Linear Algebra

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Contents

1 Sources

Sources

These notes were taking from MIT Linear Algebra taught by Professor Gilbert Strang and MIT A Vision of Linear Algbera by Professor Gilbert Strang

2 Course goals

After completing this course, you will understand:

- Systems of linear equations
- Row reduction and echelon forms
- Matrix operations, including inverses
- Block matrices
- Linear dependece and independence
- Subspaces and bases and dimensions
- Orthogonal bases and orthogonal projections
- Gram-Schmidt process
- Linear models and least-squares problems
- Determinants and their properties
- Cramer's Rule
- Eqigenvalues and eigenvectors
- Diagonalization of a matrix
- Symmetric matrices
- Positive definite matrices
- Similar matrices
- Linear transformations
- Singular value decomposition

3 Unit I: Ax = b and the Four Subspaces

3.1 The Geometry of Linear Equations

Imagine you're given n linear equations with n linear unknowns. In linear algebra you can look at (or think of) this system in one of three ways:

- Row picutre
- Column picture
- Matrix form

Example: imagine we're given this system of linear equations

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

If we think of this in **matrix form** we get...

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Which we can generalize like...

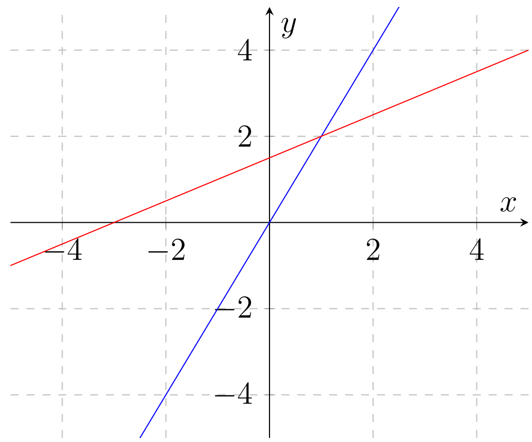
$$Ax = b$$

Where

$$\begin{aligned} A = &\text{ matrix of coefficients} \\ x = &\text{ matrix of unknowns} \\ b = &\text{ matrix of constants} \end{aligned}$$

If we think of this in **row picture form** we get a graph with x and y coordinates that satisfy each equation like...

Row picture



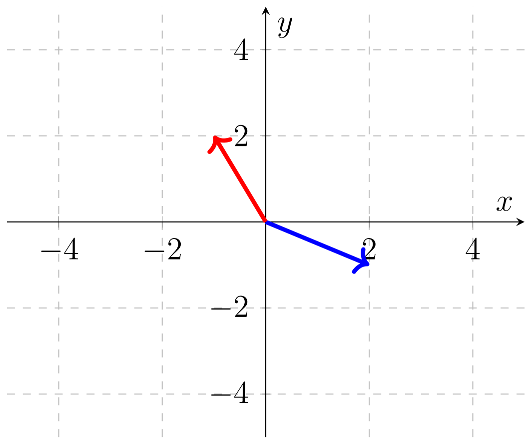
Where the **blue line** represents $2x - y = 0$ and the **red line** represents $-x + 2y = 3$

If we think of this in **column picture form** we get the following equation

$$x \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Which we refer to as the **linear combination** of the columns. This can be represented as a vector graph like so

Column picture form



Where the **blue** vector represents the column

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

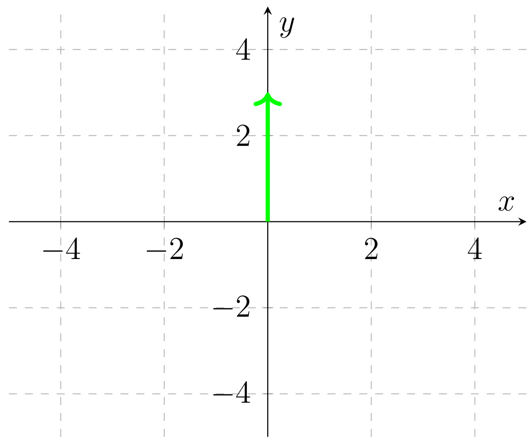
and the **red** vector represents the column

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Now we need to take a combination. Let $x = 1$ and $y = 2$

$$1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Resulting vector



What are all the combination of x and y that satisfy this equation?

Another example. Assume we're given this system of linear equations:

$$\begin{aligned} 2x - y + 0z &= 0 \\ -x + y - z &= -1 \\ 0x - 3y + 4z &= 4 \end{aligned}$$

In matrix for we get

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

In row picture form we get

Steps to solve in row picture form:

- Find all the points that would satisfy this equation
- In our example, some solutions would be:
  - $(x = 1, y = 0, z = 0)$

- $(z = 1, x = 0, y = 0)$
- $(x = 0, z = 0, y = -\frac{1}{2})$

Each row in a 3 by 3 gives us a plane in 3 dimensions. Those planes meet in one points, and that's the solution

Column picture form

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

We want to know what combination of the **left hand** vectors produces the **right hand vector**

It is obvious that the z vector is one combination that solves this system. So if we let $x = 0, y = 0, z = 1$ we have found our answer!

Now imagine the system looks like this

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

Now we've built the right hand vector by letting $x = 1, y = 1, z = 0$, and that would be one of our solutions!

Now, is it possible to **solve Ax=b for every b?**.

Changing the wording, we can ask: *do the* ***lniear combinations of the columns*** *fill 3 dimensional space?*

For the matrix A above, the answer is yes!

Let's think, when would the right hand vector not be produceable? If the 3 columns lie on the **same plane**, we would not be able to get the right vector. In other words, if one column was exactly the same as another column, we may not be able to get b

$$Ax = b$$

How to multiply a matrix by a vector?

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1(2) + 2(5) = 12 \\ 1(1) + 2(3) = 7 \end{bmatrix}$$

Ax is a combination of the columns of A