Appendix 4 Complex Phasors as Used In Electrical Circuits

If a voltage $V \cos \omega t$ is applied to a linear, time-invariant circuit containing R, L, and C in series, the equation to be solved is

$$L \frac{dI}{dt} + RI + \frac{1}{C} \int I \, dt = V_m \cos \omega t \tag{1}$$

But

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \tag{2}$$

If we assume that the current has the steady-state solution

$$I = Ae^{j\omega t} + Be^{-j\omega t} \tag{3}$$

the result of substituting in (1) in

$$j\omega L \, (Ae^{j\omega t} - Be^{-j\omega t}) + R(Ae^{j\omega t} + Be^{-j\omega t}) + \frac{1}{j\omega C} (Ae^{j\omega t} - Be^{-j\omega t}) = \frac{V_m}{2} (e^{j\omega t} + e^{-j\omega t}) \tag{4}$$

This equation can be true for all values of time only if coefficients of $e^{j\omega t}$ are the same on both sides of the equation, and similarly for $e^{-j\omega t}$.

$$A\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] = \frac{V_m}{2} \tag{5a}$$

$$B\left[R - j\left(\omega L - \frac{1}{\omega C}\right)\right] = \frac{V_m}{2} \tag{5b}$$

The complex quantity in the brackets of (5a) may be called Z and written in its equivalent form

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) = |Z|e^{j\psi}$$

where

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{6}$$

and

$$\psi = \tan^{-1}\frac{(\omega L - 1/\omega C)}{R} \tag{7}$$

Similarly,

$$R - j\left(\omega L - \frac{1}{\omega C}\right) = |Z|e^{-j\psi}$$

Then

$$A = \frac{V_{\mathrm{m}}}{2|Z|}\,e^{-j\psi}$$

$$B = \frac{V_{\mathrm{m}}}{2|Z|}\,e^{j\psi}$$

(A and B are conjugates: they have the same real parts and equal and opposite imaginary parts.) Substituting in (3),

$$I = \frac{V_{\mathrm{m}}}{|Z|}\left[\frac{e^{j(\omega t - \psi)} + e^{-j(\omega t - \psi)}}{2}\right] \tag{8}$$

By comparing with (2),

$$I = \frac{V_{\mathrm{m}}}{|Z|}\cos(\omega t - \psi) \tag{9}$$

This final result gives the desired magnitude and phase angle of the current with respect to the applied voltage. That information is contained in either constant A or constant B, and no information is given in one which is not in the other. Constant B is of necessity the conjugate of A, since this is the only way in which the two may add up to a real current, and the final exact answer for current must be real. It follows that half of the work was unnecessary. We could have started only with $V_{\mathrm{m}}e^{j\omega t}$ in place of the two-term expression which is exactly equivalent to $V_{\mathrm{m}} \cos \omega t$. For current, there would then be only

$$I = \frac{V_{\mathrm{m}}}{|Z|}\,e^{j(\omega t - \psi)} \tag{10}$$

Although this cannot actually be the expression for current, since it is a complex and not a real quantity, it contains all the information we wish to know: magnitude of current, $V/|Z|$, and its phase with respect to applied voltage, ψ. This procedure may be made exact by writing

$$V(t) = \text{Re}[V_m e^{j\omega t}] \tag{11}$$

$$I(t) = \text{Re}\left[\frac{V_m}{|Z|} e^{j(\omega t - \psi)}\right] \tag{12}$$

where Re denotes, "the real part of." Because of the inconvenience of this notation, it is usually not written explicitly but it is understood. That is, *if any single frequency sinusoid* $f(t)$ *is expressed by its magnitude and phase,* $Me^{j\theta}$, *or by its real (in-phase) and imaginary (out-of-phase) parts* $A + jB$, *the instantaneous expression may be found by multiplying by* $e^{j\omega t}$ *and taking the real part*:

$$f(t) = \text{Re}[Me^{j(\omega t + \theta)}] = \text{Re}[(A + jB)e^{j\omega t}] \tag{13}$$

So to summarize, voltage and current can be written[1]

$$V(t) = \text{Re}[V_c e^{j\omega t}] \tag{14}$$

$$I(t) = \text{Re}[I_c e^{j\omega t}] \tag{15}$$

where V_c and I_c are the complex of *phasor* representations of voltage and current respectively,

$$V_c = |V|e^{j\theta_V}, \qquad I_c = |I|e^{j\theta_I} \tag{16}$$

The differential equation (1) becomes an algebraic equation relating V_c and I_c:

$$j\omega L I_c + R I_c + \frac{I_c}{j\omega C} = V_c \tag{17}$$

from which

$$I_c = \frac{V_c}{Z} = \left(\frac{V_c}{|Z|}\right)e^{-j\psi} \tag{18}$$

so that

$$|I| = \frac{|V|}{|Z|}, \qquad \theta_I = \theta_V - \psi \tag{19}$$

where magnitude and phase of impedance Z are given by (6) and (7), respectively.

Although the subscript c is used to denote complex phasors in this development to stress their nature, the complex nature is understood from the context in most circuit analyses, without the special designation, and similarly it is understood in the phasor representation of fields used in much of this text.

[1] *An alternative form much used in the physics literature is* $V(t) = \frac{1}{2}(V_c e^{j\omega t} + \text{c.c.})$, *where* c.c. *stands for complex conjugate of the first term in brackets.*

For nonlinear or time-varying circuit elements, the complete equivalence (2) must be used if excitation is by a sinusoidal voltage, since frequency components other than ω are generated.

Power Calculations Given a sinusoidal voltage $V(t) = V_m \cos(\omega t + \phi_1)$ and a corresponding sinusoidal current $I(t) = I_m \cos(\omega t + \phi_2)$, the expression for instantaneous power is

$$W(t) = V(t)I(t) = V_m I_m \cos(\omega t + \phi_1)\cos(\omega t + \phi_2)$$

By trigonometric identities, this may be written

$$W(t) = \frac{V_m I_m}{2} [\cos(\phi_1 - \phi_2) + \cos(2\omega t + \phi_1 + \phi_2)] \tag{20}$$

The equivalent in terms of phasor current and voltage is

$$W(t) = \tfrac{1}{2} \operatorname{Re}[V_c I_c^* + V_c I_c e^{2j\omega t}] \tag{21}$$

where I_c^* denotes the complex conjugate of I_c. Often we are interested only in average power, which is given by the first terms of (20) and (21).

In another useful power calculation with complex voltage and current, $V_c I_c^*$ is found to give average power and the difference between stored energy in magnetic and electric fields. As an example, consider a simple series RLC circuit

$$\begin{aligned} V_c I_c^* &= Z I_c I_c^* = \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] I_c I_c^* \\ &= 2\left(\frac{R I_c I_c^*}{2}\right) + 4j\omega\left[\frac{L I_c I_c^*}{4} - \frac{I_c I_c^* C}{4\omega^2 C^2}\right] \end{aligned} \tag{22}$$

The real term is recognized as twice the average power dissipated in the resistor. The first term in brackets is average power stored in the inductor. (One factor of 2 comes from the form $\frac{1}{2}LI^2$, the other from the average of squared sinusoids.) The second term in brackets is the average energy stored in the capacitor since $|I_c/\omega C|$ is magnitude of capacitor voltage. Thus, (22) may be written

$$V_c I_c^* = 2W_L + 4j\omega(U_M - U_E) \tag{23}$$

where W_L is average power loss, U_M average energy in the magnetic fields of the inductor, and U_E average energy in electric fields of the capacitor. This is a special case of the complex Poynting theorem discussed in Sec. 3.13 that is applicable to any passive circuit.