOGENEOUS FIELD

If a loop of wire is moved through a region of static magnetic field which is a function of position, the flux threading the loop changes as the loop moves and an emf is generated. Consider the rectangular loop of wire (Fig. 3.3*b*) translated in the x direction with velocity v through a z-directed static magnetic field which varies linearly with x, $B_z = Cx$. If the left-hand edge is at $x = 0$ at $t = 0$, it is at $x = vt$ at time t, and the magnetic flux threading the loop is

$$\int_S \mathbf{B} \cdot d\mathbf{S} = b \int_{vt}^{vt+a} Cx\,dx = bC \left.\frac{x^2}{2}\right|_{vt}^{vt+a} = \frac{baC}{2}(2vt + a) \tag{9}$$

The induced emf is then the time rate of change of this flux,

$$-\oint \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} = baCv \tag{10}$$

This result can be checked by finding the motional field in the four sides using (6). The field $\mathbf{v} \times \mathbf{B}$ is normal to the wires along the top and bottom. On the left it is $-vCx$ and on the right side, $-vC(x + a)$. Thus the integral is

$$-\oint \mathbf{E} \cdot d\mathbf{l} = vCb(x + a) - vCbx = baCv \tag{11}$$

agreeing with the result (10).

3.4 CONSERVATION OF CHARGE AND THE CONCEPT OF DISPLACEMENT CURRENT

Faraday's law is but one of the fundamental laws for changing fields. Let us assume for the moment that certain of the laws derived for static fields in Chapters 1 and 2 can be extended simply to time-varying fields. We will write the divergence of electric and magnetic fields in exactly the same form as in statics, with the understanding that all field and source quantities are functions of time as well as of space. For the curl of electric field we take the result of Faraday's law, Eq. 3.2(5). For the curl of magnetic field, we take for the time being the form from statics, Eq. 2.7(2).

$$\nabla \cdot \mathbf{D} = \rho \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{4}$$

An elimination among these equations can be made to give an equation relating charge and current. We would expect this to show that however ρ varies with space or time, total charge is conserved. If current flows out of any volume, the amount of charge inside must decrease, and if current flows in, charge inside increases. Considering a smaller and smaller volume, in the limit the outward flow of current per unit time and per unit volume (which is recognized as the divergence of current density) must give the negative of the time rate of change of charge per unit volume at that point:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{5}$$

If, however, we take the divergence of $\mathbf{J}$ from (4),

$$\nabla \cdot \mathbf{J} = \nabla \cdot (\nabla \times \mathbf{H}) \equiv 0$$

which does not agree with the continuity argument and (5). Maxwell, by reasoning similar to this, recognized that (4), borrowed from statics, is not complete for time-varying fields. He postulated an added term $\partial \mathbf{D}/\partial t$:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{6}$$

Continuity is now satisfied, as may be shown by taking the divergence of (6) and substituting from (1):

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = -\frac{\partial \rho}{\partial t}$$

The term added to form (6) contributes to the curl of magnetic field in the same way as an actual conduction current density (motion of charges in conductors) or convection current density (motion of charges in space). Because it arises from the displacement vector $\mathbf{D}$, it has been named the *displacement current* term. There is an actual time-varying displacement of bound charges in a material dielectric, but note that displacement current can be nonzero even in a vacuum. Thus (6) could be written

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \mathbf{J}_d \tag{7}$$

where $\mathbf{J}_c$ = conduction or convection current density in amperes per square meter and $\mathbf{J}_d$ = displacement current density = $\partial\mathbf{D}/\partial t$ amperes per square meter.

The displacement current term is important within the dielectric of a capacitor whenever the capacitive voltage changes with time. It also always plays a role when moving charges induce currents in nearby electrodes. Both of these phenomena will be explored in the following section. Displacement current is negligible for many other low-frequency problems. For example, it is negligible in comparison with conduction currents in good conductors up to optical frequencies. (This point will be explored more in Sec. 3.16.) But displacement current becomes important in more and more situations as the frequency of time-varying phenomena is increased. It is essential, along with the Faraday law terms for electric field, to the understanding of all electromagnetic wave phenomena.

3.5 Physical Pictures of Displacement Current

The displacement current term enables one to explain certain things that would have proved inconsistent had only conduction or convection current been included in the magnetic field laws. Consider, for example, the circuit including the ac generator and the capacitor of Fig. 3.5*a*. Suppose that it is required to evaluate the line integral of magnetic field around the loop a–b–c–d–a. The law from statics states that the result obtained should be the current enclosed, that is, the current through any surface of which the loop is a boundary. If we take as the arbitrary surface through which current is to be evaluated one which cuts the wire A, as does S_1, a finite value is clearly obtained for the line integral. But suppose that the surface selected is one which does not cut the wire, but instead passes between the plates of the capacitor, as does S_2. If conduction current alone were included, the computation would have indicated no current passing through this surface and the result would be zero. The path around which the integral is evaluated is the same in each case, and it would be quite annoying to possess two different results. It is the displacement current term which appears at this point to

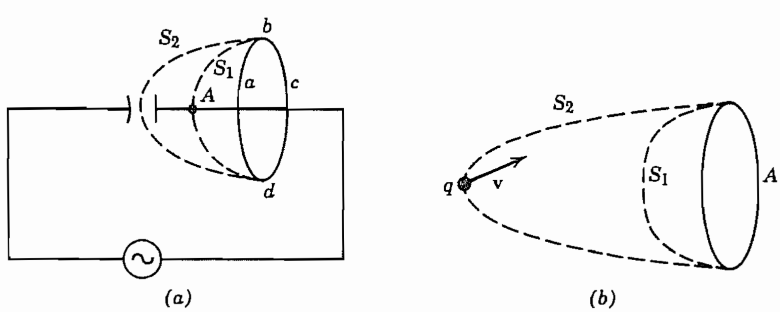

Fig. 3.5 Illustrations of how displacement current completes the circuit: (*a*) in a circuit with capacitor; (*b*) near a moving charge.

preserve the continuity of current between the plates of the capacitor, giving the same answer in either case.

To show how this continuity is preserved, consider an ideal parallel-plate capacitor of capacitance C, spacing d, area of each plate A, and applied voltage $V_0 \sin \omega t$. From circuit theory the charging current is

$$I_c = C\frac{dV}{dt} = \omega C V_0 \cos \omega t \tag{1}$$

The field inside the capacitor has a magnitude $E = V/d$, so the displacement current density is

$$J_d = \varepsilon \frac{\partial E}{\partial t} = \omega\varepsilon \frac{V_0}{d} \cos \omega t \tag{2}$$

Total displacement current flowing between the plates is the area of the plate multiplied by the density of displacement current:

$$I_d = AJ_d = \omega\left(\frac{\varepsilon A}{d}\right)V_0 \cos \omega t \tag{3}$$

The factor in parentheses is recognized as the electrostatic capacitance for the ideal parallel-plate capacitor, so (1) and (3) are equal. This value for total displacement current flowing between the capacitor plates is then exactly the same as the value of charging current flowing in the leads, calculated by the usual circuit methods above, so the displacement current does act to complete the circuit, and the same result would be obtained by the use of either S_1 or S_2 of Fig. 3.5*a*, as required.

Inclusion of displacement current is necessary for a valid discussion of another example in which a charge region q (Fig. 3.5*b*) moves with velocity **v**. If the line integral of magnetic field is to be evaluated about some loop A at a given instant, it should be possible to set it equal to the current flow for that instant through any surface of which A is a boundary. If the displacement current term were ignored, we could use any one of the infinite number of possible surfaces, as S_1, having no charge passing through, and obtain the result zero. If one of the surfaces is selected, as S_2, through which charge is passing at that instant, however, there is a contribution from convection current and a nonzero result. The apparent inconsistency is resolved when one notes that the electric field arising from the moving charge must vary with time, and thus will actually give rise to a displacement current term through both of the surfaces S_1 and S_2. The sum of displacement and convection currents for the two surfaces is the same at the given instant.

Example 3.5
Currents Induced by a Slab of Charge Moving in a Planar Diode

In a planar vacuum diode, as sketched in Fig. 3.5*c*, the cathode has been pulsed to produce a slab of charge moving from cathode to anode. The density of charge is taken

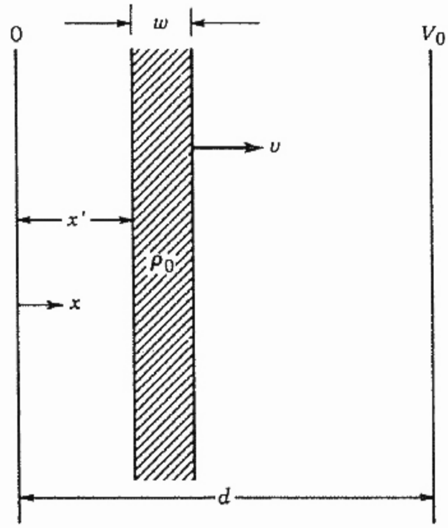

FIG. 3.5c Slab of charge moving between parallel plates.

as a uniform ρ_0. Width is w and at time t the left-hand edge is at $x = x'$ moving with velocity v. We note that the electric field E_{x1} is independent of x for $x < x'$, varies linearly for $x' < x < x' + w$, and is again independent of x for $x > x' + w$. Its integral is

$$-V_0 = E_{x1}d + \frac{\rho_0}{\varepsilon}w(d - x') - \frac{\rho_0 w^2}{2\varepsilon}$$

Then the electric fields for the three regions $0 < x < x'$, $x' < x < x' + w$, and $x' + w < x < d$ are, respectively (Prob. 3.5b),

$$E_{x1} = -\frac{V_0}{d} + \frac{\rho_0}{\varepsilon}\left[\frac{x'w}{d} + w\left(\frac{w}{2d} - 1\right)\right] \tag{4}$$

$$E_{x2} = -\frac{V_0}{d} + \frac{\rho_0}{\varepsilon}\left[x'\left(\frac{w}{d} - 1\right) + w\left(\frac{w}{2d} - 1\right) + x\right] \tag{5}$$

$$E_{x3} = -\frac{V_0}{d} + \frac{\rho_0 w}{\varepsilon d}\left(x' + \frac{w}{2}\right) \tag{6}$$

But x' is a function of time and differentiation with respect to time gives velocity v. Thus displacement current density for the three regions is

$$\varepsilon\frac{\partial E_{x1}}{\partial t} = \rho_0\frac{w}{d}\frac{\partial x'}{\partial t} = \frac{\rho_0 w v}{d} \tag{7}$$

$$\varepsilon\frac{\partial E_{x2}}{\partial t} = \rho_0 v\left(\frac{w}{d} - 1\right) \tag{8}$$

$$\varepsilon\frac{\partial E_{x3}}{\partial t} = \frac{\rho_0 w}{d}\frac{\partial x'}{\partial t} = \frac{\rho_0 w v}{d} \tag{9}$$

To the displacement current density of the region within the charge, given by (8), we add the convection current density $\rho_0 v$ so that the sum of convection and displacement currents is the same for each region, and this will also be the current per unit area induced in the plane electrodes.

3.6 MAXWELL'S EQUATIONS IN DIFFERENTIAL EQUATION FORM

Rewriting the group of equations of Sec. 3.4 with the displacement current term added, we have

$$\nabla \cdot \mathbf{D} = \rho \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{4}$$

This set of equations, together with certain auxiliary relations and definitions, is the basic set of equations of classical electricity and magnetism, governing all electromagnetic phenomena in the range of frequencies from zero through the highest-frequency radio waves (and many phenomena at light frequencies) and in the range of sizes above atomic size. The equations were first written (not in the above notation) by Maxwell in 1863 and are known as Maxwell's equations. The material in the sections preceding this should not be considered a derivation of the laws, for they cannot in any real sense be derived from less fundamental laws. Their ultimate justification comes, as with all experimental laws, in that they have predicted correctly, and continue to predict, all electromagnetic phenomena over a wide range of physical experience.

The foregoing set of equations is a set of differential equations, relating the time and space rates of change of the various field quantities at a point in space and time. The use of these will be demonstrated in the following chapters. Equivalent large-scale equations will be given in the following section.

The major definitions and auxiliary relations that must be added to complete the information are as follows:

1. *Force Law* This is, from one point of view, merely the definition of the electric and magnetic fields. For a charge q moving with velocity $\mathbf{v}$ through an electric field $\mathbf{E}$ and a magnetic field of flux density $\mathbf{B}$, the force is

$$\mathbf{f} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \quad \text{N} \tag{5}$$

2. *Definition of Conduction Current (Ohm's Law)* For a conductor,

$$\mathbf{J} = \sigma \mathbf{E} \quad \text{A/m}^2 \tag{6}$$

where σ is conductivity in siemens/meter.

3. *Definition of Convection Current* For a charge density ρ moving with velocity $\mathbf{v}_\rho$, the current density is

$$\mathbf{J} = \rho \mathbf{v}_\rho \quad \text{A/m}^2 \tag{7}$$

4. *Definition of Permittivity (Dielectric Constant)* The electric flux density $\mathbf{D}$ is related to the electric field intensity $\mathbf{E}$ by the relation

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E} \tag{8}$$

where ε_0 is the permittivity of space $\approx 8.854 \times 10^{-12}$ F/m and ε_r characterizes the effect of the atomic and molecular dipoles in the material.

As with static fields (Sec. 1.3) ε, or ε_r, is, in the most general case, anisotropic and a function of space, time, and the strength of the applied field. But for many materials it is a scalar constant, and unless specifically noted otherwise, the text will be concerned with homogeneous, isotropic, linear, and time-invariant materials for which ε is a scalar constant.

5. *Definition of Permeability* The magnetic flux density $\mathbf{B}$ is related to the magnetic intensity $\mathbf{H}$ by

$$\mathbf{B} = \mu \mathbf{H} = \mu_r \mu_0 \mathbf{H} \tag{9}$$

where μ_0 is the permeability of space $= 4\pi \times 10^{-7}$ H/m and μ_r measures the effect of the magnetic dipole moments of the atoms constituting the medium (Sec. 2.3). In general μ and μ_r are anisotropic and functions of space, time, and magnetic field strength, but unless otherwise noted they will be considered scalar constants, representing homogeneous, isotropic, linear, and time-invariant materials.

Example 3.6
Nonarbitrariness of Forms Which Satisfy Maxwell's Equations

Much of the work for the remainder of the text will be in finding forms that are solutions of Maxwell's equations. The interrelationship among electric and magnetic field components defined by Maxwell's equations means that we cannot select arbitrary functions for any one component. To illustrate the point, let us consider a capacitor formed by concentric spherical conductors with an ideal dielectric between. As in Ex. 1.4*c*, Gauss's law and symmetry give the electrostatic solution for the dielectric region as

$$\mathbf{D} = \varepsilon \mathbf{E} = \hat{\mathbf{r}} \frac{Q}{4\pi r^2} \tag{10}$$

where Q is the charge on the inner sphere. For a sinusoidally time-varying charge, one might expect the solution

$$\mathbf{E} = \hat{\mathbf{r}} \frac{Q_0}{4\pi\varepsilon r^2} \sin \omega t \tag{11}$$

A check of $\nabla\cdot(\varepsilon\mathbf{E})$ in spherical coordinates shows that it is zero, as expected for the charge-free dielectric. But consider the Maxwell equation

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{12}$$

The curl of an electric field of the form (11) is zero, so $\mathbf{H}$ can only be a function independent of time. But then the other curl equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{13}$$

cannot be satisfied since $\mathbf{J}$ is zero for the ideal dielectric, $\partial\mathbf{E}/\partial t$ by (11) is time-varying but $\nabla \times \mathbf{H}$ is independent of time. Thus (11), though it may be a useful quasistatic approximation, is not a true solution of Maxwell's equations. Proper solutions in spherical form are discussed in Chapter 10.

3.7 MAXWELL'S EQUATIONS IN LARGE-SCALE FORM

It is also convenient to have the information of Maxwell's equations in large-scale or integral form applicable to overall regions of space and paths of finite size. This is of course the type of relation that we started with in the discussion of Faraday's law (Sec. 3.2) when we derived the differential expression from it. The large-scale equivalents for Eqs. 3.6(1)–3.6(4) are

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho \, dV \tag{1}$$

$$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0 \tag{2}$$

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{dS} \tag{3}$$

$$\oint \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{\partial}{\partial t} \int_S \mathbf{D} \cdot \mathbf{dS} \tag{4}$$

Equations (1) and (2) are obtained by integrating respectively Eqs. 3.6(1) and 3.6(2) over a volume and applying the divergence theorem. Equations (3) and (4) are obtained by integrating, respectively, Eqs. 3.6(3) and 3.6(4) over a surface and applying Stokes's theorem. For example, integrating Eq. 3.6(1),

$$\int_V \nabla \cdot \mathbf{D} \, \mathrm{dV} = \int_V \rho \, dV$$

and applying the divergence theorem to the left-hand side, (1) follows. Equation (1) is seen to be the familiar form of Gauss's law utilized so much in Chapter 1. Now that we are concerned with fields that are a function of time, the interpretation is that the electric flux flowing out of any closed surface at a given instant is equal to the charge enclosed by the surface at that instant.

Equation (2) states that the surface integral of magnetic field or total magnetic flux flowing out of a closed surface is zero for all values of time, expressing the fact that magnetic charges have not been found in nature. Of course, the law does not prove that such charges will never be found; if they are, a term on the right similar to the electric charge term in (1) will simply be added, and a corresponding magnetic current term will be added to (3). We will later find situations in which fictitious magnetic charges and currents will be helpful and may be added to the equations.

Equation (3) is Faraday's law of induction, stating that the line integral of electric field about a closed path (electromotive force) is the negative of the time rate of change of magnetic flux flowing through the path. The law was discussed in some detail in Sec. 3.2.

Equation (4) is the generalized Ampère's law including Maxwell's displacement current term, and it states that the line integral of magnetic field about a closed path (magnetomotive force) is equal to the total current (conduction, convection, and displacement) flowing through the path. The physical significance of this complete law has been discussed in Secs. 3.4–3.5.

3.8 MAXWELL'S EQUATIONS FOR THE TIME-PERIODIC CASE

By far the most important time-varying case is that involving steady-state ac fields varying sinusoidally in time. Many engineering applications use sinusoidal fields. Other functions of time, such as the pulses utilized in a digitally coded system, may be considered a superposition of steady-state sinusoids of different frequencies. Fourier analysis (Fourier series for periodic functions and the Fourier integral for aperiodic functions) provides the mathematical basis for this superposition. Rather than using real sinusoidal functions directly, it is found convenient to introduce the complex exponential $e^{j\omega t}$. Electrical engineers are familiar with the advantages of this approach in the analysis of ac circuits, and physicists use the complex exponential in a variety of physical problems with sinusoidal behavior. The advantage, which comes from the fact that derivatives and integrals of $e^{j\omega t}$ are proportional to $e^{j\omega t}$ so that the function can be canceled from all equations, is even more important for the vector field problems than for scalar problems such as the circuit example. It is assumed that the reader has used this technique before in circuit analysis or other physical problems, but if review is needed, the use in analysis of a simple electrical circuit may be found in Appendix 4. Formally, the set of equations 3.6(1)–3.6(4) is easily changed to the complex form by replacing $\partial/\partial t$ by $j\omega$:

$$\nabla \cdot \mathbf{D} = \rho \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} \tag{3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} \tag{4}$$

And the auxiliary relations, Eqs. 3.6(6)–3.6(9), remain

$$\mathbf{J} = \sigma\mathbf{E} \quad \text{for conductors} \tag{5}$$

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_r\varepsilon_0\mathbf{E} \tag{6}$$

$$\mathbf{B} = \mu\mathbf{H} = \mu_r\mu_0\mathbf{H} \tag{7}$$

Equations 3.6(5) and 3.6(7) should be used with instantaneous values because of the nonlinear terms in the equations. The constitutive parameters, μ and ε, are in general functions of frequency. Materials for which frequency dependence is important are called *dispersive.*

It must be recognized that the symbols in the equations of this article have a different meaning from the same symbols used in Sec. 3.6. There they represented the instantaneous values of the indicated vector and scalar quantities. Here they represent the complex multipliers of $e^{j\omega t}$, giving the in-phase and out-of-phase parts with respect to the chosen reference. The complex scalar quantities are commonly referred to as phasors, and by analogy the complex vector multipliers of $e^{j\omega t}$ may be called vector phasors. It would seem less confusing to use a different notation for the two kinds of quantities, but one quickly runs out of symbols. The difference is normally clear from the context, and when there is danger of confusion, we will use functional notation to denote the time-varying quantities.

If we wish to obtain the instantaneous values of a given quantity from the complex value, we insert the $e^{j\omega t}$ and take the real part. For example, for the scalar ρ suppose that the complex value of ρ is

$$\rho = \rho_r + j\rho_i \tag{8}$$

where ρ_r and ρ_i are real scalars. The instantaneous value of ρ is then

$$\rho(t) = \mathrm{Re}[(\rho_r + j\rho_i)^{j\omega t}] = \rho_r \cos \omega t - \rho_i \sin \omega t \tag{9}$$

Or, alternatively, if ρ is given in magnitude and phase,

$$\rho = |\rho|e^{j\theta_\rho} \tag{10}$$

where

$$|\rho| = \sqrt{\rho_r^2 + \rho_i^2}$$

$$\theta_\rho = \tan^{-1}\frac{\rho_i}{\rho_r}$$

The true time-varying form is

$$\rho(t) = \mathrm{Re}[|\rho|e^{j(\omega t + \theta_\rho)}] = |\rho|\cos(\omega t + \theta_\rho) \tag{11}$$

For a vector quantity, such as $\mathbf{E}$, the complex value may be written

$$\mathbf{E} = \mathbf{E}_r + j\mathbf{E}_i \tag{12}$$

where $\mathbf{E}_r$ and $\mathbf{E}_i$ are real vectors. Then

$$\mathbf{E}(t) = \text{Re}[(\mathbf{E}_r + j\mathbf{E}_i)e^{j\omega t}] = \mathbf{E}_r \cos \omega t - \mathbf{E}_i \sin \omega t \tag{13}$$

Note that $\mathbf{E}_r$ and $\mathbf{E}_i$ have the same directions in space only for certain special cases. When they are in the same direction, the vector phasor (12) can be expressed as a vector "magnitude" and a scalar phase angle, but in the general case, when they are in different directions, the six scalar quantities defining the two vectors must be specified (Prob. 3.8a).

Example 3.8
A Phasor Solution of Maxwell's Equations

As an example of phasor solutions, let us consider the following fields, which we will later find to be important as standing waves:

$$B_z = -jD_0 \sin(\omega\sqrt{\mu_0\varepsilon_0}x) \tag{14}$$

$$E_y = \frac{D_0}{\sqrt{\mu_0\varepsilon_0}} \cos(\omega\sqrt{\mu_0\varepsilon_0}x) \tag{15}$$

Let us show that these do satisfy the phasor forms of Maxwell's equations. The needed rectangular coordinate components of (3) and (4), with $J = 0$, are

$$\frac{\partial E_y}{\partial x} = -j\omega B_z \tag{16}$$

$$\frac{\partial H_z}{\partial x} = \frac{1}{\mu_0}\frac{\partial B_z}{\partial x} = -j\omega\varepsilon_0 E_y \tag{17}$$

Substitution of (14) and (15) gives

$$-\frac{\omega\sqrt{\mu_0\varepsilon_0}}{\sqrt{\mu_0\varepsilon_0}} D_0 \sin(\omega\sqrt{\mu_0\varepsilon_0}x) = (-j)^2\omega D_0 \sin(\omega\sqrt{\mu_0\varepsilon_0}x) \tag{18}$$

$$-\frac{jD_0\omega\sqrt{\mu_0\varepsilon_0}}{\mu_0} \cos(\omega\sqrt{\mu_0\varepsilon_0}x) = -\frac{j\omega\varepsilon_0 D_0}{\sqrt{\mu_0\varepsilon_0}} \cos(\omega\sqrt{\mu_0\varepsilon_0}x) \tag{19}$$

The divergences of (14) and (15) are also found to be zero:

$$\frac{\partial B_z}{\partial z} \equiv 0 \text{ and } \frac{\partial E_y}{\partial y} \equiv 0 \tag{20}$$

So the forms (14) and (15) are solutions of the phasor Maxwell equations for this source-free region. If we wish the time-varying forms, we insert $e^{j\omega t}$ and take the real part:

$$B_z(x, t) = \mathrm{Re}[B_z e^{j\omega t}] = D_0 \sin(\omega\sqrt{\mu_0\varepsilon_0}x) \sin \omega t \tag{21}$$

$$E_y(x, t) = \mathrm{Re}[E_y e^{j\omega t}] = \frac{D_0}{\sqrt{\mu_0\varepsilon_0}} \cos(\omega\sqrt{\mu_0\varepsilon_0}x) \cos \omega t \tag{22}$$

Examples of Use of Maxwell's Equations

3.9 MAXWELL'S EQUATIONS AND PLANE WAVES

To make the information of Maxwell's equations still more concrete, let us show how the equations predict the propagation of uniform plane electromagnetic waves. Such waves illustrate the interplay of electric and magnetic effects and are also of great fundamental and practical importance. Let us begin from the time-varying forms of Sec. 3.6. We postulate a simple medium with constant, scalar permittivity and permeability and with no free charges and currents ($\rho = 0$, $\mathbf{J} = 0$). Maxwell's equations are then

$$\nabla \cdot \mathbf{D} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

For uniform plane waves, we assume variation in only one direction. Take this as the z direction of a rectangular coordinate system. Then $\partial/\partial x = 0$ and $\partial/\partial y = 0$. Let us start with the two curl equations (3) and (4) in rectangular coordinates. With the specialization defined above,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{leads to} \quad \begin{cases} -\dfrac{\partial E_y}{\partial z} = -\mu \dfrac{\partial H_x}{\partial t} & (5) \\[2ex] \dfrac{\partial E_x}{\partial z} = -\mu \dfrac{\partial H_y}{\partial t} & (6) \\[2ex] 0 = -\mu \dfrac{\partial H_z}{\partial t} & (7) \end{cases}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \text{leads to} \quad \begin{cases} -\dfrac{\partial H_y}{\partial z} = \varepsilon \dfrac{\partial E_x}{\partial t} & (8) \\[2ex] \dfrac{\partial H_x}{\partial z} = \varepsilon \dfrac{\partial E_y}{\partial t} & (9) \\[2ex] 0 = \varepsilon \dfrac{\partial E_z}{\partial t} & (10) \end{cases}$$

Equations (7) and (10) show that the time-varying parts of H_z and E_z are zero. Thus the fields of the wave are entirely transverse to the direction of propagation. The remaining equations break into two independent sets, with (5) and (9) relating E_y and H_x, and (6) and (8) relating E_x and H_y.

The propagation behavior is illustrated by either set. Choose the set with E_x and H_y, differentiating (6) partially with respect to z and (8) with respect to t:

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu \frac{\partial^2 H_y}{\partial z\, \partial t}, \qquad -\frac{\partial^2 H_y}{\partial t\, \partial z} = \varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

Substitution of the second equation into the first yields

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \tag{11}$$

The important partial differential equation (11) is a classical form known as the *one-dimensional wave equation*, having solutions that demonstrate propagation of a function (a "wave") in the z direction with velocity

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{12}$$

To show this, test a solution of the form

$$E_x(z, t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) \tag{13}$$

Differentiating,

$$\frac{\partial E_x}{\partial t} = f_1' + f_2' \qquad \frac{\partial E_x}{\partial z} = -\frac{1}{v} f_1' + \frac{1}{v} f_2'$$

$$\frac{\partial^2 E_x}{\partial t^2} = f_1'' + f_2'' \qquad \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2} f_1'' + \frac{1}{v^2} f_2''$$

where the prime denotes differentiation of the function with respect to the entire argument, and the double prime denotes the corresponding second derivative. Comparison of the two second derivatives shows that (11) is satisfied by such a solution with v given by (12). The first term of the solution in (13) represents a function f_1 moving in the z direction with velocity v. To show this consider the function $f_1(z)$ at various times as illustrated in Fig. 3.9. To keep on a constant reference of this wave, we must maintain

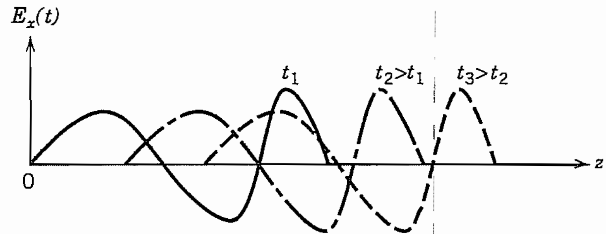

FIG. 3.9 A general wave of electric field versus distance for three different times.

the argument $t - z/v$ equal to a constant. This implies a velocity $dz/dt = v$. Similarly, to keep on a constant reference of the second term in (12) we must keep $t + z/v$ a constant, implying a velocity $dz/dt = -v$. Thus the second term represents the function f_2 traveling in the negative z direction with velocity v. These moving functions may be thought of as "waves," so that the name "wave equation" is explained.

The velocity v defined by (12) is found to be the velocity of light for the medium. In particular, for free space

$$v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = (4\pi \times 10^{-7} \times 8.85419 \times 10^{-12})^{-1/2} \\ = 2.9979 \times 10^8 \text{ m/s} \tag{14}$$

(Note that to three significant figures, this is the conveniently remembered value 3×10^8 m/s, corresponding to ε_0 taken as $1/36\pi \times 10^{-9}$ F/m.) This equivalence between the velocity of light and the predicted velocity of electromagnetic waves helped Maxwell to establish light as an electromagnetic phenomenon.

For a medium with relative permittivity ε_r and relative permeability μ_r, the velocity of the plane wave is then

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \tag{15}$$

Example 3.9
SINUSOIDAL WAVE

The most common and useful wave solution is one varying sinusoidally in space and time. Consider the function

$$E_x(z, t) = A \sin \omega\left(t - \frac{z}{v}\right) \tag{16}$$

which is a special case of (13). To show that it satisfies (11), perform the differentiations:

$$\frac{\partial^2 E_x}{\partial z^2} = -\frac{\omega^2}{v^2} A \sin \omega\left(t - \frac{z}{v}\right) \tag{17}$$

$$\mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} = -\mu\varepsilon\omega^2 A \sin \omega\left(t - \frac{z}{v}\right) \tag{18}$$

But since $v^2 = (\mu\varepsilon)^{-1}$, this is a solution. To show that it is a "wave," we see that we can stay on a maximum or crest of the function if we set

$$\omega\left(t - \frac{z}{v}\right) = (4n + 1)\frac{\pi}{2}, \qquad n = 0, 1, 2, \ldots$$

or

$$z = vt - \frac{(4n + 1)\pi v}{2\omega} \tag{19}$$

so that the crest does move in the z direction with velocity v as time progresses.

3.10 Uniform Plane Waves with Steady-State Sinusoids

To show the usefulness of the complex phasor approach for steady-state sinusoids, let us continue with this important special case. Replacement of Eqs. 3.9(6) and 3.9(8) with the complex phasor equivalents, obtained by replacing time derivatives with $j\omega$, yields

$$\frac{dE_x}{dz} = -j\omega\mu H_y \tag{1}$$

$$-\frac{dH_y}{dz} = j\omega\varepsilon E_x \tag{2}$$

Here we have utilized the total derivative with respect to z since that is now the only variable. Differentiation of (1) with respect to z and substitution of (2) yield

$$\frac{d^2 E_x}{dz^2} = -\omega^2\mu\varepsilon E_x \tag{3}$$

This is the equivalent of the wave equation, Eq. 3.9(11), but now written in phasor form. It is called a *one-dimensional Helmholtz equation.* It could also be obtained by replacing $\partial^2/\partial t^2$ with $-\omega^2$ in Eq. 3.9(11). Solution is in terms of exponentials, as can be verified by substituting in (3)

$$E_x = c_1 e^{-jkz} + c_2 e^{jkz} \tag{4}$$

where

$$k = \omega\sqrt{\mu\varepsilon} \tag{5}$$

The constant k will be met frequently in wave problems. It is a constant of the medium for a particular angular frequency ω and is frequently called the *wave number*. It may also be written in terms of the velocity v defined by Eq. 3.9(12):

$$k = \frac{\omega}{v} \tag{6}$$

The first term of (4) is one that changes its phase linearly with z, becoming increasingly negative or lagging as one moves in the positive z direction. This behavior is consistent with the interpretation that the sinusoid is traveling in the positive z direction with velocity v, resulting in a *phase constant* k rad/m. The second term of (4) is delayed (becomes more negative) in phase as one moves in the *negative* z direction and so represents a negatively traveling wave with the same phase constant.

To show the exact correspondence of this approach with that of Sec. 3.9, let us convert the phasor form to a time-varying form by the rules given in Sec. 3.8. We multiply the phasor by the exponential $e^{j\omega t}$ and take the real part of the product:

$$E_x(z, t) = \mathrm{Re}[E_x e^{j\omega t}] = \mathrm{Re}[c_1 e^{-jkz} e^{j\omega t} + c_2 e^{jkz}\, e^{j\omega t}] \tag{7}$$

For simplicity, take c_1 and c_2 to be real. Then

$$E_x(z, t) = c_1 \cos(\omega t - kz) + c_2 \cos(\omega t + kz) \tag{8a}$$

$$= c_1 \cos \omega\left(t - \frac{z}{v}\right) + c_2 \cos \omega\left(t + \frac{z}{v}\right) \tag{8b}$$

Following the interpretation of Sec. 3.9, we see two real sinusoids, the first traveling in the positive z direction with velocity v and the second traveling in the negative z direction with the same velocity. The result is then exactly as in Sec. 3.9.

Figure 3.10*a* shows the sinusoidal variation of E_x with z at a particular instant (say $t = 0$). This pattern moves to the right with velocity v if it is a positively traveling wave and to the left if it is negatively traveling. The distance between two planes with the same magnitude and direction of E_x is called *wavelength* λ and is found by the distance for which phase changes by 2π

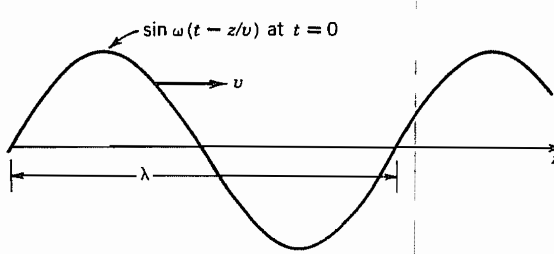

FIG. 3.10*a* Sinusoidal function plotted versus distance for one instant of time. For a positively traveling wave, the function progresses in the positive z direction with velocity v.

$$\lambda = \frac{2\pi}{k} = \frac{2\pi v}{\omega} = \frac{v}{f} \tag{9}$$

where f is frequency. Figure 3.10b shows electric field vectors of a sinusoidal wave.

Let us also look at the magnetic fields. Returning to the complex forms, we use the solution (4) in the differential equation (1):

$$H_y = -\frac{1}{j\omega\mu}\frac{dE_x}{dz} = \frac{k}{\omega\mu}[c_1 e^{-jkz} - c_2 e^{jkz}] \tag{10}$$

Using the definition of k from (5),

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}[c_1 e^{-jkz} - c_2 e^{jkz}] \tag{11}$$

The instantaneous equivalent of this is

$$H_y(z, t) = \mathrm{Re}[H_y(z)e^{j\omega t}] = \sqrt{\frac{\varepsilon}{\mu}}[c_1 \cos(\omega t - kz) - c_2 \cos(\omega t + kz)] \tag{12}$$

So E_x/H_y is $\sqrt{\mu/\varepsilon}$ for the positively traveling wave and is $-\sqrt{\mu/\varepsilon}$ for the negatively traveling wave. The consequences of these relationships for problems of wave transmission and reflection are discussed in Chapter 6.

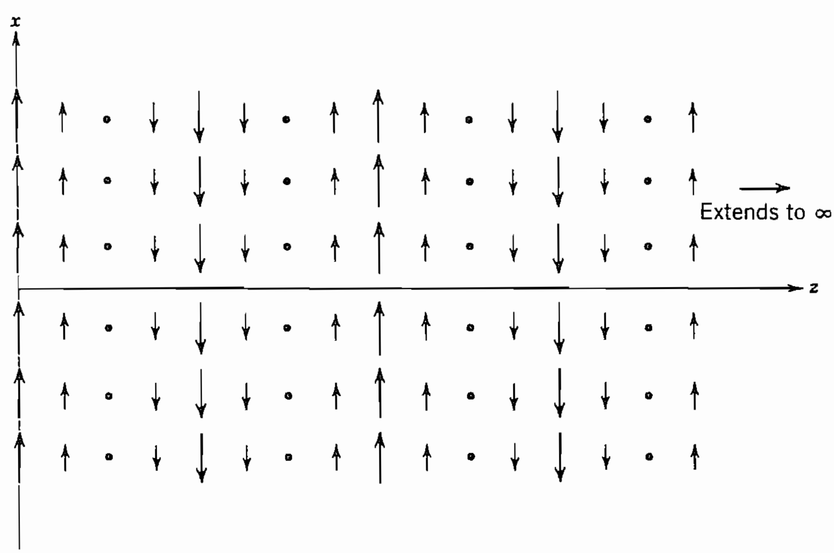

FIG. 3.10*b* Vectors showing magnitude and direction of electric field in a sinusoidal, uniform plane wave filling the half-space $0 \leq z$ for one instant of time. For a positively traveling wave, pattern moves to right with velocity $1/\sqrt{\mu\varepsilon}$.

3.11 THE WAVE EQUATION IN THREE DIMENSIONS

The one-dimensional example studied in the preceding two sections is important because it illustrates wave behavior simply, and also because it is a useful model for many important practical problems. Nevertheless we have to be concerned with wave behavior in two or three dimensions also. To derive the equation governing such phenomena, let us still specialize to simple media in which ε and μ are scalar constants and assume no free charges or convection currents within the region of concern. We may then return to the special form of Maxwell's equations given as Eqs. 3.9(1) to 3.9(4). Take the curl of Eq. 3.9(3), interchanging time and space partial derivatives:

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) \tag{1}$$

The left side is expanded by a vector identity. (See inside back cover.) The curl of magnetic field on the right side utilizes Eq. 3.9(4).

$$-\nabla^2 \mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \mathbf{E}}{\partial t}\right) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2}$$

For a source-free dielectric, $\nabla \cdot \mathbf{D} = 0$ and, if ε is not a function of space coordinates, $\nabla \cdot \mathbf{E} = 0$ also. Then

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{3}$$

This is the three-dimensional wave equation to be derived. It will be found useful in a variety of problems to be considered later, as in the analysis of propagating modes of a waveguide, resonant modes of a cavity resonator, or radiating waves from an antenna. Note that the vector equation breaks into three scalar equations, and for rectangular coordinates it separates into three scalar wave equations of the same form:

$$\nabla^2 E_x = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \tag{4}$$

and similarly for E_y and E_z. Note that if $\partial/\partial x = 0$ and $\partial/\partial y = 0$, ∇^2 is just $\partial^2/\partial z^2$ and we have the one-dimensional wave equation studied in Sec. 3.9:

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \tag{5}$$

The wave equation applies also to magnetic field for the simple medium considered here, as can be shown by taking the curl of Eq. 3.9(4) and substituting Eq. 3.9(3) to obtain

$$\nabla^2 \mathbf{H} = \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \tag{6}$$

In complex or phasor notation these reduce to three-dimensional Helmholtz equations, obtained by replacing $\partial^2/\partial t^2$ with $-\omega^2$ in (3) and (6):

$$\text{phasor forms}\begin{cases} \nabla^2\mathbf{E} = -k^2\mathbf{E} & (7)\\ \nabla^2\mathbf{H} = -k^2\mathbf{H} & (8)\\ k^2 = \omega^2\mu\varepsilon & (9)\end{cases}$$

Example 3.11
RESONANT WAVE SOLUTION FOR A RECTANGULAR BOX

We have seen that the wave equation has traveling-wave solutions. It also has standing-wave solutions under proper boundary conditions. Consider

$$E_x = C\cos k_x x \sin k_y y \sin k_z z \tag{10}$$

We use the x component of (7), expressed in rectangular coordinates:

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -k^2 E_x \tag{11}$$

Carrying out the differentiations,

$$-k_x^2E_x - k_y^2E_x - k_z^2E_x = -k^2E_x$$

or

$$k_x^2 + k_y^2 + k_z^2 = k^2 \tag{12}$$

So the phase constants in the three directions must be related by the condition (12). We shall see in Chapter 10 that this relation, combined with boundary conditions at the conducting walls, gives the conditions for resonance of waves in a rectangular cavity resonator.

3.12 POWER FLOW IN ELECTROMAGNETIC FIELDS: POYNTING'S THEOREM

The preceding sections have shown how electromagnetic waves may propagate through space or a dielectric. We know from experience that such waves can carry energy. The sun's rays, which are now known to be electromagnetic waves, warm us. The radio waves from a distant antenna bring power, admittedly small, to drive the first amplifier stage of a receiver. For lumped electrical circuits we express power through voltage and current. For electromagnetic fields, we can find a similar but more general relationship giving power and energy relationships in terms of the fields. The resulting theorem, Poynting's theorem, is one of the most fundamental and useful relationships of electromagnetic theory.

We start with the time-varying forms (Sec. 3.6) and write the two curl equations of Maxwell:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2}$$

An equivalence of vector operations (inside back cover) shows that

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) = \nabla \cdot (\mathbf{E} \times \mathbf{H}) \tag{3}$$

If products involving (1) and (2) are taken as indicated, (3) becomes

$$-\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} = \nabla \cdot (\mathbf{E} \times \mathbf{H}) \tag{4}$$

This may now be integrated over the volume of concern:

$$\int_V \left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{E} \cdot \mathbf{J}\right) dV = -\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dV$$

From the divergence theorem (Sec. 1.11), the volume integral of div($\mathbf{E} \times \mathbf{H}$) equals the surface integral of $\mathbf{E} \times \mathbf{H}$ over the boundary.

$$\int_V \left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{E} \cdot \mathbf{J}\right) dV = -\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \tag{5}$$

This is the important *Poynting's theorem* and in this form is valid for general media since we have so far made no specializations with respect to the medium. For linear, time-invariant media (5) can be recast into the form

$$\int_V \left[\frac{\partial}{\partial t}\left(\frac{\mathbf{B} \cdot \mathbf{H}}{2}\right) + \frac{\partial}{\partial t}\left(\frac{\mathbf{D} \cdot \mathbf{E}}{2}\right) + \mathbf{E} \cdot \mathbf{J}\right] dV = -\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \tag{6}$$

Problem 3.12f shows that (6) is consistent with (5) for isotropic media. Equation (6) is also valid for anisotropic media (Prob. 13.8c). The term $\varepsilon E^2/2$ was shown (Sec. 1.22) to represent the energy storage per unit volume for an electrostatic field. If this interpretation is extended by definition to any electric field,[3] the second term of (6) represents the time rate of increase of the stored energy in the electric fields of the region. Similarly, if $\mu H^2/2$ is defined as the density of energy storage for a magnetic field, the first term represents the time rate of increase of the stored energy in the magnetic fields of the region. The third term represents either the ohmic power loss if $\mathbf{J}$ is a conduction current density or the power required to accelerate charges if $\mathbf{J}$ is a convection current arising from moving charges. Both of these cases will be illustrated in the examples at the end of this section. Also, if there is an energy source, $\mathbf{E} \cdot \mathbf{J}$ is negative for that source and

[3] *For an excellent discussion of the arbitrariness of these definitions, refer to J. A. Stratton,* Electromagnetic Theory, *p. 133, McGraw-Hill, New York, 1941.*

represents energy flow out of the region. All the net energy change must be supplied externally. Thus the term on the right represents the energy flow into the volume per unit time. Changing sign, the rate of energy flow out through the enclosing surface is

$$W = \oint_S \mathbf{P} \cdot d\mathbf{S} \tag{7}$$

where

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} \tag{8}$$

and is called the Poynting vector.

Although it is known from the proof only that total energy flow out of a region per unit time is given by the total surface integral (6), it is often convenient to think of the vector $\mathbf{P}$ defined by (8) as the vector giving direction and magnitude of energy flow density at any point in space. Though this step does not follow strictly, it is a most useful interpretation and one which is justified for the majority of applications. (But see Prob. 3.12a.)

It should be noted that there are cases for which there will be no power flow through the electromagnetic field. Accepting the foregoing interpretation of the Poynting vector, we see that it will be zero when either $\mathbf{E}$ or $\mathbf{H}$ is zero or when the two vectors are mutually parallel. Thus, for example, there is no power flow in the vicinity of a system of static charges that has electric field but no magnetic field. Another very important case is that of a perfect conductor, which by definition must have a zero tangential component of electric field at its surface. Then $\mathbf{P}$ can have no component normal to the conductor and there can be no power flow into the perfect conductor.

Example 3.12a
Ohmic Loss

To demonstrate the interpretation of the theorem, let us take the simple example of a round wire carrying direct current I_z (Fig. 3.12). If R is the resistance per unit length, the electric field in the wire is known from Ohm's law to be

$$E_z = I_z R$$

The magnetic field at the surface, or at any radius r outside the wire, is

$$H_\phi = \frac{I_z}{2\pi r} \tag{9}$$

The Poynting vector $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ is everywhere radial, directed toward the axis:

$$P_r = -E_z H_\phi = -\frac{RI_z^2}{2\pi r} \tag{10}$$

We then make an integration over a cylindrical surface of unit length and radius equal

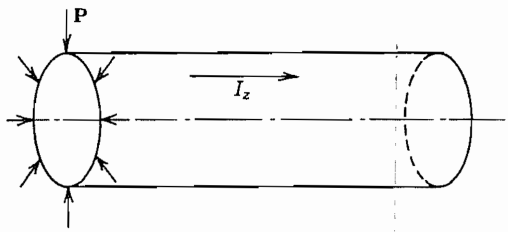

FIG. 3.12 Round wire with Poynting vector directed radially inward to supply power for ohmic losses.

to that of the wire (there is no flow through the ends of the cylinder since **P** has no component normal to the ends). All the flow is through the cylindrical surface, giving a power flow inward of amount

$$W = 2\pi r(-P_r) = I_z^2 R \tag{11}$$

We know that this result does represent the correct power flow into the conductor, being dissipated in heat. If we accept the Poynting vector as giving the correct *density* of power flow at each point, we must then picture the battery or other source of energy as setting up the electric and magnetic fields, so that the energy flows through the field and into the wire through its surface. The Poynting theorem cannot be considered a proof of the correctness of this interpretation, for it says only that the *total* power balance for a given region will be computed correctly in this manner, but the interpretation is nevertheless a useful one.

Example 3.12b
MOVING CHARGES

Let us next consider the example in which **J** is a convection current. For simplicity take a region containing particles of charge value q, mass m, and velocity $\mathbf{v}_\rho$. The convection current density is

$$\mathbf{J} = \rho \mathbf{v}_\rho = nq\mathbf{v}_\rho \tag{12}$$

where n is the density of particles. From the force law the acceleration of charges is

$$\mathbf{F} = q\mathbf{E} = m\frac{d\mathbf{v}_\rho}{dt} \tag{13}$$

and the third term in the Poynting theorem (6) is

$$\int_V \mathbf{E} \cdot \mathbf{J}\, dV = \int_V \frac{m}{q}\frac{d\mathbf{v}_\rho}{dt} \cdot (nq\mathbf{v}_\rho)\, dV = \int_V \frac{mn}{2}\frac{d(v_\rho^2)}{dt}\, dV \tag{14}$$

which we recognize to be the rate of change of the total kinetic energy of the charge group. In this example we will not try to work out the right side of (6), but the **E** in that term is related to the accelerating field, the **H** is that from the convection current, and the Poynting theorem will always be satisfied.[4]

Example 3.12c
POYNTING FLOW IN A PLANE WAVE

Finally we look at the Poynting theorem applied to the plane electromagnetic wave studied in preceding sections. The form of a sinusoidally varying wave with E_x and H_y propagating in the positive z direction was shown to be

$$E_x = E_0 \cos(\omega t - kz) \tag{15}$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}\, E_0 \cos(\omega t - kz) \tag{16}$$

The Poynting vector is then in the z direction, which is consistent with our interpretation that power is flowing in that direction:

$$P_z = E_x H_y = \sqrt{\frac{\varepsilon}{\mu}}\, E_0^2 \cos^2(\omega t - kz) \tag{17}$$

By the use of a trigonometric identity this is also

$$P_z = \sqrt{\frac{\varepsilon}{\mu}}\, E_0^2\left[\frac{1}{2} + \frac{1}{2}\cos 2(\omega t - kz)\right] \tag{18}$$

Note that there is a constant term showing that the wave carries an average power, as expected. There is also a time-varying portion representing the redistribution of stored energy in space as maxima and minima of fields pass through a given region.

3.13 POYNTING'S THEOREM FOR PHASORS

Because of the importance of phasors for sinusoidal electromagnetic fields, we need the Poynting theorem in phasor form also. It might seem that we could simply substitute in the time-varying theorem, Eq. 3.12(5), replacing $\partial/\partial t$ by $j\omega$, but this does not work since the expression is nonlinear, involving products of the fields. We start with Maxwell's equations in complex form and derive the complex Poynting theorem by steps

[4] *If the charges move through the surface surrounding the region, the net kinetic energy transport by the charge stream through the surface is also included. This is actually contained in the third term on the left as shown by L. Tonks,* Phys. Rev. ***54*** *863 (1938).*

parallel to those used for the theorem in time-varying field quantities. The two curl equations in complex phasor form are (Sec. 3.8)

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} \tag{2}$$

Consider the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}^*) \tag{3}$$

where the asterisk denotes the complex conjugate. Equations (1) and (2) may now be substituted in this identity:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot (-j\omega\mathbf{B}) - \mathbf{E} \cdot (\mathbf{J}^* - j\omega\mathbf{D}^*) \tag{4}$$

This expression is integrated through volume V and the divergence theorem utilized:

$$\begin{aligned}\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}^*)\, dV &= \oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} \\ &= -\int_V [\mathbf{E} \cdot \mathbf{J}^* + j\omega(\mathbf{H}^* \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{D}^*)]\, dV\end{aligned} \tag{5}$$

Equation (5) is the general Poynting theorem as it applies to complex phasors. To interpret, consider an isotropic medium in which all losses occur through conduction currents $\mathbf{J} = \sigma\mathbf{E}$ so that σ, μ, and ε are real scalars. Then (5) becomes

$$\oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} = -\int_V \sigma\mathbf{E} \cdot \mathbf{E}^*\, dV - j\omega \int_V [\mu\mathbf{H} \cdot \mathbf{H}^* - \varepsilon\mathbf{E} \cdot \mathbf{E}^*]\, dV \tag{6}$$

The first volume integral on the right side represents power loss in the conduction currents and is just twice the average power loss. (See Appendix 4.) Thus the real part of the complex Poynting flow on the left side can be related to this power loss. Or, interpreting the Poynting vector itself as a density of power flow as in Sec. 3.12,

$$\mathbf{P}_{\text{av}} = \tfrac{1}{2}\,\text{Re}(\mathbf{E} \times \mathbf{H}^*) \quad \text{W/m}^2 \tag{7}$$

The second volume integral on the right of (6) is proportional to the difference between average stored magnetic energy in the volume and average stored electric energy. Taking into account a factor of $\frac{1}{2}$ in the energy expressions and another $\frac{1}{2}$ for averaging of squares of sinusoids, we can then interpret the imaginary part of (6) as

$$\text{Im} \oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} = 4\omega(U_{E\text{av}} - U_{H\text{av}}) \tag{8}$$

where $U_{E\text{av}}$ is average stored energy in electric fields and $U_{H\text{av}}$ that in magnetic fields. So the imaginary part of the Poynting flow through the surface can be thought of as *reactive power* flowing back and forth to supply the instantaneous changes in net stored energy in the volume.

Example 3.13
AVERAGE POWER IN UNIFORM PLANE WAVES

To illustrate the average Poynting vector for the plane-wave case, let us take the field expressions for plane waves derived in complex form in Sec. 3.10:

$$E_x = c_1 e^{-jkz} + c_2 e^{jkz} \tag{9}$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}\,[c_1 e^{-jkz} - c_2 e^{jkz}] \tag{10}$$

The complex Poynting vector is then

$$\mathbf{E} \times \mathbf{H}^* = \sqrt{\frac{\varepsilon}{\mu}}\,[c_1 e^{-jkz} + c_2 e^{jkz}][c_1^* e^{jkz} - c_2^* e^{-jkz}]\hat{\mathbf{z}} \tag{11}$$

and the average power density, by (7), is in the z direction and equal to

$$P_{\text{av}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,[c_1 c_1^* - c_2 c_2^*] \quad \text{W/m}^2 \tag{12}$$

This equation states that the average power is simply the average power of the positively traveling wave minus that of the negatively traveling wave. The cross-product terms of (11) contribute only to reactive power, that is, to the interchange of stored energy within the wave.

3.14 CONTINUITY CONDITIONS FOR AC FIELDS AT A BOUNDARY: UNIQUENESS OF SOLUTIONS

In the study of static fields, certain boundary and continuity conditions were stated for such fields and were found essential in the solution of the field problems by the use of the differential equations. Similarly, for the use of Maxwell's equations in differential equation form, we need corresponding boundary and continuity conditions.

Consider first Faraday's law in large-scale form, Eq. 3.2(3), applied to a path formed by moving distance Δl along one side of the boundary between any two materials and returning on the other side, an infinitesimal distance into the second medium (Fig. 3.14*a*). The line integral of electric field is

$$\oint \mathbf{E} \cdot \mathbf{dl} = (E_{t1} - E_{t2})\,\Delta l \tag{1}$$

Since the path is an infinitesimal distance on either side of the boundary, it encloses zero area; therefore the contribution from changing magnetic flux is zero so long as rate of change of magnetic flux density is finite. Consequently,

$$(E_{t1} - E_{t2})\,\Delta l = 0 \quad \text{or} \quad E_{t1} = E_{t2} \tag{2}$$

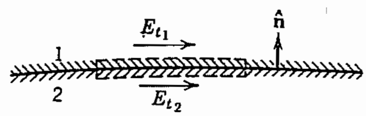

FIG. 3.14*a* Continuity of tangential electric field components at a dielectric boundary.

Similarly, the generalized Ampère law in large-scale form, Eq. 3.7(4), may be applied to a like path with its two sides on the two sides of the boundary. Again zero area is enclosed by the path, and, so long as current density and rate of change of electric flux density are finite, the integral is zero. Thus, as in (2),

$$H_{t1} = H_{t2} \tag{3}$$

Or in vector form, by use of the unit vector $\hat{\mathbf{n}}$ normal to the boundary as shown in Fig. 3.14*a*, (2) and (3) can be written as

$$\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{4}$$

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0 \tag{5}$$

Thus tangential components of electric and magnetic field must be equal on the two sides of any boundary between physically real media. The condition (3) may be modified for an idealized case such as the perfect conductor where the current densities are allowed to become infinite. This case is discussed separately in Sec. 3.15.

The integral form of Gauss's law is Eq. 3.7(1). If two very small elements of area ΔS are considered (Fig. 3.14*b*), one on either side of the boundary between any two materials, with a surface charge density ρ_s existing on the boundary, the application of Gauss's law to this elemental volume gives

$$\Delta S(D_{n1} - D_{n2}) = \rho_s \, \Delta S$$

or

$$D_{n1} - D_{n2} = \rho_s \tag{6}$$

For a charge-free boundary,

$$D_{n1} = D_{n2} \quad \text{or} \quad \varepsilon_1 E_{n1} = \varepsilon_2 E_{n2} \tag{7}$$

That is, for a charge-free boundary, normal components of electric flux density are continuous; for a boundary with charges, they are discontinuous by the amount of the surface charge density.

Since there is no magnetic charge term on the right of Eq. 3.7(2), a development corresponding to the above shows that always the magnetic flux density is continuous:

$$B_{n1} = B_{n2} \quad \text{or} \quad \mu_1 H_{n1} = \mu_2 H_{n2} \tag{8}$$

For the time-varying case, which is of greatest importance to our study, the conditions on normal components are not independent of those given for the tangential components. The reason is that the former are derived from the divergence equations (or their equivalent in large-scale form), and these may be obtained from the two curl equations

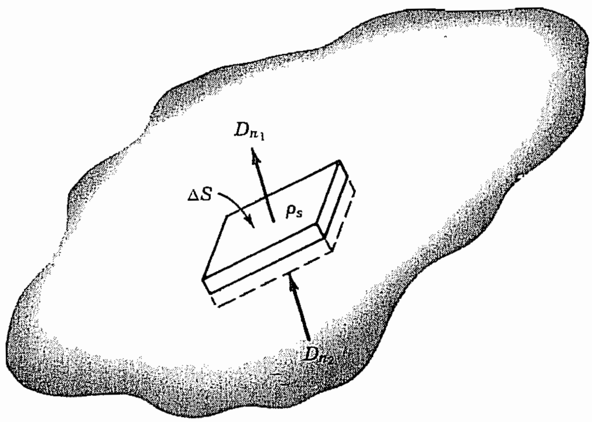

FIG. 3.14*b* Diagram showing how discontinuity in normal components of electric flux density at a boundary is related to surface charge density.

in the time-varying case (Prob. 3.6b). The conditions on tangential components were derived from the large-scale equivalents of the curl equations. Hence, for the ac solutions, it is necessary only to apply the continuity conditions on tangential components of electric and magnetic fields at a boundary between two media, and the conditions on normal components may be used as a check; if the normal components of **D** turn out to be discontinuous, (6) tells the amount of surface charge that is induced on the boundary.

Uniqueness The procedure to prove the uniqueness of solutions of Maxwell's equations follows the philosophy in Sec. 1.17. One assumes two possible solutions with the same given tangential fields on the boundary of the region of interest. The difference field is formed, and found to satisfy Poynting's theorem in the form of Eq. 3.12(5). Stratton[5] shows that for linear, isotropic (but possibly inhomogeneous) media, specification of tangential **E** and **H** on the boundary and of initial values of all fields at time zero is sufficient to specify fields uniquely within the region at all later times. The argument can be extended to anisotropic materials and certain classes of nonlinear materials, but not to materials that have multivalued relations between **D** and **E** (or **B** and **H**) or to "active" materials that produce oscillations. In steady-state problems we are not generally concerned with the specifications of initial conditions.

Although the discussion has been given for a region with closed boundaries, uniqueness arguments also apply to open regions extending to infinity, provided certain radiation conditions are satisfied by the fields. These require that the products $r\mathbf{E}$ and $r\mathbf{H}$ remain finite as r approaches infinity[6] and are satisfied by fields arising from real charge and current sources contained within a finite region. The extension to open regions is important to the potential formulation of the last part of this chapter.

[5] *J. A. Stratton,* Electromagnetic Theory, *pp. 486–488, McGraw-Hill, New York, 1941.*
[6] *S. Silver,* Microwave Antenna Theory and Design, *p. 85, IEEE Press, New York, 1984.*

3.15 Boundary Conditions at a Perfect Conductor for AC Fields

It is a good approximation in many practical problems to treat good conductors (such as copper and other metals) as though of infinite conductivity when finding the form of fields outside the conductor. We will study the effect of large but finite conductivity on fields *within* the conductor in Sec. 3.16. When we do, we will find that all fields and currents concentrate in a thin region or "skin" near the surface for time-varying fields, and this region approaches zero thickness as the conductivity approaches infinity. Thus, for the perfect conductor (infinite conductivity), we find that all fields are zero inside the conductor and any current flow must be only on the surface. The physical properties of perfect conductors are discussed in Sec. 13.4. Since the electric field is zero within the perfect conductor, continuity of tangential electric field at a boundary requires that the surface tangential electric field be zero just outside the boundary also,

$$E_t = 0 \tag{1}$$

and Eq. 3.14(6) gives the normal electric flux density as

$$D_n = \rho_s \tag{2}$$

Furthermore, since magnetic fields also vanish inside the conductor, the statement of continuity of magnetic flux lines, Eq. 3.14(8), indicates that

$$B_n = 0 \tag{3}$$

at the conductor surface. As was pointed out in the last section, however, the continuity condition on normal **B** is not independent of the condition on tangential **E** in the time-varying case. Thus, in the ac solution, (3) follows from (1), but may sometimes be useful as a check or as an alternative boundary condition.

The tangential component of magnetic field is likewise zero inside the perfect conductor but is not in general zero just outside. This discontinuity would appear to violate the condition of Eq. 3.14(3), but it will be recalled that a condition for that proof was that current density remain finite. For the perfect conductor, the finite current **J** per unit width is assumed to flow on the surface as a current sheet of zero thickness, so that current density is infinite. The discontinuity in tangential magnetic field is found by a construction similar to that of Fig. 3.14*a*. The current enclosed by the path is the current per unit width **J** flowing on the surface of the conductor perpendicular to the direction of the tangential magnetic field at the surface. Then

$$\oint \mathbf{H} \cdot \mathbf{dl} = H_t\, dl = J_s\, dl$$

or

$$J_s = H_t \quad \text{A/m} \tag{4}$$

where J_s is current per unit width, called a surface current density. The direction and sense relations for (4) are given most conveniently by the vector form of the law below.

To write the relations of (1)–(4) in vector notation, a unit vector $\hat{\mathbf{n}}$, normal to the

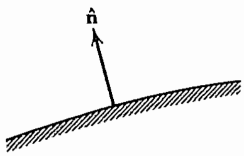

FIG. 3.15 Conducting boundary with the normal unit vector.

conductor at any given point and pointing from the conductor into the region where fields exist, is defined (Fig. 3.15). Then conditions (1)–(4) become:

$$\hat{\mathbf{n}} \times \mathbf{E} = 0 \tag{5}$$

$$\hat{\mathbf{n}} \cdot \mathbf{B} = 0 \tag{6}$$

$$\rho_s = \hat{\mathbf{n}} \cdot \mathbf{D} \tag{7}$$

$$\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H} \tag{8}$$

For an ac problem, (5) represents the only required boundary condition at a perfect conductor. Equation (6) serves as a check or sometimes as an alternative to (5). Equations (7) and (8) are used to give the charge and current induced on the conductor by the presence of the electromagnetic fields.

3.16 Penetration of Electromagnetic Fields into a Good Conductor

Maxwell's equations have been illustrated by showing the wave behavior of electromagnetic fields in good dielectrics. A second extremely important class of materials used in many electromagnetic problems is that of "good conductors." Let us examine the basic behavior of electromagnetic fields in such conductors. The development in this and the following section will be for steady-state sinusoids using phasor notation, with the usual understanding that more general time variations may be broken up into a series or continuous distribution of such sinusoids. The conductors of concern are those satisfying Ohm's law,

$$\mathbf{J} = \sigma \mathbf{E} \tag{1}$$

The constant σ is the conductivity of the conductor. At optical frequencies metals are not well represented by a real constant σ, but the approximation is valid for microwaves and millimeter waves (Sec. 13.3). Substitution of (1) into the Maxwell equation 3.8(4) gives

$$\nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\mathbf{E} \tag{2}$$

It is easy to show that the assumption of Ohm's law implies the absence of charge density. Since the divergence of the curl of any vector is zero,

$$\nabla \cdot \nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\nabla \cdot \mathbf{E} = 0$$

where we have assumed homogeneity of σ and ε. Thus

$$\nabla \cdot \mathbf{D} = \rho = 0 \tag{3}$$

The simple picture of the situation in a conductor is that mobile electrons drift through a lattice of positive ions, encountering frequent collisions. On the average, over a volume large compared with the atomic dimensions but small compared with dimensions of interest in the system under study, the net charge is zero even though some of the charges are moving through the element and causing current flow. The net movement or "drift" in such cases is found proportional to the electric field.

For metals and other good conductors, it is found that displacement current is negligible in comparison with conduction current for microwave and millimeter-wave frequencies, and in fact is not measurable until frequencies are well into the infrared. For the present we concentrate on the important cases for which $\omega\varepsilon$ in (2) is negligible in comparison with σ.

Thus, to summarize, the following specializations are appropriate to Maxwell's equations applied to good conductors, and may in fact be taken as a definition of a good conductor.

1. Conduction current is given by Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$.
2. Displacement current is negligible in comparison with current, $\omega\varepsilon << \sigma$.
3. As a consequence of (1), the net charge density is zero for homogeneous conductors.

To derive the differential equation which determines the penetration of the fields into the conductor, we first take the curl of the Maxwell curl equation for electric field, Eq. 3.8(3), and make use of a vector identity (see inside back cover) and the definition of permeability to obtain

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -j\omega\mu\nabla \times \mathbf{H} \tag{4}$$

Then using (3) and substituting (2) in (4) with displacement current neglected, we find

$$\nabla^2\mathbf{E} = j\omega\mu\sigma\mathbf{E} \tag{5}$$

Equations with forms identical to (5) can be found in a similar way for magnetic field and current density:

$$\nabla^2\mathbf{H} = j\omega\mu\sigma\mathbf{H} \tag{6}$$

$$\nabla^2\mathbf{J} = j\omega\mu\sigma\mathbf{J} \tag{7}$$

We first consider the differential equations (5)–(7) for the simple but useful example of a plane conductor of infinite depth, with no field variations along the width or length dimension. This case is frequently taken as that of a conductor filling the half-space $x > 0$ in a rectangular coordinate system with the y–z plane coinciding with the conductor surface, and is then spoken of as a "semi-infinite solid." In spite of the infinite depth requirement, the analysis of this case is of importance to many conductors of finite extent, and with curved surfaces, because at high frequencies the depth over which significant fields are concentrated is very small. Radii of curvature and conductor depth may then be taken as infinite in comparison. Moreover, any field variations along the surface due to curvature, edge effects, or variations along a wavelength are ordinarily so small compared with the variations into the conductor that they may be neglected.

For the uniform field situation shown in Fig. 3.16*a* with the electric field vector in the z direction, we assume no variations with y or z and (5) becomes

$$\frac{d^2E_z}{dx^2} = j\omega\mu\sigma E_z = \tau^2 E_z \tag{8}$$

where

$$\tau^2 \triangleq j\omega\mu\sigma \tag{9}$$

Since $\sqrt{j} = (1 + j)/\sqrt{2}$ (taking the root with the positive sign),

$$\tau = (1 + j)\sqrt{\pi f\mu\sigma} = \frac{1 + j}{\delta} \tag{10}$$

where

$$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} \quad \text{m} \tag{11}$$

A complete solution of (8) is in terms of exponentials:

$$E_z = C_1 e^{-\tau x} + C_2 e^{\tau x} \tag{12}$$

The field will increase to the impossible value of infinity at $x = \infty$ unless C_2 is zero. The coefficient C_1 may be written as the field at the surface if we let $E_z = E_0$ when $x = 0$. Then

$$E_z = E_0 e^{-\tau x} \tag{13}$$

Or, in terms of the quantity δ defined by (10) and (11),

$$E_z = E_0 e^{-x/\delta} e^{-jx/\delta} \tag{14}$$

Since the magnetic field and the current density are governed by the same differential equation as the electric field, forms identical to (14) apply; that is,

$$H_y = H_0 e^{-x/\delta} e^{-jx/\delta} \tag{15}$$

$$J_z = J_0 e^{-x/\delta} e^{-jx/\delta} \tag{16}$$

where H_0 and J_0 are the magnitudes of the magnetic field and current density at the surface.

It is evident from the forms of (14)–(16) that the magnitudes of the fields and current decrease exponentially with penetration into the conductor, and δ has the significance of the depth at which they have decreased to $1/e$ (about 36.9%) of their values at the surface, as indicated in Fig. 3.16*a*. The quantity δ is accordingly called the *depth of*

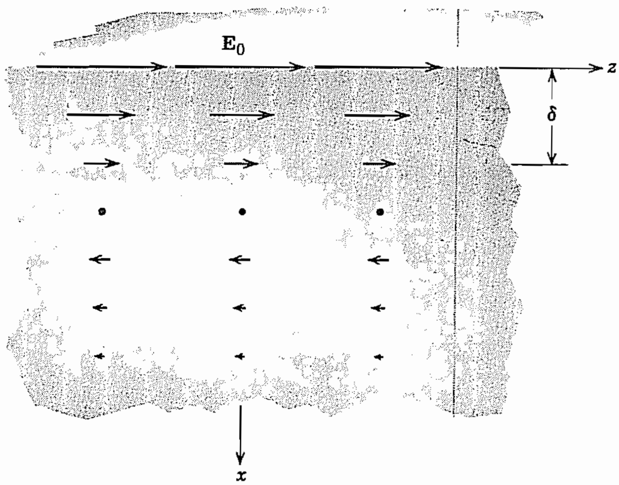

FIG. 3.16a Plane solid illustrating decay of current into conductor.

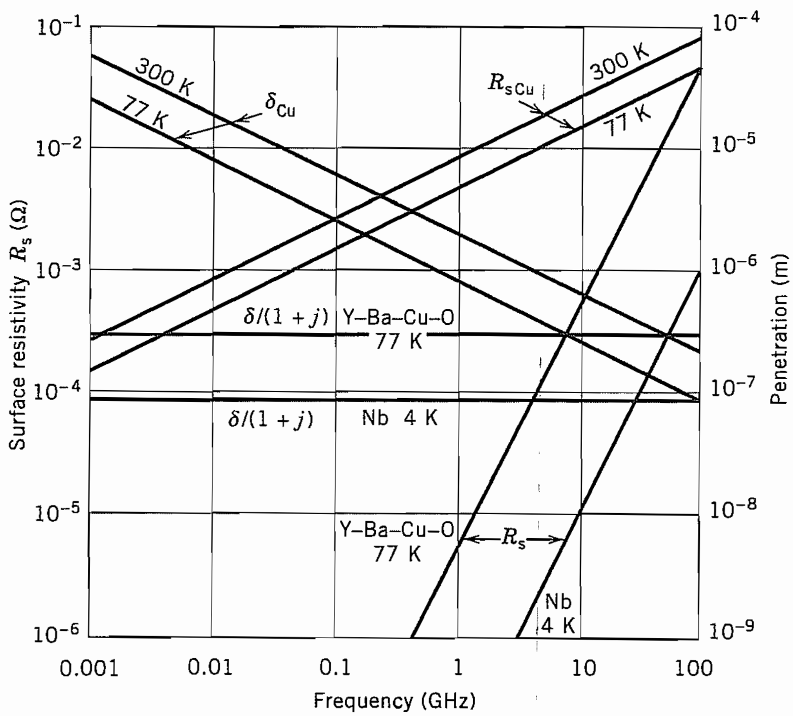

FIG. 3.16b Skin depth and surface resistance for copper at two temperatures and for two superconductors. Note that the skin depth for superconductors is (1 + j) times a real number, so penetration of fields and current density in Eqs. 3.16(14–16) have only the real exponential decay.

penetration, or *skin depth*. The phases of the current and fields lag behind their surface values by x/δ radians at depth x into the conductor. The penetration depths for copper at room temperature (300 K) and 77 K are shown in Fig. 3.16*b*. Except for ferromagnetic and ferrite materials, $\mu \approx \mu_0$.

Example 3.16

SKIN DEPTH IN AUDIO TRANSFORMER WITH IRON CORE

An audio frequency transformer has a core made of iron with $\sigma = 0.5 \times 10^7$ and $\mu = 1000\mu_0$. It is designed to work up to 15 kHz. Let us find the skin depth at this highest design frequency. From (11)

$$\begin{aligned} \delta &= (\pi \times 15 \times 10^3 \times 10^3 \times 4\pi \times 10^{-7} \times 0.5 \times 10^7)^{-1/2} \\ &= (30\pi^2 \times 10^6)^{-1/2} = 0.058 \times 10^{-3} \text{ m} = 0.058 \text{ mm} \end{aligned} \tag{17}$$

Note that this is more than 30 times smaller than for a material of the same conductivity but with a relative permeability of unity.

Advantageous electromagnetic behavior can be obtained in circumstances where cooling to cryogenic temperatures is possible if superconductors are used.[7] For reasons to be explained in Sec. 13.4, the conductivity is complex and frequency dependent, and δ is constant up to about 100 GHz at $(1 + j)$ times the value of the dc *penetration depth* λ_s. Values of δ found experimentally for niobium at 4 K and for the oxide superconductor $YBa_2Cu_3O_{7-x}$, or simply Y–Ba–Cu–O, are shown in Fig. 3.16*b* for comparison with the frequency-dependent values for copper. The oxide superconductor Y–Ba–Cu–O has an anisotropic crystal structure; it is assumed here that the highly conducting Cu–O planes are parallel to the surface. The behavior is otherwise more complicated.

3.17 INTERNAL IMPEDANCE OF A PLANE CONDUCTOR

The decay of fields into a good conductor or superconductor may be looked at as the attenuation of a plane wave as it propagates into the conductor or from the point of view that induced fields from the time-varying currents tend to counter the applied fields. The latter point of view is especially applicable to circuits, in which case we think of the field at the surface as the applied field. Currents (resulting from $\sigma\mathbf{E}$) concentrate near this surface and the ratio of surface electric field to current flow gives an *internal impedance* for use in circuit problems. By *internal*, we mean the contribution

[7] *T. Van Duzer and C. W. Turner,* Principles of Superconductive Devices and Circuits, *Sec. 3.14, Elsevier, New York, 1981. (To be reissued by Prentice Hall.)*

to impedance from the fields penetrating the conductor. This gives, in general, a resistance term and an internal inductance, the latter to be added to any external inductance contribution arising from the fields outside the conductor.

The total current flowing past a unit width on the surface of the plane conductor is found by integrating the current density, Eq. 3.16(16), from the surface to the infinite depth:

$$J_{sz} = \int_0^\infty J_z\, dx = \int_0^\infty J_0 e^{-(1+j)(x/\delta)}\, dx = \frac{J_0 \delta}{(1+j)} \tag{1}$$

The electric field at the surface is related to the current density at the surface by

$$E_{z0} = \frac{J_0}{\sigma} \tag{2}$$

Internal impedance for a unit length and unit width is defined as

$$Z_s \triangleq \frac{E_{z0}}{J_{sz}} = \frac{1+j}{\sigma\delta} \tag{3}$$

With the further definition

$$Z_s \triangleq R_s + j\omega L_i \tag{4}$$

We then have

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}} \tag{5}$$

$$\omega L_i = \frac{1}{\sigma\delta} = R_s \tag{6}$$

With σ real, the resistance and internal reactance of such a plane conductor are equal at any frequency. The internal impedance Z_s thus has a phase angle of 45 degrees. Equation (5) gives another interpretation of depth of penetration δ, for this equation shows that the skin-effect resistance of the semi-infinite plane conductor is the same as the dc resistance of a plane conductor of depth δ. That is, resistance of this conductor

Table 3.17a
Skin Effect Properties of Typical Metals

	Conductivity σ (S/m)	Depth of Penetration δ (m)	Surface Resistivity R_s (Ω)
Silver (300 K)	6.17×10^7	$0.0642 f^{-1/2}$	$2.52 \times 10^{-7} f^{1/2}$
Aluminum (300 K)	3.72×10^7	$0.0826 f^{-1/2}$	$3.26 \times 10^{-7} f^{1/2}$
Brass (300 K)	1.57×10^7	$0.127 f^{-1/2}$	$5.01 \times 10^{-7} f^{1/2}$
Copper (300 K)	5.80×10^7	$0.066 f^{-1/2}$	$2.61 \times 10^{-7} f^{1/2}$
Copper (77 K)	18×10^7	$0.037 f^{-1/2}$	$1.5 \times 10^{-7} f^{1/2}$

Table 3.17b
Skin Effect Properties of Typical Superconductors

	Complex Conductivity $\sigma = \sigma_1 - j\sigma_2$ (S/m)	Penetration Depth $\lambda_s = \delta/(1 + j)$ (m)	Surface Resistivity R_s (Ω)
$YBa_2Cu_3O_{7-x}$ (77 K)	$8.2 \times 10^6 - j20 \times 10^{17} f^{-1}$	250×10^{-9}	$40 \times 10^{-25} f^2$
Niobium (4 K)	$5.2 \times 10^6 - j175 \times 10^{17} f^{-1}$	85×10^{-9}	$1.0 \times 10^{-25} f^2$

with exponential decrease in current density is the same as though current were uniformly distributed over a depth δ.

The resistance R_s of the plane conductor for a unit length and unit width is called the *surface resistivity*. For a finite area of conductor, the resistance is obtained by multiplying R_s by length, and dividing by width since the width elements are essentially in parallel. Thus the dimension of R_s is ohms or, as it is sometimes called, ohms per square. Like the depth of penetration δ, R_s as defined by (5) is also a useful parameter in the analyses of conductors of other than plane shape, and may be thought of as a constant of the material at frequency f.

Superconductors are somewhat different from the good conductor discussed above in having a complex conductivity with the result that the surface resistance and reactance terms are not equal. But the definitions in (3) and (4) still apply. Again, $\mu \approx \mu_0$. They differ also in that R_s increases as f^2 rather than as $f^{1/2}$, as in the case of a good conductor. Values of depth of penetration (skin depth) and surface resistivity are tabulated for several metals in Table 3.17a and are plotted in Fig. 3.16*b* as functions of frequency. Table 3.17b gives experimentally derived data for the complex conductivity, penetration depth, and surface resistance for two prominent superconductors; the penetration depth and surface resistance are also plotted in Fig. 3.16*b* as functions of frequency.

Example 3.17
Approximate Internal Impedance of a Coaxial Line

The usefulness of this concept for practical problems may now be illustrated by considering the coaxial transmission line of Fig. 3.17. We select as a circuit path one which follows the outer surface of the inner conductor, AB, traversing radially across to C and then following the inner surface of the outer conductor CD, returning back radially to A. The difference between voltages V_{DA} and V_{CB} will arise in part because of the inductance calculated from flux within the path $ABCDA$, that is, the inductance external to the conductors. We consequently call this the *external inductance* and recognize it as that found for a coaxial line in Chapter 2. But there is also a voltage contribution along the path AB due to the internal impedance of the inner conductor and one along CD arising from internal impedance of the outer conductor.

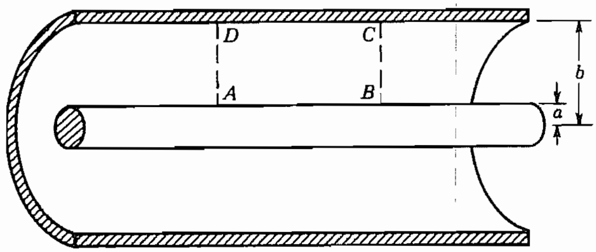

FIG. 3.17 Section of coaxial transmission line. Line integral of electric field about path $ABCD$ relates to magnetic flux associated with external inductance.

If radii of curvatures a and b are large in comparison with skin depth δ, and if thickness of the outer tubular conductor is large compared with δ, both conductors may be treated to a good degree of approximation by the planar analysis of this and the preceding section. Current concentrates on the outer surface of the inner conductor and the inner surface of the outer conductor, adjacent to the region of the fields. The inner conductor, if curvature is negligible, then appears as a plane of width equal to its circumference, $2\pi a$. Internal impedance per unit length is then

$$Z_{i1} = \frac{Z_{s1}}{2\pi a} \quad \Omega/\text{m}$$

The outer conductor, with these approximations, appears as a plane of width equal to its inner circumference, $2\pi b$. Its thickness does not enter since it is presumed much larger than δ, so that fields have died to a negligible value at the outer surface. Internal impedance per unit length from this part is then

$$Z_{i2} = \frac{Z_{s2}}{2\pi b} \quad \Omega/\text{m}$$

The sum of these two gives the total contribution to impedance from fields within the conductors and can be used in the transmission-line analysis of Chapter 5.

3.18 Power Loss in a Plane Conductor

To find average power loss per unit area of the plane conductor, we may apply the Poynting theorem of Sec. 3.13. The field components E_z and H_y produce a power flow in the x direction, or into the conductor. Utilization of the field values at the surface gives the total power flowing from the field into the conductor. In complex phasor form, using Eq. 3.13(7),

$$\mathbf{P}_L = \tfrac{1}{2}\,\text{Re}[\mathbf{E}_0 \times \mathbf{H}_0^*] = -\hat{\mathbf{x}}\tfrac{1}{2}\,\text{Re}(E_{z0}H_{y0}^*) \tag{1}$$

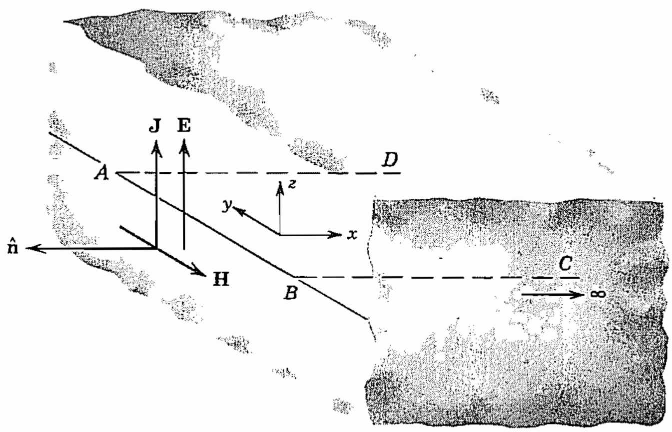

FIG. 3.18 Surface of plane conductor illustrating how magnetic field at surface relates to current flow per unit width.

The surface value of magnetic field can readily be related to the surface current, as can be seen by taking the line integral of magnetic field about some path *ABCD* of Fig. 3.18 (*C* and *D* at infinity). Since magnetic field is in the $-y$ direction for this simple case, there is no contribution to $\mathbf{H} \cdot \mathbf{dl}$ along the sides *BC* and *DA*; there is no contribution along *CD* since field is zero at infinity. Hence, for a width w,

$$\oint_{ABCD} \mathbf{H} \cdot \mathbf{dl} = \int_A^B \mathbf{H} \cdot \mathbf{dl} = -wH_{y0} \tag{2}$$

This line integral of magnetic field must be equal to the conduction current enclosed, since displacement current has been shown to be negligible in a good conductor. The current is just the width w times the current per unit width J_z. Then, utilizing (2),

$$-wH_{y0} = wJ_{sz} \quad \text{or} \quad J_{sz} = -H_{y0} \tag{3}$$

This may be written in a vector form which includes the magnitude and sense information of (3) and the fact that $\mathbf{J}$ and $\mathbf{H}$ are mutually perpendicular,

$$\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H} \tag{4}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the conductor surface, pointing into the adjoining dielectric region and $\mathbf{H}$ is the magnetic field at the surface. Note that (4) is of the same form as for perfect conductors, Eq. 3.15(8). Then using (1), (3), and Eq. 3.17(3), we obtain for power loss $W_L = |\mathbf{P}_L|$

$$W_L = \tfrac{1}{2}\,\mathrm{Re}[Z_s J_s J_s^*] = \tfrac{1}{2}\mathrm{R}_s|J_s|^2 \quad \mathrm{W/m^2} \tag{5}$$

This is a form that might have been expected in that it gives loss in terms of resistance multiplied by square of current magnitude. An alternate derivation (Prob. 3.18a) is by the integration of power loss at each point of the solid from the known conductivity and current density function.

Equation (5) will be found of the greatest usefulness throughout this text for the computation of power loss in the walls of waveguides, cavity resonators, and other electromagnetic structures. Although the walls of these structures are not plane solids of infinite depth, the results of this section may be applied for all practical purposes whenever the conductor thickness and radii of curvature are much greater than δ, depth of penetration. This includes most important cases at high frequencies. In these cases the quantities that are ordinarily known are the fields at the surface of the conductor.

Potentials for Time-Varying Fields

3.19 A Possible Set of Potentials for Time-Varying Fields

As we have seen, time-varying electromagnetic fields are related to each other and to the charge and current sources through the set of differential equations known as Maxwell's equations. It is sometimes convenient to introduce some intermediate functions, known as *potential functions*, which are directly related to the sources, and from which the electric and magnetic fields may be derived. Such functions were found useful for static fields, and in the case of the electrostatic potential, the potential itself had useful physical significance. The physical interpretation was less clear in the case of the magnetic vector potential, but it does provide a useful simplification in the analysis of some problems. In this and following sections we look for similar potential functions for the time-varying fields. It turns out that there are many possible sets. We select a commonly used set known as *retarded potentials*, which reduce to the potentials used for statics in the limit of no time variations.

We might try at first to use the forms found for statics, $\mathbf{E} = -\nabla\Phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$, with all quantities functions of time. We are faced with this problem: the electric field for time-varying conditions cannot be derived only as the gradient of scalar potential since this would require that it have zero curl, and it may actually have a nonzero curl of value $-\partial\mathbf{B}/\partial t$; it cannot be derived alone as the curl of a vector potential since this would require that it have zero divergence, and it may have a finite divergence of value ρ/ε.

Since the divergence of magnetic field is zero in the general case as it was in the static, it seems that $\mathbf{B}$ may still be set equal to the curl of some magnetic vector potential, $\mathbf{A}$.

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1}$$

This relation may now be substituted in the Maxwell equation 3.6(3) and the result written

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{2}$$

This equation states that the curl of a certain vector quantity is zero. But this is the condition that permits a vector to be derived as the gradient of a scalar, say Φ. That is,

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\Phi$$

or

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{3}$$

Equations (1) and (3) are then valid relationships between fields and potential functions $\mathbf{A}$ and Φ. Note that no specializations on the medium have been made to this point. It is found that the potential functions are most useful, though, for linear, isotropic, homogeneous media, so in the remaining part of this discussion, we take μ and ε as scalar constants appropriate to such media. With this specialization we substitute (3) in Gauss's law, Eq. 3.6(1), to obtain

$$-\nabla^2\Phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \frac{\rho}{\varepsilon} \tag{4}$$

Then, substituting $\mathbf{B} = \nabla \times \mathbf{A}$ and (3) in Eq. 3.6(4), we find

$$\nabla \times \nabla \times \mathbf{A} = \mu\mathbf{J} + \mu\varepsilon\left[-\nabla\left(\frac{\partial\Phi}{\partial t}\right) - \frac{\partial^2 \mathbf{A}}{\partial t^2}\right]$$

Using the vector identity

$$\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

this becomes

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} - \mu\varepsilon\nabla\left(\frac{\partial\Phi}{\partial t}\right) - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{5}$$

Equations (4) and (5) can be simplified by further specification of $\mathbf{A}$. That is, there are any number of vector functions whose curl is the same. One may specify also the divergence of $\mathbf{A}$ according to convenience.[8] If the divergence of $\mathbf{A}$ is chosen as[9]

[8] *Specification of divergence and curl of a vector, with appropriate boundary conditions, determines the vector uniquely through the Helmholtz theorem. See, for example, R. E. Collin,* Field Theory of Guided Waves, *2nd ed., Appendix A1, IEEE Press, Piscataway, NJ, 1991.*

[9] *This choice is known as the Lorentz condition or Lorentz gauge and leads to the symmetry of (7) and (8). Other useful gauges are the* Coulomb *and* London *gauges. See, for example, A. M. Portis,* Electromagnetic Fields: Sources and Media, *Wiley, New York, 1978. See also Prob. 3.19c.*

$$\nabla \cdot \mathbf{A} = -\mu\varepsilon \frac{\partial \Phi}{\partial t} \tag{6}$$

(4) and (5) then simplify to

$$\nabla^2 \Phi - \mu\varepsilon \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon} \tag{7}$$

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} \tag{8}$$

Thus the potentials **A** and Φ, defined in terms of the sources **J** and ρ by the differential equations (7) and (8), may be used to derive the electric and magnetic fields by (1) and (3). It is easy to see that they do reduce to the corresponding expressions of statics, for if time derivatives are allowed to go to zero, the set of equations (1), (3), (7), and (8) becomes

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon} \qquad \mathbf{E} = -\nabla \Phi \tag{9}$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{10}$$

which are recognized as the appropriate expressions from Chapters 1 and 2.

3.20 The Retarded Potentials as Integrals over Charges and Currents

The potential functions **A** and Φ for time-varying fields are defined in terms of the currents and charges by the differential equations 3.19(7) and 3.19(8). General solutions of the equation give the potentials as integrals over the charges and currents, as in the static case. The following discussion applies to the very important case of a region extending to infinity with a linear, isotropic, and homogeneous medium.

From Chapters 1 and 2 the integrals for the static potentials, which may be considered the solutions of Eqs. 3.19(9) and 3.19(10), are

$$\Phi = \int_V \frac{\rho \, dV}{4\pi\varepsilon r} \tag{1}$$

$$\mathbf{A} = \mu \int_V \frac{\mathbf{J} \, dV}{4\pi r} \tag{2}$$

A mathematical development to yield the corresponding integral solutions of the inhomogeneous wave equations 3.19(7) and 3.19(8) is given in Appendix 5. A plausibility argument is given here. The solutions are

$$\Phi(x, y, z, t) = \int_V \frac{\rho(x', y', z', t - R/v) \, dV'}{4\pi\varepsilon R} \tag{3}$$

$$\mathbf{A}(x, y, z, t) = \mu \int_V \frac{\mathbf{J}(x', y', z', t - R/v) \, dV'}{4\pi R} \tag{4}$$

where

$$v = (\mu\varepsilon)^{-1/2} \tag{5}$$

(for free space, $v = c = 2.9987 \times 10^8$ m/s) and R is the distance between source point (x', y', z') and field point (x, y, z),

$$R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2} \tag{6}$$

In the above, $t - R/v$ denotes that, for an evaluation of Φ at time t, the value of charge density ρ at time $t - R/v$ should be used. That is, for each element of charge $\rho\, dV$, the equation says that the contribution to potential is of the same form as in statics, (1), except that we must recognize a finite time of propagating the effect from the charge element to the point P at which potential is being computed, distance R away. The effect travels with velocity $v = 1/\sqrt{\mu\varepsilon}$, which, as we have seen, is just the velocity of a simple plane wave through the medium as predicted from the homogeneous wave equation. Thus, in computing the total contribution to potential Φ at a point P at a given instant t, we must use the values of charge density from points distance R away at an earlier time, $t - R/v$, since for a given element it is that effect which just reaches P at time t. A similar interpretation applies to the computation of **A** from currents in (4). Because of this "retardation" effect, the potentials Φ and **A** are called the *retarded potentials.* Once the phenomenon of wave propagation predicted from Maxwell's equations is known, this is about the simplest revision of the static formulas (1) and (2) that could be expected.

Example 3.20
FIELD FROM AN AC CURRENT ELEMENT

One of the simplest examples illustrating the meaning of this retardation, and one that will be met again in the study of radiating systems, is that of a very short wire carrying an ac current varying sinusoidally in time between two small spheres on which charges accumulate (Fig. 3.20). For a filamentary current in a small wire, the differences in

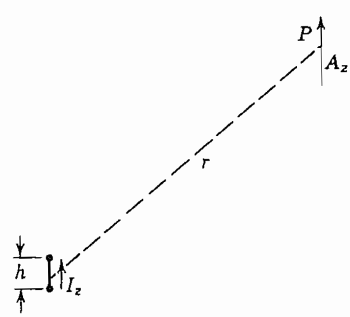

FIG. 3.20 Retarded potential from small current element.

distance from P to various points of a given cross section of the wire are unimportant, so that two parts of the volume integral in (4) may be done by integrating current density over the cross section to yield the total current in the wire. Thus, for any filamentary current,

$$\mathbf{A} = \mu \int \frac{I(t - r/v)\, d\mathbf{l}}{4\pi r} \tag{7}$$

For the particular case of Fig. 3.20, current is in the z direction only; so, by the above, $\mathbf{A}$ is also. If h is so small compared with r and wavelength, the remaining integration of (7) is performed by multiplying the current by h:

$$A_z = \frac{\mu h}{4\pi r} I_z\left(t - \frac{r}{v}\right) \tag{8}$$

Finally, if the current in the small element has the form

$$I_z = I_0 \cos \omega t \tag{9}$$

substitution in (8) gives A_z as

$$A_z = \frac{\mu h I_0}{4\pi r} \cos \omega\left(t - \frac{r}{v}\right) \tag{10}$$

From this value of $\mathbf{A}$, the magnetic and electric fields may be derived. This will be done when we return to radiation in Chapter 12.

3.21 The Retarded Potentials for the Time-Periodic Case

If all electromagnetic quantities of interest are varying sinusoidally in time, in the complex notation with $e^{j\omega t}$ understood, the set of equations 3.19(1), 3.19(3), 3.20(3), 3.20(4), and 3.19(6) becomes

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1}$$

$$\mathbf{E} = -\nabla\Phi - j\omega\mathbf{A} \tag{2}$$

$$\Phi(x, y, z) = \int_V \frac{\rho(x', y', z')e^{-jkR}\, dV'}{4\pi\varepsilon R} \tag{3}$$

$$\mathbf{A}(x, y, z) = \mu \int_V \frac{\mathbf{J}(x', y', z')e^{-jkR}\, dV'}{4\pi R} \tag{4}$$

$$\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\Phi \tag{5}$$

where $k = \omega/v = \omega\sqrt{\mu\varepsilon}$, and R is the distance between source and field points. Note that the retardation in this case is taken care of by the factor e^{-jkR} and amounts to a shift in phase of each contribution to potential according to the distance R from the contributing element to the point P at which potential is to be computed. (From here on we will frequently leave out the functional notation of coordinates, with the understanding that potentials are computed for the field point, and the integration is over all source points.)

For these steady-state sinusoids the relation between $\mathbf{A}$ and Φ in (5) fixes Φ uniquely once $\mathbf{A}$ is determined. Thus, it is not necessary to compute the scalar potential Φ separately. Both $\mathbf{E}$ and $\mathbf{B}$ may be written in terms of $\mathbf{A}$ alone:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6}$$

$$\mathbf{E} = -\frac{j\omega}{k^2}\nabla(\nabla \cdot \mathbf{A}) - j\omega\mathbf{A} \tag{7}$$

$$\mathbf{A} = \mu \int_V \frac{\mathbf{J}e^{-jkR}\, dV'}{4\pi R} \tag{8}$$

It is then necessary only to specify the current distribution over the system, to compute the vector potential $\mathbf{A}$ from it by (8), and then find the electric and magnetic fields by (6) and (7). It may appear that the effects of the charges of the system are being left out, but of course the continuity equation

$$\nabla \cdot \mathbf{J} = -j\omega\rho \tag{9}$$

relates the charges to the currents and, in fact, in this steady-state sinusoidal case, fixes ρ uniquely once the distribution of $\mathbf{J}$ is given. So an equivalent but lengthier procedure would be that of computing the charge distribution from the specified current distribution by means of the continuity equation (9), then using the complete set of equations (1) to (4).

PROBLEMS

3.2a A magnetic field of the approximate form $\mathbf{B} = \hat{\mathbf{z}}C_0 x \sin \omega t$ passes through a rectangular loop in the x–y plane following the path (0, 0) to (a, 0) to (a, b) to (0, b) to (0, 0). Show that if Faraday's law in microscopic form is used to give $\mathbf{E}$, the macroscopic form of Faraday's law is satisfied.

3.2b Find the emf around a circular loop in the $z = 0$ plane if the loop is threaded by an axial magnetic field varying with r, ϕ, and t approximately as

$$B_z = C_0 r \sin \phi \sin \omega t$$

Now find the emf for a circuit consisting of a half-circle of radius a and a straight wire from $r = a, \phi = 0$ to $r = a, \phi = \pi$.

3.2c* The betatron makes use of the electric field produced by a time-varying magnetic field in space to accelerate charged particles. Suppose that the magnetic field of a betatron has an axial component in circular cylindrical coordinates of the following form:

$$B_z(r, t) = Ctr^n \qquad (t \geq 0)$$

* *The asterisk on problems denotes ones longer or harder than the average; two asterisks denote unusually difficult or lengthy problems.*

Find the induced electric field in magnitude and direction at a particular radius a. From this find velocity v on a charge q after a time t, assuming that the charge stays in a path of constant radius a, and calculate the magnetic force on the charge. For what value(s) of n will this magnetic force just balance the centrifugal force (mv^2/a), where m is mass of the particle, so that it can remain in the path of constant radius as assumed?

3.2d Current I in a long, round wire varies sinusoidally with time. Assume no variation with z (coordinate along the wire) or ϕ (angular coordinate around the wire). Find magnetic field outside the wire, assuming it related to current flow at any instant as in the statics form. (This is called the "quasistatic" approximation.) From the differential equation form of Faraday's law find the electric field in the space outside the wire generated by the changing magnetic field of the form found. Use phasor forms and take the electric field at the surface of the wire ($r = a$) as the product of current and internal impedance Z_i per unit length. Note behavior at infinity. What is unrealistic about this model?

3.3a To demonstrate Faraday's law in class, we often move a coil by hand through the poles of a permanent magnet and observe the generated voltage on an oscilloscope. One magnet used has a flux density of 0.1 T and pole pieces about 2 cm in diameter. Estimate the velocity you can conveniently obtain by hand motion and find how many turns you need to produce peak voltages around 10 mV. Sketch the waveform expected as the coil is moved through the region between poles.

3.3b In the generator of Fig. 3.3a, the poles are reshaped so that magnetic flux density is inhomogeneous. Take the magnetic field direction as the z direction and the vertical direction of the figure as the x direction. Assume the inhomogeneous field to have a quadratic variation with x,

$$B_z = B_m\left(1 - \frac{x^2}{a^2}\right)$$

Find emf generated in the rotating loop by considering rate of change of flux, and also by use of the motional electric field in the wires. Plot the waveform of this wave in time.

3.3c In Prob. 3.3b, it may seem surprising that motional field depends only on the value of B at the instantaneous position of the conductors, whereas flux enclosed depends upon integration of field throughout the region of inhomogeneous variation, yet both give identical answers. Explain why this is so for any arbitrary variation with x.

3.3d In the generator of Fig. 3.3a, the rectangular loop is replaced by a circular loop of radius a, rotated about a line in the plane of the loop and passing through the center. This axis is normal to $\mathbf{B}$ as in Fig. 3.3a. Take B_0 as uniform and find emf generated in this loop as it is rotated with angular velocity Ω about the defined axis.

3.3e The rectangular loop of Ex. 3.3 is moved with constant velocity v in the x direction through an inhomogeneous magnetic field which varies sinusoidally with x,

$$B_z = C \sin\left(\frac{\pi x}{L}\right)$$

Find the induced emf, both from rate of change of flux and by use of the motional electric field. Find values for the special cases $a/L = \frac{1}{2}, 1, 2$.

3.3f A wire in the form of a rectangular loop with one arm at $x = 0$ extending from $y = 0$ to $y = \mathrm{b}$ and two parallel arms at $y = 0$ and $y = \mathrm{b}$ extending from $x = 0$ in the $+x$ direction as in Fig. P3.3f has static flux density B_0 in the z direction. A sliding short at

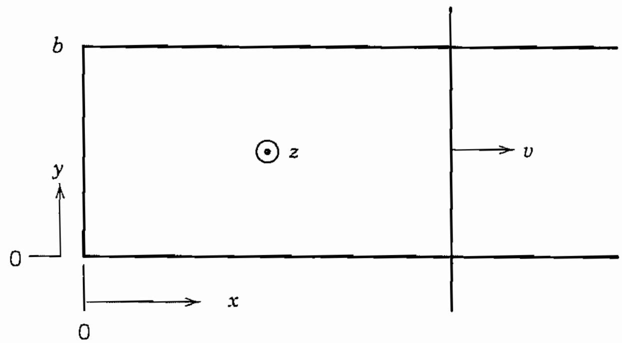

FIG. P3.3f

x moves with velocity v. Find the emf induced in the loop by rate of change of flux and by the motional method.

3.3g A long, straight wire carries a time-varying current I. A rectangular circuit of length ℓ lies in the r, z plane, with one leg distance r_1 from the axis and the other at r_2 as in Fig. P3.3g. Find the emf induced in the loop.

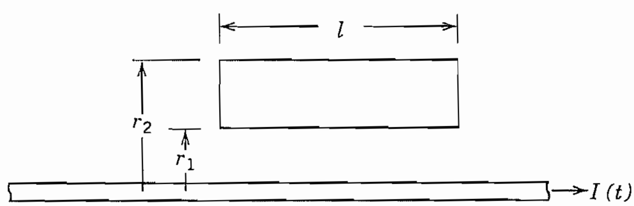

FIG. P3.3g

3.4 Conduction current density for copper with a field of 0.1 V/m (1 mV/cm) applied is 5.8×10^6 A/m^2. What number density of electrons is required to produce the same value of *convection* current density if the electrons have been accelerated in vacuum through a potential of 1 kV? What electric field magnitude would be required to produce the same magnitude of *displacement* current density in space for sinusoidally varying waves as follows: (1) a power wave of frequency 60 Hz; (2) a microwave beam of frequency 3 GHz; (3) a laser beam of wavelength 1.06 μm?

3.5a Starting from Eq. 3.4(7), prove that for a closed surface

$$\oint_S (\mathbf{J}_c + \mathbf{J}_d) \cdot d\mathbf{S} = 0$$

From this, show that the sum of convection and displacement currents is the same for both of the surfaces S_1 and S_2 in Fig. 3.5*b*. For a spherical capacitor with concentric conductors of radii a and b, with sinusoidal voltage applied between conductors, find displacement current for $a < r < b$ and show that it is equal to the charging current in the leads to the capacitor.

3.5b Obtain the expressions for electric field, Eqs. 3.5(4)–(6), from the divergence equation.

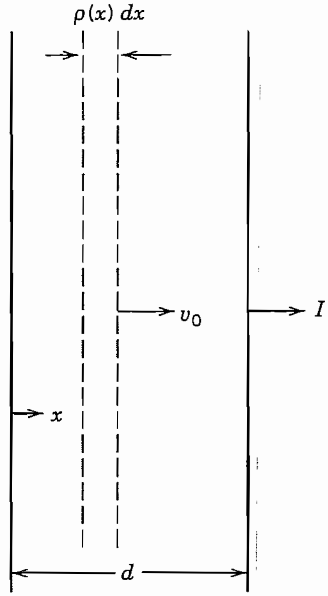

FIG. P3.5c

3.5c In a large class of electron devices, of which the klystron is a good example, current in the output circuit is induced because of the time-varying conduction current crossing the output gap. The ac current is superposed on the dc beam and moves across the gap approximately at the dc velocity v_0 of the electrons so that convection current density in phasor form may be written

$$J_c(x) = J_0 + J_1 e^{-j\omega x/v_0}$$

Suppose the output gap may be represented by parallel-plane electrodes as in Fig. P3.5c. The results of Ex. 3.5 may then be used for the induced current for each elemental slice of length dx, and total induced current for the gap may be obtained by integrating contributions over the total length d of the gap. Carry out the integration to find the induced current in the output gap and notice how it depends upon transit angle, $\omega d/v_0$.

3.6a Check the dimensional consistency of Eqs. (1) through (9) of Sec. 3.6.

3.6b Show that, if the equation for continuity of charge is assumed, the two divergence equations, 3.6(1) and (2), may be derived from the curl equations, (3) and (4), so far as ac components of the field are concerned, for regions with finite ρ and $\mathbf{J}$. This fact has made it quite common to refer to the two curl equations alone as Maxwell's equations.

3.6c Check to see which if any of following could be a field consistent with Maxwell's equations. If a special condition for ρ and $\mathbf{J}$ is needed, discuss its physical reasonableness.

(i) $\mathbf{B} = \hat{\mathbf{z}}xt$ (rectangular coordinates)
(ii) $\mathbf{E} = \hat{\mathbf{r}}C/r$ (circular cylindrical coordinates)
(iii) $\mathbf{E} = \hat{\mathbf{r}}(C/r)\cos(\omega t - \omega\sqrt{\mu\varepsilon}z)$ (circular cylindrical coordinates)

3.7a A conducting spherical balloon is charged with a constant charge Q, and its radius made to vary sinusoidally in time in some manner from a minimum value, r_{min}, to a maximum value, r_{max}. It might be supposed that this would produce a spherically symmetric, radially outward propagating electromagnetic wave. Show that this does not happen by finding the electric field at some radius $r > r_{\text{max}}$.

3.7b A capacitor formed by two circular parallel plates has an essentially uniform axial electric field produced by a voltage $V_0 \sin \omega t$ across the plates. Utilize the symmetry to find the magnetic field at radius r between the plates. Show that the axial electric field could not be exactly uniform under this time-varying condition.

3.7c Suppose that there were free magnetic charges of density ρ_m, and that a continuity relation similar to Eq. 3.4(5) applied to such charges. Find the magnetic current term that would have to be added to Maxwell's equations in such a case. Give the units of ρ_m and of magnetic current density.

3.8a Under what conditions can a complex vector quantity **E** be represented by a vector magnitude and phase angle,

$$\mathbf{E} = \mathbf{E}_0 e^{j\theta_E}$$

where $\mathbf{E}_0$ is a real vector and θ_E a real scalar?

3.8b** Consider a case in which the complex field vectors can be represented by single values of magnitude and phase:

$$\mathbf{E} = \mathbf{E}_0(x, y, z)e^{j\theta_1(x,y,z)}$$
$$\mathbf{H} = \mathbf{H}_0(x, y, z)e^{j\theta_2(x,y,z)}$$
$$\mathbf{J} = \mathbf{J}_0(x, y, z)e^{j\theta_3(x,y,z)}$$
$$\rho = \rho_0(x, y, z)e^{j\theta_4(x,y,z)}$$

Substitute in Maxwell's equations in the complex form, and separate real and imaginary parts to obtain the set of differential equations relating $\mathbf{E}_0, \mathbf{H}_0, \ldots, \theta_4$. Check the result by using the corresponding instantaneous expressions,

$$\mathbf{E}_{\text{inst.}} = \text{Re}[\mathbf{E}_0 e^{j\theta_1} e^{j\omega t}] = \mathbf{E}_0(x, y, z) \cos[\omega t + \theta_1(x, y, z)], \text{ etc.}$$

substituting in Maxwell's equations for general time variations, eliminating the time variations, and again getting the set of equations relating $\mathbf{E}_0, \ldots, \theta_4$.

3.8c Check to see which, if any, of the following could be phasor representations of fields consistent with Maxwell's equations, in a charge-free region:

(i) $\mathbf{E} = \hat{\mathbf{x}} C e^{-j\omega\sqrt{\mu\varepsilon}z}$ (rectangular coordinates)
(ii) $\mathbf{H} = \hat{\boldsymbol{\phi}}(C/r)e^{-j\omega\sqrt{\mu\varepsilon}z}$ (circular cylindrical coordinates)
(iii) $\mathbf{E} = \hat{\boldsymbol{\theta}}(C/r)e^{-j\omega\sqrt{\mu\varepsilon}r\cos\theta}$ (spherical coordinates)

3.9a Plot the sinusoidal solution 3.9(16) versus $\omega z/v$ for various times, $\omega t = 0, \pi/4, \pi/2, 3\pi/4, \pi$, and 2π, and interpret as a traveling wave.

3.9b A uniform plane wave is excited by a waveshape E_x rectangular in time. That is, $E_x = C$ for $mT < t < (m + \frac{1}{2})T$, $m = 0, 1, 2, 3$, and zero otherwise. Plot E_x versus distance z for $t = T/4, 3T/4, 5T/4, 7T/4$.

3.9c A uniform plane wave has electric field at $z = 0$ given as $E_x(0, t) = \cos \omega t + \frac{1}{2} \cos 3\omega t$. Sketch E_x versus distance for a few periods in an ideal dielectric with no dispersion. Repeat for a dielectric in which wave velocity at frequency 3ω is $\frac{1}{3}$ that at frequency ω.

3.10a In Sec. 3.10 a wave with E_x and H_y is analyzed. The other set of fields (or the other *polarization* as it will be called in Chapter 6) relates E_y and H_x. With the same assumptions as to uniformity in the x–y planes, find the wave solutions for this set in phasor form.

3.10b A radio wave that may be considered a uniform plane wave propagates through the ionosphere and interacts with some charged particles which are moving with a velocity a tenth the velocity of light in the direction normal to the magnetic field of the wave. Find the ratio of the magnetic force of the wave on the charges to the electric force.

3.10c Show that Faraday's law is satisfied in the forward-traveling plane wave with sinusoidal variations by taking a line integral of electric field from $z = 0, x = 0$ to $z = 0, x = a$ to $z = d, x = a$ to $z = d, x = 0$ back to $(0, 0)$ and relating it to the magnetic flux through that path.

3.10d Show that the generalized Ampère's law is satisfied for the forward-traveling plane wave with sinusoidal variations by taking a line integral of magnetic field from $z = 0, y = 0$ to $z = 0, y = b$ to $z = d, y = b$ to $z = d, y = 0$ back to $(0, 0)$ and relating it to displacement current through that path.

3.11a What are the relations among the constants required for each of the following to be a solution of the three-dimensional Helmholtz equation?

(i) $E_x = C \sin k_x x \sin k_y y \sin k_z z$

(ii) $E_x = C \sinh K_x x \sin k_y y \sin k_z z$

(iii) $E_x = C \sinh K_x x \sinh K_y y \sinh K_z z$

3.11b Show that the wave equation may be written directly in terms of any of the components of **H** or **E** in rectangular coordinates, or for the axial components of **H** or **E** in any coordinate system, but not for other components, such as radial and tangential components in cylindrical coordinates, or any component in spherical coordinates. That is,

$$\nabla^2 E_x = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}, \quad \nabla^2 H_z = \mu\varepsilon \frac{\partial^2 H_z}{\partial t^2}, \quad \text{etc}$$

but

$$\nabla^2 E_r \neq \mu\varepsilon \frac{\partial^2 E_r}{\partial t^2}, \quad \nabla^2 H_\phi \neq \mu\varepsilon \frac{\partial^2 H_\phi}{\partial t^2}, \quad \text{etc.}$$

3.11c Check to see under what conditions the following is a solution of the Helmholtz equation in circular cylindrical coordinates:

$$\mathbf{E} = \hat{\mathbf{r}}\left(\frac{C}{r}\right)e^{-jk_z z}$$

(Note that the vector form of ∇^2 must be used; see Prob. 3.11b.)

3.12a Describe the Poynting vector and discuss its interpretation for the case of a static point charge Q located at the center of a small loop of wire carrying direct current I.

3.12b Assuming current density constant over the conductor cross section in the Ex. 3.12a, find the Poynting vector within the wire and interpret this in terms of the distribution of dissipation.

3.12c Interpret the Poynting vector about a parallel-plate capacitor charged from zero to some final charge Q. Repeat for an inductor in which current builds up from zero to some final value. Repeat for each of these cases as charge and current is made to decay from a given value to zero. For the inductor, use a straight section of wire as example.

3.12d Show that the power flow in the uniform plane wave of Ex. 3.12c equals the product of the average energy density and the velocity v of the wave.

3.12e In each of the following, use a plane-wave model to estimate the quantities for various laser systems:

(i) A small helium–neon laser ($\lambda = 633$ nm) typically produces 1 mW in a beam 1 mm in diameter. Estimate strengths of electric and magnetic fields in the laser beam.

(ii) It is fairly easy to focus the power for a medium-power CO_2 laser ($\lambda = 10.6\ \mu$m) so that there is breakdown in air. Taking breakdown strength as at lower frequencies, about 3×10^6 V/m, estimate the power density in such a laser beam.

(iii) Very high power Nd–glass lasers ($\lambda = 1.06\ \mu$m) have been used in laser-fusion experiments. Estimate electric field strength at the target for one producing 10.2 kJ in 0.9 ns, focused to a target about 0.5 mm in diameter.

3.12f Show that Eq. 3.12(6) follows from Eq. 3.12(5) for linear, isotropic, time-invariant media.

3.13a Find the imaginary part of Eq. 3.13(11) and simplify by letting $C_1 = A_1 e^{j\phi_1}$ and $C_2 = A_2 e^{j\phi_2}$ where A_1, A_2, ϕ_1, and ϕ_2 are real. Show that the variation with z agrees with that on the right side of Eq. 3.13(8) using time-dependent **E** and **H**.

3.13b The field a large distance from a dipole radiator has the form, in spherical coordinates,

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}}\, H_\phi = \left(\frac{A}{r}\right) e^{-jkr} \sin\theta$$

Find the average power radiated through a large sphere of radius r.

3.14 Space is filled by two dielectrics, ε_1 filling the half-space $x > 0$ and ε_2 filling the half-space $x < 0$. Determine whether or not there can exist a uniform plane wave with E_x and H_y only and no variations with x or y, propagating in the z direction in this composite dielectric. The propagation factor may be e^{-jkz} with any value of k. Note that the wave, if it exists, must satisfy the wave equation in each region and the continuity conditions at the plane between the two regions.

3.16a Find the variation of an average Poynting vector for a plane wave within a good conductor and interpret.

3.16b Repeat Prob. 3.16a for an instantaneous Poynting vector.

3.17a Iron and tin have the same order of conductivity σ, around 10^7 S/m. For slab conductors of each of these used at 60 Hz, 1 kHz, and 1 MHz, find the surface resistance of the two materials if the relative permeability of the iron is 500.

3.17b Find the magnetic field **H** for any point x in the plane conductor in terms of J_0 by first finding the electric field, and then utilizing the appropriate one of Maxwell's equations to give **H**. Show that J_{sz} of Eq. 3.17(1) is equal to $-H_y$ at the surface.

3.18a The average power loss per unit volume at any point in the conductor is $|J_z|^2/2\sigma$. Show that Eq. 3.18(5) may be obtained by integrating over the conductor depth to obtain the total power loss per unit area.

3.18b A uniform plane wave of frequency 1 GHz has a power density of 1 MW/m^2 and falls upon an aluminum sheet. It can be shown that upon reflection from a good conductor, magnetic field at the surface of the conductor is essentially double that in the incident wave. Estimate the power absorbed in the aluminum per unit area and note it as a fraction of the incident power.

3.19a Show that **E** and **H** satisfy the following differential equations in a homogeneous medium containing charges and currents:

$$\nabla^2\mathbf{E} - \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{1}{\varepsilon}\nabla\rho + \mu\frac{\partial\mathbf{J}}{\partial t}$$

$$\nabla^2\mathbf{H} - \mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2} = -\nabla\times\mathbf{J}$$

3.19b A potential function commonly used in electromagnetic theory is the Hertz vector potential **Π**, so defined that electric and magnetic fields are derived from it as follows, for a homogeneous medium:

$$\mathbf{H} = \varepsilon\frac{\partial}{\partial t}\nabla\times\mathbf{\Pi}$$

$$\mathbf{E} = \nabla(\nabla\cdot\mathbf{\Pi}) - \mu\varepsilon\frac{\partial^2\mathbf{\Pi}}{\partial t^2}$$

where

$$\nabla^2\mathbf{\Pi} - \mu\varepsilon\frac{\partial^2\mathbf{\Pi}}{\partial t^2} = -\frac{\mathbf{P}}{\varepsilon}$$

and **P**, the *polarization vector* associated with sources, is so defined that

$$\mathbf{J} = \frac{\partial\mathbf{P}}{\partial t}, \qquad \rho = -\nabla\cdot\mathbf{P}$$

Show that **E** and **H** derived in this manner are consistent with Maxwell's equations.

3.19c An alternative to the *Lorentz gauge*, which defines $\nabla\cdot\mathbf{A}$ by Eq. 3.19(6), is the *Coulomb gauge* which selects it to give $\nabla\cdot\mathbf{A} = 0$. Give the differential equations relating Φ and **A** to sources ρ and **J** in this case. Discuss problems in use of this apparently simpler gauge. Note that the equations for Lorentz and Coulomb gauges become identical in the static limit and for charge-free time-varying systems.

3.19d The retarded potentials are generally used only for homogeneous media. Show the complications in attempting to extend the development to media with μ and ε functions of position.

3.20a By analogy with the integral solutions for **A** and Φ, write the integral for the Hertz vector **Π** in terms of the polarization **P**. (See Prob. 3.19b.) Note the relation between **Π** and **A** when time variations are as $e^{j\omega t}$.

3.20b* From continuity of charge, find the values of the charges that must exist at the ends of the small current element of Ex. 3.20. Find scalar potential Φ from these charges, using Eq. 3.20(3). Show that Φ and the **A** of Eq. 3.20(8) are related by the Lorentz condition 3.19(6).

3.20c Using **A** from Sec 3.20 and Φ from Prob 3.20b, find electric and magnetic fields in spherical coordinates for the small current element of Ex. 3.20, with sinusoidal current variation given by Eq. 3.20(9).