4 The Electromagnetics of Circuits

4.1 INTRODUCTION

Much of the engineering design and analysis of electromagnetic interactions are done through the mechanism of lumped-element circuits. In these, the energy-storage elements (inductors and capacitors) and the dissipative elements (resistors) are connected to each other and to sources or active elements within the circuit by conducting paths of negligible impedance. There may be mutual couplings, either electrical or magnetic, but in the ideal circuit these couplings are planned and optimized. The advantage of this approach is that functions are well separated and cause-and-effect relationships readily understandable. Powerful methods of synthesis, analysis, and computer optimization of such circuits have consequently been developed.

Most of the individual elements in an electrical circuit are small compared with wavelength so that fields of the elements are *quasistatic*; that is, although varying with time, the electric or magnetic fields have the spatial forms of static field distributions. There are important distributed effects in many real circuits, but often they can be represented by a few properly chosen lumped coupling elements. But in some circuits, of which the transmission lines are primary examples, the distributed effects are the major ones and must be considered from the beginning. In some cases in which the lumped idealizations described above do not strictly apply, lumped-element *models* can nevertheless be deduced and are useful for analysis because of the powerful circuit methods that have been developed.

We have introduced the lumped-circuit concepts, inductance and capacitance, in our studies of static fields. We have also seen how the skin effect phenomenon in conductors changes both resistance and inductance at high frequencies. We now wish to examine circuits and circuit elements more carefully from the point of view of electromagnetics. It is easy to see the idealizations required to derive Kirchhoff's laws from Maxwell's equations. It is also possible to make certain extensions of the concepts when the simplest idealizations do not apply. In particular, introduction of the retardation concepts shows that circuits may radiate energy when comparable in size with wavelength. The amount of radiated power may be estimated from these extended circuit ideas for some configurations. But for certain classes of circuits it becomes impossible to make the extensions without a true field analysis. We shall look at both types of circuits in this chapter.

The Idealizations in Classical Circuit Theory

4.2 KIRCHHOFF'S VOLTAGE LAW

Kirchhoff's two laws provide the basis for classical circuit theory. We begin with the voltage law as a way of reviewing the basic element values of lumped-circuit theory. The law states that for any closed loop of a circuit, the algebraic sum of the voltages for the individual branches of the loop is zero:

$$\sum_i V_i = 0 \tag{1}$$

The basis for this law is Faraday's law for a closed path, written as

$$-\oint \mathbf{E} \cdot \mathbf{dl} = \frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{dS} \tag{2}$$

and the definition of voltages between two reference points of the loop,

$$V_{ba} = -\int_a^b \mathbf{E} \cdot \mathbf{dl} \tag{3}$$

To illustrate the relation between the circuit expression (1) and the field expressions (2) and (3), consider first a single loop with applied voltage $V_0(t)$ and passive resistance, inductance, and capacitance elements in series (Fig 4.2*a*). A convention for positive voltage at the source is selected as shown by the plus and minus signs on the voltage generator, which means by (3) that field of the source is directed from b to a when V_0 is positive. A convention for positive current is also chosen, as shown by the arrow on $I(t)$. The interpretation of (1) by circuit theory for this basic circuit is then known to be

$$V_0(t) - RI(t) - L\frac{dI(t)}{dt} - \frac{1}{C}\int I(t)\, dt = 0 \tag{4}$$

To compare, we break the closed line integral of (2) into its contributions over the several elements:

$$-\int_a^b \mathbf{E} \cdot \mathbf{dl} - \int_b^c \mathbf{E} \cdot \mathbf{dl} - \int_c^d \mathbf{E} \cdot \mathbf{dl} - \int_d^a \mathbf{E} \cdot \mathbf{dl} = \frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{dS} \tag{5}$$

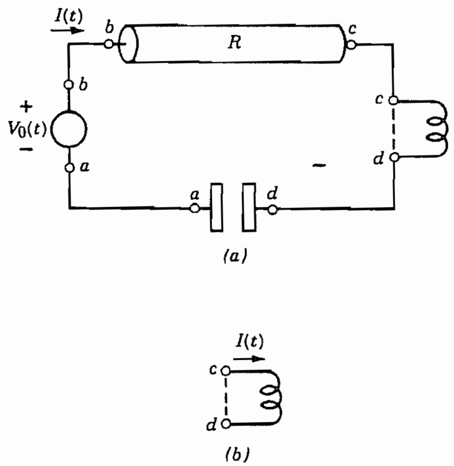

FIG. 4.2 (*a*) Series circuit with resistor, inductor, and capacitor. (*b*) Detail of inductor.

or

$$V_0(t) + V_{cb} + V_{dc} + V_{ad} = \frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{6}$$

The right side of (6) is not zero as is the right side of (4), but we recognize it as the contribution to emf generated by any rate of change of magnetic flux within the path defined as the circuit. If not entirely negligible, it can be considered as arising from an inductance of the loop which can be added to the lumped element L, or a mutually induced coupling if the flux is from an external source. Thus we will from here on consider it as negligible or included in L so that the right side of (6) is zero. (Mutual effects are added later.) We now examine separately the three voltage terms related to the passive components R, L, and C.

Resistance Element The field expression to be applied to the resistive material is the differential form of Ohm's law,

$$\mathbf{J} = \sigma\mathbf{E} \tag{7}$$

so that the voltage V_{cb} is

$$V_{cb} = -\int_b^c \mathbf{E}\cdot d\mathbf{l} = -\int_b^c \frac{\mathbf{J}}{\sigma}\cdot d\mathbf{l} \tag{8}$$

where the path is taken along some current flow path of the conductor. Conductivity σ may vary along this path. At dc or low frequencies, current I is uniformly distributed over the cross section A of the conductor, which can also vary with position. Thus

$$V_{cb} = -\int_b^c \frac{I\,dl}{\sigma A} = -IR \tag{9}$$

where

$$R = \int_b^c \frac{dl}{\sigma A} \tag{10}$$

This last is the usual dc or low-frequency resistance. The situation is more complicated at higher frequencies because of the effect of the changing magnetic fields on currents within the conductor. Current distribution over the cross section is then nonuniform, and the particular path along the conductor must be specified. In the plane skin effect analysis of Chapter 3, current was related to electric field at the surface to define a surface impedance. We shall return to this concept later in the chapter for conductors of circular cross section.

Inductance Element The voltage across the terminals of the inductive element comes from the time rate of change of magnetic flux within the inductor, shown in the figure as a coil. Assuming first that resistance of the conductor of the coil is negligible, let us take a closed line integral of electric field along the conductor of the coil, returning by the path across the terminals (Fig. 4.2*b*). Since the contribution along the part of the path which follows the conductor is zero, all the voltage appears across the terminals:

$$-\oint \mathbf{E} \cdot \mathbf{dl} = -\int_{c(\text{cond.})}^{d} \mathbf{E} \cdot \mathbf{dl} - \int_{d(\text{term.})}^{c} \mathbf{E} \cdot \mathbf{dl} = -\int_{d(\text{term.})}^{c} \mathbf{E} \cdot \mathbf{dl} \tag{11}$$

By Faraday's law, this is the time rate of change of magnetic flux enclosed:

$$-\int_{d(\text{term.})}^{c} \mathbf{E} \cdot \mathbf{dl} = -V_{dc} = \frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathbf{dS} \tag{12}$$

Inductance L is defined as the magnetic flux linkage per unit of current (Sec. 2.5)

$$L = \left[\int \mathbf{B} \cdot \mathbf{dS}\right] \Big/ I \tag{13}$$

so the voltage contributed by this term, assuming L independent of time, is

$$V_{cd} = \frac{\partial}{\partial t}(LI) = L\frac{dI}{dt} \tag{14}$$

Note that in computing flux enclosed by the path, we add a contribution each time we follow another turn around the flux. Thus for N turns, the contribution to induced voltage is just N times that of one turn, provided the same flux links each turn. This enters into the calculation of L and will be seen specifically when we find inductance of a coil.

If there is finite resistance in the turns of the coil, the second term of (11) is not zero but is the resistance of the coil, R_L, multiplied by current; therefore (11) becomes

$$-\oint \mathbf{E} \cdot \mathbf{dl} = -R_L I - V_{dc} = \frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathbf{dS}$$

or

$$V_{cd} = R_L I + L \frac{dI}{dt} \tag{15}$$

Thus, as expected, we simply add another series resistance to take care of finite conductivity in the conductors of the coil.

Capacitive Element The ideal capacitor is one in which we store only electric energy; magnetic fields are negligible so there is no contribution to voltage from changing magnetic fields but only from the charges on plates of the capacitor. The problem is then quasistatic and voltage is synonymous with potential difference between capacitor plates. So, in contrast to the inductor, we can take any path between the terminals of the capacitor for evaluation of voltage V_{da}, provided it does not stray into regions influenced by magnetic fields from other elements. We also take the definition of capacitance from electrostatics (Sec. 1.9) as the charge on one plate divided by the potential difference:

$$C = \frac{Q}{V} \tag{16}$$

Thus, from continuity,

$$I = \frac{dQ}{dt} = \frac{d}{dt}(CV_{da}) = C\frac{dV_{da}}{dt} \tag{17}$$

The last term in (17) implies a capacitance which is not changing with time. Integration of (17) with time leads to

$$V_{da} = \frac{1}{C}\int I\,dt \tag{18}$$

If the dielectric of the capacitor is lossy, there are conduction currents to add to (17), which are represented in the circuit as a conductance $G_C = 1/R_C$ in parallel with C; the value of R_C may be calculated from (10) by using conductivity of the dielectric and area of the capacitor plates.

Induced Voltages from Other Parts of the Circuit In addition to voltages induced by charges and currents of the circuit path being considered, there may be induced voltages from other portions of the circuit. In particular, if the magnetic field from one part of the circuit links another part, an induced voltage is produced through Faraday's law when this magnetic field changes with time. This coupling is represented in the circuit by means of a mutual inductor M, as shown in Fig. 4.2*c*. The value of M is defined as the magnetic flux ψ_{12} linking path 1, divided by the current I_2:

$$M = M_{12} = \frac{\psi_{12}}{I_2} \tag{19}$$

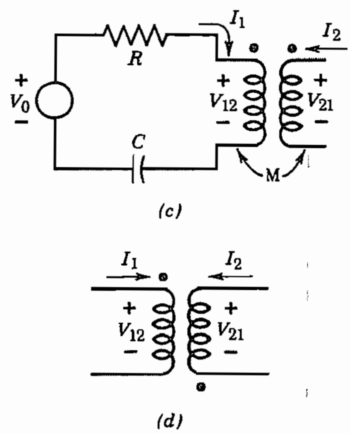

FIG. 4.2 (*c*) Circuit with a mutual inductor. (*d*) Designation of mutual coupling with negative M.

The voltage induced in the first path is then

$$V_{12} = \frac{d\psi_{12}}{dt} = M\frac{dI_2}{dt} \tag{20}$$

and the circuit equation (4) is modified to be

$$V_0 - RI_1 - L\frac{dI_1}{dt} - M\frac{dI_2}{dt} - \frac{1}{C}\int I_1\,dt = 0 \tag{21}$$

The mutual inductance M may be either positive or negative depending upon the sense of flux with respect to the defined positive reference for I_2. The sign of M is designated on a circuit diagram by the placing of dots and with sign conventions for currents and voltages as shown; those on Fig. 4.2*c* denote positive M; negative M would be designated as in Fig. 4.2*d*.

Except for certain materials (to be considered in Chapter 13) there is a reciprocal relation showing that the same M gives the voltage induced in circuit 2 by time-varying current in circuit 1:

$$V_{21} = \frac{d\psi_{21}}{dt} = M\frac{dI_1}{dt} \tag{22}$$

All mutual effects to be considered in this chapter have this reciprocal relationship.

In summary, we find that if losses in inductor and capacitor are ignored, the field approach, with understandable approximations, leads to the definitions for the three induced voltage terms for the passive elements used in the circuit approach, Eq. (4). Moreover, the definitions (10), (13), and (16) are the usual quasistatic definitions for these elements. If losses are present, a series resistance is added to L and a shunt conductance to C, again as is commonly done in the circuit approach. Coupling between circuit paths by magnetic flux adds mutual inductance elements. We next examine the Kirchhoff current law and the extension through this to multimesh circuits.

4.3 KIRCHHOFF'S CURRENT LAW AND MULTIMESH CIRCUITS

The current law of Kirchhoff states that the algebraic sum of currents flowing out of a junction is zero. Thus, referring to Fig. 4.3*a*,

$$\sum_{n=1}^{N} I_n(t) = 0 \tag{1}$$

It is evident that the idea behind this law is that of continuity of current, so we refer to the continuity equation implicit in Maxwell's equations, Eq. 3.4(5), or its large-scale equivalent:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_V \rho\, dV \tag{2}$$

If we apply this to a surface S surrounding the junction, the only conduction current flowing out of the surface is that in the wires, so the left side of (2) becomes just the algebraic sum of the currents flowing out of the wires, as in (1). The right side is the negative time rate of change of charge Q, if any, accumulating at the junction. So (2) may be written

$$\sum_{n=1}^{N} I_n(t) = -\frac{dQ(t)}{dt} \tag{3}$$

A comparison of (1) and (3) shows an apparent difference, but it is only one of interpretation. If Q is nonzero, we know that we take care of this in a circuit problem by adding one or more capacitive branches to yield the capacitive current dQ/dt at the junction. That is, in interpreting (3), the current terms on the left are taken only as convection or conduction currents, whereas in (1) displacement or capacitance currents are included. With this understanding, (1) and (3) are equivalent.

With the two laws, the circuit analysis illustrated in the preceding section can be extended to circuits with several meshes. As a simple example, consider the low-pass filter of Fig. 4.3*b* or 4.3*c*. Although currents and voltages are taken as time-varying,

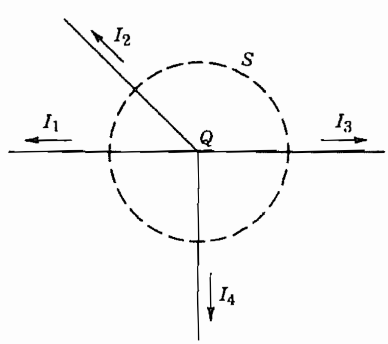

FIG. 4.3a Current flow from a junction.

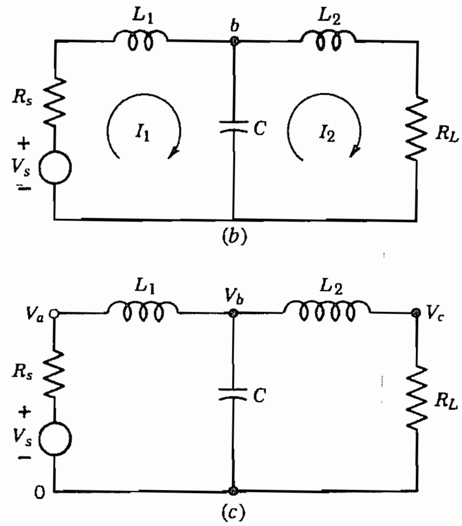

FIG. 4.3 Low-pass filter: (*b*) loop current analysis; (*c*) node voltage analysis.

we drop the functional notation for simplicity. Figure 4.3*b* illustrates the standard method utilizing mesh currents I_1 and I_2. Note that the net current through C is $(I_1 - I_2)$, which automatically satisfies the current law at node b. The voltage law is then written about each loop as follows:

$$V_s - R_s I_1 - L_1 \frac{dI_1}{dt} - \frac{1}{C}\int (I_1 - I_2)\, dt = 0 \tag{4}$$

$$-\frac{1}{C}\int (I_2 - I_1)\, dt - L_2 \frac{dI_2}{dt} - R_L I_2 = 0 \tag{5}$$

The two equations are then solved by appropriate means to give I_1 and I_2 for a given V_s.

A second standard method of circuit analysis uses node voltages V_a, V_b, and V_c as shown in Fig. 4.3*c*. These are defined with respect to some reference, here taken as the lower terminal of the voltage generator, denoted 0. Then Kirchhoff's voltage law is automatically satisfied, for if we add voltages around the first loop we have

$$V_s + (V_a - V_s) + (V_b - V_a) + (0 - V_b) \equiv 0 \tag{6}$$

Kirchhoff's current law is then applied at each of the three nodes as follows:

$$\text{Node } a\text{:} \quad \frac{V_a - V_s}{R_s} + \frac{1}{L_1}\int (V_a - V_b)\, dt = 0 \tag{7}$$

$$\text{Node } b\text{:} \quad \frac{1}{L_1}\int (V_b - V_a)\, dt + \frac{1}{L_2}\int (V_b - V_c)\, dt + C\frac{dV_b}{dt} = 0 \tag{8}$$

$$\text{Node } c\text{:} \quad \frac{1}{L_2} \int (V_c - V_b)\, dt + \frac{V_c}{R_L} = 0 \tag{9}$$

Solution of these by appropriate means yields the three node voltages in terms of the given voltage V_s. Note that no equation for the reference node need be written as it is contained in the above.

In the above we seem to be treating voltage as a potential difference when we take voltage of a node with respect to the chosen reference, but note that this is only after the circuit is defined and we are only breaking up $\int \mathbf{E} \cdot \mathbf{dl}$ into its contributions over the various branches. As illustrated in the preceding section, we do have to define the path carefully whenever there are inductances or other elements with contributions to voltage from Faraday's law.

Finally, a word about sources. The voltage generator most often met in lumped-element circuit theory is a highly localized one. For example, the electrons and holes of a semiconductor diode or transistor may induce electric fields between the conducting electrodes fabricated on the device. The entire device is typically small compared with wavelength so that the electric field, although time-varying, may be written as the gradient of a time-varying scalar potential. The integral of electric field at any instant thus yields an instantaneous potential difference V_s between the electrodes, which is the source voltage (or $V_s - IZ_s$ if current flows). The induced effects from a modulated electron stream passing across a klystron gap are similar, as are those from many other practical devices. There are interesting field problems in the analysis of induced effects from such devices, but from the point of view of the circuit designer, they are simply point sources representable by the V_s used in the circuits.

A quite different limiting case is that in which the fields driving the circuit are not localized but are distributed. An important example is that of a receiving antenna with the fields set down by a distant transmitting antenna. If voltage is taken as the line integral of electric field along the antenna, applied voltage clearly depends upon the circuit configuration and orientation with respect to the applied field. Although quite different from the case with a localized source, it is found that circuit theory is useful here also. A formulation in terms of the retarded potentials will be applied to this case in Sec. 4.11.

Current generators are natural to use as sources in place of voltage generators if emphasis is on the current induced between electrodes of the point source or small-gap device. Similarly for the distributed source, if applied magnetic field at the circuit conductor is given, induced current can be calculated and a current representation is natural. One, however, has a choice in any case since the Thévenin and Norton theorems[1] show that the two representations of Figs. 4.3*d* and 4.3*e* are equivalent with the relations

$$Y_s = Z_s^{-1}, \qquad I_s = V_s Y_s \tag{10}$$

Thus an equivalent to Fig. 4.3*c* is that of Fig. 4.3*f*, utilizing a current generator.

[1] *S. E. Schwarz and W. G. Oldham,* Electrical Engineering: An Introduction, *2nd ed., Saunders, Fort Worth, TX, 1993.*

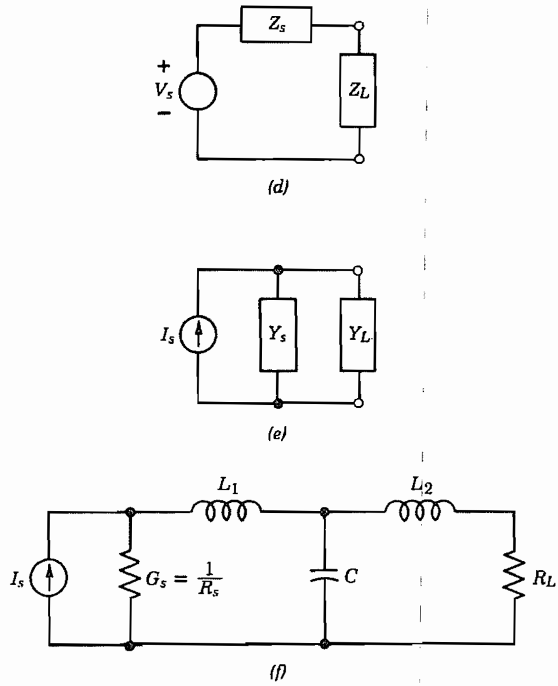

FIG. 4.3 (*d*) Thévenin circuit configuration. (*e*) Norton circuit form. (*f*) Equivalent of circuit in (*c*) using Norton source.

Skin Effect in Practical Conductors

4.4 Distribution of Time-Varying Currents in Conductors of Circular Cross Section

To study the resistive term at frequencies high enough so that current distribution is not uniform, we need to first find the current distribution. This was done in Sec. 3.16 for plane conductors. We now wish to do this for the useful case of round conductors. Recall that a good conductor is defined as one for which displacement current is negligible in comparison with conduction current so that

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma \mathbf{E} \tag{1}$$

Faraday's law equation is (in phasor form)

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{2}$$

From these two we derived the differential equation for current density, Eq. 3.16(7):

$$\nabla^2 \mathbf{J} = j\omega\mu\sigma\mathbf{J} \tag{3}$$

We now take current in the z direction and no variations with z or angle ϕ. Equation (3), expressed in circular cylindrical coordinates (inside front cover), is then

$$\frac{d^2 J_z}{dr^2} + \frac{1}{r}\frac{dJ_z}{dr} + T^2 J_z = 0 \tag{4}$$

where

$$T^2 = -j\omega\mu\sigma$$

or

$$T = j^{-1/2}\sqrt{\omega\mu\sigma} = j^{-1/2}\frac{\sqrt{2}}{\delta} \tag{5}$$

where δ is the useful parameter called "depth of penetration" or "skin depth." The differential equation (4) is a Bessel equation. Equations of this type will be studied in detail in Chapter 7, but for the present we write the two independent solutions as

$$J_z = AJ_0(Tr) + BH_0^{(1)}(Tr) \tag{6}$$

For a solid wire, $r = 0$ is included in the solution, and then it is necessary that $B = 0$ since a study of $H_0^{(1)}(Tr)$ shows that this is infinite at $r = 0$. Therefore,

$$J_z = AJ_0(Tr) \tag{7}$$

The arbitrary constant A may be evaluated in terms of current density at the surface, which is σE_0, with E_0 the surface electric field.

$$J_z = \sigma E_0 \quad \text{at} \quad r_0$$

Then (7) becomes

$$J_z = \frac{\sigma E_0}{J_0(Tr_0)} J_0(Tr) \tag{8}$$

A study of the series definitions of the Bessel functions with complex argument shows that J_0 is complex. It is convenient to break the complex Bessel function into real and imaginary parts, using the definitions

$$\text{Ber}(v) \triangleq \text{real part of } J_0(j^{-1/2}v)$$

$$\text{Bei}(v) \triangleq \text{imaginary part of } J_0(j^{-1/2}v)$$

That is,

$$J_0(j^{-1/2}v) \equiv \text{Ber}(v) + j\,\text{Bei}(v) \tag{9}$$

Ber(v) and Bei(v) are tabulated in many references.[2] Using these definitions and (5), (8) may be written

$$J_z = \sigma E_0 \frac{\text{Ber}(\sqrt{2}r/\delta) + j\,\text{Bei}(\sqrt{2}r/\delta)}{\text{Ber}(\sqrt{2}r_0/\delta) + j\,\text{Bei}(\sqrt{2}r_0/\delta)} \tag{10}$$

In Fig. 4.4*a* the magnitude of the ratio of current density to that at the outside of the wire is plotted as a function of the ratio of radius to outer radius of wire, for different values of the parameter (r_0/δ). Also, for purposes of the physical picture, these are interpreted in terms of current distribution for a 1-mm-diameter copper wire at different frequencies by the figures in parentheses.

As an example of the applicability of the plane analysis for curved conductors at high frequencies where δ is small compared with radii, we can take the present case of the round wire. If we are to neglect the curvature and apply the plane analysis, the coordinate x, distance below the surface, is $(r_0 - r)$ for a round wire. Then Eq. 3.16(16) gives

$$\left|\frac{J_z}{\sigma E_0}\right| \approx e^{-(r_0 - r)/\delta} \tag{11}$$

In Fig. 4.4*b* are plotted curves of $|J_z/\sigma E_0|$ by using this formula, and comparisons are made with curves obtained from the exact formula (10). This is done for two cases, $r_0/\delta = 2.39$ and $r_0/\delta = 7.55$. In the latter, the approximate distribution agrees well with the exact; in the former it does not. Thus, if ratio of wire radius to δ is large, it seems that there should be little error in analyzing the wire from the results developed for plane solids. This point will be pursued in impedance calculations to follow.

4.5 Impedance of Round Wires

The internal impedance (resistance and contribution to reactance from magnetic flux inside the wire) of the round wire is found from total current in the wire and the electric intensity at the surface, according to the ideas of Sec. 4.2. Total current may be obtained from an integration of current density, as for the plane conductor in Sec. 3.17; however, it may also be found from the magnetic field at the surface, since the line integral of magnetic field around the outside of the wire must be equal to the total current in the wire:

$$\oint \mathbf{H} \cdot \mathbf{dl} = I$$

[2] H. B. Dwight, *Tables of Integrals*, 3rd ed., MacMillan, New York, 1961. N. W. McLachlan, *Bessel Functions for Engineers*, 2nd ed., Oxford University Press (Clarendon), New York, 1955. M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, Schaum's Outline Series, McGraw-Hill, New York, 1968.

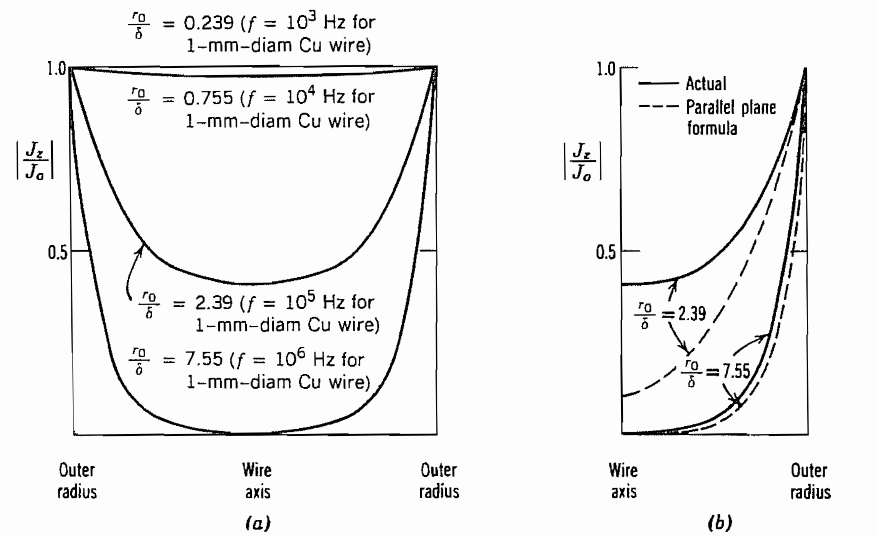

FIG. 4.4 (*a*) Current distribution in cylindrical wire for several frequencies. (*b*) Actual and approximate (parallel-plane formula) distribution in cylindrical wire. $J_0 = \sigma E_0$.

or

$$2\pi r_0 H_\phi|_{r=r_0} = I \tag{1}$$

Magnetic field is obtained from the electric field by Maxwell's equations:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{2}$$

For the round wire with no variations in z or ϕ, the fields E_z and H_ϕ alone are present, and only r derivatives remain, so (2) is simply

$$H_\phi = \frac{1}{j\omega\mu}\frac{dE_z}{dr} \tag{3}$$

An expression for current density has already been obtained in Eq. 4.4(8). Electric field is related to this through the conductivity σ:

$$E_z = \frac{J_z}{\sigma} = E_0 \frac{J_0(Tr)}{J_0(Tr_0)} \tag{4}$$

By substituting in (3) and recalling that $T^2 = -j\omega\mu\sigma$,

$$H_\phi = \frac{E_0 T}{j\omega\mu}\frac{J'_0(Tr)}{J_0(Tr_0)} = -\frac{\sigma E_0}{T}\frac{J'_0(Tr)}{J_0(Tr_0)}$$

where $J'_0(Tr)$ denotes $[d/d(Tr)]J_0(Tr)$. From (1),

$$I = -\frac{2\pi r_0 \sigma E_0}{T}\frac{J'_0(Tr_0)}{J_0(Tr_0)} \tag{5}$$

The internal impedance per unit length is defined as $Z_i \triangleq E_z(r_0)/I$. Then

$$Z_i = -\frac{TJ_0(Tr_0)}{2\pi r_0 \sigma J'_0(Tr_0)} \tag{6}$$

Note the similarity to internal impedance per square in Eq. 3.17(3).

Low-Frequency Expressions For low frequencies, Tr_0 is small and series expansions of the Bessel functions show that (6) may be expanded as

$$Z_i \approx \frac{1}{\pi r_0^2 \sigma}\left[1 + \frac{1}{48}\left(\frac{r_0}{\delta}\right)^2\right] + j\frac{\omega\mu}{8\pi} \tag{7}$$

The real or resistive part is

$$R_{\text{lf}} \approx \frac{1}{\pi r_0^2 \sigma}\left[1 + \frac{1}{48}\left(\frac{r_0}{\delta}\right)^2\right] \tag{8}$$

The first term of this expression is the dc resistance, and the second is a correction useful for r_0/δ as large as unity, that is, for radius equal to skin depth δ. The imaginary term of (7) corresponds to a low-frequency internal inductance:

$$(L_i)_{\text{lf}} \approx \frac{\mu}{8\pi} \quad \text{H/m} \tag{9}$$

The low-frequency internal inductance is the same as that found by energy methods in Sec. 2.17.

High-Frequency Expressions For high frequencies, the complex argument Tr_0 is large. It may be shown that $J_0(Tr_0)/J'_0(Tr_0)$ approaches $-j$ and the high-frequency approximation to (6) is

$$(Z_i)_{\text{hf}} = \frac{j(j)^{-1/2}}{\sqrt{2}\pi r_0 \sigma\delta} = \frac{(1+j)R_s}{2\pi r_0} \quad \Omega/\text{m} \tag{10}$$

or

$$(R)_{\text{hf}} = (\omega L_i)_{\text{hf}} = \frac{R_s}{2\pi r_0} \quad \Omega/\text{m} \tag{11}$$

So resistance and internal reactance are equal at high frequencies, and both are equal to the values for a plane solid of width $2\pi r_0$ just as assumed on physical grounds in Sec. 3.17 where $R_s = (\sigma\delta)^{-1}$.

Expression for Arbitrary Frequency To interpret (6) for arbitrary frequencies, it is useful to break into real and imaginary parts using the Ber and Bei functions, defined in Eq. 4.4(9), and their derivatives. That is,

$$\text{Ber}\, v + j\,\text{Bei}\, v = J_0(j^{-1/2}v)$$

Also let

$$\mathrm{Ber}'\, v + j\,\mathrm{Bei}'\, v = \frac{d}{dv}(\mathrm{Ber}\, v + j\,\mathrm{Bei}\, v)$$
$$= j^{-1/2} J'_0(j^{-1/2} v)$$

Then (6) may be written

$$Z_i = R + j\omega L_i = \frac{jR_s}{\sqrt{2}\pi r_0}\left[\frac{\mathrm{Ber}\, q + j\,\mathrm{Bei}\, q}{\mathrm{Ber}'\, q + j\,\mathrm{Bei}'\, q}\right]$$

where

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}} \qquad q = \frac{\sqrt{2}r_0}{\delta}$$

or

$$R = \frac{R_s}{\sqrt{2}\pi r_0}\left[\frac{\mathrm{Ber}\, q\,\mathrm{Bei}'\, q - \mathrm{Bei}\, q\,\mathrm{Ber}'\, q}{(\mathrm{Ber}'\, q)^2 + (\mathrm{Bei}'\, q)^2}\right] \quad \Omega/\mathrm{m}$$
$$\omega L_i = \frac{R_s}{\sqrt{2}\pi r_0}\left[\frac{\mathrm{Ber}\, q\,\mathrm{Ber}'\, q + \mathrm{Bei}\, q\,\mathrm{Bei}'\, q}{(\mathrm{Ber}'\, q)^2 + (\mathrm{Bei}'\, q)^2}\right] \quad \Omega/\mathrm{m} \tag{12}$$

These are the expressions for resistance and internal reactance of a round wire at any frequency in terms of the parameter q, which is $\sqrt{2}$ times the ratio of wire radius to depth of penetration. Curves giving the ratios of these quantities to the dc and to the high-frequency values as functions of r_0/δ are plotted in Figs. 4.5*a* and 4.5*b*. A careful study of these will reveal the ranges of r_0/δ over which it is permissible to use the approximate formulas for resistance and reactance.

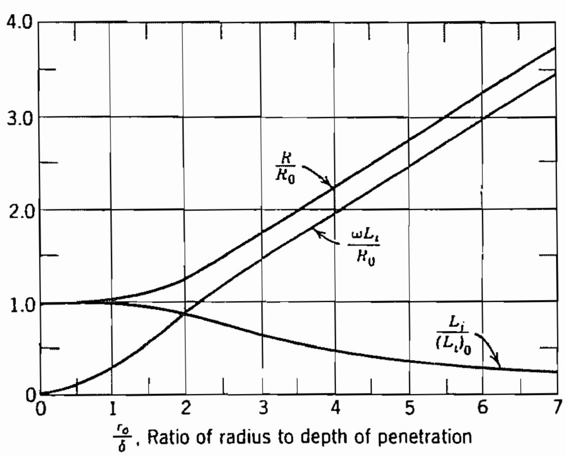

FIG. 4.5*a* Solid-wire skin effect quantities compared with dc values.

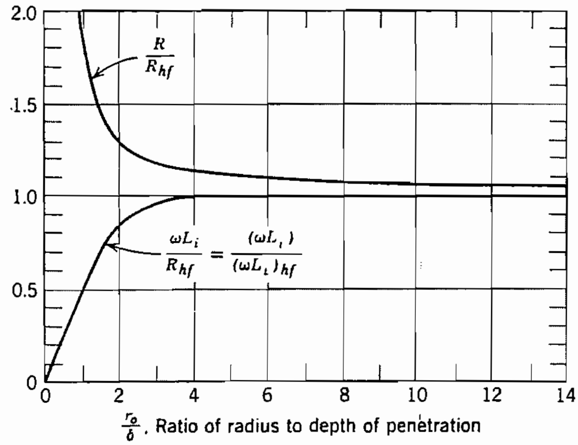

FIG. 4.5*b* Solid-wire skin effect quantities compared with values from high frequency formulas.

Calculation of Circuit Elements

4.6 Self-Inductance Calculations

Self-inductance, as defined in Chapter 2, was related to field concepts in the first part of this chapter. We have shown examples of inductance calculations for simple configurations by the method of flux linkages (Sec. 2.5) and from an energy point of view (Sec. 2.17). We now give additional examples of each method.

Example 4.6a
External Inductance of Parallel-Wire Transmission Line (Approximate)

Figure 4.6 shows two parallel conductors of radius R with their axes separated by distance $2d$. Current I flows in the z direction in the right-hand conductor and returns in the other. Magnetic field at any point (x, y) is the superposition of that from the two conductors. If conductors are far enough apart, the current distribution in either conductor is not much affected by the presence of the other, so that magnetic field from each conductor may be taken as circumferential about its axis and equal to the current divided by 2π times radius from the axis. For the $y = 0$ plane passing through the axes

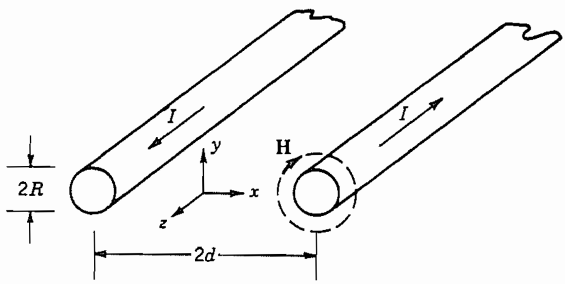

FIG. 4.6 Parallel-wire transmission line.

of the two wires, the contribution from both wires is vertical so that field, to the approximation described above, is

$$H_y(x, 0) \approx \frac{I}{2\pi(d + x)} + \frac{I}{2\pi(d - x)} \tag{1}$$

The magnetic flux between the two conductors (used in finding external inductance) is then found by integrating over this central plane. For a unit length in the z direction,

$$\begin{aligned} \psi_m &\approx \frac{\mu I}{2\pi} \int_{-(d-R)}^{(d-R)} \left[\frac{1}{d + x} + \frac{1}{d - x} \right] dx \\ &= \frac{\mu I}{2\pi} [\ln(d + x) - \ln(d - x)]_{-(d-R)}^{(d-R)} \end{aligned} \tag{2}$$

Inductance per unit length is then

$$L = \frac{\psi_m}{I} \approx \frac{\mu}{2\pi} \left[\ln\left(\frac{2d - R}{R} \right) - \ln\left(\frac{R}{2d - R} \right) \right] = \frac{\mu}{\pi} \ln\left(\frac{2d}{R} - 1 \right) \tag{3}$$

Example 4.6b
EXTERNAL INDUCTANCE OF PARALLEL-WIRE TRANSMISSION LINE (EXACT)

When spacing between conductors is comparable with wire radii, current distribution in the wires is affected and the result obtained above is modified. It can be shown either by a method of images or by conformal transformations to be described in Chapter 7 that the exact magnetic flux function ψ_m [analogous to electric flux function in Eq. 1.6(1)] and scalar magnetic potential Φ_m for this problem are

$$\psi_m = -\frac{\mu I}{4\pi} \ln\left[\frac{(x - a)^2 + y^2}{(x + a)^2 + y^2} \right] \tag{4}$$

$$\Phi_m = -\frac{I}{2\pi} \left[\tan^{-1} \frac{y}{(x - a)} - \tan^{-1} \frac{y}{(x + a)} \right] \tag{5}$$

where

$$a = \sqrt{d^2 - R^2}$$

Taking the flux difference at $x = d - R$ and $x = -d + R$ (both at $y = 0$),

$$\Delta\psi_m = \psi_m(d - R, 0) - \psi_m(-d + R, 0) \tag{6}$$

$$= -\frac{\mu I}{4\pi}\left\{\ln\left[\frac{d - R - a}{d - R + a}\right]^2 - \ln\left[\frac{-d + R - a}{-d + R + a}\right]^2\right\}$$

$$= -\frac{\mu I}{\pi}\ln\left|\frac{d - R - a}{d - R + a}\right| = -\frac{\mu I}{\pi}\ln\left|\frac{d - R - \sqrt{d^2 - R^2}}{d - R + \sqrt{d^2 - R^2}}\right| \tag{7}$$

By multiplying numerator and denominator by $[(d - R) - \sqrt{d^2 - R^2}]$, this reduces to

$$\Delta\psi_m = -\frac{\mu I}{\pi}\ln\left\{\frac{d}{R} - \sqrt{\left(\frac{d}{R}\right)^2 - 1}\right\} = \frac{\mu I}{\pi}\cosh^{-1}\left(\frac{d}{R}\right) \tag{8}$$

so

$$L = \frac{\Delta\psi_m}{I} = \frac{\mu}{\pi}\cosh^{-1}\left(\frac{d}{R}\right) \tag{9}$$

Example 4.6c
INTERNAL INDUCTANCE OF PLANE CONDUCTOR WITH SKIN EFFECT

As a third example, we utilize the energy method and calculate internal inductance. This example differs from that of Ex. 2.17,for which uniform current distribution was assumed. Moreover, we utilize the phasor forms of the skin effect formulation. The basis, as in Sec. 2.17, is the equation of the circuit form of energy storage to the field form:

$$\frac{1}{2}LI^2 = \int_V \frac{\mu}{2}H^2\,dV \tag{10}$$

The magnetic field distribution for a semi-infinite conductor with sinusoidally varying currents was found to be [Eq. 3.16(15)]

$$H_y = -\frac{\sigma\delta E_0}{(1 + j)}\,e^{-(1+j)x/\delta} \tag{11}$$

where the coordinate system of Fig. 3.16a is used. The current per unit width, J_{sz}, is just the value of H_y at the surface:

$$J_{sz} = -H_y(0) = \frac{\sigma\delta E_0}{(1 + j)} \tag{12}$$

We may now apply (10) to the calculation of L. But first we recognize (10), as written, is for instantaneous I and H. To use with phasors, we must either convert to instantaneous forms or write the equivalent of (10) for time-average stored energies. The latter procedure is simpler and we find

$$\frac{1}{4} L|I|^2 = \int_V \frac{\mu}{4} |H|^2 \, dV \tag{13}$$

where the factor of $\frac{1}{4}$ rather than $\frac{1}{2}$ on each side comes from the time average of squares of sinusoids. Taking a width w, so that current is wJ_{sz}, and a length l, and substituting (11) and (12) in (13), we obtain

$$\frac{L}{4} \frac{\sigma^2 \delta^2 E_0^2}{2} w^2 = wl \int_0^\infty \frac{\mu}{4} \frac{\sigma^2 \delta^2 E_0^2}{2} e^{-2x/\delta} \, dx$$

or

$$L = \frac{\mu l}{w} \int_0^\infty e^{-2x/\delta} \, dx = \frac{\mu l \delta}{2w} \left[-e^{-2x/\delta} \right]_0^\infty = \frac{\mu l \delta}{2w}$$

so

$$\omega L = \frac{l}{w} \cdot \frac{\omega \mu \sigma}{2\sigma} \cdot \sqrt{\frac{2}{\omega \mu \sigma}} = \frac{l}{w \sigma \delta} = \frac{R_s l}{w} \tag{14}$$

where relations for skin depth and surface resistivity have been substituted from Secs. 3.16 and 3.17. As found there, the internal reactance per square is equal to surface resistivity, R_s. This is multiplied by length l and divided by width w to give the internal reactance of the overall unit.

4.7 Mutual Inductance

The mutual inductance was defined in Sec. 4.2 as that arising from the induced voltage in one circuit due to current flowing in another circuit. We now discuss several approaches to its calculation, some of which may also be applied to calculation of self-inductance.

Flux Linkages The most direct approach is that from Faraday's law, finding the magnetic flux linking one circuit related to current in the other circuit, as in Eq. 4.2(19). Thus for two circuits 1 and 2 we write

$$M_{12} = \frac{\int_{S1} \mathbf{B}_2 \cdot d\mathbf{S}_1}{I_2} \tag{1}$$

where B_2 is the magnetic flux arising from current I_2 and integration is over the surface of circuit 1. By reciprocity $M_{21} = M_{12}$ (for isotropic magnetic materials), so the calculation may be made with the inducing current in either circuit. Consider, for example, the two parallel, coaxial conducting loops pictured in Fig. 4.7*a*. The magnetic field from a current in one loop has been found for a point on the axis in Ex. 2.3a:

$$B_z(0, d) = \frac{\mu I_2 a^2}{2(a^2 + d^2)^{3/2}} \tag{2}$$

If loop 2 is small enough compared with spacing d, this will be relatively constant over the second loop and the relation (1) gives

$$M = \frac{\pi b^2 B_z(0, d)}{I_2} = \frac{\mu \pi a^2 b^2}{2(a^2 + d^2)^{3/2}} \tag{3}$$

The exact formula is found by integrating the field over the cross section, but we will approach the exact calculation by another method.

Use of Magnetic Vector Potential Since $\mathbf{B} = \nabla \times \mathbf{A}$, application of Stokes's theorem to (1) yields an equivalent expression in terms of the magnetic vector potential:

$$M = \frac{\int_{S1} (\nabla \times \mathbf{A}_2) \cdot \mathbf{dS}_1}{I_2} = \frac{\oint \mathbf{A}_2 \cdot \mathbf{dl}_1}{I_2} \tag{4}$$

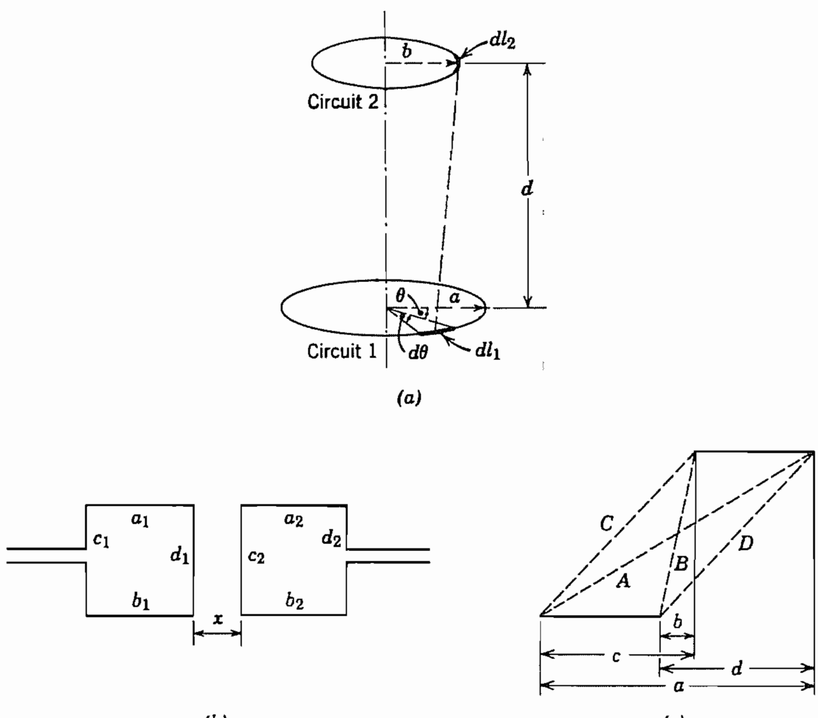

FIG. 4.7 (*a*) Two circular loops. (*b*) Two rectangular coupling loops. (*c*) Parallel current elements displaced from one another.

This form is useful in any problem for which the vector potential is more easily found than the magnetic field directly. It is especially useful for problems in which the circuit has straight-line segments or can be approximated by such segments, as in the problem of the coupling of square loops pictured in Fig. 4.7*b*. Vector potential **A** is in the direction of the current element contributing to it by Eq. 2.9(5), so the contribution to **A** from horizontal sides a_1 and b_1 is only horizontal. These sides thus contribute to mutual inductance only through integration by (4) over the horizontal parts of circuit 2, a_2 and b_2. Similarly, vertical currents in c_1 and d_1 contribute to mutual inductance only by integration over the two parallel (vertical) sides c_2 and d_2. The basic coupling element in such a configuration is then that of two parallel but displaced current elements as pictured in Fig. 4.7*c*. The contribution to mutual inductance from such elements (Prob. 4.7d) can be shown to be

$$M = \frac{\mu}{4\pi}\left\{c \ln\left[\frac{a + A}{c + C}\right] + d \ln\left[\frac{a + A}{d + D}\right] + b \ln\left[\frac{b + B}{a + A}\right] + (C + D) - (A + B)\right\} \tag{5}$$

This point of view is quite useful for qualitative thinking about couplings in a circuit as well as for quantitative analysis.

Neumann's Form Another standard form for calculation of mutual coupling of two filamentary circuits follows directly from the above. We write the vector potential **A** arising from current in circuit 2, assuming that current to be in line filaments and neglecting retardation.

$$\mathbf{A}_2 = \oint \frac{\mu I_2\, \mathbf{dl}}{4\pi R} \tag{6}$$

where R is the distance between current element $\mathbf{dl}_2$ and the field point. Substitution in (4) yields

$$M = \frac{1}{I_2}\oint\oint \frac{\mu I_2\, \mathbf{dl}_2 \cdot \mathbf{dl}_1}{4\pi R} = \frac{\mu}{4\pi}\oint\oint \frac{\mathbf{dl}_1 \cdot \mathbf{dl}_2}{R} \tag{7}$$

This standard form is due to Neumann. Note in particular its illustration of the reciprocity relation $M_{12} = M_{21}$, since integrations about circuits 1 and 2 may be taken in either order.

Example 4.7a
MUTUAL INDUCTANCE OF COAXIAL LOOPS BY NEUMANN'S FORM

For the coaxial loops of Fig. 4.7*a*, let $\mathbf{dl}_1$ be any element of circuit 1 and $\mathbf{dl}_2$ be any element of circuit 2. Then

$$\mathbf{dl}_1 \cdot \mathbf{dl}_2 = dl_2 a\, d\theta \cos \theta \tag{8}$$

$$R = \sqrt{d^2 + (a \sin \theta)^2 + (a \cos \theta - b)^2} \tag{9}$$

By substituting $\theta = \pi - 2\phi$ and

$$k^2 = \frac{4ab}{d^2 + (a + b)^2} \tag{10}$$

the integral (7) will then be found to become

$$M = \mu\sqrt{ab}k \int_0^{\pi/2} \frac{(2 \sin^2 \phi - 1)\, d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \tag{11}$$

which can be written as

$$M = \mu\sqrt{ab}\left[\left(\frac{2}{k} - k\right)K(k) - \frac{2}{k}E(k)\right] \tag{12}$$

where

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi \tag{13}$$

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \tag{14}$$

The definite integrals (13) and (14) are given in tables[3] as functions of k and are called *complete elliptic integrals* of the first and second kinds, respectively.

Example 4.7b
SELF-INDUCTANCE OF CIRCULAR LOOP THROUGH MUTUAL INDUCTANCE CONCEPTS

Neumann's form does not appear useful for the calculation of self-inductances of filamentary current paths, since radius R in (6) becomes zero at some point in the integration for such filaments. For a conductor of finite area, however, as in the round loop of wire pictured in Fig. 4.7*d*, one obtains the external contribution to self-inductance by calculating induced field at the surface of the conductor, say through the vector potential **A** as in (6). If wire radius a is small compared with loop radius r, this field is nearly the same as though current were concentrated along the center of the wire. Thus we conclude that the external inductance of the loop is well approximated by the mutual

[3] *For example, H. B. Dwight,* Tables of Integrals, *3rd ed., Macmillan, New York, 1961; or M. R. Spiegel,* Mathematical Handbook of Formulas and Tables, *Schaum's Outline Series, McGraw-Hill, New York, 1968.*

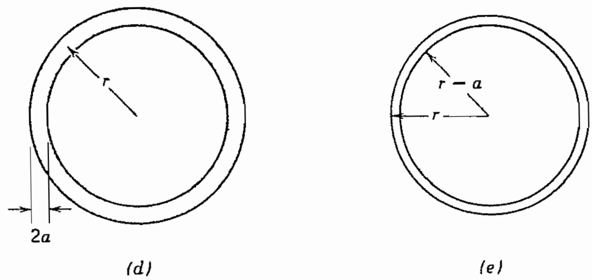

FIG. 4.7 (*d*) Conducting loop for which external self-inductance is to be found. (*e*) Filamentary loops, one through center of wire and other along inside edge, for which mutual inductance may be calculated.

inductance between the two filaments of Fig. 4.7*e*. Utilizing (12) for the mutual inductance between two concentric circles of radii r and $(r - a)$ we then have

$$L_0 = \mu(2r - a)\left[\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right]$$
$$k^2 = \frac{4r(r - a)}{(2r - a)^2} \tag{15}$$

where $E(k)$ and $K(k)$ are as defined by (13) and (14). If a/r is very small, k is nearly unity, and K and E may be approximated by

$$K(k) \cong \ln\left(\frac{4}{\sqrt{1 - k^2}}\right)$$
$$E(k) \cong 1$$

so

$$L_0 \cong r\mu\left[\ln\left(\frac{8r}{a}\right) - 2\right] \tag{16}$$

To find total L, values of internal inductance, as found in Sec. 4.5, must be added.

4.8 Inductance of Practical Coils

A study of the inductance of coils at low frequencies involves no new concepts but only new troubles because of the complications in geometry. Certain special cases are simple enough for calculation by a straightforward application of previously outlined methods. For example, for a circular coil of N turns formed into a circular cross section (Fig. 4.8*a*) we may modify the formula for a circular loop of one turn, Eq. 4.7(16),

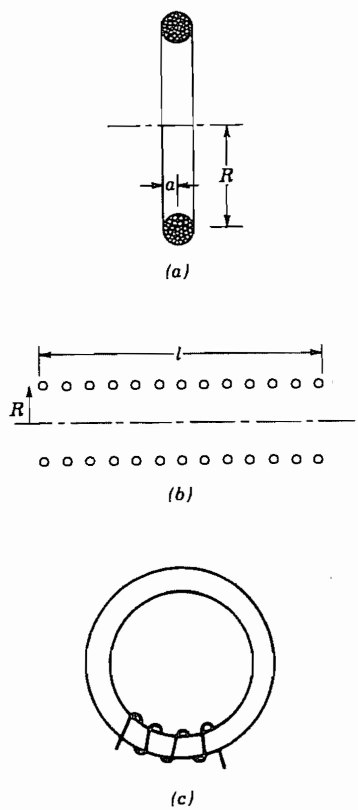

FIG. 4.8 (*a*) Coil of large radius-to-length ratio. (*b*) Solenoidal coil. (*c*) Solenoidal coil on high-permeability core.

provided the cross section is small compared with the coil radius. Magnetic field must be computed on the basis of a current NI; in addition, to compute the total induced voltage about the coil, N integrations must be made about the loop. Equation 4.7(16) is thus modified by a factor N^2. The external inductance for this coil is then

$$L_0 = N^2 R\mu\left[\ln\left(\frac{8R}{a}\right) - 2\right] \tag{1}$$

For the other extreme, the inductance of a very long solenoid (Fig. 4.8b) may be computed. If the solenoid is long enough, the magnetic field on the inside is essentially constant, as for the infinite solenoid,

$$H_z = \frac{NI}{l} \tag{2}$$

where N is the total number of turns and l the length. The flux linkage for N turns is then $N\pi R^2 \mu H_z$, and the inductance is

$$L_0 = \frac{\pi \mu R^2 N^2}{l} \tag{3}$$

For coils of intermediate length-to-radius ratio, empirical or semiempirical formulas frequently have to be used. The famous Nagaoka formula applies a correction factor F to the formula (3) for the long solenoid.[4] A simple approximate form[5] very close to this for R/l up to 2 or 3 is

$$L_0 = \frac{\pi \mu R^2 N^2}{l + 0.9R} \tag{4}$$

If a coil is wound on a toroidal core of high permeability as shown in Fig. 4.8c, the flux essentially is restricted to the core region, independent of the length of the winding. The magnetic field intensity is again given by (2) and the inductance by (3) with $l = 2\pi r_0$, where r_0 is the mean radius of the toroid.

At higher frequencies the problem becomes more complicated. When turns are relatively close together, the assumption made previously in calculating internal impedance (other portions of the circuit so far away that circular symmetry of current in the wire is not disturbed) certainly does not apply. Current elements in neighboring turns will be near enough to produce nearly as much effect upon current distribution in a given turn as the current in that turn itself. Values of skin effect resistance and internal inductance are then not as previously calculated. External inductance may also be different since changes in external fields result when current loses its symmetrical distribution with respect to the wire axis. In fact, the strict separation of internal and external inductance may not be possible for these coils, for a given field line may be sometimes inside and sometimes outside of the conductor. Finally, distributed capacitances may be important and further complicate matters (see following section).

Coils utilizing superconductors, which are materials giving zero resistance below some critical temperature near absolute zero, have become important because one can obtain with proper design very high values of uniform magnetic fields with them, without the use of iron. They may also be very efficient devices for storage of large energies. The electromagnetic principles of design are the same as given for other coils, and in fact, the approximations may be better satisfied by the thin wires typically used in superconducting magnets. The mechanical forces of the large currents must be considered in the design, and the transient behavior of a superconductor is very different from that of an ordinary conductor. Wilson[6] gives examples of various coil configurations, with reference to the background literature.

[4] *E. C. Jordan (Ed.),* Reference Data for Engineers: Radio, Electronics, Computer, and Communications, *7th ed., Howard W. Sams, Indianapolis, IN, 1985.*

[5] *H. A. Wheeler,* Proc. IRE ***16****, 1398 (1928).*

[6] *M. N. Wilson,* Superconducting Magnets, *Clarendon Press, Oxford, 1983.*

4.9 SELF AND MUTUAL CAPACITANCE

The concept of electrostatic capacitance between two conductors was introduced in Chapter 1 as the charge on one of the conductors divided by the potential difference between conductors. In the circuit analysis of Sec. 4.2 this definition was carried over as a quasistatic concept to give the usual capacitance term utilized in the analysis of circuits with time-varying excitation. Little more need be said about the simple two-conductor capacitor, but it is useful to collect expressions we have developed for some of the common capacitive elements.

1. Parallel planes with negligible fringing, A = area, d = spacing:

$$C = \frac{\varepsilon A}{d} \tag{1}$$

2. Concentric spheres of radii a and b $(b > a)$:

$$C = \frac{4\pi\varepsilon ab}{(b - a)} \tag{2}$$

3. Coaxial cylinders of radii a and b $(b > a)$:

$$C = \frac{2\pi\varepsilon}{\ln(b/a)} \quad \text{F/m} \tag{3}$$

4. Parallel cylinders with wires of radius a, with axes separated by d (to be derived in Chapter 7):

$$C = \frac{\pi\varepsilon}{\cosh^{-1}(d/2a)} \quad \text{F/m} \tag{4}$$

If there are several conductors, the electric flux from one conductor may end on several of the others and induce charge on each of those. Consider, for example, the multiconductor problem diagrammed in Fig. 4.9a. Suppose conductor 1 is raised to a positive potential with the other three bodies, 0, 2, and 3, grounded. The electric flux from 1 will divide among the other three bodies and induce negative charges on each of these. The amounts of the separate charges may be used to define capacitances C_{10}, C_{12}, and C_{13} in the circuit representation of Fig. 4.9b. Similarly, raising conductor 2 to a nonzero potential and finding induced charges on grounded conductors 0, 1, and 3 determines C_{20} and C_{23} and provides a check on C_{12}. Repetition of the process with conductor 3 at a nonzero potential gives the remaining element C_{30} and provides a check on C_{13} and C_{23}. However, it is usually not possible to measure the individual charges on the conductors which are tied together. Usually a capacitance current, dQ/dt, is measured by applying a time-varying voltage, and Q is the sum of the charges on electrodes connected together. Thus the three measurements described would yield $(C_{10} + C_{12} + C_{13})$, $(C_{20} + C_{12} + C_{23})$, and $(C_{30} + C_{13} + C_{23})$. Three additional measurements with linearly independent combinations of V_1, V_2, and V_3 are required to determine the six elements of the circuit.

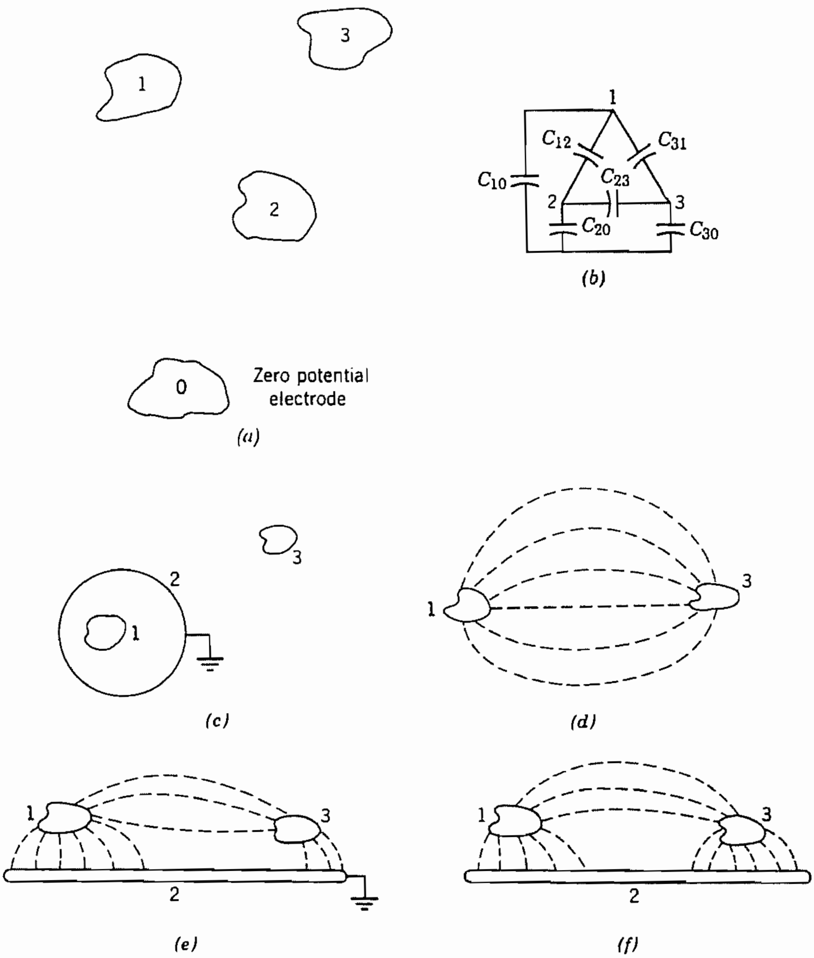

FIG. 4.9 (*a*) Four conducting bodies, one of which is chosen to have zero potential. (*b*) The equivalent circuit for (*a*). (*c*) Electrostatic shielding by a grounded sphere. (*d*) Flux lines between a pair of conductors without shielding. (*e*) Partial shielding by a grounded conducting plane. (*f*) Ungrounded nearby conductor increases coupling.

A common problem is that of decreasing the capacitive coupling between two bodies, that is, of electrostatically shielding them from one another. Consider, for example, the conductors 1 and 3 of Fig. 4.9*c*. If a grounded conductor 2 is introduced and made to surround either body 1 or 3 completely, as in Fig. 4.9*c*, it is evident that a change in potential of 3 can in no way influence the charge on 1 so that mutual capacitance $C_{13} = 0$.

More often the added conductor may not completely enclose any body, so that the capacitive coupling may not be made zero, but may only be reduced from its original value. Any finite conductor, as 2, introduced into the field acts to decrease the mutual capacitance C_{12} from its value prior to the introduction of 2, and hence provides some decrease in the capacitive coupling between 1 and 3. The reason is that fewer of the flux lines of the charge on 1 will terminate on 3 with a grounded conductor, as shown by comparing Figs. 4.9*d* and 4.9*e*. However, if 2 is not connected to the ground (the infinite supply of charge), the effect of the added electrode will be to shorten the flux lines as seen in Fig. 4.9*f*. In terms of the equivalent circuit, the effective capacitance between 1 and 3 is seen from the equivalent circuit of Fig. 4.9*b* to be given by C_{13} in parallel with C_{12} and C_{23} in series:

$$(C_{13})_{\text{eff}} = C_{13} + \frac{C_{12}C_{23}}{C_{12} + C_{23}}$$

This value is generally greater than the value of C_{13} prior to the introduction of 2 (though it need not be if 2 lies along an equipotential surface of the original field); so, if insulated from ground, the additional conductor may act to increase the effective capacitive coupling between 1 and 3. It often happens that electrodes, although grounded for direct current, may be effectively insulated or floating at high frequencies because of impedance in the grounding leads. In such cases the new electrodes do not accomplish their shielding purposes but may in fact increase capacitive coupling.

Circuits Which are Not Small Compared With Wavelength

4.10 Distributed Effects and Retardation

We now consider the generalizations to circuit theory when effects are distributed rather than lumped, and also when circuits become comparable in size with wavelength so that retardation from one part of the circuit to another must be considered. Considering first the distributed effects, we recognize that the fields contributing to circuit elements are always distributed in space and the representation by a lumped element is valid only when the region is small in comparison with wavelength and when only one type of energy storage (electric or magnetic) is important for that region. If the electric energy storage in parts of a primarily inductive element, or magnetic energy in a primarily capacitive element, becomes important, the approach through classic circuit theory is to divide into subelements that can be treated as one or the other. For example, suppose there is electric field (capacitive) coupling between the turns of the inductor of Fig. 4.10*a*. A first approximation is that of adding a capacitive element across the terminals of L to represent all the electric energy storage of the element as shown in Fig. 4.10*b*. A still better approximation is that of adding a capacitive element between each pair

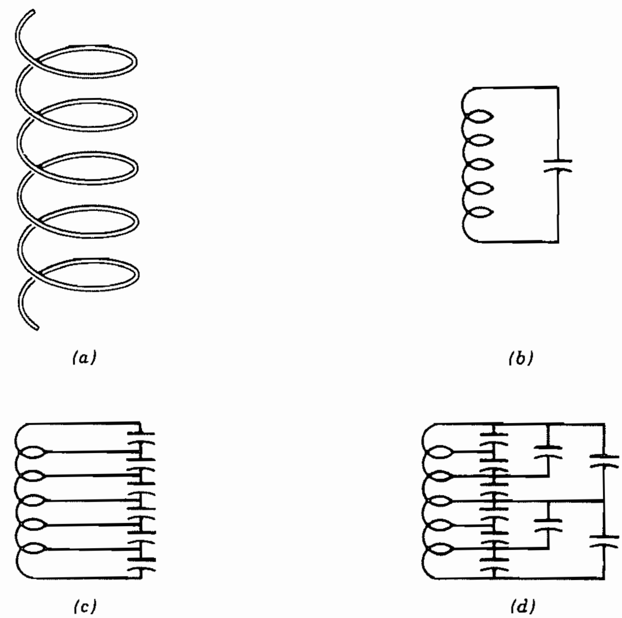

FIG. 4.10 (*a*) Coil. (*b*) Circuit with single capacitance representing electric-field coupling among turns. (*c*) Circuit representation with capacitive coupling shown between each adjacent turn. (*d*) Representation with capacitances added between nonadjacent turns.

of adjacent turns, as in Fig. 4.10*c*. But there may be coupling between nonadjacent turns and still other capacitances can be added as in Fig. 4.10*d*. The effect of these at high frequencies is to bypass some of the turns so that not all turns have the same current. This last effect would not be at all included in the simpler representation of Fig. 4.10*b*. Finally one might go to the limit and consider differential elements of the coil, attempting to find couplings to all other differential elements, to write and solve a differential equation for current distribution. This process could be carried out only for simple configurations, and even the approach through a finite number of lumped elements as in *c* or *d* becomes complicated if there are many turns.

Consider next the retardation effect arising from the finite time of propagation of electromagnetic effects across the circuit. To simplify this discussion, we consider only sinusoidal excitation so that we can define a wavelength and discuss phase relationships. More general excitations can of course be broken into a series of sinusoids through Fourier analysis. Consider, for example, the simple single-loop antenna of Fig. 4.10*e*. At low frequencies, with diameter d small in comparison with wavelength, the time of propagation of effects from one part of the loop to another is negligible. Thus, magnetic field produced by a current element at a point such as A travels to another point such as B in a negligible part of a cycle and so has negligible phase delay. The induced field from the time rate of change of the field is then 90 degrees out of phase with current in B and contributes to the inductive effect we expect for the loop at low frequencies.

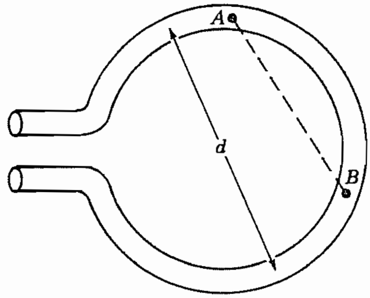

FIG. 4.10 (*e*) Loop antenna showing phase retardation between sources at A and induced fields at B when d is comparable with wavelength.

At higher frequencies, with d comparable with wavelength, the finite time of propagation about the circuit must be considered. Current at B may then not be in phase with current at A, and the magnetic field at B arising from the element at A may not be in phase with either. The time rate of change of magnetic field induces an electric field which may then be not exactly 90 degrees out of phase with I_B. If there is an in-phase component, it represents energy transfer, which turns out to be a contribution to the energy radiated by this antenna. If current distribution is known, fields throughout the circuit can be calculated and the contribution to radiated power represented in the circuit by a so-called *radiation resistance*. But to find the actual current distribution, one really needs to solve the boundary-value problem represented by the conducting loop. For some antennas or other circuits comparable in size with wavelength, it is possible to make reasonable assumptions about current distribution and extend circuit theory in this way, but the extension must be done carefully. Additional discussion of this point will be given in the next section utilizing a retarded potential formulation for circuit theory.

One important circuit having both distributed and propagation effects is the uniform transmission line. It turns out that circuit theory can be extended to this case. Agreement with field solutions is exact for perfectly conducting transmission lines and very good for lines with losses, as will be seen in Chapter 8. The circuit theory of transmission lines, to be developed in Chapter 5, is thus of very special importance.

4.11 Circuit Formulation Through the Retarded Potentials

The cause-and-effect relationships embodied in the retarded potentials of Sec. 3.19 can provide additional insights into the circuit formulation for electromagnetic problems, especially for circuits large in comparison with wavelength. This approach was first used by Carson.[7] A typical circuit follows a conductor for all or part of its path, so we start with Ohm's law in field form for a point along this path,

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} \tag{1}$$

where σ is the conductivity for the point under consideration and may vary as one moves about the circuit path. We next break up the field into an applied portion, $\mathbf{E}_0$, and an induced portion, $\mathbf{E}'$, the latter arising from the charges and the currents of the circuit itself. We also write $\mathbf{E}'$ in terms of the retarded potentials of Sec. 3.19

$$\mathbf{E}_0 + \mathbf{E}' = \mathbf{E}_0 - \nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{\mathbf{J}}{\sigma} \tag{2}$$

where $\mathbf{A}$ and Φ are given as integrals over the charges and currents of the circuit, as defined in Eqs. 3.20(3) and 3.20(4).

The term $\mathbf{J}/\sigma$ in (2) is indeterminate over nonconducting portions of the path since both J and σ are zero for insulating portions and σ is generally undefined within any localized source. We consequently integrate (2) over conducting portions of the path, obtaining a cause-and-effect relationship which can be considered the general circuit equation:

$$\int \mathbf{E}_0 \cdot d\mathbf{l} - \int \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} - \int \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} - \int \nabla\Phi \cdot d\mathbf{l} = 0 \tag{3}$$

In a conventional circuit, the first term is applied voltage, the second a resistive term, the third an inductive term, and the fourth a capacitive term. The terms are discussed separately.

Applied Voltage The first term of (3) can be identified as the applied voltage of circuit theory and is just the integral of applied electric field over the circuit path. In a circuit such as a receiving antenna (Fig. 4.11*a*), the applied field is clearly distributed over the circuit through the mechanism of the incoming electromagnetic wave and the integration of $\mathbf{E}_0$ is about the complete path:

$$V_0 = \oint \mathbf{E}_0 \cdot d\mathbf{l} \tag{4}$$

For the localized sources, discussed in Sec. 4.3, for which electric field can be considered the gradient of a scalar potential, the integration of $\mathbf{E}_0$ from 2 to 1 about the circuit of Fig. 4.11*b* is the negative of that from 1 to 2 of the source since the closed line integral of the gradient is zero. The gap in the capacitor can be ignored in this step since the localized source produces negligible field there. Thus the source voltage is

$$V_0 = \int_{2(\text{circuit})}^{1} \mathbf{E}_0 \cdot d\mathbf{l} = -\int_{1(\text{source})}^{2} \mathbf{E}_0 \cdot d\mathbf{l} \tag{5}$$

[7] J. R. Carson, Bell System Tech. J. **6**, 1 (1927).

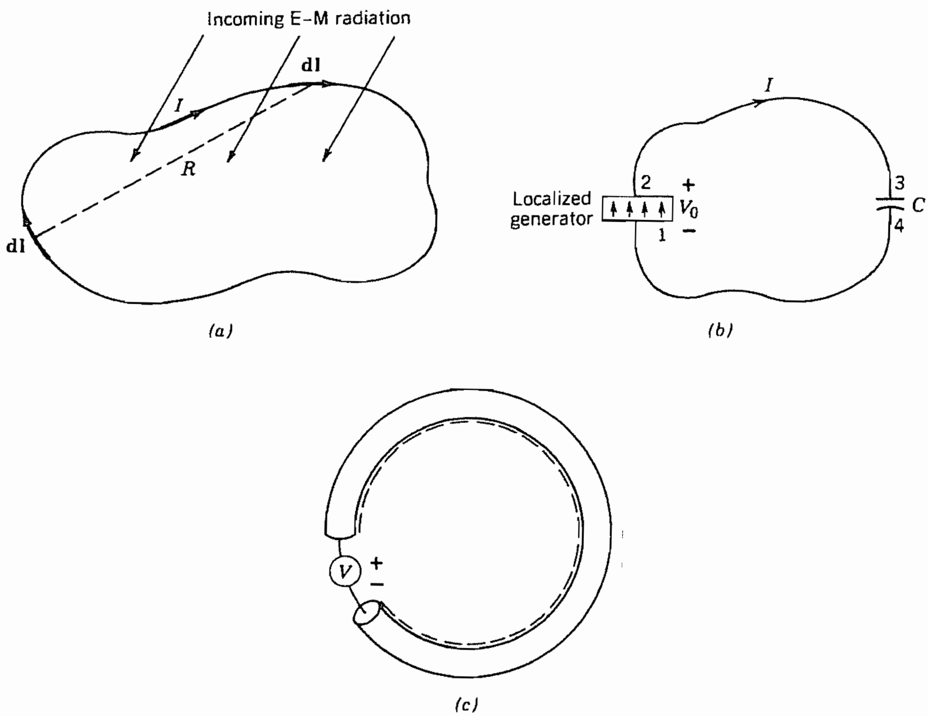

FIG. 4.11 (*a*) Closed filamentary loop excited by incoming electromagnetic wave. (*b*) Filamentary circuit with capacitor excited by a localized generator (point source). (*c*) Circular loop of round wire with circuit path along inner boundary.

In this class of problem, V_0 is independent of the circuit path, whereas in the receiving antenna class of problem discussed above, V_0 depends very much upon the circuit configuration and orientation with respect to the fields.

Internal Impedance Term The second term in (3) is of exactly the same form as the ohmic term for the resistor in the circuit example of Sec. 4.2. There we showed that in the limit of dc this corresponds to the expected resistance term. Here it is understood that σ may vary over different parts of the circuit path and the integration brings in the total resistance of the circuit path. For ac circuits it turns out that this term may also include a contribution from the inductance internal to the conductor along which the circuit path is taken, as was seen for the round wire in Sec. 4.5. Thus for the important sinusoidal case with phasor representations for currents and voltages, this term gives a complex contribution resulting from internal reactance in addition to the resistance. That is, if internal impedance per unit length is defined as the ratio of surface electric field to the total current in the conductor,

$$Z_i' = \frac{E_s}{I} \tag{6}$$

the term under consideration becomes the total internal impedance Z_i multiplied by current I:

$$\int \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} = \int \mathbf{E}_s \cdot d\mathbf{l} = I \int Z_i' \, dl = IZ_i \tag{7}$$

where the integrals are taken over the conducting portions of the circuit from 2 to 3 and 4 to 1 in Fig. 4.11*b*.

External Inductance Term The third term in (3) is the inductance term and, if the circuit path is properly selected, represents only the contribution from magnetic flux external to the conductor. Consider, for example, the loop of wire in Fig. 4.11*c*, and take the circuit path along the inner surface of the conductor. We will assume that the integral in the third term of (3) taken over the conducting portions of the circuit differs negligibly from an integral which would include the small gaps at the source and in any capacitors included in the circuit. This allows evaluation of that term with closed integrals. We take the path as stationary so that

$$\oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} = \frac{d}{dt} \oint \mathbf{A} \cdot d\mathbf{l} \tag{8}$$

From Stokes's theorem,

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \tag{9}$$

But

$$\nabla \times \mathbf{A} = \mathbf{B} \tag{10}$$

so

$$\oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{11}$$

The surface integral of (11) is the magnetic flux linking the chosen circuit, exactly as in the approach through Faraday's law in Sec. 3.2. Thus the term may be defined as an inductance term, as before, recognizing that it is the contribution from flux threading the chosen circuit path (i.e., the external inductance):

$$\oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} = L \frac{dI}{dt} \tag{12}$$

Thus this provides an alternate way of calculating inductance:

$$L = \frac{1}{I} \oint \mathbf{A} \cdot d\mathbf{l} \tag{13}$$

The above assumes the circuit small compared with wavelength so that retardation is neglected. The extensions when this assumption is not valid are discussed shortly.

Capacitive Term As with the other terms in (3) we must integrate the $\nabla\Phi$ over the conducting portions of the circuit, that is, from 2 to 3 and 4 to 1 in Fig. 4.11*b*. Here we assume that the fields arising from charges on the capacitor are negligible at the source, so we may use, as the range of integration, 4 to 3 through the source. Then, since the integral of the gradient of a scalar completely around a closed path (here including the capacitor gap) is zero, we may write

$$\int_{4(\text{circuit})}^{3} \nabla\Phi \cdot \mathbf{dl} = -\int_{3(\text{gap})}^{4} \nabla\Phi \cdot \mathbf{dl} = \Phi_3 - \Phi_4 \tag{14}$$

In a lumped capacitor this potential difference is related to charge Q through the capacitance C,

$$\Phi_3 - \Phi_4 = \frac{Q}{C} \tag{15}$$

so that this term is the capacitance term of circuit theory,

$$\int_{4(\text{circuit})}^{3} \nabla\Phi \cdot \mathbf{dl} = \frac{Q}{C} = \frac{1}{C}\int I\,dt \tag{16}$$

Circuits Comparable in Size with Wavelength The formulation in terms of retarded potentials has been shown to reduce to the usual low-frequency circuit concepts as obtained in the earlier formulation using only fields, under the same assumptions. The present formulation is attractive in that it appears more readily extendible to large-dimension circuits, such as an antenna, when retardation effects are important. To illustrate, consider the circuit of Fig. 4.11*a*, for which current is assumed concentrated in a thin wire. Let us assume that ohmic resistance is negligible and that there is no capacitor so that there is only an applied voltage and a term from the potential **A**. We also take steady-state sinusoids and phasor notation for this discussion. Thus, (3) becomes

$$\oint \mathbf{E}_0 \cdot \mathbf{dl} - j\omega \oint \mathbf{A} \cdot \mathbf{dl} = 0 \tag{17}$$

and **A**, for the filamentary current, by Eq. 3.21(4), is

$$\mathbf{A} = \oint \frac{\mu I e^{-jkR}}{4\pi R}\,\mathbf{dl}' \tag{18}$$

Substituting (18) in (17) and breaking up the exponential into its sinusoidal components, we obtain

$$\oint \mathbf{E}_0 \cdot \mathbf{dl} - j\omega \oint\oint \frac{\mu I(\cos kR - j \sin kR)}{4\pi R}\,\mathbf{dl} \cdot \mathbf{dl}' = 0 \tag{19}$$

We see that even if current I were assumed entirely in phase about the circuit, finite values of kR would lead to both real and imaginary parts of the contribution from this term. The imaginary part is the inductive reactance, as found before for this term, except now modified by the integration of the retardation term. But there is a new term in phase with I which corresponds to the energy radiated from the circuit, and can be expressed as current times a radiation resistance.

Although the general modifications for large-dimension circuits are shown by this approach, it is difficult to carry much further since we really do not know the distribution of I about the circuit, and cannot find it without a field solution of the problem. In some antennas it is possible to make reasonable guesses about the current and proceed, but it is clear that these guesses must eventually be checked through either experiment or a field analysis. Also, as we have seen, the integration is to be only over conducting surfaces to avoid the indeterminacy of the second term of (3). For many antennas, the "gaps" are larger than the conductors, and fields definitely not quasistatic in the open regions, so this further limits the applicability of this approach. A specific example will be carried further in the following section.

4.12 Circuits with Radiation

To conclude this discussion of the relationship between field theory and circuit theory, let us look at two specific circuits with radiation. As we saw in Sec. 4.11, a circuit that is not small in comparison with wavelength has retardation of induced fields from one part of the circuit to the other. The resulting phase changes produce components of induced field which are in phase with the currents, and an average power flow results. This power can be shown to be the radiation from the circuit. The phase shifts also produce some changes in the reactive impedance of the circuit, but this is usually a higher-order effect. The term we are concerned with is the integration of induced effects, the second term of Eq. 4.11(19):

$$V_{\text{induced}} = j\omega \oint\oint \frac{\mu I(\cos kR - j \sin kR)}{4\pi R} \, \mathbf{dl} \cdot \mathbf{dl}' \tag{1}$$

We illustrate this with two examples.

Example 4.12a
Small Circular Loop Antenna or Circuit

The first example is that of a circular loop, as in Fig. 4.12*a*, small enough so that current may be considered constant about the loop. If I is independent of position, it may be taken outside the integral (1) so that the induced term may be written

$$V_{\text{induced}} = (R_r + j\omega L)I \tag{2}$$

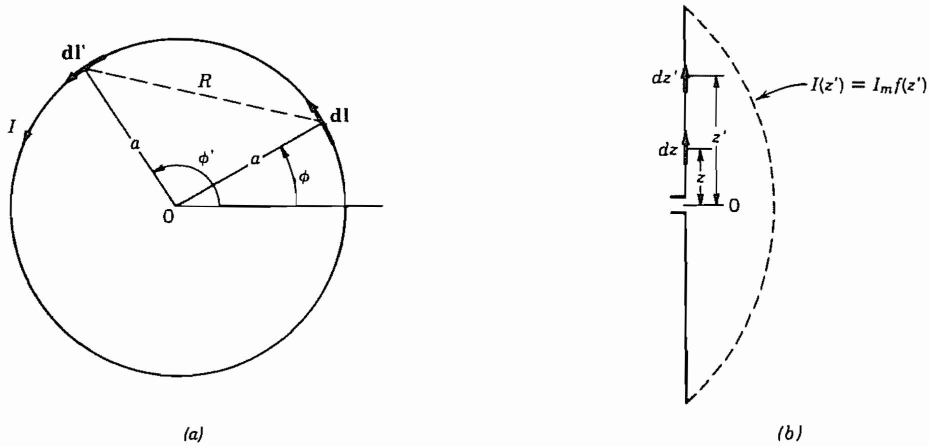

FIG. 4.12 (*a*) Circular loop with constant current I with coordinates for calculation of retardation effects. (*b*) Straight antenna of finite length with current distribution.

where

$$
\begin{aligned}
R_r &= \oint\oint \frac{\omega\mu \sin kR}{4\pi R} \, \mathbf{dl} \cdot \mathbf{dl}' \\
L &= \oint\oint \frac{\mu \cos kR}{4\pi R} \, \mathbf{dl} \cdot \mathbf{dl}'
\end{aligned}
\tag{3}
$$

The value of $\mathbf{dl}$ is $\hat{\boldsymbol{\phi}} a \, d\phi$, and that of $\mathbf{dl}'$ is $\hat{\boldsymbol{\phi}}' a \, d\phi'$. The angle between $\mathbf{dl}$ and $\mathbf{dl}'$ is $(\phi - \phi')$ and the distance R is $2a \sin[(\phi - \phi')/2]$. If the circuit is small in comparison with wavelength, $kR << 1$ and the sine term in (1) may be replaced by the first two terms of its Taylor series:

$$R_r \approx \int_0^{2\pi}\int_0^{2\pi} \frac{\omega\mu}{4\pi R} \left\{ kR - \left[\frac{k^3R^3}{3!}\right]\right\} a^2 \cos(\phi - \phi') \, d\phi \, d\phi' \tag{4}$$

The first term of (4) integrates to zero, so we see why it is necessary to retain at least two terms of the series. The second term gives

$$R_r \approx \int_0^{2\pi}\int_0^{2\pi} \frac{-4\omega\mu k^3 a^4}{24\pi} \sin^2\left(\frac{\phi - \phi'}{2}\right) \cos(\phi - \phi') \, d\phi \, d\phi' \tag{5}$$

The integrals are readily evaluated to give

$$R_r = \frac{-\omega\mu k^3 a^4}{6\pi}(-\pi^2) = \frac{\pi}{6}\left(\frac{\mu}{\epsilon}\right)^{1/2}(ka)^4 \tag{6}$$

Thus radiation resistance increases as the fourth power of the ratio of radius to wavelength (but with the understanding that this ratio is always small). For $a = 0.05\lambda$, the value is

$$R_r = \frac{120\pi^2}{6}(2\pi \times 0.05)^4 = 1.923\ \Omega \tag{7}$$

If $\cos kR$ in the expression for L is likewise expanded as a series,

$$L \approx \int_0^{2\pi}\int_0^{2\pi} \frac{\mu}{4\pi R}\left[1 - \frac{k^2R^2}{2!} + \frac{k^4R^4}{4!} + \cdots\right]a^2 \cos(\phi - \phi')\, d\phi\, d\phi' \tag{8}$$

The first term is recognized as the Neumann form for inductance of this loop (Sec. 4.7), and the remaining terms represent corrections to the inductance because of retardation. It is seldom necessary to calculate these last-mentioned corrections for circuits properly considered as lumped-element circuits.

Example 4.12b

RADIATION RESISTANCE OF A STRAIGHT ANTENNA BY CIRCUIT METHODS

As a second example, consider a straight dipole antenna as shown in Fig. 4.12*b* with current distribution

$$I(z) = I_m f(z) \tag{9}$$

where $f(z)$ is real. But here the conductor does not form a closed circuit, and as explained earlier, the Carson formulation (Sec. 4.11) only applies unambiguously over the surface of conductors. Thus we first find retarded potential **A**, which has only a z component:

$$A_z = \int_{-l}^{l} \frac{\mu I_z e^{-jkR}}{4\pi R}\, dz' = \mu I_m \int_{-l}^{l} \frac{f(z')e^{-jk|z-z'|}}{4\pi|z - z'|}\, dz' \tag{10}$$

Electric field is given in terms of **A** by Eq. 3.21(7):

$$\mathbf{E} = -j\omega\left[\mathbf{A} + \frac{1}{k^2}\nabla(\nabla \cdot \mathbf{A})\right] = -j\omega\hat{\mathbf{z}}\left[A_z + \frac{1}{k^2}\frac{\partial^2 A_z}{\partial z^2}\right] \tag{11}$$

The portion of **E** in phase with current causes the radiated power in this picture, and that clearly comes from the imaginary part of A_z. The integration of $I(z)E_{\text{in-phase}}$ over the antenna gives the total power transferred, or radiated, and this may be expressed in terms of a *radiation resistance*:

$$W = \int_{-l}^{l} I(z)(E_z)_{\text{in-phase}}\, dz = \frac{I_m^2 R_r}{2} \tag{12}$$

Thus, substituting (10) and (11), we obtain

$$R_r = \frac{2\omega\mu}{4\pi}\int_{-l}^{l} dz\, f(z) \int_{-l}^{l} f(z')\left\{\left[\frac{\sin k|z - z'|}{|z - z'|}\right] + \frac{\partial^2}{\partial z^2}\left[\frac{\sin k|z - z'|}{|z - z'|}\right]\right\} dz' \tag{13}$$

In evaluating these integrals, series expansions of the sin $k|z - z'|$ terms are often made. When carried out for the half-wave dipole [$l = \lambda/4$ and $f(z) = \cos kz$], R_r is found to be about 73.1 Ω in agreement with the value found by a Poynting integration to be utilized in Sec. 12.7. This method of finding radiation resistances of antennas is called the *induced emf method.* It is seldom easier than the Poynting integration but does show the relationship to circuit theory.

Note that for both examples, we had to assume a form for current distribution to proceed. This is a clear limitation as it can be done with reasonable confidence only in specific cases. When that is not possible, field theory must be invoked for the whole problem.

Note also that until this example, we have neglected any distributed charges along the interconnecting conductors of the circuit. Here, with current varying as $f(z)$, the continuity equation requires distributed charges, but these are taken care of by the $\nabla(\nabla \cdot \mathbf{A})$ term in (11), as shown in Sec. 3.21.

PROBLEMS

4.2a Many circuits contain nonlinear elements, that is, ones for which μ, ε, or σ, or some combination is a function of the field for at least a part of the circuit. Review the formulation of Sec. 4.2 to show this behavior explicitly. Is the general form Eq. 4.2(1) changed in such cases?

4.2b Some circuits contain time-varying elements, for which μ, ε, or σ, or a combination is a function of time for at least a part of the circuit. Discuss these cases as in Prob. 4.2a.

4.2c The sign of mutual inductance coupling is designated on a circuit diagram by the placing of black dots. With the sign convention for positive voltage and current shown in Figs. 4.2*c* and *d*, the dot location in the former denotes positive M and in the latter, negative M. Show that either can be represented by a "T-network" as in Fig. P4.2*c*, where the upper signs denote Fig. 4.2*c* and the lower, Fig. 4.2*d*.

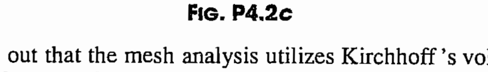

FIG. P4.2c

4.3a It has been pointed out that the mesh analysis utilizes Kirchhoff's voltage law explicitly but the current law only implicitly. Show that the current law is satisfied for each node of the circuit with mesh currents defined by Fig. 4.3*b*. Similarly show that the voltage law is satisfied by each mesh of Fig. 4.3*c*, with node voltages as shown.

4.3b The generator in the example of Figs. 4.3*b* and *c* is taken as a voltage generator in series with a source resistance. It can alternatively be taken as a current generator I_0 in parallel with a source conductance G_s. Make this substitution and write the new loop and node equations for the filter.

4.3c With a complex load impedance (admittance) connected to the source terminals as in

Figs. 4.3*d* and *e*, show that the two source representations are equivalent in producing current in and voltage across this impedance, when the conditions of Eq. 4.3(10) are satisfied.

4.3d Show that for fixed V_s and Z_s, the maximum possible power is delivered to the load when it is a "conjugate match" to Z_s, that is, $Z_L = Z_s^*$ (or $Y_L = Y_s^*$).

4.4a Make a power series expansion of Eq. 4.4(8), retaining up to quadratic terms, to show the variation of magnitude and phase with r to this approximation. Up to about what r_0/δ will this be a reasonable approximation? (See Sec. 7.14.)

4.4b Utilize the asymptotic expansions of Bessel functions to derive the approximate expression Eq. 4.4(11). What phase variation is found in this approximation? (See Sec. 7.15 or Ref. 2.)

4.4c Obtain tables of the Ber and Bei functions and plot phase of current density versus r/r_0 for $r_0/\delta = 2.39$.

4.5a Show that the ratio of very high frequency resistance to dc resistance of a round conductor of radius r_0 and material with depth of penetration δ can be written

$$\frac{R_{\text{hf}}}{R_0} = \frac{r_0}{2\delta}$$

4.5b Using the approximate formula 4.5(8), find the value of r_0/δ below which R differs from dc resistance R_0 by less than 2%. To what size wire does this correspond for copper at 10 kHz? For copper at 1 MHz? For brass at 1 MHz?

4.5c* For two z-invariant systems having the same shape of cross section and of good conductors of the same material, show that current distributions will be similar, and current densities equal in magnitude at similar points, if the applied voltage to the small system is $1/K$ in magnitude and K^2 in frequency that of the large system. Also show that the characteristic impedance of the small system will be K times that of the large system under these conditions. Check these conclusions for the case of two round wires of different radii. K is the ratio of linear dimensions ($K > 1$).

4.6a For the symmetric parallel-wire line, plot normalized external inductance, $\pi L/\mu$ versus R/d from both the approximate and exact expressions and note the range over which the approximate formula gives good results.

4.6b Derive the formula for external inductance of the coaxial line in Fig. 2.4*b* by the energy method assuming the usual situation of a material with permeability μ_0 between the electrodes.

4.6c For a parallel-plane transmission line as in Fig. 2.5*c*, find the dielectric thickness, in terms of the conductor thickness, for which the low-frequency internal inductance equals the external inductance. Take both conductor thicknesses to be the same.

4.6d A coaxial transmission line has a solid copper inner conductor of radius 0.20 cm and a tubular copper outer conductor of inner radius 1 cm, wall thickness 0.1 cm. Find the total impedance per unit length of line for a frequency of 3 GHz, including the internal impedance of both conductors.

4.6e Equivalent circuit for a differential length of coaxial transmission line. Take the lines C–B and D–A in Fig. 3.17 to be separated by a differential distance dz with z positive to the right. Write Faraday's law for the loop $ABCDA$ and use the capacitance expression given in Eq. 1.9(4) to show that the equivalent circuit shown in Fig. P4.6*e* is correct for high frequencies. (L_i is internal inductance per square and L_e is external inductance per unit length.)

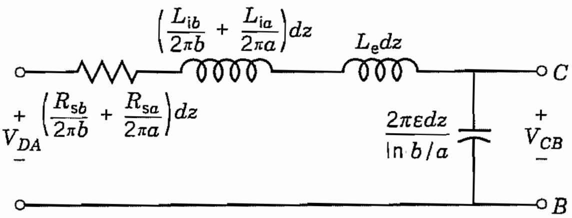

FIG. P4.6e

4.6f A frequently encountered problem in microwave circuits is the wire bonding of one part of a circuit to another as suggested in Fig. P4.6f. The configuration is usually too complex to fit any simple models, but some useful approximate formulas exist. F. E. Terman, *Radio Engineers Handbook*, McGraw-Hill, New York, 1943, gives, for round wires at high frequencies, $L = 0.20\ell[\ln(4\ell/d) - 1 + d/2\ell]$, where ℓ and d are length and diameter in millimeters, and L is in nanohenries. Estimate the inductance of the 0.5-mm-diameter wire bond in Fig. P4.6f and calculate its reactance at 1.0 GHz. Note its magnitude compared with a typical characteristic impedance of 50 Ω.

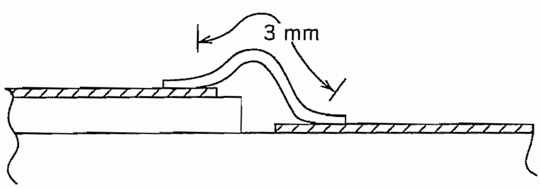

FIG. P4.6f

4.7a A coaxial line, shorted at $z = 0$, has a rectangular loop introduced for coupling, lying in a longitudinal plane with dimensions as shown in Fig. P4.7a. Find the mutual inductance between loop and transmission line assuming $d << \lambda$ so that field is essentially independent of z.

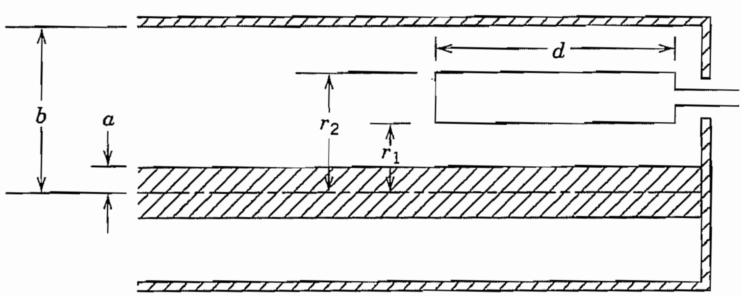

FIG. P4.7a

4.7b From tables of the complete elliptic integrals given in the references, plot the form of mutual inductance in Eq. 4.7(12) against d/a for $b/a = 1$ and for $b/a = \frac{1}{2}$.

4.7c Investigate the properties of the complete elliptic integrals for $k << 1$ and for $k \approx 1$, and obtain approximate expressions for mutual inductance for these two cases. Interpret physical meaning of these limits and compare with approximate Eq. 4.7(3).

4.7d By integration of Eq. 4.7(4), show that the contribution to mutual inductance from two parallel line segments displaced as shown in Fig. 4.7c is as given by (5).

4.7e Apply Eq. 4.7(5) to the calculation of mutual inductance between two square loops used for coupling between open-wire transmission lines as shown in Fig. 4.7*b*. The length of each side is 0.03 m; the separation x is 0.01 m. Assume that the gaps at which the lines enter are small enough to be ignored.

4.7f Plot $L_0/a\mu$ for the circular loop of round wire versus a/r from approximate and "exact" expressions and note the range of usefulness of the former. Comment on the validity of the selected mutual approach for a/r approaching unity. (Note that tables of elliptic functions are required for this comparison.)

4.7g Suppose 1-mm-diameter copper wire is formed into a single circular loop having a radius of 10 cm. A voltage generator of 1 V rms and 10 MHz is connected to an infinitesimal gap in the loop. Find the current flowing in the loop, taking into account internal impedance as well as external inductance. Justify all approximations used.

4.7h* Check Eq. 4.7(3) by taking the magnetic field of the small loop as that of a magnetic dipole and integrating flux from this over the area of the larger loop.

4.8a Plot $L_0/\mu R$ versus R/l from the expression for a long solenoid and the empirical expression 4.8(4) and compare. (If you have access to tables for the Nagaoka formula, add this curve also.)

4.8b For an ideal infinite solenoid, magnetic flux is uniform everywhere inside the solenoid. For a finite coil, as pictured in Fig. 4.8*b*, there may be more flux through the central turns of the coil than those near the ends. Explain how you make a circuit model of the coil in view of these "partial flux linkages."

4.8c* For a circular coil of square cross section, Fig. P4.8*c*, it has been shown that the largest possible inductance results when $R/s = 1.5$ for a fixed length of wire of chosen size. The value of this inductance for N turns is $L = 1.7 \times 10^{-6} RN^2$.

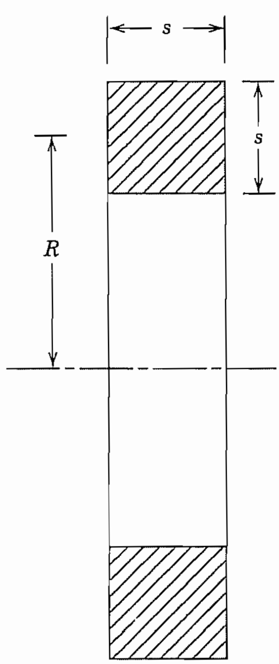

FIG. P4.8c

(i) Given 1 m of wire with cross section 1 mm^2, find the values of R, s, N, and inductance for its maximum according to this rule.

(ii) Repeat for 2 m of the same wire.

4.8d Compare the formula of Prob. 4.8c with that of Eq. 4.8(1), taking for the latter $(\text{area})^{1/2} = R/1.5$.

4.9a Discuss qualitatively the case of Fig. 4.9*e* in which two bodies, 1 and 3, which are relatively far apart have a grounded conducting plane brought in their vicinity. Give an energy argument to show that C_{13} is decreased when the plane is added.

4.9b* Suppose that the bodies 1 and 3 of Figs. 4.9*e* and *f* are spheres of radii a separated by a distance d with $a/d << 1$. If the added plane is parallel to the line joining their centers and distance b from it ($a/b << 1$), find C_{13} before and after introduction of the plane when grounded, and the effective C_{13} with the plane present and insulated from ground.

4.9c Two cylindrical conductors of radius 1 cm have their axes 4 cm apart and each axis is 4 cm above a parallel ground plane. Make rough graphical field maps and estimate the capacitances (per unit length) for this three-conductor problem. (Think about how many plots you need and the best choice of potentials for each.)

4.10a Make an order-of-magnitude estimate of the capacitance between adjacent turns of a coil as pictured in Fig. 4.10*a* if the diameter of the coil is 2 cm, the diameter of the wire 1 mm, and the spacing between turns 1 mm. Clearly state your model for the calculation.

4.10b For a coil as in Fig. 4.10*a*, in the form of a fairly open helix, it is found that the phase delay of current along the coil is well estimated by assuming propagation *along the wire* at the velocity of light in the surrounding dielectric material. For a helix of 100 turns in air, each turn 1 cm in diameter and spaced 1 mm apart on centers (wire diameter being appreciably smaller than this),

(i) Find the phase difference between current at the end and that at the beginning of the helix for such a traveling wave at $f = 150$ MHz.

(ii) Compare this with the phase difference in the retardation term, calculated along a direct path between the two ends.

4.11 We will find waves on an infinite ideal transmission line that do not radiate even though there is clearly retardation to different points along the line. Explain how this is possible.

4.12a What radius do you need to give a radiation resistance of 50 Ω from the expression derived from the small-loop circuit? Do you think the approximations reasonably satisfied for this size?

4.12b Using the method of Sec. 4.12 derive radiation resistance for a small square loop with sides d and uniform current assumed about the loop.

4.12c* As indicated in Ex. 4.12b, make a series expansion for the sine terms within the integral (retain three terms), assume $f(z') = \cos kz'$, and estimate R_r for the half-wave dipole, $l = \lambda/4$.