# 7 Two- and Three-Dimensional Boundary Value Problems

## 7.1 INTRODUCTION

In the preceding chapters, numerous special techniques have been presented for solving static and dynamic field problems. Before continuing with the important problems of wave guiding, resonance, and interaction of fields with materials and radiation, it is necessary to develop some more general and somewhat more powerful techniques of problem solution. The methods developed in this chapter will usually be illustrated first through static examples before extending to dynamic problems, and in some cases are most useful for static or quasistatic problems. Even then such solutions are of use in certain time-varying problems, as we have seen in the case of circuits and transmission lines in the preceding chapter.

The approach in this chapter is mostly through the solution of differential equations subject to boundary conditions. In certain cases the field distributions themselves are desired, but in other cases (as we saw in the calculation of circuit elements) these distributions are only steps along the way to other useful parameters.

The most general analytical method to be considered in this chapter is that of separation of variables, leading to orthogonal functions which may be superposed to represent very general field distributions. In developing this method we will spend some time on the special functions needed for circular cylindrical coordinates (Bessel functions) and for spherical coordinates (Legendre functions). A second powerful analytical method is that of conformal transformation. Although restricted to two-dimensional problems and useful primarily (but not exclusively) for solutions of Laplace's equation, it is the most convenient way of solving many problems of importance in circuits and transmission lines.

Numerical solution of field problems becomes increasingly important with the continuing advances in computing power. This is a special field in itself and a rapidly changing one, but we will give some idea of its basis and some elementary approaches to its use.

## The Basic Differential Equations and Numerical Methods

### 7.2 Roles of Helmholtz, Laplace, and Poisson Equations

We have seen how specific differential equations—the wave equation, the Helmholtz equation, and the diffusion equation—result from Maxwell's equations with certain specializations. We shall generally be concerned with such special cases, but let us look first at somewhat more general forms. We use the phasor forms, and limit ourselves to homogeneous, isotropic, and linear media. Starting with the Maxwell equation for curl **E** [Eq. 3.8(3)],

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{1}$$

The curl of this is taken and expanded (inside front cover)

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2\mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) = -j\omega\mu\nabla \times \mathbf{H} \tag{2}$$

The divergence of **E** and curl of **H** are substituted from the Maxwell equations Eqs. 3.8(1) and 3.8(4):

$$-\nabla^2\mathbf{E} + \nabla\left(\frac{\rho}{\varepsilon}\right) = -j\omega\mu[\mathbf{J} + j\omega\varepsilon\mathbf{E}]$$

or

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = j\omega\mu\mathbf{J} + \frac{1}{\varepsilon}\nabla\rho \tag{3}$$

where $k^2 = \omega^2\mu\varepsilon$. By similar operations on the curl **H** equation, we obtain

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = -\nabla \times \mathbf{J} \tag{4}$$

Equations (3) and (4) may be considered inhomogeneous Helmholtz equations. General solutions of these are difficult, but usually start from solutions of the corresponding homogeneous equations[1]

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \tag{5}$$

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = 0 \tag{6}$$

[1] *J. D. Jackson,* Classical Electrodynamics, *2nd ed., Wiley, New York, 1975.*

Many of the problems we are concerned with have no sources except on the boundaries, so the Helmholtz equations considered in this chapter are the homogeneous ones, (5) and (6).

Note that the vector equations separate simply in rectangular coordinates,

$$\nabla^2 E_x + k^2 E_x = 0 \tag{7}$$

and similarly for $E_y$, $E_z$, $H_x$, $H_y$, and $H_z$. They do not separate so simply for curvilinear coordinates, as one can see by examining the expansion for $\nabla^2$ of a vector in cylindrical and spherical coordinates (inside front cover). But for any cylindrical coordinate system, the axial component of (5) or (6) satisfies a simple Helmholtz equation,

$$\nabla^2 E_z + k^2 E_z = 0 \tag{8}$$

and similarly for $H_z$.

For quasistatic problems, the term in $k^2$ is negligible so that (5) and (6) reduce to Laplace equations:

$$\nabla^2 \mathbf{E} = 0 \tag{9}$$

$$\nabla^2 \mathbf{H} = 0 \tag{10}$$

These separate into coordinate components as discussed above. However, for quasistatic or purely static problems it is often more convenient to use the scalar potential functions defined by

$$\mathbf{E} = -\nabla\Phi, \qquad \mathbf{H} = -\nabla\Phi_m \tag{11}$$

with $\Phi$ and $\Phi_m$ satisfying Laplace equations,

$$\nabla^2\Phi = 0, \qquad \nabla^2\Phi_m = 0 \tag{12}$$

In certain cases we are concerned with static or quasistatic solutions for regions containing charges, in which case the Poisson equation applies to $\Phi$ (Sec. 1.12):

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon} \tag{13}$$

Thus the Laplace, Helmholtz, and Poisson equations govern a large number of important problems and will be the ones used for illustration of solution methods in this chapter.

**Boundary Conditions** As noted in Sec. 1.17, unique solutions of the Laplace or Poisson equation resulted if the function is specified on a boundary surrounding the region of interest. Specification of the normal derivative on such a boundary determines the solution within a constant. Section 3.14 pointed out that unique solutions of the Helmholtz equation (5) or (6) are obtained by specifying the tangential component of **E** or **H** on the closed boundary, or tangential **E** on a part of the boundary and tangential **H** on the remainder.

**Superposition** Since $\nabla^2$ is a linear operator (as are the other operators in Maxwell's equations), any two solutions are superposable and the sum is a solution provided that the medium itself is linear. We have made use of this fact previously, as in the super-

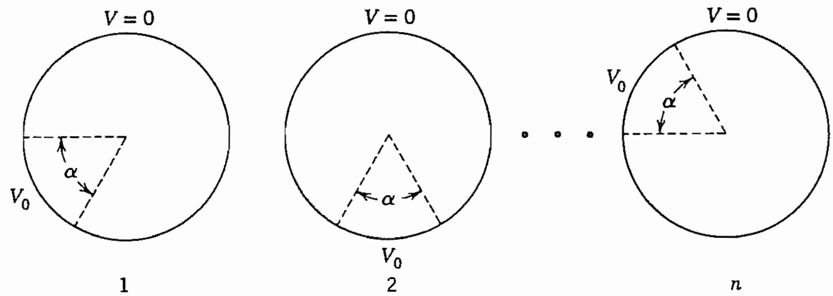



**FIG. 7.2** Series of circular cylinders with sectors of angle $\alpha$ at the potential $V_0$ oriented at multiples of $\alpha$ with respect to each other.

position of linearly polarized waves to form a circularly polarized one. We will use the principle in many future examples. Here we give an example of its use in reasoning to a simple result.

### Example 7.2
#### SOLUTION BY INVERSE APPLICATION OF SUPERPOSITION

An interesting example of the use of superposition is the solution for the potential at the center of a symmetrical structure. For example, consider a homogeneous dielectric surrounded by the circular cylinder shown in Fig. 7.2, with a potential $V_0$ applied over a portion of the boundary subtending the angle $\alpha$ and zero potential on the remainder. Suppose that $\alpha = 2\pi/n$. If the potential at the center were found for $n$ different sets of boundary conditions as shown in Fig. 7.2, where the only difference between these is that the section of the boundary to be at potential $V_0$ is rotated by the angle $k\alpha$, with $k$ an integer, the sum of the $n$ solutions would be the potential at the center of a cylinder with $V_0$ over the entire boundary; this is just $V_0$. Since every problem is identical except for a rotation by $\alpha$, which would not affect the potential at the center, the potential at the center for the original problem must be $V_0/n$. This same technique could be applied to find the potential at the center point of a square, cube, equilateral polygon, sphere, and so on, with one portion at a given potential.

## 7.3 NUMERICAL METHODS: METHOD OF MOMENTS

Easy accessibility to powerful computers has greatly expanded our ability to obtain accurate solutions for electromagnetic field problems. The range of use extends from convenient evaluation of analytic expressions, including ones for which no closed-form solutions exist, to wholly numerical solutions. Whenever it is possible to find even an approximate analytic solution, it is useful for seeing parametric dependences to gain physical insight, but more precise solutions can be obtained numerically.

We consider here only some basic methods. There are other, more specialized techniques, several of which find use in analyzing transmission structures of the kind to be studied in the following chapters.[2,3] The choice of method should be based on a tradeoff among accuracy, speed, versatility, and computer memory requirements.

The *finite-difference* method was introduced in Sec. 1.20; though simple, it has considerable range of application. A method with similar use, called the *finite-element* method, is somewhat more difficult to understand and the programming is more complex, but it has the advantage of adapting well to complex boundary shapes and also to spatially varying properties of the medium (i.e., permittivity or permeability). In both of these methods the typical calculation involves a large *banded* matrix (nonzero elements only along and near the diagonal). There are well-developed methods for inverting such a *sparse* matrix, and we saw in Sec. 1.20 an iterative method.

One may use a more computationally efficient approach, called the *method of moments*, for some problems, especially when integral quantities such as capacitance are required.[4] It is based on an integral equation rather than the differential equation on which the finite-difference and finite-element methods are based.

If charge is transferred between two conducting bodies in otherwise free space, a potential difference will exist between them and the charges will become distributed over the surfaces in such a way that the tangential electric field at the conductor surfaces is zero. This is analogous to the situation seen in the study of images in Sec. 1.18, where a point or line charge placed near a conducting surface induces surface charge on the conducting body and this cancels the tangential electric field of the source charge. Likewise, if charge is placed on an isolated conducting body, the charges will distribute themselves on the surface to eliminate the tangential electric field. The method of moments results in knowledge of the charge distribution on the surfaces and the total charge for a given potential, and hence the capacitance.

We introduce here a simple way of applying the method of moments to find static charge distributions and capacitances for two- and three-dimensional electrode systems. Some structural forms that can be treated are shown in Figs. 7.3*a*–*c*. They are shown with their surfaces subdivided into small elements to prepare for discrete numerical calculations. The surface charge density $\rho_{si}$ is assumed to be uniform over each element. The total charge ascribed to the $i$th element on a 3D structure is $\rho_{si}\Delta S_i$, where $\Delta S_i$ is the area. We will treat it as a point charge at the center of the element in making the potential calculations. The 2D structures have no variations in the axial direction and the surfaces are divided into strips of width $\Delta l_i$. The charge density $\rho_{si}$ in this case is

[2] *R. C. Boonton, Jr.*, Computational Methods for Electromagnetics and Microwaves, *Wiley, New York, 1992.*

[3] *R. Sorrentino (Ed.),* Numerical Methods for Passive Microwave and Millimeter Wave Structures, *IEEE Press, New York, 1989.*

[4] *R. F. Harrington,* Field Computation by Moment Methods, *R. E. Krieger, Malabar, FL, 1987; orig. ed., 1968.*

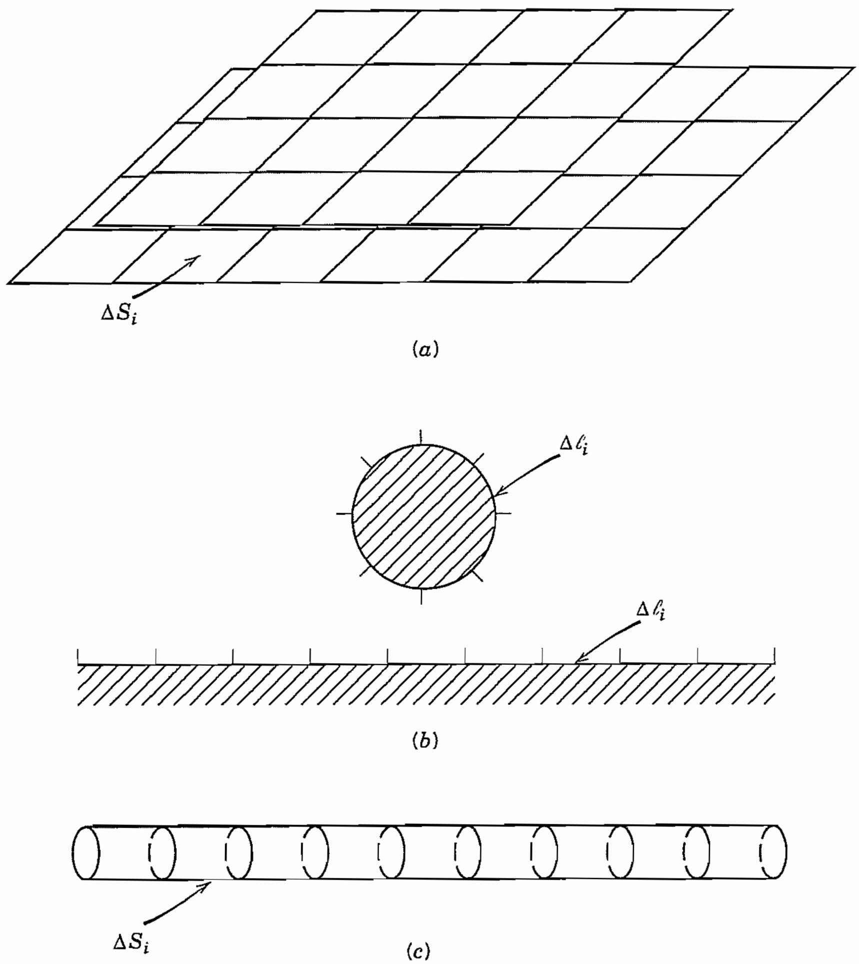



**FIG. 7.3** Examples of structures suited for evaluation by methods of moments. (*a*) Three-dimensional parallel-plate capacitor. (*b*) Round two-dimensional cylinder over a ground plane. (*c*) Isolated rod of finite length.

multiplied by $\Delta l_i$ to give the charge per unit length along the axial direction, which is represented by a line charge $q_l$ in the center of the element for the purposes of calculating potential. These point and line charges are used to calculate the potentials also at the centers of the elements.

**Three-Dimensional Structures** Potentials are calculated using the formulas for charges in free space since the conductors are accounted for by including all charges on their surfaces. The potential at the center of the $i$th element in the 3D case is written using Eq. 1.8(3) as

$$\Phi_i = \Phi_{ii} + \sum_{j \neq i}^{N} \frac{\rho_{sj} \Delta S_j}{4\pi\varepsilon |\mathbf{r}_i - \mathbf{r}_j|} \tag{1}$$

The term $\Phi_{ii}$ is the potential at the center of the $i$th element resulting from the charge on the $i$th element itself; it must be handled separately since the terms in the remainder of (1) are clearly singular when $i = j$. We find $\Phi_{ii}$ by integrating over the element. For convenience, we neglect the exact shape of the element, often a square, and replace it with a disk having the area of the element. Thus, with $r_0 = (\Delta S_i/\pi)^{1/2}$

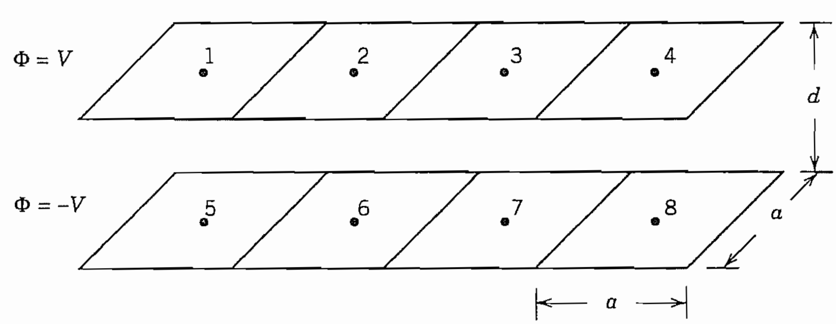
$$\Phi_{ii} = \int_0^{2\pi} d\phi \int_0^{r_0} \frac{\rho_{si} r\, dr}{4\pi\varepsilon r} = 0.282 \frac{\rho_{si}}{\varepsilon} \sqrt{\Delta S_i} \tag{2}$$

One equation of the form (1) is written for each element, thus giving a set of $N$ equations in the $N$ unknown charges in terms of the given potentials on the electrodes.

### Example 7.3a
THREE-DIMENSIONAL CAPACITOR

Let us calculate the charge distribution and capacitance of the structure in Fig. 7.3*d*. To achieve high accuracy, it is necessary to have many subdivisions of the surfaces, but here we take a very coarse grid to illustrate the procedures. It is assumed that there is negligible charge on the outer surfaces of the conductors. The potentials on the top and bottom electrodes are taken as $+V$ and $-V$, respectively. Multiplying (1), with (2) substituted, by $4\pi\varepsilon/a$, the equation for element 1 can be written as

$$4\pi(0.282)\rho_{s1} + \frac{a}{|\mathbf{r}_1 - \mathbf{r}_2|}\rho_{s2} + \cdots + \frac{a}{|\mathbf{r}_1 - \mathbf{r}_8|}\rho_{s8} = \frac{4\pi\varepsilon}{a} V \tag{3}$$




(*d*)

**FIG. 7.3*d*** Coarse subdivisions of parallel-plate capacitor for Ex. 7.3a.

Writing similar equations for the other seven subdivisions and casting in matrix form, we have

$$\begin{bmatrix} 3.54 & \dfrac{a}{|\mathbf{r}_1 - \mathbf{r}_2|} & \dfrac{a}{|\mathbf{r}_1 - \mathbf{r}_3|} & \cdots & \dfrac{a}{|\mathbf{r}_1 - \mathbf{r}_8|} \\ \dfrac{a}{|\mathbf{r}_2 - \mathbf{r}_1|} & 3.54 & \dfrac{a}{|\mathbf{r}_2 - \mathbf{r}_3|} & \cdots & \dfrac{a}{|\mathbf{r}_2 - \mathbf{r}_8|} \\ \vdots & & & & \\ \dfrac{a}{|\mathbf{r}_8 - \mathbf{r}_1|} & \dfrac{a}{|\mathbf{r}_8 - \mathbf{r}_2|} & \dfrac{a}{|\mathbf{r}_8 - \mathbf{r}_3|} & \cdots & 3.54 \end{bmatrix} \begin{bmatrix} \rho_{s1} \\ \rho_{s2} \\ \vdots \\ \rho_{s8} \end{bmatrix} = \frac{4\pi\varepsilon}{a} \begin{bmatrix} V \\ V \\ V \\ V \\ -V \\ -V \\ -V \\ -V \end{bmatrix} \tag{4}$$

The coefficients include $|\mathbf{r}_i - \mathbf{r}_j|$ and must be evaluated geometrically from Fig. 7.3*d*. The matrix could be entered into an inversion program in a computer and the charge densities found directly in terms of the potentials on the electrodes. The total charge $Q$ on one electrode is found and the capacitance is just $C = Q/2V$.

For the purpose of illustration, we will solve the problem by hand, making use of its symmetry to reduce the computational work. Symmetry dictates that the assumed uniform charge densities satisfy: $\rho_{s1} = \rho_{s4} = -\rho_{s5} = -\rho_{s8}$ and $\rho_{s2} = \rho_{s3} = -\rho_{s6} = -\rho_{s7}$. Since there are just two unknown variables, it is necessary to use only the first two rows of (4). Substituting values of $|\mathbf{r}_1 - \mathbf{r}_j|$ and $|\mathbf{r}_2 - \mathbf{r}_j|$ and using dimensions in Fig. 7.3*d*, we obtain

$$\begin{bmatrix} \left[3.87 - \dfrac{1}{(d/a)} - \dfrac{1}{\sqrt{9 + (d/a)^2}}\right] & \left[1.50 - \dfrac{1}{\sqrt{1 + (d/a)^2}} - \dfrac{1}{\sqrt{4 + (d/a)^2}}\right] \\ \left[1.50 - \dfrac{1}{\sqrt{1 + (d/a)^2}} - \dfrac{1}{\sqrt{4 + (d/a)^2}}\right] & \left[4.54 - \dfrac{1}{(d/a)} - \dfrac{1}{\sqrt{1 + (d/a)^2}}\right] \end{bmatrix} \begin{bmatrix} \rho_{s1} \\ \rho_{s2} \end{bmatrix} = \frac{4\pi\varepsilon}{a} \begin{bmatrix} V \\ V \end{bmatrix} \tag{5}$$

or taking $d/a = 0.5$, for example,

$$\begin{bmatrix} 1.54 & 0.121 \\ 0.121 & 1.65 \end{bmatrix} \begin{bmatrix} \rho_{s1} \\ \rho_{s2} \end{bmatrix} = \frac{4\pi\varepsilon V}{a} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6}$$

Inverting (6), we find

$$\rho_{s1} = 1.07\rho_{s2} = 0.605 \frac{4\pi\varepsilon V}{a} \tag{7}$$

The total charge on the top electrode is $Q = 2a^2(\rho_{s1} + \rho_{s2}) = 29.3\varepsilon aV$ and the capacitance is therefore $C = Q/2V = 14.7\varepsilon a$. Application of the method of moments with a fine grid would lead to a more accurate value for capacitance, which would be larger than the fringing-free idealization $C = \varepsilon A/d = 4\varepsilon a^2/(a/2) = 8\varepsilon a$.

**Two-Dimensional Structures** For the 2D case, we write, using Eq. 1.8(8),

$$\Phi_i = \Phi_{ii} - \sum_{j\neq i}^{N} \frac{\rho_{sj}\,\Delta l_j \ln|\mathbf{r}_i - \mathbf{r}_j|}{2\pi\varepsilon} + C_T \tag{8}$$

The distribution of charge is unaffected by the constant $C_T$ and it can be neglected (see Prob. 7.3c).

As in the 3D case, it is necessary to handle $\Phi_{ii}$ separately. The approach is to integrate the effect of the surface charge density over an assumed flat strip of width $\Delta l_i$. Thus,

$$\Phi_{ii} = -\frac{2\rho_{si}}{2\pi\varepsilon}\int_0^{\Delta l_i/2} \ln x\,dx = -\frac{\rho_{si}}{\pi\varepsilon}\left[x\ln x - x\right]_0^{\Delta l_i/2} \tag{9}$$

so that

$$\Phi_{ii} = -\frac{\rho_{si}\,\Delta l_i}{2\pi\varepsilon}\left[\ln\frac{\Delta l_i}{2} - 1\right] \tag{10}$$

**Example 7.3b**
STRIPLINE CAPACITANCE

Let us calculate the charge distribution and capacitance per unit length of a two-dimensional system of conductors, the so-called *stripline* configuration, which will be discussed in Sec. 8.6 and is shown in Fig. 7.3*e*. Here there are three conductors, with the outer ones extending to $y = \pm\infty$. The two outer ones are at the same (zero) potential and the center conductor carries a voltage $V$. Although the outer conductors extend to infinity, the surface charge decreases rapidly with $y$ beyond the edge of the center conductor. Therefore, we will cut off the outer conductor at some appropriate point

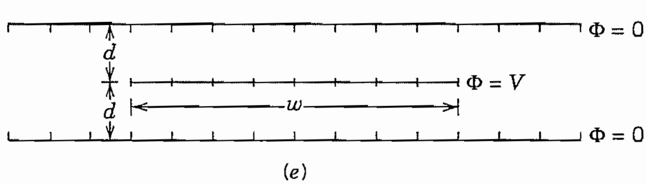



**FIG. 7.3e** Stripline structure with discretization for method of moments calculation in Ex. 7.3b.

$y_{max}$, the suitability of which could be tested by doing the problem twice with different values of $y_{max}$. We will simplify the notation by taking the widths $\Delta l$ of all segments to be the same.

The chosen segmenting is shown in Fig. 7.3*e*, where it is seen that there are 36 equal subdivisions. This is sufficiently fine for illustration, but in practice more divisions might be chosen. For this example we have 36 equations of the form (8); 8 have $\Phi = V$ and 28 have $\Phi = 0$. The equation for element 1 (on center conductor) is obtained by multiplying (8), with (10) substituted for $\Phi_{11}$, by $-2\pi\varepsilon/\Delta l$:

$$\left(\ln\frac{\Delta l}{2} - 1\right)\rho_{s1} + \ln|\mathbf{r}_1 - \mathbf{r}_2|\rho_{s2} + \cdots + \ln|\mathbf{r}_1 - \mathbf{r}_{36}|\rho_{s36} = -\frac{2\pi\varepsilon}{\Delta l}V \tag{11}$$

Or, subtracting $\rho_{si}\ln\Delta l$ from each term and summing the subtracted terms separately,

$$\begin{aligned}-(\ln 2 + 1)\rho_{s1} + \ln\frac{|\mathbf{r}_1 - \mathbf{r}_2|}{\Delta l}\rho_{s2} + \cdots + \ln\frac{|\mathbf{r}_1 - \mathbf{r}_{36}|}{\Delta l}\rho_{s36}& \\ + \ln\Delta l[\rho_{s1} + \rho_{s2} + \cdots + \rho_{s36}] &= -\frac{2\pi\varepsilon}{\Delta l}V\end{aligned} \tag{12}$$

But the total charge on the plates is zero so the term in brackets vanishes. We can write the set of equations in the form of (12) for the $N$ elements as a matrix:

$$\begin{bmatrix} -1.693 & \ln\dfrac{|\mathbf{r}_1 - \mathbf{r}_2|}{\Delta l} & \cdots & \ln\dfrac{|\mathbf{r}_1 - \mathbf{r}_{36}|}{\Delta l} \\ \ln\dfrac{|\mathbf{r}_2 - \mathbf{r}_1|}{\Delta l} & -1.693 & \cdots & \ln\dfrac{|\mathbf{r}_2 - \mathbf{r}_{36}|}{\Delta l} \\ \vdots & \vdots & & \vdots \\ \ln\dfrac{|\mathbf{r}_{11} - \mathbf{r}_1|}{\Delta l} & \ln\dfrac{|\mathbf{r}_{11} - \mathbf{r}_2|}{\Delta l} & \cdots & -1.693 \end{bmatrix} \begin{bmatrix} \rho_{s1} \\ \rho_{s2} \\ \vdots \\ \rho_{s36} \end{bmatrix} = -\frac{2\pi\varepsilon}{\Delta l} \begin{bmatrix} V \\ \vdots \\ V \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{13}$$

To obtain a numerical result, we take $w = 4d$ and $\varepsilon_r = 4$. Inversion of this matrix equation gives the charge on each element. The capacitance for the complete structure in Fig. 7.3*e* is the sum of the charges on the center conductor, found from (13), divided by $V$. Its value is $344\mu$ F/m. The value calculated analytically for an infinitely thin center conductor is found (Sec. 8.6) to be $346\mu$ F/m.

In the method of moments, the order of the matrix to be inverted is much smaller than in the finite-difference or finite-element method, but the matrix is full so that sparse matrix techniques cannot be used. An application to a time-varying radiation problem will be shown in Chapter 12.

# Method of Conformal Transformation

## 7.4 METHOD OF CONFORMAL TRANSFORMATION AND INTRODUCTION TO COMPLEX-FUNCTION THEORY

A very general mathematical attack for the two-dimensional field distribution problem utilizes the theory of functions of a complex variable. The method is in principle the most general for two-dimensional problems, and the work can be carried out to yield actual solutions for a wide variety of practical problems. For these reasons, the general method with some examples will be presented in this and the following sections.

In the theory of complex variables, we use the complex variable $Z = x + jy$, where both $x$ and $y$ are real variables. It is convenient to associate any given value of $Z$ with a point in the $x$–$y$ plane (Fig. 7.4$a$), and to call this plane the complex $Z$ plane. Of course the coordinates may also be expressed in polar form in terms of $r$ and $\theta$:

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Then

$$Z = x + jy = r(\cos\theta + j\sin\theta) = re^{j\theta} \tag{1}$$

Suppose that there is now a different complex variable $W$, where

$$W = u + jv = \rho e^{j\phi}$$

such that $W$ is some function of $Z$. This means that, for each assigned value of $Z$, there is a rule specifying a corresponding value of $W$. The functional relationship is written

$$W = f(Z) \tag{2}$$

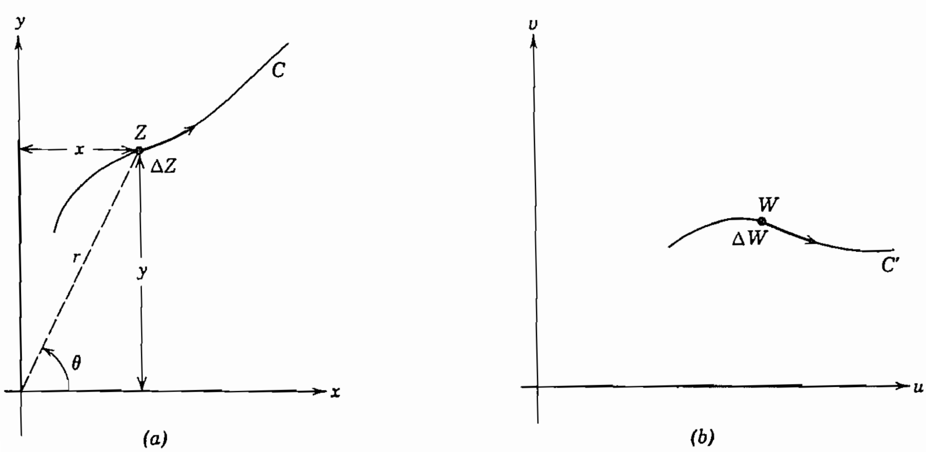



**FIG. 7.4** ($a$) $Z$ plane. ($b$) $W$ plane.

If $Z$ is made to vary continuously, the corresponding point in the complex $Z$ plane moves about, tracing out some curve $C$. The values of $W$ vary correspondingly, tracing out a curve $C'$. To avoid confusion, the values of $W$ are usually shown on a separate graph, called the complex $W$ plane (Fig. 7.4*b*).

Next consider a small change $\Delta Z$ in $Z$ and the corresponding change $\Delta W$ in $W$. The derivative of the function will be defined as the usual limit of the ratio $\Delta W/\Delta Z$ as the element $\Delta Z$ becomes infinitesimal:

$$\frac{dW}{dZ} = \lim_{\Delta Z \to 0} \frac{\Delta W}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{f(Z + \Delta Z) - f(Z)}{\Delta Z} \tag{3}$$

A complex function is said to be *analytic* or *regular* whenever the derivative defined above exists and is unique. The derivative may fail to exist at certain isolated (*singular*) points where it may be infinite or undetermined, somewhat as in real function theory. But it would appear that there is another ambiguity with respect to complex variables, since $\Delta Z$ may be taken in any arbitrary direction in the $Z$ plane from the original point. For the derivative to be unique, the ratio $\Delta W/\Delta Z$ should turn out to be independent of this direction.

If this independence of direction is to result, a necessary condition is that we obtain the same result if $Z$ is changed in the $x$ direction alone or in the $y$ direction alone. For $\Delta Z = \Delta x$,

$$\frac{dW}{dZ} = \frac{\partial W}{\partial x} = \frac{\partial}{\partial x}(u + jv) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} \tag{4}$$

For a change in the $y$ direction, $\Delta Z = j\,\Delta y$,

$$\frac{dW}{dZ} = \frac{\partial W}{\partial (jy)} = \frac{1}{j}\frac{\partial}{\partial y}(u + jv) = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y} \tag{5}$$

Two complex quantities are equal if and only if their real and imaginary parts are separately equal. Hence, (4) and (5) yield the same result if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{6}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{7}$$

These conditions, known as the *Cauchy–Riemann equations*, are then necessary conditions for $dW/dZ$ to be unique at a point and the function $f(Z)$ analytic there. It can be shown that, if they are satisfied, the same result for $dW/dZ$ is obtained for any arbitrary direction of the change $\Delta Z$, so they are also sufficient conditions.

**Example 7.4**
ANALYTICITY OF POWER FUNCTIONS

$$
\begin{aligned}
W &= Z^2 \\
u + jv = (x + jy)^2 &= (x^2 - y^2) + j2xy \\
u &= x^2 - y^2 \\
v &= 2xy
\end{aligned}
\tag{8}
$$

A check of the Cauchy–Riemann equations yields

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$$

So they are satisfied everywhere in the finite $Z$ plane, and the function is analytic everywhere there.

Actually, it is not necessary to apply the check when the functional relation is expressed explicitly between $Z$ and $W$ in terms of functions which possess a power-series expansion about the origin, as $e^Z$, sin $Z$, and so on. The reason is that each term in the series $C_n Z^n$ can be shown to satisfy the Cauchy–Riemann conditions, and consequently a series of such terms also satisfies them.

## 7.5 PROPERTIES OF ANALYTIC FUNCTIONS OF COMPLEX VARIABLES

If Eq. 7.4(6) is differentiated with respect to $x$, Eq. 7.4(7) differentiated with respect to $y$, and the resulting equations added, there results

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

Similarly, if the order of differentiation is reversed, there results

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{2}$$

These are recognized as Laplace equations in two dimensions. Thus, both the real and the imaginary parts of an analytic function of a complex variable satisfy Laplace's equation, and would be suitable for use as the potential functions for two-dimensional electrostatic problems. The manner in which these are used in specific problems and the limitations on this usefulness are demonstrated by examples in this and the next section.

For a problem in which one of the two parts, $u$ or $v$, is chosen as the potential function, the other becomes proportional to the flux function (Sec. 1.6). To show this, let us suppose that $u$ is the potential function in volts for a particular problem. The electric field, obtained as the negative gradient of $u$, yields

$$E_x = -\frac{\partial u}{\partial x}, \qquad E_y = -\frac{\partial u}{\partial y} \tag{3}$$

By the equation for the total differential, the change in $v$ corresponding to changes in the $x$ and $y$ coordinates of $dx$ and $dy$ is

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$$

But, from Cauchy–Riemann conditions, Eqs. 7.4(6) and 7.4(7),

$$-dv = \frac{\partial u}{\partial y}\,dx - \frac{\partial u}{\partial x}\,dy = -E_y\,dx + E_x\,dy$$

or

$$-\varepsilon\,dv = -D_y\,dx + D_x\,dy \tag{4}$$

By inspection of Fig. 7.5$a$, this is recognized to be just the electric flux $d\psi$ between the curves $v$ and $v + dv$, with the positive direction as shown by the arrow. Then

$$-d\psi = \varepsilon\,dv \tag{5}$$

And, except for a constant that can be set equal to zero by choosing the reference for flux at $v = 0$,

$$-\psi = \varepsilon v \quad \text{C/m} \tag{6}$$

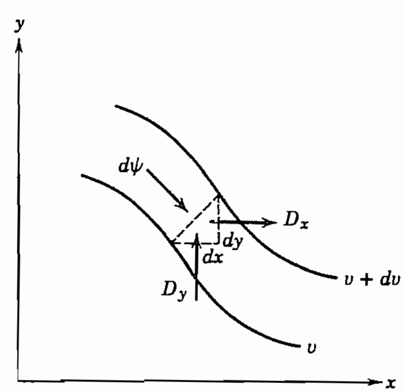



**FIG. 7.5a** Coordinates for the flux function.

Similarly, if $v$ is chosen as the potential function in volts for some problem, $\varepsilon u$ is the flux function in coulombs per meter, with proper choice of the direction for positive flux.

We have seen that either $u$ or $v$ may be used as a potential function, and then the other may be used as the flux function, since both satisfy Laplace's equation. The utility of the concept, however, hinges on being able to find the analytic function $W = f(Z)$ such that $u$ and $v$ also satisfy the boundary conditions for the problem being considered.

### Example 7.5
ELECTRODES IN PARALLEL-PLANE DIODE

As an example, suppose we desire the distribution of potentials in the $Z$ plane where the given boundary condition is

$$V = x^{4/3}, \qquad y = 0 \tag{7}$$

If we let

$$W = Z^{4/3} \tag{8}$$

it is clear that for $y = 0$, the real part of $W$ is $u = x^{4/3}$. Furthermore, we see that $dW/dZ$ exists and is unique except at $Z = 0$ (Prob. 7.4e). Thus $u$ is a suitable potential function for this problem; the real part of (8) gives the potential distribution. It is most convenient for this particular function to express $Z$ in polar coordinates

$$W = u + jv = r^{4/3}e^{j4\theta/3} \tag{9}$$

Thus

$$\begin{aligned} u &= r^{4/3} \cos \tfrac{4}{3}\theta \\ v &= r^{4/3} \sin \tfrac{4}{3}\theta \end{aligned} \tag{10}$$

Equipotentials, found by setting $u$ equal to a constant, are shown in Fig. 7.5$b$ for $u =$ 0 and 1. It is of interest that the boundary function (7) has the same form as the potential in a plane diode with the cathode at $x = 0$ and the anode potential unity at $x = 1.0$:

$$\Phi = x^{4/3}$$

Using these ideas, a plane diode can be truncated and the correct potentials produced on the free edge by placing electrodes along the equipotential lines as shown in Fig. 7.5$b$. This procedure is most important in designing electron guns with regular flow and the result is known as the Pierce gun.[5]

[5] J. R. Pierce, J. Appl. Phys. **11**, 548 (1940).

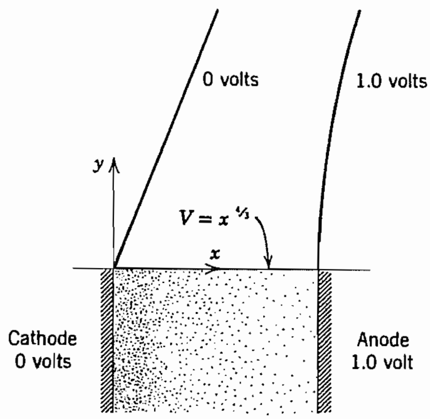



**FIG. 7.5b** Focusing for electron flow in a plane diode. The upper portion of the figure shows the electrodes outside the electron flow region.

## 7.6 Conformal Mapping for Laplace's Equation

A somewhat different point of view toward the method in Sec. 7.5 follows if we refer to the $Z$ and $W$ planes introduced in Sec. 7.4. Since the functional relationship fixes a value of $W$ corresponding to a given value of $Z$ for a given function

$$W = f(Z)$$

any point $(x, y)$ in the $Z$ plane yields some point $(u, v)$ in the $W$ plane. As this point moves along some curve $x = F(y)$ in the $Z$ plane, the corresponding point in the $W$ plane traces out a curve $u = F_1(v)$. If it should move throughout a region in the $Z$ plane, the corresponding point would move throughout some region in the $W$ plane. Thus, in general, a point in the $Z$ plane transforms to a point in the $W$ plane, a curve transforms to a curve, and a region to a region, and the function that accomplishes this is frequently spoken of as a particular *transformation* between the $Z$ and $W$ planes.

When the function $f(Z)$ is analytic, as we have seen, the derivative $dW/dZ$ at a point is independent of the direction of the change $dZ$ from the point. The derivative may be written in terms of magnitude and phase:

$$\frac{dW}{dZ} = Me^{j\alpha} \tag{1}$$

or

$$dW = Me^{j\alpha}\, dZ \tag{2}$$

By the rule for the product of complex quantities, the magnitude of $dW$ is $M$ times the magnitude of $dZ$, and the angle of $dW$ is $\alpha$ plus the angle of $dZ$. So the entire infinitesimal region in the vicinity of the point $W$ is similar to the infinitesimal region in the vicinity of the point $Z$. It is magnified by a scale factor $M$ and rotated by an angle $\alpha$. It is then evident that, if two curves intersect at a given angle in the $Z$ plane, their transformed curves in the $W$ plane intersect at the same angle, since both are rotated through the angle $\alpha$. A transformation with these properties is called a *conformal transformation.*

In particular, the lines $u$ = constant and the lines $v$ = constant in the $W$ plane intersect at right angles, so their transformed curves in the $Z$ plane must also be orthogonal (Fig. 7.6*a*). We already know that this should be so, since the constant $v$ lines have been shown to represent flux lines when the constant $u$ lines are equipotentials, and vice versa. From this point of view, the conformal transformation may be thought of as one that takes a uniform field in the $W$ plane (represented by the equispaced constant $u$ and constant $v$ lines) and transforms it so that it fits the given boundary conditions in the $Z$ plane, always keeping the required properties of an electrostatic field.

Frequently the transformation is done in steps. That is, the uniform field is transformed first into some intermediate complex plane by $Z_1 = f(W)$, then perhaps into a second intermediate complex plane $Z_2 = g(Z_1)$, and then finally into a plane $Z_3 = h(Z_2)$ in which the boundary conditions are satisfied. In general, there can be any number of steps. Of course, these functions can be combined into a single transformation, the inverse of which can then be understood on the basis of finding a function with real or imaginary part satisfying the given boundary conditions as discussed in Sec. 7.5.

There are few circumstances in which knowledge of the required boundary conditions will lead directly to the transformation that gives the solution. For help in finding the required form there are tables of conformal transformations[6] which show how one field maps into another. The mapping functions given in the tables may be used individually

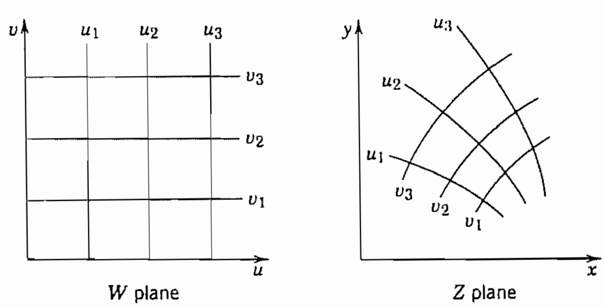



**FIG. 7.6a** A mapping of coordinate lines of the $W$ plane in the $Z$ plane.

[6] *For example, see H. Kober,* Dictionary of Conformal Representations, *Dover, New York, 1952. Also see R. Shinzinger and P. A. A. Laura,* Conformal Mapping: Methods and Applications, *Elsevier, Amsterdam, 1991.*

or combined in a series of steps to transform the uniform field into a field that fits the given problem. Some examples of the simpler transformations will be given to illustrate the method.

### Example 7.6a
### THE POWER FUNCTION: FIELD NEAR A CONDUCTING CORNER

As a basic example, consider $W$ expressed as $Z$ raised to some power:

$$W = Z^p \tag{3}$$

It is convenient to use the polar form for $Z$ [Eq. 7.4(1)]:

$$W = (re^{j\theta})^p = r^p e^{jp\theta}$$

or

$$u = r^p \cos p\theta \tag{4}$$

$$v = r^p \sin p\theta \tag{5}$$

From the conformal-mapping point of view, the field in the $W$ plane is uniform. The parallel lines of equal potential (say, $v$ equals constant) in the $W$ plane can be mapped into the $Z$ plane by setting $v$ equal to constant in (5). From the viewpoint of Sec. 7.5 one does not take explicit consideration of the existence of the $W$ plane but simply recognizes that $v$ is a solution of Laplace's equation and tries to adjust constants such that constant $v$ lines fit the equipotentials of the given problem. When only one step of transformation is required, the viewpoints are wholly equivalent.

If $v$ is chosen as the potential function, the form of one curve of constant $v$ (equipotential) is evident by inspection, for $v$ is zero at $\theta = 0$ and also at $\theta = \pi/p$. Thus, if two semi-infinite conducting planes at zero potential intersect at angle $\alpha$, where

$$p = \frac{\pi}{\alpha} \tag{6}$$

they coincide with this equipotential, and boundary conditions are satisfied. The form of the curves of constant $u$ and of constant $v$ within the angle then give the field configuration near a conducting corner. The field is assumed to result from the presence of an electrode with nonzero potential that either fits one of the constant $v$ lines or is far enough away that its shape causes no significant deviation of the $u$ and $v$ lines in the region of interest.

The equipotentials in the vicinity of the corner can be plotted by choosing given values of $v$, and plotting the polar equation of $r$ versus $\theta$ from (5) with $p$ given by (6). Similarly, the flux or field lines can be plotted by selecting several values of $u$ and plotting the curves from (4). The forms of the field, plotted in this manner, for corners with $\alpha = \pi/4$, $\pi/2$, and $3\pi/2$ are shown in Figs. 7.6$b$, 7.6$c$, and 7.6$d$, respectively.

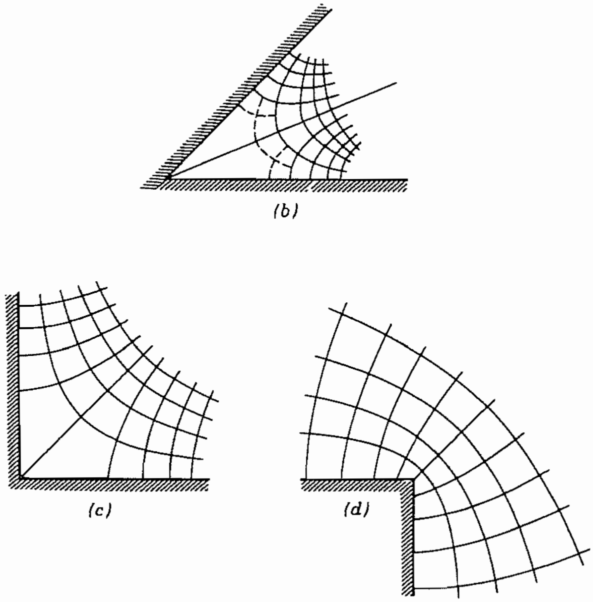



**FIG. 7.6b–d** Field near conducting corners of 45, 90, and 270 degrees.

These plots are of considerable help in judging the correct form of the field in a graphical field map having one or more conducting boundaries.

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**Example 7.6b**

THE LOGARITHMIC TRANSFORMATION: CIRCULAR CONDUCTING BOUNDARIES

Consider next the logarithmic function

$$W = C_1 \ln Z + C_2 \tag{7}$$

The logarithm of a complex number is readily found if the number is in the polar form:

$$\ln Z = \ln(re^{j\theta}) = \ln r + j\theta \tag{8}$$

so

$$W = C_1(\ln r + j\theta) + C_2$$

Take the constants $C_1$ and $C_2$ as real. Then

$$u = C_1 \ln r + C_2 \tag{9}$$

$$v = C_1\theta \tag{10}$$

If $u$ is to be chosen as the potential function, we recognize the logarithmic potential forms found previously for potential about a line charge or a charged cylinder or between coaxial cylinders. The flux function, $\psi = -\varepsilon v$, is then proportional to angle $\theta$, as it should be for a problem with radial electric field lines.

To evaluate the constants for a particular problem, take a coaxial line with an inner conductor of radius $a$ at potential zero and an outer conductor of radius $b$ at potential $V_0$. Substituting in (9), we have

$$0 = C_1 \ln a + C_2$$

$$V_0 = C_1 \ln b + C_2$$

Solving, we have

$$C_1 = \frac{V_0}{\ln(b/a)} \qquad C_2 = -\frac{V_0 \ln a}{\ln(b/a)}$$

so (7) can be written

$$W = V_0\left[\frac{\ln(Z/a)}{\ln(b/a)}\right] \tag{11}$$

or

$$\Phi = u = V_0\left[\frac{\ln(r/a)}{\ln(b/a)}\right] \quad \text{V} \tag{12}$$

$$\psi = -\varepsilon v = \frac{-\varepsilon V_0 \theta}{\ln(b/a)} \quad \text{C/m} \tag{13}$$

In the foregoing, the reference for the flux function came out automatically at $\theta = 0$. If it is desired to use some other reference, the constant $C_2$ is taken as complex, and its imaginary part serves to fix the reference $\psi = 0$.

### Example 7.6c
### The Inverse-Cosine Transformation: Hyperbolic and Elliptic Conducting Boundaries

Consider the function

$$W = \cos^{-1} Z \tag{14}$$

or

$$x + jy = \cos(u + jv) = \cos u \cosh v - j \sin u \sinh v$$

$$x = \cos u \cosh v$$

$$y = -\sin u \sinh v$$

It then follows that

$$\frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = 1 \tag{15}$$

$$\frac{x^2}{\cos^2 u} - \frac{y^2}{\sin^2 u} = 1 \tag{16}$$

Equation (15) for constant $v$ represents a set of confocal ellipses with foci at $\pm 1$, and (16) for constant $u$ represents a set of confocal hyperbolas orthogonal to the ellipses. These are plotted in Fig. 7.6*e*. With a proper choice of the region and the function (either $u$ or $v$) to serve as the potential function, the foregoing transformation could be made to give the solution to the following problems:

1. Field around a charged elliptic cylinder, including the limiting case of a flat strip
2. Field between two confocal elliptic cylinders or between an elliptic cylinder and a flat strip conductor extending between the foci
3. Field between two confocal hyperbolic cylinders or between a hyperbolic cylinder and a plane conductor extending from the focus to infinity

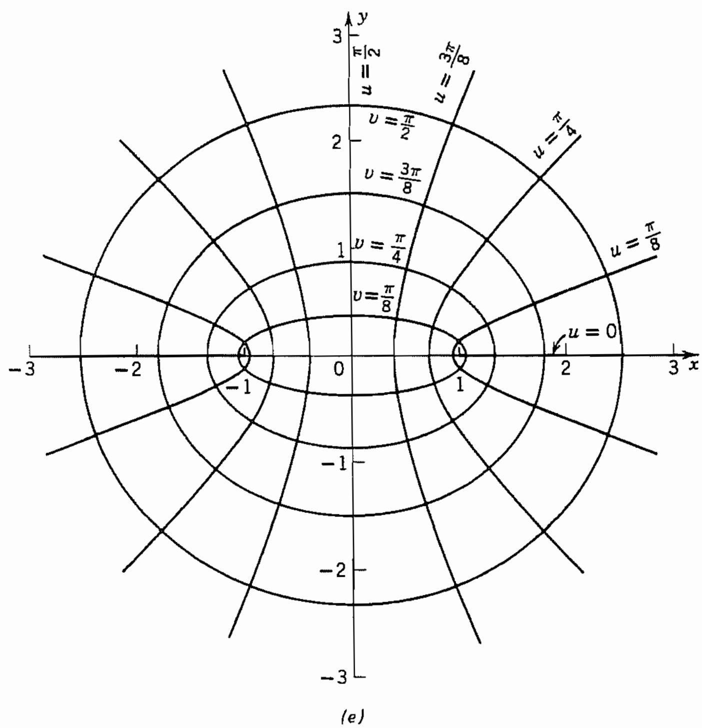



**FIG. 7.6e** Plot of the transformation $u + jv = \cos^{-1}(x + jy)$.

4. Field between two semi-infinite conducting plates, coplanar and with a gap separating them (This is a limiting case of 3.)
5. Field between an infinite conducting plane and a perpendicular semi-infinite plane separated from it by a gap

To demonstrate how the result is obtained for a particular problem, consider problem 5, illustrated by Fig. 7.6*f*. The infinite plane is taken at potential zero, and the perpendicular semi-infinite plane is taken at potential $V_0$. In using the results of the foregoing general transformation, we must now put in scale factors. To avoid confusion with the preceding, let us denote the variables for this specific problem by primes:

$$W' = C_1 \cos^{-1} kZ' + C_2 \tag{17}$$

The constant $C_1$ is inserted to fix the proper scale of potential, the constant $k$ to fix the scale of size, and the additive constant $C_2$ to fix the reference for the potential. By comparing with (14),

$$Z = kZ'$$
$$W' = C_1W + C_2$$

The constants $C_1$ and $C_2$ may be taken as real for this problem. Then

$$u' = C_1u + C_2 \tag{18}$$

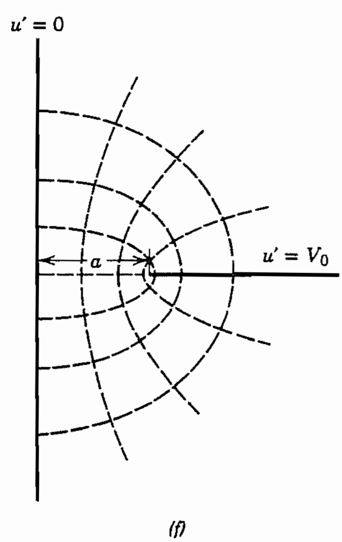



**FIG. 7.6f** Field between perpendicular planes with a finite gap.

By comparing Figs. 7.6*e* and 7.6*f*, we want $Z'$ to be $a$ when $Z$ is unity, so $k = 1/a$. Also, when $u = 0$, we want $u' = V_0$; and when $u = \pi/2$, $u' = 0$. Substitution of these values in (18) yields

$$C_1 = -\frac{2V_0}{\pi}, \qquad C_2 = V_0$$

So the transformation with proper scale factors for this problem is

$$W' = u' + jv' = V_0\left[1 - \frac{2}{\pi}\cos^{-1}\left(\frac{Z'}{a}\right)\right] \tag{19}$$

where $u'$ is the proportional function in volts, and $\varepsilon v'$ is the flux function in coulombs per meter. A few of the equipotential and flux lines with these scale factors applied are shown on Fig. 7.6*f*.

---

### Example 7.6d
PARALLEL CONDUCTING CYLINDERS

Consider next the function

$$W = K_1 \ln\left(\frac{Z - a}{Z + a}\right) \tag{20}$$

This may be written in the form

$$W = K_1[\ln(Z - a) - \ln(Z + a)]$$

By comparing with the logarithmic transformation of Ex. 7.6b which, among other things, could represent the field about a single line charge, it follows that this expression can represent the field about two line charges, one at $Z = a$ and the other of equal strength but opposite sign at $Z = -a$. However, it is more interesting to show that this form can also yield the field about parallel cylinders of any radius.

Taking $K_1$ as real,

$$u = \frac{K_1}{2}\ln\left[\frac{(x - a)^2 + y^2}{(x + a)^2 + y^2}\right] \tag{21}$$

$$v = K_1\left[\tan^{-1}\frac{y}{(x - a)} - \tan^{-1}\frac{y}{(x + a)}\right] \tag{22}$$

Thus, lines of constant $u$ can be obtained from (21) by setting the argument of the logarithm equal to a constant:

$$\frac{(x - a)^2 + y^2}{(x + a)^2 + y^2} = K_2$$

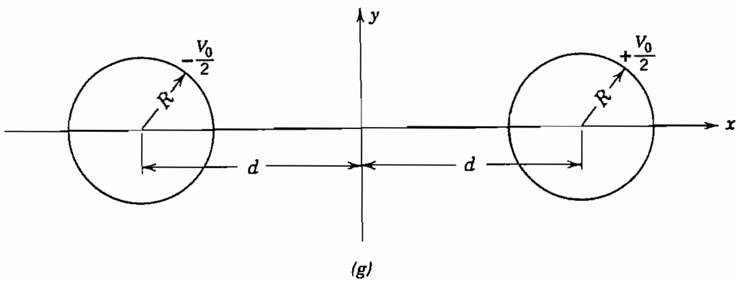



**FIG. 7.6g** Two parallel conducting cylinders.

As this may be put in the form

$$\left[x - \frac{a(1 + K_2)}{1 - K_2}\right]^2 + y^2 = \frac{4a^2 K_2}{(1 - K_2)^2} \tag{23}$$

the curves of constant $u$ are circles with centers at

$$x = \frac{a(1 + K_2)}{1 - K_2}$$

and radii $(2a\sqrt{K_2})/(1 - K_2)$. If $u$ is taken as the potential function, any one of the circles of constant $u$ may be replaced by an equipotential conducting cylinder. Thus, if $R$ is the radius of such a conductor with center at $x = d$ (Fig. 7.6g), the values of $a$ and the particular value of $K_2$ (denoted $K_0$) may be obtained by setting

$$\frac{a(1 + K_0)}{1 - K_0} = d, \qquad \frac{2a\sqrt{K_0}}{1 - K_0} = R$$

Solving,

$$a = \pm\sqrt{d^2 - R^2} \tag{24}$$

$$\sqrt{K_0} = \frac{d}{R} + \sqrt{\frac{d^2}{R^2} - 1} \tag{25}$$

The constant $K_1$ in the transformation depends on the potential of the conducting cylinder. Let this be $V_0/2$. Then, by the definition of $K_2$ ($= K_0$ on conducting cylinder) and (25),

$$\frac{V_0}{2} = K_1 \ln\sqrt{K_0} = K_1 \ln\left(\frac{d}{R} + \sqrt{\frac{d^2}{R^2} - 1}\right)$$

or

$$K_1 = \frac{V_0}{2\ln[(d/R) + \sqrt{(d^2/R^2) - 1}]} = \frac{V_0}{2\cosh^{-1}(d/R)} \tag{26}$$

Substituting in (21), the potential at any point $(x, y)$ is

$$\Phi = u = \frac{V_0}{4\cosh^{-1}(d/R)} \ln\left[\frac{(x-a)^2 + y^2}{(x+a)^2 + y^2}\right] \tag{27}$$

For $\Phi > 0$ with $x > 0$, $a < 0$ if $K_1$ is positive so the negative sign must be chosen in (24). The flux function $\psi = -\varepsilon v$ is

$$\psi = -\varepsilon v = \frac{\varepsilon V_0}{2\cosh^{-1}(d/R)}\left[\tan^{-1}\frac{y}{(x+a)} - \tan^{-1}\frac{y}{(x-a)}\right] \tag{28}$$

Although we have not put in the left-hand conducting cylinder explicitly, the odd symmetry of the potential from (27) will cause this boundary condition to be satisfied also if the left-hand cylinder of radius $R$ with center at $x = -d$ is at potential $-V_0/2$.

If we wish to use the result to obtain the capacitance per unit length of a parallel-wire line, we obtain the charge on the right-hand conductor from Gauss's law by finding the total flux ending on it. In passing once around the conductor, the first term of (28) changes by $2\pi$, and the second by zero. So

$$q = 2\pi\frac{\varepsilon V_0}{2\cosh^{-1}(d/R)} \quad \text{C/m}$$

or

$$C = \frac{q}{V_0} = \frac{\pi\varepsilon}{\cosh^{-1}(d/R)} \quad \text{F/m} \tag{29}$$

A similar procedure can be used to find the external inductance of the parallel-wire line. In that case the roles of $u$ and $v$ are opposite from the above electric field problem, with $v$ being proportional to the magnetic scalar potential. The result given in Eq. 4.6(9) for inductance is

$$L = \frac{\mu}{\pi}\cosh^{-1}\left(\frac{d}{R}\right) \quad \text{H/m} \tag{30}$$

From (29) and (30) we see that $LC = \mu\varepsilon$ as was shown to be the case for other two-conductor lines in Chapter 5. That this is a general result is shown in Sec. 8.12.

## 7.7 The Schwarz Transformation for General Polygons

In the examples in Sec. 7.6 specific functions have been set down, and the electrostatic problems solvable by these deduced from a study of their properties. In a practical problem, the reverse procedure is usually required, for the specific equipotential conducting boundaries will be given and it will be desired to find the complex function useful in solving the problem. The greatest limitation on the method of conformal transformations is that, for general shaped boundaries, there is no straightforward

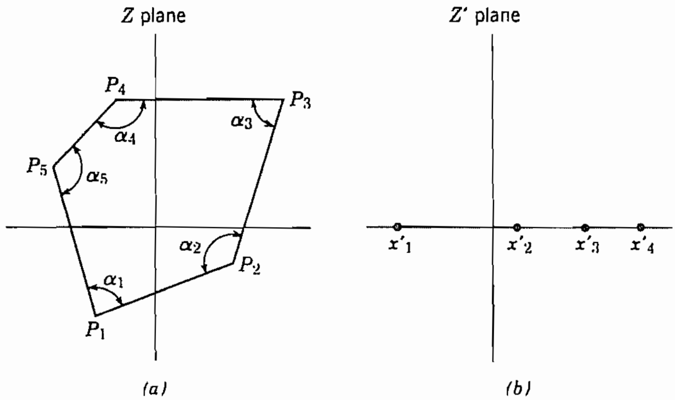



**FIG. 7.7** (*a*) General polygon in $Z$ plane. (*b*) Polygon of figure transformed into straight line in $Z'$ plane. Vertex $x_5'$ is at infinity.

procedure by which one can always arrive at the desired transformation if the two-dimensional physical problem is given. There is such a procedure, however, when the boundaries consist of straight-line sides with angle intersections.

The *Schwarz transformation* takes an arbitrary polygon in the $Z$ plane into a series of segments along the real axis in a $Z'$ plane as shown in Figs. 7.7*a* and 7.7*b*. The segments correspond to the sides of the polygon.[7] The transformation may be found by integrating the derivative:

$$\frac{dZ}{dZ'} = K(Z' - x_1')^{(\alpha_1/\pi)-1}(Z' - x_2')^{(\alpha_2/\pi)-1} \cdots (Z' - x_n')^{(\alpha_n/\pi)-1} \tag{1}$$

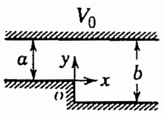
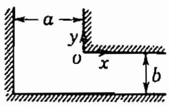
Each factor in (1) may be thought of as straightening out the boundary at one of the vertices as the transform of Ex. 7.6a did for the single corner. The setting down of (1) for a specific problem is usually easy, but the difficulties come in its integration.

Although we have spoken of the figure to be transformed as a polygon, in the practical application of the method, one or more of the vertices may be at infinity, and part of the boundary may be at a different potential from the remaining part. Then the real axis in the $Z'$ plane consists of two parts at different potentials. This latter electrostatic problem may be solved by a transformation from the $Z'$ to the $W$ plane, and thus the transformation from the $Z$ to the $W$ plane is given with the $Z'$ plane only as an intermediate step. Another sort of problem in which the method is useful is that in which a thin charged wire lies on the interior of a conducting polygon, parallel to the elements of the polygon. By the Schwarz transformation, the polygon boundary is transformed to the real axis and the wire corresponds to some point in the upper half of the $Z'$ plane. This electrostatic problem can be solved by the method of images, and so the original problem can be solved in this case also.

[7] *For more details see R. V. Churchill and J. W. Brown,* Complex Variables and Applications, *4th ed., McGraw-Hill, New York, 1984.*

**Table 7.7**

$Z = x + jy$; $W = u + jv$, where $u$ = Flux Function, $v$ = Potential

| | |
|---|---|
|   | $Z = \frac{b}{\pi}\left[\cosh^{-1}\left(\frac{\alpha^2 + 1 - 2\alpha^2 e^{kW}}{1 - \alpha^2}\right) - \alpha\cosh^{-1}\left(\frac{2e^{-kW} - (\alpha^2 + 1)}{1 - \alpha^2}\right)\right]$<br>$\alpha = \frac{a}{b}, \quad k = \frac{\pi}{V_0}$ |
|   | $Z = \frac{b}{\pi}\left[\ln\left(\frac{1 + S}{1 - S}\right) - 2\alpha\tan^{-1}\left(\frac{S}{\alpha}\right)\right]$<br>$\alpha = \frac{a}{b}, \quad S = \sqrt{\frac{e^{kW} + \alpha}{e^{kW} - 1}}$ |

Results for some important problems that have been solved by the Schwarz technique are given in Table 7.7.

**Example 7.7**

FRINGING FIELD IN PARALLEL-PLATE CAPACITOR

To illustrate how the concept of a polygon can be applied, consider the parallel-plate capacitor structure in Fig. 7.7*c* with $\Phi = V_0$ on the infinite bottom plane and $\Phi = 0$ on the plane $D$–$C$. For the purposes of the Schwarz transformation, the structure may be considered a polygon with interior angles $\alpha_1$, $\alpha_2$, and $\alpha_3$ and sides of infinite length. Application of the transformation puts all the boundaries along the real axis as in Fig. 7.7*d* where $\Phi = V_0$ for $x_2' < 0$ and $\Phi = 0$ for $x_2' > 0$. As was shown in Ex. 7.6b, a subsequent logarithmic transformation converts such a set of boundary potentials into a parallel-plane uniform field in the $W$ plane. Combining the two transformations, one

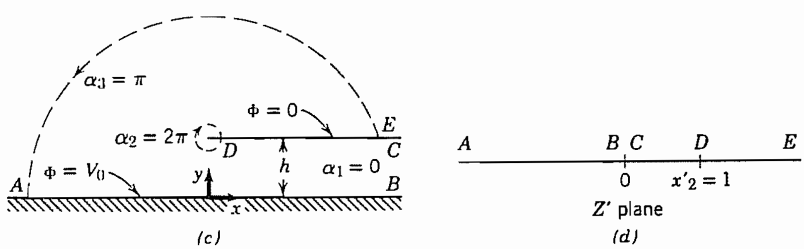



**FIG. 7.7** (*c*) Edge of parallel-plate capacitor with one plane of infinite extent (equivalent to one-half of a symmetric parallel-plate capacitor). (*d*) Transformation of the capacitor of (*c*) into a single plane.

finds that the flux lines and potential lines, $u$ and $v$, respectively, are implicitly defined in terms of position $x$, $y$ in the $Z$ plane by the relation

$$Z = \frac{h}{\pi}\left(e^{\pi W/V_0} - 1 - \frac{\pi W}{V_0} + j\pi\right) \tag{2}$$

One can select values of $u$ and $v$ and calculate the corresponding points $x$ and $y$, thus plotting out the field sketched in Fig. 1.9$a$.

## 7.8 Conformal Mapping for Wave Problems

We have seen that in conformal transformations for statics complicated boundaries are transformed to simple ones. Conformal transformations of wave problems can similarly simplify complicated boundaries.[8] The transformations can be made only in two dimensions so the fields must be independent of the third dimension. The simplification of the boundaries is also normally accompanied by increased complexity of the dielectric so this trade-off only occasionally helps.

Let us assume that there exists an analytic function $W = u + jv = f(Z) = f(x + jy)$ which transforms the given boundary shapes in the $Z$ plane to lines of constant $u$ and $v$ in the $W$ plane. We will first determine the relation between $\nabla^2_{xy}\psi$ and $\nabla^2_{uv}\psi$ in order to transform the scalar Helmholtz equation, Eq. 7.2(8),

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k^2\psi = 0 \tag{1}$$

from the $Z$ plane to the $W$ plane. To transform the derivatives, we apply the chain rule. First,

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\psi}{\partial v}\frac{\partial v}{\partial x} \tag{2}$$

Applying the chain rule a second time leads to

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2\psi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + 2\frac{\partial^2\psi}{\partial u\,\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \tag{3}$$

and similarly for $\partial^2\psi/\partial y^2$:

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\psi}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^2\psi}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + 2\frac{\partial^2\psi}{\partial u\,\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \tag{4}$$

[8] *F. E. Borgnis and C. H. Papas, in* Handbuch der Physik *(S. Flügge, Ed.), Vol. 16, p. 358, Springer Verlag, Berlin, 1958.*

Making use of the Cauchy–Riemann conditions in Eqs. 7.4(6) and 7.4(7) in the second terms of the right sides of (3) and (4) and in the last term of (4), and adding (3) and (4),

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \left(\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2}\right)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \tag{5}$$

Note from Eq. 7.4(4) and 7.4(7) that

$$\left|\frac{dW}{dZ}\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \tag{6}$$

Thus

$$\nabla^2_{xy}\psi = \left|\frac{dW}{dZ}\right|^2 \nabla^2_{uv}\psi \tag{7}$$

The quantity $|dZ/dW|$ is a scale factor which relates a differential length $|dW|$ in the $W$ plane to the corresponding length $|dZ|$ in the $Z$ plane, as we discussed in Sec. 7.6. The Helmholtz equation (1) is thus transformed to the $W$ plane giving

$$\nabla^2_{uv}\psi + \left|\frac{dZ}{dW}\right|^2 k^2\psi = 0 \tag{8}$$

Note that, in general, $|dZ/dW|$ is a function of the coordinates so that (8) is equivalent to the Helmholtz equation in an inhomogeneous medium.

Boundary conditions in the $Z$ plane consisting of zero values of $\psi$ or its normal derivatives carry over unchanged to the corresponding boundaries in the $W$ plane since the orthogonality of coordinates is conserved. If a nonzero normal derivative is specified on a boundary, the scale factor $|dW/dZ|$ enters the conversion of the boundary condition through the relation between gradients in the two planes:

$$|\nabla_{uv}\psi| = \left|\frac{dZ}{dW}\right| |\nabla_{xy}\psi| \tag{9}$$

### Example 7.8
### CURVED DIELECTRIC WAVEGUIDE

A layer of a dielectric material embedded in materials of lower permittivity can serve to guide electromagnetic waves, as will be studied in more detail in Chapter 14. The phenomenon of total internal reflection analyzed in Sec. 6.12 supplies a qualitative understanding of dielectric waveguides. Here we see how wave propagation in a curved layer, as in Fig. 7.8*a*, can be treated using conformal mapping.

The wave is assumed to be polarized with its electric field in the $z$ direction and $E_z$ is independent of $z$. Then identifying $E_z$ with $\psi$, we can write from Eq. 7.2(8), with $\partial/\partial z = 0$,

$$\nabla^2_{xy}E_z(x, y) + k^2E_z(x, y) = 0 \tag{10}$$

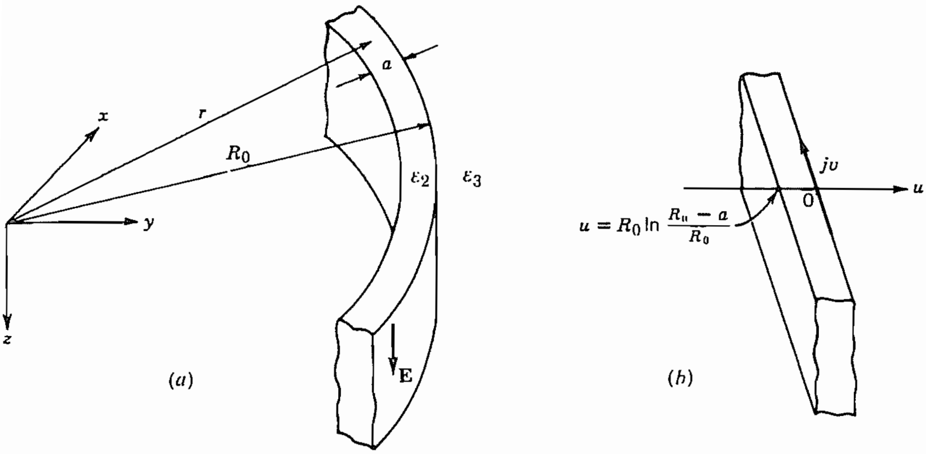



**FIG. 7.8** (*a*) Curved dielectric waveguide in *Z* plane. (*b*) Curved dielectric guide transformed into *W* plane.

Using the transformation of Ex. 7.6b in slightly different form,

$$W = R_0 \ln \frac{Z}{R_0} \tag{11}$$

for which

$$\left|\frac{dZ}{dW}\right| = e^{u/R_0} \tag{12}$$

(8) becomes

$$\nabla^2_{uv} E_z(u, v) + k^2 e^{2u/R_0} E_z(u, v) = 0 \tag{13}$$

From (11) it is easily seen that

$$u = R_0 \ln \frac{r}{R_0} \tag{14}$$

and

$$v = R_0 \theta \tag{15}$$

Therefore, the edge of the guiding layer at $r = R_0$ is the $u = 0$ line in the $W$ plane and the other edge of the guide at $r = R_0 - a$ is at $u = R_0 \ln(R_0 - a)/R_0$. The result is that the curved layer in the $Z$ plane becomes the planar region shown in Fig. 7.8*b*. Note that this layer in the $W$ plane is inhomogeneous. Equation (13) has the usual form of the Helmholtz equation only if we identify a new wave number $k'$ by

$$k' = \omega \sqrt{\mu \varepsilon \exp \frac{2u}{R_0}} \tag{16}$$

Since $k'$ is a function of $u$, one cannot substitute it directly for $k$ in the usual wave solution. However, the problem is solvable and has been used to study the leakage of energy from bends in dielectric waveguides.[9]

## Separation of Variables Method

### 7.9 Laplace's Equation in Rectangular Coordinates

One of the most powerful techniques for solution of linear partial differential equations is that of separation of variables. This leads to solutions which are products of three functions (for three-dimensional problems), each function depending upon one coordinate variable only. Such solutions might not seem very general, but they may be added to form a series which can represent very general functions. Moreover, single-product solutions of the wave equation represent modes which can propagate individually. These are of great practical importance in waveguides and resonant systems and are studied extensively in following chapters.

As the simplest example of the method of separation of variables, let us first consider two-dimensional problems in the rectangular coordinates $x$ and $y$, as we have in the transformation method of the past section. Laplace's equation in these coordinates is

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{1}$$

We wish to study product solutions of the form

$$\Phi(x, y) = X(x)Y(y) \tag{2}$$

where we see that we have a function of $x$ alone times a function of $y$ alone. From this point on $X(x)$ will be replaced by $X$ and $Y(y)$ by $Y$. Substituting in (1), we have

$$X''Y + XY'' = 0 \tag{3}$$

The double prime denotes the second derivative with respect to the independent variable in the function. Now to separate into the sum of functions of one variable only, divide (3) by (2):

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \tag{4}$$

[9] *M. Heiblum and J. H. Harris,* IEEE J. Quantum Electronics **QE-11**, *75 (1975).*

Next follows the key argument for this method. Equation (4) is to hold for all values of the variables $x$ and $y$. Since the second term does not contain $x$, and so cannot vary with $x$, the first term cannot vary with $x$ either. A function of $x$ alone which does not vary with $x$ is a constant. Similarly, the second term must be a constant. Let us denote the first as $K_x^2$ and the second as $K_y^2$. Then

$$K_x^2 + K_y^2 = 0 \tag{5}$$

and

$$\begin{aligned} X'' - K_x^2 X &= 0 \\ Y'' - K_y^2 Y &= 0 \end{aligned} \tag{6}$$

We recognize that these are in the standard form having real exponentials or hyperbolic functions as solutions. Let us write them in hyperbolic form and substitute in (2):

$$\Phi(x, y) = (A \cosh K_x x + B \sinh K_x x)(C \cosh K_y y + D \sinh K_y y) \tag{7}$$

It is clear from (5) that either $K_x^2$ or $K_y^2$ must be negative and therefore either $K_x$ or $K_y$ must be imaginary while the other is real. Furthermore, their magnitudes must be the same. Thus (7) can have either of two forms,

$$\Phi(x, y) = (A \cosh Kx + B \sinh Kx)(C \cos Ky + D' \sin Ky) \tag{8}$$

or

$$\Phi(x, y) = (A \cos Kx + B' \sin Kx)(C \cosh Ky + D \sinh Ky) \tag{9}$$

where, since $|K_x| = |K_y|$, we have used the single symbol $K$. The primes are used to indicate that the constants have changed. The choice between (8) and (9) is dictated by the nature of the boundary conditions. If the potential is required to have repeated zeros as a function of $y$, then (8) is used; if repeated zeros are specified for the $x$ variation, (9) is chosen. If the boundaries extend to infinity in one direction, real exponentials are used in place of hyperbolic functions. It may be noted from (6) that for $K_x = jK_y = 0$, the general solution has the form

$$\Phi(x, y) = (A_1 x + B_1)(C_1 y + D_1) \tag{10}$$

It is typical for product solutions that when the separation constants go to zero the functional forms of the solutions change. We will see in subsequent sections how the constants are evaluated using the boundary conditions.

For the three-dimensional case in rectangular coordinates, the procedure is simply extended. Laplace's equation is

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{11}$$

Consider solutions of the form

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \tag{12}$$

where each term on the right side is a function of just one of the independent space variables. Substituting (12) in (11), we have

$$X''YZ + XY''Z + XYZ'' = 0$$

and dividing by $\Phi$, we see that

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \tag{13}$$

We use the same argument as was used in the two-dimensional case. If the second two terms do not vary with $x$, neither can the first. Since it is a function of $x$ alone and does not vary with $x$, it must be a constant. Similar arguments apply for the second and third terms. If we let the first term be $K_x^2$, the second $K_y^2$, and the third $K_z^2$, (13) becomes

$$K_x^2 + K_y^2 + K_z^2 = 0 \tag{14}$$

and differential equations of the form (6) apply for $X$, $Y$, and $Z$. So the general solution, written as the product of $X$, $Y$, and $Z$, and sometimes called a *rectangular harmonic*, is

$$\begin{aligned}\Phi(x, y, z) = {} & [A \cosh K_x x + B \sinh K_x x][C \cosh K_y y + D \sinh K_y y] \\ & \times [E \cosh K_z z + F \sinh K_z z]\end{aligned} \tag{15}$$

It is clear that at least one of $K_x^2$, $K_y^2$, and $K_z^2$ must be negative for (14) to hold, so at least one of $K_x$, $K_y$, and $K_z$ must be imaginary. If repeated potential zeros are required in the $x$ and $y$ directions, the functions of $x$ and $y$ must be trigonometric functions so $K_x$ and $K_y$ are imaginary. There are various other combinations which may be useful. In some cases it is advantageous to replace the hyperbolic functions by real exponentials as mentioned earlier for the two-dimensional solutions.

In (15) there appear to be nine constants, to be evaluated using the six possible boundary conditions, two for each of the three coordinate directions. If, however, one divides the first bracket by $B$, the second by $D$, and the third by $F$ and multiplies the entire by $BDF$, it becomes clear that there are just four independent multiplicative constants. From (14) we see that there are only two independent separation constants so the total number of unknowns equals the number of boundary conditions.

## 7.10 Static Field Described by a Single Rectangular Harmonic

Let us see what boundaries would be required to have one of the forms of Sec. 7.9 as a solution. Take the special case of Eq. 7.9(9) with $A = 0$, $C = 0$. The product of remaining constants, $B'D$, may be denoted as a single constant $C_1$:

$$\Phi = C_1 \sin Kx \sinh Ky \tag{1}$$

It is evident from (1) that potential is zero at $y = 0$ for all $x$. Hence one boundary can be a zero-potential conducting plane at $y = 0$. Similarly, potential is zero along the plane $x = 0$ and also at other parallel planes defined by $Kx = n\pi$. Let us confine attention to the region $0 < Kx < \pi$ and $0 < y < \infty$. The intersecting zero-potential planes of interest then form a rectangular conducting trough. Let its depth in the $x$ direction be $a$. Then $Ka = \pi$ or

$$K = \frac{\pi}{a} \tag{2}$$

If there is to be a finite field in the region, there must be some electrode at a potential other than zero. Without knowing its shape for the moment, let us take the value of $y$ at which it crosses the midplane $x = a/2$ as $y = b$, and the potential of the electrode as $V_0$. Then, from (1),

$$V_0 = C_1 \sin\frac{\pi}{2}\sinh\frac{\pi b}{a} = C_1 \sinh\frac{\pi b}{a}$$

or, substituting in (1), we have

$$\Phi = \frac{V_0 \sinh(\pi y/a)}{\sinh(\pi b/a)}\sin\frac{\pi x}{a} \tag{3}$$

The potential at any point $x$, $y$ may be computed from (3). In particular, the form that the electrode at potential $V_0$ must take can be found from (3) by setting $\Phi = V_0$, yielding

$$\sinh\frac{\pi y}{a} = \frac{\sinh(\pi b/a)}{\sin(\pi x/a)} \tag{4}$$

Equation (4) can be plotted to show the form of the electrode. This is done for a value $b/a = \frac{1}{2}$ in Fig. 7.10$a$. Actually, the electrodes should extend to infinity, but if they are extended a large but finite distance, the solution studied here will represent the potential very well everywhere except near edges.

Alternatively, a straight boundary at $y = b$ could be supplied with a sinusoidal

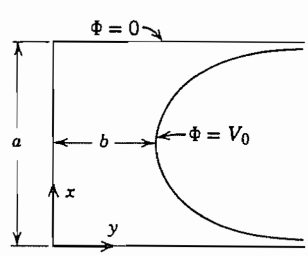



**FIG. 7.10a** Electrodes and potentials for which a single harmonic is the complete solution.

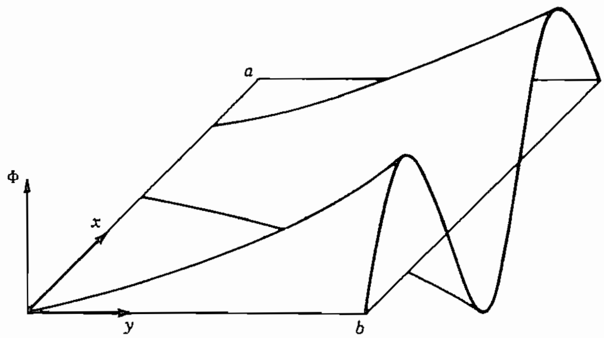



**FIG. 7.10b** Potential in a two-dimensional box with a sinusoidal distribution of potential on one side.

distribution of potential and a single harmonic would describe the potential at all points in the box. Thus, for example, if $Ka = 3\pi$ and the boundary potential were

$$\Phi(x, b) = V_0 \sin \frac{3\pi}{a} x \tag{5}$$

then the harmonic

$$\Phi = \frac{V_0 \sinh(3\pi/a)y}{\sinh(3\pi b/a)} \sin \frac{3\pi}{a} x \tag{6}$$

shown in Fig. 7.10$b$ satisfies the boundary conditions and describes the potential at all points.

## 7.11 Fourier Series and Integral

In the preceding section, we saw that a single-product solution could satisfy only very special forms of boundary conditions. For more general boundaries a sum of such solutions must be used. This is one example of situations where Fourier series or integrals are useful in forming solutions for field problems. We provide here a review of the Fourier tools with the assumption that the reader has already a measure of familiarity with them.

**Fourier Series** Fourier series are used to represent periodic functions. For the independent variable $x$, the required periodicity is expressed by

$$f(x) = f(x + L) \tag{1}$$

where $L$ is the *period* of the function. We assume that the function can be represented by a constant plus the sum of infinite series of sine and cosine functions of harmonics of a fundamental spatial frequency $k$:

$$\begin{aligned} f(x) &= a_0 + a_1 \cos kx + a_2 \cos 2kx + a_3 \cos 3kx + \cdots \\ &\quad + b_1 \sin kx + b_2 \sin 2kx + b_3 \sin 3kx + \cdots \end{aligned} \tag{2}$$

where the phase factor $k$ is related to the period $L$ in the usual way:

$$kL = 2\pi \tag{3}$$

To evaluate the unknown constants in (2) for a given function $f(x)$, we make use of so-called *orthogonality* properties of sinusoids. These are

$$\int_{-L/2}^{L/2} \cos nkx \cos mkx \, dx = 0 \qquad m \neq n \tag{4}$$

$$\int_{-L/2}^{L/2} \sin nkx \sin mkx \, dx = 0 \qquad m \neq n \tag{5}$$

and

$$\int_{-L/2}^{L/2} \sin mkx \cos nkx \, dx = 0 \qquad \begin{cases} m \neq n \\ m = n \end{cases} \tag{6}$$

However,

$$\int_{-L/2}^{L/2} \cos^2 mkx \, dx = \int_{-L/2}^{L/2} \sin^2 mkx \, dx = \frac{L}{2} \tag{7}$$

To make use of these properties, we multiply each term in (2) by cos $nkx$ and integrate over one period. Every term on the right vanishes because of the properties in (4)–(6) except the one containing cos $nkx$; that term gives $a_n L/2$ according to (7). Thus,

$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos nkx \, dx \tag{8}$$

Similarly, multiplication of (2) by sin $nkx$ with integration from $-L/2$ to $L/2$ leaves only the term involving sin $nkx$ on the right-hand side, and its coefficient, by (7), is

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin nkx \, dx \tag{9}$$

Finally, to obtain the constant term $a_0$, every term is integrated directly over a period and all the terms on the right side disappear except that containing $a_0$ so that

$$a_0 = \frac{1}{L} \int_{-L/2}^{L/2} f(x) \, dx \tag{10}$$

This merely states that $a_0$ is the average of the function $f(x)$.

For a general function, an infinite number of terms is required in the Fourier series representation. But often a sufficient degree of approximation to the desired wave shape is obtained when only a finite number of terms is used. For functions with sharp discontinuities, however, many terms may be required near the sharp corners, and the theory of Fourier series shows that the series does not converge to the function in the neighborhood of the discontinuity (Gibbs phenomenon). The derivative of the series also does not converge to the derivative of the function, but the integral of the series always converges to that of the function.

### Example 7.11a
FOURIER SERIES REPRESENTATION OF A FUNCTION OVER A FINITE INTERVAL

In static field problems, one commonly has the boundary potential specified over a finite interval, such as along a straight boundary at a constant value of one coordinate. For the purposes of matching the given boundary potential, it is desirable to express it in a Fourier series. This can be done even though the function is not periodic, having been specified only over a finite interval. The point of view is that the interval of length $a$ may be considered a period or an integral fraction of a period, and a periodic function defined to agree with the given function over the given interval, repeating itself outside that interval. A Fourier series may then be written for this periodic function which will give desired values in the interval, and although it also gives values outside the interval, that is of no consequence since the original function is not defined there.

The interval is commonly selected as a half-period since the function extended outside the interval may then be made either even or odd, and the corresponding Fourier series will then have respectively either cosine terms alone or sine terms alone. Figure 7.11$a$ shows by solid lines some possible examples of functions specified over the interval $0 < x < a$. Their extensions outside that interval as either odd or even functions are shown by the broken lines. Note that in one case the interval is $L/4$. The choice of whether to consider the function continued as an odd or an even function depends upon the form used to represent the potential in the problem. Thus, for example, in Eq. 7.10(3) the potential is expressed in terms of sin $\pi x/a$ and the appropriate series for the representation of the boundary potential will be in sines,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \tag{11}$$

where we have made use of the fact that $a = L/2$, one half-period. The coefficients are found from (9) noting that the contributions to the integral from the negative and positive intervals are equal. Thus, with $a = L/2$,

$$b_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx \tag{12}$$

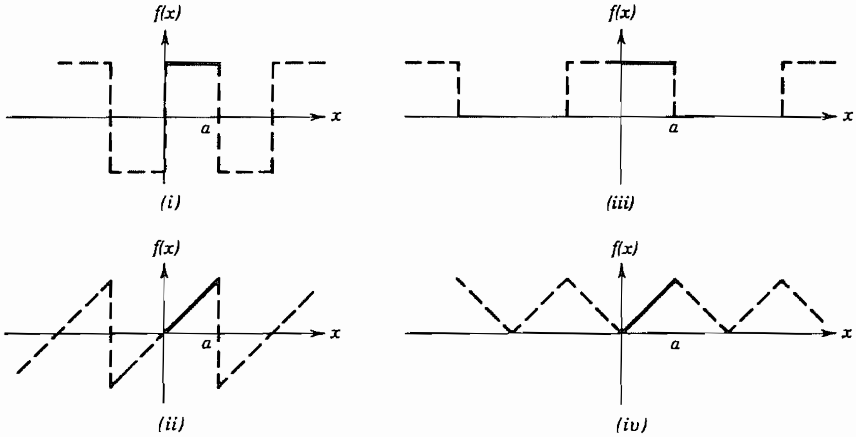



**FIG. 7.11a** Examples of functions specified over a finite interval (solid line) and odd and even continuations (broken lines).

Suppose, for example, that the specified function is $f(x) = C$ over the interval $0 < x < a$ and it is desired to expand it in a sine series. Then, (12) yields

$$b_n = \frac{2}{a}\int_0^a C \sin\frac{n\pi x}{a}\,dx = \frac{2C}{n\pi}\left[-\cos\frac{n\pi x}{a}\right]_0^a \tag{13}$$

and the series is

$$f(x) = \frac{2C}{\pi}\left[2 \sin\frac{\pi x}{a} + \frac{2}{3}\sin\frac{3\pi x}{a} + \frac{2}{5}\sin\frac{5\pi x}{a} + \cdots\right] \tag{14}$$

The series (14) has the required value $f(x) = C$ over the interval $0 < x < a$ but also represents the dashed portion of waveform (i) in Fig. 7.11$a$ outside that interval.

---

**Fourier Integral** In some problems the function of interest is defined over the entire range and is aperiodic. An example is a square function that is constant in some range $-a \leq x \leq a$ and zero elsewhere, as shown in Fig. 7.11$b$. This could be considered the limiting case of a periodic series of square pulses where the period $L$ goes to infinity. The spacing of the components $(n + 1)k - nk = 2\pi/L$ from (3) becomes vanishingly small as the period $L$ approaches infinity, and in the limit the spectrum of component sinusoidal waves becomes a continuum.[10]

[10] *R. Bracewell,* The Fourier Transform and its Applications, *2nd rev. ed., McGraw-Hill, New York, 1986.*

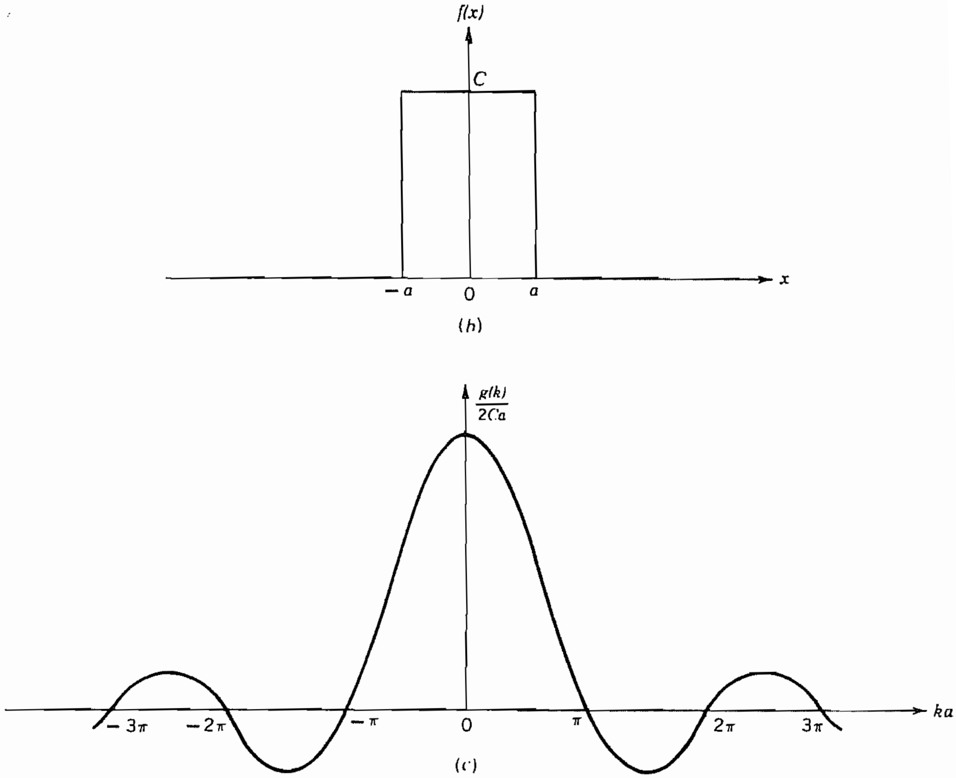



**FIG. 7.11** (*b*) Inverse Fourier transform of function in (*c*). (*c*) Fourier transform of rectangular function in (*b*). Value of $g(0)/2Ca$ is unity.

In the limiting, aperiodic case the series (2) is replaced by an integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k)e^{jkx}\,dk \tag{15}$$

and the function $g(k)$ which takes the place of $a_n$ and $b_n$ of (8)–(10) is given by[11]

$$g(k) = \int_{-\infty}^{\infty} f(x)e^{-jkx}\,dx \tag{16}$$

The theory of Fourier integrals shows that for (15) to give the same $f(x)$ that appears in (16) the function must be continuous or have only a finite number of finite discontinuities in any finite interval and must be absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty \tag{17}$$

11 *The placement of $2\pi$ in the pair (15) and (16) is arbitrary and is done in various ways in the literature.*

These conditions do not place strong limitations on the utility of the transform pair (15) and (16).

**Example 7.11b**
FOURIER TRANSFORM OF A RECTANGULAR PULSE

Using (16) we find the spectrum of spatial frequency components for the rectangular function in Fig. 7.11*b*:

$$g(k) = \int_{-a}^{a} Ce^{-jkx}\,dx = C\left[\frac{e^{-jkx}}{-jk}\right]_{-a}^{a} = 2Ca\left(\frac{\sin ka}{ka}\right) \tag{18}$$

This very important function occurs frequently in practice and is shown in Fig. 7.11*c*. The function $f(x)$ in Fig. 7.11*b* is called the *inverse* Fourier transform of that in Fig. 7.11*c*. The two are called a *transform pair.*

## 7.12 SERIES OF RECTANGULAR HARMONICS FOR TWO- AND THREE-DIMENSIONAL STATIC FIELDS

We saw in Sec. 7.10 that product solutions (harmonics) satisfying zero boundary conditions on three of the four boundaries in a two-dimensional rectangular structure can be found. However, the use of a single harmonic as the expression for the potential requires either a fourth boundary of complicated shape or a simple flat one with a sinusoidal variation of potential along it. To solve problems with an arbitrary variation of potential along a flat boundary on a coordinate line, one may use a sum of harmonics, each of which satisfies the zero conditions on three boundaries and has a weighting in the sum such that it equals the given potential at the fourth boundary. Then the given potential is expanded in a Fourier series of either sines or cosines, chosen to match the functions in the sum of harmonics. The harmonic series is evaluated at the fourth boundary and compared, term by term, with the Fourier series to evaluate the weighting coefficients in the former. These procedures are sometimes slightly modified by use of symmetries and superposition, as seen in the following examples and problems.

**Example 7.12a**
TWO-DIMENSIONAL PROBLEM WITH SPECIFIED BOUNDARY POTENTIALS

As an example of a problem which cannot be solved by using a single one of the solutions of Sec. 7.9, but can be by means of a series of these solutions, consider the two-dimensional region of Fig. 7.12*a* bounded by a zero-potential plane at $y = 0$, a

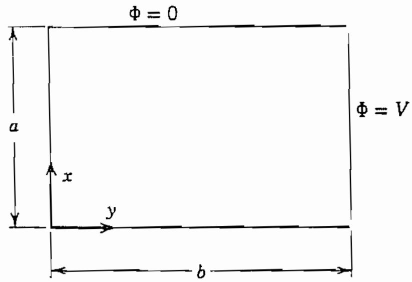



**FIG. 7.12a** Two-dimensional box for Ex. 7.12a.

zero-potential plane at $x = 0$, a parallel zero-potential plane at $x = a$, and a plane conducting lid of potential $V_0$ at $y = b$. In the ideal problem, the lid is separated from the remainder of the rectangular box by infinitesimal gaps. In a practical problem, it would only be expected that these gaps should be small compared with the rest of the box.

In selecting the proper forms from Sec. 7.9, we will choose the form having sinusoidal solutions in $x$ since potential is zero at $x = 0$ and also at $x = a$, and sinusoids have repeated zeros. So the form of Eq. 7.9(9) is suitable. Moreover, $\Phi = 0$ at $y = 0$ for all $x$ of interest, so the function of $y$ must go to zero at $y = 0$, showing that $C = 0$. Similarly, since $\Phi = 0$ at $x = 0$ for all $y$ of interest, $A = 0$. Then $\Phi$ is again zero at $x = a$, so $Ka = m\pi$, or

$$K = \frac{m\pi}{a}$$

Denoting the product of the remaining constants $B'D$ as $C_m$, we have

$$\Phi = C_m \sin \frac{m\pi x}{a} \sinh \frac{m\pi y}{a}$$

This forms satisfies the Laplace equation and the boundary conditions at $y = 0$, at $x = 0$, and at $x = a$, but a single term of this form cannot satisfy the boundary condition along the plane lid at $y = b$, as the study in Sec. 7.10 has shown. A series of such solutions also satisfies Laplace's equation and the boundary conditions at $y = 0$, at $x = 0$, and at $x = a$:

$$\Phi = \sum_{m=1}^{\infty} C_m \sin \frac{m\pi x}{a} \sinh \frac{m\pi y}{a} \tag{1}$$

For the sum (1) to give the required constant potential $V_0$ along the plane $y = b$ over the interval $0 < x < a$, we require

$$V_0 = \sum_{m=1}^{\infty} C_m \sin \frac{m\pi x}{a} \sinh \frac{m\pi b}{a}, \qquad 0 < x < a \tag{2}$$

But this is recognized as a Fourier expansion in sines of the constant function $V_0$ over the interval $0 < x < a$. This expansion was carried out in Ex. 7.11a to yield

$$f(x) = V_0 = \sum_{m=1}^{\infty} a_m \sin \frac{m\pi x}{a}, \qquad 0 < x < a \tag{3}$$

with

$$a_m = \begin{cases} \dfrac{4V_0}{m\pi}, & m \text{ odd} \\ 0, & m \text{ even} \end{cases} \tag{4}$$

Comparison of (3) with (2) shows that

$$C_m \sinh \frac{m\pi b}{a} = a_m \tag{5}$$

Substitution of the results of (5) and (4) in (1) gives

$$\Phi = \sum_{m \text{ odd}} \frac{4V_0}{m\pi} \frac{\sinh(m\pi y/a)}{\sinh(m\pi b/a)} \sin \frac{m\pi x}{a} \tag{6}$$

This series is rapidly convergent except at corners of $x \to 0$, $a$, and $y \to b$, so it can be used for reasonably convenient calculation of potential elsewhere.

We note that the evaluation of the constants in the general solution depended upon the fact that the boundary potentials were specified on surfaces in the coordinate system. Furthermore, nonzero conditions, potential or normal derivative of potential, must exist on some part of the boundary to yield a nonzero solution. As will be clarified in the next example, superposition may be used to solve problems where the boundary conditions involve several sides.

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### Example 7.12b
TWO-DIMENSIONAL PROBLEM REQUIRING SUPERPOSITION

In this example, we see a boundary potential having a Fourier series expansion which includes both trigonometric functions and a constant. Matching such a boundary condition requires the superposition of two solutions, one to match the constant and one for the trigonometric functions. Consider the problem of finding the potentials in the *conducting* rectangular solid of infinite extent in the $z$ direction shown in Fig. 7.12*b*. The surrounding region contains free space, the potential at $y = 0$ is zero, and that along the edge $y = b$ is given by $\Phi = V_0x/a$.

This problem requires a solution with repetition in the $x$ direction since the boundary conditions at the sides $x = 0$, $a$ are the same; the appropriate general form is that in Eq. 7.9(9). At $x = 0$, $a$ the $x$ component of current density must be zero since no current

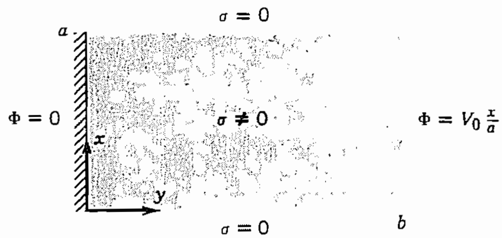


**FIG. 7.12b** Two-dimensional conductive solid embedded in a nonconductive medium.

can flow in the free space outside the conductor. Since $\mathbf{J} = \sigma\mathbf{E}$, then $E_x$ must also be zero at $x = 0, a$. Therefore, $\partial\Phi/\partial x$, given by

$$\frac{\partial\Phi}{\partial x} = K(-A \sin Kx + B' \cos Kx)(C \cosh Ky + D \sinh Ky) \tag{7}$$

must be zero at $x = 0, a$ for all $y$. Since $\cos Kx = 1$ at $x = 0$, $B' = 0$. Also, $\sin Ka = 0$ if $Ka = m\pi$ so $K = m\pi/a$. To match the boundary condition $\Phi = 0$ at $y = 0$ requires $C = 0$. Thus the potential in the $m$th harmonic is

$$\Phi_m = C_m \cos \frac{m\pi}{a} x \sinh \frac{m\pi}{a} y \tag{8}$$

The potential on the boundary at $y = b$ should be expanded in a series of cosines so that term-by-term matching with a series of terms like (8) can be done. The appropriate periodic continuation of the given boundary potential is shown in Fig. 7.11$a$(iv). It is seen to have an average value of $V_0/2$ which will be present in the series expansion. Applying Eqs. 7.11(8)–(10),

$$\Phi(x, b) = \frac{V_0}{2} - \sum_{m \text{ odd}} \frac{4V_0}{(m\pi)^2} \cos \frac{m\pi}{a} x \tag{9}$$

Matching the boundary potential (9) requires both a series of harmonics of the form in (8) and a separate solution having a constant value at $y = b$. A solution of the latter form is found from Eq. 7.9(10). The function

$$\Phi_1 = A_1 y \tag{10}$$

satisfies the boundary conditions at $x = 0, a$ and at $y = 0$. Evaluation of the constant as $V_0/2b$ gives

$$\Phi_1 = \frac{V_0 y}{2b} \tag{11}$$

The solution involving harmonic terms is found by equating a series of terms like that in (8), evaluated at the boundary $y = b$, with the series in (9):

$$\sum_m C_m \cos \frac{m\pi x}{a} \sinh \frac{m\pi b}{a} = -\sum_{m \text{ odd}} \frac{4V_0}{(m\pi)^2} \cos \frac{m\pi x}{a} \tag{12}$$

The result for the second potential is

$$\Phi_2 = -\sum_{m\ \text{odd}} \frac{4V_0}{(m\pi)^2} \frac{\sinh(m\pi y/a)}{\sinh(m\pi b/a)} \cos\frac{m\pi x}{a} \tag{13}$$

The complete solution is the superposition of (11) and (13), $\Phi = \Phi_1 + \Phi_2$.

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**Example 7.12c**

THREE-DIMENSIONAL RECTANGULAR BOX WITH POTENTIAL SPECIFIED ON ONE FACE

The method discussed above can be extended to three dimensions. As an example, we consider a box with zero potential on all faces except on the side $z = c$, where it is $V_0$. The box extends from the origin of coordinates to $x = a$, $y = b$, and $z = c$. The appropriate general form of the space harmonic is Eq. 7.9(15) with $K_x$ and $K_y$ imaginary and $K_z$ real. To meet the zero-potential boundary conditions at $x = 0$, $y = 0$, and $z = 0$, the constants $A$, $C$, and $E$ must be zero. Also, to satisfy the zero-potential condition at $x = a$ and $y = b$, the corresponding separation constants must be $m\pi/a$ and $n\pi/b$, respectively. Then from Eq. 7.9(14) we get $K_z = [(m\pi/a)^2 + (n\pi/b)^2]^{1/2}$. The general form of the potential must be a doubly infinite sum of the resulting functions:

$$\Phi = \sum_n\sum_m C_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, z \tag{14}$$

If (14) is evaluated at the boundary $z = c$, the series becomes

$$V(x, y) = \sum_n\sum_m D_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \tag{15}$$

where

$$D_{mn} = C_{mn} \sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, c \tag{16}$$

The coefficients $D_{mn}$ can be evaluated by multiplying (15) by $\sin(p\pi x/a)\sin(q\pi y/b)$ and integrating over $x$ from 0 to $a$ and over $y$ from 0 to $b$. Application of the orthogonality conditions Eqs. 7.11(5) and 7.11(7) yields

$$D_{mn} = \frac{4}{ab}\int_0^b\int_0^a V(x, y) \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, dx\, dy \tag{17}$$

For the special case of $V(x, y) = V_0$, (14), (16), and (17) give

$$\Phi = \sum_n\sum_m \frac{16V_0}{nm\pi^2} \frac{\sin(m\pi/a)x \sin(n\pi/b)y \sinh\sqrt{(m\pi/a)^2 + (n\pi/b)^2}\, z}{\sinh\sqrt{(m\pi/a)^2 + (n\pi/b)^2}\, c} \tag{18}$$

where $m$ and $n$ are odd in the summation.

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## 7.13 Cylindrical Harmonics for Static Fields

In a large class of problems of major interest, the field distribution is desired for regions with boundaries lying along the surfaces of a cylindrical coordinate system. Examples are the familiar electrostatic electron lenses found in many cathode-ray tubes or certain coaxial transmission line problems for which static solutions are useful. As has been pointed out in Sec. 7.12, the ability to evaluate the constants in product solutions depends upon having boundaries on coordinate surfaces. Therefore, the fields for this type of problem are found by separating variables in cylindrical coordinates.

A variety of types of solution are found, depending upon symmetries assumed. In general, Laplace's equation in cylindrical coordinates has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{1}$$

**Axial Symmetry with Longitudinal Invariance** In Ex. 1.8a we saw that for zero variations with $\phi$ and $z$,

$$\Phi(r) = C_1 \ln r + C_2 \tag{2}$$

**Longitudinal Invariance** It was shown in Prob. 7.9a that the solutions for zero $z$ variation, called circular harmonics, are given by

$$\Phi(r, \phi) = (C_1 r^n + C_2 r^{-n})(C_3 \cos n\phi + C_4 \sin n\phi) \tag{3}$$

Note that, for $n = 0$, axial symmetry exists but (3) breaks down and the solution is given by (2).

**Axial Symmetry** Since it is assumed that there are no variations with $\phi$, Laplace's equation (1) becomes

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{4}$$

To solve this equation, let us try to find solutions of the product form

$$\Phi(r, z) = R(r)Z(z) \tag{5}$$

Substituting in the differential equation (4), we have

$$R''Z + \frac{1}{r}R'Z + RZ'' = 0$$

where $R''$ denotes $d^2R/dr^2$, $Z''$ denotes $d^2Z/dz^2$, and so on. The variables are separated by dividing by $RZ$:

$$\frac{Z''}{Z} = -\left[\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right]$$

By the standard argument for the method of separation of variables, the left side, which is a function of $z$ alone, and the right side, which is a function of $r$ alone, must be equal to each other for all values of the variables $r$ and $z$. Both sides must then be equal to a constant. Let this constant be $T^2$. Two ordinary differential equations then result as follows:

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} = -T^2 \tag{6}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = T^2 \tag{7}$$

Equation (6) can be written as

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + T^2R = 0 \tag{8}$$

Equation (8) is the simplest form of the so-called *Bessel equation.* A sketch of the solution will be given. One method of finding a solution is to substitute a series and find the conditions on the terms of the series for it to be a valid solution of the differential equation. Thus to solve (8), the function $R$ must be assumed to be a power series in $r$:

$$R = a_0 + a_1r + a_2r^2 + a_3r^3 + \cdots$$

or

$$R = \sum_{p=0}^{\infty} a_pr^p \tag{9}$$

Substitution of this function in (8) shows that it is a solution if the constants are as follows:

$$a_p = a_{2m} = C_1(-1)^m \frac{(T/2)^{2m}}{(m!)^2}$$

($C_1$ is any arbitrary constant). That is,

$$R = C_1 \sum_{m=0}^{\infty} \frac{(-1)^m(Tr/2)^{2m}}{(m!)^2} = C_1\left[1 - \left(\frac{Tr}{2}\right)^2 + \frac{(Tr/2)^4}{(2!)^2} - \cdots\right] \tag{10}$$

is a solution to the differential equation (8). It is easy to show that (10) is convergent so that values may be calculated for any value of the argument ($Tr$). Such calculations have been made over a wide range of the values of the argument and the results are tabulated.

If $T^2$ is positive, the function defined by the series is denoted by $J_0(Tr)$ and called a Bessel function of the first kind and of zero order. This function is defined by

$$J_0(v) \triangleq 1 - \left(\frac{v}{2}\right)^2 + \frac{(v/2)^4}{(2!)^2} - \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m(v/2)^{2m}}{(m!)^2} \tag{11}$$

The particular solution (10) may then be written simply as

$$R = C_1J_0(Tr)$$

The differential equation (8) is of second order and so must have a second solution with a second arbitrary constant. (The sine and cosine constitute the two solutions for the simple harmonic motion equation.) This solution cannot be obtained by the power-series method outlined above, since a general study of differential equations would show that at least one of the two independent solutions of (8) must have a singularity at $r = 0$. Once one solution is found there is a technique for obtaining a linearly independent solution for this class of equations[12] and several different forms are possible. One form for the second solution (any of which may be called Bessel functions of second kind, order zero) easily found in tables is

$$\begin{aligned} N_0(v) &= \frac{2}{\pi} \ln\left(\frac{\gamma v}{2}\right) J_0(v) \\ &\quad - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m (v/2)^{2m}}{(m!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) \end{aligned} \tag{12}$$

The constant $\ln \gamma = 0.5772\ldots$ is Euler's constant. In general, then,

$$R = C_1 J_0(Tr) + C_2 N_0(Tr) \tag{13}$$

is the solution to (8), and the corresponding solution to (7) is

$$Z(z) = C_3 \sinh Tz + C_4 \cosh Tz \tag{14}$$

It should be noted from (12) that $N_0(Tr)$, the second solution to $R$, becomes infinite at $r = 0$, so it cannot be present in any problem for which $r = 0$ is included in the region over which the solution applies.

If $T^2$ is negative, we let $T^2 = -\tau^2$ or $T = j\tau$, where $\tau$ is real. The series (10) is still a solution and $T$ in (10) may be replaced by $j\tau$. Since all powers of the series are even, imaginaries disappear, and a new series is obtained which is also real and convergent. That is,

$$J_0(jv) = 1 + \left(\frac{v}{2}\right)^2 + \frac{(v/2)^4}{(2!)^2} + \frac{(v/2)^6}{(3!)^2} + \cdots \tag{15}$$

Values of $J_0(jv)$ may be calculated for various values of $v$ from such a series; these are also tabulated in the references and are usually denoted $I_0(v)$. Thus, one solution to (8) with $T = j\tau$ is

$$R = C_1' J_0(j\tau r) \triangleq C_1' I_0(\tau r) \tag{16}$$

There must also be a second solution which is commonly denoted $K_0(\tau r)$, so that the general solution to (8) with $T^2 = -\tau^2$ may be written

$$R = C_1' I_0(\tau r) + C_2' K_0(\tau r) \tag{17}$$

The second solution $K_0$ becomes infinite at $r = 0$ just as does $N_0$, and so is not required

[12] *See, for example, E. T. Whittaker and G. N. Watson,* A Course in Modern Analysis, *p. 369, 4th ed., University Press, Cambridge, 1927.*

in problems which include the axis $r = 0$ in the range over which the solution is to apply. The solution to the $z$ equation (7) when $T^2 = -\tau^2$ is

$$Z = C_3' \sin \tau z + C_4' \cos \tau z \tag{18}$$

Summarizing, either of the following forms satisfies Laplace's equation in the two cylindrical coordinates $r$ and $z$:

$$\Phi(r, z) = [C_1 J_0(Tr) + C_2 N_0(Tr)][C_3 \sinh Tz + C_4 \cosh Tz] \tag{19}$$

$$\Phi(r, z) = [C_1' I_0(\tau r) + C_2' K_0(\tau r)][C_3' \sin \tau z + C_4' \cos \tau z] \tag{20}$$

As was the case with the rectangular harmonics, the two forms are not really different since (19) includes (20) if $T$ is allowed to become imaginary, but the two separate ways of writing the solution are useful, as will be demonstrated in later examples. The case with no assumed symmetries is discussed in the following section.

## 7.14 BESSEL FUNCTIONS

In Sec. 7.13 an example of a Bessel function was shown as a solution of the differential equation 7.13(8) which describes the radial variations in Laplace's equation for axially symmetric fields where a product solution is assumed. This is just one of a whole family of functions which are solutions of the general Bessel differential equation.

**Bessel Functions with Real Arguments** For certain problems, as, for example, the solution for field between the two halves of a longitudinally split cylinder, it may be necessary to retain the $\phi$ variations in the equation. The solution may be assumed in product form again, $RF_\phi Z$, where $R$ is a function of $r$ alone, $F_\phi$ of $\phi$ alone, and $Z$ of $z$ alone, $Z$ has solutions in hyperbolic functions as before, and $F_\phi$ may also be satisfied by sinusoids:

$$Z = C \cosh Tz + D \sinh Tz \tag{1}$$

$$F_\phi = E \cos \nu\phi + F \sin \nu\phi \tag{2}$$

The differential equation for $R$ is then slightly different from the zero-order Bessel equation obtained previously:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(T^2 - \frac{\nu^2}{r^2}\right)R = 0 \tag{3}$$

It is apparent at once that Eq. 7.13(8) is a special case of this more general equation, that is, $\nu = 0$. A series solution to the general equation carried through as in Sec. 7.13 shows that the function defined by the series

$$J_\nu(Tr) = \sum_{m=0}^{\infty} \frac{(-1)^m (Tr/2)^{\nu+2m}}{m!\Gamma(\nu + m + 1)} \tag{4}$$

is a solution to the equation.

$\Gamma(\nu + m + 1)$ is the gamma function of $(\nu + m + 1)$ and, for $\nu$ integral, is equivalent to the factorial of $(\nu + m)$. Also for $\nu$ nonintegral, values of this gamma function are

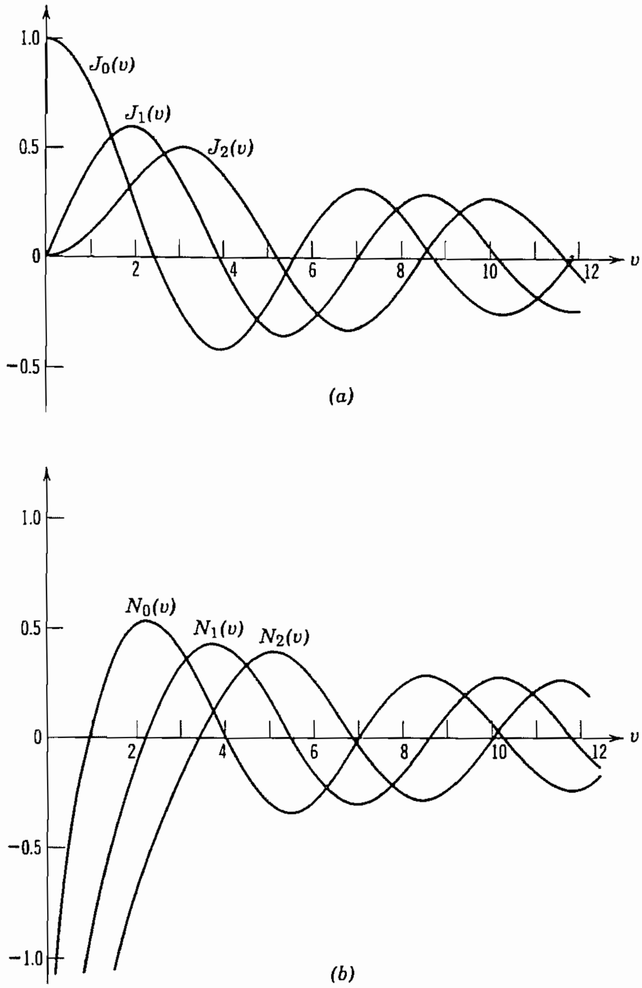


**FIG. 7.14** (*a*) Bessel functions of the first kind. (*b*) Bessel functions of the second kind.

tabulated. If $\nu$ is an integer $n$,

$$J_n(Tr) = \sum_{m=0}^{\infty} \frac{(-1)^m (Tr/2)^{n+2m}}{m!(n+m)!} \tag{5}$$

It can be shown that $J_{-n} = (-1)^n J_n$. A few of these functions are plotted in Fig. 7.14*a*. Similarly, a second independent solution[13] to the equation is

$$N_\nu(Tr) = \frac{\cos \nu\pi J_\nu(Tr) - J_{-\nu}(Tr)}{\sin \nu\pi} \tag{6}$$

[13] *If $\nu$ is nonintegral, $J_{-\nu}$ is not linearly related to $J_\nu$, and it is then proper to use either $J_{-\nu}$ or $N_\nu$ as the second solution; for $\nu$ integral, $N_\nu$ must be used. Equation (6) is indeterminate for $\nu$ integral but is subject to evaluation by usual methods.*

and $N_{-n} = (-1)^n N_n$. As may be noted in Fig. 7.14*b* these are infinite at the origin. A complete solution to (3) may be written

$$R = AJ_\nu(Tr) + BN_\nu(Tr) \tag{7}$$

The constant $\nu$ is known as the order of the equation. $J_\nu$ is then called a Bessel function of first kind, order $\nu$; $N_\nu$ is a Bessel function of second kind, order $\nu$. Of most interest for this chapter are cases in which $\nu = n$, an integer.

It is useful to keep in mind that, in the physical problem considered here, $\nu$ is the number of radians of the sinusoidal variation of the potential per radian of angle about the axis.

The functions $J_\nu(v)$ and $N_\nu(v)$ are tabulated in the references.[14,15] Some care should be observed in using these references, for there is a wide variation in notation for the second solution, and not all the functions used are equivalent, since they differ in the values of arbitrary constants selected for the series. The $N_\nu(v)$ is chosen here because it is the form most common in current mathematical physics and also the form most commonly tabulated. Of course, it is quite proper to use any one of the second solutions throughout a given problem, since all the differences will be absorbed in the arbitrary constants of the problem, and the same final numerical result will be obtained; but it is necessary to be consistent in the use of only one of these throughout any given analysis.

It is of interest to observe the similarity between (3) and the simple harmonic equation, the solutions of which are sinusoids. The difference between these two differential equations lies in the term $(1/r)(dR/dr)$ which produces its major effect as $r \to 0$. Note that for regions far removed from the axis as, for example, near the outer edge of Fig. 1.19*a*, the region bounded by surfaces of a cylindrical coordinate system approximates a cube. For these reasons, it may be expected that, away from the origin, the Bessel functions are similar to sinusoids. That this is true may be seen in Figs. 7.14*a* and *b*. For large values of the arguments, the Bessel functions approach sinusoids with magnitude decreasing as the square root of radius, as will be seen in the asymptotic forms, Eqs. 7.15(1) and 7.15(2).

**Hankel Functions** It is sometimes convenient to take solutions to the simple harmonic equation in the form of complex exponentials rather than sinusoids. That is, the solution of

$$\frac{d^2Z}{dz^2} + K^2Z = 0 \tag{8}$$

can be written as

$$Z = Ae^{+jKz} + Be^{-jKz} \tag{9}$$

[14] *E. Jahnke, F. Emde, and F. Lösch,* Tables of Higher Functions, *6th ed. revised by F. Lösch, McGraw-Hill, New York, 1960.*

[15] *M. Abramowitz and I. A. Stegun (Eds.),* Handbook of Mathematical Functions, *Dover, New York, 1964.*

where

$$e^{\pm jKz} = \cos Kz \pm j \sin Kz \tag{10}$$

Since the complex exponentials are linear combinations of cosine and sine functions, we may also write the general solution of (8) as

$$Z = A'e^{jKz} + B' \sin Kz$$

or other combinations.

Similarly, it is convenient to define new Bessel functions which are linear combinations of the $J_\nu(Tr)$ and $N_\nu(Tr)$ functions. By direct analogy with the definition (10) of the complex exponential, we write

$$H_\nu^{(1)}(Tr) = J_\nu(Tr) + jN_\nu(Tr) \tag{11}$$

$$H_\nu^{(2)}(Tr) = J_\nu(Tr) - jN_\nu(Tr) \tag{12}$$

These are called Hankel functions of the first and second kinds, respectively. Since they both contain the function $N_\nu(Tr)$, they are both singular at $r = 0$. Negative and positive orders are related by

$$H_{-\nu}^{(1)}(Tr) = e^{j\pi\nu}H_\nu^{(1)}(Tr)$$

$$H_{-\nu}^{(2)}(Tr) = e^{-j\pi\nu}H_\nu^{(2)}(Tr)$$

For large values of the argument, these can be approximated by complex exponentials, with magnitude decreasing as square root of radius. For example,

$$\underset{Tr\to\infty}{H_\nu^{(1)}(Tr)} = \sqrt{\frac{2}{\pi Tr}}\, e^{j(Tr - \pi/4 - \nu\pi/2)}$$

This asymptotic form suggests that Hankel functions may be useful in wave propagation problems as the complex exponential is in plane-wave propagation. It is also sometimes convenient to use Hankel functions as alternate independent solutions in static problems. Complete solutions of (3) may be written in a variety of ways using combinations of Bessel and Hankel functions.

**Bessel and Hankel Functions of Imaginary Arguments** If $T$ is imaginary, $T = j\tau$, and (3) becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\tau^2 + \frac{\nu^2}{r^2}\right)R = 0 \tag{13}$$

The solution to (3) is valid here if $T$ is replaced by $j\tau$ in the definitions of $J_\nu(Tr)$ and $N_\nu(Tr)$. In this case $N_\nu(j\tau r)$ is complex and so requires two numbers for each value of the argument, whereas $j^{-\nu}J_\nu(j\tau r)$ is always a purely real number. It is convenient to replace $N_\nu(j\tau r)$ by a Hankel function. The quantity $j^{\nu-1}H_\nu^{(1)}(j\tau r)$ is also purely real and so requires tabulation of only one value for each value of the argument. If $\nu$ is not an integer, $j^\nu J_{-\nu}(j\tau r)$ is independent of $j^{-\nu}J_\nu(j\tau r)$ and may be used as a second solution.

Thus, for nonintegral $\nu$ two possible complete solutions are

$$R = A_2 J_\nu(j\tau r) + B_2 J_{-\nu}(j\tau r) \tag{14}$$

and

$$R = A_3 J_\nu(j\tau r) + B_3 H_\nu^{(1)}(j\tau r) \tag{15}$$

where powers of $j$ are included in the constants. For $\nu = n$, an integer, the two solutions in (14) are not independent but (15) is still a valid solution.

It is common practice to denote these solutions as

$$I_{\pm\nu}(v) = j^{\mp\nu} J_{\pm\nu}(jv) \tag{16}$$

$$K_\nu(v) = \frac{\pi}{2} j^{\nu+1} H_\nu^{(1)}(jv) \tag{17}$$

where $v = \tau r$.

As is noted in Sec. 7.15 some of the formulas relating Bessel functions and Hankel functions must be changed for these modified Bessel functions. Special cases of these functions were seen as $I_0(\tau r)$ and $K_0(\tau r)$ in Sec. 7.13 for the axially symmetric field. The forms of $I_\nu(\tau r)$ and $K_\nu(\tau r)$ for $\nu = 0, 1$ are shown in Fig. 7.14*c*. As is suggested by these curves, the asymptotic forms of the modified Bessel functions are related to growing and decaying real exponentials, as will be seen in Eqs. 7.15(5) and 7.15(6). It is also clear from the figure that $K_\nu(\tau r)$ is singular at the origin.

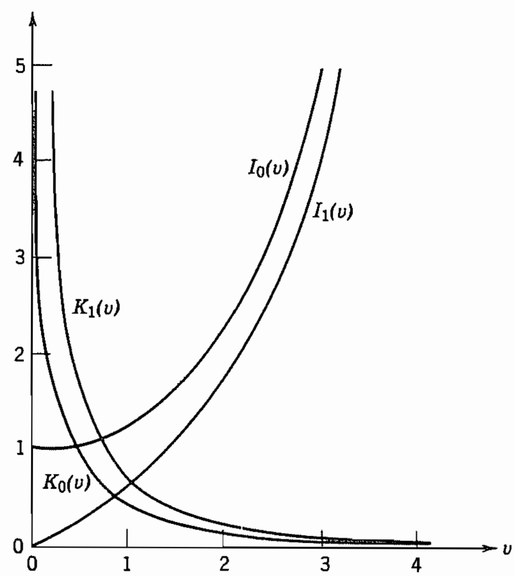


**FIG. 7.14c** Modified Bessel functions.

## 7.15 Bessel Function Zeros and Formulas[16]

The first several zeros of the low-order Bessel functions and of the derivatives of Bessel functions are given in Tables 7.15a and 7.15b, respectively.

**Table 7.15a**
**Zeros of Bessel Functions**

| $J_0$ | $J_1$ | $J_2$ | $N_0$ | $N_1$ | $N_2$ |
|---|---|---|---|---|---|
| 2.405 | 3.832 | 5.136 | 0.894 | 2.197 | 3.384 |
| 5.520 | 7.016 | 8.417 | 3.958 | 5.430 | 6.794 |
| 8.654 | 10.173 | 11.620 | 7.086 | 8.596 | 10.023 |

**Table 7.15b**
**Zeros of Derivatives of Bessel Functions**

| $J_0'$ | $J_1'$ | $J_2'$ | $N_0'$ | $N_1'$ | $N_2'$ |
|---|---|---|---|---|---|
| 0.000 | 1.841 | 3.054 | 2.197 | 3.683 | 5.003 |
| 3.832 | 5.331 | 6.706 | 5.430 | 6.942 | 8.351 |
| 10.173 | 8.536 | 9.969 | 8.596 | 10.123 | 11.574 |

### Asymptotic Forms

$$J_\nu(v) \underset{v\to\infty}{\to} \sqrt{\frac{2}{\pi v}} \cos\left(v - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{1}$$

$$N_\nu(v) \underset{v\to\infty}{\to} \sqrt{\frac{2}{\pi v}} \sin\left(v - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{2}$$

$$H_\nu^{(1)}(v) \underset{v\to\infty}{\to} \sqrt{\frac{2}{\pi v}}\, e^{j[v-(\pi/4)-(\nu\pi/2)]} \tag{3}$$

$$H_\nu^{(2)}(v) \underset{v\to\infty}{\to} \sqrt{\frac{2}{\pi v}}\, e^{-j[v-(\pi/4)-(\nu\pi/2)]} \tag{4}$$

$$\underset{v\to\infty}{j^{-\nu}J_\nu(jv)} = \underset{v\to\infty}{I_\nu(v)} \to \sqrt{\frac{1}{2\pi v}}\, e^{v} \tag{5}$$

$$\underset{v\to\infty}{j^{\nu+1}H_\nu^{(1)}(jv)} = \frac{2}{\pi} \underset{v\to\infty}{K_\nu(v)} \to \sqrt{\frac{2}{\pi v}}\, e^{-v} \tag{6}$$

[16] *More extensive tabulations are found in the sources given in footnotes 14 and 15.*

**Derivatives** The following formulas which may be found by differentiating the appropriate series, term by term, are valid for any of the functions $J_\nu(v)$, $N_\nu(v)$, $H_\nu^{(1)}(v)$, and $H_\nu^{(2)}(v)$. Let $R_\nu(v)$ denote any one of these, and $R_\nu'$ denote $(d/dv)[R_\nu(v)]$.

$$R_0' = -R_1(v) \tag{7}$$

$$R_1'(v) = R_0(v) - \frac{1}{v}R_1(v) \tag{8}$$

$$vR_\nu'(v) = \nu R_\nu(v) - vR_{\nu+1}(v) \tag{9}$$

$$vR_\nu'(v) = -\nu R_\nu(v) + vR_{\nu-1}(v) \tag{10}$$

$$\frac{d}{dv}[v^{-\nu}R_\nu(v)] = -v^{-\nu}R_{\nu+1}(v) \tag{11}$$

$$\frac{d}{dv}[v^{\nu}R_\nu(v)] = v^{\nu}R_{\nu-1}(v) \tag{12}$$

Note that

$$R_\nu'(Tr) = \frac{d}{d(Tr)}[R_\nu(Tr)] = \frac{1}{T}\frac{d}{dr}[R_\nu(Tr)] \tag{13}$$

For the $I$ and $K$ functions, different forms for the foregoing derivatives must be used. They may be obtained from these formulas by substituting Eqs. 7.14(16) and 7.14(17) in the preceding expressions. Some of these are

$$\left.\begin{aligned} vI_\nu'(v) &= \nu I_\nu(v) + vI_{\nu+1}(v) \\ vI_\nu'(v) &= -\nu I_\nu(v) + vI_{\nu-1}(v) \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} vK_\nu'(v) &= \nu K_\nu(v) - vK_{\nu+1}(v) \\ vK_\nu'(v) &= -\nu K_\nu(v) - vK_{\nu-1}(v) \end{aligned}\right\} \tag{15}$$

**Recurrence Formulas** By recurrence formulas, it is possible to obtain the values for Bessel functions of any order, when the values of functions for any two other orders, differing from the first by integers, are known. For example, subtract (10) from (9). The result may be written

$$\frac{2\nu}{v}R_\nu(v) = R_{\nu+1}(v) + R_{\nu-1}(v) \tag{16}$$

As before, $R_\nu$ may denote $J_\nu$, $N_\nu$, $H_\nu^{(1)}$, or $H_\nu^{(2)}$, but not $I_\nu$ or $K_\nu$. For these, the recurrence formulas are

$$\frac{2\nu}{v}I_\nu(v) = I_{\nu-1}(v) - I_{\nu+1}(v) \tag{17}$$

$$\frac{2\nu}{v}K_\nu(v) = K_{\nu+1}(v) - K_{\nu-1}(v) \tag{18}$$

**Integrals** Integrals that will be useful in solving later problems are given below. $R_\nu$ denotes $J_\nu$, $N_\nu$, $H_\nu^{(1)}$, or $H_\nu^{(2)}$:

$$\int v^{-\nu} R_{\nu+1}(v)\, dv = -v^{-\nu} R_\nu(v) \tag{19}$$

$$\int v^{\nu} R_{\nu-1}(v)\, dv = v^{\nu} R_\nu(v) \tag{20}$$

$$\begin{aligned}\int vR_\nu(\alpha v)R_\nu(\beta v)\, dv &= \frac{v}{\alpha^2 - \beta^2} \\ &\quad \times [\beta R_\nu(\alpha v)R_{\nu-1}(\beta v) - \alpha R_{\nu-1}(\alpha v)R_\nu(\beta v)], \quad \alpha \neq \beta\end{aligned} \tag{21}$$

$$\begin{aligned}\int vR_\nu^2(\alpha v)\, dv &= \frac{v^2}{2}[R_\nu^2(\alpha v) - R_{\nu-1}(\alpha v)R_{\nu+1}(\alpha v)] \\ &= \frac{v^2}{2}\left[R_\nu'^2(\alpha v) + \left(1 - \frac{\nu^2}{\alpha^2 v^2}\right)R_\nu^2(\alpha v)\right]\end{aligned} \tag{22}$$

## 7.16 Expansion of a Function as a Series of Bessel Functions

In Sec. 7.11 a study was made of the method of Fourier series by which a function may be expressed over a given region as a series of sines or cosines. It is possible to evaluate the coefficients in such a case because of the orthogonality property of sinusoids. A study of the integrals, Eqs. 7.15(21) and 7.15(22), shows that there are similar orthogonality expressions for Bessel functions. For example, these integrals may be written for zero-order Bessel functions, and if $\alpha$ and $\beta$ are taken as $p_m/a$ and $p_q/a$, where $p_m$ and $p_q$ are the $m$th and $q$th roots of $J_0(v) = 0$, that is, $J_0(p_m) = 0$ and $J_0(p_q) = 0$, $p_m \neq p_q$, then Eq. 7.15(21) gives

$$\int_0^a rJ_0\left(\frac{p_m r}{a}\right)J_0\left(\frac{p_q r}{a}\right) dr = 0 \tag{1}$$

So, a function $f(r)$ may be expressed as an infinite sum of zero-order Bessel functions

$$f(r) = b_1 J_0\left(p_1 \frac{r}{a}\right) + b_2 J_0\left(p_2 \frac{r}{a}\right) + b_3 J_0\left(p_3 \frac{r}{a}\right) + \cdots$$

or

$$f(r) = \sum_{m=1}^{\infty} b_m J_0\left(\frac{p_m r}{a}\right) \tag{2}$$

The coefficients $b_m$ may be evaluated in a manner similar to that used for Fourier coefficients by multiplying each term of (2) by $rJ_0(p_m r/a)$ and integrating from 0 to

$a$. Then by (1) all terms on the right disappear except the $m$th term:

$$\int_0^a rf(r)J_0\left(\frac{p_m r}{a}\right) dr = \int_0^a b_m r\left[J_0\left(\frac{p_m r}{a}\right)\right]^2 dr$$

From Eq. 7.15(22),

$$\int_0^a b_m r J_0^2\left(\frac{p_m r}{a}\right) dr = \frac{a^2}{2} b_m J_1^2(p_m) \tag{3}$$

or

$$b_m = \frac{2}{a^2 J_1^2(p_m)} \int_0^a rf(r)J_0\left(\frac{p_m r}{a}\right) dr \tag{4}$$

In the above, as in the Fourier series, the orthogonality relations enabled us to obtain coefficients of the series under the assumption that the series is a proper representation of the function to be expanded, but two additional points are required to show that the representation is valid. The series must of course converge, and the set of orthogonal functions must be *complete*, that is, sufficient to represent an arbitrary function over the interval of concern. These points have been shown for the Bessel series of (2) and for other orthogonal sets of functions to be used in this text.[17]

Expansions similar to (2) can be made with Bessel functions of other orders and types (Prob. 7.16a).

**Example 7.16**

BESSEL FUNCTION EXPANSION FOR CONSTANT IN RANGE 0 < r < A

If the function $f(r)$ in (4) is a constant $V_0$ in the range $0 < r < a$, we have

$$b_m = \frac{2V_0}{a^2 J_1^2(p_m)} \int_0^a rJ_0\left(\frac{p_m r}{a}\right) dr \tag{5}$$

Using Eq. 7.15(20) with $R = J$, $\nu = 1$, and $v = p_m r/a$, the integral in (5) becomes

$$\begin{aligned}\left(\frac{a}{p_m}\right)^2 \int_0^a \left(\frac{p_m r}{a}\right) J_0\left(\frac{p_m r}{a}\right) d\left(\frac{p_m r}{a}\right) &= \left[\left(\frac{a}{p_m}\right)^2 \left(\frac{p_m r}{a}\right) J_1\left(\frac{p_m r}{a}\right)\right]_0^a \\ &= \frac{a^2}{p_m} J_1(p_m)\end{aligned} \tag{6}$$

[17] *See, for example, E. T. Whittaker and G. N. Watson,* A Course in Modern Analysis, *4th ed., pp. 374–378, University Press, Cambridge, 1927.*

and the series expansion (2) for the constant $V_0$ is

$$f(r) = \sum_{m=1}^{\infty} \frac{2V_0}{p_m J_1(p_m)} J_0\left(\frac{p_m r}{a}\right) \tag{7}$$

or, using the values of the zeros of $J_0$ in Table 7.15a,

$$\begin{aligned} f(r) &= \frac{0.832V_0}{J_1(2.405)} J_0\left(\frac{2.405r}{a}\right) + \frac{0.362V_0}{J_1(5.520)} J_0\left(\frac{5.520r}{a}\right) \\ &\quad + \frac{0.231V_0}{J_1(8.654)} J_0\left(\frac{8.654r}{a}\right) + \cdots \end{aligned} \tag{8}$$

Further evaluation of (8) requires reference to tables in the sources given in footnotes 14 and 15 or numerical evaluation of Eq. 7.13(11).

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## 7.17 Fields Described by Cylindrical Harmonics

We will consider here the two basic types of boundary value problems which exist in axially symmetric cylindrical systems. These can be understood by reference to Fig. 7.17*a*. In one type both $\Phi_1$ and $\Phi_2$, the potentials on the ends, are zero and a nonzero potential $\Phi_3$ is applied to the cylindrical surface. In the second type $\Phi_3 = 0$ and either (or both) $\Phi_1$ or $\Phi_2$ are nonzero. The gaps between ends and side are considered negligibly small. For simplicity, the nonzero potentials will be taken to be independent of the coordinate along the surface. In the first type, a Fourier series of sinusoids is used to expand the boundary potentials as was done in the rectangular problems. In the second situation, a series of Bessel functions is used to expand the boundary potential along the radial coordinate.

**Nonzero Potential on Cylindrical Surface** Since the boundary potentials are axially symmetric, zero-order Bessel functions should be used. The repeated zeros along the $z$ coordinate dictate the use of sinusoidal functions of $z$. The potential in Eq. 7.13(20) is the appropriate form. Certain of the constants can be evaluated immediately. Since $K_0(\tau r)$ is singular on the axis, $C_2'$ must be identically zero to give a finite potential there. The cos $\tau z$ equals unity at $z = 0$ but the potential must be zero there so $C_4' = 0$. As in the problem discussed in Sec. 7.10 the repeated zeros at $z = l$ require that $\tau = m\pi/l$. Therefore the general harmonic which fits all boundary conditions except $\Phi = V_0$ at $r = a$ is

$$\Phi_m = A_m I_0\left(\frac{m\pi r}{l}\right)\sin\left(\frac{m\pi z}{l}\right) \tag{1}$$

Figure 7.17*b* shows a sketch of this harmonic for $m = 1$ and with the nonzero boundary potential on the cylinder. It is clear that we have here the problem of expanding the

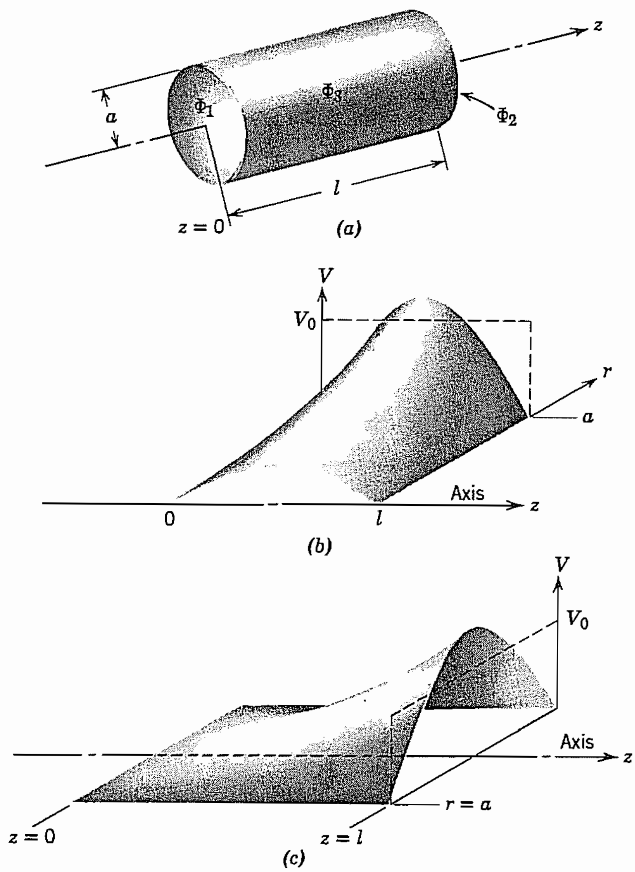


**FIG. 7.17** (*a*) Cylinder with conducting boundaries. (*b*) One harmonic component for matching boundary conditions when nonzero potential is applied to cylindrical surface in (*a*). (*c*) One harmonic component for matching boundary conditions when nonzero potential is applied to end surface in (*a*).

boundary potential in sinusoids just as in the rectangular problem of Sec. 7.12. Following the procedure used there we obtain

$$\Phi(r, z) = \sum_{m \text{ odd}} \frac{4V_0}{m\pi} \frac{I_0(m\pi r/l)}{I_0(m\pi a/l)} \sin \frac{m\pi z}{l} \tag{2}$$

**Nonzero Potential on End** In this situation if we refer to Fig. 7.17*a*, we see that $\Phi_1 = \Phi_3 = 0$ and $\Phi_0 = V_0$. In selecting the proper form for the solution from Sec. 7.13, the boundary condition that $\Phi = 0$ at $r = a$ for all values of $z$ indicates that the $R$ function must become zero at $r = a$. Thus, we select the $J_0$ functions since the $I_0$'s do not ever become zero. (The corresponding second solution, $N_0$, does not appear since potential must remain finite on the axis.) The value of $T$ in Eq. 7.13(19) is determined from the condition that $\Phi = 0$ at $r = a$ for all values of $z$. Thus, if $p_m$ is the $m$th root of $J_0(v) = 0$, $T$ must be $p_m/a$. The corresponding solution for $Z$ is in hyperbolic functions. The coefficient of the hyperbolic cosine term must be zero since $\Phi$ is zero at $z = 0$ for all values of $r$. Thus, a sum of all cylindrical harmonics with arbitrary amplitudes which satisfy the symmetry of the problem and the boundary conditions so far imposed may be written

$$\Phi(r, z) = \sum_{m=1}^{\infty} B_m J_0\left(\frac{p_m r}{a}\right) \sinh\left(\frac{p_m z}{a}\right) \tag{3}$$

One of the harmonics and the required boundary potentials are shown in Fig. 7.17*c*.

The remaining condition is that, at $z = l$, $\Phi = 0$ at $r = a$ and $\Phi = V_0$ at $r < a$. Here we can use the general technique of expanding the boundary potential in a series of the same form as that used for the potentials inside the region, as regards the dependence on the coordinate along the boundary. In Ex. 7.16 we expanded a constant over the range $0 < r < a$ in $J_0$ functions so that result can be used here to evaluate the constants in (3). Evaluating (3) at the boundary $z = l$, we have

$$\Phi(r, l) = \sum_{m=1}^{\infty} B_m \sinh\left(\frac{p_m l}{a}\right) J_0\left(\frac{p_m r}{a}\right) \tag{4}$$

Equations (4) and 7.16(7) must be equivalent for all values of $r$. Consequently, coefficients of corresponding terms of $J_0(p_m r/a)$ must be equal. The constant $B_m$ is now completely determined, and the potential at any point inside the region is

$$\Phi(r, z) = \sum_{m=1}^{\infty} \frac{2V_0}{p_m J_1(p_m) \sinh(p_m l/a)} \sinh\left(\frac{p_m z}{a}\right) J_0\left(\frac{p_m r}{a}\right) \tag{5}$$

## 7.18 Spherical Harmonics

Consider next Laplace's equation in spherical coordinates for regions with symmetry about the axis so that variations with azimuthal angle $\phi$ may be neglected. Laplace's equation in the two remaining spherical coordinates $r$ and $\theta$ then becomes (obtainable from form of inside front cover)

$$\frac{\partial^2(r\Phi)}{\partial r^2} + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) = 0 \tag{1}$$

or

$$r\frac{\partial^2 \Phi}{\partial r^2} + 2\frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r \tan\theta}\frac{\partial \Phi}{\partial \theta} = 0 \tag{2}$$

Assume a product solution

$$\Phi = R\Theta$$

where $R$ is a function of $r$ alone, and $\Theta$ of $\theta$ alone:

$$rR''\Theta + 2R'\Theta + \frac{1}{r}R\Theta'' + \frac{1}{r\tan\theta}R\Theta' = 0$$

and

$$\frac{r^2R''}{R} + \frac{2rR'}{R} = -\frac{\Theta''}{\Theta} - \frac{\Theta'}{\Theta\tan\theta} \tag{3}$$

From the previous logic, if the two sides of the equations are to be equal to each other for all values of $r$ and $\theta$, both sides can be equal only to a constant. Since the constant may be expressed in any nonrestrictive way, let it be $m(m + 1)$. The two resulting ordinary differential equations are then

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - m(m + 1)R = 0 \tag{4}$$

$$\frac{d^2\Theta}{d\theta^2} + \frac{1}{\tan\theta}\frac{d\Theta}{d\theta} + m(m + 1)\Theta = 0 \tag{5}$$

Equation (4) has a solution which is easily verified to be

$$R = C_1r^m + C_2r^{-(m+1)} \tag{6}$$

A solution to (5) in terms of simple functions is not obvious, so, as with the Bessel equation, a series solution may be assumed. The coefficients of this series must be determined so that the differential equation (5) is satisfied and the resulting series made to define a new function. There is one departure here from an exact analog with the Bessel functions, for it turns out that a proper selection of the arbitrary constants will make the series for the new function terminate in a finite number of terms if $m$ is an integer. Thus, for any integer $m$, the polynomial defined by

$$P_m(\cos\theta) = \frac{1}{2^m m!}\left[\frac{d}{d(\cos\theta)}\right]^m(\cos^2\theta - 1)^m \tag{7}$$

is a solution to the differential equation (5). The equation is known as Legendre's equation; the solutions are called Legendre polynomials of order $m$. Their forms for the first few values of $m$ are tabulated below and are shown in Fig. 7.18*a*. Since they are polynomials and not infinite series, their values can be calculated easily if desired, but values of the polynomials are also tabulated in many references.

$$\begin{aligned}
P_0(\cos\theta) &= 1\\
P_1(\cos\theta) &= \cos\theta\\
P_2(\cos\theta) &= \tfrac{1}{2}(3\cos^2\theta - 1)\\
P_3(\cos\theta) &= \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)\\
P_4(\cos\theta) &= \tfrac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)\\
P_5(\cos\theta) &= \tfrac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)
\end{aligned} \tag{8}$$

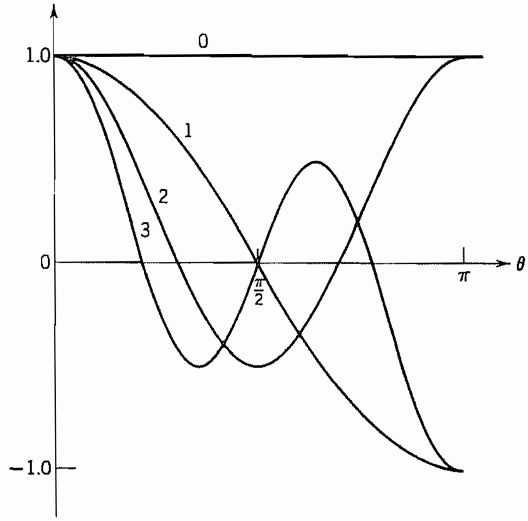


**FIG. 7.18a** Legendre polynomials.

It is recognized that $\Theta = C_1 P_m(\cos\theta)$ is only one solution to the second-order differential equation (5). There must be a second independent solution, which may be obtained from the first in the same manner as for Bessel functions, but it turns out that this solution becomes infinite for $\theta = 0$. Consequently it is not needed when the axis of spherical coordinates is included in the region over which the solution applies. When the axis is excluded, the second solution must be included. It is typically denoted $Q_n(\cos\theta)$ and tabulated in the references.[18]

An orthogonality relation for Legendre polynomials is quite similar to those for sinusoids and Bessel functions which led to the Fourier series and expansion in Bessel functions, respectively.

$$\int_0^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta = 0, \qquad m \neq n \tag{9}$$

$$\int_0^{\pi} [P_m(\cos\theta)]^2 \sin\theta \, d\theta = \frac{2}{2m+1} \tag{10}$$

It follows that, if a function $f(\theta)$ defined between the limits of 0 to $\pi$ is written as a series of Legendre polynomials,

$$f(\theta) = \sum_{m=0}^{\infty} \alpha_m P_m(\cos\theta), \qquad 0 < \theta < \pi \tag{11}$$

[18] *W. R. Smythe,* Static and Dynamic Electricity, *3rd ed., Hemisphere Publishing Co., Washington, DC, 1989.*

the coefficients must be given by the formula

$$\alpha_m = \frac{2m + 1}{2} \int_0^\pi f(\theta) P_m(\cos\theta) \sin\theta \, d\theta \tag{12}$$

## Example 7.18a
### HIGH-PERMEABILITY SPHERE IN UNIFORM FIELD

We will examine the field distribution in and around a sphere of permeability $\mu \neq \mu_0$ when it is placed in an otherwise uniform magnetic field in free space. The uniform field is disturbed by the sphere as indicated in Fig. 7.18*b*. The reason for choosing this example is threefold. It shows, first, an application of spherical harmonics. Second, it is an example of a situation in which the constants in series solutions for two regions are evaluated by matching across a boundary. Finally, it is an example of a magnetic boundary-value problem.

Since there are no currents in the region to be studied, we may use the scalar magnetic potential introduced in Sec. 2.13. The magnetic intensity is given by

$$\mathbf{H} = -\nabla \Phi_m \tag{13}$$

As the problem is axially symmetric and the axis is included in the region of interest, the solutions $P_m(\cos\theta)$ are applicable. The series solutions with these restrictions are

$$\Phi_m(r, \theta) = \sum_m P_m(\cos\theta)[C_{1m} r^m + C_{2m} r^{-(m+1)}] \tag{14}$$

The procedure is to write general forms for the potential inside and outside the sphere and match these across the boundary. Since the potential must remain finite at $r = 0$,

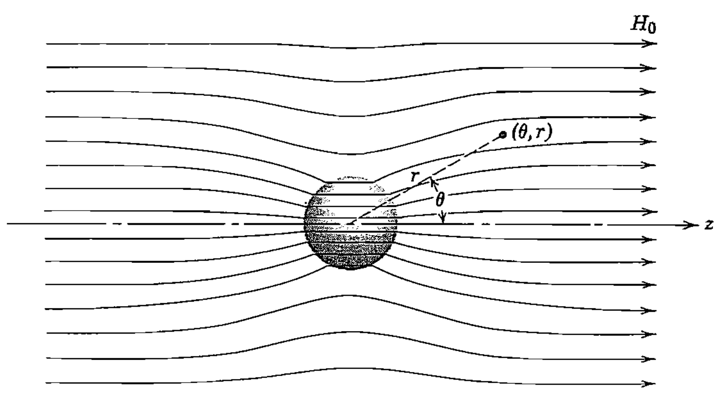


**FIG. 7.18*b*** Sphere of magnetic material in an otherwise uniform magnetic field.

the coefficients of the negative powers of $r$ must vanish for the interior. The series becomes, for the inside region,

$$\Phi_m = \sum_m A_m r^m P_m(\cos\theta) \tag{15}$$

Outside, the potential must be such that it gives a uniform magnetic field $H_0$ at infinity. The potential form which satisfies this condition is

$$\Phi_m = -H_0 r \cos\theta \tag{16}$$

That this gives a uniform field may be seen by noting that $dz = dr \cos\theta$ so

$$H_z = -\frac{\partial \Phi_m}{\partial z} = -\frac{1}{\cos\theta}\frac{\partial \Phi_m}{\partial r} = H_0 \tag{17}$$

Terms of the series (14) having negative powers of $r$ may be added to (16), since they all vanish at infinity. Then the form of the solution outside the sphere is

$$\Phi_m = -H_0 r \cos\theta + \sum_m B_m P_m(\cos\theta) \mathrm{r}^{-(m+1)} \tag{18}$$

It was pointed out in Sec. 2.14 that $\Phi_m$ is continuous across boundaries without surface currents. Therefore, the terms in (15) and (18) having the same form of $\theta$ dependence are equated, giving

$$\begin{aligned} A_0 &= B_0 a^{-1} && m = 0 \\ A_1 a &= B_1 a^{-2} - H_0 a && m = 1 \\ &\vdots \\ A_m a^m &= B_m a^{-(m+1)} && m > 1 \end{aligned} \tag{19}$$

Furthermore, the normal flux density is continuous at the boundary so

$$\mu_0 \left.\frac{\partial \Phi_m}{\partial r}\right|_{r=a+} = \mu \left.\frac{\partial \Phi_m}{\partial r}\right|_{r=a-} \tag{20}$$

Substituting (15) and (18) in (20) and equating terms with the same $\theta$ dependence, we find

$$\begin{aligned} B_0 &= 0 && m = 0 \\ \mu A_1 &= -2\mu_0 B_1 a^{-3} - \mu_0 H_0 && m = 1 \\ &\vdots \\ \mu m A_m a^{m-1} &= -\mu_0(m+1) B_m a^{-(m+2)} && m > 1 \end{aligned} \tag{21}$$

From (19) and (21) we see that $A_0 = B_0 = 0$, and that for $m > 1$, all coefficients must be zero to satisfy the two sets of conditions. The only remaining terms are those with

$m = 1$. These two equations may be solved to give $A_1$ and $B_1$ in terms of $H_0$. Substituting the results in (18) gives, for $r > a$,

$$\Phi_m = \left[\left(\frac{\mu - \mu_0}{2\mu_0 + \mu}\right)\frac{a^3}{r^3} - 1\right]H_0 r \cos\theta \tag{22}$$

from which $H$ can be found by using (13) for $r > a$. Substitution of $A_1$ into (15) gives, for $r < a$,

$$\Phi_m = -\left(\frac{3\mu_0}{2\mu_0 + \mu}\right)H_0 r \cos\theta \tag{23}$$

Applying (13), we find the field inside to be

$$\mathbf{H} = \hat{\mathbf{z}}\left(\frac{3\mu_0}{2\mu_0 + \mu}\right)H_0 \tag{24}$$

It is of interest to observe that the field inside the homogeneous sphere is uniform. Finally, multiplication of (24) by $\mu$ gives the flux density

$$\mathbf{B} = \hat{\mathbf{z}}\left(\frac{3\mu_0}{2(\mu_0/\mu) + 1}\right)H_0 \tag{25}$$

From (25) we see that for $\mu >> \mu_0$ the maximum possible value of the flux density is

$$\mathbf{B} = \hat{\mathbf{z}}3\mu_0 H_0 \tag{26}$$

---

## Example 7.18b

### EXPANSION IN SPHERICAL HARMONICS WHEN FIELD IS GIVEN ALONG AN AXIS

It is often relatively simple to obtain the field or potential along an axis of symmetry by direct application of fundamental laws, yet difficult to obtain it at any point off this axis by the same technique. Once the field is found along an axis of symmetry, expansions in spherical harmonics give its value at any other point. Suppose potential, or any component of field which satisfies Laplace's equation, is given for every point along an axis in such a form that it may be expanded in a power series in $z$, the distance along this axis:

$$\Phi\Big|_{\text{axis}} = \sum_{m=0}^{\infty} b_m z^m, \qquad 0 < z < a \tag{27}$$

If this axis is taken as the axis of spherical coordinates, $\theta = 0$, the potential off the axis may be written for $r < a$

$$\Phi(r, \theta) = \sum_{m=0}^{\infty} b_m r^m P_m(\cos\theta) \tag{28}$$

This is true since it is a solution of Laplace's equation and does reduce to the given potential (27) for $\theta = 0$ where all $P_m(\cos\theta)$ are unity.

If potential is desired outside of this region, the potential along the axis must be expanded in a power series good for $a < z < \infty$:

$$\Phi\bigg|_{\theta=0} = \sum_{m=1}^{\infty} c_m z^{-(m+1)}, \qquad z > a \tag{29}$$

Then $\Phi$ at any point outside is given by comparison with the second series of (14):

$$\Phi = \sum_{m=0}^{\infty} c_m P_m(\cos\theta) r^{-(m+1)}, \qquad r > a \tag{30}$$

For example, the magnetic field $H_z$ was found along the axis of a circular loop of wire carrying current $I$ in Sec. 2.3 as

$$H_z = \frac{a^2 I}{2(a^2 + z^2)^{3/2}} = \frac{I}{2a[1 + (z^2/a^2)]^{3/2}} \tag{31}$$

The binomial expansion

$$(1 + u)^{-3/2} = 1 - \frac{3}{2}u + \frac{15}{8}u^2 - \frac{105}{48}u^3 + \cdots$$

is good for $0 < |u| < 1$. Applied to (31), this gives for $z < a$

$$H_z\bigg|_{\text{axis}} = \frac{I}{2a}\left[1 - \frac{3}{2}\left(\frac{z^2}{a^2}\right) + \frac{15}{8}\left(\frac{z^2}{a^2}\right)^2 - \frac{105}{48}\left(\frac{z^2}{a^2}\right)^3 + \cdots\right]$$

Since $H_z$, axial component of magnetic field, satisfies Laplace's equation (Sec. 7.2), $H_z$ at any point $r$, $\theta$ with $r < a$ is given by

$$H_z(r, \theta) = \frac{I}{2a}\left[1 - \frac{3}{2}\left(\frac{r^2}{a^2}\right)P_2(\cos\theta) + \frac{15}{8}\left(\frac{r^4}{a^4}\right)P_4(\cos\theta) + \cdots\right] \tag{32}$$

## 7.19 Product Solutions for the Helmholtz Equation in Rectangular Coordinates

The technique used in the preceding sections for finding product solutions to Laplace's equation will be applied here to the scalar Helmholtz equation. Whereas the single-product solution for static problems was seen in Sec. 7.10 to be of little value, such solutions will be seen in the next chapter to be of great importance as waveguide propagation modes and will be analyzed extensively there.

Let us consider the scalar Helmholtz equation. Here we make the assumption that the dependent variable depends on $z$ in the manner of a wave, as $e^{-\gamma z}$. The variable $\psi$

remaining in the equation is, therefore, the coefficient of $e^{(j\omega t - \gamma z)}$. Written with the Laplacian explicitly in rectangular coordinates, we have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -k_c^2 \psi \tag{1}$$

where $k_c^2 = \gamma^2 + \omega^2 \mu \varepsilon$. Let us assume that the solution can be written as the product solution $\psi = X(x)Y(y)$. Substituting this form in (1),

$$X''Y + XY'' = -k_c^2 XY$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} = -k_c^2 \tag{2}$$

The primes indicate derivatives. If this equation is to hold for all values of $x$ and $y$, since $x$ and $y$ may be changed independently of each other, each of the ratios $X''/X$ and $Y''/Y$ can be only a constant. There are then several forms for the solutions, depending upon whether these ratios are taken as negative constants, positive constants, or one negative constant and one positive constant. If both are taken as negative,

$$\frac{X''}{X} = -k_x^2$$

$$\frac{Y''}{Y} = -k_y^2$$

The solutions to these ordinary differential equations are sinusoids, and by (2) the sum of $k_x^2$ and $k_y^2$ is $k_c^2$. Thus

$$\psi = XY \tag{3}$$

where

$$X = A \cos k_x x + B \sin k_x x$$

$$Y = C \cos k_y y + D \sin k_y y \tag{4}$$

$$k_x^2 + k_y^2 = k_c^2$$

Either or both of $k_x$ and $k_y$ may be imaginary in which case the corresponding sinusoid becomes a hyperbolic function. Values of the constants $k_x$ and $k_y$ are determined by conditions on $\psi$ at the boundaries in the $x$–$y$ plane. Examples of the application of these general forms will be seen extensively in the following chapter where the dependent variable $\psi$ is identified as $E_z$ or $H_z$.

## 7.20 Product Solutions for the Helmholtz Equation in Cylindrical Coordinates

In cylindrical structures, such as coaxial lines or waveguides of circular cross section, the wave components are most conveniently expressed in terms of cylindrical coordinates. Assuming that the $z$ dependence is in the waveform $e^{-\gamma z}$, the scalar Helmholtz equation becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} = -k_c^2 \psi \tag{1}$$

where $k_c^2 = \gamma^2 + \omega^2 \mu \varepsilon$. For this partial differential equation, we shall again substitute an assumed product solution and separate variables to obtain two ordinary differential equations.

Assume

$$\psi = RF_\phi$$

where $R$ is a function of $r$ alone and $F_\phi$ is a function of $\phi$ alone:

$$R''F_\phi + \frac{R'F_\phi}{r} + \frac{F''_\phi R}{r^2} = -k_c^2 RF_\phi$$

Separating variables, we have

$$r^2 \frac{R''}{R} + \frac{rR'}{R} + k_c^2 r^2 = \frac{-F''_\phi}{F_\phi}$$

The left side of the equation is a function of $r$ alone; the right of $\phi$ alone. If both sides are to be equal for all values of $r$ and $\phi$, both sides must equal a constant. Let this constant be $\nu^2$. There are then the two ordinary differential equations:

$$\frac{-F''}{F} = \nu^2 \tag{2}$$

and

$$r^2 \frac{R''}{R} + \frac{rR'}{R} + k_c^2 r^2 = \nu^2$$

or

$$R'' + \frac{1}{r}R' + \left(k_c^2 - \frac{\nu^2}{r^2}\right)R = 0 \tag{3}$$

The solution to (2) is in sinusoids. By comparing with Eq. 7.14(3) we see that solutions to (3) may be written in terms of Bessel functions of order $\nu$:

$$\psi = RF_\phi \tag{4}$$

where

$$R = AJ_\nu(k_c r) + BN_\nu(k_c r) \tag{5}$$

$$F_\phi = C \cos \nu\phi + D \sin \nu\phi$$

Either or both of the Bessel functions may be replaced by Hankel functions [Eqs. 7.14(11) and (12)] when one desires to look at waves as though propagation were in the radial direction. Thus, for example,

$$\begin{aligned} R &= A_1 H_\nu^{(1)}(k_c r) + B_1 H_\nu^{(2)}(k_c r) \\ F_\phi &= C \cos \nu\phi + D \sin \nu\phi \end{aligned} \tag{6}$$

If $k_c$ is imaginary, the ordinary Bessel functions can be replaced by the modified Bessel functions, Eqs. 7.14(16) and (17). In the examples in the following chapter, the variable $\psi$ will be identified with $E_z$ or $H_z$.

## PROBLEMS

**7.2a** Find the form of differential equation satisfied by $E_r$ in cylindrical coordinates for a charge-free, homogeneous dielectric region. Repeat for $E_\phi$. Note that these are not Laplace equations.

**7.2b** Show that none of the spherical components of electric field satisfy Laplace's equation for quasistatic problems in which $\nabla^2\mathbf{E} = 0$.

**7.2c*** Show that the rectangular component $E_z$ of electrostatic field satisfies Laplace's equation expressed in spherical coordinates.

**7.2d** Derive Laplace's equation for **H**, **A**, and $\Phi_m$ in a current-free region with static fields and for **J** and **E** in a homogeneous conductor with dc currents.

**7.2e** Use superposition to find the potential on the axis of an infinite cylinder with a potential specified as $\Phi(\phi) = V_0 \sin \phi/2$, for $0 \le \phi \le 2\pi$ on the boundary.

**7.2f** A spherical surface is at zero potential except for a sector in the region $0 < \phi < \pi/3$, $0 < \theta < \pi/2$. Find the potential at the center of the sphere.

**7.3a** Calculate the capacitance of a parallel-plate capacitor with square plates having edge length $a$ and spacing $d = a/2$ situated in free space using the method of moments. If you do the calculations by hand, divide each plate into four equal squares. If a computer program is written, run it for several subdivisions of the plates and plot the effect on capacitance.

**7.3b** Find a better approximation to the capacitance of the structure in Ex. 7.3a by subdividing each of the squares shown in Fig. 7.3*d* into four equal parts and repeating the method of moments calculation.

**7.3c** In applying the method of moments calculation to two-dimensional problems, the $\ln r_0$ term in Eq. 1.8(7) is neglected. As an illustration of the validity of this procedure, find the potential of two parallel line charges located as follows: $+q_1$ at $\phi = 0$, $r = \delta$ and $-q_1$ at $\phi = 0$, $r = 2\delta$; take the zero potential point to be $r = R$ on the $\phi = 0$ axis. Apply Eq. 1.8(7) and show that the $\ln r_0$ terms cancel to arbitrary accuracy as $R \to \infty$. How does this explain that the $\ln r_0$ terms can be neglected in the two-electrode two-dimensional method of moment problems in which the line charges have a variety of values?

**7.3d*** Write a computer program to find the stripline capacitance as in Ex. 7.3b. Extend the range included on the larger electrodes by one unit of the division in Fig. 7.3e and evaluate the effect on capacitance. Then use a subdivision of the electrodes one-half as fine as in the example. Compare the results to evaluate the importance of the grid size.

**7.4a** Check by the Cauchy–Riemann equations the analyticity of the general power term $W = C_n Z^n$ and a series of such terms,

$$W = \sum_{n=1}^{\infty} C_n Z^n$$

**7.4b** Check the following functions by the Cauchy–Riemann equations to determine if they are analytic:

$$W = \sin Z$$
$$W = e^Z$$
$$W = Z^* = x - jy$$
$$W = ZZ^*$$

**7.4c** Check the analyticity of the following, noting isolated points where the derivatives may not remain finite:

$$W = \ln Z$$
$$W = \tan Z$$

**7.4d** Take the change $\Delta Z$ in any general direction $\Delta x + j\,\Delta y$. Show that, if the Cauchy–Riemann conditions are satisfied, Eq. 7.4(3) yields the same result for the derivative as when the change is in the $x$ direction or the $y$ direction alone.

**7.4e** If by following a path around some point in the $Z$ plane, the variable $W$ takes on different values when the same $Z$ is reached, the point around which the path is taken is called a *branch point*. Evaluate $W = Z^{1/2}$ and $W = Z^{4/3}$ along a path of constant radius around the origin to show that $Z = 0$ is a branch point for these functions. Discuss the analyticity of these functions at the branch point.

**7.5a** Plot the shape of the $u = \pm 0.5$ equipotentials for the $V = x^{4/3}$, $y = 0$ boundary condition used in Ex. 7.5.

**7.5b** A thin cylindrical shell of radius $a$ has a potential described by $\Phi(a, \theta) = V_0 \cos 2\theta$. Use a method similar to that in Ex. 7.5 to find $\Phi(r,\theta)$.

**7.5c** Show that if $u$ is the potential function, the field intensity $E_y$ is equal to the imaginary part of $dW/dZ$ and $E_x$ equals the negative of the real part.

**7.5d** Use the results of Prob. 7.5c to find an expression for the slope of equipotential lines in terms of $dW/dZ$. Show that all equipotential lines except $u = 0$ are normal to the beam edge in the electron flow in Fig. 7.5*b*. ($W = Z^{4/3}$ is not analytic at $Z = 0$, as was shown in Prob. 7.4e, and the $W = 0$ line at $y = 0$ is a special case.) *Hint:* Write an expression for $du$ in terms of partial derivatives and set $du = 0$ to get relations existing along an equipotential.

**7.6a** Plot a few equipotentials and flux lines in the vicinity of conducting corners of angles $\alpha = \pi/3$ and $3\pi/4$.

**7.6b** Evaluate the constant $C_1$ and $C_2$ in the logarithmic transformation so that $u$ represents the potential function in volts about a line charge of strength $q_l$ C/m. Let potential be zero at $r = a$.

**7.6c** Show that if $v$ is taken as the potential function in the logarithmic transformation, it is applicable to the region between two semi-infinite conducting planes intersecting at an angle $\alpha$, but separated by an infinitesimal gap at the origin so that the plane at $\theta = 0$ may be placed at potential zero and the plane at $\theta = \alpha$ at potential $V_0$. Evaluate the constants $C_1$ and $C_2$, taking the reference for zero flux at $r = a$. Write the flux function in coulombs per meter.

**7.6d** Find the form of the curves of constant $u$ and constant $v$ for the functions $\sin^{-1} Z$, $\cosh^{-1} Z$, and $\sinh^{-1} Z$. Do these permit one to solve problems in addition to those from the function $\cos^{-1} Z$?

**7.6e** Apply the results of the $\cos^{-1}$ transformation to item 4 in Ex. 7.6c. Take the right-hand semi-infinite plane extending from $x = a$ to $x = \infty$ at potential $V_0$. Take the left-hand semi-infinite plane extending from $x = -a$ to $x = -\infty$ at potential zero. Evaluate the scale factors and additive constant.

**7.6f*** Apply the results of the transformation to item 2 of Ex. 7.6c. Take the elliptic cylindrical conductor of semimajor axis $a$ and semiminor axis $b$ at potential $V_0$. The inner conductor is a strip conductor extending between the foci, $x = \pm c$, where

$$c = \sqrt{a^2 - b^2}$$

Evaluate all required scale factors and constants. Find the total charge per unit length induced upon the outer cylinder and the electrostatic capacitance of this two-conductor system.

**7.6g*** Modify the derivation in Ex. 7.6d to apply to the problem of parallel cylinders of unequal radius. Take the left-hand cylinder of radius $R_1$ with center at $x = -d_1$, the right-hand cylinder of radius $R_2$ with center at $x = d_2$, and a total difference of potential $V_0$ between cylinders. Find the electrostatic capacitance per unit length in terms of $R_1$, $R_2$, and $(d_1 + d_2)$.

**7.6h** The important bilinear transformation is of the form

$$Z = \frac{aZ' + b}{cZ' + d}$$

Take $a$, $b$, $c$, and $d$ as real constants, and show that any circle in the $Z'$ plane is transformed to a circle in the $Z$ plane by this transformation. (Straight lines are considered circles of infinite radius.)

**7.6i** Consider the special case of Prob. 7.6h with $a = R$, $b = -R$, $c = 1$, and $d = 1$. Show that the imaginary axis of the $Z'$ plane transforms to a circle of radius $R$, center at the origin, in the $Z$ plane. Show that a line charge at $x' = d$ and its image at $x' = -d$ in the $Z'$ plane transform to points in the $Z$ plane at radii $r_1$ and $r_2$ with $r_1 r_2 = R^2$. Compare with the result for imaging line charges in a cylinder (Sec. 1.18).

**7.7a** Explain why a factor in the Schwarz transformation may be left out when it corresponds to a point transformed to infinity in the $Z'$ plane.

**7.7b** In Eq. 7.7(2), separate $Z$ into real and imaginary parts. Show that the boundary condition for potential is satisfied along the two conductors. Obtain the asymptotic equations for large positive $u$ and for large negative $u$, and interpret the results in terms of the type of field approached in these limits.

**7.7c*** Work the example of Prob. 7.6e by the Schwarz technique and show that the same result is obtained. This is the problem of two coplanar semi-infinite plane conductors separated by a gap $2a$, with the left-hand conductor at potential zero and the right-hand conductor at potential $V_0$.

**7.7d*** For the first example of Table 7.7, find the electrostatic capacitance in excess of what would be obtained if a uniform field existed in both of the parallel-plane regions.

**7.7e** Plot the $V_0/2$ equipotential for Ex. 7.7.

**7.8** Suppose that the wave-guiding structure in Fig. 7.8*a* is bounded on the outside by a dielectric $\varepsilon_3(r)$ which has the value $\varepsilon_2$ at $R_0$ and then decreases to an appreciably lower value as $r$ is increased. As was seen in Sec. 6.12, waves incident on a plane boundary between two dielectrics from the higher $\varepsilon$ side can be totally reflected. Find the limiting rate of decrease of $\varepsilon_3$ at $R_0$ which can permit total reflection of rays approaching the boundary, by studying the variation of the equivalent dielectric constant in the $W$ plane.

**7.9a** The so-called circular harmonics are the product solutions to Laplace's equation in the two circular cylindrical coordinates $r$ and $\phi$. Apply the basic separation of variables technique to Laplace's equation in these coordinates to yield two ordinary differential equations. Show that the $r$ and $\phi$ equations are satisfied respectively by the functions $R$ and $F_\phi$ where

$$R = C_1 r^n + C_2 r^{-n}$$

$$F_\phi = C_3 \cos n\phi + C_4 \sin n\phi$$

**7.9b** An infinite rod of a magnetic material of relative permeability $\mu_r$ lies with its axis perpendicular to the direction of a uniform magnetic field in which it is immersed. Take the rod to be of circular cross section with radius $a$ and use the expressions in Prob. 7.9a to find the fields inside and outside the rod. Note the uniformity of the field inside.

**7.10a** Plot the form of equipotentials for $\Phi = V_0/4, V_0/2$, and $3V_0/4$ for Fig. 7.10*a*.

**7.10b** Describe the electrode structure for which the single rectangular harmonic $C_1 \cosh kx \sin ky$ is a solution for potential. Take electrodes at potential $V_0$ passing through $|x| = a$ when $y = a/2$.

**7.10c** Describe the electrode structure and exciting potentials for which the single circular harmonic (Prob. 7.9a) $Cr^2 \cos 2\phi$ is a solution.

**7.11a** Obtain Fourier series in sines and cosines for the following periodic functions:

(i) A triangular wave defined by $f(x) = V_0(1 - 2x/L)$ from 0 to $L/2$ and $f(x) = V_0[(2x/L) - 1]$ from $L/2$ to $L$
(ii) A sawtooth wave defined by $f(x) = V_0 x/L$ for $0 < x < L$
(iii) A sinusoidal pulse given by $f(x) = (V_m \cos kx - V_0)$ for $-\alpha < kx < \alpha$, $f(x) = 0$, for $-\pi < kx < -\alpha$ and also for $\alpha < kx < \pi$.

**7.11b** Suppose that a function is given over the interval 0 to $a$ as $f(x) = \sin \pi x/a$. What do the cosine and sine representations yield? Explain how this single sine term can be represented in terms of cosines.

**7.11c** Find sine and cosine representations for the function $e^{kx}$ defined over the interval $0 < x < a$.

**7.11d** Plot $f(x)$ given by Eq. 7.11(14) in the neighborhood of the discontinuities using (i) five sine terms and (ii) ten sine terms and discuss differences from the rectangular function being represented.

**7.11e** A complex form of the Fourier series for a function $f(x)$ defined over the interval $0 < x < a$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n x/a}$$

Show that if this is valid, $c_n$ must be given by

$$c_n = \frac{1}{a}\int_0^a f(x)e^{-j2\pi nx/a}\,dx$$

**7.11f** Find representations for the constant $C$ over the interval $0 < x < a$ in the complex form of Prob. 7.11e, and compare the result with Eq. 7.11(14).

**7.11g** Find the Fourier integral representation for a decaying exponential, $f(x) = 0$, for $x < 0$ and $f(x) = ce^{-\alpha x}$ for $x > 0$.

**7.12a** Obtain a series solution for the two-dimensional box problem in which sides at $y = 0$ and $y = b$ are at potential zero, and end planes at $x = a$ and $x = -a$ are at potential $V_0$.

**7.12b** Find the potential distribution for the box of Prob. 7.12a with the same boundary conditions except that the potential on the side at $y = 0$ should be $V_1$ and that at $y = b$ should be $-V_1$.

**7.12c** In a two-dimensional problem, parallel planes at $y = 0$ and $y = b$ extend from $x = 0$ to $x = \infty$ and are at zero potential. The one end plane at $x = 0$ is at potential $V_0$. Obtain a series solution.

**7.12d** The fringing that occurs at the open ends of a pair of parallel plates as seen in Fig. 1.9$a$ leads to a modification of the fields between the plates from the ideal uniform distribution. Consider $x = 0$ to be the ends of the plates, which are at $y = 0, b$. The analysis of Ex. 7.7 can show that the potential between the ends of the plates may be expressed approximately as $\Phi(0, y) = V_0[(y/b) + 0.06 \sin 2\pi y/b]$. Find the distance $x$ at which the potential distribution between the plates is linear to within 1%, using the analysis of Prob. 7.12c.

**7.12e** A two-dimensional conducting rectangular solid is bounded on three sides by perfect conductors: at $y = 0$, $\Phi = 0$; at $x = 0$, $\Phi = 0$; at $y = b$, $\Phi = V_0$. It is bounded at $x = a$ by a dielectric with zero conductivity. Find an expression for the potential distribution inside the conducting solid.

**7.12f** Two concentric cylinders are located at $r = a$ and $r = b$. The inner ($r = a$) cylinder is split along its length into two halves which are at different potentials. Potential is $-V_0$ for $-\pi < \phi < 0$ and $V_0$ for $0 < \phi < \pi$. The cylinder at $r = b$ is at zero potential. Find the potential between the two cylinders.

**7.12g** The potential along the plane boundary of a half-space is in strips of width $a$ and alternates between $-V_0$ and $V_0$. Take the boundary to be at $y = 0$ and the strips to be invariant in the $z$ direction. The origin of the $x$ coordinate lies in the gap between strips so that the potential is $-V_0$ for $-a < x < 0$ and $V_0$ for $0 < x < a$. Find the potential distribution for $y \geq 0$ and determine the surface charge density along the $y = 0$ plane. Put the result in closed form (see Collin, footnote 3 of Chap. 8, p. 813) and plot for $-a < x < a$.

**7.12h*** Infinite parallel conducting plates are located at $y = 0$ and $y = a$. A conducting strip at $x = 0$, $a/2 \leq y \leq a$, $-\infty < z < \infty$, is connected to the plate at $y = a$, thus

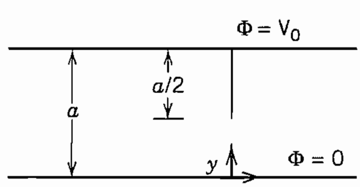


FIG. 7.12$h$

introducing additional capacitance between the plates. (See Fig. P7.12$h$.) Assume a linear potential variation for $0 \le y \le a/2$ at $x = 0$, and use superposition of boundary conditions to find an expression for the capacitance per meter in the $z$ direction added by the strip at $x = 0$.

**7.12i** Consider a rectangular prism of width $a$ in the $x$ direction and $b$ in the $y$ direction with all four sides at zero potential extending from $z = 0$ to $z = \infty$. At $z = 0$ the prism has a cap with the following potential distribution:

$$V(x, y, 0) = \begin{cases} 0 & \text{for } 0 < x < a/2, \text{ all } y \\ V_0 & \text{for } a/2 < x < a, \text{ all } y \end{cases}$$

Find the potentials within the prism.

**7.12j*** For a box as in Ex. 7.12c, find the potential distribution if the box is filled with a homogeneous, isotropic dielectric with permittivity $\varepsilon_1$ in the bottom half of the box $0 \le z \le c/2$ and free space in the remainder.

**7.13** Demonstrate that the series Eq. 7.13(10) does satisfy the differential equation 7.13(8).

**7.16a** Write a function $f(r)$ in terms of $n$th-order Bessel functions over the range 0 to $a$ and determine the coefficients.

**7.16b** Determine coefficients for a function $f(r)$ expressed over the range 0 to $a$ as a series of zero-order Bessel functions as follows:

$$f(r) = \sum_{m=1}^{\infty} c_m J_0\left(\frac{p'_m r}{a}\right)$$

where $p'_m$ denotes the $m$th root of $J'_0(v) = 0$ [i.e., $J_1(v) = 0$].

**7.17a** A cylinder divided into a set of rings with appropriately applied voltages may be used to set up a nearly uniform electric field along the axis with advantageous focusing properties for electron beams. Suppose the field at the radius $a$ of the cylinder is given approximately by $E_z(a, z) = E_0(1 + \cos \alpha z)$, where $\alpha = 2\pi/p$ and $p$ is the period of the rings. Find the potential variation along the rings ($r = a$) and for $r < a$. Determine the field on the axis and the period required to have the periodic part of the field 1% of $E_0$.

**7.17b** Show that the function

$$\Phi(r, z) = A I_0(\tau r) \cos \tau z$$

satisfies the requirement of solutions of Laplace's equation that there should be no relative maxima or minima.

**7.17c** Find the series for potential inside the cylindrical region with end plates $z = 0$ and $z = l$ at potential zero and the cylinder of radius $a$ in two parts. From $z = 0$ to $z = l/2$, it is at potential $V_0$; from $z = l/2$ to $z = l$, it is at potential $-V_0$.

**7.17d** The problem is as in Prob. 7.17c except that the cylinder is divided in three parts with potential zero from $z = 0$ to $z = b$ and also from $z = l - b$ to $z = l$. Potential is $V_0$ from $z = b$ to $z = l - b$.

**7.17e** Write the general formula for obtaining potential inside a cylindrical region of radius $a$, with zero-potential end plates at $z = 0$ and $z = l$, provided potential is given as $\Phi = f(z)$ at $r = a$.

**7.17f** Write the general formula for obtaining potential inside a cylinder of radius $a$ which, with its plane base at $z = 0$, is at potential zero, provided that the potential is given across the plane surface at $z = l$, as

$$\Phi(r, l) = f(r)$$

**7.17g** Find the potential distribution inside a cylinder with zero potential on the cylindrical surface at $r = a$, on the end plate at $z = 0$ and where $a/2 < r < a$ on the end plate at $z = l$. It also has $\Phi(r, l) = V_0$ for $0 \le r < a/2$.

**7.18a** Apply the separation of variables technique to Laplace's equation in the three spherical coordinates, $r$, $\theta$, and $\phi$, obtaining the three resulting ordinary differential equations. Write solutions to the $r$ equation and the $\phi$ equation.

**7.18b** Assume a spherical surface split into two thin hemispherical shells with a small gap between them. Assume a potential $V_0$ on one hemisphere and zero on the other and find the potential distribution in the surrounding space.

**7.18c** Write the general formulas for obtaining potential for $r < a$ and for $r > a$, when potential is given as a general function $f(\theta)$ over a thin spherical shell at $r = a$.

**7.18d** For Ex. 7.18b, write the series for $H_z$ at any point $r$, $\theta$ with $r > a$.

**7.18e** A Helmholtz coil is used to obtain very nearly uniform magnetic field over a region through the use of coils of large radius compared with coil cross sections. Consider two such coaxial coils, each of radius $a$, one lying in the plane $z = d$ and the other in the plane $z = -d$. Take the current for each coil (considered as a single turn) as $I$. Obtain the series for $H_z$ applicable to a region containing the origin, writing specific forms for the first three coefficients. Show that if $a = 2d$, the first nonzero coefficient (other than the constant term) is the coefficient of $r^4$.

**7.19** In Eqs. 7.19(3) and (4), let $\psi$ be the axial electric field component $E_z$, and simplify by taking $A$ and $C$ zero in (4). Discuss the forms of solutions and the question of finding physical boundary conditions for (i) both $k_x$ and $k_y$ real, (ii) $k_x$ real but $k_y$ imaginary, and (iii) both $k_x$ and $k_y$ imaginary. For (ii) and (iii) would physical applicability of solutions be changed if either or both of $A$ and $C$ were nonzero?